Apparent horizon and gravitational thermodynamics of the Universe: The temperature confusion, first and second laws, and extensions to modified gravity

David W. Tian

Faculty of Science, Memorial University, St. John’s, Newfoundland, Canada, A1C 5S7

Ivan Booth

Department of Mathematics and Statistics, Memorial University, St. John’s, Newfoundland, Canada, A1C 5S7

The thermodynamics of the Universe is re-studied by requiring its compatibility with the holographic-style gravitational equations which govern the dynamics of both the cosmological apparent horizon and the entire Universe. We start from the Lambda Cold Dark Matter (ΛCDM) cosmology of general relativity (GR) to establish a framework for the gravitational thermodynamics. The Clausius equation \( T_A dS_A = -A_A \psi_t \) for the isochoric process of an instantaneous apparent horizon indicates that, the Universe and its horizon entropies encode the Positive Out thermodynamic sign convention, which encourage us to adjust the traditional positive-heat-in Gibbs equation into the positive-heat-out version \( dE_m = -T_m dS_m - P_m dV \). It turns out that the standard and the generalized second laws (GSLs) of nondecreasing entropies are always respected by the event-horizon system as long as the expanding Universe is dominated by nonexotic matter \(-1 \leq w_m \leq 1\), while for the apparent-horizon simple open system the two second laws hold if \(-1 \leq w_m < -1/3\); also, the artificial local equilibrium assumption is abandoned in the GSL. All constraints regarding entropy evolution are expressed by the equation of state parameters, which show that from a thermodynamic perspective the phantom dark energy is less favored than the cosmological constant and the quintessence. Finally, the whole framework is extended from GR and ΛCDM to modified gravities with field equations \( R_{\mu\nu} - R g_{\mu\nu}/2 = 8\pi G_{\text{eff}} T_{\text{eff}}^{\mu\nu} \). Furthermore, this paper also proposes a solution to the temperature confusion and argues that the Cai-Kim temperature is more suitable than Hayward-Kodama, and both temperatures are independent of the inner or outer trappedness of the cosmological apparent horizon.

PACS numbers: 04.20.Cv , 04.50.Kd , 98.80.Jk

I. INTRODUCTION

The thermodynamics of the Universe is quite an interesting problem and has attracted a lot of discussion. Pioneering work dates back to the investigations of cosmic entropy evolutions for the de Sitter Universe [1] which is spatially flat and dominated by a positive cosmological constant, while recent studies have covered both the first and second laws of thermodynamics for the Friedmann-Robertson-Walker (FRW) Universe with a generic spatial curvature.

Recent interest on the first law of thermodynamics for the Universe was initiated by Cai and Kim’s derivation of the Friedmann equations from a thermodynamic approach [2]: this is actually a continuation of Jacobson’s work to recover Einstein’s equation from the equilibrium Clausius relation on local Rindler horizons [3], and also a part of the effort to seek the connections between thermodynamics and gravity [4] following the discovery of black hole thermodynamics [5]. For general relativity (GR), Gauss-Bonnet and Lovelock gravities, Akbar and Cai reversed the formulation in [2] by rewriting the Friedmann equations into the heat balance equation and the unified first law of thermodynamics at the cosmological apparent horizon [6]. The method of [6] was soon generalized to other theories of gravity to construct the total energy differentials by...
the corresponding modified Friedmann equations, such as scalar-tensor gravity in [7], $f(R)$ gravity in [8], the braneworld scenarios in [9, 10], the generic $f(R, \phi, \nabla_\alpha \phi \nabla^\alpha \phi)$ gravity in [11], and the Horava-Lifshitz gravity in [12]. Also, at a more fundamental level, the generic field equations of $F(R, \phi, -\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi, \mathcal{G})$ gravity are recast into the form of Clausius relation in [13].

Besides the first laws on the construction of various energy-conservation and heat-transfer equations, the entropy evolution of the Universe has also drawn plenty of attention. However, the cosmic entropies are almost exclusively studied in the generalized rather than the standard second laws. In fact, investigations via the traditional Gibbs equation $dE_m = T_m dS_m - P_m dV$ show that in GR and modified gravities, the evolution of the physical entropy $S_m$ for the matter inside the apparent and the event horizons departs dramatically from the desired nondecreasing behaviors. Thus the generalized second law (GSL) has been employed, which adds up $S_m$ with the geometrically defined entropy of the cosmological causal boundaries and anticipates the total entropy to be nondecreasing so that the standard second law could be rescued. For example, GSL has been studied in [14] for a flat Universe with multiple entropy sources (thermal, geometric, quantum etc.) by the entropy ansatz $S = |H|^\alpha$ ($\alpha > -3$), in [15] for the event-horizon system of a quintom-dominated flat Universe, and [16] for various interacting dark energy models.

Moreover, GSL has also been used as a validity constraint on modified and alternative theories of gravity. For instance, GSL has been imposed on the event-horizon system of the flat Universe of $f(R)$ gravity in [17], tentatively to the flat apparent-horizon system of generic modified gravities in [18], to the higher-dimensional Gauss-Bonnet and Lovelock gravity in [19], to the Gauss-Bonnet, Randall-Sundrum and Dvali-Gabadadze-Porrati braneworlds in [20], the Horava-Lifshitz gravity in [21], $R + f(\mathcal{G})$ gravity in [22], scalar-tensor-chameleon gravity in [23], and the self-interacting $f(R)$ gravity in [24]. Note that in the studies of GSL, the debatable “local equilibrium assumption” has been widely adopted which supposes that the matter content and the causal boundary in use (mainly the apparent or the event horizon) would have the same temperature.

Unlike laboratory thermodynamics which is a well-developed self-consistent framework, the thermodynamics of the Universe is practically a mixture of ordinary thermodynamics with analogous gravitational quantities, for which the consistency between the first and second laws and among the setups of thermodynamic functions are not yet verified. For example, the Hayward-Kodama temperature $\kappa / 2\pi$ [7, 9] or $|\kappa| / 2\pi$ [8, 10] which formally resembles the Hawking temperature of (quasi)stationary black holes [5] has been adopted in the first laws, while in GSLs both $|\kappa| / 2\pi$ [14–20, 23] and the Cai-Kim temperature [16, 21, 22] are used. Moreover, in existent literature we have noticed four conspicuous problems that are unclarified in the gravitational thermodynamics of the Universe: (1) For the Cai-Kim and the Hayward-Kodama temperatures, which one is more appropriate for the cosmic causal horizons? By solving this, the equations of total energy differential at the horizons could also be determined; (2) Is the standard second law for the physical matter content really ill-behaved and thus needs to be saved by GSL? (3) Is the local equilibrium assumption really necessary for GSL? (4) Are the thermodynamic quantities fully consistent with each other when the cosmic thermodynamics is systemized? In this paper, we will try to answer these question.

This paper is organized as follows. Starting with GR and the $\Lambda$CDM Universe, in Sec. II we derive the “holographic” gravitational equations which govern the dynamics of the apparent horizon and the cosmic spatial expansion, which imply the constraints from the EoS parameter $w_m$ on the evolution and metric signature of the apparent horizon. Section III demonstrates how these dynamical equations imply the unified first law of thermodynamics and the Clausius equation, and shows the latter encodes the positive-heat-out sign convention for the horizon entropy. In Sec. IV the Cai-Kim temperature is extensively compared with Hayward-Kodama, with the former chosen for further usage in Sec. V where we adjust the traditional Gibbs equation into the Positive Out convention to investigate the entropy evolution for the simple open systems enveloped by the apparent and event horizons. Finally the whole framework of gravitational thermodynamics is extended from $\Lambda$CDM model and GR to generic modified gravity in Sec. VI.1. Throughout this paper, we adopt the sign convention
\[ \Gamma^\gamma_{\beta\gamma} = \Gamma^\gamma_{\beta\gamma} , \quad R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} - \Gamma^\alpha_{\delta\gamma} \Gamma^\gamma_{\beta\epsilon} \] and \( R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \) with the metric signature \((-+, +++)\).

II. DYNAMICS OF THE COSMOLOGICAL APPARENT HORIZON

II.1. Apparent horizon and observable Universe

The FRW metric provides the most general description for the spatially homogeneous and isotropic Universe. In the \((t, r, \theta, \varphi)\) coordinates for an observer comoving with the cosmic Hubble flow, it has the line element (e.g. [25])

\[
\begin{align*}
\text{Eq. (1)} & \quad ds^2 = -dt^2 + \frac{a(t)^2}{1 - kr^2} dr^2 + a(t)^2 r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \\
& = h_{\alpha\beta} dx^\alpha dx^\beta + \gamma^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\end{align*}
\]

where \(a(t)\) refers to the scale factor, which is a generic function of the comoving time to be specified by the gravitational field equations. The index \(k\) denotes the normalized spatial curvature, with \(k = \{+1, 0, -1\}\) corresponding to closed, flat and open Universes, respectively. \(h_{\alpha\beta} := \text{diag}[-1, \frac{a(t)^2}{1 - kr^2}]\) represents the transverse two-metric spanned by \(x^\alpha = (t, r)\), and \(\gamma := a(t) r\) stands for the astronomical circumference/areal radius. Note that the total derivative of \(\gamma = \gamma(t, r)\) yields

\[
adr = d\gamma - H\gamma dt,
\]

where \(H\) refers to the time-dependent Hubble parameter of cosmic spatial expansion, and \(H := \frac{\dot{a}}{a}\) with the overdot denoting the derivative with respect to the comoving time \(t\). Equation (2) recasts the line element Eq. (1) into the \((t, \gamma, \theta, \varphi)\) coordinates as

\[
\begin{align*}
\text{Eq. (3)} & \quad ds^2 = \left(1 - \frac{k\gamma^2}{a^2}\right)^{-1} \left(- \left(1 - \frac{\gamma^2}{\gamma_A^2}\right) dt^2 - 2H\gamma dt d\gamma + d\gamma^2 \right) + \gamma^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\end{align*}
\]

although the comoving transverse coordinates \((t, r)\) for the FRW metric are easier to work with, we will switch to the more physical coordinates \((t, \gamma)\) whenever necessary. Based on Eq. (1), one can establish the following null tetrad adapted to the spherical symmetry and the null radial flow of the FRW spacetime,

\[
\ell^\mu = \left(1, \sqrt{1 - kr^2}, 0, 0\right), \quad n^\mu = \frac{1}{2} \left(1, -\sqrt{1 - kr^2}, 0, 0\right), \quad m^\mu = \frac{1}{\sqrt{2} \gamma} \left(0, 0, 1, \frac{i}{\sin \theta}\right),
\]

or the tetrad below with the null vectors

\[
\ell^\mu = \left(1, H\gamma + \sqrt{1 - \frac{k\gamma^2}{a^2}}, 0, 0\right), \quad n^\mu = \frac{1}{2} \left(1, H\gamma - \sqrt{1 - \frac{k\gamma^2}{a^2}}, 0, 0\right)
\]

for Eq. (3), and both tetrads have been adjusted to be compatible with the metric signature \((-+, +++)\) (e.g. Appendix B in [26]). By calculating the Newman-Penrose spin coefficients \(\rho_{\text{NP}} := -m^\mu m^\nu \nabla_\nu \ell_\mu\) and \(\mu_{\text{NP}} := m^\mu m^\nu \nabla_\nu n_\mu\) in either tetrad, the outward expansion rate \(\theta_{(\ell)} = -(\rho_{\text{NP}} + \mu_{\text{NP}})\) and the inward expansion \(\theta_{(n)} = \mu_{\text{NP}} + \bar{\mu}_{\text{NP}}\) are respectively found to be

\[
\theta_{(\ell)} = 2H + 2\gamma^{-1} \sqrt{1 - \frac{k\gamma^2}{a^2}}, \quad \theta_{(n)} = H - \gamma^{-1} \sqrt{1 - \frac{k\gamma^2}{a^2}}.
\]
For the expanding \((H > 0)\) Universe, \(\theta_{(\ell)}\) and \(\theta_{(o)}\) locate the apparent horizon \(\mathcal{T} = \mathcal{T}_A\) by the unique marginally inner trapped horizon at

\[
\mathcal{T}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} ,
\]

(7)

where \(\theta_{(\ell)} = 4H > 0, \theta_{(o)} = 0\), and also \(\partial_{\mu} \mathcal{T}\) becomes a null vector with \(g^{\mu\nu} \partial_{\mu} \mathcal{T} \partial_{\nu} \mathcal{T} = 0\) at \(\mathcal{T}_A\) [25]. Immediately the temporal derivative of Eq.(7) yields the kinematic equation

\[
\dot{\mathcal{T}}_A = -H \mathcal{T}^2_A \left( H - \frac{k}{a^2} \right) .
\]

(8)

Just like \(\mathcal{T}_A\) and \(\dot{\mathcal{T}}_A\), hereafter quantities evaluated on or related to the apparent horizon will be highlighted by the subscript \(A\).

Considering that \(\{l^\mu, n^\mu\}\) in Eqs.(4) and (5) coincide with the outgoing and ingoing tangent vector fields of the null radial congruence, ingoing signals emitted by light sources in the region \(\mathcal{T} > \mathcal{T}_A\) (where \(\theta_{(\ell)} > 0, \theta_{(o)} > 0\)) cannot travel through the marginally inner trapped apparent horizon \(\mathcal{T}_A\) to reach the comoving observer sitting at \(r = 0\). However, the region \(\mathcal{T} \leq \mathcal{T}_A\) is not necessarily the standard observable Universe in astronomy where ultrahigh redshift and visually superluminal recession can be detected [27, 28].

Note that we are working with the generic FRW metrics Eqs.(1) and (3) which allow for a nontrivial spatial curvature. This is not just for theoretical generality: in fact, astronomical observations indicate that the Universe may not be perfectly flat. For example, in the \(\Lambda\)CDM sub-model with a strict vacuum-energy condition \(w = -1\), the Wilkinson Microwave Anisotropy Probe (WMAP) nine-years data joint with other sources like the Baryon Acoustic Oscillations (BAO) evaluate the fractional energy density \(\Omega_k = -0.0027^{+0.0039}_{-0.0038}\) for the spatial curvature, independently the time-delay measurements of two strong gravitational lensing systems along with the seven-years WMAP data yield \(\Omega_k = 0.003^{+0.005}_{-0.006}\) [30], while most recently analyses based on BAO data give \(\Omega_k = -0.003 \pm 0.003\) [31].

II.2. "Holographic" dynamical equations

The matter content of the Universe is usually portrayed by a perfect-fluid type stress-energy-momentum tensor, and in the metric-independent form it reads

\[
T^{(m)}_{\mu\nu} = \text{diag}[-\rho_m, P_m, P_m, P_m] \quad \text{with} \quad P_m = w_m \rho_m ,
\]

(9)

where \(w_m := P_m/\rho_m\) refers to the equation of state (EoS) parameter. Substituting this \(T^{(m)}_{\mu\nu}\) and the metric Eq.(1) into Einstein’s equation \(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T^{(m)}_{\mu\nu}\), one obtains the first and the second Friedmann equations

\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_m \quad \text{and}
\]

\[
\dot{H} + \frac{k}{a^2} = -4\pi G (1 + w_m) \rho_m = -4\pi G h_m \quad \text{or} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi G P_m ,
\]

(10)

where \(h_m = \rho_m + P_m = (1 + w_m) \rho_m\) refers to the enthalpy density.

Primarily, the first and second Friedmann equations are respectively the first and second order differential equations of the scale factor \(a(t)\), which is the only unspecified function in the metric Eq.(1). On the other hand, recall the location and the time-derivative of the cosmological apparent horizon in Eqs. (7) and (8), and
thus Eq.\((10)\) can be rewritten into
\[
\Upsilon_A^{-2} = \frac{8\pi G}{3} \rho_m ,
\]
\[\tag{11}\]
\[
\dot{\Upsilon}_A = 4\pi G H \Upsilon_A^3 (1 + w_m) \rho_m = 4\pi G H \Upsilon_A^3 h_m ,
\]
\[\tag{12}\]
which manifest themselves as the dynamical equations of the apparent horizon. However, they have actually encoded the dynamics of spatial expansion for the entire Universe, so for this usage we will dub Eqs.\((11)\) and \((12)\) the “holographic” dynamical equations since they reflect the spirit of holography\(^1\).

Eq.\((11)\) immediately implies that, for the late-time Universe dominated by dark energy \(\rho_m = \rho_\Lambda\), the apparent horizon serves as the natural infrared cutoff for the holographic dark energy model \([33]\), in which the dark-energy density \(\rho_\Lambda^{(HG)}\) relies on the scale of the infrared cutoff \(\Upsilon_{IR}\) by \(\rho_\Lambda^{(HG)} = 3\Upsilon_{IR}^{-2} / (8\pi G)\). Moreover, with the apparent-horizon area \(A_A = 4\pi \Upsilon_A^2\), it follows from Eq.\((11)\) that
\[
\rho_m A_A = \frac{3}{2G} ,
\]
so Eq.\((12)\) can be further simplified into
\[
\dot{\Upsilon}_A = \frac{3}{2} H \Upsilon_A (1 + w_m) .
\]
\[\tag{14}\]
With the help of Eqs.\((11)\) and \((14)\), for completeness the third member (the \(P_m\) one) in Eq.\((10)\) can be directly translated into
\[
\frac{1}{\Upsilon_A^3} \left( \Upsilon_A - \frac{3}{2} H \Upsilon_A \right) = 4\pi G H P_m ,
\]
\[\tag{15}\]
and we keep it this form without further manipulations for later usage in Sec.\ III.1.

Eq.\((14)\) clearly shows that, for an expanding Universe \((H > 0)\) the apparent-horizon radius \(\Upsilon_A\) can be either expanding, contracting or even static, depending on the domain of the EoS parameter \(w_m\). In the \(\Lambda\)CDM cosmology, \(\rho_m\) could be decomposed into all possible components, \(\rho_m = \sum \rho_j^{(i)} = \rho_m(\text{baryon}) + \rho_m(\text{radiation}) + \rho_m(\text{neutrino}) + \rho_m(\text{dark matter}) + \rho_m(\text{dark energy}) + \cdots\), and the same for \(P_m\). In principle there should be an EoS parameter \(w_m^{(i)} = P_m^{(i)}/\rho_m^{(i)}\) associated to each energy component. However, practically we can regard \(w_m\) either as that of the absolutely dominating matter, or the weighted average of all relatively dominating components
\[
w_m = \frac{\sum \rho_j^{(i)}}{\rho_m} = \frac{\sum \rho_j^{(i)} P_j^{(i)}}{\rho_m P_m} = \sum \alpha_i w_m^{(i)} ,
\]
\[\tag{16}\]
with the weight coefficient given by \(\alpha_i = \rho_j^{(i)}/\rho_m\), and thus \(w_m\) varies over cosmic time scale. Then it follows from Eq.\((14)\) that:

\(^1\) We are using the word “holographic” in a generic sense as opposed to the standard terminology holographic principle in quantum gravity and string theory \([32]\).
The dominant energy condition $\rho_m \geq |P_m|$ imposes the constraint $-1 \leq w_m \leq 1$ for nonexotic matter. Here we retain the upper limit $w_m \leq 1$ but loosen the lower limit, allowing $w_m$ to cross the barrier $w_m = -1$ into the exotic phantom domain $w_m < -1$. The upper limit however is bracketed as $(\leq 1)$ to indicate that it is a physical rather than mathematical constraint.

II.3. Induced metric of the apparent horizon

The FRW metric Eq.(3) reduces to become a three-dimensional hypersurface in the $(t, \theta, \phi)$ coordinates at the apparent horizon $\Upsilon_A = \Upsilon_A(t)$, and with Eq.(14), the induced horizon metric turns out to be

$$ds^2 = (H\Upsilon_A)^{-2}((\dot{\Upsilon}_A - 2H\Upsilon_A)\dot{t}^2 + \dot{\Upsilon}_A^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2))$$

$$= \frac{9}{4}(w_m + 1)(w_m - \frac{1}{3})dt^2 + \Upsilon_A^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2). \quad (17)$$

Here the EoS parameter shows up in the coefficients of $dt^2$, and indeed the spirit of geometrodynamics allows and encourages physical parameters to directly participate in the spacetime metric, just like the mass, electric charge and angular momentum parameters in the Kerr-Newmann solution. It is easily seen that the signature of the apparent horizon solely relies on the domain of $w_m$ regardless of an expanding ($H > 0$) or contracting ($H < 0$) Universe.

(1) For $-1 < w_m < 1/3$, the apparent horizon $\Upsilon_A$ has the signature $(-,++)$ and is timelike, which shares the signature of a quasilocal *timelike membrane* in black-hole physics [38].

(2) For $w_m < -1$ or $1/3 < w_m \leq 1$, the signature is $(+,++)$ and thus $\Upsilon_A$ is spacelike. This situation has the same signature with the quasilocal *dynamical* black-hole horizons [39].

(3) For $w_m = -1$ or $w_m = 1/3$, the apparent horizon is a null surface with the signature $(0,++)$, so it coincides with the location of the cosmological event horizon $\Upsilon_E := a\int_0^\infty a^{-1}dt$ [25, 40] which by definition is a future-pointed null causal boundary, and it shares the signature of quasilocal *isolated* black-hole horizons [26].

Note that these analogies between $\Upsilon_A$ and the black-hole horizons are limited to the metric signature, while the behaviors of their $(\theta(t), \theta(r))$ and the horizon trappedness are entirely different. Among the two critical values, $w_m = -1$ corresponds to the de Sitter Universe dominated by a positive cosmological constant (or vacuum energy) [1], while $w_m = 1/3$ refers to the highly relativistic limit of $w_m$ and the EoS of radiation, with the trace of the the stress-energy-momentum tensor $g^{\mu\nu}T_{\mu\nu}^{(m)} = (3w_m - 1)\rho_m$ vanishing at $w_m = 1/3$. As will be shown later in Sec. [IV] $w_m = 1/3$ also serves as the “zero temperature divide” if the apparent-horizon temperature is measured by $\kappa/2\pi$ in terms of the Hayward-Kodama surface gravity $\kappa$. 

| $w_m$          | dominating matter                        | $\dot{\Upsilon}_A$ |
|----------------|------------------------------------------|---------------------|
| $-1/3 \leq w_m \leq 1$ and $-1 < w_m < -1/3$ | ordinary matter, and quintessence [34]  | $\dot{\Upsilon}_A > 0$, expanding |
| $w_m = -1$     | cosmological constant or vacuum energy [35] | $\dot{\Upsilon}_A = 0$, static |
| $w_m < -1$     | phantom [36]                             | $\dot{\Upsilon}_A < 0$, contracting |
II.4. Relative evolution equations

One could easily observe from Eq. (14) that for the relative evolution rate of the apparent-horizon radius $\frac{\dot{\Upsilon}_A}{\Upsilon_A}$, its ratio over that of the cosmic scale factor $\frac{\dot{a}}{a} = H$ synchronizes with the instantaneous value of the EoS parameter,

$$\frac{\dot{\Upsilon}_A}{\Upsilon_A} \bigg| \frac{\dot{a}}{a} = \frac{3}{2} (1 + w_m) ,$$

(18)

where the big slash / simply denotes “be divided by”. Equation (18) is not alone. Consider the conservation equation for the cosmic perfect fluid $\nabla_\mu T^{\mu(m)} = 0$ which arises from Einstein’s equation and the contracted Bianchi identities $\nabla_\mu G^\mu_v = 0$; only its $t$-component is nontrivial with respect to the metric Eq. (1) and leads to the continuity equation

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0 .$$

(19)

Eq. (19) immediately yields a companion for Eq. (18), as the relative evolution rate of the energy density $\frac{\dot{\rho}_m}{\rho_m}$ is normalized by $\frac{\dot{a}}{a}$ into

$$\frac{\dot{\rho}_m}{\rho_m} \bigg| \frac{\dot{a}}{a} = -3 (1 + w_m) .$$

(20)

Eqs. (18) and (20) reveal the interesting fact that throughout the history of the Universe, the relative evolution rate of the energy density $\frac{\dot{\rho}_m}{\rho_m}$ is always proportional to that of the apparent-horizon radius $\frac{\dot{\Upsilon}_A}{\Upsilon_A}$:

$$\frac{\dot{\rho}_m}{\rho_m} \bigg| \frac{\dot{\Upsilon}_A}{\Upsilon_A} = -2 .$$

(21)

In fact, integration of Eq. (21) yields $\ln \rho_m \propto -2 \ln \Upsilon_A$ and thus $\rho_m \propto \Upsilon_A^{-2}$, which matches the holographic dynamical equation (11) with the proportionality constant identified as $\frac{3}{8 \pi G}$.

III. THERMODYNAMIC IMPLICATIONS OF THE HOLOGRAPHIC DYNAMICAL EQUATIONS

Having seen the dynamical aspects of Eqs. (11), (12) and (14), we will continue to investigate their thermodynamic implications.

III.1. Unified first law of thermodynamics

The mass $M = \rho_m V$ of cosmic fluid within a sphere of radius $\Upsilon$, surface area $A = 4\pi \Upsilon^2$ and volume $V = \frac{4}{3} \pi \Upsilon^3$, can be geometrically recovered from the spacetime metric and we will identify it as the total internal energy $E$. Adopting the Misner-Sharp mass/energy [41] $E_{MS} := \frac{1}{2G} (1 - h^{\alpha\beta} \partial_\alpha \Upsilon \partial_\beta \Upsilon)$ for spherically symmetric spacetimes, Eq. (1) with $h^{\alpha\beta} = \text{diag}[-1, 1 - kr^2]$ for the Universe yields

$$E = \frac{\Upsilon^3}{2G \Upsilon_A^2} ,$$

(22)
and its equivalence with the physically defined mass $E = M = \rho_m V$ is guaranteed by Eq. (11). Equation (22) can also be reconstructed in the tetrad Eq. (4) or (5) from the Hawking energy $E_{\text{Hk}} := \frac{1}{4\pi G} \left( \int \left( -\Psi_2 - \sigma_{NP} \Lambda_{NP} + \Phi_{11} + \Lambda_{NP} \right) dA \right)^{1/2}$ for twist-free spacetimes. Immediately, the total derivative or transverse gradient of $E = E(t, r)$ is

$$dE = -\frac{1}{G} \frac{\Upsilon^3}{\Upsilon_A^3} \left( \dot{\Upsilon}_A - \frac{3}{2} H \Upsilon_A \right) dt + \frac{3}{2G} \frac{\Upsilon^2}{\Upsilon_A^2} adr \tag{23}$$

$$= -\frac{\dot{\Upsilon}_A}{G} \frac{\Upsilon^3}{\Upsilon_A^3} dt + \frac{3}{2G} \frac{\Upsilon^2}{\Upsilon_A^2} d\Upsilon , \tag{24}$$

where the relation $adr = d\Upsilon - H \Upsilon dt$ in Eq. (2) has been employed to rewrite Eq. (23) into Eq. (24), with the transverse coordinates from $(t, r)$ to $(t, \Upsilon)$. According to the holographic dynamical equations (11), (12) and (15), the energy differentials Eqs. (23) and (24) can be rewritten into

$$dE = -A \dot{\Upsilon} P \rho_m dt + A \rho_m adr \tag{25}$$

$$= -A (1 + w_m) \rho_m H \Upsilon dt + A \rho_m d\Upsilon . \tag{26}$$

Eqs. (25) and (26) can be formally compactified into

$$dE = A \psi + W dV , \tag{27}$$

where $\psi$ and $W$ are respectively the energy supply covector

$$\psi = -\frac{1}{2} \rho_m (1 + w_m) H \Upsilon dt + \frac{1}{2} \rho_m (1 + w_m) a dr \tag{28}$$

$$= -\rho_m (1 + w_m) H \Upsilon dt + \frac{1}{2} \rho_m (1 + w_m) d\Upsilon , \tag{29}$$

and the work density

$$W = \frac{1}{2} (1 - w_m) \rho_m . \tag{30}$$

Eq. (27) is exactly the unified first law of thermodynamics proposed by Hayward [43], and one can see from the derivation process that it applies to a volume of arbitrary areal radius $\Upsilon$, no matter $\Upsilon < \Upsilon_A$, $\Upsilon = \Upsilon_A$ or $\Upsilon > \Upsilon_A$. Moreover, $\psi$ and $W$ can respectively be traced back to the following scalar and covector invariants [43] of the matter tensor $T_{\mu\nu}^{(m)}$:

$$\psi_\alpha := T_{\alpha \beta \mu \nu} \partial_\beta \Upsilon + W \partial_\alpha \Upsilon \quad \text{with} \quad W := -\frac{1}{2} T_{(m) \alpha \beta} h_{\alpha \beta} , \tag{31}$$

which are valid for all spherically symmetric spacetimes besides FRW, and have Eqs. (28), (29) and (30) as their concrete components with respect to the metric Eq. (1).

### III.2. Clausius equation on the apparent horizon for an isochoric process

Having seen that the full set of holographic dynamical equations (11), (12) and (15) yield the unified first law $dE = A \psi + W dV$ for an arbitrary region in the FRW Universe, we will focus on the volume enclosed by
the apparent horizon $\Upsilon_A$. Firstly, Eq.(12) leads to
\[
\frac{\dot{\Upsilon}_A}{G} dt = A_A (1 + w_m) \rho_m H \dot{\Upsilon}_A dt ,
\]
and the left hand side can be manipulated into
\[
\frac{\dot{\Upsilon}_A}{G} dt = \frac{1}{2\pi \Upsilon_A} \left( \frac{2\pi \Upsilon_A \dot{\Upsilon}_A}{G} dt \right) = \frac{1}{2\pi \Upsilon_A} d \left( \frac{\pi \Upsilon_A^2}{G} \right) .
\]
(33)

Applying the geometrically defined Hawking-Bekenstein entropy \[5\], which is widely believed to validate for all horizons in GR (e.g. \[14\]), to the apparent horizon
\[
S_A = \frac{\pi \Upsilon_A^2}{G} = \frac{A_A}{4G} ,
\]
(34)

then employing the Cai-Kim ansatz \[2\] as the absolute temperature of the horizon,
\[
T_A = \frac{1}{2\pi \Upsilon_A} ,
\]
(35)

thus $T_A dS_A = \dot{\Upsilon}_A / G dt$ and Eq.(33) can be recast into
\[
T_A dS_A = \delta Q = - A_A \psi_t = - \left. dE \right|_{\Upsilon = 0} ,
\]
(36)

where $\psi_t$ is the $t$-component of the energy supply covector $\psi = \psi_t + \psi_\Upsilon = \psi_\alpha dx^\alpha$ in Eq.(29). Equation (36) is actually the Clausius equation for (quasi)equilibrium thermodynamic processes. Unlike nonrelativistic thermodynamics in which $\delta Q$ exclusively refers to the heat transfer (i.e. electromagnetic flow), the $\delta Q$ in Eq.(36) is used in a mass-energy-equivalence sense and denotes the generic energy flow of all types:
\[
\delta Q = \sum \delta Q^{(i)} = \delta Q(\text{radiation flux}) + \delta Q(\text{particle flux}) + \cdots .
\]
(37)

The energy-balance equation (36) implies that the region $\Upsilon \leq \Upsilon_A$ enveloped by the cosmological apparent horizon is thermodynamically an open system which exchanges both heat and matter (condensed components in the Hubble flow) with its surroundings/reservoir $\Upsilon \geq \Upsilon_A$. This is because $\Upsilon_A$ is simply a visual boundary of radial light signals for the comoving observer, rather than a real barrier for the cosmic Hubble flow.

Comparing Eq.(36) with the unified first law Eq.(27), one could find that Eq.(36) is just Eq.(27) with the two $d\Upsilon$ components removed and then evaluated at $\Upsilon_A$. Assuming that at an arbitrary moment $t = t_0$ the apparent horizon locates at $\Upsilon_{A0} := \Upsilon_A(t = t_0)$, then during the infinitesimal time interval $dt$ the horizon will move to $\Upsilon_{A0} + \dot{\Upsilon}_A dt$. Meanwhile, in the isochoric process ($d\Upsilon = 0$) for the volume of constant radius $\Upsilon_{A0}$ (i.e. a “controlled volume”), the amount of energy across the horizon $\Upsilon \equiv \Upsilon_{A0}$ during this $dt$ is just $dE = A_{A0} \psi_t$ evaluated at $t = t_0$, and for brevity we can directly drop the attendant subscript “0” since $t_0$ is arbitrary.

Finally, for the open system enveloped by the apparent horizon, we combine the Clausius equation (36) and the unified first law Eq.(27) into the total energy differential
\[
dE = A_A \psi_t dt + (A_A \psi_\Upsilon + W) d\Upsilon_A
= - T_A dS_A + \rho_m A_A d\Upsilon_A
= - T_A dS_A + \rho_m dV_A .
\]
(38)
In fact, by the continuity equation (19) one can verify that \(-T_A dS_A = V_A d\rho_m\), which agrees with the thermodynamic connotation that \(-T_A dS_A\) measures the loss of internal energy that can no longer be used to do work.

III.3. The minimum set of state functions and reversibility

Eqs. (34) and (35) clearly indicate that just like ordinary thermodynamics, the geometrically defined horizon temperature \(T_A\) and horizon entropy \(S_A\) remain as state functions, which are independent of thermodynamic processes that indeed correspond to the details of cosmic expansion \(\dot{a}(t)\) and radius evolution \(\dot{Y}_A\). Some other state functions involved here include the apparent-horizon radius \(Y_A\), the energy density \(\rho_m(t)\), the pressure \(P_m(t)\) and thus the EoS parameter \(w_m = \rho_m/P_m\). These state quantities are not totally independent as they are related with one another by the Friedmann equations (10), the holographic dynamical equation (11), and the thermodynamic relations in Secs. III.1 and III.2. Here we select the following quantities to comprise a minimum set of independent state functions for Secs. III and III.

\[
\text{Minimum set} = \{\rho_m, w_m, T_A\}.
\]  

(39)

Based on this set, the product of \(\rho_m\) and \(w_m\) yields the pressure \(P_m\). Through Eq. (13) \(\rho_m\) recovers the horizon area \(A_A\) and thus determines the entropy \(S_A\). Treating \(T_A\) as an intensive property, we do not take the approach from Eq. (11) or Eq. (13) for the recovery \(\rho_m \rightarrow A_A \rightarrow Y_A \rightarrow T_A\), and instead let \(T_A\) enter the minimum set directly as the Cai-Kim temperature ansatz.

The fact that Eq. (36) is the Clausius equation for (quasi)equilibrium or reversible thermodynamic processes without extra entropy production raises the question that, what does reversibility mean from the perspective of cosmic and apparent-horizon dynamics? From the explicit expression of the heat transfer \(\delta Q = T_A dS_A = A_A (1 + w_m) \rho_m H Y_A dt\) where the state quantity \(T_A dS_A\) is balanced by the process quantity \(\delta Q\), we naturally identify \(H\) as a process quantity; moreover, if one reverses the initial and final states of \(T_A dS_A\), the state quantities \(\{\rho_m(t), w_m, Y_A, A_A\}\) can be automatically reversed. Hence, by reversibility we mean an imaginary negation \(-H\) of the Hubble parameter that results in a spatial contraction process which directly evolves the Universe from a later state back to the earlier state of \(T_A dS_A\) without reversing the time arrow and causing energy dissipation.

[44] suggests that since the energy-matter crossing the apparent horizon for the (accelerated) expanding Universe will not come back in the future, it should cause extra entropy production, and [44] further introduced the entropy flow vector and the entropy production density for it. In fact, the reversibility of \(T_A dS_A = \delta Q\) simply allows for such a possibility in principle, rather than the realistic occurrence of the reverse process, so we believe that the entropy-production treatment in [44] is inappropriate. As will be shown later in Sec. VI.3, irreversibility and entropy production is a common feature for such (minimally coupled) modified gravities with a nontrivial effective gravitational coupling strength \((G_{\text{eff}} \neq \text{constant})\) when their field equations are cast into the GR form \(R_{\mu\nu} - R g_{\mu\nu}/2 = 8 \pi G_{\text{eff}} T^{(\text{eff})}_{\mu\nu}\), and the time evolution of \(G_{\text{eff}}\) causes irreversible energy dissipation and constitutes the only source of entropy production.

III.4. Thermodynamic sign convention

Following Secs. III.1 and III.2 we will retrieve the thermodynamic sign convention encoded in the Clausius equation \(T_A dS_A = \delta Q = -A_A \psi_t\), which will become a corner stone for studying the entropy evolution in

\[\psi_t = \frac{1}{16 \sqrt{3} \pi} \left( \frac{1}{V_A} \int_0^t \dot{Y}_A \, dt \right)\]

2 Just like the regular temperatures of thermodynamic systems, the geometric Cai-Kim temperature of the apparent horizon remains an intensive property with \(T_A = T_A(t) = 1/2 \pi Y_A(t)\); one should not treat it as an extensive property by \(T_A = T_A(V_A) = 1/(2 \pi \sqrt{V_A})\).
Firstly let’s check whether the heat flow $\delta Q$ and the isochoric energy differential $dE$ take positive or negative values. $\delta Q$ will be calculated by $T_A dS_A$, while $dE$ is to be evaluated independently via $A_A \psi_t = -A_A (1 + w_m) \rho_m H T_A dt$. Hence, in the isochoric process for an instantaneous apparent horizon $\Upsilon_A$, 

$$T_A dS_A = \frac{\dot{\Upsilon}_A}{G} dt = \frac{3}{2G} H T_A (1 + w_m) dt, \quad (40)$$

$$dE = -A_A \rho_m (1 + w_m) H T_A dt = -\frac{3}{2G} (1 + w_m) H T_A dt, \quad (41)$$

where Eqs. (13) and (14) have been used to replace $A_A \rho_m$ and $\dot{\Upsilon}_A$, respectively. For an expanding Universe ($H > 0$), this clearly shows that:

1. If the Universe is dominated by ordinary matter and quintessence, $-1 < w_m (\leq 1)$, the internal energy is decreasing $dE = A_A \psi_t < 0$, with a positive Hubble energy flow $\delta Q = T_A dS_A > 0$ going outside to the surroundings;

2. Under the dominance of the cosmological constant, $w_m = -1$ and $\{\rho_m, T_A, T_A, S_A\} =$ constant; the internal energy is unchanging, $dE = A_A \psi_t = 0$ and $\delta Q = T_A dS_A = 0$;

3. When the Universe enters the phantom-dominated state, $w_m < -1$, the internal energy increases $dE = A_A \psi_t > 0$ while $\delta Q = T_A dS_A < 0$.

Hence, based on the intuitive behaviors at the domain $-1 < w_m (\leq 1)$ for nonexotic matter, we set up the positive heat out thermodynamic sign convention for the right hand side of $dE = -\delta Q$. That is to say, heat evolved by the system takes positive values ($\delta Q_{\text{out}} = \delta Q > 0$), while heat absorbed by the system takes negative values. Obviously, this setup is totally consistent with the situations of $w_m \leq -1$. Note that because of the counterintuitive behaviors under phantom dominance, one should not take it for granted without concerning the dependence on $w_m$ that, for a spatially expanding Universe the cosmic fluid were always to flow out of the isochoric volume $V(\Upsilon = \Upsilon_0)$ with $dE = V_A d\rho_m < 0$.

### IV. SOLUTION TO THE HORIZON-TEMPERATURE CONFUSION

#### IV.1. The horizon-temperature confusion

In the thermodynamics of (quasi)stationary black holes [5], the Hawking temperature satisfies $T = \frac{k}{(2\pi)}$ based on the traditional Killing surface gravity $k$ and the Killing generators of the horizon. For the FRW Universe, one has the Hayward-Kodama inaffinity parameter $\kappa$ [43] in place of the Killing inaffinity, which yields the Hayward-Kodama surface gravity on the apparent horizon,

$$\kappa := \frac{1}{2} h^{\alpha\beta} \nabla_\alpha \nabla_\beta \Upsilon = \frac{1}{2 \sqrt{-g}} \partial_\alpha \left( \sqrt{-g} h^{\alpha\beta} \partial_\beta \Upsilon \right) \equiv -\frac{\Upsilon}{\Upsilon_A^2 \left( 1 - \frac{\dot{\Upsilon}_A}{2 H \Upsilon_A} \right)} = \frac{1}{\Upsilon_A} \left( 1 - \frac{\dot{\Upsilon}_A}{2 H \Upsilon_A} \right) \Upsilon_A, \quad (42)$$

where $h_{\alpha\beta} = \text{diag}[1, -1, -1]$ refers to the transverse two-metric in Eq. (1). Then formally following the Hawking temperature, the Hayward-Kodama temperature of the apparent horizon $\Upsilon_A$ is defined either by [7, 9]

$$\Upsilon_A := \frac{\kappa}{2\pi} = \frac{1}{2\pi \Upsilon_A} \left( 1 - \frac{\dot{\Upsilon}_A}{2 H \Upsilon_A} \right), \quad (43)$$
or \([8, 10, 18–20, 23, 24]\)

\[
T_A^{(+) := \left( \kappa \right) = \frac{1}{2\pi T_A} \left( 1 - \frac{\dot{T}_A}{2H T_A} \right) .
\]

(44)

Note that we use the symbol \((\kappa)\) to denote the partial absolute value of \(\kappa\), because existing papers have a priori abandoned the possibility of \(\dot{T}_A/(2H T_A) \geq 1\) for \(T_A^{(+)\). Equation (44) is always supplemented by the assumption \([8, 10, 18–20, 23, 24]\)

\[
\frac{\dot{T}_A}{2H T_A} < 1
\]

(45)

to guarantee a positive \(T_A^{(+)\)} which is required by the third law of thermodynamics, and even the condition \([18]\)

\[
\frac{\dot{T}_A}{2H T_A} \ll 1
\]

(46)

so that \(T_A^{(+)\)} can be approximated into the Cai-Kim temperature

\[
T_A^{(+) \approx \frac{1}{2\pi T_A} = T_A .
\]

(47)

Historically the inverse problem “from thermodynamics to gravitational equations for the Universe” \([2]\) was formulated earlier, in which the Cai-Kim temperature works perfectly for all theories gravity. Later on, the problem “from gravitational equations to thermodynamics” for FRW \([6–8]\) (as is studied in this paper) came into attention in which the Hayward-Kodama temperature seems to become effective. Considering that two different temperatures make the two mutually inverse problems asymmetric, attempts have been made to reduce the differences between them, mainly the assumptions Eq.(45) and (46).

Note that when the conditions Eqs.(46) and (47) are applied to Eq.(43), \(T_A\) would become a negative temperature. \([7]\) has suggested that one can understand this phenomenon as a consequence of the cosmological apparent horizon being inner trapped \([\theta(t) > 0, \theta(n) = 0]\), as opposed to the positive temperatures of black-hole apparent horizons which are always marginally outer trapped \([\theta(t) = 0, \theta(n) < 0]\). However, this proposal proves inappropriate; as will be shown in Sec. IV.3 the signs of \(T_A\) actually keep pace with the metric signatures rather than the inner/outer trappedness \([38, 46]\) of the horizon \(Y_A\).

IV.2. Effects of \(T_A dS_A\) and \(T_A^{(+)dS_A}\)

In Sec. III.2 we have seen \(T_A dS_A = A_A \psi_t\) for the Cai-Kim ansatz \(1/(2\pi T_A)\), and now let’s examine the effects of \(T_A\) and \(T_A^{(+)\). Given the Bekenstein-Hawking entropy \(S_A = A_A/4G\), the dynamical equation \(\dot{Y}_A = A_A H Y_A G(1 + w_m)\rho_m\) and the energy supply covector \(\psi = \psi_t + \psi_T = -(1 + w_m)\rho_m H Y_A dt + \frac{1}{2}(1 + w_m)\rho_m dY_A\), one has

\[
T_A dS_A = -\frac{\dot{Y}_A}{G} + \frac{\dot{Y}_A}{2GH Y_A} Y_A dt
\]

\[
= -A_A H Y_A (1 + w_m)\rho_m dt + \frac{1}{2}A_A (1 + w_m)\rho_m dY_A
\]

(48)

\[
= A_A \psi_t + A_A \psi_T
\]

\[
= A_A \psi .
\]
Similarly, for the $\mathcal{T}_{\lambda}^{(+)}$ defined in Eq. (44),

$$
\mathcal{T}_{\lambda}^{(+)}dS_{\lambda} = -(A_{\lambda}\psi_{t} + A_{\lambda}\psi_{r}) = -A_{\lambda}\psi.
$$

Hence, for the two terms comprising $\mathcal{T}_{\lambda}$ and $\mathcal{T}_{\lambda}^{(+)}$, $\pm \frac{1}{2\pi\Gamma_{A}}dS_{\lambda}$ is balanced by $\pm A_{\lambda}\psi_{t}$, while $\pm \frac{\dot{\psi}}{2\pi\Gamma_{A}}dS_{\lambda}$ is equal to $\pm A_{\lambda}\psi_{r}$. For the open system enveloped by the cosmological apparent horizon, combining Eqs. (48) and (49) with the unified first law Eq. (27) leads to the total energy differential

$$
dE = \mathcal{T}_{\lambda}dS_{\lambda} + WdV_{\lambda}
$$

$$
= -\mathcal{T}_{\lambda}^{(+)}dS_{\lambda} + WdV_{\lambda},
$$

as oppose to $dE = -T_{\lambda}dS_{\lambda} + \rho_{m}dV_{\lambda}$ for the Cai-Kim $T_{\lambda}$.

**IV.3. “Zero temperature divide” $w_{m} = 1/3$ and preference of Cai-Kim temperature**

Now apply the dynamical equation (14) to ($\mathcal{T}_{\lambda}$, $\mathcal{T}_{\lambda}^{(+)}$) and the assumptions in Eqs. (45) and (46). With $\dot{\Gamma}_{A} = \frac{3}{2}HT_{A}(1 + w_{m})$, the Hayward-Kodama surface gravity becomes

$$
\kappa = -\frac{1}{\Gamma_{A}}\left(1 - \frac{\dot{\Gamma}_{A}}{2HT_{A}}\right) = -\frac{3}{4\Gamma_{A}}\left(\frac{1}{3} - w_{m}\right),
$$

so it follows that

$$
\kappa \sim \begin{cases} 
> 0, & w_{m} > \frac{1}{3} \\
= 0, & w_{m} = \frac{1}{3} \\
< 0, & w_{m} < \frac{1}{3}
\end{cases}
$$

and

$$
|\kappa| \equiv \frac{1}{4\Gamma_{A}}\left|\frac{1}{3} - w_{m}\right| \equiv \begin{cases} 
\frac{3}{4\Gamma_{A}}(w_{m} - \frac{1}{3}), & w_{m} > \frac{1}{3} \\
0, & w_{m} = \frac{1}{3} \\
\frac{3}{4\Gamma_{A}}\left(\frac{1}{3} - w_{m}\right), & w_{m} < \frac{1}{3}
\end{cases}.
$$

The Hayward-Kodama temperature $\mathcal{T}_{\lambda}$ in Eq. (43) and its partially absolute value $\mathcal{T}_{\lambda}^{(+)}$ in Eq. (44) become

$$
\mathcal{T}_{\lambda} := \frac{\kappa}{2\pi} = -\frac{3}{8\pi\Gamma_{A}}\left(\frac{1}{3} - w_{m}\right) = -\frac{1}{4}\Gamma_{A}(1 - 3w_{m})
$$

$$
\mathcal{T}_{\lambda}^{(+)} := \frac{\kappa}{2\pi} = \frac{3}{8\pi\Gamma_{A}}\left(\frac{1}{3} - w_{m}\right) = \frac{1}{4}\Gamma_{A}(1 - 3w_{m}),
$$

and thus, following the discussion in Sec. III.3, fortunately $\mathcal{T}_{\lambda}$ and $\mathcal{T}_{\lambda}^{(+)}$ remain as state functions. Moreover, the supplementary assumption Eq. (45) for $\mathcal{T}_{\lambda}^{(+)} > 0$ turns out to be

$$
\frac{\dot{\Gamma}_{A}}{2HT_{A}} = \frac{3}{4}(1 + w_{m}) < 1 \implies w_{m} < 1/3.
$$

Thus the condition $\frac{\dot{\Gamma}_{A}}{2HT_{A}} \ll 1$ in Eq. (46) could be directly translated into $w_{m} \ll 1/3$, which is however inaccurate: in fact, if directly starting from Eq. (53), one will find that the approximation $\mathcal{T}_{\lambda}^{(+)} \approx T_{\lambda} = 1/(2\pi\Gamma_{A})$ requires

$$
w_{m} \to -1. \hspace{1cm} (55)
$$
Moreover, the Cai-Kim temperature makes it more practical to study the entropy evolution for the Universe, as a thermodynamic state quantity of fundamental importance, it is much more elegant and concise to work with formulations of the conjugate problems “gravity to thermodynamics” and “thermodynamics to gravity”, and as logical apparent horizon.

It is neither mathematically nor physically identical with \( w_m \ll 1/3 \) which could only be perfectly satisfied for \( w_m \to -\infty \) in the extreme phantom domain.

Based on Eqs. (51) - (55), an extensive comparison between the Hayward-Kodama temperatures \( \{ \mathcal{T}_A, \mathcal{T}_A^{(+)} \} \) with the Cai-Kim ansatz \( T_A = 1/(2\pi \Upsilon_A) \) reveals the following facts.

1. \( \mathcal{T}_A \) is negative definite for \( 1/3 < w_m (\leq 1) \), positive definite for \( w_m < 1/3 \), and \( \mathcal{T}_A \equiv 0 \) for \( w_m = 1/3 \). We will dub the special value \( w_m = 1/3 \) as the Hayward-Kodama “zero temperature divide”, which is inspired by the terminology “phantom divide” for \( w_m = -1 \) in dark-energy physics [35]. Hence, \( \mathcal{T}_A \) does not respect the third law of thermodynamics. Moreover, one has \( \mathcal{T}_A = 0 \) at \( w_m = 1/3 \) and thus \( \mathcal{T}_A dS_A = 0 \); following Eq. (48), this can be verified by

\[
A_A \psi = -A_A H \Upsilon_A (1+w_m)\rho_m dt + \frac{1}{2} A_A (1+w_m) \rho_m d\Upsilon_A
\]

\[
= A_A \rho_m (1+w_m) \left( \frac{1}{2} \Upsilon_A - H \Upsilon_A \right) dt
\]

\[
= \frac{9}{8G} H \Upsilon_A (1+w_m) (w_m - \frac{1}{3}) dt.
\]

2. The condition \( w_m < 1/3 \) for the validity of \( \mathcal{T}_A^{(+)} \) is too restrictive and unnatural, because \( w_m = 1/3 \) serves as the EoS of radiation and (1) \( w_m > 1/3 \) represents all highly relativistic energy components.

3. The equality \( \mathcal{T}_A dS_A = A_A (\psi_t + \psi_r) = -\mathcal{T}_A^{(+)} dS_A \) implies that \( \mathcal{T}_A dS_A \) and \( \mathcal{T}_A^{(+)} dS_A \) need to be balanced by \( dt \) and also the \( d\Upsilon_A \) component of \( \psi \), and thus the other \( d\Upsilon_A \) component from \( WdV_A = WA_A d\Upsilon_A \) should be nonvanishing as well. Hence, \( \mathcal{T}_A dS_A \) and \( \mathcal{T}_A^{(+)} dS_A \) always live together with \( WdV_A \) to form the total energy differential Eq. (50) rather than some Clausius-type equation \( \delta Q = \mathcal{T}_A^{(+)} dS_A = -\mathcal{T}_A dS_A = -A_A \psi \), and there exists no isochoric process (\( d\Upsilon = 0 \)) for \( \{ \mathcal{T}_A, \mathcal{T}_A^{(+)} \} \).

4. The “highly relativistic limit” \( w_m = 1/3 \) is more than the divide for negative, zero or positive Hayward-Kodama temperature \( \mathcal{T}_A \); it is also the exact divide for the induced metric of the apparent horizon to be spacelike, null or timelike, as discussed before in Sec. II.3. That is to say, the sign of the temperature synchronizes with the signature of the horizon metric. However, there are no such behaviors for analogies in black-hole physics: for example, a slowly-evolving quasilocal black-hole horizon [38, 47] can be either spacelike, null or timelike, but the horizon temperature is always positive definite regardless of the horizon signature.

5. Unlike the Cai-Kim temperature \( T_A \), the Hayward-Kodama \( \{ \mathcal{T}_A, \mathcal{T}_A^{(+)} \} \) used for the problem “from gravitational equations to thermodynamic relations for the Universe” do not work for “from thermodynamic relations to gravitational equations”. That is to say, \( \{ \mathcal{T}_A, \mathcal{T}_A^{(+)} \} \) break the symmetry between the formulations of these two mutually inverse problems.

6. The complicated expressions of \( \{ \mathcal{T}_A, \mathcal{T}_A^{(+)} \} \) in Eqs. (43), (44) and (53) look incongruous with the compact horizon entropy \( S_A = A_A/4G \).

With these considerations, we adopt the Cai-Kim ansatz \( T_A = 1/(2\pi \Upsilon_A) \) for the temperature of the cosmological apparent horizon. \( T_A \) is positive definite throughout the history of the Universe, it provides symmetric formulations of the conjugate problems “gravity to thermodynamics” and “thermodynamics to gravity”, and as a thermodynamic state quantity of fundamental importance, it is much more elegant and concise to work with. Moreover, the Cai-Kim temperature makes it more practical to study the entropy evolution for the Universe, as
will be shown in next section. This way, we believe that the temperature confusion is solved as the Cai-Kim ansatz is favored.

Furthermore, imagine a contracting Universe with $\dot{a} < 0$ and $H < 0$, and one would have a marginally outer trapped apparent horizon with $\theta(t) = 0$ and $\theta_m = 2H < 0$ at $\gamma = \gamma_A$. Hence, whether $\gamma = \gamma_A$ is outer or inner trapped only relies on the Hubble parameter to be positive or negative. In Sec. [13] we have seen that the induced-metric signature of $\gamma_A$ is independent of $H$, and neither will the Hayward-Kodama $\{\gamma_A, T_A^{(+)}\}$. Also, Eqs. (40) and (41) clearly show that, the equality $−T_A dS_A = A_A \psi_t = dE_t$ of the Cai-Kim $T_A$ validates for either $H > 0$ or $H < 0$. Hence, we further conclude that:

Neither the sign of the Hayward-Kodama nor the Cai-Kim temperature is related to the inner or outer trappedness of the cosmological apparent horizon.

V. THE (GENERALIZED) SECOND LAWS OF THERMODYNAMICS

Having studied the differential forms of the energy conservation and heat transfer, clarified the Positive Out convention and distinguished the temperature of causal boundaries, we will proceed to investigate the entropy evolution for the Universe. In existent papers, this problem is generally studied independently from the first laws, and the entropy $\bar{S}_m$ of the cosmic energy-matter content (with temperature $T_m$) is always determined by the traditional Gibbs equation $dE = T_m d\bar{S}_m − P_m dV$ (e.g. [14, 15]). It turns out that $\bar{S}_m$ is far from well-behaved, so people turn to the generalized second law (GSL) for help, which adds up $\bar{S}_m$ with the geometric entropy of the cosmological apparent or event horizons.

This popular treatment is very problematic. In fact, the equation $dE_m = T_m d\bar{S}_m − P_m dV$ encodes the “positive heat in, positive work out” convention for the physical entropy $\bar{S}_m$ and the heat transfer $T_m d\bar{S}_m$. However, as extensively discussed in Sec. [III] the geometric Bekenstein-Hawking entropy for the cosmological causal boundaries, such as $S_A = A_A/4G$ for the apparent horizon, is compatible with the totally Positive Out convention. One cannot add the traditional positive-heat-in $\bar{S}_m$ with the positive-heat-out $S_A$, and this confliction leads us to adjust the Gibbs equation into

$$dE_m = −T_m dS_m − P_m dV, \quad (57)$$

where $S_m$ is defined in the positive heat out convention favored by the Universe for consistency with the gravitational equations (11), (12) and (15). This way, one can feel free and safe to superpose the matter entropy $S_m$ and the horizon entropy $\{S_A$, etc.$\}$, and more pleasantly, and it turns out that this $S_m$ is very well behaved.

For the energy $E = M = \rho_m V$ in an arbitrary volume $V = \frac{4}{3} \pi R^3 = \frac{1}{3} A T$, Eq. (57) yields $T_m dS_m = −d(\rho_m V) − P_m dV = −V d\rho_m − (\rho_m + P_m)dV$, and thus

$$T_m dS_m = \left. 3H(\rho_m + P_m) V dt − (\rho_m + P_m) A d\gamma \right\} = \rho_m A (1 + \omega_m)(H T dt − d\gamma), \quad (58)$$

where the continuity equation (19) has been used. Based on Eq. (58), we can analyze the entropy evolution $\dot{S}_m$ for the matter inside some special radii such as the apparent and event horizons. Note that these regions are generally open thermodynamic systems with the Hubble energy flow crossing the apparent and possibly the event horizons, so one should not $a$ priori anticipate $\dot{S}_m \geq 0$; instead, we will look for the circumstances where $\dot{S}_m \geq 0$ conditionally holds.

Note that for the GSL of black holes, there is no such confliction because both the black-hole horizon entropy and the matter entropy are defined in the positive heat in convention.
V.1. The second law for the interior of the apparent horizon

For the matter inside the apparent horizon $\Upsilon = \Upsilon_A(t)$, Eq. (58) along with the holographic dynamical equations (13) and (14) yield

$$T_m dS_m^{(A)} = \rho_m A_A(1 + w_m)(HY_A - \dot{\Upsilon}_A)dt$$
$$= \frac{3}{2G}(1 + w_m)H\Upsilon_A(1 - \frac{3}{2}(1 + w_m))dt$$
$$= -\frac{9}{4G}HT_A(w_m + 1)(w_m + \frac{1}{3})dt.$$  \hspace{1cm} (59)

Apparently the second law of thermodynamics $\dot{S}_m^{(A)} \geq 0$ holds for $-1 \leq w_m \leq -1/3$. Moreover, recall that the spatial expansion of the FRW Universe satisfies

$$\ddot{a} = -\frac{4\pi G}{3}(1 + 3w_m)\rho_m,$$  \hspace{1cm} (60)

with $\ddot{a} > 0$ for $w_m < -1/3$. Hence, within GR and the $\Lambda$CDM model, we have:

The physical entropy $S_m^{(A)}$ inside the cosmological apparent horizon satisfies $\dot{S}_m^{(A)} \equiv 0$ when $w_m = -1/3$ or under the dominance of the cosmological constant $w_m = -1$, while $\dot{S}_m^{(A)} > 0$ for the stage of accelerated expansion ($\ddot{a} > 0$) dominated by quintessence $-1 < w_m < -1/3$.

Also, when $\dot{S}_m^{(A)} \geq 0$, the dominant energy condition ($\rho_m > 0$, $-1 \leq w_m \leq 1$) is respected as well.

V.2. The second law for the interior of the event horizon

For the cosmic fluid inside the future-pointed cosmological event horizon $\Upsilon_E := a \int_t^\infty a^{-1} dt$, which satisfies

$$\dot{\Upsilon}_E = H\Upsilon_E - 1.$$  \hspace{1cm} (61)

Eq. (58) leads to

$$T_m dS_m^{(E)} = \rho_m A_E(1 + w_m)(H\Upsilon_E - \dot{\Upsilon}_E)dt$$
$$= \rho_m A_E(1 + w_m)dt.$$  \hspace{1cm} (62)

Hence, we are happy to see that:

The physical entropy $S_m^{(E)}$ inside the cosmological event horizon satisfies $\dot{S}_m^{(E)} \equiv 0$ if the Universe is dominated by the cosmological constant $w_m = -1$, while $\dot{S}_m^{(E)} > 0$ for all nonexotic matter $-1 < w_m (\leq 1)$ above the phantom divide.

The importance of this result can be best seen for a closed ($k = 1$) Universe, when the event horizon $\Upsilon_E$ has a finite radius and bounds the entire spacetime. Then the physical entropy of the whole Universe is nondecreasing as long as the dominant energy condition holds $-1 \leq w_m (\leq 1)$.

Note that with the traditional Gibbs equation $dE_m = T_m d\bar{S}_m - P_m dV$ (e.g. [14, 15]) where $\bar{S}_m$ is defined in
the positive heat in convention, for the interior of the cosmological event horizon one would obtain

\[ T_m d\tilde{S}_m^{(E)} = dE_m^{(E)} + P_m dV_E = -\rho_m A_{E}(1 + w_m) dt . \]  

(63)

It would imply that \( \dot{\tilde{S}}_m > 0 \) would never be realized unless the Universe were to evolve into an exotically phantom-dominated \((w_m < -1)\) stage during its long history. We believe that Eq. (62) provides a more reasonable description for the cosmic entropy evolution than Eq. (63), especially for the spatially closed and thus finite Universe, and we regard this result as a support to the adjusted Gibbs equation (57).

V.3. GSL for the apparent-horizon system

Historically, to rescue the disastrous result of Eq. (63) in traditional studies, the \textit{generalized} second law (GSL) for the thermodynamics of the Universe was developed, which adds up the geometric entropy of the cosmological causal boundaries (e.g. \( S_A, S_E \)) to the physical entropy of the matter-energy content \( S_m \), aiming to make the total entropy nondecreasing under certain conditions. This idea is inspired by the GSL of black-hole thermodynamics [45], for which Bekenstein postulated that the black-hole horizon entropy plus the external matter entropy never decrease (for a thermodynamic closed system).

Eqs. (58) and (62) clearly indicate that the second law \( \dot{S}_m > 0 \) is well respected in our formulation, but for completeness we will still investigate the GSLs. For the simple open system consisting of the cosmological apparent horizon \( T_A \) and its interior, Eqs. (34) and (59) yields

\[ \dot{S}_m^{(A)} + \dot{S}_A = -\frac{1}{T_m} \frac{9}{4G} H T_A (w_m + 1)(w_m + \frac{1}{3}) + \frac{2\pi T_A \dot{T}_A}{G} \]  

\[ = -\frac{1}{T_m} \frac{9}{4G} H T_A (w_m + 1)(w_m + \frac{1}{3}) + \frac{1}{3} \frac{3}{2G} H T_A (w_m + 1) . \]  

(64)

In existing papers it is generally assumed that the apparent horizon would be in thermal equilibrium with the matter content and thus \( T_A = T_m \) [16, 19, 22], or \( T_m = b T_A \) (\( b = \text{constant} \)) [17, 18]. However, such assumption is too reluctant and unconvincing, so we directly move ahead from Eq. (64) without any artificial speculations relating \( T_A \) and \( T_m \).

The GSL \( \dot{S}_m^{(A)} + \dot{S}_A \geq 0 \) requires that \( \frac{1}{T_A} \frac{3}{4G} H T_A (w_m + 1) \geq \frac{1}{T_m} \frac{9}{4G} H T_A (w_m + 1)(w_m + \frac{1}{3}) \), and with \( \{H, T_A, T_A, T_m\} > 0 \) it leads to

\[ (w_m + 1) \left( \frac{T_m}{T_A} - \frac{3}{2} (w_m + \frac{1}{3}) \right) \geq 0 . \]  

(65)

or equivalently \((w_m + 1)(T_m - \frac{3}{2}(w_m + \frac{1}{3}) T_A) \geq 0 \). Hence, for the apparent-horizon system the GSL trivially holds with \( \dot{S}_m + \dot{S}_A \equiv 0 \) under the dominance of the cosmological constant \( w_m = -1 \), and:

1. For \(-1 < w_m < -1/3 \) which corresponds to an accelerated Universe dominated by quintessence, \( \dot{S}_m + \dot{S}_A > 0 \) always holds, because \( T_m / T_A > 0 \) and \( \frac{3}{2}(w_m + \frac{1}{3}) < 0 \).

2. For \(-1/3 < w_m (\leq 1) \) which corresponds to ordinary-matter dominance respecting the strong energy condition \( \rho_m + 3P_m \geq 0 \) [37], the GSL \( \dot{S}_m + \dot{S}_A \geq 0 \) conditionally holds when

\[ \frac{T_m}{T_A} > \frac{3}{2} \left( w_m + \frac{1}{3} \right) . \]  

(66)
(3) For the phantom domain \(w_m < -1\), the GSL never validates because it requires \(T_m/T_A \leq \frac{3}{2}(w_m + \frac{1}{3}) < 0\) which violates the third law of thermodynamics.

V.4. GSL for the event-horizon system

Now consider the system made up of the cosmological event horizon and its interior. The event horizon \(\Upsilon_E\) still carries the Bekenstein-Hawking entropy \(S_E = A_E/4G\) which is valid for all causal boundaries in GR (e.g. [14]), and we further assume a Cai-Kim temperature \(T_E = 1/(2\pi \Upsilon_E)\) to it. Thus, \(S_E\) and Eq. (62) lead to

\[
\dot{S}_m^{(E)} + \dot{S}_E = \frac{1}{T_m}(1 + w_m)\rho_m A_E + \frac{2\pi T_E}{G} \frac{\dot{\Upsilon}_E}{\Upsilon_E},
\]

(67)

The GSL \(\dot{S}_m^{(E)} + \dot{S}_E \geq 0\) requires \(\frac{T_m}{T_E} (H\Upsilon_E - 1) \geq -\frac{1}{2}(1 + w_m)\rho_m A_E\), and with \(\{G, T_E, T_m\} > 0\) it yields

\[
\frac{T_m}{T_E} (H\Upsilon_E - 1) \geq -(1 + w_m)G \rho_m A_E,
\]

(68)

or equivalently

\[
\frac{T_m}{T_E} (H\Upsilon_E - 1) \geq -\frac{3}{2}(1 + w_m)\frac{A_E}{A_A},
\]

(69)

as \(G \rho_m = \frac{3}{2}\frac{A_A}{A_A}\). We have to stop here for the lack of an exact expression for \(\Upsilon_E\) which relies on the concrete form of the scale factor \(a(t)\). Note that if the Universe is dominated by nonexotic matter \(-1 \leq w_m (\leq 1)\), then \(-\frac{3}{2}(1 + w_m)\frac{A_H}{A_A} \leq 0\), and a sufficient condition for \(\dot{S}_m + \dot{S}_E \geq 0\) is just \(H\Upsilon_E \geq 1\), or

\[
\Upsilon_E \geq \Upsilon_H,
\]

(70)

where \(\Upsilon_H := 1/H\) refers to the radius of the Hubble horizon [28, 40], an auxiliary scale where the recession speed reaches that of light (\(c = 1\) in our units) by Hubble’s law. Eq. (70) is a satisfactory result because it is believed to hold for the expanding FRW Universe of any spatial curvature [40].

We will end this section by some interesting comparisons. Although the apparent horizon \(\Upsilon_A\) is more compatible with the unified first law and the Clausius equation, the second law is better respected by the cosmic fluid inside the event horizon \(\Upsilon_E\). For both horizons \(\Upsilon_A\) and \(\Upsilon_E\), the second law is better formulated than the GSL. Moreover, from the standpoint of the second laws and the GSLs, the phantom \((w_m < -1)\) dark energy is definitely less favored than the cosmological constant \((w_m = -1)\) and the quintessence \((-1 \leq w_m < -1/3)\).

VI. GRAVITATIONAL THERMODYNAMICS IN ORDINARY MODIFIED GRAVITIES

For the \(\Lambda\)CDM Universe within GR, the holographic dynamical equations (11), (12) and (14) and their thermodynamic implications have been extensively discussed. In this section, we will extend this whole framework of “gravitational thermodynamics” to modified and alternative theories of relativistic gravity [48, 49].

To date, many modified theories have been developed, such as \(f(R)\) [50], quadratic [52], \(f(R,G)\) generalized Gauss-Bonnet [51], dynamical Chern-Simons [53], Brans-Dicke [54] and scalar-tensor-chameleon [23] gravities; for a quickest overview of these theories along with their detailed thermodynamic connotations, we
could obtain the modified Friedmann equations (37), which yield

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_{\text{eff}} T_{\mu\nu}^{(\text{eff})} \quad \text{with} \quad T_{\mu\nu}^{(\text{eff})} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\text{MG})}. \]  

(71)

where all terms beyond GR have been packed into \( T_{\mu\nu}^{(\text{MG})} \) and \( G_{\text{eff}} \). Here \( T_{\mu\nu}^{(\text{MG})} \) collects the modified-gravity nonlinear and higher-order effects, while \( G_{\text{eff}} \) denotes the effective gravitational coupling strength which can be directly recognized from the coefficient of the matter tensor \( T_{\mu\nu}^{(m)} \); for example [56], we have \( G_{\text{eff}} = G/f_R \) for \( f(R), \ G_{\text{eff}} = G/(1 + 2aR) \) for quadratic, \( G_{\text{eff}} = G/(f_R + 2R f_G) \) for \( f(R, G) \) generalized Gauss-Bonnet, \( G_{\text{eff}} = G \) for dynamical Chern-Simons and \( G_{\text{eff}} = G/\phi \) for Brans-Dicke gravities. Moreover, assume an effective perfect-fluid content for the total effective stress-energy-momentum tensor \( T_{\mu\nu} \),

\[ T_{\mu\nu}^{(\text{eff})} = \text{diag} \left[ -\rho_{\text{eff}}, P_{\text{eff}}, P_{\text{eff}}, P_{\text{eff}} \right] \quad \text{with} \quad P_{\text{eff}} = w_{\text{eff}} \rho_{\text{eff}}, \]  

(72)

where \( w_{\text{eff}} := P_{\text{eff}}/\rho_{\text{eff}} \) refers to the effective EoS parameter, \( \rho_{\text{eff}} = \rho_m + P_{\text{MG}} \) and \( P_{\text{eff}} = P_m + P_{\text{MG}} \).

Modified gravities aim to explain the accelerated cosmic expansion without dark-energy components, so in this section we will assume the physical matter to respect the null, weak, strong and dominant energy conditions [37], which yield \( \rho_m > 0 \) and \(-1/3 \leq w_m \leq 1\). This way, the quintessence \((-1 < w_m < -1/3\), cosmological constant \((w_m = -1)\) and most exotic phantom \((w_m < -1)\) are ruled out.

VI.1. Holographic dynamical equations in modified gravities

Substituting the FRW metric Eq. (1) and the effective cosmic fluid Eq. (72) into the field equation (71), one could obtain the modified Friedmann equations

\[ H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{eff}}}{3} \rho_{\text{eff}} \quad \text{and} \]  

\[ \dot{H} - \frac{k}{a^2} = -4\pi G_{\text{eff}} (1 + w_{\text{eff}}) \rho_{\text{eff}} = -4\pi G_{\text{eff}} h_{\text{eff}} \quad \text{or} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi G_{\text{eff}} P_{\text{eff}}, \]  

(73)

where \( h_{\text{eff}} := (1 + w_{\text{eff}})\rho_{\text{eff}} \) denotes the effective enthalpy density. Given the apparent-horizon radius \( \Upsilon_A \) and its kinematic time-derivative \( \dot{\Upsilon}_A \) in Eqs. (7) and (8), the first and second Friedmann equations can be rewritten into

\[ \Upsilon_A^{-2} = \frac{8\pi G_{\text{eff}}}{3} \rho_{\text{eff}}, \]  

(74)

\[ \dot{\Upsilon}_A = 4\pi H \Upsilon_A^3 G_{\text{eff}} (1 + w_{\text{eff}}) \rho_{\text{eff}}. \]  

(75)

Eq. (74) further implies \( A_{\text{eff}} \rho_{\text{eff}} = \frac{3}{2G_{\text{eff}}}, \) which simplifies Eq. (75) into

\[ \dot{\Upsilon}_A = \frac{3}{2} H \Upsilon_A (1 + w_{\text{eff}}). \]  

(76)
To facilitate the derivations in the next subsection, we also translate the third member in Eq. (73) into

\[
\frac{1}{\Upsilon_A^3} \left( \dot{\Upsilon}_A - \frac{3}{2} H \Upsilon_A \right) = 4\pi G \rho_{\text{eff}} .
\] (77)

Eqs. (74) – (77) constitute the full set of holographic gravitational equations for the apparent-horizon dynamics and the Universe spatial expansion in modified gravities.

VI.2. Unified first law of nonequilibrium thermodynamics

Following our previous work [56], to geometrically reconstruct the effective total internal energy \( E_{\text{eff}} \), one just needs to use \( G_{\text{eff}} \) to replace Newton’s constant \( G \) in the standard Misner-Sharp or Hawking mass in Sec. III.1, which yields

\[
E_{\text{eff}} = \frac{1}{2G_{\text{eff}}} \frac{\Upsilon^3}{\Upsilon_A^3} .
\] (78)

The total derivative of \( E_{\text{eff}} = E_{\text{eff}}(t, r) \) along with the holographic dynamical equations (74), (75) and (77) yield

\[
dE_{\text{eff}} = - \frac{1}{G_{\text{eff}}} \frac{\Upsilon^3}{\Upsilon_A^3} \left( \dot{\Upsilon}_A - \frac{3}{2} H \Upsilon_A \right) dt + \frac{3}{2G_{\text{eff}}} \frac{\Upsilon^2}{\Upsilon_A^2} d\Upsilon - \frac{\dot{G}_{\text{eff}}}{2G_{\text{eff}}} \frac{\Upsilon^3}{\Upsilon_A^3} dt
\]

\[
= -A \Upsilon H P_{\text{eff}} dt + A \rho_{\text{eff}} d\Upsilon - V \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt .
\] (80)

By the replacement \( d\Upsilon = d\Upsilon - H \Upsilon dt \), Eqs. (79) and (80) can be recast into the \((t, \Upsilon)\) transverse coordinates as

\[
dE_{\text{eff}} = - \frac{\dot{\Upsilon}_A}{G_{\text{eff}}} \frac{\Upsilon^3}{\Upsilon_A^3} dt + \frac{3}{2G_{\text{eff}}} \frac{\Upsilon^2}{\Upsilon_A^2} d\Upsilon - \frac{\dot{G}_{\text{eff}}}{2G_{\text{eff}}} \frac{\Upsilon^3}{\Upsilon_A^3} dt
\]

\[
= -A (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon dt + A \rho_{\text{eff}} d\Upsilon - V \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt .
\] (82)

Both Eqs. (80) and (82) can be compactified into the thermodynamic equation

\[
dE_{\text{eff}} = A \psi + W dV + \mathcal{E} ,
\] (83)

where \( W \) and \( \psi \) respectively refer to the effective work density and the effective energy supply covector,

\[
W = \frac{1}{2} (1 - w_{\text{eff}}) \rho_{\text{eff}} ,
\] (84)

\[
\psi = -\frac{1}{2} (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon dt + \frac{1}{2} (1 + w_{\text{eff}}) \rho_{\text{eff}} d\Upsilon
\]

\[
= - (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon dt + \frac{1}{2} (1 + w_{\text{eff}}) \rho_{\text{eff}} d\Upsilon ,
\] (85)
and similar to Sec. [III.1], \( W \) and \( \psi \) can trace back to the Hayward-type invariants \( W := -\frac{1}{2} T^{\alpha\beta}_{(e)\,\beta} \) and \( \psi_{\alpha} := T_{\alpha(e)\,\beta} \partial_{\beta} \Upsilon + W \partial_{\alpha} \Upsilon \) under spherical symmetry. The \( \mathcal{E} \) in Eq. (83) is an extensive energy term

\[
\mathcal{E} := -V \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} \, dt.
\]  

(86)

As will be shown in the next subsection, \( \mathcal{E} \) contributes to the irreversible extra entropy production, so we regard Eq. (83) as the unified first law of nonequilibrium thermodynamics [56], which is an extension of the equilibrium version Eq. (27) in GR. Moreover, it follows from the contracted Bianchi identities and Eq. (71) that \( \nabla_\mu G^\mu_\nu = 0 = 8\pi \nabla_\mu (G_{\text{eff}} T^{\mu(\text{eff})}_\nu) \), and for the FRW metric Eq. (1) it leads to

\[
\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + P_{\text{eff}}) = \frac{\dot{\mathcal{E}}}{V} = -\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}},
\]

(87)

so \( \mathcal{E} \) also shows up in the generalized continuity equation as a density dissipation effect.

VI.3. Nonequilibrium Clausius equation on the horizon

The holographic dynamical equation (75) can be slightly rearranged into \( \frac{\dot{\Upsilon}_{\text{A}}}{G_{\text{eff}}} dt = A_{\text{A}} (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon_{\text{A}} dt \), so we have

\[
\frac{1}{2\pi \Upsilon_{\text{A}}} \cdot 2\pi \Upsilon_{\text{A}} \left( \frac{\dot{\Upsilon}_{\text{A}}}{G_{\text{eff}}} dt - \frac{1}{2} \dot{\Upsilon}_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} dt \right) + \frac{1}{2\pi \Upsilon_{\text{A}}} \cdot 2\pi \Upsilon_{\text{A}} \left( \frac{1}{2} \dot{\Upsilon}_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} dt + V_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt \right)
\]

\[
= A_{\text{A}} (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon_{\text{A}} dt + V_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt.
\]

(88)

Apply the following replacement to the left hand side of Eq. (88)

\[
\frac{1}{2} \dot{\Upsilon}_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} = V_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}},
\]

(89)

whose validity is guaranteed by Eq. (74), and thus

\[
\frac{1}{2\pi \Upsilon_{\text{A}}} \frac{d}{dt} \left( \frac{\pi \Upsilon_{\text{A}}^2}{G_{\text{eff}}} \right) + \frac{1}{2\pi \Upsilon_{\text{A}}} \cdot 2\pi \Upsilon_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt = A_{\text{A}} (1 + w_{\text{eff}}) \rho_{\text{eff}} H \Upsilon_{\text{A}} dt + V_{\text{A}} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \rho_{\text{eff}} dt.
\]

(90)

Equation (90) can be compactified into the thermodynamic relation

\[
T_{\text{A}} dS_{\text{A}} + T_{\text{A}} dP_{\text{A}} S^{(A)} = \delta Q = -(A_{\text{A}} \psi_{\text{t}} + \mathcal{E}_{\text{A}}) = -dE_{\text{eff}} \bigg|_{\Upsilon_{\text{A}}=0},
\]

(91)

where \( \psi_{\text{t}} \) is just the \( t \)-component of the covector \( \psi \) in Eq. (85), \( \mathcal{E}_{\text{A}} \) is the energy dissipation term Eq. (86) evaluated at \( \Upsilon_{\text{A}} \), and \( T_{\text{A}} = \frac{1}{2\pi \Upsilon_{\text{A}}} \) denotes the Cai-Kim temperature of the apparent horizon. Here \( S_{\text{A}} \) refers to the geometrical Wald-Kodama entropy [57] associated to the dynamical apparent horizon,

\[
S_{\text{A}} = \frac{\pi \Upsilon_{\text{A}}^2}{G_{\text{eff}}} = \frac{A_{\text{A}}}{4G_{\text{eff}}} = \int \frac{dA_{\text{A}}}{4G_{\text{eff}}},
\]

(92)
where \( S_A \) takes such a compact form due to \( \Upsilon_A = \Upsilon_A(t) \) and \( G_{\text{eff}} = G_{\text{eff}}(t) \) under the maximal spatial symmetry of the Universe, while \( d_p S^{(A)} \) represents the irreversible entropy production within the apparent horizon

\[
d_p S^{(A)} = 2\pi \Upsilon_A^3 \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} dt .
\]

Hence, we regard Eq.(91) as the “nonequilibrium Clausius equation”, which still depicts the heat transfer for the isoergic process of an arbitrary instantaneous \( \Upsilon_A \). With the nonequilibrium unified first law Eq.(83), Eq.(91) can be completed into the total energy differential

\[
dE_{\text{eff}} = A_A \psi_t dt + (A_A \psi_T + W) d\Upsilon_A + E_A
\]

\[= -T_A (dS_A + d_p S^{(A)}) + \rho_{\text{eff}} dV_A .
\]

VI.4. The second law for the interior of the apparent and the event horizons

Having obtained the unified first law and the Clausius equation of nonequilibrium thermodynamics for the effective energy content in modified gravities, we will continue to investigate the entropy evolution for the physical matter content. Note that the second law of thermodynamics still applies to the physical \( \{\rho_m, P_m\} \) rather than the effective \( \{\rho_{\text{eff}}, P_{\text{eff}}\} \). For the physical energy \( E_m = \rho_m V \) contained in an arbitrary volume, the Positive-Out Gibbs equation (57) still yields 

\[
T_m dS_m = -d(\rho_m V) - P_m dV = -d\rho_m - (\rho_m + P_m) dV ,
\]

which has made use of the continuity equation (19). In fact, as proved in \[56\], under minimal geometry-matter couplings \[49, 58\] the Noether compatible definition of \( T_{\mu\nu}^{(m)} \) automatically guarantees \( \nabla^\mu T_{\mu\nu}^{(m)} = 0 \), so that the total continuity equation (87) can be decomposed into the ordinary \( \dot{\rho}_m + 3H(\rho_m + P_m) = 0 \) for the physical matter and the remaining part for the modified-gravity effect:

\[
\dot{\rho}_{(\text{MG})} + 3H \left( \rho_{(\text{MG})} + P_{(\text{MG})} \right) = -\frac{G_{\text{eff}}}{\dot{G}_{\text{eff}}} (\rho_m + \rho_{(\text{MG})}) .
\]

Hence, for the entropy \( S_m^{(A)} \) of the physical matter inside the apparent horizon \( \Upsilon = \Upsilon_A(t) \), Eq.(95) along with Eq.(76) yield

\[
T_m dS_m^{(A)} = \rho_m A_A (1 + w_m) (\Upsilon_A H - \dot{\Upsilon}_A) dt
\]

\[= \frac{3}{2} \rho_m A_A (1 + w_m) H \Upsilon_A \left( \frac{1}{3} + \frac{w_{\text{eff}}}{3} \right) dt
\]

\[= \frac{9}{2} \rho_m V_A H (1 + w_m) \left( \frac{1}{3} + \frac{w_{\text{eff}}}{3} \right) dt .
\]

where \( \rho_m A_A \) could no longer be simplified by Eq.(13) of GR. Recall that the physical matter satisfies \(-1/3 \leq w_m \leq 1 \) in modified gravities, thus:

The physical entropy \( S_m^{(A)} \) inside the cosmological apparent horizon satisfies \( S_m^{(A)} \geq 0 \) if and only if \( w_{\text{eff}} \leq -1/3 \).
Moreover, for the entropy $S_m^{(E)}$ of the physical matter inside the event horizon $\mathcal{H} = \mathcal{H}_{E}(t)$, Eq. (95) along with $\dot{T}_{E} = HT_{E} - 1$ lead to

$$T_{m}dS_{m}^{(E)} = \rho_{m}A_{E}(1 + w_{m})(HT_{E} - \dot{T}_{E})dt = \rho_{m}A_{E}(1 + w_{m})dt$$

(98)

Hence, for the FRW Universe governed by modified gravities and filled by ordinary matter $-1/3 \leq w_{m} \leq 1$:

The physical entropy $S_{m}^{(E)}$ inside the cosmological event horizon always satisfies $S_{m}^{(E)} \geq 0$ regardless of the modified-gravity theories in use.

VI.5. GSL for the apparent-horizon system

Compared with $\Lambda$CDM in Sec. VI.2 in modified gravities there are three types of entropy for the apparent-horizon system: the physical entropy $S_{m}^{(A)}$ for the internal matter content, the geometric Wald-Kodama entropy $S_{A}$ of the horizon, and the nonequilibrium entropy production. From Eqs. (91) and (97), we have

$$\dot{S}_{m}^{(A)} + \dot{S}_{A} + \dot{S}_{p}^{(A)} = -\frac{1}{2T_{m}}\frac{3}{2}\rho_{m}A_{A}(1 + w_{m})HT_{A}\left(\frac{1}{3} + w_{\text{eff}}\right) + \frac{2\pi\dot{T}_{A}}{G_{\text{eff}}} + \pi\dot{T}_{A}^{2}\frac{G_{\text{eff}}}{G_{2}}.$$

(99)

where $\dot{S}_{p}^{(A)} := d_{p}S^{(A)}/dt$. To validate the GSL $\dot{S}_{m}^{(A)} + \dot{S}_{A} + \dot{S}_{p}^{(A)} \geq 0$, the condition $\frac{\dot{T}_{A}}{G_{\text{eff}}} + \pi\dot{T}_{A}^{2}\frac{G_{\text{eff}}}{G_{2}} \geq \frac{1}{2T_{m}}\frac{3}{2}\rho_{m}A_{A}(1 + w_{m})HT_{A}\left(\frac{1}{3} + w_{\text{eff}}\right)$ should be satisfied, and with $\dot{T}_{A} = HT_{A}(1 + w_{\text{eff}})$, it leads to

$$\frac{1}{2T_{A}}\frac{3}{2}\rho_{m}A_{A}H(1 + w_{\text{eff}}) + 2\pi\dot{T}_{A}\frac{G_{\text{eff}}}{G_{2}} \geq \frac{3}{2T_{m}}\rho_{m}A_{A}H(1 + w_{m})\left(\frac{1}{3} + w_{\text{eff}}\right),$$

(100)

and thus

$$\frac{T_{m}}{T_{A}}\left(3H\left(1 + w_{\text{eff}}\right) + \frac{G_{\text{eff}}}{G_{2}}\right) \geq 3\rho_{m}A_{A}H(1 + w_{m})\left(\frac{1}{3} + w_{\text{eff}}\right),$$

(101)

where $A_{A}$ cannot be further replaced by $1/(\pi T_{A}^{2})$ to nonlinearize $T_{A}$ since $T_{A}$ is not an extensive quantity.

These results have improved the pioneering investigations in [18] for the apparent-horizon system of the flat FRW Universe in generic modified gravities, where the unified first law was not updated to nonequilibrium thermodynamics, the Hayward-Kodama temperature $T_{A}^{(+)}$ and the artificial quasi-equilibrium assumption $T_{m} = bT_{A}^{(+)}$ were employed, and the entropy production $d_{p}S^{(A)}$ was not explicitly found out. Specifically, for equilibrium theories like the dynamical Chern-Simons gravity [53, 56] with $G_{\text{eff}} = G = \text{constant}$, Eq. (101) reduces to become

$$(1 + w_{\text{eff}})\frac{T_{m}}{T_{A}} \geq \rho_{m}A_{A}G(1 + w_{m})\left(\frac{1}{3} + w_{\text{eff}}\right),$$

(102)

which appears analogous to Eq. (65) of $\Lambda$CDM.
VI.6. GSL for the event-horizon system

To study GSL for the event-horizon system, in addition to the physical entropy $S^{(E)}_m$ and the Wald-Kodama entropy $S_E = \pi \Upsilon^2_E / 4G_{\text{eff}}$, one also needs to figure out the entropy production $d_p S^{(E)}_m$ element within the event horizon $\Upsilon_E$. At first glance, the discussion in Sec. VI.3 seems only applicable to the apparent-horizon system, but Eqs. (89) and (90) clearly indicate that the entropy production $d_p S^{(A)}_m$ arises to balance the evolution effects $\dot{G}_{\text{eff}}$ in $S_A$ and $E_A$. Following this spirit it is easy to find out $d_p S^{(E)}_m = \pi \Upsilon^2_E \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}}$, and thus

$$\dot{S}^{(E)}_m + \dot{S}_E + \dot{S}_p = \frac{1}{T_m} \rho_m A_E (1 + w_m) + \frac{2\pi \Upsilon_E \dot{\Upsilon}_E}{G_{\text{eff}}} + \pi \Upsilon^2_E \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \cdot (103)$$

With $\dot{\Upsilon}_E = H\Upsilon_E - 1$ and assume $T_E = 1/(2\pi \Upsilon_E)$, the GSL $\dot{S}^{(E)}_m + \dot{S}_E + \dot{S}_p \geq 0$ requires

$$\frac{T_m}{T_E} \left( H\Upsilon_E - 1 \frac{\dot{\Upsilon}_E}{G_{\text{eff}}} + \frac{\dot{\Upsilon}_E}{2} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \right) \geq -\rho_m A_E (1 + w_m) \cdot (104)$$

Considering that in modified gravities $-1/3 \leq w_m \leq 1$, the right hand side of Eq. (104) is negative definite, so a sufficient condition to validate the GSL is

$$\frac{2}{G_{\text{eff}}} \left( \Upsilon^{-1}_H - \Upsilon^{-1}_E \right) + \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \geq 0 \cdot (105)$$

where $\Upsilon_H := 1/H$ denotes the radius of the Hubble horizon.

VII. CONCLUSIONS AND DISCUSSION

In this paper the thermodynamic implications of the holographic gravitational equations for the FRW Universe have been studied. We started from the $\Lambda$CDM model of GR to clearly build the whole framework of gravitational thermodynamics, and eventually extended it to modified gravities. A great advantage of our formulation is all constraints are expressed by the EoS parameters.

The holographic gravitational equations govern both the apparent-horizon dynamics and the cosmic spatial expansion. We have shown how they imply Hayward’s unified first law of thermodynamics $dE = A\psi + WdV$ and the isochoric-process Clausius equation $T_A dS_A = \delta Q = -A_A \psi \delta t$. Especially, the “positive heat out” sign convention for the heat transfer and the horizon entropy has been decoded.

The derivations of the Clausius equation in Sec. III.2 actually involves a long standing confusion regarding the setup of the apparent-horizon temperature, and extensive comparisons in Sec. IV have led to the argument that the Cai-Kim ansatz $T_A = 1/(2\pi \Upsilon_A)$ is more appropriate than the Hayward-Kodama temperature $T_A = \kappa / 2\pi$ and its partial absolute value $T^{(+)}_A$. Meanwhile, we have also introduced the “zero temperature divide” $w_m = 1/3$ for $T_A = \kappa / 2\pi$, and proved the signs of both temperatures are independent of the inner or outer trappedness of the apparent horizons.

With the horizon temperature and entropy clarified, the cosmic entropy evolution has been investigated. We have adjusted the traditional matter entropy and Gibbs equation into $dE_m = -T_m dS_m - P_m dV_A$ in accordance with the positive heat out convention of the horizon entropy. It turns out that the cosmic entropy is well behaved, specially for the event-horizon system, where both the second law and the GSL hold for nonexotic matter ($-1 \leq w_m \leq 1$). Also, we have clarified that the regions $\{\Upsilon \leq \Upsilon_A, T \leq T_E\}$ enveloped by the apparent
and even horizons are simple open thermodynamic systems so that one should not \textit{a priori} expect the validity of nondecreasing entropy, and abandoned the local equilibrium assumptions restricting the interior and the boundary temperatures.

Finally we have generalized the whole formulations from the $\Lambda$CDM model to ordinary modified gravities whose field equations can be compactified into the GR form $R_{\mu\nu} - Rg_{\mu\nu}/2 = 8\pi G_{\text{eff}}T_{\mu\nu}^{(\text{eff})}$. To our particular interest, we found that inside the apparent horizon the second law $\dot{S}_m \geq 0$ nontrivially holds if $w_{\text{eff}} \leq -1/3$, while inside the event horizon $\dot{S}_m \geq 0$ always validates regardless of the gravity theories in use.

There are still some interesting problems arising in this paper and yet unsolved. For example, is the apparent horizon $\Upsilon_A$ the only hologram membrane for the FRW Universe? Can the elegant relative evolution equations (18), (20) and (21) be used in astrophysical and cosmological simulations? Also, how would the cosmic entropy evolve in a contracting Universe? We hope to find out the answers in prospective studies.

ACKNOWLEDGEMENT

This work was financially supported by the Natural Sciences and Engineering Research Council of Canada.

[1] G W Gibbons, S W Hawking. Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D 15, 2738 (1977).

Emil Mottola. Thermodynamic instability of de Sitter space. Phys. Rev. D 33, 1616 (1986).
P C W Davies. Cosmological horizons and the generalized second law of thermodynamics. Class. Quant. Grav. 4, L225-L228 (1987).
P C W Davies. Cosmological horizons and entropy. Class. Quant. Grav. 5, 1349 (1988).

[2] Rong-Gen Cai, Sang Pyo Kim. First law of thermodynamics and Friedmann Equations of Friedmann-Robertson-Walker universe. Journal of High Energy Physics 2005, 050 (2005). arXiv:hep-th/0501055

[3] Ted Jacobson. Thermodynamics of spacetime: The Einstein equation of state. Phys. Rev. Lett. 75, 1260-1263 (1995). arXiv:gr-qc/9504004

[4] T. Padmanabhan. Thermodynamical aspects of gravity: New insights. Rept. Prog. Phys. 73, 046901 (2010). arXiv:0911.5004 [gr-qc]

[5] J M Bardeen, B Carter, S W Hawking. The four laws of black hole mechanics. Commun. Math. Phys. 31, 161-170 (1973).

Jacob D Bekenstein. Black holes and entropy. Phys. Rev. D 7, 2333-2346 (1973).

S W Hawking. Black hole explosions? Nature 248, 30–31 (1974).

[6] M Akbar, Rong-Gen Cai. Thermodynamic behavior of Friedmann equations at apparent horizon of FRW universe. Phys. Rev. D 75, 084003 (2007). arXiv:hep-th/0609128

[7] Rong-Gen Cai, Li-Ming Cao. Unified first law and the thermodynamics of the apparent horizon in the FRW universe. Phys. Rev. D 75, 064008 (2007). arXiv:gr-qc/0611071

[8] M Akbar, Rong-Gen Cai. Thermodynamic behavior of field equations for $f(R)$ gravity. Phys. Lett. B 648, 243–248 (2007). arXiv:gr-qc/0612089

[9] Rong-Gen Cai, Li-Ming Cao. Thermodynamics of apparent horizon in brane world scenario. Nucl. Phys. B 785, 135-148 (2007). arXiv:hep-th/0612144

Ahmad Sheykhi, Bin Wang, Rong-Gen Cai. Thermodynamical properties of apparent horizon in warped DGP braneworld. Nucl. Phys. B 779, 1-12 (2007). arXiv:hep-th/0701198

[10] Ahmad Sheykhi, Bin Wang, Rong-Gen Cai. Deep connection between thermodynamics and gravity in Gauss-Bonnet braneworlds. Phys. Rev. D 76, 023515 (2007). arXiv:hep-th/0701261

\footnote{Even the philosophical “whole Universe” would be an open system if there were matter creations which would cause irreversible extra entropy production, and one typical mechanism triggering this effect is nonminimal curvature-matter coupling.}
Kazuharu Bamba, Chao-Qiang Geng, Shinji Tsujikawa. *Equilibrium thermodynamics in modified gravitational theories*. Phys. Lett. B **688**, 101-109 (2010). arXiv:0909.2159 [gr-qc]

Rong-Gen Cai, Nobuyoshi Ohta. *Horizon thermodynamics and gravitational field equations in Horava-Lifshitz gravity*. Phys. Rev. D **81**, 084061 (2010). arXiv:0910.2307 [hep-th]

Qiao-Jun Cao, Yi-Xin Chen, Kai-Nan Shao. *Clausius relation and Friedmann equation in FRW universe model*. Journal of Cosmology and Astroparticle Physics **1005**, 030 (2010). arXiv:1001.2597 [hep-th]

Kazuharu Bamba, Chao-Qiang Geng, Shin'ichi Nojiri, Sergei D Odintsov. *Equivalence of modified gravity equation to the Clausius relation*. Europhys. Lett. **89**, 50003 (2010). arXiv:0909.4397 [hep-th]

R. Brustein. *Generalized second law in cosmology from causal boundary entropy*. Phys. Rev. Lett. **84**, 2072 (2000). arXiv:gr-qc/9904061

M.R. Setare. *Generalized second law of thermodynamics in quintom dominated universe*. Phys. Lett. B **641**, 130-133 (2006) arXiv:hep-th/0611165

K. Karami, S. Ghaffari. *The generalized second law of thermodynamics for the interacting dark energy in a non-flat FRW universe enclosed by the apparent and event horizons*. Phys. Lett. B **685**, 115-119 (2010). arXiv:0912.0363 [gr-qc]

H. Mohseni Sadjadi. *On the second law of thermodynamics in modified Gauss-Bonnet gravity*. Phys. Scripta **83**, 055006 (2011). arXiv:1009.1839 [gr-qc]

A Abdolmaleki, T Najafi, K Karami. *Generalized second law of thermodynamics in scalar-tensor gravity*. Phys. Rev. D **89**, 104041 (2014). arXiv:1401.7549 [gr-qc]

Ramon Herrera, Nelson Videla. *The generalized second law of thermodynamics for interacting f(R) gravity*. Int. J. Mod. Phys. D **23**, 1450071 (2014). arXiv:1406.6305 [gr-qc]

Abhay Ashtekar, Stephen Fairhurst, Badri Krishnan. *Isolated horizons: Hamiltonian evolution and the first law*. Phys. Rev. D **62**, 104025 (2000). gr-qc/0005083

Tamara M Davis, Charles H Lineweaver. *Expanding confusion: common misconceptions of cosmological horizons and the superluminal expansion of the universe*. Publ. Astron. Soc. Austral. **21**, 97-109 (2004). arXiv:astro-ph/0310808

Richard J Cook, M Shane Burns. *Interpretation of the cosmological metric*. Am. J. Phys. **77**, 59-66 (2009). arXiv:0803.2701 [astro-ph]

G Hinshaw, D Larson, E Komatsu, D N Spergel, C L Bennett, et al. *Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Cosmological parameter results*. Astrophys. J. Suppl. **208**, 19 (2013). arXiv:1212.5225 [astro-ph.CO]
Two accurate time-delay distances from strong lensing: Implications for cosmology. [Astrophys. J. 766, 70 (2013)]

Eric Aubourg, Stephen Bailey, Julian E. Bautista, Florian Beutler, Vaishali Bhardwaj, et al. Cosmological implications of baryon acoustic oscillation (BAO) measurements. [arXiv:1411.1074 [astro-ph.CO]]

G. 't Hooft. Dimensional reduction in quantum gravity. [arXiv:gr-qc/9310026]

R. R. Caldwell, Rahul Dave, Paul J. Steinhardt. A Phantom menace? [Phys. Lett. B 545, 23-29 (2002)]

Ivan Booth, Stephen Hawking. The cosmological constant and dark energy. [Rev. Mod. Phys. 75, 559-606 (2003)]

T. Padmanabhan. Cosmological constant: The Weight of the vacuum. [Phys. Rept. 380, 235-320 (2003)]

Abhay Ashtekar, Badri Krishnan. Dynamical horizons: Energy, angular momentum, fluxes and balance laws. [Phys. Rev. Lett. 89, 261101 (2002)]

Valerio Faraoni. Cosmological apparent and trapping horizons. [Phys. Rev. D 84, 024003 (2011)]

Charles W Misner, David H Sharp. Relativistic equations for adiabatic, spherically symmetric gravitational collapse. [Phys. Rev. 136, B571-576 (1964)]

Sean A Hayward. Gravitational energy in spherical symmetry. [Phys. Rev. D 53, 1938-1949 (1996)]

Stephen W Hawking. Gravitational radiation in an expanding universe. [J. Math. Phys. 9, 598-604 (1968)]

Sean A Hayward. Unified first law of black-hole dynamics and relativistic thermodynamics. [Class. Quant. Grav. 15, 3147-3162 (1998)]

Wang Gang, Liu Wen-Biao. Nonequilibrium thermodynamics of dark energy on cosmic apparent horizon. [Commun. Theor. Phys. 52, 383-384 (2009)]

Jacob D Bekenstein. Generalized second law of thermodynamics in black-hole physics. [Phys. Rev. D 9, 3292-3300 (1974)]

Sean A Hayward. General laws of black-hole dynamics. [Phys. Rev. D 49, 6467-6474 (1994)]

Ivan Booth, Stephen Fairhurst. The first law for slowly evolving horizons. [Phys. Rev. Lett. 92, 011102 (2004)]

Shin’ichi Nojiri, Sergei D Odintsov. Introduction to modified gravity and gravitational alternative for dark energy. [Int. J. Geom. Meth. Mod. Phys. 4, 115-146 (2007)]

Shin’ichi Nojiri, Sergei D Odintsov. Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models. [Phys. Rept. 505, 59-144 (2011)]

Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, Constantinos Skordis. Modified gravity and cosmology. [Phys. Rept. 513, 1-189 (2012)]

Kazuharu Bamba, Salvatore Capozziello, Shin’ichi Nojiri, Sergei D. Odintsov. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. [Astrophys. Space Sci. 342, 155-228 (2012)]
[49] David W Tian, Ivan Booth. Lessons from $f(R, R^2, R_n, \mathcal{L}_m)$ gravity: Smooth Gauss-Bonnet limit, energy-momentum conservation, and nonminimal coupling. Phys. Rev. D 90, 024059 (2014). arXiv:1404.7823 [gr-qc].

[50] Thomas P Sotiriou, Valerio Faraoni. $f(R)$ theories of gravity. Rev. Mod. Phys. 82, 451-497 (2010). arXiv:0805.1726 [gr-qc].

Antonio De Felice, Shinji Tsujikawa. $f(R)$ theories. Living Rev. Rel. 13, 3 (2010). arXiv:1002.3928 [gr-qc].

[51] Guido Cognola, Emilio Elizalde, Shin’ichi Nojiri, Sergei D Odintsov, Sergio Zerbini. Dark energy in modified Gauss-Bonnet gravity: late-time acceleration and the hierarchy problem. Phys. Rev. D 73, 084007 (2006). arXiv:hep-th/0601008.

[52] K S Stelle. Classical gravity with higher derivatives. Gen. Rel. Grav. 9, 353-371 (1978).

[53] R Jackiw, S Y Pi. Chern-Simons modification of general relativity. Phys. Rev. D 68, 104012 (2003). arXiv:gr-qc/0308071.

Stephon Alexander, Nicolás Yunes. Chern-Simons modified general relativity. Phys. Rept. 480, 1-55 (2009). arXiv:0907.2562 [hep-th].

[54] C Brans, R H Dicke. Mach’s principle and a relativistic theory of gravitation. Phys. Rev. 124, 925-935 (1961).

[55] Varun Sahni, Alexei Starobinsky. Reconstructing dark energy. Int. J. Mod. Phys. D 15, 2105-2132 (2006). arXiv:astro-ph/0610026.

[56] David Wenjie Tian, Ivan Booth. Friedmann equations from nonequilibrium thermodynamics of the Universe: A unified formulation for modified gravity. To appear in Phys. Rev. D 90 (2014). arXiv:1409.4278 [gr-qc].

[57] Robert M Wald. Black hole entropy is the Noether charge. Phys. Rev. D 48, 3427-3431 (1993). arXiv:gr-qc/9307038.

Ted Jacobson, Gungwon Kang, Robert C Myers. On black hole entropy. Phys. Rev. D 49, 6587-6598 (1994). arXiv:gr-qc/9312023.

Vivek Iyer, Robert M Wald. Some properties of the Noether charge and a proposal for dynamical black hole entropy. Phys. Rev. D 50, 846-864 (1994). arXiv:gr-qc/9403028.

[58] Shin’ichi Nojiri, Sergei D Odintsov. Gravity assisted dark energy dominance and cosmic acceleration. Phys. Lett. B 599, 137-142 (2004). arXiv:astro-ph/0403622.

Gianluca Allemandi, Andrzej Borowiec, Mauro Francaviglia, Sergei D Odintsov. Dark energy dominance and cosmic acceleration in first order formalism. Phys. Rev. D 72, 063505 (2005). arXiv:gr-qc/0504057.

Orfeu Bertolami, Christian G Boehmer, Tiberiu Harko, Francisco S N Lobo. Extra force in $f(R)$ modified theories of gravity. Phys. Rev. D 75, 104016 (2007). arXiv:0704.1733 [gr-qc].

Tiberiu Harko, Francisco S N Lobo. Generalized curvature-matter couplings in modified gravity. Galaxies 2, 410-465 (2014). arXiv:1407.2013.

[59] J.A.S. Lima, A.S.M. Germano. On the Equivalence of matter creation in cosmology. Phys. Lett. A 170, 373-378 (1992).

[60] Tiberiu Harko. Thermodynamic interpretation of the generalized gravity models with geometry-matter coupling. Phys. Rev. D 90, 044067 (2014). arXiv:1408.3465 [gr-qc].