Dark Matter: the Problem of Motion

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Abstract—Dark matter (DM) may be studied through the motion of objects following nongeodesic trajectories either due to the existence of an extra mass as a projection of higher dimensions onto lower ones or as motion of dipolar particles and fluids in the halos of spiral galaxies. The effect of DM has been extended nearby the core of the galaxy by means of the excess of mass appearing in the motion of fluids in the accretion disc. Nongeodesic equations and those of their deviation are derived in the presence of different classes of bimetric theories of gravity. The stability of these trajectories using the geodesic deviation technique is investigated.

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1. INTRODUCTION

Flat rotation curves of spiral galaxies cannot be explained by Newtonian or Einstein gravity since neither the former nor the latter are satisfied, and therefore such deviations from these two established theories constitute the existence of Dark Matter (DM). In our galaxy, meticulous observations have confirmed that rotation velocities range between 200 and 300 km/s if these clouds are considered to be moving in circular orbits.

Yet, there are alternative remedies related to modifying Newtonian or Einsteinian gravity by changing their corresponding gravitational potential \( \phi \) to become \( \phi = -GM[1 + \alpha \exp(-r/r_0)]/(1 + \alpha) \), such that \( \alpha = -0.9 \) and \( r_0 \approx 30 \) kpc, in order to explain the behavior of flat rotational curves of spiral galaxies [1]. Consequently, such a unique explanation of discrepancy between theory and observation is still under debate. This may lead us to revisiting the notion of DM as expressed in terms of a mass excess quantity.

This type of explanation is represented as the behavior of objects following nongeodesic equations or being shown as the projection of the fifth component on its other 4D components of the geodesic equation in space-time-matter (STM) theory [2]. Also, DM can be explained as the behavior of dipolar particles in the presence of a polarization field [3]. However, this type of description has been amended to be regarded as due to dipolar fluids connected with the effect of dark energy (DE) in the halos [4].

Thus, one must take into account that DE comprises 74% of gravitating matter of universe; while DM forms about 23% of it [5]. This may give an indication that DM particles are detected in several regions in the universe beside the halos.

Accordingly, some incidents like the excess of gamma ray radiation near the core of a galaxy can be revealed due to DM annihilation [6]. In this case, the effect of DM is observed as an excess of mass in the hydrodynamic equations for the accretion disk [7]. Moreover, it has been considered that the cause of slight deviation of perihelion motion is the influence of DM particles [8].

Now, it is essential to implement the significance of studying the problem of motion for the suspected particles or fluids in order to give a possible scenario for the behavior of DM at different scales in the universe. Accordingly, one must seek an appropriate theory of gravity able to detect its existence at different scales. One of the candidates is studying a class of bi-metric theories of gravity which is able to express strong gravitational fields like SgrA*, neutron stars and binary pulsars as well as weak gravitational fields playing the same role as general relativity [9].

From this perspective, it is vital in our study to derive the candidate equations of motion showing that the mass excess term is due to the existence of DM. Such a vital question should be addressed: What is dark matter?

In our present work, it appears possible to illustrate its cause by three different rival explanations:

(i) The existence of a scalar field associated with the Galaxy’s gravitational field [1].
(ii) The projection of higher-dimension quantities on the 4D manifold [2].

(iii) Motion of dipolar particles or fluids, as claimed, in spiral galaxies [4].

Consequently, we are going to deal with expressing the behavior of DM in terms of nongeodesic equations, derived using the Lagrangian formalism with a Bazanski-like Lagrangian [10]. This type of equation may give rise to geometrizing all trajectories associated with the appearance of DM. In other words, the appropriate path equations as described in Riemannian geometry as representing dipolar particles or fluids of the halos; and the corresponding path equations that represents the hydrostatic stream of fluids in the accretion disk due to solving the nongeodesic deviation equation, can give rise to examining the stability conditions, which means an indication of a remaining DM effect on each observed region.

On the other hand, another approach revealing the above discrepancies between theory and observation at the galactic level is due to the Modified Newtonian Gravity (MOND) [11] or its bimetric version BIMOND [12]. These types of theories are rejecting the existence of DM and DE and assert that such an anomaly is due to a deficiency in obtaining an appropriate theory of gravity able to cure the Newtonian explanation. Even though, Blanchet has regarded the MOND as a gravitational polarization effect [13].

From this perspective, we are going to derive the appropriate Bazanski-like Lagrangians [14] to examine the equivalence of nongeodesic trajectories with each of the following equations for the dipolar moments, dipolar fluids and hydrostatic streams of motion as described in general relativity (Section 2). We extend the previous equations to different versions of bimetric theories of gravity as shown in sec 3. Finally, it turns out that the problem of detecting the existence of DM is connected with studying the behavior of the stream of fluids in different gravitational fields.

This may raise the necessity to examine the stability of these systems for DM effects. This can be done by solving different deviation equations for examining the stability condition, using an independent method of coordinate transformation [15–16], which is described in Section 4.

2. DARK MATTER: EQUATIONS OF MOTION FROM DIFFERENT PERSPECTIVES

2.1. Dark Matter: Nongeodesic Equations

Dark matter can show its presence due to the excess of mass as appearing in nongeodesic trajectories. Their equations are obtained by applying the Euler-Lagrange equation to the Lagrangian [1]

\[ L = m(s)g_{\mu\nu}U^\mu \frac{D\Psi^\nu}{Ds} + m(s)U^\mu \Psi^\mu, \]  

where \( U^\mu \) is a unit tangent vector, \( \Psi^\mu \) its corresponding deviation vector, \( m(s) \) is the mass, to be considered as a function of the parameter \( s \), and \( \mu = 1, 2, 3, 4 \). According to

\[ \frac{d}{ds} \frac{\partial L}{\partial \dot{\Psi}^\alpha} - \frac{\partial L}{\partial \Psi^\alpha} = 0, \]  

one gets

\[ \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\beta\delta} U^\beta U^\delta = \frac{m(s)}{m(s)} (g^{\alpha\beta} - U^\alpha U^\beta), \]  

such that

\[ m(s) = -\nabla[g(\psi)\psi], \]

where \( g(\psi) \psi \) is a scalar function, and the right-hand side of Eq. (3) behaves as a parallel force to represent the presence of DM.

Also, the corresponding nongeodesic deviation equation is obtained by using the commutation relation for Eq. (3), i.e.,

\[ A^\mu_{\nu\rho} - A^\mu_{\rho\nu} = R^\mu_{\beta\nu\rho} A^\beta, \]

where \( A^\mu \) is an arbitrary vector, \( R^\mu_{\beta\nu\rho} \) is the curvature tensor.

Multiplying both sides by an arbitrary vectors \( U^\rho \Psi^\sigma \) and taking into consideration the condition [15]

\[ U^\mu_{\nu} \Psi^\rho = \Psi^\rho \frac{U^\mu}{U^\mu}, \]

we obtain the corresponding deviation equations

\[ \frac{D^2 \Psi^\mu}{Ds^2} = R^\mu_{\nu\rho\sigma} U^\nu U^\rho \Psi^\sigma + \left[ \frac{m(s)_{\beta}}{m(s)} (g^{\alpha\beta} - U^\alpha U^\beta) \right] \Psi^\beta. \]  

Yet, for examining the flat rotation curves, it has been found [2] by taking \( \sigma \) as a parameter describing the trajectories of particles in this region, such that \( s \sim \sigma \), that

\[ \frac{1}{m} \frac{dm}{d\sigma} \equiv \sqrt{\Lambda}/2, \]

which can be expressed as

\[ \frac{1}{m} \frac{dm}{d\sigma} \approx 2a_0/c^2, \]

where \( a_0 \) is a constant of acceleration, \( a_0 \sim 2 \times 10^{-10} \text{ m/s}^2 \), as is known in MOND, and \( c \) is the speed of light.

Accordingly, we can find that the nongeodesic equation can be related to MOND [11] in the following way:

\[ \frac{dU^\alpha}{d\sigma} + \Gamma^\alpha_{\beta\delta} U^\beta U^\delta = 2a_0 \frac{U^\beta}{c^2} (g^{\alpha\beta} - U^\alpha U^\beta), \]  

where \( \Gamma^\alpha_{\beta\delta} \) are the connection coefficients of the space-time in MOND.
where \( \dot{U}^\alpha = dx^\alpha /d\sigma \) is the associated unit tangent vector.

Consequently, the corresponding deviation equation becomes
\[
\frac{D^2 \hat{\Psi}^\mu}{D\sigma^2} = R^\mu_{\nu\rho\sigma} \dot{U}^\nu \dot{U}^\rho \hat{\Psi}^\sigma + 2 a_0 c^2 \left[ \dot{U}_\beta (g^{\alpha\beta} - \dot{U}^\alpha \dot{U}^\beta) \right] \hat{\Psi}^\rho, \tag{8}
\]
where \( \hat{\Psi}^\mu \) is the nongeodesic deviation vector.

### 2.2. Dark Matter: an Extra-Dimensional Effect

It is well known that the nongeodesic equations can be expressed as the four components of geodesic equations for a test particle [1] in a noncompact space-time, \( g_{AB,5} \neq 0 \), following from Wesson’s approach to space-time-matter [2]. Thus, the characteristics of DM can appear by solving the geodesic equation in 5 dimensions, provided that
\[
\frac{dS}{ds} = \sqrt{(1 + \epsilon \hat{\Phi}^2(U^5)^2)},
\]
such that \( \hat{\Phi} \) is a scalar function, and \( \epsilon = \pm 1 \).

Thus, it is possible to suggest the following Lagrangian:
\[
L = g_{AB} U^A \frac{D\Psi^B}{dS}, \tag{9}
\]
where \( A = 1, 2, 3, 4, 5 \). Taking a variation with respect to \( \Psi^C \) and \( U^C \), one can find the geodesic equation
\[
\frac{D U^C}{D S} = 0, \tag{10}
\]
and the geodesic deviation equation
\[
\frac{D^2 \Psi^C}{D S^2} = R_{BDE}^C U^B U^D \Psi^E. \tag{11}
\]
The force appearing in the right-hand side is expressed as the component of the fifth dimension of a 5D manifold. Accordingly, Eq. (3) may be rewritten as
\[
\frac{d^2 x^\mu}{dS^2} + \Gamma^\mu_{AB} \frac{dx^A}{dS} \frac{dx^B}{dS} = 0,
\]
\[
\frac{d^2 x^\mu}{dS^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dS} \frac{dx^\rho}{dS} = - \left( \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dS} \frac{dx^\rho}{dS} + \Gamma^\mu_{\nu\rho} \frac{dx^5}{dS} \frac{dx^5}{dS} \right).
\]
Solving Eq. (10) and considering its fifth component to be substituted in the other four components, this may be regarded as being similar to the behavior of DM particles in (3).

Thus we find that the indication of DM may be represented in terms of an excess of mass in the right-hand side of the nongeodesic equation. Such an equation is obtained as a projection of the fifth component of the geodesic equation onto its four-dimensional components.

### 2.3. Dark Matter: Equations of Motion Dipolar Moment Particles in the Halo

A rival explanation for the cause of the flat rotational curves of spiral galaxies can be found due to the presence of dipolar DM particles [3]. Such particles are not purely dipolar as they involve a monopole contribution from the stress-energy tensor obtained from the Einstein field equations. It has been proposed by Blanchet et al. to study their equations of motion composed of two sets of equations, involving \( P^\mu \), the (passive) linear momentum vector and \( \Omega^\mu \), the evolution vector, describing a microscopic (active) momentum acting as the spin tensor \( S_{\mu\nu} \) in the Papapetrou equation of motion for spinning objects [15]. These equations are obtained using the Lagrangian formalism by analogy to the their counterpart for spinning motion with precession (see the Appendix).

Thus we suggest the following Lagrangian:
\[
L = g_{\alpha\beta} P^\alpha \frac{D\Psi^\beta_{(1)}}{D S} + \Omega_{\alpha} \frac{D\Psi^\beta_{(2)}}{D S} + f_{\alpha} \Psi^\alpha_{(1)} + f_{\alpha} \Psi^\alpha_{(2)}, \tag{12}\]
in which
\[
P^\mu = \left( 2mU^\mu + \frac{D\pi^\mu}{D S} \right),
\]
where \( \pi^\mu \) is the dipole vector, \( \Psi^\mu_{(1)} \) is the nongeodesic deviation from the world line, and \( \Psi^\mu_{(2)} \) is the evolution deviation due to the dipole moment; taking the raising and lowering indices for the evolution vector with \( h_{\mu\nu} \), the projector tensor, i.e.,
\[
h_{\mu\nu} = g_{\mu\nu} - U^\mu U^\nu, \quad \Omega^\mu = h^{\mu\nu} \Omega_\nu. \tag{13}\]

Taking a variation with respect to \( \Phi^\mu_1 \) and \( \Phi^\mu_2 \) separately, we obtain the following set of equation of motion and evolution:
\[
\frac{D P^\mu}{D S} = f^\mu, \tag{14}
\]
\[
\frac{D \Omega^\mu}{D S} = f^\mu, \tag{15}
\]
such that
\[
f^\mu = 2m \frac{\bar{\pi}}{\bar{\pi}} \frac{dV}{dx} \left( \frac{\bar{\pi}}{m} \right),
\]
where $\tilde{\pi} = h^{\mu\nu}\pi_{\nu}$, and $V$ is an associated potential function in terms of the dipole vectors. The evolution equation becomes

$$\frac{D\Omega^\mu}{Ds} = \hat{f}^\mu,$$  \hspace{1cm} (16)

provided that $\hat{f} = R^\mu_{\nu\rho\sigma}{\tilde{\pi}^\sigma}U^\rho U^\nu$.

Similarly, using (A.4) and (A.5) as in [1, 2], we obtain the corresponding geodesic deviation equations:

$$\frac{D^2\Psi^\mu(1)}{DS^2} = R^\mu_{\nu\rho\sigma}U^\rho \Psi^\sigma(1) + \hat{f}^\mu_1 \Psi^\rho(1),$$  \hspace{1cm} (17)

and

$$\frac{D^2\Psi^\mu(2)}{DS^2} = R^\mu_{\nu\rho\sigma}U^\rho \Psi^\sigma(2) + \hat{f}^\mu_2 \Psi^\rho(2).$$  \hspace{1cm} (18)

Equations (16), (17) are essentially vital for examining the stability of different celestial objects in various gravitational fields due to the presence of DM particles.

2.4. Equations of Motion of Dipolar Fluid in the Halos

The involvement of a cosmological constant, a candidate DE, has a vital role in solving the mystery of dark matter. This led Blanchet et al. to reformulate the description of dipolar DM from particle contents into a fluid-like description [2]. This can be found by replacing $V$ in the equations with $W$, the effect of a polarization potential, to express the effect of DE on the system.

From this perspective, Blanchet and Le Tiec [4] have postulated that the dynamics of the dipolar fluid in a prescribed gravitational field $g_{\mu\nu}$ is derived from an action of the type

$$S = \int d^4x\sqrt{-g}L[J^\mu, \xi^\mu \dot{\xi}, g_{\mu\nu}].$$  \hspace{1cm} (19)

The density current $J^\mu$ and the polarization vector $\Pi^\mu$ are new quantities added in dipolar fluids, such that: $J^\mu = \rho U^\mu$, and $\Pi^\mu = \rho \xi^\mu$, where $\rho = 2mn$, the inertial mass density of the dipolar particles, $n$ being the number density of the dipole moment. Applying the least action principle to (19), we obtain the set of path equations

$$\frac{DK^\mu}{Ds} = \frac{f^\mu}{m},$$  \hspace{1cm} (20)

and

$$\frac{D\Omega^\mu}{Ds} = \frac{1}{\tilde{\sigma}}\nabla^\mu(W - \hat{\Pi}W) - R^\mu_{\rho\nu\lambda}w^\rho\xi^\nu K^\lambda,$$

where $\hat{f} = R^\mu_{\nu\rho\sigma} \pi^\sigma U^\rho U^\nu$. $\Pi^\mu = \frac{P^\mu}{2m}$, such that

$$R^\mu_{\rho\nu\sigma} \Pi^\rho U^\nu = \hat{f}.$$

Similarly, using (A.4) and (A.5) as in [1, 2], we obtain the path deviation and evolution deviation vector $\Psi^\mu(1)$ and evolution deviation vector $\Psi^\mu(2)$ simultaneously, provided that

$$f^\mu_1 = \Pi^\mu dW / d\Pi$$

and

$$f^\mu_2 = \frac{1}{\tilde{\sigma}}\nabla^\mu(W - \hat{\Pi}W) - R^\mu_{\rho\nu\lambda}w^\rho\xi^\nu K^\lambda.$$

Thus, using the commutation rule (A.4) and the condition (A.5), we obtain the path deviation and evolution deviation equations, respectively,

$$\frac{D^2\Psi^\mu(1)}{DS^2} = R^\mu_{\nu\rho\sigma}U^\rho \Pi^\sigma(1),$$  \hspace{1cm} (22)

and

$$\frac{D^2\Psi^\mu(2)}{DS^2} = R^\mu_{\rho\nu\lambda}U^\rho \Pi^\sigma(2) + \hat{f}^\mu \Psi^\rho(2).$$  \hspace{1cm} (23)

Using Eqs. (22) and (23), we may also examine the corresponding deviation vectors and the stability of a dipolar fluid in the halo due to the presence of DM, taking into consideration the influence of DE.

2.5. Equations of Motion of Fluids in the Accretion Disk

Due to the role of nongeodesic equations in explaining the behavior of DM particles in the accretion disk as a collisionless fluid, we are going to focus on their contribution to the mass of the accretion disc, and consequently the accretion process is less efficient than that expected from a dissipative fluid; DM gives a significant contribution to the mass of the accretion disk producing an important inflow, e.g., in
our Galaxy, a mass growth scaling as $M_{bh} = \text{const} \cdot t^{9/10}$ [16].

Thus, we can find out that the equivalence between nongeodesic motions and hydrodynamic flows appears in the set of equations
\[
\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\beta\delta} U^\beta U^\delta = f^\alpha, \tag{24}
\]
where $f^\alpha$ is described as a non-gravitational force whose vanishing turns the equation into a geodesic, which becomes
\[
\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\beta\delta} U^\beta U^\delta = \frac{1}{E + \hat{P}} h_{\alpha\beta} \hat{P}_{\beta}, \tag{25}
\]
where $\hat{P}$ is the fluid pressure, $E$ is the overall mass-energy density [7], and $\rho$ is the density. If Eq. (24) satisfies the first law of thermodynamics,
\[
\hat{P}_{\beta} = \rho c^2 \left( \frac{(E + \hat{P})}{\rho c^2} \right)_{\beta}, \tag{26}
\]
then the associated equation of motion of the fluid becomes
\[
\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\beta\delta} U^\beta U^\delta = \frac{1}{E + \hat{P}} \left( \frac{\rho c^2}{E + \hat{P}} \right)_{\beta} h_{\alpha\beta}. \tag{27}
\]

In the case of isobaric pressure, the equation of stream becomes conditionally equivalent to the geodesic one. Thus, the appearance of the extra term on the right-hand side of Eq. (4) inspired many authors to interrelate it with the problem of DM as an excess of mass due to the Lagrangian suggested by Kahil and Harko (2009) [1]:

Comparing the right-hand side of Eq. (27) to its counterpart in (3), it has been noticed that this term may perform the same function in expressing the mass-excess term which plays a vital role in expressing the involvement of DM in the equation of motion. Accordingly, we can express this link in the following relations:
\[
m(s) := \frac{\hat{P} + E}{\rho c^2}. \tag{28}
\]
Using (27), we find that
\[
\frac{1}{E + \hat{P}/\rho} \left( \frac{dE + \hat{P}/\rho}{d\sigma} \right) \approx 2a_0/c^2. \tag{29}
\]
Such a relationship that gives rise to an increase in the density of matter in the accretion disk may be regarded to emerge due to gravitational focusing of DM particles in the accretion disk [6]. However, if the parameter $s$ of the mass term and the nongeodesic equation becomes equivalent to $\sigma$ as appeared in (6), this expression gives the same results as given by MOND near the halo.

3. DARK MATTER EQUATIONS OF MOTION IN BIMETRIC THEORIES

Implementing the concept of geometrization of physics, it is essential to express the nongeodesic equations and their corresponding deviation equation to describe the behavior of particles or fluids in the presence of different bimetric gravitational fields, able to explain DM in different regions inside spiral galaxies.

3.1. Nongeodesic Trajectories in Bigravity

Hossenfelder [17] has introduced an alternative version of bimetric theory, having two different metrics $g$ and $h$ of Lorentzian signature on a manifold $M$ defining the tangential space $TM$ and co-tangential space $T^*M$, respectively. These can be obtained in terms of two types of matter and twin matter; existing individually. Each of them has its own field equations defined within Riemannian geometry.

It is well known that implementing bigravity theory, without a cosmological constants will be vital to describe the motion of dipolar objects in the halos [23]; while conformal gravity may be able to describe DM as mass excess quantities found in the accretion disk around the center of the Galaxy, the region of strong gravitational fields.

Meanwhile bimetric theories have one metric combining the two metrics, with a cosmological constant, describing a variable speed of light, replacing the DE effect in the big bang scenario [18].

From the previous versions of bimetric theories [19], we are going to present a generalized form which can present different types of path and path deviation using a bimetric theory which has two different metrics and curvatures defined in Riemannian geometry [20]. The corresponding Lagrangian can be written as [21]
\[
L = m_g g_{\mu\nu} \Psi_{\alpha} U^\mu U^\nu + m_f f_{\mu\nu} \Phi_{\mu\nu} V^\mu V^\nu + \left[ \frac{m_g(s)}{m_f(s)} (g^\alpha\beta - U^\alpha U^\beta) \right] \Psi^\rho + \left[ \frac{m_f(\tau)}{m_f(\tau)} (g^\alpha\beta - V^\alpha V^\beta) \right] \Phi^\rho. \tag{30}
\]

Thus, assuming
\[
(1) \quad \frac{d\sigma}{ds} = 0,
\]
this will give two separate sets of path equations owing to each parameter by applying the Bazanski-like Lagrangian:
\[
\frac{DU^\alpha}{DS} = \frac{m_g(s)}{m_f(s)} (g^\alpha\beta - U^\alpha U^\beta), \tag{31}
\]
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and
\begin{equation}
\frac{D V^\alpha}{D \tau} = \frac{m(f)}{m(f)(\tau)} f^\alpha - V^\alpha V^\beta,
\end{equation}
with the corresponding path deviation equations:
\begin{equation}
\frac{D^2 \Psi^\alpha}{D S^2} = R^\alpha_{\beta\gamma\delta} U^\gamma U^\beta \Psi^\delta + \left[ \frac{m(g)}{m(g)} g^\alpha^\beta - U^\alpha U^\beta \right] \Psi^\rho,
\end{equation}
and
\begin{equation}
\frac{D^2 \Phi^\alpha}{D \tau^2} = S^\alpha_{\beta\gamma\delta} V^\gamma V^\beta \Phi^\delta + \left[ \frac{m(f)}{m(f)} f^\alpha - V^\alpha V^\beta \right] \Phi^\rho.
\end{equation}
If, on the contrary,
\begin{equation}
\frac{D \tau}{D S} \neq 0 \quad [19],
\end{equation}
the two metrics can be related to each other by means of a quasi-metric [22].
\begin{equation}
\bar{g}_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu} + \alpha(g_{\mu\nu} - U_\mu U_\nu) + \alpha f(f_{\mu\nu} - V_\mu V_\nu),
\end{equation}
where \( \alpha_g \) and \( \alpha_f \) are arbitrary constants.

Such an assumption may give rise to defining the related Lagrangian of Bazanski’s flavor to describe the geodesic and geodesic deviation equations due to this version of bigravity theory.
\begin{equation}
L := \bar{g}_{\alpha\beta} U^\alpha \frac{\bar{D} \bar{\Psi}^\beta}{D S} + \bar{\Gamma}_{\beta\gamma\delta}^{\alpha} \left( \bar{g}_{\delta\beta\gamma} + \bar{g}_{\delta\beta\gamma} - \bar{g}_{\beta\gamma\delta} \right),
\end{equation}
and its corresponding Lagrangian:
\begin{equation}
L = m(S) \bar{g}_{\mu\nu} U^\mu \left( \frac{\bar{D} \bar{\Psi}^\nu}{D S} + \bar{\Gamma}_{\rho\gamma\delta}^{\mu} \bar{\Psi}^\rho \bar{U}^\gamma \bar{U}^\delta \right) + \bar{f}_\mu \bar{\Psi}^\mu.
\end{equation}
Thus, the path equation can be obtained by its variation with respect to \( \Psi^\mu \) to obtain
\begin{equation}
\frac{d \bar{U}^\alpha}{d S} + \bar{\Gamma}_{\beta\gamma\delta}^{\alpha} \bar{U}^\beta \bar{U}^\gamma = \frac{m(S)}{m(S)} \left( \bar{g}^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right),
\end{equation}
and using the commutation relation (A.14) and the condition (A.5), we obtain its corresponding deviation equation
\begin{equation}
\frac{D^2 \Phi^\alpha}{D S^2} = \bar{R}_{\nu\rho\alpha\beta} \bar{U}^\nu \bar{U}^\rho \bar{\Psi}^\sigma + \left( \frac{m(S)}{m(S)} g^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right) \bar{\Psi}^\rho,
\end{equation}
and
\begin{equation}
\hat{R}_{\mu\rho\nu\delta} = \bar{R}_{\nu\rho\alpha\beta} \bar{U}^\nu \bar{U}^\rho \bar{\Psi}^\sigma + \left( \frac{m(S)}{m(S)} g^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right) \bar{\Psi}^\rho.
\end{equation}

3.2. Equations for a Dipolar Moment in Bigravity Theory

The equation of motion for a dipolar moment in bimetric theory as a candidate to represent DM describes an interaction between ordinary and twin matter as presented by a bigravity ghost-free theory.

Accordingly, we suggest the following Lagrangian:
\begin{equation}
L = g_{\alpha\beta} \frac{D \Psi^\beta}{D S} + \frac{D \Psi^\beta}{D \tau} + f_\alpha \Psi^\alpha + \frac{D \Phi^\alpha}{D S} + \frac{\Delta \Phi^\alpha}{D \tau} + k_\alpha \Phi^\alpha + \bar{k}_\alpha \Phi^\alpha,
\end{equation}
where \( Q \) is the twin matter momentum vector, \( \Delta \) is the twin matter dipole moment vector, \( J \) is the twin nongravitational force on the dipole moment. Consequently, taking a variation with respect to \( \Psi_1, \Psi_2, \Phi_1, \) and \( \Phi_2, \) we obtain the equations for the dipolar momentum of ordinary matter, the evolution of ordinary matter, the twin dipolar moment and its evolution: Thus, for ordinary matter,
\begin{equation}
\frac{D P^\mu}{D S} = \bar{f}^\mu,
\end{equation}
and its corresponding evolution equation,
\begin{equation}
\frac{D Q^\mu}{D \tau} = k^\mu,
\end{equation}
For twin matter we obtain the equation of its dipolar moment
\begin{equation}
\frac{D Q^\mu}{D \tau} = k^\mu,
\end{equation}
where \( k^\mu \) is the corresponding nongravitational force. Also, the evolution equation of the twin dipolar moment is
\begin{equation}
\frac{D \Delta^\mu}{D \tau} = k^\mu,
\end{equation}
in which \( k^\mu \) is its associated nongravitational force.

Moreover, in order to obtain their corresponding deviation equations following the same procedures for both metrics \( g \) and \( f \) independently, we get after some manipulations the following set of deviation equations for ordinary and twin matter:

for ordinary matter,
\begin{equation}
\frac{D^2 \Psi^\mu}{D S^2} = \bar{R}_{\nu\rho\alpha\beta} \bar{U}^\nu \bar{U}^\rho \bar{\Psi}^\sigma + \left( \frac{m(S)}{m(S)} g^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right) \bar{\Psi}^\rho,
\end{equation}
and
\begin{equation}
\frac{D^2 \Psi^\mu}{D \tau^2} = \bar{R}_{\nu\rho\alpha\beta} \bar{U}^\nu \bar{U}^\rho \bar{\Psi}^\sigma + \left( \frac{m(S)}{m(S)} g^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right) \bar{\Psi}^\rho,
\end{equation}
and
\begin{equation}
\frac{D^2 \Phi^\alpha}{D S^2} = \bar{R}_{\nu\rho\alpha\beta} \bar{U}^\nu \bar{U}^\rho \bar{\Psi}^\sigma + \left( \frac{m(S)}{m(S)} g^\alpha^\beta - \bar{U}^\alpha \bar{U}^\beta \right) \bar{\Psi}^\rho.
\end{equation}
$$D^2{\Psi}_\mu = R^\mu_{\nu\rho\sigma} \Pi^\nu U^\rho \Psi_\sigma + \dot{f}_\mu \Psi_{(2)},$$

(46)

and for twin matter

$$D^2{\Phi}_\mu = S^\mu_{\nu\rho\sigma} \nabla^\nu \nabla^\rho \Phi_\sigma + k^\mu_\nu \Phi_{(1)},$$

(47)

$$D^2{\bar{\Phi}}_\mu = S^\mu_{\nu\rho\sigma} \nabla^\nu \nabla^\rho {\bar{\Phi}}_\sigma + \dot{k}^\mu_\nu {\bar{\Phi}}_{(1)},$$

(48)

where $S^\mu_{\nu\rho\sigma}$, $\nabla^\mu$, and $\dot{\Phi}$ are their associated curvature, four-velocity, the polarization vector for particles defined as twin matter, respectively.

3.3. Dipolar Fluid in Bigravity Theory

Extending the previous ideas as discussed in Subsection 3.2, to examine the existence of DM using bigravity ghost free theory and to describe both the ordinary and twin fluids simultaneously, we suggest the following Lagrangian:

$$L = g_{\mu\nu} K^\mu D\Psi_{(1)} + \Omega_{\mu} D\Psi_{(2)} + f_{\mu
u} \dot{K}^\mu D\Psi_{(1)} + \dot{\Omega}_{\mu} D\Psi_{(2)},$$

(49)

where $\dot{K}^\mu$ is the twin matter linear momentum, $\Phi_{(1)}^{\mu}$ is its deviation vector, $\dot{\Phi}_{\mu}$ is the evolution vector associated with twin matter, and $\Phi_{(2)}^{\mu}$ is the deviation vector of twin matter evolution, such that

$$\dot{f}_1 \frac{1}{\text{Halo}} \nabla \mu (W - \hat{W} W) = R^\mu_{\rho\nu\lambda} \nu^\rho K^\lambda.$$ 

Thus, taking a variation with respect to $\Psi_1$, $\Psi_2$, $\Phi_1$, and $\Phi_2$, we obtain for the ordinary fluid

$$\frac{DK^\mu}{Ds} = f_1,$$

(50)

and

$$\frac{D\Omega^\mu}{Ds} = f_2.$$ 

(51)

For the twin fluid,

$$\frac{D\dot{K}^\mu}{D\tau} = \dot{f}_1,$$

(52)

and

$$\frac{D\dot{\Omega}^\mu}{D\tau} = \frac{1}{\sigma} \nabla^\mu (W - \hat{W} W) - S^\mu_{\rho\nu\lambda} V^\rho \hat{\xi}^\nu \hat{K}^\lambda,$$

(53)

where $\hat{W}^\mu$, $\hat{\Omega}^\mu$, and $\hat{K}^\mu$ are the corresponding twin density-dependent potential, the polarization vector, and the related linear momentum vector parameterized due to the dipolar description as expressed in bigravity theory.

3.4. Nongeodesic Equations in AGN: Bigravity Theory

The bi-metric version of Eq. (4) can be obtained as the Euler–Lagrange equation from the Lagrangian

$$\tilde{L} = \tilde{g}_{\alpha\beta} \tilde{U}^\alpha D\tilde{\Psi}^\beta.$$

(54)

leading to the path equation

$$\frac{d\tilde{U}^{\alpha}}{ds} + \Gamma^\alpha_{\beta\delta} \tilde{U}^\beta \tilde{\Omega}^\delta = \tilde{m}(s) \tilde{\beta}(\tilde{g}^{\alpha\beta} - \tilde{U}^\alpha \tilde{U}^\beta),$$

(55)

and using the commutation relation (A.4) and the condition (A.5), we obtain its corresponding deviation equation

$$\frac{D^2\Psi^{\mu}}{Ds^2}\tilde{D} \tilde{g}^{\mu\nu} \tilde{U}^\mu \tilde{U}^\nu \tilde{\Psi}^{\sigma}$$

$$+ \left[ \frac{\tilde{m}(s) \tilde{\beta}(\tilde{g}^{\alpha\beta} - \tilde{U}^\alpha \tilde{U}^\beta) \right] \tilde{\Psi}^{\rho}.$$ 

(56)

4. DARK MATTER: THE STABILITY PROBLEM

4.1. Testing the Stability of Celestial Objects by the Geodesic Deviation Vector

The geodesic (or nongeodesic) deviation equations obtained from the path equation for an object, whether it is treated as a test particle or not, are inevitably used for examining the stability of the system. The term stability characterizes the amount of perturbations using the deviation vector along the course of motion and reveal the status of objects in the presence of DM.

In this present work, we are going to implement such a technique which has been applied previously in examining the stability of some cosmological models using two geometric structures [23].

Recently, this approach was modified [24], regarding the stability condition as a result of obtaining the vectorial value of the deviation vector, independent of any coordinate system, being in a covariant form, able to study the stability problem for any planetary system, and extended for examining the stability of stellar systems orbiting sources of strong gravitational fields [25].

Thus, the geodesic deviation equation (11) has the solution

$$\Psi^{\mu} = f(S) C^{\mu},$$

where $C^\alpha$ are constants and $f(S)$ is a function known from the metric. If $f(S) \rightarrow \infty$, the system becomes unstable, otherwise it is stable in a given interval [a, b].

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in which $\Psi^\alpha(S)$ behaves monotonically. These quantities can become sensors for measuring the stability of the system,

$$q := \lim_{s \to 0} \sqrt{\Psi^\alpha \Psi_\alpha}. \quad (57)$$

If $q \to \infty$, then the system is unstable, otherwise it is always stable.

Yet this condition cannot be solely satisfied if one studies the case of dipolar particles (fields).

The necessary and sufficient conditions should be related to the solution of geodesic (nongeodesic) and evolution deviation equations simultaneously, i.e.,

$$\Psi^1 = f(S)C^1, \quad (60)$$

and

$$\Psi^2 = f(S)C^2, \quad (61)$$

where $C^1$ and $C^2$ are constants, and $f(S)$ is a function known from the metric. If $f(S) \to \infty$, the system becomes unstable, otherwise it is stable in a given interval $[a, b]$ in which $\Psi^1(S)$ and $\Psi^2(S)$ behave monotonically. These quantities can become sensors for measuring the stability of the system.

Yet, these conditions can be extended to the case of a bimetric theory in the following way:

In case the case $\frac{d\tau}{ds} \neq 0$, the solution of the set of deviation equations (21) and (22) are

$$\Psi(1)a^\alpha = C^\alpha f(s), \quad (58)$$

$$\Phi(1)a^\alpha = C^\alpha f(\tau). \quad (59)$$

Thus we must obtain two stability conditions in the following way:

$$q_1 := \lim_{s \to b} \sqrt{\Psi^1 \Psi_{(1)\alpha}}. \quad (60)$$

and

$$q_2 \stackrel{def}{=} \lim_{\tau \to b} \sqrt{\Phi^1 \Phi_{(1)\alpha}}. \quad (61)$$

Meanwhile, in the case of dipolar particles in bimetric theory we get two more conditions:

$$\Psi(2)a^\alpha = C^\alpha f(s), \quad (62)$$

$$\Phi(2)a^\alpha = C^\alpha f(\tau). \quad (63)$$

In the case of the Verozub bimetric version [9], $d\tau/ds = 0$, the above two conditions for stability of a test particle reduce to a single one while four conditions for a dipole particle reduce to only two.

5. DISCUSSION AND CONCLUSION

Dark matter may be regarded either as particles or a fluid due to its detection as a source of the gravitational field. This has led many authors to reconsider it and to offer alternatives such as dipolar particles or fluids, an effect of the scalar field and its additional gravitational field, or even as a result of a projection from higher dimensions. Among the variety of definitions and notations, a class of bimetric theories of gravity has been presented to describe the status of these gravitational fields, whether it is very strong as in the core of the galaxy or a neutron star or weak as near the Sun, but still satisfying the tests of relativity. This type of theory consists in studying the motion of particles in terms of their path and deviation vectors. The use of deviation equations provides a schematic approach for estimating the stability of these systems in a covariant form, as mentioned in Section 4. It has been demonstrated that two conditions are essential in examining the stability of a test particle in a bimetric theory. As these two conditions are applied, a double effect is examined in bigravity theories. However, applying the Verozub version of bimetric gravity shows that its behavior is the same as in GR. Owing to the equation of motion, it is vital to examine the stability of these regions by solving the geodesic deviation equations, due to interrelation between geodesic deviation equations and the stability conditions.

In the present work, it has been found that nongeodesic particles, as described in a bimetric theory of gravity, may be regarded as a good representative to DM in different regions [26–29].

Nevertheless, DM has another rival explanation, to be examined near active galactic nuclei such as SgrA*, due to the excess of mass appeared in the equations of relativistic hydrodynamics (27), which is present in the nongeodesic equation (3). Also, we have a connection between the MOND parameters and the rate of mass excess term, upon parametrization shown in Eqs. (6) and (30).

Finally, we sum up that the quest of identifying precisely the nature of DM is still under debate. Yet, some authors believe that it may be regarded as a massive neutrino, a supersymmetric neutralino or even an axion [30]. The problem of motion as described in Riemannian geometry is extended to descriptions in different geometries, admitting nonvanishing curvature and torsion simultaneously.

Our work will continue to emphasize the concept of geometrization of physics in determining the existence of DM and DE by different classes of non-Riemannian geometry, as a further step in demystifying the various manifestations of DM and DE.
The Papapetrou Equation in General Relativity: Lagrangian Formalism

It is well known that the equations for spinning objects in a gravitational field have been studied extensively. This led us to suggesting the corresponding Lagrangian formalism, using a modified Bazanski Lagrangian [31] for a spinning and precessing object and their deviation equation in Riemannian geometry in the following way:

\[ L = g_{\alpha\beta} P^\alpha \frac{D\Psi^\beta}{DS} + S_{\alpha\beta} \frac{D\Psi^{\alpha\beta}}{DS} + F_\alpha \Psi^\alpha + M_{\alpha\beta} \Psi^{\alpha\beta}, \tag{A.1} \]

where \( P^\alpha = m U^\alpha + U_\beta \frac{D S_{\beta\gamma}}{DS}, \) and \( \Psi^{\mu\nu} \) is the spin deviation tensor.

Varying with respect to \( \Psi^\alpha \) and \( \Psi^{\mu\nu} \) simultaneously, we obtain

\[ \frac{DP^\mu}{DS} = F^\mu, \tag{A.2} \]
\[ \frac{DS^{\mu\nu}}{DS} = M^{\mu\nu}, \tag{A.3} \]

where \( P^\mu \) is the momentum vector, \( F^\mu = \frac{1}{2} R_{\nu\rho}^{\mu} \times S^{\nu\rho} U^\nu, \) and \( R_{\beta\rho\sigma}^{\alpha} \) is the Riemann curvature, \( \frac{D}{DS} \) is the covariant derivative with respect to a parameter \( S, S_{\alpha\beta} \) is the spin tensor, \( M^{\mu\nu} = P^\mu U^\nu - P^\nu U^\mu, \) and \( U^\alpha = \frac{dx^\alpha}{ds} \) is the unit tangent vector to the geodesic.

In equations (1) and (2) we use the identity

\[ A^{\mu}_{\nu\rho} - A^{\mu}_{\nu\rho} = R_{\beta\rho\sigma}^{\alpha} A^{\beta}, \tag{A.4} \]

where \( A^\mu \) is an arbitrary vector. Multiplying both sides of which equation?? \( \Psi^\alpha \) by arbitrary vectors \( U^\rho \Psi^\nu \) and using the condition [15].

\[ U_{\nu\rho}^{\alpha} \Psi^{\rho} = \Psi_{\nu\rho}^{\alpha} U^{\rho}, \tag{A.5} \]

and \( \Psi^\alpha \) is its deviation vector associated to the unit tangent vector \( U^\alpha \), and in a similar way:

\[ S_{\nu\rho}^{\alpha\beta} \Psi^{\rho} = \Psi_{\nu\rho}^{\alpha\beta} U^{\rho}, \tag{A.6} \]

one obtains the deviation equations [32]

\[ \frac{D^2 \Psi^\mu}{DS^2} = R_{\nu\rho\sigma}^{\mu} P^{\nu\rho} U^\rho \Psi^\sigma + F_{\nu\rho}^{\mu} \Psi^{\rho}, \tag{A.7} \]
\[ \frac{D^2 \Psi^{\mu\nu}}{DS^2} = S_{\nu\rho\sigma}^{\mu\nu} R_{\rho\sigma\epsilon}^{\nu} U^\epsilon \Psi^\rho + M_{\nu\rho}^{\mu\nu} \Psi^{\rho}. \tag{A.8} \]