Another Criterion For The Riemann Hypothesis

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Abstract

Let’s define \( \delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log x - B) \), where \( B \approx 0.2614972128 \) is the Meissel-Mertens constant. The Robin theorem states that \( \delta(x) \) changes sign infinitely often. Let’s also define \( S(x) = \theta(x) - x \), where \( \theta(x) \) is the Chebyshev function. It is known that \( S(x) \) changes sign infinitely often. We define another function \( \varpi(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B) \). We prove that when the inequality \( \varpi(x) \leq 0 \) is satisfied for some number \( x \geq 3 \), then the Riemann hypothesis should be false. The Riemann hypothesis is also false when the inequalities \( \delta(x) \leq 0 \) and \( S(x) \geq 0 \) are satisfied for some number \( x \geq 3 \) or when \( 3 \times \log x + \frac{5}{8} \times \pi \times \sqrt{x} + 1.2 \times \log \log \log x \leq 1 \) is satisfied for some number \( x \geq 13.1 \) or when there exists some number \( y \geq 13.1 \) such that for all \( x \geq y \) the inequality \( 3 \times \log x + \frac{5}{8} \times \pi \times \sqrt{x} + 1.2 \log \log \log \log x \leq 1 \) is always satisfied for some positive constant \( C \) independent of \( x \).

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1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [1]. Let \( N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n \) denotes a primorial number of order \( n \) such that \( p_n \) is the \( n \)th prime number. Say Nicolas\((p_n)\) holds provided

\[
\prod_{q \mid N_n} \frac{q}{q - 1} > e^{\gamma} \times \log \log N_n.
\]

The constant \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant, \( \log \) is the natural logarithm, and \( q \mid N_n \) means the prime number \( q \) divides to \( N_n \). The importance of this property is:

**Theorem 1.1.** [2]. Nicolas\((p_n)\) holds for all prime numbers \( p_n > 2 \) if and only if the Riemann hypothesis is true.

In mathematics, the Chebyshev function \( \theta(x) \) is given by

\[
\theta(x) = \sum_{p \leq x} \log p
\]
where \( p \leq x \) means all the prime numbers \( p \) that are less than or equal to \( x \). We know these properties for this function:

**Theorem 1.2.** \([3]\). 
\[
\lim_{x \to \infty} \frac{\theta(x)}{x} = 1.
\]

**Theorem 1.3.** \([4]\). There are infinitely many values of \( x \) such that
\[
\theta(x) > x + C \times \sqrt{x} \times \log \log x
\]
for some positive constant \( C \) independent of \( x \).

Let’s define \( S(x) = \theta(x) - x \). It is a known result that:

**Theorem 1.4.** \([5]\). \( S(x) \) changes sign infinitely often.

We also know that

**Theorem 1.5.** \([6]\). If the Riemann hypothesis holds, then
\[
\left( \frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}
\]
for all numbers \( x \geq 13.1 \).

Let’s define \( H = \gamma - B \) such that \( B \approx 0.2614972128 \) is the Meissel-Mertens constant \([7]\). We know from the constant \( H \), the following formula:

**Theorem 1.6.** \([8]\). 
\[
\sum q \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) = \gamma - B = H.
\]

For \( x \geq 2 \), the function \( u(x) \) is defined as follows
\[
u(x) = \sum_{q > x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right).
\]

Nicols showed that

**Theorem 1.7.** \([2]\). For \( x \geq 2 \):
\[
0 < u(x) \leq \frac{1}{2 \times (x-1)}.
\]

Let’s define:
\[
\delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).
\]

Robin theorem states the following result:

**Theorem 1.8.** \([9]\). \( \delta(x) \) changes sign infinitely often.

In addition, the Mertens second theorem states that:
Theorem 1.9. \cite{7}.

\[ \lim_{x \to \infty} \delta(x) = 0. \]

Besides, we use the following theorems:

Theorem 1.10. \cite{10}. For \( x > -1 \):

\[ \frac{x}{x + 1} \leq \log(1 + x) \leq x. \]

Theorem 1.11. \cite{11}. For \( x \geq 1 \):

\[ \log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}. \]

We define another function:

\[ \varpi(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right). \]

Putting all together yields the proof that the inequality \( \varpi(x) > u(x) \) is satisfied for a number \( x \geq 3 \) if and only if Nicolas\((p)\) holds, where \( p \) is the greatest prime number such that \( p \leq x \). In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

2. Results

Theorem 2.1. The inequality \( \varpi(x) > u(x) \) is satisfied for a number \( x \geq 3 \) if and only if Nicolas\((p)\) holds, where \( p \) is the greatest prime number such that \( p \leq x \).

Proof. We start from the inequality:

\[ \varpi(x) > u(x) \]

which is equivalent to

\[ \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right) > \sum_{q > x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right). \]

Let’s add the following formula to the both sides of the inequality,

\[ \sum_{q \leq x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right) \]

and due to the theorem 1.6, we obtain that

\[ \sum_{q \leq x} \log \left( \frac{q}{q - 1} \right) - \log \log \theta(x) - B > H \]

because of

\[ H = \sum_{q \leq x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right) + \sum_{q > x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right) \]

and
\[ \sum_{q \leq x} \log\left( \frac{q}{q-1} \right) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left( \log\left( \frac{q}{q-1} \right) - \frac{1}{q} \right) . \]

Let’s distribute it and remove \( B \) from the both sides:
\[ \sum_{q \leq x} \log\left( \frac{q}{q-1} \right) > \gamma + \log \log \theta(x) \]
since \( H = \gamma - B \). If we apply the exponentiation to the both sides of the inequality, then we have that
\[ \prod_{q \leq x} \frac{q}{q-1} > e^\gamma \times \log \theta(x) \]
which means that Nicolas\( (p) \) holds, where \( p \) is the greatest prime number such that \( p \leq x \). The same happens in the reverse implication. \( \square \)

**Theorem 2.2.** The Riemann hypothesis is true if and only if the inequality \( \omega(x) > u(x) \) is satisfied for all numbers \( x \geq 3 \).

**Proof.** This is a direct consequence of theorems 1.1 and 2.1. \( \square \)

**Theorem 2.3.** If the inequality \( \omega(x) \leq 0 \) is satisfied for some number \( x \geq 3 \), then the Riemann hypothesis should be false.

**Proof.** This is an implication of theorems 1.7, 2.1 and 2.2. \( \square \)

**Theorem 2.4.** If the inequalities \( \delta(x) \leq 0 \) and \( S(x) \geq 0 \) are satisfied for some number \( x \geq 3 \), then the Riemann hypothesis should be false.

**Proof.** If the inequalities \( \delta(x) \leq 0 \) and \( S(x) \geq 0 \) are satisfied for some number \( x \geq 3 \), then we obtain that \( \omega(x) \leq 0 \) is also satisfied, which means that the Riemann hypothesis should be false according to the theorem 2.3. \( \square \)

**Theorem 2.5.**
\[ \lim_{x \to \infty} \omega(x) = 0. \]

**Proof.** We know that \( \lim_{x \to \infty} \omega(x) = 0 \) for the limits \( \lim_{x \to \infty} \delta(x) = 0 \) and \( \lim_{x \to \infty} \frac{\delta(x)}{x} = 1 \). In this way, this is a consequence from the theorems 1.9 and 1.2. \( \square \)

**Theorem 2.6.** If the Riemann hypothesis holds, then
\[ \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} + \frac{1.2 \times \log x + 2}{\log \theta(x)} > 1 \]
for all numbers \( x \geq 13.1 \).

**Proof.** Under the assumption that the Riemann hypothesis is true, then we would have
\[ \prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \log x \times \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right) \]
after of distributing the terms based on the theorem 1.5 for all numbers \( x \geq 13.1 \). If we apply the logarithm to the both sides of the previous inequality, then we obtain that

\[
\sum_{q \leq x} \log \left( \frac{q}{q - 1} \right) < \gamma + \log \log x + \log \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right).
\]

That would be equivalent to

\[
\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]

where we know that

\[
\log \left( 1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right) = \frac{1}{8 \times \pi \times \sqrt{x} + 0.4} \times \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]

according to theorem 1.11 since \( \frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1 \) for all numbers \( x \geq 13.1 \). We use the theorems 1.6 and 1.7 to show that

\[
\sum_{q \leq x} \left( \log \left( \frac{q}{q - 1} \right) - \frac{1}{q} \right) = H - u(x)
\]

and \( \gamma = H + B \). So,

\[
H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]

which is the same as

\[
\begin{align*}
H - u(x) &< H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} \\

\end{align*}
\]

We eliminate the value of \( H \) and thus,

\[
-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]

which is equal to

\[
u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).
\]

We know from the theorem 2.1 that \( \nu(x) > \mu(x) \) for all numbers \( x \geq 13.1 \) and therefore,

\[
u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).
\]
Hence,
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.
\]
Suppose that \( \theta(x) = \epsilon \times x \) for some constant \( \epsilon > 1 \). Then,
\[
\log \log \theta(x) - \log \log x = \log \log(\epsilon \times x) - \log \log x
\]
\[
= \log \left( \log x + \log \epsilon \right) - \log \log x
\]
\[
= \log \log x + \log(1 + \frac{\log \epsilon}{\log x}) - \log \log x
\]
\[
= \log(1 + \frac{\log \epsilon}{\log x}).
\]
In addition, we know that
\[
\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)}
\]
using the theorem 1.10 since \( \frac{\log \epsilon}{\log x} > -1 \) when \( \epsilon > 1 \). Certainly, we will have that
\[
\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)} = \frac{\log \epsilon}{\log \theta(x)}.
\]
Thus,
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.
\]
If we add the following value of \( \frac{\log x}{\log \theta(x)} \) to the both sides of the inequality, then
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = \frac{\log \epsilon + \log x}{\log \theta(x)} = \frac{\log \theta(x)}{\log \theta(x)} = 1.
\]
We know this inequality is satisfied when \( 0 < \epsilon \leq 1 \) since we would obtain that \( \frac{\log x}{\log \theta(x)} \geq 1 \). Therefore, the proof is done.

\textbf{Theorem 2.7.} If the inequality
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} \leq 1
\]
is always satisfied for some positive constant \( C \) independent of \( x \), then the Riemann hypothesis should be false.

\textit{Proof.} This is a direct consequence of theorem 2.6.

\textbf{Theorem 2.8.} If there exists some number \( y \geq 13.1 \) such that for all \( x \geq y \) the inequality
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} \leq 1
\]
is always satisfied for some positive constant \( C \) independent of \( x \), then the Riemann hypothesis should be false.

\textit{Proof.} From the theorem 1.3, we know that there are infinitely many values of \( x \) such that
\[
\theta(x) > x + C \times \sqrt{x} \times \log \log x.
\]
for some positive constant $C$ independent of $x$. That would be equivalent to

$$\log \theta(x) > \log(x + C \times \sqrt{x} \times \log \log x)$$

and so,

$$\frac{1}{\log \theta(x)} = \frac{1}{\log(x + C \times \sqrt{x} \times \log \log x)} > \frac{1}{\log x}$$

for all numbers $x \geq 13.1$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log x)}.$$ 

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log x)} > 1$$

for those values of $x$ that comply with

$$\theta(x) > x + C \times \sqrt{x} \times \log \log x$$

due to the theorem 2.6. By contraposition, if there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log x)} \leq 1$$

is always satisfied for some positive constant $C$ independent of $x$, then the Riemann hypothesis should be false, because of there are infinitely many values of $x$ which satisfy the inequality in the theorem 1.3 and comply with $x \geq y$ no matter how big could be $y$. 

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