STABILITY OF THE CENTER OF THE SYMPLECTIC GROUP RINGS OVER
FINITE FIELDS

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ABSTRACT. We investigate the structure constants of the center \( H_n \) of the group algebra \( \mathbb{Z}[Sp_n(q)] \) over the finite field with \( q \) elements. The reflection length on the group \( GL_{2n}(q) \) induces a filtration on the algebras \( H_n \). We prove that the structure constants of the associated filtered algebra \( S_n \) are independent of \( n \). As a technical tool in the proof, we determine the growth of the centralizers under the embedding \( Sp_m(q) \subset Sp_{m+l}(q) \) and we show that the index of \( C_{Sp_m}(g) \cap C_{Sp_{m+l}}(h) \) in \( C_{Sp_{m+l}}(g) \cap C_{Sp_{m+l}}(h) \) is equal to \( q^{2d}[Sp_{r+l}(q)\|Sp_r(q)]^{-1} \) for some \( d \) and \( r \) which are uniquely determined by the conjugacy classes of \( g, h \) and \( gh \) in \( GL_{2m}(q) \).

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1. Introduction

Let \( G_1 \subset \cdots \subset G_n \subset \cdots \) be a family of finite groups and let \( \mathcal{H}_n \) denote the center of the group algebra \( \mathbb{Z}[G_n] \) for \( n \in \mathbb{N} \). The set of conjugacy classes of \( G_n \) is denoted by \( \widehat{G}_n \). For \( \lambda \in \widehat{G}_n \), the class sum \( \sum_{g \in \lambda} g \in \mathbb{Z}[G_n] \) is denoted by \( K_\lambda \). The class sums \( K_\lambda, \lambda \in \widehat{G}_n \), form a basis for \( \mathcal{H}_n \). We introduce the term saturated family to refer to the families \((G_n)_n \in \mathbb{N}\) for which non-conjugate elements of \( G_n \) remain non-conjugate in \( G_{n+1} \). Assume that the family \((G_n)_n \in \mathbb{N}\) is saturated. The embedding \( G_n \hookrightarrow G_{n+1} \) induces an injection \( \widehat{G}_n \hookrightarrow \widehat{G}_{n+1} \). Let \( G \) be the union of \( G_n \). For each \( \lambda \in \widehat{G} \), the intersection \( \lambda(n) := \lambda \cap G_n \)

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is either empty or an element of $\widehat{G}_n$, and every element of $\widehat{G}_n$ can be represented as such an intersection. Given three elements $\lambda, \mu, \eta$ in $G$, there is an $m_{\lambda,\mu,\eta} = m \in \mathbb{N}$ such that for all $n \geq m$, each of $(\lambda(n), \mu(n), \eta(n))$ are nonempty. So, for fixed $\lambda, \mu, \eta \in \widehat{G}$ and $n \geq m_{\lambda,\mu,\eta}$, the product $K_{\lambda(n)} \cdot K_{\mu(n)}$ can be written as

$$K_{\lambda(n)} \cdot K_{\mu(n)} = \sum_{\eta \in \widehat{G}} c_{\lambda,\mu,\eta}(n)K_{\eta(n)} = \sum_{\eta \in \widehat{G}, \eta(n) \neq \emptyset} c^\eta_{\lambda,\mu}(n)K_{\eta(n)}$$

where $c^\eta_{\lambda,\mu}(n) \in \mathbb{N}$, in which $c^\eta_{\lambda,\mu}(n)$ is uniquely determined as $K_{\eta(n)} \neq 0$. For a fixed $n$, the collection of $c^\eta_{\lambda,\mu}(n)$, where $\lambda(n), \mu(n), \eta(n)$ runs over $\widehat{G}_n$, are called the structure constants of the algebra $H_n$. We will call the functions $n \rightarrow c^\eta_{\lambda,\mu}(n)$ the structure functions of the family. If $|| \cdot ||$ is an $\mathbb{N}$ valued function on $G_n$ which is constant on conjugacy classes then $|| \cdot ||$ induces a function on $\widehat{G}_n$ as well. In this case, if the function is also sub-additive, in the sense that $||gh|| \leq ||g|| + ||h||$, and if $|| \cdot ||$ is invariant under the embedding $G_n \subset G_{n+1}$ then the algebra $H_n$ induces the filtered algebra $S_n$ with the same basis elements, where the multiplication is defined as

$$(1) \quad K_{\lambda(n)} \cdot K_{\mu(n)} = \sum_{||\eta|| = ||\lambda|| + ||\mu||} c^\eta_{\lambda,\mu}(n)K_{\eta(n)}.$$

When the structure functions defined via Eq. (1) of the filtered algebra of a family $(G_n)_{n \in \mathbb{N}}$ are independent of $n$, following Wan and Wang [20], we will say that the family satisfies the stability property.

For $n \in \mathbb{N}$, let $S_n$ denote the symmetric group of the set $\{1, 2, \ldots, n\}$. Farahat and Higman considered the family $(S_n)_{n \in \mathbb{N}}$ in [6] and proved that with respect to the filtration induced by reflection length, the structure constants $c^\eta_{\lambda,\mu}(n)$ of the induced filtered algebra structure on $Z(\mathbb{Z}[S_n])$ are independent of $n$. They used this result to answer the question of determining whether two representations of $S_n$ belong to the same $p$-block. In [21], as a generalization of the case considered by Farahat and Higman, Wang proved that the families generated by the wreath product $(H \wr S_n)_{n \in \mathbb{N}}$, where $H$ is a finite group, satisfy the stability property. In the case studied by Wang, when the group $H$ is a finite subgroup of $SL_2(\mathbb{C})$, the associated graded algebra of $H_n$ is isomorphic to the cohomology ring of Hilbert scheme of $n$ points on the minimal resolution of $\mathbb{C}^2/H$. Recently, in [20], Wan and Wang considered the family $(GL_n(q))_{n \in \mathbb{N}}$ and proved that this family also satisfies the stability property with respect to the filtration induced by reflection length. The result of Wan and Wang was also obtained by P.-L. Méliot in [13].

In this paper we study the family $(Sp_n(q))_{n \in \mathbb{N}}$ of symplectic groups over the finite field with $q$ elements. We introduce the set of modified symplectic partition valued functions and prove that these functions parameterize the conjugacy classes of $U_{n \in \mathbb{N}}Sp_n(q)$ and that the family $(Sp_n(q))_{n \in \mathbb{N}}$ is saturated. We consider the filtration induced from the reflection length in $GL_{2n}(q)$. The set of reflections generate $GL_{2n}(q)$ and for $U \in GL_{2n}(q)$, the minimum value of $l$ where $U$ can be written as a product of $l$ many reflections is called the reflection length of $U$ and denoted by $rl(U)$. It is constant on conjugacy classes, sub-additive function and stable under the embedding $Sp_n(q) \subset Sp_{n+1}(q)$. Therefore, for a stabilized symplectic partition valued function $\lambda$, one can talk about $||\lambda||$. With this setting, the main result is following.

**Theorem** (Stability property). [Theorem 4.30] Let $\lambda, \mu, \eta$ be three stabilized symplectic partition valued functions and assume that $||\eta|| = ||\lambda|| + ||\mu||$. Then $c^\eta_{\lambda,\mu}(n)$ is a non-negative integer independent of $n$.

We observe that all the stability properties proved so far rely on two fundamental facts: A certain action admits finitely many orbits and certain splitting of the centralizers. More precisely, in each case one first proves that a pair $(g, h) \in G \times G$ can be mapped to $G_m \times G_m$ by simultaneous conjugation, where $m$ is a fixed integer completely determined by the conjugacy classes of $g, h$ and $gh$. To prove such a result, one needs to find a so-called normal form, a formulation introduced in [20]. We will refer to the existence of normal forms as normal form theorems. Secondly, one shows that the centralizer of $g \in G_m$ splits in the centralizer of $g$ in $G_n$ for $n \geq m$, which we will call the growth of centralizers.

In the case of symplectic groups, finding a normal form can be derived from the case of general linear groups. However, the investigation of the growth of centralizers in the case of symplectic group is more complicated than the case of general linear groups, as it consists of non-linear equations. To overcome this obstacle, we first introduce a concept called primitive symplectic centralizer, and using suitable rational forms we investigate the elements in the centralizers of a unipotent element and then invoke the concept
of primitive symplectic centralizer to reduce the question of centralizer growth to a linear question. Once the degree 2 problem is reduced to a linear problem the problem becomes much more manageable. The simplified versions of these results (Proposition 5.38 and Proposition 5.41) are packed into the following:

**Theorem (Growth of centralizers).** Let $U = U_1U_2 \in \text{Sp}_m$ and $d_U$ be the dimension of the fixed space $V^U := \ker(U - I)$ of $U$, c.f. Eq. (9). Assume that there is no identity block in the Jordan form of $U$. Then for $m \leq n$ the following equalities hold:

$$|C_{\text{Sp}_n(q)}(U)| = |C_{\text{Sp}_m(q)}(U)| \cdot |\text{Sp}_{n-m}(q)| \cdot q^{2(n-m)d_U}.$$ 

If $r(U_1) + r(U_2) = r(U)$, where $rU$ denotes the reflection length, then

$$|C_{\text{Sp}_m(U_1)} \cap C_{\text{Sp}_m(U_2)}| = |C_{\text{Sp}_m(U_1)} \cap C_{\text{Sp}_m(U_2)}| \cdot |\text{Sp}_{n-m}(q)| \cdot q^{2(n-m)d_U}.$$

It is worth to mention a generalized approach to the center of the integral group rings. Namely, in terms of Gel’fand pairs. Recall that a pair of finite groups $H \subseteq G$ is called a Gel’fand pair, if the convolution algebra

$$\mathcal{H}(G, H) = \{ f : G \rightarrow \mathbb{Z} | f(gh'') = f(g), \forall h, h' \in H, \forall g \in G \}$$

of the $\mathbb{Z}$-valued functions on $G$ that are invariant on the $G$-double cosets of $G$ is commutative. Let $G$ be a finite group. If one considers the pair $(G, \text{diag}(G))$ where $\text{diag}(G) = \{(g, g) \in G \times G | g \in G \}$ then there is a $\mathbb{Z}$-algebra isomorphism

$$\mathcal{H}(G) \simeq \mathcal{H}(G \times G, \text{diag}(G)).$$

For details on this isomorphism, see [5, Proposition 1.5.22]. For an extensive study on Gel’fand pairs related to symmetric groups see [4]. Relying on this observation, one can generalize the concepts discussed earlier.

First notice that, the analogous basis elements in this case are given by the characteristic functions on $H$-double cosets of $G$. More precisely, if $\Theta$ denotes the set of $H$-double cosets of $G$, the elements

$$K_\lambda = \sum_{g \in \lambda} g$$

is an element of $\mathcal{H}(G, H)$ and the set $\{K_\lambda | \lambda \in \Theta \}$ constitute a basis for $\mathcal{H}(G, H)$. This means, if $\lambda \mu \in \Theta$ are fixed, then for all $\eta \in \Theta$, there exists unique $c^{\eta}_{\lambda, \mu} \geq 0$ such that

$$K_\lambda \cdot K_\mu = \sum_{\eta \in \Theta} c^{\eta}_{\lambda, \mu} \cdot K_\eta.$$

Consider a sequence of groups $\{G_n\}_{n \in \mathbb{N}}$ and a family of subgroups $\{H_n \leq G_n\}_{n \in \mathbb{N}}$. Let $G$ (resp. $H$) be the direct limit of $G_n$’s (resp. $H_n$’s). Then $H \leq G$. Let $\mathcal{H} = \mathcal{H}(G, H)$ (resp. $\mathcal{H}_n = \mathcal{H}(G_n, H_n)$) be the Hecke algebra corresponding to $(G, H)$ (resp. $(G_n, H_n)$). Each double coset of $H_n$ in $G_n$ extends to a unique $H_{n+1}$ double coset in $G_{n+1}$. If every distinct $H_n$ double cosets in $G_n$ remains distinct in $G_{n+1}$, then we say that the family $(G_n, H_n)$ is saturated.

Let $\Theta$ (resp. $\Theta_n$) denote the set of double cosets of $H$ (resp. $H_n$) in $G$ (resp. $G_n$) and $\Theta(n) := \{ \theta(n) := \theta \cap G_n : \theta \in \Theta \}$. If $H$-double cosets of $G$ is $H_n$-saturated then $\Theta(n) = \Theta_n$. For $n \geq 0$, one can then define $K_\lambda(n)$ for $\lambda \in \Theta$ in a similar way and introduce the structural functions $c^\eta_{\lambda, \mu}(n)$ satisfying

$$K_\lambda(n) \cdot K_\mu(n) = \sum_{\eta \in \Theta} c^\eta_{\lambda, \mu}(n) \cdot K_\eta(n).$$

In this setting, study of the structure constants of saturated families of pairs makes sense. The saturated family $(S_{2n}, B_n)$ and its structure constants are investigated in the papers [1], [3] and [17]. It turns out that, this family also satisfy the stability property, i.e. the structural functions corresponding to the top coefficients with respect to a suitable filtration are independent of $n$. For a detailed study of the pair $(S_n, B_n)$ see [4].

Finally, we recall the Frobenious formula which justifies the attention on the structure constants of the center of the integral group rings. The proof of the following theorem can be found in the appendix of [11]:

\[\text{[Details of the formula and its proof are provided here, suitable for the context of the text.]}\]
**Theorem** (Frobenious formula). Let $\lambda, \mu, \eta$ be three conjugacy classes of a finite group $G$ and let $\eta^{-1}$ be the conjugacy which consists of elements $x \in G$ where $x^{-1} \in \eta$. Then

$$c_{\lambda,\mu}^\eta(G) = \frac{|\lambda||\mu||\eta^{-1}|}{|G|} \sum_{\chi} \chi(\lambda)\chi(\mu)\chi(\eta^{-1})$$

where the sum taken over irreducible characters of $G$.

For an analogue of the Frobenious formula in the setting of Gel'fand pairs, see [18].

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2. Notations and preliminaries

In this chapter, we first introduce the notion of saturated family of groups $(G_n)_{n \in \mathbb{N}}$ and then present a systematic way of calculating structure constants in the center. In the subsequent sections, we introduce a ring, so called Farahat-Higman ring and summarize the work of Farahat and Higman.

2.1. Center of the group rings and uniformly saturated families of groups. Let $G$ be a group. Two elements $g_1, g_2 \in G$ are said to be conjugate or similar, if there exists $h \in G$ such that $h^{-1}g_1h = g_2$. The similarity relation is an equivalence relation and it is denoted by $\sim_G$. The conjugacy class of an element $g \in G$ is denoted by $g^G$ and the set of conjugacy classes of $G$ is denoted by $\mathcal{G}$. If $g \in G$ and $\lambda \in \mathcal{G}$ representing the conjugacy class of $g$, then we say that type of $g$ is $\lambda$. The center of the group algebra $\mathbb{Z}[G]$ is denoted by $\mathcal{H}(G)$. If $\lambda \in \mathcal{G}$, then the class sum

$$(2) \quad K_\lambda = \sum_{g \in \lambda} g$$

is an element of $\mathcal{H}(G)$. As $\lambda$ ranges over $\mathcal{G}$, the elements $K_\lambda$ form a basis of $\mathcal{H}(G)$ and the non-negative integers $c_{\lambda,\mu}^\eta$ defined via the equation

$$K_\lambda \cdot K_\mu = \sum_{\eta \in \mathcal{G}} c_{\lambda,\mu}^\eta K_\eta$$

are called the structure constants of $\mathcal{H}(G)$. For $A, B, C \subset G$ the fiber of $C$ in $A \times B$ is denoted by $V(A \times B : C)$ and defined by

$$V(A \times B : C) = \{(a, b) \in A \times B : ab \in C\}.$$ 

**Lemma 2.1.** Let $\lambda, \mu, \eta \in \mathcal{G}$ and $z \in \eta$. Then

$$c_{\lambda,\mu}^\eta = |V(\lambda \times \mu : \{z\})| = |V(\lambda \times \mu : \eta)||\eta|^{-1}.$$

In particular, $c_{\lambda,\mu}^\eta \in \mathbb{N}$.

**Proof.** The first equality follows from the definition of the structure constants and the basis elements $K_\eta$. In fact, the coefficient $c_{\lambda,\mu}^\eta$ is equal to the coefficient of $z$ in the expansion of the product

$$\left(\sum_{x \in \lambda} x\right) \cdot \left(\sum_{y \in \mu} y\right)$$

and it is equal to the number couples $(x, y) \in \lambda \times \mu$ which satisfy $xy = z$. Therefore $c_{\lambda,\mu}^\eta$ equals to the number of elements in $V(\alpha \times \beta : \{z\})$, which proves the first equality. The second equality follows from the first one and the set theoretic equality

$$V(A \times B : C_1 \cup C_2) = V(A \times B : C_1) \cup V(A \times B : C_2)$$

for $C_1 \cap C_2 = \emptyset$. \qed
Let $G_1 \subset G_2 \cdots \subset G_n \subset \cdots$ be an ascending chain of finite groups and let $G$ be the union of $G_n$, for $n \in \mathbb{N}$. If $x \in G_m$ and $m \leq n$, then the image of $x$ in $G_n$ is denoted by $x^n$. The family $(G_n)_{n \in \mathbb{N}}$ is said to be \textbf{saturated} if for all $x_1, x_2 \in G_m$ and for all $n \geq m$.

$$x_1 \sim_{G_n} x_2 \Leftrightarrow x_1^n \sim_{G_n} x_2^n$$

In other words, the family is said to be saturated if for all $m \in \mathbb{N}$, two non-conjugate elements in $G_m$ remains non-conjugate in $G$. For a fixed saturated family $(G_n)_{n \in \mathbb{N}}$, the algebra $\mathcal{H}(G_n)$ is simply denoted by $\mathcal{H}_n$ henceforth.

\textbf{Lemma 2.2.} Let $(G_n)_{n \in \mathbb{N}}$ be a saturated family of finite groups. The association $g^{G_m} \mapsto (g^n)^{G_n}$ defines an injection $\widehat{G}_m \hookrightarrow \widehat{G}_n$ for all $m \leq n$, thus defines a direct system. Moreover

$$\widehat{G} = \lim_{\to n} \widehat{G}_n.$$  

\textit{Proof.} The fact that $\widehat{G}_n \hookrightarrow \widehat{G}_{n+1}$ follows directly from (2.1). As each conjugacy class of $G$ is uniquely determined by an element $x \in G$ and each such element is contained in $G_n$ for some $n \in \mathbb{N}$ the natural map$$\lim_{\to n} \widehat{G}_n \longrightarrow \widehat{G}$$

is onto. As this map is induced by the limit of injective maps, it is also injective. Hence it is bijective. \hfill \square

Now we introduce some abstract notation which will have concrete meanings in each case that will be covered in the later sections. Fix a saturated family $(G_n)_{n \in \mathbb{N}}$. If $\lambda \in \widehat{G}_m$, then the image of $\lambda$ in $\widehat{G}$ is denoted by $\hat{\lambda}$. The element $\hat{\lambda}$ is called the \textbf{modification} of $\lambda$, and elements of $\widehat{G}$ are called \textbf{a modified types}. Let $\lambda \in \widehat{G}$ be a fixed modified type. The intersection $\lambda(n) := \lambda \cap G_n$, if non-empty, determines a conjugacy class in $\widehat{G}_n$. The minimal integer $l_\lambda$ for which $\lambda(l_\lambda) \neq \emptyset$ is called the \textbf{level} of $\lambda$. If $n \geq l_\lambda$ then the equality

$$\lambda(n) = \lambda$$

is a tautological consequence of the definitions. Let $\lambda \in \widehat{G}$ be a modified type. The element $\lambda(l) \in \widehat{G}_l$, where $l$ is the level of $\lambda$, is called the \textbf{completion} of $\lambda$ and denoted by $\overline{\lambda}$. For $n \geq l_\lambda$, the induced element $\lambda(n)$ is denoted by $\lambda^n$ and called the $n$-th \textbf{completion}. It is clear that $\lambda^n$ is equal to the image of $\overline{\lambda}$ in $\widehat{G}_n$ and they are both equal to $\lambda(n)$. The corresponding basis element (cf. Eq.(2)) of $\mathcal{H}_n$ determined by $\lambda(n)$ is denoted by $K_\lambda(n) \in \mathcal{H}_n$ instead of $K_{\lambda(n)}$.

Let $\lambda, \mu, \eta \in \widehat{G}$ be three modified types and let $m = \min\{l_\lambda, l_\mu, l_\eta\}$. Then for all $n \geq m$, all the three intersections $\lambda(n)$, $\mu(n)$, $\eta(n)$ are non-empty and determine elements of $\widehat{G}_n$. This means, one can form the multiplication $K_\lambda(n) \cdot K_\mu(n)$ in $\mathcal{H}_n$ for all $n \geq m$ and consider the coefficient $c_{\lambda,\mu}(n)$ of $K_\eta(n) \in \mathcal{H}_n$. We will call the resulting functions

$$n \mapsto c_{\lambda,\mu}(n)$$

\textbf{the structural functions} of $G$.

\textbf{Remark 2.3.} Using Lemma 2.1, we know that

$$c_{\lambda,\mu}(n) = |\mathbb{V}(\lambda(n) \times \mu(n) : \{z\})|$$

where $z \in \eta(m)$. But $\mathbb{V}(\lambda(n_1) \times \mu(n_1) : \{z\}) \subset \mathbb{V}(\lambda(n_2) \times \mu(n_2) : \{z\})$ for $n_1 \leq n_2$. From this, it follows that the structural functions are monotone increasing.

Now we present a certain way of calculating the structural constants, which was introduced by Farahat and Higman in \cite{6} in the context of symmetric groups. Let $G$ be a fixed group. $G$ acts on $G \times G$ with the two-fold simultaneous conjugation: For $h \in G$ and $(x, y) \in G \times G$ we set $(x, y)^h := (hxh^{-1}, hyh^{-1})$.

\textbf{Remark 2.4.} Notice that $(xy)^h$ is equal to $x^hy^h$, which means the fiber $\mathbb{V} = \mathbb{V}(\lambda \times \mu : \eta)$ is closed under two-fold conjugation, where $\lambda$, $\mu$, and $\eta$ stand for conjugacy classes. In fact, let $(x, y) \in \mathbb{V}$, i.e. the conjugacy class of $xy$ is $\eta$. Then $(x, y)^h = (hxh^{-1}, hyh^{-1})$ and $hxh^{-1}hyh^{-1} = hxyh^{-1} \sim xy$, thus $hxyh^{-1} \in \eta$. 

A saturated family of groups \( (G_n)_{n \in \mathbb{N}} \) will be called **finitely saturated** if for all \( \lambda, \mu, \eta \in \hat{G} \) the fiber set \( V = V(\lambda \times \mu : \eta) \) admits finitely many orbits with respect to the two-fold simultaneous action. We write \( V(n) \) for \( V(\lambda \times \mu : \eta) \cap G_n \times G_n \). If \( L \) is an orbit of \( V(\lambda \times \mu : \eta) \) then \( L(n) \) indicates the set \( L \cap G_n \times G_n \). A finitely saturated family will be called **uniformly saturated** if there exists \( m_L \) such that for all \( n \geq m_L \), the set \( L(n) \) is a single orbit of simultaneous conjugation action of \( G_n \) on \( G_n \times G_n \).

Next, let \( (G_n)_{n \in \mathbb{N}} \) be a uniformly saturated family of finite groups and \( \lambda, \mu, \eta \in \hat{G} \) be three stable conjugacy classes in \( G \). Assume that \( L_1, \ldots, L_s \) is the totality of orbits in \( V = V(\lambda \times \mu : \eta) \), which is finite as the family is uniformly saturated. Set \( m = \min\{l_\lambda, l_\mu, l_\eta\} \) so that for any \( n \geq m \) the intersections \( \lambda(n), \mu(n) \) and \( \eta(n) \) are non-empty and hence they determine elements of \( \hat{G}_n \). For all \( n \geq m \) the intersection \( V(n) \) of the fiber \( V \) with \( G_n \times G_n \) is equal to the disjoint union of \( L_i(n) \) and hence it follows that

\[
(3) \quad |V(n)| = \sum_{i=1}^{s} |L_i(n)|.
\]

Combining Lemma 2.1 and Eq. (3) one can deduce that

\[
(4) \quad c^\eta_{\lambda, \mu}(n) = \frac{|V(n)|}{|\eta(n)|} = \sum_{i=1}^{s} \frac{|L_i(n)|}{|\eta(n)|}
\]

Next we deal with the summands in Eq. (4). Let \( (x_i, y_i) \in L_i \). As \( x_i, y_i \in G_n \) and \( (x_i, y_i) \in V(n) \), the product \( z_i := x_iy_i \) is an element of \( \eta \cap G_n \). So \( \eta(n) \) is equal to the conjugacy class of \( z_i \) in \( G_n \), whose size is given by the usual formula:

\[
|\eta(n)| = |(z_i)^{G_n}| = |G_n/C_{G_n}(z_i)|
\]

where \( C_{G_n}(z_i) \) denotes the centralizer of \( z_i \) in \( G_n \). On the other hand, the size of \( L_i(n) \) is determined by the formula \( |G_n/\text{Stab}_{G_n}(x_i, y_i)| \) where \( \text{Stab}_{G_n}(x_i, y_i) \) denotes the stabilizer of \( (x_i, y_i) \) of the simultaneous conjugation action of \( G_n \) on \( G_n \times G_n \). But it is clear that the stabilizer of \( (x_i, y_i) \) is equal to the intersection \( C_{G_n}(x_i) \cap C_{G_n}(y_i) \). Combining all these, we find that

\[
\frac{|L_i(n)|}{|\eta(n)|} = \frac{|C_{G_n}(x_i)|}{|C_{G_n}(x_i) \cap C_{G_n}(y_i)|}
\]

and hence Eq. (4) becomes

\[
(4) \quad c^\eta_{\lambda, \mu}(n) = \sum_{i=1}^{s} \frac{|C_{G_n}(x_i)|}{|C_{G_n}(x_i) \cap C_{G_n}(y_i)|}
\]

Let us summarize the findings.

**Proposition 2.5.** Let \( (G_n)_{n \in \mathbb{N}} \) be a uniformly saturated family of groups. For each triple \( \lambda, \mu, \eta \) of modified types in \( \hat{G} \), there exists an \( m \in \mathbb{N} \) and a finitely many elements \( (x_1, y_1), \ldots, (x_s, y_s) \in \lambda(m) \times \mu(m) \) such that

1. \( x_i, y_i \in \eta(l) \) for \( i = 1, \ldots, s \).
2. For every \( n \geq m \) the structural function \( c^\eta_{\lambda, \mu}(n) \) satisfies the relation below.

\[
(5) \quad c^\eta_{\lambda, \mu}(n) = \sum_{i=1}^{s} \frac{|C_{G_n}(x_i)|}{|C_{G_n}(x_i) \cap C_{G_n}(y_i)|}
\]

Each summand on the right hand side of the above equation will be referred as the **growth of the centralizer**.

3. By the finiteness of the summation above, the growth of the structural function \( c^\eta_{\lambda, \mu}(n) \) is determined by the growth of the centralizers

\[
(5) \quad n \mapsto \frac{|C_{G_n}(x_i)|}{|C_{G_n}(x_i) \cap C_{G_n}(y_i)|}
\]

In particular, if all the functions occurring in Eq. (5) are polynomials in \( n \), then the structural function \( c^\eta_{\lambda, \mu}(n) \) is also a polynomial in \( n \).
2.2. Farahat-Higman ring. In this section, we will consider a uniformly saturated family \( (G_n)_{n \in \mathbb{N}} \) of groups which admits a certain conjugation invariant sub-additive function. More precisely, let \( (G_n)_{n \in \mathbb{N}} \) be a uniformly saturated family of groups and assume that \( G_n \) possesses a length function \( || \cdot ||_n \) with the following properties:

1. \( || \cdot ||_n \) is stable under the embedding \( G_n \subset G_{n+1} \). That is, if \( x \in G_m \) and \( n \geq m \) then
   \[
   ||x^{Tn}||_n = ||x||_m.
   \]
   Hence, \( G \) possesses a length function \( || \cdot || : G \to \mathbb{N} \) so that \( || \cdot ||_{G_n} = || \cdot ||_n \) for all \( n \in \mathbb{N} \).

2. \( || \cdot || \), and hence \( || \cdot ||_n \), is constant on the conjugacy classes.

3. \( || \cdot || \), and hence \( || \cdot ||_n \), is sub-additive. That is,
   \[
   ||xy|| \leq ||x|| + ||y||.
   \]

We will call such a family a \textbf{filtered uniformly saturated family}. Notice that, since \( || \cdot || \) is constant on the conjugacy classes, one can transfer the length function \( || \cdot || \) to \( \hat{G} \) by setting \( ||\eta|| := ||x|| \) where \( \eta \in \hat{G} \) and \( x \in \eta \) is arbitrary. Following [6] we introduce the following algebra \( S'(G) \) defined as follows: Let \( (G_n)_{n \in \mathbb{N}} \) be filtered uniformly saturated family and assume that the functions \( c^\eta_{\lambda, \mu}(n) \) are polynomials of \( n \) for all \( \lambda, \mu, \eta \). Let \( B \) be the subring of polynomials \( f(T) \in \mathbb{Z}[T] \) which maps integers to integers and consider \( S'(G) := B[K_\lambda : \lambda \in \hat{G}] \), the free polynomial algebra over the ring \( B \) with the indeterminates \( K_\lambda \in \hat{G} \), where the multiplication is defined as

\[
K_\alpha \cdot K_\mu = \sum_{\eta \in \hat{G}} c^\eta_{\alpha, \mu}(T)K_\eta.
\]

Notice that the sum is actually a finite sum, and thus, meaningful. This is an associative and commutative ring and the evaluation map \( f(T) \mapsto f(n) \) induces a surjection from \( S'(G) \) onto \( \mathcal{H}_n \). Now using the filtration, we define the induced filtered ring, called the \textbf{Farahat-Higman ring} of the uniformly saturated family and denote it by \( S(G) \) by setting:

\[
K_\alpha \cdot K_\mu = \sum_{||\eta||=||\eta'||} c^\eta_{\alpha, \mu}(T)K_\eta.
\]

Following Wan and Wang, we say that the family \( (G_n)_{n \in \mathbb{N}} \) satisfies the \textbf{stability property} if the structure constants \( c^\eta_{\alpha, \mu}(T) \) of the Farahat Higman ring are independent of \( T \), i.e. \( c^\eta_{\alpha, \mu}(T) \in \mathbb{Z} \).

2.3. An example: The uniformly saturated family \( (S_n)_{n \in \mathbb{N}} \). This section summarizes the work [6] of Farahat-Higman. The notation introduced below will be used later in the cases of the families \( (GL_n(q))_{n \in \mathbb{N}} \) and \( (SP_n(q))_{n \in \mathbb{N}} \).

We introduce the relevant notation.

1. A \textbf{partition} \( \lambda \) is a non-increasing sequence of non-negative integers \( (\lambda_1, \cdots, \lambda_r, \cdots) \) where almost all \( \lambda_i \)'s are zero.

2. The integers \( \lambda_i \) are called the \textbf{parts} of \( \lambda \) and the number of non-zero \( \lambda_i \)'s is called the \textbf{length} of \( \lambda \) and denoted by \( l = l(\lambda) \) and we write \( \lambda = (\lambda_1, \cdots, \lambda_l) \) and omit the zeros in the tail.

3. Let \( \lambda = (\lambda_1, \cdots, \lambda_r) \) be a partition. If \( m_k = |\{i : \lambda_i = k\}| \) then \( \lambda \) can be denoted as \( (1^{m_1}, \cdots, 1^{m_{\lambda_1}}) \).

4. The \textbf{weight} \( ||\lambda|| \) of a partition \( \lambda \) is defined to be the integer \( \sum_{i \in \mathbb{N}} \lambda_i \), which is well-defined as the sum is in fact over a finite set.

5. If \( ||\lambda|| = n \) then one says \( \lambda \) is a partition of \( n \) and writes \( \lambda \vdash n \). The set of partitions of \( n \) is denoted by \( \mathcal{P}_n \) and the set of all partitions is denoted by \( \mathcal{P} \) which is the union of \( (\mathcal{P}_n)_{n \in \mathbb{N}} \).

6. For \( k > 0 \), the partition \( 1_k \) is the unique partition whose non-zero parts are \( 1 \) and weight is \( k \). There is a unique partition of \( 0 \), the empty partition \( \emptyset \).

7. For two partitions \( \lambda, \mu \), their sum \( \lambda \cup \mu \) is defined to be the unique partition whose parts consists of parts of \( \lambda \) and \( \mu \).

8. For a partition \( \lambda = (\lambda_1, \cdots, \lambda_k) \), the \textbf{completion} \( \overline{\lambda} \) is the partition \( (\lambda_1 + 1, \cdots, \lambda_k + 1) \). The weight of the completion of \( \lambda \) is clearly equal to \( ||\lambda|| + l(\lambda) \).
(9) For an integer $n \geq ||\lambda|| = ||\lambda|| + l(\lambda)$ the $n$-th completion $\lambda \uparrow^n$ is the partition $\lambda \cup 1_r$, where $r = n - ||\lambda||$.

(10) If $\lambda_i \geq \mu_i$ for all $i \in \mathbb{N}$, then one defines $\lambda - \mu$ as the partition whose parts are $\lambda_i - \mu_i$. For a partition $\lambda$ with length $r$, the partition $\hat{\lambda} = \lambda - 1_r$ is called the modification of $\lambda$.

If $\lambda$ is the empty partition we still talk of the modification, completion and $n$-th completion of $\lambda$. The first two are again the empty partition and the $n$-th completion of the empty partition is clearly equal to $1_n$. Later we will introduce the notion of partition valued functions, and analogous concepts to weight, completion and modification will be introduced.

**Example 2.6.** Consider $\lambda = (4,3,3,2,1,1,1)$, a partition of $15 = 4 + 3 + 3 + 2 + 1 + 1 + 1$. The length of $\lambda$ is 7. The modification $\hat{\lambda}$ of $\lambda$ is $(4 - 1,3 - 1,3 - 1,2 - 1,1 - 1,1 - 1,1 - 1) = (3,2,2,1)$, which is a partition of 8. The completion $\overline{\lambda}$ of $\lambda$ is $(4,3,3,2)$. The 15-th completions of $\hat{\lambda}$ and $\overline{\lambda}$ are both equal to $(4,3,3,2,1,1,1)$.

Let $A$ be a subset of $\mathbb{N}$ and $g$ be a permutation of $A$. The support $[g]$ of $g$ is defined to be the subset $[g] := \{a \in A : g(a) \neq a\}$ of $A$. The group of permutations $g$ of $A$ with finite support is denoted by $S_A$. For $n \in \mathbb{N}$, let $[n]$ indicates the set $\{1,2,\ldots,n\}$. When $A = [n]$, we will follow the usual notation and simply write $S_n$ instead of $S_{[n]}$. It is well-known that the conjugacy class of an element $g \in S_n$ is completely determined by the cycle type of $g$, which determines a unique partition $\lambda^g = \lambda$ of $n$. The reflection length $l(g)$ of $g \in S_n$ is the minimal number of transpositions whose product is equal to $g$. As transpositions generate symmetric group, this definition of reflection length makes sense.

The symmetric group $S_n$ embeds in $S_{n+1}$ in a natural way. The conjugacy classes $\hat{S}_n$ of $S_n$ are in 1-1 correspondence with $P_n$. The family is clear saturated. The union of $S_n$, $n \in \mathbb{N}$, is denoted by $S_\infty$, it is the group of permutations of $\mathbb{N}$ whose supports are finite.

**Lemma 2.7.** [6] The family $(S_n)_{n \in \mathbb{N}}$ is a saturated family of groups and the bijections $\hat{S}_n \rightarrow P_n$ induce the commutative diagram below.

$\begin{array}{ccc}
\hat{S}_n & \rightarrow & P_n \\
g^\hat{\lambda} & \sim & \lambda^\hat{\lambda} \\
\downarrow & & \downarrow \\
\hat{S}_\infty & \rightarrow & P
\end{array}$

In particular, the conjugacy classes of $S_\infty$ are in 1-1 correspondence with the set of all partitions.

From the lemma it also follows that the abstract definitions of the concepts of modification, completion and $n$-th completion introduced earlier are consistent with the concrete definitions given in this section.

**Lemma 2.8.** The reflection length is constant on conjugacy classes and it is sub-additive. It is also stable under the embedding $S_m \rightarrow S_n$ for $m \leq n$. Moreover, the reflection length of $g$ is equal to the $||\lambda||$.

**Example 2.9.** Consider the permutation $g = (345)(78)$. As an element of $S_8$ and $S_{10}$, the conjugacy class of $g$ corresponds to the partitions $(3,2,1,1,1)$ and $(3,2,1,1,1,1,1)$ respectively. As an element of $S_\infty$ the conjugacy class of $g$ corresponds to the partition $(2,1)$. The completion of $(2,1)$ is $(3,2)$ whose weight is 5. The level of $g$ is also 5 which is equal to $||g||$. The reflection length of $g$ is 3 and it is equal to $||\lambda^g|| = ||(2,1)||$.

The following lemma is the normal form theorem in the context of symmetric groups whose proof is evident.

**Lemma 2.10.** Let $g, h \in S_n$ and assume that $||g \cup [h]|| = m \leq n$. Then there is an element $z$ in $S_n$ so that $(g,h)^z \in G_m \times G_m$.

**Proposition 2.11 ([6]).** (Farahat-Higman) The family $(S_n)_{n \in \mathbb{N}}$ is a uniformly saturated family of groups.

**Proof.** Let $\lambda, \mu, \eta$ be three modified types in $\hat{S}_\infty$ and consider $V = V(\lambda \times \mu : \eta)$. For $(g,h) \in V$, the number $||g \cup [h]||$ is bounded by $m := ||g|| + ||[h]||$. Hence every orbit has a representative in the finite group $G_m \times G_m$, thus there is at most finitely many orbits. (Compare with Lemma 3.22.)
Remark 2.12 (Growth of centralizers). If $g, h$ are two elements of $S_n$ then
\begin{equation}
C_{S_n}(gh) = C_{S_{\lfloor gh \rfloor}}(gh) \oplus S_{\lfloor gh \rfloor - \lfloor gh \rfloor}
\end{equation}
and hence
\begin{equation}
C_{S_n}(g) \cap C_{S_n}(h) = (C_{S_{\lfloor gh \rfloor}}(g) \cap C_{S_{\lfloor gh \rfloor}}(h)) \oplus S_{\lfloor gh \rfloor - \lfloor gh \rfloor}.
\end{equation}

Proposition 2.13 ([6]). (Farahat-Higman) For all $\lambda, \mu, \eta \in \mathcal{S}_\infty$, the structural functions $e_{\lambda, \mu}^n(n) = p_{\lambda, \mu}^n(n)$ for some polynomial $p_{\lambda, \mu}^n(t) \in \mathbb{Z}[t]$ for large $n$.

Proof. It is clear that, the index of the two groups occurring in Eq.(7) and Eq.(6) is a polynomial in $n$. In fact, if $[[gh]] = r \leq s = [[g] \cup [h]]$ then
\[
\frac{C_{S_n}(gh)}{C_{S_n}(g) \cap C_{S_n}(h)} = \frac{C_{S_{\lfloor gh \rfloor}}(gh)}{C_{S_{\lfloor gh \rfloor}}(g) \cap C_{S_{\lfloor gh \rfloor}}(h)} \cdot \frac{(n-r)!}{(n-s)!} = \frac{C_{S_{\lfloor gh \rfloor}}(gh)}{C_{S_{\lfloor gh \rfloor}}(g) \cap C_{S_{\lfloor gh \rfloor}}(h)} \cdot (n-r) \cdots (n-s+1).
\]
Since the family is uniformly saturated the result follows from Remark 2.5/3. \qed

Notice that, in the above proof, the degree of the polynomial is equal to $[[g] \cup [h]] - [[gh]]$, which is zero only if $[g] \cup [h] = [gh]$. The next lemma establishes a criteria to guarantee the equality.

Lemma 2.14. [6] Let $g, h \in S_n$. If $|\tilde{(\lambda^g)}| + |\tilde{(\lambda^h)}| = (\lambda^g \circ \lambda^h)$ then $[g] \cup [h] = [gh]$.

Proposition 2.15. [6] For $g \in S_m$ and $n \geq m$
\[
(\lambda^g)^{(n)} = \tilde{\lambda}^g.
\]
The weight of $(\tilde{\lambda}^g)$ is equal to the reflection length of $g$. Hence, if $|\lambda| > |\alpha| + |\beta|$ then $e_{\alpha, \beta}^n(n) = 0$ for all $n \in \mathbb{N}$. If the equality holds, then the polynomial $p_{\alpha, \beta}^n(t)$ is constant.

Corollary 2.16. [6] The uniformly saturated family $(S_n)_{n \in \mathbb{N}}$ satisfies the stability property.

3. The uniformly saturated family $GL_n(q)$ and the work of Wan and Wang

In this chapter, we summarize the work Stability of the centers of group algebras of $GL_n(q)$ of Wan and Wang, [20]. In the first section, following [12] and [9] we review the general theory of $GL_n(q)$ and parameterize the conjugacy classes in general linear groups over a finite field. In the second section, we closely follow [20] and construct the uniformly saturated family $GL_n(q)$. In the following sections, we present the main theorems of Wan and Wang without proofs. Some of the theorems are divided into smaller pieces because some parts will be used in the symplectic case. Some general facts concerning the centralizers of block matrices will also be discussed in as they are used in the proofs of Wan and Wang and as well as in our study concerning symplectic group rings.

3.1. Notation and preliminaries. Let $p$ be a prime and $q$ be a power of $p$. The finite field with $q$ is denoted by $\mathbb{F}_q$. The set of monic irreducible polynomials $p(t) \in \mathbb{F}_q[t] - \{t\}$ is denoted by $\Phi$. For an abstract finite dimensional vector space $V$ and $U \in GL(V)$ the residual $R^U$ and fixed space $V^U$ of $U$ are defined as
\begin{equation}
R^U = (U - 1_V)V, \quad V^U = \ker(U - 1_V).
\end{equation}
An element in $GL(V)$ is called a reflection if $\dim R^U = 1$, equivalently, codimension of $V^U$ is 1 by the equality $\dim R^U + \dim V^U = \dim V$. The reflection length $\ell(U)$ of $U \in GL(V)$ is the minimum number $r$ such that there exists a sequence of reflections of reflections $\tau_1, \ldots, \tau_r$ such that $U = \tau_1 \cdots \tau_r$.

Next we introduce the relevant combinatorial objects. These definitions will be used in symplectic group case as well.

Definition 3.1. (1) A partition valued function $\lambda$ on $\Phi$ is a function from $\Phi$ to the set of partitions $\mathcal{P}$ such that for almost all $f \in \Phi$, the image $\lambda(f)$ is the empty partition. The image will be sometimes denoted by $\lambda_f$ depending on the convenience.
(2) The **weight** $||\lambda||$ of a partition valued function $\lambda : \Phi \rightarrow \mathcal{P}$ is defined as follows:

$$||\lambda|| = \sum_{f \in \Phi} \deg(f) \cdot ||\lambda_f||$$

which makes sense as the weight of the empty partition is by definition equal to zero. The set of partition valued functions on $\Phi$ of weight $n$ is denoted by $\mathcal{P}_n(\Phi)$. The set of all partition valued functions is denoted by $\mathcal{P}(\Phi)$.

(3) The sum $\lambda \cup \mu$ of two partition valued functions $\lambda$ and $\mu$ is defined as the function sending $f$ to $\lambda(f) \cup \mu(f) = \lambda_f \cup \mu_f$.

(4) ([20]) The **unipotent part** $\lambda^e$ and non-unipotent part $\lambda^{ne}$ of $\lambda$ are defined as follows. The partition valued function $\lambda^e$ induced by the partition valued function $\lambda$ as follows:

$$\lambda^e(t - 1) = \lambda(t - 1), \quad \text{and} \quad \lambda^e(f) = \emptyset, \quad \forall f \neq t - 1.$$  

The non-unipotent part $\lambda^{ne}$ of $\lambda$ is defined as follows:

$$\lambda^{ne}(t - 1) = \emptyset, \quad \text{and} \quad \lambda^{ne}(f) = \lambda(f), \quad \forall f \neq t - 1.$$  

It is clear that, for a partition valued function $\lambda$ the equality below holds:

$$\lambda^e \cup \lambda^{ne} = \lambda.$$  

(5) A partition valued function $\lambda$ is called a **unipotent function** if it is equal to its unipotent part.

**Example 3.2.** Let $\alpha \in F_q$ be a non-square. Define $\mu \in \mathcal{P}(\Phi)$ by setting

$$\mu(t - 1) = (3, 2, 1, 1), \quad \text{and} \quad \mu(t^2 - \alpha) = (2, 2, 1),$$

and for $f \neq t - 1, t^2 - \alpha$, set $\mu(f) = \emptyset \in \mathcal{P}_0$. By definition we get

$$||\mu|| = 1 \cdot (3 + 2 + 1 + 1) + 2 \cdot (2 + 2 + 1) = 17.$$  

The unipotent part $\lambda^e$ is equal to the function which assigns $(3, 2, 1, 1)$ to $(t - 1)$ and assigns the empty partition $\emptyset$ to $f$ for all $f \in \Phi - \{t - 1\}$. The non-unipotent part $\lambda^{ne}$ of $\lambda$ is the partition valued function that assigns $(2, 2, 1)$ to $t^2 - \alpha$ and $\emptyset$ to $f$ for all $f \in \Phi - \{t^2 - \alpha\}$.

The following concepts are introduced in [20] as variants of modification, completion and $n$-th completion. Recall that the modification, completion and $n$-th completion of the empty partition were formally defined.

**Definition 3.3** (Wan-Wang). Let $\mu \in \mathcal{P}_n(\Phi)$ be a partition valued function of weight $n$. The **modification** $\overset{\circ}{\mu}$ is the partition valued function defined as the unique partition valued functions satisfying

$$\overset{\circ}{\mu}(t - 1) = \mu(t - 1) \quad \text{and} \quad \overset{\circ}{\mu}(f) = \mu(f)$$

for all $f \in \Phi - \{t - 1\}$. The **completion** $\overline{\mu}$ of $\mu$ is the partition valued function defined as the unique partition valued functions satisfying

$$\overline{\mu}(t - 1) = \mu(t - 1) \quad \text{and} \quad \overline{\mu}(f) = \mu(f)$$

for all $f \in \Phi - \{t - 1\}$. For $n \geq ||\overline{\mu}||$, define the $n$-completion $\mu^{\uparrow n} \in \mathcal{P}_n(\Phi)$ to be the unique partition valued function that satisfies

$$\mu^{\uparrow n}(t - 1) = \mu(t - 1)^{\uparrow r}$$

where $r = n - ||\mu||$ and $\mu^{\uparrow n}(f) = \mu(f)$ for all $f \neq t - 1$.

Notice that all the operations sending $\mu$ to $\overset{\circ}{\mu}$, or to $\overline{\mu}$ or to $\mu^{\uparrow n}$ affects only the unipotent part $\lambda^e$ of $\mu$.

**Example 3.4.** Let us observe the effects of the operations just introduced on the partition valued function $\mu$ of Example 3.2, which was defined as

$$\mu(t - 1) = (3, 2, 1, 1), \quad \text{and} \quad \mu(t^2 - \alpha) = (2, 2, 1),$$
and for \( f \neq t - 1, t^2 - \alpha \), set \( \mu(f) = \emptyset \in \mathcal{P}_0 \) where \( \alpha \in \mathbb{F}_q \) is a non-square. Then

\[
\begin{align*}
\hat{\mu}(t - 1) &= \mu(t - 1) = (3, 2, 1, 1) = (2, 1) \\
\hat{\mu}(t^2 - \alpha) &= \mu(t^2 - \alpha) = (2, 2, 1) \\
\hat{\mu}(f) &= \emptyset
\end{align*}
\]

for all \( f \neq t - 1, t^2 - \alpha \). The following equalities follow from the definitions.

\[
\begin{align*}
\overline{\mu}(t - 1) &= \overline{\mu}(t - 1) = (3, 2), \quad \text{and} \quad \overline{\mu}(f) = \overline{\mu}(f),
\end{align*}
\]

for all \( f \neq t - 1 \). The weight of \( \hat{\mu} \) is \( 1 \cdot (3 + 2) + 2 \cdot (2 + 2 + 1) = 15 \). Clearly, \( (\hat{\mu})^{17} = \mu \).

### 3.2. Conjugacy classes in general linear groups.

Let \( U \in GL(V) \). For \( v \in V \), the association \( v \mapsto U \cdot v \) defines an \( \mathbb{F}_q[t] \)-action on \( V \) in the following way. Define an \( \mathbb{F}_q[t] \)-module structure \((V_U, \cdot_U)\) on \( V \) by setting \( t \cdot_U v = U \cdot v \) and extending it linearly.

**Remark 3.5.** The most important property of this module is that it characterizes the conjugacy class of the defining element of the \( \mathbb{F}_q[t] \)-module. Let \( U_1, U_2 \in GL(V) \) be two \( F \)-automorphisms of \( V \) and assume that the elements \( U_1 \) and \( U_2 \) are conjugate: \( U_1 U = U U_2 \) for some \( U \in GL(V) \), which implies

\[
t \cdot_{U_1} (U_1(v)) = U_1 U(v) = U U_2(v) = U \cdot (U_2 v).
\]

As a result \( v \mapsto U(v) \) defines an \( F(t) \)-module isomorphism from \( V_{U_1} \) to \( V_{U_2} \). Let us rewrite the last inequality in a more suggestive form:

\[
\begin{array}{ccc}
V & \xrightarrow{t \cdot_{U_2}} & V \\
\downarrow U & & \downarrow U \\
V & \xrightarrow{t \cdot_{U_1}} & V
\end{array}
\]

which reads as \( V_{U_1} \) and \( V_{U_2} \) are isomorphic representation spaces of \( F[t] \). Conversely, if \( U \) is such a module isomorphism, then it is clearly a linear isomorphism which satisfies \( U_1 U = U U_2 \). As a result we have

\[
U_1^G = U_2^G \iff V_{U_1} \simeq V_{U_2},
\]

for all \( U_1, U_2 \in G \). The Eq. (11) can be stated in terms of representations. The elements \( U_1 \) and \( U_2 \) are conjugate if and only if there is an \( F[t] \)-equivariant isomorphism between \( V_{U_1} \) and \( V_{U_2} \). This interpretation will allow us to show that an equation of type

\[
XA = BX, \quad A \in \text{Mat}_{n \times n}, \quad B \in \text{Mat}_{m \times m}, \quad X \in \text{Mat}_{m \times n}
\]

admits only the trivial solution \( X = 0 \) when \( V_A \) and \( V_B \) are non-isomorphic simple modules. Of course, this is just a special case of Schur’s lemma.

Let \( U \in GL(V) \), be a fixed linear endomorphism of \( V \). Since \( \mathbb{F}_q[t] \) is a PID and \( V_U \) is a finite dimensional module, the elementary divisor theory applies and \( V_U \) admits a decomposition into primary cyclic modules where a primary cyclic \( \mathbb{F}_q[t] \) module is by definition in the following form:

\[
N_{f,i} := \mathbb{F}_q[t]/(f)^i, \quad i > 0, f \in \Phi.
\]

It is well known that the decomposition into primary cyclic modules is unique on the isomorphism class of \( V_U \) up to permuting the orders of the summands ([9, Chapter 3]). Let

\[
V_U = \bigoplus_{i=1}^{r_U} M_i
\]

be a decomposition of \( V_U \) into primary cyclic modules and for \( f \in \Phi \). For \( l \in \mathbb{N} \) define

\[
m_l^f = \#(i : M_i \simeq N_{f,i}),
\]

the number of copies of \( N_{f,i} \) in the decomposition of \( V_U \) into primary cyclic modules. As there are only finitely many such summands, \( m_l = 0 \) for almost all \( l \), in fact, for \( l > r_U \) one has \( m_l^f = 0 \). Thus, the decomposition Eq. (12) determines a partition \((1^{m_1}, \ldots, r^{m_r})\) attached to \( f \), as a result one obtains a partition valued
function $\lambda^U$ which sends $f$ to the partition $\lambda^U(f) = \lambda_f^U = (\lambda_{f,1}^U, \cdots, \lambda_{f,n}^U)$, which is defined as above. With this notation the above decomposition can be written as

$$V_U = \bigoplus_{f \in \Phi} N_f \lambda_f^U$$

where

$$N_f \lambda_f^U = \bigoplus_{i=1}^{t_f} F_q[t]/(f)^{\lambda_{f,i}}.$$ 

The weight $||\lambda^U|| = \dim V$ which follows from the fact that $\dim_{\mathbb{Z}} N_{f,i} = i \cdot \deg(f)$ together with Eq.(13). Conversely, it can be shown that for each such function $\lambda$, the corresponding $F_q[t]$-module is realized by an element $U$ of $GL(V)$. In fact, for a given polynomial $f(t) \in \Phi$ and $m \geq 1$, write $f(t)^m = t^k - a_{k-1}t^{k-1} - \cdots - a_0$, and introduce the companion matrix $J_{f,m}$ of $f^m$ by setting

$$J_{f,m} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1} \end{bmatrix}_{k \times k}$$

It is well-known that the $F_q[t]$ module defined by $J_{f,m}$ is isomorphic to

$$F_q[t]/(f(t)^m) = N_{f,m}.$$ 

So, if $\lambda_f = (\lambda_1, \cdots, \lambda_r)$ and if $J_{\lambda_f}$ denotes the block diagonal matrix $\text{diag}(J_{f_{\lambda_1}}, \cdots, J_{f_{\lambda_r}})$ then the block diagonal matrix

$$J_{\lambda} := \text{diag}(J_{\lambda_f})_{f \in \Phi}$$

is an element of the conjugacy class in $GL_n(F_q)$ that induces the partition valued function $\lambda$. This finishes the characterization of the conjugacy classes of $GL_n(q)$. Let us summarize.

**Proposition 3.6.** The association $U \mapsto \lambda^U$ defines a surjection $GL_n(F_q) \to \mathcal{P}_n(\Phi)$. Two endomorphisms $U_1, U_2 \in GL_n(F_q)$ define the same partition valued function if and only if they are conjugate in $GL_n(F_q)$. In particular, $U \mapsto \lambda^U$ induces a bijection

$$\widehat{G}_n \to \mathcal{P}_n(\Phi).$$

The basis elements of $\mathcal{H}_n$ thus can be indexed by the elements of $\mathcal{P}_n(\Phi)$.

**Remark 3.7.** Consider two primary cyclic modules $M_i = F_q[t]/(f_i^m)$, $i = 1, 2$ with distinct irreducible monic polynomials $f_1, f_2$. Then $F_q[t]$-modules $V_1$ and $V_2$ and by Schur’s lemma there is no intertwining operator between them.

The use of suitable representatives is particularly important in calculations done in [20] as well as in the symplectic group case which will be investigated later. The main importance of choosing a suitable form is that it enables one to compute the functions defined in the form $C(U_1U_2)/C(U_1) \cap C(U_2)$, cf. (5), via proving a result similar to the one presented in Remark 2.12, Eq.(6). We recall the basic result in the least explicit form, yet it will be enough for our purposes.

**Lemma 3.8.** [9, Chapter 3/10] Let $U \in \text{End}(V)$ and $m_U(t) = \prod m_i(t)^{r_i}$ be the minimal polynomial of $U$, where $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then there is a basis $B$ of $V$ such that the matrix of $U$ with respect to $B$ is in block diagonal form $\text{diag}(M_1, \cdots, M_r)$ where minimal polynomial of $M_i$ is $m_i^{r_i}(t)$.

The blocks $M_i$’s admits further decomposition into a block diagonal form, where minimal polynomial of each block of $M_i$ is a power of $m_i$. The explicit blocks can be given depending on the minimal polynomial.
Remark 3.9 (Centralizers of block diagonal matrices and Schur’s lemma). Let $U$ be an $n \times n$ invertible block diagonal matrix $\text{diag}(U_1, \cdots, U_k)$, where $U_i$ is an $n_i \times n_i$ square matrix and let $D$ be an $n \times n$ matrix. The block structure of $U$ can be used to write $D$ as a block matrix $(D_{ij})_{i,j=1}^k$, where $D_{ij}$ is an $n_i \times n_j$ matrix. The matrix $D$ commutes with $U$ if and only if the equation below holds:

$$\text{diag}(U_1, \cdots, U_k)D = D\text{diag}(U_1, \cdots, U_k),$$

which can be written in detail:

$$
\begin{bmatrix}
U_1D_{11} & U_1D_{12} & \cdots & U_1D_{1k} \\
U_2D_{21} & U_2D_{22} & \cdots & U_2D_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
U_kD_{k1} & U_kD_{k2} & \cdots & U_kD_{kk}
\end{bmatrix} =
\begin{bmatrix}
D_{11}U_1 & D_{12}U_2 & \cdots & D_{1k}U_k \\
D_{21}U_1 & D_{22}U_2 & \cdots & D_{2k}U_k \\
\vdots & \vdots & \ddots & \vdots \\
D_{k1}U_1 & D_{k2}U_2 & \cdots & D_{kk}U_k
\end{bmatrix}
$$

So, $D$ commutes with $U$ if and only if

$$U_iD_{ij}U_j^{-1} = D_{ij} \forall i,j = 1, \cdots, k.$$  

Now assume that, each $U_i$ is of the form $J_{\lambda(f_i)}$ where $f_i$ and $f_j$ are distinct irreducible polynomials for $i \neq j$. Writing Eq. (14) as $U_iD_{ij} = D_{ij}U_j$, we see that $D_{ij}$ defines an intertwining operator between $N(f_i, \lambda(f_1))$ and $N(f_j, \lambda(f_2))$. Such an operator must be zero if $f_1 \neq f_2$ according to the Remark 3.7. As a consequence, we obtain the following direct sum decomposition of the centralizer of $\text{diag}(J_{\lambda(f)})_{f \in \Phi}$:

$$C(\text{diag}(J_{\lambda(f)})_{f \in \Phi}) \cong \bigoplus_{f \in \Phi} C(J_{\lambda(f)}).$$

Remark 3.10. There are other rational forms that represent conjugacy classes. The following one will be useful in the context of symplectic groups. For $n \in \mathbb{N}$, the matrix

$$S_n =
\begin{bmatrix}
e_1 & e_2 & \cdots & e_{n-1} & e_n \\
1 & 1 & & & \\
1 & 1 & \ddots & & \\
& \ddots & \ddots & \ddots & \\
1 & 1 & \cdots & 1 & 1
\end{bmatrix}
$$

is an element of $GL_n(q)$. Its minimal polynomial is equal to $(t-1)^n$ and as an $\mathbb{F}_q[t]$-module, $V_{S_n}$ is isomorphic to $\mathbb{F}_q[t]/(t-1)^n = N_{t-1,n}$. Thus, the induced partition valued function $\lambda$ assigns the partition $(n)$ to $t-1$ and the empty partition to $t \in \Phi - \{t-1\}$. The fixed space $V^{S_n}$ of $S_n$ is generated by $e_n$, in particular, dimension of the fixed space of $S_n$ is 1.

3.3. Uniformly saturated family $GL_n(q)$. In this section, following [20] we construct the uniformly saturated family $(\text{GL}_n(q))_{n \in \mathbb{N}}$.

Definition 3.11. [20] For $m \leq n$ consider the embedding $V_m \longrightarrow V_n$ defined by the rule

$$(v_1, \cdots, v_m) \longmapsto (v_1, \cdots, v_m, \underbrace{0, \cdots, 0}_{n-m \text{ many}})$$

and identify $V_m$ with its image in $V_n$. Denote

$$V_{n-[m]} = \{(0, \cdots, 0, w_1, \cdots, w_{n-m}) : w_i \in \mathbb{F}_q \}$$

which implies $V_n = V_m \oplus V_{n-[m]}$. For $U \in \text{GL}(V_m) = \text{GL}_m(\mathbb{F}_q)$ the injection $U^\dagger \in \text{GL}(V_n)$ is defined by setting $U \oplus I_{V_{n-[m]}}$.

$$U^\dagger =
\begin{bmatrix}
U & 0 \\
0 & I_{n-m}
\end{bmatrix}.$$
The group $G_\infty = GL_\infty(q)$ is defined to be the union of $(GL_n(q))_{n \in \mathbb{N}}$.

We collect numerous results of Wan and Wang in the following lemma.

**Lemma 3.12.** [20] The following hold:

1. The family $(GL_n(q))_{n \in \mathbb{N}}$ is a saturated family.
2. The map $U \mapsto (\lambda^U)$ induces a bijection between the conjugacy classes of $GL_\infty$ and $\mathcal{P}(\Phi)$, the set of all partition valued functions. The partition $(\lambda^U)$ is called the **modified type** of $U$.
3. Let $\lambda$ be a partition valued function. Then $GL_n(q)$ contains an element whose modified type is $\lambda$ if and only if $||\lambda|| \leq n$.
4. Let $\lambda$ be a partition valued function such that $||\lambda|| = m$ and let $U \in GL_m(q)$ be an element whose stable type is $\lambda$. If $n \geq m$ then

$$\lambda^\cdot n = \lambda(U^{\cdot n})$$

**Proof.** All of the statements follows from the characterizations of conjugacy classes with partition valued functions and the definitions. 

**Example 3.13.** Let us reconsider the Example 3.4. Recall that the partition valued function $\mu$ was defined by setting

$$\mu(t - 1) = (3, 2, 1, 1), \quad \text{and} \quad \mu(t^2 - \alpha) = (2, 2, 1),$$

and for $f \neq t - 1, t^2 - \alpha$, set $\mu(f) = \emptyset \in \mathcal{P}_0$ where $\alpha \in \mathbb{F}_q - \mathbb{F}_q^2$. We already observed that $||\mu|| = 1 \cdot (3 + 2 + 1 + 1) + 2 \cdot (2 + 2) = 17$. Let $\lambda = \mu$. More precisely

$$\lambda(t - 1) = (2, 1), \quad \lambda(t^2 - \alpha) = (2, 2, 1) \quad \text{and} \quad \lambda(f) = \emptyset,$$

for all $f \neq t - 1, t^2 - \alpha$. The completion $\overline{\lambda}$ of $\lambda$ differs from $\lambda$ only on the image of $t - 1$. Applying Definition 3.3 we have $\overline{\lambda}(t - 1) = \lambda(t - 1) = (2, 1) = (3, 2)$. The weight of $\overline{\lambda}$ is $1 \cdot (3 + 2) + 2 \cdot (2 + 1 + 1) = 15$. As a result, for all $n \geq 15$, there is an element in $GL_n(q)$ whose modified type is equal to $\lambda$. Let $U \in GL_{15}(q)$ be an element whose modified type is equal to $\lambda$. Then, the partition valued function defined by $U^{17}$ is equal to $\mu$. If we denote the matrix of $U$ in $GL_{15}(q)$ again by $U$ then

$$U^{17} = \begin{bmatrix} U & 0 \\ 0 & I_2 \end{bmatrix}$$

For a modified type $\lambda \in \mathcal{P}(\Phi)$, let $\lambda(n)$ be the intersection $\mu \cap GL_n(\mathbb{F}_q)$, which is non-empty if and only if $||\lambda|| \leq n$ and let

$$K_\lambda(n) = \sum_{U \in \lambda(n)} U.$$

The sum $K_\lambda(n)$ is an element of $\mathcal{H}_n = \mathcal{H}(GL_n(\mathbb{F}_q))$, the center of the integral group algebra $\mathbb{Z}[GL_n(q)]$, as pointed earlier in the general setting of Eq.(2). Notice that, if $\lambda(n) = \emptyset$ then the above sum is over the empty set and hence equal to 0.

**Lemma 3.14.** [20], Lemma 2.3 The set $\{K_\lambda(n) : \lambda \in \mathcal{P}(\Phi), K_\lambda(n) \neq 0\}$ forms the class sum $\mathbb{Z}$-basis for the center $\mathcal{H}_n$, for each $n \geq 0$.

3.4. The growth of the centralizers. We have seen in Section 2.1, Proposition 2.5, that in order to determine the structural functions $c_{a,b}^n(g)$ one needs to study the growth of the centralizer of a fixed element as the groups enlarge. So, one needs a variant of Eq.(6).

**Remark 3.15.** Recall that if $g \in S_n$ which has no fixed points and $n \geq m$ then

$$C_{S_n}(g^{\cdot n}) = C_{S_m}(g) \oplus S_{n-m}.$$  

where, as before, $g^{\cdot n}$ is the image of $g$ under the natural identification of $S_m$ in $S_n$.  

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Remark 3.16. Let $U \in GL_n(\mathbb{F}_q)$ and $\lambda^U$ be its non-modified type. Then $\dim V^U = l(\lambda(t - 1))$. This can be seen directly from the fact that only the companion matrices belonging to $t - 1$ contributes to the 1-eigenspace and for each block, the contribution to the dimension is incremented by 1 (cf. Remark 4.1).

Let $\mu \in \mathcal{P}(\Phi), m = ||\mu||$. Assume that $U \in GL_m(q)$ whose type is $\mu$. For the matrix $U$, the following is the variant of Eq. (15). Let $\dim V^U = l(\mu_{t-1}) = d$.

Proposition 3.17. [20, Proposition 2.5] Let $n \geq m = ||\mu|| + d = ||\mu||$. Then, the centralizer $C_{GL_n(q)}(U^{\uparrow n})$ of $U^{\uparrow n} \in GL_n(\mathbb{F}_q)$ is given by

$$C_{GL_n(q)}(U^{\uparrow n}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A \in C_{GL_m(q)}(U), D \in GL_{n-m}(\mathbb{F}_q), UB = B, CU = C \right\}.$$

In particular, $A$ and $D$ are invertible and hence

$$|C_{GL_n(q)}(U^{\uparrow n})| = |C_{GL_m(q)}(U)| \cdot |GL_{n-m}| \cdot q^{2d}.$$

Proof. The second equality directly follows from the first equality and Remark 3.16. Conditions on $B$ and $C$ follows from the equality

$$\begin{bmatrix} U & 0 \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{n-m} \end{bmatrix},$$

The proof of the invertibility of $A$ and $D$ can be found in [20]. There, the authors in fact prove that

$$\det\left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \det(A) \det(D)$$

whenever $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is in the centralizer of $U$. \hfill \Box

3.5. Reflection length, modified type and the main theorems of Wan and Wang. The following Lemma is due to [8]. It is the analogue of Lemma 2.14 and used in [20] to prove a similar result to Theorem 2.10 in the case of $GL_n(q)$.

Lemma 3.18. [8, Proposition 2.9, 2.16]

(1) For $U \in GL_n(q)$, the reflection length and residual dimension are equal: $l(U) = \dim R^U = \text{codim} V_n^U$.

(2) The reflection length is sub-additive: i.e. for $U_1, U_2 \in GL_n(q)$

$$l(U_1U_2) \leq l(U_1) + l(U_2).$$

(3) If $l(U_1U_2) = l(U_1) + l(U_2)$ then

$$V_n^{U_1} \cap V_n^{U_2} = V_n^{U_1U_2} \quad \text{and} \quad V_n = V_n^{U_1} + V_n^{U_2}.$$

Lemma 3.19. [20, Lemma 3.2] The reflection length is stable under the embedding $G_m \subseteq G_n$ for all $m,n \in \mathbb{N}$ satisfying $m \leq n$. Moreover:

(1) If the modified type of $U$ is $\mu$, then $l(U) = ||\mu||$.

(2) If the modified types of $U_1, U_2, U_1U_2 \in G_\infty$ are $\lambda, \mu, \nu$ respectively, then

$$||\lambda|| + ||\mu|| \leq ||\nu||.$$

Proposition 3.17, Lemma 3.19 and Lemma 3.18 are sufficient to prove that the index function

$$n \rightarrow \frac{|C_{GL_n(q)}(U_1U_2)|}{|C_{GL_n(q)}(U_1) \cap C_{GL_n(q)}(U_2)|}$$

is independent of $n$ if

$$||\lambda|| + ||\mu|| = ||\eta||,$$

where $\lambda, \mu$ and $\eta$ are stable types of $U_1, U_2$ and $U_1U_2$, respectively. However, to prove that the structural function $c^q_{\lambda, \mu}(n)$ is indeed independent of $n$ requires to know that there are only finitely many index functions which contribute to the structural function $c^q_{\lambda, \mu}(n)$ and this is equivalent to show that the fibers $V(\lambda \times \mu : \eta)$ admits only finitely many orbits with respect to the simultaneous conjugation. Such result relies on the normal form results of Wan and Wang.
Lemma 3.20. [20] Let $U_1, U_2 \in GL_n(q)$ and $l(U_1U_2) = l(U_1) + l(U_2)$. Moreover, let $T \in GL_n(q)$ be such that
\[
TU_1U_2T^{-1} = \begin{bmatrix}
U_1 & 0 \\
0 & I_{n-l(U_1U_2)}
\end{bmatrix}
\]
then
\[
TU_1T^{-1} = \begin{bmatrix}
U_1 & 0 \\
0 & I_{n-1}
\end{bmatrix}, \quad \text{and} \quad TU_2T^{-1} = \begin{bmatrix}
U_2 & 0 \\
0 & I_{n-1}
\end{bmatrix}.
\]

Remark 3.21. Wan and Wang do not present this last lemma as an isolated entity but produce it as a by product of the proof of the proposition below. We, instead, present it independently because we will use it in the context of symplectic groups.

Proposition 3.22 (Normal Form Theorem). [20, Proposition 3.3] Let $U_1, U_2, U_1U_2 \in G_\infty$ and $\lambda, \mu, \eta$ be their modified types respectively. Suppose $||\eta|| = ||\lambda|| + ||\mu||$ and set $m = ||\nu|| + l(\nu(t-1))$. Then there exists $T \in GL_n(q)$ and $U_1, U_2 \in G_k$ such that
\[
TU_1T^{-1} = \begin{bmatrix}
U_1 & 0 \\
0 & I_{n-m}
\end{bmatrix}, \quad TU_2T^{-1} = \begin{bmatrix}
U_2 & 0 \\
0 & I_{n-m}
\end{bmatrix}, \quad TU_1U_2T^{-1} = \begin{bmatrix}
U_1U_2 & 0 \\
0 & I_{n-m}
\end{bmatrix}.
\]

Corollary 3.23. The simultaneous conjugation admits finitely many orbits. Hence $(GL_n(F_q))_{n \in \mathbb{N}}$ is a uniformly saturated family.

The following theorem is the stability property of the uniformly saturated family $(GL_n(q))_{n \in \mathbb{N}}$ and it is proved using the previous results as analogs of them used to prove the stability result for the uniformly saturated family $(S_n)_{n \in \mathbb{N}}$.

Theorem 3.24 (Stability Theorem). [20, Theorem 3.4] Let $\lambda, \mu, \eta$ be three elements of $\mathcal{P}(\Phi)$. If $||\eta|| = ||\lambda|| + ||\mu||$, then $c^q_{\lambda,\mu}(n)$ is a non-negative integer independent of $n$.

4. The case of symplectic groups: $Sp_n(q)$

In this chapter, we start dealing with the case of symplectic groups. In the first section the basics of symplectic spaces and alternating forms are discussed. In the subsequent section a detailed review of conjugacy in symplectic groups is presented. The results of the second section are used to obtain a rational form for the unipotent symplectic matrices. In the fourth section the reviewed theory is used to construct the uniformly saturated family $Sp_n(q)$. Finally, the main theorem, the stability property of center of the symplectic group rings is proved assuming Theorem 4.29 whose proof is deferred to the next chapter.

4.1. Review of symplectic groups. This section presents the basic properties of the symplectic groups $Sp_n(q)$ over finite field with $q$ elements. The main reference for this section are the books Symplectic Groups by O.T. O’Meara [15] and Linear Algebra and Geometry, a seconds course, by I. Kaplansky, [10].

Let $V$ be an $F_q$ vector space of dimension $n$, where $q$ is an odd prime power. An alternating form (or symplectic form) $Q(\cdot, \cdot)$ on $V$ is a map $V \times V \to F_q$ such that for all $u, v, w \in V$ and $a \in F_q$, the equalities
1. $Q(v, w) = -Q(w, v)$, (alternating property)
2. $Q(av + u, w) = aQ(v, w) + Q(u + w)$, (bilinearity)
hold. If $Q$ is an alternating form on $V$ then the pair $(V, Q)$ is called a symplectic space. Given two symplectic spaces $(V_i, Q_i), i = 1, 2$, over $F_q$ are called equivalent if there is a bijective linear map $\phi : V_1 \to V_2$ such that
\[
Q_2(\phi(v), \phi(w)) = Q_1(v, w),
\]
for all $v, w \in V_1$. In the case of equality $V_1 = V_2$, one speaks of the equivalency of $Q_1$ and $Q_2$ and drop the underlying vector space from the notation. As done for all bilinear forms, the effect of $Q(\cdot, \cdot)$ on $V \times V$ can be written in terms of matrices. Let $B = \{e_1, \cdots, e_n\}$ be a fixed ordered basis of $V$ and let $[S_Q]_B$ be the $n \times n$ matrix $(s_{ij})_{i,j=1}^n$ where
\[
s_{ij} = Q(e_i, e_j).
\]
The matrix \([S_Q]_B\) is a skew symmetric in the sense that, \(S_Q^\text{tr} = -S_Q\), as a consequence of the fact that \(Q\) is alternating. Let \(v, w \in V\) be two elements that are considered as column vectors written with respect to the ordered basis \(\{e_1, \ldots, e_n\}\). Then it is easily seen that

\[
Q(v, w) = v^\text{tr} \cdot [S_Q]_B \cdot w.
\]

Two elements \(v, w \in V\) are said to be **orthogonal** to each other, denoted as \(v \perp w\), if \(Q(v, w) = 0\). Similarly, two subspaces \(W_1, W_2 \subset V\) are said to be **orthogonal** to each other if for all \(w_1 \in W_1, w_2 \in W_2\), \(Q(w_1, w_2) = 0\). The orthogonality of subspaces again denoted by the notation \(W_1 \perp W_2\). For a subspace \(W \subset V\), the subspace of elements that are orthogonal to \(W\) is \(W^\perp := \{v \in V : v \perp w, \forall w \in W\}\). A symplectic space \((V, Q)\) is said to be **non-degenerate** if \(V^\perp = 0\). The non-degeneracy of a form \(Q\) is equivalent to non-vanishing of \(\det(S_Q)\), which is independent of the chosen basis. A **hyperbolic pair** \((e, f)\) with respect to \(Q\) is an element of \(V \times V\) with the property \(Q(e, f) = 1\). In this case \(e\) will be referred as the **positive** part and \(f\) will be referred as the **negative** part of the hyperbolic pair.

**Lemma 4.1.** [15, Theorem 1.1.13] Let \((V, Q)\) be a symplectic space. Then the following are equivalent:

1. \(Q\) is non-degenerate.
2. \(V\) admits an ordered basis \(\{e_1, e_2, \ldots, e_n, f_n, f_{n-1}, \ldots, f_1\}\) where \((e_i, f_i)\) is a hyperbolic pair for \(i \in \{1, \ldots, n\}\), such that \(H_i \perp H_j\) for \(i \neq j \in \{1, \ldots, n\}\), where \(H_i = \langle e_i, f_i \rangle\) is the subspace generated by the hyperbolic pair \((e_i, f_i)\). With respect to this basis the matrix of \(Q\) is equal to the block diagonal matrix

\[
Q = \begin{bmatrix}
e_1 & e_2 & \cdots & e_n & f_n & \cdots & f_2 & f_1 & 1 \ne_1 & e_2 & \cdots & e_n & f_n & \cdots & f_2 & f_1 \\
e_2 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & e_n \\
-1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & f_n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & f_2 \\
-1 & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & f_1 \\
-1 & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & e_2 \\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & e_1
\end{bmatrix}
\]

In particular, non-degenerate symplectic spaces must be even dimensional and if \(Q_1\) and \(Q_2\) are two non-degenerate symplectic forms on \(V\) then they are equivalent.

A basis \(B\) satisfying 2. of Lemma 4.1 is called a **hyperbolic basis**. In this case \(e_i\) and \(f_i\) are said to be **hyperbolic conjugates of each other**. If \(B\) is a hyperbolic basis, then \(B^+\) denote the positive parts of hyperbolic pairs in \(B\), and \(B^-\) denote the negative parts of hyperbolic pairs in \(B\).

Let \((V, Q)\) be a non-degenerate symplectic space. An element of \(U \in GL(V)\) is said to be a **symplectic transformation** if

\[
Q(Uv, Uw) = Q(v, w)
\]

for all \(v, w \in V\). The set of symplectic transformations form a group which is called the symplectic group and denoted by \(Sp(V)\). It is contained in the special linear group \(SL(V)\) of linear transformations with determinant 1 ([15], Thm. 2.1.110). For an element \(U \in GL(V)\), whether or not \(U\) is a symplectic transformation can be checked via hyperbolic bases. Let \(\{e_1, f_1, \cdots, e_n, f_n\}\) be a hyperbolic basis for \((V, Q)\) and \(U \in GL(V)\). Then \(U\) is an element of \(Sp(V)\) if and only if \(\{Ue_1, Uf_1, \cdots, Ue_n, Uf_n\}\) is a hyperbolic basis.

### 4.2. Conjugacy classes in \(Sp_n\)

In this section, the references that we follow are *On isometries of inner product space* by J. Milnor [14], and *Conjugacy Classes* by Springer-Steinberg in [2]. Since these results are not comprehensively covered in text books, we will present a thorough discussion.

Let \((V, Q)\) be a non-degenerate symplectic space of dimension \(2n\). By Proposition 3.6, conjugacy classes of \(GL(V)\) are parameterized by the partition valued functions \(\lambda : f \mapsto \lambda(f) = (\lambda_1, \cdots, \lambda_r)\) on the \(\Phi\),
which are of weight $2n$:

$$2n = ||\lambda|| = \sum_{f \in \Phi} \deg f \cdot ||\lambda(f)|| = \sum_{f \in \Phi} \deg f \cdot (\sum_{i=1}^{r_f} \lambda_i)$$

However, if one considers elements $U \in Sp(V)$, then one can not realize all the partition valued functions of weight $2n$. This is not the only obstacle. Namely, let $U_1, U_2$ be two isometries and assume that $\lambda^{U_1} = \lambda^{U_2}$. Then it is known that $U_1$ and $U_2$ are conjugate only over a suitable extension $F$ over $\mathbb{F}_q$, (cf. [10], Theorem 70, pg. 79), which means for a fixed $m$, the family $(Sp_n(q^n))_{n \in \mathbb{N}}$ is not saturated.

Let $U \in Sp(V)$ and $V_U$ denotes $\mathbb{F}_q[t]$-module whose underlying space is $V$, on which $t$ acts as $U$. i.e. $t \cdot v = Uv$. Let $m_U(t)$ denotes the minimal polynomial of $U$ and introduce the module $A(U) = \mathbb{F}_q[t]/(m_U(t))$.

From the fact that $Q(Uv, w) = Q(v, U^{-1}w)$ and the bilinearity of $Q$ it follows that for every polynomial $f \in \mathbb{F}_q(t)$ one has

$$Q(f(U)v, w) = Q(v, f(U^{-1})w). \quad (16)$$

Substituting $m_U$ in the equation Eq.(16) one gets

$$0 = Q(0 \cdot v, w) = Q(m_U(U)v, w) = Q(v, m_U(U^{-1})w), \forall v, w \in V.$$ 

Since the form $V$ is non-degenerate, it follows that $m_U(U^{-1}) = 0$ and thus the minimal polynomial of $U^{-1}$ divides that of $U$. By symmetry and the fact that both polynomials are monic, it follows that $m_U(t) = m_{U^{-1}}(t)$. As a result, the map

$$\sigma : U \mapsto U^{-1}$$

induces an isomorphism on $A(U) = \mathbb{F}_q[t]/(m_U(t))$, which is obviously an involution.

**Definition 4.2.** For $f = a_0 + a_1 t + \cdots + t^d \in \Phi$, introduce the **dual** $\overline{f} \in \mathbb{F}_q(t)$ by

$$\overline{f}(t) = \sum_{i=0}^{d} (a_i a_0^{-1} t^{d-i}). \quad (17)$$

A self-dual polynomial $f$ is called **dual-irreducible** if $f$ is either irreducible or $f = g\overline{g}$ where $g$ is an irreducible polynomial that is not self-dual. Denote the set of dual-irreducible polynomials with $\Phi^\perp$.

**Remark 4.3.** It is straightforward that $\overline{f g} = \overline{f} \overline{g}$, hence, if $f$ is an irreducible polynomial then its dual $\overline{f}$ is also irreducible. It is also clear that a self-dual polynomial is a product of dual-irreducible polynomials.

**Lemma 4.4.** If $U \in Sp_n$ then the minimal polynomial $m_U(t)$ of $U$ is self-dual. In particular, $m_U(t)$ is a product of dual-irreducible polynomials.

**Proof.** We start with noticing the following relation between the automorphism $\sigma$ of $A(U)$ sending $U$ to $U^{-1}$, and the dual operation defined on polynomials (cf. Eq.(17)):

$$\sigma(f(U)) = f(U^{-1})$$

$$= \left( \sum_{i=0}^{d} a_i U^{-i} \right) (a_0^{-1} U^d)(a_0 U^{-d})$$

$$= a_0 U^{-d} \sum_{i=0}^{d} (a_i a_0^{-1} U^{d-i})$$

$$= a_0 U^{-d} \overline{f}(U).$$

Invoking this observation in Eq. (16) and taking $f(t) = m_U(t)$ yields

$$0 = Q(m_U(U)v, w) = Q(v, a_0 U^{-d} \overline{m_U}(U)w) = Q(U^d \cdot v, a_0 \overline{m_U}(U) \cdot w).$$

As $U$ is invertible and $Q$ is non-degenerate, it follows that $\overline{m_U}(U) = 0$. The desired equality now follows from the equality of the degrees. \qed
Lemma 4.5. If $f_1$, $f_2$ are distinct monic irreducible factors of $m_U$, the minimal polynomial of $U \in Sp_n(q)$, then the generalized eigenspaces $V_{f_i} = \{ v \in V : f_i^k(U)v = 0, \text{for large } k \}$ for $i = 1, 2$ are orthogonal to each other unless $\overline{f_1} = f_2$.

Proof. Let $k$ be such that $f_i^k(U)v = 0$ for all $v \in V_{f_i}$. Then, for all $v_i \in V_{f_i}$, $i = 1, 2$ one gets

$$0 = Q(0, v_2) = Q(f_i^k(U)v_1, v_2) = Q(v_1, f_i^k(U^{-1})v_2) = Q(v_1, a_q^kU^{-dk\overline{f_i}^k}(U)v_2).$$

Next we assume that $\overline{f_1} \neq f_2$. As $\overline{f_1}$, $f_2$ are both irreducible, it follows that $\overline{f_1}^k$ and $f_2$ are coprime and there exist $h_1, h_2 \in F_q[t]$ such that $h_1\overline{f_1}^k + h_2f_2 = 1 \in F_q[t]$. As the action of $h_2f_2(U)$ on $V_{f_2}$ is zero, it follows that, on $V_{f_2}$ we have $h_1\overline{f_1}^k(U) = 1$, in particular it acts as an automorphism of $V_{f_2}$, so does $U^{-dk\overline{f_i}^k}(U)$. This finishes the proof.

Let $U \in Sp_n(q)$. Let $f(t)$ be a dual-irreducible divisor of $m_U(t)$. If $f$ is irreducible, set $W_f$ to be $V_f$ (the generalized eigenspace of $f$) and if $f = g\overline{g}$ for some irreducible non-self-dual polynomial $g$, then set $W_f$ as the subspace $V_g \oplus V_{\overline{g}}$. With this notation, the above findings can be packed into the following proposition. Recall that $\Phi^*$ is defined to be the set of dual-irreducible polynomials in $F_q[t] - \{t\}$.

Lemma 4.6. [14] For each dual-irreducible divisor $f$ of $m_U(t)$, the subspace $W_f$ is a non-degenerate symplectic space and $V$ is equal to the orthogonal sum of $W_f$'s, as $f$ ranges over dual-irreducible factors of $m_U(t)$. In particular, the restriction $U|_{W_f}$ is an isometry of $W_f$ and $V$ admits the following orthogonal sum of invariant subspaces:

$$V = \bigoplus_{\substack{\lambda(t) \in \Phi^* \setminus f(t) \in \Phi^* \setminus m_U(t) \text{ or } \deg f = 1}} W_f.$$

Proposition 4.7. [14] Let $U_1, U_2$ be two isometries of $V$. The isometries $U_1$ and $U_2$ are conjugate in $Sp_n(q)$ if and only if

1. $\lambda(U_1) = \lambda(U_2)$,
2. The isometries $(U_1)|_{W_f}$ and $(U_2)|_{W_f}$ are conjugate in $Sp(W_f)$, for $f = t \pm 1$.

In particular, the $Sp$ conjugacy class of $W_f$ for $f \neq t \pm 1$ is completely determined by the Jordan form.

Proof. For $f \neq t \pm 1$ self-dual, see the proof of Theorem 3.2 in [14]. For $f$ non-self-dual, see the second paragraph following Theorem 3.4 in ibid.

This reduces the study of conjugacy classes into the study of conjugacy classes of elements $U$ such that the polynomial $m_U(t)$ is a power of $(t \pm 1)$.

Theorem 4.8. [14, Theorem 3.2] Let $U$ be an isomorphism, and $W_{t \pm 1}$ be as in Lemma 4.6. The space $W_{t \pm 1}$ admits an orthogonal decomposition

$$V_U = W_{t \pm 1}^{1} \perp \cdots \perp W_{t \pm 1}^{r}$$

where $W_{t \pm 1}^{i}$ is a free $F_q[t]/(t \pm 1)^{m_i}$-module and $\lambda(t \pm 1) = (1^{m_1}, \ldots, r^{m_r})$.

Proof. (Sketch) Consider a not necessarily orthogonal decomposition of $V_U$ as in statement of the lemma. Then the restriction $Q_{W_{t \pm 1}^i}$ of the inner product $Q$ to $W_{t \pm 1}^i$ is non-degenerate [19, Lemma 1.4.6], [14, Theorem 3.2]. So we can consider the orthogonal decomposition of $V_U = W_{t \pm 1}^{r} \oplus (W_{t \pm 1}^{r})^\perp$ and continue by induction.

Theorem 4.9. [14, Theorem 3.4] We keep the notation and the assumptions of the previous Theorem.

1. For each $i = 1, \ldots, r$, there exists a vector space $H_i^{\pm}$ and a bilinear form $h_i^{\pm}$ on $H_i^{\pm}$, called the Wall form.
2. The dimension of $H_i^{\pm}$ is $m_i^{\pm}$, where $h_i^{\pm}$ is a non-degenerate symplectic form for odd $i$, and $h_i^{\pm}$ is a symmetric bilinear form for even $i$.
3. The equivalence classes of $(h_i^{\pm})_i$ completely determine the $Sp_m(q)$ conjugacy classes of $x|_{W_{t \pm 1}^{r}}$. 19
Following Milnor (cf. [14, Section 3]), we will recall the construction of the vector spaces \( H_i^- \) and the definition of the Wall forms \( h_i^- \) for a fixed \( i \), hence we restrict ourselves to the case \( \mu_U(U) = (t-1)^i \), i.e. to the unipotent \( U \) case. Let \( A(U):= \mathbb{F}_q[t]/(t-1)^s \) and \( \Delta = t - t^{-1} \), where \( t \) is the image of \( U \) in \( A(U) \). Introduce \( H_i^- := W_{t-1}^i/(U-I)W_{t-1}^i \). The subspace \( W_{t-1}^i \) is a free \( A(U) \)-module, hence equal to direct sum of cyclic modules \( T_1^i, \ldots, T_r^i \), for some \( r > 0 \). Since \( T_j \) is a cyclic module, there exists \( v_j \in T_j \) such that the translates \( v_j, Um_j, U^2v_j, \ldots \) generate \( T_j \). Then, it follows that \( T_j \subset H_i^- = W_{t-1}^i/(U-I)W_{t-1}^i \) is generated by \( v_j \), and hence

\[
H_i^- = \bigoplus_{j=1}^k (v_j).
\]

The association

\[
h_i^- (\overline{v}, \overline{w}) = Q(\Delta^{i-1} v, w), \forall \overline{v}, \overline{w} \in H_i^-
\]

is wellDefined and defines bilinear form on \( H_i^- \). According to the theorem, it is a symplectic non-degenerate form for odd \( i \) and symmetric non-degenerate form for even \( i \). As, over a given vector space, all non-degenerate symplectic forms are isomorphic, one can take \( h_i^- = -1 \) for \( i \) odd. Likewise, as non-degenerate symmetric bilinear forms over \( \mathbb{F}_q \) are parameterized by \( \mathbb{F}_q^2/((\mathbb{F}_q)^2)^2 \), for even \( i \) we have \( h_i^- \) is equal to \( +1 \) or \( -1 \).

**Definition 4.11.**

1. A **signed partition** is a couple \( (\lambda, h) \) such that \( \lambda = (\lambda_1, \cdots, \lambda_r) \) is an ordinary partition and \( h = (h_1, \cdots, h_r) \in \{-1, +1\}^r \) satisfying the following property: if \( \lambda_i = \lambda_j \) then \( h_i = h_j \).

2. The **weight** \( ||(\lambda, h)|| \) of a signed partition \( (\lambda, h) \) is defined as the weight \( ||\lambda|| \) of the underlying partition.

**Remark 4.12.** One can write a signed partition in the form \( \lambda = (1^{(m_1,-)}, 2^{(m_2,\pm)}, \cdots) \). For example, if \( (\lambda, h) = ((6, 6, 2, 2, 2, 1, 1, (-, -, +, +, +, +, -), -)) \) then one can write \( (\lambda, h) \) as \( (1^{(3,-)}, 2^{(4,+)}, 6^{(2,-)}) \). Also observe that the weight of a symplectic partition is always an even integer.

**Definition 4.13.**

1. A signed-partition \( (1^{(m_1,h_1)}, 2^{(m_2,h_2)}, \cdots) \) is called a **symplectic partition** if for odd \( i \), \( m_i \) is even and \( h_i = -1 \). The set of symplectic partitions is denoted by \( \mathcal{P}^s \).

2. A symplectic partition valued function (simply, a symplectic function) is a triple \( (\lambda, h^+, h^-) \), where \( \lambda \) is a partition valued function defined on \( \Phi^s \), and \( (\lambda(t-1), h^-), (\lambda(t+1), h^+) \) are symmetric partitions. The weight of such a function is defined as the weight of the underlying partition valued function. The set of symplectic partition valued functions of weight \( 2m \) is denoted by \( \mathcal{P}_2^s(\Phi^s) \) and the set of all symplectic partition valued functions is denoted by \( \mathcal{P}^s(\Phi^s) \).

With this notation, we can rephrase Theorem 4.9 as follows.

**Corollary 4.14.** [16, Theorem 1.20] The conjugacy classes in \( Sp_n(q) \) are parameterized by the symplectic partition valued functions of weight \( 2m \). If \( (\lambda, h^+, h^-) \) is the symplectic partition valued function that corresponds to the isometry \( U \), then the underlying partition valued function \( \lambda \) is equal to \( \lambda^U \), when viewed as an element of \( GL_{2m}(q) \). The symplectic function \( (\lambda, h^+, h^-) \) is called the **symplectic type** of \( U \).

### 4.3. Rational forms for unipotent blocks in \( Sp_n(q) \)

Following [7], we introduce a family of matrices what will serve as rational forms for unipotent matrices in the symplectic groups. Introduce the matrices \( S_m \) for \( m \in \mathbb{N} \) defined as follows. First recall that the matrices

\[
S_m := \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}, \quad S_m^{-1} = \begin{bmatrix}
1 & -1 & \cdots & \cdots & 1 \\
-1 & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-1 & 1 & \cdots & \cdots & 1
\end{bmatrix}.
\]
were defined earlier. Clearly, the minimal and characteristic polynomials of $S_m$ and $S_m^{-1}$ are equal to $(t-1)^n$.

Now introduce the matrices

\[ J_{2m} = \begin{pmatrix}
1 & e_2 & \cdots & e_m & f_m & \cdots & f_2 & f_1 \\
1 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1
\end{pmatrix} \]

and for $\epsilon \neq 0$

\[ J_{2m,\epsilon} = \begin{pmatrix}
1 & e_2 & \cdots & e_m & f_m & f_{m-1} & \cdots & f_1 \\
1 & 1 & \cdots & \cdots & 1 & \epsilon & \cdots & \epsilon \\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1 & \epsilon & \cdots & \epsilon & 1
\end{pmatrix} \]

written with respect to the ordered hyperbolic basis $\{e_1, e_2, \cdots, e_m, f_m, \cdots, f_2, f_1\}$. The matrices of the form $J_{2m}$ will be called $2m$-dimensional symplectic blocks and matrices of the form $J_{2m,\epsilon}$ will be called an $2m$-dimensional orthogonal blocks. The matrices $J_{2m}$ and $J_{2m,\epsilon}$ are elements of the symplectic group, which can be readily seen by checking the equality

\[ Q(C_u(J_{2m}), C_v(J_{2m})) \]

as $u, v$ ranges over $B$. The minimal polynomial of $J_{2m}$ is equal to the minimal polynomial $m_{S_m}(t) = m_{S_m^{-1}}(t) = (t-1)^n$ of $S_m$ and the minimal polynomial of $J_{2m,\epsilon}$ is equal to $(t-1)^{2m}$. In particular, 1 is the unique eigen-value in both cases. Notice also that $J_2 = I_2$ and no other $J_{2m,\epsilon}$ satisfies such an equality.

**Remark 4.15.** When $U$ is an $m \times m$ matrix, we will view $U$ as a linear operator of $V = \mathbb{F}_q^n$ in two ways: Let $v = (v_1, \cdots, v_m) \in \mathbb{F}_q^n$.

1. The association $v \mapsto v \cdot U$ is called the **right action** of $U$. The fixed space of this action is denoted by $^U V$. The following identities are obvious:

\[ J_{2m,\epsilon}^o V = \langle e_1, f_m \rangle, \quad J_{2m,\epsilon}^r V = \langle e_1 \rangle \]

2. The association $v^t \mapsto U \cdot v^t$ is called the **left action** of $U$. The fixed space of this action is denoted by $V^U$. The following identities are obvious:

\[ V^{J_{2m,\epsilon}} = \langle e_m, f_1 \rangle, \quad V^{J_{2m}} = \langle f_1 \rangle \]

In case of a symplectic block, the space $V$ splits off into two cyclic spaces with cyclic vectors $e_1$ and $f_m$. And in case of an orthogonal block, the space $V$ contains $e_1$ as a cyclic vector.

**Remark 4.16.** When the rows and columns of a matrix are labeled with bases elements, then we consider the matrix as a linear operator in two different ways, as described in the previous remark. In this case, we will consider both rows and columns of the matrix as vectors of the appropriate vector space determined by the bases.
Our next aim is to show that each symplectic unipotent conjugacy class is realized as the orthogonal sums of suitable symplectic and orthogonal blocks. To this end, we will investigate the \( F_q[t] \)-module structures on \( V \) that are induced by \( J_{2m} \) and \( J_{2m,\epsilon} \). More precisely, we will investigate the induced bilinear forms \( h_i \), as explained in Remark 4.10.

Let \( U = J_{2k+2,0} \), which acts on the symplectic space \( V_{4k+2} \). The minimal polynomial of \( U \) is \((t - 1)^{2k+1} \) and \( V_{4k+2} \) is equal to the direct sum of two cyclic \( F_q[t]/(t - 1)^{2k+1} \)-modules \( T_1 := \langle e_1, \cdots, e_{2k+1} \rangle \) and \( T_2 := \langle f_1, \cdots, f_{2k+1} \rangle \). So, \( W_{i-1}^{2k+1} = V_{4k+2} \) and \( W_i^{i-1} = 0 \) for \( i \neq 2k+1 \). The subspace \( T_1 \) (resp. \( T_2 \)) is generated by the \( U \) translates of \( e_1 \) (resp. \( f_{2k+1} \)). Recall that \( \Delta \) is defined as \( U - U^{-1} \). Thus we have

\[
\Delta = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 & 2 & 0 \\
2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & 2 & 0 & \cdots & \cdots & \cdots \\
0 & -2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & \cdots & -2 & 0 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

and hence

\[
\Delta^{2k} = \begin{bmatrix}
e_1 & \cdots & e_{2k+1} & f_{2k+1} & \cdots & f_1 \\
e_2 & \cdots & & & & \\
\vdots & & & & & \\
e_{2k+1} & & & & & \\
f_{2k+1} & & & & & \\
\vdots & & & & & \\
f_1 & & & & & \\
\end{bmatrix}
\]

The space \( H_{2k+1}^- \) is generated by \( \{2^{-2k}e_1, f_{2k+1}\} \) and

\[
h_{2k+1}^- (2^{-2k}e_1, f_{2k+1}) = Q(\Delta^{2k}(2^{-2k}e_1), f_{2k+1}) = Q(e_m, f_m) = 1.
\]

This means \( H_{2k+1}^- \) is a non-degenerate symplectic space with hyperbolic basis \( \{2^{-2k}e_1, f_{2k+1}\} \). In particular, the symplectic type \( (\lambda^U, h^+, h^-) \) of \( U \) can be described as follows. For \( f \neq t - 1 \), \( \lambda(f) = 0 \), the empty partition, and \( \lambda(t - 1) = (2k + 1, 2k + 1) = ((2k + 1)^2) \). As \( \lambda(t + 1) \) is the empty partition, \( h_i^\pm \) is a sequence of length zero. The sign corresponding to \( 2k + 1 \) is \( -1 \) as the sign is determined by the isomorphism class of \( h_{2k+1}^- \), which is a non-degenerate symplectic form. So, \( \lambda(t - 1) = ((2k + 1)^2, -1) \). With this point of view, if \( U = \bigoplus_{i=1}^r a_i J_{4i+2} \), where the direct sum is the usual orthogonal sum and \( a_i \)'s are allowed to be zero, then \( \lambda(t - 1) = (1^{(2a_{2r+1}, -1)}, \cdots, (2r + 1)^{(2a_{2r+1}, -1)}) \).

Next we consider the case \( U = J_{2k,\epsilon} \), where \( \epsilon \in \mathbb{F}_q^* \), with its action on \( V_k \). The minimal polynomial of \( U \) is \((t - 1)^{2k} \) and thus \( W_{i-1}^{i-1} = 0 \) for \( i \neq 2k \), consequently, \( W^{2k} \) is equal to the ambient space \( V_k \). The space \( V_k = W_{i-1}^{2k} \) is generated by the \( U \) translates of the cyclic vector \( e_1 \), so \( H_{2k}^- \) is generated by the image of \( e_1 \).
in $H_{2k}$. We also have

$$
\Delta = \begin{bmatrix}
e_1 & \cdots & e_{k-1} & e_k & f_k & \cdots & f_2 & f_1 \\
0 & & & & & & & \\
2 & 0 & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & 2 & 0 & & & & \\
e & \epsilon & \epsilon & 2\epsilon & 0 & & & \\
e & -2 & 0 & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\epsilon & -1 & \cdots & -2 & 0 & & & f_1
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
& & & & & & & \\
e_k \\
f_k \\
& & & & & & & \\
\vdots \\
f_1
\end{bmatrix}
$$

and

$$
\Delta^{2k-1} = \begin{bmatrix}
e_1 & \cdots & e_k & f_k & \cdots & f_1 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\epsilon'(k) & & & & & & & f_1
\end{bmatrix}
$$

where $\epsilon'(k) := -\epsilon(k) := (-1)^{k-1}2^{2k-1}\epsilon$. As a result, we have

$$
h_{2k}^-(\overline{e_1}, \overline{e_1}) = Q(\Delta(e_1), e_1) = Q(-\epsilon(k)f_1, e_1) = \epsilon(k) = (-1)^{k-1}2^{2k-1}\epsilon \neq 0.
$$

whose image $\overline{\epsilon(k)}$ in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2 = \{\pm 1\}$ is equal to the discriminant of the symmetric bilinear form $h_{2k}^-$, consequently, $h_{2k}^-$ is non-degenerate. By taking $\epsilon$ to be a 1 or a non-square, one can obtain both possible discriminant values in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$. This means, the symplectic type $(\lambda, h^+, h^-)$ of $U$ is defined as follows: $\lambda(f) = 0$ for $f \neq t - 1$ and $\lambda(t - 1) = ((2k)^{1-\epsilon(k)})$. In order to generalize as done above, consider $U = \oplus_{i=1}^r \oplus_{j=1}^{\alpha_j} J_{2t_i, t_i}$, where, as before $\alpha_i$’s are allowed to be zero. Then

$$
\lambda(t - 1) = (2^{(a_1, \Pi \equiv 1 \text{ mod } t_1)}, \ldots, 2^{(a_r, \Pi \equiv r \text{ mod } t_r)}
$$

One can combine the investigated situations immediately and derive the following proposition:

**Proposition 4.17.** [7, Proposition 2.3] Let $U \in Sp(V_m)$ be a unipotent matrix. Then there is a hyperbolic basis $V_m$ so that the matrix of $U$ in this basis is equal to the orthogonal sum of suitable symplectic and orthogonal unipotent blocks.

**Remark 4.18.** When considering matrices, we will always label rows and columns with basis elements, hence each matrix will determine a unique endomorphism. So, if $M$ is a matrix, then one can check whether $M$ is an isometry or not by checking the equality

$$
Q(u, v) = Q((M(u), M(v))
$$

where $u, v$ range over the basis set that is used to label rows and columns. One can also decide whether $M$ is an isometry or not, by considering the matrix of $Q$ (again denoted by $Q$) with respect to the basis used to label $M$. Indeed, the question of $M$ being an isometry is equivalent to the equality

$$
M^t Q M = Q.
$$
The matrix of $Q$ with respect to the basis used in the definition of symplectic/orthogonal blocks is the following:

$$
\begin{bmatrix}
e_1 & \cdots & e_k & f_k & \cdots & f_1 \\
\vdots & & & & \vdots & & \vdots \\
1 & & & & 1 & e_1 \\
\vdots & & & & \vdots & & \vdots \\
-1 & & & & f_k & f_k \\
\vdots & & & & \vdots & & \vdots \\
-1 & & & & & & f_1
\end{bmatrix}
$$

4.4. The uniformly saturated family $(Sp_n(q))_{n \in \mathbb{N}}$. Let $V_\infty$ be an infinite dimensional $\mathbb{F}_q$-vector space. We will consider $V_\infty$ with the ordered basis $B = \{e_1, f_1, \ldots, e_m, f_m, \ldots\}$ and the subspace generated by $B_m = \{e_1, f_1, \ldots, e_n, f_n\}$ will be denoted as $V_m$. The hyperbolic conjugate of $w \in B$ is denoted by $w'$. We endow $V$ with the unique symplectic structure where $V_m$ is a non-degenerate symplectic space and $B_m = \{e_1, f_1, \ldots, e_n, f_n\}$ is a hyperbolic basis. For $m \leq n$, the orthogonal complement $V_m^\perp$ of $V_m$ in $V_n$ is denoted by $V_{m,n}$ and its hyperbolic basis $\{e_{m+1}, f_{m+1}, \ldots, e_n, f_n\}$ is denoted by $B_{m,n}$. The inclusion $V_m \subset V_n$ induces an embedding from

$$
(\cdot)^\perp_n : GL_{2m}(q) \longrightarrow GL_{2n}(q)
$$

which carries $Sp_m(q)$ into $Sp_n(q)$ and thus defines a direct system of groups. The direct limit of this system will be denoted by $Sp_\infty(q)$ and referred as the infinite symplectic group. The similar map from $GL_m(q)$ to $GL_n(q)$ is defined in [20] and it is denoted by $U \longmapsto U^{\perp n}$. It is clear that the map $(\cdot)^{2n}$ from $GL_{2m}(q)$ to $GL_{2n}(q)$ coincides with the map $(\cdot)^{\perp n}$ defined above. The group $GL_\infty(q)$ is defined in the same manner.

Recall that the weight of a symplectic function on $\Phi^*$ was defined as the weight of the underlying partition valued function. The modification operation $\circ$, completion $\pi$ and $n$-th completion $\overline{\pi}^n$ are defined in a similar way. In particular, let $(\lambda, h^+, h^-)$ be a symplectic function.

**Definition 4.19.** The weight $||(\lambda, h^+, h^-)||$ of $(\lambda, h^+, h^-)$ is by definition

$$(\lambda, h^+, h^-) = ||\lambda||.$$ 

The set of symplectic functions of weight $2n$ is denoted by $\mathcal{P}_{2n}(\Phi^*)$. The set of all symplectic functions is denoted by $\mathcal{P}(\Phi^*)$. For $(\lambda, h^+, h^-) \in \mathcal{P}_{2n}(\Phi^*)$ the modification $(\lambda, h^+, h^-)$ is defined by setting

$$(\lambda, h^+, h^-)^\circ = (\lambda, h^+, h^-)$$

where $h^+ = h^+_i$ and $h^- = h^-_i$ is defined as follows. First recall that $(\lambda(t-1), h^-)$ is by definition a symplectic partition. As a result, it can be written as $(1^{(m_1, \epsilon_1)}, 2^{(m_2, \epsilon_2)}, \ldots, r^{(m_r, \epsilon_r)})$ where $\epsilon_i = \pm 1$ and for odd $i$, $m_i$ is even and $\epsilon_i = 1$. The modified partition $\tilde{\lambda}$ is then equal to $(1^{m_2}, \ldots, (r-1)^{m_r})$. So we define $h^- = (h_1, \ldots, h_{r-1})$ where $h_i = \epsilon_{i+1}$ for $i = 1, \ldots, r-1$. In particular, the resulting signed partition $(\tilde{\lambda}, h^-)$ can be written as $(1^{m_2}, \ldots, (r-1)^{m_r})$. Clearly, the resulting signed partition is in general not a symplectic partition. Likewise,

$$(\lambda, h^+, h^-)^{\perp n} = (\lambda, h^+, h^-)$$

where $h^+ = h^+_i$ and $h^- = h^-_i$. Finally, the $n$-completion $(\lambda, h^+, h^-)^{\overline{\pi} n}$ of $(\lambda, h^+, h^-)$ is defined by the rule

$$(\lambda, h^+, h^-)^{\overline{\pi} n} = (\lambda^{2n}, h^+^{\overline{\pi}}, h^-^{\overline{\pi}})$$

where $h^+^{\overline{\pi}} = h^+$ and $h^-^{\overline{\pi}}$ is defined similarly. In fact, consider $(\lambda(t-1), h^-) = (1^{(m_1, \epsilon_1)}, 2^{(m_2, \epsilon_2)}, \ldots, r^{(m_r, \epsilon_r)})$. Then we define $h^-^{\overline{\pi}}$ the sequence $\pm 1$ so that the equality $(\lambda(t-1)^{\overline{\pi} n}, h^-^{\overline{\pi} n}) = (1^{(m_0, -1)}, 2^{(m_1, \epsilon_1)}, \ldots, (r+1)^{(m_r, \epsilon_r)})$ holds, where $r = n - ||\lambda||$. The unipotent and non-unipotent blocks are defined analogously.
**Remark 4.20.** Note that, unlike the maps \( \lambda \mapsto \bar{\lambda} \) and \( \lambda \mapsto \lambda^{\uparrow 2n} \), the modification operator \((\cdot)\) does not map \( \mathcal{P}(\Phi^s) \) to itself as the weight of the resulting function may fail to be even. The set of **modified symplectic functions** \( \mathcal{P}^{st} \) is defined as the image \( \mathcal{P}^{s}(\Phi) \) of \( \lambda \mapsto \bar{\lambda} \). Clearly in this case \( \lambda \mapsto \bar{\lambda} \) maps the modified symplectic functions to the symplectic functions.

If \( U \in \text{Sp}_m(q) \) and \( ^s\lambda^U = (\lambda, h^+, h^-) \) is the symplectic type of \( U \), then it follows that

\[
(20) \quad ^s\lambda^{(U^{\uparrow n})} = (\lambda^{\uparrow 2n}, h^+, h^{-\uparrow n}).
\]

where the operation \( \lambda \mapsto \lambda^{\uparrow 2n} \) for partition valued functions was described in Remark 3.3. Relying on this observation we follow the idea of the definition given in [20] and introduce the map

\[
U \mapsto ^s\lambda^U = \in \mathcal{P}^{st}(\Phi)
\]

and called the image function **modified symplectic type of** \( U \).

**Remark 4.21** (Reflection length). Let \( G \) be an abstract group and \( R \subset G \) be a set of elements that generates \( G \) as a monoid. The length \( l(g) \) of \( g \in G \) with respect to \( R \) is defined to be the minimum of

\[
\{ l \in \mathbb{N} : g = r_1 r_2 \cdots r_l, \ r_i \in R \}.
\]

Such a function is clearly a sub-additive function. If \( R \) is closed under conjugation then \( l \) is invariant on the conjugacy classes. In the case of symplectic groups, the set \( R \) is taken to be transvections in general, which are by definition reflections of determinant 1. In this case, the relation between reflection length and residual space of an element \( g \in \text{Sp}_n(q) \) is as follows, (cf [15], Thm. 2.1.11):

1. If \( g \) is an involution then \( l(g) = \dim R^g + 1 \).
2. If \( g \) is not an involution then \( l(g) = \dim R^g \).

This means, the reflection length on \( \text{Sp}_n(q) \) induced by transvections is not consistent with the weight of the stable type. As a result, we will be considering \( \text{Sp}_n(q) \) with the reflection length induced from \( GL_{2n}(q) \).

**Lemma 4.22.**

1. The family \( \{\text{Sp}_n(q)\}_{n \in \mathbb{N}} \) is a saturated family.
2. The map \( U \mapsto ^s\lambda^U \) induces a bijection between the conjugacy classes of \( \text{Sp}_\infty(q) \), and the set of all stabilized symplectic functions \( \mathcal{P}^{st} \).
3. Let \( \lambda \in \mathcal{P}^{st} \) be a modified symplectic function. Then \( \text{Sp}_m(q) \) contains an element whose symplectic stable type is \( \lambda \) if and only if \( ||\bar{\lambda}|| \leq 2m \).
4. Let \( \lambda \in \mathcal{P}^{st} \) be a modified symplectic function such that \( ||\bar{\lambda}|| = 2m \). Let \( U \in \text{Sp}_m(q) \) be an element whose modified type is \( \lambda \) and \( n \) be an integer greater than \( m \). Then

\[
\lambda^{\uparrow n} = \lambda^{(U^{\uparrow n})}
\]

where \( \lambda^{\uparrow n} \) denotes the image of \( \lambda \) in \( \widehat{\text{Sp}}_n(q) \).
5. Reflection length remains unchanged under the embedding \( \text{Sp}_m(q) \hookrightarrow \text{Sp}_m(q) \) and it is equal to the weight of the stable type.

**Proof.**

1. By Eq.(20) one can see that non-conjugate elements in \( \text{Sp}_m(q) \) remain non-conjugate in \( \text{Sp}_n(q) \) for \( m \leq n \) which proves the first claim.
2. The fact that \( U \mapsto ^s\lambda^U \) defines a well-defined map from \( \widehat{\text{Sp}}_\infty(q) \) to \( \mathcal{P}^{st}(\Phi) \) follows from Eq.(20) and the rest follows from Theorem 4.14.
3. and 4. are formal consequences of the definitions.
5. Follows from the fact that the weight of the symplectic stable type is equal to the weight of the stable type and Lemma 3.19/1.

The following two lemmas are symplectic analogous of Lemma 3.19 and Lemma 3.18.

**Lemma 4.23.** [8, Proposition 2.9, 2.16]

1. For \( U \in \text{Sp}_n(q) \) the reflection length \( rl(U) \) is equal to the codim \( V^U_n \).
Proof. (of 4.23 and 4.24) Use Lemma 3.18 and Lemma 3.19 and the fact that the reflection length on $Sp_m(q)$ is the reflection length induced by $GL_{2m}(q)$ and along with the fact that weight of a symplectic function is equal to the weight of the underlying partition valued function. □

We end this section following the lines of [20] in the context of symplectic groups. Let $\lambda = (\lambda, h^+, h^-)$ be a stabilized symplectic function and let $\lambda$ also denote the conjugacy class in $Sp_{\infty}(q)$ which corresponds to $\lambda$. Let $n$ be a positive integer. Then

$$\lambda(n) := Sp_n \cap \lambda \neq \emptyset \iff \|\lambda\| \leq 2n,$$

in which case we set

$$K_\lambda(n) = \sum_{g \in \lambda(n)} g.$$

$K_\lambda(n)$ is an element of $H_n := \mathcal{H}(Sp_n(q))$, the center of the integral group algebra $\mathbb{Z}[Sp_n(q)]$. Notice that if $\lambda(n) = \emptyset$ then the above sum is over the empty set and hence equal to 0.

Lemma 4.25. The set $\{K_\lambda(n) \neq 0 : \lambda \in \mathcal{P}(\Phi)\}$ forms the class sum $\mathbb{Z}$-basis for the center $H_n$, for each $n \geq 0$.

4.5. Structure constants of $H_n$ and the main theorems. We start with proving the normal form theorem (cf. Proposition 3.22) in the context of symplectic groups. This will allow us to deduce that the simultaneous conjugation admits finitely many orbits.

Proposition 4.26 (Normal Form Theorem). Let $U_1, U_2, U_1U_2 \in Sp_n(q)$ and $\lambda, \mu, \eta$ be their modified symplectic types respectively. Suppose $||\eta|| = ||\lambda|| + ||\mu||$ and $||\eta|| = 2m$. There exists $T \in Sp_n(q)$ and $U_1, U_2 \in Sp_m(q)$ such that

$$TU_1T^{-1} = \begin{bmatrix} U_1 & 0 \\ 0 & I_{2n-2m} \end{bmatrix}, \quad TU_2T^{-1} = \begin{bmatrix} U_2 & 0 \\ 0 & I_{2n-2m} \end{bmatrix}$$

and

$$TU_1U_2T^{-1} = \begin{bmatrix} U_1U_2 & 0 \\ 0 & I_{2n-2m} \end{bmatrix}.$$

Proof. We will use Lemma 3.20 as it is used in the proof of Prop. 3.22 in [20]. Since the modified symplectic type of $U_1U_2$ is $\eta$, and $||\eta|| = 2m$, it follows that there exists a symplectic transformation $U_\eta \in Sp_m(q)$ which is conjugate to $U_1U_2$, hence there exists an element $T$ in $Sp_n(q)$ so that the matrix of $TU_1U_2T^{-1}$ is equal to the matrix $U_\eta^T$:

$$TU_1U_2T^{-1} = U_\eta^T = \begin{bmatrix} U_\eta & 0 \\ 0 & I_{2n-2m} \end{bmatrix}.$$

Considering $U_1, U_2, U_1U_2$ as elements of $GL_{2n}(q)$ and using the fact that the weight of the symplectic partition valued function and the weight of the ordinary partition valued function defined by the same element are equal, we may apply Lemma 3.20 to the triple $U_1, U_2, U_1U_2$, from which the result follows. □

Let $Z = Z(\lambda \times \beta : \eta)$ be the set of elements $(U_1, U_2) \in \lambda \times \beta$ such that $U_1U_2 \in \eta$. The group $Sp_\infty(q)$ acts on $Z$ by simultaneous conjugation, which is defined by the rule $T \cdot (U_1, U_2) := (TU_1T^{-1}, TU_2T^{-1})$, for $T \in Sp_\infty(q)$.
Corollary 4.27. The set \( Z \) admits finitely many orbits with respect to the simultaneous conjugation.

Proof. Follows directly from the normal form theorem as each orbit contains a representative in \( Sp_m(q) \), which is a finite set. \( \square \)

By the proposition, up to conjugation, we may assume that \( U_1, U_2 \) and \( U = U_1 U_2 \) are all contained in \( Sp_m(q) \). Let \( d \) be the dimension of the fixed space of \( U_\eta \).

Corollary 4.28. Let \( L_1, \cdots, L_k \) be the totality of orbits in \( Z = Z(\lambda \times \beta : \eta) \) and \( (U_{1i}, U_{2i}) \cap Sp_m \times Sp_m \). Let \( (U_{1i}, U_{2i}) \in L_i \) and \( U_i = U_{1i} U_{2i} \) for \( i = 1, \cdots, k \). Then for \( n \geq m \)

\[
(22) \quad \eta_{\lambda, \mu}(n) = \sum_{i=1}^{k} \frac{C_{Sp_m(q)}(U_i^{\uparrow n})}{C_{Sp_m(U_{1i}^{\uparrow n})} \cap C_{Sp_m(U_{2i}^{\uparrow n})}}
\]

where \( \eta_{\lambda, \mu}(n) \geq 0 \) is the coefficient of \( K_\eta(n) \) satisfying

\[
K_\lambda(n) \cdot K_\mu(n) = \sum_{\eta \in \mathcal{P}(\mathfrak{f})} \eta_{\lambda, \mu}(n) \cdot K_\eta(n)
\]

Proof. For \( i, j = 1, \cdots, k \), the elements \( U_i \) and \( U_j \) are conjugate to each other and together conjugate to \( U \), so one can take \( U_i = U \). This means, \( Z(n) := Z \cap Sp_n(q) \times Sp_n(q) \) is in fact the set of \( (x, y) \in \lambda \times \mu \) such that \( xy \in (U^{\uparrow n})^{Sp_n} \), hence \( \eta_{\lambda, \mu}(n) = \frac{Z(n)}{\eta} \). Order of the orbit of \((U_{1i}, U_{2i})\) is equal to \( Sp_n(q)/Stab(U_{1i}, U_{2i}) \), where \( Stab(U_{1i}, U_{2i}) \) is the stabilizer of \((U_{1i}, U_{2i})\) under the simultaneous conjugation. The cardinality of the stabilizer is clearly equal to \( C_{Sp_m(U_{1i}^{\uparrow n})} \cap C_{Sp_m(U_{2i}^{\uparrow n})} \). \( \square \)

Theorem 4.29 (Growth of centralizers). For \( m \leq n \) the following equalities hold:

\[
(23) \quad |C_{Sp_m(q)}(U^{\uparrow n})| = |C_{Sp_m(q)}(U)| \cdot |Sp_m(q)| \cdot q^{2(n-m)d}.
\]

and

\[
(24) \quad |C_{Sp_m(U_1^{\uparrow n})} \cap C_{Sp_m(U_2^{\uparrow n})}| = |C_{Sp_m(U_1)} \cap C_{Sp_m(U_2)}| \cdot |Sp_m(q)| \cdot q^{2(n-m)d}.
\]

Proof. See the next chapter. \( \square \)

The following theorem is the stability theorem in the case of symplectic groups. We present it in the form given in [20].

Theorem 4.30 (Stability Theorem). Let \( \lambda, \mu, \eta \) be three modified symplectic functions and assume that \( ||\eta|| = ||\lambda|| + ||\mu|| \). Then \( \eta_{\lambda, \mu}(n) \) is a non-negative integer independent of \( n \).

Proof. Substituting the order formulas \((23)\) and \((24)\) in the equation given in Corollary 4.28 we see that each summand in the right hand side of the Eq. \((22)\) is equal to

\[
\frac{|C_{Sp_m(q)}(U_i)|}{|C_{Sp_m(U_1^{\uparrow n})} \cap C_{Sp_m(U_{2i}^{\uparrow n})}|}
\]

which is independent of \( n \). \( \square \)

5. PROOF OF CENTRALIZER GROWTH THEOREM

In this chapter, we will prove the Theorem 4.29, which was the main ingredient of the proof of the Theorem 4.30.

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5.1. Generic matrices and symplectic equations. Let $F$ be an arbitrary field and $n, m \in \mathbb{N}$ be positive integers. The set of $n \times m$ matrices whose entries are in $F[x_{ij}]$, $i = 1, \ldots, n; j = 1, \ldots, m$ is called the $n \times m$ generic matrices. Let $S = \{i_{1,j_1}, \ldots, i_{r,j_r}\}$ be a set of indices. A generic matrix with free indices in $S$ is a generic $n \times m$ matrix $D(S) = D(d_{ij})_{i,j}$ such that $d_{ij} = x_{ij}$ if $(i, j) \in S$ and $d_{ij} \in F$ if $(i, j) \notin S$. By substituting elements from $F$ to the variables in $S$, each generic matrix $D(S)$ with free variables in $S$ defines a function from $F^S$ to $\text{Mat}_{n \times m}(F)$. If $\overline{\sigma} \in F^S$, the image of $\overline{\sigma}$ under this map is denoted by $D(\overline{\sigma})$ and each matrix in the image of a generic matrix $D$ is called a realization of $D$. In the case of $S = \{(i, j) = i = 1, \ldots, n; j = 1, \ldots, m\}$ there is a unique generic matrix, the universal generic matrix $X$. For example, if $S = \{(1, 1), (2, 2)\}$, then

$$\begin{bmatrix}
11 & 3 \\
2 & 22
\end{bmatrix}$$

is a generic $2 \times 2$ matrix with respect to $S$. Then the realization $D(5, 7)$ of $D$ is

$$\begin{bmatrix}
5 & 3 \\
2 & 7
\end{bmatrix}$$

Let $f$ be a function of the entries of $D$. Then one can define a function $f^D$ on the set of realizations of $D$. For example $\det^D$ for $D$ introduced above is given by the following formula:

$$\det^D(x_{11}, x_{22}) = x_{11}x_{22} - 6.$$ 

Recall our conventions on the labeling of the rows and columns of matrices. We now insist on the condition that when the matrix is square, the labeling of rows and columns will be assumed to be done with respect to the same ordered basis. For example if $X$ is the $2n \times 2n$ generic matrix and $B = \{e_1, f_1, \ldots, e_n, f_n\}$ is an hyperbolic basis for $V$, then columns and rows of the $X$ are indexed by the basis elements preserving their orders. So, an entry of $X$ is of the following form: $x_{uv}$ where $u, v \in B$. To be even more concrete, we present the following example.

**Example 5.1.** Assume that $X$ is the $4 \times 4$ universal generic matrix and the indexing of its columns (and hence its rows) is $e_1, e_2, f_2, f_1$. Then we write the universal matrix $X$ as

$$X = \begin{pmatrix}
e_1 & e_2 & f_2 & f_1 \\
e_1, e_1 & e_1, e_2 & e_1, f_2 & e_1, f_1 \\
e_2, e_1 & e_2, e_2 & e_2, f_2 & e_2, f_1 \\
e_1, f_1 & e_1, f_2 & e_1, f_2 & e_1, f_1
\end{pmatrix}.$$

The $uv$-th symplectic equation $SE(u, v, B)$ with respect to the fixed hyperbolic basis with a prescribed ordering, which concerns the entries of $u$-th and $v$-th columns of $X$, is defined as follows:

$$\sum_{i=1}^{n} x_{ei,u}x_{f_i,v} - \sum_{i=1}^{n} x_{f_i,u}x_{ei,v} = Q(u, v).$$

Observe that the left hand side of the equation is nothing but the formal image of $Q(C_u(X), C_v(X))$. In fact, by considering matrices with labeled rows and columns, we will view the columns of matrices as elements in the image vector space, and we will often identify the column and the vector defined by the column (depending on the labeling). For example symplectic equation $SE(e_1, f_2)$ for $X$ above can be calculated by treating the entries as coefficients of basis vectors. That is

$$0 = Q(e_1, f_2) = Q(X(e_1), X(f_2)) = Q(x_{e_1, e_1}e_1 + x_{e_2, e_1}e_2 + x_{f_1, e_1}f_1, x_{e_1, f_2}e_1 + x_{e_2, f_2}e_2 + x_{f_2, f_2}f_2 + x_{f_1, f_2}f_1) = x_{e_1,f_1}x_{f_1,f_2} + x_{e_2,f_1}x_{f_2,f_2} - x_{f_2,e_1}x_{e_2,f_2} - x_{f_1,e_1}x_{e_1,f_2}$$

The set of all symplectic equations $SE(u, v, B)$, $u, v \in B$ is called the symplectic equations with respect to $B$ and denoted by $SE(B)$. 

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Remark 5.2. Symplectic equations can be considered for generic matrices with free variables. For example, consider the the following $4 \times 4$ generic matrix $D(S)$ with free variables in $S = \{(e_2, e_1), (e_2, f_2)(f_1, f_2)\}$

$$D(S) = \begin{pmatrix} e_1 & e_2 & f_2 & f_1 \\ 3 & 0 & 4 & 0 \\ x_{e_2e_1} & 3 & x_{e_2f_2} & -4 \\ 0 & 0 & 6 & 0 \\ 1 & 0 & x_{f_1f_2} & 6 \end{pmatrix}$$

Then the symplectic equations with respect to $D(S)$ are obtained by specifying entries of $D(S)$ in the symplectic equations and the will be denoted again by $E(u, v)$ when the basis $B$ and $D$ are fixed.

1. The equation $SE(e_1, f_2)$ is obtained by considering the equality

$$Q(e_1, f_2) = Q(C_{e_1}(D(S)), C_{f_2}(D(S)))$$

hence $E(e_1, f_2)$ is 0 = $3x_{f_1f_2} + 6x_{e_2e_1} - 4$, or simply

$$4 = 3x_{f_1f_2} + 6x_{e_2e_1}.$$  

2. The equations $SE(e_1, e_2)$ and $E(f_2, f_1)$ can be computed similarly and they are simply $0 = 0$.

3. Finally, the equation $SE(e_1, f_1)$ is

$$1 = Q(e_1, f_1) = 3 \cdot 6 + x_{e_2e_1} \cdot 0 - 0 \cdot (-4) - 1 \cdot 0 = 18.$$  

This means that there is no symplectic realization $M$ of $D(S)$.

Using this terminology, there is a tautological result concerning the symplectic transformations which we record as the next lemma. It will be beneficial in the calculation of the growth of the centralizers of unipotent elements.

Lemma 5.3. Let $(V, Q)$ be a non-degenerate symplectic space and $B$ an hyperbolic basis with a prescribed order. Let $U \in GL(V)$. Then, $U \in Sp(V)$ if and only if the columns of $U$ satisfy the symplectic equations $SE(B)$.

We end this section with inducing the question of the growth of the centralizer of a general symplectic matrix $U$ case to the unipotent $U$ case:

Remark 5.4 (Growth depends on the unipotent block). Let $U$ be a symplectic transformation whose non-modified type is the symplectic partition valued function $(\lambda, h^+, h^-)$ of weight $2m$. Then, by Lemma 4.6, we may assume that $U$ has the form

$$U = \begin{bmatrix} U^{\lambda^+} & 0 \\ 0 & U^\lambda \end{bmatrix}$$

where the type of $U^{\lambda^+}$ is $\lambda^+$, the type of $U^\lambda$ is $\lambda$, and the diagonal sum of the matrices is an orthogonal sum. From this we conclude that the minimal polynomial of $U^\lambda$ is a power of $t - 1$ and the minimal polynomial of $U^{\lambda^+}$ is coprime to $t - 1$. Now we consider the embedding of $U$ into $Sp_n(q)$ for some $n > m$ and an element $D$ from the centralizer of $U^{\lambda^+}$ and writing it in the block form of $U^{\lambda^+}$ yields the following equality:

$$\begin{bmatrix} U^{\lambda^+} & 0 \\ 0 & U^\lambda \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} U^{\lambda^+} & 0 \\ 0 & U^\lambda \end{bmatrix}$$

Then one obtains the following equality of matrices:

$$\begin{bmatrix} U^{\lambda^+}D_{11} & U^{\lambda^+}D_{12} & U^{\lambda^+}D_{13} \\ UD_{21} & UD_{22} & UD_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} D_{11}U^{\lambda^+} & D_{12}U^\lambda & D_{13} \\ D_{21}U^{\lambda^+} & D_{22}U^\lambda & D_{23} \\ D_{31}U^{\lambda^+} & D_{32}U^\lambda & D_{33} \end{bmatrix}$$

From this, it follows that each $D_{ij}$ is an intertwining operator between $F_q[t]$-modules. However, as pointed out earlier in Remark 3.7 and Remark 3.9, an intertwining operator between two modules with distinct primary cyclic parts must be zero. Since the primary cyclic parts of the modules defined by $U^\lambda$ and $I_{2n-2m}$
are all of type \( \mathbb{F}_q[t]/(t-1)^r \) for some \( r \geq 1 \) and the primary cyclic parts of the modules defined by \( U_{\lambda^n} \) are all of type \( \mathbb{F}_q[t]/(f)^r \) for some \( f \neq t - 1 \) and \( r \geq 1 \) it follows that the intertwining operators \( D_{12}, D_{13}, D_{21}, D_{31} \) are all zero. As a result

\[
D = \begin{bmatrix}
D_{11} & 0 & 0 \\
0 & D_{22} & D_{23} \\
0 & D_{32} & D_{33}
\end{bmatrix}
\]

where \( D_{11} \) is in the centralizer of \( U_{\lambda^n} \) and \( \begin{bmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{bmatrix} \) is in the centralizer of \( U_{\lambda} \). This means, in order to investigate the growth of the centralizer of a symplectic matrix \( U \) under the embedding \( U \mapsto U^{\uparrow \uparrow n} \), it is sufficient to consider the same question for the unipotent block of \( U \).

5.2. Unipotent Matrix Actions. In this section, we introduce an action of \( \text{Mat}_{n \times n} \times \text{Mat}_{m \times m} \) on \( \text{Mat}_{n \times m} \) as follows. For every square matrix \( A \in \text{Mat}_{n \times n}, B \in \text{Mat}_{m \times m} \) and \( M \in \text{Mat}_{n \times m} \) put

\[
(A,B) \cdot M = A B M
\]

We will introduce some terminology concerning the fixed points of a fixed \( (A,B) \in \text{Mat}_{n \times n} \times \text{Mat}_{m \times m} \) which is similar to the concept of symplectic equations introduced earlier. Taking \( M \) as the generic matrix \( X \) and writing

\[
AXB - X = 0
\]

induces a homogeneous system of linear equations in the variables \( x_{ij}, i = 1, \ldots, n, j = 1, \ldots, m \), which will be denoted by \( E(A,B;x_{ij}) \). Clearly, each solution of the system \( E(A,B;x_{ij}) \) defines a fixed point of \( (A,B) \). An index \((r,k)\) is called a free index with respect to \( (A,B) \), if \( x_{rk} \) does not appear in the system \( E(A,B;x_{ij}) \) of linear equations induced by Eq.(28), in which case we refer to \( x_{rk} \) as a free variable with respect to \( (A,B) \), or simply a free variable. This means, if \( M \in \text{Mat}_{m \times n} \) then the condition of \( M \) being a fixed point can be checked without knowing \( m_{rk} \), so the following definition makes sense: A generic fixed point of \( (A,B) \) with respect to a set \( S \) of free indices is a generic matrix \( D(S) \) with free variables in \( S \) where \( D(\overline{\pi}) \) is a fixed point of \( (A,B) \) for every \( \overline{\pi} \in F^S \).

Example 5.5. Let \( A = B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \). Then the equation Eq.(28) reads as

\[
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} =
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}
\]

Direct multiplication yields

\[
\begin{bmatrix}
x_{11} + x_{12} \\
x_{21} + x_{22} + x_{11} + x_{12}
\end{bmatrix} =
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}
\]

Therefore, the induced homogeneous system \( E(A,B;x_{ij}) = E(A,B;x_{11},x_{12},x_{21},x_{22}) \) of linear equations is

\[
x_{12} = 0 \\
x_{11} + x_{22} = 0
\]

This means, the only free index with respect to \( (A,B) \) is \((2,1)\). The matrix

\[
D(x_{12}) = \begin{bmatrix} 1 & 0 \\ x_{21} & -1 \end{bmatrix}
\]

is thus a generic fixed point of \( (A,B) \) and the realization \( D(2) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \) of \( D \) is an actual fixed point of \( (A,B) \).

Lemma 5.6. Let \( A \in \text{Mat}_{n \times n} (\mathbb{F}_q), B \in \text{Mat}_{m \times m} (\mathbb{F}_q) \) and let \( S \) be the set of free indices induced by \( (A,B) \). If \( GFix(A,B) \) denotes the set of generic fixed points of \( (A,B) \) and \( Fix(A,B) \) denotes the set of fixed points of \( (A,B) \) then

\[
|Fix(A,B)| = |GFix(A,B)| \cdot q^{|S|}.
\]
Proof. Follows from the definitions.

The last lemma will be useful when considering the growth of the centralizer of elements under the natural embedding $Sp_m(q) \hookrightarrow Sp_n(q)$ for $m \leq n$, where the next lemma will be useful when considering the intersection of centralizers of two matrices. An $n \times m$ matrix whose only non-zero is 1 and placed at the $(r, k)$ will be denoted by $1_{r,k}$. Observe that in the notation there is no reference to the size, but in each case, it will be determined by the context.

**Lemma 5.7.** An index $(r, k)$ is a free index with respect to $(A, B)$ if and only if the matrix $1_{r,k}$ is a fixed point of $(A, B)$.

**Proof.** (⇒) Assume that $(r, k)$ is a free index. Then the linear system of equations $E(A, B; x_{i,j})$ induced by $(A, B)$ is homogeneous and $x_{r,k}$ does not appear in these equations. As every homogeneous system of linear equations admits the trivial solution, $1_{r,k}$ is a fixed point of $(A, B)$.

(⇐) Assume that $(r, k)$ is not free and let

$$\alpha x_{r,k} + \text{other terms with various variables } x_{i,j} = 0$$

where $\alpha \neq 0$. But in this situation the previous equation becomes $1 = 0$ as the all the variables are equal to zero except $x_{r,k}$, which is absurd. □

Now we will restrict the previous action to a certain subset $U_n$ of unipotent matrices in $\text{Mat}_{n \times n}$ for which we will be able to determine the free indices precisely. We define $U_n$ as the set of unipotent matrices $U$ of size $n$ which satisfy the following properties: $U$ is lower triangular and the subdiagonal entries of $U$ are all non-zero. Hence, elements of $U_n$ are of the following form:

$$U = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{nn-1} & 1
\end{pmatrix}$$

where $u_{ii-1} \neq 0$ for $i = 2, \ldots, n$.

**Remark 5.8.**

1. Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis and suppose that the rows and columns of the matrix $U \in U_n$ are indexed by $B$. Then $V^U = \langle e_n \rangle$ and $U^V = \langle e_1 \rangle$.

2. Moreover, a symplectic block $J_{3m+2}$ is a diagonal sum of two matrices from $U_{2m+1}$ and an orthogonal block $J_{3m}$ is an element of $U_{2m}$.

For $n, m \in \mathbb{N}$, one can restrict the previous action to $U_n \times U_m$. This action will be called the unipotent action. We are interested in the free indices of $(U_1, U_2)$ with $U_1 \in U_n, U_2 \in U_m$ the unipotent action. So let us fix $U_1$ and $U_2$. Observe that $U_m$ is closed under inversion and hence $U_2^{-1} \in U_m$. So we may write

$$U_1 = \begin{pmatrix}
e_1^1 e_2^1 e_3^1 \cdots e_n^1 \\
1 & 0 & 0 & \cdots & 0 \\
u_{21} & 1 & 0 & \cdots & 0 \\
u_{31} & u_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{nn-1} & 1
\end{pmatrix}, \quad U_2^{-1} = \begin{pmatrix}
e_1^2 e_2^2 e_3^2 \cdots e_m^2 \\
1 & 0 & 0 & \cdots & 0 \\
v_{21} & 1 & 0 & \cdots & 0 \\
v_{31} & v_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{mm-1} & 1
\end{pmatrix}$$

Consider an $n \times m$ matrix $M$. Then the rows of $M$ will be labeled with $B_1$ and the columns of $M$ will be labeled with $B_2$.

**Lemma 5.9.** The index $(n, 1)$ is the unique free index of the unipotent pair $(U_1, U_2)$. In general, $(e_n^1, e_1^2)$ is the unique free index.
Proof. Let $X$ be the generic $n \times m$ matrix. By direct multiplication we calculate the $ij$-th entry of $U_1 X$ and $XU_2^{-1}$ and obtain

\[ u_{i1}x_{1j} + \cdots + u_{i(i-1)}x_{i-1j} + x_{ij} = (U_1X)_{ij} \]
\[ = (XU_2^{-1})_{ij} \]
\[ = x_{ij} + x_{ij+1}v_{ij+1} + \cdots + x_{im}v_{mj} \]

As the subdiagonal entries of $U_1$ and $U_2^{-1}$ are non-zero, it follows that, in the linear equation induced by the $ij$-th position, the coefficients of $x_{i-1j}$ and $x_{ij+1}$ are non-zero, hence they can not be free. On the other hand, the equation (29) shows that, in the equation induced by the $ij$-th position, none of the entries below or on the right of $ij$-th position occurs. This proves the claim concerning the index $(n, 1)$. \]

Remark 5.10. The claim that the index $(e_n^1, e_1^2)$ is free can be proved using the description of the eigenvectors of $U_1$ and $U_2$, which were determined in Remark 5.8. Thus we have

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
u_{21} & 1 & 0 & \cdots & 0 \\
u_{31} & u_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_{n1} & u_{n2} & \cdots & u_{n(n-1)} & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
v_{21} & 1 & 0 & \cdots & 0 \\
v_{31} & v_{32} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{mm-1} & 1
\end{bmatrix}
\]

This means $1_n$ is a solution of $U_1 X U_2 = X$. By the Lemma 5.7, $(n, 1)$ is a free index. This observation, i.e. proving an index is free by means of 1-eigen-vectors, will be useful when considering the intersection of two centralizers in the symplectic group.

Lemma 5.11. In a generic fixed point $D$ of unipotent action (hence in all fixed points), the first row is zero, except possibly for the first entry. This row is called the leading row of $D$. The basis element $e_1^2$ corresponding to this row is called the leading basis element.

Proof. The first row of $U_1 X U_2$ can be directly computed, hence we can consider the first row of $U_1 X U_2$ and $X$. By doing so, one obtains the following system of equations that a generic fixed point must satisfy:

\[
x_{1m} = x_{1m} \\
x_{1m-1} = x_{1m-1} + v_{mm-1} x_{1m} \\
x_{1m-2} = x_{1m-2} + v_{mm-2} x_{1m-1} + v_{mm-2} x_{1m} \\
\vdots \\
x_{12} = x_{12} + \sum_{j=3}^{m} v_{j2} x_{1j} \\
x_{11} = x_{11} + \sum_{j=2}^{m} v_{j1} x_{1j}
\]

Since the subdiagonal entries are non-zero, it follows from the second equation that $x_{1m} = 0$. Using this fact in the third equation yields

\[
x_{1m-2} = x_{1m-2} + v_{mm-2} x_{1m-1}.
\]

As $v_{mm-2}$ is a subdiagonal entry, it is non-zero and hence $x_{1m-1} = 0$. Clearly, this procedure can be iterated until the last equation, which proves the lemma. \]
As a result, a generic fixed point \( D(x_{n1}) \) of \((A, B) \in \mathcal{U}_n \times \mathcal{U}_m \) is of the following form:

\[
D(x_{n1}) = \begin{bmatrix}
  d_{11} & 0 & \cdots & 0 \\
  d_{21} & d_{22} & \cdots & d_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,m} \\
  x_{n1} & d_{n2} & \cdots & d_{nm}
\end{bmatrix} \leftarrow \text{leading row}
\]

where for every \( \alpha \in \mathbb{F}_q \) the matrix \( D(\alpha) \) obtained by substituting \( \alpha \) in \( x_{n1} \) is a fixed point of \((A, B)\) under the unipotent action. The row (resp. column) containing the free index will be called the \textit{pivotal row} (resp. \textit{leading column}). For a generic fixed point \( D \), the element in the intersection of the leading row and leading column will be called the \textit{leading element}. Hence, in the above example, the leading element of \( D(x_{n1}) \) is \( d_{11} \in F \).

Now we generalize these notions to the diagonal sum of matrices. Let \( A = \text{diag}(A_1, \cdots, A_{r_1}) \) and \( B = \text{diag}(B_1, \cdots, B_{r_2}) \) be two \( n \times n \) matrices where each block \( A_i \) (resp. \( B_i \)) of \( A \) (resp. \( B \)) are contained in \( \mathcal{U} = \bigcup_{j \geq 1} \mathcal{U}_j \). A fixed point \( D \) of \((A, B)\) is subject to the homogeneous system of linear equations \( E \), which is defined by the following equation:

\[
AXB = X.
\]

Let the sizes of \( A_i \) and \( B_j \) be \( a_i \) and \( b_j \) respectively, for \( i = 1, \cdots, r_1; j = 1, \cdots, r_2 \). And let \( X_{ij} \) be the block form of \( X \) that is induced from the block forms of \( A \) and \( B \). More precisely, the \( X_{ij} \) is a \( a_i \times b_j \) matrix. It is then clear that, the homogeneous system of equations \( E \) is equal to the union of homogeneous system of equations \( E_{ij} \) defined by the equation:

\[
A_i X_{ij} B_j = X_{ij}.
\]

But this means, if \( D \) is a fixed point of \((A, B)\) then each \( D_{ij} \) is a fixed point of a certain unipotent action, and hence, one can talk about pivotal row, leading column and leading row of \( X_{ij} \). It is also clear that each \( E_{ij} \) contains distinct variables, as a result, an indeterminate \( x_{uv} \) can occur in at most one system of equations \( E_{ij} \). In particular, the set equality concerning linear equations below holds:

\[
E = \bigcup_{i=1,\cdots,r_1; j=1,\cdots,r_2} E_{ij}.
\]

It is also clear that each \( E_{ij} \) contains distinct variables, as a result, an indeterminate \( x_{uv} \) can occur in at most one system of equations \( E_{ij} \). Call this system of equations \( E(x_{uv}) \). It is then clear that \( x_{uv} \) does not occur in the homogeneous system of linear equations induced from \( AXB - X = 0 \) if and only if it does not appear in \( E(x_{uv}) \), i.e. it is a free variable of the equation \( E(x_{uv}) \). Relying on this observation, we define the \textit{set of free variables} of \( E \) as the union of the set of free variables of \( E_{ij} \).

From our previous work, we know that the unique free variable of \( A_i X_{ij} B_j \) is the the variable placed in the position \((a_i, 1)\). So, if we consider two blocks \( X_{i1j}, X_{i2j} \) in the same column, then, their free variables are contained in the same column of \( X \), i.e. leading column of \( X_{i1j} \) and \( X_{i2j} \) are contained in the same column of \( X \). As a result, one can talk about the leading columns of \( X \). In fact, the same kind of work can be done for leading rows and pivotal rows as well. Finally, a matrix \( D \) is called a \textit{generic fixed point} of \((A, B)\), if \( D_{ij} \) is a generic fixed point of \((A_i, B_j)\).

5.3. **Centralizers of unipotent elements.** In this section, we start working with our original setting. Let \( U \) be a unipotent matrix in \( Sp_m(q) \) where \( \eta \) is the modified symplectic type \( U \) and \( 2m = ||\eta|| \). By Theorem 4.8, it follows that \( V_m = E_1 \perp \cdots \perp E_r \), where \( E_i \)'s are non-degenerate symplectic spaces that are invariant under \( U \). Moreover, Proposition 4.17 allows us, up to conjugation we may assume

\[
U = \text{diag}(U_1, \cdots, U_r)
\]

and \( U_{|E_i} = U_i \neq I \) and that \( U_1, \cdots, U_k \) are symplectic unipotent blocks and \( U_{k+1}, \cdots, U_r \) are orthogonal unipotent blocks. The ordered basis of \( E_i \) that is used to index the columns and rows of \( U_i \) is
$B_i = \{ e_{i1}, \ldots, e_{in_i}, f_{in_i}, \ldots, f_{i1} \}$. The set $\mathcal{B}_m = \cup_{i=1}^r B_i$ forms a hyperbolic basis for $V_m$. We also fix $X = (x_{uv})_{u,v \in \mathcal{B}_m}$, the $2m \times 2m$ matrix where $x_{uv}$ is an indeterminate over $\mathbb{F}_q$. As in the previous section, we consider $X$ as a block matrix $(X_{ij})_{i,j=1,\ldots, r}$, which is induced by the block form of $U$.

Note that the matrix $U^{-1}$ is an element of $\mathcal{U}_m$, and it is a again a block diagonal matrix with the same block diagonal structure. Clearly the splitting $V_m = E_1 \perp \cdots \perp E_r$ is preserved by $U^{-1}$. We will label the rows and columns of $U^{-1}$ again labeled with the elements of $\mathcal{B}_m$. A generic fixed point $D$ of $(U, U^{-1})$ will be called a \textbf{generic centralizer} of $U$. Finally, let $d$ be the dimension $\dim V^U = \dim U V$.

\textbf{Proposition 5.12.} Let $D$ be a generic centralizer of $U$ and let $D_{ij}$ be the blocks of $D$ induced by the block structure of $U$. Then:

1. If $U_i$ and $U_j$ are both orthogonal, then the block $D_{ij}$ of the generic solution is of the following form:

$$D_{ij} = \begin{bmatrix} e_{j1} & \cdots & \cdots & f_{j1} \\ a_{e_{i1}e_{j1}} & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ x_{f_{i1}f_{j1}} & * & \cdots & * \\ 1. \text{ cl.} & 1. \text{ cl.} & \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ f_{i1} \\ \vdots \end{bmatrix} \begin{bmatrix} \leftarrow \text{ leading row} \\ \leftarrow \text{ pivotal row} \end{bmatrix}$$

where $a_{e_{i1}e_{j1}}$ is the leading term of $D_{ij}$.

2. If $U_i = J_{2s}$, $U_j = J_{2r}$ are both symplectic, then the block $D_{ij}$ is of the following form:

$$D_{ij} = \begin{bmatrix} e_{j1} & \cdots & \cdots & e_{jn_j} & f_{jn_j} & \cdots & \cdots & f_{j1} \\ a_{e_{i1}e_{j1}} & 0 & \cdots & 0 & a_{e_{i1}f_{jn_j}} & 0 & \cdots & 0 \\ * & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * & \cdots & * \\ x_{e_{i1}f_{jn_j}} & * & \cdots & * & x_{e_{i1}f_{jn_j}} & * & \cdots & * \\ a_{f_{jn_j}f_{j1}} & 0 & \cdots & 0 & a_{f_{jn_j}f_{j1}} & 0 & \cdots & 0 \\ * & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * & \cdots & * \\ x_{f_{j1}f_{i1}} & * & \cdots & * & x_{f_{j1}f_{i1}} & * & \cdots & * \\ 1. \text{ cl.} & 1. \text{ cl.} & \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ f_{i1} \\ \vdots \end{bmatrix} \begin{bmatrix} \leftarrow \text{ leading row} \\ \leftarrow \text{ pivotal row} \end{bmatrix}$$

$$D_{ij} = \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ f_{i1} \\ \vdots \end{bmatrix} \begin{bmatrix} \leftarrow \text{ leading row} \\ \leftarrow \text{ pivotal row} \end{bmatrix}$$
As pointed out earlier, the homogeneous system of equations induced by the equality $UXU^{-1} - X = 0$ is equal to the disjoint union of the homogeneous system of equations induced by $U_i X_i U^{-1}_i - X_{ij} = 0$. So, one can consider blocks individually. All cases are similar. We will just prove the last two cases. Let $D = D_{ij}$.

Write the matrices $A_j$ and $B_j$ as block matrices as follows:

$$
D_{ij} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad D_{ji} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}
$$

where $A_i$'s are $s \times 2r$ matrices and $B_i$'s are $2s \times r$ matrices. Using the fact that $U_i$ is a block diagonal matrix, one can write equation (30) as follows:

$$
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} S_s & 0 \\ 0 & S_s^{-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} J_{2r,e}^{-1} = \begin{bmatrix} S_s A_1 J_{2r,e}^{-1} \\ S_s^{-1} A_2 J_{2r,e}^{-1} \end{bmatrix}
$$

and

$$
\begin{bmatrix} B_1 & B_2 \end{bmatrix} = J_{2r,e} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} S_s & 0 \\ 0 & S_s^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} J_{2r,e} B_1 S_s^{-1} & J_{2r,e} B_2 S_s \end{bmatrix}.
$$

Proof. As pointed out earlier, the homogeneous system of equations induced by the equality $UXU^{-1} - X = 0$ is equal to the disjoint union of the homogeneous system of equations induced by $U_i X_i U^{-1}_i - X_{ij} = 0$. So, one can consider blocks individually. All cases are similar. We will just prove the last two cases. Let $U_i = J_{2s} = diag(S_s, S_s^{-1})$ and $U_j = J_{2r,e}$. Recall that, for $s > 0$, the matrix $S_s$ is defined as follows.

$$
S_s := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

The blocks $D_{ij}$ and $D_{ji}$ are subject to the equations

$$
(30) \quad D_{ij} = U_i D_{ij} U_j^{-1}, \quad D_{ji} = U_j D_{ji} U_i^{-1}.
$$

Write the matrices $D_{ij}$ and $D_{ji}$ as block matrices as follows:

$$
D_{ij} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad D_{ji} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}
$$

where $A_i$'s are $s \times 2r$ matrices and $B_i$'s are $2s \times r$ matrices. Using the fact that $U_i$ is a block diagonal matrix, one can write equation (30) as follows:

$$
\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} S_s & 0 \\ 0 & S_s^{-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} J_{2r,e}^{-1} = \begin{bmatrix} S_s A_1 J_{2r,e}^{-1} \\ S_s^{-1} A_2 J_{2r,e}^{-1} \end{bmatrix}
$$

and

$$
\begin{bmatrix} B_1 & B_2 \end{bmatrix} = J_{2r,e} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} S_s & 0 \\ 0 & S_s^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} J_{2r,e} B_1 S_s^{-1} & J_{2r,e} B_2 S_s \end{bmatrix}.
$$
This means, $A_1, A_2, B_1, B_2$ are all fixed points of the unipotent action. As a result, the top rows of $A_1, A_2, B_1, B_2$ are zero except possibly for the first entries. The claim concerning the indices of the free variables follows from Lemma 5.9 and Lemma 5.11.

\[\Box\]

**Definition 5.13.** The set of basis elements that corresponds to a leading row (resp. pivotal row) is called a **leading basis** (resp. **pivotal basis**) element. The set of leading (resp. pivotal) basis elements is denoted with $B_{\text{lead}}$ (resp. $B_{\text{pivot}}$). In detail:

\[B_{\text{lead}} = \{e_{i1} : i = 1, \ldots, k, k + 1, \ldots r\} \cup \{f_{in_k} : i = 1, \ldots, k\} \subset B.\]

and

\[B_{\text{pivot}} = \{f_{i1} : i = 1, \ldots, k, k + 1, \ldots r\} \cup \{e_{in_k} : i = 1, \ldots, k\} \subset B.\]

Bearing in mind the block form of $U$ and using Remark 4.15 we see that the subset $B_{\text{lead}}$ is a basis of the fixed subspace $U V_m$, i.e. the fixed space of the map defined by multiplication by $U$ on the right. Likewise, the subset $B_{\text{pivot}}$ is a basis of the fixed subspace $V_m U$, i.e. the fixed space of the map defined by multiplication by $U$ on the left, equivalently, the fixed space of the map defined by multiplication by $U^t$ on the left. The subspace of $V_m$ generated by $B_{\text{lead}} \cup B_{\text{pivot}}$ is denoted by $F_U$.

**Lemma 5.14.** Keeping the notation $U = \text{diag}(U_1, \ldots, U_r)$, cf. Eq. 5.3, we have the following.

1. The subspaces $U V$ and $V U$ are generated by $B_{\text{lead}}$ and $B_{\text{pivot}}$.
2. The set hyperbolic conjugates of elements of $B_{\text{pivot}}$ is equal to $B_{\text{lead}}$ and the cardinality of both of these sets are equal to $d$, dimension of the fixed space of $U$.
3. The subspaces $U V$ and $V U$ are totally isotropic.
4. The subspace $F_U = V U \oplus U V$ is a non-degenerate symplectic space, and it splits in $V_m$:

\[V_m = F_U \perp (F_U)^\perp\]

We will write $F_U \perp$ in place of $(F_U)^\perp$. As a result, if $C \in V_m$ then $C = C^{F_U} + C^{F_U \perp}$, where $C^{F_U} \in F_U$, $C^{F_U \perp} = F_U \perp$ and $Q(C^{F_U}, C^{F_U \perp}) = 0$.

**Proof.**

1. The fact that the subspaces $U V$ and $V U$ are generated by $B_{\text{lead}}$ and $B_{\text{pivot}}$ is already discussed in the previous paragraph.
2. This follows from the explicit determination of the blocks of a generic element $D$ in the centralizer of $U$, as given in Proposition 5.12.
3.4 Follows from 2.

\[\Box\]

**Remark 5.15.** Notice that $|B_{\text{lead}}| = |B_{\text{pivot}}| = \dim V U = \dim U V$. We also observe that, the set of leading basis elements is equal to the set of basis elements that corresponds to the leading columns. From this we conclude that, an index $(u, v)$ is a free index if and only if $(u, v) \in B_{\text{pivot}} \times B_{\text{lead}}$.

**Definition 5.16.**

1. A $2m \times 2m$ matrix $D = (d_{uv})_{u,v \in B}$ will be called a **primitive matrix** if $d_{uv} = x_{uv}$ for $(u, v) \in B_{\text{pivot}} \times B_{\text{lead}}$, and $d_{uv} \in F_q$ for $(u, v) \notin B_{\text{pivot}} \times B_{\text{lead}}$. In particular, if $v \notin B_{\text{lead}}$ then the column $C_v(D)$ defines a unique element of $V_m$.
2. A square matrix whose entries are indexed by $B_{\text{pivot}} \times B_{\text{lead}}$ will be called a **free-index matrix**.
3. For a free-index matrix $A = (a_{uv})_{(u,v) \in B_{\text{pivot}} \times B_{\text{lead}}}$, substituting $a_{uv}$ for $x_{uv}$ defines an element $Mat_{2m \times 2m}(F_q)$ which is denoted by $D(A)$. The matrix $D(A)$ is called a realization of $D$.
4. The map given by the rule $M = (m_{uv})_{u,v \in B} \mapsto M_{\text{pivot}} := (m_{uv})_{(u,v) \in B_{\text{pivot}} \times B_{\text{lead}}}$ is denoted by $M \mapsto M_{\text{pivot}}$. The submatrix $M_{\text{pivot}}$ of $M$ will be referred as the **pivotal submatrix** of $M$.
5. The **leading submatrix** $M_{\text{lead}}$ of a matrix $M = (m_{uv})_{u,v \in B}$ (which can be a primitive matrix as well) is defined as the matrix $M_{\text{lead}} = (m_{uv})_{u,v \in B_{\text{lead}}}$. If $M$ is a realization of $D$ then $M_{\text{lead}} = D_{\text{lead}}$ and $D_{\text{pivot}} = (x_{uv})_{(u,v) \in B_{\text{pivot}} \times B_{\text{lead}}}$. Entries of $D_{\text{lead}}$ (or $M_{\text{lead}}$) will be referred as **leading entries** of $D$ (or $M$).
6. The column $C_v$ of $M$ or $D$ will be called a **leading column** for $v \in B_1$.
7. If $A = (a_{uv})_{u \in B_1, v \in B_1}$ is a free-indexed $d \times d$ matrix, then $\overline{A} = (\overline{a_{uv}})_{u,v \in B}$ where $\overline{a_{uv}} = a_{uv}$ if $(u,v) \in B_{\text{pivot}} \times B_{\text{lead}}$ and $\overline{a_{uv}} = 0$ if $(u,v) \notin B_{\text{pivot}} \times B_{\text{lead}}$. 

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(8) Let $u, v$ be two basis elements and $D$ be a primitive centralizer of $U$. We introduce the notation

$$\phi_{uv} = (Q(C_u(D), C_v(D)))_{U}$$

and

$$\omega_{uv} = (Q(C_u(D), C_v(D)))_{U^\perp}$$

where $(Q(C_u(D), C_v(D)))_{U}$ is an element of the symplectic space $F_U = (B_{pivot}) \oplus (B_{lead})$ and

$(Q(C_u(D), C_v(D)))_{U^\perp}$ is an element of the orthogonal complement $(F_U)^\perp$ of $F_U$.

**Remark 5.17.** Let $A$ be a free-index-matrix and consider $\overline{A}$. Then by definition of free indices and Lemma 5.14 it follows that the columns of $\overline{A}$ are eigenvectors of $U$ and rows of $\overline{A}$ are eigenvectors of $U^t$.

**Lemma 5.18.** Let $D$ be a primitive matrix with respect to $U$. If $A$ is a free-index-matrix such that $D(A)$ is in the centralizer of $U$ then $D(B)$ is in the centralizer of $U$ for all free-index-matrix $B$.

**Proof.** This follows directly from the definition of a free index. That is, the entries $m_{uv}$ of $M = D(A)$ do not occur in the equations $UMU^{-1} - M = 0$ for $(u, v) \in B_{pivot} \times B_{lead}$.

A primitive matrix $D$ is called a **primitive centralizer** of $U$ if a realization $D(A)$ (hence all realizations) of $D$ commutes with $U$.

**Lemma 5.19.** Let $D$ be a primitive centralizer of $U$, $u \in B_{lead}$ be a leading basis element and $R_u$ be the row of $D$ corresponding to $u$. Then all the entries of $R_u$ is zero except the leading entries $d_{uv}$, i.e. $d_{uv} = 0$ for $v \notin B_{lead}$. In short, if $u \in B_{lead}$ and $v \notin B_{lead}$ then $d_{uv} = 0$.

**Proof.** This is a reformulation of Lemma 5.11.

**Example 5.20.** Consider the block diagonal matrix $U$ whose diagonal entries are $J_6$ and $J_{4,\epsilon}$ with $\epsilon \neq 0$ and let $D$ be a primitive centralizer of $U$. Write $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ where $D_{11}$ is a $6 \times 6$ matrix. Then $UXU^{-1} = X$ implies

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} J_6 & 0 \\ 0 & J_{4,\epsilon} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} J_6^{-1} & 0 \\ 0 & J_{4,\epsilon}^{-1} \end{bmatrix} = \begin{bmatrix} J_6D_{11}J_6^{-1} & J_{4,\epsilon}^{-1}D_{21}J_{4,\epsilon}^{-1} \\ J_{4,\epsilon}^{-1}D_{22}J_{4,\epsilon}^{-1} & J_6D_{12}J_6^{-1} \end{bmatrix}$$

By the Proposition 5.12 it follows that $D$ is of the following type:

$$D = \begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} & f_{12} & f_{11} & e_{21} & e_{22} & f_{22} & f_{21} \\
\epsilon_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{40} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{60} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

where, for each choice of $x_{ij}$, the resulting matrix commutes with $U$. Clearly, the set of pivotal basis elements is $B_{pivot} = \{e_{13}, f_{11}, f_{21}\}$, and the set of leading basis elements is $B_{lead} = \{\epsilon_{11}, f_{13}, e_{21}\}$. Consider the vectors $C_{f13}(D)$ and $C_{e22}(D)$. Then we have the following equalities:

$$C_{f13} = a_{12}\epsilon_{11} + x_{12}e_{13} + a_{22}f_{13} + x_{22}f_{11} + a_{32}e_{21} + x_{32}f_{21}$$

$$C_{f13}^{-1} = d_{24}e_{12} + d_{54}f_{12} + d_{84}e_{22} + d_{94}f_{22}$$

Likewise we have the following equalities:

$$C_{e22} = 0e_{11} + d_{38}e_{13} + 0f_{13} + d_{68}f_{11} + a_{33}e_{21} + d_{10}f_{21}$$

$$C_{e22}^{-1} = d_{28}e_{12} + d_{58}f_{12} + d_{88}e_{22} + d_{98}f_{22}$$

This means

$$Q(C_{f13}, C_{e22}) = Q(C_{f13}^{-1}, C_{e22}^{-1}) + Q(C_{f13}^{-1}, C_{e22}^{-1}) \in \mathbb{F}_q$$
Now we will investigate several cases of inner-products. In order to simplify the notation, we will use \( F \) or \( u \) as a result, we introduce the concept of the primitive symplectic centralizer. Not guaranteed as the conditions for being an isometry involves equations with the indeterminates \( \delta_{ij} \).

Remark 5.21. Let \( M \) be a primitive matrix and for \( m \in B \), denote the hyperbolic conjugate of \( w \) with \( w' \). For \( u \in B \), using Lemma 5.14, we write \( C_u(D) = C_u^F(D) + C_u^{F\perp}(D) \), where the summands are orthogonal to each other. If \( u \in B_I \) then

\[
C_u^F = \sum_{w \in B_I} d_{wu} \cdot w + \sum_{w \in B_p} x_{wu} \cdot w
\]

and

\[
C_u^{F\perp} = \sum_{w \in B - (B_I \cup B_p)} d_{wu} \cdot w \in \mathbb{F}_q.
\]

If \( u \in B - B_I \) then

\[
C_u^F = \sum_{w \in B_I} d_{wu} \cdot w + \sum_{w \in B_p} d_{wu} \cdot w
\]

and

\[
C_u^{F\perp} = \sum_{w \in B - (B_I \cup B_p)} d_{wu} \cdot w \in \mathbb{F}_q
\]

Now we will investigate some cases of inner-products.

**Case 1:** \( u, v \in B_I \). In this case, the inner product \( Q(C_u^F, C_v^F) \) can be written as:

\[
Q(C_u^F, C_v^F) = Q(\sum_{w \in B_I} d_{wu} \cdot w + \sum_{w \in B_I} x_{wu} \cdot w', \sum_{w \in B_I} d_{wu} \cdot w + \sum_{w \in B_I} x_{wu} \cdot w')
\]

and as \( B_p \) consists of hyperbolic conjugates of the elements of \( B_I \), using the last equation we get

\[
Q(C_u(D), C_v(D)) = \sum_{w \in B_I} \delta_w d_{wu} \cdot x_{wu} + \sum_{w \in B_I} \delta_w x_{wu} \cdot d_{wu} + Q(C_u^{F\perp}(D), C_v^{F\perp}(D))
\]

as

\[
Q(C_{f_{13}}^F, C_{e_{22}}^F) = a_{12}d_{68} + x_{12}0 - a_{22}d_{38} - x_{22}0 + a_{32}d_{10,8} - x_{32}0 = a_{12}d_{68} - a_{22}d_{38} + a_{32}d_{10,8} \in \mathbb{F}_q
\]

and

\[
Q(C_{f_{13}}^{F\perp}, C_{e_{22}}^{F\perp}) = (d_{24}d_{58} - d_{54}d_{28} + d_{84}d_{58} - d_{94}d_{88}) \in \mathbb{F}_q.
\]

Consider the matrices \( D_{\text{pivot}} \) and \( D_{\text{lead}} \) along with the matrix \( \sigma \) which is introduced as:

\[
D_{\text{lead}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{and} \quad D_{\text{pivot}} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix},
\]

where instead of labeling elements w.r.t the corresponding pivotal basis elements \( e_{11}, f_{13}, e_{21} \); the usual labeling of entries are used. We observe that

\[
Q(C_{f_{13}}^F, C_{e_{22}}^F) = (D_{\text{lead}}^\sigma D_{\text{pivot})_{23}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} (D_{\text{lead}}^\sigma D_{\text{pivot})_{32}}
\]

Each realization \( M \) of a primitive centralizer \( D \) of \( U \) is a true centralizer of \( U \). However, it is not always the case that \( M \in Sp_m(q) \). Even existence of a realization \( M \) of \( D \) which is an element of \( Sp_m(q) \) is not guaranteed as the conditions for being an isometry involves equations with the indeterminates \( x_{uv} \). As a result, we introduce the concept of **primitive symplectic centralizer** of \( U \). First we make some observations. In order to simplify the notation, we will use \( B_I \) and \( B_p \) instead of \( B_{\text{lead}} \) and \( B_{\text{pivot}} \) respectively.
where the $\delta_{w} = Q(w, w') = \pm 1$. Clearly, $Q(C_u^{F^+}, C_v^{F^+}) \in \mathbb{F}_q$.

**Case 2:** $u, v \in B - B_1$. In this case, $C_u(D)$ and $C_v(D)$ defines an element of $V_m$ and hence $Q(C_u(D), C_v(D)) \in \mathbb{F}_q$.

**Case 3:** $u \in B_1, v \notin B_1$. In this case we have

\begin{equation}
Q(C_u(D), C_v(D)) = \sum_{w \in B_1} \delta_{w} d_{wu} d_{wv} + \sum_{w \in B_1} \delta_{w} x_{w} u d_{wv} + Q(C_u^{F^+}(D), C_v^{F^+}(D)) \in \mathbb{F}_q.
\end{equation}

Notice that only the second summand contains indeterminates. However, since $w \in B_1$ and $v \notin B_1$ by Lemma 5.19 we get $d_{wv} = 0$, hence the summand involving the indeterminates vanishes and thus, in this last case, the inner product is a scalar. Recall that we write $\phi_{uv}$ to indicate the inner product $Q(C_u^F, C_v^F)$ and $\omega_{uv}$ to indicate the inner product $Q(C_u^{F^+}, C_v^{F^+})$. We also introduce the matrices

\begin{align*}
\Phi_D &= (\phi_{uv})_{u,v \in B_1}, \\
\Omega_D &= (\omega_{uv})_{u,v \in B_1}.
\end{align*}

Let $u, v \in B$ and $C_u(D), C_v(D)$ be two columns of a primitive centralizer $D$ of $U$. We want to consider the equality

\[ Q(C_u(D), C_v(D)) = Q(u, v). \]

**Case 1:** $u, v \in B - B_1$. By Remark 5.21 it follows that $Q(C_u(D), C_v(D)) \in \mathbb{F}_q$. As a result, the above equality can be checked directly.

**Case 2:** $u \in B_1, v \in B - B_1$. Then the inner product $Q(C_u(D), C_v(D))$ is given by Eq.(32) above. This means, the inner product $Q(C_u(D), C_v(D))$ does not involve indeterminates and the above equation can be checked directly.

Observe that, these equalities hold for $D$ if and only if they hold for one (hence for any) realizations of $D$. As a result we obtain the following:

**Lemma 5.22.** Let $D$ be a primitive centralizer of $U$ and $M$ be a realization of $D$. If $M \in Sp_m(q)$ then the following hold:

1. $Q(C_u(D), C_v(D)) = Q(u, v)$ for all $(u, v) \in B \times B - B_1 \times B_1$.
2. $D_{\text{lead}} = M_{\text{lead}}$ is invertible.

**Proof.** The first assertion is already dealt prior to the lemma. By the Lemma 5.19, the leading rows of $M$ and $M_{\text{lead}}$, when considered as vectors, define the same elements in $V_m$. Hence, a non-trivial linear relation between the rows of $M_{\text{lead}}$ yields a non-trivial linear relation between the rows of $M$. As $M$ is invertible, this can not be the case. \qed

In the light of the lemma, we say that a primitive centralizer $D$ of $U$ is a **primitive symplectic centralizer** of $U$ if $D$ satisfies the conditions 1. and 2. of Lemma 5.22. By definition, for a fixed primitive symplectic centralizer $D$ of $U$ and its realization $M$ of $D$, it follows that $M$ is an element of $Sp_m(q)$ if and only $Q(C_u(M), C_v(M)) = Q(u, v) = 0$ for $u, v \in B_1$ as elements of $B_1$ are orthogonal to each other by Lemma 5.14. Using the matrices $\Phi$ and $\Omega$ introduced in (33), this observation can be rephrased as follows:

**Lemma 5.23.** Let $M$ be a realization of a primitive symplectic centralizer $D$ of $U$. Then $M \in Sp_m(q)$ if and only if

\[ \Phi_M = -\Omega_M. \]

**Proof.** Follows from the fact that $Q(C_u(M), C_v(M)) = \phi_{uv}(M) + \omega_{uv}(M)$ for $u, v \in B_1$. \qed

**Proposition 5.24.** There exists an invertible matrix $\sigma$ such that

\[ Q(C_u^F, C_v^F) = (D^\dagger_{\text{lead}} \cdot \sigma \cdot M_{\text{pivot}})_{uu} - (D^\dagger_{\text{lead}} \cdot \sigma \cdot M_{\text{pivot}})_{vu} \]

for all $u, v \in B_1$. In particular, $\Phi_M = (D^\dagger_{\text{lead}} \cdot \sigma \cdot M_{\text{pivot}}) - (D^\dagger_{\text{lead}} \cdot \sigma \cdot M_{\text{pivot}})^t$.

We need two lemmas:
Lemma 5.25. Let \((V, Q)\) be a symplectic space with a hyperbolic basis \(B = \{e_1, f_1, \ldots, e_k, f_k\}\) and let \(v_1, \ldots, v_k\) be arbitrary elements of \(V\), written as column vectors:

\[
v_1 = \begin{bmatrix} v_{1e_1} \\ v_{1f_1} \\ \vdots \\ v_{1e_k} \\ v_{1f_k} \end{bmatrix}, \quad v_2 = \begin{bmatrix} v_{2e_1} \\ v_{2f_1} \\ \vdots \\ v_{2e_k} \\ v_{2f_k} \end{bmatrix}, \quad \ldots, \quad v_k = \begin{bmatrix} v_{ke_1} \\ v_{kf_1} \\ \vdots \\ v_{ke_k} \\ v_{kf_k} \end{bmatrix}.
\]

Let \(v_i^\ast\) (resp. \(v_i^\dagger\)) be the \(k\)-tuple vector obtained from \(v_i\) by keeping tuples indexed by the basis vectors \(P_i = \{e_1, \ldots, e_k\}\) (resp. \(P_i = \{f_1, \ldots, f_k\}\)) for \(i = 1, \ldots, k\) and removing the other entries. Let \(T_1\) and \(T_2\) be the set of \(k \times k\) matrices whose \(i\)-th column is \(v_i^\ast\) and \(v_i^\dagger\) respectively. Then

\[
Q(v_i, v_j) = (T_1^t T_2)_{ij} - (T_1^t T_2)_{ji}
\]

**Proof.** This follows from direct calculation. The \(i\)-th row of \(T_1^t\) is \((v_{ie_1}, \ldots, v_{ie_k})^T\) and the \(j\)-th column of \(T_2\) is \((v_{jf_1}, \ldots, v_{jf_k})\) and hence the right hand side of the above equation is

\[
(v_{ie_1}, \ldots, v_{ie_k}) \cdot (v_{j_1f_1} \ldots v_{j_kf_k}) = (v_{ie_1}, \ldots, v_{ie_k}) \cdot (v_{j_1f_1} \ldots v_{j_kf_k})
\]

which is clearly equal to the inner product

\[
Q(v_i, v_j) = Q(v_{ie_1}, v_{if_1} + v_{ie_2} e_k + v_{if_k} f_k, v_{je_1}, v_{jf_1} + v_{je_2} e_k + v_{jf_k} f_k).
\]

Next we assume that \(P_1, P_2\) is an arbitrary partition of \(B\) so that none of the hyperbolic pairs \(e_j, f_j\) fall into the same \(P_i\). Observe that the partition above satisfies this property. We call such a partition isotropic. Finally, a square matrix \(\sigma\) is called a signed permutation matrix if each row and each column has only one non-zero entry which is either \(1\) or \(-1\).

**Corollary 5.26.** Let \((V, Q)\), \(B\) be an arbitrary hyperbolic basis in an arbitrary order, and \(v_1, \ldots, v_k\) be as above. Let \(P_1, P_2\) be an isotropic partition of \(B\) and \(T_1, T_2\) be defined in the manner described in the previous lemma. Then, there is a \(k \times k\) signed permutation matrix \(\sigma\) such that

\[
Q(v_i, v_j) = Q((T_1^t \sigma T_2)_{ij} - (T_1^t \sigma T_2)_{ji})
\]

**Proof.** Multiplication with a permutation matrix on the left acts on the rows the of matrix. Let \(g\) be the permutation of \(P_2\) so that the \(i\)-th element of \(P_1\) and \(g \cdot P_2\) form hyperbolic pairs and let \(\sigma_1\) be the corresponding permutation matrix. Let \(\sigma_2\) be the diagonal matrix with entries \(\pm 1\) where \((ij) - \theta \text{th entry is 

Next we assume that \(P_1, P_2\) is an arbitrary partition of \(B\) so that none of the hyperbolic pairs \(e_j, f_j\) fall into the same \(P_i\). Observe that the partition above satisfies this property. We call such a partition isotropic. Finally, a square matrix \(\sigma\) is called a signed permutation matrix if each row and each column has only one non-zero entry which is either \(1\) or \(-1\).

**Corollary 5.26.** Let \((V, Q)\), \(B\) be an arbitrary hyperbolic basis in an arbitrary order, and \(v_1, \ldots, v_k\) be as above. Let \(P_1, P_2\) be an isotropic partition of \(B\) and \(T_1, T_2\) be defined in the manner described in the previous lemma. Then, there is a \(k \times k\) signed permutation matrix \(\sigma\) such that

\[
Q(v_i, v_j) = Q((T_1^t \sigma T_2)_{ij} - (T_1^t \sigma T_2)_{ji})
\]

**Proof.** Multiplication with a permutation matrix on the left acts on the rows the of matrix. Let \(g\) be the permutation of \(P_2\) so that the \(i\)-th element of \(P_1\) and \(g \cdot P_2\) form hyperbolic pairs and let \(\sigma_1\) be the corresponding permutation matrix. Let \(\sigma_2\) be the diagonal matrix with entries \(\pm 1\) where \((ij) - \theta \text{th entry is 

Recall that if \(A = (a_{uv})_{u,v \in B_i}\) is a free-indexed \(d \times d\) matrix, then \(A = (\pi_{uv})_{u,v \in B_i}\) was defined as by the rule \(\pi_{uv} = a_{uv}\) if \((u, v) \in B_p \times B_i\) and \(\pi_{uv} = 0\) otherwise.

**Proposition 5.27.** Let \(D\) be a primitive symplectic centralizer and \(M\) be a realization of \(D\). Then the following are equivalent:

1. \(M\) is a symplectic matrix.
2. The free-indexed matrix \(D_{\text{lead}}^t \cdot \sigma \cdot M_{\text{pivot}}\) satisfies the equation
   \[
   D_{\text{lead}}^t \cdot \sigma \cdot M_{\text{pivot}} - (D_{\text{lead}}^t \cdot \sigma \cdot M_{\text{pivot}})^t = -\Omega.
   \]
3. \(D_{\text{lead}}^t \cdot \sigma \cdot M_{\text{pivot}} = S - \Omega/2\) where \(S\) is a free-indexed symmetric matrix and \(\Omega/2 = (\omega_{uv}/2)_{u,v \in B_i}\).
(4) There exists a symmetric matrix $S$ such that
\[ M_{\text{pivot}} = (D_{\text{lead}}^r \cdot \sigma)^{-1} \cdot S - (D_{\text{lead}}^l \cdot \sigma)^{-1} \cdot \Omega/2. \]

As a result, for each primitive symplectic centralizer $D$, there exists a realization $M$ of $D$ which is an isometry. In fact, there exists $q^{\frac{d^2+1}{2}}$ many symplectic realizations of $D$ and they are of the form
\[ M + (D_{\text{lead}}^r \cdot \sigma)^{-1} \cdot S \]

where $S$ is an $d \times d$ symmetric matrix.

Proof. Write $T$ in place of $D_{\text{lead}}^l \cdot \sigma \cdot M_{\text{pivot}}$. From Lemma 5.23 it follows that $M$ is an isometry if and only if $\Phi = -\Omega$. Hence the equivalence of (1) and (2) follows from Proposition 5.24 which states that $\Phi = T - T^t$. Assuming (2) and taking $S = (T + T^t)/2$ yields (3). Conversely, assume that $T = S - \Omega/2$ with symmetric $S$. This implies $T^t = S + \Omega/2$ as $\Omega$ is an anti-symmetric matrix. As a result, $T - T^t = -\Omega$, which is the statement of (2). The equivalence of (3) and (4) follows from the fact that $D_{\text{lead}}$ and $\sigma$ are invertible matrices.

\[ \square \]

5.4. Growth of centralizers. We keep our assumptions on $U$, $\eta$ and $V_m$ and consider $V_m \subset V_n$. The hyperbolic basis for $V_{m,n}$ is denoted by $B_{m,n} = \{e_1, f_1, \ldots, e_{n-m}, f_{n-m}\}$. Thus, the union of the hyperbolic bases $B_r$, $i = 1, \ldots, r$ is equal to $B_m$ and $B_{m,n} \cup B_n = B_0$ is a hyperbolic basis of $V_n$. As before, rows and columns of the matrices in $GL_{2n}(q)$ are indexed by the basis $B$. If $u \in B$ and $M \in GL_{2n}(q)$ then $C_u(M)$ denotes column of $M$ which corresponds to basis element $u$. Finally, recall that $B_l$ generates $UV$ and $B_p$ generates $V^U$ and these bases form hyperbolic conjugates of each other. Next consider $U^{\uparrow \uparrow} = U \perp I_{2(n-m)} \in S_{p_{n}}(q)$. An element $M \in GL_{2n}(q)$ will be considered as a block matrix of the form $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, where $M_{11}$ is an $2m \times 2m$ matrix.

We recall Theorem 3.17 in this context.

Proposition 5.28. [20, Proposition 2.5] The centralizer $C_{GL_{2n}(q)}(U^{\uparrow \uparrow})$ of $U^{\uparrow \uparrow} \in GL_{2n}(F_q)$ is given by
\[ C_{GL_{2n}(q)}(U^{\uparrow \uparrow}) = \{ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} | M_{11} \in C_{GL_{2m}(q)}(U), M_{22} \in GL_{2(n-m)}(q), UM_{12} = M_{12}, M_{21}U = M_{21} \} \]

The columns of $M_{12}$ and rows of $M_{21}$ are indexed by the elements of $B_{m,n}$. Moreover, the columns of $M_{12}$ (resp. rows of $M_{21}$) are elements of $V^U$ (resp. $UV$). By Lemma 5.14, it follows that, for $v \in B_{m,n}$, the $v$-th column $C_v(M_{12})$ (resp. row $R_v(M_{21})$) of $M_{12}$ (resp. $M_{21}$) are of the form
\[ C_v(M_{12}) = \sum_{w \in B_m} m_{wv} \cdot w = \sum_{w \in B_p} m_{wv} \cdot w \]

and
\[ R_v(M_{21}) = \sum_{w \in B_m} m_{vw} \cdot w = \sum_{w \in B_l} m_{vw} \cdot w \]

respectively, as $V^U$ is generated by $B_p$ and $UV$ is generated by $B_l$. From these equations we get the following:

Lemma 5.29. Columns of $M_{12}$ are orthogonal to each other. Moreover, $R_u(M_{12}) = 0$ if $u$ is not a pivotal basis element and $C_v(M_{21}) = 0$ if $v$ is not a leading basis element.

Proof. Let $v_1, v_2 \in B_{m,n}$. The inner product of $C_{v_1}(M_{12})$ and $C_{v_2}(M_{12})$ is the sum of products of the form $\delta_w m_{wv_1} m_{wv_2}$ where $w \in B_m$ and $w'$ is the hyperbolic conjugate of $w$. So, one of the factor must be zero, as $w \in B_p$ implies $w' \notin B_p$, and thus $m_{wv_2} = 0$.

We will call an $n \times n$ matrix $D = \begin{bmatrix} D_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a primitive centralizer of $U^{\uparrow \uparrow}$ if $D_{11}$ is a primitive centralizer of $U$, entries of $M_{12}, M_{21}$ and $M_{22}$ are in $F_q$; and
\[ M_{22} \in GL_{2(n-m)}(F_q), \quad UM_{12} = M_{12}, \quad M_{21}U = M_{21}. \]
Example 5.30. Let us revisit the block diagonal matrix $U$ whose diagonal entries are $J_6$ and $J_{4,\epsilon}$ with $\epsilon \neq 0$ of Example 5.12. We consider generic fixed points of $U^{\uparrow n}$. By Lemma 5.29, $M_{21}$ and $M_{12}$ are of the form

$$M_{21} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & f_{13} & f_{12} & f_{11} & e_{21} & e_{22} & f_{22} & f_{21} \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 \end{bmatrix} \quad \leftarrow e_1$$

$$M_{12} = \begin{bmatrix} e_{11} & f_{11} & e_{22} & f_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix} \quad \leftarrow e_1$$

and $D_{11}$ is a primitive centralizer of $U$, i.e. $D_{11}$ is of the following form

$$D_{11} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & f_{13} & f_{12} & f_{11} & e_{21} & e_{22} & f_{22} & f_{21} \\ a_{11} & d_{12} & a_{13} & d_{13} & d_{12} & d_{13} & a_{12} & a_{13} & d_{12} & d_{13} \\ d_{32} & d_{33} & d_{22} & d_{23} & d_{22} & d_{23} & d_{32} & d_{33} & d_{22} & d_{23} \\ a_{21} & d_{22} & a_{23} & d_{23} & a_{22} & a_{23} & d_{22} & d_{23} & a_{22} & a_{23} \\ d_{31} & d_{32} & d_{33} & d_{32} & d_{33} & d_{32} & d_{33} & d_{32} & d_{33} & d_{32} \\ d_{51} & d_{52} & d_{53} & d_{52} & d_{53} & d_{52} & d_{53} & d_{52} & d_{53} & d_{52} \\ a_{51} & d_{52} & a_{53} & d_{53} & a_{52} & a_{53} & d_{52} & d_{53} & a_{52} & a_{53} \\ d_{81} & d_{82} & d_{83} & d_{82} & d_{83} & d_{82} & d_{83} & d_{82} & d_{83} & d_{82} \\ d_{91} & d_{92} & d_{93} & d_{92} & d_{93} & d_{92} & d_{93} & d_{92} & d_{93} & d_{92} \\ \end{bmatrix} \quad \begin{array}{c} e_{11} \leftarrow \text{leading b.} \\ e_{12} \leftarrow \text{pivotal b.} \\ f_{11} \leftarrow \text{leading b.} \\ f_{12} \leftarrow \text{leading b.} \\ f_{13} \leftarrow \text{leading b.} \\ f_{14} \leftarrow \text{leading b.} \\ f_{15} \leftarrow \text{leading b.} \\ f_{16} \leftarrow \text{leading b.} \\ f_{17} \leftarrow \text{leading b.} \\ f_{18} \leftarrow \text{leading b.} \\ f_{19} \leftarrow \text{leading b.} \\ f_{20} \leftarrow \text{pivotal b.} \end{array}$$

Finally, $M_{22}$ is an arbitrary invertible $4 \times 4$ matrix.

In order to determine the true definition of primitive symplectic centralizer of $U^{\uparrow n}$ we will investigate the equation $Q(C_u(D), C_v(D)) = Q(u, v)$ with $u, v \in B$ for a fixed primitive centralizer $D$ of $U^{\uparrow n}$ and a realization $M$ of $D$. Since $V_n = V_m \perp V_{m,n}$, each column vector $C_u$ of $D$ (or $M$) admits a sum $C_u(M_{12}) + C_u(M_{22})$ where $C_u(M_{12}) \in V_m$ and $C_u(M_{22}) \in V_{m,n}$. As a consequence $Q(C_u, C_v) = Q(C_u(M_{12}), C_v(M_{12})) + Q(C_u(M_{22}), C_v(M_{22}))$. Recall that $V_m$ also admits the orthogonal decomposition $F_U \perp F_U^\perp$, c.f. Lemma 5.14.

Case 1: $u, v \in B_{m,n}$.

**Lemma 5.31.** $Q(C_u(D), C_v(D)) = Q(u, v)$ for all $u, v \in B_{m,n}$ if and only if $M_{22} \in Sp_{n-m}(q)$. In particular, if $M \in Sp_n(q)$ then $M_{22} \in Sp_{n-m}(q)$.

**Proof.** As discussed above, a column $C_u(D)$ for $u \in B_{m,n}$ is equal to $C_u(M_{12}) + C_u(M_{22})$ and the summands are orthogonal to each other. So by Lemma 5.29 it follows that $Q(C_u(D), C_v(D)) = Q(C_u(M_{22}), C_v(M_{22}))$. This proves the assertion. \hfill \square

Case 2: $u \in B_m \setminus B_1$, $v \in B_{m,n}$. In this case, as $u \perp v$, the equation under discussion becomes

$$Q(C_u(M_{11}), C_v(M_{12})) + Q(C_u(M_{21}), C_v(M_{22})) = 0.$$
Since \( u \) is not leading, by Lemma 5.29, the column \( C_u(M_{21}) \) is the zero vector. As a consequence, the second inner-product vanishes automatically. So, consider \( C_u(M_{11}) = \sum_{w \in B_m} m_{wu} \cdot w \) and \( C_v(M_{12}) = \sum_{w \in B_m} m_{wv} \cdot w \). The inner product of these elements is given by

\[
Q(C_u(M_{11}), C_v(M_{12})) = \sum_{w \in B_m} \delta_w m_{wu} m_{wv}
\]

where \( w' \) is the hyperbolic conjugate of \( w \) and \( \delta_w \) is equal to \( Q(w, w') \). But by Lemma 5.29, \( m_{wv} = 0 \) if \( w' \notin B_p \), hence the above sum becomes

\[
\sum_{w \in B_l} \delta_w m_{wu} m_{wv}.
\]

as the factor \( m_{wu} = 0 \) if \( w \in B_l \) and \( u \notin B_l \). Hence, the above summation vanishes. This proves the following:

**Lemma 5.32.** For \( u \in B_m - B_l \) and \( v \in B_{m,n} \) the equality below holds.

\[
Q(C_u(M), C_v(M)) = 0.
\]

**Case 3:** \( u \in B_l \), \( v \in B_{m,n} \). The equation under discussion is again

\[
Q(C_u(M_{11}), C_v(M_{12})) + Q(C_u(M_{21}), C_v(M_{22})) = 0.
\]

Let \( M_{12,pivot} \) be the \( d \times 2(n - m) \) matrix obtained by the rows of \( M_{12} \) that correspond to the pivotal basis elements in \( B_m \), i.e. keeping the possible non-zero entries. So, the rows of \( M_{12,pivot} \) are indexed by \( B_p \) and columns are indexed by \( B_{m,n} \). Observe that the vectors induced by the columns of \( M_{12} \) and \( M_{12,pivot} \) are the same, as the removed entries are all zero. As discussed in the proof of Proposition 5.24, the first inner product \( Q(C_u(M_{11}), C_v(M_{12})) \) is equal to the product of the \( u \)-th row of \( (M_{11})_{lead} \cdot \sigma \) with \( C_v(M_{12}) = C_v(M_{12,pivot}) = \sum_{u \in B_p} m_{wu} \cdot u \). Thus, fixing \( u \) and letting \( u \) ranges over \( B_l \) and writing \( C_v(M_{12}) \) as a \( d \times 1 \) column vector, the above equation can be written as a matrix product:

\[
(M_{11})_{lead}^t \cdot \sigma \cdot C_v(M_{12,pivot}) = \begin{bmatrix}
Q(C_{u_1}(M_{21}), C_v(M_{22})) \\
Q(C_{u_2}(M_{21}), C_v(M_{22})) \\
\vdots \\
Q(C_{u_n}(M_{21}), C_v(M_{22}))
\end{bmatrix}
\]

where \( u_1, \ldots, u_d \in B_l \). Since \( (M_{11})_{lead} \) and \( \sigma \) are invertible matrices, it follows that \( C_v(M_{12,pivot}) \), and hence \( C_v(M_{12}) \), is uniquely determined by \( M_{lead} \), \( M_{21} \) and \( M_{22} \). The only non-zero entries of \( M_{12} \) correspond to pivotal basis elements and thus we denote the matrix obtained by the entries of \( M_{12} \) that are not contained in a leading row by \( (M_{12})_{lead} \), which is an \( h \times k \) matrix. Likewise, we denote the matrix obtained by removing the columns of \( M_{21} \) that do not correspond to a pivotal row is denoted by \( (M_{21})_{pivot} \). With these notations we get the following.

**Lemma 5.33.** \( C_u(M) \perp C_v(M) \) for all \( u \in B_l \) and for all \( v \in B_{m,m} \) if and only if

\[
(M_{11}^t)_{lead} \cdot \sigma \cdot (M_{12,pivot}) = Q(C_u(M_{21}), C_v(M_{22})) \in B_{p,v} \in B_{m,m}.
\]

**Proof.** Notice also that the right hand side of (34) is uniquely determined by \( C_v \), as \( M_{11}^t \cdot \sigma \) is invertible. □

**Case 4:** \( u, v \in B_l \). As before, the equations under discussion becomes

\[
Q(C_u(M_{11}), C_v(M_{12})) + Q(C_u(M_{21}), C_v(M_{22})) = 0, \quad u, v \in B_l
\]
since the leading basis elements are orthogonal to each other.

**Lemma 5.34.** If \( M \in Sp_n(q) \) then \( D_{11} \) is a primitive symplectic centralizer of \( U \).

**Proof.** Let \( u, v \in B_m \) and assume that \( u \) is not leading. Writing \( C_u(D) = C_u(D_{11}) + C_u(M_{21}) \), \( C_v(D) = C_v(D_{11}) + C_v(M_{21}) \) and using the fact that the summands are orthogonal to each other along with the fact that \( C_u(M_{21}) = 0 \), it follows that

\[
Q(u, v) = Q(C_u(M), C_v(M)) = Q(C_u(D), C_v(D)) = Q(C_u(D_{11}), C_v(D_{11}))
\]
\( \forall u \in B_1, v \in B_m, \) as \( M \) is in an isometry. As a result, \( D_{11} \) is a primitive symplectic centralizer.

With these observations, the following definition makes sense.

**Definition 5.35.** A primitive centralizer \( D = \begin{bmatrix} D_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \) of \( U^{\uparrow n} \) is called a primitive symplectic centralizer of \( U^{\uparrow n} \) if \( D_{11} \) is a primitive symplectic centralizer of \( U \), \( M_{22} \in Sp_{n-m}(q) \) and \( M_{12} \) satisfy the equation in Lemma 5.33.

Let \( D \) be a primitive symplectic centralizer of \( U^{\uparrow n} \) and \( M \) be a realization of \( D \). Notice that \( M \) is automatically contained in the centralizer of \( U \).

**Lemma 5.36.** \( M \in Sp_n(q) \) if and only if 
\[
Q(C_u(M), C_v(M)) = 0
\]
for all \( u, v \in B_1 \).

**Proof.** According to the discussion prior to the definition of primitive symplectic centralizer of \( U^{\uparrow n} \), we have 
\[
Q(C_u(M), C_v(M)) = Q(u, v) \text{ for all } u, v \in B \times B - B_1 \times B_1.
\]

As we have done in the previous section, we will write \( C_u(D) \) as a sum of orthogonal vectors. \( V_n \) is equal to the orthogonal sum \( V_m \oplus V_{m,n} \) and \( V_m \) is equal to the orthogonal sum of \( F_U \) and \( (F_U)^- \). So, each leading column vector \( C_u(D) \) can be written as an orthogonal sum
\[
C_u(D) = C_u(D_{11})^F + C_u(D_{11})^{F^+} + C_u(M_{21})
\]
where \( C_u(\cdot)^F \) and \( C_u(\cdot)^{F^+} \) were defined in Lemma 5.14. By the last lemma, \( M \in Sp_n(q) \) if and only if 
\[
0 = Q(C_u(D_{11})^F, C_u(D_{11})^F) + Q(C_u(D_{11})^{F^+}, C_u(D_{11})^{F^+}) + Q(C_u(M_{21}), C_v(M_{21}))
\]
or equivalently
\[
\Phi = -\Omega - Q(C_u(M_{21}), C_v(M_{21}))_{u,v \in B_1}.
\]

The following lemma can be proved in the same way Proposition 5.27 is proved.

**Lemma 5.37.** \( M \) is a symplectic matrix if and only if there exists a symmetric matrix \( S \) such that
\[
(M_{11})_{\text{pivot}} = M_{\text{pivot}} = (D^{\uparrow r}_{\text{lead}} \cdot \sigma)^{-1} \cdot (S - \Omega/2 - Q(C_u(M_{21}), C_v(M_{21})))_{u,v \in B_1}/2).
\]

Combining all, we get the following variant of Proposition 3.17 which is proved in [20]:

**Proposition 5.38.** The centralizer of \( U^{\uparrow n} \) in \( Sp_n(q) \) admits the following description:
\[
C_{Sp_n(q)}(U^{\uparrow n}) = \{ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in GL_{2n}(q) \mid M_{22} \in Sp_{n-m}(q), U M_{12} = M_{12}, U M_{21} = M_{21},
\]
\[
M_{12,\text{lead}} = ((M_{11})_{\text{lead}} \cdot \sigma)^{-1} Q(C_u(M_{21}), C_v(M_{22}))_{u,v \in B_{m,n}}
\]
\[
M_{11} := M_{11} + ((M_{11})_{\text{lead}} \cdot \sigma)^{-1} Q(C_u(M_{21}), C_v(M_{21}))_{u,v \in B_1}/2 \in C_{GL_{2m}(q)}(U) \cap Sp_m(q) \}
\]

Equivalently
\[
C_{Sp_n(q)}(U^{\uparrow n}) = \{ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in C_{GL_2(n)}(U^{\uparrow n}) \mid M_{22} \in Sp_{n-m}(q),
\]
\[
M_{12,\text{lead}} = ((M_{11})_{\text{lead}} \cdot \sigma)^{-1} Q(M_{21}, M_{22})
\]
\[
M_{11} := M_{11} + ((M_{11})_{\text{lead}} \cdot \sigma)^{-1} Q(C_u(M_{21}), C_v(M_{21}))_{u,v \in B_1}/2 \in C_{GL_{2m}(q)}(U) \cap Sp_m(q) \}
\]

In particular, if \( U \in Sp_n(q) \) is an arbitrary isometry whose modified symplectic type is \( \lambda \) and \( \| \lambda \| = 2m \), then
\[
|C_{Sp_n(U^{\uparrow n})}| = |C_{Sp_n(U)}| \cdot |Sp_{n-m}(q)| \cdot q^{2x(n-m)}.
\]

**Proof.** \( (\subseteq) \) Let \( M \in C_{Sp_n(q)}(U^{\uparrow n}) \). By Lemma 5.31 \( M_{22} \in Sp_{n-m}(q) \). The equalities \( U M_{12} = M_{12} \) and \( M_{21} U = M_{21} \) follow from Proposition 3.17. The equality
\[
M_{12,\text{lead}} = ((M_{11})_{\text{lead}} \cdot \sigma)^{-1} Q(C_u(M_{21}), C_v(M_{22}))_{u \in B_1, v \in B_{m,n}}
\]

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follows from the Lemma 5.33. The only difference between $M_{11}$ and $M'_{11}$ occur in the free indices, hence $M'_{11}$ is also in the centralizer of $U$ by Proposition 5.12. By Lemma 5.37,

$$(M_{11})_{\text{pivot}} = M_{\text{pivot}} = (D_{\text{lead}}^{r} \cdot \sigma)^{-1} \cdot (S - \Omega/2 - Q(C_{u}(M_{21}), C_{v}(M_{21})))_{u,v \in B_{1}}/2. $$

As a result $M'_{11}$ satisfies the 4th of Proposition 5.27 hence $M'_{11}$ is an isometry and commutes with $U$.

(≥) Let $M$ be an element of the right handside. The last condition ensures that $M'_{11}$ is in the centralizer of $U$, and hence as above, $M_{11}$ is in the centralizer of $U$. The first three conditions now ensure that $M$ is in the centralizer of $U^\perp$. The fact that $M$ is an isometry is a consequence of the previous investigations.

The second set equality follows from the first one, as the defining conditions of the second set implies that $M$ is an isometry, as dealt in the preceding discussion. Now consider equality concerning the cardinalities. First assume that $U$ is a unipotent element. Then the equality follows from the previous set equality as the $M_{12}$ is uniquely determined by $M_{11}, M_{21}$ and $M_{22}$, and the number of possible $M_{21}$ matrices is $q^{2h(n-m)}$ as $h$ is the dimension of the 1-eigenspace of $U$. For general $U$, the result follows from Remark 3.7.

Now assume that $U_{1}, U_{2} \in Sp_{m}(q)$ where $\lambda$ and $\mu$ are their modified symplectic types. Moreover, assume that $U_{1}U_{2} = U = J_{F}$ and $\|\eta\| = \|\lambda\| + \|\mu\|$.

**Lemma 5.39.** The following equality holds:

$$C_{GL_{n}(q)}(U_{1}^{\perp}) \cap C_{GL_{n}(q)}(U_{2}^{\perp}) = \left\{ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \mid \begin{align*} M_{11} & \in C_{GL_{m}(q)}(U_{1}) \cap C_{GL_{m}(q)}(U_{2}), \\
M_{22} & \in GL_{2(n-m)}(\mathbb{F}_{q}), \\
UM_{12} & = M_{12}, M_{21}U = M_{21} \end{align*} \right\}. $$

**Proof.** For $i = 1, 2$, Proposition 3.17 implies that $M \in C_{GL_{2m}(q)}(U_{i}^{\perp})$ if and only if the following hold:

1. $M_{11} \in C_{GL_{2m}(q)}(U_{i})$
2. Columns of $M_{12}$ consist of eigen-vectors of $U_{i}$,
3. Columns of $M'_{21}$ consist of eigen-vectors of the $U_{i}^{\perp}$.

Let $V^{U_{1}}, V^{U_{2}}, V^{U}$ denote the fixed spaces of $U_{1}, U_{2}$ and $U$, respectively. By Lemma 3.18/3 we know that

$$V^{U_{1}} \cap V^{U_{2}} = V^{U}, \quad V^{U_{1}} \cap V^{U_{2}} = V^{U^{t}}$$

as reflection length of $U_{i}$ and $U_{i}^{\perp}$ are same. Now assume that $M$ is contained in the intersection. Then by 1., $M_{11} \in C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$. Conversely, assume that $M$ is contained in the intersection. Then $M_{11} \in C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$. As columns of $M_{12}$ (respectively rows of $M_{21}$) consists of elements of $V^{U} = V^{U_{1}} \cap V^{U_{2}}$ (respectively $V^{U^{t}} = V^{U_{1}} \cap V^{U_{2}}$) it follows that $M \in C_{GL_{2m}(q)}(U_{1}^{\perp}) \cap C_{GL_{2m}(q)}(U_{2}^{\perp})$ by Lemma 3.17.

**Lemma 5.40.** Let $A \in C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$ and $B \in GL_{2m}(q)$. Let $C = (c_{uv})_{u,v \in B_{m}} = A - B$. Assume that $c_{uv} = 0$ if $(u,v) \notin B_{e_{1}} \times B_{e_{1}}$. Then $B \in C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$ as well.

**Proof.** All the entries of $C$ except $C_{pivot}$ are zero. We know from Remark 5.17 that each column (resp. row) of $C$ is then a 1-eigenvector of $U$ (resp. $U^{t}$). Invoking 3.18/3 we see that each column (resp. row) of $C$ is then a 1-eigenvector of $U_{1}$ and $U_{2}$ (resp. $U_{1}^{\perp}$ and $U_{2}^{\perp}$). This means, $C$ is contained in $C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$. Now the result follows from the fact that $A \in C_{GL_{2m}(q)}(U_{1}) \cap C_{GL_{2m}(q)}(U_{2})$ and $B = A - C$.

**Proposition 5.41.** Let $C_{\mu, \lambda}(n)$ denote the intersection $C_{Sp_{m}(q)}(U_{1}^{\perp}) \cap C_{Sp_{m}(q)}(U_{2}^{\perp})$ for $n \geq m$. Then the set equality

$$C_{\mu, \lambda}(n) = \left\{ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in GL_{2n}(q) \mid M_{22} \in Sp_{n-m}(q), UM_{12} = M_{12}, M_{21}U = M_{21}, \\
M_{12, \text{lead}} = ((M'_{11})^{\ast}_{\text{lead}} \cdot \sigma)^{-1} Q(M_{21}, M_{22}) \\
M'_{11} = M_{11} + ((M'_{11})^{\ast}_{\text{lead}} \cdot \sigma)^{-1} Q(C_{u}(M_{21}), C_{v}(M_{21}))_{u,v \in B_{1}}/2 \in C_{\mu, \lambda}(m) \right\}$$

holds for $n \geq m$. In particular, if $U, U_{1}, U_{2} \in Sp_{m}(q)$ are isometries and the modified symplectic type of $U$ is $\lambda$ with $\|\lambda\| = 2m$ and $U_{1}U_{2} = U$, then

$$|C_{Sp_{m}(q)}(U_{1}^{\perp}) \cap C_{Sp_{m}(q)}(U_{2}^{\perp})| = |C_{Sp_{m}(q)}(U_{1}) \cap C_{Sp_{m}(q)}(U_{2})| \cdot |Sp_{n-m}(q)| \cdot q^{2h(n-m)}. $$
Proof. Let $M \in C_{SP_\sigma(q)}(U_1) \cap C_{SP_\sigma(q)}(U_2)$. Then $M \in C_{SP_\sigma(q)}(U^{m_1})$ as $U_1 U_2 = U$. So by Proposition 5.38, the assertions $M_{22} \in SP_{m_1}(q), U M_{12} = M_{12}, M_{21} U = M_{21}$, and $(M_{12})_{lead} = ((M'_{11})_{lead} \cdot \sigma)^{-1}Q(M_{21}, M_{22})$ follows immediately. By Lemma 5.39, $M_{11}$ is an element of $C_{GL_{2m_1}(q)}(U_1) \cap C_{GL_{2m_1}(q)}(U_2)$ and by Lemma 5.40, $M'_{11} \in C_{GL_{2m_1}(q)}(U_1) \cap C_{GL_{2m_1}(q)}(U_2)$. As argued in Proposition 5.38, $M'_{11}$ is an isometry. The converse containment follows from direct calculation using the discussion concerning the sufficiency conditions for $M$ being an isometry. 

\[
\begin{align*}
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\end{align*}
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