Contributions to the Dark Matter 3-Point Function from the Radiation Era

A. Liam Fitzpatrick$^{1,2}$, Leonardo Senatore$^{2,3,4}$, and Matias Zaldarriaga$^{2,3}$

$^1$ Physics Department
Boston University, Boston, MA 02215, USA

$^2$ Jefferson Physical Laboratory
Harvard University, Cambridge, MA 02138, USA

$^3$ Center for Astrophysics
Harvard University, Cambridge, MA 02138, USA

$^4$ School of Natural Sciences
Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract

We consider the contribution to the three-point function of matter density fluctuations from nonlinear growth after modes re-enter the horizon, and discuss effects that must be included in order to predict the three-point function with an accuracy comparable to primordial nongaussianities with $f_{NL} \sim \text{few}$. In particular, we note that the shortest wavelength modes measured in galaxy surveys entered the horizon during the radiation era, and, as a result, the radiation era modifies their three-point function by a magnitude equivalent to $f_{NL} \sim \mathcal{O}(4)$. On longer wavelengths, where the radiation era is negligible, we find that the corrections to the nonlinear growth from relativistic effects become important at the level $f_{NL} \sim \text{few}$. We implement a simple method for numerically calculating the three-point function, by solving the second-order equations of motion for the perturbations with the first order perturbations providing a source.

1 Introduction

The spectrum of initial density perturbations from inflation is known to be nearly gaussian. Should deviations from a gaussian distribution be detected, however, they will imply potentially powerful constraints on models of inflation. In particular, the shape and size of the three-point function of primordial density perturbations in many models is predicted to be near current experimental limits, and is one of the best observables for distinguishing different inflationary models [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] as well as its (perhaps less compelling) alternatives [13]. The best measurements so far come from cosmic microwave background (CMB) data [14, 15, 16], but large-scale structure (LSS) measurements are improving and will become comparable. The shortest wavelength measured by LSS is smaller than for the CMB, and it is a three-dimensional map instead of a two-dimensional one. Therefore LSS measurements potentially include many more modes. However, they also therefore involve modes that entered the horizon at earlier times, when the universe was radiation dominated.
Our purpose is to quantify the effect of radiation on non-Gaussianities in LSS measurements in comparison to that from primordial non-Gaussianities.

The significance of the radiation era is due to the nonlinear growth of non-Gaussianities, which occurs even for Gaussian initial conditions [17, 18]. The leading contributions have been studied in the perturbation theory (PT) formalism, dropping the radiation component of the universe and using Newtonian gravity [19, 20, 21]. While this approach has been very successful, the future improvement of observational data will necessitate knowing the theoretical error arising from the PT formalism assumptions. In fact, the late-time contribution to non-Gaussianities from PT is orders of magnitude larger than that expected from primordial non-Gaussianity [22], and even relatively small corrections to the PT ansatz may swamp the signal we are interested in.

Experimental limits on non-Gaussianities are generically given in terms of a scalar variable $f_{\text{NL}}$ [23, 24] which in the so-called “local” ansatz parameterizes the deviation of the Newtonian potential from a Gaussian variable $\Phi_g$ as

$$\Phi = \Phi_g + f_{\text{NL}} (\Phi_g^2 - \langle \Phi_g^2 \rangle)$$

(1)

There are other possible shapes from models of inflation, but we will use this one for comparison. We find that radiation effects on the LSS are comparable in size to $f_{\text{NL}} \sim 4$. Current LSS experimental limits are still consistent with a Gaussian distribution and have 95% confidence limits of $-29 < f_{\text{NL}} < 69$ [25]. So far, they are nicely consistent with the limits from the CMB [26], which, when combined together, give the constraint $-1 < f_{\text{NL}} < 63$ at 95% confidence level.

2 Estimates

2.1 PT Formalism

We are ultimately interested in calculating the difference between the three-point function from the PT formalism and from a universe with a radiation component, and comparing this difference to the three-point function from primordial non-Gaussianities. Consider first what kind of effects we expect for small vs. large wavelength modes. For very large scales that cross the horizon deep in the matter dominated region, we do not expect any growth in $\delta$ before the matter-dominated region begins and thus we can essentially treat the entire history of the universe as matter dominated. However, for short wavelength modes, growth has already begun deep in the radiation dominated region and it is not immediately clear whether this can give a large correction or not to the distribution of perturbations. The perturbations are very small during the radiation dominated era, and this implies that non-Gaussianities generated during this region will be very small. That is, non-Gaussianities from gravitational instability are generated because the first order fluctuations are a source for the second order fluctuations:

$$\delta^{(2)} \sim \delta^{(1)^2} ,$$

(2)
and thus this generates non-gaussianity dominantly at recent times. In this section, we will 
estimate the effect of radiation by considering how a large distortion in $\dot{\delta}$ during the radiation era affects the three-point function at late times.

We first recall a few results. The method of PT is essentially to perform an expansion in the scale factor $a$. At second order,

$$\delta(a, k) = \delta_1(k) a + \frac{1}{2} \delta_2(k) a^2,$$

$$\delta_2(k) = \int \frac{d^3q_1 d^3q_2}{(2\pi)^3} \delta(k - q_1 - q_2) F_2(q_1, q_2) \delta_1(q_1) \delta_1(q_2).$$

In Newtonian gravity with matter only and no decaying mode, one obtains

$$\frac{1}{2} F_2(q_1, q_2) = \frac{5}{7} + \frac{\hat{q}_1 \cdot \hat{q}_2}{2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} (\hat{q}_1 \cdot \hat{q}_2)^2.$$

Thus, as a function of $a$, the three-point function is

$$\langle \delta k_1 \delta k_2 \delta k_3 \rangle(a) \sim a^4 P^2 F_2 + a^3 B_1(k_1, k_2, k_3),$$

where $P(B_1)$ is the two-point (three-point) function of $\delta_1$, and thus $B_1$ encodes the three-point function initial condition. Modifying the evolution history up to a scale factor around $a_{eq}$ will modify $B_1$ proportionally by $a_{eq}$. For instance, suppressing all growth of nongaussianity in this simple approximation up until $a_{eq}$, so that $\langle \delta k_1 \delta k_2 \delta k_3 \rangle(a_{eq}) = 0$, imposes

$$B_1 \sim -a_{eq} P^2 F_2.$$

We therefore ought to expect corrections to the three-point function today proportional to $a_{eq}$.

Such a small correction matters because at late times and short wavelengths, the nongaussianity from nonlinear growth (i.e. from $F_2$) swamps the contribution from initial conditions. Within linear theory, $\delta_k(a) = M_k(a) \Phi_k^{\text{prim}}$,

$$M_k(a) = -\frac{3}{5} \frac{k^2 T(k)}{\Omega_m H_0^2} D_1(a),$$

where $D_1(a)$ is the growing mode for $\delta$ and $T(k)$ is the transfer function. Under the local ansatz $\Phi^{\text{prim}} = \Phi_g + f_{NL} \Phi_g^2$, this leads to a dark matter three-point function proportional to $f_{NL}$:

$$B_L(k_1, k_2, k_3) = M_{k_1} M_{k_2} M_{k_3} 2 f_{NL} (P_{\Phi}(k_1) P_{\Phi}(k_2) + \text{cyc.}).$$

The full three-point function is then approximately the sum of the contribution $B_L$ from linear growth and the contribution $B_G$ from non-linear growth. For the sake of this approximation, we will take isosceles triangle configurations $|k_1| = |k_2| = y |k_3|$, in which case

---

1. The reader familiar with the PT literature should note the additional factor of $\frac{1}{2}$ in the expression for $F_2$. This is a consequence of the additional factor of $\frac{1}{2}$ in our convention for $\delta^{(2)}$.

2. Power spectra for fluctuations other than $\delta$ will be denoted with a subscript to indicate the fluctuation, e.g. $P_{\Phi}(k)$ for $\Phi$ fluctuations.
\[ \frac{1}{2} F_2(k_1, k_2) = \frac{1 + 3y^2}{14y^2} \] and \[ \frac{1}{2} F_2(k_1, k_3) = \frac{1}{2} F_2(k_2, k_3) = \frac{1}{28} (13 - \frac{5}{y^2}). \] Note that \( y \geq \frac{1}{2} \). We will also ignore the scalar tilt \( n_s \), so \( P_\Phi(k) \propto k^{-3} \). Now compare this to a correction in the nonlinear bispectrum of the size we expect from modifying the model at scale factor \( a_{eq} \), i.e.

\[ \delta B_G \approx a_{eq} F_2(k_1, k_2) P(k_1) P(k_2) + \text{cycl.:} \]

\[ \frac{\delta B_G}{B_L} \approx \frac{a_{eq} M_{k_3}}{2f_{NL}} \left( \frac{2y^3 F_2(k_1, k_3) + F_2(k_1, k_2) (M_{k_3})^2}{2y^3 + 1} \right). \]  \tag{9}

We can simplify this in various limits. The transfer functions \( T(k) \) simplify at large and at small \( k \) in a flat (matter + radiation) universe:

\[ T(k \ll k_{eq}) = 1, \]  \tag{10}

\[ T(k \gtrsim 15k_{eq}) = \frac{45}{2} \Omega_m H_0^2 \frac{k^2}{a_{eq}} \log(k/6k_{eq}). \]  \tag{11}

In the limit of squeezed triangles \( y \gg 1 \), we arrive at the following estimate

\[ \frac{\delta B_G}{B_L} \approx \frac{a_{eq} M_{k_3}}{f_{NL}} \frac{13}{28} \left\{ \begin{array}{ll}
\frac{-0.56}{f_{NL} \Omega_m} \left( \frac{k_3}{k_{eq}} \right)^2 & k_3 \ll k_{eq} \\
\frac{-0.34}{f_{NL} \Omega_m} \log(k_3/6k_{eq}) & k_3 \gg k_{eq}
\end{array} \right\}, \quad y \gg 1. \]  \tag{12}

Note that all dependence on the shorter wavelength modes \( k_1, k_2 \) drops out. Thus, even if \( k_1, k_2 \) enter the horizon deep in the radiation era, the 3-pt function is unaffected by the radiation corrections as long as \( k_3 \) enters during matter dominance (MD), where \( \delta B_G \) is suppressed by \( (k_3/k_{eq})^2 \); however, the 3-pt function is affected if \( k_3 \) enters during the radiation era.

We next turn to the limit of equilateral triangles, \( y = 1, k \equiv k_i \):

\[ \frac{\delta B_G}{B_L} \approx \frac{a_{eq} M_k}{f_{NL}} \frac{2}{7} \left\{ \begin{array}{ll}
\frac{-0.34}{f_{NL} \Omega_m} \left( \frac{k}{k_{eq}} \right)^2 & k \ll k_{eq} \\
\frac{-0.9}{f_{NL} \Omega_m} \log(k/6k_{eq}) & k \gg k_{eq}
\end{array} \right\}, \quad y = 1. \]  \tag{13}

In other words, modifying the matter content of the universe at \( a_{eq} \) near or after matter-radiation equality alters the bispectrum roughly corresponding to \( f_{NL} \sim \text{few} \) for modes that entered the horizon somewhat before matter-radiation equality. We will need a more precise calculation of nonlinear growth near matter-dominination equality in order to predict the bispectrum at order \( f_{NL} \sim \text{few} \).

### 2.2 Radiation Effect in Newtonian Gravity

By the previous arguments, we expect that the dominant contribution from radiation will come from the era near the transition between matter and radiation, because at this time, \( \delta \) is as large as possible before matter dominates. Short wavelength modes, which by previous arguments will get the largest contribution, will therefore already be inside the horizon at
that point and we should be able to approximate the effect on them using Newtonian gravity. In this limit, we will be able to obtain an analytic approximation.

The equations of motion for the dark matter perturbations in Newtonian gravity are (e.g. \[18\])

\[
\delta'_k + i k V_k = - \left(1 + \frac{q_1 \cdot q_2}{q_2^2}\right) \delta_{q_1 q_2} V_{q_2},
\]

\[
 i k V_k' + \mathcal{H} i k V_k + \frac{3}{2} \mathcal{H}^2 \Omega_m(\eta) \delta_k = \frac{1}{2} k^2 \hat{q}_1 \cdot \hat{q}_2 V_{q_1} V_{q_2}.
\]

Primes denote derivatives with respect to \(\eta\). The behavior of the background is that of a universe with matter and radiation with no cosmological constant, which can be written in terms of \(a_{eq} = \frac{\Omega_{m,0}}{\Omega_m}\) as \[27\]

\[
a = \frac{a_{eq}}{a_{eq}} = 2\alpha \eta + \alpha^2 \eta^2, \quad \alpha = \frac{H_0}{2\sqrt{a_{eq} (1 + a_{eq})}} = \sqrt{2} - 1, \]

and \(\Omega_m(\eta) = \Omega_m_{\text{rad}} = \frac{1}{1 + a_{eq} \eta}\). It is straightforward to solve equations (\[14\] \[15\]) at linear order. The solution normalized so that \(\delta^{(1)} = 1\) today is

\[
\delta^{(1)}_g = \frac{2 + 3\alpha \eta (2 + \alpha \eta)}{2 + 3\alpha \eta_0 (2 + \alpha \eta_0)}, \quad i k V^{(1)}_g = - \frac{6\alpha(1 + \alpha \eta)}{2 + 3\alpha \eta_0 (2 + \alpha \eta_0)}.
\]

Taking the first order solutions to contain only a growing mode, we obtain an analytic solution for the second order modes, which for brevity we do not write out. Instead we expand in \(1/\alpha\), and obtain the PT result equation (\[5\]) plus corrections. The leading correction is enhanced by log\((a_{eq})\):

\[
\frac{1}{2} F_{2, \text{rad}} = \frac{1}{2} F_{2, \text{PT}} - \frac{1}{2} a_{eq} \log(a_{eq}^{-1}) \frac{1}{3 \sqrt{q_1^2 q_2^2}} (k^2 - (q_1 + q_2)^2) (k^2 - (q_1 - q_2)^2) + \ldots
\]

\[
= \frac{1}{2} F_{2, \text{PT}} + \frac{1}{2} a_{eq} \log(a_{eq}^{-1}) \frac{4}{35} (1 - (\hat{q}_1 \cdot \hat{q}_2)^2) + \ldots.
\]

For example, on equilateral configurations (i.e. \(k = q_1 = q_2\)), we have \(F_{2, \text{rad}} - F_{2, \text{PT}} \approx a_{eq} \frac{3}{20} \log(a_{eq}^{-1}) F_{2, \text{PT}}\), which is \(\approx 1.2 a_{eq} F_{2, \text{PT}}\) for \(a_{eq} = 3 \times 10^{-4}\). As in the case without radiation, the kernel \(F_{2, \text{rad}}\) is invariant under rescalings \((k, q_1, q_2) \to \lambda(k, q_1, q_2)\), and all scale-dependence of \(\delta^{(2)}\) comes from the scale-dependence of \(\delta^{(1)}\). Note that the leading term vanishes on “squashed” triangle configurations \(k = |q_1 \pm q_2|\), and in this case the subleading, non-log-enhanced contribution dominates. However, the non-log-enhanced contribution is more sensitive to the behavior of the linear solutions during the radiation era, and in order to get an accurate estimate valid at large \(k\) we would have to include some effects we have neglected.

\[3\] We take the initial conditions at arbitrarily small \(\eta\), so the decaying mode is projected out and the solution is the growing mode.
In the above approximation, we included radiation only in the background dependence, i.e. in $\mathcal{H}(\eta)$. This approximation is exact in the limit $k \gg k_{\text{eq}}$. A more careful treatment would include the effect of the radiation density perturbations as well, which would enhance the evolution of the Newtonian potential $\Phi$ at early times. One might then obtain an analytic approximation by including the backreaction of only the radiation on the Newtonian potential at early times and only the matter at later times, and matching the two solutions in an intermediate regime. One would expect in this way to see an enhancement of the final perturbation size, similar to results from the analogous method in linear perturbation theory [28]. However, we instead will now turn to a numeric calculation that will give the fluctuations at second order for all wavelengths.

3 Method

In this section we will describe our method for calculating the effects of radiation on the dark matter density three-point function at late times, based on an expansion at second order in the fluctuations. That is, we split all perturbations $\mathcal{O} = \mathcal{O}^{(1)} + \frac{1}{2} \mathcal{O}^{(2)}$ into a first order piece $\mathcal{O}^{(1)}$ and a second order piece $\mathcal{O}^{(2)}$. For closely related work on second order corrections to cosmological perturbations see e.g. [29, 30, 31, 32, 33, 34]. Since we are interested in following the dark matter distribution, we can approximate the baryon-photon fluid as a fluid with $w = \frac{1}{3}$, and otherwise neglect $\Omega_b$ for simplicity, taking a universe of two perfect fluids composed respectively of collisionless matter and radiation. In this approach, the second order perturbations are sourced due to nonlinear interactions by the first order ones, and thus are quadratic in them, $\mathcal{O}^{(2)} \sim \mathcal{O}^{(1)} \mathcal{O}^{(1)}$. This leads to a nonvanishing three-point function even for purely gaussian initial conditions, and this contribution should be removed in order to reconstruct the primordial nongaussianity. In particular, we are interested in the kernel $F_2$ in Fourier space:

$$\frac{1}{2} \delta^{(2)}_{\vec{k}}(\eta) = \frac{1}{2} \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^3} \delta(\vec{k} - \vec{q}_1 - \vec{q}_2) F_2(\vec{q}_1, \vec{q}_2; \eta) \delta^{(1)}_{\vec{q}_1}(\eta_0) \delta^{(1)}_{\vec{q}_2}(\eta_0),$$

(19)

where $\eta$ is conformal time and we choose $\eta_0 = \eta_{\text{today}}$ in order to make contact with existing literature. We compute $F_2$ in a straightforward manner for any $q_1, q_2$ by inputting the first order solutions into the equations of motion as sources and solving numerically for the second order terms. This procedure could be carried out iteratively to higher orders as well by inputting the first and second order solutions as sources at third order, etc.

We also may neglect all vector and tensor degrees of freedom. The reason is that vector and tensor degrees of freedom vanish at first order in the initial scalar perturbations, arising only at second order. Therefore, we may decompose any equation with a vector index into its scalar $\mathcal{O}_s$ and vector $\mathcal{O}_v^i$ piece. In Fourier space, the most general equation at second order is

$$(aq_1^i + bq_2^i)\mathcal{O}_s^{(1)}(q_1)\mathcal{O}_s^{(1)}(q_2) + k^i \mathcal{O}_s^{(2)}(k) + \mathcal{O}_v^{(2)}(k) = 0,$$

(20)
where \( k^i \mathcal{O}_v^i(k) = 0 \). By contracting this equation with \( k^i \), we therefore project out all vector modes and obtain a second order equation for just the scalar modes. A similar argument applies to the tensor fluctuations that arise from primordial scalar fluctuations. In addition, tensor modes have their own primordial fluctuations, whose size is typically suppressed with respect to the scalar fluctuations in models of inflation. More important, their two-point function with the scalar modes necessarily vanishes by rotational invariance, and thus they give no contribution to the matter three-point function. We may therefore neglect them as well.

The computation is simplest in conformal Newtonian gauge, where the metric is diagonal. Specifically, we will take
\[
\omega_i = 0, \quad \chi_{ij} = 0,
\]
so that \( \omega, \chi \) contain no scalar modes. We may therefore neglect \( \omega \) and \( \chi \) completely for following the scalar modes at second order.

The energy-momentum tensor is that for a fluid of matter, which is pressureless up to negligible \( \mathcal{O}(\frac{T}{m}) \) corrections, and a fluid of radiation \( w = 1/3 \):
\[
T_{\nu,m}^\mu = \rho_m u^\mu_m u_{\nu,m}, \quad T_{\nu,r}^\mu = \rho_r \left( \frac{4}{3} u^\mu_r u_{\nu,r} + \frac{1}{3} \delta^\mu_{\nu} \right), \quad u^\mu_{m,r} = \left( \frac{\epsilon^\nu}{a} \left( 1 + \frac{\nu^2_{m,r}}{2} \right), \frac{\epsilon^{\nu}}{a} v^\nu_{m,r} \right).
\]
(23)

We will denote the longitudinal piece of velocity as \( v^i = \hat{k}^i V \). We obtain the equations of motions from Einstein’s equation \( G_{\nu}^\mu = 8\pi G_N T_{\nu}^\mu \) and the conservation equations \( \nabla_{\mu} T_{\nu,m,r}^\mu = 0, \nabla_{\mu} (\rho_m u_{m}^{\mu}) = 0 \), which is valid because matter and radiation interact only gravitationally.

It is straightforward to derive the Einstein tensor exactly in conformal Newtonian gauge with only scalar modes. We write it down for reference in appendix A. There are two components of the Einstein equation that will be useful to us. The first is the time-time component, \( a^2 G^0_0 = 3H^2 T^0_0 / \rho \), which in Fourier space is
\[
k^i \Phi^{(2)}_k + 3H \Phi_k^{(2)} + 3\mathcal{H}^2 \Psi_k^{(2)} + \frac{3}{2} \mathcal{H}^2 \left( \frac{\bar{\rho}_m}{\rho} \delta^{(2)}_m + \frac{\bar{\rho}_r}{\rho} \delta^{(2)}_r \right) = -S_3 \equiv \tag{24}
\]
\[
\left( q_1 \cdot q_2 - 4q_2^2 + 6\mathcal{H}^2 \right) \Phi^{(1)}_{q_1} \Phi^{(1)}_{q_2} + 3 \mathcal{H} \Phi^{(1)}_{q_1} \Phi^{(1)}_{q_2} + 12 \mathcal{H} \Phi^{(1)}_{q_1} \Phi^{(1)}_{q_2} \]
\[-3\mathcal{H}^2 q_1 \cdot q_2 \left( \frac{\bar{\rho}_m}{\rho} V_{q_1,m} V_{q_2,m} + \frac{4}{3} \frac{\bar{\rho}_r}{\rho} V_{q_1,r} V_{q_2,r} \right).
\]
Primes denote \( \eta \) derivatives, and \( \mathcal{H} = \frac{\dot{a}}{a} \). In the above equation, an integral \((2\pi)^{-3} \int d^3q_1 d^3q_2 \delta(k - q_1 - q_2) \) is implied over terms quadratic in the first order perturbations. In general, we mean for such an integral to be implicit in any equation in Fourier space with terms quadratic in the first order perturbations. The second useful component is the shear piece, specifically \( (k^i \hat{k}^j - \frac{1}{3} \delta_{ij}^k) G^{i,j} \), which projects out the \( \delta_{ij} \) piece in \( G^{i,j} \). The resulting equation of motion is
\[
\frac{1}{3} k^2 (-\Phi^{(2)} + \Psi^{(2)}) = \frac{1}{3} k^2 S_4(k, \eta) \equiv \tag{25}
\]
We now turn to the conservation equations for matter. The first comes from $\nabla_\mu (\rho_m u^\mu_m) = 0$, the second from $\nabla_\mu T^\mu_{i,m} = 0$:

$$
\delta''_m + i k V''_m - 3 \Phi''_m = S_1 (k, \eta) \equiv 2 \delta^{(1)}_{q_1,m} \delta^{(1)}_{q_2,m} - 2 i (q_1 \cdot \hat{q}_2) \delta^{(1)}_{q_1,m} V^{(1)}_{q_2,m} - 4 i q_2 \Phi^{(1)}_{q_1} V^{(1)}_{q_2,m} + 2 i (\hat{q}_1 \cdot q_2) V^{(1)}_{q_1,m} \Phi^{(1)}_{q_2} - 2 (\hat{q}_1 \cdot \hat{q}_2) V^{(1)}_{q_1,m} V^{(1)}_{q_2,m} ,
$$

$$
V'_m + \mathcal{H} V''_m + i k \Psi^{(2)} = S_2 (k, \eta) \equiv 2 (\hat{q}_1 \cdot \hat{q}_2) \Phi^{(1)}_{q_1} V^{(1)}_{q_2,m} - i k (\hat{q}_1 \cdot \hat{q}_2) V^{(1)}_{q_1,m} V^{(1)}_{q_2,m} - 2 i k \Phi^{(1)}_{q_1} \Phi^{(1)}_{q_2} .
$$

where we have simplified the sources $S_i$ by imposing the first-order equations of motion and symmetrizing in $q_1, q_2$ when convenient.

We take the two radiation equations of motion from $\nabla_\mu T^\mu_{i,r} = 0$ and $\nabla_\mu T^\mu_{0,r} = 0$:

$$
\delta''_r - 4 \Phi''_r + \frac{4 i k}{3} V''_r = S_6 \equiv - \frac{4}{3} i \hat{q}_1 \cdot q_2 V^{(1)}_{q_1,r} \delta^{(1)}_{q_2,r} + \frac{16}{3} i \hat{q}_1 \cdot q_2 V^{(1)}_{q_1,r} \Phi^{(1)}_{q_2} - \frac{16}{3} i q_2 \Phi^{(1)}_{q_1} V^{(1)}_{q_2,r} + 2 \delta^{(1)}_{q_1,r} \delta^{(1)}_{q_2,r} ,
$$

$$
V'_r + \frac{i k}{4} \delta''_r + i k \Psi^{(2)} = S_7 \equiv \frac{i}{3 k} (2 q_1 q_2 + (q_1^2 + q_2^2 - 3 k^2) \hat{q}_1 \cdot \hat{q}_2) V^{(1)}_{q_1,r} V^{(1)}_{q_2,r} + \frac{i k}{4} \delta^{(1)}_{q_1,r} \delta^{(1)}_{q_2,r} + 4 \hat{q}_2 \cdot \hat{k} \Phi^{(1)}_{q_1} V^{(1)}_{q_2,r} .
$$

The behavior of the background is that of a universe with matter and radiation, as in equation (16). In addition to the above equations of motion, we require the equations for the first order perturbations. The equations of motion for the first order perturbations are identical in form to those for the second order solutions, with the sources $S_i$ set to zero. We solve these numerically.

The initial conditions for the fluctuations depends on physics before they reenter the horizon. There is both a contribution from before a mode exits the horizon during inflation that depends on the inflationary model as well as a possible contribution due to light degrees of freedom that perturb the reheating surface. Outside the horizon, it is useful to work with the metric of scalar modes can be set by comparing with $\zeta$ gauge, which in the absence of spatial gradients is defined at nonlinear order by

$$
ds^2 = a^2 (\eta) \left(-d\eta^2 + e^{2\zeta} dx^i dx^i\right) .
$$

For the minimal model of inflation, $\zeta$ is initially an approximately gaussian field with a three-point function that is suppressed by a slow-roll parameter [11, 21, 33]. We therefore choose $\zeta^{(2)} = 0$ initially, in order to separate out the effects due to nonlinear growth at late times. This condition in turn sets the initial conditions on most of the remaining perturbations
through the equations of motion. Let us first consider how it sets \( \Psi^{(2)} \) and \( \Phi^{(2)} \) initially. \( \Psi^{(2)} \) and \( \Phi^{(2)} \) are related to each other through the constraint equation (25), which enforces

\[
\Psi^{(2)} = \Phi^{(2)} + S_4.
\]  

At nonlinear order, transforming from \( \zeta \) gauge (30) to Newtonian gauge shows that \( \zeta = -\Phi - \frac{1}{2} \Psi \) during the radiation era in the absence of spatial gradients. Thus, at \( a \ll a_{eq}, k \eta \ll 1 \)

\[
\Phi = -\frac{2}{3} \zeta - \frac{1}{6} S_4,
\]  

and therefore, we have \( \frac{1}{2} \Phi^{(2)} = -\frac{1}{6} S_4 \) initially.

The initial conditions for the perturbations therefore depend on the behavior of the source terms at early times \( a \ll a_{eq} \) during the radiation era and long wavelengths \( k \eta \ll 1 \), because as above the equations of motion each contain perturbations without time derivatives. At early times, these perturbations contain only a growing mode, which is usually constant outside the horizon, and the equations of motion turn into an algebraic relations between the perturbations and the sources. This is also what happens in the linear theory, except that the source terms \( S_i \) are not present in that case. In the long-wavelength early-time limit, we will see that we need to use only \( S_3 \rightarrow -6H^2 \Phi^{(1)} q_1 q_2 \) and \( S_4 \rightarrow -\frac{9}{k^2} \left( \hat{k} \cdot q_1 \hat{k} \cdot q_2 - \frac{1}{2} q_1 \cdot q_2 \right) \Phi^{(1)} q_1 q_2 \) (using \( iV^{(1)} = \frac{1}{2} k \eta \Phi^{(1)} \) initially). Equation (24) and (25) together imply the initial condition for \( \delta_r^{(2)} \):

\[
\delta_r^{(2)} = -\frac{2}{3H^2} S_3 - 2S_4 - 2\Phi^{(2)}.
\]  

The second order velocities \( V_m^{(2)}, V_r^{(2)} \), start off negligible at early times; if needed, they can be read off of equations (27,29) by taking \( V^{(2)} \propto \eta \) at small \( \eta \).

Finally, we need to set the initial condition for \( \delta_m^{(2)} \). It does not appear in the equations of motion at early times, because it always has a time derivative acting on it or else is multiplied by \( (\dot{\rho}_m/\bar{\rho}) \), which vanishes in the infinite past. We will assume adiabatic initial conditions, where there is just a single light scalar degree of freedom outside the horizon (i.e. no entropic modes). In this case, we may set the initial condition on \( \delta_m^{(2)} \) by the fact that all scalar fluctuations arise from the time-shift of their background. More precisely, for adiabatic initial conditions, all scalar fluctuations outside the horizon arise from a single scalar fluctuation, which may be parameterized as the pion \( \delta t \) for the spontaneously broken time translations \( \delta_t \). Thus, for either matter or radiation, \( \bar{\rho} + \delta \rho = \bar{\rho}(t + \delta t) = \bar{\rho}(t) + \delta t \bar{\rho}(t) + \frac{1}{2} \delta t^2 \bar{\rho}(t) + \ldots \). At first order, \( \delta t^{(1)} = \frac{\delta \rho}{\bar{\rho}} \) is therefore equal for matter and radiation, and this implies the usual adiabatic relation \( \delta_m = \frac{3}{4} \delta_r \) initially. At second order, one must equate \( \delta t^{(2)} = \frac{\delta \rho^{(2)}}{\bar{\rho}} - \bar{\rho} \left( \frac{\delta \rho^{(1)}}{\bar{\rho}} \right)^2 \), and therefore the initial condition for \( \delta_m^{(2)} \) is

\[
\delta_m^{(2)} = \frac{3}{4} \delta_r^{(2)} - \frac{3}{16} \delta_r^{(1)2}.
\]  

(34)
In principle, the equations of motion \((24-29)\) and the initial conditions are all that is needed to find the second order kernel on any scale. In practice, however, we also must specify the gauge in which \(\delta_m^{(2)}\) is actually measured in observations. While it is straightforward now to calculate \(\delta_m^{(2)}\) in conformal Newtonian gauge, it is unlikely that this is the quantity we are interested in. We shall not determine in this paper which is the correct gauge for \(\delta_m\) corresponding to measurements (or, put differently, what is the correct gauge-invariant observable measured in surveys). Rather, we shall treat gauge dependence as an additional source of uncertainty. What this means requires some care, as one can always choose the gauge \(\delta_m = 0\), where the matter density provides the clock for the coordinate system. We shall avoid such extreme gauge choices and instead consider the change in \(\delta_m^{(2)}\) going from Newtonian to synchronous gauge. On subhorizon scales \(k/H \gg 1\), perturbations become increasingly local and insensitive to the gauge choice, and \(\delta\) from one gauge to the next changes as \(\sim H^2 k^2 \delta\). For example, under a time diffeomorphism \(\eta \rightarrow \eta + \alpha, \Psi \) changes at first order by \(\Psi \rightarrow \Psi - H\alpha - \alpha'\), and \(\delta\) changes at second order by \(\delta \rightarrow \delta + 3H\alpha(1 + \delta) + \ldots\). Thus inside the horizon during the matter era, a gauge transformation that changes the metric \(\Psi\) by \(\mathcal{O}(1)\) has \(H\alpha \sim \Psi\), and consequently \(\delta^{(2)} \sim 3\Psi^{(1)}\delta^{(1)} \sim 3\frac{(\delta^{(1)})^2}{M_k} \sim 5\frac{H^2}{k^2}(\delta^{(1)})^2\). Thus the physical significance of contributions to \(F_2\) that behave parametrically as \(H^2/k^2\) is unclear, and much additional work is required to understand these contributions.

4 Results

4.1 Numeric Results

The second order kernel \(F_2\) is a function of two three-momenta \(\vec{q}_1\) and \(\vec{q}_2\) and is thus naively a function of six variables. However, due to rotational invariance, it is in fact a function of only three variables, which we will choose to be \(k, x_1 \equiv \frac{\vec{q}_1}{k}\), and \(x_2 \equiv \frac{\vec{q}_2}{k}\). In the limit of short wavelengths and negligible radiation, \(F_2\) approaches the PT result of equation \((5)\), which is scale invariant and thus depends only on \(x_1\) and \(x_2\). Note that \(1 + x_2 > x_1 > 1 - x_2\) by momentum conservation and we may take \(x_1 > x_2\) without loss of generality by symmetry. The kernel \(F_2(q_1, q_2)\) enters the three-point function through

\[
\langle \delta_{k_1} \delta_{k_2} \delta_{k_3} \rangle = \delta \left( \sum_i \vec{k}_i \right) (2\pi)^3 \left( F_2(k_1, k_2) P_{k_1} P_{k_2} + \text{cyc.} \right)
\]

\[
= B_G(k_1, k_2, k_3) \times (2\pi)^3 \delta \left( \sum_k \vec{k}_i \right),
\]

(35)

and in the three-point function one may further take \(k_1 > k_2 > k_3\) without loss of generality. It is convenient to introduce the reduced three-point function

\[
Q(k_1, k_2, k_3) = \frac{B_G(k_1, k_2, k_3)}{P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_1)P(k_3)}.
\]

(36)
Figure 1: The kernel $x_1^{-1} x_2^{-1} Q(x_1, x_2)$ for $k$ fixed at $k = 11 k_{eq}$. The left plot is from our numeric computation, and is indistinguishable by eye from the PT limit given in equation (5). The right plot is the difference between the numeric result and the PT result, rescaled by $a_{eq}$ because by equation (6), $a_{eq}$ controls the size of the radiation correction.

$Q(k_1, k_2, k_3)$ is identical to $F_2(k, k)$ on equilateral triangles. Unlike $F_2$, though, it is symmetric under permutations of $k_1, k_2, k_3$, and thus takes into account some cancellations in the three-point function between $F_2(k_1, k_2)$ and its permutations.

A plot of $x_1^{-1} x_2^{-1} Q(x_1, x_2)$ from our numeric computation with $k$ fixed at $11 k_{eq}$ is plotted in Figure 1 along with a plot of the difference between our numeric result and the PT result. The observed value of $k_{eq}$ is $0.013 h \text{Mpc}^{-1}$. At $k = 11 k_{eq}$, the difference is very close to that obtained in Newtonian gravity with radiation, equation (18), shown in Figure 2. This indicates that $k / k_{eq} \gtrsim 10$ is already large enough that the approximation used in section 2.2 is good.

The result from our numeric method takes into account not only radiation corrections but also corrections from general relativity (GR), the leading contributions of which are post-Newtonian (PN) corrections. The scale-dependence of these two effects is completely different. GR effects are important on larger scales and behave parametrically as $(H/k)^2$ compared to the PT kernel. We can partially isolate such effects by using our numeric method with $\Omega_r = 0$ (equivalently, $a_{eq} = 0$), and taking the initial conditions appropriate for a matter-only universe. In fact, in this case the second order perturbations may be solved for analytically, as in [30, 31, 34], and seen explicitly to be suppressed by $(H/k)^2$ with respect to the leading correction.

---

4 We define the “difference” $\delta Q$ as the change from corrections to $F_2$, and not from corrections to the power spectrum $P(k)$. Explicitly,

$$\delta Q = \frac{\delta F_2(k_1, k_2) P(k_1) P(k_2) + \delta F_2(k_2, k_3) P(k_2) P(k_3) + \delta F_2(k_3, k_1) P(k_3) P(k_1)}{P(k_1) P(k_2) + P(k_2) P(k_3) + P(k_3) P(k_1)} ,$$

$$\delta F_2(q_1, q_2) = F_{2, \text{exact}} - F_{2, \text{PT}} .$$

(37)
Figure 2: The log-enhanced difference between the PT kernel and the kernel from using Newtonian gravity with a background radiation component. The analytic result is given in equation (18).

piece, as shown in appendix A, (eq. 60):

\[
\frac{1}{2} \delta_k^{(2)} = \left[ \left( \beta_k - \alpha_k \right) + \frac{\beta_k}{2} \left( \hat{q}_1 \cdot \hat{q}_2 \right) \right] \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \alpha_k \left( \hat{q}_1 \cdot \hat{q}_2 \right)^2 + \gamma_k \left( \frac{q_1}{q_2} - \frac{q_2}{q_1} \right)^2 \left( \frac{\delta q_1^{(1)}}{1 + \frac{H}{k^2}} \right) \left( \frac{\delta q_2^{(1)}}{1 + \frac{H}{k^2}} \right),
\]

\[\alpha_k = \frac{2}{7} + \frac{59H^2}{14k^2} + \frac{45H^4}{2k^4}, \quad \beta_k = 1 - \frac{H^2}{2k^2} + \frac{54H^4}{k^4}, \quad \gamma_k = -\frac{3H^2}{2k^2} + \frac{9H^4}{2k^4}. \]  

(38)

As we have discussed, however, such effects are gauge-dependent and require more care in order to be written in terms of present-day observables. The effect of radiation, on the other hand, is nearly independent of the magnitude of \(k\) for modes that entered the horizon in the radiation era and approaches a constant correction at small scales. In order to give a sense of the scale-dependence, we may fix the shape of the triangle formed by \(k, q_1, \) and \(q_2\). In Figure 3, we take \(k_2 = k_3 = yk, k_1 = k\) and plot the correction to \(Q\) as a function of \(k/H\). For comparison, we also show the correction in a matter-only universe as well as the leading correction (18) in Newtonian gravity when radiation is included in the background (but radiation perturbations are neglected). For our choice of parameters, \(k_{eq}/H \approx 80\), and the radiation correction starts to dominate over PN effects between about \(k_{eq}\) and 10 \(k_{eq}\), depending on the triangle shape. The small parameter suppressing the PN effects is \((H/k)^2\), and thus they are larger than the radiation effects when \((H/k)^2 \gtrsim a_{eq} \sim (H/k_{eq})^2\), i.e. when \(k \lesssim k_{eq}\). The radiation effects and PN effects have opposite sign for equilateral and squeezed triangles, and thus the total correction crosses through zero in the transition region. One can see that the second order perturbation \(\delta^{(2)}\) approaches our simple approximation (eq. 18). In fact, from Figure 4 we see that the corrections over the entire range of \(k\) are well-approximated by the sum of the radiation corrections (eq. 18) and the GR corrections (eq. 60): \(\delta Q \approx \delta Q_{2,rad} + \delta Q_{2,GR}\). Note that, although the radiation correction to \(Q_2\) grows as a function of redshift a little slower than \((1+z)\), the correction from primordial nongaussianities
Figure 3: The correction $|\delta Q|$ for $k_1 = k_2 = yk_3, k_3 \equiv k$ with $y$ fixed at $\frac{1}{2}, 1, 10$, from left to right, respectively. The correction is shown for a) a matter-only universe ($a_{eq} = 0$), red, dotted, b) the exact numeric correction, black, solid, and c) the difference in Newtonian gravity between a universe with matter and one with matter + radiation (blue, dot-dashed, for which the log-enhanced piece of equation 18 vanishes when $y = \frac{1}{2}$; the size is independent of $k$ because it is the difference between two results within Newtonian gravity). The black dashed line is the exact radiation effect, defined as the difference between the exact numeric correction and the matter-only correction. The exact numeric calculation has $a_{eq} = 3 \times 10^{-4}$, in which case $k_{eq}/H \approx 80$.

has nearly the same $z$ dependence and their ratio is nearly flat, as shown in Figure 5.

Finally, we show the comparison between the corrections to $\delta Q$ and the contribution from primordial nongaussianities with $f_{NL} = 1$ in Figure 6. The correction from radiation becomes the dominant correction only around $k \gtrsim 3k_{eq}$. At $k = 10k_{eq} = 0.13 h {\text{Mpc}}^{-1}$, which is comparable to the largest $k$ that next-generation LSS observations will be able to resolve [17, 22], the correction from radiation is comparable to $f_{NL} \approx 4.6$ for equilateral triangles.

4.2 Including $\Lambda$

So far, we have considered a universe with $\Omega_m + \Omega_r = 1$, which is only a good approximation up until $\Omega_\Lambda$ grows to be non-negligible. To calculate the effect of radiation in our universe with vacuum energy $\Lambda \neq 0$, one simply includes the effect of $\Lambda$ on the background, which modifies $H(\eta)$. Otherwise, the equations of motion for the linear and second order solutions are unchanged.

In a universe with matter and $\Lambda$, the Newtonian equations of motion do not admit an exact solution which is separable, i.e. of the form $\delta_k^{(2)}(\eta) \sim F_2(q_1, q_2)\delta_1(q_1)\delta_1(q_2)(D_+^{(1)}(\eta))^2$. It has been argued that, to good approximation, they are almost separable (see e.g. [18]), and $D_+^{(1)}(\eta)$ takes the form of the (matter + $\Lambda$) linear solutions whereas $F_2(q_1, q_2)$ is the same
as the PT result. A better approximation is to calculate \( \delta^{(2)} \) exactly in Newtonian gravity, matching the linear solutions onto the matter era growing modes \( \delta \propto a \) at \( a \ll 1 \). When we compute \( \delta^{(2)} \) numerically in this way, we find that the result is exactly matched by the form

\[
\frac{1}{2} F_2(q_1, q_2; \Omega_m) = \epsilon(\Omega_m) + \frac{q_1 \cdot q_2}{2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + (1 - \epsilon(\Omega_m)) (q_1 \cdot q_2)^2 ,
\]

where \( \epsilon(\Omega_m) \) is determined numerically. A parameterization of \( \epsilon(\Omega_m) \) that turns out to work well is

\[
\epsilon_{\text{exact}}(\Omega_m) \approx \epsilon_{\text{fit}}(\Omega_m) = \frac{5}{7} + c (1 - \Omega_m^{1/15}) , \quad c = 0.023969 .
\]

As we show in Figure 7, this parameterization fits the exact numeric \( \epsilon(\Omega_m) \) to better than 0.0035% for \( 0.2 < \Omega_m < 1 \).

The exact Newtonian result provides a much better approximation to the second order density perturbations than the PT result does. Since the linear modes are normalized at their current values, the PT three-point function today is insensitive to the value of \( \Omega_m \). For comparison, in Figure 8 we show the difference between the PT result and the exact result for the reduced three-point function, as well as the difference between the Newtonian result and the exact result. Including \( \Lambda \), the Newtonian result is still only off by about the equivalent of \( f_{\text{NL}} \approx 4 \), due to radiation. However, the unmodified PT result of equation (5) is off by over an order of magnitude more.
Figure 5: The correction $|\delta Q|$ relative to the primordial contribution with $f_{NL} = 1$ on equilateral triangles as a function of redshift $z$. The wavenumber in the left plot is, from top to bottom, $k = k_{eq} \times \{30, 10, 4, 3, 2, 3\}$. As $k$ decreases from $30k_{eq}$, the GR corrections grow and approximately cancel the radiation corrections near $k \sim 2k_{eq}$. At smaller $k$, the correction from GR is well-approximated by the matter-only analytic result \((59)\). The right plot shows the exact result relative to $\delta Q_{f_{NL}=1}$ for $k = 0.4k_{eq}$, where essentially all the correction is from GR. The reason the correction in the left plot decreases at large $z$ is that $\delta/\phi$ gets smaller in the past, so the contribution from $f_{NL}$ is bigger in terms of $\delta$.

Figure 6: The corrections $|\delta Q|$ for $k_1 = k_2 = yk_3 \equiv yk$ with $y$ fixed at $\frac{1}{2}, 1, 10$, for left, middle right, respectively, compared to the contribution from primordial nongaussianities with $f_{NL} = 1$. The exact numeric correction to the PT kernel is dashed, whereas the contribution from $f_{NL}$ is shown in solid.

4.3 Leading Effect of Non-Gaussianities on Galaxy Power Spectrum

Ultimately, the dark matter three-point function must be related to observables. In \([25]\), it was proposed that non-gaussianities in the dark matter distribution might best be measured through the additional scale-dependence they induced in the galaxy two-point function. This
Figure 7: Comparison of the exact numeric result for $\epsilon(\Omega_m)$ compared with the parameterization $0.023969(1 - \Omega_m^{1/15})$ for a flat universe with $\Lambda$ and matter only.

Figure 8: The corrections in a flat universe with $\Omega_{m,0} = 0.27, \Omega_{r,0} = 3 \times 10^{-4}\Omega_{m,0}$ to $|\delta Q|$ for $k_1 = k_2 = yk_3 \equiv yk$ with $y$ fixed at $\frac{1}{2}, 1, 10$, for left, middle, right, respectively, compared to the contribution from primordial nongaussianities with $f_{NL} = 1$. The black, solid line is the difference between the exact result and the PT result. The blue, dot-dashed line is the difference between the exact result and the Newtonian approximation of equation (39). The contribution from $f_{NL} = 1$ is shown in dashed, black. The straight red, dotted line is the Newtonian radiation correction of section 2.2, including the non-log-enhanced piece. The PT result is very good on squashed triangles, but off by more than an order of magnitude more than the Newtonian approximation on equilateral and squeezed triangles at large $k$.

approach has the advantage that it is possible to write the effect in terms of quantities that may be studied with exactly gaussian statistics.

The reason for the leading effect on the two-point function is that non-gaussianities cor-
relate the long- and short-wavelength fluctuations. With Gaussian fields, a long-wavelength fluctuation looks like a background fluctuation that increases or decreases the amount of short-wavelength fluctuation needed to form a bound object [38]. However, when different modes are correlated, a long-wavelength fluctuation tends to increase or decrease the actual size of the short-wavelength fluctuation. More precisely, with \( k_L \ll k_S \), \( \delta_{k_L} \) is a long-wavelength mode that affects the variance of the short-wavelength mode \( \delta_{k_S} \):

\[
\frac{\partial}{\partial \delta_{-k_L}} \langle \delta_{k_1} \delta_{k_2} \rangle = \left( (\delta F_2(k_1, k_L) + \frac{2f_{NL}}{M_{k_L}})P_{k_1} + (\delta F_2(k_2, k_L) + \frac{2f_{NL}}{M_{k_L}})P_{k_2} \right) \times \delta \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_L \right),
\]

(41)

where the subscript \( 0 \) means that the expectation value is taken with \( \delta_{k_L} \) equal to zero, and where we have included the contribution from \( f_{NL} \) through \( \delta^{(2)}_{k_S} = M_{k_S}(\Phi^{(1)}_{k_S}+f_{NL}(\Phi^{(1)}_{\Phi^{(1)}})_{k_S}) \) in linear perturbation theory, with the * indicating a convolution. We are also concentrating on the difference between the effect due to radiation and general relativity with respect to the standard newtonian treatment. The value of the short-wavelength mode variance determines the local value of the rms density fluctuations, i.e. \( \sigma_8 \), and thus the total effect of the long wavelength perturbation is (see for a more precise derivation [25, 39]):

\[
\frac{dn}{d\delta_{-k_L}} = \left( \frac{dn}{d\delta_{-k_L}} \right)_0 + \left( \frac{1}{2P_{k_S}} \frac{\partial}{\partial \delta_{-k_L}} \langle \delta_{k_1} \delta_{k_2} \rangle \right) \frac{\partial n}{\partial \log \sigma_8}. \tag{42}
\]

The point of this expression is that the non-gaussianity induces a bias, and that this can be expressed in terms of the unknown quantities \( (dn/d\delta_{-k_L})_0 \) and \( \partial n/\partial \log \sigma_8 \) that may be calculated in Newtonian gravity with (matter+\( \Lambda \)) only and \( f_{NL} = 0 \). In the squeezed limit, the correction to \( dn/d\delta_{k_L} \) from radiation approaches the value \( 8/35 \cdot a_{eq} \log a_{eq} \sin^2 \theta \) times \( \partial n/\partial \log \sigma_8 \) for \( P(k) \propto k^{-3} \) in the approximation of (18), where \( \theta \) is the angle between \( k_L \) and \( k_2 \). For isosceles triangles in the squeezed limit, \( \theta = \pi/2 \); the more precise derivation in [39] demonstrates that \( \theta \) should be averaged over in the bias with measure \( \sin \theta d\theta \). When \( k_L \gtrsim 10k_{eq} \), the contribution to the same quantity from \( f_{NL} \) is \( 2f_{NL}/M_{k_L} \) times \( \partial n/\partial \log \sigma_8 \), which is a significantly smaller contribution for large \( k \) and \( f_{NL} = 1 \). However, the contribution from \( f_{NL} \) is \( k \)-dependent whereas the leading radiation contribution is not. Thus, the leading radiation contribution appears as just an overall rescaling of the galaxy bias and does not contaminate the signal for \( f_{NL} \) when \( f_{NL} \) is fit to match the shape of the galaxy power spectrum. Such contamination occurs only at sub-leading order and is comparable to \( f_{NL} \approx 1 \). This can be seen in Figure 9, where we show \( \delta Q \), which is proportional to the bias, after subtracting off the limiting value of \( \delta Q \) at large \( k \), and we compare it with the analogous effect from \( f_{NL} = 1 \). Since the effect is order one, as we explained before, it becomes comparable to the GR effects that we expect to affect the distribution of collapsed objects at this order.

5 Discussion

Our main goal was to quantify the contribution from radiation to nongaussianities in the dark matter distribution compared with those from primordial nongaussianities. We have found
Figure 9: The corrections $|\delta Q| - |\delta Q_{k \gg k_{eq}}|$ for $k_1 = k_2 = 10k_3$, compared to the contribution from primordial non-Gaussianities with $f_{NL} = 1$. The correction is shown (solid, black) for the exact numeric result with $\Omega_{m,0} = 0.27, \Omega_{r,0} = 3 \times 10^{-4} \Omega_{m,0}$. The contribution from $f_{NL}$ is shown in blue, dashed.

that the effect on the three-point function for modes with $k \approx 10 k_{eq}$ is comparable to $f_{NL} \approx 4$. At shorter wavelengths, the contribution from primordial non-Gaussianities shrinks like $1/\log k$ while the contribution from radiation becomes scale-independent. Non-Gaussianities may also be measured through the additional scale-dependence they induce on the power spectrum. Such scale dependence arising from radiation corrections is comparable to $f_{NL} = 1$. We have also compared the size of primordial non-Gaussianities with the relativistic corrections in Newtonian gauge to the nonlinear growth of non-Gaussianities. Such corrections are negligible at large $k \gg k_{eq}$ but at smaller than $k \lesssim k_{eq}$ are comparable to $f_{NL} \approx 4$ on equilateral triangles. However, at small $k$ the effect on $\delta \rho/\rho$ of a gauge transformation from Newtonian to synchronous gauge becomes large and parametrically the same as the relativistic corrections. Thus, to clearly interpret this result, $\delta \rho/\rho$ must be related to present-day observables.

We have made a number of approximations to simplify the analysis. We neglected baryons and neutrinos, as well as higher order photon moments. We also have focussed on the dark matter density perturbations, which must eventually be related to visible objects. Furthermore, the density field $\rho(x)$ we have been using is the local inertial density, which will receive volume and redshift distortions from the local metric and peculiar velocity, as well as distortions along the line-of-sight. If observations of non-Gaussianities are to be interpreted with uncertainty less than $\Delta f_{NL} \lesssim 4$, then such distortions most likely must be understood and quantified as well.
Acknowledgments

We thank S. Tassev for sharing preliminary results on second order perturbations and D. Baumann for comments on the draft. ALF was partially supported by DOE grant DE-FG02-01ER-40676, and an NSF graduate research fellowship. LS was supported in part by the National Science Foundation under Grant No. PHY-0503584. MZ was supported by NASA NNG05GJ40G and NSF AST-0506556 as well as the David and Lucile Packard, Alfred P. Sloan and John D. and Catherine T. MacArthur foundations.

Appendix

A Second Order $\delta \rho / \rho$ in a Matter-Only Universe

The form of the second order perturbations in a matter-only universe has been previously derived in [30, 31] using the formalism we adopt in this paper, and also independently in [34] using an action approach. The result can be written analytically in this case. For the convenience of the reader, we present and comment on the results in our formalism.

In addition to setting $\Omega_r = 0, \Omega_m = 1$ in our equations of motion and ignoring the radiation perturbations, one must also choose appropriate initial conditions. We separate out the propagation from the initial conditions by using the method of Green’s functions, as follows. First, it is useful to supplement the equations of motion we have so far with the $G_{i0}$ component of Einstein’s equation

$$\mathcal{H} k^2 \Psi^{(2)} + k^2 \Phi^{(2)} - \frac{3}{2} \mathcal{H}^2 i k V^{(2)} = -S_5(k, \eta) \equiv 2(k \cdot q_2) \Psi^{(1)}_{q_1} \Psi^{(1)}_{q_2} + 3 \mathcal{H}^2 i (k \cdot \hat{q}_1) V^{(1)}_{q_1} \delta^{(1)}_{q_2} .$$

For reference, we present all components of the Einstein tensor for a metric in Newtonian gauge with only scalar modes:

$$a^2 G^0_0 = -e^{-2\Psi} 3(\mathcal{H} - \Phi')^2 + e^{2\Phi} ((\partial \Phi)^2 - 2\partial^2 \Phi) ,$$

$$a^2 G^i_j = \delta^i_j \left[ -e^{-2\Psi} ((\mathcal{H} - \Phi')(\mathcal{H} - 3\Phi' - 2\Psi') + 2(\mathcal{H}' - \Phi'')) + e^{2\Phi} ((\partial \Psi)^2 + \partial^2 \Psi - \partial^2 \Phi) \right] + e^{2\Phi} \left[ -\partial_i \Psi \partial_j \Psi + \partial_i \Phi \partial_j \Phi - 2 \partial_i \Phi \partial_j \Psi - \partial_i \partial_j (-\Phi + \Psi) \right] ,$$

$$G_{i0} = 2(\mathcal{H} - \Phi') \partial_i \Psi + 2 \partial_i \Phi' .$$

It is now straightforward to obtain a second order equation for $\delta^{(2)}$ sourced by the first order perturbations. We first use equation (25) to eliminate $\Psi^{(2)} = S_4 + \Phi^{(2)}$. Then from (24,43), we have

$$V^{(2)} = \frac{i 2 k^2 S_3 - 6 \mathcal{H} S_5 + 3 k^2 \mathcal{H}^2 \delta^{(2)} + 2 k^4 \Phi^{(2)}}{9 k \mathcal{H}^3} .$$

Eliminating $\Phi^{(2)}$ from equations (26,27), we obtain $\Phi^{(2)}$ in terms of $\delta^{(2)}, \delta'^{(2)}, S_i$, which upon substitution back into (26) gives the desired equation for $\delta^{(2)}$. Using that in a matter-only
universe, \( \mathcal{H} = 2/\eta \), we obtain

\[
\delta^{(2)\mu} + \frac{1296 + 72k^2\eta^2 - 2k^4\eta^4}{216\eta + 18k^2\eta^2 - k^4\eta^4} \delta^{(2)\nu} + \frac{6k^2(-42 + k^2\eta^2)}{216 + 18k^2\eta^2 - k^4\eta^4} \delta^{(2)} = \mathcal{F}[S_i],
\]

(48)

where \( \mathcal{F}[S_i] \) is a function of the source terms above, and therefore quadratic in the first order solutions. It is fairly long, but we will see that the final answer simplifies considerably when we substitute the first order solutions. The above equation is a second order sourced equation of motion for \( \delta^{(2)} \) and thus its solution is given by

\[
\delta^{(2)} = c_1^{(2)} \delta_1(\eta) + c_2^{(2)} \delta_2(\eta) + \int_0^\eta \mathcal{F}[S_i](\eta')G(\eta, \eta')d\eta',
\]

(50)

where \( G(\eta, \eta') \) is the Green’s function

\[
G(\eta, \eta') = \frac{\delta_1(\eta)\delta_2(\eta') - \delta_1(\eta')\delta_2(\eta)}{\delta_1(\eta')\delta_2(\eta') - \delta_1(\eta)\delta_2(\eta')},
\]

(51)

and \( \delta_1, \delta_2 \) are the growing and decaying mode homogeneous solutions

\[
\delta_1(\eta) = 12 + (k\eta)^2,
\]

(52)

\[
\delta_2(\eta) = \frac{-18 + (k\eta)^2}{(k\eta)^5}.
\]

(53)

The homogeneous solutions for \( \delta^{(2)} \) are the same as the homogeneous (growing and decaying) mode solutions for \( \delta^{(1)} \), and indeed for \( \delta^{(n)} \) at any order, since their equations of motion only differ by the source term \( \mathcal{F} \).

At first order, the equations of motion may easily be solved, and one finds that \( \Phi \) is constant and the matter perturbation growing mode is

\[
\delta^{(1)}_k = -(12 + \eta^2 k^2) \left( \frac{\Phi}{6} \right),
\]

(54)

\[
V^{(1)}_k = -(2ik\eta) \left( \frac{\Phi}{6} \right).
\]

(55)

\footnote{Explicitly,}

\[
\mathcal{F}[S_i] = \left( k^2 \eta \left( k^2 \eta^2 + 18 \right) \left( k^4 \eta^4 - 18k^2\eta^2 - 216 \right) \right)^{-1} \times
\]

\[
\left( -\eta^9 S_4(\eta)k^{12} + \eta^9 S_3(\eta)k^{10} + \eta^9 S_1(\eta)k^{10} + 1188\eta^3 S_4(\eta)k^8 - 6\eta^8 S_5(\eta)k^8
\]

\[
-6\eta^8 S_3(\eta)k^8 - 108\eta^6 S_5(\eta)k^7 - 3\eta^9 S_5(\eta)k^7 - 1188\eta^3 S_3(\eta)k^6 + 3888\eta^3 S_4(\eta)k^6
\]

\[
+540\eta^7 S_1(\eta)k^6 + 18\eta^8 S_3(\eta)k^6 - 3888\eta^7 S_3(\eta)k^4 + 4288\eta^7 S_4(\eta)k^4 + 9672\eta^4 S_1(\eta)k^4 + 58320\eta^2 S_2(\eta)k^4 + 1620\eta^5 S_3(\eta)k^4
\]

\[
-2(\eta^8 k^8 - 1188k^4\eta^4 - 3888\eta^2 k^2 + 116640) S_2(\eta)k^2 + 58320i\eta^2 S_2(\eta)k^2
\]

\[
-2(\eta^8 k^8 - 1188k^4\eta^4 - 3888\eta^2 k^2 + 116640) S_2(\eta)k^2 + 1620\eta^5 S_3(\eta)k^2
\]

\[
+62208\eta^2 S_3(\eta)k^2 + 93312\eta^2 S_3(\eta)k^2 + 419904 S_1(\eta)k^2 - 38880\eta^3 S_3(\eta)k^2 - 1664\eta^3 S_3(\eta)k^2
\]

\[
-9720\eta^4 S_3''(\eta)k^2 + 419904i S_2(\eta)k - 279936 S_2(\eta) - 419904\eta S_3(\eta) + 69984\eta^2 S_3''(\eta)
\]

(49)
Upon substituting these into the source term $\mathcal{F}$, the Green’s function integral may be performed analytically:

$$
\frac{1}{2} \int_0^\eta \mathcal{F}[S](\eta')G(\eta, \eta')d\eta' = \left[ \left( \frac{k^4}{14} + \frac{3}{28} (q_1^2 + q_2^2) k^2 - \frac{5}{28} (q_1^2 - q_2^2)^2 \right) \eta^4 
+ \left( -\frac{23k^2}{7} - \frac{167}{14} (q_1^2 + q_2^2) + \frac{45(q_1^2 - q_2^2)^2}{14k^2} \right) \eta^2 \right] \left( \Phi_p^{(1)}(q_1)\Phi_p^{(1)}(q_2) \right)
$$

(56)

where we have factored out $\Phi^2/36$ in order to more easily compare the leading term with the PT kernel. To obtain the correct initial conditions, we take the early-time super-horizon limit $\eta \to 0, k \to 0$ as before (see paragraph preceding equation (33)):

$$
\delta_m^{(2)} = -\frac{2}{3\mathcal{H}^2}S_3 - 2S_4 - 2\Phi^{(2)}, \\
\psi^{(2)} = \Phi^{(2)} + S_4.
$$

(57)

Outside the horizon during the matter era, $\zeta = -\Phi - \frac{2}{3}\Phi = -\frac{5}{3}\Phi - \frac{1}{3}S_4$ and thus $\Phi^{(2)} = -\frac{2}{3}S_4$ initially. Using the limiting values of $S_3 = -6\mathcal{H}^2\Phi_q^{(1)}$, $S_4 = -\frac{10}{k^2}((\dot{k}q_1\dot{k}q_2 - \frac{1}{3}q_1q_2)\Phi_q^{(1)}$, we obtain the initial condition for $\delta^{(2)}$:

$$
\delta^{(2)}_{\text{init}} = \left( \frac{5k^4 - 3(q_1^2 - q_2^2)^2}{k^4} \right) \Phi_q^{(1)}\Phi_q^{(1)}.
$$

(58)

Since the initial conditions are set at $\eta \to 0$, there is no component of decaying mode in equation (50), and thus $c_1 = \delta^{(2)}_{\text{init}}/12$. Putting everything together, we finally obtain

$$
\frac{1}{2}\delta^{(2)} = \left[ \left( \frac{k^4}{14} + \frac{3}{28} (q_1^2 + q_2^2) k^2 - \frac{5}{28} (q_1^2 - q_2^2)^2 \right) \eta^4 
+ \left( \frac{59k^2}{14} - \frac{125}{14} (q_1^2 + q_2^2) - \frac{9(q_1^2 - q_2^2)^2}{7k^2} \right) \eta^2 
+ \left( 90 + 36(q_1^2 + q_2^2) - 54(q_1^2 - q_2^2)^2 \right) \right] \left( \Phi_p^{(1)}(q_1)\Phi_p^{(1)}(q_2) \right)
$$

(59)

This agrees with the expression for $\delta\rho/\rho$ one may obtain from the second-order metric in (34). We may rewrite this in a slightly different form to more easily compare with the PT result of equation (5), which corresponds to the $k\eta \to \infty$ limit:

$$
\frac{1}{2}\delta^{(2)} = \left[ (\beta_k - \alpha_k) + \frac{\beta_k}{2}(q_1\cdot q_2) \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \alpha_k(q_1\cdot q_2)^2 + \gamma_k \left( \frac{q_1}{q_2} - \frac{q_2}{q_1} \right)^2 \right]
$$

$$
\times \left( \frac{\delta^{(1)}_{q_1}}{1 + \frac{3\mathcal{H}^2}{q_1^2}} \frac{\delta^{(1)}_{q_2}}{1 + \frac{3\mathcal{H}^2}{q_2^2}} \right),
$$

$$
\alpha_k = \frac{2}{7} + \frac{59\mathcal{H}^2}{14k^2} + \frac{45\mathcal{H}^4}{2k^4}, \quad \beta_k = 1 - \frac{\mathcal{H}^2}{2k^2} + \frac{54\mathcal{H}^4}{k^4}, \quad \gamma_k = -\frac{3\mathcal{H}^2}{2k^2} + \frac{9\mathcal{H}^4}{2k^4}.
$$

(60)
A.1 Squeezed Limit of $\delta^{(2)}$

We note one immediate check of equation (59) based on the “squeezed” limit $q_1 \sim k \gg q_2$, at leading order. In this limit, $\Phi_{q_2}$ is a very long wavelength mode whose only physical effect is to modify the physical time and wavelength of the short wavelength mode $[1, 36, 37]$. More precisely, coordinates $(\eta, x)$ with a long wavelength background $\Phi_{k_l}$ are equivalent to the modified coordinates $(\eta', x')$, where

$$a(\eta')dx' = (1 - \Phi_{k_l})a(\eta)dx ,$$
$$a(\eta')d\eta' = (1 + \Phi_{k_l})a(\eta)d\eta .$$

(61)  
(62)

During the matter-era, $a(\eta) \propto \eta^2$, and thus

$$\eta' = \eta(1 + \frac{1}{3}\Phi_{k_l}) ,$$
$$x' = x(1 - \frac{5}{3}\Phi_{k_l}) .$$

(63)  
(64)

The effect of this coordinate redefinition on the short wavelength matter density $\rho_{k_s}$ is

$$\rho_{k_s} \rightarrow \rho_{k_s} + \frac{1}{3}\Phi_{k_l} \frac{\partial \rho_{k_s}}{\partial \log \eta} + \frac{5}{3}\Phi_{k_l} \frac{\partial \rho_{k_s}}{\partial \log k_s}$$

$$= \bar{\rho}\delta_{k_s} + \rho\frac{1}{3}\Phi_{k_l} \left( -3\eta\mathcal{H}\delta_{k_s} + \eta\delta_{k_s}' \right) + \frac{5}{3}\Phi_{k_l}\bar{\rho}_s \delta_{k_s} \frac{\partial \delta_{k_s}}{\partial k_s}$$

$$= \bar{\rho} \left( \delta + 2(72 - 6k_s^2\eta^2) \left( \frac{\Phi_{k_l}\Phi_{k_s}}{36} \right) \right) .$$

(65)

In terms of the explicit expression for $\delta^{(2)}$ (eq. 59), the effect of the long wavelength mode $\Phi_{k_l}$ is obtained from $\frac{1}{2}\delta^{(2)}(q_1 = k_s, q_2 = k_l) + \frac{1}{2}\delta^{(2)}(q_1 = k_l, q_2 = k_s)$ in the limit $k_l \ll k_s$, and we find

$$\frac{1}{2}\delta^{(2)}_{k_s} \rightarrow 2(72 - 6k_s^2\eta^2) \left( \frac{\Phi_{k_l}\Phi_{k_s}}{36} \right) ,$$

(66)

which indeed agrees with equation (65).

References

[1] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305 (2003) 013 [astro-ph/0210603].

[2] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, “Second-order cosmological perturbations from inflation,” Nucl. Phys. B 667, 119 (2003) [arXiv:astro-ph/0209156].

[3] N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, “Ghost inflation,” JCAP 0404, 001 (2004) [hep-th/0312100]. L. Senatore, “Tilted ghost inflation,” Phys. Rev. D 71 (2005) 043512 [astro-ph/0406187].
[4] M. Alishahiha, E. Silverstein and D. Tong, “DBI in the sky,” Phys. Rev. D 70, 123505 (2004) [hep-th/0404084].

[5] X. Chen, M. x. Huang, S. Kachru and G. Shiu, “Observational signatures and non-Gaussianities of general single field inflation,” JCAP 0701 (2007) 002 [hep-th/0605045].

[6] M. Zaldarriaga, “Non-Gaussianities in models with a varying inflaton decay rate,” Phys. Rev. D 69, 043508 (2004) [astro-ph/0306006].

[7] P. Creminelli, L. Senatore and M. Zaldarriaga, “Estimators for local non-Gaussianities,” JCAP 0703 (2007) 019 [astro-ph/0606001].

[8] C. Cheung, P. Creminelli, L. Fitzpatrick, J. Kaplan and L. Senatore “The Effective Field Theory of Inflation,” arXiv:0709.0293 [hep-th].

[9] P. Creminelli, M. A. Luty, A. Nicolis and L. Senatore, “Starting the universe: Stable violation of the null energy condition and non-standard cosmologies,” JHEP 0612 (2006) 080 [hep-th/0606090].

[10] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, “k-inflation,” Phys. Lett. B 458 (1999) 209 [hep-th/9904075].

[11] E. D. Stewart and D. H. Lyth, “A More accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation,” Phys. Lett. B 302 (1993) 171 [gr-qc/9302019].

[12] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, “Ghost condensation and a consistent infrared modification of gravity,” JHEP 0405 (2004) 074 [hep-th/0312099].

[13] P. Creminelli and L. Senatore, “A smooth bouncing cosmology with scale invariant spectrum,” JCAP 0711 (2007) 010 [arXiv:hep-th/0702165].

[14] D. N. Spergel et al., “Wilkinson Microwave Anisotropy Probe (WMAP) three year results: Implications for cosmology,” astro-ph/0603449.

[15] P. Creminelli, L. Senatore, M. Zaldarriaga and M. Tegmark, “Limits on $f_{NL}$ parameters from WMAP 3yr data,” JCAP 0703 (2007) 005 [astro-ph/0610600].

[16] D. Babich and M. Zaldarriaga, “Primordial Bispectrum Information from CMB Polarization,” Phys. Rev. D 70 (2004) 083005 [astro-ph/0408455].

[17] E. Sefusatti and E. Komatsu, “The bispectrum of galaxies from high-redshift galaxy surveys: Primordial non-Gaussianity and non-linear galaxy bias,” Phys. Rev. D 76, 083004 (2007) [arXiv:0705.0343 [astro-ph]].

[18] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro, “Large-scale structure of the universe and cosmological perturbation theory,” Phys. Rept. 367, 1 (2002) [arXiv:astro-ph/0112551].

[19] M. H. Goroff, B. Grinstein, S. J. Rey and M. B. Wise, “Coupling of Modes of Cosmological Mass Density Fluctuations,” Astrophys. J. 311, 6 (1986).

[20] B. Jain and E. Bertschinger, “Second order power spectrum and nonlinear evolution at high redshift,” Astrophys. J. 431, 495 (1994) [arXiv:astro-ph/9311070].

[21] N. Makino, M. Sasaki and Y. Suto, “Analytic approach to the perturbative expansion of nonlinear gravitational fluctuations in cosmological density and velocity fields,” Phys. Rev. D 46, 585 (1992).

[22] R. Scoccimarro, E. Sefusatti and M. Zaldarriaga, “Probing Primordial Non-Gaussianity with Large-Scale Structure,” Phys. Rev. D 69, 103513 (2004) [arXiv:astro-ph/0312286].

[23] E. Komatsu et al. [WMAP Collaboration], “First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Tests of Gaussianity,” Astrophys. J. Suppl. 148 (2003) 119 [arXiv:astro-pl/0302223].
[24] D. Babich, P. Creminelli and M. Zaldarriaga, “The shape of non-Gaussianities,” JCAP 0408 (2004) 009 [astro-ph/0405356].
[25] A. Slosar, C. Hirata, U. Seljak, S. Ho and N. Padmanabhan, “Constraints on local primordial non-Gaussianity from large scale structure,” JCAP 0808, 031 (2008) [arXiv:0805.3580 [astro-ph]].
[26] K. M. Smith, L. Senatore and M. Zaldarriaga, “Optimal limits on $J_{NL}^{\text{local}}$ from WMAP 5-year data,” arXiv:0901.2572 [astro-ph].
[27] U. Seljak, “A Two fluid approximation for calculating the cosmic microwave background anisotropies,” Astrophys. J. 435, L87 (1994) [arXiv:astro-ph/9406050].
[28] S. Dodelson, “Modern Cosmology,” Amsterdam, Netherlands: Academic Pr. (2003) 440 p
[29] N. Bartolo, S. Matarrese and A. Riotto, “Evolution of second-order cosmological perturbations and non-Gaussianity,” JCAP 0401, 003 (2004) [arXiv:astro-ph/0309692].
[30] N. Bartolo, S. Matarrese and A. Riotto, “The Full Second-Order Radiation Transfer Function for Large-Scale CMB Anisotropies,” JCAP 0605, 010 (2006) [arXiv:astro-ph/0512481].
[31] S. Matarrese, S. Mollerach and M. Bruni, “Second-order perturbations of the Einstein-de Sitter universe,” Phys. Rev. D 58, 043504 (1998) [arXiv:astro-ph/9707278].
[32] N. Bartolo, S. Matarrese and A. Riotto, “CMB Anisotropies at Second-Order II: Analytical Approach,” JCAP 0701, 019 (2007) [arXiv:astro-ph/0610110].
[33] N. Bartolo, S. Matarrese and A. Riotto, “Cosmic Microwave Background Anisotropies up to Second Order,” arXiv:astro-ph/0703496.
[34] L. Boubekeur, P. Creminelli, J. Norena and F. Vernizzi, “Action approach to cosmological perturbations: the 2nd order metric in matter dominance,” JCAP 0808, 028 (2008) [arXiv:0806.1016 [astro-ph]].
[35] D. S. Salopek and J. R. Bond, “Nonlinear evolution of long wavelength metric fluctuations in inflationary models,” Phys. Rev. D 42 (1990) 3936.
[36] P. Creminelli and M. Zaldarriaga, “Single field consistency relation for the 3-point function,” JCAP 0410, 006 (2004) [astro-ph/0407059].
[37] C. Cheung, A. L. Fitzpatrick, J. Kaplan and L. Senatore, “On the consistency relation of the 3-point function in single field inflation,” JCAP 0802, 021 (2008) [arXiv:0709.0295 [hep-th]].
[38] A. Cooray and R. K. Sheth, “Halo models of large scale structure,” Phys. Rept. 372, 1 (2002) arXiv:astro-ph/0206508.
[39] S. Matarrese and L. Verde, “The effect of primordial non-Gaussianity on halo bias,” Astrophys. J. 677 (2008) L77 [arXiv:0801.4826 [astro-ph]].