CRITICAL ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. In this unpublished note, we sketch an idea of using a three-piece mollifier to slightly improve the known percentages of zeros and simple zeros of the Riemann zeta-function on the critical line. This uses the recent result of Bettin et al. on the twisted fourth moment of the Riemann zeta-function.

1. Introduction

Let \( \zeta(s) \) be the Riemann zeta-function. Let \( N(T) \) denote the number of zeros of \( \zeta(s) \), \( \rho = \beta + i\gamma \), with \( 0 < \gamma \leq T \) counted with multiplicity. Also let \( N_0(T) \) denote the number of such critical zeros with \( \beta = 1/2 \) and \( N^*_0(T) \) denote the number of such critical zeros with \( \beta = 1/2 \) and being simple. Define \( \kappa \) and \( \kappa^* \) by

\[
\kappa = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}, \quad \kappa^* = \liminf_{T \to \infty} \frac{N^*_0(T)}{N(T)}.
\]

Selberg [11] was the first to prove that a positive proportion of zeros lie on the critical line. Following the approach of Levinson [10] and the observation of Heath-Brown [9], it is now known that \[ \kappa > 0.410 \] and \[ \kappa^* > 0.405 \].

Remark 1.1. Bui et al. [3] showed that \( \kappa > 0.4105 \) and \( \kappa^* > 0.40582 \). Shortly after that, Feng [8] put a paper on arXiv claiming that \( \kappa > 0.4173 \) and \( \kappa^* > 0.4075 \). Feng also used Levinson’s method, but instead chose the mollifier as a sum of various pieces of different shapes \( \sum_{n \leq T} \vartheta_1 + \ldots + \sum_{n \leq T} \vartheta_l \). At this point, Feng took \( \vartheta_1 = \ldots = \vartheta_l = 4/7 - \varepsilon \). The choice \( \vartheta_1 = 4/7 - \varepsilon \) was from Conrey [4], but it was not clear why, say, \( \vartheta_2 < 4/7 \) is admissible. Conrey then emailed Feng requesting for the verification of that statement. Feng later agreed that that was a mistake, and subsequently replaced it with the second version. In this updated version [8; version 2], which is the same as the published one [7], he chose \( \vartheta_1 = 4/7 - \varepsilon \) and \( \vartheta_2 = \ldots = \vartheta_l = 1/2 - \varepsilon \). With this Feng obtained \( \kappa > 0.4128 \), but no claim on the lower bound for \( \kappa^* \). Feng still did not give any explanation as to why, say, the range \( \vartheta_2 < 1/2 \) is admissible. This is doubtful and can be problematic. Note that if one just applies the result of Balasubramanian et al. [1] on the twisted second moment of the Riemann zeta-function to, say, the cross term \( \int |\zeta(1/2 + it)|^2 \sum_{n \leq T} \sum_{n \leq T} dt \), then one needs \( \vartheta_1 + \vartheta_2 < 1 \). So without extra work, one can only take \( \vartheta_1 = 4/7 - \varepsilon \) and \( \vartheta_2 = \ldots = \vartheta_l = 3/7 - \varepsilon \) in Feng’s paper. This numerically leads to the above bound \( \kappa > 0.410725 \) in [1].

In this paper we shall prove

Theorem 1.1. We have

\[
\kappa > 0.410918 \quad \text{and} \quad \kappa^* > 0.40589.
\]

Remark 1.2. Rigourously speaking this is not yet a theorem. Here we assume a result on the twisted third moment of the Riemann zeta-function (see Theorem 5.1 for the precise
statement). We leave that unproved. It is possible that the ideas of Bettin et al. work in this context as well.

2. Reduction to mean-value theorems

2.1. The mollifier. To get lower bounds for $N_0(T)$ and $N_0^*(T)$ it suffices to consider a certain mollified second moment of the Riemann zeta-function and its derivatives. This is well-known, so we shall simply state the conclusion.

Let $Q(x)$ be a real polynomial satisfying $Q(0) = 1$, and define

$$V(s) = Q\left( -\frac{1}{\mathcal{L}} \frac{d}{ds} \right) \zeta(s),$$

where for large $T$ we denote

$$\mathcal{L} = \log T.$$

Suppose $\psi(s)$ is a “mollifier”. Littlewood’s lemma and the arithmetic-mean, geometric-mean inequality give

$$\kappa \geq 1 - \frac{1}{R} \log \left( \frac{1}{T} \int_1^T |V\psi(\sigma_0 + it)|^2 \, dt \right) + o(1),$$

where $\sigma_0 = 1/2 - R/\mathcal{L}$, and $R$ is a bounded positive real number to be chosen later. Actually, by choosing $Q(x)$ to be a linear polynomial, we obtain a lower bound for the proportion of simple zeros, $\kappa^*$. We also require the condition that

$$P_3(0) = P''_2(0) = P''_2(0) = 0. $$

For the third mollifier, we take

$$\psi_3(s) = \zeta(s + \frac{3}{2} - \sigma_0) \sum_{n=1}^{\infty} \frac{\mu_2(n) \mu_3(\log \frac{y_3}{\log \frac{y_3}{y_3}})}{n^{s-1}},$$

where $P_3(x) = \sum c_j x^j$ is a certain polynomial satisfying $P_3(0) = P''_3(0) = P''_3(0) = 0$. We also require the condition that $P_1(1) + P_3(1) = 1$ (see the below remark). Throughout the paper we denote $y_1 = T^{\vartheta_1}$, $y_2 = T^{\vartheta_2}$ and $y_3 = T^{\vartheta_3}$, where $0 < \vartheta_3 < \vartheta_2 < \vartheta_1 < 1$ (we shall see later what conditions are required on $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$). Note that formally

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n^s} = \frac{1}{\zeta(s)},$$

which explains why $\psi_3(s)$ may also be a useful choice of a mollifier.

It was originally an idea of K. Soundararajan to make use of the twisted fourth moment of the Riemann zeta-function and use a two-piece mollifier of the form $\psi_1 + \psi_3$. The author learned about this from Brian Conrey, Chris Hughes and K. Soundararajan. However, the implied proportion of critical zeros using this two-piece mollifier has never been worked out explicitly.
Remark 2.1. We can apply Levinson’s method (see, for example, Appendix A of [5]) to our choice of \( \psi = \psi_1 + \psi_2 + \psi_3 \). It is important to note that when applying Littlewood’s lemma, we need to estimate the integral on the right side of a rectangle. Assume that \( \psi_1(s) + \psi_2(s) \) can be expressed as a Dirichlet series, \( \psi_1(s) + \psi_2(s) = \sum_n \alpha(n)n^{-s} \), that integral is negligible given that \( \alpha(1) = 1 \). That is why we require \( P_1(1) + P_3(1) = 1 \).

2.2. A smoothing argument. It simplifies some calculations to smooth out the integral in (2). We introduce a smooth function \( w(t) \) with the following properties

i) \( 0 \leq w(t) \leq 1 \) for all \( t \in \mathbb{R} \),

ii) \( w \) has compact support in \([T/4, 2T] \),

iii) \( w^{(j)}(t) \ll \Delta^{-j} \) for each \( j = 0, 1, 2, \ldots \), where \( \Delta = T \mathcal{L}^{-1} \).

Theorem 2.1. Suppose \( \vartheta_1 < 4/7, \vartheta_2 < 1/2 \) and \( \vartheta_3 < 1/4 \). Then we have

\[
\int_{-\infty}^{\infty} |V\psi(\sigma_0 + it)|^2 w(t) dt = c(P_1, P_2, P_3, Q, R, \vartheta_1, \vartheta_2, \vartheta_3) \hat{w}(0) + O_\varepsilon(T \mathcal{L}^{-1+\varepsilon}),
\]

where \( c(P_1, P_2, P_3, Q, R, \vartheta_1, \vartheta_2, \vartheta_3) = c_1 + c_2 + c_3 + 2c_{12} + 2c_{23} + 2c_{31} \), and the \( c_i \) and \( c_{ij} \) are given below by (4) and (9).

2.3. Numerical evaluations. It is a standard exercise to deduce from Theorem 2.1 the unsmoothed version

\[
\int_{1}^{T} |V\psi(\sigma_0 + it)|^2 dt = c(P_1, P_2, P_3, Q, R, \vartheta_1, \vartheta_2, \vartheta_3) T + O_\varepsilon(T \mathcal{L}^{-1+\varepsilon}).
\]

Hence

\[
\kappa \geq 1 - \frac{\log c(P_1, P_2, P_3, Q, R, \vartheta_1, \vartheta_2, \vartheta_3)}{R} + o(1).
\]

Using Mathematica, with \( R = 1.26 \),

\[
Q(x) = 0.49069 + 0.61077(1 - 2x) - 0.14199(1 - 2x)^3 + 0.04054(1 - 2x)^5,
\]

\[
P_1(x) = 0.83516 + 0.09758x^2 - 0.29393x^3 + 0.73372x^4 - 0.3753x^5,
\]

\[
P_2(x) = 0.0237x^3 - 0.00744x^4 + 0.00174x^5
\]

and

\[
P_3(x) = 0.00155x^4 - 0.00013x^5,
\]

we get \( \kappa > 0.410918 \). To get \( \kappa^* > 0.40589 \) we take \( R = 1.12 \), \( Q(x) = 1 - 1.03232x \),

\[
P_1(x) = 0.82653x + 0.02626x^2 - 0.00774x^3 + 0.34803x^4 - 0.19371x^5,
\]

\[
P_2(x) = 0.0324x^3 - 0.00759x^4 + 0.00742x^5
\]

and

\[
P_3(x) = 0.00094x^4 - 0.00031x^5.
\]

3. The mean-value results

Writing \( \psi = \psi_1 + \psi_2 + \psi_3 \) and opening the square, we get

\[
\int |V\psi|^2 w = \int |V\psi_1|^2 w + \int |V\psi_2|^2 w + \int |V\psi_3|^2 w + 2\text{Re} \left\{ \int |V|^2 \psi_1 \overline{\psi_2 w} \right\} + 2\text{Re} \left\{ \int |V|^2 \psi_2 \overline{\psi_3 w} \right\} + 2\text{Re} \left\{ \int |V|^2 \psi_3 \overline{\psi_1 w} \right\}
\]

\[
= I_1 + I_2 + I_3 + 2\text{Re}(I_{12}) + 2\text{Re}(I_{23}) + 2\text{Re}(I_{31}).
\]

We shall compute the integrals in turn. It turns out that \( I_{12}, I_{23} \) and \( I_{31} \) are asymptotically real.
3.1. **The main terms.** First we quote Theorem 2 of Conrey [4]. Conrey’s theorem is for the unsmoothed version, but the following smoothed version follows easily from that.

**Theorem 3.1** (Conrey). Suppose \( \vartheta_1 < 4/7 \) and \( P_1(0) = 0 \). Then we have

\[
\int_{-\infty}^{\infty} |V \psi_1(\sigma_0 + it)|^2 \omega(t) \, dt = c_1 \hat{\omega}(0) + O_\varepsilon(T \mathcal{L}^{-1+\varepsilon}),
\]

where

\[
c_1 = P_1(1)^2 + \frac{1}{4i} \int_0^1 \int_0^1 e^{2Rt} \left( Q(t)P_1'(u) + \vartheta_1Q'(t)P_1(u) + \vartheta_1 RQ(t)P_1(u) \right)^2 \, dt \, du. \tag{4}
\]

The terms \( I_{12} \) and \( I_2 \) are evaluated in [3].

**Theorem 3.2** (Bui, Conrey and Young). Suppose \( \vartheta_2 < \vartheta_1 < 4/7 \) and \( P_1(0) = 0 = P_2(0) = P_2''(0) \). Then we have

\[
\int_{-\infty}^{\infty} |V^2 \psi_2(\sigma_0 + it)\omega(t) \, dt = c_{12} \hat{\omega}(0) + O_\varepsilon(T \mathcal{L}^{-1+\varepsilon}),
\]

where

\[
c_{12} = \frac{4\vartheta_2}{v_1^2} \frac{d^2}{dx_1dx_2} \left[ \int_0^1 \int_{t_1+t_2 \leq u} \frac{1}{4i} e^{R(1-\vartheta_1(x_1-x_2) + \vartheta_2(t_1-t_2))} (1 - u)Q(\vartheta_1 x_1 + \vartheta_2 t_1)
\]

\[
Q(1 + \vartheta_1 x_2 - \vartheta_2 t_2)P_1 \left( x_1 + x_2 + 1 - \frac{\vartheta_1}{\vartheta_1} (1 - u) \right) P_2''(u - t_1 - t_2) \, dt_1 \, dt_2 \, du \right]_{x_1 = x_2 = 0}. \tag{5}
\]

**Theorem 3.3** (Bui, Conrey and Young). Suppose \( \vartheta_2 < \frac{1}{2} \) and \( P_2(0) = P_2'(0) = P_2''(0) = 0 \). Then we have

\[
\int_{-\infty}^{\infty} |V \psi_2(\sigma_0 + it)|^2 \omega(t) \, dt = c_2 \hat{\omega}(0) + O_\varepsilon(T \mathcal{L}^{-1+\varepsilon}),
\]

where

\[
c_2 = \frac{2}{3\vartheta_2} \frac{d^4}{dx_1^2dx_2^2} \left[ \int_{[0,1]^4} e^{-\vartheta_2 R(x_1 + x_2 - t_1(x_1 + u) - t_2(x_2 + u)) + 2Rt_3(1 + \vartheta_2(x_1 + x_2 - t_1(x_1 + u) - t_2(x_2 + u)))}
\]

\[
Q(\vartheta_2(-x_1 + x_2 + t_2(x_2 + u))) + t_3 \left( 1 + \vartheta_2(x_1 + x_2 - t_1(x_1 + u) - t_2(x_2 + u)) \right)
\]

\[
Q(\vartheta_2(-x_2 + t_1(x_1 + u)) + t_3 \left( 1 + \vartheta_2(x_1 + x_2 - t_1(x_1 + u) - t_2(x_2 + u)) \right))
\]

\[
P_2''((x_1 + u)(1 - t_1)) P_2''((x_2 + u)(1 - t_2)) dt_1 dt_2 dt_3 du \right]_{x_1 = x_2 = 0}. \tag{6}
\]

We are left to evaluate \( I_3, I_{23} \) and \( I_{31} \), which are:

\[
I_3 = \int_{-\infty}^{\infty} |V(\sigma_0 + it)|^2 \zeta(\frac{1}{2} + it)^2 \sum_{m \leq y_3} \mu_2(m) P_3 \left( \frac{\log y_3 / m}{\log y_3} \right) \sum_{l \leq y_3} \mu_2(l) P_3 \left( \frac{\log y_3 / l}{\log y_3} \right) \omega(t) \, dt,
\]

\[
I_{23} = \int_{-\infty}^{\infty} |V(\sigma_0 + it)|^2 \zeta(\frac{1}{2} + it) \sum_{mn \leq y_2} \frac{\mu_2(m) P_2 \left( \frac{\log y_2 / mn}{\log y_2} \right)}{m^{1/2 + it} n^{1/2 - it}} \sum_{l \leq y_3} \frac{\mu_2(l) P_3 \left( \frac{\log y_3 / l}{\log y_3} \right)}{l^{1/2 - it}} \omega(t) \, dt
\]
and

\[ I_{31} = \int_{-\infty}^{\infty} |V(\sigma + it)|^2 \zeta(\frac{1}{2} + it) \sum_{m \leq y_1} \frac{\mu(m) P_1(\log y_1/m)}{m^{1/2 + it}} \sum_{l \leq y_3} \frac{\mu_2(l) P_3(\log y_3/l)}{l^{1/2 + it}} w(t) dt. \]

**Theorem 3.4.** Suppose \( \vartheta_3 < 1/4 \) and \( P_3(0) = P_3''(0) = P_3''(0) = 0 \). Then we have

\[ I_3 = c_3 \hat{w}(0) + O(\varepsilon (T \mathcal{L}^{-1+\varepsilon})), \]

where

\[
\begin{align*}
c_3 &= \frac{1}{128^3} \frac{d^8}{dx_1^2 dx_2^2 dx_3^2 dx_4^2} \int_{[0,1]^5} e^{-\vartheta_3 R(x_2 + x_3) + R t_1 (1-t) + \vartheta_3 (x_1 + x_3) + R t_3 (1-t) (1+\vartheta_3 (x_2 + x_4))} \\
&
\times e^{R t_3 (t_1 + \vartheta_3 (-x_1 + x_2 + (x_1 + x_3) t_1)) + R t_4 (t_2 + \vartheta_3 (x_1 - x_2 + (x_1 - x_2 + x_2 + x_4) t_2))} \\
&\quad \left( t_1 - t_2 + \vartheta_3 \left( -x_1 + x_2 + (x_1 + x_3) t_1 - (x_2 + x_4) t_2 \right) \right) \\
&\quad \left( t_1 - t_2 + \vartheta_3 \left( -x_3 + x_4 + (x_1 + x_3) t_1 - (x_2 + x_4) t_2 \right) \right) \\
&\quad \left( 1 + \vartheta_3 (x_1 + x_3) \right) \left( 1 + \vartheta_3 (x_2 + x_4) \right) (1-u)^3 \\
&\quad Q \left( -\vartheta_3 x_2 + t_2 (1-t_3) \left( 1 + \vartheta_3 (x_2 + x_4) \right) + t_3 \left( t_1 + \vartheta_3 \left( -x_1 + x_2 + (x_1 + x_3) t_1 \right) \right) \right) \\
&\quad Q \left( -\vartheta_3 x_3 + t_1 (1-t_4) \left( 1 + \vartheta_3 (x_1 + x_3) \right) + t_4 \left( t_2 + \vartheta_3 (x_3 - x_4 + (x_2 + x_4) t_2) \right) \right) \\
&\quad P_3 (x_1 + x_2 + u) P_3 (x_3 + x_4 + u) dt_1 dt_2 dt_3 dt_4 du \Bigg|_{x=0}. \tag{7}
\end{align*}
\]

**Theorem 3.5.** Suppose \( \vartheta_2 < 1/2, \vartheta_3 < 1/4 \) and \( P_2(0) = P_2''(0) = P_2''(0) = 0 = P_3(0) = P_3'(0) = P_3''(0) = 0 \). Then we have

\[ I_{23} = c_{23} \hat{w}(0) + O(\varepsilon (T \mathcal{L}^{-1+\varepsilon})), \]

where

\[
\begin{align*}
c_{23} &= \frac{2}{32^3} \frac{d^6}{dx_1^2 dx_2^2 dx_3^2 dx_4^2} \int_{[0,1]^5} e^{-R (\vartheta_2 x_1 + \vartheta_3 x_2) + R t_2 (1-t_3-t_4) (\vartheta_2 (1+x_1) - \vartheta_3 (1-u))} \\
&\times e^{R t_3 (1+t_4) (1+\vartheta_2 x_1 + \vartheta_3 x_2) + R (1-t_4) \left( \vartheta_3 (x_2 - x_3) - t_1 (2t_2 - 1) (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) \right)} \\
&\quad \left( -\vartheta_3 (x_2 - x_3) + t_1 (2t_2 - 1) (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) \right) \\
&\quad + t_3 \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) \right) \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) \right) t_1 \left( x_1 + 1 - \frac{\vartheta_1}{\vartheta_2} (1-u) \right)^2 (1-u)^3 \\
&\quad Q \left( -\vartheta_3 x_2 + t_1 t_2 (1-t_3 t_4) (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) + t_3 t_4 (1+\vartheta_2 x_1 + \vartheta_3 x_2) \\
&\quad + (1-t_4) (\vartheta_3 (x_2 - x_3) - t_1 (2t_2 - 1) (\vartheta_2 (1+x_1) - \vartheta_3 (1-u))) \right) \\
&\quad Q \left( -\vartheta_2 x_1 + t_3 \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 (\vartheta_2 (1+x_1) - \vartheta_3 (1-u)) \right) \right) \\
&\quad P_2'' \left( (x_1 + 1 - \frac{\vartheta_1}{\vartheta_2} (1-u)) (1-t_1) \right) P_3 (x_2 + x_3 + u) dt_1 dt_2 dt_3 dt_4 du \Bigg|_{x=0}. \tag{8}
\end{align*}
\]
Theorem 3.6. Suppose $\vartheta_1 < 4/7$, $\vartheta_3 < 1/4$ and $P_1(0) = 0 = P_3(0) = P_3'(0)$. Then we have
\[ I_{31} = c_{31} \hat{w}(0) + O_\varepsilon(TLC^{-1+\varepsilon}), \]
where
\[ c_{31} = \frac{1}{2\pi} \frac{d^4}{dx_1dx_2dx_3} \left[ \int_{\{0,1\}^3} e^{-R(\vartheta_1 x_2 + \vartheta_3 x_3) + R_1(1 + t_2)(1 + \vartheta_1 x_1 + \vartheta_3 x_3) - \vartheta_1 R_2(x_1 - x_2)} \right. \]
\[ \left( - \vartheta_1 (x_1 - x_2) + t_1 (1 + \vartheta_1 x_1 + \vartheta_3 x_3) \right) \left( 1 + \vartheta_1 x_1 + \vartheta_3 x_3 \right)(1 - u) \]
\[ Q \left( - \vartheta_1 x_2 + t_1 t_2 (1 + \vartheta_1 x_1 + \vartheta_3 x_3) - \vartheta_1 t_2 (x_1 - x_2) \right) Q \left( - \vartheta_3 x_3 + t_1 (1 + \vartheta_1 x_1 + \vartheta_3 x_3) \right) \]
\[ P_1 \left( x_1 + x_2 + 1 - \frac{\vartheta_1}{x_1} (1 - u) \right) P_3(x_3 + u) dt_1 dt_2 du \bigg|_{x=0}. \tag{9} \]

3.2. The shift parameters. Rather than working directly with $V(s)$, we shall instead consider the following three general integrals
\[ I_3(\alpha, \beta, \gamma, \delta) = \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta + it) \zeta(\frac{1}{2} + \gamma - it) \zeta(\frac{1}{2} + \delta - it) \]
\[ \sum_{m \leq y_3} \frac{\mu_2(m) P_3 \left( \frac{\log y_3/m}{\log y_2} \right)}{m^{1/2+it}} \sum_{l \leq y_3} \frac{\mu_2(l) P_3 \left( \frac{\log y_3/l}{\log y_3} \right)}{l^{1/2-it}} w(t) dt, \tag{10} \]
\[ I_{23}(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta + it) \zeta(\frac{1}{2} + \gamma - it) \]
\[ \sum_{m \leq y_2} \frac{\mu_2(m) P_3 \left( \frac{\log y_2/\log y_2}{\log y_2} \right)}{m^{1/2+it}} \sum_{l \leq y_3} \frac{\mu_2(l) P_3 \left( \frac{\log y_3/l}{\log y_3} \right)}{l^{1/2-it}} w(t) dt \tag{11} \]
and
\[ I_{31}(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta + it) \zeta(\frac{1}{2} + \gamma - it) \]
\[ \sum_{m \leq y_1} \frac{\mu_2(m) P_3 \left( \frac{\log y_1/m}{\log y_1} \right)}{m^{1/2-it}} \sum_{l \leq y_3} \frac{\mu_2(l) P_3 \left( \frac{\log y_3/l}{\log y_3} \right)}{l^{1/2+it}} w(t) dt. \tag{12} \]

Our main goal in the rest of the paper is to prove the following lemmas.

Lemma 3.1. Suppose $\vartheta_3 < 1/4$ and $P_3(0) = P_3'(0) = P_3''(0) = P_3'''(0) = 0$. Then we have
\[ I_3(\alpha, \beta, \gamma, \delta) = c_3(\alpha, \beta, \gamma, \delta) \hat{w}(0) + O_\varepsilon(TLC^{-1+\varepsilon}), \]
uniformly for $\alpha, \beta, \gamma, \delta \ll T^{-1}$, where $c_3(\alpha, \beta, \gamma, \delta)$ is given by
\[ \frac{1}{12d^4} \frac{d^4}{dx_1dx_2dx_3dx_4} \left[ \int_{\{0,1\}^2} \frac{d^4}{dx_2dx_3dx_4} \right. \]
\[ \left( T^{t_1-t_2} y_3^x \sum_{x_1+x_2+(x_1+x_3) t_1-(x_2+x_4) t_2} - (\alpha+\gamma) t_3 \left( T^{t_1-t_2} y_3^{x_3+x_4+(x_1+x_3) t_1-(x_2+x_4) t_2} \right)^{-(\delta-\gamma) t_4} \right. \]
\[ \left( t_1-t_2+\vartheta_3 \left( -x_1+x_2+(x_1+x_3) t_1-(x_2+x_4) t_2 \right) \right) \]
\[ \left( t_1-t_2+\vartheta_3 \left( -x_3+x_4+(x_1+x_3) t_1-(x_2+x_4) t_2 \right) \right) (1+\vartheta_3(x_1+x_3)) \]
\[ (1+\vartheta_3(x_2+x_4))(1-u)^3 P_3(x_1+x_2+u) P_3(x_3+x_4+u) dt_1 dt_2 dt_3 dt_4 du \bigg|_{x=0}. \tag{13} \]
Remark 3.1. In the special case $\alpha = \beta = \gamma = \delta = 0$, Lemma 3.1 agrees with the mollified fourth moment of the Riemann zeta-function predicted by Conrey and Snaith using the ratios conjecture [6 Theorem 6.1].

Lemma 3.2. Suppose $\psi_2 < 1/2$, $\psi_3 < 1/4$ and $P_2(0) = P_2'(0) = P_3'(0) = P_3''(0) = 0$. Then we have

$$I_2(\alpha, \beta, \gamma) = c_{23}(\alpha, \beta, \gamma) \hat{w}(0) + O_{\varepsilon}(TL^{-1+\varepsilon}),$$

uniformly for $\alpha, \beta, \gamma \ll L^{-1}$, where $c_{23}(\alpha, \beta, \gamma)$ is given by

$$\frac{2}{3!^2} \frac{d^6}{dx_1^2 dx_2^2 dx_3^2} \left[ \int_{[0,1]^3} \frac{\gamma}{y_2} \left( ax_1 + \beta x_2 + \gamma x_3 \right) \left( (ax_1 + \beta x_2 + \gamma x_3) t_1 \right) \left( t_2 \right) \left( t_3 \right) \right]$$

Lemma 3.3. Suppose $\psi_1 < 4/7$, $\psi_3 < 1/4$ and $P_1(0) = 0$. Then we have

$$I_3(\alpha, \beta, \gamma) = c_{31}(\alpha, \beta, \gamma) \hat{w}(0) + O_{\varepsilon}(TL^{-1+\varepsilon}),$$

uniformly for $\alpha, \beta, \gamma \ll L^{-1}$, where $c_{31}(\alpha, \beta, \gamma)$ is given by

$$\frac{1}{\psi_1} \frac{d^4}{dx_1 dx_2 dx_3} \left[ \int_{[0,1]^3} \frac{\gamma}{y_1} \left( ax_1 + \beta x_2 + \gamma x_3 \right) \left( (ax_1 + \beta x_2 + \gamma x_3) t_1 \right) \left( t_2 \right) \left( t_3 \right) \right]$$

We now prove that Theorems 3.4 and 3.6 follow from Lemmas 3.1 and 3.3 respectively. Let $I_*$ denote either $I_{23}$ or $I_{31}$. We first note that

$$I_3 = Q \left( \frac{-1}{L} \frac{d}{d\beta} \right) Q \left( \frac{-1}{L} \frac{d}{d\gamma} \right) I_3(\alpha, \beta, \gamma, \delta) \bigg|_{\alpha=\delta=0, \beta=\gamma=-R/L}$$

and

$$I_* = Q \left( \frac{-1}{L} \frac{d}{d\beta} \right) Q \left( \frac{-1}{L} \frac{d}{d\gamma} \right) I_*(\alpha, \beta, \gamma) \bigg|_{\alpha=0, \beta=\gamma=-R/L}$$

We then argue that we can obtain either $c_3$ or $c_*$ by applying the above differential operator to the corresponding $c_3(\alpha, \beta, \gamma, \delta)$ or $c_*(\alpha, \beta, \gamma)$. Since $I_3(\alpha, \beta, \gamma, \delta)$, $I_*(\alpha, \beta, \gamma)$, $c_3(\alpha, \beta, \gamma, \delta)$ and $c_*(\alpha, \beta, \gamma)$ are holomorphic with respect to $\alpha, \beta, \gamma, \delta$ small, the derivatives appearing in (16) and (17) can be obtained as integrals of radii $\sim L^{-1}$ around the points $\alpha = \delta = 0, \beta = \gamma = -R/L$, using Cauchy’s integral formula. Since the error terms hold uniformly on
these contours, the same error terms that hold for $I_3(\alpha, \beta, \gamma, \delta)$ and $I_4(\alpha, \beta, \gamma)$ also hold for $I_3$ and $I_4$.

Next we check that applying the above differential operator to $c_3(\alpha, \beta, \gamma, \delta)$ and $c_4(\alpha, \beta, \gamma)$ does indeed give $c_3$ and $c_4$. Notice the formula

$$Q\left(-\frac{1}{\mathcal{L}} \frac{d}{d\beta}\right) X^{-\beta} = Q\left(\frac{\log X}{\mathcal{L}}\right) X^{-\beta}. \quad (18)$$

Using (18) and (15), we have

$$Q\left(-\frac{1}{\mathcal{L}} \frac{d}{d\beta}\right) Q\left(-\frac{1}{\mathcal{L}} \frac{d}{d\gamma}\right) c_{31}(0, \beta, \gamma) = \frac{1}{\sqrt{1}} \frac{d^2}{dx_1 dx_2 dx_3} \int_{[0,1]^3} y_1^{\beta x_2} y_3^{\gamma x_3} (T y_1 x_1) - \gamma t_1$$

$$(y_1^{-x_1+x_2} (T y_1 x_1) \gamma t_1) - (\vartheta_3 x_3 + t_1(1 + \vartheta_1 x_1 + \vartheta_3 x_3)) (1 + \vartheta_1 x_1 + \vartheta_3 x_3)$$

$$(1 - u) Q\left(-\vartheta_1 x_2 - \vartheta_1 t_2(x_2 - x_1) + t_1 t_2(1 + \vartheta_1 x_1 + \vartheta_3 x_3)\right)$$

$$Q\left(-\vartheta_3 x_3 + t_1(1 + \vartheta_1 x_1 + \vartheta_3 x_3)\right) P_1\left(x_1 + x_2 + 1 - \frac{\vartheta_3 x_3}{\vartheta_1}(1 - u)\right) P_3(x_3 + u) dt_1 dt_2 du \bigg|_{x=0}.$$  

Setting $\beta = \gamma = -R/\mathcal{L}$ and simplifying gives (9). A similar argument produces (8) from (14) and produces (7) from (13).

We prove Lemma 3.3 in Section 5, Lemma 3.2 in Section 6 and Lemma 3.1 in Section 7.

4. Various Lemmas

4.1. The Euler-Maclaurin formula. The following two lemmas are easy consequences of the Euler-Maclaurin formula, see [3] Lemmas 4.4 and 4.6.

Lemma 4.1. Suppose $y_2 \leq y_1$, $|z| \ll (\log y_2)^{-1}$, and that $f_1$ and $f_2$ are smooth functions. Then we have

$$\sum_{n \leq y_2} \frac{d_k(n)}{n} \left(\frac{y_2}{n}\right)^z f_1\left(\frac{\log y_1}{\log y_2}\right) f_2\left(\frac{\log y_2}{\log y_2}\right) = \frac{(\log y_2)^k}{(k-1)!} \int_0^1 y_2^{z u} (1-u)^{k-1} f_1\left(1 - \frac{(1-u) \log y_2}{\log y_1}\right) f_2(u) du + O\left((\log y_2)^{k-1}\right).$$

Lemma 4.2. Suppose $-1 \leq \sigma \leq 0$. Then we have

$$\sum_{n \leq y_1} \frac{d_k(n)}{n} \left(\frac{y_1}{n}\right)^\sigma \ll (\log y_1)^{k-1} \min \{|\sigma|^{-1}, \log y_1\}.$$  

4.2. Mellin pairs. By convention, we set $P_j(x) = 0$ for $j = 1, 2, 3$ and $x \leq 0$. Note that with this definition we have

$$P_1\left(\frac{\log y_1/n}{\log y_1}\right) = \sum_j \frac{a_j}{(\log y_1)^{j+1}} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_1}{n}\right)^u \frac{du}{u^{j+1}} \quad (19)$$

for all $n$. Similar expressions holds for $P_2\left(\frac{\log y_2/n}{\log y_2}\right)$ and $P_3\left(\frac{\log y_3/n}{\log y_3}\right)$.

5. Proof of Lemma 3.3

5.1. Reduction to a contour integral. We shall used the following unproved twisted third moment of the Riemann zeta-function.
**Theorem 5.1.** Suppose \( H \) and \( K \) satisfy \( H \leq T^{4/7-\varepsilon} \) and \( K \leq T^{1/4-\varepsilon} \), or \( HK \leq T^{3/4-\varepsilon} \) and \( K \leq T^{1/2-\varepsilon} \). Then we have

\[
\sum_{h \leq H, k \leq K} \frac{a_h a_k}{\sqrt{hk}} \int_{-\infty}^{\infty} \left( \frac{1}{2} + \alpha + it \right) \zeta \left( \frac{1}{2} + \beta + it \right) \zeta \left( \frac{1}{2} + \gamma - it \right) \left( \frac{h}{k} \right)^i w(t) dt
\]

\[
= \sum_{h \leq H, k \leq K} \frac{a_h a_k}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \left\{ Z_{\alpha,\beta,\gamma}(h,k) + \left( \frac{t}{2\pi} \right)^{-(\alpha+\gamma)} Z_{-\alpha,\beta,-\gamma}(h,k) \right. \\
+ \left. \left( \frac{t}{2\pi} \right)^{-(\beta+\gamma)} Z_{-\alpha,-\beta,\gamma}(h,k) \right\} dt + O_\varepsilon(T^{1-\varepsilon})
\]

uniformly for \( \alpha, \beta, \gamma \ll \mathcal{L}^{-1} \), where

\[
Z_{\alpha,\beta,\gamma}(h,k) = \sum_{kab = h} \frac{1}{a^{1/2+\alpha}b^{1/2+\beta}c^{1/2+\gamma}}.
\]

Recall that \( I_{31}(\alpha, \beta, \gamma) \) is defined by (12). We have

\[
I_{31}(\alpha, \beta, \gamma) = I_{31}^{1} + I_{31}^{2} + I_{31}^{3} + O_\varepsilon(T^{1-\varepsilon}),
\]

where

\[
I_{31}^{1} = \hat{w}(0) \sum_{l,m} \frac{\mu(m)\mu_2(l)}{\sqrt{lm}} P_l \left( \frac{\log y_1/m}{\log y_1} \right) P_3 \left( \frac{\log y_3/l}{\log y_3} \right) \sum_{lmb = mc} \frac{1}{a^{1/2+\alpha}b^{1/2+\beta}c^{1/2+\gamma}}.
\]

\( I_{31}^{2} \) is obtained by multiplying \( I_{31}^{1} \) with \( T^{-(\alpha+\gamma)} \) and changing the shifts \( \alpha \leftrightarrow -\gamma \), \( \gamma \leftrightarrow -\alpha \), and \( I_{31}^{3} \) is obtained by multiplying \( I_{31}^{1} \) with \( T^{-(\beta+\gamma)} \) and changing the shifts \( \beta \leftrightarrow -\gamma \), \( \gamma \leftrightarrow -\beta \).

We shall first work on \( I_{31}^{1} \). In view of (19) we get

\[
I_{31}^{1} = \hat{w}(0) \sum_{i,j} \frac{a_i c_j j!}{(\log y_1)^i (\log y_3)^j} \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_1^i y_3^j d\frac{u}{y_1} d\frac{v}{y_3}.
\]

The Euler product implies that

\[
\sum_{lmb = mc} \frac{\mu(m)\mu_2(l)}{m^{1/2+u}a^{1/2+v}b^{1/2+\alpha}c^{1/2+\gamma}} = A(\alpha, \beta, \gamma, u, v) \frac{(1+\alpha+\gamma)(1+\beta+\gamma)\zeta(1+u+v)^2}{\zeta(1+\alpha+u)(1+\beta+u)\zeta(1+\gamma+v)^2},
\]

(20)

where \( A(\alpha, \beta, \gamma, u, v) \) is an arithmetical factor converging absolutely in a product of half-planes containing the origin. Hence

\[
I_{31}^{1} = \hat{w}(0) (1+\alpha+\gamma)(1+\beta+\gamma) \sum_{i,j} \frac{a_i c_j j!}{(\log y_1)^i (\log y_3)^j} J_{i,j},
\]

(21)

where

\[
J_{i,j} = \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_1^i y_3^j \frac{A(\alpha, \beta, \gamma, u, v)\zeta(1+u+v)^2}{\zeta(1+\alpha+u)(1+\beta+u)\zeta(1+\gamma+v)^2} d\frac{u}{y_1} d\frac{v}{y_3}.
\]

\( \text{Theorem 5.1 is likely to hold for larger ranges of } H \text{ and } K. \) These conditions, however, suffice for our purposes.
Using the Dirichlet series for $\zeta(1 + u + v)^2$ and reversing the order of summation and integration, we obtain

$$J_{i,j} = \sum_{n \leq g_3} \frac{d(n)}{n} \int \int \frac{(y_1/n)^u}{\zeta(1 + \alpha + u)\zeta(1 + \beta + u)\zeta(1 + \gamma + v)^2 u^{v+1}} du \ dv$$

(22)

Note that here we are able to restrict the sum over $n$ to $n \leq g_3$ by moving the $v$-integral far to the right. We now move the contours of integration to $\Re(u) = \Re(v) \approx L^{-1}$. Bounding the integrals trivially shows that $J_{i,j} \ll L^{i+j-2}$. Hence from the Taylor series $A(\alpha, \beta, \gamma, u, v) = A(0, 0, 0, 0, 0) + O(L^{-1}) + O(|u| + |v|)$, we can replace $A(\alpha, \beta, \gamma, u, v)$ by $A(0, 0, 0, 0, 0)$ in $J_{i,j}$ with an error of size $O(L^{i+j-3})$. By letting $\alpha = \beta = \gamma = u = v = s$ in (20), it is easy to verify that $A(0, 0, 0, 0, 0) = 1$. The $u$ and $v$ variables in (22) are now separated so that

$$J_{i,j} = \sum_{n \leq g_3} \frac{d(n)}{n} M_i(\alpha, \beta) M_j(\gamma) + O(L^{i+j-3}),$$

(23)

where

$$M_i(\alpha, \beta) = \frac{1}{2\pi i} \int \int \frac{d^2}{\zeta(1 + \alpha + u)\zeta(1 + \beta + u)\zeta(1 + \gamma + v)^2 u^{v+1}}$$

and

$$M_j(\gamma) = \frac{1}{2\pi i} \int \int \frac{d^2}{\zeta(1 + \gamma + v)^2}$$

The expression $M_i(\alpha, \beta)$ was evaluated in [3] Lemma 5.7

$$M_i(\alpha, \beta) = \frac{1}{j! (\log y_1)^2} d^2 x_1 d^2 x_2 \left[ y_1^{x_1 + \beta x_2} (x_1 + x_2) \log y_1 + \log y_1/n \right]_{x_1=x_2=0} + O(L^{j-3}).$$

(24)

We evaluate $M_j(\gamma)$ with the following lemma.

**Lemma 5.1.** Suppose $j \geq 2$. Then for some $\nu \approx (\log \log y_3)^{-1}$ we have

$$M_j(\gamma) = \frac{1}{j! (\log y_3)^2} d^2 x_3 \left[ y_3^{x_3} (x_3 \log y_3 + \log y_3/n)^j \right]_{x_3=0} + O(L^{j-3}) + O_\epsilon \left( \left( \frac{y_3}{n} \right)^{-\nu} \right).$$

**Proof.** Let $Y = o(T)$ be a large parameter to be chosen later. By Cauchy’s theorem, $M_j(\gamma)$ is equal to the residue at $v = 0$ plus integrals over the line segments $C_1 = \{ v = L^{-1} + it, t \in \mathbb{R}, |t| \geq Y \}$, $C_2 = \{ v = \sigma \pm iy, -\log T \leq \sigma \leq L^{-1} \}$ and $C_3 = \{ v = -\log y + it, |t| \leq Y \}$, where $\epsilon$ is some fixed positive constant such that $\zeta(1 + \gamma + v)$ has no zeros in the region on the right hand side of the contour determined by the $C_j$. Furthermore, we require that for such $\epsilon$ we have $1/\zeta(\sigma + it) \ll (2 + |t|)$ in this region (see [12] Theorem 3.11). Then the integral over $C_1$ is

$$\ll \left( \log Y \right)^2 / Y^j \ll_\epsilon Y^{-2+\epsilon},$$

since $j \geq 2$. The integral over $C_2$ is

$$\ll \left( \log Y \right)^{j+1} / Y^j \ll_\epsilon Y^{-3+\epsilon}.$$ 

Finally, the contribution from $C_3$ is

$$\ll \left( \log Y \right)^j \left( \frac{y_3}{n} \right)^{-c / \log Y} \ll_\epsilon \left( \frac{y_3}{n} \right)^{-c / \log Y} Y^\epsilon.$$

Choosing $Y \approx L$ gives an error so far of size $O_\epsilon \left( \left( y_3/n \right)^{-\nu} L^\epsilon \right) + O_\epsilon (L^{-2+\epsilon})$.

For the residue at $v = 0$, we write this as

$$\frac{1}{2\pi i} \int \int \frac{d^2}{\zeta(1 + \gamma + v)^2}$$
where the contour is a circle of radius $\propto \mathcal{L}^{-1}$ around the origin. This integral is trivially bounded by $O(\mathcal{L}^{-2})$, hence by taking the first term in the Taylor series of $\zeta(1 + \gamma + \nu)$ we get

$$\text{Res}_{n=0} = \frac{\gamma^2(\log y_3/n)^j}{j!} + \frac{2\gamma(\log y_3/n)^{j-1}}{(j-1)!} + \frac{(\log y_3/n)^{j-2}}{(j-2)!} + O(\mathcal{L}^{j-3}).$$

The above main term can be written in a compact form

$$\frac{1}{j!(\log y_3)^2} \frac{d^2}{dx_3^2} \left[ y_3^{\gamma x_3} \left( x_3 \log y_3 + \log y_3/n \right)^j \right]_{x_3=0}$$

and the lemma follows. \hfill \square

In view of (23), (24) and Lemma 5.1 we get

$$\int \left( x_1 + x_2 + 1 - \frac{\varphi_1}{y_1}(1-u) \right)^i (x_3 + u)^j du \left|_{x=0} \right. + O_\varepsilon(\mathcal{L}^{-2+\varepsilon}) + O(\mathcal{L}^{i+j-3}).$$

Using Lemmas 23 and 24 this is equal to

$$\int_0^1 (1-u) \left( x_1 + x_2 + 1 - \frac{\varphi_1}{y_1}(1-u) \right)^i (x_3 + u)^j du \left|_{x=0} \right. + O_\varepsilon(\mathcal{L}^{i+j-3}).$$

As $j \geq 2$, putting this back to (21) we get

$$I_{31}^1 = \frac{\tilde{w}(0)}{\log y_1^2} \zeta(1 + \alpha + \gamma) \zeta(1 + \beta + \gamma) \frac{d^4}{dx_1 dx_2 dx_3^2} \left[ y_1^{\alpha x_1 + \beta x_2} y_3^{\gamma x_3} \right] \int_0^1 (1-u) P_1 \left( x_1 + x_2 + 1 - \frac{\varphi_1}{y_1}(1-u) \right) P_3(x_3 + u) du \left|_{x=0} \right. + O_\varepsilon(\mathcal{L}^{i-1+\varepsilon}).$$

5.2. Deduction of Lemma 3.3 Combining $I_{31}^1$, $I_{31}^2$ and $I_{31}^3$ we have

$$I_{31} = \frac{\tilde{w}(0)}{\log y_1^2} \frac{d^4}{dx_1 dx_2 dx_3^2} \left[ \int_0^1 U_1(x)(1-u) P_1 \left( x_1 + x_2 + 1 - \frac{\varphi_1}{y_1}(1-u) \right) P_3(x_3 + u) du \right] \left|_{x=0} \right. + O_\varepsilon(\mathcal{L}^{i-1+\varepsilon}),$$

where

$$U_1 = \frac{y_1^{\alpha x_1 + \beta x_2} y_3^{-\gamma x_3}}{(\alpha + \gamma)(\beta + \gamma)} - \frac{T^{-(\alpha+\gamma)} y_1^{\gamma x_1 + \beta x_2} y_3^{-\alpha x_3}}{(\alpha + \gamma)(\beta - \alpha)} - \frac{T^{-(\beta+\gamma)} y_1^{\alpha x_1 - \gamma x_2} y_3^{-\beta x_3}}{(\alpha - \beta)(\beta + \gamma)}.$$

Using the identity

$$\frac{1}{(\alpha + \gamma)(\beta + \gamma)} = \frac{1}{(\alpha + \gamma)(\beta - \alpha)} + \frac{1}{(\alpha - \beta)(\beta + \gamma)}$$

and the integral formula

$$\frac{1 - z^{-(\alpha+\gamma)}}{\alpha + \beta} = (\log z) \int_0^1 z^{-(\alpha+\beta)t} dt,$$

(25)
we can write
\[
U_1 = \frac{y^\alpha x_1 + \beta x_2 y_3^\gamma x_3 - T^{-(\alpha+\gamma)} y_1^{-\gamma x_1 + \beta x_2} y_3^{-\alpha x_3} - y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 - T^{-(\beta+\gamma)} y_1^\alpha x_1 - y_3^\alpha x_3}{(\alpha+\gamma)(\beta-\alpha)} - \frac{\alpha x_1 + \beta x_2 y_3^\gamma x_3 - T^{-(\alpha+\gamma)} y_1^{-\gamma x_1 + \beta x_2} y_3^{-\alpha x_3} - y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 - T^{-(\beta+\gamma)} y_1^\alpha x_1 - y_3^\alpha x_3}{(\beta-\alpha)(\beta+\gamma)}
\]
\[
= \frac{\mathcal{L}}{\beta-\alpha} (1 + \vartheta_1 x_1 + \vartheta_3 x_3) y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 \int_0^1 (T y_1 x_3)^{-(\alpha+\gamma)t_1} dt_1
\]
Hence
\[
I_{31} = \frac{\hat{w}(0) \mathcal{L}}{(\beta-\alpha)(\log y_1)^2} d^4 \int_0^1 y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\alpha+\gamma)t_1} (1 + \vartheta_1 x_1 + \vartheta_3 x_3)
\]
\[
(1-u) P_t (x_1 + x_2 + 1 - \frac{\vartheta_1}{\alpha} (1-u)) P_3 (x_3 + u) dt_1 du \bigg|_{z=0}
\]
\[
- \frac{\hat{w}(0) \mathcal{L}}{(\beta-\alpha)(\log y_1)^2} d^4 \int_0^1 y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\beta+\gamma)t_1} (1 + \vartheta_1 x_2 + \vartheta_3 x_3)
\]
\[
(1-u) P_t (x_1 + x_2 + 1 - \frac{\vartheta_1}{\beta} (1-u)) P_3 (x_3 + u) dt_1 du \bigg|_{z=0} + O_\varepsilon (T \mathcal{L}^{-1+\varepsilon}).
\]
Changing the roles of the variables \(x_1\) and \(x_2\) in the second term we obtain
\[
I_{31} = \frac{\hat{w}(0) \mathcal{L}}{(\beta-\alpha)(\log y_1)^2} d^4 \int_0^1 \int_0^1 V_1 (x_1 + x_2 + 1 - \frac{\vartheta_1}{\alpha} (1-u)) P_3 (x_3 + u) dt_1 du \bigg|_{z=0} + O_\varepsilon (T \mathcal{L}^{-1+\varepsilon}),
\]
where
\[
V_1 = \frac{y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\alpha+\gamma)t_1} - y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\beta+\gamma)t_1}}{\beta-\alpha}.
\]
Using (23) again we have
\[
V_1 = y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\alpha+\gamma)t_1} \left(1 - \frac{(y_1 x_1 + x_2 (T y_1 x_3) t_1)^{-(\beta-\alpha)}}{\beta-\alpha}\right)
\]
\[
= \mathcal{L} \left( - \vartheta_1 (x_1 - x_2) + t_1 (1 + \vartheta_1 x_1 + \vartheta_3 x_3) \right) y_1^\alpha x_1 + \beta x_2 y_3^\gamma x_3 (T y_1 x_3)^{-(\alpha+\gamma)t_1}
\]
\[
\int_0^1 (y_1 x_1 + x_2 (T y_1 x_3) t_1)^{-(\beta-\alpha) t_2} dt_2.
\]
Lemma 3.3 follows from this and (26).

6. PROOF OF LEMMA 3.2

6.1. Reduction to a contour integral. Recall that \(I_{23}(\alpha, \beta, \gamma)\) is defined by (11). We have
\[
I_{23}(\alpha, \beta, \gamma) = I_{23}^1 + I_{23}^2 + I_{23}^3,
\]
where
\[
I_{23}^1 = \hat{w}(0) \sum_{l,m,n} \mu_2(m) \mu_2(l) P_2 \left( \log \frac{y_2}{mn} \right) P_3 \left( \log \frac{y_3}{l} \right) \sum_{a,b,c=1} \frac{1}{a^{1/2} + b^{1/2} + c^{1/2} + \gamma},
\]
\( I_{23}^1 \) is obtained by multiplying \( I_{23}^3 \) with \( T^{-(\alpha + \gamma)} \) and changing the shifts \( \alpha \leftrightarrow -\gamma, \gamma \leftrightarrow -\alpha \), and \( I_{23}^3 \) is obtained by multiplying \( I_{23}^1 \) with \( T^{-(\beta + \gamma)} \) and changing the shifts \( \beta \leftrightarrow -\gamma, \gamma \leftrightarrow -\beta \).

We first work on \( I_{23}^1 \). In view of (19) we get

\[
I_{23}^1 = \hat{w}(0) \sum_{i,j} \frac{b_i c_j j!}{(\log y_2)^i (\log y_3)^j} \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_2^u y_3^v \mu_2(m) \mu_2(l) (mn)^{1/2+u1/2+v1/2+\alpha b_1/2+\beta c_1/2+\gamma} \quad du \quad dv.
\]

The arithmetical sum is

\[
\sum_{mab=\text{lnc}} \frac{\mu_2(m) \mu_2(l)}{(mn)^{1/2+u1/2+v1/2+\alpha b_1/2+\beta c_1/2+\gamma}} = B(\alpha, \beta, \gamma, u, v) \frac{\zeta(1 + \alpha + \gamma) \zeta(1 + \beta + \gamma) \zeta(1 + \alpha + u) \zeta(1 + \beta + u) \zeta(1 + u + v)^4}{\zeta(1 + \alpha + v)^2 \zeta(1 + \beta + v)^2 \zeta(1 + \gamma + u)^2 \zeta(1 + 2u)^2},
\]

where \( B(\alpha, \beta, \gamma, u, v) \) is an arithmetical factor converging absolutely in a product of half-planes containing the origin. So

\[
I_{23}^1 = \hat{w}(0) \zeta(1 + \alpha + \gamma) \zeta(1 + \beta + \gamma) \sum_{i,j} \frac{b_i c_j j!}{(\log y_2)^i (\log y_3)^j} K_{i,j},
\]

where

\[
K_{i,j} = \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_2^u y_3^v \frac{B(\alpha, \beta, \gamma, u, v) \zeta(1 + \alpha + u) \zeta(1 + \beta + u) \zeta(1 + u + v)^4}{\zeta(1 + \alpha + v)^2 \zeta(1 + \beta + v)^2 \zeta(1 + \gamma + u)^2 \zeta(1 + 2u)^2} \quad du \quad dv.
\]

Using the Dirichlet series for \( \zeta(1 + u + v)^4 \) and changing the order of summation and integration, we have

\[
K_{i,j} = \sum_{n \leq y_3} \frac{d_i(n)}{n} \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} \left( \frac{y_2}{n} \right)^u \left( \frac{y_3}{n} \right)^v \frac{B(\alpha, \beta, \gamma, u, v) \zeta(1 + \alpha + u) \zeta(1 + \beta + u)}{\zeta(1 + \alpha + v)^2 \zeta(1 + \beta + v)^2 \zeta(1 + \gamma + u)^2 \zeta(1 + 2u)^2} \quad du \quad dv.
\]

Note that here we are able to restrict the sum over \( n \) to \( n \leq y_3 \) by moving the \( v \)-integral far to the right. We now move the contours of integration to \( \text{Re}(u) = \text{Re}(v) < L^{-1} \). Bounding the integrals trivially shows that \( K_{i,j} \ll L^{i+j-2} \). Hence from the Taylor series \( B(\alpha, \beta, \gamma, u, v) = B(0, 0, 0, 0, 0) + O(L^{-1}) + O(|u| + |v|) \), we can replace \( B(\alpha, \beta, \gamma, u, v) \) by \( B(0, 0, 0, 0, 0) \) in \( K_{i,j} \), with an error of size \( O(L^{i+j-3}) \). By letting \( \alpha = \beta = \gamma = u = v = s \) in (27), it is easy to verify that \( B(0, 0, 0, 0, 0) = 1 \). The \( u \) and \( v \) variables in (27) are now separated so that

\[
K_{i,j} = \sum_{n \leq y_3} \frac{d_i(n)}{n} N_i(\alpha, \beta, \gamma) N_j(\alpha, \beta) + O(L^{i+j-3}),
\]

where

\[
N_i(\alpha, \beta, \gamma) = \frac{1}{2\pi i} \int_{(L^{-1})} \left( \frac{y_2}{n} \right)^u \zeta(1 + \alpha + u) \zeta(1 + \beta + u) \quad du
\]

and

\[
N_j(\alpha, \beta) = \frac{1}{2\pi i} \int_{(L^{-1})} \left( \frac{y_3}{n} \right)^v \zeta(1 + \alpha + v)^2 \zeta(1 + \beta + v)^2 \quad dv.
\]

We evaluate \( N_i(\alpha, \beta, \gamma) \) and \( N_j(\alpha, \beta) \) with the following lemmas.
Lemma 6.1. Suppose \( i \geq 3 \). Then we have

\[
N_i(\alpha, \beta, \gamma) = \frac{4}{(i-2)!L} \oint_{\gamma x_1=0} \frac{d^2}{dx_1^2} \left[ \int_0^1 \int_0^1 y_2^{\gamma x_1-\left( (\alpha(1-t_2)+\beta t_2) t_1 x_1 \right)} \zeta(\frac{y_2}{n})^{-\left( (\alpha(1-t_2)+\beta t_2) t_1 \right)} \right] dt_1 dt_2_{x_1=0} + O(L^{i-3}).
\]

Proof. An argument on the level of the prime number theorem shows that \( N_i(\alpha, \beta, \gamma) \) is equal to \( \frac{4}{(i-2)!L} \oint_{\gamma x_1=0} \frac{d^2}{dx_1^2} \left[ \int_0^1 \int_0^1 y_2^{\gamma x_1-\left( (\alpha(1-t_2)+\beta t_2) t_1 x_1 \right)} \zeta(\frac{y_2}{n})^{-\left( (\alpha(1-t_2)+\beta t_2) t_1 \right)} \right] dt_1 dt_2_{x_1=0} + O(L^{i-3}). \)

We write the residue at \( u = 0 \) as

\[
\frac{1}{2\pi i} \oint_{q} \frac{\zeta(1+\alpha+u) \zeta(1+\beta+u)}{(1+\gamma+u)^2 \zeta(1+2u)^2} \frac{du}{u^{i+1}},
\]

where \( q = y_2/n \) and the contour is a circle of radius \( \asymp L^{-1} \) around the origin. This integral is trivially bounded by \( O(L^{i-2}) \). Hence by taking the first terms in the Taylor series of the zeta-functions we have

\[
N_i(\alpha, \beta, \gamma) = \frac{4}{(i-2)!L} \oint_{\gamma x_1=0} \frac{d^2}{dx_1^2} \left[ \int_0^1 \int_0^1 y_2^{\gamma x_1} N(x_1) \right]_{x_1=0} du + O(L^{i-3}).
\]

We next use the identity

\[
(\gamma + u)^2 = \frac{1}{(\log y_2)^2} \frac{d^2}{dx_1^2} \left( y_2^{\gamma x_1} \right)_{x_1=0},
\]

to write

\[
N_i(\alpha, \beta, \gamma) = \frac{4}{(i-2)!L} \oint_{\gamma x_1=0} \frac{d^2}{dx_1^2} y_2^{\gamma x_1} N(x_1) \left|_{x_1=0} \right. du + O(L^{i-3}).
\]

Taking the power series gives

\[
N(x_1) = \sum_{k=0}^{\infty} \frac{(x_1 \log y_2 + \log q)^k}{k!} \int_{\gamma x_1=0} \frac{u^{k-i+1}}{(\alpha + u)(\beta + u)} du.
\]

As there are three poles inside the contour, it is slightly easier to compute the residue at infinity. In other words, changing the variable \( u \to 1/u \) yields

\[
N(x_1) = \sum_{k=0}^{\infty} \frac{(x_1 \log y_2 + \log q)^k}{k!} \int_{\gamma x_1=0} \frac{u^{-k-1}}{(1+\alpha u)(1+\beta u)} du.
\]

In view of the power series of \((1+\alpha u)^{-1}\) and \((1+\beta u)^{-1}\), we have

\[
N(x_1) = \sum_{k=0}^{\infty} \frac{(x_1 \log y_2 + \log q)^k}{k!} \sum_{l_1,l_2 \geq 0} (-\alpha)^{l_1} (-\beta)^{l_2} \frac{1}{2\pi i} \oint u^{l_1+l_2+k-1} du.
\]

The integral picks out the terms \( k = i + l_1 + l_2 \), giving

\[
N(x_1) = (x_1 \log y_2 + \log q)^i \sum_{l_1,l_2 \geq 0} (-\alpha)^{l_1} (-\beta)^{l_2} \frac{1}{2\pi i} \oint u^{l_1+l_2+k-1} du.
\]

We now separate the variables \( l_1, l_2 \) and \( i \) by using the standard beta function and its integral representation

\[
\frac{1}{(i+l_1+l_2)!} = B(i-1,l_1+l_2+2) \frac{1}{(i-2)!(l_1+l_2+1)!}
\]
we have

\[ \frac{1}{(i-2)!l_1l_2!} \int_0^1 \int_0^1 (1-t_1)t_1^{i-2}t_1^{i-2}t_2 dt_1 dt_2. \]

Putting this into (32) we obtain

\[
N(x_1) = \frac{(x_1 \log y_2 + \log q)^i}{(i-2)!} \int_0^1 \int_0^1 (1-t_1)^{i-2}t_1^{i-2}t_2 \exp \left\{-\alpha(x_1 \log y_2 + \log q)t_1(1-t_2) - \beta(x_1 \log y_2 + \log q)t_2 \right\} dt_1 dt_2,
\]

and the lemma follows.

\[\square\]

**Lemma 6.2.** Suppose \( j \geq 4 \). Then for some \( \nu \asymp (\log \log y_3)^{-1} \) we have

\[
N_j(\alpha, \beta) = \frac{1}{j!(\log y_3)^4} \frac{d^4}{dx_2^2dx_3^2} \left[ y_3^{\alpha x_2+\beta x_3} \left( (x_2 + x_3) \log y_3 + \log y_3/n \right)^j \right]_{x_2=x_3=0} + O(L^{-j-5}) + O\left( \left( \frac{y_3}{n} \right)^{-\nu} \right).
\]

**Proof.** Similarly to Lemma 5.1, \( N_j(\alpha, \beta) \) equals the residue at \( v = 0 \) plus an error of size \( O\left( (y_3/n)^{-\nu} L^\varepsilon \right) + O(L^{-2+\varepsilon}). \)

For the residue at \( v = 0 \), we write this as

\[
\frac{1}{2\pi i} \int \frac{(y_3/n)^v}{(y_3/n)^{1+\alpha+\beta}} \frac{dv}{(1+\alpha+\beta)^2(1+\beta+\nu^2)}
\]

where the contour is a circle of radius \( \asymp L^{-1} \) around the origin. This integral is trivially bounded by \( O(L^{-1}) \). Hence by taking the first terms in the Taylor series of the zeta-functions we have

\[
\text{Res}_{v=0} = \frac{1}{2\pi i} \int \frac{(y_3/n)^v}{(y_3/n)^{1+\alpha+\beta}} \frac{dv}{(1+\beta+\nu^2)} + O(L^{-j-5}).
\]

The above integral can be written in a compact form as

\[
\frac{1}{j!(\log y_3)^4} \frac{d^4}{dx_2^2dx_3^2} \left[ y_3^{\alpha x_2+\beta x_3} \left( (x_2 + x_3) \log y_3 + \log y_3/n \right)^j \right]_{x_2=x_3=0},
\]

and the lemma follows. \(\square\)

In view of (30) and Lemmas 6.1 and 6.2 we get

\[
K_{i,j} = \frac{4(\log y_2)^{i-2}(\log y_3)^{j-4}}{(i-2)!j!} \int_0^1 \int_0^1 \frac{d^6}{dx_1^2dx_2^2dx_3^2} \left[ \frac{1}{y_3} \frac{d_4(n)}{n} \frac{y_3}{n} \left( \frac{y_3}{n} \right)^{-\nu} \sum_{n \leq y_3} \frac{d_4(n)}{n} \frac{y_3}{n} \right] + O(L^{i+j-3}).
\]

Using Lemmas 4.1 and 4.2 the \( O \)-terms are \( \ll \varepsilon L^{i+1+\varepsilon} + L^{i+j-3} \), while the first term is

\[
\frac{2(\log y_2)^{i-2}(\log y_3)^j}{3(i-2)!j!} \frac{d^6}{dx_1^2dx_2^2dx_3^2} \int_0^1 \int_0^1 \int_0^1 \gamma x_1 - \left( \alpha(1-t_2)+\beta t_2 \right) t_1 x_1 y_3^{\alpha x_2+\beta x_3} (1-t_1)^{i-2} t_1 \]

while
Applying (25) leads to

\[
\left( x_1 + 1 - \frac{\partial}{\partial u} (1-u) \right) \left( x_2 + x_3 + u \right)^{j} dt_1 dt_2 du \bigg|_{u=0} + O(\mathcal{L}^{i+j-3}).
\]

Putting this back to (25) and using the assumption that \( j \geq 4 \) we get

\[
I_{23}^{1} = \frac{2\tilde{w}(0)}{3(\log y_2)^2} \zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma) \frac{d^6}{dx_1^2 dx_2^2 dx_3^2} \left[ \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} U_2(x) t_1 \left( x_1 + 1 - \frac{\partial}{\partial u} (1-u) \right)^2 (1-u)^3 \right. \\
\left. P'_2 \left( (x_1 + 1 - \frac{\partial}{\partial u} (1-u))(1-t_1) \right) P_3(x_2 + x_3 + u) dt_1 dt_2 du \bigg|_{u=0} + O_{x}(\mathcal{L}^{-1+\epsilon}). \right.
\]

6.2. Deduction of Lemma 3.2 Combining \( I_{23}^{1}, I_{23}^{2} \) and \( I_{23}^{3} \) we have

\[
I_{23}^{3} = \frac{2\tilde{w}(0)}{3(\log y_2)^2} \frac{d^6}{dx_1^2 dx_2^2 dx_3^2} \left[ \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} U_2(x) t_1 \left( x_1 + 1 - \frac{\partial}{\partial u} (1-u) \right)^2 (1-u)^3 \right. \\
\left. P'_2 \left( (x_1 + 1 - \frac{\partial}{\partial u} (1-u))(1-t_1) \right) P_3(x_2 + x_3 + u) dt_1 dt_2 du \bigg|_{u=0} + O_{x}(\mathcal{L}^{-1+\epsilon}), \right.
\]

where

\[
U_2 = \frac{\gamma_{x_1} \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1+x_1) \alpha x_2 + \beta x_3 + \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1-u)}{y_2} \\
- T_{-(\alpha+\gamma)} y_2 \frac{-\alpha x_1 - (-\gamma(1-t_2) + \beta t_2) t_1(1+x_1) - \gamma x_2 + \beta x_3 + (-\gamma(1-t_2) + \beta t_2) t_1(1-u)}{(\alpha+\gamma)(\beta-\alpha)} \\
+ T_{-(\beta+\gamma)} y_2 \frac{-\beta x_1 - (\alpha(1-t_2) - \gamma t_2) t_1(1+x_1) \alpha x_2 - \gamma x_3 + (\alpha(1-t_2) - \gamma t_2) t_1(1-u)}{(\alpha-\beta)(\beta+\gamma)}. \\
\]

As in the previous section we can write

\[
U_2 = \frac{\gamma_{x_1} \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1+x_1) \alpha x_2 + \beta x_3 + \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1-u)}{y_2} \\
\left( \frac{1 - (T y_2^{-1} t_1(1-t_2)(1+x_1) x_3 + t_1(1-t_2)(1-u))}{(\alpha+\gamma)} \right) \\
\frac{\beta - \alpha}{\alpha + \gamma} \\
\left( \frac{1 - (T y_2^{-1} t_1(1-t_2)(1+x_1) x_3 + t_1(1-t_2)(1-u))}{(\beta+\gamma)} \right). \\
\]

Applying (25) leads to

\[
U_2 = \frac{\mathcal{L}}{\beta - \alpha} \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1(1-t_2) \left( \vartheta_2(1+x_1) - \vartheta_3(1-u) \right) \right) \\
\gamma_{x_1} \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1+x_1) \alpha x_2 + \beta x_3 + \left( \alpha(1-t_2) + \beta t_2 \right) t_1(1-u) \\
\left. \int_{0}^{1} T y_2^{-1} t_1(1-t_2)(1+x_1) x_3 + t_1(1-t_2)(1-u) \right)^{(\alpha+\gamma)} t_3 dt_3 \\
- \frac{\mathcal{L}}{\beta - \alpha} \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 \left( \vartheta_2(1+x_1) - \vartheta_3(1-u) \right) \right). \\
\]
Changing $t_2 \to 1 - t_2$ in the first term, and changing the roles of the variables $x_2$ with $x_3$ in the second term yields

$$I_{23} = \frac{2\hat{\omega}(0)\mathcal{L}}{3(\log y_2)^2} \frac{d^6}{dx_1^2 dx_2^2 dx_3^2} \left[ \int_{[0,1]^4} V_2(x) \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 (\vartheta_2 (1 + x_1) - \vartheta_3 (1 - u)) \right) t_1 \left( x_1 + 1 - \frac{\vartheta_2}{\vartheta_3} (1 - u) \right)^2 (1 - u)^3 P_2'' \left( x_1 + 1 - \frac{\vartheta_2}{\vartheta_3} (1 - u) \right) (1 - t_1) \right]_{\varepsilon=0} + O_{\varepsilon}(T\mathcal{L}^{-1+\varepsilon}),$$

where

$$V_2 = \frac{1}{\beta - \alpha} \left( \gamma x_1 - (\alpha t_2 + \beta (1-t_2)) t_1 (1+x_1) \right) \frac{\alpha x_2 + \beta x_3 + (\alpha t_2 + \beta (1-t_2)) t_1 (1-u)}{y_3} \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\alpha+\gamma)t_3} - \gamma x_1 - (\alpha (1-t_2) + \beta t_2) t_1 (1+x_1) \frac{\alpha x_3 + \beta x_2 + (\alpha (1-t_2) + \beta t_2) t_1 (1-u)}{y_3} \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\beta+\gamma)t_3}.$$

Using (25) again implies that $V_2$ equals

$$\begin{align*}
\gamma x_1 - (\alpha t_2 + \beta (1-t_2)) t_1 (1+x_1) & \frac{\alpha x_2 + \beta x_3 + (\alpha t_2 + \beta (1-t_2)) t_1 (1-u)}{y_3} \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\alpha+\gamma)t_3} \\
& \left( 1 - \left( y_2 (2t_2 - 1)(1+x_1) - x_2 + x_3 - t_1 (2t_2 - 1)(1-u) \right) \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\beta+\gamma)t_3} \right) \left( 1 - \frac{\vartheta_2}{\vartheta_3} (1 - u) \right)^2 (1 - u)^3 P_2'' \left( x_1 + 1 - \frac{\vartheta_2}{\vartheta_3} (1 - u) \right) (1 - t_1) \\
& = \mathcal{L} \left( - \vartheta_3 (x_2 - x_3) + t_1 (2t_2 - 1) (\vartheta_2 (1 + x_1) - \vartheta_3 (1 - u)) \right) + t_3 \left( 1 + \vartheta_2 x_1 + \vartheta_3 x_2 - t_1 t_2 (\vartheta_2 (1 + x_1) - \vartheta_3 (1 - u)) \right) t_1 \left( x_1 + 1 - \frac{\vartheta_2}{\vartheta_3} (1 - u) \right)^2 (1 - u)^3 \mathcal{L} \left( x_2 + x_3 - t_1 (2t_2 - 1)(1-u) \right) \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\beta+\gamma)t_3} \int_0^1 \left( T y_2 x_1 - t_1 t_2 (1+x_1) x_2 + t_1 t_2 (1-u) \right)^{-(\beta+\gamma)t_3} dt_4,
\end{align*}$$

and the lemma follows.

7. Proof of Lemma 3.1

7.1. Reduction to a contour integral. We first state the twisted fourth moment of the Riemann zeta-function [2].

**Theorem 7.1** (Bettin, Bui, Li and Radziwill). Suppose $H, K \leq T^{1/4-\varepsilon}$. Then we have

$$\sum_{k \leq H, \delta} a_k \overline{a_k} \int_{-\infty}^{\infty} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta + it) \zeta(\frac{1}{2} + \gamma - it) \zeta(\frac{1}{2} + \delta - it) \left( \frac{H}{K} \right)^{it} w(t) dt$$
planes containing the origin. Hence

\[ Z_{\alpha,\beta,\gamma,\delta}(h,k) = \sum_{k=1}^{\infty} \frac{a_k}{h^k} \int_{-\infty}^{\infty} w(t) \left\{ Z_{\alpha,\beta,\gamma,\delta}(h,k) + \left( \frac{t}{2\pi} \right)^{-\alpha} Z_{\gamma,\beta,-\alpha,\delta}(h,k) ight. \\
+ \left( \frac{t}{2\pi} \right)^{-\alpha+\delta} Z_{-\beta,\gamma,-\alpha,\delta}(h,k) + \left( \frac{t}{2\pi} \right)^{-\beta+\gamma} Z_{\alpha,-\gamma,-\beta,\delta}(h,k) \\
+ \left( \frac{t}{2\pi} \right)^{-\beta+\delta} Z_{\alpha,-\delta,-\beta,\alpha}(h,k) + \left( \frac{t}{2\pi} \right)^{-\alpha+\beta+\gamma+\delta} Z_{-\gamma,-\delta,-\alpha,-\beta}(h,k) \right\} dt + O_\varepsilon(T^{1-\varepsilon}) \]

uniformly for \( \alpha, \beta, \gamma, \delta \ll L^{-1} \), where

\[ Z_{\alpha,\beta,\gamma,\delta}(h,k) = \sum_{k=abcd} \frac{1}{a^{1/2+\alpha}b^{1/2+\beta}c^{1/2+\gamma}d^{1/2+\delta}}. \]

Recall that \( I_3(\alpha, \beta, \gamma, \delta) \) is defined by (10). We write

\[ I_3(\alpha, \beta, \gamma, \delta) = I_3^1 + I_3^2 + I_3^3 + I_3^4 + I_3^5 + O_\varepsilon(T^{1-\varepsilon}) \]

correspondingly to the decomposition in Theorem (7.1). We first work on \( I_3^1 \), which is equal to

\[ \tilde{w}(0) \sum_{l,m} \frac{\mu_2(m)\mu_2(l)}{\sqrt{4\pi m}} P_3(\frac{\log y_m}{\log y_3}) P_3(\frac{\log y_l}{\log y_3}) \sum_{mabcd} \frac{1}{a^{1/2+\alpha}b^{1/2+\beta}c^{1/2+\gamma}d^{1/2+\delta}}. \]

In view of (11) we get

\[ I_3^1 = \tilde{w}(0) \sum_{l,m} \frac{c_ic_j l^j}{(\log y_3)^{i+j}} \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_3^{u+v} \sum_{mabcd} \frac{\mu_2(m)\mu_2(l)}{m^{1/2+u}a^{1/2+v}b^{1/2+\alpha}c^{1/2+\beta}d^{1/2+\gamma}e^{1/2+\delta}} du \ dv, \]

The arithmetical sum is

\[ \sum_{mabcd} \frac{\mu_2(m)\mu_2(l)}{m^{1/2+u}a^{1/2+v}b^{1/2+\alpha}c^{1/2+\beta}d^{1/2+\gamma}e^{1/2+\delta}} = C(\alpha, \beta, \gamma, \delta, u, v) \frac{(1+\alpha+\gamma)(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}{\zeta(1+\alpha+v)\zeta(1+\beta+v)^2\zeta(1+\gamma+u)^2\zeta(1+\delta+u)^2}, \]

where \( C(\alpha, \beta, \gamma, \delta, u, v) \) is an arithmetical factor converging absolutely in a product of half-planes containing the origin. Hence

\[ I_3^1 = \tilde{w}(0) \zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta) \sum_{i,j} \frac{c_ic_j l^j}{(\log y_3)^{i+j}} L_{i,j}, \]

where

\[ L_{i,j} = \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_3^{u+v} \frac{C(\alpha, \beta, \gamma, \delta, u, v)\zeta(1+u+v)^4}{\zeta(1+\alpha+v)^2\zeta(1+\beta+v)^2\zeta(1+\gamma+u)^2\zeta(1+\delta+u)^2} \frac{du}{u^{i+1}} \frac{dv}{v^{j+1}}. \]

Using the Dirichlet series for \( \zeta(1+u+v)^4 \) and reversing the order of summation and integration, we have

\[ L_{i,j} = \sum_{n \leq y_3} \frac{d_4(n)}{n} \left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} \frac{y_3^{u+v}}{u^{i+1} v^{j+1}} \frac{C(\alpha, \beta, \gamma, \delta, u, v)\zeta(1+\alpha+v)^2\zeta(1+\beta+v)^2\zeta(1+\gamma+u)^2\zeta(1+\delta+u)^2}{u^{i+1} v^{j+1}} \frac{du}{u^{i+1}} \frac{dv}{v^{j+1}}. \]

Here we are able to restrict the sum over \( n \) to \( n \leq y_3 \) by moving the \( u, v \)-integrals far to the right. We now move the contours of integration to \( \text{Re}(u) = \text{Re}(v) \gg L^{-1} \). Bounding the
integrals trivially shows that $L_{i,j} \ll \mathcal{L}^{i-j-4}$. As before we can replace $C(\alpha, \beta, \gamma, \delta, u, v)$ by $C(0, 0, 0, 0, 0, 0)$ in $L_{i,j}$ with an error of size $O(\mathcal{L}^{i-j-5})$. By letting $\alpha = \beta = \gamma = \delta = u = v = s$ in (33), it is easy to verify that $C(0, 0, 0, 0, 0, 0) = 1$. The $u$ and $v$ variables in (33) are now separated so that

$$L_{i,j} = \sum_{n \leq y_3} \frac{d_4(n)}{n} N_i(\gamma, \delta) N_j(\alpha, \beta) + O(\mathcal{L}^{i-j-5}),$$

where the function $N_j(\alpha, \beta)$ is defined in (31). Using Lemma 5.2 we obtain

$$L_{i,j} = \frac{(\log y_3)^{i+j-8}}{6! j!} \frac{d^8}{dx_1^2 dx_2^2 dx_3^2 dx_4^2} \left[ \int_0^1 \frac{y_3^{\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4}}{\log y_3} (1-u)^3 (x_3 + x_4 + u)^i (x_1 + x_2 + u)^j \right]_{x=0} + O(\mathcal{L}^{i-j-5}).$$

In view of Lemmas 4.1 and 4.2 the $O$-terms are $\ll \mathcal{L}^{-1+\epsilon} + \mathcal{L}^{-j-1+\epsilon} + \mathcal{L}^{i-j-5}$, while the first term is $

\frac{(\log y_3)^{i+j-4}}{6! j!} \frac{d^8}{dx_1^2 dx_2^2 dx_3^2 dx_4^2} \left[ \int_0^1 \frac{y_3^{\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4}}{\log y_3} (1-u)^3 (x_3 + x_4 + u)^i (x_1 + x_2 + u)^j \right]_{x=0} + O(\mathcal{L}^{i-j-5}).$
We write
\[
\frac{y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}}{(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)} = \frac{y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}}{(\alpha + \gamma)(-\gamma + \delta)(\beta - \alpha)(\beta + \delta)} - \frac{y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}}{(-\gamma + \delta)(\alpha + \delta)(\beta + \gamma)(\beta - \alpha)}
\]
and
\[
T^{-(\alpha + \beta + \gamma + \delta)} y_3^{-\gamma x_1-\delta x_2-\alpha x_3-\beta x_4} = \frac{T^{-(\alpha + \beta + \gamma + \delta)} y_3^{-\gamma x_1-\delta x_2-\alpha x_3-\beta x_4}}{(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)} - \frac{T^{-(\alpha + \beta + \gamma + \delta)} y_3^{-\gamma x_1-\delta x_2-\alpha x_3-\beta x_4}}{(-\gamma + \delta)(\alpha + \delta)(\beta + \gamma)(\beta - \alpha)}.
\]
Notice that we can change the roles of \(x_1\) with \(x_2\), or of \(x_3\) with \(x_4\) in any term of \(U(x)\) without affecting the value of \(I_3(\alpha, \beta, \gamma, \delta)\) in (36). Applying both changes to the last term in (37), we can replace \(U(x)\) with
\[
\frac{y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}}{(-\gamma + \delta)(\beta - \alpha)} \left(1 - (T y_3^{x_1+x_3})^{-(\alpha+\gamma)}\right) \left(1 - (T y_3^{x_2+x_4})^{-(\beta+\delta)}\right)
\]
and
\[
\frac{y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}}{(-\gamma + \delta)(\beta - \alpha)} \left(1 - (T y_3^{x_1+x_3})^{-(\alpha+\delta)}\right) \left(1 - (T y_3^{x_2+x_4})^{-(\beta+\gamma)}\right).
\]
Using (29) we then get
\[
I_3(\alpha, \beta, \gamma, \delta) = \frac{\mathcal{L}^2 \hat{w}(0)}{6(\log y_3)^4} \int \frac{d^6}{[0,1]^3} \left[ V_1(x_1, t_1, t_2) - V_2(x_1, t_1, t_2) \right]
\]
\[
(1 - u)^3 P_3(x_1 + x_2 + u) P_3(x_3 + x_4 + u) dt_1 dt_2 du \bigg|_{u=0} + O(\mathcal{L}^{-1+\varepsilon}),
\]
where
\[
V_1 = y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_1+x_3})^{-(\alpha+\gamma)} (T y_3^{x_2+x_4})^{-(\beta+\delta)} \left(1 + \vartheta_3(x_1 + x_3)\right) \left(1 + \vartheta_3(x_2 + x_4)\right)
\]
and
\[
V_2 = y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_1+x_4})^{-(\alpha+\delta)} (T y_3^{x_2+x_3})^{-(\beta+\gamma)} \left(1 + \vartheta_3(x_1 + x_4)\right) \left(1 + \vartheta_3(x_2 + x_3)\right).
\]
Again notice that \(I_3(\alpha, \beta, \gamma, \delta)\) is unchanged if we swap any of these pairs of variables \(x_1 \leftrightarrow x_2\), \(x_3 \leftrightarrow x_4\) and \(t_1 \leftrightarrow t_2\) in \(V_1(\mathbf{x}, t_1, t_2)\) or \(V_2(\mathbf{x}, t_1, t_2)\). We next replace \(V_1 - V_2\) in the integrand with
\[
\frac{1}{2} \left( V_1(x_1, x_2, x_3, x_4, t_1, t_2) - V_2(x_2, x_1, x_3, x_4, t_2, t_1) - V_2(x_1, x_2, x_3, x_4, t_1, t_2) + V_1(x_2, x_1, x_3, x_4, t_2, t_1) \right),
\]
which is
\[
\frac{1}{2} \left( (1 + \vartheta_3(x_1 + x_3)) (1 + \vartheta_3(x_2 + x_4)) \right) \left( y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_1+x_3})^{-(\alpha+\gamma)} (T y_3^{x_2+x_4})^{-(\beta+\delta)} \right)
\]
\[
- y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_2+x_4})^{-(\alpha+\delta)} (T y_3^{x_1+x_3})^{-(\beta+\gamma)} \left(1 + \vartheta_3(x_2 + x_4)\right) \left(1 + \vartheta_3(x_1 + x_3)\right)
\]
\[
- y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_1+x_4})^{-(\alpha+\delta)} (T y_3^{x_2+x_3})^{-(\beta+\gamma)} \left(1 + \vartheta_3(x_1 + x_4)\right) \left(1 + \vartheta_3(x_2 + x_3)\right)
\]
\[
+ y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4} (T y_3^{x_1+x_4})^{-(\alpha+\gamma)} (T y_3^{x_2+x_3})^{-(\beta+\delta)} \left(1 + \vartheta_3(x_1 + x_4)\right) \left(1 + \vartheta_3(x_2 + x_3)\right)
\]
\[
= \frac{1}{2} \left( (1 + \vartheta_3(x_1 + x_3)) (1 + \vartheta_3(x_2 + x_4)) \right) y_3^{\alpha x_1+\beta x_2+\gamma x_3+\delta x_4}
\]
\[(T y_3^{x_1+x_3})^{-(\alpha+\gamma)} (T y_3^{x_2+x_4})^{-(\beta+\delta)} \left(1 - \left(T^{t_1-t_2} y_3^{-(x_1+x_3)t_1-(x_2+x_4)t_2}\right)^{-(\beta-\gamma)}\right)^{(\beta-\gamma)}
\]

Using (25) again and simplifying we obtain Lemma 3.1.

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