GRAVITATIONAL ENERGY-MOMENTUM IN MAG

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Energy-momentum (and angular momentum) for the Metric-Affine Gravity theory is considered from a Hamiltonian perspective (linked with the Noether approach). The important roles of the Hamiltonian boundary term and the many choices involved in its selection—which give rise to many different definitions—are emphasized. For each choice one obtains specific boundary conditions along with a value for the quasilocal, and (with suitable asymptotic behavior) total (Bondi and ADM) energy-momentum and angular momentum. Applications include the first law of black hole thermodynamics—which identifies a general expression for the entropy. Prospects for a positive energy proof are considered and quasilocal values for some solutions are presented.

1 Gravitational Energy-Momentum

Energy-momentum is a fundamental conserved quantity which is associated with the symmetry of space-time geometry. In the modern view space-time geometry is dynamic and this is the basis for our gravity theories. The primary source of gravity is the energy-momentum density for matter and all other interaction fields. But these sources can exchange energy-momentum with the gravitational field locally, which leads to the expectation that gravity should also have its own local energy-momentum density.

While total energy-momentum is well defined (for gravitating systems with suitable asymptotics) standard techniques for identifying a local “gravitational energy-momentum density” gave only various, noncovariant, reference frame dependent pseudotensors, which cannot give a well defined localization. This can be understood in terms of the equivalence principle, which implies that the gravitational field cannot be detected at a point.

It is now believed that the proper idea is quasi-local (i.e., associated with a closed 2-surface) energy-momentum. The many proposals and approaches have been referred to elsewhere. Among the various criteria for a good quasilocal energy-momentum expression that have been advocated we typically find good limits: including ADM (spatial infinity), Bondi (null infinity), weak field and flat spacetime. However it has been observed that there are an infinite number of expressions satisfying such requirements. Hence additional principles and criteria are very much needed.

2 Hamiltonian approach

One approach is to regard energy as the value of the Hamiltonian. The gravitational Hamiltonian (for a finite region $\Sigma$),

$$H(N) = \int_\Sigma N^\mu H_\mu + \oint_{S=\partial \Sigma} B(N),$$

(1)
depends on a displacement vector field $N$ and includes a spatial hypersurface and a spatial 2-boundary term. It turns out that the boundary term plays a very important role, giving both the quasilocal values and the boundary conditions.

For our purpose differential form notation has several advantages, which are associated with the (generalized Stokes) boundary theorem, spacetime projection via pullback and the interior product, and a neat representation of geometric objects. Consider a first order Lagrangian for a k-form field

$$\mathcal{L} = d\varphi \wedge p - \Lambda.$$  \hspace{1cm} (2)

The general variational formula

$$\delta \mathcal{L} = d(\delta \varphi \wedge p) + \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p$$ \hspace{1cm} (3)

implicitly defines the pairs of first order field equations. Local diffeomorphism invariance requires this relation to be identically satisfied for $\delta = \mathcal{L}_N$, the Lie derivative. Consequently, since $\mathcal{L}_N \equiv i_N d + d i_N$ on the components of forms,

$$\mathcal{L}_N \mathcal{L} = d i_N \mathcal{L} \equiv d(\mathcal{L}_N \varphi \wedge p) + \mathcal{L}_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_N p.$$  \hspace{1cm} (4)

Hence the Hamiltonian 3-form,

$$\mathcal{H}(N) := \mathcal{L}_N \varphi \wedge p - i_N \mathcal{L},$$ \hspace{1cm} (5)

satisfies the differential identity

$$d \mathcal{H}(N) \equiv (\text{terms proportional to field equations}).$$ \hspace{1cm} (6)

Rearranging (5) using (3) gives an expression of the form

$$\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d \mathcal{B}(N).$$ \hspace{1cm} (7)

Upon substitution of $d \mathcal{H}(N) = d(N^\mu \mathcal{H}_\mu) = dN^\mu \wedge \mathcal{H}_\mu + N^\mu d \mathcal{H}_\mu$ into the differential identity (6), the coefficient of $dN^\mu$ gives an algebraic identity

$$\mathcal{H}_\mu \equiv (\text{terms proportional to field equations}).$$ \hspace{1cm} (8)

Hence “on shell” (i.e., when the field equations are satisfied) the Hamiltonian 3-form $\mathcal{H}(N)$ is a “conserved current”. The “conserved” value of the Hamiltonian (the integral of (5), having the aforementioned form (7))—since $\mathcal{H}_\mu$ vanishes “on shell”—depends only on the spatial 2-boundary term, which thus determines the quasilocal energy-momentum and angular momentum.

However, as with other Noether currents, $\mathcal{H}(N)$ is not unique. We can add to it a total differential (without changing the Hamiltonian equations of motion or the conservation property). This amounts to modifying $\mathcal{B}(N)$, allowing one to “improve” the quasilocal expression. Indeed in many cases (including General Relativity) it is necessary to adjust $\mathcal{B}$. Fortunately, $\mathcal{B}$ is not arbitrary. A further principle of the formalism controls its form: one should choose the Hamiltonian boundary term $\mathcal{B}$ so that the boundary term in $\delta \mathcal{H}$ vanishes, when the desired fields are held fixed (“controlled”) on $S$ (as discussed elsewhere in detail, technically this is necessary in order for the Hamiltonian to be differentiable). There is thus a nice division: the Hamiltonian density $\mathcal{H}_\mu$ determines the evolution and
constraint equations, the boundary term \( B \) determines the boundary conditions and the quasilocal energy-momentum.

Along with this Hamiltonian variation boundary principle we have advocated an additional criterion, namely covariance. For each dynamical field we found\(^3\) using symplectic techniques\(^8\) that there are only two covariant choices for \( B \):

\[
B_\varphi(N) = i_N \varphi \wedge \Delta p - (-1)^k \Delta \varphi \wedge i_N \bar{p},
\]

\[
B_p(N) = i_N \bar{\varphi} \wedge \Delta p - (-1)^k \Delta \varphi \wedge i_N p,
\]

here \( \Delta \varphi := \varphi - \bar{\varphi} \), and \( \Delta p := p - \bar{p} \) where \( \bar{\varphi}, \bar{p} \) represent reference values. The associated Hamiltonian variations have the form

\[
\delta H_\varphi(N) = \text{field eqn terms} + di_N (\delta \varphi \wedge \Delta p),
\]

\[
\delta H_p(N) = \text{field eqn terms} - di_N (\Delta \varphi \wedge \delta p),
\]

revealing a boundary symplectic structure, which yields the associated boundary conditions: respectively Dirichlet or Neumann “control mode”. Only for \( B_\varphi \) and \( B_p \) are the Hamiltonian variation boundary terms projections of 4-covariant expressions. Note that, just as in thermodynamics (with enthalpy, Gibbs, Helmholtz, \ldots), there are various kinds of energy corresponding to different boundary conditions.

Specifying the quasilocal boundary term \( B(N) \) involves choices including

- the **representation**, (i.e., the dynamic variables) e.g., the metric, orthonormal frame, connection, spinors.

- the **control mode**: the boundary conditions, essentially Dirichlet or Neumann.

- the **reference configuration**: e.g., Minkowski, de Sitter, Friedmann-Robertson-Walker cosmology, Schwarzschild. The meaning is that all quasilocal quantities vanish when the field has the reference values, so it determines the zero of energy etc.

- the **displacement vector field** \( N \): Which timelike displacement gives the energy? Which spatial displacement gives the momentum? Which rotational displacement gives the angular momentum?

### 3 Metric Affine Gravity

We now apply these ideas to the Metric Affine Gravity Theory (MAG)\(^1\)\(^2\)\(^3\). The geometric potentials are the metric coefficients \( g_{\mu\nu} \), the coframe 1-form \( \vartheta^\alpha \), and the connection 1-form \( \Gamma^\alpha_{\beta\gamma} \). The associated field strengths are

\[
Dg_{\mu\nu} := dg_{\mu\nu} - \Gamma^\gamma_{\mu\nu} g_{\gamma\nu} - \Gamma^\gamma_{\nu\mu} g_{\gamma\nu},
\]

\[
T^\alpha := D\vartheta^\alpha := d\vartheta^\alpha + \Gamma^\alpha_{\beta\gamma} \wedge \vartheta^\beta,
\]

\[
R^\alpha_{\beta} := d\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta}.
\]

the **non-metricity** 1-form, the **torsion** 2-form, and the **curvature** 2-form, respectively.

Independent variation with respect to the potentials \((g, \vartheta, \Gamma)\) and conjugate momenta \((\pi, \tau, \rho)\) of the “first order” (source free) MAG Lagrangian 4-form:

\[
\mathcal{L} := Dg_{\mu\nu} \wedge \pi^{\mu\nu} + T^\alpha \wedge \tau_{\alpha} + R^\alpha_{\beta} \wedge \rho_{\alpha}^\beta - \Lambda(g, \vartheta, \pi, \tau, \rho),
\]
yields

\[ \delta \mathcal{L} = d(\delta g_{\mu\nu} \pi^{\mu\nu} + \delta \theta^\alpha \wedge \tau_\alpha + \delta \Gamma^\alpha_\beta \wedge \rho_\alpha^\beta) \]

\[ + \delta g_{\mu\nu} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} + \delta \theta^\alpha \wedge \frac{\delta \mathcal{L}}{\delta \theta^\alpha} + \delta \Gamma^\alpha_\beta \wedge \frac{\delta \mathcal{L}}{\delta \Gamma^\alpha_\beta} \]

\[ + \frac{\delta \mathcal{L}}{\delta \pi^{\mu\nu}} \wedge \delta \pi^{\mu\nu} + \frac{\delta \mathcal{L}}{\delta \tau_\alpha} \wedge \delta \tau_\alpha + \frac{\delta \mathcal{L}}{\delta \rho_\alpha^\beta} \wedge \delta \rho_\alpha^\beta, \quad (17) \]

which implicitly defines the first order equations (the detailed form is not needed here).

We decompose the Lagrangian according to

\[ \mathcal{L} \equiv dt \wedge i_N \mathcal{L} \]

\[ = dt \wedge (\mathcal{L}_N g_{\mu\nu} \pi^{\mu\nu} + \mathcal{L}_N \theta^\alpha \wedge \tau_\alpha + \mathcal{L}_N \Gamma^\alpha_\beta \wedge \rho_\alpha^\beta - \mathcal{H}(N)). \quad (18) \]

to find the covariant Hamiltonian 3-form. Explicitly, it has the standard form \([7]\)

where

\[ N^\mu \mathcal{H}_\mu \equiv i_N \Lambda + D g_{\mu\nu} \wedge i_N \pi^{\mu\nu} - T^\alpha \wedge i_N \tau_\alpha - R^\alpha_\beta \wedge i_N \rho_\alpha^\beta \]

\[ - i_N \theta^\alpha \wedge D \tau_\alpha - i_N \Gamma^\alpha_\beta (D \rho_\alpha^\beta - g_{\alpha\varepsilon} \pi^{\varepsilon\beta} - g_{\mu\alpha} \pi^{\mu\beta} + \theta^\beta \wedge \tau_\alpha), \quad (19) \]

\[ \mathcal{B}(N) \equiv i_N \theta^\alpha \tau_\alpha + i_N \Gamma^\alpha_\beta \rho_\alpha^\beta. \quad (20) \]

We then replace the Hamiltonian boundary term \([20]\) by choosing one of the covariant quasilocal boundary expressions for the MAG

\[ \mathcal{B}(N) = \left\{ \begin{array}{l}
-\Delta g_{\mu\nu} i_N \pi^{\mu\nu} \\
-\Delta g_{\mu\nu} i_N \pi^{\mu\nu}
\end{array} \right\} + \left\{ \begin{array}{l}
 i_N \theta^\alpha \Delta \tau_\alpha + \Delta \theta^\alpha \wedge i_N \pi^{\alpha} \\
i_N \theta^\beta \Delta \tau_\alpha + \Delta \theta^\beta \wedge i_N \tau_\alpha
\end{array} \right\} + \left\{ \begin{array}{l}
\bar{D}_\beta N^\alpha \Delta \rho_\alpha^\beta + \Delta \Gamma^\alpha_\beta \wedge i_N \bar{\rho}_\alpha^\beta \\
\bar{D}_\beta N^\alpha \Delta \rho_\alpha^\beta + \Delta \Gamma^\alpha_\beta \wedge i_N \bar{\rho}_\alpha^\beta
\end{array} \right\}, \quad (21) \]

where the upper (lower) line in each bracket is to be selected if the field (momentum) is controlled. Again there are several kinds of energy, each corresponds to the work done in a different (ideal) physical process.

A technical point here is that we replaced the \(i_N \Gamma\) terms using the identity

\[ \mathcal{L}_N \theta^\alpha \equiv DN^\alpha + i_N T^\alpha - i_N \Gamma^\alpha_\beta \theta^\beta \equiv \bar{D} N^\alpha - i_N \Gamma^\alpha_\beta \theta^\beta, \quad (22) \]

and then dropped the non-covariant, frame gauge dependent \(\mathcal{L}_N \theta \Delta \rho\) terms, to obtain fully covariant expressions. These covariant end results follow directly from a different treatment of the connection\([21]\).

With standard flat asymptotics:

\[ N^\alpha \sim (\text{constant} + O(1/r))^+ + (\epsilon^\alpha_\beta x^\beta + O(1)), \quad (23) \]

\[ \{\Delta g, \Delta \theta, \Delta \rho\} \sim O^+(1/r) + O^-(1/r^2), \quad (24) \]

\[ \{\Delta \pi, \Delta \tau, \Delta \Gamma\} \sim O^-(1/r^2) + O^+(1/r^3), \quad (25) \]

we obtain asymptotically (at spatial infinity) finite values for the quasilocal quantities and an automatically vanishing boundary term in \(\delta \int \mathcal{H}(N)\).

These quasilocal expressions have good correspondence limits to special cases of the MAG including GR, the Poincaré Gauge Theory, and the teleparallel theory. The latter has recently had a revival largely because of new hopes for its utility regarding energy-momentum localization.
4 Applications

In conclusion we here briefly consider several applications of our MAG Hamiltonian boundary term quasilocal energy-momentum expressions.

- **Black hole thermodynamics**: By choosing the boundary on the horizon and at infinity we get the first law and a generalized expression for the entropy:
  \[ T \delta S = \oint_{H} \kappa \epsilon^{\alpha \beta} \delta \rho_{\alpha \beta}, \]  
  where \( \kappa \) is the surface gravity and \( \epsilon^{\alpha \beta} \) is the binormal to the horizon.

- **A positive energy proof?** The formalism gives the necessary expressions, but one must consider each distinct parameter choice separately. The prospects are very poor in general, but not bad for a few special cases with limited \( R^2 \) terms, e.g., just \( R_{\alpha \alpha} \) or the scalar or pseudoscalar curvature squared.

- **Positive total energy test**: This test, based on the fundamental requirement that gravity should be purely attractive, is expected to give severe constraints on the parameters—in principle—however it requires a lot of effort to get a result.

- **Quasilocal quantities for exact solutions**: We calculated the quasilocal energy for some exact MAG solutions. For the first solution found by Tresguerres and the first solution we found, the frame (with vanishing cosmological constant for simplicity) is
  \[ \vartheta^0 = f dt, \quad \vartheta^r = f^{-1} dr, \quad \vartheta^\theta = r d\theta, \quad \vartheta^\phi = r \sin \theta d\phi, \]  
  where \( f^2 = 1 - 2m/r + b_{10} N_0^2/(2\kappa a_0 r^2) \). Details of the necessary parameter restrictions and expressions for the torsion and nonmetricity are given in the cited works. Using a Minkowski reference geometry and analytic matching (the simplest but probably not the most physical choice) for the quasilocal energy we found, for Dirichlet and Neumann boundary conditions respectively,  
  \[ E = a_0 r (1 - f^{-1}) + b f'(f^2 - 1)(1 + n_0^2 f^{-2}), \]  
  \[ E = a_0 r (f - 1) + 2b f'(1 - f)(f + n_0^2 f^{-2}), \]  
  for our solution and
  \[ E = a_0 r (1 - f^{-1}) + b f'(f^2 - 1)(1 + n_0^2 f^{-2}) + b n_0^2 f^{-2}(1 - f)/r, \]  
  \[ E = a_0 r (f - 1) + 2b f'(1 - f)(f + n_0^2 f^{-2}) + b n_0^2 (f^{-1} - 1)/r, \]
  for the Tresguerres solution. For the interesting special case which has \( f = 1 - m/r \), we found
  \[ E = a_0 r (1 - f^{-1}) \quad \text{and} \quad E = a_0 r (f - 1), \]  
  for Dirichlet and Neumann boundary conditions respectively. These few examples are representative of our findings for other solutions. All of our results have the expected asymptotic limit: \(-a_0 m\); however much more work will be needed to appreciate the physical significance of the detailed shape of these quasilocal energy distributions.
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