Classification of Rank-One Submanifolds

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Abstract. We study ruled submanifolds of Euclidean space. First, to each (parametrized) ruled submanifold $\sigma$, we associate an integer-valued function, called degree, measuring the extent to which $\sigma$ fails to be cylindrical. In particular, we show that if the degree is constant and equal to $d$, then the singularities of $\sigma$ can only occur along an $(m - d)$-dimensional “striction” submanifold. This result allows us to extend the standard classification of developable surfaces in $\mathbb{R}^3$ to the whole family of flat and ruled submanifolds without planar points, also known as rank-one: an open and dense subset of every rank-one submanifold is the union of cylindrical, conical, and tangent regions.

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1. Introduction and Main Result

Developable surfaces in $\mathbb{R}^3$ are classical objects in differential geometry, enjoying a rich collection of properties. For instance, they have zero Gaussian curvature; they are envelopes of one-parameter families of planes; they are ruled, with tangent plane stable along the rulings.

Developable surfaces started to be studied in depth during the eighteenth century (even before the term “differential geometry” was coined [14]), when Leonhard Euler and Gaspard Monge got interested in their properties and classification. Their efforts led to the following, nowadays well-known, theorem.

**Theorem 1.1** ([13, section 3.24]). An open and dense subset of every developable surface is a union of planar regions, cones, cylinders, and tangent surfaces of space curves.

In arbitrary dimension and codimension, a broad generalization of the notion of developable surface (without planar points) is that of rank-one submanifold. Let $M$ be an $m$-dimensional submanifold of $\mathbb{R}^{m+n}$. We say that $M$ is rank-one if the rank of the Gauss map $M \to G(m, \mathbb{R}^{m+n})$ is constant and equal to one; equivalently, if $M$ is both flat and $(m - 1)$-ruled.

Rank-one—and, more generally, constant-rank—submanifolds received a great deal of attention during the 1960s [19, p. ix], yet they continue to be the subject of intensive study; see, e.g., [3,5,11,16,18,22].

It is therefore natural to ask whether a statement similar to that of Theorem 1.1 might hold for rank-one submanifolds [17]. In this paper, we answer such question affirmatively, by showing that there are only three types of rank-one submanifolds: cylindrical, conical, and tangent.

**Definition 1.2** (Cylinder). Given a smooth unit-speed curve $\gamma: I \to \mathbb{R}^{m+n}$ and an $(m - 1)$-tuple of orthonormal vectors $X_1, \ldots, X_{m-1}$, the map

$$I \times \mathbb{R}^{m-1} \to \mathbb{R}^{m+n}$$

$$(t, u^1, \ldots, u^{m-1}) \mapsto \gamma(t) + u^1X_1 + \cdots + u^{m-1}X_{m-1}$$

is called a cylindrical submanifold or, more simply, a cylinder.

**Definition 1.3** (Cone). Let $U \subset \mathbb{R}^{m-1}$, and let $\psi: U \to \mathbb{R}^{m+n}$ be a smooth map whose differential has rank $m - 2$ everywhere. Given a smooth unit vector field $X_\psi$ along $\psi$ such that

$$X_\psi \notin \text{span} \left( \frac{\partial \psi}{\partial u^1}, \ldots, \frac{\partial \psi}{\partial u^{m-1}} \right),$$

the map

$$U \times \mathbb{R} \to \mathbb{R}^{m+n}$$

$$(u^1, \ldots, u^m) \mapsto \psi(u^1, \ldots, u^{m-1}) + u^mX_\psi(u^1, \ldots, u^{m-1})$$

is called a conical submanifold or, more simply, a cone.
Definition 1.4 (Tangent submanifold). Let $\xi: U \to \mathbb{R}^{m+n}$ be an $(m - 1)$-dimensional regular submanifold of $\mathbb{R}^{m+n}$. Given a smooth unit vector field $X_\xi$ along $\xi$ such that
\[
\frac{\partial \xi}{\partial u^1} \wedge \cdots \wedge \frac{\partial \xi}{\partial u^{m-1}} \wedge X_\xi = 0,
\]
the map \[U \times \mathbb{R} \to \mathbb{R}^{m+n} \]
\[(u^1, \ldots, u^m) \mapsto \xi(u^1, \ldots, u^{m-1}) + u^m X_\xi(u^1, \ldots, u^{m-1})\]
is called a tangent submanifold of $\xi$.

Theorem 1.5. An open and dense subset of every rank-one submanifold is a union of cylindrical, conical, and tangent submanifolds. Conversely, any cylindrical and any singular ruled submanifold of degree one, as defined in Sect. 2, is rank-one.

We emphasize that Theorem 1.5 does not involve any assumption on the codimension of $M$. On the other hand, it is worth noting that when $M$ is rank-one, the first normal space (span of the image of the second fundamental form) is everywhere of dimension one; hence a rank-one submanifold is necessarily the image of an isometric composition [6], as explained by the following theorem.

Theorem 1.6 ([7,10]). Suppose that $M$ is rank-one. If the first normal bundle is parallel in the normal connection, then $M$ is contained in an $(m + 1)$-dimensional totally geodesic submanifold of $\mathbb{R}^{m+n}$. On the other hand, if the first normal bundle is not parallel, then $M$ is contained in a rank-one hypersurface of $\mathbb{R}^{m+n}$ as a totally geodesic submanifold.

Recall that rank-one hypersurfaces can be locally described in terms of the Gauss parametrization; see [4].

The rest of the paper, which is devoted to proving Theorem 1.5, is organized as follows. After discussing some background material, in Sect. 3 we generalize the classic theory of noncylindrical ruled surfaces in $\mathbb{R}^3$ to arbitrary dimension and codimension; in particular, we verify that, under some reasonable assumption, the singularities of a ruled submanifold always concentrate along a lower dimensional striction submanifold. In Sect. 4 we then specialize our theory to the rank-one case and show that a noncylindrical rank-one submanifold is necessarily singular. Finally, by applying the results obtained earlier, in Sect. 5 we prove Theorem 1.5.

The classification problem for rank-one submanifold was also studied by Ushakov in his Ph.D. thesis [20]. The author came up with a classification theorem [20, Theorem 2] that looks quite similar to ours (although the techniques used in the proof do not). Unfortunately we were not able to fully understand such result and therefore cannot provide a comparison with Theorem 1.5.
2. Preliminaries

In this section we discuss some preliminaries. Standard references for submanifolds are [1,8] and [15, Chapter 8].

2.1. Distribution Along Curves

Let \( \gamma : I \to \mathbb{R}^{m+n} \) be a smooth regular curve. Without loss of generality, we may assume \( \gamma \) to be unit-speed. A distribution of rank \( r \) along \( \gamma \) is a rank-\( r \) subbundle of the ambient tangent bundle \( T_{\mathbb{R}^{m+n}}|_{\gamma} \) over \( \gamma \). Recall that

\[
T_{\mathbb{R}^{m+n}}|_{\gamma} = \bigcup_{t \in I} T_{\gamma(t)} \mathbb{R}^{m+n}.
\]

Let \( D \) be a distribution of rank \( r \) along \( \gamma \), and let \( D^\perp \) be the distribution of rank \( m+n-r \) along \( \gamma \) whose fiber at \( t \) is the orthogonal complement of \( D_t \) in \( \mathbb{R}^{m+n} \). For each \( t \in I \), define a map \( \rho_t : D_t \to D_t^\perp \) by

\[
v \mapsto \pi^\perp \frac{dv}{dt}(t),
\]

where \( v \) is extended arbitrarily to a smooth local section of \( D \), and where \( \pi^\perp \) denotes orthogonal projection onto \( D^\perp \).

**Definition 2.1.** The rank of \( \rho_t \) is called the degree of the distribution \( D \) at \( t \). We say that \( D \) is of degree \( d \) if \( d_t := \text{rank } \rho_t = d \) for all \( t \in I \).

**Remark 2.2.** Note that \( d_t \leq \min(r, m+n-r) \).

Now, let \((E_1, \ldots, E_r)\) be a frame for \( D \), i.e., an \( r \)-tuple of smooth vector fields along \( \gamma \) forming a basis of \( D_t \) for all \( t \in I \). In particular, suppose that \((E_1, \ldots, E_r)\) is parallel with respect to the induced connection on \( D \), so that \( \pi^\perp(\dot{E}_1) = \cdots = \pi^\perp(\dot{E}_r) = 0 \); here \( \pi^\perp \) is the orthogonal projection onto \( D \). Then

\[
\text{rank } \rho_t = \dim \text{span}(\dot{E}_1(t), \ldots, \dot{E}_r(t)).
\]

2.2. Ruled Submanifolds

A (parametrized) submanifold of \( \mathbb{R}^{m+n} \) is a smooth map \( V \to \mathbb{R}^{m+n} \), where \( V \) is a subset of \( \mathbb{R}^m \). A point where the differential is injective is called regular; otherwise it is singular. A submanifold is itself called regular if all its points are regular, and singular otherwise.

**Remark 2.3.** A submanifold, even when regular, may have self-intersections in its image.

**Definition 2.4.** Given a smooth unit-speed curve \( \gamma : I \to \mathbb{R}^{m+n} \) and a smooth orthonormal frame \((X_j)_{j=1}^{m-1}\) along \( \gamma \), the submanifold

\[
\sigma : I \times \mathbb{R}^{m-1} \to \mathbb{R}^{m+n} \quad (t, u^1, \ldots, u^{m-1}) \mapsto \gamma(t) + u^1 X_1(t) + \cdots + u^{m-1} X_{m-1}(t)
\]
is called a ruled submanifold of $\mathbb{R}^{m+n}$. We say that $\gamma$ is a directrix of $\sigma$. Moreover, for any fixed $t$, the $(m-1)$-dimensional affine subspace of $\mathbb{R}^{m+n}$ spanned by $X_1(t), \ldots, X_{m-1}(t)$ is called a ruling of $\sigma$.

The simplest examples of ruled submanifolds are the cylinders (Definition 1.2). A cylinder is a ruled submanifold whose rulings are all parallel (in the usual Euclidean sense). On the other hand, a ruled submanifold is said to be noncylindrical if the degree of the ruling distribution $\text{span}(X_j)^{m-1}_{j=1}$ is nonzero for all $t \in I$. Similarly, we speak of ruled submanifolds of degree $d$ if the degree of the ruling distribution is constant and equal to $d$.

Recall that the first normal space of a Riemannian submanifold at a point $p$ is the linear subspace of the normal space spanned by the image of the second fundamental form at $p$. For a ruled submanifold, the degree of the ruling distribution and the dimension of the first normal space are closely related, as the next lemma shows.

Lemma 2.5. Let $\sigma$ be a ruled submanifold. If $\sigma$ is of degree $d$, then, at any point $p \in \sigma$, the first normal space $N^1_p \sigma$ satisfies

$$d - 1 \leq \dim N^1_p \sigma \leq d + 1.$$  

Proof. Let $X_0 = \dot{\gamma}$, and let $\Pi$ be the second fundamental form of $\sigma$. Since $\Pi$ vanishes on each ruling, the first normal space is spanned by

$$\Pi(x_0, x_0), \ldots, \Pi(x_0, x_{m-1});$$

here $x_0 = X_0|_p$, $x_1 = X_1|_p$, etc. Recall that

$$\Pi(X_0, X_j) = \pi^\perp_{\sigma} \dot{X}_j = \pi^\perp_{\sigma} \rho(X_j),$$

where $\pi^\perp_{\sigma}$ denotes orthogonal projection onto $N\sigma$.

Assume that $\sigma$ is of degree $d$. Since

$$\text{span} \left( \rho_t(x_1), \ldots, \rho_t(x_{m-1}) \right) \subset N_p \sigma \oplus \left( T_p M \cap \left( \text{span}(x_j)^{m-1}_{j=1} \right) ^\perp \right),$$

it follows that

$$d - 1 \leq \dim \text{span} \left( \Pi(x_0, x_1), \ldots, \Pi(x_0, x_{m-1}) \right) \leq d,$$

from which one concludes that

$$d - 1 \leq \dim \text{span} \left( \Pi(x_0, x_0), \ldots, \Pi(x_0, x_{m-1}) \right) \leq d + 1.$$  

□

3. The Striction Submanifold

The purpose of this section is to extend the notion of line of striction of a ruled surface in $\mathbb{R}^3$ to arbitrary dimension and codimension; see, e.g., [9, section 3-5] for the classical theory.
To begin with, let us assume that \( \sigma \) is a ruled submanifold. The classical line of striction is defined under the assumption that the surface \( (t, u) \mapsto \gamma(t) + uX(t) \) is noncylindrical, i.e., that \( \dot{X} \) never vanishes. Here we shall require that, for all \( t \in I \),

\[
\begin{align*}
\dim \text{span}(\rho_t X_1(t), \ldots, \rho_t X_{m-1}(t)) &= d > 0, \\
\dim \text{span}(\rho_t X_{m-d}(t), \ldots, \rho_t X_{m-1}(t)) &= d;
\end{align*}
\]

in other words, that both distributions \( \mathcal{D} = \text{span}(X_j)_{j=1}^{m-1} \) and \( \text{span}(X_h)_{h=m-d}^{m-1} \) are of degree \( d > 0 \). Note that, by Remark 2.2, we have the inequality

\[
0 < d \leq \min(m - 1, n + 1).
\]

In this setting, we are going to search for an \( (m - d) \)-dimensional submanifold \( \beta = \beta(t, u^1, \ldots, u^{m-d-1}) \), lying in the image of \( \sigma \) as a graph over \( I \times \mathbb{R}^{m-d-1} \), such that \( \langle \partial \beta / \partial t, \rho X_h \rangle = 0 \) for all \( h = m - d, \ldots, m - 1; \) such submanifold is called the striction submanifold of \( \sigma \). Note that \( \beta \) is contained in the image of \( \sigma \) as a graph over \( I \times \mathbb{R}^{m-d-1} \) if and only if there are smooth functions \( u^{m-d}, \ldots, u^{m-1} \) such that

\[
\beta(t, u^1, \ldots, u^{m-d-1}) = \gamma(t) + u^1 X_1(t) + \cdots + u^{m-d-1} X_{m-d-1}(t)
\]

\[
+ u^{m-d}(t, u^1, \ldots, u^{m-d-1}) X_{m-d}(t) + \cdots
\]

\[
+ u^{m-1}(t, u^1, \ldots, u^{m-d-1}) X_{m-1}(t).
\]

We thus compute

\[
\frac{\partial \beta}{\partial t} = \dot{\beta} = \dot{\gamma} + u^1 \dot{X}_1 + \cdots + u^{m-d-1} \dot{X}_{m-d-1}
\]

\[
+ u^{m-d} \dot{X}_{m-d} + u^{m-d} \dot{X}_{m-d} + \cdots
\]

\[
+ u^{m-1} \dot{X}_{m-1} + u^{m-1} \dot{X}_{m-1},
\]

which implies that

\[
\langle \dot{\beta}, \rho X_h \rangle = \langle \dot{\gamma}, \rho X_h \rangle + u^1 \langle \dot{X}_1, \rho X_h \rangle + \cdots + u^{m-1} \langle \dot{X}_{m-1}, \rho X_h \rangle.
\]

Letting

\[
A = \left( \begin{array}{c}
\langle \dot{X}_{m-d}, \rho X_{m-d} \rangle \\
\vdots \\
\langle \dot{X}_{m-1}, \rho X_{m-1} \rangle
\end{array} \right) \in \mathbb{R}^{d \times d}
\]

and

\[
b = - \left( \begin{array}{c}
\langle \gamma, \rho X_{m-d} \rangle + u^1 \langle \dot{X}_1, \rho X_{m-d} \rangle + \cdots + u^{m-d-1} \langle \dot{X}_{m-d-1}, \rho X_{m-d} \rangle \\
\vdots \\
\langle \gamma, \rho X_{m-1} \rangle + u^1 \langle \dot{X}_1, \rho X_{m-1} \rangle + \cdots + u^{m-d-1} \langle \dot{X}_{m-d-1}, \rho X_{m-1} \rangle
\end{array} \right) \in \mathbb{R}^{d \times 1},
\]
where \( A = A(t) \) and \( b = b(t, u^1, \ldots, u^{m-d-1}) \), our problem is equivalent to the linear system of equations

\[
A \begin{pmatrix} u^{m-d} \\ \vdots \\ u^{m-1} \end{pmatrix} = b \tag{1}
\]

in the \( d \) unknowns \( u^{m-d}, \ldots, u^{m-1} \). By assumption, the matrix \( A \) has full rank, and so \( A^{-1}b \) is the unique solution.

It remains to show that the corresponding solution \( \beta \) is well-defined, meaning that its image does not depend on the choice of the curve \( \gamma \). To this end, suppose that, for another choice of \( \gamma \), the striction submanifold is different from the one defined by \( A^{-1}b \). Then there are two striction submanifolds, say \( \beta_1 \) and \( \beta_2 \). It follows that there exist two \( d \)-tuples \((u^{m-d}_1, \ldots, u^{m-1}_1)\), \((u^{m-d}_2, \ldots, u^{m-1}_2)\) of functions such that

\[
\beta_\ell(t, u^1, \ldots, u^{m-d-1}) = \gamma(t) + u^1 X_1(t) + \cdots + u^{m-d-1} X_{m-d-1}(t) \\
\quad + u^{m-d}_\ell(t, u^1, \ldots, u^{m-d-1}_\ell) X_{m-d}(t) + \cdots \\
\quad + u^{m-1}_\ell(t, u^1, \ldots, u^{m-d-1}_\ell) X_{m-1}(t),
\]

with \( \ell = 1, 2 \). This implies that the solution of (1) is not unique, which is a contradiction.

Having verified that the striction submanifold is well-defined, we explain its significance by proving the following proposition.

**Proposition 3.1.** The only singular points of \( \sigma \), if any, are along its striction submanifold. In particular, \( \sigma \) is singular if and only if

\[
\dot{\beta}(t, u^1, \ldots, u^{m-d-1}) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0
\]

for some \((t, u^1, \ldots, u^{m-d-1}) \in I \times \mathbb{R}^{m-d-1}\).

**Proof.** First of all, we express \( \sigma \) in terms of the striction submanifold:

\[
\sigma(t, u^1, \ldots, u^{m-1}) = \beta(t, u^1, \ldots, u^{m-d-1}) + u^{m-d} X_{m-d}(t) + \cdots + u^{m-1} X_{m-1}(t).
\]

To find the singularities of \( \sigma \), we compute its partial derivatives, take their wedge product \( Z \), and then set it equal to zero.

The partial derivatives of \( \sigma \) with respect to \( t \) and \( u^j \) are

\[
\frac{\partial \sigma}{\partial t} = \dot{\beta} + u^{m-d} \dot{X}_{m-d} + \cdots + u^{m-1} \dot{X}_{m-1}
\]

and

\[
\frac{\partial \sigma}{\partial u^j} = \begin{cases} X_j + \frac{\partial u^{m-d}}{\partial u^j} X_{m-d} + \cdots + \frac{\partial u^{m-1}}{\partial u^j} X_{m-1} & \text{if } j \in \{1, \ldots, m-d-1\}, \\ X_j & \text{if } j \in \{m-d, \ldots, m-1\}, \end{cases}
\]
respectively. Hence, computing their wedge product gives
\[ Z = \left( \dot{\beta} + u^{m-d} \dot{X}_{m-d} + \cdots + u^{m-1} \dot{X}_{m-1} \right) \wedge \left( X_1 + \frac{\partial u^{m-d}}{\partial u^1} X_{m-d} + \cdots + \frac{\partial u^{m-1}}{\partial u^1} X_{m-1} \right) \wedge \cdots 
\wedge \left( X_{m-d-1} + \frac{\partial u^{m-d}}{\partial u^{m-d-1}} X_{m-d} + \cdots + \frac{\partial u^{m-1}}{\partial u^{m-d-1}} X_{m-1} \right) \wedge X_{m-d} \wedge \cdots \wedge X_{m-1} 
\]
\[ = \left( \dot{\beta} + u^{m-d} \dot{X}_{m-d} + \cdots + u^{m-1} \dot{X}_{m-1} \right) \wedge X_1 \wedge \cdots \wedge X_{m-1}. \]

We claim that \( Z = 0 \) exactly when
\[
\begin{cases}
m^{d} = \cdots = u^{m-1} = 0, \\
\dot{\beta} \wedge X_1 \wedge \cdots \wedge X_{m-1} = 0.
\end{cases}
\]

To verify the claim, note that \( Z(t, u^1, \ldots, u^{m-d-1}) = 0 \) if and only if
\[
\dot{\beta}(t, u^1, \ldots, u^{m-d-1}) + u^{m-d} \dot{X}_{m-d}(t) + \cdots + u^{m-1} \dot{X}_{m-1}(t) = a_1 X_1(t) + \cdots + a_{m-1} X_{m-1}(t)
\]
for some scalars \( a_1, \ldots, a_{m-1} \), i.e.,
\[
\dot{\beta}(t, u^1, \ldots, u^{m-d-1}) = a_1 X_1(t) + \cdots + a_{m-1} X_{m-1}(t) 
- u^{m-d} \dot{X}_{m-d}(t) - \cdots - u^{m-1} \dot{X}_{m-1}(t).
\]

On the other hand, by definition of \( \beta \), we have \( \langle \dot{\beta}, \rho X_{m-d} \rangle = \cdots = \langle \dot{\beta}, \rho X_{m-1} \rangle = 0 \). Hence (2) implies
\[
\begin{cases}
u^{m-d} \langle X_{m-d}(t), \rho_t X_{m-d}(t) \rangle + \cdots + u^{m-1} \langle X_{m-1}(t), \rho_t X_{m-d}(t) \rangle = 0, \\
\vdots \\
u^{m-d} \langle \dot{X}_{m-d}(t), \rho_t X_{m-1}(t) \rangle + \cdots + u^{m-1} \langle \dot{X}_{m-1}(t), \rho_t X_{m-1}(t) \rangle = 0.
\end{cases}
\]

This defines a homogenous linear system of full rank, which has unique (trivial) solution; thus \( u^{m-d} = \cdots = u^{m-1} = 0 \). \( \square \)

Remark 3.2. Suppose that \( \rho_t X_j(t) \neq 0 \). Since the vectors \( \dot{\beta}(t, u^1, \ldots, u^{m-d-1}), X_1(t), \ldots, X_{m-1}(t) \) are in the orthogonal complement of \( \rho_t X_j(t) \), the condition
\[
\dot{\beta}(t, u^1, \ldots, u^{m-d-1}) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0
\]
is equivalent to
\[
\dot{X}_j(t) \wedge \dot{\beta}(t, u^1, \ldots, u^{m-d-1}) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0.
\]

Remark 3.3. The proof of Proposition 3.1 shows that the dimension of the image of \( d\sigma \) at a singular point is \( m - 1 \). Since, whenever \( u^{m-d} = \cdots = u^{m-1} = 0 \),
\[
Z = \dot{\beta} \wedge \frac{\partial \beta}{\partial u^1} \wedge \cdots \wedge \frac{\partial \beta}{\partial u^{m-d-1}} \wedge X_{m-d} \wedge \cdots \wedge X_{m-1},
\]
it follows that the image of $d\beta(t,u)$ has dimension at least equal to $m - d - 1$ for all $(t,u) \in I \times \mathbb{R}^{m-d-1}$.

4. Rank-One Submanifolds

Here we study an important subclass of ruled submanifolds, called rank-one.

**Definition 4.1.** Let $\alpha$ be an $m$-dimensional submanifold of $\mathbb{R}^{m+n}$. The *relative nullity index* of $\alpha$ at a regular point $q$ is the dimension of the kernel of the second fundamental form of $\alpha$ at $q$ [2]. We say that $\alpha$ is a *rank-one* submanifold if it is ruled, and the relative nullity index at $q$ is equal to $m - 1$ for all regular points $q \in \alpha$.

**Remark 4.2.** The kernel of the second fundamental form of $\alpha$, also known as the *relative nullity distribution*, coincides with the kernel of the differential of the Gauss map $\alpha \to G(m, \mathbb{R}^{m+n})$ [8, Exercise 1.24].

**Remark 4.3.** Requiring that $\alpha$ be ruled in Definition 4.1 is only done for ease of presentation, as we previously defined ruled submanifolds as foliated by complete leaves.

The next result characterizes rank-one submanifolds in two different ways.

**Theorem 4.4** ([21], [12, Lemma 3.1]). If $\sigma$ is a ruled submanifold without planar points, then the following statements are equivalent:

1. $\sigma$ is rank-one.
2. The induced metric on $\sigma$ is flat.
3. All tangent spaces along (the regular points of) any fixed ruling can be canonically identified with the same linear subspace of $\mathbb{R}^{m+n}$.

According to Theorem 4.4, a ruled submanifold $\sigma$ that is free of planar points is rank-one exactly when the span of its partial derivatives is independent of $u^1, \ldots, u^{m-1}$. As explained in [18, section 3], this condition is equivalent to $\dot{X}_j$ being tangent to $\sigma$ for all $j = 1, \ldots, m - 1$.

**Lemma 4.5** ([18, Corollary 3.8]). If $\sigma: (t, u) \mapsto \gamma(t) + u^j X_j(t)$ is rank-one, then the following system of equations holds:

$$
\begin{aligned}
\dot{X}_1 \wedge \dot{\gamma} \wedge X_1 \wedge \cdots \wedge X_{m-1} &= 0, \\
&\vdots \\
\dot{X}_{m-1} \wedge \dot{\gamma} \wedge X_1 \wedge \cdots \wedge X_{m-1} &= 0.
\end{aligned}
$$

Conversely, suppose that $\sigma$ has no planar point and is regular along $\gamma$. If (3) holds, then $\sigma$ is rank-one.
This lemma implies that a noncylindrical rank-one submanifold is necessarily of degree one, because (3) forces \( \rho_t X_j(t) \) to lie in the orthogonal complement of \( D_t = \text{span}(X_j(t))_{j=1}^{m-1} \) in the tangent space of \( \sigma \), which is a subspace of dimension one.

A further consequence of Lemma 4.5 is that noncylindrical rank-one submanifolds necessarily have singularities.

**Proposition 4.6.** If \( \sigma \) is rank-one and noncylindrical, then all points of the striction hypersurface are singular. Conversely, if \( \sigma \) is a ruled submanifold of degree one that is singular at \( \beta(t, u^1, \ldots, u^{m-2}) \) for all \( t \in I \) and some \( (u^1, \ldots, u^{m-2}) \in \mathbb{R}^{m-2} \), then it is rank-one.

**Proof.** Once and for all, suppose that \( d = 1 \). Then

\[
\dot{\beta}(t, u^1, \ldots, u^{m-2}) = \dot{\gamma}(t) + u^1 \dot{X}_1(t) + \cdots + u^{m-2} \dot{X}_{m-2}(t) \\
+ \dot{u}^{m-1}(t, u^1, \ldots, u^{m-2})X_{m-1}(t) \\
+ u^{m-1}(t, u^1, \ldots, u^{m-2}) \dot{X}_{m-1}(t).
\]

Thus, when \( \sigma \) is rank-one,

\[
\dot{X}_j(t) \wedge \dot{\beta}(t, u^1, \ldots, u^{m-2}) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0, \tag{5}
\]

which, by Remark 3.2, implies

\[
\dot{\beta} \wedge X_1 \wedge \cdots \wedge X_{m-1} = 0.
\]

Conversely, if \( \sigma \) is singular at \( \beta(t, u^1, \ldots, u^{m-2}) \), then (5) holds. Substituting (4), we obtain

\[
\dot{X}_j(t) \wedge \dot{\gamma}(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0,
\]

as desired. \( \square \)

## 5. Proof of Theorem 1.5

We may now finalize the proof of our main result, Theorem 1.5 in the introduction.

Suppose that \( \sigma \) is a noncylindrical rank-one submanifold. Then

\[
\sigma(t, u^1, \ldots, u^{m-1}) = \beta(t, u^1, \ldots, u^{m-2}) + u^{m-1}X_{m-1}(t)
\]

and, by Proposition 4.6, all points of the striction hypersurface \( \beta \) are singular points of \( \sigma \); besides, by Remark 3.3, \( \dim d\sigma = m - 1 \) along \( \beta \).

We shall distinguish two cases:

1. The striction hypersurface \( \beta \) is regular. Then \( \sigma \) is a tangent submanifold of \( \beta \).
2. The striction hypersurface is singular everywhere, i.e., \( \dim d\beta(t, u) = m - 2 \) for all \( (t, u) \in I \times \mathbb{R}^{m-2} \). Then \( \sigma \) is a conical submanifold.

Conversely, if \( \sigma \) is a conical or a tangent submanifold of degree one, then it is ruled and singular. Hence, by Proposition 4.6, it is rank-one.
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Declarations

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