A novel teleparallel dark energy model

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Abstract. Although equivalent to general relativity, teleparallel gravity is conceptually speaking a completely different theory. In this theory, the gravitational field is described by torsion, not by curvature. By working in this context, a new model is proposed in which the four-derivative of a canonical scalar field representing dark energy is nonminimally coupled to the “vector torsion”. This type of coupling is motivated by the fact that, in teleparallel gravity, the scalar field couples to torsion through its four-derivative. It is found that the current state of accelerated expansion of the Universe corresponds to a late-time attractor that can be (i) a dark-energy-dominated de Sitter solution ($\omega_\phi = -1$), (ii) a quintessence-type solution with $\omega_\phi \geq -1$, or (iii) a phantom-type $\omega_\phi < -1$ dark energy.

1 Introduction

Like the other fundamental interactions of nature, gravitation can be described in terms of a gauge theory, the so-called Teleparallel Equivalent of General Relativity or also known as Teleparallel Gravity (TG), which attributes gravitation to torsion [1–3]. A crucial concept of gravitation is that the metric tensor itself defines neither curvature nor torsion. In fact, curvature and torsion are properties of connections, and many different connections, with different curvature and torsion tensors, can be defined on the very same metric spacetime. A general Lorentz connection has 24 independent components, and thus it is seen that any gravitational theory in which the source is the 10 components symmetric energy-momentum tensor will not be able to determine uniquely the connection. The teleparallel connection and the Levi-Civita (or Christoffel) connection are the only two choices respecting the correct number of degrees of freedom of gravitation—all other choices will include additional degrees of freedom. The former may be considered a kind of “dual” to the latter in the sense that, whereas the teleparallel connection has vanishing curvature and non-vanishing torsion, the Levi-Civita connection has vanishing torsion and non-vanishing curvature [1, 2].

On the other hand, from cosmic observations of Supernovae Ia (SNe Ia) [4], cosmic microwave background (CMB) radiation [5], large scale structure (LSS) [6], baryon acoustic oscillations (BAO) [7], and weak lensing [8], it is seen that the Universe is currently in a phase of accelerated expansion. Such phase is generally assumed to be driven by a peculiar form of energy, called dark energy, which in turn can be assumed to be generated by a scalar field with negative pressure. A cosmological constant is simpler and more natural than a scalar field, and could be considered as an alternative model. However, extreme fine tuning and coincidence problems make it quite problematic [9, 10]. Other models have also been proposed, like for example $f(R)$ theories [11, 12].

Considering that scalar fields can interact with other fields, such as the gravitational sector
of the theory, following the same spirit of scalar-tensor theories we can consider a nonminimal coupling between the scalar field and gravity. Many authors have studied models with a scalar field nonminimally coupled to gravity in the framework of GR \[13–31\]. Recently, it has been considered, in analogy with a similar construction in GR, a nonminimally coupled scalar field in the context of TG by adding a term $f(\phi) T$, with $f(\phi)$ a function of the scalar field and $T$ the so-called torsion scalar. This theory, which addresses the dark energy problem, has been called “teleparallel dark energy” (TDE) \[32–41\]. On the other hand, as is well-known, in the context of TG, although a scalar field itself does not feel gravity, its four-derivative (which is a vector field) interacts with the vector part of torsion \[2\]. Inspired in this property, the purpose of this paper is to study a new dark energy model in which the four-derivative of the scalar field couples nonminimally to the vector part of torsion. Throughout the paper we adopt natural units $c = 1$ such that $\kappa^2 = 8\pi G$; we use a metric with signature $(+, -, -, -)$.

2 The model

The torsion tensor can be decomposed into three components, irreducible under the global Lorentz group: there will be a vector

$$V_\mu = T^\nu_{\nu\mu},$$

an axial part

$$A^\mu = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma},$$

and a purely tensor part

$$T^\lambda_{\mu\nu} = \frac{1}{2} (T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu}) + \frac{1}{6} (g_{\nu\lambda} V_\mu + g_{\mu\nu} V_\lambda) - \frac{1}{3} g_{\nu\lambda} V_\mu,$$

that is, a tensor with vanishing vector and axial parts. These components are usually called “vector torsion”, “axial torsion” and “pure tensor torsion” \[1\]. When considering a scalar field, since it couples to curvature, it must also couple to torsion because the gravitational interaction can be described alternatively in terms of curvature or torsion. Also, since the scalar field interacts with torsion through its four-derivative (vector field), when a nonminimal coupling to gravitation is allowed the four-divergence of the scalar field can be coupled with the vector torsion. Thus, we consider the following action for the nonminimally coupled quintessence field

$$S = \int d^4x \ h \left[ \frac{T}{2\kappa^2} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \eta f(\phi) \partial_\mu \phi \partial^\mu \phi \right] + S_m(\psi_m, h^a_\mu),$$

where $h = \det(h^a_\mu) = \sqrt{-g}$ ( $h^a_\mu$ are the orthonormal components of the tetrad), $\mathcal{L}_G = \frac{h T}{2\kappa^2}$ is the lagrangian of teleparallelism and $S_m(\psi_m, h^a_\mu)$ is the matter action (see Refs. \[1, 2, 37\]). Moreover, the parameter $\eta$ is a dimensionless constant and $f(\phi)$ is an arbitrary function of the scalar field with units of mass. It is interesting to note that under a conformal rescaling of the metric \[32\], the two models, TDE scenario \[32\] and the model in action \[1\], “new teleparallel dark energy” (NTDE), are mathematically related by defining a transformed scalar field, potential and adding an explicit coupling between the scalar field and matter. However, although mathematically related by a conformal transformation, the two models are physically different (see for example Refs. \[12, 35\]). Here we do not consider an explicit coupling with
matter, but this could be considered in future studies. On the other hand, it may be important to mention that the nonminimal coupling term in action \( \text{(4)} \) has appeared before in other contexts; metric-scalar gravity with torsion \cite{46}, conformally invariant teleparallel gravity \cite{47} and \( F(T) \) theories \cite{48, 49}, when studying conformal transformations and conformal symmetry. In what follows we calculate from the action (4) the total energy momentum tensor and the motion equation for the dark sector. The energy momentum tensor associated with the scalar field is calculated as

\[
\Theta_{\rho}^\alpha = -\frac{1}{h} \delta S_{\phi} \delta h_{\alpha} = \eta \left[ f(\phi) \left( V^\rho \partial_\rho \phi + \nabla_\alpha \partial^\rho \phi - h_{\alpha}^\rho \nabla_\mu \partial_\mu \phi \right) + f,_{\phi} \left( \partial_\rho \phi \partial_\alpha \phi - h_{\alpha}^\rho \partial_\mu \phi \partial_\mu \phi \right) \right] - \frac{h_{\alpha}^\rho}{2} \partial_\mu \phi \partial_\rho \phi - V(\phi) + \partial_\alpha \phi \partial^\rho \phi, \quad (5)
\]

where \( \nabla^\mu \) is the covariant derivative in the teleparallel connection \cite{1, 2} and \( f,_{\phi} \equiv \frac{df}{d\phi} \). The symmetric part is

\[
\Theta_{(\mu\nu)} = \eta \left[ f(\phi) \left( V_{(\mu} \partial_{\nu)} \phi + \nabla_{(\mu} \partial_{\nu)} \phi - g_{\mu\nu} \nabla_t \partial^t \phi \right) + f,_{\phi} \left( \partial_{\mu} \phi \partial_{\nu} \phi - g_{\mu\nu} \partial_t \phi \partial_t \phi \right) \right] - g_{\mu\nu} \left( \frac{1}{2} \partial_\tau \phi \partial^\tau \phi - V(\phi) \right) + \partial_\mu \phi \partial_\nu \phi, \quad (6)
\]

while the anti-symmetric part is given by

\[
\Theta_{[\mu\nu]} = \frac{2\kappa^2}{h} \eta f(\phi) \partial_\epsilon \phi \sigma^\epsilon_{\mu\nu}, \quad (7)
\]

where \( \sigma_{\mu}^{\nu} \) is the Spin Tensor of the gravitational field, which is defined as \cite{50}

\[
\sigma^{\nu}_{\lambda\gamma} \equiv - \frac{\partial L_G}{\partial h_{\lambda}^{\alpha}} \frac{\delta h_{\sigma}^{\alpha}}{\delta \epsilon^{\gamma}} = \frac{h}{\kappa^2} S_{[\lambda\gamma]}^{\mu} = - \frac{h}{4\kappa^2} \left( T^{\mu}_{\lambda\gamma} + \delta^{\mu}_{\lambda} \nabla_{\lambda} - \delta^{\mu}_{\lambda} \nabla_{\lambda} \right), \quad (8)
\]

where we have used

\[
\frac{\delta h_{\alpha}^{\alpha}}{\delta \epsilon^{\alpha\beta}} = \frac{1}{2} \left( g_{\beta\sigma} h_{\alpha}^{\sigma} - g_{\alpha\sigma} h_{\beta}^{\sigma} \right), \quad (9)
\]

with \( \delta \epsilon^{\alpha\beta} \) an infinitesimal anti-symmetric (Lorentz) tensor and \( S_{[\lambda\gamma]}^{\mu} \) the anti-symmetric part of the superpotential \cite{41}. However, the antisymmetric part (7) is not relevant on cosmological scales where there is homogeneity and isotropy. Varying the action with respect to the scalar field we find the motion equation

\[
\nabla_\mu \partial^\mu \phi - \partial_\mu \phi V^\mu + \eta f(\phi) \left( \nabla_\mu V^\mu - V_\mu \partial^\mu \phi \right) + V_\phi = 0, \quad (10)
\]

which is written in terms of the covariant derivative of the teleparallel connection and \( V_\phi \equiv \frac{dV}{d\phi} \). The first two terms reflect more once the fact already mentioned that the scalar field interacts with torsion through its four-derivative. We will now go to consider the spatially flat FLRW background \cite{37, 39}

\[
h_{\mu}^{\alpha}(t) = \text{diag}(1, a(t), a(t), a(t)), \quad (11)
\]

and we will study the cosmological implications for our model. As usual, we will work in the preferred class of frames where this background is a solution of the gravitational field equation (see Ref. \cite{37}).
By imposing the flat FLRW geometry \((11)\) in \((6)\) we obtain for the energy density
\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) - 3 \eta f(\phi) H \dot{\phi},
\]
and for the pressure density
\[
p_\phi = \frac{1}{2} \left(1 + 2 \eta f,\phi \right) \dot{\phi}^2 - V(\phi) + \eta f(\phi) \ddot{\phi}.
\]
On the other hand, imposing the same background \((11)\) in the motion equation \((10)\) we find
\[
\ddot{\phi} + 3 H \dot{\phi} - 3 \eta \left(\dot{H} + 3 H^2\right) f(\phi) + V,\phi = 0.
\]
This is the evolution equation for the scalar field and can alternatively be written in the standard form \(\dot{\rho}_\phi + 3 H (1 + \omega_\phi) \rho_\phi = 0\) with \(\omega_\phi \equiv \frac{p_\phi}{\rho_\phi}\) the equation-of-state parameter.

3 Cosmological dynamics

To study the cosmological dynamics of the model, we introduce the following dimensionless variables:
\[
x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6} H}, \quad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3} H}, \quad u \equiv \kappa f, \quad \lambda \equiv -\frac{V,\phi}{\kappa V}, \quad \alpha \equiv f,\phi.
\]
In terms of these dimensionless variables, the fractional energy densities \(\Omega_\phi\) and \(\Omega_m\) for the scalar field and background matter are given by
\[
\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3 H^2} = x^2 + y^2 - \sqrt{6} \eta u x, \quad \Omega_m \equiv \frac{\kappa^2 \rho_m}{3 H^2} = 1 - \Omega_\phi,
\]
respectively. By using the physical condition \(0 \leq \Omega_\phi \leq 1\) in equation \((16)\), the range in the phase space for the variables \(x, u\) and \(y\) is constrained.

On the other hand, the equation of state of the field \(\omega_\phi\) reads
\[
\omega_\phi = \left(1 + 2 \eta \alpha\right) x^2 - y^2 + \eta u \left(-\sqrt{6} x + \eta u (3 - s) + \lambda y^2\right) x^2 + y^2 - \sqrt{6} \eta u x.
\]
The effective equation of state \(\omega_{eff}\) is given by
\[
\omega_{eff} \equiv \frac{p_m + p_\phi}{\rho_m + \rho_\phi} = (\gamma - 1) \left[1 - \left(x^2 + y^2 - \sqrt{6} \eta u x\right)\right] + (1 + 2 \eta \alpha) x^2 - y^2 + \eta u \left(-\sqrt{6} x + \eta u (3 - s) + \lambda y^2\right),
\]
where we have defined \(\gamma \equiv 1 + \omega_m\) and the accelerated expansion occurs for \(\omega_{eff} < -\frac{1}{3}\). Also, it is defined the parameter
\[
s \equiv -\frac{\dot{H}}{H^2} = \left(\frac{2}{3} + \eta^2 u^2\right)^{-1} \left[\gamma - (\gamma - 1) \left(x^2 + y^2 - \sqrt{6} \eta u x\right) + (1 + 2 \eta \alpha) x^2 + \eta u \eta u - x\right].
\]
The dynamical system of ordinary differential equations (ODE) for the model is written as

\[ x' = (3 - s) \left( -x + \frac{\sqrt{6}}{2} \eta u \right) + \frac{\sqrt{6}}{2} \lambda y^2, \]  

(20)

\[ y' = \left( s - \frac{\sqrt{6} \lambda x}{2} \right) y, \]  

(21)

\[ u' = \sqrt{6} \alpha x, \]  

(22)

\[ \lambda' = -\sqrt{6} (\Gamma - 1) \lambda x^2, \]  

(23)

\[ \alpha' = \sqrt{6} \Pi x. \]  

(24)

In these equations primes denote derivative with respect to the so-called e-folding time \( N \equiv \ln a \). Also, we have defined the parameters

\[ \Pi \equiv \kappa f(\phi), \quad \Gamma \equiv \frac{V V_{\phi \phi}}{V^2_{,\phi}}. \]  

(25)

From now we concentrate on exponential scalar field potential of the form \( V(\phi) = V_0 e^{-\lambda \kappa \phi} \), such that \( \lambda \) is a dimensionless constant, that is, \( \Gamma = 1 \) (equivalently, we could consider potentials satisfying \( \lambda \equiv -\frac{V_{,\phi}}{\kappa V} \approx \text{const} \), which is valid for arbitrary but nearly flat potentials [53, 54]). We study two simple cases: one where \( \alpha \) is a constant and another where \( \alpha \) depends on \( u \). In the first case, it is considered a nonminimal coupling function \( f(\phi) \propto \phi \) and thus \( \alpha = \text{const} \) and \( \Pi = 0 \). Finally, we consider a dynamically changing \( \alpha(u) \). If \( u(\phi) \equiv \kappa f(\phi) \) is a general function, with inverse function \( \phi(u) = f^{-1}(u/\kappa) \) thus \( \alpha(\phi) \) and \( \Pi(\phi) \) can be expressed in terms of \( u \) (see [37, 39, 56]). In this form, the dynamical system of (ODE) (20)–(24) is a dynamical autonomous system and we can obtain the fixed points or critical points \((x_c, y_c, u_c)\) by imposing the conditions \( x_c' = y_c' = u_c' = 0 \). From the definition (15), \( x_c, y_c, u_c \) should be real, with \( y_c \geq 0 \). To study the stability of the critical points, we substitute linear perturbations, \( x \to x_c + \delta x, y \to y_c + \delta y, \) and \( u \to u_c + \delta u \) around each critical point and linearize them. The eigenvalues of the perturbations matrix \( M \), namely, \( \mu_1, \mu_2 \) and \( \mu_3 \), determine the conditions of stability of the critical points [9, 12]: (i) Stable node: \( \mu_1 < 0, \mu_2 < 0 \) and \( \mu_3 < 0 \). (ii) Unstable node: \( \mu_1 > 0, \mu_2 > 0 \) and \( \mu_3 > 0 \). (iii) Saddle point: one or two of the three eigenvalues are positive and the other negative. (iv) Stable spiral: The determinant of the matrix \( M \) is negative and the real parts of \( \mu_1, \mu_2 \) and \( \mu_3 \) are negative. A critical point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii). The Universe will eventually enter these attractor solutions regardless of the initial conditions.

4 Constant \( \alpha \)

4.1 Critical points

In this section we consider a nonminimal coupling function \( f(\phi) \propto \phi \) such that \( \alpha \) is a constant.

The critical points are presented in the Table 1 along with properties in Table 2. The critical point I.a is a matter-dominated solution (\( \Omega_m = 1 \)) with equation of state type cosmological constant \( \omega_\phi = -1 \), that exists for all values. The point I.b is also a matter-dominated solution.
that exists for \( u_c \in \mathbb{R} \) and \( \omega_m = \omega_{\text{eff}} = 1 \). On the other hand, the fixed point \( I.c \) correspond to a dark-energy-dominated de Sitter solution with \( \Omega_\phi = 1 \) and \( \omega_\phi = \omega_{\text{eff}} = -1 \). This point exists for all values. It is a viable cosmological solution to describe the current accelerated expansion of our Universe.

Table 1: Critical points for \( \alpha = \text{const} \).

| Name | \( x_c \) | \( y_c \) | \( u_c \) | \( \Omega_\phi \) | \( \omega_\phi \) | \( \omega_{\text{eff}} \) |
|------|-------|-------|-------|----------|----------|----------|
| I.a  | 0     | 0     | 0     | 0        | -1       | \( \gamma - 1 \) |
| I.b  | 0     | 0     | \( u_c \) | 0        | \( \eta \lambda u_c - 1 \) | 1        |
| I.c  | 0     | 1     | -\( \frac{\lambda}{3 \eta} \) | 1        | -1       | -1       |

Table 2: Stability properties, and conditions for acceleration and existence for fixed points in Table 1.

| Name | Stability | Acceleration | Existence |
|------|-----------|--------------|-----------|
| I.a  | Unstable  | No           | All values|
| I.b  | Unstable  | No           | \( \omega_m = 1 \) |
| I.c  | Stable node or Stable spiral, or Saddle | All values | All values |

Figure 1: Evolution of \( \Omega_m \) (dashed), \( \Omega_\phi \) (dotdashed), \( \omega_\phi \) (dotted) and \( \omega_{\text{eff}} \) (solid) with \( \gamma = 1 \), \( \lambda = 0.8 \), \( \alpha = 1 \) and \( \eta = -0.2 \). It was considered the initial conditions \( x_i = 1 \times 10^{-8} \), \( y_i = 1.7 \times 10^{-6} \) and \( u_i = 1 \times 10^{-7} \). The system asymptotically evolves toward the values \( \Omega_\phi = 1 \), \( \Omega_m = 0 \) and \( \omega_\phi = \omega_{\text{eff}} = -1 \). Also, we have \( \Omega_\phi \approx 0.72 \), \( \Omega_m \approx 0.28 \), \( \omega_\phi \approx -0.92 \) and \( \omega_{\text{eff}} = -0.66 \) at the present epoch \( N = \log_{10} (a) \approx 4 \).

4.2 Stability

Substituting the linear perturbations, \( x \to x_c + \delta x \), \( y \to y_c + \delta y \), and \( u \to u_c + \delta u \) into the autonomous system \([20] \quad [24] \) and linearize them, we find the follows components for the matrix
of linear perturbations $\mathcal{M}$

$$\mathcal{M}_{11} = \frac{\eta u_c (\lambda y_c^2 - \sqrt{6} (2 \alpha \eta - 3 \gamma + 6) x_c) - \gamma y_c^2 + 3 (2 \alpha \eta - \gamma + 2) x_c^2 - 3 (2 - \gamma)}{\eta^2 u_c^2 + \frac{2}{3}} + 3 (2 - \gamma), \quad (26)$$

$$\mathcal{M}_{12} = \frac{2 (\eta \lambda u_c - \gamma) x_c + \sqrt{6} \gamma \eta u_c + \frac{2 \sqrt{6}}{3} \lambda}{\eta^2 u_c^2 + \frac{2}{3}} y_c, \quad (27)$$

$$\mathcal{M}_{13} = -\frac{\eta ((2 \lambda x_c + \sqrt{6} \gamma) y_c^2 - \sqrt{6} (2 \alpha \eta - 3 \gamma + 6) x_c^2 + \sqrt{6} (2 - \gamma))}{2 (\eta^2 u_c^2 + \frac{2}{3})} + \frac{2 \eta}{3} [\eta u_c (3 \gamma x_c - \sqrt{6} \lambda) y_c^2 - 3 (2 \alpha \eta - \gamma + 2) x_c^2 + 9 (2 - \gamma) x_c] + \left(2 \lambda x_c + \sqrt{6} \gamma \right) y_c^2 - \sqrt{6} (2 \alpha \eta - 3 \gamma + 6) x_c^2 + \sqrt{6} (2 - \gamma) \left(\eta^2 u_c^2 + \frac{2}{3}\right)^{-2}, \quad (28)$$

$$\mathcal{M}_{21} = \frac{2 (2 \alpha \eta - \gamma + 2) x_c - \sqrt{6} (2 - \gamma) \eta u_c}{\eta^2 u_c^2 + \frac{2}{3}} y_c - \sqrt{6} \lambda y_c, \quad (29)$$

$$\mathcal{M}_{22} = \frac{\eta u_c (3 \lambda y_c^2 - \sqrt{6} (2 - \gamma) x_c) - 3 \gamma y_c^2 + (2 \alpha \eta - \gamma + 2) x_c^2 + \gamma - 2}{\eta^2 u_c^2 + \frac{2}{3}} - \sqrt{6} \lambda x_c - \frac{6}{2}, \quad (30)$$

$$\mathcal{M}_{23} = \frac{2 \eta y_c (3 \eta u_c (\gamma y_c^2 + (-2 \alpha \eta + \gamma - 2) x_c^2 - \gamma + 2) + 2 \lambda y_c^2 - 2 \sqrt{6} (2 - \gamma) x_c)}{3 (\eta^2 u_c^2 + \frac{2}{3})^2} - \frac{\eta y_c (\lambda y_c^2 - \sqrt{6} (2 - \gamma) x_c)}{\eta^2 u_c^2 + \frac{2}{3}}, \quad (31)$$

$$\mathcal{M}_{31} = \sqrt{6} \alpha, \quad \mathcal{M}_{32} = 0, \quad \mathcal{M}_{33} = 0. \quad (32)$$

The eigenvalues of the matrix $\mathcal{M}$, for each critical point, are as follows:

- **Point I.a:**
  \[\mu_{1,2} = \frac{3}{4} (2 - \gamma) \left(-1 \pm \sqrt{1 + \frac{8 \eta \alpha}{2 - \gamma}}\right), \quad \mu_3 = \frac{3 \gamma}{2} \quad (33)\]

- **Point I.b:**
  \[\mu_{1,2} = 0, \quad \mu_3 = 3. \quad (34)\]

- **Point I.c:**
  \[\mu_{1,2} = \frac{3}{2} \left(-1 \pm \sqrt{1 + \frac{24 \eta \alpha}{\lambda^2 + 6}}\right), \quad \mu_3 = -3 \gamma. \quad (35)\]
The point I.a is always unstable (since $\mu_3 > 0$). The point I.b. is also unstable for all values. Finally, the critical point I.c is stable node for $-\lambda^2/24 + 6/24 < \eta \alpha < 0$. (36)

For $\eta \alpha > 0$ it is a saddle point. On the other hand, when

$$\eta \alpha < \frac{-\lambda^2 + 6}{24},$$

thus $\mu_1$ and $\mu_2$ are complex with real part negative. In this case, the determinant of the matrix of perturbations $\det \mathcal{M}|(x_\ast, y_\ast, u_\ast) = \frac{162\gamma \eta \alpha}{4 + \lambda^3}$ is negative and the point I.c is a stable spiral. So, we find that the fixed point I.c is an attractor (for conditions (36) or (37)) and a viable cosmological solution to explain the late-time accelerated expansion. The Universe will eventually enter this solution regardless of the initial conditions. In the Fig. 1 it is shown as the Universe tends asymptotically to the dark-energy-dominated de Sitter solution I.c, passing through the matter-dominated solution I.a.

5 Dynamically changing $\alpha$

5.1 Critical points

Following Refs. [37, 39, 56], let us consider a general nonminimal coupling function $f(\phi)$ such that $\alpha$ can be expressed in terms of $u$ and $\alpha(u) \to \alpha(u_\ast) = 0$ when $(x, y, u) \to (x_\ast, y_\ast, u_\ast)$ (we note that $(x_\ast, y_\ast, u_\ast)$ is a fixed point of the system). The field $\phi$ rolls down toward $\pm \infty$ ($x > 0$ or $x < 0$) with $f(\phi) \to 1/\kappa$ and $u_\ast = 1$. The fixed points are presented in the Table 3, and the properties in Table 4.

The point II.a is not realistic because $\Omega_{\phi} = -\frac{3\gamma^2}{2} < 0$. The fixed points II.b and II.c are both scalar-field-dominated solutions with $\Omega_{\phi} = 1$ and equation of state type “stiff matter” $\omega_{\phi} = 1$. These points exist for all values. The fixed point II.d is a scaling solution that exists for $\eta \lambda < \gamma < 2$. This point is a realistic solution when

$$\frac{2 (3 \gamma - \lambda^2)}{3 (\gamma + 2)} \leq \eta \lambda \leq \frac{2 \gamma}{\gamma + 2},$$

since in this case $0 \leq \Omega_{\phi} \leq 1$. For nonrelativistic matter $\gamma = 1$ thus $\frac{2}{3} - \frac{2 \lambda^2}{9} \leq \eta \lambda \leq \frac{2}{3}$. On the other hand, in the case of relativistic matter (radiation) $\gamma = 4/3$ we have that $\frac{4}{3} - \frac{\lambda^2}{9} \leq \eta \lambda \leq \frac{4}{3}$. Just like the above fixed points II.a-II.c, the point II.d is not viable to explain a late-time acceleration. However, since point II.d is a scaling solution, this can be used to provide the cosmological evolution in which the energy density of the scalar field decreases proportionally to that of the background fluid in either a radiation or matter-dominated era. Finally, the point II.e is also a scalar-field-dominated solution, but different to II.b and II.c, this point is a viable solution to explain the late-time cosmic acceleration. This point exists for $\eta \lambda \leq 1 - \frac{\lambda^2}{\gamma}$. The equation of state is given by

$$\omega_{\phi} = \omega_{\text{eff}} = \frac{2 \lambda^2 + 3 (3 \eta \lambda - 2)}{3 (2 - \eta \lambda)},$$

(39)
and the accelerated expansion occurs for \( \eta \lambda < \frac{1}{2} - \frac{\lambda^2}{3} \). From (39) we have that \( \omega_\phi \geq -1 \) if \( \eta \lambda \geq -\frac{\lambda^2}{3} \). Moreover, it is also possible that this solution to be phantom, that is \( \omega_\phi < -1 \), if \( \eta \lambda < -\frac{\lambda^2}{3} \). Is worth highlighting that unlike the dark energy models with phantom or ghost scalar field [9,10], the present model is devoid of any quantum instability [57].

Table 3: Critical points for dynamically changing \( \alpha(u) \).

| Name  | \( x_c \)   | \( y_c \) | \( u_c \) | \( \Omega_\phi \) | \( \omega_\phi \) | \( \omega_{eff} \) |
|-------|-------------|----------|----------|----------------|----------------|----------------|
| II.a  | \( \sqrt{\frac{6}{\eta}} \) | 0 | 1 | \( \frac{3 \eta^2}{2} \) | \( \gamma - 1 \) | \( \gamma - 1 \) |
| II.b  | \( \sqrt{\frac{6}{\eta}} - \sqrt{1 + \frac{3 \eta^2}{2}} \) | 0 | 1 | 1 | 1 | 1 |
| II.c  | \( \sqrt{\frac{6}{\eta}} + \sqrt{1 + \frac{3 \eta^2}{2}} \) | 0 | 1 | 1 | 1 | 1 |
| II.d  | \( \sqrt{\frac{6}{\eta}} \frac{\sqrt{3 \sqrt{2 - \gamma} \sqrt{\gamma - \eta \lambda}}}{\sqrt{2 |\lambda|}} \) | \( \frac{3 (\gamma (2 - \eta \lambda) - 2 \eta \lambda)}{2 \lambda \gamma} \) | \( \gamma - 1 \) | \( \gamma - 1 \) |
| II.e  | \( \sqrt{\frac{6 (3 \eta + \lambda)}{3 (2 - \eta \lambda)}} \) | \( \sqrt{\frac{6 (1 - \eta \lambda - \lambda^2 \sqrt{\frac{2 + \eta^2}{2 - \eta \lambda}})}{2 - \eta \lambda}} \) | 1 | 1 | [39] | [39] |

Table 4: Stability properties, and conditions for acceleration and existence of the fixed points in Table 3.

| Name  | Stability                      | Acceleration | Existence               |
|-------|--------------------------------|--------------|-------------------------|
| II.a  | Stable node or Saddle          | No           | No, if 0 \( \leq \Omega_\phi \leq 1 \) |
| II.b  | Unstable node or Saddle \( (\gamma = 1) \) | No           | All values              |
| II.c  | Unstable node or Saddle \( (\gamma = 1) \) | No           | All values              |
| II.d  | Stable node or Saddle or Stable Spiral \( (\gamma = 1) \) | No           | Eq. (38)                |
| II.e  | Stable node or Saddle          | \( \eta \lambda < \frac{1}{2} - \frac{\lambda^2}{4} \) | \( \eta \lambda \leq 1 - \frac{\lambda^2}{6} \)  |

5.2 Stability

For dynamically changing \( \alpha(u) \) the components of the matrix of perturbation \( \mathcal{M} \) are written as

\[
\mathcal{M}_{11} = \frac{3 (\eta (\lambda y_c^2 - 3 \sqrt{6} (2 - \gamma) x_c) - \gamma y_c^2 + 3 (2 - \gamma) (x_c^2 - 1))}{3 \eta^2 + 2} + 3 (2 - \gamma), \tag{40}
\]

\[
\mathcal{M}_{12} = \frac{\sqrt{6} (\sqrt{6} (\eta \lambda - \gamma) x_c + 2 \lambda + 3 \gamma \eta) y_c}{3 \eta^2 + 2}, \tag{41}
\]
\[ \mathcal{M}_{13} = -\frac{1}{2} [3 \eta \left( (2 \lambda x_c + \sqrt{6} \gamma) y_c^2 - 4 \tau_c x_c^3 - \sqrt{6} (2 - \gamma) (3 x_c^2 - 1) \right) - \\
4 \left( 3 \gamma x_c - \sqrt{6} \lambda \right) y_c^2 - 4 \sqrt{6} \tau_c x_c^2 + 12 (2 - \gamma) x_c (x_c^2 - 3)] (3 \eta^2 + 2)^{-1} + \\
2 [3 \eta \left( (2 \lambda x_c + \sqrt{6} \gamma) y_c^2 - \sqrt{6} (2 - \gamma) (3 x_c^2 - 1) \right) - 2 \left( 3 \gamma x_c - \sqrt{6} \lambda \right) y_c^2 + \\
6 (2 - \gamma) x_c (x_c^2 - 3)] (3 \eta^2 + 2)^{-2} - \sqrt{6} \tau_c x_c^2, \] (42)

\[ \mathcal{M}_{21} = \frac{3 (2 - \gamma)(2 x_c - \sqrt{6} \eta) y_c}{3 \eta^2 + 2} - \frac{\sqrt{6} \lambda y_c}{2}, \] (43)

\[ \mathcal{M}_{22} = \frac{3 \left( \eta \left( 3 \lambda y_c^2 - \sqrt{6} (2 - \gamma) x_c \right) - 3 \gamma y_c^2 + (2 - \gamma) (x_c^2 - 1) \right)}{3 \eta^2 + 2} - \frac{\sqrt{6} \lambda x_c - 6}{2}, \] (44)

\[ \mathcal{M}_{23} = \frac{12 y_c \left( \eta \left( \lambda y_c^2 - \sqrt{6} (2 - \gamma) x_c \right) - \gamma y_c^2 + (2 - \gamma) (x_c^2 - 1) \right)}{(3 \eta^2 + 2)^2} + \\
3 y_c \left( \eta \left( -\lambda y_c^2 + 2 \tau_c x_c^2 + \sqrt{6} (2 - \gamma) x_c \right) + 2 \gamma y_c^2 - 2 (2 - \gamma) (x_c^2 - 1) \right), \] (45)

\[ \mathcal{M}_{31} = \mathcal{M}_{32} = 0, \quad \mathcal{M}_{33} = \sqrt{6} \tau_c x_c. \] (46)

Here \( \tau_c \) is defined by \( \tau_c \equiv \frac{d^2u}{du^2} \bigg|_{u=u_c}. \) The eigenvalues of the matrix \( \mathcal{M} \) for each critical point, are as follows:

- **Point II.a:**
  \[ \mu_1 = \frac{3 (\gamma - \eta \lambda)}{2}, \quad \mu_2 = -\frac{3 (2 - \gamma)}{2}, \quad \mu_3 = 3 \eta \tau_c. \] (47)

- **Point II.b \((\gamma = 1): \)\n  \[ \mu_1 = \frac{\lambda}{2} \left( \sqrt{9 \eta^2 + 6} - (3 \eta - \frac{6}{\lambda}) \right), \quad \mu_2 = 3, \quad \mu_3 = \tau_c \left( 3 \eta - \sqrt{9 \eta^2 + 6} \right). \] (48)

- **Point II.c \((\gamma = 1): \)\n  \[ \mu_1 = -\frac{\lambda}{2} \left( \sqrt{9 \eta^2 + 6} + 3 \eta - \frac{6}{\lambda} \right), \quad \mu_2 = 3, \quad \mu_3 = \tau_c \left( 3 \eta + \sqrt{9 \eta^2 + 6} \right). \] (49)

- **Point II.d \((\gamma = 1): \)\n  \[ \mu_{1,2} = \frac{3}{4} \left( -1 \pm \sqrt{1 + Y} \right), \quad \mu_3 = \frac{3 \tau_c}{\lambda}, \quad Y = \frac{8 (\eta \lambda - 1) (2 \lambda^2 + 9 \eta \lambda - 6)}{(3 \eta^2 + 2) \lambda^2}. \] (50)

- **Point II.e:**
  \[ \mu_1 = \frac{3 (\gamma + 2) \eta \lambda + 2 (\lambda^2 - 3 \gamma)}{2 - \eta \lambda}, \quad \mu_2 = \frac{6 \eta \lambda + \lambda^2 - 6}{2 - \eta \lambda}, \quad \mu_3 = \frac{2 (\lambda + 3 \eta) \tau_c}{2 - \eta \lambda}. \] (51)

The point II.a is a stable node for \( \eta \lambda > \gamma \) and \( \eta \tau_c < 0 \), whereas it is a saddle point for \( \eta \lambda < \gamma \) and/or \( \eta \tau_c > 0 \). In any case, it is not realistic solution since \( \Omega_\phi < 0 \). When the background
In the left panel, for \(\alpha(u) = 1 - u\) (red line, starting at \(\alpha = 1\)), we have chosen \(\lambda = 0.8\) and \(\eta = -0.2\), with initial conditions \(x_i = 1 \times 10^{-8}, y_i = 1.7 \times 10^{-6}\) and \(u_i = 1 \times 10^{-8}\). The present epoch \((N = \log_{10}(a) \approx 4)\) corresponds to \(\Omega_\phi \approx 0.72, \Omega_m \approx 0.28, \omega_\phi \approx -0.92\) and \(\omega_{eff} \approx -0.66\). The Universe asymptotically evolves toward \(\Omega_\phi = 1\) and \(\omega_\phi = \omega_{eff} = -0.95\).

In the right panel, for \(\alpha(u) = -1 + u\) (red line, starting at \(\alpha = -1\)), we have chosen \(\lambda = 0.3\) and \(\eta = -1.4\), with initial conditions \(x_i = 1 \times 10^{-9}, y_i = 2.16 \times 10^{-6}\) and \(u_i = 1.1 \times 10^{-7}\). At the present time, also with \(\Omega_\phi \approx 0.72\) and \(\Omega_m \approx 0.28\), we have \(\omega_\phi \approx -1.05\) and \(\omega_\phi \approx -0.75\). In this case the evolution converges to \(\Omega_\phi = 1\) and \(\omega_\phi = \omega_{eff} = -1.32\).

Since the determinant of the matrix of perturbations \(\text{det} M|_{(x_c,y_c,u_c)} = -\frac{27}{16} \frac{\lambda}{Y}\) is negative, in this case the point II.d is a stable spiral. Finally, for \(\eta \lambda < 1 - \frac{\lambda^2}{6}\), the point II.e is a stable node if

\[
\eta \lambda < \frac{2 (3 \gamma - \lambda^2)}{3 (\gamma + 2)} \quad \text{and} \quad (\lambda + 3 \eta) \tau_c < 0,
\]

(55) otherwise it is a saddle point. Whenever accelerated expansion occurs, \(\eta \lambda < \frac{1}{2} - \frac{\lambda^2}{4}\), (and satisfying the constraint (55)) this fixed point is a stable node and therefore a attractor. Just like the point I.c, the fixed point II.e is also a viable cosmological solution to explain the
current phase of accelerated expansion. In Fig. 2 the Universe tends asymptotically to the solution II.e for dynamically changing $\alpha(u)$ with accelerated expansion, first passing through the matter-dominated solution I.a with constant $\alpha$. In the left panel, we consider for simplicity the function $f(\phi) = \frac{1}{\kappa} (1 - e^{-\kappa \phi})$ such that $\alpha(u) = 1 - u$ and $\tau_c = -1$. Similarly, in the right panel, it is considered the function $f(\phi) = \frac{1}{\kappa} (1 + e^{\kappa \phi})$ such that $\alpha(u) = -1 + u$ and $\tau_c = 1$. Other functions may also be considered provided that $\alpha(u_c) = 0$ and $\tau_c < 0$ or $\tau_c > 0$ (according to (55)).

6 Concluding remarks

A novel model has been proposed, named “new teleparallel dark energy” (NTDE), in which it is allowed a nonminimal coupling between the scalar field of quintessence and gravity, represented by torsion. As is well-known [1, 2], the scalar field couples to torsion through its four-derivative—which is a vector field. It is then natural to consider a nonminimal coupling of the four-derivative of the scalar field and the vector part of torsion. It is important to note that unlike the TDE scenario of Ref. [32], which was proposed by following an analogy with the nonminimally coupled scalar field in GR, in the present model the nonminimal coupling is proposed according to a conceptually different description of the gravitational field in TG [1–3]. Let us stress that NTDE does not have an analogue in GR, since in GR the gravitational field cannot be decomposed in a form analogous to the decomposition of torsion in TG (see Eqs. (1), (2), and (3)). It is also important to observe that once the nonminimal coupling is switched on, the scalar field becomes coupled, through its four-derivative, to the spin tensor of the gravitational field (see Eq. (7)). This is due to that in TG, besides a well-defined local energy-momentum density [1, 2, 58], the gravitational field also has a well-defined spin current density. The same happens in TDE, but until now had not been given an interpretation. In both models, the coupling to the gravitational spin tensor becomes negligible on cosmological scales.

By studying the dynamics of the model, we have found the critical points, presented in Tables 1 and 3. In Table 1 the final attractor of the Universe is a dark-energy-dominated de Sitter solution I.c, with $\Omega_\phi = 1$ and $\omega_\phi = \omega_{\text{eff}} = -1$. On the other hand, in Table 3 the final attractor is a scalar-field-dominated solution II.e, also with $\Omega_\phi = 1$, but in this case either $\omega_\phi = \omega_{\text{eff}} \geq -1$ or $\omega_\phi < -1$ in which case it represents a phantom-type solution. However, unlike the dark energy models with phantom (ghost) field [9, 10], here the phantom regime ($\omega_\phi < -1$) is described without the problem of quantum instability [57]. Additionally, unlike of the TDE scenario, here the phantom Universe is an attractor solution for the cosmological dynamics. The fixed points I.c and II.e are viable cosmological solutions to explain the current accelerated expansion of the Universe.

It is interesting to remark that the models TDE and NTDE are mathematically related through a conformal transformation. This can be seen by defining transformed scalar field and potential, and by adding an explicit coupling between the scalar field and matter. However, as already pointed out in the case of scalar-tensor theories and coupled dark energy in GR, where this type of mathematical relationship also occurs [12, 13], the two models are physically different. Furthermore, differently from coupled dark energy in GR, here we have taken the freedom to exclude a possible explicit coupling between the scalar field and matter (although this could be considered in a future work).
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