Conductance statistics near the Anderson transition

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I. INTRODUCTION

Since the pioneering work of the Anderson\(^1\) we know that disorder localizes electrons. At zero temperature any small amount of the disorder localizes all electrons in the one-dimensional systems\(^2\). In higher dimension \((d > 2)\) for systems with orthogonal symmetry) weak disorder does not destroy the metallic regime. Only when the strength of the disorder increases over a certain specific value (critical disorder), the electron becomes localized. This phenomenon - transition from the metallic to the insulating regime due to an increase of disorder - is called Anderson transition\(^3\).

Scaling theory of the Anderson transition uses the conductance \(g\) as the order parameter. It is supposed\(^5\) that the system size dependence of the conductance is determined only by the value of the conductance itself:

\[
\frac{\partial \ln g}{\partial \ln L} = \beta(g)
\]

where \(\beta(g)\) is an analytical function of \(g\). \(\beta\) is positive (negative) in the metallic (localized) regimes, respectively. For dimension \(d > 2\) the function \(\beta(g)\) changes its sign, being positive for \(g \gg 1\) and negative in the limit \(g \to -\infty\). There is an unstable fixed point \(g_c\) defined as the solution of \(\beta(g = g_c) = 0\). System-size independent critical conductance \(g_c\) represents the critical point of the Anderson transition.

Relation (1) contains no information about the microscopic structure of the model. This means that the Anderson transition is universal. The form of the \(\beta\) function is determined only by the physical symmetry and dimension of the system\(^8\).

Soon after the formulation of the scaling theory of localization it became clear that the conductance \(g\) is not a self-averaged quantity. Reproducible fluctuations of the conductance were found both in the metallic and in the insulating regimes\(^9,10\). The knowledge of the mean value \(\langle g \rangle\) is therefore not sufficient for complete description of the transport properties. One has to deal with the conductance distribution \(P_\epsilon(g)\)\(^11,12\) or, equivalently, with all cumulants of the conductance. This is easier in the metallic regime, where \(P(g)\) is Gaussian and the conductance fluctuations are universal\(^13-21\) and independent on the value of the mean conductance and/or the system size. The width of the distribution depends only on the dimension, physical symmetry of the system\(^13\) and on the boundary conditions\(^22,23\). In the insulator, conductance wildly fluctuates within the ensemble of macroscopically equivalent ensembles. It is the logarithm of the conductance which is distributed normally in the limit of large system size\(^11,12,17,24\).

At present we have no complete analytical theory able to describe the conductance statistics at the critical point. Analytical results are known only for systems of dimension \(d = 2 + \epsilon (\epsilon \ll 1)\)\(^12,25,26\). Expression for conductance cummulants\(^25\) enabled to estimate the shape of the critical conductance distribution \(P_c(g)\)\(^26\). \(P_\epsilon(g)\) is system size independent and Gaussian in the neighbor of the mean value \(\langle g \rangle \sim \epsilon^{-1}\). The distribution possesses power law tail \(P_c(g) \sim g^{-1/2}\epsilon^2\) for \(g \to \infty\), and the delta-function peak \(\delta(g)\). These results can not be applied to three dimensional (3D) system \((\epsilon = 1)\) where \(\langle g \rangle \sim 1^{27}\). Numerical simulations are therefore crucial in 3D system. The first systematic numerical analysis of the conductance statistics in 3D was done in\(^27\) and was followed by a series of papers\(^28-34\). In 2D systems, critical conductance distribution was numerically studied in the regime of quantum Hall effect\(^35\) and in systems with the spin-orbit interaction\(^28,36,37\).

This paper reviews our recent numerical data for the conductance distribution. We address the question of the shape, universality and the scaling of the critical conductance distribution.

In numerical calculation of the conductance we suppose that two opposite sites of the sample are connected to semi-infinite perfect leads and use the multichannel Landauer formula\(^38\) which relates the conductance \(g\) (in units \(2e^2/h\)) to the transmission matrix \(t\):

\[
g = \text{Tr} t^L t^R = \sum_{i=1}^{N_{\text{open}}} \cosh^{-2} \frac{z_i}{2}
\]

In (2) we introduced the variables \(z_i, i = 1, 2 \ldots N_{\text{open}}, (z_1 < z_2 < \ldots)\) which parametrize the eigenvalues of the matrix \(t^L t^R\). In the limit \(L_L >> L_I, z_i\) converges to \(2L_L/\lambda_i\) where \(\lambda_i\) is the ith localization length of the quasi-one dimensional (quasi-1D) system\(^17,27\). \(N_{\text{open}}\) is the number of open channels.

Owing to relation (2), the analysis of the conductance can be reduced to the calculation of the eigenvalues of the matrix \(t^L t^R\). The general formula for \(t\) was derived in\(^39,40\).

According to the scaling theory, critical exponents of the Anderson transition as well as conductance statistics do not depend on the microscopic details of the model. In numerical simulations, we consider the Anderson Hamiltonian

\[
\mathcal{H} = W \sum_n \varepsilon_n c_n^\dagger c_n + \sum_{[nn']} \tau_{nn'} c_n^\dagger c_{n'}.
\]


In (3) n counts the sites of the d-dimensional lattice and \([nn']\) means nearest neighbor sites. The hopping term \(\tau\) equals to unity for orthogonal systems, unless the anisotropy is considered. When spin orbit scattering is considered, \(\tau\) is a \(2 \times 2\) matrix. Parameter \(W\) measures the strength of the disorder. For box distribution of random energies \(\varepsilon_n\) \((P(\varepsilon) = 1\) for \(|\varepsilon| \leq 1/2\) and \(P(\varepsilon) = 0\) otherwise), the 3D Anderson model (3) exhibits Anderson transition at \(W_c = 16.5\).

II. FINITE SIZE SCALING

As it was discussed in the Section I, the conductance \(G\) is not the self-averaged quantity. To avoid statistical fluctuations, quasi-1D systems were introduced and the localization length \(\gamma\) is calculated instead of the conductance. In the limit \(L/\gamma \rightarrow \infty\) (\(L\) and \(\gamma\) are the length and the width of the system, respectively) \(\gamma\) is a self-averaged quantity which converges to its mean value. Finite size scaling is then used for the analysis of the disorder and the system width dependence of \(\gamma\). It is assumed that the variable \(\Lambda = \gamma/L\) is a function of only one parameter: \(\Lambda(\gamma/L) = \Lambda(\gamma/W)\). Here \(\gamma = \xi(W)\) is the correlation length which diverges in the vicinity of the critical point \(\xi(W) = |W - W_c|^{-\nu}\). Critical exponents \(\nu\) and \(s = (d-2)/\nu\) characterize the critical behavior of the localization length and of the conductance, respectively.

Finite size scaling analysis of the quasi-1D data enabled to test the universality of the Anderson transition and provided us with the more accurate estimation of the critical exponent \(\nu \approx 1.54\).

A. Scaling of the mean conductance

Verification of the scaling theory of the localization requires the proof of the universal scaling of the mean conductance and of the entire conductance distribution in the critical regime. Single parameter scaling of various mean values, \(\langle g \rangle\), \(\exp(\ln g)\), and \(1/(g^{-1})\), was proved numerically for the 3D Anderson model. Statistical ensembles of more than \(N_{stat} \geq 10^6\) cubes of the size from \(4^3\) to \(18^3\) were collected for various values of the disorder \(W\). Fig. 1 shows typical data for the system size dependence of the mean conductance. In agreement with (1), \(\langle g \rangle\) increases (decreases) with the system size in the metallic (localized) regime. By the use of the general fitting procedure, the critical disorder and the critical exponent \(\nu = 1.54\) were obtained.

Data in fig. 1 confirm that the variance, \(\text{var}(g) = \langle (g^2) \rangle - \langle g \rangle^2\), is a unambiguous function of the mean \(\langle g \rangle\) in the critical regime. This supports, but still does not prove the single parameter scaling theory. General proof of the theory requires verification of the single parameter scaling of all conductance cummulants. This is numerically impossible since higher cummulants are fully determined by rare events with very large values of the conductance.

B. Scaling of the conductance distribution

As higher cummulants are not treatable numerically, we test the scaling of the conductance distribution by the analysis of the scaling of percentiles \(g_\alpha\). Percentile \(g_\alpha\) is defined as

\[
\alpha = \int_0^{g_\alpha} P(g) dg.
\]

Owing to (4), the probability to find \(g < g_\alpha\) equals to \(\alpha\). Of course, the percentile \(g_\alpha\) is a function of disorder and system size: \(g_\alpha = g_\alpha(L, W)\). Single parameter scaling of percentiles has been proved for several values of \(\alpha\).
Suppose that $g_\alpha$ and $g_\beta$ ($\alpha < \beta$) obey the single parameter scaling. Then $g_\gamma$ ($\alpha < \gamma < \beta$) scales, too. Therefore, in contrast to the analysis of the conductance cummulants, it is enough to analyze only a few percentiles. Next, if $g_\alpha$ and $g_\beta$ scale, then the difference $g_\beta - g_\alpha$ scales. Scaling of percentiles assures thus the scaling of the entire conductance distribution. Of course, this analysis is not applicable to the limit $\alpha \to 1$, because available statistical ensembles are never big enough to provide us with sufficient information about the tail of the distribution.

### III. CRITICAL CONDUCTANCE DISTRIBUTION

At present, we have no analytical description of the critical conductance distribution in 3D systems. Analytical results were obtained only for the conductance cummulants in the dimension $d = 2 + \epsilon$ close to the lower critical dimension ($\epsilon \ll 1$). In spite of the non-universality of higher order conductance cummulants

\[
(\delta g^n) = \begin{cases} 
\epsilon^{n-2} & n < n_0 = \epsilon^{-1} \\
\sim L^{\epsilon n - n} & n > \epsilon^{-1}
\end{cases}
\]  

the critical distribution $P_c(g)$ was shown to be universal and $L$-independent in the limit $L \to \infty$. However, theoretical analysis of the form of the critical distribution, is applicable only in the limit of very small $\epsilon$.

#### A. The form of the critical conductance distribution

All what we know about the $P_c(g)$ in 3D is based on the numerical data. In Fig. 2 we present $P_c(g)$ for the 3D Anderson model. Data confirm that the critical conductance distribution is system size independent, as required. The shape of $P_c(g)$ differs considerably from the conductance distribution in the metallic and in the insulating regimes. To explain the typical properties of the critical conductance distribution, we use our knowledge about statistical properties of parameters $z$.
course, on the length of the system. We believe that these non-universalities could be compensated by the change of another model parameter (see, for instance37, where the anizotropy is compensated by the length of the system).

D. Dimension dependence

Right figure in Fig. 2 compares the critical conductance distribution for 3D and 4D cubes. As supposed, the maximum of \( P_c(g) \) for 4D is shifted toward smaller conductances, because the critical disorder increases as the spatial dimension increases and higher disorder means lower mean conductance5,26. Qualitatively, however, both distributions are very similar: \( P_c(\ln g) \) decreases as \( \exp[-\ln^2 g] \) for \( \ln g \to -\infty \) and possesses the non-analyticity at \( \ln g = 0 \). This similarity is not surprising, because the form of the distribution is determined mostly by the statistics of \( z_1 \) and \( z_2 \), which are qualitatively similar in 3D and 4D10.

More interesting is the investigation of the \( P_c(g) \) in the systems of dimension 2 + \( \epsilon \)51. As we are not able to create the \( d \)-dimensional hyper-cubes with non-integer \( d \) in computers, we simulated the transport on bifractal lattices52,51. Bifractals are linear along the propagation direction and possess the fractal lattice in the cross section (fig. 5). We proved that the critical exponent \( \nu \) depends only on the spectral dimension of the lattice. Mean conductance, var \( g \) and the critical distribution \( P_c(g) \) depend, however, on the lattice topology. For instance, fig. 5 shows that bifractals A and B have the same spectral dimension, but different critical distribution. This is the reason why obtained data can not be used for the verification of relations (5).

In Fig. 5 we present \( P_c(g) \) for three different bifractals. As expected, \( \langle g \rangle \) increases and the distribution converges to Gaussian when \( \epsilon \to 0 \). However, we found neither the \( \delta \) - function peak at \( g = 0 \) nor the power-law tail of the distribution for \( g \gg \langle g \rangle \), predicted by the theory26.

IV. CONDUCTANCE DISTRIBUTION IN NON-CRITICAL REGIME

Although we have no analytical theory of the conductance statistics in the critical regime, we can learn some typical properties of the conductance distribution from the analysis of the quasi-1D weakly disordered systems48. Starting from the Dorokhov - Mello-Pereyra-Kumar equation15 for the probability distribution of parameters \( z \), the conductance distribution \( P(g) \) can be calculated. \( P(g) \) depends on the length of the system.
tentatively similar to the critical conductance distribution. In contrast to the metallic regime, the spectrum of $z$ becomes dimension dependent in the critical region. Both quasi-1D and 3D-dimensional numerical studies confirmed that at the critical point

$$\langle z_i \rangle \sim i$$

($d > 2$). Owing to (6), the difference $\Delta = \langle z_2 - z_1 \rangle$ is smaller than $\langle z_1 \rangle$ in 3D, while it equals to $\langle z_1 \rangle$ in the quasi-1D system. The contribution of the second channel is therefore more important in 3D than in the quasi-1D. This explains longer tail of the distribution in the 3D system (fig. 6b).

It is commonly believed that the distribution $P(\ln g)$ is Gaussian in the insulating phase, independently on the dimension of the system. This is, however, not true. The spectrum of $z$ depends namely on the dimension of the system also in the localized regime. For the 3D systems it was proved numerically that the difference $\Delta = \langle z_2 - z_1 \rangle$ is constant, independent on the disorder and on the system size. Therefore the second channel influences always the statistics of the conductance. More than its contribution to the value of the conductance it is important that constant value of $\Delta$ prevents the distribution $P(z_1)$ to develop into the Gaussian form. While the values $z_1 \ll \langle z_1 \rangle$ are still possible, the probability to find systems with much higher values $z_1 \gg \langle z_1 \rangle$ is strongly suppressed. The distribution $P(z_1)$ is therefore not symmetric. The same is true for the distribution $P(\ln g)$ (Fig. 7) which possesses a long tail for small values of $|\ln g|$ and decreases much faster than Gaussian for $|\ln g| \to \infty$.

Note that in weakly disordered quasi-1D systems $\Delta \sim \langle z_1 \rangle$. The distance $\langle z_2 - z_1 \rangle$ is much larger than the width of the distribution $P(z_1)$. Higher channels therefore do not influence the distribution of $z_1$ and $P(\ln g)$ is Gaussian.

V. CONCLUSION

We reviewed recent progress in numerical studies of the statistics of the conductance in the critical regime. Numerical analysis confirms that the conductance distribution in 3D Anderson model obeys single parameter scaling. Analysis of the statistics of the eigenvalues of the transmission matrix enables us to understand the main features of the conductance distribution in the critical regime. Although we still have no analytical description of the conductance statistics in the critical regime, we hope that results of numerical experiments will inspire theoreticians to formulate the general analytical theory of the Anderson transition.
