Cosmology with exponential potentials

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We examine in the context of general relativity the dynamics of a spatially flat Robertson-Walker universe filled with a classical minimally coupled scalar field $\phi$ of exponential potential $V(\phi) \sim \exp(-\mu \phi)$ plus pressureless baryonic matter. This system is reduced to a first-order ordinary differential equation for $\Omega_{\phi}(w_{\phi})$ or $q(w_{\phi})$, providing direct evidence on the acceleration/deceleration properties of the system. As a consequence, for positive potentials, passage into acceleration not at late times is generically a feature of the system for any value of $\mu$, even when the late-times attractors are decelerating. Furthermore, the structure formation bound, together with the constraints $\Omega_{m0} \approx 0.25 - 0.3$, $-1 \leq w_{\phi0} \leq -0.6$ provide, independently of initial conditions and other parameters, the necessary condition $0 < \mu \leq 1.6\sqrt{8\pi G N}$, while the less conservative constraint $-1 \leq w_{\phi} \leq -0.93$ gives $0 < \mu \lesssim 0.7\sqrt{8\pi G N}$. Special solutions are found to possess intervals of acceleration. For the almost cosmological constant case $w_{\phi} \approx -1$, the general relation $\Omega_{\phi}(w_{\phi})$ is obtained. The generic (non-linearized) late-times solution of the system in the plane $(w_{\phi}, q)$ is also derived.

I. INTRODUCTION

Scalar-field cosmologies continuously receive through time special attention. The interest on scalar fields has been stimulated when it has been realized that a scalar field might be responsible for inflation of the very early universe. If its self-interaction potential energy density is sufficiently flat, it can violate the strong energy condition and drive the universe into an accelerated expansion. Even if the potential is too steep to drive inflation, such models still have important cosmological consequences. Recent observational evidence shows that the energy density of the universe today may be dominated by a homogeneous component with negative pressure (quintessence)\textsuperscript{2}. Slowly rolling self-interacting scalar fields may provide in the late-times evolution of the universe a dynamical mechanism to achieve a small effective cosmological constant.

Models with a variety of self-interaction potentials have been studied and the cosmological conclusions are naturally strongly model dependent. There are attempts to base the choice of the potential on some fundamental dynamical considerations. A class of potentials that is commonly investigated and which arises in a number of physical situations has an exponential dependence on the scalar field. Such potentials are motivated by higher-dimensional supergravity, superstring theory, or M-theory that have been compactified to an effective four-dimensional theory\textsuperscript{2}. Also, higher-order gravity theories are conformally equivalent to general relativity plus a scalar field with potential which asymptotically tends to an exponential form\textsuperscript{2}.

Since the first inflationary cosmological models were presented, most calculations have been performed using one of a variety of approximations. While these are usually sufficient and often unavoidable, there is considerable interest in scalar field cosmologies that can be solved exactly. Exponential potentials have been extensively considered in cosmology by various authors, basically in Robertson-Walker models with no other source besides the scalar field\textsuperscript{3}. Solutions with anisotropic/inhomogeneous geometries were obtained in\textsuperscript{4}. When a perfect fluid is added, only a few solution are known (see references\textsuperscript{5}-\textsuperscript{10} for relevant works). In the context of ekpyrotic or pre-big bang collapse, exponential potentials were analyzed in\textsuperscript{11}, while non-minimal couplings of scalar fields with exponential potentials have been considered in\textsuperscript{12}. Attention has also been directed to the possible effect of such a scalar field on the growth of large-scale structure in the universe\textsuperscript{13}. Other recent studies on exponential potentials have appeared in\textsuperscript{14}.

In the present paper, we examine the dynamics of a spatially flat Robertson-Walker universe filled with a classical minimally coupled scalar field of exponential potential plus pressureless baryonic matter (luminous and nonluminous) non-interacting with the field. Clearly, this model cannot be used from the beginning of the universe evolution, but only after decoupling of radiation and dust. Thus, we do not take into account inflation, creation of matter, nucleosynthesis, e.t.c. By performing changes of phase-space variables, we reduce the system of Einstein equations into a first order nonlinear differential equation, which can be expressed as an equation for $\Omega_{\phi}(w_{\phi})$ or $q(w_{\phi})$. Solving this equation leads to integration of the whole system. Furthermore, studying this with respect to its acceleration properties, as well as with respect to standard observational constraints, restrictions on the parameter determining the steepness of...
the potential are obtained. For the almost cosmological constant case $w_\phi \approx -1$, the solution $\Omega_\phi(w_\phi)$ is obtained. The late-times behaviour $\Omega_\phi(w_\phi)$ of the system is also found.

II. GENERAL ANALYSIS OF THE MODEL

We consider the action

$$S = \frac{1}{2k^2} \int R \sqrt{-g} \, d^4x + \int L_{\text{mat}} \sqrt{-g} \, d^4x$$

$$- \int \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V \right) \sqrt{-g} \, d^4x,$$  

(1)

with the term $L_{\text{mat}}$ corresponding to the perfect fluid, and we adopt the Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j,$$  

(2)

where $\gamma_{ij}$ is a maximally symmetric 3-dimensional metric, and $k = -1, 0, 1$ parametrizes the spatial curvature. The Einstein equations for the system of a perfect fluid and a scalar field is equivalent to the following set of equations

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0$$

(3)

$$\dot{\rho}_\phi + 6 \frac{\dot{a}}{a} (\rho_\phi - V) = 0$$

(4)

$$H^2 = \frac{\dot{a}^2}{a^2} = \beta (\rho + \rho_\phi) - \frac{k}{a^2},$$

(5)

where $\beta = \kappa^2/3 = 8\pi G_N/3$, a dot means differentiation with respect to proper time $t$, and $\rho_\phi$ is the energy density of the scalar field defined by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi).$$

(6)

Since the perfect fluid and the scalar field are non-interacting, each separately satisfies its own conservation equation (6), respectively. Equation (11) is equivalent to the classical equation of motion $\Delta \phi = V'(\phi)$ of the field $\phi$, where $\Delta$ is the 4-dimensional Laplacian of the metric (12) and a prime denotes differentiation with respect to $\phi$. From Eq. (11), it arises that the function $\rho_\phi(t)$ is monotonically decreasing (increasing) for an expanding (contracting) universe.

From Eqs. (3), (4), (6), one can derive the Raychaudhuri equation

$$\dot{H} = -\frac{3\beta}{2} (\rho + p + \dot{\phi}^2) + \frac{k}{a^2},$$

(7)

which is also written in the form

$$\frac{\ddot{a}}{a} = \beta \left( 3V - 2\rho_\phi - \frac{1 + 3w}{2} \rho \right),$$

(8)

where $w = p/\rho$ characterizes the perfect fluid.

Defining the quantity

$$\varrho = \rho^2 + \rho_\phi,$$

(9)

we obtain from Eqs. (3), (11) that

$$\dot{\varrho} + 6 \frac{\dot{a}}{a} (\varrho - V + \rho p) = 0.$$  

(10)

We will specify to a flat universe ($k = 0$) with dust matter ($w = 0$). Equation (10), due to (5), is written for an expanding universe as

$$\sqrt{\rho_\phi - V} \dot{\varrho}' = \mp 3\sqrt{2\beta} \left( \sqrt{\varrho - \rho_\phi + \rho_\phi} \right)^{1/2} (\varrho - V),$$

(11)

where $\dot{\varrho}' = \pm \sqrt{2(\rho_\phi - V)}$. Similarly, Eq. (11) takes the form

$$\rho_\phi' = \mp 3\sqrt{2\beta} \left( \sqrt{\varrho - \rho_\phi + \rho_\phi} \right)^{1/2} \sqrt{\rho_\phi - V}.$$  

(12)

Dividing Eqs. (11) and (12), we get

$$\frac{\rho_\phi'}{\varrho'} = \frac{\rho_\phi - V}{\varrho - V}.$$  

(13)

The system of Eqs. (11)-(13) is symmetric under time reversal $t \rightarrow -t$, thus a contracting universe arises from an expanding one. Additionally, the stable fixed points of the system for an expanding universe become unstable fixed points in a contracting universe and vice-versa, thus we do not discuss the collapsing behaviour further.

We consider a potential with an exponential-like form

$$V(\phi) = V_0 e^{-\mu \phi},$$

(14)

and define

$$R = \frac{\rho_\phi}{\varrho}, \quad Q = \frac{\sqrt{\varrho - \rho_\phi}}{\sqrt{\rho_\phi - V}}.$$  

(15)

The system of Eqs. (12), (13) is written equivalently as

$$Q = \frac{(R' - \mu R)^2}{18\beta (R - 1)} - R,$$

(16)

$$\left( \ln Q \right)' = \frac{R' + \mu R - 2\mu}{2(R - 1)},$$

(17)

under the conditions $\pm V_0 (R' - \mu R) < 0$, $(R' - \mu R)^2 > 18\beta R(R - 1)$. One characteristic of this system is that, due to the exponential potential, it does not contain the independent variable $\phi$ explicitly. Since the system of Eqs. (10)- (14) is symmetric under the simultaneous change $\mu \rightarrow -\mu$, $\phi \rightarrow -\phi$, we will consider, without loss of generality, only $\mu > 0$. Combining Eqs. (10), (17), we can eliminate $Q$, and obtain

$$R' - \mu R = 0.$$  

(18)
or
\[ 4(R - 1)R'' - 3R^2 - 2\mu R(R - 3)R' - (R - 2) \left( (18\beta - \mu^2)R - 18\beta \right) = 0. \tag{19} \]

Equation (18) is equivalent to \( \dot{\rho}_\phi = -\rho \), which gives from Eq. (6) a static universe \( a = \text{constant} \), and is not of further interest here. Making the transformation
\begin{equation}
\chi = R - 1, \quad \psi = \frac{\mu}{R}, \tag{20}
\end{equation}
the second order differential equation (18) reduces to the following first order differential equation in \( \psi(\chi) \):
\begin{equation}
4\chi \frac{d\psi}{d\chi} = (\chi - 1)(1 - \nu^{-1}\chi)\psi^3 - 2(\chi - 2)\psi^2 - 3\psi, \tag{21}
\end{equation}
where \( \nu^{-1} + 1 = 18\beta/\mu^2 \). In terms of \( \chi, \psi \), the previous constraints take the form \( \pm V_0(\psi^{-1} - \chi - 1) < 0 \), \( (\psi^{-1} - \chi - 1)^2 > (\nu^{-1} + 1)\chi(\chi + 1) \).

In terms of the original variables, it is
\begin{equation}
\chi = \frac{\dot{\phi}^2}{2V}, \quad \psi = \mu \left( \frac{d\chi}{d\phi} \right)^{-1}, \tag{22}
\end{equation}
and thus \( V_0\chi > 0 \). Finding solutions of Eq. (24) is the starting point for running the equations backwards to find solutions of the whole system. Equation (24) belongs to the class of Abel equations. There is no known systematic way for solving such sort of equations. For the same problem with the one examined here, another Abel equation, inequivalent to (24), was obtained in Ref. [8].

The difference is that there, a transformation of the configuration variables has been performed.

Given a solution \( \psi(\chi) \) of Eq. (24), one can find all the relevant physical quantities in parametric form. It is obvious that \( \chi \) is not in general a global time parameter, but it remains a good time parameter, as long as \( \chi(t) \) is monotonically increasing or decreasing. So, the scalar field \( \phi(\chi) \) is found from Eq. (22) as
\begin{equation}
\phi(\chi) = \phi_1 + \mu^{-1} \int \psi(\chi) d\chi, \tag{23}
\end{equation}
where \( \phi_1 \) is integration constant. Then, the scale factor (normalized to unity today) is expressed as
\begin{equation}
a^3(\chi) = \frac{1}{V_0} \frac{\rho_0(\nu^{-1} + 1)\chi e^{\mu\phi(\chi)}}{(\psi^{-1} - \chi - 1)^2 - (\nu^{-1} + 1)\chi(\chi + 1)}, \tag{24}
\end{equation}
with \( \rho_0 \) the present value of \( \rho \), and the Hubble parameter is
\begin{equation}
H^2(\chi) = \frac{\mu^2 V_0}{18\chi} e^{-\mu\phi(\chi)(\psi^{-1} - \chi - 1)^2}. \tag{25}
\end{equation}

The luminosity distance \( d_L = (1 + z) \int_{t_0}^t dt/a \), which is directly measured, is given by
\begin{equation}
d_L(z) = \pm \frac{1 + z}{\sqrt{2\mu \rho_0^3/3}} \int_x^{x_0} \frac{\psi(\chi)}{\sqrt{V_0 \chi}} e^{\mu\phi(\chi)} \left\{ V_0 \left[ (\psi^{-1} - \chi - 1)^2 - (\nu^{-1} + 1)\chi - 1 \right] \right\}^{1/2} d\chi, \tag{26}
\end{equation}
where the redshift \( z \) is defined by \( 1 + z = 1/a \), and \( \chi_0 \) denotes the value of \( \chi \) today. The flatness parameters \( \Omega_m = \beta p/H^2 \), \( \Omega_\phi = \beta \rho_\phi/H^2 \) are found as
\begin{equation}
\Omega_\phi(\chi) = \frac{(\nu^{-1} + 1)\chi(\chi + 1)}{(\psi^{-1} - \chi - 1)^2} = 1 - \Omega_m(\chi). \tag{27}
\end{equation}

Similarly, the energy density and the pressure \( \rho_\phi = (\dot{\phi}^2/2) - V \) of the scalar field are given respectively by \( \rho_\phi(\chi) = V_0(\chi + 1)e^{-\mu\phi(\chi)} \), \( p_\phi(\chi) = V_0(\chi - 1)e^{-\mu\phi(\chi)} \). Besides these parametric expressions, a physically interesting point is the determination of the equation of state \( \rho_\phi = p_\phi(\rho_\phi) \) of the scalar field, or equivalently of the function \( w_\phi(\rho_\phi) \), where \( w_\phi = p_\phi/\rho_\phi \) is the state parameter of the scalar field, given by
\begin{equation}
w_\phi = \frac{\chi - 1}{\chi + 1} \Leftrightarrow \chi = \frac{1 + w_\phi}{1 - w_\phi}. \tag{28}
\end{equation}
The inequality \( V_0\chi > 0 \) translates to \( |w_\phi| < 1 \) for \( V_0 > 0 \), and \( |w_\phi| > 1 \) for \( V_0 < 0 \). Then, it is found that
\begin{equation}
\rho_\phi(w_\phi) = \frac{2V_0}{e^{w_\phi} - 1} = \frac{1}{2} - \frac{3(\nu^{-1} + 1)(1 - \chi)}{2(\psi^{-1} - \chi - 1)^2} \frac{1}{1 + 3w_\phi \Omega_\phi}. \tag{29}
\end{equation}

In this language, Eq. (24) becomes a second order differential equation for \( \ln \rho_\phi(w_\phi) \). Concerning the problem of acceleration, we can find the decelerating parameter \( q = -\ddot{a}/aH^2 \) from Eqs. (24), (25), (30) as
\begin{equation}
q = 1 - \frac{3(\nu^{-1} + 1)(1 - \chi)}{2(\psi^{-1} - \chi - 1)^2} \frac{1}{1 + 3w_\phi \Omega_\phi}. \tag{30}
\end{equation}

Since the variables \( \chi, \psi \) are phase space variables, by studying Eq. (24), it is straightforward from Eqs. (30) to get information on the acceleration/deceleration properties of the cosmological system. It is seen from the first equality of Eq. (30) that acceleration is possible only for \( -1 < w_\phi < 0 \), and thus, it is possible only for \( V_0 > 0 \). Additionally, for \( V_0 > 0 \), since \( 0 < \Omega_\phi < 1 \), it arises from the second equality of Eq. (30) that acceleration is possible only for \( -1 < w_\phi < -1/3 \). This is an exact result, without the usual assumption on the dominance of the quintessence component in the energy density, and of course, it is only a necessary condition. Finally, the relation between \( \chi \) and \( t \) is found by
\begin{equation}
t(\chi) = t_\chi \pm \frac{1}{\sqrt{2\mu}} \int_{x_0}^{x} \frac{\psi(\chi)}{\sqrt{V_0 \chi}} e^{\mu\phi(\chi)/2} d\chi, \tag{31}
\end{equation}
with \( t_\chi \) integration constant. We observe that the number of essential integration constants of the system is three, namely \( \rho_0, \phi_1 \), and the one contained in \( \psi(\chi) \) from integration of Eq. (24). This is also predicted from the beginning, since the system consists of two first-order equations, (3), and one second-order equation (14), giving four integration constants, from where one of them is taken out due to the time translation.

In Ref. [5], for \( V_0 > 0 \) and
\begin{equation}
x = \sqrt{\frac{\beta}{2}} \phi, \quad y = \frac{\sqrt{\beta} \psi}{H}, \tag{32}
\end{equation}
the system of Eqs. (33)-(35) was written as a plane-autonomous system
\[
\frac{dx}{d\ln a} = -3x + \frac{\mu}{2\beta}y^2 + \frac{3}{2}x(1 + x^2 - y^2) \quad (33)
\]
\[
\frac{dy}{d\ln a} = -\frac{\mu}{2\beta}xy + \frac{3}{2}y(1 + x^2 - y^2). \quad (34)
\]

It was further found that for any value of \( \mu \), there are three kinds of fixed points, characterized by the values of
\[(x, y): (i) (\pm 1, 0), (ii) (0, 0), (iii) (\mu, \sqrt{18\beta - \mu^2})/3\sqrt{2}\beta \]
for \( \mu^2 < 18\beta \) and \( 3\sqrt{\beta}(1, 1)/\sqrt{2}\mu \) for \( \mu^2 > 9\beta \). The fixed points (i) are unstable nodes or saddle points, the fixed point (ii) is a saddle point, while the fixed points (iii) are stable points or saddles. More specifically, for any value of \( \mu (\mu^2 \neq 9\beta, 72\beta/7) \) there exists one stable fixed point, which is accelerating for \( \mu^2 < 6\beta \), and decelerating for \( \mu^2 > 6\beta \). For \( V_0 < 0 \), generic solutions recollapse to a singularity after a finite lifetime. The relation between the variables \( x, y \) and \( \chi, \psi \) can be found from the previous definitions as
\[
\chi = x^2/y^2, \quad \psi^{-1} = 1 + x(x - 3\sqrt{2\beta/\mu})y^{-2}. \quad (35)
\]

III. COSMOLOGICAL IMPLICATIONS

We are not in position to find the generic solution of Eq. (21), and then, study from Eq. (30) the accelerating properties of the system. However, if \( f = \Omega_m/\Omega_{\phi} = (1/\Omega_{\phi}) - 1 \) is the fraction of the flatness parameters, we are able to study the accelerating properties of the system by deriving from Eqs. (21), (30) the following first order differential equation for the function \( f(w_{\phi}) \)
\[
\frac{w_{\phi}^2 - 1}{w_{\phi}(f + 1)} \frac{df}{dw_{\phi}} = \frac{\epsilon}{\sqrt{(\nu^{-1} + 1)(w_{\phi} + 1)(f + 1)}} \times \left[ \frac{(1 - \nu^{-1}) - (1 + \nu^{-1})w_{\phi}}{\sqrt{2\nu^2\epsilon\nu^{-1}(w_{\phi} + 1)(f + 1)}} - 1 \right], \quad (36)
\]
where \( \epsilon = sgn(V_0(1 - w_{\phi})) \), and then, \( 2g(w_{\phi}) = 1 + 3w_{\phi}/(1 + f(w_{\phi})) \). For \( V_0 > 0 \), due to the above discussion on the fixed points, it is obvious that all the solutions of the system start their evolution with deceleration at \( \chi = +\infty (w_{\phi} = 1) \). Concerning the parameter \( \chi \), we can obtain its time evolution as \( \dot{\chi} = \dot{\phi} + 3H\phi = \pm\mu\sqrt{2V_{\chi}^\chi\psi^{-1}} \), thus, for \( V_0 > 0 \) and \( \phi < 0 \), the function \( \chi(t) \) (and also the function \( w_{\phi}(t) \)) is monotonically decreasing. This means that all the solutions emanating from the repeller \((-1, 0)\) have an initial period, characterized by \( \phi < 0 \), where \( \chi(t) \) is monotonically decreasing from \( \chi = +\infty \) to \( \chi = 0 \) (and also \( w_{\phi} \) monotonically decreases from \( w_{\phi} = 1 \) to \( w_{\phi} = -1 \)). We have plotted the solutions of Eq. (36) for these initial time intervals in Figure 1, shown by the family of “circular” lines, where an indicative value of the parameter \( \nu = 3/4 \) has been used. The deceleration parameter \( g(w_{\phi}) \) is shown in the same figure by the inclined lines passing through the point \( 1/2 \) of the vertical axis, providing direct evidence on the acceleration/deceleration properties of the system. Lines of the same indication in Figure 1 correspond to the same solution, while time increases from the right to the left along the curves. Using a first-order differential equation, since only one initial value enters, there is the merit of representing the whole space of solutions on a plane diagram. It is seen that there is a subclass of solutions which enter from initial deceleration to an acceleration era. However, there are still solutions with \( w_{\phi} < -1/3 \) which remain decelerated. It is easy to check from Eq. (36) that the structure of Figure 1 is typical for any value of \( \nu \). Thus, even for \( \nu > 1/2 \), the late-times attractor is decelerating, there are certain classes of solutions which have finite accelerating intervals not at late times. Actually, the supernovae data favour that the present-day accelerated expansion is a recent phenomenon \( \Omega_{\phi} \approx 0.5 \). Nucleosynthesis predictions \( \Omega_{\phi} \), on the other hand, claim that at 95 per cent confidence level, it is \( \Omega_{\phi}(z = 10^{10}) \leq 0.045 \). Besides this, during galaxy formation epoch \((z = 2 - 4)\), the quintessence density parameter is \( \Omega_{\phi} < 0.05 \). This is an additional constraint, applicable in the dust regime, which restricts the viable solutions. For instance, for the indicative value \( \nu = 3/4 \) of Figure 1, no accelerating solution can accommodate this bound, which means that the present accelerating era, if possible, has to be located in some future time having \( \phi > 0 \). For different values of \( \nu \), however, the above bound may match with the present-day acceleration, already in the epoch with \( \phi < 0 \).

FIG. 1: The class of all solutions with \( V_0 > 0, \phi < 0 \) for an indicative value of \( \nu = 3/4 \). Family of “circular” lines represent \((\Omega_m/\Omega_{\phi})(w_{\phi})\), while inclined lines passing through the point \( 1/2 \) of the vertical axis represent \( g(w_{\phi}) \). Lines of the same indication correspond to the same solution. The scale factor increases from the right to the left along the curves. A subclass of solutions show passage from deceleration to acceleration not at late times. Similar portraits occur for all values of \( \nu \).

In order to pass into the region with \( \phi > 0 \), an appropriate time variable, covering a region with both signs of \( \phi \), is the parameter \( \Omega_m \), or equally well \( \Omega_{\phi} \), or also \( f \) (we are going to use \( f \)). This is due to that, in general,
\[ \dot{\Omega}_m = 3\beta^2\rho V(\chi - 1)/H^3, \]

and, thus, for \( V_0 > 0, \chi < 1 \)
(or \(-1 < w_\phi < 0\)), the functions \( \Omega_m(t), f(t) \) are monotonically decreasing. Equation 35 can, then, be easily inverted, and considered as a first order differential equation for \( w_\phi(f) \), and then, \( 2q(f) = 1 + 3w_\phi(f)/(1 + f) \).

The solutions \( w_\phi(f) \) (“lower” curves of the same indication), \( q(f) \) (“upper” curves) are shown in Figure 2 for the value \( \nu = 3/4 \), where time increases from the right to the left along the curves. Lines of the same indication in Figure 2 correspond to the same solution, and also, they are the continuations of the corresponding curves of Figure 1. It is well known that a set of complementary observations indicate that the matter energy density \( \Omega_{m0} \) is no more than 0.3 [18], while observations of the higher peaks of the cosmic background radiation give \( \Omega_{m0} = 0.25 \) [19]. A conservative estimation for the present quintessence equation of state is \(-1 \leq w_{\phi 0} \leq -0.6 \) [18]. It is now a simple task, using figures similar to Figures 1 and 2, to determine for any particular value of \( \nu \), if there are accelerating solutions satisfying these two constraints on \( \Omega_{m0}, w_{\phi 0} \), together with the previous one from structure formation. The result of this analysis is that the above bounds are satisfied only for \( \nu \leq 3/4 \), with agreement to previous studies on exponential potentials, based on numerical analysis of the full second-order equations of the system [20]. This range of \( \nu \) corresponds to a scalar field dominated attractor \( \Omega_\phi = 1 \), with limiting value \( w_\phi = (1 - \nu^{-1})/(1 + \nu^{-1}) \). Although there is no explicit dependence on the scale factor \( a \) or the redshift \( z \) in Eq. 35, and therefore in Figures 1 and 2, however, in the previous analysis there is the merit of providing direct information on the viability of whole spaces of solutions, without particular reference to integration constants or other parameters of the system. Additionally, the non-dependence of \( w_\phi \) on \( z \) is profitable for a quantitative treatment, as it is known that \( w_\phi \) changes very rapidly between the redshifts of various supernovae data and the present.

### A. Approximate solution for \( w_\phi \approx -1 \)

The values \( w_\phi \approx -1 \) are of particular interest, since firstly, it is known [21] to cover various orders of magnitude in scale factor during the cosmic evolution, and secondly, for a class of potentials larger than those examined here, a combined analysis [10] including all three peaks in the Boomerang data of the CMBR spectra, and the Type Ia supernovae, gave as best fit \( \Omega_{\phi 0} = 0.75, -1 \leq w_{\phi 0} \leq -0.93 \), thus, behaviour similar to a cosmological constant. Considering this restrictive bound for \( w_{\phi 0} \), and following the previous analysis, we are restricted to \( \nu \leq 0.08 \) as necessary condition for the viability of the exponential model. In this case, we are in position to find the relation \( \Omega_\phi(w_\phi) \). Making the transformation

\[ \xi = \eta \chi^{-1}, \quad \Upsilon = (\eta \chi)^{-3/4} \psi^{-1}, \quad (37) \]

where \( \eta = sgn(V_0) \), Eq. (21) takes the form

\[ \frac{d\Upsilon}{d\xi} = \frac{2\xi - \eta \Upsilon}{2\xi^5/4} + \frac{(1 - \eta \xi)(\eta - \nu^{-1}\xi^{-1})}{4\sqrt{\xi}}. \quad (38) \]

For \( \chi \approx 0 \), Eq. (23) is approximated as

\[ \frac{d\Upsilon}{d\zeta} + \frac{4}{\sqrt{3}} \Upsilon + \zeta \approx 0, \quad (39) \]

where

\[ \zeta = \frac{3\eta + 2\xi}{2\sqrt{3} \xi^3} = \frac{3 + 2}{2\sqrt{3} (\eta \chi)^3} = \frac{1}{2} w_\phi \frac{1 - w_\phi}{1 + w_\phi} \cdot \frac{\xi}{\Omega_\phi}. \quad (40) \]

The solution of Eq. (39) is

\[ c_1|\Upsilon + \sqrt{3} \zeta|^3 = c_2|\Upsilon + \frac{1}{\sqrt{3}} \zeta|, \quad (41) \]

where \( c_1, c_2 \) are integration constants with \( |c_1| + |c_2| > 0 \). In terms of the physical variables \( w_\phi, \Omega_\phi \), Eq. (41) takes the form

\[ \frac{c_1}{|1 + w_\phi| \sqrt{1 - w_\phi^2}} \frac{1 + w_\phi}{2} = \epsilon \sqrt{2(\nu^{-1} + 1)} \sqrt{1 + w_\phi} \frac{1}{\Omega_\phi} \quad (42) \]

which is the first integral between \( w_\phi, \Omega_\phi \) for \( w_\phi \approx -1 \). This equation represents the “bottom” parts of the lines \( w_\phi(f) \) in Figure 2. Eq. (42) is a cubic for \( 1/\sqrt{\Omega_\phi} \), and can, therefore, be solved for \( \Omega_\phi(w_\phi) \). Then, from Eq. (27), (35) one gets the expression \( \psi(\chi) \), and plugging into Eqs. (29)-(31) all the relevant quantities can be obtained, at least as quadratures.

![Figure 2: Class of solutions with \( V_0 > 0, -1 < w_\phi < 0 \) for an indicative value of \( \nu = 3/4 \). “Lower” lines of the same indication represent \( w_\phi(f) \), while “upper” ones represent \( q(f) \). Lines of the same indication correspond to the same solution, and they are the continuations of the corresponding curves of Figure 1. The scale factor increases from the right to the left along the curves. Similar portraits occur for all values of \( \nu \).](image-url)
B. Special solutions

For two particular values of $\nu$, we can immediately find from Eq. (21) some special solutions of the system

\[
\nu = 1/3 \quad : \quad \psi = 3/(1 - 3\chi) \quad (43)
\]
\[
\nu = 9/7 \quad : \quad \psi = 3/(1 - \chi). \quad (44)
\]

The cases $\nu = 1/3, 9/7$ correspond to $\mu^2 = 9\beta/2, 81\beta/8$, for which values the general solution of the system we discuss has been found in [1] (for $V_0 > 0$), and [8] respectively. For the solution (43) with $V_0 > 0$ it arises that $\dot{\phi} > 0$. Additionally, we can see that the function $\chi(t)$ (and also $w_\phi(t)$) is monotonically increasing with $0 < \chi < 1/3$, and that this solution has an eternal acceleration for $\chi > (7 - \sqrt{33})/24$. Explicitly, for the solution (43) for both signs of $V_0$ the various quantities are given in parametric form from Eqs. (24) - (30) as follows

\[
\phi = \phi_1 - \mu^{-1}\ln(1 - 3\chi) = \phi_1 - \mu^{-1}\ln\left(2 + w_\phi - 1\right) \quad (45)
\]
\[
a^3 = \frac{9\rho_0e^{\mu\phi_1}}{V_0(1 - 3\chi)^2} = \frac{9\rho_0e^{\mu\phi_1}}{V_0(1 + w_\phi)^2} \quad (46)
\]
\[
\Omega_\phi = \frac{9\chi(1 + \chi)}{(1 + 3\chi)^2} = \frac{9(1 + w_\phi)}{(2 + 2w_\phi)^2} \quad (47)
\]
\[
\rho_\phi = -\frac{2V_0}{\mu\phi_1}(1 - w_\phi)^2 \quad (48)
\]
\[
p_\phi = p_\phi - \frac{2V_0}{\mu\phi_1}\left(1 + \sqrt{1 - \frac{3e^{\mu\phi_1}}{V_0}\rho_\phi}\right) \quad (49)
\]
\[
q = \frac{1}{2} - \frac{27\chi(1 - \chi)}{2(1 + 3\chi)^2} = \frac{1}{2} + \frac{27w_\phi(1 + w_\phi)}{4(2 + 2w_\phi)^2} \quad (50)
\]
\[
t = t_1 \frac{6e^{\mu\phi_1}}{\sqrt{2}\mu V_0} = \frac{1}{1 - 3\chi} = \frac{t_1 + 3e^{\mu\phi_1}}{\mu V_0}\sqrt{-\frac{V_0(1 + w_\phi)}{1 + 2w_\phi}} \quad (51)
\]

\[
\leftrightarrow \quad w_\phi = \frac{9e^{\mu\phi_1} + \mu^2V_0(t - t_1)^2}{9e^{\mu\phi_1} + 2\mu^2V_0(t - t_1)^2}. \]

where $\rho_0, \phi_1, t_1$ are integration constants. The ± sign of Eq. (50) is independent of that appeared throughout discriminating the cases with $\phi$ positive or negative. For $V_0 > 0$, since $p_\phi > 0$, from Eq. (49) it is $w_\phi < -1/2$, and the relevant sign in Eq. (49) is the negative one. In agreement with the previous discussion, this solution starts decelerating with $w_\phi \approx -1$, and as $w_\phi$ increases monotonically with time, the universe accelerates to $w_\phi = -1/2$. Actually, since $\delta_0 = 31 - 3(8\Omega_0 + 3\sqrt{9 - 8\Omega_0})$, we have acceleration for any $\Omega_\phi \geq 0.37$. We can also see that this solution possesses an event horizon [22], since $\int dt/a(t) \sim \chi^{1/6}/2F(1/6,5/6,7/6,3\chi)$, and the limit for $\chi \to 1/3$ of this function is finite.

The solution (43) can be easily seen that it does not have accelerating intervals, so it is not of special interest here.

C. Late-times approximation

Since equation (21) contains phase-space variables, it is suitable for approximating the general solution around the fixed points of the system. We give for $V_0 > 0$ two more equivalent forms of Eq. (21), which will be used to obtain the general late-times evolution of the system. Defining

\[
\tilde{\chi} = \frac{\varepsilon}{\chi - \nu}, \quad Y = \chi^{-3/4}\psi^{-1}, \quad (52)
\]

Eq. (21) is written as

\[
Y \frac{dY}{d\chi} = \frac{\varepsilon(2 - \nu)\tilde{\chi} - 1}{2\chi^{5/4}((\nu\chi + \varepsilon)^2/4 + \chi^{5/4}(\nu\chi + \varepsilon)^3/2}, \quad (53)
\]

where $\varepsilon = 1$ (resp. $-1$) for $\chi > \nu$ (resp. $\chi < \nu$). For $\chi \approx \nu$,\n
\[
\tilde{\chi} = \frac{\varepsilon}{\chi - \nu}, \quad Y = \chi^{-3/4}\psi^{-1}, \quad (54)
\]

Eq. (21) takes the form

\[
Y \frac{dY}{d\chi} = \frac{\varepsilon\tilde{\chi} - 1}{2\chi^{5/4}(\tilde{\chi} + \varepsilon)^3/2}Y + \frac{(\nu - 1)\tilde{\chi} - \varepsilon}{4\nu\chi^{5/4}((\tilde{\chi} + \varepsilon)^3/2}, \quad (55)
\]

where, as before, $\varepsilon = 1$ (resp. $-1$) for $\chi > 1$ (resp. $\chi < 1$).

For $\mu^2 < 9\beta$, Eq. (55) is approximated close to the fixed point (iii) as

\[
Y \frac{dY}{du} + \frac{2 - \nu}{\sqrt{1 - \nu}}Y + u \approx 0, \quad (56)
\]

where

\[
u = \frac{2\sqrt{1 - \nu}}{3(2 - \nu)} \left[3\nu + 2\frac{\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (57)
\]
\[
= \frac{2\sqrt{1 - \nu}}{3(2 - \nu)} \left[3\nu + 2\frac{\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (58)
\]
\[
= \frac{2\sqrt{1 - \nu}}{3(2 - \nu)} \left[3\nu + 2\frac{\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (59)
\]
\[
= \frac{2\sqrt{1 - \nu}}{3(2 - \nu)} \left[3\nu + 2\frac{\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (60)
\]

while for $\mu^2 > 9\beta$, Eq. (55) is approximated close to the fixed point (iii) as

\[
Y \frac{dY}{du} + \frac{\nu}{\nu - 1}Y + u \approx 0, \quad (61)
\]

where

\[
u = \frac{2}{3}\sqrt{1 - \nu} \left[5 - \frac{5\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (62)
\]
\[
= \frac{2}{3}\sqrt{1 - \nu} \left[5 - \frac{5\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (63)
\]
\[
= \frac{2}{3}\sqrt{1 - \nu} \left[5 - \frac{3\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (64)
\]
\[
= \frac{2}{3}\sqrt{1 - \nu} \left[5 - \frac{5\chi^{3/4}}{\chi^{1/4}(\nu\chi + \varepsilon)^3/4}\right] \quad (65)
\]
For \( \mu^2 < 9\beta \), the solution of Eq. \((56)\) is
\[
\begin{align*}
    c_1 |Y + \sqrt{1 - \nu u}|^{1-\nu} &= c_2 |Y + \frac{1}{\sqrt{1 - \nu}} u|,
\end{align*}
\]
where \( c_1, c_2 \) are integration constants with \( |c_1| + |c_2| > 0 \). Concerning Eq. \((61)\), its solution for \( 9\beta < \mu^2 < 72\beta/7 \) is
\[
\begin{align*}
    c_3 |Y + (\sigma - \tau) v|^\sigma - \tau &= c_4 |Y + (\sigma + \tau) v|^\sigma + \tau,
\end{align*}
\]
while, for \( \mu^2 > 72\beta/7 \) the solution is
\[
\begin{align*}
    \ln |Y^2 + 2\sigma vY + v^2| - \frac{2\sigma}{\sqrt{1 - \sigma}} \arctan \frac{Y + \sigma v}{\sqrt{1 - \sigma}} v = c.
\end{align*}
\]
In the above two equations, \( c_3, c_4, c \) are integration constants with \( |c_3| + |c_4| > 0 \), and \( \sigma = 1/2\sqrt{1 - \nu^{-1}} \),
\[
\tau = (\sqrt{4 - 3\nu}/4)(\nu - 1).
\]
We can write \( \Omega_\phi \) from Eq. \((55)\) in terms of \( x, y \) as
\[
Y = \left( \frac{x}{x} \right)^{3/2} \left[ 1 + \left( x - \sqrt{\nu - 1} + 1 \right) \frac{x}{y^2} \right],
\]
and thus, replacing \( Y, u, v \) in terms of \( x, y \) from Eq. \((69)\), \((50)\), \((51)\) respectively, Eqs. \((66)\), \((67)\), \((68)\) become late-times first integrals of the system in the plane \( (\phi/H, \sqrt{V/H}) \), approximating respectively, for \( \mu^2 < 9\beta \) a scalar field dominated attractor (node), for \( 9\beta < \mu^2 < 72\beta/7 \) a node “scaling” attractor, and for \( \mu^2 > 72\beta/7 \) a spiral “scaling” attractor.

Furthermore, we can write \( \Omega_\phi \) from Eq. \((24)\), \((30)\) in terms of \( w_\phi, \Omega_\phi \) or \( w_\phi, q \) as
\[
\begin{align*}
    \Omega_\phi &= \frac{\sqrt{2}}{(1 - w_\phi)^{1/4}} \left( \sqrt{\frac{2}{1 + w_\phi}} - \frac{\nu^{-1} + 1}{\Omega_\phi} \right),
\end{align*}
\]
\[
\begin{align*}
    \Omega_\phi &= \frac{\sqrt{2}}{(1 - w_\phi)^{1/4}} \left( \sqrt{\frac{2}{1 + w_\phi}} - \frac{3(\nu^{-1} + 1)w_\phi}{2q - 1} \right).
\end{align*}
\]
Thus, replacing \( Y, u, v \) in terms of \( w_\phi, \Omega_\phi \) from Eqs. \((70)\), \((60)\), \((65)\) respectively, Eqs. \((66)\), \((67)\), \((68)\) become late-times first integrals of the system between the variables \( w_\phi, \Omega_\phi \).

Finally, \( \Omega_\phi \) can be written from its definition as \( \Omega_\phi = 1 - \beta \rho_0 (1 + z)^3 / H^2 \), and thus, \( Y \) is written as
\[
\begin{align*}
    Y &= \frac{\sqrt{2}}{(1 - w_\phi)^{1/4}} \left( \sqrt{\frac{2}{1 + w_\phi}} - \frac{\nu^{-1} + 1}{1 - \beta \rho_0 (1 + z)^3 H^{-2}} \right),
\end{align*}
\]
Replacing \( Y \) from Eq. \((72)\), Eqs. \((66)\), \((67)\), \((68)\) become late-times first integrals of the system between the variables \( w_\phi, (1 + z)^3 / H^2 \).

**IV. CONCLUSIONS**

In conclusion, we have studied how long a scalar field of exponential potential can play the role of quintessence in a realistic universe filled with pressureless baryonic matter. By deriving and studying directly the first order differential equation of the deceleration parameter \( q \) with respect to the state parameter of the scalar field \( w_\phi \), we found that there exist whole classes of solutions which enter into accelerating eras not at late times, independently of the parameter \( \mu \) determining the steepness of the potential. Moreover, the condition \( w_\phi < -1/3 \) does not necessarily imply acceleration. Observational constraints of \( \Omega_m, w_\phi \) provide immediate bounds on \( \mu \) for the viable solutions, irrespectively of initial conditions or other parameters, in accordance – as it has been claimed \cite{201} – with all the current observations. Using the same first order equation, we have obtained the solution \( \Omega_\phi (w_\phi) \) for the almost cosmological constant case \( w_\phi \approx -1 \), and the generic late-times solution of the system in the plane of the variables \( (\phi/H, \sqrt{V/H}), (w_\phi, \Omega_\phi), (w_\phi, q), (w_\phi, (1 + z)^3 / H^2) \).

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