Roper resonances in chiral quark models

B. Golli\textsuperscript{1} and S. Širca\textsuperscript{2}

\textsuperscript{1} Faculty of Education, University of Ljubljana and J. Stefan Institute, 1000 Ljubljana, Slovenia
\textsuperscript{2} Faculty of Mathematics and Physics, University of Ljubljana and J. Stefan Institute, 1000 Ljubljana, Slovenia

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Abstract. We derive a method to calculate the multi-channel K matrix applicable to a broad class of models in which mesons linearly couple to the quark core. The method is used to calculate pion scattering amplitudes in the energy region of low-lying P11 and P33 resonances. A good agreement with experimental data is achieved if in addition to the elastic channel we include the $\pi\Delta$ and $\sigma N$ ($\sigma\Delta$) channels where the $\sigma$-meson models the correlated two-pion decay. We solve the integral equation for the K matrix in the approximation of separable kernels; it yields a sizable increase of the widths of the $\Delta(1232)$ and the $N(1440)$ resonances compared to the bare quark values.

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1 Introduction

Among the low-lying nucleon excitations, the P11 Roper resonance $N(1440)$ as well as its counterpart in the P33 partial wave, the $\Delta(1600)$, play a special role due to their relatively low masses and due to the rather peculiar behavior of the scattering and electro-excitation amplitudes. The constituent quark model (CQM) in which the excited states are treated as bound states yields partial decay widths which are generally considerably smaller than the experimental values [1–8] unless all widths are scaled by a factor of 2–3 in order to fit the experimental width of the $\Delta(1232)$. This indicates that the structure of the Roper can not be explained by a simple excitation of the quark core (like most of the other low-lying states) and that other degrees of freedom need to be included.

Several attempts to understand and explain the nature of the Roper resonance have been proposed and discussed. They include the study of the Roper resonance on the lattice [9–12], investigations in different quark models, like the models based on $qqqq\overline{q}$ configurations [13–15] or those incorporating the meson cloud consisting primarily of pions and $\sigma$-mesons [16–23], as well as in models with hybrid ($qqgg$ or glueball) configurations [24, 25]. Pion-nucleon scattering in the region of the Roper resonance has also been studied in the framework of chiral perturbation theory [26–28]. The scattering amplitudes and the decay rates have been well established in certain phenomenological approaches, for example in energy-independent partial-wave analysis of $\pi N \rightarrow \pi\pi N$ scattering [29–31], energy-dependent analysis [32, 33], model-independent analysis [34], as well as in dynamical coupled-channel models [35, 36] and in effective-Lagrangian models (see, for example, [37–44]). Such models typically contain a large number of ingredients (e.g. coupling constants, cut-offs, meson and baryon states) in order to parameterize the observed scattering (or electro-production) amplitudes and to simultaneously fit many measured observables. However, because of the multitude of model ingredients, it is usually difficult to determine which degrees of freedom are truly relevant for a particular resonance and to obtain insight into its quark structure. Furthermore, a similar level of agreement with data can be achieved using rather disparate sets of parameters; using results of quark-model calculations may offer an important guidance in particular to the choice of the form factors of resonant states.

The first aim of the present work is to develop a method which would allow solutions obtained in a broad class of quark-model calculations using bound-state boundary conditions to be incorporated into a dynamical framework that could be used to predict observables measured in meson scattering experiments as well as in electro-weak production of mesons on the nucleon. The interplay of different baryonic and mesonic degrees of freedom in such processes can in many cases considerably alter the results obtained by simply calculating the transition form-factors of the baryons in the underlying quark model.

In addition, our goal is to set up a computational scheme beyond the simple approximation for the K matrix and connect the observables extracted in our analysis of the relevant degrees of freedom to the predictions of the underlying quark model, which would eventually give us a chance to discriminate between different models.

Our method is not intended to be a competitive approach to those using effective Lagrangians. Rather, the method complements them by establishing a link between the effective models and the underlying baryon structure.
Its main distinctions with respect to the (non-dynamical) effective Lagrangian approaches are: (i) baryons are treated as composite particles from the very beginning; the strong and electro-weak form-factors are derived from baryon internal structure and not inserted a posteriori; as a consequence the method introduces a much smaller number of free parameters; (ii) the meson cloud around baryons is included in a consistent way also in the asymptotic states; (iii) through the Kohn variational principle for the $K$ matrix it allows one to determine the internal structure of the baryon resonances which may substantially differ from the structure determined in the calculations using bound-state boundary conditions. On the other hand, because of the complex structure of the baryons in the model, we are not able to treat a very large set of ingredients. Consequently, the approach in its present form cannot be used at higher energies where many channels open.

In several aspects the present method is similar to the method of solving the integral equation for the $T$ matrix described in [35] and used in [36]. In both approaches the coupled-channel equations are solved beyond the usual Born approximation including the effect of the meson cloud in the baryon; as a consequence, in both cases the expression of the $T$ matrix ($K$ matrix) involves the physical nucleon rather than a structureless particle of effective Lagrangian approaches. The starting points are however different: the approach of [35] starts from baryon-meson and meson-meson Lagrangians and using the unitary transformation method derives a model Hamiltonian with dynamically generated baryon-meson vertices which can be related to the vertices calculated in quark models. Our approach is complementary in that it starts from the quark-model type calculation and extends the many-body wave-function such that it properly includes the asymptotic boundary conditions of a particular channel. The deficiency of our approach in the present form is the lack of meson self-interactions.

Our future plan is to extend the approach to include meson photo- and electro-production which requires the inclusion of new channels. In order not to deal with an excessively large number of ingredients in that case, the aim of the present analysis is also to reveal which degrees of freedom are most relevant for an explanation of the structure and the dynamics of the low-lying resonances in the P11 and P33 partial waves and to pin down the parameters governing these resonances.

In our previous work [45] we have developed a method to calculate the scattering and the pion electro-production amplitudes in models in which the pions linearly couple to the quark core. In such models it is possible to find the exact expression for the $T$ matrix as well as for the $K$ matrix without explicitly specifying the form of the asymptotic states. The method has been successfully applied to the calculation of the phase shift and the pion electro-production amplitudes for the P33 partial wave by considering only the elastic channel. In this paper we extend the method to calculate directly the multi-channel $K$-matrix. The resulting matrix is symmetric and real, thus ensuring unitarity. Since we are now considering processes at relatively higher energies we generalize the approach by adopting fully relativistic kinematics.

The first attempt to treat dynamically the resonant pion scattering in a quark model with excitations of the quark core has been done in the framework of the Cloudy Bag Model (CBM). For the P33 partial wave [19,23] the experimental phase shift in the energy range of the $\Delta(1232)$ resonance has been well reproduced. In the $N(1440)$ case the width of the resonance has been considerably underestimated [17,19,22] except in the case of the coupled-channel calculation [21] using the $K$-matrix approach involving the $\pi\Delta$ and $\pi$-Roper inelastic channels in the static approximation. The latter approach is similar to the method of our sect. 3.3 except that we use relativistic kinematics.

In the present work we restrict ourselves to a class of Hamiltonians with linear meson-baryon coupling, neglecting meson self-interaction. We are particularly interested in the interplay of the degrees of freedom involving the $\Delta$ isobar and the $\sigma$-meson which we expect to play the dominant role in the dynamics of the Roper resonances; in this respect our work is similar to that of Ref. [39, 46]. We were, however, not able to confirm the most intriguing conclusion found in [39] that the formation of the $N(1440)$ resonance can be explained without introducing a bare Roper state.

In sect. 2 we introduce a general form of models in which mesons linearly couple to the quark core. The input to our computational scheme are the matrix elements of the meson interaction between the quark states. We construct the $K$ matrix and the corresponding principal-value states which incorporate in a systematic way the many-quark quasi-bound states describing the excitations of the quark core. We assume that the decay into two pions which dominates the inelastic processes in the lower energy region proceeds through a two-body decay involving either an unstable isobar which in turn decays into the nucleon and the second pion or an unstable meson and the nucleon.

In sect. 3 we show how a quasi-bound state calculated in the underlying quark model can be inserted in a state that obeys proper scattering boundary conditions. We derive a set of coupled integral equations for the meson amplitudes and the parameters governing the strength of the quasi-bound states which are responsible for the resonant parts of the scattering amplitudes. We first solve the set in a simple approximation equivalent to the so-called Born approximation for the $K$ matrix widely used in various multi-channel approaches. In order to find the solution beyond the Born approximation we introduce an approximation which makes the kernels separable, however, ensuring that they reduce to the exact form when evaluated on-shell, as well as preserving the proper symmetries of the $K$ matrix.

The results of our analysis using the quark wave-functions from the Cloudy Bag Model are presented in sect. 4. We analyze to what extent various approximations are able to explain the resonant behavior of the P11 and P33 partial amplitudes. We stress the important role of the
background terms originating from the crossed \((u\text{-channel})\) processes, and show that the inclusion of the \(\sigma N\) channel considerably improves the results in the P11 case. The meson-baryon coupling constants needed to fit the scattering amplitudes in the Born approximation for the K matrix are generally considerably larger than those predicted by the quark model. Solving the integral equation for the K matrix beyond the Born approximation we show that the resulting dressing of the vertices as well as the influence of the neighboring resonances can explain the strong enhancement of the coupling constants.

2 Coupled-channel K-matrix formalism

2.1 The K matrix for models with linear meson-quark coupling

We consider here a rather broad class of models in which the mesons linearly couple to the three-quark core. The part of the Hamiltonian referring to mesons can be written as

\[
H_x = \int dk \sum_{lmt} \left\{ \omega_k a^\dagger_{lmt}(k)a_{lmt}(k) + \left[ V_{lmt}(k)a^\dagger_{lmt}(k) + V^\dagger_{lmt}(k) a_{lmt}(k) \right] \right\},
\]

where \(a^\dagger_{lmt}(k)\) is the creation operator for a \(l\)-wave meson with the third component of spin \(m\) and \(-\) in the case of isovector mesons \(-\) the third component of isospin \(t\). In the case of the \(p\)-wave pions, the source can be cast in the form

\[
V_{lmt}(k) = -v(k) \sum_{i=1}^3 \sigma^i_m \tau^i_t,
\]

with \(v(k)\) depending on a particular quark model and containing the information about the underlying quark structure. (We omit the index \(l\) from the pion operators.) We assume that the interaction \(V(k)\) can generate bare quark states with quantum numbers different from the ground state by flipping the spin and isospin of the quarks, and furthermore, excite quarks to higher spatial states. In particular, the state with the flipped spins and isospin corresponds to the \(\Delta(1232)\) isobar, while exciting one quark from the \(1s\) to the \(2s\) state generates an excited threenuclear meson that is associated with the Roper resonance \(N(1440)\) or \(\Delta(1600)\).

Chew and Low [48] considered a model similar to (1) except that they did not allow for excitations of the nucleon core. They showed that the T matrix for \(\pi N\) scattering was proportional to \(\langle \Psi^- (W) | V_{lmt}(k) | \Psi^0_N \rangle\), where \(\Psi^- (W)\) were the incoming states. In general, the corresponding formula for the K matrix cannot be written in such a simple form. However, in the \(JT\) basis, in which the K and T matrices are diagonal, it is possible to express the K matrix as

\[
K_{\pi N \pi N}(k, k_0) = -\pi \sqrt{\frac{\omega_k E_N}{k W}} \langle \Psi^N_{J^T} (W) | | V(k) | | \Psi^0_N \rangle\).
\]

The corresponding principal-value (PV) state obeys a similar equation as the in- and out-going states in the Chew-Low model and takes the form:

\[
| \Psi^N_{S^T} (W) \rangle = N_0 \left\{ \left[ a^\dagger (k_0) | \Psi^0_N \rangle \right]^J^T \right. - \frac{P}{H - W} \left. \left[ V(k_0) | \Psi^0_N \rangle \right]^J^T \right\},
\]

where \([ ]^J^T\) denotes coupling to good \(J\) and \(T\), and

\[
N_0 = \sqrt{\frac{\omega_0 E_N}{k_0 W}}.
\]

We work in the center-of-mass system, \(W\) is the invariant energy of the system, \(\omega_0\) and \(k_0 = \sqrt{\omega_0^2 - m_N^2}\) are the energy and momentum of the pion, and \(E_N\) is the nucleon energy:

\[
\omega_0 = W - E_N = \frac{W^2 - M_N^2 + m_N^2}{2W}, \quad E_N = \sqrt{M_N^2 + k_0^2}.
\]

It is worthwhile to notice that a PV state is a superposition of incoming and outgoing waves (i.e. can be regarded as a standing wave); the in- and out-states become meaningful only when the T and the S matrix are constructed from the K matrix:

\[
T = \frac{K}{(1 - iK)}, \quad S = 1 + 2iT.\]

2.2 Channels including the pion and unstable isobars

We now extend the above formulas to the multichannel case. We apply the usual approach used in phenomenological analyses (e.g. [29]) in which it is assumed that the two-pion decay proceeds through some intermediate unstable particle, either a meson or a baryon.

Let us first consider the situation in which the intermediate particle is a baryon, \(B\), (e.g. the \(\Delta\) isobar which dominates at low energies) which in turn decays into the nucleon and the second pion. The decays into the nucleon and an unstable meson will be treated in the next subsection.

In analogy with (4) we introduce the PV state corresponding to the \(\pi B\) channel as

\[
| \Psi^B_{J^T}(W, M) \rangle = N_1 \left\{ \left[ a^\dagger (k_1) | \Psi^0_B (M) \rangle \right]^J^T \right. - \frac{P}{H - W} \left. \left[ V(k_1) | \Psi^0_B (M) \rangle \right]^J^T \right\}.
\]

Here \(k_1\) is the momentum of the first pion; the energy \(E\) of the isobar \(B\) and the energy of the first pion are related through

\[
E = \frac{W^2 + M^2 - m_B^2}{2W}, \quad \omega_1 = W - E,
\]

\[
J = J^T = 1, \quad T = 0.
\]

\[
S = 1 + 2i (2T).\]
where $M$ is the mass of $B$, i.e. the invariant mass of the nucleon and the second pion, and

$$N_1 = \sqrt{\frac{\langle \omega_1 W \rangle}{k_1 W}}.$$ (9)

The normalization of the PV states (4) and (7) states is dictated by their respective first terms which represent a free pion and the nucleon or the $B$ isobar. The states $\Psi^\alpha, \alpha = N, (B, M), \ldots$ are then normalized as

$$\langle \Psi^\alpha (W) | \Psi^\beta (W') \rangle = \delta (W - W') [\delta_{\alpha, \beta} + K^2 \delta_{\alpha, \beta}].$$

(see e.g. [47], eq. (7.28)). In the case of the $\pi N$ channel $\alpha = N$ is a single index, while in the $\pi B$ channel it includes also the invariant mass $M$, so $\delta_{\alpha, \beta}$ should be interpreted as $\delta (M - M')$. The PV states are not orthonormal; the orthonormal states corresponding to intermediate isobars is discussed in Appendix B.

For a process in which the initial pion-nucleon system with invariant mass $W$ decays into the (first) pion with momentum $k$ and the pion-nucleon system with the quantum numbers of the intermediate baryon $B$, we write

$$K^{JT}_{\pi B\pi N} (k, k_0, M) = -\pi \sqrt{\frac{\langle \omega_k E \rangle}{k W}} \langle \Psi^N (W) | V (k) | \tilde{\Psi}_B (M) \rangle.$$

(11)

The $\pi B$ to $\pi N$ transition matrix element of the multichannel K matrix is

$$K^{JT}_{\pi N \pi B} (k, k_1, M) = -\pi \sqrt{\frac{\langle \omega_k E \rangle}{k W}} \langle \Psi^B (W, M) | V (k) | \Psi^N \rangle.$$

(12)

The matrix element corresponding to the $\pi B$ to $\pi B'$ transition (e.g. $\pi (k_1) + \Delta (M) \rightarrow \pi (k_1) + \Delta (M')$) is given by

$$K^{JT}_{\pi B' \pi B} (k, k_1, M', M) = -\pi \sqrt{\frac{\langle \omega_k E \rangle}{k W}} \langle \Psi^B (W, M) | V (k) | \tilde{\Psi}_B' (M') \rangle.$$ (13)

### 2.3 Channels including the $\sigma$-meson

We consider here the situation in which the decay proceeds through an unstable meson and a baryon. We derive the expression for the K matrix in the simplest case of the $\sigma N$ channel (in the P11 wave) and the $\Delta \pi$ channel (in the P33 wave) which dominate the inelastic processes in the energy range below $\sim 1700$ MeV for these two partial waves. The extension of the method to other unstable mesons is straightforward but more complicated because of a larger number of involved channels.

The $\sigma$-meson appears as a chiral partner of the pion in several versions of chiral quark models which also provide the form of its coupling to the quark core, e.g. in the linear $\sigma$-model with quarks [50–52] or in various bosonized versions of the Nambu–Jona-Lasinio model [53–55]. In the non-linear realizations of the models its coupling is realized by two correlated pions, see e.g. [18, 37].

In this work we use a purely phenomenological approach and take the form (1) for the $\sigma$-meson coupling. We assume that only $s$-wave $\sigma$-mesons couple to the quark core such that the interaction vertex takes the form:

$$\tilde{V}_\mu (k) = V^\mu (k) w_\sigma (\mu), \quad V^\mu (k) = G_\sigma \frac{k}{\sqrt{2\omega_{\mu k}}}.$$  (14)

Apart from the momentum $k$, the one-$\sigma$-meson states are labeled by the invariant mass of the two-pion system, $2m_\sigma < \mu < \infty$. Here $\omega_{\mu k} = k^2 + \mu^2$ and $w_\sigma (\mu)$ is a normalized mass distribution function centered around the nominal value of the $\sigma$-meson mass with the corresponding width, modeling the resonant decay into two pions.

In analogy with (3), (11), (12), and (13), we first introduce the $\sigma$-meson with the energy and momentum $\omega_\mu$ and $k_\mu$ and the nucleon with the energy $E_N^0 = \sqrt{M_N^2 + k_\mu^2}$ on the one side, and on the other side either the pion ($\omega_0, k_0$) and the nucleon, the pion ($\omega_1, k_1$) and the intermediate baryon with the invariant mass $M$, or another $\sigma$-meson and the nucleon:

$$K^{\frac{1}{2} \frac{1}{2}}_{\sigma N \pi N} (W, \mu) = -\pi N_0 \langle \Psi^N (W) | \tilde{V}_\mu (k_\mu) | \Psi^N \rangle,$$

$$K^{\frac{1}{2} \frac{1}{2}}_{\sigma N \pi N} (W, \mu) = -\pi N_0 \langle \Psi^N (W) | \tilde{V}_\mu (k_\mu) | \Psi^N \rangle,$$

$$K^{\frac{1}{2} \frac{1}{2}}_{\sigma N \pi B} (W, M, \mu, \mu') = -\pi N_0 \langle \Psi^B (W, M) | \tilde{V}_\mu (k_\mu) | \Psi^N \rangle,$$

$$K^{\frac{1}{2} \frac{1}{2}}_{\sigma N \pi B} (W, M, \mu, \mu') = -\pi N_0 \langle \Psi^B (W, M) | \tilde{V}_\mu (k_\mu) | \Psi^N \rangle,$$

and

$$\omega_\mu = \frac{W^2 - M_N^2 + \mu^2}{2W}, \quad E_N^0 (k_\mu) = W - \omega_\mu. \quad (15)$$

In the P33 partial wave the decay into two correlated $l = 0$ pions can proceed only through the intermediate $s$-wave $\sigma$-meson and the $\Delta$ isobar. In this case the channel is labeled by the invariant mass of the two correlated pions, $\mu$, and the invariant mass of the pion-nucleon system from the $\Delta$ resonance, $M$. The corresponding elements of the
K matrix contain four invariant masses which is rather difficult to handle computationally. In sect. 3.4 we discuss a method which simplifies the calculation by averaging over the $\Delta$ invariant mass such that the dependence on $M$ is eliminated. The matrix elements then assume the same form as (14) with $N$ replaced by $\Delta$, $M_N$ by $M$, and $E_N^\mu$ by $E^\mu = \sqrt{M^2 + k^2_\mu}$.

In the P11 and P33 case the $\rho$-meson can be included in a similar way but since the experimental data [56] indicate that the contribution of the $\rho$-meson is almost negligible in the energy region under consideration we do not include it in our calculation.

### 2.4 Constructing the T and the S matrix

The multichannel K matrix acquires the following form

$$
\begin{array}{ccc}
K_{NN} & K_{NB}(M') & K_{N\sigma}(\mu') \\
K_{BN}(M) & K_{BB}(M, M') & K_{B\sigma}(M', \mu) \\
K_{N\sigma}(\mu) & K_{\sigma B}(\mu, M) & K_{\sigma \sigma}(\mu, \mu')
\end{array}
$$

where we have used a shorthand notation $B$ for the $\pi B$ channel and $\sigma$ for either the $\sigma N$ or the $\sigma \Delta$ channel. The $T$ matrix is related to the K matrix through (6). In the inelastic channel the matrix elements depend on the continuous variable ($M$ or $\mu$), yielding a set of coupled integral (Heitler) equations:

$$
T_{NN} = K_{NN} + i T_{NN} K_{NN}
$$

$$
+ i \sum_B \int_{M_N + m_e}^{W - m_e} dM T_{NB}(M) K_{BN}(M)
$$

$$
+ \int_{2m_n}^{W - M_N} d\mu T_{N\sigma}(\mu) K_{N\sigma}(\mu),
$$

$$
T_{NB}(M) = K_{NB}(M) + i T_{NB} K_{NB}(M)
$$

$$
+ i \sum_B \int_{M_N + m_e}^{W - m_e} dM' T_{NB'}(M') K_{BB}(M', M)
$$

$$
+ \int_{2m_n}^{W - M_N} d\mu T_{N\sigma}(\mu) K_{B\sigma}(\mu, M),
$$

$$
T_{N\sigma}(\mu) = K_{N\sigma}(\mu) + i T_{N\sigma} K_{N\sigma}(\mu)
$$

$$
+ i \sum_B \int_{M_N + m_e}^{W - m_e} dM T_{NB}(M) K_{B\sigma}(M, \mu)
$$

$$
+ \int_{2m_n}^{W - M_N} d\mu' T_{N\sigma}(\mu') K_{\sigma \sigma}(\mu', \mu').
$$

In the P33 case the nucleon mass $M_N$ in the integral over $\mu$ is replaced by the $W$-dependent averaged invariant mass of the intermediate $\Delta$.

The unitarity of the S matrix it fulfilled provided the K matrix is real and symmetric which is especially important when we use approximate methods; in such a case it is considerably more advantageous to use a certain ansatz for the K matrix (or, equivalently for the principal-value state) rather than for the T matrix since in the latter case the unitarity has to be enforced at each step of the calculation. Let us remark that for a general chiral quark model, the K matrix and the corresponding principal-value state can be calculated variationally using the Kohn variational principle

$$
\langle \delta \Psi^P | H - W | \Psi^P \rangle = 0,
$$

where $\Psi^P$ is a suitably chosen trial state.

### 3 The integral equations for the K matrix

#### 3.1 Ansätze for the channel PV states

In the formal expressions for the PV states (4) and (7) the interaction $V(k)$ generates bare quark states with quantum numbers different from the ground state, as well as superpositions of bare quark states and one or more mesons. We choose a particular ansatz which implies the proper asymptotic behavior of different channels consisting of a meson and a baryon carrying its own meson cloud. The form of the pion state in a $\pi B'$ channel can be read-off from the general relation (55) holding for the eigenstate of the Hamiltonian (1). Multiplying (55) for $\Psi_A = \Psi_{B'}$ by $\langle \Psi_B(k) \rangle$ we obtain

$$
(\omega_k + E_{B'}(k) - W)(\tilde{\Psi}_{B'}(k)|a_m(k)|\Psi_{B'}(W))
$$

$$
= -\langle \tilde{\Psi}_{B'}(k)|V_{m}(k)|\Psi_{B'}(W)\rangle,
$$

with $E_{B'}(k) = \sqrt{M^2 + k^2}$, where $M'$ is either the mass of a stable baryon or the invariant mass of the $\pi N$ subsystem corresponding to the intermediate baryon $B'$. For the elastic channel we assume the following ansatz:

$$
|\Psi_{B'}^\mu(W)\rangle = N_0 \left\{ |a_k(k_0)|\Psi_N(k_0)|^JT + \sum_R c_R^\mu(W)|\Phi_R\rangle \right\}
$$

$$
+ \int dk \frac{\chi_{JT}^N(k, k_0)}{\omega_k + E_N(k_0) - W}[a_k(k)|\Psi_N(k)|]^JT
$$

$$
+ \sum_B \int dM \int dk \frac{\chi_{JT}^B(k, k_0, M)}{\omega_k + E_B(k) - W}[a_k(k)|\tilde{\Psi}_{B}(M)|]^JT
$$

$$
+ \int d\mu \int dk \frac{\chi_{JT}^\sigma(k, k_0, \mu)}{\omega_{\mu k} + E_{JT}(k) - W}[b^\dagger_{\mu}(k)|\tilde{\Phi}_{JT}\rangle. \right\}
$$

The first term, as discussed in the previous section, defines the channel and determines the normalization, the second term is the sum over bare quark states, denoted as $\tilde{\Psi}_R$, with quantum numbers of the channel. The amplitudes $\chi$ are proportional to the amplitudes (18); the first term corresponds to the one-pion state on top of the ground state, the terms in the sum to the one-pion states around different excited states, and the last term to the one-$\sigma$ state around either the nucleon (P11) or the $\Delta$ (P33), with $E_{JT}(k)$ denoting the energy of the respective baryon. Above the one- and two-pion thresholds these amplitudes are related to the elastic and inelastic elements of the on-shell K matrix:

$$
K_{NN}(W) = \pi N_0^2 \chi_{JT}^N(k_0, k_0),
$$

$$
K_{BN}(W, M) = \pi N_0 N_B \chi_{JT}^B(k_1, k_0, M),
$$

$$
K_{N\sigma}(W, \mu) = \pi N_0 N_{\sigma} \chi_{JT}^\sigma(\mu_1, k_0, \mu),
$$

$$
K_{BB}(M, M') = \pi N_B^2 \chi_{JT}^B(k_1, k_0, M),
$$

$$
K_{B\sigma}(M', \mu) = \pi N_B N_{\sigma} \chi_{JT}^\sigma(\mu_1, k_0, \mu'),
$$

$$
K_{\sigma \sigma}(\mu, \mu') = \pi N_\sigma^2 \chi_{JT}^\sigma(\mu_1, \mu_2).
$$


In the P11 case, one of the $\Phi_R$ states is the nucleon.

The ansatz can be simplified by realizing that the main contribution to the integrals over the invariant mass $M$, in (19), comes from $M$ close to $M_B$, i.e. the resonant energy of the isobar $B$. In appendix B we show that in such a case one can write the state $\tilde{\psi}_B$ in a simplified form

$$\hat{\psi}_B(M) \equiv w_B(M)|\tilde{\psi}_B(M)\rangle,$$

(21)

where $\tilde{\psi}_B$ is dominated by the bare quark configuration and only weakly depends on $M$. The function $w_B(M)$ can be identified with the mass distribution function $\sigma(M)$ multiplied by the kinematic corrections – the Blatt-Weisskopf barrier-penetration factor – that ensure proper threshold behavior (see e.g. [29]). Using this approximation the integration over $M$ in (19) selects $M = M_B$ and similarly $\mu = m_\sigma$ in the last term, yielding

$$|\psi_{JT}^N(W)\rangle = N_0 \left\{ [a^\dagger(k_0)|\psi_N(k_0)\rangle]^{JT} + \sum R c_R^N(W)|\Phi_R\rangle + \int dk \frac{\chi_{NN}^T(k, k_0)}{\omega_k + E_N(k) - W} [a^\dagger(k)|\psi_N(k)\rangle]^{JT} + \sum B \int dk \frac{\psi_B^T(k, k_0, M_B)}{\omega_k + E_B(k) - W} [a^\dagger(k)|\tilde{\psi}_B(M_B)\rangle]^{JT} + \int dk \frac{\chi_{NT}^T(k, k_0, m_\sigma)}{\omega_k + E_T(k) - W} b^\dagger(k)|\tilde{\psi}_{JT}\rangle \right\},$$

(22)

where $\omega_k = \sqrt{k^2 + m_\sigma^2}$. We have introduced $\chi_{JT}^B = w_B(M)\tilde{\chi}_{JT}^B$ and $\chi_{JT}^N = w_\sigma(\mu)\tilde{\chi}_{JT}^N$.

The inelastic channels corresponding to the $\pi B$ and $\sigma N$ channels can be written in compact forms

$$|\psi_{JT}^B(W, M)\rangle = N_B w_B(M) \left\{ [a^\dagger(k_1)|\tilde{\psi}_B(M)\rangle]^{JT} + \sum R c_R^B(W, M)|\Phi_R\rangle + \int dk \frac{\chi_{BT}^B(k, k_1, M)}{\omega_k + E_N(k) - W} [a^\dagger(k)|\psi_N(k)\rangle]^{JT} + \sum B' \int dk \frac{\chi_{BT}^B(k, k_1, M', k)}{\omega_k + E_B(k) - W} [a^\dagger(k)|\tilde{\psi}_B(M_B)\rangle]^{JT} + \int dk \frac{\chi_{JT}^B(k, k_1, m_\sigma, M)}{\omega_k + E_T(k) - W} a^\dagger(k)|\tilde{\psi}_{JT}\rangle \right\},$$

(23)

and

$$|\psi_{JT}^N(W, \mu)\rangle = N_\sigma w_\sigma(\mu) \left\{ [a^\dagger(k_\mu)|\tilde{\psi}_N(M)\rangle]^{JT} + \sum R c_R^N(W, \mu)|\Phi_R\rangle + \int dk \frac{\chi_{NT}^N(k, k_\mu, \mu)}{\omega_k + E_N(k) - W} [a^\dagger(k)|\psi_N(k)\rangle]^{JT} + \sum B' \int dk \frac{\chi_{NT}^N(k, k_\mu, M_B', \mu)}{\omega_k + E_B(k) - W} [a^\dagger(k)|\tilde{\psi}_N(M_B')\rangle]^{JT} + \int dk \frac{\chi_{JT}^N(k, k_\mu, m_\sigma, \mu)}{\omega_k + E_T(k) - W} a^\dagger(k)|\tilde{\psi}_{JT}\rangle \right\}. $$

(24)

Here we have assumed the following factorization

$$\chi_{JT}^{NT} = w_H(m_H)w_H(m'_H)\chi_{JT}^{H}, \quad c_H = w_H(m_H)c_H^R,$$

(25)

where $H$ stands for either the $\pi B$ channels or $\sigma N$, and $m_H$ is the invariant mass (either $M$ or $\mu$). Above the two-pion threshold the meson amplitudes are related to the $K$ matrix by

$$K^{HH}(W, m'_H, m_H) = \pi N_H N_H \chi_{JT}^{HH}(k_H', k_H, m'_H, m_H).$$

(26)

The requirement that the $K$ matrix be symmetric imposes the constraint $\chi_{JT}^{HH}(k', k) = \chi_{JT}^{HH}(k, k')$ on the pion amplitudes. The ansatz therefore ensures that the scattering amplitudes are directly proportional to the on-shell $K$ matrix needed in the equation for the $T$ matrix (16). For the off-shell matrices, equality (26) is not fulfilled in general.

### 3.2 Derivation of the coupled integral equations

The equations for the pion amplitudes $\chi$ and the coefficients $c_R$ in the ansatz (19), (23) and (24) are derived either from (17) or directly from the commutation relations (55) which hold for our particular choice of the quark-pion interaction.

By requiring stationarity with respect to the variation of the coefficients $c_R^N$, $c_R^B$ and $c_R^N$ we get

$$\left(W - M_R^B\right)c_R^B(W) = V_{NR}(k_0) + \int dk \frac{V_{NR}(k)\chi_{JT}^N(k, k_0)}{\omega_k + E_N(k) - W} + \sum B' \int dk \frac{V_{B'R}(k)\chi_{JT}^B(k, k_0, M_B')}{\omega_k + E_B(k) - W} + \int dk \frac{V_{m_\sigma}(k)\chi_{JT}^N(k, k_0, m_\sigma)}{\omega_k + E_T(k) - W},$$

(27)

$$\left(W - M_R^B\right)c_R^B(W, M) = V_{NR}(k_1) + \int dk \frac{V_{NR}(k)\chi_{JT}^B(k, k_1, M)}{\omega_k + E_N(k) - W} + \sum B' \int dk \frac{V_{B'R}(k)\chi_{JT}^B(k, k_1, M, M_B')}{\omega_k + E_B(k) - W} + \int dk \frac{V_{m_\sigma}(k)\chi_{JT}^B(k, k_1, m_\sigma, M)}{\omega_k + E_T(k) - W},$$

(28)

$$\left(W - M_R^N\right)c_R^N(W, \mu) = V_{\sigma R}(k_\mu) + \int dk \frac{V_{\sigma R}(k)\chi_{JT}^N(k, k_\mu, \mu)}{\omega_k + E_N(k) - W} + \sum B' \int dk \frac{V_{B'R}(k)\chi_{JT}^N(k, k_\mu, M_B', \mu)}{\omega_k + E_B(k) - W} + \int dk \frac{V_{\sigma R}(k)\chi_{JT}^N(k, k_\mu, m_\sigma, \mu)}{\omega_k + E_T(k) - W}.$$
Here
\[ V_{NR}(k) = \langle \Phi_R | V(k) | \Phi_N \rangle = Z_{NR}^{1/2} \langle \Phi_R | V(k) | \Phi_N \rangle, \]
\[ V_{BR}^M(k) = \langle \Phi_R | V(k) | \tilde{\Psi}_B(M) \rangle = Z_{BR}^{1/2} \langle \Phi_R | V(k) | \Phi_B \rangle, \]
\[ V_{NR}^M(k) = \langle \Phi_R | V^M(k) | \tilde{\Psi}_N \rangle = Z_{NR}^{1/2} \langle \Phi_R | V^M(k) | \Phi_N \rangle, \]
where \( Z_B \) is the wave-function normalization, while \( \langle \Phi_R | V(k) | \Phi_N \rangle \) and \( \langle \Phi_R | V^M(k) | \Phi_B \rangle \) are obtained from the underlying quark model. In the P33 case \( V_{NR}^M(k) \) is replaced by \( V_{NR}^M(k) \).

In the P11 case, one of the states \( \Phi_R \) is replaced by the (exact) ground state in which the requirement of stationarity is equivalent to the requirement that the channel states are orthogonal to the ground state for \( W > M_N \). In this case the mass of the bare state \( M_0^B \) in (27), (28) and (29) is replaced by the ground-state mass \( M_N \), while the matrix elements are given by

\[ V_{NR}(k) \rightarrow (W - M_N) \frac{\langle \Phi_N | V(k) | \Phi_N \rangle}{\omega_k + E_N(k) - W}, \]
\[ V_{BR}^M(k) \rightarrow (W - M_N) \frac{\langle \Phi_N | V(k) | \tilde{\Psi}_B(M) \rangle}{\omega_k + E_B(k) - W}. \]

Requiring stationarity with respect to pion amplitudes leads to the familiar Lippmann-Schwinger equation for the \( K \) matrix. The equation for the \( \chi_{T\bar{T}}^{NN} \) amplitude which is related to the elastic part of the \( K \) matrix is obtained from (18) for \( B = B' = N \). Using our ansatz (22) and taking \( M_1 = M_T = \frac{1}{2} \), we obtain, after multiplying (18) by \( C_{1\frac{1}{2}-m'1m} C_{T\frac{1}{2}-t1t} \) and summing over \( m \) and \( t \),

\[ \chi_{T\bar{T}}^{NN}(k, k_0) = K_{KNN}(k, k_0) - \sum_R c_R^N(W) V_{NR}(k) \]
\[ + \int dk' \frac{K_{NN}(k, k') \chi_{T\bar{T}}^{NN}(k', k_0)}{\omega_k + E_N(k') - W} \]
\[ + \sum_B \int dk' \frac{K_{MB}(k, k') \chi_{B\bar{B}}^{NN}(k', k_0, M_B)}{\omega_k + E_B(k') - W} \]
\[ + \int dk' \frac{K_{MN}(k, k') \chi_{T\bar{T}}^{NN}(k', k_0, m_{\sigma})}{\omega_k + E_T(k') - W}. \]

where we have introduced the kernels

\[ K_{MB}^{NN}(k, k') = - \sum_{mtm't'} \langle \Psi_N(k) | a^\dagger_{m't'}(k') \rangle \]
\[ \times \left[ \left( \frac{V_{mt}^M(k) + (\omega_k + E_N(k) - W) a_{mt}(k)}{m_{\sigma}} \right) \tilde{\Psi}_B(M) \right] \]
\[ \times C_{J\frac{1}{2}-m'1m} C_{T\frac{1}{2}-t1t'} C_{T\frac{1}{2}-m'1m} C_{T\frac{1}{2}-t1t}. \]

For \( B = N \), \( \tilde{\Psi}_B(M) \) reduces to \( \Psi_N \) and \( M \) to \( M_N \).

For the general amplitude involving the \( \pi B \) channels we use our ansatz (23) which yields, after using (25) and canceling \( w_B(M)w_B(M') \) on both sides,

\[ \chi_{T\bar{T}}^{BB}(k, k_1, M_1, M) = K_{MB}^{BB}(k, k_1) - \sum_R c_R^B(W, M) V_{BR}^M(k) \]
\[ + \int dk' \frac{K_{BB}^{B'R}(k, k') \chi_{T\bar{T}}^{BB}(k', k_1, M_{B'}, M)}{\omega_k + E_{B'}(k') - W} \]
\[ + \int dk' \frac{K_{BB}^{B'}(k, k') \chi_{T\bar{T}}^{BB}(k', k_1, m_{\sigma}, M)}{\omega_k + E_{T'}(k') - W}. \]

The form of (34) justifies the factorization (25) for the amplitude \( \chi_{T\bar{T}}^{BB} \). (The expression for \( K_{MM'}^{\sigma\sigma} \) used in our calculation is given in sect. 3.3.)

Equations (32) and (34) imply the following form for the pion \( \chi \) amplitudes:

\[ \chi_{T\bar{T}}^{NN}(k, k_0) = - \sum_R c_R^N(W) V_{NR}(k) + D_{NN}^{NN}(k, k_0), \]
\[ \chi_{T\bar{T}}^{BB}(k, k_1, M', M) = - \sum_R c_R^B(W, M) V_{BR}^M(k) \]
\[ + D_{MM'}^{BB}(k, k_1), \]

where \( \Psi \) are the dressed vertices and \( D \) are the background parts of the amplitudes.

The amplitudes involving the \( \sigma \)-meson fulfill the same type of integral equations as the pion amplitudes with the kernels given in Appendix C. They assume the forms:

\[ \chi_{T\bar{T}}^\sigma(k, k_1, M, \mu) = - \sum_R c_R^{\sigma}(W, \mu) V_{BR}^M(k) + D_{MM'}^{\sigma\sigma}(k, k_1), \]
\[ \chi_{T\bar{T}}^\sigma(k, k_1, M, \mu) = - \sum_R c_R^{\sigma}(W, M) V_{BR}^M(k) + D_{MM'}^{\sigma\sigma}(k, k_1), \]
\[ \chi_{T\bar{T}}^\sigma(k, k_1, M, \mu) = - \sum_R c_R^{\sigma}(W, \mu) V_{BR}^M(k) + D_{MM'}^{\sigma\sigma}(k, k_1). \]

3.3 The Born approximation for the K matrix

The Born approximation for the \( K \) matrix consists in neglecting the terms in (27), (28), (29), (32), and (34) involving the integrals. The \( K \) matrix is then constructed from the meson amplitudes (35)–(39) using (20) and (26),
and substituting the dressed vertices $V_{BB}$ by the corresponding bare vertices $V_{BB}$ as well as $D$ by $K$. The expressions for $K^{HH}$ are derived in Appendix C: note that they involve only the on-shell amplitudes which are not influenced by the approximation of separable kernels. They acquire the form

$$K^{BB'}_{MM'}(k_1, k_1') = \sum_{B'} f_{BB'}^{BB} \frac{2M_{BB'} V_{BB'}^M(k_1') V_{BB'}^{M'}(k_1)}{2\omega_{\mu} + M_{BB'}^2 - M^2 - m_{\pi}^2}$$

and (in the P11 case)

$$K^{B\sigma}_{MM}(k_1, k_\mu) = \sum_{B'} 2M_{BB'} V_{BB'}^\mu(k_\mu) V_{BB'}^{N}(k_1)$$

$$= K^{\sigma}_{BM}(k_1, k_\mu)$$

$$K^{\sigma\sigma}_{\mu\mu}(k_\mu, k'_\mu) = \sum_{B} 2M_{BB'} V_{BB'}^{\mu_\mu}(k_\mu) V_{BB'}^{\mu_\mu}(k_\mu)$$

(40)

The symmetry in the background terms follows from the symmetry of the denominator in the $\omega$-channel, e.g. $2E_N\omega_\mu - M_N - m_\pi^2 = 2E_N\omega_\mu - M_N - m_{\pi}^2$. Because we deal with s-wave scattering, the direct term (i.e. the first term in (37)) referring to the nucleon pole and the background part for $B = N$ almost cancel and can be dropped from the above sums. In the P33 case the decay into two correlated $l = 0$ pions proceeds through the $\sigma\Delta$ channel. As discussed in sect. 2.3 it is sensitive to average over the $\Delta$ invariant mass such that the matrix elements depend only on the invariant mass of the two-pion system. The averaging is discussed in the following.

3.4 Averaging over invariant masses

The averaging over the $\Delta$ invariant mass in the $\sigma\Delta$ channel implies that the matrix elements in the kernels assume the same form as (40) with $N$ replaced by $\Delta$, and $V_{NN}(k)$ by the averaged interaction matrix element defined as

$$\bar{V}_{\Delta B}^{\mu}(k_\mu)^2 = \frac{\bar{\tilde{k}}_\mu W}{\omega_{\mu} E^{\mu}} \int_{M_N+m_\pi}^{W-m_\pi} dM \omega_{\Delta}(M)^{2} N_{\mu}^{\sigma} V_{\Delta B}^{\mu}(k_\mu)^2$$

(41)

while the denominator in $K^{\sigma\sigma}$ contains $\bar{\omega}_\mu$, $\bar{k}_\mu$ and $E^\mu$ evaluated by (15) in which $M_N$ is replaced by the averaged invariant mass $\bar{M}$.

The averaged invariant masses $\bar{M}$ and $\bar{\mu}$ are found by suitable smooth numerical approximations approaching the nominal hadron masses for large $W$, while remaining close to either $M_N + m_\pi$ or $2m_\pi$, for $W$ slightly above the two pion threshold.

The approximation of averaging over the invariant mass as in (41) can be applied to other matrix elements, as well as to the $K$ matrix and the $T$ matrix themselves. For the decay of a resonance $B'$ into a pion and an unstable isobar $B$ which in turn decays into the nucleon and the second pion, we introduce

$$\bar{V}_{BB'}(\bar{k}_1)^2 = \frac{\bar{\tilde{k}}_1 W}{\omega_1 E} \int_{M_N+m_\pi}^{W-m_\pi} dM w_B(M)^{2} N_{\mu}^{2} V_{BB'}^{\mu}(k_\mu)^2$$

(42)

where $\bar{\omega}_1$ and $E$ are those of (8) evaluated at $M = \bar{M}$.

Similarly, for the decay of a resonance $B'$ through a baryon $B$ and a $\sigma$-meson which in turn decays into two pions, the matrix element averaged over the meson invariant mass reads

$$\bar{V}_{BB'}(\bar{k})^2 = \frac{\bar{W}}{\omega E} \int_{2m_\pi}^{W-M_B} d\mu w_{\sigma}(\mu)^{2} N_{\mu}^{2} V_{BB'}^{\mu}(k_\mu)^2$$

where $\bar{\omega} = (W^2 - M_B^2 + \bar{\mu}^2)/2W$, $\bar{E} = W - \bar{\omega}$ and $\bar{k}^2 = \bar{\omega}^2 + \bar{\mu}^2$. The averaging procedure turns the integral Heitler equation (16) into a set of algebraic equations. Such an approximation does not influence the elastic channel since the matrix elements involving unstable hadrons in this channel always appear only under the integral; in inelastic channels this means that the $M$-dependent amplitudes are replaced by some averaged value. Identical averaging of amplitudes (42) is used in phenomenological analyses of $\pi N \rightarrow \pi \pi N$ reactions proceeding through the unstable intermediate hadron (see e.g. [29]).

3.5 Solving the integral equations in the approximation of separable kernels

In this section we solve the set of coupled integral equations (27), (28), (32), (34), beyond the Born approximation for the $K$ matrix. The solution for the vertices yields a considerable enhancement with respect to their bare values while the solution for the coefficients $c_R$ involves a considerable mixing of different ‘bare’ resonances (denoted by $R$) appearing in our ansatz. The method yields simultaneously the position of the resonance as well as the pertinent wave-function renormalization. Since the quark-$\sigma$ vertex is not as well determined as the quark-$\pi$ vertex, we treat the $\sigma$-meson vertices only in the Born approximation discussed above.

Inserting the ansätze (35)–(39) into the set of coupled equations we obtain

$$V_{NR}(k) = V_{NR}(k) + \int dk' K^{NN}(k, k') V_{NR}(k')$$

$$+ \sum_{B'} \int dk' K^{BB'}_{MM'}(k, k') V_{BR}^{MM'}(k')$$

$$= V_{NR}(k) + \int dk' K^{BB'}_{MM'}(k, k') V_{BR}^{MM'}(k')$$

(43)

$$V_{BR}(k) = V_{BR}(k) + \int dk' K^{NN}(k, k') V_{NR}(k')$$

$$+ \sum_{B'} \int dk' K^{BB'}_{MM'}(k, k') V_{BR}^{MM'}(k')$$

= V_{BR}(k) + \int dk' K^{BB'}_{MM'}(k, k') V_{BR}^{MM'}(k')$$

(44)

The background parts $D^{NN}(k, k_0)$ and $D^{BB}_{MM}(k, k_1)$ obey the integral equations of the type (43) with the non-homogeneous terms $K^{NN}(k, k_0)$ and $K^{BB}_{MM}(k, k_1)$, respectively,
while the $\mathbf{D}^{k,k_0}_M$ and $\mathbf{D}^{k,k_1}_M$ satisfy the integral equations of the type (44) with the non-homogeneous terms $\mathbf{K}^{k,k_0}_M$ and $\mathbf{K}^{k,k_1}_M$, respectively.

In appendix C we introduce several approximations which enable us to write the kernels in separable form:

$$
\mathbf{K}^{k,k'}_N(k) = \sum_{B'} f^B_{N,k'} \frac{M^B_{N,k}}{E_N} (\omega_0 + \varepsilon^B_{N})
\times \frac{\mathcal{V}^B_{N,N}(k') \mathcal{V}^B_{N,N}(k)}{(\omega_k + \varepsilon^B_{N}) (\omega_k + \varepsilon^B_{N}(M))},
$$

\(45\)

$$
\mathbf{K}^{k,k'}_M(k) = \sum_{B'} f^B_{N,k'} \frac{M^B_{N,k}}{E_N} (\omega_1 + \varepsilon^B_{N})
\times \frac{\mathcal{V}^B_{N,N}(k') \mathcal{V}^B_{N,N}(k)}{(\omega_k + \varepsilon^B_{N}) (\omega_k + \varepsilon^B_{N}(M))}
= \mathbf{K}^{k,k'}_M(k',k),
$$

\(46\)

$$
\mathbf{K}^{k,k'}_M(k) = \sum_{B'} f^B_{N,k'} \frac{M^B_{N,k}}{E_N} (\omega_1 + \varepsilon^B_{N}(M))
\times \frac{\mathcal{V}^{M,B}_{B',B'}(k') \mathcal{V}^{M,B}_{B',B'}(k)}{(\omega_k + \varepsilon^B_{N}(M))(\omega_k + \varepsilon^B_{N}(M'))},
$$

\(47\)

where $\varepsilon^N_{B} = (M^2_{B} - M^2_{N} - m^2_{B})/2E_N$ and $\varepsilon^B_{N}(M) = (M^2_{B} - M^2 - m^2_{B})/2E_N$ while $f_{AB}$ are given by (65). Here $M_{B}$ stands for the nominal (fixed) mass of the isobar $B$, while the invariant mass $M$ pertinent to isobar $B$ is a variable. Using the separable kernels we are able to solve the system exactly. This is important from the numerical point of view since it is now possible to control the principal value integration over the poles of the kernel and thus avoid possible numerical instabilities.

For the coefficients $c^k_{R'}$ (here $H$ denotes $\pi N$, $\pi B$, $\pi B$ channels) we obtain a set of algebraic equations

$$
\sum_{R'} A_{R,R'}(W)c^k_{R'}(W,m_H) = b^k_{R}(m_H),
$$

\(48\)

where

$$
A_{R,R'} = (W - M^0_{R}) \delta_{R,R'} + \sum_{B'} \int dk \mathcal{V}^{B}_{B,R'}(k) \mathcal{V}^{M,B}_{B,R'}(k) \omega_k + E_{B}(k) - W,
$$

$$
b^k_{R} = \mathcal{V}^{M,B}_{B,R}(k_1) + \sum_{B'} \int dk \mathcal{D}^{B}_{M,M,B'}(k,k_1) \mathcal{V}^{M,B'}_{B',B}(k) \omega_k + E_{B}(k) - W,
$$

$$
b^k_{R'} = \mathcal{V}^{M,B}_{B,R'}(k_1),
$$

\(49\)

Here the sum over $B'$ includes also the ground state, in the P33 case $V_{\pi R}$ is replaced by $V_{\pi R}$. The equalities between $b$ and $\mathcal{V}$ can be proved by iterating equations for $\mathbf{D}$ and $\mathcal{V}$.

Using the vector notation $\mathbf{c}^H \equiv [c^H_{R}, c^H_{R'}, \ldots]^T$ and $\mathcal{V}^H \equiv [\mathcal{V}^H_{\pi N}, \mathcal{V}^H_{\pi B}, \ldots]^T$, the solution of (48) can be written in the form $\mathbf{c}^H = \mathbf{A}^{-1} \mathbf{V}^H$. The zeros of $\mathbf{A}$ occur at the positions of the poles of the $K$ matrix related to the resonance $R$; we denote the corresponding energies by $M_R$. The procedure to determine the coefficients $c^H_R$ is then the following: we first determine the zeros of the $A$-matrix determinant; by adjusting the energies of the bare states, $M^B_{R}$, we can force the poles of the $K$ matrix to acquire some desired values. (Note that in the case of several channels and strong background they do not coincide with the corresponding experimental values.) Diagonalizing the $A$ matrix, $\mathbf{U} \mathbf{A} \mathbf{U}^T = \mathbf{D}$, we write

$$
\mathbf{D} = \text{diag}[(\lambda_R, \lambda_{R'}, \ldots)]
$$

\(50\)

which defines the wave function normalization $Z_R$ pertinent to the resonance $R$. The solution can now be cast in the form

$$
\mathbf{c}^H = \mathbf{U}^T \mathbf{D}^{-1} \mathbf{U} \mathbf{V}^H.
$$

Finally, the resonant part of the $\chi$ amplitudes appearing in the expression for the $K$ matrix (e.g. (35)) takes the form

$$
\chi^H \chi^H = -\mathbf{V}^H \mathbf{c}^H - \mathbf{V}^H \mathbf{U}^T \mathbf{D}^{-1} \mathbf{U} \mathbf{V}^H
$$

\(51\)

The interpretation of (52) is that the resonant states $\mathcal{R}, \mathcal{R}', \ldots$ are not eigenstates of $H$ and therefore mix:

$$
\tilde{\phi}_R = \sum_{R'} u_{R,R'} \phi_{R'},
$$

\(52\)

where

$$
\tilde{\varphi}_R = \sum_{R'} u_{R,R'} \varphi_{R'},
$$

The exception is the ground state – which by assumption is the eigenstate of $H$ – for which the mixing of other resonances due to (30) and (31) vanishes at the nucleon pole ($W = M_N$) and does not affect the $\pi N$ coupling constant.

Note that neglecting the off-diagonal terms $A_{R,R'}$ in (49) the expression for $Z_R$ in the vicinity of the resonance assumes the familiar form

$$
Z_R(M_R) = 1 - \frac{d}{dW} \Sigma_R(W) \bigg|_{W=M_R}.
$$

\(53\)

4 Results

4.1 The parameters of the Cloudy Bag Model

We analyze the capability of our approach in the framework of the Cloudy Bag Model (CBM) as one of the most popular examples of quark-pion dynamics. The Hamiltonian of the model has the form (1) and (2) with

$$
v(k) = \frac{1}{2f} \frac{k^2}{\sqrt{12\pi^2 \omega_k}} \omega^0_{\text{MIT}} j_1(kR) k^2 - \omega^0_{\text{MIT}} - \frac{1}{kR},
$$

\(54\)
assuming three quarks in the 1s state. The parameter \( f \) corresponds to the pion decay constant \( f_\pi \), and \( \omega^0_{\text{MIT}} = 2.043 \). It is a known drawback of the model that the \( \pi NN \) coupling constant is underestimated, irrespectively of the bag radius, if \( f \) is fixed to the experimental value \( f_\pi = 93 \) MeV. We therefore adopt the conventional smaller value of \( f = 76 \) MeV which reproduces the \( \pi NN \) coupling constant. The free parameters of the model are the bag radius \( R \) and the energies of the bare quark states corresponding to the nucleon and the excited states. We have also considered alternative forms of the \( k \)-dependence which avoid the typical oscillations due to the sharp cut-off at the bag surface but have found almost no change in the final results. We use the same bag radius for the excited states as for the ground state; as a consequence, the matrix elements of the quark-pion interaction between the quark configurations with different spatial structure are all proportional to \( v(k) \):

\[
\langle \Phi_B' | V | \Phi_B \rangle = r_{BB'} r_q v(k) \langle J_{B'}, T_B' \rangle \sum_{i=1}^3 \sigma^i r^i |J_B, T_B \rangle,
\]

where \( r_q = 1 \) if both \( B \) and \( B' \) are in the \( (1s)^3 \) configuration, \( r_q = r_\omega \) for the transition between the \( (1s)^2 (2s)^1 \) configuration and the ground state, and \( r_q = \frac{2}{3} + r_\omega^2 \) if both \( B \) and \( B' \) are in the \( (1s)^2 (2s)^1 \) configuration. Here

\[
r_\omega = \frac{1}{\sqrt{2}} \left[ \frac{\omega^0_{\text{MIT}} (\omega^0_{\text{MIT}} - 1)}{\omega^0_{\text{MIT}} (\omega^0_{\text{MIT}} - 1)} \right]^{1/2},
\]

with \( \omega^0_{\text{MIT}} = 5.396 \). The parameter \( r_{BB'} \) in (53) allows us to tune the chosen coupling constant relative to its \( SU(6) \) value. The value \( r_{NN} = 1 \) is fixed by our choice of \( f \). We assume \( r_{BB'} = r_{BB} \).

In the P11 case the sum over \( R \) in (22), (23) and (24) includes besides the nucleon, the Roper \( N (1440) \) and the \( N (1710) \), and in the P33 case, the \( \Delta (1232), \Delta (1600) \) and \( \Delta (1920) \). We do not include further intermediate states and channels since our present goal is to find a pattern common to the low-lying Roper-like resonances. This limits the validity of our approach to energies below ~ 1700 MeV. The approach can be extended in a straightforward way by including higher intermediate states as well as other channels.

In sect. 4.2 we first discuss the results of the Born approximation for the K matrix, and in sect. 4.3 the results when the integral equation for the K matrix is solved. The parameters in the two cases are displayed in Table 1. The parameters \( M_\Delta, m_\sigma \) and \( \Gamma_\sigma \) are kept fixed: \( M_\Delta \) at the experimental position of the pole of the K matrix, while from the recent analysis of Leutwyler \[57\] we take \( m_\sigma = 450 \) MeV and \( \Gamma_\sigma = 550 \) MeV. The values for \( M_R, M_{\Delta^*} \) and \( m_\rho \), and for \( G_\pi \) are free in the Born approximation as well as in the full calculation. The parameters \( r_{BB'} \) defined in (53) are free in the Born approximation; in the full calculation they are kept at the values predicted by the quark model except for the value of the \( \pi R \) coupling. The \( r_{\pi BB} \) correspond to the averaged values of the dressed vertices and are explained in sect. 4.3.

### Table 1. The model parameters used in sect. 4.2 (Born) and sect. 4.3 (Full) for the P11 and P33 partial waves. \( M_B \) are the positions of the K-matrix poles, \( M_{B^*} \) corresponds to \( N(1710) \). The parameter \( r_{BB'} \) is defined in (53) and \( r_{\pi BB'} \) refers to the values used in the kernels in the full calculation.

| Parameter | \( P_{11} \) | \( P_{33} \) | \( P_{11} \) | \( P_{33} \) |
|-----------|-----------|-----------|-----------|-----------|
| \( R \)   | 0.83 fm   |           |           |           |
| \( M_\Delta \) | 1232 MeV | 1232 MeV |           |           |
| \( M_R \) | 1520 MeV |           |           |           |
| \( M_{\Delta^*} \) | 1780 MeV |           |           |           |
| \( M_{B^*} \) | 1870 MeV |           |           |           |
| \( m_\pi \) | 450 MeV |           |           |           |
| \( \Gamma_\pi \) | 550 MeV |           |           |           |
| \( G_\sigma \) | 0.96 | 0.99 |           |           |
| \( r_{\pi NN} \) | 1.30 | 1.12 | 1.00 | 1.00 |
| \( r_{\pi NN} \) | 1.30 | 1.12 | 1.00 | 1.00 |

#### 4.2 The Born approximation

We first analyze the P33 partial amplitudes. The results turn out to be almost insensitive to the value of the bag radius so the only parameters to adjust are the positions of the resonances \( M_\Delta \) and \( M_{\Delta^*} \) and the relative coupling strengths \( r_{\pi \Delta} \) and \( r_{\pi \Delta^*} \). (Note that our value for \( M_{\Delta^*} \) is the position of the K-matrix pole and should not be identified with the nominal value of the resonance invariant mass.)

In order to investigate the importance of different degrees of freedom we include in the first step only the \( \pi N \) and the \( \pi \Delta \) channels without the background. In this case the model reproduces the amplitudes at lower energies provided we take a value for \( r_{\pi N} \) which is substantially larger than unity (Fig. 1). When the background is included, the agreement considerably improves except for the energies close to the \( \Delta (1600) \) resonance. This is most notably seen in Fig. 2 in which the inelasticity exhibits a qualitatively different behavior compared to the case with no background, and becomes consistent with the experimental data up to \( W \approx 1700 \) MeV. A small kink around 1700 MeV is an indication of the \( \Delta (1600) \) resonance.

Including the \( \sigma \Delta \) and the \( \pi R \) channels gives an almost perfect fit to the experimental amplitudes also in the vicinity of the \( \Delta (1600) \) resonance, washing out almost completely the signature of that resonance in the phase shift (Fig. 1).

Turning to the case of the P11 partial wave we note that including only the \( \pi N \) and \( \pi \Delta \) channels without the background the Born approximation fails to reproduce the amplitudes determined in the partial wave analysis even if we considerably increase the values of some coupling constants and take \( r_{\pi NN} = 2.20 \) and \( r_{\pi \Delta R} = 1.55 \) (Figs. 1 and 4). Adding the background yields the correct behavior of the amplitudes below the two-pion threshold. In order
to reproduce the amplitudes in the region of the Roper resonance we have to keep the large value for \( r_{\Delta R} \), still, the approximation does not reproduce the rapid rise of the inelasticity just above the two-pion threshold nor the property that it remains close to unity even well above the Roper resonant energy (Fig. 2). Including the \( \sigma N \) channel reproduces the threshold behavior and considerably improves the agreement at higher energies.

Let us stress that in order to reproduce the widths of the \( N(1440) \) and \( \Delta(1232) \) resonances, the respective \( \pi NR \) and \( \pi N\Delta \) coupling constants have to assume considerably larger values than the corresponding bare-quark values. The choice of the model parameters \( r_{BB'} \) (53) that drive these constants turns out to be quite different for the P33 and the P11 case. A possible solution to these inconsistencies is presented in the next section.

### 4.3 Results of the full calculation

The results presented here have been obtained using the same set of parameters as in the Born approximation for
the bag radius, the mass and the width of the \(\sigma\)-meson, and the position of the K-matrix poles (see Table 1). Similar results are obtained for 0.75 \(\text{fm} < R < 1.0 \text{ fm}\) as well as for 400 \(\text{MeV} < m_{\sigma} < 550 \text{ MeV}\) provided the coupling constants are slightly readjusted. We keep the same set of parameters for both partial waves; we allow only for a slight deviation in the parameters entering the kernels.

We first investigate the relation between the bare matrix elements of the baryon-meson interaction (53) and the corresponding dressed values \(V_{BB'}\) which are solutions of the integral equations (43) and (44). In the case of the elastic channel we introduce the ratio

\[
r_{\pi NB} = \frac{V_{NB}(k_0)}{V_{NB}(k_0)}
\]

measuring the renormalization of the bare vertex. In the case of the \(N(1440)\) and the \(\Delta(1232)\) we find that this ratio exhibits a relatively strong energy dependence (Fig. 5) and yields a substantial enhancement of the bare coupling constant over a broad energy range. The enhancement is consistent with the value of the corresponding coupling constant used in our analysis in the Born approximation.

The formulas (43) and (44) represent a system of coupled non-linear integral equations for the dressed vertices since the kernels (45) – (47) themselves contain these vertices. We have not attempted to solve the system exactly but have substituted the dressed vertices appearing in the kernels by \(V_{BB'}(k) = \bar{r}_{\pi BB'} V_{BB'}(k)\) with suitably chosen values for \(\bar{r}_{\pi BB'}\). We have adjusted these values by averaging the corresponding solutions for the dressed vertices in the relevant energy range, allowing for small variations to obtain better overall fits. This approximation is justified because the contribution from the integrals in (43) and (44) turns out to be less important compared to the leading term. Also, many of the dressed vertices only negligibly influence the result such that we can put the corresponding \(\bar{r}_{\pi BB'} = 1\). In Table 1 only a few relevant cases are listed; in all other cases \(\bar{r}_{\pi BB'} = 1\).

In the P11 wave the calculated amplitudes closely follow the experimental values in the energy range from the threshold up to \(W \sim 1700 \text{ MeV}\) (Figs. 4 and 2). In particular, we should stress the excellent agreement of the amplitudes slightly above the \(2\pi\) threshold where the inelasticity is dominated by the \(\sigma N\) channel. Compared to the results of the Born approximation we notice an improvement at very low energies as well as at those above the resonance, which is a consequence of the energy-dependent dressing.

Above \(W \sim 1700 \text{ MeV}\) the imaginary part of the amplitude as well as the inelasticity drop rapidly. The simple model involving only the Roper-like bound state and the nucleon breaks down here and the effects of other resonances as well as other channels such as the \(pN\) and \(\eta N\) may become more important. To investigate this point we have included in our calculation the next P11 excitation, the \(N(1710)\), and assumed that it couples only to the \(\sigma N\) channel, representative of a generic \(\pi N\) decay with \(\sim 80\%\) of the value of \(G_\sigma\). (The experimental branching fraction for \(N(1710) \to N \pi \pi\) is \((40–90)\%\).) A better agreement at higher energies is obtained, although a similar effect can be achieved by decreasing the bag radius (or increasing the cut-off parameter in general) which makes a model-independent analysis less reliable.

In Fig. 6 we display the imaginary parts of the off-diagonal T-matrix elements corresponding to the processes \(\pi N \to \pi \Delta\) and \(\pi N \to \sigma N\) calculated with the \(N(1710)\) included. The amplitudes are averaged over the range of unstable hadron invariant masses as described in sect. 3.4. In comparison to the values extracted from a recent partial-wave analysis of \(\pi N \to \pi N\) and \(\pi \pi N\) [29], our model overestimates the \(\sigma N\) decay probability and underestimates the \(\pi \Delta\) channel. The disagreement is a consequence of the destructive interference between the nucleon and the bare Roper in this channel at higher \(W\). In the Born approximation, a better agreement in both
Inelastic channels is obtained. Had we departed from our standard set of parameters chosen to yield a good overall description of various amplitudes in both partial waves, or by including further channels in the integral equations, an improvement can be achieved in the full calculation as well. (Note also the large systematic scatter of data.)

The full calculation for the P33 partial wave does not significantly improve the results of the Born approximation which are anyway satisfactory throughout the energy range from the threshold to \( W \sim 1700 \) MeV. It does, however, explain the strong enhancement of the bare quark-model \( \pi N \Delta \) coupling of the Born approximation. Above \( W \sim 1800 \) MeV the agreement is lost even if we include another bare resonant state corresponding to the \( \Delta(1910) \), indicating the need to include further channels not present in our analysis.

![Fig. 7](image.png)

**Fig. 7.** The real and imaginary parts of the T matrix and inelasticity for the P11 partial wave for different bag radii: \( R = 1 \) fm (dotted lines), \( R = 0.9 \) fm (dashed lines), \( R = 0.83 \) fm (solid lines), and \( R = 0.77 \) fm (dashed-dotted lines).

In order to test the sensitivity of the results we have varied the parameters appearing in Table 1. Varying the bag radius and the parameters \( r_{NB} \) we find that in the P33 case the results are almost independent of the bag radius while a few percent change in \( r_{NB} \)‘s may improve the agreement. In the P11 case, the results are more sensitive to the choice of the bag radius (Fig. 7). For \( R \sim 0.8 \) fm the agreement at larger energies improves; for \( R \lesssim 0.75 \) fm the real part of the amplitude comes too close to zero and the phase shift loses its characteristic resonant behavior. The optimal value of \( r_{NN} \) increases by \( \sim 5 \% \) for \( R = 1.0 \) fm and decreases by \( \sim 10 \% \) for \( R = 0.77 \) fm.

In Table 1 we list the positions of poles of the K matrix which are free parameters in our approach. Alternatively, we could have used the masses of the bare quark states entering our ansaetze for the PV states (22), (23) and (26); these masses are needed as the pole values through the procedure described in sect. 3.5. From the values of the bare masses we may obtain an indication of how sensible is the comparison of baryon masses calculated in various quark models (with no mesons included) and the experimental positions of the corresponding resonances. We are not able to give a conclusive answer since the result is strongly sensitive on some quantities (in particular the bag radius and the \( \pi RR \) coupling constant) which only weakly influence the behaviour of the scattering amplitudes and are therefore not well fixed in our calculation. Nevertheless, we can conclude that the self-energy corrections and the effects of resonance mixing most strongly affect the ground state, lowering its mass compared to the bare mass by 310 MeV at \( R_{bag} = 0.83 \) fm (or by 180 MeV at \( R_{bag} = 1.0 \) fm). The bare delta-nucleon mass difference turns out to be smaller than the experimental one by about 90 MeV; the remaining difference can be attributed to the chromomagnetic interaction. In the case of the Roper resonance the bare mass splitting is smaller by 120 MeV at \( R_{bag} = 0.83 \) fm (or by 60 MeV at \( R_{bag} = 1.0 \) fm). Since our value for the K matrix pole is typically 100 MeV higher than the experimental value, the resulting bare Roper-nucleon mass splitting remains in the ballpark of admissible values. A similar conclusion holds for the \( \Delta(1600) \) where the bare mass splitting is typically 200 MeV lower than the value deduced from the K matrix pole.

5 Conclusions and perspectives

We have developed a general method to incorporate the excited baryons represented as quasi-bound quark-model states into a dynamical framework with correct unitarity and symmetry requirements and proper boundary conditions. To illustrate the key points, we have used the Cloudy Bag Model, although the method is applicable to more sophisticated models. In fact, the only information needed as input to our computational scheme are the matrix elements of the meson interaction between the quark states, \( \Phi(k)|V|\Phi(k = 0) \). In a more ambitious computational scheme, the method can be extended to allow for the readjustment of the intrinsic three-quark wavefunction to the scattering boundary conditions by using the Kohn variational principle.

We have shown that an intricate interplay of the \( \pi N, \pi \Delta \) and \( \sigma N \) degrees of freedom governs the elastic and inelastic pion-nucleon scattering in the energy range from the threshold up to \( W \sim 1700 \) MeV. The model explains at least qualitatively the behavior of the amplitudes in the vicinity of the \( \Delta(1600) \) which can be regarded as the P33-wave counterpart of the \( N(1440) \) in the P11 wave. We have studied in a systematic way the role of the background processes which turn out to be important to qualitatively reproduce the experimental data throughout the first and the second resonance region.

We have described the correlated two-pion decay in the relative s-wave by the \( \sigma \)-meson. In spite of this purely phenomenological approach, our results show that this degree of freedom is crucial to explain certain features of the scattering amplitudes, in particular the inelasticity for the P11 partial wave just above the two-pion threshold which rapidly rises from zero to unity and remains close to this.
value in a broad energy range. Because of the $s$-wave nature of the $\sigma$-meson coupling compared to the $p$-wave coupling of the pion to the $\Delta$, the two-pion decay is dominated by the $\sigma N$ channel in the energy region slightly above the two-pion threshold. This offers the possibility to determine the strength of the coupling in a model-independent way, which is generally not the case at higher energies where the same level of agreement can be obtained in a broad range of parameters. Our calculation indicates that a low mass of the $\sigma$-meson ($m_\sigma \approx 450$ MeV) is preferable over the larger mass typically obtained in partial-wave analyses.

We have designed a framework to numerically solve the integral equation for the $K$ matrix by approximating the kernels with a separable form which preserves the symmetries of the matrix and thus ensures unitarity. The important outcome of this calculation is a substantial increase of the quark-model pion-baryon couplings explaining the large width of the Roper and possibly other resonances. Such a strong enhancement could — at least partially — accommodate relatively weak pion-baryon coupling strengths predicted in constituent quark models. This shows, on the one hand, that one can explain the nature of the Roper resonances without invoking “exotic” degrees of freedom mentioned in the Introduction and, on the other hand, establishes one of the benchmarks for an assessment of the underlying quark models. Still, the scattering analysis alone should not be expected to provide a definitive selection criterion, because a consistent description, as shown in the example above, can be achieved within a relatively broad range of parameters. A future analysis devoted to pion electro-production could provide a more complete set of criteria. The application of the method to calculate the electro-production amplitudes will be treated in a separate paper.

### A Evaluation of one- and two-pion matrix elements

We derive some expressions for the matrix elements of the pion field between the eigenstates of the Hamiltonian with the pion part given by (1). If $\Psi_A$ is an eigenstate then

$$(\omega_k + H - E_A) a_{m\ell}(k) |\Psi_A\rangle = - V^\dagger_{m\ell}(k) |\Psi_A\rangle,$$  \hspace{1cm} (55)

$$(\omega_k + \omega'_k + H - E_A) a_{m\ell}(k) a_{m\ell'}(k') |\Psi_A\rangle$$

$$= - (V^\dagger_{m\ell}(k) a_{m\ell'}(k') + V^\dagger_{m\ell'}(k') a_{m\ell}(k')) |\Psi_A\rangle,$$  \hspace{1cm} (56)

Multiplying (55) by an eigenstate of (1) we obtain

$$\langle \Psi_B(k) | a_{m\ell}(k) |\Psi_A\rangle = \nu \delta(\omega_k + E_B(k) - E_A)$$

$$\frac{(\bar{\Psi}_B(k) V^\dagger_{m\ell}(k) |\Psi_A\rangle)}{(\omega_k + E_B(k) - E_A)} ,$$  \hspace{1cm} (57)

where $\nu$ is an arbitrary constant; it is used to determine the normalization of the channel states (19), (23) and (24).

Taking for $\Psi_A$ either (19) or (23), multiplying (56) by an eigenstate with momentum $k + k'$, and neglecting the terms with two or more pions, we obtain

$$(\omega_k + \omega'_k + E_B(k + k') - W) \langle \bar{\Psi}_B(k + k') a_{m\ell}(k) |\Psi_B'(k')\rangle$$

$$= - (\bar{\Psi}_B(k + k') |V^\dagger_{m\ell}(k) |\Psi_B'(k')\rangle).$$  \hspace{1cm} (58)

Note that when $B'$ is on-shell, $E_{B'} = W - \omega'_k$, and (58) reduces to (57).

### B Derivation of the mass distribution function

We derive here the expression for the mass distribution function $w_B(M)$ in (21) for the case of the $\pi\Delta$ channel which dominates the two-pion decay through the intermediate baryon. In this case we can assume that the intermediate $\Delta$ decays only into a pion and the nucleon. (This assumption is justified since experimentally the elastic channel remains the dominant process also well above the two-pion threshold.) Then the $K$ matrix becomes a scalar (denoted as $K_\Delta$) and the orthonormalized state in (10) assumes the form

$$|\bar{\Psi}_\Delta(M)\rangle = \frac{1}{\sqrt{1 + K_\Delta(M)^2}} \sqrt{\frac{2}{k_2 M}} c_\Delta^N(M) \frac{1}{z_\Delta} \bar{\Psi}_\Delta(M)\rangle,$$

$$\equiv w_\Delta(M)|\bar{\Psi}_\Delta(M)\rangle.$$  \hspace{1cm} (59)

where $M$ is the invariant mass of the intermediate $\Delta$, while $\omega_2$ and $k_2$ are the energy and momentum of the second pion. From (35) we see that close to the resonance the pion amplitude behaves as $\chi^N_\Delta(k, k_2) \approx - c^N_\Delta(M) V_N\Delta(k)$ and from (36) we find $\chi^N_\Delta(k, k_2, M_B) \approx - c^N_\Delta(M) V^{N\Delta}_B(k)$, with $c^N_\Delta \propto (M - M_\Delta)^{-1}$. The state $\bar{\Psi}_\Delta(M)$ introduced in (59) can then be written in the form

$$|\bar{\Psi}_\Delta(M)\rangle = z_\Delta \left\{ |\Psi_\Delta\rangle - \int dk \frac{V_N\Delta(k)}{\omega_k + E_N(k) - M} |a^1(k)\langle |\Psi_N\rangle|^{3/2}$$

$$- \sum_B \int dk \frac{V^{N\Delta}_B(k)}{\omega_k + E_B(k) - M} |a^1(k)\langle |\Psi_B\rangle|^{3/2} \right\}. \hspace{1cm} (60)$$

Note that in the weak-coupling limit the form (60) corresponds to the usual perturbative expression for the $\Delta$ state, with $|\Psi_N$ and $|\bar{\Psi}_\Delta$ replaced by the corresponding bare quark states. It is dominated by the bare-quark state $|\Psi_\Delta$. This is another reason for choosing the particular form of factorization in (59).

From (51) and (26) we have ($K_\Delta \equiv K^{3/2}_{N\Delta}$)

$$K_\Delta(M) = \pi \frac{\omega_2 M_N}{k_2 M} \frac{V_N\Delta(k_2)^2}{Z_\Delta(M)(M_\Delta - M)} + \ldots \hspace{1cm} (61)$$
for $M \approx M_\Delta$, where the terms denoted by \ldots vanish at the resonance; $Z_{\Delta}(W)$ is defined in (50). From (59) it then follows

$$w_\Delta(M) = \frac{K_\Delta}{z_\Delta} \frac{1}{\sqrt{1 + K_\Delta^2}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} \frac{1}{\sqrt{\omega_2 M_N}} .$$

At this point we can use the expression for the K matrix obtained numerically or use a suitable parameterization either of the computed form or of the experimental data. The simplest choice is

$$K_\Delta = \frac{C}{M_\Delta - M}, \quad C = \frac{\omega_2 M_N}{k_2 M_\Delta} N_\Delta(k_2^2) Z_{\Delta}(M_\Delta) ,$$

where $\omega_2^2 = (M_\Delta^2 - M_N^2 + m_\pi^2)/2M_\Delta$ and the residue $C$ (corresponding to $\frac{1}{2}f$) is assumed to be $W$-independent; the second equality follows from (61). From (21) and (60) we have

$$w_\Delta(M) = \frac{1}{\sqrt{\omega_2 M_N}} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} \sqrt{\omega_2 M_N} .$$

For sufficiently small values of $C$, $w_\Delta(M)$ in (63) is strongly peaked around $M_\Delta$ and has a unit integral provided that $z_\Delta = Z_{\Delta}(M_\Delta) = 1$, hence

$$w_\Delta(M)^2 \to \delta(M - M_\Delta), \quad C \to 0.$$  

In this limit we obtain the usual wave-function normalization of (60), i.e. $z_\Delta = Z_{\Delta}(M_\Delta)^{-1/2}$.

In a more precise calculation we use a better approximation for the K matrix, $K_\Delta = C/(M_\Delta - M) + D$, which turns out to give a very good approximation for $M$ from the threshold to values well above the resonance. Note, however, that the parameter $D$ should not be identified with the background contribution since in the approximate formula (62) the coefficient $C$ is kept fixed while in (61) all terms exhibit strong $k_2$ (or, equivalently, $M$) dependence. The coefficient $D$ is chosen such as to compensate this dependence. In our calculation we use $C = 55$ MeV and for $D$ a constant value $-0.41$ below $M \sim 1400$ MeV and, above it, a value that smoothly approaches zero.

C Evaluating the kernels

To evaluate the kernel (33) in the integral equation for $\chi_{JT}$ amplitudes we insert a complete set of states

$$1 = |\Psi_N\rangle\langle\Psi_N| + \sum_B |\bar{\Psi}_B(M)\rangle\langle\bar{\Psi}_B(M)|,$$

where the sum implies also the integral over invariant masses and momenta. Next we use the (adjoint of) (57) and (58). Following Ericson and Weise (see [49], sect. 2.5.3.) we substitute the momenta of the intermediate baryon by a suitably chosen average over $k + k'$ denoted as $\bar{k}$. The kernel then takes the form:

$$\mathcal{K}_{MNB}^{NB}(k, k') = \int d\omega_k \langle \bar{\Psi}_B(k) | \bar{\Psi}_B(k') \rangle$$

$$\approx \sum_B f_B \langle \bar{\Psi}_B(k) | \bar{\Psi}_B(k') \rangle$$

where

$$f_B = \sqrt{(2J_B + 1)(2J_B + 1)(2T_B + 1)}$$

The matrix elements are those entering the expression for the $\chi$’s in (35)-(36), $\langle \bar{\Psi}_N(k') | \bar{\Psi}_B(k') \rangle = w_B(k')$ and $\langle \bar{\Psi}_B(k) | \bar{\Psi}_B(k') \rangle = w_B(k') M_B^N$. We can now approximately perform the integration over $M'$ assuming (64):

$$\mathcal{K}_{MN}^{NB}(k, k') \approx \int d\omega_k \langle \bar{\Psi}_B(k) | \bar{\Psi}_B(k') \rangle$$

$$= \sum_B f_B \langle \bar{\Psi}_B(k) | \bar{\Psi}_B(k') \rangle$$

In order to build up a feasible computational scheme we approximate the kernel with a separable expression, a relativistic extension of the approximation used in Ref. [23]:

$$\frac{1}{\omega_k + \omega'_k + E_B(k) - W} \approx \frac{1}{\omega_k + \omega'_k + E_B(k) - W} \approx \frac{1}{\omega_k + \omega'_k + E_B(k) - W} \approx \frac{1}{\omega_k + \omega'_k + E_B(k) - W} .$$

Here $k_1$, $\omega_1$ are the on-shell pion momentum and energy in the $\pi B$ channel satisfying $W = \omega_1 + E(k_1) = \omega_1 + E_N(k_0).$ The approximation on the RHS coincides with the exact expression on the LHS when either of the two pions is on-shell, i.e. when either $\omega_k = \omega_0$ or $\omega'_k = \omega_0$. This can be easily seen by writing $W = \omega_0 + E_N(k_0)$ on the LHS and $W = \omega_1 + E(k_1)$ on the RHS in the first case and vice versa in the second one.) When both pions are on-shell, the denominator can be cast in the form

$$\frac{1}{\omega_1 + E_B(k) - E_N(k_0)} \approx \frac{E_B(k) + E_N(k_0) - \omega_1}{M_B^2 + 2\omega_1 E_N(k_0) - M_\Delta^2} .$$

We have assumed $\langle k_0 : k_1 \rangle = 0$ which is essentially the same approximation suggested in [49] and the denominator acquires the characteristic $u$-channel form. Since we describe the resonance in terms of the wave function rather than bispinors, our numerator differs from the correct relativistic expression in the $u$-channel (see e.g. [49]). Since our expression is anyway approximate, we replace the factor $E_B(k) + E_N(k_0) - \omega_1 = E_B(k) + E_B(k_1) - \omega_0$ in our numerator by the correct relativistic expression $2M_B^N$. 

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In the general case we find:
\[
\mathcal{K}_{MM'}^{BB'}(k, k') = \sum_{B'\nu} f_{BB'}^{\nu} \frac{V_B^{M'}(k') V_{B'}^{M}(k)}{\omega_k + \omega_k' + E_{B'}(k) - W},
\]
where the sum includes also the ground states. We now approximate
\[
\frac{1}{\omega_k + \omega_k' + E_{B'}(k) - W} \approx \frac{\omega_1 + \omega_1' + E_{B'}(k) - W}{(\omega_k' + E_{B'}(k) - E(k_1))(\omega_k + E_{B'}(k) - E'(k_1))}.
\]
Here \(k_1, \omega_1, k_1', \omega_1'\) are the on-shell pion momenta and energies satisfying \(W = \omega + E(k_1) = \omega_1 + E'(k_1), E(k_1) = \sqrt{\mathbf{p}_\pi^2 + m_\pi^2}, E'(k_1') = \sqrt{\mathbf{p}_\pi'^2 + m_\pi^2}\). The approximation on the RHS of (67) coincides with the exact expression on the LHS of (67) when either \(p_\pi = 0\), or \(\omega_1 = \omega_1'\). When on-shell, the kernels are proportional to the \(u\)-channel background elements of the \(K\) matrix. We now use the same approximation as in (66) for \(k\) as well as for the numerator; the final forms are given by (45)–(47).

In the case of the kernels involving the \(\sigma\)-meson we find
\[
\mathcal{K}_{MN}^{B\sigma}(k, k') = \sum_{B'} \frac{\omega_{nk}}{\omega_{nk} + \omega_k + E_{B'}(k) - W} V_{B'}^{M}(k') V_{B'}^{N}(k) = \mathcal{K}_{MN}^{\sigma\mu}(k, k'),
\]
\[
\mathcal{K}_{MN}^{\sigma\nu}(k, k') = \sum_{B'} \frac{\omega_{nk}}{\omega_{nk} + \omega_k + E_{B'}(k) - W} V_{B'}^{N}(k') V_{B'}^{M}(k).
\]
where in the \(P11\) case the sum over \(B'\) includes only \(J = \frac{3}{2}\) isobars; in the \(P11\) case the \(N\) in the above expression is replaced by \(\Delta\) while the sum over \(B'\) includes only \(J = \frac{1}{2}\) isobars.

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