Extended phase space thermodynamics for Bardeen black holes in massive gravity

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Abstract

This paper presents an exact solution of Bardeen black hole in presence of massive gravity, which is characterized by the additional parameter $m$. Here, we focus on 4D Bardeen AdS massive black hole solutions as example to discuss the thermodynamical properties such as temperature, entropy and specific heat at constant pressure. We also study the critical behaviour of Bardeen AdS massive black holes by considering the cosmological constant $\Lambda$ as thermodynamical variable (pressure) as well as the parameter associated with the nonlinear electrodynamics. We calculate the critical value of pressure and temperature and study the effect of magnetic charge $e$ and mass parameter $m$. It is seen that the thermodynamical volume are the independent of mass parameter $m$. The critical values are the one in which phase transition takes place and the nature of mass parameter $m$ and magnetic charge $e$ are opposite to each other. The critical temperature and pressure were highly sensitive for these parameters.

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I. INTRODUCTION

Black holes, coined by Wheeler [1] were generally considered as rather esoteric objects of purely theoretical interest and little physical relevance. It is well known that a black hole is thermodynamical system, there exist fundamental connections between the thermodynamics and general relativity since the discovery of black hole entropy by Bekenstein [2–6] and Hawking [7]. During the past years the development of Maldasena conjecture (AdS/CFT correspondence) [8], has attracted significant attention to AdS black holes. The properties of black hole physics have changed dramatically and subsequent studies found that the AdS black hole behave like the Van der Walls fluid.

The first regular black hole was proposed by Bardeen [9] which means there is no singularity. Later on an exact solution of Bardeen black hole was given by Ayon-Beato and Garcia [10–12] wherein general relativity coupled to nonlinear electrodynamics. Thereafter, more intense emphasis was given by researchers investing the regular black holes and more regular black holes were discovered recently [13–22]. But most of these solutions are based on Bardeen’s Model [9]. In later years the generalized solutions of Bardeen black hole model were developed which includes Bardeen de Sitter black hole [23, 24], rotating Bardeen solution [25], noncommutative Bardeen solution [26], higher dimensional black holes [27] and EGB black holes [28–32] etc. However, many singular black hole solutions with massive gravity [33–36] are already present in the literature, but they are not regular black hole solutions. Therefore, the aim of this work is to get 4D spherically symmetric Bardeen-like black hole solution in massive gravity in AdS spacetimes viz., Bardeen AdS massive metric. It is shown that the Bardeen AdS massive metric is an exact black hole solution in AdS spacetime thereby generalizing the Bardeen black hole solution [37] which is encompassed as a special case. We analyze their thermodynamical properties and also perform a phase structure analysis of the Bardeen AdS massive black holes.

In the extended phase space cosmological constant ($\Lambda$) is identified as thermodynamical pressure ($P = -\Lambda/8\pi$) and its conjugate quantity is thermodynamical volume [38, 39]. Hawking and Page [40] have first studied the phase transition between the AdS black hole and thermal AdS space and also confinement/deconfinement phase of gauge field was studied by Witten [41]. The phase transition behaviour of charged AdS black hole and the Van der Walls liquid-gas system were studied by Chamblin [42, 43]. In fact, phase transition plays an
important role to investigate the thermodynamical properties of the objects at the critical point. This consideration has been investigated for different type of black holes [44–67].

The paper is organized as follows. In Section II we find the exact solution of Bardeen black hole in the presence of massive gravity and studied its physical and thermodynamical properties. Section III, gives the investigation of phase structure for Bardeen AdS black holes in massive gravity. The thermodynamical stability and phase diagrams are studied in Sec IV. Finally, we discuss our result and conclusions in Section V. (Here we use the units $G_4 = k_B = c = 1$).

II. BARDEEN BLACK HOLE SOLUTION IN MASSIVE GRAVITY

The Einstein- Hilbert action coupled to nonlinear electrodynamics in the presence of massive gravity with negative cosmological constant [68] is given by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - 2\Lambda + m^2 \sum_i c_i \mathcal{U}_i(g, f) - \frac{1}{4\pi} \mathcal{L}(F) \right], \quad (1)$$

where $R$ is scalar curvature, $\Lambda$ is a cosmological constant, $m$ is a parameter of massive gravity, $c_i$ are constants, $f$ is symmetric tensor and $\mathcal{U}_i$ are polynomials of eigenvalues of the $(4 \times 4)$ matrix $\mathcal{K}_\mu^\nu = \sqrt{g^{\mu\alpha}} f_{\alpha\nu}$ which can be written as

$$\begin{align*}
\mathcal{U}_1 &= [\mathcal{K}], \\
\mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\
\mathcal{U}_3 &= [\mathcal{K}]^3 - 3 [\mathcal{K}] [\mathcal{K}^2] + 2 [\mathcal{K}^3], \\
\mathcal{U}_4 &= [\mathcal{K}]^4 - 6 [\mathcal{K}^2] [\mathcal{K}]^2 + 8 [\mathcal{K}^3] [\mathcal{K}] + 3 [\mathcal{K}^2]^2 - 6 [\mathcal{K}^4],
\end{align*} \quad (2)$$

with $[\mathcal{K}] = [\mathcal{K}_\mu^\nu]$ and $\mathcal{L}(F)$ is the function of $F \equiv F_{\mu\nu}F^{\mu\nu}$ be the electromagnetic field tensor which is the generalization of Maxwell field and it reduces to Maxwell field in the weak field limit. We consider Lagrangian density of the non-linear electromagnetic field as [12]

$$\mathcal{L}(F) = \frac{1}{2se^2} \left( \frac{\sqrt{2e^2 F}}{1 + \sqrt{2e^2 F}} \right)^{\frac{5}{2}}, \quad (3)$$

where $\mathcal{L}(F)$ is the Lagrangian density of Bardeen source [12] and parameters $M$ and $e$ connected with $s$ by relation $s = e/2M$. We would like to consider the spherically-symmetric
static metric of the spacetime, given by the following line element

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \quad \text{with} \quad f(r) = 1 - \frac{2m(r)}{r}. \quad (4) \]

We consider the following metric ansatz \[68\] for the reference metric as

\[ f_{\mu\nu} = \text{diag}(0, 0, c^2h_{ij}), \quad (5) \]

where \( c \) is the positive constant. Using the metric ansatz (5), one can easily write \( \mathcal{U}_i \) as

\[ \mathcal{U}_1 = \frac{2}{r}, \quad \mathcal{U}_2 = \frac{2}{r^2}, \quad \mathcal{U}_3 = 0 \quad \text{and} \quad \mathcal{U}_4 = 0. \quad (6) \]

Using the action (1) and variation of this action with respect to the metric tensor \((g_{\mu\nu})\) and the electromagnetic potential \((A_\mu)\), respectively, leads to

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + m^2\chi_{\mu\nu} = T_{\mu\nu} \equiv 2 \left[ \frac{\partial \mathcal{L}(F)}{\partial F} F^{\mu\rho}F_\rho^\nu - g_{\mu\nu}\mathcal{L}(F) \right], \]

\[ \nabla_\mu \left( \frac{\partial \mathcal{L}(F)}{\partial F'} F^{\mu\nu} \right) = 0, \quad \text{and} \quad \nabla_\mu (\ast F^{\mu\nu}) = 0, \quad (7) \]

where \( T_{\mu\nu} \) is the energy momentum tensor, \( G_{\mu\nu} \) and \( \chi_{\mu\nu} \) are the Einstein tensor and massive gravity tensor respectively \[68\] are given as

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad (8) \]

\[ \chi_{\mu\nu} = -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2). \quad (9) \]

The non-vanishing components of \( F_{ab} \) are \( F_{\theta\phi} = g(r) \sin \theta \) and with potential \( A_\phi = -g(r) \cos \theta \), reads \[12\]. The non-zero components of energy momentum tensor is given by,

\[ T^t_t = T^r_r = \frac{8Me^2}{(r^2 + e^2)^{\frac{3}{2}}}, \quad (10) \]

\[ T^\theta_\theta = T^\phi_\phi = \frac{8Me^2(r^2 - 4)}{(r^2 + e^2)^{\frac{3}{2}}}. \quad (11) \]

Using the Eq. (4) the \((r, r)\) components of the Eq. (7) are

\[ m'(r) + \frac{3r^2}{2l^2} + m^2 \left( \frac{cc_1 r^2}{2} + c^2 c_2 \right) = \frac{2Me^2}{(r^2 + e^2)^{\frac{3}{2}}}, \quad (12) \]

where the prime is the first derivatives with respect to \( r \). We can obtain the metric function \( f(r) \) using the Eq. (12). Integrating the Eq. (12) in the limit \( r \to \infty \),

\[ m(r) + \frac{r^3}{2l^2} + m^2 \left( \frac{cc_1 r^2}{2} + c^2 c_2 \right) = M \quad (13) \]
by substituting $m(r)$ in $f(r)$ Eq. (4), then the solution becomes

$$f(r) = 1 - \frac{2Mr^2}{(r^2 + e^2)^{\frac{3}{2}}} + \frac{r^2}{l^2} + m^2 \left( c^2 c_2 + \frac{cc_1 r}{2} \right), \quad (14)$$

This solutions describe the four dimensional $AdS$ Bardeen black hole in the presence of massive gravity and it is characterized by the mass $M$, cosmological constant $\Lambda = -3/l^2$, magnetic charge $e$, and mass parameter $m$. In the absence of magnetic charge $e$ this black hole solution interpolated with 4D $AdS$ massive black hole [34],

$$f(r) = 1 - \frac{2Mr^2}{r} + \frac{r^2}{l^2} + m^2 \left( c^2 c_2 + \frac{cc_1 r}{2} \right), \quad (15)$$

for $m = 0$, it reduces to $AdS$ Bardeen black hole as [37]

$$f(r) = 1 - \frac{2GMr^2}{(r^2 + e^2)^{\frac{3}{2}}} + \frac{r^2}{l^2}, \quad (16)$$

and for $e = 0, m = 0$ it gives to $AdS$ Schwazshild black hole. It would be more convenient to study the horizon structure. The horizon of the black hole can be achieved when $f(r_+)$ = 0.

$$1 - \frac{2GMr^2}{(r^2 + e^2)^{\frac{3}{2}}} + \frac{r^2}{l^2} + m^2 \left( c^2 c_2 + \frac{cc_1 r}{2} \right) = 0 \quad (17)$$

This equation can not be solved analytically. The numerical analysis of the $f(r_+)$ = 0 on the varying the massive gravity parameters is depicted in the Fig. [3]. The numerical analysis of $f(r_+)$ = 0 reveals that it is possible to find non-vanishing value of magnetic monopole charge ($e$), cosmological constant ($\Lambda$) and massive parameter ($m$) for which metric function $f(r)$ is minimum, i.e, $f(r_+)$ = 0 this will give four roots $r_+, r_-, r_c$ and $r_m$ which correspond to the Cauchy horizon, event horizon, cosmological horizon and fourth horizon due to mass parameter ($m$) of the black holes respectively. The size of black hole increase with decrease the massive parameter ($m$) with constant magnetic charge ($e$).

The horizon limit of the Bardeen $AdS$ massive black hole are:

1) The size of horizon increase with decrease the magnetic charge and the magnetic charge does not affect the cosmological and massive horizon.

2) The cosmological and massive horizon increases with the mass parameter $m$ and no cosmological and massive horizon for $m < 1.125$.

3) The Cauchy and event horizon independent of mass parameter $m$ and the cosmological and massive horizon independent of magnetic charge $e$. 

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Now, let us study the nature of singularity structure at $r = 0$. It becomes useful to consider the curvature invariants of the spacetime such as the Ricci square ($R_{ab}R^{ab}$), and Kretschmann scalars ($R_{abcd}R^{abcd}$)

$$
\lim_{r \to 0} R_{ab}R^{ab} = \frac{36}{l^2} - \frac{144M}{e^3} \left( \frac{1}{l^2} - \frac{M}{e^3} + \frac{m^2 cc_1}{4} \right) + \frac{42m^2 c^2}{e^2} \left( \frac{m^2 c_2 c^2}{e^2} + \frac{2c_2}{l^2} + \frac{5c_2^2}{12} + \frac{c^4 c_2^2}{e^2} \right),
$$

$$
\lim_{r \to 0} R_{abcd}R^{abcd} = \frac{12}{l^4} - \frac{48M}{e^3} \left( \frac{1}{l^2} + \frac{M}{e^3} + \frac{5m^2 c_2 c^2}{6e^2} \right) + \frac{7m^4 c^2}{e^2} \left( \frac{c_1^2}{e^2} + \frac{6c_2^2 c^2}{7e^2} + \frac{2e^2 c^2 c_2^2}{m^2 l^2} + \frac{4c_2}{m^2 l^2} \right).
$$

(18)

These curvature invariants show that the solution (14) is regular everywhere including origin ($r = 0$). The singularity of the solution is removed due to the presence of non-linear source (3).

III. THERMODYNAMICS

In this section, we explore the thermodynamics of the Bardeen-AdS massive black hole solutions (14). The thermodynamical quantities temperature ($T_+$), entropy ($S_+$) and heat capacity ($C_+$) associated with the black hole solutions. The mass of the Bardeen-massive-AdS black hole is

$$
M_+ = \frac{(r_+^2 + c_2^2)^{\frac{3}{2}}}{2r_+^2} \left[ 1 + \frac{r_+^2}{l^2} + m^2 \left( c_2^2 c_2 + \frac{cc_1 r_+}{2} \right) \right].
$$

(19)
and the mass of AdS Bardeen black hole [37] for \( m \to 0 \), one recovers mass of the AdS Schwarzschild black hole from (19), when \( m = 0, e = 0 \). The temperature of the black hole associated with it, known as Hawking temperature. Hawking temperature of the black hole can be defined by

\[
T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \partial \sqrt{-g} g^{\mu\nu} \xi_{\mu} \xi_{\nu} |_{r=r_+},
\]

where \( \kappa \) is surface gravity, \( \xi^\mu = \partial^\mu t \) is a Killing vector for static spherically symmetric case.

Now, on inserting the temperature \( T_+ \) associated with the Bardeen massive black hole can be calculated as

\[
T_+ = \frac{1}{4\pi r_+(e^2 + r_+^2)} \left[ r_+^2 - e^2 + \frac{m^2}{2} \left( c_2 e^2 r_+ (2r_+^2 - e^2) + 2r_+ c c_1 (r_+^2 - e^2) \right) + \frac{3r_+^2}{l^2} \right],
\]

and the AdS Bardeen black hole [37], when \( m \to 0 \), one recovers from (21) the temperature of Bardeen black hole [28]

\[
T_+ = \frac{1}{4\pi r_+} \left( \frac{r_+^2 - e^2}{r_+^2 + e^2} \right).
\]

In turns the temperature reduces to \( T_+ = 1/4\pi r_+ \) of the Schwarzschild black holes, can be obtained by taking \( e = 0 \) in (22). The Fig. 3 displays the behaviour of temperature for various values of \( e \) and \( m \), which shows that the temperature grows to a maximum \( T_{max} \) then it drop to minimum value and increase again. The maximum temperature \( T_{max} \) depends on charge \( e \) and is shown in the Table II. Thus a remarkable result is that, unlike the Schwarzschild black hole, for the radius corresponding to the temperature of Bardeen AdS massive black hole increases with \( e \) and decrees with the mass parameter \( m \) (See in the Fig. 4). A maximum of Hawking temperature occurs at the critical radius shown in Table II. It turns out that the maximum temperature decreases with increase in the values of \( e \) and \( m \) which diverges when the horizon radius shrinks to zero.

One can easily obtain the following formula by using the first law of thermodynamics \( dM_+ = T_+ dS_+ + \phi dE \) for the entropy of black hole

\[
S_+ = \int T_+^{-1} dM_+ = \int T_+^{-1} \frac{dM_+}{dr_+} dr_+.
\]

Now, substituting the value of \( M_+ \) and \( T_+ \) form Eqs. (19), (21) into Eq. (23), we obtained the entropy of Bardeen AdS-massive black hole as

\[
S_+ = \pi \left[ (r_+^2 - 2e^2)(r_+^2 + e^2)^{1/2} + 3e^2 r_+ \log[r_+ + (r_+^2 + e^2)^{1/2}] \right].
\]
FIG. 2: The plot of temperature vs horizon radius $r_+$ with different values of magnetic charge $e$ (Left) and different values of mass parameter $m$ (Right) with fixed value of $c = 1, c_1 = -1, c_2 = 1$ and $l = 20.$

| $m=0.85$ | $e=0.40$ |
|----------|----------|
| $e = 0.55$ | $m = 0.25$ |
| $e = 0.57$ | $m = 0.50$ |
| $e = 0.592$ | $m = 0.75$ |
| $e = 0.63$ | $m = 1.0$ |
| $r_+^T$ | $T_{+}^{Max}$ |
| 1.485 | 0.0296 |
| 1.531 | 0.0281 |
| 1.1578 | 0.0266 |
| 1.647 | 0.0243 |

TABLE I: Maximum Hawking temperature ($T_{+}^{max}$) at critical radius ($r_+^T$) for the 4D AdS Bardeen massive black hole.

Thus, the area law is no longer valid for regular black holes. We can also be derived the temperature according to the entropy using the first law of thermodynamics.

The thermodynamic or local stability of a black hole is performed by studying the behaviour of its heat capacity ($C_+$). The positive ($C_+ > 0$) heat capacity indicates that the black hole is stable; when it is negative ($C_+ < 0$), the black hole is said to be unstable [27–29]. The heat capacity of the black hole is defined as

$$C_+ = \frac{\partial M_+}{\partial T_+} = \left( \frac{\partial M_+}{\partial r_+} \right) \left( \frac{\partial r_+}{\partial T_+} \right).$$

The heat capacity of Bardeen -AdS-massive black hole, by using Eqs. (19), (21), and (25), we obtained

$$C_+ = \frac{\pi (e^2 + r_+)^{5/2}(6r_+^4 + 2l^2(2r_+^2 - e^2) + m^2l^2(2c_2^2c_2r_+^2 + 2c_1r_+^3 - 4c_2^2c_2e^2 - cc_1e^2))}{r_+(2e^2l^2 + 7e^2l^2r_+^2 + 9e^2r_+^4 - l^2r_+^4 + 3r_+^6)} + m^2l^2(2c_2^2c_2e^4 + 7c_2^2c_2r_+^2 + cc_1e^2r_+^3 - c^2c_2r_+^4).$$

(26)
FIG. 3: The plot of specific heat vs horizon radius with fixed value of $c = 1$, $c_1 = -1$, $c_2 = 1$, $e = 0.40$ and $l = 20$.

It can be seen clearly, that the heat capacity depends on the charge $e$, mass parameter $m$, and cosmological constant $\Lambda$. The heat capacity is plotted in Fig. 3 for different values of $e$. The heat capacity is positive (negative) $r_+ < r_C$ ($r_+ > r_C$) suggesting thermodynamic stability (instability) of smaller (larger) regular black holes. The heat capacity is discontinuous at $r_+ = r_C$ which means the second order phase transition occurs $[20, 40]$. Interestingly, the discontinuity of the heat capacity occurs at $r_+ = 1.439$, point at which point the Hawking temperature has the maximum value $T_+ = 0.0296$ for $e = 0.55$. Hence the phase transition occurs from the lower to higher mass black holes corresponding from positive to negative heat capacity of the black hole. In the absence of mass parameter $m$, it reduce to the expression for heat capacity of Bardeen black hole $[28]$. The heat capacity for Schwarzschild black hole $C_+ = -2\pi r_+^3$. in the limit $m = 0$. 


IV. THERMODYNAMIC STABILITY AND PHASE DIAGRAMS

Now we study the phase transition of the Bardeen AdS massive black hole by using the P-V criticality and phase diagrams. The thermodynamical relation between the pressure and cosmological constant given as

$$ P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}. \quad (27) $$

The mass of the black hole interpretes enthalpy of thermodynamical system. The equation of state can by obtained by using the Eq. (21) and Eq. (27) as

$$ P_+ = \frac{1}{2r_+T_+} \left(1 + \frac{e^2}{r_+^2}\right) - \frac{1}{4\pi r_+^2} \left(1 - \frac{e^2}{2r_+^2}\right) + \frac{m^2}{16\pi r_+^4} \left(cc_1r_+(e^2 - 2r_+^2) + 2c^2 c_2(r_+^2 - 2e^2)\right), \quad (28) $$

and the corresponding thermodynamics volume is

$$ V = \left(\frac{\partial H_+}{\partial P_+}\right)_{S_+, e} = \frac{4\pi}{3} \left(r_+^2 + e^2\right)^{\frac{3}{2}}. \quad (29) $$

At the inflection point, we can determine the critical temperature and critical pressure by following equations

$$ \frac{\partial P_+}{\partial r_+} = 0, \quad \frac{\partial^2 P_+}{\partial r_+^2} = 0. \quad (30) $$

Substitution Eq. (28) into Eq. (30), we find the critical points and the horizon radius satisfy the following equation

$$ \frac{24e^2 + 30e^2r_+^2 - 2r_+^4 + m^2[2c_2c^2(24e^4 + 30r_+^4e^2 - 2r_+^4) + 9cc_1e^2r_+^3]}{128\pi^3r_+^8(e^2 + r_+^2)(3e^2 + r_+^2)}. \quad (31) $$

The Eq. (31) can not be solved analytically, so we can calculate the critical radius $r_+$, critical pressure $P_+$ and temperature $T_+$ numerically and the numerical results are presented in Table III and Table IIII for different value of mass parameter $m$ and magnetic charge $e$. We can see that the critical radius $r_+$ increases with magnetic charge $e$ and decrease with the mass parameter $m$. The universal ratio $P_c r_c / T_c$ are increasing function of the magnetic charge $e$ and massive parameter $m$. In order to obtained the phase transition of the black holes an analog with the vander Walls phase transition, we can identify the free energy of the black hole. to see the effect of magnetic charge $e$ and mass parameter $m$ on the phase structure of the system, we fix the magnetic charge $e$ first and vary the mass parameter $m$. The critical pressure and critical temperature increase with critical radius (see Fig. [I]) and the opposite nature show when we fix the mass parameter $m$ and vary of magnetic charge $e$. 

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TABLE II: The table for critical temperature $T_C$, critical pressure $P_C$ and $P_C r_C/T_C$ corresponding different value of mass $m$ with fixed value of $c = 1, c_1 = -1, c_2 = 1, e = 0.1$.

| $m$ | $r_C$  | $T_C$  | $P_C$  | $P_C r_C/T_C$ |
|-----|--------|--------|--------|---------------|
| 0   | 0.3970 | 0.2513 | 0.1161 | 0.1821        |
| 1   | 0.3864 | 0.4425 | 0.2489 | 0.2173        |
| 2   | 0.3803 | 1.0174 | 0.6488 | 0.2425        |
| 3   | 0.3782 | 1.9758 | 1.3158 | 0.2518        |
| 4   | 0.3774 | 3.3178 | 2.2496 | 0.2558        |
| 5   | 0.3770 | 5.0432 | 3.4504 | 0.2579        |

TABLE III: The table for critical temperature $T_C$, critical pressure $P_C$ and $P_C r_C/T_C$ corresponding to the different value of magnetic charge $e$ with fixed value of $c = 1, c_1 = -1, c_2 = 1, m = 1$.

| $e$ | $r_C$  | $T_C$  | $P_C$  | $P_C r_C/T_C$ |
|-----|--------|--------|--------|---------------|
| 0.1 | 0.3864 | 0.4425 | 0.2489 | 0.1821        |
| 0.2 | 0.7524 | 0.1917 | 0.0667 | 0.2617        |
| 0.3 | 1.099  | 0.1083 | 0.0317 | 0.3216        |
| 0.4 | 1.427  | 0.06697| 0.0191 | 0.4069        |
| 0.5 | 1.738  | 0.04231| 0.0131 | 0.5381        |

In $T_+ - r_+$ plots we can see that the three black hole (small, intermediate and large) for the $P < P_c$, $P = P_c$ and $P > P_c$ for Bardeen AdS massive black hole with the same magnetic charge $e$ and the mass parameter $m$ for a certain range of temperature. The small and large are stable but intermediate black hole is unstable, since the heat capacity is negative (see Fig. 3). When $T_+ < T_*$ the small black hole and $T_+ > T_*$ corresponding to large black hole due to small free energy. We can transit one phase to other phase at critical temperature due to same free energy. In Fig. 4 (middle) isotherms represents the first order $T_+ < T_c$ and second order phase transition $T_+ = T_*$, which is obtained from the free energy diagram (see Fig. 4 (lower)) and the corresponding temperature is $T_* = 0.335$. In $G_+ - T_+$ plots the appearance of characteristic swallow tail in show that the obtained values are critical ones in which the phase transition take place. In Fig. 4 (lower), we can see that swallow tail
shape $P < P_c$, for the first order phase transition and $P = P_c = 0.0249$ for the second order phase transition.

FIG. 4: The plots of pressure and temperature vs horizon radius and Gibbs free energy ($G_+$) vs temperature ($T_+$) for $e = 0.1$ (left) and $e = 0.5$ (right) with fixed values of $c = 1, c_1 = -0.75, c_2 = 0.75$ and $m = 1$.

Fig. 5 shows the effects of variation of mass parameter $m$ and magnetic charge $e$ on critical values. The isobars are increase function of mass parameter $m$ and decreasing
FIG. 5: The plots of pressure and temperature vs horizon radius and Gibbs free energy ($G_+$) vs temperature ($T_+$) for $e = 0.1$ (left) and $e = 0.5$ (right) with fixed values of $c = 1, c_1 = -0.75, c_2 = 0.75$ and $m = 1$.

with the magnetic charge $e$ respectively. The gap between the two sub-critical bar are the increases mass parameter $m$ as well as magnetic charge $e$. The sub-critical isobar is the region, where the phase transition take place. The only difference is that Gibbs free energy
of two phases are increasing function of massive parameter $m$ and decreasing function of magnetic charge $e$. Interestingly, the universal relation $P_c r_c / T_c$, shows that the the effects of variation of mass parameter $m$ and magnetic charge $e$ are decreasing function massive parameter whereas critical temperature and pressure are increasing function of it.

V. CONCLUSIONS

In this paper, we have presented exact solution of Bardeen AdS black holes in massive gravity with a negative cosmological constant, thereby generalizing bardeen black holes which are included as a special case ($e = 0, l^2 \rightarrow \infty$). The AdS Bardeen black holes are characterized by analyzing horizons, which at most could be four, viz. inner Cauchy, event, cosmological and massive horizons. We have analysed the thermodynamical properties and phase structure of Bardeen AdS massive black holes. Despite complicated solutions, exact expression for the thermodynamical quantities like the black hole mass, Hawking temperature, entropy and free energy at event horizon $r_+$ are obtained. The thermodynamical stability of the black holes is also analysed by studying the heat capacity. The entropy (21) of the black holes is modified due to the magnetic charge $e$ and the mass parameter $m$ resulting area law $S = A/4$ no longer valid. The phase transition is detectable by the divergence of the heat capacity ($C_+$) at a critical radius $r_c$ (changes with $e, m$), such that the black hole is stable with positive heat capacity ($C_+ > 0$), and unstable with negative heat capacity ($C_+ < 0$).

The phase transition of Bardeen AdS massive black holes have also been studied in the extended phase by considering the cosmological constant as thermodynamic pressure. It is seen that the thermodynamic volume is independent of mass parameter $m$ and magnetic charge $e$. The phase diagram shows that the obtained critical values of pressure and temperature are one in which phase transition take place. Interestingly, it is pointed out that the nature of massive parameter $m$ and magnetic charge $e$ are opposite to each other. The critical temperature and pressure were found highly sensitive to the variation parameters ($m$ and $e$).

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