SYMPLECTIC STRUCTURES ON MODULI SPACES OF PARABOLIC HIGGS BUNDLES AND HILBERT SCHEME

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Abstract. Parabolic triples of the form \((E_*, \theta, \sigma)\) are considered, where \((E_*, \theta)\) is a parabolic Higgs bundle on a given compact Riemann surface \(X\) with parabolic structure on a fixed divisor \(S\), and \(\sigma\) is a nonzero section of the underlying vector bundle. Sending such a triple to the Higgs bundle \((E_*, \theta)\) a map from the moduli space of stable parabolic triples to the moduli space of stable parabolic Higgs bundles is obtained. The pull back, by this map, of the symplectic form on the moduli space of stable parabolic Higgs bundles will be denoted by \(d\Omega'\). On the other hand, there is a map from the moduli space of stable parabolic triples to a Hilbert scheme \(\text{Hilb}^\delta(Z)\), where \(Z\) denotes the total space of the line bundle \(K_X \otimes \mathcal{O}_X(S)\), that sends a triple \((E_*, \theta, \sigma)\) to the divisor defined by the section \(\sigma\) on the spectral curve corresponding to the parabolic Higgs bundle \((E_*, \theta)\). Using this map and a meromorphic one–form on \(\text{Hilb}^\delta(Z)\), a natural two–form on the moduli space of stable parabolic triples is constructed. It is shown here that this form coincides with the above mentioned form \(d\Omega'\).

1. Introduction

In [2], we proved that the two–form on the moduli space of triples of the form \((E, \theta, \sigma)\), where \((E, \theta)\) is a stable Higgs bundle on Riemann surface \(X\) and \(\sigma\) a nonzero section of \(E\), obtained by pulling back the natural symplectic form on the moduli space of stable Higgs bundles coincides with the pullback of the symplectic structure on the Hilbert scheme of zero dimensional subschemes of the total space of of the holomorphic cotangent bundle \(K_X\) of \(X\). Our aim here is to establish an analogous result in the context of parabolic triples.

Let \(X\) be a compact connected Riemann surface of genus \(g\). Fix a finite subset \(S\) of \(X\). A parabolic vector bundle of rank two with parabolic structure over \(S\) consists of a holomorphic vector bundle \(E\), a line \(F_s \subset E_s\), and a real number \(0 < \lambda_s < 1\) for each \(s \in S\). A Higgs field on this parabolic bundle is a holomorphic section \(\theta\) of the \(\text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)\) such that \(\theta(s)\) is nilpotent with respect to the flag \(0 = F^0 \subset F^1 = F_s \subset F^2 = E_s\) of the fiber \(E_s\) for each \(s \in S\).

We fix the parabolic weights \(\{\lambda_s\}_{s \in S}\) and consider parabolic Higgs bundles of rank two and fixed positive degree \(d\), with \(d > 6(g - 1) + \#S\). Let \(\mathcal{M}^s_H\) denote the moduli space of stable parabolic Higgs bundles. For such stable parabolic Higgs bundles the dimension of the space of all holomorphic sections of the underlying vector bundle remains constant
over the moduli space (Lemma 3.1). Therefore, there is a projective bundle

\[ \phi : \mathbb{P}_H \rightarrow \mathcal{M}_H^* \]

whose fiber over any point representing a parabolic Higgs bundle \(((E, \{F_s\}), \theta)\) is the projective space \(\mathbb{P}H^0(X, E)\). In other words, \(\mathbb{P}_H\) is the moduli space of triples of the form \(((E, \{F_s\}), \theta, \sigma)\), where \(((E, \{F_s\}), \theta)\) is a stable parabolic Higgs bundle and \(\sigma\) is a nonzero section of \(E\). See [1] and [5], where the notion of triples were introduced, for a detailed study and many interesting results on triples.

The moduli space \(\mathcal{M}_H^*\) has a natural holomorphic one–form, which we call \(\Omega\), such that \(d\Omega\) is a symplectic form on \(M^s_H\). The total space of the holomorphic cotangent bundle of the moduli space of stable parabolic bundles sits inside \(\mathcal{M}_H^*\) as a Zariski open dense subset. The restriction of \(\Omega\) to this open subset coincides with the canonical one–form on the total space of any cotangent bundle. The main result proved here relates the form \(\phi^*\Omega\) on \(\mathbb{P}_H\) with a certain one–form on a Hilbert scheme.

Let \(Z\) denote the surface defined by the total space of the line bundle \(K_X \otimes \mathcal{O}_X(S)\) over \(X\). Given a stable parabolic Higgs bundle \(((E, \{F_s\}), \theta)\), there is a spectral curve, which is a divisor on \(Z\) and a rank one torsionfree sheaf \(\mathcal{L}\) on it such that \(\gamma_*\mathcal{L} \cong E\), where \(\gamma\) is the projection map of the spectral curve to \(X\). Since \(\gamma\) is a finite map, we have \(H^0(X, E)\) identified with the space of all sections of \(\mathcal{L}\). Considering the divisor of the section of \(\mathcal{L}\) corresponding to \(\sigma \in H^0(X, E) \setminus \{0\}\) we get a map from \(\mathbb{P}_H\) to the Hilbert scheme \(\text{Hilb}^\delta(Z)\) of zero dimensional subschemes of \(Z\) of length \(\delta = d + 2(g - 1) + \#S\). This map, which we will denote by \(f\), sends a parabolic triple \(((E, \{F_s\}), \theta, \sigma)\) to the divisor on the spectral curve for \(((E, \{F_s\}), \theta)\) defined by \(\sigma\). Note that using the inclusion map of a spectral curve in \(Z\), a divisor on a spectral curve is a zero dimensional subscheme of \(Z\). (See Section 3.1 for the details.)

Using the fact that \(Z\) is the total space of \(K_X \otimes \mathcal{O}_X(S)\), there is a natural meromorphic one–form \(\Omega_\delta\) on \(\text{Hilb}^\delta(Z)\). The pullback \(f^*\Omega_\delta\) is a holomorphic one–form on \(\mathbb{P}_H\).

We prove that \(f^*\Omega_\delta\) coincides with \(\phi^*\Omega\) (Theorem 3.2).

Although we have restricted ourselves to rank two parabolic bundles, the extension of Theorem 3.2 to higher rank case is straight forward. The reason for restriction to rank two case is the ensuing notational simplification.

2. Preliminaries

2.1. Parabolic Higgs bundles. Let \(X\) be a compact connected Riemann surface of genus \(g\). Fix a finite subset

\[ S := \{s_1, s_2, \cdots, s_n\} \subset X. \]

If \(g = 0\), then take \(n \geq 4\), and \(n \geq 1\) if \(g = 1\).
A parabolic vector bundle of rank two over $X$ with parabolic structure over $S$ consists of the following [8]:

1. a holomorphic vector bundle $E$ of rank two over $X$;
2. for each point $s \in S$, a line $F_s \subset E_s$ of the fiber $E_s$;
3. for each point $s \in S$, a real number $\lambda_s$ with $0 < \lambda_s < 1$.

The numbers $\{\lambda_s\}$ are called parabolic weights. See [8] for the definition of parabolic (semi)stability.

We fix real numbers $\{\lambda_s\}_{s \in S}$ and an integer $d$. Henceforth, by a parabolic vector bundle we will always mean a parabolic vector bundle over $X$ of rank two and degree $d$ with parabolic structure over $S$ and with parabolic weight $\lambda_s$ for each $s \in S$.

Let $K_X$ denote the holomorphic cotangent bundle of $X$. A Higgs structure on a parabolic vector bundle $E := (E, \{F_s\})$ is a holomorphic section

\[ \theta \in H^0(X, \text{End}(E) \otimes K_X \otimes O_X(S)) \]

with the property that for each $s \in S$, the image of the homomorphism

\[ \theta(s) : E_s \rightarrow (E \otimes K_X \otimes O_X(S))_s \]

is contained in the subspace $F_s \otimes (K_X \otimes O_X(S))_s \subset (E \otimes K_X \otimes O_X(S))_s$ and $\theta(s)(F_s) = 0$ [6], [3], [4]. In other words, $\theta(s)$ is nilpotent with respect to the flag $0 \subset F_s \subset E_s$.

A parabolic Higgs bundle $(E_s, \theta)$ as above is called stable if for every line subbundle $L$ of $E$ with

\[ \theta(L) \subset L \otimes K_X \otimes O_X(S) \subset E \otimes K_X \otimes O_X(S) \]

the following inequality is satisfied

\[ \text{degree}(L) + \sum_{s \in S'} \lambda_s < \frac{\text{par-deg}(E_s)}{2}, \]

where $S' := \{s \in S \mid L_s = F_s\}$; if the strict inequality is replaced by partial inequality, then $(E_s, \theta)$ is called semistable.

The moduli space of semistable parabolic Higgs bundles will be denoted by $\mathcal{M}_H$. It is an irreducible normal quasiprojective variety, and $\mathcal{M}_H^s$ is a Zariski open smooth subvariety of it.

2.2. Symplectic structure on moduli space. The moduli space $\mathcal{M}_H^s$ has a natural holomorphic symplectic structure [3] Section 6]; we will briefly recall it here.

Take a stable parabolic Higgs bundle $(E_s, \theta) := ((E, \{F_s\}), \theta)$ represented by a point in $\mathcal{M}_H^s$. Define

\[ \text{End}^1(E) \subset \text{End}(E) \]

by the condition that for any $s \in S$ and $v \in \text{End}^1(E)_s$ we have $v(F_s) \subset F_s$. Let

\[ \text{End}^0(E) \subset \text{End}^1(E) \]
be defined by the condition $v(F_s) = 0$. Consider the two term complex $C$, of sheaves defined by

$$
C_0 := \text{End}^1(E) \xrightarrow{[-\theta]} C_1 := \text{End}^0(E) \otimes K_X \otimes \mathcal{O}_X(S)
$$

where $C_i$ is at the $i$th position.

The tangent space $T_{(E, \theta)} \mathcal{M}^s_H$ of the variety $\mathcal{M}^s_H$ at the point represented by the parabolic Higgs bundle $(E, \theta)$ is identified with the hypercohomology $H^1(C)$ \cite[Lemma 6.1]{3}. Consider the homomorphism

$$
\begin{array}{c}
\text{End}^1(E) \xrightarrow{[-\theta]} C_1 \\
\downarrow \\
0
\end{array}
$$

This induces a homomorphism

$$
(2.3) \quad f : H^1(C) \longrightarrow H^1(X, \text{End}^1(E)).
$$

Now observe that $\text{End}^1(E)^* \cong \text{End}^0(E) \otimes \mathcal{O}_X(S)$ with the duality pairing defined by

$$(\omega, \alpha) \mapsto \text{trace}(\omega \alpha) \in \mathbb{C},$$

where $\alpha \in \text{End}^1(E)_x$ and $\omega \in (\text{End}^0(E) \otimes \mathcal{O}_X(S))_x$, and $x$ is any point of $X$. Now the Serre duality says

$$
H^1(X, \text{End}^1(E))^* \cong H^0(X, \text{End}^0(E) \otimes K_X \otimes \mathcal{O}_X(S)).
$$

Consequently, we have a functional $\theta' \in H^1(C)^*$ defined by $\beta \mapsto (\theta, f(\beta))$, where $f$ is constructed in (2.3).

Let $\Omega$ denote the one–form on the variety $\mathcal{M}^s_H$ that sends any tangent vector $\beta \in T_{(E, \theta)} \mathcal{M}^s_H$, where $(E, \theta) \in \mathcal{M}^s_H$, to $\theta'(\beta)$ constructed above.

The two–form $d\Omega$ is a symplectic form on $\mathcal{M}^s_H$. The restriction of $d\Omega$ to the Zariski open subset of $\mathcal{M}^s_H$ defined by the total space of the cotangent bundle of the moduli space of parabolic bundles coincides with the canonical symplectic structure on cotangent bundles. (See \cite{3}, \cite{4}.)

2.3. Spectral data for parabolic Higgs bundles. Let $(E, \theta)$ be a parabolic Higgs bundle. So we have $\text{trace}(\theta) \in H^0(X, K_X)$ and

$$
\text{trace}(\theta^2) \in H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S))
$$

since $\theta(s)$ is nilpotent for each $s \in S$. Set

$$
(2.4) \quad \mathcal{H} := H^0(X, K_X) \oplus H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).
$$

Hitchin defined a map

$$
(2.5) \quad \psi : \mathcal{M}_H \longrightarrow \mathcal{H}
$$

that sends any semistable parabolic Higgs bundle $(E, \theta)$ to $(\text{trace}(\theta), \text{trace}(\theta^2)) \in \mathcal{H}$ \cite{4}, \cite{5}, \cite{6}. This map $\psi$ is known as the Hitchin map, and $\mathcal{H}$ is known as the Hitchin space.
Let $Z$ denote the total space of the line bundle $K_X \otimes \mathcal{O}_X(S)$, which is a quasiprojective complex surface. The natural projection of $Z$ to $X$ will be denoted by $\gamma$. Since $S$ is an effective divisor, there is a natural homomorphism from $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$ to $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$; this homomorphism will be denoted by $q$.

Take any point $(\alpha, \beta) \in \mathcal{H}$. Consider the map from $Z$ to the total space of the line bundle $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$ defined by

\[(2.6) \quad z \mapsto z \otimes z + q(z \otimes \alpha(\gamma(z))) + q(\beta(\gamma(z))) \in (K_X^{\otimes 2} \otimes \mathcal{O}_X(2S))_{\gamma(z)}.
\]

The inverse image (in $Z$) by this map of the zero section of $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$ is the spectral curve associated to the point $(\alpha, \beta)$ of the Hitchin space (see [7], [4]).

For any point $h = (\alpha, \beta) \in \mathcal{H}$, the corresponding spectral curve will be denoted by $Y_h$. The restriction to $Y_h$ of the projection $\gamma$ to $X$ will also be denoted by $\gamma$. Note that the map

\[(2.7) \quad \gamma : Y_h \longrightarrow X
\]

is of degree two. This map is evidently ramified over every point of $S$.

Given a parabolic Higgs bundle, there is a corresponding spectral curve and a torsionfree sheaf on it [7], [4]. To describe the construction, take a semistable parabolic Higgs bundle $(E_s, \theta) \in \mathcal{M}_H$ on $X$. The spectral curve associated to it is the one defined by $\psi((E_s, \theta)) \in \mathcal{H}$ by the above construction, where $\psi$ is the Hitchin map defined in (2.5). Denote the point $\psi((E_s, \theta))$ by $h$.

There is a torsionfree sheaf $\mathcal{L}$ of rank one on the spectral curve $Y_h$ such that

\[(2.8) \quad \gamma_* \mathcal{L} \cong E
\]

(the underlying vector bundle of the parabolic bundle), where $\gamma$ as in (2.7). The spectral curve can be thought of as the eigenvalues of the endomorphism $\theta$. The sheaf $\mathcal{L}$ is defined by the corresponding eigenvectors (see [7] for the details). Since the map $\gamma$ is ramified over any point of $s \in S$, the direct image $\gamma_* \mathcal{L}$ has a filtration over $s$. This filtration is defined by the order of the vanishing (at $\gamma^{-1}(s)$) of a (locally defined) section of $\mathcal{L}$. In the isomorphism of $\gamma_* \mathcal{L}$ with $E$, the filtration of $\gamma_* \mathcal{L}$ at any $s \in S$ coincides with the filtration $F_s \subset E_s$ for the parabolic structure.

3. Parabolic triples and Hilbert scheme

In Section 2.1 we fixed the degree of a parabolic vector bundle to be $d$.

Henceforth, we will assume that the integer $d$, which is the degree of a parabolic vector bundle, satisfies the condition

\[d > 6g - 6 + n\]

where $n = \# S$. 
Lemma 3.1. For any semistable parabolic Higgs bundle \((E_*, \theta) \in \mathcal{M}_H\) over \(X\), we have \(H^1(X, E) = 0\), where \(E\) is the underlying vector bundle. Consequently, \(\dim H^0(X, E) = d + 2(1 - g)\).

Proof. Take any \((E_*, \theta) \in \mathcal{M}_H\). Let \(E\) be the underlying vector bundle for the parabolic vector bundle \(E_*\). Since \(H^1(X, E) \cong H^0(X, E^* \otimes K_X)^*\), it suffices to show \(H^0(X, E^* \otimes K_X) = 0\). Assume that \(\tau \in H^0(X, E^* \otimes K_X) \setminus \{0\}\) is a nonzero section.

So \(\tau\) defines a nonzero homomorphism from \(E\) to \(K_X\), which will be denoted by \(\bar{\tau}\). The kernel of \(\tau\) is a torsionfree coherent subsheaf of \(E\). Therefore, it defines a line bundle over \(X\), which will be denoted by \(L\). Now we have a diagram

\[
\begin{array}{c}
L \\ \downarrow \theta \\
\end{array} \quad \begin{array}{c}
E \\ \downarrow \tau \\
K_X \\
\end{array}
\]

We will show that the composition \((\bar{\tau} \otimes \text{Id}) \circ \theta \circ \iota = 0\). To prove this, first note that the top sequence in (3.1) shows that

\[
\deg(L) \geq \deg(E) - \deg(K_X) = d - 2g + 2.
\]

On the other hand, \(\deg(K_X^{\otimes 2} \otimes \mathcal{O}_X(S)) = 2g - 4 + n\), where \(n\) is the cardinality of the set \(S\). Since \(d\) is assumed to be at least \(6g - 5 + n\), we have

\[
\deg(L) > \deg(K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).
\]

Consequently, there is no nonzero homomorphism from \(L\) to \(K_X^{\otimes 2} \otimes \mathcal{O}_X(S)\). In particular, the composition \((\bar{\tau} \otimes \text{Id}) \circ \theta \circ \iota = 0\).

Let \(L'\) denote the line subbundle of \(E\) generated by \(L\) (note that \(\iota\) may not be fiberwise injective). Since \((\bar{\tau} \otimes \text{Id}) \circ \theta \circ \iota = 0\), it follows immediately, that \(\theta(L') \subseteq L' \otimes K_X \otimes \mathcal{O}_X(S)\). Finally, we have

\[
\deg(L') \geq \deg(L) \geq d - 2g + 2 = \frac{d}{2} + \frac{1}{2}(d - 4g + 4) > \frac{d}{2} + \frac{n}{2} = \frac{\text{par-deg}(E_*)}{2}
\]

where the second inequality was obtained in (3.2) and the third one follows from the assumption \(d > 6g - 6 + n\); the first inequality follows from the fact that there is a nonzero homomorphism from \(L\) to \(L'\). The above inequality shows that the line subbundle \(L'\) of \(E\) contradicts the semistability condition of the parabolic Higgs bundle \((E_*, \theta)\).

Therefore, we have \(H^1(X, E) = 0\). Now the Riemann–Roch says that \(\dim H^0(X, E) = d + 2(1 - g)\). This completes the proof of the lemma. \(\square\)

The above lemma says that \(\dim H^0(X, E)\) remains constant over \(\mathcal{M}_H^s\). Therefore, there is a natural projective bundle

\[
\phi : \mathbb{P}_H \rightarrow \mathcal{M}_H^s
\]
of relative dimension \(d - 2g + 1\) such that the fiber over any point \((E_*, \theta) \in \mathcal{M}_H\) is the projective space \(\mathbb{P}H^0(X, E)\) consisting of all lines in \(H^0(X, E)\); as before, \(E\) denotes the underlying vector bundle for the parabolic bundle \(E_*\).

Therefore, \(\mathbb{P}_H\) is the moduli space of triples of the form \((E_*, \theta, \sigma)\), where \((E_*, \theta)\) is a stable parabolic Higgs bundle and \(\sigma \in H^0(X, E) \setminus \{0\}\) a nonzero section.

By a parabolic triple we will mean a triple \((E_*, \theta, \sigma)\) of the above type. Consequently, \(\mathbb{P}_H\) is the moduli space of all parabolic triples.

In Section 2.2 we defined a symplectic structure \(d\Omega\) on \(\mathcal{M}_s^H\). Define the algebraic one–form

\[
\Omega' := \phi^*\Omega
\]

on \(\mathbb{P}_H\), where \(\phi\) is defined in (3.3). So,

\[
d\Omega' = \phi^*d\Omega
\]

is the pullback to \(\mathbb{P}_H\) of the symplectic form \(\Omega\) on \(\mathcal{M}_s^H\).

3.1. Parabolic triples and spectral data. Take a parabolic triple \((E_*, \theta, \sigma) \in \mathbb{P}_H\). Its image \(\psi((E_*, \theta)) \in \mathcal{H}\) will be denoted by \(h\), where \(\psi\) is the Hitchin map defined in (2.5).

As we saw in Section 2.3, the parabolic Higgs bundle \((E_*, \theta)\) gives a spectral curve \(Y_h\) and a torsionfree sheaf \(L\) of rank one on \(Y_h\). It was noted in (2.8) that \(\gamma_\ast L \cong E\), where \(\gamma\) is the projection of \(Y_h\) to \(X\). Now, since \(\gamma\) is a finite map, the natural homomorphism

\[
H^i(Y_h, L) \longrightarrow H^i(X, \gamma_\ast L) = H^i(X, E)
\]

is an isomorphism for all \(i \geq 0\). Therefore, \(\mathbb{P}H^0(Y_h, L) \cong \mathbb{P}H^0(X, E)\). The point in \(\mathbb{P}H^0(Y_h, L)\) corresponding the point \(\sigma \in \mathbb{P}H^0(X, E)\) will be denoted by \(\sigma'\). In particular, \(\sigma'\) is a divisor on \(Y_h\).

We will calculate the degree of the divisor defined by \(\sigma'\). The Hitchin map \(\psi\) in (2.3) is a complete integrable system for the symplectic structure \(d\Omega\) on \(\mathcal{M}_H\) and the fiber of \(\psi\) over any \(h' \in \mathcal{H}\) is the Jacobian of the corresponding spectral curve \(Y_{h'}\). Consequently, the genus of \(Y_{h'}\) coincides with \(\text{dim} \mathcal{M}_H/2 = 4g - 3 + n\). Since

\[
\text{degree}(L) - (4g - 3 + n) + 1 = \chi(L) = \chi(E) = d + 2(1 - g)
\]

we conclude that \(\text{degree}(L) = d + n + 2(g - 1)\). Hence the degree of the divisor defined by the section \(\sigma'\) of \(L\), which coincides with the degree of \(L\), is \(d + n + 2(g - 1)\), where \(n = \#S\).

Set \(\delta = d + n + 2(g - 1)\). Let \(\text{Hilb}^\delta(Z)\) denote the Hilbert scheme of zero dimensional subschemes of the surface \(Z\) (the total space of \(K_X \otimes O_X(S)\)) of length \(\delta\).

The divisor of \(\sigma'\) is a zero dimensional subscheme of \(Y_h\) of length \(\delta\). Taking the image of \(\sigma'\) by the inclusion map of the spectral curve \(Y_h\) in \(Z\), we get a zero dimensional subscheme
of $Z$ of length $\delta$. Therefore, we have an element of $\text{Hilb}^\delta(Z)$. Let

$$f : \mathbb{P}_H \longrightarrow \text{Hilb}^\delta(Z)$$

be the map that sends any parabolic triple to the zero dimensional subscheme of $Z$ defined by the divisor of the corresponding section on the spectral curve for the underlying parabolic Higgs bundle. In other words, $f$ sends any $(E_*, \theta, \sigma) \in \mathbb{P}_H$ to the element of $\text{Hilb}^\delta(Z)$ defined by the divisor of $\sigma'$ on $Y_h$.

### 3.2. Forms on moduli of triples.

Using the map $f$ defined in (3.5) we will construct an algebraic one–form on $\mathbb{P}_H$, and for that we will first show the existence of a natural meromorphic one–form on $\text{Hilb}^\delta(Z)$.

We start by observing that the variety $Z$ has a natural meromorphic one–form with pole, of order at most one, along the divisor $\gamma^{-1}(S)$, where $\gamma$, as before is the projection of $Z$ to $X$. The $\gamma^*\mathcal{O}_X(S)$ valued one–form sends any tangent vector $v \in T_zZ$ to $z \otimes d\gamma(z)(v)$, where $d\gamma(z) : T_zZ \longrightarrow T_{\gamma(z)}X$ is the differential of $\gamma$ at $z$. Note that since $z$ is an element of the fiber $(K_X \otimes \mathcal{O}_X(S))_{\gamma(z)}$, the tensor product $z \otimes d\gamma(z)(v)$ gives an element of the fiber $(\mathcal{O}_X(S))_{\gamma(z)}$ after contracting $(K_X)_{\gamma(z)}$ with $T_{\gamma(z)}X$. Since $S$ is an effective reduced divisor, a $\gamma^*\mathcal{O}_X(S)$ valued one–form on $Z$ is a meromorphic one–form on $Z$ with a pole along $\gamma^{-1}(S)$ of order at most one.

The meromorphic one–form on $Z$ defined above will be denoted by $\Omega_Z$. Using $\Omega_Z$, a meromorphic one–form will be constructed on Hilbert scheme $\text{Hilb}^k(Z)$ of zero dimensional subschemes of $Z$ of length $k$, where $k \geq 1$.

Consider the Zariski open dense subset $U_k \subset \text{Hilb}^k(Z)$ consisting of distinct $k$ points of $Z$. Let

$$z = \{z_1, z_2, \ldots, z_k\} \in \text{Hilb}^k(Z)$$

be a point of $U_k$, that is, all $z_i$ are distinct. Then we have

$$T_z \text{Hilb}^k(Z) = \bigoplus_{i=1}^k T_{z_i}Z.$$
as the above sum makes sense on the complement. More precisely, the pole of the one–
form on $U_k$ is over this divisor, and the order of the pole is one. Since $U_k$ is a Zariski
open dense subset of $\text{Hilb}^k(Z)$, the meromorphic one–form on $U_k$ defines a meromorphic
one–form on $\text{Hilb}^k(Z)$.

The meromorphic one–form on $\text{Hilb}^k(Z)$ defined above will be denoted by $\Omega_k$. Note that
$\text{Hilb}^1(Z) = \mathbb{Z}$ and $\Omega_1 = \Omega_Z$.

Consider the meromorphic one–form $f^*\Omega_\delta$ on $\mathbb{P}_H$, where the map $f$ is defined in (3.5).
It was noted in Section 2.3 that for any spectral curve $Y_h$ and any $s \in S$, we have
$\gamma^{-1}(s) \cap Y_h = \{0\}$ (that is, the spectral curve is totally ramified over $s$ and passes
through 0). From this it follows immediately that $f^*\Omega_\delta$ is a holomorphic one–form on $\mathbb{P}_H$.
Indeed, for the origin $0 \in (K_X \otimes \mathcal{O}_X(S))_x$, where $x \in X$, the form $\Omega_Z$ vanishes at 0.
Therefore, $f^*\Omega_\delta$ is a holomorphic one–form on $\mathbb{P}_H$.

Recall the one–form $\Omega' = \phi^*\Omega$ on $\mathbb{P}_H$ constructed in (3.4).

**Theorem 3.2.** The one–form $f^*\Omega_\delta$ on $\mathbb{P}_H$ coincides with the one–form $\Omega'$.
In particular, $df^*\Omega_\delta$ coincides with the pullback $\phi^*d\Omega$ of the symplectic form $d\Omega$ on $\mathcal{M}_H^s$.

This theorem will be proved in the next section.

4. Identification of one–forms

We start with the following lemma.

**Lemma 4.1.** There is a holomorphic one–form $\omega$ on $\mathcal{M}_H^s$ such that $\phi^*\omega$ coincides with
$f^*\Omega_\delta$, where $\phi$ is the projection defined in (3.3).

**Proof.** Recall that $\mathcal{P}_H$ is a projective bundle over $\mathcal{M}_H^s$ with $\phi$ being the projection map.
Since there is no nonzero holomorphic one–form on a projective space, the pullback of
$f^*\Omega_\delta$ by the inclusion map of a fiber of $\phi$ must vanish identically.

Let $T^\phi \subset T\mathcal{P}_H$ be the relative tangent bundle. In other words, $T^\phi$ is the kernel of the
differential $d\phi : T\mathcal{P}_H \rightarrow \phi^*T\mathcal{M}_H^s$. Since the evaluation of $f^*\Omega_\delta$ on $T^\phi$ vanishes, there
is a homomorphism

$$\omega' : T\mathcal{P}_H/T^\phi \rightarrow \mathcal{O}_{\mathcal{P}_H}$$

such that $f^*\Omega_\delta$ coincides with the composition of the natural projection of $T\mathcal{P}_H$ to
$T\mathcal{P}_H/T^\phi$ with the homomorphism $\omega'$; here $\mathcal{O}_{\mathcal{P}_H}$ is the structure sheaf of $\mathcal{P}_H$, or equivalently,
the sheaf defined by the trivial line bundle.

For any point $\zeta \in \mathcal{M}_H^s$, the restriction of $T\mathcal{P}_H/T^\phi$ to the fiber $\phi^{-1}(\zeta)$ is a trivial
vector bundle. In fact, $T\mathcal{P}_H/T^\phi$ is identified with the pullback $\phi^*T\mathcal{M}_H^s$. Since $\phi^{-1}(\zeta)$ is
a compact and connected, the homomorphism $\omega'(p)$ is independent of $p \in \phi^{-1}(\zeta)$ (with
$\zeta$ fixed). In other words, there is a holomorphic one–form $\omega$ on $\mathcal{M}_H^s$ such that $\omega'$ is the
pullback of $\omega$. This completes the proof of the lemma. \qed
Recall the one–form $\Omega$ on $M_sH$ constructed in Section 2.2. Since $\Omega' = \phi^*\Omega$ (see (3.4)), in view of Lemma 4.1, to prove Theorem 3.2 it suffices to establish that the two one–forms $\omega$ and $\Omega$ on $M_sH$ coincide.

Consider the Hitchin map $\psi$ defined in (2.5). We want to show that there is a holomorphic one–form $\Omega_H$ on the Hitchin space $H$ such that

\begin{equation}
\Omega - \omega = \psi^*\Omega_H.
\end{equation}

To prove the existence of such a form $\Omega_H$, take a point $h \in H$ such that the corresponding spectral curve $Y_h$ is smooth. We recall that there is a nonempty Zariski open subset $U$ of $H$ such that for any point $h' \in U$ the corresponding spectral curve $Y_{h'}$ is smooth.

The fiber $\psi^{-1}(h)$ is identified with the Picard variety $J_h := \text{Pic}^{d+2(1-g)}(Y_h)$ of degree $d + 2(1-g)$ line bundles on $Y_h$. (The degree $d + 2(1-g)$ was computed in the proof of Lemma 3.1.) Let $j_h : J_h \rightarrow M_sH$ be the inclusion map of the fiber of $\psi$. From the constructions of $\Omega$ and $\omega$ it follows that $j_h^*\Omega = j_h^*\omega$.

Therefore, exactly as in the proof of Lemma 4.1 we conclude that for any point $z \in \psi^{-1}(h)$, the homomorphism

\begin{equation}
(\Omega - \omega)(z) : T_zM_sH \rightarrow \mathbb{C}
\end{equation}

factors through the projection $d\psi(z) : T_zM_sH \rightarrow T_{\psi(z)}H$ defined the differential of $\psi$ at the point $h$. Consequently, there is a holomorphic one–form $\Omega_H$ on $U$ such that $\Omega - \omega = \psi^*\Omega_H$ on $\psi^{-1}(U)$, where $U$, as before, is the open subset of $H$ defined by the points corresponding to smooth spectral curves. Since $\Omega - \omega$ on $\psi^{-1}(U)$ extends to $M_sH$, $U$ is a Zariski open dense subset of $H$ and the map $\psi$ is a submersion everywhere, it follows immediately that $\Omega_H$ extends to $H$ and the equality in (4.1) is valid on $M_sH$.

**Lemma 4.2.** The one–form $\Omega_H$ on $H$ vanishes identically.

**Proof.** Recall that $H = H^0(X, K_X) \oplus H^0(X, K_X^2 \otimes O_X(S))$. Set $H' := H \setminus \{0\}$, the nonzero vectors. For any nonzero complex number $c$, consider the automorphism of $H'$ that sends any point $(\alpha, \beta)$ to $(c\alpha, c^2\beta)$, where $\alpha \in H^0(X, K_X)$ and $\beta \in H^0(X, K_X^2 \otimes O_X(S))$.

So we have a free action of $\mathbb{C}^*$ on $H'$ defined this way. The quotient space $Q := H'/\mathbb{C}^*$ is a weighted projective space. Let

\begin{equation}
\rho : H' \rightarrow Q
\end{equation}

be the quotient map.

In Section 2.3 given a point of $H$ we constructed a spectral curve, which is a divisor on $Z$, the total space of $K_X \otimes O_X(S)$. We want to describe the above action of $\mathbb{C}^*$ on $H'$ in terms of spectral curves. On $Z$ there is an action of $\mathbb{C}^*$ defined by the condition

- $\mathbb{C}^*$ acts on $K_X$ by $c \cdot K_X = cK_X$.
- $\mathbb{C}^*$ acts on $O_X(S)$ by $c \cdot O_X(S) = c^{-1}O_X(S)$.

The action of $\mathbb{C}^*$ on $K_X \otimes O_X(S)$ is the product of these actions.
that the action of any \( c \in \mathbb{C}^* \) sends a point \( z \) to \( cz \), where the scalar multiplication is defined by the vector space structure of the fibers of the line bundle \( K_X \otimes \mathcal{O}_X(S) \). It is easy to see that for any \( h \in \mathcal{H}' \) and \( c \in \mathbb{C}^* \), the spectral curve corresponding to the point \( ch \) coincides with the image of the spectral curve corresponding to the point \( h \) by the automorphism of \( Z \) defined action of \( c \).

On the other hand, the meromorphic one–form \( \Omega_Z \) on \( Z \) (constructed in Section 3.2) evidently has the property that it vanishes along the orbits of \( \mathbb{C}^* \) on \( Z \). In other words, for the projection \( \gamma \) of \( Z \) to \( X \), the pullback of \( \Omega_Z \) by the inclusion map of a fiber of \( \gamma \) vanishes identically. Furthermore, for any \( c \in \mathbb{C}^* \), if \( T_c \) denotes the automorphism of \( Z \) defined by the multiplication by \( c \), then \( T_c^* \Omega_Z = c \Omega_Z \). From these observations it follows immediately that there is a one–form \( \Omega_Q \) on the weighted projective space \( Q \) such that \( \rho^* \Omega_Q = \Omega_H \) on \( \mathcal{H}' \), where \( \rho \) is the projection in (4.2).

A weighted projective space does not admit any nonzero holomorphic one–form. Hence we have \( \Omega_Q = 0 \). Since \( \rho^* \Omega_Q = \Omega_H \), it follows immediately that \( \Omega_H = 0 \), and the proof of the lemma is complete.

The above results clearly combine together in imply Theorem 3.2.

**Proof of Theorem 3.2.** Lemma 4.2 (4.1) together imply that \( \Omega = \omega \) on \( \mathcal{M}_H^s \). So,
\[
\Omega' := \phi^* \Omega = \phi^* \omega.
\]
Now Lemma 4.1, which says that \( \phi^* \omega = f^* \Omega_\delta \), completes the proof of the theorem.

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