Martingales in Homogeneous spaces

Simão N. Stelmastchuk
Departamento de Matemática, Universidade Estadual do Paraná,
84600-000 - União da Vitória - PR, Brazil. e-mail: simnaos@gmail.com

Abstract
Let \( G/H \) be a reductive homogeneous space and \( \nabla_{G/H} \) a \( G \)-invariant connection. Our interesse is to study \( \nabla_{G/H} \)-martingales in \( G/H \). In fact, we yields a correspondence between \( \nabla_{G/H} \)-martingales and local martingales \( m \), where \( m \) is the subspace of Lie algebra \( g \) such that \( g = h \oplus m \) such that \( Ad(H)(m) \subset m \). Here \( h \) is the Lie subalgebra of \( H \). As application we show that martingales in the sphere \( S^n \) are in 1-1 correspondence with local martingales in \( \mathbb{R}^n \).

Key words: Homogeneous space; martingales; stochastic analysis on manifolds

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1 Introduction
Let \( G \) be a Lie Group and \( H \) closed Lie subgroup. In this work we consider the reductive homogeneous spaces. It means that \( g, h \) are Lie algebras of \( G \) and \( H \), respectively, and there exists a subspace \( m \) of \( g \) such that \( g = h \oplus m \) and \( Ad(H)(m) \subset m \). Our intention is to study the martingales in \( G/H \) with respect to \( G \)-invariant connections. A first study in this direction was done by M. Arnaudon in [3], where he characterized the martingales with respect the canonical connection in \( G/H \) in function of local martingales in \( m \). The reader can see that his strategy was used the stochastic exponential in the sense of Stratonovich (see for example [8]) to show this.

In our paper, being natural to see \( \pi: G \to G/H \) as submersion, furthermore, as principal fiber bundle, our idea is given a \( G \)-invariant connection \( \nabla_{G/H} \) on \( G/H \) and to construct a desirable connection \( \nabla^G \) on \( G \) such that \( \pi: G \to G/H \) is an affine submersion with horizontal distribution. It means that \( \pi \ast (\nabla^G_{A^h}B^h) = \nabla_{G/H}^X B \), where \( X, Y \) are vector fields on \( G/H \) and \( A^h, B^h \) are their lifts to \( G \), respectively. The last definition was introduced by N. Abe and K. Hasewaga in [1].

Take the connections \( \nabla_{G/H} \) and \( \nabla^G \) as above. Following the natural idea of projecting the horizontal geodesics of \( G \) in geodesics of \( G/H \) we wish to project horizontal \( \nabla^G \)-martingales in \( \nabla_{G/H} \)-martingales. To make the role of geodesics in \( G \) we will use the Itô exponential on \( G \), which was introduced by author in [15]. Given a local martingale \( M \) in \( g \) the Itô exponential \( X = e^G(M) \) with respect to \( \nabla^G \) is the solution of the stochastic differential equation in Itô sense:

\[
d^{\nabla^G} X_t = L_{(X_t)}(e) dM, \quad X_0 = e.
\]

In context proposed until here, our main Theorem says:
Theorem: Let $G/H$ a reductive homogeneous space $G/H$. Let $\nabla^{G/H}$ and $\nabla^G$ connections on $G/H$ and $G$, respectively, such that $\pi$ is an affine submersion with horizontal distribution. If $X_t$ is a $\nabla^{G/H}$-martingale in $G/H$, then it is written as $\pi \circ e^G(M)$, where $M$ is a local martingale in $\mathfrak{m}$.

The hypothesis of Theorem is satisfied in many examples of homogeneous spaces, which we give in this work. However, a special application is the sphere. Viewing the sphere $S^n$ as homogeneous space we show that the martingales in sphere are in 1-1 correspondence with local martingales in $\mathbb{R}^n$.

2 Stochastic calculus

In this work we use freely the concepts and notations of P. Protter [12], P. Meyer [10], M. Emery [6] and [7], S. Kobayashi and N. Nomizu [9] and J. Cheeger and D.G. Ebin [5]. We suggest the reading of [4] for a complete survey about the objects of this section. From now on the adjective smooth means $C^\infty$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for example [6]). Our basic assumption is that every stochastic process is continuous.

Let $M$ be a smooth manifold and $X_t$ a continuous stochastic process with values in $M$. We call $X_t$ a semimartingale if, for all $f$ smooth function, $f(X_t)$ is a real semimartingale.

Let $M$ be a smooth manifold with connection $\nabla^M$. Let $X$ be a continuous semimartingale with values in $M$, $\theta$ a section of $T^*M$ and $b$ a section of $T^{(2,0)}M$. We denote by $\int_0^t \theta d\nabla^M X$ the Itô integral of $\theta$ along $X$ and by $\int_0^t b d(X,X)$ the quadratic integral of $b$ along $X$. We recall that $X$ is a $\nabla$-martingale if and only if $\int_0^t \theta d\nabla^M X$ is a local martingale for any $\theta \in \Gamma(T^*M)$.

Let $M$ and $N$ be smooth manifolds endowed with connections $\nabla^M$ and $\nabla^N$, respectively, and $F : M \rightarrow N$ a smooth map. P. Catuogno in [4] shows the following version for Itô formula in smooth manifolds, which will be said geometric Itô formula:

$$\int_0^t \theta dN(F(X)) = \int_0^t F^*\theta dM X + \frac{1}{2} \int_0^t \beta^*_F \theta (dX, dX), \quad (1)$$

where $\beta_F$ is the second fundamental form of $F$ and $\theta \in \Gamma(T^*N)$.

From the above formula, it follows that $F$ is an affine map if it and only if sends $\nabla^M$-martingales to $\nabla^N$-martingales.

3 Connections on homogeneous spaces

Let $H$ be a closed Lie subgroup of $G$. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of $G$ and $H$, respectively. We assume that the homogeneous space $G/H$ is reductive, that is, there is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. Let $\pi$ be the natural mapping of $G$ onto the space $G/H$ of the cosets $gH$, $g \in G$. Also, for each $a \in G$ we define $\tau_a : G/H \rightarrow G/H$ by $\tau_a(gH) = agH$, the left translation. If $a \in G$ and $L_a$ are the left translation on $G$, then $\pi \circ L_a = \tau_a \circ \pi$. 

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The differential of \( \pi \) at \( e \) shows that \( \ker(d\pi)_e = \mathfrak{h} \). Since \( d\pi \) is onto we get the canonical isomorphism \( m \cong T_e(G/H) \).

As the left translation \( L_g \) is a diffeomorphism, for every \( g \in G \), we have

\[
T_g G = (L_g)_e \mathfrak{h} \oplus (L_g)_e m.
\]

Thus, writing

\[
TG_h := \{(L_g)_e \mathfrak{h}; \forall g \in G\} \text{ and } TG_m := \{(L_g)_e m; \forall g \in G\}
\]

follows that \( TG = TG_h \oplus TG_m \).

Let us denote the Maurer-Cartan form on \( G \) as \( \omega \). Theorem 11.1 in [9] shows that the principal fiber bundle \( G(G/H, H) \) has the vertical part of the Maurer-Cartan as a connection form with respect to decomposition \( \mathfrak{g} = \mathfrak{h} \oplus m \). In other words, \( TG_m \) is a connection in \( G(G/H, H) \). The horizontal lift from \( G/H \) to \( G \) is denoted by \( \mathcal{H} \) and the horizontal projection of \( TG \) into \( TG_m \) is written as \( \mathfrak{h} \).

Let \( A \in m \). The left invariant vector field \( \tilde{A} \) on \( G \) is denoted by \( \tilde{A}(g) = L_g A \) and the \( G \)-invariant vector field \( \pi_* A \) on \( G/H \) is defined by \( \pi_* A = \pi g_* A \). It is clear that \( \tilde{A} \) is a horizontal vector field on \( G \).

It is well-known, see Theorem 8.1 in [11], that for each \( G \)-invariant connection \( \nabla^{G/H} \) is associated to a unique \( Ad(H) \)-invariant bilinear map \( \beta : m \times m \rightarrow m \), that is,

\[
\beta(Ad(H)(A), Ad(H)(B)) = Ad(H)\beta(A, B), \quad A, B \in m.
\]

This correspondence is given by

\[
(\nabla^{G/H}_{A} B)_{\circ} = \beta(A, B), \quad A, B \in m.
\]

Since we are interested on martingales in \( G/H \), our idea is choose a good connection \( \nabla^{G} \) such that it is horizontally projected over \( \nabla^{G/H} \). In other words, we choose \( \nabla^{G} \) in the way that \( \pi : G \rightarrow G/H \) is an affine submersion with horizontal distribution. This definition was given by N. Abe and H. Hasegawa in [11] and it means the following. Taking \( A, B \in m \) we yields the left invariant vectors fields \( A, B \) on \( G \) and the \( G \)-invariant vector fields \( \pi_* A, \pi_* B \) on \( G/H \). It is clear that \( \tilde{A} \), \( \tilde{B} \) are horizontal and \( \pi_*(\tilde{A}) = \pi_* A \) and \( \pi_*(\tilde{B}) = \pi_* B \). In other words, \( \tilde{A}, \tilde{B} \) are horizontal lift of \( A, B \), respectively. Therefore \( \pi \) is an affine submersion with horizontal distribution if

\[
\mathfrak{h}(\nabla^{G}_{A} \tilde{B}) = \mathcal{H}(\nabla^{G/H}_{\pi_* A} B_*).
\]

A natural way to construct a connection \( \nabla^{G} \) from \( \nabla^{G/H} \) such that \( \pi \) is affine submersion with horizontal distribution is to extend \( \beta \) to a bilinear map \( \alpha \) to \( \mathfrak{g} \times \mathfrak{g} \) into \( \mathfrak{g} \) such that \( \alpha(A, B) = \beta(A, B) \) for \( A, B \in m \). Thus, there exists a left invariant connection \( \nabla^{G} \) on \( G \) such that

\[
(\nabla^{G}_{A} \tilde{B})(e) = \alpha(A, B), \quad X, Y \in \mathfrak{g}.
\]

We prove some geometric necessary facts.

**Proposition 3.1** Let \( \nabla^{G/H}, \nabla^{G} \) be connections such that \( \pi \) is an affine submersion with horizontal distribution.
1. If $f \in C^\infty(G/K)$ then
\[
Hess^G(f \circ \pi)(\tilde{A}, \tilde{B}) = Hess^{G/H}(f)(\pi(g))(A_\ast, B_\ast),
\]
for $A, B \in \mathfrak{m}$.

2. If $A, B \in \mathfrak{m}$ then
\[
\beta_\pi(\tilde{A}, \tilde{B}) = 0,
\]
where $\beta_\pi$ is the second fundamental form of $\pi$.

**Proof:** 1. For $A, B \in \mathfrak{m}$, $\pi_\ast(g)(\tilde{A}(g)) = A_\ast(\pi(g))$ and $\pi_\ast(g)(\tilde{B}(g)) = B_\ast(\pi(g))$ for all $g \in G$. By definition of hessiano, for every $f \in C^\infty(G/K)$,
\[
Hess^{G/H}(f(\pi(g)))(A_\ast, B_\ast) = A_\ast(\pi(g))(B_\ast f) - df(\nabla^G_H A_\ast B_\ast)(\pi(g))
\]
\[= \tilde{A}(g)B(f \circ \pi) - (f \circ \pi)(\nabla^G_H \tilde{A} \tilde{B})(g)
\]
\[= Hess^G(f \circ \pi)(\tilde{A}, \tilde{B}).
\]

2. Given $A, B \in \mathfrak{m}$ we have, by definition of the second fundamental form,
\[
\beta_\pi(\tilde{A}, \tilde{B}) = \nabla^G_{\pi, \tilde{A}} \pi_\ast \tilde{B} - \pi_\ast \nabla^G_{\tilde{A}} \tilde{B}.
\]
Being $\pi$ an affine submersion with horizontal distribution, we obtain
\[
\beta_\pi(\tilde{A}, \tilde{B}) = \nabla^G_{A_\ast} B_\ast - \nabla^G_{A_\ast} B_\ast = 0.
\]

### 4 Martingales in homogeneous space

We endow $G$ with a left invariant connection $\nabla^G$ and $\mathfrak{g}$ with a flat connection $\nabla^\mathfrak{g}$. In [15], the author defines the Itô stochastic exponential with respect to $\nabla^G$ and $\nabla^\mathfrak{g}$ as the solution of the Itô stochastic differential equation
\[
d\nabla^G X_t = L(\nabla^G)\theta_X dM, \quad X_0 = e,
\]
where $M$ is a semimartingale in $\mathfrak{g}$. For simplicity, we call $e^G(M)$ of Itô exponential. In [15], we have the following results about Itô exponential

**Theorem 4.1** Given a semimartingale $X$ in $G$, there exists a unique semimartingale $M$ in $\mathfrak{g}$ such that $X = e^G(M)$.

**Theorem 4.2** Let $\nabla^G$ be a connection on $G$. The $\nabla^G$-martingale in $G$ are exactly the process $e^G(M)$ where $M$ is a local martingale on $\mathfrak{g}$.

Before we work with martingales in $G/H$ it is necessary to develop a result in the Lie group $G$. It is related with the left translate of semimartingales by a random variable with values in $G$. In consequence, we see that the set of martingales in $G$ with respect to a left invariant connection do not change if we translate it to left by a random variable with values in $G$.

**Proposition 4.3** Let $G$ be a Lie group and $\nabla^G$ a left-invariant connection on $G$. If $Y_t$ is a semimartingale on $G$ and $\xi$ is a random variable with values in $G$, then, for $\theta$ 1-form on $G$,
\[
\int \theta d\nabla^G \xi Y_t = \int (L_\xi^\theta) d\nabla^G Y_t.
\]
Proof: We begin denoting the product on Lie group $G$ by $m$. Let $\theta$ be a 1-form on $G$. As a function to $m$, the Itô integral along $\xi Y_t$ is writing as

$$\int \theta d^{\nabla^G} \xi Y_t = \int \theta d^{\nabla^G} m(\xi, Y_t).$$

The geometric Itô formula \[1\] gives

$$\int \theta d^{\nabla^G} \xi Y_t = \int \theta d^{\nabla^G} \pi Y_t + \int \theta d^{\nabla G}(\xi, Y_t) + 1/2 \int \beta_m^* \theta(d(\xi, Y_t), d(\xi, Y_t)).$$

From Proposition 3.15 in \[7\] we see that

$$\int \theta d^{\nabla^G} \xi Y_t = \int (R^*_t \theta) d^{\nabla^G} \xi + \int (L^*_t \theta) d^{\nabla^G} Y_t + 1/2 \int \beta_m^* \theta(d(\xi, Y_t), d(\xi, Y_t)).$$

Then

$$\beta_m(0, Y) = \nabla_{m_*(0, Y)} m_* (0, Y) - m_*(\nabla^{G \times G}(0, Y))$$

where in forth equality we use the fact that $\nabla^G$ is a left invariant connection.

Thus we get

$$\int \theta d^{\nabla^G} \xi Y_t = \int (L^*_t \theta) d^{\nabla^G} Y_t + 1/2 \int \beta_m^* \theta(d(\xi, Y_t), d(\xi, Y_t)).$$

We claim that the $\beta_m(d(\xi, Y_t), d(\xi, Y_t))$ is null. In fact, let $0 \in T_0 G$ and $Y_a$ a left invariant vector field on $G$. Here, 0 is the vector associated to the constant process $\xi$. Then

$$\beta_m(0, Y) = \nabla_{m_*(0, Y)} m_* (0, Y) - m_*(\nabla^{G \times G}(0, Y))$$

$$= \nabla_{m_*(0, Y)} m_* (0, Y) - m_*(\nabla^{G \times G}(0, Y))$$

$$= \nabla_{m_*(0, Y)} m_* (0, Y) - m_*(\nabla^{G \times G}(0, Y))$$

$$= 0,$$

where in forth equality we use the fact that $\nabla^G$ is a left invariant connection.

Thus we get

$$\int \theta d^{\nabla^G} \xi Y_t = \int (L^*_t \theta) d^{\nabla^G} Y_t.$$
Proof: Let \( X_t \) be a \( \nabla^{G/H} \)-martingale and \( Y_t \) its horizontal lift to \( G \). Taking a 1-form \( \theta \) on \( G/H \) follows

\[
\int \theta d^{G/H} Z_t = \int \theta d^{G/H} \tau_{Y_0^{-1}} X_t = \int \theta d^{G/H} \tau_{Y_0^{-1}} \pi(Y_t) = \int \theta d^{G/H} \pi(L_{Y_0^{-1}} Y_t).
\]

From the geometric Itô formula (1) and Proposition 3.1 we see that

\[
\int \theta d^{G/H} Z_t = \int \pi^* \theta d^{G/H} (L_{Y_0^{-1}} Y_t) + \int \pi_* \hat{\theta} \pi (d(L_{Y_0^{-1}} Y_t), d(L_{Y_0^{-1}} Y_t)) = \int \pi^* \theta d^G (L_{Y_0^{-1}} Y_t).
\]

Proposition 4.3 now assures that

\[
\int \theta d^{G/H} Z_t = \int \theta \pi^* \pi_* (L_{Y_0^{-1}} Y_t) = \int \theta \tau_{Y_0^{-1}} \pi^* d^G X_t.
\]

Again, from geometric Itô formula (1) and Proposition 3.1 we conclude that

\[
\int \theta d^{G/H} Z_t = \int \theta \tau_{Y_0^{-1}} d^{G/H} \pi(Y_t) = \int \theta \tau_{Y_0^{-1}} d^{G/H} X_t.
\]

Since \( X_t \) is \( \nabla^{G/H} \)-martingale, it follows that \( Z_t \) is a \( \nabla^{G/H} \)-martingale.

Proposition above allows considering \( \nabla^{G/H} \)-martingales with initial condition \( o \), that is, we can consider only the \( \nabla^{G/H} \)-martingales \( X_t \) with \( X_0 = o \), where \( o = H \) is the origin in \( G/H \).

Lemma 4.6 Let \( G/H \) a reductive homogeneous space \( G/H \). Let \( \nabla^{G/H} \) and \( \nabla^G \) connections on \( G/H \) and \( G \), respectively, such that \( \pi \) is an affine submersion with horizontal distribution. If \( U_t \) is a horizontal parallel stochastic transport along \( X_t \), then \( \pi_*(U_t) \) is a parallel stochastic transport along the semimartingale \( \pi(X_t) \) in \( G/K \).

Proof: It is sufficient to show that \( \pi_*(U_t) \) satisfies the formula of the parallel stochastic transport, see for instance (8.11) in [6]. Consider \( f \in C^\infty(G/K) \). Applying this formula we obtain that

\[
(\pi_* U_t) f + (\pi_* U_0) f = U_t (f \circ \pi) + U_0 (f \circ \pi) = \int Hess(f \circ \pi)(U_t, \delta X_t) = \int Hess(f)(\pi_* U_t, \delta \pi(X_t)),
\]

where we used the Proposition 3.1 in the later equality. It follows immediately that \( \pi_*(U_t) \) is parallel stochastic transport along \( \pi(X_t) \).

Theorem 4.7 Let \( G/H \) a reductive homogeneous space \( G/H \). Let \( \nabla^{G/H} \) and \( \nabla^G \) connections on \( G/H \) and \( G \), respectively, such that \( \pi \) is an affine submersion with horizontal distribution. If \( X_t \) is a \( \nabla^{G/H} \)-martingale in \( G/H \), then it is written as \( \pi \circ e^G(M) \), where \( M \) is a local martingale in \( m \).
Proof: Let $X_t$ be a $\nabla^{G/H}$-martingale in $G/H$ and $Y_t$ its horizontal lift in $G$. Consider a 1-form $\theta$ in $T^*(G/K)$. Since $\pi$ is an affine submersion with horizontal distribution, from Proposition 2.1 and the geometric Itô formula (11) we obtain

$$\int \theta d^{G/H}X_t = \int \theta d^{G/H}(\pi(Y_t)) = \int (\pi^*\theta)d^{G}Y_t = \int \theta \pi_*d^{G}Y_t,$$

where we used that $Y_t$ is a horizontal semimartingale in $G$. Hence

$$d^{G/H}X_t = \pi_*d^{G}Y_t.$$ 

Let $\{H_1, \ldots, H_n\}$ be a basis on $\mathfrak{g}$. Choose $\{H_\kappa, \kappa = 1, \ldots, r\}$ such that it is a basis of $\mathfrak{m}$. By Theorem 4.1, there is a unique semimartingale $N$ in $\mathfrak{g}$ such that $d^G Y_t = L_{Y_t} dN$. If we write $N = \sum_{\kappa=1}^r N^\kappa H_\kappa + \sum_{j=r+1}^n N^j H_j$, then $d^G Y_t = dN^\kappa U_t^\kappa + dN^j U_t^j$, where $U_t^i = L_{Y_t} H_i, i = 1, \ldots, n$. It is obvious that $\sum_{\kappa=1}^r N^\kappa H_\kappa$ is a semimartingale in $\mathfrak{m}$ and that

$$d^{G/H}X_t = \pi_* (dN^\kappa U_t^\kappa) = dN^\kappa \pi_*(U_t^\kappa). \quad (3)$$

The set $\{U^1, \ldots, U^n\}$ is a moving frame along $Y_t$ (see [6] for the definition of moving frame). Hence $\{\pi_*(U^1), \ldots, \pi_*(U^n)\}$ is a moving frame along $X_t$, by Lemma above. Let us denote by $\{\eta_1, \ldots, \eta_r\}$ the dual basis of $\{\pi_*(U^\kappa), \kappa = 1, \ldots, r\}$ along $X_t$. Define $M_t = \sum_{l=1}^r M_t^l H_l$ a semimartingale in $\mathfrak{m}$, where $M_t^l = \int \eta_l d^{G/H}X_t$. For every $l = 1, \ldots, r$, we claim that $M_t^l = N_t^l$. In fact,

$$M_t^l = \int \eta_l d^{G/H}X_t = \int \eta_l dN_t^\kappa \pi_*(U_t^\kappa) = \int dN_t^\kappa \eta_l \pi^* U_t^\kappa = \int dN_t^l = N_t^l.$$

It follows that $N_t = M_t + \sum_{l=r+1}^n N_t^l H_l$. From this and (3) we conclude that $d^{G/H}X_t = \pi_* (L_{Y_t} dM_t)$, and also that

$$d^{G/H}X_t = \pi_{Y_t} dM_t.$$ \quad (4)

The semimartingale $M_t$ above is called the lifting of $X_t$ in $\mathfrak{m}$ (see [6] for this definition). From the stochastic differential equation (11) we conclude directly that $X_t$ is a $\nabla^{G/H}$-martingale if, and only if, $M_t$ is a local martingale in $\mathfrak{m}$. Theorem is proved if we see that $Y_t = e^G(M_t)$.

Remark 1 In the proof of the Theorem above, we founded a semimartingale $Y_t = e^G(M_t)$. Since $M$ is a local martingale in $\mathfrak{m}$, we can consider $M$ as local martingale in $\mathfrak{g}$. Therefore $Y_t$ is a $\nabla^G$-martingale, which follows from Theorem 1.2 Furthermore, in terms of theory of connections, $Y_t$ can be consider as a horizontal martingale in $G$.

Remark 2 From the proof of Theorem 1.2 we have that a semimartingale $X_t$ in $G/H$ satisfies the Itô stochastic differential equation

$$d^{G/H}X_t = \pi_{Y_t} dM_t, \quad X_0 = 0,$$ \quad (5)

where $M_t$ is a semimartingale in $\mathfrak{m}$ and $o = H$. 

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Example 4.1 K. Nomizu in [11] defined by canonical affine connection of the second kind the connection $\nabla^{G/H}$ which has the connection function $\beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ given by $\beta(A, B) = 0$, for $A, B \in \mathfrak{m}$. We extend $\beta$ for a connection function $\alpha(A, B) = 0$, for $A, B \in \mathfrak{g}$. Then, the connection $\nabla^G$ is given by $\nabla^G_A B = 0$. With these connections, it is clear that $\pi : G \to G/H$ is an affine submersion with horizontal distribution. Theorem 4.4 assures that for each $\nabla^{G/H}$-martingale $X$ there exists a local martingale in $\mathfrak{m}$ such that $X_t = \pi \circ e^G(M_t)$. This result was first proved by M. Arnaudon in [3]. As a particular case of this example we have the Symmetric Spaces which admits a $G$-invariant metrics (see Theorem 3.3, chapter XI, in [12]).

Example 4.2 K. Nomizu in [11] called the canonical affine connection of the first kind the connection $\nabla^G$ which has the connection function $\beta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defined as $\beta(A, B) = \frac{1}{2}[A, B]_\mathfrak{m}$. The natural way to extend $\beta$ to $\alpha$ is to take $\alpha(A, B) = \frac{1}{2}[A, B]_\mathfrak{g}$, for $A, B \in \mathfrak{g}$. In accordance to correspondence between connections on $G/H$ and $G$ and connections functions $\beta$ and $\alpha$, respectively, $\nabla^G_A B = \frac{1}{2}[A, B]_\mathfrak{m}$ and $\nabla^G B = \frac{1}{2}[A, B]_\mathfrak{g}$. It follows directly that $\pi : G \to G/H$ is an affine submersion with distribution horizontal. Therefore every $\nabla^{G/H}$-martingale $X_t$ is written as $X_t = \pi \circ e^G(M_t)$, where $M_t$ is a local martingale in $\mathfrak{m}$, which follows from Theorem 4.4.

Example 4.3 A class of homogeneous space that satisfy the Example above are the normal homogeneous spaces. Following Definition 6.60 in [13], a Riemannian homogeneous space $M = G/H$ is called normal homogeneous if there exists a bi-invariant metric on $G$ such that $\pi_* \mathfrak{e}$ maps the orthogonal complement $\mathfrak{h}^\perp \mathfrak{g}$ isometrically to $\mathfrak{m}_{\pi(e)}$. It is know that Levi-Civita connection on $G$ is given by $\nabla^G_A B = \frac{1}{2}[A, B]_\mathfrak{g}$, for $A, B$. In the other side, it is possible to show that the Levi-Civita connection on $G/H$ is given by $\nabla^G_A B = \frac{1}{2}[A, B]_\mathfrak{m}$, for $A, B \in \mathfrak{m}$ (see proposition 6.62 in [13]). In fact, every normal homogenous space is naturally reductive (see page 220 in [15] or [2]).

Example 4.4 A example more general than above is the following. Let $M = G/H$ be a homogeneous space. We admit that $M$ has a $G$-invariant metric $\langle \cdot, \cdot \rangle$. Using Theorem 3.36 in [13] we obtain a left invariant metric $\langle \cdot, \cdot \rangle$ on $G$ such that $\pi : G \to G/H$ is a Riemannian submersion. Theorem 4.7 assures that every $\nabla^{G/H}$-martingale $X_t$ is written as $X_t = \pi \circ e^G(M_t)$, where $M_t$ is a local martingale in $\mathfrak{m}$.

5 Martingales in sphere

Let $S^n$ be a sphere $n$-dimensional in $\mathbb{R}^n$. We can write $S^n$ as a normal homogeneous space in the following way. In [13], we found in Example 6.61(a) that if we define a bi-invariant metric on $SO(n+1)$ by $< U, V > = \frac{1}{2} tr(U^t V) = -B(U, V)/(2n-2)$, $n \geq 2$, $B$ is the Killing form, then $S^n = SO(n+1)/SO(n)$ is a normal homogeneous space. Furthermore, the normal homogeneous metric on $S^n = SO(n+1)/SO(n)$ is the usual metric on $S^n$. It directly follows that $SO(n+1)/SO(n)$ is a reductive homogeneous space. The reductive decomposition is given by $\mathfrak{o}(n+1) = \mathfrak{o}(n) + \mathfrak{m}$, where $\mathfrak{m}$ is the subspace of all $n \times n$
matrices of the form
\[
\begin{pmatrix}
0 & -x^t \\
x & 0_n
\end{pmatrix},
\]
where \( x = (x_1, \ldots, x_n) \) is a column vector in \( \mathbb{R}^n \) and \( 0_n \) the \( n \times n \) zero matrix. It is clear that \( m \) is isomorphic to \( \mathbb{R}^n \). Let us denote such isomorphism by \( \phi : m \to \mathbb{R}^n \). It is immediate that a semimartingale \( \xi \) in \( \mathbb{R}^n \) is a local martingale if and only if \( \phi(\xi) = M \) is a local martingale in \( m \).

**Theorem 5.1** Let \( S^n \) be a sphere \( n \)-dimensional in \( \mathbb{R}^n \) with its usual metric induced of \( \mathbb{R}^{n+1} \). There is a 1-1 correspondence between matingales in \( S^n \) and local martingales in \( \mathbb{R}^n \).

**Proof:** Let \( X_t \) be a \( \nabla S^n \)-martingale in \( S^n \), where \( \nabla S^n \) is the Levi-Civita connection. Theorem 4.7 yields a unique local martingale in \( \mathbb{m} \) such that \( X_t = \pi \circ e^G(M) \), where \( \nabla^G \) is the Levi-Civita connection on \( SO(n+1) \). Using the isomorphism \( \phi : \mathbb{R}^n \to \mathbb{m} \) defined above we see that \( M = \phi(\xi) \), where \( \xi \) is the unique local martingale in \( \mathbb{R}^n \) that satisfies such relation. It follows that \( X_t \) is unique related with \( \xi \), and the proof is complete.

By Remark 2 we know that a \( \nabla S^n \)-martingale \( X_t \) satisfies the Itô stochastic differential equation
\[
d^{S^n}X_t = \tau_{Y_t}dM_t, \quad X_0 = \delta',
\]
where \( M_t \) is a local martingale in \( \mathbb{m} \) and \( \delta' = (1, 0, \ldots, 0) \). In the other hand, there exists a unique local martingale \( \xi \) such that \( M = \phi(\xi) \). So, for a 1-form \( \theta \) we can compute
\[
\int \theta d^{S^n}X_t = \int \theta \tau_{Y_t}dM_t = \int \theta \tau_{Y_t}d\phi(\xi)_t = \int \theta \tau_{Y_t} \phi_{\xi}, d\xi_t,
\]
where we used the geometric Itô formula (1) in the last equality. Thus \( X_t \) satisfies the following Itô differential equation
\[
d^{S^n}X_t = \tau_{Y_t} \phi_{\xi}d\xi_t, \quad \xi_0 = (0, 0, \ldots, 0).
\]

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