A characterization of ramification groups of local fields with imperfect residue fields

Takeshi Saito

March 1, 2022

Abstract

We give a characterization of ramification groups of local fields with imperfect residue fields, using those for local fields with perfect residue fields. As an application, we reprove an equality of ramification groups for abelian extensions defined in different ways.

Let $K$ be a henselian discrete valuation field. Let $L$ be a Galois extension and let $G = \text{Gal}(L/K)$ be the Galois group. In the classical case where the residue field $F$ of $K$ is perfect, the lower numbering filtration $(G_{i,\text{cl}})$ indexed by integers $i \geq 0$ is defined by $G_{i,\text{cl}} = \text{Ker}(G \to \text{Aut}(O_L/m_i^{i+1}))$. Further, the upper numbering filtration $(G_{r,\text{cl}})$ indexed by rational numbers $r > 0$ is defined using the Herbrand function to renumber the lower numbering [8, Chapitre IV, Section 3].

In the general case where the residue field need not be perfect, an upper numbering filtration $(G_{r,\text{cl}})$ indexed by rational numbers $r > 0$ is defined first in [1] using rigid geometry and later in [6] purely in the language of schemes. In the classical case where $F$ is perfect, they are related to each other by the relation $G_r = G_{r-1,\text{cl}}$.

We give an axiomatic characterization of the filtration $(G_{r,\text{cl}})$. The axiom has two conditions. The first condition is the relation in the classical case above. The second condition is the compatibility with tangentially dominant extensions. A similar approach reducing to the classical case was proposed in [3].

For a discrete valuation field $K$, the tangent space at an algebraic closure $\bar{F}$ of the residue field $F$ is defined as an $\bar{F}$-vector space using the cotangent complex. In the classical case where the residue field is perfect, it is nothing but the scalar extension of the Zariski tangent space defined as the dual $\text{Hom}_F(m_K/m_K^2, \bar{F})$. An extension of discrete valuation fields is said to be tangentially dominant if the induced morphism on the tangent spaces is dominant (Definition 2.1.2). An unramified extension is tangentially dominant and a tangentially dominant extension has ramification index 1.

The uniqueness is a consequence of the existence of tangentially dominant extension with perfect residue field. The existence follows from the functorial properties of the filtration $(G_{r,\text{cl}})$.

For $r > 1$, the graded quotient $\text{Gr}_r G = G_r / G_{r+1}$ is defined by $G_{r+1} = \bigcup_{s \geq r} G^s$ and is an $F_p$-vector space. A canonical injection

(0.1) $\text{Hom}(\text{Gr}_r G, F_p) \to \text{Hom}_F(m_K^r/m_K^{r+1}, H_1(L_{\bar{F}/S}))$
as a generalization of a non-logarithmic variant of the refined Swan conductor defined by Kato in [1]. We also give an axiomatic characterization of this morphism, similar to the characterization for \( G_r \) itself.

As an application of the characterizations, we give a new proof of the equality of two filtrations for abelian extensions in positive characteristic. By the Hasse-Arf theorem, the filtration \((G^n)\) defined in [1] is in fact indexed by integers \( n > 1 \) for abelian extensions. The other filtration is the filtration \((G^\text{nr}_{\text{Ma}})\) defined by Matsuda in [5] as a modification of that defined by Kato in [1]. The equality was proved in [2] except for the smallest index \( n = 2 \) and the remaining case was proved by Yatagawa in [9]. The equality is proved by verifying that the filtration \((G^\text{nr}_{\text{Ma}})\) satisfies the same axiom. We also prove that the injection \((0.1)\) equals the morphism \( rsw' \) defined in [5] and [9], as a modification of the refined Swan conductor defined in [1].

A variant \((G^\text{nr}_{\text{log}})\) of the upper numbering filtration \((G^\text{nr})\) called the logarithmic upper numbering filtration is also defined in [1]. In the case where the ramification index \( e_{L/K} \) is 1, the two filtrations are the same: \( G^\text{nr} = G^\text{nr}_{\text{log}} \). If \( K' \) is a log smooth extension of \( K \) and \( L' = LK' \), the canonical injection \( G' = \text{Gal}(L'/K') \to G = \text{Gal}(L/K) \) is known to induce isomorphisms \( G^\text{nr}_{\text{log}} = G^\text{nr}_{\text{log}} \) for \( e = e_{K'/K} \). Further if the ramification index \( e_{K'/K} \) is divisible by \( e_{L/K} \), we have \( e_{L'/K'} = 1 \). Thus, a characterization of \((G^\text{nr})\) gives an indirect characterization \((G^\text{nr}_{\text{log}})\).

The author thanks the referee for careful reading and the suggestion to include comments on the logarithmic filtration. The research is partially supported by Grant-in-Aid (B) 19H01780.

## 1 Totally ramified case

Let \( K \) be a henselian discrete valuation field. Let \( L \) be a totally ramified Galois extension of \( K \) and let \( G = \text{Gal}(L/K) \) be the Galois group. For a rational number \( r > 1 \), the upper ramification group \((G^r)\) defined in [1, Definition 3.4] equals the subgroup defined in [8, Chapitre IV, Section 3] denoted \( G^r_{\text{cl}} \), by [1, Proposition 3.7 (3)].

Assume that \( L \) is wildly ramified and let \( r > 1 \) be the largest rational number such that the subgroup \((G^r)\) of the wild inertia subgroup \( P \subset G \) is non-trivial. Let \( E \) be the residue field and \( e = e_{L/K} \) be the ramification index. We give a description of the canonical injection

\[
(1.1) \quad G^{r'} = \text{Hom}_{\mathbb{F}_p}(G^r, \mathbb{F}_p) \to \text{Hom}_E(m^{e(r-1)}_L/m^{e(r-1)+1}_L, E)
\]

for the \( \mathbb{F}_p \)-vector space \( G^r \), in the case where \( L \) is totally ramified over \( K \). The injection \((1.1)\) is a special case of \((0.1)\).

We begin with a description of extensions of vector spaces over a field of characteristic \( p > 0 \) by \( \mathbb{F}_p \)-vector spaces.

**Lemma 1.1.** Let \( F \) be a field of characteristic \( p > 0 \). Let \( G \subset F \) be a finite subgroup of the additive group. Then, the polynomial

\[
(1.2) \quad a_1 = \frac{\prod_{\sigma \in G}(X - \sigma)}{\prod_{\sigma \in G, \sigma \neq 0}(-\sigma)}
\]
\[ T = \text{Spec} \pi \text{ and the constant term } \phi \text{ isomorphism } E \text{ denote the residue field. The minimal polynomial } \text{ that } \]

\[ f \text{ is an Eisenstein polynomial, the reduced closed fiber } f/\alpha \text{ is connected, then the morphism } [H]: G^\vee \rightarrow E^\vee \text{ is an injection.} \]

**Proof.** 1. By \([7, \text{Lemma 2.1.5}]\), \(a = \prod_{\sigma \in G}(X - \sigma) \in F[X] \) is an additive separable polynomial such that (1.3) with \(a_1\) replaced by \(a\) is exact. Since the coefficient in \(a\) of degree 1 is \(\prod_{\sigma \in G, \sigma \neq 0}(\sigma)\), the assertion follows. \(\square\)

Let \(K\) be a henselian discrete valuation field and \(L\) be a totally ramified Galois extension of degree \(e\) of Galois group \(G\). Let \(\alpha \in L\) be a uniformizer and let \(E = F\) denote the residue field. The minimal polynomial \(f \in \mathcal{O}_K[X]\) is an Eisenstein polynomial and the constant term \(\pi = f(0)\) is a uniformizer of \(K\). We define a closed immersion \(T = \text{Spec} \mathcal{O}_L \rightarrow Q = \text{Spec} \mathcal{O}_K[X]\) by sending \(X\) to \(\alpha\). For a rational number \(r > 1\) such that \(\text{er} \in \mathbb{Z}\), define a dilatation

\[ Q^{[r]}_T = \text{Spec} \mathcal{O}_L[X] \left[ \frac{f}{\alpha^{er}} \right] \rightarrow Q_T = \text{Spec} \mathcal{O}_L[X]. \]

The generator \(f\) of the kernel \(I = \text{Ker}(\mathcal{O}_K[X] \rightarrow \mathcal{O}_L)\) defines a basis over \(\mathcal{O}_L\) of the conormal module \(N_{T/Q} = I/I^2\) and \(\alpha^{er}\) defines a basis of the \(E\)-vector space \(m_L^{er}/m_L^{er+1}\). As subspaces of \(N_{E/Q} = J/J^2\) for \(J = \text{Ker}(\mathcal{O}_K[X] \rightarrow E) = (X, f) = (X, \pi)\), we have an equality

\[ N_{T/Q} \otimes_{\mathcal{O}_L} E = m_K/m_K^2 \]

since \(f\) is an Eisenstein polynomial. The basis \(f\) of \(N_{T/Q} \otimes_{\mathcal{O}_L} E\) corresponds to the uniformizer \(\pi = f(0) \in m_K/m_K^2\).

By sending \(S\) to \(f/\alpha^{er}\), we define an isomorphism \(\mathcal{O}_L[X, S]/(f - \alpha^{er}S) \rightarrow \mathcal{O}_L[X][f/\alpha^{er}]\). Since \(f\) is an Eisenstein polynomial, the reduced closed fiber \(Q^{[r]}_E = \text{Spec} (\mathcal{O}_L[X][f/\alpha^{er}] \otimes_{\mathcal{O}_L} E)_{\text{red}}\) is identified with \(\text{Spec} E[S]\). By this identification and (1.3), we define an isomorphism

\[ Q^{[r]}_E \rightarrow \text{Hom}_E(m_L^{er}/m_L^{er+1}, N_{T/Q} \otimes_{\mathcal{O}_L} E)^\vee \rightarrow \text{Hom}_E(m_L^{er}/m_L^{er+1}, m_K/m_K^2)^\vee = m_L^{(r-1)}/m_L^{(r-1)+1} \]
of smooth group schemes of dimension 1 over E.

Let \( Q^{(r)}_T \rightarrow Q^{(r)}_T \) be the normalization and define a section \( T \rightarrow Q_T \) to be the unique lifting of the section \( T \rightarrow Q_T \) defined by sending \( X \) to \( \alpha \). Let \( Q^{(r)}_E \) denote the reduced part of the closed fiber \( Q^{(r)}_T \times_T \text{Spec } E \) and let \( Q^{(r)}_E \subset Q^{(r)}_E \) denote the connected component containing the image of the closed point of \( T \) by the section \( T \rightarrow Q^{(r)}_T \).

**Proposition 1.2.** Let \( K \) be a henselian discrete valuation field with residue field \( F \) of characteristic \( p > 0 \). Let \( L \) be a totally ramified Galois extension of degree \( n = e \) with residue field \( E = F \) and let \( G = \text{Gal}(L/K) \) be the Galois group. Let \( \alpha \in L \) be a uniformizer and let \( f \in \mathcal{O}_K[X] \) be the minimal polynomial. Decompose \( f = \prod_{i=1}^n (X - \alpha_i) \) so that \( \alpha_n = \alpha \) and \( \text{ord}_L(\alpha_i - \alpha_n) \) is increasing in \( i \).

1. Let \( r > 1 \) be the largest rational number such that \( G^r \neq 1 \). Then, we have

\[
(1.7) \quad e_r = \text{ord}_L f'(\alpha) + \text{ord}_L(\alpha_{n-1} - \alpha_n).
\]

Define an injection \( \beta : G^r \rightarrow G_a \) by \( \beta(\sigma) \equiv \frac{\sigma(\alpha) - \alpha}{\alpha_{n-1} - \alpha_n} \mod m \) and an additive polynomial \( b_1 \in E[X] \) by \( b_1 = \prod_{\sigma \in G^r} (X - \beta(\sigma)) / \prod_{\sigma \in G^r, \sigma \neq 1} (-\beta(\sigma)) \). Define an isomorphism \( G_a \rightarrow m^{e(r-1)}_L / m^{e(r-1)+1}_L \) by \( f(\alpha)(\alpha_{n-1} - \alpha_n) / f(0) \in m^{e(r-1)}_L \) and identify \( Q^{(r)}_E \) with \( m^{e(r-1)}_L / m^{e(r-1)+1}_L \) by the isomorphism \( (1.6) \). Then, there exists an isomorphism

\[
(1.8) \quad 0 \rightarrow G^r \quad \rightarrow Q^{(r)}_E \quad \rightarrow Q^{(r)}_E \quad \rightarrow 0
\]

of exact sequences.

2. Let \( i > 0 \) be the largest integer such that \( G_{i, \text{cl}} = \text{Ker}(G \rightarrow \text{Aut}(\mathcal{O}_L/m_{i+1}^{\#})) \neq 1 \). Then, we have

\[
(1.9) \quad i = \text{ord}_L(\alpha_{n-1} - \alpha_n) - 1.
\]

Let \( K \subset M \subset L \) be the intermediate extension corresponding to \( G_{i, \text{cl}} \subset G \) and let \( U_i^L = 1 + m_i^L \subset L^X \) and \( U_i^M = 1 + m_i^M \subset M^X \) be the multiplicative subgroups. Let \( N_i^L : U_i^L / U_i^{i+1} \rightarrow U_i^M / U_i^{i+1} \) denote the morphism induced by the norm \( N_{L/M} : L^X \rightarrow M^X \) and \( T_i^L : U_i^L / U_i^{i+1} = m_i^L / m_i^{i+1} \rightarrow U_i^M / U_i^{i+1} = m_i^M / m_i^{i+1} \) be the isomorphism induced by the trace \( \text{Tr}_{L/M} : L \rightarrow M \). Define an isomorphism \( G_a \rightarrow U_i^L / U_i^{i+1} \) by sending 1 to the class of \( \alpha_{n-1}/\alpha_n \in U_i^L \). Then, the diagram

\[
(1.10) \quad 0 \rightarrow G_{i, \text{cl}} \quad \rightarrow G_{i, \text{cl}} \quad \rightarrow G_a \quad \rightarrow G_a \quad \rightarrow 0
\]

is an isomorphism of exact sequences.

**Proof.** 1. We have \((1.7) \) by [7] Lemma 3.3.1.5. We have a commutative diagram \((1.8) \) with \( b_1 \) and \( f'(\alpha)(\alpha_{n-1} - \alpha_n) \) replaced by \( b = \prod_{\sigma \in G^r} (X - \beta(\sigma)) \) and \( c = \prod_{i=1}^m (\alpha_n - \alpha_i) \cdot (\alpha_{n-1} - \alpha_n)^{n-m} \) for \( m = \#G - \#G^r \) by [7]
Lemma 3.3.1.1, since the canonical isomorphism $N_{T/Q} \to N_{E/S} \otimes E$ maps $f$ to $f(0)$. Since $b = \prod_{\sigma \in G', \sigma \neq 1} (-\beta(\sigma)) \cdot b_1$ and $c = \prod_{\sigma \in G', \sigma \neq 1} (-\beta(\sigma)) \cdot f'(\alpha)(\alpha_{n-1} - \alpha_n)$, we obtain (1.8).

2. Since $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ and $\text{ord}_L(\alpha_i - \alpha_n)$ is increasing, the equality (1.9) follows from the definition of $G_{i,cl}$.

By [8, Chapitre V, Proposition 8, Section 6], the morphism $(T^{i})^{-1} \circ N^{i}: U_{L}^{i}/U_{L}^{i+1} \to U_{L}^{i}/U_{L}^{i+1}$ is defined by a separable additive polynomial such that the coefficient of degree 1 is 1 and the upper line of (1.10) is exact. Since $\sigma(\alpha)/\alpha = 1 + (\sigma(\alpha) - \alpha_n)/(\alpha_{n-1} - \alpha_n) \cdot (\alpha_{n-1}/\alpha_n - 1)$, the left square is commutative. Since the left square is commutative, the right square is also commutative by the uniqueness of $b_1$. \hfill \Box

**Corollary 1.3.** 1. We have

(1.11) $e_r = \text{ord}_L f'(\alpha) + (i + 1)$.

2. There exists an isomorphism

$$
\begin{array}{cccccc}
0 & \longrightarrow & G^r & \longrightarrow & G^{(r)_o} & \longrightarrow & m_{L}^{e(\alpha-\alpha_n)+1} / m_{L}^{e(\alpha-\alpha_n)+1} & \longrightarrow & 0 \\
0 & \longrightarrow & G_{i,cl} & \longrightarrow & U_{L}^{i}/U_{L}^{i+1} & \xrightarrow{(T^{i})^{-1} \circ N^{i}} & U_{L}^{i}/U_{L}^{i+1} & \longrightarrow & 0 \\
\end{array}
$$

of exact sequences.

**Proof.** 1. The equality (1.11) follows from (1.7) and (1.9).

2. Combining (1.8) and (1.10), we obtain (1.11). \hfill \Box

By Lemma 1.12, the extension in the upper line of (1.12) defines a canonical injection

(1.13) $G^{r^\vee} = \text{Hom}_{F_p}(G^r, F_p) \to \text{Hom}_{E}(m_{L}^{e(\alpha-\alpha_n)+1} / m_{L}^{e(\alpha-\alpha_n)+1}, E)$.

Assume that the residue field of $F$ is perfect and let $L$ be a Galois extension of $K$. Let $K^{ur} \subset L$ denote the maximum unramified extension corresponding to the inertia subgroup $I \subset G$. For a rational number $r > 1$, we apply the construction of (1.13) to the totally ramified extension $M \subset L$ of $K^{ur}$ corresponding to $G^{r^+} = \bigcup_{s>r} G^s \subset I \subset G$ and to $H^r = \text{Gr}^r G = G^r / G^{r^+} \subset H = \text{Gal}(M/K^{ur}) = I / G^{r^+}$. Let $e' = e_{M/K}$ be the ramification index and $E' \subset E$ be the residue field of $M$. We obtain an injection

(1.14) $(G^{r^\vee} G^{r})^\vee = \text{Hom}_{F_p}(G^{r^\vee} G, F_p) \to \text{Hom}_{E'}(m_{M}^{e(\alpha-\alpha_n)+1} / m_{M}^{e(\alpha-\alpha_n)+1}, E')$

$\subset \text{Hom}_{E}(m_{L}^{e(\alpha-\alpha_n)+1} / m_{L}^{e(\alpha-\alpha_n)+1}, E)$.

For abelian extensions, we have the Hasse-Arf theorem.

**Theorem 1.4 ([8, Chapitre V, Section 7, Théorème 1]).** Let $K$ be a henselian discrete valuation field with perfect residue field and let $L$ be a finite abelian extension of $K$. Let $n \geq 1$ be an integer and $r$ be a rational number satisfying $n < r \leq n + 1$. Then, we have $G^r = G^{r^+}$.
2 Tangent spaces and a characterization of ramification groups

Definition 2.1 ([7, Definition 1.1.8]). Let $K$ be a discrete valuation field, $S = \text{Spec} \mathcal{O}_K$ and $F$ be the residue field.

1. For an extension $E$ of $F$, let $L_{E/S}$ denote the cotangent complex and we call the spectrum

\begin{equation}
\Theta_{K,E} = \text{Spec} S(H_1(L_{E/S}))
\end{equation}

of the symmetric algebra over $E$ the tangent space of $S$ at $E$.

2. If $\mathcal{O}_K \to \mathcal{O}_{K'}$ is a faithfully flat morphism of discrete valuation rings, we say that $K'$ is an extension of discrete valuation fields of $K$. We say that an extension $K'$ of discrete valuation fields of $K$ is tangentially dominant if, for a morphism $\bar{F} \to \bar{F}'$ of algebraic closures of the residue fields, the morphism

\[ S(H_1(L_{F/S})) \to S(H_1(L_{F'/S'})) \]

is an injection.

The morphism

\begin{equation}
m_K/m_K^2 \otimes_F \bar{F} = H_1(L_{F/S}) \otimes_F \bar{F} \to H_1(L_{\bar{F}/S})
\end{equation}

defined by the functoriality of cotangent complexes is an injection by [7, Proposition 1.1.3.1]. The injection (2.2) is an isomorphism if $F$ is perfect. The distinguished triangle

\begin{equation}
L_{S/\mathbb{Z}} \otimes_{\bar{O}_S} \bar{F} \to L_{\bar{F}/\mathbb{Z}} \to L_{\bar{F}/S} \to H_1(L_{\bar{F}/S'})
\end{equation}

defines a canonical surjection such that the composition with (2.2) is induced by \[ \text{d}: m_K/m_K^2 \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \bar{F} \]

[7, Proposition 1.7.3] such that the composition with (2.2) is induced by \[ d: m_K/m_K^2 \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} F \]. If $K$ is of characteristic $p > 0$, (2.3) is an isomorphism by [7, Proposition 1.1.7.3]. If $K'$ is a tangentially dominant extension of $K$, the morphism $H_1(L_{\bar{F}/S}) \to H_1(L_{\bar{F}/S'})$ is an injection.

Proposition 2.2 ([7, Proposition 1.1.10]). Let $K \to K'$ be an extension of discrete valuation fields. We consider the following conditions:

1. The ramification index $e_{K'/K}$ is 1 and $F' = \mathcal{O}_{K'}/m_{K'}$ is a separable extension of $F = \mathcal{O}_K/m_K$.

2. The extension $K'$ is tangentially dominant over $K$.

3. The ramification index $e_{K'/K}$ is 1.

Then, we have the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Theorem 2.3. Let $r > 1$ be a rational number. For finite Galois extensions $L$ of henselian discrete valuation fields $K$, there exists a unique way to define a normal subgroup $G^r$ of the Galois group $G = \text{Gal}(L/K)$ satisfying the following conditions:

1. If the residue field of $K$ is perfect, then $G^r = G^r_{cl}^{-1}$.

2. Let $K'$ be a tangentially dominant extension of $K$. Then the natural injection $G' = \text{Gal}(L'/K') \to G$ for $L' = LK'$ induces an isomorphism $G'^r \to G^r$.
For a separable closure $\bar{K}$ of $K$, extend the normalized discrete valuation $\text{ord}_K$ to $\bar{K}$. For a rational number $r$, set $m_K^r = \{ x \in \bar{K} \mid \text{ord}_K x \geq r \}$ and $m_K^{r+} = \{ x \in \bar{K} \mid \text{ord}_K x > r \}$. The quotient $m_K^r / m_K^{r+}$ is a vector space of dimension 1 over the residue field $F$. For $r > 1$, define $G^{r+} = \bigcup_{s > r} G^s$ and $\text{Gr} \bar{G}G = G^r / G^{r+}$.

**Theorem 2.4.** Let $r > 1$ be a rational number. For finite Galois extensions $L$ of henselian discrete valuation fields $K$, for morphisms $L \rightarrow \bar{K}$ to separable closures over $K$ and for the residue field $F$ of $\bar{K}$, there exists a unique way to define an injection

\[(2.4) \quad \text{Hom}(\text{Gr} \bar{G}G, F_p) \rightarrow \text{Hom}_F(m_K^r / m_K^{r+}, H_1(L_F/S)).\]

satisfying the following conditions:

1. Assume that the residue field of $K$ is perfect. Let $E$ be the residue field of $L$, $e = e_{L/K}$ be the ramification index and identify $\text{Hom}_E(m_{L}^{e(r-1)} / m_{L}^{e(r-1)+1}, E)$ with a subgroup of $\text{Hom}_F(m_K^r / m_K^{r+}, H_1(L_F/S))$ by the injection $m_K^r / m_K^{r+} \rightarrow H_1(L_F/S)$ (2.2). Then, the diagram

\[
\begin{array}{ccc}
\text{Hom}(\text{Gr} \bar{G}G, F_p) & \longrightarrow & \text{Hom}_F(m_K^r / m_K^{r+}, H_1(L_F/S)) \\
\| & & \| \\
\text{Hom}(\text{Gr} \bar{G}G, F_p) & \longrightarrow & \text{Hom}_E(m_{L}^{e(r-1)} / m_{L}^{e(r-1)+1}, E)
\end{array}
\]

is commutative.

2. Let $K'$ be a tangentially dominant extension of $K$, let $\bar{K} \rightarrow \bar{K}'$ be a morphism of separable closures extending $L \rightarrow L' = LK'$ and let $\bar{F} \rightarrow \bar{F}'$ be the morphism of residue fields. Then, for the natural injection $G' = \text{Gal}(L'/K') \rightarrow G$, the diagram

\[
\begin{array}{ccc}
\text{Hom}(\text{Gr} \bar{G}G', F_p) & \longrightarrow & \text{Hom}_F(m_K^r / m_K^{r+}, H_1(L_F/S)) \\
\downarrow & & \downarrow \\
\text{Hom}(\text{Gr} \bar{G}G', F_p) & \longrightarrow & \text{Hom}_F(m_K^r / m_K^{r+}, H_1(L_{\bar{F}'}/S')).
\end{array}
\]

is commutative.

The uniqueness is a consequence of the following existence of a tangentially dominant extension with perfect residue field.

**Proposition 2.5 ([7] Proposition 1.1.12).** Let $K$ be a discrete valuation field. Then, there exists a tangentially dominant extension $K'$ of $K$ such that the residue field $F'$ is perfect.

**Proof of Theorem 2.3.** We show the uniqueness. By Proposition 2.5 there exists a tangentially dominant extension $K'$ of $K$ with perfect residue field. Let $G' = \text{Gal}(L'/K') \rightarrow G$ be the natural injection for $L' = LK'$. Then, by the conditions (1) and (2), the subgroup $G' \subset G$ is the image of $G^{e-1} \subset G'$.

To show the existence, it suffices to prove that the subgroup $G' \subset G$ defined in [1] satisfies the conditions (1) and (2). The equality $G' = G^{e-1}$ is proved in [1] Proposition 3.7 (3)]. The condition (2) is satisfied by [7] Proposition 4.2.4 (1)].

**Proof of Theorem 2.4.** We show the uniqueness. If the residue field is perfect, the morphism (2.4) is uniquely determined by the commutative diagram (2.5) since its right vertical arrow is an injection induced by the injection (2.2). In general, by Proposition 2.5.
there exists a tangentially dominant extension $K'$ of $K$ with perfect residue field. Then, the morphism $(2.4)$ is uniquely determined by the commutative diagram $(2.6)$ since its right vertical arrow is an injection.

To show the existence, it suffices to prove that the morphism $[7] (4.20)$ satisfies the conditions (1) and (2). Assume that the residue field is perfect. To show the commutative diagram $(2.5)$, we may assume that $G^r = 1$ and $G^rG = G^r$ by the construction of the morphisms. Then, since the construction of $(1.13)$ is a special case of $[7] (4.20)$, the condition (1) is satisfied. The condition (2) follows from $[7] (4.19)$.

3 Abelian extensions

Theorem 3.1. Let $r > 1$ be a rational number.

1. For finite abelian extensions $L$ of henselian discrete valuation fields $K$, there exists a unique way to define a normal subgroup $G^r$ of the Galois group $G = \text{Gal}(L/K)$ satisfying the following conditions:
   (1) If the residue field of $K$ is perfect, then $G^r = G^r_{cl}$.
   (2) Let $K'$ be a tangentially dominant extension of $K$. Then the natural injection $G' = \text{Gal}(L'/K') \to G$ for $L' = LK'$ induces an isomorphism $G'^r \to G^r$.

2. Let $L$ be a finite abelian extension of a henselian discrete valuation field $K$ and let $n \geq 1$ be the integer satisfying $n < r \leq n + 1$. Then, we have $G^r = G^{n+1}$.

Theorem 3.2. Let $n > 1$ be an integer. For finite abelian extensions $L$ of henselian discrete valuation fields $K$, for morphisms $L \to \bar{K}$ to separable closures over $K$ and for the residue fields $\bar{F}$ of $\bar{K}$, there exists a unique way to define an injection

\[ \text{Hom}(G^n/G^{n+1}, \bar{F}_p) \to \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{F'/S})). \]

satisfying the following conditions:

(1) Assume that the residue field of $K$ is perfect and let the notation be as in Theorem 2.4 (1). Then the diagram

\[
\begin{array}{ccc}
\text{Hom}(G^n/G^{n+1}, \bar{F}_p) & \to & \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{F'/S})) \\
\| & & \uparrow \\
\text{Hom}(G^n_{cl}/G^{n+1}_{cl}, \bar{F}_p) & \to & \text{Hom}_F(m^n_K/m^{n+1}_K, E)
\end{array}
\]

is commutative.

(2) Let $K'$ be a tangentially dominant extension of $K$ and let the notation be as in Theorem 2.4 (2). Then, the diagram

\[
\begin{array}{ccc}
\text{Hom}(G^n/G^{n+1}, \bar{F}_p) & \to & \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{F'/S})) \\
\downarrow & & \downarrow \\
\text{Hom}(G^n/G^{n+1}, \bar{F}_p) & \to & \text{Hom}_{F'}(m^n_{K'/F'/S'} m^{n+1}_{K'/F'/S'}, H_1(L_{F'/S'}))
\end{array}
\]

is commutative.

Proof of Theorem 3.1. 1. is proved in the same way as Theorem 2.3.

2. By 1, this follows from the Hasse-Arf theorem Theorem 1.4. □
Proof of Theorem 3.2. This is a special case of Theorem 2.1.

Assume that $K$ is a henselian discrete valuation field of equal characteristic $p > 0$ and let $L$ be a finite abelian extension. Then, by the Hasse-Arf theorem Theorem 3.1.2 and by the isomorphism $H_1(L_{F/S}) \rightarrow \Omega_{O_F}^1 \otimes_{O_K} F$ (2.3), for an integer $n > 0$, the injection (3.1) defines an injection

$$\text{Hom}(G^n/G^{n+1}, F_p) \rightarrow \text{Hom}_F(m_K^n/m_K^{n+1}, \Omega_{O_K}^1 \otimes_{O_K} F).$$

A decreasing filtration $(G^n_{Ma})$ indexed by integers $n > 0$ is defined in [4, Definition 3.1] as a non-logarithmic modification of a filtration $(G^n_{Ka})$ defined in [4, Definition (2.1)]. Further, for an integer $n > 0$, a canonical morphism

$$rsw': \text{Hom}(G^n_{Ma}/G^n_{Ma}, F_p) \rightarrow \text{Hom}_F(m_K^n/m_K^{n+1}, \Omega_{O_K}^1 \otimes_{O_K} F)$$

is defined in [5, Definition 3.2.5] except the case $p = 2$, $n = 2$ and in [9, Definition 1.18] in the exceptional case $p = 2$, $n = 2$, as a modification of the refined Swan conductor defined in [4, Corollary (5.2)].

As an application, we give a new proof of the equalities of the two filtrations and the two morphisms, different from that in [2] and [9].

Corollary 3.3. Let $L$ be an abelian extension of a henselian discrete valuation field $K$ of equal characteristic $p > 0$ and let $n > 1$ be an integer.

1. ([2, Théorème 9.10 (i)] for $n > 1$, [9, Theorem 3.1] for $n$ general) We have an equality $G^n = G^n_{Ma}$ of subgroups of $G$.

2. ([2, Théorème 9.10 (ii)] for $n > 1$, [9, Corollary 2.13] for $n$ general) The injection (3.1) is the same as $rsw'$ (3.5).

The following proof is by the reduction to the logarithmic variant [2, Théorème 9.11] in the classical case where the residue field is perfect.

Proof. It suffices to show that the filtration $(G^n_{Ma})$ and the morphism $rsw'$ satisfy the conditions in Theorems [3.1] and [3.2].

We show that the conditions (1) are satisfied. Assume that the residue field $F$ is perfect. Then, we have $G^n = G^{n-1}_{cl}$ and $G^n_{Ma} = G^{n-1}_{Ma}$. Since $G^n_{Ka} = G^{n-1}_{cl}$ in this case by [2, Théorème 9.11 (i)], the condition (1) in Theorem 3.1 is satisfied.

We identify $\Omega_{O_K}^1 \otimes_{O_K} F$ with $m_K^2/m_K^2$ by $d: m_K^2/m_K^2 \rightarrow \Omega_{O_K}^1 \otimes_{O_K} F$ and $\text{Hom}_F(m_K^n/m_K^{n+1}, \Omega_{O_K}^1 \otimes_{O_K} F)$ with $\text{Hom}_F(m_K^{n-1}/m_K^{n}, F)$ by the induced isomorphism. Then, the morphism (3.5) is identified with the morphism

$$rsw: \text{Hom}(G^{n-1}_{cl}/G^{n}_{cl}, F_p) \rightarrow \text{Hom}_F(m_K^{n-1}/m_K^{n}, F)$$

defined in [4, Corollary (5.2)]. Since the morphism (3.6) equals (3.4) by [2, Théorème 9.11 (ii)], the condition (1) in Theorem 3.2 is satisfied.

We show that the conditions (2) are satisfied. For an extension $K'$ of henselian discrete valuation field of ramification index 1, the diagram

$$\begin{array}{c}
\text{Hom}(G^n_{Ma}/G^{n+1}_{Ma}, F_p) \\
\downarrow
\end{array} \xrightarrow{rsw'} \begin{array}{c}
\text{Hom}_F(m_K^n/m_K^{n+1}, \Omega_{O_K}^1 \otimes_{O_K} F) \\
\downarrow
\end{array}$$

$$\begin{array}{c}
\text{Hom}(G^n_{Ma}/G^{n+1}_{Ma}, F_p) \\
\downarrow
\end{array} \xrightarrow{rsw'} \begin{array}{c}
\text{Hom}_F(m_{K'}^n/m_{K'}^{n+1}, \Omega_{O_{K'}}^1 \otimes_{O_{K'}} F')
\end{array}$$

(3.7)
is commutative. Hence the condition (2) in Theorem 3.2 is satisfied.

If $K'$ is tangentially dominant over $K$, then the morphism $\Omega^1_{O_K} \otimes_{O_K} F \to \Omega^1_{O_{K'}} \otimes_{O_{K'}} F'$ is an injection. Hence by the commutative diagram (3.7), the morphism $G^n/G^{n+1} \to G^n/G^{n+1}$ is a surjection. By the descending induction on $n$, the condition (2) in Theorem 3.1 is satisfied.

References

[1] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, Amer. J. of Math., 124.5 (2002), 879-920.

[2] A. Abbes, T. Saito, *Analyse micro-locale ℓ-adique en caractéristique $p > 0$: le cas d’un trait*, Publ. RIMS 45 (2009), no. 1, 25-74.

[3] J. Borger, *Conductors and the moduli of residual perfection*, Math. Ann., 329 No. 1, (2004), 1-30.

[4] K. Kato, *Swan conductors for characters of degree one in the imperfect residue field case*, Contemporary Math. 83 (1989), 101–131.

[5] S. Matsuda, *On the Swan conductor in positive characteristic*, Amer. J. Math. 119 (1997), no. 4, 705–739.

[6] T. Saito, *Ramification groups of coverings and valuations*, Tunisian J. of Math., Vol. 1, No. 3, 373-426, 2019.

[7] T. Saito, *Graded quotients of ramification groups of local fields with imperfect residue fields*, [arXiv:2004.03768](https://arxiv.org/abs/2004.03768).

[8] J-P. Serre, *Corps Locaux*, Hermann, Paris, 1968.

[9] Y. Yatagawa, *Equality of two non-logarithmic ramification filtrations of abelianized Galois group in positive characteristic*, Doc. Math. 22 (2017), 917–952.
A characterization of ramification groups of local fields with imperfect residue fields

Takeshi Saito

Abstract

We give a characterization of ramification groups of local fields with imperfect residue fields, using those for local fields with perfect residue fields. As an application, we reprove an equality of ramification groups for abelian extensions defined in different ways.

Let $K$ be a henselian discrete valuation field. Let $L$ be a Galois extension and let $G = \text{Gal}(L/K)$ be the Galois group. In the classical case where the residue field $F$ of $K$ is perfect, the lower numbering filtration $(G_{i,cl})$ indexed by integers $i \geq 0$ is defined by $G_{i,cl} = \text{Ker}(G \to \text{Aut}(\mathcal{O}_L/m_i^{i+1}L))$. Further, the upper numbering filtration $(G_{r,cl})$ indexed by rational numbers $r > 0$ is defined using the Herbrand function to renumber the lower numbering [8, Chapitre IV, Section 3].

In the general case where the residue field need not be perfect, an upper numbering filtration $(G^r)$ indexed by rational numbers $r > 0$ is defined first in [1] using rigid geometry and later in [6] purely in the language of schemes. In the classical case where $F$ is perfect, they are related to each other by the relation $G^r = G^r_{cl}$.

We give an axiomatic characterization of the filtration $(G^r)$. The axiom has two conditions. The first condition is the relation in the classical case above. The second condition is the compatibility with tangentially dominant extensions. A similar approach reducing to the classical case was proposed in [3].

For a discrete valuation field $K$, the tangent space at an algebraic closure $\bar{F}$ of the residue field $F$ is defined as an $\bar{F}$-vector space using the cotangent complex. In the classical case where the residue field is perfect, it is nothing but the scalar extension of the Zariski tangent space defined as the dual $\text{Hom}_F(m_K/m_K^2, \bar{F})$. An extension of discrete valuation fields is said to be tangentially dominant if the induced morphism on the tangent spaces is dominant (Definition 2.1.2). An unramified extension is tangentially dominant and a tangentially dominant extension has ramification index 1.

The uniqueness is a consequence of the existence of tangentially dominant extension with perfect residue field. The existence follows from the functorial properties of the filtration $(G^r)$.

For $r > 1$, the graded quotient $\text{Gr}^r G = G^r/G^{r+}$ is defined by $G^{r+} = \bigcup_{s>r} G^s$ and is an $F_p$-vector space. A canonical injection

$$\text{Hom}(\text{Gr}^r G, F_p) \to \text{Hom}_F(m_K^r/m_K^{r+}, H_1(L_{F/S}))$$  \hspace{1cm} (1)
is defined in \([7, (4.20)]\), as a generalization of a non-logarithmic variant of the refined Swan conductor defined by Kato in [4]. We also give an axiomatic characterization of this morphism, similar to the characterization for \(G^r\) itself.

As an application of the characterizations, we give a new proof of the equality of two filtrations for abelian extensions in positive characteristic. By the Hasse-Arf theorem, the filtration \((G^n)\) defined in [1] is in fact indexed by integers \(n > 1\) for abelian extensions. The other filtration is the filtration \((G^n_{\text{Ma}})\) defined by Matsuda in [5] as a modification of that defined by Kato in [4]. The equality was proved in [2] except for the smallest index \(n = 2\) and the remaining case was proved by Yatagawa in [9]. The equality is proved by verifying that the filtration \((G^n_{\text{Ma}})\) satisfies the same axiom. We also prove that the injection (1) equals the morphism \(rsw'\) defined in [5] and [9], as a modification of the refined Swan conductor defined in [4].

A variant \((G^r_{\text{log}})\) of the upper numbering filtration \((G^r)\) called the logarithmic upper numbering filtration is also defined in [1]. In the case where the ramification index \(e_{L/K}\) is 1, the two filtrations are the same: \(G^r = G^r_{\text{log}}\). If \(K'\) is a log smooth extension of \(K\) and \(L' = LK'\), the canonical injection \(G' = \text{Gal}(L'/K') \to G = \text{Gal}(L/K)\) is known to induce isomorphisms \(G^r_{\text{log}} = G^r_{\text{log}}\) for \(e = e_{K'/K}\). Further if the ramification index \(e_{K'/K}\) is divisible by \(e_{L/K}\), we have \(e_{L'/K'} = 1\). Thus, a characterization of \((G^r)\) gives an indirect characterization \((G^r_{\text{log}})\).

**Acknowledgement**  The author thanks the referee for careful reading and the suggestion to include comments on the logarithmic filtration. The research is partially supported by Grant-in-Aid (B) 19H01780.

1 **Totally ramified case**

Let \(K\) be a henselian discrete valuation field. Let \(L\) be a totally ramified Galois extension of \(K\) and let \(G = \text{Gal}(L/K)\) be the Galois group. For a rational number \(r > 1\), the upper ramification group \(G^r\) defined in [1] Definition 3.4 equals the subgroup defined in [8, Chapitre IV, Section 3] denoted \(G^{r-1}_{\text{cl}}\), by [1] Proposition 3.7 (3]).

Assume that \(L\) is wildly ramified and let \(r > 1\) be the largest rational number such that the subgroup \(G^r\) of the wild inertia subgroup \(P \subset G\) is non-trivial. Let \(E\) be the residue field and \(e = e_{L/K}\) be the ramification index. We give a description of the canonical injection

\[
G^r = \text{Hom}_{F_p}(G^r, F_p) \to \text{Hom}_E(m^e_{L/(r-1)})/m^e_{L(r-1)+1}, E)
\]

for the \(F_p\)-vector space \(G^r\), in the case where \(L\) is totally ramified over \(K\). The injection (2) is a special case of [1].

We begin with a description of extensions of vector spaces over a field of characteristic \(p > 0\) by \(F_p\)-vector spaces.

**Lemma 1.1** Let \(F\) be a field of characteristic \(p > 0\).

1. Let \(G \subset F\) be a finite subgroup of the additive group. Then, the polynomial

\[
a_1 = \frac{\prod_{\sigma \in G}(X - \sigma)}{\prod_{\sigma \in G, \sigma \neq 0}(-\sigma)}
\]

(3)
since $f$ is an Eisenstein polynomial such that the coefficient of degree 1 is 1 and that the sequence

$$0 \to G \to \mathbf{G}_a \xrightarrow{a_1} \mathbf{G}_a \to 0 \quad (4)$$

is exact.

2. ([4, Proposition 2.1.6 (2)⇒(3)]) Let $E$ be an $F$-vector space of finite dimension and let $0 \to G \to H \to E \to 0$ be an extension of $E$ by an $F_p$-vector space $G$ of finite dimension, as smooth group schemes over $F$. Define a morphism

$$[H]: G^\vee = \text{Hom}_{F_p}(G, F_p) \to \text{Ext}(E, F_p) = E^\vee = \text{Hom}_F(E, F) \quad (5)$$

by sending a character $\chi: G \to F_p$ to the linear form $f: E \to F$ such that there exists a commutative diagram

$$\begin{array}{cccccc}
0 & \to & G & \to & H & \to & E & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_p & \to & \mathbf{G}_a & \xrightarrow{x^p-x} & \mathbf{G}_a & \to & 0.
\end{array}$$

If $H$ is connected, then the morphism $[H]: G^\vee \to E^\vee$ is an injection.

**Proof** 1. By [4, Lemma 2.1.5], $a = \prod_{\sigma \in G}(X - \sigma) \in F[X]$ is an additive separable polynomial such that [4] with $a_1$ replaced by $a$ is exact. Since the coefficient in $a$ of degree 1 is $\prod_{\sigma \in G, \sigma \neq 0}(-\sigma)$, the assertion follows.

Let $K$ be a henselian discrete valuation field and $L$ be a totally ramified Galois extension of degree $e$ of Galois group $G$. Let $\alpha \in L$ be a uniformizer and let $E = F$ denote the residue field. The minimal polynomial $f \in \mathcal{O}_K[X]$ is an Eisenstein polynomial and the constant term $\pi = f(0)$ is a uniformizer of $K$. We define a closed immersion $T = \text{Spec} \mathcal{O}_L \to Q = \text{Spec} \mathcal{O}_K[X]$ by sending $X$ to $\alpha$. For a rational number $r > 1$ such that $er \in \mathbb{Z}$, define a dilatation

$$Q_T[r] = \text{Spec} \mathcal{O}_L[X][f/\alpha^{er}] \to Q_T = \text{Spec} \mathcal{O}_L[X].$$

The generator $f$ of the kernel $I = \text{Ker}(\mathcal{O}_K[X] \to \mathcal{O}_L)$ defines a basis over $\mathcal{O}_L$ of the conormal module $N_{T/Q} = I/I^2$ and $\alpha^{er}$ defines a basis of the $E$-vector space $m^e_L/m^{e+1}_L$. As subspaces of $N_{E/Q} = J/J^2$ for $J = \text{Ker}(\mathcal{O}_K[X] \to E) = (X, f) = (X, \pi)$, we have an equality

$$N_{T/Q} \otimes_{\mathcal{O}_L} E = m_K/m^2_K \quad (6)$$

since $f$ is an Eisenstein polynomial. The basis $f$ of $N_{T/Q} \otimes_{\mathcal{O}_L} E$ corresponds to the uniformizer $\pi = f(0) \in m_K/m^2_K$.

By sending $S$ to $f/\alpha^{er}$, we define an isomorphism $\mathcal{O}_L[X, S]/(f - \alpha^{er}S) \to \mathcal{O}_L[X][f/\alpha^{er}]$. Since $f$ is an Eisenstein polynomial, the reduced closed fiber $Q_E[r] = \text{Spec} (\mathcal{O}_L[X][f/\alpha^{er}] \otimes_{\mathcal{O}_L} E)_{\text{red}}$ is identified with $\text{Spec} E[S]$. By this identification and (6), we define an isomorphism

$$Q_E[r] \to \text{Hom}_E(m^e_L/m^{e+1}_L, N_{T/Q} \otimes_{\mathcal{O}_L} E) \to \text{Hom}_E(m^e_L/m^{e+1}_L, m_K/m^2_K) = m^e_L/m^{e(r-1)+1}_L \quad (7)$$
of smooth group schemes of dimension 1 over \( E \).

Let \( Q_T^{(r)} \rightarrow Q_T^{(r)} \) be the normalization and define a section \( T \rightarrow Q_T^{(r)} \) to be the unique lifting of the section \( T \rightarrow Q_T \) defined by sending \( X \) to \( \alpha \). Let \( Q_E^{(r)} \) denote the reduced part of the closed fiber \( Q_T^{(r)} \times_T \text{Spec} \ E \) and let \( Q_E^{(r)} \subset Q_E^{(r)} \) denote the connected component containing the image of the closed point of \( T \) by the section \( T \rightarrow Q_T^{(r)} \).

**Proposition 1.2** Let \( K \) be a henselian discrete valuation field with residue field \( F \) of characteristic \( p > 0 \). Let \( L \) be a totally ramified Galois extension of degree \( n = e \) with residue field \( E = F \) and let \( G = \text{Gal}(L/K) \) be the Galois group. Let \( \alpha \in L \) be a uniformizer and let \( f \in \mathcal{O}_K[X] \) be the minimal polynomial. Decompose \( f = \prod_{i=1}^{n}(X - \alpha_i) \) so that \( \alpha_n = \alpha \) and \( \text{ord}_E(\alpha_i - \alpha_n) \) is increasing in \( i \).

1. Let \( r > 1 \) be the largest rational number such that \( G^r \neq 1 \). Then, we have

\[
er = \text{ord}_L f'(\alpha) + \text{ord}_L (\alpha_{n-1} - \alpha_n).
\]

Define an injection \( \beta: G^r \rightarrow G_a \) by \( \beta(\sigma) \equiv \frac{\sigma(\alpha) - \alpha}{\alpha_{n-1} - \alpha_n} \mod m_L \) and an additive polynomial \( b_1 \in E[X] \) by \( b_1 = \prod_{\sigma \in G^r}(X - \beta(\sigma))/\prod_{\sigma \in G^r, \sigma \neq 1}(-\beta(\sigma)) \). Define an isomorphism \( G_a \rightarrow mL^{(r-1)}/m_L^{(r-1)+1} \) by \( f'(\alpha)(\alpha_{n-1} - \alpha_n)/f(0) \in mL^{(r-1)} \) and identify \( Q_E^{(r)} \) with \( mL^{(r-1)}/m_L^{(r-1)+1} \) by the isomorphism \( [7] \). Then, there exists an isomorphism

\[
0 \longrightarrow G^r \longrightarrow Q_E^{(r)} \longrightarrow Q_E^{(r)} \longrightarrow 0
\]

of exact sequences.

2. Let \( i > 0 \) be the largest integer such that \( G_{i,cl} = \ker(G \rightarrow \text{Aut}(\mathcal{O}_L/m_L^{i+1})) \neq 1 \). Then, we have

\[
i = \text{ord}_L (\alpha_{n-1} - \alpha_n) - 1.
\]

Let \( K \subset M \subset L \) be the intermediate extension corresponding to \( G_{i,cl} \subset G \) and let \( U_i^1 = 1 + mL_i \subset L^\times \) and \( U_M^i = 1 + m_M^i \subset M^\times \) be the multiplicative subgroups. Let \( N_i: U_i^L/U_i^{i+1} \rightarrow U_M^i/U_M^{i+1} \) denote the morphism induced by the norm \( N_{L/M}: L^\times \rightarrow M^\times \) and \( T_i: U_i^L/U_i^{i+1} = mL_i/m_L^{i+1} \rightarrow U_M^i/U_M^{i+1} = mL_i/m_L^{i+1} \) be the isomorphism induced by the trace \( T_{L/M}: L \rightarrow M \).

Define an isomorphism \( G_a \rightarrow U_i^L/U_i^{i+1} \) by sending \( 1 \) to the class of \( \alpha_{n-1}/\alpha_n \in U_i^L \). Then, the diagram

\[
0 \longrightarrow G_{i,cl} \overset{\sigma \rightarrow \sigma(\alpha)/\alpha}{\longrightarrow} U_i^L/U_i^{i+1} \overset{(T_i)^{-1} \circ N_i}{\longrightarrow} U_i^L/U_i^{i+1} \longrightarrow 0
\]

is an isomorphism of exact sequences.

**Proof** 1. We have \([8]\) by \([7]\) Lemma 3.3.1.5].
We have a commutative diagram (9) with $b_1$ and $f'(\alpha)(\alpha_{n-1} - \alpha_n)$ replaced by $b = \prod_{\sigma \in G^r} (X - \beta(\sigma))$ and $c = \prod_{i=1}^m (\alpha_n - \alpha_i) \cdot (\alpha_{n-1} - \alpha_n)^{n-m}$ for $m = \#G - \#G^r$ by [7, Lemma 3.3.1.1], since the canonical isomorphism $N_T/Q \to N_E/S \otimes E$ maps $f$ to $f(0)$. Since $b = \prod_{\sigma \in G^r, \sigma \neq 1} (-\beta(\sigma)) \cdot b_1$ and $c = \prod_{\sigma \in G^r, \sigma \neq 1} (-\beta(\sigma)) \cdot f'(\alpha)(\alpha_{n-1} - \alpha_n)$, we obtain (9).

2. Since $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ and $\text{ord}_L(\alpha_i - \alpha_n)$ is increasing, the equality (10) follows from the definition of $G_{i, \ell}$.

By [8, Chapitre V, Proposition 8, Section 6], the morphism $(T^i)^{-1} \circ N^i: U_L/U_L^{i+1} \to U_L/U_L^{i+1}$ is defined by a separable additive polynomial such that the coefficient of degree 1 is 1 and the upper line of (11) is exact. Since $\sigma(\alpha)/\alpha = 1 + (\sigma(\alpha) - \alpha_n)/(\alpha_{n-1} - \alpha_n) \cdot (\alpha_{n-1}/\alpha_{n-1})$, the left square is commutative. Since the left square is commutative, the right square is also commutative by the uniqueness of $b_1$.

\begin{corollary}
1. We have
\[ er = \text{ord}_L f'(\alpha) + (i + 1). \] (12)

2. There exists an isomorphism
\begin{equation}
\begin{array}{ccccccccc}
0 & \longrightarrow & G^r & \longrightarrow & G_E^{(r)0} & \longrightarrow & m_L^{(r-1)} / m_L^{(r-1)+1} & \longrightarrow & 0 \\
\| & & \| & \| & & \| & \| & & \\
0 & \longrightarrow & G_{i, \ell} & \longrightarrow & U_L/U_L^{i+1} & \stackrel{(T^i)^{-1} \circ N^i}{\longrightarrow} & U_L/U_L^{i+1} & \longrightarrow & 0
\end{array}
\end{equation} (13)
of exact sequences.
\end{corollary}

\begin{proof}
1. The equality (12) follows from (8) and (10).

2. Combining (9) and (11), we obtain (13).
\end{proof}

By Lemma 11.2, the extension in the upper line of (13) defines a canonical injection
\begin{equation}
G^{r^\vee} = \text{Hom}_{F_p}(G^r, F_p) \to \text{Hom}_E(m_L^{(r-1)} / m_L^{(r-1)+1}, E). \tag{14}
\end{equation}

Assume that the residue field of $F$ is perfect and let $L$ be a Galois extension of $K$. Let $K^{ur} \subset L$ denote the maximum unramified extension corresponding to the inertia subgroup $I \subset G$. For a rational number $r > 1$, we apply the construction of (14) to the totally ramified extension $M \subset L$ of $K^{ur}$ corresponding to $G^{r^+} = \bigcup_{s > r} G^s \subset I \subset G$ and to $H^r = G^{r^+}/G^r \subset H = \text{Gal}(M/K^{ur}) = I/G^{r^+}$. Let $e' = e_{M/K}$ be the ramification index and $E' \subset E$ be the residue field of $M$. We obtain an injection
\begin{equation}
(G^{r^\vee}G)^\vee = \text{Hom}_{F_p}(G^{r^\vee}G, F_p) \to \text{Hom}_{E'}(m_M^{e'(r-1)} / m_M^{e'(r-1)+1}, E') \subset \text{Hom}_E(m_L^{e(r-1)} / m_L^{e(r-1)+1}, E). \tag{15}
\end{equation}

For abelian extensions, we have the Hasse-Arf theorem.

\begin{theorem} \textbf{([8 Chapitre V, Section 7, Théorème 1])}
Let $K$ be a henselian discrete valuation field with perfect residue field and let $L$ be a finite abelian extension of $K$. Let $n \geq 1$ be an integer and $r$ be a rational number satisfying $n < r \leq n + 1$. Then, we have $G^r = G^{n+1}$.
\end{theorem}
2 Tangent spaces and a characterization of ramification groups

Definition 2.1 ([7, Definition 1.1.8]) Let $K$ be a discrete valuation field, $S = \text{Spec} \mathcal{O}_K$ and $F$ be the residue field.

1. For an extension $E$ of $F$, let $L_{E/S}$ denote the cotangent complex and we call the spectrum
   \[ \Theta_{K,E} = \text{Spec} S(H_1(L_{E/S})) \] (16)
of the symmetric algebra over $E$ the tangent space of $S$ at $E$.

2. If $\mathcal{O}_K \to \mathcal{O}_{K'}$ is a faithfully flat morphism of discrete valuation rings, we say that $K'$ is an extension of discrete valuation fields of $K$. We say that an extension $K'$ of discrete valuation fields of $K$ is tangentially dominant if, for a morphism $\bar{F} \to \bar{F}'$ of algebraic closures of the residue fields, the morphism
   \[ S(H_1(L_{\bar{F}/S})) \to S(H_1(L_{\bar{F}'/S}')) \]
is an injection.

The morphism
   \[ \frac{m_K}{m_K^2} \otimes_F \bar{F} = H_1(L_{F/S}) \otimes_F \bar{F} \to H_1(L_{\bar{F}/S}) \] (17)
defined by the functoriality of cotangent complexes is an injection by [7, Proposition 1.1.3.1]. The injection (17) is an isomorphism if $F$ is perfect. The distinguished triangle
   \[ L_{S/Z} \otimes_{\mathcal{O}_S} \bar{F} \to L_{\bar{F}/Z} \to L_{\bar{F}/S} \]
defines a canonical surjection
   \[ H_1(L_{F/S}) \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \bar{F} \] (18)
[7, Proposition 1.1.7.3] such that the composition with (17) is induced by $d: \frac{m_K}{m_K^2} \to \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \bar{F}$. If $K$ is of characteristic $p > 0$, (18) is an isomorphism by [7, Proposition 1.1.7.3]. If $K'$ is a tangentially dominant extension of $K$, the morphism $H_1(L_{\bar{F}/S}) \to H_1(L_{\bar{F}'/S'})$ is an injection.

Proposition 2.2 ([7, Proposition 1.1.10]) Let $K \to K'$ be an extension of discrete valuation fields. We consider the following conditions:

1. The ramification index $e_{K'/K}$ is 1 and $F' = \mathcal{O}_{K'}/m_{K'}$ is a separable extension of $F = \mathcal{O}_K/m_K$.
2. The extension $K'$ is tangentially dominant over $K$.
3. The ramification index $e_{K'/K}$ is 1.

Then, we have the implications $(1) \Rightarrow (2) \Rightarrow (3)$.

Theorem 2.3 Let $r > 1$ be a rational number. For finite Galois extensions $L$ of henselian discrete valuation fields $K$, there exists a unique way to define a normal subgroup $G^r$ of the Galois group $G = \text{Gal}(L/K)$ satisfying the following conditions:

1. If the residue field of $K$ is perfect, then $G^r = G_{cl}^{-1}$.
2. Let $K'$ be a tangentially dominant extension of $K$. Then the natural injection $G' = \text{Gal}(L'/K') \to G$ for $L' = LK'$ induces an isomorphism $G'^r \to G^r$. 

For a separable closure $\bar{K}$ of $K$, extend the normalized discrete valuation $\text{ord}_K$ to $\bar{K}$. For a rational number $r$, set $m_r^K = \{ x \in \bar{K} \mid \text{ord}_K x \geq r \} \supset m_r^{+} = \{ x \in \bar{K} \mid \text{ord}_K x > r \}$. The quotient $m_r^K / m_r^+\bar{K}$ is a vector space of dimension 1 over the residue field $\bar{F}$. For $r > 1$, define $G^r = \bigcup_{s > r} G^s$ and $G^r G = G^r / G^{r+}$. 

**Theorem 2.4** Let $r > 1$ be a rational number. For finite Galois extensions $L$ of henselian discrete valuation fields $K$, for morphisms $L \to \bar{K}$ to separable closures over $K$ and for the residue field $\bar{F}$ of $\bar{K}$, there exists a unique way to define an injection

$$\text{Hom}(Gr^r G, F_p) \to \text{Hom}_F (m_r^K / m_r^+\bar{K}, H_1(L_E/S)).$$

(19)

satisfying the following conditions:

1. Assume that the residue field of $K$ is perfect. Let $E$ be the residue field of $L$, $e = e_{L/K}$ be the ramification index and identify $\text{Hom}_E (m_{L}^{e(r-1)}/m_{L}^{e(r-1)+1}, E)$ with a subgroup of $\text{Hom}_F (m_r^K/m_r^+\bar{K}, H_1(L_E/S))$ by the injection $m_r^K/m_r^+\bar{K} \to H_1(L_E/S)$ (17). Then, the diagram

$$\begin{array}{ccc}
\text{Hom}(Gr^r G, F_p) & \longrightarrow & \text{Hom}_F (m_r^K/m_r^+\bar{K}, H_1(L_E/S)) \\
\downarrow & & \uparrow \\
\text{Hom}(Gr^r G, F_p) & \longrightarrow & \text{Hom}_E (m_{L}^{e(r-1)}/m_{L}^{e(r-1)+1}, E)
\end{array}$$

(20)

is commutative.

2. Let $K'$ be a tangentially dominant extension of $K$, let $\bar{K} \to \bar{K}'$ be a morphism of separable closures extending $L \to L' = LK'$ and let $\bar{F} \to \bar{F}'$ be the morphism of residue fields. Then, for the natural injection $G' = \text{Gal}(L'/K') \to G$, the diagram

$$\begin{array}{ccc}
\text{Hom}(Gr^r G, F_p) & \longrightarrow & \text{Hom}_F (m_r^K/m_r^+\bar{K}, H_1(L_E/S)) \\
\downarrow & & \downarrow \\
\text{Hom}(Gr^r G', F_p) & \longrightarrow & \text{Hom}_{F'} (m_r^{K'/}\bar{K}', H_1(L_{E'/S'}))
\end{array}$$

(21)

is commutative.

The uniqueness is a consequence of the following existence of a tangentially dominant extension with perfect residue field.

**Proposition 2.5** (17 Proposition 1.1.12) Let $K$ be a discrete valuation field. Then, there exists a tangentially dominant extension $K'$ of $K$ such that the residue field $\bar{F}'$ is perfect.

**Proof of Theorem 2.3** We show the uniqueness. By Proposition 2.5, there exists a tangentially dominant extension $K'$ of $K$ with perfect residue field. Let $G' = \text{Gal}(L'/K') \to G$ be the natural injection for $L' = LK'$. Then, by the conditions (1) and (2), the subgroup $G^r \subset G$ is the image of $G_{cl}^{r-1} \subset G'$.

To show the existence, it suffices to prove that the subgroup $G^r \subset G$ defined in [1] satisfies the conditions (1) and (2). The equality $G^r = G_{cl}^{r-1}$ is proved in [1 Proposition 3.7 (3)]. The condition (2) is satisfied by [1 Proposition 4.2.4 (1)].

□
Proof of Theorem 2.4 We show the uniqueness. If the residue field is perfect, the morphism \( (19) \) is uniquely determined by the commutative diagram \( (20) \) since its right vertical arrow is an injection induced by the injection \( (17) \). In general, by Proposition 2.5, there exists a tangentially dominant extension \( K' \) of \( K \) with perfect residue field. Then, the morphism \( (19) \) is uniquely determined by the commutative diagram \( (21) \) since its right vertical arrow is an injection.

To show the existence, it suffices to prove that the morphism \( [7, (4.20)] \) satisfies the conditions (1) and (2). Assume that the residue field is perfect. To show the commutative diagram \( (20) \), we may assume that \( Gr_r + 1 = 1 \) and \( Gr_r G = G_r \) by the construction of the morphisms. Then, since the construction of \( (14) \) is a special case of \( [7, (4.20)] \), the condition (1) is satisfied. The condition (2) follows from \( [7, (4.19)] \).

\[ \square \]

3 Abelian extensions

Theorem 3.1 Let \( r > 1 \) be a rational number.

1. For finite abelian extensions \( L \) of henselian discrete valuation fields \( K \), there exists a unique way to define a normal subgroup \( G^r \) of the Galois group \( G = \text{Gal}(L/K) \) satisfying the following conditions:
   (1) If the residue field of \( K \) is perfect, then \( G^r = G^r_{cl} \).
   (2) Let \( K' \) be a tangentially dominant extension of \( K \). Then the natural injection \( G' = \text{Gal}(L'/K') \to G \) for \( L' = LK' \) induces an isomorphism \( G^r \to G^r \).

2. Let \( L \) be a finite abelian extension of a henselian discrete valuation field \( K \) and let \( n \geq 1 \) be the integer satisfying \( n < r \leq n + 1 \). Then, we have \( G^r = G^{n+1} \).

Theorem 3.2 Let \( n > 1 \) be an integer. For finite abelian extensions \( L \) of henselian discrete valuation fields \( K \), for morphisms \( L \to \bar{K} \) to separable closures over \( K \) and for the residue fields \( \bar{F} \) of \( \bar{K} \), there exists a unique way to define an injection

\[ \text{Hom}(G^n/G^{n+1}, F_p) \to \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{\bar{F}}/S)). \] (22)

satisfying the following conditions:

(1) Assume that the residue field of \( K \) is perfect and let the notation be as in Theorem 2.4 (1). Then the diagram

\[ \text{Hom}(G^n/G^{n+1}, F_p) \longrightarrow \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{\bar{F}}/S)) \]

is commutative.

(2) Let \( K' \) be a tangentially dominant extension of \( K \) and let the notation be as in Theorem 2.4 (2). Then, the diagram

\[ \text{Hom}(G^n/G^{n+1}, F_p) \longrightarrow \text{Hom}_F(m^n_K/m^{n+1}_K, H_1(L_{\bar{F}}/S)) \]

is commutative.
is commutative.

**Proof of Theorem 3.1**
1. is proved in the same way as Theorem 2.3.
2. By 1, this follows from the Hasse-Arf theorem Theorem 1.4.

**Proof of Theorem 3.2**
This is a special case of Theorem 2.4.

Assume that $K$ is a henselian discrete valuation field of equal characteristic $p > 0$ and let $L$ be a finite abelian extension. Then, by the Hasse-Arf theorem Theorem 3.1 and by the isomorphism $H_1(L/F) \to \Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F}$ (18), for an integer $n > 0$, the injection (22) defines an injection

$$\text{Hom}(G^n/G^{n+1}, F_p) \to \text{Hom}_F(m^n_K/m^{n+1}_K, \Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F}).$$

(25)

A decreasing filtration $(G^a_{Ma})$ indexed by integers $a > 0$ is defined in [5, Definition 3.1.1] as a non-logarithmic modification of a filtration $(G^a_{Ka})$ defined in [4, Definition (2.1)]. Further, for an integer $n > 0$, a canonical morphism

$$\text{rsw}' : \text{Hom}(G^a_{Ma}/G^{a+1}_{Ma}, F_p) \to \text{Hom}_F(m^n_K/m^{n+1}_K, \Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F})$$

(26)

is defined in [5, Definition 3.2.5] except the case $p = 2, n = 2$ and in [9, Definition 1.18] in the exceptional case $p = 2, n = 2$, as a modification of the refined Swan conductor defined in [4, Corollary (5.2)].

As an application, we give a new proof of the equalities of the two filtrations and the two morphisms, different from that in [2] and [9].

**Corollary 3.3** Let $L$ be an abelian extension of a henselian discrete valuation field $K$ of equal characteristic $p > 0$ and let $n > 1$ be an integer.

1. ([2, Théorème 9.10 (i)] for $n > 1$, [2, Theorem 3.1] for $n$ general) We have an equality $G^n = G^n_{Ma}$ of subgroups of $G$.
2. ([2, Théorème 9.10 (ii)] for $n > 1$, [9, Corollary 2.13] for $n$ general) The injection (25) is the same as rsw' (26).

The following proof is by the reduction to the logarithmic variant [2, Théorème 9.11] in the classical case where the residue field is perfect.

**Proof** It suffices to show that the filtration $(G^a_{Ma})$ and the morphism rsw' satisfy the conditions in Theorems 3.1 and 3.2.

We show that the conditions (1) are satisfied. Assume that the residue field $F$ is perfect. Then, we have $G^n = G^{n-1}_{cl}$ and $G^n_{Ma} = G^{n-1}_{Ka}$. Since $G^{n-1}_{Ka} = G^{n-1}_{cl}$ in this case by [2, Théorème 9.11 (i)], the condition (1) in Theorem 3.1 is satisfied.

We identify $\Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F}$ with $m_K/m_K^2$ by $d: m_K/m_K^2 \to \Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F}$ and $\text{Hom}_F(m^n_K/m^{n+1}_K, \Omega^1_{\overline{O}_K} \otimes_{\overline{O}_K} \overline{F})$ with $\text{Hom}_F(m^{n-1}_K/m^n_K, \overline{F})$ by the induced isomorphism. Then, the morphism (26) is identified with the morphism

$$\text{rsw} : \text{Hom}(G^{n-1}_{cl}/G^n_{cl}, F_p) \to \text{Hom}_F(m^{n-1}_K/m^n_K, F)$$

(27)
defined in [4, Corollary (5.2)]. Since the morphism (27) equals (25) by [2, Théorème 9.11 (ii)], the condition (1) in Theorem 3.2 is satisfied.

We show that the conditions (2) are satisfied. For an extension $K'$ of henselian discrete valuation field of ramification index 1, the diagram

\[
\begin{array}{ccc}
\Hom(G_m^{n+1}/G_m^n, \mathbb{F}_p) & \xrightarrow{\text{rsw'}} & \Hom_{\mathbb{F}}(m^n_{K'}/m^{n+1}_{K'}, \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \bar{F}) \\
\downarrow & & \downarrow \\
\Hom(G_m^{n+1}/G_m^n, \mathbb{F}_p) & \xrightarrow{\text{rsw'}} & \Hom_{\mathbb{F}'}(m^n_{K'}/m^{n+1}_{K'}, \Omega^1_{\mathcal{O}_{K'}} \otimes_{\mathcal{O}_{K'}} \bar{F}')
\end{array}
\]

is commutative. Hence the condition (2) in Theorem 3.2 is satisfied.

If $K'$ is tangentially dominant over $K$, then the morphism $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \bar{F} \to \Omega^1_{\mathcal{O}_{K'}} \otimes_{\mathcal{O}_{K'}} \bar{F}'$ is an injection. Hence by the commutative diagram (28), the morphism $G^n_m/G^{n+1}_m \to G^n/G^{n+1}$ is a surjection. By the descending induction on $n$, the condition (2) in Theorem 3.1 is satisfied.

□

References

[1] A. Abbes, T. Saito, Ramification of local fields with imperfect residue fields, Amer. J. of Math., 124.5 (2002), 879-920.

[2] A. Abbes, T. Saito, Analyse micro-locale ℓ-adique en caractéristique $p > 0$: le cas d’un trait, Publ. RIMS 45 (2009), no. 1, 25-74.

[3] J. Borger, Conductors and the moduli of residual perfection, Math. Ann., 329 No. 1, (2004), 1-30.

[4] K. Kato, Swan conductors for characters of degree one in the imperfect residue field case, Contemporary Math. 83 (1989), 101–131.

[5] S. Matsuda, On the Swan conductor in positive characteristic, Amer. J. Math. 119 (1997), no. 4, 705–739.

[6] T. Saito, Ramification groups of coverings and valuations, Tunisian J. of Math., Vol. 1, No. 3, 373-426, 2019.

[7] T. Saito, Graded quotients of ramification groups of local fields with imperfect residue fields, arXiv:2004.03768.

[8] J-P. Serre, CORPS LOCAUX, Hermann, Paris, 1968.

[9] Y. Yatagawa, Equality of two non-logarithmic ramification filtrations of abelianized Galois group in positive characteristic, Doc. Math. 22 (2017), 917–952.

School of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan

E-mail: t-saito@ms.u-tokyo.ac.jp