Abstract

We study a class of infinite horizon impulse control problems with execution delay when the dynamics of the system is described by a general adapted stochastic process. The problem is solved by means of probabilistic tools relying on the notion of Snell envelope and infinite horizon reflected backward stochastic differential equations. This allows us to establish the existence of an optimal strategy over all admissible strategies.

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1 Introduction

Impulse control is one of the main topics in the control theory that has attracted a lot of research activity since it has a wide range of applications including mathematical finance, insurance, economics, etc. It was first introduced by Bensoussan and Lions (1984) and afterwards a large and growing literature has developed on the subject.

Several works were developed in the Markovian case using tools from dynamic programming and quasi variational inequalities, see e.g. [9, 3, 12, 13, 4] among many others. The first attempt to study the non-Markovian case was achieved by Djehiche et al. [6] by using probabilistic tools. Their approach relies on the Snell envelope notion and the reflected backward stochastic differential equations (BSDEs for short) to solve impulse control problems over a finite time horizon. We also refer to Hdhiri and Karouf [10] for the risk-sensitive case.

In this work, we study an infinite horizon impulse control with execution delay, i.e. there is a fixed lag of time $\Delta$ between the time of decision-making and the time when the execution is performed. We mention the work by Robin [15] for the impulse control with delay only in one pending order during the horizon time. Bayraktar and Egami [2] adopt the same framework of the previous paper for infinite horizon case, where they assume the magnitude of the impulse is...
chosen at the time of execution. Under restrictive assumptions on the controlled state process, Bar-Ilan and Sulem [1] study an infinite horizon impulse control with an arbitrary number of pending orders. Øksendal and Sulem [15] study also the problem with execution delay when the underlying process is a jump-diffusion. Hdhiri and Karouf [14] consider a finite horizon impulse control problem with execution delay where they use the same probabilistic tools of [9], such as the Snell envelope notion and reflected BSDEs to solve the problem. Due to the delay $\Delta > 0$, when the horizon is finite, this problem turns into the backward resolution of a finite number of optimal stopping problems [4, 13, 10].

The main contribution of the present work is a solution to an infinite horizon impulse control problem with execution delay for a wide class of stochastic processes adapted to the Brownian filtration that are not necessarily Markovian. On the other hand, the running reward functional is not only a deterministic function of the underlying process but may also be random. Our method relies on constructing an approximation scheme for the value function in terms of a sequence of solutions of infinite horizon reflected BSDEs. Contrarily to the finite horizon case, the problem now cannot be reduced to the backward resolution of a finite optimal stopping problems. The main point is related to the proof of continuity property of the value function of the problem.

The procedure of finding a sequence of optimal stopping times can be divided into a sequence of steps as follows. Given an initial time $t$, we find the first time $\tau_1$ where it is optimal to intervene and the denote the corresponding optimal impulse $\beta_1^*$. Note that this is the first optimal stopping time after the initial time when the controller may intervene. Next, the execution time is not instantaneous, but after a lag of time $\Delta$. Next, we proceed to find the first time $\tau_1 + \Delta$ where it is optimal to intervene. This will give the optimal stopping time $\tau_2$ and the corresponding optimal impulse $\beta_2^*$. We continue this procedure over and over again.

The paper is organized as follows. In Section 2, we provide some preliminaries and recall the existence and uniqueness result for solutions to infinite horizon reflected BSDEs. The link with the Snell envelope of processes is also given. In Section 3, we formulate the impulse control problem. In Section 4, we construct an approximation scheme for the value function of the control problem, relying on the infinite horizon reflected BSDEs and the Snell envelope. Section 5, is devoted to establish the existence of an optimal impulse control over strategies with a limited number of impulses. In Section 6, we prove the continuity property of the value function and exhibit an optimal impulse control over all admissible strategies. Finally, in Section 7, we extend the study to the risk-sensitive case or exponential utilities. We finish with a small appendix where we present the Snell envelope properties and the notion of predictable and optional projections.

2 Preliminary results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{t \geq 0}$. We denote by $(\mathcal{F}_t^B := \sigma\{B_s, s \leq t\})_{t \geq 0}$ the natural filtration of $B$ and $(\mathcal{F}_t)_{t \geq 0}$ its completion with the $\mathbb{P}$-null sets of $\mathcal{F}$ and $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. Let $\mathcal{P}$ be the $\sigma$-algebra on $\Omega \times [0, \infty]$ of $\mathcal{F}_t$-progressively measurable sets. Next let us introduce the following spaces:

i) $L^2 = \{\eta : \mathcal{F}_\infty - \text{measurable random variable, s.t. } \mathbb{E}[|\eta|^2] < \infty\}$;

ii) $H^{2,m} = \{(v_t)_{0 \leq t < \infty} : \mathcal{F}_t$-progressively measurable, $\mathbb{R}^m$-valued process s.t. $\mathbb{E}\left[\int_0^\infty |v_s|^2 ds\right] < \infty\} (m \geq 1)$;

iii) $S^2 = \{(y_t)_{0 \leq t < \infty} : \mathcal{F}_t$-progressively measurable process s.t. $\mathbb{E}\left[\sup_{0 \leq t < \infty} |y_t|^2\right] < \infty\}$;
Theorem 2.1. We say that the triple of progressively measurable processes \((Y_t, Z_t, K_t)_{t \geq 0}\) is a solution of the infinite horizon BSDE associated with \((g, \xi, L)\),

\[
\begin{align*}
Y_t &= \xi + \int_t^\infty g(s, Y_s, Z_s) \, ds + K_\infty - K_t - \int_t^\infty Z_s \, dB_s, \quad t \geq 0; \\
Y_t &\geq X_t, \quad \forall t \geq 0, \text{ and } \int_0^\infty (Y_t - X_t) \, dK_t = 0.
\end{align*}
\]  

We have the following existence and uniqueness result of the solution of (2.1).

Theorem 2.1 (\(\mathbb{R}\)). Assume that:

(i) \(\xi\) is \(\mathcal{F}_\infty\)-measurable and belongs to \(L^2\), the process \(X := (X_t)_{t \geq 0}\) belongs to \(S_c^2\) and such that \(\limsup_{t \to +\infty} X_t \leq \xi\) \(\mathbb{P}\)-a.s.

(ii) The coefficient \(g\) is a map from \([0, \infty) \times \Omega \times \mathbb{R}^{1+d}\) to \(\mathbb{R}\) verifying:

(a) The process \((g(t, 0, 0))_{t \geq 0}\) belongs to \(\mathcal{H}^{2, d}\);

(b) There exist two positive deterministic borelian functions \(u_1\) and \(u_2\) from \(\mathbb{R}^+\) into \(\mathbb{R}^+\) such that \(\int_0^\infty u_1(t) \, dt < \infty, \int_0^\infty u_2^2(t) \, dt < \infty\) and for every \((y, z)\) and \((y', z')\) in \(\mathbb{R}^{1+d}\)

\[
\mathbb{P}\text{-a.s., } |g(t, y, z) - g(t, y', z')| \leq u_1(t)|y - y'| + u_2(t)|z - z'|, \quad t \in [0, \infty).
\]

Then there exists a triple of processes \((Y, Z, K)\) which satisfies (2.1) and the following representation holds true:

\[
\forall t \geq 0, \quad Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left[ \int_t^\tau g(s, Y_s, Z_s) \, ds + X_\tau 1_{[\tau < \infty]} + \xi 1_{[\tau = \infty]} |\mathcal{F}_t] \right].
\]  

Furthermore, for any \(t \geq 0\), the stopping time

\[
D_t = \begin{cases} 
\inf \{ s \geq t, Y_s \leq X_s \} & \text{if finite,} \\
+\infty & \text{otherwise,}
\end{cases}
\]

is optimal after \(t\) in the sense that

\[
Y_t = \mathbb{E} \left[ \int_t^{D_t} g(s, Y_s, Z_s) \, ds + X_{D_t} 1_{[D_t < \infty]} + \xi 1_{[D_t = \infty]} |\mathcal{F}_t \right]. \quad \square
\]  

3 Formulation of the impulse problem with delay

Let \(L = (L_t)_{t \geq 0}\) be a stochastic process that describes the evolution of a system which we assume \(\mathcal{P}\)-measurable, with values in \(\mathbb{R}^d\) and such that \(\mathbb{E} \int_0^\infty |L_s|^2 \, ds < \infty\). An impulse control is a sequence of pairs \(\delta = (\tau_n, \xi_n)_{n \geq 1}\) in which \((\tau_n)_{n \geq 1}\) is a sequence of \(\mathcal{F}_t\)-stopping times such that \(0 \leq \tau_1 \leq \cdots \leq \tau_n \cdots\) \(\mathbb{P}\)-a.s. and \((\xi_n)_{n \geq 1}\) a sequence of random variables with values in a finite subset \(U\) of \(\mathbb{R}^d\) such that \(\xi_n\) is \(\mathcal{F}_{\tau_n}\)-measurable. Considering the subset \(U\) finite is in line with the fact that, in practice, the controller has only access to limited resources which allows him to exercise impulses of finite size.
For any $n \geq 1$, the stopping time $\tau_n$ stands for the $n$-th time where the controller makes the decision to impulse the system with a magnitude equal to $\xi_n$ and which will be executed after a time lag $\Delta$. Therefore, we require that $\tau_{n+1} - \tau_n \geq \Delta$, $\mathbb{P}$-a.s. and then we obviously have $\lim_{n \to +\infty} \tau_n = +\infty$.

The sequence $\delta = (\tau_n, \xi_n)_{n \geq 1}$ is said to be an admissible strategy of impulse control, and the set of admissible strategies will be denoted by $\mathcal{A}$.

When the decision maker implements the strategy $\delta = (\tau_n, \xi_n)_{n \geq 1}$, the controlled process $L^\delta = (L_t^\delta)_{t \geq 0}$ is defined as follows:

$$L_t^\delta = \begin{cases} L_t & \text{if } 0 \leq t < \tau_1 + \Delta, \\ L_t + \xi_1 + \cdots + \xi_n & \text{if } \tau_n + \Delta \leq t < \tau_{n+1} + \Delta, n \geq 1, \end{cases}$$

or in a compact form

$$\forall t \geq 0, \quad L_t^\delta = L_t + \sum_{n \geq 1} \xi_n 1_{[\tau_n + \Delta \leq t]}.$$ 

On the other hand, when the strategy $\delta$ is implemented, the associated total discounted expected payoff (the reward function) is given by:

$$J(\delta) := \mathbb{E} \left[ \int_0^{\infty} e^{-rs} h(s, L_s^\delta) \, ds - \sum_{n \geq 1} e^{-r(\tau_n + \Delta)} \psi(\xi_n) \right], \quad (3.1)$$

where:

i) $h$ is a non-negative function which stands for the instantaneous reward and $r$, the discount factor, is a positive real constant.

ii) $\psi$ is the cost of making an impulse or intervention and it has the form

$$\psi(\xi) = k + \phi(\xi),$$

where $k$ (resp. $\phi$) is a positive constant (resp. non-negative function) and it stands for the fixed (resp. variable) part of the cost of making an intervention.

The objective is to find an optimal strategy $\delta^* = (\tau_n^*, \xi_n^*)_{n \geq 1}$, i.e. which satisfies

$$J(\delta^*) = \sup_{\delta \in \mathcal{A}} J(\delta). \quad \square$$

**Remark 3.1.** The process $L$ can take the following form:

$$L_t = x + \int_0^t b(s, \omega) \, ds + \int_0^t \sigma(s, \omega) \, dB_s, \quad t \geq 0,$$ 

where $b$ (resp. $\sigma$) is a process of $\mathcal{H}^{2,1}$ (resp. $\mathcal{H}^{2,d}$). Then $L$ is an Itô process which is not Markovian, therefore the standard methods [3, 12, 13], etc. based on the Markovian properties do not apply.

Next throughout this paper, we make the following assumptions.

**Assumption 3.1.** i) The functions $h : [0, +\infty) \times \Omega \times \mathbb{R}^l \to [0, +\infty)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable and uniformly bounded by a constant $\gamma$ in all its arguments i.e.,

$$\forall (t, w, x) \in [0, +\infty) \times \Omega \times \mathbb{R}^l, \quad 0 \leq h(t, w, x) \leq \gamma.$$ 

ii) $\phi$ is a non-negative function defined on $U$. Note that since $U$ is finite then $\phi(\xi)$ is obviously bounded for any $\xi$ random variable with values in $U$. 

4
4 Iterative scheme

In this section, we consider an iterative scheme which relies on infinite horizon reflected BSDEs in order to find an optimal strategy that maximizes the total discounted expected reward \( \mathbb{E}[\sum_{t=0}^{\infty} e^{-rt} c(t, \xi(t))] \). Let \( \nu \) be an \( \mathcal{F}_t \)-stopping time and \( \xi \) a finite \( \mathcal{F}_\nu \)-random variable, i.e., \( \text{card}(\xi(\Omega)) < \infty \).

Next, let \((Y^0_n(\nu, \xi), Z^i_n(\nu, \xi))_{t \geq 0}\) be the solution in \( \mathcal{S}^2 \times \mathcal{H}^{2,d} \) of the following standard BSDE with infinite horizon:

\[
Y^0_t(\nu, \xi) = \int_t^\infty e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds - \int_t^\infty Z^0_s(\nu, \xi) dB_s, \quad t \geq 0.
\]

(4.3)

The solution of (4.3) exists and is unique thanks to the result by Chen [16], Theorem 1. In addition, the process \( Y^0(\nu, \xi) \) satisfies, for any \( t \geq 0 \),

\[
Y^0_t(\nu, \xi) = \mathbb{E} \left[ \int_t^\infty e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds | \mathcal{F}_t \right].
\]

(4.4)

We will now define \( Y^n(\nu, \xi) \) for \( n \geq 1 \), iteratively in the following way. For any \( n \geq 1 \), let \((Y^n(\nu, \xi), Z^n(\nu, \xi), K^n(\nu, \xi))\) be a triple of processes of \( \mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{S}^2 \) which satisfies: \( \forall t \geq 0 \),

\begin{enumerate}
  \item \( Y^n_t(\nu, \xi) = \int_t^\infty e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds + K^n_t(\nu, \xi) - \int_t^\infty Z^n_s(\nu, \xi) dB_s \),
  \item \( Y^n_t(\nu, \xi) \geq \mathbb{E} \left[ \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds | \mathcal{F}_t \right] + \max_{\beta \in U} \left\{ \mathbb{E} \left[ -e^{-r(t+\Delta)} \psi(\beta) + Y^n_{t+\Delta}(\nu, \xi + \beta) | \mathcal{F}_t \right] \right\} 
\]
  \item \( \int_0^\infty (Y^n_t(\nu, \xi) - O^n_t(\nu, \xi)) dK^n_t(\nu, \xi) = 0 \).
\end{enumerate}

(4.5)

Note that once \( Y^{n-1}(\nu, \xi) \) is defined, the process \( O^n_t(\nu, \xi) \) is defined through the optional projections of the non-adapted process \( \left( \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds \right)_{t \geq 0} \) and \( (-e^{-r(t+\Delta)} \psi(\beta) + Y^{n-1}_{t+\Delta}(\nu, \xi + \beta))_{t \geq 0} \) (see Part (II) in the appendix for more details).

We have the following properties of the processes \( Y^n(\cdot, \cdot), n \geq 1 \).

**Proposition 4.1.** For any \( n \geq 1 \), the triple \((Y^n(\nu, \xi), Z^n(\nu, \xi), K^n(\nu, \xi))\) is well-posed and satisfies, for all \( t \geq 0 \),

\[
Y^n_t(\nu, \xi) = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E} \left[ \int_t^\tau e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds + O^n_s(\nu, \xi) | \mathcal{F}_t \right].
\]

(4.6)

Moreover we have: \( i) \) For all \( t \geq 0 \)

\[
0 \leq Y^n_t(\nu, \xi) \leq \frac{\gamma}{r} e^{-rt}.
\]

(4.7)

\( ii) \forall n \geq 0 \) and \( t \geq 0 \),

\[
Y^n_t(\nu, \xi) \leq Y^{n+1}_t(\nu, \xi).
\]

(4.8)

**Proof.** It will be obtained by induction. Let \( \nu \) be a stopping time, \( \xi \) a generic \( \mathcal{F}_\nu \)-measurable random variable. As previously noticed, for \( n = 0 \), the pair \((Y^0(\nu, \xi), Z^0(\nu, \xi))_{t \geq 0}\) exists, belongs to \( \mathcal{S}^2 \times \mathcal{H}^{2,d} \) and satisfies (4.7) since \( 0 \leq h \leq \gamma \).

Let us now deal with the case \( n = 1 \). First note that the process \( O^1(\nu, \xi) \) belongs to \( \mathcal{S}^2 \) (by Appendix, Part (II)) and \( \lim_{t \to \infty} O^1_t(\nu, \xi) = 0 \). Actually this holds true since \( Y^0(\nu, \xi) \) is continuous and \( \lim_{t \to \infty} Y^0_t(\nu, \xi) = 0 \) by (4.7). Therefore the triple of processes \((Y^1(\nu, \xi), Z^1(\nu, \xi), K^1(\nu, \xi))\)
is well defined through the BSDE \((4.3)\) and by \((2.2)\) verifies \((4.6)\). Finally for \(t \geq 0\),

\[
O_t^1(\nu, \xi) = \mathbb{E} \left[ \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} \, ds \bigg| \mathcal{F}_t \right] + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(t+\Delta)}(-\psi(\beta)) + Y_{t+\Delta}^0(\nu, \xi + \beta) \bigg| \mathcal{F}_t \right] \right\}
\]

\[
\leq \mathbb{E} \left[ \frac{\gamma}{r} \{ e^{-rt} - e^{-r(t+\Delta)} \} - ke^{-r(t+\Delta)} + \frac{\gamma}{r} e^{-r(t+\Delta)} \bigg| \mathcal{F}_t \right]
\]

\[
\leq \frac{\gamma}{r} e^{-rt}. \quad (4.9)
\]

Again by the characterization \((4.6)\), we have:

\[
\forall t \geq 0, \; Y_t^1(\nu, \xi) = \text{ess sup}_{\tau \in T_n} \mathbb{E} \left[ \int_t^\tau e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} \, ds + O_t^1(\nu, \xi) \bigg| \mathcal{F}_t \right]. \quad (4.10)
\]

Therefore,

\[
0 \leq Y_t^1(\nu, \xi) \leq \text{ess sup}_{\tau \in T_n} \mathbb{E} \left[ \int_t^\tau \frac{\gamma}{r} e^{-rs} \, ds + \frac{\gamma}{r} e^{-r\tau} \bigg| \mathcal{F}_t \right] \leq \text{ess sup}_{\tau \in T_n} \mathbb{E} \left[ \frac{\gamma}{r} (e^{-rt} - e^{-r\tau}) + \frac{\gamma}{r} e^{-r\tau} \bigg| \mathcal{F}_t \right] = \frac{\gamma}{r} e^{-rt}.
\]

Next let us assume that for some \(n\) the triple \((Y^n(\nu, \xi), Z^n(\nu, \xi), K^n(\nu, \xi))\) is well-posed and \((4.6)-(4.7)\) hold true. The process \(O^{n+1}(\nu, \xi)\) belongs to \(S_2^\infty\) as the predictable projection of a continuous process and \(\lim_{n \to \infty} O^{n+1}(\nu, \xi) = 0\) by \((4.7)\) which is valid for \(n\) by the induction hypothesis. Therefore the triple \((Y^{n+1}(\nu, \xi), Z^{n+1}(\nu, \xi), K^{n+1}(\nu, \xi))\) is well-posed by the BSDE \((4.5)\) and by \((2.2)\) verifies \((4.6)\). Finally the fact that \(Y^{n+1}(\nu, \xi)\) verifies \((4.7)\) can be obtained as for \(Y^1(\nu, \xi)\) since \(O^{n+1}(\nu, \xi)\) verifies \((4.9)\). The induction is now complete.

Finally we have also \((4.8)\) by comparison of solutions of reflected BSDEs since we obviously have \(Y^0(\nu, \xi) \leq Y^1(\nu, \xi)\) and we conclude by using an induction argument. 

**Remark 4.2.** Since \(\text{card}(\xi(\Omega))\) is finite, then \(\xi\) takes only a finite number of values \(k_1, \cdots, k_m\), then using the uniqueness it follows immediately that, for any \(t \geq \nu\),

\[
Y_t^n(\nu, \xi) = \sum_{k=1}^m Y_t^n(\nu, k) 1_{\{\xi = k_i\}}.
\]

Therefore it is enough to know \(Y_t^n(\nu, \xi)\), for any constant \(\theta \in \xi(\Omega)\), to deduce \(Y_t^n(\nu, \xi)\).

**Proposition 4.2.** Let \(\nu\) be a stopping time and \(\xi\) an \(\mathcal{F}_\nu\)-measurable random variable, then:

i) The sequence \((Y^n(\nu, \xi))_{n \geq 0}\) converges increasingly and pointwisely \(\mathbb{P}\)-a.s. to a càdlàg process \(Y(\nu, \xi)\) which satisfies, for all \(t \geq 0\),

\[
Y_t(\nu, \xi) = \text{ess sup}_{\tau \in T_n} \mathbb{E} \left[ \int_t^\tau e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} \, ds + O_t(\nu, \xi) \bigg| \mathcal{F}_t \right], \quad (4.11)
\]

where

\[
O_t(\nu, \xi) = \mathbb{E} \left[ \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} \, ds \bigg| \mathcal{F}_t \right] + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(t+\Delta)}(-\psi(\beta)) + Y_{t+\Delta}(\nu, \xi + \beta) \bigg| \mathcal{F}_t \right] \right\}.
\]

ii) If \(\nu'\) is a stopping time satisfying \(\nu \leq \nu'\), then \(\mathbb{P}\)-a.s., \(Y_t(\nu, \xi) = Y_t(\nu', \xi)\) for all \(t \geq \nu'\).
Proof. i) From Proposition 4.1, we have that the sequence \( (Y_t^n(\nu, \xi))_{n \geq 0} \) is increasing and satisfies, for any \( n \geq 0 \),

\[
0 \leq Y_t^n(\nu, \xi) \leq \frac{\gamma}{r} e^{-rt}.
\]

Then, taking the limit as \( n \to \infty \), we obtain that the sequence \( (Y_t^n(\nu, \xi))_{n \geq 0} \) converges to the \( \mathcal{P} \)-measurable process \( Y(\nu, \xi) \) satisfying

\[
\forall t \geq 0, \quad 0 \leq Y_t(\nu, \xi) \leq \frac{\gamma}{r} e^{-rt}. \tag{4.12}
\]

Let us now show that \( (Y_t(\nu, \xi))_{t \geq 0} \) is càdlàg. We have from the expression (4.6) that the process \( Y_t(\nu, \xi) = \int_0^t e^{-rs} h(s, L_s + \xi) 1_{[s \geq 0]} ds \) is a continuous supermartingale which converges increasingly and pointwisely to the process \( \left( Y_t(\nu, \xi) = \int_0^t e^{-rs} h(s, L_s + \xi) 1_{[s \geq 0]} ds \right)_{t \geq 0} \), which is càdlàg, as a limit of increasing sequence of continuous supermartingales (for further details, see Dellacherie and Meyer Vol. B, pp. 86). In particular \( (Y_t(\nu, \xi))_{t \geq 0} \) is càdlàg. Therefore the process \( (O_t(\nu, \xi))_{t \geq 0} \) is also càdlàg (see Part (II) in Appendix). To complete the proof, it is enough to use point v) of Part (I) in Appendix.

ii) We proceed by induction on \( n \). Since the solution of the BSDE

\[
Y_t^n(\nu, \xi) = \int_t^\infty e^{-rs} h(s, L_s + \xi) 1_{[s \geq 0]} ds - \int_t^\infty Z_s^n(\nu, \xi) dB_s, \quad t \geq 0,
\]

is unique, it follows that, for any \( \xi \in \mathcal{F}_n \), \( Y_t^n(\nu, \xi) = Y_t^n(\nu', \xi) \) for any \( t \geq \nu' \). Suppose now that the property is also valid for some \( n \), then \( O_{t+1}^n(\nu, \xi) = O_{t+1}^n(\nu', \xi) \), \( \forall t \geq \nu' \). Also by uniqueness of the solution of (4.13), for any \( n \geq 0 \) and \( t \geq 0 \), we have the following equality

\[
Y_t^{n+1}(\nu, \xi) = Y_{t+1}^{n+1}(\nu', \xi), \quad t \geq \nu'.
\]

Hence, the property holds true for any \( n \geq 0 \), therefore by taking the limit as \( n \to +\infty \), we obtain the proof of the claim. \( \square \)

5 Infinite delayed impulse control with a finite number of interventions

In this section, we consider the case when the controller is allowed to make use of a finite number \( n \geq 1 \) of interventions at most. So let us define the set of bounded (by \( n \)) strategies by

\[
\mathcal{A}_n := \{(\tau_k, \xi_k)_{k \geq 0} \in \mathcal{A}, \text{ such that } \tau_n + \Delta = +\infty, \mathbb{P} \text{-a.s.}\}.
\]

\( \mathcal{A}_n \) is the set of strategies where only \( n \) impulses at most are allowed. The following is the main result of this section.

**Proposition 5.3.** Let \( n \geq 1 \) be fixed. Then there exists a strategy \( \delta_n^* \) which belongs to \( \mathcal{A}_n \) such that

\[
Y_0^n(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta) = J(\delta_n^*),
\]

i.e., \( \delta_n^* \) is optimal in \( \mathcal{A}_n \).

**Proof.** We first define the strategy \( \delta_n^* \). Let \( \tau_0^n \) be the stopping time defined as:

\[
\tau_0^n = \begin{cases} 
\inf\{ s \in [0, \infty), \ O_s^n(0, 0) \geq Y_s^n(0, 0) \} & \text{if finite,} \\
+\infty & \text{otherwise.}
\end{cases}
\]

Then, we define the optimal strategy as:

\[
\delta_n^*(\tau_k, \xi_k) = \begin{cases} 
\delta_n(\tau_k, \xi_k) & \text{if } \tau_k + \Delta \leq \tau_0^n \\
-\Delta & \text{otherwise.}
\end{cases}
\]

We then have that the set of \( \delta_n^* \) is a subset of \( \mathcal{A}_n \). Therefore, we need to prove that it is the optimal choice. This is done by showing that the value function satisfies

\[
Y_0^n(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta) = J(\delta_n^*).
\]

This is done by constructing a sequence of strategies \( \delta_k \) such that

\[
Y_0^n(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta) = J(\delta_k),
\]

where the optimal value function is achieved by \( \delta_k \). This is done by induction on \( k \). For \( k = 1 \), we have

\[
Y_0^n(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta) = J(\delta_1),
\]

where the optimal value function is achieved by \( \delta_1 \). For \( k > 1 \), we have

\[
Y_0^n(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta) = J(\delta_{k-1}),
\]

where the optimal value function is achieved by \( \delta_{k-1} \). Therefore, the optimal strategy is achieved by \( \delta_n^* \). \( \square \)
Then

\[ O^n_{\tau_0^0}(0, 0) := 1_{[\tau_0^0 < \infty]} \left\{ \mathbb{E} \left[ \int_{\tau_0^0}^{\tau_0^0 + \Delta} e^{-rs} h(s, L_s) ds \mid \mathcal{F}_{\tau_0^0} \right] \right\}, \]

\[ + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(\tau_0^0 + \Delta)} (-\psi(\beta)) + Y^n_{\tau_0^0 + \Delta}(0, \beta) \mid \mathcal{F}_{\tau_0^0} \right] \right\}, \]

\[ = 1_{[\tau_0^0 < \infty]} \left\{ \mathbb{E} \left[ \int_{\tau_0^0}^{\tau_0^0 + \Delta} e^{-rs} h(s, L_s) ds \mid \mathcal{F}_{\tau_0^0} \right] \right\}, \]

\[ + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(\tau_0^0 + \Delta)} (-\psi(\beta)) + Y^n_{\tau_0^0 + \Delta}(0, \beta) \mid \mathcal{F}_{\tau_0^0} \right] \right\}, \]

since, as mentioned previously, \( Y^n_{\tau_0^0 + \Delta}(0, \beta) = Y^n_{\tau_0^0 + \Delta}(0, \beta) \). Therefore, as \( U \) is finite, there exists \( \beta_0^n \) with values in \( U \), \( \mathcal{F}_{\tau_0^0^0} \)-measurable such that

\[ O^n_{\tau_0^0}(0, 0) = 1_{[\tau_0^0 < \infty]} \left\{ \mathbb{E} \left[ \int_{\tau_0^0}^{\tau_0^0 + \Delta} e^{-rs} h(s, L_s + \xi) ds - e^{-r(\tau_0^0 + \Delta)} \psi(\beta_0^n) + Y^n_{\tau_0^0 + \Delta}(0, \beta_0^n) \mid \mathcal{F}_{\tau_0^0^0} \right] \right\}. \]

Now, for any \( k \in \{1, \ldots, n - 1\} \), once \((\tau_{k-1}^n, \beta_{k-1}^n)\) defined, we define \( \tau_k^n \) by

\[ \tau_k^n = \inf \left\{ s \geq \tau_{k-1}^n + \Delta, O^{n-k}_{\tau_{k-1}^n}(\tau_{k-1}^n, \beta_{k-1}^n) - \beta_{k-1}^n) \geq Y^{n-k}_{s}(\tau_{k-1}^n, \beta_{k-1}^n) \right\}. \]

and \( \beta_k^n \) such that

\[ O^{n-k}_{\tau_k^n}(\tau_{k-1}^n, \beta_0^n + \cdots + \beta_{k-1}^n) = \mathbb{E} \left[ \int_{\tau_k^n}^{\tau_k^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n + \cdots + \beta_{k-1}^n) ds \right] \]

\[ - e^{-r(\tau_k^n + \Delta)} \psi(\beta_k^n) + Y^{n-k-1}_{\tau_k^n + \Delta}(\tau_{k-1}^n, \beta_0^n + \cdots + \beta_{k-1}^n + \beta_k^n) \mid \mathcal{F}_{\tau_k^n} \]

where we have used the equality \( Y^{n-k-1}_{\tau_k^n + \Delta}(\tau_{k-1}^n, \beta_0^n + \cdots + \beta_{k-1}^n + \beta_k^n) = Y^{n-k-1}_{\tau_k^n + \Delta}(\tau_k^n, \beta_0^n + \cdots + \beta_{k-1}^n + \beta_k^n). \)

Now let us show that \( \delta_n^n \) is optimal. First note that from the characterisation (4.6), we have that

\[ Y^n_0(0, 0) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E} \left[ \int_0^\tau e^{-rs} h(s, L_s) ds + O^n_\tau(0, 0) \right]. \]

Moreover, since the process \( O^n_\tau(0, 0) \) is continuous on \([0, \infty) \) \( (O^n_\infty(0, 0) = \lim_{t \to \infty} O^n_t(0, 0) = 0) \), then the stopping time \( \tau_0^n \) is optimal after 0. It follows that

\[ Y^n_0(0, 0) = \mathbb{E} \left[ \int_0^{\tau_0^n} e^{-rs} h(s, L_s) ds + O^n_{\tau_0^n}(0, 0) \right]. \] (5.13)

But,

\[ O^n_{\tau_0^n}(0, 0) := 1_{[\tau_0^n < \infty]} \left\{ \mathbb{E} \left[ \int_{\tau_0^n}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds \mid \mathcal{F}_{\tau_0^n} \right] \right\}, \]

\[ + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(\tau_0^n + \Delta)} (-\psi(\beta)) + Y^n_{\tau_0^n + \Delta}(0, \beta) \mid \mathcal{F}_{\tau_0^n} \right] \right\}, \]

\[ = 1_{[\tau_0^n < \infty]} \left\{ \mathbb{E} \left[ \int_{\tau_0^n}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds \mid \mathcal{F}_{\tau_0^n} \right] \right\}, \]

\[ + \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(\tau_0^n + \Delta)} (-\psi(\beta)) + Y^n_{\tau_0^n + \Delta}(\tau_0^n, \beta) \mid \mathcal{F}_{\tau_0^n} \right] \right\}, \]

\[ = 1_{[\tau_0^n < \infty]} \mathbb{E} \left[ \int_{\tau_0^n}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) + Y^n_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) \mid \mathcal{F}_{\tau_0^n} \right]. \]
The previous equality combined with (5.13) gives

\[ Y_0^n(0,0) = E \left[ \int_0^{\tau_0^n} e^{-rs} h(s, L_s) ds + \int_{\tau_0^n}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) + Y_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) \right]. \]

Henceforth,

\[ Y_0^n(0,0) = E \left[ \int_0^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) + Y_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) \right]. \] (5.14)

By using again (4.60), we obtain

\[ Y_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) = \operatorname{ess sup}_{\tau \in \mathcal{F}_{\tau_0^n + \Delta}} E \left[ \int_{\tau_0^n + \Delta}^{\tau} e^{-rs} h(s, L_s + \beta_0^n) ds + O_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) |\mathcal{F}_{\tau_0^n + \Delta} \right], \]

and \( \tau_1^n \) is an optimal stopping time after \( \tau_0^n + \Delta \). Then,

\[
Y_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) = E \left[ \int_{\tau_0^n + \Delta}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n) ds + O_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) |\mathcal{F}_{\tau_0^n + \Delta} \right]
\]

\[ = E \left[ \int_{\tau_0^n + \Delta}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n) ds + E \left[ \int_{\tau_0^n + \Delta}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n) ds |\mathcal{F}_{\tau_0^n + \Delta} \right] \right] \]

Therefore,

\[
Y_{\tau_0^n + \Delta}(\tau_0^n, \beta_0^n) = E \left[ \int_{\tau_0^n + \Delta}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n) ds - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) \right]
\]

\[ + Y_{\tau_1^n + \Delta}(\tau_1^n, \beta_0^n + \beta_1^n) |\mathcal{F}_{\tau_0^n + \Delta} \right]. \] (5.15)

Now, inserting (5.15) in (5.14), we obtain

\[ Y_0^n(0,0) = E \left[ \int_0^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds + \int_{\tau_0^n + \Delta}^{\tau_0^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n) ds \right.
\]

\[ - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) - e^{-r(\tau_0^n + \Delta)} \psi(\beta_0^n) + Y_{\tau_1^n + \Delta}(\tau_1^n, \beta_0^n + \beta_1^n) \right]. \]

Repeat this reasoning as many times as necessary to obtain

\[ Y_0^n(0,0) = E \left[ \int_0^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds + \sum_{1 \leq k \leq n - 1} \int_{\tau_{k-1}^n + \Delta}^{\tau_k^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n + \cdots + \beta_{k-1}^n) ds \right.
\]

\[ - \sum_{k=0}^{n-1} e^{-r(\tau_k^n + \Delta)} \psi(\beta_k^n) + Y_{\tau_{n-1}^n + \Delta}(\tau_{n-1}^n, \beta_0^n + \cdots + \beta_{n-1}^n) \right]. \] (5.16)

Therefore, in view of (4.41), we have

\[ Y_{\tau_{n-1}^n + \Delta}(\tau_{n-1}^n, \beta_0^n + \cdots + \beta_{n-1}^n) = E \left[ \int_{\tau_{n-1}^n + \Delta}^{\infty} e^{-rs} h(s, L_s + \beta_0^n + \cdots + \beta_{n-1}^n) ds |\mathcal{F}_{\tau_{n-1}^n + \Delta} \right]. \]

By inserting the last term in (5.16), we obtain

\[ Y_0^n(0,0) = E \left[ \int_0^{\tau_0^n + \Delta} e^{-rs} h(s, L_s) ds + \sum_{k \geq 1} \int_{\tau_{k-1}^n + \Delta}^{\tau_k^n + \Delta} e^{-rs} h(s, L_s + \beta_0^n + \cdots + \beta_{k-1}^n) ds \right.
\]

\[ - \sum_{k \geq 0} e^{-r(\tau_k^n + \Delta)} \psi(\beta_k^n) \right] = J(\delta_n) \right]. \]
where we have set $\tau_n^* = +\infty$, $P$-a.s. Next, it remains to show that the strategy $\delta_n^*$ is optimal over $A_n$, i.e., $J(\delta_n^*) \geq J(\delta'_n)$ for any $\delta_n \in A_n$.

Indeed, let $\delta'_n = (\tau'_n, \beta'_k)_{k \geq 0}$ be a strategy of $A_n$ (then $\tau'_n = +\infty$, $P$-a.s.). The definition of the Snell envelope allows us to write

$$Y_0^n(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_n^*} e^{-rs} h(s, L_s) \, ds + O_{\tau_n^*}(0, 0) \right],$$

where

$$O_{\tau_n^*}(0, 0) = \mathbb{E} \left[ \int_{\tau_n^*}^{\tau_n^* + \Delta} e^{-rs} h(s, L_s) \, ds | \mathcal{F}_{\tau_n^*} \right]$$

$$+ \max_{\beta \in U} \left\{ \mathbb{E} \left[ -e^{-s(\tau_n^* + \Delta)} \psi(\beta) + Y_{\tau_n^* + \Delta}^{n-1}(\tau_n^*, \beta) | \mathcal{F}_{\tau_n^*} \right] \right\}$$

$$\geq \mathbb{E} \left[ \int_{\tau_n^*}^{\tau_n^* + \Delta} e^{-rs} h(s, L_s) \, ds - e^{-s(\tau_n^* + \Delta)} \psi(\beta_n^m) + Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) | \mathcal{F}_{\tau_n^*} \right],$$

since $Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m)$ for any $\beta \in U$. It yields

$$Y_0^n(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_n^* + \Delta} e^{-rs} h(s, L_s) \, ds - e^{-s(\tau_n^* + \Delta)} \psi(\beta_n^m) + Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) \right].$$

On the other hand, we have

$$Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) = \mathbb{E} \sup_{\tau \geq \tau_n^* + \Delta} \left[ \int_{\tau_n^*}^{\tau + \Delta} e^{-rs} h(s, L_s + \beta_n^m) \, ds + O_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) | \mathcal{F}_{\tau_n^* + \Delta} \right]$$

$$\geq \mathbb{E} \left[ \int_{\tau_n^* + \Delta}^{\tau + \Delta} e^{-rs} h(s, L_s + \beta_n^m) \, ds + O_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) | \mathcal{F}_{\tau_n^* + \Delta} \right]$$

$$\geq \mathbb{E} \left[ \int_{\tau_n^* + \Delta}^{\tau + \Delta} e^{-rs} h(s, L_s + \beta_n^m) \, ds - e^{-s(\tau_n^* + \Delta)} \psi(\beta_n^m) + Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m) | \mathcal{F}_{\tau_n^* + \Delta} \right].$$

Therefore,

$$Y_0^n(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_n^* + \Delta} e^{-rs} h(s, L_s) \, ds + \int_{\tau_n^* + \Delta}^{\tau_n^* + \Delta} e^{-rs} h(s, L_s + \beta_n^m) \, ds \right]$$

$$- e^{-s(\tau_n^* + \Delta)} \psi(\beta_n^m) - e^{-s(\tau_n^* + \Delta)} \psi(\beta_n^m) + Y_{\tau_n^* + \Delta}(\tau_n^*, \beta_n^m + \beta_n^m) \right].$$

Repeat this reasoning many times, we get:

$$Y_0^n(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_n^* + \Delta} e^{-rs} h(s, L_s) \, ds + \sum_{1 \leq k \leq n-1} \int_{\tau_n^* + \Delta}^{\tau_k^* + \Delta} e^{-rs} h(s, L_s + \beta_n^m) \, ds \right]$$

$$+ \int_{\tau_n^* + \Delta}^{+\infty} e^{-rs} h(s, L_s + \beta_n^m) \, ds - \sum_{k=0}^{n-1} e^{-s(\tau_k^* + \Delta)} \psi(\beta_k^m) \right]$$

$$= J(\delta'_n).$$

Hence,

$$Y_0^n(0, 0) = J(\delta_n^*) \geq J(\delta'_n)$$

which implies that strategy $\delta_n^*$ is optimal.
6 Impulse control problem in the general case

In this section, we consider the case when the number of interventions is not limited, i.e., the controller can intervene as many times as she wishes. But here before establishing the existence of the optimal control over all admissible strategies, we need the continuity property of the limiting process \((Y_t(\nu, \xi))_{t \geq 0}\) which is an important point.

**Proposition 6.4.** The process \((Y_t(\nu, \xi))_{t \geq 0}\) of \(\Omega_1\) is continuous.

**Proof.** First, note that the process \((O_t(\nu, \xi))_{t \geq 0}\) is càdlàg since \(Y(\nu, \xi)\) is so by (i) of Proposition 4.2 and Appendix, Part (II). Next, let \(T\) be a predictable stopping time such that \(\Delta_T Y(\nu, \xi) < 0\). By Part (I) of the Appendix, the process \((O_t(\nu, \xi))_{t \geq 0}\) has a negative jump at \(T\) and \(O_T-(\nu, \xi) = Y_T-(\nu, \xi)\). We then have:

\[
O_T-(\nu, \xi) - O_T(\nu, \xi) = \max_{\beta \in U} \left\{ \mathbb{E} \left[ -e^{-r(T+\Delta)}\psi(\beta) + Y(T+\Delta)-(\nu, \xi + \beta) \big| \mathcal{F}_T \right] \right\}
- \max_{\beta \in U} \left\{ \mathbb{E} \left[ -e^{-r(T+\Delta)}\psi(\beta) + Y_{T+\Delta}(\nu, \xi + \beta) \big| \mathcal{F}_T \right] \right\}
\leq \max_{\beta \in U} \left\{ \mathbb{E} \left[ Y(T+\Delta)-(\nu, \xi + \beta) - Y_{T+\Delta}(\nu, \xi + \beta) \big| \mathcal{F}_T \right] \right\}
= \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_{T+\Delta}(\xi + \beta)} \left( Y(T+\Delta)-(\nu, \xi + \beta) - Y_{T+\Delta}(\nu, \xi + \beta) \right) \big| \mathcal{F}_T \right] \right\},
\]

where for any predictable stopping time \(T \geq \nu\) and \(\xi\) an \(\mathcal{F}_T\)-measurable r.v.,
\(A_T(\xi) := \{ \omega \in \Omega, \Delta_T Y(\nu, \xi) < 0 \}\) which belongs to \(\mathcal{F}_T\). Thus
\[
1_{A_T(\xi)} \{ O_T-(\nu, \xi) - O_T(\nu, \xi) \} \leq \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_T(\xi)} \times 1_{A_{T+\Delta}(\xi + \beta)} \left( Y(T+\Delta)-(\nu, \xi + \beta) - Y_{T+\Delta}(\nu, \xi + \beta) \right) \big| \mathcal{F}_T \right] \right\}.
\] (6.18)

We note that there exists at least one \(\beta \in U\) such that the right-hand side is positive. Otherwise the left-hand side is null and this is contradictory. Since it holds that \(Y_{T+\Delta}(\nu, \xi + \beta) \geq O_{T+\Delta}(\nu, \xi + \beta)\) and on the set \(A_{T+\Delta}(\xi + \beta), Y(T+\Delta)-(\nu, \xi + \beta) = O(T+\Delta)-(\nu, \xi + \beta)\). Therefore, \(6.18\) implies
\[
1_{A_T(\xi)} \{ O_T-(\nu, \xi) - O_T(\nu, \xi) \} \leq \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_T(\xi)} \times 1_{A_{T+\Delta}(\xi + \beta)} \left( O(T+\Delta)-(\nu, \xi + \beta) - O_{T+\Delta}(\nu, \xi + \beta) \right) \big| \mathcal{F}_T \right] \right\}
- \mathbb{E} \left[ 1_{A_T(\xi)} \times \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_{T+\Delta}(\xi + \beta)} \left( O(T+\Delta)-(\nu, \xi + \beta) - O_{T+\Delta}(\nu, \xi + \beta) \right) \big| \mathcal{F}_{T+\Delta} \right] \right\} \big| \mathcal{F}_T \right] \right\}
\leq \mathbb{E} \left[ 1_{A_T(\xi)} \times \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_{T+\Delta}(\xi + \beta)} \left( O(T+\Delta)-(\nu, \xi + \beta) - O_{T+\Delta}(\nu, \xi + \beta) \right) \big| \mathcal{F}_{T+\Delta} \right] \right\} \big| \mathcal{F}_T \right] \right\}
\leq \mathbb{E} \left[ 1_{A_T(\xi)} \times O_{T+\Delta}(\nu, \xi + \beta_1) \big| \mathcal{F}_{T+\Delta} \right] \big| \mathcal{F}_T \right] \right\},
\]

where \(\beta_1\) is a r.v. \(\mathcal{F}_{T+\Delta}\)-measurable valued in \(U\). Note that, as above, the left-hand is not null. Next as \(A_T(\xi)\) is also \(\mathcal{F}_{T+\Delta}\)-measurable then,
\[
1_{A_T(\xi)} \{ O_T-(\nu, \xi) - O_T(\nu, \xi) \} \leq \mathbb{E} \left[ 1_{A_T(\xi)} \times 1_{A_{T+\Delta}(\xi + \beta_1)} \left( O(T+\Delta)-(\nu, \xi + \beta_1) - O_{T+\Delta}(\nu, \xi + \beta_1) \right) \big| \mathcal{F}_T \right] \right\}.
\] (6.19)
Repeating this reasoning several times yields

\[
1_{A_T(\xi)}(O_T(\nu, \xi) - O_T(\nu, \xi)) \leq \mathbb{E}\left[ 1_{A_T(\xi)} \left( \prod_{k=1}^{k=n} 1_{A_{T+k\Delta}(\xi+\beta_1+\cdots+\beta_k)} \times \left( O_{T+n\Delta}^{-}(\nu, \xi + \beta_1 + \cdots + \beta_n) - O_{T+n\Delta}^{-}(\nu, \xi + \beta_1 + \cdots + \beta_n) \right) \right) \mid F_T \right],
\]

where the random variables \( \beta_k \) are valued in \( U \) and \( F_{T+k\Delta} \)-measurable. But the left-hand side converges to 0, \( \mathbb{P} \)-a.s. when \( n \to +\infty \). Indeed, by using (4.7) for any \( \nu \) and \( \xi \), we have

\[
|O_t(\nu, \xi)| \leq \frac{1}{r} \left( e^{-rt} - e^{-r(t+\Delta)} \right) + \|\psi\| e^{-r(t+\Delta)} + \frac{1}{r} e^{-r(t+\Delta)}, \quad \forall t \geq 0,
\]

and then \( \lim_{t \to +\infty} O_t(\nu, \xi) = 0 \) uniformly with respect to \( \nu \) and \( \xi \). Thus

\[
1_{A_T(\xi)}(O_T(\nu, \xi) - O_T(\nu, \xi)) = 0,
\]

which is contradictory and then the process \( Y(\nu, \xi) \) is continuous.

**Theorem 6.1.** Let us assume that Assumption (3.1) hold and let us define the strategy \( \delta^* = (\tau_n^*, \beta_n^*)_{n \geq 0} \) by

\[
\tau_0^* = \begin{cases} 
\inf\{s \in [0, \infty); O_s(0, 0) \geq Y_s(0, 0)\}, \\
+\infty, \quad \text{otherwise}
\end{cases}
\]

and \( \beta_0^* \) an \( F_{\tau_0^*} \)-r.v. such that

\[
O_{\tau_0^*}(0, 0) := \mathbb{E}\left[ \int_{\tau_0^*}^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^*+\Delta)} \psi(\beta_0^*) + Y_{\tau_0^*+\Delta}(\tau_0^*, \beta_0^*) \mid F_{\tau_0^*} \right].
\]

For any \( n \geq 1 \),

\[
\tau_n^* = \inf \left\{ s \geq \tau_{n-1}^* + \Delta, O_s(\tau_{n-1}^*, \beta_{n-1}^* + \beta_{n-1}^*) \geq Y_s(\tau_{n-1}^*, \beta_{n-1}^* + \beta_{n-1}^*) \right\},
\]

and \( \beta_n^* \) the \( U \)-valued \( F_{\tau_n^*} \)-measurable r.v. such that

\[
O_{\tau_n^*}(\tau_{n-1}^*, \beta_{n-1}^* + \cdots + \beta_{n-1}^*) = \mathbb{E}\left[ \int_{\tau_n^*}^{\tau_n^*+\Delta} e^{-rs} h(s, L_s + \beta_{n-1}^* + \cdots + \beta_{n-1}^*) ds \\
- e^{-r(\tau_n^*+\Delta)} \psi(\beta_n^*) + Y_{\tau_n^*+\Delta}(\tau_n^*, \beta_n^* + \cdots + \beta_{n-1}^* + \beta_{n-1}^*) \mid F_{\tau_n^*} \right].
\]

Then, the strategy \( \delta^* = (\tau_n^*, \beta_n^*)_{n \geq 0} \) is optimal for the impulse control problem, i.e.,

\[
Y_0(0, 0) = \sup_{\delta \in \mathcal{A}} J(\delta) = J(\delta^*).
\]

**Proof.** We first prove that \( Y_0(0, 0) = J(\delta^*) \).

To begin with let us point out that we have:

\[
Y_0(0, 0) = \text{ess sup}_{\tau \in T_0} \mathbb{E}\left[ \int_0^\tau e^{-rs} h(s, L_s) ds + O_{\tau}(0, 0) \right].
\]

Since \( Y(\nu, \xi) \) is continuous and \( (O_t(0, 0))_{t \geq 0} \) is so, then, for any stopping time \( \nu \) and any \( F_\nu \)-measurable r.v. \( \xi \), the stopping time \( \tau_0^* \) is optimal after 0. This yields

\[
Y_0(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^*} e^{-rs} h(s, L_s) ds + O_{\tau_0^*}(0, 0) \right] \tag{6.22}
\]
where

$$\text{O}_{\tau_0}^* (0, 0) = \mathbb{E} \left[ \int_{\tau_0}^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds | \mathcal{F}_{\tau_0} \right]$$

$$+ \max_{\beta \in U} \left\{ \mathbb{E} \left[ e^{-r(\tau_0 + \Delta)} (\psi(\beta)) + Y_{\tau_0 + \Delta} (0, \beta) | \mathcal{F}_{\tau_0} \right] \right\}$$

$$= \mathbb{E} \left[ \int_{\tau_0}^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0^*) + Y_{\tau_0 + \Delta} (\tau_0^*, \beta_0^*) | \mathcal{F}_{\tau_0} \right].$$

The second equality is valid thanks to Proposition 4.2-ii) since \( Y_{\tau_0 + \Delta} (0, \beta) = Y_{\tau_0 + \Delta} (\tau_0^*, \beta), \) \( \forall \beta \in U. \) Combining this with (6.22), yields

$$Y_0 (0, 0) = \mathbb{E} \left[ \int_0^{\tau_0} e^{-rs} h(s, L_s) ds + \int_{\tau_0}^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0^*) + Y_{\tau_0 + \Delta} (\tau_0^*, \beta_0^*) \right]$$

$$= \mathbb{E} \left[ \int_0^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0^*) + Y_{\tau_0 + \Delta} (\tau_0^*, \beta_0^*) \right].$$

On the other hand, we have that

$$Y_{\tau_0 + \Delta} (\tau_0^*, \beta_0^*) = \text{ess sup}_{\tau \in \tau_0 + \Delta} \mathbb{E} \left[ \int_{\tau_0 + \Delta}^{\tau_0 + \Delta} e^{-rs} h(s, L_s + \beta_0^*) ds + O_r (\tau_0^*, \beta_0^*) | \mathcal{F}_{\tau_0 + \Delta} \right].$$

As the stopping time \( \tau_1^* \) is optimal after \( \tau_0^* + \Delta, \) then

$$Y_{\tau_0 + \Delta} (\tau_0^*, \beta_0^*) = \mathbb{E} \left[ \int_{\tau_0}^{\tau_1^*} e^{-rs} h(s, L_s + \beta_0^*) ds + O_r (\tau_1^*, \beta_0^*) | \mathcal{F}_{\tau_0 + \Delta} \right]$$

$$= \mathbb{E} \left[ \int_{\tau_0 + \Delta}^{\tau_1^*} e^{-rs} h(s, L_s + \beta_0^*) ds - e^{-r(\tau_1^* + \Delta)} \psi(\beta_1^*) + Y_{\tau_1^* + \Delta} (\tau_1^*, \beta_0^* + \beta_1^*) | \mathcal{F}_{\tau_0 + \Delta} \right].$$

We plug the last quantity in the previous one to obtain

$$Y_0 (0, 0) = \mathbb{E} \left[ \int_0^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds + \int_{\tau_0 + \Delta}^{\tau_1^*} e^{-rs} h(s, L_s + \beta_0^*) ds \right.$$  

$$\left. - e^{-r(\tau_0 + \Delta)} \psi(\beta_0^*) - e^{-r(\tau_1^* + \Delta)} \psi(\beta_1^*) + Y_{\tau_1^* + \Delta} (\tau_1^*, \beta_0^* + \beta_1^*) \right].$$

Now, we use the same reasoning as many times as necessary to get

$$Y_0 (0, 0) = \mathbb{E} \left[ \int_0^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds + \sum_{1 \leq k \leq n-1} \int_{\tau_{k-1}^* + \Delta}^{\tau_k^* + \Delta} e^{-rs} h(s, L_s + \beta_0^* + \cdots + \beta_{k-1}^*) ds \right.$$  

$$\left. - \sum_{k=0}^{n-1} e^{-r(\tau_k^* + \Delta)} \psi(\beta_k^*) + Y_{\tau_n^* + \Delta} (\tau_n^*, \beta_0^* + \cdots + \beta_n^*) \right]. \quad (6.23)$$

But by (11.12), \( \lim_{n \to \infty} Y_{\tau_n^* + \Delta} (\tau_n^*, \beta_0^* + \cdots + \beta_n^*) = 0, \) therefore take the limit w.r.t \( n \) in the left hand-side of the previous equality to obtain that,

$$Y_0 (0, 0) = J(\delta^*).$$

To proceed, we prove that the strategy \( \delta^* = (\tau_n^*, \beta_n^*)_{n \geq 0} \) is optimal for the general impulse control problem, i.e. \( J(\delta^*) \geq J(\delta') \) for any \( \delta' = (\tau_n^*, \beta_n^*)_{n \geq 0} \text{ in } A. \) The definition of the Snell envelope allows us to write

$$Y_0 (0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_0'} e^{-rs} h(s, L_s) ds + O_\tau (0, 0) \right].$$
But, we have
\[
O_{\tau_0}(0, 0) \geq \mathbb{E} \left[ \int_{\tau_0}^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0') + Y_{\tau_0 + \Delta}(\tau_0', \beta_0') | \mathcal{F}_{\tau_0} \right]
\]
which yields
\[
Y_0(0, 0) \geq \mathbb{E} \left[ \int_{0}^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0') + Y_{\tau_0 + \Delta}(\tau_0', \beta_0') \right]. \tag{6.24}
\]
Next
\[
Y_{\tau_0 + \Delta}(\tau_0', \beta_0') = \mathbb{E} \sup_{\tau \in T_{\tau_0 + \Delta}} \mathbb{E} \left[ \int_{\tau_0 + \Delta}^{\tau} e^{-rs} h(s, L_s + \beta_0') ds + O_{\tau}(\tau_0', \beta_0') | \mathcal{F}_{\tau_0 + \Delta} \right]
\]
\[
\geq \mathbb{E} \left[ \int_{\tau_0 + \Delta}^{\tau_1 + \Delta} e^{-rs} h(s, L_s + \beta_0') ds + O_{\tau_1}(\tau_0', \beta_0') | \mathcal{F}_{\tau_0 + \Delta} \right]
\]
\[
\geq \mathbb{E} \left[ \int_{\tau_0 + \Delta}^{\tau_1 + \Delta} e^{-rs} h(s, L_s + \beta_0') ds - e^{-r(\tau_1 + \Delta)} \psi(\beta_1') + Y_{\tau_1 + \Delta}(\tau_1', \beta_0' + \beta_1') | \mathcal{F}_{\tau_0 + \Delta} \right].
\]
Therefore,
\[
Y_0(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds + \int_{\tau_0 + \Delta}^{\tau_1 + \Delta} e^{-rs} h(s, L_s + \beta_0') ds - e^{-r(\tau_0 + \Delta)} \psi(\beta_0') - e^{-r(\tau_1 + \Delta)} \psi(\beta_1') + Y_{\tau_1 + \Delta}(\tau_1', \beta_0' + \beta_1') \right].
\]
By repeating this argument many times, we obtain
\[
Y_0(0, 0) \geq \mathbb{E} \left[ \int_0^{\tau_0 + \Delta} e^{-rs} h(s, L_s) ds + \sum_{1 \leq k \leq n-1} \int_{\tau_{k-1} + \Delta}^{\tau_k + \Delta} e^{-rs} h(s, L_s + \beta_0' + \cdots + \beta_{k-1}') ds - \sum_{k=0}^{n} e^{-r(\tau_{k-1} + \Delta)} \psi(\beta_k') + Y_{\tau_n + \Delta}(\tau_n', \beta_0' + \cdots + \beta_n') \right].
\]
Finally, taking the limit as \( n \to +\infty \), yields
\[
Y_0(0, 0) \geq \mathbb{E} \left[ \int_0^{+\infty} e^{-rs} h(s, L_s') ds - \sum_{n \geq 0} e^{-r(\tau_n + \Delta)} \psi(\beta_n') \right] = J(\delta')
\]
since \( \lim_{n \to +\infty} Y_{\tau_n + \Delta}(\tau_n', \beta_0' + \cdots + \beta_n') = 0 \). Hence, the strategy \( \delta^* \) is optimal. \( \square \)

### 7 Risk-sensitive impulse control problem

In this section, we extend the previous results to the risk-sensitive case where the controller has a utility function which is of exponential type. In order to tackle this problem we do not use BSDEs, as in the previous section, but instead, the Snell envelope which is more appropriate. A similar version of this problem is considered in Hdhiri et al. \(^{10}\) in the case when the horizon is finite.

When the decision maker implements a strategy \( \delta = (\tau_n, \xi_n)_{n \geq 0} \), the payoff is given by
\[
J(\delta) := \mathbb{E} \left[ \exp \left\{ \int_0^{+\infty} e^{-rs} h(s, L_s') ds - \sum_{n \geq 0} e^{-r(\tau_n + \Delta)} \psi(\xi_n) \right\} \right], \tag{7.25}
\]
where $\theta > 0$ is the risk-sensitive parameter. Hereafter, for sake of simplicity, we will treat only the case $\theta = 1$ since the other cases are treated in a similar way.

We proceed by recasting the risk-sensitive impulse control problem into an iterative optimal stopping problem, and by exploiting the Snell envelope properties, we shall be able to characterize recursively an optimal strategy to this risk-sensitive impulse control problem.

### 7.1 Iterative optimal stopping and properties

Let $\nu$ be a stopping time and $\xi$ an $\mathcal{F}_\nu$-measurable random variable, we introduce the sequence of processes $(Y^n_t(\nu, \xi))_{n \geq 0}$ defined recursively by

$$Y^n_0(\nu, \xi) = \mathbb{E} \left[ \exp \left\{ \int_t^{t+\infty} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds \right\} \bigg| \mathcal{F}_t \right], \quad t \geq 0, \quad (7.26)$$

and, for $n \geq 1$,

$$Y^n_t(\nu, \xi) = \operatorname{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \exp \left\{ \int_t^{\tau+\infty} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds \right\} O^n_t(\nu, \xi) | \mathcal{F}_t \right], \quad t \geq 0, \quad (7.27)$$

where

$$O^n_t(\nu, \xi) = \max_{\beta \in U} \left\{ \mathbb{E} \left[ \exp \left\{ \int_t^{\tau+\infty} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds - e^{-r(t+\Delta)} \psi(\beta) \right\} \right] \right\} \nu(\nu, \xi + \beta) | \mathcal{F}_t \right].$$

Then the sequence of processes $(Y^n_t(\nu, \xi))_{n \geq 0}$ enjoys the following properties:

**Proposition 7.5.**

i) For any $n \in \mathbb{N}$, the process $Y^n(\nu, \xi)$ belongs to $\mathcal{S}_e^2$ and verifies $\lim_{t \to +\infty} Y^n_t(\nu, \xi) = 1$.

ii) The sequence of processes $(Y^n(\nu, \xi))_{n \geq 0}$ satisfies: $\mathbb{P}$ a.s, for any $t \geq 0$,

$$0 \leq Y^n_t(\nu, \xi) \leq Y^{n+1}_t(\nu, \xi) \leq \exp\left( \frac{\gamma e^{-rt}}{r} \right). \quad (7.28)$$

Moreover the process $Y_t(\nu, \xi) = \lim_{n \to \infty} Y^n_t(\nu, \xi)$, $t \geq 0$, is càdlàg and satisfies

$$\mathbb{P} - \text{a.s.} \forall t \geq 0, 0 \leq Y_t(\nu, \xi) \leq \exp\left( \frac{\gamma e^{-rt}}{r} \right). \quad (7.29)$$

Finally, it holds that

$$Y_t(\nu, \xi) = \operatorname{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \exp \left\{ \int_t^\tau e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds \right\} O_t(\nu, \xi) | \mathcal{F}_t \right], \quad (7.30)$$

where

$$O_t(\nu, \xi) := \max_{\beta \in U} \left\{ \mathbb{E} \left[ \exp \left\{ \int_t^{\tau+\infty} e^{-rs} h(s, L_s + \xi) 1_{[s \geq \nu]} ds - e^{-r(t+\Delta)} \psi(\beta) \right\} Y_{t+\Delta}(\nu, \xi + \beta) | \mathcal{F}_t \right\} \right\}.$$

iii) For any two stopping times $\nu$ and $\nu'$ such that $\nu \leq \nu'$ and $\xi$ an $\mathcal{F}_\nu$-measurable, we have

$$\mathbb{P} - \text{a.s.}, \forall t \geq \nu', Y_t(\nu, \xi) = Y_t(\nu', \xi).$$

**Proof.** Let $\nu$ be a stopping time and $\xi$ is an $\mathcal{F}_\nu$-measurable random variable.

i) We will show by induction that for each $n \geq 0$, $Y^n(\nu, \xi)$ belongs to $\mathcal{S}_e^2$, verifies $\lim_{t \to +\infty} Y^n_t(\nu, \xi) = 1$ and $\mathbb{P}$ a.s, for any $t \geq 0$, for any $\nu, \xi$,

$$0 \leq Y^n_t(\nu, \xi) \leq \exp\left( \frac{\gamma e^{-rt}}{r} \right).$$
Let us start with the case \( n = 0 \). Through the definition of \( Y^0(\nu, \xi) \) in [12], we have
\[
\lim_{t \to +\infty} Y^0_t(\nu, \xi) = 1 \quad \text{since} \quad h \quad \text{is bounded. On the other hand,}
\]
\[
E \left[ \sup_{t \geq 0} \left| Y^0_t(\nu, \xi) \right|^2 \right] = E \left[ \sup_{t \geq 0} \left| E \left[ \exp \left\{ \int_t^{+\infty} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] \right|^2 \right] 
\leq E \left[ \sup_{t \geq 0} \exp \left\{ 2 \int_t^{+\infty} \gamma e^{-rs} ds \right\} \right] = E \left[ \exp \left\{ 2 \int_0^{+\infty} \gamma e^{-rs} ds \right\} \right] = \exp \left( \frac{2\gamma}{\nu} \right),
\]
since \( h \) is uniformly bounded by \( \gamma \) (Assumption 3.1). In addition, we note that: \( \forall t \geq 0, \)
\[
Y^0_t(\nu, \xi) = E \left[ \exp \left\{ \int_t^{+\infty} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] 
= E \left[ \exp \left\{ \int_0^{+\infty} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} \exp \left\{ - \int_0^t e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} \right].
\]
As martingales w.r.t. the Brownian filtration are continuous then clearly \( Y^0(\nu, \xi) \) is continuous. Thus \( Y^0(\nu, \xi) \) belongs to \( \mathcal{S}^2 \). Finally
\[
0 \leq Y^0_t(\nu, \xi) = E \left[ \exp \left\{ \int_t^{+\infty} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] \leq \exp \left( \frac{\gamma e^{-rt}}{\nu} \right).
\]
Thus the property holds for \( n = 0 \). Assume now that it holds for some \( n \geq 1 \). First note that since \( \forall \nu \geq 0, \)
\[
0 \leq Y^{n+1}_t(\nu, \xi) = \text{ess sup}_{\tau \in T_t} E \left[ \exp \left\{ \int_0^{\tau} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] 
= \text{ess sup}_{\tau \in T_t} E \left[ \exp \left\{ \int_0^{\tau} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} \right] \times \max_{\beta \in U} \left\{ E \left[ \exp \left\{ \int_0^{\tau+\Delta} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds - e^{-r(\tau+\Delta)} \psi(\beta) \right\} Y^0_{\tau+\Delta}(\nu, \xi) | F_t \right] \right\} | F_t \right]
\leq \text{ess sup}_{\tau \in T_t} E \left[ \exp \left( \frac{\gamma e^{-rt} - e^{-r(\tau+\Delta)}}{F_t} + \frac{\gamma e^{-r(\tau+\Delta)}}{\nu} \right) \right] | F_t \right] = \exp \left( \frac{\gamma e^{-rt}}{\nu} \right).
\]
It implies that
\[
\lim_{t \to \infty} \text{sup}_{n \geq 1} Y^{n+1}_t(\nu, \xi) \leq 1.
\]
On the other hand
\[
Y^{n+1}_t(\nu, \xi) = \text{ess sup}_{\tau \in T_t} E \left[ \exp \left\{ \int_0^{\tau} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] 
\geq \lim_{t \to +\infty} E \left[ \exp \left\{ \int_0^{+\infty} e^{-rs}h(s, L_s + \xi)1_{[s \geq \nu]} ds \right\} | F_t \right] = Y^0_t(\nu, \xi).
\]
since \( \lim_{t \to \infty} O^0_{T+}(\nu, \xi) = 1 \) by the induction hypothesis. Thus
\[
\liminf_{t \to \infty} Y^{n+1}_t(\nu, \xi) \geq \lim_{t \to \infty} Y^0_t(\nu, \xi) = 1.
\]
This combined with the above estimates yield
\[
\lim_{t \to \infty} Y^{n+1}_t(\nu, \xi) = 1.
\]
It remains to show that $Y^{n+1}(\nu, \xi)$ belongs to $S^2$. With the above estimates, it is enough to show that it is continuous. First note that the process

$$
\Theta_t^{n+1} = \exp \left\{ \int_0^t e^{-rs} h(s, L_s + \xi^1) 1_{\{s \geq \nu\}} ds \right\} O_t^{n+1}(\nu, \xi), t \geq 0,
$$

is continuous on $[0, +\infty)$ therefore its Snell envelope is also continuous on $[0, +\infty]$, i.e., $Y_t^{n+1}(\nu, \xi) \exp \{ \int_0^t h(s, L_s + \xi^1) 1_{\{s \geq \nu\}} ds \}, t \geq 0$, is continuous on $[0, +\infty]$ and then $Y^{n+1}(\nu, \xi)$ is continuous on $[0, +\infty]$. The proof of the claim is now complete.

To show that $\mathbb{P}$-a.s. $\forall t \geq 0$,

$$
Y_t^n(\nu, \xi) \leq Y_t^{n+1}(\nu, \xi)
$$

it is enough to use an induction argument and to take into account that $\mathbb{P}$-a.s., $\forall \xi$ an $\mathcal{F}_t$-r.v., $\forall t \geq 0$,

$$
Y_t^1(\nu, \xi) \geq Y_t^0(\nu, \xi)
$$

(7.31)
since for any $T \geq t$,

$$
Y_t^1(\nu, \xi) \geq \mathbb{E} \left[ \exp \left\{ \int_t^T e^{-rs} h(s, L_s + \xi^1) 1_{\{s \geq \nu\}} ds \right\} O_T^1(\nu, \xi) | \mathcal{F}_t \right].
$$

Take now the limit when $T \to \infty$ to obtain (7.31) since $\lim_{T \to +\infty} O_T^1(\nu, \xi) = 1$.

Next for $t \geq 0$ let us set $Y_t(\nu, \xi) = \lim_{n \to \infty} Y_t^n(\nu, \xi)$. Therefore $Y_t(\nu, \xi)$ verifies (7.29) by taking the limit in (7.28). Now $(Y_t^n(\nu, \xi) \exp \{ \int_0^t h(s, L_s + \xi) 1_{\{s \geq \nu\}} ds \})_{t \geq 0}$ is a bounded increasing sequence of continuous supermartingales, then its limit is càdlàg and then $Y(\nu, \xi)$ is càdlàg. Finally by Part (I)-v) of Appendix the process $O(\nu, \xi)$ is càdlàg and $O^n(\nu, \xi) \to O(\nu, \xi)$, therefore by the continuity of the Snell enveloppe through càdlàg processes, $Y(\nu, \xi)$ verifies (7.30).

iii) To show that for any two stopping times $\nu$ and $\nu'$ such that $\nu \leq \nu'$ and $\xi$ an $\mathcal{F}_t$-measurable, we have $\mathbb{P}$-a.s.

$$
Y_t(\nu, \xi) = Y_t(\nu', \xi), \quad \forall t \geq \nu'
$$

it is enough to show that $\forall n \geq 0$

$$
Y_t^n(\nu, \xi) = Y_t^n(\nu', \xi).
$$

But this property is obtained by an induction. Actually for $n = 0$ this property is valid in view of the definition of $Y_t^0(\nu, \xi)$ and since $1_{\{s \geq \nu\}} = 1_{\{s \geq \nu'\}}$ if $s \geq t \geq \nu' \geq \nu$. Next assume that the property is valid for some $n$. Therefore for any $\beta \in U$ (constant), by induction hypothesis

$$
\mathbb{E} \left[ \exp \left\{ \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi^1) 1_{\{s \geq \nu\}} ds - e^{-r(t+\Delta)} \psi(\beta) \right\} \times Y_{t+\Delta}^n(\nu, \xi + \beta) \right| \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \left\{ \int_t^{t+\Delta} e^{-rs} h(s, L_s + \xi^1) 1_{\{s \geq \nu\}} ds - e^{-r(t+\Delta)} \psi(\beta) \right\} \times Y_{t+\Delta}^n(\nu', \xi + \beta) \right| \mathcal{F}_t \right].
$$

Take now the supremum over $\beta \in U$ to obtain $O_t^{n+1}(\nu, \xi) = O_t^{n+1}(\nu', \xi)$, thus $Y_t^{n+1}(\nu, \xi) = Y_t^{n+1}(\nu', \xi)$. The proof is now complete.

**Remark 7.3.** As in Proposition 5.3, we can show in the same way that for any $n \geq 0$, there exists a strategy $\delta_n^*$ which belongs to $A_n$ such that

$$
Y_0^n(0, 0) = \sup_{\delta \in A_n} J(\delta) = J(\delta_n^*),
$$

i.e., $\delta_n^*$ is optimal in $A_n$. 

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7.2 The optimal strategy for the risk-sensitive problem

We now deal with the issue of existence of an optimal strategy for the risk-sensitive impulse control problem with delay. The main difficulty is related to continuity of the process $Y(\nu, \xi)$. Once this property is established we exhibit an optimal startegy and show that $Y(0, 0)$ is the value function of the control problem. So first let us focus on the continuity property.

**Proposition 7.6.** Let us assume that Assumption $\text{(7.30)}$ hold. Then the process $(Y(\nu, \xi))_{t \geq 0}$ defined in (7.30) is continuous.

**Proof.** This proof is similar to the one of Proposition 6.4. First let us notice that the process $(O_t(\nu, \xi))_{t \geq 0}$ is càdlàg since $Y(\nu, \xi)$ is so (see Appendix Part (II)). Next, let $T$ be a predictable stopping time such that $\Delta_T Y(\nu, \xi) < 0$. It implies that the process $(O_t(\nu, \xi))_{t \geq 0}$ has a negative jump at $T$ and $O_T^-(\nu, \xi) = Y_T^-(\nu, \xi)$ (see Appendix, Part (I)). Therefore,

\[
O_T^-(\nu, \xi) - O_T(\nu, \xi) = \max_{\beta \in U} \left\{ E \left[ \exp \left\{ \int_T^{T+\Delta} e^{-rs} h(s, L_s + \xi) 1_{\{s \geq \nu\}} ds - e^{-r(T+\Delta)} \psi(\beta) \right\} \times Y(T+\Delta)^-(\nu, \xi + \beta) | F_T \right] \right\}
\]

\[
- \max_{\beta \in U} \left\{ E \left[ \exp \left\{ \int_T^{T+\Delta} e^{-rs} h(s, L_s + \xi) 1_{\{s \geq \nu\}} ds - e^{-r(T+\Delta)} \psi(\beta) \right\} \times Y(T+\Delta)(\nu, \xi + \beta) | F_T \right] \right\}
\]

\[
\leq \max_{\beta \in U} \left\{ E \left[ 1_{A_T(\xi + \beta)} \exp \left\{ \int_T^{T+\Delta} e^{-rs} h(s, L_s + \xi) 1_{\{s \geq \nu\}} ds - e^{-r(T+\Delta)} \psi(\beta) \right\} \right. \right\}
\]

\[
\times \left. \left( Y(T+\Delta)^-(\nu, \xi + \beta) - Y(T+\Delta)(\nu, \xi + \beta) \right) | F_T \right] \right\}
\]

\[
\leq \max_{\beta \in U} \left\{ E \left[ 1_{A_T(\xi + \beta)} \exp \left\{ \int_T^{T+\Delta} e^{-rs} ds - e^{-r(T+\Delta)} \gamma ds \right\} \right. \right\}
\]

\[
\times \left. \left( Y(T+\Delta)^-(\nu, \xi + \beta) - Y(T+\Delta)(\nu, \xi + \beta) \right) | F_T \right] \right\}
\]

\[
\leq \max_{\beta \in U} \left\{ E \left[ 1_{A_T(\xi + \beta)} \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right. \right\}
\]

\[
\times \left. \left( Y(T+\Delta)^-(\nu, \xi + \beta) - Y(T+\Delta)(\nu, \xi + \beta) \right) | F_T \right] \right\}
\]

where for any predictable stopping time $T \geq \nu$ and $\xi$ an $F_\nu$-measurable r.v.

$A_T(\xi) := \{ \omega \in \Omega, \Delta_T Y(\nu, \xi) < 0 \}$ which belongs to $F_T$. Therefore

\[
1_{A_T(\xi)} \{ O_T^-(\nu, \xi) - O_T(\nu, \xi) \} \leq \max_{\beta \in U} \left\{ E \left[ 1_{A_T(\xi)} \times 1_{A_T(\xi + \beta)} \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right. \right\}
\]

\[
\times \left. \left( Y(T+\Delta)^-(\nu, \xi + \beta) - Y(T+\Delta)(\nu, \xi + \beta) \right) \right\} \right\}.
\]

(7.32)

Let us notice that there exists at least one $\beta \in U$ such that the right-hand side is positive. Otherwise the left-hand side is null and this is a contradiction. Since we have that $Y(T+\Delta)(\nu, \xi + \beta) \geq O_{T+\Delta}(\nu, \xi + \beta)$ and on the set $A_{T+\Delta}(\xi + \beta)$, $Y(T+\Delta)^-(\nu, \xi + \beta) = O(T+\Delta)^-(\nu, \xi + \beta)$. 

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Therefore (7.32) implies
\[
1_{A_T(\xi)} \{ O_T^-(\nu, \xi) - O_T(\nu, \xi) \} \\
\leq \max_{\beta \in U} \left\{ \mathbb{E} \left[ 1_{A_T(\xi)} \times 1_{A_{T+\Delta(\xi+\beta)}} \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right| \mathcal{F}_T \right\} \\
\times \left( O_{(T+\Delta)}(\nu, \xi + \beta) - O_{T+\Delta}(\nu, \xi + \beta) \right) \right\} \left| \mathcal{F}_T \right\} \\
\leq \mathbb{E} \left[ 1_{A_T(\xi)} \times \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right| \mathcal{F}_T \right\} \\
\times \frac{\exp \left\{ \gamma (e^{-rT} - e^{-r(T+\Delta)}) \right\} \times \mathbb{E} \left[ 1_{A_{T+\Delta(\xi+\beta_1)}} \{ O_{(T+\Delta)}(\nu, \xi + \beta_1) \} \mathcal{F}_{T+\Delta} \} \} \left| \mathcal{F}_T \right\},
\]
where \( \beta_1 \) is a r.v. \( \mathcal{F}_{T+\Delta} \)-measurable valued in \( U \). Note that, as previously, the left-hand side is not null. Next, since we have that \( A_T(\xi) \) and \( \left( \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right) \) are also \( \mathcal{F}_{T+\Delta} \)-measurable then
\[
1_{A_T(\xi)} \{ O_T^-(\nu, \xi) - O_T(\nu, \xi) \} \\
\leq \mathbb{E} \left[ 1_{A_T(\xi)} \times 1_{A_{T+\Delta(\xi+\beta_1)}} \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+\Delta)}) \right\} \right| \mathcal{F}_T \right\} \\
\times \left\{ O_{(T+\Delta)}(\nu, \xi + \beta_1) - O_{T+\Delta}(\nu, \xi + \beta_1) \right\} \right\} \left| \mathcal{F}_T \right\}. \tag{7.33}
\]
Now we repeat this reasoning many times, we obtain
\[
1_{A_T(\xi)} \{ O_T^-(\nu, \xi) - O_T(\nu, \xi) \} \\
\leq \mathbb{E} \left[ 1_{A_T(\xi)} \left\{ \prod_{k=1}^{k=n} 1_{A_{T+n\Delta(\xi+\beta_1+\ldots+\beta_n)}} \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+n\Delta)}) \right\} \right\} \right| \mathcal{F}_T \right\} \\
\times \left( O_{(T+n\Delta)}(\nu, \xi + \beta_1 + \ldots + \beta_n) - O_{T+n\Delta}(\nu, \xi + \beta_1 + \ldots + \beta_n) \right) \right\} \right\} \left| \mathcal{F}_T \right\}, \tag{7.34}
\]
where the random variables \( \beta_k \) are valued in \( U \) and \( \mathcal{F}_{T+k\Delta} \)-measurable. But the left-hand side converges to \( 0 \) \( \mathbb{P} \) - a.s. when \( n \to +\infty \), since for any \( \nu \) and \( \xi \), by using (4.7), we have for all \( t \geq 0 \)
\[
|O_t(\nu, \xi)| \leq \exp \left\{ \frac{\gamma}{r} (e^{-rT} - e^{-r(T+n\Delta)} + ||\psi|| e^{-r(T+n\Delta)} + \frac{\gamma}{r} e^{-r(T+n\Delta)}) \right\}, \tag{7.35}
\]
and then \( \lim_{t \to +\infty} O_t(\nu, \xi) = 1 \) uniformly with respect to \( \nu \) and \( \xi \). By taking the limit w.r.t \( n \) in (7.34) we obtain,
\[
1_{A_T(\xi)} \{ O_T^-(\nu, \xi) - O_T(\nu, \xi) \} = 0,
\]
which is a contradiction and then the process \( Y(\nu, \xi) \) is continuous.

We are now ready to give the main result of this section.

**Theorem 7.2.** Assume that Assumption 3.1 hold. Let us define the strategy \( \delta^* = (\tau_n^*, \beta_n^*)_{n \geq 0} \) by:
\[
\tau_0^* = \begin{cases} 
\inf \{ s \in [0, \infty); O_\delta(0, 0) \geq Y_\delta(0, 0) \} & \text{if finite,} \\
+\infty & \text{otherwise}
\end{cases}
\]
and \( \beta_0^* \) is an \( \mathcal{F}_{\tau_0^*} \)-r.v. valued in \( U \) such that
\[
O_{\tau_0^*}(0, 0) := \mathbb{E} \left[ \exp \left\{ \int_{\tau_0^*}^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^*+\Delta)} \psi(\beta_0^*) \right\} \right| \mathcal{F}_{\tau_0^*} \right\}.
\]
For \( n \geq 1 \),
\[
\tau^*_n = \inf \left\{ s \geq \tau^*_{n-1} + \Delta, O_s(\tau^*_{n-1}, \beta^*_0 + \cdots + \beta^*_{n-1}) \geq Y_s(\tau^*_{n-1}, \beta^*_0 + \cdots + \beta^*_{n-1}) \right\},
\]
and \( \beta^*_n \) is an \( \mathcal{F}_{\tau^*_n} \)-r.v. valued in \( U \) such that
\[
O_{\tau^*_n}(\tau^*_{n-1}, \beta^*_0 + \cdots + \beta^*_{n-1}) = E \left[ \exp \left\{ \int_{\tau^*_n}^{\tau^*_{n+1}} e^{-r s} h(s, L_s) ds \right\} \right].
\]

But since for any \( \nu, \xi \), the process \( Y(\nu, \xi) \) is continuous, then the stopping time \( \tau^*_n \) is optimal after 0. This yields
\[
Y_0(0, 0) = E \left[ \exp \left\{ \int_0^{\tau^*_0} e^{-r s} h(s, L_s) ds \right\} O_{\tau^*_0}(0, 0) \right].
\]

Note that we have used the point iii) of Proposition 7 in the last equality to replace \( Y_{\tau^*_0 + \Delta}(0, \beta^*_0) \) with \( Y_{\tau^*_0 + \Delta}(\tau^*_0, \beta^*_0) \). Hence
\[
Y_0(0, 0) = E \left[ \exp \left\{ \int_0^{\tau^*_0 + \Delta} e^{-r s} h(s, L_s) ds - e^{-r(\tau^*_0 + \Delta)} \psi(\beta^*_0) \right\} Y_{\tau^*_0 + \Delta}(\tau^*_0, \beta^*_0) \right].
\]

Similarly, we have that
\[
Y_{\tau^*_0 + \Delta}(\tau^*_0, \beta^*_0) = E \left[ \exp \left\{ \int_{\tau^*_0 + \Delta}^{\tau^*_1 + \Delta} e^{-r s} h(s, L_s + \beta^*_0) ds - e^{-r(\tau^*_0 + \Delta)} \psi(\beta^*_1) \right\} Y_{\tau^*_1 + \Delta}(\tau^*_1, \beta^*_0 + \beta^*_1) \right].
\]
Replacing this in \( Y_0(0, 0) \), it follows that
\[
Y_0(0, 0) = E \left[ \exp \left\{ \int_0^{\tau^*_0 + \Delta} e^{-r s} h(s, L_s) ds + \int_{\tau^*_0 + \Delta}^{\tau^*_1 + \Delta} e^{-r s} h(s, L_s + \beta^*_0) ds \right. \right. \\
- e^{-r(\tau^*_0 + \Delta)} \psi(\beta^*_0) - e^{-r(\tau^*_0 + \Delta)} \psi(\beta^*_1) \left\} Y_{\tau^*_1 + \Delta}(\tau^*_1, \beta^*_0 + \beta^*_1) \right].
\]
Repeating this argument as many times as necessary, we obtain that

\[ Y_0(0, 0) = \mathbb{E} \left[ \exp \left\{ \int_0^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds + \sum_{1 \leq k \leq n} \int_{\tau_{k-1}^*+\Delta}^{\tau_k^*+\Delta} e^{-rs} h(s, L_s + \beta^*_0 + \cdots + \beta^*_{k-1}) ds \\
- \sum_{k=0}^n e^{-r(\tau^*_k+\Delta)} \psi(\beta^*_k) \right\} Y_{\tau^*_n+\Delta}(\tau^*_n, \beta^*_0 + \cdots + \beta^*_n) \right]. \] (7.38)

But since \( \mathbb{P}\{\tau^*_n \geq n \Delta\} = 1 \) then \( \mathbb{P}\)-a.s. the series \( \sum_{n \geq 0} e^{-r \tau^*_n} \psi(\beta^*_n) \) is convergent and \( |\sum_{n \geq 0} e^{-r \tau^*_n} \psi(\beta^*_n)| \leq C \) for some constant \( C \). On the other by (7.29) and monotonicity we have,

\[ Y^0_{\tau^*_n+\Delta}(\tau^*_n, \beta^*_0 + \cdots + \beta^*_n) \leq Y_{\tau^*_n+\Delta}(\tau^*_n, \beta^*_0 + \cdots + \beta^*_n) \leq \exp \left( \frac{\gamma e^{-r(\tau^*_n+\Delta)}}{r} \right). \]

As

\[ \lim_{n \to \infty} Y^0_{\tau^*_n+\Delta}(\tau^*_n, \beta^*_0 + \cdots + \beta^*_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} \exp \left( \frac{\gamma e^{-r(\tau^*_n+\Delta)}}{r} \right) = 1 \]

then

\[ \lim_{n \to \infty} Y_{\tau^*_n+\Delta}(\tau^*_n, \beta^*_0 + \cdots + \beta^*_n) = 1. \]

Take now the limit w.r.t \( n \) in the right-hand side of (7.38) to obtain that \( Y_0(0, 0) = J(\delta^*) \).

**Step 2:** \( J(\delta^*) \geq J(\delta') \) for any other strategy \( \delta' = (\tau'_{n}, \beta'_{n})_{n \geq 0} \in \mathcal{A} \).

We have that

\[ Y_0(0, 0) \geq \mathbb{E} \left[ \exp \left\{ \int_0^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds \right\} O_{\tau_0^*}(0, 0) \right]. \]

and

\[ O_{\tau_0^*}(0, 0) \geq \mathbb{E} \left[ \exp \left\{ \int_{\tau_0^*}^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^*+\Delta)} \psi(\beta_0') \right\} Y_{\tau_0^*+\Delta}(\tau_0^*, \beta_0') | \mathcal{F}_{\tau_0^*} \right] \]

since by Proposition 7.5(iii), \( Y_{\tau_0^*+\Delta}(0, \beta_0') = Y_{\tau_0^*+\Delta}(\tau_0^*, \beta_0') \). From where, we get that

\[ Y_0(0, 0) \geq \mathbb{E} \left[ \exp \left\{ \int_0^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds - e^{-r(\tau_0^*+\Delta)} \psi(\beta_0') \right\} Y_{\tau_0^*+\Delta}(\tau_0^*, \beta_0') \right]. \] (7.39)

By a similar way,

\[ Y_{\tau_0^*+\Delta}(\tau_1', \beta_0') = \text{ess sup}_{\tau \in \tau_{\tau_0^*+\Delta}} \mathbb{E} \left[ \exp \left\{ \int_{\tau_0^*+\Delta}^{\tau} e^{-rs} h(s, L_s + \beta_0') ds \right\} O_{\tau}(\tau_0^*, \beta_0') | \mathcal{F}_{\tau_0^*+\Delta} \right] \]

\[ \geq \mathbb{E} \left[ \exp \left\{ \int_{\tau_0^*+\Delta}^{\tau_1'} e^{-rs} h(s, L_s + \beta_0') ds \right\} O_{\tau_1'}(\tau_0^*, \beta_0') | \mathcal{F}_{\tau_0^*+\Delta} \right] \]

\[ \geq \mathbb{E} \left[ \exp \left\{ \int_{\tau_0^*+\Delta}^{\tau_1'+\Delta} e^{-rs} h(s, L_s + \beta_0') ds - e^{-r(\tau_1'+\Delta)} \psi(\beta_1') \right\} Y_{\tau_1'+\Delta}(\tau_1', \beta_0' + \beta_1') | \mathcal{F}_{\tau_0^*+\Delta} \right]. \]

Therefore,

\[ Y_0(0, 0) \geq \mathbb{E} \left[ \exp \left\{ \int_0^{\tau_0^*+\Delta} e^{-rs} h(s, L_s) ds + \int_{\tau_0^*+\Delta}^{\tau_1'+\Delta} e^{-rs} h(s, L_s + \beta_0') ds \\
- e^{-r(\tau_0^*+\Delta)} \psi(\beta_0') - e^{-r(\tau_1'+\Delta)} \psi(\beta_1') \right\} Y_{\tau_1'+\Delta}(\tau_1', \beta_0' + \beta_1') \right]. \]
Repeat this argument many times, it follows that
\[
Y_0(0,0) \geq \mathbb{E}\left[ \exp \left\{ \int_0^{\tau_0^+ + \Delta} e^{-rs} h(s, L_s) ds + \sum_{1 \leq k \leq n} \int_{\tau_{k-1}^+ + \Delta}^{\tau_k^+ + \Delta} e^{-rs} h(s, L_s + \beta_0 + \cdots + \beta_{k-1}) ds \right. \\
- \sum_{k=0}^{n} e^{-r(\tau_k^+ + \Delta) \psi(\beta_k')} \left. Y_{\tau_k^+ + \Delta}(\tau_{n}, \beta_0' + \cdots + \beta_n') \right]\right].
\]

Now, we take the limit as \( n \to +\infty \) in the right hand-side of this inequality to obtain that
\[
Y_0(0,0) \geq \mathbb{E}\left[ \exp \left\{ \int_0^{+\infty} e^{-rs} h(s, L_s^\prime) ds - \sum_{n \geq 0} e^{-r(\tau_n^+ + \Delta) \psi(\beta_n')} \right\} \right] = J(\delta')
\]
since the series is convergent and bounded and, as above, \( \lim_{n \to \infty} Y_{\tau_n^+ + \Delta}(\tau_n', \beta_0' + \cdots + \beta_n') = 1 \).

This latter point can be obtained by \( (7.29) \) and the fact that \( Y(\nu, \xi) \geq Y^0(\nu, \xi) \). Therefore \( Y_0(0,0) \geq J(\delta') \). Thus, we conclude that for any arbitrary strategy \( \delta \) in \( \mathcal{A} \), we have that
\[
Y_0(0,0) = J(\delta^*) = \sup_{\delta \in \mathcal{A}} J(\delta)
\]
which means that \( \delta^* \) is optimal.

8 Appendix

Part (I): Snell envelope.

Let \( U \) be an \( \mathcal{F}_t \)-adapted càdlàg process which belongs to class \([D]\), i.e. the random variables set \( \{U_\theta, \theta \in \mathcal{T} \} \) is uniformly integrable. The Snell envelope of the process \( U \) denoted by \( SN(U) \) is the smallest càdlàg super-martingale which dominates \( U \). It exists and verifies:

i) \( \forall t \geq 0, \ SN_t(U) := \text{ess sup}_{\theta \in \mathcal{T}_t} \mathbb{E}[U_\theta | \mathcal{F}_t] \). \hfill (8.40)

ii) \( \lim_{t \to \infty} SN_t(U) = \limsup_{t \to \infty} U_t \).

iii) The jumping times of \( (SN_t(U))_{t \geq 0} \) are predictable and verify \( \{ \Delta(SN_t(U)) < 0 \} \subset \{ SN_{t-}(U) = U_{t-} \} \cap \{ \Delta_k(U) < 0 \} \). \( \square \)

iv) If \( U \) has only positive jumps on \( [0, \infty] \), then \( SN(U) \) is a continuous process on \( [0, \infty] \). Moreover, if \( \theta \) is a \( \mathcal{F}_t \)-stopping time and, \( \tau_\theta^0 = \inf\{s \geq \theta, SN(U)_s \leq U_s\} \) \((+\infty \text{ if empty})\), then \( \tau_\theta^0 \) is optimal after \( \theta \), i.e.,
\[
SN(U)_\theta = \mathbb{E}[SN(U)_{\tau_\theta^0} | \mathcal{F}_\theta] = \mathbb{E}[U_{\tau_\theta^0} | \mathcal{F}_\theta] = \text{ess sup}_{\tau \geq \theta} \mathbb{E}[U_{\tau} | \mathcal{F}_\theta]. \hfill (8.41)
\]

v) If \( (U_n)_{n \geq 0} \) and \( U \) are càdlàg processes of class \([D]\) and such that the sequence of process \( (U_n)_{n \geq 0} \) converges increasingly and pointwisely to \( U \), then \( (SN(U_n))_{n \geq 0} \) converges increasingly and pointwisely to \( SN(U) \).

For further reference and details on the Snell envelope, we refer to \( \mathbb{[7]} \) or \( \mathbb{[5]} \).

Part (II): Optional and predictable projections

Let \( X := (X_t)_{t \geq 0} \) be a measurable bounded process.
i) There exists an optional (resp. predictable) process $Y$ (resp. $Z$) such that:

$$E[X_T1_{\{T<\infty\}}|F_T] = Y_T1_{\{T<\infty\}}, \ P-a.s. \ for \ any \ stopping \ time \ T$$

(resp.

$$E[X_T1_{\{T<\infty\}}|F_{T^-}] = Z_T1_{\{T<\infty\}}, \ P-a.s. \ for \ any \ predictable \ stopping \ time \ T).$$

The process $Y$ (resp. $Z$) is called the optional (resp. predictable) projection of the process $X$.

ii) If $X$ is càdlàg, then $Y$ is also càdlàg.

iii) Since the filtration $(F_t)_{t\geq0}$ is Brownian then $F_{T^-} = F_T$ and the processes $Y$ and $Z$ are indistinguishable. In particular, the optional projection of a bounded continuous process is also continuous. Finally for any predictable stopping time $T$

$$E[\Delta_T X | F_T] = \Delta_T Z, \ P-a.s.$$ 

For more details one can see ([5], pp.113, ).

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