Abstract. This paper is a contribution to the theory of finite semigroups and their classification in pseudovarieties, which is motivated by its connections with computer science. The question addressed is what role is played by the consideration of an order compatible with the semigroup operation. In the case of unions of groups, so-called completely regular semigroups, the problem of which new pseudovarieties appear in the ordered context is solved. As applications, it is shown that the lattice of pseudovarieties of ordered completely regular semigroups is modular and that taking the intersection with the pseudovariety of bands defines a complete endomorphism of the lattice of all pseudovarieties of ordered semigroups.

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1. Introduction

Semigroup theory has developed in several fronts, as semigroups appear naturally in many contexts. On the algebraic front, varieties of semigroups, in the sense of universal algebra, have received considerable attention in the literature. Thus, the study of the lattice of all varieties of semigroups is one of the
classical topics in semigroup theory. One of the first results in this direction was a complete description of the lattice of all varieties of bands (semigroups in which all elements are idempotent) which was described independently by Birjukov [8], Fennemore [11,12] and Gerhard [13]. Bringing groups into play, a natural generalization of both classes is the class of completely regular semigroups (meaning semigroups in which every element lies in a subgroup) which also forms a variety in a suitable algebraic signature, that of unary semigroups. Consequently, intensive attention was also paid to the study of the lattice of varieties of completely regular semigroups. To recall the basic contributions to that theory one should mention the book by Petrich and Reilly [22] and a series of seminal papers by Polák [25–27].

With the applications of semigroup theory in the algebraic theory of regular languages via Eilenberg’s correspondence, the focus was shifted in part to the theory of pseudovarieties of finite semigroups, that is, classes of finite semigroups that are closed under taking homomorphic images, subsemigroups, and finite direct products. Some of the results, for example the mentioned description of the lattice of all varieties of bands, were translated automatically to the case of pseudovarieties due to the local finiteness property. However, many results required the development of new techniques. For example, results concerning pseudovarieties of finite completely regular semigroups were obtained by Trotter and the first author in [6] using profinite techniques; such techniques evolved in the aftermath of Reiterman’s [29] use of profinite semigroups to describe pseudovarieties and were extensively developed by the first author. For an overview of known results in this area we refer to the books on the theory of finite semigroups by the first author [1] and Rhodes and Steinberg [30].

Another step in the applications in the theory of regular languages was a refinement of Eilenberg’s correspondence due to Pin [23], which describes what classes of languages are obtained when closure under complementation is discarded. The theory of finite ordered semigroups introduced there may be viewed as a generalization of the theory of finite semigroups. Simply put, ordered semigroups are semigroups enriched by a partial order which is compatible with the multiplication, also called a stable partial order. So, in particular, every semigroup together with the equality relation is an ordered semigroup. Furthermore, every pseudovariety of semigroups \( V \) determines a pseudovariety of ordered semigroups \( V^o \) by taking all possible stable partial orders on every semigroup from the original pseudovariety \( V \). The pseudovarieties of the form \( V^o \) are denoted by the same symbol as the original pseudovariety \( V \); we call them *selfdual*, since they are characterized by the property of being closed under taking dually ordered semigroups. Denote the lattice of all subpseudovarieties of ordered semigroups of the pseudovariety \( U \) by \( \mathcal{L}_o(U) \), and the lattice of all subpseudovarieties of semigroups of the pseudovariety \( V \) by \( \mathcal{L}(V) \). Then, by the previous observations, \( \mathcal{L}(V) \) may be viewed as a sublattice of \( \mathcal{L}_o(V) \). Notice also that the ordered version of the theory of profinite semigroups was introduced by Pin and Weil in [24] and further developed in the literature.
One can say that, besides the results in connection with the theory of varieties of regular languages, a systematic study of pseudovarieties of ordered semigroups was not so far significantly developed. We can mention only the result by Emery [10], who described the lattice of all pseudovarieties of ordered normal bands $\mathcal{L}_o(\text{NB})$, that is, bands satisfying the identity $x y z x = x z y x$. Furthermore, Kuříl [18] studied the lattice of all pseudovarieties of ordered bands. Although his paper deals with varieties of ordered bands, it yields the same result for pseudovarieties, because bands are locally finite. The description of that lattice was possible since he proved that there are just a few pseudovarieties of ordered bands which are not selfdual. More precisely, all of them are subpseudovarieties of the pseudovariety of normal bands. Thus, by the description of the lattice of pseudovarieties of (unordered) bands and by the result of Emery, [18] yields a complete description of the lattice $\mathcal{L}_o(\mathcal{B})$.

Our research is inspired by results from [18], as we have tried to extend the results to a richer class. Another pseudovariety without new subpseudovarieties of ordered semigroups is the pseudovariety of all groups $G$, because each finite ordered group is only trivially ordered. More formally, we have $\mathcal{L}_o(G) = \mathcal{L}(G)$.

The main result of this paper is the following statement.

**Theorem 1.** Let $V$ be a pseudovariety of ordered completely regular semigroups containing all semilattices. Then $V$ is selfdual.

Hence, one needs to deal with pseudovarieties of ordered completely regular semigroups which do not contain all semilattices. It is not difficult to show that such pseudovarieties contain only normal orthogroups, which are completely regular semigroups in which idempotents form normal bands. Thus, we can state that all pseudovarieties of ordered completely regular semigroups which are not selfdual are contained in $\mathcal{L}_o(\text{NOCR})$, where NOCR is the pseudovariety of all finite normal orthogroups. We also prove the following result.

**Theorem 2.** The lattice $\mathcal{L}_o(\text{NOCR})$ is isomorphic to the direct product of the lattices $\mathcal{L}_o(\text{NB})$ and $\mathcal{L}(G)$.

Of course, we cannot say that we completely described the lattice $\mathcal{L}_o(CR)$, because there are parts which are not known, namely $\mathcal{L}(CR)$ and particularly $\mathcal{L}(G)$. The lattice $\mathcal{L}(CR)$ has been extensively studied and much of the deeper results about it are based on Polák’s work [25–27]. When the idempotents form a subsemigroup, we say that the semigroup is orthodox. Orthodox completely regular semigroups are also known as orthogroups. They form a pseudovariety, denoted OCR. Polák’s methods yield a complete description of the lattice $\mathcal{L}(OCR)$ in terms of $\mathcal{L}(G)$ [20]. Thus, up to a knowledge of the lattice of pseudovarieties of groups, the lattice of pseudovarieties of ordered semigroups $\mathcal{L}_o(OCR)$ is described. In any case, the current knowledge about $\mathcal{L}(CR)$ is sufficient to obtain the following applications.
Theorem 3. The lattice $\mathcal{L}_o(CR)$ is modular.

Theorem 4. The correspondence $V \mapsto V \cap B$ defines a complete endomorphism of the lattice $\mathcal{L}_o(S)$.

The structure of the paper is the following. Section 2 fixes notation and recalls some notions and results from the literature. Then Theorem 1 is proved in Sect. 3 while Theorem 2 is proved in Sect. 4. Applications, including Theorems 3 and 4 are given in Sect. 5.

2. Preliminaries and Notation

We assume that the reader is familiar with the basic concepts from the theory of semigroups. In particular, we need some results from profinite theory of finite semigroups (see [1,3]) for which the ordered version was introduced in [24]. Besides these theories we also recall other notions.

For a quasiorder $\sigma$ on the set $M$, that is, a reflexive and transitive binary relation, we write also $a \sigma b$ to express $(a, b) \in \sigma$. Both infix and suffix notation are also used for a partial order, which is an antisymmetric quasiorder. For a quasiorder $\sigma$, we denote by $\sigma^d$ the dual quasiorder, that is, $\sigma^d = \{(a, b) \in M \times M \mid (b, a) \in \sigma\}$. Then $\sigma \cap \sigma^d$ is an equivalence relation on $M$, which is denoted by $\tilde{\sigma}$; the corresponding quotient set is denoted by $M/\tilde{\sigma}$ and its elements by $[a]$ for each $a \in M$. Moreover, the quasiorder $\sigma$ naturally induces a partial order on the quotient set $M/\tilde{\sigma}$, denoted by $\leq_\sigma$; it is such that $[a] \leq_\sigma [b]$ if and only if $(a, b) \in \sigma$.

A quasiorder $\leq$ on a semigroup $S$ is stable if, for every $a, s, t \in S$, the following condition holds: if $s \leq t$ then $sa \leq ta$ and $as \leq at$. By an ordered semigroup we mean a semigroup $(S, \cdot)$ which is equipped with a stable partial order. Homomorphisms of ordered semigroups are homomorphisms of semigroups which are isotone. Similar to the unordered case, pseudovarieties of ordered semigroups are classes of finite ordered semigroups closed under taking homomorphic images, (ordered) subsemigroups and finite products. For a pseudovariety $V$ of finite ordered semigroups, we denote by $V^d$ the pseudovariety of all dually ordered semigroups, i.e., $(S, \cdot, \leq) \in V^d$ if and only if $(S, \cdot, \geq) \in V$. We say that $V$ is selfdual if $V = V^d$.\footnote{It is perhaps more common to consider in the purely algebraic context the dual of a semigroup $(S, \cdot)$ to be $(S, *)$ where $a * b = b \cdot a$. This is not the notion of duality concerning us here.} Such a pseudovariety of ordered semigroups is also characterized by the property that $(S, \cdot, \leq) \in V$ implies $(S, \cdot, =) \in V$: the identity mapping is a homomorphism from the ordered semigroup $(S, \cdot, =)$ onto an arbitrary ordered semigroup $(S, \cdot, \geq)$, while the diagonal mapping embeds $(S, \cdot, =)$ into the product $(S, \cdot, \leq) \times (S, \cdot, \geq)$. If a pseudovariety of semigroups $V$ is given, then we may consider the (selfdual) pseudovariety consisting of all ordered semigroups which are members of $V$.
equipped with every possible stable partial order. Conversely, if \( V \) is a selfdual pseudovariety of ordered semigroups then we may consider the pseudovariety consisting of all semigroups \((S, \cdot)\) such that \((S, \cdot, =) \in V\). These constructions give a one-to-one correspondence between selfdual pseudovarieties of ordered semigroups and pseudovarieties of semigroups. Note that usually the same symbol is used to denote both a pseudovariety of semigroups and the corresponding selfdual pseudovariety of ordered semigroups.

By a **profinite semigroup** we mean a compact semigroup which is residually finite as a topological semigroup, finite semigroups being viewed as discrete topological spaces. Just as pseudovarieties (of semigroups) are defined by sets of *pseudoidentities* [29], pseudovarieties of ordered semigroups are defined by *pseudoinequalities* [19,24]. Syntactically, both pseudoidentities and pseudoinequalities are pairs of *pseudowords*, the difference lies in the way they are interpreted. In a sense pseudowords are terms in an enriched language where not only the semigroup operation is allowed; more formally, pseudowords are members of a finitely generated free profinite semigroup. Often, we only need an extra unary operation, represented by the pseudoword \( x^\omega \); in general, for an element \( s \) of a profinite semigroup, \( s^\omega \) is the only idempotent in the closed subsemigroup generated by \( s \). We also write \( s^{\omega+1} \) instead of \( s^{\omega}s \). It lies in the maximal subgroup with idempotent \( s^\omega \); its inverse in that group is denoted \( s^{\omega-1} \). For a set \( \Sigma \) of pseudoidentities or pseudoinequalities, \( [\Sigma] \) is the class consisting of all finite semigroups that satisfy all elements of \( \Sigma \) in the sense that, under arbitrary evaluation of the variables, the two sides are, respectively, equal or in increasing order.

Completely regular semigroups may be viewed as semigroups with a unary operation of inversion in the unique maximal subgroup containing a given element. As such, they form a variety, which is defined by the identities

\[
x(yz) = (xy)z, \quad xx^{-1}x = x, \quad (x^{-1})^{-1} = x, \quad xx^{-1} = x^{-1}x.
\]

It is common to write \( s^0 \) for the product \( ss^{-1} \). In a completely regular profinite semigroup, note that \( s^{-1} = s^{\omega-1} \) and \( s^0 = s^\omega \).

We extend the notion of ordered semigroup to unary semigroups by requiring that not only the partial order be stable under multiplication but also under the unary operation. Note that this is not the common practice in the theory of ordered groups, where inversion is not required to preserve the order. But, it is natural to assume it as a generalization of the finite case, where inversion is given by a power. Also, when studying varieties of ordered algebras, it is natural to require that all basic operations preserve the order [9]. In particular, note that, for a stable quasiorder \( \rho \) on a completely regular semigroup the conditions \( s \rho s^0 \) and \( s^0 \rho s \) are equivalent: taking inverses in \( s \rho s^0 \), we get \( s^{-1} \rho s^0 \), which, multiplying by \( s \), yields \( s^0 \rho s \); the same steps also establish the converse.
All finite completely regular semigroups form the pseudovariety \( CR = \{ x^{\omega + 1} = x \} \) which contains all finite orthogroups, which constitute the pseudovariety \( OCR = \{ x^{\omega + 1} = x, x^{\omega} y^{\omega} = (x^{\omega} y^{\omega})^{\omega} \} \). The latter in turn contains the pseudovariety of all finite bands \( B = \{ x^2 = x \} \) and, in particular, all finite semilattices \( SI = \{ x^2 = x, xy = yx \} \) and all finite normal bands \( NB = \{ x^2 = x, xyzx = xzyx \} \). The pseudovariety \( OCR \) also contains the pseudovariety of all finite groups \( G = \{ x^{\omega} = 1 \} \). We also follow the standard notation where \( \Omega \) denotes the pseudovariety of all semigroups whose subgroups belong to a given pseudovariety of groups \( H \). We let \( CR(H) = CR \cap \Omega \).

Our aim is a description of all subpseudovarieties of ordered semigroups of the pseudovariety \( CR \). By Birkhoff’s basic results on universal algebra, varieties of algebras can be uniquely determined by fully invariant congruences on the term algebra over a countable set of variables. A similar concept of fully invariant stable quasiorder is possible in the case of varieties of ordered algebras. In particular, a proof technique in [18] is based on manipulating the fully invariant stable quasiorders corresponding to the varieties of ordered bands. However, the situation in finite universal algebra is more complicated. One needs to study relatively free profinite semigroups over a finite set of variables and, moreover, one needs to consider them for arbitrarily large finite sets. We use the notation \( \overline{\Omega}_X S \) from [1] for the free profinite semigroup over a given finite set of generators \( X \) in the pseudovariety of all semigroups \( S \). Usually, we use the natural number \( n \) as an index in \( \overline{\Omega}_X S \) instead of \( X \) when \( X = \{ x_1, \ldots, x_n \} \). Then the inclusion \( \overline{\Omega}_n S \subseteq \overline{\Omega}_m S \) may be used freely whenever \( n \leq m \). The same structures are also considered as free profinite ordered semigroups. We associate with a pseudovariety \( V \) of finite ordered semigroups a system of relations \( (\rho_{\varphi,n})_n \) on the profinite semigroups \( \overline{\Omega}_n S \) where each \( \rho_{\varphi,n} \) is given by the formula

\[
\rho_{\varphi,n} = \{ (u, v) \in \overline{\Omega}_n S \times \overline{\Omega}_n S \mid V \models u \leq v \};
\]

note that we write \( V \models u \leq v \) to mean that every member of \( V \) satisfies the pseudoinequality \( u \leq v \). It is known that each \( \rho_{\varphi,n} \) in this system is a closed stable quasiorder. It is also fully invariant, but this property is satisfied by the whole system in the following sense, which we introduce for a general system of relations.

By a system of relations \( (\rho_n)_n \) we mean a family of relations indexed by the positive integers such that \( \rho_n \) is a relation on \( \overline{\Omega}_n S \) for every \( n \). We say that the system \( (\rho_n)_n \) is fully invariant if, for each continuous homomorphism \( \varphi : \overline{\Omega}_n S \rightarrow \overline{\Omega}_m S \) and \( u \rho_m v \), we have \( \varphi(u) \rho_m \varphi(v) \). Additionally, if every relation \( \rho_n \) in this system is a closed stable quasiorder on \( \overline{\Omega}_n S \), then we call the system \( \rho = (\rho_n)_n \) a fully invariant system of closed stable quasiorders. It is not clear whether every such system \( \rho \) determines a pseudovariety \( V \) such that \( \rho = (\rho_{\varphi,n})_n \) (see [5] for a discussion concerning a related conjecture). To explain what kind of property is potentially missing, first denote \( \hat{\rho} = (\hat{\rho}_n)_n \), where \( \hat{\rho}_n \) is the equivalence relation corresponding to \( \rho_n \). Then, for each \( n \) we
consider $\Omega_n S/\widehat{\rho}_n$, which is a compact ordered semigroup, where the partial order is $\leq_{\rho_n}$. If for each $n \geq 1$ the ordered semigroup $\Omega_n S/\widehat{\rho}_n$ is residually finite, then $\rho = \rho_V$, where $V$ is given as the class of all finite ordered semigroups which are finite quotients of some ordered semigroup $(\Omega_n S/\widehat{\rho}_n, \leq_{\rho_n})$. If this property is true then we call $\rho$ a complete system of pseudoinequalities. Note that, for the corresponding pseudovariety $V$ of ordered semigroups, we may also write $V = [\rho]$ to mean $V = [\rho_n : n \geq 1]$. We talk about a complete system of pseudoidentities when $V$ is a selfdual pseudovariety of ordered semigroups. Notice also that $V$ is selfdual if and only if each $\rho_n$ is symmetric.

The question whether the ordered semigroup $\Omega_n S/\widehat{\rho}_n$ is residually finite seems to be difficult in general, but it is trivial in the case when $\Omega_n S/\widehat{\rho}_n$ is itself finite. This simple observation is sufficient for our considerations, because it turns out to be enough to work with systems $\rho = (\rho_n)$ for which each equivalence relation $\widehat{\rho}_n$ has finite index. Note that this condition is equivalent to the pseudovariety $[\rho]$ being locally finite.

For two fully invariant systems of closed stable quasiorders $\rho$ and $\sigma$, we write $\rho \subseteq \sigma$ if $\rho_n \subseteq \sigma_n$ holds for every $n \geq 1$. For specific pseudovarieties $V$, such as CR, H, CR(H), B, SI etc., we will use the symbol $\sim_V$ instead of $\rho_V$. In general, if we want to express that a pair $u,v \in \Omega_n S$ is $\rho_n$-related then we may omit the index $n$ which is clear from the context and we may simply write $u \rho v$. Note that $\rho_V \subseteq \rho_{V'}$ whenever $W \subseteq V$. In this way, we may write, e.g., $\sim_{CR} \subseteq \sim_{OCR} \subseteq \sim_B \subseteq \sim_{SI}$.

We recall from the literature the definition of the 0 and 1 functions, which take values on pseudowords and are fundamental in the theory of completely regular semigroups. Notice that these functions may be defined in a more general setting; however, in our contribution, we use them just for the free profinite semigroups $\Omega X S$ over the finite set $X$. For an arbitrary $u \in \Omega X S$, we denote by $c(u)$ the smallest set $A \subseteq X$ such that $u$ belongs to the closed subsemigroup of $\Omega X S$ generated by $A$. In other words, it is the set of all letters occurring in $u$, i.e., the set of all $x \in X$ such that there exist $u', u'' \in \Omega X S^1$ satisfying $u = u'xu''$; for this reason, it is called the content of $u$. We extend the previous definition of the content function $c$ to the case of the free profinite monoid $\Omega X S^1$ by putting $c(1) = \emptyset$. Recall that $u \sim_{SI} v$ if and only if $c(u) = c(v)$. Now, for every $u \in \Omega X S$, there are unique $u', u'' \in \Omega X S^1$ and $x \in X$ such that $u = u'xu''$, $x \not\in c(u')$ and $c(u') \cup \{x\} = c(u)$. The pseudoword $u'$ in this unique factorization is denoted by $0(u)$ and $x$ in the same factorization is denoted by $\overline{u}(u)$. Note that for $u$ such that $|c(u)| = 1$ we have $0(u) = 1$. By left to right duality we obtain the definition of $1(u)$ and $\overline{u}(u)$. See [6] for further details.

A fundamental tool in semigroup theory is given by Green’s relations [14]. For two elements $s$ and $t$ of a semigroup, we write $s \leq_J t$, $s \leq_R t$, $s \leq_\mathcal{D} t$ if, respectively, the ideal, the left ideal, the right ideal generated by $s$ is contained in that generated by $t$. These are quasiorders on the semigroup,
whose intersection with their duals are denoted, respectively, \( J, L, \) and \( R \). The intersection of \( R \) and \( L \) is denoted \( H \). A semigroup is said to be \emph{stable} if \( s \ J \) st implies \( s \ R \) st and \( s \ J \) ts implies \( s \ L \) ts. It is well known that, in a stable semigroup, the equivalence relation \( J \) is the join of \( R \) and \( L \); moreover, the main semigroups of interest in this paper, namely profinite semigroups and completely regular semigroups, are stable [15].

Given a congruence \( \theta \) on a semigroup \( S \) and one of Green’s relations \( \mathcal{H} \), we say that the elements \( s \) and \( t \) of \( S \) are \( \mathcal{H} \)-equivalent modulo \( \theta \) if their \( \theta \)-classes are \( \mathcal{H} \)-equivalent in the quotient semigroup \( S/\theta \). It is well known that, if \( V \) is a pseudovariety between \( Sl \) and \( CR \), two pseudowords \( u, v \in \Omega_X S \) are \( J \)-equivalent modulo \( V \) if and only if \( c(u) = c(v) \) (see [1, Theorem 8.1.7]); it follows that, in this case, the relation \( u \ R \ v \) holds modulo \( V \) if so does \( 0(u) = 0(v) \) (which entails the equality \( \overline{0}(u) = \overline{0}(v) \) since we are assuming that \( c(u) = c(v) \)), and the relation \( u \ L \ v \) holds modulo \( V \) if so does \( 1(u) = 1(v) \).

The following is an easy consequence of [21, Proposition 2.2] which gives the analogous result for varieties.

**Proposition 2.1.** Let \( H \) be a locally finite pseudovariety of groups. Then \( CR(H) \) is a locally finite pseudovariety of semigroups.

It is well known that every pseudovariety is the directed union of its subpseudovarieties generated by a single semigroup, which are locally finite. For this purpose, it suffices to consider a countable sequence of all (up to isomorphism) finite members of the pseudovariety and then for each natural number \( n \) to take the pseudovariety generated by the direct product of the first \( n \) semigroups in the sequence. Such an expression of \( G \) is used later.

### 3. Pseudovarieties Above Semilattices

In this section, we prove that above the pseudovariety \( Sl \) all pseudovarieties of ordered completely regular semigroups are selfdual.

So, let \( V \) be a given pseudovariety of ordered completely regular semigroups with the corresponding fully invariant system of closed stable quasiorders \( \rho_V \). Our goal is to show that all relations \( (\rho_V)_n \) are symmetric, which is equivalent with the selfduality of \( V \).

Following the basic idea of the paper [18], the proof of symmetry proceeds by induction with respect to the content of pseudowords. For a pair of pseudowords \( u, v \) that are \( (\rho_V)_n \)-related, we consider the pair of pseudowords \( 0(u), 0(v) \) (and, dually, also the pair \( 1(u) \) and \( 1(v) \)), for which we want to invoke an induction assumption. However, \( 0(u) \) and \( 0(v) \) may not be \( (\rho_V)_{n-1} \)-related, so we need to introduce another system of relations (for \( 0 \), and a dual one for \( 1 \)) which takes the role of \( \rho_V \) in the induction process. Ideally, this new system of relations would be a complete system of pseudoinequalities, corresponding to another pseudovariety of ordered completely regular semigroups. But, this is not the case in general, which leads to some extra technical difficulties.
We start by describing the new pair of systems of relations formally. For a complete system of pseudoinequalities $\rho$, we consider the system of relations $\rho^0 = (\rho^0_n)_n$ where each $\rho^0_n$ is defined as follows. On the set $\overline{\mathcal{P}}_n S$ we let $u \rho^0_n v$ if there are pseudowords $u', v' \in \overline{\mathcal{P}}_{n+1} S$ such that $u' \rho^0_{n+1} v'$, $0(u') = u$ and $0(v') = v$. Furthermore, we let $\rho^{oc} = (\rho^{oc}_n)_n$ where each $\rho^{oc}_n$ is the intersection $\rho^0_n \cap (\sim_{Sl})_n$. Dually, we define $\rho^1$ and $\rho^{1c}$ if we consider the function 1 instead of 0.

**Lemma 3.1.** Let $\rho$ be a complete system of pseudoinequalities such that $\rho \subseteq \sim_{Sl}$ and let $u, v \in \overline{\mathcal{P}}_X S$ be pseudowords such that $c(u) = c(v)$. If $0(u) \rho^0 0(v)$, then there exist $x, x' \in X$ and $s, s' \in \overline{\mathcal{P}}_X S^1$ such that, for $u' = 0(u)xs$ and $v' = 0(v)x's'$, the following properties hold:

\begin{align*}
&u' \rho v', \quad 0(u') = 0(u), \quad 0(v') = 0(v) \tag{1} \\
&c(u) = c(u') = c(v'), \quad \overline{u}(u) = x, \quad \overline{u}(v) = x'. \tag{2}
\end{align*}

**Proof.** From the assumption that $0(u) \rho^0 0(v)$ we know that there exist $x, x' \in X$ and $s, s' \in \overline{\mathcal{P}}_X S^1$ such that, for $u' = 0(u)xs$ and $v' = 0(v)x's'$, the conditions in (1) hold. We distinguish two cases. At first, we assume that $c(0(u)) = c(0(v))$. Then $x$ is equal to $x'$ because $\rho \subseteq \sim_{Sl}$. Since $\rho$ is fully invariant and $c(u) = c(v)$, by applying, if needed, a suitable substitution, we may assume that $x$ belongs to $c(u) \setminus c(0(u))$. Consequently, the conditions in (2) hold. Secondly, we assume that $c(0(u)) \neq c(0(v))$. Then $x$ occurs in $v'$ and, therefore, in $0(v)$. In a similar way we get that $x'$ occurs in $0(u)$. Since $c(u') = c(v')$, we deduce (2) again. \(\square\)

The following result relates $\rho^0$ and $\rho^1$ with the restriction of $\rho$ to the set of idempotents, the so-called trace of $\rho$ \cite{22}.

**Lemma 3.2.** Let $\rho$ be a complete system of pseudoinequalities such that $\sim_{CR} \subseteq \rho \subseteq \sim_{Sl}$ and let $u, v \in \overline{\mathcal{P}}_X S$ be arbitrary pseudowords. Then $w^\omega \rho v^\omega$ if and only if

\[0(u) \rho^0 0(v), \quad 1(u) \rho^1 1(v), \quad \text{and} \quad u \sim_{Sl} v. \tag{3}\]

**Proof.** Suppose first that $w^\omega \rho v^\omega$. Since, in general, $w$ and $w^\omega = ww^{\omega-1} = w^{\omega-1}w$ have the same values for each of the functions $c$, 0, and 1, and $\rho \subseteq \sim_{Sl}$, we obtain the conditions in (3).

Conversely, suppose that (3) holds. Let $u', v' \in \overline{\mathcal{P}}_X S$ be given by Lemma 3.1 and let $u'', v'' \in \overline{\mathcal{P}}_X S$ be given by the dual version of Lemma 3.1, concerning $\rho^1$, so that there are $y, y' \in X$ and $t, t' \in \overline{\mathcal{P}}_X S^1$ such that $u'' = ty1(u)$, $v'' = t'y1(v)$, and

\begin{align*}
&u'' \rho v'', \quad 1(u'') = 1(u), \quad 1(v'') = 1(v) \tag{4} \\
&c(u) = c(u'') = c(v''), \quad \overline{t}(u) = y, \quad \overline{t}(v) = y'. \tag{5}
\end{align*}

From the first conditions in (1) and (4), we deduce that

\[(u'u'')^\omega \rho (v'v'')^\omega. \tag{6}\]
But, by (2) and (5), the idempotents \((u'u'')o\) and \(u'o\) have the same content and, therefore, they are \(J\)-equivalent modulo \(\sim_{CR}\). By (1) and (4), they also have the same values under each of the functions 0 and 1, which entails that they are \(H\)-equivalent modulo \(\sim_{CR}\), whence they are in the same \(\sim_{CR}\)-class. For the same reason, we also have \((v'v'')o\sim_{CR}v'o\). Combining these observations with (6), we obtain the required relation \(u'o\rho v'o\) since \(\sim_{CR}\subseteq\rho\). □

The following lemma enables us to prove that the relations \(\rho_n\) are symmetric by a certain inductive argument using the symmetry of both \(\rho_0\) and \(\rho_1\).

It is the order analog for finite semigroups of a key result in the theory of completely regular semigroups describing a congruence in terms of its kernel (the set of elements congruent to an idempotent) and trace [22, Lemma VI.3.1].

**Lemma 3.3.** Let \(S\) be a completely regular semigroup and let \(\rho\) be a stable quasiorder on \(S\). Then, the following conditions are equivalent for arbitrary \(J\)-equivalent elements \(s, t \in S\):

(i) \(s \rho t\);
(ii) \(s^0 \rho t^0\) and \(st^{-1} \rho (st^{-1})^0\);
(iii) \(s^0 \rho t^0\) and \(t^{-1}s \rho (t^{-1}s)^0\);
(iv) \(s^0 \rho t^0\) and \(s^{-1}t \rho (s^{-1}t)^0\);
(v) \(s^0 \rho t^0\) and \(ts^{-1} \rho (ts^{-1})^0\).

**Proof.** We establish only the equivalence (i) ⇔ (ii) as the equivalence of (i) with each of the remaining conditions may be proved similarly.

Suppose first that \(s \rho t\). Since \(\rho\) is stable, it follows that \(s^0 \rho t^0\). On the other hand, multiplying on the right \(s \rho t\) by \(t^{-1}\), we obtain \(st^{-1} \rho (st^{-1})^0\). Further multiplying on the left by \((st^{-1})^0\), we deduce that \(st^{-1} \rho (st^{-1})^0\).

For the converse, assuming (ii), note that \(s = s^0ss^0 \rho t^0st^0 = t^0st^{-1}t \rho t^0(st^{-1})^0t\). Since \(t^0\) and \((st^{-1})^0\) are \(J\)-equivalent idempotents that admit as a common suffix the element \(t\) of the same \(J\)-class, by stability of \(S\), they are \(L\)-equivalent modulo \(\sim_{CR}\). Hence, we have \(t^0(st^{-1})^0t = t^0t = t\) and, therefore, \(s \rho t\). □

The following lemma gives a characterization of a complete system of pseudoinequalities \(\rho\) in terms of \(\rho_0\), \(\rho_1\), content and kernel.

**Lemma 3.4.** Let \(\rho\) be a complete system of pseudoinequalities such that \(\sim_{CR} \subseteq \rho \subseteq \sim_{Sl}\) and let \(u, v \in \Omega X S\) be arbitrary pseudowords. The following conditions are equivalent:

(i) \(u \rho v\);
(ii) \(0(u) \rho^0 0(v), 1(u) \rho^1 1(v), u \sim_{Sl} v, \) and \(uv^{1-} \rho (uv^{1-})o\);
(iii) \(0(u) \rho^0 0(v), 1(u) \rho^1 1(v), u \sim_{Sl} v, \) and \(vu^{1-} \rho (vu^{1-})o\).

**Proof.** The result is an immediate consequence of Lemmas 3.2 and 3.3. □
Let $H = V \cap G$. Thus, we have the inclusions $\sim_{\text{CR}(H)} \subseteq \rho_V \subseteq \sim_{\text{SI}}$. In the next lemma, we need the assumption that $H$ is locally finite which, by Proposition 2.1, entails that $V$ is locally finite. The lemma describes basic properties of the relation $\rho^\text{oc}$ which is equal to $\rho^0$ under the assumption that $\rho^0 \subseteq \sim_{\text{SI}}$. It enables an induction argument on the content in the main statement of this section.

**Lemma 3.5.** Let $H$ be a locally finite pseudovariety of groups and $\rho$ be a complete system of pseudoinequalities such that $\sim_{\text{CR}(H)} \subseteq \rho \subseteq \sim_{\text{SI}}$. Then $\rho^\text{oc}$ is a complete system of pseudoinequalities which contains $\rho$.

**Proof.** First, we prove that for each $n$ the relation $\rho^\text{oc}_n$ is stable. Let $u \rho^\text{oc}_n v$ be a pair of pseudowords and $w$ be another pseudoword and assume that $u, v, w \in \bigcap_n S$. Consider the set $A = \{uw, \ldots, x_n\}$. There exist $k \in \{1, \ldots, n+1\}$ and $s, s' \in \bigcap_{n+1} S^1$ such that

$$ux_k s_{\rho_{n+1}} vx_k s'_1, \quad x_k \notin c(u) = c(v), \quad x_k = \overline{0}(ux_k s) = \overline{0}(vx_k s').$$

Using the assumption that $\rho$ is a fully invariant system, we may assume that $k = n + 1$. Since $\rho_{n+1}$ is stable, we get $wux_{n+1}s_{\rho_{n+1}} wvx_{n+1}s'$ from which $wu \rho_{n+1}^\text{oc} wv$ follows. To prove $uw \rho_{n+1}^\text{oc} wv$, we need to substitute $wux_{n+1}$ for $x_{n+1}$ in $wux_{n+1} \rho_{n+1} wvx_{n+1}s'$ first, which is possible because $\rho$ is fully invariant.

Next, we prove that, for each $n$, the relation $\rho^\text{oc}_n$ is a quasiorder. The relation $\rho^\text{oc}_n$ is clearly reflexive. To prove transitivity, assume that we have $u \rho^\text{oc}_n v$ and $v \rho^\text{oc}_n w$. Since $\rho^\text{oc}_n \subseteq \sim_{\text{SI}}$ we have $c(u) = c(v) = c(w)$. Thus, there are $k \in \{1, \ldots, n+1\}$ and $s, s' \in \bigcap_{n+1} S^1$ such that

$$ux_k s_{\rho_{n+1}} vx_k s'_1, \quad x_k \notin c(u) = c(v), \quad x_k = \overline{0}(ux_k s) = \overline{0}(vx_k s').$$

We have similarly a letter $x_t$ and pseudowords $t, t' \in \bigcap_{n+1} S^1$ for the relation of pseudowords $v \rho^\text{oc}_n w$. Since $\rho$ is fully invariant, we may assume that $x_t = x_k$. Thus, we get

$$vx_k t_{\rho_{n+1}} wx_k t', \quad x_k \notin c(v) = c(w), \quad x_k = \overline{0}(vx_k t) = \overline{0}(wx_k t').$$

If we use stability of $\rho$, we get $(ux_k s)^\omega_{\rho_{n+1}} (vx_k s'^1)^\omega_{\rho_{n+1}} (wx_k t)^\omega_{\rho_{n+1}} (wx_k t')^\omega$. If we multiply the first relation by $(vx_k t)^\omega$ on the right, we obtain $(ux_k s)^\omega(vx_k t)^\omega_{\rho_{n+1}} (vx_k s'^1)^\omega(vx_k t)^\omega$. Since $0((ux_k s')^\omega) = 0((vx_k t)^\omega) = v$, the pseudowords $(vx_k s')^\omega$ and $(vx_k t)^\omega$ are $\mathcal{R}$-equivalent idempotents modulo $\sim_{\text{CR}(H)}$. It follows that their product $(vx_k s')^\omega(vx_k t)^\omega$ is $\sim_{\text{CR}(H)}$-equivalent to the right factor $(vx_k t)^\omega$. Thus, we have $(ux_k s)^\omega(vx_k t)^\omega_{\rho_{n+1}} (ux_k s)^\omega(vx_k t)^\omega$, because $\sim_{\text{CR}(H)} \subseteq \rho$. By transitivity of $\rho$ and from $(ux_k s)^\omega_{\rho_{n+1}} (wx_k t')^\omega$, we get $(ux_k s)^\omega(vx_k t)^\omega_{\rho_{n+1}} (wx_k t')^\omega$. By applying the 0 function, we may deduce that $u \rho_{n+1}^\text{oc} w$.

The next step is to show that, for each $n$, the relation $\rho^\text{oc}_n$ is closed. Let $(u_k, v_k)_k$ be a sequence of pairs of pseudowords in $\rho^\text{oc}_n$ such that $\lim u_k = u$ and $\lim v_k = v$. Since $u_k (\sim_{\text{SI}}) v_k$ holds for every $k$, we deduce that $u (\sim_{\text{SI}}) v$. By
the assumption that \( u_k \rho_n^0 v_k \), there are \( \ell_k \in \{1, \ldots, n + 1 \} \) and pseudowords \( s_k, s'_k \in \overline{\Omega}_{n+1}S^1 \) such that

\[
 u_k x_{\ell_k} s_k \rho_{n+1} v_k x_{\ell_k} s'_k, \, x_{\ell_k} \notin c(u_k) = c(v_k), \, x_{\ell_k} = \overline{0}(u_k x_{\ell_k} s_k) = \overline{0}(v_k x_{\ell_k} s'_k).
\]

We consider an infinite set \( I \) of indices such that \( c(u_k) = c(u) \) for every \( k \in I \). Then, we may assume that, for all \( k \in I \), the variable \( x_{\ell_k} \) is equal to \( x_{n+1} \) because \( \rho_{n+1} \) is fully invariant. Now, from the sequence of pairs \( (u_k x_{\ell_k} s_k, v_k x_{\ell_k} s'_k)_{k \in I} \) we can choose a convergent subsequence; denote by \( (\overline{u}, \overline{v}) \) its limit. Since \( \rho_{n+1} \) is a closed relation, we obtain \( \overline{u} \rho_{n+1} \overline{v} \). It follows that \( u = 0(\overline{u}) \rho_n^0 0(\overline{v}) = v \) because multiplication is a continuous operation on \( \overline{\Omega}_{n+1}S \).

We also need to prove that \( \rho^{0C} \) is fully invariant. Assume that we have \( u \rho_n^{0C} v \) and consider a continuous homomorphism \( \varphi : \overline{\Omega}_nS \to \overline{\Omega}_mS \). Again, \( \rho^{0C} \subseteq \sim_{SI} \) gives \( c(u) = c(v) \) and, moreover, there are \( s, s' \in \overline{\Omega}_{n+1}S^1 \) such that

\[
 u x_{n+1}s \rho_{n+1} v x_{n+1}s', \, x_{n+1} \notin c(u) = c(v), \, x_{n+1} = \overline{0}(u x_{n+1}s) = \overline{0}(v x_{n+1}s').
\]

Consider the continuous homomorphism \( \varphi' : \overline{\Omega}_{n+1}S \to \overline{\Omega}_{m+1}S \) defined by the rules \( \varphi(x_{n+1}) = x_{m+1} \) and \( \varphi'(x_i) = \varphi(x_i) \) for all indices \( i \in \{1, \ldots, n\} \). It follows that \( \varphi'(u x_{n+1}s) \rho_{m+1} \varphi'(v x_{n+1}s') \). Since \( 0(\varphi'(u x_{n+1}s)) = \varphi'(u) = \varphi(u) \) and, similarly, we have \( 0(\varphi'(v x_{n+1}s')) = \varphi(v) \), we obtain \( \varphi(u) \rho_m \varphi(v) \). We have proved that \( \rho^{0C} \) is fully invariant.

Finally, assuming that \( u \rho_n v \), we see that \( u \) and \( v \) have the same content and \( u x_{n+1} \rho_{n+1} v x_{n+1} \). Hence, \( u \rho_n^{0C} v \) and we get the inclusion \( \rho_n \subseteq \rho_n^{0C} \). So, we have proved that \( \rho^{0C} \) is a fully invariant system of closed stable quasiorders such that \( \rho \subseteq \rho^{0C} \). Since, for each \( n \), the equivalence relation \( \sim_n \) has finite index, the same is true for \( \sim_n^{0C} \). Consequently, the quotient \( \overline{\Omega}_nS/\rho_n^{0C} \) is finite and \( \rho^{0C} \) is a complete system of pseudoinequalities.

Note that the inclusion \( \rho \subseteq \rho^0 \), which was proved in the previous lemma, only requires the assumption that \( \rho \) is a complete system of pseudoinequalities.

Lemma 3.5 may be applied in the case where \( \rho^0 \subseteq \sim_{SI} \). Since the constructed system of relations \( \rho^0 \) may not be contained in \( \sim_{SI} \), we need to understand what happens in such a case.

**Lemma 3.6.** Let \( \rho \) be a complete system of pseudoinequalities such that \( \sim_{CR} \subseteq \rho \subseteq \sim_{SI} \). If \( \rho^0 \) is not contained in \( \sim_{SI} \), then all relations \( \rho_n^0 \) are symmetric.

**Proof.** Let \( u \rho_n^0 v \) hold for a given arbitrary pair of pseudowords from \( \overline{\Omega}_nS \). We want to show that also \( v \rho_n^0 u \). Assume for the moment, that \( c(u) \neq c(v) \). Since we obtain the pair \((u, v)\) by application of the function \( 0 \) on a certain pair of \( \rho_{n+1}\)-related pseudowords which have the same content, there are indices \( k, \ell \in \{1, \ldots, n\} \) such that

\[
 c(u x_k) = c(v x_\ell), \quad 0(u x_k) = u \quad \text{and} \quad 0(v x_\ell) = v.
\]
Note that, whenever $c(u) = c(v)$, one can take for $x_k$ and $x_\ell$ the letter $x_{n+1}$ and (7) also holds. The pair of pseudowords $ux_k, vx_\ell$ is useful in the following considerations.

We distinguish two cases. First, assume that there is a pair of pseudowords $s$ and $t$ such that $s \rho_m t$, the first letter of $s$ is $y$, the first letter of $t$ is $z$, and that these letters are different elements from $\{x_1, \ldots, x_m\}$. Now, we substitute in $s$ and $t$ the pseudoword $vx_\ell$ for $y$, the pseudoword $ux_k$ for $z$, and $x_k$ for other variables. In this way, we obtain a continuous homomorphism $\varphi : \overline{\mathcal{L}}_m S \to \overline{\mathcal{L}}_{n+1} S$ and we get $\varphi(s) \rho_{n+1} \varphi(t)$. Now, we see that $0(\varphi(s)) = v$ and $0(\varphi(t)) = u$, which gives $v \rho^n u$. Thus, we are done in this case.

In the second case, we assume that all $\rho$-related pseudowords have the same first letter. In particular, we may assume that the first letter of $u$ and $v$ is $x_j$. However, there are $s$ and $t$ such that $s \rho_m t$ and $c(0(s)) \neq c(0(t))$. Then, $\overline{0}(s) = y \neq z = \overline{0}(t)$ hold for some letters $y, z \in \{x_1, \ldots, x_m\}$. We substitute in both $s$ and $t$ the pseudowords $x_j^\omega vx_\ell$ for $z$, $x_j^\omega ux_k$ for $y$ and $x_j^\omega$ for other variables. In this way, we obtain a continuous homomorphism $\varphi : \overline{\mathcal{L}}_m S \to \overline{\mathcal{L}}_{n+1} S$. Then, for the resulting $\rho$-related pseudowords $\varphi(s)$ and $\varphi(t)$, we have $0(\varphi(s)) = x_j^\omega v$ and $0(\varphi(t)) = x_j^\omega u$ by condition (7). Since $\sim_{\text{CR}} \subseteq \rho$, $x_j^\omega+1$ is $\tilde{\rho}$-related to $x_j$. Hence, the initial factor $x_j^\omega$ in both $\varphi(s)$ and $\varphi(t)$ can be removed because both $v$ and $u$ start with $x_j$. In this way, we obtain certain $\rho$-related pseudowords $s'$ and $t'$ such that $0(\varphi(s')) = v$ and $0(\varphi(t')) = u$ which gives $v \rho^n u$. \hfill \Box

We may now prove the claimed symmetry in the locally finite case.

**Proposition 3.7.** Let $\mathcal{H}$ be a locally finite pseudovariety of groups. Let $\rho$ be a complete system of pseudoequalities such that $\sim_{\text{CR}(\mathcal{H})} \subseteq \rho \subseteq \sim_\mathcal{S}$. Then $\rho$ is a complete system of pseudoidentities.

**Proof.** We need to prove that, for every positive integer $n$, the relation $\rho_n$ is symmetric, i.e., we want to prove the implication

$$\forall u, v : u \rho_n v \implies v \rho_n u. \tag{8}$$

This implication is slightly informal as it is not clear how $n$ is quantified. At first, we need to clarify the relationship between the pseudowords $u, v$ and the index $n$. We know that $u \in \overline{\mathcal{L}}_n S$ implies $u \in \overline{\mathcal{L}}_m S$ for every $m > n$. Therefore, for a pair of pseudowords $u \in \overline{\mathcal{L}}_n S$ and $v \in \overline{\mathcal{L}}_m S$, there is a minimum $k$ such that $u, v \in \overline{\mathcal{L}}_k S$, which we denote $n_{u,v}$ for the purpose of this proof. Thus, the index $n$ in the implication (8) is meant as this minimum index. Since $n$ depends on $u$ and $v$, we need to further clarify from which set these pseudowords are taken. So, we denote $\overline{\mathcal{L}}$ the union of all $\overline{\mathcal{L}}_n S$. Now we may write (8) in the following refined form:

$$\forall u, v \in \overline{\mathcal{L}} : u \rho_{n_{u,v}} v \implies v \rho_{n_{u,v}} u.$$

Moreover, we prove this statement for all possible $\rho$ satisfying the assumptions. For that purpose, we denote by $\overline{\mathcal{L}}_\mathcal{H}$ the class of all complete systems of
pseudoinequalities $\rho$ such that $\sim_{\text{CR}(H)} \subseteq \rho \subseteq \sim_{\text{SI}}$. Note that the assumption $\sim_{\text{CR}(H)} \subseteq \rho$ immediately gives that $\rho_n$ has finite index for every $n$. Now, we are ready to improve (8) into the final precise form:

$$\forall u, v \in \mathcal{I}, \quad \forall \rho \in \Gamma_H : u \rho_{n_{u,v}} v \implies v \rho_{n_{u,v}} u. \quad (9)$$

We prove this statement by induction with respect to the size of $c(u)$.

Let us assume that $|c(u)| = 1$ and let $n = n_{u,v}$. Then, from the assumption that $\rho \subseteq \sim_{\text{SI}}$, we obtain the equalities $c(u) = c(v) = \{x_n\}$. Since $\text{CR}(H)$ is locally finite, there exist positive integers $a$ and $b$ such that $u \sim_{\text{CR}(H)} x_n^a$ and $v \sim_{\text{CR}(H)} x_n^b$. As $\sim_{\text{CR}(H)} \subseteq \rho$, we deduce that $x_n^a \rho x_n^b$. Raising to the power $\omega - 1$ and multiplying by $x_n^a$, we obtain $x_n^b \rho x_n^a$, whence $v \rho u$.

Assume next that $|c(u)| = k > 1$ and that the statement (9) holds for every $u$ containing less than $k$ letters. Since $u \rho v$, we get by Lemma 3.4 that

$$0(u) \rho^0 0(v), \quad 1(u) \rho^1 1(v), u \sim_{\text{SI}} v, \quad uv^{\omega-1} \rho (uv^{\omega-1})^\omega$$

and

$$uv^{\omega-1} \rho (vu^{\omega-1})^\omega. \quad (10)$$

We want to use the same lemma which also gives the reverse implication. To prove that $v \rho u$, we need to check that $v$ and $u$ may be interchanged in (10).

To prove the implication $0(u) \rho^0 0(v) \Rightarrow 0(v) \rho^0 0(u)$, we distinguish two cases. If $\rho^0 \subseteq \sim_{\text{SI}}$ then we have $\rho^0 = \rho^0c \in \Gamma_H$ by Lemma 3.5. We may now use the induction assumption, namely that the statement (9) is valid for the pair of pseudowords $0(u), 0(v) \in \mathcal{I}$. If $\rho^0$ is not contained in $\sim_{\text{SI}}$ then $\rho^0$ is symmetric by Lemma 3.6. So, in both cases, we obtain $0(v) \rho^0 0(u)$. The implication $1(u) \rho^1 1(v) \Rightarrow 1(v) \rho^1 1(u)$ follows dually from all appropriate dual versions of the lemmas. Finally, we may conclude that $v \rho u$ by Lemma 3.4. \qed

We are now ready to establish the main result, which is presented as Theorem 1 in the introduction.

**Theorem 3.8.** Let $\mathcal{V}$ be a pseudovariety of ordered completely regular semigroups containing all semilattices. Then $\mathcal{V}$ is selfdual.

**Proof.** Let $(H_i)_{i \geq 1}$ be a sequence of locally finite pseudovarieties of groups such that $H_i \subseteq H_j$ for $i < j$ and $\bigcup_{i \geq 1} H_i = G$. Consider, for every $i \geq 1$, the pseudovariety of ordered completely regular semigroups $V_i = \mathcal{V} \cap \text{CR}(H_i)$. Clearly, we have $\text{SI} \subseteq V_i \subseteq \text{CR}(H_i)$. Next, consider the corresponding complete system of pseudoinequalities $\rho_{V_i}$ for which we have $\sim_{\text{CR}(H_i)} \subseteq \rho_{V_i} \subseteq \sim_{\text{SI}}$. Then, by Proposition 3.7, $\rho_{V_i}$ is a complete system of pseudoidentities. In other words, $V_i$ is selfdual.

Now, we have

$$\bigcup_{i \geq 1} V_i = \bigcup_{i \geq 1} \left( \mathcal{V} \cap \text{CR}(H_i) \right) = \mathcal{V} \cap \bigcup_{i \geq 1} \text{CR}(H_i) = \mathcal{V} \cap \text{CR} = \mathcal{V}.$$

Finally, the statement of the theorem follows from the observation that every directed union of selfdual pseudovarieties is a selfdual pseudovariety. \qed
4. Normal Orthogroups

As defined in the introduction, a normal orthogroup is a completely regular semigroup for which the subsemigroup of idempotents is a normal band. The pseudovariety of all finite normal orthogroups is thus given by

\[ \text{NOCR} = [x^{\omega+1} = x, \ x^{\omega} y^{\omega} = (x^{\omega} y^{\omega})^{\omega}, \ x^{\omega} y^{\omega} x^{\omega} z^{\omega} x^{\omega} = x^{\omega} z^{\omega} x^{\omega} y^{\omega} x^{\omega}] \]

A structural description of normal orthogroups is well known: they are strong normal bands of groups, see, e.g., [22, Sect. IV.2]. Notice that one of the consequences of this description is the equality \( \text{NOCR} = \text{NB} \vee \text{G} \). We show that, more generally, for every pseudovariety \( V \) of ordered normal orthogroups, we have the join decomposition \( V = (V \cap \text{NB}) \vee (V \cap \text{G}) \). Since it turns out that a pseudovariety of ordered completely regular semigroups not containing \( \text{Sl} \) is contained in \( \text{NOCR} \), this reduces the problem of describing all non-selfdual pseudovarieties of ordered normal orthogroups to that of identifying all non-selfdual pseudovarieties of ordered normal bands, which was done by Emery [10].

Some of the following results are well known and so their proofs might be omitted (for example Lemma 4.2), however we keep most details to make the paper self-contained. In comparison with [22], we want to modify the results in three directions. Firstly, we consider ordered semigroups and secondly we work within the theory of pseudovarieties of finite semigroups. The third direction, which is only implicitly contained in [22, Theorem IV.2.7], is the relativization with respect to the fixed pseudovariety of groups.

Throughout this section, \( V \) denotes a pseudovariety of ordered semigroups. Before we introduce the results concerning ordered normal orthogroups, we explain the importance of the pseudovariety \( \text{NOCR} \) for our discussions.

**Lemma 4.1.** Let \( V \subseteq \text{CR} \) be such that \( \text{Sl} \nsubseteq V \). Then \( V \models x^{\omega} y^{\omega} x^{\omega} \leq x^{\omega} \) or \( V \models x^{\omega} \leq x^{\omega} y^{\omega} x^{\omega} \). In both cases \( V \subseteq \text{NOCR} \).

**Proof.** We assume that \( \rho_V \nsubseteq \sim_{\text{Sl}} \). This means that there is \( (u, v) \in \rho_V \) such that there is a variable \( y^{\omega} \) that occurs in just one of the pseudowords \( u \) and \( v \).

If we substitute \( y^{\omega} \) for the variable \( y \) and \( x^{\omega} \) for all other variables and if we multiply the resulting pseudoinequality by \( x^{\omega} \) from both sides, we obtain either \( x^{\omega} y^{\omega} x^{\omega} \rho_V x^{\omega} \) or \( x^{\omega} \rho_V x^{\omega} y^{\omega} x^{\omega} \). This gives the first part of the statement.

Now assume that \( x^{\omega} y^{\omega} x^{\omega} \rho_V x^{\omega} \) as the second case is dual. Multiplying by \( y^{\omega} \) yields the relation \( (x^{\omega} y^{\omega})^2 \rho_V x^{\omega} y^{\omega} \). Multiplying by \( x^{\omega} y^{\omega} \) and using transitivity, we deduce that \( (x^{\omega} y^{\omega})^3 \rho_V x^{\omega} y^{\omega} \). Inductively, it follows that \( (x^{\omega} y^{\omega})^n \rho_V x^{\omega} y^{\omega} \) for every \( n \geq 1 \). The left side of this relation converges to \( (x^{\omega} y^{\omega})^{\omega} \). Since \( \rho_V \) is closed, we deduce that \( (x^{\omega} y^{\omega})^{\omega} \rho_V x^{\omega} y^{\omega} \). As \( \rho_V \) is a stable quasiorder and \( \sim_{\text{CR}} \subseteq \rho_V \), we obtain \( V \models (x^{\omega} y^{\omega})^{\omega} = x^{\omega} y^{\omega} \). Since every product of idempotents is idempotent modulo \( V \), we get

\[
    x^{\omega} y^{\omega} x^{\omega} z^{\omega} x^{\omega} \rho_V x^{\omega} y^{\omega} x^{\omega} z^{\omega} x^{\omega} \cdot x^{\omega} y^{\omega} x^{\omega} z^{\omega} x^{\omega} = (11)
\]
Lemma 4.3. Let \( x^\omega y^\omega x^\omega \rho_N x^\omega \) and \( x^\omega z^\omega x^\omega \rho_N x^\omega \) in the prefix and the suffix of the right hand side of (11) we get \( y^\omega z^\omega y^\omega z^\omega \). Exchanging \( y \) and \( z \), we obtain \( x^\omega y^\omega x^\omega z^\omega x^\omega \rho_N x^\omega y^\omega x^\omega z^\omega x^\omega \) which implies \( V \subseteq \text{NOCR} \). □

We recall some basic facts concerning normal orthogroups. We start by recalling the solution of the word problem for normal bands which is well known: for the pseudowords \( u, v \), we have \( \text{NB} \models u = v \) if and only if \( c(u) = c(v) \), the first letter in \( u \) is the same as the first letter in \( v \) and the last letter in \( u \) is the same as the last letter in \( v \). This property implies, for example, that \( \text{NOCR} \models x^\omega y^\omega x^\omega z^\omega x^\omega = x^\omega y^\omega z^\omega x^\omega \). We use such pseudoidentities freely. Other useful pseudoidentities are mentioned in the following lemma.

Lemma 4.2. The pseudovariety \( \text{NOCR} \) satisfies the pseudoidentities \( (xy)^\omega = x^\omega y^\omega \) and \( x^\omega y^\omega zx^\omega = x^\omega yzx^\omega \).

Proof. Clearly, the pseudowords \( x^\omega y^\omega \) and \( xy \) are \( \mathcal{J} \)-related modulo \( \sim_{\text{NOCR}} \). Since we can interchange the placing of idempotents inside a product of idempotents we have \( (xy)^\omega x^\omega y^\omega \sim_{\text{NOCR}} x^\omega (xy)^\omega x^\omega y^\omega \sim_{\text{NOCR}} x^\omega x^\omega (xy)^\omega y^\omega = (xy)^\omega \). In the same way we get \( x^\omega y^\omega (xy)^\omega \sim_{\text{NOCR}} (xy)^\omega \). Thus, modulo \( \sim_{\text{NOCR}} \), we have \( (xy)^\omega \leq_{\mathcal{J}} x^\omega y^\omega \) and also \( (xy)^\omega \leq_{\mathcal{J}} x^\omega y^\omega \). By stability of profinite semigroups, we deduce that \( (xy)^\omega \) and \( x^\omega y^\omega \) are \( \mathcal{H} \)-related modulo \( \sim_{\text{NOCR}} \). Since both are idempotents, they are equal modulo \( \sim_{\text{NOCR}} \).

We may derive the following sequence of pseudoidentities which are valid in \( \text{NOCR} \):

\[
x^\omega y^\omega x^\omega z^\omega x^\omega = (x^\omega y)^{\omega+1} x^\omega (zx^\omega)^{\omega+1} = x^\omega y \cdot x^\omega \cdot (x^\omega y)^{\omega} x^\omega (zx^\omega)^{\omega} \cdot x^\omega \cdot z^\omega x^\omega = x^\omega y \cdot x^\omega \cdot (x^\omega y)^{\omega} (zx^\omega)^{\omega} x^\omega \cdot z^\omega x^\omega = (x^\omega y)^{\omega+1} (zx^\omega)^{\omega+1} = x^\omega yzx^\omega.
\]

□

The following lemma is a natural variation of the important statement that, in every normal orthogroup, the \( \mathcal{H} \)-relation is a congruence.

Lemma 4.3. Let \( S \) be an ordered normal orthogroup. Then the mapping \( \varphi : S \rightarrow E(S) \) given by the rule \( \varphi(a) = a^\omega \) is a homomorphism of ordered semigroups. Moreover, if \( \beta : S \rightarrow T \) is a surjective homomorphism such that \( T \) is an ordered band, then \( \beta \) can be factorized through \( \varphi \).

Proof. The property \( \varphi(st) = \varphi(s) \cdot \varphi(t) \), for every pair \( s, t \) of elements of \( S \), follows from Lemma 4.2. If \( s \leq t \) then \( s^\omega \leq t^\omega \) follows. Hence, \( \varphi \) is also an isotone mapping.

Now, for \( s \in S \) we have \( \beta(s) = \beta(s^2) \). The second part follows from the fact that \( \beta \circ \varphi = \beta \), because \( \beta(\varphi(s)) = \beta(s^2) = \beta(s) \). □
Before we exhibit another canonical surjective homomorphism from an ordered normal orthogroup, we introduce some important examples of ordered normal orthogroups. The first example of a normal orthogroup is a 0-group, that is, a group $G$ enriched by a zero element $0$. We denote this semigroup $G^0$. Since we are interested in ordered semigroups, we point out that the partial order that we consider on $G^0$ is the equality. However, there are other possible stable partial orders on this semigroup, which we denote in a different way. The ordered semigroup $G^\top$ is the set $G \cup \{\top\}$ together with the operation $\cdot$ which is an extension of the multiplication on $G$ such that the element $\top$ is a zero element which is the maximum element with respect to the partial order $\leq$. Notice that all the other elements are incomparable, because they are members of the group. One can see that $G^\top$ is a normal orthogroup satisfying the pseudoinequality $x^\omega \leq x^\omega y^\omega x^\omega$. We denote the dual ordered semigroup of $G^\top$ as $G^\bot$. This means that the special element $\bot$ in $G^\bot$ is a zero with respect to the multiplication and it is the minimum element with respect to the partial order. Thus, $G^\bot$ satisfies the pseudoinequality $x^\omega y^\omega x^\omega \leq x^\omega$.

If we take $G$ to be a trivial group consisting of the idempotent element $1$, then we obtain in the previous construction an ordered semilattice $\{1\}^\top = U^+ = \{1, \top\}$ which satisfies the identity $x \leq xy$. One can show that $S\uparrow^+ = [x^2 = x, xy = yx, x \leq xy] = [x^2 = x, x \leq yxy]$ is the pseudovariety of ordered semigroups generated by $U^+$: indeed, in every non-trivial ordered semilattice $S$ satisfying $x \leq xy$ we can choose two distinct elements $a, b$ such that $a \neq ab$ and consider the subsemigroup $\{a, ab\}$ which is isomorphic to $U^+$. By the description, contained in Proposition 4.4 below, of the lattice of all pseudovarieties of ordered normal bands by Emery [10], we also see that $S\uparrow^+$ is a minimal pseudovariety of ordered semigroups. The other minimal pseudovarieties in the lattice are the dual pseudovariety $S\uparrow^-$, which is generated by the dual ordered semigroup $U^- = \{1, \bot\}$, and the well known pseudovarieties of left zero semigroups $LZ = [xy = x]$ and right zero semigroups $RZ = [xy = y]$. The pseudovariety $LZ$ is generated by a two element semigroup $L = \{a, b\}$ with the multiplication given by the rules $aa = ab = a$, $ba = bb = b$. Although one may order this semigroup by $a < b$ to obtain an ordered semigroup $L^<$, this homomorphic image of $L$ generates the same pseudovariety, because $L$ is isomorphic to the subsemigroup of the product $L^< \times L^<$ consisting of two incomparable elements $(a, b)$ and $(b, a)$. Finally, from left-right duality we get that $RZ$ is also generated by a single two-element right zero semigroup $R$.

Now, we are ready to recall the description of the lattice of all pseudovarieties of ordered normal bands [10], in terms of inequalities and also of minimal generators, which explains that the lattice is isomorphic to the 4th power of a two-element lattice.

**Proposition 4.4 ([10]).** The lattice of all pseudovarieties of ordered normal bands is presented on Fig. 1. The characterization of every pseudovariety by an inequality is given inside $B$, that is, the identity $x^2 = x$ is satisfied too.
Corollary 4.5. The lattice of all pseudovarieties of ordered normal bands is isomorphic to the set of all subsets of the set \{U^+, U^-, L, R\} partially ordered by the inclusion relation.

Proof. A formal proof needs to check which inequalities are satisfied in the semigroups \(U^+, U^-, L\) and \(R\). We omit such technical computations and show just one example. The inequality \(x \leq xy\) is satisfied by \(U^+\) and \(L\) and it is not satisfied by \(U^-\) and \(R\). \(\square\)

Now, we return to our aim, namely to the description of all pseudovarieties of ordered normal orthogroups by means of a join decomposition. We start with one particular case.

Lemma 4.6. Let \(V \subseteq \text{NOCR}\) be such that \(V \models x^\omega = x^\omega y^\omega x^\omega\). Then, we have \(V = (V \cap \text{NB}) \lor (V \cap G)\).

Proof. Let \(S \in \text{NOCR}\) be such that \(S \models x^\omega = x^\omega y^\omega x^\omega\). This means that \(E(S)\) is a rectangular band. In particular, \(S\) is a completely simple semigroup. We choose one idempotent \(e \in E(S)\) and consider \(H_e\), the \(\mathcal{H}\)-class of \(e\). Now, we consider the mapping \(\psi : S \rightarrow H_e\) given by the rule \(\psi(s) = ese\). Clearly, the mapping \(\psi\) is isotone. Moreover, for every \(s, t \in S\) we have \(\psi(s) \cdot \psi(t) = ese \cdot ete = este = \psi(st)\) by Lemma 4.2. Thus \(\psi\) is a homomorphism of ordered semigroups.

Next, we consider the mapping \(\alpha : S \rightarrow E(S) \times H_e\) given by the rule \(\alpha(s) = (\varphi(s), \psi(s))\), where \(\varphi : S \rightarrow E(S)\) is described in Lemma 4.3. Since both \(\varphi\) and \(\psi\) are homomorphisms, \(\alpha\) is a homomorphism as well. We claim that \(\alpha\) is injective. Let \(s\) and \(t\) be such that \(\alpha(s) = \alpha(t)\). Then \(s^\omega = \varphi(s) = \varphi(t) = t^\omega\) gives that \(s\) and \(t\) belong to the same \(\mathcal{H}\)-class. By Green’s Lemmas
the mapping \( s \mapsto ese \) is an injective mapping from \( H_{s^\omega} \) onto \( H_e \). Hence, \( s = t \) and the claim is proved.

Since both mappings \( \varphi \) and \( \psi \) are surjective, it follows that \( E(S) \in V \cap NB \) and \( H_e \in V \cap G \). Thus, we obtain \( S \in (V \cap NB) \cup (V \cap G) \), which completes the proof. \( \square \)

Now, we extend the conclusion of the previous lemma to more pseudovarieties.

**Lemma 4.7.** Let \( V \subseteq NOCR \) be such that \( V \models x^\omega \leq x^\omega y^\omega x^\omega \). Then \( V = (V \cap NB) \cup (V \cap G) \).

**Proof.** If \( V \models x^\omega = x^\omega y^\omega x^\omega \) then we can apply Lemma 4.6. So, assume that there is a semigroup \( T \) in \( V \) which does not satisfy the pseudoidentity \( x^\omega = x^\omega y^\omega x^\omega \). Thus, there are \( e, f \in E(T) \) such that \( e \neq efe, e \leq efe \). Hence, \( U^+ \) is isomorphic to the ordered subsemigroup \( \{ e, efe \} \) of \( T \), and therefore \( U^+ \) belongs to \( V \).

Now, let \( S \) be an arbitrary ordered semigroup from \( V \). Let \( e \) be an arbitrary idempotent in \( S \) and \( H_e \) be the subgroup of \( S \) formed by the \( \mathcal{H} \)-class of \( e \). Recall that \( V \subseteq NOCR \) implies \( (ese)^\omega = es^\omega e \), for every \( s \in S \). So, assuming \( es^\omega e = e \), we obtain \( ese \in H_e \). Hence we can define the mapping \( \psi_e : S \rightarrow H_e^\top \) given by the rule

\[
\psi_e(x) = \begin{cases} 
eq e \rightarrow e, & \text{if } e x^\omega e = e \\
\top, & \text{otherwise.}
\end{cases}
\]

We claim that \( \psi_e \) is a homomorphism from the ordered semigroup \( S \) to the ordered normal orthogroup \( H_e^\top \).

Let \( s, t \in S \) be arbitrary elements. Since \( es^\omega e \cdot et^\omega e = es^\omega t^\omega e = e(s + t)^\omega e \), we deduce that \( e(s + t)^\omega e = e \) if and only if both equalities \( es^\omega e = e \) and \( et^\omega e = e \) are valid. Therefore, the disjunction \( \psi_e(s) = \top \) or \( \psi_e(t) = \top \) is equivalent to \( \psi_e(st) = \top \). And, furthermore, if \( \psi_e(st) \neq \top \), i.e., \( \psi_e(st) = ese \), then \( \psi_e(s) = ese \) and \( \psi_e(t) = ete \). In this case, we get \( \psi_e(s) \cdot \psi_e(t) = ese \cdot ete = este \) by Lemma 4.2. Thus, in every case \( \psi_e(st) = \psi_e(s) \cdot \psi_e(t) \).

Now, assume that \( s, t \in S \) are such that \( s \leq t \). We need to show that \( \psi_e(s) \leq \psi_e(t) \), which is trivially satisfied if \( \psi_e(t) = \top \). So, assume that \( et^\omega e = e \) and \( \psi_e(t) = ete \). From \( s \leq t \) we get \( ese \leq ete \). Thus, \( es^\omega e = (ese)^\omega \leq (ete)^\omega = et^\omega e = e \). Since \( S \) satisfies the pseudoidentity \( x^\omega \leq x^\omega y^\omega x^\omega \), we also get \( e \leq es^\omega e \). Hence, \( es^\omega e = e \) and we obtain \( \psi_e(s) = ese \). Now, the inequality \( \psi_e(s) \leq \psi_e(t) \) follows from \( s \leq t \). We proved the claim.

Note that \( \psi_e \) is surjective if and only if \( e \) does not belong to the minimal \( \mathcal{J} \)-class of \( S \). If \( e \) belongs to the minimal \( \mathcal{J} \)-class, then \( \psi_e(S) = H_e \). These observations are not used in what follows, because we use another argument which ensures that \( H_e^\top \in V \). We just notice that \( H_e^\top \) may be seen as a homomorphic image of the product \( H_e \times U^+ \) using the homomorphism \( \beta : H_e \times U^+ \rightarrow H_e^\top \) mapping \( (g, 1) \mapsto g \), \( (g, \top) \mapsto \top \) for each \( g \in H_e \). Since we have \( U^+ \in V \cap NB \) and \( H_e \in V \cap G \), we see that \( H_e^\top \in (V \cap NB) \cup (V \cap G) \).
In the rest of the proof, we show that $S$ may be reconstructed from these ordered normal orthogroups and from the ordered normal band $E(S) \in V \cap NB$. To prove this, we use also the homomorphism $\varphi : S \to E(S)$ described in Lemma 4.3.

We consider the following homomorphism 
\[ \alpha : S \to E(S) \times \prod_{e \in E(S)} H^\top_e, \]
where, for every $s \in S$, we put $\alpha(s) = (\varphi(s), (\psi_e(s))_{e \in E(S)})$. Since $\varphi$ and all $\psi_e$'s are homomorphisms of ordered semigroups, the mapping $\alpha$ is also a homomorphism. We show that $\alpha$ is injective. Let $s, t \in S$ be such that $\alpha(s) = \alpha(t)$. Then, $\varphi(s) = s^\omega = t^\omega = \varphi(t)$ is an idempotent from $E(S)$ which we denote by $f$. Then, we have also $\psi_f(s) = \psi_f(t)$, from which we deduce that $fs = ft$. Since $f = s^\omega = t^\omega$ we get $s = s^{\omega + 1} = fs = ft = t^{\omega + 1} = t$ and we proved that $\alpha$ is injective.

Since $H^\top_e \in (V \cap NB) \vee (V \cap G)$, for every $e \in E(S)$, we get $S \in (V \cap NB) \vee (V \cap G)$ as well. As $S$ was an arbitrary member of $V$ we have thus proved that $V \subseteq (V \cap NB) \vee (V \cap G)$. \hfill \Box

Let $RB = [xyx = x, x^2 = x]$ be the pseudovariety of all finite rectangular bands. We are now ready for establishing a more precise version of Theorem 2.

**Theorem 4.8.** The mapping $\iota : L_o(NOCR) \to L_o(Sl) \times L(RB) \times L(G)$ given by the rule $\iota(V) = (V \cap Sl, V \cap RB, V \cap G)$ is a lattice isomorphism.

**Proof.** From the structure of the lattice $L_o(NB)$, we know that the correspondence $V \mapsto (V \cap Sl, V \cap RB)$ defines a lattice isomorphism $L_o(NB) \to L_o(Sl) \times L(RB)$. We consider the mapping $\gamma : L_o(NB) \times L(G) \to L_o(NOCR)$ given by the rule $\gamma(V, H) = V \vee H$. Clearly, both mappings $\iota$ and $\gamma$ are isotone mappings between the ordered sets $L_o(NOCR)$ and $L_o(NB) \times L(G)$. We show that they are mutually inverse mappings. This implies that they are isomorphisms of ordered sets and therefore also lattice isomorphisms.

To show that $\gamma \circ \iota$ is the identity mapping, we need to prove that $(V \cap NB) \lor (V \cap G) = V$ for every $V \subseteq NOCR$. This is true if $V \models x^\omega \leq x^\omega y^\omega x^\omega$, by Lemma 4.7. Clearly, one may use the dual version of the lemma if $V \models x^\omega y^\omega x^\omega \leq x^\omega$. So, we may assume that $V$ does not satisfy any of these two pseudoinequalities. Thus, by Lemma 4.1 we know that $Sl \subseteq V$. From Theorem 3.8 we get that $V$ is selfdual. This means that if we consider an arbitrary ordered semigroup $(S, \cdot, \leq) \in V$, then the pseudovariety $V$ contains also the ordered semigroup $(S, \cdot, =)$. Since $(S, \cdot, =)$ is a homomorphic image of $(S, \cdot, =)$ we may deal only with the case of unordered semigroups.

So, let $(S, \cdot, =) \in V$ be arbitrary. Now, it is possible to modify the proof of Lemma 4.7 in such a way that $H^\top_e$ is replaced by $H^0_e$. Since the partial order on $S$ is equality, the mapping $\psi_e$ is trivially isotone. Moreover, $V$ contains $Sl$ and, therefore, $H^0_e \in (V \cap NB) \lor (V \cap G)$. Thus, we conclude that $S \in (V \cap NB) \lor (V \cap G)$. We have proved the required equality $(V \cap NB) \lor (V \cap G) = V$. 

\( \Box \)
To show that $\iota \circ \gamma$ is the identity mapping, we need to show that $(V \lor H) \cap NB = V$ and $(V \lor H) \cap G = H$ for every pair $V \subseteq NB$ and $H \subseteq G$.

For the first equality, one direction, namely $(V \lor H) \cap NB \supseteq V$, is trivial. Let $S$ be an arbitrary member of $(V \lor H) \cap NB$. This means that there are $S_1 \in V$, $S_2 \in H$ and there is an ordered subsemigroup $T$ of $S_1 \times S_2$ such that there is a surjective homomorphism of ordered semigroups $\beta : T \to S$. Since $S_1 \times S_2 \in V \lor H \subseteq NB \lor G \subseteq NOCR$, by Lemma 4.3 we know that the restriction of $\beta$ to $E(T)$ is also a surjective homomorphism of ordered semigroups. Hence, we may assume that $T$ is an ordered band. Since $S_2$ is a group, we see that $T$ is isomorphic to a subsemigroup of $S_1$. Hence, we have $T \in V$ and, consequently, $S \in V$ and the equality $(V \lor H) \cap NB = V$ is proved.

From the well known fact that, for a surjective homomorphism $\beta : T \to G$, where $T$ is a finite semigroup and $G$ is a group, there is a subgroup $H$ of $T$ such that $\beta(H) = G$, it follows that the mapping $V \mapsto V \cap G$ is a complete homomorphism $L_o(S) \to L(G)$. This was observed in [7, Theorem 3.1] for the unordered case, but the easy proof is the same for the ordered case. In particular, we have $(V \lor H) \cap G = H$ and the proof is complete. \hfill $\blacksquare$

5. Applications

We gather in this section several applications of our results.

Selfduality of pseudovarieties of ordered completely regular semigroups may now be characterized by very simple conditions.

Theorem 5.1. Let $V$ be a pseudovariety of ordered completely regular semigroups. Then the following conditions are equivalent:

(i) the pseudovariety $V$ is selfdual;
(ii) the intersection $V \cap NB$ is selfdual;
(iii) the intersection $V \cap Sl$ is selfdual;
(iv) either both or none of the ordered semilattices $U^+$ and $U^-$ belong to $V$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from the fact that the intersection of selfdual pseudovarieties is also selfdual. On the other hand, (iii) $\Rightarrow$ (iv) is an immediate consequence of Corollary 4.5.

To prove that (iv) $\Rightarrow$ (i), note first that, if both $U^+$ and $U^-$ belong to $V$, then $V$ contains $Sl$; hence, $V$ is selfdual by Theorem 3.8. On the other hand, if neither $U^+$ nor $U^-$ belong to $V$, then $V$ is contained in $NOCR$ by Lemma 4.1 and $V = (V \cap Sl) \lor (V \cap RB) \lor (V \cap G)$ by Theorem 4.8; by Corollary 4.5, we conclude that $V \cap Sl$ is the trivial pseudovariety and, therefore, $V$ is selfdual. \hfill $\blacksquare$

Given a pseudovariety of ordered semigroups $V$, note that $V \lor V^d$ is the least selfdual pseudovariety containing $V$. We call it the selfdual closure of $V$ and denote it $\overline{V}$. 

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Proposition 5.2. The selfdual closure of a pseudovariety $V$ of ordered completely regular semigroups is the join $V \vee V'$ where $V'$ is the selfdual closure of $V \cap Sl$.

Proof. If $V$ contains $Sl$, then $V \vee V' = V$ is selfdual by Theorem 3.8. Otherwise, by Theorem 4.8 we have a decomposition $V = (V \cap Sl) \vee (V \cap RB) \vee (V \cap G)$, where the only term that may not be selfdual is $V \cap Sl$. Since the selfdual closure of $V \cap Sl$ is precisely $V'$, the result is now immediate. \hfill \Box

Following the terminology of [4], we say that a pseudovariety of semigroups is order primitive if it is not generated by a proper subpseudovariety of ordered semigroups; equivalently, a pseudovariety is order primitive if it is not the selfdual closure of a non-selfdual pseudovariety of order semigroups. We also recall the Krohn-Rhodes complexity pseudovarieties [16]. Denote by $A$ the pseudovariety consisting of all finite aperiodic semigroups, that is, finite semigroups all of whose subgroups are trivial. Note that $CR \cap A = B$.

Given pseudovarieties of semigroups $V$ and $W$, $V \ast W$ denotes the pseudovariety of semigroups generated by all semidirect products $S \ast T$ with $S \in V$ and $T \in W$. It is well known that this defines an associative operation on $\mathcal{L}(S)$. The complexity pseudovarieties $C_n$ are defined recursively by $C_0 = A$ and $C_{n+1} = C_n \ast G \ast A$. From the Krohn-Rhodes decomposition theorem it follows that the ascending chain $C_n$ covers $S$. For much more on the Krohn-Rhodes complexity, see [30].

Theorem 5.3. Let $V \in \mathcal{L}(CR)$. Then $V$ is not order primitive if and only if $V \subseteq NOCR$ and $V \cap B$ lies in the interval $[Sl, NB]$.

Proof. Suppose first that $V = \tilde{W}$ with $V \neq W$. In particular, $W$ is not selfdual and so, by Theorem 3.8 and Lemma 4.1, we must have $V \subseteq NOCR$, so that $V \cap B \subseteq NB$. If $Sl$ is not contained in $V$, then neither $U^+$ nor $U^-$ can belong to $W$; by Theorem 5.1, it then follows that $W$ must be selfdual, which contradicts our assumptions. Hence, the inclusion $Sl \subseteq V$ holds.

For the converse, consider the pseudovariety $U = V \cap B$. By inspection of Fig. 1, the assumption that $U$ belongs to the interval $[Sl, NB]$ implies that $U = \tilde{X}$, where $X = (U \cap RB) \vee Sl^+$. From Theorem 4.8, it follows that $V = \tilde{W}$, where $W = X \vee (V \cap G)$. Moreover, $W$ is not selfdual by Theorems 4.8 and 5.1. Hence, $V$ is not order primitive. \hfill \Box

The following result settles one of the problems left open in [4, Table 3].

Corollary 5.4. For every $n \geq 0$, the pseudovariety $CR \cap C_n$ is order primitive.

Proof. It suffices to note that $C_n$ contains $B$, so that $CR \cap C_n$ fails both criteria of Theorem 5.3. \hfill \Box

The following extends to the ordered case a result of Pastijn [20].
Theorem 5.5. The lattice $L_o(CR)$ is modular.

Proof. We must show that the pentagon

$$\begin{array}{c}
Y \\
V \\
U \\
W \\
X
\end{array}$$

does not appear as a sublattice of $L_o(CR)$. Suppose, on the contrary, that it does appear. The idea of the proof is to show that it is possible to render selfdual all vertices of the pentagon still retaining a pentagon, which leads to a contradiction since the lattice $L(CR)$ is modular [20, Corollary 8].

We first observe that $Y$ cannot belong to $L_o(NOCR)$ since this lattice is modular as it is isomorphic to a product of modular lattices by Theorem 4.8. Note also that either at least one of $U$ and $V$ is not selfdual or $W$ is not selfdual, but not both conditions can hold: otherwise, either all vertices of the pentagon are selfdual or they all belong to $L_o(NOCR)$. Moreover, the (left: $U$, $V$; or right: $W$) side of the pentagon where non-selfduality is not present must contain $SI$ while, on the other side, none of the vertices contains $SI$: first, if on both sides we would have pseudovarieties not containing $SI$, then the pentagon would be found within $L_o(NOCR)$ by Lemma 4.1; second, if $SI$ is contained in at least one pseudovariety on each side, then it is contained in $X$ and the pentagon would be placed in $L(CR)$ by Theorem 3.8. The same argument shows that, $U$ is selfdual if and only if so is $V$. In any case, $X$ cannot be selfdual.

Let $RG = RB \lor G$, the pseudovariety of all finite so-called rectangular groups. We replace in the pentagon each non-selfdual vertex by its intersection with $RG$. We show that this produces a sublattice of $L(CR)$ which is still a pentagon.

Consider first the case where $U$, $V$, and $X$ are not selfdual. Then, these three pseudovarieties are replaced by their intersections with $RG$. Doing so, the new bottom pseudovariety is still the only intersection of the two new sides. On the other hand, as $W$ contains $SI$, each of $U$ and $V$ is contained in the join with $W$ of its intersection with $RG$; hence, $Y$ remains the only join of the new two sides. Thus, we obtain a pentagon, except if there is some side which collapses. The only possible collapse would come from the identification of two vertices being intersected with $RG$. But, since $U$, $V$, and $X$ are not selfdual and form a chain, their intersections with $SI$ must be the same. Hence, by Theorem 4.8, their intersections with $RG$ must remain distinct.

Finally, consider the case where $W$ is not selfdual. Then, both $W$ and $X$ are replaced by their intersections with $RG$. The argument in the preceding paragraph then allows us to show that the modified pentagon is still a pentagon. Thus, in all cases, we reach the announced contradiction, thereby proving the theorem. □
We next recall the following result of Reilly and Zhang.

**Theorem 5.6** ([28]). *The correspondence* \( V \mapsto V \cap B \) *defines a complete endomorphism of the lattice* \( \mathcal{L}(S) \).

As a step in the proof that intersection with \( B \) remains a complete endomorphism of the lattice \( \mathcal{L}_o(S) \), we first show that that is the case for the restriction to \( \mathcal{L}_o(CR) \).

Since \( NOCR = NB \lor G \) and taking the intersection with \( B \) determines a complete endomorphism of \( \mathcal{L}(S) \), it follows that \( V \cap B = V \cap NB \) for every \( V \in \mathcal{L}_o(NOCR) \). We use this property freely for the rest of the paper.

**Lemma 5.7.** *Let* \( V \) *be a pseudovariety of ordered completely regular semigroups. Then, we have* \( \widetilde{V} \cap B = \widetilde{V} \cap NB \).

**Proof.** If \( V \) is selfdual, then so is \( V \cap B \) and this is the value of both sides of the equality in the statement of the lemma. If \( V \) is not selfdual then \( \tilde{V} \) is contained in \( NOCR \). By Proposition 5.2, since \( V \) is not selfdual, we have \( \tilde{V} = V \lor SI \). We thus obtain the following chain of equalities:

\[
\widetilde{V} \cap B = \tilde{V} \cap NB = (V \cap NB) \lor SI = (V \lor SI) \cap NB = \tilde{V} \cap NB = \tilde{V} \cap B.
\]

Next, we use the method of Reilly and Zhang [28] to produce complete endomorphisms of the lattice \( \mathcal{L}_o(CR) \). Following the terminology in that paper, we denote by \( DCh \) the class of finite completely regular semigroups \( S \) such that the partially ordered set \( S/\mathcal{J} \) is a chain. The next two lemmas are borrowed from [28]. The first is the core of the argument.

**Lemma 5.9** ([28, Lemma 3.10]). *Let* \( S \in DCh \), \( T \in S \), *and* \( \varphi : T \to S \) *be a surjective homomorphism. Then there exists a subsemigroup* \( R \) *of* \( T \) *such that* \( R \in DCh \) *and the restriction* \( \varphi|_R \) *is surjective.*
For a class $\mathcal{C}$ of finite ordered semigroups, denote by $\langle \mathcal{C} \rangle_o$ the pseudovariety of ordered semigroups generated by $\mathcal{C}$.

**Lemma 5.10** (see [28, Lemma 3.11]). Let $U, V \in \mathcal{L}_o(S)$. If $S \in \text{DCh}$ is such that $S \in U \lor V$, then $S \in \langle U \cap \text{DCh} \rangle_o \lor \langle V \cap \text{DCh} \rangle_o$.

**Proof.** The proof of [28, Lemma 3.11] carries through unchanged to the ordered case; it is a simple application of Lemma 5.9. □

We may now state the ordered version of the main result of [28, Theorem 3.12].

**Theorem 5.11.** Let $W \in \mathcal{L}_o(\text{CR})$ have the following properties:

(i) the correspondence $V \mapsto V \cap W$ defines an endomorphism of the lattice $\mathcal{L}_o(\text{CR})$;

(ii) if $V \in \mathcal{L}_o(W)$ then we have $V = \langle V \cap \text{DCh} \rangle_o$.

Then the correspondence $V \mapsto V \cap W$ defines a complete endomorphism of the lattice $\mathcal{L}_o(S)$.

**Proof.** Taking into account Lemma 5.10, the proof of [28, Theorem 3.12] carries over unchanged to the ordered case to show that the correspondence $V \mapsto V \cap W$ defines an endomorphism of the lattice $\mathcal{L}_o(S)$. That it is actually a complete endomorphism follows from the order analog of [28, Lemma 3.2], whose proof requires no essential changes. □

The following in an immediate application of Theorems 5.8 and 5.11.

**Corollary 5.12.** The correspondence $V \mapsto V \cap B$ defines a complete endomorphism of the lattice $\mathcal{L}_o(S)$.

**Proof.** It only remains to verify the hypothesis (ii) of Theorem 5.11, namely that $V = \langle V \cap \text{DCh} \rangle_o$ for every $V \in \mathcal{L}_o(B)$. This is well known in case $V$ is selfdual. For each of the eight non-selfdual $V \in \mathcal{L}_o(B)$, which appear in Fig. 1, we gave in Sect. 4 generating sets consisting of elements of DCh, namely a suitable subset of the set $\{U^+, U^-, L, R\}$.

Adopting the terminology of [30, Definition 6.1.5], we say that an element $a$ of a lattice is strictly finite join irreducible (sfji) if $a = b \lor c$ implies $a = b$ or $a = c$; and we say that $a$ is finite join irreducible (fji) if $a \leq b \lor c$ implies $a \leq b$ or $a \leq c$.

The following result solves a problem left open in [4, Table 3].

**Corollary 5.13.** The pseudovariety $B$ is fji in the lattice $\mathcal{L}_o(S)$.

**Proof.** It is easy to see that, if $V$ is sfji and intersection with $V$ distributes over finite joins, then $V$ is fji. On the other hand, it follows from Kuřil’s characterization of $\mathcal{L}_o(B)$ [18] that $B$ is sfji in $\mathcal{L}_o(S)$. □
6. Final Remarks

Although we have dealt in this paper mostly with (pro)finite semigroups, Theorems 1, 2, and 3 extend with somewhat simpler proofs to the case of varieties of ordered completely regular semigroups. In particular, there is no need to take care of reducing to the locally finite case in the proof of the variety analog of Theorem 1.

It would be of interest to extend our results to the non-regular analog of finite completely regular semigroups, namely the finite semigroups in which every regular element lies in a group. They form a pseudovariety which is commonly denoted DS in the literature and that was first considered by Schützenberger [31] in connection with formal language theory. By Green’s Lemmas, DS may alternatively be characterized as consisting of all finite semigroups in which regular \( J \)-classes are subsemigroups. The class DO of all finite semigroups whose regular \( J \)-classes form orthodox subsemigroups may be easier to handle as it is much better understood [2]. Because of connections with logic and formal language theory [17, 32], it is of particular interest to consider the pseudovariety DA, which consists of all finite semigroups whose regular elements are idempotents. It is shown in [4, Table 3] that \( DS \cap H \) is fji in \( L_o(S) \), where \( H \) is an arbitrary nontrivial pseudovariety of groups; in contrast, it is not known whether the same holds for \( DO \cap H \) and for DA. The lack of torsion makes the approach of [4] fail for subpseudovarieties of DO but may facilitate the extension of the methods in the present paper.

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