STEIN’S METHOD FOR TEMPERED STABLE DISTRIBUTIONS

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Abstract. In this article, we develop Stein characterization for two-sided tempered stable distribution. Stein characterizations for normal, gamma, Laplace, and variance-gamma distributions already known in the literature follow easily. One can also derive Stein characterizations for more difficult distributions such as the distribution of product of two normal random variables, a difference between two gamma random variables. Using the semigroup approach, we obtain estimates of the solution to Stein equation. Finally, we apply these estimates to obtain error bounds in the Wasserstein-type distance for tempered stable approximation in three well-known problems: comparison between two tempered stable distributions, Laplace approximation of random geometric sums, and six moment theorem for the symmetric variance-gamma approximation of functionals of double Wiener-Itô integrals. We also compare our results with the existing literature.

1. Introduction

Stein’s method introduced by Charles Stein [39] is a powerful approach for deriving bounds for normal approximation. The method is based on the simple fact that, any real-valued random variable $Z$ has $\mathcal{N}(0, 1)$ distribution, if and only if

$$E(f'(Z) - Zf(Z)) = 0,$$

where $f$ is any real-valued absolutely continuous function such that $E|f'(Z)| < \infty$. This characterization leads us to the Stein equation

$$f'(x) - xf(x) = h(x) - Eh(Z), \quad (1)$$

where $h$ is a real-valued test function. Replacing $x$ with a random variable $Y$ and taking expectations on both sides of (1) gives

$$E(f'(Y) - Yf(Y)) = Eh(Y) - Eh(Z). \quad (2)$$

This equality (2) plays a crucial role in Stein’s method. The $\mathcal{N}(0, 1)$ distribution is characterized by (1) such that the problem of bounding the quantity $|Eh(Y) - Eh(Z)|$ depends on smoothness of the solution to (1) (see Section 2.2 of [9]), and behavior of $Y$. For more details on Stein’s method, we refer to the reader the monograph [36].

Over the years, Stein’s method has become one of the most popular tool for deriving bounds on the distance between two distributions and approximations to other classical distributions (see, [8, 17, 22, 31]). Stein’s method for various families of distributions is also a topic of keen interest for researchers (see, for example, Pearson [43], variance-gamma [18, 19, 20], discrete Gibbs measure [13, 29] family). Recently, Arras and Houdré [3, 4], Chen et. al. [10, 11], Upadhye and Barman [44], Xu [45] have developed Stein’s method for stable distributions. It is clear

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from the above articles that the derivation of Stein’s method for the family of stable distributions is not straightforward due to the lack of symmetry and heavy-tailed behavior of stable distributions. One of the major obstacles in developing the method is the moments of stable distribution do not exist whenever the stability parameter $\alpha \in (0, 1]$. To overcome these issues, different approaches and various assumptions are used to derive Stein’s method for the family of stable distributions.

Tempered stable distributions (TSD) were first introduced by Koponen [27] by tempering the tail properties of the stable distributions. TSD has mean, variance, exponential moments, and each TSD converges weakly to the stable distribution, whenever the tempering parameters tend to zero. For more details on TSD, we refer to the reader [26]. Therefore TSD is an interesting family of probability distributions for researchers in probability theory as well as financial mathematics, see [6, 7, 35, 42].

Küchlar and Tappe [28] define two-sided and one-sided TSD as a six-parameter and three-parameter family of probability distributions, respectively. Again, TSD include many sub-families of distributions, such as CGMY, KoBol, bilateral-gamma, also the variance-gamma distributions, and as the special or limiting cases, the normal, gamma, Laplace, product of two normal and difference of two gamma distributions. Researchers in probability theory have widely studied the Stein’s method for normal [39], gamma [31], Laplace [34], product-normal [23] and variance-gamma [19, 20] distributions. Therefore, it is of interest to develop the Stein’s method for TSD and see its relation for the distributions mentioned above.

In this article, we obtain a Stein characterization for two-sided TSD using the characteristic function (cf) approach. It enables us to give the Stein characterizations for normal, gamma, Laplace, product of two normal, difference of two gamma, and variance-gamma distributions from the existing literature. Further, it also enables us to give new Stein characterizations for truncated Lévy flight, CGMY, KoBol, and bilateral-gamma distributions. Next, we prove the existence of an additive size bias distribution for the one-sided case of TSD, in particular, the gamma distribution. Using the semigroup approach, we solve our Stein equation. We also derive some interesting estimates of the solution to Stein equation. Finally, we apply our estimates to obtain error bounds in the Wasserstein-type distance for tempered stable approximation in three well-known problems: comparison between two TSD, Laplace approximation of random geometric sums, and six moment theorem for the symmetric variance-gamma approximation of functionals of double Wiener-Itô integrals. We also compare our results with the existing literature.

The organization of this article is as follows. Section 2 introduces some notations and preliminaries. In Section 3 we state our results and their relevance to the existing literature on Stein characterization for TSD, in particular, for a sub-family of TSD, namely, the variance-gamma distributions (VGD). We solve our Stein equation by the semigroup approach. We also find estimates of the solution to Stein equation. In Section 4 we discuss three applications of our results.

2. Notations and Preliminaries

In this section, we review some preliminaries and known results used to develop Stein’s method for TSD. Let us first discuss the large family of distributions, namely TSD.
2.1. Tempered stable distributions. We first define the TSD and its related properties.

Definition 2.1. ([28, p.2]) A random variable $X$ having cf

$$\phi(z) = \exp \left( \int_{\mathbb{R}} (e^{izu} - 1) \nu(du) \right), \quad z \in \mathbb{R}, \quad (3)$$

and the Lévy measure

$$\nu(du) = \left( \frac{\alpha^+}{u^{1+\beta^+}} e^{-\lambda^+u} 1_{(0,\infty)}(u) + \frac{\alpha^-}{|u|^{1+\beta^-}} e^{-\lambda^-|u|} 1_{(-\infty,0)}(u) \right) du \quad (4)$$

is said to follow two-sided TSD with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^- \in (0, \infty)$, and $\beta^+, \beta^- \in [0, 1)$, and it is denoted by $X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-)$.

Definition 2.2. ([28, p.2]) A random variable $X$ having cf (3) and the Lévy measure \( \nu(du) = \frac{\alpha^+}{u^{1+\beta^+}} e^{-\lambda^+u} 1_{(0,\infty)}(u) du \) is said to follow one-sided TSD with positive support and parameters $\alpha^+, \lambda^+ \in (0, \infty)$ and $\beta^+ \in [0, 1)$, and it is denoted by $X \sim \text{TSD}_1(\alpha^+, \beta^+, \lambda^+)$.

Observe that, for $\alpha^- \to 0^+$, the Definition 2.1 reduces to Definition 2.2 which is the limiting distribution of $X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-)$.

Definition 2.3. ([28, p.2]) A random variable $X$ having cf (3) and the Lévy measure \( \nu(du) = \frac{\alpha^-}{u^{1+\beta^-}} e^{-\lambda^-u} 1_{(-\infty,0)}(u) du \) is said to follow one-sided TSD with negative support and parameters $\alpha^+, \lambda^+ \in (0, \infty)$ and $\beta^+ \in [0, 1)$, and it is denoted by $X \sim \text{TSD}_2(\alpha^-, \beta^-, \lambda^-)$.

Observe that, for $\alpha^+ \to 0^+$, the Definition 2.1 reduces to Definition 2.3 which is the limiting distribution of $X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-)$.

Remark 2.4. (i) With an appropriate choice of parameters, TSD cover truncated Lévy flight, CGMY, KoBol, variance-gamma, bilateral-gamma distributions and others in the existing literature (see, [28] for more details).

(ii) Let $X \sim \text{TSD}_1(\alpha^+, 0, \lambda^+)$ with parameters $\alpha^+, \lambda^+ \in (0, \infty)$. Then, $X \sim \text{Gamma}(\alpha^+, \lambda^+)$. 

(iii) Let $X \sim \text{TSD}_2(\alpha^-, 0, \lambda^-)$ with parameters $\alpha^-, \lambda^- \in (0, \infty)$. Then, $-X \sim \text{Gamma}(\alpha^-, \lambda^-)$. 

(iv) It is known that density function of stable distributions can not be written in closed form for each $\alpha \in (0, 1)$, where $\alpha$ is the stability parameter (see, p.33, [1]). Indeed, non-Gaussian stable distributions have heavy tails, and they are asymptotically equivalent to Pareto distribution (see, [16]).

(v) TSD are designed by tempering the tail properties of the stable distributions (see, Remark 2.3, [28]). However, the density function of TSD may not be available in closed form but, Küchler and Tappe [Section 7, [28]] have shown the existence of density function for each TSD with “nice” asymptotic properties (see, [28] Proposition 7.2 and Theorem 7.7).

2.2. Variance-gamma distributions. Next, we discuss an important subclass of TSD, namely VGD. Let us define the various characterizations for VGD in terms of its cf.
Next, we list the special and limiting cases of VGD (see, [19] for more details).

**Definition 2.5.** (Küchler and Tappe [28]) A random variable $X$ with cf given by (8), and the Lévy measure
\[
\nu_{VGD}(du) = \left(\frac{\alpha}{u}e^{-\lambda^+ u}1_{(0,\infty)}(u) + \frac{\alpha}{|u|}e^{-\lambda^-|u|}1_{(-\infty,0)}(u)\right)du
\]
is said to follow a VGD with parameters $\alpha, \lambda^+, \lambda^- \in (0, \infty)$, and it is denoted by $X \sim VGD_0(\alpha, \lambda^+, \lambda^-)$.

Note here that, $VGD_0(\alpha, \lambda^+, \lambda^-) \overset{d}{=} \text{TSD}(\alpha, 0, \lambda^+, \alpha, 0, \lambda^-)$, where $\overset{d}{=}$ denotes equality in distribution.

**Definition 2.6.** (Finlay and Seneta [15]) A random variable $X$ having cf
\[
\phi_{VGD_1}(z) = \left(1 - iz\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right) + \frac{z^2}{\lambda^+ \lambda^-}\right)^{-\alpha}, \ z \in \mathbb{R}
\]
is said to follow a VGD with parameters $\alpha, \lambda^+, \lambda^- \in (0, \infty)$, and it is denoted by $X \sim VGD_1(\alpha, \lambda^+, \lambda^-)$.

**Definition 2.7.** A random variable $X$ having cf
\[
\phi_{VGD_2}(z) = \left(1 - i2\theta z + \sigma^2 z^2\right)^{-\frac{\alpha}{2}}, \ z \in \mathbb{R}
\]
is said to follow a VGD with parameters $\sigma^2, \ r \in (0, \infty)$ and $\theta \in \mathbb{R}$, and it is denoted by $X \sim VGD_2(\sigma^2, \ r, \ \theta)$.

Definition 2.7 is used later for obtaining a Stein identity for VGD. In the following remark, we discuss relationship of the above representations with each other.

**Remark 2.8.** Note that, the cf representations (5) and (6) (for $\nu_{VGD}$ defined in Definition 2.5) are exactly same by suitably adjusting the parameters, and by using Frullani’s improper integral [2] formula. For more details about this integral, we refer the reader to Appendix A. Again, substituting $\frac{1}{\lambda^+} = \sigma^2, \ (\frac{1}{\lambda^+} - \frac{1}{\lambda^-}) = 2\theta$, and $\alpha = \frac{\alpha}{2}$ in (5), we get (6).

Next, we list the special and limiting cases of VGD (see, [19] for more details).

(O1) Let $\sigma^2 > 0$ and a random variable $X_r$ has distribution $VGD_2(\frac{\sigma^2}{2}, r, 0)$ with cf (8). Then, $X_r$ weakly converges to $N(0, \sigma^2)$, whenever $r \to \infty$.

(O2) Let $\alpha, \lambda > 0$ and a random variable $X_\sigma$ has distribution $VGD_2(\sigma^2, 2\alpha, (2\lambda)^{-1})$ with cf (8). Then, $X_\sigma$ weakly converges to $\text{Gamma}(\alpha, \lambda)$, whenever $\sigma \to 0$.

(O3) Let $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are two independent normal random variables. Then, $XY \sim VGD_2(\sigma_X^2 \sigma_Y^2, 1, 0)$.

(O4) Let $\sigma^2 > 0$, then the distribution of $VGD_2(\sigma^2, 2, 0)$ has Laplace($0, \sigma^2$) distribution.

2.3. Function spaces and probability metrics. Next, we define a function space and suitable probability metric required to develop Stein’s method for TSD. Let $S(\mathbb{R})$ be the Schwartz space defined by
\[
S(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \lim_{|x| \to \infty} |x^m \frac{d^n}{dx^n} f(x)| = 0, \text{ for all } m, n \in \mathbb{N} \right\},
\]
where $C^\infty(\mathbb{R})$ is the class of infinitely differentiable functions on $\mathbb{R}$. It is important to note that the Fourier transform on $S(\mathbb{R})$ is automorphism onto itself. This enables us to identify the elements of dual space $S^*(\mathbb{R})$ with $S(\mathbb{R})$. In particular,
if \( f \in S(\mathbb{R}) \), and \( \hat{f}(u) = \int_\mathbb{R} e^{-iux} f(x) dx, \ u \in \mathbb{R} \), then \( \hat{f}(u) \in S(\mathbb{R}) \). Similarly, if \( \hat{f}(u) \in S(\mathbb{R}) \), and \( f(x) = \int_\mathbb{R} e^{iux} \hat{f}(u) du, \ x \in \mathbb{R} \), then \( f(x) \in S(\mathbb{R}) \), see [41].

Finally, we define Wasserstein-type distance, see [3]. Let

\[
H_r = \left\{ h : \mathbb{R} \to \mathbb{R} \mid h \text{ is } r \text{ times differentiable and } \| h^{(k)} \| \leq 1, k = 0, 1, \ldots, r \right\},
\]

where \( h^{(k)} \), \( k = 1, \ldots, r \), is the \( k \)-th derivative of \( h \), with \( h^{(0)} = h \) and \( \| f \| = \sup_{x \in \mathbb{R}} |f(x)| \). Then, for any two random variables \( Y \) and \( Z \) the distance is given by

\[
d_{W_r}(Y, Z) := \sup_{h \in H_r} \left| \mathbb{E}[h(Y)] - \mathbb{E}[h(Z)] \right|.
\]

We use this distance for studying TSD approximation problems. Note that, \( d_{W_r} \) has the following order relationship with the classical Wasserstein distance \( W_1 \).

\[
d_{W_r}(Y, Z) \leq d_{W_1}(Y, Z) \leq W_1(Y, Z) \leq W_p(Y, Z), \quad r, p \geq 1.
\]

We use this relationship and discuss the consequences of our results in Section 4.

3. Results

In this section, we present components of Stein’s method for TSD.

3.1. Stein characterization. First, we present a Stein characterization for TSD.

**Theorem 3.1.** Let \( X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-) \). Then,

\[
\mathbb{E} \left( X f(X) - \int_\mathbb{R} f(X + u) \nu(du) \right) = 0, \ \ f \in S(\mathbb{R}). \tag{7}
\]

*Proof.* Recall first that, for \( X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-) \), cf is given by (3) with the Lévy measure (4). Taking logarithms on both sides of (3), and differentiating with respect to \( z \), we have

\[
\phi'(z) = i \int_\mathbb{R} x e^{izx} F_X(dx) \phi(z). \tag{8}
\]

Let \( F_X \) be the distribution function (cumulative distribution function) of \( X \). Then,

\[
\phi(z) = \int_\mathbb{R} e^{izx} F_X(dx) \text{ and } \phi'(z) = i \int_\mathbb{R} x e^{izx} F_X(dx). \tag{9}
\]

Using (8) in (9) and rearranging the integrals, we have

\[
0 = i \int_\mathbb{R} x e^{izx} F_X(dx) - i \int_\mathbb{R} u e^{izu} \nu(du) \phi(z)
= \int_\mathbb{R} x e^{izx} F_X(dx) - \int_\mathbb{R} u e^{izu} \nu(du) \phi(z) \tag{10}
\]

The second integral of (10) can be written as

\[
\left( \int_\mathbb{R} u e^{izu} \nu(du) \right) \phi(z) = \int_\mathbb{R} \int_\mathbb{R} u e^{izu} e^{izx} F_X(dx) \nu(du)
= \int_\mathbb{R} \int_\mathbb{R} u e^{iz(u+x)} \nu(du) F_X(dx)
\]
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{isy} \nu(du) F_X(d(y - u)) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{isz} \nu(du) F_X(d(x - u)) = \int_{\mathbb{R}} e^{isz} \int_{\mathbb{R}} u F_X(d(x - u)) \nu(du). \quad (11) \]

Substituting (11) in (10), we have

\[ 0 = \int_{\mathbb{R}} x e^{izx} F_X(dx) - \int_{\mathbb{R}} e^{izx} \int_{\mathbb{R}} u F_X(d(x - u)) \nu(du) = \int_{\mathbb{R}} e^{izx} \left( x F_X(dx) - \int_{\mathbb{R}} u F_X(d(x - u)) \nu(du) \right). \quad (12) \]

On applying Fourier transform to (12), multiplying with \( f \in S(\mathbb{R}) \) and integrating over \( \mathbb{R} \), we get

\[ \int_{\mathbb{R}} f(x) \left( x F_X(dx) - \int_{\mathbb{R}} u F_X(d(x - u)) \nu(du) \right) = 0. \quad (13) \]

The second integral of (13) can be seen as

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} uf(x) F_X(d(x - u)) \nu(du) = \int_{\mathbb{R}} \int_{\mathbb{R}} uf(y + u) F_X(dy) \nu(du) = \int_{\mathbb{R}} \int_{\mathbb{R}} uf(x + u) F_X(dx) \nu(du) = E \left( \int_{\mathbb{R}} uf(X + u) \nu(du) \right). \quad (14) \]

Substituting (14) in (13), we have

\[ E \left( X f(X) - \int_{\mathbb{R}} f(X + u) \nu(du) \right) = 0. \]

Hence the theorem is proved. \( \square \)

**Remark 3.2.** One can also prove the converse of Theorem 3.1 by choosing \( f(x) = e^{isz} \), where \( s, x \in \mathbb{R} \) in (7). We refer to the reader Appendix A for proof of the converse of Theorem 3.1. We derive the characterizing (Stein) identity (7) for TSD using the Lévy-Khinchine representation of the cf. Also, observe that several classes of distributions such as variance-gamma, bilateral-gamma, CGMY, and KoBol can be viewed as TSD. Stein characterization for these classes of distributions can be easily derived using (7).

Note that, from Definition 2.5 and Theorem 3.1 for \( f \in S(\mathbb{R}) \) a Stein identity for \( \text{VDG}_0(\alpha, \lambda^+, \lambda^-) \) is

\[ \mathbb{E} X f(X) = \mathbb{E} \left( \int_{\mathbb{R}} f(X + u) \nu_{\text{VDG}}(du) \right) = \alpha \mathbb{E} \int_{0}^{\infty} \left( e^{-\lambda^+ u} f(X + u) - e^{-\lambda^- u} f(X - u) \right) du. \quad (15) \]
Next, we establish a Stein characterization for VGD\(_2(\sigma^2, r, \theta)\).

**Corollary 3.3.** Let \( X \sim \text{VGD}_2(\sigma^2, r, \theta) \) with \( c_f(\sigma) \). Then,
\[
\mathbb{E}(\sigma^2 X f''(X) + (\sigma^2 r + 2\theta X) f'(X) + (r \theta - X) f(X)) = 0, \quad f \in S(\mathbb{R}).
\]  

**Proof.** Applying integration by parts formula twice on the right hand side of (15)

Next observe that
\[
\mathbb{E}X f(X) = \alpha \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) \mathbb{E}f(X)
+ \alpha \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) \mathbb{E} \int_0^\infty \left( e^{-\lambda^+ u} f'(X + u) - e^{-\lambda^- u} f'(X - u) \right) du
+ \alpha \frac{\lambda^-}{\lambda^+} \mathbb{E} \int_0^\infty e^{-\lambda^+ u} f'(X + u) du + \alpha \frac{\lambda^+}{\lambda^-} \mathbb{E} \int_0^\infty e^{-\lambda^- u} f'(X - u) du
= \alpha \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) \mathbb{E}f(X)
+ \alpha \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) \mathbb{E} \int_0^\infty \left( e^{-\lambda^+ u} f'(X + u) - e^{-\lambda^- u} f'(X - u) \right) du
+ \frac{2\alpha}{\lambda^+ \lambda^-} \mathbb{E}f'(X) + \frac{\alpha}{\lambda^+ \lambda^-} \mathbb{E} \int_0^\infty \left( e^{-\lambda^+ u} f''(X + u) - e^{-\lambda^- u} f''(X - u) \right) du.
\]  

Next observe that

(a) \( \mathbb{E}X f'(X) = \alpha \mathbb{E} \int_0^\infty \left( e^{-\lambda^+ u} f'(X + u) - e^{-\lambda^- u} f'(X - u) \right) du \),  

(b) \( \mathbb{E}X f''(X) = \alpha \mathbb{E} \int_0^\infty \left( e^{-\lambda^+ u} f''(X + u) - e^{-\lambda^- u} f''(X - u) \right) du \),

as \( f \in S(\mathbb{R}) \). Now, applying (18a) and (18b) on (17), we get

\[
\mathbb{E} \left( \frac{1}{\lambda^+ \lambda^-} X f''(X) + \left( \frac{2\alpha}{\lambda^+ \lambda^-} + \Lambda X \right) f'(X) + (\alpha \Lambda - X) f(X) \right) = 0,
\]

where \( \Lambda = \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) \), \( f \in S(\mathbb{R}) \). Setting the parameters
\[
\frac{1}{\lambda^+ \lambda^-} = \sigma^2, \quad \Lambda = \left( \frac{1}{\lambda^+} - \frac{1}{\lambda^-} \right) = 2\theta, \quad \alpha = \frac{r}{2^2},
\]

we get the our desired conclusion. \( \square \)

Next, we compare our characterization with some well-known Stein characterizations in literature.

**Remark 3.4.** (i) Our Stein characterization matches exactly with Stein characterization given in Gaunt [19], whenever the location parameter \( \mu = 0 \). In general, Gaunt [19] uses the density approach developed in [10], and the density of VGD is usually written in terms of modified Bessel function. Therefore, the derivation of Stein characterization using density approach is quite lengthy (see [19]). However, we show that using cf approach, the derivation of Stein characterization is quick and easy to understand.
(ii) We also observe that a Stein identity for $VGD_2(\frac{\sigma^2}{r}, r, 0)$ is given by
\[
E\left(\frac{\sigma^2}{r} X f''(X) + \sigma^2 f'(X) - X f(X)\right) = 0,
\]
which in the limit $r \to \infty$ is the Stein identity for classical $\mathcal{N}(0, \sigma^2)$.

(iii) Taking $r = 1, \sigma^2 = \sigma_X^2 \sigma_Y^2$ and $\theta = 0$, the Stein identity reduces to
\[
E\left(\sigma_X^2 \sigma_Y^2 (X f''(X) + f'(X)) - X f(X)\right) = 0,
\]
which is the Stein identity for products of independent $\mathcal{N}(0, \sigma_X^2)$ and $\mathcal{N}(0, \sigma_Y^2)$, see [24].

(iv) We can also deduce Stein identities for symmetrized-gamma or symmetric case of variance-gamma, Laplace, gamma distributions using Corollary 3.3.

Next, we state a corollary for one-sided TSD, which provides a Stein characterization for gamma distribution.

**Corollary 3.5.** Let $X \sim TSD_1(\alpha, 0, \lambda)$. Then,
\[
E X f(X) = E X E \left(f(X) + \frac{1}{\lambda} f'(X + Y)\right), \quad f \in S(\mathbb{R}),
\]
where $Y$ is a random variable having exponential distribution with parameter $\lambda$, independent of $X$.

**Proof.** Let $X \sim TSD_1(\alpha, 0, \lambda)$ with $\alpha, \lambda > 0$. Then by Remark 2.4, $X$ has the gamma distribution with parameters $\alpha$ and $\lambda$. Following steps similar to the proof of Theorem 3.1, one can find a Stein identity for $TSD_1(\alpha, 0, \lambda)$ in the form
\[
E X f(X) = \alpha E \left(\int_0^{\infty} e^{-\lambda u} f(X + u) du\right), \quad f \in S(\mathbb{R}).
\]
(20)

Note that, $EX = \frac{\alpha}{\lambda}$. Applying integration by parts formula on the right hand side of (20), we have
\[
E X f(X) = E \left(\frac{\alpha}{\lambda} f(X) + \frac{\alpha}{\lambda} \int_0^{\infty} e^{-\lambda u} f'(X + u) du\right)
= E X E \left(f(X) + \int_0^{\infty} e^{-\lambda u} f'(X + u) du\right)
= E X E \left(f(X) + \frac{1}{\lambda} f'(X + Y)\right),
\]
where $Y$ is exponential random variable with parameter $\lambda$, independent of $X$.
Hence the result. \[\square\]

**Remark 3.6.**
(i) Note that Corollary 3.5 claims the existence of an additive exponential size-bias (see, [9]) distribution for the gamma distribution.

(ii) The Stein characterization for gamma distribution is first introduced by Luk ([31, Subsection 2.2]) using Barbour generator approach [5] without additive size-bias distribution. The Stein identity given in ([31, Lemma 2.9]) is for $\chi^2_{2(n+1)}$ distribution with additive size-bias distribution. Under the assumptions of Luk [31], both identities can be retrieved from Corollary 3.5.
3.2. Stein equation. Note that, from Theorem 3.1, for any \( f \in \mathcal{S}(\mathbb{R}) \), \( A_X(f)(x) := -xf(x) + \int_{\mathbb{R}} f(x + u)\nu(du) \) is a Stein operator for TSD. Observe also that, \( A_X \) is an integral operator, where domain of the operator is \( \mathcal{F}_X = \mathcal{S}(\mathbb{R}) \) (see, [44] for more details). For more general discussion on domain of operators, we refer the reader to [11] and references therein. As mentioned in Section 1, the next step in Stein’s method is to set a Stein equation. For any \( X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-) \) and \( h \in \mathcal{H}_r \) (see, Subsection 2.3) with \( \mathbb{E}h(X) < \infty \), a Stein equation for TSD is given by

\[
A_X(f)(x) = h(x) - \mathbb{E}(h(X)).
\]  

To solve (21), we apply the semigroup approach. The semigroup approach for solving the Stein equation is developed by Barbour [5], and Arras and Houdré [3] generalized it for infinitely divisible distributions with the finite first moment. Following Barbour’s approach [5], we choose a family of operators \((P_t)_{t \geq 0}\), for all \( x \in \mathbb{R} \), as

\[
P_t(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{izxe^{-t}} \frac{\phi(\xi)}{\phi(e^{-t}\xi)} d\xi, \quad f \in \mathcal{F}_X. \tag{22}
\]

Note here that, one can define a cf, for all \( z \in \mathbb{R} \), and \( t \geq 0 \), by

\[
\phi_t(z) := \frac{\phi(z)}{\phi(e^{-t}z)} = \int_{\mathbb{R}} e^{izu} \mathbb{E}X(\xi) (du), \tag{23}
\]

where \( \mathbb{E}X(\xi) \) is the distribution function of \( X(\xi) \) and \( \phi \) is the cf of TSD given in [3]. The property given in (23) is also known as self-decomposability (see, [37]). Using this property, we get

\[
P_t(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(z) e^{izxe^{-t}} e^{izu} \mathbb{E}X(\xi) (du) dz
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(z) e^{iz(u+xe^{-t})} \mathbb{E}X(\xi) (du) dz
= \int_{\mathbb{R}} f(u+xe^{-t}) \mathbb{E}X(\xi) (du), \tag{24}
\]

where the last step follows by applying inverse Fourier transform.

**Proposition 3.7.** The family of operators \((P_t)_{t \geq 0}\) given in (22) is a \( \mathbb{C}_0 \)-semigroup on \( \mathcal{F}_X \).

For details of the proof, we refer the reader to Appendix A.

Next, we establish an infinitesimal generator of the semigroup \((P_t)_{t \geq 0}\).

**Lemma 3.8.** Let \((P_t)_{t \geq 0}\) be a \( \mathbb{C}_0 \) semigroup defined in (22). Then, its generator \( T \) is given by

\[
T(f)(x) = -xf'(x) + \int_{\mathbb{R}} f'(x + u)\nu(du), \quad f \in \mathcal{S}(\mathbb{R}). \tag{25}
\]

**Proof.** For all \( f \in \mathcal{S}(\mathbb{R}) \),

\[
T(f)(x) = \lim_{t \to 0^+} \frac{1}{t} (P_t(f)(x) - f(x))
= \frac{1}{2\pi} \lim_{t \to 0^+} \int_{\mathbb{R}} \hat{g}(z)e^{izx} \frac{1}{t} (e^{iz(e^{-t}-1)}\phi_t(z) - 1) dz.
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(z)e^{izx} \left( -x + \int_{\mathbb{R}} e^{izu}u\nu(du) \right) (iz)dz \text{ (using Prop. A.3)}
\]

\[
= -xf'(x) + \int_{\mathbb{R}} f'(x + u)u\nu(du),
\]

where the last equality follows by applying inverse Fourier transform. This completes the proof. \hfill \square

Observe that, for any \( f \in \mathcal{S}(\mathbb{R}) \),

\[
\mathcal{T}f(x) = -xf'(x) + \int_{\mathbb{R}} f'(x + u)u\nu(du) = \mathcal{A}_X(f')(x).
\]

Next, we provide the solution to our Stein equation (21).

**Theorem 3.9.** Let \( X \sim \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-) \). Then for \( h \in \mathcal{H}_r \), the function \( f_h : \mathbb{R} \to \mathbb{R} \) defined by

\[
f_h(x) := -\int_0^\infty e^{-t} \int_{\mathbb{R}} h'(xe^{-t} + y)F_X(dy)dt,
\]

(26)

solves (21).

**Proof.** To prove this theorem, we use the connection between the operators \( \mathcal{A}_X \) and \( \mathcal{T} \). We write,

\[
\mathcal{A}f_h(x) = -xf_h(x) + \int_{\mathbb{R}} f_h(x + u)u\nu(du)
\]

\[
= \mathcal{T}(g_h)(x), \text{ (where } g_h(x) = -\int_0^\infty (P_t(h)(x) - \mathbb{E}h(X)) dt, \ h \in \mathcal{H}_r \)
\]

\[
= -\int_0^\infty \mathcal{T}P_t(h)(x)dt
\]

\[
= -\int_0^\infty \frac{d}{ds}P_t(h)(x)dt
\]

\[
= P_0h(x) - P_\infty h(x)
\]

\[
= h(x) - \mathbb{E}h(X) \text{ (by Proposition 3.7)}.
\]

Hence, \( f_h \) is the solution to (21).

Using some standard argument, one can show that \( g_h \) is well-defined and \( g_h'(x) = f_h(x), \ x \in \mathbb{R} \). For details of the proof, we refer the reader to Appendix A. \hfill \square

### 3.3. Properties to the solution.

The next step is to estimate the properties of \( f_h \). In the following theorem, we establish estimates of \( f_h \), which play a crucial role in the TSD approximation problems. Gaunt [19, 20] and Döbler et. al. [12] propose various methods for bounding the solution to the Stein equations that allow them to derive properties of the solution to the Stein equation, in particular for a subfamily of TSD, namely the variance-gamma. However, we derive the properties of the solution to the Stein equation for TSD using its self-decomposable property.

**Theorem 3.10.** For \( h \in \mathcal{H}_4 \), let \( f_h \) be defined in (26). Then,

\[
\|f_h\| \leq \|h^{(1)}\|, \quad \|f_h'\| \leq \frac{1}{2}\|h^{(2)}\|, \quad \|f_h''\| \leq \frac{1}{3}\|h^{(3)}\|, \quad \|f_h'''\| \leq \frac{1}{4}\|h^{(4)}\|.
\]

(27)
For any \( x, y \in \mathbb{R} \),
\[
\|f''_h(x) - f''_h(y)\| \leq \frac{\|h^{(4)}\|}{4} |x - y|.
\]  
(28)

**Proof.** Recall the definition of \((P_t)_{t \geq 0}\),
\[
P_t f(x) = \int_{\mathbb{R}} f(y + e^{-t}x) F_{X(t)}(dy), ~ f \in \mathcal{F}_X,
\]
where \( F_{X(t)} \) is the distribution function of \( X(t) \). Thus, for \( h \in \mathcal{H}_4 \),
\[
\frac{d}{dx}(P_t(h)(x)) = e^{-t} \int_{\mathbb{R}} h^{(1)}(xe^{-t} + y) F_{X(t)}(dy),
\]
\[
\frac{d^2}{dx^2}(P_t(h)(x)) = e^{-2t} \int_{\mathbb{R}} h^{(2)}(xe^{-t} + y) F_{X(t)}(dy),
\]
\[
\frac{d^3}{dx^3}(P_t(h)(x)) = e^{-3t} \int_{\mathbb{R}} h^{(3)}(xe^{-t} + y) F_{X(t)}(dy),
\]
and
\[
\frac{d^4}{dx^4}(P_t(h)(x)) = e^{-4t} \int_{\mathbb{R}} h^{(4)}(xe^{-t} + y) F_{X(t)}(dy).
\]

Let
\[
f_h(x) = - \int_0^\infty e^{-t} \int_{\mathbb{R}} h'(xe^{-t} + y) F_{X(t)}(dy) dt.
\]
It can be easily seen that \( f_h \) is thrice differentiable. Hence, \( \|f_h\| \leq \|h^{(1)}\|, ~ \|f'_h\| \leq \frac{1}{2}\|h^{(2)}\|, ~ \|f''_h\| \leq \frac{1}{3}\|h^{(3)}\|, ~ \|f'''_h\| \leq \frac{1}{4}\|h^{(4)}\| \).

Now observe that, for any \( x, y \in \mathbb{R} \) and \( h \in \mathcal{H}_4 \),
\[
|f''_h(x) - f''_h(y)| \leq \int_0^\infty e^{-3t} \int_{\mathbb{R}} \left| h^{(3)}(xe^{-t} + z) - h^{(3)}(ye^{-t} + z) \right| F_{X(t)}(dz) dt
\]
\[
\leq \int_0^\infty e^{-3t} \int_{\mathbb{R}} \|h^{(4)}\| |x - y| e^{-t} F_{X(t)}(dz) dt
\]
\[
= \|h^{(4)}\| |x - y| \int_0^\infty e^{-4t} dt
\]
\[
= \frac{\|h^{(4)}\|}{4} |x - y|,
\]
the desired conclusion follows. \( \square \)

### 4. Applications

In this section, we present three applications of our estimates.

#### 4.1. Comparison between two TSD

As a first application of our estimates, we derive a simple error bound for approximation between TSD. We refer the reader to [30] for a number of similar bounds for comparison of uni-variate distributions. First, we establish a corollary to Theorem 3.10, which is used in deriving an upper bound in the Wasserstein-type distance between two TSD.
Corollary 4.1. For \( h \in \mathcal{H}_3 \), let \( f_h \) be defined in (20). Let \( \alpha^+ = \alpha^- = \alpha, \beta^+ = \beta^- = 0 \) and \( \lambda^+ = \lambda^- = \lambda \). Then, for any \( x \in \mathbb{R} \)

\[
\|xf_h''(x)\| \leq 2 \left( \|h^{(2)}\| + \frac{\alpha}{3\lambda}\|h^{(3)}\| \right). 
\]  

(29)

Proof. As \( f_h \) solves (21), thus we have

\[
-xf_h(x) + \int_{\mathbb{R}} f_h(x + u)\nu(du) = h(x) - \mathbb{E}h(X).
\]  

(30)

Recall that, the Lévy measure for \( \text{TSD}(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-) \) given in (4). Assume that \( \alpha^+ = \alpha^- = \alpha, \beta^+ = \beta^- = 0, \lambda^+ = \lambda^- = \lambda \) and differentiating (30) twice with respect to \( x \), we have

\[
-xf_h''(x) = h^{(2)}(x) + 2f_h'(x) - \int_{\mathbb{R}} f_h''(x + u)\nu(du)
\]

\[
= h^{(2)}(x) + 2f_h'(x) - \alpha \int_{0}^{\infty} e^{-\lambda u} (f_h''(x + u) - f_h''(x - u)) \, du.
\]  

(31)

Using (27), we have

\[
\|xf_h''(x)\| \leq 2\|h^{(2)}\| + \frac{2\alpha}{3\lambda}\|h^{(3)}\| \int_{0}^{\infty} e^{-\lambda u} \, du
\]

\[
= 2\|h^{(2)}\| + \frac{2\alpha}{3\lambda}\|h^{(3)}\|,
\]

the desired conclusion follows.

\[
\square
\]

In the following theorem, we establish a simple error bound for approximation between two TSD.

Theorem 4.2. Let \( X \sim \text{TSD}(\alpha_1, 0, \lambda_1; \alpha_1, 0, \lambda_1) \) and \( Y \sim \text{TSD}(\alpha_2, 0, \lambda_2; \alpha_2, 0, \lambda_2) \). Then, for \( \lambda_1 > 1 \)

\[
d_{W_2}(Y, X) \leq \frac{\lambda_1^2}{\lambda_1^2 - 1} \left( \left| \frac{\alpha_1}{\lambda_1^2} - \frac{\alpha_2}{\lambda_2^2} \right| + 2 \left( 1 + \frac{\alpha_1}{3\lambda_1} \right) \left| \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} \right| \right).
\]

Proof. By (21), we can write the Stein equation for \( X \sim \text{TSD}(\alpha_1, 0, \lambda_1; \alpha_1, 0, \lambda_1) \) as

\[
h(x) - h(X) = -xf(x) + \int_{\mathbb{R}} f(x + u)\nu(du)
\]

\[
= -xf(x) + \alpha_1 \int_{0}^{\infty} (f(x + u) - f(x - u)) e^{-\lambda_1 u} \, du
\]  

(32)

Thus, we have

\[
\mathbb{E} (h(Y) - h(X)) = \mathbb{E} \left( -Yf_h(Y) + \alpha_1 \int_{0}^{\infty} (f_h(Y + u) - f_h(Y - u)) e^{-\lambda_1 u} \, du \right)
\]

\[
= \mathbb{E} \left( -Yf_h(Y) + \frac{\alpha_1}{\lambda_1} \int_{0}^{\infty} (f_h'(Y + u) + f_h'(Y - u)) e^{-\lambda_1 u} \, du \right)
\]

\[
= \mathbb{E} \left( -Yf_h(Y) + \frac{2\alpha_1}{\lambda_1} f_h'(Y) + \frac{\alpha_1}{\lambda_1^2} \int_{0}^{\infty} (f_h''(Y + u) + f_h''(Y - u)) e^{-\lambda_1 u} \, du \right)
\]
\[ \begin{align*}
&= \mathbb{E} \left( -Y f_h(Y) + \frac{2\alpha_1}{\lambda_1^2} f'_h(Y) + \frac{1}{\lambda_1^2} Y f''_h(Y) \right) \\
&\quad + \frac{1}{\lambda_1^2} \mathbb{E} \left( -Y f''_h(Y) + \alpha_1 \int_0^\infty \left( f''_h(Y + u) - f''_h(Y - u) \right) e^{-\lambda_1 u} du \right) \\
&\quad + \frac{1}{\lambda_1^2} \mathbb{E} \left( -Y f''_h(Y) + \alpha_1 \int_0^\infty \left( f''_h(Y + u) - f''_h(Y - u) \right) e^{-\lambda_1 u} du \right)
\end{align*} \]

Taking \( \sup_{h \in H_3} \) on both side of (33) and rearranging the terms, we have

\[ \left( 1 - \frac{1}{\lambda_1^2} \right) d_{W_3}(Y, X) \leq \sup_{h \in H_3} \left| \mathbb{E} \left( -Y f_h(Y) + \frac{2\alpha_1}{\lambda_1^2} f'_h(Y) + \frac{1}{\lambda_1^2} Y f''_h(Y) \right) \right| \]

Note that, \( Y \sim \text{TSD}(\alpha_2, 0, \lambda_2; \alpha_2, 0, \lambda_2) \). Hence, following steps similar to proof of Corollary 3.3, we get

\[ \mathbb{E} \left( -Y f_h(Y) + \frac{2\alpha_2}{\lambda_2^2} f'_h(Y) + \frac{1}{\lambda_2^2} Y f''_h(Y) \right) = 0 \]

Using (35) in (34), we have

\[ \left( 1 - \frac{1}{\lambda_1^2} \right) d_{W_3}(Y, X) \leq \sup_{h \in H_3} \left| \mathbb{E} \left( -Y f_h(Y) + \frac{2\alpha_1}{\lambda_1^2} f'_h(Y) + \frac{1}{\lambda_1^2} Y f''_h(Y) \right) \right| \]

\[ = \mathbb{E} \left( -Y f_h(Y) + \frac{2\alpha_2}{\lambda_2^2} f'_h(Y) + \frac{1}{\lambda_2^2} Y f''_h(Y) \right) \]

\[ \leq \sup_{h \in H_3} \left| \mathbb{E} \left[ \left( \frac{2\alpha_1}{\lambda_1^2} - \frac{2\alpha_2}{\lambda_2^2} \right) f'_h(Y) \right] + \mathbb{E} \left[ \left( \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} \right) Y f''_h(Y) \right] \right| \]

Assume that \( \|h^{(k)}\| \leq 1 \) for \( k = 2, 3 \). Then, by using (27) and (29) in (36), we get our desired result.

\[ \square \]

**Remark 4.3.** Note that, if \( \alpha_1 = \alpha_2 \) and \( \lambda_1 = \lambda_2 > 1 \), this bound is equal to zero. This shows that \( Y \overset{d}{=} X \).

**4.2. Rate of convergence for the Laplace approximation of random geometric sums.** Recall that the distribution of a random variable \( X \) with \( \text{cf} \psi_1(z) = \frac{1}{1 + \alpha z}, \ z \in \mathbb{R} \), where \( \lambda > 0 \), is called the Laplace distribution (we write \( X \sim \text{Laplace}(0, \frac{1}{\lambda}) \)). Note here that, \( X \sim \text{Laplace}(0, \frac{1}{\lambda}) \overset{d}{=} \text{TSD}(1, 0, \lambda; 1, 0, \lambda) \). Let \( N_p \) be a Geo\((p)\) random variable with \( \text{cf} \psi_2(z) = \frac{p}{1-(1-p)e^z}, \ z \in \mathbb{R} \). In the following theorem, we present a well-known limit theorem concerning to geometric sum of i.i.d random variables.

**Theorem 4.4.** ([25], p. 2) Let \( (Y_n)_{n \geq 1} \) be a sequence of i.i.d random variables with zero mean and variance \( \frac{2}{\lambda^2} \in (0, \infty) \) and let \( N_p \sim \text{Geo}(p) \) be independent of \( Y_i \) with probability mass function \( \mathbb{P}(N_p = k) = p(1-p)^k; \ k = 0, 1, 2, \ldots, 0 < p < 1 \). Then, \( S_p := \sqrt{p} \sum_{i=1}^{N_p} Y_i \overset{d}{=} \text{Laplace}(0, \frac{1}{\lambda}), \ as \ p \to 0 \).
Next, we define centered equilibrium distribution, which plays an important role in the Laplace approximation for geometric random sum.

**Definition 4.5.** (p.9) For any non-degenerate random variable $Y$ with mean zero and variance $\frac{1}{\alpha^2} \in (0, \infty)$, we say that the random variable $Y^L$ has the centered equilibrium distribution with respect to $Y$ if

$$Eg(Y) - g(0) = \frac{1}{\alpha^2} E (g''(Y^L)),$$  
(37)

for all twice differentiable functions $g$ such that $g, g'$ and $g''$ are bounded. We can the map $Y \to Y^L$ the centered equilibrium transformation.

Note here that, if we consider $g(x) = xf(x) \in S(\mathbb{R})$, then (37) becomes

$$\text{EY} f(Y) = \frac{1}{\alpha^2} E (Y^L f''(Y^L) + 2f'(Y^L)).$$  
(38)

To derive upper bound for the Laplace approximation in the Wasserstein-type distance, we need the following lemma.

**Lemma 4.6.** Let $(Y_n)_{n \geq 1}$ be a sequence of i.i.d random variables such that $\text{E}Y_i = 0$ and $\text{E}Y_i^2 = \frac{1}{\alpha^2} \in (0, \infty)$ and $\sup_{i \in \mathbb{N}} \text{E}|Y_i|^3 = \rho < \infty$, and let $N_p \sim \text{Geo}(p)$ be a random variable independent of $Y_i$. Then

1. For $S_p := \sqrt{p} \sum_{k=1}^{N_p} Y_k$, the variable with centered equilibrium distribution has the form $S_p^L = \sqrt{p} \left( \sum_{k=1}^{N_p} Y_k + Y_{N_p+1}^L \right)$.

2. $\text{E} |S_p - S_p^L| = \frac{\sqrt{2p}}{6} \rho$.

The proof of this lemma follows by similar computations [38, Lemma 4.1] and [34, Proposition 3.4], respectively.

Next, we establish a corollary to Theorem 3.10 which is used in deriving rate of convergence for the Laplace approximation of geometric sums.

**Corollary 4.7.** For $h \in \mathcal{H}_4$, let $f_h$ be defined in [20]. Let $\alpha^+ = \alpha^- = 1$, $\beta^+ = \beta^- = 0$ and $\lambda^+ = \lambda^- = \lambda$. Define $A_0 f_h(x) = xf_h(x)$. Then, for any $x, y \in \mathbb{R}$

$$\| (A_0 f_h(x))^n - (A_0 f_h(y))^n \| \leq \frac{\|h^{(3)}\| + \|h^{(4)}\|}{2\lambda} |x - y|.$$  
(39)

**Proof.** Recall that, the Lévy measure for TSD$(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-)$ given in [4]. Let $\alpha^+ = \alpha^- = 1$, $\beta^+ = \beta^- = 0$ and $\lambda^+ = \lambda^- = \lambda$. Define $A_0 f_h(x) = xf_h(x)$. Then, differentiating (30) twice with respect to $x$, we have

$$(A_0 f_h(x))^n = -h^{(2)}(x) + \int_0^\infty e^{-\lambda u} (f_h''(x + u) - f_h''(x - u)) \, du.$$  

For any $x, y \in \mathbb{R}$,

$$\| (A_0 f_h(x))^n - (A_0 f_h(y))^n \| \leq \|h^{(2)}(x) - h^{(2)}(y)\| + \left| \int_0^\infty e^{-\lambda u} (f_h''(x + u) - f_h''(y + u)) \, du \right|$$

$$+ \left| \int_0^\infty e^{-\lambda u} (f_h''(x - u) - f_h''(y - u)) \, du \right|$$
Theorem 4.8. Let \((Y_n)_{n \geq 1}\) be a sequence of i.i.d random variables with \(\mathbb{E} Y_i = 0\), \(\mathbb{E} Y_i^2 = \frac{2}{\lambda^2} \in (0, \infty)\) and let \(N_p \sim \text{Geo}(p)\) be independent of \(Y_i\) with probability mass function \(\mathbb{P}(N_p = k) = p(1-p)^k; \ k = 0, 1, 2, \ldots, 0 < p < 1\). Also let \(X \sim \text{Laplace}(0, \frac{1}{\lambda^2})\) and denote \(S_p := \sqrt{\frac{N_p}{\lambda}} Y_i\). Then, for any \(\lambda > 1\),
\[
d_{W_4}(S_p, X) \leq \frac{\rho \lambda (2\lambda^2 + 1)}{12(\lambda^2 - 1)} p^{\frac{1}{2}}.
\]

Proof. Recall first that \(\text{Laplace}(0, \frac{1}{\lambda^2}) \overset{d}{=} \text{TSD}(1, 0; \lambda; 1, 0, \lambda)\). Thus, using (21), we have
\[
\mathbb{E} (h(S_p) - h(X)) = \mathbb{E} \left( -S_p f_h(S_p) + \int_{\mathbb{R}} f_h(S_p + u) \nu(du) \right)
\]
\[
= \mathbb{E} \left( -S_p f_h(S_p) + \int_0^{\infty} (f_h(S_p + u) - f_h(S_p - u)) e^{-\lambda u} du \right)
\]
\[
= \mathbb{E} \left( -S_p f_h(S_p) + \frac{1}{\lambda} \int_0^{\infty} \left( f'_h(S_p + u) + f'_h(S_p - u) \right) e^{-\lambda u} du \right)
\]
\[
= \mathbb{E} \left( -S_p f_h(S_p) + \frac{2}{\lambda^2} f'_h(S_p) + \frac{1}{\lambda^2} \int_0^{\infty} \left( f''_h(S_p + u) - f''_h(S_p - u) \right) e^{-\lambda u} du \right)
\]
\[
= \mathbb{E} \left( -S_p f_h(S_p) + \frac{2}{\lambda^2} f'_h(S_p) + \frac{1}{\lambda^2} S_p f''_h(S_p) \right)
\]
\[
+ \frac{1}{\lambda^2} \mathbb{E} \left( -S_p f''_h(S_p) + \int_0^{\infty} \left( f''_h(S_p + u) - f''_h(S_p - u) \right) e^{-\lambda u} du \right)
\]
\[
= \mathbb{E} \left( -S_p f_h(S_p) + \frac{2}{\lambda^2} f'_h(S_p) + \frac{1}{\lambda^2} S_p f''_h(S_p) \right)
\]
\[
+ \frac{1}{\lambda^2} \mathbb{E} \left( -S_p f''_h(S_p) + \int_{\mathbb{R}} f''_h(S_p + u) \nu(du) \right)
\]
(42)

Taking \(\sup_{h \in \mathcal{H}_4}\) on both side of (42) and rearranging the terms, we have
\[
\left( 1 - \frac{1}{\lambda^2} \right) d_{W_4}(S_p, X) \leq \sup_{h \in \mathcal{H}_4} \left| \mathbb{E} \left( -S_p f_h(S_p) + \frac{2}{\lambda^2} f'_h(S_p) + \frac{1}{\lambda^2} S_p f''_h(S_p) \right) \right|
\]
(43)

Using \[38\] in \[43\], we have
\[
d_{W_4}(S_p, X) \leq \sup_{h \in \mathcal{H}_4} \frac{\lambda^2}{\lambda^2 - 1} \left( \mathbb{E} \left[ \frac{2}{\lambda^2} f'_h(S_p) + \frac{1}{\lambda^2} f''_h(S_p) \right] - \mathbb{E} \left[ \frac{2}{\lambda^2} f'_h(S_p^L) + \frac{1}{\lambda^2} f''_h(S_p^L) \right] \right)
\]
Thus, using (21), we have

\[ E \left[ 2f'_{h}(S_{p}) + f''_{h}(S_{p}) \right] - E \left[ 2f'_{h}(S_{p}^{L}) + f''_{h}(S_{p}^{L}) \right] \]

(44)

Assume that \( \|h^{(k)}\| \leq 1 \) for \( k = 3, 4 \). Then, by using (39) on (44), we have

\[ d_{W_{3}}(S_{p}, X) \leq \frac{1}{\lambda^{2} - 1} \left( 1 + \frac{1}{2\lambda} \right) E \left| S_{p} - S_{p}^{L} \right| \]
\[ \leq \frac{2\lambda + 1}{2\lambda(\lambda^{2} - 1)} \frac{\lambda^{2}}{6} \rho p^{\frac{1}{2}} \quad \text{(by Lemma 11.6)} \]
\[ = \frac{\rho(2\lambda + 1)}{12(\lambda^{2} - 1)} p^{\frac{1}{2}}, \]

the desired conclusion follows. \( \square \)

**Remark 4.9.** The reference [34] shows that \( d_{BL}(S_{p}, X) \leq C_{1}(\lambda, \rho)p^{\frac{1}{2}} \), where \( C_{1}(\lambda, \rho) \) is some positive constant depends on \( \lambda \) and \( \rho \), and \( d_{BL} \) denotes the bounded Lipschitz distance. Gaunt [20] also proves that \( W_{1}(S_{p}, X) \leq C_{2}(\lambda, \rho)p^{\frac{1}{2}} \), where \( C_{2}(\lambda, \rho) \) is a positive constant depends on \( \lambda \) and \( \rho \). In comparison with the rates derived in [34, 20], we see that the \( O(p^{\frac{1}{2}}) \) rate in (11) is optimal.

### 4.3. Six moment theorem for the symmetric variance gamma approximation of double Wiener-Itô integrals.

Recently, Eichelsbacher and Thäle [14] extended the Malliavin-Stein method for variance gamma distributions. Here, we obtain an upper bound for the symmetric variance gamma approximation of general functional of an isonormal Gaussian process in the Wasserstein-type distance. We also prove the six moment theorem for the symmetric variance gamma approximation of double Wiener-Itô integrals.

Let us first introduce some notation (see, the book [32] for detailed discussion). Let \( \mathbb{D}^{p,q} \) be the Banach space of all functions in \( L^{q}(\gamma) \), where \( \gamma \) is the standard Gaussian measure, whose Malliavin derivative up to order \( p \) also belong to \( L^{q}(\gamma) \). Let \( \mathbb{D}^{\infty} \) be the class of infinitely many times Malliavin differentiable random variables. We also introduce the well-known \( \Gamma \)-operators (see, [33]). For a random variable \( G \in \mathbb{D}^{\infty} \), we define \( \Gamma_{1}(G) = G \), and for every \( j \geq 2 \),

\[ \Gamma_{j}(G) = \langle DG, -DL^{-1}\Gamma_{j-1}(G) \rangle_{\mathcal{B}}. \]

Here \( D \) is the Malliavin derivative, \( L^{-1} \) is the pseudo-inverse of the infinitesimal generator of the Ornstein-Uhlenbeck semigroup, and \( \mathcal{B} \) is a real separable Hilbert space. Lastly, for \( f \in \mathcal{B}^{\circ 2} \), we write \( I_{2}(f) \) for the double Wiener-Itô integral of \( f \).

**Theorem 4.10.** Let \( G \in \mathbb{D}^{2,4} \) be such that \( \mathbb{E}(G) = 0 \) and let \( X \sim VGD_{1}(0, \alpha, \lambda, \lambda) \). Then, for \( \lambda > 1 \)

\[ d_{W_{3}}(G, X) \leq \frac{\lambda^{2}}{3(\lambda^{2} - 1)} E \left| \frac{1}{\lambda^{2}} G - \Gamma_{3}(G) \right| + \frac{\lambda^{2}}{2(\lambda^{2} - 1)} \left| \frac{2\alpha}{\lambda^{2}} - \mathbb{E}(\Gamma_{2}(G)) \right| \]

(45)

**Proof.** Recall first that \( VGD_{1}(0, \alpha, \lambda, \lambda) \overset{d}{=} \text{TSD}(\alpha, 0, \lambda; \alpha, 0, \lambda) \). Thus, using (21), we have

\[ \mathbb{E}(h(G) - h(X)) = \mathbb{E} \left( -Gf_{h}(G) + \int_{\mathbb{R}} f_{h}(G + u) \nu(du) \right) \]
\[ = \mathbb{E} \left( -Gf_h(G) + \alpha \int_0^\infty (f_h(G + u) - f_h(G - u)) e^{-\lambda u} du \right) \]
\[ = \mathbb{E} \left( -Gf_h(G) + \frac{\alpha}{\lambda^2} \int_0^\infty (f_h'(G + u) + f_h'(G - u)) e^{-\lambda u} du \right) \]
\[ = \mathbb{E} \left( -Gf_h(G) + \frac{2\alpha}{\lambda^2} f_h'(G) + \frac{\alpha}{\lambda^2} \int_0^\infty (f_h''(G + u) - f_h''(G - u)) e^{-\lambda u} du \right) \]
\[ = \mathbb{E} \left( -Gf_h(G) + \frac{2\alpha}{\lambda^2} f_h'(G) + \frac{1}{\lambda^2} Gf_h''(G) \right) + \frac{1}{\lambda^2} \mathbb{E} \left( -Gf_h''(G) + \alpha \int_0^\infty (f_h''(G + u) - f_h''(G - u)) e^{-\lambda u} du \right) \]
\[ = \mathbb{E} \left( -Gf_h(G) + \frac{2\alpha}{\lambda^2} f_h'(G) + \frac{1}{\lambda^2} Gf_h''(G) \right) + \frac{1}{\lambda^2} \mathbb{E} \left( -Gf_h''(G) + \int \frac{1}{\lambda^2} Gf_h''(G + u) \nu(du) \right) \] (46)

Taking \( \sup_{h \in \mathcal{H}_3} \) on both side of (46) and rearranging the terms, we have
\[ \left( 1 - \frac{1}{\lambda^2} \right) d_{W_3}(G, X) \leq \sup_{h \in \mathcal{H}_3} \left| \mathbb{E} \left( -Gf_h(G) + \frac{2\alpha}{\lambda^2} f_h'(G) + \frac{1}{\lambda^2} Gf_h''(G) \right) \right| . \] (47)

Let \( f : \mathbb{R} \to \mathbb{R} \) be a twice differentiable function with bounded first and second derivative. Then, it was shown in the proof of [14, Theorem 4.1] that
\[ \left| \mathbb{E} \left( -Gf(G) + \frac{2\alpha}{\lambda^2} f'(G) + \frac{1}{\lambda^2} f''(G) \right) \right| = \left| \mathbb{E} \left( f''(G) \left( \frac{1}{\lambda^2} - \Gamma_3(G) \right) + f'(G) \left( \frac{2\alpha}{\lambda^2} - \mathbb{E} \left( \Gamma_2(G) \right) \right) \right) \right| \leq \| f'' \| \mathbb{E} \left( \frac{1}{\lambda^2} - \Gamma_3(G) \right) + \| f' \| \mathbb{E} \left( \frac{2\alpha}{\lambda^2} - \mathbb{E} \left( \Gamma_2(G) \right) \right) \] (48)

Using (48) in (47), and rearranging the terms, we have
\[ d_{W_3}(G, X) \leq \frac{\lambda^2}{\lambda^2 - 1} \left( \| f'' \| \mathbb{E} \left( \frac{1}{\lambda^2} - \Gamma_3(G) \right) + \| f' \| \mathbb{E} \left( \frac{2\alpha}{\lambda^2} - \mathbb{E} \left( \Gamma_2(G) \right) \right) \right) . \] (49)

Assume that \( \| h^{(k)} \| \leq 1 \), for \( k = 2, 3 \). Then, by using the estimates (47) in (49), we get our desired result. \( \square \)

**Remark 4.11.** Eichelsbacher and Thäle [14, Theorem 4.1] provide an upper bound in the Wasserstein distance for variance gamma approximation of general functionals of an isonormal Gaussian process in terms of two constants. In [14], authors did not mention about these constants explicitly. Recently, Gaunt [20, 21] establish an upper bound in the Wasserstein distance for variance gamma approximation of general functionals of an isonormal Gaussian process, and obtain explicit constants in the main result of the article [14, Theorem 4.1]. Note that, our bound in the Wasserstein-type distance also include the explicit constants for the symmetric variance gamma approximation.
The following corollary immediately follows for the symmetric variance-gamma approximation of double Wiener-Itô integrals, that leads to the six moment theorem.

**Corollary 4.12.** Let $G_n = I_2(f_n)$ with $f_n \in S^\circ 2$, $n \geq 1$ and $X \sim VGD_1(0, \alpha, \lambda, \lambda)$. Then, for $\lambda > 1$

$$d_{W_3}(G_n, X) \leq \frac{\lambda^2}{3(\lambda^2 - 1)} \left( \frac{1}{120} \kappa_6(G_n) - \frac{1}{3\lambda^2} \kappa_4(G_n) + \frac{1}{4} (\kappa_3(G_n))^2 + \frac{1}{\lambda^2} \kappa_2(G_n) \right)^{\frac{1}{2}}$$

$$+ \frac{\lambda^2}{2(\lambda^2 - 1)} \left| \frac{2\alpha}{\lambda^2} - \kappa_2(G_n) \right|.$$  \hspace{1cm} (50)

**Proof.** It is shown in [33, Lemma 4.2 and Theorem 4.3] that

$$E(\Gamma_2(G_n)) = \kappa_2(G_n).$$  \hspace{1cm} (51)

It is also justified in [14, Theorem 4.1] that

$$E \left| \frac{1}{\lambda^2} G_n - \Gamma_3(G_n) \right| \leq \left( E \left( \frac{1}{\lambda^2} G_n - \Gamma_3(G_n) \right)^2 \right)^{\frac{1}{2}}.$$  \hspace{1cm} (52)

It is also shown in the proof of [14, Theorem 5.8] that

$$E \left( \frac{1}{\lambda^2} G_n - \Gamma_3(G_n) \right)^2 = \frac{1}{120} \kappa_6(G_n) - \frac{1}{3\lambda^2} \kappa_4(G_n) + \frac{1}{4} (\kappa_3(G_n))^2 + \frac{1}{\lambda^2} \kappa_2(G_n).$$  \hspace{1cm} (53)

Using (52) in (53), we get

$$E \left| \frac{1}{\lambda^2} G_n - \Gamma_3(G_n) \right| \leq \left( \frac{1}{120} \kappa_6(G_n) - \frac{1}{3\lambda^2} \kappa_4(G_n) + \frac{1}{4} (\kappa_3(G_n))^2 + \frac{1}{\lambda^2} \kappa_2(G_n) \right)^{\frac{1}{2}}.$$  \hspace{1cm} (54)

Using (53) and (54) in the RHS of (45), we get (50), as desired. \hfill \Box

**Remark 4.13.** Eichelsbacher and Thäle [14, Corollary 5.10] provide an upper bound in the Wasserstein distance for variance gamma approximation of double Wiener-Itô integrals in terms of two constants, and first six cumulants. In [14], authors did not mention about these constants explicitly. Recently, Gaunt [20, 21] establish an upper bound in the Wasserstein distance for variance gamma approximation of double Wiener-Itô integrals, and obtain explicit constants in the main result of the article [14, Corollary 5.10]. Note that, our bound in the Wasserstein-type distance also include the explicit constants, and the first six cumulants for the symmetric variance gamma approximation.

**Appendix A.**

In this section, we derive some properties of TSD, and prove some results that are used in the previous sections. We also prove the results that we stated in the main text without proofs.

A.1. **Self-decomposability.** Here, we discuss self-decomposable property of TSD that are used to estimate the properties of the solution to Stein equation.

**Lemma A.1.** Let $X \sim TSD(\alpha^+, \beta^+, \lambda^+; \alpha^-, \beta^-, \lambda^-)$. Then, $X$ has the self-decomposable property.
Proof. In view of Lévy measure (4), we can express the cf of TSD (3) as

$$\phi(t) = \exp \left\{ \int_{\mathbb{R}} \left( e^{itu} - 1 \right) \frac{k(u)}{u} \, du \right\}, \quad t \in \mathbb{R}, \quad (A.1)$$

where $k : \mathbb{R} \to \mathbb{R}$ denotes the function

$$k(u) = \frac{\alpha^+}{|u|^\beta} e^{-\lambda^+ u} 1_{(0, \infty)}(u) - \frac{\alpha^-}{|u|^\beta} e^{-\lambda^- u} 1_{(-\infty, 0)}(u), \quad u \in \mathbb{R}. \quad (A.2)$$

Note that $k \geq 0$ on $(0, \infty)$ and $k \leq 0$ on $(-\infty, 0)$. Again, $k$ is strictly decreasing on $(-\infty, 0)$ and $(0, \infty)$. It is an immediate consequence of (Sato [37, Corollary 15.11]) that TSD are self-decomposable.

A.2. Frullani improper integral. Here we discuss a special type of improper integral, so-called Frullani integral that is used to prove similarity of various cf representations of VGD. We prove a lemma on this integral

**Lemma A.2.** Let a function $g : (0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$ and let $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to \infty} g(x)$ exist finitely, and the limiting values are $g_0$ and $g_\infty$ respectively. Then

$$\int_0^\infty \frac{g(ax) - g(bx)}{x} \, dx = (g_0 - g_\infty) \log \left( \frac{b}{a} \right),$$

where $b, a > 0$.

**Proof.** We prove this lemma by extending the integrand, and using Fubini’s theorem. Note that

$$\int_0^\infty \frac{g(ax) - g(bx)}{x} \, dx = \int_0^\infty \int_0^a g'(xt) \frac{1}{x} \, dt \, dx = \int_0^a \int_0^\infty \frac{g'(xt)}{x} \, dt \, dx = \int_0^a (g_\infty - g_0) \frac{1}{x} \, dx = (g_0 - g_\infty) \log \left( \frac{b}{a} \right).$$

Hence the lemma is proved.

As for example, consider the integral $I = \int_0^\infty \frac{e^{-\lambda_1 x} - e^{-\lambda_2 x}}{x} \, dx$, where $\lambda_1, \lambda_2 > 0$.

Then by the above lemma one can easily obtain $I = \log \left( \frac{\lambda_2}{\lambda_1} \right)$.

Next, we prove a technical result used in the previous section.

**Proposition A.3.** Let $x, z \in \mathbb{R}$. Then, for all $t \geq 0$,

$$\lim_{t \to 0^+} \frac{1}{t} \left( e^{izx(e^{-t} - 1)} - 1 \right) = \left( -x + \int_{\mathbb{R}} u e^{izu} \nu(du) \right)(iz). \quad (A.3)$$

**Proof.** Recall from Section 3 if $X$ be a tempered stable random variable, we write

$$\phi_t(z) = \frac{\phi(z)}{\phi(e^{-iz})} = \exp \left( \int_{\mathbb{R}} \left( e^{izu} - e^{iue^{-iz}} \right) \nu(du) \right), \quad t \geq 0 \text{ (see [23]).}$$
Now, let us consider LHS of (A.3),

\[ \lim_{t \to 0^+} \frac{1}{t} \left( e^{iz(x e^{-t} - 1)} \phi_t(z) - 1 \right) = \lim_{t \to 0^+} \frac{1}{t} \left( \exp \left( iz(x e^{-t} - 1) + \int_{\mathbb{R}} (e^{izu} - e^{iue^{-t}}) \nu(du) \right) - 1 \right) = \lim_{t \to 0^+} \frac{1}{t} \left( \exp (A + iB) - 1 \right), \tag{A.4} \]

where

\[ A = \int_{\mathbb{R}} (\cos(uz) - \cos(uz e^{-t})) \nu(du) \text{ and} \]

\[ B = \left( xz(e^{-t} - 1) + \int_{\mathbb{R}} (\sin(uz) - \sin(uz e^{-t})) \nu(du) \right). \]

Applying Euler’s formula for complex exponential to (A.4), and rearranging the limits, we have

\[ \lim_{t \to 0^+} \frac{1}{t} \left( e^{izx(e^{-t} - 1)} \phi_t(z) - 1 \right) = \lim_{t \to 0^+} \frac{e^A \cos(B) - 1}{t} + i \lim_{t \to 0^+} \frac{e^A \sin(B)}{t}. \tag{A.5} \]

It is easy to show that at \( t = 0 \), \( e^A \cos(B) - 1 = 0 \) and \( e^A \sin(B) = 0 \). Thus, on applying L’Hospital rule on (A.5), taking limit as \( t \) tend to \( 0^+ \), and using dominated convergence theorem, we have

\[ \lim_{t \to 0^+} \frac{1}{t} \left( e^{izx(e^{-t} - 1)} \phi_t(z) - 1 \right) = \left( \int_{\mathbb{R}} iu \sin(zu) \nu(du) - x + \int_{\mathbb{R}} u \cos(zu) \nu(du) \right) \left( iz \right) = \left( -x + \int_{\mathbb{R}} u \cos(zu) + i \sin(zu) \nu(du) \right) \left( iz \right) = \left( -x + \int_{\mathbb{R}} u e^{izu} \nu(du) \right) \left( iz \right) \]

This completes the proof. \( \square \)

### A.3. Further proofs.

#### A.3.1. Proof of the converse of Theorem 3.1

For any \( s \in \mathbb{R} \), let \( f(x) = e^{isx} \), \( x \in \mathbb{R} \), then (7) becomes

\[ \mathbb{E} X e^{isX} = \mathbb{E} \int_{\mathbb{R}} e^{is(X + u)} \nu(du) = e^{isX} \int_{\mathbb{R}} e^{isu} \nu(du). \]

Setting \( \phi(s) = \mathbb{E} e^{isX} \), Then

\[ \phi'(s) = i\phi(s) \int_{\mathbb{R}} e^{isu} \nu(du). \tag{A.6} \]

Integrating out the real and imaginary parts of (A.6) leads, for any \( t \geq 0 \), to

\[ \phi(t) = \exp \left( i \int_{0}^{t} \int_{\mathbb{R}} e^{isu} \nu(du) ds \right) \]
where \( \mathcal{P} \) defined as Dirac-

A similar computation can be done for \( t \leq 0 \). Hence the converse part of the theorem is proved.

**A.3.2. Proof of Proposition 3.7** For each \( f \in \mathcal{F}_X \), it is easy to show that \( P_0 f(x) = f(x) \) and \( \lim_{t \to \infty} P_t(f)(x) = \int_R f(x) F_X(dx) \). Now, for any \( s, t \geq 0 \), we have

\[
\phi_{t+s}(z) = \frac{\phi(z)}{\phi(e^{-(t+s)} z)} = \frac{\phi(e^{-s} z)}{\phi(e^{-(t+s)} z)} = \phi_s(z) \phi_t(e^{-s} z) \tag{A.7}
\]

Using \([A.7]\), we have

\[
LHS = P_{t+s}(f)(x) = \frac{1}{2\pi} \int_R \hat{f}(z) e^{izxe^{-(t+s)}} \phi_{t+s}(z) \, dz
= \frac{1}{2\pi} \int_R \hat{f}(z) e^{izxe^{-(t+s)}} \phi_s(z) \phi_t(e^{-s} z) \, dz. \tag{A.8}
\]

We need to show that \( P_{t+s}(f)(x) = P_t(P_s f)(x) \) for all \( f \in \mathcal{F}_X \). Let \( \delta \) be the Dirac-\( \delta \) measure.

**RHS = \( P_t(P_s f)(x) \)**

\[
= \frac{1}{2\pi} \int_R \overline{P_s(f)(z)} e^{izxe^{-t}} \phi_t(z) \, dz
= \frac{1}{2\pi} \int_R \left( \int_R e^{-ivz} P_s(f)(v) \, dv \right) e^{izxe^{-t}} \phi_t(z) \, dz
= \frac{1}{(2\pi)^2} \int_R \left( \int_R e^{-ivz} \left( \int_R \hat{f}(w) e^{iuv} \phi_s(w) \, dw \right) \, dv \right) e^{izxe^{-t}} \phi_t(z) \, dz
= \frac{1}{(2\pi)^2} \int_R \hat{f}(w) \phi_s(w) \int_R e^{izxe^{-t}} \phi_t(z) \left( \int_R e^{iv(e^{-s} w - z)} \, dv \right) \, dz \, dw
= \frac{1}{(2\pi)^2} \int_R \hat{f}(w) \phi_s(w) \int_R e^{izxe^{-t}} \phi_t(z) 2\pi \delta(e^{-s} w - z) \, dz \, dw
= \frac{1}{2\pi} \int_R \hat{f}(w) \phi_s(w) e^{izxe^{-t}} \phi_t(e^{-s} w) \, dz
= \frac{1}{2\pi} \int_R \hat{f}(z) e^{izxe^{-(t+s)}} \phi_s(z) \phi_t(e^{-s} z) \, dz
= P_{t+s}(f)(x) = LHS \text{ (from } [A.8]),
\]

and the desired conclusion follows.

**A.3.3. Remaining proof of Theorem 3.5** Let us consider a function \( g_h : \mathbb{R} \to \mathbb{R} \) defined as

\[
g_h(x) = -\int_0^\infty (P_t(h)(x) - Eh(X)) \, dt, \ h \in H_r,
\]

where \( (P_t)_{t \geq 0} \) is the semigroup defined in \([22]\).
Using (21), we have

\[
|P_t(h)(x) - \mathbb{E}h(X)| = \left| \int_{\mathbb{R}} h(y + e^{-t}x) F_{X(t)}(dy) - \int_{\mathbb{R}} h(y) F_X(dy) \right|
\]

\[
= \left| \int_{\mathbb{R}} (h(y + e^{-t}x) - h(y)) F_{X(t)}(dy) \right|
+ \int_{\mathbb{R}} h(y) F_{X(t)}(dy) - \int_{\mathbb{R}} h(y) F_X(dy) \right|
\]

\[
\leq e^{-t|x|} h'(1) + \int_{\mathbb{R}} \phi_t(z) - \phi(z) dz
\]

using (A.10) in (A.9), one can easily show

\[
\leq e^{-t|x|} |h(1)| + \int_{\mathbb{R}} \phi_t(z) - \phi(z) dz
\]

(A.9)

Now, let us calculate an upper bound between the difference of two characteristic functions \(\phi_t\) and \(\phi\). For all \(t > 0\) and \(z \in \mathbb{R}\),

\[
|\phi_t(z) - \phi(z)| = \left| \frac{\phi(z)}{\phi(e^{-t}z)} - \phi(z) \right| \leq |\phi(\epsilon^{t} z) - 1| = |e^{\omega(z)} - 1|,
\]

where \(\omega(z) = \int_{\mathbb{R}} (e^{izx} - 1) \nu(du)\). Note that the function \(z \rightarrow e^{\omega(z)}\) is a characteristic function for all \(s \in (0, \infty)\). Thus, for all \(z \in \mathbb{R}\) and \(t > 0\),

\[
|\phi_t(z) - \phi(z)| \leq \left| \int_{0}^{1} \frac{d}{ds}(\exp(s\omega(z)))ds \right|
\]

\[
\leq |\omega_t(z)|
\]

\[
\leq \max\{e^{-t\beta^+}, e^{-t\beta^-}\} \int_{\mathbb{R}} (e^{izu} - 1) \tilde{\nu}(du)
\]

\[
\leq \max\{e^{-t\beta^+}, e^{-t\beta^-}\} C (1 + |z|^2), \; C > 0,
\]

(A.10)

where \(\tilde{\nu}(du) = \left(\frac{\alpha^+}{\alpha^+ + \beta^+} \mathbf{1}_{(-\infty, 0)}(u) + \frac{\alpha^-}{\alpha^- + \beta^-} \mathbf{1}_{(-\infty, 0)}(u)\right) du\), and the last inequality is followed by [1] p.30, Ex. 1.2.16. Using (A.10) in (A.9), one can easily show that \(\int_{0}^{\infty} |P_t(h)(x) - \mathbb{E}h(X)| dt < \infty\). Hence, \(g_h(x)\) is well-defined. By dominated convergence theorem, we see that \(g_h\) is differentiable and

\[
g'_h(x) = -\lim_{\zeta \rightarrow \infty} \frac{d}{dx} \int_{0}^{\zeta} (P_t(h)(x) - \mathbb{E}h(X)) dt
\]

\[
= -\lim_{\zeta \rightarrow \infty} \frac{d}{dx} \int_{0}^{\zeta} \left( \int_{\mathbb{R}} h(xe^{-t} + u) F_{X(t)}(du) \right) dt
\]

\[
= -\int_{0}^{\infty} e^{-t} \int_{\mathbb{R}} h'(u + xe^{-t}) F_{X(t)}(du) dt = f_h(x),
\]

the desired conclusion follows.

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