Size of local finite field Kakeya sets

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Abstract. Let \( F \) be a finite field consisting of \( q \) elements and let \( n \geq 1 \) be an integer. In this paper, we study the size of local Kakeya sets with respect to subsets of \( F^n \) and obtain upper and lower bounds for the minimum size of a (local) Kakeya set with respect to an arbitrary set \( T \subseteq F^n \).

Keywords: Local Kakeya Sets, Minimum Size, Probabilistic Method

1 Introduction

The study of finite field Kakeya sets is of interest from both theoretical and application perspectives. Letting \( F \) be a finite field containing \( q \) elements and \( n \geq 1 \) be an integer, Wolff [1] used counting arguments and planes to estimate that the minimum size of a global Kakeya set covering all vectors in \( F^n \) grows at least as \( q^{n/2} \). Later Dvir [2] used polynomial methods to obtain sharper bounds (of the form \( C \cdot q^n \)) on the minimum size of global Kakeya sets and for further improvements in the multiplicative constant \( C \), we refer to Saraf and Sudan [3].

In this paper, we are interested in studying local Kakeya sets with respect to subsets of \( F^n \). Specifically, in Theorem 1, we obtain upper and lower bounds for the minimum size of a Kakeya set with respect to a subset \( T \subseteq F^n \).

The paper is organized as follows. In Section 2, we describe local Kakeya sets and prove our main result (Theorem 1) regarding the minimum size of a local Kakeya set.

2 Local Kakeya sets

Let \( F \) be a finite field containing \( q \) elements and for \( n \geq 1 \) let \( F^n \) be the set of all \( n \)-tuple vectors with entries belonging to \( F \).

We say that a set \( K \subseteq F^n \) is a Kakeya set with respect to the vector \( x = (x_1, \ldots, x_n) \in F^n \) if there exists \( y = y(x) \in F^n \) such that the line

\[
L(x, y) := \bigcup_{a \in F} \{y + a \cdot x\} \subseteq K,
\]

where \( a \cdot x := (ax_1, \ldots, ax_n) \). For a set \( T \subseteq F^n \), we say that \( K \subseteq F^n \) is a Kakeya set with respect to \( T \) if \( K \) is a Kakeya set with respect to every vector \( x \in T \).

The following result describes the minimum size of local Kakeya sets.
Theorem 1. Let $\mathcal{T} \subseteq \mathbb{F}^n$ be any set with cardinality $\# \mathcal{T}$ an integer multiple of $q - 1$ and let $\theta(\mathcal{T})$ be the minimum size of a Kakeya set with respect to $\mathcal{T}$. We then have that

$$q \sqrt{M} + \min\left(0, q - \sqrt{M}\right) \leq \theta(\mathcal{T}) \leq q^n \left(1 - \left(1 - \frac{1}{q^{n-1}}\right)^{M-1}\right)$$

(2.2)

where $M := \frac{\# \mathcal{T}}{q - 1}$.

For example suppose $M = \epsilon \cdot \left(\frac{q^n - 1}{q - 1}\right)$ for some $0 < \epsilon \leq 1$. From the lower bound in (2.2), we then get that $\theta(\mathcal{T})$ grows at least of the order of $q^n/2$. Similarly, using the fact that $1 - x \geq e^{-x - x^2}$ for $0 < x \leq \frac{1}{2}$, we get that

$$\left(1 - \frac{1}{q^{n-1}}\right)^{M-1} \geq \exp\left(-\frac{M - 1}{q^{n-1}}\left(1 + \frac{1}{q^{n-1}}\right)\right) \geq e^{-\Delta},$$

where $\Delta := \frac{\epsilon}{q - 1} \left(1 + \frac{1}{q^{n-1}}\right)$. From (2.2) we then get that

$$\theta(\mathcal{T}) \leq q + q^n \left(1 - e^{-\Delta}\right).$$

In what follows we prove the lower bound and the upper bound in Theorem 1 in that order.

Proof of Lower Bound in Theorem 1

The proof of the lower bound consists of two steps. In the first step, we extract a subset $\mathcal{N}$ of $\mathcal{T}$ containing vectors that are non-equivalent. In the next step, we then use a high incidence counting argument similar to Wolff (1999) and estimate the number of vectors in a Kakeya set $\mathcal{K}$ with respect to $\mathcal{N}$.

Step 1: Say that vectors $x_1, x_2 \in \mathbb{F}^n$ are equivalent if $x_1 = a \cdot x_2$ for some $a \in \mathbb{F} \setminus \{0\}$. We first extract a subset of vectors in $\mathcal{T}$ that are pairwise non-equivalent. Pick a vector $x_1 \in \mathcal{N}$ and throw away all the vectors in $\mathcal{T}$ that are equivalent to $x_1$. Next, pick a vector $x_2$ in the remaining set and again throw away the vectors that are equivalent to $x_2$. Since we throw away at most $q - 1$ vectors in each step, after $r$ steps, we are left with a set of size at least $\# \mathcal{T} - r(q - 1)$. Thus the procedure continues for

$$M = \frac{\# \mathcal{T}}{q - 1}$$

(2.3)

steps, assuming henceforth that $M$ is an integer.

Let $\mathcal{N} = \{x_1, \ldots, x_M\} \subseteq \mathcal{T}$ be a set of size $M$ and let $\mathcal{K}$ be a Kakeya set with respect to $\mathcal{N}$, of minimum size. By definition (see (2.1)), there are vectors $y_1, \ldots, y_M$ in $\mathbb{F}^n$ such that the line $L(x_i, y_i) \subseteq \mathcal{K}$ for each $1 \leq i \leq M$. Moreover, since $\mathcal{K}$ is of minimum size, we must have that

$$\mathcal{K} = \bigcup_{j=1}^{M} \{L(x_i, y_i)\}.$$
**Step 2:** To estimate the number of distinct vectors in $K$, suppose first that each vector in $K$ belongs to at most $t$ of the lines in $\{L(x_i, y_i)\}_{1 \leq i \leq M}$. Since each line $L(x_i, y_i)$ contains $q$ vectors, we get that the total number of vectors in $K$ is bounded below by

$$\#K \geq \frac{Mq}{t}. \quad (2.5)$$

Suppose now that there exists a vector $v$ in $K$ that belongs to at least $t + 1$ lines $L_1, \ldots, L_{t+1} \subseteq \bigcup_{1 \leq i \leq M} \{L(x_i, y_i)\}$. Because the vectors $\{x_i\}_{1 \leq i \leq M}$ are not equivalent, any two lines $L(x_i, y_i)$ and $L(x_j, y_j)$ must have at most one point of intersection. To see this is true suppose there were scalars $a_1 \neq a_2$ and $b_1 \neq b_2$ in $\mathbb{F}$ such that

$$y_1 + a_1 \cdot x_1 = y_2 + b_1 \cdot x_2$$

for $i = 1, 2$. Subtracting the equations we would then get $(a_1 - a_2) \cdot x_1 = (b_1 - b_2) \cdot x_2$, contradicting the fact that $x_1$ and $x_2$ are not equivalent.

From the above paragraph, we get that any two lines in $\{L_i\}_{1 \leq i \leq t+1}$ have exactly one point of intersection, the vector $v$. Since each line $L_i$ contains $q$ vectors, the total number of vectors in $\{L_i\}_{1 \leq i \leq t+1}$ equals $(q - 1)(t + 1) + 1$, all of which must be in $K$. From (2.5), we therefore get that

$$\#K \geq \min\left(\frac{Mq}{t}, (q - 1)(t + 1) + 1\right)$$

and setting $t = \sqrt{M}$, we get

$$\#K \geq q\sqrt{M} + \min\left(0, q - \sqrt{M}\right).$$

From the expression for $M$ in (2.3), we then get (2.2). \hfill \blacksquare

**Proof of Upper Bound in Theorem 1**

We use the probabilistic method. Let $\{x_1, \ldots, x_M\}$ be the set of non-equivalent vectors obtained in Step 1 in the proof of the lower bound with $M = \frac{\#T}{q - 1}$ (see (2.3)). Let $Y_1, \ldots, Y_M$ be independently and uniformly randomly chosen from $\mathbb{F}^n$ and for $1 \leq i \leq M$, set

$$S_i := \bigcup_{j=1}^{i} \{L(x_j, y_j)\},$$

where $L(x, y)$ is the line containing the vectors $x$ and $y$ as defined in (2.1).

By construction, the set $S_M$ forms a Kakeya set with respect to $T$. To estimate the expected size of $S_M$, we use recursion. For $1 \leq i \leq M$, let $\theta_i := \mathbb{E}\#S_i$ be the expected size of $S_i$. Given $S_{i-1}$, the probability that a vector
chosen from $\mathbb{F}^n$, uniformly randomly and independent of $\mathcal{S}_{i-1}$, belongs to the set $\mathcal{S}_{i-1}$ is given by $p_i := \frac{\theta_i}{q^n}$. Therefore

$$
\mathbb{E}\# \left( L(\mathbf{x}_i, \mathbf{Y}_i) \cap \mathcal{S}_{i-1} \right) = q \cdot \mathbb{E} \left( \frac{\#\mathcal{S}_{i-1}}{q^n} \right)
$$

and so

$$
\theta_i = \theta_{i-1} + q \left( 1 - \frac{\theta_{i-1}}{q^n} \right) = \theta_{i-1} \left( 1 - \frac{1}{q^{n-1}} \right) + q. \quad (2.6)
$$

Letting $a = 1 - \frac{1}{q^n}$, and using (2.6) recursively, we get

$$
\theta_i = a^{i-1} \cdot \theta_1 + q \cdot (1 + a + \ldots + a^{i-2}) = a^{i-1} \cdot \theta_1 + \frac{q(1 - a^{i-1})}{1 - a}.
$$

Using $\theta_1 = q$, we then get that

$$
\theta_M = a^{M-1} \cdot q + q^n \left( 1 - \left( 1 - \frac{1}{q^{n-1}} \right)^{M-1} \right)
\leq q + q^n \left( 1 - \left( 1 - \frac{1}{q^{n-1}} \right)^{M-1} \right).
$$

This implies that there exists a Kakeya set with respect to $\mathcal{T}$ of size at most $\theta_M$. ■

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