Non-Hermitian Hamiltonian approach to quantum transport in disordered networks with sinks: validity and effectiveness

Giulio G. Giusteri,1, 2 Francesco Mattiotti,1 and G. Luca Celardo1, 3

1 Dipartimento di Matematica e Fisica and Interdisciplinary Laboratories for Advanced Materials Physics, Università Cattolica del Sacro Cuore, via Musei 41, I-25121 Brescia, Italy
2 International Research Center on Mathematics & Mechanics of Complex Systems, via XIX marzo 1, I-04012 Cisterna di Latina, Italy
3 Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via Bassi 6, I-27100, Pavia, Italy

(Dated: November 4, 2014)

We investigate the validity of the non-Hermitian Hamiltonian approach in describing quantum transport in disordered tight-binding networks connected to external environments, acting as sinks. Usually, non-Hermitian terms are added to such networks to summarize the effects of the coupling to the sinks. Our general conclusion is that the non-Hermitian approach is valid when the energy dependence of the coupling to the external environments is smooth in the energy range spanned by the eigenstates of the network. As a specific example, we consider a ring of sites connected to a central one-dimensional lead. The lead acts as a sink which absorbs the excitation initially present in the ring. The coupling strength to the lead controls the opening of the ring system. This model has been widely discussed in literature in the context of light-harvesting systems. Here we derive a non-Hermitian model, able to describe the transport properties of the system, and we discuss its limit of validity, by a comparison with the analysis of the full Hermitian model. We also analyze the effect of static disorder on transport, showing that a non-Hermitian model is still valid in presence of static disorder when the intensity of the latter is less then the energy band in the lead. Under that condition, we show that results about the interplay of opening and disorder, previously obtained within the non-Hermitian Hamiltonian approach, remain valid when the full Hermitian model in presence of disorder is considered.

PACS numbers: 71.35.-y, 72.15.Rn, 05.60.Gg

I. INTRODUCTION

Open quantum systems are nowadays at the center of many research fields in physics, ranging from quantum computing to transport in nano- and meso-scale solid state system as well as biological aggregates. In particular, charge/excitation transport in the quantum coherent regime can be considered one of the central subjects in modern solid-state physics [1, 2]. Transport properties depend strongly on the degree of openness of the system. In important applications, the effect of the opening is large, and cannot be treated perturbatively. The analysis of open quantum systems beyond the perturbative regime is often difficult due to the presence of infinitely many degrees of freedom. Thus, a consistent way to take the effect of the opening into account for arbitrary coupling strength between the system and the external world is highly desirable.

In a typical situation, we have a discrete quantum system coupled to an external environment characterized by a continuum of states. Elimination of the continuum leads to an effective non-Hermitian Hamiltonian. This approach to open quantum systems has been shown to be a very effective tool in dealing also with the strong coupling regime [3, 7]. The non-Hermitian Hamiltonian approach offers several advantages: (i) it reduces an infinite dimensional problem to a finite dimensional one; (ii) it allows to compute conductance and the whole time-evolution of the relevant subsystem; (iii) the effects of interference between discrete states and the continuum, such as Superradiance or Fano resonances can be easily analyzed. Moreover, this approach can be considered a matrix generalization of the Fermi Golden Rule [8].

Tight-binding networks are often considered in literature to model quantum transport and their coupling with external environments, acting as sinks, is taken into account by adding non-Hermitian terms to the Hamiltonian [9]. Indeed, non-Hermitian models are more and more used to describe trapping or loss of excitation into transport channels of complex biological aggregates [9–12], but a proper justification of the employed non-Hermitian model is often overlooked.

Together with the coupling to a sink, such networks are usually coupled to other environments, which induce different kinds of disorder: static disorder (space-dependent) and dynamical disorder (time-dependent). When disorder is added to the system to take into account the effect of other environments, the strength of the coupling to the sink is usually assumed to be unaffected by the presence of disorder.

This assumption has been used both when dealing with dynamical disorder [13, 14] and with static disorder [12, 15, 16]. Specifically, some of the authors of this paper have previously analyzed the interplay of opening and static disorder in paradigmatic models of quantum transport, such as one-dimensional and three-
dimensional tight-binding models. Within the framework of the non-Hermitian Hamiltonian approach, it was found a novel cooperative regime characterized by the presence of subradiant hybrid states. Moreover, cooperative robustness to disorder has been shown to play an important role in the dynamics of quantum systems with sinks. Those results were obtained assuming the coupling to the sinks to be independent of the disorder strength, but this assumption must fail for large enough disorder.

In this paper we consider a tight-binding network composed by a ring-like structure coupled to a semi-infinite lead (Fig. 1). This model has been discussed in several publication in literature due to its relevance to light-harvesting complexes and to proposals of bio-engineered devices for photon sensing. Here we derive a non-Hermitian Hamiltonian able to describe the transport properties of the model. By comparing the results of the full Hermitian model with the results obtained with the non-Hermitian model we will elucidate the limit of validity of the assumption usually made in literature and mentioned above.

The purpose of the present paper is twofold: first, we want to assess the limit of validity for the use of a non-Hermitian Hamiltonian to model the transport properties in presence of a sink; secondly, we want to ascertain the reliability of this approach in reproducing the physics of the full Hermitian system in presence of disorder, with particular reference to the existence of subradiant hybrid states and cooperative robustness to static disorder.

In Sec. [II] we introduce the non-Hermitian Hamiltonian approach to open quantum systems; in Sec. [III] we present our Hermitian model and we derive the corresponding non-Hermitian Hamiltonian, showing, in Sec. [IV], the effects of Superradiance in such a system. We then analyze the validity and effectiveness of the non-Hermitian model in reproducing the dynamics of the Hermitian system, in both absence (Sec. [V]) and presence (Sec. [VI]) of diagonal disorder. A summary of the results and their implications for the modeling of quantum sinks is given in the concluding section.

II. DERIVATION OF THE NON-HERMITIAN HAMILTONIAN

In this Section we present a standard derivation of the non-Hermitian effective Hamiltonian. Alternative derivations can also be found in Refs. [3, 21].

Let us consider a discrete quantum system $A$, interacting with another system $B$, which represents the environment. We assume that the subspace $A$ is spanned by $N_A$ discrete states $|i\rangle$, while the subsystem $B$ represents the environment with states $|c, E\rangle$, where $c = 1, \ldots, M$ is a discrete quantum number, labeling the decay channels, and $E$ is another discrete quantum number, representing the energy (we will take the continuum limit of this quantum number later).

In order to derive the effective non-Hermitian Hamiltonian which describes the intrinsic system $A$, let us consider the projectors, within the Hilbert space of the total system $A + B$, on the two subsystems:

$$P_A = \sum_{i=1}^{N_A} |i\rangle\langle i|, \quad P_B = \sum_{c=1}^{M} \sum_{E=1}^{N_B} |c, E\rangle\langle c, E|.$$

Under the orthogonality conditions $\langle i | j \rangle = \delta_{ij}$, $\langle c, E | c', E' \rangle = \delta_{c,c'} \delta_{E-E'}$, $\langle i|E\rangle = 0$, we can rewrite the total Hamiltonian of the system as

$$H = H_0 + V = \begin{pmatrix} H_{AA} & 0 \\ 0 & H_{BB} \end{pmatrix} + \begin{pmatrix} 0 & H_{AB} \\ 0 & 0 \end{pmatrix},$$

where $H_{AA} = P_A H P_A$, $H_{AB} = P_A H P_B$ and similarly for the other terms.

We can now define the unperturbed propagator $G_0(x) = (x - H_0)^{-1}$ and the total propagator $G(x) = (x - H)^{-1}$, related by the Dyson equation

$$G(x) = G_0(x) + G_0(x) V G(x),$$

which gives rise to the following coupled equations for $G_{AA} = P_A G P_A$ and $G_{BA} = P_B G P_A$:

$$\begin{align*}
G_{AA} &= G_{0A} + G_{0A} H_{AB} G_{BA}, \\
G_{BA} &= G_{0B} H_{BA} G_{AA}.
\end{align*}$$

By substitution we obtain

$$G_{AA} = G_{0A} + G_{0A} H_{AB} G_{0B} H_{BA} G_{AA}$$

and, taking into account that $G_{0B} = (x - H_{BB})^{-1}$, we have

$$G_{AA}(x) = \frac{1}{x - H_{AA} - H_{AB} \frac{1}{x - H_{BB}} H_{BA}}.$$}

From this expression we obtain an effective Hamiltonian, which defines the propagator over the subspace $A$ and takes the form

$$H_{eff}(x) = H_{AA} + H_{AB} \frac{1}{x - H_{BB}} H_{BA}.$$
Taking the continuum limit
\[ \sum_{c,E} \rightarrow \sum_{c} \int \rho(E) \, dE \]
and using the identity
\[ \frac{1}{x - x_0} = \text{PV} \left( \frac{1}{x - x_0} \right) \pm i\pi \delta(x - x_0), \]
the non-Hermitian Hamiltonian can be written as
\[ H_{\text{eff}}^\pm(x) = H_{AA} + \Delta(x) \pm \frac{i}{2} Q(x), \quad (5) \]
where
\[ Q_{ij}(x) = 2\pi \sum_c \int A_i^*(E)(A_j^*(E))^* \rho(E) \delta(x - E) \, dE \]
and
\[ \Delta_{ij}(x) = \sum_c \text{PV} \int \frac{A_i^*(E)(A_j^*(E))^* \rho(E)}{x - E} \, dE, \quad (6) \]
with \( \rho(E) \) the density of states in the environment \( B \).

The ambiguity in the sign of the last term in Eq. (5), producing two distinct forms of the effective Hamiltonian, comes from the fact that the propagator \( G_{BB}^0 \), which appears in Eq. (8), can be associated with either the forward or the backward evolution: the minus sign gives the forward-in-time evolution, i.e.
\[ \theta(t - t_0)U(t, t_0) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left[ -\frac{i}{\hbar} x(t - t_0) \right] x - H_{\text{eff}}(x) \, dx, \quad (8) \]
while the plus sign gives the backward-in-time evolution, i.e.
\[ \theta(t_0 - t)U(t, t_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left[ -\frac{i}{\hbar} x(t - t_0) \right] x - H_{\text{eff}}(x) \, dx. \quad (9) \]
Note that \( U(t, t_0) \) is the projection through \( P_A \) of the full evolution operator of the system \( A + B \). Thus, if the initial state of the total system has components only on the intrinsic system \( A \), its evolution under the operator \( U(t, 0) \) in Eq. (10) gives the projection of the wave function of the total system over the intrinsic system.

From now on we will use the notation \( H_{\text{eff}}(x) \) for \( H_{\text{eff}}^+(x) \), referring to \( H_{\text{eff}}(x) \) as the effective Hamiltonian. To actually compute the evolution of an initial state, it is convenient to make use of the \((x\)-dependent\) basis of eigenstates of \( H_{\text{eff}}(x) \). Since \( H_{\text{eff}}(x) \) is in general non-Hermitian, due to the presence of the decay operator \( Q(x) \), its eigenvalues
\[ \mathcal{E}_r(x) = E_r(x) - \frac{i}{2} \Gamma_r(x), \quad r = 1, \ldots, N_A, \]
are complex, and it has left and right eigenstates
\[ H_{\text{eff}}(x)|r, x\rangle = \mathcal{E}_r(x)|r, x\rangle, \quad \langle \tilde{r}, x|H_{\text{eff}}(x) = \langle \tilde{r}, x|\mathcal{E}_r(x), \]
which are bi-orthogonal, i.e. the identity operator is given by
\[ 1 = \sum_r |r, x\rangle\langle \tilde{r}, x|. \]

Note that, when \( H_{\text{eff}}(x) \) is symmetric, \( \langle \tilde{r}, x| \) equals the transpose of \( |r, x\rangle \), and not its Hermitian conjugate, as it happens in the case of Hermitian Hamiltonian operators.

Assuming now \( t > 0 \), the evolution operator on states of the intrinsic system \( A \) can be written as
\[ U(t, 0) = \frac{i}{2\pi} \sum_r \int_{-\infty}^{\infty} e^{-\frac{1}{\hbar} \pi t^2 |r, x\rangle\langle \tilde{r}, x|} \, dx. \quad (10) \]
Due to the coupling between the intrinsic system and the environment, the total probability for an initially intrinsic state to remain in \( A \) may not be conserved in time, this is why the evolution operator \( U \) is, in general, non-unitary. This property can be already gathered from Eq. (10), but it will become more evident in the next Section.

In the case \([H_{\text{eff}}(x), H_{\text{eff}}(x)] = 0\) (which is always true in the case \( N_A = 1 \)) we can write the evolution operator in a particularly useful form. Indeed, we can define, for any \( r = 1, \ldots, N_A \),
\[ G_r^+(x) = \frac{|r, x\rangle\langle \tilde{r}, x|}{x - E_r(x) + \frac{1}{2} \Gamma_r(x)}, \]
and express the propagator in the form
\[ G(x) = \sum_r (G_r^+(x) - G_r^-(x)) \]
\[ = \sum_r \frac{-i \Gamma_r(x)|r, x\rangle\langle \tilde{r}, x|}{[x - E_r(x)]^2 + \frac{1}{4} \Gamma_r(x)^2}, \quad (11) \]
so that the evolution operator on states of the intrinsic system \( A \) reads
\[ U(t, 0) = \frac{1}{2\pi} \sum_r \int_{-\infty}^{\infty} e^{-\frac{1}{\hbar} \pi t^2 \Gamma_r(x)|r, x\rangle\langle \tilde{r}, x|} \, dx. \quad (12) \]
This form of the evolution operator will be used in the next sections.

The effective non-Hermitian Hamiltonian, Eqs. (5-7) can be greatly simplified if the \( x \)-dependence can be neglected. The validity of such an approximation is a delicate issue, nevertheless, in many relevant situations, it can be proven to be a very good approximation [8]. Roughly speaking, this approximation is valid when \( \Delta_{ij}(x) \) and \( Q_{ij}(x) \) are smooth functions of \( x \) in the physically relevant energy range, determined by the eigenvalues of the Hamiltonian of the closed system. In such a case, the non-Hermitian Hamiltonian behaves as an effective Hamiltonian, and allows us to compute the time evolution of the projection of the total wave function on the intrinsic system, see Eq. (10). Indeed, we can expand any initial state of the intrinsic system, over the eigenstates of the effective Hamiltonian and its time
We consider here a simple model with \( N_R \) two-level systems arranged in a ring and coupled to a common decay channel, the sink, which we model with a one-dimensional lead.

Such a ring-like structure has been considered in several papers \[12\] \[13\] \[17\] \[20\] as a paradigmatic model to describe different systems, such as molecular J-aggregates \[22\], bio-inspired devices for photon sensing \[13\] and efficient light-harvesting systems \[19\]. In particular it has been often considered in the frame of exciton transport in natural photosynthetic systems, where chlorophyll molecules aggregate in ring-like structures around a reaction center, representing a central core absorber, where the excitation can be trapped \[23\]. Chlorophyll molecules are able to absorb photons and can be modeled as two-level system. Under low light intensity, only one excitation is considered and the molecular aggregate becomes equivalent to a tight-binding model where one particle can hop from site to site.

We first introduce a Hermitian model to describe the transport of excitation from the ring-like structure to the central core absorber represented by a lead, as described in Fig. 1. Note that also in Ref. \[18\] the central core absorber of the photon-sensing device was modeled by a lead.

Specifically, we consider a ring with \( N_R \) sites, connected with nearest-neighbor coupling \( \Omega \), described by the tight binding Hamiltonian

\[
H_R = \Omega \sum_{\langle r,r' \rangle} (|r\rangle \langle r'| + |r'\rangle \langle r|) , \tag{14}
\]

where the sum runs over the pairs of neighboring sites. In what follows we will measure energies in units of cm\(^{-1}\) and times in ps. This choice, common in models for molecular aggregates, corresponds to set \( 1/\hbar = 0.06 \pi \) cm/ps.

Each site of the ring is connected, through the tunneling amplitude \( \Omega_{RL} \), to the first site of a lead, described by a linear chain of \( N_L \) resonant sites with nearest-neighbor coupling \( \Omega_L \). The Hamiltonian for the lead is

\[
H_L = \Omega_L \sum_{j=1}^{N_L-1} (|\ell_j\rangle \langle \ell_{j+1}| + |\ell_{j+1}\rangle \langle \ell_j|) , \tag{15}
\]

and the interaction between the ring and the lead is described by

\[
V_{RL} = \Omega_{RL} \sum_{r=1}^{N_R} (|r\rangle \langle \ell_1| + |\ell_1\rangle \langle r|) , \tag{16}
\]

so that the total Hamiltonian of the system, written on the site basis

\[
\{ |r\rangle, |\ell_j\rangle, r = 1, \ldots, N_R, j = 1, \ldots, N_L \} , \tag{17}
\]

reads

\[
H = H_R + V_{RL} + H_L . \tag{18}
\]

One can imagine that, when \( N_L \) is large enough, the lead represents a good sink, in that it absorbs most of the excitation present in the system.

### A. The non-Hermitian Hamiltonian

Since our main focus is on the decay of the excitation from the ring and not on its dynamics in the lead, we will now derive an effective Hamiltonian for the subsystem formed by the ring, summarizing into a non-Hermitian term the effects of the subsystem represented by the lead. In this derivation we will follow the procedure described in Sec. II.

The eigenvalues of the lead Hamiltonian are given by

\[
E_q = -2\Omega_L \cos \frac{\pi q}{N_L+1} , \quad q = 1, \ldots, N_L , \tag{19}
\]

and the components on the site \( |\ell_j\rangle \) of the lead eigenstates read

\[
\langle \ell_j | \psi_q \rangle = \sqrt{\frac{2}{N_L+1}} \sin \frac{\pi j q}{N_L+1} .
\]

From Eq. \[19\] and recalling that

\[
\sin \frac{\pi q}{N_L+1} = \sqrt{1 - (E_q/2\Omega_L)^2} ,
\]
we can derive the density of states \( \rho(E) = dq/dE \) as

\[
\rho(E) = \frac{N_L + 1}{2\pi\Omega L} \frac{1}{\sqrt{1 - (E/2\Omega L)^2}}.
\]

We can now compute the matrix elements connecting the site \( r \) of the ring with the eigenstates of the lead with energy \( E \):

\[
A_r(E) = \langle r | H | \ell \rangle \langle \ell | \psi(E) \rangle
= \Omega R L \sqrt{\frac{2}{N_L + 1}} \sqrt{1 - (E/2\Omega L)^2}.
\]

(20)

In the limit \( N_L \to \infty \) (semi-infinite lead) we have a continuum of states in the lead, obtaining

\[
A_r(E)A_r(E)^* \rho(E) = \frac{\Omega^2 R L}{2\pi\Omega L} \sqrt{1 - (E/2\Omega L)^2}.
\]

(21)

We can then define the transition matrix \( Q(x) \), see Eq. (5), by

\[
Q_{rr'}(x) = \left\{ \begin{array}{ll}
\gamma \sqrt{1 - \frac{x^2}{4\Omega_L^2}} & \text{for } x \in [-2\Omega_L, 2\Omega_L], \\
0 & \text{otherwise},
\end{array} \right.
\]

(22)

where we introduced the opening strength

\[
\gamma = \frac{2\Omega^2 R L}{\Omega L}.
\]

(23)

By using Eq. (21) we can also derive the expression for the matrix \( \Delta(x) \), see Eq. (7), thus obtaining the effective Hamiltonian

\[
H_{\text{eff}}(x) = H_R + \Delta(x) - \frac{i}{2} Q(x).
\]

(24)

Note that the matrix elements \( Q_{rr'}(x) \) and \( \Delta_{rr'}(x) \) do not depend on \( r \) or \( r' \). This fact implies that they commute and they both have only one non-zero eigenvalue. The state corresponding to that eigenvalue is the fully symmetric state

\[
|S\rangle = \frac{1}{\sqrt{N_R}} \sum_r |r\rangle,
\]

(25)

which is also an eigenstate of \( H_R \), corresponding to the maximum energy \( 2\Omega \). The remaining \( N_R - 1 \) eigenstates of \( Q(x) \) and \( \Delta(x) \) are degenerate and can always (i.e. for any \( x \)) be chosen to match those of \( H_R \) orthogonal to \( |S\rangle \). For this reason, \( Q(x) \) and \( \Delta(x) \) commute with \( H_R \).

The only non-zero eigenvalue of \( Q(x) \) is given by

\[
\Gamma_{sr}(x) = \left\{ \begin{array}{ll}
N_R \gamma \sqrt{1 - \frac{x^2}{4\Omega_L^2}} & \text{for } x \in [-2\Omega_L, 2\Omega_L], \\
0 & \text{otherwise},
\end{array} \right.
\]

(26)

and the only non-zero eigenvalue of \( \Delta(x) \) is

\[
\Delta_{sr}(x) = \gamma N_R \Psi \int_{-2\Omega L}^{2\Omega L} \sqrt{1 - E^2/4\Omega_L^2} \ dx - E.
\]

(27)

From the foregoing facts, we obtain the important consequence that we can diagonalize the effective Hamiltonian \( H_{\text{eff}}(x) \) on the \( x \)-independent basis of eigenstates of \( H_R \). The only eigenvalue of the intrinsic system (the ring) which is modified by the opening is

\[
E_{sr} = 2\Omega + \Delta_{sr}(x) - \frac{i}{2} \Gamma_{sr}(x),
\]

while the others are

\[
E_r = 2\Omega \cos \frac{2\pi r}{N_R}, \quad r = 1, \ldots, N_R - 1,
\]

(28)

which coincide with the eigenvalues of \( H_R \) [13].

Remarkably, we are in the peculiar situation in which only one ring state (\( |S\rangle \), Eq. (25)) is coupled to the lead, and the number of relevant degrees of freedom, as far as decay properties are concerned, may look already dramatically reduced to 1. Nevertheless, the dependency on \( x \) of \( \Gamma_{sr} \) and \( \Delta_{sr} \), keeps the actual number of degrees of freedom infinite.

The time-evolution operator for the sole ring state, \( |S\rangle \), which is coupled to the lead is given by

\[
U_S(t, 0) = \frac{1}{2\pi} \int_{-2\Omega L}^{2\Omega L} e^{\frac{i}{\hbar} \int_{-2\Omega L}^{2\Omega L} e^{\frac{i}{\hbar} \Gamma_{sr}(x)} dx} dx,
\]

(29)

while the ring states which are orthogonal to \( |S\rangle \) are effectively decoupled from the lead. For those states the opening term in the effective Hamiltonian vanish, and it is trivially, but exactly, \( x \)-independent: those states will never decay.

To complete the wanted dimensional reduction, we then need to derive an \( x \)-independent (or energy-independent) approximation of \( H_{\text{eff}}(x) \), Eq. (24).

Now, if we send \( \Omega_L \to \infty \) and \( \Omega R L \to \infty \) keeping \( \gamma \) fixed (wide-band limit), we clearly obtain an exact energy-independence with

\[
\Delta_{sr}(x) \to 0 \quad \text{and} \quad \Gamma_{sr}(x) \to \gamma N_R.
\]

(28)

With those assumptions we get

\[
U_S(t, 0) = \exp \left( \frac{2\Omega t}{\hbar} - \frac{\gamma N_R t}{2\hbar} \right),
\]

(29)

and the effective energy-independent non-Hermitian Hamiltonian describing the evolution of the intrinsic sys-
tem, the ring, becomes

$$H_{\text{eff}} = H_R - i\frac{\gamma}{2} O,$$

(30)

where $O$ is a full matrix with all entries equal to 1, and the components of $H_{\text{eff}}$ on the ring-site basis read

$$(H_{\text{eff}})_{rr'} = (H_R)_{rr'} - i\frac{\gamma}{2}.$$

Accordingly, the evolution operator on the whole intrinsic subspace is given by

$$U(t, 0) = e^{-iH_{\text{eff}}t/\hbar}.$$  

(31)

In summary, the effective non-Hermitian model, depicted in Fig. 2, is given by an open ring of $N_R$ resonant sites equally coupled, with strength $\gamma$, to a common decay channel, in which the excitation can be lost. We will analyze in Sec. V below the limit of validity for Case 1 above, showing that the non-Hermitian Hamiltonian just derived contains, together with the Hamiltonian of the closed ring $H_R$, another term $O$, which is a full matrix, representing the decay matrix. Such a matrix also represents a long-range hopping between the sites of the ring, mediated by the coupling of the sites of the ring to the common decay channel in the lead. This long-range hopping will be relevant to understand the interplay of opening and disorder discussed in Sec. VI.

IV. SUPERRADIANCE IN TRANSPORT

A very important phenomenon that can be easily analyzed by means of the energy-independent effective Hamiltonian is Superradiance. It is the cooperative effect which produces a strong inhomogeneity in the decay rates of the states of the intrinsic subsystem: some states, named superradiant, display large decay rates, while the decay of some other states is very slow, sometimes even negligible. Indeed, we already noticed that the ring subsystem is, for any $\gamma \neq 0$, in a superradiant regime, with a single superradiant state $|S\rangle$, Eq. (23), absorbing all the decay width $\gamma N_R$, and $N_R - 1$ subradiant states with vanishing decay widths.

The roots of this effect lie in the interference due to the competition of different states of the intrinsic subsystem to decay in the same channel in the continuum. Considering a generic situation, let us assume that we can obtain an $x$-independent form of the terms $Q$ and $\Delta$ of Eqs. (6) and (7), respectively. Necessarily, the effective $Q$ possesses a factorized structure, since it is derived by the tensor product of the (rectangular) transition matrices $A^r_i$. It thus can have only as many non-zero eigenvalues as the number $M$ of decay channels. In the energy-independent approximation, $\Delta$ usually present the same factorized structure of $Q$, and thus $[\Delta, Q] = 0$.

We must now distinguish two situations:

1. When $[H_{AA}, Q] = 0$, the eigenvalues $E_r$ of $H_{\text{eff}}$ are given by the sum of those of $H_{AA}$, $\Delta$, and $-(i/2)Q$, so that we can have at most $M$ non-vanishing decay widths, while $N_A - M$ eigenstates of $H_{\text{eff}}$ are not decaying at all.

2. When $[H_{AA}, Q] \neq 0$, we encounter an additional effect, named Superradiance transition. Indeed, the relative energy scale of the opening term $\Delta - (i/2)Q$ with respect to that of the intrinsic Hamiltonian $H_{AA}$ becomes important in determining how the decay width is distributed among the eigenstates of $H_{\text{eff}}$: when the opening is weak the eigenstates will be close to the eigenstates of $H_{AA}$ and all of them will have a similar decay rate; on the other hand, when the opening is strong, the eigenstates will approach those of $\Delta - (i/2)Q$, and only $M$ of them will have a significant decay width. We then see a transition from a non-superradiant regime (weak opening) to a superradiant regime (strong opening). This transition is not present in Case 1 above, where we are in a superradiant regime for any opening strength.

One could also consider the case $[\Delta, Q] \neq 0$, but under this condition it is not possible to predict the behavior of the system regarding Superradiance on general grounds, and we need to look at the eigenvalues of the specific $H_{\text{eff}}$ at hand.

As already mentioned, our tight-binding model offers a paradigmatic realization of Case 1 above, showing that the symmetry of the coupling between each ring site and the sink is responsible for the effective perfect segregation of the decay widths. Moreover, the Superradiance transition introduced in Case 2 plays a fundamental role in determining the dynamics of our system when static disorder is added, and Sec. VI is devoted to the analysis of such effect in both the Hermitian and the non-Hermitian models introduced above.

A. Superradiant decay

Here we consider the effects of Superradiance on the decay of states which are initially excited on the ring, showing that the non-Hermitian model correctly reproduces the dynamics of the full Hermitian system. Given an initial ring state $|\psi_0\rangle$, we consider the survival probability

$$P(t) = \sum_{r=1}^{N_R} |\langle r | \psi(t) \rangle|^2,$$

(32)

computing the time evolution in both the Hermitian model, Eq. (18),

$$|\psi^H(t)\rangle = e^{-iHt/\hbar} |\psi_0\rangle,$$

(33)

and the non-Hermitian one, Eq. (30),

$$|\psi^{\text{eff}}(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi_0\rangle.$$  

(34)
the two models will depart from each other, due to the fact that, in our simulations, both $\Omega$ and $N_L$ are finite. In Sec. V we will analyze in detail the critical times up to which the agreement persists.

V. LIMIT OF VALIDITY OF THE NON-HERMITIAN MODEL: NON-EXPONENTIAL DECAY

The non-Hermitian Hamiltonian, Eq. (30), constitutes a great simplification of the full Hermitian problem, since it eliminates the infinite number of degrees of freedom of the lead. As it was shown in the previous Sections, the non-Hermitian Hamiltonian becomes exact for infinite length and infinite energy band in the lead.

In this Section we want to clarify the effect of $\Omega_L$ and $N_R$, being finite on the validity of the non-Hermitian approach. We will restrict our attention to the survival probability $P(t)$ computed for the superradiant initial state $|S\rangle$ of Eq. (25), for which the non-Hermitian Hamiltonian predicts an exponential decay with a decay width given by $N_R\gamma$. For all the other initial states, orthogonal to the state $|S\rangle$, the non-Hermitian Hamiltonian predicts that they are subradiant and do not decay at all. Since we have only one level, $|S\rangle$, coupled to the lead, the problem of the validity of the non-Hermitian Hamiltonian approach is strongly connected with the problem of the validity of the Fermi Gold Rule, which predicts an exponential decay of the survival probability of a single unstable quantum state coupled with a continuum of states [24–26]. As in the case of the Fermi Golden Rule, we will show that the exponential behavior is typically valid for intermediate times, while for both short and long times the decay is not exponential.

A. Finite-size effects

For finite lead length the decay of the excitation from the ring will not be irreversible, since the excitation can bounce back at the end of the lead, inducing a revival of the survival probability. Thus, we can expect deviations of the Hermitian evolution from the exponential decay due to the reflection of the wave-packet at the end of the lead. This bouncing effect is always present in any finite-size sink and it is known in literature as mesoscopic echo [28].

We can analytically estimate the bouncing time $t_B$ assuming that the excitation, after leaving the ring, goes through the lead with a certain group velocity $v_g$, and bounces back once reached the end of the lead.

If we consider the site index on the lead as a spatial position, we can describe the eigenstates of the lead as plane waves

$$\psi_q(j) \propto \sin(k(q)j), \quad q, j = 1, \ldots, N_L,$$  (36)
with wave numbers
\[ k(q) = \frac{\pi q}{N_L + 1}, \quad k \in (0, \pi). \]  
(37)

From this point of view, a superposition of lead eigenstates forms a wave-packet. The group velocity of this wave-packet is given by
\[ v_g = \frac{\partial \omega(k)}{\partial k} \bigg|_{\bar{k}} = \frac{1}{\hbar} \frac{\partial E(k)}{\partial k} \bigg|_{\bar{k}}, \]  
(38)

where \( \bar{k} \) is the mean wave number of the waves that form the wave-packet. Using the wave numbers of the eigenstates defined above, we can write the energies of the lead as
\[ E(k) = -2\Omega_L \cos k, \]  
(39)

so that the group velocity becomes
\[ v_g = \frac{2\Omega_L}{\hbar} \sin \bar{k}. \]  
(40)

The excitation has to go through the entire lead twice before reaching again the starting point. From Eq. (40), we see that the maximum velocity of a wave-packet is \( v_g = 2\Omega_L \), hence we can expect the agreement between the Hermitian evolution and the non-Hermitian one to persist up to the time
\[ t_B = \frac{2N_L}{v_g} = \frac{\hbar N_L}{\Omega_L}. \]  
(41)

In order to check our estimate for \( t_B \), in Fig. 4 we compare the superradiant decay \( \exp(-N_R\gamma t) \), produced by the non-Hermitian model (black curve), with the Hermitian evolution (symbols) computed for different values of the lead size \( N_L \). As \( N_L \) increases, the agreement between the Hermitian and the non-Hermitian evolution persists up to a critical time after which we have deviations from the exponential decay and a revival of the survival probability. For small values of \( N_L \), the agreement time increases linearly with \( N_L \) and it is well estimated by the values of \( t_B \), see vertical arrows. On the other side, for larger values of \( N_L \), the agreement time becomes independent of the length of the lead. This suggests that a different effect cause the departure of \( P(t) \) from the exponential decay.

B. Finite-bandwidth effects

From Fig. 4 we notice that, when we increase \( N_L \) above a certain value, the agreement time does not improve, even if the bouncing time increases. Indeed, the large-size regime is characterized by an \( N_L \)-independent agreement time, marking the transition from the superradiant decay to a much slower one. The origin of this brake in the decay is very general and it is due to the presence of a finite energy band in the lead, whose bandwidth equals \( 4\Omega_L \), see discussion in Ref. [24].

Indeed from Eq. (27) we see that the time evolution of the superradiant state is given by the Fourier transform of the function
\[ \mathcal{L}(x) = \frac{1}{\pi} \frac{\Gamma_{sr}(x)/2}{|x - 2\Omega - \Delta_{sr}(x)|^2 + \frac{\gamma}{2}\Gamma_{sr}(x)^2}. \]  
(42)

In the limit \( \Omega_L \to \infty \), \( \Gamma_{sr} \) and \( \Delta_{sr} \) do not depend on \( x \) (see discussion in Sec. III). Moreover, the limits of integration go to infinity. Thus, in this limit, the time evolution of the superradiant state is the Fourier transform of a Lorentzian function, which gives an exponential decay. On the other side, for finite bandwidth in the lead, we can expect deviations from the exponential decay due to two reasons: the Lorentzian function is now distorted due the fact that both \( \Gamma_{sr} \) and \( \Delta_{sr} \) depend on \( x \) and the limits of integration do not go to infinity anymore.

As it is shown in Appendix A, under the conditions
\[ \Omega \ll \Omega_L, \quad \Gamma_{sr} = N_R\gamma \ll 4\Omega_L, \]  
(43)

the transition amplitude \( A_t(E) \) and the density of states \( \rho(E) \), see Eq. (21), are very smooth function of the energy in an energy range determined by the decay width \( \Gamma_{sr} \) and centered at the energy of the initial state \( 2\Omega \). This fact implies that the function \( \mathcal{L}(x) \) is well approximated by a Lorentzian function around its maximum value which occurs near the energy of the initial state. In such a case, the main deviations from the exponential decay are due solely to the truncation of the function \( \mathcal{L}(x) \) outside the energy band of the lead.

Specifically, in Appendix A, we show that the decay of \( P(t) \) is exponential between two time-scales \( t_0 \) and \( t_S \).
and we have:

\[
P(t) \approx \begin{cases} 
1 - \frac{\Omega^2 + N\hbar^2 t^2}{\hbar^2} & \text{for } t < t_0, \\
e^{-N\hbar \gamma t/h} & \text{for } t_0 < t < t_S, \\
\text{const.}/t^3 & \text{for } t > t_S.
\end{cases}
\tag{44}
\]

From Eq. (44), the decay is initially quadratic in time, as predicted by perturbation theory, and the transition to the exponential decay occurs at a time \( t_0 \) given by

\[
t_0 = \frac{\hbar}{2\Omega L},
\tag{45}
\]

which has been derived in Appendix A and, with a more heuristic approach, in Appendix B.

The quadratic initial decay given by the full Hermitian model is shown in Fig. 5 (symbols) for different values of \( \Omega_L \). In the same figure, the short-time analytic estimate (curves) given in Eq. (44) is shown to be a good estimate of the initial behavior of the Hermitian system up to the time \( t_0 \), marked with arrows in Fig. 5.

The critical time for which we have deviations from the exponential decay for long times has been derived in Appendix A, where it is shown that, for \( t > t_S \), the decay is not exponential anymore, but the survival probability decays as a power law, and we have:

\[
t_S \propto \frac{2\hbar}{\gamma N_R} \log \frac{4\Omega L}{\gamma N_R}.
\tag{46}
\]

In Fig. 6 the logarithmic dependence of the agreement time on \( \Omega_L \) is shown to agree with numerical results. Note that the power-law decay \( P(t) \propto t^{-3} \) for \( t > t_S \) is in agreement with numerical results, see dashed line in Fig. 7.

We also observe that both the finite-size and finite-bandwidth effects disappear in the thermodynamic limit (\( N_L \to \infty, \Omega_L \to \infty, \rho(E) = 1/2\pi \)) considered in Sec. II during the derivation of \( H_{\text{eff}} \), since both \( t_B \) and \( t_S \) grow to infinity, while \( t_0 \) goes to zero.

The fact that the exponential decay is valid in an intermediate time scale, is not a peculiar feature of our model, but it is a general feature of the decay of an unstable quantum state in the continuum, see the discussion about the validity of the Fermi Golden Rule in Ref. [25].

C. Strong deviations from exponential decay

Until now we have analyzed the short and long time deviations from the exponential decay, assuming that the function in Eq. (42) can be well approximated by a Lorentzian function away from the edges (which is true under the conditions given in Eq. (43)). Here we briefly analyze what happens if \( \Omega \) and \( \Gamma_{\text{er}} \) are not small compared to \( \Omega_L \), so that the conditions in Eq. (43) are no longer satisfied. In Fig. 7 we show the survival probability starting from the state \( |S \rangle \) for different values of the ratio \( \Omega/\Omega_L \). The exponential decay predicted by the non-Hermitian Hamiltonian is well satisfied up to large values of \( \Omega/\Omega_L \). Indeed, it is still a good approximation for \( \Omega/\Omega_L \sim 1/2 \). While, as we further increase this ratio, the decay is strongly suppressed. In particular, when \( \Omega > \Omega_L \), the initial state energy lies outside the energy...
VI. THE EFFECTS OF STATIC DISORDER

In this Section we aim at studying the effects of static diagonal disorder on the transport properties of the system under consideration. Note that we add disorder only in the ring, leaving the lead unchanged. Such a disorder is modeled by position-dependent, but time-independent, fluctuations of the ring site energies, that is we added to $H$ the term

$$D = \sum_{r=1}^{N_R} \epsilon_r |r\rangle \langle r|,$$

(47)

where $\epsilon_r$ are independent random variables uniformly distributed on $[-W/2, W/2]$, and $W$ represents the disorder strength.

It is well known that static diagonal disorder induces Anderson localization: the eigenstates of the system become exponentially localized, thus preventing transport. The critical disorder strength in one-dimensional aggregates for such a localization to occur is given by

$$W_{\text{loc}} \approx \frac{100}{\sqrt{N}},$$

(48)

where $N$ is the system size [15]. Since the superradiant state $|S\rangle$ is a fully extended state on the ring, diagonal disorder will tend to suppress Superradiance in transport by inducing the localization of eigenstates. On the other side, disorder will destroy the perfect symmetry of the ring which produces zero decay widths of the subradiant states. Thus, disorder will not only decrease the decay width of the superradiant state, but it will also increase the decay widths of the subradiant states. This situation was already analyzed by means of the non-Hermitian Hamiltonian approach in Refs [15, 16], where it was shown that, upon increasing the disorder strength, the decay widths of the subradiant and superradiant states become the same, and equal to $\gamma$ for $W > W_{\text{sr}}$. Hence, $W_{\text{sr}}$ represents the critical disorder strength above which Superradiance is quenched. Note that opening and disorder have competing effects, since the opening induces a long-range hopping among the sites of the ring, as it is clearly seen from the structure of the full matrix $O$ in Eq. [30] which can be expected to contrast the localization effects of disorder.

In presence of disorder we do not have only one state coupled to the lead as in the previous section, but we have a genuine many-level problem. Here we analyze the effect of static disorder under the conditions given in Eq. [43], under which the non-Hermitian Hamiltonian was valid for the case without disorder.

The non-trivial competition between opening and disorder has been analyzed in Refs [15, 16], within the framework of the non-Hermitian Hamiltonian approach to open quantum systems. The analysis was performed assuming the coupling $\gamma$ to the continuum to be independent of the disorder strength. Such an assumption is often used in literature and greatly simplifies the cal-

band of the lead and thus the survival probability decays very slowly.

In Fig. 8 we show the survival probability starting from the state $|S\rangle$ for different values of the ratio $\Gamma_{\text{sr}}/4\Omega_L$. In this case, deviations from the exponential decay predicted by the non-Hermitian Hamiltonian start already when $\Gamma_{\text{sr}}/4\Omega_L = 0.1$, thus showing that the agreement between the Hermitian and the non-Hermitian model is very sensitive to the decay width of the initial state. The strong oscillations which can be seen in Fig. 8 are due to the fact that, for large $\Gamma_{\text{sr}}$, the coupling $\Omega_{RL}$ between the ring and the first lead site is large.

![FIG. 7. (Color online) Survival probability, $P(t)$ is plotted vs time $t$, for different ratios $\Omega/\Omega_L$. The results obtained with the full Hermitian model (symbols) are compared with the result predicted by the non-Hermitian model (dashed curve). Data shown refer to the case $N_R = 10$, $\Omega_{RL} = 10$, $N_L = 1000$, $\Omega_L = 100$, and different values of $\Omega$.](image_url)

![FIG. 8. (Color online) Survival probability, $P(t)$ is plotted vs the rescaled time $t^* = \Gamma_{\text{sr}}/h = N_R \gamma / h$ time, for different values of the ratio $\Gamma_{\text{sr}}$ over the energy bandwidth in the lead $4\Omega_L$. The results obtained with the full Hermitian model (solid curves) are compared with the result predicted by the non-Hermitian model (dashed curve). Data shown refer to the case $N_R = 10$, $\Omega = 1$, $N_L = 1000$, $\Omega_L = 100$, and different values of $\Omega_{RL}$.](image_url)
culations. On the other side, one can expect the presence of diagonal disorder to affect the outcome of the reduction procedure leading to $H_{\text{eff}}$, Eq. (30). For instance, the coupling to the continuum will in general depend on the disorder strength. In order to understand this point, one can consider only one site coupled to a lead with an energy bandwidth of $4\Omega_L$. If we assume the opening strength to the lead to be independent of disorder, the non-Hermitian Hamiltonian of this system reads $H_{\text{eff}} = E_0 + \epsilon_0 = i\gamma/2$, which implies that the survival probability decays exponentially as $e^{\gamma t/\hbar}$ for any value of the disorder strength $W$. Clearly, this cannot be true when $W \gg 4\Omega_L$, since in that case the probability of the initial state to be outside the energy band of the lead will be large and the transport will be consequently suppressed. One of the purposes of this section is to clarify up to which critical value $W_{\text{map}}$ of the disorder strength one can faithfully reproduce the dynamics of the full Hermitian system with the non-Hermitian Hamiltonian, under the condition that the opening parameter $|\gamma|$, Eq. (23), is independent of $W$.

The other purpose of this Section is to see whether the important effects, found in Refs. [15,16], are indeed present in the full Hermitian model of transport considered in this paper. The two main findings of Refs. [15,16] can be summarized as follows: (i) Cooperative robustness to disorder. For large enough opening strength the critical disorder $W_{\text{sr}}$ needed to quench superradiance increases linearly with the system size. (ii) Subradiant hybrid regime. In the superradiant regime the response of the superradiant and subradiant subspaces to disorder is very different. While superradiant states display robustness to disorder by remaining extended up to $W_{\text{sr}}$, subradiant states show strong signatures of localization. Indeed, they have hybrid features displaying both an exponentially localized peak and a uniform delocalized plateau.

A. Comparison between Hermitian and non-Hermitian models in presence of disorder

To assess the effectiveness of the non-Hermitian description, under the assumption that the coupling to the continuum is independent of disorder, we will first study the survival probability $P(t)$ of finding the excitation on the ring at time $t$, comparing both the results given by the Hermitian and the non-Hermitian model in presence of disorder.

We first consider the symmetric state $|S\rangle$, Eq. (25), as initial state. As we already noticed, in absence of disorder the $P(t)$ of this state decays exponentially with a decay width equal to $N_R\gamma$. Due to the presence of disorder the symmetric state $|S\rangle$ will not correspond to the state with the largest decay width anymore, but it will be a superposition of superradiant and subradiant states. Thus we can expect a fast initial decay (faster then $\hbar/\gamma$), corresponding to the superradiant components, followed by a slow decay (slower then $\hbar/\gamma$), due to the subradiant components. For large disorder superradiance will be destroyed and the non-Hermitian model predicts that all the states will have the same decay width, equal to $\gamma$, so that in this regime the survival probability will decay as $e^{-\gamma t/\hbar}$. As for the full Hermitian model, we can expect that the decay for large disorder will be much slower than $\hbar/\gamma$, due to the fact that disorder will send the energies of the ring outside the energy band of the lead.

The comparison of the survival probability for a symmetric initial state $|S\rangle$, between the Hermitian and the non-Hermitian model, as we vary the disorder strength $W$, is shown in Fig. 9. When the disorder strength is small, both models agree: we have a bi-exponential behavior of the $P(t)$ in which a fast decay (with rate $N_R\gamma/\hbar$) is followed by a much slower decay, almost negligible for very small $W$. An analytic formula for the $P(t)$ in this regime has been obtain in Ref. [15, Eq. (16)]. Upon increasing the disorder strength $W$ the fast decay stops earlier and earlier and the subsequent slow decay becomes a little faster. Note that the agreement between the Hermitian evolution (symbols) and the non-Hermitian one (curves) is very good up to large values of the disorder strength $W$. When the two predictions become dramatically different. Indeed, for large disorder, the non-Hermitian model shows a decay rate of $\gamma/\hbar$ (see dashed curve in Fig. 9), while the decay of the Hermitian model is strongly suppressed (see circles corresponding to $W = 1000$ in Fig. 9).
FIG. 10. (Color online) Average survival probability $P(t)$, starting from a randomly generated ring state, computed for different values of the disorder strength $W$. A perfect agreement between the Hermitian model (symbols) and the non-Hermitian one (curves) appears, except for the largest considered value, $W = 1000$ (blue), in which case the prediction of the non-Hermitian model (curve) is remarkably different from the Hermitian one (circles). As a dashed curve we plotted the exponential decay $e^{-\gamma t/\hbar}$ corresponding to the independent site limit of the non-Hermitian model. Parameters are $N_R = 10$, $\Omega = 1$, $\Omega_{RL} = 10$, $\Omega_L = 100$, $N_L = 250$.

The non-Hermitian model is effective in reproducing the dynamics of the Hermitian model, up to rather large values of the disorder strength $W$, also for generic initial states, not only for the fully symmetric state $|S\rangle$. To show this, we computed the survival probability $P(t)$ starting from a randomly generated ring state. Note that a random initial state will have mainly components on the subradiant subspace, so that we can expect that disorder will increase the transport efficiency. For large disorder the non-Hermitian model predicts an exponential decay $P(t) = e^{-\gamma t/\hbar}$, while we can expect a much slower decay from the full Hermitian model.

The average $P(t)$ computed for different values of $W$ is shown in Fig. 10, where a perfect agreement between the Hermitian model (symbols) and the non-Hermitian one (curves) appears, except for very large values of disorder. Indeed, while the decay of $P(t)$ in the non-Hermitian case is faster and faster as disorder increases, approaching the limiting decay rate $\gamma/\hbar$, for the Hermitian case the decay has a non-monotone behavior with the disorder strength, since it increases for small disorder and it is strongly suppressed for large disorder.

What is the origin of the different behavior between the Hermitian and the non-Hermitian model in the large-disorder regime? Can we estimate the critical value of the disorder strength $W_{\text{map}}$ up to which the non-Hermitian model faithfully reproduces the Hermitian one?

To answer this questions, we analyze the agreement between the two models looking at the transport efficiency $\eta$, defined as

$$\eta = 1 - P(t = \hbar/\gamma).$$

Note that $\eta$ is the probability that the excitation has escaped into the lead within the time $\hbar/\gamma$. If $P(t)$ decays with a rate $\gamma/\hbar$, corresponding to that of non-interacting decaying sites, $\eta$ assumes the value $1 - 1/e$. Hence, a value of $\eta$ greater than $1 - 1/e$ signals a superradiant cooperative decay, while a value of $\eta$ smaller than that threshold signals a subradiant decay.

In Fig. 11 we show the efficiency $\eta$ varying the coupling $\Omega_L$ in the lead (and accordingly modifying $\Omega_{RL}$ and $N_L$ to keep the decay rate $\gamma$ fixed and remove the bouncing effect). The results of the non-Hermitian model are shown as full curves, while those of the Hermitian model are shown as symbols. In the upper panel we consider the fully symmetric initial state $|S\rangle$ of Eq. (25), while
in the lower panel we consider the fully antisymmetric state $|AS\rangle$ of Eq. (35). For zero disorder the state $|S\rangle$ is superradiant and the state $|AS\rangle$ is subradiant with zero decay width.

For the non-Hermitian case the behavior of $\eta$ is independent of $\Omega_L$, since we kept $\gamma$ fixed: the efficiency of the symmetric state (Fig. 11, upper panel) decreases with the disorder strength, asymptotically approaching the value $1 - 1/e$, which would be the efficiency of non-interacting decaying sites, while the efficiency of the anti-symmetric state (Fig. 11, lower panel) increases with the disorder up to the same limiting value.

As for the Hermitian model, it is in perfect agreement with the non-Hermitian one for small disorder strength, while, for strong disorder, the efficiency goes to zero. Most importantly, we notice that the agreement between the Hermitian and the non-Hermitian model is excellent up to a critical disorder strength (vertical dashed lines).

From this point of view the problem of mapping a Hermitian model into a non-Hermitian one in presence of disorder is similar to the problem of the validity of the energy-independent approximation discussed in Sec. III. It is remarkable that the two models agree up to $W \approx W_{\text{map}}$, when the energy band in the ring is one half of the energy band in the lead. This result is consistent with the results of Fig. 7 where we analyzed the validity of the energy-independent approximation, varying the energy of the initial state.

In the following subsections we will analyze whether the interesting effects found in the non-Hermitian model and described at the beginning of this section (cooperative robustness and subradiant hybrid states) can be found also in the Hermitian model for $W < W_{\text{map}}$.

**B. Cooperative robustness to disorder**

As already mentioned, disorder will quench superradiance and the critical disorder $W_{\text{sr}}$ at which this occurs has been computed in Ref. [15 Eq. (11)], for the non-Hermitian model, assuming a disorder independent opening strength. For the sake of clarity we report below that

$$W_{\text{map}} \approx 2\Omega_L.$$  \hspace{1cm} (50)

Note that $W_{\text{map}}$ tends to infinity in the thermodynamic limit $N_L \to \infty$, $\Omega_L \to \infty$, $\rho(E) = 1/2\pi$.

The estimate just given of $W_{\text{map}}$ confirms the heuristic argument given at the beginning of this section about the origin of the different behavior of the two models: when $W \sim \Omega_L$ the energies of the states of the ring become comparable with the energy band in the lead, thus producing a suppression of transport. Such a suppression is completely neglected in the non-Hermitian model which is derived by assuming an infinite energy band in the lead, as explained in Sec. [III].

For the non-Hermitian model (symbols) the agreement between the two models persists up to $W = W_{\text{map}}$ (see vertical arrow). Note that the agreement does not depend on the ring size, as implied by the foregoing discussion.

To illustrate this effect we plotted in Fig. 12 the transport efficiency $\eta_{\text{sr}}$ versus disorder computed taking as initial state the symmetric state $|S\rangle$, for different ring sizes $N_R$. The results for the non-Hermitian model (full curves) are compared with the results for the Hermitian model (symbols). The agreement between the two models persists up to $W_{\text{map}}$ (see vertical arrow). We stress that the linear growth of $W_{\text{sr}}$ with the ring size $N_R$, Eq. (52), is a manifestation of cooperative robustness to disorder.

We stress that the linear growth of $W_{\text{sr}}$, as $N_R \gamma / 4\Omega$, Eq. (51), reduces to

$$W_{\text{sr}} = \sqrt{3N_R \gamma}.$$  \hspace{1cm} (52)

FIG. 12. (Color online) The efficiency $\eta_{\text{sr}}$, Eq. (49), computed starting from the symmetric state of Eq. (25), is plotted versus the disorder strength $W$. The size $N_R$ of the ring has been varied, keeping fixed $\Omega = 1$, $\Omega_L = 10$, $\Omega_L = 100$, and $N_L = 100$. Symbols are obtained with the Hermitian model, while solid curves with the non-Hermitian one. The dashed horizontal line indicates the non-interacting sites efficiency $1 - 1/e$, asymptotically approached by the non-Hermitian evolution. The vertical dashed lines mark the Superradiance transition $W_{\text{sr}}$, Eq. (52), and the arrow indicates $W_{\text{map}}$, Eq. (50).
FIG. 13. (Color online) Probability of being on a ring site at distance $d$ from site 1, obtained by the long-time evolution of an excitation initially localized on site 1, for different values of the ring size $N_R$. The wave function $|\psi^*|^2$ is normalized by setting to 1 the probability of being on the ring. We construct the long-time shape of the probability by letting the system evolve until a steady configuration is reached. We chose the disorder strength $W = 10$ in a regime where Anderson localization should be achieved, while Superradiance is not yet destroyed, that is $W_{\text{loc}} < W < W_{\text{sr}}$. Parameters are $\Omega = 1$, $\Omega_{RL} = 10$, $\Omega_L = 100$, $\gamma = 2$, and $N_L = 100$. The agreement between the Hermitian model $H + D$ (circles) and the non-Hermitian one $H_{\text{eff}} + D$ (solid curves) is very good. The exponential peak on the initially excited site corresponds to the one obtained for a closed ring ($\Omega_{RL} = \gamma = 0$), indicated by the black dashed curve. Dotted horizontal lines mark the values $0.38/N_R$ and have been drawn to highlight the scaling of the plateau with the ring size.

C. Hybrid subradiant states

In Ref. [15, 16] it was shown that the superradiant state does not localize at the finite-size Anderson transition, $W_{\text{loc}}$ Eq. (45), but it starts to localize only above the superradiant transition, $W_{\text{sr}}$. On the other side, subradiant states feel the Anderson transition in way similar to that of the states of the closed system. Specifically, it was shown that, for $W_{\text{loc}} < W < W_{\text{sr}}$, subradiant states display a hybrid nature, with an exponentially localized peak and an extended plateau. The persistence of signatures of Anderson localization in the subradiant regime is somehow surprising: since in this regime the opening is large, one could expect that the long-range coupling induced by the opening would destroy localization. This regime was named subradiant hybrid regime in Ref. [16].

To show that this regime is present also in the Hermitian model, we cannot follow the same procedure that was followed in Ref. [15, 16], where the structure of the eigenstates of the effective Hamiltonian was analyzed. On the other side we can analyze the long-term dynamics of a state initially localized on a single site of the ring. This state has a small overlap with the superradiant state. That component will decay fast, and the dynamics will bring the system in the subradiant subspace with a much slower decay. Thus we can expect that the hybrid structure of the subradiant states will reveal itself in the long-time form of the wave-function.

In order to show this point, in Fig. 13 we plot the probability of being on the ring site $r$, obtained by the long-time evolution $\psi(t=\infty)$ of an excitation localized on site 1. We chose the disorder strength $W$ in a regime where Anderson localization should be achieved, while Superradiance is not yet destroyed, that is $W_{\text{loc}} < W < W_{\text{sr}}$. The probability plotted in Fig. 13 is normalized by setting to 1 the probability of being on the ring.

In the localized regime and in absence of the coupling with the lead (closed model) the diffusion of an excitation initially placed on one site would be suppressed, resulting in a long-time probability distribution exponentially localized on the initial site (see dashed curve in Fig. 13).

On the other side, in the open model we obtain an hybrid state, characterized by an exponential peak on the initial site and a fully extended plateau on the other sites. The important features of this hybrid structure are: (i) the exponential peak coincides with the one obtained in a closed ring (for which $\Omega_{RL} = \gamma = 0$); (ii) the probability on the extended plateau decreases as $1/N_R$ as we increase the ring size. Again we observe that the non-Hermitian model (solid curves) is in very good agreement with the Hermitian one (circles), thus proving that the presence of hybrid subradiant states, described in [16], is a genuine feature of the full Hermitian model from which $H_{\text{eff}}$ is deduced. Importantly, in the limit $N_R \to \infty$, the subradiant states become fully localized. For a more detailed discussion of the origin of this regime see Ref. [16].

VII. CONCLUSIONS

We analyze the problem of describing the transport properties of quantum networks coupled to external environments acting as a sink, in the sense that they absorb the excitation from the network in an irreversible way. To this end, we analyze a paradigmatic model for quantum transport. Our tight-binding model consists of a network of sites arranged in a ring and connected to a central lead. We derive an energy-independent non-Hermitian model which greatly simplifies the analysis of its transport properties. This non-Hermitian model retains only the degrees of freedom of the ring, summarizing the coupling with the infinite degrees of freedom of the lead into non-Hermitian opening terms, which induce a decay of the probability to be on the ring. Such non-Hermitian terms can be obtained from the same quantities which are used in the Fermi Golden Rule: the transition amplitudes from the discrete states of the quantum network to the continuum of states in the external sinks, and the density of states in the sinks.

Such a kind of non-Hermitian models are widely used in literature, but the problem of their validity is often
overlooked. Here, by comparing the results of the full Hermitian model with those given by the non-Hermitian one, we demonstrate that the energy independent non-Hermitian Hamiltonian approach is valid in the regime of large energy band in the lead. Under that condition, we show that the interesting effects usually described with the non-Hermitian model, such as Super and Subradiance in transport, are present also in the full Hermitian model.

We also consider transport from the ring in presence of static disorder. We discuss the validity of the assumption that the opening strength to the continuum is independent of disorder, which is often used in literature since it greatly simplifies the problem. We show that the non-Hermitian Hamiltonian, with opening terms independent of disorder, is able to describe transport in the full Hermitian model up to a critical disorder for which the energy range in the ring is comparable to the energy band in the lead. In this regime we were able to confirm the existence of the interesting effects predicted within the non-Hermitian Hamiltonian approach also in the full Hermitian model. Indeed, superradiant states are cooperatively robust to disorder, while subradiant states show a different behavior, displaying a hybrid nature, due to the interplay of disorder and opening.

Our results have a wide range of applicability: if we consider a generic quantum network of sites coupled to an external lead, the energy-independent non-Hermitian Hamiltonian approach is valid under the condition that the energy band in the lead is much larger than the energy range of the network. In the case of generic external environments acting as sinks, the same approach is effective when the transition amplitudes from the network states to the sink states are smooth functions of the energy, in the range determined by the eigenvalues of the disordered network (including also its decays widths).

ACKNOWLEDGMENTS

We acknowledge useful discussions with A. Biella, F. Borgonovi, R. Fazio, L. Kaplan, F. Izrailev, S. Pascazio, and H. M. Pastawski.

Appendix A

In order to estimate the effect of the finite bandwidth on the decay, we consider a different approximation of the time-evolution operator of Eq. (27), slightly more refined than the one leading to $\mathcal{H}_{\text{eff}}$ and Eq. (29). We assume the bandwidth to be finite, but large enough to justify the following approximation: we consider the transition amplitude $A_r(E)$ and the density of states $\rho(E)$, see Eq. (21), to be constant in the finite energy band $[-2\Omega_L, 2\Omega_L]$. Specifically, we assume that $A_r(E)(A_r(E)^\dagger)\rho(E) = \gamma/2\pi$ inside the energy band of the lead and zero outside. We can now substitute in Eq. (27) the limiting values given by Eq. (28), that corresponds to choosing

$$\Gamma_{sr}(x) = \begin{cases} \gamma N_R & \text{for } x \in [-2\Omega_L, 2\Omega_L], \\ 0 & \text{otherwise}, \end{cases}$$

Moreover we also assume $\Delta_{sr}(x) = 0$. The evolution operator of Eq. (27) becomes then

$$U_S(t, 0) \approx \frac{1}{2\pi Z} \int_{-2\Omega_L}^{2\Omega_L} e^{-\int_{0}^{t} \gamma N_R} \ dx,$$

that is the Fourier transform of a truncated Lorentzian, suitably normalized by means of the factor $Z$ to ensure that $U_S(0, 0) = 1$. The evolution will be well approximated by an exponential only for intermediate times and we will have deviations both for small and large times, due to the truncation at the edges of the energy band of the lead.

Some remarks on the accuracy of the approximation leading to Eq. (A2) are in order. Given the choice of $\Gamma_{sr}(x)$ in Eq. (A1), it is possible to explicitly compute the energy shift

$$\Delta_{sr}(x) = \gamma \log \left| \frac{x + 2\Omega_L}{x - 2\Omega_L} \right|.$$

The latter function is odd, with derivative

$$\Delta'(0) = \frac{\gamma}{2\pi\Omega_L},$$

and slowly divergent as $x$ approaches the edges of the band. We then see that, by setting $\Delta_{sr}(x) = 0$, we obtain an integrand in Eq. (A2) significantly distorted if compared to the exact one (Eq. (27)) only in a neighborhood of the edges of the bandwidth. To minimize the effects of such a distortion, it is then crucial that the maximum point of the exact integrand function lies far enough from the edges of the energy band of the lead. Moreover we need the decay width of the superradiant state to be much smaller than the energy band in the lead. Since the position the maximum point is determined (to leading order) by the average energy of the Superradiant state $\langle S \mid H \mid S \rangle = 2\Omega$, we obtain the conditions:

$$\Omega \ll \Omega_L, \quad \Gamma_{sr} = N_R \gamma \ll 4\Omega_L.$$

These conditions are necessary for the approximation leading to Eq. (A2) to be accurate.

Starting from the evolution operator obtained in Eq. (A2), we can now give an estimate of the times $t_0$ and $t_S$ at which the decay of the survival probability $P(t)$, computed with the Hermitian evolution of $\langle S \rangle$, changes from the quadratic behavior to the exponential decay predicted by the non-Hermitian model ($t_0$) and from the exponential decay to a power-law decay ($t_S$).

The evolution operator $U_S(t, 0)$ of Eq. (A2) is given by the Fourier transform of a Lorentzian function multiplied by a rectangular function with support on $[-2\Omega_L, 2\Omega_L]$. Recalling that the Fourier transform of that rectangular
function is given by
\[ \sin \left( \frac{2\Omega L}{\hbar} t \right) \]
and that the Fourier transform of a product is the convolution of the Fourier transforms, Eq. (A2) becomes
\[ U_S(t,0) = \int_{-\infty}^{+\infty} \frac{\sin \left( \frac{2\Omega L}{\hbar} \tau \right)}{Z\pi \tau} e^{-\frac{\pi i}{\hbar} (t-\tau)} e^{-\frac{\gamma N}{2} |\tau|} d\tau. \]

Now, since
\[ \lim_{\omega \to \infty} \frac{\sin(\omega \tau)}{\pi \tau} = \delta(\tau) \]
for any \( \omega \) in the sense of distributions, if we consider in Eq. (A5) the wide-band limit \( \Omega_L \to \infty \), we immediately recover the evolution given by Eq. (29) for any time \( t \). On the other hand, the effects of a finite bandwidth strongly modify the decay at both small and large times.

Under the assumption of Eq. (A4), we can set \( \Omega \approx 0 \) and neglect the oscillating term \( e^{-\frac{2\pi i}{\hbar} (t-\tau)} \), so that \( U_S(t,0) \) reduces to the convolution of the exponential decay \( e^{-\frac{\gamma N}{2} |\tau|} \) with the kernel
\[ K(\tau) = \frac{\sin \left( \frac{2\Omega L}{\hbar} \tau \right)}{Z\pi \tau}. \]

The normalization factor \( Z \), needed to compensate the approximation \( \Delta(x) \approx 0 \) already introduced in Eq. (A2), can be easily found by applying the normalization condition \( U_S(0,0) = 1 \).

The fact that the small-time decay is quadratic can be easily seen by considering the parity of \( e^{-\frac{\gamma N}{2} |\tau|} \) and \( K(\tau) \): since they are both even functions, their derivatives are odd and
\[ \int_{-\infty}^{+\infty} \frac{\sin \left( \frac{2\Omega L}{\hbar} \tau \right)}{Z\pi \tau} \frac{d}{dt} \left( e^{-\frac{\gamma N}{2} |\tau|} \right) \bigg|_{\tau=0} d\tau = 0. \]

Consequently, the derivative of \( U_S(t,0) \) vanishes for \( t = 0 \) and the decay is quadratic. This is true for any finite value of \( \Omega_L \), but we see a sharp transition to a linear short-time decay in the limit \( \Omega_L \to \infty \).

An intuitive explanation of that transition can be given with the aid of Fig. 14. In the first panel we plotted \( e^{-\frac{\gamma N}{2} |\tau|} \) and \( K(\tau-t) \) for \( t = 0 \). The evolution at each time \( t \) is given by the integral of the product of the exponential and the kernel \( K \) and the dominant contribution for \( t = 0 \) comes from the shaded region in Fig. 14 first panel. The amplitude of this region is twice the inverse of the oscillation frequency \( 2\Omega_L/\hbar \). As we increase \( t \) of a small amount \( dt \), since the shaded region lies on both sides of the peak of the exponential function, the variation in the integral is of order \( O(dt^2) \), producing a quadratic decay. This is no longer true in the limit \( \Omega_L \to \infty \), since \( K \) tends to a Dirac function whose support can lie only on one side of the peak, so that the variation of the convolution integral is of order \( O(dt) \), entailing a linear small-time decay.

From analogous considerations we can estimate \( t_0 \) as the time at which the relevant region (shaded region in Fig. 14 second panel) lies only on one side of the peak of the exponential. Since the peak of the kernel \( K(t-\tau) \) is at \( \tau = t \), we have
\[ t_0 = \frac{\hbar}{2\Omega_L}. \]

For even larger times, together with the previously described exponential term (right shaded region in Fig. 14 third panel), a second term contributes to the convolution integral (left shaded region in Fig. 14 third panel). The first involves the central part of the kernel \( K \) and the tail of the exponential function, while the second involves the tail of the kernel \( K \) and the central part of the exponential function. The first contribution is again proportional to \( e^{-\frac{\gamma N}{2} |\tau|} \) and the second one to \( h/(2\Omega_L t) \). When the term involving the tail of \( K \) is dominant, we have a power-law decay. Hence, we can estimate the transition time \( t_S \) as the time at which the two contributions
are comparable by setting 
\[ e^{-\frac{\gamma N_R}{2\hbar} t_S} \approx \frac{\hbar}{2\Omega_L t_S}, \]
which leads to the equation 
\[ \frac{\gamma N_R}{2\hbar} t_S = \log \frac{4\Omega_L}{\gamma N_R} + \log \frac{\gamma N_R}{2\hbar} t_S \]
and, neglecting the last logarithmic term, to the estimate 
\[ t_S \propto \frac{2\hbar}{\gamma N_R} \log \frac{4\Omega_L}{\gamma N_R}. \quad (A8) \]

The exponent of the power-law decay, being determined by the long-time behavior of the convolution kernel \( K \), is strongly dependent on how the Lorentzian density of Eq. (A2) is deformed to be zero outside the energy band of the lead. Indeed, the sharp truncation considered above, given by the definition of \( \Gamma_{sr} \) in Eq. (A1), corresponds to multiplying the Lorentzian with a rectangular function, that produces a \( 1/t \) decay due to the form of the kernel \( K \) of Eq. (A6).

Nevertheless, we can easily understand the effect of a different deformation: If we multiply the Lorentzian function in Eq. (A2) by a compactly supported function which goes to zero as \( (x - E_{\text{edge}})^2 \) in proximity of the edges of the energy band, by a well-known result in Fourier analysis, we will obtain a convolution kernel \( K \) which decays as \( 1/t^{p+1} \) for large times. Such a modification does not affect any of the foregoing results, but produces a long-time decay of the probability amplitude proportional to \( 1/t^{p+1} \). Consequently, the survival probability \( P(t) \) will decay as \( 1/t^{(p+1)} \).

If we consider now the detailed structure of \( \Gamma_{sr} \) in the finite-bandwidth case, Eq. (A5), we see that it goes to zero in proximity of the edges of the energy band with exponent \( p = 1/2 \). This implies a decay \( 1/t^{1/2} \) of the convolution kernel and the decay \( 1/t^{3/2} \) of the survival probability \( P(t) \), which was indeed found in the numerical results shown in Fig. 11.

Appendix B

To determine the behavior for very short times we will follow now a different approach, more heuristic than the one used in Appendix A. If we consider an initial state on the ring, then it “becomes aware” of the presence of the lead only after some time. In particular, we can expect the initial dynamics to be determined by the interaction of the ring with the first site of the lead. If we had \( \Omega_L = 0 \), the fully symmetric ring state \( |S\rangle \) would be only coupled to the first lead site, and its dynamics would be determined by the \( 2 \times 2 \) Hamiltonian
\[ \begin{pmatrix} 2\Omega \sqrt{N_R \Omega_{RL}} & 0 \\ \sqrt{N_R \Omega_{RL}} & 0 \end{pmatrix}, \]
which has eigenvalues
\[ \lambda_{1,2} = \Omega \pm \sqrt{\Omega^2 + N_R \Omega_{RL}^2}. \]
Consequently, the \( P(t) \) associated with \( |S\rangle \) would be
\[ P(t) = \cos^2 \left( \frac{\lambda_1 - \lambda_2}{2\hbar} t \right), \]
and we obtain the following estimate for the short-time decay of the survival probability of the superradiant state:
\[ P(t) \approx 1 - \frac{\Omega^2 + N_R \Omega_{RL}^2}{\hbar^2} t^2. \quad (B1) \]

According to the foregoing argument, the time \( t_0 \) up to which Eq. (B1) can be a good approximation of the dynamics should decrease upon increasing the coupling \( \Omega_L \) within the lead. Indeed, the value \( t_0 = \hbar/2\Omega_L \), presented in Eq. (A7), gives a good estimate of this threshold.
A 37, 3835 (1988); F. C. Spano, J. R. Kuklinski and S. Mukamel, J. Chem. Phys. 94, 7534 (1991); M. J. Stephen, J. Chem. Phys. 40, 669 (1964); R. H. Lehmberg, Phys. Rev. A 2, 883 (1970).

[14] G. L. Celardo, P. Poli, L. Lussardi and F. Borgonovi, Phys. Rev. B 90, 085142 (2014).

[15] G. L. Celardo, G. G. Giusteri and F. Borgonovi, Phys. Rev. B 90, 075113 (2014).

[16] G. L. Celardo, A. Biella, L. Kaplan, F. Borgonovi Fortschr. Phys. 61, No. 2-3, 250-260 (2013); A. Biella, F. Borgonovi, R. Kaiser, G. L. Celardo, EPL 103 57009 (2013).

[17] J. M. Moix, M. Khasin, and J. Cao, New J. Phys., 15, 085010 (2013).

[18] K. D. B. Higgins, S. C. Benjamin, T. M. Stace, G. J. Milburn, B. W. Lovett, E. M. Gauger, arXiv:1306.1483.

[19] M. Sarovar, K. B. Whaley, New J. Phys. 15, 013030 (2013).

[20] F. C. Spano and S. Mukamel, J. Chem. Phys. 91, 683 (1989).

[21] Messiah, Quantum Mechanics, Vol II, chap. XXI.13, North Holland Publishing Company Amstredam (1962).

[22] H. Fidder, J. Knoester and D. A. Wiersma, J. Chem. Phys. 95, 7880 (1991); J. Moll, S. Daehne, J. R. Durrant and D. A. Wiersma, J. Chem. Phys. 102, 6362 (1995).

[23] X. Hu, T. Ritz, A. Damjanovic, K. Schulten, J. Phys. Chem. B 101, 3854 (1997).

[24] P. Facchi and S. Pascazio, La Regola d’oro di Fermi, Quaderni di Fisica Teorica, Bibliopolis (1999).

[25] A. Peres, Annals of Physics 129, 33 (1980).

[26] H. M. Pastawski, Physica B 398, 278 (2007); E. Rufeil Fiori, H. M. Pastawski, Chem. Phys. Lett. 420, 35 (2006).

[27] S. Bochner, K. Chandrasekharan, Fourier Transforms, Princeton University Press (1949).