Effects of node position on diffusion and trapping efficiency for random walks on fractal scale-free trees

Junhao Peng\textsuperscript{1,2} and Guoai Xu\textsuperscript{3}

\textsuperscript{1} College of Math and Information Science, Guangzhou University, Guangzhou 510006, People’s Republic of China
\textsuperscript{2} Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, People’s Republic of China
\textsuperscript{3} State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, People’s Republic of China

E-mail: pengjh@gzhu.edu.cn and xga@bupt.edu.cn

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Abstract. We study unbiased discrete random walks on fractal scale-free trees (FSFTs) based on their self-similar structure and the relations between random walks and electrical networks. Firstly, we provide new methods to derive analytic solutions of the mean first-passage time (MFPT) for any pair of nodes, the mean trapping time (MTT) for any target node and the mean diffusing time (MDT) for any starting node. Then, using the MTT and the MDT as the measures of trapping efficiency and diffusion efficiency respectively, we analyze the effect of a trap’s position on the trapping efficiency and the effect of the starting position on the diffusion efficiency. Comparing the trapping efficiency and diffusion efficiency among all nodes of an FSFT, we find the best (or worst) trapping sites and the best (or worst) diffusing sites. Our results show that the node at the center of the FSFT is the best trapping site, but it is also the worst diffusing site. The nodes that are farthest from the two hubs are the worst trapping sites, but they are also the best diffusion sites. Comparing the maxima of the MTT and MDT with their minima, we find that the maximum of the MTT is about \((20m^2 + 32m + 12)/(4m^2 + 4m + 1)\) times the minimum of the MTT, whereas the maximum of the MDT is almost equal to the minimum of the MDT. These results show that the
position of the target node has a large effect on the trapping efficiency, but the position of the starting node has almost no effect on the diffusion efficiency. We also conducted numerical simulations which showed they are in good agreement with the derived results.

**Keywords:** dynamical processes (experiment), dynamical processes (theory), stochastic processes (theory), network dynamics

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1. Introduction

Random walks on fractals, which can be applied as models for transport in disordered media [1, 2], have attracted a great deal of interest [3]–[6]. The range of applicability and physical interest is enormous [7]–[13].

A basic quantity relevant to random walks is the trapping time or mean first-passage time (MFPT), which is the expected number of steps to hit the target node (or trap) for the first time, for a walker starting from a source node. It is a quantitative indicator to characterize the transport efficiency, and many other quantities can be expressed in terms of it. By locating the target node (or trap) at one special node and averaging the MFPTs over all the starting nodes, we get the mean trapping time (MTT) for the special node. By locating the source node at one special node and then averaging the MFPTs over all the target nodes, we obtain the mean diffusing time (MDT) for the special node. Both the MTT and the MDT vary with the position of the node and they can be used as the measures of trapping efficiency and diffusion efficiency for network nodes respectively. By comparing the MTT and MDT among all the network nodes, we can find the effects of node position on the trapping efficiency and diffusion efficiency. The nodes that have the minimum MTT (or the maximum MTT) are the best (or worst) trapping sites and the nodes that have the minimum MDT (or maximum MDT) are the best (or worst) diffusion sites.

In the past few years, the MFPT for random walks on fractals has been extensively studied [6], [14]–[19]. For example, the MTT for some special nodes has been obtained for different fractals (or networks), such as Sierpinski gaskets [14], Apollonian networks [20], pseudofractal scale-free webs [21], deterministic scale-free graphs [22] and some special trees [23]–[28]. The MDT for some special nodes has been obtained for exponential treelike networks [28], scale-free Koch networks [29] and deterministic scale-free graphs [30]. There have also been some works focusing on the global mean first-passage time (GMFPT), i.e., the average of MFPTs over all pairs of nodes; these results were obtained for some special trees [24]–[26], [28, 31, 32] and dual Sierpinski gaskets [33].

However, the previous results for the MTT and MDT are restricted to only some special nodes and we cannot compare the trapping efficiency and diffusion efficiency among all the network nodes. It is still difficult to derive the analytic solutions of the MTT for any target node (or trap) and the MDT for any source node in these networks.

In 2007, Rozenfeld et al [34] introduced two deterministic networks (i.e., (u, v)-flowers and (u, v)-trees) which could mimic real networks [35]. Subsequently, random walks on this kind of network attracted a great deal of interest [34, 36, 37]. Recursive fractal scale-free trees (FSFTs) are variants of (u, v)-trees [38]. For random walks on FSFTs, the MTT for the hubs and the GMFPT have been obtained [38], the MTT for some low-generation nodes can also be derived by the methods of [37]. However, analytic calculations of the MFPT for any pair of nodes, the MTT for any target node and the MDT for any source node are still unresolved.

In this paper, based on the self-similar structure of the FSFT and the relations between random walks and electrical networks [39, 40], we provide new methods to derive analytic solutions of the MFPT for any pair of nodes, the MTT for any target node and the MDT for any starting node.
Effects of node position on diffusion and trapping efficiency in FSF trees

Figure 1. The iterative construction method of the FSFT. For each edge of $G(t-1)$, we replace it by a cluster on the right-hand side of the arrow, where large green solid squares stand for the new external nodes, the small blue solid square stands for the new internal node and the solid circles represent the original nodes.

Furthermore, using the MTT and the MDT as the measures of trapping efficiency and diffusion efficiency respectively, we compare the trapping efficiency and diffusion efficiency among all nodes of the FSFT and find the best (or worst) trapping sites and the best (or worst) diffusing sites. Our results show that the central node of the FSFT is the best trapping site, but it is also the worst diffusing site, and that the nodes that are the farthest from the two hubs are the worst trapping sites, but they are also the best diffusion sites. Comparing the maxima of the MTT and MDT with their minima, we find that the maximum of the MTT is about $(20m^2 + 32m + 12)/(4m^2 + 4m + 1)$ times the minimum of the MTT, whereas the maximum of the MDT is almost equal to the minimum of the MDT. These results show that the position of the target node has a large effect on the trapping efficiency, but the position of the source node has almost no effect on the diffusion efficiency.

The method we present can also be used to solve the problem of the MFPT on other self-similar trees.

2. Brief introduction to the FSFT

The recursive fractal scale-free trees (FSFTs) we consider can be constructed iteratively [38, 41]. For convenience, we call the times of iteration the generations of the FSFT and denote by $G(t)$ the FSFT of generation $t$. For $t = 0$, $G(0)$ is an edge connecting two nodes. For $t > 0$, $G(t)$ is obtained from $G(t-1)$ by performing the following operations on every edge, as shown in figure 1. Replace the edge by a path two links long, with the two endpoints of the path being the same endpoints as those of the original edge and the new node having an initial degree 2 (called an internal node) being in the middle of the path; then attach $m$ new nodes with initial degree 1 (called external nodes) to each endpoint of the path.

The FSFT $G(t)$ can also be constructed by another method, highlighting the self-similarity that is shown in figure 2 [38]. The FSFT $G(t)$ is composed of $2m + 2$ copies, called subunits, of $G(t-1)$ which are connected to one another by its two hubs (i.e., nodes with the highest degree).

This type of network presents the following interesting structural features. They are scale free [41, 42] and fractal with the fractal dimension $d_f = \ln(2m+2)/\ln 2$ [7, 8, 34]. The total number of edges $E_t$ and the total number of nodes $N_t$ [38, 41] satisfy

$$E_t = (2m+2)^t,$$

$$N_t = (2m+2+1)^t.$$
Figure 2. Alternative construction of the FSFT which highlights the self-similarity. The FSFT of generation $t$, denoted by $G(t)$, is composed of $2m + 2$ copies, called subunits, of $G(t - 1)$ which are labeled as $G_0(t), G_1(t), G_2(t), \ldots, G_{2m+1}(t)$, and connected to one another at its two hubs $A$ and $B$.

$N_t = 1 + E_t = 1 + (2m + 2)^t$. \hfill (2)

3. Formulation of the problem

Let $F(x, y)$ denote the MFPT from nodes $x$ to $y$ in the FSFT $G(t)$ and $\Omega$ denote the node set of $G(t)$; the sum

$$k(x, y) = F(x, y) + F(y, x)$$

is called the commute time and the MFPT can be expressed in term of commute times \cite{39},

$$F(x, y) = \frac{1}{2} \left( k(x, y) + \sum_{u \in \Omega} \pi(u)[k(y, u) - k(x, u)] \right), \hfill (3)$$

where $\pi(u) = d_u/2E_t$ is the stationary distribution for random walks on the FSFT and $d_u$ is the degree of node $u$.

If we view the networks under consideration as electrical networks by considering each edge to be a unit resistor and let $\Psi_{xy}$ denote the effective resistance between two nodes $x$ and $y$ in the electrical networks, we have \cite{39}

$$k(x, y) = 2E_t \Psi_{xy}, \hfill (4)$$

where $E_t$ is the total number of edges of $G(t)$. Since the FSFT we study is a tree, the effective resistance between any two nodes is exactly the shortest path length between the two nodes. Hence,

$$\Psi_{xy} = L_{xy}, \hfill (5)$$

where $L_{xy}$ denotes the shortest path length between nodes $x$ and $y$. Thus,

$$k(x, y) = 2E_t L_{xy}. \hfill (6)$$
Substituting \( k(x, y) \) from equation (6) in equation (3), we obtain

\[
F(x, y) = E_t \left( L_{xy} + \sum_{u \in \Omega} \pi(u)L_{yu} - \sum_{u \in \Omega} \pi(u)L_{xu} \right).
\]

(7)

Averaging the MFPTs over all the starting nodes and all target nodes, we obtain the MTT and MDT. That is to say, if we define

\[
T_y = \frac{1}{E_t} \sum_{x \in \Omega, x \neq y} F(x, y),
\]

(8)

\[
D_x = \frac{1}{E_t} \sum_{y \in \Omega, y \neq x} F(x, y),
\]

(9)

\( T_y \) is just the MTT for target node \( y \) and \( D_x \) is just the MDT for starting node \( x \). Let

\[
S_x = \sum_{y \in \Omega} L_{xy},
\]

(10)

\[
W_x = \sum_{u \in \Omega} \pi(u)L_{xu} = \frac{1}{2E_t} \sum_{u \in \Omega} (L_{xu}d_u),
\]

(11)

\[
\Sigma = \sum_{u \in \Omega} \left( \pi(u) \sum_{x \in \Omega} L_{xu} \right);
\]

(12)

we find that the MFPT from \( x \) to \( y \) can be rewritten as

\[
F(x, y) = E_t(L_{xy} + W_y - W_x).
\]

(13)

Substituting \( F(x, y) \) from equation (13) in equations (8) and (9), we obtain

\[
T_y = S_y + N_tW_y - \Sigma,
\]

(14)

\[
D_x = S_x + \Sigma - N_tW_x.
\]

(15)

Hence, if we can calculate \( \Sigma \) and \( S_x, W_x \) for any node \( x \), we can calculate the MFPT for any pair of nodes \((x, y)\), and the MTT and MDT for any node \( x \). Although it is difficult to calculate these quantities for a general tree, we present methods to calculate these quantities for the FSFT based on its self-similar structure. Therefore, we can calculate the MFPT for any pair of nodes and the MTT and MDT for any node.

4. Detailed methods for calculating the MFPT, MTT and MDT

4.1. Method for calculating \( S_x \) and \( W_x \)

For convenience, we classify the nodes of \( G(t) \) into different levels. Nodes that are generated before \( k \) (including \( k \)) times of iteration are said to belong to level \( k \) in this paper. Thus, nodes that belong to level \( k \) also belong to levels \( k + 1, k + 2, \ldots, t \). For example, in the second-generation FSFT with \( m = 2 \), which is shown in figure 3, the level
Figure 3. The construction of the FSFT of generation 2 for the limiting case of $m = 2$. The level information of the nodes is as follows: nodes represented by black solid squares belong to levels 0, 1, 2; nodes represented by large red circles belong to levels 1, 2; nodes represented by small blue circles belong to level 2. The subunit represented by the blue dotted line is labeled by a sequence $\{2, 4\}$.

Figure 4. The construction of $\Lambda_{k-1}$ and the relation between the value of $i_k$ and the location of subunit $\Lambda_k$ in $\Lambda_{k-1}$. The subunits represented by blue dashed edges are the subunits $\Lambda_k$ corresponding to the values of $i_k$ below, whose two hubs are labeled as $A_k, B_k$.

For convenience, we label the two hubs of subunit $\Lambda_k$ as $A_k, B_k$ and build the mapping between hubs of $\Lambda_{k-1}$ and hubs of $\Lambda_k$ as shown in figure 4. The hub of $\Lambda_{k-1}$ labeled as $i_k = 0$ is mapped to the hub of $\Lambda_k$ labeled as $i_k = 1, 2, \ldots, 2m + 1$.

For example, in the FSFT of generation 2 shown in figure 3, the subunit represented by the blue dashed edge, which is a subunit of level 2, is labeled by a sequence $\{2, 4\}$. We also classify the subunits of $G(t)$ into different levels and let $\Lambda_k$ denote the subunit of level $k$ ($k \geq 0$). In this paper, $G(t)$ is said to be a subunit of level 0. For any $k \geq 0$, the $2m + 2$ subunits of $\Lambda_k$ are said to be subunits of level $k + 1$. Thus, any edge of $G(t)$ is a subunit of level $t$ and $\Lambda_k$ is a copy of the FSFT with generation $t - k$. Similarly to the method of [37], we label the subunit $\Lambda_k$ ($1 \leq k \leq t$) by a sequence $\{i_1, i_2, \ldots, i_k\}$, where $i_j$ ($1 \leq j \leq k$) labels its position in its parent subunit $\Lambda_{j-1}$. Figure 4 shows the construction of $\Lambda_{k-1}$ and the relation between the value of $i_k$ and the location of subunit $\Lambda_k$ in $\Lambda_{k-1}$: all subunits $\Lambda_k$ are represented by edges, the ones represented by blue dashed edges are the subunits $\Lambda_k$ corresponding to values of $i_k = 0, 1, 2, \ldots, 2m + 2$. For example, in the FSFT of generation 2 shown in figure 3, the subunit represented by the blue dashed edge, which is a subunit of level 2, is labeled by a sequence $\{2, 4\}$. For convenience, we label the two hubs of subunit $\Lambda_k$ as $A_k, B_k$ and build the mapping between hubs of $\Lambda_{k-1}$ and hubs of $\Lambda_k$ as shown in figure 4.
\(A_{k-1}\) is also a hub of \(\Lambda_k\) labeled as \(A_k\), while \(i_k = 0, 1, 2, \ldots, m\). The hub of \(A_{k-1}\) labeled as \(B_{k-1}\) is also a hub of \(\Lambda_k\) labeled as \(B_k\), while \(i_k = m + 1, m + 2, \ldots, 2m + 1\).

For any \(k \geq 0\), we define
\[
S^{(k)} = \begin{pmatrix} S_{A_k} \\ S_{B_k} \end{pmatrix}, \quad W^{(k)} = \begin{pmatrix} W_{A_k} \\ W_{B_k} \end{pmatrix}.
\]

As derived in appendix A, for any \(k > 0\), the \(S^{(k)}\) satisfy the following recursion relations:
\[
S^{(k)} = \mathcal{M}_{ik} S^{(k-1)} + \mathcal{V}^{(k)}_{ik}, \quad i_k = 0, 1, 2, \ldots, 2m + 1,
\]
where
\[
\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \quad \mathcal{V}^{(k)}_{0} = \begin{pmatrix} 0 \\ -2^{t-k} (2m + 2)^{t-k} \end{pmatrix}, \quad i_k = 1, 2, \ldots, m,
\]
\[
\mathcal{M}_{ik} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{V}^{(k)}_{ik} = \begin{pmatrix} 0 \\ \eta_k \end{pmatrix}, \quad i_k = m + 1, m + 2, \ldots, 2m,
\]
\[
\mathcal{M}_{2m+1} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{V}^{(k)}_{2m+1} = \begin{pmatrix} -2^{t-k} (2m + 2)^{t-k} \\ 0 \end{pmatrix},
\]
and \(\eta_k = 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k}]\).

As derived in appendix B, for any \(k > 0\), the \(W^{(k)}\) satisfy the following recursion relations:
\[
W^{(k)} = \mathcal{M}_{ik} W^{(k-1)} + \mathcal{U}^{(k)}_{ik}, \quad i_k = 0, 1, 2, \ldots, 2m + 1,
\]
where the \(\mathcal{M}_{ik}(i_k = 0, 1, 2, \ldots, 2m + 1)\) are given by equations (18)–(21) and the \(\mathcal{U}_{ik}(i_k = 0, 1, 2, \ldots, 2m + 1)\) are given by
\[
\mathcal{U}^{(k)}_{0} = \begin{pmatrix} 0 \\ -2^{t-k} (2m + 2)^{-k} \end{pmatrix}, \quad \mathcal{U}^{(k)}_{2m+1} = \begin{pmatrix} -2^{t-k} (2m + 2)^{-k} \\ 0 \end{pmatrix},
\]
\[
\mathcal{U}^{(k)}_{ik} = \begin{pmatrix} 0 \\ 2^{t-k} [1 - (2m + 2)^{-k}] \end{pmatrix}, \quad i_k = 1, 2, \ldots, m,
\]
\[
\mathcal{U}^{(k)}_{ik} = \begin{pmatrix} 0 \\ 2^{t-k} [1 - (2m + 2)^{-k}] \end{pmatrix}, \quad i_k = m + 1, m + 2, \ldots, 2m.
\]

Using equation (17) repeatedly, for any \(k \geq 1\), we obtain
\[
S^{(k)} = \mathcal{M}_{ik} S^{(k-1)} + \mathcal{V}^{(k)}_{ik}
\]
\[
= \mathcal{M}_{ik} \mathcal{M}_{ik-1} S^{(k-2)} + \mathcal{M}_{ik} \mathcal{V}^{(k-1)}_{ik-1} + \mathcal{V}^{(k)}_{ik}
\]
\[
= \ldots
\]
\[
= \mathcal{M}_{ik} \mathcal{M}_{ik-1} \cdots \mathcal{M}_{i1} S^{(0)} + \sum_{l=1}^{k-1} \mathcal{M}_{ik} \mathcal{M}_{ik-1} \cdots \mathcal{M}_{i_l+1} \mathcal{V}^{(l)}_{i_l} + \mathcal{V}^{(k)}_{ik}
\]

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Similarly, using equation (22) repeatedly, for any \( k \geq 1 \), we get

\[
W^{(k)} = M_{ik}M_{ik-1}\cdots M_{i1}W^{(0)} + \sum_{l=1}^{k-1} M_{ik}M_{ik-1}\cdots M_{il+1}U_{il} + U_{ik}^k. \tag{27}
\]

For \( S^{(0)} \) and \( W^{(0)} \), it is straightforward that

\[
S^{(0)} = (S_{A_0}, S_{B_0})^T = S_{A_0}(1, 1)^T, \tag{28}
\]

\[
W^{(0)} = (W_{A_0}, W_{B_0})^T = W_{A_0}(1, 1)^T, \tag{29}
\]

where \((x, y)^T\) is the transpose of vector \((x, y)\) and \(S_{A_0}\) and \(W_{A_0}\) are \(S_x\) and \(T_x\) for nodes of level 0 respectively, which have been derived in appendix C.

Noticing that any edge of \( G(t) \) is a subunit of level \( t \), its two end nodes are just its two hubs. If we know the label sequence \( \{i_1, i_2, \ldots, i_t\} \) for any edge of \( G(t) \), we can exactly calculate \( S^{(t)} \) and \( W^{(t)} \) for its two end nodes. Hence, we can derive the expression of \( S_x \) and \( W_x \) for any node \( x \) of \( G(t) \).

### 4.2. Exact calculation of \( \Sigma \)

We find that

\[
\Sigma = \sum_{u \in \Omega} (\pi(u) \sum_{x \in \Omega} L_{xu}) = \frac{1}{2E_t} \sum_{u \in \Omega} (d_u S_u) \tag{30}
\]

and \( \sum_{u \in \Omega} (d_u S_u) \) is just the summation of \( S_x \) for the two end nodes of every edge of \( G(t) \).

(Note that for node \( x \) which is the intersection of \( n \) edges, \( S_x \) will be counted \( n \) times.) Because any edge of \( G(t) \) is a subunit of level \( t \), which is in one to one correspondence with a sequence \( \{i_1, \ldots, i_t\} \), its two end nodes are also its two hubs labeled as \( A_t, B_t \).

Thus,

\[
\sum_{u \in \Omega} (d_u S_u) = \sum_{\{i_1, \ldots, i_t\}} \left( \sum_{\{i_1, \ldots, i_t\}} S^{(t)} \right). \tag{31}
\]

For the right-hand side of the equation, the second summation is run over all the subunits of level \( t \) (i.e., let \( \{i_1, \ldots, i_t\} \) run over all the possible values); the first summation just adds the two entries of \( \sum_{\{i_1, \ldots, i_t\}} S^{(t)} \) together.

Making use of the following identity:

\[
\sum_{\{i_1, \ldots, i_l\}} \sum_{l=1}^{t-1} = \sum_{l=1}^{t-1} \sum_{\{i_1, \ldots, i_l\}}
\]

and defining

\[
M_{\text{tot}} = \sum_{i=0}^{2m+1} M_i, \tag{32}
\]

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\[ \psi_{\text{tot}}^l = \sum_{i=0}^{2m+1} \psi_{i}^l, \quad (33) \]

we have

\[ \sum_{\{i_1, \ldots, i_t\}} M_{i_1} M_{i_{t-1}} \cdots M_{i_{t+1}} \psi_{i_t} = (2m + 2)^{t-1} M_{\text{tot}}^{-1} \psi_{\text{tot}}^l. \quad (34) \]

Thus,

\[ \sum_{\{i_1, \ldots, i_t\}} S^{(t)} = \sum_{\{i_1, \ldots, i_t\}} \left[ M_{i_1} M_{i_{t-1}} \cdots M_{i_t} S^{(0)} + \sum_{l=1}^{t-1} M_{i_1} M_{i_{t-1}} \cdots M_{i_{t-1}} \psi_{i_l} + \psi_{i_t} \right] \]

\[ = M_{\text{tot}}^{l} S^{(0)} + \sum_{l=1}^{t-1} (2m + 2)^{l-1} M_{\text{tot}}^{-1} \psi_{\text{tot}}^l + (2m + 2)^{t-1} \psi_{\text{tot}}^l \]

\[ = M_{\text{tot}}^{l} S^{(0)} + \sum_{l=1}^{t-1} (2m + 2)^{l-1} M_{\text{tot}}^{-1} \psi_{\text{tot}}^l. \quad (35) \]

Substituting \( M_i \) from equations (18) to (21) in equation (32), and orthogonally decomposing \( M_{\text{tot}} \), we obtain

\[ \begin{pmatrix} m + 3/2 & m \\ \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix}. \quad (36) \]

Therefore,

\[ \begin{pmatrix} \sqrt{2} & 2 \\ \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix}. \quad (37) \]

Similarly, we get

\[ \psi_{\text{tot}}^l = [m 2^{t-k}(2m + 2)^{t} - 2^{t}(m + 1)(2m + 2)^{t-k}] \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (38) \]

Thus,

\[ \begin{pmatrix} \sqrt{2} & 2 \\ \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} S_{A_0} \]

\[ = (2m + 2)^{t} S_{A_0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (39) \]

\[ \text{doi:10.1088/1742-5468/2014/04/P04032} \]
and

\[ \sum_{l=1}^{t} (2m + 2)^{l-1} M_{\text{tot}}^{l-1} \psi_{\text{tot}}^l = \sum_{l=1}^{t} (2m + 2)^{l-1} \left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \left( \begin{array}{c} (2m + 2)^{l-1} \\ 0 \end{array} \right) \right\} \]

\times \left( \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \]

\[ = \sum_{l=1}^{t} (2m + 2)^{l-1} \left[ m2^{l-1}(2m + 2)^{t} - 2^l(m + 1)(2m + 2)^{t-l} \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = \left[ m(2m + 2)^{t-1}(2^t - 1) - (m + 1)(2m + 2)^{t-1}(4m + 4)^t - \frac{1}{4m + 3} \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (40) \]

Inserting equations (39), (40) and (C.2) into equation (35), we obtain

\[ \sum_{\{i_1, \ldots, i_t\}} S^{(t)} = M_{\text{tot}}^{t} S^{(0)} + \sum_{l=1}^{t} (2m + 2)^{l-1} M_{\text{tot}}^{l-1} \psi_{\text{tot}}^l \]

\[ = \left\{ (2m + 2)^{2t} + (2^t - 1)(3m + 1)(2m + 2)^{2t-1} \right. \]

\[ - \frac{m + 1}{4m + 3} (2m + 2)^{2t-1} [(4m + 4)^t - 1] \left\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (41) \]

Replacing \( \sum_{\{i_1, \ldots, i_t\}} S^{(t)} \) from equation (41) in equation (31), we get

\[ \Sigma = \frac{1}{2E_t} \sum_{u \in \Omega} (d_u S_u) \]

\[ = \left( 3m + 1 - \frac{m + 1}{4m + 3} \right) 2^t(2m + 2)^{t-1} - \frac{m - 1}{2m + 2}(2m + 2)^t + \frac{m + 1}{(4m + 3)(2m + 2)}. \quad (42) \]

4.3. Examples

We first calculate the MTT or MDT for nodes of level 0 which are labeled as \( A_0 \) and \( B_0 \). Inserting equations (C.2) and (C.4) and (42) into equations (14) and (15), we obtain the MTT and MDT for nodes \( A_0 \) and \( B_0 \),

\[ T_{B_0} = T_{A_0} = S_{A_0} + N_t W_{A_0} - \Sigma \]

\[ = \frac{2m + 2}{4m + 3} 2^t(2m + 2)^{t-1} + \frac{2m + 1}{2m + 2} 2^t - \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}. \quad (43) \]
\[ D_B = D_A = S_A + \Sigma - N_i W_A \]
\[ = \frac{1}{m+1}(2m+2)^l + \frac{6m^2 + 6m + 1}{(4m+3)(m+1)}2^l(2m+2)^t \]
\[ - \frac{2m+1}{2m+2}2^l + \frac{4m^2 + 4m + 1}{(4m+3)(2m+2)}. \] (44)

In general, for any node \( x \) of \( G(t) \), it must be a hub of a certain subunit \( \Lambda_k (1 \leq k \leq t) \). Given the label sequences \( \{i_1, i_2, \ldots, i_k\} \) of the corresponding subunit, we can derive the MTT and MDT for the node \( x \). For any node pair \((x, y)\), if we know the label sequences for the corresponding subunits whose hubs include \( x \) or \( y \), we can calculate the MFPT between the two nodes. In order to explain our methods, we select two special subunits \( \Lambda_k \) and \( \Lambda_j \).

For the first subunit \( \Lambda_k \) (with \( k \) even), its labels satisfy \( i_l = 1 \) for odd \( l \) and \( i_l = m+1 \) for even \( l \). Thus, it is labeled as \( \{1, m+1, 1, m+1, \ldots, 1, m+1\} \). According to the structure of the FSFT, one of its hubs belongs to level \( k - 1 \), which is the farthest node from nodes \( A_0 \) and \( B_0 \) among all nodes of level \( k - 1 \); the other hub belongs to level \( k \), which is the farthest node from nodes \( A_0 \) and \( B_0 \) among all nodes of level \( k \). The reason for selecting this subunit is that we can derive the MTT or MDT for the farthest node from nodes \( A_0 \) and \( B_0 \) among all nodes of level \( k \) (\( 1 \leq k \leq t, k \) can be odd because \( k - 1 \) is odd). For convenience, we denote by \( H_k \) the farthest nodes from nodes \( A_0 \) and \( B_0 \) among all nodes of level \( k \). For example, in the special FSFT of generation 2, \( H_1 \) and \( H_2 \) are labeled in figure 3. Let \( i_1 = 1, i_2 = m + 1, \ldots, i_{k - 1} = 1, i_k = m + 1 \) in equations (26) and (27); we obtain

\[ S_{H_k} = S_{A_0} + (4m + 4)^l \frac{4m + 2}{4m + 3} - 2^{t-k}(2m + 2)^t + \frac{(4m + 4)^{t-k}}{4m + 3}, \] (45)
\[ W_{H_k} = W_{A_0} + 2^l \frac{4m + 2}{4m + 3} - 2^{t-k} + \frac{2^l}{(4m + 4)^k(4m + 3)}. \] (46)

The detailed derivation of equations (45) and (46) is shown in appendix D.

For the second subunit \( \Lambda_j \) (\( 2 \leq j \leq t \)), its labels satisfy that \( i_l = 0 \) and \( i_l = 2m + 1 \) for \( l > 1 \). Thus, it is labeled as \( \{0, 2m + 1, \ldots, 2m + 1\} \). One of its hubs is the center node \( O_0 \) (see figure 6); the other one belongs to level \( j \), which is the nearest internal node from the center node \( O_0 \) among all internal nodes of level \( k \). The reason for selecting this subunit is that we can derive the MTT or MDT for the nearest internal node from the center node \( O_0 \) among all internal nodes of level \( k \). For convenience, we denote by \( Q_j \) the nearest internal node to the center node \( O_0 \) among all internal nodes of level \( j \). For example, in the special FSFT of generation 2, \( Q_2 \) is labeled in figure 3. Let \( i_1 = 0, i_2 = 2m + 1, \ldots, i_{j - 1} = 2m + 1, i_j = 2m + 1 \) in equations (26) and (27); we have

\[ S_{Q_j} = S_{A_0} - (4m + 4)^{t-j - 1} + \frac{2m}{2m + 1}2^{t-j}(2m + 2)^{t-j} + \frac{(4m + 4)^{t-j}}{2m + 1}, \] (47)
\[ W_{Q_j} = W_{A_0} - \frac{2^{t-j} - 2^{t-k}}{2m + 2} - \frac{2^{t-k}}{2m + 1} \left( \frac{1}{2m + 2} - \frac{1}{(2m + 2)^k} \right). \] (48)
The detailed derivation of equations (47) and (48) is shown in appendix E. As derived in appendix F, the distance between \( H_k \) and \( Q_j \) is

\[
L_{H_kQ_j} = 2^t - 2^{t-k} + 2^{t-1} - 2^{k-j}.
\]

Substituting \( W_x, W_y, L_{xy} \) from equations (46), (48) and (49) respectively in equation (13), we obtain the MFPT from \( H_k \) to \( Q_j \),

\[
F(H_k, Q_j) = (4m + 4)^{t-1} \frac{8m^2 + 14m + 7}{4m + 3} + \frac{(4m + 4)^{t-j}}{2m + 1}
- (2m + 2)^{t-1}2^{t-j} \frac{4m^2 + 4m + 2}{2m + 1} - \frac{(4m + 4)^{t-k}}{4m + 3}.
\]

Similarly, the MFPT from \( Q_j \) to \( H_k \) is

\[
F(Q_j, H_k) = (4m + 4)^{t-1} \frac{40m^2 + 70m + 29}{4m + 3} - 2^{t-k+1} (2m + 2)^t - \frac{(4m + 4)^{t-j}}{2m + 1}
- (2m + 2)^{t-1}2^{t-j} \frac{4m^2 + 8m + 2}{2m + 1} + \frac{(4m + 4)^{t-k}}{4m + 3}.
\]

Inserting equations (45), (46) and (42) into equations (14) and (15), we obtain the MTT and MDT for node \( H_k \),

\[
T_{H_k} = (4m + 4)^t \frac{10m + 6}{4m + 3} - 2^{t-k+1} (2m + 2)^t - 2^{t-k} + \frac{2(4m + 4)^{t-k}}{4m + 3}
+ 2^t \frac{16m^2 + 22m + 7}{(4m + 3)(2m + 2)} + \frac{2^{t-j}}{(4m + 3)^2(4m + 3)} - \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)^2}.
\]

\[
D_{H_k} = (4m + 4)^t \frac{6m^2 + 6m + 1}{(4m + 3)(m + 1)} + \frac{(2m + 2)^t}{m + 1} - \frac{2^{t-k}}{(2m + 2)^2(4m + 3)}
- 2^t \frac{16m^2 + 22m + 7}{(4m + 3)(2m + 2)} + \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}.
\]

Similarly, the MTT and MDT for node \( Q_j \) are

\[
T_{Q_j} = (4m + 4)^t \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)} + \frac{2^t 4m + 1}{4m + 4} + 2^{t-j} (2m + 2)^{t-1} \frac{4m}{2m + 1}
+ \frac{2(4m + 4)^{t-j}}{2m + 1} + \frac{2^{t-j} 2m}{(2m + 2)(2m + 1)} + \frac{2^{t-j}}{(2m + 2)^2(2m + 1)}
- \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}.
\]

\[
D_{Q_j} = (4m + 4)^t \frac{12m^2 + 12m + 2}{(4m + 3)(2m + 2)} + \frac{(2m + 2)^t}{m + 1} - \frac{2^t 4m + 1}{4m + 4}
- \frac{2^{t-j} 2m}{(2m + 2)(2m + 1)} - \frac{2^{t-j}}{(2m + 2)^2(2m + 1)} + \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}.
\]
We simulated random walks on the FSFT (with $m = 2$, $t = 4$ and a total number of nodes of 1297) to test our results. Locating the target (or starting) node at $H_k (k = 1, 2, \ldots, t)$ (or $Q_j, j = 2, \ldots, t$), simulating random walks starting from (or ending at) all other nodes 1 time, writing down the first-passage times (FPTs) from the starting nodes to the target nodes, and averaging the FPTs over all the possible starting (target) nodes, we obtain the MTT (or MDT) for these nodes. The results of a 100 times simulation for nodes $H_k (k = 1, 2, 3, 4)$ are given in figure 5, which shows good agreement. Figure 5(b) also shows that the position has almost no effect on the MDT because the four lines for $\text{MDT}_{H_k} (k = 1, 2, 3, 4)$ are almost overlapped. Averaging the 100 times simulation results and comparing them with the derived results, we find that the relative error is less than $10^{-3}$.

We also simulated the MFPT between $H_k (k = 1, 2, \ldots, t)$ and $Q_j (j = 2, \ldots, t)$ for 100 000 times and calculated the relative error; we found that the relative error was less than $10^{-4}$.

Numerical simulations for other cases, with $m = 1$ or 2 and $t = 2, 3, 4$ or 5, were also conducted. The results are similar to the case with $m = 2$ and $t = 4$. For larger $m$ and $t$, simulation is difficult because it is quite time-consuming.

5. Effect of node position on trapping efficiency for random walks on the FSFT

We derive the relations between the $T_x$ for nodes of level $k$ and that for nodes of level $k + 1$, and compare the $T_x$ for nodes of adjacent levels.

Considering any subunit of level $k$ as shown in figure 6, it is composed of $2m + 2$ subunits of level $k + 1$ (black oval with solid line). Its two hubs (i.e., $A_k$ and $B_k$) are the only two nodes of level $k$; its nodes of level $k + 1$ are hubs of its $2m + 2$ subunits of level
Figure 6. The construction of subunit $A_k$. It is composed of $2m + 2$ subunits of level $k + 1$ labeled as $G_0, G_1, \ldots, G_{2m+1}$. The nodes labeled as $A_k, B_k, O_k, C_k$ and $R_k$ are all hubs of the $2m + 2$ subunits.

$k + 1$ (i.e., $A_k, B_k, O_k, C_k$ and $R_k$). Assuming that the $T_x$ for nodes of level $k$ (i.e., $T_{A_k}, T_{B_k}$) are known, we will analyze the $T_x$ for node $x$ of level $k + 1$ (i.e., $O_k, C_k$ and $R_k$).

For any $k \geq 0$, it is easy to obtain the following equations due to equations (14), (A.4), (A.5), (A.10), (B.4), (B.5) and (B.10):

$$T_{O_k} = S_{O_k} - \Sigma + N_i W_{O_k}$$
$$= \frac{1}{2} (S_{A_k} + S_{B_k}) - 2^{t-k-1} (2m + 2)^{t-k-1} - \Sigma$$
$$+ N_i \left\{ \frac{1}{2} (W_{A_k} + W_{B_k}) - 2^{t-k-1} (2m + 2)^{-k-1} \right\}$$
$$= \frac{1}{2} (T_{A_k} + T_{B_k}) - 2^{t-k} (2m + 2)^{t-k-1} - 2^{t-k-1} (2m + 2)^{-k-1},$$

(56)

$$T_{C_k} = S_{C_k} - \Sigma + N_i W_{C_k}$$
$$= S_{A_k} + 2^{t-k-1} [(2m + 2)^t - (2m + 2)^{t-k-1}] - \Sigma$$
$$+ N_i \left\{ W_{A_k} - 2^{t-k-1} [1 - (2m + 2)^{-k-1}] \right\}$$
$$= T_{A_k} + 2^{t-k} [(2m + 2)^t - (2m + 2)^{t-k-1}] + 2^{t-k-1} [1 - (2m + 2)^{-k-1}],$$

(57)

and

$$T_{R_k} = T_{B_k} + 2^{t-k} [(2m + 2)^t - (2m + 2)^{t-k-1}] + 2^{t-k-1} [1 - (2m + 2)^{-k-1}].$$

(58)

Note that $T_{A_0} = T_{B_0}$ and let $k = 0$ in equations (56)–(58); we find

$$T_{O_0} < T_{A_0} = T_{B_0} < T_{C_0} = T_{R_0}.$$  

(59)

For $k \geq 1$, it is easy to derive from equations (57) and (58) that

$$T_{C_k} > T_{A_k} \quad \text{and} \quad T_{R_k} > T_{B_k}.$$  

(60)

As proved in appendix G, for $k \geq 1$,

$$\min \{T_{A_k}, T_{B_k}\} < T_{O_k} < \max \{T_{A_k}, T_{B_k}\}.$$  

(61)

Therefore, for $k \geq 1$,

$$\min \{T_{A_k}, T_{B_k}\} = \min \{T_{A_k}, T_{B_k}, T_{O_k}, T_{C_k}, T_{R_k}\}.$$  

(62)

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Let $\Omega_k$ denote set for nodes of level $k$ and note that $A_k$ and $B_k$ are the only two nodes of level $k$ in $\Lambda_k$; \{$A_k, B_k, O_k, C_k, R_k$\} represents all nodes of level $k + 1$ in $\Lambda_k$; equation (62) implies that
\[
\min\{T_x : x \in \Omega_k\} = \min\{T_x : x \in \Omega_{k+1}\}, \quad k \geq 1.
\] (63)

However, equation (59) shows that
\[
T_{O_0} = \min\{T_x : x \in \Omega_1\} < \min\{T_x : x \in \Omega_0\}.
\] (64)

Thus,
\[
T_{O_0} = \min\{T_x : x \in \Omega_1\} = \min\{T_x : x \in \Omega_2\} = \min\{T_x : x \in \Omega\}.
\] (65)

Let $k = 0$ and replace $T_{A_0}$ and $T_{B_0}$ with equation (43) in equation (56); we obtain the minimum of the MTT,
\[
T_{O_0} = \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}2^t(2m + 2)^t + \frac{4m + 1}{4m + 4}2^t - \frac{4m^2 + 4m + 1}{(4m + 3)(2m + 2)}. \tag{66}
\]

The result for $T_{O_0}$ is consistent with that derived in [38], which shows the correctness of our methods.

For the maximum of the MTT, we can derive from equations (59)–(61) that
\[
\max\{T_x : x \in \Omega_k\} < \max\{T_x : x \in \Omega_{k+1}\}, \quad k \geq 0.
\] (67)

We can also derive from equations (57) and (58) that the nodes with maximum MTT among nodes of level $k + 1$ are the nodes that are directly connected to the nodes of level $k$ with maximum MTT among all nodes of level $k$. According to the structure of the FSFT, the nodes with maximum MTT among all nodes of level $k$ are the nodes which are farthest from the nodes of level 0. Let $T_{\max}^k$ denote the maximum of the MTT among nodes of level $k$; it is straightforward that
\[
T_{\max}^k = T_{A_0}.
\]

For $k \geq 1$, we can also obtain from equations (57) and (58) that
\[
T_{\max}^{k+1} = T_{\max}^k + 2^{t-k}[2(2m + 2)^t - (2m + 2)^{t-k-1}] + 2^{t-k-1}[1 - (2m + 2)^{-k-1}]. \tag{68}
\]

Using equation (68) repeatedly and replacing $T_{\max}^0$ with equation (43), we obtain
\[
T_{\max}^t = T_{\max}^0 + \sum_{k=1}^t \{2^{t-k+1}[(2m + 2)^t - (2m + 2)^{t-k}] + 2^{t-k}[1 - (2m + 2)^{-k}]\}
\]
\[
= (2m + 2)^t\frac{10m^2 + 6}{4m + 3} - 2(2m + 2)^t + 2^t\frac{16m^2 + 22m + 7}{(4m + 3)(2m + 2)}
\]
\[
+ \frac{1}{(2m + 2)^t(4m + 3)} - \frac{12m^2 + 14m + 3}{(4m + 3)(2m + 2)}. \tag{69}
\]

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Because all nodes of $G(t)$ belong to level $t$, $T_{\text{max}}^t$ is the maximum of the MTT among all nodes of FSFT $G(t)$. Comparing $T_{\text{max}}^t$ with $T_{O_0}$, when $t \to \infty$, we have
\[
\frac{T_{\text{max}}^t}{T_{O_0}} \approx \frac{20m^2 + 32m + 12}{4m^2 + 4m + 1},
\]
which shows that the maximum of the MTT is about $(20m^2 + 32m + 12)/(4m^2 + 4m + 1)$ times the minimum of the MTT. Thus, the position of the target node has a large effect on the trapping efficiency.

6. Effect of node position on diffusion efficiency for random walks on the FSFT

Similarly to the analysis of trapping efficiency, we first derive the relation between $D_x$ for nodes of level $k$ and that for nodes of level $k+1$, and then compare $D_x$ for nodes of adjacent levels; finally, we find and compare the minimum and maximum of the MDT among all nodes of the FSFT.

Considering any subunit of level $k$ ($k \geq 0$) as shown in figure 6, it is easy to obtain the following equations due to equations (15), (A.4), (A.5), (A.10), (B.4), (B.5) and (B.10):
\[
D_{O_k} = \frac{1}{2}(D_{A_k} + D_{B_k}) + 2^{t-k-1}(2m+2)^{-k-1},
\]
\[
D_{C_k} = D_{A_k} - 2^{t-k-1}[1 - (2m+2)^{-k-1}],
\]
and
\[
D_{R_k} = D_{B_k} - 2^{t-k-1}[1 - (2m+2)^{-k-1}].
\]
Note that $D_{A_0} = D_{B_0}$ and let $k = 0$ in equations (71)–(73); we find
\[
D_{O_0} > D_{A_0} = D_{B_0} > D_{C_0} = D_{R_0}.
\]
For $k \geq 1$, it is easy to derive from equations (72) and (73) that
\[
D_{C_k} < D_{A_k} \quad \text{and} \quad D_{R_k} < D_{B_k}.
\]
As proved in appendix H, for $k \geq 1$,
\[
\min\{D_{A_k}, D_{B_k}\} < D_{O_k} < \max\{D_{A_k}, D_{B_k}\}.
\]
Therefore, for $k \geq 1$,
\[
\max\{D_{A_k}, D_{B_k}\} = \max\{D_{A_k}, D_{B_k}, D_{O_k}, D_{C_k}, D_{R_k}\}.
\]
Because $A_k$ and $B_k$ are the only two nodes of level $k$ in $\Lambda_k$ and $\{A_k, B_k, O_k, C_k, R_k\}$ represents all nodes of level $k+1$ in $\Lambda_{k+1}$, equation (77) implies that
\[
\max\{D_x : x \in \Omega_k\} = \max\{D_x : x \in \Omega_{k+1}\}, \quad k \geq 1.
\]
However, equation (74) leads to
\[
D_{O_0} = \max\{D_x : x \in \Omega_1\} > \min\{D_x : x \in \Omega_0\}.
\]
Thus,
\[ D_{O_0} = \max \{ D_x : x \in \Omega_1 \} = \max \{ D_x : x \in \Omega_t \} = \max \{ D_x : x \in \Omega \}. \] (80)

Let \( k = 0 \) and replace \( D_A^0 \) and \( D_B^0 \) with equation (44) in equation (71); we obtain the maximum of the MDT,
\[
D_{O_0} = \frac{1}{m+1} (2m+2)^t + \frac{6m^2 + 6m + 1}{(4m+3)(m+1)} 2^t (2m+2)^t \\
- \frac{4m+1}{4m+4} 2^t + \frac{4m^2 + 4m + 1}{(4m+3)(2m+2)}. \] (81)

For the minimum of the MDT, we can derive from equations (74)–(76) that
\[
\min \{ D_x : x \in \Omega_k \} > \min \{ D_x : x \in \Omega_{k+1} \}, \quad k \geq 0. \] (82)

Similarly to the analysis of the maximum of the MTT, we find that the nodes with minimum MDT among the nodes of level \( k \) are just the nodes that have maximum MTT among the nodes of level \( k \). Let \( D_{k_{\min}} \) denote the minimum of the MDT among nodes of level \( k \); it is straightforward that
\[
T_{0_{\min}} = D_{A_0}. \]

We can also obtain from equations (72) and (73) that
\[
D_{k_{\min}}^{k+1} = D_{k_{\min}}^k - 2^{t-k-1} [1 - (2m+2)^{-k-1}]. \] (83)

Using equation (83) repeatedly and replacing \( D_{0_{\min}} \) with equation (44), we obtain the minimum of the MDT among all nodes of the FSFT,
\[
D_{t_{\min}} = D_{0_{\min}} - \sum_{k=1}^{t} \{ 2^{t-k} [1 - (2m+2)^{-k}] \} \\
= D_{0_{\min}} - \sum_{k=1}^{t} 2^{t-k} + 2^t \sum_{k=1}^{t} \frac{4m+4}{(4m+3)^{t-k}} \\
= \frac{(2m+2)^t}{m+1} + (2m+2)^t \frac{6m^2 + 6m + 1}{(4m+3)(m+1)} - 2^t \frac{16m^2 + 22m + 7}{(4m+3)(2m+2)} \\
- \frac{1}{(2m+2)^t(4m+3)} + \frac{4m^2 + 4m + 1}{(4m+3)(2m+2)} + 1. \] (84)

Comparing \( D_{O_0} \) with \( D_{t_{\min}}' \), when \( t \to \infty \), we have
\[
\frac{D_{O_0}}{D_{t_{\min}}'} \approx 1, \] (85)

which implies that the difference between the maximum and the minimum of the MDT is quite small. Thus, the position of the starting node has almost no effect on the diffusion efficiency.

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7. Conclusion

In this paper, we have provided general methods to calculate the MFPT for any pair of nodes, the MTT for any target node and the MDT for any source node. Using the MTT and the MDT as the measures of trapping efficiency and diffusion efficiency respectively, we compare the trapping efficiency and diffusion efficiency among all nodes of the FSFT and find the best (or worst) trapping sites and the best (or worst) diffusing sites. Our results show that the trap’s position has a large effect on the trapping efficiency, but the position of the starting node has almost no effect on the diffusion efficiency. Our method was based on the self-similar structure of the FSFT and the unique properties of the tree. It can also be used to solve the problem of the MFPT on other self-similar trees.

Having obtained the MFPT for any pair of nodes, one can further analyze the distribution of the first-passage time (FPT). Recently, the asymptotic form of the full distribution of the first-passage time (FPT) in confined media has been given in [43, 44], when the total number of nodes $N_t \to \infty$. For deterministic fractals, if the MFPT and GMFPT can be calculated, the asymptotic distribution of the FPT for large $N_t$ can also be derived [43, 44]. We have provided methods to calculate the MFPT for any pair of nodes of the FSFT, so the GMFPT can also be calculated. Therefore, one can derive the distribution of the FPT for any pair of nodes. One can further analyze the uniformity distribution $P(\omega)$ according to the methods of [45]. However, the MFPT is just the first moment for the distribution of the FPT; it is not always a sufficient measure to characterize the first-passage dynamics of a system. Explicit derivation of the distribution of the FPT for any $N_t$ is still an interesting unresolved problem.

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Appendix A. Derivation of equation (17)

Considering any subunit of level $k - 1$ as shown in figure A.1, its two hubs are the only two nodes of level $k - 1$; its nodes of level $k$ are hubs of its $2m + 2$ subunits of level $k$ (i.e., $A$, $B$, $O$ and $C_i$ ($i = 1, 2, \ldots, 2m$) in figure A.1). Assuming that the $S_x$ for a node of level $k - 1$ (i.e., $S_A$, $S_B$) is known, we will analyze the $S_x$ for node $x$ of level $k$ (i.e., $O$ and $C_i$, $i = 1, 2, \ldots, 2m$).

Let

$$S_x(i) = \sum_{y \in G_i} L_{xy}, \quad i = 0, 1, \ldots, 2m + 3, \quad (A.1)$$

where ‘$y \in G_i$’ means that $y$ belongs to the node set of $G_i$. Thus,

$$S_x = \sum_{i=0}^{2m+3} S_x(i) - mL_xA - mL_xB - L_{xO}. \quad (A.2)$$

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Firstly, we calculate $S_{C_1}$. $C_1$ and $A$ are the two hubs of $G_1$ which is a subunit of level $k$. The distance between $C_1$ and $A$ is $L_{AC_1} = 2^{t-k}$; the total number of nodes of $G_1$ is $N_t - k$. We also find $S_{C_1}(1) = S_A(1)$ and for node $y \in G_i (i \neq 1)$, $L_{yC_1} = L_{yA} + L_{AC_1}$. Thus,

$$S_{C_1} = \sum_{i=0}^{2m+3} S_{C_1}(i) - mL_{AC_1} - mL_{BC_1} - L_{OC_1}$$

$$= S_A(1) + \sum_{i \neq 1} \sum_{y \in G_i} (L_{yA} + L_{AC_1}) - (4m + 2)L_{AC_1}$$

$$= \sum_{i=0}^{2m+3} S_A(i) + \sum_{i \neq 1} \sum_{y \in G_i} L_{AC_1} - (4m + 2)L_{AC_1}$$

$$= S_A + mL_{BA} + L_{OA} + [(2m + 1)N_{t-k} + N_{t} - N_{t-k+1} - (4m + 2)]L_{AC_1}$$

$$= S_A + 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k}].$$  \hfill (A.3)

Similarly, for $i = 0, 1, \ldots, m$,

$$S_{C_i} = S_A + 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k}].$$  \hfill (A.4)

For $i = m + 1, m + 2, \ldots, 2m + 1$,

$$S_{C_i} = S_B + 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k}].$$  \hfill (A.5)

Noticing that node $O$ is one hub of $G_0$ and $G_{2m+1}$, we find $L_{AO} = L_{BO} = 2^{t-k}$; $S_O(0) = S_A(0)$ and

$$S_O(0) = \sum_{y \in G_0} L_{yO} = \sum_{y \in G_0} (L_{yB} - L_{BO})$$

$$= S_B(0) - N_{t-k}L_{BO}.$$  \hfill (A.6)
Therefore,
\[ S_O(0) = \frac{1}{2} [S_A(0) + S_B(0)] = \frac{N_{l-k} L_{BO}}{2}. \]  
(A.7)

By symmetry,
\[ S_O(2m + 1) = \frac{1}{2} [S_A(2m + 1) + S_B(2m + 1)] - \frac{N_{l-k} L_{AO}}{2}. \]  
(A.8)

For any node \( y \in G_i \) (\( i = 1, 2, \ldots, m, 2m + 2 \)), \( L_yO = L_yA + L_yB \) and \( L_yO = L_yB - L_AO \), thus \( L_yO = \frac{1}{2}(L_yA + L_yB) \). By symmetry, \( L_yO = \frac{1}{2}(L_yA + L_yB) \) also holds for any node \( y \in G_i \) (\( i = m + 1, m + 2, \ldots, 2m, 2m + 3 \)). Therefore, for any \( i = 1, 2, \ldots, 2m, 2m + 2, 2m + 3, \)
\[ S_O(i) = \sum_{y \in G_i} L_yO = \frac{1}{2} (L_yA + L_yB) \]
\[ = \frac{1}{2} [S_A(i) + S_B(i)]. \]  
(A.9)

Hence,
\[ S_O = \sum_{i=0}^{2m+3} S_O(i) - mL_{AO} - mL_{BO} \]
\[ = \frac{1}{2} \sum_{i=0}^{2m+3} [S_A(i) + S_B(i)] - N_{l-k} L_{BO} - 2mL_{BO} \]
\[ = \frac{1}{2} (S_A + S_B) - 2^{i-k}(2m + 2)^{t-k}. \]  
(A.10)

If we label the two hubs of \( \Lambda_k \) as \( A_k, B_k \), we have \( A_{k-1} \equiv A, B_{k-1} \equiv B \). We have the following mapping between hubs of \( \Lambda_{k-1} \) and hubs of \( \Lambda_k \):
\begin{align*}
A_k & \equiv A, \quad B_k \equiv O, \quad i_k = 0, \\
A_k & \equiv O, \quad B_k \equiv A, \quad i_k = 2m + 1, \\
A_k & \equiv A, \quad B_k \equiv C_{i_k}, \quad i_k = 1, 2, \ldots, m, \\
A_k & \equiv C_{i_k}, \quad B_k \equiv B, \quad i_k = m + 1, m + 2, \ldots, 2m.
\end{align*}  
(A.11)

For \( i_k = 0 \),
\[ S^{(k)} = \begin{pmatrix} S_A \\ S_O \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} S_A \\ S_B \end{pmatrix} + \begin{pmatrix} 0 \\ -2^{i-k}(2m + 2)^{t-k} \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} S^{(k-1)} + \begin{pmatrix} 0 \\ -2^{i-k}(2m + 2)^{t-k} \end{pmatrix}. \]  
(A.12)

Therefore, equation (17) holds for \( i_k = 0 \). Similarly, we can verify that equation (17) holds for \( i_k = 1, 2, \ldots, 2m + 1 \).

**Appendix B. Derivation of equation (22)**

Considering any subunit of level \( k - 1 \) as shown in figure A.1, assuming \( S_x \) for a node of level \( k - 1 \) (i.e., \( W_A, W_B \)) is known, we will analyze \( W_x \) for node \( x \) of level \( k \) (i.e., \( O \) and \( C_i \), \( i = 1, 2, \ldots, 2m \)). Let

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\[ W_x(i) = \frac{1}{2E_t} \sum_{y \in G_i} (L_{xy} d_y), \quad i = 0, 1, \ldots, 2m + 3; \quad (B.1) \]

we have

\[ W_x = \sum_{i=0}^{2m+3} W_x(i). \quad (B.2) \]

Note that the degrees for nodes that are the intersection of two subgraphs were counted respectively in every subgraph and the degree for the node in \( \Lambda_{k-1} \) is just the summation of the degrees for the nodes in all the subgraphs \( \Lambda_k \). For example, the degree of node \( O \) in \( \Lambda_{k-1} \) is just the summation of the degree for \( O \) in subgraphs \( G_0 \) and \( G_{2m+1} \).

It is easy to obtain that \( W_{C_1}(1) = W_A(1) \). For node \( y \in G_i \) (\( i \neq 1 \)), \( L_y C_1 = L_y A + L_{AC_1} \); thus,

\[
W_{C_1} = \sum_{i=0}^{2m+3} W_{C_1}(i)
= W_A(1) + \frac{1}{2E_t} \sum_{i \neq 1} \sum_{y \in G_i} (L_y A + L_{AC_1}) d_y
= \sum_{i=0}^{2m+3} W_A(i) + \frac{1}{2E_t} \sum_{i \neq 1} \sum_{y \in G_i} d_y L_{AC_1}
= W_A + \frac{1}{2E_t} [2E_t - 2E_t-k] L_{AC_1}
= W_A + 2^{t-k}[1 - (2m + 2)^{-k}]. \quad (B.3)
\]

Similarly, for \( i = 0, 1, \ldots, m, \)

\[
W_{C_i} = W_A + 2^{t-k}[1 - (2m + 2)^{-k}]. \quad (B.4)
\]

For \( i = m + 1, m + 2, \ldots, 2m + 1, \)

\[
W_{C_i} = W_B + 2^{t-k}[1 - (2m + 2)^{-k}]. \quad (B.5)
\]

Because node \( O \) is one hub of \( G_0 \) and \( G_{2m+1} \), it is straightforward that \( L_{AO} = L_{BO} = 2^{t-k} \). We also find that \( W_O(0) = W_A(0) \) and

\[
W_O(0) = \frac{1}{2E_t} \sum_{y \in G_0} L_y O d_y
= \frac{1}{2E_t} \sum_{y \in G_0} (L_y B - L_{BO}) d_y
= W_B(0) - 2^{t-k}(2m + 2)^{-k}. \quad (B.6)
\]

Thus,

\[
W_O(0) = \frac{1}{2} [W_A(0) + W_B(0)] - 2^{t-k-1}(2m + 2)^{-k}. \quad (B.7)
\]

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By symmetry,
\[ W_O(2m+1) = \frac{1}{2} [W_A(2m+1) + W_B(2m+1)] - 2^{t-k-1}(2m+2)^{-k}. \] (B.8)

For any \( i = 1, 2, \ldots, 2m, 2m+2, 2m+3 \), if node \( y \in G_t \), we have \( L_{yO} = 1/2(L_{yA} + L_{yB}) \).

Therefore,
\[ W_O(i) = \frac{1}{2} [W_A(i) + W_B(i)] . \] (B.9)

Replacing \( W_O(i) \) from equations (B.7)–(B.9) in equation (B.2), we obtain
\[ W_O = \frac{1}{2}(W_A + W_B) - 2^{t-k}(2m+2)^{-k}. \] (B.10)

Similarly to appendix A, if we label the two hubs of \( \Lambda_k \) as \( A_k, B_k \) and let
\[ \mathcal{W}^{(k)} \equiv \frac{W_{A_k}}{W_{B_k}} \], (B.11)
we can verify that equation (22) holds for \( i_k = 0, 1, 2, \ldots, 2m + 1 \).

**Appendix C. Derivation of \( S_{A_0} \) and \( W_{A_0} \)**

\( A_0 \) is one of the two nodes of level 0 (i.e., \( A, B \) in figure 2); it is also one of the two hubs of \( G(t) \). In order to tell the difference in \( S_{A_0} \) (and \( W_{A_0} \)) for FSFTs of different generations \( t \), let \( S_{A_0}^t \) and \( W_{A_0}^t \) denote the \( S_{A_0} \) and \( W_{A_0} \) in the FSFT of generation \( t \). It is straightforward that \( S_{A_0}^0 = 1 \) and \( W_{A_0}^0 = \frac{1}{2} \). For \( t > 1 \), \( S_{A_0}^t \) satisfies the following recursion relation:
\[ S_{A_0}^t = (m + 1)S_{A_0}^{t-1} + S_{A_0}^{t-1} + 2^{t-1}(N_{t-1} - 1) + m[S_{A_0}^{t-1} + 2^t(N_{t-1} - 1)]. \] (C.1)

On the right-hand side of the equation, the first item represents the summation for the shortest path length between node \( A_0 \) and nodes in the subunit \( G_i(t) \) \((i = 0, 1, \ldots, m)\); the second item represents the summation for the shortest path length between node \( A_0 \) and nodes in the subunit \( G_{2m+1}(t) \), the third item represents the summation for the shortest path length between node \( A_0 \) and nodes in the subunit \( G_i(t) \) \((i = m + 1, m + 2, \ldots, 2m)\). Noting that \( N_{t-1} = (2m+1)^{t-1} + 1 \), we obtain
\[
S_{A_0} = S_{A_0}^t = (2m + 2)S_{A_0}^{t-1} + (2m + 1)2^{t-1}(2m+2)^{t-1}
= (2m + 2)^2S_{A_0}^{t-2} + (2m + 1)(2m+2)^{t-1} \left[ 2^{t-2} + 2^{t-1} \right]
= \ldots
= (2m + 2)^tS_{A_0}^0 + (2m + 1)(2m+2)^{t-1} \left[ 2^0 + 2^1 + \cdots + 2^{t-1} \right]
= (2m + 2)^t + (2m + 1)(2m+2)^{t-1}(2^t - 1). \] (C.2)

Similarly, we find that \( W_{A_0}^t \) satisfies the following recursion relation:
\[
W_{A_0}^t = \frac{1}{2m+2} \left\{ (m + 1)W_{A_0}^{t-1} + W_{A_0}^{t-1} + 2^{t-1} + m[W_{A_0}^{t-1} + 2^t] \right\} . \] (C.3)

\textit{doi:10.1088/1742-5468/2014/04/P04032}
Hence,

\[
W_{A_0} = W_{A_0}^t = W_{A_0}^{t-1} + \frac{2m + 1}{2m + 2} 2^{t-1}
\]

\[
\quad = \ldots
\]

\[
= W_{A_0}^0 + \frac{2m + 1}{2m + 2} [2^0 + 2^1 + \ldots + 2^{t-1}]
\]

\[
= \frac{1}{2} + \frac{2m + 1}{2m + 2} (2^t - 1).
\]

\[\text{(C.4)}\]

**Appendix D. Derivation of equations (45) and (46)**

It is easy to verify that

\[
M_{i_k} M_{i_k-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = M_{m+1}.
\]

\[\text{(D.1)}\]

\[
M_{m+1} M_1 = M_1, \quad M_1^n = M_1, \quad M_{m+1}^n = M_{m+1}.
\]

\[\text{(D.2)}\]

Let \(i_1 = 1, i_2 = m + 1, \ldots, i_{k-1} = 1, i_k = m + 1\) in equations (26) and (27); noticing the mapping \(A_k \equiv H_k\) and \(B_k \equiv H_{k-1}\), for any even \(k > 0\),

\[
(S_{H_k}, S_{H_{k-1}})^T = S^{(k)}
\]

\[
= M_{i_k} M_{i_{k-1}} \ldots M_{i_k} S^{(0)} + \sum_{l=1}^{k-1} M_{i_k} M_{i_{k-1}} \ldots M_{i_{l+1}} V_l^{(0)} + V_{i_k}^{(0)}
\]

\[
= M_1 S^{(0)} + M_{m+1} \sum_{l=1}^{k-1} V_l^{m+1} + M_1 \sum_{l=1}^{k-1} V_l^{m+1} + V_{m+1}^{k}
\]

\[
= \left\{ S_{A_0} + \sum_{l=1}^{k-1} 2^{t-l} [(2m + 2)^l - (2m + 2)^t] \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + V_{m+1}^{k}
\]

\[\text{(D.3)}\]

where \(\gamma_k = S_{A_0} + (4m + 4)^t ((4m + 2)/(4m + 3)) - (4m + 4)^t-k ((2m + 2)^k - 1/(4m + 3))\). Hence, one gets \(S_{H_k} = \gamma_k\) and equation (45) is obtained.

Similarly, we can also derive

\[
W^{(k)} = \begin{pmatrix} \lambda_k \\ \lambda_{k-1} \end{pmatrix},
\]

\[\text{(D.4)}\]

where \(\lambda_k = W_{A_0} + 2^t ((4m + 2)/(4m + 3)) - 2^{t-k} + 2^t/(4m + 4)^k (4m + 3)\). Hence, \(W_{H_k} = \lambda_k\) and equation (46) is obtained.

doi:10.1088/1742-5468/2014/04/P04032
Appendix E. Derivation of equations (47) and (48)

It is easy to verify that
\[
M_{2m+1}^k = \begin{pmatrix} 1/2^k & 1 - 1/2^k \\ 0 & 1 \end{pmatrix}.
\]

(E.1)

Let \( i_1 = 0, i_2 = 2m + 1, \ldots, i_{k-1} = 2m + 1, i_j = 2m + 1 \) in equations (26) and (27); noticing the mapping \( A_k \equiv Q_j \) and \( B_j \equiv O_0 \), we obtain
\[
(S_Q, S_O)^T \equiv S^{(j)}
\]

\[
= M_{2m+1}^{j-1} M_0 S^{(0)} + \sum_{l=2}^{j-1} M_{2m+1}^{j-l} V_0^l + \sum_{l=2}^{j} M_{2m+1}^{j-l} V_{2m+1}^l + V_{2m+1}^j
\]

\[
= \begin{pmatrix} S_{A_0} - (4m + 4)^{t-1} + 2t-j(2m + 2)^{t-1} \\ S_{A_0} - (4m + 4)^{t-1} \end{pmatrix} - \begin{pmatrix} 2t-j \sum_{l=2}^{j} (2m + 2)^{t-l} \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} tS_{A_0} - (4m + 4)^{t-1} + \frac{2m}{2m + 1} 2t-j(2m + 2)^{t-1} + \frac{(4m + 4)^{t-j}}{2m + 1} \\ S_{A_0} - (4m + 4)^{t-1} \end{pmatrix}.
\]

(E.2)

Therefore, equation (47) is obtained. Similarly, equation (48) can also be derived.

Appendix F. Derivation of equation (49)

For any subunit \( \Lambda_k \), \( L_{A_k B_k} = 2^{t-k} \) and \( L_{A_k O_k} = 2^{t-k-1} \). Because \( Q_j \) and \( O_0 \) are the two hubs of a subunit \( \Lambda_j \), \( H_k \) and \( H_{k-1} \) are the two hubs of a subunit \( \Lambda_k \), we have \( L_{Q_j O_0} = 2^{t-j} \) and \( L_{H_k H_{k-1}} = 2^{t-k} \). Hence,
\[
L_{H_k Q_j} = L_{H_k A_0} + L_{A_0 O_0} - L_{Q_j O_0}
\]

\[
= L_{H_k H_{k-1} A_0} + L_{A_0 O_0} - L_{Q_j O_0}
\]

\[
= 2^{t-k} + L_{H_{k-1} A_0} + 2^{t-1} - 2^{t-j}
\]

\[
= 2^{t-k} + 2^{t-k+1} + L_{H_{k-2} A_0} + 2^{t-1} - 2^{t-j}
\]

\[
= 2^{t-k} + 2^{t-k+1} + \cdots + L_{H_1 A_0} + 2^{t-1} - 2^{t-j}
\]

\[
= 2^{t-k} + 2^{t-k+1} + \cdots + 2^{t-1} + 2^{t-1} - 2^{t-j}
\]

\[
= 2^t - 2^{t-k} + 2^{t-1} - 2^{k-j}.
\]

(F.1)

Appendix G. Proof of equation (61)

For any \( k \geq 1 \), according the following mappings for nodes of \( \Lambda_k \) and \( \Lambda_{k+1} \):

\[
\begin{align*}
A_{k+1} & \equiv A_k, & B_{k+1} & \equiv O_0 & i_{k+1} = 0 \\
A_{k+1} & \equiv O_k, & B_{k+1} & \equiv B_k & i_{k+1} = 2m + 1 \\
A_{k+1} & \equiv A_k, & B_{k+1} & \equiv C_k & i_{k+1} = 1, 2, \ldots, m \\
A_{k+1} & \equiv R_k, & B_{k+1} & \equiv B_k & i_{k+1} = m + 1, m + 2, \ldots, 2m,
\end{align*}
\]

(G.1)

\[\text{doi:10.1088/1742-5468/2014/04/P04032} \]
we have

\[ T_{B_{k+1}} - T_{A_{k+1}} = \begin{cases} T_{O_k} - T_{A_k} & i_{k+1} = 0 \\ T_{B_k} - T_{O_k} & i_{k+1} = 2m + 1 \\ T_{C_k} - T_{A_k} & i_{k+1} = 1, 2, \ldots, m \\ T_{B_k} - T_{R_k} & i_{k+1} = m + 1, m + 2, \ldots, 2m. \end{cases} \]  

(G.2)

Replacing \( T_{O_k}, T_{C_k} \) and \( T_{R_k} \) with equations (56)–(58) respectively, we get

\[ T_{B_{k+1}} - T_{A_{k+1}} = \begin{cases} \frac{1}{2}(T_{B_k} - T_{A_k}) - \theta_k & i_{k+1} = 0 \\ \frac{1}{2}(T_{B_k} - T_{A_k}) + \theta_k & i_{k+1} = 2m + 1 \\ \xi_k - \xi_k & i_{k+1} = 1, 2, \ldots, m \\ -\xi_k & i_{k+1} = m + 1, m + 2, \ldots, 2m, \end{cases} \]

(G.3)

where \( \theta_k = 2^{t-k}(2m + 2)^{t-k-1} + 2^{t-k-1}(2m + 2)^{-k-1} \) and \( \xi_k = 2^{t-k}[(2m + 2)^{t} - (2m + 2)^{t-k-1}] + 2^{t-k-1}[1 - (2m + 2)^{-k-1}]. \)

For any \( n \geq 1 \), we find

\[
|T_{B_n} - T_{A_n}| \geq 2^{t-n+1}(2m + 2)^{t-n} + 2^{t-n}(2m + 2)^{-n},
\]

(G.4)

\[
|T_{B_n} - T_{A_n}| \leq 2^{t-n+1}[(2m + 2)^{t} - (2m + 2)^{t-n}] + 2^{t-n}[1 - (2m + 2)^{-n}].
\]

(G.5)

Equations (G.4) and (G.5) are proved by mathematical induction as follows.

Let \( k = 0 \) in equation (G.3); we obtain

\[
|T_{B_1} - T_{A_1}| = \begin{cases} 2^{t}(2m + 2)^{t-1} + 2^{t-1}(2m + 2)^{-1} & i_1 = 0 \\ 2^{t}(2m + 2)^{t-1} + 2^{t-1}(2m + 2)^{-1} & i_1 = 2m + 1 \\ 2^{t}[(2m + 2)^{t} - (2m + 2)^{t-1}] + 2^{t-1}[1 - (2m + 2)^{-1}] & \text{others}. \end{cases}
\]

(G.6)

Thus, equations (G.4) and (G.5) hold for \( n = 1 \).

Assuming that equations (G.4) and (G.5) hold for certain \( n = k \geq 1 \), we now prove that equations (G.4) and (G.5) also hold for \( n = k + 1 \).

According to equation (G.2), \( T_{B_{k+1}} - T_{A_{k+1}} \) has \( 2m + 2 \) cases due to the different values of \( i_{k+1} \). In the case of \( i_{k+1} = 1, 2, \ldots, 2m \), it is easy to verify that equations (G.4) and (G.5) hold for \( n = k + 1 \) according to equation (G.3).

In the case of \( i_{k+1} = 0 \), let \( n = k \) in equation (G.4) and substitute \( T_{B_k} - T_{A_k} \) with the right-hand side of equation (G.4); we obtain

\[
T_{B_{k+1}} - T_{A_{k+1}} = \frac{1}{2}(T_{B_k} - T_{A_k}) - 2^{t-k}(2m + 2)^{t-k-1} - 2^{t-k-1}(2m + 2)^{-k-1} \n
\geq \frac{1}{2} [2^{t-k+1}(2m + 2)^{t-k} + 2^{t-k}(2m + 2)^{-k}] - 2^{t-k}(2m + 2)^{t-k-1} - 2^{t-k-1}(2m + 2)^{-k-1} \\
> 2^{t-k}(2m + 2)^{t-k-1} + 2^{t-k-1}(2m + 2)^{-k-1}.\]

(G.7)
Let \( n = k \) in equation (G.5) and substitute \( T_{A_k} - T_{B_k} \) with the right-hand side of equation (G.5); we have

\[
|T_{B_{k+1}} - T_{A_{k+1}}| = \frac{1}{2}(T_{B_k} - T_{A_k}) - 2^{t-k}(2m + 2)t^{k-1} - 2^{t-k}(2m + 2)^{-k-1} \\
\leq \frac{1}{2} [2^{t-k+1}(2m + 2)^t - (2m + 2)^{t-k} + 2^{t-k}[1 - (2m + 2)^{-k}]] \\
+ 2^{t-k}(2m + 2)^{t-k-1} + 2^{t-k}(2m + 2)^{-k-1} \\
= 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k} + (2m + 2)^{t-k-1}] \\
+ 2^{t-k-1}[1 - (2m + 2)^{-k} + (2m + 2)^{-k-1}] \\
< 2^{t-k}[(2m + 2)^t - (2m + 2)^{t-k-1} + 2^{t-k-1}[1 - (2m + 2)^{-k-1}]].
\]

Therefore, equations (G.4) and (G.5) hold for \( n = k + 1 \) in the case of \( i_{k+1} = 0 \).

Similarly, we can prove that they both hold for \( n = k + 1 \) in the case of \( i_{k+1} = 2m + 1 \). Therefore, for all the \( 2m + 2 \) cases of \( T_{B_{k+1}} - T_{A_{k+1}} \), equations (G.4) and (G.5) hold for \( n = k + 1 \), which leads to them both holding for any \( n \geq 1 \).

We now come back to prove equation (61). Without loss of generality, we assume \( T_{B_k} \geq T_{A_k} \). Therefore,

\[
T_{O_k} - T_{A_k} = \frac{1}{2}(T_{B_k} - T_{A_k}) - 2^{t-k}(2m + 2)t^{k-1} - 2^{t-k}(2m + 2)^{-k-1} \\
\geq \frac{1}{2} [2^{t-k+1}(2m + 2)^t - 2^{t-k}(2m + 2)^{-k}] \\
\quad - 2^{t-k}(m + 2)^{t-k-1} - 2^{t-k-1}(2m + 2)^{-k-1} \\
> 0
\]

and

\[
T_{B_k} - T_{O_k} = \frac{1}{2}(T_{B_k} - T_{A_k}) + 2^{t-k}(2m + 2)t^{k-1} + 2^{t-k}(2m + 2)^{-k-1} > 0.
\]

Therefore, equation (61) holds when \( T_{B_k} \geq T_{A_k} \). Similarly, we can prove that equation (61) holds when \( T_{B_k} \leq T_{A_k} \).

**Appendix H. Proof of equation (76)**

For any \( k \geq 1 \), according the mappings for nodes of \( \Lambda_k \) and \( \Lambda_{k+1} \) as shown in equation (G.1), we have

\[
D_{B_{k+1}} - D_{A_{k+1}} = \begin{cases} 
D_{O_{k+1}} - D_{A_k} & i_{k+1} = 0 \\
D_{B_k} - D_{O_{k+1}} & i_{k+1} = 2m + 1 \\
D_{C_k} - D_{A_k} & i_{k+1} = 1, 2, \ldots, m \\
D_{B_k} - D_{R_k} & i_{k+1} = m + 1, m + 2, \ldots, 2m.
\end{cases}
\]

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Replacing $D_{O_k}$, $D_{C_k}$ and $D_{R_k}$ with equations (71)–(73) respectively, we have

\[ D_{B_{k+1}} - D_{A_{k+1}} = \begin{cases} 
\frac{1}{2} (D_{B_k} - D_{A_k}) + 2^{t-k-1} (2m+2)^{-k-1} & i_{k+1} = 0 \\
\frac{1}{2} (D_{B_k} - D_{A_k}) - 2^{t-k-1} (2m+2)^{-k-1} & i_{k+1} = 2m + 1 \\
-2^{t-k-1} [1 - (2m+2)^{-k-1}] & i_{k+1} = 1, 2, \ldots, m \\
2^{t-k-1} [1 - (2m+2)^{-k-1}] & i_{k+1} = m + 1, m + 2, \ldots, 2m.
\end{cases} \quad (H.2) \]

For any $k \geq 1$, we find

\[ |D_{B_k} - D_{A_k}| \geq 2^{t-k} (2m+2)^{-k}, \quad (H.3) \]
\[ |D_{B_k} - D_{A_k}| \leq 2^{t-k} [1 - (2m+2)^{-k}], \quad (H.4) \]

which are proved by mathematical induction as follows.

Let $k = 0$ in equation (H.2); we obtain

\[ D_{B_1} - D_{A_1} = \begin{cases} 
2^{t-1} (2m+2)^{-1} & i_1 = 0 \\
-2^{t-1} (2m+2)^{-1} & i_1 = 2m + 1 \\
2^{t-1} [1 - (2m+2)^{-1}] & i_1 = 1, 2, \ldots, m \\
-2^{t-1} [1 - (2m+2)^{-1}] & i_1 = m + 1, m + 2, \ldots, 2m.
\end{cases} \quad (H.5) \]

Therefore, it is easy to verify that equations (H.3) and (H.4) hold for $k = 1$.

Assuming that (H.3) and (H.4) hold for certain $k \geq 1$, we now prove that equations (G.4) and (G.5) also hold for $k + 1$.

According to equation (H.1), $D_{B_{k+1}} - D_{A_{k+1}}$ has $2m + 2$ cases due to the different values of $i_{k+1}$. In the case of $i_{k+1} = 1, 2, \ldots, 2m$, it is easy to verify that equations (H.3) and (H.4) hold for $k + 1$ due to equation (H.2).

In the case of $i_{k+1} = 0$, substituting $D_{B_k} - D_{A_k}$ with the right-hand side of equation (H.4) in equation (H.2), we get

\[ |D_{B_{k+1}} - D_{A_{k+1}}| = |\frac{1}{2} (D_{B_k} - D_{A_k}) + 2^{t-k-1} (2m+2)^{-k-1}| \\
\leq \frac{1}{2} [2^{t-k} [1 - (2m+2)^{-k}]] + 2^{t-k-1} (2m+2)^{-k-1} \\
< 2^{t-k-1} [1 - (2m+2)^{-k-1}]. \quad (H.6) \]

Substituting $D_{A_k} - D_{B_k}$ from the right-hand side of equation (H.3) in equation (H.2), we have

\[ |D_{B_{k+1}} - D_{A_{k+1}}| = |\frac{1}{2} (D_{B_k} - D_{A_k}) + 2^{t-k-1} (2m+2)^{-k-1}| \\
\geq \frac{1}{2} [2^{t-k} (2m+2)^{-k}] - 2^{t-k-1} (2m+2)^{-k-1} \\
> 2^{t-k-1} (2m+2)^{-k-1}. \quad (H.7) \]

Therefore, equations (H.3) and (H.4) hold for $k + 1$ in the case of $i_{k+1} = 0$. Similarly, we can prove that they both hold for $k + 1$ in the case of $i_{k+1} = 2m + 1$. Therefore, for all
the $2m + 2$ cases of $D_{B_{k+1}} - D_{A_{k+1}}$, equations (H.3) and (H.4) hold for $k + 1$, which leads to them both holding for any $k \geq 1$.

We now come back to prove equation (76). Without loss of generality, assuming $D_{B_k} \geq D_{A_k}$, thus

$$D_{O_k} - D_{A_k} = \frac{1}{2} (D_{B_k} - D_{A_k}) + 2^{t-k-1} (2m + 2)^{-k-1} > 0,$$  \hspace{1cm} (H.8)

and equation (H.3) implies

$$D_{B_k} - D_{O_k} = \frac{1}{2} (D_{B_k} - D_{A_k}) - 2^{t-k-1} (2m + 2)^{-k-1}$$

$$\geq \frac{1}{2} [2^{t-k}(2m + 2)^{-k}] - 2^{t-k-1} (2m + 2)^{-k-1}$$

$$> 0.$$  \hspace{1cm} (H.9)

Therefore, equation (76) holds when $D_{B_k} \geq D_{A_k}$. Similarly, we can prove that equation (76) holds when $D_{B_k} \leq D_{A_k}$.

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