On Vertex Conditions In Elastic Beam Frames: Analysis on Compact Graphs

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Abstract

We consider three-dimensional elastic frames constructed out of Euler-Bernoulli beams and describe extension of matching conditions by relaxing the vertex-rigidity assumption and the case in which concentrated mass may exists. This generalization is based on coupling an (elastic) energy functional in terms of field’s discontinuities at a vertex along with purely geometric terms derived out of first principles. The corresponding differential operator is shown to be self-adjoint. Although for planar frames with a class of rigid-joints the operator decomposes into a direct sum of two operators, this property only holds for a special class of the proposed model. Application of theoretical results is then discussed in details for compact frames embedded in Euclidean spaces with different dimensions. This includes extension of the established results for rigid-joint case on exploiting the symmetry present in a frame and decomposing the operator by restricting it onto reducing subspaces corresponding to irreducible representations of the symmetry group. Derivation of characteristic equation based on the idea of geometric-free local spectral basis and enforcing geometry of the graph into play by an appropriate choice of the coefficient set will be discussed. Finally, we prove the limit conditions in parameter space which results in decomposing of vector-valued beam Hamiltonian to a direct sum of scalar-valued ones.

1 Introduction

Lattice materials are cellular structures obtained by tessellating a unit cell comprising a few beams or bars. Recently, the meta-material concept has been extended to materials showing novel mechanical behavior, e.g., the so-called auxetic materials exhibit the unusual mechanical property of having a negative Poisson’s ratio opening novel applications in molecular scales of crystallizing systems and to much larger scales in sound damping structures by band-gap opening.

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From theoretical point of view, elastic deformation of continuous bodies may be generally studied using the theory of elasticity. However, simplification on analysis can be reached by utilizing the kinematics of deformation and making some assumptions on the resulting strains. Under such simplification assumptions, modeling variety of natural and engineered tessellated lattices can generally be studied under beam theories. Under Euler-Bernoulli beam model, each beam is described by an energy functional which involves four degrees of freedom for every infinitesimal element along the beam: lateral (2 degrees of freedom), axial, and angular displacements, see Figure 1. At a joint, these four functions, supported on the beams involved, must be related via (matching) transmission conditions that take into account the physics of a joint, see [5,7,23] for more details. All of these make modeling of such structures made of joined together beams a topic of natural interest for engineering research and, more recently, for mathematicians working on differential equations defined on special metric spaces, namely quantum graphs [5,21,24,36].

This is equivalent to modeling the lattice as a discrete graph with set of vertices and edges joining them in physical space. More formally, a quantum graph is a metric graph equipped with a differential operator “Hamiltonian” and suitable vertex matching conditions. We refer to the work [6] for detail background along this line.

Recently, extension of the results to higher-order Hamiltonian gains interest by the more theoretically oriented communities. As an example, in the work [24,45] vertex conditions for differential operator of fourth derivative on the planar metric graph are presented. It is shown that such conditions corresponding to the free movement of beams and depending on the angles among them in the equilibrium state. Non-trivial extension of this result to the three dimensional case is presented in [5] in which joint conditions has been derived out of first principles. This extension is mainly based on coupling all degrees of freedom at the so called rigid-joint, i.e., joint in which the displacement and rotation vector fields are remained continuous at vertex set. It is shown that for rigid vertex model, field’s coupling is inevitable even in the planar lattice case. This makes the corresponding Hamiltonian a vector valued self-adjoint operator on graph with physically interpretable vertex conditions, namely, equilibrium of net forces and moments applied on the vertex.

Problem Statement. Main purpose of current manuscript is to answer the following problems

(i) how to generalize vertex model by relaxing the rigidity assumption so that discontinuity of fields are admissible in the solution space?

(ii) under which (parametric) limits the vector-valued beam Hamiltonian on planar graphs decomposes to a direct sum of scalar-valued ones?

1Most inclusive classical beam models are the Euler-Bernoulli and Timoshenko beam theories.

2These questions have been raised in the outlook Section of recent work [5].
Answering the problems above will provide a more complete picture on analysis of such continua. Most activities along this line have been focused on applied directions, while more theoretical works are mainly limited to Schrödinger-type operator or higher order operators applied on simple geometry, e.g. a serially-connected beams connected by vibrating point masses [4,11,13,27,29,30,34,37,45]. Thereby, this manuscript is written with a hope of accommodating a step further towards physically sound analysis of observed phenomenon in the recently developed (meta) materials, and understanding the role of vertex transmission conditions on spectrum of Hamiltonian defined on metric graphs, see Section 5 for potential extension and application of the current work.

This paper is structured as follows: in Section 2, after some preliminary discussion, we put forward a simple, general and compact description of the generalized semi-rigid vertex model. Definition of this type of joint is based on triggering both geometric description of the deformed frame (similar to the rigid-joint model) as well as inclusion of an energy functional in terms of displacement and rotation discontinuities, see Definition 2. In a special case of vanishing energy at the vertex, our result will be equivalent to the rigid-joint model derived out of purely geometric first principles in [5], see Definition 1. Next, formal derivation of self-adjoint Hamiltonian on graph with corresponding vertex conditions will be presented, see Theorem 2.2. This will be followed by decomposition of the Hamiltonian on planar graphs, see Corollary 2.4. In Section 3, we discuss in details application of our theoretical framework on two compact frames, namely cantilevered beam and a three dimensional example in which the symmetry can be exploited to decompose the operator by restricting it onto reducing subspaces corresponding to irreducible representations of the symmetry group, see Theorem 3.1. Focus of Section 4 will be on spectral analysis of 3-star planar finite graphs. This will be done by introducing set of local spectral basis defined on an interval [0,1] followed by derivation of purely geometric basis. These two unrelated quantities then will be combined to construct characteristic equation corresponding the eigenvalue problem on such class of graphs, see Proposition 4.1. This concludes answering question (i) in problem statement, see Theorem 2.2. At the end of last Section we will formally answer question (ii) in the problem statement, see Theorem 4.7. We conclude this manuscript by presenting a partial list of directions for further mathematical investigations together with additional references.

2 Elasticity on Beam Frame

2.1 Preliminary

In this Section we first briefly review parameterization of a beam’s elastic deformation. This requires introducing two vector valued quantities, namely displacement and rotation vectors defined out of geometric description of beams, followed by recalling the notation of a rigid-vertex, see definition 1, at which edges are met and has been developed in former work [5]. Main contribution in this Section will be on generalization of the rigid-vertex model to a semi-rigid one, see definition 2.

2.1.1 Parameterization of Beam Deformation

According to the Euler–Bernoulli hypothesis, which states that “plane sections remain plane,” the geometry of the spatial beam is described by the centroid line and a family of the corresponding cross-sections. A fixed spatial basis with orthonormal base vectors \( \{ \vec{E}_1, \vec{E}_2, \vec{E}_3 \} \) is introduced, \(^3\)The term centroid indicates that this line is the locus of the centers of mass of the cross-sections.
which span the physical space (three dimensional Euclidean space) in which the beam is embedded. Moreover a family of orthonormal basis \( \{i, j, k\} \), called the cross-section basis or the material basis is employed to describe the orientation of the cross section of beam. The deformed configuration of the beam can be fully described by the position vector \( \vec{g}(x) \) with \( x \) representing the arc-length coordinate of the reference configuration, along with the family of orthonormal basis \( \{\vec{i}(x), \vec{j}(x), \vec{k}(x)\} \) which describe the orientation of the cross sections in the deformed configuration, see [5] for detail setup. The displacement vector \( \vec{g}(x) \) in the reference basis of the undeformed beam decomposes as

\[
\vec{g}(x) := u(x)\vec{i} + w(x)\vec{j} + v(x)\vec{k}.
\]

The component \( u(x) \) is called the axial displacement while \( w(x) \) and \( v(x) \) are lateral displacements. Introducing in-axis angular displacement of the form \( \eta(x) := \vec{j}(x) \cdot \vec{k} \), then for small deformation regime, linear part of rotation vector \( \vec{\omega}(x) \) has representation, see Lemma 3.3 in [5] for this derivation

\[
\vec{\omega}(x) = \eta(x)\vec{i} - v'(x)\vec{j} + w'(x)\vec{k}.
\]

Consider now several beams, labeled \( e_1, \ldots, e_n \) coupled at a joint. The type of matching vertex conditions there has a central role on type of coupling among the edges and thereby on global characteristic of the frame. Next, we will generalize the notion of rigid-joint model in [5].

\subsection{Generalization of Joint Model}

A beam frame can be described as a geometric graph \( \Gamma = (V, E) \), where \( V \) denotes the set of vertices and \( E \) the set of edges. The vertices \( v \in V \) correspond to joints and edges \( e \in E \) are the beams. Each edge \( e \) is a collection of the following information: origin and terminus vertices \( v^o_e, v^t_e \in V \), length \( \ell_e \) and the local basis \( \{i_e, j_e, k_e\} \). Describing the vertices \( V \) as points in \( \mathbb{R}^3 \) also fixes the length \( \ell_e \) and the axial direction \( i_e \) (from origin to terminus). We will use the sign indicator \( s^p_e \) which is defined to be \(-1\) when \( v = v^o_e \) and \(+1\) if \( v = v^t_e \) and \( 0 \) otherwise. This sign convention is consistent with the sign of outward normal derivative vector of an external surface applied in continuum mechanics literature, e.g. see [15]. In this paper an edge \( e \) incident to \( v \) will be denoted by notation \( e \sim v \), and moreover set

\[
\vec{g}_e(v) := \lim_{x \to 0} \vec{g}_e(x), \quad \vec{\omega}_e(v) := \lim_{x \to 0} \vec{\omega}_e(x).
\]

Associated to each vertex \( v \) are two unknown vectors \( \vec{g}^o_v \in \mathbb{C}^3 \) and \( \vec{\omega}^o_v \in \mathbb{C}^3 \) with their components written with respect to the global coordinate system \( \{E_1, E_2, E_3\} \). Next we will define two class of vertex matching conditions which posses a central role in the remaining sections.

**Definition 1.** A joint \( v \) is called rigid, if for all \( e \sim v \) the displacement and rotation on beams satisfy continuity conditions

\[
\begin{align*}
  u_e(v)i_e + w_e(v)j_e + v_e(v)k_e &= \vec{g}^o_v \\
  \eta_e(v)i_e - v'_e(v)j_e + w'_e(v)k_e &= \vec{\omega}^o_v
\end{align*}
\]

Observe that rigid-vertex model in the Definition 1 is derived based on purely geometric description of a frame. However, in order to extend this definition to a semi-rigid one, it will be required to include energies developed due to discontinuity of fields at the vertex set. Let, denote by \( S^m_+ \) to be set of real non-negative valued symmetric positive definite matrices of size \( n \).
Definition 2. A joint \( \mathbf{v} \) with assigned set of (stiffness) matrices \( \{ \mathcal{K}_{ge}^{(v)} \}_{e \sim v} \) and \( \{ \mathcal{K}_{\omega e}^{(v)} \}_{e \sim v} \) elements of \( S_3^3 \) is called semi-rigid, if for each edge \( e \sim v \) it satisfies

\[
\begin{align*}
    u_e(\mathbf{v})\tilde{t}_e + w_e(\mathbf{v})\tilde{j}_e + v_e(\mathbf{v})\tilde{k}_e &+ s^e(\mathcal{K}_{ge}^{(v)})^{-1}(c_e u'_e(\mathbf{v})\tilde{t}_e - b_e w''_e(\mathbf{v})\tilde{j}_e - a_e v''_e(\mathbf{v})\tilde{k}_e) = \tilde{g}_e^o \\
    \eta_e(\mathbf{v})\tilde{t}_e - v'_e(\mathbf{v})\tilde{j}_e + w'_e(\mathbf{v})\tilde{k}_e &+ s^e(\mathcal{K}_{\omega e}^{(v)})^{-1}(d_e \eta'_e(\mathbf{v})\tilde{t}_e - a_e v''_e(\mathbf{v})\tilde{j}_e + b_e w''_e(\mathbf{v})\tilde{k}_e) = \tilde{\omega}_e^o
\end{align*}
\] (5a)

We stress that presence of higher-order derivative in the definition of semi-rigid joint model in (5) compare to the rigid one in (4) has a direct connection to the energy at the vertex due to discontinuity of displacement and rotation vectors. In fact, let denote by set of matrices \( \{ \mathcal{B}_e \}_{e \in \mathcal{E}} \) with \( \mathcal{B}_e \in SO(3) \) transforming vector presentation in global coordinate system to the local one associated to an edge \( e \). Moreover, let denote by

\[
f_e(\mathbf{v}) := s^e(c_e u'_e\tilde{t}_e - b_e w''_e\tilde{j}_e - a_e v''_e\tilde{k}_e), \quad m_e(\mathbf{v}) := s^e(d_e \eta'_e\tilde{t}_e - a_e v''_e\tilde{j}_e + b_e w''_e\tilde{k}_e)
\] (6)

to be force and movement vectors corresponding to an edge \( e \sim v \) evaluated (in the limit sense) at \( \mathbf{v} \), then the corresponding developed force (moment) has a (linear-spring) form

\[
\begin{align*}
    f_e(\mathbf{v}) &= \mathcal{K}_{ge}^{(v)}(\mathcal{B}_e\tilde{g}_e^o - \tilde{g}_e(\mathbf{v})), \\
    m_e(\mathbf{v}) &= \mathcal{K}_{\omega e}^{(v)}(\mathcal{B}_e\tilde{\omega}_e^o - \tilde{\omega}_e(\mathbf{v}))
\end{align*}
\]

Sum of these (or net) forces and moments, classically appear from operator-theoretic derivations to guarantee self-adjointness of the resulting operator which contains physically well-known interpretation, i.e. balance of net force and moments. Interested reader may refer to the work [5] for detail discussion on derivation of self-adjoint operator corresponding rigid-joint model. However, in the semi-rigid case, individual force and moments corresponding to each edge appears directly in the definition of vertex model. Physical interpretation of each constituent terms in (5) are as follows:

(i) In condition (5a), \( c_e u'_e \) is the force developed in direction \( \tilde{t}_e \) due to in-axis tension of the edge \( e \), while \( a_e v''_e \) and \( b_e w''_e \) are shear forces developed inside the edge in the directions \( \tilde{k}_e \) and \( \tilde{j}_e \), respectively.

(ii) In condition (5b), \( d_e \eta'_e \) is the moment associated with angular displacement developed in direction \( \tilde{t}_e \) due to in-axis rotation of the edge \( e \), while \( a_e v''_e \) and \( b_e w''_e \) represent bending moments of the edge \( e \) in the directions \( \tilde{j}_e \) and \( \tilde{k}_e \), respectively.

We finally remark that the vector quantities in (5a)–(5b) remain invariant under local change in edge’s coordinate system. Next, we will present energy form on graph and its corresponding Hamiltonian for a generally three dimensional frames with semi-rigid joint model specified above.

### 2.2 Variational and Differential Formulations

In the context of the kinematic Euler–Bernoulli assumptions for beam, no pre-stress, or external force, the strain energy of the an edge \( e \) is expressed as

\[
U^{(e)}(x) = \frac{1}{2} \int_x (a_e(x)|v''_e(x)|^2 + b_e(x)|w''_e(x)|^2 + c_e(x)|u''_e(x)|^2 + d_e(x)|\eta'_e(x)|^2) dx
\] (7)

The integration here is over the beam \( e \), parameterized by the arc-length \( x \in [0, \ell_e] \). Throughout the rest of manuscript we assume each beam in our frame is homogeneous in the axial direction
and hence $a_e(x) \equiv a_e$ and so on. Extension of all results to variable stiffness is straightforward.

Regarding energy at the vertex set, let $\{K'_{y_2}\}_{e \sim v}$ and $\{K'_{\omega_2}\}_{e \sim v}$ be a family of stiffness-matrices in $S^3_\Gamma$ associate to edges $e \sim v$, then energy at $v$ due to the discontinuity of displacement and rotation fields is a functional of a form

$$U^{(v)} = \frac{1}{2} \sum_{e \sim v} \left( \|B_e \tilde{g}^\circ_e - \tilde{g}^\circ_e(v)\|^2_{K_e} + \|B_e \tilde{\omega}^\circ_e - \tilde{\omega}^\circ_e(v)\|^2_{K_e} \right)$$

Above, we use the convention of weighted induced norm in which for a vector $\vec{u} = u_k \tilde{u}_k$ and a matrix $\mathcal{K} = \kappa_{ij} \tilde{u}_i \tilde{u}_j$ defined with respect to an orthonormal basis $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, then $\|\vec{u}\|_\mathcal{K}^2 = \langle \vec{u}, \vec{u} \rangle_\mathcal{K} = \tilde{u}_i \kappa_{ij} \tilde{u}_j$. Combining energy functional corresponding to the deformation on edge set $\mathcal{E}$ and discontinuity at vertex set $\mathcal{V}$ respectively defined in (7) and (8), then total energy of the beam is expressed as

$$U^{(\Gamma)} := \sum_{e \in \mathcal{E}} U^{(e)} + \sum_{v \in \mathcal{V}} U^{(v)}$$

### 2.2.1 Quadratic Form and Self-Adjoint Operator

In this manuscript we will apply the convention that $\mathcal{H}^s(e)$ is the Sobolev space of $s$-times weakly differentiable functions on an edge $e$ whose derivatives up to order $s$ are in $L^2(e)$. The norm $\|u\|_{\mathcal{H}^s}$ for $u$ in the Sobolev space $\mathcal{H}^s(e)$ is equivalent to the norm $\|(I - \partial_x^2)^{s/2} u\|_{L^2(e)}$ in the Lebesgue space $L^2(e)$, e.g. see [8]. Let define underlying Hilbert space, i.e. in the terminology of Gelfand triples, the pivot space

$$\mathcal{L}^2(\Gamma) := \prod_{e \in \mathcal{E}} L^2(e) \times \prod_{e \in \mathcal{E}} L^2(e) \times \prod_{e \in \mathcal{E}} L^2(e) \times \prod_{e \in \mathcal{E}} L^2(e) \times \prod_{v \in \mathcal{V}} \mathbb{C}^3(v) \times \prod_{v \in \mathcal{V}} \mathbb{C}^3(v)$$

For any elements of the form $(\Psi, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ) \in \mathcal{L}^2(\Gamma)$ with $\Psi = [v, w, u, \eta]^T$, inner product on $\mathcal{L}^2(\Gamma)$ is defined as

$$\langle (\Psi, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ), (\tilde{\Psi}, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ) \rangle_{\mathcal{L}^2(\Gamma)} = \sum_{e \in \mathcal{E}} \langle \Psi_e, \tilde{\Psi}_e \rangle_{L^2(e)} + \sum_{v \in \mathcal{V}} \langle \tilde{g}_0^\circ, \tilde{g}_0^\circ \rangle_{m_v} + \sum_{v \in \mathcal{V}} \langle \tilde{\omega}_0^\circ, \tilde{\omega}_0^\circ \rangle_{m_v}$$

where $m_v$ is the magnitude (generally non-zero) of concentrated mass at the vertex $v$. Above, the first norm on the right-hand side of (11) is over edges, i.e.

$$\langle \Psi_e, \tilde{\Psi}_e \rangle_{L^2(e)} := \langle v_e, \tilde{v}_e \rangle_{L^2(e)} + \langle w_e, \tilde{w}_e \rangle_{L^2(e)} + \langle u_e, \tilde{u}_e \rangle_{L^2(e)} + \langle \eta_e, \tilde{\eta}_e \rangle_{L^2(e)}$$

Additionally, denote by $\mathcal{H}_S(\Gamma)$ to be a Hilbert space defined as

$$\mathcal{H}_S(\Gamma) := \prod_{e \in \mathcal{E}} \mathcal{H}^2(e) \times \prod_{e \in \mathcal{E}} \mathcal{H}^2(e) \times \prod_{e \in \mathcal{E}} \mathcal{H}^1(e) \times \prod_{e \in \mathcal{E}} \mathcal{H}^1(e) \times \prod_{v \in \mathcal{V}} \mathbb{C}^3(v) \times \prod_{v \in \mathcal{V}} \mathbb{C}^3(v)$$

This space is equipped with the norm induced by inner product of constituent Sobolev norms, namely for elements $(\Psi, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ) \in \mathcal{H}_S(\Gamma)$, then

$$\langle (\Psi, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ), (\tilde{\Psi}, \tilde{g}_0^\circ, \tilde{\omega}_0^\circ) \rangle_{\mathcal{H}_S(\Gamma)} = \sum_{e \in \mathcal{E}} \left( \langle v_e, \tilde{v}_e \rangle_{\mathcal{H}^2(e)} + \langle w_e, \tilde{w}_e \rangle_{\mathcal{H}^2(e)} + \langle u_e, \tilde{u}_e \rangle_{\mathcal{H}^1(e)} + \langle \eta_e, \tilde{\eta}_e \rangle_{\mathcal{H}^1(e)} \right)$$

$$+ \sum_{v \in \mathcal{V}} \langle \tilde{g}_0^\circ, \tilde{g}_0^\circ \rangle_{m_v} + \sum_{v \in \mathcal{V}} \langle \tilde{\omega}_0^\circ, \tilde{\omega}_0^\circ \rangle_{m_v}$$
Define a Sesquilinear form \( S : \mathcal{L}^2(\Gamma) \times \mathcal{L}^2(\Gamma) \to \mathbb{C} \) as
\[
S[(\Psi, g^\circ_0, \omega^\circ_0), (\tilde{\Psi}, \tilde{g}^\circ_0, \tilde{\omega}^\circ_0)] := \sum_{e \in \mathcal{E}} S^{(e)}[(\Psi, \tilde{\Psi})] + \sum_{v \in \mathcal{V}} S^{(v)}[(g^\circ_0, \omega^\circ_0), (\tilde{g}^\circ_0, \tilde{\omega}^\circ_0)]
\]
(13)
constructed out of a form associate to the edge set
\[
S^{(e)} := \int_e \left( a_e v''_e \overline{u'}_e + b_e w'_e \overline{w'}_e + c_e w'_e \overline{w'}_e + d_e u'_e \overline{u'}_e \right) dx
\]
(14)
and vertex set of the underlying graph
\[
S^{(v)} := \sum_{e \sim v} \langle B_v g^\circ_0 - \tilde{g}^\circ_e(v), B_v \tilde{g}^\circ_0 - \tilde{g}^\circ_e(v) \rangle_{K_{\tilde{g}^\circ_e}} + \sum_{e \sim v} \langle B_v \omega^\circ_0 - \tilde{\omega}^\circ_e(v), B_v \tilde{\omega}^\circ_0 - \tilde{\omega}^\circ_e(v) \rangle_{K_{\tilde{\omega}^\circ_e}}
\]
(15)
Following proposition provides a formal mathematical description of the energy form on graph with semi-rigid joint model at the vertex set. Its proof is similar to the proof of Theorem 3.1 in [5] with slight modification and we will present it here for sake of completeness. First let us stress that Equation (10) specifies the underlying inner product, while (12) prescribes the correct smoothness requirements on the individual fields. The sesquilinear form (13) is decomposed to the (virtual) energies at edge (14) and vertex (15) sets, where the latter one is absent in the case of rigid-vertex assumption.

**Proposition 2.1.** Energy functional (9) of the beam frame with free semi-rigid joints is the quadratic form corresponding to the positive closed sesquilinear form (13) densely defined on the Hilbert space \( L^2(\Gamma) \) stated in (12) with domain of \( S \) in (13) consisting of vectors \((\Psi, g^\circ_0, \omega^\circ_0) \in H_S(\Gamma)\).

**Proof of Proposition 2.1.** For (fixed) real-valued symmetric positive definite matrices \( \{K^{(v)}_{\tilde{g}^\circ_e}\}_{e \sim v} \) and \( \{K^{(v)}_{\tilde{\omega}^\circ_e}\}_{e \sim v} \), the sesquilinear form \( S \) in (13) is obviously positive and symmetric,
\[
S[(\Psi, g^\circ_0, \omega^\circ_0), (\Psi, g^\circ_0, \omega^\circ_0)] \geq 0
\]
\[
S[(\Psi, g^\circ_0, \omega^\circ_0), (\tilde{\Psi}, \tilde{g}^\circ_0, \tilde{\omega}^\circ_0)] = S[(\tilde{\Psi}, \tilde{g}^\circ_0, \tilde{\omega}^\circ_0), (\Psi, g^\circ_0, \omega^\circ_0)]
\]
(16)
Next we establish that \( S \) is closed, i.e. the domain \( D(S) \), the subspace of \( H_S(\Gamma) \), is complete with respect to the norm
\[
\| (\Psi, g^\circ_0, \omega^\circ_0) \|_S := \| (\Psi, g^\circ_0, \omega^\circ_0) \|_{L^2(\Gamma)} + S[(\Psi, g^\circ_0, \omega^\circ_0), (\Psi, g^\circ_0, \omega^\circ_0)]
\]
(17)
First, note that the space \( H_S(\Gamma) \) is complete with respect to its norm, the sum of Sobolev norms of individual Sobolev spaces and weighted norms on \( \mathbb{C}^3 \). Therefore \( D(S) \) is a closed subspace of \( H_S(\Gamma) \) and thus also complete with respect to the norm of \( H_S(\Gamma) \). Since for components \( v \) and \( w \) inequality
\[
\| f' \|_{L^2(e)} \leq \alpha \left( |e|^{-1} \| f \|_{L^2(e)} + |e| \| f'' \|_{L^2(e)} \right)
\]
holds for some \( \alpha > 0 \) independent of \( f \), along with application of bounds on vertex values of components of \( \Psi \) and their derivatives (see [9, 21]), then
\[
|f(v_e)|^2 \leq 2|e|^{-1} \| f \|^2_{L^2(e)} + |e| \| f' \|^2_{L^2(e)}, \quad \text{for any } f \in H^1(e),
\]
\[
|f'(v_e)|^2 \leq 2|e|^{-1} \| f' \|^2_{L^2(e)} + |e| \| f'' \|^2_{L^2(e)}, \quad \text{for any } f \in H^2(e),
\]
(19)
(20)
where \( |e| \) is the length of the edge \( e \) and \( v_e \) is one of its endpoints. Taking into account that matrices \( \{K^{(v)}_{\tilde{g}^\circ_e}\}_{e \sim v} \) are elements of \( S^3_+ \) along with the relations for vectors \( \tilde{g}^\circ_0 \) and \( \tilde{\omega}^\circ_0 \) in terms of fields stated in (5), then \( S \)-norm (17) is equivalent to the Sobolev norm of \( H_S(\Gamma) \).
Stem on the semi-boundedness and closeness of the form $\mathcal{S}$, it corresponds to a self-adjoint differential operator or Hamiltonian on the metric graph. Main result of this paper stated in the following Theorem will characterize this differential operator and its domain.

**Theorem 2.2.** Energy form (9) on a beam frame with free semi-rigid joints corresponds to the non-negative self-adjoint operator $A: L^2(\Gamma) \to L^2(\Gamma)$ with compact resolvant. Operator $A$ on set of edges $e \in \mathcal{E}$ and vertices $v \in \mathcal{V}$ of the graph acting as

$$
\begin{pmatrix}
\frac{v_e}{w_e} \\
\frac{u_e}{\eta_e}
\end{pmatrix}_{e \in \mathcal{E}} \mapsto \begin{pmatrix}
\left(\begin{array}{c}
a_e v_e''' \\
b_e w_e'''
\end{array}\right) \\
\left(\begin{array}{c}
c_e u_e'' \\
d_e \eta_e''
\end{array}\right)
\end{pmatrix}_{e \in \mathcal{E}} \left(\begin{pmatrix}
\tilde{g}_v^0 \\
\tilde{\omega}_v^0
\end{pmatrix}_{v \in \mathcal{V}}
\right)
$$

where above $\tilde{F}_v$ and $\tilde{M}_v$ are respectively the net forces and moments at vertex $v$, i.e.

$$
\tilde{F}_v := \sum_{e \sim v} s_e^0 (c_e u_e j_e - b_e w_e' j_e - a_e v_e' k_e), \quad \tilde{M}_v := \sum_{e \sim v} s_e^0 (d_e \eta_e' k_e + b_e w_e' k_e)
$$

Domain of the operator $A$ consists of the functions $(\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0)$ belong to

$$
\mathcal{H}_A(\Gamma) := \prod_{e \in \mathcal{E}} H^4(e) \times \prod_{e \in \mathcal{E}} H^4(e) \times \prod_{e \in \mathcal{E}} H^2(e) \times \prod_{v \in \mathcal{V}} C^3(v) \times \prod_{v \in \mathcal{V}} C^3(v)
$$

that satisfy at each vertex $v$ and for all $e \sim v$, non-homogeneous Robin conditions (5a) and (5b).

**Proof of Theorem 2.2.** Proof of the Theorem is based on extension of theoretical result in [37] towards vector-valued Hamiltonian and adaptation of arguments in [5] by generalized of vertex model. The reason for $A$ to be a self-adjoint operator with a compact resolvant, is that it is the Friedrichs extension of the triple $(L^2(\Gamma), \mathcal{H}_S(\Gamma), \mathcal{S})$ which can be applied here due to the observations in Proposition 2.1. Thus the operator $A_S$ is self-adjoint with domain

$$
\mathcal{D}(A_S) = \left\{ (\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0) \in \mathcal{H}_S(\Gamma) : \exists (\hat{\Psi}, \tilde{g}_v^0, \tilde{\omega}_v^0) \in L^2(\Gamma) \text{ so that } \right\}
$$

$$
\mathcal{S}[(\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0), (\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0)] = \{ (\tilde{g}_v^0, \tilde{\omega}_v^0) \}_{L^2(\Gamma)} \forall (\hat{\Psi}, \tilde{g}_v^0, \tilde{\omega}_v^0) \in \mathcal{H}_S(\Gamma)
$$

Thereby, two parts integration in the expression of the sesquilinear form $\mathcal{S}$ leads to, $(\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0) \in \mathcal{D}(A_S)$ if and only if $(\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0) \in \mathcal{H}_S(\Gamma)$ and there exists $(\hat{\Psi}, \tilde{g}_v^0, \tilde{\omega}_v^0) \in L^2(\Gamma)$ such that for any $(\Psi, \tilde{g}_v^0, \tilde{\omega}_v^0) \in \mathcal{H}_S(\Gamma)$, then

$$
\sum_{e \in \mathcal{E}} \int_{e} \left( v_e''' \tilde{v}_e + b_e w_e''' \tilde{w}_e - c_e u_e'' \tilde{u}_e - d_e \eta_e'' \tilde{\eta}_e \right) dx + \sum_{v \in \mathcal{V}} \sum_{e \sim v} \left( \langle B_e \tilde{g}_v^0 - \tilde{g}_e, B_e \tilde{\omega}_v^0 - \tilde{\omega}_e \rangle_{K(e)} \right) + 
\sum_{v \in \mathcal{V}} \sum_{e \sim v} \left( \langle B_e \tilde{\omega}_v^0 - \tilde{\omega}_e, B_e \tilde{\omega}_v^0 - \tilde{\omega}_e \rangle_{K(e)} + \sum_{v \in \mathcal{V}} B_v(\Psi, \hat{\Psi}) = \sum_{v \in \mathcal{V}} \langle \tilde{g}_v^0, \tilde{\omega}_v^0 \rangle_{m_v} + \sum_{v \in \mathcal{V}} \langle \tilde{\omega}_v^0, \tilde{\omega}_v^0 \rangle_{m_v}
$$

Above $B_v$ is a boundary term arises from integration by-parts, and has a form (evaluated at $v$)

$$
B_v(\Psi, \hat{\Psi}) := \sum_{e \sim v} (a_e v_e'' \tilde{v}_e - a_e v_e''' \tilde{v}_e + b_e w_e''' \tilde{w}_e - b_e w_e''' \tilde{w}_e + c_e u_e'' \tilde{u}_e + d_e \eta_e'' \tilde{\eta}_e)
$$
Applying the fact that set of \( \{ \tilde{e}, \tilde{j}_e, \tilde{\omega}_e \} \) are orthogonal basis on edge \( e \), along with realization of displacement vector \( \tilde{g}_e(v) \) stated in (1), then

\[
c_eu'_e \tilde{u}_e - b_e w'''_e \tilde{w}_e - a_e v'''_e \tilde{v}_e = \langle c_eu'_e \tilde{t}_e - b_e w'''_e \tilde{j}_e - a_e v'''_e \tilde{k}_e, \tilde{g}_e \rangle
\]

Next, we will apply the form of force and moment stated in (6). Adding and subtracting \( B_e \tilde{g}_o \) from the right-hand side of the above expression, summing over \( e \sim v \), along with the fact that for each \( e \sim v \), property \( \langle \tilde{f}_e, B_e \tilde{g}_o \rangle = \langle B_e \tilde{f}_e, \tilde{g}_o \rangle \) holds, then

\[
\sum_{e \sim v} (c_eu'_e \tilde{u}_e - b_e w'''_e \tilde{w}_e - a_e v'''_e \tilde{v}_e) = \langle \tilde{F}_e, \tilde{g}_o \rangle - \sum_{e \sim v} \langle \tilde{f}_e, B_e \tilde{g}_o - \tilde{g}_e \rangle
\]  
\( (26) \)

Above we use the fact that vector \( \tilde{g}_o \) is characteristic of \( v \) and independent from \( e \sim v \), along with the property of summing the vectors in first entry of right-hand side is in global-coordinate system,

\[
\sum_{e \sim v} \langle B^T_e \tilde{f}_e, \tilde{g}_o \rangle = \sum_{e \sim v} \langle B^T_e \tilde{f}_e, \tilde{g}_o \rangle = \langle \tilde{F}_v, \tilde{g}_o \rangle
\]  
\( (27) \)

Similar steps maybe applied along with realization of rotation vector \( \tilde{\omega}_e(v) \) stated in (2) to represent

\[
\sum_{e \sim v} (d_e \eta'_e \bar{\bar{h}}_e + b_e w'''_e \bar{\bar{w}}_e + a_e v'''_e \bar{\bar{v}}_e) = \langle \tilde{M}_e, \tilde{\omega}_o \rangle - \sum_{e \sim v} \langle \tilde{m}_e, B_e \tilde{\omega}_o - \tilde{\omega}_e \rangle
\]  
\( (28) \)

This along with (26) and (28) will be applied in (25) to obtain the boundary term

\[
B_v(\Psi, \tilde{\Psi}) = \langle \tilde{F}_v, \tilde{g}_o \rangle - \sum_{e \sim v} \langle \tilde{f}_e, B_e \tilde{g}_o - \tilde{g}_e \rangle + \langle \tilde{M}_v, \tilde{\omega}_o \rangle - \sum_{e \sim v} \langle \tilde{m}_e, B_e \tilde{\omega}_o - \tilde{\omega}_e \rangle
\]

Application of \( B_v(\Psi, \tilde{\Psi}) \) above along with identity \( \langle \bar{u}, \bar{v} \rangle = \langle K^{-1} \bar{u}, \bar{v} \rangle \) for \( \bar{u}, \bar{v} \in \mathbb{C}^3 \), then (24) reduces to

\[
\sum_{e \in E} \int_e \left( a_e v''_e \bar{\bar{v}}_e + b_e w'''_e \bar{\bar{w}}_e - c_e u''_e \bar{\bar{u}}_e - d_e \eta''_e \bar{\bar{h}}_e \right) dx + \sum_{v \in V} \left( \langle \tilde{F}_v, \tilde{g}_o \rangle + \sum_{e \sim v} \langle \tilde{f}_e, \tilde{g}_o \rangle \right) - \sum_{u \in V} \sum_{e \sim v} \left( \langle K^{\omega}_g \rangle^{-1} \tilde{f}_e + \tilde{g}_e - B_e \tilde{g}_o \right) + \langle \tilde{M}_v, \tilde{\omega}_o \rangle - \sum_{v \in V} \sum_{e \sim v} \left( \langle K^{\omega}_g \rangle^{-1} \tilde{m}_e + \tilde{\omega}_e - B_e \tilde{\omega}_o \right)
\]

But, by the Definition 2, properties \( (K^{\omega}_g)^{-1} \tilde{f}_e + \tilde{g}_e - B_e \tilde{g}_o = 0 \) and \( (K^{\omega}_g)^{-1} \tilde{m}_e + \tilde{\omega}_e - B_e \tilde{\omega}_o = 0 \) hold. Moreover, applying \( \langle \tilde{F}_v, \tilde{g}_o \rangle = \langle m^{-1} \tilde{F}_v, \tilde{g}_o \rangle_{m_0} \) and \( \langle \tilde{M}_v, \tilde{\omega}_o \rangle = \langle m^{-1} \tilde{M}_v, \tilde{\omega}_o \rangle_{m_0} \), then (24) reduces to

\[
\sum_{e \in E} \int_e \left( a_e v''_e \bar{\bar{v}}_e + b_e w'''_e \bar{\bar{w}}_e - c_e u''_e \bar{\bar{u}}_e - d_e \eta''_e \bar{\bar{h}}_e \right) dx + \sum_{v \in V} \left( \langle m^{-1} \tilde{F}_v, \tilde{g}_o \rangle_{m_0} + \sum_{e \sim v} \langle m^{-1} \tilde{f}_e, \tilde{g}_o \rangle_{m_0} \right) - \sum_{v \in V} \sum_{e \sim v} \left( \langle m^{-1} \tilde{m}_e, \tilde{\omega}_o \rangle_{m_0} + \sum_{e \sim v} \langle m^{-1} \tilde{\omega}_e, \tilde{\omega}_o \rangle_{m_0} \right)
\]

This then implies that the expression for the operator \( A_S \) which coincides with that of \( A \). Both domains also coincide. Finally, the equivalence of norms between \( \| \cdot \|_S \) and \( H_S(\Gamma) \) stated in Proposition 2.1 proves the positivity of operator \( A \).
Remark 2.2. We stress that the net forces and moments at vertex \( \mathbf{v} \) in the statement of Theorem 2.2 are respectively of the form \( \mathbf{F}_v := \sum_{e \ni v} \mathbf{f}_e \) and \( \mathbf{M}_v := \sum_{e \ni v} \mathbf{m}_e \) constructed out of edgewise counterparts defined in (6). These two quantities encapsulate important physical meaning, namely, they respectively model dynamics of net forces and moments developed at a vertex with concentrated mass and due to a relative displacement and torsion of edges adjacent to the vertex. This generalizes the result in [5] in which for the case of massless rigid-joint, the two conditions are vanishing. But presence of concentrated mass causes apprentice of spectral dependent vertex conditions.

2.2.2 Planar Frame

A particularly well-studied case in the literature is the planar frame with rigid joints, which in its undeformed configuration is embedded in a two dimensional plane. It has been shown that in this situation the Hamiltonian of the frame decouples into two operators, one linking out-of-plane with angular displacements and the other one linking in-plane with axial displacements, see Corollary 4.2 in [5] for detail. Next, we will discuss role of semi-rigid joint on validity of a similar decoupling property. Without loss of generality assume that this plane has normal vector \( \vec{E}_3 \). We also assume that for all edges \( e \), vector \( \vec{k}_e \) is chosen to be the same as the global vector \( \vec{E}_3 \). Following definition will characterize class of (vertex) stiffness matrices which preserves decoupling of modes similar to rigid-joint case. In this section for simplicity we assume global basis are the standard ones, i.e. \( \vec{E}_i \) has only non-zero entry at position \( i \).

Definition 3. Matrix \( \mathbf{K} \in S^2_+ \) is called \( \vec{k} \)-plane preserving if it has a form

\[
\mathbf{K} := \begin{pmatrix} \kappa_1 & \kappa & 0 \\ \kappa & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} \tag{29}
\]

Let denote by \( d_\kappa := \kappa_1 \kappa_2 - \kappa^2 \), then observe that for any vector \( \vec{x} = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \) and \( \mathbf{K} \) of the form (29), then

\[
\mathbf{K}^{-1} \vec{x} = \begin{pmatrix} \kappa_1 \kappa_2 \kappa^2 \end{pmatrix} = d_\kappa^{-1}(\kappa_2 x_1 - \kappa x_2) \vec{i} + d_\kappa^{-1}(\kappa_1 x_2 - \kappa x_1) \vec{j} + d_\kappa^{-1} x_3 \vec{k} \tag{30}
\]

suppresses a presence of \( x_1 \), and \( x_2 \) in the \( \vec{k} \)-direction. We stress that \( \det(\mathbf{K}) = \kappa_3 d_\kappa \in \mathbb{R}_+ \) and applying property \( \kappa_3 > 0 \), thereby \( d_\kappa \in \mathbb{R}_+ \). Special member of \( \vec{k} \)-plane preserving matrices are the diagonal ones with positive entries. In this case, (30) reduces to

\[
\mathbf{K}^{-1} \vec{x} = \begin{pmatrix} \kappa_1 \kappa_2 \kappa^2 \end{pmatrix} = \kappa_1^{-1} x_1 \vec{i} + \kappa_2^{-1} x_2 \vec{j} + \kappa_3^{-1} x_3 \vec{k} \tag{31}
\]

Following corollary shows that under class of \( \vec{k} \)-plane preserving stiffness matrices, the operator decomposes into a direct sum of two operators, one coupling out-of-plane to angular displacements and the other coupling in-plane with axial displacements. Observe that for planar graphs and the convention discussed above, vectors \( \vec{i} \) and \( \vec{j} \) can be written of size two. Prior to state the desired result, let denote by \( \mathcal{P} := (\vec{E}_1, \vec{E}_2) \) to be 3 x 2 matrix, and moreover let \( \mathcal{Q} := \vec{E}_3 \).

Corollary 2.4. Free planar network of beams with semi-rigid joints equipped with family of \( \vec{k} \)-preserving stiffness matrices \( \{ \mathbf{K}^{(v)}_{\mathbf{e}} \}_{e \ni \mathbf{v}} \) and \( \{ \mathbf{K}^{(w)}_{\mathbf{e}} \}_{e \ni \mathbf{w}} \) is described by Hamiltonian \( \mathcal{A} = \mathcal{A}^{(\text{out})} \oplus \mathcal{A}^{(\text{in})} \) where \( \mathcal{A}^{(\text{out})} \) is a differential operator acting as

\[
\left\{ \begin{pmatrix} \mathbf{v}_e \\ \eta_e \end{pmatrix}_{e \in \mathcal{E}}, \left( \mathbf{v}^{(v)}_{\mathbf{e}} \right)_{\mathbf{e} \in \mathcal{V}}, \left( \mathbf{w}^{(w)}_{\mathbf{e}} \right)_{\mathbf{e} \in \mathcal{V}} \right\} \mapsto \left\{ \begin{pmatrix} \mathbf{a}_e \mathbf{v}_e''' \\ -d_e \eta_e'' \end{pmatrix}, \left( \mathbf{m}_e^{-1} \mathbf{F}^{(\text{out})}_{\mathbf{e}} \right)_{\mathbf{e} \in \mathcal{V}}, \left( \mathbf{m}_e^{-1} \mathbf{M}^{(\text{out})}_{\mathbf{e}} \right)_{\mathbf{e} \in \mathcal{V}} \right\} \tag{32}
\]

10
on functions in the space \( \prod_{e \in \mathcal{E}} \mathcal{H}^4(e) \times \prod_{e \in \mathcal{E}} \mathcal{H}^2(e) \times \prod_{v \in \mathcal{V}} \mathbb{C}(v) \times \prod_{v \in \mathcal{V}} \mathbb{C}^2(v) \) satisfying at each vertex \( v \in \mathcal{V} \) and for all \( e \sim v \) the primary conditions

\[
v_e(v) - s_e^p \left( Q^T \mathcal{K}_{ge}^{(v)} Q \right)^{-1} a_e v_e'' = v_e^o \quad (33a)
\]
\[
\eta_e(v)\vec{i}_e - v_e'(v)\vec{j}_e + s_e^p \left( P^T \mathcal{K}_{we}^{(v)} P \right)^{-1} \left( d_e \eta_e \vec{i}_e - a_e v_e'' \vec{j}_e \right) = \vec{\omega}_e^o \quad (33b)
\]

with net forces and moments respectively of a form

\[
F_{v_{\text{out}}} := - \sum_{e \sim v} s_e^p a_e v_e'' , \quad \vec{M}_{v_{\text{out}}} := \sum_{e \sim v} s_e^p (d_e \eta_e \vec{i}_e - a_e v_e'' \vec{j}_e) \quad (34a)
\]

The operator \( A^{(in)} \) acts as

\[
\left\{ \left( \begin{array}{c} w_e \\ u_e \end{array} \right) \right\}_{e \in \mathcal{E}} , \left( \bar{g}_e^o \right)_{v \in \mathcal{V}} , \left( \omega_v^o \right)_{v \in \mathcal{V}} \right\} \mapsto \left\{ \left( \begin{array}{c} b_e w_e'' \\ -c_e u_e' \end{array} \right) , \left( m_{v_{\text{in}}}^{-1} \vec{F}_{v_{\text{in}}} \right)_{v \in \mathcal{V}} , \left( m_{v_{\text{in}}}^{-1} M_{v_{\text{in}}} \right)_{v \in \mathcal{V}} \right\} \quad (35)
\]

on functions in the space \( \prod_{e \in \mathcal{E}} \mathcal{H}^4(e) \times \prod_{e \in \mathcal{E}} \mathcal{H}^2(e) \times \prod_{v \in \mathcal{V}} \mathbb{C}(v) \times \prod_{v \in \mathcal{V}} \mathbb{C}^2(v) \) satisfying at each vertex \( v \) and for all \( e \sim v \) the primary conditions conditions

\[
u_e(v)\vec{i}_e + w_e(v)\vec{j}_e + s_e^p \left( P^T \mathcal{K}_{we}^{(v)} P \right)^{-1} \left( c_e u_e' \vec{i}_e - b_e w_e'' \vec{j}_e \right) = \bar{g}_e^o \quad (36a)
\]
\[
w_e(v) + s_e^p \left( Q^T \mathcal{K}_{we}^{(v)} Q \right)^{-1} b_e w_e'' = \omega_v^o \quad (36b)
\]

with net forces and moments respectively of a form

\[
\vec{F}_{v_{\text{in}}} := \sum_{e \sim v} s_e^p (c_e u_e' \vec{i}_e - b_e w_e'' \vec{j}_e) , \quad M_{v_{\text{in}}} := \sum_{e \sim v} s_e^p b_e w_e'' \quad (37a)
\]

Proof of Corollary 2.4. The differential expression for the operator \( A \) is already in the “block-diagonal” form, see (21), so it remains to show that the vertex conditions decompose as described. But that follows directly from projecting conditions (5a), (5b) and later (38) onto the common normal \( \vec{k} \) and onto its orthogonal complement along with application of definition (3). \( \square \)

3 Eigenvalue problem on Compact Graph

3.1 Preliminary

Our aim in the remaining Sections is to characterize the spectrum \( \sigma(A) \) of operator \( A \) acting on compact graphs \(^4\). According to the Proposition 2.1 this spectrum is positive and discrete. As it is common in literature, we shall rewrite the eigenvalue problem into an equivalent matrix differential value problem, known as “characteristic” or “secular” equation [3,26]. The most direct approach is to solve the eigenvalue equation \( A \Psi = \lambda \Psi \) component-wise on every edge before applying conditions at vertices. We will show detail of this derivation for a simple 1D example (known as Cantilevered beam) in which discontinuity of fields are admissible. However, problems can quickly become computationally overwhelming due to large number of degrees of freedom associated to each beam. Techniques based on representation theory of finite groups maybe applied on a highly symmetric

\(^4\)In our setting compact frames means: that \( \mathcal{E} \) is a finite set and each edge \( e \in \mathcal{E} \) has finite length.
graph example to decompose the original problem into a sum of operators each corresponding to a particular class of vibrational modes, e.g. see [2, 5]. We will discuss extension of this result for the case in which concentrated mass exists at the semi-rigid joint. Another approach to deal with more general problems (e.g. absence of symmetry) is to employ numerical schemes such as finite element method, e.g. see [1]. The price for that however, is losing the analytic form of secular form and requirements of dense frame’s discretization for capturing high energy modes. Stem on the techniques applied in this framework, we will discuss the idea of constructing geometric-free local spectral basis set together with enforcing geometry of compact graph into play by representation the global solution as a linear combination of them. This will be the topic of next Section and a departure point for the forthcoming work on spectral analysis of periodic frames equipped with general class of vertex matching models [17].

Here, we recall that for the general semi-rigid vertex model, at each $v \in V$ and for all $e \sim v$, conditions (5) should be enforced. Moreover, presence of concentrated mass ($m_v > 0$) will turn the net force and moment conditions depending on the eigenvalue, namely at each $v \in V$ properties

$$
\sum_{e \sim v} s_v^e (c_e u_e^e \vec{r}_e - b_e w_e^e \vec{r}_e - a_e v_e^e \vec{r}_e) = \lambda m_v \vec{g}_v^o,
$$

(38a)

$$
\sum_{e \sim v} s_v^e (d_e \eta_e^e \vec{r}_e - a_e v_e^e \vec{r}_e + b_e w_e^e \vec{r}_e) = \lambda m_v \vec{w}_v^o.
$$

(38b)

are imposed. Finally, it maybe interesting to observe that vertex conditions (5a) and (38a) can be represented in a well-known compact format established for class of second-order Schrödinger operators, e.g. see [6]. For $e \sim v$, let $\vec{g}_e$ and $\vec{f}_e$ be vectors of size 3 with entries of displacement and force vectors corresponding $e_v$ in local coordinate of $e$. Similar definition holds for vectors $\vec{w}_v$ and $\vec{m}_e$. Introduce matrices

$$
A_v := \begin{pmatrix}
B_1 & -B_2 & 0 & \cdots & 0 & 0 \\
0 & B_2 & -B_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & B_{n-1} & -B_n \\
-\lambda m_v B_1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},
B_v := \begin{pmatrix}
s_1 K_1^{-1} & -s_2 K_2^{-1} & 0 & \cdots & 0 & 0 \\
s_2 K_2^{-1} & -s_3 K_3^{-1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s_{n-1} K_{n-1}^{-1} & -s_n K_n^{-1} \\
s_1 B_1 & s_2 B_2 & s_3 B_3 & \cdots & s_{n-1} B_{n-1} & s_n B_n
\end{pmatrix}
$$

where $B_i$ is the transformation matrix of an edge $e_i$ introduced in Section 2 and 0 is zero-block matrix of size 3 $\times$ 3. Moreover, let denote by $g^{(v)} := [\vec{g}_1 \ \vec{g}_2 \ \cdots \ \vec{g}_n]^T$ and $f^{(v)} := [\vec{f}_1 \ \vec{f}_2 \ \cdots \ \vec{f}_n]^T$ to be vectors of size $(3d_v) \times 1$, with $d_v$ stands for degree of vertex $v$. Then conditions (5a) and (38a) can be recast as a single equation $A_v g^{(v)} + B_v f^{(v)} = 0$. Similar result should hold for rotation and moments at $v$ by relation $A_v \omega^{(v)} + B_v m^{(v)} = 0$. These two systems can be then merged as

$$
\begin{pmatrix}
A_v & 0 \\
0 & A_v
\end{pmatrix}
\begin{pmatrix}
g^{(v)} \\
\omega^{(v)}
\end{pmatrix} +
\begin{pmatrix}
B_v & 0 \\
0 & B_v
\end{pmatrix}
\begin{pmatrix}
f^{(v)} \\
m^{(v)}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

(39)

which is in the form of a well-known general (homogeneous) vertex condition. Observe that $A_v$ and $B_v$ are $(3d_v) \times (3d_v)$ matrices. Due to the (Block) diagonal structure of (39), in order to guarantee that the correct number of independent conditions (equal to $6d_v$) is imposed, rank of the $3d_v \times 6d_v$ global pair matrices $(A_v, B_v)$ must be equal to $3d_v$, i.e., maximal. It is easy to check that this property holds for the proposed general class of semi-rigid joint model and the case in which concentrated mass exists.
3.2 Example of One Dimensional Graph

Consider beam frame depicted in Figure 2 consisting of two edges $e_1, e_2$ meeting at the central free semi-rigid joint $v_c$. The beams are oriented from the beam's ends to the joint, see the local basis of each beam. In the case of rigid central vertex, and by merging the two edges to one edge, see Remark 4.4 in [5], this frame is known as cantilevered beam and all fields will be decoupled. However, this decoupling for semi-rigid vertex is only the case if stiffness matrix in (38) is diagonal. For simplicity and comparison among the two types of join models, here we assume diagonal vertex stiffness and will investigate the role of different parameters on eigenfunctions corresponding to the in-plane lateral displacement field $w(x)$. General eigenvalue problem for $e = 1, 2$ then has a form

$$b_e w_e'''(x) = \lambda w_e(x)$$

subjected to fixed boundary condition at $v_1$, i.e. $w_1(0) = w'_1(0) = 0$ and free type at $v_2$, namely $w_2''(0) = w_2'''(0) = 0$, see Table 1 in [5] for general boundary types. Applying properties $\bar{j}_2 = -\bar{j}_1$ and $\bar{k}_2 = \bar{k}_1$ along with (5) and (38), at the central vertex $v_c$ displacement (and rotation) relations

$$w_1(\ell_1) - \kappa_{g_1}^{-1} b_1 w_1'''(\ell_1) = -w_2(\ell_2) + \kappa_{g_2}^{-1} b_2 w_2'''(\ell_2) = w_c^0$$

along with force (and moment) constraints

$$-b_1 w_1'''(\ell_1) + b_2 w_2'''(\ell_2) = \lambda m_c w_c^0$$

should be satisfied. In the rigid-vertex case, i.e. letting $\kappa_{g_1}^{-1}, \kappa_{v_1}^{-1} \rightarrow 0$, and by setting parameters $b_i = 1$, then eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ satisfy characteristic equation

$$\cosh(\mu_n L) \cos(\mu_n L) + 1 = 0$$

with $L := \ell_1 + \ell_2$ and $\mu_n := \lambda_n^{1/4}$. Corresponding normalized eigenfunctions then have a form

$$w^{(n)}(x) = A \left[ \cosh(\mu_n x) - \cos(\mu_n x) - \frac{\cosh(\mu_n L) + \cos(\mu_n L)}{\sinh(\mu_n L) + \sin(\mu_n L)} \left( \sinh(\mu_n x) - \sin(\mu_n x) \right) \right]$$

Figure 2: Geometry of 1D graph in its equilibrium with (generally) semi-rigid central vertex $v_c$.

For the semi-rigid central joint model, imposing the boundary conditions at $v_1, v_2$, then general solution to the eigenvalue problem (40) on each edge reduces to

$$w_1^{(n)}(x) = A_1^{(n)}(\sinh(\mu_n x) - \sin(\mu_n x)) + B_1^{(n)}(\cosh(\mu_n x) - \cos(\mu_n x)),$$

For example see wikipedia page.
\[ w_2^{(n)}(x) = A_2^{(n)}(\sinh(\mu_n x) + \sin(\mu_n x)) + B_2^{(n)}(\cosh(\mu_n x) + \cos(\mu_n x)). \]  

(45b)

Applying vertex conditions (41) and (42) in general solutions (45), eigenvalue problem reduces to finding the \((n\text{-dependent})\) coefficient vector \([A_1, B_1, A_2, B_2, w_v^o, \omega_v^o]^T\) in the kernel of the \(6 \times 6\) matrix \(M(\lambda)\) given by

\[
\left( \begin{array}{ccccccc}
S_{\mu_{1_1}}^{-} - \mu^3 \kappa_{1_1}^{-1} C_{\mu_{1_1}}^{+} & C_{\mu_{1_2}}^{-} - \mu^3 \kappa_{1_2}^{-1} S_{\mu_{1_2}}^{+} & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
S_{\mu_{2_1}}^{+} - \mu^3 \kappa_{2_1}^{-1} C_{\mu_{2_1}}^{+} & C_{\mu_{2_2}}^{-} - \mu^3 \kappa_{2_2}^{-1} S_{\mu_{2_2}}^{+} & 1 & 0 & C_{\mu_{2_1}}^{-} \omega & -S_{\mu_{2_1}}^{+} & \mu m_v C_{\mu_{2_1}}^{-} \\
S_{\mu_{2_1}}^{+} & C_{\mu_{2_2}}^{-} & 0 & 0 & -\mu^2 m_v & C_{\mu_{2_1}}^{-} \\
S_{\mu_{2_1}}^{+} & C_{\mu_{2_2}}^{-} & 0 & 0 & -1 & \mu m_v C_{\mu_{2_1}}^{-} \\
S_{\mu_{2_1}}^{+} & C_{\mu_{2_2}}^{-} & 0 & 0 & -\mu^2 m_v & C_{\mu_{2_1}}^{-}
\end{array} \right)
\]

with the corresponding eigenvalue \(\lambda_n\) being a solution of (40). Above (and in the rest of paper) we used the following abbreviations to make the matrix presentation more compact

\[
S_\gamma := \sin(\gamma), \quad C_\gamma := \cos(\gamma), \quad S^\pm_\gamma := \sinh(\gamma) \pm \sin(\gamma), \quad C^\pm_\gamma := \cosh(\gamma) \pm \cos(\gamma),
\]

(46)

Figure 3 plots eigenfunctions corresponding to the first two eigenvalues of \(M\) for different set of parameters. Compare to the cantilevered beam (44), it is clear that how introducing semi-rigid central vertex will cause discontinuity of \(w(x)\) and its derivative.

Figure 3: Plots of first two eigenfunctions \(w^{(1)}(x)\) and \(w^{(2)}(x)\) in global coordinate for different values of concentrated mass \(m_v\), and vertex stiffness including \(\kappa_{1}^{-1}\) and \(\kappa_{2}^{-1}\).

Referring to the simple example above, it is clear that deriving analytical solutions for more complicated graph in which all degrees of freedom active is not a desirable (and maybe computationally feasible) task. In the work [5], application of group representation theory to classify eigenmodes and simplifying analysis of graphs with a high degree of symmetry for class of rigid-vertex model has been discussed. Main goal of next example is to extend this result to semi-rigid joint model equipped with concentrated mass at a vertex.

### 3.3 Example of Three Dimensional Graph.

Let \(\Gamma\) be the graph formed by three leg beams \(e_1, e_2, e_3\) and a vertical beam \(e_0\), all joining at the central vertex \(v_c\), see Figure 4. The structure and its material parameters are assumed to be symmetric with respect to rotation by \(2\pi/3\) around the vertical axis and with respect to reflection swapping a pair of the leg beams, i.e.
Assumption 1. The symmetric graph $\Gamma$ satisfies

(i) Beam’s material parameters satisfy $a_0 = b_0$ and that the leg beams are identical with principal axes of inertia that may be chosen to have $\vec{j}_s \perp \vec{E}_3$ for $s = 1, 2, 3$.

(ii) Displacement and rotation stiffness corresponding semi-rigid central vertex $v_c$, for vertical edge is a pair $(\kappa_{g_0}, \kappa_{w_0})$ and for the leg edges satisfy $(\kappa_{g_s}, \kappa_{w_s}) = (\kappa_{g}, \kappa_{w})$ for $s = 1, 2, 3$.

The leg ends $v_1, v_2, v_3$ are fixed (vanishing displacement and angular displacement), vertex $v_c$ is a free semi-rigid joint and the end $v_0$ of the vertical beam is free. Our main result of this Section is a decomposition of the operator $\mathcal{A}$ into a direct sum of four self-adjoint operators.

**Theorem 3.1** (Domain Decomposition). The Hamiltonian operator $\mathcal{A}$ of the beam frame $\Gamma$ is reduced by the decomposition

$$\mathcal{L}^2(\Gamma) = \mathcal{H}_{\text{id}} \times \mathcal{H}_{\text{alt}} \times \mathcal{H}_\omega \times \mathcal{H}_\chi,$$

where the disjoint subspaces consist of elements $\bar{\Psi} \equiv (\Psi, \bar{g}_{v_c}^\circ, \bar{\omega}_{v_c}^\circ)$ coinciding as set with

$$\mathcal{H}_{\text{tri}} := \{ \bar{\Psi} \in \mathcal{L}^2(\Gamma) : v_0 = w_0 = 0, w_s = \eta_s = 0, u_1 = u_2 = u_3, \bar{g}_{v_c}^\circ \cdot \vec{E}_1 = 0, \bar{g}_{v_c}^\circ \cdot \vec{E}_2 = 0, \bar{\omega}_{v_c}^\circ = 0 \},$$

$$\mathcal{H}_{\text{alt}} := \{ \bar{\Psi} \in \mathcal{L}^2(\Gamma) : v_0 = w_0 = u_0 = 0, u_s = \pi_s = 0, w_1 = w_2 = w_3, \eta_1 = \eta_2 = \eta_3, \bar{g}_{v_c}^\circ = 0, \bar{\omega}_{v_c}^\circ \cdot \vec{E}_1 = 0, \bar{\omega}_{v_c}^\circ \cdot \vec{E}_2 = 0 \},$$

$$\mathcal{H}_\omega := \{ \bar{\Psi} \in \mathcal{L}^2(\Gamma) : 0 = \eta_0 = 0, b_0 w_0 = ia_0 v_0, \Psi_3 = \omega \Psi_2 = \omega^2 \Psi_1, \bar{g}_{v_c}^\circ \cdot \vec{E}_3 = 0, \bar{\omega}_{v_c}^\circ \cdot \vec{E}_3 = 0, \bar{g}_{v_c}^\circ \cdot \vec{E}_2 = i(\bar{g}_{v_c}^\circ \cdot \vec{E}_1), \bar{\omega}_{v_c}^\circ \cdot \vec{E}_1 = -i(\bar{\omega}_{v_c}^\circ \cdot \vec{E}_2) \} = \mathcal{H}_\chi,$$

with $s \in \{1, 2, 3\}$ labels the legs, $\Psi_s := (v_s, u_s, u_s, \eta_s)^T$, and $\omega = e^{2\pi i / 3}$.

**Proof of Theorem 3.1.** Decomposition of domain is based on the proof of Theorem 5.1 in [5]. In details, the properties of $\Psi_s := [v_s, u_s, u_s, \eta_s]^T$ in each irreducible subspace is based on condition (i) in the Assumption 1. Moreover, properties on the vectors $\bar{g}_{v_c}^\circ$ and $\bar{\omega}_{v_c}^\circ$ is based on the edge’s force and moments relations (5a) and (5b) repetitively by setting $c = c_0$ along with application of condition (ii) in the Assumption 1. \hfill $\Box$

**Remark 3.2.** A decomposition $\mathcal{H} = \prod_{\rho} \mathcal{H}_{\rho}$ is reducing for an operator $\mathcal{A}$ if $\mathcal{A}$ is invariant on each of the subspaces and the operator domain $\text{Dom}(\mathcal{A})$ is “aligned” with respect to the decomposition, namely

$$\text{Dom}(\mathcal{A}) = \prod_{\rho} \left( \mathcal{H}_{\rho} \cap \text{Dom}(\mathcal{A}) \right)$$

This means that every aspect of the spectral data of the operator $\mathcal{A}$ is the union of the spectral data corresponding to a restricted subspace, interested reader may refer to the work [5] for detailed background along this line.
Abstract result of Theorem 3.1 maybe applied to explicitly characterize eigenfunctions belonging to each irreducible subspace. In the next example we will restrict such analysis to the case $\mathcal{A}_\omega = \mathcal{A}|_{\mathcal{H}_\omega}$ and will leave similar calculations for the subspaces $\mathcal{H}_{tri}$ and $\mathcal{H}_{alt}$ to the interested reader.

**Characteristic equation for $\mathcal{A}_\omega$ operator.** Restricting to the $\mathcal{H}_\omega$ subspace, imposing free vertex condition at $\mathbf{v}_0$ and fixed vertex conditions at $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then general solution set is of the form

$$
\begin{align*}
v(x) &= A_v (\sinh(\mu x) - \sin(\mu x)) + B_v (\cosh(\mu x) - \cos(\mu x)), \\
iw(x) &= A_w (\sinh(\mu x) - \sin(\mu x)) + B_w (\cosh(\mu x) - \cos(\mu x)), \\
u(x) &= A_u \sin(\beta x), \\
i\eta(x) &= A_\eta \sin(\beta x), \\
v_0(x) &= A_0 (\sinh(\mu x) + \sin(\mu x)) + B_0 (\cosh(\mu x) + \cos(\mu x)),
\end{align*}
$$

where $\mu = (\lambda/a)^{1/4}$, and $\beta = (\lambda/d)^{1/2}$. By the assumption that discontinuity of fields are only admissible at $\mathbf{v}_c$, derivation of vertex conditions are discussed in the Appendix. Constraining to the case in which $\kappa_{\omega_1}^{-1}, \kappa_{\omega_1}^{-1} \to 0$, and assuming unit material parameters $a = a_0 = d = 1$, then application of set linearly independent conditions at the central vertex $\mathbf{v}_c$ summarized in remark 5.1, eigenvalue problem reduces to finding the ($n$-dependent) coefficient vector $[A_v, B_v, A_w, B_w, A_u, A_\eta, A_0, B_0, (\vec{g}_{\omega_c}^\circ \cdot \vec{E}_1), (\vec{g}_{\omega_c}^\circ \cdot \vec{E}_2)]^T$ in the kernel of the $10 \times 10$ matrix $\mathbb{M}(\lambda; \kappa_{\omega_0}^{-1}, \kappa_{\omega_0}^{-1})$ decomposed as

$$
\begin{align*}
\mathbb{M}(\lambda; \kappa_{\omega_0}^{-1}, \kappa_{\omega_0}^{-1}) &= \mathbb{M}_v(\lambda) + \mathbb{M}_{g_0}(\lambda; \kappa_{\omega_0}^{-1}) + \mathbb{M}_{w_0}(\lambda; \kappa_{\omega_0}^{-1}) \\
&= \mathbb{M}_v(\lambda) + \mathbb{M}_{g_0}(\lambda; \kappa_{\omega_0}^{-1}) + \mathbb{M}_{w_0}(\lambda; \kappa_{\omega_0}^{-1})
\end{align*}
$$

(48)

Above, modification towards admissible discontinuity of displacement and rotation fields at $\mathbf{v}_c$ are given by matrices $\mathbb{M}_{g_0}$ and $\mathbb{M}_{w_0}$, respectively defined as

$$
\begin{align*}
\mathbb{M}_{g_0}(\lambda; \kappa_{\omega_0}^{-1})_{p,q} &= -\kappa_{\omega_0}^{-1} \mu^3 (C_{\mu}^- \delta_{q,7} + S_{\mu}^+ \delta_{q,8}) \delta_{7,p} \\
\mathbb{M}_{w_0}(\lambda; \kappa_{\omega_0}^{-1})_{p,q} &= +\kappa_{\omega_0}^{-1} \mu^2 (S_{\mu}^- \delta_{q,7} + C_{\mu}^- \delta_{q,8}) \delta_{8,p}
\end{align*}
$$

(49a) (49b)
while matrix $\mathbb{M}_r(\lambda)$ governs the problem with rigid joint model $\mathbf{v}_e$ with entries\(^6\)

\[
\begin{pmatrix}
\mu C^-_\mu & \mu S^+_\mu & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
S^-_\mu C_\alpha & C^-_\mu S_\alpha & 0 & 0 & -S^2_\mu C_\alpha & 0 & 0 & 0 & 1 \\
S^-_\mu C_\alpha & C^-_\mu S_\alpha & 0 & 0 & +S^2_\mu S_\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & S^-_\mu & C^-_\mu & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \mu C^-_\mu S_\alpha & S^+_\mu S_\alpha & 0 & -S^2_\mu C_\alpha & 0 & 0 & 1 \\
0 & 0 & \mu C^-_\mu S_\alpha & S^+_\mu S_\alpha & 0 & +S^2_\mu S_\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{2} C^+_\mu S_\alpha \quad \frac{3}{2} \mu S^-_\mu S_\alpha \quad \frac{3}{2} \mu C^+_\mu \quad \frac{3}{2} S^+_{\mu} \quad \frac{3}{2} \mu S^-_\mu \quad \frac{3}{2} C^+_{\mu} \quad \frac{3}{2} C^-_{\mu} \quad \frac{3}{2} C^-_{\mu} \quad 0 \\
\frac{3}{2} S^+_{\mu} \quad \frac{3}{2} \mu C^-_\mu \quad \frac{3}{2} S^-_\mu \quad \frac{3}{2} \mu S^-_\mu \quad \frac{3}{2} C^-_\mu \quad \frac{3}{2} C^-_\mu \quad \frac{3}{2} S^-_\mu \quad \frac{3}{2} \mu S^-_\mu \quad -\frac{3}{2} \mu^2 m_\alpha \\
\end{pmatrix}
\]

4 Local Spectral Basis and Graph Geometry

4.1 Preliminary

The goal of this Section is to develop characteristics equation corresponding to an eigenvalue problem on 3-star planar graph $\Gamma$, see Figure 5. Unlike the former examples, here this task will be done by introducing geometric-free local spectral basis applicable to each edge of graph, while geometry of $\Gamma$ will be later encoded in the coefficient set of constructed fields. This method is general and clearly can be applied to derive characteristic equation derived in the former examples. But we keep this ordering of exhibition to emphasize the benefits of each method. In contrast with the commonly applied (fixed) local basis in numerical methods, e.g. finite element method [1], here each basis solves an eigenvalue problem on a fixed segment $[0, 1]$ with appropriate boundary conditions. This can be considered as an extension of the framework applied in analysis of periodic graphs in [28], by considering coupled fields and enriching solution’s domain in which discontinuity of fields are admissible. For the sake of clearness, discussion will be limited to a special case in which the boundary vertices $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are fixed. More general cases will be a straightforward extension and will be presented on analysis of periodic graphs in the forthcoming manuscript [17].

Consider a planar frame depicted in Figure 5 consisting of three beams $e_1, e_2, e_3$ meeting at the free central semi-rigid joint $\mathbf{v}_e$. The beams are oriented from the fixed ends to the joint; the local basis of each beam is shown in the figure. Introducing $\Psi_e(x) := [v_e(x), \eta_e(x)]^T$, we are interested on eigenvalue problem (32) on this graph. The vertex conditions in Corollary 2.4 implies that at each vertex $\mathbf{v}$ and for each $e \sim \mathbf{v}$, subjected to displacement-force condition

\[
v_e(\mathbf{v}) + s^0_e a_e \kappa^{-1}_v v''_e(\mathbf{v}) = v^0_e \tag{50}
\]

and rotation-moment conditions

\[
\eta_e(\mathbf{v}) + s^0_e d_e \kappa^{-1}_w \eta'_e(\mathbf{v}) = \omega^0_e \cdot \mathbf{e}_e, \quad -v'_e(\mathbf{v}) - s^0_e a_e \kappa^{-1}_v v''_e(\mathbf{v}) = \omega^0_e \cdot \mathbf{j}_e \tag{51}
\]

Moreover, net-force and moments at $\mathbf{v}$ implies that

\[
-\sum_{e \sim \mathbf{v}} s^0_e a_e v''_e(\mathbf{v}) = \lambda \mathbf{m}_e v^0_e, \quad \sum_{e \sim \mathbf{v}} s^0_e (d_e \eta'_e(\mathbf{v}) \mathbf{i}_e - a_e v''_e(\mathbf{v}) \mathbf{j}_e) = \lambda \mathbf{m}_e \omega^0_e \tag{52}
\]

\(^6\)See section 5.3.3 in [5] for an alternative derivation of characteristic equation corresponding rigid-joint model.
In this Section we will assume that boundary vertices \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) are rigid, i.e. the vectors \( v_s(0) = v_{s0}^o \) and \( \tilde{\omega}_s(0) = \tilde{\omega}_{s0}^o \) for \( s = 1, 2, 3 \). Semi-rigid vertex model is assumes at the central vertex \( \mathbf{v}_c \), on which conditions (50) and (51) for unit length edges, \( \ell_e = 1 \), reduce to

\[
v_1(1) + a\kappa_{g_v}^{-1}v''_1(1) = v_2(1) + a\kappa_{g_v}^{-1}v''_2(1) = v_3(1) + a\kappa_{g_v}^{-1}v''_3(1) = v_{v_0}^o , \tag{53}
\]

and

\[
\eta_1(1) + d\kappa_{\omega_v}^{-1}\eta'_1(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{\iota}_1 , \quad -v'_1(1) - a\kappa_{\omega_v}^{-1}v''_1(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{j}_1 \tag{54a}
\]

\[
\eta_2(1) + d\kappa_{\omega_v}^{-1}\eta'_2(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{\iota}_2 , \quad -v'_2(1) - a\kappa_{\omega_v}^{-1}v''_2(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{j}_2 \tag{54b}
\]

\[
\eta_3(1) + d\kappa_{\omega_v}^{-1}\eta'_3(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{\iota}_3 , \quad -v'_3(1) - a\kappa_{\omega_v}^{-1}v''_3(1) = \tilde{\omega}^{\circ}_{v_0} \cdot \tilde{j}_3 \tag{54c}
\]

Expansion of conditions (52) edgewise, then at the central vertex

\[
-a\nu''_1(1) - a\nu''_2(1) - a\nu''_3(1) = \lambda m_{v_0} v_{v_0}^o , \tag{55a}
\]

\[
d\eta'_1(1)\tilde{\iota}_1 + d\eta'_2(1)\tilde{\iota}_2 + d\eta'_3(1)\tilde{\iota}_3 - a\nu'_1(1)\tilde{j}_1 - a\nu'_2(1)\tilde{j}_2 - a\nu'_3(1)\tilde{j}_3 = \lambda m_{v_0} \tilde{\omega}^{\circ}_{v_0} \tag{55b}
\]

Next we will construct geometric-free local spectral basis (eigenvalue dependent) for each field \( \eta(x) \) and \( v(x) \) separately.

Figure 5: Geometry of 3-star planar graph in its equilibrium state. Variation of angles \( \delta_1 \) and \( \delta_2 \) span all possible geometries corresponding to this class of graphs.

### 4.2 Local Representation

#### 4.2.1 Spectral Basis For Out of Plane Displacement

Let denote by \( \Sigma_v^D \) to be the spectrum of an operator \( A_v v(x) = av'''(x) \) on an interval \([0, 1]\) with fixed and Robin boundary conditions respectively at \( x = 0 \) and \( x = 1 \), i.e. \( v(x) \) satisfying

\[
v(0) = 0, \quad v'(0) = 0, \quad v(1) + a\kappa_{g_v}^{-1}v''(1) = 0, \quad v'(1) + a\kappa_{g_v}^{-1}v''(1) = 0. \tag{56}
\]

Imposing boundary conditions (56), then set \( \Sigma_v^D \) is equivalent to

\[
\Sigma_v^D := \left\{ \lambda \in \mathbb{R} \mid D_v(\lambda; \kappa_{g_v}^{-1}, \kappa_{\omega_v}^{-1}, a) = 0 \right\} \tag{57}
\]

where for \( \mu_a := (\lambda/a)^{1/4} \), function \( D_v(\lambda; \kappa_{g_v}^{-1}, \kappa_{\omega_v}^{-1}, a) \) is of a form

\[
D_v = (S^-_\mu - a\kappa_{g_v}^{-1}\mu^3C^+_\mu)(S^+_\mu + a\kappa_{\omega_v}^{-1}\mu C^+_\mu) - (C^-_\mu - a\kappa_{g_v}^{-1}\mu^3S^-_\mu)(C^+_\mu + a\kappa_{\omega_v}^{-1}\mu S^+_\mu) \tag{58}
\]
For the class of fixed boundary conditions at both boundary vertices, \( D_v = 2(\cosh(\mu_a) \cos(\mu_a) + 1) \) which is well known characteristic equation\(^7\) for the so-called clamped cantilevered beam with unit length, see (43) for the case of fixed/free boundaries counterpart. If \( \lambda \notin \Sigma_v^D \cup \{0\} \), there exist four linearly independent solutions \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) on \([0, 1]\) of

\[
\mathcal{A}_v \phi_k(x) = a \phi_k'''(x) = \lambda \phi_k(x)
\]  

(59)
such that at the boundary points satisfying

\[
\begin{align*}
\phi_1(0) &= 1, & \phi_1'(0) &= 0, & \phi_1(1) - a \kappa_{a_1}^{-1} \phi_1''(1) &= 0, & \phi_1'(1) + a \kappa_{a_1}^{-1} \phi_2''(1) &= 0, \\
\phi_2(0) &= 0, & \phi_2'(0) &= 1, & \phi_2(1) - a \kappa_{a_1}^{-1} \phi_2''(1) &= 0, & \phi_2'(1) + a \kappa_{a_1}^{-1} \phi_3''(1) &= 0, \\
\phi_3(0) &= 0, & \phi_3'(0) &= 0, & \phi_3(1) - a \kappa_{a_1}^{-1} \phi_3''(1) &= 1, & \phi_3'(1) + a \kappa_{a_1}^{-1} \phi_4''(1) &= 0, \\
\phi_4(0) &= 0, & \phi_4'(0) &= 0, & \phi_4(1) - a \kappa_{a_1}^{-1} \phi_4''(1) &= 0, & \phi_4'(1) + a \kappa_{a_1}^{-1} \phi_4''(1) &= 1.
\end{align*}
\]

(60)

Thereby, for \( \lambda \notin \Sigma_v^D \cup \{0\} \) and for each \( k = 1, \ldots, 4 \), general solution \( \phi_k(x) \) is of a form

\[
\phi_k(x) = A_k \sinh(\mu_a x) + B_k \sin(\mu_a x) + C_k \cosh(\mu_a x) + D_k \cos(\mu_a x)
\]

(61)

Imposing vertex conditions (60) determines the local basis \( \phi_k(x) \), e.g.

\[
\begin{align*}
\phi_3(x) &= \frac{1}{D_v(\lambda; \kappa_{a_1}^{-1}, \kappa_{\omega_a}^{-1}, a)} \left( (S_{\mu_a}^+ + a \kappa_{\omega_a}^{-1} \mu_a C_{\mu_a}^+) S_{\mu_a}^- - (C_{\mu_a}^- + a \kappa_{\omega_a}^{-1} \mu_a S_{\mu_a}^+) C_{\mu_a}^- \right), \\
\phi_4(x) &= \frac{-\mu_a^{-1}}{D_v(\lambda; \kappa_{a_1}^{-1}, \kappa_{\omega_a}^{-1}, a)} \left( (C_{\mu_a}^- - a \kappa_{\omega_a}^{-1} \mu_a S_{\mu_a}^-) S_{\mu_a}^- - (S_{\mu_a}^- - a \kappa_{\omega_a}^{-1} \mu_a C_{\mu_a}^+) C_{\mu_a}^- \right).
\end{align*}
\]

(62)

where the notations above are introduced in (46). This set of basis will be later applied on writing the general solution for fields \( v_v(x) \). We stress that no-information regarding the geometry exists in the derived basis set as they are purely local.

### 4.2.2 Spectral Basis For Angular Displacement

Similar concept can be applied to derive local basis for \( \eta(x) \) field. This will be similar to the derivation in [28] with the difference of Robin type boundary condition on one end. In fact, let denote by \( \Sigma_\eta^D \) to be the solution to an eigenvalue problem \( \mathcal{A}_\eta \eta(x) = -d \eta''(x) = \lambda \eta(x) \) on interval \([0, 1]\) with boundary conditions

\[
\eta(0) = 0, \quad \eta(1) + d \kappa_{\omega_\eta}^{-1} \eta'(1) = 0
\]

(63)

Imposing the fixed/Robin boundary conditions (63), this set \( \Sigma_\eta^D \) is equivalent to

\[
\Sigma_\eta^D := \{ \lambda \in \mathbb{R} : D_\eta(\lambda; \kappa_{\omega_\eta}^{-1}, d) = 0 \}
\]

(64)

where for \( \beta_d := (\lambda/d)^{1/2} \), function \( D_\eta(\lambda; \kappa_{\omega_\eta}^{-1}, d) \) is of a form

\[
D_\eta := \sin(\beta_d) + d \kappa_{\omega_\eta}^{-1} \beta_d \cos(\beta_d)
\]

(65)

\(^7\)For example see wikipedia page.
If \( \lambda \notin \Sigma^D_\eta \cup \{0\} \), there exist two linearly independent solutions \( \psi_1(x) \) and \( \psi_2(x) \), depending on \( \lambda \), on \([0, 1]\) of eigenvalue problem

\[
A_\eta \psi(x) = -d\psi''(x) = \lambda \psi(x)
\]

(66)
such that at boundary points of the interval satisfy

\[
\begin{align*}
\psi_1(0) &= 1, & \psi_1(1) + d\kappa^{-1}_\omega \psi_1'(1) &= 0, \\
\psi_2(0) &= 0, & \psi_2(1) + d\kappa^{-1}_\omega \psi_2'(1) &= 1.
\end{align*}
\]

(67)

For the case that \( \lambda \notin \Sigma^D_\eta \cup \{0\} \), general solution of eigenvalue problem (66) can be written as

\[
\psi_k(x) = A_k \sin(\beta dx) + B_k \cos(\beta dx)
\]

(68)

Imposing vertex conditions (67), then for such \( \lambda \)'s, basis functions can be explicitly determined, e.g.

\[
\psi_2(x) = \frac{1}{D_\eta(\lambda; \kappa^{-1}_\omega, d)} \sin(\beta dx)
\]

(69)

Figure 6 plots functions \( \psi_2(x) \), \( \phi_3(x) \), and \( \phi_4(x) \) corresponding to the first two elements of sets \( \Sigma^D_\eta \) and \( \Sigma^D_v \), respectively. One may observes that if \( \lambda \in \Sigma^D_v \cup \{0\} \), then solution \( v_e(x) \) for all \( e \) will be decoupled and can be determined explicitly, e.g. see (62). Similar concept holds for \( \eta_e(x) \).

4.3 Geometry via Global Representation

In the following we will construct solution \( \Psi(x) \) to the coupled eigenvalue problem on the 3-star graph using the local basis developed above. Key idea on bringing geometry of the graph into play is by writing the solution for fields as a linear combination of local basis in such a way that they automatically satisfy vertex conditions (50) and (51). We will assume that the \( (n\text{-dependent}) \) basis functions \( \phi_k(x) \) and \( \psi_k(x) \) are lifted to each of the edges in graph \( \Gamma \), using the described before identifications of these edges with the segment \([0, 1]\). Abusing notations, we will use the same names \( \phi_k \) and \( \psi_k \) for the lifted functions. Denoting by

\[
\Sigma^D := \Sigma^D_v \cup \Sigma^D_\eta \cup \{0\},
\]

(70)
then for $\lambda \not\in \Sigma^D$ we can use (59) and (66) to represent any solution $\Psi(x)$ of eigenvalue problem (32) on each edge $e$. In detail, solution $v_s(x)$ for edge $e_s$ and $s = 1, 2, 3$ will be represented as

$$
\begin{align*}
 v_1(x) &= v_{1o}\phi_1(x) - (\vec{\omega}_{1o} \cdot \vec{j}_1)\phi_2(x) + v_{1c}\phi_3(x) - (\vec{\omega}_{1c} \cdot \vec{j}_1)\phi_4(x), \\
v_2(x) &= v_{2o}\phi_1(x) - (\vec{\omega}_{2o} \cdot \vec{j}_2)\phi_2(x) + v_{2c}\phi_3(x) - (\vec{\omega}_{2c} \cdot \vec{j}_2)\phi_4(x), \\
v_3(x) &= v_{3o}\phi_1(x) - (\vec{\omega}_{3o} \cdot \vec{j}_3)\phi_2(x) + v_{3c}\phi_3(x) - (\vec{\omega}_{3c} \cdot \vec{j}_3)\phi_4(x).
\end{align*}
$$

while solutions $\eta_s(x)$ are of the form

$$
\begin{align*}
 \eta_1(x) &= (\vec{\omega}_{1o} \cdot \vec{i}_1)\psi_1(x) + (\vec{\omega}_{1c} \cdot \vec{i}_1)\psi_2(x), \\
 \eta_2(x) &= (\vec{\omega}_{2o} \cdot \vec{i}_2)\psi_1(x) + (\vec{\omega}_{2c} \cdot \vec{i}_2)\psi_2(x), \\
 \eta_3(x) &= (\vec{\omega}_{3o} \cdot \vec{i}_3)\psi_1(x) + (\vec{\omega}_{3c} \cdot \vec{i}_3)\psi_2(x).
\end{align*}
$$

Interested reader may try to check vertex conditions (50) and (51) by replacing the representations (71) and (72) along with applying boundary properties of local spectral basis stated in (60) and (67). We stress that that for a fixed graph’s geometry, the unknown coefficients consist of scalar quantity $v_{1o}$ and vector $\vec{\omega}_{1c}$ of size two. The latter has representation in the global coordinate as

$$
\vec{\omega}_{1c} = (\vec{\omega}_{1c} \cdot \vec{E}_1)\vec{E}_1 + (\vec{\omega}_{1c} \cdot \vec{E}_2)\vec{E}_2.
$$

Moreover, for $\ell = 1, 2$, let introduce two vectors $I_{E\ell}$ and $J_{E\ell}$ which encode geometric information of the graph through

$$
I_{E\ell} := [(\vec{E}_\ell \cdot \vec{i}_1), (\vec{E}_\ell \cdot \vec{i}_2), (\vec{E}_\ell \cdot \vec{i}_3)]^T, \quad J_{E\ell} := [\vec{E}_\ell \cdot \vec{j}_1, (\vec{E}_\ell \cdot \vec{j}_2), (\vec{E}_\ell \cdot \vec{j}_3)]^T
$$

Application of presentations (71) and (72) in the net force and moment relations (52), reduces the problem to the following set of linearly independent conditions at the central vertex $v_c$ of a form

$$
\begin{align*}
 -3a\phi''''(1)v_{1o} + a(1^TJ_{E1})\phi''''(1)(\vec{\omega}_{1c} \cdot \vec{E}_1) + a(1^TJ_{E2})\phi''''(1)(\vec{\omega}_{1c} \cdot \vec{E}_2) &= \lambda m_{1c}v_{1c}, \\
 -a(1^TJ_{E1})\phi''(1)v_{1o} + a(J_{E1}^TJ_{E1})\phi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_1) + a(J_{E2}^TJ_{E2})\phi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_2) + d(I_{E1}^TI_{E1})\psi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_1) + d(I_{E2}^TI_{E2})\psi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_2) &= \lambda m_{1c}(\vec{\omega}_{1c} \cdot \vec{E}_1),
\end{align*}
$$

and

$$
\begin{align*}
 -a(1^TJ_{E2})\phi''''(1)v_{1o} + a(J_{E2}^TJ_{E1})\phi''''(1)(\vec{\omega}_{1c} \cdot \vec{E}_1) + a(J_{E2}^TJ_{E2})\phi''''(1)(\vec{\omega}_{1c} \cdot \vec{E}_2) + d(I_{E2}^TI_{E1})\psi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_1) + d(I_{E2}^TI_{E2})\psi''(1)(\vec{\omega}_{1c} \cdot \vec{E}_2) &= \lambda m_{1c}(\vec{\omega}_{1c} \cdot \vec{E}_2).
\end{align*}
$$

These then reduce the eigenvalue problem to finding the $(n$-dependent$)$ coefficient vector $[v_{1o}$, $(\vec{\omega}_{1c} \cdot \vec{E}_1), (\vec{\omega}_{1c} \cdot \vec{E}_2)]$ in the kernel of the $3 \times 3$ matrix $M(\lambda)$ defined as

$$
M(\lambda; \kappa_{g_{v}}, \kappa_{w_{v}}, \kappa_{w_{c}}, m_{v}) := aM_{v}(\lambda; \kappa_{g_{v}}, \kappa_{w_{v}}, \kappa_{w_{c}}) + dM_{\eta}(\lambda; \kappa_{w_{c}}) - \lambda m_{v}I
$$

with constituent matrices

$$
\begin{align*}
M_{v} := \\
\begin{pmatrix}
-\phi''''(1)1^T & \phi''''(1)1^T & \phi''''(1)1^T \\
-\phi''''(1)1^T & \phi''''(1)1^T & \phi''''(1)1^T \\
-\phi''''(1)1^T & \phi''''(1)1^T & \phi''''(1)1^T
\end{pmatrix}, \quad M_{\eta} := \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & \psi''(1)I_{E1}I_{E1} & \psi''(1)I_{E1}I_{E2} \\
0 & \psi''(1)I_{E2}I_{E1} & \psi''(1)I_{E2}I_{E2}
\end{pmatrix}
\end{align*}
$$

where $1$ is a vector of size 3 with unit entries. The graph’s geometric information in (75) can be further factorized from the local basis set. We will summarize the derivations above in the following result.
Proposition 4.1. Eigenvalues $\lambda \notin \Sigma^D$ corresponding to Hamiltonian (32) on 3-star planar graph with (rigid) fixed boundary conditions and semi-rigid central vertex is the solution of characteristic equation $\det(M(\lambda)) = 0$ where

$$
M(\lambda) = -a\left( \phi''(1)G_0 + \phi''(1)G_1 - \phi''(1)G_1^T - \phi''(1)G_J \right) + dv_2(1)G_I - \lambda m_v I \tag{76}
$$

with $G_0$ is a $3 \times 3$ matrix will all its entries zero except $(G_0)_{11} = 1T1$. Moreover, $M$ is constructed out of local spectral basis defined in (59), (66), and purely geometric (global) basis

$$
G_1 := \begin{pmatrix} 1 & T & J_{E_1} & 0 & 0 \\ T & J_{E_2} & 0 & 0 \end{pmatrix}, \quad G_J := \begin{pmatrix} 0 & J_{E_1}^T J_{E_1} & J_{E_1}^T J_{E_2} \\ 0 & J_{E_2}^T J_{E_1} & J_{E_2}^T J_{E_2} \end{pmatrix}, \quad G_I := \begin{pmatrix} 0 & 0 & 0 \\ 0 & T & I_{E_1} & I_{E_2} \end{pmatrix}
$$

with their entries defined in (74).

Example 1. Next we will show application of Proposition 4.1 on solving an eigenvalue problem for a selected geometry. Consider 3-star graph in Figure 5 with the choice of (clockwise) angles $\delta_1 = \pi/2$ and $\delta_2 = \pi$. In this case the edge’s local coordinate basis satisfy

$$
\vec{i}_1 = -\vec{j}_2 = \vec{j}_3 = \vec{E}_1 = [1 \ 0]^T, \quad \vec{j}_1 = \vec{i}_2 = -\vec{i}_3 = \vec{E}_2 = [0 \ 1]^T
$$

Straightforward calculations shows that $T J_{E_1} = 0$, $T J_{E_2} = 1$, and moreover

$$
I_{E_1}^T I_{E_1} = J_{E_2}^T J_{E_2} = 1, \quad I_{E_1}^T I_{E_2} = J_{E_1}^T J_{E_2} = 0, \quad I_{E_2}^T I_{E_2} = J_{E_1}^T J_{E_1} = 2
$$

Applying this set of geometric information in (76), the eigenvalue problem is equivalent on finding ($n$-dependent) coefficient vector $[v_c^o, (\vec{\omega}_c^o \cdot \vec{E}_1), (\vec{\omega}_c^o \cdot \vec{E}_2)]^T$ in the kernel of matrix $M(\lambda)$ of the form

$$
\begin{pmatrix}
3a\phi''(1) + \lambda m_v & 0 & -a\phi''(1) \\
0 & 2a\phi''(1) + dv_2(1) - \lambda m_v & 0 \\
-a\phi''(1) & 0 & a\phi''(1) + 2dv_2(1) - \lambda m_v
\end{pmatrix}
$$

The fact that $M(\lambda)$ is singular at an eigenvalue $\lambda \notin \Sigma^D$ implies that the spectrum of operator $A$ can be further factorized as $\sigma(A) = \Sigma^D \cup \Sigma_1 \cup \Sigma_2$ where $\Sigma^D$ is defined in (70) and

$$
\Sigma_1 := \left\{ \lambda \in \mathbb{R}_+ : 2a\phi''(1) + dv_2(1) - \lambda m_v = 0 \right\} \tag{77a}
$$

$$
\Sigma_2 := \left\{ \lambda \in \mathbb{R}_+ : (3a\phi''(1) + \lambda m_v)(a\phi''(1) + 2dv_2(1) - \lambda m_v) - a^2\phi''(1)\phi''(1) = 0 \right\} \tag{77b}
$$

Figure (7) plots the first three eigenfunctions $v^{(n)}(x)$ for the rigid-vertex model, i.e. $\kappa_{\omega}^{-1} \to 0$, and material parameters $d = 1, d = 10^5$. In these plots, the colorbar reflects value of torsion $\eta^{(n)}(x)$ along each edge. As it is expected, although these plots coincide with the ones reported in [5], but the characteristics equation is different in multiple ways, interested reader may compare Example 5.2 in [5] and Proposition 4.1 above.

Figure 8 plots the eigenfunctions for materials parameter $a = 1, d = 10^5$ while semi-rigid property of central vertex comes to play by setting $\kappa_{\omega}^{-1} = 1$. While (one-by-one) pictorial comparison of middle pictures in the above Figures may not reveal noticeable different, but in a neighborhood of central vertex $v_c$ the behavior is changed considerably, see magnified rectangular windows in plots above on the deformation’s mode in a neighborhood of the central vertex.
It has been discussed in [5] that for the rigid-vertex case, the above results will not match with the formulation [24] in which only scalar field \( v(x) \) exists in the Hamiltonian. Taking edge’s angular displacement stiffness \( d \) to 0 or to \( \infty \) will activate only eigenfunctions \( \eta^{(n)}(x) \) or suppresses rotation of (with normal vector \( \vec{E}_3 \)) tangent plane at the central vertex respectively. The former is due to decreasing graph’s resistance to the angular displacement while the latter is causes by enforcing small torsion of edges near the vertex due to high resistance of graph to this mode of deformation. Although limit of \( d \to \infty \) is necessary (and not sufficient) on coinciding the two models in [24] and [5], but the key observation here is to (only) relax rigidity of vertex to the angular displacement (torsion) by letting \( \kappa_{\omega \eta} \to 0 \). This makes application of proposed generalized vertex conditions in this manuscript an essential tool for answering problem (ii) stated in the introduction Section.

### 4.4 Joint Model on Fields Decoupling

Rest of the manuscript will be dedicated to a formal proof on decomposing vector-valued Hamiltonian to a set of scalar ones discussed above. First, observe that for a diagonal form of matrix \( \mathcal{K}_{\omega \epsilon}^{(v)} \) in
(33b), and by an assumption that \( m_o = 0 \), then operator \( \mathcal{A}^{(\text{out})} \) reduces to coupled system

\[
\begin{pmatrix} v_e \\ \eta_e \end{pmatrix} \mapsto \begin{pmatrix} a_e v''_e \\ -d_e \eta''_e \end{pmatrix}
\]

(78)

acting on functions in the space \( \prod_{e \in E} \mathcal{H}^4(e) \times \prod_{e \in E} \mathcal{H}^2(e) \) satisfying at a vertex \( v \) with degree \( n_v \), the primary conditions

\[
v_1 - s^p_1 a_1 \kappa^{-1}_{\infty} v''_1 = \cdots = v_{n_v} - s^p_{n_v} a_{n_v} \kappa^{-1}_{\infty} v''_{n_v}
\]

\[
(\eta_1 + s^p_1 d_1 \kappa^{-1}_{\infty} \eta''_1)j_1 - (v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v'_{1})j_1 = \cdots = (\eta_3 + s^p_3 d_3 \kappa^{-1}_{\infty} \eta''_3)j_3 - (v'_3 + s^p_{n_v} a_{n_v} \kappa^{-1}_{\infty} v''_{n_v})j_3
\]

along with conjugate ones

\[
s^p_1 a_1 v''_1 + \cdots + s^p_{n_v} a_{n_v} v''_{n_v} = 0
\]

\[
s^p_1 (d_1 \eta''_1j_1 - a_1 v''_1j_1) + \cdots + s^p_{n_v} (d_{n_v} \eta''_{n_v}j_{n_v} - a_{n_v} v''_{n_v}j_{n_v}) = 0
\]

Following Lemma reformulates the second constraint in primary conditions above which will be essential for rest of discussion.

**Lemma 4.2.** At each \( v \), the primary vertex condition

\[
(\eta_e + s^p_e d_e \kappa^{-1}_{\infty} \eta''_e)j_e - (v'_e + s^p_e a_e \kappa^{-1}_{\infty} v''_e)j_e = \tilde{\omega}^\circ_e
\]

holds for all \( e \sim v \) if and only if

\[
(\vec{j} \cdot \vec{i}_e) (v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1) + (\vec{j} \cdot \vec{i}_e) (v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2) + (\vec{j} \cdot \vec{i}_e) (v'_3 + s^p_3 a_3 \kappa^{-1}_{\infty} v''_3) = 0, \quad (82a)
\]

\[
(\vec{j} \cdot \vec{i}_e) (v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1) - (\vec{j} \cdot \vec{i}_e) (v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2) + (\vec{j} \cdot \vec{i}_e) (\eta_e + s^p_e d_e \kappa^{-1}_{\infty} \eta''_e) = 0. \quad (82b)
\]

**Proof of Lemma 4.2.** Establishing the above result is based on the proof of Lemma 4.3 in [5]. The main difference here is to replace fields with the extended ones introduced in this manuscript, e.g. \( v'_e \) in [5] will be replaced by \( v'_e + s^p_e a_{e} \kappa^{-1}_{\infty} v''_e \) and so on.

**Remark 4.3.** Observe that for the case \( n_v = 2 \), condition (82a) is trivial resulting \( v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1 = v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1 \) and \( v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2 = v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2 \). Thereby without loss of generality we assume that \( n_v \geq 3 \). Recalling definitions of vectors \( \tilde{\omega}^\circ_e(v) \) and \( \tilde{m}^\circ_e(v) \) in (2) and (6) respectively, then condition (82a) can be expressed as

\[
\left( (\vec{i}_1(v) + \kappa^{-1}_{\infty} \vec{m}_1(v) \times \vec{i}_1) \times (\vec{i}_2(v) + \kappa^{-1}_{\infty} \vec{m}_2(v) \times \vec{i}_2) \right) \cdot (\vec{i}_e(v) + \kappa^{-1}_{\infty} \vec{m}_e(v) \times \vec{i}_e) = 0
\]

for all \( e \sim v \). In fact, setting \( w'_e = 0 \) in the representation of \( \tilde{\omega}^\circ_e(v) \) and \( \tilde{m}^\circ_e(v) \), then (83) has a form

\[
\left( (\vec{i}_1(v) + v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1 \vec{k}) \times (\vec{i}_2(v) + v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2 \vec{k}) \right) \cdot (\vec{i}_e(v) + v'_e + s^p_e a_e \kappa^{-1}_{\infty} v''_e \vec{k}) = 0
\]

(83)

Now, applying properties \( (\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_e = 0 \), \( (\vec{i}_1 \times \vec{k}) \cdot \vec{k} = 0 \), and \( (\vec{k} \times \vec{i}_2) \cdot \vec{k} = 0 \) in (84) implies that

\[
\left( v'_1 + s^p_1 a_1 \kappa^{-1}_{\infty} v''_1 \right) \left( \vec{k} \times \vec{i}_2 \right) \cdot \vec{i}_e + \left( v'_2 + s^p_2 a_2 \kappa^{-1}_{\infty} v''_2 \right) \left( \vec{i}_1 \times \vec{k} \right) \cdot \vec{i}_e + \left( v'_3 + s^p_3 a_3 \kappa^{-1}_{\infty} v''_3 \right) \left( \vec{i}_1 \times \vec{i}_2 \right) \cdot \vec{k} = 0
\]

which is condition (82a) as it has been claimed.
Remark 4.4. Letting $\vec{m}_e(v) := \vec{i}_e + \kappa_{\omega_e}^{-1} \vec{m}_e \times \vec{i}_e$, observe that (82a) has another representation of a form
\[
(\vec{t}_1(v) \times \vec{t}_2(v)) \cdot \vec{t}_e(v) + (\vec{m}_1(v) \times \vec{m}_2(v)) \cdot \vec{m}_e(v) = 0
\] (85)
for all $e \sim v$. For the special class of rigid-joints, i.e. the case $\kappa_{\omega_e}^{-1} \to 0$, (85) reduces to the condition that all vectors $\{\vec{t}_e(v)\}_{e \sim v}$ lie in the same plane, i.e. $\langle \vec{t}_1(v) \times \vec{t}_2(v) \rangle \cdot \vec{t}_e(v) = 0$ for all $e \sim v$. This is an important observation that irrespective to the value of vertex's torsional stiffness $\kappa_{\omega_e}$, local planar structure of an unreformed graph will be conserved under frame deformation.

Now, let introduce
\[
D_e(v, \eta) := \frac{(\vec{j}_e \cdot \vec{j}_1)}{(\vec{j}_1 \cdot \vec{i}_2)} (v''_1 + s^p_2 a_2 \kappa_{\omega_e}^{-1} v''_2) - \frac{(\vec{j}_2 \cdot \vec{j}_e)}{(\vec{j}_1 \cdot \vec{i}_2)} (v''_1 + s^p_1 a_1 \kappa_{\omega_e}^{-1} v''_1) - \eta_e
\] (86)
Observe that by Remark 4.3, the expression (86) is well-defined. This then allows to write condition (82b) in terms of rotational moment $\eta'_e$ at vertex $v$ of a form
\[
s^p_e d_e \eta'_e = \kappa_{\omega_e} D_e(v, \eta)
\] (87)
Applying (87) in the second constraint of conjugate condition implies that
\[
s^p_e a_1 v''_1 j_1 + \cdots + s^p_n a_n v''_n j_n = -\kappa_{\omega_e} (D_1(v, \eta) v''_1 + \cdots + D_n(v, \eta) v''_n) = 0
\] (88)
Since the differential expression for the operator $A^{(\text{out})}$ is already in the “block-diagonal” form, applying result of Lemma 4.2 and (88), then under an appropriate limit, the operator $A^{(\text{out})}$ will be decomposed to two scalar-valued operators. We summarize this result in the following Proposition.

Proposition 4.5. Under a limit $\kappa_{\omega_e} \to 0$, the operator $A^{(\text{out})}$ in (78) is decomposed as
\[
A^{(\text{out})} = A^{(\text{out})}_v \oplus A^{(\text{out})}_\eta
\] (89)
with $A^{(\text{out})}_v : v_e \mapsto a_e v''_e$, and satisfying at each (internal) vertex $v \in V$ with a degree $n_v$, conditions
\[
v_1 - s^p_1 a_1 \kappa_{\omega_e}^{-1} v''_1 = \cdots = v_{n_v} - s^p_n a_n \kappa_{\omega_e}^{-1} v''_n,
\] (90a)
\[
(\vec{j}_2 \cdot \vec{i}_e)(v''_1 + s^p_2 a_2 \kappa_{\omega_e}^{-1} v''_2) + (\vec{j}_1 \cdot \vec{i}_e)(v''_1 + s^p_2 a_2 \kappa_{\omega_e}^{-1} v''_2) + (\vec{j}_1 \cdot \vec{i}_2) (v''_1 + s^p_1 a_1 \kappa_{\omega_e}^{-1} v''_1) = 0,
\] (90b)
\[
s^p_1 a_1 v''_1 + \cdots + s^p_n a_n v''_n = 0,
\] (90c)
\[
s^p_1 a_1 v''_1 j_1 + \cdots + s^p_n a_n v''_n j_n = 0.
\] (90d)
Moreover, $A^{(\text{out})}_\eta : \eta_e \mapsto -d_e \eta''_e$ satisfies at each (internal) vertex $v \in V$ condition
\[
d_e \eta''_e = 0.
\] (91)
Similar decomposition result of a form stated in the Proposition 4.5 can be established for $(w, u)$ fields corresponding $A^{(\text{in})}$ operator by letting $\kappa_{gu} \to 0$. Avoiding detail calculation this result is summarized in the following Proposition.
Proposition 4.6. Under a limit \( \kappa_{gu} \to 0 \), the operator \( A^{(in)} \) is decomposed as

\[
A^{(in)} = A^{(in)}_w \oplus A^{(in)}_u
\]

with \( A^{(in)}_w : w_e \mapsto b_e w_e'''' \), and satisfying at each (internal) vertex \( v \in \mathcal{V} \) with a degree \( n_v \), conditions

\[
\begin{align*}
(\vec{j}_2 \cdot \vec{i}_e)(w_1 - s^0_1 b_1 \kappa^{-1}_{\omega w} w_1'') + (\vec{j}_e \cdot \vec{i}_1)(w_2 - s^0_2 b_2 \kappa^{-1}_{\omega w} w_2'') + (\vec{j}_1 \cdot \vec{i}_2)(w_e - s^0_e b_e \kappa^{-1}_{\omega w} w_e'') &= 0, \\
& w_1' + s^0_1 b_1 \kappa^{-1}_{\omega w} w_1'' = \cdots = w_{n_v} + s^0_{n_v} b_{n_v} \kappa^{-1}_{\omega w} w_{n_v}'', \\
s^0_1 b_1 w_1'' j_1 + \cdots + s^0_{n_v} b_{n_v} w_{n_v}'' j_{n_v} &= 0, \\
s^0_1 a_1 v_1'' + \cdots + s^0_{n_v} a_{n_v} v_{n_v}'' &= 0.
\end{align*}
\]

Moreover, \( A^{(in)}_u : u_e \mapsto -c_e u_e'' \) satisfies at each (internal) vertex \( v \in \mathcal{V} \) condition

\[
c_e u_e' = 0.
\]

Combining of Propositions 4.5 and 4.6, thereby we proved the following Theorem, see question (ii) of problem statement in Section 1.

Theorem 4.7. Under limit \((\kappa_{\eta \eta}, \kappa_{gu}) \to (0,0)\), vector-valued beam Hamiltonian on planar frames is decomposed as direct sum

\[
A = \left( A^{(out)}_w \oplus A^{(out)}_{\eta} \right) \bigoplus \left( A^{(in)}_w \oplus A^{(in)}_u \right)
\]

with each scalar-valued Hamiltonian is defined in Propositions 4.5 and 4.6.

Remark 4.8. As a final remark, observe that for a 3-star planar graph and in the limiting case of \((\kappa_{gu}, \kappa_{\omega v}) \to (0,0)\), vertex conditions (90a) are of the form reported for scalar-valued beam operator [24]. Thereby, this special case is a sub-class of the proposed coupled Hamiltonian under certain limits of parameter space of semi-rigid joint derived in this manuscript.

5 Outlook

Interesting problem is to mathematically investigate the validity of frame model as a structure composed of one-dimensional segments. This could be of interest to dwell on an alternative approach to model junctions between beams which consists in an asymptotic analysis for three dimensional plate models when the thickness tends to zero. There is a significant mathematical literature on this question for second-order operators see, for example, [4, 7, 10, 11, 21, 22, 38, 40, 44], with a variety of operators arising in the limit. Results in this line for the case of fourth-order equations is expected to be of interest to engineering communities working on analysis of 3D structural, e.g. see [14, 32, 33], as well as more theoretically oriented research communities, e.g. see [19, 38, 39].

Recently, full description of spectra corresponding to the scalar valued fourth-order (Schrödinger) operator on a so called class of hexagonal lattices has been discussed in [16]. The result of Theorem 4.7 states that scalar-valued Hamiltonian applied in [16] is a special case of vector-valued one. Spectral analysis of coupled Hamiltonian on periodic beam lattices equipped with the (general) vertex model proposed in this manuscripts may be of interest and provides a more complete picture on spectral analysis of such continua, see [17, 18, 31, 35].
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Appendix

Derivation of characteristic equation (48). Following Remark 3.2, the graph $\Gamma$ is invariant under the following geometric transformations and their products: $R$ acting as the rotation counterclockwise by $\theta = 2\pi/3$ around the axis $-\vec{i}_0$, and $F$ acting as the reflection with respect to the plane spanned by $\vec{i}_1$ and $\vec{i}_0$. These transformations generate the group $G = D_3$, the dihedral group of degree 3, according to the presentation

$$G = \langle R, F : R^3 = I, F^2 = I, FRFR = I \rangle$$

(96)

Let us now fix the following local bases: take $\vec{j}_1$ to be orthogonal to the plane spanned by $\vec{i}_0$ and $\vec{i}_1$; let $\vec{j}_0 = \vec{j}_1$, see Figure 4; this determines $\vec{k}_0$ and $\vec{k}_1$. More specifically,

$$\vec{i}_0 = -\vec{E}_3, \quad \vec{j}_0 = \vec{E}_2, \quad \vec{k}_0 = \vec{E}_1$$

(97a)

$$\vec{i}_1 = \cos(\alpha)\vec{E}_1 + \sin(\alpha)\vec{E}_3, \quad \vec{j}_1 = \vec{E}_2, \quad \vec{k}_1 = -\sin(\alpha)\vec{E}_1 + \cos(\alpha)\vec{E}_3$$

(97b)

We further assume

$$\vec{i}_2 = R\vec{i}_1, \quad \vec{j}_2 = R\vec{j}_1, \quad \vec{k}_2 = R\vec{k}_1, \quad \text{and} \quad \vec{i}_3 = R\vec{i}_2, \quad \vec{j}_3 = R\vec{j}_2, \quad \vec{k}_3 = R\vec{k}_2.$$ (98)

With a slight abuse of notation by using the same letters for the matrices realizing, we obtain the following geometric representation of $G$,

$$R = \begin{pmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ (99)

Next we will discuss application of domain decomposition stated in Theorem 3.1. We start with the coordinate decomposition of vector $\vec{g}_{v_c}^\circ$ in the global coordinate system

$$\vec{g}_{v_c}^\circ = (\vec{g}_{v_c}^\circ \cdot \vec{E}_1)\vec{E}_1 + (\vec{g}_{v_c}^\circ \cdot \vec{E}_2)\vec{E}_2 + (\vec{g}_{v_c}^\circ \cdot \vec{E}_3)\vec{E}_3$$

(100)

and similarly for vector $\vec{\omega}_{v_c}^\circ$ as

$$\vec{\omega}_{v_c}^\circ = (\vec{\omega}_{v_c}^\circ \cdot \vec{E}_1)\vec{E}_1 + (\vec{\omega}_{v_c}^\circ \cdot \vec{E}_2)\vec{E}_2 + (\vec{\omega}_{v_c}^\circ \cdot \vec{E}_3)\vec{E}_3$$

(101)

Irreducible Representation $H_\omega$. Stating with the conditions

$$\begin{pmatrix} R\vec{g}_0 \\ R\vec{g}_3 \\ R\vec{g}_1 \\ R\vec{g}_2 \end{pmatrix} = \omega \begin{pmatrix} \vec{g}_0 \\ \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta_0 \\ \eta_3 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \omega \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}.$$ (102)
and by referring to the Theorem 3.1, for beam $e_0$ properties

$$u_0 = 0, \quad \eta_0 = 0, \quad b_0 w_0 = i a_0 v_0,$$  \hspace{1cm} (103)

holds, while on the other beams

$$v_2 = \omega v_1, \quad w_2 = \omega w_1, \quad u_2 = \omega u_1, \quad \eta_2 = \omega \eta_1,$$  \hspace{1cm} (104)

$$v_3 = \bar{\omega} v_1, \quad w_3 = \bar{\omega} w_1, \quad u_3 = \bar{\omega} u_1, \quad \eta_3 = \bar{\omega} \eta_1.$$  \hspace{1cm} (105)

Moreover, in the space $H_{c_i}$, the two vectors $\vec{g}^o_{c_e}$ and $\bar{\omega}^o_{c_e}$ satisfy $\vec{g}^o_{c_e} \cdot \vec{E}_3 = 0$ and $\bar{\omega}^o_{c_e} \cdot \vec{E}_3 = 0$. Next we will apply the vertex conditions (5) and (38) on the appropriate fields. By symmetry reduction in (104), we will denote by $u(x) := u_s(x)$ for $s = 1, 2, 3$, and similarly for the rest of the fields on (leg) edges. Setting $e = e_0$ in (5a) and applying the property $u_0 \equiv 0$, then

$$w_0(\ell_0)\tilde{j}_0 + v_0(\ell_0)\bar{k}_0 + \kappa_{g_0}^{-1}(-b_0 w''_0(\ell_0)\tilde{j}_0 - a_0 v'''_0(\ell_0)\bar{k}_0) = \bar{g}^o_{c_e}.$$  \hspace{1cm} (106)

Representation of local basis in the global coordinate in (97) along with expansion (100) implies two conditions

$$v_0(\ell_0) - a_0 \kappa_{g_0}^{-1} v'''_0(\ell_0) = \bar{g}^o_{c_e} \cdot \vec{E}_1,$$  \hspace{1cm} (107a)

$$w_0(\ell_0) - b_0 \kappa_{g_0}^{-1} w'''_0(\ell_0) = \bar{g}^o_{c_e} \cdot \vec{E}_2.$$  \hspace{1cm} (107b)

Setting $e = e_1$, then (5a) is equivalent to condition

$$u(\ell)\tilde{j}_1 + w(\ell)\bar{j}_1 + v(\ell)\bar{k}_1 + \kappa_{g_1}^{-1}(cu'(\ell)\tilde{i}_1 - bw'''(\ell)\bar{j}_1 - av'''(\ell)\bar{k}_1) = \bar{g}^o_{c_e}.$$  \hspace{1cm} (108)

Following similar steps by applying representation of local coordinates in the global one, then one realizes three independent conditions as

$$(u(\ell) + c \kappa_{g_1}^{-1} u'(\ell)) \cos(\alpha) - (v(\ell) - a \kappa_{g_1}^{-1} v''(\ell)) \sin(\alpha) = \bar{g}^o_{c_e} \cdot \vec{E}_1,$$  \hspace{1cm} (109a)

$$w(\ell) - b \kappa_{g_1}^{-1} w'''(\ell) = \bar{g}^o_{c_e} \cdot \vec{E}_2,$$  \hspace{1cm} (109b)

$$(u(\ell) + c \kappa_{g_1}^{-1} u'(\ell)) \sin(\alpha) + (v(\ell) - a \kappa_{g_1}^{-1} v''(\ell)) \cos(\alpha) = 0.$$  \hspace{1cm} (109c)

Same types of analysis can be applied to utilize the implication of vertex condition (5b). In fact by setting $e = e_0$, then

$$+ w'_0(\ell_0) + b_0 \kappa_{\omega_0}^{-1} w''_0(\ell_0) = \bar{\omega}^o_{c_e} \cdot \vec{E}_1,$$  \hspace{1cm} (110a)

$$- v'_0(\ell_0) - a_0 \kappa_{\omega_0}^{-1} v''_0(\ell_0) = \bar{\omega}^o_{c_e} \cdot \vec{E}_2.$$  \hspace{1cm} (110b)

Moreover, for edge $e_1$ the vertex condition (5b) turns to the following three independent conditions

$$(\eta(\ell) + d \kappa_{\omega_1}^{-1} \eta'(\ell)) \cos(\alpha) - (w'(\ell) + b \kappa_{\omega_1}^{-1} w''(\ell)) \sin(\alpha) = \bar{\omega}^o_{c_e} \cdot \vec{E}_1,$$  \hspace{1cm} (111a)

$$- v(\ell) - a \kappa_{\omega_1}^{-1} v''(\ell)) = \bar{\omega}^o_{c_e} \cdot \vec{E}_2,$$  \hspace{1cm} (111b)

$$(\eta(\ell) + d \kappa_{\omega_1}^{-1} \eta'(\ell)) \sin(\alpha) + (w'(\ell) + b \kappa_{\omega_1}^{-1} w''(\ell)) \cos(\alpha) = 0.$$  \hspace{1cm} (111c)

It remains to apply the dynamics of vertex $v_c$ in (38). Expansion of the net force in (38a) is equivalent to condition

$$cu'(\ell)\tilde{i}_1 + \omega \tilde{j}_2 + \bar{\omega} \tilde{j}_3 - bw'''(\ell)\tilde{j}_1 + \omega \bar{j}_2 + \bar{\omega} \bar{j}_3 - av'''(\ell)\bar{k}_1 + \omega \bar{k}_2 + \bar{\omega} \bar{k}_3$$

$$- b_0 w'''_0(\ell_0)\tilde{j}_0 - a_0 v'''_0\bar{k}_0 = \lambda m_{v_c} \bar{g}^o_{c_e}.$$  \hspace{1cm} (112)
But due to expansion (97), identities

$$(\cos(\alpha))^{-1}(\vec{r}_1 + \omega \vec{r}_2 + \bar{\omega} \vec{r}_3) = i(\vec{j}_1 + \omega \vec{j}_2 + \bar{\omega} \vec{j}_3) = -(\sin(\alpha))^{-1}(\vec{k}_1 + \omega \vec{k}_2 + \bar{\omega} \vec{k}_3) = \frac{3}{2}(\vec{E}_1 + i\vec{E}_2)$$

holds. Expansion of $\vec{g}_{v_c}^\circ$ in the global coordinate along with factorization of terms in direction of $\vec{E}_1$ turns to the vertex condition

$$\frac{3}{2} cu'(\ell) \cos(\alpha) + \frac{3}{2} b iv''''(\ell) + \frac{3}{2} a v''''(\ell) \sin(\alpha) - a_0 v''''(\ell_0) = \lambda m_{v_c}(\vec{g}_{v_c}^\circ \cdot \vec{E}_1) \quad (113)$$

We stress that condition corresponding direction $\vec{E}_2$ is linearly dependent to the one stated in (113), see Remark 5.1 for details. Application of the vertex condition (38b) is equivalent to condition

$$d\eta'(\ell)(\vec{r}_1 + \omega \vec{r}_2 + \bar{\omega} \vec{r}_3) - av''''(\ell)(\vec{j}_1 + \omega \vec{j}_2 + \bar{\omega} \vec{j}_3) + bw''''(\ell)(\vec{k}_1 + \omega \vec{k}_2 + \bar{\omega} \vec{k}_3)$$

$$- a_0 v''''(\ell_0) \vec{j}_0 + b_0 w''''(\ell_0) \vec{k}_0 = \lambda m_{v_c}(\vec{\omega}_{v_c} \cdot \vec{E}_2)$$

which by following similar steps in derivation of (113) is reduces to a single independent condition

$$\frac{3}{2} din'(\ell) \cos(\alpha) - \frac{3}{2} b iv''''(\ell) \sin(\alpha) - \frac{3}{2} a v''''(\ell) - a_0 v''''(\ell_0) = \lambda m_{v_c}(\vec{\omega}_{v_c} \cdot \vec{E}_2) \quad (114)$$

With the restrictions given by (103) and (104), the eigenvalue problem is fully determined by functions $v, w, u$ and $\eta$ on the base edges and the function $v_0$ defined on vertical edge. Following Remark will identify set of linearly independent conditions out of the ones have been derived above.

**Remark 5.1.** Applying the relations $\vec{g}_{v_c}^\circ \cdot \vec{E}_2 = +i(\vec{g}_{v_c}^\circ \cdot \vec{E}_1)$ and $\vec{\omega}_{v_c} \cdot \vec{E}_1 = -i(\vec{\omega}_{v_c} \cdot \vec{E}_2)$ for subspace $\mathcal{H}_\omega$ stated in the Theorem 3.1, then set of linearly independent conditions is equivalent to (107a), (109), (110b), (111), along with net force and moment conditions in (113) and (114) respectively.

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