Quantum of action in entangled relativity

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In this letter, I show that Planck’s quantum of action \( \hbar \) varies proportionally to Newton’s constant \( G \) in entangled relativity, which manifests an explicit connection between the quantum and the gravitational worlds. On the other hand, the quantum parameter that appears in the phase of the path integral formulation of the theory is a quantum of energy squared \( \epsilon^2 \)—instead of a quantum of action. I show that the value of this quantum of energy is set to be the Planck energy by the semiclassical limit of the theory for which the gravitational interaction can safely be assumed to be classical and to vary slowly with respect to the time-scale of quantum phenomena. It follows that there is no fundamental notion of elementary units of time and space in this theory, providing an interesting circumstantial evidence that this theory might be well behaved at the Planck energy scale.

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Entangled relativity is a general theory of relativity that requires the existence of matter in order (even) to be defined, therefore realizing Einstein’s original idea that a satisfying theory of relativity cannot allow for the existence of vacuum solutions [1–6]. Indeed, vacuum solutions imply that inertia—which is defined from the metric tensor in a relativistic theory—could be defined in the total absence of matter, which would de facto violate the principle of relativity of inertia [1–6] that Einstein named Mach’s principle in [2]. Despite its very unusual action—see Eq. (1) below—entangled relativity has been shown to possess general relativity as a limit in fairly generic (classical) situations [7–11], which indicates that, at least up to further scrutiny, the theory might be viable from an observational point of view.

The action of entangled relativity reads as follows [12]

\[
S = \frac{\xi}{2c} \int d^4x \frac{\mathcal{L}_m^2(f, g)}{R(g)},
\]

where the constant \( \xi \) has the dimension of the coupling constant between matter and geometry in general relativity (that is, \( \kappa_{GR} = 8\pi G/c^4 \), where \( G \) is Newton’s constant and \( c \) the speed that defines null spacetime cones), \( R \) is the usual Ricci scalar that is constructed upon the metric tensor \( g_{\mu\nu} \), \( d^4x := \sqrt{|g|} d^4x \) is the spacetime volume element, with \( |g| \) the metric \( g \) determinantal, and \( \mathcal{L}_m \) is the Lagrangian density of matter fields \( f \)—which could be the current standard model of particle physics Lagrangian density, but most likely a completion of it.\(^1\) \( \mathcal{L}_m \) depends on both the fields \( f \) themselves and the spacetime metric \( g \) according to the comma-goes-to-semicolons rule [13]. Let us note that the value of \( \xi \) does not have any impact on the classical field equations.

Let us now write the path integral formulation of the theory:

\[
Z = \int Dg \prod_i Df_i \exp \left[ -\frac{i}{2c^2} \int d^4x \frac{\mathcal{L}_m^2(f, g)}{R(g)} \right],
\]

where \( \epsilon \) is a constant to be determined, with the units of an energy. From Eq. (1), one could naively think that \( \epsilon^2 = c\hbar/\xi \), but we will see that it is not the case. In fact, \( \xi \) is only useful to define the theory from the historical notion of an action—to which the least action principle could be applied. But we know from modern physics that only the quantum phase in the path integral formulation of the theory is actually relevant, because the least action principle is just one of the consequences of the path integral. Hence, only the value of \( \epsilon \) is relevant for the theory as a whole, that is, for both its classical and quantum sides—such that \( \epsilon \), rather than \( \xi \), is a fundamental parameter in this theory. Therefore, for now-on, I will no-longer talk about \( \xi \).

One can check that entangled relativity cannot be defined if matter is not defined from the beginning.\(^2\) As a consequence, Eq. (2) forces spacetime to be permeated with matter fields—notably from the expected quantum vacuum fields—and therefore the theory satisfies Einstein’s original wish to have a theory of relativity that do not allow for pure vacuum solutions.\(^3\)

The quantum field theory defined in Eq. (2) remains to be studied in general, but its semiclassical limit that

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\(^1\) Entangled relativity was presented for the first time in [7], but the quantum nature of \( \xi \) (erroneously identified as being \( \kappa \) in [7]) was not realized until [12]. Otherwise, the theory was named for the first time in [8]. The name follows from the fact that matter and curvature cannot be treated separately at the level of the action in this framework—such that they are entangled, in the etymological sense. Hence, it has nothing to do with quantum entanglement, a priori.

\(^2\) That is, it is not defined if \( \mathcal{L}_m = \emptyset \) in Eq. (2).

\(^3\) In the sense of a total absence of matter fields. The quantum vacuum on the other hand is permeated with matter fields whenever \( \mathcal{L}_m \neq \emptyset \) in the path integral.
assumes a classical background is pretty enlightening already—notably in order to understand why it can recover the usual features of the Core theory of physics—that is, general relativity in addition to the standard model of particle physics [14]—despite its unusual nonlinear form. Indeed, at the background level, the following phases are equivalent [7, 15]

\[-\frac{1}{2\epsilon^2} \int d^4x \frac{\mathcal{L}^2_m(f, g)}{R(g)} = \frac{1}{\epsilon^2} \int d^4x \frac{1}{\kappa} \left( \frac{R(g)}{2\kappa} + \mathcal{L}_m(f, g) \right),\]

provided that \(\mathcal{L}_m \neq 0,^4\) and where \(\kappa\) is a scalar-field. Obviously, the value of \(\epsilon\) does not impact the classical limit of the theory. The classical equivalence between the original \(f(R, \mathcal{L}_m)\) theory on the left-hand-side of Eq. (3) and the Einstein-dilaton theory on the right-hand-side, stems from a very well known fact: non-linear algebraic functions of the Ricci scalar in the action are equivalent to having an additional scalar degree-of-freedom with gravitational strength [16, 17]. As a consequence, it indicates that the theory defined in Eq. (3) should be immune to the Ostrogradskian instability and to the Cauchy problem despite not being of second order—just as \(f(R)\) theories [17, 18]. Eq. (3) also shows that, at the classical level, Newton’s constant

\[G := \kappa \delta^4/(8\pi)\]

is actually not a constant but a field. Otherwise, whereas \(\kappa = -R/T\) in general relativity, one can check that \(\kappa = -R/\mathcal{L}_m^o\) from the right-hand-side of Eq. (3), where \(\mathcal{L}_m^o\) is the on-shell value of \(\mathcal{L}_m\). For instance, for a pure magnetic field \(\vec{B}\), \(\mathcal{L}_m^o \propto B^2\) on-shell [19], whereas for a barotropic perfect fluid for which the rest mass density is conserved, it has been found that \(\mathcal{L}_m^o = -\rho\) on-shell, where \(\rho\) is the total energy density of the fluid [20]. In [21], it is argued that for a soliton with fixed rest mass and structure, the average on-shell Lagrangian is \(\mathcal{L}_m^o = T\), where \(T\) is the trace of the stress-energy tensor. This discussion has then been extended to other situations in [22]. Otherwise, one can argue from [23] that for composite particles in the standard model of physics—such as atomic nuclei—one has \(\mathcal{L}_m^o = T\), where \(T\) is the total trace of the fields that constitute the composite particles—that is, the classical part of the trace in addition to all the relevant quantum trace anomalies:

\[\mathcal{L}_m^o = -\frac{\beta_\epsilon(e)}{2\epsilon} F_{\mu\nu}F^{\mu\nu} - \frac{\beta_3(g_3)}{2g_3} G_{\mu\nu}^a G^{\mu\nu}_a - \sum_{i=e, u, d} (1 + \gamma_m) m_i \bar{\psi}_i \psi_i,\]

where \(F_{\mu\nu}\) is the Faraday tensor, \(G_{\mu\nu}^a\) is the gluons tensor, \(e\) and \(g_3\) are respectively the photons and the gluons coupling constants, \(\beta_\epsilon(e) = \lambda \partial \ln e / \partial \lambda\) and \(\beta_3(g_3) = \lambda \partial \ln g_3 / \partial \lambda\) are their respective beta functions relative to the quantum scale invariance violation, where \(\lambda\) is the energy scale of the considered physical processes, \(m_i\) is the fermion mass, \(\psi_i\) their spinor, and \(\gamma_m = -\lambda \partial \ln m / \partial \lambda\) is the beta function relative to the dimensional anomaly of the fermion masses coupled to the gluons.\(^5\) This is consistent with the fact that the mass of a composite object equals the total trace of the fields that constitute the particle—see, for instance, Chapter III–4 in [25]. It is due to the constraint that the internal stresses all vanish, and it remains true even if some of the internal forces do not contribute a priori to the trace—such as classical electromagnetic forces, which contribute to the trace through the condition that internal stresses all vanish, while being itself traceless [26].

As usual in \(f(R)\) theories, the trace of the metric field equation yields to a differential equation on the gravitational scalar degree-of-freedom, which here reads

\[3\kappa^2 \Box \kappa^{-2} = \kappa \left( T - \mathcal{L}_m^o \right).\]

Hence, whenever \(\mathcal{L}_m^o = T\), the scalar degree-of-freedom is not sourced. The consequence of this intrinsic decoupling [27, 28] is that the theory behaves as general relativity—that is \(\kappa \approx \text{constant} \approx \text{much whenever } \mathcal{L}_m^o \approx T\), which actually occurs in many expected situations of the observable universe. It notably follows that, at the classical level, this theory predicts that the post-Newtonian parameters \(\gamma\) and \(\beta\) are both equal to one [27, 29]—as in general relativity—that the value of Newton’s constant \(G\) freezes at least at the beginning of the matter era [11, 28]; that neutron stars are not much different from the ones of general relativity [8, 9], nor seem to be black-holes [10], nor even gravitational-waves [30, 31]. The trace Eq. (6) of the metric field equation imposes a differential condition on the ratio between \(\mathcal{L}_m^o\) and \(R\), the same way the trace of Einstein’s equation imposes an algebraic condition on the ratio between \(T\) and \(R\). These ratio simply give the amplitude with which matter curves spacetime at the classical level, either in the theory defined in Eq. (1) or in general relativity. Indeed, the metric field equation from both sides of Eq. (3) reads [7, 15]

\[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} + \kappa^2 (\nabla_\nu \nabla_\mu - g_{\mu\nu} \Box) \kappa^{-2},\]

and the (non-)conservation of the stress-energy tensor reads

\[\nabla_\sigma \left( \kappa^{-1} T^{\mu\sigma} \right) = \mathcal{L}_m^o \nabla^\mu \kappa^{-1}.\]

\(^4\)If \(\mathcal{L}_m = 0\) in the right-hand-side of Eq. (3), then it does not correspond to entangled relativity, as defined in the left-hand-side, since the latter is not even defined for \(\mathcal{L}_m = 0\).

\(^5\)Note that at the energy scale of a hadron, the effect of the weak interactions, as well as the effect of the heavy quarks have been integrated out [24].
such that one recovers general relativity minimally coupled to matter fields in the $\kappa = \text{constant}$ limit—which, in many situations, simply follows from the intrinsic decoupling mentioned above. Near to this limit, Eq. (2) can be approximated as

$$Z \approx \int Dg \prod_i Df_i \exp \left[ \frac{i}{\kappa \epsilon^2} \int d^3 x \left( \frac{R(g)}{2\kappa} + \mathcal{L}_m(f, g) \right) \right],$$

as long as the spatio-temporal scales of the phenomena considered are small with respect to the spatio-temporal variations of the background value of $\kappa$. If $\mathcal{L}_m$ also is the standard model of particle physics in this limit, then Eq. (9) turns out to be nothing but the Core theory of physics, with

$$\hbar c := \kappa \epsilon^2,$$  \hspace{1cm} (10)

notably meaning that one has $\hbar \propto G$ effectively. Hence, from the constant $\kappa$ limit of the theory, one deduces that the only parameter of the theory in Eq. (2)$^6$ is the Planck energy $\epsilon$, defined as

$$\epsilon = \sqrt{\frac{\hbar c}{\kappa}}.$$  \hspace{1cm} (11)

More importantly, this shows that the Core theory of physics might simply just correspond to a limit of entangled relativity Eq. (2). Also, let us stress that neither Planck’s constant $\hbar$, nor Newton’s constant $G$, are actually constant—as both can vary and are proportional to each other ($\hbar \propto G$) at the semiclassical limit for which the background value of $\kappa$ does not vary much. This seems to manifest an explicit connection between the quantum and the gravitational worlds in entangled relativity.

Indeed, the only two universal constants that appear in the definition of the theory in Eq. (2) are the causal structure constant $c$ and the Planck energy $\epsilon$.$^7$ It directly follows that the Planck time and length are not fundamental constants in entangled relativity, such that there is no fundamental notion of elementary units of time and space in this theory. This provides an interesting circumstantial evidence that this theory might be well behaved at the Planck energy scale, notably because there is a priori no reason to expect anything special happening to the continuum structure of spacetime at any scale— unlike what one expects in the Core theory of physics at the Planck length and time [32–38].

Another potential interesting aspect is that Eq. (10) shows that the reason the universe seems to be semiclassical at the macroscopic scales on the one hand, and the reason why gravity seems to be so weak at the macroscopic scales on the other hand, share the same origin in entangled relativity—because the classical limit $\hbar \to 0$ corresponds to the weak gravity limit $\kappa \to 0$. Thus, to the question “why does the universe behave semiclassically rather than fully quantum-mechanically?”$^8$, one could answer “because gravity is weak,” and vice versa.

But if entangled relativity is indeed a completion of general relativity, Eq. (9) implies that quantum properties should depend on the localisation in spacetime through the dependence of $\kappa$. For instance, the canonical commutation relation should read

$$[\hat{A}, \hat{B}] \approx i\frac{\kappa \epsilon^2}{c},$$

where $\hat{A}$ and $\hat{B}$ are two arbitrary canonical conjugate quantities, as long as the spatio-temporal scales of the quantum phenomena considered are small with respect to the spatio-temporal variations of the background value of $\kappa$ in the semiclassical limit of the theory. This also seems to indicate that in entangled relativity, Heisenberg’s uncertainty principle is only valid at the constant $\kappa$ limit of the theory, and that it might be replaced by something different at the full quantum level depicted by Eq. (2).

Experimentally testing the variation of $\hbar$ will strongly depend on the level of variation of the background value of $\kappa$ for a given phenomenon in the semiclassical limit, and its impact on observables. All of which will require demanding theoretical work in order to be determined rigorously. One difficulty notably comes from the intrinsic decoupling [27–29] of the scalar degree of freedom in the classical field equations mentioned above—which freezes its dynamics for fairly generic situations [7–11]. Indeed, the exact amplitude of the decoupling critically depends on the on-shell value of the matter Lagrangian density that appears in the field equations, which is difficult to compute from first principles for the complicated fields that notably make celestial bodies [8]. However, there is no decoupling for (unbounded)$^9$ magnetic fields, since $\mathcal{L}^\circ_m(\propto B^2) \neq 0$ for a magnetic field. That means that the energy of (unbounded) magnetic fields can source the equation of the gravitational scalar-field $\kappa$ Eq. (6), although the amplitude of its (classical) perturbation will be as weak as the metric (classical) perturbation that is created by magnetic fields. Hence, one

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$^6$ Or in Eq. (3).

$^7$ $c$ is often referred to as the speed of light.

$^8$ One may argue that decoherence is the major answer to this question. But let us note that the decoherence time scale is proportional to $\hbar^2$ [39], such that the small value of $\hbar$, really is the main reason why our universe behaves semiclassically rather than quantum-mechanically at the macroscopic scales—as one would expect directly from the path integral formulation of the theory.

$^9$ When a magnetic (or electric) field is bounded, such as in a nucleus for instance, then it contributes to the total trace of the bound object due to the constraint that the internal stresses all vanish [26].
can expect $\kappa$—and therefore $\hbar$—to vary very slightly for intense (or, perhaps, spatially large) magnetic fields—or for other situations that are yet to be found. This means that, at least in principle, this unique prediction of the theory may be probed at the experimental or observational level.

Otherwise, if entangled relativity is indeed a completion of general relativity, it might also open up new avenues in order to explain the (arguable) mysteries of gravity (or, equivalently in the present context, to the semiclassical behavior of our universe): indeed, our semiclassical universe might have started of from a primordial (fully) quantum universe where a fluctuation of $\kappa$ could have led this field to a value close enough to zero for the universe to start to behave semiclassically from this seed. In other words, our semiclassical universe would take its roots from a patch of a primordial quantum universe that simply randomly started to behave semiclassically. This patch would define the initial conditions of our semiclassical universe. This hypothesis would give a somewhat satisfactory explanation to the apparent weakness of the gravitational phenomenon (or, equivalently in the present context, to the reason $\hbar$ is so small that our universe behaves semiclassically rather than fully quantum-mechanically).

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