SL(2,R) CHERN-SIMONS THEORIES WITH RATIONAL CHARGES
AND 2-DIMENSIONAL CONFORMAL FIELD THEORIES

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ABSTRACT
We present a hamiltonian quantization of the $SL(2, R)$ 3-dimensional Chern-Simons theory with fractional coupling constant $k = s/r$ on a space manifold with torus topology in the “constrain-first” framework. By generalizing the “Weyl-odd” projection to the fractional charge case, we obtain multi-components holomorphic wave functions whose components are the Kac-Wakimoto characters of the modular invariant admissible representations of $\hat{A}_1$ current algebra with fractional level. The modular representations carried by the quantum Hilbert space satisfy both Verlinde’s and Vafa’s constraints coming from conformal field theory. They are the “square-roots” of the representations associated to the conformal $(r, s)$ minimal models. Our results imply that Chern-Simons theory with $SO(2, 2)$ as gauge group, which describes 2 + 1-dimensional gravity with negative cosmological constant, has the modular properties of the Virasoro discrete series. On the way, we show that the 2-dimensional counterparts of Chern-Simons $SU(2)$ theories with half-integer charge $k = p/2$ are the modular invariant $D_{p+1}$ series of $\hat{A}_1$ current algebra of level $2p - 2$.
1. Introduction

Three-dimensional Chern-Simons topological gauge theory with $SL(2, R)$ as gauge group has attracted considerable interest for various reasons including its relationship to both 2-dimensional and 3-dimensional quantum gravity.

Chern-Simons gauge theories with non-compact gauge groups are not expected to present any special pathology as 3-dimensional quantum field theories. Their Hamiltonian being identically zero and their action being linear in time derivatives, one expects on general grounds that they define perfectly unitary quantum theories. Therefore it appears that their 2-dimensional counterparts cannot be the non-unitary Wess-Zumino-Witten models on non-compact group manifolds: if 2-dimensional quantum field theories associated to Chern-Simons theories with non-compact gauge group do exist, they are likely to represent some, possibly yet unknown, generalization of current algebra constructions. Understanding such generalization is another motivation to study $SL(2, R)$ Chern-Simons gauge theory.

Unfortunately, the extension of the Hamiltonian quantization techniques which allowed a non-perturbative solution of Chern-Simons theories with compact gauge groups to theories with non-compact gauge groups is revealed to be problematic. Canonical quantization in the holomorphic “quantize-first” scheme has been essential for establishing the correspondence between 3-dimensional Chern-Simons theory with compact gauge groups and 2-dimensional current algebras, but this approach is not viable for the real non-compact $SL(2, R)$ due to the lack of a gauge invariant polarization. Analyses based on polarizations which are not gauge invariant provided some intriguing information about $SL(2, R)$ Chern-Simons theory, but were difficult to carry out at explicit and less formal levels and were limited to the case of trivial space topology. Recent perturbative computations have stressed the substantially novel features that non-compact gauge groups introduce into the quantization of topological Chern-Simons theories.

In this paper we will present a canonical quantization of Chern-Simons theory with $SL(2, R)$ as gauge group in the “constrain-first” framework. This approach, being gauge invariant ab initio, avoids the difficulties of non-gauge invariant polarizations affecting the “quantize-first” method. We will limit ourselves
to the case when the “space” manifold $\Sigma$ is a 2-dimensional torus; such limitation has been sufficient to unravel the underlying 2-dimensional current algebra structure in the compact case.

The starting point in the “constrain-first” approach is the classical gauge invariant phase space $\mathcal{M}$, the space of flat gauge connections on the space manifold $\Sigma$. Since $\mathcal{M}$ is finite-dimensional, the canonical quantization problem actually has a finite number of degrees of freedom. However, the fact that $\mathcal{M}$ is not in general a smooth manifold, makes its quantization rather non-standard. Even in the case of compact gauge groups, $\mathcal{M}$ has singularities of finite order which are associated to important quantum-mechanical effects, such as the “shift” of the central charge in the Sugawara construction for 2-dimensional algebras [12]. When the gauge group is $SL(2, R)$, the singularities of $\mathcal{M}$ are of a more general type, as we will shortly see: they play a central role in the quantization of the $SL(2, R)$ Chern-Simons theory which we consider here.

When $\Sigma$ is a torus, the problem of quantizing $\mathcal{M}$ is reduced to the problem of quantizing the moduli space of flat-connections of an abelian gauge group [11]. This makes the computation for genus one drastically simpler than for higher genus, where non-abelian Chern-Simons theory appears to be vastly more complex than abelian. On the other hand, the factorization properties of 2-dimensional conformal field theories suggest that the torus topology already contains most, if not all, of the complexities of higher genus. The solution of this apparent paradox is that $\mathcal{M}$ for a torus is almost the space of flat connections of an abelian group, but not quite: it is the space of abelian flat connections modulo the action of a discrete group whose fixed points give rise to orbifold singularities. It is only here that the quantization of non-abelian Chern-Simons theory (with compact gauge group) for genus one differs from the computationally trivial abelian case. Thus, in some sense, the singularities of $\mathcal{M}$ for genus one must encode much of the information about the theory on higher genus surfaces, at least for compact non-abelian gauge groups. That this remains true for non-compact gauge groups like $SL(2, R)$ is plausible, though yet to be proven.

In comparison with $\mathcal{M}_{SU(2)}$, the distinctive feature of the phase space $\mathcal{M}_{SL(2, R)}$ is its non-compactness. Even if one restricts oneself (as we will essentially do in this paper) to the sector of $\mathcal{M}_{SL(2, R)}$ corresponding to flat connections
which lie in the “compact” Cartan subgroup of $SL(2, R)$, one has to deal effectively with the non-compact smooth manifold obtained by deleting the singularities from the compact non-smooth phase space. One consequence of the non-compactness of the (effective) phase space is that the integrality condition on the Chern-Simons charge $k$ \cite{1},\cite{9},\cite{10} disappears. Another consequence is that infinite-dimensional Hilbert spaces emerge, in general, upon quantization. However, if one takes $k$ to be rational, the Hilbert space of quantum states becomes finite-dimensional \cite{13}. Therefore, it is reasonable to think that theories with rational $k$ correspond to rational conformal 2-dimensional field theories, or to some “deformation” of them, which, when $k$ becomes an integer, reduce to the familiar non-abelian current algebras. Recently, abelian Chern-Simons theories with rational charge $k = p/q$ have been investigated because of their (possible) relevance to the theory of quantum Hall effect and to a new mechanism for superconductivity \cite{14}. Wave functions are represented by $q$-dimensional multiplets of theta-functions of level $pq$, \cite{15},\cite{16} which can be thought of as describing holomorphic sections of a holomorphic “line bundle” with rational Chern class $p/q$ on the non-compact phase space. When $k$ is an integer, by projecting the abelian Hilbert space to the “Weyl” odd sector, one obtains the non-abelian wave functions, that is the Kac-Weyl characters of the integrable current algebra representations \cite{11},\cite{12}. In this paper we will show that the appropriate generalization of such projection to the fractional charge case leads to the modular invariant Kac-Wakimoto \cite{17} characters of the (non-integrable and non-unitary) representations of $\hat{A}_1$ current algebra with fractional admissible level. We will also discover that the modular representations acting on the Hilbert space of states of the $SL(2, R)$ Chern-Simons theory are identical to one of the two factors into which the modular representations of the conformal minimal models factorize. This suggests that the 2-dimensional counterpart of the $SL(2, R)$ Chern-Simons theory might be non-conformal. It also implies that Chern-Simons theory with gauge group $SL(2, R) \times SL(2, R) \approx SO(2, 2)$ and rational charges $(k, \frac{1}{4k})$ has exactly the modular properties of the conformal minimal models. Since $SO(2, 2)$ Chern-Simons theory describes $2 + 1$ dimensional gravity with negative cosmological constant, it seems therefore supported Witten’s speculation \cite{7} that gravity in $2 + 1$ dimensions and the Virasoro discrete series are related.
2. Geometric Quantization of $\mathcal{M}_{SL(2,R)}$

Flat $SL(2,R)$ connections on a torus correspond to pairs $(g_1, g_2)$ of commuting $SL(2,R)$ elements, modulo overall conjugation in $SL(2,R)$. $g_1$ and $g_2$ represent the holonomies of the flat connections around the two non-trivial cycles of the torus. $SL(2,R)$ has a non-trivial $\mathbb{Z}_2$ center and $SO(1,2) \approx SL(2,R)/\mathbb{Z}_2$. Therefore, $\mathcal{M}$ is a four-cover of the space $\mathcal{M}'$ of $SO(1,2)$ flat connections, since to each $SO(1,2)$ flat connection correspond four $SL(2,R)$ flat connections whose holonomies differ by elements of the center $\mathbb{Z}_2$. It is convenient to describe $\mathcal{M}$ in terms of the simpler $\mathcal{M}'$. Let us think of $SL(2,R)$ as the group of unimodular $2 \times 2$ real matrices. The basic fact of $\mathcal{M}$ (or $\mathcal{M}'$) is that it is the union of three “sectors”

$$\mathcal{M} = \bigcup_{i=1,2,3} \mathcal{M}_i,$$

where the $\mathcal{M}_i$'s, $i = 1, 2, 3$, are the space of $SL(2,R)$ flat connections whose respective holonomies have two imaginary (and conjugate) eigenvalues ($i = 1$), two eigenvectors with real (and reciprocal) eigenvalues ($i = 2$), and one single eigenvector with unit eigenvalue ($i = 3$).

When $i = 1$, both holonomies can be simultaneously brought by conjugation into the compact $U(1)$ subgroup of $SL(2,R)$. Therefore $\mathcal{M}_1 \approx T^{(1)}$, the two dimensional torus. Let us introduce the real coordinates $(\theta_1, \theta_2)$ for $\mathcal{M}_1$. In our normalization, the periodic coordinates $\theta_{1,2}$ lie in the unit real interval when the gauge group is $SO(1,2)$; for $SL(2,R)$, these take values in the enlarged interval of length 2.

For $i = 2$, the holonomies can be conjugated into a diagonal form. However, one can still conjugate diagonal holonomies by an element of the gauge group which permutes the eigenvalues. Therefore, when the gauge group is $SO(1,2)$, $\mathcal{M}_2 \approx R^{(2)}/\mathbb{Z}_2$. If $(x,y)$ are cartesian coordinates on the real plane $R^{(2)}$, the $\mathbb{Z}_2$ action is the reflection around the origin, mapping $(x,y)$ onto $(-x,-y)$. If the gauge group is $SL(2,R)$, $\mathcal{M}_2$ consists of four copies of $R^{(2)}/\mathbb{Z}_2$.

Finally, when $i = 3$, holonomies can be conjugated into an upper triangular form with units on the diagonal. Conjugation allows one to rescale the (non-vanishing) elements in the upper right corner by an arbitrary positive number,
Thus, $\mathcal{M}_3 \approx S^1$, the real circle. Being odd-dimensional, $S^1$ cannot be a genuine non-degenerate symplectic space. In fact, the symplectic form on the space of flat connections coming from the Chern-Simons action, when pushed down to $\mathcal{M}_3$ vanishes identically. $\mathcal{M}_3$ represents a “null” direction for the symplectic form of the $SL(2, R)$ Chern-Simons theory, reflecting the indefiniteness of the $SL(2, R)$ Killing form. Since $\mathcal{M}_3$ is a disconnected piece of the total phase space $\mathcal{M}$ (or $\mathcal{M}'$), it is consistent to consider the problem of quantizing $\mathcal{M}_1 \cup \mathcal{M}_2$ independently of $\mathcal{M}_3$. After all, modding out by the “null” directions (such as those originated by gauge symmetries), is the common recipe for dealing with degenerate symplectic forms. Hence, in what follows we concentrate on $\mathcal{M}_1 \cup \mathcal{M}_2$, though it is conceivable that the “light-like” sector $\mathcal{M}_3$ merits further investigation.

To summarize, the space of gauge flat connections on the torus (disregarding $\mathcal{M}_3$) looks as follows: a torus ($\mathcal{M}_1$) with planes ($\mathcal{M}_2$) “attached” to it at the points $z_s$ in a discrete set $\mathcal{N} \equiv \mathcal{M}_1 \cap \mathcal{M}_2$, representing flat connections with holonomies in the center of the gauge group. For the $SO(1, 2)$ case, $\mathcal{N}$ contains a single point, whose $\mathcal{M}_1$ and $\mathcal{M}_2$ coordinates are $(\theta_1^{(s)}, \theta_2^{(s)}) = (0, 0)$ and $(x^{(s)}, y^{(s)}) = (0, 0)$, respectively. When the gauge group is $SL(2, R)$, $\mathcal{N}$ consists of four points, with $(\theta_1^{(s)}, \theta_2^{(s)}) = (\pm 1, \pm 1)$. The $\mathcal{M}_2$ planes are “folded” by the $Z_2$ reflections around the points in $\mathcal{N}$.

The distinctive feature of classical phase space $\mathcal{M}$ is that it ceases to be a smooth manifold around the points in $\mathcal{N}$. The quantization of the classical phase space $\mathcal{M}$ involves considering smooth functions or smooth sections of appropriate line bundles on $\mathcal{M}$, raising the question of the meaning of “smooth” sections on a non-smooth manifold such as $\mathcal{M}$. Our strategy is to consider first the smooth, non-compact manifold $\mathcal{M}/\mathcal{N}$ obtained by deleting the singular points in $\mathcal{N}$. $\mathcal{M}/\mathcal{N}$ consists of two disconnected smooth components, $\mathcal{M}_1/\mathcal{N}$ and $\mathcal{M}_2/\mathcal{N}$. We will then consider quantizations of $\mathcal{M}_1/\mathcal{N}$ and $\mathcal{M}_2/\mathcal{N}$ which admit sections that can be “glued” at the points in $\mathcal{N}$. The final Hilbert space will be the span of those “glued” sections. Our “intuitive” approach could conceivably be substantiated with more rigorous methods of algebraic geometry.

We will perform the quantization of $\mathcal{M}_1/\mathcal{N}$ and $\mathcal{M}_2/\mathcal{N}$ in the holomorphic scheme [9]-[11] since, as is familiar from the study of the compact Chern-Simons
theory, $\mathcal{M}$ admits a natural family $\mathcal{T}$ of Kähler polarizations. $\mathcal{T}$ is the Siegel upper complex plane, because the choice of a complex structure on the 2-dimensional space manifold $\Sigma$ induces a complex structure on the space of connections on $\Sigma$ and, by projection, on $\mathcal{M}$. For $\tau \in \mathcal{T}$, let us introduce holomorphic coordinates on both $\mathcal{M}_1/\mathcal{N}$

\begin{equation}
  z = \theta_1 + \tau \theta_2, \quad \bar{z} = \theta_1 + \bar{\tau} \theta_2, \quad (\theta_1, \theta_2) \in \mathcal{M}_1/\mathcal{N}
\end{equation}

and $\mathcal{M}_2/\mathcal{N}$

\begin{equation}
  z = x + \tau y, \quad \bar{z} = x + \bar{\tau} y, \quad (x, y) \in \mathcal{M}_2/\mathcal{N}.
\end{equation}

Then the symplectic form which descends from the Chern-Simons action with charge $k$

\begin{equation}
  S = \frac{k}{4\pi} \int_{\Sigma \times \mathbb{R}^1} \langle A, dA + \frac{1}{3} A \wedge A \rangle
\end{equation}

can be written both on $\mathcal{M}_1$ and $\mathcal{M}_2$ in the coordinates systems (2) and (3) as follows:

\begin{equation}
  \omega = \frac{ik\pi}{2\tau_2} dz \wedge d\bar{z} \equiv i\partial \partial K, \quad \tau_2 \equiv Im \tau
\end{equation}

where $K$ is the Kähler potential which we choose to be

\begin{equation}
  K = \frac{k\pi}{4\tau_2} (z - \bar{z})^2.
\end{equation}

In the context of Kähler quantization, the Hilbert space of quantum states is the span of square integrable holomorphic sections of a holomorphic line bundle with hermitian structure whose curvature two-form is the symplectic form $\omega$ in (5).

The quantization of $\mathcal{M}_2/\mathcal{N}$ is rather straightforward. Since $\mathcal{M}_2/\mathcal{N}$ is not simply connected, the holomorphic wave functions $\psi(z)$ can acquire an arbitrary phase $e^{2\pi i \theta_i}$ when moving around the singular points $z_i = 0$ of $\mathcal{N}$. The Böhm-Aharonov phases $e^{2\pi i \theta_i}$ should be regarded as free parameters of the quantization. A further two-fold ambiguity of the $\mathcal{M}_2/\mathcal{N}$ quantization stems from the fact that the gauge invariant $\mathcal{M}_2/\mathcal{N}$ is the quotient of the complex plane (with the origin deleted) by the action of the reflection around the origin. Thus, physical wave
functions should be invariant under the action of the unitary operator implementing the reflection around the origin. Since there are two ways of implementing reflections according to the “intrinsic” parity of the wave functions, one concludes that the wave functions on each of the four “sheets” of \( \mathcal{M}_2/\mathcal{N} \) are

\[
\psi^{(\vartheta, \pm)}(z) = z^{\vartheta} \chi^{(\pm)}(z),
\]

where \( \chi^{(\pm)}(z) \) is holomorphic, even (odd) around the origin, and each choice of \((\vartheta, \pm)\) is associated to a different quantum Hilbert space \( \mathcal{H}^{(\vartheta, \pm)}_{\mathcal{M}_2/\mathcal{N}} \).

Let us now turn to \( \mathcal{M}_1/\mathcal{N} \). The crucial difference between quantum mechanics on the non-compact \( \mathcal{M}_1/\mathcal{N} \) and on the compact torus \( \mathcal{M}_1 \) originates from the fact that the homotopy group \( \pi_1(\mathcal{M}_1/\mathcal{N}) \) is non-abelian:

\[
ab = ba\delta, \quad [a, \delta] = [b, \delta] = 0,
\]

where \( a \) and \( b \) are the non-trivial cycles of the compact torus and \( \delta = \prod_i \delta_i \) is the product of the cycles \( \delta_i \) surrounding the singularities \( z_i \) in \( \mathcal{N} \). In this case, quantum states are represented by multi-components wave functions \( \Psi(z) = (\psi^\alpha(z)) \), \( \alpha = 0, 1, ..., q - 1 \), transforming in some irreducible unitary, \( q \)-dimensional representation of the homotopy group \( \pi_1(\mathcal{M}_1/\mathcal{N}) \). Let us consider a basis for such representation which diagonalizes the \( \delta_i \)’s. For the representation to be finite-dimensional and irreducible, the \( \delta_i \)’s must be represented by rational phases. Moreover, we take all \( \delta_i \)’s to be the same, since we require that modular transformations (which mix the singular points in \( \mathcal{N} \)) act on the Hilbert space of wave functions. In conclusion we take \( \delta = exp(-2\pi ip/q) \) with \( p \) integer, coprime with \( q \).

In the holomorphic quantization scheme, wave functions \( \Psi(z) \) should be holomorphic and, in the trivialization corresponding to (8), should have the periodicity properties of theta functions with fractional “level” \( k \):

\[
\Psi(z + 2m + 2n\tau) = exp(-2\pi ik\tau n^2 - 2\pi ikzn)a^m b^n \Psi(z),
\]

where \( a \) and \( b \) are \( q \times q \) unitary matrices which provide a representation of homotopy relations (8). Note that on the compact torus \( \mathcal{M}_1 \), \( a \) and \( b \) would be
one-dimensional phases and the consistency (cocycle) condition for the transition functions in (9) would require $2k$ to be an integer \([9]-[11]\). In our case, the consistency condition coming from (8) relates the Chern-Simons charge $k$ to the monodromy of the wave functions around the singular points:

$$e^{2\pi i 2k} = e^{2\pi ip/q}.$$  

(10)

Therefore, we restrict ourselves henceforth to the case of $k$ rational:

$$2k = 2s/r = p/q,$$  

(11)

with $s$ and $r$ integers, relatively prime, and $r$ chosen to be positive. It should be stressed that the restriction to $k$ rational is motivated by the interest to investigate the connection between $SL(2, R)$ Chern-Simons theory and 2-dimensional rational conformal field theories. When $k$ is irrational one expects an infinite-dimensional Hilbert space of holomorphic wave functions: an interesting possibility, which we do not pursue here.

The holomorphic components $\psi^\alpha(z)$ of the wave functions $\Psi(z)$ can be thought of as representing holomorphic sections of the holomorphic line bundle $L^{(k)}_o$ on the non-compact $M_1/N$ with fractional “Chern-class” $p/q$. Locally, a section $\psi$ of $L^{(k)}_o$ would be given by the $q$-root of a theta function of level $p$. $\psi$ would have non-trivial monodromy $\delta = e^{2\pi ip/q}$ around the singular points in $N$, but would be single-valued when holomorphically extended to a $q$-cover $\tilde{M}_1$ of the torus $M_1$. If $M_1$ is the complex torus with modular parameter $\tau$, the $q$-cover $\tilde{M}_1$ is a torus with modular parameter $q\tau$. The $q$ components $\psi^\alpha(z)$ of the wave function $\Psi(z)$ should be identified with the different holomorphic extensions of $\psi$ to $\tilde{M}_1$: they should therefore be theta functions of level $q \times p/q = p$ on the torus with modular parameter $q\tau$. We will verify that this is in fact the case. In the following however we will simply think of $\Psi(z)$ as hoomorphic sections of a vector bundle on the compact torus $M_1$ with fibers of dimensions $q$.

It follows from (9) that inequivalent holomorphic quantizations of $M_1/N$ with the same $k$ are in one-to-one correspondence with classes of inequivalent, unitary and irreducible representations of the ‘t Hooft algebra

$$ab = bae^{2\pi ip/q}.$$  

(12)
Let us denote by \( R_{p/q}^{(\vartheta_a, \vartheta_b)} \) the following \( q \)-dimensional representation of (12):

\[
\begin{align*}
(a)_{\alpha\beta} &= e^{2\pi i \vartheta_a} e^{-2\pi i p/q \alpha} \delta_{\alpha, \beta} \\
(b)_{\alpha\beta} &= e^{2\pi i \vartheta_b} \delta_{\alpha, \beta+1}, \quad \alpha, \beta = 0, 1, \ldots, q-1.
\end{align*}
\]

(13)

It is easy to check that the “characteristics” \((\vartheta_a, \vartheta_b)\) modulo \((m/q, n/q)\) (with \(m, n\) relative integers) label all the inequivalent unitary irreducible representations of (12). The space of classes of inequivalent (irreducible, unitary) representations of (12) is therefore isomorphic to a 2-dimensional torus \( T_{p/q} \).

It is has been stated [9]-[10] that for the modular group to act on the Hilbert space of holomorphic wave functions (3) one needs both \( pq \) even and the characteristics \( \vartheta_a, \vartheta_b \equiv 0 \mod 1/q \). Since this is not quite correct, let us pause to discuss the issue of modular invariance in some detail. (See also [13]-[15].) Let us denote by \( s, t, c \) the following external automorphisms of the ‘t Hooft algebra (12):

\[
\begin{align*}
s &: \begin{cases} 
a \to b^{-1} \\
b \to a \end{cases} \\
t &: \begin{cases} 
a \to a \\
b \to e^{-i\pi p/q} ab \end{cases} \\
c &: \begin{cases} 
a \to a^{-1} \\
b \to b^{-1} \end{cases}
\end{align*}
\]

(14)

One can to verify that \( s, t, c \) satisfy the modular group relations, \( s^2 = c \) and \((st)^3 = 1\) and that the “conjugation” operator \( c \) commutes with the modular group generators, \( sc = cs, tc = ct \). The automorphisms \( s, t, c \) map representations of (12) onto generically inequivalent representations; therefore they induce a non-trivial action on the torus \( T_{p/q} \); the space of classes of inequivalent (irreducible, unitary) representations of the ‘t Hooft algebra (12). This action, however, is not the “standard” action of the modular group on the 2-dimensional torus, which is linear and homogenous in the coordinates \((q\vartheta_a, q\vartheta_b)\). Denoting by \( s_*, t_*, c_* \) the action of \( s, t, c \) induced on \( T_{p/q} \) one can explicitly calculate from (13) that \( t_* \) has an inhomogenous term:

\[
\begin{align*}
c_* &: (q\vartheta_a, q\vartheta_b) \to (-q\vartheta_a, -q\vartheta_b) \\
s_* &: (q\vartheta_a, q\vartheta_b) \to (-q\vartheta_b, q\vartheta_b) \\
t_* &: (q\vartheta_a, q\vartheta_b) \to (q\vartheta_a, q\vartheta_a + q\vartheta_b + pq/2),
\end{align*}
\]

(15)
where \( q^a, b \) are real numbers modulo integers. The vector space of a representation \( R_{p/q}^{(a, b)} \), belonging in an equivalence class which is invariant under \( s, t, c \), carries a (unitary) representation of the modular group whose generators we will denote by \( S, T \) and \( C \). Such a representation \( R_{p/q}^{(a, b)} \) defines through (9) a vector space of holomorphic wave functions \( \Psi(z) \) which supports a (unitary) representation of the modular group. The generators of this modular representation will be indicated below by \( U(s), U(t) \) and \( U(c) \). From (15) it follows that if \( pq \) is even the only (up to equivalence) modular invariant quantization corresponds to the (equivalence class of the) \( R_{p/q}^{(0, 0)} \) representation of the ‘t Hooft algebra (12), a fact already recognized in the earlier literature on Chern-Simons theory [9], [10]. However eq. (15) also implies that modular invariance can be maintained for \( pq \) odd as well by choosing a representation of the ‘t Hooft algebra in the equivalence class of \( R_{p/q}^{(1/2, 1/2)} \). This was first realized in [13] in the context of the abelian Chern-Simons theory. We will show in the following that in the non-abelian theory the choice \( k = p/2 \) with \( pq = p \) odd (disregarded in [3], [10] on modular invariance grounds) does actually lead to the characters forming the \( D_{p+1} \) series of modular invariants for \( \hat{A}_1 \) current algebra [19]. Since these conformal models are well-defined on Riemann surfaces of arbitrary topology it is likely that a modular invariant quantization of Chern-Simons theory with \( k \) integer and odd, extending to all genuses the quantization that we will exhibit here for the torus topology, does exist.

In geometric quantization, in order to implement canonical transformations which do not leave the polarization invariant (such as modular transformations), the wave functions \( \Psi_\tau(z) \) are also regarded as dependent on the polarization \( \tau \in \mathcal{T} \). The \( \tau \) dependence is determined by the requirement that quantum Hilbert spaces \( \mathcal{H}_\tau \) relative to different \( \tau \)’s be unitarily equivalent with respect to the hermitian forms

\[
\left( \Psi^{(1)}_\tau, \Psi^{(2)}_\tau \right) = \int_{\mathcal{M}_1} dz \, d\bar{z} \, \tau^{-1/2} e^{-\frac{k}{12}(z-\bar{z})^2} \Psi^{(1)}_\tau(\bar{z})^* \Psi^{(2)}_\tau(z) \quad (16)
\]

associated to the Kähler structure (3). This implies that the wave functions \( \Psi_\tau(z) \) should be parallel with respect to a flat, unitary connection on the vector bundle with base \( \mathcal{T} \) and fibers \( \mathcal{H}_\tau \) [11]. One has now to distinguish the cases when \( pq \) is
even or odd. For $pq$ even, an orthonormal $p$-dimensional basis for the $q$-components parallel wave functions $\Psi(z)$ of the quantization of $\mathcal{M}_1/\mathcal{N}$ is given by:

$$(\Psi_N(\tau; z))^\alpha \equiv \psi_N^\alpha(\tau; z) = \theta_{qN+pa,pq/2}(\tau; z/q), \quad N = 0, 1, \ldots, p-1,$$

where the $\theta_{n,m}(\tau; z)$ ($n$ integer modulo $2m$) are level $m$ SU(2) theta functions:

$$\theta_{n,m}(\tau; z) \equiv \sum_{j \in \mathbb{Z}} e^{2\pi im\tau(j + \frac{n}{2m})^2 + 2\pi iz(j + \frac{n}{2m})}.$$

For $pq$ odd, we have seen that modular invariance requires the representation of the 't Hooft algebra (12) to be (equivalent to) $R_{p/q}(1/2,1/2)$. With this choice, an orthogonal $p$-dimensional basis of parallel holomorphic wave functions is:

$$(\Psi_N(\tau; z))^\alpha = (-1)^q N + p\alpha \sum_{j \in \mathbb{Z}} e^{i\pi pq\tau(j + N/p + q/\alpha + 1/2)^2 + i\pi p(z-q)(j + N/p + q/\alpha + 1/2)}$$

$$= \theta_{q(2N+p)+2p\alpha,2pq}(\tau; z/2q) - \theta_{q(2N-p)+2p\alpha,2pq}(\tau; z/2q),$$

$$-p/2 < N < p/2. \quad (18)$$

Among classical canonical transformations, reflections $\hat{c}$ around the singular points in $\mathcal{N}$

$$\hat{c} : z \to -z. \quad (19)$$

are of special interest for our purposes. $\hat{c}$ will be implemented on the Hilbert space of wave functions $\Psi(z)$ by a unitary operator $U(\hat{c})$:

$$U(\hat{c}) : \Psi(z) \to C\Psi(-z), \quad (20)$$

where $C$ is a $q \times q$ unitary matrix acting on the “internal” indices $\alpha$, which implements the automorphism $c$ defined in (14) on the vector space of $c_\ast$-invariant representations of (12):

$$Ca^m b^n = a^{-m} b^{-n} C. \quad (21)$$

For $a, b$ in both the representation $R_{p/q}^{(0,0)}$ (when $pq$ is even) and $R_{p/q}^{(1/2,1/2)}$ (when $pq$ odd) the solution of (21) is:

$$(C)_{\alpha,\beta} = \delta_{\alpha,-\beta}. \quad (22)$$
The Hilbert spaces $\mathcal{H}_{\mathcal{M}_1/N}$ spanned by the sections (17) and (18) split under the action of $U(c)$ into “even” and “odd” subspaces $\mathcal{H}^\pm_{\mathcal{M}_1/N}$. For $pq$ even an orthogonal parallel basis of $\mathcal{H}^\pm_{\mathcal{M}_1/N}$ is

$$\Psi^\alpha_N = \psi^{\alpha,(\pm)}_N(\tau; z) = \theta_{qN+pa,pq/2}(\tau; z/q) \pm \theta_{-qN+pa,pq/2}(\tau; z/q),$$

while for $pq$ odd one has:

$$\Psi^\alpha_N = (\theta_{q(2N+p)+2pa,2pq}(\tau; z/2q) \pm \theta_{-q(2N+p)+2pa,2pq}(\tau; z/2q)) \mp (\theta_{q(p-2N)+2pa,2pq}(\tau; z/2q) \pm \theta_{-q(p-2N)+2pa,2pq}(\tau; z/2q)).$$

Were we simply trying to quantize $\mathcal{M}_1/N$, we would keep both the even and the odd sector since canonical transformations (19) in the $\mathcal{M}_1$ sector do not correspond to gauge transformations of the original Chern-Simons $SL(2, R)$ theory. However, we really want to quantize the union $\mathcal{M}_1 \cup \mathcal{M}_2$. There are no “rigorous” ways to quantize a phase space consisting of different branches with a non-zero intersection. Phase spaces of this sort have appeared in the context of 2-dimensional gravity in [21]. It seems reasonable to think of a wave function on the union $\mathcal{M}_1 \cup \mathcal{M}_2$ as a pair $(\psi_1, \psi_2)$ of wave functions, with $\psi_1 \in \mathcal{H}_{\mathcal{M}_1/N}$ and $\psi_2 \in \mathcal{H}_{\mathcal{M}_2/N}$, “agreeing” in some sense on the intersection $N$. Our proposal is that $\psi_1$ and $\psi_2$ should have the same behaviour around the points in $N$. Since $\psi_1$ and $\psi_2$ are represented by holomorphic functions, this implies that the pair $(\psi_1, \psi_2)$ should be determined uniquely by $\psi_1$ and that most of the states $\psi_2$ in the infinite-dimensional $\mathcal{H}_{\mathcal{M}_2/N}$ should be discarded. Moreover, the Böhm-Aharonov phase $e^{2\pi i \theta}$ in the $\mathcal{M}_2$ branch should coincide with the analogous quantity $e^{2\pi ip/q}$ in the $\mathcal{M}_1$ sector. However, all states $\psi_2$ in $\mathcal{H}_{\mathcal{M}_2/N}$ have the same behaviour under reflections $\hat{c}$ around singular points, since $\hat{c}$ corresponds to a gauge transformation of the Chern-Simons theory in the $\mathcal{M}_2$ branch. This should put a restriction on the states $\psi_1$, which, in order to “agree” with $\psi_2$, should also have definite parity under $\hat{c}$. In conclusion the (only) rôle of $\mathcal{M}_2$ should be “transmitting” to $\mathcal{M}_1$ the definite $\hat{c}$-parity projection. The phase space $\mathcal{M}_1 \cup \mathcal{M}_2$ admits therefore two inequivalent quantizations, with Hilbert spaces isomorphic to $\mathcal{H}^\pm_{\mathcal{M}_1/N}$. A similar ambiguity is present in the $SU(2)$ case [12], but it is the “odd” quantization which is related to 2-dimensional conformal field theories for generic $k$. In fact, only the
“odd” projection gives positive integer fusion rules for $k$ generic, suggesting that this is the quantization of the Chern-Simons theory on the torus which generalizes, in some appropriate sense, to higher genus space manifolds [11]. In our case as well, “odd” quantization gives positive, integer fusion rules for generic $k$, as we will shortly see, though we do not yet know its 2-dimensional interpretation.

Wave functions in $\mathcal{H}_{M_1/N}$ are related to the Kac-Wakimoto characters [17] of irreducible, modular invariant representations of $SL(2,R)$ current algebra with fractional central charge $m \equiv t/u$ ($t, u$ coprime integer relative numbers, $u$ positive) satisfying the admissibility condition

$$2u + t - 2 \geq 0. \quad (25)$$

The Kac-Wakimoto characters are defined as follows:

$$\chi_{j(N',\alpha')}^{(m,\tau)}(z) = \text{tr}_{\mathcal{H}_{j,m}} e^{2\pi i \tau L_0 + 2\pi iz J_3}, \quad (26)$$

where $\mathcal{H}_{j,m}$ is the highest weight irreducible representation of $SL(2,R)$ current algebra with level $m$ and spin $j$. $j = j(N',\alpha')$ ranges over the following set:

$$j = 1/2(N' - \alpha'(m + 2)), \quad N' = 1, 2, ..., 2u + t - 1, \quad \alpha' = 0, 1, ... u - 1. \quad (27)$$

In order to exhibit the explicit relation between Kac-Wakimoto characters and Chern-Simons wave functions $\Psi(z)$ one has to distinguish the cases when:

(i) $p$ is even and $q$ is odd, so that $p = 2s$ and $q = r$;

(ii) $p$ is odd and $q$ is even, so that $p = s$ and $r = 2q$ is a multiple of 4;

(iii) both $p$ and $q$ are odd, so that $p = s$ and $r = 2q \equiv 2 \mod 4$.

In case (i) the “odd” orthogonal wave functions in (23) can be written in terms of the Kac-Wakimoto characters $\chi_{j,m}$ of level $m$ given by:

$$m + 2 = k, \quad (28)$$

i.e. $u = q = r$ and $p/2 = s = 2u + t$. The explicit relation is:

$$\frac{\psi_{\alpha}^N(-)(\tau; z)}{\Pi(\tau; z)} = \begin{cases} 
\chi_{j(N,2\alpha);m}(\tau; z) & \text{if } \alpha \in \{0, 1, \ldots, (r-1)/2\} \\
\chi_{j(s-N,2\alpha-r);m}(\tau; z) & \text{if } \alpha \in \{(r+1)/2, \ldots, r-1\},
\end{cases} \quad (29)$$

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where $\Pi(\tau; z)$ is the Kac-Wakimoto denominator:

$$
\Pi(\tau; z) = \theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z).
$$

(30)

$\Pi(\tau; z)$ is holomorphic and non-vanishing on $\mathcal{M}_1/\mathcal{N}$. Therefore, the wave functions $\Psi_{N}^{(-)}(\tau; z)$ and the wave functions

$$
\Psi_{N}(\tau; z) = \frac{\Psi_{N}^{(-)}(\tau; z)}{\Pi(\tau; z)}
$$

appearing in (29), describe equivalent wave functions on $\mathcal{M}_1/\mathcal{N}$, related to each others by a Kähler transformation

$$
K \to K + f(z) + f^*(\bar{z}),
$$

(31)

with $f(z) = -\ln(\Pi(\tau; z))$. Eq.(31) introduces in the scalar product (16) a factor which has been interpreted as the jacobian of a certain change of integration variables in the path integral formulation of the Chern-Simons theory [9].

Eq.(29) implies that to each quantum state emerging out of the quantization of the $SL(2, R)$ Chern-Simons theory with fractional $2k$ there corresponds a multiplet of $SL(2, R)$ current algebra characters, rather than one single character as in the $SU(2)$ case when $2k$ is integer. One does not expect, therefore, that the hypothetical 2-dimensional theory underlying 3-dimensional $SL(2, R)$ Chern-Simons theory has full $SL(2, R)$ current algebra symmetry. One might speculate that such a theory could be obtained from some coset construction of $SL(2, R)$ current algebra, though not a standard coset as will become apparent from the analysis of the modular transformation properties which will be studied in the following section.

If (ii) holds, “odd” wavefunctions in (23) are expressible in terms of Kac-Wakimoto characters of level

$$
m + 2 = 4k
$$

(i.e. $u = q/2 = r/4$ and $p = s = 2u + t$), as it follows from the identities:

$$
\frac{\psi_{N}^{\alpha,(-)}(\tau; z)}{\Pi(\tau; z)} = \begin{cases} 
\chi_{j(2N,\alpha);m}(\tau; z) & \text{if } \alpha \in \{0, 1, \ldots, r/4 - 1\} \\
\chi_{j(s-2N,\alpha-r/4);m}(\tau; z) & \text{if } \alpha \in \{r/4, \ldots, r/2 - 1\}.
\end{cases}
$$

(33)
Finally, if (iii) is true, the level $m$ of the $SL(2, R)$ current algebra is still given by eq.(32), but $u = q = r/2$ and $2u + t = 2p = 2s$. The relation between wave functions and characters becomes:

$$
\frac{\psi^\alpha_{-}(\tau; z)}{\Pi(\tau; z)} = \chi_{j(p+2N,\alpha);m}(\tau; z/2) + \chi_{j(p-2N,\alpha);m}(\tau; z/2).
$$

(34)

In all cases (i)-(iii), the Kac-Wakimoto admissibility condition is equivalent to the statement that “odd” projection $H_{M_1/N}$ be non-empty (i.e. $s \geq 2$).

When the Chern-Simons charge $k$ is an integer, i.e. $r = 1 = q$ and $p = 2k = 2s$ (case (i)), eq.(29) reduces to the well-established [9]-[10] identification between Chern-Simons wave functions and integrable $\hat{A}_1$ Kac-Weyl characters of level $m = k - 2$ forming the diagonal modular invariant $A_{k-1}$ series of the classification of Cappelli et al.[19].

Wave functions are one-dimensional vectors of holomorphic functions also if $k = p/2$ is half-integer (and $p = s$ odd, $q = r/2 = 1$). This case belongs in (iii), therefore the level $m = 2(p - 1)$ of the current algebra is a multiple of 4. Eq.(34) becomes:

$$
\psi'_{N}(\tau; z) = \frac{\psi^\alpha_{-}(\tau; z)}{\Pi(\tau; z)} = \chi_{p+2N;2(p-1)}(\tau; z/2) + \chi_{p-2N;2(p-1)}(\tau; z/2),
$$

(35)

from which one sees that wave functions are precisely those linear combinations of $\hat{A}_1$ characters $\chi_{n;2(p-1)}$ of level $2(p - 1)$ which form the $D_{p+1}$-series of [19].
3. Modular Transformations

Let us use \( \hat{s} \) and \( \hat{t} \) to denote the canonical transformations of the classical phase space \( \mathcal{M}_1/\mathcal{N} \) which generate the modular group \( SL(2, \mathbb{Z}) \) of the torus

\[
\hat{s} : (\tau, z) \to (-1/\tau, z/\tau) \\
\hat{t} : (\tau, z) \to (\tau + 1, z),
\]

and satisfy the relations:

\[
\hat{s}^2 = \hat{c}, \quad (\hat{s}\hat{t})^3 = 1.
\]

\( \hat{s} \) and \( \hat{t} \) will be represented on the space of the multi-components wave functions \( (\Psi(z))^\alpha \equiv \psi^\alpha(\tau; z) \) by means of unitary operators \( U(s) \) and \( U(t) \):

\[
U(s) : \Psi(\tau; z) \to (S^{-1}\Psi)(-1/\tau; z/\tau) \\
U(t) : \Psi(\tau; z) \to (T^{-1}\Psi)(\tau + 1; z),
\]

where \( S \equiv (S)^{\alpha\beta} \) and \( T \equiv (T)^{\alpha\beta} \) are unitary \( q \times q \) matrices acting on the “internal” indices and implementing the modular transformations \( (14) \) on the representation space the ‘t Hooft algebra \( (12) \). Choosing the \( R_{p/q}^{(0,0)} \) representation of \( (12) \) when \( pq \) is even and \( R_{p/q}^{(1/2,1/2)} \) when \( pq \) is odd, one obtains the following expressions for matrices \( T \) and \( S \):

\[
(T^{(p;q)})^{\alpha\beta} = \delta_{\alpha,\beta}(-1)^p e^{2\pi i \frac{p^2q}{2} \alpha^2 - 2\pi i \theta(p;q)/3} \\
(S^{(p;q)})^{\alpha\beta} = \frac{1}{\sqrt{q}} e^{2\pi i \frac{p^2}{q} \alpha \beta},
\]

\( \alpha, \beta = 0, 1, \ldots, q - 1. \)

The phase \( \theta(p;q) \) in \( (39) \) is determined from the \( SL(2, \mathbb{Z}) \) relation \( (ST)^3 = 1 \), which gives:

\[
e^{2\pi i \theta(p;q)} = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} (-1)^{pqn} e^{2\pi i \frac{p}{q} n^2}.
\]

When \( p = 1 \) (and \( q \) is even) this is the celebrated Gauss sum \([22]\), and it is well-known that \( \theta(1;2K) \equiv 1/8 \) mod 1, agreeing with the fact that the conformal central charge of a free compactified 2-dimensional scalar field is one. In fact,
$R^{(1;2K)}$ is the representation of the modular group associated to the conformal blocks of a 2-dimensional scalar field compactified on a circle of radius $R^2 = 2m/n$ with $m$ and $n$ integers and $K = mn$. $R^{(1;2K)}$ is also the representation of the modular group that one obtains upon quantization of the abelian Chern-Simons theory on a torus [9]-[10]. For $p \neq 1$, the sum in (40) is a generalized Gauss sum which has not yet appeared in conformal field theory and which we calculate in the Appendix. Some properties of $\theta(p; q)$ follow immediately from its definition (40):

\[ \theta(p+2q; q) \equiv \theta(p; q) \mod 1 \]

\[ \theta(p'; q) \equiv \theta(p; q) \mod 1, \text{ if } p' \equiv pn^2 \mod 2q \]

for $n$ integer. The explicit formula for $\theta(p; q)$ derived in the Appendix implies that

\[ e^{8\pi i \theta(p; q)} = -1, \]

i.e., that the allowed values for $\theta(p; q)$ are $\pm 1/8$ and $\pm 3/8$ (mod 1).

The representations of the modular group acting on the quantum Hilbert spaces $\mathcal{H}_{M_1/N}$ and $\mathcal{H}_{CS}$ can now be derived from the modular properties of the theta functions in (17),(18),(23),(24) and from the representation (39) acting on the “internal” indices. $\mathcal{H}_{M_1/N}$ carries the $p-$dimensional representation $R^{(q;p)}$ “dual” to the representation $R^{(q;p)}$ defined in (39):

\[ T_{N,M}^{(q;p)} = (-1)^{Npq} e^{2\pi i \frac{q^2}{p}N^2 - 2\pi i \theta(q;p)/3} \delta_{N,M} \]

\[ S_{N,M}^{(q;p)} = \frac{1}{\sqrt{p}} e^{2\pi i \frac{q}{p}NM} N, M = 0, 1, ..., p-1. \]

This representation is equivalent to the representation of the modular group obtained in [13],[15] by quantizing abelian Chern-Simons theory with fractional coupling constant. For $q \not\equiv n^2 \mod 2p$ its interpretation in terms of 2-dimensional conformal field theories is still obscure. We concentrate, however, on the representations carried by $\mathcal{H}_{CS} \equiv \mathcal{H}_{M_1/N}^{\pm}$. Note that $(S^{(q;p)})^2 = C$, with $(C)_{N,M} = \delta_{N,-M}$ being the “charge conjugation” matrix. Since $C$ commutes with the matrices in (43), $R^{(q;p)}$ decomposes into two representations $R^{(q;p)}_{\pm}$, even and odd under $C$:

\[ R^{(q;p)} = R^{(q;p)}_{+} \oplus R^{(q;p)}_{-}, \]

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where \( R_{\pm}^{(q;p)} \) is \( p/2 \pm 1 \) dimensional if \( p \) is even (i.e., if \( r \) is odd) and \( (p \pm 1)/2 \)-dimensional if \( p \) is odd (i.e., if \( r \) is even). Since \((S_{\pm}^{(q;p)})^2 = -1\), it is convenient to define a \( \tilde{R}_{\pm}^{(q;p)} \) by \( \tilde{S}_{\pm}^{(q;p)} = -iS_{\pm}^{(q;p)} \) and \( \tilde{T}_{\pm}^{(q;p)} = iT_{\pm}^{(q;p)} \) such that the charge conjugation matrix is equal to the identity. When \( q = 1 \) the “odd” representation \( \tilde{R}_{\pm}^{(1;p)} \) is the one associated to modular invariants of \( \hat{A}_1 \) current algebra (to the diagonal \( A_{p/2-1} \) series of level \( p/2 - 2 \) if \( p \) is even, to the \( D_{p+1} \) series of level \( 2p - 2 \) if \( p \) is odd.). The fusion rules associated to the “even” representation are not positive and integer-valued for generic \( p \), suggesting that the “even” quantization does not extend to Chern-Simons theories defined on higher genus surfaces [11],[12]. The same remains true for generic \( q \). This motivates the choice \( H_{CS} = H_{\hat{A}_1/N} \) carrying the modular representation \( R_{CS}^{(s/r)} \equiv \tilde{R}_{\pm}^{(q;p)} \) which has the following explicit matrix representation:

\[
\begin{align*}
T_{N,M}^{CS} &= i(-1)^Npq e^{2\pi i\frac{q}{p}N^2 - 2\pi i\theta(q;p)/3} \delta_{N,M} \\
S_{N,M}^{CS} &= \frac{1}{\sqrt{p}} \sin 2\pi i\frac{q}{p}NM \quad N, M = 1, ..., [(p - 1)/2],
\end{align*}
\]

(45)

where \([x]\) is the largest integer \( \leq x \). Therefore, the central charge \( c \) and the conformal dimensions \( h_N \) of a hypothetical 2-dimensional conformal field theory underlying the 3-dimensional \( SL(2,R) \) Chern-Simons theory should satisfy the equation

\[
h_N - c/24 = N^2/4k - \theta(q;p)/3 + 1/4 \mod 1. \quad (46)
\]

If such theory were unitary and had a unique identity operator corresponding to the conformal block labelled by \( \bar{N} \in \{1, 2, ..., [(p - 1)/2]\} \), one would have:

\[
\begin{align*}
c &= 2 - 12\bar{N}^2q/p + 8\theta(q;p) \mod 8 \\
h_N &= \frac{q}{2p}(N^2 - \bar{N}^2) \mod 1.
\end{align*}
\]

(47)

Since eqs. (29) and (33) express the multi-component Chern-Simons wave functions in terms of Kac-Wakimoto \( SL(2,R) \) characters with level \( m \) given by Eqs. (28) and (32), the modular representation \( R_{CS}^{(s/r)} \) is related, when \( pq \) is even, to the Kac-Wakimoto representation \( R_{KW}^{(m)} \) through the equation:

\[
R_{KW}^{(m)} = R^{(pq)} \otimes R_{CS}^{(s/r)}. \quad (48)
\]
When $pq$ is odd, a similar equation holds with the l.h.s. given by the modular representation acting on the Kac-Wakimoto characters which appear in (34) and which define a generalization of the $D$-series to the fractional level case. Eq. (48) encodes the relationship between Chern-Simons theories and Wess-Zumino-Witten models when $2k$ is fractional. For integer $2k$ (i.e., for $q = 1$), the left factor on the r.h.s. of (48) is trivial, and one obtains the well-established correspondence between Chern-Simons states and current algebra blocks. For fractional $2k$, Eq. (48) can be phrased by saying that the 2-dimensional theory underlying $SL(2, R)$ Chern-Simons theory is the “quotient” of $SL(2, R)$ current algebra by some yet unknown generalization of the gaussian model whose modular properties are given by $R_{(p;q)}$.

It was discovered in [23] that the Kac-Wakimoto characters are related by means of a certain projection to the Rocha-Caridi characters of the $c < 1$ conformal discrete series. This suggests that the modular representation $R_{CS}^{(s/r)}$ in (45) has something to do with the representation $R_{Vir}^{(r,s)}$ relative to the $(r,s)$ minimal model of Belavin-Polyakov-Zamolodchikov. This in fact turns out to be the case and one can establish, when $pq$ is even, the following equation:

$$R_{Vir}^{(r,s)} = R_{CS}^{(r/4s)} \otimes R_{CS}^{(s/r)} ,$$

where $r$ must be chosen odd. (This is always possible since $r$ and $s$ are coprime integers: however, the r.h.s. of the Eq.(49) is not symmetric under the interchange of $s$ and $r$ if one of them is even. The equation as written is not valid for $r$ even.) In order to understand how (49) comes about, let us consider the abelian Chern-Simons theory with even integer charge $K = pq$ whose algebra of observables $O_K$ is generated by the holonomies $A$ and $B$ around the non-trivial cycles of the torus $[[9],[10]]$:

$$AB = BAe^{2\pi i /K}.$$  

(50)

The quantum Hilbert space is K-dimensional and spanned by $SU(2)$ theta functions $\theta_{\lambda,K/2}$ (with $\lambda \in Z_K$) of level $K/2$. It carries the representation $R^{(1;K)}$ of the modular group. Now the crucial fact is that

$$O_K \approx O_{p/q} \times O_{q/p},$$

(51)
where \( O_{p/q} (O_{q/p}) \) is a 't Hooft algebra defined as in (50), with \( \tilde{A} \equiv A^p, \tilde{B} \equiv B^p \quad (\tilde{A} \equiv A^q, \tilde{B} \equiv B^q) \) and \( O_{p/q}, O_{q/p} \) commute among themselves. Therefore \( R^{(1,K)} \) factorizes:

\[
R^{(1,K)} = R^{(p;q)} \otimes R^{(q;p)}. \tag{52}
\]

The 't Hooft algebra \( O_K \) is invariant under a conjugation \( C_K, C_K : A \rightarrow A^{-1}, B \rightarrow B^{-1} \), which is in the commutant of the representation \( R^{(1,K)} \) and, therefore, can be represented by a diagonal matrix in the representation space of \( R^{(1,K)} \). In the holomorphic representation of \( O_K \) spanned by the theta-functions \( \theta_{\lambda,K/2} \), the operator \( C_K \) acts as follows:

\[
C_K : \lambda \rightarrow -\lambda. \tag{53}
\]

\( R^{(1,K)} \) decomposes into two modular representations, “even” and “odd” under \( C_K \) and the “odd” representation is associated to non-abelian current algebra (of level \( K-2 \)), as mentioned above. The new fact which occurs when \( K = pq \) is not prime is that, because of the decomposition (51), the group \( W_K \) of conjugations of \( O_K \) is enlarged to a four element group \( Z_2 \otimes Z_2 \), generated by the conjugations \( C_{p/q} \) and \( C_{q/p} \) of the algebras \( O_{p/q} \) and \( O_{q/p} \), with \( C_K = C_{p/q} C_{q/p} = C_{q/p} C_{p/q} \).

Decomposing \( \lambda \in Z_K \approx Z_p \otimes Z_q \) in terms of \( M \in Z_q \) and \( N \in Z_p \), \( \lambda \equiv Mp - Nq \), the action of the conjugation operators is:

\[
C_{p/q} : \lambda \rightarrow \bar{\lambda} \equiv Mp + Nq
\]

\[
C_{q/p} : \lambda \rightarrow -\bar{\lambda}. \tag{54}
\]

The existence of extra conjugation operators opens the possibility of considering several kinds of projections of the representation \( R^{(1,K)} \) according to the values of \( C_{p/q} \) and \( C_{q/p} \). Projecting to the odd sector of a single conjugation operator (let us say \( C_{q/p} \)), one obtains the modular representation of the Kac-Wakimoto of level \( m \) given by Eqs.(28),(32), as apparent from (48). Considering instead the subrepresentation which is completely anti-symmetric with respect to the whole conjugation group \( W_K \) of \( O_K \), one obtains the representation of the modular group relative to the \((r,s)\) minimal models, where \( r, s \) are defined through \( K/2 = pq/2 \equiv rs \). In fact, the completely anti-symmetric holomorphic wave functions are:

\[
\chi_{\lambda}(z;\tau) = \theta_{\lambda,K/2} - \theta_{\bar{\lambda},K/2} + \theta_{-\lambda,K/2} - \theta_{-\bar{\lambda},K/2}, \tag{55}
\]
with $\lambda \in \Gamma$, where $\Gamma \subset Z_K$ is any fundamental set for the action of $W_K$ on $Z_K$. $\chi_{\lambda}(0;\tau)$ are nothing but the numerators of the Rocha-Caridi characters of the completely degenerate representations of the $c < 1$ $(r,s)$ minimal models. Since $R_{CS}^{(s/r)}$ is the odd projection (with respect to $C_{q/p}$) of $R^{(q/p)}$, this establishes Eq. (49).

Loosely speaking, Eq. (49) tells us that the hypothetical 2-dimensional conformal field theory corresponding to $SL(2,R)$ Chern-Simons topological theory with fractional charge $k = s/r$ can be regarded as the “square root” of conformal $(r,s)$ minimal models. More precisely, Eq. (49) states that Chern-Simons theory with gauge group $SL(2,R) \times SL(2,R)$ and charges $(k,1/4k)$ with $k = s/r$ rational has the modular properties of the $(r,s)$ minimal model if $r$ is odd, and of the $(r/4,s)$ minimal model if $r$ is even (in which case it must be a multiple of 4).

The fact that the states of $SL(2,R)$ Chern-Simons theory are labeled by one of two indices appearing in minimal models formulas, seems to suggest that they correspond to Virasoro representations at the “boundary” of the Kac table. The conformal dimensions $\Delta_{N,0}$ with $N = 1, \ldots, s-1$ of the degenerate “boundary” representations of the minimal $(r,s)$ model satisfy the equation

$$\Delta_{N,0} - c/24 = rN^2/4s - 1/24, \quad (56)$$

which looks almost the same as the corresponding equation (46) for the Chern-Simons theory, were it not for the phase $\theta(q;p)/3$ not equal, for generic $(r,s)$, to $1/24$ . When considering the tensor product of two Chern-Simons representations, as in (49), the two phases $\theta(q;p)/3$ and $\theta(p;q)/3$ add up to produce the $1/24$ required by the Kac formula. Actually, degenerate Virasoro representations at the boundary of the Kac table are not closed under modular transformations, so that the disagreement between (46) and (56) is not truly surprising. However, considering also the apparently important role that boundary representations play in the context of string theory in $c < 1$ conformal backgrounds, we feel that the closeness between (56) and (46) may nevertheless be significant.

The algebraic data needed to reconstruct a 2-dimensional rational conformal field theory is not exhausted by the modular representation of the conformal blocks
of the genus one partition function. The representations of the modular group associated with conformal field theories, together with the braid matrices, should satisfy a set of rather restrictive identities, known as “the polynomial equations” \[24\]. In the Chern-Simons framework, the derivation of the braid matrices would require the solution of the theory on a space-manifold with the topology of a sphere with punctures, a problem which we did not address here. The polynomial equations imply however certain necessary, but not sufficient, conditions for the representation of the modular group of the conformal blocks of the identity operator on the torus. The most celebrated among these conditions, due to E. Verlinde \[25\], requires that the numbers \(N_{ijk}\), defined in terms of the modular matrix \(S\) as

\[N_{ijk} = \sum_n S_{im} S_{jn} S_{kn} S_{0n}, \tag{57}\]

be positive and integer, since they are interpreted as the fusion rules of the conformal field theory. From the expression for \(S_{CS}^{(s/r)}\), derivable from (43)-(45), one obtains the same (positive and integer) fusion rules as for the \(SU(2)\) Wess-Zumino model of level \(\ell = \frac{p-3}{2}\):

\[N_{CS}^{s} = N_{ij}^{SU(2)_{\frac{p-3}{2}}}. \tag{58}\]

(58) can be thought of as the “square-root” of the Virasoro minimal models fusion rules, in agreement with (49).

Modular invariance of 4-point correlation functions of primary operators on the sphere gives rise to another necessary condition for the \(h_p\’s\) in (47) to be the spectrum of dimensions of a conformal field theory \[26\]. This condition requires that

\[\langle \alpha_i \alpha_j \alpha_k \alpha_l \rangle^{N_{ijkl}} = \prod_r \alpha_r^{N_{ijkl,r}}, \tag{59}\]

where \(\alpha_i = \exp(2\pi i h_i)\) and

\[N_{ijkl,r} = N_{ijr} N_{klr} + N_{jkr} N_{ril} + N_{ikr} N_{rlj}. \]

One can verify that (59) is indeed satisfied by the dimensions in (47) and the fusion rules in (58).
The fact that the representations $R_{CS}^{(s/r)}$ of the modular group satisfy both Verlinde and Vafa conditions is quite remarkable, and makes $R_{CS}^{(s/r)}$ interesting from the point of view of conformal field theories regardless of the Chern-Simons framework which we adopted to derive them. $R_{CS}^{(s/r)}$ is invariant for $q \to q + 2p$ and representations $R_{CS}^{(s/r)}$ and $R_{CS}^{(s/r')}$ with the same $p$ for which

$$q' \equiv qn^2 \mod 2p,$$

with $n$ integer, are unitarily equivalent (as it follows from (51)). Thus, for each given $p$, as $q$ varies one obtains a finite number of inequivalent modular representations $R_{CS}^{(s/r)}$ one for each equivalence class in $\mathbb{Z}_{2p}/\sim$, where $\sim$ is the equivalence relation in $\mathbb{Z}_{2p}$ defined by (60).

The equivalence class of $q = 1$ corresponds to the usual modular representations relative to $\hat{A}_1$ current algebra with integer level. The $R_{CS}^{(s/r)}$’s for low values of $s$ (and $q \not\sim 1$) coincide with modular group representations associated with Wess-Zumino-Witten models on group manifolds other than $SU(2)$. For example, for $s = 3$, $R_{CS}^{(3/r)}$ is 2-dimensional and, according to the value of $r$, coincides either with the representation of the modular group associated to $SU(2)$ or with that associated with $(E_7)_1$. For $s = 4$, $R_{CS}^{(s/r)}$ is 3-dimensional, and as $r$ runs over the integers comprime with $s = 4$ modulo $16 = 4s$, one obtains the representations of the modular group of $SO(2N + 1)_1$ with $N = 2, 3, ..., 7$ and of the Ising model (for $r = 3$). But already for $s = 5$, the representations $R_{CS}^{(s/r)}$ for $q \not\sim 1$, do no longer appear to be equivalent to modular group representations coming from current algebras.

Unlike in the compact case, the geometric quantization of $SL(2, R)$ Chern-Simons theory does not provide the explicit functions of Teichmüller space which transform according to the representation $R_{CS}^{(s/r)}$ of the modular group and which are identifiable with the Virasoro characters of an underlying 2-dimensional conformal theory. Actually, although conditions (57) and (59) are both satisfied, this 2-dimensional “object”, at least for generic $k$, might belong to a class of quantum field theories more general than the conformal family — a class of theories for which concepts like holomorphic blocks, modular invariance and fusion rules would be still be meaningful. It is tempting to speculate that the 2-dimensional
theories associated to $SL(2, R)$ Chern-Simons theories are related to quantum deformations of $SL(2, R)$ Kac-Moody algebras with quantum parameter $q = e^{4\pi i k}$, to which several concepts of conformal field theories extend.

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Appendix A.

The following theta function

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau}$$  \hfill (A.1)

is a holomorphic modular form of weight 1/2 for the subgroup $\Gamma_\theta$ of $SL(2, \mathbb{Z})$ \cite{20}:

$$\theta(\alpha(\tau)) = e^{2\pi i \phi(\alpha)} \left( \frac{c\tau + d}{i} \right)^{1/2} \theta(\tau),$$  \hfill (A.2)

where $\alpha \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$, with $c > 0$, $\alpha(\tau) \equiv \frac{a\tau + b}{c\tau + d}$ and $\phi(\alpha)$ is a phase defining the multiplier system of $\theta(\tau)$. We will also need the weight 1/2 modular form

$$\tilde{\theta}(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi n^2 \tau}$$  \hfill (A.3)

whose multiplier system is $e^{2\pi i \tilde{\phi}(\alpha)} = e^{2\pi i \phi(\alpha \alpha^{-1} \circ t^{-1})}$, with $\alpha \in t^{-1} \Gamma_\theta t$. Let us consider the limit of (A.1) and (A.3) for

$$\tau \to p/q + i\epsilon, \quad \epsilon \to 0^+, \quad q > 0.$$  \hfill (A.4)

From the definitions (A.1), (A.3), one derives the asymptotic expressions:

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi \frac{p}{q} n^2} e^{-\pi n^2 \epsilon} e^{\tau \to p/q + i\epsilon} \approx e^{i\pi \frac{p}{q} n^2} \sum_{n^2 \leq \frac{1}{\sqrt{\epsilon}}} e^{i\pi \frac{p}{q} n^2} \approx \frac{1}{\sqrt{q}\epsilon} \frac{1}{\sqrt{q}} \sum_{n=0}^{n=q-1} e^{i\pi \frac{p}{q} n^2}$$  \hfill (A.5)
and

$$\tilde{\theta}(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi \frac{p}{q} n^2} e^{-\pi n^2 \epsilon \tau \rightarrow p/q + i\epsilon} \frac{1}{\sqrt{q \epsilon}} \frac{1}{\sqrt{\eta}} \sum_{n=0}^{n=q-1} (-1)^n e^{i\pi \frac{p}{q} n^2}. \quad (A.6)$$

When $pq$ is even one can find (not uniquely) an element $\alpha \in \Gamma_\theta$, such that

$$\alpha = \begin{pmatrix} a & b \\ q & -p \end{pmatrix}. \quad (A.7)$$

When $pq$ is odd a matrix $\alpha$ satisfying (A.7) can be found in the modular subgroup $t^{-1} \Gamma_\theta t$. In the limit (A.4) one has

$$e^{2\pi i \alpha(\tau)} \rightarrow 0$$

with $\alpha$ satisfying (A.7), so that either

$$\theta(\alpha(\tau)) \rightarrow 1$$

(for $pq$ even) or

$$\tilde{\theta}(\alpha(\tau)) \rightarrow 1$$

(for $pq$ odd). Therefore the modular properties (A.2) imply the asymptotic expression

$$\theta(\tau) \tau \rightarrow p/q + i\epsilon \approx \frac{1}{\sqrt{q \epsilon}} e^{-2\pi i \phi(\alpha)}, \quad (A.8)$$

if $pq$ is even, and

$$\tilde{\theta}(\tau) \tau \rightarrow p/q + i\epsilon \approx \frac{1}{\sqrt{q \epsilon}} e^{-2\pi i \tilde{\phi}(\alpha)}, \quad (A.9)$$

if $pq$ is odd. Comparing with (A.5) and (A.6) one concludes that

$$e^{2\pi i \theta(p;q)} = e^{-2\pi i \phi(\alpha)} \quad \text{(A.10)}$$

if $pq$ is even, and

$$e^{2\pi i \tilde{\theta}(p;q)} = e^{-2\pi i \tilde{\phi}(\alpha)} = e^{-2\pi i \phi(t \alpha t^{-1})} \quad \text{(A.11)}$$

if $pq$ is odd.
θ(τ) is related to the Dedekind function η(τ) by means of the Gauss identity

\[ \theta(\tau) = \frac{\eta^2(\tau+1)}{\eta(\tau + 1)}, \]  

which allows one to express the multiplier system \( e^{2\pi i \phi(\alpha)} \) in terms of the multiplier system of the Dedekind function. The latter involves the Dedekind symbol \( S(p; q) \) which is defined for relatively prime numbers \( p, q \) as follows:

\[ S(p; q) \equiv \sum_{n=1}^{q-1} ((n/q - 1/2))((np/q - 1/2)), \]  

with \( ((x)) \equiv x \text{ modulo integers} \) and \(-1/2 \leq ((x)) \leq 1/2\). One derives in this way

\[ e^{2\pi i \theta(p; q)} = e^{2\pi i (1/2S(p; q) - S(p+q; 2q))} \]  

if \( pq \) is even, and

\[ e^{2\pi \theta(p; q)} = e^{2\pi i \theta(p+q; q)} = e^{2\pi i (1/2S(p; q) - S(p; 2q))} \]  

if \( pq \) is odd. (Notice that if \( p, q \) are coprime \( p + q, q \) are coprime too.)
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