Coherence Trade-off relations in Multipartite systems

Chandrashekar Radhakrishnan\textsuperscript{1,2}, Manikandan Parthasarathy\textsuperscript{3}, Segar Jambulingam\textsuperscript{3} and Tim Byrnes\textsuperscript{1,2,4,5}.

\textsuperscript{1} New York University, 1555 Century Avenue, Pudong, Shanghai 200122, China
\textsuperscript{2} NYU-ECNU Institute of Physics at NYU Shanghai, 3663 Zhongshan Road North, Shanghai 200062, China
\textsuperscript{3} Department of Physics, Ramakrishna Mission Vivekananda College, Mylapore, Chennai 600004, India
\textsuperscript{4} Department of Physics, New York University, New York 10002, USA
\textsuperscript{5} National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

E-mail: cr2442@nyu.edu

Abstract. Quantum coherence is investigated using a new measure with metric properties and entropic nature and decomposed into local and intrinsic contributions. The trade-off relation between these contributions as well as their distribution properties are studied for simple tripartite systems and the more complex spin chain model. We find that the coherence changes its nature with respect to the parameters of the quantum state under investigation.

1. Introduction

Quantifying coherence in the framework of quantum information theory was introduced in [1] and further developed in [2, 3]. The fundamental definitions of incoherent states, incoherent operations and maximally coherent states and the set of properties a functional should satisfy to be considered as a coherence measure were discussed in [1]. All these developments, were necessitated due to the importance of quantum coherence in a wide variety of fields [4, 5, 6, 7]. Coherence is a basic quantum property like entanglement, but it differs from entanglement on the basis of two features viz

(i) Coherence is a basis dependent quantity.
(ii) Coherence can arise due to intra-qubit nature as well as inter-qubit correlations.

In this work we delve on the second feature and formulate measuring schemes to estimate the two different forms of coherence originating from the inter-qubit and intra-qubit nature, as well as the total coherence of the system. Further we investigate how coherence is shared between the subsystems in multipartite systems. Distribution of coherence is discussed through the introduction of monogamy type of relations for coherence. Finally, these concepts are applied to study the Heisenberg \textit{XXZ}-spin chain model.

Two different coherence measures namely the relative entropy of coherence and the $\ell_1$ norm were used in [1]. While the relative entropy is an entropic measure, the $\ell_1$-norm is geometric in nature with formal distance properties. A function $d$ over a set $X$ is called a distance if $\forall x, y \in X$ it satisfies the properties (i) $d(x, y) > 0 \ \forall x \neq y$ and $d(x, x) = 0$ (Positivity) (ii) $d(x, y) = d(y, x)$ (Symmetry). In addition if $d$ satisfies $d(x, y) + d(y, z) \geq d(x, z)$ i.e., the triangle...
inequality, then \( d \) is a metric over the space \( X \). We can immediately see that the relative entropy is not symmetric and hence is not a distance, but the \( \ell_1 \) measure is a distance since it satisfies the axioms described above. A new coherence measure with both entropic and geometric nature is introduced using the quantum version of the Jensen-Shannon divergence (QJSD):

\[
\mathcal{J}(\rho, \sigma) = \frac{1}{2}[S(\rho||\rho + \sigma)/2 + S(\sigma||\rho + \sigma)/2].
\]

The QJSD is a distance measure and bounded \( 0 \leq \mathcal{J} \leq 1 \), but it does not obey the triangle inequality. On the contrary, the square root of QJSD is a distance measure obeying the triangle inequality (metric) for all pure states [8, 9]. Though there is no proof that the mixed states obey triangle inequality, numerical studies up to five qubits [10] indicate its validity.

2. Local, Intrinsic and Total coherence: Definition and Properties

The basic properties a functional \( C \) should satisfy to be considered as a quantum coherence are as follows: (i) \( C \geq 0 \) and \( C \equiv 0 \) iff \( \rho \in \mathcal{I}^{(b)} \), where \( \mathcal{I}^{(b)} \) is the set of incoherent states. (ii) \( C(\rho) \) is invariant under unitary transformations, (iii) \( C(\rho) \) is monotonic under incoherent completely positive trace preserving (ICPTP) map, as well as under selective incoherent measurements on average. (iv) \( C(\rho) \) is convex i.e., does not increase under mixing of quantum states. In terms of a distance measure the coherence in an arbitrary state is the distance to the closest incoherent state

\[
C(\rho) \equiv \min_{\sigma \in \mathcal{I}^{(b)}} D(\rho, \sigma),
\]

In general the incoherent state \( \sigma = \sum_k p_k |b_k\rangle\langle b_k| \) where \( |b_k\rangle \) are the fixed basis and \( p_k \) are the probabilities. For a multipartite system, the incoherent state \( \sigma = \sum_k p_k \tau_{k,1}^{(b)} \otimes \cdots \otimes \tau_{k,N}^{(b)} \), where \( \tau_{k,n}^{(b)} \) is the incoherent state on the subsystem \( n \) i.e., \( \tau_{k,n}^{(b)} = \sum_k p_k |b_{k,n}\rangle\langle b_{k,n}| \).

At this juncture we need to notice in a bipartite system, there are two types of coherent states viz. \( |0\rangle - |1\rangle \langle 0| - |1\rangle \) and \( |0\rangle|0\rangle - |1\rangle|1\rangle \). In the former type the coherence is on each qubit and in the latter one it is collective in nature. A schematic sketch describing the local coherence, intrinsic coherence and the total coherence are given in Fig. 1(a). The intrinsic coherence can be computed by relaxing the basis constraint. The definition of intrinsic coherence is

\[
C_I(\rho) \equiv \min_{\sigma \in \mathcal{I}_S} D(\rho, \sigma_S),
\]

where \( \mathcal{I}_S \) is the set of incoherent state, and it is not in the same basis as \( \rho \). Since the contribution is independent of the basis choice, it is intrinsic in nature and it is equal to the entanglement in the system. The coherent that is localized on the individual qubits - local coherence can be computed using the relation

\[
C_L(\rho) \equiv D(\sigma_S^{\min}, \rho^d),
\]

where \( \sigma_S^{\min} \) and \( \rho^d \) are the minimum solutions of (3) and (2) respectively. Based on the triangle inequality we can notice that

\[
C \leq C_L + C_I.
\]

A very nice example corroborating the coherence decompositions is given by the ground state of the \( N = 2 \) Ising model described by the Hamiltonian

\[
H = \lambda \sigma_1^z \sigma_2^z + J(\sigma_1^x + \sigma_2^x) + \epsilon \lambda(\sigma_1^z + \sigma_2^z).
\]
Figure 1. Quantum coherence in multipartite system. (a) The total coherence $C$ with contributions from local coherence $C_L$ and the Intrinsic coherence $C_I$. (b) Definition of various coherences as the distance between the states.

The parameters $J$ and $\lambda$ are the coupling parameters and $\epsilon$ is a symmetry breaking term. Using the square root of QJSD defined via the expression

$$D(\rho, \sigma) = \sqrt{J(\rho, \sigma)} = \sqrt{S\left(\frac{\rho + \sigma}{2}\right) - \frac{S(\rho)}{2} - \frac{S(\sigma)}{2}},$$

we numerically estimate the values of $C_L$, $C_I$ and $C$ and the results are given in Fig 2 (a). We observe that the local coherence and the intrinsic coherence are complementary to each other. In the limit $J \ll \lambda$ the coherence is intrinsic in nature since the ground state approaches a Bell state in the $J = 0$ and $\epsilon \rightarrow 0$ limit. In the limit $J \ll \lambda$ the coherence is localized with each spin since for $\lambda = 0$ the ground state is $(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)$. Also we notice that the total coherence is less than the sum of the local and intrinsic coherence as suggested in Eq. (5).

3. Shareability and Distribution of Quantum coherence:

The shareability and distribution of quantum coherence in multipartite system is explored in this section. The tripartite coherence in a system $\rho_{123}$ may be decomposed in any one of the following forms

$$C_{123} \leq C_1 + C_2 + C_3 + C_{1:2:3}, \quad C_{123} \leq C_1 + C_2 + C_3 + C_{2:3} + C_{1:23},$$

$$C_{1:23} \geq C_{1:2} + C_{1:3}.$$ (7)

where $C_n$ is the local coherence of the $n^{th}$ subsystem obtained from the reduced density matrix $\rho_n$ and $C_{1:2:3}$ is the intrinsic coherence i.e., $C_I(\rho_{123})$. There are many such equivalent decompositions from which we can conclude that

$$C_{1:23} \simeq C_{2:3} + C_{1:23} \simeq C_{1:2} + C_{1:2:3} \simeq C_{1:3} + C_{1:3:2}.$$ (8)

Eq. (8) gives us a basic idea about how quantum coherence is shared between the qubits. To understand the distribution of coherence we naturally need to define the monogamy of coherence similar to the monogamy of entanglement [11, 12]. In a maximally coherent tripartite system $\rho_{123}$, the coherence between the system 1 and the bipartition 23 is related to the coherence between the subsystems 1 and 2 as well the coherence between 1 and 3 through the inequality

$$C_{1:23} \geq C_{1:2} + C_{1:3}.$$ (9)

If the inequality is obeyed the system is called monogamous and if not it is referred to as polygamous system. For a multipartite system the inequality is $C_{1:2...N} \geq \sum_{n=1}^{N} C_{1:n}$ and we define the measure

$$M = \sum_{n=2}^{N} C_{1:n} - C_{1:2...N}.$$ (10)
which is monogamous for $M \leq 0$ and polygamous $M > 0$. Thus from the monogamy concept we get to know whether the coherence is distributed in a bipartite fashion or in a multipartite fashion.

4. Application to Multipartite systems

4.1. Tripartite systems

The tripartite states can be divided into the GHZ and the $W$ class which are unrelated under local operations and classical communication [13]. (i) GHZ class: The coherence of the mixed GHZ states defined via the expression $\rho_{GHZ} = \frac{1}{2} I + \mu |GHZ\rangle \langle GHZ|$ where $|GHZ\rangle = \cos \phi |000\rangle + \sin \phi |111\rangle$ is plotted in Fig 2(b) for $\phi \in [0, 2\pi]$ and $0 \leq \mu \leq 1$. From the plot we observe that the local coherence $C_n$ and the bipartitions $C_{mn}$ is always zero. Hence we conclude that the intrinsic coherence in this system pertains to situations where all the three sites are involved. The total coherence is equal to $C = C_{1:2:3}$ and is also equal to the bipartition $C_{1:m:n}$. (ii) $W$ class: The generalized $W$ state is defined as $|W\rangle = \sin \theta \cos \phi |010\rangle + \sin \theta \sin \phi |010\rangle + \cos \theta |001\rangle$ with $0 \leq \phi < 2\pi$ and $0 \leq \theta \pi$. In Fig 2(c) we plot the coherences from the various contributions corresponding to $\theta = \pi/4$. We notice that the coherences $C_{1:3}$ and $C_{2:3}$ show complementary behaviour as the system oscillates between the maximally entangled Bell state between sites 13 $|\phi = n\pi\rangle$ and 23 $|\phi = (n+1/2)\pi\rangle$ and the third site is always decoupled. The intrinsic coherence $C_{1:2:3}$ is always a constant. Thus we notice that the $W$ states show trade-off relations analogous to the Ising model. The GHZ states which are coherently connected in a tripartite manner have a monogamous behaviour whereas the $W$ state which has a bipartite nature of coherence, exhibits polygamy over the entire range of the generalization parameters.

4.2. Spin Chain

The nature of quantum states in many-body systems can be understood from the study of quantum coherence in the systems. Towards this end we investigate the one dimensional Heisenberg XXZ model whose Hamiltonian is

$$H = J \sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z),$$

(11)

where $J$ is the nearest neighbor spin coupling and $\Delta$ is the anisotropy parameter. When $J > 0$, the system exhibits a phase transition from the ferromagnetic axial regime to the antiferromagnetic regime at $\Delta = -1$. We estimate the various types of coherences as shown in Fig 2(d) using exact diagonalization techniques. The ferromagnetic ground state is unique and so in the ferromagnetic ($\Delta < -1$) limit all the coherences vanish. In the limit $\delta \gg 1$ the

Figure 2. Coherence measured using QJSD for (a) $N = 2$ site Ising model with $\epsilon = 0.2$; (b) Werner GHZ state (c) $W$ state with $\theta = \pi/4$; (d) $N = 10$ site $XXZ$ spin chain model with $J = 1$. Inset: Monogamy of the $XXZ$ spin chain.
ground state is a superposition of the Neel states \( |0101\ldots01\rangle + |1010\ldots10\rangle / \sqrt{2} \) and coherence saturates to the Bell state value \( C = C_{1:2..N} \sim 0.56 \). The local coherence \( C_n \) is always zero due the spin flip symmetry and the intrinsic coherence between any two sites \( C_{1:n} \) decreases with the increase in distance between them. But at \( \Delta = -1 \) they all converge to the same value since phase transition increases the overall effect of coherence in the system.

The Heisenberg spin chain exhibits monogamous behaviour for \( \Delta > 2.9 \) and polygamous behaviour \(-1 < \Delta < 2.9 \) as shown in Fig 2(d) inset. In the \( \Delta \gg 1 \) limit when the ground state is a Néel state the coherence is entirely due the distribution \( C_{1:2..N} \). This is due to the fact that the Néel state is the same as the GHZ state. The dominant off-diagonal terms \( \sigma^x_n \sigma^x_{n+1} + \sigma^y_n \sigma^y_{n+1} = 2(\sigma^+_n \sigma^-_{n+1} + \sigma^-_n \sigma^+_{n+1}) \) tend to create coherence on nearby sites when \( \Delta \) is small and hence the system displays polygamous behaviour in this limit. By redistributing quantum coherence from the relatively local sites to the genuinely multipartite form the anisotropy parameter switches the spin chain between polygamous and monogamous nature.

5. Conclusions
Quantum coherence is quantified using a distance measure with metric properties which enables its decomposition into Local and Intrinsic contributions. The nature of coherence enables us to determine the type of state under consideration i.e., whether it is a genuinely multipartite state or more bipartite in nature. The coherence transitions from local to intrinsic nature in the Ising model. In the case of Heisenberg XXZ model, there is a cross over from the monogamous to the polygamous behaviour as the anisotropy parameter is varied. Application of these concepts to the fields of condensed matter physics and quantum metrology will further our understanding of quantum phase transition [14], frustration effects in magnetism [15] and may also lead to interferometric advantages [16, 17].

6. Acknowledgments
The work is supported by the Shanghai Research Challenge Fund, New York University Global seed Grants for Collaborative Research, National Natural Science Foundation of China Grant No. 61571301, and the Thousand Talents Program for Distinguished Young Scholars.

References
[1] Baumgratz T, Cramer M and Plenio M B 2014 Phys. Rev. Lett 113 140401
[2] Girolami D 2014 Phys. Rev. Lett 113 170401
[3] Streltsov A, Singh U, Dhar H S, Bera M N and Adesso G 2015 Phys. Rev. Lett 115 020403
[4] Narasimhachar V S and Gour G 2015 Nature Communications 6 7689
[5] Lostaglio M, Korzekwa K, Jennings D and Rudolph T 2015 Phys. Rev. X 5 021001
[6] Engel G S 2007 Nature 446 782
[7] Romero E et. al 2014 Nature Physics 10 676
[8] Brie J and Harremoes P 2009 Phys. Rev. A 79 052311
[9] Majtey A P, Lamberti P W and Prato D P 2005 Phys. Rev. A 72 052310
[10] Lamberti P W, Majtey A P, Borrás A, Casas M and Plastino A 2008 Phys. Rev. A 77 052311
[11] Coffman V, Kundu J and Wootters W K 2000 Phys. Rev. A 61 052306
[12] Osborne T J and Verstraete F 2006 Phys. Rev. Lett. 96 220503
[13] Diír W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[14] Buluta I and Nori F 2009 Science 326 108
[15] Ferraro A, García-Saez A and Acín A 2007 Phys. Rev. A 76 052321
[16] Higgins B L, Barry D W, Bartlett S D, Wiseman H M and Pyrde G J 2007 Nature 450 393
[17] Sahota J and Quesada N 2015 Phys. Rev. A 91 013808