Domination in graphs with bounded propagation: algorithms, formulations and hardness results

Ashkan Aazami
aaazami@uwaterloo.ca
Department of Combinatorics and Optimization
University of Waterloo, Waterloo, ON N2L 3G1, Canada

February 17, 2008

Abstract

We introduce a hierarchy of problems between the DOMINATING SET problem and the POWER DOMINATING SET (PDS) problem called the $\ell$-round power dominating set ($\ell$-round PDS, for short) problem. For $\ell = 1$, this is the DOMINATING SET problem, and for $\ell \geq n - 1$, this is the PDS problem; here $n$ denotes the number of nodes in the input graph. In PDS the goal is to find a minimum size set of nodes $S$ that power dominates all the nodes, where a node $v$ is power dominated if (1) $v$ is in $S$ or it has a neighbor in $S$, or (2) $v$ has a neighbor $u$ such that $u$ and all of its neighbors except $v$ are power dominated. Note that rule (1) is the same as for the DOMINATING SET problem, and that rule (2) is a type of propagation rule that applies iteratively. The $\ell$-round PDS problem has the same set of rules as PDS, except we apply rule (2) in “parallel” in at most $\ell - 1$ rounds. We prove that $\ell$-round PDS cannot be approximated better than $2^{\log^{1-\epsilon} n}$ even for $\ell = 4$ in general graphs. We provide a dynamic programming algorithm to solve $\ell$-round PDS optimally in polynomial time on graphs of bounded tree-width. We present a PTAS (polynomial time approximation scheme) for $\ell$-round PDS on planar graphs for $\ell = O\left(\frac{\log n}{\log \log n}\right)$. Finally, we give integer programming formulations for $\ell$-round PDS.

1 Introduction

The POWER DOMINATING SET problem (PDS, for short) is a covering problem in which the goal is to “power dominate” (cover) all the nodes of a given undirected graph $G$ by picking as few nodes as possible. There are two rules for power dominating the nodes; the first one has a “local” effect but the second one allows a type of indirect propagation. More precisely, given a set of nodes $S$, the set of nodes that are power dominated by $S$, denoted $\mathcal{P}(S)$, is obtained as follows

(R1) if node $v$ is in $S$, then $v$ and all of its neighbors are in $\mathcal{P}(S)$;

(R2) (propagation) if node $v$ is in $\mathcal{P}(S)$, one of its neighbors $w$ is not in $\mathcal{P}(S)$, and all other neighbors of $v$ are in $\mathcal{P}(S)$, then $w$ is inserted into $\mathcal{P}(S)$.

The PDS problem is to find a node-set $S$ of minimum size that power dominates all the nodes (i.e., find $S \subseteq V$ with $|S|$ minimum such that $\mathcal{P}(S) = V$). For example, consider the planar graph $G = (V,E)$ in Figure 1. The graph is obtained by taking the union of $m$ paths on $k + 1$ nodes that all meet at a common node $v$; note that $|V| = k \cdot m + 1$. It is easy to check that the size of a minimum PDS in $G$ is
by taking \( S = \{v\} \) we get \( \mathcal{P}(S) = V \). In more detail, by applying (R1) we power dominate \( v \) and all of its neighbors in the set \( X_1 \); after that, each node in the set \( X_1 \) has exactly one neighbor that is not power dominated yet, namely, its neighbor in \( X_2 \); thus, we can sequentially apply propagation rule (R2) to each node in \( X_1 \) (in any order) to power dominate all the nodes in \( X_2 \); continuing in this way, all the nodes will be power dominated eventually.

The PDS problem arose in the context of monitoring electric power networks. A power network contains a set of nodes and a set of edges connecting the nodes. A power network also contains a set of generators, which supply power, and a set of loads, where the power is directed to. In order to monitor a power network we need to measure all the state variables of the network by placing measurement devices. A Phasor Measurement Unit (PMU) is a measurement device placed on a node that has the ability to measure the voltage of the node and the current phase of the edges connected to the node; PMUs are expensive devices. The goal is to install the minimum number of PMUs such that the whole system can be monitored. These units have the capability of monitoring remote elements via propagation (as in Rule 2); see Brueni [1], Baldwin et al. [2], and Mili et al. [3]. Most measurement systems require one measurement device per node, but this does not apply to PMUs; hence, PMUs give a huge advantage. To see this in more detail consider a power network \( G = (V, E) \), and assume that the resistances of the edges in the power network are known, and the goal is to measure the voltages of all nodes. For simplicity, assume that there are no generators and loads. By placing a PMU at node \( v \) we can measure the voltage of \( v \) and the electrical current on each edge incident to \( v \). Next, by using Ohm’s law we can compute the voltage of any node in the neighborhood of \( v \) (Rule 1). Now assume that the voltage on \( v \) and all of its neighbors except \( w \) is known. By applying ohm’s law we can compute the current on the edges incident to \( v \) except \( \{v, w\} \). Next by using Kirchhoff’s law we compute the current on the edge \( \{v, w\} \). Finally, applying Ohm’s law on the edge \( \{v, w\} \) gives us the voltage of \( w \) (Rule 2).

PMUs are used to monitor large system stability and to give warnings of system-wide failures. PMUs have become increasingly popular for monitoring power networks, and have been installed by several electricity companies since 1988 [4, 5]. For example, the USA Western States Coordinating Council (WSCC) had installed around 34 PMUs by 1999 [5]. By now, several hundred PMUs have been installed world wide [6]. Some researchers in electrical engineering regard PMUs as the most important device for the future of power systems [7].

The PDS problem is \( \text{NP} \)-hard even when the input graph is bipartite [8]. For further references for the PDS problem please see [8, 9, 10, 11, 12, 13, 14, 15]. PDS is a generalization of the \textbf{Dominating Set} problem. A \textbf{Dominating Set} of a graph \( G = (V, E) \) is a set of nodes \( S \) such that every node in the graph is either in \( S \) or has a neighbor in \( S \). The problem of finding a \textbf{Dominating Set} of minimum size in a given graph \( G \) has been studied extensively in the past 20 years, see the books by Haynes et al. [16, 17]. The \textbf{Dominating Set} problem is a well-known \( \text{NP} \)-hard problem [18].
simple greedy algorithm achieves a logarithmic approximation guarantee, [19], and modulo the \( P \neq NP \) conjecture, no polynomial time algorithm gives a better approximation guarantee, [20, 21, 22].

In this paper we introduce a hierarchy of problems between DOMINATING SET and PDS, by adding a parameter \( \ell \) to PDS which restricts the number of "parallel" rounds of propagation that can be applied. The rules are the same as PDS, except we try to apply the propagation rule in parallel as much as possible. In the first round we apply the rule (R1) to all the nodes in \( S \), and for the rest of the rounds we only consider "parallel" application of the propagation rule (R2). In every "parallel" round we power dominate all the new nodes that can be power dominated by applying the propagation rule to all of the nodes that are power dominated in the previous "parallel" rounds. Given a parameter \( \ell \), the \( \ell \)-round PDS problem is the problem in which we want to power dominate all of the nodes in at most \( \ell \) parallel rounds. Clearly, the \( \ell \)-round PDS problem for \( \ell = 1 \) is exactly the DOMINATING SET problem, and for a graph \( G \) with \( n \) nodes the \( \ell \)-round PDS problem for \( \ell \geq n - 1 \) is exactly the PDS problem. The notion of parallel propagation comes from the fact that changes in the electrical network propagate in parallel and not sequentially. A feasible solution for the PDS problem provides a plan for installing monitoring devices to monitor the whole power network, but it does not provide any guarantees on the time-lag between a fault in the network and its detection. Deducing information through a parallel round of propagation takes one unit of time and in some applications we want to detect a failure in the network after at most \( \ell \) units of time. The addition of the parameter \( \ell \) achieves this time constraint.

The practical motivation for the \( \ell \)-round PDS problem has been explained above. In addition, there are some theoretical motivations. Although the PDS problem has been studied since 1993, there are very few algorithmic results (including approximation algorithms). The \( \ell \)-round PDS problem serves as a unified model for studying the PDS problem and the DOMINATING SET problem. The introduction of the parameter \( \ell \) allows us to examine the complexity of the problem in terms of \( \ell \): how does the threshold for the hardness of approximation vary in terms of \( \ell \)? The hardness threshold is logarithmic for \( \ell = 1 \) (the DOMINATING SET problem) and it is \( \Omega(2^{\log^{1-\epsilon} n}) \) for \( \ell = n - 1 \) (the PDS problem). Is the latter hardness threshold valid for constant \( \ell \)? Moreover, for planar graphs there is extensive recent literature on PTASs (polynomial time approximation schemes) for the DOMINATING SET problem and its variants, but these results do not apply to the PDS problem. A major open question in the area is whether there exists a PTAS for the PDS problem on planar graphs. One avenue that may lead to advances on this question is to design a PTAS for the \( \ell \)-round PDS problem on planar graphs, for small values of \( \ell \). Integer programming formulations for the PDS problem have been studied, and we give a new formulation in the last part of this paper. Our formulation is based on the notion of parallel propagation in \( \ell \) rounds. We first give a formulation for the \( \ell \)-round PDS problem, and then modify it to get another formulation for the PDS problem.

### 1.1 Our main results

We initiate the study of a natural extension of the PDS problem and prove the following main results.

- For general graphs, we present a reduction from the MinRep problem to the \( \ell \)-round PDS problem which shows that \( \ell \)-round PDS for \( \ell \geq 4 \) cannot be approximated better than \( 2^{\log^{1-\epsilon} n} \), unless \( NP \subseteq DTIME(n^{\text{polylog}(n)}) \). We use a reduction similar to one that has been used to prove the same hardness result for the PDS and the directed PDS problems in a paper jointly authored with M. Stilp [13].

- We provide a dynamic programming algorithm to solve the \( \ell \)-round PDS problem optimally in polynomial time on graphs of bounded tree-width. This dynamic programming algorithm is
based on our reformulation for the $\ell$-round PDS problem. This reformulation is an extension of the one introduced by Guo et al. [10] for PDS. Guo et al. [10] gave a new formulation for PDS in terms of “valid orientation” of the edges; they use this formulation to design a dynamic programming algorithm to solve the PDS problem optimally in linear time on graphs of bounded tree-width.

- We focus on planar graphs, and give a PTAS for $\ell$-round PDS for $\ell = O\left(\frac{\log n}{\log \log n}\right)$. Baker’s PTAS [23] for the DOMINATING SET problem on planar graphs is a special case of our result with $\ell = 1$, and no similar result of this type was previously known for $\ell > 1$. We also show that the $\ell$-round PDS problem in planar graphs is $\text{NP}$-hard for all $\ell \geq 1$. Note that our PTAS does not apply to PDS in general, because the running time is super-polynomial for $\ell = \omega\left(\frac{\log n}{\log \log n}\right)$.

- Finally we study integer programming formulations for $\ell$-round PDS.

Here is a brief discussion on the relation between this paper and some previous joint work with M. Stilp on the PDS problem [13]. Although the results in [13] do not imply any of the results in this paper, there are two topics that uses similar methods: (1) The hardness result for $\ell$-round PDS (Theorem 5.1) extends the construction used to prove the hardness of directed PDS in [13], (2) The dynamic programming algorithm for $\ell$-round PDS (Section 3) and the algorithm for directed PDS in [13] are both based on reformulation of the problems that extend the methods of Guo et al. [10].

## 2 Preliminaries

Most of the graphs that we consider here are undirected. Given a graph $G = (V, E)$, we denote the number of nodes in the graph by $n$. Sometimes the graphs that we consider have some directed edges in addition to undirected edges. Given such a graph, we denote by $d^-(v)$ and $d^+(v)$ the number of directed edges with $v$ as the head and the tail respectively. Let $\hat{G}$ denote the underlying undirected graph obtained from $G$ by ignoring the direction of the directed edges. Then the closed neighborhood of a node $u$ in $G$ is defined by $N[u] = \left\{ v : \{u, v\} \in E(\hat{G}) \right\} \cup \{u\}$. Now we define the “parallel” propagation rule formally.

**Definition 2.1** Given a graph $G = (V, E)$ and a subset of nodes $S \subseteq V$, the set of nodes that can be power dominated by applying at most $k$ rounds of parallel propagation, denoted by $\mathcal{P}^k(S)$, is defined recursively as follows:

$$
\mathcal{P}^k(S) = \begin{cases} 
\bigcup_{v \in S} N[v] & k = 1 \\
\mathcal{P}^{k-1}(S) \cup \left\{ v : (u, v) \in E, N[u] \setminus \{v\} \subseteq \mathcal{P}^{k-1}(S) \right\} & k \geq 2
\end{cases}
$$

We can now define the $\ell$-round PDS problem formally.

**Definition 2.2** ($\ell$-round PDS) Given a parameter $\ell$, the $\ell$-round PDS problem is the problem in which we are given a graph $G = (V, E)$ and the goal is to find a minimum size subset of nodes $S \subseteq V$, such that $\mathcal{P}^\ell(S) = V$.

Given a graph $G$ and a parameter $\ell$, we denote by $\text{Opt}_\ell(G)$ the size of the optimal solution for the $\ell$-round PDS problem. It is easy to see that the size of an optimal solution for $\ell$-round PDS does not increase by increasing the value of the parameter $\ell$; to see this, consider an optimal solution $S^*$ for $\ell$-round PDS and note that it is also a feasible solution for $(\ell + 1)$-round PDS. This proves the following property of the $\ell$-round PDS problem.
**Proposition 2.3** Let \( G = (V, E) \) be a graph with \( n \) nodes, then for any parameter \( 1 \leq \ell \leq \ell' \leq n \) we have \( \text{opt}_{\ell'}(G) \leq \text{opt}_{\ell}(G) \).

The \( \ell \)-round PDS problem is very different from the Dominating Set problem, even for \( \ell = O(1) \). The following result shows that the size of an optimal solution for the Dominating Set problem can be much bigger than the size of an optimal solution for the 2-round PDS problem.

**Proposition 2.4** Let \( \ell \geq 1 \) be a given parameter, the ratio between the size of the optimal solution for \( \ell \)-round PDS and \((\ell+1)\)-round PDS can be \( \Theta\left(\frac{n}{\ell}\right) \) even on planar graphs.

**Proof:** Consider the graph \( G \) that is obtained by taking \( m \) paths, \( P_1, \ldots, P_m \), each of length \( \ell + 1 \) that share a common node \( v \) as shown in Figure 1 (with \( k = \ell + 1 \)). Note that \( G \) has \( n = m \cdot (\ell + 1) + 1 \) nodes. It is easy to check that \( v \) can power dominate the entire graph in exactly \( \ell + 1 \) parallel rounds; in the first round \( v \) and \( X_1 \) are power dominated, in the second round \( X_2 \) is power dominated, and so on. This shows that \( \text{opt}_{\ell+1}(G) = 1 \). Next we prove that the size of the optimal solution for \( \ell \)-round PDS is at least \( m = \frac{n-1}{\ell+1} = O\left(\frac{n}{\ell}\right) \). For the sake of contradiction assume that it is not, so there is an optimal solution \( S^* \) such that \( |S^*| < m \). This implies that there is a path \( P_i \) that has no node in \( S^* \) except possibly the node \( v \). This means that the last node \( u_i \) on the path \( P_i \) (in \( X_k \)) cannot be power dominated since there are only \( \ell \) parallel rounds and the distance between \( v \) and \( u_i \) is \( \ell + 1 \). This contradicts the fact that \( S^* \) is a feasible solution for \( \ell \)-round PDS, so \( \text{opt}_{\ell}(G) \geq m \). Also note that by taking \( S \) to be the set of middle nodes of the paths \( P_1, \ldots, P_m \) we can power dominate all of the nodes in \( \ell \) rounds (in fact, in \( \frac{\ell}{2} \) rounds). This shows that \( \text{opt}_{\ell}(G) = m \), and therefore the ratio between \( \text{opt}_{\ell}(G) \) and \( \text{opt}_{\ell+1}(G) \) is \( m = O\left(\frac{n}{\ell}\right) \). \( \square \)

The \( \ell \)-round PDS problem can be generalized as follows. Given a subset \( V' \) of the nodes, find a minimum size set of nodes that power dominates the nodes in \( V' \) in at most \( \ell \) parallel rounds. We will use this generalization in our PTAS in Section 4.

**Definition 2.5 (Generalized \( \ell \)-round PDS) An instance of the generalized \( \ell \)-round PDS problem is given by a pair \((G, V')\), where \( G = (V, E) \) is an undirected graph and \( V' \subseteq V \), and the goal is to find a minimum size set of nodes \( S \) such that \( V' \subseteq P^\ell(S) \).**

### 3 Dynamic programming for \( \ell \)-round PDS

In this section we provide a dynamic programming algorithm to solve the generalized \( \ell \)-round PDS problem optimally. Our dynamic programming algorithm is based on tree decompositions [24]. The running time of this algorithm is exponential both in the tree-width of the given graph and the logarithm of the parameter \( \ell \). Therefore, we obtain a polynomial time algorithm for graphs of bounded tree-width. Our dynamic programming algorithm is based on our reformulation for the \( \ell \)-round PDS problem. This reformulation is an extension of a new formulation for PDS introduced by Guo et al. [10]. Our dynamic programming is similar to the dynamic programming for the Dominating Set problem given in [25] and the dynamic programming for PDS given in [10].

**Definition 3.1** A tree decomposition of a graph \( G = (V, E) \) is a pair \( \langle \{X_i \subseteq V \mid i \in I\}, T \rangle \), where \( T = (I, F) \) is a tree, satisfying the following properties:

1. \( \bigcup_{i \in I} X_i = V \);
2. For every edge \(\{u,v\} \in E\) there exists an \(i \in I\) such that \(\{u,v\} \subseteq X_i\);

3. For all \(i, j, k \in I\), if \(j\) is on the unique path from \(i\) to \(k\) in \(T\), then we have: \(X_i \cap X_k \subseteq X_j\).

The width of \(\langle\{X_i | i \in I\}, T\rangle\) is defined as \(\max_{i \in I}|X_i| - 1\). The tree-width of \(G\) is defined as the minimum width over all tree decompositions. The nodes of \(T\) are called \(T\)-nodes, and each \(X_i\) is called a bag.

For designing a dynamic programming algorithm based on tree decomposition it is usually easier to work with nice tree decompositions [26] that have a simple structure defined as follows.

Definition 3.2 A tree decomposition \(\langle\{X_i | i \in I\}, T\rangle\) is called a nice tree decomposition if the following conditions are satisfied:

1. \(T\) is rooted, at \(r\), and every node of \(T\) has at most 2 children;
2. If a node \(i \in I\) has two children \(j\) and \(k\), then \(X_i = X_j = X_k\) (in this case \(i\) is called a Join Node);
3. If a node \(i\) has one child \(j\), then either of the following holds:
   
   (a) \(X_j \subset X_i\) and \(|X_i \setminus X_j| = 1\) (in this case \(i\) is called an Insert Node).
   
   (b) \(X_i \subset X_j\) and \(|X_j \setminus X_i| = 1\) (in this case \(i\) is called a Forget Node).

A tree decomposition can be transformed into a nice tree decomposition [26]. Therefore, to design a dynamic programming algorithm based on tree decomposition, we can assume that we are given a nice tree decomposition of the input graph.

Lemma 3.3 ([26]) Given a tree decomposition of a graph \(G\) that has width \(k\) and \(O(n)\) nodes, where \(n\) is the number of nodes of \(G\), a nice tree decomposition of \(G\) that also has width \(k\) and \(O(n)\) nodes can be found in time \(O(n)\).

Now we provide a reformulation, called timed-orientation, for (generalized) \(\ell\)-round PDS in terms of the orientation of the edges and labeling of the nodes. This reformulation makes it possible to design a dynamic programming algorithm to solve the (generalized) \(\ell\)-round PDS problem optimally in polynomial time on graphs of bounded tree-width. In this formulation we orient the edges in order to show the direction that the propagation rule is applied; if the node \(w\) is power dominated by applying the propagation rule on \(v\) then we orient the edge \(\{v, w\}\) from \(v\) toward \(w\). Moreover, if \(w\) is not power dominated through \(v\) we will leave the edge \(\{v, w\}\) undirected. In order to keep track of the round in which every node is power dominated, we introduce a time vector \(\{t_v : v \in V\}\). The round in which \(v\) is power dominated is denoted by \(t_v\) and it can take any value from the set \(\{0, \ldots, \ell\} \cup \{+\infty\}\); the \(+\infty\) is used to denote a node that will not be power dominated (this is needed for generalized \(\ell\)-round PDS). Consider a directed edge \((u, v)\). There are two cases: either \(t_v = 1\) or \(t_v > 1\). In the former case \(u\) should be in the (optimal) solution, since \(t_v = 1\) means \(v\) is power dominated at the first round by applying the domination rule. In the latter case \(v\) and all of its neighbors except \(v\) (i.e \(N[u] \setminus v\)) should be power dominated before we can apply the propagation rule. In fact, \(v\) will be power dominated right after the round where the last node in \(N[u] \setminus v\) is power dominated (see the property P5 in the following definition).
**Definition 3.4** (valid timed-orientation) Let \( \langle G, V' \rangle \) be an instance of the generalized \( l \)-round PDS problem where \( G = (V, E) \) is a graph and \( V' \subseteq V \) is a subset of nodes. A valid timed-orientation for \( \langle G, V' \rangle \) is a graph \( G_o = (V, E_d \cup E_u) \) such that for every \( \{u, v\} \in E \) either there is a directed edge \( (u, v) \) or \( (v, u) \) in \( E_d \) or an undirected edge \( \{u, v\} \) in \( E_u \), together with the time vector \( \{t_v : v \in V\} \) (possible values for \( t_v \) are \( \{0, 1, \ldots, \ell\} \cup \{+\infty\} \)) that satisfies the following properties:

- (P1) \( \forall v \in V' : 0 \leq t_v \leq \ell \),
- (P2) \( \forall v \in V : 1 \leq t_v \leq \ell \Rightarrow d^-(v) = 1 \),
- (P3) \( \forall v \in V : t_v = +\infty \Rightarrow d^-(v) = d^+(v) = 0 \),
- (P4) \( \forall v \in V : t_v = 0 \Rightarrow d^-(v) = 0 \),
- (P5) \( \forall (u, v) \in E_d : t_v = \begin{cases} 1 & \text{if } t_u = 0 \\ 1 + \max\{t_w : w \in N[u] - v\} & \text{otherwise.}\end{cases} \)

The set \( O = \{v \in V : t_v = 0\} \) is called the origin of the valid timed-orientation.

Now we show that the existence of a valid timed-orientation with \( S \) as the origin is equivalent to having \( S \) as a feasible solution to the \( \ell \)-round PDS problem.

**Lemma 3.5** Let \( G = (V, E) \) be an undirected graph, and \( \langle G, V' \rangle \) be an instance of the generalized \( \ell \)-round PDS problem. Then \( S \subseteq V \) is the origin of a valid timed-orientation if and only if \( V' \subseteq P^l(S) \).

**Proof:** Suppose \( S \subseteq V \) power dominates \( V' \) in at most \( \ell \) parallel rounds. Now we construct a valid timed-orientation with \( S \) as the origin. We orient the edges in the same way as the propagation rule applies. Consider an edge \( \{u, v\} \) and assume that \( v \) is power dominated by applying power domination rules to \( u \). Then we orient the edge \( \{u, v\} \) from \( u \) toward \( v \). Let \( E_d \) be the set of oriented edges and let \( E_u \) be the rest of the edges. This defines the graph \( G_o = (V, E_d \cup E_u) \). Now we define the time vector. First, for any node \( v \) that is not power dominated, set \( t_v = +\infty \). Second, for every \( v \in S \), set \( t_v = 0 \). Finally, define \( t_v \) for the rest of the nodes as \( t_v = \min\{r \geq 1 : v \in P^r(S)\} \). It is straightforward to check that the above orientation and the time vector satisfy all of the properties P1 to P5 in the definition of the valid timed-orientation. Therefore, there is a valid timed-orientation with \( S \) as its origin.

Now assume that \( G \) has a valid timed-orientation with \( S \) as the origin. So there is an oriented graph \( G_o = (V, E_d \cup E_u) \) and a time vector \( \{t_v : v \in V\} \) that satisfy the properties P1 to P5 of the valid timed-orientation. Define \( V_r = \{v \in V : 0 \leq t_v \leq r\} \). We now prove by induction that \( V_r \subseteq P^r(S) \) which implies that \( V_l \subseteq P^l(S) \). First note that this proves the lemma, since any node \( v \in V' \) has \( 0 \leq t_v \leq \ell \) by the property P1. Note that any node \( v \) with \( t_v = 0 \) is in \( S \), so it is immediately power dominated. Also any node \( v \) with \( t_v \) is power dominated (by the domination rule) in the first round, since by P2 it has an incoming directed edge, say \( (w, v) \), and by P5 the node \( w \) is in \( S \). Hence, \( V_l \subseteq P^1(S) \). Now assume that the induction hypothesis holds for \( r = k \) (where \( k \geq 1 \)), that is \( V_k \subseteq P^k(S) \). We will show that it also holds for \( r = k + 1 \). Consider a node \( v \) such that \( t_v = k + 1 \). By P2 it has exactly one incoming edge, say \( (u, v) \in E_d \). The node \( u \) has \( t_u \geq 1 \) (since if \( t_u = 0 \) then P5 implies \( t_v = 1 \)), so by the property P5 any \( w \in N[u] - v \) has \( t_w \leq k \). Therefore, \( N[u] \setminus \{v\} \subseteq P^k(S) \). This is correct for any \( v \) with \( t_v = k + 1 \), so \( X = \{v \in V : t_v = k + 1\} \subseteq \{v : (u, v) \in E, N[u] \setminus \{v\} \subseteq P^k(S)\} \). Therefore, by the definition of the parallel propagation \( X \) can be power dominated in one parallel round, so we have \( V_{k+1} = V_k \cup X \subseteq P^{k+1}(S) \). This proves the induction step and completes the proof. \( \square \)
Theorem 3.6 Given a pair \( \langle G, V' \rangle \) where \( G = (V, E) \) is a graph with tree-width \( k \) and \( V' \subseteq V \), a minimum size set \( S \subseteq V \) such that \( V' \subseteq \mathcal{P}^k(S) \) can be obtained in time \( O(e^{m_e + k \log \ell} \cdot |V|) \), for some global constant \( c \), where \( m_e \) is the maximum number of edges that a bag can have.

Proof: Please refer to Appendix A.2 for the details of the dynamic programming.

In general graphs the number of edges in a bag is at most \( \binom{k+1}{2} \) but in planar graphs, it is linear in the number of nodes, where \( k \) denotes the tree-width. Therefore, the above theorem implies the following corollary.

Corollary 3.7 Given a pair \( \langle G, V' \rangle \) where \( G = (V, E) \) is a planar graph with tree-width \( k \) and \( V' \subseteq V \), a minimum size set \( S \subseteq V \) such that \( V' \subseteq \mathcal{P}^k(S) \) can be obtained in time \( O(e^{k \log \ell} \cdot |V|) \), for some global constant \( c \).

4 \( \ell \)-round PDS on planar graphs

In this section we present a PTAS (polynomial time approximation scheme\(^1\)) for the \( \ell \)-round PDS problem on planar graphs when \( \ell = O\left(\frac{\log n}{\log \log n}\right) \). Baker’s PTAS [23] for the DOMINATING SET problem on planar graphs is a special case of our result with \( \ell = 1 \), but there are no previous results of this type for \( \ell > 1 \). Our PTAS works in the same fashion as Baker’s PTAS, but our analysis and proofs are novel contributions of this paper. Demaine and Hajiaghayi [27] recently used bidimensionarity theory to design PTASs for some variants of the DOMINATING SET problem on planar graphs (such as CONNECTED DOMINATING SET), but their methods do not apply to the \( \ell \)-round PDS problem because the relevant parameter is not bidimensional. We also have the following NP-hardness result for \( \ell \)-round PDS on planar graphs.

Proposition 4.1 For any \( \ell \geq 1 \) the \( \ell \)-round PDS problem is an NP-hard problem even on planar graphs.

Proof: We use a modification of the reduction that has been used to prove that the PDS problem on planar graphs is NP-hard [9, 10]. Please refer to Appendix A.1 for more details.

Now we describe our PTAS for the \( \ell \)-round PDS problem on planar graphs, when \( \ell \) is small. First we provide some useful definitions and notations. Consider an embedding of a planar graph \( G \). We define the nodes at level \( i \) denoted by \( L_i \) as follows [23]. Let \( L_1 \) be the set of nodes on the exterior face in the given embedding of \( G \). For \( i > 1 \), the set \( L_i \) is defined as the set of nodes on the exterior face of the graph induced on \( V \setminus \bigcup_{j=i-1}^{i-1} L_j \). We denote by \( L(a, b) = \bigcup_{i=a}^{b} L_i \) the set of nodes at levels \( a \) through \( b \). A planar graph is called \( k \)-outerplanar if it has an embedding where no node is at level greater than \( k \). For example, consider the graph in Figure 2. Clearly, the graph is a 2-outerplanar graph. The set \( L_1 = \{u_1, u_2, \ldots, u_8\} \) is the set of nodes at level 1 and the set \( L_2 = \{v_1, v_2, \ldots, v_8\} \) is the set of nodes at level 2. Given a graph \( G = (V, E) \) and \( V' \subseteq V \), we denote the subgraph induced on \( V' \) by \( G[V'] \).

Before describing our PTAS, let us look at Baker’s PTAS [23] for the DOMINATING SET problem on planar graphs. Given the parameter \( \epsilon = \frac{1}{k} \), Baker’s algorithm finds a feasible solution with size within \((1 + \epsilon)\) times the optimal value. This algorithm considers \( k \) different decompositions \( D_1, \ldots, D_k \)

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\(^1\)A polynomial time approximation scheme (PTAS) is an algorithm that given any fixed \( \epsilon > 0 \) provides a solution with cost within \((1 + \epsilon)\) times the optimal value in polynomial time.
of the nodes of $G$ and then finds a feasible solution for each of them. The $i$th decomposition consists of blocks with $k + 1$ consecutive levels. The $j$th block in decomposition $D_i$ contains nodes of levels $jk + i$ through $(j + 1)k + i$ (note that each $D_i$ is obtained from $D_1$ by shifting the levels). Also note that every two consecutive blocks in a given decomposition share a common level. Next, the algorithm solves the DOMINATING SET problem optimally on each block of $D_i$. This is possible since each block is $(k + 1)$-outerplanar; hence, it has tree-width at most $3(k + 1) - 1$ and the DOMINATING SET problem can be solved optimally by dynamic programming [23] (the result in Section 3 for $\ell = 1$ also shows this fact). Then, it takes the union of the optimal solutions for the blocks in $D_i$ to obtain a feasible solution for the DOMINATING SET problem in the graph $G$. Let $S_i$ denote this feasible solution. Finally, the algorithm outputs the solution that has minimum size, i.e. $\min_{i=1,\ldots,k} |S_i|$, among all $k$ decompositions. It is not hard to argue that the size of this solution is within $\frac{k + 1}{k} = 1 + \epsilon$ times the optimal value. The key property that is needed for this argument is the fact that consecutive blocks share a common level.

Now let us describe our PTAS informally. Consider a parameter $k$, that is a function of the parameter $\ell$ and the approximation factor $(1 + \epsilon)$; $k$ will be defined later in the formal description of our algorithm. We decompose the graph in $k$ different ways, $D_1, \ldots, D_k$. In each decomposition the graph is decomposed into blocks of $k + 4\ell - 2$ consecutive levels. The $j$th block in $D_i$ is defined as $B_{i,j} = L(jk + i - 2\ell + 1, (j + 1)k + i - 1 + 2\ell - 1)$. We denote the $k$ middle levels of $B_{i,j}$ by $C_{i,j} = L(jk + i, (j + 1)k + i - 1)$. In our PTAS, for each decomposition $D_i$, we optimally solve instances of the generalized $\ell$-round PDS problem ($\mathcal{I}_{i,j} = \langle G[B_{i,j}], C_{i,j} \rangle$) defined for each block in $D_i$. Note that each instance $\mathcal{I}_{i,j}$ can be optimally solved by using the dynamic programming algorithm given in Section 3. Let $O_{i,j}$ denote the optimal solution for this instance. Then we take the union of the solutions corresponding to blocks in $D_i$, $\Pi_i = \cup_{j \geq 0} O_{i,j}$. By doing this for all $k$ decompositions we get $k$ different feasible solutions for the original graph $G$. Finally, we choose the solution with minimum size among these $k$ solutions. We will see that the $2\ell - 1$ extra levels around the $k$ middle levels and the common levels between consecutive blocks plays an important role in the feasibility and the near optimality of the final output of the algorithm.

To make the role of common levels clear, consider an instance of 4-round PDS shown in Figure 2. Assume that we partition the planar graph into two levels. The first level is the outer cycle and the second level is the inner cycle. It is easy to check that the size of an optimal solution in any one of these partitions is 1. For example $\{u_1\}$ and $\{v_5\}$ are optimal solutions for partition 1 and partition 2 respectively. But if we consider the original graph $G$, it is straightforward to check that $S = \{u_1\} \cup \{v_5\}$ is not a feasible solution for the instance $G$ of 4-round PDS; the set of nodes that can be power dominated in at most four parallel rounds is $\mathcal{P}^4(S) = \{u_1, u_2, u_5, u_8, v_1, v_4, v_5, v_6\}$. Note that the propagation rule was applied in the subgraph but not in the original graph, so in the subgraph $u
may have all but one node of $N[u]$ in $P^i(S)$ but this need not hold for the original graph (see $u_8$, for example).

To prevent such a problem, we need to consider extra levels around each block as we did in the blocks $B_{i,j}$ above. We will show that a feasible solution, found by our algorithm, for the instance $I_{i,j}$ will power dominate at least its $k$ middle levels in the original graph $G$. This implies that the union of the solutions for the blocks in a given decomposition will be a feasible solution for $G$, since the union of the $k$ middle levels of the blocks covers all the nodes in $G$.

**Algorithm 1** PTAS for $\ell$-round PDS

1: Given a planar embedding of the graph $G$, and the parameter $0 < \epsilon \leq 1$.
2: Let $k = 4 \cdot \lceil \frac{\ell}{\epsilon} \rceil$.
3: for $i = 1$ to $k$ do
4:  for all $j \geq 0$ do
5:    Solve “generalized” $\ell$-round PDS on $\langle G[B_{i,j}], C_{i,j} \rangle$
6:    Let $O_{i,j}$ be an optimal solution for $\langle G[B_{i,j}], C_{i,j} \rangle$
7:  end for
8:  $\Pi_i = \bigcup_{j \geq 0} O_{i,j}$
9: end for
10: $r \leftarrow \text{argmin} \{ |\Pi_i| : i = 1, \ldots, k \}$
11: Output $\Pi_{O} = \Pi_r$.

**Theorem 4.2** Let $\ell$ be a given parameter, where $\ell = O(\frac{\log n}{\log \log n})$. Then Algorithm 1 is a PTAS for the $\ell$-round PDS problem on planar graphs.

**Proof:** Let $\Pi^*$ be an optimal solution for $\ell$-round PDS in $G$. To prove the theorem, it is enough to prove the following two claims: 1) $O_{i,j}$ is a feasible solution for the instance $\langle G, C_{i,j} \rangle$ of the generalized $\ell$-round PDS problem, 2) $\Pi^* \cap B_{i,j}$ is a feasible solution for $\langle G[B_{i,j}], C_{i,j} \rangle$. First let us see how the theorem follows from the above claims. The first claim shows that $\Pi_i$, for each $i$, is a feasible solution for the $\ell$-round PDS problem for $G$, since $\bigcup_{j \geq 0} C_{i,j} = V$. The second claim shows that $|O_{i,j}| \leq |\Pi^* \cap B_{i,j}|$, so $|\Pi_i| \leq \sum_j |\Pi^* \cap B_{i,j}|$. In the right hand side we counted the nodes in the optimal solution twice on $4\ell - 2$ common levels between any two consecutive blocks. By considering all values of parameter $\ell$, $1 \leq i \leq k$, we can find an $i$ such that the number of double counted nodes in the optimal solution $\Pi^*$ is at most $\frac{4\ell - 2}{k} |\Pi^*|$. This implies that $|\Pi_O| \leq (1 + \frac{4\ell - 2}{k}) |\Pi^*|$, so by setting $k = 4 \cdot \lceil \frac{\ell}{\epsilon} \rceil$ we get a $(1 + \epsilon)$-approximation algorithm.

Now we analyze the running time of the algorithm. In step (5) we solve an instance of the generalized $\ell$-round PDS problem. The graph in this instance has $k + 4\ell - 2$ levels, so it is a $(k + 4\ell - 2)$-outerplanar graph. It is known that the tree-width of any $d$-outerplanar graph is at most $3d - 1$ [28]. Therefore step (5) of our algorithm can be done in $O(\ell \epsilon \log \ell)$ time by Corollary 3.7, since the graph has tree-width at most $3(k + 4\ell) \leq 12(\lceil \frac{\ell}{\epsilon} \rceil + \ell)$. This shows that $\ell$ should be $O(\frac{\log n}{\log \log n})$ in order to have a polynomial time algorithm and in this case step (5) can be done in $n^{O(\frac{\ell}{\epsilon})}$ time. Also note that the value of $j$ can be at most $\frac{\ell}{\epsilon}$, since the number of levels of a planar graph is at most $n$ (the number of nodes) and each $C_{i,j}$ has $k$ levels. This shows that the algorithm executes step (5) at most $k \times \frac{\ell}{\epsilon} = n$ times. Also the steps (8) and (10) can be done in polynomial time. Therefore the running time of the algorithm is polynomial in the number of nodes of $G$ for a fixed $\epsilon$.

Now we prove the two claims stated above: 1) $O_{i,j}$ is a feasible solution for $\langle G, C_{i,j} \rangle$, 2) $\Pi^* \cap B_{i,j}$ is a feasible solution for $\langle G[B_{i,j}], C_{i,j} \rangle$. 


Proof of the first claim: We know that $O_{i,j}$ is a feasible solution for $\langle G[B_{i,j}], C_{i,j} \rangle$. Let $t_v$ denote the round in which $v$ was power dominated in $G[B_{i,j}]$. So any node $v \in C_{i,j}$ and possibly some nodes $v \in B_{i,j} \setminus C_{i,j}$ satisfy $0 \leq t_v \leq \ell$. For simplicity we use $L^s$ to denote the levels $L(jk + i - 2(\ell - s), (j + 1)k + i - 1 + 2(\ell - s))$ for any $s \geq 1$, and also $L^0 = B_{i,j}$. Observe that $L^{s+1}$ (for $s \geq 1$) is obtained from $L^s$ by deleting the first two levels of $L^s$ and the last two levels of $L^s$.

First note that by taking $O_{i,j}$ all nodes $v$ with $t_v = 0$ in $L^0$ are power dominated in the graph $G$. Now we claim that the following statement is correct: for each $s$, $1 \leq s \leq \ell$, all nodes $v$ with $t_v \leq s$ in $L^s$ are power dominated in the graph $G$. We prove the statement by induction on $s$. The base case $s = 1$ is trivial by applying the first rule of PDS, since any node $v \in L^1$ with $t_v = 1$ had a neighbor $u \in L^0$ with $t_u = 0$. Therefore all nodes with $t_v \leq 1$ in $L^1$ are power dominated. Assume that the statement is correct for all $s < s'$. Consider a node $v$ with $t_v = s'$ which lies in $L^{s'}$, and assume that it was power dominated by applying the propagation rule on $u$. It is easy to see that $u$ is inside the levels $L(jk + i - 2(\ell - s') - 1, (j + 1)k + i - 1 + 2(\ell - s') + 1)$, and therefore all neighbors of $u$ are inside the levels $L(jk + i - 2(\ell - s') - 2, (j + 1)k + i - 1 + 2(\ell - s') + 2) = L^{s'-1}$ (see Figure 3). As $v$ was power dominated by $u$, any $w \in N[u] - v$ satisfies $t_w \leq s' - 1$. By the induction hypothesis any node $w \in N[u] - v$ is already power dominated in the graph $G$. The propagation rule can be applied to $u$ to power dominate $v$. This completes the induction step and proves the statement. An important property that should be preserved is that all nodes $v$ with $t_v = s'$ should be power dominated in parallel. It is easy to see that in the above proof any such node can be power dominated in parallel since they were power dominated in parallel in the induced graph $G[B_{i,j}]$.

To prove the first claim it is enough to note that the set $L^\ell$ is exactly $C_{i,j}$, so all the nodes in $C_{i,j}$ with $0 \leq t_v \leq \ell$ (which is exactly all the nodes in $C_{i,j}$) can be power dominated in the graph $G$.

![Figure 3: Induction step](image)

Proof of the second claim: We know that $\Pi^*$ is a feasible solution for $G$. Let $t_v$ denote the round in which node $v \in B_{i,j}$ was power dominated in $G$. So each node $v \in B_{i,j}$ satisfies: $0 \leq t_v \leq \ell$. Define $L^s$ as before. The same induction hypothesis as above will prove the statement: for each $s$, $0 \leq s \leq \ell$, all nodes $v$ with $t_v \leq s$ in $L^s$ can be power dominated in the induced subgraph $G[B_{i,j}]$. The proof is similar to the first claim. Note that the set $L^\ell$ is exactly $C_{i,j}$, so all the nodes in $C_{i,j}$ can be power dominated in $G[B_{i,j}]$ in at most $\ell$ parallel rounds.\[\square\]

5 Hardness of $\ell$-round PDS

In this section we prove the following result by a reduction from the Minrep problem. The reduction given here is similar to the reduction used to prove the same hardness of approximation for the Directed PDS problem [13].

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Theorem 5.1 The $\ell$-round PDS problem for any $\ell \geq 4$ cannot be approximated within $2^{\log^{1-\epsilon} n}$ ratio, for any fixed $\epsilon > 0$, unless $\text{NP} \subseteq \text{DTIME}(n^{\text{polylog}(n)})$.

In the MinRep problem we are given a bipartite graph $G = (A, B, E)$ with a partitioning of $A$ and $B$ into equal size subsets, say $A = \bigcup_{i=1}^{q_A} A_i$ and $B = \bigcup_{j=1}^{q_B} B_j$, where $|A_i| = m_A = \frac{|A|}{q_A}$ and $|B_j| = m_B = \frac{|B|}{q_B}$. This partitioning naturally defines a super bipartite graph $\mathcal{H} = (A, B, \mathcal{E})$. The super nodes of $\mathcal{H}$ are $A = \{A_1, A_2, \ldots, A_{q_A}\}$ and $B = \{B_1, B_2, \ldots, B_{q_B}\}$, and the super edges are $\mathcal{E} = \{A_iB_j | \exists a \in A_i, b \in B_j : \{a, b\} \in E(G)\}$. We say that a super edge $A_iB_j$ is covered by $\{a, b\} \in E(G)$ if $a \in A_i$ and $b \in B_j$. The goal in MinRep is to pick the minimum number of nodes $A' \cup B' \subseteq V(G)$ from $G$ to cover all the super edges in $\mathcal{H}$. The following theorem states the hardness of the MinRep problem [29].

Theorem 5.2 [29] The MinRep problem cannot be approximated within the ratio $2^{\log^{1-\epsilon} n}$, for any fixed $\epsilon > 0$, unless $\text{NP} \subseteq \text{DTIME}(n^{\text{polylog}(n)})$, where $n = |V(G)|$.

The reduction: Theorem 5.1 is proved by a reduction from the MinRep problem. In the following we create an instance $\overline{G} = (\overline{V}, \overline{E})$ of $\ell$-round PDS from a given instance $G = (A, B, E)(\mathcal{H} = (A, B, \mathcal{E}))$ of the MinRep problem.

1. Add a new node $w^*$(master node) to the graph $G$, and add an edge between $w^*$ and all the other nodes in $G$. Also add three new nodes $w_1^*, w_2^*, w_3^*$ and connect them to $w^*$.

2. \forall i \in \{1, \ldots, q_A\}, j \in \{1, \ldots, q_B\}$ do the following:
   
   (a) Let $E_{ij} = \{e_1, e_2, \ldots, e_\kappa\}$ be the set of edges between $A_i = \{a_{i1}, \ldots, a_{im_A}\}$ and $B_j = \{b_{j1}, \ldots, b_{j_{m_B}}\}$ in $G$, where $\kappa$ is the number of edges between $A_i$ and $B_j$.
   
   (b) Remove $E_{ij}$ from $G$.
   
   (c) Let the edge $e_\ell \in E_{ij}$ be incident to $a_{i\ell}$ and $b_{j\ell}$ (in $G$). In this labeling for simplicity the same node might get different labels. Let $\mathcal{D}_{ij}$ be the graph in Figure 4 (a dashed line shows an edge between a node and the master node $w^*$). Make $\lambda = 4$ new copies of the graph $\mathcal{D}_{ij}$ and then identify nodes $a_{i\ell}$’s, $b_{j\ell}$’s with the corresponding nodes in $A_i$ and $B_j$ (in $G$). Note that the $\lambda$ copies are sharing the same set of nodes, $A_i$ and $B_j$, but other nodes are disjoint.

3. Let $\overline{G} = (\overline{V}, \overline{E})$ be the obtained graph (See Figure 5 for an illustration).

The analysis: The next lemma shows that the size of an optimal solution in $\ell$-round PDS is exactly one more than the size of an optimal solution in the MinRep instance. The number of nodes in the constructed graph is at most $|V(\overline{G})| \leq 4 + |V(G)| + 10\lambda |E(G)|$. This shows that the above reduction is a gap preserving reduction from MinRep to $\ell$-round PDS with the same gap (hardness ratio) as the MinRep problem. Therefore the following lemma will complete the proof of the above theorem. As we mentioned above, the reduction given here is similar to the one used for proving the hardness of the directed PDS problem [13]. One important part of the above construction (see Figure 4) is the gadget on the set of nodes $\{\alpha, \beta, \gamma\}$. Note that there should be such a gadget between the center node in $\mathcal{D}_{ij}$ and each $u_\alpha$ and $v_\alpha$; in Figure 4 not all of the gadgets are shown (for example between $u_2$ and the center node). This gadget introduces direction into undirected construction, it allows the propagation in only one direction. After the center node of $D_{i,j}$ is power dominated, all of the other nodes in $D_{i,j}$ get power dominated (by the propagation rule). On the other hand, the power domination cannot propagate through the gadget in the other direction (toward the center node).
Lemma 5.3 The pair $(A^*, B^*)$ is an optimal solution to the instance $G = (A, B, E)$ of the MinRep problem if and only if $\Pi^* = A^* \cup B^* \cup \{w^*\} \subseteq V(\overline{G})$ is an optimal solution to the instance $\overline{G}$ of $\ell$-round PDS (for all $\ell \geq 4$).

Proof: The node $w^*$ should be in any optimal solution in order to power dominate $w^*_1, w^*_2, w^*_3$, since otherwise we need to have at least 2 nodes from the set $\{w^*_1, w^*_2, w^*_3\}$ to get a feasible solution. By picking $w^*$, all the nodes in $A \cup B$ (and also the nodes inside $D_{ij}$'s that are the neighbors of $w^*$) will be power dominated.

Assume that $A^* \cup B^*$ is an optimal solution for the MinRep instance $G$. Now let us show that $\Pi = A^* \cup B^* \cup \{w^*\}$ is a feasible solution to the PDS instance $\overline{G}$. As described above, the nodes in $A \cup B$ and some nodes inside $D_{ij}$'s are power dominated by $w^*$. Consider a super edge $A_iB_j$ in $\mathcal{H}$. Note
that the set $A^* \cup B^*$ covers all the super edges in $H$. So there exists an edge $e_q = \{a_{i_q}, b_{j_q}\} \in E(G)$ such that $a_{i_q} \in A^*$ and $b_{j_q} \in B^*$, which covers the super edge $A_iB_j$. Since $a_{i_q}$ and $b_{j_q}$ are in the set $\Pi$ they will power dominate their neighbors, $u_q$ and $v_q$, in all of the 4 copies of $D_{ij}$ in $\overline{G}$. After $u_q$ and $v_q$ are power dominated, the node $d_w$ will power dominate the center node in $D_{ij}$. It is easy to check that after the center node is power dominated, all of the nodes in $D_{ij}$ will be power dominated (through the gadgets on the nodes $\alpha, \beta, \gamma$). This shows that $\Pi$ is a feasible solution for PDS in $\overline{G}$. Also it is straightforward to check that such a solution will power dominate the entire graph in at most 4 parallel rounds. Therefore, $\text{opt}_{\ell}(\overline{G}) \leq |A^* \cup B^*| + 1$.

By Proposition 2.3 the size of an optimal solution for PDS is a lower bound on the size of an optimal solution for $\ell$-round PDS. So it is enough to prove that the above upper bound is also a lower bound for PDS in $\overline{G}$. Let $\Pi^* \subseteq V(\overline{G})$ be an optimal solution for PDS. As we saw above, $w^*$ should be in any optimal solution for PDS. Now define $A' = A \cap \Pi^*$ and $B' = B \cap \Pi^*$. First we prove that any optimal solution of PDS only contains nodes from $A \cup B \cup \{w^*\}$, and then we show that $(A', B')$ covers all the super edges. Suppose for the contradiction that $\Pi^*$ contains some nodes which are not in $A \cup B \cup \{w^*\}$. So there are some $D_{ij}$'s that cannot be power dominated completely by $\Pi^* \cap (A \cup B \cup \{w^*\})$. By symmetry all the 4 copies of $D_{ij}$ are not completely power dominated. So the optimal solution $\Pi$ needs to have at least one node from at least 3 of the 4 copies, and the remaining one might be power dominated by applying the propagation rule. By removing these 3 nodes from $\Pi^*$ and adding $a_{i_q} \in A_i$ and $b_{j_q} \in B_j$ to $\Pi^*$ for some arbitrary edge $e_q = \{a_{i_q}, b_{j_q}\} \in E(G)$ we can power dominate all of the 4 copies of $D_{ij}$. This is a contradiction with the optimality of $\Pi^*$. This proves that any optimal solution will consist of nodes only from $A \cup B \cup \{w^*\}$. To show that $(A', B')$ covers all the super edges, it is enough to note the following: suppose no node from the inside of any copies of $D_{ij}$ is in the optimal solution; then any $D_{ij}$ can be power dominated only by taking both end points of an edge between the corresponding partitions $(A_i, B_j)$. This shows that the size of an optimal solution for PDS on $\overline{G}$ is at least the size of an optimal solution for the MinRep problem on $G$ plus 1. This completes the proof of the lemma. 

\[\square\]

6 Integer programming formulations for $\ell$-round PDS

In this section we present an integer programming (IP) formulation for the $\ell$-round PDS problem, and then we present a related integer programming formulation for the original PDS problem. Finally, we consider LP relaxations of these two IPs, and we show that they both have integrality gap of $\Omega(n)$.

6.1 An integer programming for $\ell$-round PDS

Here we consider an IP formulation for the $\ell$-round PDS problem. Given an undirected graph $G = (V, E)$ and a parameter $1 \leq \ell \leq n$, where $n = |V|$, define the set of parallel rounds $T = \{1, \ldots, \ell\}$. The variables in the IP formulation are as follows. Let $S^*$ be the optimal solution. We have a binary variable $x_v$ for each node $v$, that is equal to 1 if and only if the node $v$ is in $S^*$ ($S^* = \{v \in V : x_v = 1\}$). For each node $v$ and a parallel round $t \in T$ we have a binary variable $z^t_v$, where $z^t_v = 1$ means that the node $v$ is power dominated on (or before) the parallel round $t$. For each edge $\{u, v\} \in E$ and a parallel round $t \in T$ we have binary variables $Y^t_{u \rightarrow v}$ and $Y^t_{v \leftarrow u}$, where $Y^t_{u \rightarrow v} = 1$ means that $u$ can power dominate $v$ at the parallel round $t + 1$. Before stating the IP formulation formally, we describe the constraints informally. We have a set of constraints for the termination condition saying that every node should be power dominated at the end of the last round (see (1)). The second set of constraints are for power dominating all the nodes in the closed neighborhood of a node that is in the optimal
solution (see (2)). We have another set of constraints for each edge \( \{u, v\} \in E \) that checks if the propagation rule (R2) can be applied to \( u \) in order to power dominate \( v \) in the next round; node \( u \) is ready to power dominate \( v \) only if \( u \) and all of its neighbors except \( v \) are already power dominated (see (3)). The last set of constraints are for checking if node \( v \) is power dominated at time \( t \); node \( v \) is power dominated only if either it is in the optimal solution or at least one of its neighbors can power dominate \( v \) at time \( t - 1 \) (see (4)). Once \( v \) is power dominated, it should remain power dominated. The term \( +x_v \) in the right hand side of constraint (4) is needed to ensure that the \( z \) variables are monotone, that is, \( z_{v}^{t+1} \geq z_{v}^{t} \).

\[
\text{(IP}_\ell\text{)} \quad \min \sum_v x_v \\
\text{s.t.} \\
(1) \quad 1 \leq z_{v}^{\ell} \quad \forall v \in V \\
(2) \quad z_{v}^{1} \leq \sum_{u \in N[v]} x_u \quad \forall v \in V \\
(3) \quad Y_{u \to v}^{t} \leq z_{w}^{t} \quad \forall (u, v) \in E, \forall w \in N[u] \setminus \{v\}, \forall t \in T \\
(4) \quad z_{v}^{t} \leq \sum_{u \in N(v)} Y_{u \to v}^{t-1} + x_v \quad \forall v \in V, \forall t \in T \setminus \{1\} \\
(5) \quad \text{all variables are binary}
\]

To get an LP relaxation, we relax the variables to be non-negative instead of being binary. It turns out that this LP relaxation is very weak. We can add the following valid constraints that make the LP stronger\(^2\). The first one forces the number of power dominated nodes at the first round to be at least equal to the size of the smallest closed neighborhood, and the second one forces the number of power dominated nodes to increase by at least one at each round.

\[
(6) \quad \delta(G) + 1 \leq \sum_v z_{v}^{1} \\
(7) \quad \sum_v z_{v}^{t-1} + 1 \leq \sum_v z_{v}^{t} \quad \forall t \in T \setminus \{1\}
\]

We now prove that the new LP relaxation has integrality gap of \( \Omega(n) \).

**Theorem 6.1** Let \( \ell \) be a given parameter, where \( \ell = \Omega(\log n) \). Then the LP relaxation for the \( \ell \)-round PDS problem has an integrality gap of \( \Omega(n) \) even on planar graphs.

**Proof:** Consider the graph \( G \) that is obtained from the cycle on \( m \) nodes, \( C_m \), by creating a new node \( v' \) for each node \( v \) and connecting it by an edge to the original node (adding the edge \( \{v, v'\} \) to \( G \)) (see Figure 6). The graph \( G \) has \( n = 2m \) nodes and \( 2m \) edges. It follows from the proof of Theorem 4.1 that the size of a minimum power dominating set in \( G \) is at least \( \lceil \frac{n}{3} \rceil \) and the optimal solution power dominates the graph \( G \) in two parallel rounds. Therefore the size of any optimal solution for

\(^2\)These constraints increase the optimum value of the LP on the cycle with 9 nodes, \( C_9 \), for \( \ell = 3 \) from 0.6 to 1.
\( \ell \)-round PDS is at least \( \lceil n/6 \rceil \) for any \( \ell \geq 2 \). Now we show that the LP relaxation has the optimum value of \( O(1) \), and this completes the proof. Let \( U \) denote the set of nodes in \( C_m \) in the graph \( G \), and \( V \) denote the set of nodes of degree 1 in \( G \) (the newly introduced nodes). We assign value \( \alpha \geq 0 \) to all of the variables corresponding to the nodes in \( U \), and value \( \beta \geq 0 \) to all of the other variables; \( \forall u \in U : x_u = \alpha \) and \( \forall v \in V : x_v = \beta \). Before giving a feasible solution with the objective value of \( O(1) \) (for the LP based on \( \alpha, \beta \)), we compute the value of the variables based on \( \alpha \) and \( \beta \) for the first few rounds.

If we apply the set of constraints (2) to all the nodes we get: \( \forall u \in U : z_u^1 \leq 3\alpha + \beta \) and \( \forall v \in V : z_v^1 \leq \alpha + \beta \). Let \( u, v, w \) be the nodes of \( G \) as shown in Figure 6. By symmetry, we get 3 different types of \( Y \) variables: \( Y^t_{u-v}, Y^t_{v-u} \) and \( Y^t_{w-u} \). Again the symmetry of the graph and the constraints imply that the value of \( Y \) only depends on its type. It is easy to check that the set of constraints (3) give the following inequalities: \( Y^1_{u-v} \leq 3\alpha + \beta (= z_u^1), Y^1_{v-u} \leq \alpha + \beta (= z_v^1) \) and \( Y^1_{w-u} \leq \alpha + \beta (= z_v^1) \).

Next we apply constraint (4) and we get: \( z_u^2 \leq (\alpha + \beta) + 2 \times (\alpha + \beta) + \alpha = 4\alpha + 3\beta (= 3z_v^1 + \alpha) \) and \( z_v^2 \leq 3\alpha + \beta + \beta (= z_u^1 + \beta) \). We can continue in this way, until we reach the parallel round \( t = \ell \).

It is easy to prove by induction that the following assignment for \( Y \) and \( z \) variables satisfy the set of constraints (2) to (4): \( Y^t_{u-v} = z_u^t, Y^t_{v-u} = Y^t_{w-u} = z_v^t \), where \( z_u \) and \( z_v \) are defined recursively as follows:

\[
\begin{align*}
z_u^{t+1} &= \alpha + 3z_v^t \quad (E1) \\
z_v^{t+1} &= \beta + z_u^t \quad (E2)
\end{align*}
\]

Note that we assign the same value, \( z_u^t \), to all nodes in \( U \), and the same value, \( z_v^t \), to all nodes in \( V \). Also we assign the same value for all \( Y \) variables of the same type. By combining the above two recursive equations \( E1 \) and \( E2 \), we get the following independent recursive definition for \( z_u \):

\[
\begin{align*}
z_u^{t+2} &= \alpha + 3\beta + 3z_u^t \quad (E3) \\
z_u^1 &= 3\alpha + \beta \quad (E4) \\
z_u^2 &= 4\alpha + 3\beta \quad (E5)
\end{align*}
\]

It remains to check if the above assignments also satisfy the set of constraints (6) and (7).

\[
\begin{align*}
(6): m \cdot (3\alpha + \beta) + m \cdot (\alpha + \beta) &\geq 2 \Rightarrow m \cdot (4\alpha + 2\beta) \geq 2 \Rightarrow 2\alpha + \beta \geq \frac{1}{m} = \frac{2}{n} \\
(7): m \cdot z_{u}^{t-1} + m \cdot z_{v}^{t-1} &\leq m \cdot z_{u}^{t-1} + m \cdot \beta + m \cdot 3z_{v}^{t-1} + m \cdot \alpha \Rightarrow \alpha + \beta \geq \frac{1}{m} = \frac{2}{n}
\end{align*}
\]

Therefore, in order to satisfy the set of constraints (6) and (7), take \( \alpha, \beta \geq 0 \) be real numbers such that \( \alpha + \beta = \frac{1}{m} \). Equation \( E3 \) for any \( t \geq 1 \) implies that \( z_u^{t+2} \geq 3 \times z_u^t \). By solving this recursive inequality we get \( z_u^{2k} \geq z_u^1 \cdot 3^k \geq \frac{1}{m} \cdot 3^k \). Hence, for \( k = \lceil \log_3 m \rceil \) we get \( z_u^{2k} \geq 1 \), and consequently we get \( z_u^{2k+1} \geq z_u^{2k} \geq 1 \) by Equation \( E2 \). Therefore by taking \( \ell = 2 \cdot \lceil \log_3 m \rceil + 1 \), the set of constraints (1) is also satisfied. Note that the above feasible solution has the objective value of \( m \times (\alpha + \beta) = 1 \), since \( |U| = |V| = m \). This shows that the LP has the optimum value of \( O(1) \) and this completes the proof. \( \square \)

### 6.2 An integer programming formulation for PDS

As we mentioned before, the \( \ell \)-round PDS problem for \( \ell \geq n - 1 \) is the PDS problem. Here we consider an integer programming (IP) formulation for the PDS problem that is different from the IP
formulation considered in the previous section for \( \ell \)-round PDS. The IP formulation here is based on finding an ordering in which the nodes are power dominated in the optimal solution. Given an undirected graph \( G = (V, E) \) with \( n \) nodes, let \( T = \{1, 2, \ldots, n\} \). In this IP, all of the variables have the same definition and meaning as before except \( z^t_v \). Here, \( z^t_v \) indicates the round in which \( v \) is power dominated. The variable \( z^t_v \) is equal to 1 if and only if \( v \) is power dominated at round \( t \). Before stating the IP formulation for the PDS problem formally, we describe the set of constraints informally. There are two sets of constraints for presenting the ordering in which the nodes are power dominated (see (1) and (2)). There is another set of constraints for checking if node \( u \) can power dominate node \( v \) at round \( t \). Node \( u \) can power dominate \( v \) only if either \( u \) is in the optimal solution or all of the nodes in its closed neighborhood except \( v \) are already power dominated (see (3)). The last set of constraints is to check if a node can be power dominated at round \( t \). Node \( v \) can be power dominated at time \( t \) only if either it is in the optimal solution or at least one of its neighbors was ready to power dominate it at the end of the previous round.

\[
\begin{align*}
(\text{IP}_O) \quad & \min \sum_v x_v \\
\text{s.t.} \quad & \sum_t z^t_v = 1 \quad \forall v \in V(G) \\
& \sum_v z^t_v = 1 \quad \forall t \in T \\
& Y^t_{u \rightarrow v} \leq \sum_{t' = 1}^t z^t_{w} + x_u \quad \forall (u, v) : \{u, v\} \in E(G), \forall t \in T \setminus \{n\}, \forall w \in N[u] - v \\
& z^t_v \leq \sum_{u \in N(v)} Y^{t-1}_{u \rightarrow v} + x_v \quad \forall v \in V(G), \forall t \in T \setminus \{1\} \\
& \text{(5) All variables are binary}
\end{align*}
\]

It is easy to verify that the above IP formulates the PDS problem. Now we consider the LP relaxation that is obtained by relaxing the integrality of variables to nonnegativity constraints. We add the following valid inequality to the above IP to get a stronger LP. This new inequality forces a power domination step to occur at each round.

\[
1 \leq \sum_{\{u, v\} \in E} Y^t_{u \rightarrow v} \quad \forall t \in T \setminus \{n\}
\]
Now we show that this LP relaxation has a big integrality gap.

**Theorem 6.2**  *The LP relaxation for PDS has an integrality gap of $\Omega(n)$.*

**Proof:** Consider the graph $G = (V, E)$ that is obtained from the cycle on $m$ nodes, $C_m$, by attaching a node of degree one to each node of the cycle (see Figure 6). Clearly, $G$ has $n = 2 \times m$ nodes. It is easy to check that the size of an optimal solution for PDS on $G$ is at least $\frac{2}{n} = \frac{n}{n} = \Omega(n)$.

Now we show that the LP relaxation has an optimum value of $O(1)$. The graph $G$ has $n$ nodes and $n$ edges. Assign value $\frac{1}{n}$ to each variable $x_v$ and $z_v^t$ and value $\frac{1}{2n}$ to each variable $Y_t^{u \rightarrow v}$. It is easy to check that all of the constraints are satisfied by this assignment. The objective value of this assignment is $n \times \frac{1}{n} = 1$, and this implies that the optimum value is $O(1)$. $\square$
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Appendix

A.1 Proofs

Proof of Proposition 4.1: It is known that Dominating Set is an NP-hard problem on planar graphs [18], so we just need to prove the theorem for \( \ell \geq 2 \). We use (almost) the same reduction that has been used to prove the NP-hardness of PDS on planar graphs [9, 10]. Given a planar graph \( G = (V, E) \) and a parameter \( \ell \geq 2 \) construct a graph \( G' \), an instance of \( \ell \)-round PDS, as follows: for each node \( v \in V \) create a path \( P_v \) of length \( \ell - 1 \) and identify an end point of \( P_v \) with the node \( v \). This transformation preserves the planarity, and the theorem follows from the fact that the size of an optimal solution for the Dominating Set problem on \( G \) is equal to the size of an optimal solution for the \( \ell \)-round PDS problem on \( G' \). The proof goes in the same way as the proof for the NP-hardness of PDS by Kenis et al. [9] and Guo et al. [10].

Assume that \( S \) is an optimal solution for the Dominating Set problem in \( G \). Now it is easy to see that \( S \) power dominates \( G' \) in exactly \( \ell \) parallel rounds; the set \( S \) in one parallel round power dominates all nodes in \( V \) and then in the remaining \( \ell - 1 \) parallel rounds each node \( v \in V \) starts to power dominate the nodes on its attached path, \( P_v \).

Now assume that \( S' \) is an optimal solution for the \( \ell \)-round PDS problem on \( G' \). It is easy to see that there is such an \( S' \) that contains only nodes of degree at least 3, so consider that \( S' \). Assume that \( S' \) is not a dominating set for \( G \), so there is a node \( v \in V \) that is not power dominated in the first parallel round. This means that \( v \) is power dominated by applying the propagation rule to one of its neighbors say \( u \). It is not hard to see that \( u \in V \). Let \( u' \) be the neighbor of \( u \) in the path \( P_v \). Since \( u \) is not in \( S' \), \( u' \) should also be power dominated through \( u \) by applying the propagation rule. This means that \( u \) power dominates both \( v \) and \( u' \), but this is impossible. Therefore \( S' \) should be a dominating set for \( G \).

In the reduction for the NP-hardness of PDS in [9, 10] only an edge, that is, a path of length 1, is attached to each node. Here, we attach a path of length \( \ell - 1 \) to make an instance such that the optimal solution needs exactly \( \ell \) parallel rounds to power dominate the entire graph.

A.2 Dynamic programming

Our dynamic programming is based on valid timed-orientations which is an extension of the new formulation for PDS introduced by Guo et al. [10]. It is similar to the dynamic programming for the Dominating Set problem given in [25] and the dynamic programming for PDS given in [10].

Fix a parameter \( \ell \) and consider the \( \ell \)-round PDS problem. Assume that the graph \( G = (V, E) \) and \( V' \subseteq V \) and a nice tree decomposition \( \langle \{X_i \subseteq V \mid i \in I\}, T = (I, F) \rangle \) of \( G \) with tree-width \( k \) are given as input. Let \( T_i \) denote the subtree of \( T \) rooted at node \( i \in I \), and \( Y_i \) denote the set \( (\bigcup_{j \in V(T_i)} X_j) \setminus X_i \). Furthermore, let \( G_i \) be the subgraph induced on \( Y_i \cup X_i \), i.e. \( G_i = G[Y_i \cup X_i] \). Also denote by \( G'_i \) the subgraph induced on \( X_i \). Let \( n_i \) and \( m_i \) be the number of nodes and the number of edges in \( G'_i \) respectively. The dynamic programming works on the bottom-up fashion. On each bag \( X_i \), it considers all valid timed-orientations of the subgraph \( G_i \) and stores the number of origins together with the orientation on the edges of \( X_i \) as the states of the bag \( X_i \). In the other words, the valid timed-orientation is stored through the states of the bag.

The state of a bag: The state \( s \) for a bag \( X_i \) defines the orientation of the edges inside \( G'_i \), the time label assigned to the nodes of \( X_i \), the number of directed edges from \( v \in X_i \) to all nodes in \( Y_i \), and the number of origins together with the orientation on the edges of \( X_i \) as the states of the bag \( X_i \).
and also the maximum of the time-label assigned to the neighbors of \( v \) in the set \( Y_i \). In a bag state \( s \) we denote the state of an edge \( e = \{u, v\} \in E(G'_i) \) by \( s(e) \), the time-label assigned to \( v \in X_i \) by \( s_t(v) \), the number of incoming edges from \( Y_i \) to \( v \) by \( s_-(v) \), the number of outgoing edges from \( v \) to \( Y_i \) by \( s_+(v) \) and the maximum of the time-label assigned to the neighbors of \( v \) in \( Y_i \) by \( s_y(v) \). Let \( e = \{u, v\} \) be an edge in \( G'_i \), then \( s(e) \) takes one of the following 3 values: \( "u \rightarrow v" \), \( "v \rightarrow u" \) or \( "\bot" \); where the first two values shows the direction of the edge \( e \) in the valid timed-orientation and the third one indicates that \( e \) is left undirected. Consider a node \( v \in X_i \), \( s_t(v) \) takes a value from \( \{0\} \cup \{1, 2, \ldots, \ell\} \cup \left\{ \hat{1}, \hat{2}, \ldots, \hat{\ell} \right\} \cup \{+\infty\} \). The node \( v \) with \( s(v) = \hat{a} \) means that we still ask for a propagation rule to be applied to a neighbor of \( v \) and power dominate it, and \( s(v) = a \) shows that the node \( v \) is already power dominated at the current stage of the algorithm. We use \( \|s(v)\| \) to denote the integer value of the label \( s(v) \) ignoring the hat notation (e.g. \( \|2\| = 2 \)). Also \( s_-(v) \) takes a value from \( \{0, 1\} \), \( s_+(v) \) takes a value from \( \{0, 1, 2\} \) and \( s_y(v) \) takes a value from \( \{0, 1, \ldots, \ell\} \cup \{+\infty\} \). Values of 0 or 1 for \( s_-(v) \) and \( s_+(v) \) shows the exact number of incoming/outgoing edges to/from \( v \), but 2 means that there are at least 2 edges. Let us denote by \( \mathcal{S}_i \) the set of all possible states for the bag \( X_i \). It is straightforward to check that the number of bag states for \( X_i \) is \( |\mathcal{S}_i| = 3^{m_i} \times (2\ell + 2)^{n_i} \times 5^{n_i} \times (\ell + 2)^{n_i} \). Note that a node cannot have \( s_-(v) = 1 \) and \( s_+(v) = 2 \) at the same time (this follows easily from Definition 3.4), so \( s_-(v) \) and \( s_+(v) \) have 5 different combinations.

For each bag \( X_i \) we will compute and store a mapping \( A_i : \mathcal{S}_i \rightarrow \mathbb{N} \cup \{+\infty\} \). For a bag state \( s \in \mathcal{S}_i \), the value \( A_i(s) \) shows the minimum number of origins in the optimal valid timed-orientation of the subproblem induced on \( G_i \) under the restriction that the orientation of edges and labeling of nodes in \( X_i \) is defined by the state \( s \). A bag state \( s \in \mathcal{S}_i \) for the bag \( X_i \) is called \textit{invalid} if

\[
\begin{align*}
&\text{(-P1) } \left( \exists v \in V' \cap X_i : s_t(v) = +\infty \right) \lor \\
&\text{(-P2) } \left( \exists v \in X_i : (1 \leq \|s_t(v)\| \leq \ell) \land (d^-_t(v) + s_-(v) > 1) \right) \lor \\
&\text{(-P3) } \left( \exists v \in X_i : s_t(v) = +\infty \land (d^-_t(v) + s_-(v) + s_+(v) + s_y(v) \geq 1) \right) \lor \\
&\text{(-P4) } \left( \exists v \in X_i : s_t(v) = 0 \land d^-_t(v) + s_-(v) \geq 1 \right) \lor \\
&\text{(-P5) } \left( \exists e = \{u, v\} \in E(G'_i) : s(e) = "u \rightarrow v" \land (\|s_t(v)\| = 1 \land s_t(u) \neq 0) \lor (\|s_t(v)\| > 1 \land s_t(v) < 1 + \max \{s_y(u) \cup \{s(w) : w \in N_i[u] - v\}\}) \lor \\
&\left( \exists v \in X_i, \exists a \in \left\{ \hat{1}, \ldots, \hat{\ell} \right\} : (s_t(v) = a \land d^-_t(v) + s_-(v) \neq 0) \right) \lor \\
&\left( \exists a \in \{1, \ldots, \ell\} : s_t(v) = a \land d^-_t(v) + s_-(v) = 0 \right)
\end{align*}
\]

where \( d^-_t(u) \), \( d^+_t(u) \), and \( N_i[u] \) denote respectively the in-degree, out-degree, and closed neighborhood of \( u \) in the graph that is obtained from \( G'_i \) by orienting edges according to the state \( s \). Recall that the in-degree/out-degree shows the number of directed incoming/outgoing edges, but for the closed neighborhood we consider the undirected graph \( G'_i \). Informally a bag state is invalid if it either violates any one of the valid timed-orientation’s properties (P1 to P5), or it cannot be extended to a valid timed-orientation. Now we describe our dynamic programming:

**Step 1:** (Initialization) In this step for each leaf node \( i \) in the tree \( T \), we define (initialize) the mapping \( A_i \) for each \( s \in \mathcal{S}_i \) as follows:

\[
A_i(s) = \begin{cases} 
+\infty & \text{if either } s \text{ is invalid or } (\exists v \in X_i : s_-(v) + s_+(v) + s_y(v) \neq 0) \\
|\{v \in X_i : s(v) = 0\}| & \text{otherwise}
\end{cases}
\]

**Step 2:** (Bottom-Up Computation) In this step we compute in the bottom-up fashion from leaves to root the mapping corresponding to each bag in the tree. Recall that the tree nodes have three types:
JOIN NODE, INSERT NODE, FORGET NODE. In the following we describe how to compute \( A_i \) in each of these three cases. In each point of the algorithm we preserve the following invariant: for each \( s \in S_i \) there exists a valid timed-orientation for \( G_i \) which is compatible with \( s \) and has minimum number of origins equal to \( A_i(s) \) where the power domination of all nodes in \( G_i \) are justified except for the ones in \( \{ v \in X_i : s(v) = \hat{a} \text{ for some } a \in \{1, \ldots, \ell\} \} \).

**Forget Node:** Suppose \( i \) is a forget node with child \( j \), and assume that \( X_j = X_i \cup \{x\} \). The bag states \( s \in S_i \) and \( s' \in S_j \) are called \textit{forget-compatible} and denoted by \( s \sim s' \), if

(F1) \( \forall e \in E(G'_j) : s(e) = s'(e) \),

(F2) \( \forall v \in V(G'_j) : s_t(v) = s'_t(v) \)

(F3) \( \forall v \in V(G'_j) : s_-(v) = s'_-(v) + [s'({x, v}) = "x → v"] \land s_+(v) = s'_+(v) + [s'({x, v}) = "v → x"] \)

(F4) \( s'(x) \in \{0, 1, \ldots, \ell\} \cup \{+\infty\} \)

(F5) \( \forall v \in V(G'_j) : s_y(v) = \begin{cases} \max\{s'_y(v), s'_t(x)\} & \text{if } \{x, v\} \in E(G'_j) \\ s'_y(v) & \text{otherwise} \end{cases} \),

where \( |P| \) is equal to 1 if \( P \) is a true statement and 0 otherwise. Now we compute the mapping \( A_i \) for the bag \( X_i \) as follows: \( \forall s \in S_i \)

\[
A_i(s) = \begin{cases} +\infty & \text{if } s \text{ is invalid} \\ \min\{A_j(s') : s' \in S_j, s \sim s'\} & \text{otherwise} \end{cases}
\]

Note that since \( x \) is not in \( X_i \), by property (3) of the tree decomposition (Definition 3.1) it will never appears in any bag in the rest of the algorithm. This implies that the power domination of \( x \) should be justified within \( G_j \).

**Insert Node:** Suppose \( i \) is an insert node with child \( j \), and assume that \( X_i = X_j \cup \{x\} \). We introduce a mapping \( \phi : S_i \rightarrow S_j \) in the following way. We map a given state \( s \in S_i \) to \( s' = \phi(s) \in S_j \) in the following way:

(I1) \( \forall e \in E(G'_j) : s'(e) = s(e) \)

(I2) \( \forall v \in V(G'_j) : s'_t(v) = \begin{cases} \hat{a} & \text{if } s_t(v) = a \text{ and } [s({x, v}) = "x → v"] = 1 \\ s_t(v) & \text{otherwise} \end{cases} \)

(I3) \( \forall v \in V(G'_j) : s'_-(v) = s_-(v), s'_+(v) = s_+(v), s'_y(v) = s_y(v) \)

We now compute the mapping \( A_i \) for the bag \( X_i \) as follows: \( \forall s \in S_i \)

\[
A_i(s) = \begin{cases} +\infty & \text{if } s \text{ is invalid or } s_-(x) + s_+(x) + s_y(x) \neq 0 \\ A_j(\phi(s)) + [s(x) = 0] & \text{otherwise} \end{cases}
\]

Again note that, since \( x \) appears in \( X_i \) but not in \( X_j \) by property (3) of tree decomposition it is the first time that it appears in the subtree rooted at node \( i \). Also note that \( x \) cannot have a neighbor in \( Y_i \).

**Join Node:** Suppose \( i \) is a join node with \( j \) and \( k \) as its children, and assume that \( X_i = X_j = X_k \). We say \( s' \in S_j \) and \( s'' \in S_k \) are \textit{join-compatible} with \( s \in S_i \) and denote it by \( s \sim (s', s'') \), if the following conditions hold:
Informally $s$ is join-compatible with the pair of states $(s', s'')$ if they are assigning the same label to the nodes and the same orientation to the edges, and also if a node is power dominated it should be justified in either $s'$ or $s''$. We now compute the mapping $A_i$ for the bag $X_i$ as follows: $\forall s \in S_i$

$$A_i(s) = \begin{cases} +\infty & \text{s is invalid} \\ \min \left\{ A_j(s') + A_k(s'') - |\{v \in X_i : s(v) = 0\}| : s' \in S_j, s'' \in S_k, s \sim (s', s'') \right\} & \text{otherwise} \end{cases}$$

**Step 3: (At root $r$)** Let $r$ be the root of the tree decomposition $T$. Finally we compute the number of origins in the optimal solution for $\ell$-round PDS on the instance $\langle G, V' \rangle$ in the following way:

$$\min \{ A_r(s) : s \in S_r, \forall v \in X_r : s(v) \in \{0, 1, \ldots, \ell\} \cup \{+\infty\} \}.$$ 

Recall that the number of bag states is $|S_i| = 3^{m_i} \times (2\ell + 2)^{n_i} \times 5^{n_i} \times (\ell + 2)^{n_i}$. Let $m_e$ be the maximum number of edges that a bag can have, and also note that the maximum number of nodes is $(k + 1)$. It is easy to check that each step of the dynamic programming can be computed in time $O(c^{m_e+k \log \ell})$, for some global constant $c$. This shows that the total running time of our algorithm is $O(c^{m_e+k \log \ell} \cdot |V|)$. 

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