Abstract Geometric Algebra. Orthogonal and Symplectic Geometries

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Abstract

Our main interest in this paper is chiefly concerned with the conditions characterizing orthogonal and symplectic abstract differential geometries. A detailed account about the sheaf-theoretic version of the symplectic Gram-Schmidt theorem and of the Witt's theorem is also given.

Key Words: Orthosymmetric $\mathcal{A}$-bilinear forms, sheaf of $\mathcal{A}$-radicals, convenient $\mathcal{A}$-modules.

Introduction

Abstract Differential Geometry (acronym, ADG) offers a new approach to classical Differential Geometry (on smooth manifolds). This new approach differs from the classical way of understanding the geometry of smooth manifolds, differential spaces à la Mostow [15], à la Sikorski [17], and the likes, in the sense that, for instance, differential spaces in general are governed by new classes of “smooth” functions, whereas in ADG the structural sheaf of functions characterizing a differential space (in the terminology of ADG, a

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differential triad), is replaced instead by an arbitrary sheaf of algebras $\mathcal{A}$, based on a topological space $X$, whose role is just to parametrize $\mathcal{A}$. The same (sheaf of) algebras may in some cases contain a tremendous amount of singularities, while still retaining the classical character of a differential mechanism, yet without any underlying (smooth) manifold: see e.g. Mallios[9], Mallios[11]. This results to significant potential applications, even to quantum gravity (ibid.). We may also point out that the main moral of ADG is the functorial mechanism of (classical) calculus, cf. Mallios[10], viz. Physics is $\mathcal{A}$-invariant regardless of what $\mathcal{A}$ is.

Yet, a particular instance of the above that also interests us here is the standard Symplectic Differential Geometry (on manifolds), where a special important issue is the so-called orbifolds theory; see e.g. Mallios [9] Vol. II, Chapt. X; Section 3a] concerning its relation with ADG, or da Silva[18] for the classical case. The following constitutes a sheaf-theoretic fundamental prelude with a view towards potential applications of ADG, the whole set-up being in effect a Lagrangian perspective. In particular, one of the goals of this paper consists in trying to generalize primarily the symplectic Gram-Schmidt theorem and the Witt theorem for isometric symplectic convenient $\mathcal{A}$-modules, see e.g. Crumeyrolle[4], as well as some other results, necessary for the setting of the aforesaid sheaf-theoretic version, in terms of $\mathcal{A}$-modules (see below) of both orthogonal and symplectic geometries. Most of the concepts of the latter version are defined on the basis of the classical ones; see, for instance, Artin[2], Crumeyrolle[4], Lang[8]. Our main reference, throughout the present account, is Mallios[9], which may be useful for the basics of ADG.

This is a continuation of work done by Mallios and Ntumba [12], [13], and [14].

**Convention:** Throughout the paper, $X$ will denote an arbitrary topological space and the pair $(X, \mathcal{A})$ a fixed $\mathbb{C}$-algebraized space, cf. Mallios[9], p. 96]; all $\mathcal{A}$-modules are understood to be defined on $X$.

For easy reference, we recall a few basic definitions.

Let $(X, \mathcal{A})$ be a $\mathbb{C}$-algebraized space, that is the pair $(X, \mathcal{A})$ consists of a topological space $X$ and a (preferably unital and commutative) sheaf of
\(\mathbb{C}\)-algebras \(A \equiv (A, \tau, X)\). A sheaf of \(A\)-modules (or an \(A\)-module) on \(X\), is a sheaf \(E \equiv (E, \pi, X)\), on \(X\), such that the following hold:

- \(E\) is a sheaf of abelian groups.
- For every point \(x \in X\), the corresponding stalk \(E_x\) of \(E\) is a (left) \(A_x\)-module.
- The \emph{exterior module multiplication} in \(E\), viz. the map
  \[ A \circ E \to E : (a, z) \mapsto a \cdot z \in E_x \subseteq E, \]
  with \(\tau(a) = \pi(z) = x \in X\), is \emph{continuous}.

On another hand, suppose given a presheaf of \(\mathbb{C}\)-algebras \(A \equiv (A(U), \tau^U)\) and a presheaf of abelian groups \(E \equiv (E(U), \rho^U)\), both on a topological space \(X\) such that

- \(E(U)\) is a (left) \(A(U)\)-module, for every open set \(U\) in \(X\).
- For any open sets \(U, V\) in \(X\), with \(V \subseteq U\),
  \[ \rho^U_V (a \cdot s) = \tau^U_V (a) \cdot \rho^U_V (s), \]
  for any \(a \in A(U)\) and \(s \in E(U)\). We call such a presheaf \(E\) a \emph{presheaf of \(A(U)\)-modules} on \(X\), or simply an \(A\)-presheaf on \(X\).

These two notions relate to one-another in the sense that the sheafification of a presheaf of \(A(U)\)-modules on a topological space \(X\) is an \(A\)-module. See Mallios [9 (1.54)].

## 1 Symplectic Gram-Schmidt theorem

**Lemma 1.1** Let \([ (E, F; \phi); A]\) be a pairing of \(A\)-modules. Then, \(\phi\) induces an \(A\)-morphism, viz.

\[ \phi^* : F \to E^* := \mathcal{H}om_A(E, A), \]
see Mallios \[9\] p.133; (6.3)p.134; (6.8)p.135], given by

\[
\phi^E_U(t)(s) := \phi_V(s, \sigma^U_V(t)) \equiv \phi_V(s, t|_V),
\]

where \(U\) is open in \(X\), \(t \in \mathcal{F}(U)\), \(s \in \mathcal{E}(V)\) and the \(\sigma^U_V\) the restriction maps of the presheaf of sections of \(\mathcal{F}\). Likewise, \(\phi\) gives rise to a similar \(A\)-morphism:

\[
\phi^\mathcal{F} : \mathcal{E} \longrightarrow \mathcal{F}^*.
\]

**Proof.** Assume that \((\mathcal{E}^*(U), \kappa^U_V)\) is the presheaf of sections of \(\mathcal{E}^*\). For \(\phi^E\) to be an \(A\)-morphism, we must have

\[
\kappa^U_V \circ \phi^E_U = \phi^E_V \circ \sigma^U_V,
\]

for any open subsets \(U, V\) of \(X\) such that \(V \subseteq U\). In fact, fix \(U\) and \(V\). For \(t \in \mathcal{F}(U)\) and \(s \in \mathcal{E}(W)\), where \(W \subseteq V\) is an open subset of \(X\), \(\kappa^U_V(\phi^E_U(t))(s) = \phi_W(s, t|_W)\). On the other hand, \(\phi^E_V(t|_V)(s) = \phi_W(s, t|_W)\).

The preceding shows the correctness of our assertion regarding the map \(\phi^E\), to this effect still, see Mallios \[9\] (13.19) p.75 and (6.5) p. 27. In a similar way, one shows that \(\phi^\mathcal{F}\) is an \(A\)-morphism. \(\blacksquare\)

Linked with Lemma 1.1 is an important concept, which we now introduce.

**Definition 1.1** Let \([(\mathcal{E}, \mathcal{F}; \phi); A]\) be a pairing of \(A\)-modules \(\mathcal{E}\) and \(\mathcal{F}\), and \(\phi^E\) and \(\phi^\mathcal{F}\) be the induced \(A\)-morphisms, according to Lemma 1.1. By the **orthogonal** of \(\mathcal{E}\) (resp. \(\mathcal{F}\), denoted \(\mathcal{E}^\perp\) (resp. \(\mathcal{F}^\perp\)), we mean the kernel of \(\phi^E\) (resp. \(\phi^\mathcal{F}\)), (see Mallios \[9\] p.108 for the kernel of an \(A\)-morphism). \(\phi\) is said to be **non-degenerate** if \(\mathcal{E}^\perp = \mathcal{F}^\perp = 0\), and **degenerate** otherwise.

**Lemma 1.2** Let \([(\mathcal{E}, \mathcal{F}; \phi); A]\) be a pairing of \(A\)-modules. Then, \(\mathcal{E}^\perp\) (resp. \(\mathcal{F}^\perp\)) is a sub-\(A\)-module of \(\mathcal{F}\) (resp. \(\mathcal{E}\)).

**Proof.** The proof follows Mallios \[9\] (2.10) p. 108. \(\blacksquare\)
Lemma 1.3 If $[(\mathcal{E}, \mathcal{F}; \phi); \mathcal{A}]$ is a pairing of free $\mathcal{A}$-modules, then for every open subset $U$ of $X$,

$$\mathcal{E}^\perp(U) = \mathcal{E}(U)^\perp, \quad \mathcal{F}^\perp(U) = \mathcal{F}(U)^\perp,$$

where

$$\mathcal{E}(U)^\perp := \{ t \in \mathcal{F}(U) : \phi_V(\mathcal{E}(U), t) = 0 \}$$

and similarly

$$\mathcal{F}(U)^\perp := \{ t \in \mathcal{E}(U) : \phi_V(t, \mathcal{F}(U)) = 0 \}.$$

Proof. That $\mathcal{E}^\perp(U) \subseteq \mathcal{E}(U)^\perp$ is clear. Now, let $\mathcal{E}(U)^\perp$ and $\{ e_i^U \}_{i=1}^n$ be a canonical basis of $\mathcal{E}(U)$. Since $\phi_V(e_i^U, t)|_V = \phi_V(e_i^U|_V, t|_V) = 0$ and $\{ e_i^U|_V \}_{i=1}^n$ being a canonical basis of $\mathcal{E}(V)$, we have $\phi_V(s, t|_V) = 0$, for any $s \in \mathcal{E}(V)$. Therefore, $\mathcal{E}(U)^\perp \subseteq \mathcal{E}^\perp(U)$, and hence the equality $\mathcal{E}^\perp(U) = \mathcal{E}(U)^\perp$.

The second equality is shown in a similar way. \[\square\]

Scholium 1.1 For the particular case where $\phi$ is an $\mathcal{A}$-bilinear form on an $\mathcal{A}$-module $\mathcal{E}$, we denote by $\mathcal{E}^\perp$ the left $\mathcal{A}$-orthogonal of $\mathcal{E}$, whereas $\mathcal{E}^\top$ will be its right $\mathcal{A}$-orthogonal. So, for any open subset $U$ of $X$, one has

$$\mathcal{E}^\perp(U) = \{ t \in \mathcal{E}(U) : \phi_V(\mathcal{E}(V), t|_V) = 0, \text{ for all open } V \subseteq U \},$$

and similarly

$$\mathcal{E}^\top(U) = \{ t \in \mathcal{E}(U) : \phi_V(t|_V, \mathcal{E}(V)) = 0, \text{ for all open } V \subseteq U \}.$$  

Thus, for the particular case where $\mathcal{F} = \mathcal{E}$ in Definition 1.1, one gets

$$\mathcal{E}^\perp := \ker \phi^\mathcal{E} \subseteq \mathcal{E} \quad \text{and} \quad \mathcal{E}^\top := \ker \phi^\mathcal{E} \subseteq \mathcal{E}.$$  

Refer to $\mathcal{E}^\perp(U)$ and $\mathcal{E}^\top(U)$ above, for every open $U \subseteq X$, to understand the nuance between $\mathcal{E}^\perp$ and $\mathcal{E}^\top$.

Lemma 1.4 Let $\phi$ be a non-degenerate $\mathcal{A}$-bilinear form on an $\mathcal{A}$-module $\mathcal{E}$. Then the mappings $\perp \equiv \perp(\phi)$, $\top \equiv \top(\phi)$ have the following properties:
(1) (a) If \( G \subseteq H \), then \( G^\perp \supseteq H^\perp \)
(b) If \( G \subseteq H \), then \( G^\top \supseteq H^\top \)

(2) (c) \( (G + H)^\perp = G^\perp \cap H^\perp \)
(d) \( (G + H)^\top = G^\top \cap H^\top \)

for all sub-\(A\)-modules \(G\) and \(H\) of \(E\).

**Proof.** Assertion (1) is clear. For Assertion (2), we have for every open subset \(U\) of \(X\) and \(t \in (G + H)^\perp(U)\) if and only if \(\phi_V((G + H)(V), t|_V) = \phi_V(G(V), t|_V) + \phi_V(H(V), t|_V) = 0\), where \(V\) is an arbitrary open subset contained in \(U\). But if \(\phi_V(G(V), t|_V) + \phi_V(H(V), t|_V) = 0\) and similarly \(\phi_V(H(V), t|_V) = 0\); therefore \((G + H)^\perp \subseteq G^\perp \cap H^\perp\). Conversely, let \(t \in \mathcal{E}(U)\) such that \(t \in (G^\perp \cap H^\perp)(U) := G^\perp(U) \cap H^\perp(U)\). Therefore, for every open \(V \subseteq U\), \(\phi_V(G(V), t|_V) = 0\) and \(\phi_V(H(V), t|_V) = 0\). Thus, \(\phi_V(G(V)+H(V), t|_V) := \phi_V((G+H)(V), t|_V) = 0\); hence \(G^\perp \cap H^\perp \subseteq (G+H)^\perp\). Part (d) of Assertion (2) is proved in a similar way. 

This particular case, in Scholium \[1.1\] will allow us to define later an important instance that orthogonality \((\perp, \top)\) presents: orthosymmetry. For the classical case, cf. Gruenberg-Weir \[7, p. 97\]. For the moment, it is appropriate to state the analogue of the symplectic Gram-Schmidt theorem. See de Gosson \[6, p.12\] for the classical result. But first, we need the following scholium.

**Scholium 1.2** For the purpose of Theorem \[1.3\] below, we assume that the pair \((X, \mathcal{A})\) is an ordered algebraized space with \(\mathcal{A}\) a unital \(\mathbb{C}\)-algebra sheaf. Furthermore, the order of \((X, \mathcal{A})\) is such that every nowhere-zero section of \(\mathcal{A}\) is invertible, viz. if \(s \in \mathcal{A}(U)\), where \(U\) is open in \(X\), is such that \(s|_V(V) \neq 0\) for every open \(V \subseteq U\), then \(s \in \mathcal{A}(U)^\bullet \cong \mathcal{A}^\bullet(U)\) (\(\mathcal{A}^\bullet\) denotes the sheaf generated by the complete presheaf \(U \longmapsto \mathcal{A}(U)^\bullet\)), where \(U\) runs over the open subsets of \(X\), and \(\mathcal{A}(U)^\bullet \cong \mathcal{A}^\bullet(U)\) consists of the invertible elements of the unital \(\mathbb{C}\)-algebra \(\mathcal{A}(U)\); cf. Mallios \[9 pp 282, 283\]).

**Definition 1.2** Let \(\mathcal{E}\) be an \(\mathcal{A}\)-module. A symplectic \(\mathcal{A}\)-morphism (or symplectic \(\mathcal{A}\)-form) on \(\mathcal{E}\) is an \(\mathcal{A}\)-bilinear form \(\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}\) which is
• skew-symmetric (one also says antisymmetric):

\[ \phi_U(r, s) = -\phi_U(s, r) \]

for any \( r, s \in \mathcal{E}(U) \) and open subset \( U \subseteq X \)

(equivalently, in view of the bilinearity of \( \phi : \phi_U(r, r) = 0 \)

and \( U \) open in \( X \))

• non-degenerate:

\[ \phi_U(r, s) = 0 \]

for all \( s \in \mathcal{E}(U) \) if and only if \( r = 0 \).

A symplectic \( \mathcal{A} \)-module is a self-pairing \( (\mathcal{E}, \phi) \), where \( \phi \) is a symplectic \( \mathcal{A} \)-form.

**Theorem 1.1** Let \( (\mathcal{E}, \phi) \) be a free \( \mathcal{A} \)-module of rank \( 2n \), \( \phi : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A} \) a non-zero skew-symmetric non-degenerate \( \mathcal{A} \)-bilinear form, and \( I \) and \( J \) two (possibly empty) subsets of \( \{1, \ldots, n\} \). Moreover, let \( A = \{ r_i \in \mathcal{E}(U) : i \in I \} \) and \( B = \{ s_j \in \mathcal{E}(U) : j \in J \} \) such that

\[ \phi_U(r_i, r_j) = \phi_U(s_i, s_j) = 0, \quad \phi_U(r_i, s_j) = \delta_{ij}, \quad (i, j) \in I \times J. \]  

Then, there exists a basis \( \mathcal{B} \) of \( (\mathcal{E}(U), \phi_U) \) containing \( A \cup B \).

**Proof.** We have three cases. With no loss of generality, we assume that \( U = X \).

1. **Case: \( I = J = \emptyset \)** Since \( \mathcal{A}^{2n} \neq 0 \) (we already assumed that \( \mathcal{C} \equiv \mathcal{C}_X \subseteq \mathcal{A} \)), there exists an element

\[ 0 \neq r_1 \in \mathcal{E}(X) \cong \mathcal{A}^{2n}(X) \cong \mathcal{A}(X)^{2n} \]

(take e.g. the image (by the isomorphism \( \mathcal{E}(X) \cong \mathcal{A}^{2n}(X) \)) of an element in the canonical basis of (sections) of \( \mathcal{A}^{2n}(X) \)). There exists a section \( \mathbf{s}_1 \in \mathcal{E}(X) \) such that \( \phi_V(\mathbf{r}_1|_V, \mathbf{s}_1|_V) \neq 0 \) for any open subset \( V \) in \( X \) (such a section \( \mathbf{s}_1 \) exists; indeed, if there is no section \( \mathbf{s}_1 := a_1e_1 + \ldots + a_{2n}e_{2n} \),

where \( (e_i)_{1 \leq i \leq 2n} \) is a canonical basis of \( \mathcal{E}(X) \), such that \( \phi_V(\mathbf{r}_1|_V, \mathbf{s}_1|_V) \neq 0 \) for any open \( V \subseteq X \), then there exists an open subset \( W \) of \( X \) such that
\(\phi_W(r_1|_W, e_i|_W) = 0\). But this is impossible since \((e_i|_W)_{1 \leq i \leq 2n}\) is a basis of \(\mathcal{E}(W)\) and \(\phi_W\) is non-degenerate. Hence, based on the hypothesis on \(A\) (cf. Scholium 1.2), \(\phi_X(r_1, s_1)\) is invertible in \(A(X)\). Putting \(s_1 := u^{-1}s_1\), where \(u \equiv \phi_X(r_1, s_1) \in A(X)\), one gets

\[\phi_X(r_1, s_1) = 1.\]

Now, let us consider

\[S_1 := [r_1, s_1],\]

that is, the \(A(X)\)-plane, spanned by \(r_1\) and \(s_1\) in \(\mathcal{E}(X)\), along with its orthogonal complement in \(\mathcal{E}(X)\), i.e.,

\[S_1^\perp \equiv T_1 := \{t \in \mathcal{E}(X) : \phi_X(t, z) = 0, \text{ for all } z \in S_1\}.\]

The sections are linearly independent, for if \(s_1 = ar_1\), with \(a \in A(X)\), then

\[1 = \phi_X(r_1, s_1) = \phi_X(r_1, ar_1) = a\phi_X(r_1, r_1) = 0,\]

a contradiction. So, \([r_1, s_1]\) is a basis of \(S_1\). Furthermore, we prove that

\[(i) \quad S_1 \cap T_1 = 0, \quad (ii) \quad S_1 + T_1 = \mathcal{E}(X).\]

Indeed, \((i)\) since \(\phi_X(r_1, s_1) \neq 0\), we have \(S_1 \cap T_1 = 0\). On the other hand, \((ii)\) for every \(z \in \mathcal{E}(X)\), one has

\[z = (-\phi_X(z, r_1)s_1 + \phi_X(z, s_1)r_1) + (z + \phi_X(z, r_1)s_1 - \phi_X(z, s_1)r_1),\]

with

\[-\phi_X(z, r_1)s_1 + \phi_X(z, s_1)r_1 \in S_1,\]

and

\[z + \phi_X(z, r_1)s_1 - \phi_X(z, s_1)r_1 \in T_1.\]

Thus,

\[\mathcal{E}(X) = S_1 \oplus T_1.\]

The restriction \(\phi_1 \equiv \phi_{1,X}\) of \(\phi_X\) to \(T_1\) is non-degenerate, because if \(z_1 \in T_1\) is such that \(\phi_1(z_1, z) = 0\) for all \(z \in T_1\), then \(z_1 \in T_1^\perp\) and hence \(z_1 \in T_1 \cap T_1^\perp = S_1^\perp \cap T_1^\perp = (S_1 + T_1)^\perp = \mathcal{E}(X)^\perp = 0\), (the second equality derives from Lemma 1.4); so \(z_1 = 0\). \((T_1, \phi_1)\) is thus a symplectic free \(A(X)\)-module of
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\[ \text{rank } 2(n - 1). \] Repeating the construction above \( n - 1 \) times, we obtain a strictly decreasing sequence

\[ (\mathcal{E}(X), \phi_X) \supseteq (T_1, \phi_1) \supseteq \cdots \supseteq (T_{n-1}, \phi_{n-1}) \]

of symplectic free \( \mathcal{A}(X) \)-modules with rank \( T_k = 2(n - k), k = 1, \ldots, n - 1 \), and also an increasing sequence

\[ \{r_1, s_1\} \subseteq \{r_1, r_2; s_1, s_2\} \subseteq \cdots \subseteq \{r_1, \ldots, r_n; s_1, \ldots, s_n\} \]

of gauges; each satisfying the relations (1).

(2) Case \( I = J \neq \emptyset \). We may assume without loss of generality that \( I = J = \{1, 2, \ldots, k\} \), and let \( S \) be the subspace spanned by \( \{r_1, \ldots, r_k; s_1, \ldots, s_k\} \). Clearly, \( \phi_X|_S \) is non-degenerate; by Adkins-Weintraub [1, Lemma (2.31), p.360], it follows that \( S \cap S^\perp = 0 \). On the other hand, let \( z \in \mathcal{E}(X) \). One has

\[ z = (-\sum_{i=1}^{k} \phi_X(z, r_i)s_i + \sum_{i=1}^{k} \phi_X(z, s_i)r_i) + (\sum_{i=1}^{k} \phi_X(z, r_i)s_i - \sum_{i=1}^{k} \phi_X(z, s_i)r_i), \]

with

\[-\sum_{i=1}^{k} \phi_X(z, r_i)s_i + \sum_{i=1}^{k} \phi_X(z, s_i)r_i \in S,\]

and

\[ z + \sum_{i=1}^{k} \phi_X(z, r_i)s_i - \sum_{i=1}^{k} \phi_X(z, s_i)r_i \in S^\perp. \]

Thus,

\[ \mathcal{E}(X) = S \oplus S^\perp. \]

Based on the hypothesis on \( S_1 \) the restriction \( \phi_X|_S \) is a symplectic \( \mathcal{A} \)-bilinear form. It is also easily seen that the restriction \( \phi_{X,S^\perp} \) is skew-symmetric. Moreover, since \( S \oplus S^\perp \) and \( \mathcal{E}(X)^\perp = 0 \), if there exist \( z_1 \in S^\perp \) such that \( \phi_X(z_1, z) = 0 \) for all \( z \in S^\perp \), then \( z_1 \in \mathcal{E}(X)^\perp = 0 \), i.e., \( z_1 = 0 \). Thus, \( \phi_X|_{S^\perp} \) is non-degenerate and hence a symplectic \( \mathcal{A} \)-form. Applying Case (1), we obtain a symplectic basis of \( S^\perp \), which we denote as

\[ \{r_{k+1}, \ldots, r_n; s_{k+1}, \ldots, s_n\}. \]
Then,
\[ \mathfrak{B} = \{ r_1, \ldots, r_n; s_1, \ldots, s_n \} \]
is a symplectic basis of \( \mathcal{E}(X) \) with the required property.

(3) Case \( J \setminus I \neq \emptyset \) (or \( I \setminus J \neq \emptyset \)). Suppose that \( k \in J \setminus I \); since \( \phi_X \) is non-degenerate there exists \( \overline{r}_k \in \mathcal{E}(X) \) such that \( \phi_X(\overline{r}_k, s_k) \neq 0 \) in the sense that \( \phi_V(\overline{r}_k|_V, s_k|_V) \neq 0 \) for any open \( V \subseteq X \). In other words, the section \( v \equiv \phi_X(\overline{r}_k, s_k) \in \mathcal{A}(X) \) is nowhere zero, and is therefore invertible by virtue of the property of the \( \mathbb{C} \)-algebra sheaf \( \mathcal{A} \), as indicated in Scholium 1.2. So, if \( r_k := v^{-1} \overline{r}_k \), we have \( \phi_X(r_k, s_k) = 1 \). Next, let us consider the sub-\( \mathcal{A}(X) \)-module \( R \), spanned by \( r_k \) and \( s_k \), viz. \( R = [r_k, s_k] \). As in Case (1), we have
\[ \mathcal{E}(X) = R \oplus R^\perp. \]
Clearly, for every \( i \in I \), \( r_i \in R^\perp \). To show this, fix \( i \) in \( I \), and assume that \( r_i = a_k + bs_k + x \), where \( a, b \in \mathcal{A}(X) \) and \( x \in R^\perp \). So, one has
\[ 0 = \phi_X(r_i, s_k) = a, \quad 0 = \phi_X(r_i, r_k) = b, \]
which corroborates the claim that \( r_i \in R^\perp \) for all \( i \in I \). On the other hand, let us consider the sub-\( \mathcal{A}(X) \)-module, \( P \), generated by \( r_k \) and \( s_k \), viz. \( P = [r_k, s_k] \). As in Case (2), one shows that
\[ \mathcal{E}(X) = P \oplus P^\perp. \]
Since \( r_k \in \mathcal{E}(X) \), there exists \( a_j \in \mathcal{A}(X) \) such that
\[ r_k = \sum_{j \in J} a_j s_j + x, \]
where \( x \in P^\perp \). For any \( j \neq k \) in \( J \), one has \( \phi_X(r_k, s_j) = 0 \). Thus, we have found a section \( r_k \in \mathcal{E}(X) \) such that \( \phi_X(r_i, r_k) = 0 \) for any \( i \in I \) and \( \phi_X(r_k, s_j) = \delta_{kj} \) for any \( j \in J \). Then \( A \cup B \cup \{ r_k \} \) is a family of linearly independent sections: the equality
\[ a_k r_k + \sum_{i \in I} a_i r_i + \sum_{j \in J} b_j s_j = 0 \]
implies that \( a_k = a_i = b_j = 0 \). Repeating this process as many times as necessary, we are lead back to Case (2), and the proof is finished.

Referring to Theorem 1.1, the basis \( \mathfrak{B} \) is called a symplectic \( \mathcal{A}(U) \)-basis of \( (\mathcal{E}(U), \phi_U) \).
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Corollary 1.1 If \((\mathcal{E}, \phi)\) is a symplectic free \(\mathcal{A}\)-module of rank \(2n\), then, for every open \(U \subseteq X\),
\[
\mathcal{E}(U) = H_1^U \oplus \cdots \oplus H_n^U,
\]
where \(H_1^U, \ldots, H_n^U\) are pairwise orthogonal non-isotropic two-dimensional sub-
\(\mathcal{A}(U)\)-modules.

Proof. The proof is similar to a good extent to the first part of the proof of
Theorem 1.1. In fact, let \(U\) be an open subset of \(X\) and \(r_1 \in \mathcal{E}(U)\), a nowhere-
zero section. There exists a section \(s_1 \in \mathcal{E}(U)\) such that \(\phi_V(r_1|_V, s_1|_V) \neq 0\) for
any open \(V \subseteq U\). Clearly, \(r_1, s_1\) must be linearly independent, and the sub-
\(\mathcal{A}(U)\)-module \(H_1^U := [r_1, s_1]\), spanned by \(r_1\) and \(s_1\), is non-isotropic.
As in the proof of Theorem 1.1, Case (1), one has
\[
\mathcal{E}(U) = H_1^U \oplus H_1^\perp.
\]
The restriction \(\phi_{H_1^\perp} \equiv (\phi_U)|_{H_1^\perp}\) of \(\phi_U\) to \(H_1^\perp\) is non-degenerate, because if
\(t \in H_1^\perp\) is such that \(\phi_{H_1^\perp}(t, z) = \phi_U(t, z) = 0\) for all \(z \in H_1^\perp\), then \(t \in H_1^\perp = (H_1^\perp)^\perp\) and hence \(t \in H_1^\perp \cap H_1^\perp = (H_1 + H_1^\perp)^\perp = \mathcal{E}(U)^\perp = 0\), which implies
that \(t = 0\). Thus, \((H_1^\perp, \phi_{H_1^\perp})\) is a symplectic free \(\mathcal{A}(U)\)-module of rank \(2(n-1)\).
Next, take a nowhere-zero \(r_2 \in H_1^\perp\); since \(\phi_U(r_2, r_1) = \phi_U(r_2, s_1) = 0\), there
exists a section \(s_2 \in H_1^\perp\) such that \(\phi_V(r_2|_V, s_2|_V) \neq 0\) for any open \(V \subseteq U\).
As above, one has
\[
H_1^\perp = H_2^\perp + H_2^\perp,
\]
where \(H_2 := [r_2, s_2]\). The direct decomposition sum of \(\mathcal{E}(U)\) follows by re-
peating the construction above \(n - 2\) times.

Each sub-\(\mathcal{A}(U)\)-module \(H_i^U\) in Corollary 1.1 has an ordered basis \((r_i, s_i)\)
such that \((\phi_U(r_i, s_i))|_V \equiv (\phi_V(r_i|_V, s_i|_V) := a_i|_V \neq 0\) for any open subset \(V\)
of \(U\). Then, based on the hypothesis that every nowhere-zero section of \(\mathcal{A}\) is
invertible, see Scholium 1.2, the restriction of \(\phi_U\) to \(H_i^U\) with respect to the
basis \((r_i, a_i^{-1}s_i)\) has matrix
\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
Hence, we have
Corollary 1.2 If \((E, \phi)\) is a symplectic free \(A\)-module of rank \(2n\), then for every open subset \(U\) of \(X\), there exists an ordered basis of \(E(U)\) with respect to which \(\phi_U\) has matrix

\[
A_{2n}^U = \begin{pmatrix}
0 & 1 & & & \\
-1 & 0 & & & \\
& & \ddots & & \\
& & & 0 & 1 \\
& & & -1 & 0
\end{pmatrix}.
\]

Moreover, symplectic \(A\)-modules of the same rank are isometric.

2 Orthosymmetric \(A\)-bilinear forms

Definition 2.1 An \(A\)-bilinear form \(\phi : E \oplus E \longrightarrow A\) on an \(A\)-module \(E\) is called orthosymmetric if the following is true:

\[
\phi_U(r, s) = 0 \quad \text{is equivalent to} \quad \phi_U(s, r) = 0,
\]

for all \(r, s \in E(U)\), with \(U\) any open subset of \(X\).

It is clear that if \(\phi\) is orthosymmetric, then \(\perp \equiv \perp(\phi) \equiv \top(\phi) \equiv \top\), i.e. \(F^\perp = F^\top\) for any sub-\(A\)-module \(F\) of \(E\). Moreover, if \(\phi\) is symmetric or skew-symmetric, then \(\phi\) is orthosymmetric. The following theorem shows that the converse of the preceding statement is true on every open subset of \(X\).

Theorem 2.1 Let \(E\) be an \(A\)-module and \(\phi \equiv (\phi_U) : E \oplus E \longrightarrow A\) an orthosymmetric \(A\)-bilinear form. Then, componentwise \(\phi\) is either symmetric or skew-symmetric.

Proof. Let \(U\) be an open subset of \(X\), and \(r, s, t \in E(U)\). Clearly, we have

\[
\phi_U(r, \phi_U(r, s)t) - \phi_U(r, \phi_U(r, s)t) = \phi_U(r, t)\phi_U(r, s) - \phi_U(r, s)\phi_U(r, t) = 0,
\]
but
\[ \phi_U(r, \phi_U(r, t)s - \phi_U(r, s)t) = 0 \]

is equivalent to
\[ \phi_U(\phi_U(r, t)s - \phi_U(r, s)t, r) = 0; \]

thus we obtain
\[ \phi_U(r, t)\phi_U(s, r) = \phi_U(r, s)\phi_U(t, r). \tag{3} \]

For \( t = r \), \( \phi_U(r, r)\phi_U(s, r) = \phi_U(r, s)\phi_U(r, r) \).
If
\[ \phi_V(r|_V, s|_V) \neq \phi_V(s|_V, r|_V), \text{ for any open } V \subseteq U, \tag{4} \]

then
\[ \phi_U(r, r) = 0. \]

(We note in passing that (4) suggests that both \( \phi_V(r|_V, s|_V) \) and \( \phi_V(s|_V, r|_V) \) are nowhere zero on \( V \), because if, for instance, \( \phi_V(r|_V, s|_V)(x) = 0 \) for some \( x \in V \) then \( \phi_V(r|_V, s|_V) = 0 \) on some open neighborhood \( R \subseteq V \) of \( x \) (cf. Mallios [9, (3.7), p.13]), i.e., assuming that \( \rho_R^U \) and \( \sigma_R^U \) are the restriction maps for the presheaves of sections of \( E \) and \( \mathcal{A} \), respectively, we have
\[ \sigma_R^U(\phi_U(s, r)) = \phi_R(\rho_R^U(s), \rho_R^U(r)) \equiv \phi_R(s|_R, r|_R) = 0, \]
which, by hypothesis, is equivalent to \( \phi_R(r|_R, s|_R) = 0 \). That is a contradiction to (4).

Similarly, as
\[ \phi_U(s, \phi_U(s, t)r) - \phi_U(s, \phi_U(s, r)t) = 0, \]

which, obviously, leads to
\[ \phi_U(s, t)\phi_U(r, s) = \phi_U(s, r)\phi_U(t, s), \tag{5} \]

one has, for \( t = s \),
\[ \phi_U(s, s)\phi_U(r, s) = \phi_U(s, r)\phi_U(s, s). \]

Using (4), we have
\[ \phi_U(s, s) = 0. \]
We actually have more than just what we have obtained so far. Indeed, if (4) holds, then $\phi_U(t, t) = 0$ for all $t \in \mathcal{E}(U)$. We prove this statement as follows.

(A) Let $\phi_V(r|_V, t|_V) \neq \phi_V(t|_V, r|_V)$ for any open $V \subseteq U$. Since

$$\phi_U(t, r)\phi_U(s, t) = \phi_U(t, s)\phi_U(r, t), \quad (6)$$

by putting $s = t$, we have $\phi_U(t, t) = 0$.

(B) Suppose that there exists an open $W \subseteq U$ such that $\phi_W(r|_W, t|_W) = \phi_W(t|_W, r|_W)$. Then, by virtue of (3) and since $\phi_W(r|_W, s|_W) \neq \phi_W(s|_W, r|_W)$ everywhere on $W$, it follows that

$$\phi_W(r|_W, t|_W) = 0.$$ 

On the other hand, suppose that $\phi_V(s|_V, t|_V) \neq \phi_V(t|_V, s|_V)$ for any open $V \subseteq U$. Putting $r = t$ in (3), one gets $\phi_U(t, t) = 0$. Now, assume that there exists an open $T \subseteq U$ such that $\phi_T(s|_T, t|_T) = \phi_U(t|_T, s|_T)$ and for any open subset $V \subseteq U \setminus T$, where $T$ is the closure of $T$ in $X$, $\phi_V(s|_V, t|_V) \neq \phi_V(t|_V, s|_V)$. By virtue of (3) and of

$$\phi_T(s|_T, r|_T) \neq \phi_T(r|_T, s|_T),$$

it follows that

$$\phi_T(s|_T, t|_T) = \phi_T(t|_T, s|_T) = 0.$$ 

Hence,

$$\phi_T(r|_T + t|_T, s|_T) = \phi_T(r|_T, s|_T) \neq \phi_T(s|_T, r|_T) = \phi_T(s|_T, r|_T + t|_T),$$

and if we substitute $r|_T + t|_T$ and $s|_T$ for $t|_V$ and $r|_V$ respectively in (A), we get

$$\phi_T(r|_T + t|_T, r|_T + t|_T) = 0.$$ 

But $\phi_T(r|_T, r|_T) = 0$ (since $\phi_U(r, r) = 0$ and $T \subseteq U$ is open), then if $\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T) = 0$, one has

$$\phi_T(t|_T, t|_T) = 0. \quad (7)$$

If $\phi_T(r|_T, t|_T) \neq 0 \neq \phi_T(t|_T, r|_T)$ everywhere on $T$, and $\phi_T(r|_T, t|_T) \neq \phi_T(t|_T, r|_T)$, we deduce from (3), by putting $s = t$, $\phi_T(t|_T, t|_T) = 0$. If instead we have $\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T)$, we will end up with

$$\phi_T(r|_T, t|_T) = \phi_T(t|_T, r|_T) = 0.$$
which leads to (7) as previously shown. Next, \( \phi_V(s|V,t|V) \neq \phi_V(t|V,s|V) \) for every open \( V \subseteq U \setminus \overline{T} \), so \( \phi_V(t|V,t|V) = 0 \) for every such \( V \); coupling the latter observation with (7) and the fact that sections are continuous, one gets in this case too that \( \phi_V(t,t) = 0 \).

We have shown that there are only two cases: either \( \phi_U(r,r) = 0 \) for all \( r \in \mathcal{E}(U) \), or for some \( r \in \mathcal{E}(U) \), \( \phi_U(r,r) \neq 0 \), from which we deduce that \( \phi_U(r,s) = \phi_U(s,r) \) for all \( r, s \in \mathcal{E}(U) \).

Finally, we notice in ending the proof that if \( \phi_U(r,r) = 0 \) for all \( r \in \mathcal{E}(U) \), then

\[
\phi_U(r,s) = -\phi_U(s,r)
\]

for all \( r, s \in \mathcal{E}(U) \). ■

**Scholium 2.1** In connection with the proof of Theorem 2.1, if there exists an open subset \( L \subseteq T \) such that \( \phi_L(r|L,t|L) = \phi_L(t|L,r|L) = 0 \) and \( \phi_V(r|V,t|V) \neq \phi_V(t|V,r|V) \) for every \( V \subseteq T \setminus \overline{L} \), where \( \overline{L} \) is the closure of \( L \) in \( X \), then \( \phi_L(t|L,t|L) = 0 \) and \( \phi_V(t|V,t|V) = 0 \) for every open \( V \subseteq T \setminus \overline{L} \). Hence, \( \phi_T(t|T,t|T) = 0 \).

Referring still to Theorem 2.1 if \( \phi_U \) is symmetric, the geometry is called **orthogonal**. If \( \phi_U \) is skew-symmetric, the geometry is called **symplectic**. No other case can occur if \( \phi \) must be orthosymmetric. A *pairing* \((\mathcal{E}, \phi)\) is called **symmetric** if every \( \phi_U \) is symmetric, and **skew-symmetric** if every \( \phi_U \) is skew-symmetric.

**Definition 2.2** Let \((\mathcal{E}, \phi) \equiv [(\mathcal{E}, \phi); \mathcal{A}] \equiv [((\mathcal{E}, \mathcal{E}); \phi); \mathcal{A}]\) be a self-pairing of an \( \mathcal{A} \)-module \( \mathcal{E} \), where \( \phi \) is orthosymmetric. Then, by the **radical** of \( \mathcal{E} \), we mean the **orthogonal** \( \mathcal{E}^\perp \). If \( \mathcal{F} \) is a sub-\( \mathcal{A} \)-module of \( \mathcal{E} \), the **radical**, \( \text{rad} \mathcal{F} \), of \( \mathcal{F} \) is defined as \( \mathcal{F} \cap \mathcal{F}^\perp \). If \( \text{rad} \mathcal{F} = 0 \), \( \mathcal{F} \) is said to be **non-isotropic**; otherwise, it is called **isotropic**.

**Lemma 2.1** Let \((\mathcal{E}, \phi)\) be an \( \mathcal{A} \)-module and \( \mathcal{F} \) a sub-\( \mathcal{A} \)-module of \( \mathcal{E} \). If \( \phi \) is orthosymmetric and \( \mathcal{E} = \mathcal{F} \oplus \mathcal{E}^\perp \), then \( \mathcal{F} \) is non-isotropic.
Proof. Let $U$ be an open subset of $X$, and $r \in F^\perp(U)$, i.e. $\phi_V(F(V),r|_V) = 0$ for any open $V \subseteq U$. But $\phi_V(E^\perp(V),r|_V) = \phi_V(E^\top(V),r|_V) = 0$ for any open $V \subseteq U$, because $E^\perp = E^\top$, and therefore

$$\phi_V(F(V) + E^\perp(V),r|_V) = \phi_V(F(V),r|_V) = 0$$

for any open $V \subseteq U$. Hence, $r \in E^\perp(U)$. We have thus $F^\perp(U) \subseteq E^\perp(U)$, so that $F(U) \cap F^\perp(U) = (F \cap F^\perp)(U) := (\text{rad}(F))(U) = 0$.

Definition 2.3 Let $E$ be an $A$-module. An $A$-endomorphism $\phi \in \text{End} E$ is called $A$-involution if $\phi^2 = \text{Id}_E$. An $A$-projection is an $A$-endomorphism $p \in \text{End} E$ such that $p^2 = p$, in other words $p$ is idempotent. The $A$-morphism $q \equiv \text{Id}_E - p$ is clearly an $A$-projection; $p$ and $q$ are called supplementary $A$-projections.

Lemma 2.2 Let $(E,\phi)$ be a free $A$-module of finite rank. Then, every non-isotropic free sub-$A$-module $F$ of $E$ is a direct summand of $E$; viz.

$$E = F \perp F^\perp.$$
for all $s \in F(V)$ and where $V$ is open in $U$. But for every $t \in E(U)$, $p(t)$ is unique, therefore $p(r + t) = p(r) + p(t)$. Likewise, one shows that for all $\alpha \in A(U)$, $p(\alpha t) = \alpha p(t)$. The observation undertaken about $p$ means that $p : E(U) \rightarrow E(U)$ is $A(U)$-linear. Next, since $p^2 = p$, then the $A(U)$-morphism $p : E(U) \rightarrow F(U)$ is an $A(U)$-projection. Furthermore, since

$$\phi_V((t - p(t))|_V, s) = \phi_V(t|_V - p(t)|_V, s) = 0$$

for all $t \in E(U)$ and $s \in F(V)$, with $V$ open in $U$, the supplementary $A(U)$-projection $q := I - p$ is such that for all $t \in E(U)$, $q(t) \equiv (I - p)(t) \in F(U)$, i.e. $q$ maps $E(U)$ on $F(U)$. Hence, every element $t \in E(U)$, where $U$ runs over the open subsets of $X$, may be written as

$$t = p(t) + (t - p(t))$$

with $p(t) \in F(U)$ and $t - p(t) \in F(U)$, thus

$$E(U) = F(U) \oplus F(U) = (F \oplus F)(U)$$

within $A(U)$-isomorphisms (see cf. Mallios [9, relation (3.14), p.122] for the $A(U)$-isomorphism $F(U) \oplus F(U) = (F \oplus F)(U)$). Finally, since $F$ is non-isotropic, it follows that

$$E(U) = (F \perp F)(U)$$

for every open $U \subseteq X$. Thus, we reach the sought $A$-isomorphism of the lemma. ■

**Definition 2.4** A convenient $A$-module is a self-pairing $(E, \phi)$, where $E$ is a free $A$-module of finite rank and $\phi$ an orthosymmetric $A$-bilinear form, such that the following conditions are satisfied.

1. If $F$ is a free sub-$A$-module of $E$, then the orthogonal $F^\perp$ and the radical $\text{rad} F$ are free sub-$A$-modules of $E$.

2. Every free sub-$A$-module $F$ of $E$ is orthogonally reflexive, i.e. $(F^\perp)^\perp \equiv F = F$. 

3. The intersection of any two free sub-$A$-modules of $E$ is a free sub-$A$-module.
Lemma 2.3 If \((\mathcal{E}, \phi)\) is a convenient \(\mathcal{A}\)-module, then, given any two free sub-\(\mathcal{A}\)-modules \(\mathcal{G}\) and \(\mathcal{H}\) of \(\mathcal{E}\), one has
\[
(\mathcal{G} \cap \mathcal{H})^\perp = \mathcal{G}^\perp + \mathcal{H}^\perp.
\]

Proof. By virtue of Lemma 1.4, we have
\[
\left(\mathcal{G}^\perp + \mathcal{H}^\perp\right)^\perp = \left(\mathcal{G}^\perp + (\mathcal{H}^\perp)^\perp\right)
\]
whence
\[
\mathcal{G}^\perp + \mathcal{H}^\perp = (\mathcal{G}^\perp + \mathcal{H}^\perp)^\perp = (\mathcal{G} \cap \mathcal{H})^\perp.
\]

Lemma 2.4 If \((\mathcal{E}, \phi)\) is a convenient \(\mathcal{A}\)-module and \(\mathcal{F}\) a non-isotropic free sub-\(\mathcal{A}\)-module of \(\mathcal{E}\), then \((\mathcal{F}, \tilde{\phi})\), where \(\tilde{\phi} := \phi|_\mathcal{F}\), is a convenient \(\mathcal{A}\)-module.

Proof. Let \(\perp(\tilde{\phi})\) and \(\perp(\phi)\) denote orthogonality with respect to \(\tilde{\phi}\) and \(\phi\) respectively. Let \(\mathcal{G}\) and \(\mathcal{H}\) be sub-\(\mathcal{A}\)-modules of \(\mathcal{F}\).

1. That \(\mathcal{G}^\perp(\tilde{\phi})\) and \(\text{rad}_{\tilde{\phi}}\mathcal{G}\) are free sub-\(\mathcal{A}\)-modules is clear. Indeed,
\[
\mathcal{G}^\perp(\tilde{\phi}) = \mathcal{G}^\perp(\phi) \cap \mathcal{F}
\]
and
\[
\text{rad}_{\tilde{\phi}}\mathcal{G} := \mathcal{G} \cap \mathcal{G}^\perp(\tilde{\phi}) = \mathcal{G} \cap (\mathcal{G}^\perp(\phi) \cap \mathcal{F}) = (\mathcal{G} \cap \mathcal{G}^\perp(\phi)) \cap \mathcal{F} =: \text{rad}_{\phi}\mathcal{G} \cap \mathcal{F}.
\]

2. By an easy calculation, we have
\[
\mathcal{G}^\perp(\tilde{\phi})^\perp = (\mathcal{G}^\perp(\tilde{\phi}))^\perp \cap \mathcal{F}
\]
\[
= (\mathcal{G}^\perp(\phi) \cap \mathcal{F})^\perp \cap \mathcal{F}
\]
\[
= (\mathcal{G}^\perp(\phi)^\perp \cap (\mathcal{F}^\perp(\phi))) \cap \mathcal{F}
\]
\[
= (\mathcal{G} \cap \mathcal{F}) + (\mathcal{F}^\perp(\phi) \cap \mathcal{F})
\]
\[
= \mathcal{G} \cap \mathcal{F}
\]
\[
= \mathcal{G}
\]
(3) Immediate. ■

We now turn to the following theorem.

**Theorem 2.2** Let \((\mathcal{E}, \phi)\) be a non-isotropic skew-symmetric convenient \(A\)-module, and \(\mathcal{F}\) a totally isotropic sub-\(A\)-module of rank \(k\). Then, there is a non-isotropic sub-\(A\)-module \(\mathcal{H}\) of \(\mathcal{E}\) of the form

\[ \mathcal{H} = \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_k, \]

where if \(\mathcal{F}(U) = [r_{1,U}, \ldots, r_{k,U}]\) with \(U\) an open subset of \(X\), then \(r_{i,U} \in \mathcal{H}_i(U)\) for \(1 \leq i \leq k\).

**Proof.** Suppose that \(k = 1\), i.e. \(\mathcal{F} \cong A\). If \(\mathcal{F}(X) = [r_X]\) with \(r_X \in \mathcal{E}(X)\) a nowhere-zero section, then for every open \(U \subseteq X, \mathcal{F}(U) = [r_U]\), where \(r_U = r_X|_U\). Since \(\phi_X\) is non-degenerate, there exists a nowhere-zero section \(s_X \in \mathcal{E}(X)\) such that \(\phi_U(r_X|_U, s_X|_U) \neq 0\) for every open \(U \subseteq X\). The correspondence

\[ U \mapsto \mathcal{H}(U) := [r_U, s_U] \equiv [r_X|_U, s_X|_U], \]

where \(U\) runs over the open sets in \(X\), along with the obvious restriction maps, yields a complete presheaf of \(A\)-modules on \(X\). Clearly, the pair \((\mathcal{H}, \tilde{\phi})\), where \(\tilde{\phi}\) is the \(A\)-bilinear morphism \(\tilde{\phi} : \mathcal{H} \oplus \mathcal{H} \rightarrow A\) such that

\[ (r_U, s_U) \mapsto \tilde{\phi}_U(r_U, s_U) := \phi_U(r_U, s_U), \]

is non-isotropic. Hence, the theorem holds for the case \(k = 1\). Let us now proceed by induction to \(k > 1\). To this end, put \(\mathcal{F}_{k-1} \cong A^{k-1}\) and \(\mathcal{F}_k := \mathcal{F} \cong A^k\). Then, \(\mathcal{F}_{k-1} \not\subsetneq \mathcal{F}_k\), so \(\mathcal{F}_k^\perp \not\subsetneq \mathcal{F}_{k-1}^\perp\). Since orthogonal of free sub-\(A\)-modules in a convenient \(A\)-module are free sub-\(A\)-modules, the inclusion \(\mathcal{F}_k^\perp \not\subsetneq \mathcal{F}_{k-1}^\perp\) implies that, if \(\mathcal{F}_{k-1}^\perp \cong A^m\) and \(\mathcal{F}_k^\perp \cong A^n\) with \(n < m\), then \(\mathcal{F}_{k-1}^\perp \setminus \mathcal{F}_k^\perp \cong A^{m-n}\). For every open \(U \subseteq X\), pick \(s_{k,U} \in \mathcal{F}_{k-1}^\perp(U)\), put \(\mathcal{H}_k(U) = [r_{k,U}, s_{k,U}]\). The correspondence

\[ U \mapsto \mathcal{H}_k(U), \]

where \(U\) is open in \(X\), along with the obvious restriction maps, is a complete presheaf of \(A(U)\)-modules. Since \(\phi_U(r_{i,U}, s_{k,U}) = 0\) for \(1 \leq i \leq k - 1\),
φ_U(r_{k,U}, s_{k,U}) \neq 0. Hence, \mathcal{H}_k(U) is a non-isotropic \mathcal{A}(U)-plane containing r_{k,U}. By Lemma 2.2, \mathcal{E} = \mathcal{H}_k \perp \mathcal{H}_k^\perp. Since r_{k,U}, s_{k,U} \in \mathcal{F}_{k-1}(U), \mathcal{H}_k(U) \subseteq \mathcal{F}_{k-1}^\perp(U) for every open \ U \subseteq X; so \mathcal{H}_k \subseteq \mathcal{F}_{k-1}^\perp, which in turn implies that \mathcal{F}_{k-1} \subseteq \mathcal{H}_k^\perp. Apply an inductive argument to \mathcal{F}_{k-1} regarded as a sub-\mathcal{A}-module of the non-isotropic skew-symmetric convenient \mathcal{A}-module \mathcal{H}_k^\perp. 

We are now set for the analog of the Witt’s theorem; to this end we assume that \((X, \mathcal{A})\) is an algebraized space satisfying the condition of Scholium 1.2. For the classical Witt’s theorem, see Adkins-Weintraub [11, pp 368-387], Artin [2, pp 121, 122], Berndt [3, p 21], Crumeyrolle [4, pp 11, 12], Deheuvels [5, pp 148, 152], Lang [8, pp 591, 592], O’Meara [16, p 9].

**Theorem 2.3 (Witt’s Theorem)** Let \(\mathcal{E} \equiv (\mathcal{E}, \phi)\) and \(\mathcal{E}' \equiv (\mathcal{E}', \phi')\) be isometric non-isotropic skew-symmetric convenient \mathcal{A}-modules, \(\mathcal{F} \equiv (\mathcal{F}, \tilde{\phi})\), where \(\tilde{\phi} := \phi|_{\mathcal{F}}, a free sub-\mathcal{A}\)-module of \(\mathcal{E}\), and \(\sigma \equiv (\sigma_U) : \mathcal{F} \longrightarrow \mathcal{E}'\) an \(\mathcal{A}\)-isometry of \(\mathcal{F}\) into \(\mathcal{E}'\). Then, \(\sigma\) extends to an \(\mathcal{A}\)-isometry of \(\mathcal{E}\) onto \(\mathcal{E}'\).

**Proof.** Since \(\mathcal{E}\) is convenient and \(\mathcal{F}\) is a free sub-\(\mathcal{A}\)-module of \(\mathcal{E}\), there exists a free sub-\(\mathcal{A}\)-module of \(\mathcal{E}\) such that \(\mathcal{F} = \mathcal{G} \perp \operatorname{rad} \mathcal{F}\), where if \(\mathcal{F}\) and \(\operatorname{rad} \mathcal{F}\) are \(\mathcal{A}\)-isomorphic to \(\mathcal{A}^k\) and \(\mathcal{A}^l\) respectively, then \(\mathcal{G}\) is \(\mathcal{A}\)-isomorphic to \(\mathcal{A}^{k-l}\). By Lemma 1.4(1), \(\mathcal{F}^\perp \subseteq \mathcal{G}^\perp\); since \(\mathcal{G}^\perp\) is non-isotropic and skew-symmetric, and \(\operatorname{rad} \mathcal{F}\) is a totally isotropic free sub-\(\mathcal{A}\)-module, by applying Theorem 2.2, we see that there is a free sub-\(\mathcal{A}\)-module \(\mathcal{H}\) of \(\mathcal{G}^\perp\) of the form

\[
\mathcal{H} := \mathcal{H}_1 \perp \cdots \perp \mathcal{H}_l
\]

in which each \(\mathcal{H}_i\) is a non-isotropic free sub-\(\mathcal{A}\)-module of rank 2 and such that if

\[
(\operatorname{rad} \mathcal{F})(U) = [r_{1,U}, \ldots, r_{l,U}],
\]

where \(U\) is an open subset of \(X\), then \(r_{i,U} \in \mathcal{H}_i(U)\) with \(i = 1, \ldots, l\). Since \(\mathcal{H}\) is non-isotropic it splits \(\mathcal{G}^\perp: \mathcal{G}^\perp = \mathcal{H} \perp \mathcal{J};\) in fact, \(\mathcal{J} \cong \mathcal{H}^\perp\) (see Lemma 2.2). Hence,

\[
\mathcal{E} = \mathcal{G}^\perp \perp \mathcal{G} = \mathcal{H} \perp \mathcal{J} \perp \mathcal{G},
\]

within \(\mathcal{A}\)-isomorphisms respectively. Put \(\mathcal{F}' := \sigma(\mathcal{F}), \mathcal{G}' := \sigma(\mathcal{G})\) and \(r_{i,U}' := \sigma_U(r_{i,U}), 1 \leq i \leq l,\) for every open \(U \subseteq X\). Now, let us fix \(U\) in the topology of \(X\). Clearly,

\[
\mathcal{F}'(U) = \{t \in \mathcal{E}'(U) : \phi'_U(\sigma_U(s), t) = 0, s \in \mathcal{F}(U)\}.
\]
and
\[ \mathcal{F}(U)^\perp = \{ z \in \mathcal{E}(U) : \phi_U(s, z) = 0, \ s \in \mathcal{F}(U) \}. \]

For every \( z \in \mathcal{F}(U)^\perp \), we have for all \( s \in \mathcal{F}(U) \)
\[ \phi'_U(\sigma_U(s), \sigma_U(z)) = \phi_U(s, z) = 0; \]
we thus deduce that
\[ \sigma_U(\mathcal{F}^\perp(U)) = \sigma_U(\mathcal{F}(U)^\perp) \subseteq \mathcal{F}'(U)^\perp = \mathcal{F}^\perp(U). \]

hence,
\[ \sigma_U(\text{rad } \mathcal{F}(U)) := \sigma_U(\mathcal{F}(U) \cap \mathcal{F}(U)^\perp) = \sigma_U(\mathcal{F}(U)) \cap \sigma_U(\mathcal{F}(U)^\perp), \text{ since } \sigma_U \text{ is an } \mathcal{A}(U)-\text{isomorphism} \]
\[ \subseteq \mathcal{F}'(U) \cap \mathcal{F}'(U)^\perp = \text{rad } \mathcal{F}'(U) := \text{rad } \sigma_U(\mathcal{F}(U)). \]

Conversely, let \( \text{rad } \sigma_U(\mathcal{F}(U)) := \sigma_U(\mathcal{F}(U)) \cap \sigma_U(\mathcal{F}(U)^\perp) \). As \( \sigma_U \) is an \( \mathcal{A}(U) \)-isomorphism there exists a unique \( s \in \mathcal{F}(U) \) such that \( t = \sigma_U(s) \). But
\[ 0 = \phi'_U(\sigma_U(s), \sigma_U(s)) = \phi_U(r, s) \]
for every \( r \in \mathcal{F}(U) \). Consequently, \( s \in \mathcal{F}(U)^\perp \). Thus,
\[ s \in \mathcal{F}(U) \cap \mathcal{F}(U)^\perp =: \text{rad } \mathcal{F}(U); \]

hence
\[ t \in \sigma_U(\text{rad } \mathcal{F}(U)), \]

from which we deduce that
\[ \text{rad } \sigma_U(\mathcal{F}(U)) \subseteq \sigma_U(\text{rad } \mathcal{F}(U)). \]

The end result of this argument is that
\[ \text{rad } \sigma_U(\mathcal{F}(U)) = \sigma_U(\text{rad } \mathcal{F}(U)). \]

Since \( U \) is arbitrary, it follows that
\[ \text{rad } \mathcal{F}' \equiv \text{rad } \sigma(\mathcal{F}) = \sigma(\text{rad } \mathcal{F}) \cong \mathcal{A}^l. \]

Since \( \sigma \) is an \( \mathcal{A} \)-isomterry, we obtain that
\[ \mathcal{F}' := \sigma(\mathcal{F}) = \sigma(\mathcal{G} \perp \text{rad } \mathcal{F}) = \mathcal{G} \perp \text{rad } \mathcal{F}' \]
is a radical splitting of $\mathcal{F}'$. Repeating the early argument, we have

$$\mathcal{E}' = \mathcal{H}' \perp \mathcal{J}' \perp \mathcal{G}'$$

in which

$$\mathcal{H}' = \mathcal{H}_1' \perp \cdots \perp \mathcal{H}_l'$$

with each $\mathcal{H}_i'$ a non-isotropic free sub-$\mathcal{A}$-module of rank 2 such that if $(\text{rad} \mathcal{F}')(U) = [r_{i,U}']$, where $U$ is open in $X$, then $r_{i,U}' \in \mathcal{H}_i'(U)$ for every $1 \leq i \leq l$. Suppose for every $i = 1, \ldots, l$, $\mathcal{H}_i(U) = [r_{i,U}, s_{i,U}]$ and $\mathcal{H}_i'(U) = [r_{i,U}', s_{i,U}']$. Let $\alpha = (\alpha_U) : \mathcal{H} \rightarrow \mathcal{H}'$ be an $\mathcal{A}$-morphism, given by the prescription

$$\alpha_U(r_{i,U}) = r_{i,U}' \quad \text{and} \quad \alpha_U(s_{i,U}) = s_{i,U}'$$

for every open $U \subseteq X$ and $i = 1, \ldots, l$. That $\alpha$ is an $\mathcal{A}$-isomorphism is clear.

Next, observe that for every open $U \subseteq X$ and $i = 1, \ldots, l$, since $\phi_U$ and $\phi_U'$ are non-degenerate, $\phi_U(r_{i,U}, s_{i,U})$ and $\phi_U'(r_{i,U}', s_{i,U}')$ are nowhere zero sections; consequently based on the hypothesis regarding the coefficient algebra sheaf $\mathcal{A}$, $\phi_U(r_{i,U}, s_{i,U})$ and $\phi_U'(r_{i,U}', s_{i,U}')$ are invertible. It is clear that for every open $U \subseteq X$ and $i = 1, \ldots, l$,

$$\mathcal{H}_i(U) = [r_{i,U}', s_{i,U}' \phi_U(r_{i,U}, s_{i,U})(\phi_U'(r_{i,U}', s_{i,U}'))^{-1}]$$

The $\mathcal{A}$-morphism $\beta \equiv (\beta_U) : \mathcal{H} \rightarrow \mathcal{H}'$ given by

$$\beta_U(r_{i,U}) = r_{i,U}' \quad \text{and} \quad \beta_U(s_{i,U}) = s_{i,U}' \phi_U(r_{i,U}, s_{i,U})(\phi_U'(r_{i,U}', s_{i,U}'))^{-1}$$

is clearly an $\mathcal{A}$-isomorphism such that

$$\phi_U'(\beta_U(r_{i,U}), \beta_U(s_{i,U})) = \phi_U(r_{i,U}, s_{i,U});$$

in other words, $\beta$ is an $\mathcal{A}$-isometry of $\mathcal{H}$ onto $\mathcal{H}'$. Furthermore, $\beta$ agrees with $\sigma$ on each $r_{i,U}$, and hence on rad $\mathcal{F}$. Also, the given $\sigma$ carries $\mathcal{G}$ onto $\mathcal{G}'$ isomorphically. Hence $\sigma$ extends to an $\mathcal{A}$-isometry of $\mathcal{H} \perp \mathcal{G}$ onto $\mathcal{H}' \perp \mathcal{G}'$.

Now, rank $(\mathcal{E}) = \text{rank } (\mathcal{E}')$; hence rank $(\mathcal{J}) = \text{rank } (\mathcal{J}')$; hence by Corollary [1.2] there is an $\mathcal{A}$-isometry of $\mathcal{J}$ onto $\mathcal{J}'$. Hence, finally, $\sigma$ extends to an isometry of $\mathcal{E} = (\mathcal{H} \perp \mathcal{G}) \perp \mathcal{J}$ onto $\mathcal{E}' = (\mathcal{H}' \perp \mathcal{G}') \perp \mathcal{J}'$. ■
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