We investigate the connection between the NSVZ and the DRED forms of the gauge 
\( \beta \)-function in an \( N = 1 \) supersymmetric gauge theory. We construct a coupling constant 
redefinition that relates the two forms up to four loops. By abelian calculations, we are 
able to infer the complete non-abelian form of \( \beta_g^{(3)DRED} \), and also \( \beta_g^{(4)DRED} \) except for 
one undetermined parameter.
1. Introduction

An all-orders formula for the gauge $\beta$-function in an $N = 1$ supersymmetric gauge theory was presented some years ago. This result (which we shall call $\beta_{g}^{NSVZ}$) originally appeared (for the special case of no chiral superfields) in Ref. [1], and was subsequently generalised, using instanton calculus, in Ref. [2]. (See also Ref. [3].) For a recent discussion emphasising the importance of holomorphy, see [4].

Recently the renormalisation group fixed points of $\beta_{g}^{NSVZ}$ have become important in the study of duality (for a review, see Ref. [5]). An interesting question, therefore, is as follows: given the renormalisation scheme dependence of $\beta$-functions beyond one loop, in which scheme is the NSVZ result valid? For instance, will calculations using standard dimensional reduction (DRED) give the NSVZ result? The DRED result certainly agrees with the NSVZ result at one and two loops; moreover certain properties of $\beta_{g}^{DRED}$ at higher loops are consistent with the NSVZ result. Namely, $\beta_{g}^{DRED}$ is known to vanish at three loops for a one-loop finite theory [6] [7] and furthermore, if we specialise to the $N = 2$ case, $\beta_{g}$ and the anomalous dimensions of the chiral superfields vanish beyond one loop [8]. One might accordingly be tempted to speculate that DRED will reproduce the NSVZ formula to all orders. However, in a recent note [9] we showed that this is not the case; at three loops the DRED result is related to the NSVZ result by a coupling constant redefinition. In the present paper we shall give more details of this calculation and also extend the result to four loops, at least in the abelian case.

Before proceeding, however, it is worthwhile emphasising the following point. It is sometimes asserted that the perturbative coefficients of $\beta_{g}$ are quite arbitrary beyond two loops, so that, for example, all contributions at three and more loops can be transformed to zero. We shall see, however, that in the general case (with a superpotential) the nature of possible changes in $\beta_{g}$ due to redefinitions $\delta g$ which are manifestly gauge-invariant analytic functions of $g$ and the Yukawa couplings $Y^{ijk}$ is heavily constrained. (One has also the freedom to make redefinitions $\delta Y^{ijk}$, but, as we shall see, these are not germane to the issue of whether the NSVZ and DRED results are equivalent.)

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1 We will discuss later whether this result will apply in schemes other than DRED.
2. The three-loop calculation

The Lagrangian $L_{\text{SUSY}}(W)$ for an $N = 1$ supersymmetric theory is defined by the superpotential

$$W = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j.$$  \hfill (2.1)

$L_{\text{SUSY}}$ is the Lagrangian for the $N = 1$ supersymmetric gauge theory, containing the gauge multiplet $V$ and a multiplet of chiral superfields $\Phi_i$ with component fields $\{\phi_i, \psi_i\}$, transforming as a representation $R$ of the gauge group $G$. We assume that there are no gauge-singlet fields. The $\beta$-functions for the Yukawa couplings $\beta_Y^{ijk}$ are given by

$$\beta_Y^{ijk} = Y^{p(ij)} Y^{k} p = Y^{ijp} Y^{k} p + (k \leftrightarrow i) + (k \leftrightarrow j),$$  \hfill (2.2)

where $\gamma$ is the anomalous dimension for $\Phi$. The one-loop results for the gauge coupling $\beta$-function $\beta_g$ and for $\gamma$ are given by

$$16\pi^2 \beta_g^{(1)} = g^3 Q, \quad \text{and} \quad 16\pi^2 \gamma^{(1)i} = P^i,$$  \hfill (2.3)

where

$$Q = T(R) - 3C(G), \quad \text{and} \quad P^i = \frac{1}{2} Y^{ikl} Y^{jkl} - 2g^2 C(R)^i_j.$$  \hfill (2.4a)

Here $Y_{jkl} = (Y^{jkl})^*$, and

$$T(R)\delta_{AB} = \text{tr}(R_A R_B), \quad C(G)\delta_{AB} = f_{ACD} f_{BCD} \quad \text{and} \quad C(R)^i_j = (R_A R_A)^i_j.$$  \hfill (2.5)

The two-loop $\beta$-functions for the dimensionless couplings were calculated in Refs. [1], [11], [13]:

$$(16\pi^2)^2 \beta_g^{(2)} = 2g^5 C(G)Q - 2g^3 r^{-1} C(R)^i_j P^i_j,$$  \hfill (2.6a)

$$(16\pi^2)^2 \gamma^{(2)i} = [- Y_{jmn} Y^{mpi} - 2g^2 C(R)^p_j \delta^i_n] P^m_p + 2g^4 C(R)^i_j Q,$$  \hfill (2.6b)

where $Q$ and $P^i_j$ are given by Eq. (2.4), and $r = \delta_{AA}$.

In our notation the NSVZ formula for $\beta_g$ is

$$\beta_{g}^{\text{NSVZ}} = \frac{g^3}{16\pi^2} \left[ \frac{Q - 2r^{-1} \text{tr}[\gamma C(R)]}{1 - 2C(G)g^2(16\pi^2)^{-1}} \right],$$  \hfill (2.7)
which leads to
\[
(16\pi^2)^3 \beta^{(3)}_{gNSVZ} = 4g^7QC(G)^2 - 4g^5C(G)r^{-1}16\pi^2\text{tr} \left[ \gamma^{(1)}C(R) \right]
- 2g^3r^{-1}(16\pi^2)^2\text{tr} \left[ \gamma^{(2)}C(R) \right].
\] (2.8)

As mentioned in the Introduction, one would like to know in which renormalisation scheme this result is valid, since \(\beta\)-functions are scheme dependent. It is easily seen from Eqs. (2.3) and (2.6) that the NSVZ result coincides with the results of DRED up to two loops. Will DRED reproduce the NSVZ result to all orders? There are two pieces of evidence which appear to favour this conjecture. Firstly, \(\beta^{(3)}_{gDRED}\) has been explicitly shown to vanish for a one-loop finite theory\(^\text{1}\), i.e. one for which \(P = Q = 0\), and it is clear from Eqs. (2.8) and (2.6) that this property is shared by \(\beta^{(3)}_{gNSVZ}\). A second piece of evidence comes from specialising to the \(N = 2\) case. In \(N = 1\) language, an \(N = 2\) theory is defined by the superpotential
\[
W = \sqrt{2}g\eta_A\chi^i S^{ji} \xi_j
\] (2.9)

where \(\eta, \chi\) and \(\xi\) transform according to the adjoint, \(S^*\) and \(S\) representations respectively. The set of chiral superfields \(\chi, \xi\) is called a hypermultiplet. \(N = 2\) theories have one-loop divergences only\(^\text{2}\); using DRED we may therefore expect that \(\beta_g\) and the anomalous dimension of both the \(\eta\) and the hypermultiplet should vanish beyond one loop. \(^\text{3}\) We see that the NSVZ result of Eq. (2.7) is consistent with this property; we have
\[
P_{\eta AB} = Qg^2\delta_{AB}
\]
\[
P_\chi = P_\xi = 0,
\] (2.10)

so that if \(\gamma\) vanishes beyond one loop then Eq. (2.7) reduces to \(\beta^{NSVZ}_g = \frac{g^3}{16\pi^2}Q\), which is of course the one-loop result. In particular, since from Eq. (2.6b) we have \(\gamma^{(2)} = 0\) for \(N = 2\), we have \(\beta^{(3)NSVZ}_g = 0\) for an \(N = 2\) theory. Nevertheless, despite these indications that the NSVZ result might coincide exactly with that obtained using DRED, we shall now show that in fact the NSVZ and DRED results part company at three loops. We shall see that they are related by a coupling constant redefinition corresponding to a change of renormalisation scheme. We shall calculate \(\beta^{(3)DRED}_g\) explicitly in the abelian case and

\(^2\) Clearly the absence of divergences beyond one loop implies that \(\beta_g\) vanishes beyond one loop as long as minimal subtraction is employed; higher order contributions to the \(\beta_g\) can be invoked by making finite subtractions, or equivalently by a redefinition \(g \rightarrow g + \delta g\).
construct the coupling constant redefinition which effects the transition to the NSVZ result. We shall then extend this redefinition to the full non-abelian case by exploiting the known $N = 2$ properties of the DRED result, and hence we shall deduce the full non-abelian $\beta_g^{(3)DRED}$.

We calculate $\beta_g$ by computing the divergences in the vector field two-point function, using the super-Feynman gauge. This is sufficient since in the abelian case the background superfield calculation and the normal superfield calculation are identical. The relevant diagrams are shown in Fig. 1. The shaded blobs represent one-loop self-energy insertions. We use dimensional reduction (DRED) with minimal subtraction, setting $\epsilon = 4 - d$. The divergent part of each individual diagram, after performing all the appropriate subtractions for divergent sub-diagrams, will be expressible as a combination $A\Pi_{1/2} + B\Pi_0$, where $A$ and $B$ are analytic in $\frac{1}{\epsilon}$ and the coupling constants, and $\Pi_{1/2}$ and $\Pi_0$ are projection operators defined in the Appendix. with $\Pi_{1/2} + \Pi_0 = -\partial^2$. Upon adding the results for all diagrams, the total will only involve $\Pi_{1/2}$, reflecting the transversality of the vector propagator. We use a convenient and efficient short-cut to calculate the total coefficient for $\Pi_{1/2}$ without calculating the full contribution for each diagram. The idea is the following: To obtain the total coefficient of $\Pi_{1/2}$, it will be sufficient to know the difference $B - A$ for each diagram. Upon summing over all diagrams, the sum over the $B$s will give zero just leaving the sum over the $A$s, which is what we want. The point is that we can obtain the combination $2(B - A)$ simply by adding up the divergent contribution to $D^2\bar{D}^2$ from each diagram, regarding the derivatives as exactly anticommuting. (So that $D^2\bar{D}^2$, $\bar{D}^2D^2$ and $D^\alpha\bar{D}^2D_\alpha$ would each count the same.) Each diagram may start with up to 10 $D$s and 10 $\bar{D}$s (all on internal lines). For each diagram, we manipulate the supercovariant derivatives $D$ and $\bar{D}$ using integration by parts and the anticommutation relations Eq. (A.1) until we obtain a set of diagrams each of which contains 8 $D$s and 8 $\bar{D}$s, possibly together with some ordinary derivatives. During this process we avoid integrating any $D$ or $\bar{D}$ onto an external line. The 3 $d^8\theta$ integrals will absorb 6 $D$s and 6 $\bar{D}$s, leaving 2 $D$s and 2 $\bar{D}$s which will contribute to $\Pi_{1/2}$ or $\Pi_0$. We are only interested in knowing the contribution to $D^2\bar{D}^2$, so from this point on we can treat the $D$s and $\bar{D}$s as exactly anticommuting. We now use integration by parts to arrange for each loop to contain 2 $D$s and 2 $\bar{D}$s, and integrate the remaining 2 $D$s and 2 $\bar{D}$s onto an external line, writing them in the form $D^2\bar{D}^2$. We can now do the $\theta$ integrals, leaving us with a momentum integral. We evaluate the momentum integral, subtract its subdivergences using minimal subtraction and finally obtain the divergent contribution to $D^2\bar{D}^2$. The reader might like to check the extent to
which this trick simplifies the one-loop (or even two-loop) calculation. The disadvantage, obviously, is that we lose the nice check afforded by the cancellation of the Π₀ terms.

The results calculated according to the above procedure for each diagram in Fig. 1 are shown in Table 1. The momentum integrals for each diagram can be expressed (using integration by parts) in terms of a basic set depicted in Fig. 2, which can be evaluated with their subtractions once and for all. In Fig. 2, a dot on a line represents a squared propagator, and arrows represent linear momentum factors in the numerator, the momenta which correspond to a pair of arrows being contracted together. Each diagram in Fig. 2 represents a Feynman integral of logarithmic overall divergence; there are no external lines because we have set the external momentum zero. This means that at least one propagator must be given a mass, since otherwise there is no scale defined and of course dimensional regularisation cannot be employed: the overall logarithmic ultra-violet and infra-red divergences would cancel. (Alternatively an external momentum may be “threaded” in an arbitrary way through the diagram). Also, any explicit infra-red divergence corresponding to a squared propagator must also be regularised by introducing a mass; such a mass will often serve to define the scale, too. Sometimes an alternative δ-function infra-red regulator can be useful[14], whereby instead of

\[ \frac{1}{(k^2)^2} \rightarrow \frac{1}{(k^2 + m^2)^2} \quad \text{or} \quad \frac{1}{k^2(k^2 + m^2)} \]

one has

\[ \frac{1}{(k^2)^2} \rightarrow \frac{1}{(k^2)^2} + \frac{2}{4 - d}\delta^{(4)}(k). \]

(Note that we have chosen to perform the diagrammatic calculation in Euclidean space, for which bookkeeping of factors of i etc. is easier.) After subtraction of ultra-violet subdivergences, the result for each diagram is independent both of the means by which a scale is introduced and the method used to regulate infra-red divergences. (Note that this statement is true only of the subtracted diagram.) The second column of Table 1 shows the expression for each momentum integral in terms of those in Fig. 2. (The presence of a zero in this column may indicate that the D-algebra for the diagram gave zero; or alternatively, that the momentum integral reduced to a “factorised” form which can be shown[13] to produce no simple pole after subtraction.) Each diagram in Fig. 1 also corresponds to a product of group matrices and Yukawa couplings which may be expressed in terms of
basic invariants, and which are given in the third column of Table 1 (including also the symmetry factor for the diagram). The invariants \( X_1, X_2, X_3 \) and \( X_4 \) are given by

\[
X_1 = g^2 Y^{klm} P_{lm} C(R)^n Y_{kn} = g^2 \text{tr} [S_4 C(R)] \\
X_2 = g^4 Y^{klm} C(R)^n C(R)^p Y_{kn} = g^4 \text{tr} [S_1 C(R)] , \\
X_3 = g^4 \text{tr}[PC(R)^2], \quad X_4 = g^2 \text{tr}[P^2 C(R)].
\]

(See Appendix A for the definitions of \( S_1, S_4 \) and other notational details). Upon adding all the results, we find

\[
<VV>_{\text{pole}} = 2r^{-1} \left\{ (2A - C)(X_1 + 2X_3 - 2g^6 Q \text{tr}[C(R)^2]) + (2B + C - 2D)X_4 \right\}.
\]

(2.14)

We note that all contributions involving the diagram \( E \) have cancelled; this will be important, since \( E \) is the only diagram out of our basic set in Fig. 2 whose simple pole involves \( \zeta(3) \). We also note that the invariant \( X_2 \) does not appear in the final result, and therefore \( <VV>_{\text{pole}} \) (and hence \( \beta_g \)) vanish in the one-loop finite case, when \( P = Q = 0 \).

It is interesting that we can draw this conclusion without evaluating any of the Feynman integrals. The \( \beta \)-function is derived from the simple poles in the subtracted diagrams. We find

\[
A_{\text{simple}} = \frac{4}{3} \frac{1}{(16\pi^2)^3 \epsilon}, \quad B_{\text{simple}} = -\frac{2}{3} \frac{1}{(16\pi^2)^3 \epsilon}, \\
C_{\text{simple}} = \frac{2}{3} \frac{1}{(16\pi^2)^3 \epsilon}, \quad D_{\text{simple}} = -\frac{2}{3} \frac{1}{(16\pi^2)^3 \epsilon},
\]

and hence we have (note that with our conventions, at \( L \) loops \( \beta_g^{(L)} \) differs from the corresponding simple pole contribution to \( <VV>_{\text{pole}} \) by a factor of \( \frac{4L}{4} \))

\[
(16\pi^2)^3 \beta_g^{DRED} = r^{-1} g \left\{ 3X_1 + 6X_3 + X_4 - 6g^6 Q \text{tr}[C(R)^2] \right\},
\]

(2.15)

where \( r = \delta_{AA} \). On the other hand, the NSVZ result for \( \beta_g^{(3)} \) in the abelian case, obtained by setting \( C(G) = 0 \) in Eq. (2.7), is simply \( \beta_g^{(3)NSVZ} = -2 \frac{g^3}{16\pi^2} r^{-1} \text{tr}[\gamma^{(2)} C(R)] \), and so we have

\[
(16\pi^2)^3 \beta_g^{(3)NSVZ} = r^{-1} g \left\{ 2X_1 + 4X_3 - 4g^6 Q \text{tr}[C(R)^2] \right\}.
\]

(2.16)

We note that \( X_4 \) appears only in the DRED result Eq. (2.16), while the other terms appear in the same combination \( X_1 + 2X_3 - 2g^6 Q \text{tr}[C(R)^2] \) in both Eqs. (2.16) and (2.17). We

\footnote{Of course there is no non–trivial one-loop finite abelian theory; we are here really speaking of the corresponding terms in the full non-abelian case.}
also note that the same combination already appeared in Eq. (2.14) before evaluating the momentum integrals.

We now wish to show that the two results Eqs. (2.16) and (2.17) are in fact related by a coupling constant redefinition, equivalent to a change of renormalisation scheme. In general, a redefinition $\delta g$ of $g$ induces a change $\delta \beta_g$ of $\beta_g$ given (to lowest order in $g$) by

$$
\delta \beta_g(g, Y) = \left[ \beta_Y \frac{\partial}{\partial Y} + \beta_Y^* \frac{\partial}{\partial Y^*} + \beta_g \frac{\partial}{\partial g} \right] \delta g - \delta g \frac{\partial}{\partial g} \beta_g. 
$$

(2.18)

In particular, if we choose

$$
\delta g = -(16\pi^2)^{-2} \frac{1}{2} r^{-1} g^3 \text{tr}[PC(R)],
$$

(2.19)

then the resulting $\delta \beta_g$ is given by

$$(16\pi^2)^3 \delta \beta_g = r^{-1} g \left(-X_1 - 2X_3 - X_4 + 2g^6 \text{Qtr}[C(R)^2]\right).$$

(2.20)

We easily see that then

$$
\beta_g^{(3)NSVZ} = \beta_g^{(3)DRED} + \delta \beta_g.
$$

(2.21)

It is obvious from this analysis that, as we already mentioned in the Introduction, it is quite non-trivial that $\beta_g^{(3)NSVZ}$ and $\beta_g^{(3)DRED}$ may be related in this way. Hence this provides a strong check on the validity of the NSVZ result.

In these considerations we explored redefinitions of the gauge coupling $g$ only; what of redefinitions of the Yukawa couplings $Y^{ijk}$? Under an arbitrary redefinition

$$
Y^{ijk} \rightarrow (Y^{ijk})',
$$

(2.22)

we have at once that

$$
\beta_g'(Y', g) = \beta_g(Y, g) \quad \text{and} \quad \gamma'(Y', g) = \gamma(Y, g).
$$

(2.23)

It follows that if $\beta_g(Y, g)$ and $\gamma(Y, g)$ satisfy Eq. (2.7), then $\beta_g'(Y, g)$ and $\gamma'(Y, g)$ do likewise. This means that a redefinition $Y \rightarrow Y'$ has no effect on the question of whether $(\beta_g, \gamma)$ obey the NSVZ condition. This does not mean that redefinitions of $Y$ have no significance, however; for example, in Ref. [16], we showed how to construct a redefinition of $Y$ corresponding to a change to a scheme such that $\gamma^{(3)} = 0$ for a one-loop finite theory. We will return to this redefinition in the next section.

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We turn now to the non-abelian case. The crucial observation is that $\delta g$ as defined in Eq. (2.19) does not vanish for an $N = 2$ theory in general (though it does in the abelian case, as may be easily verified). There is, however, an obvious generalisation of it to the non-abelian case, to wit

$$\delta g = (16\pi^2)^{-2}\frac{1}{2} g^3 \left[ r^{-1} \text{tr} [PC(R)] - g^2 QC(G) \right]$$

(2.24)

where we have reversed the overall sign (compared to Eq. (2.19)) because we plan to use this $\delta g$ to go back from $\beta_{g}^{(3)NSVZ}$ to $\beta_{g}^{(3)DRED}$. It is easy to verify that Eq. (2.24) leads to $\delta g = 0$ in the $N = 2$ case. Is this the only possible extension of $\delta g$ to the non-abelian case? We are constrained by the following requirements:

1. $\delta g = 0$ for a one-loop finite theory. This is because we know that both $\beta_{g}^{(3)NSVZ}$ and $\beta_{g}^{(3)DRED}$ vanish in the one-loop finite case. This is manifest in the NSVZ case, and was explicitly verified in Ref. [3] for the DRED case. (See also Ref. [4].) In the three-loop case the relevant $\beta$-functions in Eq. (2.18) are one-loop. Since these vanish in this case, to produce a vanishing $\delta \beta$, $\delta g$ must vanish also.

2. $\delta g = 0$ for a $N = 2$ theory. In the $N = 2$ case we know that, as discussed earlier, $\beta_{g}^{(3)NSVZ} = \beta_{g}^{(3)DRED} = 0$. Clearly we therefore require $\delta \beta = 0$; $\delta g = 0$ ensures this, since the operator $\left[ \beta_Y \frac{\partial}{\partial Y} + \beta_Y^* \frac{\partial}{\partial Y^*} + \beta_g \frac{\partial}{\partial g} \right]$ just corresponds to multiplication by a factor in the $N = 2$ case.

3. Eq. (2.19) must hold in the abelian case.

4. The resulting terms in $\delta \beta_g$ must correspond to possible 1PI Feynman graphs.

It is easy to convince oneself that Eq. (2.24) represents the only possible transformation (up to an overall constant, which we have fixed by the abelian calculation). Our result for $\beta_g^{(3)DRED}$ in the non-abelian case is therefore [5]:

\[
(16\pi^2)^3 \beta_g^{(3)DRED} = r^{-1}g \left\{ 3X_1 + 6X_3 + X_4 - 6g^6 Q \text{tr}[C(R)^2] \\
- 4g^4 C(G) \text{tr}[PC(R)] \right\} + g^7 QC(G) [4C(G) - Q] \\
- 3r^{-1}g^3 Y \varepsilon_{ikm} Y_{jkn} P^m_n C(R)^{j_i} + 6r^{-1}g^5 \text{tr} \left[ PC(R)^2 \right] \\
+ r^{-1}g^3 \text{tr} \left[ P^2 C(R) \right] - 6r^{-1}Qg^7 \text{tr} \left[ C(R)^2 \right] - 4r^{-1}g^5 C(G) \text{tr} \left[ PC(R) \right] \\
+ g^7 QC(G) \left[ 4C(G) - Q \right].
\]

(2.25)
In the general case, no-one has explicitly computed $\beta_y^{(3)DRED}$; there does, however, exist a calculation in the special case of a theory without any chiral superfields by Avdeev and Tarasov [17]. For this special case it is easy to see from Eq. (2.25) that we obtain

$$(16\pi^2)^3 \beta_g^{(3)DRED} = -21g^7C(G)^3$$

which precisely agrees with the result of Ref. [17]. Note that from Eq. (2.28) we have that

$$(16\pi^2)^3 \beta_g^{(3)NSVZ} = -12g^7C(G)^3. \tag{2.27}$$

This difference was first remarked upon in Ref. [1]. The fact that we successfully reproduce Eq. (2.26) is an excellent check of both our abelian calculation and our coupling constant redefinition. It is intriguing to note that this redefinition, as defined in Eq. (2.24), can be written:

$$\delta g = -\frac{1}{4} \beta_y^{(2)}. \tag{2.28}$$

This of course suggests the possibility of generalising $\delta g$ to all orders, and hence deriving the all-orders $\beta_g^{(3)DRED}$. We have been unable to provide such a construction, however. In the next section we proceed to four loops to test whether the simplicity of our result Eq. (2.28) is sustained.

3. The four-loop calculation

In the last section we were able to show that the coupling constant redefinition $\delta g$ relating $\beta_g^{(3)DRED}$ to $\beta^{(3)NSVZ}$ uniquely determined (up to an overall constant) without any further calculation. Unfortunately this property does not extend to four loops, as is easily seen as follows. From Eq. (2.25) let us rewrite $\beta_g^{(3)DRED}$ in the form

$$\beta_g^{(3)DRED} = -4\Delta_1 + 3\Delta_2 + \Delta_3 \tag{3.1}$$

where

$$(16\pi^2)^3 \Delta_1 = g^5C(G) [r^{-1} \text{tr} [PC(R)] - g^2QC(G)] \tag{3.2a}$$

$$(16\pi^2)^3 \Delta_2 = r^{-1} \text{tr} [g^3S_4C(R) - 2g^7QC(R)^2 + 2g^5PC(R)^2] \tag{3.2b}$$

$$(16\pi^2)^3 \Delta_3 = g^3r^{-1} \text{tr} [P^2C(R)] - g^7Q^2C(G). \tag{3.2c}$$
The corresponding formula for $\beta^{(3)NSVZ}_g$ is

$$\beta^{(3)NSVZ}_g = -4\Delta_1 + 2\Delta_2. \quad (3.3)$$

The purpose of these decompositions is that each of the $\Delta_i$ represents a candidate for $\delta g$ satisfying the requirements we formulated in the previous section; and, indeed, the $\Delta_i$ are the only such candidates. In particular in the $N = 2$ case we have $\Delta_1 = \Delta_2 = \Delta_3 = 0$. Therefore we may anticipate that to relate $\beta^{(4)DRED}_g$ to $\beta^{(4)NSVZ}_g$ we may need to make a redefinition of the form $\delta g = \sum \alpha_i \Delta_i$ where the $\alpha_i$ are as yet undetermined coefficients. We will also, of course, need to take into account the effect on the four-loop $\beta$-functions of the $O(g^5)$ redefinition discussed in the last section. It is immediately clear that we shall be unable to determine the coefficient $\alpha_1$ from an abelian calculation, since $\Delta_1$ vanishes in the abelian case. Using Eq. (2.18), we obtain the leading-order change in $\beta_g$ due to each of the $\Delta_i$, as follows:

$$(16\pi^2)^4 \delta \beta_g(\Delta_1) = 2g^5C(G)r^{-1}\left\{ \text{tr} [S_4C(R)] + \text{tr}[P^2C(R)] + 2g^2\text{tr}[PC(R)^2] \\
+ g^2Q\text{tr}[PC(R)] - 2g^4Q\text{tr}[C(R)^2] \right\} - 4g^9Q^2C(G)^2$$

$$(16\pi^2)^4 \delta \beta_g(\Delta_2) = 2g^3r^{-1}\text{tr} [(S_7 + 2S_8 + Y^*S_4Y)C(R) + S_5P] \\
+ 4g^5r^{-1}\text{tr} \left[ \{ S_4C(R) + S_5 + P^2C(R) \} C(R) \right] \\
+ 4Qg^7r^{-1}\text{tr} [PC(R)^2 - S_1C(R)] + 8g^7r^{-1}\text{tr}[PC(R)^3] \\
- 8Qg^9r^{-1}\text{tr} [C(R)^3 + QC(R)^2]$$

$$(16\pi^2)^4 \delta \beta_g(\Delta_3) = 4g^3r^{-1}\text{tr} \left[ P^3C(R) + 2g^2P^2C(R)^2 - 2g^4QPC(R)^2 + S_5P \right] \\
- 4g^9Q^3C(G).$$

It is straightforward to verify that each of the $\delta \beta_g(\Delta_i)$ vanishes for $N = 2$ supersymmetry; manifestly they also vanish in a one-loop finite theory. We must be careful about the logic of our procedure here, because we know that $\gamma^{(3)DRED}$ does not vanish in a one-loop finite theory [18], so there is no reason to expect that $\beta^{(4)DRED}_g$ will either (and, indeed, it does not). There is, however, a redefinition $Y \rightarrow Y + \delta Y$ corresponding to a change to a renormalisation scheme (DRED', say) such that $\gamma^{(3)DRED'}$ does vanish in the one-loop finite case[19]. In this scheme $\beta^{(4)DRED'}_g$ will also vanish, relying on the
that in an $n$-loop finite theory $\beta^{(n+1)}_g = 0$. Suppose that we have found a $\delta g$ which transforms $\beta^{(4)}_{g\,DRED}$ to the NSVZ form $\beta^{(4)}_{g\,NSVZ}$ (i.e. consistent with Eq. (2.7)). (We must take into account here the fact that $\gamma^{(3)}_{NSVZ}$ differs from $\gamma^{(3)}_{DRED}$, owing to the redefinition of $g$ which takes us from DRED to the NSVZ form at the three-loop level—see later.) Applying the above $\delta Y$ to $\gamma^{(3)}_{NSVZ}$ and $\beta^{(4)}_{g\,NSVZ}$ yields a new $\gamma^{(3)}_{NSVZ'}$ and $\beta^{(4)}_{g\,NSVZ'}$ which are also of the NSVZ form, recalling that a redefinition of $Y$ does not affect whether or not $\beta_g$ satisfies Eq. (2.7). Our redefinition $\delta g$ also transforms $\beta^{(4)}_{g\,DRED'}$ into $\beta^{(4)}_{g\,NSVZ'}$, which both vanish in the one-loop finite case, and hence we conclude that $\delta g$ must itself vanish in the one-loop finite case.

From Eq. (2.7), we have that

$$(16\pi^2)^4 \beta^{(4)}_{g\,NSVZ} = 8g^9 QC(G)^3 - 8g^7 C(G)^2 r^{-1} 16\pi^2 \text{tr} \left[ \gamma^{(1)}(C(R)) \right]$$

$$- 4g^5 C(G) r^{-1} (16\pi^2)^2 \text{tr} \left[ \gamma^{(2)}(C(R)) \right]$$

$$- 2g^3 r^{-1} (16\pi^2)^3 \text{tr} \left[ \gamma^{(3)}_{NSVZ}(C(R)) \right].$$

(3.7)

Now in this equation, $\gamma^{(1)}$ and $\gamma^{(2)}$ are unambiguous, being defined in Eq. (2.3) and (2.66) respectively, but we must be careful about $\gamma^{(3)}$.

From Ref. [16] we have the result for $\gamma^{(3)}_{DRED}$:

$$(16\pi^2)^3 \gamma^{(3)}_{DRED} = \kappa \left\{ g^6 \left[ 12C(R)C(G)^2 - 2C(R)^2 C(G) - 10C(R)^3 - 4C(R)\Delta(R) \right] + g^4 \left[ 4C(R)S_1 - C(G)S_1 + S_2 - 5S_3 \right] + g^2 Y^* S_1 Y + \frac{1}{4} M \right. + g^2 \left[ C(R)S_4 - 2S_5 - S_6 \right] - g^4 \left[ PC(R)C(G) + 5PC(R)^2 \right] + 4g^6 QC(G)C(R) \right\} + 2Y^* S_4 Y - \frac{1}{2} S_7 - S_8 + g^2 \left[ 4C(R)S_4 + 4S_5 \right] + g^4 \left[ 8C(R)^2 P - 2QC(R)P - 4QS_1 - 10r^{-1} \text{Tr} \left[ PC(R) \right] C(R) \right] + g^6 \left[ 2Q^2 C(R) - 8C(R)^2 Q + 10QC(R)C(G) \right]$$

(3.8)

where $\kappa = 6\zeta(3)$. Group theoretic factors are defined in Appendix A.

Now in the previous section we constructed a $\delta g$ that related $\beta^{(3)}_{g\,DRED}$ to $\beta^{(3)}_{g\,NSVZ}$, Eq. (2.24). As we mentioned earlier, this redefinition affects $\gamma$ too, so that

$$\gamma^{(3)}_{NSVZ} = \gamma^{(3)}_{DRED} - 4g\delta g C(R)(16\pi^2)^{-1}$$

$$= \gamma^{(3)}_{DRED} - 2(16\pi^2)^{-3} g^4 \left[ r^{-1} \text{tr} \left[ PC(R) \right] - g^2 QC(G) \right] C(R).$$

(3.9)

This completes the definition of $\beta^{(4)}_{g\,NSVZ}$. We anticipate that it will be related to $\beta^{(4)}_{g\,DRED}$ as follows:
\[
\beta_g^{(4)DRED} = \beta_g^{(4)NSVZ} + \sum_i \alpha_i \delta \beta_g(\Delta_i) + \Omega
\]  
(3.10)

where \( \Omega \) is the change in \( \beta_g^{(4)} \) due to the redefinition in Eq. (2.24),

\[
\Omega = \left[ \beta_Y^{(2)} \frac{\partial}{\partial Y} + \beta_Y^{(2)} \frac{\partial}{\partial Y^*} + \beta_g^{(2)} \frac{\partial}{\partial g} \right] \delta g - \delta g \frac{\partial}{\partial g} \beta_g^{(2)}.
\]  
(3.11)

(There are, of course, terms of \( O((\delta g)^2) \), but these are \( O(g^{11}) \) and hence affect \( \beta_g^{(5)} \).)

Substituting for \( \beta_Y^{(2)} \) and \( \beta_g^{(2)} \) we obtain:

\[
(16\pi^2)^4 \Omega = g^3 r^{-1} \{ 2Q g^4 \text{tr}[PC(R)^2 + g^2 C(R)^3] - 2g^2 \text{tr}[P^2 C(R)^2 + g^2 PC(R)^3] \\
+ 2g^4 Q \text{tr}[S_1 C(R)] - \text{tr}[Y^* S_4 Y C(R)] \\
- \text{tr} \left[ 2g^2 (S_5 + S_4 C(R)) C(R) + S_5 P \right] \}. 
\]  
(3.12)

Given an explicit calculation of \( \beta_g^{(4)DRED} \), we would be able to test the validity of the construction Eq. (3.10); and if it proved to be valid, determine the \( \alpha_i \). This calculation is beyond our strength; we have, however, performed a partial calculation consisting of the contributions to \( \beta_g^{(4)DRED} \) in the abelian theory of \( O(g^3 Y^6) \). From Eqs. (3.4)–(3.6) it is clear that this will enable us both to test whether our construction works with regard to such terms and also to determine \( \alpha_2 \) and \( \alpha_3 \), thus fixing \( \beta_g^{(4)DRED} \) apart from a single unknown parameter (\( \alpha_1 \)). The calculations were performed using the methods explained earlier in the three-loop case. The relevant diagrams are shown in Fig. 3, and the results are given in Table 2. As before, the momentum integrals can be expressed in terms of a convenient basis, depicted in Fig. 4. The simple poles for each of these (subtracted) diagrams are given by

\[
F_{\text{simple}} = \frac{1}{(16\pi^2)^4 \epsilon} \left( \frac{1}{2} - \zeta(3) \right), \quad G_{\text{simple}} = \frac{1}{(16\pi^2)^4 \epsilon} \left( -1 + \zeta(3) \right), \\
H_{\text{simple}} = \frac{1}{(16\pi^2)^4 \epsilon} \left( \frac{5}{6} + \zeta(3) \right), \quad I_{\text{simple}} = \frac{5}{2} \frac{1}{(16\pi^2)^4 \epsilon}, \\
J_{\text{simple}} = -\frac{2}{3} \frac{1}{(16\pi^2)^4 \epsilon}, \quad K_{\text{simple}} = -\frac{5}{6} \frac{1}{(16\pi^2)^4 \epsilon}, \\
L_{\text{simple}} = \frac{1}{2} \frac{1}{(16\pi^2)^4 \epsilon}, \quad M_{\text{simple}} = \frac{11}{6} \frac{1}{(16\pi^2)^4 \epsilon}, \\
N_{\text{simple}} = -\frac{1}{6} \frac{1}{(16\pi^2)^4 \epsilon}, \quad P_{\text{simple}} = -\frac{1}{6} \frac{1}{(16\pi^2)^4 \epsilon}, \\
Q_{\text{simple}} = -\frac{2}{3} \frac{1}{(16\pi^2)^4 \epsilon}, \quad R_{\text{simple}} = -\frac{1}{2} \frac{1}{(16\pi^2)^4 \epsilon}.
\]  
(3.13)
It is interesting to note that $\zeta(3)$ dependence only appears in those diagrams which contain a “figure-of-eight” sub-diagram. This could perhaps be explained by invoking recent proposals which relate the appearance of transcendental numbers in Feynman integrals to knot theory\textsuperscript{[19]}. On adding the results for all the diagrams, we find

$$
(16\pi^2)^4 \beta_g^{(4)DRED} = \frac{1}{2} g^3 r^{-1} \text{tr} \left[ (2 P^3 - 2 S_8 - 19 Y^* S_4 Y - S_7) C(R) \right] \\
- \frac{1}{2} \kappa g^3 r^{-1} \text{tr}[MC(R)] - \frac{5}{3} g^3 r^{-1} \text{tr}[S_5 P] \\
+ \cdots \quad \text{(terms involving } C(G), \text{ and of order } g^5 Y^4, g^7 Y^2 \text{ and } g^9.\text{)}
$$

(3.14)

The coefficient of the $\text{tr}[MC(R)]$ term was derived not by direct calculation, but by exploiting the fact that, as we observed earlier, the coupling constant redefinition which makes $\gamma^{(3)}$ vanish in the one-loop finite case should also make $\beta^{(4)}_g$ vanish. It was shown in Ref.\textsuperscript{[16]} that the redefinition

$$
(16\pi^2)^2 \delta Y^{ijk} = \frac{1}{4} \kappa Y^{ilm} Y^{jpq} Y^{krs} Y_{lpr} Y_{mqs} + O(g^2)
$$

(3.15)

makes $\gamma^{(3)}$ vanish in the one-loop finite case. Since a change $\delta Y$ induces a leading-order change in $\beta_g$ of the form

$$
\delta \beta_g = - \left[ \delta Y, \frac{\partial}{\partial Y^*} + \delta Y^*, \frac{\partial}{\partial Y} \right] \beta_g^{(2)}
$$

(3.16)

we can deduce the coefficient of $\text{tr}[MC(R)]$ to be as given. This indirect deduction was necessitated because the direct calculation of this coefficient proved very involved. One might feel slightly uneasy because it is not clear that the theorem that $\beta_g^{(n+1)}$ vanishes in an $n$-loop finite theory should be valid in an arbitrary renormalisation scheme. To allay these doubts, we have explicitly computed the coefficients of $\text{tr}[C(R)^2 \Delta(R)]$ and $\text{tr}[Y^* S_1 Y C(R)]$ in $\beta_g^{(4)DRED}$ in the one-loop finite case, and checked that they agree with those obtained by the same indirect argument.

Now comparing the result Eq. (3.14) with Eqs. (3.4)–(3.7) and (3.12), we see that we require

$$
\alpha_2 = -\frac{2}{3}, \quad \alpha_3 = \frac{1}{6}.
$$

(3.17)

We note that it is non-trivial that a solution exists at all for $\alpha_2$ and $\alpha_3$, since we need to reproduce six terms in Eq. (3.14) and we have only two free parameters $\alpha_2$ and $\alpha_3$ to adjust. Note, however, that the coefficient of $\text{tr}[MC(R)]$ automatically satisfies Eq. (3.7),
irrespective of the values of \(\alpha_2\) and \(\alpha_3\); this is essentially guaranteed, as we know that we can redefine \(Y\) so that the \(M\) terms vanish in both \(\gamma^{(3)}\) and \(\beta^{(4)}_g\) (hence trivially satisfying Eq. (3.7) as far as these terms are concerned), and we also know, as mentioned earlier, that redefinitions of \(Y\) have no effect on whether Eq. (3.7) is satisfied.

As we have indicated, several miracles were required to facilitate our construction. It is perhaps disappointing, however, that it remains unclear how the redefinition we have checked using Eq. (3.16) might have been proportional to \(\beta^{(3)}_{gNSVZ}\) but it is easy to see from Eqs. (3.1) and (3.3) that this doesn’t work.

The final complete result is

\[
(16\pi^2)^4 \beta^{(4)DRED}_g = g^3 r^{-1} \left\{ -\frac{1}{2} \kappa M - \frac{10}{3} Y^* S_4 Y + \frac{2}{3} P^3 - \frac{1}{3} S_7 - \frac{2}{3} S_8 \right\} C(R) - \frac{2}{3} S_5 P
\]

\[
+ g^5 r^{-1} \left\{ -2\kappa Y^* S_1 Y - (2\kappa + \frac{38}{3}) C(R) S_4 + (4\kappa - \frac{38}{3}) S_5 + 2\kappa S_6
\]

\[
- \frac{10}{3} P^2 C(R) + (2\alpha_1 + 4) C(G) S_4 + 2\alpha_1 C(G) P^2 \right\} C(R)
\]

\[
+ g^7 r^{-1} \left\{ -8\kappa C(R) S_1 + 2\kappa C(G) S_1 - 2\kappa S_2 + 10\kappa S_3
\]

\[
+ 24r^{-1} \left[ PC(R) C(R) + (10\kappa - \frac{76}{3}) PC(R)^2 + 2 PC(R) + \frac{38}{3} Q S_1
\]

\[
+ (2\kappa + 4\alpha_1 + 8) C(G) PC(R) + 2\alpha_1 QC(G) P - 8 C(G)^2 P \right\} C(R)
\]

\[
+ g^9 r^{-1} \left[ \frac{76}{3} QC(R)^3 + \frac{4}{3} Q^2 C(R)^2 - (8\kappa + 4\alpha_1 + 32) QC(G) C(R)^2
\]

\[
- 24\kappa C(G)^2 C(R)^2 + 4\kappa C(G) C(R)^3 + 20\kappa C(R)^4 + 8\kappa \Delta(R) C(R)^2
\]

\[
+ g^9 \left[ 8 QC(G)^3 - 4\alpha_1 Q^2 C(G)^2 - \frac{2}{3} Q^3 C(G) \right] \right\} C(R)
\]

(3.18)

We see that in a one-loop finite theory (\(P = Q = 0\), we have \(\beta^{(4)DRED}_g \neq 0\); but, as noted earlier, it can be transformed to zero by means of the redefinition \(Y \to Y + \delta Y\), where \(\delta Y\) is the transformation that makes \(\gamma^{(3)DRED}\) vanish in such a theory. We gave the \(O(Y^5)\) term in this \(\delta Y\) in Eq. (3.17); the full expression is [16]

\[
(16\pi^2)^3 \delta Y^{ijk} = \kappa \left\{ \frac{1}{4} Y^{ilm} Y^{jpq} Y^{krs} Y_{lpr} Y_{mqs} + \frac{1}{2} g^2 S_1 (i_m Y^{jk})m
\]

\[
+ g^4 \left[ C(R)^{i_m C(R)^j} Y^{kn} - \frac{1}{2} Y^{mn} C(R)^{i_m C(R)^{kn} - \Delta(R) Y^{ijk}
\]

\[
- \frac{1}{2} C(G) C(R)^{i_m Y^{jk})m + 3C(G)^2 Y^{ijk} \right\} \right\}.
\]

(3.19)

This redefinition defines the DRED’ scheme which we introduced previously; correspondingly, the same redefinition applied to \(\beta^{(4)NSVZ}_g\), as given in Eq. (3.7), will define the NSVZ’ scheme for which (given \(P = Q = 0\)) \(\beta^{(4)NSVZ'}_g = 0\) likewise (as can readily be checked using Eq. (3.10)).
4. Conclusions

We have explicitly constructed the coupling constant redefinitions that relate $\beta_g^{NSVZ}$ to $\beta_g^{DRED}$ up to and including four loops, except for one undetermined parameter. The fact that this construction was possible demonstrates that the renormalisation scheme in which the $NSVZ$ form is valid is perturbatively related to the conventional DRED scheme. As a by-product of our investigations we have obtained $\beta_g^{(3)DRED}$ and $\beta_g^{(4)DRED}$ for a general non-abelian theory, except for the dependence of $\beta_g^{(4)DRED}$ on the same undetermined parameter.

The fixed points of $\beta_g^{NSVZ}$, which satisfy the equation

$$2r^{-1}\text{tr}[\gamma C(R)] = Q$$

are significant in dual gauge theories. Now the existence of a fixed point is preserved under a coupling constant redefinition $g \rightarrow g'(g)$, as long as the function $g'(g)$ is differentiable at the fixed point. Our demonstration that the NSVZ and DRED schemes are perturbatively related therefore suggests that there is a fixed point of $\beta_g^{DRED}$ corresponding to any fixed point of $\beta_g^{NSVZ}$. It would have been interesting had we been able to construct the $DRED \leftrightarrow NSVZ$ redefinition to all orders; unfortunately, however, to the extent that we have pursued the perturbative form, there is no indication of an all-orders result. It is tantalising, in this regard, that in the development of background-field covariant superfield Feynman rules [20] it appears that beyond one loop, loops of $\epsilon$-scalars play a crucial role in the relevant counter-terms. Such a loop provides a factor of $\epsilon$, so that the simple pole in $\epsilon$ thus depends on what would have been the double pole had the $\epsilon$-scalars been ordinary. Given that all higher order poles in $\epsilon$ are determined in terms of the simple pole by the ’t Hooft consistency conditions [21], one might have hoped to proceed to an all-orders construction of $\beta_g^{DRED}$. In this endeavour we have not been successful.

Nevertheless, we feel that the exercise has been worthwhile. We have demonstrated beyond all reasonable doubt that there does exist a scheme in which the $NSVZ$ $\beta$-function is valid. Our result for $\beta_g^{(3)DRED}$, in conjunction with the result for $\gamma^{(3)DRED}$ from Ref. [10], is in any event of interest phenomenologically [22], especially, perhaps, in post-post-modern theories with additional matter content (see for example [23]).

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Appendix A. Notation and Conventions

Here we give some details concerning our notation and various useful formulae. With our conventions, the $D$-algebra in Minkowski space is

$$\{D_\alpha, \bar{D}_\dot{\alpha}\} = \frac{1}{2} i \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu,$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_\dot{\alpha}, \bar{D}_\dot{\beta}\} = 0,$$  \hspace{1cm} (A.1)

where $\sigma^\mu = (I, \sigma^i)$, $\sigma^i$ being the Pauli matrices. It is then easy to verify that the projection operators

$$\Pi_0 = D^2 \bar{D}^2 + \bar{D}^2 D^2,$$

$$\Pi_\frac{1}{2} = -2 D^\alpha \bar{D}^2 D_\alpha$$  \hspace{1cm} (A.2)

satisfy

$$\Pi_\frac{1}{2} + \Pi_0 = -\partial^2$$  \hspace{1cm} (A.3)

as stated in Section 2. We use standard superfield Feynman rules, as described, for example, in Ref. [24], with minor changes of normalisation consequent upon our conventions in Eq. (A.1); the only unusual feature being the way we use transversality of the vector self-energy to simplify the $D$-algebra, as described in Section 2. A useful identity in manipulation of supergraphs (which follows immediately from Eqs. (A.2), (A.3)) is

$$D^2 \bar{D}^2 D^2 = -\partial^2 D^2, \quad \bar{D}^2 D^2 \bar{D}^2 = -\partial^2 \bar{D}^2.$$  \hspace{1cm} (A.4)

Gauge invariance of the superpotential means that the Yukawa couplings $Y^{ijk}$ satisfy the equation

$$Y^{m(ij} R_{A_m}^k = 0$$  \hspace{1cm} (A.5)

From this identity it is straightforward to prove that:

$$Y^{imn} R_{A_m}^j R_{A_n}^k = \frac{1}{2} [Y^{mjk} C(R)^{i} m - Y^{imk} C(R)^{j} m - Y^{ijm} C(R)^{k} m].$$  \hspace{1cm} (A.6)

Eq. (A.3) is rather similar to momentum conservation at a three-point vertex; correspondingly, Eq. (A.6) is analogous to the simple identity

$$p.q = \frac{1}{2} [(p + q)^2 - p^2 - q^2]$$  \hspace{1cm} (A.7)
Eq. (A.6) is very useful in dealing with the group theoretic factors. It is always possible, in the calculations we have performed, to push $R_A$‘s around so as to produce the quadratic Casimir $C(R)$ (though there is no reason to expect this property to persist to arbitrary orders). In this process, much labour involving dummy indices is avoided by the adoption of a diagrammatic notation where $Y$ or $Y^*$ are represented by vertices, index contractions $\delta^{ij}$ by propagators, and $R_A$ by “mass insertions”.

We use the following definitions:

\[ S^i_{1j} = Y^{imn}C(R)^p_mY_{jpn} \quad (A.8a) \]
\[ (Y^*S_1Y)^i_{j} = Y^{imn}S_1^p_mY_{jpn} \quad (A.8b) \]
\[ S^i_{2j} = Y^{imn}C(R)^p_mC(R)^q_nY_{jpq} \quad (A.8c) \]
\[ S^i_{3j} = Y^{imn}(C(R)^2)^p_mY_{jpn} \quad (A.8d) \]
\[ S^i_{4j} = Y^{imn}P^p_mY_{jpn} \quad (A.8e) \]
\[ (Y^*S_4Y)^i_{j} = Y^{imn}S_4^p_mY_{jpn}. \quad \] (A.8f)
\[ S^i_{5j} = Y^{imn}C(R)^p_mP^q_pY_{jmq} \quad (A.8g) \]
\[ S^i_{6j} = Y^{imn}C(R)^p_mP^q_nY_{jpq} \quad (A.8h) \]
\[ S^i_{7j} = Y^{imn}P^p_mP^q_nY_{jpq} \quad (A.8i) \]
\[ S^i_{8j} = Y^{imn}(P^2)^p_mY_{jpn} \quad (A.8j) \]
\[ \Delta(R) = \sum_\alpha C(R_\alpha)T(R_\alpha) \quad (A.8k) \]
\[ M^i_{j} = Y^{ikl}Y^{kmn}_{lrs}Y^{pmr}Y^{qns}Y_{jpq} \quad (A.8l) \]

In Eq. (A.8k) the sum over $\alpha$ is a sum over irreducible representations.
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| Diagram | Integrals | Group factor |
|---------|-----------|--------------|
| a       | 0         |              |
| b       | $C$       | $2X_4 + 4X_3 - 2X_1$ |
| c       | $A$       | $4X_1$       |
| d       | $B$       | $2X_4$       |
| e       | $B$       | $4X_4$       |
| f       | $2A + 2B$ | $-16X_3$     |
| g       | $A$       | $-8X_3$      |
| h       | $A$       | $16X_3$      |
| i       | $B$       | $16X_3$      |
| j       | $B$       | $16X_3$      |
| k       | $-2A + \frac{1}{2}C$ | $8g^6\text{tr}[C(R)^2]$ |
| l       | $-\frac{1}{2}A + \frac{1}{2}B$ | $16g^6\text{tr}[C(R)^2]$ |
| m       | $-\frac{1}{2}A + \frac{1}{2}B$ | $-32g^6\text{tr}[C(R)^2]$ |
| n       | $B$       | $8g^6\text{tr}[C(R)^2]$ |
| o       | $-B - 2D$ | $8X_3 + 2X_4 + 8g^6\text{tr}[C(R)^3]$ |
| p       | 0         |              |
| q       | $A$       | $-8X_2 + 24X_3 + 48g^6\text{tr}[C(R)^3]$ |
| r       | $2A + 2D$ | $4X_2 - 12X_3 - 24g^6\text{tr}[C(R)^3]$ |
| s       | $2A - 2D - E$ | $4X_2 - 12X_3 - 24g^6\text{tr}[C(R)^3]$ |
| t       | $A$       | $8X_3 - 8X_2 + 16g^6\text{tr}[C(R)^3]$ |
| u       | 0         |              |
| v       | $E$       | $4X_2 - 4X_3 - 8g^6\text{tr}[C(R)^3]$ |
| w       | $2A + B + 2D$ | $8X_3 + 16g^6\text{tr}[C(R)^3]$ |
| x       | 0         |              |
| y       | $4A - C - E$ | $8X_3 + 16g^6\text{tr}[C(R)^3]$ |
| z       | $A$       | $-16X_3 - 32g^6\text{tr}[C(R)^3]$ |

Table 1: Three-loop contributions to $<VV>_{\text{pole}}$ for the diagrams in Figure 1. Each contribution is obtained by multiplying the simple pole from the momentum integral in the first column by the group theory factor in the second column, and by $r^{-1}$. 
| Diagram | Integrals     | Group factor                                |
|---------|--------------|---------------------------------------------|
| aa      | $A$          | $16X_3 + 32g^6\text{tr}[C(R)^3]$           |
| bb      | $2A + 2B$    | $-32g^6\text{tr}[C(R)^3]$                  |
| cc      | $B$          | $16g^6\text{tr}[C(R)^3]$                   |
| dd      | $B$          | $16g^6\text{tr}[C(R)^3]$                   |
| ee      | $A$          | $32g^6\text{tr}[C(R)^3]$                   |
| ff      | $B + E$      | $-32g^6\text{tr}[C(R)^3]$                  |
| gg      | $E$          | $16g^6\text{tr}[C(R)^3]$                   |
| hh      | $4A + 3B - 2D$ | $8g^6\text{tr}[C(R)^3]$               |
| ii      | $2B - 8D + 2E$ | $8g^6\text{tr}[C(R)^3]$              |
| jj      | $4B + C + 2D + E$ | $16g^6\text{tr}[C(R)^3]$            |
| kk      | $B + E$      | $16g^6\text{tr}[C(R)^3]$                   |
| ll      | $B + E$      | $-32g^6\text{tr}[C(R)^3]$                  |
| mm      | $B$          | $32g^6\text{tr}[C(R)^3]$                   |
| nn      | $A$          | $32g^6\text{tr}[C(R)^3]$                   |
| pp      | $2A + 2B$    | $-32g^6\text{tr}[C(R)^3]$                  |
| qq      | $B$          | $-16g^6\text{tr}[C(R)^3]$                  |
| rr      | $2A - 2D$    | $-16g^6\text{tr}[C(R)^3]$                  |
| ss      | $A$          | $-32g^6\text{tr}[C(R)^3]$                  |
| tt      | $A$          | $32g^6\text{tr}[C(R)^3]$                   |
| uu      | $B$          | $16g^6\text{tr}[C(R)^3]$                   |
| vv      | 0            |                                            |
| ww      | 0            |                                            |
| xx      | 0            |                                            |

*Table 1 (continued)*
| Diagram | Integrals | Group Factor |
|---------|-----------|--------------|
| a       | $4P - 2R + 2J$ | $\text{tr}[S_7C(R) - 2S_5P]$ |
| b       | $F + 2N$ | $4\text{tr}[P^3C(R)]$ |
| c       | $J + 2Q - L$ | $2\text{tr}[S_7C(R)]$ |
| d       | $J + 2P$ | $8\text{tr}[S_5P] - 4\text{tr}[S_7C(R)]$ |
| e       | $-I$ | $4\text{tr}[Y^*S_4YC(R)]$ |
| f       | $-M$ | $2\text{tr}[-Y^*S_4YC(R) + S_5P]$ |
| g       | 0 | |
| h       | 0 | |
| i       | 0 | |
| j       | 0 | |
| k       | $-L$ | $-4\text{tr}[S_7C(R)]$ |
| l       | $-G$ | $2\text{tr}[(P^3 - S_8)C(R)]$ |
| m       | $-K$ | $2\text{tr}[S_7C(R)]$ |
| n       | $-H$ | $4\text{tr}[S_8C(R)]$ |
| p       | $-J$ | $8\text{tr}[S_5P]$ |
| q       | $-J$ | $4\text{tr}[S_5P]$ |
| r       | $-F$ | $4\text{tr}[P^3C(R)]$ |
| s       | $-F$ | $4\text{tr}[P^3C(R)]$ |

Table 2: Four-loop contributions to $<VV>_{\text{pole}}$ for the diagrams in Figure 2. Each contribution is obtained by multiplying the simple pole from the momentum integral in the first column by the group theory factor in the second column, and by $r^{-1}g^2$. Note that only contributions including terms of $O(Y^6)$ have been retained in the group theory factors.
Fig. 1. Three-loop vector self-energy diagrams
Fig. 1. (Continued)
Fig. 2. Three-loop momentum integrals
Fig. 3. Four-loop vector self-energy diagrams
Fig. 3. (Continued)
Fig. 4. Four-loop momentum integrals