A Primal-Dual Smoothing Framework for Max-Structured Non-Convex Optimization

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We propose a primal-dual smoothing framework for finding a near-stationary point of a class of non-smooth non-convex optimization problems with max-structure. We analyze the primal and dual gradient complexities of the framework via two approaches, i.e., the dual-then-primal and primal-the-dual smoothing approaches. Our framework improves the best-known oracle complexities of the existing method, even in the restricted problem setting. As an important part of our framework, we propose a first-order method for solving a class of (strongly) convex-concave saddle-point problems, which is based on a newly developed non-Hilbertian inexact accelerated proximal gradient algorithm for strongly convex composite minimization that enjoys duality-gap convergence guarantees. Some variants and extensions of our framework are also discussed.

Key words: non-convex optimization; primal-dual smoothing; convex-concave saddle-point problems; non-Hilbertian inexact accelerated proximal gradient; stochastic optimization

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1. Introduction. We consider a class of non-convex non-smooth optimization problems, where the non-convex function has a max-structure. Let us first formally state the problem.

1.1. Problem statement. Let $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ and $(\mathcal{Y}, \| \cdot \|_\mathcal{Y})$ be finite-dimensional real normed spaces, with dual spaces denoted by $(\mathcal{X}^*, \| \cdot \|_{\mathcal{X}^*})$ and $(\mathcal{Y}^*, \| \cdot \|_{\mathcal{Y}^*})$, respectively. Let us consider the following optimization problem:

$$ q^* := \min_{x \in \mathcal{X}} \left\{ q(x) := f(x) + r(x) \right\}, \quad \text{where} \quad f(x) := \max_{y \in \mathcal{Y}} \Phi(x, y) - g(y). \quad (1.1) $$

In (1.1), the function $r : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is a closed and convex function with domain $\text{dom} \ r := \{x \in \mathcal{X} : r(x) < +\infty\}$. For convenience, define $\mathcal{X}' := \text{dom} \ r$, and we assume that both $\mathcal{X}'$ and $\mathcal{Y} \subseteq \mathcal{Y}$ are nonempty, closed and convex sets, and additionally, $\mathcal{Y}$ is bounded. The function $g : \mathcal{Y} \to \mathbb{R}$ is convex and continuous on $\mathcal{Y}$. We do not require $r$ or $g$ to be differentiable, but instead assume
that both \( r \) and \( g \) are “simple” in the sense that certain associated Bregman proximal projection (BPP) problems (formally introduced in Section 2.3) are easily solvable. Let \( \mathcal{X}' \subseteq \mathcal{X} \) and \( \mathcal{Y}' \subseteq \mathcal{Y} \) be some open sets that contain \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. We let the function \( \Phi : \mathcal{X}' \times \mathcal{Y}' \to \mathbb{R} \) be jointly continuous on \( \mathcal{X}' \times \mathcal{Y}' \) and \( \Phi(x, \cdot) \) be concave on \( \mathcal{Y} \), for any \( x \in \mathcal{X} \). In addition, we let \( \Phi \) satisfy the following assumptions.

**Assumption 1.1 (Smoothness of \( \Phi(\cdot, y) \)).** For any \( y \in \mathcal{Y} \), \( \Phi(\cdot, y) \) is (Fréchet) differentiable on \( \mathcal{X}' \), with the gradient at \( x \in \mathcal{X}' \) denoted by \( \nabla_x \Phi(x, y) \). Furthermore, there exist Lipschitz parameters \( L_{xx}, \tilde{L}_{xy} < +\infty \) such that for any \( x, x' \in \mathcal{X} \) and any \( y, y' \in \mathcal{Y} \),

\[
\|\nabla_x \Phi(x, y) - \nabla_x \Phi(x', y)\|_x \leq L_{xx} \|x - x'\|_x, \tag{1.2}
\]

\[
\|\nabla_x \Phi(x, y) - \nabla_x \Phi(x, y')\|_x \leq \tilde{L}_{xy} \|y - y'\|_y. \tag{1.3}
\]

**Assumption 1.2 (Weak Convexity of \( \Phi(\cdot, y) \)).** For any \( y \in \mathcal{Y} \), \( \Phi(\cdot, y) \) is \( \gamma \)-weakly convex on \( \mathcal{X} \) for some \( \gamma \in (0, L_{xx}] \), i.e., for any \( x, x' \in \mathcal{X} \),

\[
\Phi(x', y) - \Phi(x, y) - \langle \nabla_x \Phi(x, y), x' - x \rangle \geq -\left(\frac{\gamma}{2}\right) \|x' - x\|_x^2. \tag{1.4}
\]

**Assumption 1.3 (Smoothness of \( \Phi(x, \cdot) \)).** For any \( x \in \mathcal{X} \), \( \Phi(x, \cdot) \) is (Fréchet) differentiable on some open set \( \mathcal{Y}' \supseteq \mathcal{Y} \), with the gradient at \( y \in \mathcal{Y} \) denoted by \( \nabla_y \Phi(x, y) \). Furthermore, there exist Lipschitz parameters \( \tilde{L}_{yx}, L_{yy} < +\infty \) such that for any \( x, x' \in \mathcal{X} \) and \( y, y' \in \mathcal{Y} \),

\[
\|\nabla_y \Phi(x, y) - \nabla_y \Phi(x', y)\|_y \leq \tilde{L}_{yx} \|x - x'\|_x, \tag{1.5}
\]

\[
\|\nabla_y \Phi(x, y) - \nabla_y \Phi(x, y')\|_y \leq L_{yy} \|y - y'\|_y. \tag{1.6}
\]

Before introducing some applications of the problem in (1.1), we make some remarks. First, note that in Assumption 1.2, the reason we have \( \gamma \leq L_{xx} \) is due to (1.2) in Assumption 1.1. Specifically, from (1.2), by the descent lemma (see e.g., Peypouquet [24, Lemma 1.30]), we see that

\[
|\Phi(x', y) - \Phi(x, y) - \langle \nabla_x \Phi(x, y), x' - x \rangle| \leq (L_{xx}/2) \|x' - x\|_x^2, \quad \forall x, x' \in \mathcal{X}, \ \forall y \in \mathcal{Y}, \tag{1.7}
\]

which implies (1.4) with \( \gamma = L_{xx} \). However, as we will see later, our proposed smoothing framework can take advantage of the situation where \( \gamma \ll L_{xx} \). Second, we unify the “cross” Lipschitz parameters \( \tilde{L}_{xy} \) and \( \tilde{L}_{yx} \) into a single one, by defining a new parameter

\[
L_{xy} := \max\{\tilde{L}_{xy}, \tilde{L}_{yx}\},
\]

so that both (1.3) and (1.5) hold with \( L_{xy} \). (Indeed, under certain regularity conditions of \( \Phi \), the tightest choices of \( \tilde{L}_{xy} \) and \( \tilde{L}_{yx} \) coincide, and we can set the value to be \( L_{xy} \).) Finally, for well-posedness, we will always assume that the optimal value of (1.1) is finite, namely \( q^* > -\infty \).
1.2. Applications. The problem in (1.1) has many applications, from which we detail three.

Example 1.1 (Distributionally Robust Learning). In learning theory, an important problem is population risk minimization (PRM), which reads

$$\min_{x \in X} \mathbb{E}_{\xi \sim p}[\ell(x, \xi)].$$  (1.8)

In (1.8), the optimization variable $x$ represents (the coefficients of) the model that one intends to learn, and typically constrained in some closed and convex set $X \neq \emptyset$. Since we have some uncertainty about the problem data, we model it as a random variable $\xi$ with distribution $p$ and support $\Xi$. Given a model $x$ under a realization of the problem data $\xi$, the loss function $\ell : X \times \Xi \to \mathbb{R}$ returns loss $\ell(x, \xi)$, which then gives the population risk $\mathbb{E}_{\xi \sim p}[\ell(x, \xi)]$ after taking expectation. The optimal solution of (1.8) then represents the learned model. To improve the statistical properties of the learned model (e.g., unbiasedness or sparsity), one typically adds a regularizer $\tau : \mathcal{X} \to \mathbb{R}$ to the objective function in (1.8), and solves the regularized PRM problem instead:

$$\min_{x \in X} \mathbb{E}_{\xi \sim p}[\ell(x, \xi)] + \tau(x).$$  (1.9)

For simplicity, we will consider the case where $\xi$ can only take finitely many values, and denote the support $\Xi := \{\xi_1, \ldots, \xi_n\}$. The formulation in (1.8) or (1.9) implicitly assumes that the distribution $p$ is known exactly, which is not the case in many circumstances. However, for most of these situations, we know an uncertainty set $\mathcal{P}$ that $p$ belongs to (by using either prior knowledge or certain estimation procedures). Some typical examples of $\mathcal{P}$ include

$$\mathcal{B}_{TV}(\bar{p}, \alpha) := \{p \in \Delta_n : d_{TV}(p, \bar{p}) \leq \alpha\} \quad \text{or} \quad \mathcal{B}_{W_2}(\bar{p}, \alpha) := \{p \in \Delta_n : d_{W_2}(p, \bar{p}) \leq \alpha\},$$

where $\Delta_n := \{p \in \mathbb{R}^n : p \geq 0, \sum_{i=1}^n p_i = 1\}$ denotes the (standard) unit simplex in $\mathbb{R}^n$, $\bar{p} \in \Delta_n$ denotes the nominal distribution, $\alpha > 0$ denotes the “radius” of $\mathcal{P}$, and $d_{TV}$ and $d_{W_2}$ denote the total variation and 2-Wasserstein distances, respectively. Based on $\mathcal{P}$, we solve the following distributionally robust regularized PRM instead:

$$\min_{x \in \mathcal{X}} \max_{p \in \mathcal{P}} \sum_{i=1}^n p_i \ell(x, \xi_i) + \tau(x).$$  (1.10)

We note that (1.10) fits into the template in (1.1) if we define $\Phi : (x, p) \mapsto \sum_{i=1}^n p_i \ell(x, \xi_i)$, for $x \in \mathcal{X}$ and $p \in \mathcal{P}$, and $r := \tau + \iota_{\mathcal{X}}$, where $\iota_{\mathcal{X}}$ denotes the indicator function of $\mathcal{X}$, namely

$$\iota_{\mathcal{X}}(x) = \begin{cases} 0, & x \in \mathcal{X} \\ +\infty, & x \notin \mathcal{X} \end{cases}.$$  (1.11)
In the learning problems, we usually assume that $\ell(\cdot, \xi)$ is $L_\xi$-smooth on $\mathcal{X}$, namely $\ell(\cdot, \xi)$ is differentiable on some open set $\mathcal{X}' \supseteq \mathcal{X}$ with $L_\xi$-Lipschitz gradient on $\mathcal{X}$. As such, $\Phi(\cdot, p)$ satisfies (1.2) with $L_{xp} := \sup_{p \in P} \sum_{i=1}^{n} p_{i} L_{\xi_{i}} < +\infty$. In addition, we see that

$$\|\nabla_{x} \Phi(x, p) - \nabla_{x} \Phi(x, p')\|_{*} \leq L_{xp}\|p - p'\|_{1},$$

where $L_{xp} := \sup_{x \in \mathcal{X}} \max_{i \in [n]} \|\nabla_{x} \ell(x, \xi_{i})\|_{*}$. Clearly, $L_{xp} < +\infty$ if $\mathcal{X}$ is bounded. If $\mathcal{X}$ is unbounded, we may still have $L_{xp} < +\infty$ — for example, this happens if $\ell(x, \xi) = l((\xi, x))$, where $l : \mathbb{R} \to \mathbb{R}$ has bounded derivative on $\mathbb{R}$. Examples of such $l$ include the Huber loss function or the quadratically smoothed hinge loss function [32]. The above shows that (1.10) satisfies Assumption 1.1. Using similar reasoning, we see that Assumptions 1.2 and 1.3 are satisfied as well.

**Example 1.2 (Minimizing maximum of smooth functions).** Given $n$ functions $\{f_{i} : \mathcal{X} \to \mathbb{R}\}_{i=1}^{n}$ such that each $f_{i}$ is $L_{i}$-smooth on $\mathcal{X}$, we aim to minimize their point-wise maximum, namely

$$\min_{x \in \mathcal{X}} [f(x) := \max_{i \in [n]} f_{i}(x) = \max_{p \in \Delta_{n}} \sum_{i=1}^{n} p_{i} \ell_{i}(x)], \quad \forall x \in \mathcal{X}. \quad (1.12)$$

If we let $\Phi(x, p) := \sum_{i=1}^{n} p_{i} \ell_{i}(x)$ for $x \in \mathcal{X}$ and $p \in \Delta_{n}$ and $r := \ell_{X}$, then (1.12) fits into the template in (1.1). Therefore, following the discussions in Example 1.1, we see that (1.12) also satisfies Assumptions 1.1 to 1.3.

**Example 1.3 (Dual problem of composite optimization).** Consider the following composite optimization problem, where one wishes to solve

$$\min_{x \in \mathcal{X}} h(c(x)) + \varphi(x), \quad (1.13)$$

where $c : \mathcal{X} \to \mathbb{R}^{n}$ is (Fréchet) differentiable on $\mathcal{X}$ with Jacobian at $x \in \mathcal{X}$ denoted by $J_{c}(x)$, and $h : \mathcal{X} \to \mathbb{R}$ is a closed and convex function and is Lipschitz on $\mathcal{X}$. We assume that $J_{c}$ is Lipschitz on $\mathcal{X}$, namely there exists $b < +\infty$ such that $\|J_{c}(x) - J_{c}(x')\| \leq b\|x - x'\|$, for any $x, x' \in \mathcal{X}$. In addition, $\varphi : \mathcal{X} \to \mathbb{R}$ is some regularizer. As detailed in Davis and Drusvyatskiy [7], the problem in (1.13) has many applications, e.g., robust phase retrieval [3] and covariance matrix estimation [4]. Using Fenchel duality, we can rewrite (1.13) as

$$\min_{x \in \mathcal{X}} \max_{y \in \text{dom} h^{*}} \sum_{i=1}^{n} y_{i} c_{i}(x) - h^{*}(y) + r(x), \quad (1.14)$$

where for each $i \in [n]$, $c_{i} : \mathcal{X} \to \mathbb{R}$ denotes the $i$-th component of the (vector-valued) function $c$. Additionally, by the Lipschitz continuity of $h$, we see that $\text{dom} h^{*} \subseteq \mathbb{R}^{n}$ is nonempty, convex and bounded, and is also closed in many cases of interest, e.g., $h$ is any norm function. Note that by defining $\Phi(x, y) := \sum_{i=1}^{n} y_{i} c_{i}(x)$, $\varphi := r + \ell_{X}$ and $g := h^{*}$, (1.14) fits into the template in (1.1), and since $\Phi(\cdot, \cdot)$ takes a similar form to that in Example 1.1, (1.14) also satisfies Assumptions 1.1 to 1.3.
1.3. Convergence criterion: near-stationary point. Since the objective function in (1.1) is non-convex, without any additional assumptions on the problem structure, it is generally NP-hard to obtain an approximate optimal solution of (1.1) for any desired accuracy. Therefore, recent research (see e.g., Davis et al. [8], Davis and Drusvyatskiy [7], Davis and Grimmer [9]) has been focusing on finding an $\varepsilon$-near-stationary ($\varepsilon$-NS) point of (1.1), which we introduce informally below under the simpler setting that $X$ is Hilbertian, namely the norm $\|\cdot\|$ can be induced by some inner product. The formal definition of $\varepsilon$-NS point for the general normed space involves the notion of Bregman divergence and is deferred to Section 2.4. Let us first define the proximal point at $x$ with function $q$ and step-size $0 < \lambda < \gamma^{-1}$ as
\[
\text{prox}(q, x, \lambda) := \arg\min_{x' \in X} q(x') + (2\lambda)^{-1}\|x' - x\|^2.
\] (1.15)
(Note that as we will see in Section 4, the condition $0 < \lambda < \gamma^{-1}$ ensures that the minimization problem in (1.15) is strongly convex and hence admits a unique optimal solution.) We call $x \in X$ an $\varepsilon$-NS point of (1.1), if there exists some $0 < \lambda < \gamma^{-1}$ such that
\[
\|\lambda^{-1}(x - \text{prox}(q, x, \lambda))\| \leq \varepsilon.
\] (1.16)
As we will see later (cf. Lemma 4.1), if (1.16) holds, then $\text{prox}(q, x, \lambda)$ is an $\varepsilon$-approximate-stationary ($\varepsilon$-AS) point of (1.1), meaning that there exists a Fréchet subgradient (defined in Section 2.1) of $q$ at $\text{prox}(q, x, \lambda)$ with norm no larger than $\varepsilon$. Since the normalized distance from $x$ to $\text{prox}(q, x, \lambda)$, namely $\|\lambda^{-1}(x - \text{prox}(q, x, \lambda))\|$, is no larger than $\varepsilon$, we call $x$ an $\varepsilon$-NS point of (1.1).

1.4. Measure of computational cost. In this work, we will develop a first-order method to find an $\varepsilon$-NS point of (1.1). The main computational cost of our method occurs in two aspects:
\begin{itemize}
  \item i) computing the primal gradient $\nabla_x \Phi(x, y)$ and the dual gradient $\nabla_y \Phi(x, y)$ at $(x, y) \in X \times Y$,
  \item ii) solving certain BPP problems involving the “simple” non-smooth functions $r$ and $g$.
\end{itemize}
Indeed, due to the proximal-gradient nature of our method, the numbers of solved BPP problems involving $r$ and $g$ are constant multiples (in fact, at most two) of the numbers of computed primal and dual gradients in our method, respectively. Due to this reason, we measure the computational cost of our method by the complexities of the computed primal and dual gradients, which we call primal gradient complexity and dual gradient complexity, respectively. Note that we distinguish between the primal and dual gradient complexities, instead of combining them together, mainly because in certain scenarios, the cost of computing the primal and dual gradients can be different, and/or the cost of solving the BPP problems involving $r$ and involving $g$ can also be different. In these situations, distinguishing between the primal and dual gradient complexities allows a more accurate characterization of the computational cost of certain first-order method designed to find an $\varepsilon$-NS point of (1.1).
1.5. Related work. Let us review the representative works in the literature, all of which focus on the Hilbertian setting, namely both $X$ and $Y$ are finite-dimensional real Hilbert spaces.

**Weakly convex optimization (WCO).** Note that from Assumption 1.2, we can easily show that $f$ is $\gamma$-weakly convex on $\mathcal{X}$ (cf. Lemma 3.3). Therefore, the problem in (1.1) belongs to the class of WCO problems, which has been studied in several works recently. Davis and Grimmer [9] propose a proximal point method (PPM) for finding an $\varepsilon$-NS point of (1.1) (with $r = \iota_X$), where each proximal sub-problem is solved inexactly by the subgradient method (where the subgradient is in the sense of Fréchet; see Section 2.1). As another approach, Davis and Drusvyatskiy [7] propose to find an $\varepsilon$-NS point of (1.1) using the proximal subgradient method directly, without leveraging the PPM framework. Additionally, both works consider the stochastic setting, where the subgradient of $f$ can only be accessed through its unbiased stochastic estimator with finite second moment. Despite the ingenuity and success of these methods, a standing assumption is that at any $x \in \mathcal{X}$, a subgradient of $f$ can be easily obtained. However, in the case of (1.1), as we shall see in Lemma 3.2, computing a subgradient of $f$ generally requires solving the dual maximization problem in the definition of $f$ exactly, which may not be possible or accomplished easily at least. As such, these methods may not be readily applicable to our problem in (1.1).

**WCO with max-structure.** In the case where $f$ has the max-structure as in (1.1) with $g \equiv 0$, Kong and Monteiro [13] propose an accelerated inexact PPM to find an $\varepsilon$-NS point of the smoothed version of (1.1), i.e., $\min_{x \in \mathcal{X}} f_\rho(x) + r(x)$, where $f_\rho$ is a smooth approximation of $f$ and is given by $f_\rho : x \mapsto \max_{y \in Y} \Phi(x, y) - (2\rho)^{-1} \|y - \bar{y}\|^2$ for all $x \in \mathcal{X}'$ and some $\bar{y} \in Y$. However, similar to the works [7, 9] reviewed above, the authors assume that the gradient of $f_\rho$ can be obtained easily. As we will see in Lemma 3.5, this amounts to assume that the maximization problem in the definition of $f_\rho$ can be solved exactly and easily, which may not be the case in general, especially when the structure of either $\Phi(x, \cdot)$ or $Y$ (or both) is complicated. In another work, Thekumparampil et al. [30] propose a PPM-based approach to find an $\varepsilon$-NS point of (1.1) by further assuming that $r \equiv 0$. Unlike the aforementioned works, they do not assume certain dual maximization problem can be solved exactly. Instead, they assume the smoothness properties of $\Phi(\cdot, \cdot)$ as in Assumptions 1.1 and 1.3, and solve the proximal sub-problem by combining the Mirror-Prox method [17] and the accelerated gradient method [18]. However, the analysis in this work critically leverage the inner-product-inducibility of the norm $\|\cdot\|$ (namely $\|x\|^2 = \langle x, x \rangle$), and it is not clear (at least to us) how to generalize this approach to the non-Hilbertian setting.

**$\delta$-saddle-stationary ($\delta$-SS) point.** As a final note, since (1.1) can also be viewed as a non-convex-concave minimax optimization problem, i.e., $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x) + \Phi(x, y) - g(y)$, there exist several works (e.g., Nouiehed et al. [22], Lu et al. [16], Ostrovskii et al. [23]) that aim to find a $\delta$-SS
Table 1. Comparison of primal and dual gradient complexities with Thekumparampil et al. [30] to find an ε-NS point of (1.1), in the restricted case where \( r \equiv 0, g \equiv 0 \) and both \( \mathbb{X} \) and \( \mathbb{Y} \) are Hilbert spaces. Note that \( 0 < \gamma \leq L_{xx} \).

| Algorithms | Primal Oracle Comp. | Dual Oracle Comp. |
|------------|---------------------|-------------------|
| Theku. et al. [30] | \( O\left((L_{xx} + L_{xy} + L_{yy})\varepsilon^{-3}\ln^2(\varepsilon^{-1})\right) \) | \( O\left((L_{xx} + L_{xy} + L_{yy})\varepsilon^{-3}\ln^2(\varepsilon^{-1})\right) \) |
| Our framework (Algo. 1) | \( O\left(\sqrt{\gamma}L_{xx}(\sqrt{L_{yy}} + L_{xy})\varepsilon^{-3}\ln^2(\varepsilon^{-1})\right) \) | \( O\left(\gamma(\sqrt{L_{yy}} + L_{xy})\varepsilon^{-3}\ln(\varepsilon^{-1})\right) \) |

point of this minimax problem for any given \( \delta > 0 \) (see Lu et al. [16, Section III-A] for the definition of \( \delta \)-SS point). In the Hilbertian setting, it can be shown that one can obtain this point from an \( \varepsilon \)-NS point of (1.1), and vice versa, by properly choosing \( \varepsilon \) and \( \delta \) (see e.g., Lin et al. [15, Proposition 4.12]). However, since we are mainly interested in the minimization problem in (1.1), rather than the above-mentioned minimax problem, we only focus on the \( \varepsilon \)-NS point in this work.

1.6. Main contributions. Our main contributions are threefold.

First, we propose a primal-dual smoothing framework (namely Algorithm 1) for finding an \( \varepsilon \)-NS point of (1.1). We analyze the primal and dual gradient complexities of our framework using two approaches: (i) the dual-then-primal smoothing approach and (ii) the primal-then-dual smoothing approach. To the best of our knowledge, our framework is the first one that finds an \( \varepsilon \)-NS point of (1.1) under the non-Hilbertian setting. Even under the Hilbertian setting and the restrictive case where both \( r \equiv 0 \) and \( g \equiv 0 \), the primal and dual gradient complexities of our framework are better than the those of Thekumparampil et al. [30], and the improvement is especially significant in the regime where \( \gamma \ll L_{xx} \) (recall that \( 0 < \gamma \leq L_{xx} \)) — see Table 1 for details.

Second, as an important part of our framework, we propose an efficient method for solving a class of (strongly) convex-concave saddle-point problems (SPPs) with primal strong convexity (cf. Section 6). As the workhorse of this method, we develop a non-Hilbertian inexact accelerated proximal gradient (APG) method for strongly convex composite optimization (cf. Section 5) that enjoys certain duality-gap convergence guarantees, and appears to be the first of its kind in the literature. We believe that this inexact APG method may be of independent interest.

Third, we provide a variant and an extension of our framework (cf. Section 7). We first consider the case where \( f \) has a “simple” dual structure, and corresponds to the same assumption in Kong and Monteiro [13]. In this case, we show that the primal gradient complexity for finding an \( \varepsilon \)-NS point of (1.1) has order \( O(\varepsilon^{-3}\ln(\varepsilon^{-1})) \), which recovers the result in Kong and Monteiro [13] up to a logarithmic factor. Secondly, we extend our framework to the stochastic case, where the gradients \( \nabla_x \Phi(x, y) \) and \( \nabla_y \Phi(x, y) \) are only accessible through their stochastic unbiased estimators. Although such an extension is rather straightforward, we show that primal and dual gradient complexities of this extension indeed match the best-known in the literature (see e.g., Rafique et al. [25]).
1.7. Organization. Our work mainly consists of three parts.

The first part includes Sections 2 and 3, and lays the foundation of the whole work. Specifically, in Section 2, we introduce several important notions in non-convex analysis and non-Hilbertian optimization (such as Bregman divergence and Bregman proximal projection). In Section 3, we develop several important lemmas that characterize the (sub-)differential and convexity properties of the function $f$ and its smooth approximation.

The second part consists of Sections 4 to 6, and forms the main body of this work. Its organization is illustrated in Figure 1. Specifically, in Section 4, we propose our primal-dual smoothing framework and analyze its number of iterations. We also analyze the primal and dual gradient complexities of this framework based on the the complexity results of a sub-problem solver that we will develop in Section 6. This sub-problem solver is essentially an efficient first-order method for solving a class of (strongly) convex-concave SPPs. This method critically leverage a non-Hilbertian inexact APG method that is developed in Section 5.

The last part consists of Sections 7 and 8, wherein we discuss some variants and extensions of our framework, and conclude by pointing out some open problems and future research directions.

2. Preliminaries. We introduce several important notions that will be used in our analysis. Throughout this work, for any nonempty set $\mathcal{X} \subseteq \mathbb{X}$, we denote its interior by $\text{int} \mathcal{X}$, its boundary by $\text{bd} \mathcal{X}$ and its closed convex hull by $\text{clconv} \mathcal{X}$. Also, define the extended real line $\mathbb{R} := \mathbb{R} \cup \{+\infty\}$.

2.1. Directional derivative, Fréchet subdifferential and gradient. Following Kruger [14], given any function $f : \mathcal{W} \to \mathbb{R}$, where $\mathcal{W} \subseteq \mathbb{X}$ is a nonempty open set, define its (Hadamard) directional derivative at $x \in \mathcal{W}$ in the direction of $d \in \mathbb{X}$ as

$$f'(x; d) = \lim_{t \downarrow 0, t \to d} \frac{f(x + td) - f(x)}{t},$$

whenever the limit exists. If the limit in (2.1) exists for every $d \in \mathbb{X}$, then $f$ is directionally differentiable at $x$. If $f$ is directionally differentiable at each $x \in \mathcal{W}$, then we say that $f$ is directionally differentiable on $\mathcal{W}$.
We define the Fréchet subdifferential of $f$ at $x \in \mathcal{W}$ as

$$
\partial f(x) := \left\{ \xi \in X^*: \lim_{h \to 0, h \in X^*} \frac{f(x+h) - f(x) - \langle \xi, h \rangle}{\|h\|} \geq 0 \right\}. \tag{2.2}
$$

In other words, $\xi \in \partial f(x)$ if and only if $f(x+h) \geq f(x) + \langle \xi, h \rangle + o(\|h\|)$. We call the elements in $\partial f(x)$ the Fréchet subgradients of $f$ at $x$. Note that $\partial f(x)$ is closed and convex, and if $f$ is convex, then $\partial f(x)$ is the (convex) subdifferential of $f$ at $x$. We say $f$ is Fréchet subdifferentiable at $x$ if $\partial f(x) \neq \emptyset$, and Fréchet subdifferentiable on $\mathcal{W}$ if $\partial f(x) \neq \emptyset$ for all $x \in \mathcal{W}$.

We define the gradient (or Fréchet derivative) of $f$ at $x \in \mathcal{W}$, denoted by $\nabla f(x)$, as the unique element in $X^*$ that satisfies

$$
\lim_{h \to 0, h \in \mathcal{W}} \frac{f(x+h) - f(x) - \langle \nabla f(x), h \rangle}{\|h\|} = 0. \tag{2.3}
$$

In other words, $f(x+h) = f(x) + \langle \nabla f(x), h \rangle + o(\|h\|)$. From the definitions in (2.1) and (2.3), we see that $f$ is differentiable at $x$ if and only if $d \mapsto f'(x; d)$ is a linear function on $X$, and in this case, we have $f'(x; d) = \langle \nabla f(x), d \rangle$ for all $d \in X$. We say that $f$ is (Fréchet) differentiable at $x \in X$ if $\nabla f(x)$ exists, and (Fréchet) differentiable on $\mathcal{W}$ if $\nabla f(x)$ exists for all $x \in \mathcal{W}$.

### 2.2. Distance generating function and Bregman divergence.

Let $\mathcal{U}$ be a nonempty, convex and closed set in a finite-dimensional real normed space $U$. We call $\omega_{\mathcal{U}}: \mathcal{U} \to \mathbb{R}$ a distance generating function (DGF) on $\mathcal{U}$ if it is continuous and 1-strongly-convex on $\mathcal{U}$ and essentially smooth, i.e., it is continuously differentiable on the interior of its domain (denoted by $\text{int dom } \omega_{\mathcal{U}}$) and for any sequence $\{u_k\}_{k \geq 0} \subseteq \text{int dom } \omega_{\mathcal{U}}$ that converges to a boundary point $u \in \text{bd dom } \omega_{\mathcal{U}}$, we have $\|\nabla \omega_{\mathcal{U}}(u_k)\|$, $\frac{k}{k \to +\infty} \to +\infty$. Based on $\omega_{\mathcal{U}}$, let us define its induced Bregman divergence as

$$
D_{\omega_{\mathcal{U}}}(u', u) := \omega_{\mathcal{U}}(u') - \omega_{\mathcal{U}}(u) - \langle \nabla \omega_{\mathcal{U}}(u), u' - u \rangle, \quad \forall u' \in \text{dom } \omega_{\mathcal{U}}, \forall u \in \text{int dom } \omega_{\mathcal{U}}. \tag{2.4}
$$

Define $\mathcal{U}^\circ := \mathcal{U} \cap \text{int dom } \omega_{\mathcal{U}}$. Since $\omega_{\mathcal{U}}$ is 1-strongly-convex on $\mathcal{U}$, we have

$$
D_{\omega_{\mathcal{U}}}(u', u) \geq (1/2)\|u' - u\|^2, \quad \forall u' \in \mathcal{U}, \forall u \in \mathcal{U}^\circ. \tag{2.5}
$$

### 2.3. Bregman proximal projection (BPP).

For any $u \in \mathcal{U}^\circ$ and any convex and closed function $\varphi: \mathcal{U} \to \mathbb{R}$, define the BPP of $u$ onto $\mathcal{U}$ under $\varphi$ and the DGF $\omega_{\mathcal{U}}$ (associated with dual vector $\xi \in \mathbb{U}^*$ and step-size $\lambda > 0$) as the following mapping:

$$
u \mapsto u^+ := \arg\min_{u' \in \mathcal{U}} \varphi(u') + \langle \xi, u' \rangle + \lambda^{-1}D_{\omega_{\mathcal{U}}}(u', u) = \arg\min_{u' \in \mathcal{U}} \varphi(u') + \langle \xi, u' \rangle + \lambda^{-1}\omega_{\mathcal{U}}(u') \quad \text{where} \quad \xi := \xi - \lambda^{-1}\nabla \omega_{\mathcal{U}}(u). \tag{2.7}
$$

Note that if $\inf_{u \in \mathcal{U}} \varphi(u) > -\infty$, then the minimization problem in (2.7) always has a unique solution in $\mathcal{U}^\circ$ (cf. Zhao [33, Lemma A.1]). We say that the function $\varphi$ has an easily computable BPP on $\mathcal{U}$ if there exists a DGF $\omega_{\mathcal{U}}$ on $\mathcal{U}$ such that the minimization problem in (2.7) has a unique and easily computable solution in $\mathcal{U}^\circ \cap \text{dom } \varphi$, for any $\xi \in \mathbb{U}^*$ and $\lambda > 0$. For further discussions on BPP, we refer readers to Nesterov [20] and Juditsky and Nemirovski [12].
2.4. $\varepsilon$-NS point. Let us formally define the $\varepsilon$-NS point when $X$ is a normed space, using the notions of DGF and BPP in Sections 2.2 and 2.3, respectively. Let $\omega_X : X \to \mathbb{R}$ be a DGF on $X$ (cf. Section 2.2). Throughout this work, we assume that $\omega_X$ satisfy the following additional properties:

(i) it is twice continuously differentiable on the interior of its domain (i.e., $\text{int dom} \omega_X$),
(ii) $X \subseteq \text{int dom} \omega_X$ (so that $X^o := X \cap \text{int dom} \omega_X = X$),
(iii) its gradient $\nabla \omega_X$ is $\beta_X$-Lipschitz on $X$, where $\beta_X \in [1, +\infty)$.

(Note that if $X$ is bounded, then property (iii) is implied by properties (i) and (ii), together with the 1-strong-convexity of $\omega_X$ on $X$.) Using a simpler form of BPP in (2.7) with $\xi = 0$, we can define the proximal point at $x \in \text{int dom} \omega_X$ with function $q$ and step-size $0 < \lambda < \gamma^{-1}$ as

$$
\text{prox}(q, x, \lambda) := \arg \min_{x' \in X} q(x') + \lambda^{-1} D_{\omega_X}(x', x) = \arg \min_{x' \in X} q(x') + \lambda^{-1} D_{\omega_X}(x', x),
$$

where (2.9) follows from that $\text{dom} q = \text{dom} r = X$. (Note that as we will see in Section 4, the condition $0 < \lambda < \gamma^{-1}$ ensures that the minimization problem in (2.9) is strongly convex and hence admits a unique optimal solution.) We call $x \in X$ an $\varepsilon$-NS point of (1.1), if there exists some $0 < \lambda < \gamma^{-1}$ such that

$$
\|\lambda^{-1}(x - \text{prox}(q, x, \lambda))\| \leq \varepsilon / \beta_X.
$$

Note that the above definition recovers the one when $X$ is Hilbertian, as introduced in Section 1.4. Specifically, if we let $X = X$ and $\omega_X = (1/2) \cdot \| \cdot \|^2$, then $D_{\omega_X}(x', x) = (1/2) \| x' - x \|^2$ and $\beta_X = 1$, and hence (2.10) reduces to (1.16). Similar to the discussions in Section 1.4, if (2.10) holds, then $\text{prox}(q, x, \lambda)$ is an $\varepsilon$-AS point of (1.1). Since $\beta_X \geq 1$, the normalized distance $\|\lambda^{-1}(x - \text{prox}(q, x, \lambda))\|$ is no larger than $\varepsilon$. As a result, $x$ is called an $\varepsilon$-NS point of (1.1).

3. Important Lemmas. Our algorithmic framework, both in terms of its development and analysis, critically leverage the following lemmas that characterize certain (sub-)differential and convexity properties of the function $f$ (defined in (1.1)) and its smooth approximation. Recall from Assumption 1.1 that $\Phi(\cdot, y)$ is differentiable on the open set $\mathcal{X}' \supseteq \mathcal{X}$ for all $y \in \mathcal{Y}$.

3.1. Lemmas on $f$. Let us begin by characterizing the directional derivative of $f$ (recall that $f$ is a non-smooth and non-convex function). Our characterization can be regarded as a particular version of Danskin’s Theorem [6].

**Lemma 3.1.** The function $f$ in (1.1) is directionally differentiable on the open set $\mathcal{X}'$. For any $x \in \mathcal{X}'$ and $d \in \mathbb{R}$, its directional derivative $f'(x; d)$ can be characterized as

$$
f'(x; d) = \sup_{y \in \mathcal{Y}^*(x)} \langle \nabla_x \Phi(x, y), d \rangle, \quad \text{where} \quad \mathcal{Y}^*(x) := \arg \max_{y \in \mathcal{Y}} \Phi(x, y) - g(y) \neq \emptyset.
$$
In particular, if $\mathcal{Y}^*(x)$ is a singleton, i.e., $\mathcal{Y}^*(x) = \{ y^*(x) \}$, then $f$ is differentiable at $x$ and
\[
\nabla f(x) = \nabla_x \Phi(x, y^*(x)).
\] (3.2)

**Proof.** See Appendix A.

Next, we show the local Lipschitz continuity of $f$, and characterize its Fréchet subdifferential.

**Lemma 3.2.** The function $f$ is locally Lipschitz on $\mathcal{X}$. In addition, for any $x \in \mathcal{X}'$, the Fréchet subdifferential $\partial f(x) = \text{clconv} \{ \nabla_x \Phi(x, y) : y \in \mathcal{Y}^*(x) \}$.

**Proof.** See Appendix B.

Finally, let us make a simple observation about the weak convexity of $f$, based on Lemma 3.2.

**Lemma 3.3.** The function $f$ is $\gamma$-weakly convex on $\mathcal{X}$, namely, for any $x \in \mathcal{X}$, we have
\[
f(x + d) - f(x) - \langle \xi, d \rangle \geq -(\gamma/2) \| d \|^2, \quad \forall \xi \in \partial f(x).
\] (3.3)

**Proof.** Fix any $x \in \mathcal{X}$ and any $d \in \mathcal{X}$. By Assumption 1.2, we have that for any $y \in \mathcal{Y}^*(x)$,
\[
f(x + d) - f(x) - \langle \nabla_x \Phi(x, y), d \rangle \geq \Phi(x + d, y) - \Phi(x, y) - \langle \nabla_x \Phi(x, y), d \rangle \geq -(\gamma/2) \| d \|^2.
\] (3.4)

By taking convex combination and limit on (3.4) if necessary, we see that
\[
f(x + d) - f(x) - \langle \xi, d \rangle \geq -(\gamma/2) \| d \|^2, \quad \forall \xi \in \text{clconv} \{ \nabla_x \Phi(x, y) : y \in \mathcal{Y}^*(x) \}
\]
where (a) follows from Lemma 3.2. This completes the proof. $\square$

### 3.2. Lemmas on the dually smoothed $f$.

Let $\omega_\mathcal{Y} : \mathcal{Y} \to \overline{\mathbb{R}}$ be a DGF on $\mathcal{Y}$ (cf. Section 2.2), and define the $\rho$-dually-smoothed $f$ as
\[
f_\rho(x) = \max_{y \in \mathcal{Y}} \left[ \phi_\rho^D(x, y) := \Phi(x, y) - g(y) - \rho \omega_\mathcal{Y}(y) \right], \quad \forall x \in \mathcal{X}',
\] (3.5)
where $\rho > 0$ is called the dual smoothing parameter. Define the range of $\omega_\mathcal{Y}$ on $\mathcal{Y}$ as
\[
R_\mathcal{Y}(\omega_\mathcal{Y}) := \sup_{y \in \mathcal{Y}} |\omega_\mathcal{Y}(y)|,
\] (3.6)
and we have $R_\mathcal{Y}(\omega_\mathcal{Y}) < +\infty$ since $\mathcal{Y}$ is compact and $\omega_\mathcal{Y}$ is continuous on $\mathcal{Y}$. Clearly, with this parameter, we can uniformly bound the point-wise difference between $f_\rho$ and $f$: for any $x \in \mathcal{X}$,
\[
|f_\rho(x) - f(x)| \leq \sup_{y \in \mathcal{Y}} |\phi_\rho^D(x, y) - (\Phi(x, y) - g(y))| = \sup_{y \in \mathcal{Y}} |\rho \omega_\mathcal{Y}(y)| = \rho R_\mathcal{Y}(\omega_\mathcal{Y}).
\] (3.7)

In addition, let us define the unique solution to the maximization problem in (3.5) as $y_\rho^*(x)$, namely
\[
y_\rho^*(x) := \arg \max_{y \in \mathcal{Y}} \phi_\rho^D(x, y), \quad \forall x \in \mathcal{X}'.
\] (3.8)

Based on these definitions, let us first show that the mapping $y_\rho^* : \mathcal{X}' \to \mathcal{Y}$ is Lipschitz on $\mathcal{X}$, even if $\mathcal{Y}$ is unbounded. (To be clear, all the other results in this paper still assume the boundedness of $\mathcal{Y}$, unless otherwise mentioned.)
Lemma 3.4. Regardless of whether $\mathcal{Y}$ is bounded, the mapping $y^*_\rho$ is $(L_{xy}/\rho)$-Lipschitz on $\mathcal{X}$.

Proof. See Appendix C. □

Based on Lemma 3.4, we can show the smoothness of $f_\rho$ on $\mathcal{X}$.

Lemma 3.5. The function $f_\rho$ is differentiable on $\mathcal{X}'$ and $\nabla f_\rho(x) = \nabla_x \Phi(x, y^*_\rho(x))$ for all $x \in \mathcal{X}'$. In addition, the gradient $\nabla f_\rho : \mathcal{X}' \rightarrow \mathcal{X}^*$ is $L_\rho$-Lipschitz on $\mathcal{X}$ with $L_\rho := L_{xx} + L_{xy}^2/\rho$.

Proof. See Appendix D.

Remark 3.1. Two remarks are in order. First, when the function $(x, y) \mapsto \Phi(x, y)$ is bilinear, i.e., $\Phi(x, y) = \langle Ax, y \rangle$ for some linear operator $A : \mathcal{X} \rightarrow \mathcal{Y}^*$, we have $L_{xx} = 0$ and $L_{xy} = \|A\|_{op}$, i.e., the operator norm of $A$. Thus we exactly recover the result in Nesterov [20, Theorem 1]. Second, note that compared to similar statements about the Lipschitz continuities of $y^*_\rho$ and $\nabla f_\rho$, e.g., Sinha et al. [28, Lemma 1], Lemma 3.5 does not require any differentiability assumptions on the function $\Phi(x, \cdot) : \mathcal{Y}' \rightarrow \mathbb{R}$ for any $x \in \mathcal{X}$.

Next, using the same reasoning as in Lemma 3.3, we have the following lemma.

Lemma 3.6. The function $f_\rho$ is $\gamma$-weakly convex on $\mathcal{X}$.

Finally, we prove a uniform bound on the distance between $\nabla f_\rho(x)$ and $\partial f(x)$ over $x \in \mathcal{X}$, for any $\rho > 0$. Given the normed space $(\mathcal{U}, \| \cdot \|)$ as in Section 2.2, for any point $u \in \mathcal{U}$ and any nonempty set $\mathcal{U} \subseteq \mathcal{U}$, define the distance from $u$ to $\mathcal{U}$ as

$$\text{dist}(u, \mathcal{U}) := \inf_{u' \in \mathcal{U}} \|u - u'\|. \quad (3.9)$$

Additionally, let us define the diameter of $\mathcal{Y}$ as

$$D_Y := \sup_{y, y' \in \mathcal{Y}} \|y - y'\| < +\infty. \quad (3.10)$$

Lemma 3.7. For any $\rho > 0$ and any $x \in \mathcal{X}$, we have

$$\text{dist}(\nabla f_\rho(x), \partial f(x)) \leq L_{xy} \text{dist}(y^*_\rho(x), \mathcal{Y}^*(x)) \leq L_{xy} D_Y. \quad (3.11)$$

Proof. See Appendix E. □

4. A primal-dual smoothing framework for finding an $\varepsilon$-NS point of (1.1). As the name suggests, our framework utilizes both ideas of primal smoothing and dual smoothing. The notion of dual smoothing has been introduced in Section 3.2, and let us now introduce primal smoothing. For concreteness, we use the objective function $q$ in (1.1) as an example. From Lemma 3.3 and the convexity of the function $r$ on $\mathcal{X}$, we know that $q$ is $\gamma$-weakly convex on $\mathcal{X}$.
Lemma 3.1, we see that the function $q_{prox}$.

For any $x \in \mathcal{X}$ and any $0 < \lambda < \gamma^{-1}$, let $q^\lambda(x)$ be the optimal value of the minimization problem in (2.9), namely

$$q^\lambda(x) := \min_{x' \in \mathcal{X}} [Q^\lambda(x'; x) := q(x') + \lambda^{-1} D_{\omega_{\mathcal{X}}}(x', x)]$$

$$= Q^\lambda(\text{prox}(q, x, \lambda); x), \quad (4.1)$$

where (4.2) follows from the definition of $\text{prox}(q, x, \lambda)$ in (2.9). (Note that since $\lambda^{-1} > \gamma$ and that $\omega_{\mathcal{X}}$ is 1-strong convex on $\mathcal{X}$, the function $Q^\lambda(\cdot; x)$ is $(\lambda^{-1} - \gamma)$-strongly convex on $\mathcal{X}$, and hence $\text{prox}(q, x, \lambda)$ in (2.9) is indeed unique.) Using the definition of $D_{\omega_{\mathcal{X}}}(x', x)$ in (2.4) and invoking Lemma 3.1, we see that the function $q^\lambda : \text{int dom } \omega_{\mathcal{X}} \to \mathbb{R}$ is differentiable on $\text{int dom } \omega_{\mathcal{X}} \supseteq \mathcal{X}$, and

$$\nabla q^\lambda(x) = \nabla^2 \omega_{\mathcal{X}}(x) \lambda^{-1} (x - \text{prox}(q, x, \lambda)), \quad \forall x \in \text{int dom } \omega_{\mathcal{X}}, \quad (4.3)$$

For this reason, we can regard the operation in (4.1), which transforms $q$ to $q^\lambda$, as the primal smoothing procedure on $q$, and call the resulting function $q^\lambda$ the $\lambda$-primally-smoothed $q$.

Now, let us define $q_\rho := f_\rho + r$ for some $\rho > 0$, where $f_\rho$ is the $\rho$-dually-smoothed $f$. As such, $q_\rho$ can be regarded as the $\rho$-dually-smoothed $q$. Similar to the above, we can define the $\lambda$-primally-smoothed $q_\rho$ for some $0 < \lambda < \gamma^{-1}$, denoted by $q_\rho^\lambda : \text{int dom } \omega_{\mathcal{X}} \to \mathbb{R}$, as

$$q_\rho^\lambda(x) := \min_{x' \in \mathcal{X}} [Q_\rho^\lambda(x'; x) := q_\rho(x') + \lambda^{-1} D_{\omega_{\mathcal{X}}}(x', x)]$$

$$= Q_\rho^\lambda(\text{prox}(q_\rho, x, \lambda); x), \quad \text{ where } \text{prox}(q_\rho, x, \lambda) := \arg \min_{x' \in \mathcal{X}} Q_\rho^\lambda(x'; x). \quad (4.5)$$

The minimization problem in (4.4) indeed suggests an iterative scheme for finding an $\varepsilon$-NS of (1.1), which forms the basis of our framework shown in Algorithm 1. Specifically, we start with any point in $\mathcal{X}$, and in each iteration, we approximately solve the minimization problem in (4.4) with some accuracy $\eta > 0$, and denote this $\eta$-optimal solution as $x_{k+1}$ (cf. Step 2). Throughout all the iterations, we fix the primal smoothing parameter $\lambda \in (0, \gamma^{-1})$ and the dual smoothing parameters $\rho = 2\eta/R_y(\omega_y)$, where $R_y(\omega_y)$ is defined in (3.6). The appropriate choices of $\eta$ and $\lambda$
will become apparent after our analysis. We terminate the algorithm once the distance between two successive iterates falls below $\sqrt{2\eta/(\lambda^{-1} - \gamma)}$ (see the termination criterion in (4.6)).

Before ending the description of our framework, let us notice that the termination criterion (4.6) is easily checkable — in fact, it is solely based on the distances between successive iterates. This is in contrast to the convergence criteria in previous works (e.g., Kong and Monteiro [13] and Thekumparampil et al. [30]) that involve quantities like $q(x_k)$ or $\nabla f_p(x_k)$, whose evaluation typically requires solving certain dual optimization problems. Hence these convergence criteria are harder to check than ours.

In the following, we analyze Algorithm 1 using two different approaches. Let $K \geq 1$ denote the iteration that Algorithm 1 terminates, so that Algorithm 1 outputs $x_K$. For either approach, we derive the choice of $\eta$ (as a function of $\lambda$, $\varepsilon$ and $\beta_X$) such that

$$
\|x_K - \text{prox}(q, x_K, \lambda)\| \leq \varepsilon\lambda/\beta_X, \tag{4.7}
$$

which implies that $x_K \in \mathcal{X}$ is an $\varepsilon$-NS point of (1.1) (cf. Section 2.4). Before presenting our analysis, we first show that if (4.7) holds, then i) $\text{prox}(q, x_K, \lambda)$ is an $\varepsilon$-AS point, namely, there exists a Fréchet subgradient of $q$ at $\text{prox}(q, x_K, \lambda)$ with norm not exceeding $\varepsilon$ and ii) the gradient of $q^\lambda$ at $x_K$ has a small norm (not exceeding $\varepsilon$), where $q^\lambda$ is the $\lambda$-primally-smoothed $q$ as defined in (4.1).

**Lemma 4.1.** For any $\varepsilon > 0$ and any $0 < \lambda < \gamma^{-1}$, if $x \in \mathcal{X}$ satisfies $\|x - \text{prox}(q, x, \lambda)\| \leq \varepsilon\lambda/\beta_X$, then we have

$$
dist(0, \partial q(\text{prox}(q, x, \lambda))) \leq \varepsilon \quad \text{and} \quad \|\nabla q^\lambda(x)\|_* \leq \varepsilon. \tag{4.8}
$$

**Proof.** Applying the first-order optimality condition to the definition of $\text{prox}(q, x, \lambda)$ (cf. (2.9)), we have that for any $0 < \lambda < \gamma^{-1},$

$$
\lambda^{-1}(\nabla \omega_X(x) - \nabla \omega_X(\text{prox}(q, x, \lambda))) \in \partial q(\text{prox}(q, x, \lambda)). \tag{4.9}
$$

As a result, using the $\beta_X$-Lipschitz continuity of $\nabla \omega_X$ on $\mathcal{X}$, we have

$$
dist(0, \partial q(\text{prox}(q, x, \lambda))) = \inf_{\xi \in \partial q(\text{prox}(q, x, \lambda))} \|\xi\|_* \\
\leq \lambda^{-1}\|\nabla \omega_X(x) - \nabla \omega_X(\text{prox}(q, x, \lambda))\|_* \\
\leq \lambda^{-1}\beta_X\|x - \text{prox}(q, x, \lambda)\| \leq \varepsilon.
$$

Using the $\beta_X$-Lipschitz continuity of $\nabla \omega_X$ on $\mathcal{X}$ again, we can easily show that the operator norm of $\nabla^2 \omega(x) : \mathcal{X} \to \mathcal{X}^*$ (denoted by $\|\nabla^2 \omega(x)\|_\text{op}$) is uniformly bounded on $\mathcal{X}$ by $\beta_X$, namely

$$
\|\nabla^2 \omega(x)\|_\text{op} := \sup_{z \in \mathcal{X}} \|\nabla^2 \omega(x)z\|_* \leq \beta_X, \quad \forall x \in \mathcal{X}. \tag{4.10}
$$

Therefore, by the definition of $\nabla q^\lambda(x)$ in (4.3), we have

$$
\|\nabla q^\lambda(x)\|_* \leq \lambda^{-1}\|\nabla^2 \omega_X(x)\|_\text{op}\|x - \text{prox}(q, x, \lambda)\| \leq \lambda^{-1}\beta_X\|x - \text{prox}(q, x, \lambda)\| \leq \varepsilon. \quad \Box
$$
4.1. Approach I: Dual-then-primal smoothing. We analyze Algorithm 1 by regarding it as an inexact proximal-point framework for finding a near-stationary point of $q_\rho$, namely the $\rho$-dually-smoothed $q$. Indeed, we first show that $x_K \in X$ is a near-stationary point of $q_\rho$, by bounding $\|x_K - \text{prox}(q_\rho, x_K, \lambda)\|$, and then bound the distance between the proximal points $\text{prox}(q, x_K, \lambda)$ and $\text{prox}(q_\rho, x_K, \lambda)$, namely $\|\text{prox}(q, x_K, \lambda) - \text{prox}(q_\rho, x_K, \lambda)\|$. These two bounds together yield a bound on $\|x_K - \text{prox}(q, x_K, \lambda)\|$, thereby showing that $x_K$ is a near-stationary point of $q$. These steps are formalized below.

**Lemma 4.2.** In Algorithm 1, for any $\eta > 0$, we have $\|x_K - \text{prox}(q_\rho, x_K, \lambda)\| \leq 2\sqrt{2\eta/(\lambda^{-1} - \gamma)}$.

**Proof.** By the fact that $\text{prox}(q_\rho, x_K, \lambda) = \arg\min_{x' \in X} Q^\lambda_{\rho}(x'; x_K)$ (cf. (4.5)) and the $(\lambda^{-1} - \gamma)$-strong-convexity of the function $Q^\lambda_{\rho}(\cdot; x_K)$ on $X$, we have

$$\frac{\lambda^{-1} - \gamma}{2} \|x_{K+1} - \text{prox}(q_\rho, x_K, \lambda)\|^2 \leq Q^\lambda_{\rho}(x_{K+1}; x_K) - Q^\lambda_{\rho}(\text{prox}(q_\rho, x_K, \lambda); x_K) \leq \eta,$$  

(4.11)

where the second inequality follows from the condition in Step 2 and (4.2). This implies that $\|x_{K+1} - \text{prox}(q_\rho, x_K, \lambda)\| \leq \sqrt{2\eta/(\lambda^{-1} - \gamma)}$. On the other hand, by the definition of $K$, we have $\|x_{K+1} - x_K\| \leq \sqrt{2\eta/(\lambda^{-1} - \gamma)}$. This completes the proof. □

**Lemma 4.3.** For any $\eta > 0$ and any $x \in X$, we have

$$\|\text{prox}(q_\rho, x, \lambda) - \text{prox}(q, x, \lambda)\| \leq 2\sqrt{\rho R^2(\omega_\gamma)/(\lambda^{-1} - \gamma)}.$$  

(4.12)

In particular, if $x = x_K$ and $\rho = 2\eta/R^2(\omega_\gamma)$ (as in Algorithm 1), then we have

$$\|\text{prox}(q_\rho, x_K, \lambda) - \text{prox}(q, x_K, \lambda)\| \leq 2\sqrt{2\eta/(\lambda^{-1} - \gamma)}.$$  

(4.13)

**Proof.** By the fact that $\text{prox}(q, x, \lambda) = \arg\min_{x' \in X} Q^\lambda(x'; x)$ (cf. (2.8)) and the $(\lambda^{-1} - \gamma)$-strong-convexity of $Q^\lambda(\cdot; x)$ on $X$, we have

$$\frac{\lambda^{-1} - \gamma}{2} \|\text{prox}(q_\rho, x, \lambda) - \text{prox}(q, x, \lambda)\|^2 \leq Q^\lambda(\text{prox}(q_\rho, x, \lambda); x) - Q^\lambda(\text{prox}(q, x, \lambda); x).$$  

(4.14)

In addition,

$$Q^\lambda(\text{prox}(q_\rho, x, \lambda); x) - Q^\lambda(\text{prox}(q_\rho, x, \lambda); x) = f(\text{prox}(q_\rho, x, \lambda)) - f_\rho(\text{prox}(q_\rho, x, \lambda)) \leq \rho \Omega_\gamma(\omega_\gamma),$$  

(4.15)

$$Q^\lambda_{\rho}(\text{prox}(q_\rho, x, \lambda); x) - Q^\lambda_{\rho}(\text{prox}(q_\rho, x, \lambda); x) \leq Q^\lambda_{\rho}(\text{prox}(q, x, \lambda); x) - Q^\lambda_{\rho}(\text{prox}(q, x, \lambda); x)$$

$$= f_\rho(\text{prox}(q_\rho, x, \lambda)) - f(\text{prox}(q_\rho, x, \lambda)) \leq \rho \Omega_\gamma(\omega_\gamma),$$  

(4.16)

where (a) and (c) follow from (3.7) and (b) follows from (4.5). Now, by combining (4.14), (4.15) and (4.16), we complete the proof. □
Combining Lemmas 4.2 and 4.3, we see that
\[ \|x_K - \text{prox}(q, x_K, \lambda)\| \leq 4\sqrt{2}\eta/\left(\lambda^{-1} - \gamma\right), \tag{4.17} \]
and hence we have the following theorem.

**Theorem 4.1.** In Algorithm 1, for any \( \epsilon > 0 \), if we set the accuracy parameter
\[ \eta = \epsilon^2\lambda(1 - \gamma\lambda)/(32\beta_X^2), \tag{4.18} \]
then \( \|x_K - \text{prox}(q, x_K, \lambda)\| \leq \epsilon\lambda/\beta_X \), meaning that \( x_K \) is an \( \epsilon \)-NS point of (1.1).

### 4.2. Approach II: Primal-then-dual smoothing.

Alternatively, we can directly view Algorithm 1 as an inexact proximal-point framework for finding a near-stationary point of (1.1). Specifically, we will show that Step 2 in Algorithm 1 implies that \( x_{k+1} \) is also an approximate solution for the optimization problem in (4.1), namely
\[ Q^\lambda(x_{k+1}; x_k) \leq Q^\lambda(\text{prox}(q, x_k, \lambda); x_k) + 5\eta, \quad \forall \ k \geq 1. \tag{4.19} \]

The reason that we (approximately) solve the “dually smoothed” optimization problem in (4.4), instead of that in (4.1), is because the former enjoys certain smoothness properties and hence can be more efficiently solved using (primal-dual) first-order methods (cf. Section 6). Then, based on (4.19), we can easily arrive at (4.17). The details are shown below.

**Lemma 4.4.** In Algorithm 1, for all \( k \geq 1 \), we have (4.19) and
\[ \|x_{k+1} - \text{prox}(q, x_k, \lambda)\| \leq \sqrt{10}\eta/\left(\lambda^{-1} - \gamma\right). \tag{4.20} \]

**Proof.** Note that by the definitions of \( q^\lambda \) and \( q^\rho \) in (4.1) and (4.4), respectively, for any \( x \in \mathcal{X} \), we have
\[ \sup_{x' \in \mathcal{X}} |Q^\lambda(x'; x) - Q^\lambda(x'; x)| = \sup_{x' \in \mathcal{X}} |f(x') - f(x')| \leq \rho R_Y(\omega_Y), \tag{4.21} \]
\[ |q^\lambda(x) - q^\rho(x)| \leq \sup_{x' \in \mathcal{X}} |Q^\lambda(x'; x) - Q^\lambda(x'; x)| \leq \rho R_Y(\omega_Y), \tag{4.22} \]
where the inequality in (4.21) follows from (3.7). By Step 2 and (4.22), we have
\[ Q^\rho(x_{k+1}; x_k) \leq q^\rho(x_k) + \eta \leq q^\lambda(x_k) + \rho R_Y(\omega_Y) + \eta. \tag{4.23} \]

On the other hand, by (4.21), we have
\[ Q^\lambda(x_{k+1}; x_k) - \rho R_Y(\omega_Y) \leq Q^\lambda(x_{k+1}; x_k). \tag{4.24} \]
Combining (4.23) and (4.24), and using that \( \rho = 2\eta/R_y(\omega_y) \), we have
\[
Q^\lambda(x_{k+1}; x_k) \leq q^\lambda(x_k) + 2\rho R_y(\omega_y) + \eta = Q^\lambda(\text{prox}(q, x_k, \lambda); x_k) + 5\eta. \tag{4.25}
\]
By the fact that \( \text{prox}(q, x_k, \lambda) = \arg \min_{x' \in \mathcal{X}} Q^\lambda(x'; x_k) \) (cf. (2.8)) and the \((\lambda^{-1} - \gamma)\)-strong-convexity of \( Q^\lambda(\cdot; x) \) on \( \mathcal{X} \), we therefore have
\[
\|x_{k+1} - \text{prox}(q, x_k, \lambda)\| \leq \sqrt{10\eta/(\lambda^{-1} - \gamma)}. \tag{4.25}
\]
Now, from (4.6), we know that \( x_K \) satisfies \( \|x_{K+1} - x_K\| \leq \sqrt{2\eta/(\lambda^{-1} - \gamma)} \). This, together with Lemma 4.4, implies (4.17), which then leads to Theorem 4.1.

4.3. Bound on the number of iterations of Algorithm 1. The simple structure of Algorithm 1 enables us to easily derive a bound on the number of iterations of Algorithm 1.

**Theorem 4.2.** For any \( \lambda \in (0.8\gamma^{-1}, \gamma^{-1}) \), Algorithm 1 terminates in at most
\[
\left\lceil \frac{q(x_1) - q^*}{\left(\frac{\gamma \lambda}{1-\gamma \lambda} - 4\right) \eta} \right\rceil + 1 \tag{4.26}
\]
iterations, where \( q^* > -\infty \) is the optimal value of (1.1).

**Proof.** Note that if (4.6) is not satisfied at iteration \( k \), then \( \|x_{k+1} - x_k\| > \sqrt{2\eta/(\lambda^{-1} - \gamma)} \). Therefore, using (2.5) and (4.22), we have
\[
Q^\lambda(x_{k+1}; x_k) \geq q_\rho(x_{k+1}) + (2\lambda)^{-1}\|x_{k+1} - x_k\|^2 \geq q(x_{k+1}) - \rho R_y(\omega_y) + \eta/(1 - \gamma \lambda). \tag{4.27}
\]
On the other hand, using Step 2 and (4.22), we have
\[
Q^\lambda(x_{k+1}; x_k) \leq q_\rho(x_k) + \eta \leq q^\lambda(x_k) + \rho R_y(\omega_y) + \eta \leq q(x_k) + \rho R_y(\omega_y) + \eta, \tag{4.28}
\]
where the last step follows from \( q^\lambda(x) \leq Q^\lambda(x; x) = q(x) \) for all \( x \in \mathcal{X} \). Combining (4.27) and (4.28), and plugging in the value \( \rho = 2\eta/R_y(\omega_y) \), we have
\[
q(x_{k+1}) \leq q(x_k) + 2\rho R_y(\omega_y) - \frac{\gamma \lambda}{1 - \gamma \lambda} \eta = q(x_k) - \left(\frac{\gamma \lambda}{1 - \gamma \lambda} - 4\right) \eta. \tag{4.29}
\]
Summing over \( k = 1, \ldots, K - 1 \), we have
\[
\left(\frac{\gamma \lambda}{1 - \gamma \lambda} - 4\right) \eta(K - 1) \leq q(x_1) - q(x_K) \leq q(x_1) - q^*. \tag{4.30}
\]
Rearranging and we complete the proof. \( \square \)

Based on Theorem 4.1 and Theorem 4.2, we have the following corollary.

**Corollary 4.1.** For any \( \lambda \in (0.8\gamma^{-1}, \gamma^{-1}) \) and any \( \varepsilon > 0 \), if we set \( \eta \) as in (4.18), then Algorithm 1 returns an \( \varepsilon \)-NS point of (1.1) in no more than \( \bar{K} \) iterations, where
\[
\bar{K} := \left\lceil \frac{32\beta^2(\varepsilon x_1) - q^*)}{5\varepsilon^2 \lambda (\gamma \lambda - 0.8)} \right\rceil + 1. \tag{4.31}
\]
4.4. Choice of $\lambda$ and the primal and dual gradient complexities of Algorithm 1.

From Corollary 4.1, we see that in order to reduce the bound on the number of iterations of Algorithm 1 (namely $\bar{K}$), we should choose $\lambda$ to be as close as $\gamma^{-1}$ as possible. However, note that in Step 2 we need to solve the minimization problem in (4.4) with accuracy $\eta$. Since the choice of the accuracy parameter $\eta$ in (4.18) is proportional to $\lambda(1 - \gamma\lambda)$, and the strong-convexity parameter of the function $Q^\lambda_\rho(\cdot; x_k)$ is $\lambda^{-1} - \gamma$, if $\lambda$ is close to $\gamma^{-1}$, then the strong-convexity parameter becomes very small (in other words, the problem becomes ill-conditioned) and the accuracy becomes very high. Intuitively, this should result in a high computational cost in Step 2, and we formalize this intuition below.

Primal and dual gradient complexities for implementing Step 2. Indeed, since $\lambda < \gamma^{-1}$ and $\omega_\chi$ and $\omega_\gamma$ are 1-strongly-convex on $\chi$ and $\gamma$, respectively, we observe that the minimization problem (4.4) in Step 2 is indeed a (strongly) convex-concave SPP:

$$\min_{x \in \chi} \max_{y \in \gamma} r(x) + \lambda^{-1} D_{\omega_\chi}(x; x_k) + \Phi(x, y) - g(y) - \rho \omega_\gamma(y).$$  \hspace{1cm} (4.32)

In the next two sections (namely Sections 5 and 6), we will develop an efficient first-order method for solving a general class of (strongly) convex-concave SPP that subsumes (4.32) as a special case. Specifically, this method finds $x_{k+1} \in \chi$ that satisfies the $\eta$-optimality condition in Step 2 with primal gradient complexity

$$C_p(\eta) = O \left( \left( \sqrt{\frac{L_{yy}}{\rho}} + \frac{L_{xy}}{\sqrt{(\lambda^{-1} - \gamma)\rho}} \right) \sqrt{\frac{L_{xx} + \gamma \beta_{\chi}}{\lambda^{-1} - \gamma}} \ln^2 \left( \frac{1}{\eta \rho} \right) \right),$$ \hspace{1cm} (4.33)

and dual gradient complexity

$$C_d(\eta) = O \left( \left( \sqrt{\frac{L_{yy}}{\rho}} + \frac{L_{xy}}{\sqrt{(\lambda^{-1} - \gamma)\rho}} \right) \ln \left( \frac{1}{\eta \rho} \right) \right),$$ \hspace{1cm} (4.34)

where the primal and dual gradient complexities are defined in Section 1.4. From (4.33) and (4.34), it is clear that if $\lambda$ is close to $\gamma^{-1}$, then both the primal and dual gradient complexities for solving (4.4) with accuracy $\eta$ becomes very high. In fact, to reduce both $C_p(\eta)$ and $C_d(\eta)$, we wish to choose $\lambda$ as small as possible.

The analysis above reveals a trade-off in the choice of $\lambda$, that is, between reducing the number of iterations of Algorithm 1 and reducing the computational cost of implementing Step 2. From Theorem 4.1 (and Corollary 4.1), we know the legitimate range of $\lambda$ is $(0.8\gamma^{-1}, \gamma^{-1})$, and hence a natural choice of $\lambda$ would be the mid-point of this interval, namely $\lambda = 0.9\gamma^{-1}$. Based on this choice of $\lambda$, as well as the choice of $\eta$ in Theorem 4.1, we have the following result.
Corollary 4.2. For any $\varepsilon > 0$, if we choose $\lambda = 0.9\gamma^{-1}$ and set $\eta$ as in (4.18), then Algorithm 1 finds an $\varepsilon$-NS point of (1.1) with primal gradient complexity

$$T_p(\varepsilon) = O\left(\sqrt{\gamma L_{xx}}(\sqrt{L_{yy} \gamma + L_{xy}})\varepsilon^{-3} \ln(\varepsilon^{-1})^2\right)$$

and dual gradient complexity

$$T_d(\varepsilon) = O\left(\gamma(\sqrt{L_{yy} \gamma + L_{xy}})\varepsilon^{-3} \ln(\varepsilon^{-1})\right).$$

Proof. Indeed, if we substitute the value of $\rho$ as in Algorithm 1, the value of $\eta$ as in (4.18) and $\lambda = 0.9\gamma^{-1}$ into the definitions of $C_p(\eta)$ and $C_d(\eta)$ in (4.33) and (4.34), respectively, then we have

$$C_p(\eta) = O\left(\sqrt{L_{xx} / \gamma}((\sqrt{L_{yy} \gamma + L_{xy}})\varepsilon^{-1} \ln(\varepsilon^{-1})^2\right) \quad \text{and} \quad C_d(\eta) = O\left(\gamma((\sqrt{L_{yy} \gamma + L_{xy}})\varepsilon^{-1} \ln(\varepsilon^{-1})\right).$$

Furthermore, Corollary 4.1 states that if $\eta$ is set as in (4.18), then in order for Algorithm 1 to return an $\varepsilon$-NS point of (1.1), the number of iterations is bounded by $\bar{K} = O(\varepsilon^{-2} \gamma)$ (since $\lambda = 0.9\gamma^{-1}$). Combining this bound with (4.37), we complete the proof. $\square$

As promised above, in the next two sections (namely Sections 5 and 6), we will develop an efficient first-order method for solving the SPP in (4.32). Specifically, Section 5 is devoted to a new non-Hilbertian inexact APG method for strongly convex composite optimization, which forms the basis of the actual first-order method for solving (4.32) that will be developed in Section 6.

5. A non-Hilbertian inexact APG method. Let us consider the following (strongly) convex optimization problem:

$$(P): \quad P^* := \min_{u \in \mathcal{U}} \left\{ P(u) := h(u) + \zeta(u) + \mu \omega_{\mathcal{U}}(u) \right\},$$

where $\mathcal{U}$ is a nonempty, convex and closed set in the normed space $\mathcal{U}$ as given in Section 2.2, the function $h$ is $L_h$-smooth on $\mathcal{U}$, namely it is differentiable on some open set $\mathcal{U}' \supset \mathcal{U}$ and $\nabla h$ is $L_h$-Lipschitz on $\mathcal{U}$, and the function $\zeta$ has an easily computable BPP on $\mathcal{U}$ with DGF $\omega_{\mathcal{U}}$ (cf. Section 2.3). Both functions $h$ and $\zeta$ are convex on $\mathcal{U}$. In addition, by the 1-strong-convexity of $\omega_{\mathcal{U}}$ on $\mathcal{U}$, the objective function $P$ is $\mu$-strongly convex on $\mathcal{U}$, where $\mu \geq 0$. We assume that (5.1) has an optimal solution $u^*$, which necessarily lies in $\mathcal{U}^0(= \mathcal{U} \cap \text{int dom } \omega_{\mathcal{U}})$, and hence $P(u^*) = P^*$. (Note that if $\mu > 0$, $u^* \in \mathcal{U}^0$ is guaranteed to exist and is unique.)

In particular, we are interested in the case where $h$ has the following max-structure:

$$h(u) = \max_{v \in \mathcal{V}} \bar{\Psi}(u, v), \quad \forall u \in \mathcal{U'},$$

where $\mathcal{V}$ is a nonempty, convex and closed set in the normed space $\mathcal{V}$ as given in Section 2.2.
where $\mathcal{V}$ is a nonempty, compact and convex set contained in some open set $\mathcal{V}'$ and $\bar{\Psi}: \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is jointly continuous on $\mathcal{U}' \times \mathcal{V}'$. In addition, for any $v \in \mathcal{V}$, $\bar{\Psi}(\cdot, v)$ is convex on $\mathcal{U}$ and differentiable on $\mathcal{U}'$, and for any $u, u' \in \mathcal{U}$ and $v, v' \in \mathcal{V}$, we have

$$\|\nabla_u \bar{\Psi}(u, v) - \nabla_u \bar{\Psi}(u', v)\|_* \leq L_{uu} \|u - u'\|, \quad (5.3)$$

$$\|\nabla_u \bar{\Psi}(u, v) - \nabla_u \bar{\Psi}(u, v')\|_* \leq L_{uv} \|v - v'\|. \quad (5.4)$$

Also, $\bar{\Psi}(u, \cdot)$ is $\rho$-strongly concave on $\mathcal{V}$ for any $u \in \mathcal{U}$ and some $\rho > 0$. Under these structural assumptions on $\bar{\Psi}$, from Lemma 3.5, we see that $h$ is indeed $L_h$-smooth on $\mathcal{U}$ with $L_h := L_{uu} + L_{uv}^2 / \rho$. Also, the max-structure of $h$ enables us to write the dual problem associated with (5.2) as follows:

$$(D) : \Xi^* := \max_{v \in \mathcal{V}} \{ \Xi(v) := \inf_{u \in \mathcal{U}} \bar{\Psi}(u, v) + \zeta(u) + \mu \omega_U(u) \}. \quad (5.5)$$

(Note that from Sion’s minimax theorem [29], we know that strong duality holds between $(P)$ and $(D)$, namely $P^* = \Xi^*$.) Accordingly, let us define the duality gap

$$\Delta(u, v) := P(u) - \Xi(v), \quad \forall (u, v) \in \mathcal{U} \times \mathcal{V}. \quad (5.6)$$

The usefulness of the max-structure in (5.2) will become clear in Section 6.

Indeed, in our setting, a typical choice to solve (5.1) is the non-Hilbertian proximal gradient methods and its accelerated variants (see e.g., Nesterov [20], Tseng [31]). These methods assume that the gradient of $h$ at any $u \in \mathcal{U}$ can be obtained exactly. However, this can be restrictive in some scenarios where computing the gradient involves conducting certain simulations or solving certain optimization problems, which is precisely the case in Section 6. In this section, we are interested in the scenario where the value and gradient of $h$ at $u \in \mathcal{U}$ together have certain non-zero but controllable error $\delta$, and satisfies the $(\delta, \bar{L})$-inexact model as in Devolder et al. [11], which will be reviewed shortly in Section 5.1. Our purpose in this section is to develop an inexact non-Hilbertian APG method for finding an approximately optimal solution of (5.1) under this inexact model.

Before presenting our method, let us remark that although the inexact APG methods for strongly convex optimization problems have been well studied in the Hilbertian setting (see e.g., Devolder et al. [11, 10], Schmidt et al. [27]), the study in the non-Hilbertian setting has been rather scarce. Indeed, when $\mathcal{X}$ is Hilbertian, the analyses in the various previous works critically leverage several special properties of the Hilbertian distance $D_{\omega_{\mathcal{X}}}(y, x) = (1/2)\|y - x\|_{\mathcal{X}}^2$ (where $\omega_{\mathcal{X}} = (1/2)\|\cdot\|_{\mathcal{X}}^2$), including symmetry and inner-product inducibility. Therefore, these analyses cannot be straightforwardly extended to the non-Hilbertian setting, and different techniques have to be developed. Another attractive feature of our method is that convergence guarantees on the duality gap in (5.6) can be obtained when $\mathcal{U}$ is bounded (in addition to those on the primal optimality gap). To our knowledge, such guarantees have been rarely studied in the literature of inexact proximal gradient methods, even in the Hilbertian setting.
Specifically, let $\delta$, the $(\delta, \bar{L})$-inexact model on $\mathcal{U}$, be an approximate solution of the maximization problem in (5.2). As shown in Lemma 5.1, $(\Psi(u, \bar{\nu}), \nabla_u \Psi(u, \bar{\nu}))$ is indeed a $(\delta, \bar{L})$-FOA of $h$ at $u$.

**Lemma 5.1.** For any $u \in \mathcal{U}$ and $\delta > 0$, let $\bar{\nu} \in \mathcal{V}$ satisfy that $h(u) - \Psi(u, \bar{\nu}) \leq \delta / 2$, then we have

\begin{align*}
    h(u') & \geq \Psi(u, \bar{\nu}) + \langle \nabla_u \Psi(u, \bar{\nu}), u' - u \rangle, \quad \forall u' \in \mathcal{U}, \\
    h(u') & \leq \Psi(u, \bar{\nu}) + \langle \nabla_u \Psi(u, \bar{\nu}), u' - u \rangle + (\bar{L}/2) \|u' - u\|^2 + \delta, \quad \forall u' \in \mathcal{U},
\end{align*}

where $\bar{L} = 2L_h$. In words, $(\Psi(u, \bar{\nu}), \nabla_u \Psi(u, \bar{\nu}))$ is a $(\delta, 2L_h)$-FOA of $h$ at $u$.

**Proof.** The proof can be regarded as an extension of that in Devolder et al. [11, Section 3.2], and is shown in Appendix F. \qed

Finally, let us remark that there exist many more scenarios where $h$ is equipped with the $(\delta, \bar{L})$-inexact model. For details, we refer readers to Devolder et al. [11, Section 2.3 and Section 3].

### 5.2. Algorithm statement

Our inexact non-Hilbertian APG method is shown in Algorithm 2. The design of this method leverages Nesterov’s famous estimate sequence framework [19], and in particular the version proposed in Nesterov [20] (which results in an exact non-Hilbertian APG method for solving (5.1) with $\mu = 0$).

Now, let us make some comments on Algorithm 2. First, let us focus on the choices of the input. Intuitively, the approximation errors $\{\delta_t\}_{t \geq 0}$ will accumulate along the iterations of Algorithm 2, in a way that depends on the weights $\{\alpha_t\}_{t \geq 0}$. Therefore, the choices of $\{\delta_t\}_{t \geq 0}$ should depend on the accuracy of the approximate solution of (5.1) that we wish to find. The choice of $\bar{L}$ is made such that $(\hat{h}(u_t), \hat{\nabla}h(u_t))$ is a $(\delta_t, \bar{L})$-FOA of $h$ at $u_t$ for all $t \geq 0$, and hence depends on the way that we find the first-order approximation $(\hat{h}(u_t), \hat{\nabla}h(u_t))$. Lastly, the weights $\{\alpha_t\}_{t \geq 0}$ control both
The convergence rate of Algorithm 2 as well as the accumulation rate of the approximation errors, and their choices will be made clear in Section 5.3.

Next, let us focus on the solving the sub-problems in Algorithm 2. Indeed, the sub-problems occur in three places: i) finding \( z_0 \) in the initialization phase, ii) finding \( \pi_{t+1} \) in (5.10) and iii) finding \( w_{t+1} \) in (5.14). Using the definition of the Bregman divergence \( D_{\omega_{t}}(\cdot,\cdot) \) in (2.4), we see that all of the three sub-problems share the same form below:

\[
\begin{align*}
    u^* := & \arg \min_{u \in \mathcal{U}} \zeta(u) + \langle \xi, u \rangle + \alpha^{-1} \omega_{t}(u) \quad \text{for some } \xi \in \mathcal{U}^* \text{ and } \alpha > 0, \quad (5.17) \\
    \end{align*}
\]

which is a BPP problem associated with \( \zeta \) and the DGF \( \omega_{t} \) (cf. (2.7) in Section 2.3). As assumed at the beginning of Section 5, the solution of this problem is easily computable. In the following, let us provide some examples to justify this assumption.

Some “easy” examples of (5.17). First, note that if \( \mathcal{U} \) is Hilbertian with inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \| \cdot \| \), and \( \mathcal{U} = \mathcal{U} \), we can take \( \omega_{t} = (1/2) \| \cdot \|^2 \), so that the problem in (5.17) becomes the usual proximal minimization problem associated with the function \( \zeta \). As such, we will provide two simple examples below where \( \mathcal{U} \) is non-Hilbertian and \( \zeta \equiv 0 \) (or equivalently, \( \zeta = t_{\mathcal{U}} \) and \( \mathcal{U} = \mathcal{U} \)). For more examples, we refer readers to Nesterov [20], Juditsky and Nemirovski [12].
(E1) Let \( U = (\mathbb{R}^n, \|\cdot\|_1) \) with \( \|u\|_1 := \sum_{i=1}^{n} |u_i| \), \( \mathcal{U} = \Delta_n \) and \( \omega_u(u) = \sum_{i=1}^{n} u_i \ln u_i \). From Nesterov [20], we know that \( \omega_u \) is 1-strongly convex on \( \Delta_n \) with respect to \( \|\cdot\|_1 \), and the minimization problem in (5.17), commonly referred to as the “entropic projection” problem, has the following closed-form solution:

\[
    u_i^* = \exp(-\alpha \xi_i) / \sum_{j=1}^{n} \exp(-\alpha \xi_j), \quad \forall i \in [n],
\]

(E2) Let \( U = (\mathbb{R}^n, \|\cdot\|_p) \), where \( p \in (1, 2] \) and \( \|u\|_p := \left( \sum_{i=1}^{n} |u_i|^p \right)^{1/p} \). Consequently, \( \mathcal{U}^* = (\mathbb{R}^n, \|\cdot\|_q) \), where \( q := 1/(1-p^{-1}) \in [2, +\infty) \). Let \( \mathcal{U} = \mathbb{R}^n_+ \) and \( \omega_u(u) = (1/2) \|u\|_p^2 \), and (5.17) becomes

\[
    u^* := \arg \min_{u \geq 0} \langle \xi, u \rangle + (2\alpha)^{-1} \|u\|_p^2.
\]

Note that \( \omega_u \) is \((p-1)\)-strongly convex with respect to \( \|\cdot\|_p \) on \( \mathbb{R}^n \) (cf. Ben-Tal et al. [1, Section 8]). In addition, let us observe that the minimization problem in (5.19) can be solved in closed-form. Indeed, from the KKT conditions, we easily see that if \( \xi_i \geq 0 \), then \( u_i^* = 0 \), for all \( i \in [n] \). Therefore, without loss of generality, let us assume \( \xi < 0 \), and rewrite (5.19) as

\[
    (t^*, \overline{\pi}^*) := \arg \min_{t \geq 0} \arg \min_{\overline{\pi} \geq 0, \|\overline{\pi}\|_p = 1} t\langle \xi, \overline{\pi} \rangle + (2\alpha)^{-1}t^2.
\]

Note that we can recover \( u^* \) from \((t^*, \overline{\pi}^*)\) by letting \( u^* = t^* \overline{\pi}^* \). Clearly, since \( \xi < 0 \), we have \( \overline{\pi}^*_i = (|\xi_i|/\|\xi\|_q)^{q/p} \) for \( i \in [n] \) and hence \( \langle \xi, \overline{\pi}^* \rangle = -\|\xi\|_q < 0 \). Based on this, we then have

\[
    t^* = \arg \min_{t \geq 0} \left( 2\alpha \right)^{-1}t^2 - \|\xi\|_q t = \alpha \|\xi\|_q.
\]

(As a side note, note that the approach above can also be used to derive the closed-form solution of (5.19) without the nonnegativity constraint on \( u \).

Finally, let us observe that all the iterates generated in Algorithm 2 (including \( \{u_t\}_{t \geq 0}, \{\overline{\pi}_t\}_{t \geq 0}, \{z_t\}_{t \geq 0} \) and \( \{\overline{\pi}_t\}_{t \geq 0} \)) lie in \( \mathcal{U}^o \). Indeed, since \( z_0 \) is the output of a BPP problem, we know that \( z_0 \in \mathcal{U}^o \) (cf. Section 2.3), and the aforementioned observation simply follows from induction.

5.3. Convergence results of Algorithm 2. Let us present the choices of \( \{\alpha_t\}_{t \geq 0} \) and the associated convergence results of Algorithm 2 for both the non-strongly-convex (\( \mu = 0 \)) and strongly-convex (\( \mu > 0 \)) cases. We will focus on analyzing the strongly-convex case since it is the one that will be used in Section 6. Based on the max-structure of \( h \) in (5.2), we let \( \{v_t\}_{t \geq 0} \subseteq \mathcal{V} \) be any sequence that satisfies

\[
    h(u_t) - \overline{\Psi}(u_t, v_t) \leq \delta_t / 2, \quad \forall t \geq 0.
\]

Indeed, from Lemma 5.1, we know that \( (h(u_t), \nabla h(u_t)) = (\overline{\Psi}(u_t, v_t), \nabla_u \overline{\Psi}(u_t, v_t)) \) is a \((\delta_t, 2L_h)\)-FOA of \( h \) at \( u_t \), for all \( t \geq 0 \). In addition, define another sequence \( \{\overline{\pi}_t\}_{t \geq 0} \subseteq \mathcal{V} \) such that

\[
    \overline{\pi}_t := A_t^{-1} \sum_{i=0}^{t} \alpha_i v_i, \quad \forall t \geq 0.
\]
Namely, \( \{\overline{\nu}_t\}_{t \geq 0} \) is the weighted average of \( \{\nu_t\}_{t \geq 0} \). Our results below not only concern the convergence of the primal optimality gap \( \{P(z^t) - P^*\}_{t \geq 0} \), but also the convergence of the duality gap \( \{\overline{\Delta}(z_t, \overline{v}_t)\}_{t \geq 0} \) (cf. (5.6)), in the case where \( \mathcal{U} \) is bounded.

**Theorem 5.1 (The case \( \mu = 0 \)).** In Algorithm 2, if \( \mu = 0 \), then we can choose

\[
\alpha_0 = 1 \quad \text{and} \quad \alpha_t = (2t + 3)/4, \quad \forall t \geq 1,
\]

and under such choices, we have

\[
P(z_t) - P^* \leq \frac{4\bar{L}D_{\omega,t}(u_0, u_0)}{(t + 2)^2} + \frac{\sum_{i=0}^{t}(i + 2)^2\delta_i}{(t + 2)^2}, \quad \forall t \geq 0.
\]

In addition, if \( \mathcal{U} \) is bounded and \( (\hat{h}(u_t), \nabla h(u_t)) = (\tilde{\Psi}(u_t, v_t), \nabla_u \tilde{\Psi}(u_t, v_t)) \) for all \( t \geq 0 \), then we can choose \( \bar{L} = 2L_h \) and obtain

\[
\overline{\Delta}(z_t, \overline{v}_t) \leq \frac{8L_h\Omega_{\omega,t}(u_0)}{(t + 2)^2} + \frac{\sum_{i=0}^{t}(i + 2)^2\delta_i}{(t + 2)^2}, \quad \forall t \geq 0,
\]

where

\[
\Omega_{\omega,t}(u_0) := \max_{u \in \mathcal{U}} D_{\omega,t}(u, u_0) < +\infty.
\]

**Proof.** See Appendix G. \( \square \)

**Theorem 5.2 (The case \( \mu > 0 \)).** In Algorithm 2, if \( \mu > 0 \), then we can choose

\[
\alpha_0 = 1 \quad \text{and} \quad \alpha_t = (1 + \sqrt{\theta})^{-t} - 1/\sqrt{\theta}, \quad \forall t \geq 1, \quad \text{where} \quad \theta := \mu/\bar{L}.
\]

Under such choices, we have

\[
P(z_t) - P^* \leq (1 + \sqrt{\theta})^{-t}L D_{\omega,t}(u_0, u_0) + (1 + \sqrt{\theta})^{-t}\sum_{i=0}^{t}(1 + \sqrt{\theta})^i\delta_i, \quad \forall t \geq 0.
\]

In addition, if \( \mathcal{U} \) is bounded and \( (\hat{h}(u_t), \nabla h(u_t)) = (\tilde{\Psi}(u_t, v_t), \nabla_u \tilde{\Psi}(u_t, v_t)) \) for all \( t \geq 0 \), then we can choose \( \bar{L} = 2L_h \) and obtain

\[
\overline{\Delta}(z_t, \overline{v}_t) \leq 2(1 + \sqrt{\theta})^{-t}L_h\Omega_{\omega,t}(u_0) + (1 + \sqrt{\theta})^{-t}\sum_{i=0}^{t}(1 + \sqrt{\theta})^i\delta_i, \quad \forall t \geq 0,
\]

where \( \Omega_{\omega,t}(u_0) < +\infty \) is defined in (5.26).

**Proof.** See Appendix G. \( \square \)

Let us consider the simple case where the sequence of errors \( \{\delta_i\}_{i \geq 0} \) is uniformly bounded by \( \delta > 0 \) (namely, \( \delta_i \leq \delta \) for all \( t \geq 0 \)). If \( \mathcal{U} \) is bounded and \( (\hat{h}(u_t), \nabla h(u_t)) = (\tilde{\Psi}(u_t, v_t), \nabla_u \tilde{\Psi}(u_t, v_t)) \) for all \( t \geq 0 \), then (5.29) becomes

\[
\overline{\Delta}(z_t, \overline{v}_t) \leq 2(1 + \sqrt{\theta})^{-t}L_h\Omega_{\omega,t}(u_0) + (1 + 1/\sqrt{\theta})\delta.
\]
We observe that the right-hand side of (5.30) consists of two terms: the first term linearly decreases in \( t \) at rate \((1 + \sqrt{\theta})^{-1}\), and the second term, which represents the accumulated errors resulted from the approximate gradients \( \{\hat{\nabla} h(u_t)\}_{t \geq 0} \), is a constant in \( t \) and proportional to \( \delta \). Consequently, to find the number of iterations \( t \) needed to ensure \( \Delta(z_t, \overline{\nu}_t) \leq \epsilon \), we can properly choose \( \delta \) such that the second term \((1 + 1/\sqrt{\theta})\delta \leq \epsilon/2\), and then find \( t \) needed such that the first term
\[
2(1 + \sqrt{\theta})^{-t} L_h \Omega_{\omega_d}(u_0) \leq \epsilon/2.
\]

Of course, if \( \mathcal{U} \) is unbounded, then based on (5.28), we can use the same reasoning to find the number of iterations \( t \) needed to ensure \( P(z_t) - P^* \leq \epsilon \). This is formalized in the corollary below.

**Corollary 5.1.** In Algorithm 2, if \( \mu > 0 \), choose \( \{\alpha_t\}_{t \geq 0} \) as in (5.27). Fix any \( \epsilon > 0 \) and let \( \{\delta_t\}_{t \geq 0} \) satisfy that
\[
\delta_t \leq \delta := \frac{\epsilon}{2(1 + \sqrt{L/\mu})}, \quad \forall t \geq 0.
\]
Under such choices, we have that \( P(z_t) - P^* \leq \epsilon \) for all \( t \geq \bar{t}_p \), where
\[
\bar{t}_p := \left[ \left( \sqrt{\frac{L}{\mu}} + 1 \right) \ln \left( \frac{2LD_{\omega_d}(u^*, u_0)}{\epsilon} \right) \right].
\]
In addition, if \( \mathcal{U} \) is bounded and \((\hat{h}(u_t), \hat{\nabla} h(u_t)) = (\Psi(u_t, v_t), \nabla_a \Psi(u_t, v_t)) \) for all \( t \geq 0 \), then choose \( L = 2L_h \) and we have that \( \Delta(z_t, \overline{\nu}_t) \leq \epsilon \) for all \( t \geq \bar{t}_d \), where
\[
\bar{t}_d := \left[ \left( \sqrt{\frac{2L_h}{\mu}} + 1 \right) \ln \left( \frac{4L_h \Omega_{\omega_d}(u_0)}{\epsilon} \right) \right].
\]

**5.4. An adaptive stopping criterion when \( \mu > 0 \).** We derive a sufficient condition to certify the \( \epsilon \)-optimality of the iterates \( \{w_t\}_{t \geq 0} \) generated in Algorithm 2, in the case where \( \mu > 0 \). The purpose of developing this condition is to provide an adaptive stopping criterion that allows us to terminate Algorithm 2 “early”, which we explain below.

Indeed, from Corollary 5.1, we know that if the sequence of approximation errors \( \{\delta_t\}_{t \geq 0} \) satisfy (5.31), then in the worst-case, we have \( P(z_t^*) - P^* \leq \epsilon \) after \( \bar{t}_p \) iterations, where \( \bar{t}_p \) is defined in (5.32). However, in certain cases, stopping Algorithm 2 after \( \bar{t}_p \) iterations can be quite conservative, as some iterates in Algorithm 2 (such as \( z_t, u_t \) or \( w_t \)) may already be \( \epsilon \)-optimal for some \( t \ll \bar{t}_p \). Therefore, in this section, we will derive an easy-to-check condition so that as soon as it is satisfied at the \( t \)-th iteration, we can stop Algorithm 2 and conclude that \( P(w_{t+1}) - P^* \leq \epsilon \).

In addition, this condition also mitigates the situation where \( \mathcal{U} \) is unbounded and it is difficult to estimate the quantity \( D_{\omega_d}(u^*, u_0) \) that appears in the definition of \( \bar{t}_p \), which prohibits us from running Algorithm 2 for a fixed number of iterations.
To state our stopping criterion, given any \( u, \overline{u} \in \mathcal{U}^o \), let us first define
\[
u^+ := \arg\min_{\nu' \in \mathcal{U}} \langle \nabla h(\overline{u}) + e(\overline{u}), u' \rangle + \zeta(u') + \mu \omega_L(u') + \lambda^{-1} D_{\omega_L}(u', u), \tag{5.34}\]
where \( e(\overline{u}) \in \mathcal{U}^* \) denotes the error on the gradient \( \nabla h(\overline{u}) \) and \( \lambda > 0 \). Based on \( u, \overline{u}, u^+ \in \mathcal{U}^o \), define
\[
G := L_h(u^+ - \overline{u}) \quad \text{and} \quad \overline{G} := \lambda^{-1} (\nabla \omega_X(u) - \nabla \omega_X(u^+)). \tag{5.35}
\]
The following lemma is crucial to establish our stopping criterion.

**Lemma 5.2.** We have
\[
P(u^+) - P^* \leq 3 \left( \| \overline{G} \|_2^2 + \| G \|_2^2 + \| e(\overline{u}) \|_2^2 \right) / (2 \mu). \tag{5.36}
\]

**Proof.** See Appendix H. □

Now, let us observe that (5.34) has exactly the same form as (5.14), by letting \( \overline{u} := u_{t+1}, \ u := \overline{u}_{t+1}, \ u^+ := w_{t+1}, \ \lambda := \alpha_{t+1} / (A_t \mu + \tilde{L}) \) and
\[
e(u_{t+1}) := \nabla h(u_{t+1}) - \nabla h(u_{t+1}). \tag{5.37}
\]
Similar to (5.35), let us define
\[
G_t := L_h(w_{t+1} - u_{t+1}) \quad \text{and} \quad \overline{G}_t := (A_t \mu + \tilde{L})^{-1} \alpha_{t+1} (\nabla \omega_L(\overline{u}_{t+1}) - \nabla \omega_L(w_{t+1})). \tag{5.38}
\]

Based on Lemma 5.2, we easily have the following stopping criterion.

**Theorem 5.3.** In Algorithm 2, for any \( t \geq 0 \), if
\[
\| \overline{G}_t \|_2^2 + \| G_t \|_2^2 \leq \mu \epsilon / 3 \quad \text{and} \quad \| e(u_{t+1}) \|_* \leq \sqrt{\mu \epsilon / 3}, \tag{5.39}
\]
than \( P(w_{t+1}) - P^* \leq \epsilon \).

Let us make several comments about the stopping criterion in Theorem 5.3. First of all, the objects \( e(\overline{u}), G \) and \( \overline{G} \) (cf. (5.34) and (5.35)) together can be regarded as an extension to the *proximal gradient mapping* proposed in Nesterov [21], in the following two senses. First, we allow the error term \( e(\overline{u}) \) to exist in the gradient \( \nabla h(\overline{u}) \), and second, the projection in (5.34) can be non-Hilbertian (namely, we do not restrict \( \omega = (1/2) \| \cdot \|^2 \) for some Hilbertian norm \( \| \cdot \| \)). Next, let us focus on the implementation of this criterion, which requires i) computing two additional sequences \( \{ G_t \}_{t \geq 0} \) and \( \{ \overline{G}_t \}_{t \geq 0} \) and ii) ensuring that \( \| e(u_{t+1}) \|_* \leq \sqrt{\mu \epsilon / 3} \). For the first requirement, note that all the quantities appearing in the definitions of \( G_t \) and \( \overline{G}_t \) (cf. (5.38)) have already been computed in Algorithm 2, and so computing the additional sequences \( \{ G_t \}_{t \geq 0} \) and \( \{ \overline{G}_t \}_{t \geq 0} \) only slightly increases the computational cost of Algorithm 2 at each iteration. The second requirement can be accomplished by properly choosing the approximation error \( \delta_{t+1} \) and the approximate smoothness parameter \( \tilde{L} \), which we will discuss below.
5.4.1. Estimating $\|e(u)\|_*$. Let us illustrate two situations where we can ensure $\|e(u)\|_* \leq \epsilon_e := \sqrt{\mu e}/3$ for some $u \in \mathcal{U}^o$, where $e(u) := \nabla h(u) - \nabla h(u)$ denotes the error on the approximate gradient $\nabla h(u)$. The first situation is more general, and includes any $(\delta, \bar{L})$-inexact model with “extended domain”. The second situation is simpler but more restrictive, as it specifically makes use of the max-structure of $h$ as in (5.2).

Situation I: $(\delta, \bar{L})$-inexact model with “extended domain”. Let us slightly extend the definition of $(\delta, \bar{L})$-inexact model in Definition 5.1, in the sense that (5.7) holds for all $u \in \bar{U}$, where $\bar{U}$ is a closed convex set with nonempty interior such that $\mathcal{U}^o \subseteq \text{int}(\bar{U})$ — we shall call this the “extended” $(\delta, \bar{L})$-inexact model. Let $B_{\|\cdot\|}(u, r) := \{u' \in \bar{U} : \|u' - u\| \leq r\}$ denote the $\|\cdot\|$-ball centered at $u \in \mathcal{U}^o$ with radius $r$, and define the distance from $u$ to $\text{bd}\bar{U}$ (i.e., the boundary of $\bar{U}$) as

$$d(u) := \text{dist}(u, \text{bd}\bar{U}) := \sup \{r \geq 0 : B_{\|\cdot\|}(u, r) \subseteq \bar{U}\}. \tag{5.40}$$

Using the same argument as in Devolder et al. [11, Section 2.2], if $(\hat{h}(u), \nabla \hat{h}(u))$ satisfies the “extended” $(\delta, \bar{L})$-inexact model, then

$$\|e(u)\|_* \leq \begin{cases} (\bar{L}/2)d(u) + \delta/d(u), & \text{if } 0 < d(u) \leq \sqrt{2\delta/L} \\ \sqrt{2\bar{L}\delta}, & \text{if } d(u) > \sqrt{2\delta/L} \end{cases}. \tag{5.41}$$

Based on (5.41), it is simple to ensure $\|e(u)\|_* \leq \epsilon_e$ by properly choosing $\delta$, as shown below.

**Proposition 5.1.** Let $u \in \mathcal{U}^o$ and $(\hat{h}(u), \nabla \hat{h}(u))$ satisfies the “extended” $(\delta, \bar{L})$-inexact model. Then $\|e(u)\|_* \leq \epsilon_e$ if

$$\delta := (\epsilon_e/2) \min\{d(u), \epsilon_e/\bar{L}\}. \tag{5.42}$$

In particular, if $\bar{U} = \mathbb{U}$, then $\delta = \epsilon_e^2/(2\bar{L}) = \mu e/(6\bar{L})$.

**Proof.** If $d(u) \leq \epsilon_e/\bar{L}$, then $\delta = \epsilon_e d(u)/2 \leq \bar{L}d(u)^2/2$ (or $d(u) \leq \sqrt{2\delta/\bar{L}}$), and hence $\|e(u)\|_* \leq (\bar{L}/2)d(u) + \delta/d(u) \leq \epsilon_e/2 + \epsilon_e/2 = \epsilon_e$. If $d(u) > \epsilon_e/\bar{L}$, then $\delta = \epsilon_e^2/(2\bar{L}) < \bar{L}d(u)^2/2$ (or $d(u) > \sqrt{2\delta/\bar{L}}$), and hence $\|e(u)\|_* \leq \sqrt{2\bar{L}\delta} = \epsilon_e$. If $\mathcal{U} = \mathbb{U}$, then $d(u) = +\infty$ and $\delta = \epsilon_e^2/(2\bar{L})$. \hfill $\Box$

Situation II: $h$ has the max-structure in (5.2). In this case, let $\hat{v} \in \mathbb{V}$ satisfy that $h(u) - \bar{\Psi}(u, \hat{v}) \leq \delta/2$. From Lemma 5.1, we already know that $(\bar{h}(u), \nabla \bar{h}(u)) = (\bar{\Psi}(u, \hat{v}), \nabla u \bar{\Psi}(u, \hat{v}))$ is a $(\delta, 2Lh)$-FOA of $h$ at $u$. In fact, the error $e(u) = \nabla u \bar{\Psi}(u, \hat{v}) - \nabla h(u)$ can also be easily bounded.

**Lemma 5.3.** If $\hat{v} \in \mathbb{V}$ satisfies that $h(u) - \bar{\Psi}(u, \hat{v}) \leq \delta/2$, then we have

$$\|e(u)\|_* = \|\nabla h(u) - \nabla \bar{h}(u)\|_* = \|\nabla u \bar{\Psi}(u, \hat{v}) - \nabla h(u)\|_* \leq L_{uu} \sqrt{\delta/\rho}. \tag{5.43}$$

As a result, we have $\|e(u)\|_* \leq \epsilon_e$ if $\delta \leq \epsilon_e^2 \rho/L_{uu}$. \hfill $\Box$

**Proof.** See Appendix F. \hfill $\Box$
6. An efficient first-order method for solving convex-concave SPPs. Based on the non-Hilbertian inexact APG method developed in Section 5, we are ready to develop a first-order method for solving a class of (strongly) convex-concave SPPs that subsumes the one in (4.32) as a special case. We analyze the primal and dual gradient complexities of this method, which enable us to derive the primal and dual gradient complexities of Algorithm 1 in Section 4.4.

6.1. Problem setup. Let us consider the following (strongly) convex-concave SPP:

$$\min_{x \in X} \max_{y \in Y} \left\{ S(x, y) := \mu \omega_X(x) + r(x) + \Psi(x, y) - g(y) - \rho \omega_Y(y) \right\},$$  \hspace{1cm} (6.1)

where $\mu > 0$, $\rho > 0$, $\Psi : X' \times Y' \to \mathbb{R}$ is jointly continuous on $X' \times Y'$ and convex-concave on $X \times Y'$, namely $\Psi(\cdot, y)$ is convex on $X$ for any $y \in Y$ and $\Psi(x, \cdot)$ is concave on $Y$ for any $x \in X$. In addition, $\Psi$ shares the same smoothness assumptions as $\Phi$ in Assumptions 1.1 and 1.3, except that in (1.2), the smoothness parameter $L_{xx}$ is replaced by a larger one $L'_{xx}$.

Before proceeding further, let us mention that the (strongly) convex-concave SPP in (4.32), which is solved in Step 2 of Algorithm 1, is a special case of the SPP in (6.1). Indeed, in (4.32), using the definition of the Bregman divergence $D_{\omega_X}(\cdot, \cdot)$ (cf. (2.4)), we can write

$$\lambda^{-1} D_{\omega_X}(x; x_k) + \Phi(x, y) \leq (\lambda^{-1} - \gamma) \omega_X(x) + \gamma \omega_Y(x) - \lambda^{-1} \langle \nabla \omega_X(x_k), x \rangle + \Phi(x, y),$$ \hspace{1cm} (6.2)

where $\leq$ omits the terms that are constant w.r.t. the optimization variable $x$. Since $\omega_X$ is 1-strongly convex on $X$ and $\Phi(\cdot, y)$ is $\gamma$-weakly convex on $X$, we see that $\Psi(\cdot, y)$ is convex on $X$. In addition, from the $\beta_X$-smoothness of $\omega_X$ and $L_{xx}$-smoothness of $\Phi(\cdot, y)$ on $X$, we have $L'_{xx} := L_{xx} + \gamma / \beta_X$.

Now, substitute (6.2) into (4.32) and we see that (4.32) falls under the problem class in (6.1).

Next, let us write down the primal and dual problems associated with (6.1):

Primal : \hspace{1cm} $p^* := \min_{x \in X} \left\{ p(x) := f(x) + r(x) + \mu \omega_X(x) \right\},$  \hspace{1cm} (6.3)

Dual : \hspace{1cm} $d^* := \max_{y \in Y} \left\{ d(y) := \pi(y) - g(y) - \rho \omega_Y(y) \right\},$  \hspace{1cm} (6.4)

where the functions $f : X' \to \mathbb{R}$ and $\pi : Y' \to \mathbb{R}$ are defined as

$$f(x) := \max_{y \in Y} \left\{ \psi^D(x, y) := \Psi(x, y) - g(y) - \rho \omega_Y(y) \right\}, \hspace{1cm} \forall x \in X',$$ \hspace{1cm} (6.5)

$$\pi(y) := \min_{x \in X} \left\{ \psi^P(x, y) := \Psi(x, y) + r(x) + \mu \omega_X(x) \right\}, \hspace{1cm} \forall y \in Y'.$$ \hspace{1cm} (6.6)

We call $p : X \to \mathbb{R}$ in (6.3) and $d : Y \to \mathbb{R}$ in (6.4) the primal and dual functions, respectively.

Next, we state several facts about the SPP in (6.1) and its associated primal and dual problems in (6.3) and (6.4), respectively. These facts will be useful in our algorithmic development. First, note that due to the 1-strong-convexity of $\omega_X$ and $\omega_Y$ on $X$ and $Y$, respectively, the primal function
p and dual function \( d \) are \( \mu \)-strongly-convex and \( \rho \)-strongly-concave on \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. As such, both the primal and dual problems (in (6.3) and (6.4)) have unique optimal solutions, which we denote by \( x^* \in \mathcal{X} \) and \( y^* \in \mathcal{Y} \), respectively. Since \( S : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) in (6.1) is convex-concave and jointly continuous on \( \mathcal{X} \times \mathcal{Y} \), together with the compactness of \( \mathcal{Y} \), we can invoke Sion’s minimax theorem [29] to conclude that \( p^* = d^* \). Hence, the SPP in (6.1) has a unique saddle point \((x^*, y^*) \in \mathcal{X} \times \mathcal{Y}\), which by definition satisfies that

\[
S(x^*, y) \leq S(x^*, y^*) \leq S(x, y^*), \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.
\]

Also, notation-wise, let us denote the unique optimal solutions of (6.5) and (6.6) as \( y^*(x) \) and \( x^*(y) \), respectively, namely,

\[
y^*(x) := \arg \max_{y \in \mathcal{Y}} \psi^D(x, y) \quad \text{and} \quad x^*(y) := \arg \min_{x \in \mathcal{X}} \psi^P(x, y), \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y},
\]

and from (6.7), we easily see that

\[
y^*(x^*) = y^* \quad \text{and} \quad x^*(y^*) = x^*.
\]

In addition, by Lemma 3.4, we see that the mapping \( y^* : \mathcal{X}' \to \mathcal{Y} \) is \( (L_{xy}/\rho) \)-Lipschitz on \( \mathcal{X} \) and similarly, the mapping \( x^* : \mathcal{Y}' \to \mathcal{X} \) is \( (L_{xy}/\mu) \)-Lipschitz on \( \mathcal{Y} \).

Finally, let us show that the function \( \pi : \mathcal{Y} \to \mathbb{R} \) in (6.6) is smooth on \( \mathcal{Y} \). At this point, it is tempting to conclude this property directly from Lemma 3.5. However, this approach would require the boundedness of \( \mathcal{X} \), which does not necessarily hold. To circumvent this difficulty, let us first note that the function \( \pi \) in (6.6) can be equivalently written as

\[
\pi(y) = \min_{x \in \overline{\mathcal{X}}} \psi^P(x, y), \quad \text{where} \quad \overline{\mathcal{X}} := \text{clconv} \ x^*(\mathcal{Y})
\]

and \( x^*(\mathcal{Y}) := \{ x^*(y) : y \in \mathcal{Y} \} \subseteq \mathcal{X} \) consists of all the optimal solutions \( x^*(y) \) of (6.6) given \( y \in \mathcal{Y} \). Since \( \mathcal{Y} \) is compact and \( x^* \) is continuous on \( \mathcal{Y} \), we see that \( x^*(\mathcal{Y}) \) is compact, and hence its closed convex hull \( \overline{\mathcal{X}} \subseteq \mathcal{X} \) in (6.10) is convex and compact. Now, based on the new definition of \( \pi \) in (6.10), we can invoke Lemma 3.5 to conclude the following.

**Lemma 6.1.** The function \( \pi \) is differentiable on \( \mathcal{Y} \) with \( \nabla \pi(y) = \nabla_y \Psi(x^*(y), y) \). In addition, \( \nabla \pi : \mathcal{Y} \to \mathbb{R}^* \) is \( L_\pi \)-Lipschitz on \( \mathcal{Y} \), where \( L_\pi := L_{yy} + L_{xy}^2/\mu \).

Let us define the duality gap associated with (6.1) as

\[
\Delta(x, y) = p(x) - d(y), \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.
\]

In the following, we propose a non-Hilbertian accelerated dual inexact gradient method for solving (6.1). Specifically, for any \( \eta > 0 \), we aim to find a primal-dual pair \((x, y) \in \mathcal{X} \times \mathcal{Y}\) such that

\[
\Delta(x, y) \leq \eta.
\]
6.2. A dual non-Hilbertian inexact APG method. The idea of this method is conceptually very simple: From Lemma 6.1, we know that the function $\pi$ is $L_\pi$-smooth on $\mathcal{Y}$, and hence we can apply the non-Hilbertian accelerated APG method (namely Algorithm 2) as developed in Section 5 to the dual maximization problem in (6.4). (Note that the adaptation of Algorithm 2 to maximization problems are straightforward.) For each $t \geq 0$, given $y_t \in \mathcal{Y}$, to find the first-order approximation $(\hat{\pi}(y_t), \hat{\nabla}\pi(y_t))$ as in (5.12), we can first find $\hat{x}(y_t) \in X$ such that

$$
\psi^P(\hat{x}(y_t), y_t) - \pi(y_t) \leq \bar{\epsilon}/2 \text{ for some } \bar{\epsilon} > 0,
$$

and then let $(\hat{\pi}(y_t), \hat{\nabla}\pi(y_t)) = (\Psi(\hat{x}(y_t), y_t), \nabla_y \Psi(\hat{x}(y_t), y_t))$. Indeed, observe that the problem in (6.6) has the same form as the one in (5.1), and hence it can then be solved by the “exact” version of Algorithm 2 (since the gradient $\nabla_x \Psi(x, y)$ can be computed exactly at any $x \in \mathcal{X}$). In addition, let us define the weighted average of $\{\hat{x}(y_t^i)\}_{t \geq 0}$ as

$$
\tau_t := A_t^{-1} \sum_{i=0}^{t} \alpha_i \hat{x}(y_t^i), \quad \forall t \geq 0.
$$

The structure of this method is illustrated in Figure 2. From Corollary 5.1, we immediately have the following result.

**Corollary 6.1.** If we apply Algorithm 2 to solve the dual problem in (6.4), by choosing $\{\alpha_t\}_{t \geq 0}$ as in (5.27) with $\theta = \rho/(2L_\pi)$, $(\hat{\pi}(y_t), \hat{\nabla}\pi(y_t)) = (\Psi(\hat{x}(y_t), y_t), \nabla_y \Psi(\hat{x}(y_t), y_t))$ for all $t \geq 0$ and

$$
\bar{\epsilon} = \frac{\eta}{2(1 + \sqrt{2L_\pi/\rho})},
$$

then for any starting point $y_0 \in \mathcal{Y}$, Algorithm 2 generates a sequence $\{\bar{y}_t\}_{t \geq 0} \subseteq \mathcal{Y}$ such that

$$
\Delta(\tau_t, \bar{y}_t) \leq \eta \text{ for all } t \geq t_d(\eta),
$$

where

$$
t_d(\eta) := \left\lceil \sqrt{\frac{2L_\pi}{\rho} + 1} \ln \left( \frac{4L_\pi \Omega_{\omega\mathcal{Y}}(y_0)}{\eta} \right) \right\rceil.
$$

and $\Omega_{\omega\mathcal{Y}}(y_0) := \max_{y \in \mathcal{Y}} D_{\omega\mathcal{Y}}(y; y_0) < +\infty$. 

---

**Figure 2.** Illustration of the dual non-Hilbertian inexact APG method.
Remark 6.1. Two remarks are in order. First, since each iteration of Algorithm 2 only involves computing one dual gradient $\nabla y \Psi(x, y)$, from Corollary 6.1, we see that to find a primal-dual pair $(x, y) \in X \times Y$ such that $\Delta(x, y) \leq \eta$, the number of computed dual gradients in the dual inexact APG method is no more than $t_d(\eta)$. Second, if the structure of either $Y$ or $\omega_Y$ (or both) is relatively simple (e.g., Example (E1)), the quantity $\Omega_{\omega_Y}(y_0)$ can be easily estimated. In addition, we can stop the method early (i.e., before $t_d(\eta)$ iterations) as soon as the adaptive stopping criterion as described in Theorem 5.3 is satisfied.

Next, from Corollary 5.1, we know that for any $t \geq 0$, to find $\hat{x}(y_t) \in X$ that satisfies (6.12), the number of iterations of the “exact” version of Algorithm 2 does not exceed
\[
\left\lceil \frac{(L'_{xx} + 1) \ln \left( \frac{4L'_{xx}D_{\omega_X}(x^*(y_t), x_0)}{\bar{\epsilon}} \right)}{\sqrt{L_{yy}/\rho + L_{xy}/\sqrt{\mu \rho}}} \right\rceil,
\]
(6.16)
where $x_0 \in X^\circ$ denotes the starting point. Note that in the above, the quantity $D_{\omega_X}(x^*(y_t), x_0)$ depends on $y_t \in Y$, which is inconvenient for our analysis. Therefore, let us upper bound it by
\[
\Gamma_{\omega_X}(x_0) := \sup_{y \in Y} D_{\omega_X}(x^*(y), x_0) = \sup_{x \in x^*(Y)} D_{\omega_X}(x, x_0) < +\infty,
\]
(6.17)
which is independent of $y \in Y$. (Note that the finiteness of $\Gamma_{\omega_X}(x_0)$ follows from the compactness of $x^*(Y) \subseteq X$ and the continuity of $\omega_X$ on $X$.) By substituting the value of $\bar{\epsilon}$ in (6.14) and using $D_{\omega_X}(x^*(y_t), x_0) \leq \Gamma_{\omega_X}(x_0)$, the quantity in (6.16) can be upper bounded by
\[
t_p(\eta) := \left\lceil \frac{(L'_{xx} + 1) \ln \left( \frac{8(1 + \sqrt{2L_{yy}/\rho})L'_{xx}\Gamma_{\omega_X}(x_0)\eta}{\bar{\epsilon}} \right)}{\sqrt{L_{yy}/\rho + L_{xy}/\sqrt{\mu \rho}}} \right\rceil.
\]
(6.18)
Consequently, we have the following corollary.

Corollary 6.2. Under the setting of Corollary 6.1, to find a primal-dual pair $(x, y) \in X \times Y$ such that $\Delta(x, y) \leq \eta$, the number of primal gradients $\nabla x \Psi(x, y)$ computed does not exceed $t_d(\eta)t_p(\eta)$, where $t_d(\eta)$ and $t_p(\eta)$ are defined in (6.15) and (6.18), respectively.

Remark 6.2. To find $\hat{x}(y_t) \in X$ satisfying (6.12), we can stop the “exact” version of Algorithm 2 as soon as the adaptive stopping criterion in Theorem 5.3 is satisfied. Note that in this case, since there are no gradient errors, the stopping criterion in (5.39) simplifies to $\|G_t\|^2 + \|G_t\|^2 \leq \mu \epsilon/3$, and we no longer need to estimate $\|\epsilon(u)\|$, as in Section 5.4.1.

Finally, let us state the dual and primal gradient complexities of the dual inexact APG method.

Corollary 6.3. The dual inexact APG method finds a primal-dual pair $(x, y) \in X \times Y$ that satisfies $\Delta(x, y) \leq \eta$ with dual gradient complexity
\[
C_d(\eta) = O \left( \left( \sqrt{L_{yy}/\rho + L_{xy}/\sqrt{\mu \rho}} \right) \ln \left( \frac{1}{\eta \rho} \right) \right)
\]
(6.19)
and primal gradient complexity
\[
\overline{C}_p(\eta) = O \left( \left( \sqrt{\frac{L_{yy}}{\rho}} + \frac{L_{xy}}{\sqrt{\mu \rho}} \right) \sqrt{\frac{L'_{xx}}{\mu}} \ln^2 \left( \frac{1}{\eta \rho} \right) \right). \tag{6.20}
\]

**Remark 6.3.** As mentioned at the beginning of Section 6, the SPP in (6.1) encompasses that in (4.32) as special case, with
\[
\mu := \lambda^{-1} - \gamma \quad \text{and} \quad L'_{xx} := L_{xx} + \gamma \beta \chi. \tag{6.21}
\]
Therefore, if we apply the dual inexact APG method to the SPP in (4.32), then the dual and primal gradient complexities would be the same as \(\overline{C}_d(\eta)\) and \(\overline{C}_p(\eta)\) in Corollary 6.3 above, with \(\mu\) and \(L'_{xx}\) are replaced by their values in (6.21).

7. Extensions and discussions. Indeed, our primal-dual smoothing framework in Algorithm 1 is fairly flexible, and depending on different problem assumptions, it can easily accommodate several variants and extensions. Let us discuss two of them in Sections 7.1 and 7.2. In addition, in Section 7.1, we also provide a simple variant of the non-Hilbertian inexact APG method in Algorithm 2 that only involves solving one BPP problem at each iteration.

7.1. “Simple” dual structure. Indeed, in the cases where the dual maximization problem in the definition of \(f\) in (1.1) has a simple form, we may be able to find a suitable DGF \(\omega_Y\) such that the dual maximization problem that defines \(\tilde{f}\) in (6.5) can be easily solved, without appealing to (iterative) first-order methods. For example, in Example 1.2 in Section 1.2, if we fix any \(x \in X\) and let \(c := (\ell_i(x))_{i=1}^n\), then the maximization problem in (1.12) can be written as \(\max_{p \in \Delta_n} c^T p\). In this case, it is natural to choose \(\omega_Y(p) = \sum_{i=1}^n p_i \ln p_i\) for \(p \in \mathbb{R}_+^n\), and then the maximization problem in (6.5) becomes \(\max_{p \in \Delta_n} c^T p - \rho \omega_Y(p)\). As described in (E1) in Section 5.2, this “entropic projection” problem has a simple closed-form solution that can be computed in \(O(n)\) time.

Indeed, from Lemma 3.5, we know that in the cases above, the gradient of \(\tilde{f}\) at any \(x \in X\) can be easily computed, and \(\tilde{f}\) is \((L'_{xx} + L^2_{xy}/\rho)\)-smooth on \(X\). Therefore, we can directly apply the “exact” version of Algorithm 2 to the primal problem in (6.3) and obtain an \(\eta\)-optimal solution \(x \in X\). From Corollary 5.1, we know that the primal gradient complexities of this scheme is
\[
O \left( \left( \sqrt{\frac{L'_{xx}}{\mu}} + \frac{L_{xy}}{\sqrt{\mu \rho}} \right) \ln^2 \left( \frac{1}{\eta \rho} \right) \right). \tag{7.1}
\]
Based on this, we can easily analyze the primal gradient complexity of Algorithm 1 for finding an \(\varepsilon\)-NS point of (1.1).
Corollary 7.1. For any $\varepsilon > 0$, if we choose $\lambda = 0.9\gamma^{-1}$ and set $\eta$ as in (4.18), then Algorithm 1 finds an $\varepsilon$-NS point of (1.1) with primal gradient complexity

$$O\left((\sqrt{\gamma L_{xx}} + L_{xy}\gamma\varepsilon^{-1})\varepsilon^{-2}\ln(\varepsilon^{-1})\right).$$

(7.2)

Note that due to the “simple” dual structure, we neither need to assume any differentiability and smoothness properties of $\Phi(x, \cdot)$ (or equivalently, $\Psi(x, \cdot)$) on $\mathcal{Y}$, and nor need to analyze the dual gradient complexity of Algorithm 1.

7.2. The stochastic setting. Our primal-dual smoothing framework (i.e., Algorithm 1) can be easily extend to the stochastic setting, where at any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we only have access to the primal gradient $\nabla_x \Phi(x, y)$ and the dual gradient $\nabla_y \Phi(x, y)$ via their unbiased stochastic estimators, denoted by $\tilde{\nabla}_x \Phi(x, y)$ and $\tilde{\nabla}_y \Phi(x, y)$, respectively. Specifically, for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we assume that $\tilde{\nabla}_x \Phi(x, y)$ and $\tilde{\nabla}_y \Phi(x, y)$ satisfy the following conditions:

$$\mathbb{E} [\tilde{\nabla}_x \Phi(x, y)] = \nabla_x \Phi(x, y), \quad \mathbb{E} [\tilde{\nabla}_y \Phi(x, y)] = \nabla_y \Phi(x, y),$$

(7.3)

$$\mathbb{E} [\|\tilde{\nabla}_x \Phi(x, y) - \nabla_x \Phi(x, y)\|^2] \leq \sigma_x^2, \quad \mathbb{E} [\|\tilde{\nabla}_y \Phi(x, y) - \nabla_y \Phi(x, y)\|^2] \leq \sigma_y^2,$$

(7.4)

where both $\sigma_x^2, \sigma_y^2 < +\infty$. Note that the conditions in (7.4) indicate that both (stochastic) gradient estimators $(x, y) \mapsto \tilde{\nabla}_x \Phi(x, y)$ and $(x, y) \mapsto \tilde{\nabla}_y \Phi(x, y)$ have uniformly bounded variances over $\mathcal{X} \times \mathcal{Y}$.

Similar to the deterministic setting, our goal in the stochastic setting is to find an $\varepsilon$-NS point of (1.1), but in the sense of expectation. Specifically, we aim to find a random point $x \in \mathcal{X}^o$ such that $\mathbb{E}[\|x - \text{prox}(q, x, \lambda)\|] \leq \varepsilon\lambda / \beta_X$ (cf. (2.10)). Under this goal, we provide a stochastic extension of Algorithm 1, which is shown in Algorithm 3. Compared to Algorithm 1, there are two major differences. First, we modify the inexact criterion in Step 2 such that it holds in expectation. Second, instead of terminating Algorithm 3 adaptively using the criterion in (4.6), we run Algorithm 3 for a pre-determined number of iterations $K$. This is because a single realization of $x_k$, in general, do not provide useful information in bounding $\mathbb{E}[\|x_k - \text{prox}(q, x_k, \lambda)\|]$. Consequently, in Algorithm 3, we do not output the last iterate $x_K$ as in Algorithm 1, but rather a random point uniformly sampled from $\{x_1, \ldots, x_K\}$. The convergence guarantee of Algorithm 3 is shown in the following theorem.

Theorem 7.1. Fix any $\varepsilon > 0$. In Algorithm 3, given any $\Delta_q(x_1) \geq q(x_1) - q^*$ and $\lambda \in (0, \gamma^{-1})$, if we let

$$\eta = \varepsilon^2 \lambda / (12\beta_X^2) \quad \text{and} \quad K = 8\beta_X^2 \Delta_q(x_1) / (\lambda \varepsilon^2),$$

(7.5)

then we have $\mathbb{E}[\|x_{out} - \text{prox}(q, x_{out}, \lambda)\|] \leq \varepsilon\lambda / \beta_X$.

Proof. See Appendix I. $\square$
Algorithm 3 Stochastic primal dual smoothing framework

**Input:** Accuracy parameter \( \eta > 0 \), smoothing parameters \( \lambda \in (0, \gamma^{-1}) \) and \( \rho = \eta / (4R_y(\omega_y)) \)

**Initialize:** \( x_1 \in \mathcal{X} \)

For \( k = 1, \ldots, K \):

Find (a random point) \( x_{k+1} \in \mathcal{X}^\circ \) such that \( \mathbb{E}[Q^\lambda_\rho(x_{k+1}; x_k) | x_k] \leq q^\lambda_\rho(x_k) + \eta \).

**Output:** \( x_{\text{out}} = x_k \), where \( k \sim \text{Unif}\{1, \ldots, K\} \) (namely the uniform distribution over \( \{1, \ldots, K\} \) )

**Remark 7.1.** Note that in Theorem 7.1, we need to estimate an upper bound of \( q(x_1) - q^* \), i.e., \( \Delta_q(x_1) \), which amounts to estimating a lower bound of \( q^* \). Note that in many real-world problems, the function \( q \), as a loss or cost function, is nonnegative on \( \mathcal{X} \). Therefore, we can simply let \( \Delta_q(x_1) = q(x_1) \) in this case.

Let us briefly analyze the primal and dual gradient complexities for Algorithm 3 to find an \( \varepsilon \)-NS point of (1.1) in expectation. For simplicity we only focus on the dependence of these complexities on the accuracy parameter \( \varepsilon \). Indeed, in each iteration of Algorithm 3, there exist many stochastic first-order methods (e.g., Chen et al. [5, Algorithm 1] and Zhao [33, Algorithm 1]) that we can use to find the desired \( x_{k+1} \in \mathcal{X}^\circ \), and the primal and dual gradient complexities of these methods all share the same order, i.e., \( O((\rho \eta)^{-1}) \). From the choices of \( \rho \) in Algorithm 3 and \( \eta \) in Theorem 7.1, we have \( O((\rho \eta)^{-1}) = O(\eta^{-2}) = O(\varepsilon^{-4}) \). In addition, since the total number of iterations \( K = O(\varepsilon^{-2}) \) (cf. Theorem 7.1), the primal and dual gradient complexities of Algorithm 3 are of order \( O(\varepsilon^{-6}) \), which indeed match the state-of-the-art (see e.g., Rafique et al. [25]).

7.3. A simple variant of Algorithm 2. Let us observe that at each iteration in Algorithm 2, we need to solve two BPP problems associated with \( \zeta \) and the DGF \( \omega_U \) in (5.10) and (5.14), respectively. One may naturally wonder if it is possible to solve only one BPP problem at each iteration, and this leads to Algorithm 4. Indeed, this algorithm is simpler than Algorithm 2, in the sense that it does not involve the sequence \( \{w_t\}_{t \geq 0} \), and hence only involves solving one BPP problem at each iteration. The design and analysis of Algorithm 4 are almost identical to those of Algorithm 2, since both algorithms can be derived from the estimate sequence framework in Nesterov [20], and they differ only in one step of the derivation. As such, the convergence guarantees of Algorithm 4 are similar to those of Algorithm 2 in Section 5.3, and we leave the details to the readers.

8. Conclusion and future work. In this work, we have proposed a primal-dual smoothing framework for finding an \( \varepsilon \)-NS point of a class of non-smooth non-convex optimization problems in (1.1). As a contribution of independent interest, we have developed a non-Hilbertian inexact APG method for the strongly convex composite optimization problems in (5.1). There are some problems left open and we wish to consider them in future work.
Algorithm 4 A simple variant of Algorithm 2

**Input, Define & Initialize:** Same as Algorithm 2. Additionally, let $\bar{u}_0 = z_0$.

**Repeat** (until some convergence criterion is satisfied)

- $u_{t+1} := (1 - \tau_{t+1})z_t + \tau_{t+1}\bar{u}_t \quad (7.6)$
- Compute $\hat{\nabla}h(u_{t+1})$ where $(\hat{h}(u_{t+1}), \hat{\nabla}h(u_{t+1}))$ is a $(\delta_{t+1}, \bar{L})$-FOA of $h$ at $u_{t+1} \quad (7.7)$
- $s_{t+1} := s_t + \alpha_{t+1}\hat{\nabla}h(u_{t+1}) \quad (7.8)$
- $\pi_{t+1} := \arg\min_{u \in \mathcal{U}} \left< s_{t+1}, u \right> + A_{t+1}(\zeta(u) + \mu\omega_{t}(u)) + \bar{L}D_{\omega_{t}}(u, u_0) \quad (7.9)$
- $z_{t+1} := (1 - \tau_{t+1})z_t + \tau_{t+1}\bar{u}_{t+1} \quad (7.10)$
- $t := t + 1 \quad (7.11)$

First, the lower complexity bound for finding an $\varepsilon$-NS point of (1.1) is not known yet. Establishing this bound with dependence on the problem parameters (including $L_{xx}$, $L_{xy}$, $L_{yy}$ and $\gamma$) and the accuracy $\varepsilon$ can be useful to understand the “optimality” of the existing methods (including ours).

Second, as detailed in Section 7.2, a straightforward extension of our framework to the stochastic setting can find an $\varepsilon$-NS point of (1.1) in expectation with primal and dual gradient complexities both of order $O(\varepsilon^{-6})$. It seems that this result can be further improved using “cleverer” strategies, and developing these strategies would be an interesting direction for future research.

Third, in Section 2.3, the additional assumptions (ii) and (iii) that we place on the DGF $\omega_X$ appear to be somewhat stringent. In fact, the well-known example in (E1) does not satisfy either assumption. Note that assumption (iii) amounts to assuming that $\|\nabla^2\omega_X(x)\|$ (namely, the operator norm of $\nabla^2\omega_X(x)$) is uniformly bounded by $\beta_X < +\infty$ over $x \in X$. Without assumption (ii), this may fail even when $X$ is bounded, which is precisely due to the potential “blow-up” behavior of $\nabla^2\omega_X(\cdot)$ near the boundary of $\text{dom}\,\omega_X$. The failure of assumption (iii) poses serious challenges in designing optimization algorithms for finding an $\varepsilon$-NS point of (1.1) with complexity guarantees.

We believe that addressing this problem will have far-reaching impact in the broader context of non-Euclidean non-convex optimization, and it is very worthwhile to pursue this problem in the future.

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Appendix A: Proof of Lemma 3.1. Fix any $x \in \mathcal{X}$ and any $d \in \mathbb{X}$. Consider any sequences $\{t_n\}_{n \geq 0} \subseteq \mathbb{R}$ and $\{d_n\}_{n \geq 0} \subseteq \mathbb{X}$ such that $t_n \downarrow 0$ and $d_n \to d$. Since $\mathcal{X} \subseteq \mathcal{X}'$ and $\mathcal{X}'$ is open, then there exists some $N \geq 0$ such that for all $n \geq N$, $x + t_n d_n \in \mathcal{X}'$. Note that due to the compactness of $\mathcal{Y}$, for all $x \in \mathcal{X}'$, the set of dual optimal solutions $\mathcal{Y}^*(x)$ (as defined in (3.1)) is nonempty. By the definition of $f$ in (1.1), for any $y \in \mathcal{Y}^*(x)$ and $n \geq N$, we have

$$\frac{f(x + t_n d_n) - f(x)}{t_n} \geq \frac{\Phi(x + t_n d_n, y) - \Phi(x, y)}{t_n}.$$

(A.1)

Therefore, we have for any $y \in \mathcal{Y}^*(x)$,

$$\liminf_{n \to +\infty} \frac{f(x + t_n d_n) - f(x)}{t_n} \geq \liminf_{n \to +\infty} \frac{\Phi(x + t_n d_n, y) - \Phi(x, y)}{t_n} = \liminf_{n \to +\infty} \frac{t_n \langle \nabla_x \Phi(x, y), d_n \rangle + o(t_n \|d_n\|)}{t_n} = \langle \nabla_x \Phi(x, y), d \rangle.$$

As such, we have

$$\liminf_{n \to +\infty} \frac{f(x + t_n d_n) - f(x)}{t_n} \geq \sup_{y \in \mathcal{Y}^*(x)} \langle \nabla_x \Phi(x, y), d \rangle.$$

(A.2)

Next, let $\{x_n\}_{n \geq 0} \subseteq \mathcal{X}'$ be any sequence such that $x_n \to x$. We aim to show that if $y_n \in \mathcal{Y}^*(x_n)$ for all $n \geq 0$ and $y_n \to y \in \mathcal{Y}$, then $y \in \mathcal{Y}^*(x)$. Indeed, by definition,

$$\limsup_{n \to +\infty} f(x_n) = \limsup_{n \to +\infty} \Phi(x_n, y_n) - g(y_n) \geq [a] \Phi(x, y) - g(y) \leq f(x),$$

(A.3)

where (a) follows from the joint continuity of $\Phi$ on $\mathcal{X}' \times \mathcal{Y}$ (cf. Section 1.1) and the continuity of $g$ on $\mathcal{Y}$, and (b) follows from the definition of $f$ in (1.1) and that $y \in \mathcal{Y}$. On the other hand, since $\Phi(\cdot, y)$ is continuous on $\mathcal{X}$ for any $y \in \mathcal{Y}$, $f$ is clearly lower semicontinuous. As a result, we have

$$\limsup_{n \to +\infty} f(x_n) \geq \liminf_{n \to +\infty} f(x_n) \geq f(x).$$

(A.4)

Combining (A.3) and (A.4), we have $f(x) = \Phi(x, y) - g(y)$, implying that $y \in \mathcal{Y}^*(x)$. As a result, if we let $x_n := x + t_n d_n$ (so that $x_n \to x$) and $y_n \in \mathcal{Y}^*(x + t_n d_n)$, then any limit point of $\{y_n\}_{n \in \mathbb{N}}$ (which exists since $\mathcal{Y}$ is compact) belongs to $\mathcal{Y}^*(x)$. Consequently, we have

$$\limsup_{n \to +\infty} \frac{f(x + t_n d_n) - f(x)}{t_n} \leq \limsup_{n \to +\infty} \frac{\Phi(x + t_n d_n, y_n) - \Phi(x, y_n)}{t_n} = \limsup_{n \to +\infty} \frac{t_n \langle \nabla_x \Phi(x, y_n), d_n \rangle + o(t_n \|d_n\|)}{t_n} \leq \sup_{y \in \mathcal{Y}^*(x)} \langle \nabla_x \Phi(x, y), d \rangle.$$

(A.5)

Combining (A.2) and (A.5), we see that for all $x \in \mathcal{X}'$ and $d \in \mathbb{X}$, $f'(x; d)$ exists and

$$f'(x; d) = \lim_{n \to +\infty} \frac{f(x + t_n d_n) - f(x)}{t_n} = \sup_{y \in \mathcal{Y}^*(x)} \langle \nabla_x \Phi(x, y), d \rangle.$$

$\square$
Remark A.1. Our proof of Lemma 3.1 can be regarded as a simplified version of that for Bernard and Rapaport [2, Theorem D1]. This is because we assume the Fréchet differentiability of \( \Phi(\cdot, y) \), which is stronger than the notion of Gâteaux differentiability of \( \Phi(\cdot, y) \) assumed in [2, Theorem D1]. However, note that our result is also stronger, namely we show that \( f \) is (Hadamard) directionally differentiable, whereas [2, Theorem D1] only shows that \( f \) is Gâteaux directionally differentiable, a notion weaker than (Hadamard) directional differentiability.

Appendix B: Proof of Lemma 3.2 Fix any \( x \in X \) and consider its compact neighborhood \( V(x) \) in \( X \), namely \( x \in V(x) \subseteq X \). Define \( M_{V(x)} := \sup_{(z,y) \in V(x) \times Y} \| \nabla_x \Phi(z, y) \|_* \), and note that \( M_{V(x)} < +\infty \) since \( \nabla_x \Phi(\cdot, \cdot) \) is jointly (Lipschitz) continuous on \( X \times Y \) (which follows from Assumption 1.1) and \( V(x) \) is compact. For any \( x', x'' \in V(x) \), we have

\[
|f(x') - f(x'')| = | \sup_{y \in Y} [ \Phi(x', y) - g(y)] - \sup_{y \in Y} [ \Phi(x'', y) - g(y)] |
\leq \sup_{y \in Y} | \Phi(x', y) - \Phi(x'', y) |
\leq \sup_{y \in Y} \int_0^1 \langle \nabla_x \Phi(x'' + t(x' - x''), y), x' - x'' \rangle \, dt 
\leq \sup_{y \in Y} \int_0^1 \| \nabla_x \Phi(x'' + t(x' - x''), y) \|_* \| x' - x'' \| 
\leq M_{V(x)} \| x' - x'' \|. \tag{B.1}
\]

This shows that \( f \) is \( M_{V(x)} \)-Lipschitz on \( V(x) \).

Next, let us show \( \partial f(x) = \text{clconv} \{ \nabla_x \Phi(x, y) : y \in V^*(x) \} \). For notational convenience, let

\[
\mathcal{A}(x) := \{ \nabla_x \Phi(x, y) : y \in V^*(x) \}. \tag{B.2}
\]

Fix any \( x \in X' \) and any \( d \in X \). We first show that \( \text{clconv} \mathcal{A}(x) \subseteq \partial f(x) \). To see this, if \( y \in V^*(x) \), then

\[
f(x + d) - f(x) - \langle \nabla_x \Phi(x, y), d \rangle \geq \Phi(x + d, y) - \Phi(x, y) - \langle \nabla_x \Phi(x, y), d \rangle = o(\|d\|),
\]

and hence \( \nabla_x \Phi(x, y) \in \partial f(x) \). Since \( \partial f(x) \) is closed and convex, we have \( \text{clconv} \mathcal{A}(x) \subseteq \partial f(x) \). Then, by (3.1), we have

\[
f'(x; d) = \sup_{\xi \in \mathcal{A}(x)} \langle \xi, d \rangle = \sup_{\xi \in \text{clconv} \mathcal{A}(x)} \langle \xi, d \rangle \leq \sup_{\xi \in \partial f(x)} \langle \xi, d \rangle. \tag{B.3}
\]

On the other hand, for any \( \xi \in \partial f(x) \), by the definition in (3.1), we have

\[
f'(x; d) = \lim_{t \downarrow 0, d' \rightarrow d} \frac{f(x + td') - f(x)}{t} \geq \lim_{t \downarrow 0, d' \rightarrow d} \frac{t \langle \xi, d' \rangle + o(t \|d'\|)}{t} = \langle \xi, d \rangle.
\]

This shows that \( f'(x; d) \geq \sup_{\xi \in \partial f(x)} \langle \xi, d \rangle \). This, together with (B.3), implies that

\[
f'(x; d) = \sup_{\xi \in \partial f(x)} \langle \xi, d \rangle = \sup_{\xi \in \text{clconv} \mathcal{A}(x)} \langle \xi, d \rangle. \tag{B.4}
\]
Since both $\partial f(x)$ and $\text{clconv} \mathcal{A}(x)$ are closed and convex sets and share the same support function $d \mapsto f'(x;d)$, they share the same indicator function (cf. Rockafellar [26, Theorem 13.2]) and hence $\partial f(x) = \text{clconv} \mathcal{A}(x)$. □

Appendix C: Proof of Lemma 3.4. Consider any $x, x' \in \mathcal{X}$. By the $\rho$-strong concavity of $\phi^D_{\rho}(x, \cdot)$ on $\mathcal{Y}$ (for any $x \in \mathcal{X}'$), we have

$$\begin{align*}
(\rho/2)\|y^\rho_\star(x') - y^\rho_\star(x)\|^2 &\leq \phi^D_\rho(x, y^\rho_\star(x)) - \phi^D_\rho(x, y^\rho_\star(x')) \\
(\rho/2)\|y^\rho_\star(x') - y^\rho_\star(x)\|^2 &\leq \phi^D_\rho(x', y^\rho_\star(x')) - \phi^D_\rho(x', y^\rho_\star(x)).
\end{align*}$$

As a result,

$$\begin{align*}
\|y^\rho_\star(x') - y^\rho_\star(x)\|^2 &\leq \rho^{-1}(\phi^D_\rho(x, y^\rho_\star(x)) - \phi^D_\rho(x, y^\rho_\star(x'))) + \phi^D_\rho(x', y^\rho_\star(x')) - \phi^D_\rho(x', y^\rho_\star(x)) \\
&\leq \rho^{-1} \int_0^1 \|\nabla_x \phi^D_\rho(x' + t(x-x'), y^\rho_\star(x)) - \nabla_x \phi^D_\rho(x' + t(x-x'), y^\rho_\star(x'))\| \|x-x'\| dt \\
&\leq \rho^{-1} \int_0^1 \|\nabla_x \phi^D_\rho(x' + t(x-x'), y^\rho_\star(x)) - \nabla_x \phi^D_\rho(x' + t(x-x'), y^\rho_\star(x'))\| \|x-x'\| dt \\
&\leq (L_{xy}/\rho)\|y^\rho_\star(x) - y^\rho_\star(x')\|. \quad (C.1)
\end{align*}$$

If $y^\rho_\star(x) = y^\rho_\star(x')$, then we trivially have $\|y^\rho_\star(x) - y^\rho_\star(x')\| \leq (L_{xy}/\rho)\|x-x'\|$. Otherwise, this can be obtained by dividing both sides of (C.1) by $\|y^\rho_\star(x) - y^\rho_\star(x')\|$.

Appendix D: Proof of Lemma 3.5. The differentiability of $f_\rho$ on $\mathcal{X}'$ directly follows from the $\rho$-strong concavity of $\phi^D_{\rho}(x, \cdot)$ on $\mathcal{Y}$ (for any $x \in \mathcal{X}'$) and Lemma 3.1, from which we also see that $\nabla f_\rho(x) = \nabla_x \phi(x, y^\rho_\star(x))$. Consequently, we have

$$\begin{align*}
\|\nabla f_\rho(x') - \nabla f_\rho(x)\|_* &= \|\nabla_x \phi^D_{\rho}(x', y^*(x')) - \nabla_x \phi^D_{\rho}(x, y^*(x))\|_* \\
&\leq \|\nabla_x \phi^D_{\rho}(x', y^*(x')) - \nabla_x \phi^D_{\rho}(x, y^*(x'))\|_* + \|\nabla_x \phi^D_{\rho}(x, y^*(x')) - \nabla_x \phi^D_{\rho}(x, y^*(x))\|_* \\
&\leq L_{xx}\|x-x'\| + L_{xy}\|y^*(x) - y^*(x')\| \\
&\leq (L_{xx} + L_{xy}^2/\rho)\|x-x'\|,
\end{align*}$$

where the last step follows from Lemma 3.4.

Appendix E: Proof of Lemma 3.7. From Lemmas 3.2 and 3.5, we know that $\mathcal{A}(x) \subseteq \partial f(x)$ (where $\mathcal{A}(x)$ is defined in (B.2) and $\nabla f_\rho(x) = \nabla_x \phi(x, y^\rho_\star(x))$, and hence

$$\begin{align*}
\text{dist} (\nabla f_\rho(x), \partial f(x)) &\leq \text{dist} (\nabla_x \phi(x, y^\rho_\star(x)), \mathcal{A}(x)) \\
&= \inf_{y \in \mathcal{Y}^*(x)} \|\nabla_x \phi(x, y^\rho_\star(x)) - \nabla_x \phi(x, y)\|_* \\
&\leq L_{xy} \inf_{y \in \mathcal{Y}^*(x)} \|y^\rho_\star(x) - y\| \\
&= L_{xy} \text{dist} (y^\rho_\star(x), \mathcal{Y}^*(x)),
\end{align*}$$

where (a) follows from Assumption 1.1. This shows the first inequality in (3.11). Also, as both $y^\rho_\star(x) \in \mathcal{Y}$ and $\mathcal{Y}^*(x) \subseteq \mathcal{Y}$, we have $\text{dist} (y^\rho_\star(x), \mathcal{Y}^*(x)) \leq D_\mathcal{Y}$, which shows the second inequality.
Appendix F: Proof of Lemmas 5.1 and 5.3. Given any \( u \in \mathcal{U} \), let \( v^*(u) \in \mathcal{V} \) be the optimal solution of the maximization problem in (5.2), and hence \( h(u) = \Psi(u, v^*(u)) \). From the \( \rho \)-strong concavity of \( \bar{\Psi}(u, \cdot) \), we have
\[
\frac{\rho}{2} \|v^*(u) - \hat{v}\|^2 \leq h(u) - \bar{\Psi}(u, \hat{v}) \leq \delta/2 \quad \implies \quad \|v^*(u) - \hat{v}\|^2 \leq \delta/\rho. \tag{F.1}
\]
Therefore, we have
\[
h(u') = \bar{\Psi}(u', v^*(u')) \overset{(a)}{=} \bar{\Psi}(u', \hat{v}) \overset{(b)}{=} \bar{\Psi}(u, \hat{v}) + \langle \nabla_u \bar{\Psi}(u, \hat{v}), u' - u \rangle,
\]
where (a) follows from (5.4) and (b) follows from (F.1). This proves Lemma 5.3. Consequently,
\[
\|\nabla h(u) - \nabla_u \bar{\Psi}(u, \hat{v})\|^2 \overset{(a)}{=} \|\nabla_u \bar{\Psi}(u, v^*(u)) - \nabla_u \bar{\Psi}(u, \hat{v})\|^2 \leq L_{uv}^2 \|v^*(u) - \hat{v}\|^2 \overset{(b)}{\leq} \left( \frac{L_{uv}^2}{\rho} \right) \delta, \tag{F.3}
\]
where (a) follows from (5.4) and (b) follows from (F.1). This proves Lemma 5.3. Consequently,
\[
\langle \nabla h(u) - \nabla_u \bar{\Psi}(u, \hat{v}), u' - u \rangle \leq \langle \nabla h(u) - \nabla_u \bar{\Psi}(u, \hat{v})\|^2 + (L_h/2)\|u' - u\|^2 \overset{(a)}{\leq} \left( \frac{\delta}{2} \right) \left( \frac{L_{uv}^2}{\rho} \right) + \left( \frac{L_h}{2} \right)\|u' - u\|^2 \overset{(b)}{\leq} \frac{\delta}{2} + \left( \frac{L_h}{2} \right)\|u' - u\|^2, \tag{F.4}
\]
where (a) follows from (F.3) and (b) follows from \( L_h = L_{uu} + L_{uv}^2/\rho \geq L_{uv}^2/\rho \). Now, by substituting (F.4) into (F.2), we have (5.8).

Appendix G: Proof of Theorems 5.1 and 5.2. Let us begin our proof by defining the sequence of functions \( \{\psi_t : \mathcal{U} \to \mathbb{R}\}_{t \geq 0} \) such that for all \( t \geq 0 \),
\[
\psi_t(u) := \sum_{i=0}^t \alpha_i \hat{h}(u_i) + \langle \hat{\nabla} h(u_i), u - u_i \rangle + \zeta(u) + \mu \omega_l(u) + \bar{L} D_{w_l}(u, u_0), \quad \forall u \in \mathcal{U}. \tag{G.1}
\]
Indeed, the functions \( \{\psi_t\}_{t \geq 0} \) play pivotal roles in analyzing Algorithm 2. In addition, let us define
\[
\psi^*_t := \min_{u \in \mathcal{U}} \psi_t(u), \quad \forall t \geq 0. \tag{G.2}
\]
Our proof can be streamlined into the following three lemmas. The first lemma below establishes a lower bound of \( \psi^*_t \), for all \( t \geq 0 \).
Lemma G.1. If $\alpha_0 = 1$ and

$$(A_t \mu + \bar{L})A_{t+1} \geq \bar{L} \alpha_{t+1}^2, \quad \forall t \geq 0,$$  

(G.3)

then we have that for all $t \geq 0$,

$$\psi^*_t \geq A_t P(z_t) - E_t, \quad \text{where} \quad E_t := \sum_{i=0}^t A_i \delta_i.$$  

(G.4)

Proof. Let us show (G.4) using induction. When $t = 0$, from the initialization in Algorithm 2, we see that $z_0 := \arg \min_{u \in U} \psi_0(u)$ and hence

$$\psi^*_0 = \psi_0(z_0) = \alpha_0 \hat{h}(z_0) + \langle \nabla h(z_0), u - z_0 \rangle + \zeta(z_0) + \mu \omega_U(z_0) + \bar{L} D_{\omega_U}(u, z_0)$$

$$\geq \hat{h}(z_0) + \langle \nabla h(z_0), u - z_0 \rangle + (\bar{L}/2)\|u - z_0\|^2 + \zeta(z_0) + \mu \omega_U(z_0)$$

$$\geq h(z_0) - \delta_0 + \zeta(z_0) + \mu \omega_U(z_0) \equiv P(z_0) - A_0 \delta_0,$$

where in (a) we use $\alpha_0 = 1$ and the 1-strong convexity of $\omega_U$ on $U$, in (b) we use the second inequality in (5.7) and in (c) we use $A_0 = \alpha_0 = 1$. Now, suppose that (G.4) holds for some $t \geq 0$, and let us show that (G.4) holds for $t + 1$. First, observe that for any $u \in U$, we have

$$\psi_{t+1}(u) = \psi_t(u) + \alpha_{t+1} \hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), u - u_{t+1} \rangle + \zeta(u) + \mu \omega_U(u)$$

$$\geq \psi^*_t + (A_t \mu + \bar{L}) D_{\omega_U}(u, \bar{u}_{t+1}) + \alpha_{t+1} \hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), u - u_{t+1} \rangle + \zeta(u) + \mu \omega_U(u)$$

$$\geq \psi^*_t + (A_t \mu + \bar{L}) D_{\omega_U}(w_{t+1}, \bar{w}_{t+1})$$

$$+ \alpha_{t+1} \hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), w_{t+1} - u_{t+1} \rangle + \zeta(w_{t+1}) + \mu \omega_U(w_{t+1}),$$  

(G.5)

where (a) follows from $\bar{u}_{t+1} = \arg \min_{u \in U} \psi_t(u)$, since $s_t = \sum_{i=0}^t \alpha_i \hat{\nabla} h(u_i)$ (cf. (5.10)) and (b) follows from the definition of $w_{t+1}$ in (5.14). Using the induction hypothesis, we have

$$\psi^*_t \geq A_t \hat{h}(h(z_t)) + \zeta(z_t) + \mu \omega_U(z_t) - E_t$$

$$\geq A_t \hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), z_t - u_{t+1} \rangle + \zeta(z_t) + \mu \omega_U(z_t) - E_t,$$  

(G.6)

where (a) follows from the first inequality in (5.7). Combining (G.5) and (G.6), we have

$$\psi_{t+1}(u) \geq A_{t+1} \hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), A_t z_t + \alpha_{t+1} w_{t+1} - A_{t+1} u_{t+1} \rangle$$

$$+ \{A_t \zeta(z_t) + \mu \omega_U(z_t)\} + \alpha_{t+1} \{\zeta(w_{t+1}) + \mu \omega_U(w_{t+1})\} + (A_t \mu + \bar{L}) D_{\omega_U}(w_{t+1}, \bar{w}_{t+1}) - E_t$$

$$\geq A_{t+1} \{\hat{h}(u_{t+1}) + \langle \nabla h(u_{t+1}), (1 - \tau_{t+1}) z_t + \tau_{t+1} w_{t+1} - u_{t+1} \rangle$$

$$+ (1 - \tau_{t+1}) \zeta(z_t) + \mu \omega_U(z_t)\} + \tau_{t+1} \{\zeta(w_{t+1}) + \mu \omega_U(w_{t+1})\}$$

$$+ ((A_t \mu + \bar{L})/(2A_{t+1})) \|w_{t+1} - \bar{w}_{t+1}\|^2 \} - E_t$$

$$\geq A_{t+1} \{h(u_{t+1}) + \langle \nabla h(u_{t+1}), z_t - u_{t+1} \rangle + \zeta(z_t) + \mu \omega_U(z_t)\} - E_t.$$
\[(b)\] 
\[
A_{t+1} \{ \hat{h}(u_{t+1}) + (\nabla h(u_{t+1}), z_{t+1} - u_{t+1}) + \zeta(z_{t+1}) + \mu \omega u(z_{t+1}) \\
+ ((A_t \mu + \bar{L})/(2A_{t+1}^\tau_{t+1}^2))\|z_{t+1} - u_{t+1}\|^2 \} - E_t
\]
\[(c)\] 
\[
A_{t+1} \{ \hat{h}(u_{t+1}) + (\nabla h(u_{t+1}), z_{t+1} - u_{t+1}) + (\bar{L}/2)\|z_{t+1} - u_{t+1}\|^2 + \zeta(z_{t+1}) + \mu \omega u(z_{t+1}) \} - E_t
\]
\[(d)\] 
\[
A_{t+1} \{ \hat{h}(z_{t+1}) - \delta_{t+1} + \zeta(z_{t+1}) + \mu \omega u(z_{t+1}) \} - E_t
\]
\[= A_{t+1} P(z_{t+1}) - E_{t+1}, \quad \forall u \in U,\] (G.7)

where in (a) we use \(\tau_{t+1} = \alpha_{t+1}/A_{t+1}\) and the 1-strong convexity of \(\omega u\) on \(U\), in (b) we use the definition of \(z_{t+1}\) in (5.15), in (c) we use \(z_{t+1} - u_{t+1} = \tau_{t+1}(w_{t+1} - u_{t+1})\), which follows from (5.11) and (5.15), and
\[
\frac{A_t \mu + \bar{L}}{A_{t+1}^\tau_{t+1}^2} = \frac{(A_t \mu + \bar{L})A_{t+1}}{\alpha_{t+1}^2} \geq \bar{L},
\]
which follows from (G.3), and in (d) we use the second inequality in (5.7). We finish the induction by minimizing \(\psi_{t+1}\) over \(U\) on the left-hand side of (G.7). \(\Box\)

At this point, a natural question would be whether the choices of \(\{\alpha_t\}_{t \geq 0}\) in Theorems 5.1 and 5.2 satisfy the condition (G.3). This is confirmed in the next lemma.

**Lemma G.2.** The choices of \(\{\alpha_t\}_{t \geq 0}\) in (5.23) when \(\mu = 0\) and in (5.27) when \(\mu > 0\) lead to the following values of \(\{A_t\}_{t \geq 0}\):
\[
A_t = \begin{cases} 
(t+2)^2/4, & \text{when } \mu = 0, \\
(1 + \sqrt{\theta})^t, & \text{when } \mu > 0,
\end{cases} \quad \forall t \geq 0.
\] (G.8)

In addition, the condition in (G.3) is satisfied in both cases (i.e., \(\mu = 0\) and \(\mu > 0\)).

**Proof.** Let us first focus on the case where \(\mu = 0\), and show \(A_t = (t+2)^2/4\) for all \(t \geq 0\) via induction. Clearly, this holds when \(t = 0\) as \(A_0 = \alpha_0 = 1\). Suppose this is true for some \(t \geq 0\). Then
\[
A_{t+1} = A_t + \alpha_{t+1} = (t+2)^2/4 + (2t+5)/4 = (t+3)^2/4.
\] (G.9)
This completes the induction. Note that when \(\mu = 0\), the condition in (G.3) simplifies to \(\alpha_t^2 \leq A_t\) for all \(t \geq 1\), which clearly holds as \(\alpha_t = (t+1.5)/2\) and \(A_t = (t+2)^2/4\) for all \(t \geq 1\). Next, let us show \(A_t = (1 + \sqrt{\theta})^t\) for all \(t \geq 0\) when \(\mu > 0\). Again, we prove this using induction. Note that this clearly holds when \(t = 0\) as \(A_0 = \alpha_0 = 1\). Suppose this is true for some \(t \geq 0\). Then we have
\[
A_{t+1} = A_t + \alpha_{t+1} = (1 + \sqrt{\theta})^t + (1 + \sqrt{\theta})^t \sqrt{\theta} = (1 + \sqrt{\theta})^{t+1},
\] (G.10)
and this finishes the induction. Using (G.8) and the monotonicity of \(\{A_t\}_{t \geq 0}\), we have
\[
(A_t \mu + \bar{L})A_{t+1} = \bar{L}(1 + \theta A_t) A_{t+1} \geq \bar{L} \theta A_t A_{t+1} \geq \bar{L} \theta A_t^2 = \bar{L} \theta (1 + \sqrt{\theta})^{2t} = \bar{L} \alpha_{t+1}^2, \quad \forall t \geq 0,
\]
which means that the condition in (G.3) is satisfied. We hence complete the proof. \(\Box\)
Lastly, the third lemma below establishes two upper bounds of \( \psi_t^* \), for all \( t \geq 0 \). The first one involves the optimal value \( P^* \) and \( D_{\omega_t}(u^*, u_0) \), which is the Bregman “distance” from any optimal solution \( u^* \) to \( u_0 \). The second one involves the dual function \( \Xi \) and \( \Omega_{\omega_t}(u^0) \) (cf. (5.26)), which is the Bregman “distance” from the furthest point \( u \in U \) to \( u_0 \), and is finite when \( U \) is bounded.

**Lemma G.3.** For any \( t \geq 0 \), we have

\[
\psi_t^* \leq A_t P^* + \bar{L} D_{\omega_t}(u^*, u_0). \tag{G.11}
\]

In addition, if \( U \) is bounded and \((\hat{h}(u_t), \hat{\nabla} h(u_t)) = (\bar{\Psi}(u_t, v_t), \nabla_u \bar{\Psi}(u_t, v_t)) \) for some \( \{v_t\}_{t \geq 0} \subseteq V \) (which need not satisfy (5.21)), then for any \( t \geq 0 \), we have

\[
\psi_t^* \leq A_t \Xi(\bar{\tau}_t) + \bar{L} \Omega_{\omega_t}(u_0), \tag{G.12}
\]

where \( \bar{\tau}_t \) is defined in (5.22) and \( \Omega_{\omega_t}(u_0) < +\infty \) is defined in (5.26).

**Proof.** Since \((\hat{h}(u_t), \hat{\nabla} h(u_t))\) is a \((\delta_t, \bar{L})\)-FOA of \( h \) at \( u_t \) for all \( t \geq 0 \), by using the first inequality in (5.7), we have that for all \( u \in U \),

\[
\psi_t^* \leq \psi_t(u) \leq \sum_{i=0}^t \alpha_i(h(u) + \zeta(u) + \mu_{\omega_i}(u)) + \bar{L} D_{\omega_i}(u, u_0) = A_t P(u) + \bar{L} D_{\omega_t}(u, u_0), \tag{G.13}
\]

Substitute \( u = u^* \) into (G.13) and we obtain (G.11). If \((\hat{h}(u_t), \hat{\nabla} h(u_t)) = (\bar{\Psi}(u_t, v_t), \nabla_u \bar{\Psi}(u_t, v_t)) \) for all \( t \geq 0 \), then using the convexity of \( \bar{\Psi}(\cdot, \cdot) \), we have

\[
\hat{h}(u_t) + \langle \hat{\nabla} h(u_t), u - u_t \rangle = \bar{\Psi}(u_t, v_t) + \langle \nabla_u \bar{\Psi}(u_t, v_t), u - u_t \rangle \leq \bar{\Psi}(u, v_t), \quad \forall t \geq 0. \tag{G.14}
\]

Based on this, and by using the concavity of \( \bar{\Psi}(u, \cdot) \), we have

\[
\psi_t(u) \leq \sum_{i=0}^t \alpha_i \bar{\Psi}(u, v_i) + A_t(\zeta(u) + \mu_{\omega_t}(u)) + \bar{L} D_{\omega_t}(u, u_0) \\
\leq A_t(\bar{\Psi}(u, \bar{\tau}_t) + \zeta(u) + \mu_{\omega_t}(u)) + \bar{L} \Omega_{\omega_t}(u_0). \tag{G.15}
\]

Now, by taking infimum of \( u \) over \( U \) on both sides of (G.15), we obtain (G.12). \( \square \)

Based on the three lemmas above, the proofs of Theorems 5.1 and 5.2 are immediate. Indeed, by combining (G.6) and (G.13), we obtain the convergence rate of the primal optimality gap \( P(z_t) - P^* \), namely

\[
P(z_t) - P^* \leq \frac{\bar{L} D_{\omega_t}(u^*, u_0)}{A_t} + \frac{\sum_{i=0}^t A_i \delta_i}{A_t}. \tag{G.16}
\]

In addition, if we let the sequence \( \{v_t\}_{t \geq 0} \) in Lemma G.3 satisfy (5.21), then from Lemma 5.1, we know that \((\hat{h}(u_t), \hat{\nabla} h(u_t)) = (\bar{\Psi}(u_t, v_t), \nabla_u \bar{\Psi}(u_t, v_t)) \) is a \((\delta_t, \bar{L})\)-FOA of \( h \) at \( u_t \) with \( \bar{L} = 2L_h \).
Therefore, we can combine (G.6) and (G.12) to obtain the convergence rate of the duality gap $\Delta(z_t, \pi_t) = P(z_t) - \Xi(\pi_t)$ (cf. (5.6)), namely

$$
\bar{\Delta}(z_t, \pi_t) \leq \frac{2L_h \Omega_{z_k}(u_0)}{A_t} + \sum_{i=0}^{t} A_{t_i} \delta_{t_i}.
$$

(G.17)

Now, to show Theorem 5.1, we can simply substitute the values of $\{A_{t_i}\}_{t \geq 0}$ when $\mu = 0$ (cf. (G.8)) into (G.16) and (G.17) above. We can also show Theorem 5.2 in the same way, except that the values of $\{A_{t_i}\}_{t \geq 0}$ when $\mu > 0$ (cf. (G.8)) are substituted.

**Appendix H: Proof of Lemma 5.2.** The first-order optimality condition of (5.34) yields

$$
0 \in \partial(\zeta + t\mu)(u^+) + \nabla h(\pi) + e(\pi) + \mu \nabla \omega_{\mu t}(u^+) - \overline{G}
$$

$$
\iff \overline{G} - \nabla h(\pi) - e(\pi) \in \partial(\zeta + t\mu)(u^+) + \mu \nabla \omega_{\mu t}(u^+)
$$

$$
\iff [\xi := \overline{G} + \nabla h(u^+) - \nabla h(\pi) - e(\pi)] \in \nabla h(u^+) + \partial(\zeta + t\mu)(u^+) + \mu \nabla \omega_{\mu t}(u^+) = \partial P(u^+),
$$

By the $\mu$-strong convexity of $P$ on $\mathcal{U}$, we have

$$
P^* = P(u^*) \geq P(u^+) + \langle \xi, u^* - u^+ \rangle + (\mu/2)\|u^* - u^+\|^2 \geq P(u^+) - \|\xi\|^2/(2\mu). \quad \text{(H.1)}
$$

On the other hand, we have $\|\nabla h(u^+) - \nabla h(\pi)\|_\star \leq L_h \|u^+ - \pi\| = \|G\|$, and hence

$$
\|\xi\|^2 \leq 3(\|\overline{G}\|^2 + \|\nabla h(u^+) - \nabla h(\pi)\|^2 + \|e(\pi)\|^2) \leq 3(\|\overline{G}\|^2 + \|G\|^2 + \|e(\pi)\|^2). \quad \text{(H.2)}
$$

By combining (H.1) and (H.2), we arrive at (5.36).

**Appendix I: Proof of Theorem 7.1.** Using the same arguments that lead to (4.25), we have that for any $k \geq 1$,

$$
\mathbb{E}[Q^\lambda(x_{k+1}; x_k) | x_k] \leq q^\lambda(x_k) + 2\rho R_{\gamma}(\omega_{\gamma}) + \eta = Q^\lambda(\text{prox}(q, x_k, \lambda); x_k) + 3\eta/2, \quad \text{(I.1)}
$$

where the equality follows from (4.2) and $\rho = \eta/(4R_{\gamma}(\omega_{\gamma}))$ (cf. Algorithm 3). By the $(2\lambda)^{-1}$-strong-convexity of $Q^\lambda(\cdot; x_k)$ on $\mathcal{X}$, we have

$$
Q^\lambda(x_k; x_k) - Q^\lambda(\text{prox}(q, x_k, \lambda); x_k) \geq (4\lambda)^{-1} \|\text{prox}(q, x_k, \lambda) - x_k\|^2.
$$

Combining (I.1) and (I.2), we have

$$
\mathbb{E}[Q^\lambda(x_{k+1}; x_k) | x_k] + (4\lambda)^{-1} \|\text{prox}(q, x_k, \lambda) - x_k\|^2 \leq Q^\lambda(x_k; x_k) + 3\eta/2. \quad \text{(I.3)}
$$

Since $Q^\lambda(x_{k+1}; x_k) = q(x_{k+1}) + \lambda^{-1} D_{\omega_k}(x_{k+1}; x_k) \geq q(x_{k+1})$ and $Q^\lambda(x_k; x_k) = q(x_k)$, we have

$$
\mathbb{E}[q(x_{k+1}) | x_k] + (4\lambda)^{-1} \|\text{prox}(q, x_k, \lambda) - x_k\|^2 \leq q(x_k) + 3\eta/2. \quad \text{(I.4)}
$$
If we telescope (I.4) over $k = 1, \ldots, K$, then we have
\[
E[q(x_{K+1})] + (4\lambda)^{-1}\sum_{k=1}^{K}E[\|\text{prox}(q, x_k, \lambda) - x_k\|^2] \leq q(x_1) + 3K\eta/2. \tag{I.5}
\]

Using the definition of $x_{\text{out}}$ in Algorithm 3 and the fact that $E[q(x_{K+1})] \geq q^*$, we have
\[
E[\|\text{prox}(q, x_{\text{out}}, \lambda) - x_{\text{out}}\|^2] = (1/K)\sum_{k=1}^{K}E[\|\text{prox}(q, x_k, \lambda) - x_k\|^2] \\
\leq 4\lambda(q(x_1) - q^*)/K + 6\lambda\eta. \tag{I.6}
\]

Taking square root on both sides of (I.6) and using the choices of $\eta$ and $K$ in (7.5), we have
\[
\sqrt{E[\|\text{prox}(q, x_{\text{out}}, \lambda) - x_{\text{out}}\|^2]} \leq \sqrt{4\lambda(q(x_1) - q^*)/K + 6\lambda\eta} \leq \varepsilon\lambda/\beta X. \tag{I.7}
\]

Since the function $a \mapsto \sqrt{a}$ is concave on $\mathbb{R}_+$, we have
\[
E[\|\text{prox}(q, x_{\text{out}}, \lambda) - x_{\text{out}}\|] \leq \sqrt{E[\|\text{prox}(q, x_{\text{out}}, \lambda) - x_{\text{out}}\|^2]}. \tag{I.8}
\]

Combining (I.7) and (I.8), we complete the proof.

References

[1] Ben-Tal A, Margalit T, Nemirovski A (2001) The ordered subsets mirror descent optimization method with applications to tomography. *SIAM J. Optim.* 12(1):79–108.

[2] Bernhard P, Rapaport A (1995) On a theorem of danskin with an application to a theorem of von neumann-sion. *Nonlinear Anal.* 24(8):1163–1181.

[3] Candès E, Strohmer T, Voroninski V (2013) Phaselift: exact and stable signal recovery from magnitude measurements via convex programming. *Communications on Pure and Applied Mathematics* 66(8):1241–1274.

[4] Chen Y, Chi Y, Goldsmith AJ (2015) Exact and stable covariance estimation from quadratic sampling via convex programming. *IEEE Trans. Inf. Theory* 61(7):4034–4059.

[5] Chen Y, Lan G, Ouyang Y (2017) Accelerated schemes for a class of variational inequalities. *Math. Program.* 165(1):113–149.

[6] Danskin JM (1967) *The theory of Max-Min and its application to weapons allocation problems* (Springer-Verlag).

[7] Davis D, Drusvyatskiy D (2019) Stochastic model-based minimization of weakly convex functions. *SIAM J. Optim.* 29(1):207–239.

[8] Davis D, Drusvyatskiy D, MacPhee KJ (2018) Stochastic model-based minimization under high-order growth. [http://www.optimization-online.org/DB_HTML/2018/07/6690.html](http://www.optimization-online.org/DB_HTML/2018/07/6690.html).

[9] Davis D, Grimmer B (2019) Proximally guided stochastic subgradient method for nonsmooth, nonconvex problems. *SIAM J. Optim.* 29(3):1908–1930.
[10] Devolder O, Glineur F, Nesterov Y (2013) First-order methods with inexact oracle: the strongly convex case. CORE Discussion Paper (2013/16).

[11] Devolder O, Glineur F, Nesterov Y (2014) First-order methods of smooth convex optimization with inexact oracle. *Math. Program.* 146:37–75.

[12] Juditsky A, Nemirovski A (2012) First-order methods for nonsmooth convex large-scale optimization, I: General purpose methods. *Optimization for Machine Learning.* 121–148 (MIT Press).

[13] Kong W, Monteiro RDC (2019) An accelerated inexact proximal point method for solving nonconvex-concave min-max problems. arXiv:1905.13433.

[14] Kruger AY (2003) On Fréchet subdifferentials. *J. Math. Sci.* 116(3):3325–3358.

[15] Lin T, Jin C, Jordan MI (2019) On gradient descent ascent for nonconvex-concave minimax problems. arXiv:1906.00331.

[16] Lu S, Tsaknakis I, Hong M, Chen Y (2019) Hybrid block successive approximation for one-sided non-convex min-max problems: Algorithms and applications. arXiv:1902.08294.

[17] Nemirovski A (2005) Prox-method with rate of convergence $O(1/t)$ for variational inequalities with lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM J. Optim.* 15(1):229–251.

[18] Nesterov Y (1983) A method of solving a convex programming problem with convergence rate $o(1/k^2)$. *Soviet Mathematics Doklady* 27(2):372–376.

[19] Nesterov Y (2004) *Introductory Lectures on Convex Optimization: A Basic Course* (Springer).

[20] Nesterov Y (2005) Smooth minimization of non-smooth functions. *Math. Program.* 103(1):127–152.

[21] Nesterov Y (2013) Gradient methods for minimizing composite functions. *Math. Program.* 140(1):125–161.

[22] Nouiehed M, Sanjabi M, Huang T, Lee JD, Razaviyayn M (2019) Solving a class of non-convex min-max games using iterative first order methods. *Advances in Neural Information Processing Systems 32*, 14934–14942.

[23] Ostrovskii DM, Lowy A, Razaviyayn M (2020) Efficient search of first-order nash equilibria in nonconvex-concave smooth min-max problems. arXiv:2002.07919.

[24] Peypouquet J (2015) *Convex optimization in normed spaces : theory, methods and examples* (Springer).

[25] Rafique H, Liu M, Lin Q, Yang T (2018) Non-convex min-max optimization: Provable algorithms and applications in machine learning. arXiv:1810.02060.

[26] Rockafellar RT (1970) *Convex analysis* (Princeton University Press).

[27] Schmidt M, Roux NL, Bach FR (2011) Convergence rates of inexact proximal-gradient methods for convex optimization. *Proc. NIPS*, 1458–1466.
[28] Sinha A, Namkoong H, Duchi J (2017) Certifying some distributional robustness with principled adversarial training. arXiv:1710.10571.

[29] Sion M (1958) On general minimax theorems. Pacific J. Math. 8(1):171–176.

[30] Thekumparampil KK, Jain P, Netrapalli P, Oh S (2019) Efficient algorithms for smooth minimax optimization. Proc. NIPS, 12680–12691.

[31] Tseng P (2008) On accelerated proximal gradient methods for convex-concave optimization. Technical report, University of Washington, Seattle.

[32] Zhang T (2004) Solving large scale linear prediction problems using stochastic gradient descent algorithms. Proc. ICML, 919–926.

[33] Zhao R (2019) Optimal stochastic algorithms for convex-concave saddle-point problems. arXiv:1903.01687.