SPATIAL AND TEMPORAL DYNAMICS OF AN ALMOST PERIODIC REACTION-DIFFUSION SYSTEM FOR WEST NILE VIRUS

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Abstract. In current paper, we put forward a reaction-diffusion system for West Nile virus in spatial heterogeneous and time almost periodic environment with free boundaries to investigate the influences of the habitat differences and seasonal variations on the propagation of West Nile virus. The existence, uniqueness and regularity estimates of the global solution for this disease model are given. Focused on the effects of spatial heterogeneity and time almost periodicity, we apply the principal Lyapunov exponent $\lambda(t)$ with time $t$ to get the initial infected domain threshold $L^*$ to analyze the long-time dynamical behaviors of the solution for this almost periodic West Nile virus model and give the spreading-vanishing dichotomy regimes of the disease. Especially, we prove that the solution for this West Nile virus model converges to a time almost periodic function locally uniformly for $x$ in $\mathbb{R}$ when the spreading occurs, which is driven by spatial differences and seasonal recurrence. Moreover, the initial disease infected domain and the front expanding rate have momentous impacts on the permanence and extinction of the epidemic disease. Eventually, numerical simulations identify our theoretical results.

1. Introduction

West Nile virus (WNv) causes mosquito-borne epidemic diseases seriously threatening people’s lives by invading people’s nervous system. Since the West Nile virus broke out in New York in 1999, it has become endemic all over the United States. It was estimated that 7 million human infections had occurred from 1999 to 2019, making it the main mosquito-borne virus infectious disease in America ([2]). In recent years,

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the infection of WNv has spread from North America to Europe, bringing about considerable deaths. In order to supply feasible measures to prevent and control the propagations of WNv, it is pretty worthwhile to apply mathematical models to investigate the long-time spreading dynamics of the WNv.

West Nile virus spreads mainly through mosquitoes as the vectors and biting birds as the hosts. Concentrated on the temporal transmission of the WNv, there have been many works by ordinary differential equations to explore the existence and stability of the equilibrium, and introduce the basic reproduction number as a threshold value to study the transmission dynamics of WNv, such as [3, 4, 5, 6, 7] and references therein.

The free migration movements of the infected bird populations and mosquitoes populations are usually random, so the spatial diffusion term should be in consideration. Therefore, only using ordinary differential systems to describe the spatial propagation of the West Nile virus is no more suitable. In view of the spatial heterogeneity, Allen et al. [8] studied the following SIS reaction-diffusion model in 2008,

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_S \Delta S - \beta(x) SI + \gamma(x) I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial I}{\partial t} &= d_I \Delta I + \beta(x) SI - \gamma(x) I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial S}{\partial n} &= \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align*}
\]

where domain \( \Omega \in \mathbb{R}^k (k \geq 1) \) is bounded with smooth boundary \( \partial \Omega \); \( S(x, t) \) and \( I(x, t) \) are the population densities of susceptible and infected individuals at position \( x \) and time \( t \); positive constants \( d_S \) and \( d_I \) represent diffusion rates for the susceptible and infected populations; \( \beta(x) \) is the disease transmission rate at position \( x \) and \( \gamma(x) \) is the disease recovery rate at position \( x \), both of which are positive Hölder continuous functions, respectively. They studied the effects of the heterogeneous media and the individual movement of susceptible and infected populations on the permanence and eradication of the disease and obtained the global dynamics of model (1.1) by basic reproduction number \( R_0 \). Lewis et al. [9] investigated the spreading speed of the WNv by a reaction-diffusion system. Maidana and Yang [10] used the traveling wave solution of the WNv model to study the spatial spreading of the disease across North America.

The infected boundaries driven by birds and mosquitoes migrating from one habitat to another change with respect to time. Thus, applying the fixed studying domain is not appropriate. Free boundaries conditions have largely attracted lots of concentrations recently and they are frequently used in biological mathematical models, for instance, [11, 14, 12, 13, 15, 16, 17]. In view of the moving infected boundaries, Lin and Zhu [18] investigated a reaction-diffusion system to explore the spatial spreading of WNv using free boundaries to represent the disease spreading fronts. Tarboush et al. [19] studied a WNv model which incorporates a Partial differential equation and an ordinary differential equation with moving boundaries. Cheng and Zheng [20] considered a reaction-advection-diffusion WNv model with double free boundaries and studied the influence of advection term on the boundary asymptotic spreading speeds.

In reality, the outbreak of the disease is not always caused by single factor. Apart from the spatial heterogeneity, the temporal heterogeneity caused by alternations of seasonality is also a significant factor in influencing the propagation of the disease. Peng and Zhao [21] investigated the model (1.1) in a time-periodic heterogeneous environment which the transmission rate \( \beta(x, t) \) and recovery rate \( \gamma(x, t) \) are periodic for time \( t \). Zhang and Wang [22] studied a diffusive SIR time periodic system and investigated the spatial dynamics of this epidemic model. Shan et al. [23] investigated a periodic compartmental WNv model with time delay
and obtained the effects of seasonal recurrent phenomena on the spreading and recurrence of the epidemic disease.

From a biological view, the effects of the alternation of seasons on the disease transmission rate, disease recovery rate and the disease death rate are not same. Thus, these parameter periods for the epidemic model are usually different. Therefore, we had to look for more reasonable mathematical model. Considering the differences of the periodic coefficients, it is significant to study the time almost periodic system. Shen and Yi [24] studied the convergence of the positive solution for almost periodic models of Fisher and Kolmogorov type. Huang and Shen [25] investigated the spreading dynamics of KPP models in time almost periodic and space periodic environment and gave the estimates of the spreading speed. Wang and Zhao [26] discussed the basic reproduction ratio \( R_0 \) and obtained its computing formula for almost periodic compartmental ordinary differential epidemic models. Wang et al. [27] investigated a reaction-diffusion SIS model in a time almost periodic environment and gave the estimates of the spreading speed. Wang and Zhao [26] discussed the persistence or extinction of the solution for epidemic model. Recently, Qiang et al. [28] studied a nonlocal reaction-diffusion model with time delay in almost periodic media and discussed the threshold dynamics using the upper Lyapunov exponent.

However, there are few studies on mosquito-borne diseases using almost periodic system. For the sake of better exploring the mechanisms of the disease outbreak and more reasonably describing the transmission rules of WNv, almost periodic mathematical biology models incorporate spatial heterogeneity with time almost periodicity should be vitally considered to study the propagation of WNv. Motivated by the previous rules of WNv, almost periodic mathematical biology models incorporate spatial heterogeneity with time better exploring the mechanisms of the disease outbreak and more reasonably describing the transmission of the solution for epidemic model. Recently, Qiang et al. [28] studied a nonlocal reaction-diffusion model with time delay in almost periodic media and discussed the threshold dynamics using the upper Lyapunov exponent.

\[
\begin{align*}
U_t &= D_1 U_{xx} + \alpha_1(x,t) \beta \frac{N_2 - U}{N_1} V - \gamma(x,t) U, \quad g(t) < x < h(t), t > 0, \\
V_t &= D_2 V_{xx} + \alpha_2(x,t) \beta \frac{N_2 - V}{N_1} U - d(x,t) V, \quad g(t) < x < h(t), t > 0, \\
U(x,t) &= V(x,t) = 0, \quad x = h(t) \text{ or } x = g(t), t > 0, \\
h(0) &= h_0, \quad h'(t) = -\mu U_x(h(t),t), \quad t > 0, \\
g(0) &= -h_0, \quad g'(t) = -\mu U_x(g(t),t), \quad t > 0, \\
U(x,0) &= U_0(x), V(x,0) = V_0(x), \quad -h_0 \leq x \leq h_0,
\end{align*}
\]

where \( U(x,t) \) and \( V(x,t) \) are the densities of infected bird populations and mosquito populations at location \( x \) and time \( t \), respectively; \( N_1 \) and \( N_2 \) are the total population capacities of the birds and mosquitoes; \( D_1 \) and \( D_2 \) are the diffusion rates of the birds and mosquitoes, respectively; \( \alpha_1(x,t) \) and \( \alpha_2(x,t) \) are the WNv transmission probabilities per bite to birds and mosquitoes at location \( x \) and time \( t \); \( \beta \) is the biting rate of mosquitoes to birds; \( \gamma(x,t) \) is the recovery rate of birds from infection at location \( x \) and time \( t \); \( d(x,t) \) is the death rate of the mosquitoes at location \( x \) and time \( t \). The moving region \( (g(t),h(t)) \) is the infected domain of WNv. Meanwhile, we suppose that the double free boundaries submit to classical Stefan conditions obeying the Fick’s first law, that is, \( g'(t) = -\mu U_x(g(t),t) \) and \( h'(t) = -\mu U_x(h(t),t) \), where \( \mu \) is positive. Moreover, we assume that \( \alpha_1(x,t), \alpha_2(x,t), \gamma(x,t), d(x,t) \in C^{2+\alpha_0,1+\frac{\alpha_0}{2}}(\mathbb{R} \times [0, \infty)) \) are positive bounded functions for some \( \alpha_0 \in (0, 1) \), and uniformly almost periodic in \( t \). What is more, \( \alpha_1(x,t), \alpha_2(x,t), \gamma(x,t), d(x,t) \) have positive super and lower bound.
In order to simplify the number of parameters in this model, denote

\[(1.3) \quad a_1(x, t) := \frac{\alpha_1(x, t)\beta}{N_1}, a_2(x, t) := \frac{\alpha_2(x, t)\beta}{N_1}, d_1(x, t) := \gamma(x, t), d_2(x, t) := d(x, t),\]

then \(a_1(x, t), a_2(x, t), d_1(x, t), d_2(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R} \times [0, \infty))\) for any \(\alpha \in (0, \alpha_0).\)

On the basis of the previous simplifications and assumptions, we are going to investigate the following simplified WNv system,

\[(1.4) \quad \begin{align*}
U_t &= D_1 U_{xx} + a_1(x, t) (N_1 - U) V - d_1(x, t) U, \quad g(t) < x < h(t), \quad t > 0, \\
V_t &= D_2 V_{xx} + a_2(x, t) (N_2 - V) U - d_2(x, t) V, \quad g(t) < x < h(t), \quad t > 0, \\
U(x, t) &= V(x, t) = 0, \quad x = h(t) \text{ or } x = g(t), \quad t > 0, \\
h(0) &= h_0, \quad h'(t) = -\mu U_x(h(t), t), \quad t > 0, \\
g(0) &= -h_0, \quad g'(t) = -\mu U_x(g(t), t), \quad t > 0, \\
U(x, 0) &= U_0(x), \quad V(x, 0) = V_0(x), \quad -h_0 \leq x \leq h_0.
\end{align*}\]

For the convenience of studying, we make the following assumptions about the initial functions \(U_0\) and \(V_0,\)

\[(1.5) \quad \begin{align*}
U_0(x) &\in C^2([-h_0, h_0]), \quad U_0(\pm h_0) = 0, \quad 0 < U_0(x) \leq N_1 \text{ in } (-h_0, h_0), \\
V_0(x) &\in C^2([-h_0, h_0]), \quad V_0(\pm h_0) = 0, \quad 0 < V_0(x) \leq N_2 \text{ in } (-h_0, h_0).
\end{align*}\]

In this paper, our primary purpose is to research a reaction-diffusion WNv model with moving infected regions \((g(t), h(t))\) in the spatial heterogeneous and time almost periodic media, and discuss the effects of the spatial heterogeneity and time almost periodicity on the spreading and vanishing of the epidemic disease. In view of the biological reality, this WNv model \((1.4)\) is first proposed to incorporate the spatial heterogeneity with time almost periodicity in studying epidemic disease. We first give the global existence, uniqueness and regularity estimates of the solution, the method of which is not trivially similar to homogeneous WNv models (See Theorems 2.1, 3.1 and 3.4). In view of spatial variants with seasonal changes, the virus transmission rate \((\alpha_1(x, t), \alpha_2(x, t))\) between mosquitoes and birds, birds recovery rate \((\gamma(x, t))\) from infection and mosquitoes death rate \((d(x, t))\) all depend on location \(x\) and time \(t\), which is more consistent with the disease spreading reality. Moreover, since the coefficients are heterogeneous and the boundary is moving, we introduce the principal Lyapunov exponent \(\lambda(t)\) with respect to time \(t\) (See section 4) to get the initial infected domain \(L^*\) as a threshold value and we obtain the spreading-vanishing dichotomy regimes of West Nile virus (See Theorem 2.2) using it. We prove that the eventually infected domain is no more than \(2L^*\) when the vanishing occurs. Importantly, we prove that the solution for system \((1.4)\) converges to a time almost periodic function for fixed \(x\) in bounded subsets of \(\mathbb{R}\) when the spreading occurs, whose asymptotic behavior is very different from other homogeneous WNv models, the solution of which converges to a positive constant equilibrium, such as \([3, 9, 13]\). Our results show that the spatial heterogeneity and temporal almost periodicity driven by spatial differences and seasonal recurrence lead to the cyclic appearance of the cases of infection. Moreover, the initial West Nile virus infected domain and the front expanding rate have momentous impacts on the permanence and extinction of the epidemic disease. These results are useful for people to understand the spreading dynamics of the disease with spatial and seasonal diversity and implement measures to prevent and control the transmission of the infectious diseases.

The rest of the paper is arranged as follows. In section 2 we first prepare some preliminaries and assumptions, then present the main results. In section 3 we provide a detailed proof of the global existence,
uniqueness and regularity estimates of the solution for problem (1.4) in the time almost periodic and spatial heterogeneous environment which is not trivial. In section 4 considering the spatial heterogeneity and time almost periodicity, we introduce the principal Lyapunov exponent and obtain some vital properties of this threshold value. In section 5 we apply the principal Lyapunov exponent to obtain the sufficient conditions for persistence and eradication of the epidemic disease and explore the long-time asymptotic behaviors of the solution for heterogeneous system (1.4) by applying a different method from other homogeneous and periodic WNv models. In section 6 we make several numerical simulations to identify our theory results, then present some discussions and biological meanings about our analytical results for WNv model.

2. Preliminaries and Main Results

In the section, we make some preparations and display our main results.

2.1. Preliminaries.

Firstly, we recall several definitions about almost periodic function from Section 2.1 of [24] or Section 3 of [29].

Definition 2.1. (i) A function $f \in C(\mathbb{R}, \mathbb{R}^k)(k \geq 1)$ is called an almost periodic function if for any $\epsilon > 0$, the set

$$T(f, \epsilon) := \{ \tau \in \mathbb{R} \mid |f(t + \tau) - f(t)| < \epsilon \text{ for any } t \in \mathbb{R} \}$$

is relatively dense in $\mathbb{R}$. We say a matrix function $A(x,t)$ is almost periodic if every entry of it is almost periodic.

(ii) A function $f \in C(\mathbb{R} \times \mathbb{R})$ is uniformly almost periodic in $t$ if $f(x, \cdot)$ is almost periodic for every $x \in \mathbb{R}$, and $f$ is uniformly continuous on $E \times \mathbb{R}$ for any compact set $E \subset \mathbb{R}$.

(iii) A function $f(x,t,u,v) \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^k)(m,n,k \geq 1)$ is uniformly almost periodic in $t$ with $x \in \mathbb{R}$ and $(u,v)$ in bounded sets if $f$ is uniformly continuous for $t \in \mathbb{R}, x \in \mathbb{R}$ and $(u,v)$ in bounded sets and $f(x,t,u,v)$ is almost periodic in $t$ for every $x \in \mathbb{R}, u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Definition 2.2. (iv) The hull of a uniformly almost periodic matrix $A(x,t)$ is defined by

$$H(A) = \{ B(\cdot, \cdot) \mid \exists t_n \to \infty, \text{such that } B(x,t + t_n) \to A(x,t) \text{ uniformly for } t \in \mathbb{R}, \quad x \text{ in bounded sets} \}.$$ 

(v) The hull of a uniformly almost periodic matrix $F(x,t,u,v)$ is defined by

$$H(F) = \{ G(\cdot, \cdot, \cdot, \cdot) \mid \exists t_n \to \infty, \text{such that } G(x,t + t_n,u,v) \to F(x,t,u,v) \text{ uniformly for } t \in \mathbb{R}, (x,u,v) \text{ in bounded sets} \}.$$ 

In general, the system (1.4) can be seen as the special form of the following system,

\begin{equation}
\begin{aligned}
u_t &= D_1u_{xx} + f_1(x,t,u,v), & g(t) < x < h(t), & t > 0, \\
v_t &= D_2v_{xx} + f_2(x,t,u,v), & g(t) < x < h(t), & t > 0, \\
u(x,t) &= v(x,t) = 0, & x = h(t) \text{ or } x = g(t), & t > 0, \\
h(0) &= h_0, & h'(t) = -\mu u_x(h(t),t), & t > 0, \\
g(0) &= -h_0, & g'(t) = -\mu u_x(g(t),t), & t > 0, \\
u(x,0) &= u_0(x), & v(x,0) = v_0(x), & -h_0 \leq x \leq h_0,
\end{aligned}
\end{equation}

(2.1)
where initial data \((u_0, v_0)\) satisfy \([1,5]\), and \(f_i(x, t, u, v)\) satisfies the following conditions for \(i = 1, 2\).

\(H1\) \(f_i(x, t, u, v) \in C^1(\mathbb{R}^4)\), \(Df_i(x, t, u, v) = (\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial u}, \frac{\partial f_i}{\partial v})\) is bounded for \((x, t) \in \mathbb{R} \times \mathbb{R}\) and \((u, v)\) in bounded sets.

\(H2\) There exist positive constants \(M\) and \(N\) such that

\[
\sup_{x \in \mathbb{R}, t \in \mathbb{R}, u \geq M, v \in \mathbb{R}} f_1(x, t, u, v) < 0, \quad \sup_{x \in \mathbb{R}, t \in \mathbb{R}, u \in \mathbb{R}, v \geq N} f_2(x, t, u, v) < 0,
\]

\[
\sup_{x \in \mathbb{R}, t \in \mathbb{R}, u \geq 0, v \geq 0} \frac{\partial f_1}{\partial u}(x, t, u, v) < 0, \quad \sup_{x \in \mathbb{R}, t \in \mathbb{R}, u \geq 0, v \geq 0} \frac{\partial f_2}{\partial v}(x, t, u, v) < 0.
\]

\(H3\) \(f_i\) and \(Df_i\) are uniformly almost periodic in \(t \in \mathbb{R}\) with \(x \in \mathbb{R}\) and \((u, v)\) in bounded sets.

\(H4\) Let

\[
F(x, t, u, v) = \begin{pmatrix} f_1(x, t, u, v) \\ f_2(x, t, u, v) \end{pmatrix}.
\]

For any given sequences \(\{x_n\} \subset \mathbb{R}\) and \(\{G_n\} \subset H(F)\), there exist subsequences \(\{x_{n_k}\} \subset \{x_n\}\) and \(\{G_{n_k}\} \subset \{G_n\}\) such that \(\lim_{k \to \infty} G_{n_k}(x + x_{n_k}, t, u, v)\) exists for \(t \in \mathbb{R}\) uniformly and \((x, u, v)\) in bounded sets.

In this paper, we take \(G, \alpha, H, C, A, L > 0\) such that \(\lim_{k \to \infty} G_{n_k}(x + x_{n_k}, t, u, v)\) exists for \(t \in \mathbb{R}\) uniformly and \((x, u, v)\) in bounded sets.

Define matrix function \(A(x, t)\) by

\[
A(x, t) := \begin{pmatrix} \frac{\partial f_1(x, t, u, 0)}{\partial u} & \frac{\partial f_1(x, t, 0, v)}{\partial v} \\ \frac{\partial f_2(x, t, u, 0)}{\partial u} & \frac{\partial f_2(x, t, 0, v)}{\partial v} \end{pmatrix} = \begin{pmatrix} -d_1(x, t) & a_1(x, t)N_1 \\ a_2(x, t)N_2 & -d_2(x, t) \end{pmatrix}.
\]

Moreover, we assume that \(A(x, t)\) satisfies

\(H5\) There exists some \(L^* > 0\) such that \(\inf_{\hat{x} \in \mathbb{R}, L \geq L^*} \lambda(A(\cdot + \hat{x}, \cdot), L) > 0\).

Where \(\lambda(A, L)\) is the principal Lyapunov exponent and \(L^*\) is a constant dependent on \(a_i(x, t), N_i, d_i(x, t)\) for \(i = 1, 2\), which will be explicitly explained in section 4.

2.2. Main results.

Next we will present our main results for problem \([1,4]\). In Section 3, we will prove that \(h'(t) > 0\) and \(g'(t) < 0\) in \(t \in (0, +\infty)\). Therefore, we denote \(g_\infty := \lim_{t \to \infty} g(t)\), and \(h_\infty := \lim_{t \to \infty} h(t)\). Further, we can obtain that \(g_\infty \in [−\infty, 0]\) and \(h_\infty \in (0, \infty]\).

**Theorem 2.1** (Existence and uniqueness). Assuming any given initial functions \((U_0, V_0)\) satisfy \([1,5]\). For any \(\alpha \in (0, \alpha_0)\), there exists a time \(T\) such that the system \([1,4]\) admits a unique global solution \((U, V; g, h) \in (C^{2+\alpha,1+\frac{\alpha}{2}}([g(t), h(t)] \times (0, T]))^2 \times (C^{1+\alpha/2}((0, T]))^2\), where \(T\) is dependent on \(\alpha, h_0, ||U_0||_{C^2([-h_0, h_0])}\) and \(||V_0||_{C^2([-h_0, h_0])}\).

**Remark 2.1.** Actually, the solution for system \([1,4]\) uniquely exists for all \(t \in (0, \infty)\) (See Theorem 3.4).

In order to investigate the asymptotic dynamics of system \([1,4]\), we first introduce the following system,

\[
\begin{align*}
U_t &= D_1U_{xx} + a_1(x, t)(N_1 - U)V - d_1(x, t)U, \quad -\infty < x < \infty, \quad t > 0, \\
V_t &= D_2V_{xx} + a_2(x, t)(N_2 - V)U - d_2(x, t)V, \quad -\infty < x < \infty, \quad t > 0.
\end{align*}
\]
Then we get

**Theorem 2.2** (Spreading-vanishing dichotomy). Supposing (H1) – (H5) hold and the initial functions \((U_0, V_0)\) satisfy \([1,5]\). Let \((U, V; U_0, V_0, h_0)\) be the solution of \([1,4]\), for such \(L^*\) in (H5), the following spreading-vanishing dichotomy regimes hold:

Either

1. Vanishing: \(h_\infty - g_\infty \leq 2L^*\) and
   \[
   \lim_{t \to +\infty} U(x, t; U_0, V_0, h_0) = 0, \quad \lim_{t \to +\infty} V(x, t; U_0, V_0, h_0) = 0
   \]
   uniformly in \(x \in [g_\infty, h_\infty]\);

or

2. Spreading: \(h_\infty - g_\infty = \infty\) and
   \[
   \lim_{t \to +\infty} U(x, t; U_0, V_0, h_0) - U^*(x, t) = 0, \quad \lim_{t \to +\infty} V(x, t; U_0, V_0, h_0) - V^*(x, t) = 0
   \]
   locally uniformly for \(x\) in \(\mathbb{R}\), where \((U^*(x, t), V^*(x, t))\) is the unique positive time almost periodic solution of \([2,5]\).

**Theorem 2.3** (Spreading-vanishing threshold). Supposing (H1) – (H5) hold. For any given \(h(0), g(0)\) and the initial functions \((U_0, V_0)\) satisfying \([1,5]\). Let \((U, V; U_0, V_0, g, h)\) be the solution of \([1,4]\), for such \(L^*\) in (H5), the followings hold.

1. If \(\lambda(0) > 0\), then \(h(0) - g(0) \geq 2L^*\), further, \(h_\infty - g_\infty = \infty\), thus, the spreading occurs;
2. If \(\lambda(0) < 0\), then there exists a constant \(\mu^* \geq 0\) such that the spreading occurs when \(\mu > \mu^*\) and vanishing occurs when \(0 < \mu \leq \mu^*\).

**Remark 2.2.** The explicit explanation for principal Lyapunov exponent \(\lambda(t)\) can refer to section 4. The above theorem gives the sufficient conditions about the spreading and vanishing of the disease. The threshold constant \(L^*\) determines the uniform persistence or extinction of WNv by influencing the sign of the principal Lyapunov exponent (See section 5).

**Remark 2.3.** In his paper, we always suppose that (H5) holds. According to Theorem 2.3 and Lemma 5.1, considering the meaning of biology, we can explain this assumption in the sense that the living habitat at remote distance is in high-risk of infection by the disease.

3. **Existence and Uniqueness**

In this section, we will show the existence and uniqueness of the global solution for system \([1,4]\). Since the system \([1,4]\) can be regarded as a special case of the system \([2,1]\). We only need to give a explicit proof for system \([2,1]\). Although there are similar results about the solution for epidemic models with constant coefficients, the proofs of the \([2,1]\) in heterogeneous environment can not easily obtained by analogy. Therefore, we provide a detailed proof according to Theorem 1.1 (30).

**Theorem 3.1.** Assume that (H1) – (H4) hold. For any \(\alpha \in (0, \alpha_0)\) and any given \((u_0, v_0)\) satisfying \([1,5]\), there exists \(T > 0\) such that the system \([2,1]\) admits a unique solution \((u, v, g, h)\) in \((C^{1+\alpha, (1+\alpha)/2}(D_T))^2 \times (C^{1+\alpha/2}(0, T))^2\), where \(D_T = \{(x, t) \in \mathbb{R}^2 \mid x \in [g(t), h(t)], t \in [0, T]\}\), and \(T\) is only dependent on \(\alpha, h_0, \|u_0\|_{C^2([-h_0, h_0])}\) and \(\|v_0\|_{C^2([-h_0, h_0])}\).
Proof. We divide this proof into two steps.

**Step 1** The local existence of the solution for problem (2.1).

Let

\[ y = \frac{2x}{h(t) - g(t)} - \frac{h(t) + g(t)}{h(t) - g(t)}, \]

(3.1)

\[ m(y, t) = u(x, t), n(y, t) = v(x, t), \]

\[ f_1(y, t, m, n) = f_1(x, t, u, v), \]

\[ f_2(y, t, m, n) = f_2(x, t, u, v), \]

then direct calculation gives

\[ \frac{\partial y}{\partial x} = \frac{2}{h(t) - g(t)} := \sqrt{A(y, g(t), h(t))}, \]

\[ \frac{\partial y}{\partial t} = \frac{-y(h'(t) - g'(t)) + (h'(t) + g'(t))}{h(t) - g(t)} := B(y, g(t), g'(t), h(t), h'(t)), \]

(3.2)

and \((m, n)\) satisfy the following system,

\[
\begin{aligned}
    m_t - D_1 Am_y + Bm_y &= \tilde{f}_1(y, t, m, n), \quad y \in (-1, 1), 0 < t \leq T, \\
    n_t - D_2 An_y + Bn_y &= \tilde{f}_2(y, t, m, n), \quad y \in (-1, 1), 0 < t \leq T, \\
    m(\pm 1, t) = 0, n(\pm 1, t) = 0, \quad 0 < t \leq T, \\
    m(y, 0) = u_0(h_0y), n(y, 0) = v_0(h_0y), \quad y \in [-1, 1].
\end{aligned}
\]

(3.3)

Meanwhile, \(h(t)\) and \(g(t)\) satisfy

\[
\begin{aligned}
    h(0) &= h_0, h'(t) = -\mu \frac{2}{n(t) - g(t)} m_y(1, t), \quad 0 < t \leq T, \\
    g(0) &= -h_0, g'(t) = -\mu \frac{2}{h(t) - g(t)} m_y(-1, t), \quad 0 < t \leq T.
\end{aligned}
\]

(3.4)

Next, we will show the existence of solution for (3.3) with (3.4).

Let

\[ h^* = -\mu u_0'(h_0), g^* = -\mu u_0'(-h_0), \]

\[ T_0 = \min \left\{ 1, \frac{h_0}{2(2 + h^*)}, \frac{h_0}{2(2 - g^*)} \right\}, \]

(3.5)

\[ \Gamma = \{ h_0, h^*, g^*, ||h_0||_{C^2([-h_0, h_0])}, ||v_0||_{C^2([-h_0, h_0])} \} \]

\[ \Theta_T = \{ (h, g) \in (C^1([-0, T]))^2 \mid h(0) = h_0, g(0) = -h_0, h'(0) = h^*, g'(0) = g^*, \]

\[ ||h' - h^*||_{L^\infty} \leq 1, ||g' - g^*||_{L^\infty} \leq 1 \}, \]

then \(h^* > 0, g^* < 0\) and \(\Theta_T\) is a bounded closed convex subset of \((C^1([-0, T]))^2\) for any \(0 < T \leq T_0\).

Let

\[ \Theta_{T_0}^* = \{ (h, g) \in (C^1([0, T_0]))^2 \mid h(0) = h_0, h'(0) = h^*, g'(0) = g^*, ||h' - h^*||_{L^\infty} \leq 2, \]

\[ ||g' - g^*||_{L^\infty} \leq 2 \} . \]
For any given \((g,h) \in \Theta_T\), we can extend \(h\) and \(g\) such that \((g,h) \in \Theta_{T_0}\). Hence, if \((g,h) \in \Theta_T\), then \((g,h) \in \Theta_{T_0}\). And \(h(t)\) and \(g(t)\) satisfy

\[
|h(t) - h_0| \leq T_0\|h'|_\infty \leq T_0(2 + h^*) \leq \frac{h_0}{2},
\]

\[
|g(t) - (-h_0)| \leq T_0\|g'|_\infty \leq T_0(2 + g^*) \leq T_0(2 - g^*) \leq \frac{h_0}{2}
\]

for any \(t \in [0, T_0]\), then \(h(t) \in \left[\frac{h_0}{2}, \frac{3h_0}{2}\right]\) and \(g(t) \in \left[\frac{-3h_0}{2}, -\frac{h_0}{2}\right]\) in \([0, T_0]\). Hence, the transformations \((3.1)\) and \((3.2)\) are well defined for \(t \in [0, T_0]\). Applying the standard parabolic equation theory \((3.1)\), there exists a \(T_* \in (0, T_0]\) such that there is a unique solution \((\overline{m}(y, t), \overline{n}(y, t)) \in (C^{1+\alpha, \frac{1}{2+\alpha}}(\Delta_{T_*}))^2\) for problem \((3.3)\) with \(T_*\) dependent on \(\Gamma, \alpha, ||u_0||_\infty\) and \(||v_0||_\infty\). And there exists a positive constant \(C_1(\Gamma, \alpha, T_*, T_-1)\) such that

\[
||\overline{m}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\Delta_{T_*})} + ||\overline{n}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\Delta_{T_*})} \leq C_1(\Gamma, \alpha, T_*, T_-1),
\]

where \(\Delta_{T_*} = [-1, 1] \times [0, T_*]\). In view of the choice of \(\Gamma\) in \((3.5)\), \(T_*\) is only dependent on \(\Gamma\) and \(\alpha\). Hence,

\[
||\overline{m}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\Delta_{T_*})} + ||\overline{n}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\Delta_{T_*})} \leq C_1,
\]

with \(C_1\) dependent on \(\Gamma, \alpha\). Moreover, for \(0 < T < T_*\), it follows

\[
||\overline{m}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\mathcal{D}_T)} + ||\overline{n}||_{C^{1+\alpha, \frac{1}{2+\alpha}}(\mathcal{D}_T)} \leq C_1.
\]

Since \((\overline{m}(y, 0)\) and \(\overline{n}(y, 0)\)) are more than but not identically equal to \(0\) for \(y \in [-1, 1]\), \(\tilde{f}_i(y, t, 0, 0) \geq 0\), \(a_i N_i \geq 0\) on \([-1, 1] \times [0, T]\) and \(\tilde{f}_i(y, t, m, n)\) satisfies \((H1)\) for \(i = 1, 2\), by the maximum principle (resp. Positivity Lemma, \(32\)), then \((\overline{m}, \overline{n}) > 0\) for \((y, t) \in (-1, 1) \times (0, T]\).

Consider that the solution \((\overline{m}, \overline{n})\) depends continuously on the initial data \((g,h) \in \Theta_T\). Let

\[
\overline{h}(t) = h_0 - \mu \int_0^t \frac{2}{h(s) - g(s)} \overline{m}_y(1, s) ds, \quad \overline{g}(t) = -h_0 - \mu \int_0^t \frac{2}{h(s) - g(s)} \overline{m}_g(1, s) ds,
\]

for \(t \in [0, T]\), then \((\overline{g}, \overline{h})\) depend on \((g, h) \in \Theta_T\) and

\[
\overline{h}(0) = h_0, \overline{h}'(0) = h^*, \overline{h}'(t) > 0, \overline{g}(0) = -h_0, \overline{g}'(0) = g^*, \overline{g}'(t) < 0.
\]

Moreover, it follows

\[
\overline{h}'(t) \in C^2([0, T]), ||\overline{h}'(t)||_{C^2([0, T])} \leq C_2,
\]

\[
\overline{g}'(t) \in C^2([0, T]), ||\overline{g}'(t)||_{C^2([0, T])} \leq C_2,
\]

for some \(C_2\) dependent on \(\Gamma, \alpha\).

Define \(\mathcal{F} : \mathcal{D}_1 \times \mathcal{D}_2 \times \Theta_T \rightarrow C(\Delta_T) \times C(\Delta_T) \times (C^1([0, T]))^2\) by

\[
\mathcal{F}(m, n, g, h) = (\overline{m}, \overline{n}, \overline{g}, \overline{h}),
\]

where

\[
\mathcal{D}_1 = \{m \in C(\Delta_T)|m(y, 0) = u_0(h_0y), ||m - u_0||_{C(\Delta_T)} \leq 1\},
\]

\[
\mathcal{D}_2 = \{n \in C(\Delta_T)|n(y, 0) = v_0(h_0y), ||n - v_0||_{C(\Delta_T)} \leq 1\}.
\]

It is easily to see that \(\mathcal{F}(m, n, g, h) = (m, n, g, h)\) if and only if \((m, n, g, h)\) is the solution of \((3.3)\) with \((3.4)\).
Combining (3.7) and (3.9), it follows
\[
\begin{align*}
\|\cdot\|_{C^2([0,T])} & \leq \|\cdot\|_{C^2([0,T])} T^{\frac{\alpha}{2}} + \|\cdot\|_{C^2([0,T])} T^{\frac{\alpha}{2}} \\
& \leq C_1 T^{\frac{\alpha}{2}}
\end{align*}
\]
and
\[
\begin{align*}
\|\cdot\|_{C^2([0,T])} & \leq \|\cdot\|_{C^2([0,T])} T^{\frac{\alpha}{2}} \leq C_2 T^{\frac{\alpha}{2}}, \\
\|\cdot\|_{C^2([0,T])} & \leq \|\cdot\|_{C^2([0,T])} T^{\frac{\alpha}{2}} \leq C_2 T^{\frac{\alpha}{2}}.
\end{align*}
\]
Therefore, if we take \( T = \min \left\{ 1, \frac{b_0}{1 + b'}(\frac{b_0}{2(2-g')}, C', \frac{1}{2(1+\alpha)}), C', \frac{1}{2(1+\alpha)} \right\}, \) then \( F \) maps \( D_1 \times D_2 \times \Theta_T \) into itself.
Further, we can get that \( F \) is compact. Applying the Schauder fixed point theorem to \( F \), there exists a solution \( (m, n, g, h) \in D_1 \times D_2 \times \Theta_T \). Applying the Schauder estimates, \( (m, n, g, h) \in (C^{1+\alpha,\frac{\alpha}{2}}([-1,1] \times [0, T]))^2 \times C^{1+\alpha,\frac{\alpha}{2}}([0, T])^2 \). Hence, the problem (2.1) has a solution \( (u, v, g, h) \) in \( (C^{1+\alpha,\frac{\alpha}{2}}([g(t), h(t)] \times [0, T]))^2 \times C^{1+\alpha,\frac{\alpha}{2}}([0, T])^2 \).

**Step 2** The uniqueness of the solution for problem (2.1).
Assume that \((u_i, v_i, g, h)(i=1,2) \in D_1 \times D_2 \times \Theta_T\) are the two solutions of (2.1) for \( 0 < T \ll 1 \). Applying the strong maximum principle to \( u_i \), we can get that \( u_i(x,t) > 0 \) for \( x \in (g(t), h(t)) \) and \( 0 < t < T \). In view of \( u_i(t, h(t)) = 0, u_i(t, g(t)) = 0 \), it follows \( u_{ix}(t, h(t)) < 0, u_{ix}(t, g(t)) > 0 \) for \( i = 1, 2 \), which implies \( h'(t) > 0, g'(t) < 0 \) for \( t \in (0, T) \), then we can suppose that
\[
h_0 \leq h(t) \leq h_0 + 1, -h_0 - 1 \leq g(t) \leq -h_0
\]
for \( t \in [0, T] \), and
\[
u_i \leq \|u_0\|_{\infty} + 1, v_i \leq \|v_0\|_{\infty} + 1
\]
in \([g(t), h(t)] \times [0, T]\) for \( i = 1, 2 \).

As in transformations (3.1), take
\[
m_i(y, t) = u_i(x, t), n_i(y, t) = v_i(x, t)
\]
for \( i = 1, 2 \), then \((y, t) \in [-1,1] \times [0, T]\).

Let
\[
m = m_1 - m_2, n = n_1 - n_2, h = h_1 - h_2, g = g_1 - g_2,
\]
direct calculation gives the following system,
\[
\begin{align*}
m_t & = D_1 A_1(y, t)m_{yy} + B_1(y, t)m_y - a_1(y, t)m - \tilde{a}_1(y, t)n, \\
& = D_1 (A_1 - A_2)m_{yy} + (B_2 - B_1)m_{yy} + b_1(y, t)\frac{y(h-g)+h+g}{2}, \
& \quad y \in (-1, 1), 0 < t \leq T, \\
n_t & = D_2 A_1(y, t)n_{yy} + B_1(y, t)n_y - a_2(y, t)m - \tilde{a}_2(y, t)n, \\
& = D_2 (A_1 - A_2)n_{yy} + (B_2 - B_1)n_{yy} + b_2(y, t)\frac{y(h-g)+h+g}{2}, \
& \quad y \in (-1, 1), 0 < t \leq T, \\
m(\pm1, t) & = 0, n(\pm1, t) = 0, \\
m(0, t) & = n(0, t) = 0,
\end{align*}
\]
for \( i = 1, 2 \), and \( y \in (-1, 1), t \in [0, T] \).
with
\[ h'(t) = \mu \left( \frac{2}{h_2(t) - g_2(t)} m_{2y}(1, t) - \frac{2}{h_1(t) - g_1(t)} m_{1y}(1, t) \right), \]
\[ g'(t) = \mu \left( \frac{2}{h_2(t) - g_2(t)} m_{2y}(-1, t) - \frac{2}{h_1(t) - g_1(t)} m_{1y}(-1, t) \right), \]
for \( 0 < t \leq T, h(0) = 0, g(0) = 0, i = 1, 2, \) where
\[ A_i(y, t) = \frac{4}{(h_i(t) - g_i(t))^2}, \]
\[ B_i(y, t) = \frac{y(h_i'(t) - g_i'(t)) + (h_i'(t) + g_i'(t))}{h_i(t) - g_i(t)}, \]
\[ b_i(y, t) = \int_0^1 f_{ix}(t, H(h_1, h_2, g_1, g_2, s), m_2, n_2) ds, \]
\[ \tilde{a}_i(y, t) = \int_0^1 f_{in}(t, \frac{y(h_1 - g_1) + (g_1 + n_1)}{2}, m_2, n_2 + s(m_1 - m_2)) ds, \]
\[ a_i(y, t) = \int_0^1 f_{in}(t, \frac{y(h_1 - g_1) + (h_1 + g_1)}{2}, m_2 + s(m_1 - m_2), n_1) ds, \]
\[ H(h_1, h_2, g_1, g_2) = \frac{y(h_2 + s(h_1 - h_2) - (g_2 + s(g_1 - g_2)) + (h_2 + s(h_1 - h_2) + (g_2 + s(g_1 - g_2))))}{2}. \]
In view of \((H1) - (H5)\), we can get \( a_i, \tilde{a}_i, b_i \in L^\infty(\Delta_T) \) for \( i = 1, 2 \) with \( ||a_i(y, t)||_{L^\infty}, ||\tilde{a}_i(y, t)||_{L^\infty} \) and \( ||b_i(y, t)||_{L^\infty} \) dependent on \( h_0, ||u_0||_{L^\infty} \) and \( ||v_0||_{L^\infty} \). In view of \((3.7) - (3.11)\), applying \( L^p \) theory for parabolic equations and Sobolev imbedding theorem to system \((3.12)\), there are positive constants \( C_3, C_4 \) and \( C_5 \) which depend on \( \Gamma, \alpha \) such that
\[ ||m||_{C^{i+\alpha, \frac{1+\alpha}{2}}(\Delta_T)} + ||n||_{C^{i+\alpha, \frac{1+\alpha}{2}}(\Delta_T)} \leq C_3(D1)(||h_1 - g_1||^{-2} - h_2 - g_2)^2) m_{2yy}||c(\Delta_T)\]
\[ + ||(h_1 - g_1) ||_{C(\Delta_T)} + ||y(h_1' - g_1') + (h_1' + g_1') - (h_2' - g_2') \frac{y(h_2' - g_2') + (h_2' + g_2')} {h_2(t) - g_2(t)} m_{2yy}||c(\Delta_T)\]
\[ + C_3(D2)(||h_1 - g_1||^{-2} - h_2 - g_2)^2) m_{2yy}||c(\Delta_T) + ||(h_1' - g_1') + (h_1' + g_1') - (h_2' - g_2') \frac{h_1(t) - g_1(t)} {h_2(t) - g_2(t)} m_{2yy}||c(\Delta_T)\]
\[ \leq C_4(||h||_{C^i([0, T]))} + ||g||_{C^i([0, T]))} + ||h - g||_{C^i([0, T]))} + ||h + g||_{C^i([0, T]))} \]
\[ \leq C_5(||h||_{C^i([0, T]))} + ||g||_{C^i([0, T]))}. \]
Applying the proofs of \((5.4.3)\) and Theorem 5.5.4 \((33)\) to \( m_y(y, t) \) and \( n_y(y, t) \), without needing to expand \( m \) and \( n \) to a larger domain, we obtain that there exists a positive constant \( \tilde{C}_1 \) independent of \( T^{-1} \) such that
\[ ||m||_{C^{i+\alpha, \frac{1+\alpha}{2}}(\Delta_T)} + ||n||_{C^{i+\alpha, \frac{1+\alpha}{2}}(\Delta_T)} \leq \tilde{C}_1(||m||_{C^{i+\alpha, \frac{1+\alpha}{2}}(\Delta_T)}, \]
where \( [\cdot] \) is the Hölder seminorm. Therefore, according to \((3.14)\) and the above inequalities, it follows that
\[ ||m||_{C^{i+\frac{1}{2}}(\Delta_T)} \leq \tilde{C}_1 C_5(||h||_{C^i([0, T]))} + ||g||_{C^i([0, T]))}, \]
\[ ||n||_{C^{i+\frac{1}{2}}(\Delta_T)} \leq \tilde{C}_1 C_5(||h||_{C^i([0, T]))} + ||g||_{C^i([0, T]))}. \]
Combining (3.13) and (3.15), there is \( C_6 \) dependent on \( \Gamma, \alpha \) such that
\[
[h']_{C^2((0,T),\mathbb{R})} \leq \mu \left[ \frac{2}{h_1 - g_1} m_g(1,t) c_{\gamma^2((0,T),\mathbb{R})} \right] + \mu \left[ \frac{2}{h_1 - g_1} - \frac{2}{h_2 - g_2} \right] m_g(1,t) c_{\gamma^2((0,T),\mathbb{R})}
\]
\[
\leq C_6 (|h|_{C^1([0,T])} + |g|_{C^1([0,T])})
\]
and
\[
[g']_{C^2((0,T),\mathbb{R})} \leq \mu \left[ \frac{2}{h_1 - g_1} m_g(-1,t) c_{\gamma^2((0,T),\mathbb{R})} \right] + \mu \left[ \frac{2}{h_1 - g_1} - \frac{2}{h_2 - g_2} \right] m_g(-1,t) c_{\gamma^2((0,T),\mathbb{R})}
\]
\[
\leq C_6 (|h|_{C^1([0,T])} + |g|_{C^1([0,T])}).
\]
Since \( h(0) = h'(0) = 0 \) and \( g(0) = g'(0) = 0 \), then
\[
|h - h(0)|_{C^1([0,T])} \leq 2|h' - h'(0)|_{C^1([0,T])} T^\frac{\alpha}{2} \leq \hat{C}_5 |h|_{C^1([0,T])} T^\frac{\alpha}{2},
\]
\[
|g - g(0)|_{C^1([0,T])} \leq 2|g' - g'(0)|_{C^1([0,T])} T^\frac{\alpha}{2} \leq \hat{C}_6 |g|_{C^1([0,T])} T^\frac{\alpha}{2}.
\]
Therefore, if \( T \) is small enough, then \( h = 0 \) and \( g = 0 \), which implies \( m = 0 \) and \( n = 0 \). Thus, the local existence and uniqueness of the solution have been proved.

**Proof of Theorem 3.1** Let \( f_i \) be defined by (2.3) for \( i = 1, 2 \). In view that \( a_1(x,t), a_2(x,t), d_1(x,t), d_2(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]) \) for any \( \alpha \in (0,\alpha_0) \) and have positive upper and lower bound, it follows that \( f_i(x,\cdot, U, V) \in C^{1+\frac{\alpha}{2}}((0,T))(i = 1, 2) \) for the \( T \) in Theorem 3.1.

Make the transformations as (3.1), combining (3.7) with (3.9), then, it can be obtained that
\[
\tilde{f}_i(y,t) := f_i \left( \frac{y(h(t) - g(t)) + (h(t) + g(t))}{2}, t, m(y,t), n(y,t) \right) \in C^{1+\frac{\alpha}{2}}([-1,1] \times [0,T]).
\]
Using the Schauder theory for parabolic equations to system (3.3) and (3.4), we can get that \((m,n,g,h) \in (C^{2+\alpha,1+\frac{\alpha}{2}}([-1,1] \times (0,T)))^2 \times C^{1+\frac{\alpha}{2}}((0,T))^2 \). Since the system (1.4) can be regarded as the special case of (2.1), and satisfies all of the assumptions in Theorem 3.1, thus, the system (1.4) admits a unique solution \((U, V; g, h) \in (C^{2+\alpha,1+\frac{\alpha}{2}}(0,T)) \times (0,T))^2 \times C^{1+\frac{\alpha}{2}}((0,T))^2 \). Thus, the local existence and uniqueness of solution for system (1.4) are proved.

For the convenience of later proof, we provide the following Comparison Principle in order to estimate the boundness of \( U(x,t) \), \( V(x,t) \) for system (1.4) and the free boundaries \( x = g(t), x = h(t) \). The lemma is similar to Lemma 3.5 in [13].

**Lemma 3.2** (Comparison Principle). Assume that \( T \in (0, +\infty), \tilde{h}(t), \tilde{g}(t) \in C^1([0,T]), \tilde{U}, \tilde{V} \in C(D_T^+ \cap C^{2,1}(D_T^-)) \) with \( 0 < \tilde{U} \leq N_1, 0 < \tilde{V} \leq N_2 \) and \((\tilde{U}, \tilde{V}; \tilde{h}, \tilde{g})\) satisfies
\[
\begin{cases}
\tilde{U}_t - D_1 \tilde{U}_{xx} \geq a_1(x,t)(N_1 - \tilde{U}) - d_1(x,t) \tilde{U}, & \tilde{g}(t) < x < \tilde{h}(t), \ 0 < t < T, \\
\tilde{V}_t - D_2 \tilde{V}_{xx} \geq a_2(x,t)(N_2 - \tilde{V}) - d_2(x,t) \tilde{V}, & \tilde{g}(t) < x < \tilde{h}(t), \ 0 < t < T, \\
\tilde{U}(x,t) \geq 0, \tilde{V}(x,t) \geq 0, & x = \tilde{g}(t) \text{ or } \tilde{h}(t), \ 0 < t < T, \\
\tilde{h}(0) \geq h_0, \tilde{h}(t) \geq -\mu \tilde{U}_x(\tilde{h}(t), t), & 0 < t < T, \\
\tilde{g}(0) \leq -h_0, \tilde{g}(t) \leq -\mu \tilde{U}_x(\tilde{g}(t), t), & 0 < t < T, \\
\tilde{U}(x,0) \geq U_0(x), \tilde{V}(x,0) \geq V_0(x), & -h_0 \leq x \leq h_0,
\end{cases}
\]
then the solution \((U, V; g, h)\) of (1.4) satisfies
\[
\begin{align*}
\mathcal{U}(x, t) & \geq U(x, t), \quad \mathcal{V}(x, t) \geq V(x, t), \\
\mathcal{H}(t) & \geq h(t), \quad g(t) \geq \mathcal{G}(t), \quad \text{for } g(t) \leq x \leq h(t), t \in (0, T],
\end{align*}
\]
where \(D_T^* = \{(x, t) \in \mathbb{R}^2 \mid x \in (\mathcal{G}(t), \mathcal{H}(t)), t \in (0, T]\}.

\textbf{Remark 3.1.} If \((\mathcal{U}, \mathcal{V}, \mathcal{G}, \mathcal{H})\) satisfies the conditions of Lemma 3.2, then it is called the upper solution of (1.4). The corresponding lower solution can be similarly defined by reversing the above inequalities.

In order to extend the local solution of (1.4) to all \(t \in (0, \infty)\), according to Lemma 2.2 in [14] or Lemma 2.5 and Lemma 2.6 in [20], we give the rough estimates about the upper and lower bound of \(U(x, t), V(x, t), g'(t)\) and \(h'(t)\).

\textbf{Lemma 3.3.} Assume that \(T \in (0, +\infty)\). Let \((U, V; g, h)\) be a solution of (1.4) for \(t \in (0, T]\), then there exists a positive constant \(C > 0\) independent of \(T\) such that
\[
\begin{align*}
0 < U(x, t) & \leq N_1, 0 < V(x, t) \leq N_2, \quad \text{for } g(t) < x < h(t), 0 < t \leq T, \\
0 < h'(t) & \leq C, -C \leq g'(t) < 0, \quad \text{for } 0 < t \leq T.
\end{align*}
\]

Now we turn to show the global existence of the solution for problem (1.4).

\textbf{Theorem 3.4.} For any given initial data \((U_0, V_0)\) satisfying (1.5), the unique solution \((U, V; g, h)\) of system (1.4) exists for all \(t \in (0, \infty)\).

\textbf{Proof.} Now we aim to show that the solution for system (1.4) can extend to all \(t \in (0, \infty)\). If the maximal existence interval of the solution is \([0, T_{max})\), then we will show \(T_{max} = +\infty\). On the contrary, assuming that \(T_{max} < +\infty\). According to Lemma 3.3, we can get that \(U(x, t) \leq N_1, V(x, t) \leq N_2\) for \((x, t) \in [g(t), h(t)] \times [0, T_{max})\). Moreover, for the above positive constant \(C\) in Lemma 3.3 independent on \(T_{max}\), it holds that \(0 < h'(t), -g'(t) \leq C\), follows \(h_0 \leq h(t) \leq h_0 + CT_{max}\) and \(-h_0 - CT_{max} \leq g(t) \leq -h_0\) for \(t \in [0, T_{max})\).

As in transformation (3.1), take \(m(y, t) = U(x, t), n(y, t) = V(x, t)\). For any given \(T < T_{max}\), applying the \(L^p\) theory to (1.4), there exists a positive constant \(C_1 (\Gamma, N_1, N_2, T_{max})\) independent of \(T\) such that
\[
\|m\|_{W_p^{2,1}(\Delta_T)} + |n|_{W_p^{2,1}(\Delta_T)} \leq \tilde{C}_1 (\Gamma, N_1, N_2, T_{max}),
\]
thus, \((m, n) \in (W_p^{2,1}(\Delta_T^*))^2\) for \(p > \frac{3}{1-\alpha}\) and
\[
\begin{align*}
\|m\|_{W_p^{2,1}(\Delta_{T_{max}})} + |m|_{C^{1+\frac{2}{p}(\Delta_{T_{max}})} = \tilde{C}_1 (\Gamma, N_1, N_2, T_{max}),
\end{align*}
\]
In view of (3.13), we can get \((h, g) \in (C^{1+\frac{2}{p}}(0, T_{max})))^2\) and
\[
\|h\|_{C^{1+\frac{2}{p}}([0, T_{max}])} \leq C_2 (\Gamma, N_1, N_2, T_{max}), \quad \|g\|_{C^{1+\frac{2}{p}}([0, T_{max}])} \leq \tilde{C}_2 (\Gamma, N_1, N_2, T_{max}).
\]
Applying the Schauder theory to (1.4), we can get that
\[
(m, n) \in (C^{2+\alpha,1+\frac{2}{p}}([-1, 1] \times (0, T_{max})))^2,
\]
and it holds that
\[
\|m\|_{C^{2+\alpha,1+\frac{2}{p}}([-1, 1] \times [\varepsilon, T_{max}])} + \|n\|_{C^{2+\alpha,1+\frac{2}{p}}([-1, 1] \times [\varepsilon, T_{max}])} \leq \tilde{C}_3 (\epsilon, \Gamma, N_1, N_2, T_{max})
\]
where \(\tilde{C}_3\) is an independent constant on \(\varepsilon\).
for any small $0 < \varepsilon \ll T_{max}$. Therefore, $(U, V) \in (C^{2+\alpha,1+\beta}_x([g(t), h(t)] \times (0, T_{max}))^2$ and

$$
(3.22) \quad \|U\|_{C^{2+\alpha,1+\beta}_x([g(t), h(t)] \times [\varepsilon, T_{max}])} + \|V\|_{C^{2+\alpha,1+\beta}_x([g(t), h(t)] \times [\varepsilon, T_{max}])} \leq C_3 (\varepsilon, \Gamma, N_1, N_2, T_{max}).
$$

Thus, the system (1.4) admits a solution $(U, V; g, h)$ on $(0, T_{max})$. Take $\{T_n\} \subset (0, T_{max})$ such that $T_n \to T_{max}$ as $n \to \infty$. Let $T_n$ and $(U(x, T_n), V(x, T_n); g(T_n), h(T_n))$ be the initial state. By Theorem 3.1, there is a constant $t_0$ small enough dependent on

$$
g(T_n), g'(T_n), h(T_n), h'(T_n), \|U(\cdot, T_n)\|_{C^{2+\alpha,1+\beta}_x([g(T_n), h(T_n)])}, \|V(\cdot, T_n)\|_{C^{2+\alpha,1+\beta}_x([g(T_n), h(T_n)])}
$$

such that problem (1.4) admits a unique solution $(U_n, V_n; g_n, h_n)$ for $t \in [T_n, T_n + t_0]$. Considering the uniqueness of the solution for (1.4), it follows that the solution $(U, V; g, h) = (U_n, V_n; g_n, h_n)$ for $T_n \leq t < \min \{T_n + t_0, T_{max}\}$, which implies that the solution $(U, V; g, h)$ for (1.4) can be extended to $[0, T_{max} + t_0]$. In view of (3.21) and (3.22), $t_0$ can be taken independent of $n$ such that $T_n + t_0 > T_{max}$, which is contradict to the choice of $T_{max}$. Thus, this theorem has been proved. □

4. Principal Lyapunov Exponent

In order to investigate the global dynamics for model (1.3), considering the spatial heterogeneity and temporal almost periodicity, we first introduce the principal Lyapunov exponent and explore several valuable properties, which will be frequently used in later studies.

For any given $L > 0$ and the uniformly almost periodic matrix function $A(x, t)$ defined by (2.4), consider the following equation,

$$
(4.1) \quad \begin{cases} 
I_t = D(x, D)I + A(x, t)I, & -L < x < L, \ t > 0, \\
I(-L, t) = I(L, t) = 0, & t > 0,
\end{cases}
$$

where $-D(x, D)$ is a second-order strongly elliptic differential operator matrix of diagonal type with $D(x, D) = (D_i \partial_{x_i})$ for $i = 1, 2$.

Let $X \hookrightarrow C^2([-L, L]) \times C^2([-L, L])$ be the fractional power space (Chapter 1, [34]) with respect to the sectorial operator $-D(x, D)$ with homogeneous Cauchy boundary conditions, where $\mathcal{D}(-D(x, D)) = \{(u, v) \in C^2([-L, L])^2 \mid u(\pm L) = v(\pm L) = 0\}$. By the standard semigroup theory (35), for any $I_0 \in X$, there exists a unique solution $I(t, \cdot; I_0, A)$ of (4.1) satisfying $I(0, \cdot; I_0, A) = I_0(\cdot)$.

**Definition 4.1 (DEFINITION 4.3, Part II, [36]).** We define the principal Lyapunov exponent $\lambda(A, L)$ of (4.1) as

$$
\lambda(A, L) = \lim_{t \to +\infty} \sup_{\varepsilon} \frac{\ln \|\Phi(A, t)\|_X}{t},
$$

where $\Phi(A, t)$ satisfies $\Phi(A, t)I_0 = I(t, \cdot; I_0, A)$ for $I_0 \in X$.

Assume that $f_i (i = 1, 2)$ satisfies (H1) - (H4), then $g_i \in H(f_i)$ satisfies (H1) - (H4). Applying the standard semigroup theory for parabolic equations, for any given $g_i \in H(f_i)$ and $(U_0, V_0) \in X^+$, there exists a unique solution $(U(\cdot, t; U_0, V_0, g_1, g_2), V(\cdot, t; U_0, V_0, g_1, g_2))$ for the following equation

$$
(4.2) \quad \begin{cases} 
U_t = D_1 U_{xx} + g_1(x, t, U, V), & -L < x < L, \ t > 0, \\
V_t = D_2 V_{xx} + g_2(x, t, U, V), & -L < x < L, \ t > 0, \\
(U(x, t) = V(x, t) = 0, & x = -L \text{ or } x = L, \ t > 0,
\end{cases}
$$

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for all \( t > 0 \) with \( \mathcal{U}(.0; U_0, V_0, g_1, g_2) = U_0(x), \mathcal{V}(.0; U_0, V_0, g_1, g_2) = V_0(x) \), where \( X^+ = \{(u, v) \in X \mid (u, v) \geq 0\}, X^{++} = \text{Int}(X^+) \).

Further, the system [4.2] generates a skew-product semiflow

\[
\Pi_t : X^+ \times H(f_1) \times H(f_2) \rightarrow X^+ \times H(f_1) \times H(f_2), \quad t \geq 0
\]

(4.3)

\[
(U_0, V_0, g_1, g_2) \mapsto (U(\cdot, t; U_0, V_0, g_1, g_2), V(\cdot, t; U_0, V_0, g_1, g_2), g_1 \cdot t, g_2 \cdot t),
\]

where \( g_i \cdot t(x, \cdot, U, V) = g_i(x, \cdot + t, U, V), i = 1, 2 \). It can be easily seen that \( \Pi_t \) is continuous and compact by Lemma 3.3.

Next we introduce the definition of continuous separation for skew-product semiflow.

**Definition 4.2** (Definition 3.11, [37]). The skew-product semiflow (4.3) is said to admit a continuous separation if there are subspaces \( \{X_1(G)\}_{G \in H(F)} \) and \( \{X_2(G)\}_{G \in H(F)} \) with the following properties:

1) \( X = X_1(G) \oplus X_2(G) (G \in H(F)) \) and \( X_1(G), X_2(G) \) vary continuously for \( G \in H(F) \);

2) \( X_1(G) = \text{span}\{I(G)\}, \) where \( I(G) \in X^{++} \) and \( \|I(G)\| = 1 \) for \( G \in H(F) \);

3) \( X_2(G) \cap X^+ = \{0\} \) for every \( G \in H(F) \);

4) \( \Phi(G, t)X_1(G) = X_1(G \cdot t) \) and \( \Phi(t, G)X_2(G) \subset X_2(G \cdot t) \) for any \( t > 0 \) and \( G \in H(F) \);

5) There are \( K_1 > 0 \) and \( \sigma > 0 \) such that for any \( G \in H(F) \) and \( w \in X_2(G) \) with \( \|w\| = 1 \),

\[
\|\Phi(G, t)w\| \leq K_1 e^{-\sigma t}\|\Phi(G, t)I(G)\|, \quad t > 0.
\]

**Lemma 4.1.** Assume that \( \lambda(A(x, t), L) \) is the principal Lyapunov exponent of (4.1), then it is monotonically increasing of \( L \in (0, \infty) \).

**Proof.** According to Lemma 4.5 (Part III, [36]), the skew-product semiflow \( \Pi_t \) generated by (4.3) is strongly monotone in the sense that

\[
(U(.t, U_0, V_0, g_1, g_2), U(\cdot, t, U_0, V_0, g_1, g_2)) \in X^{++}
\]

for any \( t > 0, (U_0, V_0) \in X^+, g_i \in H(f_i) (i = 1, 2) \). Thus, by Theorem 4.4 of [36], the skew-product semiflow (4.3) admits a continuous separation, then there exists \( I_L : H(A) \rightarrow X^{++} \) with \( I_L = (U_L, V_L) \) satisfying the following properties:

(a) \( I_L \) is continuous and \( \|I_L(\tilde{A})\| = 1 \) for any \( \tilde{A} \in H(A) \);

(b) \( \lambda(A, L) = \lim_{t \rightarrow \infty} \frac{\ln\|I_L(A^{1-t}I_L(A))\|}{t} = \lim_{t \rightarrow \infty} \frac{\ln\|\Phi(\tilde{A}, t)I_L(\tilde{A})\|}{t} \) for any \( \tilde{A} \in H(A) \).

Assume that \( I(x, t, I_{L_i}, A) \) for \( i = 1, 2 \) are the solutions for (4.2) with \( L = L_1, L_2 \), respectively. Without loss of generation, supposing \( 0 < L_1 < L_2 \), then there is small \( \tau > 0 \) such that \( \tau I_{L_1} \) uniformly for \( x \in [-L_1, L_1] \). According to Comparison Principle, \( I(x, t, I_{L_2}, A) \geq I(x, t, \tau I_{L_1}, A) \) for \( x \in [-L_1, L_1] \). In
view of (a) and (b), for any $\hat{A} \in H(A)$, it holds that

\[
\lambda(A, L_2) = \lim_{t \to \infty} \frac{\ln ||I(\cdot, t, I_{L_2}, \hat{A})||}{t} \\
\geq \lim_{t \to \infty} \frac{\ln ||I(\cdot, t, \tau I_{L_1}, \hat{A})||}{t} \\
= \lim_{t \to \infty} \frac{\ln ||\Phi(\hat{A}, t)\tau I_{L_1}(\hat{A})||}{t} \\
= \lim_{t \to \infty} \frac{\ln(\tau + \ln ||I(\cdot, t, I_{L_1}, \hat{A})||)}{t} \\
= \lambda(A, L_1).
\]

(4.4)

Thus, our proof is completed. □

Considering that the infected domain $(g(t), h(t))$ is moving with respect to time $t$, we introduce the corresponding principal Lyapunov exponent

\[
\lambda(t) := \lambda(A, \frac{h(t) - g(t)}{2}), \quad t \geq 0
\]

for the following system

\[
\begin{cases}
I_t = D(x, D)I + A(x, t)I, \quad g(t) < x < h(t), \quad t > 0 \\
I(h(t), t) = I(g(t), t) = 0, \quad t > 0,
\end{cases}
\]

(4.5)

where $-D(x, D)$ is a second-order strongly elliptic differential operator matrix of diagonal type with $D(x, D) = (D_i \partial_i)$ for $i = 1, 2$. In view of the Lemmas 3.3 and 4.1, we can easily give the following result.

**Theorem 4.2.** $\lambda(t)$ is monotonically increasing about $t$.

5. **The Long-Time Dynamics of WNv**

In this section, we will discuss the long-time dynamical behaviors of the solution for (1.4) and investigate the conditions determining the spreading permanently or vanishing eventually for this disease.

Firstly, we give the following definitions of vanishing and spreading for WNv.

**Definition 5.1.** The disease is vanishing if $h_\infty - g_\infty < \infty$ and

\[
\lim_{t \to +\infty} ||U(\cdot, t)||_{C(g(t), h(t))} = 0, \quad \lim_{t \to +\infty} ||V(\cdot, t)||_{C(g(t), h(t))} = 0;
\]

The disease is spreading if $h_\infty - g_\infty = \infty$ and

\[
\lim_{t \to +\infty} \inf_{t \to +\infty} ||U(\cdot, t)||_{C(g(t), h(t))} > 0, \quad \lim_{t \to +\infty} \inf_{t \to +\infty} ||V(\cdot, t)||_{C(g(t), h(t))} > 0.
\]

Next, for system (1.2), we recall a result from [38] (Theorem A) which will be applied in proving Theorem 2.2 and Theorem 4.3.

**Lemma 5.1.** Let matrix function $A(x, t)$ be defined by (2.4). For any given $g_i \in H(f_i)$ for $i = 1, 2$. Let $(U(\cdot, t; U_0, V_0, g_1, g_2), V(\cdot, t; U_0, V_0, g_1, g_2))$ be the solution of (4.2), then the followings hold.
(1) If $\lambda(A, L) < 0$, then

$$
\lim_{t \to \infty} ||U(.; t; U_0, V_0, g_1, g_2)|| = 0, \lim_{t \to \infty} ||V(.; t; U_0, V_0, g_1, g_2)|| = 0
$$

uniformly for $g_i \in H(f_i)$. Further, $\lim_{t \to \infty} ||U(.; s + t; U_0, V_0, s)|| = 0$ and $\lim_{t \to \infty} ||U(.; s + t; U_0, V_0, s)|| = 0$

uniformly for $s \in \mathbb{R}$.

(2) If $\lambda(A, L) > 0$, there exist $U_L : H(f_1) \times H(f_2) \to C([-L, L])$ and $V_L : H(f_1) \times H(f_2) \to C([-L, L])$ such that $U_L(\cdot, g_1, g_2)$ and $V_L(\cdot, g_1, g_2)$ are continuous for $g_i \in H(f_i)$ and $U(.; t; U_L, V_L, g_1, g_2) = U_L(g_1 \cdot t, g_2 \cdot t)(\cdot)$, $V(.; t; U_L, V_L, g_1, g_2) = V_L(g_1 \cdot t, g_2 \cdot t)(\cdot)$. Meanwhile, it holds that

$$
\lim_{t \to \infty} ||U(.; s + t; U_0, V_0, g_1, g_2)|| = 0,
\lim_{t \to \infty} ||V(.; s + t; U_0, V_0, g_1, g_2)|| = 0
$$

uniformly in $g_i \in H(f_i)$ for any $(U_0, V_0) \in X^+ \setminus \{0\}$. Further, $U_L^*(x, t) := U_L(f_1 \cdot t, f_2 \cdot f_2)(x)$ and $V_L^*(x, t) := V_L(f_1 \cdot t, f_2 \cdot f_2)(x)$ are uniformly almost periodic in $t \in \mathbb{R}$. Moreover, for any $(U_0, V_0) \in X^+ \setminus \{0\}$, it holds that

$$
\lim_{t \to \infty} ||U(.; s + s; U_0, V_0, s)|| = 0, \lim_{t \to \infty} ||V(.; s + s; U_0, V_0, s)|| = 0
$$

uniformly for $s \in \mathbb{R}$, where $U(.; s + s; U_0, V_0, s) = U(.; t; U_0, V_0, f_1 \cdot s, f_2 \cdot s), V(.; s + s; U_0, V_0, s) = V(.; t; U_0, V_0, f_1 \cdot s, f_2 \cdot s)$.

**Lemma 5.2.** Assume that (H1) – (H5) hold. Take $L \geq L^*$, then

$$
\inf_{\substack{x \in [-L, L], \tilde{x} \in \mathbb{R} \ni \in H(f_i), g_i \in H(f_i)}} U^*(x, 0, \tilde{x}, g_1, g_2, L) > 0, \inf_{\substack{x \in [-L, L], \tilde{x} \in \mathbb{R} \ni \in H(f_i), g_i \in H(f_i)}} V^*(x, 0, \tilde{x}, g_1, g_2, L) > 0,
$$

for $i = 1, 2$. Where $(U^*(x, t; \tilde{x}, g_1, g_2, L), V^*(x, t; \tilde{x}, g_1, g_2, L))$ is the unique positive almost periodic solution of the following system,

$$
\begin{align*}
U_t &= D_1 U_{xx} + g(x + \tilde{x}, t, U, V), \quad -L < x < L, \tilde{x} \in \mathbb{R}, t > 0, \\
V_t &= D_2 V_{xx} + g(x + \tilde{x}, t, U, V), \quad -L < x < L, \tilde{x} \in \mathbb{R}, t > 0, \\
U(x, t) &= V(x, t) = 0, \quad x = -L \text{ or } x = L, t > 0.
\end{align*}
$$

(5.1)

Indeed, we can see that

$$(U^*(x, t; \tilde{x}, g_1, g_2, L), V^*(x, t; \tilde{x}, g_1, g_2, L)) = (U^*(x, 0; \tilde{x}, g_1 \cdot t, g_2 \cdot t, L), V^*(x, 0; \tilde{x}, g_1 \cdot t, g_2 \cdot t, L)).$$

Proof. The proof of this lemma can refer to Lemma 4.1 in [39], it can be proved by making a minor modification, so we omit the detailed proof.

Considering the dependence of boundary functions $g(t)$ and $h(t)$ on $\mu$, denote

$$h_\mu(t) := h(t) = h(t; U_0, V_0, h_0), \quad g_\mu(t) := g(t) = g(t; U_0, V_0, h_0)$$

with $h(0) = h_0, g(0) = -h_0$. Then the following result holds.

**Lemma 5.3.** For all $t > 0$, $h_\mu(t)$ is strictly monotonically increasing in $\mu$, and $g_\mu(t)$ is strictly monotonically decreasing in $\mu$.  

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Proof. We will prove this lemma mainly by Comparison Principle. Assume that \((U_1, V_1; g_{\mu_1}, h_{\mu_1})\) and \((U_2, V_2; g_{\mu_2}, h_{\mu_2})\) are the two solutions for problem (1.4). For simplification, we only need to compare \(h_{\mu_1}(t)\) with \(h_{\mu_2}(t)\), then we can similarly obtain the strict monotonicity of \(g_{\mu}(t)\).

Without loss of generality, assume that \(0 < \mu_1 < \mu_2\), then

\[
(5.2) \quad h'_{\mu_1}(t) = -\mu_1 U_{1x}(h_{\mu_1}(t), t) < -\mu_2 U_{1x}(h_{\mu_1}(t), t).
\]

By Lemma 3.2 it follows \(h_{\mu_1}(t) \leq h_{\mu_2}(t)\) for all \(t \in [0, \infty)\).

Now it is our turn to prove that \(h_{\mu_1}(t) < h_{\mu_2}(t)\) in \([0, \infty)\). On the contrary, assume that positive time \(T^*\) is the first time such that \(h_{\mu_1}(t) < h_{\mu_2}(t)\) for \(t \in (0, T^*)\) and \(h_{\mu_1}(T^*) = h_{\mu_2}(T^*)\), then

\[
(5.3) \quad h'_{\mu_1}(T^*) \geq h'_{\mu_2}(T^*).
\]

Let \(\Sigma_{T^*} := \{(x, t) \in \mathbb{R}^2 | 0 \leq x < h_{\mu_1}(t), 0 < t \leq T^*\}\). Applying the strong maximum principle to \(U_1\) and \(U_2\), it follows that \(U_1(x, t) < U_2(x, t)\) in \(\Sigma_{T^*}\). Let \(H(x, t) = U_2(x, t) - U_1(x, t)\), then \(H(x, t) > 0\) for \((x, t) \in \Sigma_{T^*}\) and \(H(h_{\mu_1}(T^*), T^*) = 0\). Then, we can get that \(H_x(h_{\mu_1}(T^*), T^*) < 0\). In view of \((U_1)_x(h_{\mu_1}(T^*), T^*) < 0\) and \(\mu_1 < \mu_2\), then

\[
-\mu_1 (U_1)_x(h_{\mu_1}(T^*), T^*) < -\mu_2 (U_2)_x(h_{\mu_2}(T^*), T^*).
\]

Therefore, \(h'_{\mu_1}(T^*) < h'_{\mu_2}(T^*)\), which yields a contradiction to (5.3). Thus, \(h_{\mu}(t)\) is strictly monotonically increasing about \(\mu\) for all \(t > 0\).

Similarly, we can get that \(-g_{\mu_1}(t) < -g_{\mu_2}(t)\) for all \(t > 0\). Therefore, our proof is completed. \(\square\)

In the rest of this section, for any given \((U_0, V_0)\) satisfying (1.5), let

\[
(U(x, t; U_0, V_0, h_0), V(x, t; U_0, V_0, h_0))
\]

denote the solution of system (1.4) with

\[
U(x, 0; U_0, V_0, h_0) = U_0, V(x, 0; U_0, V_0, h_0) = V_0, h(0) = h_0, g(0) = -h_0.
\]

**Theorem 5.4.** If \(h_\infty - g_\infty < \infty\), then \(\lim_{t \to \infty} h'(t, U_0, V_0, h_0) = 0\), \(\lim_{t \to \infty} g'(t, U_0, V_0, h_0) = 0\).

**Proof.** Now we are going to prove the case of \(h'(t, U_0, V_0, h_0)\). On the contrary, assume that there exists a positive sequence \(\{t_n\}\) with \(\lim_{n \to \infty} t_n = \infty\) such that

\[
(5.4) \quad \lim_{n \to \infty} h'(t_n, U_0, V_0, h_0) > 0.
\]

Let

\[
h_n(t) = h(t + t_n, U_0, V_0, h_0), \quad \text{for} \quad t \geq 0,
\]

then \(\lim_{n \to \infty} h_n(t) = h_\infty\) uniformly for \(t \geq 0\). According to Lemma 3.3 we can get that \(\{h'_n(t)\}\) is uniformly bounded and equicontinuous on \([0, \infty)\). By Arzela-Ascoli theorem, there exists \(h^*(t)\) such that \(\lim_{n \to \infty} h'_n(t) = h^*(t)\) uniformly in any bounded sets of \([0, \infty)\). Since \(\lim_{n \to \infty} h_n(t) = h_\infty < \infty\), then \(h^*(t) \equiv 0\), which implies that \(\lim_{n \to \infty} h'(t_n, U_0, V_0, h_0) = 0\). It is a contradiction to (5.4). Similarly, \(\lim_{t \to \infty} g'(t, U_0, V_0, h_0) = 0\). \(\square\)

**Theorem 5.5.** Assume that (H1) \(\sim\) (H5) hold. If \(h_\infty - g_\infty < \infty\), then

\[
\lim_{t \to +\infty} U(x, t; U_0, V_0, h_0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} V(x, t; U_0, V_0, h_0) = 0
\]

uniformly in \(x \in [g_\infty, h_\infty]\). That is, the disease will vanish.
Proof. Let \( f_i \) be defined as in \([2,3]\) for \( i = 1, 2 \), then \( f_1 \) and \( f_2 \) satisfy \((H1) - (H4)\) and \( A(x, t) \) defined by \([2,4]\) satisfies \((H5)\). If \( h_\infty - g_\infty < \infty \), it is easily to show that \( h_\infty < \infty \) and \( g_\infty > -\infty \).

According to regularity and the priori estimates about parabolic equations \([34]\), considering the system \([14]\), for any given sequence \( \{t_n\} \) satisfying \( t_n \to \infty \) as \( n \to \infty \), there exists a subsequence \( \{t_{n_k}\} \) satisfying \( t_{n_k} \to \infty \) as \( k \to \infty \), \((\tilde{U}^*(x, t), \tilde{V}^*(x, t)) \in (C([g_\infty, h_\infty] \times \mathbb{R}))^2 \) and \( g_i^* \in H(f_i) \) for \( i = 1, 2 \) such that \( f_i \cdot t_{n_k} \to g_i^* \),

\[
\lim_{k \to \infty} ||U(\cdot, t + t_{n_k}; U_0, V_0, h_0) - \tilde{U}^*(\cdot, t)||_{C^1([g(t+tn_k), h(t+tn_k)])} = 0
\]

and

\[
\lim_{k \to \infty} ||V(\cdot, t + t_{n_k}; U_0, V_0, h_0) - \tilde{V}^*(\cdot, t)||_{C^1([g(t+tn_k), h(t+tn_k)])} = 0,
\]

where \((\tilde{U}^*(x, t), \tilde{V}^*(x, t))\) is the entire solution for the following system,

\[
\begin{align*}
U_t &= D_1 U_{xx} + f_1(x, t, U, V), & g_\infty < x < h_\infty, \\
V_t &= D_2 V_{xx} + f_2(x, t, U, V), & g_\infty < x < h_\infty, \\
U(x, t) &= V(x, t) = 0, & x = g_\infty \text{ or } x = h_\infty.
\end{align*}
\]

Next we accomplish the proof of this theorem by two steps.

Step 1 To show \( h_\infty - g_\infty \leq 2\lambda^* \) following from \( h_\infty - g_\infty < \infty \).

On the contrary, assume that \( h_\infty - g_\infty \in (2\lambda^*, \infty) \), then there exist \( t^* > 0 \) and \( \epsilon > 0 \) such that \( h(t) - g(t) > h_\infty - g_\infty - 2\epsilon > 2\lambda^* \) for \( t \geq t^* \), thus, by \((H5)\) and Theorem \([4,2]\) \( \lambda(t) > 0 \). For the following system

\[
\begin{align*}
U_t &= D_1 U_{xx} + f_1(x, t, U, V), & g_\infty + \epsilon < x < h_\infty - \epsilon, & t > 0, \\
V_t &= D_2 V_{xx} + f_2(x, t, U, V), & g_\infty + \epsilon < x < h_\infty - \epsilon, & t > 0, \\
U(x, t) &= V(x, t) = 0, & x = g_\infty + \epsilon \text{ or } x = h_\infty - \epsilon, & t > 0,
\end{align*}
\]

by Comparison Principle, we can get that

\[
U(\cdot, t + t^*; U_0, 0, h_0) \geq \tilde{U}(\cdot, t + t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*)
\]

and

\[
U(\cdot, t + t^*; U_0, 0, h_0) \geq \tilde{V}(\cdot, t + t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*).
\]

Where

\[
(\tilde{U}(\cdot, t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*), \tilde{V}(\cdot, t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*))
\]

is the solution of \([5,8]\) with

\[
\begin{align*}
\tilde{U}(\cdot, t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*)) &= U(\cdot, t^*; U_0, 0, h_0), \\
\tilde{V}(\cdot, t^*; U(\cdot, t^*; U_0, 0, h_0), V(\cdot, t^*; U_0, 0, h_0), t^*)) &= V(\cdot, t^*; U_0, 0, h_0).
\end{align*}
\]

In view of Lemma \([5,1]\) the system \([5,8]\) admits a positive almost time periodic solution \((U_\epsilon(x, t), V_\epsilon(x, t))\).

Moreover, for any \((U_0, V_0) \in X^{++}\), it holds that

\[
\lim_{t \to \infty} ||\tilde{U}(\cdot, t + t^*; U_0, V_0, t^*) - U_\epsilon(\cdot, t + t^*)|| = 0
\]
and

\[(5.10) \lim_{t \to \infty} ||\hat{V}(\cdot, t + t^*; U_0, V_0, t^*) - V(\cdot, t + t^*)|| = 0.\]

By Comparison Principle, combining (5.9) and (5.10), we get

\[\hat{U}^*(x, t) > 0, \hat{V}^*(x, t) > 0, x \in (g_\infty, h_\infty), t \in \mathbb{R},\]

which implies \(\hat{U}_x^*(h_\infty, t) < 0, \hat{V}_x^*(h_\infty, t) < 0.\) Therefore,

\[\lim_{t \to \infty} \sup U_x(h(t), t; U_0, V_0, h_0) < 0,\]

it implies

\[\lim_{t \to \infty} h'(t) = \lim_{t \to \infty} -\mu U_x(h(t), t; U_0, V_0, h_0) > 0,\]

which is contradict to Theorem 5.4. Thus, we can obtain that \(h_\infty - g_\infty < \infty\) gives \(h_\infty - g_\infty \leq 2L^*\).

**Step 2** To show that if \(h_\infty - g_\infty < \infty\), then

\[(5.11) \lim_{t \to \infty} ||U(\cdot, t; U_0, V_0, h_0)||_{C([g(t), h(t))]} = 0, \lim_{t \to \infty} ||V(\cdot, t; U_0, V_0, h_0)||_{C([g(t), h(t))]} = 0.\]

Let

\[
\begin{align*}
\tilde{u}_0(x) &= \begin{cases} U_0(x), & \text{for } -h_0 \leq x \leq h_0, \\ 0, & \text{for } |x| > h_0. \end{cases} \\
\tilde{v}_0(x) &= \begin{cases} V_0(x), & \text{for } -h_0 \leq x \leq h_0, \\ 0, & \text{for } |x| > h_0. \end{cases}
\end{align*}
\]

Assume that \((\tilde{u}(x, t), \tilde{v}(x, t))\) is the solution of the problem

\[
\begin{align*}
\tilde{u}_t &= D_1 \tilde{u}_{xx} + f_1(x, t, \tilde{u}, \tilde{v}), & & g_\infty < x < h_\infty, t > 0, \\
\tilde{v}_t &= D_2 \tilde{v}_{xx} + f_2(x, t, \tilde{u}, \tilde{v}), & & g_\infty < x < h_\infty, t > 0, \\
\tilde{u}(g_\infty, t) &= \tilde{u}(h_\infty, t) = 0, & & t > 0, \\
\tilde{v}(g_\infty, t) &= \tilde{v}(h_\infty, t) = 0, & & t > 0, \\
\tilde{u}(x, 0) &= \tilde{u}_0(x), & & g_\infty \leq x \leq h_\infty.
\end{align*}
\]

Applying the Lemma 3.2, we can get that

\[\tilde{u}(x, t) \geq U(x, t; U_0, V_0, h_0) \geq 0, \tilde{v}(x, t) \geq V(x, t; U_0, V_0, h_0) \geq 0, \text{ for } x \in [g(t), h(t)], t > 0.\]

If \(h_\infty - g_\infty < 2L^*\), assuming \((H5)\), then \(\lambda \left( A, \frac{h_\infty - g_\infty}{2} \right) < 0.\) By Lemma 5.1 \(\lim_{t \to \infty} (\tilde{u}, \tilde{v}) = (0, 0)\) uniformly for \(x \in [g_\infty, h_\infty].\) Hence,

\[\lim_{t \to \infty} ||U(\cdot, t; U_0, V_0, h_0)||_{C([g(t), h(t))]} = 0, \lim_{t \to \infty} ||V(\cdot, t; U_0, V_0, h_0)||_{C([g(t), h(t))]} = 0.\]

If \(h_\infty - g_\infty = 2L^*\), without loss of generality, assume that \(\lim_{t \to \infty} ||U(\cdot, t; U_0, V_0, h_0)||_{C([g(t), h(t))]} \neq 0,\) then there exist a sequence \(\{s_n\}\) with \(s_n \to \infty\) as \(n \to \infty\), \((\hat{U}^*(x), \hat{V}^*(x))\) with \(U^* \geq, \neq 0\) and \(\hat{g}_1 \in H(f_1)\) such that \(\lim_{n \to \infty} f_1 \cdot s_n = \hat{g}_1^*\) and

\[
\begin{align*}
\lim_{n \to \infty} ||U(\cdot, s_n; U_0, V_0, h_0) - \hat{U}^*(\cdot)||_{C([g(s_n), h(s_n))]} = 0, \\
\lim_{n \to \infty} ||V(\cdot, s_n; U_0, V_0, h_0) - \hat{V}^*(\cdot)||_{C([g(s_n), h(s_n))]} = 0.
\end{align*}
\]
It follows that \((U(\cdot, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*)) , V(\cdot, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*))\) is the entire solution for the following equation,

\[
\begin{align*}
    u_t &= D_1 u_{xx} + \bar{g}_1^*(x, t, u, v), \quad g_\infty < x < h_\infty, \\
    v_t &= D_2 v_{xx} + \bar{g}_2^*(x, t, u, v), \quad g_\infty < x < h_\infty, \\
    u(g_\infty, t) &= u(h_\infty, t) = 0, \\
    v(g_\infty, t) &= v(h_\infty, t) = 0.
\end{align*}
\]  
(5.12)

Applying Hopf lemma to \(U(h_\infty, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*)\) and \(U(g_\infty, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*)\), we can get that

\[
U_x(h_\infty, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*) < 0, \quad U_x(g_\infty, t; \bar{U}^*, \bar{V}^*, \bar{g}_1^*, \bar{g}_2^*) > 0,
\]

which implies

\[
\lim_{n \to \infty} h'(\bar{s}_n) = -\lim_{n \to \infty} \mu U_x(h(\bar{s}_n), \bar{s}_n; U_0, V_0, h_0) > 0
\]

and

\[
\lim_{n \to \infty} g'(\bar{s}_n) = -\lim_{n \to \infty} \mu U_x(g(\bar{s}_n), \bar{s}_n; U_0, V_0, h_0) < 0.
\]

This is contradict to Theorem 5.4. Thus, our proof is completed. \(\Box\)

**Remark 5.1.** From the proof of the above theorem, we can obtain that the densities of infected populations will decay to 0 and the eventually infected domain is no more than \(2L^*\) when the disease vanishes.

The following theorem gives the long-time asymptotic behavior as the spreading happens, which is the sharp distinction for our spatial heterogeneous and time almost periodic WNv model.

**Theorem 5.6.** Assume that (H1)-(H5) hold. For any given \(h_0\) and \((U_0, V_0)\) satisfying (1.5), let

\[
(U(x, t; U_0, V_0, h_0), V(x, t; U_0, V_0, h_0))
\]

be the solution for (1.4). If \(h_\infty - g_\infty = \infty\), then

\[
\lim_{t \to +\infty} U(x, t; U_0, V_0, h_0) - U^*(x, t) = 0, \quad \lim_{t \to +\infty} V(x, t; U_0, V_0, h_0) - V^*(x, t) = 0
\]

locally uniformly for \(x \in \mathbb{R}\), where \((U^*(x, t), V^*(x, t))\) is the unique positive almost periodic solution of the system (2.8). That is, the disease will spread.

**Proof.** Firstly, we aim to show that \(h_\infty = \infty\) and \(g_\infty = -\infty\) when \(h_\infty - g_\infty = \infty\). On the contrary, assume that \(g_\infty = -\infty\) and \(h_\infty < \infty\). According to Theorem 5.4, it is easily to yield a contradiction to \(\lim_{t \to \infty} h'(t) = 0\). Therefore, \(g_\infty = -\infty\) and \(h_\infty = \infty\).

Next we will prove (5.13). Let \(U_0 := N_1, V_0 := N_2\), then by Comparison Principle, \(U(x, t; N_1, N_2, f_1: (-t), f_2: (-t))\) and \(V(x, t; N_1, N_2, f_1: (-t), f_2: (-t))\) decrease in \(t \in \mathbb{R}\). Take

\[
U^*(f_1, f_2)(x) := \lim_{t \to -\infty} U(x, t; N_1, N_2, f_1: (-t), f_2: (-t)), \quad V^*(f_1, f_2)(x) := \lim_{t \to -\infty} V(x, t; N_1, N_2, f_1: (-t), f_2: (-t))
\]

for \(x \in \mathbb{R}\). And it follows

\[
U(\cdot, t; U^*(f_1, f_2)(x), V^*(f_1, f_2)(x), f_1, f_2) = U^*(f_1 \cdot t, f_2 \cdot t)(\cdot),
\]

\[
V(\cdot, t; U^*(f_1, f_2)(x), V^*(f_1, f_2)(x), f_1, f_2) = V^*(f_1 \cdot t, f_2 \cdot t)(\cdot),
\]

where \((U(x, t; N_1, N_2, f_1, f_2), V(x, t; N_1, N_2, f_1, f_2))\) is the solution for (4.2) for \(U_0 = N_1, V_0 = N_2\) and \(L = \infty\). Let \(U_L(f_1, f_2)(x)\) and \(V_L(f_1, f_2)(x)\) be in Lemma 5.1, then for any fixed \(x, U_L(f_1, f_2)(x)\) and \(V_L(f_1, f_2)(x)\)
are increasing in \( L \). Applying the Comparison Principle and Lemma 5.2 we can obtain that

\[
\lim_{L \to \infty} U_L(f_1, f_2)(x) = U^*(f_1, f_2)(x), \quad \lim_{L \to \infty} V_L(f_1, f_2)(x) = V^*(f_1, f_2)(x)
\]

locally uniformly for \( x \in \mathbb{R} \).

For any \( T > 0 \) satisfying \( h(T) - g(T) > 2L^* \), denote \( U(\cdot, T; U_0, V_0, h_0) := U(\cdot, T) \) and \( V(\cdot, T; U_0, V_0, h_0) := V(\cdot, T) \), we can get

\[
U(x, t + T; U_0, V_0, h_0) \geq U_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) \quad \text{for} \quad t \geq 0,
\]

\[
V(x, t + T; U_0, V_0, h_0) \geq V_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) \quad \text{for} \quad t \geq 0,
\]

where \((U_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T), V_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T))\) is the solution of following system

\[
\begin{align*}
U_t &= D_1 U_{xx} + f_1 \cdot T(x, t, U, V), \quad g(T; U_0, V_0, h_0) < x < h(T; U_0, V_0, h_0), \quad t > 0, \\
V_t &= D_2 V_{xx} + f_2 \cdot T(x, t, U, V), \quad g(T; U_0, V_0, h_0) < x < h(T; U_0, V_0, h_0), \quad t > 0, \\
U(x, t) &= V(x, t) = 0, \quad x = g(T; U_0, V_0, h_0) \text{ or } x = h(T; U_0, V_0, h_0), \quad t > 0
\end{align*}
\]

with \( L = \frac{h(T; U_0, V_0, h_0) - g(T; U_0, V_0, h_0)}{2} \)

\[
U_L(x, 0; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) = U(x, T; U_0, V_0, h_0)
\]

and

\[
V_L(x, 0; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) = V(x, T; U_0, V_0, h_0).
\]

According to Lemma 5.1,

\[
\lim_{t \to \infty} U_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) - U_L(f_1 \cdot (t + T), f_2 \cdot (t + T))(x) = 0,
\]

\[
\lim_{t \to \infty} V_L(x, t; U(\cdot, T), V(\cdot, T), f_1 \cdot T, f_2 \cdot T) - V_L(f_1 \cdot (t + T), f_2 \cdot (t + T))(x) = 0
\]

uniformly for \( x \) in \([g(T; U_0, V_0, h_0), h(T; U_0, V_0, h_0)]\). In view of (5.14)

\[
\lim_{L \to \infty} U_L(f_1 \cdot (t + T), f_2 \cdot (t + T))(x) = U^*(f_1 \cdot (t + T), f_2 \cdot (t + T))(x),
\]

\[
\lim_{L \to \infty} V_L(f_1 \cdot (t + T), f_2 \cdot (t + T))(x) = V^*(f_1 \cdot (t + T), f_2 \cdot (t + T))(x)
\]

uniformly for \( x \) in any bounded sets of \( \mathbb{R} \). By Comparison Principle,

\[
(U(x, t; U_0, V_0, h_0), V(x, t; U_0, V_0, h_0)) \geq (U_L(x, t; U_0, V_0, h_0), V_L(x, t; U_0, V_0, h_0))
\]

uniformly for \((x, t) \in [g(t), h(t)] \times [0, \infty)\). Then we can get

\[
\lim_{t \to \infty} U(x, t; U_0, V_0, h_0) - U^*(f_1 \cdot t, f_2 \cdot t)(x) = 0
\]

and

\[
\lim_{t \to \infty} V(x, t; U_0, V_0, h_0) - V^*(f_1 \cdot t, f_2 \cdot t)(x) = 0
\]

locally uniform for \( x \in \mathbb{R} \). Therefore, \((U^*(f_1 \cdot t, f_2 \cdot t)(x), V^*(f_1 \cdot t, f_2 \cdot t)(x))\) is the solution for (2.5). Applying the similar method in Proposition 4.1(3) of [39], the uniqueness of the solution can be easily proved. Further,
by Lemma 5.2 we can obtain that

\begin{equation}
\inf_{x \in \mathbb{R}, t \in \mathbb{R}^+} U^*(f_1 \cdot t, f_2 \cdot t)(x) > 0, \quad \inf_{x \in \mathbb{R}, t \in \mathbb{R}^+} V^*(f_1 \cdot t, f_2 \cdot t)(x) > 0.
\end{equation}

Take \( U^*(x, t) = U^*(f_1 \cdot t, f_2 \cdot t)(x) \), \( V^*(x, t) = V^*(f_1 \cdot t, f_2 \cdot t)(x) \). It is only necessary to prove that \( U^*(f_1 \cdot t, f_2 \cdot t)(x) \) and \( V^*(x, t) = V^*(f_1 \cdot t, f_2 \cdot t)(x) \) are uniformly almost periodic in \( t \in \mathbb{R} \) with \( x \) in bounded subsets of \( \mathbb{R} \). Since \( f_i(x, t, U, V) \) is uniformly almost periodic in \( t \) with \( x \in \mathbb{R} \) and \((U, V)\) in bounded subsets of \( \mathbb{R}^2 \) for \( i = 1, 2 \), according to Theorems 1.17 and 2.10 ([10]), for any sequences \( \{a_n\} \subset \mathbb{R} \) and \( \{b_n\} \subset \mathbb{R} \), there exist \( \{t_n\} \subset \{a_n\} \) and \( \{s_n\} \subset \{b_n\} \) such that

\[
\lim_{n \to \infty} f_i(x, t + t_n + s_n, U, V) = \lim_{n \to \infty} f_i(x, t + t_n + s_m, U, V)
\]

for \((x, t, U, V) \in \mathbb{R}^4, i = 1, 2 \). Assume that

\[
\lim_{n \to \infty} f_i(x, t + t_n + s_n, U, V) = f_i^*(x, t, U, V), \quad \lim_{n \to \infty} f_i(x, t + s_n, U, V) = f_i^{**}(x, t, U, V).
\]

Then we can get that

\[
\lim_{m \to \infty} U(x, t + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = U^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x),
\]

\[
\lim_{m \to \infty} V(x, t + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = V^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x)
\]

uniformly for \( x \) in bounded sets of \( \mathbb{R} \). Further, it follows that

\[
\lim_{n \to \infty} \lim_{m \to \infty} U(x, t + t_n + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = U^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x),
\]

\[
\lim_{n \to \infty} \lim_{m \to \infty} V(x, t + t_n + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = V^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x)
\]

uniformly for \( x \) in bounded sets of \( \mathbb{R} \). Moreover,

\[
\lim_{n \to \infty} U(x, t + t_n + s_n, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = U^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x)
\]

\[
\lim_{n \to \infty} V(x, t + t_n + s_n, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) = V^*(f_1^{**} \cdot t, f_2^{**} \cdot t)(x)
\]

uniformly for \( x \) in bounded sets of \( \mathbb{R} \). Thus,

\[
\lim_{n \to \infty} \lim_{m \to \infty} U(x, t + t_n + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2)
\]

\[
= \lim_{n \to \infty} U(x, t + t_n + s_n, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2),
\]

\[
\lim_{n \to \infty} \lim_{m \to \infty} V(x, t + t_n + s_m, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2)
\]

\[
= \lim_{n \to \infty} V(x, t + t_n + s_n, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2).
\]
According to the regularity and priori estimates for parabolic differential equations,

\[ U(x, t, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) \quad \text{and} \quad V(x, t, U^*(f_1, f_2), V^*(f_1, f_2), f_1, f_2) \]

are uniformly continuous for \((x, t) \in \mathbb{R}^2\), applying Theorems 1.17 and 2.10 (H10), it follows that \(U^*(f_1 \cdot t, f_2 \cdot t)\) and \(V^*(f_1 \cdot t, f_2 \cdot t)\) are almost periodic in \(t \in \mathbb{R}\) uniformly with \(x\) in bounded sets of \(\mathbb{R}\). Therefore, our proof is completed. \(\Box\)

Proof of Theorem 2.2. Assume that \((H1)\)-(H5) hold. For any given \(g(0), h(0)\) and initial functions \((U_0, V_0)\) satisfying (1.5). Let

\[
(U(x, t; U_0, V_0, g, h), V(x, t; U_0, V_0, g, h))
\]

be the solution of system (1.4). It is easy to see that either \(\frac{h_0}{1}\) satisfies (1.5). Let

\[
\Lambda := \{\mu \mid h_0(\infty) - g_0(\infty) < \infty\}, \nu := \sup \Lambda.
\]

Proof of Theorem 2.3. (1) Assume that \((H5)\) holds, considering that \(h(t)\) is increasing and \(g(t)\) is decreasing, if \(\lambda(0) > 0\), then \(h(0) - g(0) \geq 2L^*\), and

\[
(5.19) \quad h_0 - g_0 > h(0) - g(0) \geq 2L^*.
\]

Further, we can get that \(\lambda(A, \frac{h_{\infty} - g_{\infty}}{2}) > 0\). According to Theorem 2.2 we can obtain that \(h_0 - g_0 = \infty\). Therefore, the disease is spreading.

(2) Assume that \(h(0) - g(0) < 2L^*\). Denote \(h_\mu(\infty) := \lim_{t \to \infty} h_\mu(t), g_\mu(\infty) := \lim_{t \to \infty} g_\mu(t)\).

Let

\[
(5.20) \quad \Lambda := \{\mu \mid h_\mu(\infty) - g_\mu(\infty) < \infty\}, \nu := \sup \Lambda.
\]
If $\Lambda$ is an empty set, then $h_\mu(\infty) - g_\nu(\infty) = \infty$ for all $\mu > 0$. In this case, $\mu^* = 0$ satisfies the conditions. If $\Lambda$ is a nonempty set, we first prove that there exists a $T > h_\mu(\infty) - g_\nu(\infty) = \infty$. Then there exists a $T > 0$ such that $h_\mu(T) - g_\nu(T) > 2L^*$. In view of the continuous dependence of $h_\mu$ and $g_\nu$ on $\mu$, there is a $\varepsilon > 0$ small enough such that $h_\mu(T) - g_\nu(T) > 2L^*$ for any $\mu \in [\nu - \varepsilon, \nu + \varepsilon]$. Therefore, we have

$$h_\mu(\infty) - g_\nu(\infty) = \lim_{t \to \infty} h_\mu(t) - g_\nu(t) > h_\mu(T) - g_\nu(T) > 2L^*, \quad \mu \in [\nu - \varepsilon, \nu + \varepsilon].$$

According to (5.19), we obtain that $h_\mu(\infty) - g_\nu(\infty) = \infty$, which implies that $\Lambda \cap [\nu - \varepsilon, \nu + \varepsilon]$ is an empty set. It is contradict to (5.20). Thus, we have proved that $h_\mu(\infty) - g_\nu(\infty) < \infty$.

When $\mu > \nu$, we claim that $h_\mu(\infty) - g_\nu(\infty) = \infty$. On the contrary, assume that $h_\mu(\infty) - g_\nu(\infty) < \infty$, then $\mu \leq \nu$, which is a contradiction. Therefore, by Theorem 2.2, the spreading happens.

When $\mu \leq \nu$, by the Lemma 5.3 we can obtain

$$h_\mu(t) - g_\nu(t) \leq h_\mu(t) - g_\nu(t) \text{ for all } t \in (0, +\infty).$$

Moreover, $h_\mu(\infty) - g_\nu(\infty) \leq h_\mu(\infty) - g_\nu(\infty) < \infty$, thus, by Theorem 2.2, the vanishing happens. In this case, we can take $\mu^* = \nu$. Therefore, our proof is completed. $\Box$

**Remark 5.4.** When the initial infected domain is smaller than $2L^*$, for any given initial functions $(U_0, V_0)$, the spreading or vanishing of the epidemic disease mainly depend on the front expanding rate $\mu$.

6. Simulations and Discussions

6.1. Simulations.

In this subsection, we make some numerical simulations about our WNv model. Since the parameters $a_i(x, t)$ and $d_i(x, t)(i = 1, 2)$ are positive almost periodic functions, and the double boundaries are moving, the classical numerical simulation methods is not proper. We use the implicit finite difference scheme developed in [42] for numerical simulations about the free boundaries problems to make some simulations to identify our results.

Fix the parameter values in system (1.4) as follows, the explicit biological interpretations of which can be seen from Komar et al. [2], Wonham et al. [3] or Lewis et al. [9],

$$D_1 = 3, \quad D_2 = 0.125, \quad N_1 = 1, \quad N_2 = 20, \quad \beta = 0.6,$$

$$a_1(x, t) = 0.88 \times (1 + 0.56 \times \cos\left(\frac{t}{2}\right)) + 0.088 \times \frac{2 + x}{1 + x^2} \cos x, x \in \mathbb{R}, t \geq 0,$$

$$a_2(x, t) = 0.16 \times (1 + 0.2 \times \cos\left(\frac{\pi t}{3}\right)) + 0.024 \times \frac{1 + x}{1 + x^2} \cos x, x \in \mathbb{R}, t \geq 0,$$

$$d_1(x, t) = 0.1 \times (1 + 0.3 \times \sin\left(\frac{t}{3}\right)) + 0.02 \times \frac{2 + x}{1 + x^2} \sin x, x \in \mathbb{R}, t \geq 0,$$

$$d_2(x, t) = 0.029 \times (1 + 0.1 \times \sin\left(\frac{\pi t}{2}\right)) + 0.0016 \times \frac{1 + x}{1 + x^2} \sin x, x \in \mathbb{R}, t \geq 0.$$

Take the following initial functions:

$$U_0(x) = \begin{cases} 
0.1 \times \cos\left(\frac{\pi x}{2h_0}\right), & \text{for } -h_0 \leq x \leq h_0, \\
0, & \text{for } |x| > h_0.
\end{cases}$$

(6.1)
\( V_0(x) = \begin{cases} 
2 \times \cos\left(\frac{\pi x}{\theta h_0}\right), & \text{for } -h_0 \leq x \leq h_0, \\
0, & \text{for } |x| > h_0.
\end{cases} \)

Moreover, in order to simplify the simulations, we take \( h(0) = h_0, g(0) = -h_0 \).

6.1.1. *The effect of the initial infected domain.*

Fix \( \mu = 0.1 \), take \( h_0 = 2.0, 1.0, 0.6, 0.5 \), respectively. As is shown in Fig.1, the WNv is spreading and the solution for system (1.4) converges to a positive heterogeneous steady state in Fig.1(a) and Fig.1(b), which indicates that the recurrent appearance of the cases of infection; while the WNv is vanishing and the solution for system (1.4) decays to 0 in Fig.1(c) and Fig.1(d). The Theorem 2.3 (1) implies that the disease will spread when \( h(0) - g(0) \geq 2L^* \). For given initial functions \((U_0, V_0)\) in (6.1) and (6.2), according to our simulations, we can deduce that the initial infected domain threshold \( L^* \in (0.6, 1.0) \).

Moreover, it indicates that the initial infected domain is larger, the eventual infected density of populations is bigger from Fig.1(a) and Fig.1(b), which conforms to the disease propagation mechanism.

\[ \begin{array}{cccc}
(a) h_0 = 2.0 & (b) h_0 = 1.0 & (c) h_0 = 0.6 & (d) h_0 = 0.5 \end{array} \]

**Figure 1.** Fix \( \mu = 0.1 \), the Fig.1(a) and Fig.1(b) show that the WNv is spreading and the solution converges to a positive solution when \( h_0 = 2.0, 1.0 \), respectively, while the Fig.1(c) and Fig.1(d) show that the WNv is vanishing and the solution decays to 0 when \( h_0 = 0.6, 0.5 \), respectively.
6.1.2. The effect of the infected domain boundary expanding rate.

Fix \( h_0 = 0.6 \), take \( \mu = 0.1, 0.2 \), respectively. As in shown in Fig. 2, the WNv is spreading when \( \mu = 0.2 \), while the WNv is vanishing when \( \mu = 0.1 \). The Theorem 2.3(2) implies that the disease will spread when \( \mu > \mu^* \) and the disease will vanish when \( \mu \leq \mu^* \). For given initial functions \((U_0, V_0)\) and \( h_0 \), in view of our simulations, we can deduce that the expanding capacity rate threshold \( \mu^* \in (0.1, 0.2) \). According to this result, people can implement effective treatments to control the boundary expanding rate to restrain the propagation of the epidemic disease.

![Figure 2](image_url)

**Figure 2.** Fix \( h_0 = 0.6 \), the Fig. 2(a) shows that the WNv is spreading and the solution converges to a positive heterogeneous steady state when \( \mu = 0.2 \), while the Fig. 2(b) shows that the WNv is vanishing and the solution decays to 0 when \( \mu = 0.1 \).

6.2. Discussions.

Neuroinvasive disease caused by West Nile virus is one of the most serious epidemic diseases and has brought considerable deaths since it occurred. In order to supply feasible measures to predict and control the spreading of the epidemic disease, it is urgent to investigate the propagation mechanisms for WNv. In order to describe the transmission of WNv more reasonably, almost periodic mathematical biology models should be importantly considered. In this paper, we mainly propose a new reaction-diffusion WNv model (1.4) with moving infected domains \((g(t), h(t))\) in the spatial heterogeneous and time almost periodic environment and explore the long-time asymptotic dynamical behaviors of the solution for this model.

Firstly, considering the spatial heterogeneity and time almost periodicity, we prove the global existence, uniqueness and get the regularity estimates of solution for system (1.4), which is not trivial to obtain. Next, we define the principal Lyapunov exponent \( \lambda(A, L) \) and \( \lambda(t) \) with respect to time \( t \) and get some analytic properties of it. Moreover, we give the initial infected domain critical size \( L^* \) using the principal Lyapunov exponent. In this paper, under the assumption of \( \lambda(A, L) > 0 \) for \( L \geq L^* \), we obtain the following results: if \( \lambda(t_0) > 0 \) for some \( t_0 \geq 0 \), that is \( h(t_0) - g(t_0) \geq 2L^* \), then \( h_\infty - g_\infty = \infty \) and the disease will spread no matter how big the diffusion rates and the initial data are; if \( h(0) - g(0) < 2L^* \), there exists a threshold value \( \mu^* \geq 0 \) which represents the infected region expanding capacity. When \( \mu > \mu^* \), the disease will spread and the disease will vanish when \( \mu \leq \mu^* \). What is most important, assuming \((H1)-(H5)\), we obtain the long-time dynamical behaviors of WNv model by giving the spreading-vanishing dichotomy regimes of system (1.4).
When the disease is vanishing, the densities \((U(x,t,g,h), V(x,t,g,h))\) of infected birds and mosquitoes will asymptotically converge to 0 uniformly for \(x \in [g_\infty, h_\infty]\) and the eventually infected domain is no more than \(2L^*\). When the disease is spreading, the densities \((U(x,t,g,h), V(x,t,g,h))\) of infected birds and mosquitoes will converge to a positive almost periodic solution \((U^*(x,t), V^*(x,t))\) of system (2.5) uniformly for \(x\) in any compact subsets of \(\mathbb{R}\). The asymptotic behavior of the solution when spreading occurs is largely different from the other homogeneous WNv models. This result is advantageous to study the cyclic outbreak laws of the West Nile virus caused by environmental differences and seasonal changes.

In view of the biological reality, our WNv model (1.4) is first proposed incorporate the spatial heterogeneity with time almost periodicity, which is more reasonable. Meanwhile, we discuss the explicit dynamical behaviors by mathematical techniques, which can be used to investigate other mosquito-borne epidemic models. Moreover, our techniques in studying almost periodic systems different from other homogeneous and periodic systems can be applied in other almost periodic equations. Our methods using principal Lyapunov exponent can also applied to investigate other epidemic models.

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