A compactness result for scalar-flat metrics on low dimensional manifolds with umbilic boundary

Marco G. Ghimenti · Anna Maria Micheletti

Received: 22 April 2020 / Accepted: 28 February 2021 / Published online: 9 June 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
Let \((M, g)\) be a compact Riemannian \(n\)-dimensional manifold with umbilic boundary. It is well known that, under certain hypothesis, in the conformal class of \(g\) there are scalar-flat metrics that have \(\partial M\) as a constant mean curvature hypersurface. In this paper we prove that these metrics are a compact set in the case of low dimensional manifolds, that is \(n = 6, 7, 8\), provided that the Weyl tensor is always not vanishing on the boundary.

Mathematics Subject Classification 35J65 · 53C21

1 Introduction
Let \((M, g)\) be a \(n\)-dimensional \((n \geq 3)\) compact Riemannian manifold with boundary \(\partial M\). In [11, 12] J. Escobar investigated the question if \(M\) can be conformally deformed to a scalar flat manifold with boundary of constant mean curvature hypersurface. This problem is particularly interesting because it is a higher-dimensional generalization of the well known Riemann mapping Theorem and it is equivalent to finding positive solutions to a linear equation on the interior of \(M\) with a critical nonlinear boundary condition of Neumann type:

\[
\begin{align*}
L_g u &= 0 \quad \text{in } M \\
B_g u + (n-2)u^{\frac{n}{n-2}} &= 0 \quad \text{on } \partial M.
\end{align*}
\]

Here \(L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g\) where \(-\Delta_g\) is the Laplace-Beltrami operator on \((M, g)\) and \(R_g\) the scalar curvature of \(M\) and \(B_g = -\frac{\partial}{\partial v} - \frac{n-2}{2} h_g\), where \(v\) is the outward normal to \(\partial M\) and \(h_g\) is the mean curvature of the boundary.
The existence of solutions is established by Escobar [11], Marques [26], Almaraz [3], Chen [9], Mayer and Ndiaye [25]. Once the existence of solutions of (1.1) is settled, it is natural to study the compactness of the full set of solutions. Define

$$ Q(M, \partial M) := \inf \left\{ Q(u) : u \in H^1(M), u \not\equiv 0 \text{ on } \partial M \right\}, $$

where

$$ Q(u) := \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g + \int_{\partial M} |u|^\frac{2(a-1)}{n-2} d\sigma_g}{\left( \int_{\partial M} |u|^\frac{2(a-1)}{n-2} d\sigma_g \right)^\frac{n-2}{n-1}}. $$

When $Q(M, \partial M) = -\infty$ the solution does not exist, while if $-\infty < Q(M, \partial M) \leq 0$ the solution is unique up to a constant factor. The situation turns out to be delicate if $Q(M, \partial M) > 0$ and the underlying manifold is not the euclidean ball (in the case of the euclidean ball the set of solution is known to be non compact). In this case multiple solutions exist, and Felli and Ould Ahmedou in [13,14] proved compactness for $n = 3$ when $M$ has umbilic boundary and for any dimension $n \geq 4$ in the case of locally conformally flat manifolds with umbilic boundary. Later, Almaraz, Queiroz and Wang removed the hypothesis of umbilic boundary in dimension $n = 3$ in [4]. If the dimension of the manifold is $n \geq 7$ and the trace-free second fundamental form in non zero everywhere on $\partial M$, Almaraz in [1] proved compactness. Very recently, Kim, Musso and Wei [22] showed that compactness continues to hold when $n = 4, 5$ without further hypothesis, and when $n = 6, 7$ and the trace-free second fundamental form in non zero everywhere on $\partial M$.

Compactness was proved also by the authors in [15] for manifold with umbilic boundary when $n = 8$ and the Weyl tensor of the boundary is always different from zero, or if $n > 8$ and the Weyl tensor of $M$ is always different from zero on the boundary. An example of non compactness is given for $n \geq 25$ and manifolds with umbilic boundary in [2]. We recall that the boundary of $M$ is called umbilic if the trace-free second fundamental form of $\partial M$ is zero everywhere.

In the present work we are interested to extend the result of [15] to dimension $n = 6, 7, 8$ when the Weyl tensor of $M$ is always different from zero on the boundary. Namely we want to prove compactness of the set of positive solutions to

$$ \begin{align*}
L_g u &= 0 \quad \text{in } M \\
B_g u + (n-2)u^p &= 0 \quad \text{on } \partial M
\end{align*} $$

where $1 < p \leq \frac{n}{n-2}$ and the boundary of $M$ is umbilic. Our main result is the following.

**Theorem 1** Let $(M, g)$ a smooth, $n$-dimensional Riemannian manifold of positive type with regular umbilic boundary $\partial M$. Suppose that $n = 6, 7, 8$ and that the Weyl tensor $W_g$ is not vanishing on $\partial M$. Then, given $\tilde{p} > 1$, there exists a positive constant $C$ such that, for any $p \in \left[ \tilde{p}, \frac{n}{n-2} \right]$ and for any $u > 0$ solution of (1.2), it holds

$$ C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\alpha}(M)} \leq C $$

for some $0 < \alpha < 1$. The constant $C$ does not depend on $u$, $p$.

Our strategy follows the argument of the seminal paper of Khuri Marques and Schoen [23]. A crucial step is to provide a sharp correction term (see Sect. 2.2) for the usual approximation of a rescaled solution by a bubble around an isolated simple blow up point. This sharp...
correction term is a solution of a suitable linearized equation (see 2.17). The assumption of the umbilicity of the boundary forces us to deal with higher order terms in the expansion of the metric tensor, and this makes the proof of the result technically hard. Moreover, it determines the right hand side of the Eq. (2.17), which gives the aforementioned correction term.

Another crucial step relies on a classical local argument with a Pohozaev type identity and we need a local Pohozaev sign condition which is essential for the proof. In the case of low dimensional manifolds this requires a very accurate pointwise estimate of the correction term which seems not to have an explicit form in the case of boundary Yamabe problem. This process is somewhat inspired by the strategy used by Kim, Musso and Wei [22] to estimate the correction term on low dimensional manifold with non umbilic boundary.

The paper is organized as follows: in Sect. 2 we provide some necessary preliminary notions; in particular in Sect. 2.1 we introduce some type of blow up points and in Sect. 2.2 we define the correction term. Section 3 contains an accurate description of the correction term, and the Pohozaev sign condition is studied in Sect. 4, for the case \( n = 7, 8 \), and in Sect. 5, for the case \( n = 6 \). The proof of Theorem 1 is shown in Sect. 6. Some technical proofs are postponed to the Appendix.

## 2 Preliminaries and notations

### Remark 2

We collect here our main notations. We will use the indices \( 1 \leq i, j, k, m, p, r, s, t, \tau \leq n - 1 \) and \( 1 \leq a, b, c, d \leq n \). Moreover we use the Einstein convention on repeated indices. We denote by \( g \) the Riemannian metric, by \( R_{abcd} \) the full Riemannian curvature tensor, by \( R_{ab} \) the Ricci tensor and by \( R \) the scalar curvature of \((M, g)\); moreover the Weyl tensor of \((M, g)\) will be denoted by \( W_g \). The bar over an object (e.g. \( \bar{W}_g \)) will mean the restriction to this object to the metric of \( \partial M \). Finally, on the half space \( \mathbb{R}^n_+ = \{ y = (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n, y_n \geq 0 \} \) we set \( B_r(y_0) = \{ y \in \mathbb{R}^n, |y - y_0| \leq r \} \) and \( B^+_r(y_0) = B_r(y_0) \cap \{ y_n > 0 \} \). When \( y_0 = 0 \) we will use simply \( B_r = B_r(0) \) and \( B^+_r = B^+_r(0) \). On the half ball \( B^+_r \) we set \( \partial' B^+_r = B^+_r \cap \partial \mathbb{R}^n_+ = B^+_r \cap \{ y_n = 0 \} \) and \( \partial^+ B^+_r = \partial B^+_r \cap \{ y_n > 0 \} \). On \( \mathbb{R}^n_+ \) we will use the following decomposition of coordinates: \((y_1, \ldots, y_{n-1}, y_n) = (\tilde{y}, y_n) = (z, t) \) where \( \tilde{y}, z \in \mathbb{R}^{n-1} \) and \( y_n, t \geq 0 \).

Fixed a point \( q \in \partial M \), we denote by \( \psi_q : B^+_r \to M \) the Fermi coordinates centered at \( q \). We denote by \( B^+_r(q, r) \) the image of \( B^+_r \). When no ambiguity is possible, we will denote \( B^+_r(q, r) \) simply by \( B^+_r(q, r) \), omitting the chart \( \psi_q \).

We recall that \( \omega_{n-2} \) is the \( n - 1 \) dimensional spherical element.

On the boundary \( \partial M \) of \( M \), it is well known the existence of the Fermi coordinates. Since \( \partial M \) is umbilic, the use of Fermi coordinates and of a suitable conformal metric simplifies much of the future computations.

Given \( q \in \partial M \) there exists a conformally related metric \( \tilde{g}_q = \Lambda_q g \) such that some geometric quantities at \( q \) have a simpler form which will be summarized in the next claim. We also know that \( \Lambda_q(q) = 1, \frac{\partial \Lambda_q}{\partial y_n}(q) = 0 \) for all \( k = 1, \ldots, n - 1 \). In order to simplify notations, we will omit the tilde symbol and we will omit the Fermi conformal coordinates \( \psi_q : B^+_r \to M \) whenever it is not needed, so we will write \( y \in B^+_r \) instead of \( \psi_q(y) \in M \), \( 0 \) instead of \( q = \psi_q(0) \), \( u \) instead of \( u \circ \psi_q \) and so on.

### Remark 3

In Fermi conformal coordinates around \( q \in \partial M \), it holds (see [26, Prop. 3.1])

\[
|\det g_q(y)| = 1 + O(|y|^N) \quad \text{for any given } N \text{ large} \tag{2.1}
\]
which, choosing $N = 5$, leads to

$$|h_{ij}(y)| = O(|y^4|) \quad |h_g(y)| = O(|y^4|). \quad (2.2)$$

In [26, eq. (2.7)] it is proved

$$g_{ij}^{ij}(y) = \delta_{ij} + \frac{1}{3} \bar{R}_{ikjl} y_k y_l + R_{ninj} y_n^2$$

$$+ \frac{1}{6} \bar{R}_{ikjl,mp} \frac{1}{15} \bar{R}_{iksl} \bar{R}_{jmsp} y_k y_l y_m y_p$$

$$+ \left( \frac{1}{2} R_{ninj,kl} + \frac{1}{3} \text{Sym}_{ij}(\bar{R}_{iksl} R_{nsnj}) \right) y_j y_k y_l$$

$$+ \frac{1}{3} R_{ninj,nk} y_n y_k + \frac{1}{12} \left( R_{ninj,nn} + 8 R_{ninj} R_{nsnj} \right) y_n^4 + O(|y|^5) \quad (2.3)$$

and in [26, eq (2.4)] and in [3, Lemma 2.4] are shown the following identities

$$\bar{R}_{gs}^q(y) = O(|y|^2) \quad \text{and} \quad \partial^2_{ii} \bar{R}_{gs}^q = -\frac{1}{6} |W|^2. \quad (2.4)$$

The following formulas are all contained in [26, Proposition 3.2]

$$\partial^2_{ii} \bar{R}_{gs}^q = -2 R_{ninj} R_{ninj} - 2 R_{ninj,ij} = -2 R_{ninj}^2 - 2 R_{ninj,ij} \quad (2.5)$$

$$\bar{R}_{kl} = R_{nn} = R_{nk} = R_{nn,kk} = 0 \quad (2.6)$$

$$R_{ninj,nn} = -2 R_{ninj}^2 \quad (2.7)$$

All the quantities in the Remark are calculated in $q \in \partial M$, unless otherwise specified.

We set $U(y) := \frac{1}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^\frac{n-1}{2}}$ to be the standard bubble. The function $U$ solves the problem

$$\begin{cases}
\Delta U = 0 & \text{in } \mathbb{R}_+^n \\
\frac{\partial U}{\partial n} + (n - 2) U \frac{\partial}{\partial n} = 0 & \text{on } \partial \mathbb{R}_+^n.
\end{cases} \quad (2.8)$$

**Remark 4** Let $f : \mathbb{R} \times \mathbb{R}_+^+ \to \mathbb{R}$ be a smooth integrable function and fix a $c \geq 0$. We have the following integral identities

$$R_{ninj} \int_{\partial \mathbb{R}_+^n} f(|\bar{y}|, c) y_i y_j d\bar{y} = 0 \quad (2.9)$$

$$\bar{R}_{t\tau sp} \int_{\# \mathbb{R}_+^n} f(|\bar{y}|, c) y_t y_\tau y_s y_p d\bar{y} = 0 \quad (2.10)$$

$$R_{ninj} \bar{R}_{t\tau sp} \int_{\partial \mathbb{R}_+^n} f(|\bar{y}|, c) y_i y_j y_t y_\tau y_s y_p d\bar{y} = 0 \quad (2.11)$$

$$\bar{R}_{ijk\ell} \bar{R}_{t\tau sp} \int_{\# \mathbb{R}_+^n} f(|\bar{y}|, c) y_i y_j y_k y_\ell y_t y_\tau y_s y_p d\bar{y} = 0 \quad (2.12)$$

$$R_{ninj} R_{nknl} \int_{\partial \mathbb{R}_+^n} f(|\bar{y}|, c) y_i y_j y_k y_l d\bar{y} = \frac{2}{3} R_{ninj}^2 \int_{\partial \mathbb{R}_+^n} f(|\bar{y}|, c) y_i^4 d\bar{y}$$
\[ = \frac{2}{n^2 - 1} R^2_{nij} \int_{\partial \mathbb{R}^n_+} f(|\bar{y}|, c)|\bar{y}|^4 d\bar{y} \quad (2.13) \]

**Proof** The first identity follows from the fact that \( R_{nn} = 0 \), while (2.10), (2.11) and (2.12) follow from that and from the symmetries of \( \bar{R}_{ijkl} \). For the last formula we have, using symmetry again, that

\[
R_{nij} R_{nknl} \int_{\partial \mathbb{R}^n_+} f(|\bar{y}|, c) y_i y_j y_k y_l d\bar{y} = 2 R^2_{nij} \int_{\partial \mathbb{R}^n_+} f(|\bar{y}|, c) y^2_i y^2_j d\bar{y}
\]

and we conclude by the elementary identities

\[
3 \int_{\mathbb{R}^2} f(x^2 + y^2)x^2 y^2 dxdy = \int_{\mathbb{R}^2} f(x^2 + y^2)x^4 dxdy
\]

and

\[
\int_{\partial \mathbb{R}^n_+} f(|\bar{y}|, c)|\bar{y}|^4 d\bar{y} = \frac{3}{n^2 - 1} \int_{\partial \mathbb{R}^n_+} f(|\bar{y}|, c)|\bar{y}|^4 d\bar{y}.
\]

\[\square\]

**Remark 5** We collect here some results contained in [1, Lemma 9.4] and in [1, Lemma 9.5]. The proof is by direct computation. For \( m > k + 1 \)

\[
\int_{0}^{\infty} \frac{t^k dt}{(1 + t)^m} = \frac{k!}{(m - 1)(m - 2) \cdots (m - 1 - k)}
\]

\[\int_{0}^{\infty} \frac{dt}{(1 + t)^m} = \frac{1}{m - 1} \quad (2.14)\]

Moreover, set, for \( \alpha, m \in \mathbb{N} \),

\[ I^\alpha_m := \int_{0}^{\infty} \frac{s^\alpha ds}{(1 + s^2)^m} \]

it holds

\[
I^\alpha_m = \frac{2m}{\alpha + 1} I^{\alpha+2}_{m+1} \text{ for } \alpha + 1 < 2m
\]

\[
I^\alpha_m = \frac{2m}{2m - \alpha - 1} I^{\alpha}_{m+1} \text{ for } \alpha + 1 < 2m
\]

\[
I^\alpha_m = \frac{2m - \alpha - 3}{\alpha + 1} I^{\alpha+2}_{m} \text{ for } \alpha + 3 < 2m.
\]

\[ (2.15) \]

### 2.1 Blow up points and the Khuri-Marques-Schoen scheme

The operators \( L_g \) and \( B_g \) are conformally invariant. Then it is more convenient to consider the following family of problems that are slightly general, but in which all the terms are invariant with respect to conformal transformations:

\[
\begin{cases}
L_g u = 0 & \text{in } M \\
B_g u + (n - 2) f_i^{-t_i} u^{p_i} = 0 & \text{on } \partial M.
\end{cases}
\]

\[ (2.16) \]
where \( p_i \in \left[ \tilde{p}, \frac{n}{n-2} \right] \) for some fixed \( \tilde{p} > 1 \), \( \tau_i = \frac{n}{n-2} - p_i \), \( f_i \to f \) in \( C^1_{\text{loc}} \) for some positive function \( f \) and \( g_i \to g_0 \) in the \( C^3_{\text{loc}} \) topology.

First, we collect the definition of various type of blow up points.

**Definition 6** We say that \( x_0 \in \partial M \) is a blow up point for the sequence \( u_i \) of solutions of (2.16) if there is a sequence \( u_i \) such that

1. \( x_i \to x_0 \);
2. \( x_i \) is a local maximum point of \( u_i|_{\partial M} \);
3. \( u_i(x_i) \to +\infty \).

Shortly we say that \( x_i \to x_0 \) is a blow up point for \( \{u_i\}_i \).

We say that \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) if \( x_i \to x_0 \) is a blow up point for \( \{u_i\}_i \) and there exist two constants \( \rho, C > 0 \) such that

\[
u_i(x) \leq C\tilde{d}_g(x,x_i)^{-\frac{1}{\tilde{p}-1}} \quad \text{for all } x \in \partial M \setminus \{x_i\}, \quad \tilde{d}_g(x,x_i) < \rho.
\]

Here \( \tilde{g} \) denotes the metric on the boundary induced by \( g \) and \( \tilde{d}_g(\cdot, \cdot) \) is the geodesic distance on the boundary between two points.

Finally, given \( x_i \to x_0 \) an isolated blow up point for \( \{u_i\}_i \), and given \( \psi_i : B^+_\rho(0) \to M \) the Fermi coordinates centered at \( x_i \), we define the spherical average of \( u_i \) as

\[
\tilde{u}_i(r) = \frac{2}{\omega_{n-1} r^{n-1}} \int_{\partial^+ B^+_r} u_i \circ \psi_i d\sigma_r
\]

and

\[
w_i(r) := r^{\frac{1}{\tilde{p}-1}} \tilde{u}_i(r)
\]

for \( 0 < r < \rho \).

We say that \( x_i \to x_0 \) is an isolated simple blow up point for \( \{u_i\}_i \) solutions of (2.16) if \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) and there exists \( \rho \) such that \( w_i \) has exactly one critical point in the interval \((0, \rho)\).

It is standard to prove the following proposition (see, for example [1,13,15,23])

**Proposition 7** Let \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) and \( \rho \) as in Definition 6. We set

\[
v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{1-p_i} y), \quad \text{for } y \in B^+_{\rho M_i^{p_i-1}}(0), \quad \text{where } M_i := u_i(x_i)
\]

Then, given \( R_i \to \infty \) and \( \beta_i \to 0 \), up to subsequences, we have

\[
|v_i - U|_{C^2(B^+_{R_i}(0))} < \beta_i \quad \text{and} \quad \lim_{i \to \infty} p_i = \frac{n}{n-2}.
\]

Furthermore, if \( x_i \to x_0 \) is an isolated simple blow up point for \( \{u_i\}_i \), then there exist \( C, \rho > 0 \) such that

1. \( M_i u_i (\psi_i(y)) \leq C|y|^{2-n} \quad \text{for all } y \in B^+_{\rho}(0) \setminus \{0\};
2. \( M_i u_i (\psi_i(y)) \geq C^{-1} G_i(y) \quad \text{for all } y \in B^+_{\rho}(0) \setminus B^+_{\rho}(0) \quad \text{where } r_i := R_i M_i^{1-p_i} \text{ and } G_i \text{ is the Green’s function which solves}
\]

\[
\begin{cases}
L_{\bar{g}_i} G_i = 0 \quad \text{in } B^+_{\rho}(0) \setminus \{0\} \\
G_i = 0 \quad \text{on } \partial^+ B^+_{\rho}(0) \\
B_{\bar{g}_i} G_i = 0 \quad \text{on } \partial' B^+_{\rho}(0) \setminus \{0\}
\end{cases}
\]
and $|y|^{n-2} G_i(y) \to 1$ as $|y| \to 0$.

he usual strategy to prove compactness of solutions of Yamabe problems dates back to Schoen program, which was performed by Li and Zhu in [24], by Marques in [27], and finally achieved in the seminal Khuri Marques and Schoen paper [23]. The idea is to prove firstly that only isolated simple blow up points may occur, then, to give a precise description of the asymptotic profile of a rescaled solution around an isolated simple blow up points. Finally, to rule out the possibility of having isolated simple blow up points.

The key tool to accomplish these steps is a sign estimates of a Pohozaev type formula for a blowing up sequence of solutions that we recall here. This version of Pohozaev identity was previously used in [1,13]

**Theorem 8 (Pohozaev Identity)** Let $u$ a $C^2$-solution of the following problem

$$
\begin{cases}
L_g u = 0 & \text{in } B_r^+ \\
B_g u + (n - 2) f^{-\tau} u^p = 0 & \text{on } \partial B_r^+
\end{cases}
$$

for $B_r^+ = \psi^{-1}(B^+_g(q, r))$ for $q \in \partial M$, with $\tau = \frac{n}{n-2} - p > 0$. Let us define

$$P(u, r) := \int_{\partial^+ B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r + \frac{r(n-2)}{p+1} \int_{\partial (\partial^+ B_r^+)} f^{-\tau} u^{p+1} d\bar{\sigma}
$$

and

$$\bar{P}(u, r) := \int_{\partial^+ B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r + \frac{r(n-2)}{p+1} \int_{\partial (\partial^+ B_r^+)} f^{-\tau} u^{p+1} d\bar{\sigma}
$$

Then

$$\bar{P}(u, r) = P(u, r)
$$

**2.2 A sharp approximation of blow up points**

To describe the asymptotic profile of a rescaled solution around an isolated simple blow up point in the case of manifolds with umbilic boundary we introduce the function $\gamma_q = \gamma$ which solves

$$
\begin{cases}
-\Delta \gamma = \frac{1}{4} \tilde{R}_{ijkl}(q) y_k y_l + R_{nij}(q) y_i^2 & \text{on } \mathbb{R}^n_+ \\
\frac{\partial \gamma}{\partial y_n} = -n U \frac{\gamma}{\partial y_n} & \text{on } \partial \mathbb{R}^n_+
\end{cases}
$$

In [16] and in [15] the authors prove the following lemma.
Lemma 9 Assume $n \geq 5$. Given a point $q \in \partial M$, there exists a solution $\gamma : \mathbb{R}^n_+ \to \mathbb{R}$ of the linear problem (2.17).

In addition it holds

$$|\nabla^\tau \gamma(y)| \leq C(1 + |y|)^{4-\tau-n} \text{ for } \tau = 0, 1, 2;$$

$$\int_{\mathbb{R}^n_+} \gamma \Delta \gamma dy \leq 0;$$

$$\int_{\partial \mathbb{R}^n_+} U \frac{n}{n-2}(t, z) \gamma(t, z) dz = 0;$$

$$\gamma(0) = \frac{\partial \gamma}{\partial y_1}(0) = \cdots = \frac{\partial \gamma}{\partial y_{n-1}}(0) = 0.$$  

Let $x_i \to x_0$ an isolated simple blow up point for $u_i$ of solutions of (2.16). Set

$$v_i(y) := \delta_i^{|1-n|} u_i(\delta_i y) \text{ for } y \in B^+_R(0)$$

we know that $v_i$ satisfies

$$\begin{cases}
L_{\hat{g}_i} v_i = 0 & \text{in } B^+_R(0) \\
B_{\hat{g}_i} v_i + (n - 2) \hat{f}_{\tau_i} v_i^{p_i} = 0 & \text{on } \partial B^+_R(0)
\end{cases}$$

where $\hat{g}_i(y) := \hat{g}_i(\delta_i y) = \Lambda_{\delta_i}^\frac{4}{n-2}(\delta_i y)g(\delta_i y), \hat{f}_i(y) = f_i(\delta_i y), f_i = \Lambda_{x_i} f \to \Lambda_{x_0} f$ and $\tau_i = \frac{n}{n-2} - p_i$.

Using the term $\gamma$ we are able to give a good estimate of the rescaled solution $v_i$ around the isolated blow up point $x_i \to x_0$. Indeed we have the following proposition.

Proposition 10 Assume $n \geq 6$. Let $\gamma$ be defined in (2.17). There exist $R, C > 0$ such that

$$|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C \delta_i^3 (1 + |y|)^{5-n}$$

$$\left| \frac{\partial}{\partial j} \left( v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \right| \leq C \delta_i^3 (1 + |y|)^{4-n}$$

$$\left| y_n \frac{\partial}{\partial n} \left( v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \right| \leq C \delta_i^3 (1 + |y|)^{5-n}$$

$$\left| \frac{\partial^2}{\partial j \partial k} \left( v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \right| \leq C \delta_i^3 (1 + |y|)^{3-n}$$

for $|y| \leq \frac{R}{2\delta_i}$.

This proposition has been proved in [15, Proposition 12] for the case $n \geq 8$, but can be extended with minor changes to dimension $n = 6, 7$, so we will omit the proof.

3 A characterization of function $\gamma$

In this section we give a an accurate description of a solution $\gamma$ of (2.17), similarly to [22]. First we split

$$\gamma = \Phi + E$$

Springer
where $\Phi = \Phi_1 + \Phi_2$ is a polynomial function and $\Phi_1, \Phi_2$ solve, respectively

$$-\Delta \Phi_1 = R_{nij}(q) y_i y_j \partial_{ij} U \quad \text{on } \mathbb{R}^n_+$$

$$-\Delta \Phi_2 = \frac{1}{3} R_{ikjl}(q) y_k y_l \partial_{kl} U \quad \text{on } \mathbb{R}^n_+$$

while $E$ is an harmonic function solving

$$\begin{cases}
-\Delta E = 0 & \text{on } \mathbb{R}^n_+ \\
\lim_{y_n \to 0} \frac{\partial E}{\partial y_n} = -n U \frac{\partial}{\partial y_n} E - q & \text{on } \partial \mathbb{R}^n_+ ,
\end{cases}$$

with $q = \frac{\partial \Phi}{\partial y_n} + nU \frac{\partial}{\partial y_n} \Phi$.

**Lemma 11** For $n = 5$ or $n \geq 7$ the function

$$\Phi_2 = \frac{1}{3} R_{ikjl}(q) y_i y_j y_k y_l \left\{ \frac{n - 2}{6(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+2}{2}}} + a_1 \frac{n(n^2 - 4)(n + 4)}{(n - 6)(n - 4)} \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+6}{2}}} \right\}$$

solves (3.2) for any $a_1 \in \mathbb{R}$.

**Lemma 12** For $n = 5$ or $n \geq 7$ the function

$$\begin{align*}
\Phi_1 &= R_{nij}(q) y_i y_j \left\{ \frac{1}{12(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+2}{2}}} + \frac{n - 2}{6} \frac{1 + y_n^2 - y_n}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n}{2}}} \\
&\quad + a_1 \frac{n(n^2 - 4)}{(n - 4)(n - 6)} \left[ \frac{(n + 4)}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+6}{2}}} - \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+4}{2}}} \right] \\
&\quad + a'_1 \frac{n(n - 2)}{n - 4} \left[ \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+2}{2}}} - 2n(n + 2) \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+4}{2}}} \right] \\
&\quad + a''_1 n(n - 2) \left[ \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+2}{2}}} \right] \right\}
\end{align*}$$

solves (3.1) for any $a_1, a'_1, a''_1 \in \mathbb{R}$.

The proof of these two results is postponed in the appendix.

For our purpose will be sufficient to fix $a_1 = a'_1 = 0$. This allows also to extend the previous results for $n = 6$, as we summarize hereafter.

**Corollary 13** For $n \geq 5$ the functions

$$\begin{align*}
\hat{\Phi}_1 &:= R_{nij}(q) y_i y_j \left\{ \frac{1}{12(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n+2}{2}}} + \frac{n - 2}{6} \frac{1 + y_n^2 - y_n}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n}{2}}} \right\} \\
\hat{\Phi}_2 &:= \frac{1}{3} R_{ikjl}(q) y_i y_j y_k y_l \left\{ \frac{n - 2}{6(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n}{2}}} \right\}
\end{align*}$$

solve respectively (3.1) and (3.2).

**Proof** For $n = 5$ and $n \geq 7$ the result is proved in the appendix, in the proofs of Lemmas 11 and 12. For $n = 6$, notice that both functions $\hat{\Phi}_1, \hat{\Phi}_2$ are well defined Then the claim follows by direct computation.

\( \square \)
4 Case \( n = 7, 8 \)

In [15] it is proved that, if \( x_i \rightarrow x_0 \) is isolated simple blow-up point for \( u_i \), then, for \( n \geq 7 \) it holds

\[
P(u_i, r) \geq R(U, U) + R(U, \delta_i^2 \gamma) + R(\delta_i^2 \gamma, U) + O(\delta_i^{n-2})
\]

\[
\geq \delta_i^4 \int_{\mathbb{R}^n_+} \frac{(n-2)\omega_{n-2} L_i^2}{(n-1)(n-3)(n-5)(n-6)} \left[ \frac{(n-2)}{6} |\bar{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{nlnj}(x_i) \right]
\]

\[
- 2\delta_i^4 \int_{\mathbb{R}^n_+} \gamma_j \Delta \gamma_j d\gamma + o(\delta_i^4).
\]

(4.1)

where

\[
R(u, v) := - \int_{\mathbb{R}^n_+} \left( y^b \partial_b u + \frac{n-2}{2} u \right) \left[ (L_{\hat{g}_i} - \Delta) v \right] d\gamma.
\]

(4.2)

and \( \hat{g}_i(y) := \Lambda_{n-2}^4 (\delta_i) g(\delta_i y) \).

This, for \( n = 7 \), becomes

\[
P(u_i, r) \geq \delta_i^4 \omega_7 L_i^7 \left[ \frac{25}{288} |\bar{W}(x_i)|^2 - \frac{5}{36} R_{7\gamma j}(x_i) \right] - 2\delta_i^4 \int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\gamma + o(\delta_i^4),
\]

(4.3)

and, for \( n = 8 \),

\[
P(u_i, r) \geq \delta_i^4 \omega_8 L_i^8 \left[ \frac{1}{35} |\bar{W}(x_i)|^2 + \frac{1089}{34020} R_{8\gamma j}(x_i) \right] + o(\delta_i^4),
\]

(4.4)

The proof of (4.1) can be found in [15, Prop. 14].

In this section we will prove the following result

**Lemma 14** Let \( x_i \rightarrow x_0 \) is an isolated simple blow-up point for \( u_i \) solution of (2.16) then it holds

\[
P(u_i, r) \geq \delta_i^4 \omega_7 L_i^7 \left[ \frac{25}{288} |\bar{W}(x_i)|^2 + \frac{7}{54} R_{7\gamma j}(x_i) \right] + o(\delta_i^4) \text{ for } n = 7;
\]

(4.5)

\[
P(u_i, r) \geq \delta_i^4 \omega_8 L_i^8 \left[ \frac{1}{35} |\bar{W}(x_i)|^2 + \frac{1089}{34020} R_{8\gamma j}(x_i) \right] + o(\delta_i^4) \text{ for } n = 8.
\]

(4.6)

**4.1 A crucial estimate**

To prove Theorem 1 it will be necessary to estimate the value of \( -\int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\gamma d\gamma_n \) in order to obtain that the right hand sides of (4.3) and of (4.4) are positive. By the description of \( \gamma \) in terms of \( E \) and \( \Phi \), we can simplify this integral term as following.

**Lemma 15** If \( n \geq 7 \) we have

\[
- \int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\gamma d\gamma_n = \int_{\partial \mathbb{R}^n_+} q \Phi d\gamma + \int_{\partial \mathbb{R}^n_+} q E d\gamma - \int_{\mathbb{R}^n_+} \Phi \Delta \Phi d\gamma d\gamma_n.
\]

**Proof** We get, since \( E \) is harmonic, and integrating by parts, that

\[
- \int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\gamma d\gamma_n = - \int_{\mathbb{R}^n_+} (E + \Phi) \Delta \Phi d\gamma d\gamma_n
\]
Now, keeping in mind that \( q = \frac{\partial \Phi}{\partial y_n} + nU \pi^\frac{2}{n-2} \Phi \) and equation (3.3) we have

\[
- \int_{\partial \mathbb{R}^n_+} \partial_n E \Phi d\tilde{y} + \int_{\partial \mathbb{R}^n_+} E \partial_n \Phi d\tilde{y} = \int_{\partial \mathbb{R}^n_+} (nU \pi^\frac{2}{n-2} E + q) \Phi d\tilde{y} + \int_{\partial \mathbb{R}^n_+} E (q - nU \pi^\frac{2}{n-2} \Phi) d\tilde{y}
\]

and we get the result.

\[\square\]

**Lemma 16** If \( n \geq 7 \) we have

\[
\int_{\partial \mathbb{R}^n_+} q E d\tilde{y} = \int_{\mathbb{R}^n_+} |\nabla E|^2 d\tilde{y} - n \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E^2 d\tilde{y} \geq 0.
\]

**Proof** First of all, by (3.3), integrating by parts we have

\[
0 = \int_{\mathbb{R}^n_+} -E \Delta E d\tilde{y} - n \int_{\partial \mathbb{R}^n_+} |\nabla E|^2 d\tilde{y} - n \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E^2 d\tilde{y} - \int_{\partial \mathbb{R}^n_+} q E d\tilde{y}
\]

which proves the first equality. Notice that \( E \in D^{1,2}(\mathbb{R}^n_+) \) by difference, since \( \gamma, \Phi \in D^{1,2}(\mathbb{R}^n_+) \) if \( n > 6 \). Here \( D^{1,2}(\mathbb{R}^n_+) = \{ u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+) : \nabla u \in L^2(\mathbb{R}^n_+) \} \).

To conclude we argue as in [22, Lemma 4.6]. Firstly, observe that, since \( q = \frac{\partial (\Phi_1 + \Phi_2)}{\partial y_n} + nU \pi^\frac{2}{n-2} (\Phi_1 + \Phi_2) \), by Lemma 11, Lemma 12, and in light of identities (2.9), (2.10) we immediately get

\[
\int_{\partial \mathbb{R}^n_+} q U d\tilde{y} = 0.
\]

Now, we use \( E \) and \( U \) as test functions respectively in equation (2.8) and in equation (3.3), obtaining

\[
(n - 2) \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E d\tilde{y} = \int_{\mathbb{R}^n_+} \nabla U \nabla E d\tilde{y} - n \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E \tilde{y} + \int_{\partial \mathbb{R}^n_+} q U d\tilde{y}
\]

\[
= n \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E d\tilde{y},
\]

thus \( \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E d\tilde{y} = 0 \). At this point we can conclude the proof of the Lemma. In fact, it is well known that the function \( U \) is minimizer for

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} |\nabla u|^2 dy - \frac{(n - 2)^2}{2n - 2} \int |u|^{\frac{2n-2}{n-2}} dy
\]

on the Nehari manifold \( \mathcal{M} := \{ u \in D^{1,2}(\mathbb{R}^n_+) : \| u \|_{D^{1,2}}^2 = (n - 2)|u|^{\frac{2n-2}{n-2}} \} \). Since \( E \in D^{1,2}(\mathbb{R}^n_+) \), \( \int_{\mathbb{R}^n_+} \nabla U \nabla E dy = 0 \) and \( \int_{\partial \mathbb{R}^n_+} U \pi^\frac{2}{n-2} E d\tilde{y} = 0 \), we have that \( E \in T_U \mathcal{M} \) and
we can compute

\[
0 \leq \left. \frac{d^2}{dt^2} J(U + tE) \right|_{t=0} = \int_{\mathbb{R}^n_+} |\nabla E|^2 d\bar{y} dy_n - n \int_{\partial \mathbb{R}^n_+} U \frac{\pi^2}{2} E^2 d\bar{y}
\]

which ends the proof.

We can further simplify the estimate for \(-\int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\bar{y} dy_n\).

**Lemma 17** If \(n \geq 7\) we have

\[-\int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\bar{y} dy_n \geq \int_{\partial \mathbb{R}^n_+} q \Phi d\bar{y} - \int_{\mathbb{R}^n_+} \Phi \Delta \Phi d\bar{y} dy_n - \int_{\partial \mathbb{R}^n_+} \Phi \Delta \Phi_1 d\bar{y} dy_n.\]

**Proof** Combining Lemma 15 and Lemma 16 we have that

\[-\int_{\mathbb{R}^n_+} \gamma \Delta \gamma d\bar{y} dy_n \geq \int_{\partial \mathbb{R}^n_+} q \Phi d\bar{y} - \int_{\partial \mathbb{R}^n_+} \Phi \Delta \Phi d\bar{y} dy_n - \int_{\partial \mathbb{R}^n_+} \Phi \Delta \Phi_1 d\bar{y} dy_n.\]

At this point we can prove immediately by (2.11) that

\[\int_{\partial \mathbb{R}^n_+} \frac{\partial \Phi_1}{\partial y_n} \Phi_1 d\bar{y} = \int_{\partial \mathbb{R}^n_+} \frac{\partial \Phi_2}{\partial y_n} \Phi_1 d\bar{y} = \int_{\partial \mathbb{R}^n_+} n U \frac{\pi^2}{2} \Phi_1 \Phi_2 d\bar{y} = 0\]

and by (2.12) that

\[\int_{\partial \mathbb{R}^n_+} \frac{\partial \Phi_2}{\partial y_n} \Phi_2 d\bar{y} = \int_{\partial \mathbb{R}^n_+} n U \frac{\pi^2}{2} \Phi_2^2 d\bar{y} = 0.\]

Now, taking in account equation (3.2), we have

\[-\int_{\mathbb{R}^n_+} \Phi \Delta \Phi_2 d\bar{y} dy_n = \frac{1}{3} \int_{\mathbb{R}^n_+} \Phi \tilde{R}_{ijkl} y_k y_l \partial^2_{ij} U\]

\[= \frac{n(n-2)}{3} \int_{\mathbb{R}^n_+} \Phi \tilde{R}_{ijkl} y_k y_l y_j (|\bar{y}|^2 + (1 + y_n)^2)^{-\frac{n}{2}} = 0\]

again by (2.11) and (2.12). Similarly we prove that \(-\int_{\mathbb{R}^n_+} \Phi_2 \Delta \Phi_1 d\bar{y} dy_n = 0\) and we conclude the proof.

**4.2 Case \(n = 7\)**

In this case we can take \(a_1 = a_1' = a_2' = 0\) in the expression of \(\tilde{\Phi}_1\) given in Lemma 12, so we set

\[\tilde{\Phi}_1 = R_{nilj} y_i y_j A(|\bar{y}|, y_n),\]

where

\[A(|\bar{y}|, y_n) := \frac{1}{12(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n-2}{2}}} + \frac{n-2}{6} \frac{1 + y_n^2 - y_n}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n}{2}}}\]

and we have the final result of this subsection.
Lemma 18 If $n \geq 7$ we have

\[-\int_{\partial R^+} \gamma \Delta \gamma d\tilde{y}d\gamma \geq \frac{2}{n^2 - 1} R^2_{\text{nij}} \left[ \int_{\partial R^+} A(\|\tilde{y}\|, 0) \frac{\partial}{\partial y_n} A(\|\tilde{y}\|, y_n)_{y_n=0} |\tilde{y}|^4 d\tilde{y} \right. \]

\[+ n \int_{\partial R^+} A(\|\tilde{y}\|, 0)^2 |\tilde{y}|^4 d\tilde{y} + n(n - 2) \left. \int_{\partial R^+} A(\|\tilde{y}\|, y_n) \frac{\partial}{\partial y_n} \frac{\partial}{\partial y_n} \frac{\partial}{\partial y_n} \frac{\partial}{\partial y_n} |\tilde{y}|^4 y_n^2 d\tilde{y} d\gamma \right) \tag{4.7} \]

in addition for $n = 7$

\[-\int_{\partial R^+} \gamma \Delta \gamma d\tilde{y} \geq \frac{29}{432} \omega_5 t_7^2 R^2_{77}. \]

Proof We have, by (2.13)

\[\int_{\partial R^+} \frac{\partial \Phi_1}{\partial y_n} \Phi_1 d\tilde{y} \frac{\partial}{\partial y_n} = \frac{2}{n^2 - 1} R^2_{\text{nij}} \int_{\partial R^+} A(\|\tilde{y}\|, 0) \frac{\partial}{\partial y_n} A(\|\tilde{y}\|, y_n)_{y_n=0} |\tilde{y}|^4 d\tilde{y} \]

Similarly we have

\[n \int_{\partial R^+} U \frac{n^2}{\gamma} \Phi_1 \Phi_1 d\tilde{y} = n \int_{\partial R^+} A(\|\tilde{y}\|, 0)^2 |\tilde{y}|^2 + 1 R_{\text{nij}} y_i y_j R_{\text{nlk}} y_l y_k d\tilde{y} \]

Finally, using (3.1) and (2.13) we have

\[\int_{\partial R^+} \Phi_1 \Delta \Phi_1 d\tilde{y} d\gamma = \int_{\partial R^+} A(\|\tilde{y}\|, y_n) R_{\text{nij}} y_i y_j R_{\text{nlk}} y_l y_k U d\tilde{y} d\gamma \]

which, in light of Lemma 17, proves the first claim.

To conclude the proof we will have to estimate several integral quantities involving the functions $A(\|\tilde{y}\|, y_n)$ and its derivative

\[\frac{\partial}{\partial y_n} A(\|\tilde{y}\|, y_n)_{y_n=0} = -\frac{5}{4} \left( 1 + |\tilde{y}|^2 \right) - \frac{35}{6} \left( 1 + |\tilde{y}|^2 \right)^{-\frac{5}{2}} \]

which we compute below. Notice also that, by change of variables, we have

\[\int_{\partial R^+} |\tilde{y}|^4 d\tilde{y} = \omega_5 t_7^9 \text{ and } \int_{\partial R^+} |\tilde{y}|^4 y_n^{\beta} d\tilde{y} d\gamma = \omega_5 t_7^9 \int_0^\infty \frac{t^\beta dt}{(1 + t)^{2\alpha-10}}. \]
Keeping in mind (2.15) we have
\[
\int_{\partial \mathbb{R}^2_+} A \frac{\partial}{\partial y_n} A |\tilde{y}|^4\, d\tilde{y} = -\omega_5 \frac{85}{24} I^9_7. \tag{4.8}
\]
and
\[
7 \int_{\partial \mathbb{R}^2_+} A^2 |\tilde{y}|^4 \left( \frac{1 + |\tilde{y}|^2}{1 + |\tilde{y}|^2} \right)^2 d\tilde{y} = \omega_5 \frac{191}{72} I^9_7. \tag{4.9}
\]
Finally, in light of (2.14), we have
\[
35 \int_{\mathbb{R}^2} A |\tilde{y}|^4 \frac{y_n^2}{(1 + y_n^2 + |\tilde{y}|^2)^2} d\tilde{y} = \omega_5 \frac{5}{2} I^9_7. \tag{4.10}
\]
By (4.8), (4.9) and (4.10) we get the proof.

**Proof of first claim of Lemma 14** By (4.3), (2.15), and by Lemma 18 we immediately get (4.5).

---

**4.3 Case \( n = 8 \)**

For \( n = 8 \) we want to repeat the same strategy used for \( n = 7 \). Unfortunately, taking all the coefficients equal to zero in \( \tilde{\Phi}_1 \) does not prove the sign condition. For this case thus we consider
\[
\tilde{\Phi}_1 = R_{ninj} y_i y_j A(|\tilde{y}|, y_n, b),
\]
where
\[
A(|\tilde{y}|, y_8, b) := \frac{1}{12(|\tilde{y}|^2 + (1 + y_8)^2)^3} + \frac{1 + y_n^2 - y_n}{(|\tilde{y}|^2 + (1 + y_8)^2)^2} + \frac{b}{(|\tilde{y}|^2 + (1 + y_8)^2)^2}.
\]

**Lemma 19** For \( n = 8 \) and \( b = -2 \) we have
\[
- \int_{\mathbb{R}^8_+} \gamma \Delta y d\gamma y \geq \frac{121}{13601} \omega_6 I_{10}^8 R_{8i8j}^2.
\]

**Proof** We can recast (4.7) for \( n = 8 \) obtaining
\[
- \int_{\mathbb{R}^8_+} \gamma \Delta y d\gamma y \geq \frac{2}{63} R_{8i8j}^2 \left[ \int_{\partial \mathbb{R}^8_+} A(|\tilde{y}|, 0, b) \left. \frac{\partial A}{\partial y_8} A(|\tilde{y}|, y_8, b) \right|_{y_8=0} |\tilde{y}|^4 \, d\tilde{y} \Bigg] + 8 \int_{\partial \mathbb{R}^8_+} \frac{A(|\tilde{y}|, 0, b)^2 |\tilde{y}|^4}{|\tilde{y}|^2 + 1} d\tilde{y} + 48 \int_{\mathbb{R}^8_+} \frac{A(|\tilde{y}|, y_8, b) |\tilde{y}|^4 y_8^2}{(|\tilde{y}|^2 + (1 + y_8)^2)^3} d\tilde{y} \right]. \tag{4.11}
\]

We have
\[
\frac{1}{\omega_6} \int_{\partial \mathbb{R}^8_+} A \left. \frac{\partial A}{\partial y_8} \right|_{y_8=0} |\tilde{y}|^4 \, d\tilde{y} = \left[ -\frac{21}{4} - \frac{35}{12} b - \frac{35}{64} b^2 \right] I_{10}^8, \tag{4.12}
\]
and
\[
\frac{8}{\omega_6} \int_{\partial \mathbb{R}^8_+} A^2 \frac{|\tilde{y}|^4}{|\tilde{y}|^2 + 1} \, d\tilde{y} = I_{10}^8 \left[ \frac{221}{54} + \frac{85}{36} b + \frac{7}{16} b^2 \right]. \tag{4.13}
\]

\( \Box \) Springer
and
\[
\frac{48}{\omega_6} \int_{\mathbb{R}^8_+} A(|\vec{y}|, y_8, b)|\vec{y}|^4 y_8^2 d\vec{y} dy_8 = I_8^{10} \left[ \frac{5}{6} + b \frac{5}{144} \right].
\] (4.14)

So by (4.12), (4.13) and (4.14), the inequality (4.11) becomes
\[
- \int_{\mathbb{R}^8_+} \nu \Delta \gamma d\vec{y} dy_8 \geq \frac{2}{63} R^2_{8i8_j} \omega_6 I_8^{10} \left[ \frac{35}{108} - \frac{25}{48} b - \frac{7}{64} b^2 \right]
\]
which for \( b = -2 \) gives the claim. \( \square \)

**Proof of second claim of Lemma 14** By (4.3), (2.15), and by Lemma 19 we immediately get (4.6). \( \square \)

## 5 Case \( n = 6 \)

When dealing with low dimensions, often it is convenient to work in cylindrical sets
\[
D^+_r := [0, r] \times B^6_r \subset \mathbb{R}^6_+
\]
instead of spheres \( B^+_r = B^6_r \cap \mathbb{R}^6_+ \). In the limit \( r \to \infty \) the difference between the two approaches is of higher order, but the boundary of \( D^+_r \) is easier to manage. So, we compute the Pohozaev identity on cylindrical sets. Again, as in [15, Proposition 14] we have that, if \( x_i \to x_0 \) is isolated simple blow-up point for \( u_i \), then
\[
P(u_i, r) \geq R(U, U) + R(U, \delta^2_i \gamma, U) + R(\delta^2_i \gamma, U) + O(\delta^4_i)
\] (5.1)

where \( R(u, v) \) in this case is
\[
R(u, v) := -\int_{D^+_r} \left( \frac{y^\beta \partial_h u + \frac{n-2}{2} u}{(1+y_n)^\alpha + 1} \right) [(L_{\delta_i} - \Delta) v] dy.
\]

Throughout this section we will prove the following lemma.

**Lemma 20** Let \( x_i \to x_0 \) be an isolated simple blow-up point for \( u_i \) solution of (2.16) then it holds
\[
P(u_i, r) \geq R(U, U) + R(U, \delta^2_i \gamma_x) + R(\delta^2_i \gamma_x, U) + O(\delta^4_i)
\]
\[
= \omega_4 I_6^6 \delta^4_i \log \left( \frac{1}{\delta_i} \right) \left[ \frac{8}{45} \tilde{W}(x_i) \right]^2 + \frac{8}{15} R^2_{8i8_k}(x_i) + O(\delta^4_i)
\] (5.2)

**Remark 21** We recall the following elementary identity, obtained by change of variables
\[
\int_0^r \int_{B_r^{-1}} \frac{|\vec{y}|^\beta d\vec{y}}{(1+y_n)^2 + |\vec{y}|^2} = \int_0^r \frac{(1+y_n)^{\beta+n-1} dy_n}{(1+y_n)^{2\alpha}} \int_{B_r^{-1}} \frac{|\vec{y}|^\beta d\vec{y}}{1 + |\vec{y}|^2} \quad (5.3)
\]
and, finally, that
\[
\int_0^{r/\delta} \frac{y_n^{\alpha}}{(1+y_n)^{\alpha+1}} = \log \left( \frac{1}{\delta} \right) + O(1), \quad \text{and} \quad \int_0^{r/\delta} \frac{y_n^{\alpha+2} - y_n^\alpha}{(1+y_n)^{\alpha+3}} = \log \left( \frac{1}{\delta} \right) + O(1)
\]
for \( \alpha = 0, 2, 4 \).
With these premises we have the following result (in order to simplify notation, we denote $\delta$ for $\delta_i$ and $q$ for $x_i$).

**Lemma 22** We have

$$R(U, U) = \omega_4 I_8^6 \delta^4 \log \left( \frac{L}{\delta} \right) \left[ \frac{8}{45} |\hat{W}(q)|^2 - \frac{16}{15} R_{nins}^2 \right] + O(\delta^4).$$

**Proof** The proof is similar to [15, Lemma 15]. We focus here on the main differences, omitting the standard calculations. By definition of $L_{\delta_i}$ we have

$$R(U, U) = \frac{(n - 2)^2}{2} \int_{D_{r^2-1}} \frac{|y|^2 - 1}{[(1 + y_n)^2 + |\vec{y}|^2]^{n+1}} n y_i y_j \left( g^{ij}(\delta y) - \delta^{ij} \right) dy$$

$$- \frac{(n - 2)^2}{2} \int_{D_{r^2-1}} \frac{|y|^2 - 1}{[(1 + y_n)^2 + |\vec{y}|^2]^{n}} \left( g^{ij}(\delta y) - 1 \right) dy$$

$$- \frac{(n - 2)^2}{2} \int_{D_{r^2-1}} \frac{|y|^2 - 1}{[(1 + y_n)^2 + |\vec{y}|^2]^{n}} \delta \delta_i g^{ij}(\delta y) y_j dy$$

$$- \frac{(n - 2)^2}{8(n - 1)} \int_{D_{r^2-1}} \frac{|y|^2 - 1}{[(1 + y_n)^2 + |\vec{y}|^2]^{n-1}} \delta^2 R_g(\delta y) dy + O(\delta^4)$$

$$=: A_1 + A_2 + A_3 + A_4 + O(\delta^4).$$

Using the symmetries of the curvature tensor and the expansion of the metric we have that, for $n = 6$,

$$A_1 = \delta^4 \frac{24}{5} R_{nins}^2 \int_{D_{r^2-1}} \frac{|y|^2 - 1|\vec{y}|^2 y_n^2}{[(1 + y_n)^2 + |\vec{y}|^2]^7} dy$$

$$+ \delta^4 \frac{48}{35} R_{nins, ji} \int_{D_{r^2-1}} \frac{|y|^2 - 1|\vec{y}|^4 y_n^2}{[(1 + y_n)^2 + |\vec{y}|^2]^7} dy + O(\delta^4). \quad (5.4)$$

$$A_2 + A_3 = -\delta^4 4 R_{nins}^2 \int_{D_{r^2-1}} \frac{|y|^2 - 1 y_n^4}{[(1 + y_n)^2 + |\vec{y}|^2]^6} dy$$

$$- \delta^4 \frac{8}{5} R_{nins, ij} \int_{D_{r^2-1}} \frac{|y|^2 - 1|\vec{y}|^2 y_n^2}{[(1 + y_n)^2 + |\vec{y}|^2]^6} dy + O(\delta^4). \quad (5.5)$$

and

$$A_4 = \delta^4 \frac{1}{150} |\hat{W}(q)|^2 \int_{D_{r^2-1}} \frac{|y|^2 - 1|\vec{y}|^2}{[(1 + y_n)^2 + |\vec{y}|^2]^5} dy$$

$$+ \delta^4 \frac{2}{5} R_{nins}^2 \int_{D_{r^2-1}} \frac{|y|^2 - 1 y_n^2}{[(1 + y_n)^2 + |\vec{y}|^2]^5} dy$$

$$+ \delta^4 \frac{2}{5} R_{nins, ij} \int_{D_{r^2-1}} \frac{|y|^2 - 1 y_n^2}{[(1 + y_n)^2 + |\vec{y}|^2]^5} dy + O(\delta^4). \quad (5.6)$$

Now, using (5.3) we have
Again, we follow the main lines of [15, Lemma 16]. We have by definition of $R(u, v)$, by (2.3) and (2.18), that

$$R(U, \delta^2 \gamma) + R(\delta^2 \gamma, U) = -\delta^4 \int_{D_{r^2-1}^+} \gamma \Delta \gamma dy + O(\delta^4).$$

**Proof** Again, we follow the main lines of [15, Lemma 16]. We have by definition of $R(u, v)$, by (2.3) and (2.18), that

$$R(U, \delta^2 \gamma) + R(\delta^2 \gamma, U) = -\delta^4 \int_{D_{r^2-1}^+} (y_b \partial_b U + 2U) \left[ \frac{1}{3} \bar{R}_{ijk} \gamma_k y_l + R_{nj} y_n^2 \right] \partial_i \partial_j \gamma dy$$

$$- \delta^4 \int_{D_{r^2-1}^+} y_b \partial_b \gamma \left[ \frac{1}{3} \bar{R}_{ijk} y_k y_l + R_{nj} y_n^2 \right] \partial_i \partial_j U dy$$

$$- \delta^4 \int_{D_{r^2-1}^+} 2\gamma \left[ \frac{1}{3} \bar{R}_{ijk} \gamma_k y_l + R_{nj} y_n^2 \right] \partial_i \partial_j U dy + O(\delta^4).$$

By (2.17), immediately we have

$$A_3 = 2 \int_{D_{r^2-1}^+} \gamma \Delta \gamma. \quad (5.10)$$
Integrating by parts, and recalling that the index $b = 1, \ldots, n$ while $i, j, k, l, s = 1, \ldots, n - 1$, we have

\[
A_2 = 6 \int_{D^+_r} \gamma \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U dy + \int_{D^+_r} y_b y \partial_b \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U dy + \int_{D^+_r} y_b y \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_b \partial_i \partial_j U dy + \int_{D^+_r} y_b y \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_b \partial_i \partial_j U dy + \int_{\partial B^5_{r/\delta}} y_s v_s y \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U d\sigma dy.
\]

Now, we estimate the boundary terms. On $\partial B^5_{r/\delta}$ we have $y_s v_s = |\bar{y}| = r/\delta$. Taking in account (2.18) we get

\[
\left| \int_{0}^{r/\delta} \int_{\partial B^5_{r/\delta}} y_s v_s y \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U d\sigma dy \right| \leq C \int_{0}^{r/\delta} \int_{\partial B^5_{r/\delta}} \left( \frac{r}{\delta} \right)^{-5} d\sigma dy = O(1).
\]

Similarly we obtain

\[
\left| \int_{B^5_{r/\delta}} y_n y_q \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U \right|_{y_n = \frac{r}{5}} d \bar{y} = O(1).
\]

Moreover,

\[
\int_{D^+_r} y_b y \partial_b \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U dy = \int_{D^+_r} y_s y \partial_s \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l \right] \partial_i \partial_j U dy + \int_{\partial B^5_{r/\delta}} y_n y \left[ R_{ninj} y_n^2 \right] \partial_i \partial_j U dy = 2 \int_{D^+_r} \gamma \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_i \partial_j U dy = -2 \int_{D^+_r} \gamma \Delta \gamma,
\]

and, using (2.17) for the first term of (5.11) we have

\[
A_2 = -8 \int_{D^+_r} \gamma \Delta \gamma dy + \int_{D^+_r} y_b y \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_b \partial_i \partial_j U dy + O(1).
\]

For the term $A_1$ we integrate by parts twice. As before, all the boundary terms are estimated by a constant number. So, using the symmetries of the curvature tensor we have, after the first integration,

\[
A_1 = \int_{D^+_r} (\partial_i U + y_b \partial_i \partial_b U + 2 \partial_i U) \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_j \gamma dy + \int_{D^+_r} (y_b \partial_i U + 2 U) \partial_i \left[ \frac{1}{3} \bar{R}_{ijkl} y_k y_l + R_{ninj} y_n^2 \right] \partial_j \gamma dy + O(1)
\]

\[\copyright\] Springer
\[ = \int_{D_{r_\delta}^{+}} (3\partial_t U + y_b \partial_t \partial_b U) \left[ \frac{1}{3} \bar{R}_{kjl} y_k y_l + R_{nij} y_n^2 \right] \partial_j \gamma dy + O(1) \]

And, integrating again,

\[ A_1 = -\int_{D_{r_\delta}^{+}} (4\partial_j \partial_t U + y_b \partial_j \partial_t \partial_b U) \left[ \frac{1}{3} \bar{R}_{kjl} y_k y_l + R_{nij} y_n^2 \right] \gamma dy + O(1) \]

\[ = 4 \int_{D_{r_\delta}^{+}} \gamma \Delta \gamma dy - \int_{D_{r_\delta}^{+}} y_b \partial_j \partial_t \partial_b U \left[ \frac{1}{3} \bar{R}_{kjl} y_k y_l + R_{nij} y_n^2 \right] \gamma dy + O(1). \]

Adding \( A_1, A_2 \) and \( A_3 \) we get the proof. \( \square \)

**Lemma 24** We have

\[ \int_{D_{r_\delta}^{+}} \gamma \Delta \gamma dy = \int_{D_{r_\delta}^{+}} \Phi_1 \Delta \Phi_1 dy + O(1). \]

**Proof** We have, integrating by parts, and since \( E \) is harmonic

\[ \int_{D_{r_\delta}^{+}} \gamma \Delta \gamma dy = \int_{D_{r_\delta}^{+}} (\Phi + E) \Delta (\Phi + E) dy = \int_{D_{r_\delta}^{+}} (\Phi + E) \Delta \Phi dy \]

\[ = -\int_{D_{r_\delta}^{+}} \nabla (\Phi + E) \nabla \Phi dy + \int_{\partial B_{r_\delta}^{+}} (\Phi + E) \nabla \Phi \cdot v d\sigma dy_n. \]

(5.12)

By the decay of \( \gamma \) in (2.18) and by the explicit expression of \( \Phi \) in Corollary 13, we obtain

\[ |\nabla^\gamma E(y) \leq C(1 + |y|)^{4-\tau-n} \]

and the same holds for \( \Phi \). At this point we can easily see that

\[ \left| \int_{0}^{r/\delta} \int_{\partial B_{r_\delta}^{+}} (\Phi + E) \nabla \Phi \cdot v d\sigma dy_n \right| = O(1). \]

We can integrate again by parts in (5.12) and, keeping in mind that again the boundary term is estimate by a constant and that \( E \) is harmonic, we get

\[ \int_{D_{r_\delta}^{+}} \gamma \Delta \gamma dy = -\int_{D_{r_\delta}^{+}} \nabla (\Phi + E) \nabla \Phi dy + O(1) \]

\[ = \int_{D_{r_\delta}^{+}} \Delta (\Phi + E) \Phi dy + O(1) = \int_{D_{r_\delta}^{+}} \Phi \Delta \Phi dy + O(1). \]

Now we proceed in Lemma 17, using (2.11) and (2.12) to prove that

\[ \int_{D_{r_\delta}^{+}} \Phi \Delta \Phi dy = \int_{D_{r_\delta}^{+}} \Phi_1 \Delta \Phi_1 dy \]

and concluding the proof. \( \square \)

**Lemma 25** We have

\[ R(U, \delta^2 \gamma) + R(\delta^2 \gamma, U) = \omega_4 I_0^6 \delta^4 \log \left( \frac{r}{\delta} \right) \frac{24}{15} R_{nins}^2 + O(\delta^4). \]
Proof By Lemma 23 and Lemma 24, taking into account (2.13), we have

\[ R(U, \delta^2 \gamma) + R(\delta^2 \gamma, U) = -2\delta^4 \int_{D_{r^2-1}^+} \Phi_1 \Delta \Phi_1 dy + O(\delta^4) \]

\[ = \frac{8}{35} \delta^4 R_{\text{inis}}^2 \left[ \int_{D_{r^2-1}^+} \frac{|\tilde{y}|^4 y_n^2 dy}{(|\tilde{y}|^2 + (1 + y_n)^2)^6} + 8 \int_{D_{r^2-1}^+} \frac{|\tilde{y}|^4 (y_n^2 + y_n^4 - y_n^3) dy}{(|\tilde{y}|^2 + (1 + y_n)^2)^7} \right] \]

\[ + O(\delta^4) \]

\[ =: \frac{8}{35} \delta^4 R_{\text{inis}}^2 [B_1 + B_1] + O(\delta^4). \]

By (5.3) we have

\[ B_1 = \int_0^{r_0/\delta} \frac{y_n^2 dy_n}{(1 + y_n)^3} \int_{B_{r_0/\delta}} \frac{|\tilde{y}|^4 d\tilde{y}}{(1 + |\tilde{y}|^2)^6} = \log \left( \frac{1}{\delta} \right) I_6^8 \]

\[ B_2 = 8 \int_0^{r_0/\delta} \frac{y_n^2 + y_n^4 - y_n^3 dy_n}{(1 + y_n)^3} \int_{B_{r_0/\delta}} \frac{|\tilde{y}|^4 d\tilde{y}}{(1 + |\tilde{y}|^2)^7} = 8 \log \left( \frac{1}{\delta} \right) I_7^8 \]

and, by (5) we get the proof. \( \square \)

Proof of Lemma 20. Lemma 22 and Lemma 25 lead us to (5.2). \( \square \)

6 Proof of the main result

We start proving the following Weyl vanishing property.

Proposition 26 Let \( n \geq 6 \) and let \( x_i \to x_0 \) be an isolated simple blow up point for \( u_i \) solution of (2.16) Then

\[ W(x_0) = 0. \]

Proof By a direct application of Proposition 7 to the definition of \( \bar{P} \) we have

\[ \bar{P}(u_i, r) \leq C \delta_i^{n-2}. \]

By Theorem 8, \( \bar{P}(u_i, r) \) equals \( P(u_i, r) \), which, in turn, is bounded from below by the estimates (4.5), (4.6) and (5.2). As a consequence it holds

\[ |\bar{W}(x_i)|^2 + R_{\text{inis}}^2(x_i) \leq \begin{cases} 
C \delta_i^2 & \text{for } n = 8 \\
C \delta_i & \text{for } n = 7 \\
-C (\log(\delta_i))^{-1} & \text{for } n = 6
\end{cases} \]

which gives the result. In fact \( W(x_0) = 0 \) if and only if both \( \bar{W} \) and \( R_{\text{inis}}^2 \) vanish at \( x_0 \). \( \square \)

Now we give a series of results whose proofs are very similar to the ones contained in [15], so we will omit them.

First, we can rule out the possibility to have isolated blow up points which are not simple. As in the previous proposition, for the proof it is crucial that \( P(u_i, r) \) is strictly positive when \( |W(x_0)| \neq 0 \), which we have proved in equations (4.5) and (5.2).

Proposition 27 Assume \( n \geq 6 \). Let \( x_i \to x_0 \) be an isolated simple blow up point for \( u_i \) solution of (2.16). Assume \( |W(x_0)| \neq 0 \). Then \( x_0 \) is isolated simple.
Next, we can prove a splitting lemma.

**Proposition 28** Assume $n \geq 6$. Given $\beta > 0$ and $R > 0$ there exist two constants $C_0, C_1 > 0$ (depending on $\beta$, $R$ and $(M, g)$) such that, if $u$ is a solution of

\[
\begin{aligned}
L_g u &= 0 \quad \text{in } M \\
B_g u + (n-2) f^{-\tau} u^p &= 0 \quad \text{on } \partial M
\end{aligned}
\]

and $\max_{\partial M} u > C_0$, then $\tau := \frac{n-2}{n-2} - p < \beta$ and there exist $q_1, \ldots, q_N \in \partial M$, with $N = N(u) \geq 1$ with the following properties: for $j = 1, \ldots, N$

1. Set $r_j := R u(q_j)^{1-p}$, then $\{B_j \cap \partial M\}_j$ are a disjoint collection;
2. we have $\left| u(q_j)^{-1} u(\psi_j(y)) - U(u(q_j)^{p-1} y) \right|_{C^2(B_{r_j}^{+})} < \beta$ (here $\psi_j$ are the Fermi coordinates at point $q_j$);
3. we have

\[
\begin{aligned}
\min_{i \neq j} d_{\bar{g}}(x_i(u), q_j(u)) \geq C_0 \text{ for any } j \neq k.
\end{aligned}
\]

Here $d_{\bar{g}}$ is the geodesic distance on $\partial M$.

Assume also $W(x) \neq 0$ for any $x \in \partial M$. Then there exists $d = d(\beta, R)$ such that, for any solution of (6.1) with $\max_{\partial M} u > C_0$, we have

\[
\min_{i \neq j} d_{\bar{g}}(q_i(u), q_j(u)) \geq d.
\]

Now we can prove our main result.

**Proof of Theorem 1.** We proceed by contradiction, supposing that there exists a sequence of solutions $\{u_i\}_i$ of problems (2.16) and that $x_i \to x_0$ is a blow up point for $u_i$. Let $q_1(u_i), \ldots, q_N(u_i) \in \partial M$ the sequence of points given by proposition 28. We can prove that $d_{\bar{g}}(x_i, q_k(u_i)) \to 0$ for some sequence of $k_i$. So $q_k \to x_0$ is a blow up point for $u_i$. Now by propositions 28 and 27 we have that $q_k \to x_0$ is an isolated simple blow up point for $u_i$. Then, by Proposition 26, this should imply that $|W(x_0)| = 0$ which contradicts our hypotheses, proving the theorem.

\[\square\]

### 7 Appendix: proofs of Lemma 11 and Lemma 12

We recall a result contained in [22].

**Lemma 29** Suppose $n = 5$ or $n \geq 7$. We have

1. The function

\[
\Phi_0 := \frac{1}{4(n-6)} \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n-6}{2}}} + \frac{a_1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n-2}{2}}} + a_2
\]

for $a_1, a_2 \in \mathbb{R}$ satisfies

\[
-\Delta \Phi_0 := \frac{1}{(|\bar{y}|^2 + (1 + y_n)^2)^{\frac{n-4}{2}}}
\]

(7.1)
(2) The function \( \Phi_1 := \frac{1}{4(n-4)} \frac{y_n+1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} + a_1 \frac{y_n+1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} = -\left( \frac{1}{n-4} \right) \partial_n \Phi_0, \) for \( a_1 \in \mathbb{R} \) satisfies

\[- \Delta \Phi_1 := \frac{y_n+1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} \quad (7.2)\]

(3) The function \( \Phi_2 := \frac{1}{2(n-4)} \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} + a_2 \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} + a_2', \) for \( a_2, a_2' \in \mathbb{R} \) satisfies

\[- \Delta \Phi_2 := \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} \quad (7.3)\]

**Proof** The first claim is proved in [22, Lemma A.1] (in particular in formula (A.2)). The second claim is proved again in [22, Lemma A.1], while the last claim corresponds to [22, Lemma A.2].

**Lemma 30** Let \( n \geq 5 \). The function

\[ \Phi_0 = \begin{cases} 
\frac{1}{6(n-8)} \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-8}{2}}} + a_1 \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} + a_2 & \text{for } n \neq 8 \\
-\frac{1}{12} \log(|\overline{y}|^2+(1+y_n)^2) + a_1 \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} + a_2 & \text{for } n = 8
\end{cases}\]

for \( a_1, a_2 \in \mathbb{R} \) satisfies

\[- \Delta \Phi_0 := \frac{1}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-6}{2}}} \quad (7.4)\]

**Proof** By change of variables we have that

\[- \Delta \tilde{\Phi}_0(\tilde{y}, y_n-1) = \frac{1}{(|\overline{y}|^2+y_n^2)^{\frac{n-6}{2}}} = \frac{1}{r^{n-6}}, \quad (7.5)\]

where \( r := \sqrt{|\overline{y}|^2+y_n^2} \). So, in spherical coordinates, set \( \varphi_0(r) = \tilde{\Phi}_0(\tilde{y}, y_n-1) \), (7.5) becomes

\[- \varphi_0'' - \frac{n-1}{r} \varphi_0' = \frac{1}{r^{n-6}} \quad (7.6)\]

and one can check that

\[ \varphi_0(r) = \begin{cases} 
\frac{1}{6(n-8)} \frac{1}{r^{n-8}} + \frac{a_1}{r^{n-2}} + a_2 & \text{for } n \neq 8 \\
-\frac{1}{6} \log r + \frac{a_1}{r^2} + a_2 & \text{for } n = 8
\end{cases} \]

solves (7.6).

**Lemma 31** Let \( n = 5 \) or \( n \geq 7 \). Set \( \beta_{kl} := \frac{\partial^2 \Phi_0}{(n-6)(n-4)} + \frac{\Phi_0}{(n-3)} \delta_{kl} \). Then

\[- \Delta \beta_{kl} = y_k y_l U. \quad (7.7)\]

**Proof** By (7.4) we have \( -\Delta \beta_{kl} = \frac{(n-6)(n-4) y_k y_l}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} - \frac{(n-6) \delta_{kl}}{(|\overline{y}|^2+(1+y_n)^2)^{\frac{n-2}{2}}} \) and by (7.1) we get the result.

Now we can achieve the prove of the two lemmas.
Proof of Lemma 11. Since $\bar{R}_{ijk} = 0$ have $\Phi_2 := \frac{1}{3} \bar{R}_{ijkl}(q) \partial_{ij}^2 \left( \frac{\partial_x^2 \Phi_0}{(n-6)(n-4)} \right) = \frac{1}{3} \bar{R}_{ijkl}(q) \partial_{ij}^2 \beta_{kl}$. Thus, by (7.7), we have

$$-\Delta \Phi_2 = \frac{1}{3} \bar{R}_{ijkl}(q) \partial_{ij}^2 (-\Delta \beta_{kl}) = \frac{1}{3} \bar{R}_{ijkl}(q) \partial_{ij}^2 (y_k y_l U) \frac{1}{3} \bar{R}_{ijkl}(q) y_k y_l \partial_{ij}^2 U$$

using the symmetry of the curvature tensor.

Proof of Lemma 12. By Lemma 11 and Lemma 30 we have that

$$-\Delta \left[ \frac{\partial_x^2 \Phi_0}{n(n-6)(n-4)} + \frac{\Phi_0}{n-4} + \Phi_2 - 2\Phi_1 \right] = \gamma^2_n U,$$

so $\Phi_1 = R_{nij}(q) \partial_{ij}^2 \left[ \frac{\partial_x^2 \Phi_0}{(n-6)(n-4)} + \frac{\Phi_0}{n-4} + \Phi_2 - 2\Phi_1 \right]$. The claim follows by direct computation.

References

1. Almaraz, S.: A compactness theorem for scalar-flat metrics on manifolds with boundary. Calc. Var. 41, 341–386 (2011)
2. Almaraz, S.: Blow-up phenomena for scalar-flat metrics on manifolds with boundary. J. Differ. Equ. 251(7), 1813–1840 (2011)
3. Almaraz, S.: An existence theorem of conformal scalar-flat metrics on manifolds with boundary. Pac. J. Math. 248, 1–22 (2010)
4. Almaraz, S., Queiroz, O., Wang, S.: A compactness theorem for scalar-flat metrics on 3-manifolds with boundary. J. Funct. Anal. 277, 1813–1840 (2011)
5. Aubin, T.: Equations differentielles non lineaires et probleme de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. 55, 269–296 (1976)
6. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer, Berlin (1998)
7. Brendle, S.: Convergence of the Yamabe flow in dimension 6 and higher. Invent. Math. 170, 541–576 (2007)
8. Chen, S.S.: Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions, arxiv:0912.1302
9. Escobar, J.: Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature and boundary. Ann. Math. 136, 1–50 (1992)
10. Escobar, J.: Sharp constant in a Sobolev trace inequality. Indiana Univ. Math. J. 37, 687–698 (1988)
11. Felli, V.: Ahmedou, M Ould: Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries. Math. Ann. 322, 667–699 (2002)
12. Felli, V.: Ahmedou, M Ould: A geometric equation with critical nonlinearity on the boundary. Pac. J. Math. 218, 75–99 (2005)
13. Ghimenti, M.G., Micheletti, A.M.: Compactness for conformal scalar-flat metrics on umbilic boundary manifolds 200, 30 p. (2020)
14. Ghimenti, M., Micheletti, A.M., Pistoia, A.: Blow-up phenomena for linearly perturbed Yamabe problem on manifolds with umbilic boundary. J. Differ. Equ. 267, 587–618 (2019)
15. Giraud, G.: Sur la probleme de Dirichlet generalise. Ann. Sci. École Norm. Sup. 46, 131–145 (1929)
16. Han, Z.C., Li, Y.: The Yamabe problem on manifolds with boundary: existence and compactness results. Duke Math. J. 99, 489–542 (1999)
17. Hebey, E., Vaugon, M.: Le probleme de Yamabe equivariant. Bull. Sci. Math. 117, 241–286 (1993)
22. Kim, S., Musso, M., Wei, J.: Compactness of scalar-flat conformal metrics on low-dimensional manifolds with constant mean curvature on boundary. arXiv:1906.01317
23. Khuri, M., Marques, F., Schoen, R.: A compactness theorem for the Yamabe problem. J. Differ. Geom. 81, 143–196 (2009)
24. Li, Y.Y., Zhu, M.: Yamabe type equations on three dimensional Riemannian manifolds. Commun. Contemp. Math. 1, 1–50 (1999)
25. Mayer, M., Ndiaye, C.B.: Barycenter technique and the Riemann mapping problem of Cherrier-Escobar. J. Differ. Geom. 107(3), 519–560 (2017)
26. Marques, F.: Existence results for the Yamabe problem on manifolds with boundary. Indiana Univ. Math. J. 54, 1599–1620 (2005)
27. Marques, F.: A priori estimates for the Yamabe problem in the non-locally conformally flat case. J. Differ. Geom. 71, 315–346 (2005)
28. Marques, F.: Compactness and non compactness for Yamabe-type problems. Prog. Nonlinear Differ. Equ. Appl. 86, 121–131 (2017)
29. Schoen, R., Zhang, D.: Prescribed scalar curvature on the n-sphere. Calc. Var. Partial Differ. Equ. 4, 1–25 (1996)
30. Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. 20, 479–495 (1984)
31. Trudinger, N.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Annali Scuola Norm. Sup. Pisa 22, 265–274 (1968)
32. Yamabe, H.: On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12, 21–37 (1960)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.