LOJASIEWICZ–SIMON GRADIENT INEQUALITIES FOR COUPLED
YANG-MILLS ENERGY FUNCTIONALS

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Abstract. We prove Łojasiewicz–Simon gradient inequalities for coupled Yang-Mills energy functionals using Sobolev spaces which impose minimal regularity requirements on pairs of connections and sections. The Łojasiewicz–Simon gradient inequalities for coupled Yang-Mills energy functionals generalize that of the pure Yang-Mills energy functional due to the first author [26, Theorem 23.9] for base manifolds of arbitrary dimension and due to Råde [52, Proposition 7.2] for dimensions two and three.

Contents
1. Introduction 2
1.1. A brief history of Łojasiewicz–Simon gradient inequalities and their applications to gradient flows and energy gaps in geometric analysis 3
1.2. Łojasiewicz–Simon gradient inequalities for analytic functionals on Banach spaces 4
1.3. Łojasiewicz–Simon gradient inequality for coupled Yang-Mills $L^2$-energy functionals
1.3.1. Łojasiewicz–Simon gradient inequality for boson and fermion coupled Yang-Mills $L^2$-energy functionals 6
1.3.2. Łojasiewicz–Simon gradient inequality for the Yang-Mills-Higgs $L^2$-energy functional 10
1.3.3. Łojasiewicz–Simon gradient inequality for the Seiberg-Witten $L^2$-energy functionals 11
1.3.4. Łojasiewicz–Simon gradient inequality for the non-Abelian monopole $L^2$-energy functionals 12
1.4. Applications of the Łojasiewicz–Simon gradient inequality for the coupled Yang-Mills energy functionals 13
1.5. Automorphisms and transformation to Coulomb gauge 13
1.5.1. Transformation to Coulomb gauge 14
1.5.2. Real analytic Banach manifold structures on quotient spaces 15
1.6. Outline of the monograph 16
1.7. Notation and conventions 16
1.8. Acknowledgments 17
2. Existence of Coulomb gauge transformations for connections and pairs 17
1. Introduction

Our primary goal in this work is to prove Lojasiewicz–Simon gradient inequalities for coupled Yang-Mills $L^2$-energy functionals. A key feature of our results is that we use systems of Sobolev norms that are as weak as possible. This property turns out to be essential in their application to the analysis of gradient flows or energy gaps. Those are typical examples of applications of Lojasiewicz–Simon gradient inequalities in geometric analysis, as illustrated by results of the first author in [26, 25]. Our gradient inequalities use $W^{1,p}$ Sobolev norms for coupled Yang-Mills pairs over manifolds of arbitrary dimension $d \geq 2$, including the case $p = d/2$, where the Sobolev exponent is borderline (or critical) in sense that we explain later in this Introduction.
In the remainder of our Introduction, we outline in Section 1.1 the history of Lojasiewicz–Simon gradient inequalities and survey their applications in geometric analysis, mathematical physics, and applied mathematics. In Section 1.2, we review our abstract Lojasiewicz–Simon gradient inequality for an analytic functional on a Banach space. We state our results on Lojasiewicz–Simon gradient inequalities for coupled Yang-Mills $L^2$-energy functionals in Section 1.3. Unlike in the case of the harmonic map energy functional considered in [32], one must restrict the Hessian of a coupled Yang-Mills energy functional to a suitable slice for the action of the group of gauge transformations in order to obtain an elliptic operator that has the Fredholm property required by our abstract Lojasiewicz–Simon gradient inequality [32, Theorem 2]. In order to obtain the strongest possible version of the resulting Lojasiewicz–Simon gradient inequality for a coupled Yang-Mills energy functional, we therefore need to prove existence of a global transformation to Coulomb gauge valid for borderline Sobolev exponents — going beyond standard results described in [24, 37] or previous results due to the first author [28] — and we state the required theorem in Section 1.5.

1.1. A brief history of Lojasiewicz–Simon gradient inequalities and their applications to gradient flows and energy gaps in geometric analysis. Since its discovery by Lojasiewicz in the context of analytic functions on Euclidean spaces [72, Proposition 1, p. 92] and subsequent generalization by Simon to a class of analytic functionals on certain Hölder spaces [86, Theorem 3], the Lojasiewicz–Simon gradient inequality has played a significant role in analyzing questions such as a) global existence, convergence, and analysis of singularities for solutions to nonlinear evolution equations that are realizable as gradient-like systems for an energy functional, b) uniqueness of tangent cones, and c) energy gaps and discreteness of energies. For applications of the Lojasiewicz–Simon gradient inequality to gradient flows in geometric analysis, beginning with the harmonic map energy functional, we refer to Irwin [59], Kwon [67], Liu and Yang [71], Simon [87], and Topping [92]; for gradient flow for the Chern-Simons functional, see Morgan, Mrowka, and Ruberman [76]; for gradient flow for the Yamabe functional, see Brendle [12, Lemma 6.5 and Equation (100)] and Carlatto, Chodosh, and Rubinstein [15]; for Yang-Mills gradient flow, we refer to our monograph [26], Råde [82], and Yang [101]; for mean curvature flow, we refer to the survey by Colding and Minicozzi [23]; and for Ricci curvature flow, see Ache [3], Haslhofer [51], Haslhofer and Müller [52], and Kroncke [65, 66].

For applications of the Lojasiewicz–Simon gradient inequality to proofs of global existence, convergence, convergence rate, and stability of non-linear evolution equations arising in other areas of mathematical physics (including the Cahn-Hilliard, Ginzburg-Landau, Kirchoff-Carrier, porous medium, reaction-diffusion, and semi-linear heat and wave equations), we refer to the monograph by Huang [57] for a comprehensive introduction and to the articles by Chill [17, 18], Chill and Fiorenza [19], Chill, Haraux, and Jendoubi [20], Chill and Jendoubi [21, 22], Feireisl and Simondon [34], Feireisl and Takáč [35], Grasselli, Wu, and Zheng [43], Haraux [46], Haraux and Jendoubi [47, 48, 49], Haraux, Jendoubi, and Kavian [50], Huang and Takáč [58], Jendoubi [60], Rybka and Hoffmann [84, 85], Simon [86], and Takáč [90]. For applications to fluid dynamics, see the articles by Feireisl, Laurençot, and Petzeltová [33], Frigeri, Grasselli, and Krejčí [39], Grasselli and Wu [42], and Wu and Xu [102].

For applications of the Lojasiewicz–Simon gradient inequality to proofs of energy gaps and discreteness of energies for Yang-Mills connections, we refer to our articles [29]. A key feature
of our version of the Lojasiewicz–Simon gradient inequality for the pure Yang-Mills energy functional [26, Theorem 23.9] is that it holds for $W^{1,p}$ Sobolev norms. Those norms are considerably weaker than the $C^{2,\alpha}$ Hölder norms originally employed by Simon in [86, Theorem 3] and this affords considerably greater flexibility in applications. For example, when $(X, g)$ is a closed, four-dimensional, Riemannian manifold, the $W^{1,2}$ Sobolev norm on (bundle-valued) one-forms is (in a suitable sense) quasi-conformally invariant with respect to conformal changes in the Riemannian metric $g$. In particular, that observation is exploited in our proof of [25, Theorem 1], which asserts discreteness of $L^2$ energies of Yang-Mills connections on arbitrary $G$-principal bundles over $X$, for any compact Lie structure group $G$.

1.2. Lojasiewicz–Simon gradient inequalities for analytic functionals on Banach spaces. There are essentially three approaches to establishing a Lojasiewicz–Simon gradient inequality for a particular energy functional arising in geometric analysis or mathematical physics: 1) establish the inequality from first principles, 2) adapt the argument employed by Simon in the proof of his [86, Theorem 3], or 3) apply an abstract version of the Lojasiewicz–Simon gradient inequality for an analytic or Morse-Bott functional on a Banach space. Most famously, the first approach is exactly that employed by Simon in [86], although this is also the avenue followed by Kwon [67], Liu and Yang [71] and Topping [92, 93] for the harmonic map energy functional and by Råde for the Yang-Mills energy functional. Occasionally a development from first principles may be necessary, as discussed by Colding and Minicozzi in [23]. However, in almost all of the examples cited in Section 1.1 one can derive a Lojasiewicz–Simon gradient inequality for a specific application from an abstract version for an analytic or Morse-Bott functional on a Banach space. For this strategy to work well, one desires an abstract Lojasiewicz–Simon gradient inequality with the weakest possible hypotheses and a proof of such a gradient inequality (quoted as Theorem 1 here) was the one of the goals of our article [32].

We now recall from [32] the following generalization of Simon’s infinite-dimensional version [86, Theorem 3] of the Lojasiewicz gradient inequality [72]. As we explained in detail in [32], Theorem 1 generalizes Huang’s [57, Theorems 2.4.2 (i) and 2.4.5] and other previously published versions of the Lojasiewicz–Simon gradient inequality for analytic functionals on Banach spaces.

We begin with the concept of a gradient map [57, Section 2.1B], [6, Section 2.5].

**Definition 1.1** (Gradient map). (See [57, Definition 2.1.1].) Let $\mathcal{U} \subset \mathcal{X}$ be an open subset of a Banach space, $\mathcal{X}$, and let $\mathcal{F} \subset \mathcal{X}$ be a Banach space with continuous embedding, $\mathcal{F} \subseteq \mathcal{X}^*$. A continuous map, $\mathcal{M} : \mathcal{U} \to \mathcal{F}$, is called a gradient map if there exists a $C^1$ function, $\mathcal{E} : \mathcal{U} \to \mathbb{R}$, such that

$$\mathcal{E}'(x)v = \langle v, \mathcal{M}(x) \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \forall x \in \mathcal{U}, \quad v \in \mathcal{X},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{X} \times \mathcal{X}^*}$ is the canonical bilinear form on $\mathcal{X} \times \mathcal{X}^*$. The real-valued function, $\mathcal{E}$, is called a potential for the gradient map, $\mathcal{M}$.

When $\mathcal{F} = \mathcal{X}^*$ in Definition 1.1 then the differential and gradient maps coincide.

Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}^*$ denote its continuous dual space. We call a bilinear form $b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, definite if $b(x, x) \neq 0$ for all $x \in \mathcal{X} \setminus \{0\}$. We say that a continuous embedding of a Banach space into its continuous dual space, $j : \mathcal{X} \to \mathcal{X}^*$, is definite if the

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1Unless stated otherwise, all Banach spaces are considered to be real in this monograph.
pullback of the canonical pairing, \( \mathcal{X} \times \mathcal{X} \ni (x, y) \mapsto \langle x, j(y) \rangle_{\mathcal{X} \times \mathcal{X}^*} \to \mathbb{R} \), is a definite bilinear form.

**Theorem 1** (Lojasiewicz–Simon gradient inequality for analytic functions on Banach spaces). (See [32, Theorem 2].) Let \( \mathcal{X} \) and \( \mathcal{X}' \) be Banach spaces with continuous embeddings, \( \mathcal{X} \subset \mathcal{X}' \subset \mathcal{X}^* \), and such that the embedding, \( \mathcal{X} \subset \mathcal{X}^* \), is definite. Let \( \mathcal{U} \subset \mathcal{X}' \) be an open subset, \( \mathcal{E} : \mathcal{U} \to \mathbb{R} \) a \( C^2 \) function with real analytic gradient map, \( \mathcal{M} : \mathcal{U} \to \mathcal{X}' \), and \( x_\infty \in \mathcal{U} \) a critical point of \( \mathcal{E} \), that is, \( \mathcal{M}(x_\infty) = 0 \). If \( \mathcal{M}'(x_\infty) : \mathcal{X} \to \mathcal{X}' \) is a Fredholm operator with index zero, then there are constants, \( Z \in (0, \infty) \), and \( \sigma \in (0, 1) \), and \( \theta \in [1/2, 1) \), with the following significance. If \( x \in \mathcal{U} \) obeys

\[
\|x - x_\infty\|_\mathcal{X} < \sigma,
\]

then

\[
\|\mathcal{M}(x)\|_{\mathcal{X}'} \geq Z|\mathcal{E}(x) - \mathcal{E}(x_\infty)|^\theta.
\]

**Remark 1.2** (Comments on the embedding hypothesis in Theorem 1). The hypothesis in Theorem 1 on the continuous embedding, \( \mathcal{X} \subset \mathcal{X}^* \), is easily achieved given a continuous embedding of \( \mathcal{X}' \) into a Hilbert space \( \mathcal{H} \).

**Remark 1.3** (On the choice of Banach spaces in applications of Theorem 1). The hypotheses of Theorem 1 are designed to give the most flexibility in applications of a Lojasiewicz–Simon gradient inequality to analytic functionals on Banach spaces. An example of a convenient choice of Banach spaces modeled as Sobolev spaces, when \( \mathcal{M}'(x_\infty) \) is realized as an elliptic partial differential operator of order \( m \), would be

\[
\mathcal{X} = W^{k,p}(X; V), \quad \mathcal{X}' = W^{k-m,p}(X; V), \quad \text{and} \quad \mathcal{X}^* = W^{-k,p'}(X; V),
\]

where \( k \in \mathbb{Z} \) is an integer, \( p \in (1, \infty) \) a constant with dual Hölder exponent \( p' \in (1, \infty) \) defined by \( 1/p + 1/p' = 1 \), while \( X \) is a closed Riemannian manifold of dimension \( d \geq 2 \) and \( V \) is a Riemannian vector bundle with a compatible connection, \( \nabla : C^\infty(X; V) \to C^\infty(X; T^*X \otimes V) \), and \( W^{k,p}(X; V) \) denote Sobolev spaces defined in the standard way [4]. When the integer \( k \) is chosen large enough, the verification of analyticity of the gradient map, \( \mathcal{M} : \mathcal{U} \to \mathcal{X}' \), is straightforward. Normally, that is the case when \( k \geq m + 1 \) and \((k - m)p > d \) or \( k - m = d \) and \( p = 1 \), since \( W^{k-m,p}(X; \mathbb{C}) \) is then a Banach algebra by [4, Theorem 4.39]. If the Banach spaces are instead modeled as Hölder spaces, as in Simon [80], a convenient choice of Banach spaces would be

\[
\mathcal{X} = C^{k,\alpha}(X; V), \quad \text{and} \quad \mathcal{X}' = C^{k-m,\alpha}(X; V),
\]

where \( \alpha \in (0, 1) \) and \( k \geq m \), and these Hölder spaces are defined in the standard way [5]. Following Remark 1.2, the definiteness of the embedding \( C^{k,\alpha}(X; V) = \mathcal{X} \subset \mathcal{X}^* \) in this case is the achieved by observing that \( C^{k,\alpha}(X; V) \subset L^2(X; V) \).

We refer the reader to [32, Theorem 4] for a statement and proof of our abstract Lojasiewicz–Simon gradient inequality for Morse-Bott functionals on Banach spaces.

Theorem 1 appears to us to be the most widely applicable abstract version of the Lojasiewicz–Simon gradient inequality that we are aware of in the literature. However, for applications where \( \mathcal{M}'(x_\infty) \) is realized as an elliptic partial differential operator of even order, \( m = 2n \), and the nonlinearity of the gradient map is sufficiently mild, it often suffices to choose \( \mathcal{X} \) to be the
Banach space, $W^{n,2}(X; V)$, and choose $\mathcal{X} = \mathcal{X}^*$ to be the Banach space, $W^{-n,2}(X; V)$. The distinction between the differential, $\mathcal{E}(x) \in \mathcal{X}^*$, and the gradient, $\mathcal{M}(x) \in \mathcal{X}$, then disappears. Similarly, the distinction between the Hessian, $\mathcal{E}''(x_\infty) \in (\mathcal{X} \times \mathcal{X})^*$, and the Hessian operator, $\mathcal{M}'(x_\infty) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$, disappears. Finally, if $\mathcal{E} : \mathcal{X} \supset \mathcal{U} \rightarrow \mathbb{R}$ is real analytic, then the simpler Theorem 2 is often adequate for applications.

**Theorem 2** (Lojasiewicz–Simon gradient inequality for analytic functions on Banach spaces). (See [32] Theorem 1.) Let $\mathcal{X} \subset \mathcal{X}^*$ be a continuous, definite embedding of a Banach space into its dual space. Let $\mathcal{U} \subset \mathcal{X}$ be an open subset, $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$ be an analytic function, and $x_\infty \in \mathcal{U}$ be a critical point of $\mathcal{E}$, that is, $\mathcal{E}'(x_\infty) = 0$. Assume that $\mathcal{E}''(x_\infty) : \mathcal{X} \rightarrow \mathcal{X}^*$ is a Fredholm operator with index zero. Then there are constants $Z \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathcal{U}$ obeys

\[
\|x - x_\infty\|_{\mathcal{X}} < \sigma,
\]

then

\[
\|\mathcal{E}'(x)\|_{\mathcal{X}^*} \geq Z|\mathcal{E}(x) - \mathcal{E}(x_\infty)|^\theta.
\]

1.3. **Lojasiewicz–Simon gradient inequality for coupled Yang-Mills $L^2$-energy functionals.** In this subsection, we summarize consequences of Theorem 1 for coupled Yang-Mills $L^2$-energy functionals.

1.3.1. **Lojasiewicz–Simon gradient inequality for boson and fermion coupled Yang-Mills $L^2$-energy functionals.** We begin with a definition (due to Parker [79]) of two coupled Yang-Mills energy functionals.

**Definition 1.4** (Boson and fermion coupled Yang-Mills energy functionals). [79] Section 2] Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ a smooth principal $G$-bundle over $X$, and $E$ be a complex finite-dimensional $G$-module equipped with a $G$-invariant Hermitian inner product, $g : G \rightarrow \text{Aut}_\mathbb{C}(E)$ be a unitary representation [14] Definitions 2.1.1 and 2.16, and $E = P \times_g E$ be a smooth Hermitian vector bundle over $X$, and $m$ and $s$ be smooth real-valued functions on $X$.

We define the **boson coupled Yang-Mills $L^2$-energy functional** by

\[
\mathcal{E}_g(A, \Phi) := \frac{1}{2} \int_X \left( |F_A|^2 + |\nabla_A \Phi|^2 - m|\Phi|^2 - s|\Phi|^4 \right) \, d\text{vol}_g,
\]

for all smooth connections, $A$ on $P$, and smooth sections, $\Phi$ of $E$, where

$\nabla_A : \mathcal{C}^\infty(X; E) \rightarrow \mathcal{C}^\infty(T^*X \otimes E)$,

is the covariant derivative induced on $E$ by the connection $A$ on $P$ and $F_A \in \Omega^2(X; \text{ad}P)$ is the curvature of $A$ and $\text{ad}P := P \times_{\text{ad}} g$ denotes the real vector bundle associated to $P$ by the adjoint representation of $G$ on its Lie algebra, $\text{Ad} : G \ni u \rightarrow \text{Ad}_u \in \text{Aut}(g)$, with fiber metric defined through the Killing form on $g$.

Suppose that $X$ admits a spin$^c$ structure comprising a Hermitian vector bundle $W$ over $X$ and a **Clifford multiplication map**, $c : T^*X \rightarrow \text{End}_\mathbb{C}(W)$, thus

\[
c(\alpha)^2 = -g(\alpha, \alpha) \text{id}_W, \quad \forall \alpha \in \Omega^1(X),
\]
and
\[ D_A := c \circ \nabla_A : C^\infty(X; W \otimes E) \to C^\infty(X; W \otimes E), \]
is the corresponding \textit{Dirac operator} \cite[Appendix D]{69}, \cite[Sections 1.1 and 1.2]{66}, where \( \nabla_A \) denotes the covariant derivative induced on \( \otimes^n (T^* X) \otimes E \) (for \( n \geq 0 \)) and \( W \otimes E \) by the connection \( A \) on \( P \) and Levi-Civita connection for the metric \( g \) on \( TX \).

We define the \textit{fermion coupled Yang-Mills} \( L^2 \)-\textit{energy functional} by
\[ F_g(A, \Psi) := \frac{1}{2} \int_X \left( |F_A|^2 + \langle \Psi, D_A \Psi \rangle - m |\Psi|^2 \right) \, d\text{vol}_g, \]
for all smooth connections, \( A \) on \( P \), and smooth sections, \( \Psi \) of \( W \otimes E \).

We recall from \cite[Corollary D.4]{69} that a closed orientable smooth manifold \( X \) admits a \( spin^c \) structure if and only if the second Stiefel-Whitney class \( w_2(X) \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \) is the mod 2 reduction of an integral class. One calls \( W \) the \textit{fundamental spinor bundle} and it carries irreducible representations of \( Spin^c(d) \); when \( X \) is even-dimensional, there is a splitting \( W = W^+ \oplus W^- \) and Clifford multiplication restricts to give \( \rho : T^* X \to \text{Hom}_{C}(W^+, W^-) \) \cite[Definition D.9]{69}.

Although initially defined for smooth connections and sections, the energy functionals \( E_g \) and \( F_g \) in Definition \ref{def1} extend to the case of Sobolev connections and sections of class \( W^{1,2} \).

A short calculation shows that the gradient of the boson coupled Yang-Mills energy functional \( E_g \) in \ref{eq1} with respect to the \( L^2 \) metric on \( C^\infty(X; \Lambda^1 \otimes adP \oplus E) \),
\[ \langle \mathcal{M}_g(A, \Phi), (a, \phi) \rangle_{L^2(X, g)} := \frac{d}{dt} E_g(A + ta, \Phi + t\phi) \bigg|_{t=0} = E'_g(A, \Phi)(a, \phi), \]
for all \( (a, \phi) \in C^\infty(X; \Lambda^1 \otimes adP \oplus E) \), is given by
\[ \langle \mathcal{M}_g(A, \Phi), (a, \phi) \rangle_{L^2(X, g)} = (d^*_AF_A, a)_{L^2(X)} + \text{Re}(\nabla_A^* \nabla_A \Phi, \phi)_{L^2(X)} + \text{Re}(\nabla_A \Phi, \rho(a) \Phi)_{L^2(X)} - \text{Re}(m \Phi, \phi)_{L^2(X)} - 2 \text{Re} \int_X s|\Phi|^2(\Phi, \phi) \, d\text{vol}_g, \]
where \( d^*_A = d_A^* : \Omega^l(X; adP) \to \Omega^{l-1}(X; adP) \) is the \( L^2 \) adjoint of the exterior covariant derivative \( d_A : \Omega^l(X; adP) \to \Omega^{l+1}(X; adP) \), for integers \( l \geq l' \). We call \( (A, \Phi) \) a \textit{boson Yang-Mills pair} (with respect to the Riemannian metric \( g \) on \( X \)) if it is a critical point for \( E_g \), that is, \( \mathcal{M}_g(A, \Phi) = 0 \).

Similarly, one finds that the gradient of the fermion coupled Yang-Mills energy functional \( F_g \) in \ref{eq1} with respect to the \( L^2 \) metric on \( C^\infty(X; \Lambda^1 \otimes adP \oplus W \otimes E) \),
\[ \langle \mathcal{M}_g(A, \Psi), (a, \psi) \rangle_{L^2(X, g)} := \frac{d}{dt} F_g(A + ta, \Psi + t\psi) \bigg|_{t=0} = F'_g(A, \Psi)(a, \psi), \]
for all \( (a, \psi) \in C^\infty(X; \Lambda^1 \otimes adP \oplus W \otimes E) \), is given by
\[ \langle \mathcal{M}_g(A, \Psi), (a, \psi) \rangle_{L^2(X, g)} = (d^*_AF_A, a)_{L^2(X)} + \text{Re}(D_A \Psi - m \Psi, \psi)_{L^2(X)} + \frac{1}{2} \text{Re}(\Psi, \rho(a) \Psi)_{L^2(X)}, \]
where the action of \( a \in \Omega^1(X; \text{ad}P) \) on \( \Psi \in C^\infty(X; W \otimes E) \) is defined by
\[
\rho(\alpha \otimes \xi)(\phi \otimes \eta) := c(\alpha)\phi \otimes g_\alpha(\xi)\eta,
\]
for all \( \alpha \in \Omega^1(X) \), \( \xi \in C^\infty(X; \text{ad}P) \), \( \phi \in C^\infty(X; W) \), and \( \eta \in C^\infty(X; E) \), where \( g_\alpha : g \to \text{End}_C(E) \) is the representation of the Lie algebra induced by the representation \( g : G \to \text{End}_C(E) \) of the Lie group.

We call \( (A, \Psi) \) a fermion Yang-Mills pair (with respect to the Riemannian metric \( g \) on \( X \)) if it is a critical point for \( \mathcal{F}_g \), that is, \( \mathcal{M}_g(A, \Psi) = 0 \).

Note that both the boson and fermion coupled Yang-Mills energy functionals reduce to the pure Yang-Mills energy functional when \( \Phi \equiv 0 \) or \( \Psi \equiv 0 \), respectively,
\[
\mathcal{E}_g(A) := \frac{1}{2} \int_X |F_A|^2 \, dv_g,
\]
and \( A \) is a Yang-Mills connection (with respect to the Riemannian metric \( g \) on \( X \)) if it is a critical point for \( \mathcal{E}_g \), that is,
\[
\mathcal{M}_g(A) = d_A^g F_A = 0.
\]

Given a Hermitian or Riemannian vector bundle, \( V \), over \( X \) and covariant derivative, \( \nabla_A \), which is compatible with the fiber metric on \( V \), we denote the Banach space of sections of \( V \) of Sobolev class \( W^{k,p} \), for any \( k \in \mathbb{N} \) and \( p \in [1, \infty] \), by \( W^{k,p}_A(X; V) \), with norm,
\[
\|v\|_{W^{k,p}_A(X)} := \left( \sum_{j=0}^k \int_X |\nabla^j_A v|^p \, dv_g \right)^{1/p},
\]
when \( 1 \leq p < \infty \) and
\[
\|v\|_{W^{k,\infty}_A(X)} := \sum_{j=0}^k \text{ess sup}_X |\nabla^j_A v|,
\]
when \( p = \infty \), where \( v \in W^{k,p}_A(X; V) \). If \( k = 0 \), then we denote \( \|v\|_{W^{0,p}_A(X)} = \|v\|_{L^p(X)} \). For \( p \in [1, \infty) \) and nonnegative integers \( k \), we use \[4\] Theorem 3.12 (applied to \( W^{k,p}_A(X; V) \) and noting that \( X \) is a closed manifold) and Banach space duality to define
\[
W^{-k,p'}_A(X; V) := \left( W^{k,p}_A(X; V) \right)^*,
\]
where \( p' \in (1, \infty) \) is the dual exponent defined by 
\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]
Elements of the Banach space dual \( (W^{k,p}_A(X; V))^* \) may be characterized via \[4\] Section 3.10 as distributions in the Schwartz space \( \mathcal{S}'(X; V) \) \[4\] Section 1.57.

As our first application of Theorem 1, we have the following generalization of \[26\] Theorem 23.9 from the case of the pure Yang-Mills energy functional \([1.13]\), when \( p = 2 \) and \( X \) has dimension \( d = 2, 3, \) or \( 4 \), and Råde’s \[82\] Proposition 7.2, when \( p = 2 \) and \( X \) has dimension \( d = 2 \) or \( 3 \). Because gauge transformations of class \( W^{2,2} \) are continuous when \( d = 2 \) or \( 3 \) and standard versions of the slice theorem \[24\] Proposition 2.3.4, \[37\] Theorem 3.2, \[68\] Theorem 10.4 for the action of gauge transformations are applicable, the proof of the analogue of Theorem 3 for the pure Yang-Mills \( L^2 \)-energy functional due to Råde is simpler for \( d = 2, 3 \) and \( p = 2 \) \[82\] Proposition 7.2.
Theorem 3 (Lojasiewicz–Simon gradient inequality for the boson coupled Yang-Mills energy functional). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group, \(P\) be a smooth principal \(G\)-bundle over \(X\), and \(E = P \times_g \mathbb{E}\) be a smooth Hermitian vector bundle over \(X\) defined by a finite-dimensional unitary representation, \(g : G \to \text{Aut}_\mathbb{C}(\mathbb{E})\). Let \(A_1\) be a \(C^\infty\) reference connection on \(P\), and \(\Phi(\Phi_\infty)\) a boson coupled Yang-Mills pair on \((P, E)\) for \(g\) of class \(W^{1,q}\), with \(q \in [2, \infty)\) obeying \(q > d/2\). If \(p \in [2, \infty)\) obeys \(d/2 \leq p \leq q\), then the gradient map,

\[
\mathcal{M}_g : (A_1, 0) + W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E),
\]

is real analytic and there are constants \(Z \in (0, \infty)\), and \(\sigma \in (0, 1]\), and \(\theta \in [1/2, 1)\), depending on \(A_1\), \((A_\infty, \Phi_\infty)\), \(g\), \(G\), \(p\), and \(q\) with the following significance. If \((A, \Phi)\) is a \(W^{1,q}\) Sobolev pair on \((P, E)\) obeying the Lojasiewicz–Simon neighborhood condition,

\[
(1.16)\quad \| (A, \Phi) - (A_\infty, \Phi_\infty) \|_{W^{-1,p}_{A_1}(X)} < \sigma,
\]

then the boson coupled Yang-Mills energy functional \((1.6)\) obeys the Lojasiewicz–Simon gradient inequality

\[
(1.17)\quad \| \mathcal{M}_g(A, \Phi) \|_{W^{-1,p}_{A_1}(X)} \geq Z|\mathcal{E}_g(A, \Phi) - \mathcal{E}_g(A_\infty, \Phi_\infty)|^\theta.
\]

The statement of Theorem 3 simplifies with the addition of the rather mild assumption that \(A_1 = A_\infty\) and that \((A_\infty, \Phi_\infty)\) is \(C^\infty\) (which can be assumed, modulo a \(W^{2,q}\) gauge transformation, provided by the regularity Theorem 2.23).

Corollary 4 (Lojasiewicz–Simon gradient inequality for the boson coupled Yang-Mills energy functional). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group, \(P\) be a smooth principal \(G\)-bundle over \(X\), and \(E = P \times_g \mathbb{E}\) be a smooth Hermitian vector bundle over \(X\) defined by a finite-dimensional unitary representation, \(g : G \to \text{Aut}_\mathbb{C}(\mathbb{E})\). Let \((A_\infty, \Phi_\infty)\) a smooth boson coupled Yang-Mills pair for \(g\) on \((P, E)\). If \(p \in [2, \infty)\) obeys \(p \geq d/2\), then the gradient map,

\[
\mathcal{M}_g : (A_\infty, 0) + W^{1,p}_{A_\infty}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to W^{-1,p}_{A_\infty}(X; \Lambda^1 \otimes \text{ad} P \oplus E),
\]

is real analytic and, for \(d/2 < q < \infty\) obeying \(q > p\), there are constants \(Z \in (0, \infty)\), and \(\sigma \in (0, 1]\), and \(\theta \in [1/2, 1)\), depending on \((A_\infty, \Phi_\infty)\), \(g\), \(G\), \(p\), and \(q\) with the following significance. If \((A, \Phi)\) is a \(W^{1,q}\) Sobolev pair on \((P, E)\) that obeys the Lojasiewicz–Simon neighborhood condition,

\[
(1.18)\quad \| (A, \Phi) - (A_\infty, \Phi_\infty) \|_{W^{-1,p}_{A_\infty}(X)} < \sigma,
\]

then the boson coupled Yang-Mills energy functional \((1.6)\) obeys the Lojasiewicz–Simon gradient inequality,

\[
(1.19)\quad \| \mathcal{M}_g(A, \Phi) \|_{W^{-1,p}_{A_\infty}(X)} \geq Z|\mathcal{E}_g(A, \Phi) - \mathcal{E}_g(A_\infty, \Phi_\infty)|^\theta.
\]

Similarly, for the fermion coupled Yang-Mills energy functional, we have the

Theorem 5 (Lojasiewicz–Simon gradient inequality for the fermion coupled Yang-Mills energy functional). Assume the hypotheses of Theorem 3 except that we require that \(X\) admit a spin\(^c\) structure \((\rho, W)\), replace the role of \(\mathcal{E}_g\) in \((1.6)\) by \(\mathcal{F}_g\) in \((1.8)\), and replace the role of the pair \((A, \Phi)\) and critical point \((A_\infty, \Phi_\infty)\) of \(\mathcal{E}_g\) by the pair \((A, \Psi)\) and critical point \((A_\infty, \Psi_\infty)\) of
\( \mathcal{F}_g \), where \( \Psi \) and \( \Psi_\infty \) are sections of \( W \otimes E \). Then the conclusions of Theorem 3 hold mutatis mutandis.

**Remark 1.5** (Lojasiewicz–Simon gradient inequality for coupled Yang-Mills energy functionals on quotient spaces). We recall that the space of all smooth connections on \( P \) is an affine space, \( \mathcal{A}(p) = A_1 + \Omega^1(X; \text{ad } P) \). While the energy functionals \( E_g \) and \( F_g \) in Definition 1.4 were initially defined on affine spaces modeled on \( C^\infty(X; \Lambda^1 \otimes \text{ad } P \oplus E) \) or \( C^\infty(X; \Lambda^1 \otimes \text{ad } P \oplus W \otimes E) \), the functionals are invariant under the action of the group of gauge transformations, \( \text{Aut}(P) \), and thus descend to the corresponding quotient spaces. The resulting configuration spaces may be given the structure of smooth Banach manifolds in a standard way [24, Sections 4.2.1], [37, Chapter 3], [44, Section 1] and, with minor modifications of standard proofs, the structure of real analytic Banach manifolds as discussed in Section 1.5.2.

**Remark 1.6** (Lojasiewicz–Simon gradient inequality for the Yang-Mills energy functional over a Riemann surface). When \( d = 2 \), it is known that the pure Yang-Mills energy functional obeys the Morse-Bott condition in the sense of [32, Definition 1.9] and so by [? , Theorem 4] (our abstract Lojasiewicz–Simon gradient inequality for Morse-Bott functionals on Banach spaces), one has the optimal Lojasiewicz–Simon exponent, \( \theta = 1/2 \).

We have chosen to derive the Lojasiewicz–Simon gradient inequalities (in Theorems 3 and 5) for two specific coupled Yang-Mills energy functionals, motivated by physical considerations, namely the properties of regularity, naturality, and conformal invariance (in dimension four) described by Parker in [79, Section 2]. However, it is clear from the proofs of Theorems 3 and 5 that one can expect the same conclusions for any \( L^2 \)-energy functional on pairs of connections and sections with the same nonlinearity structure. Indeed, proofs of such results can be obtained by simple modifications of our proof of the Lojasiewicz–Simon gradient inequality for the boson coupled Yang-Mills energy functional, just as we do in this article for the case of the fermion coupled Yang-Mills energy functional.

### 1.3.2. Lojasiewicz–Simon gradient inequality for the Yang-Mills-Higgs \( L^2 \)-energy functional.

A well-known example in complex differential geometry of a coupled Yang-Mills \( L^2 \)-energy functional is the Yang-Mills-Higgs functional, which we now describe. See Bradlow [9, 10], Bradlow and García-Prada [11], Hitchin [53], Hong [54], Li and Zhang [70], and Simpson [88] for additional details and further references.

We shall follow the description by Bradlow and García-Prada [11, Section 3], but refer the reader to Hong [54], Li and Zhang [70], and the cited references for variants of the Yang-Mills-Higgs functional described here. Let \( E \) be a complex vector bundle with Hermitian metric \( H \) over a compact Kähler manifold \((X, \omega)\). Let \( \mathcal{A}_H \) denote the affine space of smooth connections on \( E \) that are unitary (that is, compatible with the metric \( H \)), and \( \Omega^0(X; E) \) denote the vector space of smooth sections of \( E \), and \( \tau \in \mathbb{R} \).

One defines the Yang-Mills-Higgs \( L^2 \)-energy functional on \((A, \Phi) \in \mathcal{A}_H \times \Omega^0(X; E)\) by

\[
\mathcal{E}_{H, \omega, \tau}(A, \Phi) := \frac{1}{2} \int_X \left( |F_A|^2 + 2|\nabla_A \Phi|^2 + |\Phi \otimes \Phi^* - \tau \text{id}_E|^2 \right) \, d\text{vol}_\omega,
\]

where \( \Phi^* := \langle \cdot, \Phi \rangle_H \), the dual of \( \Phi \) with respect to the metric \( H \).
By definition, a Yang-Mills-Higgs pair \((A, \Phi)\) is a critical point of the Yang-Mills-Higgs functional \(\mathcal{E}_{H,\omega,\tau}\), so \(\mathcal{M}_{H,\omega,\tau}(A, \Phi) = 0\), or equivalently \((A, \Phi)\) satisfies the second-order Yang-Mills-Higgs equations (the Euler-Lagrange equations defined by the functional (1.20)). A calculation reveals that a pair is an absolute minimum of \(\mathcal{E}_{H,\omega,\tau}\) if and only if it obeys the first-order vortex equations,

\[
\begin{align*}
F_A^{0,2} &= 0, \\
\bar{\partial} \Phi &= 0, \\
\Lambda F_A &= \sqrt{-1} (\Phi \otimes \Phi^* - \tau \text{id}_E),
\end{align*}
\]

where \(\Lambda F_A\) denotes contraction of \(F_A\) with \(\omega\). Let \(u(E) \subset \text{End}_C(E)\) denote the subbundle of skew-Hermitian endomorphisms of \(E\).

The proof of Theorem 3 carries over mutatis mutandis to give

**Theorem 6** (Lojasiewicz–Simon gradient inequality for the Yang-Mills-Higgs \(L^2\)-energy functional). Let \(X\) be a compact, Kähler manifold of complex dimension \(n \geq 1\) and \(E\) be a complex vector bundle with Hermitian metric \(H\) over \(X\). Let \(A_1\) be a smooth reference connection on the principal frame bundle for \(E\). Assume that \(d = 2n \geq 2\) and \(p \in (1, \infty)\) obey one of the conditions in Theorem 3. Then the gradient map,

\[\mathcal{M}_{H,\omega,\tau} : (A_1, 0) + W_{A_1}^{1,p}(X; \Lambda^1 \otimes u(E) \oplus E) \to W_{A_1}^{-1,p}(X; \Lambda^1 \otimes u(E) \oplus E),\]

is real analytic and the remaining conclusions of Theorem 3 hold mutatis mutandis for the Yang-Mills-Higgs functional (1.20).

1.3.3. Lojasiewicz–Simon gradient inequality for the Seiberg-Witten \(L^2\)-energy functionals. For another example of a coupled Yang-Mills energy functional whose absolute minima can be readily identified, we consider the Seiberg-Witten equations.

Expositions of the Seiberg-Witten equations are now provided by many authors but, for the sake of consistency, we shall follow our development in [31]. Let \((\rho, W)\) denote a spin\(^c\) structure on a four-dimensional manifold, \(X\), with Riemannian metric, \(g\). We recall from [31] Equation (2.55) that a pair \((B, \Psi)\), comprising a spin\(^c\) connection, \(B\), on \(W = W^+ \oplus W^-\) and a section, \(\Psi\), of \(W^+\) is a Seiberg-Witten monopole if

\[
\begin{align*}
\text{tr}(F_B^-) - \rho^{-1}(\Psi \otimes \Psi^*)_0 &= 0, \\
D_B \Psi &= 0,
\end{align*}
\]

recalling that \(\rho : \Lambda^+ \cong su(W^+)\) is the isomorphism of Riemannian vector bundles induced by Clifford multiplication, \(D_B : C^\infty(X; W^+) \to C^\infty(X; W^-)\) is the Dirac operator, and \((\cdot)_0\) denotes the trace-free part of \(\Psi \otimes \Psi^* \in \text{End}_C(W^+)\). We have restricted \(B\) to \(W^+\), so \(F_B \in C^\infty(X; u(W^+) \otimes \Lambda^+) = \Omega^+(X; u(W^+))\) and \(\text{tr}(F_B^+) \in C^\infty(X; i\Lambda^+) = \Omega^+(X; i\mathbb{R})\), using the fiberwise trace homomorphism, \(\text{tr} : u(W^+) \to i\mathbb{R}\). The Seiberg-Witten equations (1.22) are a system of first-order partial differential equations in \((B, \Psi)\) and thus cannot be the Euler-Lagrange equations of any action functional. However, as we recall from [55, 61, 77], Seiberg-Witten monopoles have a variational interpretation by an argument which is the reverse of those provided by Bradlow and García-Prada [11, Section 3] or Hong [54, Section 1] in their derivations of the vortex equations or Li and Zhang [70, Section 1] for the Hermitian-Einstein equations.
Thus, from [55] Equation (1.6) or [77] Proposition 2.1.4], the Seiberg-Witten \( L^2 \)-energy functional is

\[
\mathcal{E}_g(B, \Psi) = \int_X \left( |\nabla B \Psi|^2 + \frac{1}{2} |\text{tr}(F_B)|^2 + \frac{R}{4} |\Psi|^2 + \frac{1}{8} |\Psi|^2 \right) \, d\text{vol}_g + 2\pi^2 c_1(W^+)^2,
\]

where \( c_1(W^+)^2 := \int_X c_1(W^+)^2 \). The topological term, \( 2\pi^2 c_1(W^+)^2 \), is independent of the pair \((B, \Psi)\) and does not affect the critical points. In particular,

\[
\mathcal{E}_g(B, \Psi) \geq 2\pi^2 c_1(W^+)^2,
\]

and a pair \((B, \Psi)\) is a Seiberg-Witten monopole if and only if equality is achieved.

Hong and Schabrun derive a version of the Lojasiewicz–Simon gradient inequality [55] Lemma 5.3] based in part on an earlier proof due to Wilkin for the Yang-Mills-Higgs functional over a Riemann surface [101] Proposition 3.5]. However, the proof of Theorem 3 carries over \textit{mutatis mutandis} to give

**Theorem 7** (Lojasiewicz–Simon gradient inequality for the Seiberg-Witten \( L^2 \)-energy functional). Let \((X, g)\) be a closed, four-dimensional, oriented, Riemannian smooth manifold with spin\(^c\) structure \((\rho, W)\). Let \(B_1\) be a smooth reference spin\(^c\) connection on \(W\). Assume that \(p \in (1, \infty)\) obeys the hypotheses of Theorem 3 with \(d = 4\). Then the gradient map,

\[
\mathcal{M}_g : (B_1, 0) + W^{1,p}_B(X; i\Lambda^1 \oplus W^+) \to W^{-1,p}_B(X; i\Lambda^1 \oplus W^+),
\]

is real analytic and the remaining conclusions of Theorem 3 hold \textit{mutatis mutandis} for the Seiberg-Witten functional [123].

1.3.4. \textit{Lojasiewicz–Simon gradient inequality for the non-Abelian monopole \( L^2 \)-energy functionals.} For our final example of a coupled Yang-Mills energy functional whose absolute minima can be readily identified, we have the non-Abelian monopoles arising in the work of the first author and Leness [31], Okonek and Teleman [78], and Pidstrigatch and Tyurin [80].

Following [31], we consider pairs \((A, \Phi)\) obeying

\[
(F^+_A)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_0 = 0,
\]

\[
D_A \Phi = 0,
\]

where \(A\) is a unitary connection on a Hermitian vector bundle, \(E\), with curvature \(F_A \in C^\infty(X; \Lambda^2 \otimes u(E)) = \Omega^2(X; u(E))\) and \((F^+_A)_0 \in C^\infty(X; \Lambda^+ \otimes su(E)) = \Omega^+(X; su(E))\), while \(\rho : \Lambda^+ \otimes su(E) \cong su(W^+) \otimes su(E)\) is the isomorphism of Riemannian vector bundles induced by Clifford multiplication, \(D_A : C^\infty(X; W^+ \otimes E) \to C^\infty(X; W^+ \otimes E)\) is the Dirac operator, and \((\cdot)_0\) denotes the trace-free part of \(\Phi \otimes \Phi^* \in \text{End}_C(W^+ \otimes E)\). Let \(su(E) \subset \text{End}_C(E)\) denote the subbundle of skew-Hermitian, trace-zero endomorphisms of \(E\).

By extending the derivations of the Seiberg-Witten \( L^2 \)-energy functional in [55] or [77], we find that the \textit{non-Abelian monopole \( L^2 \)-energy functional} is

\[
\mathcal{E}_g(A, \Phi) = \int_X \left( |\nabla A \Phi|^2 + \frac{1}{2} |F_A|^2 + \frac{R}{4} |\Phi|^2 + \frac{1}{8} |\Phi|^2 \right) \, d\text{vol}_g - 4\pi^2 c^2(E) + \frac{1}{2} ||F^+_A||^2_{L^2(X)} - 2 ||F^+_A||^2_{L^2(X)}.
\]
The connections $A_e$ on $\det E$ and $A_w$ on $\det W^+$ are fixed, with no dynamical role, so the true variables in the SO(3)-monopole equations are the SO(3) connection $\hat{A}$ induced by $A$ on the bundle $su(E)$ and the (spinor) section $\Phi$ of $W^+ \otimes E$. The action functional, $\mathcal{E}_g(A, \Phi)$, again has a universal lower bound and is achieved if and only if $(A, \Phi)$ is a non-Abelian monopole, namely a solution to (1.24). Again the proof of Theorem 3 carries over mutatis mutandis to give

**Theorem 8** (Lojasiewicz–Simon gradient inequality for the non-Abelian monopole $L^2$-energy functional). Let $(X, g)$ be a closed, four-dimensional, oriented, Riemannian, smooth manifold with spin$^c$ structure $(\rho, W)$. Let $E$ be a Hermitian vector bundle over $X$, and $A_e$ be a smooth connection on $\det E$, and $B$ be a smooth spin$^c$ connection on $W$, and $A_1$ be a smooth reference connection on $E$ inducing $A_e$ on $\det E$. Assume that $p \in (1, \infty)$ obeys the hypotheses of Theorem 3 with $d = 4$. Then the gradient map,

$$\mathcal{M}_g : (A_1, 0) + W^1_{A_1}(X; \Lambda^1 \otimes su(E) \oplus W^+ \otimes E) \to W^{-1}_{A_1}(X; \Lambda^1 \otimes su(E) \oplus W^+ \otimes E),$$

is real analytic and the remaining conclusions of Theorem 3 hold mutatis mutandis for the non-Abelian monopole functional (1.25).

[TODO - $W^+$ and not $W^-$ above?]

Our interest in Lojasiewicz–Simon gradient inequalities for coupled Yang-Mills and harmonic map energy functionals is motivated by the wealth of potential applications. We shall survey some of those applications below.

### 1.4. Applications of the Lojasiewicz–Simon gradient inequality for the coupled Yang-Mills energy functionals.

In [26], we apply the Lojasiewicz–Simon gradient inequality for the pure Yang-Mills energy functional [26, Theorem 23.9] to prove global existence, convergence, convergence rate, and stability results for solutions $A(t)$ to the associated gradient flow,

$$\frac{\partial A}{\partial t} = -\mathcal{M}_g(A(t)), \quad A(0) = A_0,$$

that is,

$$\frac{\partial A}{\partial t} = -d^\rho_{A(t)} F_{A(t)}, \quad A(0) = A_0.$$

Given our Lojasiewicz–Simon gradient inequalities for the boson and fermion coupled Yang-Mills energy functionals, Theorems 3 and 5, the main conclusions in [26] for pure Yang-Mills gradient flow should extend easily to the more general case of coupled Yang-Mills gradient flows.

In [29] and [25], we applied the Lojasiewicz–Simon gradient inequality to prove an energy gap result for Yang-Mills connections with small $L^d/2$ energy and, more generally, discreteness of $L^2$ energies of Yang-Mills connections over closed Riemannian smooth manifolds when $d = 4$. The proofs of those results should extend without difficulty to the case of solutions to the coupled Yang-Mills equations.

### 1.5. Automorphisms and transformation to Coulomb gauge.

For some energy functionals, the associated Hessian is already an elliptic second-order partial differential operator on a Sobolev space, but for others the Hessian is only elliptic when combined with a type of Coulomb gauge condition [24, 37] and it is only then that one can apply Theorem 1. For example, in the first category, one has the harmonic map energy and Yamabe functionals, while in the second category one has the Yang-Mills and coupled Yang-Mills energy functionals.
The Yang-Mills energy functional is invariant under the action of gauge transformations (or bundle automorphisms) and so, in principle, one can always find a gauge transformation to produce the required Coulomb gauge condition with the aid of a slice theorem. However, in order to prove the most useful version of the Lojasiewicz–Simon gradient inequality, it is convenient to have a stronger version of the slice theorem for the action of the group of gauge transformations, going beyond the usual statements found in standard references such as Donaldson and Kronheimer [24] or Freed and Uhlenbeck [37] and proved by applying the Implicit Function Theorem. One stronger version of a slice theorem, valid in dimension four, was proved by the first author as [28, Theorem 1.1] but it nevertheless falls short of what we need for our application to the proofs of our Lojasiewicz–Simon gradient inequalities (even when translated to the setting of pairs). Thus, a second purpose of this article is to prove a stronger version of [28, Theorem 1.1] for both connections and pairs rather than just connections as in [28], but using standard Sobolev norms with borderline Sobolev exponents rather than the critical exponent norms employed in [28] and valid in all dimensions.

1.5.1. Transformation to Coulomb gauge. We first state the desired result for connections and then its analogue for pairs.

**Theorem 9** (Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ connections that are $W^{1,d/2}$ close to a reference connection). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A_1$ is a $C^\infty$ connection on $P$, and $A_0$ is a Sobolev connection on $P$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then there exists a constant $\zeta = \zeta(A_0, A_1, g, G, p, q) \in (0, 1]$ with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ that obeys

\[
\|A - A_0\|_{W^{1,p}_A(X)} < \zeta,
\]

then there exists a gauge transformation $u \in \text{Aut}(P)$ of class $W^{2,q}$ such that

\[
d^*_A(u(A) - A_0) = 0,
\]

and

\[
\|u(A) - A_0\|_{W^{1,p}_A(X)} < 2N\|A - A_0\|_{W^{1,p}_A(X)},
\]

where $N = N(A_0, A_1, g, G, p, q) \in [1, \infty)$ is the constant in the forthcoming Proposition 2.11.

For a description of the action of the group of gauge transformations in Theorem 9 and the definition of the Coulomb gauge condition for connections, we refer the reader to Section 2.6, and for an explanation of the remainder of the notation in Theorem 9 we refer the reader to Section 1.3.4.

The essential point in Theorem 9 is that the result holds for the critical exponent, $p = d/2$ with $d \geq 3$, when the Sobolev space $W^{2,p}(X)$ fails to embed in $C(X)$ (see [11, Theorem 4.12]) and a proof of Theorem 9 by the Implicit Function Theorem in the case $p > d/2$ fails when $p = d/2$. In this situation, a $W^{2,d/2}$ gauge transformation $u$ of $P$ is not continuous, the set $\text{Aut}^{2,d/2}(P)$ of $W^{2,d/2}$ gauge transformations is not a manifold, and $\text{Aut}^{2,d/2}(P)$ cannot act smoothly on the affine space $\mathscr{A}^{1,d/2}(P)$ of $W^{1,d/2}$ connections on $P$. When $d = 4$ and $p \geq 2$, this phenomenon is discussed by Freed and Uhlenbeck in [37, Appendix A].
The proof of Theorem 9 adapts mutatis mutandis to establish the following refinement of the Proposition 2.8] and Theorem 4.1.

**Theorem 10** (Existence of $W^{2,q}$ Coulomb gauge transformations for $W^{1,q}$ pairs that are $W^{1,d}$ close to a reference pair). Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = \mathbb{P} \times_y \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \to \text{Aut}_\mathbb{C}(\mathbb{E})$. If $A_1$ is a $C^\infty$ connection on $P$, and $(A_0, \Phi_0)$ is a Sobolev pair on $(P,E)$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then there exists a constant $\zeta = \zeta(A_1, A_0, \Phi_0, g, G, p, q) \in (0,1]$ with the following significance. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P,E)$ that obeys

\[(1.27) \quad \|(A, \Phi) - (A_0, \Phi_0)\|_{W^{1,p}_A(X)} < \zeta,\]

then there exists a gauge transformation $u \in \text{Aut}(P)$ of class $W^{2,q}$ such that

\[d^*_{A_0, \Phi_0}(u(A, \Phi) - (A_0, \Phi_0)) = 0,\]

and

\[\|u(A, \Phi) - (A_0, \Phi_0)\|_{W^{1,p}_A(X)} < 2N\|(A, \Phi) - (A_0, \Phi_0)\|_{W^{1,p}_A(X)},\]

where $N = N(A_1, A_0, \Phi_0, g, G, p, q) \in [1, \infty)$ is the constant in the forthcoming Proposition 2.19.

For a description of the action of the group of gauge transformations in Theorem 10 and the definition of the Coulomb gauge condition for pairs, we refer the reader to Section 2.8.

1.5.2. **Real analytic Banach manifold structures on quotient spaces.** In order to establish the analyticity of the pure or coupled Yang-Mills $L^2$-energy functionals on affine spaces of $W^{1,q}$ connections $\mathcal{A}^{1,q}(P)$ or pairs $\mathcal{B}^{1,q}(P,E)$, respectively, it is not necessary to know that their quotient spaces with respect to the action of the group $\text{Aut}^{2,q}(P)$ of gauge transformations are analytic Banach manifolds. Nevertheless, because this readily follows from the proofs of Theorem 9 and Theorem 10, respectively, we include the relevant statements here for the case of connections, noting that the analogous statements for pairs are similar.

Theorem 9 provides the essential ingredient one needs to show not only that the quotient space $\mathcal{B}(P) := \mathcal{A}^{1,q}(P) / \text{Aut}^{2,q}(P)$ is a $C^\infty$ but also a real analytic Banach manifold away from orbits $[A] = \{u(A) : u \in \text{Aut}^{2,q}(P)\}$ corresponding to $W^{1,q}$ connections $A$ on $P$ whose stabilizers (or isotropy groups), $\text{Stab}(A) := \{u \in \text{Aut}^{2,q}(P) : u(A) = A\}$, are non-minimal, that is, contain the Center($G$) as a proper subgroup. To show that

\[\mathcal{B}^*(P) = \{A \in \mathcal{A}^{1,q}(P) : \text{Stab}(A) = \text{Center}(G)\} / \text{Aut}^{2,q}(P),\]

is a $C^\infty$ Banach manifold, one only needs the ‘easy case’ of Theorem 9 where $p = q$, as the condition $q > d/2$ ensures that the proofs using $H^{k+1}(X)$ Sobolev spaces (with $d = 4$ and $k \geq 2$) due to Donaldson and Kronheimer [24] Sections 4.2.1 and 4.2.2 or Freed and Uhlenbeck [37, pp. 48-51] apply mutatis mutandis. We have the following analogue of [24 Proposition 4.2.9], [37, Corollary, p. 50], for real analytic Banach manifolds and $X$ of dimension $d \geq 2$ rather than $C^\infty$ Hilbert manifolds and $X$ of dimension four.
Corollary 11 (Real analytic Banach manifold structure on the quotient space of $W^{1,q}$ connections). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $q$ obey $d/2 < q < \infty$. If $A_1$ is a $C^\infty$ reference connection on $P$ and $[A] \in \mathcal{B}(P)$, then there is a constant $\varepsilon = \varepsilon(A_1, [A], g, G, q) \in (0, 1]$ with the following significance. If

$$B_A(\varepsilon) := \left\{ a \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P) : d^* a = 0 \text{ and } \|a\|_{W^{1,q}_{A_1}(X)} < \varepsilon \right\},$$

then the map,

$$\pi_A : B_A(\varepsilon)/\text{Stab}_A \ni [a] \mapsto [A + a] \in \mathcal{B}(P),$$

is a homeomorphism onto an open neighborhood of $[A] \in \mathcal{B}(P)$. For $a \in B_A(\varepsilon)$, the stabilizer of $a$ in $\text{Stab}_A$ is naturally isomorphic to that of $\pi_A(a)$ in $\text{Aut}^{2,q}(P)$. In particular, the inverse coordinate charts, $\pi_A$, determine real analytic transition functions for $\mathcal{B}^s(G)$, giving it the structure of a real analytic Banach manifold, and each map $\pi_A$ is a real analytic diffeomorphism from the open subset of points $[a] \in B_A(\varepsilon)/\text{Stab}_A$ where $\pi_A(a)$ has stabilizer isomorphic to $\text{Center}(G)$. As in \cite[p. 328]{91}, one may consider the quotient space of framed connections modulo gauge transformations, $\mathcal{B}(P) := (\mathcal{A}^{1,q}(P) \times P_{x_0})/\text{Aut}^{2,q}(P)$, for some fixed base point $x_0 \in X$, and now the obvious analogue of Theorem 9 shows that $\mathcal{B}(P)$ is a real analytic Banach manifold.

Corollary 11 may be easily extended to the setting of pairs by applying Theorem 10 in place of Theorem 9. We leave such extensions to the reader, but refer to \cite[Theorem 4.2]{79} and \cite[Proposition 2.8]{30} for statements and proofs of $C^\infty$ Banach manifold structures for quotient spaces of pairs.

1.6. Outline of the monograph. To apply Theorem 1 to pure or coupled Yang-Mills energy functionals and obtain the best possible results in those applications, one requires the global Coulomb gauge constructions provided by Theorems 9 or 10 and those results are proved in Section 2. In Section 3 we derive Łojasiewicz–Simon gradient inequalities for the coupled Yang-Mills energy functionals, proving Theorems 3, 5, 6, 7, 8, and Corollary 11. Appendix A establishes the Fredholm properties and computes the index of an elliptic partial differential operator with smooth coefficients acting on Sobolev spaces, Appendix B discusses the equivalence of Sobolev norms defined by Sobolev and smooth connections, and Appendix C establishes the Fredholm properties and computes the index of a Hodge Laplacian with smooth and Sobolev coefficients.

1.7. Notation and conventions. For the notation of function spaces, we follow Adams and Fournier \cite{4}, and for functional analysis, Brezis \cite{13} and Rudin \cite{83}. We let $\mathbb{N} := \{0, 1, 2, 3, \ldots \}$ denote the set of non-negative integers. We use $C = C(\ast, \ldots, \ast)$ to denote a constant which depends at most on the quantities appearing on the parentheses. In a given context, a constant denoted by $C$ may have different values depending on the same set of arguments and may increase from one inequality to the next. If $\mathcal{X}, \mathcal{Y}$ is a pair of Banach spaces, then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of all continuous linear operators from $\mathcal{X}$ to $\mathcal{Y}$. We denote the continuous dual space of $\mathcal{X}$ by $\mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{R})$. We write $\langle x, \alpha \rangle_{\mathcal{X} \times \mathcal{X}^*}$ for the pairing between $\mathcal{X}$ and its dual space, where $x \in \mathcal{X}$ and $\alpha \in \mathcal{X}^*$. If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then its adjoint is denoted by $T^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$, where $(T^* \beta)(x) := \beta(Tx)$ for all $x \in \mathcal{X}$ and $\beta \in \mathcal{Y}^*$. 
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2. Existence of Coulomb gauge transformations for connections and pairs

In Section 2.6, we prove our refinement, Theorem 9, of the standard construction of a $W^{2,q}$ Coulomb gauge transformation $u$, with $q > d/2$, for a $W^{1,q}$ connection $A$ on a principal $G$-bundle $P$ over a closed Riemannian smooth manifold of dimension $d \geq 2$. We extend this result in Section 2.8 to the action of gauge transformations on affine spaces of $W^{1,q}$ pairs, obtaining our refinement, Theorem 10 of the standard constructions of Coulomb gauge transformations in that context due to Parker [79] and the first author and Leness [30]. Finally, in Section 2.9 we extend known regularity results for solutions to the Yang-Mills equations in dimensions greater than or equal to two and coupled Yang-Mills equations in dimension four to the case of solutions to the coupled Yang-Mills equations in dimensions greater than or equal to two.

2.1. Action of Sobolev gauge transformations on Sobolev connections. Suppose that $P$ is a smooth principal $G$-bundle over a smooth manifold $X = P/G$ of dimension $d$, where $P \times G \to P$ is a right action of $G$ on $P$. For $q > d/2$, let $\text{Aut}^{2,q}(P)$ denote the Banach Lie group of Sobolev $W^{2,q}$ automorphisms (or gauge transformations) of $P$ [24, Section 2.3.1], [37, Appendix A and p. 32 and pp. 45–51], [38, Section 3.1.2]. We recall that there is a smooth left action,

$$\text{Aut}^{2,q}(P) \times P \to P,$$

which commutes with the right action of $G$ on $P$. This induces a smooth right (affine) action on the affine space $\mathcal{A}^{1,q}(P)$ of Sobolev $W^{1,q}$ connections on $P$,

$$\mathcal{A}^{1,q}(P) \times \text{Aut}^{2,q}(P) \ni (A, u) \to u(A) \in \mathcal{A}^{1,q}(P),$$

(2.1)

defined by pull-back,

$$u(A) := u^* A, \quad \forall u \in \text{Aut}^{2,q}(P) \text{ and } A \in \mathcal{A}^{1,q}(P).$$

The constraint $q > d/2$ is required to ensure that $W^{2,q}(X) \subset C(X)$ by the Sobolev Embedding [4 Theorem 4.12] and thus $u \in \text{Aut}^{2,q}(P)$ is a continuous gauge transformation of $P$ and that $W^{2,q}(X)$ is a Banach algebra by [4 Theorem 4.39].

Given a $W^{1,q}$ connection $A_0$ on $P$, the standard construction of a slice for the action of $\text{Aut}^{2,q}(P)$ on $\mathcal{A}^{1,q}(P)$ provides constants $\varepsilon = \varepsilon(A_0, g, P) \in (0, 1]$ and $C = C(A_0, g, P) \in [1, \infty)$ such that if $A$ is close to $A_0$ in the sense that,

$$\|A - A_0\|_{W^{1,q}_A(X)} < \varepsilon,$$
then there exists \( u \in \text{Aut}^{2,q}(P) \) such that \( u(A) \) is in \textit{Coulomb gauge relative to} \( A_0 \), that is,
\[
d_A^* u(A) - A_0 = 0,
\]
and \( u(A) \) is close to \( A_0 \),
\[
\|u(A) - A_0\|_{W^{1,q}_0(X)} < C\varepsilon.
\]
For example, see [23, Proposition 2.3.4], [37, Theorem 3.2], [68, Theorem 10.4] or [28, Theorem 1.1] for statements of the Slice Theorem and their proofs using the Implicit Function Theorem for smooth maps of Banach spaces.

Our Theorem 9 relaxes the condition that \( A \) be \( W^{1,q}_{A_0} \) close to \( A \) for \( q > d/2 \) to \( W^{1,p}_{A_0} \) close for \( p \) obeying \( d/2 \leq p \leq q \) when \( d \geq 3 \) and provides \( W^{1,p} \) bounds for \( u(A) - A_0 \) in terms of \( A - A_0 \). This is significant since \( \text{Aut}^{2,q}(P) \) is not a smooth manifold and the action (2.1) cannot be smooth when \( q = d/2 \), so the Implicit Function Theorem does not apply.

2.2. \textit{A priori} estimates for Laplace operators with Sobolev coefficients and existence and uniqueness of strong solutions. Before proceeding to the proof of Theorem 9 we begin with some preparatory lemmata and remarks that have some interest in their own right. Standard theory for existence and uniqueness of strong solutions to (scalar) second-order elliptic partial differential equations, such as [40, Chapter 9], requires that the second-order coefficients be continuous and the lower-order coefficients be bounded. Here, we observe that one can relax those requirements on the lower-order coefficients and accommodate the setting we employ in this article.

For a smooth connection \( A \) on \( P \) and integers \( l \geq 0 \), we let
\[
\Delta_A = d_A^* d_A + d_A d_A^* \quad \text{on} \quad \Omega^l(X; \text{ad} P)
\]
denote the \textit{Hodge Laplace operator}. Our proof of Theorem 9 will require \textit{a priori} \( L^p \) estimates, existence and uniqueness results, Fredholm properties, and Hodge decompositions involving the Hodge Laplacian (2.2) when \( A \) is a \( W^{1,q}_P \) Sobolev connection. When \( A \) is a \( C^\infty \) connection and we restrict our attention to \( p = 2 \), those properties are immediate consequences of more general results (for example, see Gilkey [11]) for elliptic operators on sections of vector bundles over closed manifolds.

**Proposition 2.1** (\textit{A priori} \( L^p \) estimate for a Laplace operator with Sobolev coefficients). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, and \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( l \geq 0 \) be an integer. If \( A \) is a \( W^{1,q}_P \) connection on \( P \) with \( q > d/2 \), and \( A_1 \) is a \( C^\infty \) connection on \( P \), and \( p \) obeys \( d/2 \leq p \leq q \), then
\[
\Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P)
\]
is a bounded operator. If in addition \( p \in (1, \infty) \), then there is a constant \( C = C(A, A_1, g, G, l, p, q) \in [1, \infty) \) such that
\[
\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C \left( \|\Delta_A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right), \quad \forall \xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P).
\]

**Remark 2.2** (Regularity of distributional solutions to elliptic partial differential equations). Suppose as in the hypotheses of Proposition 2.1 that \( A_1 \) is a smooth connection on \( P \). By analogy...
with \cite{74} Definition 2.56, we call \( \xi \in L^1(X; \Lambda^l \otimes \text{ad} P) \) a distributional solution to the equation
\[
(\xi, \Delta_A \eta)_{L^2(X)} = 0, \quad \forall \eta \in C^\infty(X; \Lambda^l \otimes \text{ad} P).
\]
In the case of the scalar Laplace operator on functions, \( C^\infty \)-smoothness of distributional solutions is provided by Wegl's Lemma \cite{105} Theorem 18.G. More generally, the \( C^\infty \)-smoothness of a solution \( u \in L^1_{\text{loc}}(\Omega) \) to a scalar (second-order) elliptic equation on an open subset \( \Omega \subset \mathbb{R}^d \) is a consequence of regularity theory for solutions in \( H^s_{\text{loc}}(\Omega) \), for \( s \in \mathbb{R} \) \cite{36} Theorem 6.33. Such regularity results extend to the case of elliptic systems (see \cite{26} and references therein) and so we conclude that if \( \xi \) is a distributional solution to the equation \( \Delta_A \xi = 0 \), then \( \xi \in C^\infty(X; \Lambda^l \otimes \text{ad} P) \).

Proposition 2.3 implies that the domain of the unbounded operator \( \Delta_A \) on \( L^p(X; \Lambda^l \otimes \text{ad} P) \) is
\[
\mathcal{D}_p(\Delta_A) = W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P).
\]
(We omit the subscript \( p \) when that is clear from the context.) In order to give criteria for when the term \( ||\xi||_{L^p(X)} \) can be eliminated from the right-hand side of the a priori estimate \cite{24}, we need to analyze the spectrum of the Hodge Laplacian with Sobolev coefficients. The forthcoming Proposition 2.3 is an analogue of \cite{40} Theorem 8.6, for a scalar, second-order, strictly elliptic equation in divergence form with homogeneous Dirichlet boundary condition over a bounded domain \( \Omega \subset \mathbb{R}^d \). However, it is not a direct consequence since the first and zeroth-order coefficients of the Laplace operator \( \Delta_A \) on \( L^p(X; \text{ad} P) \) are not necessarily bounded unless \( q > d \), which we do not wish to assume, for a \( W^{1,q} \) connection \( A \) on \( P \).

Let \( \mathcal{X} \) be a Banach space and \( T : \mathcal{X} \to \mathcal{X} \) a bounded operator. Recall from \cite{83} Definition 4.17 (c)] that the spectrum, \( \sigma(T) \), of \( T \) is the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda \) is not invertible. Thus \( \lambda \in \sigma(T) \) if and only if at least one of the following two statements is true: a) The range of \( T - \lambda \) is not all of \( \mathcal{X} \), or b) \( T - \lambda \) is not one-to-one. In the latter case, \( \lambda \) is an eigenvalue of \( T \); the corresponding eigenspace is \( \text{Ker}(T - \lambda) \); each \( x \in \text{Ker}(T - \lambda) \) (except \( x = 0 \)) is an eigenvector of \( T \) and satisfies the equation \( Tx = \lambda x \).

If \( T : \mathcal{D}(\mathcal{X}) \subset \mathcal{X} \to \mathcal{X} \) is a closed operator with dense domain, \( \mathcal{D}(\mathcal{X}) \subset \mathcal{X} \), then we say that \( \lambda \notin \sigma(T) \) if the operator \( T - \lambda : \mathcal{D}(\mathcal{X}) \to \mathcal{X} \) has a bounded inverse and otherwise that \( \lambda \in \sigma(T) \) \cite{52} Section 5.6, p. 357]. \cite{83} Section III.6.1, pp. 174–175].

**Proposition 2.3** (Spectral properties of a Laplace operator with Sobolev coefficients). Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( l \geq 0 \) be an integer. If \( A \) is a \( W^{1,q} \) connection on \( P \) with \( d/2 < q < \infty \), and \( A_1 \) is a \( C^\infty \) reference connection on \( P \), and \( p \in (1, \infty) \) obeys \( d/2 \leq p \leq q \), then the spectrum, \( \sigma(\Delta_A) \), of the unbounded operator,
\[
\Delta_A : \mathcal{D}(\Delta_A) \subset L^p(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P),
\]
is countable without accumulation points, consisting of non-negative, real eigenvalues, \( \lambda \), with finite multiplicities, \( \dim \ker(\Delta_A - \lambda) \).

**Proof.** Corollary \cite{82} implies that the operator,
\[
\Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P),
\]
is Fredholm and, setting \( K := \ker(\Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P)) \), that
\[
\Delta_A : K^\perp \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to K^\perp \cap L^p(X; \Lambda^l \otimes \text{ad} P),
\]
is invertible. We denote the Green’s operator of $\Delta_A$ by
\[ G_A : L^p(X; \Lambda^I \otimes \text{ad}P) \rightarrow W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P), \]
so that $G_A \Delta_A = 1 - \Pi_A$, where $\Pi_A : W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P) \rightarrow K$ is $L^2$-orthogonal projection, and $\Delta_A G_A = 1 - \Pi_A$, where $\Pi_A : L^p(X; \Lambda^I \otimes \text{ad}P) \rightarrow K$ again denotes $L^2$-orthogonal projection.

The Sobolev embedding, $W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P) \subseteq L^p(X; \Lambda^I \otimes \text{ad}P)$, is compact by [4, Theorem 6.3] and hence the composition of $G_A$ with this embedding,
\[ G_A : L^p(X; \Lambda^I \otimes \text{ad}P) \rightarrow L^p(X; \Lambda^I \otimes \text{ad}P), \]
is compact by [13, Proposition 6.3]. But then [83, Theorem 4.25] implies that preceding operator has the spectral properties, aside from reality and non-negativity, described in the conclusion of Proposition 2.3.

To relate the spectra of $G_A$ and $\Delta_A$, observe that for any $\lambda \in \mathbb{C} \setminus \{0\}$ and $\chi \in K_+ \cap L^p(X; \Lambda^I \otimes \text{ad}P)$, the equation,
\[ (\Delta_A - \lambda)\xi = \chi, \]
for $\xi \in K_+ \cap W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P)$ is equivalent to the equation,
\[ (G_A \Delta_A - \lambda G_A)\xi = G_A\chi, \]
that is,
\[ (G_A - \lambda^{-1})\xi = -\lambda^{-1}G_A\chi. \]
In other words, $\lambda \neq 0$ is in the spectrum of $\Delta_A$ if and only if $\lambda^{-1}$ is in the spectrum of $G_A$.

To see that the eigenvalues of $G_A$ are real as claimed, note that $\Delta_A$ has $L^2$-adjoint, $\Delta_A^* = \Delta_A$, and so $G_A$ has $L^2$-adjoint, $G_A^* = G_A$, and the operator,
\[ G_A : L^2(X; \Lambda^I \otimes \text{ad}P) \rightarrow L^2(X; \Lambda^I \otimes \text{ad}P), \]
is bounded and self-adjoint. Thus, $\sigma(G_A) \subset \mathbb{R}$ by [84, Theorem VI.8] and hence $\sigma(\Delta_A) \subset \mathbb{R}$.

Finally, to see that the eigenvalues of $\Delta_A$ are non-negative, observe that if $\Delta_A \xi = \lambda \xi$ for $\lambda \in \sigma(\Delta_A) \setminus \{0\}$ and $\xi \in W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P) \setminus \{0\}$, then $\lambda \|\xi\|_{L^2}^2 = \langle \lambda \xi, \xi \rangle_{L^2(X)} = \langle \Delta_A \xi, \xi \rangle_{L^2(X)} = \langle d_A \xi, d_A^* \xi \rangle_{L^2(X)} + \langle d_A^* \xi, d_A \xi \rangle_{L^2(X)} \geq 0$ by (2.2), and thus $\lambda \geq 0$, as claimed. \hfill \Box

\begin{remark}[Spectral properties of a Laplace operator with Sobolev coefficients on $L^p$ spaces and compact perturbations] We recall from Weyl’s Theorem [63, Theorem IV.5.35] that if $T$ is a closed operator on a Banach space $\mathcal{X}$ and $K$ is an operator on $\mathcal{X}$ that is compact relative to $T$, then $T$ and $T + K$ have the same essential spectrum. In particular, under the hypotheses of Corollary 2.3 the operator,
\[ \Delta_A - \Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^I \otimes \text{ad}P) \rightarrow L^p(X; \Lambda^I \otimes \text{ad}P), \]
is compact by the proof of that corollary. Therefore, the essential spectrum of $\Delta_A$ as an unbounded operator on $L^p(X; \Lambda^I \otimes \text{ad}P)$ is empty and hence the spectrum of $\Delta_A$ consists purely of real eigenvalues with finite multiplicity, since the same is true of $\Delta_{A_1}$. These observations could be used to give an alternative proof of Proposition 2.3 in place of the one that we provide.
\end{remark}

\begin{corollary}[A priori $L^p$ estimate for a Laplace operator with Sobolev coefficients] Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $l \geq 0$ be an integer. If $A$ is a $W^{1,q}$ connection
on $P$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ connection on $P$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then the kernel $\text{Ker} \, \Delta_A \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P)$ of the operator \( \Delta_A \) is finite-dimensional and

\[
\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_A \xi\|_{L^p(X)}, \quad \forall \xi \in (\text{Ker} \, \Delta_A)^\perp \cap W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P),
\]

where $\perp$ denotes $L^2$-orthogonal complement and $C = C(A, A_1, g, l, p, q) \in [1, \infty)$.

Before proceeding to the proofs of these results proper, we begin with the

**Lemma 2.6 (A priori $L^p$ estimate for a Laplace operator with smooth coefficients).** Assume the hypotheses on $A_1$, $d$, $G$, $l$, $P$, and $(X, g)$ in Proposition 2.4 and let $p \in (1, \infty)$. If $A$ is $C^\infty$, then there is a constant $C = C(A, A_1, g, l, p) \in [1, \infty)$ such that

\[
\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C \left( \|\Delta_A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right), \quad \forall \xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P).
\]

**Proof.** Suppose first that $\Delta_g$ is the Laplace-Beltrami operator on $C^\infty(X)$ defined by the Riemannian metric $g$. The a priori $L^p$ estimate for $\Delta_g$ analogous to \( \Delta_A \) can be obtained from the $a$ priori interior $L^p$ estimate provided by [40, Theorem 9.11] for a scalar, second-order, strictly elliptic operator with $C^\infty$ coefficients defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with the aid of a $C^\infty$ partition of unity subordinate to a finite set of coordinate charts covering the closed manifold, $X$. For the general case, one first chooses in addition a set of local trivializations for $\Lambda^l \otimes \text{ad} P$ corresponding to the coordinate neighborhoods, after shrinking those neighborhoods if needed. The Bochner-Weitzenböck formula \([37, \text{Equation (C.7)}], \ [68, \text{Equation (II.1)}]\) for $\Delta_A$ implies that $\Delta_A - \nabla^*_A \nabla_A$ is a first-order differential operator with $C^\infty$ coefficients and that $\Delta_A$ has principal symbol given by the $C^\infty$ Riemannian metric $g$ times the identity endomorphism of $\Lambda^l \otimes \text{ad} P$. (In fact, $\Delta_A = \nabla^*_A \nabla_A$ when $l = 0$.) The (first-order) covariant derivative of $\xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P)$ may be estimated with the following analogue of the interpolation inequality \([40, \text{Theorem 7.27}]\), valid for $p \in [1, \infty)$,

\[
\|\nabla_A \xi\|_{L^p(X)} \leq \varepsilon \|\xi\|_{W^{2,p}_{A_1}(X)} + C\varepsilon^{-1}\|\xi\|_{L^p(X)},
\]

where $C = C(A_1, g) \in [1, \infty)$ and $\varepsilon$ is any positive constant. The conclusion now follows by combining the preceding observations and using rearrangement with small $\varepsilon$ to remove the term $\|\nabla_A \xi\|_{L^p(X)}$ from the right-hand side. \( \square \)

We can now proceed to the

**Proof of Proposition 2.4**. We choose a $C^\infty$ connection, $A_s$, on $P$ that we regard as a smooth approximation to $A$. We write $A = A_s + a$, with $a \in W^{1,q}_{A_1}(X; \Lambda^l \otimes \text{ad} P)$ and a bound $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ with small constant $\varepsilon \in (0, 1]$ to be chosen during the proof, and write $A_s = A_1 + a_1$, where $a_1 \in C^\infty(X; \Lambda^l \otimes \text{ad} P)$ may be ‘large’. We expand $\Delta_A = \Delta_{A_s} + a$ to give

\[
\Delta_A \xi = \Delta_{A_s} \xi + \nabla_A a \times \xi + a \times \nabla_A \xi + a \times a \times \xi,
\]

and thus, for $\xi \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P)$,

\[
\Delta_A \xi = \Delta_{A_s} \xi + \nabla_{A_s} a \times \xi + a_1 \times a \times \xi + a \times \nabla_{A_1} \xi + a \times a \times \xi,
\]
Finally, when $q > d$ and (2.9) again holds; when any $t$ Theorem 4.12] to give which also yields (2.9).

Equation (6.20), immediate from [4, Theorem 4.12], we need $(\Delta X)^{1/2}$ yields

$$\|\Delta A - \Delta A_r\|_{L^p(X)} \leq z \left( \|\nabla A_t a\|_{L^q(X)} \|\xi\|_{L^r(X)} + \|a \times \nabla A_t \xi\|_{L^p(X)} + \|a\|_{W^{2,p}_{A_t}(X)} \|\xi\|_{L^p(X)} + \|a\|_{L^p(X)} \right),$$

where $z = z(g, G, l) \in [1, \infty)$. To ensure a continuous Sobolev embedding $W^{1,p}(X) \subset L^d(X)$ by [4, Theorem 4.12], we need $p^* = dp/(d - p) \geq d$, that is, $p \geq d - p$ or $p \geq d/2$, which we assume in our hypotheses.

To ensure a continuous Sobolev embedding $W^{1,q}(X) \subset L^{d/2}(X)$ when $q < d$, we need $q^* = dq/(d - q) \geq 2q$, that is, $d \geq 2d - 2q$ or $2q \geq d$ or $q \geq d/2$, which follows from our hypothesis that $q \geq d/2$; when $q \geq d$, the fact that $W^{1,q}(X) \subset L^{d/2}(X)$ is a continuous Sobolev embedding is immediate from [4, Theorem 4.12].

Consequently, by the preceding continuous Sobolev embeddings and the Kato Inequality [37, Equation (6.20)],

$$\|\Delta A - \Delta A_r\|_{L^p(X)} \leq z \left( \|\nabla A_t a\|_{L^q(X)} \|\xi\|_{L^r(X)} + \|a \times \nabla A_t \xi\|_{L^p(X)} + \|a\|_{L^p(X)} \right),$$

where $z = z(g, G, l, p, q) \in [1, \infty)$.

When $q < d$, we recall from [4, Theorem 4.12] that there is a continuous embedding $W^{1,q}(X) \subset L^{q^*}(X)$, where $q^* = dq/(d - q)$. Hence, $1/q^* = 1/q - 1/d$ or $1/q = 1/q^* + 1/d$ and so, using $p \leq q$,

$$\|a \times \nabla A_t \xi\|_{L^p(X)} \leq z \|a \times \nabla A_t \xi\|_{L^{q^*}(X)} \leq z \|a\|_{L^{q^*}(X)} \|\nabla A_t \xi\|_{L^{d}(X)},$$

and therefore, by the preceding continuous Sobolev embeddings,

$$(2.9) \quad \|a \times \nabla A_t \xi\|_{L^p(X)} \leq C \|a\|_{W^{1,q}_{A_t}(X)} \|\nabla A_t \xi\|_{W^{1,p}_{A_t}(X)},$$

where $C = C(g, G, l, p, q) \in [1, \infty)$. When $q = d$ and $d/2 \leq p < d$, we can define $t \in [d, \infty)$ by $1/p = 1/t + 1/d$ and apply the continuous Sobolev embedding $W^{1,d}(X) \subset L^t(X)$ from [4, Theorem 4.12] to give

$$\|a \times \nabla A_t \xi\|_{L^p(X)} \leq z \|a\|_{L^t(X)} \|\nabla A_t \xi\|_{L^{d}(X)} \leq \|a\|_{W^{1,q}_{A_t}(X)} \|\nabla A_t \xi\|_{W^{1,p}_{A_t}(X)},$$

and (2.9) again holds; when $q = d = p$, we can simply use the embedding $W^{1,d}(X) \subset C(X)$ for any $t \in [1, \infty)$ and observe that (2.9) holds from

$$\|a \times \nabla A_t \xi\|_{L^d(X)} \leq z \|a\|_{L^{2d}(X)} \|\nabla A_t \xi\|_{L^{2d}(X)} \leq \|a\|_{W^{1,d}_{A_t}(X)} \|\nabla A_t \xi\|_{W^{1,d}_{A_t}(X)}.$$
Combining our previous $L^p$ bound for $(\Delta_A - \Delta_{A_s}) \xi$ with the inequality \eqref{eq:2.9} gives

\begin{equation}
(\Delta_A - \Delta_{A_s}) \xi \leq z \left( \|\nabla A_1 a\|_{L^p(X)} + \|a\|_{W^{1,q}_{A_1}(X)}^2 \right) \|\xi\|_{W^{2,p}_{A_1}(X)}
+ z \|a_1\|_{C(X)} \|a\|_{W^{1,q}_{A_1}(X)} \|\xi\|_{W^{1,p}_{A_1}(X)}
+ z \|a\|_{W^{1,q}_{A_1}(X)} \|\nabla A_1 \xi\|_{W^{1,p}_{A_1}(X)},
\end{equation}

where $z = z(g, G, l, p, q) \in [1, \infty)$. Combining the preceding bound with the \textit{a priori} estimate \eqref{eq:2.6} for $\Delta_{A_s}$ provided by Lemma \ref{lem:2.6},

\[ \|\xi\|_{W^{2,p}_{A_1}(X)} \leq C_0 \left( \|\Delta_{A_s} \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right), \]

with constant denoted by $C_0 = C_0(A_1, A_s, g, p) \in [1, \infty)$ for clarity, yields

\[ \|\xi\|_{W^{2,p}_{A_1}(X)} \leq C_0 \left( \|\Delta A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right) + z \left( \|a\|_{W^{1,q}_{A_1}(X)} + \|a\|_{W^{1,q}_{A_1}(X)}^2 \right) \|\xi\|_{W^{2,p}_{A_1}(X)}
+ z \|a_1\|_{C(X)} \|a\|_{W^{1,q}_{A_1}(X)} \|\xi\|_{W^{1,p}_{A_1}(X)}. \]

We can choose $a = A - A_s$ so that $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ for small constant $\varepsilon \in (0, 1]$, but we are not at liberty to choose $a_1 = A_s - A_1$ to be $W^{1,q}_{A_1}(X)$-small. Thus in our forthcoming rearrangement arguments we first apply the interpolation inequality \eqref{eq:2.7},

\[ \|\nabla A_1 \xi\|_{L^p(X)} \leq \delta \|\xi\|_{W^{2,p}_{A_1}(X)} + C_1 \delta^{-1} \|\xi\|_{L^p(X)}, \]

where $C_1 = C_1(A_1, g) \in [1, \infty)$ and $\delta = \delta(A_1, \|A_s - A_1\|_{C(X)}, g, G, l, p, q) \in (0, 1]$ is a constant chosen small enough that

\[ \delta z \|a_1\|_{C(X)} \leq 1/2, \]

and thus, for a constant $C_2 = C_2(A_1, A_s, g, G, l, p, q) \in [1, \infty)$,

\[ \|\xi\|_{W^{2,p}_{A_1}(X)} \leq 2C_0 \left( \|\Delta A \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right)
+ 2z \left( \|a\|_{W^{1,q}_{A_1}(X)} + \|a\|_{W^{1,q}_{A_1}(X)}^2 \right) \|\xi\|_{W^{2,p}_{A_1}(X)} + C_2 \|\xi\|_{L^p(X)}. \]

Provided $\|a\|_{W^{1,q}_{A_1}(X)} \leq \varepsilon$ and we choose $\varepsilon \equiv \varepsilon(g, G, l, p, q) = 1/(8z) \in (0, 1]$ in the preceding inequality, rearrangement yields the desired estimate \eqref{eq:2.4}. Our proof of \eqref{eq:2.4} also verifies that the operator $\Delta_A$ in \eqref{eq:2.3} is bounded since $\Delta_{A_s}$ is bounded with the same domain and range spaces.

Next, we have the

\textbf{Proof of Corollary \ref{cor:2.3}} The finite-dimensionality of Ker $\Delta_A \cap W^{2,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P)$ follows from Corollary \ref{cor:C.2}. We observe that, by increasing the constant $C$ as needed, the term $\|\xi\|_{L^p(X)}$ appearing on the right-hand side of the inequality \eqref{eq:2.4} can be replaced by $\|\xi\|_{L^2(X)}$. This is clear when $p \leq 2$, while if $p > 2$, we can choose $s \in (p, \infty)$ and apply the interpolation inequality \cite{Hoffman} Equation \eqref{eq:7.10},

\[ \|\xi\|_{L^p(X)} \leq \delta \|\xi\|_{L^s(X)} + \delta^{-p} \|\xi\|_{L^2(X)}, \]
for \( \nu := (1/2 - 1/p)/(1/p - 1/s) > 0 \) and arbitrary positive \( \delta \). Because \( p \geq d/2 \), we have a continuous Sobolev embedding \( W^{2,p}(X) \subset L^p(X) \) as already observed, so
\[
\| \xi \|_{L^p(X)} \leq C_1 \delta \| \xi \|_{W^{2,p}_{\Lambda_1^d}(X)} + \delta^{-\nu} \| \xi \|_{L^2(X)},
\]
where \( C_1 = C_1(A_1, g, l, p) \in [1, \infty) \). Hence, for \( \delta(A_1, g, l, p) \in (0, 1] \) given by \( \delta = 1/(2C_1) \), we can use rearrangement in (2.4) to replace \( \| \xi \|_{L^p(X)} \) by \( \| \xi \|_{L^2(X)} \). Therefore, the estimate (2.4) implies
\[
(2.11) \quad \| \xi \|_{W^{2,p}_{\Lambda_1^d}(X)} \leq C \left( \| \Delta A \xi \|_{L^p(X)} + \| \xi \|_{L^2(X)} \right), \quad \forall \xi \in W^{2,p}_{\Lambda_1^d}(X; \Lambda^l \otimes \text{ad}P).
\]
Proposition 2.3 implies that the spectrum \( \sigma(\Delta_A) \) of \( \Delta_A \) on \( L^2(X; \Lambda^l \otimes \text{ad}P) \) consists purely of non-negative eigenvalues and is discrete with no accumulation points. Let \( \mu[A] \) denote the least positive eigenvalue of \( \Delta_A \) on \( L^2(X; \Lambda^l \otimes \text{ad}P) \) and recall from [16] Rayleigh’s Theorem, p. 16 (or more generally [89] Theorem 6.5.1), applied to the Green’s operator \( G_A \) for \( \Delta_A \) that
\[
\mu[A] = \inf_{\xi \in (\ker \Delta_A)^\perp} \frac{\langle \xi, \Delta_A \xi \rangle_{L^2(X)}}{\| \xi \|_{L^2(X)}^2},
\]
where \( (\ker \Delta_A)^\perp \) denotes the \( L^2 \)-orthogonal complement of \( \ker \Delta_A \subset H^1_{\Lambda_1^d}(X; \Lambda^l \otimes \text{ad}P) \), with equality achieved in the infimum if and only if \( \xi \) is an eigenvector with eigenvalue \( \mu[A] \). Therefore, if \( \xi \in (\ker \Delta_A)^\perp \cap W^{2,p}_{\Lambda_1^d}(X; \Lambda^l \otimes \text{ad}P) \), then
\[
\| \xi \|_{L^2(X)} \leq \mu[A]^{-1} \| \Delta_A \xi \|_{L^2(X)}.
\]
Hence, the inequality (2.5) follows from the preceding eigenvalue bound and (2.11) when \( p \geq 2 \). For the case \( d/2 \leq p < 2 \) (which forces \( d = 2 \) or 3), we observe that
\[
\| \xi \|_{L^2(X)}^2 \leq \frac{1}{\mu[A]} \langle \Delta_A \xi, \xi \rangle_{L^2(X)} \leq \frac{1}{\mu[A]} \| \Delta_A \xi \|_{L^p(X)} \| \xi \|_{L^{p'}(X)},
\]
where \( p' \) is defined by \( 1/p + 1/p' = 1 \), and so
\[
\| \xi \|_{L^2(X)} \leq \frac{1}{\sqrt{\mu[A]}} \| \Delta_A \xi \|_{L^p(X)} \| \xi \|_{L^{p'}(X)} \leq \frac{1}{\frac{1}{2\sqrt{\mu[A]}} \left( \delta^{-1} \| \Delta_A \xi \|_{L^p(X)} + \delta \| \xi \|_{L^{p'}(X)} \right)},
\]
for arbitrary positive \( \delta \). We have a continuous Sobolev embedding \( W^{2,p}(X) \subset L^{p'}(X) \) by [1] Theorem 4.12 if \( p' \leq p^* = dp/(d - 2p) \) or \( 1/p' = 1 - 1/p \geq 1/p^* = 1 - 2/d \) or \( 1 + 2/d \geq 2/p \) or \( (d + 2)/d \geq 2/p \), that is, if \( p \geq 2d/(d + 2) \). Moreover, \( 2d/(d + 2) \leq d/2 \) if and only if \( 4 \leq d + 2 \), that is, \( d \geq 2 \). Hence, for \( d \geq 2 \) and \( p \geq d/2 \), we have the bound
\[
\| \xi \|_{L^{p'}(X)} \leq C_2 \| \xi \|_{W^{2,p}_{\Lambda_1^d}(X)},
\]
where \( C_2 = C_2(A_1, g, l, p) \in [1, \infty) \). Combining the preceding inequalities for the case \( d/2 \leq p < 2 \) gives
\[
\| \xi \|_{L^2(X)} \leq \frac{1}{2\sqrt{\mu[A]}} \left( \delta^{-1} \| \Delta_A \xi \|_{L^p(X)} + \delta C_2 \| \xi \|_{W^{2,p}_{\Lambda_1^d}(X)} \right).
\]
Combining the preceding inequality with (2.11) and applying rearrangement by choosing \( \delta = \sqrt{\mu[A]}/(CC_2) \) yields the desired the inequality (2.5) for this case too. \( \square \)
2.3. Regularity for distributional solutions to an elliptic equation with Sobolev coefficients. We shall need to address a complication that arises when establishing regularity for distributional solutions to an elliptic equation with Sobolev coefficients. We shall confine our discussion to the Hodge Laplace operator, though one can clearly establish more general results of this kind.

Lemma 2.7 (Regularity for distributional solutions to an equation defined by the Hodge Laplace operator for a Sobolev connection). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group and \(P\) be a smooth principal \(G\)-bundle over \(X\). Let \(A_1\) be a \(C^\infty\) connection on \(P\), and \(A\) be a \(W^{1, q}\) connection on \(P\) with \(d/2 < q < \infty\), and \(l \geq 0\) be an integer. If \(q' \in (1, \infty)\) is the dual exponent defined by \(1/q + 1/q' = 1\) and \(\eta \in L^{q'}(X; \Lambda^l \otimes \text{ad}P)\) is a distributional solution\(^2\) to
\[
\Delta_A \eta = 0,
\]
then \(\eta \in W^{2, q}_{A_1}(X; \Lambda^l \otimes \text{ad}P)\).

Proof. We recall from \((C.3)\) in the proof of Corollary \((C.2)\) (with \(p = q\)) that
\[
\Delta_A - \Delta_{A_1} : W^{1,u}_{A_1}(X; \Lambda^l \otimes \text{ad}P) \to L^q(X; \Lambda^l \otimes \text{ad}P)
\]
is a bounded operator, where allowable values of \(u \in (1, \infty)\) are given by
\[
(2.12) \quad u = \begin{cases} d + \varepsilon & \text{if } d/2 < q < d, \\ 2d & \text{if } q = d, \\ q & \text{if } q > d, \end{cases}
\]
and \(\varepsilon \in (0, 1]\) is chosen small enough that \(q^* = dq/(d - q) \geq d + \varepsilon\), which is possible when \(q > d/2\). Consequently, the dual operator,
\[
\Delta_A - \Delta_{A_1} : L^{q'}(X; \Lambda^l \otimes \text{ad}P) \to W^{-1,u'}_{A_1}(X; \Lambda^l \otimes \text{ad}P),
\]
is bounded, where the dual exponent \(u' \in (1, \infty)\) is defined by \(1/u + 1/u' = 1\).

We write \(\Delta_A = \Delta_{A_1} + (\Delta_A - \Delta_{A_1})\), set \(\alpha := (\Delta_{A_1} - \Delta_A)\eta \in W^{-1,u'}_{A_1}(X; \Lambda^l \otimes \text{ad}P)\), and observe that \(\eta\) is a distributional solution to
\[
(2.13) \quad \Delta_{A_1} \eta = \alpha.
\]
We now appeal to regularity for distributional solutions to an equation (namely, \((2.13)\)) defined by an elliptic operator \(\Delta_{A_1}\) with \(C^\infty\) coefficients (see Remark \((2.2)\) and \((2.6)\) to conclude that \(\eta \in W^{1,u}_{A_1}(X; \Lambda^l \otimes \text{ad}P)\). The range of exponents \(u \in (d, \infty)\) given by \((2.12)\) ensures that \(\eta \in C(X; \Lambda^l \otimes \text{ad}P)\), since \(W^{1,u}(X) \subset C(X)\) by \((4)\) Theorem 4.12) when \(u > d\). Moreover, the estimate \((C.5)\) for \((\Delta_{A_1} - \Delta_A)\eta\) in terms of \(a = A - A_1\) and \(\eta\) ensures that
\[
(\Delta_{A_1} - \Delta_A)\eta \in L^q(X; \Lambda^l \otimes \text{ad}P).
\]
In particular, \(\alpha \in L^q(X; \Lambda^l \otimes \text{ad}P)\) and regularity for solutions to an elliptic equation (that is, \((2.13)\)) with \(C^\infty\) coefficients implies that \(\eta \in W^{2,q}_{A_1}(X; \Lambda^l \otimes \text{ad}P)\), as desired. \(\square\)

\(^2\) In the sense of Remark \((2.2)\)
2.4. Surjectivity of a perturbed Laplace operator. We now consider surjectivity properties of a perturbation of a Laplace operator, namely

**Lemma 2.8** (Surjectivity of a perturbed Laplace operator). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group and \(P\) be a smooth principal \(G\)-bundle over \(X\). Let \(A_1\) be a \(C^\infty\) connection on \(P\) and \(A\) be a \(W^{1,q}\) connection on \(P\) with \(d/2 < q < \infty\). Then there is a constant \(\delta = \delta(A, g) \in (0, 1]\) with the following significance. If \(a \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P)\) obeys

\[
\|a\|_{L^q(X)} < \delta \quad \text{when} \quad d \geq 3 \quad \text{or} \quad \|a\|_{L^q(X)} < \delta \quad \text{when} \quad d = 2,
\]

then the operator,

\[
d_A^*d_{A+a} : (\text{Ker} \, \Delta_A)^\perp \cap W^{2,q}_{A_1}(X; \text{ad}P) \to (\text{Ker} \, \Delta_A)^\perp \cap L^q(X; \text{ad}P),
\]

is well-defined and surjective.

**Remark 2.9** (Comparison with the argument due to Donaldson and Kronheimer). Our proof of Lemma 2.8 is based on a similar argument arising in Donaldson and Kronheimer [24, p. 66], as part of their version of the proof of Uhlenbeck’s local Coulomb gauge-fixing theorem [24, Theorem 2.3.7]. We make some adjustments to their argument for reasons which we briefly explain here. When \(B\) is a Sobolev connection matrix [24, p. 66], one has to take into account the possibility that their operator \(d^*d_B\), for a \(g\)-valued Sobolev one-form \(B\) over \(S^4\), could have dense range but still fail to be surjective, so one would first have to verify, for example, that \(d^*d_B\) has closed range. Similarly, while elliptic regularity ensures that if \(\eta \in \mathcal{L}^2(S^4; \text{ad}P) \cap \text{Ker} \, d^*d_B\) then \(\eta\) is \(C^\infty(S^4; \text{ad}P)\) when \(B\) is \(C^\infty\), those regularity issues become more subtle when \(B\) is merely a Sobolev one-form.

**Remark 2.10** (Dual spaces and direct sums of subspaces of Banach spaces). Our proof of Lemma 2.8 is clarified by a few observations concerning the dual space of a finite-dimensional subspace \(K\) of a Banach space \(\mathcal{K}\) that is continuously embedded in a Hilbert space, \(\mathcal{H}\). Since \(K\) has finite dimension, it has a closed complement, \(\mathcal{K}_0 \subset \mathcal{K}\), such that \(\mathcal{K} = \mathcal{K}_0 \oplus K\) (vector space direct sum) by [33, Definition 4.20 and Lemma 4.21(a)]. We may also simply define \(\mathcal{K}_0 := K_\perp \cap \mathcal{K}\), where \(K_\perp \subset \mathcal{H}\) is the orthogonal complement of \(K \subset \mathcal{H}\) and a closed subspace of \(\mathcal{H}\) [33, Theorem 12.4]. Therefore, \(K_\perp \cap \mathcal{K} \subset \mathcal{K}\) is also a closed subspace and a closed complement for \(K \subset \mathcal{K}\) in the sense of [33, Definition 4.20]. In summary:

\[
\mathcal{K} = \mathcal{K}_0 \oplus K, \quad \text{with} \quad \mathcal{K}_0 := K_\perp \cap \mathcal{K},
\]

is an orthogonal direct sum of a finite-dimensional and a closed subspace.

Recall that if \(M \subset \mathcal{K}\) is any subspace, then \(M^\circ := \{\alpha \in \mathcal{K}^* : \langle u, \alpha \rangle_{\mathcal{K} \times \mathcal{K}^*} = 0, \forall u \in M\}\) denotes the annihilator of \(M\) in \(\mathcal{K}^*\) [33, Section 4.6]. Because \(\mathcal{K} = \mathcal{K}_0 \oplus K\), we have \(\mathcal{K}^* = \mathcal{K}_0^* \oplus K^\circ\), the direct sum of the annihilators in \(\mathcal{K}^*\) of the subspaces \(\mathcal{K}_0\) and \(K\) of \(\mathcal{K}\).

Because \(K \subset \mathcal{K}\) and \(\mathcal{K}_0 \subset \mathcal{K}\) are closed subspaces, we have \(K^\circ \cong (\mathcal{K} / K)^*\) and \(\mathcal{K}_0^\circ \cong K^*\) by [33, Theorem 4.9]. Since \(\mathcal{K} = \mathcal{K}_0 \oplus K\), then \(\mathcal{K} / K \cong \mathcal{K}_0\) and \(\mathcal{K} / \mathcal{K}_0 \cong K\), so \(K^\circ \cong \mathcal{K}_0^*\) and \(\mathcal{K}_0^* \cong K^\circ\), where the final isomorphism follows from the fact that \(K\) is finite-dimensional. In particular, \(\mathcal{K}^* \cong K^* \oplus \mathcal{K}_0^* \cong K \oplus \mathcal{K}_0^*\). Since \(K^*\) is finite-dimensional, then \(K^* \subset \mathcal{K}^*\) is a closed subspace by [13, Proposition 11.1] and so the complement \(\mathcal{K}_0^* \subset \mathcal{K}^*\) is a closed subspace
by [83, Definition 4.20 and Lemma 4.21]. In summary:

\[(2.17) \quad \mathcal{X}^* = \mathcal{X}^*_0 \oplus K,\]

is a direct sum of a finite-dimensional and a closed subspace.

**Proof of Lemma 2.8.** Note that when \(a = 0\), the operator \((2.15)\) is invertible by Corollary C.2.

To see that the operator \((2.15)\) is well-defined, observe that if \(\xi = d_A^* d_A \alpha \in L^q(X; adP)\) for some \(\alpha \in W^2,q(X; adP)\) and \(\eta \in \text{Ker} \Delta_A \cap L^q(X; adP)\), then

\[(\xi, \eta)_{L^2(X)} = (d_A^* d_A + a \chi, \eta)_{L^2(X)} = (d_A^* d_A \chi, d_A \eta)_{L^2(X)} = 0,
\]

since \(\Delta_A \eta = d_A^* d_A \eta = 0\) and thus \((d_A^* d_A \eta, \eta)_{L^2(X)} = \|d_A \eta\|^2_{L^2(X)} = 0\), so that \(d_A \eta = 0\). Thus,

\[(2.18) \quad \text{Ran}(d_A^* d_A + a : W^2,q(X; adP) \to L^q(X; adP)) \subset (\text{Ker} \Delta_A) \perp \cap L^q(X; adP),
\]

and so the operator \((2.15)\) is well-defined.

It is convenient to abbreviate \(\mathcal{X} := W^2,q(X; adP)\), and \(K := \text{Ker} \Delta_A \cap W^2,q(X; adP)\), and \(\mathcal{Y} := L^q(X; adP)\), and \(\mathcal{H} := L^2(X; adP)\), and denote \(T = d_A^* d_A + a\) in \((2.15)\). The kernel \(K \subset \mathcal{X}\) is finite-dimensional by Corollary C.2 and its \(L^2\)-orthogonal complements \(K^\perp \cap \mathcal{X}\) and \(K^\perp \cap \mathcal{Y}\) and \(K^\perp\) provide closed complements of \(K\) in \(\mathcal{X}\) and \(\mathcal{Y}\) and \(\mathcal{H}\), respectively. Similarly, we let \(K^\perp \cap \mathcal{Y}^*\) and \(K^\perp \cap \mathcal{X}^*\) denote the closed complements of \(K^* \cong K\) in \(\mathcal{H}^* \cong \mathcal{H}\) and \(\mathcal{Y}^*\) and \(\mathcal{X}^*\), respectively, where \(\mathcal{Y}^* = L^q(X; adP)\), with \(q' \in (1, \infty)\) defined by \(1/q + 1/q' = 1\), and \(\mathcal{X}^* = W^{−2,q}(X; adP)\).

If \(N \subset \mathcal{Y}^*\) is any subspace, we recall that the annihilator [83, Section 4.6] of \(N\) in \(\mathcal{Y}\) is

\[\mathcal{Y}^\circ := \{\xi \in \mathcal{Y} : (\xi, \alpha)_{\mathcal{Y} \times \mathcal{Y}^*} = 0, \forall \alpha \in \mathcal{Y}^*\},\]

and \((\cdot, \cdot)_{\mathcal{Y} \times \mathcal{Y}^*} : \mathcal{Y} \times \mathcal{Y}^* \to \mathbb{R}\) is the canonical pairing. The operator, \(T : \mathcal{X} \to \mathcal{Y}\), is Fredholm by Lemma C.4, and we can identify its range using

\[
\text{Ran}(T : \mathcal{X} \to \mathcal{Y}) = \overline{\text{Ran}(T : \mathcal{X} \to \mathcal{Y})} \quad (\text{by closed range})
\]

\[= \mathcal{Y}^\circ \text{Ker}(T^* : \mathcal{Y}^* \to \mathcal{X}^*) \quad (\text{by [13, Corollary 2.18 (iv)])}).
\]

Therefore, we have shown that

\[
\text{Ran}(T : \mathcal{X} \to \mathcal{Y}) = \mathcal{Y}^\circ \text{Ker}(T^* : \mathcal{Y}^* \to \mathcal{X}^*),
\]

and so \(\text{Ran}(T : \mathcal{X} \to \mathcal{Y}) = \mathcal{Y}\) if and only if \(\text{Ker}(T^* : \mathcal{Y}^* \to \mathcal{X}^*) = 0\). Similarly, because \(T : \mathcal{X} \to \mathcal{Y}\) has closed range and \(\text{Ran}(T : \mathcal{X} \to \mathcal{Y}) \subset K^\perp \cap \mathcal{Y}\) by \((2.18)\), the operator,

\[
T : K^\perp \cap \mathcal{X} \to K^\perp \cap \mathcal{Y},
\]

also has closed range and we obtain

\[
\text{Ran}(T : K^\perp \cap \mathcal{X} \to K^\perp \cap \mathcal{Y}) = \mathcal{Y}^\circ \text{Ker}(T^* : K^\perp \cap \mathcal{Y}^* \to K^\perp \cap \mathcal{X}^*).
\]

Consequently,

\[
(2.19) \quad \text{Ran}(T : K^\perp \cap \mathcal{X} \to K^\perp \cap \mathcal{Y}) = K^\perp \cap \mathcal{Y}
\]

\[\iff \text{Ker}(T^* : K^\perp \cap \mathcal{Y}^* \to K^\perp \cap \mathcal{X}^*) = 0.
\]
If \( T^* : K^+ \cap \mathcal{Y}^* \to K^+ \cap \mathcal{X}^* \) were not injective, there would be a non-zero \( \eta \in K^+ \cap \mathcal{Y}^* \) such that \( T^* \eta = 0 \). In other words, because \( T^* = d^*_A + a_A \), there would be a non-zero \( \eta \in K^+ \cap L^q(X; \text{ad} P) \) such that

\[
d^*_A + a_A \eta = 0 \in W^{2,q}_A(X; \text{ad} P),
\]

that is, \( (d^*_A + a_A \eta)(\xi) = 0 \) for all \( \xi \in K^+ \cap W^{2,q}_A(X; \text{ad} P) \) or equivalently,

\[
\langle \chi, d^*_A + a_A \eta \rangle_{\mathcal{X} \times \mathcal{Y}^*} = \langle \eta, d^*_A + a_A \chi \rangle_{\mathcal{Y} \times \mathcal{X}^*} = (\eta, d^*_A + a_A \chi)_{L^2(X)} = 0,
\]

\( \forall \chi \in K^+ \cap W^{2,q}_A(X; \text{ad} P) \).

Lemma 2.7 implies that \( \eta \in W^{2,q}_A(X; \text{ad} P) \). Observe that, by writing \( d_{A+a} = d_A + [a, \chi] \),

\[
0 = (d^*_A + a_A \chi, \eta)_{L^2(X)} = (d_{A+a} \chi, d_A \eta)_{L^2(X)} = (d_{A+a} \chi, d_A \eta)_{L^2(X)} + ([a, \chi], d_A \eta)_{L^2(X)}.
\]

Since \( \eta \perp \text{Ker} \Delta_A \cap W^{2,q}_A(X; \text{ad} P) \) and letting \( \mu[A] \) denote the least positive eigenvalue of the Laplace operator \( \Delta_A \) on \( L^2(X; \text{ad} P) \) provided by Proposition 2.8, we have

\[
\mu[A] \leq \frac{(\eta, d^*_A + a_A \eta)_{L^2(X)}}{||\eta||_{L^2(X)}^2} = \frac{||d_A \eta||_{L^2(X)}^2}{||\eta||_{L^2(X)}^2}
\]

and thus,

\[
||\eta||_{L^2(X)} \leq \mu[A]^{-1/2} ||d_A \eta||_{L^2(X)}.
\]

Hence, we obtain

\[
(2.20) \quad ||\eta||_{W^{1,2}_A(X)} \leq C_1 ||d_A \eta||_{L^2(X)},
\]

for a constant \( C_1 = C_1(A, g) = 1 + \mu[A]^{-1/2} \in [1, \infty) \). For \( d \geq 3 \) and using \( 1/2 = (d-2)/2d + 1/d \) and the continuous multiplication \( L^{2d/(d-2)}(X) \times L^d(X) \to L^2(X) \), we see that

\[
||(a, \chi), d_A \eta)_{L^2(X)} || \leq ||(a, \chi)_{L^2(X)}|| ||d_A \eta||_{L^2(X)}
\]

\[
\leq ||a||_{L^d(X)} ||\chi||_{L^{2d/(d-2)}(X)} ||d_A \eta||_{L^2(X)}
\]

\[
\leq C_2 ||a||_{L^d(X)} ||\chi||_{W^{1,2}_A(X)} ||d_A \eta||_{L^2(X)},
\]

where the constant \( C_2 = C_2(g) \in [1, \infty) \) is the norm of the continuous Sobolev embedding \( W^{1,2}(X) \subset L^{2d/(d-2)}(X) \) provided by [4, Theorem 4.12]. Hence, setting \( \chi = \eta \) and applying the a priori estimate \( (2.20) \), the preceding identity and inequalities yield

\[
||d_A \eta||_{L^2(X)}^2 = ||(a, \chi), d_A \eta)_{L^2(X)} ||^2
\]

\[
\leq C_1 C_2 ||a||_{L^d(X)} ||d_A \eta||_{L^2(X)}^2,
\]

and so, if \( \eta \neq 0 \), we have

\[
||a||_{L^d(X)} \geq C_1 C_2,
\]

contradicting our hypothesis (2.14) that \( ||a||_{L^d(X)} < \delta \), with \( \delta \) small.
For the case $d = 2$, we instead use the continuous multiplication $L^4(X) \times L^4(X) \to L^2(X)$ and continuous Sobolev embedding $W^{1,2}(X) \subset L^r(X)$ for $1 \leq r < \infty$ provided by [3] Theorem 4.12. In particular, using the embedding $W^{1,2}(X) \subset L^4(X)$ with norm $C_3 = C_3(g) \in [1, \infty)$, we obtain

$$\|d_A \eta\|_{L^2(X)} ^2 \leq |\langle [a, \eta], d_A \eta \rangle|_{L^2(X)} \leq \| [a, \eta] \|_{L^2(X)} \| d_A \eta \|_{L^2(X)} \leq \| a \|_{L^4(X)} \| \eta \|_{L^4(X)} \| d_A \eta \|_{L^2(X)} \leq C_3 \| a \|_{L^4(X)} \| \eta \|_{W^{1,2}_A(X)} \| d_A \eta \|_{L^2(X)} \leq C_1 C_3 \| a \|_{L^4(X)} \| d_A \eta \|_{L^2(X)}^2 \quad \text{(by (2.20)),}$$

which again yields a contradiction to our hypothesis (2.14), just as in the case $d \geq 3$. \hfill \Box

2.5. **A priori estimates for Coulomb gauge transformations.** We now establish a generalization of [28, Lemma 6.6], which is in turn an analogue of [34, Lemma 2.3.10]. We allow $p \geq 2$ and any $d \geq 2$ rather than assume $d = 4$, as in [28, Lemma 6.6], but we use standard Sobolev norms rather than the ‘critical exponent’ Sobolev norms employed in the statement and proof of [28, Lemma 6.6], since we do not seek an explicit optimal dependence of constants on the reference connection, $A_0$.

Before stating our generalization of [28, Lemma 6.6], we digress to recall from [38, p. 231], that a gauge transformation, $u \in \text{Aut}(P)$, may be viewed as a section of the fiber bundle $\text{Ad}P := P \times_{Ad} G \to X$, where we denote $\text{Ad}(g) : G \ni h \to g^{-1}hg \in G$, for all $h \in G$. With the aid of a choice of a unitary representation, $\rho : G \subset \text{Aut}_C(\mathbb{E})$, we may therefore consider $\text{Ad}P$ to be a subbundle of the Hermitian vector bundle $P \times_{\rho} \text{End}_C(\mathbb{E})$. We can alternatively replace $\rho : G \to \text{Aut}_C(\mathbb{E})$ by $\text{Ad} : G \to \text{Aut}(g)$ and $\text{End}_C(\mathbb{E})$ by $\text{End}(g)$. A choice of connection $A$ on $P$ induces covariant derivatives on all associated vector bundles, such as $E = P \times_{\rho} \mathbb{E}$ and $P \times_{\rho} \text{End}_C(\mathbb{E})$ or $\text{ad}P = P \times_{\text{ad}} g$ and $P \times_{\text{Ad}} \text{End}(g)$. We can thus define Sobolev norms of sections of $\text{Ad}P$, generalizing the construction of Freed and Uhlenbeck in [37, Appendix A].

A similar construction is described by Parker [79, Section 4], but we note that while the center of $\text{Aut}(P)$ — which is given by $P \times_{\text{Ad}} \text{Center}(G)$ — acts trivially on $\mathcal{A}(P)$, it does not act trivially on $C^\infty(X; E)$.

**Proposition 2.11** (A priori $W^{1,p}$ estimate for $u(A) - A_0$ in terms of $A - A_0$). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. Let $A_1$ be a $C^\infty$ connection on $P$, and $A_0$ be a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$, and $p \in (1, \infty)$ obey $d/2 \leq p \leq q$, and $\delta \in (0, 1]$. Then there are constants $N \in [1, \infty)$ and $\varepsilon = \varepsilon(A_0, A_1, g, G, p, q) \in (0, 1]$ (with dependence on $p$ replaced by dependence on $\delta$ when $p = d$) with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ and $u \in \text{Aut}(P)$ is a gauge transformation of class $W^{2,q}$ such that

$$d^*_A (u(A) - A_0) = 0,$$

then the following hold. If

$$\|A - A_0\|_{L^\delta(X)} \leq \varepsilon \quad \text{and} \quad \|u(A) - A_0\|_{L^\delta(X)} \leq \varepsilon,$$

in applications of Proposition 2.11 we can choose $\delta = 1$ without loss of generality.
where

\[ s(p) := \begin{cases} 
  d & \text{if } p < d, \\
  d + \delta & \text{if } p = d, \\
  p & \text{if } p > d,
\end{cases} \]

then

\[ \|u(A) - A_0\|_{L^p(X)} \leq N\|A - A_0\|_{W^{1,p}_A(X)}, \]

where \( N = N(A_0, A_1, g, G, p, q) \). If in addition \( A \) obeys

\[ \|A - A_0\|_{W^{1,p}_A(X)} \leq M, \]

for some constant \( M \in [1, \infty) \), then

\[ \|u(A) - A_0\|_{W^{1,p}_A(X)} \leq N\|A - A_0\|_{W^{1,p}_A(X)}, \]

where \( N = N(A_0, A_1, g, G, M, p, q) \).

**Proof.** Following the convention of [95, p. 32] for the action of \( u \in \text{Aut}(P) \) on connections \( A \) on \( P \) and setting \( B := u(A) \) for convenience, we have

\[ B - A_0 = u^{-1}(A - A_0)u + u^{-1}d_{A_0}u. \]

Our task is thus to estimate the term \( d_{A_0}u \). Rewriting the preceding equality gives a first-order, linear elliptic equation in \( u \) with \( W^{1,q} \) coefficients,

\[ d_{A_0}u = u(B - A_0) - (A - A_0)u. \]

Corollary [C.2] implies that the kernel,

\[ K := \text{Ker} \left( \Delta_{A_0} : W^{2,q}_{A_1}(X; \text{ad}P) \to L^q(X; \text{ad}P) \right), \]

is finite-dimensional. Let

\[ \Pi : L^2(X; \text{ad}P) \to K \subset L^2(X; \text{ad}P) \]

denote the \( L^2 \)-orthogonal projection and denote

\[ \gamma := \Pi u \in K \subset W^{2,q}_{A_1}(X; \text{ad}P), \]
\[ u_0 := u - \gamma \in K^\perp \cap W^{2,q}_{A_1}(X; \text{ad}P), \]

where \( \perp \) is \( L^2 \)-orthogonal complement. We may assume without loss of generality that we have a unitary representation, \( G \subset U(n) \). Recall that (due to [99, Equation (6.2)]),

\[ d_{A_0}^* = (-1)^{-d(k+1)+1} \ast d_{A_0} \ast \text{ on } \Omega^k(X; \text{ad}P), \]

---

\footnote{The use of the constant \( M \) could be avoided if we replaced the right-hand-side of Inequality [95] by \( N(1 + \|A - A_0\|_{W^{1,p}_A(X)})\|A - A_0\|_{W^{1,p}_A(X)} \).}
where \( * : \Omega^k(X) \rightarrow \Omega^{d-k} \) is the Hodge star operator on \( k \)-forms. Because \( d^*_{A_0}(B - A_0) = 0 \) and \( d_{A_0} u = d_{A_0} u_0 \), an application of \( d^*_{A_0} \) to (2.28) yields
\[
d^*_{A_0} d_{A_0} u_0 = -*(d_{A_0} u \wedge *(B - A_0)) + u d^*_{A_0} (B - A_0) - (d^*_{A_0} (A - A_0)) u + *(A - A_0) \wedge d_{A_0} u_0.
\]

We shall use the bound \( \|u\|_{C(X)} \leq 1 \) for any \( u \in \text{Aut}(P) \) of class \( W^{2,q} \), implied by the fact that the representation for \( G \) is unitary. Recall that \( \Delta_{A_0} = d^*_{A_0} d_{A_0} \). We first consider the case \( p < d \) and setting \( p^* = dp/(d - p) \), we have \( 1/p = 1/d + 1/p^* \) and the continuous multiplication \( L^d(X) \times L^{p^*}(X) \rightarrow L^p(X) \), so
\[
\|\Delta_{A_0} u_0\|_{L^p(X)} \leq \|d_{A_0} u_0\|_{L^{p^*}(X)} \|B - A_0\|_{L^d(X)} + \|d^*_{A_0} (A - A_0)\|_{L^{p^*}(X)} \|u\|_{C(X)}
+ \|A - A_0\|_{L^d(X)} \|d_{A_0} u_0\|_{L^{p^*}(X)}
\leq \left( \|B - A_0\|_{L^d(X)} + \|A - A_0\|_{L^d(X)} \right) \|d_{A_0} u_0\|_{L^{p^*}(X)}
+ \|d^*_{A_0} (A - A_0)\|_{L^p(X)}.
\]

Second, for the case \( p > d \), we use the continuous multiplication \( L^p(X) \times L^{\infty}(X) \rightarrow L^p(X) \), so
\[
\|\Delta_{A_0} u_0\|_{L^p(X)} \leq \left( \|B - A_0\|_{L^p(X)} + \|A - A_0\|_{L^p(X)} \right) \|d_{A_0} u_0\|_{L^{\infty}(X)}
+ \|d^*_{A_0} (A - A_0)\|_{L^p(X)}.
\]

Third, for the case \( p = d \), we can instead use \( r \in (d, \infty) \) defined by \( 1/d = 1/(d + \delta) + 1/r \) and the resulting continuous multiplication, \( L^{d+\delta}(X) \times L^{r}(X) \rightarrow L^{d}(X) \), to give
\[
\|\Delta_{A_0} u_0\|_{L^d(X)} \leq \left( \|B - A_0\|_{L^{d+\delta}(X)} + \|A - A_0\|_{L^{d+\delta}(X)} \right) \|d_{A_0} u_0\|_{L^{r}(X)}
+ \|d^*_{A_0} (A - A_0)\|_{L^d(X)}.
\]

By [3] Theorem 4.12, we have continuous Sobolev embeddings,
\[
W^{1,p}(X) \subset \begin{cases} L^{dp/(d-p)}(X) & \text{if } 1 \leq p < d, \\ L^r(X) & \text{if } p = d \text{ and } 1 \leq r < \infty, \\ C(X) & \text{if } p > d. \end{cases}
\]

Therefore, the Sobolev Embedding Theorem and Kato Inequality [37] Equation (6.20) give, for \( r \in (d, \delta) \) determined by \( \delta \) as above,
\[
\|d_{A_0} u_0\|_{L^{p^*}(X)} \leq C_0 \|d_{A_0} u_0\|_{W^{1,p}_{A_1}(X)}, \quad p < d,
\]
\[
\|d_{A_0} u_0\|_{L^{r}(X)} \leq C_0 \|d_{A_0} u_0\|_{W^{1,d}_{A_1}(X)}, \quad p = d,
\]
\[
\|d_{A_0} u_0\|_{L^{\infty}(X)} \leq C_0 \|d_{A_0} u_0\|_{W^{1,p}_{A_1}(X)}, \quad p > d,
\]
where \( C_0 = C_0(g,p) \) or \( C_0(g,\delta) \in [1, \infty) \) is bounded below by the norm of the Sobolev embedding (2.30). Writing \( A_0 = A_1 + a_0 \), for \( a_0 \in W^{1,q}_{A_1}(X; A^1 \otimes \text{ad} P) \), so \( d_{A_0} u_0 = d_{A_1} u_0 + [a_0, u_0] \), we see
that
\[ \| \nabla A_1 d_{A_0} u_0 \|_{L^p(X)} \leq \| \nabla A_1 d_{A_1} u_0 \|_{L^p(X)} + \| \nabla A_1 [a_0, u_0] \|_{L^p(X)} \]
\[ \leq \| \nabla^2 A_1 u_0 \|_{L^p(X)} + \| \nabla A_1 a_0 \times u_0 + a_0 \times \nabla A_1 u_0 \|_{L^p(X)}. \]

By hypothesis, we have \( p = d/2 < q \) or \( d/2 < p \leq q \), and thus, defining \( r \in [p, \infty] \) by \( 1/p = 1/q + 1/r \) and using the continuous Sobolev embeddings \( W^{1,p} \subset L^{2p}(X) \) and \( W^{2,p}(X) \subset L^r(X) \) (with norm bounded above by \( C_0 \) for \( p \geq d/2 \)),
\[ \| \nabla A_1 d_{A_0} u_0 \|_{L^p(X)} \leq \| \nabla^2 A_1 u_0 \|_{L^p(X)} + z \| \nabla A_1 a_0 \|_{L^s(X)} \| u_0 \|_{L^r(X)} + z \| a_0 \|_{L^{2p}(X)} \| \nabla A_1 u_0 \|_{L^{2p}(X)} \]
\[ \leq \| \nabla^2 A_1 u_0 \|_{L^p(X)} + z C_0 \| \nabla A_1 a_0 \|_{L^s(X)} \| u_0 \|_{W^{2,p}(X)} + z C_0^2 \| a_0 \|_{W^{1,p}(X)} \| \nabla A_1 u_0 \|_{W^{1,p}(X)} \]
\[ \leq \| \nabla^2 A_1 u_0 \|_{L^p(X)} + z C_0 \| a_0 \|_{W^{1,q}(X)} \| u_0 \|_{W^{2,p}(X)} + z C_0^2 \| a_0 \|_{W^{1,q}(X)} \| u_0 \|_{W^{2,p}(X)}, \]
where \( z = z(g) \in [1, \infty) \) and now \( C_0 = C_0(A_1, g, p, q) \in [1, \infty) \). By substituting the preceding bound into (2.31), we find that
\[ \| d_{A_0} u_0 \|_{L^p(X)} \leq C_1 \| u_0 \|_{W^{2,p}(X)}, \quad p < d, \]
\[ \| d_{A_0} u_0 \|_{L^p(X)} \leq C_1 \| u_0 \|_{W^{2,1}(X)}, \quad p = d, \]
\[ \| d_{A_0} u_0 \|_{L^p(X)} \leq C_1 \| u_0 \|_{W^{2,p}(X)}, \quad p > d, \]
for a constant \( C_1 = C_1(A_1, g, p, q, \| a_0 \|_{W^{1,q}(X)}) \in [1, \infty) \), with dependence on \( p \) replaced by \( \delta \) when \( p = d \). By combining the preceding three cases \( (p < d, \text{ and } p = d, \text{ and } p > d) \), we obtain
\[ \| \Delta A_0 u_0 \|_{L^p(X)} \leq C_1 \left( \| B - A_0 \|_{L^s(X)} + \| A - A_0 \|_{L^s(X)} \right) \| u_0 \|_{W^{2,p}(X)} \]
\[ + \| d_{A_0}^* (A - A_0) \|_{L^p(X)}, \]
where \( s = s(p) \) is as in (2.23). From the \( a \text{ priori} \) estimate (2.25) in Corollary 2.25 — and noting that this lemma also holds for \( \text{AdP} \) in place of \( \text{adP} \) via the definition of Sobolev norms of \( u \in \text{Aut}(P) \) described earlier — we have the \( a \text{ priori} \) estimate,
\[ \| u_0 \|_{W^{2,p}(X)} \leq C_3 \| \Delta A_0 u_0 \|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q, \]
where \( C_3 = C_3(A_1, A_0, g, G, p, q) \in [1, \infty) \). Substituting the \( a \text{ priori} \) estimate (2.33) into our \( L^p \) bound (2.32) for \( \Delta A_0 u_0 \) gives, for \( s \) as in (2.24),
\[ \| \Delta A_0 u_0 \|_{L^p(X)} \leq C_1 C_3 \left( \| B - A_0 \|_{L^s(X)} + \| A - A_0 \|_{L^s(X)} \right) \| \Delta A_0 u_0 \|_{L^p(X)} \]
\[ + \| d_{A_0}^* (A - A_0) \|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q. \]
Provided
\[ \| B - A_0 \|_{L^s(X)} \leq 1/(4C_1 C_3) \quad \text{and} \quad \| A - A_0 \|_{L^s(X)} \leq 1/(4C_1 C_3), \]
as assured by (2.22), then rearrangement in the preceding inequality yields
\[ \| \Delta A_0 u_0 \|_{L^p(X)} \leq 2 \| d_{A_0}^* (A - A_0) \|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q. \]
Therefore, by combining the inequalities (2.35) and (2.33) we find that
\begin{equation}
\|u_0\|_{W^{2,p}(X)} \leq 2C_3\|d^*_A(A - A_0)\|_{L^p(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q.
\end{equation}
Using \(d_{A_0}u = d_{A_0}u_0\) and (2.27) and the facts that \(|u| \leq 1\) and \(|u^{-1}| \leq 1\) on \(X\), we obtain
\begin{equation}
\|B - A_0\|_{L^p(X)} \leq \|A - A_0\|_{L^p(X)} + \|d_{A_0}u_0\|_{L^p(X)}.
\end{equation}
From (2.37) and (2.36), we see that
\begin{equation}
\|B - A_0\|_{L^p(X)} \leq \|A - A_0\|_{L^p(X)} + \|u_0\|_{W^{1,p}(X)}
\leq \|A - A_0\|_{L^p(X)} + 2C_3\|d^*_A(A - A_0)\|_{L^p(X)}.
\end{equation}
Using \(A_0 = A_1 + a_0\), we have \(d^*_A(A - A_0) = d^*_A(A - A_0) + a_0 \times (A - A_0)\) and
\begin{equation}
\|d^*_A(A - A_0)\|_{L^p(X)} \leq z\|A - A_0\|_{W^{1,p}(X)} + z\|a_0\|_{L^p(X)}\|A - A_0\|_{L^p(X)}.
\end{equation}
Applying the continuous Sobolev embedding, \(W^{1,p}(X) \subset L^2(X)\), with norm \(C_0 = C_0(g, p) \in [1, \infty)\) and the Kato Inequality [37, Equation (6.20)],
\begin{equation}
\|d^*_A(A - A_0)\|_{L^p(X)} \leq z\|A - A_0\|_{W^{1,p}(X)} + zC_0^2\|a_0\|_{W^{1,p}(X)}\|A - A_0\|_{W^{1,p}(X)}.
\end{equation}
Thus,
\begin{equation}
\|B - A_0\|_{L^p(X)} \leq C_4\|A - A_0\|_{W^{1,p}(X)}, \quad \text{for } 1 < p < \infty \text{ and } d/2 \leq p \leq q,
\end{equation}
where \(C_4 = C_4(A_0, A_1, g, G, p, q) \in [1, \infty)\), giving the desired \(L^p\) estimate (2.24) for \(B - A_0\).

We now estimate the \(L^p\) norms of the covariant derivatives of the right-hand side of the identity (2.27). Considering the term \(u^{-1}d_{A_0}u\) in the right-hand side of the identity (2.27) and recalling that \(\nabla_{A_0}u = d_{A_0}u = d_{A_0}u_0\), we have
\[\nabla_{A_0}(u^{-1}d_{A_0}u_0) = -u^{-1}(\nabla_{A_0}u_0)u^{-1} \otimes d_{A_0}u_0 + u^{-1}\nabla_{A_0}d_{A_0}u_0.\]
First, if \(d/2 \leq p < d\) and using the continuous multiplication, \(L^p(X) \times L^d(X) \to L^p(X),\)
\begin{equation}
\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq \|\nabla_{A_0}u_0\|_{L^p(X)}\|\nabla_{A_0}u_0\|_{L^d(X)} + \|\nabla_{A_0}u_0\|_{L^p(X)}
\leq C_0^2\|\nabla_{A_0}u_0\|_{W^{1,p}(X)} + \|\nabla_{A_0}u_0\|_{L^p(X)}
\leq C_0^2\|u_0\|_{W^{2,p}(X)} + \|u_0\|_{W^{2,p}(X)}.
\end{equation}
Second, if \(p = d\) and using the continuous multiplication, \(L^{2d}(X) \times L^{2d}(X) \to L^d(X),\)
\begin{equation}
\|\nabla_{A_0}(u^{-1}d_{A_0}u_0)\|_{L^d(X)} \leq \|\nabla_{A_0}u_0\|_{L^{2d}(X)}\|\nabla_{A_0}u_0\|_{L^{2d}(X)} + \|\nabla_{A_0}u_0\|_{L^d(X)}
\leq C_0^2\|\nabla_{A_0}u_0\|_{W^{1,d}(X)} + \|\nabla_{A_0}u_0\|_{L^d(X)}
\leq C_0^2\|u_0\|_{W^{2,d}(X)} + \|u_0\|_{W^{2,d}(X)}.
\end{equation}
Third, if $p > d$,\
\[
\|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq \|\nabla A_0u_0\|_{L^\infty(X)}\|\nabla A_0u_0\|_{L^p(X)} + \|\nabla^2 u_0\|_{L^p(X)} \\
\leq C_6^2\|\nabla A_0u_0\|^2_{W^{1,p}_0(X)} + \|\nabla^2 u_0\|_{L^p(X)} \\
\leq C_6^2\|u_0\|^2_{W^{2,p}_0(X)} + \|u_0\|_{W^{2,p}_0(X)}.
\]

Thus, by combining the three preceding cases and applying Lemma \[B.2\] (3),\
\[
(2.40) \quad \|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq C_6^2 C_0^2\|u_0\|^2_{W^{1,p}_0(X)} + C_6\|u_0\|_{W^{2,p}_0(X)}, \quad \text{for } d/2 \leq p \leq q,
\]
where $C_6 = C_6(A_0, A_1, g, p, q) \in [1, \infty)$ is the constant in Lemma \[B.2\] (3). By combining the bound (2.40) for $\|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)}$ with (2.36), we find that\
\[
\|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq 2C_6^2 C_0^2 C_3|d_{A_0}^*(A - A_0)|_{L^p(X)}^2 + 2C_6 C_3|d_{A_0}^*(A - A_0)|_{L^p(X)}^2,
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$.

But\
\[
(2.41) \quad |d_{A_0}^*(A - A_0)|_{L^p(X)} \leq z\|A - A_0\|_{W^{1,p}_0(X)},
\]
for a generic constant $z = z(g) \in [1, \infty)$, and by Lemma \[B.2\] (3),\
\[
(2.42) \quad \|A - A_0\|_{W^{1,p}_0(X)} \leq C_7\|A - A_0\|_{W^{1,p}_0(X)},
\]
where $C_7 = C_7(A_0, A_1, g, p) \in [1, \infty)$ is the constant in Lemma \[B.2\] (3), and because $A$ is now assumed to obey (2.28), that is,\
\[
\|A - A_0\|_{W^{1,p}_0(X)} \leq M,
\]
we obtain\
\[
\|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq 2C_6^2 C_0^2 C_3(zC_6 C_7 C_0^2 M + 1)|d_{A_0}^*(A - A_0)|_{L^p(X)}^2,
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$, and thus by (2.41) and (2.42),\
\[
(2.43) \quad \|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)} \leq 2z^2 C_6 C_7 C_3(zC_6 C_7 C_0^2 M + 1)|A - A_0|_{W^{1,p}_0(X)}^2,
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$.

Considering the term $u^{-1}(A - A_0)u$ in the right-hand side of (2.27), we discover that\
\[
\nabla A_0(u^{-1}(A - A_0)u) = -u^{-1}(\nabla A_0 u)u^{-1}(A - A_0)u + u^{-1}(\nabla A_0(A - A_0))u \\
+ u^{-1}(A - A_0) \otimes \nabla A_0 u.
\]
Noting that $\nabla A_0 u = \nabla A_0 u_0$ and $\|u\|_{C(X)} \leq 1$, the preceding identity gives, for $d/2 \leq p < d$,\
\[
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq 2\|\nabla A_0 u_0\|_{L^p(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)} \\
\leq 2C_6^2\|\nabla A_0 u_0\|_{W^{1,p}_0(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)} \\
\leq 2C_6^2\|u_0\|_{W^{2,p}_0(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)}.
\]
Second, for the case $p = d$,
\[
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^d(X)} \leq 2\|\nabla A_0 u\|_{L^2d(X)}\|A - A_0\|_{L^2d(X)} + \|\nabla A_0(A - A_0)\|_{L^d(X)} \\
\leq 2C_0^2\|\nabla A_0 u\|_{W^{1,d}_A(X)}\|A - A_0\|_{W^{1,d}_A(X)} + \|\nabla A_0(A - A_0)\|_{L^d(X)} \\
\leq 2C_0^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A - A_0\|_{W^{1,d}_{A_0}(X)} + \|\nabla A_0(A - A_0)\|_{L^d(X)}.
\]

Third, for the case $d < p < \infty$,
\[
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq 2\|\nabla A_0 u\|_{L^\infty(X)}\|A - A_0\|_{L^p(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)} \\
\leq 2C_0\|\nabla A_0 u\|_{W^{1,d}_A(X)}\|A - A_0\|_{L^p(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)} \\
\leq 2C_0\|u_0\|_{W^{2,p}_{A_0}(X)}\|A - A_0\|_{L^p(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)}.
\]

Hence, the combination of the preceding three cases gives
\[
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq 2C_0^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A - A_0\|_{W^{1,p}_{A_0}(X)} + \|\nabla A_0(A - A_0)\|_{L^p(X)},
\]
for $p < \infty$ and $d/2 \leq p \leq q$.

and applying Lemma B.2 Items (3) and (5),
\[
\tag{2.44}
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq 2C_0C_7C_6^2\|u_0\|_{W^{2,p}_{A_0}(X)}\|A - A_0\|_{W^{1,p}_{A_0}(X)} + C_7\|A - A_0\|_{W^{1,p}_{A_0}(X)},
\]
for $p < \infty$ and $d/2 \leq p \leq q$.

Therefore, combining the inequalities (2.36) and (2.44) yields
\[
\tag{2.45}
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq 4C_0C_7C_6^2C_3\|d_{A_0}(A - A_0)\|_{L^p(X)}\|A - A_0\|_{W^{1,p}_{A_0}(X)} + C_7\|A - A_0\|_{W^{1,p}_{A_0}(X)},
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$.

From (2.25) and (2.41), we see that (2.45) simplifies to give
\[
\tag{2.46}
\|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} \leq (4zC_0C_7C_6^2C_3M + 1)C_7\|A - A_0\|_{W^{1,p}_{A_0}(X)},
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$.

From the identity (2.27) (noting again that $d_{A_0} = d_{A_0}u_0$) we have
\[
\|\nabla A_0(B - A_0)\|_{L^p(X)} \leq \|\nabla A_0(u^{-1}(A - A_0)u)\|_{L^p(X)} + \|\nabla A_0(u^{-1}d_{A_0}u_0)\|_{L^p(X)}.
\]
Combining the preceding estimate with the inequalities (2.43) and (2.46) gives
\[
\tag{2.47}
\|\nabla A_0(B - A_0)\|_{L^p(X)} \leq C_5\|A - A_0\|_{W^{1,p}_{A_0}(X)},
\]
for $1 < p < \infty$ and $d/2 \leq p \leq q$,
with a constant $C_5 = C_5(A_0, A_1, g, G, M, p, q) \in \mathbb{R}$.

Finally, from (2.39) and (2.47) and Lemma B.2 (3) we obtain the desired $W^{1,p}_{A_0}$ bounds (2.22) for $u(A) - A_0$ in terms of $A - A_0$, recalling that $B = u(A)$, with large enough constant $N = N(A_0, A_1, g, G, M, p, q) \in \mathbb{R}$ and under the hypothesis (2.22) with small enough constant $\epsilon = \epsilon(A_0, A_1, g, G, p, q) \in (0, 1]$.

The proof of Proposition 2.11 also yields the following useful
Lemma 2.12 (\textit{A priori} $W^{2,p}$ estimate for a $W^{2,q}$ gauge transformation $u$ intertwining two $W^{1,q}$ connections). Assume the hypotheses of Proposition 2.11, excluding those on the connection $A$. Then there is a constant $C = C(A_0, A_1, g, G, p, q) \in [1, \infty)$ with the following significance. If $A$ obeys the hypotheses of Proposition 2.11 and $u \in \text{Aut}^{2,q}(P)$ is the resulting gauge transformation, depending on $A$ and $A_0$, such that
\[ d_{A_0}^{*}(u(A) - A_0) = 0, \]
then
\[ \|u\|_{W^{2,p}(X)} \leq C. \]

\textbf{Proof.} Write $u = u_0 + \gamma$ as in the proof of Proposition 2.11, with $u_0 \in (\text{Ker } \Delta_{A_0})'$ and $\gamma \in \text{Ker } \Delta_{A_0}$, and observe that
\[ \|u\|_{W^{2,p}(X)} \leq C \left(\|\Delta_{A_0} u\|_{L^p(X)} + \|u\|_{L^p(X)}\right) \] (by Proposition 2.11)
\[ = C \left(\|\Delta_{A_0} u_0\|_{L^p(X)} + \|u\|_{L^p(X)}\right) \]
\[ \leq C \left(\|d_{A_0}^{*}(A - A_0)\|_{L^p(X)} + \|u\|_{L^p(X)}\right) \] (by (2.35))
\[ \leq C \left(1 + \|A_0 - A_1\|_{W^{1,p}_{A_1}(X)}\right)\|A - A_0\|_{W^{1,p}_{A_1}(X)} + C\|u\|_{L^p(X)} \] (by (2.38))
\[ \leq C \left(1 + \|A_0 - A_1\|_{W^{1,p}_{A_1}(X)}\right)(\varepsilon + C \text{Vol}_g(X)^{1/p}), \]
where the last inequality follows from (2.22) with $\varepsilon \in (0,1]$, the Sobolev embedding $W^{1,p}(X) \subset L^q(X)$ given by [4, Theorem 4.12], the Kato Inequality [37, Equation (6.20)], and the fact that $|u| \leq 1$ pointwise. This completes the proof. \qed

2.6. Existence of Coulomb gauge transformations for connections. Finally, we can proceed to the

\textit{Proof of Theorem 9.} We shall apply the method of continuity, modeled on the proofs of [35, Theorem 2.1] due to Uhlenbeck and [21, Proposition 2.3.13] due to Donaldson and Kronheimer. For a related application of the method of continuity, see the proof [28, Theorem 1.1] due to the first author.

We begin by defining a one-parameter family of $W^{1,q}$ connections by setting
\[ A_t := A_0 + t(A - A_0), \quad \forall t \in [0,1], \]
and observe that their curvatures are given by
\[ F(A_t) = F_{A_0} + td_{A_0}(A - A_0) + \frac{t^2}{2}[A - A_0, A - A_0], \]
and they obey the bounds
\[ \|F(A_t)\|_{L^q(X)} \leq \|F_{A_0}\|_{L^q(X)} + \|d_{A_0}(A - A_0)\|_{L^q(X)} + \frac{1}{2}\|A - A_0\|_{L^q(X)}^2 \]
\[ \leq \|F_{A_0}\|_{L^q(X)} + \|d_{A_0}(A - A_0)\|_{L^q(X)} + c\|A - A_0\|_{W^{1,q}_{A_1}(X)}^2, \]
with $c = c(g,q) \in [1, \infty)$ and where we use the Sobolev embedding, $W^{1,q}(X) \subset L^q(X)$ [4, Theorem 4.12] and the Kato Inequality [37, Equation (6.20)] to obtain the last inequality. Note
that we have a continuous embedding, \( W^{1,q}(X) \subset L^{2q}(X) \), when \( 2q \leq q^* := dq/(d - q) \), that is \( 2d - 2q \leq d \) or \( q \geq d/2 \), as implied by our hypotheses. Therefore,

\[
\|F(A_t)\|_{L^n(X)} \leq K, \quad \forall t \in [0, 1],
\]

for \( K = K(A, A_0, G, g, q) \in [1, \infty) \), noting that \( F_{A_0} \in L^q(X; A^2 \otimes \text{ad} P) \) since \( A_0 \) is of class \( W^{1,q} \).

Let \( S \) denote the set of \( t \in [0, 1] \) such that there exists a \( W^{2,q} \) gauge transformation \( u_t \in \text{Aut}(P) \) with the property that

\[
d^*_{A_0}(u_t(A_t) - A_0) = 0 \quad \text{and} \quad \|u_t(A_t) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_t - A_0\|_{W^{1,p}_{A_1}(X)},
\]

where \( N \) is the constant in Proposition 2.11. Clearly, \( 0 \in S \) since the identity automorphism of \( P \) is the required gauge transformation in that case, so \( S \) is non-empty. As usual, we need to show that \( S \) is an open and closed subset of \([0, 1]\).

**Step 1** (\( S \) is open). To prove openness, we shall adapt the argument of Donaldson and Kronheimer in [21] Section 2.3.8. We apply the Implicit Function Theorem to the gauge fixing equation,

\[
d^*_{A_0}(u_t(A_t) - A_0) = d^*_{A_0} (u_t^{-1}(A_t - A_0)u_t + u_t^{-1}d_{A_0}u_t) = 0.
\]

As usual, we denote \( B_t = u_t(A_t) \) for convenience, for \( t \in S \). Let \( t_0 \in S \), so we have

\[
d^*_{A_0}(u_{t_0}(A_{t_0}) - A_0) = 0 \quad \text{and} \quad \|u_{t_0}(A_{t_0}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\|A_{t_0} - A_0\|_{W^{1,p}_{A_1}(X)}.
\]

Our task is to show that \( t_0 + s \in S \) for \(|s|\) sufficiently small, that is, there exists \( u_{t_0+s} \in \text{Aut}^{2,q}(P) \) such that the preceding two properties hold with \( t_0 \) replaced by \( t_0 + s \). By the hypothesis (1.26) of Theorem 9, we have \( \|A - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta \) and thus, since \( A_{t_0} - A_0 = A_0 + t_0(A - A_0) - A_0 = t_0(A - A_0) \) and \( t_0 \in [0, 1] \), we see that

\[
\|A_{t_0} - A_0\|_{W^{1,p}_{A_1}(X)} < \zeta,
\]

and so

\[
\|u_{t_0}(A_{t_0}) - A_0\|_{W^{1,p}_{A_1}(X)} < 2N\zeta.
\]

It will be convenient to define \( a \in W^{1,q}_{A_1}(X; A^1 \otimes \text{ad} P) \) by

\[
(2.49) \quad u_{t_0}(A_{t_0}) = : A_0 + a,
\]

The preceding inequality ensures that

\[
(2.50) \quad \|a\|_{W^{1,p}_{A_1}(X)} < 2N\zeta.
\]

We shall seek a solution \( u_{t_0+s} \in \text{Aut}^{2,q}(P) \) to the gauge-fixing equation,

\[
d^*_{A_0}(u_{t_0+s}(A_{t_0+s}) - A_0) = 0.
\]

In particular, we shall seek a solution in the form

\[
u_{t_0+s} = e^{\chi_s}u_{t_0}, \quad \text{for} \ \chi_s \in W^{2,q}_{A_1}(X; \text{ad} P),
\]

so the gauge-fixing equation becomes

\[
(2.51) \quad d^*_{A_0}(e^{\chi_s}u_{t_0}(A_{t_0+s}) - A_0) = 0.
\]
For $s \in \mathbb{R}$, it will be convenient to define $b_s \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$ by

$$u_{t_0}(A_{t_0+s}) = A_0 + a + b_s. \tag{2.52}$$

We can determine $b_s$ explicitly using $A_{t_0+s} = A_0 + (t_0 + s)(A - A_0)$, so that

$$u_{t_0}(A_{t_0+s}) - A_0 = u_{t_0}^{-1}((t_0 + s)(A - A_0))u_{t_0} + u_{t_0}^{-1}d_{A_0}u_{t_0}$$

$$= u_{t_0}^{-1}(t_0(A - A_0))u_{t_0} + u_{t_0}^{-1}d_{A_0}u_{t_0} + u_{t_0}^{-1}(s(A - A_0))u_{t_0}$$

$$= u_{t_0}(A_{t_0}) - A_0 + su_{t_0}^{-1}(A - A_0)u_{t_0}$$

$$= a + su_{t_0}^{-1}(A - A_0)u_{t_0} \quad \text{(by (2.49))},$$

and thus

$$b_s = su_{t_0}^{-1}(A - A_0)u_{t_0}, \quad s \in \mathbb{R}. \tag{2.53}$$

Note that $u_{t_0} \in \text{Aut}^{2,q}(P)$ and so we have the estimate

$$\|b_s\|_{W^{1,q}_{A_1}(X)} \leq |s|\|C_0\|A - A_0\|_{W^{1,q}_{A_1}(X)}, \quad s \in \mathbb{R}, \tag{2.54}$$

for $C_0 = C_0(A_0, g, q, t_0) \in [1, \infty)$. In particular, $b_s \to 0$ strongly in $W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$ as $s \to 0$.

The gauge fixing equation (2.51) takes the form

$$e^{\chi_s}u_{t_0}(A_{t_0+s}) - A_0 = e^{\chi_s}(A_0 + a + b_s) - A_0$$

$$= e^{-\chi_s}(a + b_s)e^{\chi_s} + e^{-\chi_s}d_{A_0}(e^{\chi_s}).$$

The equation to be solved is then $H(\chi, b) = 0$, where

$$H(\chi, b) := d_{A_0}^* \left( e^{-\chi}(a + b)e^\chi + e^{-\chi}d_{A_0}(e^\chi) \right). \tag{2.55}$$

For any $q > d/2$, the expression (2.55) for $H$ defines a smooth map,

$$H : (\text{Ker } \Delta_{A_0})^\perp \cap W^{2,q}_{A_1}(X; \text{ad} P) \times W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P) \to (\text{Ker } \Delta_{A_0})^\perp \cap L^q(X; \text{ad} P). \tag{2.56}$$

Here, we note that if $\xi = d_{A_0}^*a_\xi$ for some $a_\xi \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then $\xi \perp \text{Ker } \Delta_{A_0}$, as implied by the preceding expression for $H$. Indeed, for any $\gamma \in \text{Ker } \Delta_{A_0}$,

$$(\xi, \gamma)_{L^2(X)} = (d_{A_0}^*a_\xi, \gamma)_{L^2(X)} = (a_\xi, d_{A_0}\gamma)_{L^2(X)} = 0.$$ 

Hence, the image of $H$ is contained in $(\text{Ker } \Delta_{A_0})^\perp \cap L^q(X; \text{ad} P)$.

The Implicit Function Theorem asserts that if the partial derivative,

$$(D_1 H)_{(0,0)} : (\text{Ker } \Delta_{A_0})^\perp \cap W^{2,q}_{A_1}(X; \text{ad} P) \to (\text{Ker } \Delta_{A_0})^\perp \cap L^q(X; \text{ad} P),$$

is surjective, then for small $b \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$ there is a small solution $\chi \in W^{2,q}_{A_1}(X; \text{ad} P)$ to $H(\chi, b) = 0$ and that is $L^2$-orthogonal to $\text{Ker } \Delta_{A_0}$.

Now the linearization, $(D_1 H)_{(0,0)}$, of the map $H$ at the origin $(0,0)$ with respect to variations in $\chi$ is given by

$$(D_1 H)_{(0,0)}\chi = d_{A_0}^*d_{A_0} + a\chi.$$
But \( \|a\|_{W_{A_1}^{1,p}(X)} < 2N\zeta \) by (2.50) and because of the continuous Sobolev embeddings provided by [4, Theorem 4.12],
\[
W^{1,p}(X) \subset L^d(X), \quad \text{for } d \geq 3 \text{ and } p \geq d/2, \\
W^{1,p}(X) \subset L^4(X), \quad \text{for } d = 2 \text{ and } p \geq 2,
\]
we obtain,
\[
\|a\|_{L^d(X)} < 2C_1N\zeta \quad \text{when } d \geq 3 \quad \text{and} \quad \|a\|_{L^4(X)} < 2C_1N\zeta \quad \text{when } d = 2,
\]
where \( C_1 = C_1(g,p) \in [1,\infty) \) is the norm of the Sobolev embedding employed. By the hypothesis of Theorem 4, we can choose \( \zeta \in (0,1] \) as small as desired. Hence, the operator \( d_{A_0}^*d_{A_0+a} \) is surjective by Lemma 2.8.

To summarize, we have shown that if \( |s| \) is small and \( t_0 \in S \), then there exists \( u_{t_0+s} \in \text{Aut}^{2,q}(P) \) such that
\[
d_{A_0}^*(u_{t_0+s}(A_{t_0+s}) - A_0) = 0.
\]
It remains to check that the following norm condition holds,
\[
(2.57) \quad \|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W_{A_1}^{1,p}(X)} < 2N \|A_{t_0+s} - A_0\|_{W_{A_1}^{1,p}(X)},
\]
for small enough \( |s| \), to conclude that \( t_0 + s \in S \). To see this, we first note that since \( A_{t_0+s} = A_0 + (t_0 + s)(A - A_0) \), we have
\[
\|A_{t_0+s} - A_0\|_{W_{A_1}^{1,p}(X)} = (t_0 + s)\|A - A_0\|_{W_{A_1}^{1,p}(X)} < (t_0 + s)\zeta \quad \text{by (1.20)},
\]
and thus, for \( t_0 + s \leq 1 \) and \( \zeta \leq \varepsilon/C_1 \),
\[
\|A_{t_0+s} - A_0\|_{W_{A_1}^{1,p}(X)} \leq \varepsilon/C_1,
\]
and thus,
\[
\|A_{t_0+s} - A_0\|_{L^q(X)} \leq \varepsilon, \quad \text{if } p < d, \\
\|A_{t_0+s} - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon, \quad \text{if } p = d, \\
\|A_{t_0+s} - A_0\|_{L^p(X)} \leq \varepsilon, \quad \text{if } p > d,
\]
where \( \varepsilon \) is the constant in Proposition 2.11 and \( C_1 = C_1(g,p) \) or \( C_1(\delta,p) \in [1,\infty) \) is the norm (provided by [4, Theorem 4.12]) of the continuous Sobolev embedding, \( W^{1,p}(X) \subset L^d(X) \) when \( d/2 \leq p < d \) and \( W^{1,p}(X) \subset L^{d+\delta}(X) \) when \( p = d \). This verifies the hypotheses (2.22) and (2.25) of Proposition 2.11 for \( A_{t_0+s} - A_0 \) (in place of \( A - A_0 \) in the statement of that proposition).

On the other hand,
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W_{A_1}^{1,p}(X)} \\
= \|e^{xs}u_{t_0}(A_{t_0+s}) - A_0\|_{W_{A_1}^{1,p}(X)} \\
\leq \|e^{xs}u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W_{A_1}^{1,p}(X)} + \|u_{t_0}(A_{t_0+s}) - A_0\|_{W_{A_1}^{1,p}(X)} \\
= \|e^{xs}u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W_{A_1}^{1,p}(X)} + \|a + b_s\|_{W_{A_1}^{1,p}(X)} \quad \text{(by (2.52))},
\]
The inequalities (1.26), (2.50), and (2.54) (which also holds with \( p \) in place of \( q \)) yield the bound
\[
\|a + b_s\|_{W^{1,p}\left(A_1\right)} \leq \|a\|_{W^{1,p}\left(A_1\right)} + \|b_s\|_{W^{1,p}\left(A_1\right)} < 2N\zeta + |s|C_0\|A - A_0\|_{W^{1,p}\left(A_1\right)} \\
\leq 2N\zeta + |s|C_0\zeta \\
\leq \varepsilon/(2C_1),
\]
for small enough \( \zeta \). (Note that we could also have used \( \|b_s\|_{W^{1,q}\left(A_1\right)} \leq C_0\|A - A_0\|_{W^{1,q}\left(A_1\right)} \) and the continuous embedding \( W^{1,q}(X) \subset W^{1,p}(X) \) and rely on our freedom to also choose \( |s| \) small.) But if \( |s| \) is small then so is \( \|b_s\|_{W^{1,q}\left(A_1\right)} \) by (2.54) and hence \( \|\chi_s\|_{W^{2,q}(X)} \) is small by the Implicit Function Theorem\(^5\) and so we may assume that
\[
\|e^{\chi_s}u_{t_0}(A_{t_0+s}) - u_{t_0}(A_{t_0+s})\|_{W^{1,p}\left(A_1\right)} \leq \varepsilon/(2C_1),
\]
for small enough \( |s| \). Collecting the preceding inequalities gives
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}\left(A_1\right)} \leq \varepsilon/C_1,
\]
and thus,
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^q(X)} \leq \varepsilon, \quad \text{if } p < d,
\]
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^q(X)} \leq \varepsilon, \quad \text{if } p = d,
\]
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{L^p(X)} \leq \varepsilon, \quad \text{if } p > d,
\]
verifying the hypotheses (2.22) and (2.25) of Proposition 2.11 for \( u_{t_0+s}(A_{t_0+s}) - A_0 \) (in place of \( u(A) - A_0 \) in the statement of that proposition).

Hence, Proposition 2.11 yields the bound
\[
\|u_{t_0+s}(A_{t_0+s}) - A_0\|_{W^{1,p}\left(A_1\right)} \leq N\|A_{t_0+s} - A_0\|_{W^{1,p}\left(A_1\right)} < 2N\|A_{t_0+s} - A_0\|_{W^{1,p}\left(A_1\right)}.
\]
This verifies the norm condition (2.57) and we conclude that \( t_0 + s \in S \) and so \( S \) is open.

**Step 2** (\( S \) is closed). For closedness, we adapt the argument in [24 Section 2.3.7]. Set \( B_t := u_t(A_t) \) for \( t \in S \) and observe that the inequality (2.38) yields
\[
||F(B_t)||_{L^q(X)} = ||F(u_t(A_t))||_{L^q(X)} = ||u_t(F(A_t))||_{L^q(X)} = ||F(A_t)||_{L^q(X)} \leq K, \quad \forall t \in S.
\]
Let \( \{t_m\}_{m \in \mathbb{N}} \subset S \) be a sequence and suppose that \( t_m \to t_\infty \in [0,1] \) as \( m \to \infty \). Since \( q > d/2 \) by hypothesis, the Uhlenbeck Weak Compactness [95 Theorem 1.5 = 3.6] (see also [100 Theorem 7.1] for a recent exposition) implies that there exists a subsequence \( \{m'\} \subset \{m\} \) and, after relabeling, a sequence of \( W^{2,q} \) gauge transformations, \( \{u_{m'}\}_{m \in \mathbb{N}} \subset \text{Aut}(P) \) and a \( W^{1,q} \) connection \( B_\infty \) on \( P \) such that, as \( m \to \infty \), we have
\[
B_{t_m} \to B_\infty \quad \text{weakly in } W^{1,q}(X; \Lambda^1 \otimes \text{ad}P).
\]

---

\(^5\)The Quantitative Inverse Function Theorem — see, for example, [11 Proposition 2.5.6] — can be used to give a precise bound on \( \chi_s \) given a bound on \( b_s \).
Hence, for $1 \leq r < q^* := dq/(d - q)$ and $W^{1,q}(X) \subseteq L^r(X)$, there exists a subsequence $\{m''\} \subset \{m\}$ such that, after again relabeling, as $m \to \infty$ we have

$$B_{t_m} \to B_\infty \text{ strongly in } L^r(X; \Lambda^1 \otimes \text{ad}P).$$

The $W^{2,q}$ gauge transformations $u_t$ intertwine the $W^{1,q}$ connections $A_t$ and $B_t$ via the relation,

$$B_t = A_0 + u_t^{-1}(A - A_0)u_t + u_t^{-1}d_{A_0}u_t,$$

and thus

$$d_{A_0}u_t = u_tB_t + u_tA_0 + (A - A_0)u_t, \quad \forall t \in [0,1].$$

Because $A_{t_m} \to A_\infty := A_0 + t_\infty(A - A_0)$ strongly in $W^{1,q}$ and $B_{t_m} \to B_\infty$ weakly in $W^{1,q}$ and $B_{t_m} \to B_\infty$ strongly in $L^r$ as $m \to \infty$, there exists a $W^{2,q}$ gauge transformation $u_\infty \in \text{Aut}(P)$ such that, as $m \to \infty$,

$$u_{t_m} \rightharpoonup u_\infty \text{ weakly in } W^{2,q}_{A_1}(X; \text{Ad}P) \quad \text{and} \quad u_{t_m} \to u_\infty \text{ strongly in } W^{1,r}_{A_1}(X; \text{Ad}P).$$

In particular,

$$B_\infty = u_\infty(A_\infty) \quad \text{and} \quad d_{A_0}^*(u_\infty(A_\infty) - A_0) = 0.$$

The Coulomb gauge condition follows from the fact that, for any $\xi \in \mathcal{C}^\infty(X; \text{ad}P)$, we have

$$0 = \lim_{m \to \infty} (d_{A_0}^*(u_{t_m}(A_{t_m}) - A_0), \xi)_{L^2(X)} = \lim_{m \to \infty} (u_{t_m}(A_{t_m}) - A_0, d_{A_0}\xi)_{L^2(X)} = (u_\infty(A_\infty) - A_0, d_{A_0}\xi)_{L^2(X)} = (d_{A_0}^*(u_\infty(A_\infty) - A_0), \xi)_{L^2(X)}.$$
and combining this inequality with the preceding inequality yields
\[ \|u_{t_m}(A_{t_m}) - A_0\|_{W^{1,p}(X)} < 2N\zeta, \quad \forall m \in \mathbb{N}. \]
Since \( u_\infty(A_\infty) \) is the weak limit of \( u_{t_m}(A_{t_m}) \) in \( W^{1,q}_A(X; \Lambda^1 \otimes \text{ad}P) \), we have
\[ \|u_\infty(A_\infty) - A_0\|_{W^{1,p}(X)} \leq \liminf_{m \to \infty} 2N\|A_{t_m} - A_0\|_{W^{1,p}(X)}. \]
But \( A_{t_m} \to A_\infty \) strongly in \( W^{1,q}_A(X; \Lambda^1 \otimes \text{ad}P) \) by construction of the path \( A_t \) and as \( p \leq q \) by hypothesis and \( \|A_{t_m} - A_0\|_{W^{1,p}(X)} < \zeta \) for all \( m \in \mathbb{N} \), then
\[ \|A_t - A_0\|_{W^{1,p}(X)} = \lim_{m \to \infty} \|A_{t_m} - A_0\|_{W^{1,p}(X)} \leq \zeta. \]
Combining the preceding inequalities yields
\[ \|u_\infty(A_\infty) - A_0\|_{W^{1,p}(X)} \leq 2N\zeta. \]
We choose \( \zeta \in (0,1] \) small enough that \( \zeta \leq \varepsilon/C_1 \) and \( 2N\zeta \leq \varepsilon/C_1 \), where \( C_1 \) is the norm of the Sobolev embedding \( W^{1,p}(X) \subset L^q(X) \) when \( p \neq d \) or \( L^{d+\delta}(X) \) when \( p = d \) as in Step 1 and then observe that
\[ \|A_{t_\infty} - A_0\|_{W^{1,p}(X)} \leq \varepsilon/C_1 \quad \text{and} \quad \|u_\infty(A_\infty) - A_0\|_{W^{1,p}(X)} < \varepsilon/C_1. \]
The first inequality above verifies the hypothesis (2.25) of Proposition 2.11 Moreover,
\[ \|A_{t_\infty} - A_0\|_{L^q(X)} \leq \varepsilon \quad \text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^q(X)} \leq \varepsilon, \quad \text{if} \ p < d, \]
\[ \|A_{t_\infty} - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon \quad \text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^{d+\delta}(X)} \leq \varepsilon, \quad \text{if} \ p = d, \]
\[ \|A_{t_\infty} - A_0\|_{L^p(X)} \leq \varepsilon \quad \text{and} \quad \|u_\infty(A_{t_\infty}) - A_0\|_{L^p(X)} \leq \varepsilon, \quad \text{if} \ p > d, \]
which verifies the hypothesis (2.22) of Proposition 2.11 on norms (for \( A_{t_\infty} - A_0 \) in place of \( A - A_0 \) and \( u_\infty(A_{t_\infty}) - A_0 \) in place of \( u(A) - A_0 \) in the statement of that proposition). Since \( d^*_A(u_\infty(A_\infty) - A_0) = 0 \), as required by (2.21), Proposition 2.11 implies that
\[ \|u_\infty(A_\infty) - A_0\|_{W^{1,p}(X)} \leq N\|A_{t_\infty} - A_0\|_{W^{1,p}(X)} < 2N\|A_{t_\infty} - A_0\|_{W^{1,p}(X)}. \]
Thus, \( t_\infty \in S \) and so \( S \) is closed.

Consequently, \( S \subset [0,1] \) is non-empty and open and closed by the preceding two steps, so \( S = [0,1] \) and this completes the proof of Theorem 9. \( \square \)

2.7. Real analytic Banach manifold structure on the quotient space of connections. The statements and proofs of Lemmata 2.13 and 2.14 would follow standard lines (see Gilkey [41, Theorem 1.5.2], for example) if the operators
\[ d_A : \Omega^l(X; \text{ad}P) \to \Omega^{l+1}(X; \text{ad}P), \quad l \geq 0, \]
had \( C^\infty \) coefficients, rather than Sobolev coefficients as we allow here, and formed an elliptic complex, rather than only satisfying \( d_A \circ d_A = F_A \).
Lemma 2.13 (Continuous operators on $L^p$ spaces and $L^2$-orthogonal decompositions). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A$ is a connection on $P$ of class $W^{1,q}$ with $q \geq d/2$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $l \geq 1$ is an integer, and $p$ obeys $d/2 \leq p \leq q$, then the operator

$$d_A^* : W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P) \to L^p(X; \Lambda^{l-1} \otimes \text{ad } P),$$

is continuous and, if in addition $q > d/2$, then the operator

$$d_A : W^{2,p}_{A_1}(X; \Lambda^{l-1} \otimes \text{ad } P) \to W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P),$$

is also continuous, and there is an $L^2$-orthogonal decomposition,

$$W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P) = \text{Ker} \left( d_A^* : W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P) \to L^p(X; \Lambda^{l-1} \otimes \text{ad } P) \right)$$

$$\oplus \text{ Ran } \left( d_A : W^{2,p}_{A_1}(X; \Lambda^{l-1} \otimes \text{ad } P) \to W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P) \right).$$

Proof. If $\xi \in W^{2,p}_{A_1}(X; \Lambda^{l-1} \otimes \text{ad } P)$ and we write $A = A_1 + a$, with $a \in W^{1,q}_{A_1}(X; \Lambda^l \otimes \text{ad } P)$, then $d_A \xi = d_A^* \xi + [a, \xi]$ and using the fact that $W^{1,p}(X) \subset L^{2p}(X)$ for any $p \geq d/2$ by \cite{4} Theorem 4.12] and applying the Kato Inequality \cite{37} Equation (6.20),

$$\|d_A \xi\|_{L^p(X)} \leq z \left( \|\nabla A_1 \xi\|_{L^p(X)} + \|a\|_{L^{2p}(X)} \|\xi\|_{L^{2p}(X)} \right)$$

$$\leq z \left( \|\nabla A_1 \xi\|_{L^p(X)} + \|a\|_{W^{1,q}_{A_1}(X)} \|\xi\|_{W^{1,p}_{A_1}(X)} \right)$$

$$\leq z \left( 1 + \|a\|_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{W^{1,p}_{A_1}(X)},$$

where $z = z(g, G, p, q) \in [1, \infty)$ and we use the fact that $q \geq p$. Similarly,

$$\|d_A^* \xi\|_{L^p(X)} \leq z \left( 1 + \|a\|_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{W^{1,p}_{A_1}(X)}$$

and so the operator $d_A^* : W^{1,p}_{A_1}(X; \Lambda^l \otimes \text{ad } P) \to L^p(X; \Lambda^{l-1} \otimes \text{ad } P)$ is continuous.

Moreover, defining $r \in [p, \infty]$ by $1/p = 1/q + 1/r$, we recall that by \cite{4} Theorem 4.12] we have i) $W^{2,p}(X) \subset L^r(X)$ for any $r \in [1, \infty)$ when $p = d/2$, and ii) $W^{2,p}(X) \subset L^\infty(X)$ when $p > d/2$. Thus, using

$$\nabla A_1 d_A \xi = \nabla A_1 d_A^* \xi + \nabla A_1 a \times \xi + a \times \nabla A_1 \xi,$$

we see that

$$\|\nabla A_1 d_A \xi\|_{L^p(X)} \leq z \left( \|\nabla^2 A_1 \xi\|_{L^p(X)} + \|\nabla A_1 a\|_{L^q(X)} \|\xi\|_{L^r(X)} + \|a\|_{L^{2p}(X)} \|\nabla A_1 \xi\|_{L^{2p}(X)} \right)$$

$$\leq z \left( \|\nabla^2 A_1 \xi\|_{L^p(X)} + \|\nabla A_1 a\|_{L^q(X)} \|\xi\|_{W^{2,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)} \|\nabla A_1 \xi\|_{W^{1,p}_{A_1}(X)} \right)$$

$$\leq z \left( 1 + \|a\|_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{W^{2,p}_{A_1}(X)}.$$
$d \geq 2$, so we have $W^{1,p}(X) \subset L^2(X)$ for all $p \geq d/2$ and $d \geq 2$. Using $\perp$ to denote $L^2$-orthogonal complement and $\oplus$ to denote $L^2$-orthogonal decomposition, we have

$$W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P) = \left(\text{Ran} \left( d_A : W^{2,p}_{A_1}(X; \Lambda^{t-1} \otimes \text{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P) \right) \right) \perp \text{Ran} \left( d_A : W^{2,p}_{A_1}(X; \Lambda^{t-1} \otimes \text{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P) \right).$$

For all $\eta \in W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P)$ and $\xi \in W^{2,p}_{A_1}(X; \Lambda^{t+1} \otimes \text{ad}P)$ we have

$$(\eta, d_A \xi)_{L^2(X)} = (d_A^* \eta, \xi)_{L^2(X)}$$

and so $\eta \perp \text{Ran}(d_A : W^{2,p}_{A_1}(X; \Lambda^{t-1} \otimes \text{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P))$ if and only if $\eta \in \text{Ker}(d_A^* : W^{1,p}_{A_1}(X; \Lambda^t \otimes \text{ad}P) \to L^p(X; \Lambda^{t-1} \otimes \text{ad}P))$. This concludes the proof of the lemma.

Although not required by the proofs of Lemma 2.13 or Corollary 11 it is useful to note that the operator $d_A$ in that statement has closed range.

**Lemma 2.14 (Closed range operators on $L^p$ spaces).** Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$. If $A$ is a connection on $P$ of class $W^{1,q}$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $p$ obeys $d/2 \leq p \leq q$, then the operator

$$d_A : W^{2,p}_{A_1}(X; \text{ad}P) \to W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P),$$

have closed range.

**Proof.** Note that the two operators in the statement of the lemma are bounded by Lemma 2.13. Let $\{\chi_n\}_{n \in \mathbb{N}} \subset W^{2,p}_{A_1}(X; \text{ad}P)$ and suppose that $d_A \chi_n \to \xi \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P)$ as $n \to \infty$. Thus

$$d_A^* d_A \chi_n = \Delta_A \chi_n \to d_A^* \xi \in L^p(X; \text{ad}P)$$

as $n \to \infty$. We may assume without loss of generality that $\{\chi_n\}_{n \in \mathbb{N}} \subset (\text{Ker} \Delta_A)^\perp$, where $\perp$ denotes $L^2$-orthogonal complement, and so the a priori estimate (2.5) in Corollary 2.5 then implies that

$$\|\chi_n - \chi_m\|_{W^{2,p}_{A_1}(X)} \leq C\|\Delta_A (\chi_n - \chi_m)\|_{L^p(X)}, \quad \forall n, m \in \mathbb{N}.$$

Hence, the sequence $\{\chi_n\}_{n \in \mathbb{N}}$ is Cauchy in $W^{2,p}_{A_1}(X; \text{ad}P)$ and thus $\chi_n \to \chi \in W^{2,p}_{A_1}(X; \text{ad}P)$ and $d_A \chi_n \to d_A \chi \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P)$ as $n \to \infty$. Therefore, $d_A$ on $W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P)$ has closed range.

We are now ready to complete the

**Proof of Corollary 11.** Every compact Lie group has a compatible structure of a real analytic manifold [14, Section III.4, Exercise 1] and this structure is unique by [37, Theorem 2.11.3]. In particular, the exponential map is a real analytic diffeomorphism from an open neighborhood of the origin in the Lie algebra $\mathfrak{g}$ onto an open neighborhood of the identity in $G$. We recall from [37, Proposition A.2] that $\text{Aut}^{k+1,2}(P)$ may be given the structure of a Hilbert Lie group when $k \geq 2$ and, because $W^{2,q}(X)$ (with $q > d/2$) and $H^{k+1}(X) = W^{k+1,2}(X)$ are Banach algebras and contained in $C(X)$ (the Banach algebra of continuous functions on $X$), the same arguments show that $\text{Aut}^{2,q}(P)$ may be given the structure of a $C^\infty$ Banach Lie group and that both $\text{Aut}^{2,q}(P)$ and $\text{Aut}^{k,2}(P)$ may be given the structure of real analytic manifolds.
According to [37, Proposition A.3], the (right) action of $\text{Aut}^{k+1,2}(P)$ on $A^{k,2}(P)$ is $C^\infty$ when $k \geq 2$ and the same proof applies mutatis mutandis to show that this action is real analytic and that the action $[2.1]$ of $\text{Aut}^{2,q}(P)$ on $A^{1,q}(P)$ is not only $C^\infty$ but also real analytic.

The only additional ingredient one needs to show that $B^*(P)$ is real analytic is the observation that the map $H$ defined in (2.55) and (2.56) is real analytic and thus, rather than apply the customary $C^\infty$ Inverse Function Theorem one can instead apply its real analytic counterpart [32, Section 2.1.1] to show that for each $A_0 \in A^{1,q}(P)$, the map defined in the statement of [24, Theorem 3.2],

$$
(2.58) \quad \Xi_{A_0} : A^{1,q}(P) \ni \partial A_0 \ni A \mapsto (u(A) - A_0, A)
$$

is a real analytic diffeomorphism onto an open neighborhood of $(0, \text{id}_P)$, for a small enough open neighborhood $\partial A_0$ of a $W^{1,q}$ connection $A$ on $P$ and the gauge transformation $u$ is produced by Theorem [9] so $u(A)$ is in Coulomb gauge with respect to $A_0$. The open neighborhood $\partial A_0$ may be chosen to be $\text{Aut}^{2,q}(P)$-invariant and the map $\Xi_{A_0}$ is $\text{Aut}^{2,q}(P)$-equivariant. The proof that the quotient $B(P)$ is a Hausdorff topological space follows mutatis mutandis either by adapting the proof of [37, Corollary, p. 50] or by adapting the proof of [24, Lemma 4.2.4], using the observation the $L^2$ distance function,

$$
(2.59) \quad \text{dist}_{L^2}(\{A\}, [B]) := \inf_{u \in \text{Aut}^{2,q}(P)} \|u(A) - B\|_{L^2(X)},
$$

is a metric on $B(P)$ and, in particular, that the quotient topology is metrizable. This completes the proof of Corollary [11].

2.8. Existence of Coulomb gauge transformations for pairs. We now adapt the construction of Section 2.6 to the case of pairs. In [30, p. 208], we employed a left action of $\text{Aut}(P)$ on the affine space of pairs, $A(P) \times C^\infty(X; E)$, so $\text{Aut}(P)$ acts on $A(P)$ by pushforward (consistent with Donaldson and Kronheimer [24]) and on $C^\infty(X; E)$ in the usual way, which is a left action. Here, to be consistent with Section 2.6 we shall use the opposite convention and continue to let $\text{Aut}(P)$ act on $A(P)$ by pullback (consistent with Freed and Uhlenbeck [37] and Uhlenbeck [95]) and use inversion to define a right action on $C^\infty(X; E)$, so that

$$
(2.60) \quad u(A, \Phi) := (u^*A, u^{-1}\Phi), \quad \forall A \in A(P), \Phi \in C^\infty(X; E), \text{ and } u \in \text{Aut}(P),
$$
giving a smooth (affine) right action,

$$
A(P) \times C^\infty(X; E) \times \text{Aut}(P) \to A(P) \times C^\infty(X; E).
$$

Passing to Banach space completions, but temporarily suppressing the $W^{1,q}$ reference connection $A_0$ (for $q > d/2$) from our notation, the differential of the smooth map,

$$
\text{Aut}^{2,q}(P) \ni u \mapsto u(A, \Phi) \in A^{1,q}(P) \times W^{1,q}(X; E),
$$
at $\text{id}_P \in \text{Aut}^{2,q}(P)$ is given by

$$
(2.60) \quad W^{2,q}_{A_1}(X; \text{ad}P) \ni \xi \mapsto d_{A,\Phi}\xi := (d_A\xi, -\xi\Phi) \in W^{1,q}_{A_1}(X; \text{ad}P) \oplus W^{1,q}_{A_1}(X; E),
$$
using $u = e^x$ for $u$ near $\id_P$; compare [30] Proposition 2.1. We say that a $W^{1,q}$ pair $(A, \Phi)$ is in
\textit{Coulomb gauge relative to $(A_0, \Phi_0)$} if
\begin{equation}
\label{eq:61}
d_{A_0,0}^\ast d_{A_0,0} ((A, \Phi) - (A_0, \Phi_0)) = 0,
\end{equation}
As in the case of Theorem 9, the proof of Theorem 10 is facilitated by preparatory lemmata and
a proposition, which we now state. For convenience, we define
\begin{equation}
\label{eq:62}
\Delta_{A_0,0} := d_{A_0,0}^\ast d_{A_0,0} \quad \text{on $W^{2,q}_{A_1}(X; \ad P)$}.
\end{equation}
The proofs of Propositions 2.1 and 2.3 and Corollary 2.5 adapt \textit{mutatis mutandis} to establish the
following analogues for pairs, specialized to the case $l = 0$.

\textbf{Proposition 2.15} \textit{(A priori $L^p$ estimate for a Laplace operator with Sobolev coefficients). Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie
group and $P$ be a smooth principal $G$-bundle over $X$, and $E = \mathbb{P} \times \mathbb{P} \mathbb{E}$ be a smooth Hermitian vector
bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho : G \to \Aut_C(\mathbb{E})$. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $q > d/2$, and $A_1$ is a $C^\infty$
connection on $P$, and $p$ obeys $d/2 \leq p \leq q$, then
\begin{equation}
\label{eq:63}
\Delta_{A,\Phi} : W^{2,p}_{A_1}(X; \ad P) \to L^p(X; \ad P)
\end{equation}
is a bounded operator. If in addition $p \in (1, \infty)$, then there is a constant $C = C(A, \Phi, A_1, g, G, p, q) \in [1, \infty)$
such that
\begin{equation}
\label{eq:64}
\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C \left( \|\Delta_{A,\Phi} \xi\|_{L^p(X)} + \|\xi\|_{L^p(X)} \right), \quad \forall \xi \in W^{2,p}_{A_1}(X; \ad P).
\end{equation}

\textbf{Proposition 2.16} \textit{(Spectral properties of a Laplace operator with Sobolev coefficients). Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie
group and $P$ be a smooth principal $G$-bundle over $X$, and $E = \mathbb{P} \times \mathbb{P} \mathbb{E}$ be a smooth Hermitian vector
bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho : G \to \Aut_C(\mathbb{E})$. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$
reference connection on $P$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then the spectrum, $\sigma(\Delta_{A,\Phi})$, of the unbounded operator
\begin{equation}
\label{eq:65}
\Delta_{A,\Phi} : D(\Delta_{A,\Phi}) \subset L^p(X; \ad P) \to L^p(X; \ad P),
\end{equation}
is countable without accumulation points, consisting of non-negative, real eigenvalues, $\lambda$, with
finite multiplicities, $\dim \ker(\Delta_{A,\Phi} - \lambda)$.}

\textbf{Corollary 2.17} \textit{(A priori $L^p$ estimate for a Laplace operator with Sobolev coefficients). Let $(X,g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie
group and $P$ be a smooth principal $G$-bundle over $X$, and $E = \mathbb{P} \times \mathbb{P} \mathbb{E}$ be a smooth Hermitian vector
bundle over $X$ defined by a finite-dimensional unitary representation, and $\varrho : G \to \Aut_C(\mathbb{E})$. If $(A, \Phi)$ is a $W^{1,q}$ pair on $(P,E)$ with $d/2 < q < \infty$, and $A_1$ is a $C^\infty$
connection on $P$, and $p \in (1, \infty)$ obeys $d/2 \leq p \leq q$, then the kernel $\ker \Delta_{A,\Phi} \cap W^{2,p}_{A_1}(X; \ad P)$ of the operator (2.62) is
finite-dimensional. Moreover,
\begin{equation}
\label{eq:66}
\|\xi\|_{W^{2,p}_{A_1}(X)} \leq C \|\Delta_{A,\Phi} \xi\|_{L^p(X)}, \quad \forall \xi \in (\ker \Delta_{A,\Phi})^{\perp} \cap W^{2,p}_{A_1}(X; \ad P),
\end{equation}
where $^{\perp}$ denotes $L^2$-orthogonal complement and $C = C(A, \Phi, A_1, g, G, p, q) \in [1, \infty)$.

The proof of Lemma 2.8 adapts \textit{mutatis mutandis} to establish the following analogue for pairs.
Lemma 2.18 (Surjectivity of a perturbed Laplace operator). Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group and \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times G \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( \rho : G \to \text{Aut}(E) \). Let \( A_1 \) be a \( C^\infty \) connection on \( P \) and \((A, \Phi)\) be a \( W^{1,q} \) pair on \((P, E)\) with \( d/2 < q < \infty \). Then there is a constant \( \delta = \delta(A, \Phi, g) \in (0, 1] \) with the following significance. If \( (a, \phi) \in W^{1,q}_A(X; A^1 \otimes \text{ad}P \oplus E) \) obeys
\[
\| (a, \phi) \|_{L^d(X)} < \delta \quad \text{when} \quad d \geq 3 \quad \text{or} \quad \| (a, \phi) \|_{L^4(X)} < \delta \quad \text{when} \quad d = 2,
\]
then the operator,
\[
d^*_A,\Phi d_{A+a,\Phi+\phi} : (\text{Ker } \Delta_{A,\Phi})^\perp \cap W^{2,q}_A(X; \text{ad}P) \to (\text{Ker } \Delta_{A,\Phi})^\perp \cap L^q(X; \text{ad}P),
\]
is surjective.

Finally, the proof of Proposition 2.11 adapts mutatis mutandis to establish the following (simplified) analogue for pairs.

Proposition 2.19 (A priori \( W^{1,p} \) estimate for \( u(A, \Phi) - (A_0, \Phi_0) \) in terms of \( (A, \Phi) - (A_0, \Phi_0) \)). Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group and \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times G \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( \rho : G \to \text{Aut}(E) \). Let \( A_1 \) be a \( C^\infty \) connection on \( P \), and \((A_0, \Phi_0)\) be a \( W^{1,q} \) pair on \((P, E)\) with \( d/2 < q < \infty \) and \( p \in (1, \infty) \) obey \( d/2 \leq p \leq q \). Then there are constants \( N = N(A_1, A_0, \Phi_0, g, G, p, q) \in [1, \infty) \) and \( \varepsilon = \varepsilon(A_1, A_0, \Phi_0, g, G, p, q) \in (0, 1] \) with the following significance. If \( (A, \Phi) \) is a \( W^{1,q} \) pair on \((P, E)\) and \( u \in \text{Aut}(P) \) is a gauge transformation of class \( W^{2,q} \) such that
\[
d^*_A,\Phi_0 (u(A, \Phi) - (A_0, \Phi_0)) = 0,
\]
then the following hold. If \( (A, \Phi) \) and \( u(A, \Phi) \) obey
\[
\| (A, \Phi) - (A_0, \Phi_0) \|_{W^{1,p}_A(X)} \leq \varepsilon \quad \text{and} \quad \| u(A, \Phi) - (A_0, \Phi_0) \|_{W^{1,p}_A(X)} \leq \varepsilon,
\]
then
\[
\| u(A, \Phi) - (A_0, \Phi_0) \|_{W^{1,p}_A(X)} \leq N \| (A, \Phi) - (A_0, \Phi_0) \|_{W^{1,p}_A(X)}.
\]

Proof of Theorem 11. Given these preliminaries, Corollary 2.17 Lemma 2.18 and Proposition 2.19 the proof of Theorem 11 follows mutatis mutandis from that of Theorem 9. \( \square \)

The proof of Proposition 2.19 yields the following analogue of Lemma 2.12 for pairs.

Lemma 2.20 (A priori \( W^{2,p} \) estimate for a \( W^{2,q} \) gauge transformation \( u \) intertwining two \( W^{1,q} \) pairs). Assume the hypotheses of Proposition 2.14 (excluding those on the pair \( (A, \Phi) \)). Then there is a constant \( C = C(A_0, \Phi_0, A_1, g, G, p, q) \in [1, \infty) \) with the following significance. If \( (A, \Phi) \) obeys the hypotheses of Proposition 2.19 and \( u \in \text{Aut}^{2,q}(P) \) is the resulting gauge transformation, depending on \( (A, \Phi) \) and \((A_0, \Phi_0)\), such that
\[
d^*_A,\Phi_0 (u(A, \Phi) - (A_0, \Phi_0)) = 0,
\]
then
\[
\| u \|_{W^{2,p}_A(X)} \leq C.
\]
While not required for the proof of Theorem 10 this is a convenient point at which to note that the proof of Lemma 2.13 (specialized to the case \( l = 1 \)) adapts \textit{mutatis mutandis} to give the following analogue for pairs.

**Lemma 2.21** (Continuous operators on \( L^p \) spaces and \( L^2 \)-orthogonal decompositions). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times \mathbb{E} \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( \varrho : G \to \text{Aut}_\mathbb{C}(\mathbb{E}) \). If \((A, \Phi)\) is a Sobolev pair on \((P, E)\) of class \( W^{1,q} \) with \( q \geq d/2 \), and \( A_1 \) is a \( C^\infty \) reference connection on \( P \), and \( p \) obeys \( d/2 \leq p < q \), then the operator

\[
d_{A,\Phi}^* : W^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to L^p(X; \text{ad}P \oplus E),
\]

is continuous and, if in addition \( q > d/2 \), then the operator

\[
d_{A,\Phi} : W^{2,p}(X; \text{ad}P \oplus E) \to W^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]

is also continuous and there is an \( L^2 \)-orthogonal decomposition,

\[
W^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) = \text{Ker} \left( d_{A,\Phi}^* : W^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to L^p(X; \text{ad}P \oplus E) \right)
\oplus \text{Ran} \left( d_{A,\Phi} : W^{2,p}(X; \text{ad}P \oplus E) \to W^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \right).
\]

**2.9. Regularity for solutions to the Yang-Mills and coupled Yang-Mills equations.** It is well-known that techniques due to Uhlenbeck [96, 95] can be used to show that, given a weak solution to the Yang-Mills equation, there exists a gauge transformation such that the gauge-transformed solution is smooth. We give a proof of a similar fact here that generalizes easily to the case of coupled Yang-Mills equations.

We have the following generalization of [79, Theorem 5.3], due to Parker, and [30, Proposition 3.7], due to the author and Leness, from the case of \( d = 4 \) to arbitrary \( d \geq 2 \).

**Theorem 2.22** (Regularity for solutions to the Yang-Mills equation). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, and \( P \) be a smooth principal \( G \)-bundle over \( X \). If \( d/2 < q < \infty \) and \( A \) is a \( W^{1,q} \) connection on \( P \) that is a weak solution to the Yang-Mills equation with respect to the Riemannian metric \( g \), then there exists a \( W^{2,q} \) gauge transformation \( u \in \text{Aut}(P) \) such that \( u(A) \) is a \( C^\infty \) Yang-Mills connection on \( P \).

**Proof.** We proceed as in the proof of [30, Proposition 3.7] and note that the affine space \( \mathcal{A}(P) \) of \( C^\infty \) connections on \( P \) is dense in the affine space \( \mathcal{A}^{1,q}(P) \) of \( W^{1,q} \) connections on \( P \) and so there exists a \( C^\infty \) connection \( A_0 \) on \( P \) such that

\[
\|A - A_0\|_{W^{1,q}_A(X)} < \zeta,
\]

where \( \zeta = \zeta(A_0, A_1, g, G, q) \in (0, 1] \) is the constant in Theorem 9 and \( A_1 \) is any fixed \( C^\infty \) reference connection on \( P \). Hence, there is a \( W^{2,q} \) gauge transformation \( u \in \text{Aut}(P) \) such that \( u(A) \) obeys

\[
d_{A_0}^*(u(A) - A_0) = 0,
\]
and
\[ \|u(A) - A_0\|_{W^1,q_A(X)} < 2N\|A - A_0\|_{W^{1,q}_A(X)} < 2N\zeta, \]
where \( N = N(A_0, A_1, g, G, q) \in [1, \infty) \) is the constant in Proposition 2.11. Hence, we may assume without loss of generality that \( A \) is in Coulomb gauge with respect to \( A_0 \) and that \( a := A - A_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \) is a weak solution to
\[ d^*_{A_0 + a} F_{A_0 + a} = 0 \quad \text{and} \quad d^*_{A_0} a = 0, \]
and thus a weak solution to the quasi-linear, second-order elliptic system,
\[
(\Delta_{A_0} + \lambda_0) a + a \times \nabla_{A_0} a + a \times a \times a = \lambda a - F_{A_0},
\]
where \( \Delta_{A_0} = d^*_{A_0} d_{A_0} + d_{A_0} d^*_{A_0} \) is the usual Hodge Laplacian on \( \Omega^1(X; \text{ad}P) \) and \( \lambda_0 > 0 \) is any positive constant. We recall from [26] that, for any \( A \in \Lambda^1 \otimes \text{ad}P \), the right-hand side of Equation (2.69) belongs to \( W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \) by hypothesis on \( A \) and the fact that \( A_0 \) is \( C^\infty \). If \( q > d \), then \( W^{1,q}(X) \subset C(X) \) by [42 Theorem 4.12] and so
\[ a \times \nabla_{A_0} a + a \times a \times a \in L^q(X; \Lambda^1 \otimes \text{ad}P). \]
But then existence and uniqueness of solutions in \( W^{2,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \) to (2.69), given a source term in \( L^q(X; \Lambda^1 \otimes \text{ad}P) \), implies that \( a \in W^{2,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \). Using the fact that \( W^{k,q}(X) \) is a Banach algebra when \( kq > d \) and invertibility of (2.70), we can iterate in the usual way to show that \( a \in W^{k+2,q}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \) for all integers \( k \geq 0 \) and hence that \( a \in C^\infty(X; \Lambda^1 \otimes \text{ad}P) \).

Therefore, it suffices to consider the case \( d/2 < q < d \). If \( q^* = dq/(d - q) \) and \( r \in [1, q) \) obeys \( 1/q^* + 1/q \leq 1/r \), then there is a continuous multiplication map \( L^{q^*}(X) \times L^q(X) \to L^r(X) \) and a continuous Sobolev embedding \( W^{1,q}(X) \subset L^{q^*}(X) \) by [42 Theorem 4.12]. Thus, \( a \times \nabla_{A_0} a \in L^r(X; \Lambda^1 \otimes \text{ad}P) \) if
\[ 1/r \geq (d - q)/(dq) + 1/q = (2d - q)/(dq), \]
that is, \( r \leq dq/(2d - q) \).

Similarly, we have a continuous multiplication map \( L^{3s}(X) \times L^{3s}(X) \times L^{3s}(X) \to L^s(X) \) for any \( s \in [1, \infty] \) and a continuous Sobolev embedding \( W^{1,q}(X) \subset L^{3s}(X) \) provided \( 3s \leq q^* \), that is, \( s \leq dq/(3(d - q)) \) and for this choice of \( s \), we have \( a \times a \times a \in L^s(X; \Lambda^1 \otimes \text{ad}P) \).

We now observe that \( r \leq s \) when \( q \geq d/2 \), as we assume by hypothesis, if we choose \( r = dq/(2d - q) \) and \( s = dq/(3d - dq) \). Indeed, we then have
\[ r \leq s \iff dq/(2d - q) \leq dq/(3d - dq) \iff 3d - 3q \leq 2d - q \iff d \leq 2q. \]
Hence, for \( r = dq/(2d - q) \), we have
\[ a \times \nabla_{A_0} a + a \times a \times a \in L^r(X; \Lambda^1 \otimes \text{ad}P), \]
and thus elliptic regularity theory for (2.69) implies that \( a \in W^{2,r}_{A_1}(X; \Lambda^1 \otimes \text{ad}P) \), similar to the case \( q > d \). By [42 Theorem 4.12], we have a continuous Sobolev embedding, \( W^{2,r}(X) \subset W^{1,r^*} \), where \( r^* = dr/(d - r) \), and thus we obtain
\[ a \in W^{1,r^*}_{A_1}(X; \Lambda^1 \otimes \text{ad}P). \]
In the limiting case \( q = d/2 \) we would have \( r = (d^2/2)/(2d - d/2) = d/3 \) and in the limiting case \( q = d \) we would have \( r = d \), so \( r \in (d/3, d) \) and thus \( r^* \in (d/2, \infty) \). In particular,

\[
r^* = \frac{dr}{d - r} = \frac{d^2q/(2d - q) - dq/(2d - q)}{d - dq/(2d - q)} = \frac{dq}{1 - q/(2d - q)} = \frac{dq}{2(d - q)}.
\]

We may write \( r^* = q + \delta \), where \( \delta = \delta(d, q) \) is defined by

\[
\delta := r^* - q = \frac{dq}{2(d - q)} - q = \frac{dq - 2(d - q)q}{2(d - q)} = \frac{2q^2 - dq}{2(d - q)} = \frac{q(2q - d)}{2(d - q)},
\]

and thus \( \delta(d, q) > 0 \) for \( d/2 < q < d \). Consequently, we see a regularity improvement and because \( \delta(d, q) \) is an increasing function of \( q \), only finitely many iterations of this regularity improvement are required to give \( r^* > d \), at which point we can apply the regularity argument for the case \( q > d \) to again conclude that \( a \in C^\infty(X; \Lambda^1 \otimes \text{ad}P) \).

The proof of Theorem 2.22 adapts mutatis mutandis to give the following generalization of [79, Theorem 5.3], due to Parker, and [30, Proposition 3.7], due to the author and Leness, from the case of \( d = 4 \) to arbitrary \( d \geq 2 \).

**Theorem 2.23** (Regularity for solutions to the boson coupled Yang-Mills equations). Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times_g \mathbb{E} \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( g : G \rightarrow \text{Aut}_c(\mathbb{E}) \). If \( d/2 < q < \infty \) and \( (A_\infty, \Phi_\infty) \) is a \( W^{1, q} \) pair on \( (P, E) \) that is a critical point of the boson coupled Yang-Mills energy functional (1.6), then there exists a \( W^{2, q} \) gauge transformation \( u \in \text{Aut}(P) \) such that \( u(A_\infty, \Phi_\infty) \) is a \( C^\infty \) pair on \( (P, E) \).

**Proof.** We proceed as in the proofs of [79, Theorem 5.3] and [30, Proposition 3.7] and note that the affine space \( \mathcal{A}(P) \times C^\infty(E) \) of \( C^\infty \) pairs is dense in the affine space \( \mathcal{A}^{1, q}(P) \times W^{1, q}(X; E) \) of \( W^{1, q} \) pairs and so there exists a \( C^\infty \) pair \( (A_0, \Phi_0) \) on \( (P, E) \) such that

\[
\|(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1, q}_A(X)} < \zeta,
\]

where \( \zeta = \zeta(A_1, A_0, \Phi_0, q, G, q) \in (0, 1] \) is the constant in Theorem 11 and \( A_1 \) is any fixed \( C^\infty \) reference connection on \( P \). Hence, there is a gauge transformation \( u \in \text{Aut}(P) \) of class \( W^{2, q} \) such that \( u(A_\infty, \Phi_\infty) \) obeys

\[
d_{A_0, \Phi_0}^* (u(A_\infty, \Phi_\infty) - (A_0, \Phi_0)) = 0,
\]

and

\[
\|u(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1, q}_A(X)} < 2N\|(A_\infty, \Phi_\infty) - (A_0, \Phi_0)\|_{W^{1, q}_A(X)} < 2N\zeta,
\]

where \( N = N(A_1, A_0, \Phi_0, g, G, q) \in [1, \infty) \) is the constant in Proposition 2.19. Hence, we may assume without loss of generality that \( (A_\infty, \Phi_\infty) \) is in Coulomb gauge with respect to \( (A_0, \Phi_0) \) and that \( (a, \phi) := (A_\infty, \Phi_\infty) - (A_0, \Phi_0) \in W^{1, q}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E) \) is a weak solution to the
quasi-linear, second-order elliptic system,
\[
d^*_{A_0} d_{A_0} a + \nabla^*_{A_0} \nabla_{A_0} \phi + a \times \nabla_{A_0} a + a \times \nabla_{A_0} \phi + \phi \times \nabla_{A_0} \phi \\
+ a \times a \times a + a \times a \times \phi + a \times \phi \times \phi \\
+ m \phi + s \Phi_0 \times \Phi_0 \times \phi + s \Phi_0 \times \phi \times \phi + s \phi \times \phi \times \phi = f(m, s, A_0, \Phi_0),
\]
where we employ the expression (1.10) for the gradient of the affine space of \( W^{d_k,p} \) pairs on \((X, E)\), where \( k \in \mathbb{Z} \) is a positive integer and \( p \in (1, \infty) \); we write \( \mathcal{D}(P, E) := \mathcal{A}(P) \times C^\infty(X; E) \) for the affine space of \( C^\infty \) pairs on \((P, E)\).

3.1. Lojasiewicz–Simon gradient inequality for boson coupled Yang-Mills energy functional. For any \( C^\infty \) reference connection \( A_1 \) on \( P \), let
\[
\mathcal{D}^{k,p}(P, E) := \mathcal{A}^{k,p}(P) \times W^{k,p}_{A_1}(X; E),
\]
denote the affine space of \( W^{k,p} \) pairs on \((P, E)\), where \( k \in \mathbb{Z} \) is a positive integer and \( p \in (1, \infty) \); we write \( \mathcal{D}(P, E) := \mathcal{A}(P) \times C^\infty(X; E) \) for the affine space of \( C^\infty \) pairs on \((P, E)\).

3.1.1. Analyticity of the boson coupled Yang-Mills energy functional and its gradient map on the Sobolev space of pairs. We begin by establishing the following result on analyticity of the boson coupled Yang-Mills \( L^2 \)-energy functional, \( \mathcal{E} : x_\infty + \mathcal{H} \to \mathbb{R} \) (where \( x_\infty = (A_\infty, \Phi_\infty) \) is a critical point); this also serves as a stepping stone towards the proof that its gradient map, \( \mathcal{H} : x_\infty + \mathcal{H} \to \mathcal{H} \) is real analytic for suitable choices of the Banach spaces, \( \mathcal{H} \) and \( \mathcal{H} \), in Theorem [14].

Proposition 3.1 (Analyticity of the boson coupled Yang-Mills \( L^2 \)-energy functional on the affine space of \( W^{1,p} \) pairs). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times_{\varrho} \mathbb{E} \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( \varrho : G \to \text{Aut}_G(\mathbb{E}) \), and \( A_1 \) be a \( C^\infty \) connection on \( P \), and \( m, s \in C^\infty(X) \). If \( 4d/(d+4) \leq p < \infty \), then the function,
\[
\mathcal{E} : \mathcal{D}^{1,p}(P, E) \to \mathbb{R},
\]
is real analytic, where \( \mathcal{E} \) is as in (1.6).
Proof. We fix a pair \((A, \Phi) \in \mathcal{P}_1^p(P, E)\) and write \((A, \Phi) = (A_1, \Phi_1) + (a_1, \phi_1)\), where \((a_1, \phi_1) \in W_{A_1}^1(P; \Lambda^1 \otimes \text{ad} P \oplus E)\). We will show that \(\mathcal{E}\) is analytic at \((A, \Phi)\). For \((a, \phi) \in W_{A_1}^1(P; \Lambda^1 \otimes \text{ad} P \oplus E)\), we write \(A = A_1 + a_1 + a\) and expand

\[
F_{A+a} = F_{A_1+a_1+a} = F_{A_1} + d_{A_1}(a_1 + a) + (a_1 + a) \times (a_1 + a)
\]

and

\[
\nabla_{A+a}(\Phi + \phi) = \nabla_{A+a_1+a}(\Phi + \phi) = \nabla_{A_1}(\Phi + \phi) + \nabla a_1(\Phi + \phi) + \phi(a_1 + a)(\Phi + \phi).
\]

Using the definition (1.6) of \(\mathcal{E}\), we compute

\[
2\mathcal{E}(A + a, \Phi + \phi) = T_1 + T_2 + T_3,
\]

where the terms \(T_i := T_i(a, \phi)\), for \(i = 1, 2, 3\), are given by

\[
T_1 := \|F_{A_1}\|_{L^2(X)}^2 + \|d_{A_1}(a_1 + a)\|_{L^2(X)}^2
+ \|\nabla A_1(\Phi + \phi)\|_{L^2(X)}^2 + \|\nabla a_1(\Phi + \phi)\|_{L^2(X)}^2
+ 2\|d_{A_1}(a_1 + a) \times (a_1 + a)\|_{L^2(X)}^2,
\]

and

\[
T_2 := \|\nabla A_1(\Phi + \phi)\|_{L^2(X)}^2 + \|\nabla a_1(\Phi + \phi)\|_{L^2(X)}^2
+ \|\nabla A_1(\Phi + \phi)\|_{L^2(X)}^2 + \|\nabla a_1(\Phi + \phi)\|_{L^2(X)}^2
+ 2\|d_{A_1}(a_1 + a) \times (a_1 + a)\|_{L^2(X)}^2,
\]

and

\[
T_3 := - \int_X (m|\Phi + \phi|^2 + s|\Phi + \phi|^4) \, d\text{vol}_g.
\]

Hence, we can write the difference as

\[
2\mathcal{E}(A + a, \Phi + \phi) - 2\mathcal{E}(A, \Phi) = T'_1 + T'_2 + T'_3,
\]

where the difference terms \(T'_i := T_i(a, \phi) - T_i(0,0)\), for \(i = 1, 2, 3\), are given by

\[
T'_1 = \|d_{A_1} a\|_{L^2(X)}^2 + 2\|d_{A_1} a_1, d_{A_1} a\|_{L^2(X)} + (a \times (a_1 + a), (a_1 + a) \times (a_1 + a))\|_{L^2(X)}^2
+ 2\|d_{A_1} a_1, (a_1 + a) \times (a_1 + a)\|_{L^2(X)} + (d_{A_1} a_1, (a_1 + a) \times (a_1 + a))\|_{L^2(X)}^2,
\]

and

\[
T'_2 = \|\nabla A_1 a\|_{L^2(X)}^2 + 2\|\nabla A_1 a_1, a_1, a\|_{L^2(X)} + (a \times (a_1 + a), (a_1 + a) \times (a_1 + a))\|_{L^2(X)}^2
+ 2\|\nabla A_1 a_1, (a_1 + a) \times (a_1 + a)\|_{L^2(X)} + (d_{A_1} a_1, (a_1 + a) \times (a_1 + a))\|_{L^2(X)}^2.
\]
and
\[
T'_3 = \int_X \left( m|\phi|^2 + 2m \operatorname{Re}(\Phi, \phi) + s|\phi|^4 + 4s(\operatorname{Re}(\Phi, \phi))^2 + 4s(|\Phi|^2 + |\phi|^2) \operatorname{Re}(\Phi, \phi) \right) \, d\nu_g \\
+ \int_X 2s|\Phi|^2 |\phi|^2 \, d\nu_g.
\]

To see the origin of the expression (3.22) for \( T'_2 \), we observe that
\[
T'_2 := \|\nabla_{A_1+g+a}(\Phi + \phi)\|_{L^2(X)}^2 - \|\nabla_{A_1+g+a} \Phi\|_{L^2(X)}^2 \\
= \|\nabla_{A_1}(\Phi + \phi) + g(a_1 + a)(\Phi + \phi)\|_{L^2(X)}^2 - \|\nabla_{A_1} \Phi + g(a_1) \Phi\|_{L^2(X)}^2 \\
= \|\nabla_{A_1}(\Phi + \phi)\|_{L^2(X)}^2 - \|\nabla_{A_1} \Phi\|_{L^2(X)}^2 \\
+ \|g(a_1 + a)(\Phi + \phi)\|_{L^2(X)}^2 - \|g(a_1) \Phi\|_{L^2(X)}^2 \\
+ 2 \operatorname{Re}(\nabla_{A_1} (\Phi + \phi), g(a_1 + a)(\Phi + \phi))_{L^2(X)} - 2 \operatorname{Re}(\nabla_{A_1} \Phi, g(a_1) \Phi)_{L^2(X)} \\
= T'_{21} + T'_{22} + T'_{23}.
\]

For the first term, we have
\[
T'_{21} := \|\nabla_{A_1}(\Phi + \phi)\|_{L^2(X)}^2 - \|\nabla_{A_1} \Phi\|_{L^2(X)}^2 = \|\nabla_{A_1} \phi\|_{L^2(X)}^2 + 2 \operatorname{Re}(\nabla_{A_1} \Phi, \nabla_{A_1} \phi)_{L^2(X)}.
\]

For the second term, we see that
\[
T'_{22} := \|g(a_1 + a)(\Phi + \phi)\|_{L^2(X)}^2 - \|g(a_1) \Phi\|_{L^2(X)}^2 \\
= \|g(a_1 + a) \Phi + g(a_1 + a) \phi\|_{L^2(X)}^2 - \|g(a_1) \Phi\|_{L^2(X)}^2 \\
= \|g(a) \Phi\|_{L^2(X)}^2 + \|g(a_1 + a) \phi\|_{L^2(X)}^2 + 2 \operatorname{Re}(g(a_1) \Phi, g(a) \phi)_{L^2(X)} \\
+ 2 \operatorname{Re}(g(a_1 + a) \Phi, g(a_1 + a) \phi)_{L^2(X)}.
\]

For the third term, we have
\[
T'_{23} := 2 \operatorname{Re}(\nabla_{A_1} (\Phi + \phi), g(a_1 + a)(\Phi + \phi))_{L^2(X)} - 2 \operatorname{Re}(\nabla_{A_1} \Phi, g(a_1) \Phi) \\
= 2 \operatorname{Re}(\nabla_{A_1} \Phi, g(a_1) \Phi + g(a) \Phi + g(a_1 + a) \phi)_{L^2(X)} + 2 \operatorname{Re}(\nabla_{A_1} \phi, g(a_1 + a) (\Phi + \phi))_{L^2(X)} \\
- 2 \operatorname{Re}(\nabla_{A_1} \Phi, g(a_1) \Phi)_{L^2(X)} \\
= 2 \operatorname{Re}(\nabla_{A_1} \Phi, g(a) \Phi + g(a_1 + a) \phi)_{L^2(X)} + 2 \operatorname{Re}(\nabla_{A_1} \phi, g(a_1 + a) (\Phi + \phi))_{L^2(X)}.
\]

By adding the preceding terms, we obtain the expression (3.22) for \( T'_2 \).

To complete the proof of analyticity of \( \mathcal{E} \) at \( (A, \Phi) \), we observe that there is a continuous Sobolev embedding, \( W^{1,p}(X) \subset L^4(X) \), when \( p^* = dp/(d-p) \ge 4 \) by [4] Theorem 4.12, that is, \( dp \ge 4(d-p) \) or \( p(d+4) \ge 4d \). Hence, we obtain a continuous multilinear map, \( \otimes_{i=1}^4 W^{1,p}(X) \rightarrow L^1(X) \), by combining the Sobolev embedding \( W^{1,p}(X) \rightarrow L^4(X) \) with the continuous multiplication map, \( \otimes_{i=1}^4 L^4(X) \rightarrow L^1(X) \). Combining these observations with the Kato Inequality [37] Equation (6.20)], we obtain an estimate of the form,
\[
|\mathcal{E}(A + a, \Phi + \phi) - \mathcal{E}(A, \Phi)| \le |T'_1| + |T'_2| + |T'_3|,
\]
where, for a constant $C = C(g, G) \in [1, \infty)$,

$$C^{-1}|T_1| \leq \|a\|^2_{W^{1,p}_{A_1}(X)} + \|a_1\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)})^3$$

$$+ \|F_{A_1}\|_{L^2(X)} \|a\|_{W^{1,p}_{A_1}(X)} + \|F_{A_1}\|_{L^2(X)} \|a\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)})$$

$$+ \|a\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)})^2$$

$$+ \|a_1\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}),$$

noting that $W^{1,p}(X) \subset L^2(X)$ if $dp/(d - p) \geq 2$, that is, $dp \geq 2(d - p)$ or $p(d + 2) \geq 2d$ or $p \geq 2d/(d + 2)$, and

$$C^{-1}|T_2| \leq \|\phi\|^2_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)} (\|\phi\|_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)})$$

$$+ \left(\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}\right)^2 \|\phi\|^2_{W^{1,p}_{A_1}(X)} + \|a_1\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} \|\phi\|^2_{W^{1,p}_{A_1}(X)}$$

$$+ \|\phi\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}) (\|\phi\|_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)})$$

$$+ \|\phi\|^2_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} (\|a_1\|_{W^{1,p}_{A_1}(X)} + \|a\|_{W^{1,p}_{A_1}(X)}),$$

and

$$C^{-1}|T_3| \leq \|\phi\|^2_{W^{1,p}_{A_1}(X)} + \|\phi\|_{W^{1,p}_{A_1}(X)} \|a\|_{W^{1,p}_{A_1}(X)} + \|\phi\|^4_{W^{1,p}_{A_1}(X)} + \|\phi\|^2_{W^{1,p}_{A_1}(X)} \|\phi\|^2_{W^{1,p}_{A_1}(X)}$$

$$+ \|\phi\|^3_{W^{1,p}_{A_1}(X)} \|\phi\|^2_{W^{1,p}_{A_1}(X)} + \|\phi\|^3_{W^{1,p}_{A_1}(X)} \|\phi\|^2_{W^{1,p}_{A_1}(X)}.$$

Note that $4d/(d + 4) \geq 2d/(d + 2)$, so the condition $p \geq 2d/(d + 2)$ is assured by the stronger $p \geq 4d/(d + 4)$. Thus, $\mathcal{E}'(A + a, \Phi + \phi)$ is a polynomial of degree four in the variable $(a, \phi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E)$. This completes the proof of Proposition 3.1. \hfill \Box

We now verify the formula (1.10) for the differential $\mathcal{E}'(A, \Phi)$ and gradient $\mathcal{M}(A, \Phi)$.

**Lemma 3.2** (Differential and gradient of the boson coupled Yang-Mills $L^2$-energy functional). Assume the hypotheses of Proposition 3.1 with the dual Hölder exponent $p'$ in the range $1 < p' \leq 4d/(3d - 4)$ determined by $4d/(d + 4) \leq p < \infty$ and $1/p + 1/p' = 1$. Then the expression for $\mathcal{E}'(A, \Phi) \in (W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E))^*$ and $\mathcal{M}(A, \Phi) \in W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E)$ is given by
\[ \mathcal{E}'(A, \Phi)(a, \phi) = \left( (a, \phi), \mathcal{M}(A, \Phi) \right)_{L^2(X)} \]

\[ = (d_A^* F_a, a)_{L^2(X)} + \Re(\nabla_A^* \nabla_A \Phi, \phi)_{L^2(X)} + \Re(\nabla_A \Phi, g(a) \Phi)_{L^2(X)} \]

\[ - \Re(m \Phi, \phi)_{L^2(X)} - 2 \Re \int_X s |\Phi|^2 \langle \Phi, \phi \rangle \, d\text{vol}_g, \]

\[ \forall (a, \phi) \in W^{1,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E). \]

**Proof.** We establish (1.10) by extracting the terms that are linear in \((a, \phi)\) from the expressions for \(T^i, \) for \(i = 1, 2, 3, \) arising in the proof of Proposition 3.1. We compute \(\mathcal{E}'(A + ta, \Phi + t\phi)\) using the identities,

\[ F_{A + ta} = F_A + td_a a + \frac{t^2}{2} [a_a] \quad \text{and} \quad \nabla_{A + ta}(\Phi + t\phi) = \nabla_A(\Phi + t\phi) + g(ta)(\Phi + t\phi), \]

to obtain

\[ \mathcal{E}'(A + ta, \Phi + t\phi) = \frac{1}{2} \int_X |F_A + td_a a + \frac{t^2}{2} [a_a]|^2 \, d\text{vol}_g \]

\[ + \frac{1}{2} \int_X \left| \nabla_A \Phi + t(\nabla_A \phi) + g(a) \Phi + t^2 g(a) \phi \right|^2 \, d\text{vol}_g \]

\[ - \frac{1}{2} \int_X (m|\Phi + t\phi|^2 + s|\Phi + t\phi|) \, d\text{vol}_g, \]

that is,

\[ \mathcal{E}'(A + ta, \Phi + t\phi) = \frac{1}{2} \int_X \left( |F_A|^2 + 2t \langle F_A, d_A a \rangle \right) \, d\text{vol}_g \]

\[ + \frac{1}{2} \int_X \left( |\nabla_A \Phi|^2 + 2t \Re(\nabla_A \Phi, \nabla_A \phi + g(a) \Phi) \right) \, d\text{vol}_g \]

\[ - \frac{1}{2} \int_X m \left( |\Phi|^2 + 2t \Re(\Phi, \phi) \right) \, d\text{vol}_g \]

\[ - \frac{1}{2} \int_X s \left( |\Phi|^4 + 4t |\Phi|^2 \Re(\Phi, \phi) \right) \, d\text{vol}_g + \text{higher powers of } t. \]

This yields the expression (1.10) for \(\mathcal{E}'(A, \Phi)(a, \phi)\) and completes the proof of Lemma 3.2. \(\square\)

It is convenient to define, for \(p \in (1, \infty)\) and dual exponent \(p' \in (1, \infty)\) determined by \(1/p + 1/p' = 1,\) the following Banach spaces,

\[ \mathfrak{X} := W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \quad \text{and} \quad \mathfrak{X}^* = W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E), \]

\[ \tilde{\mathfrak{X}} := W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E). \]

Note that for \(p \geq 2,\) the inclusion, \(\tilde{\mathfrak{X}} \subset \mathfrak{X}^*,\) is a continuous embedding of Banach spaces.

The expression (1.10) defines the gradient as a map, \(\mathcal{M} : (A_1, 0) + \mathfrak{X} \to \tilde{\mathfrak{X}},\) where \(\mathcal{M}(A, \Phi) \in \tilde{\mathfrak{X}}\) acts on \((b, \varphi) \in \mathfrak{X}\) by \(L^2\) inner product via the inclusion \(\tilde{\mathfrak{X}} \subset \mathfrak{X}^*.\) We now have the
Proposition 3.3 (Analyticity of the gradient map for the boson coupled Yang-Mills $L^2$-energy functional). Assume the hypotheses of Proposition [3.1] and, in addition, that $p \geq d/2$. Then

$$ \mathcal{M} : (A_1, 0) + \mathfrak{X} \to \mathfrak{X} $$

is a real analytic map, where $\mathcal{M}$ is as in (1.10) and $\mathfrak{X}$ and $\mathfrak{X}$ are as in (3.3).

Proof. As usual, given $p \in (1, \infty)$, we let $p' \in (1, \infty)$ denote the dual Hölder exponent defined by $1/p + 1/p' = 1$. By [4, Theorem 4.12], we have a continuous Sobolev embedding, $W^{1,p'}(X; \mathbb{C}) \subset L^r(X; \mathbb{C})$, when

1. $1 < p' < d$ and $1 \leq r \leq (p')^* := dp'/d - p'$, or
2. $p' = d$ and $1 \leq r < \infty$, or
3. $d < p' < \infty$ and $1 \leq r \leq \infty$.

In each case, we obtain a continuous Sobolev embedding, $L^{r'}(X; \mathbb{C}) \subset W^{-1,p}(X; \mathbb{C})$, by duality and density of the continuous embedding, $W^{1,p'}(X; \mathbb{C}) \subset L^r(X; \mathbb{C})$, where $1/r + 1/r' = 1$ and $1/p + 1/p' = 1$. We shall require $p', r < \infty$ in order to appeal to the dualities, $(W^{1,p}(X; \mathbb{C}))^* = W^{-1,p}(X; \mathbb{C})$ and $(L^r(X; \mathbb{C}))^* = L^{r'}(X; \mathbb{C})$; these dualities fail when $p' = \infty$ (and thus $p = 1$) or $r = \infty$ (and thus $r' = 1$).

If we write $A = A_1 + a_1$, for $a_1 \in W_{A_1}^{1,p}(X; \Lambda^1 \otimes \text{ad}P)$, then from the formula (1.10) for $\mathcal{M}(A, \Phi)$ we have the formal expression,

$$ \mathcal{M}(A_1 + a_1, \Phi) = d_{A_1}^a d_{A_1} a_1 + \nabla_{A_1}^a \nabla_{A_1} \Phi + d_{A_1}^a F_{A_1} + F_{A_1} \times a_1 $$

$$ + \nabla_{A_1} a_1 \times a_1 + \nabla_{A_1} a_1 \times \Phi + \nabla_{A_1} a_1 \times a_1 $$

$$ + \nabla_{A_1} \Phi \times a_1 + \nabla_{A_1} a_1 \times a_1 + \nabla_{A_1} a_1 \times \Phi + a_1 \times a_1 \times a_1 \times a_1. $$

Observe that

$$ W_{A_1}^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \ni (a_1, \Phi) \to d_{A_1}^a d_{A_1} a_1 + \nabla_{A_1}^a \nabla_{A_1} \Phi \in W_{A_1}^{-1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E), $$

is analytic and the term $d_{A_1}^a F_{A_1}$ is constant with respect to $(a_1, \Phi)$. Therefore, to prove that $\mathcal{M}$ is analytic, it suffices to prove the

Claim 3.4. Continue the preceding notation. Then

$$ W_{A_1}^{1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \ni (a_1, \Phi) $$

$$ \mapsto \mathcal{M}(A_1 + a_1, \Phi) - d_{A_1}^a d_{A_1} a_1 + \nabla_{A_1}^a \nabla_{A_1} \Phi - d_{A_1}^a F_{A_1} \in L^{r'}(X; \Lambda^1 \otimes \text{ad}P \oplus E), $$

is a cubic polynomial in $(a_1, \Phi)$ and its first-order covariant derivatives with respect to $\nabla_{A_1}$, with universal coefficients (depending at most on $g$ and $G$).

Proof of Claim 3.4. We compute an $L^{r'}$ bound for each term in equation (3.4) for $\mathcal{M}(A, \Phi)$; we consider the cases $p' < d$, $p' = d$, and $p' > d$ separately, recalling that $A = A_1 + a_1$.

Step 1 ($L^{r'}$ estimates for $F_{A_1} \times a_1$ and $\nabla_{A_1} \Phi \times a_1$). We claim that

$$ \|F_{A_1} \times a_1\|_{L^{r'}(X)} \leq z \|F_{A_1}\|_{L^p(X)} \|a_1\|_{W_{A_1}^{1,p}(X)}, $$

$$ \|\nabla_{A_1} \Phi \times a_1\|_{L^{r'}(X)} \leq z \|\Phi\|_{W_{A_1}^{1,p}(X)} \|a_1\|_{W_{A_1}^{1,p}(X)}, $$

where $z = z(g, p) \in [1, \infty)$. 
We consider each of these two subcases in turn.

which yields (3.6) for all $d$.

$W^{1,p}(X) \subset L^q(X)$, for $p^* = dp/(d-p)$ if $p < d$; b) $W^{1,p}(X) \subset L^q(X)$ for any $q \in [1, \infty)$ if $p = d$; and c) $W^{1,p}(X) \subset L^\infty(X)$ if $p > d$. We consider each of these three subcases in turn.

Suppose $d/(d-1) < p < d$. In order to have a continuous multiplication map, $L^p(X) \times L^q(X) \to L^{r'}(X)$, we require that the inequality $1/p + 1/p^* \leq 1/r'$ holds, that is

$$1/p + 1/(dp/(d-p)) \leq 1/r' = 1 - 1/r = 1 - (dp - d)/dp,$$

or equivalently,

$$1/p + (d-p)/(dp) \leq 1 - (1/p - 1/d),$$

namely,

$$1/p + 1/p - 1/d \leq 1/p + 1/d.$$ Therefore, $1/p \leq 2/d$ or $p \geq d/2$, as we assumed in our hypotheses. Thus,

$$\| F_{A_1} \times a_1 \|_{L^{r'}(X)} \leq z \| F_{A_1} \|_{L^p(X)} \| a_1 \|_{L^{p^*}(X)} \leq z \| F_{A_1} \|_{L^q(X)} \| a_1 \|_{W^{1,p}(X)},$$

which yields the first inequality in (3.6) for all $d \geq 2$ in this subcase.

Suppose $p = d$. Since we also needed $p < d/(d-1)$ in this case, then $d$ must obey $d > d/(d-1)$ or $d - 1 > 1$ or $d \geq 3$ for this subcase. (If $d = 2$, then $p = d = 2$ forces $p' = 2$, so the subcase $p' < d$ and $p = d$ cannot occur.) We now only need a continuous multiplication map, $L^d(X) \times L^q(X) \to L^{r'}(X)$, for large enough $q \in [1, \infty)$ and this requires only that $r' < d$, that is, $1/r' = 1 - 1/r > 1/d$ or

$$1 - (dp - d - p)/dp = 1/p + 1/d = 2/d > 1/p = 1/d,$$

which holds for all positive $d$. Thus,

$$\| F_{A_1} \times a_1 \|_{L^{r'}(X)} \leq z \| F_{A_1} \|_{L^p(X)} \| a_1 \|_{L^q(X)} \leq z \| F_{A_1} \|_{L^q(X)} \| a_1 \|_{W^{1,p}(X)},$$

which yields (3.6) for all $d \geq 3$ in this subcase.

Suppose $p > d$. We also needed $p < d/(d-1)$ in this case, but that holds for any $d \geq 2$ for this subcase since $d \geq d/(d-1)$ for any $d \geq 2$. We now only need a continuous multiplication map, $L^p(X) \times L^\infty(X) \to L^{r'}(X)$, and this requires only that $r' < p$, that is, $1/r' = 1 - 1/r \geq 1/p$ or

$$1 - (dp - d - p)/dp = 1/p + 1/d \geq 1/p,$$

which holds for all positive $d$. Thus,

$$\| F_{A_1} \times a_1 \|_{L^{r'}(X)} \leq z \| F_{A_1} \|_{L^p(X)} \| a_1 \|_{L^\infty(X)} \leq z \| F_{A_1} \|_{L^p(X)} \| a_1 \|_{W^{1,p}(X)},$$

which yields (3.6) for all $d \geq 2$ in this subcase.

Case 2 ($p' = d$). We can choose $r \in [1, \infty)$ arbitrarily large in the continuous Sobolev embedding, $W^{1,p'}(X) \subset L^{r'}(X)$, and so we may choose $r' > 1$ arbitrarily small. Since $p' = p/(p-1) = d$ for this case, then $p = dp - d$ or $p(d - 1) = d$ and so $p = d/(d-1) \leq d$. If $d = 2$, then $p = 2$ while if $d \geq 3$, then $p < d$, in which case $p^* = dp/(d-p) = (d^2/(d-1))/((d-d)/(d-1)) = d/(d-2)$. By [4] Theorem 4.12 we have continuous Sobolev embeddings, a) $W^{1,p}(X) \subset L^p(X)$ for $p < d$ and $p^* = d/(d-2)$ with $d \geq 3$; and b) $W^{1,p}(X) \subset L^q(X)$ for any $q \in [1, \infty)$ if $p = d = 2$ and $p' = 2$. We consider each of these two subcases in turn.
Suppose \(d \geq 3\). In order to have a continuous multiplication map, \(L^p(X) \times L^{p'}(X) \to L'^{r'}(X)\), we require that the inequality \(1/p + 1/p' \leq 1/r'\) holds. But \(p > 1\) by hypothesis and \(p' = d/(d-2) > 1\) for \(d \geq 3\) or \(p = 2\) for \(d = 2\). Hence, this multiplication map is continuous for small enough \(r' > 1\). Thus,

\[
\|F_{A_1} \times a_1\|_{L'^{r'}(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{L^{p'}(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{W^{1,p}_{A_1}(X)},
\]

which again yields (3.6) for the subcase \(d \geq 3\).

Suppose \(d = 2\). Then \(p = 2 = p'\) and we have a continuous multiplication map, \(L^p(X) \times L^q(X) \to L'^{r'}(X)\), for any small enough \(r' > 1\) and large enough \(q \in [1, \infty)\). Thus,

\[
\|F_{A_1} \times a_1\|_{L'^{r'}(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{L^q(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{W^{1,p}_{A_1}(X)},
\]

which again yields (3.6) for the subcase \(d = 2\).

**Case 3 \((p' > d)\)**. We can again choose \(r \in [1, \infty)\) arbitrarily large in the continuous Sobolev embedding, \(W^{1,p'}(X) \subset L^r(X)\), and so we may choose \(r' > 1\) arbitrarily small. (We refrain from choosing \(r = \infty\) because continuity of the Sobolev embedding, \(W^{1,p'}(X) \subset L^\infty(X)\), does not imply continuity of \(L^1(X) \subset W^{-1,p}(X)\).) Since \(p' = p/(p-1) > d\) for this case, then \(p > dp - d\) or \(p(d - 1) < d\) and so \(p < d/(d-1) \leq d\). We therefore have \(p^* = dp/(d-p)\) and a continuous Sobolev embedding, \(W^{1,p'}(X) \subset L^{p^*}(X)\).

In order to have a continuous multiplication map, \(L^p(X) \times L^{p'}(X) \to L'^{r'}(X)\), we require that the inequality \(1/p + 1/p^* \leq 1/r'\) holds. But \(p > 1\) by hypothesis and \(p^* = dp/(d-p) > dp/d = p > 1\) for all \(d \geq 2\). Hence, this multiplication map is continuous for small enough \(r' > 1\) and

\[
\|F_{A_1} \times a_1\|_{L'^{r'}(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{L^{p^*}(X)} \leq z\|F_{A_1}\|_{L^p(X)}\|a_1\|_{W^{1,p}_{A_1}(X)},
\]

which again yields (3.6) for this case.

An argument identical to that for (3.6) gives

\[
\|\nabla A_1 \Phi \times a_1\|_{L'^{r'}(X)} \leq z\|\nabla A_1 \Phi\|_{L^p(X)}\|a_1\|_{W^{1,p}_{A_1}(X)},
\]

proving (3.7). This completes Step 1.

**Step 2** \((L'^r \text{ estimates for } \nabla A_1 \Phi \times \Phi \text{ and } \nabla A_1 a_1 \times \Phi \text{ and } \nabla A_1 a_1 \times a_1)\). We claim that

\[
\|\nabla A_1 \Phi \times \Phi\|_{L'^{r'}(X)} \leq z\|\Phi\|_{W^{1,p}_{A_1}(X)^2}^2,
\]

(3.8)

\[
\|\nabla A_1 a_1 \times \Phi\|_{L'^{r'}(X)} \leq z\|\Phi\|_{W^{1,p}_{A_1}(X)}\|a_1\|_{W^{1,p}_{A_1}(X)},
\]

(3.9)

\[
\|\nabla A_1 a_1 \times a_1\|_{L'^{r'}(X)} \leq z\|a_1\|_{W^{1,p}_{A_1}(X)}^2,
\]

(3.10)

where \(z = z(g,p) \in [1, \infty)\).

From the proof of (3.6) in Step 1 we have

\[
\|\Phi \times \nabla A_1 \Phi\|_{L'^{r'}(X)} \leq z\|\Phi\|_{W^{1,p}_{A_1}(X)}\|\nabla A_1 \Phi\|_{L^p(X)},
\]

and this gives (3.8); identical arguments give (3.9) and (3.10). This completes Step 2.

Note that \(\|m\|_{C(X)}\) and \(\|s\|_{C(X)}\) are finite by hypothesis.
Step 3 \((L^r)\) estimate for \(m\Phi\). We have
\[
\|m\Phi\|_{L^r(X)} \leq \|m\|_{C(X)} \|\Phi\|_{L^r(X)},
\]
and because \(r' \leq p\) by inspection of each of the three cases, \(p' < d\) and \(p' = d\) and \(p' > d\), and subcases \((p < d\) and \(p = d\) and \(p > d\) where applicable) we obtain
\[
\|m\Phi\|_{L^r(X)} \leq z\|m\|_{C(X)} \|\Phi\|_{L^p(X)},
\]
as desired.

Step 4 \((L^r)\) estimates for \(s|\Phi|^2\). We claim that
\[
\|s|\Phi|^2\|_{L^r(X)} \leq z\|s\|_{C(X)} \|\Phi\|^2_{W^{1,p}(X)},
\]
where \(z = z(g,p) \in [1,\infty)\).

From the proof of \((3.6)\) in Step 1 we have
\[
\|s|\Phi|^2\|_{L^r(X)} \leq z\|s\|_{C(X)} \|\Phi\|^2_{L^p(X)} \|\Phi\|_{W^{1,p}(X)}
\]
Moreover, we have
\[
\|\Phi\|^2_{L^p(X)} = \|\Phi\|^2_{L^{2p}(X)} \leq z\|\Phi\|^2_{W^{1,p}(X)}.
\]
Combining the preceding inequalities yields \((3.12)\).

Step 5 \((L^r)\) estimates for \(a_1 \times a_1 \times a_1 \times a_1 \times \Phi\) and \(a_1 \times \Phi \times \Phi\). We claim that
\[
\|a_1 \times \Phi \times \Phi\|_{L^r(X)} \leq z\|a_1\|_{W^{1,p}(X)} \|\Phi\|^2_{W^{1,p}(X)},
\]
\[
\|a_1 \times a_1 \times \Phi\|_{L^r(X)} \leq z\|a_1\|_{W^{1,p}(X)} \|\Phi\|^2_{W^{1,p}(X)},
\]
\[
\|a_1 \times a_1 \times a_1\|_{L^r(X)} \leq z\|a_1\|^3_{W^{1,p}(X)},
\]
where \(z = z(g,p) \in [1,\infty)\).

From the proof of \((3.6)\) in Step 1 we have
\[
\|a_1 \times \Phi \times \Phi\|_{L^r(X)} \leq z\|a_1\|_{W^{1,p}(X)} \|\Phi\|_{L^p(X)}.
\]
Continuity of the multiplication, \(L^{2p}(X) \times L^{2p}(X) \to L^p(X)\), and continuity of the Sobolev embedding, \(W^{1,p}(X) \subset L^{2p}(X)\), valid for any \(p \geq d/2\) (which we assume by hypothesis), gives
\[
\|\Phi \times \Phi\|_{L^p(X)} \leq z\|\Phi\|^2_{L^{2p}(X)} \leq z\|\Phi\|^2_{W^{1,p}(X)}.
\]
Combining the preceding inequalities yields \((3.13)\). The proofs of the estimates \((3.14)\) and \((3.15)\) are the same as that of \((3.13)\). This completes Step 5.

The estimates obtained in each of the preceding steps show that the map \((3.5)\) has the properties asserted in the statement of Claim \((3.4)\) and this completes its proof.

In particular, Claim \((3.4)\) implies that
\[
\mathcal{M}(A_1 + a_1, \Phi) \in L^r(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]
is a continuous cubic polynomial in \((a_1, \Phi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E)\), with universal coefficients depending at most on \(g\) and \(G\). Because

\[
L'(X; \Lambda^1 \otimes \text{ad}P \oplus E) \subset W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]

is a continuous Sobolev embedding by our choice of \(r'\), we see that

\[
\mathcal{M}(A_1 + a_1, \Phi) \in W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]

is a cubic polynomial in \((a_1, \Phi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E)\), again with universal coefficients depending at most on \(g\) and \(G\). This completes the proof of Proposition 3.3 \qed

3.1.2. Fredholm and index properties of the Hessian operator for the boson coupled Yang-Mills energy functional on the Sobolev space of pairs. Consider the Hessian map, \(\mathcal{M}'(A, \Phi) : \mathfrak{X} \rightarrow \mathfrak{X}^*\).

**Lemma 3.5** (Hessian and Hessian operator for the boson coupled Yang-Mills \(L^2\)-energy functional). Assume the hypotheses of Proposition 3.7. Then we have the schematic formula for \(\mathcal{M}(A, \Phi) \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)\) and \(\mathcal{M}''(A, \Phi) \in (\mathfrak{X} \times \mathfrak{X})^*\) given by

\[
\mathcal{M}'(A, \Phi)(a, \phi) = d_A^* d_A a + \nabla_A^1 \nabla_A \phi + F_A \times a + \nabla_A^1 (h(a) \Phi) + \Phi \times \nabla_A \phi
\]

\[
- \rho(a)^2 \nabla_A \Phi + \nabla_A \Phi \times \phi + h(a) \Phi \times \Phi
\]

\[\text{where}
\]

\[
\mathcal{M}''(A, \Phi)(a, \phi)(b, \varphi) = \langle (b, \varphi), \mathcal{M}'(A, \Phi)(a, \phi) \rangle_{L^2(X)}, \quad \forall (a, \phi), (b, \varphi) \in \mathfrak{X}.
\]

**Proof.** Let \((a_i, \phi_i) \in \mathfrak{X}\), for \(i = 1, 2\). We compute the terms in \(\langle \mathcal{M}(A + ta_2, \Phi + t\phi_2), (a_1, \phi_1) \rangle_{L^2(X)}\) that are linear in \(t\) using the expression (1.10) for the gradient. First,

\[
(F_{A + ta_2}, d_{A + ta_2} a_1)_{L^2(X)} = \left( F_A + td_A a_2 + t^2 2 [a_2, a_2], d_A a_1 + t[a_2, a_1] \right)_{L^2(X)}
\]

\[
= (F_A, d_A a_1)_{L^2(X)} + t(F_A, [a_2, a_1])_{L^2(X)} + t(d_A a_2, d_A a_1)_{L^2(X)} + O(t^2).
\]

Second,

\[
(\nabla_{A + ta_2} (\Phi + t\phi_2), \nabla_{A + ta_2} \phi_1)_{L^2(X)}
\]

\[
= \langle (\nabla_A + t\rho(a_2))(\Phi + t\phi_2), \nabla_A \phi_1 + t\rho(a_2) \phi_1 \rangle_{L^2(X)}
\]

\[
= (\nabla_A \Phi, \nabla_A \phi_1)_{L^2(X)} + t(\nabla_A \phi_2, \nabla_A \phi_1)_{L^2(X)} + t(\rho(a_2) \Phi, \nabla_A \phi_1)_{L^2(X)}
\]

\[+ t(\nabla_A \Phi, \rho(a_2) \phi_1)_{L^2(X)} + O(t^2).
\]

Third,

\[
(\nabla_{A + ta_2} (\Phi + t\phi_2), \rho(a_1)(\Phi + t\phi_2))_{L^2(X)}
\]

\[
= \langle (\nabla_A + t\rho(a_2))(\Phi + t\phi_2), \rho(a_1)(\Phi + t\phi_2) \rangle_{L^2(X)}
\]

\[
= (\nabla_A \Phi, \rho(a_1) \Phi)_{L^2(X)} + t(\nabla_A \phi_2, \rho(a_1) \Phi)_{L^2(X)} + t(\rho(a_2) \Phi, \rho(a_1) \Phi)_{L^2(X)}
\]

\[+ t(\nabla_A \Phi, \rho(a_1) \phi_2)_{L^2(X)} + O(t^2).
\]
Fourth,
\[(m(\Phi + t\phi_2), \phi_1)_{L^2(X)} = (m\Phi, \phi_1)_{L^2(X)} + t(m\phi_2, \phi_1)_{L^2(X)}\].

Fifth,
\[\int_X s|\Phi + t\phi_2|^2 (\Phi + t\phi_2, \phi_1) \, d \text{vol}_g = \int_X s(|\Phi|^2 + 2t \text{Re}(\Phi, \phi_2) + t^2|\phi_2|^2) (\Phi + t\phi_2, \phi_1) \, d \text{vol}_g \]
\[= \int_X s|\Phi|^2 (\Phi, \phi_1) \, d \text{vol}_g \]
\[+ t \int_X (s|\phi_2|^2 (\phi_2, \phi_1) + 2s \text{Re}(\Phi, \phi_2) (\Phi, \phi_1)) \, d \text{vol}_g + O(t^2)\].

By subtracting \( (\mathcal{M}'(A, \Phi)(a_2, \phi_2), (a_1, \phi_1))_{L^2(X)} \), collecting all the first-order terms in \( t \), and reversing the roles of \( (a_1, \phi_1) \) and \( (a_2, \phi_2) \), we see that
\[ (\mathcal{M}'(A, \Phi)(a_1, \phi_1), (a_2, \phi_2))_{L^2(X)} = (d_Aa_1, d_Aa_2)_{L^2(X)} + 2(F_A, [a_1, a_2])_{L^2(X)} \]
\[+ \text{Re}(\nabla A\phi_1, \nabla A\phi_2)_{L^2(X)} \]
\[+ \text{Re}((\rho(a_1)\Phi, \nabla A\phi_2)_{L^2(X)} + (\rho(a_2)\Phi, \nabla A\phi_1)_{L^2(X)}) \]
\[+ \text{Re}(\nabla A\Phi, \rho(a_1)\phi_2 + \rho(a_2)\phi_1)_{L^2(X)} \]
\[+ \text{Re}(\rho(a_1)\Phi, \rho(a_2)\Phi)_{L^2(X)} \]
\[= \int_X ((m + 2s|\Phi|^2) (\phi_1, \phi_2) + 4s(\Phi, \phi_1)(\Phi, \phi_2)) \, d \text{vol}_g, \]

By now viewing \( \mathcal{M}'(A, \Phi)(a_1, \phi_1) \) as an element of \( X^* \), we obtain the expression \( 3.16 \) \( \square \).

When \( (A, \Phi) \) is a \( C^\infty \) pair, we shall need to compare \( \mathcal{M}'(A, \Phi) \) with the \( L^2 \)-self-adjoint, second-order partial differential operator,
\[(3.18)\quad \mathcal{M}'(A, \Phi) + d_{A, \Phi}d_{A, \Phi}^* : C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E), \]
in order to prove that \( \mathcal{M}'(A, \Phi) \) is Fredholm with index zero upon restriction to
\[(3.19)\quad \mathcal{X}^* := \text{Ker} \left(d_{A, \Phi}^* : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to L^p(X; \text{ad}P) \right). \]

We recall from \( 2.60 \) that
\[d_{A, \Phi}\xi = (d_A\xi, -\xi\Phi), \quad \forall \xi \in C^\infty(X; \text{ad}P), \]
with \( L^2 \)-adjoint,
\[(3.20)\quad d_{A, \Phi}^*(a, \phi) = d_A^*a - \langle \phi, \cdot\Phi \rangle^*, \quad \forall (a, \phi) \in C^\infty(\Lambda^1 \otimes \text{ad}P \oplus E), \]
for every \( (a, \phi) \in C^\infty(\Lambda^1 \otimes \text{ad}P \oplus E) \), where the section \( \langle \phi, \cdot\Phi \rangle^* \) of \( \text{ad}P \) is defined by
\[\langle \phi, \cdot\Phi \rangle^* : C^\infty(X; \text{ad}P) \to \langle \phi, \Phi \rangle_{L^2(X)} = (\phi, \xi\Phi)_{L^2(X)}, \quad \forall \xi \in C^\infty(X; \text{ad}P). \]

According to Lemma \( 2.21 \), the operator,
\[d_{A, \Phi}^* : W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to L^p(X; \text{ad}P), \]
is bounded when \( (A, \Phi) \) is a \( W^{1,q} \) pair with \( q \geq d/2 \) and \( p \) obeys \( d/2 \leq p \leq q \); therefore \( \mathcal{X}^* \) in \( 3.19 \) is a Banach space since it is a closed subspace of the Banach space \( \mathcal{X} = W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \).
When \( (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E) \), the expression (3.16) yields, after formally expanding \( \nabla^*_A(\varrho(a)\Phi) = \nabla_A a \times \Phi + a \times \nabla_A \Phi + a \times \Phi, \)

\[
\mathcal{M}'(A, \Phi)(a, \phi) = d_A^*a d_A a + \nabla^*_A \nabla_A \phi + F_A \times a + \nabla_A a \times \Phi + a \times \nabla_A \Phi + \Phi \times \nabla_A \phi \\
+ \varrho(a)\Phi + \nabla_A \Phi \times \phi + \varrho(a)\Phi \times \Phi - (m + 2s|\Phi|^2)\phi - 4s\langle \Phi, \phi \rangle \Phi. 
\] (3.21)

To determine the Fredholm property and index of \( \mathcal{M}'(A, \Phi) \) upon restriction to a Coulomb-gauge slice, we shall need the following consequence of Theorem A.1.

**Proposition 3.6** (Fredholm and index zero property of the augmented Hessian operator). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group. Let \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( E = P \times_{\varrho} \mathbb{E} \) be a smooth Hermitian vector bundle over \( X \) defined by a finite-dimensional unitary representation, \( \varrho : G \to \text{Aut}_\mathbb{C}(\mathbb{E}) \), and \( A \) is a \( C^\infty \) reference connection on \( P \). If \((A, \Phi)\) is a \( C^\infty \) pair on \((P, E)\) and \( k \in \mathbb{Z} \) is an integer and \( 1 < p < \infty \), then the following operator is Fredholm with index zero,

\[
\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^* : W^{k+2,p}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to W^{k,p}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E). 
\] (3.22)

**Proof.** We can compare the principal symbol of the connection Laplacian, \( \nabla_A^* \nabla_A \), and Hodge Laplacian, \( \Delta_A = d_A^* d_A + d_A d_A^* \) (2.2), on \( C^\infty(X; \Lambda^1 \otimes \text{ad}P) = \Omega^1(X; \text{ad}P) \) using the Bochner-Weitzenböck formula [71, 8], [37 Appendix C], [68 Appendix II] and [103]. From [68 Corollary II.2], one has

\[
\Delta_A a = \nabla_A^* \nabla_A a + \text{Ric}_g \times a + F_A \times a, \quad \forall a \in \Omega^1(X; \text{ad}P),
\] (3.23)

where \( \text{Ric}_g \) denotes the Ricci curvature tensor of the Riemannian metric \( g \) on the manifold \( X \) of dimension \( d \geq 2 \) and, as usual, we employ \( \times \) to denote any universal bilinear expression with constant coefficients depending at most on the Lie group, \( G \). In particular, \( \Delta_A \) is a second-order, elliptic partial differential operator on \( C^\infty(X; \Lambda^1 \otimes \text{ad}P) \) with \( C^\infty \) coefficients and scalar principal symbol given by the Riemannian metric \( g \) on \( T^*M \).

From the expression (3.21) for \( \mathcal{M}'(A, \Phi) \) and (2.60) for \( d_{A,\phi} \) and (3.20) for \( d^{*}_{A,\phi} \), we see that

\[
\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^* = \Delta_A \oplus \nabla_A^* \nabla_A + \text{Lower-order terms}.
\]

Thus, \( \mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^* \) is an elliptic, second-order partial differential operator on \( C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E) \) with \( C^\infty \) coefficients and principal symbol given by the Riemannian metric \( g \) on \( T^*M \).

The expression (3.16) for \( \mathcal{M}'(A, \Phi) \) implies that, for any \( (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E) \),

\[
(\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^*) - (\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^*)^* (a, \phi) \\
= F_A \times a + \nabla^*_A(\varrho(a)\Phi) + \Phi \times \nabla_A \phi - \varrho(a)^* \nabla_A \Phi + \nabla_A \Phi \times \phi + \varrho(a)\Phi \times \Phi \\
- (m + 2s|\Phi|^2)\phi - 4s\langle \Phi, \phi \rangle \Phi.
\]

Consequently, the following expression defines a first-order partial differential operator,

\[
\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^* - (\mathcal{M}'(A, \Phi) + d_{A,\phi}d_{A,\phi}^*)^* 
\]

and Theorem [A.1] implies that the operator (3.22) is Fredholm with index zero. \( \square \)
3.1.3. Gradient map and Hessian operator for the boson coupled Yang-Mills energy functional on a Coulomb-gauge slice. Suppose that $(A_\infty, \Phi_\infty)$ is a $C^\infty$ pair on $(P, E)$, recall that $x$ is as in (3.3) and $\mathcal{X}$ is as in (3.19), and define
\begin{align}
\tilde{x} &:= W_{A_1}^{-1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E), \tag{3.24}
\tilde{\mathcal{X}} &:= \text{Ker} \left( d^*_{A_\infty, \Phi_\infty} : W_{A_1}^{-1,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to W_{A_1}^{-2,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \right). \tag{3.25}
\end{align}

Because $(W^{1,p}(X; \mathbb{C}))^* = W^{-1,p'}(X; \mathbb{C})$ and $p \geq p'$ for all $p \geq 2$, we see that
$$\tilde{\mathcal{X}} \subset \mathcal{X}^*, \quad \forall p \geq 2.$$ For a $C^\infty$ pair $(A, \Phi)$ on $(P, E)$, the Hessian operator, $\mathcal{M}'(A, \Phi) : \mathcal{X} \to \mathcal{X}^*$, is defined by the schematic expression (3.16) and related to the Hessian, $\mathcal{E}''(A, \Phi) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, by
\begin{align*}
\mathcal{E}''(A, \Phi)((a, \phi), (b, \varphi)) &= \langle (b, \varphi), \mathcal{M}'(A, \Phi)(a, \phi) \rangle_{\mathcal{X} \times \mathcal{X}^*}, \\
&= \langle (b, \varphi), \mathcal{M}'(A, \Phi)(a, \phi) \rangle_{\mathcal{X}^*}, \quad \forall (a, \phi), (b, \varphi) \in \mathcal{X},
\end{align*}
where we define
\begin{equation}
\mathcal{H} := L^2(X; \Lambda^1 \otimes \text{ad}P \oplus E). \tag{3.26}
\end{equation}

According to Theorem [A.1], the elliptic, linear, second-order partial differential operator,
\begin{equation*}
d^*_{A, \Phi}d_{A, \Phi} : W_{A_1}^{k+2,p}(X; \text{ad}P) \to W_{A_1}^{k,p}(X; \text{ad}P),
\end{equation*}
is Fredholm for any $k \in \mathbb{Z}$ and $p \in (1, \infty)$ with kernel,
\begin{equation*}
K := \text{Ker} \left( d^*_{A, \Phi}d_{A, \Phi} : C^\infty(X; \text{ad}P) \to C^\infty(X; \text{ad}P) \right),
\end{equation*}
and range,
\begin{equation*}
\text{Ran} \left( d^*_{A, \Phi}d_{A, \Phi} : W_{A_1}^{k+2,p}(X; \text{ad}P) \to W_{A_1}^{k,p}(X; \text{ad}P) \right) = K^\perp \cap W_{A_1}^{k,p}(X; \text{ad}P),
\end{equation*}
where $\perp$ denotes $L^2$-orthogonal complement. Hence, the operator,
\begin{equation*}
d^*_{A, \Phi}d_{A, \Phi} : K^\perp \cap W_{A_1}^{k+2,p}(X; \text{ad}P) \to K^\perp \cap W_{A_1}^{k,p}(X; \text{ad}P),
\end{equation*}
is invertible, with inverse
\begin{equation*}
(d^*_{A, \Phi}d_{A, \Phi})^{-1} : K^\perp \cap W_{A_1}^{k,p}(X; \text{ad}P) \to K^\perp \cap W_{A_1}^{k+2,p}(X; \text{ad}P).
\end{equation*}

We define the Green’s operator,
\begin{equation*}
G_{A, \Phi} : W_{A_1}^{k,p}(X; \text{ad}P) \to W_{A_1}^{k+2,p}(X; \text{ad}P),
\end{equation*}
for the Laplacian, $d^*_{A, \Phi}d_{A, \Phi}$, by setting
\begin{equation*}
G_{A, \Phi} \xi := \begin{cases} 
(d^*_{A, \Phi}d_{A, \Phi})^{-1} \xi, & \forall \xi \in K^\perp \cap W_{A_1}^{k,p}(X; \text{ad}P), \\
0, & \forall \xi \in K.
\end{cases}
\end{equation*}

For any $k \in \mathbb{Z}$ and $p \in (1, \infty)$, we now let
\begin{equation}
\Pi_{A_\infty, \Phi_\infty} : W_{A_1}^{k,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to \text{Ker} d^*_{A_\infty, \Phi_\infty} \cap W_{A_1}^{k,p}(X; \Lambda^1 \otimes \text{ad}P \oplus E). \tag{3.27}
\end{equation}
denote the $L^2$-orthogonal projection onto the slice through $(A_\infty, \Phi_\infty)$. Because
\[
W_{A_1}^{k,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E)
= \text{Ker} \, d^*_{A_\infty, \Phi_\infty} \cap W_{A_1}^{k,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E)
\]
\[
\oplus \text{Ran} \left( d^*_{A_\infty, \Phi_\infty} : W_{A_1}^{k+1,p}(X; \text{ad} P \oplus E) \to W_{A_1}^{k,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \right)
\]
is an $L^2$-orthogonal direct sum, we see that
\[
(3.28) \quad \Pi_{A_\infty, \Phi_\infty} = \text{id} - d^*_{A_\infty, \Phi_\infty} G_{A_\infty, \Phi_\infty} d^*_{A_\infty, \Phi_\infty}.
\]
Because of the invariance of the energy functional, $\mathcal{E} : \mathcal{P}(P, E) \to \mathbb{R}$, in \([1,3]\) with respect to the action of $\text{Aut}(P)$ on $C^\infty$ pairs, $\mathcal{P}(P, E) = \mathcal{A}(P) \times C^\infty(X; E)$, we have the identity
\[
(3.29) \quad \mathcal{E}(u(A, \Phi)) = \mathcal{E}(A, \Phi), \quad \forall u \in \text{Aut}(P) \text{ and } (A, \Phi) \in \mathcal{P}(P, E).
\]
Note that if $u(t) \in \text{Aut}(P)$ is a family of gauge transformations depending smoothly on $t \in \mathbb{R}$ such that $u(0) = \text{id}_P$, then the identity \((3.29)\) implies that
\[
\mathcal{E}'(A, \Phi)(d_{A, \Phi} \xi) = \left. \frac{d}{dt} \mathcal{E}(u(t)(A, \Phi)) \right|_{t=0} = 0, \quad \forall (A, \Phi) \in \mathcal{P}(P, E),
\]
where
\[
d_{A, \Phi} \xi = \left. \frac{d}{dt} u(t)(A, \Phi) \right|_{t=0} \in C^\infty(X; \Lambda^1 \otimes \text{ad} P \oplus E),
\]
with $\xi = \dot{u}(0) \in C^\infty(X; \text{ad} P) = T_{\text{id}_P} \text{Aut}(P) = T_{\text{id}_P} C^\infty(X; \text{Ad} P)$. Before considering higher-order derivatives of the identity \((3.29)\) with respect to $u \in \text{Aut}(P)$, we digress to discuss the Chain Rule for maps of Banach spaces.

If $F : \mathcal{Y} \to \mathcal{Z}$ and $G : \mathcal{X} \to \mathcal{Y}$ are $C^\infty$ maps of Banach spaces, then the Chain Rule gives
\[
(3.30) \quad (F \circ G)'(x) = F'(G(x)) \circ G'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Z}), \quad \forall x \in \mathcal{X},
\]
\[
(3.31) \quad (F \circ G)''(x) = F''(G(x)) \circ G'(x)^2 + F'(G(x)) \circ G''(x) \in \mathcal{L}(\mathcal{X} \times \mathcal{X}, \mathcal{Z}), \quad \forall x \in \mathcal{X}.
\]
More explicitly, if $u, v \in \mathcal{X}$, then
\[
(F \circ G)'(x)(u) = F'(G(x))(G'(x)(u)),
\]
\[
(F \circ G)''(x)(u, v) = F''(G(x))(G'(x)(u), G'(x)(v)) + F'(G(x))(G''(x)(u, v)).
\]
The expression for the Hessian of the composition simplifies when $F'(y) = 0$ at $y = G(x)$ to give
\[
(3.32) \quad (F \circ G)''(x) = F''(G(x)) \circ G'(x)^2 \in \mathcal{L}(\mathcal{X} \times \mathcal{X}, \mathcal{Z}).
\]
This ends our digression on the Chain Rule for $C^\infty$ maps of Banach spaces.

By computing the first-order differential with respect to $u$ of the expression \((3.29)\) at $\text{id}_P \in \text{Aut}(P)$ in directions $\xi \in T_{\text{id}_P} \text{Aut}(P) = C^\infty(X; \text{ad} P)$ and recalling the definition \((2.61)\) of $d_{A, \Phi} \xi$, we see that the first-order differential of $\mathcal{E}$ and its gradient map obey
\[
(3.33) \quad \mathcal{E}'(A, \Phi)(d_{A, \Phi} \xi) = 0 = (d_{A, \Phi} \xi, \mathcal{M}(A, \Phi))_{L^2(\mathcal{X})}, \quad \forall \xi \in C^\infty(X; \text{ad} P),
\]
and thus
\[
d^*_{A, \Phi} \mathcal{M}(A, \Phi) = 0, \quad \forall (A, \Phi) \in \mathcal{P}(P, E).
\]
Next, we observe that (3.21) implies that the differential of the energy functional satisfies
\[(3.34)\quad \mathcal{E}'(u(A, \Phi))(u(a, \phi)) = \mathcal{E}'(A, \Phi)(a, \phi),\]
\[\forall u \in \text{Aut}(P) \text{ and } (A, \Phi) \in \mathcal{P}(P, E) \text{ and } (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E).\]
By computing the first-order differentials with respect to \(u\) of the expression (3.34) at \(id_P \in \text{Aut}(P)\) in directions \(\xi \in T_{id_P} \text{Aut}(P) = C^\infty(X; \text{ad}P)\) and recalling the definition (2.61) of \(d_A \Phi\), we see that the gradient map and second-order differential obey
\[\mathcal{E}''(A, \Phi) (d_A \Phi \xi, (a, \phi)) + \mathcal{E}'(A, \Phi)(d_A \Phi \xi) = 0,\]
and thus, by (3.33) and the fact that the Hessian \(\mathcal{E}''(A, \Phi)\) is a symmetric operator,
\[\mathcal{E}''(A, \Phi) (d_A \Phi \xi, (a, \phi)) = 0 = \mathcal{E}''(A, \Phi) (d_A \Phi \xi, (a, \phi)),\]
\[\forall (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E).\]
Hence, by the relation (3.17) between \(\mathcal{E}''(A, \Phi)\) and \(\mathcal{M}'(A, \Phi)\),
\[(d_A \Phi \xi, \mathcal{M}'(A, \Phi)(a, \phi))_{L^2(X)} = 0 = ((a, \phi), \mathcal{M}'(A, \Phi)d_A \Phi \xi)_{L^2(X)},\]
\[\forall (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E).\]
Consequently, the Hessian operator satisfies
\[(3.35a)\quad d_A^* \mathcal{M}'(A, \Phi)(a, \phi) = 0, \quad \forall (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E).\]
\[(3.35b)\quad \mathcal{M}'(A, \Phi)d_A \Phi \xi = 0, \quad \forall \xi \in C^\infty(X; \text{ad}P).\]
Thus, for any \(k \in \mathbb{Z}\) and \(p \in (1, \infty)\) when \((A_\infty, \Phi_\infty)\) is a \(C^\infty\) pair, the Hessian operator,
\[\mathcal{M}'(A_\infty, \Phi_\infty) : W^{k+2,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \rightarrow W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),\]
naturally restricts to the Coulomb-gauge slice domain and range defined by \((A_\infty, \Phi_\infty)\),
\[(3.36)\quad \mathcal{M}'(A_\infty, \Phi_\infty) : \text{Ker} \, d_{A_\infty, \Phi_\infty}^* \cap W^{k+2,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \rightarrow \text{Ker} \, d_{A_\infty, \Phi_\infty}^* \cap W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),\]
when the Hessian operator is also computed at the pair \((A_\infty, \Phi_\infty)\). The gradient map,
\[\mathcal{M} : (A_\infty, \Phi_\infty) + W^{k+2,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \rightarrow W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),\]
restricts to the slice domain and range with the \(L^2\)-orthogonal projection, \(\Pi_{A_\infty, \Phi_\infty}\),
\[(3.37)\quad \mathcal{M} \equiv \Pi_{A_\infty, \Phi_\infty} \mathcal{M} : (A_\infty, \Phi_\infty) + \text{Ker} \, d_{A_\infty, \Phi_\infty}^* \cap W^{k+2,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \rightarrow \text{Ker} \, d_{A_\infty, \Phi_\infty}^* \cap W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E).\]
The definition (3.27) of the \(L^2\)-orthogonal projection, \(\Pi_{A, \Phi}\), and the relation (3.33) yield
\[\Pi_{A, \Phi} \mathcal{M}(A, \Phi) = \mathcal{M}(A, \Phi), \quad \forall (A, \Phi) \in \mathcal{P}(P, E).\]
In the definition (3.37), we suppress dependence on the pair \((A_\infty, \Phi_\infty)\) in the choice of Coulomb-gauge slice from the notation for \(\mathcal{M}\), which we may regard as a lift to a coordinate chart on an open neighborhood of the point \([A, \Phi] \in \mathcal{C}(P, E) := \mathcal{P}(P, E)/\text{Aut}(P)\) of the gradient map on the quotient.
We apply Proposition 3.6 to prove the Fredholm and index zero properties of the Hessian operator.

**Proposition 3.7** (Fredholm and index zero properties of the Hessian operator for the boson coupled Yang-Mills $L^2$-energy functional on a Coulomb-gauge slice). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times \mathbb{R}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $g : G \to \text{Aut}_\mathbb{C}(\mathbb{E})$, and $A_1$ is a $C^\infty$ reference connection on $P$, and $m, s \in C^\infty(X)$. If $(A, \Phi)$ is a $C^\infty$ pair on $(P, E)$ and $k \in \mathbb{Z}$ is an integer and $1 < p < \infty$, then the following operator is Fredholm with index zero,

$$\mathcal{M}'(A, \Phi) : \ker d^*_{A, \Phi} \cap \mathcal{W}^{k+2,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to \ker d^*_{A, \Phi} \cap \mathcal{W}^{k,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E).$$

**Proof.** To reduce notational clutter, we abbreviate the domain and range Sobolev spaces by

$$\mathcal{W}^{k+2,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \quad \text{and} \quad \mathcal{W}^{k,p}(X; \Lambda^1 \otimes \text{ad} P \oplus E).
$$

We shall adapt the argument of Råde [82, p. 148]. Observe that $(3.38)$ implies that the operator $(3.40)$ is clearly bounded. For injectivity, if $\eta = d_{A, \Phi} \xi \in \ker d^*_{A, \Phi} \cap \mathcal{W}^{k+2,p}$ for $\xi \in \mathcal{W}^{k+3,p}$ and $d^*_{A, \Phi} d_{A, \Phi} \xi = 0$, then $(d^*_{A, \Phi} d_{A, \Phi})^2 \xi = 0$. We may assume without loss of generality that $\xi \perp \ker d^*_{A, \Phi}$ and because $d^*_{A, \Phi} d_{A, \Phi} : \mathcal{W}^{k+2,p} \to \mathcal{W}^{k,p}$ is Fredholm with index zero by Theorem A.1, the following operator is invertible,

$$d_{A, \Phi} : \ker d^*_{A, \Phi} \cap \mathcal{W}^{k+2,p} \to (\ker d^*_{A, \Phi} \cap \mathcal{W}^{k+2,p}) \perp \mathcal{W}^{k,p}.$$ 

Indeed, the operator $(3.40)$ is clearly bounded. For injectivity, if $\eta = d_{A, \Phi} \xi \in \ker d^*_{A, \Phi} \cap \mathcal{W}^{k+2,p}$ for $\xi \in \mathcal{W}^{k+3,p}$ and $d^*_{A, \Phi} d_{A, \Phi} \xi = 0$, then $(d^*_{A, \Phi} d_{A, \Phi})^2 \xi = 0$. We may assume without loss of generality that $\xi \perp \ker d^*_{A, \Phi}$ and because $d^*_{A, \Phi} d_{A, \Phi} : \mathcal{W}^{k+2,p} \to \mathcal{W}^{k,p}$ is Fredholm with index zero by Theorem A.1, the following operator is invertible,

$$d^*_{A, \Phi} d_{A, \Phi} : (\ker d^*_{A, \Phi} d_{A, \Phi}) \perp \mathcal{W}^{k+2,p} \to (\ker d^*_{A, \Phi} d_{A, \Phi}) \perp \mathcal{W}^{k,p}.$$ 

Thus, $\xi = 0$ and the operator $(3.40)$ is injective. For surjectivity, suppose $\chi = d_{A, \Phi} \zeta \perp \ker d^*_{A, \Phi} \cap \mathcal{W}^{k,p}$ for $\zeta \in \mathcal{W}^{k+1,p}$. We may again assume without loss of generality that $\zeta \perp \ker d^*_{A, \Phi}$ and because $d^*_{A, \Phi} d_{A, \Phi} : \mathcal{W}^{k+2,p} \to \mathcal{W}^{k,p}$ has closed range (because it is a Fredholm operator), this implies that the operator $(3.40)$ is surjective and thus invertible by the Open Mapping Theorem.
According to Proposition 3.6 the operator (3.38) is Fredholm. Consequently, \( \mathcal{M}'(A, \Phi) : \text{Ker} \, d_{A,\Phi}^* \cap W^{k+2,p} \to \text{Ker} \, d_{A,\Phi}^* \cap W^{k,p} \) is Fredholm by virtue of the direct sum decomposition (3.39) and invertibility of the operator (3.40). We compute indices,

\[
\text{Index}
\begin{pmatrix}
\mathcal{M}'(A, \Phi) + d_{A,\Phi}d_{A,\Phi}^*
\end{pmatrix}
: W^{k+2,p} \to W^{k,p}
\]

\[
= \text{Index}
\begin{pmatrix}
\mathcal{M}'(A, \Phi) & 0 \\
0 & d_{A,\Phi}d_{A,\Phi}^*
\end{pmatrix}
: (\text{Ker} \, d_{A,\Phi}^* \oplus \text{Ran} \, d_{A,\Phi}) \cap W^{k+2,p}
\]

\[
\to \text{Ker} \, d_{A,\Phi}^* \oplus \text{Ran} \, d_{A,\Phi} \cap W^{k,p}
\]

\[
= \text{Index}
\begin{pmatrix}
\mathcal{M}'(A, \Phi) : \text{Ker} \, d_{A,\Phi}^* \cap W^{k+2,p} \to \text{Ker} \, d_{A,\Phi}^* \cap W^{k,p}
\end{pmatrix}
+ \text{Index}
\begin{pmatrix}
d_{A,\Phi}d_{A,\Phi}^* : \text{Ran} \, d_{A,\Phi} \cap W^{k+2,p} \to \text{Ran} \, d_{A,\Phi} \cap W^{k,p}
\end{pmatrix}.
\]

Therefore, because \( \text{Index}(d_{A,\Phi}d_{A,\Phi}^* : \text{Ran} \, d_{A,\Phi} \cap W^{k+2,p} \to \text{Ran} \, d_{A,\Phi} \cap W^{k,p}) = 0, \)

\[
\text{Index}
\begin{pmatrix}
\mathcal{M}'(A, \Phi) : \text{Ker} \, d_{A,\Phi}^* \cap W^{k+2,p} \to \text{Ker} \, d_{A,\Phi}^* \cap W^{k,p}
\end{pmatrix}
= \text{Index}
\begin{pmatrix}
\mathcal{M}'(A, \Phi) + d_{A,\Phi}d_{A,\Phi}^* : W^{k+2,p} \to W^{k,p}
\end{pmatrix}.
\]

But the latter index is zero by Proposition 3.6 and this completes the proof of Proposition 3.7. \( \square \)

3.1.4. Analyticity of the gradient map for the boson coupled Yang-Mills \( L^2 \)-energy functional on a Coulomb-gauge slice. Suppose \((A_\infty, \Phi_\infty)\) and \((A, \Phi)\) are \( C^\infty \) pairs on \((P, E)\) and recall from (1.9) that the first-order differential and gradient map of the boson coupled Yang-Mills \( L^2 \)-energy functional (1.6) are related by

\[
\mathcal{E}'(A, \Phi)(a, \phi) = ((a, \phi), \mathcal{M}(A, \Phi))_{L^2(X)}, \quad \forall \, (a, \phi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E).
\]

If we now restrict \((a, \phi)\) to a pair in \( \text{Ker} \, d_{A,\Phi_\infty}^* \cap C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E) \), then the preceding relation yields

\[
\mathcal{E}'(A, \Phi)(a, \phi) = ((a, \phi), \Pi_{A,\Phi_\infty} \mathcal{M}(A, \Phi))_{L^2(X)} = ((a, \phi), \hat{\mathcal{M}}(A, \Phi))_{L^2(X)},
\]

\[
\forall \, (a, \phi) \in \text{Ker} \, d_{A,\Phi_\infty}^* \cap C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]

where \( \Pi_{A,\Phi_\infty} \) is the \( L^2 \)-orthogonal projection (3.27) onto the Coulomb-gauge slice through \((A_\infty, \Phi_\infty)\) and we appeal to the definition (3.37) of \( \hat{\mathcal{M}} \). Consequently,

\[
(3.41) \quad \hat{\mathcal{M}} = \Pi_{A,\Phi_\infty} \mathcal{M} : (A_\infty, \Phi_\infty) + \text{Ker} \, d_{A,\Phi_\infty}^* \cap W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E)
\]

\[
\to \text{Ker} \, d_{A,\Phi_\infty}^* \cap W^{-1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E),
\]

is the gradient map for the restriction,

\[
\mathcal{E} : (A_\infty, \Phi_\infty) + \text{Ker} \, d_{A,\Phi_\infty}^* \cap W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad}P \oplus E) \to \mathbb{R}.
\]
of the boson coupled Yang-Mills $L^2$-energy functional to the Coulomb-gauge slice through $(A_\infty, \Phi_\infty)$. Note that the relations (3.35) imply that the Hessian operator (namely, the derivative of the gradient map, $\mathcal{H}$) at $(A_\infty, \Phi_\infty)$ simplifies to

$$\mathcal{H}'(A_\infty, \Phi_\infty) = (\Pi_{A_\infty, \Phi_\infty} \circ \mathcal{H})'((A_\infty, \Phi_\infty) : \text{Ker} d_{A_\infty, \Phi_\infty}^* \cap W^{1,p}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E) \rightarrow \text{Ker} d_{A_\infty, \Phi_\infty}^* \cap W^{-1,p}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E).$$

Proposition 3.3 yields the

**Corollary 3.8** (Analyticity of the gradient map for the boson coupled Yang-Mills $L^2$-energy functional on a Coulomb-gauge slice). Assume the hypotheses of Proposition 3.3 and let $(A_\infty, \Phi_\infty)$ be a $C^\infty$ pair on $(P, E)$. Then the following map is analytic,

$$\text{Ker} d_{A_\infty, \Phi_\infty}^* \cap W^{1,p}_A \ni (a_1, \phi_1) \mapsto \mathcal{H}(A_\infty + a_1, \Phi_\infty + \phi_1) = \Pi_{A_\infty, \Phi_\infty} \mathcal{H}(A_\infty + a_1, \Phi_\infty + \phi_1) \in \text{Ker} d_{A_\infty, \Phi_\infty}^* \cap W^{-1,p}_A,$$

where we abbreviate

$$W^{\pm 1,p}_A = W^{\pm 1,p}_A(X; \Lambda^1 \otimes \text{ad}P \oplus E).$$

**Proof.** The conclusion follows from Proposition 3.3 and the fact that the operators, $\Pi_{A_\infty, \Phi_\infty}$, define continuous projections.

### 3.1.5. Estimates for gauge transformations intertwining two pairs

We shall require the

**Lemma 3.9** (Estimate for the action of a $W^{2,q}$ gauge transformation intertwining two $W^{1,q}$ pairs). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_{\varrho} E$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\varrho : G \rightarrow \text{Aut}_C(E)$, and $A_1$ is a $C^\infty$ reference connection on $P$, $q > d/2$ and $p$ obeys $d/2 \leq p \leq q$, then there is a constant $C = C(g, p) \in [1, \infty)$ with the following significance. If $(A, \Phi)$ and $(A', \Phi')$ are $W^{1,q}$ pairs on $(P, E)$ and $u \in \text{Aut}^{2,q}(P)$, then

$$\|u(A, \Phi) - u(A', \Phi')\|_{W^{1,p}_A(X)} \leq C \left(1 + \|u\|_{W^{2,p}_A(X)}\right) \|(A, \Phi) - (A', \Phi')\|_{W^{1,p}_A(X)},$$

and

$$\|(A, \Phi) - (A', \Phi')\|_{W^{1,p}_A(X)} \leq C \left(1 + \|u\|_{W^{2,p}_A(X)}\right) \|u(A, \Phi) - u(A', \Phi')\|_{W^{1,p}_A(X)}.$$ 

If in addition $p \geq 4/3$ when $d = 2$ and $p' \in [1, \infty)$ is the dual Hölder exponent defined by $1/p + 1/p' = 1$, and $(a, \phi) \in W^{1,p'}_A(X; \Lambda^1 \otimes \text{ad}P)$, then

$$\|u(a, \phi)\|_{W^{1,p'}_A(X)} \leq C \left(1 + \|u\|_{W^{2,p}_A(X)}\right) \|(a, \phi)\|_{W^{1,p'}_A(X)},$$

and

$$\|(a, \phi)\|_{W^{1,p'}_A(X)} \leq C \left(1 + \|u\|_{W^{2,p}_A(X)}\right) \|u(a, \phi)\|_{W^{1,p'}_A(X)}.$$
Proof. Recall that \( u(A) - A_1 = u^{-1}(A - A_1)u + u^{-1}d_{A_1}u \) by (2.27) and similarly for \( A' \), so
\[
u(A) - u(A') = u^{-1}(A - A_1)u - u^{-1}(A' - A_1)u = u^{-1}(A - A')u,
\]
and thus,
\[
u(A, \Phi) - u(A', \Phi') = (u^{-1}(A - A')u, u(\Phi - \Phi')).
\]
Therefore, writing \( a := A - A' \in W_{A_1}^{1,p}(X; \Lambda^1 \otimes \text{ad}P) \) for convenience,
\[
\nabla_{A_1}(u(A) - u(A')) = -u^{-1}(\nabla_{A_1}u)u^{-1}au + u^{-1}(\nabla_{A_1}a)u + u^{-1}(\nabla_{A_1}u).
\]
By taking \( L^p \) norms and using the pointwise bound \( |u| \leq 1 \), the Sobolev embedding \( W^{1,p}(X) \subset L^{2p}(X) \) (valid for \( p \geq d/2 \), and the Kato Inequality [37, Equation (6.20)], we obtain
\[
\|\nabla_{A_1}(u(A) - u(A'))\|_{L^p(X)} \leq 2\|\nabla_{A_1}u\|_{L^{2p}(X)}\|a\|_{L^{2p}(X)} + \|\nabla_{A_1}a\|_{L^p(X)} \leq C\|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}a\|_{L^p(X)}.
\]
Similarly, \( \nabla_{A_1}(u(\Phi - \Phi')) = (\nabla_{A_1}u)(\Phi - \Phi') + u(\nabla_{A_1}(\Phi - \Phi')) \) and
\[
\|\nabla_{A_1}(u(\Phi - \Phi'))\|_{L^p(X)} \leq \|\nabla_{A_1}u\|_{L^{2p}(X)}\|\Phi - \Phi'\|_{L^{2p}(X)} + \|\nabla_{A_1}(\Phi - \Phi')\|_{L^p(X)} \leq \|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)}\|\Phi - \Phi'\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}(\Phi - \Phi')\|_{L^p(X)}.
\]
By combining the preceding estimates, we obtain the first inequality; the second inequality is proved by a symmetric argument.

For the third inequality, we note that \( u(a, \phi) = (u^{-1}au, u(\Phi)) \) and use the identity,
\[
\nabla_{A_1}(u(a)) = -u^{-1}(\nabla_{A_1}u)u^{-1}au + u^{-1}(\nabla_{A_1}a)u + u^{-1}(\nabla_{A_1}u).
\]
We make the

**Claim 3.10.** Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \) and \( p \in [d/2, \infty], \) with \( p \geq 4/3 \) if \( d = 2 \). If \( p' \in [1, \infty) \) is the dual Hölder exponent defined by \( 1/p + 1/p' = 1 \), then there is a continuous Sobolev multiplication map,
\[
W^{1,p'}(X) \times W^{1,p}(X) \to L^{p'}(X).
\]

Given Claim 3.10, we have
\[
\|\nabla_{A_1}(u(a))\|_{L^{p'}(X)} \leq z\|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)}\|a\|_{W^{1,p}_{A_1}(X)} + \|\nabla_{A_1}a\|_{L^{p'}(X)};
\]
for \( z = z(g, p) \in [1, \infty) \), while
\[
\|u(a)\|_{L^{p'}(X)} = \|a\|_{L^{p'}(X)}.
\]
Thus,
\[
\|u(a)\|_{W^{1,p'}_{A_1}(X)} \leq z \left( 1 + \|\nabla_{A_1}u\|_{W^{1,p}_{A_1}(X)} \right) \|a\|_{W^{1,p'}_{A_1}(X)},
\]
with the analogous estimate for \( \|u(\phi)\|_{W^{1,p'}_{A_1}(X)} \). This yields the third inequality and symmetry yields the fourth inequality.
Proof of Claim 3.10. A multiplication, $L^r(X) \times L^s(X) \to L^{p'}(X)$, is continuous provided $r, s \in [p', \infty]$ obey $1/r + 1/s \leq 1/p'$. If these exponents $r, s$ also yield continuous Sobolev embeddings,

$$W^{1,p'}(X) \subset L^r(X) \quad \text{and} \quad W^{1,p}(X) \subset L^s(X),$$

then we obtain the desired continuous Sobolev multiplication map (8.3.2). To confirm the existence of suitable exponents, $r, s$, we shall consider the cases $p < d$, $p = d$, and $p > d$ separately.

**Case 1 ($p < d$).** Choose $s = p^* = dp/(d-p)$ to give the continuous Sobolev embedding, $W^{1,p}(X) \subset L^{p^*}(X)$, provided by [4, Theorem 4.12]. We required that $s \in [p', \infty]$, so $p$ must obey $p^* \geq p'$, that is

$$dp/(d-p) \geq p/(p-1),$$

or $p > 1$ and $d(p-1) \geq d-p$ or $p(d+1) \geq 2d$ and so we require that $p \geq 2d/(d+1)$ and $p > 1$. Note that $d/2 \geq 2d/(d+1) \quad \iff \quad d+1 \geq 4$, that is, $d \geq 3$; this is why for $d = 2$ we augment the hypothesis $p \geq d/2$ in Claim 3.10 by $p \geq 4/3$. Because $p < d$, we also have $p' = p/(p-1) > d/(d-1)$. We consider three subcases, depending on whether $p' < d$, $p' = d$, or $p' > d$.

Suppose $p' > d$. Then we have a continuous Sobolev embedding, $W^{1,p'}(X) \subset L^\infty(X)$, by [4, Theorem 4.12] and because $p^* \geq p'$ when $p \geq 2d/(d+1)$ and $p > 1$, we may choose any $r \in [1, \infty]$ large enough that $1/r + 1/p^* \leq 1/p'$.

Suppose $p' = d$. Then we have a continuous Sobolev embedding, $W^{1,p'}(X) \subset L^r(X)$, by [4, Theorem 4.12] for any $r \in [1, \infty)$ and because $p^* > p'$ when $p > 2d/(d+1)$ and $p > 1$, we may choose $r$ large enough that $1/r + 1/p^* \leq 1/p'$. (The condition $p' > d/(d-1)$ and $p' = d$ forces $d \geq 3$, so the subcase $p < d$ and $p' = d$ cannot occur for $d = 2$.)

Suppose $p' < d$. Then the condition $p' > d/(d-1)$ implies that $d/(d-1) < d$, that is, $d - 1 > 1$ or $d > 3$. (The subcase $p < d$ and $p' < d$ cannot occur for $d = 2$.) We choose $r = (p')^* = dp'/(d-p') = d(p/(p-1))/(d-p)/(p-1) = dp/(dp-d-p)$ and use the continuous Sobolev embedding, $W^{1,p'}(X) \subset L^{(p')^*}(X)$, provided by [4, Theorem 4.12]. To have a continuous multiplication, $L^{(p')^*}(X) \times L^{p^*}(X) \to L^{p'}(X)$, we see that $p > 1$ must obey

$$1/(p')^* + 1/p^* \leq 1/p',$$

that is,

$$(dp-d-p)/dp + (d-p)/p \leq (p-1)/p,$$

or

$$d(p-1) - p + d - p \leq d(p-1),$$

or $d \leq 2p$, that is, $p \geq d/2$. Combining the conclusions of the three subcases verifies the case $p < d$.

**Case 2 ($p = d$).** For any $s \in [1, \infty)$, we have a continuous Sobolev embedding, $W^{1,p}(X) \subset L^s(X)$. We also required that $s \in [p', \infty]$, so $p$ and $s$ must obey $p' \leq s < \infty$. Because $p = d$, we have $p' = p/(p-1) = d/(d-1)$, so $p' = d$ for $d = 2$ and $1 < p' < d$ for $d \geq 3$.

Suppose $p' = d$ and $d = 2$. Then $W^{1,p'}(X) \subset L^r(X)$ is a continuous Sobolev embedding for any $r \in [1, \infty)$. We also required that $r \in [p', \infty]$ and further restrict to $r \in (p', \infty)$ since $s$ is finite. In particular, we may choose $r, s \in (p', \infty)$ such that $1/r + 1/s \leq 1/p'$. 

Suppose $p' < d$ and $d \geq 3$. Then $W^{1,p'}(X) \subset L^r(X)$ is a continuous Sobolev embedding for $r = (p')^* = dp/(dp - d - p)$. We also required that $r \in [p', \infty]$ and further restrict to $r \in (p', \infty)$ since $s$ is finite, so $p$ must obey $p' < (p')^*$, that is

$$p/(p-1) < dp/(dp - d - p),$$

or $p > 1$ and $dp - d - p < d(p-1)$, or simply $p > 1$. In particular, we may choose $r = (p')^* \in (p', \infty)$ and then $s \in (p', \infty)$ large enough that $1/r + 1/s \leq 1/p'$.

Combining the conclusions of each of the two subcases verifies the case $p = d$.

Case 3 $(p > d)$. For any $s \in [1, \infty]$, we have a continuous Sobolev embedding, $W^{1,p}(X) \subset L^s(X)$. We also required that $s \in [p', \infty]$, so $p$ and $s$ must obey $p' \leq s \leq \infty$. Because $p > d$, we have $p' = p/(p-1) < d/(d-1)$, so $1 < p' < d$ for $d \geq 2$. Thus, $W^{1,p'}(X) \subset L^r(X)$ is a continuous Sobolev embedding for $r = (p')^* = dp/(dp - d - p)$. We also required that $r \in [p', \infty]$, so $p$ must obey $p' \leq (p')^*$, which holds for any $p > 1$ from our analysis of the case $p = d$. In particular, we may choose $r = (p')^* \in [p', \infty)$ and then $s \in (p', \infty]$ large enough that $1/r + 1/s \leq 1/p'$. This verifies the case $p > d$.

Combining these three cases completes the proof of Claim \ref{claim3.10}.

The third and fourth inequalities follow from Claim \ref{claim3.10} as described earlier, so this completes the proof of Lemma \ref{lemma3.9}.

3.1.6. Completion of the proof of Theorem \ref{thm3} We can now proceed to the proof of Theorem \ref{thm3}.

Proof of Theorem \ref{thm3} We first consider the simpler case where the pair $(A, \Phi)$ is in Coulomb gauge relative to the critical point $(A_\infty, \Phi_\infty)$ and then consider the general case.

Case 1 ($(A, \Phi)$ in Coulomb gauge relative to $(A_\infty, \Phi_\infty)$). By hypothesis, $(A_\infty, \Phi_\infty)$ is a $W^{1,q}$ pair that is a critical point for the functional $\mathcal{E}$ in \ref{eq1.6}. By the regularity Theorem \ref{thm2.23} there exists a $W^{2,q}$ gauge transformation $u_\infty$ such that $u_\infty(A_\infty, \Phi_\infty)$ is a $C^\infty$ pair. In particular, $u_\infty(A_\infty, \Phi_\infty)$ is a $W^{2,q}$ pair and $u_\infty(A, \Phi)$ is in Coulomb gauge relative to $u_\infty(A_\infty, \Phi_\infty)$. Following \ref{claim3.19} and \ref{claim3.25}, we choose the Banach spaces,

$$\mathcal{X} = \text{Ker} \left( d^*_u(A_\infty, \Phi_\infty) : W^{1,p}_A(X; \Lambda^1 \otimes \text{adP} \oplus E) \to L^p(X; \text{adP}) \right),$$

$$\mathcal{Z} = \text{Ker} \left( d^*_u(A_\infty, \Phi_\infty) : W^{-1,p}_A(X; \Lambda^1 \otimes \text{adP} \oplus E) \to W^{-2,p}_A(X; \text{adP}) \right).$$

Hence, $\mathcal{X} \subset \mathcal{Z}$ is a continuous embeddings of Banach spaces and

$$\mathcal{X}^* = \text{Ker} \left( d^*_u(A_\infty, \Phi_\infty) : L^p(X; \text{adP}) \to W^{-1,p}_A(X; \Lambda^1 \otimes \text{adP} \oplus E) \right).$$

We observe that $\mathcal{Z} \subset \mathcal{X}^*$ is a continuous embedding of Banach spaces when $W^{-1,p}(X; \mathbb{C}) \subset W^{-1,p'}(X; \mathbb{C})$ is a continuous Sobolev embedding and thus when $p$ obeys $p \geq 2$.

Proposition \ref{prop3.7} implies that the Hessian operator with $x_\infty = u_\infty(A_\infty, \Phi_\infty)$,

$$\mathcal{M}^t(x_\infty) : \mathcal{X} \to \mathcal{Z},$$

is Fredholm with index zero while Corollary \ref{cor3.3} implies that the gradient map,

$$\mathcal{M} : x_\infty + \mathcal{X} \to \mathcal{Z},$$
is analytic, where we recall from (3.37) that
\[ \hat{\mathcal{M}} = \Pi_{u_{\infty}(A_{\infty}, \Phi_{\infty})}. \]

Hence, Theorem 1 implies that there exist constants \( Z' \in (0, \infty) \) and \( \sigma' \in (0, 1] \) (depending on \( (A_1, \Phi_1) \), and \( u_{\infty}(A_{\infty}, \Phi_{\infty}) \), and \( g, G, p, P \)) such that if
\[ \|u_{\infty}(A, \Phi) - u_{\infty}(A_{\infty}, \Phi_{\infty})\|_{W^{1,p}_{A_1}(X)} < \sigma', \]
then
\[ |\mathcal{E}(u_{\infty}(A, \Phi)) - \mathcal{E}(u_{\infty}(A_{\infty}, \Phi_{\infty}))|^\theta \leq Z'\|\hat{\mathcal{M}}(u_{\infty}(A, \Phi))\|_{W^{-1,p}_{A_1}(X)}. \]

By Lemma 3.9 there exists \( C_1 = C_1(A_1, g, p, u_{\infty}) = C_1(A_1, A_{\infty}, \Phi_{\infty}, g, p) \in [1, \infty) \) so that
\[ \|u_{\infty}(A, \Phi) - u_{\infty}(A_{\infty}, \Phi_{\infty})\|_{W^{1,p}_{A_1}(X)} \leq C_1\|(A, \Phi) - (A_{\infty}, \Phi_{\infty})\|_{W^{-1,p}_{A_1}(X)}. \]

More explicitly, Lemma 3.9 gives \( C_1 = C(1 + \|u_{\infty}\|_{W^{-1,p}_{A_1}(X)}) \), where \( C = C(g, p) \in [1, \infty) \). Therefore, setting \( \sigma := C_1^{-1}\sigma' \), we see that if \( (A, \Phi) \) obeys the Lojasiewicz–Simon neighborhood condition (1.16), namely
\[ \|(A, \Phi) - (A_{\infty}, \Phi_{\infty})\|_{W^{1,p}_{A_1}(X)} < \sigma, \]
then (3.43) holds and thus also (3.44). Moreover,
\[ \|\hat{\mathcal{M}}(u_{\infty}(A, \Phi))\|_{W^{-1,p}_{A_1}(X)} = \|\hat{\mathcal{M}}(u_{\infty}(A, \Phi))\|_{(W^{1,p}_{A_1}(X))^\ast}, \]
\[ = \sup \left\{ |\hat{\mathcal{M}}(u_{\infty}(A, \Phi))(a, \phi)| : \|u_{\infty}(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq 1 \right\} \]
\[ = \sup \left\{ |\hat{\mathcal{M}}(A, \Phi)(a, \phi)| : \|u_{\infty}(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq 1 \right\} \text{ (by gauge invariance),} \]
where the supremum is over all pairs, \( (a, \phi) \in W^{1,p}_{A_1}(X; A^1 \otimes \text{ad} P) \), obeying the inequality. But
\[ \|(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq C_1\|u_{\infty}(a, \phi)\|_{W^{1,p}_{A_1}(X)} \]
by Lemma 3.9 and therefore,
\[ \left\{ (a, \phi) \in W^{1,p}_{A_1}(X; A^1 \otimes \text{ad} P) : \|u_{\infty}(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq 1 \right\} \]
\[ \subset \left\{ (a, \phi) \in W^{1,p}_{A_1}(X; A^1 \otimes \text{ad} P) : C_1^{-1}\|(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq 1 \right\}. \]

Combining the preceding equality and inequality yields,
\[ \|\hat{\mathcal{M}}(u_{\infty}(A, \Phi))\|_{W^{-1,p}_{A_1}(X)} \leq \sup \left\{ |\hat{\mathcal{M}}(A, \Phi)(a, \phi)| : C_1^{-1}\|(a, \phi)\|_{W^{1,p}_{A_1}(X)} \leq 1 \right\} \]
\[ = C_1\|\hat{\mathcal{M}}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)}. \]
Substituting the preceding inequality into (3.44) yields
\[ |\mathcal{E}(A, \Phi) - \mathcal{E}(A_\infty, \Phi_\infty)|^\theta = |\mathcal{E}(u_\infty(A, \Phi)) - \mathcal{E}(u_\infty(A_\infty, \Phi_\infty))|^{\theta} \quad \text{(by gauge invariance)} \]
\[ \leq Z' \|\mathcal{M}(u_\infty(A, \Phi))\|_{W^{-1,p}_{A_1}(X)} \quad \text{(by (3.44))} \]
\[ \leq Z'C_1\|\mathcal{M}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)}. \]

But \( \mathcal{M}(A, \Phi) = \Pi_{u_\infty(A_\infty, \Phi_\infty)} \mathcal{M} \) and because the projection,
\[ \Pi_{u_\infty(A_\infty, \Phi_\infty)} : W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to W^{k,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus E), \]
is bounded with norm one (for any \( k \in \mathbb{Z} \) and \( 1 < p < \infty \)), then
\[ \|\mathcal{M}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)} \leq \|\mathcal{M}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)}. \]

Hence, the Lojasiewicz–Simon gradient inequality (1.17) holds for the pairs \((A, \Phi)\) and \((A_\infty, \Phi_\infty)\) with constants \((Z, \theta, \sigma)\), where \( Z := C_1Z' \).

Case 2 \(((A, \Phi)\) not in Coulomb gauge relative to \((A_\infty, \Phi_\infty)\)). Let \( \zeta = \zeta(A_1, A_\infty, \Phi_\infty, g, G, p, q) \in (0, 1] \) and \( N = N(A_1, A_\infty, \Phi_\infty, g, G, p, q) \in [1, \infty) \) denote the constants in Theorem 10 and choose \( \zeta_1 \in (0, \zeta] \) small enough that \( 2N\zeta_1 < \sigma_1 \), where we now use \( \sigma_1 \) to denote the Lojasiewicz–Simon constant from Case 1. If \((A, \Phi)\) obeys
\[ \|(A, \Phi) - (A_\infty, \Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < \zeta_1, \]
then Theorem 10 provides \( u \in \text{Aut}^2(P) \), depending on the pair \((A, \Phi)\), such that
\[ d_{A_\infty, \Phi_\infty}(u(A, \Phi) - (A_\infty, \Phi_\infty)) = 0, \]
\[ \|u(A, \Phi) - (A_\infty, \Phi_\infty)\|_{W^{1,p}_{A_1}(X)} < 2N\zeta_1 < \sigma. \]

By applying Case 1 to the pairs \( u(A, \Phi) \) and \((A_\infty, \Phi_\infty)\), we obtain
\[ |\mathcal{E}(u(A, \Phi)) - \mathcal{E}(A_\infty, \Phi_\infty)|^{\theta} \leq C_1Z'||\mathcal{M}(u(A, \Phi))\|_{W^{-1,p}_{A_1}(X)}. \]

Estimating as in Case 1 with \( u \) replacing \( u_\infty \), we see that
\[ \|\mathcal{M}(u(A, \Phi))\|_{W^{-1,p}_{A_1}(X)} \leq C_2\|\mathcal{M}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)}, \]
where \( C_2 = C(1 + \|u\|_{W^{2,p}_{A_1}(X)}) \) and \( C = C(g, p) \in [1, \infty) \). According to Lemma 2.20 we have
\[ \|u\|_{W^{2,p}_{A_1}(X)} \leq C_3, \]
where \( C_3 = C_3(A_\infty, \Phi_\infty, A_1, g, G, p, q) \in [1, \infty) \). By combining the preceding inequalities, we obtain
\[ |\mathcal{E}(A, \Phi) - \mathcal{E}(A_\infty, \Phi_\infty)|^{\theta} = |\mathcal{E}(u(A, \Phi)) - \mathcal{E}(A_\infty, \Phi_\infty)|^{\theta} \quad \text{(by gauge invariance)} \]
\[ \leq C_1Z'||\mathcal{M}(u(A, \Phi))\|_{W^{-1,p}_{A_1}(X)} \]
\[ \leq C_1C(1 + C_3)Z'||\mathcal{M}(A, \Phi)\|_{W^{-1,p}_{A_1}(X)}. \]

Hence, we obtain the Lojasiewicz–Simon gradient inequality (1.17) with constants \((Z, \theta, \sigma)\), where we now choose \( Z = C_1C(1 + C_3)Z' \) and \( \sigma = \zeta_1 \).
This completes the proof of Theorem $\text{3}$.

Remark 3.11 (On the proof of Theorem $\text{3}$ for $p = 2$ and $d = 2, 3, 4$). As we discussed prior to the statement of Theorem $\text{2}$ that version of the abstract Lojasiewicz–Simon gradient inequality, while considerably more restrictive, has the advantage that, when applicable, its hypotheses are much easier to verify than those of Theorem $\text{1}$. Thus, for $2 \leq d \leq 4$, we may choose $p = 2$ and observe that the condition $d/2 \leq p \leq q$ is met for $q > 2$. With the notation used in the proof of Theorem $\text{1}$ we can then choose

$$\mathcal{X} = \text{Ker} \left( d^*_{u_{\infty}(A_{\infty}, \Phi_{\infty})}: W_{A_1}^{1,2}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to L^2(X; \Lambda^1 \otimes \text{ad} P \oplus E) \right),$$

with dual space,

$$\mathcal{X}^* = \text{Ker} \left( d^*_{u_{\infty}(A_{\infty}, \Phi_{\infty})}: W_{A_1}^{-1,2}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \to W_{A_1}^{-2,2}(X; \Lambda^1 \otimes \text{ad} P \oplus E) \right).$$

Proposition $\text{3.1}$ implies that the restriction of the energy functional to the Coulomb-gauge slice,

$$\mathcal{E}: x_{\infty} + \mathcal{X} \to \mathbb{R},$$

is real analytic, where $x_{\infty} := u_{\infty}(A_{\infty}, \Phi_{\infty})$. Proposition $\text{3.7}$ implies that the Hessian operator,

$$\mathcal{E}''(x_{\infty}) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*),$$

is Fredholm with index zero. Case $\text{1}$ of the proof of Theorem $\text{3}$ now follows easily, where the gauge transformation, $u_{\infty}$, is again chosen via Theorem $\text{2.23}$ so that $u_{\infty}(A_{\infty}, \Phi_{\infty})$ is a $C^\infty$ pair. The difficult third and fourth inequalities in the statement of Lemma $\text{3.9}$ are not required. Case $\text{2}$ of the proof of Theorem $\text{3}$ is unchanged.

3.2. Lojasiewicz–Simon gradient inequality for fermion coupled Yang-Mills energy functional. We assume the notation and conventions of Section $\text{3.1}$. By analogy with Proposition $\text{3.1}$ we establish the forthcoming Proposition $\text{3.12}$ for the analyticity of the fermion coupled Yang-Mills $L^2$-energy functional, $\mathcal{F}$. Again, this serves as a stepping stone towards the proof that its gradient map, $\mathcal{M}: x_{\infty} + \mathcal{X} \to \mathcal{F}$, is real analytic for suitable choices of Banach spaces as in Theorem $\text{1}$.

Proposition 3.12 (Analyticity of the fermion coupled Yang-Mills $L^2$-energy functional). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $(\rho, W)$ be a spin$^c$ structure on $X$, and $G$ be a compact Lie group, $P$ be a smooth principal $G$-bundle over $X$, and $E = P \times_\rho \mathbb{E}$ be a smooth Hermitian vector bundle over $X$ defined by a finite-dimensional unitary representation, $\rho: G \to \text{Aut}_\mathbb{C}(\mathbb{E})$, and $A_1$ be a smooth reference connection on $P$, and $m \in C^\infty(X)$. If $4d/(d+4) \leq p < \infty$, then the functional $\text{(1.8)},$

$$\mathcal{F}: \mathcal{A}^{1,p}(P) \times W_{A_1}^{1,p}(X; W \otimes E) \to \mathbb{R},$$

is real analytic.

Proof. We prove analyticity at a point $(A, \Psi)$ and write $A = A_1 + a_1$, where $a \in W_{A_1}^{1,q}(X; \Lambda^1 \otimes \text{ad} P)$. For any $(a, \psi) \in \mathcal{X}$, we have

$$F_{A+a} = F_{A_1} + d_{A_1}(a_1 + a) + \frac{1}{2}[a_1 + a, a_1 + a],$$

$$D_{A+a}(\Psi + \psi) = D_{A_1}(\Psi + \psi) + \rho(a_1 + a)(\Psi + \psi).$$
By definition (1.12) of the fermion coupled Yang-Mills $L^2$-energy functional, we obtain
\[ 2 \mathcal{F}(A + a, \Psi + \psi) = T_1 + T_2 + T_3, \]
where the curvature term,
\[ T_1 := (F_{A+a}, F_{A+a})_{L^2(X)}, \]
has the same expansion as the corresponding term $T_1$ in Proposition 3.12 for the boson coupled Yang-Mills $L^2$-energy functional, while
\[
T_2 := (\Psi, D_{A_1} \Psi)_{L^2(X)} + (\Psi, D_{A_1} \psi)_{L^2(X)} + (\psi, D_{A_1} \Psi)_{L^2(X)} + (\psi, D_{A_1} \psi)_{L^2(X)}
\]
and thus,
\[
T_3 := \int_X m \left( |\Psi|^2 + \langle \Psi, \psi \rangle + \langle \psi, \Psi \rangle + |\psi|^2 \right) \, d \text{vol}_g.
\]
The terms in the expression for the difference,
\[ 2 \mathcal{F}(A + a, \Psi + \psi) - 2 \mathcal{F}(A, \Psi) = T'_1 + T'_2 + T'_3, \]
are organized in such a way that
\[ T'_1 := (F_{A+a}, F_{A+a})_{L^2(X)} - (F_A, F_A)_{L^2(X)} \]
has the same expansion as the corresponding term $T'_1$ for the boson coupled Yang-Mills $L^2$-energy functional in the proof of Proposition 3.12 and the remaining terms are given by
\[
T'_2 := 2 \text{Re}(\Psi, D_{A_1} \psi)_{L^2(X)} + (\psi, D_{A_1} \psi)_{L^2(X)} + 2 \text{Re}(\Psi, \rho(a_1 + a) \psi)_{L^2(X)} + (\psi, \rho(a_1 + a) \psi)_{L^2(X)},
\]
\[
T'_3 := \int_X m \left( 2 \text{Re}(\Psi, \psi) + |\psi|^2 \right) \, d \text{vol}_g.
\]
The proof of analyticity of $\mathcal{F}$ at $(A, \Psi)$ now follows by adapting mutatis mutandis the arguments used to prove Proposition 3.12. \hfill \Box

We now verify the formula (1.12) for the differential $\mathcal{E}'(A, \Phi)$ and gradient $\mathcal{M}(A, \Phi)$.

**Lemma 3.13** (Differential and gradient of the fermion coupled Yang-Mills $L^2$-energy functional). Assume the hypotheses of Proposition 3.12 with the dual Hölder exponent $p'$ in the range $1 < p' \leq 4d/(3d - 4)$ determined by $4d/(d + 4) \leq p < \infty$ and $1/p + 1/p' = 1$. Then the expression for $\mathcal{E}'(A, \Psi) \in (W^{1,p}(X; \Lambda^1 \otimes \text{ad} P \oplus W \otimes E))^*$ and $\mathcal{M}(A, \Psi) \in W^{-1,p'}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus W \otimes E)$ is given by (1.12), namely
\[
\mathcal{E}'(A, \Psi)(a, \psi) = ((a, \psi), \mathcal{M}(A, \Psi))_{L^2(X)}
\]
\[
= (d'' A F_A, a)_{L^2(X)} + \text{Re}(D_A \Psi - m \Psi, \psi)_{L^2(X)} + \frac{1}{2} (\Psi, \rho(a) \Psi)_{L^2(X)},
\]
\[
\forall (a, \psi) \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P \oplus W \otimes E).\]
Proof. It suffices to extract the terms that are linear in \((a, \psi)\) from the expressions for 
\[T'_1 = 2(F_A, d_A a)_{L^2(X)} + (d_A a, d_A a)_{L^2(X)}\]
and \(T'_2\) and \(T'_3\) arising in the proof of Proposition 3.12. 

Remark 3.14 (Pointwise self-adjointness and reality). The fact that the term \(\langle \Psi, \rho(a) \Psi \rangle\) appearing in (1.12) is real could be inferred indirectly by noting the origin of this term and the fact that the Dirac operator, \(D_A\), is self-adjoint. To see directly that \(\langle \Psi, \rho(a) \Psi \rangle\) is real, recall that Clifford multiplication is skew-Hermitian, so \(c(\alpha)^* = -c(\alpha) \in \text{End}_C(W)\) for all \(\alpha \in \Omega^1(X)\) (for example, see [45, p. 49]) while if \(\xi \in \mathfrak{g}\), then \(\varrho_s(\xi)^* = -\varrho_s(\xi)\) since we assume that Lie structure group, \(G\), of \(P\) acts on the complex, finite-dimensional vector space \(E\) via a unitary representation, \(\varrho : G \to \text{End}_C(E)\), and \(\varrho_s : \mathfrak{g} \to \text{End}_C(E)\) is the induced representation of the Lie algebra, \(\mathfrak{g}\). Hence, given \(\alpha \otimes \xi \in C^\infty(T^*X \otimes \text{ad}P) = \Omega^1(X; \text{ad}P)\) and recalling that \(E = P \times_\varrho E\), then \(\rho(\alpha \otimes \xi) = c(\alpha) \otimes \varrho_s(\xi) \in \text{End}_C(W \otimes E)\) obeys
\[\rho(\alpha \otimes \xi)^* = c(\alpha)^* \otimes \varrho_s(\xi)^* = c(\alpha) \otimes \varrho_s(\xi) = \rho(\alpha \otimes \xi).
\]
In particular, \(\rho(a) \in \text{End}_C(W \otimes E)\) satisfies \(\rho(a)^* = \rho(a)\) for all \(a \in \Omega^1(X; \text{ad}P)\).

We now compute the Hessian operator, \(\mathcal{M}'(A, \Psi)\), at a \(C^\infty\) pair \((A, \Psi) \in \mathcal{A}(P) \times C^\infty(X; W \otimes E)\). The gradient, \(\mathcal{M}(A, \Psi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus W \otimes E)\) in (1.12), may be written as
\[\mathcal{M}(A, \Psi) = d_A^* F_A + \frac{1}{2} (\langle (D_A - m) \Psi \cdot + \cdot (D_A - m) \Psi \rangle + \frac{1}{2} \rho^{-1}(\Psi \otimes \Psi^*),
\]
where the terms involving \(\Psi\) in this expression for \(\mathcal{M}(A, \Psi)\) are defined by the \(L^2\)-pairings,
\[\text{Re}(D_A \Psi \cdot \Psi)_{L^2(X)} = \frac{1}{2} \left(\langle (D_A - m) \Psi, \Psi \rangle_{L^2(X)} + \langle (D_A - m) \Psi, \Psi \rangle_{L^2(X)}\right),
\]
\[\frac{1}{2} \langle \Psi, \rho(a) \rho(a)^* \rangle_{L^2(X)} = \frac{1}{2} \langle \Psi \otimes \Psi^*, \rho(a)^* \rangle_{L^2(X)} = \frac{1}{2} \langle \rho^{-1}(\Psi \otimes \Psi^*), a \rangle_{L^2(X)},
\]
\[\forall (a, \psi) \in C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus W \otimes E).
\]
Taking the derivative of the gradient \(\mathcal{M}(A, \Psi)\) in (3.45) with respect to \((A, \Psi)\) in the direction \((a, \psi)\) yields
\[\mathcal{M}'(A, \Psi)(a, \psi) = d_A^* d_A a + \frac{1}{2} (D_A \psi \cdot + \cdot D_A \psi) - \frac{1}{2} (m \psi \cdot + \cdot m \psi)
\]
\[+ (a \wedge \cdot)^* F_A + \frac{1}{2} (\rho(a) \Psi \cdot + \cdot \rho(a) \Psi) + \frac{1}{2} \rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*).
\]
By virtue of (3.46) we may view the Hessian operator for \(\mathcal{F}\) at a \(C^\infty\) pair \((A, \Psi)\) as an elliptic, linear, second-order partial differential operator, 
\[\mathcal{M}'(A, \Psi) : C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus W \otimes E) \to C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus W \otimes E).
\]
The differential,
\[d_{A, \Psi} : C^\infty(X; \text{ad}P) \to C^\infty(X; \Lambda^1 \otimes \text{ad}P \oplus W \otimes E),
\]
is defined just as in (2.60) except that we replace \(E\) by \(W \otimes E\). We can now simply make the following observation to conclude the proof of Theorem 3.5.

Proof of Theorem 3.5. The argument applies mutatis mutandis the corresponding steps used to prove Theorem 3.3 and all required intermediate results established in Section 3.1. \(\square\)
Appendix A. Fredholm and index properties of elliptic operators on Sobolev spaces

We note the following corollary of the standard Fredholm property \([41\text{ Lemma 1.4.5}], \ [56\text{ Theorem 19.2.1}]\) of elliptic pseudo-differential operators on Sobolev spaces \(W^{k+m,p}(M;V)\) when \(k \in \mathbb{Z}\) and \(p = 2\) and which may be compared with \([98\text{ Theorem 5.7.2}]\). We include a detailed proof of Theorem A.1 because we are unaware of a published reference. The statement of Theorem A.1 both generalizes \([56\text{ Theorem 19.2.1}]\) by allowing \(1 < p < \infty\) rather than \(p = 2\) and specializes \([56\text{ Theorem 19.2.1}]\) by restricting to partial differential operators (thus of integer order \(m\)) and restricting \(k\) to be an integer. In our application, we shall only need the case \(m = 2\), but we provide the more general version for the benefit of other applications. We leave it to the reader to consider extensions to the general case of pseudo-differential operators with real order \(m\) on \(W^{k,p}\) spaces with real \(k\).

**Theorem A.1** (Fredholm property for elliptic partial differential operators on Sobolev spaces). (See \([26\text{ Lemma 41.1}]\).) Let \(V\) and \(W\) be finite-rank, smooth vector bundles over a closed, smooth manifold, \(M\), and \(k \in \mathbb{Z}\) an integer, and \(p \in (1, \infty)\). If \(P : C^\infty(M;V) \to C^\infty(M;W)\) is an elliptic partial differential operator of integer order \(m \geq 1\) and with \(C^\infty\) coefficients, then \(P : W^{k+m,p}(M;V) \to W^{k,p}(M;W)\) is a Fredholm operator with index,

\[
\text{Index } P = \dim \text{Ker } (P : C^\infty(M;V) \to C^\infty(M;W)) - \dim \text{Ker } (P^* : C^\infty(M;W^*) \to C^\infty(M;V^*)) ,
\]

where \(P^*\) is the formal \((L^2)\) adjoint of \(P\), and has range,

\[
\text{Ran } (P : W^{k+m,p}(M;V) \to W^{k,p}(M;W)) = (K^*)\perp \cap W^{k,p}(M;W),
\]

where \(\perp\) denotes \(L^2\)-orthogonal complement and

\[
K = \text{Ker } (P^* : C^\infty(M;W^*) \to C^\infty(M;V^*)) \cong K^* \subset C^\infty(M;W).
\]

If \(V = W^*\) and \(P - P^* : C^\infty(M;V) \to C^\infty(M;W)\) is a differential operator of order \(m - 1\), then \(\text{Index } P = 0\).

**Remark A.2** (On the index of elliptic operators with scalar principal symbol). We recall from \([94\text{ Proposition 2.4}]\) that if \(E\) is a smooth complex vector bundle over a closed, smooth manifold, \(M\), and \(P : C^\infty(M;E) \to C^\infty(M;E)\) is an elliptic differential operator whose principal symbol is a scalar multiple of the identity (as often occurs in Geometric Analysis), then \(\text{Index } P = 0\). However, if \(P : C^\infty(M;E) \to C^\infty(M;E)\) is more generally an elliptic pseudo-differential operator, then \(P\) need not have index zero \([44\text{ Example 2.2}]\).

We refer to Gilkey \([41]\) and Hörmander \([56]\) for the definitions (over domains \(\Omega \subseteq \mathbb{R}^d\)) of a differential operator \(P\) of integer order \(m \geq 1\) \([41\text{ Section 1.1}]\), a pseudo-differential operator \(P\) of order \(m \in \mathbb{R}\) \([41\text{ Section 1.2.1}]\), the vector spaces \(\Psi^m\) of pseudo-differential operators of order \(m \in \mathbb{R}\) and \(\Psi^{-\infty} = \bigcap_{m \in \mathbb{R}} \Psi^m\) of infinitely smoothing pseudo-differential operators \([41\text{ Section 1.2.1}]\), the \(L^2\)-adjoint \(P^*\) and composition \(PQ\) of pseudo-differential operators \([41\text{ Section 1.2.2}]\), the extension of the definition of pseudo-differential operators on complex-valued functions over domains in \(\mathbb{R}^d\) to sections of finite-rank vector bundles over domains in \(\mathbb{R}^d\) \([41\text{ Section 1.2.7}]\),

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6Gilkey and Hörmander allow \(m, k \in \mathbb{R}\), as we could also in Theorem A.1, but we shall omit this refinement.
elliptic pseudo-differential operators on sections of finite-rank vector bundles over domains in $\mathbb{R}^d$ [41, Section 1.3.1], and elliptic pseudo-differential operators on sections of finite-rank vector bundles over closed manifolds [41, Sections 1.3.3 and 1.3.5]. Similarly, for $s \in \mathbb{R}$, we refer to Gilkey [41] and Hörmander [56] for the definitions of Sobolev spaces $W^{s,2}(\mathbb{R}^d; \mathbb{C})$ [41, Section 1.1.3], $W^{s,2}(\Omega; \mathbb{C})$ [41, Section 1.2.6], and $W^{s,2}(M; \mathbb{C})$ and $W^{s,2}(M; V)$ [41, Sections 1.3.4 and 1.3.5].

When $k \in \mathbb{N}$ is a non-negative integer and $1 \leq p \leq \infty$, we use the standard definitions (without appealing to symbols or the symbol calculus) of Sobolev spaces, $W^{k,p}(\Omega; \mathbb{R})$ and $W_0^{k,p}(\Omega; \mathbb{R})$, in Adams and Fournier [4] Section 3.2 for domains $\Omega \subseteq \mathbb{R}^d$ and Aubin [5] Section 2.3] and Gilkey [41, Sections 1.3.4 and 1.3.5] for their extensions to functions on closed manifolds, $W^{k,p}(M; \mathbb{R})$, and sections of finite-rank vector bundles over closed manifolds, $W^{k,p}(M; V)$. For integers $k < 0$ and $1 < p < \infty$, we follow [4] Sections 3.9 to 3.14] and by analogy define

$$W^{k,p}(M; V) := (W^{-k,p'}(M; V))^*,$$

where the dual Hölder exponent, $1 < p' < \infty$, is defined by $1/p + 1/p' = 1$ and $(W^{-k,p'}(M; V))^*$ is the continuous dual of the Banach space, $W^{-k,p'}(M; V)$.

For a differential operator, $P$, of integer order $m \geq 1$, any integer $k \in \mathbb{Z}$ and $v \in W^{k+m,p}(M; V)$, one may define $Pv \in (C^\infty(M; V))^*$ in the sense of distributions [4, Section 1.62] by

$$(Pv)(w) := (v, P^*w)_{L^2(M; W^*)}, \quad \forall w \in C^\infty(M; W^*),$$

where the $L^2$-adjoint, $P^* : C^\infty(M; W^*) \to C^\infty(M; V^*)$, is also a differential operator of order $m \geq 1$. For integer $k \geq 0$, then $Pv$ has its usual meaning and we have $Pv \in W^{k,p}(M; V)$, with

$$P : W^{k+m,p}(M; V) \to W^{k,p}(M; W),$$

defining a bounded operator by definition of the Sobolev spaces, $W^{k+m,p}(M; V)$. Similarly, for integers $k \leq -m$, we may define $Pv \in W^{k,p}(M; V)$ by

$$(Pv)(w) := (v, P^*w)_{L^2(M; W^*)}, \quad \forall v \in W^{k+m,p}(M; V) \text{ and } w \in W^{-k,p'}(M; V),$$

noting that $P^* : W^{-k,p'}(M; W^*) \to W^{-k-m,p'}(M; V^*)$ is a bounded operator and so $P^*w \in W^{-k-m,p'}(M; V^*)$ with its usual meaning.

Finally, for $m \geq 2$ and integers $k$ in the range $-m+1 \leq k \leq -1$, we choose a Riemannian metric, $g$, on $M$, a connection, $\nabla : C^\infty(M; V) \to C^\infty(M; T^*M \otimes V)$, and form the augmented connection Laplace operator,

$$\nabla^* \nabla + 1 : C^\infty(M; V) \to C^\infty(M; V).$$

The resulting operator, $\nabla^* \nabla + 1 : W^{l+2,p}(M; V) \to W^{l,p}(M; V)$, is invertible for integers $l \geq 0$ and $1 < p < \infty$. By duality (as above) it extends to an invertible operator, $\nabla^* \nabla + 1 : W^{l+2,p}(M; V) \to W^{l,p}(M; V)$, for integers $l \leq -2$ and $1 < p < \infty$ and because it has divergence form, it also defines an invertible operator, $\nabla^* \nabla + 1 : W^{l,p}(M; V) \to W^{-l,p}(M; V)$, for $1 < p < \infty$. (Observe that the bilinear map,

$$W^{l,p}(M; V) \times W^{l,p'}(M; V) \ni (u, v) \mapsto ((\nabla^* \nabla + 1)u, v)_{L^2(X)},$$

is continuous since

$$|((\nabla^* \nabla + 1)u, v)_{L^2(X)}| \leq \|\nabla u\|_{L^p(X)} \|\nabla v\|_{L^{p'}(X)} + \|u\|_{L^p(X)} \|v\|_{L^{p'}(X)},$$
We therefore obtain a continuous, linear map,
\[ \nabla^* \nabla + 1 : W^{1,p}(M; V) \to (W^{1,p'}(M; V^*))^* = W^{-1,p}(M; V), \]
which is clearly injective. The map is surjective by [4] Sections 3.5–3.14] because every \( \alpha \in (W^{1,p'}(M; V^*))^* \) is represented by
\[ \alpha(v) = (\nabla v, u_1)_{L^2(X)} + (v, u_2)_{L^2(X)}, \]
for \( u_i \in L^p(M; V) \) for \( i = 1, 2 \). If \( p = 2 \), then we find that \( u_2 = u \) and \( u_1 = \nabla u \) for \( u \in W^{1,2}(M; V) \) (compare [40, Section 8.2]) and thus also for \( 2 \leq p < \infty \); duality yields the case \( 1 < p \leq 2 \). Hence, the map is an isomorphism by the Open Mapping Theorem.

For \( m = 2n \) and \( n \in \mathbb{N} \), we may thus define \( Pv \in W^{k,p}(M; V) \) by
\[ (Pv)(w) := ((\nabla^* \nabla + 1)^{m/2} v, P^* w)_{L^2(M; V^*)}, \]
\[ \forall v \in W^{k+m,p}(M; V) \text{ and } w \in W^{-k+m,p'}(M; V), \]
noting that \((\nabla^* \nabla + 1)^{m/2} v \in W^{k,p}(M; V)\) and \( P^* w \in W^{-k,p'}(M; V)\). For \( m = 2n + 1 \) and \( n \in \mathbb{N} \), one must first define the square root \([4] (\nabla^* \nabla + 1)^{1/2} : W^{l+1,p}(M; V) \to W^{l,p}(M; V)\), for integers \( l \in \mathbb{Z} \) and \( 1 < p < \infty \); see Feehan [26] and references cited therein for a survey of approaches to the definition of the fractional powers, \((\nabla^* \nabla + 1)^s\), for \( s \in \mathbb{R} \). Once the square root of the augmented connection Laplace operator is defined, then the definition of \( Pv \in W^{k,p}(M; V) \) is identical to the case \( m = 2n \).

*Proof of Theorem* [A.1] We use [41] Lemma 1.3.6] to find a pseudo-differential operator \( S : C^\infty(M; W) \to C^\infty(M; V) \) of order \(-m\) so that
\[ SP - I \in \Psi^{-\infty}(M; V) \quad \text{and} \quad PS - I \in \Psi^{-\infty}(M; W), \]
where \( \Psi^{-\infty}(M; W) \) and \( \Psi^{-\infty}(M; V) \) are the vector spaces of infinitely smoothing pseudo-differential operators [41] Sections 1.2 and 1.3].

The operator \( P : W^{k+m,p}(M; V) \to W^{k,p}(M; W) \) is continuous since \( P \) is an elliptic partial differential operator of order \( m \geq 1 \) and by definition of the Sobolev space \( W^{k+m,p}(M; V) \). Combining the preceding observation with the a priori elliptic estimate (see [26] Theorem 14.60], for example),
\[ \|v\|_{W^{k+m,p}(M)} \leq C \left( \|v\|_{W^{k,p}(M)} + \|Pv\|_{W^{k,p}(M)} \right), \]
implies that the expression \( \|v\|_{W^{k,p}(M)} + \|Pv\|_{W^{k,p}(M)} \) defines a norm for \( v \in C^\infty(M; V) \) which is equivalent to the standard norm on \( W^{k+m,p}(M; V) \). Furthermore, the operators
\[ SP - I : W^{k,p}(M; V) \to W^{k+1,p}(M; V), \]
\[ PS - I : W^{k,p}(M; W) \to W^{k+1,p}(M; W), \]
\[7\]This could be accomplished geometrically with the aid of a Dirac operator, \( D : C^\infty(M; V) \to C^\infty(M; V) \), when \( V \) and \( W \) are spinor bundles and \( M \) admits a spin^c structure [69].
are continuous because, denoting $H^l(M; V) = W^{l,2}(M; V)$ and $H^l(M; W) = W^{l,2}(M; W)$, the operators

$$SP - I : H^l(M; V) \to H^{l+n}(M; V),$$
$$PS - I : H^l(M; W) \to H^{l+n}(M; W),$$

are continuous by [11] Lemma 1.3.5, for any integers $l \in \mathbb{Z}$ and $n \in \mathbb{N}$, and the standard Sobolev Embedding [11] Theorem 4.12 and duality [11] Sections 3.5–3.14, which provide continuous embeddings, $W^{k,p}(M; V) \subset H^l(M; V)$ for sufficiently small $l \leq l_0(d, k, p)$ and $H^{l+n}(M; W) \subset W^{k+1,p}(M; W)$ for sufficiently large $n \geq n_0(d, k, p)$, where $d$ is the dimension of $M$. (The finite integers $l_0(d, k, p)$ and $n_0(d, k, p)$ can of course be determined explicitly from the full statement of the Sobolev Embedding [11] Theorem 4.12, but their precise values are unimportant here.) Thus,

$$\|Sw\|_{W^{k+m,p}(M; W)} \leq C \left( \|PSw\|_{W^{k,p}(M; W)} + \|w\|_{W^{k,p}(M; W)} \right) \leq C \left( \|(PS - I)w\|_{W^{k,p}(M; W)} + \|w\|_{W^{k,p}(M; W)} \right) \leq C \|w\|_{W^{k,p}(M; W)} \quad \text{(by continuity of $PS - I$ on $W^{k,p}(M; W)$)},$$

and so the operator $S : W^{k,p}(M; V) \to W^{k+m,p}(M; W)$ is continuous.

The embeddings $W^{k+1,p}(M; V) \subset W^{k,p}(M; V)$ and $W^{k+1,p}(M; W) \subset W^{k,p}(M; W)$ are compact by the Rellich-Kondrachov Theorem [11] Theorem 6.3 when $k \geq 0$ and when $k \leq -1$ using duality [13] Theorem 6.4 and compactness of the embeddings $W^{-k,p}(M^*; V^*) \subset W^{-k-1,p}(M^*; V^*)$ and $W^{-k,p}(M; W^*) \subset W^{-k-1,p}(M; W^*)$. Hence, the operators (now viewed as compositions with compact embeddings),

$$SP - I : W^{k,p}(M; V) \to W^{k,p}(M; V),$$
$$PS - I : W^{k,p}(M; W) \to W^{k,p}(M; W),$$

are compact by [13] Proposition 6.3. Thus, $P : W^{k+m,p}(M; V) \to W^{k,p}(M; W)$ is Fredholm by [11] Section 1.4.2, Definition, p. 38).

Because $P : W^{k+m,p}(M; V) \to W^{k,p}(M; W)$ is Fredholm, its index may be computed by [2] Lemma 4.38,

$$\text{Index } P = \dim \ker \left( P : W^{k+m,p}(M; V) \to W^{k,p}(M; W) \right) - \dim \ker \left( P^* : W^{-k,p'}(M^*; W^*) \to W^{-k-m,p'}(M^*; W^*) \right).$$

If $w \in (\ker P^*) \cap W^{-k,p'}(M; W^*)$, then $w \in (\ker P^*) \cap H^l(M; W^*)$ for all integers $l \leq l_0(d, k, p)$, where $l_0$ is given by the Sobolev Embedding [11] Theorem 4.12, and consequently (since the Banach space dual $P^*$ is defined by the realization of the formal adjoint and $P^*$ is thus an elliptic partial differential operator of order $m$) we see that $w \in (\ker P^*) \cap C^\infty(M; W^*)$ by elliptic regularity [11] Lemma 1.3.2 and Section 1.3.5. Of course, if $v \in (\ker P) \cap W^{k,m,p}(M; V)$, then $v \in (\ker P) \cap C^\infty(M; V)$ by the same argument. This yields the stated formula for the index of $P : W^{k+m,p}(M; V) \to W^{k,p}(M; W)$. 

If $K \subset W^{-k,p'}(M;W^*)$ is any subspace, we recall that the annihilator \[83\] Section 4.6] of $K$ in $(W^{-k,p'}(M;W^*))^* = W^{k,p}(M;W)$ is

$$K^\circ = \{ w \in W^{k,p}(M;W) : \langle \alpha, w \rangle = 0, \forall \alpha \in K \},$$

and $\langle \cdot, \cdot \rangle : W^{-k,p'}(M;W^*) \times (W^{-k,p'}(M;W^*))^* \to \mathbb{R}$ is the canonical pairing. We can therefore identify the range of $P : W^{k+m,p}(M;V) \to W^{k,p}(M;W)$ using

$$\text{Ran} \left( P : W^{k+m,p}(M;V) \to W^{k,p}(M;W) \right)$$

$$= \text{Ran} \left( P : W^{k+m,p}(M;V) \to W^{k,p}(M;W) \right) \quad \text{(by closed range)}$$

$$= \ker \left( P^* : W^{-k,p'}(M;W^*) \to W^{-k-m,p'}(M;V^*) \right)^\circ \quad \text{(by \[13\] Corollary 2.18 (iv)).}$$

If $K := \ker \left( P^* : W^{-k,p'}(M;W^*) \to W^{-k-m,p'}(M;V^*) \right)$, then $K \subset C^\infty(M;W^*)$, as we observed by elliptic regularity and

$$K^\circ = \{ w \in W^{k,p}(M;W) : w(\alpha) = 0, \forall \alpha \in K \}$$

$$= \{ w \in W^{k,p}(M;W) : \alpha(w) = 0, \forall \alpha \in K \} \quad \text{(by } (W^{k,p}(M;W))^* \cong W^{k,p}(M;W))$$

$$= \{ w \in W^{k,p}(M;W) : \iota(\kappa)(w) = 0, \forall \kappa \in K^* \}$$

$$= \{ w \in W^{k,p}(M;W) : (w, \kappa)_{L^2(M;W)} = 0, \forall \kappa \in K^* \}$$

$$= (K^*)^\perp \cap W^{k,p}(M;W),$$

where $K^* \subset C^\infty(M;W)$ and $\iota : K^* \cong K$ is the canonical isomorphism, $\kappa \mapsto \iota(\kappa) = (\cdot, \kappa)_{L^2(M;W)}$. This yields the claimed identification of the range of $P$.

Noting that $(W^{-k,p'}(M;V))^* = W^{k,p'}(M;V^*)$ and $(W^{-k-m,p'}(M;W))^* = W^{k+m,p}(M;W^*)$, we see that the operator,

$$P^* : W^{k+m,p}(M;W^*) \to W^{k,p}(M;V^*),$$

is Fredholm by \[41\] Lemma 1.4.3, since the same is true for the operator,

$$P : W^{-k,p'}(M;V) \to W^{-k-m,p'}(M;W).$$

If $V = W^*$, we may write $P^* = P + (P^* - P)$ and observe that $P^* - P : W^{k+m,p}(M;V) \to W^{k+1,p}(M;W)$ is bounded since $P^* - P$ has order $m - 1$. The inclusion, $W^{k+1,p}(M;W) \subset W^{k,p}(M;W)$, is compact by \[41\] Theorem 6.3, so $P^* - P : W^{k+m,p}(M;V) \to W^{k,p}(M;W)$ is compact. Hence, $\text{Index}(P^*) = \text{Index} P$ by \[56\] Corollary 19.1.8]. But we also have $\text{Index} P^* = -\text{Index} P$ by \[41\] Lemma 1.4.4 (a)] and thus $\text{Index} P = 0$. \[\square\]

The following corollary is partially based on Abramovich and Aliprantis \[2\] Theorem 4.46], Melrose \[75\] Lecture 9, Proposition 18] and Treves \[94\] Theorem 2.4].

**Corollary A.3** (Green’s operators for elliptic partial differential operators on Sobolev spaces). Assume the hypotheses of Theorem A.7 and let $G : W^{k,p}(M;W) \to W^{k+m,p}(M;V)$ be the Green’s operator for $P$ defined by

$$Gw := \begin{cases} P^{-1}w, & w \in \text{Ran} P \subset W^{k,p}(M;W), \\ 0, & w \in (\text{Ran} P)^\perp. \end{cases}$$
Then $G : C^\infty(M; W) \to C^\infty(M; V)$ is an elliptic pseudo-differential operator of order $-m$ with

$$GP = \id - \Pi_1 \quad \text{and} \quad PG = \id - \Pi_2,$$

where $\Pi_1$ is the $L^2$-orthogonal projection onto $\Ker P \subset W^{k+m,p}(M; V)$ and $\Pi_2$ is the $L^2$-orthogonal projection onto $(\Ran P)^\perp \subset W^{k,p}(M; W)$. Moreover, if $k \geq 0$, the following operator is bounded,

$$G : W^{k,p}(M; W) \to W^{k+m,p}(M; V).$$

**Proof.** From Melrose [75, Lecture 6, Proposition 11] or [75, Lecture 7, Theorem 3] or Treves [94, Theorem 2.4], there exists an elliptic pseudo-differential operator, $G : C^\infty(M; W) \to C^\infty(M; V)$, of integer order $-m$ such that $\id - GP$ and $\id - PG$ are infinitely smoothing operators. We then appeal to Melrose [75, Lecture 9, Proposition 18] to conclude that $\id - GP$ and $\id - PG$ can be identified as the stated $L^2$-orthogonal projections.

For $k \geq 0$ and any $w \in W^{k,p}(M; W)$, we have

$$\|Gw\|_{W^{k+m,p}(M)} \leq C \left( \|PGw\|_{W^{k,p}(M)} + \|w\|_{W^{k,p}(M)} \right) \quad \text{(by [26, Theorem 14.60])}$$

$$\leq C \left( \|(\id - \Pi_2)w\|_{W^{k,p}(M)} + \|w\|_{W^{k,p}(M)} \right)$$

$$\leq C \|w\|_{W^{k,p}(M)},$$

and so the conclusion on boundedness follows. \qed

**Appendix B. Equivalence of Sobolev norms defined by Sobolev and smooth connections**

Suppose that $(X, g)$ is a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ is a compact Lie group, and $P$ is a smooth principal $G$-bundle over $X$. In standard references for gauge theory [24, 37], it is generally assumed in the construction of Sobolev completions of spaces such as $\Omega^1(X; \ad P)$ that one defines Sobolev norms using a covariant derivative $\nabla_A$ determined by a connection $A$ on $P$ that is smooth or of class $W^{k,p}$ for $p \geq 1$ and an integer $k \geq 1$ large enough that $kp > d$ or even $kp \gg d$. However, in this article, we often consider connections $A$ with more borderline regularity, for example of class $W^{1,q}$ for $q > d/2$, and in that situation, one must exercise care in the definition of Sobolev spaces using such connections. Lemmas B.1 and B.2 provide some guidance.

**Lemma B.1** (Second-order Kato inequality and second-order Sobolev norms). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $V = P \times_g \mathcal{V}$ be a smooth Riemannian vector bundle over $X$ defined by a finite-dimensional, orthogonal representation, $\rho : G \to \text{Aut}_\mathbb{R}(\mathcal{V})$. Then there exists a constant $C = C(g, q) \in [1, \infty)$ with the following significance. If $A$ is a $W^{1,q}$ connection on $P$ with $q > d/2$, then for all $v \in C^\infty(X; V)$,

$$\|v\|_{C^2(X)} \leq C \|v\|_{W^{2,q}_A(X)}.$$

**Proof.** The first-order analogue of (B.1), namely,

$$\|v\|_{C^1(X)} \leq \kappa_1 \|v\|_{W^{1,q}_A(X)},$$
when \( q > d \) and \( \kappa_1 = \kappa_1(g) \in [1, \infty) \) is the norm of the Sobolev embedding \( W^{1,q}(X) \subset C(X) \), is an immediate consequence of the pointwise first-order Kato inequality, \( |\nabla v| \leq |\nabla_A v| \) from [37] Inequality (6.20), in turn a consequence of the compatibility of the fiber metric on \( V \) with \( \nabla_A \).

We first note that, for \( f \in C^\infty(X; \mathbb{R}) \), the norm
\[
\|f\|_{W^{2,q}(X)} = \|\nabla^2 f\|_{L^q(X)} + \|\Delta f\|_{L^q(X)} + \|f\|_{L^q(X)}
\]
is equivalent (with respect to a constant depending at most on \( (g, q) \)) by virtue of [40] Theorem 9.11 to
\[
\|f\|_{W^{2,q}(X)} = \|\nabla f\|_{L^q(X)} + \|f\|_{L^q(X)}
\]
where \( \Delta \) is the Laplace operator defined by the Riemannian metric \( g \) on \( X \). Now recall the pointwise identity [37] Equation (6.18),
\[
\Delta |v|^2 = 2(\nabla_A \nabla_A v, v) - 2|\nabla_A v|^2.
\]
Hence, letting \( \kappa_2 = \kappa_2(g) \in [1, \infty) \) denote the norm of the Sobolev embedding \( W^{2,q}(X) \subset C(X) \),
\[
\|v\|_{C(X)} = \|v\|_{L^2(X)}
\]
\[
\leq \kappa_2 \|v\|_{W^{2,q}(X)}
\]
\[
= \kappa_2 \left( \|\Delta|v|^2\|_{L^q(X)} + \|v|^2\|_{L^q(X)} \right)
\]
\[
\leq \kappa_2 \left( 2\|v\|_{C(X)} \|\nabla_A \nabla_A v\|_{L^q(X)} + 2\|\nabla_A v\|_{L^{2q}(X)} + \|v\|_{C(X)} \|v\|_{L^q(X)} \right).
\]
Recall that \( W^{1,q}(X) \subset L^{2q}(X) \), for \( q < d \), if and only if \( 2q \leq q^* = dq/(d-q) \), that is, \( 2(d-q) \leq d \) or \( q \geq d/2 \); the embedding is immediate from [41] Theorem 4.12 when \( q \geq d \). Thus, applying the first-order Kato Inequality and the preceding Sobolev embedding for functions,
\[
\|v\|_{L^{2q}(X)} \leq \kappa_1 \left( \|\nabla v\|_{L^q(X)} + \|v\|_{L^q(X)} \right) \leq \kappa_1 \left( \|\nabla_A v\|_{L^q(X)} + \|v\|_{L^q(X)} \right),
\]
we obtain
\[
\|v\|_{C(X)}^2 \leq \kappa_2 \left( 2\|v\|_{C(X)} \|\nabla_A \nabla_A v\|_{L^q(X)} + 2\kappa_1 \left( \|\nabla_A v\|_{L^q(X)} + \|v\|_{L^q(X)} \right)^2
\]
\[
+ \|v\|_{C(X)} \|v\|_{L^q(X)} \right).
\]
We now use Young’s Inequality \( 2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2 \) from [40] Inequality (7.8) and rearrangement with a suitably small and universal \( \varepsilon \) to give
\[
\|v\|_{C(X)}^2 \leq C^2 \left( \|\nabla_A \nabla_A v\|_{L^q(X)}^2 + \|\nabla_A v\|_{L^q(X)}^2 + \|\nabla_A v\|_{L^q(X)}^2 + \|v\|_{L^q(X)}^2 \right)
\]
where \( C = C(g, q) \in [1, \infty) \). We simplify the right-hand side in the preceding inequality via
\[
\|\nabla_A \nabla_A v\|_{L^q(X)} \leq z \left( \|\nabla_A v\|_{L^q(X)} + \|\nabla_A v\|_{L^q(X)} + \|v\|_{L^q(X)} \right),
\]
where \( z \) is a constant depending at most on the Riemannian metric on \( X \). The desired Sobolev inequality [41] now follows by taking square roots. \( \square \)

**Lemma B.2** (Equivalence of Sobolev norms defined by Sobolev and smooth connections). Let \( (X, g) \) be a closed, Riemannian, smooth manifold of dimension \( d \geq 2 \), and \( G \) be a compact Lie group, and \( P \) be a smooth principal \( G \)-bundle over \( X \), and \( V = P \times_g V \) be a smooth Riemannian vector bundle over \( X \) defined by a finite-dimensional, orthogonal representation, \( \rho : G \to \text{Aut}_\mathbb{R}(V) \), and \( q > d/2 \) and \( p \) obey \( d/2 \leq p \leq q \). Let \( A_1 \) be a \( C^\infty \) connection on \( P \), and \( A_0 \) be a Sobolev connection on \( P \), and \( a_0 := A_0 - A_1 \).


(1) There exists $C = C(g, p) \in [1, \infty)$ such that, if $a_0 \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then

$$\|\xi\|_{L^r(X)} \leq \frac{C}{\|\xi\|_{W^{1,p}_{A_0}(X)^{\Lambda^1 \otimes P}}}, \quad \text{for} \begin{cases} 1 \leq r \leq \frac{dp}{d - p} & \text{if } p < d, \\ 1 \leq r < \infty & \text{if } p = d, \\ r = \infty & \text{if } p > d, \end{cases}$$

for all $\xi \in C^\infty(X; V)$; moreover, there exists $C = C(A_1, g, p) \in [1, \infty)$ such that

$$\|\xi\|_{L^r(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)}, \quad \text{for} \begin{cases} 1 \leq r \leq \frac{dp}{d - 2p} & \text{if } p < d/2, \\ 1 \leq r < \infty & \text{if } p = d/2, \\ r = \infty & \text{if } p > d/2, \end{cases}$$

for all $\xi \in C^\infty(X; V)$.

(2) If $a_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then there exists $C = C(g, q) \in [1, \infty)$ such that, for all $\xi \in C^\infty(X; V)$,

$$\|\xi\|_{C(X)} \leq C\|\xi\|_{W^{2,q}_{A_0}(X)}.$$

(3) If $a_0 \in W^{1,p}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then there exists $C = C(g, p, \|a_0\|_{W^{1,p}_{A_1}(X)}) \in [1, \infty)$ so that

$$C^{-1}\|\xi\|_{W^{1,p}_{A_1}(X)} \leq \|\xi\|_{W^{1,p}_{A_1}(X)} \leq C\|\xi\|_{W^{1,p}_{A_1}(X)}, \quad \forall \xi \in C^\infty(X; V).$$

(4) If $a_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then there exists $C = C(g, p, q, \|a_0\|_{W^{1,q}_{A_1}(X)}) \in [1, \infty)$ so that

$$C^{-1}\|\xi\|_{W^{2,q}_{A_1}(X)} \leq \|\xi\|_{W^{2,q}_{A_1}(X)} \leq C\|\xi\|_{W^{2,q}_{A_1}(X)}, \quad \forall \xi \in C^\infty(X; V).$$

(5) If $a_0 \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then there exists $C = C(g, p, q, \|a_0\|_{W^{1,q}_{A_1}(X)}) \in [1, \infty)$ so that

$$\|\xi\|_{W^{2,p}_{A_0}(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)}, \quad \forall \xi \in C^\infty(X; V).$$

(6) If $a_0 \in W^{2,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then there exists $C = C(g, p, q, \|a_0\|_{W^{2,q}_{A_1}(X)}) \in [1, \infty)$ so that

$$C^{-1}\|\xi\|_{W^{1,p}_{A_1}(X)} \leq \|\xi\|_{W^{2,p}_{A_1}(X)} \leq C\|\xi\|_{W^{2,p}_{A_1}(X)}, \quad \forall \xi \in C^\infty(X; V).$$

\textit{Proof.} Item (1) is a well-known consequence of the Sobolev Embedding \cite[Theorem 4.12]{M} for scalar functions and the Kato Inequality \cite[Equation (6.20)]{M} in the case of the embedding $W^{1,p}(X) \subset L^r(X)$. Item (2) restates the conclusion of Lemma B.1.

For Item (3), we use $\nabla A_0 \xi = \nabla A_1 \xi + [a_0, \xi]$ and estimate

$$\|\nabla A_1 \xi\|_{L^p(X)} \leq \|\nabla A_0 \xi\|_{L^p(X)} + \|a_0, \xi\|_{L^p(X)} \leq \|\xi\|_{W^{1,p}_{A_0}(X)} + \|a_0\|_{L^{2p}(X)} \|\xi\|_{L^{2p}(X)} \leq C(1 + \|a_0\|_{W^{1,p}_{A_1}(X)}) \|\xi\|_{W^{1,p}_{A_1}(X)},$$

where we used the continuous Sobolev embedding $W^{1,p}(X) \subset L^{2p}(X)$ for $p \geq d/2$ and Item (1) to obtain the last inequality. Here, $z = z(g) \in [1, \infty]$ and $C \in [1, \infty)$ has the stated dependencies. The analogous estimate with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument.
For Item (1), we first write
\[ \nabla^2_{A_1} \xi = \nabla^2_{A_0} \xi + \nabla_{A_0} a_0 \times \xi + a_0 \times \nabla_{A_0} \xi + a_0 \times a_0 \times \xi. \]
Taking $L^q$ norms of both sides of (B.2), we see that
\[
\| \nabla^2_{A_1} \xi \|_{L^q(X)} \leq \| \nabla^2_{A_0} \xi \|_{L^q(X)} + \| \nabla_{A_0} a_0 \times \xi \|_{L^q(X)} + \| a_0 \times \nabla_{A_0} \xi \|_{L^q(X)} + \| a_0 \times a_0 \times \xi \|_{L^q(X)},
\]
and thus, for $z = z(q) \in [1, \infty)$,
\[ \| \nabla^2_{A_1} \xi \|_{L^q(X)} \leq \| \nabla^2_{A_0} \xi \|_{L^q(X)} + z \| \nabla_{A_0} a_0 \|_{L^q(X)} \| \xi \|_{C(X)} + z \| a_0 \|_{W^{2,q}(X)} \| \nabla_{A_0} \xi \|_{L^q(X)} + z \| a_0 \|_{L^2} \| \xi \|_{C(X)} .
\]
By Item (2), we have
\[ \| \nabla_{A_0} a_0 \|_{L^q(X)} \leq \| a_0 \|_{W^{1,q}_{A_0}(X)} \leq C \| a_0 \|_{W^{1,q}_{A_1}(X)},
\]
and by Item (1) and the fact that $W^{1,q}(X) \subset L^{2q}(X)$ for $q > d/2$, we obtain
\[ \| \nabla_{A_0} \xi \|_{L^{2q}(X)} \leq C \| \xi \|_{W^{2,q}_{A_0}(X)}.
\]
Similarly, Item (2) gives
\[ \| \xi \|_{C(X)} \leq C \| \xi \|_{W^{2,q}_{A_0}(X)}.
\]
By substituting the preceding inequalities into (B.3), we find that
\[ \| \xi \|_{W^{2,q}_{A_1}(X)} \leq C \| \xi \|_{W^{2,q}_{A_0}(X)},
\]
where $C \in [1, \infty)$ has the stated dependencies. The analogous inequality with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument.

For Item (5), define $r \in [p, \infty]$ by $1/p = 1/q + 1/r$, recall that $p = d/2 < q$ or $d/2 < p \leq q$, interchange the roles of $A_0$ and $A_1$ in (B.2), and take $L^p$ norms to give
\[
\| \nabla^2_{A_0} \xi \|_{L^p(X)} \leq \| \nabla^2_{A_1} \xi \|_{L^p(X)} + \| \nabla_{A_1} a_0 \times \xi \|_{L^p(X)} + \| a_0 \times \nabla_{A_1} \xi \|_{L^p(X)} + \| a_0 \times a_0 \times \xi \|_{L^p(X)}
\]
\[
\leq \| \nabla^2_{A_1} \xi \|_{L^p(X)} + z \| \nabla_{A_1} a_0 \|_{L^q(X)} \| \xi \|_{L^r(X)} + z \| a_0 \|_{L^{2p}(X)} \| \nabla_{A_1} \xi \|_{L^p(X)} + z \| a_0 \|_{L^2} \| \xi \|_{L^r(X)}
\]
\[
\leq \| \nabla^2_{A_1} \xi \|_{L^p(X)} + C \| a_0 \|_{W^{1,q}_{A_1}(X)} \| \xi \|_{W^{2,p}_{A_1}(X)} + C \| a_0 \|_{W^{1,p}_{A_1}(X)} \| \nabla_{A_1} \xi \|_{W^{2,p}_{A_1}(X)}
\]
\[
+ C \| a_0 \|_{W^{1,q}_{A_1}(X)} \| \xi \|_{W^{2,p}_{A_1}(X)},
\]
where $z = z(q) \in [1, \infty)$ and, to obtain the last inequality, we use the continuous Sobolev embeddings $W^{1,p}(X) \subset L^{2p}(X)$ and $W^{1,q}(X) \subset L^{2q}(X)$ for $d/2 \leq p \leq q$ and Item (1) together with the continuous Sobolev embedding $W^{2,p}(X) \subset L^r(X)$, for $r \in [1, \infty)$ if $p = d/2$ and $r = \infty$ if $p > d/2$. Therefore, we obtain
\[ \| \xi \|_{W^{2,p}_{A_0}(X)} \leq C \| \xi \|_{W^{2,p}_{A_1}(X)},
\]
where $C \in [1, \infty)$ has the stated dependencies.
For Item (6), we take $L^p$ norms of (B.2) and use $\nabla A_0 a_0 = \nabla A_1 a_0 + [a_0, a_0]$ to give
\[
\|\nabla^2 A_1 \xi\|_{L^p(X)} \leq \|\nabla^2 A_0 \xi\|_{L^p(X)} + \|\nabla A_0 a_0 \times \xi\|_{L^p(X)} + \|a_0 \times \nabla A_0 \xi\|_{L^p(X)}
+ \|a_0 \times \nabla A_0 \xi\|_{L^p(X)}
\leq \|\nabla^2 A_0 \xi\|_{L^p(X)} + z\|\nabla A_1 a_0\|_{L^{2p}(X)}\|\xi\|_{L^{2p}(X)} + \|a_0\|_{C(X)}\|\nabla A_0 \xi\|_{L^p(X)}
+ 2z\|a_0\|_{C(X)}\|\xi\|_{L^p(X)}
\leq C\|\xi\|_{W^{2p,\infty}_{A_1}(X)},
\]
where $C \in [1, \infty)$ has the stated dependencies. The analogous inequality with the roles of $A_0$ and $A_1$ reversed follows by a symmetric argument.

**APPENDIX C. FREDHOLM AND INDEX PROPERTIES OF A HODGE LAPLACIAN WITH SOBOLEV COEFFICIENTS**

In this section we include proofs of results regarding the Fredholm properties of the Hodge Laplace operators encountered in Sections 2 and 3 that would be standard if the operator had smooth coefficients and acted on $L^2$ rather than $L^p$ Sobolev spaces as we allow here.

When $A$ is a smooth connection, the Hodge Laplace operator, $\Delta_A$, in (2.2) is an elliptic, second-order partial differential operator with smooth coefficients that is $L^2$-self-adjoint and so Theorem A.1 immediately provides the

**Proposition C.1** (Fredholm and index zero properties of a Laplace operator with smooth coefficients). Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and $G$ be a compact Lie group, and $P$ be a smooth principal $G$-bundle over $X$, and $l \geq 0$ an integer. If $A$ and $A_1$ are $C^\infty$ connections on $P$ and $1 < p < \infty$, then the operator,
\[
\Delta_A : W^{2p,\infty}_{A_1}(X; \Lambda^l \otimes \text{ad}P) \to L^p(X; \Lambda^l \otimes \text{ad}P),
\]
is Fredholm with index zero and closed range $K^\perp \cap L^p(X; \Lambda^l \otimes \text{ad}P)$, where $\perp$ denotes $L^2$-orthogonal complement and $K \subset W^{2p,\infty}_{A_1}(X; \Lambda^l \otimes \text{ad}P)$ is the kernel of $\Delta_A$ in (C.1).

Proposition C.1 and a compact operator perturbation argument provides following useful generalization from the case of a $C^\infty$ to a $W^{1,q}$ connection $A$.

**Corollary C.2** (Fredholm and index zero properties of a Laplace operator with Sobolev coefficients). Assume the hypotheses of Proposition C.1, but allow $A$ to be a $W^{1,q}$ connection with $d/2 < q < \infty$ and restrict $p \in (1, \infty)$ so that $d/2 \leq p \leq q$. Then the operator,
\[
\Delta_A : W^{2p,\infty}_{A_1}(X; \Lambda^l \otimes \text{ad}P) \to L^p(X; \Lambda^l \otimes \text{ad}P),
\]
is Fredholm with index zero and closed range $K^\perp \cap L^p(X; \Lambda^l \otimes \text{ad}P)$, where $\perp$ denotes $L^2$-orthogonal complement and $K \subset W^{2p,\infty}_{A_1}(X; \Lambda^l \otimes \text{ad}P)$ is the kernel of $\Delta_A$ in (C.2).
Proof. By hypothesis, $A_1$ is a $C^\infty$ connection on $P$. We write $A = A_1 + a$, for $a \in W^{1,q}_{A_1}(X; \Lambda^L \otimes \text{ad}P)$ and proceed by modifying the derivation of the $L^p$-bound (2.10) for $(\Delta_A - \Delta_{A_1})\xi$ in the proof of Proposition 2.1 to show that, for suitable $u \in [1, \infty)$ and choosing $A_3 = A_1$, the operator
\begin{equation}
\Delta_A - \Delta_{A_1} : W^{1,q}_{A_1}(X; \Lambda^L \otimes \text{ad}P) \rightarrow L^p(X; \Lambda^L \otimes \text{ad}P)
\end{equation}
is bounded and, because the Sobolev embedding $W^{2,p}(X) \subseteq W^{1,q}(X)$ will be compact by the Rellich-Kondrachov [4] Theorem 6.3, then the following composition of that compact embedding and the preceding bounded operator,
\begin{equation}
\Delta_A - \Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^L \otimes \text{ad}P) \rightarrow L^p(X; \Lambda^L \otimes \text{ad}P)
\end{equation}
is compact by [4] Proposition 6.3.

By retracing the steps in the proof of Proposition 2.1 we find that, for $r \in [p, \infty]$ defined by $1/p = 1/q + 1/r$,
\begin{equation}
\|\Delta_A - \Delta_{A_1}\|_{L^p(X)} \leq \|a\|_{W^{1,q}_{A_1}(X)} + \|a\|_{W^{1,q}_{A_1}(X)}^2 \|\xi\|_{L^p(X)} + \|\xi\|_{L^p(X)}^2 \|a\|_{W^{1,q}_{A_1}(X)} \|\xi\|_{W^{1,q}_{A_1}(X)},
\end{equation}
where i) $s = d$ when $q < d$, or ii) $s = d$ when $q = d$ and $p < d$, or iii) $s = 2d$ when $q = d = p$, or iv) $s = p$ when $q > d$. We consider each of these cases separately.

Case 1 ($p = d/2 < q < d$ and $s = d$). We have $r \in [p, \infty]$ since $p < q$ for this case and a continuous embedding $W^{1,d}(X) \subseteq L^r(X)$ by [4] Theorem 4.12, so the operator (C.3) is bounded for $u = d$. Moreover, $W^{2,p}(X) \subseteq W^{1,d}(X)$ is compact by [4] Theorem 6.3 provided $p^* = dp/(d - p) \geq d$ or equivalently $p \geq d/2$, which holds by hypothesis and the operator (C.4) is compact for this case.

Case 2 ($d/2 < p \leq q < d$ and $s = d$). We have $r \in [p, \infty]$ since $p < q$ for this case and a continuous embedding $W^{1,d+\varepsilon}(X) \subseteq L^r(X)$ by [4] Theorem 4.12 for any $\varepsilon > 0$. Also, we have a compact embedding $W^{2,p}(X) \subseteq W^{1,d+\varepsilon}(X)$ by [4] Theorem 6.3 provided $p^* = dp/(d - p) \geq d + \varepsilon$; because $dp/(d - p) > d$ when $p > d/2$, a choice of $\varepsilon \in (0, 1]$ is always possible in this case. Hence, the operator (C.3) is bounded for $u = d + \varepsilon$ and the operator (C.4) is compact for this case.

Case 3 ($d/2 \leq p < q = d$ and $s = d$). We have $r \in [p, \infty]$ since $p < q$ for this case and a continuous embedding $W^{1,d}(X) \subseteq L^r(X)$ by [4] Theorem 4.12. Also, we have a compact embedding $W^{2,p}(X) \subseteq W^{1,d}(X)$ by [4] Theorem 6.3 since $p \geq d/2$. Hence, the operator (C.3) is bounded for $u = d$ and the operator (C.4) is compact for this case.

Case 4 ($p = q = d$ and $s = 2d$). We have $r = \infty$ since $p = q$ for this case and a continuous embedding $W^{1,d+\varepsilon}(X) \subseteq L^\infty(X)$ by [4] Theorem 4.12 for any $\varepsilon > 0$ and in particular for $\varepsilon = d$. Also, we have a compact embedding $W^{2,p}(X) \subseteq W^{1,\varepsilon}(X)$ by [4] Theorem 6.3 for any $\varepsilon \in [1, \infty]$ since $p = d$ and in particular for $\varepsilon = 2d$. Hence, the operator (C.3) is bounded for $u = 2d$ and the operator (C.4) is compact for this case.

Case 5 ($d/2 \leq p < q$ and $q > d$ and $s = p$). We have $r \in [p, \infty)$ and a continuous embedding $W^{1,d}(X) \subseteq L^r(X)$ by [4] Theorem 4.12. Also, we have a compact embedding $W^{2,p}(X) \subseteq W^{1,d}(X)$ by [4] Theorem 6.3 since $p \geq d/2d$. Hence, the operator (C.3) is bounded for $u = d$ and the operator (C.4) is compact for this case.
Case 6 \((p = q = s > d)\). We have \(r = \infty\) and a continuous embedding \(W^{1,p}(X) \subset L^\infty(X)\) by [4, Theorem 4.12]. Also, we have a compact embedding \(W^{2,p}(X) \subset W^{1,p}(X)\) by [4, Theorem 6.3]. Hence, the operator (C.3) is a first-order differential operator. The conclusions now follow from Theorem A.1. Proposition C.3 implies that the operator
\[
\Delta_{A_1} : W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P)
\]
is Fredholm with index zero while the operator \(\Delta_A - \Delta_{A_1}\) in (C.4) is compact from each of the preceding cases, so the operator (C.2) is Fredholm with index zero by [56, Corollary 19.1.8].

The identification of the range of the operator (C.2) follows \textit{mutatis mutandis} the proof of the corresponding fact in the statement of Theorem A.1. The only difference, after noting that \(\Delta_A^* = \Delta_A\) and \(p \leq q \implies q' \leq p'\) and thus \(L^{p'}(X) \subset L^{q'}(X)\), is that we appeal to the following regularity result for distributional solutions, \(b \in L^{p'}(X; \Lambda^l \otimes \text{ad} P)\), to an elliptic linear partial differential equation, \(\Delta_A b = 0\), with Sobolev rather than \(C^\infty\) coefficients,
\[
\operatorname{Ker} \left( \Delta_A : L^{p'}(X; \Lambda^l \otimes \text{ad} P) \to W^{-2,p'}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \right) = \operatorname{Ker} \left( \Delta_A : W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P) \to L^p(X; \Lambda^l \otimes \text{ad} P) \right).
\]
Indeed, because \(b \in L^{p'}(X; \Lambda^l \otimes \text{ad} P)\) by the preceding remarks, Lemma 2.7 implies that \(b \in W^{2,q}_{A_1}(X; \Lambda^l \otimes \text{ad} P)\) and, in particular, \(b \in W^{2,p}_{A_1}(X; \Lambda^l \otimes \text{ad} P)\) since \(p \leq q\). This completes the proof of Corollary C.2.

We shall also need to consider Fredholm properties of the perturbed Laplace operator,
\[
d_A^* d_{A+a} : C^\infty(X; \text{ad} P) \to C^\infty(X; \text{ad} P),
\]
when a \(C^\infty\) connection, \(A\), one-form, \(a \in \Omega^l(X; \text{ad} P)\), and Fréchet space, \(\Omega^l(X; \text{ad} P)\), are replaced by suitable Sobolev counterparts. As usual, we begin with the simpler case of smooth coefficients.

**Lemma C.3** (Fredholm and index zero properties of a perturbed Laplace operator with smooth coefficients). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group and \(P\) be a smooth principal \(G\)-bundle over \(X\). If \(A\) and \(A_1\) are \(C^\infty\) connections on \(P\) and \(a \in C^\infty(X; \Lambda^1 \otimes \text{ad} P)\) and \(1 < q < \infty\), then the operator,
\[
d_A^* d_{A+a} : W^{2,q}_{A_1}(X; \text{ad} P) \to L^q(X; \text{ad} P),
\]
is Fredholm with index zero.

**Proof.** We observe that \(d_A^* d_{A+a} : C^\infty(X; \text{ad} P) \to C^\infty(X; \text{ad} P)\) is an elliptic, second-order partial differential operator such that
\[
d_A^* d_{A+a} - d_A^* d_A = d_A^*[a, \cdot] - [a, \cdot]^* d_A : C^\infty(X; \text{ad} P) \to C^\infty(X; \text{ad} P)
\]
is a first-order differential operator. The conclusions now follow from Theorem A.1.

**Lemma C.4** (Fredholm and index zero properties of a perturbed Laplace operator with Sobolev coefficients). Let \((X, g)\) be a closed, Riemannian, smooth manifold of dimension \(d \geq 2\), and \(G\) be a compact Lie group and \(P\) be a smooth principal \(G\)-bundle over \(X\). If \(A_1\) is a \(C^\infty\) connection
on $P$ and $A$ is a $W^{1,q}$ connection on $P$ with $d/2 < q < \infty$ and $a \in W^{1,q}_{A_1}(X; \Lambda^1 \otimes \text{ad} P)$, then the operator,
\[ d_A^* d_{A+a} : W^{2,q}_{A_1}(X; \text{ad} P) \to L^q(X; \text{ad} P), \]
is Fredholm with index zero.

Proof. The argument is almost identical to the proof of Corollary C.2. Write $A = A_1 + a_1$ and observe that
\[ d_A^* d_{A+a} = d_{A_1}^* d_{A_1} + d_{A_1}^*[a_1 + a, \cdot] + [a_1, \cdot]^* d_{A_1} + [a_1, \cdot]^*[a_1 + a, \cdot]. \]
By again retracing the steps in the proof of Proposition 2.1, we find that, for convergence rates, and stability of gradient flows defined by an energy function, Banach spaces with gradient maps valued in Hilbert spaces
\[ \text{observe that Theorem D.1 is a variant of Theorem 1 from } [32]. \]

Proof. The argument is almost identical to the proof of Corollary C.2. Write $A = A_1 + a_1$ and observe that
\[ d_A^* d_{A+a} = d_{A_1}^* d_{A_1} + d_{A_1}^*[a_1 + a, \cdot] + [a_1, \cdot]^* d_{A_1} + [a_1, \cdot]^*[a_1 + a, \cdot]. \]
By again retracing the steps in the proof of Proposition 2.1, we find that, for $r \in [p, \infty]$ defined by $1/p = 1/q + 1/r$ and $\xi \in W^{2,q}_{A_1}(X; \text{ad} P),$
\[ \|(d_A^* d_{A+a} - d_{A_1}^* d_{A_1})\xi\|_{L^p(X)} \]
\[ \leq z \left( \|a\|_{W^{1,q}_{A_1}(X)} + \|a_1\|_{W^{1,q}_{A_1}(X)} + \|a_1\|_{W^{1,q}_{A_1}(X)} \|a\|_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{L^p(X)} \]
\[ + z \left( \|a_1\|_{W^{1,q}_{A_1}(X)} + \|a\|_{W^{1,q}_{A_1}(X)} \right) \|\xi\|_{W^{1,q}_{A_1}(X)}, \]
with the values of $s$ specified in the proof of Corollary C.2. The remainder of the proof of Corollary C.2 now applies to show that the operator C.7 is Fredholm with index zero. □

Appendix D. Convergence of gradient flows under the validity of the Lojasiewicz–Simon gradient inequality

While Theorem 1 has important applications to proofs of global existence, convergence, convergence rates, and stability of gradient flows defined by an energy function, $\mathcal{E} : \mathcal{X} \supset \mathcal{U} \to \mathbb{R}$, with gradient map, $\mathcal{M} : \mathcal{X} \supset \mathcal{U} \to \mathcal{Y},$ (see [27, Section 2.1] for an introduction and Simon [80] for his pioneering development), the gradient inequality (1.3) is most useful when it has the form,
\[ \|\mathcal{M}(x)\|_{\mathcal{H}} \geq Z|\mathcal{E}(x) - \mathcal{E}(x_{\infty})|^{\theta}, \quad \forall x \in \mathcal{U} \text{ with } \|x - x_{\infty}\|_{\mathcal{X}} < \sigma, \]
where $\mathcal{H}$ is a Hilbert space and the Banach space, $\mathcal{X}$, is a dense subspace of $\mathcal{H}$ with continuous embedding, $\mathcal{X} \subset \mathcal{H}$, and so $\mathcal{H}^* \subset \mathcal{X}^*$ is also a continuous embedding. To that end, we recall the following variant of Theorem 1 from [32].

Theorem D.1 (Generalized Lojasiewicz–Simon gradient inequality for analytic functions on Banach spaces with gradient maps valued in Hilbert spaces). (See [32, Theorem 3].) Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be Banach spaces with continuous embeddings, $\mathcal{X} \subset \tilde{\mathcal{X}} \subset \mathcal{X}^*$. Let $\mathcal{U} \subset \mathcal{X}$ be an open subset, $\mathcal{E} : \mathcal{U} \to \mathbb{R}$ be an analytic function, and $x_{\infty} \in \mathcal{U}$ be a critical point of $\mathcal{E}$, that is, $\mathcal{E}'(x_{\infty}) = 0$. Let $\mathcal{H}$ be a Hilbert space and let
\[ \mathcal{X} \subset \mathcal{U} \subset \mathcal{H} \quad \text{and} \quad \tilde{\mathcal{X}} \subset \mathcal{H} \subset \mathcal{X}^*, \]
be continuous embeddings of Banach spaces such that the compositions,
\[ \mathcal{X} \subset \mathcal{U} \subset \mathcal{H} \quad \text{and} \quad \tilde{\mathcal{X}} \subset \mathcal{H} \subset \mathcal{X}^*, \]
induce the same embedding, \( \mathcal{X} \subset \mathcal{H} \). Let \( \mathcal{M} : U \to \tilde{\mathcal{X}} \) be a gradient map for \( \mathcal{E} \) in the sense of Definition 1.1. Suppose that for each \( x \in U \), the bounded linear operator,

\[
\mathcal{M}'(x) : \mathcal{X} \to \tilde{\mathcal{X}},
\]

has an extension to a bounded linear operator,

\[
\mathcal{M}_1(x) : \mathcal{I} \to \mathcal{H},
\]

and such that the following map is continuous,

\[
U \ni x \mapsto \mathcal{M}_1(x) \in \mathcal{L}(\mathcal{I}, \mathcal{H}).
\]

If \( \mathcal{M}'(x_\infty) : \mathcal{X} \to \tilde{\mathcal{X}} \) and \( \mathcal{M}_1(x_\infty) : \mathcal{I} \to \mathcal{H} \) are Fredholm operators with index zero, then there are constants, \( Z \in (0, \infty) \) and \( \sigma \in (0, 1] \) and \( \theta \in [1/2, 1) \), with the following significance. If \( x \in U \) obeys

\[
(D.1) \quad \|x - x_\infty\|_\mathcal{X} < \sigma,
\]

then

\[
(D.2) \quad \|\mathcal{M}(x)\|_\mathcal{H} \geq Z|\mathcal{E}(x) - \mathcal{E}(x_\infty)|^\theta.
\]

Let us now briefly explain why Theorem D.1 is so useful in applications to questions of global existence and convergence of a strong solution, that is, \( u \in C([0, T); \mathcal{X}) \) with time derivative \( \dot{u} \in C((0, T); \mathcal{H}) \) (for \( T \in (0, \infty) \)), to the Cauchy problem for the gradient system

\[
(D.3) \quad \dot{u}(t) = -\mathcal{M}(u(t)) \quad \text{in} \; \mathcal{H}, \quad t \in (0, T), \quad u(0) = u_0.
\]

The importance of a geometric version of Theorem D.1 to a more specific setting in geometric analysis was famously pioneered by Simon in [86], generalizing a result of Lojasiewicz for gradient flows in Euclidean spaces [73].

A weak solution to the gradient system for \( \mathcal{E} \) has the form \( u \in C([0, T); \mathcal{X}) \) with time derivative \( \dot{u} \in C((0, T); \mathcal{X}^*) \), obeying

\[
(D.4) \quad \dot{u}(t) = -\mathcal{E}'(u(t)) \quad \text{in} \; \mathcal{X}^*, \quad t \in (0, T), \quad u(0) = u_0.
\]

To illustrate the application of Theorem D.1 we include from [27] a proof of a simplified version of our [27, Proposition 24.12] that yields convergence, \( u(t) \to u_\infty \) in \( \mathcal{H} \) as \( t \to \infty \) for a global strong solution, \( u \in C([0, \infty); \mathcal{X}) \cap C^1([0, \infty); \mathcal{H}) \) to (D.3), when \( \mathcal{M} \) and \( \mathcal{E} \) obey the version of Lojasiewicz–Simon gradient inequality in (D.2).

The statement and proof of Huang’s Proposition D.2 are closely modeled on Huang’s [57, Proposition 3.3.2], but for the gradient system (D.3) in a Hilbert space. By contrast, Huang’s version allows apparently more general weak gradient-like differential inequalities in Banach spaces, namely [57, Equation (3.10a) or Equation (3.10')] with auxiliary conditions such as those in his [57, Equation (3.10b)] or [57, Equation (3.10')]. However, examples satisfying Huang’s gradient-like differential inequalities and auxiliary conditions appear to us to be difficult to find except when they reduce to a pure gradient system (D.3) in a Hilbert space or Simon’s gradient-like system [86, Equation (3.1)] in a Hilbert space,

\[
\dot{u}(t) = -\mathcal{M}(u(t)) + \mathcal{R}(t) \quad \text{in} \; \mathcal{H}, \quad t \in (0, T), \quad u(0) = u_0,
\]
where \( \mathcal{R} \in C((0, \infty); \mathcal{H}) \) obeys a decay condition (as \( t \to \infty \)) implying Huang’s [57, Equation (3.10b)] or Simon’s hypothesis in [86, Equation (3.1)],

\[
\|\mathcal{R}(t)\|_\mathcal{H} \leq \alpha \|\dot{u}(t)\|_\mathcal{H},
\]

where \( \alpha \in (0, 1) \) is a constant.

**Proposition D.2** (Convergence of gradient flow under the validity of the Lojasiewicz–Simon gradient inequality). Let \( \mathcal{U} \) be an open subset of a Banach space, \( \mathcal{X} \), that is continuously embedded and dense in a Hilbert space, \( \mathcal{H} \). Let \( \mathcal{E} : \mathcal{U} \subset \mathcal{X} \to \mathbb{R} \) be a \( C^1 \) function on an open subset, \( \mathcal{U} \subset \mathcal{X} \), with gradient map \( \mathcal{M} : \mathcal{U} \subset \mathcal{X} \to \mathcal{H} \), and \( x_\infty \in \mathcal{U} \) be a critical point of \( \mathcal{E} \), that is, \( \mathcal{E}'(x_\infty) = 0 \). Let \( u \in C([0,\infty);\mathcal{U}) \cap C^1([0,\infty);\mathcal{H}) \) be a strong solution to \( \text{(D.3)} \) such that

\[
\inf\{\|\mathcal{E}(u(t))\| : t \geq 0\} > -\infty.
\]

If \( \mathcal{E} \) and \( \mathcal{M} \) satisfy a Lojasiewicz–Simon gradient inequality \( \text{(D.2)} \) in the orbit \( O(u) = \{u(t) : t \geq 0\} \), that is,

\[
\|\mathcal{M}(u(t))\|_\mathcal{H} \geq Z|\mathcal{E}(u(t)) - \mathcal{E}(x_\infty)|^\theta, \quad \forall t \geq 0,
\]

for constants \( Z \in (0, \infty) \) and \( \theta \in [1/2, 1) \), then

\[
\int_0^\infty \|\dot{u}(t)\|_\mathcal{H} dt \leq \int_{\mathcal{E}_\infty}^\infty \frac{1}{Z|s - \mathcal{E}(x_\infty)|^{\theta}} ds < \infty,
\]

where \( \mathcal{E}_\infty := \lim_{t \to \infty} \mathcal{E}(u(t)) \in \mathbb{R} \), and thus

\[
u(t) \to u_\infty \quad \text{in} \quad \mathcal{H}, \quad \text{as} \ t \to \infty,
\]

for some \( u_\infty \in \mathcal{H} \).

**Proof.** The function \( [0, \infty) \ni t \mapsto \mathcal{E}(u(t)) \in \mathbb{R} \) is \( C^1 \) by direct calculation and obeys (see [57, Proposition 3.1.2])

\[
-\frac{d}{dt} \mathcal{E}(u(t)) = -\mathcal{E}'(u(t))(\dot{u}(t)) \quad \text{(Chain Rule)}
\]

\[
= -\langle \dot{u}(t), \mathcal{M}(u(t)) \rangle_{\mathcal{X} \times \mathcal{X}^*} \quad \text{(by Definition \[11\])}
\]

\[
= -\langle \dot{u}(t), \mathcal{M}(u(t)) \rangle_\mathcal{H} \quad \text{(by} \mathcal{H} \subset \mathcal{X}^* \text{ via} \mathcal{H} \ni h \mapsto (., h)_\mathcal{H} \ni \mathcal{H}^*)
\]

\[
= \|\mathcal{M}(u(t))\|^2_\mathcal{H} = \|\mathcal{M}(u(t))\|_\mathcal{H} \|\dot{u}(t)\|_\mathcal{H} \geq 0, \quad \forall t \in (0, \infty) \quad \text{(by} \text{D.3).}
\]

Hence, \( \dot{\mathcal{E}}(u(t)) \) is a nonincreasing and uniformly bounded function of \( t \in [0, \infty) \) by \( \text{(D.5)} \), so \( \mathcal{E}_\infty = \lim_{t \to \infty} \mathcal{E}(u(t)) \) exists, as asserted by the proposition. Set \( H(t) := \mathcal{E}(u(t)) \), for all \( t \in [0, \infty) \), and observe that \( H(t) \) is monotone and absolutely continuous on \([0, \infty)\) and obeys, by the preceding equality,

\[
-\frac{d}{dt} H(t) = \|\mathcal{M}(u(t))\|_\mathcal{H} \|\dot{u}(t)\|_\mathcal{H}, \quad \forall t \in [0, \infty).
\]

Let \( \phi : \mathbb{R} \to \mathbb{R} \) be the function defined by \( \phi(s) = Z|s - \mathcal{E}(x_\infty)|^\theta \), for all \( s \in \mathbb{R} \), and let \( \Phi : \mathbb{R} \to \mathbb{R} \) be the absolutely continuous function given by

\[
\Phi(x) := \int_0^x \frac{1}{\phi(s)} ds = \int_{\mathcal{E}_\infty}^x \frac{1}{Z|s - \mathcal{E}(x_\infty)|^{\theta}} ds, \quad \forall x \in \mathbb{R},
\]
where \( \lim_{t \to \infty} H(t) = \mathcal{E}_\infty \). The function \( \Phi \) is differentiable a.e. on \( \mathbb{R} \) with \( \Phi'(x) = 1/\phi(x) \) for a.e. \( x \in \mathbb{R} \). According to [57, Lemma 3.2.1], the composition \( \Phi \circ H \) is absolutely continuous on \([0, \infty)\) and there holds

\[ \frac{d}{dt} \Phi(H(t)) = \frac{H'(t)}{\phi(H(t))}, \quad \forall t \in \Lambda, \]

where \( \Lambda \subset [0, \infty) \) is such that the complement, \([0, \infty) \setminus \Lambda \), has zero Lebesgue measure.

For any \( t \in \Lambda \), we have two possibilities: either i) \( \|\mathcal{M}(u(t))\|_{\mathcal{H}} = 0 \), or ii) \( \|\mathcal{M}(u(t))\|_{\mathcal{H}} > 0 \). For Case i), we observe that the Lojasiewicz–Simon gradient inequality (D.6) takes the shape,

\[ \phi(H(t)) = Z|\mathcal{E}(u(t)) - \mathcal{E}(x_{\infty})|^\theta \leq \|\mathcal{M}(u(t))\|_{\mathcal{H}}, \]

and so

\[ -\frac{d}{dt} \Phi(H(t)) = -\frac{H'(t)}{\phi(H(t))} \quad \text{(by (D.9))} \]

\[ = \frac{\|\mathcal{M}(u(t))\|_{\mathcal{H}} \|\dot{u}(t)\|_{\mathcal{H}}}{\phi(H(t))} \quad \text{(by (D.8))} \]

\[ \geq \frac{\|\mathcal{M}(u(t))\|_{\mathcal{H}} \|\dot{u}(t)\|_{\mathcal{H}}}{\|\mathcal{M}(u(t))\|_{\mathcal{H}}} \quad \text{(by (D.10))} \]

\[ = \|\dot{u}(t)\|_{\mathcal{H}}, \]

that is,

\[ -\frac{d}{dt} \Phi(H(t)) \geq \|\dot{u}(t)\|_{\mathcal{H}}. \]

Therefore, by the non-negativity of the function \(-d\Phi(H(t))/dt\), combined with the fact that \([0, \infty) \setminus \Lambda \) has Lebesgue measure zero, we obtain the estimate,

\[ -\frac{d}{dt} \Phi(H(t)) \geq \|\dot{u}(t)\|_{\mathcal{H}}, \quad \text{a.e. } t \in [0, \infty), \]

for both Cases i) and ii). Integration and the fact that \( \lim_{t \to \infty} H(t) = \mathcal{E}_\infty \) yields

\[ \int_0^\infty \|\dot{u}(t)\|_{\mathcal{H}} \, dt \leq \Phi(H(0)) - \lim_{t \to \infty} \Phi(H(t)) = \Phi(H(0)) - \Phi(\mathcal{E}_\infty) = \Phi(H(0)). \]

By the definitions of \( \Phi(x) \) and \( H(t) \), this is (D.7), since

\[ \Phi(H(0)) = \int_{\mathcal{E}_\infty}^{H(0)} \frac{1}{\phi(s)} \, ds = \int_{\mathcal{E}_\infty}^{\mathcal{E}(u(0))} \frac{1}{Z|s - \mathcal{E}(x_{\infty})|^\theta} \, ds. \]

The final convergence assertion follows from the fact that

\[ u(t_n) - u(t_m) = \int_{t_m}^{t_n} \dot{u}(t) \, dt, \quad \forall t_m, t_n \in [0, \infty), \]

and thus, for any unbounded sequence of times, \( \{t_n\}_{n=1}^\infty \subset [0, \infty) \), the sequence of points, \( \{u(t_n)\}_{n=1}^\infty \subset \mathcal{H} \), is Cauchy in \( \mathcal{H} \) and thus converges to a limit, \( u_\infty \in \mathcal{H} \), as \( t \to \infty \), independent of the choice, \( \{t_n\}_{n=1}^\infty \). This completes the proof of Proposition D.2. \qed

The convergence of \( u(t) \) in \( \mathcal{H} \) provided by Proposition D.2 can be improved with the assumption of the following
Hypothesis D.3 (Regularity and a priori interior estimate for a trajectory). (See [27] Hypothesis 24.10.) Let $C_1$ and $\rho$ be positive constants and let $T \in (0, \infty]$. Given $u \in C([0, \infty) ; \mathcal{X}) \cap C^1([0, \infty) ; \mathcal{H})$, we say that $\dot{u} : [0, T) \to \mathcal{X}$ obeys an a priori interior estimate on $[0, T]$ if, for every $S \geq 0$ and $\delta > 0$ obeying $S + \delta \leq T$, the map $\dot{u} : [S + \delta, T) \to \mathcal{X}$ is Bochner integrable and there holds

\[
\int_{S+\delta}^{T} \| \dot{u}(t) \|_X \, dt \leq C_1(1 + \delta^{-\rho}) \int_{S}^{T} \| \dot{u}(t) \|_{\mathcal{H}} \, dt.
\]

Given Hypothesis D.3, the bound (D.7) in Proposition D.2 improves to

\[
\int_{\delta}^{\infty} \| \dot{u} \|_X \, dt \leq C_1(1 + \delta^{-\rho}) \int_{s_{\infty}}^{E(u(0))} \frac{1}{c|s - E(x_{\infty})|^\rho} \, ds < \infty,
\]

for all $\delta > 0$, and so the convergence in $\mathcal{H}$ improves to $u(t) \to u_\infty$ in $\mathcal{X}$ as $t \to \infty$.

The application of Proposition D.2 and similar results to proofs of global existence, convergence, convergence rates, and stability of solutions to (D.3) are described at length in [27, Section 2.1].

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