Higher Schwarzian Derivative and Dirichlet Morrey Space

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Abstract. We treat the logarithmic derivative model and Schwarzian derivative model of the Dirichlet-Morrey Teichmüller space. It is shown that the higher Bers maps, induced by the higher Schwarzian differential operators, are holomorphic in Dirichlet-Morrey Teichmüller space. It is also shown that the logarithmic derivative model of this Teichmüller space is connected.

1. Introduction

Let \( \mathbb{D} = \{ z : |z| < 1 \} \) be the unit disc in the extended complex plane \( \hat{\mathbb{C}} \). Let \( \mathbb{D}' \) be the exterior of \( \mathbb{D} \) and \( S^1 = \partial \mathbb{D} \) be the boundary of \( \mathbb{D} \). Denote by \( M(\mathbb{D}') \) the open unit ball of the Banach space \( L^\infty(\mathbb{D}') \) of all Beltrami differentials \( \mu(z) \) on \( \mathbb{D}' \). It is well known that for each \( \mu(z) \in M(\mathbb{D}') \), there exists a unique quasiconformal mapping \( f_\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) whose complex dilatation is equal to \( \mu \) in \( \mathbb{D}' \) and is zero in \( \mathbb{D} \), normalized by

\[
 f_\mu(0) = (f_\mu)'(0) - 1 = (f_\mu)'(0) = 0,
\]

(see [1] [3]). Two Beltrami coefficients \( \mu_1 \) and \( \mu_2 \) in \( M(\mathbb{D}') \) are said to be Teichmüller equivalent, denoted by \( \mu_1 \sim \mu_2 \), if \( f^{\mu_1}(\mathbb{D}) = f^{\mu_2}(\mathbb{D}) \). The universal Teichmüller space \( T \) is defined as \( T = M(\mathbb{D}')/\sim \), where \([\mu]\) is the Teichmüller equivalent class containing \( \mu \in M(\mathbb{D}') \).

The Schwarzian derivative \( S_f \) of a conformal mapping \( f \) in \( \mathbb{D} \) is defined by

\[
 S_f = \left( N_f \right)' - \frac{1}{2} \left( N_f \right)^2,
\]

where \( N_f = (\log f)' \).

Denote by \( B_\alpha(\mathbb{D}) \) the Banach space of all holomorphic functions \( \varphi \) in \( \mathbb{D} \) with the following finite norm

\[
 \| \varphi \|_n = \sup_{z \in \mathbb{D}} |\varphi(z)|(1 - |z|^2)^n, \quad n = 1, 2, \ldots.
\]

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It is well known that the Bers projection
\[ \beta_3 : M(D^\nu) \to B_2(D), \quad \beta_3(\mu) = S_{f_{|1r}}, \]
is a holomorphic split submersion from \( M(D^\nu) \) onto its image, which descends down to the Bers embedding \( B : T \to B_2(D) \). Via the Bers embedding, \( T \) carries a natural complex Banach manifold structure so that \( \Phi : M(D^\nu) \to T \) is a holomorphic split submersion which sends \( \mu \) to the equivalent class \([\mu]\) (see [17], [18]).

It is of interest to embed the universal Teichmüller space onto an open subset of some complex Banach space of holomorphic functions in \( D \) in terms of some general differential operators. Krushkal considered in [16] some nonlinear differential operators of higher order of the form
\[ P_n(f) = F \left( \frac{f''(z)}{f'(z)}, \frac{f'''(z)}{f'(z)}, \ldots, \frac{f^{(n)}(z)}{f'(z)}, f''(z), \ldots, f^{(0)}(z) \right), \quad z \in D, \]
where \( F \) is an analytic function of its arguments \( n \geq 2 \). It was proved [16] that the map \( P_n : M(D^\nu) \to B_{n-1}(D) \), which is defined by the correspondence of \( \mu \in M(D^\nu) \) to \( P_n(f^\nu) \in B_{n-1}(D) \), \( n \geq 3 \), is holomorphic.

Schippers considered in [20] some other nonlinear differential operators. For \( n \geq 3 \), define \( \sigma_3(f) = S_f \) and
\[ \sigma_{n+1}(f)(z) = \sigma_n(f)(z) - (n-1)N_f(z)\sigma_n(f)(z). \]

For more general differential operators, we refer the reader to [2], [13], [15], [23] and references therein.

Buss [7] proved that the higher Bers map \( \beta_n : M(D^\nu) \to B_{n-1}(D) \), which is defined by the correspondence of \( \mu \in M(D^\nu) \) to \( \sigma_n(f^\nu) \in B_{n-1}(D) \), \( n \geq 3 \), is holomorphic.

**Theorem 1.1.** [7] Let \( n \geq 3 \). The higher Bers map \( \beta_n : M(D^\nu) \to B_{n-1}(D) \) is holomorphic. The differential \( D_\nu \beta_n \) at the origin is given by the following correspondence
\[ \nu \mapsto \frac{(-1)^n n!}{\pi} \int_{D^\nu} \frac{\nu(w)}{(z-w)^{n+1}}dudv, \]
which induces a bounded surjective operator from \( L^\nu(D^\nu) \) onto \( B_{n-1}(D) \).

It should be pointed out that the case \( n = 3 \) is the classical result of Bers [6]. The higher Bers maps on Weil-Petersson and BMO Teichmüller space were also investigated recently by the authors (see [25] [26]). In this paper, we will treat the higher Bers maps on the Dirichlet-Morrey Teichmüller space.

Let
\[ S_D(I) = \{ r\zeta \in D : 1 - |r| \leq r < 1, \zeta \in I \} \]
denote the Carleson square in \( D \) and
\[ S_{D^\nu}(I) = \{ r\zeta \in D^\nu : 1 \leq r < 1 + |r|, \zeta \in I \} \]
denote the Carleson square in \( D^\nu \), where \( I \) be an open sub-arc of \( S^1 \). For \( 0 < q < \infty \), a non-negative Borel measure \( \mu \) on \( D \) is called \( q \)-Carleson measure if
\[ ||\mu||_{D,q} := \sup_{I \subset \partial D} \frac{\mu(S_D(I))}{|I|^q} < \infty. \]
Replacing \( S_D(I) \) by \( S_{D^\nu}(I) \), we can define \( q \)-Carleson measure on \( D^\nu \) similarly. Clearly, \( \mu \) is the classical Carleson measure for the case \( q = 1 \). Denote by \( CM_q(D) \) the set of all \( q \)-Carleson measures on \( D \) and \( CM_q(D^\nu) \) the set of all \( q \)-Carleson measures on \( D^\nu \). It is well known that a non-negative Borel measure \( \mu \) belongs to \( CM_q(D) \) if and only if
\[ \sup_{I \subset \partial D} \int_{D} \frac{(1 - |r|^2)^q}{|I - r\zeta|^q}d\mu(z) < \infty, \]

(4)
where \( p \in (0, \infty) \) (see [30]).

The Bloch space \( \mathcal{B} \) consists of all analytic functions \( f \) in \( \mathbb{D} \) so that

\[
||f||_{\mathcal{B}} := \sup_{z \in \mathbb{D}} |f'(z)(1 - |z|^2)| < \infty.
\]

The little Bloch space \( \mathcal{B}_0 \), a closed subspace of \( \mathcal{B} \), consists of functions \( f \in \mathcal{B} \) such that

\[
\lim_{|z| \to 1} |f'(z)(1 - |z|^2)| = 0.
\]

Let \( 0 \leq p < \infty \), the weighted Dirichlet space \( \mathcal{D}^p(\mathbb{D}) \) is the set of all analytic functions \( f \) in \( \mathbb{D} \) for which

\[
||f||_{\mathcal{D}^p(\mathbb{D})}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^p dm(z) < \infty,
\]

where \( dm(z) \) denotes the normalized Lebesgue area measure.

For \( 0 < \lambda, p \leq 1 \), the Dirichlet-Morrey space \( \mathcal{D}^\lambda_p(\mathbb{D}) \), introduced recently in [12], consists of those analytic functions \( f \) in \( \mathbb{D} \) such that

\[
||f||_{\mathcal{D}^\lambda_p(\mathbb{D})} = \sup_{a \in \mathbb{D}} \left( 1 - |a|^2 \right)^{\frac{\lambda - 1}{p - 1}} \|f \circ \varphi_a - f(a)\|_{\mathcal{D}^p} < \infty,
\]

where \( \varphi_a(z) = \frac{a-z}{1-\overline{a}z}, z \in \mathbb{D}, a \in \mathbb{D} \). Some basic properties of Dirichlet-Morrey space were characterized in [12].

The authors obtained in [24] the following result.

**Theorem 1.2.** [24] Suppose that \( f \) is a bounded univalent function in \( \mathbb{D} \) and \( \log f' \in \mathcal{B}_0, 0 < \lambda < 1 \) and \( 0 < p \leq 1 \). Then the following statements are equivalent:

1. \( \log f' \in \mathcal{D}^\lambda_p(\mathbb{D}) \);
2. \( |f(z)|^2 (1 - |z|^2)^p dm(z) \in CM_{p,1}(\mathbb{D}) \);
3. \( f \) can be extended to a quasiconformal mapping in the extended plane \( \hat{\mathbb{C}} \) such that its complex dilatation \( \mu \) satisfies

\[
\lim_{|z| \to 1} \mu(z) = \frac{|z|^\lambda}{(1 - |z|^2)^{p-1}} dm(z) \in CM_{p,1}(\mathbb{D}^\prime).
\]

Denote by \( \mathcal{L}(\mathbb{D}^\prime) \) the Banach space of all essentially bounded measurable functions \( \mu \) on \( \mathbb{D}^\prime \) each of which induces a \( p\lambda \)-Carleson measure \( \eta_\mu = \frac{|\mu(z)|}{(1 - |z|^2)^{p-1}} dm(z) \). The norm on \( \mathcal{L}(\mathbb{D}^\prime) \) is defined as

\[
\|\mu\|_{\mathcal{L}} = \|\mu\|_{\infty} + \|\eta_\mu\|_{CM_{p,1}} < \infty.
\]

Let \( \mathfrak{M}(\mathbb{D}^\prime) = M(\mathbb{D}^\prime) \cap \mathcal{L}(\mathbb{D}^\prime) \). Dirichlet-Morrey Teichmüller space \( T_{\mathfrak{M},p,1} \) is defined as \( \mathfrak{M}(\mathbb{D}^\prime)/\sim \), where \( \sim \) denotes the Teichmüller equivalent relation defined as above.

We use \( \mathcal{N}_{p,1,n}(\mathbb{D}) (n \geq 3) \) to denote the space of all analytic functions \( f \) in \( \mathbb{D} \) with the norm

\[
||f||_{\mathcal{N}_{p,1,n}}^2 = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{2n-4+4p} \left( \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right)^p dm(z) < \infty.
\]

In this paper, we shall prove the following

**Theorem 1.3.** Let \( n \geq 3 \). The higher Bers map \( \beta_n : \mathfrak{M}(\mathbb{D}^\prime) \to \mathcal{N}_{p,1,n}(\mathbb{D}) \) is well defined and holomorphic. The differential \( D_0 \beta_n \) at the origin is given by the following correspondence

\[
\mu \mapsto \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}} \frac{\mu(w)}{(z - w)^{n+1}} dudv.
\]
We also consider the pre-logarithmic derivative model of the Dirichlet-Morrey Teichmüller space. Let us first recall some notions and definitions.

Let \( S_Q \) be the class of all univalent analytic functions \( f \) in \( \mathbb{D} \), which can be extended to a quasiconformal mapping in \( \mathbb{C} \), normalized by \( f(0) = f'(0) - 1 = 0 \). Then the universal Teichmüller space can be described as \( T(1) = \{ \log f' : f \) belongs to \( S_Q \} \). It is well known that \( T(1) \) is a disconnected subset of Bloch space \( \mathcal{B} \), and \( T_0 = \{ \log f' \in T(1) : f(\theta) = \infty, \theta \in [0,2\pi) \} \) are connected components of \( T(1) \) (see [32]).

In recent years, the pre-logarithmic derivative model of the universal Teichmüller space and its subspaces have been much investigated (See [4] [5] [8] [9] [10] [14] [21] [22] [11] [27] [28] [29] [32]).

We consider the pre-logarithmic derivative model \( T_{DM}^0(1) \) of Dirichlet-Morrey Teichmüller space, which is defined as

\[
T_{DM}^0(1) = \{ \log f' : f \in S_Q \text{ and } \log f' \in \mathcal{B}_0 \cap \mathcal{D}_A' \}.
\]

We endow the space \( \mathcal{B}_0 \cap \mathcal{D}_A' \) the following norm

\[
\| \psi \|_{\mathcal{B}_0, \mathcal{D}_A'} = \| \psi \|_{\mathcal{B}} + \| \psi \|_{\mathcal{D}_A'(\mathbb{D})}.
\]

Let \( T_{DM}^0(1) = \{ \log f' \in T_{DM}^0(1) : f(\mathbb{D}) \) is bounded \}. We obtain the following

**Theorem 1.4.** \( T_{DM}^0(1) \) is connected in \( \mathcal{B}_0 \cap \mathcal{D}_A'(\mathbb{D}) \).

Throughout this paper, we use the notation \( a \leq b \) to denote that there is a constant \( C > 0 \) such that \( a \leq Cb \), and the notation \( a \equiv b \) to indicate that \( a \leq b \leq a \).

### 2. Proof of Theorem 1.3

We shall prove Theorem 1.3 in this section. Some lemmas are needed. The following result gives some higher derivative characterizations of \( \mathcal{D}_A'(\mathbb{D}) \) (see [12]).

**Lemma 2.1.** [12] Let \( f \) be an analytic function on \( \mathbb{D} \) and \( 0 < p, \lambda \leq 1 \). Then \( d\mu(z) = |f(z)|^\lambda(1 - |z|^2)^p \mathcal{d}m(z) \) is a \( p\lambda \)-Carleson measure if and only if \( dv(z) = |f'(z)|^\lambda(1 - |z|^2)^{p+2} \mathcal{d}m(z) \) is a \( p\lambda \)-Carleson measure. Furthermore,

\[
\sup_{z \in \mathbb{D}} (1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} |f(z)|^\lambda(1 - |z|^2)^p \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^p \mathcal{d}m(z) =
\]

\[
\sup_{z \in \mathbb{D}} (1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} |f'(z)|^\lambda(1 - |z|^2)^{p+2} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^{p+2} \mathcal{d}m(z).
\]

We also need the following result (see [31]).

**Lemma 2.2.** [31] Suppose that \( k > -1, r, t > 0, \) and \( r + t - k > 2 \). If \( t < k + 2 < r \), then there exists a universal constant \( C > 0 \) such that for all \( z, \zeta \in \mathbb{D} \),

\[
\int_{\mathbb{D}} \frac{(1 - |w|^2)^k}{|1 - \bar{w}z|^t |1 - \bar{w}\zeta|^r} \mathcal{d}m(w) \leq C \frac{(1 - |z|^2)^{2+k-r}}{|1 - \bar{z}\zeta|^t}.
\]

where \( w = u + iv \).

We now show that the higher Bers map is well defined on Dirichlet-Morrey Teichmüller space.

**Lemma 2.3.** Let \( n \geq 3 \). If \( \mu \in \mathfrak{M}(\mathcal{D}') \), then \( \sigma_n(f^\mu) \in \mathcal{N}_{p,\lambda,n}(\mathbb{D}) \).
Proof. We will prove this Lemma by using mathematical induction. It follows from Corollary 2.5 in [24] that if $\mu \in \mathfrak{M}(\mathbb{D}^r)$, then $\sigma_3(f^\mu)(z) \in \mathcal{N}_{p,1,3}$. Now suppose that $\sigma_n(f^\mu) \in \mathcal{N}_{p,1,n}$, $n \geq 3$, we shall prove that $\sigma_{n+1}(f^\mu) \in \mathcal{N}_{p,1,n+1}$.

Indeed, by Lemma 2.1, we have

$$\sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p |\sigma_n(f^\mu)(z)|^2 (1-|z|^2)^{2n-4+2p} dm(z) < \infty.$$  \hspace{1cm} (5)

Observing that

$$\sigma_{n+1}(f^\mu)(z) = \sigma_n(f^\mu)(z) - (n-1)\mathcal{N}_{p,\nu}(z)\sigma_n(f^\mu)(z), n \geq 3,$$

we deduce that

$$|\sigma_{n+1}(f^\mu)(z)| \leq |\sigma_n(f^\mu)(z)| + |(n-1)\mathcal{N}_{p,\nu}(z)\sigma_n(f^\mu)(z)|.$$  \hspace{1cm} (6)

Noting that $f^\mu$ is a univalent analytic function in $\mathbb{D}$, we conclude from [19] that

$$\sup_{z \in \mathbb{D}}|\mathcal{N}_{p,\nu}(z)|(1-|z|^2) \leq 6.$$  \hspace{1cm} (7)

Consequently, combing (5), (6) with (7) gives

$$\sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p |\sigma_{n+1}(f^\mu)(z)|^2 (1-|z|^2)^{2n-4+4p} dm(z)$$

$$\leq \sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p |\sigma_n(f^\mu)(z)|^2 (1-|z|^2)^{2n-4+2p} dm(z)$$

$$+ \sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p |\sigma_n(f^\mu)(z)|^2 (1-|z|^2)^{2n-4+4p} dxdy$$

$$< \infty.$$  \hspace{1cm} (8)

This implies that $\sigma_{n+1}(f^\mu) \in \mathcal{N}_{p,1,n+1}$. The proof of Lemma 2.3 is completed. \hfill \Box

The following result shows that the Bers map $\beta_3 : \mathfrak{M}(\mathbb{D}^r) \rightarrow \mathcal{N}_{p,1,3}(\mathbb{D})$ is Lipschitz continuous.

**Lemma 2.4.** Let $0 < p, \lambda \leq 1$. For any $\mu, \nu \in \mathfrak{M}(\mathbb{D}^r)$, the following inequality holds.

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{N}_{p,1,3}} \leq \|\mu - \nu\|_L.$$

**Proof.** In [4], it is proved that for any two elements $\mu, \nu \in M(\mathbb{D}^r)$,

$$\|\beta_3(\mu) - \beta_3(\nu)\|^2 (1-|z|^2) \leq \int_{\mathbb{D}} \frac{|\mu(\zeta) - \nu(\zeta)|^2 + \|\mu - \nu\|_L^2 |\mu(\zeta)|^2}{|\zeta - z|^4} dm(\zeta).$$

Therefore,

$$\|\beta_3(\mu) - \beta_3(\nu)\|^2_{\mathcal{N}_{p,1,3}} = \sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p \|\beta_3(\mu) - \beta_3(\nu)\|^2 (1-|z|^2)^{2+2p} dm(z)$$

$$\leq \sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p \|\mu(\zeta) - \nu(\zeta)\|^2 |\zeta - z|^4 dm(\zeta)(1-|z|^2)^p \left(1 - |\dot{a}\|^2 \right)^p dm(z)$$

$$+ \|\mu - \nu\|_L^2 \sup_{z \in \mathbb{D}}(1-|z|^2)^{(1-\lambda)} \int_{\mathbb{D}} \left(1 - |\dot{a}\|^2 \right)^p \|\mu(\zeta)|^2 |\zeta - z|^4 dm(\zeta)(1-|z|^2)^p \left(1 - |\dot{a}\|^2 \right)^p dm(z).$$
Consequently, by a change of variable $\zeta = \frac{1}{\varphi}$, we get
\[
\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{N}_{1,3}}^2 \leq \sup_{a \in \mathcal{D}} (1 - |a|^2)^{\gamma_{1,3}} \int_{\mathcal{D}} \frac{|\mu(\frac{1}{\varphi}) - \nu(\frac{1}{\varphi})|^2}{(1 - |\varphi|^2)^{2\gamma}} \left( \frac{1 - |a|^2}{|1 - \overline{a}\varphi|} \right)^{2\gamma} \, dudv \\
\times \int_{\mathcal{D}} \left( 1 - |z|^2 \right)^\gamma (1 - |\varphi|^2)^{2\gamma} |1 - \overline{a}\varphi|^2 |1 - \overline{\varphi}|^{2\gamma} \, dxdy \\
+ \|\mu - \nu\|_{\infty}^2 \sup_{a \in \mathcal{D}} (1 - |a|^2)^{\gamma_{1,3}} \int_{\mathcal{D}} \frac{|\mu(\frac{1}{\varphi})|^2}{(1 - |\varphi|^2)^{2\gamma}} \left( \frac{1 - |a|^2}{|1 - \overline{a}\varphi|} \right)^{2\gamma} \, dudv \\
\times \int_{\mathcal{D}} \left( 1 - |z|^2 \right)^\gamma (1 - |\varphi|^2)^{2\gamma} |1 - \overline{a}\varphi|^2 |1 - \overline{\varphi}|^{2\gamma} \, dxdy.
\]
(8)

In [24], we have proved that if the complex dilatation $\mu$ satisfies
\[
\frac{|\mu(z)|^2}{(1 - |z|^2)^{2\gamma}} \, dm(z) \in CM_{p,1}(\mathbb{D}),
\]
then
\[
\frac{|\mu(\frac{1}{\varphi})|^2}{(1 - |\varphi|^2)^{2\gamma}} \, dm(z) \in CM_{p,1}(\mathbb{D}).
\]
(9)

Therefore, combing (4), (8) with (9) and using Lemma 2.2 yields
\[
\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{N}_{1,3}} \leq \|\mu - \nu\|_{L^2}.
\]
This completes the proof of Lemma 2.4. \qed

It should be pointed out that the case $p = 1, \lambda = 1$ has been proved in [22].

We are now in a position to prove Theorem 1.3.

Proof. We first show that the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}) \rightarrow \mathcal{N}_{p,1,n}(\mathbb{D})$ is continuous. For simplicity of notations, for any $\mu, \nu \in \mathfrak{M}(\mathbb{D})$, we use $f$ to denote the quasiconformal mapping whose complex dilatation is equal to $\mu$ in $\mathbb{D}$ and is zero in $\mathcal{D}$, and $g$ to denote the quasiconformal mapping whose complex dilatation is equal to $\nu$ in $\mathbb{D}$ and is zero in $\mathcal{D}$, both normalized
\[
f(0) = f'(0) - 1 = f''(0) = 0 \quad \text{and} \quad g(0) = g'(0) - 1 = g''(0) = 0.
\]

By the definition of the higher Schwarzian derivative, we have
\[
\|\sigma_{n+1}(f) - \sigma_{n+1}(g)\|_{\mathcal{N}_{p,1,n+1}} \leq \|\sigma'_n(f) - \sigma'_n(g)\|_{\mathcal{N}_{p,1,n+1}} \\
+ (n - 1)\|N_f \sigma_n(f) - N_g \sigma_n(g)\|_{\mathcal{N}_{p,1,n+1}}.
\]
(10)

It follows from Lemma 2.1 that
\[
\|\sigma'_n(f) - \sigma'_n(g)\|_{\mathcal{N}_{p,1,n+1}} \approx \|\sigma_n(f) - \sigma_n(g)\|_{\mathcal{N}_{p,1,n}}.
\]
(11)

Note that
\[
|N_f \sigma_n(f) - N_g \sigma_n(g)| \leq |N_f| \|\sigma_n(f) - \sigma_n(g)\| + |\sigma_n(g)| |N_f - N_g|.
\]
(12)

We conclude from (11) and (12) that
\[
\|N_f \sigma_n(f) - N_g \sigma_n(g)\|_{\mathcal{N}_{p,1,n+1}} \leq \|N_f\|_{\mathcal{B}} \|\sigma_n(f) - \sigma_n(g)\|_{\mathcal{N}_{p,1,n}} \\
+ \|\sigma_n(g)\|_{\mathcal{N}_{p,1,n}} |N_f - N_g|_{\mathcal{B}}.
\]
(13)
By Theorem 3.1 in Chapter II in [17], there is a constant $C > 0$ such that
\[ ||N_f - N_g||_\infty \leq C||\mu - \nu||_\infty. \]  

(14)

Consequently, combining (7), (10), (13) with (14) yields
\[ ||\sigma_{n+1}(f) - \sigma_{n+1}(g)||_{N_{\nu,\lambda}} \leq ||\sigma_n(f) - \sigma_n(g)||_{N_{\nu,\lambda}} + ||\mu - \nu||_\infty. \]

Repeating this process $n - 3$ times gives
\[ ||\sigma_{n+1}(f) - \sigma_{n+1}(g)||_{N_{\nu,\lambda}} \leq ||\sigma_3(f) - \sigma_3(g)||_{N_{\nu,\lambda}} + ||\mu - \nu||_\infty. \]

By Lemma 2.4, we get
\[ ||\sigma_{n+1}(f) - \sigma_{n+1}(g)||_{N_{\nu,\lambda}} \leq ||\mu - \nu||_\mathcal{L}. \]

This implies that the higher Bers map $\beta_n : \mathcal{H}(\mathbb{D}^*) \rightarrow \mathcal{N}_{p,n}(\mathbb{D})$ is continuous.

We now turn to show that the higher Bers map $\beta_n : \mathcal{H}(\mathbb{D}^*) \rightarrow \mathcal{N}_{p,n}(\mathbb{D})$ is holomorphic. Since we have proved that $\beta_n$ is continuous, it is sufficient to show that for any $\mu \in \mathcal{H}(\mathbb{D}^*)$ and $v \in \mathcal{L}(\mathbb{D}^*)$, $\beta_n(\mu + tv)$ is holomorphic in a small neighborhood of $t = 0$ in the complex plane. Since $\mu \in \mathcal{H}(\mathbb{D}^*)$, there exists a positive constant $\epsilon$ such that for any $t$ with $|t| < 2\epsilon$,
\[ ||\mu + tv||_\infty < 1 \quad \text{and} \quad ||\mu + tv||_\mathcal{L} < \infty. \]

For simplicity of notations, we use $\psi(t)$ to denote $\beta_n(\mu + tv)$. For fixed $z \in \mathbb{D}$, the function $\psi(t)$ is holomorphic in $|t| < 2\epsilon$. For $|t| < \epsilon$, $|\psi(t)| < \epsilon$, it follows from Cauchy formula that
\[
\left| \frac{\psi(t)(z) - \psi(t_0)(z)}{t - t_0} - \frac{d}{dt}\big|_{t=t_0} \psi(t)(z) \right| = \frac{|t - t_0|}{2\pi} \left| \int_{|s| = 2\epsilon} \frac{\psi(s)(z)}{s - t} ds \right| \leq \frac{|t - t_0|}{2\pi \epsilon^3} \int_{|s| = 2\epsilon} |\psi(s)(z)||ds|.
\]

(15)

Using Fubini theorem yields
\[
(1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - \overline{z}z} \right)^p \left| \psi(t)(z) - \psi(t_0)(z) \right|^2 - \frac{d}{dt}\big|_{t=t_0} \psi(t)(z) \right| \left( 1 - |z|^2 \right)^{2n+4p} dxdy
\leq (1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - \overline{z}z} \right)^p \left| t - t_0 \right|^2 \frac{d}{dt}\big|_{t=t_0} \psi(t)(z) \right| \left( 1 - |z|^2 \right)^{2n+4p} dxdy
\leq |t - t_0|^2 (1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} \int_{|s| = 2\epsilon} |\psi(s)(z)|^2 |ds| (1 - |z|^2)^{2n+4p} dxdy
= |t - t_0|^2 \int_{|s| = 2\epsilon} (1 - |a|^2)^{(1 - \lambda)} \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - \overline{z}z} \right)^p |\psi(s)(z)|^2 (1 - |z|^2)^{2n+4p} dxdy |ds|
\leq |t - t_0|^2.
\]

This implies that the limit
\[
\lim_{t \to t_0} \psi(t)(z) = \frac{d}{dt}\big|_{t=t_0} \psi(t)(z)
\]
exists in $\mathcal{N}_{p,n}(\mathbb{D})$. Thus, we conclude that $\beta_n : \mathcal{H}(\mathbb{D}^*) \rightarrow \mathcal{N}_{p,n}(\mathbb{D})$ is holomorphic.

Furthermore, Buss proved in Theorem 3.4 in [7] that
\[
\frac{d}{dt}\big|_{t=0} \psi(t)(z) = \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}} \frac{\mu(w)}{(z - w)^{n+1}} dudv.
\]

The proof follows. \(\square\)
3. The connectivity of $T^0_{\Delta M(p)}(1)$

In this section, we shall prove Theorem 1.4. Let $r > 1$ and $\Delta_r = \{z : |z| < r\}$. A Beltrami differential $\mu(z) \in M(D^\prime)$ is called a vanishing Beltrami differential if for any $\epsilon > 0$, there exists $r > 1$ such that $\|\mu\|_{\Delta_r} < \epsilon$. Denote by $M^0(D^\prime)$ the collection of all vanishing Beltrami differentials.

Let $\mathcal{M}^0(D^\prime) = \mathcal{M}(D^\prime) \cap M^0(D^\prime)$. For each $\mu(z) \in \mathcal{M}^0(D^\prime)$, there exists a unique quasiconformal mapping $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined as in the introduction such that $f^\mu(D)$ is bounded. Define the pre-Bers projection mapping $L_\mu : \mathcal{M}^0(D^\prime) \to D^\prime_4(\mathbb{D})$ by setting $L_\mu(\mu) = \log(f^\mu)'$. To prove Theorem 1.4, we need the following result which has its own interest.

**Proposition 3.1.** The pre-Bers projection mapping $L_\mu : \mathcal{M}^0(D^\prime) \to D^\prime_4(\mathbb{D})$ is well defined and holomorphic.

**Proof.** For any $\mu \in \mathcal{M}^0(D^\prime) \subset M^0(D^\prime)$, it follows from [5] that $\log(f^\mu)' \in \mathcal{B}_0$. It also follows from Theorem 1.2 that $\log(f^\mu)' \in D^\prime_4(\mathbb{D})$. Therefore, the pre-Bers projection mapping $L_\mu : \mathcal{M}^0(D^\prime) \to D^\prime_4(\mathbb{D})$ is well defined. To prove that $L_\mu : \mathcal{M}^0(D^\prime) \to D^\prime_4(\mathbb{D})$ is holomorphic, we first show that it is continuous. For $\mu, \nu \in \mathcal{M}^0(D^\prime)$, it follows from Theorem 3.1 in Chapter II in [17] that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{(f^\mu)'''}{(f^\nu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \leq \|\mu - \nu\|_{\infty}.$$

By Lemma 2.4, we have

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{B}^3_{\mathbb{D}}} \leq \|\mu - \nu\|_{\mathbb{C}}.$$

Therefore, from Lemma 2.1, we get

$$\|\log(f^\mu)' - \log(f^\nu)'\|_{L_2(D)}^2 \approx \sup_{\mathbb{D}} (1 - |z|^2)^{p(1-\lambda)} \left| \frac{(f^\mu)'''}{(f^\mu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \left| (1 - |z|^2)^{p\lambda} \right| ^2 dxdy$$

$$\approx \sup_{\mathbb{D}} (1 - |z|^2)^{p(1-\lambda)} \left| \frac{(f^\mu)'''}{(f^\mu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \left| (1 - |z|^2)^{p\lambda} \right| ^2 dxdy$$

$$\leq \sup_{\mathbb{D}} (1 - |z|^2)^{p(1-\lambda)} \left| \frac{(f^\mu)'''}{(f^\mu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \left| (1 - |z|^2)^{p\lambda} \right| ^2 dxdy$$

$$+ \sup_{\mathbb{D}} (1 - |z|^2)^{p(1-\lambda)} \left| \frac{(f^\mu)'''}{(f^\mu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \left| (1 - |z|^2)^{p\lambda} \right| ^2 dxdy$$

$$\leq \|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{B}^3_{\mathbb{D}}}$$

$$+ \sup_{\mathbb{D}} (1 - |z|^2)^{p(1-\lambda)} \left| \frac{(f^\mu)'''}{(f^\mu)''} - \frac{(f^\nu)'''}{(f^\nu)''} \right| \left| (1 - |z|^2)^{p\lambda} \right| ^2 dxdy$$

This implies that $L_\mu : \mathcal{M}^0(D^\prime) \to D^\prime_4(\mathbb{D})$ is continuous.

Similar to the proof of Theorem 1.3, it remains to show that for any $\mu \in \mathcal{M}^0(D^\prime)$ and $\nu \in \mathcal{L}(D^\prime)$, $L_\mu(\mu + \nu t)$ is holomorphic in a small neighborhood of $t = 0$ in the complex plane. Chose a positive constant $\epsilon$ such that for any $t$ with $|t| < 2\epsilon$, $\|\mu + \nu t\|_{\infty} < 1$ and $\|\mu + \nu t\|_{\mathbb{C}} < \infty$. We abbreviate the function $L_\mu(\mu + \nu t)$ by $\phi(t)$. For fixed $z \in \mathbb{D}$, the function $\phi(t)$ is holomorphic in $|t| < 2\epsilon$ (see [18]) and (15) still holds for $\phi(t)$. 

Thus, by Fubini theorem, we deduce that
\[
(1 - |a|^2)^{(1 - 1)} \int_D \left| \frac{\phi(t)(z) - \phi(t_0)(z)}{t - t_0} - \frac{d}{dt}|_{t=t_0}\phi(t)(z) \right| (1 - |z|^2)^p \left( \frac{1 - |a|^2}{1 - |z|^2} \right)^p dxdy
\]
\[
\leq (1 - |a|^2)^{(1 - 1)} \frac{|t - t_0|^2}{4\pi^2 e^\mu} \int_D \left( \int_{|s|=\rho_1} |\phi(s)(z)| |ds| \right)^2 (1 - |z|^2)^p \left( \frac{1 - |a|^2}{1 - |z|^2} \right)^p dxdy
\]
\[
\leq (1 - |a|^2)^{(1 - 1)} |t - t_0|^2 \int_D \int_{|s|=\rho_1} |\phi(s)(z)|^2 |ds| (1 - |z|^2)^p \left( \frac{1 - |a|^2}{1 - |z|^2} \right)^p dxdy|ds|
\]
\[
= |t - t_0|^2 \int_{|s|=\rho_1} (1 - |a|^2)^{(1 - 1)} \int_D |\phi(s)(z)|^2 (1 - |z|^2)^p \left( \frac{1 - |a|^2}{1 - |z|^2} \right)^p dxdy|ds|
\]
\[
\leq |t - t_0|^2.
\]
Therefore, we deduce that the limit
\[
\lim_{t \to t_0} \frac{\phi(t) - \phi(t_0)}{t - t_0} = \frac{d}{dt}|_{t=t_0}\phi(t)
\]
exists in $D^p_A(D)$. This implies that $L_0 : \mathfrak{M}^0(D^*) \to D^p_A(D)$ is holomorphic and the proof follows.

We now state the proof of Theorem 1.4.

**Proof.** Let $f^* \in T^0_{DM}(1)$. By Theorem 1.2, $f$ can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation $\mu$ satisfies $|\mu(z)| \leq 1 / (2\pi z^2 - 1) dxdy \in CM_{\mu_\lambda}(D)$. Let $f^*$ be the quasiconformal mapping in $\bar{C}$ with $f^{-1}(\infty) = (f^*)^{-1}(\infty)$ and $\bar{f}^* = t \mu \bar{f}$. We now prove the path $t \mapsto \log(f^*)$, $0 \leq t \leq 1$ is continuous in $\mathfrak{R} \cap D^p_A(D)$.

For $f^1$, $f^2$, we conclude from Proposition 3.1 that
\[
\| \log(f^1) - \log(f^2) \|_{D^p_A(D)} \leq \| t_1 - t_2 \| \cdot \| \mu \|_{L^1}.
\]

On the other hand, by (14) (see Theorem 3.1 in Chapter II in [17]), we get
\[
\| \log(f^1) - \log(f^2) \|_{\mathfrak{R}} \leq \| t_1 - t_2 \| \cdot \| \mu \|_{L^\infty}.
\]

Thus, we deduce that
\[
\| \log(f^1) - \log(f^2) \|_{\mathfrak{R}, D^p_A(D)} \leq \| t_1 - t_2 \| \cdot \| \mu \|_{L^1}.
\]

This means that the path $t \mapsto \log(f^*)$, $0 \leq t \leq 1$ is continuous in $\mathfrak{R} \cap D^p_A(D)$.

Therefore, each $f^* \in T^0_{DM}(1)$ can be connected by a continuous path in $\mathfrak{R} \cap D^p_A(D)$ to a Möbius transformation $\gamma$ with $\log \gamma' \in T^0_{DM}(1)$. Observe that $\gamma(D)$ is bounded, it follows that the path $t \mapsto \log \gamma'$ connects the point $0$ to the point $0$ in $T^0_{DM}(1)$, where $\gamma' = \gamma'(\rho z)$. This implies that $T^0_{DM}(1) = \{ f^* \in T^0_{DM}(1) : f(D) \text{ is bounded} \}$ is connected in $\mathfrak{R} \cap D^p_A(D)$.

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