Negative curvature in automorphism groups of one-ended hyperbolic groups

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Abstract

In this article, we show that some negative curvature may survive when taking the automorphism group of a finitely generated group. More precisely, we prove that the automorphism group Aut(G) of a one-ended hyperbolic group G which is not virtually a surface group turns out to be acylindrically hyperbolic. As a consequence, given an automorphism \( \varphi \in \text{Aut}(G) \), we deduce that the semidirect product \( G \rtimes_{\varphi} \mathbb{Z} \) is acylindrically hyperbolic if and only if \( \varphi \) has infinite order in \( \text{Out}(G) \).

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1 Introduction

Finding some information about the automorphism group is one the most natural questions we can ask about a given group. It is also a particularly difficult one. The main reason is that, typically, it is much more difficult to work with the automorphism group than the group itself. As a consequence, being able to read properties of the automorphism group directly from the group may be quite useful.

Taking the point of view of geometric group theory, one of the most efficient methods to study a group is to exhibit some phenomena of negative curvature. So a natural but vague question is the following: does the negatively-curved geometry of a group survive when taking its automorphism group?

Such a question does not seem to be so unreasonable. For instance, the study of (outer) automorphism groups of free groups and of surface groups, two of the most studied families of groups in geometric group theory, is fundamentally based on the negatively-curved geometries of free groups and surface groups. Inspired by these two examples, we ask the following more precise question:
**Question 1.1.** Is the automorphism group of a finitely generated acylindrically hyperbolic group acylindrically hyperbolic as well?

The class of acylindrically hyperbolic groups, as defined in [Osi16], encompass many interesting families of groups, including in particular non-elementary hyperbolic and relatively hyperbolic groups, most 3-manifold groups, groups of deficiency at least two, many small cancellation groups, the Cremona group of birational transformations of the complex projective plane, many Artin groups, and of course the two examples mentioned above, namely outer automorphism groups of free groups (of rank at least three) and mapping class groups of (non-exceptional) surfaces. Nevertheless, being acylindrically hyperbolic provides valuable information on the group, as many aspects of the theory of hyperbolic and relatively hyperbolic groups can be generalised in the context of acylindrical hyperbolicity. These include various algebraic, model-theoretic, and analytic properties; small cancellation theory; and group theoretic Dehn surgery. We refer to [Osi17] and references therein for more information.

This article is dedicated to the proof of the following statement, which provides a partial positive answer to Question 1.1:

**Theorem 1.2.** The automorphism group of any one-ended hyperbolic group which is not virtually a surface group is acylindrically hyperbolic.

Let us mention two applications of this statement.

**Corollary 1.3.** Let $X$ be a compact $n$-dimensional Riemannian manifold of negative curvature, $n \geq 3$. Then the automorphism group $\text{Aut}(\pi_1(X))$ is acylindrically hyperbolic.

*Proof.* The fundamental group $\pi_1(X)$ is a hyperbolic group whose boundary is an $(n-1)$-sphere. Because $n \geq 3$, it must be one-ended and it cannot be virtually a surface group (otherwise its boundary would be a circle). So Theorem 1.2 applies.

In our next application, random groups are defined following [Cha95].

**Corollary 1.4.** Almost surely, the automorphism group of a random group is acylindrically hyperbolic.

*Proof.* Following [Cha95, Théorème 4.18], almost surely a random group is a hyperbolic group whose boundary is a Menger curve. Such a group must be one-ended and it cannot be virtually a surface group (otherwise its boundary would be a circle). So Theorem 1.2 applies.

We emphasize that, although the structure of outer automorphism groups of one-ended hyperbolic groups is pretty well-understood (see [DG11, Corollary 4.7]), it is not sufficient to deduce our theorem. In fact, outer automorphism groups are rarely acylindrically hyperbolic. Also, many of the properties which follow from the acylindrical hyperbolicity cannot be deduced from the study of the outer automorphism group.

Our proof of Theorem 1.2 is based on JSJ decompositions of hyperbolic groups as constructed in [Bow98]. Such a decomposition exists if our group $G$ is not virtually a surface group and if $\text{Out}(G)$ is infinite. Next, the argument goes as follows:

- The starting point of our argument is the well-known fact that the canonicity of the JSJ decomposition of a one-ended hyperbolic group $G$ implies that the automorphism group $\text{Aut}(G)$ naturally acts on the associated Bass-Serre tree $T$.

- Next, we observe that $G$ acts on the JSJ tree $T$ with WPD elements. By applying a small cancellation theorem of [DGO17], it follows that there exists some $g \in G$ such that the normal closure $\langle g \rangle$ is free and intersects trivially vertex- and edge-stabilisers of $T$. 

• The key point of the argument is to show that there exists a finitely generated free subgroup \( H \leq \langle g \rangle \) which is not elliptic in any splitting of \( G \) with virtually cyclic edge-groups. As a consequence of Whitehead’s work \([Whi36]\), we know that there exists some \( h \in H \) such that \( H \) does not split freely relatively to \( \langle h \rangle \).

• By looking at the action \( \text{Aut}(G) \sim T \), it turns out that, if the inner automorphism \( \iota(h) \) is not WPD with respect to \( \text{Aut}(G) \sim T \), then there must exist infinitely many pairwise non-conjugate automorphisms sending a fixed power of \( h \) to the same element.

• By applying Paulin’s construction \([Pau91]\), one gets an action of \( G \) on some real tree, namely one of the asymptotic cones of \( G \), with respect to which \( g \) is elliptic. As a consequence of Rips’ theory, as exposed in \([Gui08]\), it follows that \( G \) splits relatively to \( \langle h \rangle \) over a virtually cyclic subgroup.

• The consequence is that \( H \) must split freely relatively to \( \langle h \rangle \), which is impossible by definition of \( h \). Therefore, the inner automorphism \( \iota(h) \) turns out to be a WPD element with respect to \( \text{Aut}(G) \sim T \), proving the acylindrical hyperbolicity of \( \text{Aut}(G) \).

As it can be seen, two points are fundamental in our strategy: the JSJ decomposition has to be canonical, and a sequence of pairwise non-conjugate automorphisms has to lead to a splitting of the group. We expect that our arguments still hold when these two ingredients are present, leading to the acylindrical hyperbolicity of the automorphism group. Many JSJ decompositions of many different kinds of groups can be found in the literature, but very few are known to be canonical. And the combination of Paulin’s construction with Rips’ theory is essentially the only known method to construct splittings from sequences of automorphisms, but most of the time it is a strategy which is difficult to apply outside the world of hyperbolic groups where asymptotic cones are not real trees. Nevertheless, part of these arguments has been applied to some right-angled Artin groups in \([Gen18]\), and we expect that the same strategy will allow us to extend Theorem 1.2 to toral relatively hyperbolic groups.

Interestingly, Theorem 1.2 provides non-trivial information on cyclic extensions of hyperbolic groups. In fact, in full generality, the acylindrical hyperbolicity of the automorphism group implies that many cyclic extensions of the group must be acylindrically hyperbolic as well. More precisely:

**Theorem 1.5.** Let \( G \) be a group whose center is finite and whose automorphism group is acylindrically hyperbolic. For every automorphism \( \varphi \in \text{Aut}(G) \), the semidirect product \( G \rtimes_\varphi \mathbb{Z} \) is acylindrically hyperbolic if and only if \( \varphi \) has infinite order in \( \text{Out}(G) \).

Because non-elementary hyperbolic groups have finite centers, the following statement is an immediate consequence of Theorems 1.3 and 1.2:

**Corollary 1.6.** Let \( G \) be a one-ended hyperbolic group which is not virtually a surface group, and let \( \varphi \in \text{Aut}(G) \) be an automorphism. Then \( G \rtimes_\varphi \mathbb{Z} \) is acylindrically hyperbolic if and only if \( \varphi \) has infinite order in \( \text{Out}(G) \).

A similar statement holds for free groups \([Gho18]\) (up to finite index) and for some right-angled Artin groups \([Gen18]\). We do not know any counterexample to this statement in the context of arbitrary finitely generated acylindrically hyperbolic groups. (Such a counterexample would provide a negative answer to Question 1.1.)

Let us conclude this introduction with a few remarks. First, it is worth noticing that Question 1.1 has a negative answer if the group is not assumed to be finitely generated, as
shown by [GM18, Remark 4.10]. Next, Theorem 1.2 does not apply to all non-elementary hyperbolic groups: what about infinitely-ended hyperbolic groups and groups which are virtually surface groups? In the former case, the hyperbolic group splits over a finite subgroup, but we cannot expect to make the automorphism group act on the associated Bass-Serre tree since automorphism groups of free products may satisfy strong fixed-point properties [Var18, Var14, KNO17] including Serre’s property FA [CV96, Led18]. So a different approach is needed here. Nevertheless, we expect that the automorphism group of any finitely generated group (not necessarily hyperbolic) splitting over a finite subgroup is acylindrically hyperbolic (or virtually cyclic if the initial group was virtually cyclic). About virtually surface groups, we also need to find a different approach since their automorphism groups may also satisfy Serre’s property FA [CV96]. But, the case of surface groups is well-known. Indeed, if \( \Sigma_g \) denotes an orientable and closed surface of genus \( g \geq 2 \), then the automorphism group \( \text{Aut}(\pi_1(\Sigma_g)) \) coincides with the mapping class group \( \text{Mod}^{\pm}(\Sigma_g, 1) \), whose acylindrical hyperbolicity can be obtained thanks to its action on the curve graph of \( \Sigma_g, 1 \) (see [PS17] and references therein for more information). The picture for virtually surface groups is less clear, although it may be expected that automorphism groups remain acylindrically hyperbolic.

**Organisation of the paper.** In Section 2 we collect the few statements about JSJ decompositions of hyperbolic groups which will be needed in the paper. In Section 3, we state and prove a sufficient condition for an element of a hyperbolic group which is loxodromic in the JSJ tree to define a WPD element of the automorphism group via the inner automorphism associated to it. And in Section 4, we construct a relative splitting of the group when this condition fails. Finally, Section 5 is dedicated to the proof of Theorem 1.2 and Section 6 to Theorem 1.5.

**Acknowledgment.** ...

### 2 JSJ decompositions of hyperbolic groups

Our study of automorphism groups of one-ended hyperbolic groups is based on the notion of **JSJ decompositions**. Initially introduced by Sela in [Sel97], we use the construction given by Bowditch in [Bow98]. A simplified version of [Bow98, Theorem 0.1] is:

**Theorem 2.1.** Let \( G \) be a one-ended hyperbolic group which is not virtually a surface group. Then there is a canonical non-trivial splitting of \( G \) as a finite graph of groups such that each edge-group is virtually infinite cyclic and such that there exist two types of vertex-groups:

- vertex-groups of type 1 are virtually free;
- and vertex-groups of type 2 are quasiconvex subgroups not of type 1.

Moreover, two vertex-groups of the same type are not adjacent, and any virtually cyclic subgroup on which \( G \) splits can be conjugate into an edge-group or a vertex-group of type 1.

(To make the link with Bowditch’s statement, notice that a hyperbolic group is a cocompact Fuchsian group if and only if it is virtually a surface group; see [BK02, Theorem 5.4] and the related discussion. Moreover, as discussed in the introduction of [Bow98], the boundary of a one-ended hyperbolic group is always locally connected.)

We refer to this decomposition of \( G \) as its **JSJ decomposition**, and to the associated Bass-Serre tree as its **JSJ tree**. The uniqueness of the decomposition follows from the
Proposition 2.2. Let $G$ be a one-ended hyperbolic group which is not virtually a surface group. Denote by $T$ the JSJ tree of $G$. The center $Z(G)$ of $G$ is included into the kernel of the action $G \curvearrowright T$, and, if we identify canonically the quotient $G/Z(G)$ with the subgroup of inner automorphisms $	ext{Inn}(G) \leq \text{Aut}(G)$, then the action $G/Z(G) \curvearrowright T$ extends to an action $\text{Aut}(G) \curvearrowright T$ via:

$$
\begin{align*}
\text{Aut}(G) &\rightarrow \text{Isom}(T) \\
\varphi &\mapsto (x \mapsto \text{vertex whose stabiliser is } \varphi(\text{stab}(x)))
\end{align*}
$$

The fact that the action of the center $Z(G)$ of $G$ on the JSJ tree is trivial follows from the observation that $Z(G)$ acts trivially on the boundary of $G$. (More generally, the kernel of the action $G \curvearrowright \partial G$ turns out to coincide with the unique maximal finite normal subgroup of $G$.)

We emphasize that the JSJ decomposition may be trivial. Indeed, taking a one-ended hyperbolic group with a finite outer automorphism group, the JSJ decomposition must be trivial since such a group does not split over a virtually cyclic subgroup according to [BF95]. But this obstruction turns out to be the only situation where the JSJ decomposition is trivial.

Lemma 2.3. Let $G$ be a one-ended hyperbolic group which is not virtually a surface group. If $\text{Out}(G)$ is infinite, then the JSJ decomposition of $G$ is non-trivial. As a consequence, the action of $G$ on its JSJ tree admits loxodromic isometries.

Proof. The first remark is that, as a consequence of [Bow98, Theorem 5.28], the action $G \curvearrowright T$ is minimal. Therefore, in order to deduce that the JSJ decomposition is not trivial, it is sufficient to show that the decomposition contains two vertex-groups of different types. Since $G$ is not virtually free, we already know that the JSJ decomposition contains a vertex-group of type 2. Next, According to [BF95], if $\text{Out}(G)$ is infinite then $G$ has to split over a virtually cyclic subgroup. But such a subgroup must be included into a vertex-group of type 1 of the JSJ decomposition (up to conjugation). This concludes the proof.

In our argument, the following observation will be fundamental:

Proposition 2.4. Let $G$ be a one-ended hyperbolic group which is not virtually a surface group. Any element $g \in G$ defining a loxodromic isometry of the JSJ tree $T$ must be WPD with respect to $G \curvearrowright T$.

Recall that, given a group $G$ acting on metric space $X$ by isometries, an element $g \in G$ is WPD if, for every $x \in X$ and every $\epsilon > 0$, there exists some $N \geq 1$ such that the set

$$\{h \in G \mid d(x, hx) \leq \epsilon \text{ and } d(g^N x, hg^N x) \leq \epsilon\}$$

is finite. WPD elements are fundamental in the theory of acylindrically hyperbolic groups since we can define acylindrically hyperbolic groups as non-virtually cyclic groups admitting actions on hyperbolic spaces with at least one WPD element [Osi16].

In the case of a trees, WPD elements can be characterised in an easier way. In the sequel, we will often use the following statement without mentioning it.

Lemma 2.5. Let $G$ be a group acting on a simplicial tree $T$, and $g \in G$ a loxodromic isometry. Then $g$ is WPD if and only if there exist two points $x, y \in \text{axis}(g)$ such that the intersection $\text{stab}(x) \cap \text{stab}(y)$ is finite.
The implication is proved in \cite[Corollary 4.3]{MO15}. The converse is an immediate consequence of the definition. Now, let us go back to Proposition \ref{prop:inner-automorphisms}. 

\textit{Proof of Proposition \ref{prop:inner-automorphisms}.} Let $T$ denote the JSJ tree of $G$. Suppose that $g \in G$ is loxodromic and fix an edge $e$ in its axis. If $\text{stab}(e) \cap \text{stab}(ge)$ is infinite, then $g$ belongs to the commensurator of $\text{stab}(e)$. But, since $\text{stab}(e)$ is virtually cyclic, it has finite-index in its commensurator, so there must exist some power $s \geq 1$ such that $g^s$ belongs to $\text{stab}(e)$. This contradicts the fact that $g$ is a loxodromic isometry of $T$, concluding the proof of four proposition. \hfill \square

\section{Inner automorphisms as generalised loxodromic elements}

We saw in the previous section that the automorphism group $\text{Aut}(G)$ of our hyperbolic group $G$ acts naturally on the JSJ tree of $G$. A natural strategy to prove that $\text{Aut}(G)$ is acylindrically hyperbolic is to show that it contains WPD elements with respect to this action. The most natural attempt in this direction is to start with WPD elements of $G$ and to look at the corresponding inner automorphisms. The next statement provides a sufficient criterion to determine when such an inner automorphism defines a WPD element with respect to the action of $\text{Aut}(G)$ on the JSJ tree.

\textbf{Proposition 3.1.} \textit{Let $G$ be a one-ended hyperbolic group which is not virtually a surface group. Let $T$ denote the JSJ tree of $G$. Suppose that $g \in G$ is WPD with respect to $G \acts T$ but that $\iota(g^s)$ is not WPD with respect to $\text{Aut}(G) \acts T$ for every $s \geq 1$. Then there exist $s \geq 1$ and infinitely many pairwise non-conjugate automorphisms $\varphi_1, \varphi_2, \ldots \in \text{Aut}(G)$ such that $\varphi_i(g^s) = \varphi_j(g^s)$ for every $i, j \geq 1$.}

\textit{Proof.} Fix some element $g \in G$ which is WPD with respect to $G \acts T$, and let $\gamma \subset T$ denote its axis. Fix an edge $e \in \gamma$ and let $Z$ denote its $G$-stabiliser. Since $Z$ has finite index in its normaliser $N(Z)$, the orbit $N(Z) \cdot e$ must be finite. Let $C$ denote the convex hull of this orbit; it is a finite subtree. It follows from the fact that $g$ is WPD with respect to $G \acts T$ that there exists some $s \geq 1$ such that $\text{stab}_C(g) \cap N(Z) \cdot e$ is finite. As

$$N(Z) \cap g^s N(Z) g^{-s} \subset \text{stab}_C(g) \cap \text{stab}_C(g^s C),$$

we deduce that $N(Z) \cap g^s N(Z) g^{-s}$ must be finite. Now, we know by assumption that $\iota(g^s)$ is not WPD with respect to $\text{Aut}(G) \acts T$, so the intersection

$$P := \text{stab}_{\text{Aut}(G)}(e) \cap \text{stab}_{\text{Aut}(G)}(g^{2s} e)$$

must be infinite. First, since

$$P \cap \text{Inn}(G) = \{\iota(h) \mid h \in \text{stab}_C(e) \cap \text{stab}_C(g^{2s} e)\} \subset \{\iota(h) \mid h \in \text{stab}_C(e) \cap \text{stab}_C(g^s e)\},$$

we notice that $P \cap \text{Inn}(G)$ is finite, so that $P$ has infinite image in $\text{Out}(G)$. Fix a sequence of automorphisms $\varphi_1, \varphi_2, \ldots \in P$ which are pairwise non-conjugate.

Next, fix some $\varphi \in P$. Because $\varphi$ belongs to $\text{stab}_{\text{Aut}(G)}(e)$, we know that $\varphi(Z) = Z$; because $\varphi$ belongs to $\text{stab}_{\text{Aut}(G)}(g^s e)$, we know that

$$g^s Zg^{-s} = \varphi(g^s Z g^{-s}) = \varphi(g^s) \varphi(Z) \varphi(g^{-s}) = \varphi(g^s) Z \varphi(g^{-s}),$$

hence $\varphi(g^s) = g^n$ for some $n \in N(Z)$; and because $\varphi$ belongs to $\text{stab}_{\text{Aut}(G)}(g^{2s} e)$, we know that

$$g^{2s} Z g^{-2s} = \varphi(g^{2s} Z g^{-2s}) = \varphi(g^s) \varphi(g^s) \varphi(Z) \varphi(g^s)^{-1} \varphi(g^s)^{-1} = g^n g^n Z n^{-1} g^{-s} n^{-1} g^{-s} = g^n g^n Z g^{-s} n^{-1} g^{-s},$$

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hence \( n \in g^sN(Z)g^{-s} \). Thus, we have proved that
\[
P \subseteq \{ \varphi \in \text{Aut}(G) \mid \varphi(g^s) \in g^s(N(Z) \cap g^sN(Z)g^{-s}) \}.
\]
But we know that the intersection \( N(Z) \cap g^sN(Z)g^{-s} \) is finite, so we can find a subsequence in \( \varphi_1, \varphi_2, \ldots \in P \) all of whose automorphisms send \( g^s \) to the same element. \( \square \)

4 Relative splittings from Paulin’s construction

In this section, we are interested in understanding what happens when Proposition 3.1 applies, or more precisely, when there exists an element of the group which is fixed by infinitely many pairwise non-conjugate automorphisms. Our main result in this direction is the following:

Proposition 4.1. Let \( G \) be a hyperbolic group and \( g \in G \) an infinite-order element. Suppose that there exist infinitely many pairwise non-conjugate automorphisms \( \varphi_1, \varphi_2, \ldots \in \text{Aut}(G) \) such that \( \varphi_i(g) = \varphi_j(g) \) for every \( i, j \geq 1 \). Then \( G \) splits relatively to \( \langle g \rangle \) over a virtually cyclic subgroup.

Recall that a group \( G \) splits relatively to a subgroup \( H \) if \( H \) can be conjugate into one of the factors of the splitting. Proposition 4.1 is clearly inspired by [BF95, Corollary 1.3], and similarly our proof is based on Paulin’s construction [Pau91]. First of all, we need to recall basic definitions related to asymptotic cones of groups.

Asymptotic cones. An ultrafilter \( \omega \) over a set \( S \) is a collection of subsets of \( S \) satisfying the following conditions:

- \( \emptyset \notin \omega \) and \( S \in \Omega \);
- for every \( A, B \in \omega \), \( A \cap B \in \omega \);
- for every \( A \subseteq S \), either \( A \in \omega \) or \( A^c \in \omega \).

Basically, an ultrafilter may be thought of as a labelling of the subsets of \( S \) as “small” (if they do not belong to \( \omega \)) or “big” (if they belong to \( \omega \)). More formally, notice that the map
\[
\Psi(S) \to \{0, 1\}, \quad A \to \begin{cases} 0 & \text{if } A \notin \omega \\
1 & \text{if } A \in \omega \end{cases}
\]
defines a finitely additive measure on \( S \).

The easiest example of an ultrafilter is the following. Fixing some \( s \in S \), set \( \omega = \{ A \subseteq S \mid s \in A \} \). Such an ultrafilter is called principal.

Now, fix a metric space \( (X, d) \), a non-principal ultrafilter \( \omega \) over \( \mathbb{N} \), a scaling sequence \( \epsilon = (\epsilon_n) \) satisfying \( \epsilon_n \to 0 \), and a sequence of basepoints \( o = (o_n) \in X^{\mathbb{N}} \). A sequence \( (r_n) \in \mathbb{R}^\mathbb{N} \) is \( \omega \)-bounded if there exists some \( M \geq 0 \) such that \( \{ n \in \mathbb{N} \mid |r_n| \leq M \} \in \omega \) (ie., if \( |r_n| \leq M \) for “\( \omega \)-almost all \( n \)”). Set
\[
B(X, \epsilon, o) = \{ (x_n) \in X^{\mathbb{N}} \mid (\epsilon_n \cdot d(x_n, o_n)) \text{ is } \omega \text{-bounded} \}.
\]

We may define a pseudo-distance on \( B(X, \epsilon, o) \) as follows. First, we say that a sequence \( (r_n) \in \mathbb{R}^\mathbb{N} \) \( \omega \)-converges to a real \( r \in \mathbb{R} \) if, for every \( \epsilon > 0 \), \( \{ n \in \mathbb{N} \mid |r_n - r| \leq \epsilon \} \in \omega \). If so, we write \( r = \lim_{\omega} r_n \). Then, our pseudo-distance is
\[
B(X, \epsilon, o)^2 \to [0, +\infty), \quad \lim_{\omega} \epsilon_n \cdot d(x_n, y_n).
\]
Notice that the \( \omega \)-limit always exists since the sequence which is considered is \( \omega \)-bounded.
**Definition 4.2.** The *asymptotic cone* \( \text{Cone}_\omega(X, \epsilon, o) \) of \( X \) is the metric space obtained by quotienting \( B(X, \epsilon, o) \) by the relation: \( (x_n) \sim (y_n) \) if \( d((x_n), (y_n)) = 0 \).

The picture to keep in mind is that \((X, \epsilon_n \cdot d)\) is a sequence of spaces we get from \( X \) by “zooming out”, and the asymptotic cone if the “limit” of this sequence. Roughly speaking, the asymptotic cones of a metric space are asymptotic pictures of the space. For instance, any asymptotic cone of \( \mathbb{Z}^2 \), thought of as the infinite grid in the plane, is isometric to \( \mathbb{R}^2 \) endowed with the \( \ell^1 \)-metric; and the asymptotic cones of a simplicial tree (and more generally of any Gromov-hyperbolic space) are real trees.

**Paulin’s construction.** This paragraph is dedicated to the main construction of [Pau91], allowing us to construct an action of a group on one of its asymptotic cones thanks to a sequence of pairwise non-conjugated automorphisms. As our language is different (but equivalent), we give a self-contained exposition of the construction below.

Let \( G \) be a finitely generated group with a fixed generating set \( S \) and let \( \varphi_1, \varphi_2, \ldots \in \text{Aut}(G) \setminus \{\text{Id}\} \) be a collection of automorphisms. The goal is to construct a non-trivial action of \( G \) on one of its asymptotic cones from the sequence of twisted actions

\[
\begin{align*}
G & \to \text{Isom}(\text{Cone}(G)) \\
g & \mapsto (h \mapsto \varphi_n(g) \cdot h), \quad n \geq 1.
\end{align*}
\]

For every \( n \geq 1 \), set \( \lambda_n = \min_{x \in G} \max_{s \in S} d(x, \varphi_n(s) \cdot x) \); notice that \( \lambda_n \geq 1 \) since \( \varphi_n \) is different from the identity. We suppose that \( \lambda_n \xrightarrow{n \to \infty} +\infty \). Now, fixing some non-principal ultrafilter \( \omega \) over \( \mathbb{N} \) and some sequence \( o = (o_n) \in G^\mathbb{N} \) satisfying, for every \( n \geq 1 \), the equality \( \max_{s \in S} d(o_n, \varphi_n(s) \cdot o_n) = \lambda_n \), notice that the map

\[
\begin{align*}
G & \to \text{Isom}(\text{Cone}(G)) \\
g & \mapsto ((x_n) \mapsto (\varphi_n(g) \cdot x_n))
\end{align*}
\]

defines an action by isometries on \( \text{Cone}(G) := \text{Cone}_\omega(G, (1/\lambda_n), o) \). The only fact to verify is that, if \( g \in G \) and if \((x_n)\) defines a point of \( \text{Cone}(G) \), then so does \((\varphi_n(g) \cdot x_n)\). Writing \( g \) as product of generators \( s_1 \cdots s_r \) of minimal length, we have

\[
\frac{1}{\lambda_n} d(o_n, \varphi_n(g) \cdot x_n) \leq \frac{1}{\lambda_n} \sum_{i=1}^r d(o_n, \varphi_n(s_i) \cdot x_n)
\]

\[
\leq \frac{1}{\lambda_n} \sum_{i=1}^r (d(o_n, \varphi_n(s_i) \cdot o_n) + d(o_n, x_n))
\]

\[
\leq \|g\| \left(1 + \frac{1}{\lambda_n} d(o_n, x_n)\right)
\]

Consequently, if the sequence \((d(o_n, x_n))\) is \( \omega \)-bounded, ie., if \((x_n)\) defines a point of \( \text{Cone}(G) \), then the sequence \((d(o_n, \varphi_n(g) \cdot x_n))\) is \( \omega \)-bounded as well, ie., \((\varphi_n(g) \cdot x_n)\) also defines a point of \( \text{Cone}(G) \).

**Fact 4.3.** The action \( G \rightharpoonup \text{Cone}(G) \) does not fix a point.

Suppose that \( G \) fixes a point \((x_n)\) of \( \text{Cone}(G) \). Then, for \( \omega \)-almost all \( n \) and all \( s \in S \), the inequality

\[
\frac{1}{\lambda_n} d(x_n, \varphi_n(s) \cdot x_n) \leq \frac{1}{2}
\]

holds, hence

\[
\lambda_n \leq \max_{s \in S} d(x_n, \varphi_n(s) \cdot x_n) \leq \lambda_n/2,
\]

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which is impossible.

The conclusion is that we can associate a fixed-point free action of \( G \) on one of its asymptotic cones from an infinite collection of automorphisms, provided that our sequence \( (\lambda_n) \) tends to infinity. So the natural question is now: when does it happen?

**Fact 4.4.** If the automorphisms \( \varphi_1, \varphi_2, \ldots \) of \( G \) are pairwise non-conjugate, then the equality \( \lim_{\omega} \lambda_n = +\infty \) holds.

Suppose that the \( \omega \)-limit of \( (\lambda_n) \) is not infinite. So there exists a subsequence \( (\lambda_{\sigma(n)}) \) which is bounded above by some constant \( R \). So, for every \( n \geq 1 \) and every \( s \in S \), one has

\[
d\left(1, o_{\sigma(n)}^{-1} \varphi_{\sigma(n)}(u)o_{\sigma(n)} \right) = d\left(o_{\sigma(n)}, \varphi_{\sigma(n)}(u)o_{\sigma(n)} \right) \leq \lambda_n \leq R,
\]

ie., \( o_{\sigma(n)}^{-1} \varphi_{\sigma(n)}(u)o_{\sigma(n)} \in B(1, R) \). Since the ball \( B(1, R) \) is finite, it follows from the pigeonhole principle that there exist two distinct \( m, n \geq 1 \) such that

\[
o_{\sigma(n)}^{-1} \varphi_{\sigma(n)}(u)o_{\sigma(n)} = o_{\sigma(m)}^{-1} \varphi_{\sigma(m)}(u)o_{\sigma(m)}
\]

for every \( s \in S \). It follows that \( \varphi_{\sigma(n)} \) and \( \varphi_{\sigma(m)} \) are conjugate. This concludes the proof of our fact.

So the point of Paulin’s construction is that we can associate a fixed-point free action of \( G \) on one of its asymptotic cones from an infinite subset of \( \text{Out}(G) \).

**Rips’ machinery.** Typically, when looking at an action on an asymptotic cone, the objective is to get an action on a real tree in order to construct, thanks to Rips’ machinery, an action on a simplicial tree, or equivalently according to Bass-Serre theory, a splitting of the corresponding group. This strategy is illustrated by the following statement, which is a special case of [Gui08, Corollary 5.2]. We recall that a group \( G \) splits relatively to a subgroup \( H \) if \( G \) decomposes as an HNN extension or an amalgamated product such that \( H \) is included into a factor, or equivalently such that \( H \) fixes a vertex in the associated Bass-Serre tree.

**Theorem 4.5.** Let \( G \) be a finitely generated group acting fixed-point freely and minimally on a real tree \( T \), and \( H \subset G \) a subgroup fixing a point of \( T \). Suppose that:

- arc-stabilisers are finitely generated;
- a non-decreasing sequence of arc-stabilisers stabilises;
- for every arc \( I \), there does not exist \( g \in G \) satisfying \( g \cdot \text{stab}(I) \cdot g^{-1} \subset \text{stab}(I) \).

Assuming that \( T \) is not a line, then \( G \) splits relatively to \( H \) over a cyclic-by-(arc-stabiliser) subgroup.

**Proof of the main proposition.** We are now ready to prove Proposition 4.1. The strategy is to find an action on a real tree thanks to Paulin’s construction and next to apply Theorem 4.5.

**Proof of Proposition 4.1.** Fix a finite generating set \( S \) of \( G \), a non-principal ultrafilter \( \omega \) over \( \mathbb{N} \), and, for every \( n \geq 1 \), set \( \lambda_n = \min \max_{x \in G} d(x, \varphi_n(s) \cdot x) \). According to Fact 4.4 one has \( \lim_{\omega} \lambda_n = +\infty \). It follows from Paulin’s construction explained above that there exists some \( o = (o_n) \in G^\mathbb{N} \) such that \( G \) acts on the asymptotic cones \( \text{Cone}(G) = \text{Cone}_{\omega}(G, (1/\lambda_n), o) \) via

\[
\begin{align*}
G & \to \text{Isom}(\text{Cone}(G)) \\
g & \mapsto ((x_n) \mapsto (\varphi_n(g) \cdot x_n))
\end{align*}
\]
Moreover, this action is fixed-point free according to Fact 4.3. Notice that Cone\((G)\) is a real tree since \(G\) is hyperbolic.

**Claim 4.6.** *The element \(g\) fixes a point in Cone\((G)\).*

Let \(h \in G\) be such that \(\varphi_i(g) = h\) for every \(i \geq 1\). Because \(h\) has infinite order, there exists a quasi-geodesic \(\gamma \subset G\) and some constant \(D \geq 0\) such that
\[
|d(x, hx) - \|h\| - 2d(x, \gamma)| \leq D
\]
for every \(x \in G\), where \(\|h\| := \min\{d_S(x, hx) \mid x \in G\}\). As a consequence, for every \(n \geq 1\), one has
\[
d_S(o_n, \varphi_n(g) \cdot o_n) = d_S(o_n, h \cdot o_n) \geq \|h\| + 2d_S(o_n, p_n) - D
\]
where \(p_n\) denotes a point of \(\gamma\) minimising the distance to \(o_n\). Since \(g \cdot o = (\varphi_n(g) \cdot o_n)\) defines a point of Cone\((S(\Gamma))\), it follows from the previous inequality that the sequence \((d_S(o_n, p_n) / \lambda_n)\) is \(\omega\)-bounded, so that \(p = (p_n)\) defines a point of Cone\((S(\Gamma))\). We have
\[
d_{Cone}(p, g \cdot p) = \lim_{\omega} \frac{1}{\lambda_n} d_S(p_n, \varphi_n(g) \cdot p_n) = \lim_{\omega} \frac{1}{\lambda_n} d_S(p_n, h \cdot p_n)
\]
\[
\leq \lim_{\omega} \frac{1}{\lambda_n} (\|h\| + D) = 0.
\]
Thus, we have proved that \(g\) fixes a point of Cone\((G)\), concluding the proof of our claim.

**Claim 4.7.** *Arc-stabilisers in Cone\((G)\) are virtually cyclic.*

For a proof of this claim, we refer to [Pau91, Proposition 2.4] whose arguments can be easily adapted to the language of asymptotic cones.

Let \(T \subset Cone(G)\) be a subtree on which \(G\) acts minimally. Notice that \(T\) cannot be a line since otherwise the commutator subgroup of \(G\) would be abelian, implying that \(G\) is solvable, and thus contradicting the fact that \(G\) is a non-elementary hyperbolic group. Consequently, Theorem 4.5 applies, and we conclude that \(G\) must split relatively to \(\langle g \rangle\) over a virtually cyclic subgroup.

\[\square\]

## 5 Proof of the main theorem

Our last section is dedicated to the proof of Theorem 1.2, namely:

**Theorem 5.1.** *Let \(G\) be a one-ended hyperbolic group which is not virtually a surface group. Then \(\text{Aut}(G)\) is acylindrically hyperbolic.*

Before turning to the proof of Theorem 5.1, let us state a few results which will be needed. The first result we need is [DGO17, Theorem 8.7].

**Theorem 5.2.** *Let \(G\) be a group acting on a hyperbolic space \(X\). If \(g \in G\) is WPD, then there exists some \(s \geq 1\) such that the normal closure \(\langle g^s \rangle\) is a free subgroup all of whose non-trivial element are loxodromic isometries of \(X\).*

Next, we will need some information on free splittings of finitely generated free groups. The following statement is essentially a consequence of Whitehead’s work [Whi36]. We refer to [CM15, Lemma 5.3] for a proof.

**Proposition 5.3.** *Let \(F\) be a finitely generated free group. There exists some \(g \in F\) such that \(F\) does not split freely relatively to \(\langle g \rangle\).*

We are finally ready to prove Theorem 5.1.
**Proof of Theorem 5.1.** Let $G$ be a one-ended hyperbolic which is not virtually a surface group, and let $T$ denote its JSJ tree.

Suppose first that no element of $G$ defines a loxodromic isometry of $T$. Lemma 2.3 implies that $\text{Out}(G)$ must be finite. Consequently, $\text{Aut}(G)$ contains the subgroup of inner automorphisms as a finite-index subgroup isomorphic to the quotient of $G$ by its center. Since the center of a non-elementary hyperbolic group must be finite, we conclude that $\text{Aut}(G)$ must be a non-elementary hyperbolic group as well. In particular, it has to be acylindrically hyperbolic.

From now on, suppose that $G$ contains elements which are loxodromic isometries of $T$. As a consequence of Proposition 2.4 there exists some $g \in G$ which is WPD with respect to the action $G \curvearrowright T$, and, as a consequence of Theorem 5.2 up to replacing $g$ with one of its powers we may suppose that $\langle g \rangle$ is a free subgroup intersecting trivially vertex- and edge-stabilisers of $T$. Fix a finite generating set $S$ of $G$, and consider the subgroup

$$H = \{xgx^{-1}, \ x \in S \cup \{1\} \} \leq \langle g \rangle.$$ 

According to Proposition 5.3 there exists some $h \in H$ such that $H$ does not split freely relatively at $\langle h \rangle$.

Fix a splitting of $G$ over a virtually cyclic subgroup, and let $T'$ denote the associated Bass-Serre tree.

**Claim 5.4.** The subgroup $H$ does not fix a point of $T'$.

If $h$ is a loxodromic isometry of $T'$, then the conclusion is clear. So suppose that $h$ is elliptic. Notice that $h$ cannot stabilise an edge since it follows from Theorem 2.1 that $H$ intersects trivially any virtually cyclic subgroup of $G$ over which $G$ splits. So the fixed-point set of $h$ has to be reduced to a single vertex, say $p \in T'$. Now, we distinguish two cases.

Suppose first that there exists $x \in S$ such that $x$ is a loxodromic isometry of $T'$. Then $h$ and $xhx^{-1}$ are two elliptic isometries whose fixed-point sets are disjoint, so that it follows from [Ser03] Proposition I.6.26 that $h \cdot xhx^{-1}$ is a loxodromic element of $T'$. Because this element belongs to $H$, we conclude that $H$ does not fix a point of $T'$.

Now, suppose that any element of $S$ is elliptic in $T'$. Since $G$ does not fix a point of $T'$ and that $S$ generates $G$, we know that there exists $x \in S$ which does not fix the unique vertex $p$ fixed by $h$. Let $q \in T'$ denote a vertex fixed by $x$. Notice that, because $h$ does not stabilise an edge, the vertex $p$ separates $q$ and $h \cdot q$. As a consequence, if the fixed-point sets of $x$ and $hx^{-1}h^{-1}$ intersect, then $p$ must be fixed by either $x$ or $hx^{-1}h^{-1}$. We already know that $x$ does not fix $p$. But, in the latter case, we deduce that

$$x^{-1} \cdot p = h^{-1} \cdot hx^{-1}h^{-1} \cdot h \cdot p = h^{-1} \cdot h^{-1}x^{-1}h^{-1} \cdot p = h^{-1} \cdot p = p,$$

which is impossible. Therefore, the fixed-point sets of $x$ and $hx^{-1}h^{-1}$ must be disjoint. It follows from [Ser03] Proposition I.6.26 that $x \cdot hx^{-1}h^{-1}$ is a loxodromic element of $T'$. Because this element belongs to $H$, we conclude that $H$ does not fix a point of $T'$.

It concludes the proof of our claim, i.e., $H$ acts fixed-point freely on $T'$.

Now, notice that $H$ has trivial edge-stabilisers in $T'$. Indeed, according to Theorem 2.1 any edge-stabiliser of $T'$ must fix a point in $T$, so that the conclusion follows from the fact that $H$ intersects trivially vertex- and edge-stabilisers of $T$. The conclusion is that $H$ acts on $T'$ fixed-point freely and with trivial edge-stabilisers. By definition of the element $h \in H$, we deduce that $h$ is a loxodromic isometry of $T'$.
Thus, we have proved that $h$ cannot be elliptic in the Bass-Serre tree associated to a
splitting of $G$ over a virtually cyclic subgroup. The same conclusion holds for any non-
trivial power of $h$. Therefore, for any $s \geq 1$, $G$ does not split relatively to $(h^s)$ over a
virtually cyclic subgroup. Notice that, as a consequence of Proposition 2.4, $h$ is WPD
with respect to $G \sim T$, so that Proposition 3.1 applies. We deduce from Proposition 4.1
that the inner automorphism $\iota(h)$ is WPD with respect to the action $\text{Aut}(G) \sim T$.

Therefore, $\text{Aut}(G)$ is either acylindrically hyperbolic or virtually cyclic. But $\text{Inn}(G) \cong G/\langle G \rangle$ is not virtually cyclic since $G$ is one-ended, so $\text{Aut}(G)$ cannot be virtually
cyclic. We conclude that $\text{Aut}(G)$ is acylindrically hyperbolic as desired. \hfill \square

6 Applications to extensions of hyperbolic groups

The goal of this section is to prove Theorem 1.5 namely:

**Theorem 6.1.** Let $G$ be a group whose center is finite and whose automorphism group
is acylindrically hyperbolic. For every automorphism $\varphi \in \text{Aut}(G)$, the semidirect product
$G \rtimes \mathbb{Z}$ is acylindrically hyperbolic if and only if $\varphi$ has infinite order in $\text{Out}(G)$.

We begin by proving a stronger version of the converse:

**Proposition 6.2.** Let $G$ be a group whose center is finite and whose automorphism
group is acylindrically hyperbolic. If $H \leq \text{Aut}(G)$ satisfies $H \cap \text{Inn}(G) = \{1\}$, then the
semidirect product $G \rtimes H$ is acylindrically hyperbolic.

**Proof.** Notice that $\langle \text{Inn}(G), H \rangle = \text{Inn}(G) \rtimes H$ since $\text{Inn}(G)$ is a normal subgroup and that
$H \cap \text{Inn}(G) = \{1\}$. On the other hand, because the center $Z(G)$ of $G$ is a characteristic
normal subgroup, we have the short exact sequence

$$1 \to Z(G) \to G \to (G/Z(G)) \to H \to 1.$$ 

By noticing that $\varphi \cdot \iota(g) \cdot \varphi^{-1} = \iota(\varphi(g))$ for every $g \in G$, we deduce that the canonical
isomorphism

$$
\begin{align*}
\left\{ \begin{array}{c}
G/Z(G) \\
g
\end{array} \right. & \rightarrow \text{Inn}(G) \\
\rightarrow & \iota(g)
\end{align*}
$$

induces an isomorphism $(G/Z(G)) \rtimes H \rightarrow \text{Inn}(G) \rtimes H$. Because being acylindrically
hyperbolic is stable under quotients with finite kernels and extensions with finite kernels
(according to [MO17, Lemma 1]), it follows that $G \rtimes H$ is acylindrically hyperbolic if
and only if so is $(\text{Inn}(G), H)$.

So it remains to show that $(\text{Inn}(G), H)$ is acylindrically hyperbolic. Notice that, as a
consequence of [Osi16, Lemma 7.2], any normal subgroup of an acylindrically hyperbolic
group must contain at least one of the generalised loxodromic elements of the group. Therefore, $\text{Aut}(G)$ must contain a generalised loxodromic element which is an inner
automorphism. It follows that $(\text{Inn}(G), H)$ contains a generalised loxodromic element
in its own right, i.e., $(\text{Inn}(G), H)$ is acylindrically hyperbolic or virtually cyclic. But $\text{Inn}(G)$ cannot be virtually cyclic because it is a normal subgroup of an acylindrically
hyperbolic group, again according to [Osi16, Lemma 7.2], so the desired conclusion holds. \hfill \square

Now we are ready to prove our theorem.

**Proof of Theorem 1.5.** If $\varphi$ has infinite order in $\text{Out}(G)$ then $\langle \varphi \rangle \cap \text{Inn}(G) = \{1\}$, so that Proposition 6.2 applies and implies that $G \rtimes \mathbb{Z}$ is acylindrically hyperbolic. If $\varphi$
has finite order in $\text{Out}(G)$, say $n$, then $G \rtimes \mathbb{Z}$ is a finite-index subgroup of $G \rtimes \mathbb{Z}$
isomorphic to $G \rtimes_{\text{Id}} \mathbb{Z} = G \oplus \mathbb{Z}$ (since the semi-direct product depends only on the image of the automorphism in $\text{Out}(G)$). But the direct sum of two infinite groups is never acylindrically hyperbolic according to [Osi16 Corollary 7.3], and being acylindrically hyperbolic is stable under taking finite-index subgroup, so we conclude that $G \rtimes_{\varphi} \mathbb{Z}$ cannot be acylindrically hyperbolic.

Because non-elementary hyperbolic groups have finite centers, our next statement follows immediately from Theorems 1.5 and 1.2:

**Corollary 6.3.** Let $G$ be a one-ended hyperbolic group which is not virtually a surface group, and let $\varphi \in \text{Aut}(G)$ be an automorphism. Then $G \rtimes_{\varphi} \mathbb{Z}$ is acylindrically hyperbolic if and only if $\varphi$ has infinite order in $\text{Out}(G)$.

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