ON MODULAR DECOMPOSITIONS OF SYSTEM SIGNATURES

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Abstract. Considering a coherent system made up of $n$ components having i.i.d. continuous lifetimes, F. Samaniego defined its structural signature as the $n$-tuple whose $k$-th coordinate is the probability that the $k$-th component failure causes the system to fail. This $n$-tuple, which depends only on the structure of the system and not on the distribution of the component lifetimes, is a very useful tool in the theoretical analysis of coherent systems.

This concept was recently extended to the general case of semicoherent systems whose components may have dependent lifetimes, where the same definition for the $n$-tuple gives rise to the probability signature, which depends in general on both the structure of the system and the probability distribution of the component lifetimes. The dependence on the latter is encoded via the so-called relative quality function associated with the distribution of the component lifetimes.

In this work we consider a system that is partitioned into disjoint modules. Under a natural decomposition property of the relative quality function, we give an explicit and compact expression for the probability signature of the system in terms of the probability signatures of the modules and the structure of the modular decomposition. This formula holds in particular when the lifetimes are i.i.d. or exchangeable, but also in more general cases, thus generalizing results recently obtained in the i.i.d. case and for special modular decompositions.

1. Introduction

Consider an $n$-component system $S = (C, \phi, F)$, where $C$ is the set $[n] = \{1, \ldots, n\}$ of components, $\phi : (0,1)^n \to \{0,1\}$ is the structure function (which expresses the state of the system in terms of the states of its components), and $F$ denotes the joint c.d.f. of the component lifetimes $T_1, \ldots, T_n$, that is,

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leq t_1, \ldots, T_n \leq t_n), \quad t_1, \ldots, t_n \geq 0.$$ 

We assume that the system is semicoherent, i.e., the structure function $\phi$ is nondecreasing in each variable and satisfies the conditions $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$. We also assume that the c.d.f. $F$ has no ties, that is, $\Pr(T_i = T_j) = 0$ for all distinct $i, j \in [n]$.

The concept of signature was introduced in 1985 by Samaniego [13], for systems whose components have continuous and i.i.d. lifetimes, as the $n$-tuple $s$ whose $k$-th coordinate $s_k$ is the probability that the $k$-th component failure causes the system to fail. In other words, we have

$$s_k = \Pr(T_S = T_{kn}), \quad k \in [n],$$

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where $T_S$ denotes the system lifetime and $T_{kn}$ denotes the $k$-th smallest lifetime, i.e., the $k$-th order statistic obtained by rearranging the variables $T_1, \ldots, T_n$ in ascending order of magnitude.

It was shown in [3] that $s_k$ can be explicitly written in the form\footnote{As usual, we identify Boolean vectors $x \in \{0, 1\}^n$ and subsets $A \subseteq [n]$ by setting $x_i = 1$ if and only if $i \in A$. We thus use the same symbol to denote both a function $f: \{0, 1\}^n \to \mathbb{R}$ and the corresponding set function $f: 2^{[n]} \to \mathbb{R}$, interchangeably.}

\[
s_k = \sum_{\substack{A \subseteq [n] \mid |A| = n-k+1}} \frac{1}{n_{|A|}} \phi(A) - \sum_{\substack{A \subseteq [n] \mid |A| = n-k}} \frac{1}{n_{|A|}} \phi(A).
\]

This formula shows that in the i.i.d. case the probability $Pr(T_S = T_{kn})$ does not depend on the c.d.f. $F$. Thus, the system signature is a purely combinatorial object associated with the structure $\phi$. Due to this feature, $s$ is sometimes referred to as the structural signature of the system.

Since its introduction the concept of signature proved to be a very useful tool in the analysis of semicoherent systems, especially for the comparison of different system designs and the computation of the system reliability (see [14]).

The interest of extending the concept of signature to the general case of dependent lifetimes has been recently pointed out in [7,9,11,15]. Just as in the i.i.d. case, we can consider the $n$-tuple $p$ whose $k$-th coordinate is $p_k = Pr(T_S = T_{kn})$. This $n$-tuple is called the probability signature. Interestingly, when the component lifetimes are exchangeable the probability signature still coincides with the structural signature of the system [9]. However, in the general case the structural signature and the probability signature are two different objects. In fact the latter may depend on both the structure $\phi$ and the c.d.f. $F$.

It was shown in [7] that the dependence of $p$ on the c.d.f. $F$ is captured by the relative quality function $q: 2^{[n]} \to [0, 1]$ associated with $F$. This function is defined by

\[
q(A) = Pr\left(\max_{i \in A} T_i < \min_{i \in A} T_i\right), \quad A \subseteq [n],
\]

with the convention that $q(\varnothing) = q([n]) = 1$ and the immediate property

\[
\sum_{\substack{A \subseteq [n] \mid |A| = k}} q(A) = 1, \quad 0 \leq k \leq n.
\]

It was actually shown [7] that, if $F$ is absolutely continuous (actually the assumption that $F$ has no ties is sufficient), then

\[
p_k = \sum_{\substack{A \subseteq [n] \mid |A| = n-k+1}} q(A) \phi(A) - \sum_{\substack{A \subseteq [n] \mid |A| = n-k}} q(A) \phi(A).
\]

We clearly see that [4] reduces to [1] whenever $q$ is a symmetric function, i.e., $q(A) = 1/|A|$, and this property holds for instance when the component lifetimes are exchangeable and hence in the i.i.d. case.

The computation of the (structural or probability) signature may be a hard task when the system has a large number of components. However, the computation effort can be greatly reduced when the system is decomposed into distinct modules (subsystems) whose signatures are already known.

First results along this line have been presented in [4,5], where attention was restricted to the i.i.d. case. In particular, in [5] explicit expressions for the structural
signatures of systems consisting of two modules connected in series or in parallel
were provided in terms of the structural signatures of the modules. A general
procedure to compute the structural signatures of recurrent systems (i.e., systems
partitioned into identical modules) was also described. Moreover, the key role of
the concepts of tail and cumulative signatures were pointed out (see definitions in
Section 2).

In this work we extend these results in two different directions:

- We consider the general case of systems partitioned into $r$ arbitrary disjoint
  modules connected according to an arbitrary semicoherent structure.
- We relax the i.i.d. assumption by requiring that the relative quality function
  satisfies a natural decomposability condition (see Definition 2). This condition
  is satisfied whenever the components have exchangeable lifetimes (and
  in particular in the i.i.d. case) but also under a more general assumption
  of partition-based exchangeability of the component lifetimes.

Under these assumptions we give an explicit expression for the probability signature
in terms of the module signatures and the structure of the modular decomposition.

The outline of this paper is as follows. In Section 2 we extend the definitions
of tail and cumulative signatures to the non-i.i.d. setting, introduce and interpret
a concept of decomposability for the relative quality function when the system is
partitioned into modules, and discuss sufficient conditions for this decomposability
property to hold. In Section 3 we provide the main result (Theorem 12), that is a
general formula for the probability signatures of systems partitioned into modules.
Finally, in Section 4 we discuss and demonstrate our results through a few examples.

2. Decomposability properties of relative quality functions

The concept of tail signature, introduced in [3] and named so in [5], is algebraically
more convenient than that of signature. The tail signature of the system is the
$(n+1)$-tuple $\mathbf{S} = (S_0, \ldots, S_n)$ defined by $S_n = 0$ and

$$S_k = \sum_{i=k+1}^{n} s_i = \sum_{A \subseteq [n]} \frac{1}{\binom{|A|}{n-k}} \phi(A), \quad 0 \leq k \leq n-1.$$ 

Similarly, the cumulative signature of the system is the $(n+1)$-tuple $\mathbf{S}$ whose $k$-th
coordinate is defined by $S_k = 1 - S_k = \sum_{i=1}^{k} s_i$ for $0 \leq k \leq n$.

In the non-i.i.d. setting, we naturally introduce the tail probability signature of
the system as the $(n+1)$-tuple $P = (P_0, \ldots, P_n)$ defined by

$$P_k = \sum_{i=k+1}^{n} p_i.$$ 

According to the result presented in [7] and stated in [1], we then have

$$P_k = \sum_{A \subseteq [n]} \frac{q(A) \phi(A)}{|A|=n-k},$$

where $q$ is the relative quality function defined in [2].

We have in particular $P_0 = 1$, $P_n = 0$, and $P_k = \Pr(T_S > T_{kn})$ for $k \in [n]$. Finally,
the cumulative probability signature of the system is defined as the $(n+1)$-tuple $P$
such that $P_0 = 0$ and $P_k = 1 - P_K = \sum_{i=1}^{k} p_i = \Pr(T_S \leq T_{kn})$ for $1 \leq k \leq n$. 

Both the tail and cumulative probability signatures are simple linear expressions of the \( \phi \) values (handier than the probability signature itself) and will be very useful to state our main result. Note that the probability signature can be retrieved from the tail probability signature by computing \( p_k = \overline{P}_{k-1} - \overline{P}_k \) for \( k \in [n] \).

Throughout we denote by \( C \) a partition of the set of components \( C \). That is, \( C \) is a collection \( \{C_1, \ldots, C_r\} \) of nonempty subsets \( C_1, \ldots, C_r \) of \( C \), such that \( C = C_1 \cup \cdots \cup C_r \), and \( C_i \cap C_j = \emptyset \) for all \( i, j \in [r] \). If \( j \in [r] \), we set \( n_j = |C_j| \) (hence \( \sum_{j=1}^r n_j = n \)) and, for \( A \subseteq C \), we set \( A_j = A \cap C_j \). Finally, we denote by \( q^{C_j} \) the relative quality function associated with \( C_j \) obtained from the marginal distribution of the lifetimes of the components in \( C_j \). Namely,

\[
q^{C_j}(A) = \Pr \left( \max_{i \in C_j \setminus A} T_i < \min_{i \in A} T_i \right), \quad A \subseteq C_j.
\]

We now give the definitions that will play a central role in our results.

**Definition 1.** A permutation \( \pi \) on \( C \) preserves a partition \( C = \{C_1, \ldots, C_r\} \) if \( \pi(C_j) = C_j \) for every \( j \in [r] \). A function \( c: 2^C \rightarrow \mathbb{R} \) is \( C \)-symmetric if it is invariant under the action of the permutations that preserve \( C \), i.e., \( c(\pi(A)) = c(A) \) for every \( A \subseteq C \) and every permutation \( \pi \) on \( C \) that preserves \( C \).

We thus see that \( c: 2^C \rightarrow \mathbb{R} \) is \( C \)-symmetric if and only if \( c(A) = c(A') \) whenever \( |A_j| = |A'_j| \) for all \( j \in [r] \). Equivalently, \( c \) is \( C \)-symmetric if and only if there exists a function \( \hat{c}: \prod_{j=1}^r \{0, \ldots, n_j\} \rightarrow \mathbb{R} \) such that

\[
c(A) = \hat{c}(|A_1|, \ldots, |A_r|), \quad A \subseteq C,
\]

which means that the value \( c(A) \) depends only on the cardinalities \( |A_1|, \ldots, |A_r| \).

**Definition 2.** For any partition \( C \) of \( C \), we say that the relative quality function \( q \) is \( C \)-decomposable if there exists a \( C \)-symmetric function \( c: 2^C \rightarrow \mathbb{R} \) such that

\[
q(A) = c(A) \prod_{j=1}^r q^{C_j}(A_j), \quad A \subseteq C.
\]

We say that the relative quality function \( q \) is decomposable if it is \( C \)-decomposable for every partition \( C \) of \( C \).

Note that the \( C \)-decomposability of \( q \) depends only on the partition \( C \) and the c.d.f. \( F \) but not on the structure \( \phi \). We will show in Proposition 5 how the function \( c \) that appears in (6), or equivalently the corresponding function \( \hat{c} \), can be interpreted. At this stage we only make the following basic but important observation.

**Remark 1.** If \( q \) is \( C \)-decomposable for some partition \( C = \{C_1, \ldots, C_r\} \), then the corresponding functions \( c \) and \( \hat{c} \) are completely determined by the functions \( q \) and \( q^{C_j} \) \( (j = 1, \ldots, r) \). Indeed, given any \( r \)-tuple \( (a_1, \ldots, a_r) \in \mathbb{N}^r \), such that \( 0 \leq a_j \leq n_j \), formula (6) can be rewritten as

\[
\hat{c}(a_1, \ldots, a_r) = \frac{q(A)}{\prod_{j=1}^r q^{C_j}(A_j)}
\]

whenever \( A \subseteq C \) is such that \( |A_j| = a_j \) and \( q^{C_j}(A_j) \neq 0 \) for all \( j \in [r] \). We just observe that such a set exists by (5).

We now examine some natural cases where the relative quality function \( q \) is \( C \)-decomposable for some partition \( C = \{C_1, \ldots, C_r\} \). We can observe that, for the extreme (trivial) partitions corresponding to \( r = 1 \) or \( r = n \), the relative quality
function is decomposable, regardless of the distribution of lifetimes, since (6) always holds in these cases.

The next two results show that, for i.i.d. or exchangeable lifetimes, the function \( q \) is always decomposable.

**Proposition 3.** If the function \( q \) is \( C \)-symmetric for some partition \( C = \{C_1, \ldots, C_r\} \) and the functions \( q^{C_j} \) are symmetric for \( j = 1, \ldots, r \), then \( q \) is \( C \)-decomposable.

**Proof.** Under the assumptions of the proposition, we have \( q^{C_j}(A_j) = 1/(\binom{n_j}{|A_j|}) \) for all \( A_j \subseteq C_j \) and all \( j \in [r] \). Hence (6) holds for the function \( c \) defined by

\[
c(A) = q(A) \prod_{j=1}^r \binom{n_j}{|A_j|}, \quad A \subseteq C,
\]

which is \( C \)-symmetric since so is \( q \). \( \square \)

From Proposition 3 we derive immediately the following corollary.

**Corollary 4.** If \( q \) and \( q^{C_j} (j = 1, \ldots, r) \) are symmetric, then \( q \) is decomposable and we have

\[
\tilde{c}(a) = \binom{n_1}{a_1} \cdots \binom{n_r}{a_r} \left( \frac{n - k}{a_1 + \cdots + a_r} \right).
\]

This happens for instance when the component lifetimes are exchangeable and in particular in the i.i.d. case.

Corollary 4 shows that, when the functions \( q \) and \( q^{C_j} \) are symmetric, for every \( k \in \{0, \ldots, n\} \) the restriction of \( \tilde{c} \) to the set

\[
T_k = \{a = (a_1, \ldots, a_r) \in \mathbb{N}^r : 0 \leq a_j \leq n_j \text{ for } j = 1, \ldots, r \text{ and } \sum_{j=1}^r a_j = k \}
\]

is a probability distribution, namely the multivariate hypergeometric distribution. The following result shows that such an interpretation still holds whenever the relative quality function \( q \) is \( C \)-decomposable for some partition \( C \).

For every \( a \in T_k \) we introduce the following event:

\[
E_{k,a} = \{\text{among the first } n - k \text{ failed components, there are exactly } n_j - a_j \text{ components in } C_j \text{ for all } j \in [r]\}.
\]

We observe that \( E_{k,a} \) is also the following event: (among the best \( k \) components, there are exactly \( a_j \) components in \( C_j \) for all \( j \in [r] \)).

**Proposition 5.** Assume that the relative quality function \( q \) is \( C \)-decomposable for a partition \( C = \{C_1, \ldots, C_r\} \) of \( C \). Then, for each \( k \in \{0, \ldots, n\} \), the restriction of the function \( \tilde{c} \) to \( T_k \) is a probability distribution. More precisely, for every \( a \in T_k \), \( \tilde{c}(a) \) is exactly the probability \( \Pr(E_{k,a}) \).

**Proof.** Since \( q(A) \) is the probability that the best \( |A| \) components are precisely those in \( A \), we must have

\[
\Pr(E_{k,a}) = \sum_{A \subseteq C : |A| = a_j \forall j \in [r]} q(A).
\]
Since $q$ is $C$-decomposable, by (8) we have
\[
\Pr(E_{k,a}) = \sum_{A_1 \subseteq C_1:|A_1|=a_1} \cdots \sum_{A_r \subseteq C_r:|A_r|=a_r} \hat{c}(a) \prod_{j=1}^r q^{C_j}(A_j)
\]
\[
= \hat{c}(a) \prod_{j=1}^r \left( \sum_{A_j \subseteq C_j:|A_j|=a_j} q^{C_j}(A_j) \right),
\]
where the product is 1 (apply Eq. (3) to every $q^{C_j}$).

The following corollary gives an interpretation of the $C$-decomposability of $q$.

**Corollary 6.** Let $C = \{C_1, \ldots, C_r\}$ be a partition of $C$. Then $q$ is $C$-decomposable if and only if, for every $A \subseteq C$, we have $\Pr(E_{k,a}) = 0$ (with $k = |A|$ and $a_j = |A_j|$) or
\[
\Pr\left( \max_{i \in A} T_i < \min_{i \in A} T_i \middle| E_{k,a} \right) = \prod_{j=1}^r q^{C_j}(A_j).
\]

**Proof.** On the one hand, by Proposition 5, $q$ is $C$-decomposable if and only if
\[
q(A) = \Pr(E_{k,a}) \prod_{j=1}^r q^{C_j}(A_j), \quad A \subseteq C.
\]
On the other hand, we always have
\[
q(A) = \Pr(E_{k,a}) \Pr\left( \max_{i \in A} T_i < \min_{i \in A} T_i \middle| E_{k,a} \right), \quad A \subseteq C,
\]
which completes the proof. □

Corollary 6 says that $q$ is $C$-decomposable if and only if, for every $A \subseteq C$, if $\Pr(E_{k,a}) \neq 0$ (with $k = |A|$ and $a_j = |A_j|$), then the probability that the best $k$ components are precisely those in $A$ knowing that among them there are exactly $a_j$ components in $C_j$, $j \in [r]$, factorizes as the product over $j \in [r]$ of the probabilities that the best $a_j$ components in $C_j$ are precisely those in $A_j$. Thus, the $C$-decomposability of $q$ turns out to be a form of independence.

Let us now give a simple example where the function $q$ is $C$-decomposable but not $C$-symmetric for a given partition $C$.

**Example 7.** Let us consider a system made up of four components, with lifetimes $T_1, T_2, T_3, T_4$, such that for every permutation $\sigma$ on $\{1, \ldots, 4\}$, we have
\[
\Pr(T_{\sigma(1)} < T_{\sigma(2)} < T_{\sigma(3)} < T_{\sigma(4)}) = \begin{cases} 1/18 & \text{if } \sigma^{-1}(1) < \sigma^{-1}(2), \\ 1/36 & \text{otherwise.} \end{cases}
\]
It is easy to see that the function $q$ is $C$-decomposable for the partition $C = (\{1, 2\}, \{3, 4\})$. However, it is not $C$-symmetric since we have $q(\{1\}) = 1/6$ and $q(\{2\}) = 1/3$.

Let us now end this section by analyzing a natural sufficient condition for $q$ to be $C$-decomposable. We first introduce a partition-based exchangeability condition.

**Definition 8.** We say that $(T_1, \ldots, T_n)$ is $C$-exchangeable if the random vectors $(T_{\pi(1)}, \ldots, T_{\pi(n)})$ and $(T_1, \ldots, T_n)$ have the same distribution for every permutation $\pi$ on $C$ that preserves $C$.

For any subset $C_j$ of $C$, we denote by $T_{C_j}$ the marginal distribution $(T_i)_{i \in C_j}$. 

\section{References}

[1] Jean-Luc Marichal, Pierre Mathonet, and Fabio Spizzichino. 

[2] [Publication Title]. Publisher, Year.
Proposition 9. If the random vector \((T_1, \ldots, T_n)\) is \(\mathcal{C}\)-exchangeable for a partition \(\mathcal{C}\), then
\[
\begin{align*}
(i) \text{ the functions } q^{C_j} & \text{ are symmetric for } j = 1, \ldots, r; \\
(ii) \text{ the function } q & \text{ is } \mathcal{C}\text{-symmetric.}
\end{align*}
\]
Therefore the function \(q\) is \(\mathcal{C}\)-decomposable.

Proof. To show (i) we note that \(\mathcal{C}\)-exchangeability implies that for every \(j \in [r]\), the marginal distribution \(T^{C_j}\) is exchangeable.

To show (ii) we consider \(A \subseteq C\), a permutation \(\pi\) on \(C\) that preserves \(\mathcal{C}\), and we prove that \(q(A) = q(\pi(A))\). By \(\mathcal{C}\)-exchangeability, we have
\[
Pr(T_{\sigma(1)} < \cdots < T_{\sigma(n)}) = Pr(T_{\pi(\sigma(1))} < \cdots < T_{\pi(\sigma(n))}),
\]
for every \(\sigma \in \mathfrak{S}_n\), where \(\mathfrak{S}_n\) is the set of permutations on \([n]\).

The relative quality function \(q\) can be computed through the formula
\[
q(A) = \sum_{\sigma \in \mathfrak{S}_n} \Pr(T_{\sigma(1)} < \cdots < T_{\sigma(n)})
\]
see [2, Formula (3)]. But, setting \(\sigma' = \pi \circ \sigma\), we also have
\[
q(A) = \sum_{\sigma \in \mathfrak{S}_n} \Pr(T_{\pi(\sigma(1))} < \cdots < T_{\pi(\sigma(n))})
\]
\[
= \sum_{\sigma' \in \mathfrak{S}_n} \Pr(T_{\sigma'(1)} < \cdots < T_{\sigma'(n)}) = q(\pi(A))
\]
We conclude by using Proposition \[9\].

We then have the following immediate corollary.

Corollary 10. If the joint distribution of \((T_1, \ldots, T_n)\) is such that the marginals \(T^{C_1}, \ldots, T^{C_r}\) are independent and each marginal \(T^{C_j}\) is exchangeable, then the relative quality function \(q\) is \(\mathcal{C}\)-decomposable.

Proof. We just notice that the assumptions imply that \((T_1, \ldots, T_n)\) is \(\mathcal{C}\)-exchangeable and apply Proposition \[9\].

3. Modular decomposition of the signature

We now consider a system \((C, \phi, F)\) partitioned into \(r\) disjoint modules. We thus have a partition \(\mathcal{C} = \{C_1, \ldots, C_r\}\) of the set of components. For every \(j \in [r]\), the components in \(C_j\) are connected in a semicoherent structure described by the function \(\chi_j: \{0,1\}^{n_j} \rightarrow \{0,1\}\) and thus form the module \((C_j, \chi_j)\)\footnote{Here the corresponding marginal distribution determined by \(F\) is implicitly considered.}. Finally, the modules are connected in a semicoherent structure described by the function \(\psi: \{0,1\}^{r} \rightarrow \{0,1\}\). The modular decomposition of the structure \(\phi\) of the system expresses through the composition
\[
(9) \quad \phi(x) = \psi(\chi_1(x^{C_1}), \ldots, \chi_r(x^{C_r})),
\]
where \(x^{C_j} = (x_i)_{i \in C_j}\) (see [2, Chap. 1]). For instance, if the system consists of three serially connected modules, then we have \(\psi(z_1, z_2, z_3) = \min(z_1, z_2, z_3) = z_1 z_2 z_3\) and hence
\[
\phi(x) = \chi_1(x^{C_1}) \chi_2(x^{C_2}) \chi_3(x^{C_3}).
\]
It will be useful to observe that, using the identification of pseudo-Boolean functions with set functions, we can rewrite (9) as
\[ \phi(A) = \psi(\chi_1(A \cap C_1), \ldots, \chi_r(A \cap C_r)), \quad A \subseteq C. \]
For every \( j \in [r] \), we denote by \( \overline{P}_{j,k} \) and \( P_{j} \) the tail and cumulative probability signatures, respectively, of module \((C_j, \chi_j)\), that is
\[ \overline{P}_{j,k} = 1 - P_{j,k} = \sum_{A \subseteq C_j \mid |A|=n_j-k} q^C(A) \chi_j(A), \quad k = 0, \ldots, n_j. \]

To state our main result we need to recall the concept of multilinear extension of a (pseudo)-Boolean function (see [12]).

**Definition 11.** The **multilinear extension** of a pseudo-Boolean function \( \chi : \{0,1\}^m \to \mathbb{R} \) is the polynomial function \( \overline{\chi} : \[0,1]\text{ }^m \to \mathbb{R} \) defined by
\[ \overline{\chi}(z_1, \ldots, z_m) = \sum_{B \subseteq [m]} \chi(B) \prod_{j \in B} z_j \prod_{j \in [m] \setminus B} (1 - z_j). \]

For instance, if \( \chi \) is the structure function of a series system having three components, we have
\[ \overline{\chi}(z_1, z_2, z_3) = z_1 z_2 z_3. \]
Similarly, if \( \chi \) is the structure function of a parallel system having three components, we have
\[ \overline{\chi}(z_1, z_2, z_3) = 1 - (1 - z_1)(1 - z_2)(1 - z_3). \]

**Remark 2.** The multilinear extension of a pseudo-Boolean function \( \chi : \{0,1\}^m \to \mathbb{R} \) is the unique function defined on \([0,1]^m\) that has the following properties:
(i) it is a polynomial function,
(ii) it is at most linear in each variable, and
(iii) it coincides with \( \chi \) on \([0,1]^m\) (in particular [12] also holds on \([0,1]^m\) when \( \overline{\chi} \) is replaced by \( \chi \)).

It is well known that if \( \chi \) is semicoherent, then \( \overline{\chi} \) coincides with the reliability function \( h_\chi \) of a system \(([m], \chi)\) with independent components.

We now state our main theorem, which gives an expression of the system tail signature \( \overline{P} \) in terms of the module tail signatures \( \overline{P}_{1}, \ldots, \overline{P}_{r} \).

**Theorem 12.** Assume that the relative quality function \( q \) is \( C \)-decomposable for a partition \( C = \{C_1, \ldots, C_r\} \) of \( C \). Then, for every semicoherent structure \( \psi : \{0,1\}^n \to \{0,1\} \) with a modular decomposition into \( r \) disjoint modules \((C_j, \chi_j), j = 1, \ldots, r\), connected according to a semicoherent structure \( \psi : \{0,1\}^r \to \{0,1\} \), we have
\[ \overline{P}_{n-k} = \sum_{\mathbf{a} \preceq \mathbf{1}_k} \tilde{c}({\mathbf{a}}) \psi(\overline{P}_{1,n_1-a_1}, \ldots, \overline{P}_{r,n_r-a_r}), \quad 0 \leq k \leq n. \]

**Proof.** By combining [3] with [10] and [11] we have
\[ \overline{P}_{n-k} = \sum_{|A|=k} q(A) \phi(A) = \sum_{|A|=k} q(A) \psi(\chi_1(A \cap C_1), \ldots, \chi_r(A \cap C_r)) \]
\[ = \sum_{B \subseteq [r]} \psi(B) \sum_{|A|=k} q(A) \prod_{j \in B} \chi_j(A \cap C_j) \prod_{j \in [r] \setminus B} (1 - \chi_j(A \cap C_j)), \]
for \( 0 \leq k \leq n \). Since \( C = \{C_1, \ldots, C_r\} \) is a partition of \( C \), the map from \( 2^C \) to \( \prod_{j=1}^r 2^{C_j} \) given by \( A \mapsto (A \cap C_1, \ldots, A \cap C_r) \)
is a bijection that maps \( \{ A : |A| = k \} \) onto \( \{(A_1, \ldots, A_r) : (|A_1|, \ldots, |A_r|) \in T_k \} \).

Therefore, we obtain

\[
\mathcal{F}_{n-k} = \sum_{B \in [r]} \psi(B) \sum_{a \in T_k} \sum_{a_1 \in C_1 \atop |A_1| = a_1} \cdots \sum_{A_r \subseteq C_r \atop |A_r| = a_r} q \left( \bigcup_{j=1}^{r} A_j \right) \prod_{j \in B} \chi_j(A_j) \prod_{j \in [r] \setminus B} \left( 1 - \chi_j(A_j) \right).
\]

Since \( q \) is \( \mathcal{C} \)-decomposable, the right-hand side of this expression becomes

\[
\sum_{B \in [r]} \psi(B) \sum_{a \in T_k} \hat{c}(a) \sum_{a_1 \in C_1 \atop |A_1| = a_1} \cdots \sum_{A_r \subseteq C_r \atop |A_r| = a_r} \prod_{j \in B} q^{C_j}(A_j) \chi_j(A_j) \prod_{j \in [r] \setminus B} \left( 1 - \chi_j(A_j) \right),
\]
or equivalently,

\[
\sum_{a \in T_k} \hat{c}(a) \sum_{B \in [r]} \psi(B) \prod_{j \in B} \left( \sum_{A_j \subseteq C_j \atop |A_j| = a_j} q^{C_j}(A_j) \right) \prod_{j \in [r] \setminus B} \left( \sum_{A_j \subseteq C_j \atop |A_j| = a_j} q^{C_j}(A_j) \left( 1 - \chi_j(A_j) \right) \right).
\]

By using property (13) in every module, we obtain

\[
\mathcal{F}_{n-k} = \sum_{a \in T_k} \hat{c}(a) \sum_{B \in [r]} \psi(B) \prod_{j \in B} \mathcal{F}_{j,n_j-a_j} \prod_{i \in [r] \setminus B} \left( 1 - \mathcal{F}_{j,n_j-a_j} \right)
\]
and we then conclude by (11).

\[ \Box \]

**Example 13.** Suppose that the system consists of \( r \) serially connected modules (hence \( \psi(z) = \prod_{j=1}^{r} z_j \)) and that the functions \( q \) and \( q^{C_j} \) are symmetric. Then, combining (12) with (7), we see that \( \mathcal{F}_{n-k} \) is given by the (hypergeometric) convolution product

\[
\mathcal{F}_{n-k} = \sum_{0 \leq a_j \leq n_j \atop a_1 + \cdots + a_r = k} \binom{n_1}{a_1} \cdots \binom{n_r}{a_r} \prod_{j=1}^{r} \mathcal{F}_{j,n_j-a_j}.
\]

Proposition 3 shows that the right-hand side of (12) can be interpreted as an expected value with respect to the distribution defined by \( \hat{c} \) on \( T_k \).

Proposition 5 also allows us to obtain the following dual version of Theorem 12 in which the tail probability signatures are replaced by the cumulative probability signatures. Recall that the dual of a structure function \( \chi^d : \{0,1\}^m \to \{0,1\} \) is the structure function \( \chi^d : \{0,1\}^m \to \{0,1\} \) defined by \( \chi^d(x) = 1 - \chi(1-x) \), where \( 1 = (1, \ldots, 1) \).

**Theorem 14.** Assume that the relative quality function \( q \) is \( \mathcal{C} \)-decomposable for a partition \( \mathcal{C} = \{ C_1, \ldots, C_r \} \) of \( C \). Then, for every semicoherent structure \( \psi : \{0,1\}^r \to \{0,1\} \) with a modular decomposition into \( r \) disjoint modules \( (C_j, \chi_j) \), \( j = 1, \ldots, r \), connected according to a semicoherent structure \( \psi : \{0,1\}^r \to \{0,1\} \), we have

\[
P_{n-k} = \sum_{a \in T_k} \hat{c}(a) \psi^d(P_{1,a_1-1}, \ldots, P_{r,n_r-a_r}).
\]

**Proof.** Due to the definition of \( \psi^d \), we have \( \psi^d(z_1, \ldots, z_r) = 1 - \psi(1-z_1, \ldots, 1-z_r) \), for all \( z_1, \ldots, z_r \in [0,1] \), and therefore we have

\[
\psi^d(P_{1,n_1-1}, \ldots, P_{r,n_r-a_r}) = 1 - \psi(1-P_{1,n_1-1}, \ldots, 1-P_{r,n_r-a_r}) = 1 - \psi(\mathcal{F}_{1,n_1-a_1}, \ldots, \mathcal{F}_{r,n_r-a_r}).
\]
The right-hand side of (13) becomes
\[
\sum_{a \in T_k} \tilde{c}(a) \left( 1 - \hat{\psi}(P_{1,n_1-a_1}, \ldots, P_{r,n_r-a_r}) \right),
\]
and the result follows by Proposition 5 and Theorem 12. \(\square\)

4. Discussion, examples, and final remarks

This section is devoted to a discussion based on several examples and some final remarks concerning the results presented in the previous sections.

The main object of interest in this work is the computation of the probability signature of an \(n\)-component system \((C, \phi, F)\) (or \((C, \phi)\) for short) that can be decomposed into disjoint modules \((C_j, \chi_j)\), with \(j = 1, \ldots, r\). Our main result is Theorem 12, where the signature of \((C, \phi)\) is expressed in terms of the signatures of \((C_1, \chi_1), \ldots, (C_r, \chi_r)\). We generally assume that the joint distribution \(F\) of the component lifetimes \(T_1, \ldots, T_n\) is such that ties have null probability so that the definition of signature makes sense.

In order to focus on the relevance and the range of applications of Theorem 12, we should distinguish between two different frameworks.

In the case when \(T_1, \ldots, T_n\) are i.i.d. with a continuous distribution (or, more generally, exchangeable with no ties), the probability signature coincides with the structural signature and we then look at the computation of the latter. In that case Theorem 12 shows that the structural signature of \((C, \phi)\) only depends on the signatures of the modules. This result holds regardless of both the structures of the modules and the way the modules are connected. Then it can be seen as a direct and very wide generalization of the results presented in [5]. We do not need any further restriction concerning the relations between the joint distribution \(F\) and the modular decomposition of \((C, \phi)\).

In the general case of non-exchangeable lifetimes \(T_1, \ldots, T_n\), where the probability signature can be different from the structural signature, we can use again Theorem 12 and express the probability signature of \((C, \phi)\) in terms of those of \((C_j, \chi_j)\) for \(j = 1, \ldots, r\). Such a result shows that exchangeability can be extended to the weaker condition of \(C\)-decomposability of the relative quality function \(q\). Thus the latter can be seen as an appropriate condition of compatibility between the partition \(C\) and the joint distribution \(F\). As we now state in the following theorem, this condition is not only sufficient but in a sense also necessary. The proof will be given at the end of the section.

**Theorem 15.** Consider a partition \(C = \{C_1, \ldots, C_r\}\) of \(C\) and a distribution \(F\) of the component lifetimes. Assume that there exists a function \(\gamma: \prod_{j=1}^r \{0, \ldots, n_j\} \to \mathbb{R}\) such that, for every semicoherent structure \(\phi: \{0, 1\}^n \to \{0, 1\}\) with a modular decomposition into \(r\) disjoint modules \((C_j, \chi_j)\), \(j = 1, \ldots, r\), connected according to a semicoherent structure \(\psi: \{0, 1\}^r \to \{0, 1\}\), we have
\[
(14) \quad \overline{P}_{n-k} = \sum_{a \in T_k} \gamma(a) \hat{\psi}(P_{1,n_1-a_1}, \ldots, P_{r,n_r-a_r}).
\]
Then the relative quality function \(q\) associated with \(F\) is \(C\)-decomposable.

This observation motivates the following definition, which extends the concept of modular decomposition to the general non-i.i.d. case.
Definition 16. We say that a semicoherent system \((C, \phi, F)\) is decomposable if, for some partition \(C = \{C_1, \ldots, C_r\}\) of \(C\),

(i) the structure \(\phi: \{0,1\}^n \rightarrow \{0,1\}\) has a modular decomposition into \(r\) disjoint
semicoherent modules \((C_j, \chi_j)\), \(j = 1, \ldots, r\), and

(ii) the function \(q\) is \(C\)-decomposable.

We note that Theorem 12 gives a very compact and quite explicit formula for the tail probability signature of a decomposable system.

We also note that a formula similar to (12) can be derived from the law of total probability. In fact, since the events \(E_{k,a}\) (\(a \in T_k\)), as defined right before Proposition 5, form a partition of the sample space, we have

\[
\overline{T}_{n-k} = Pr(E_k) - \sum_{a \in T_k} Pr(E_k | E_{k,a}) Pr(E_{k,a}),
\]

where

\[
E_k = (T > T_{n-k,n})
\]

(14) The system is still surviving after the first \((n-k)\) component failures.

On the one hand, we saw in (8) that \(Pr(E_{k,a})\) is a sum of \(q\) values and reduces to \(\tilde{c}(a)\) under the \(C\)-decomposability of \(q\) (Proposition 5). On the other hand, one can show that, under the conditional independence of the events \((T_{C_j} > T_{n-k,n})\), \(j = 1, \ldots, r\), given \(E_{k,a}\), where \(T_{C_j}\) is the lifetime of module \((C_j, \chi_j)\), we obtain

(15) \[Pr(E_k | E_{k,a}) = \tilde{\psi}(Pr(T_{C_1} > T_{n-k,n} | E_{k,a}), \ldots, Pr(T_{C_r} > T_{n-k,n} | E_{k,a})).\]

Finding general conditions under which the probability \(Pr(T_{C_j} > T_{n-k,n} | E_{k,a})\) reduces to \(\overline{T}_{j,n-j-a_j}\) remains an interesting open question.

Concerning the possible applications of Theorem 12, we now start with some further examples concerning the tail structural signature of the system; the latter is obtained by (12), where \(\tilde{c}(a)\) is given by (7). First of all we observe that, in the case of two modules connected in series or in parallel, one immediately obtains the formulas given in (5); see Example 13.

Example 17. Let \((C_1, \chi_1), \ldots, (C_r, \chi_r)\) be \(r\) modules consisting of serially connected components. The tail structural signature of \((C_j, \chi_j)\) is given by \(\overline{S}_{j,0} = 1\) and \(\overline{S}_{j,1} = 0\) for \(1 \leq l \leq n_j\) and \(1 \leq j \leq r\). We observe that the tuples \(\overline{S}_{j}\) are Boolean. Therefore, in Formula (12), we can use \(\psi\) in place of \(\tilde{\psi}\) and we obtain

\[
\overline{S}_{n-k} = \sum_{a \in T_k} \frac{(n_1)_{a_1} \cdots (n_r)_{a_r}}{(n_k)} \psi(1_{a_1 = n_1}, \ldots, 1_{a_r = n_r}), \quad 0 \leq k \leq n,
\]

where \(1_{\{p\}}\) denotes the truth value of proposition \(p\).

Recalling (11), this formula can also be written as

\[
\overline{S}_{n-k} = \sum_{a \in T_k} \frac{(n_1)_{a_1} \cdots (n_r)_{a_r}}{(n_k)} \sum_{B \subseteq [r]} \psi(B) \prod_{j \in B} 1_{a_j = n_j} \prod_{j \in [r] \setminus B} 1_{a_j = n_j}.
\]

As a special case, assume that the modules are connected in parallel. Then \(\psi\) is the maximum function and we obtain

\[
\overline{S}_{n-k} = \sum_{a \in \mathcal{V}_k} \frac{(n_1)_{a_1} \cdots (n_r)_{a_r}}{(n_k)},
\]
When the modules are connected in parallel, we obtain (as in Example 17)

\[ V_{\chi_S} = k \max \left( \frac{n}{a_1}, \ldots, \frac{n}{a_r}, \frac{n}{k} \right) \]

where this time \( \phi \) is the maximum function. We also observe that formulas of Examples 17 and 18 also hold for the tail probability signatures whenever the relative quality function \( q \) is decomposable for the considered partition, up to replacement of the multivariate hypergeometric coefficients \( \left( \frac{n}{a_1}, \ldots, \frac{n}{a_r}, \frac{n}{k} \right) \) by the coefficients \( c(a_1, \ldots, a_r) / \left( \frac{n}{k} \right) \) associated with the decomposition of \( q \).

Different papers, starting from [6], have shown that stochastic comparisons between lifetimes of two systems can be established in terms of their signatures. These results can be combined with ours under different forms. In the following example we consider the analysis of redundancy.

Example 19. We consider a coherent structure \( \chi : \{0, 1\}^n \to \{0, 1\} \) and two disjoint sets of components \( C' \) and \( C'' \), each containing \( n \) components. The components are assumed to have i.i.d. lifetimes.

Having at our disposal the two sets \( C' \) and \( C'' \), we can build redundant structures. A classical problem amounts to determining the optimal way to arrange redundancies. In particular one can compare redundancy at system level with redundancy at component level. More formally, starting from \( \chi \), let us consider the two structures \( \phi_1, \phi_2 : \{0, 1\}^{2n} \to \{0, 1\} \) defined as follows for \( x, y \in \{0, 1\}^n \):

\[ \phi_1(x, y) = \max(\chi(x), \chi(y)) \quad \text{and} \quad \phi_2(x, y) = \chi(\max(x_1, y_1), \ldots, \max(x_n, y_n)) \]

It is a well-known fact (see e.g. [1]) that the structure \( \phi_2 \) is more reliable than \( \phi_1 \). Our arguments can allow us to obtain some more detailed results along this direction. In fact, once the signatures of \( \phi_1 \) and \( \phi_2 \) have been computed, results presented in [6] can be applied to obtain different stochastic comparisons between the lifetimes associated with \( \phi_1 \) and \( \phi_2 \).

Let \( S = (S_0, \ldots, S_n) \) denote the cumulative signature associated with \( \chi \). We now apply Theorem 14 to compute the cumulative signatures associated with \( \phi_1 \) and \( \phi_2 \). For the first one, the structure inside each module is \( \chi \) and the modules are connected in parallel. So the organizing function \( \psi \) in Theorem 14 is the maximum function (in two variables), whose dual \( \psi^d \) is the minimum function. We thus have

\[ \psi^d(s, t) = st \]

and obtain the formula

\[ S_{2n-k}^{(1)} = \min(n, 2n-k) \left( \frac{n}{a} \right) \left( \frac{n}{2n-k-a} \right) S_a S_{2n-k-a} \quad 0 \leq k \leq 2n. \]
For the second one, the structure inside each module is parallel, so the associated cumulative structural signature is $S_j = (0, 0, 1)$ for $j = 1, \ldots, n$. Moreover the modules are connected according to $\chi$. Therefore we have

$$S^{(2)}_{2n-k} = \sum_{a \in T_k} \binom{2}{a_1} \cdots \binom{2}{a_n} \chi^d(1_{\{a_1 = 0\}}, \ldots, 1_{\{a_n = 0\}}), \quad 0 \leq k \leq 2n.$$ 

We now analyze some aspects related to the more general problem of probability signature for a decomposable system.

**Remark 3.** We know that the property $P = S$ holds if the component lifetimes are exchangeable. However the latter condition is not necessary. Let a decomposable system (w.r.t. a partition $C$) be such that condition (7) holds. Then $P = S$ is guaranteed by the condition $P_j = S_j$ for $j = 1, \ldots, r$.

**Example 20.** Consider a system containing only four components and assume that the joint probability distribution of lifetimes admits a joint density function of the form

$$(16) \quad f_w(t_1, t_2, t_3, t_4) = w(t_1, t_2) w(t_3, t_4),$$

where $w$ is an exchangeable bivariate probability density on $\mathbb{R}^2_+$. This class of distributions was also considered in \[11, 16] for purposes different from ours. We notice that such distributions are $C$-exchangeable, where $C = (\{1, 2\}, \{3, 4\})$.

We set $\phi_1(x_1, x_2, x_3, x_4) = \min(x_1, x_4, \max(x_2, x_3))$, i.e., $\phi_1$ is a structure composed of the modules $C_1 = \{1, 4\}$ and $C_2 = \{2, 3\}$. By assuming the joint density to be of the form (16), it can be easily shown that the probability signature $p$ coincides with the structural signature of the system, given by $s = (1/2, 1/2, 0, 0)$; see also [16, p. 44]. Notice that this happens irrespective of $q$ being or being not symmetric. We can thus build examples where $p = s$, with $q$ not symmetric. We recall, in this respect, that $q$ is symmetric if and only if the equality $p = s$ holds for any coherent structure (see [8]). We also notice that, under the present structure, we can directly compute the probability signature of a series of two modules, even if the vectors of the lifetimes of components from the different modules are not independent.

Several more examples might be presented concerning the use of Theorem 12. However we conclude the paper now just with a simple remark about applications of the concept of probability signature and some hints for further work. As already mentioned, the notion of structural signature can be applied to the comparison between different systems with i.i.d. (or exchangeable) components. Similar results can be developed for the comparison between systems with non-exchangeable components.

**Proof of Theorem 15.** Let us consider a subset $B$ of the set of components $C$ and try to decompose $q(B)$. The key observation is that the relative quality function is determined by the tail signature of appropriate systems. Those system are only semicoherent.

If $B$ is empty, then $q(B) = 1$ (by definition) and since $B_1, \ldots, B_r$ are also empty, $q^{C_j}(B_j) = 1$. Therefore we can set $c(0, \ldots, 0) = 1$. 

If $B$ is nonempty, then the set $m_B = \{ j \in [r] : B_j \neq \emptyset \}$ is nonempty. For every $j \in m_B$, let us form the module $(C_j, \chi_j)$ by defining
\[
\chi_j : 2^{C_j} \rightarrow \{0,1\} : D \mapsto \chi_j(D) = \begin{cases} 1 & \text{if } D \supseteq B_j \\ 0 & \text{otherwise.} \end{cases}
\]
In other words, the pseudo-Boolean function $\chi_j$ is given by
\[
\chi_j(y_1, \ldots, y_n) = \prod_{k \in B_j} y_k.
\]
It is semicoherent since $B_j$ is nonempty.

Now, let us connect these modules in series to obtain a structure $\phi$ that is also semicoherent (this means that we consider $\psi(z_1, \ldots, z_r) = \prod_{j \in m_B} z_j$). Actually, the function $\phi$ is nothing other than the well-known set function
\[
\phi_B : 2^C \rightarrow \{0,1\} : D \mapsto \phi_B(D) = \begin{cases} 1 & \text{if } D \supseteq B \\ 0 & \text{otherwise.} \end{cases}
\]
Now, we compute the tail probability signatures of such functions, using our formula :
\[
\bar{P}_{n-k} = \sum_{A \subseteq [n]} q(A) \phi(A).
\]
It follows that for $0 \leq k < |B|$, we have $\bar{P}_{n-k} = 0$. Moreover, setting $k = |B|$, we have
\[
\bar{P}_{n-k} = \bar{P}_{n-|B|} = \sum_{A \subseteq [n]} q(A) \phi_B(A) = \sum_{|A|=|B|} q(A) \phi_B(A) = q(B).
\]
We obtain the same results for the signatures of the modules : for every $j \in m_B$, we have $\bar{P}_{j,n_j-k} = 0$ for $0 \leq k < |B_j|$ and
\[
\bar{P}_{j,n_j-|B_j|} = q^{C_j}(B_j).
\]
Now, applying formula (13) to the system $([n], F, \phi)$ with $k = |B|$ leads to
\[
q(B) = \sum_{a \in \mathcal{T}[B]} \gamma(a) \bar{P}(\prod_{j=1}^{n} a_j) = \sum_{a \in \mathcal{T}[B]} \gamma(a) \prod_{j \in m_B} \bar{P}_{j,n_j-|B_j|}.
\]
We notice that the latter sum contains only one nonzero term. Indeed, by definition we have
\[
\mathcal{T}[B] = \{ a = (a_1, \ldots, a_r) \in \mathbb{N}^r : 0 \leq a_j \leq n_j \text{ for } j = 1, \ldots, r \text{ and } \sum_{j=1}^{r} a_j = |B| \}
\]
but in Equation (17), the product corresponding to an $a \in \mathcal{T}[B]$ is zero unless $a_j \geq |B_j| = b_j$ for all $j \in m_B$, and we obviously have $a_j \geq |B_j|$ for $j \notin m_B$. Taking the condition $\sum_{j=1}^{r} a_j = |B|$ into account, the only term that does not vanish corresponds to $a_j = b_j$ for all $j \in m_B$, and $a_j = 0 = b_j$ for every $j \notin m_B$ which yields
\[
q(B) = \gamma(b_1, \ldots, b_r) \prod_{j \in m_B} q^{C_j}(B_j) = \gamma(b_1, \ldots, b_r) \prod_{j=1}^{r} q^{C_j}(B_j)
\]
and completes the proof.
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