SYNCHRONIZATION ANALYSIS OF DRIVE-RESPONSE MULTI-LAYER DYNAMICAL NETWORKS WITH ADDITIVE COUPLINGS AND STOCHASTIC PERTURBATIONS

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Abstract. This paper concerns the synchronization of a kind of drive-response multi-layer dynamical networks with additive couplings and stochastic perturbations. Multi-layer networks are a kind of complex networks with different layers, which consist of different kinds of interactions or multiple subnetworks. Additive couplings are designed to capture the different layered connections. In this paper, two pinning controllers are designed to guarantee the synchronization of the stochastic multi-layer network. One is the state-feedback pinning controller with constant control gains. The other one is the adaptive pinning controller with adaptive control gains. It is worthwhile to mention that our assumptions on the activation functions satisfy a generalized Lipschitzian condition which are weaker than those in the previous works. Moreover, as we prove, only selected part of the nodes to be controlled are enough to guarantee that the drive system and response network can be stochastically synchronized. Finally, an example and its simulations are presented to show the feasibility effectiveness of our control schemes.

1. Introduction. It is known that complex networks can exhibit chaotic behaviors if the complex networks are subjected to an unpredictable perturbation. To control the chaos arising in the neural networks, studies on the synchronization of chaos neural networks have been performed in a lot of literature. Chaos synchronization was tracked back to the pioneering work of Pecora and Carroll [33]. From then on, various different types of synchronization and control schemes have been proposed.
in the last decades. For examples, adaptive synchronization (see e.g. [14, 25, 53]), synchronization based on impulsive control (e.g. Guan et al. [8], Li et al. [16], Li et al. [17], Li et al. [18], Li et al. [19], Lu et al. [25], Lu et al. [26], Lu et al. [27], Stamov and Stamova [36], Stamova [37], Yang et al. [49], Yu and Yu [50], Zhang et al. [54]), global synchronization (see e.g. [6]), inphase and antiphase synchronization (see e.g. [35]), finite-time synchronization (e.g. Liu et al. [21], Liu et al. [22], Liu et al. [23], and Liu et al. [24]), synchronization based on periodically intermittent adaptive control (Guo et al. [9], Wang et al. [42]), pinning synchronization (Li et al. [20]), pinning synchronization based on adaptive control (Wang et al. [43]), lag synchronization (e.g. [47]), event-based synchronization ([38]), consensus with antagonistic interactions ([28]) and so on.

However, most of the synchronization schemes and control methods were designed for the traditional networks. In the real environment, the complex network is more complicated. For example, when we consider a social network between two high schools, it is easy to know the relationships and interactions inside each school. But there are relationships and interactions between two schools such as being friends, family members. Figure 1 is presented to show the relationships and interactions in a two-layer network (e.g. two schools).

![Figure 1. an example of multi-layer network with 2 layers.](image)

To mimic more realistic relationships and interactions in real world, it is great significant to propose the multi-layer networks. Multi-layer networks take different kinds of interactions or multiple subnetworks into considerations. A specific kind of interaction corresponds to a specific layer of the multilayer networks, which forms multiple network topology. Dynamic multilayer networks were considered in [39, 10]. Synchronization of multi-agent systems with additive coupling was proposed in [10]. To study the multiple network topologies, He et al. [10] proposed and studied the
following system with additive couplings.

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + f(t, x_i(t)) \\
&+ \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} D_k(x_j(t - \tau) - x_i(t - \tau)).
\end{align*}
\]

However, realistic environments are usually subjected to the unpredicted external perturbations, which may cause a great uncertainties. This disturbance can arise from various types of environmental noise, mathematically interpreted by the so-called white noise. White noise can be regarded as the derivative of a stochastic nature of Brownian motion. These uncertain disturbances and Brownian motion can be modelled by the stochastic differential equations (SDE). Stochastic modelling plays a great role in life science and industry. When we consider the effect of the unpredictable disturbance on a complex network, it is necessary to take the stochastic effects into consideration. Studies on the stochastic complex systems (including biological model or neural networks) were performed in the literature (see e.g. [34, 51, 9, 50, 32, 42, 43, 46, 38, 23, 13, 44, 48]).

In [56], the authors investigated a leader-following synchronization problem for multi-layer network with additive couplings. Up to now, we do not find any result considering the outer synchronization problem for multi-layer network with stochastic perturbations. Therefore, in this paper, we will study a stochastic multi-layer dynamic network with time-varying delays and additive couplings. Next section is given to model description. In section 3, two pinning controllers are designed to guarantee the synchronization of the multi-layer network. One is the state-feedback controller with constant control gains. The other is the state-feedback controller with adaptive control gains. We also show that only part of the nodes to be controlled is enough. Finally, an example and its simulations are given to show the effectiveness of our control schemes.

2. Model description, assumptions and preliminary lemmas. Notations. \(\mathbb{R}^+, \mathbb{R}^n\) and \(\mathbb{R}^{n \times m}\) denote the set of all nonnegative real numbers, the \(n\)-dimensional Euclidean space and the set of all \(n \times m\) real matrices, respectively. \(I_n\) denotes the identity matrix with order \(n\). The superscript \(T\) represents the transpose operation to a corresponding matrix. For two symmetric matrices \(P\) and \(Q\), \(P - Q > 0\) and \(P > Q\) mean that the matrix \(P - Q\) is positive definite; \(P - Q \geq 0\) and \(P \geq Q\) mean that the matrix \(P - Q\) is nonnegative definite.

In this paper, we propose the synchronization schemes for a kind of drive-response multi-layer dynamical networks with additive couplings and stochastic perturbations. Two pinning controllers are designed. One is the state-feedback controller with constant control gains. The other is the state-feedback controller with adaptive control gains. We also show that only part of the nodes to be controlled is enough. The drive system is proposed as the following multi-layer dynamic network with additive couplings [52, 10, 43], where each node is driven by the following equation,

\[
\begin{align*}
\dot{x}_i(t) &= \left(-Dx_i(t) + B\hat{f}_i(x_i(t)) + F\hat{g}_i(x_i(t - \tau_1(t))) + J_i \\
&+ \sum_{k=1}^{M} h_k \sum_{j=1}^{N} a_{ij}^{(k)} S_k(x_j(t - \tau_2(t)) - x_i(t - \tau_2(t)))\right)dt,
\end{align*}
\]

\(i = 1, 2, \cdots, N\), where \(N\) is the number of coupled nodes, \(x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n\) represents the state vector of the \(i\)-th node at time \(t\), \(B, D, F \in \mathbb{R}^{n \times n}\) are constant matrices, \(\hat{f}_i(\cdot) = (\hat{f}_{i1}(\cdot), \hat{f}_{i2}(\cdot), \cdots, \hat{f}_{in}(\cdot))^T, \hat{g}_i(\cdot) = (\hat{g}_{i1}(\cdot), \hat{g}_{i2}(\cdot), \cdots, \hat{g}_{in}(\cdot))^T \in \mathbb{R}^n\) are continuous vector functions, \(J_i\) is a vector of constant external
input for the \(i\)th node, \(h_k \geq 0\) is the strength of coupling contributed by the \(k\)-th layer, \(A^{(k)} = (a^{(k)}_{ij}) \in \mathbb{R}^{N \times N}\) is the outer coupling matrix (symmetric adjacency matrix), \(S_k \in \mathbb{R}^{n \times n}\) is the \(k\)th layer inner coupling matrix (which may not be symmetric) and \(\tau_1(t), \tau_2(t)\) are the transmission delays.

Moreover, the response system is introduced as the following system with stochastic perturbations

\[
dy_i(t) = \left( -Dy_i(t) + B\hat{f}_i(y_i(t)) + F\hat{g}_i(y_i(t - \tau_1(t))) + J_i + c_i(t) \
+ \sum_{k=1}^{M} h_k \sum_{j=1}^{N} a^{(k)}_{ij} S_k(y_j(t - \tau_2(t)) - y_i(t - \tau_2(t))) dt \right.
\]

\[
+ \sigma_i(t, y_i(t) - x_i(t)), y_i(t - \tau_1(t)) - x_i(t - \tau_1(t)), \]

\[
y_i(t - \tau_2(t)) - x_i(t - \tau_2(t)), d\mathcal{W}(t),
\]

\[i = 1, 2, \ldots, N, \] where \(y_i = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T\) is the state vector of \(i\)th node at time \(t\), \(c_i(t)\) is the control input to be designed for the \(i\)th node, \(W(t)\) is a standard \(m\)-dimensional Wiener process and is defined on a probability space \((\Omega, \mathcal{F}, P)\) with the natural filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) generated by \(\{W(s), 0 \leq s \leq t\}\), \(\sigma_i(\cdot) \in \mathbb{R}^{n \times m}\) is the noise intensity matrix.

Letting \(e_i(t) = y_i(t) - x_i(t)\), i.e., \((e_{i1}(t), e_{i2}(t), \ldots, e_{in}(t))^T = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T - (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T\) be the synchronization error between the \(i\)th node of the drive system and that of the response system, we obtain the following dynamical system describing the synchronization error,

\[
d\epsilon_i(t) = \left( -De_i(t) + Bf_i(e_i(t)) + Fg_i(e_i(t - \tau_1(t))) + c_i(t) \
+ \sum_{k=1}^{M} h_k \sum_{j=1}^{N} a^{(k)}_{ij} S_k(e_j(t - \tau_2(t)) - e_i(t - \tau_2(t))) dt \right.
\]

\[
+ \sigma_i(t, e_i(t) - e_i(t - \tau_1(t))), e_i(t - \tau_2(t)))d\mathcal{W}(t),
\]

\[i = 1, 2, \ldots, N, \] where \(f_i(e_i(t)) = \hat{f}_i(y_i(t)) - \hat{f}_i(x_i(t)) = \hat{f}_i(x_i(t) + e_i(t)) - \hat{f}_i(x_i(t))\), \(g_i(e_i(t)) = \hat{g}_i(y_i(t)) - \hat{g}_i(x_i(t)) = \hat{g}_i(x_i(t) + e_i(t)) - \hat{g}_i(x_i(t)).\) Moreover,

\[
\frac{1}{N} \sum_{j=1}^{N} a^{(k)}_{ij} (e_j(t - \tau_2(t)) - e_i(t - \tau_2(t)))
\]

\[
= \sum_{j=1}^{N} a^{(k)}_{ij} e_j(t - \tau_2(t)) - \sum_{j=1}^{N} a^{(k)}_{ij} e_i(t - \tau_2(t)) = -\sum_{j=1}^{N} a^{(k)}_{ij} e_j(t - \tau_2(t)).
\]

where

\[
i^{(k)}_{ij} = -a^{(k)}_{ij}, \quad \text{for} \ i \neq j, \ \text{and} \ i^{(k)}_{ii} = \sum_{j=1,j \neq i}^{N} a^{(k)}_{ij}.
\]

For each \(k\), \(L^{(k)} = (l^{(k)}_{ij})_{N \times N}\) is the so-called Laplacian matrix [31] which satisfies the diffusion property \((\sum_{j=1}^{N} l^{(k)}_{ij} = 0)\), is nonnegative definite and has a maximum eigenvalue \(\lambda^{(k)} > 0\).
Therefore, this dynamical system can be written as the following compact form,

\[
de(t) = \left( - (I_N \otimes D) e(t) + (I_N \otimes B) f(e(t)) + (I_N \otimes F) g(e(t - \tau_1(t))) \right) + C(t) - \sum_{k=1}^{M} h_k(L^{(k)} \otimes S_k)e(t - \tau_2(t)) \right) dt \\
+ \sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) dW(t),
\]

where \( e(t) = (e^T_1(t), e^T_2(t), \cdots, e^T_N(t)) \), \( f(e(t)) = (f^T_1(e_1(t)), \cdots, f^T_N(e_N(t))) \), \( g(e(t)) = (g^T_1(e_1(t)), g^T_2(e_2(t)), \cdots, g^T_N(e_N(t))) \), \( C(t) = (c^T_1(t), c^T_2(t), \cdots, c^T_N(t)) \), and \( \sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) = (\sigma^T_1(t, e_1(t), e_1(t - \tau_1(t)), e_1(t - \tau_2(t))), \cdots, \sigma^T_N(t, e_N(t), e_N(t - \tau_1(t)), e_N(t - \tau_2(t))) \).

Moreover, for nonnegative function \( V(t, e) \), which is defined on \( \mathbb{R}^+ \times \mathbb{R}^{Nn} \), differentiable in \( t \) and continuously twice differentiable in \( e \), we define the following notations.

\[
V_i(t, e) = \frac{\partial V(t, e)}{\partial e_{ii}}, \quad V_{ce}(t, e) = \left( \frac{\partial V(t, e)}{\partial e_{i1}}, \cdots, \frac{\partial V(t, e)}{\partial e_{in}} \right), \quad V_{cc} = \left( \frac{\partial^2 V(t, e)}{\partial e_{ik}\partial e_{jl}} \right).
\]

Then we also define \([29, 30]\) an operator \( \mathcal{L} \) along the trajectory of the system (5) as follows.

\[
\mathcal{L} V(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) \\
= V_i(t, e(t)) + V_{ce}(t, e(t)) \left[ - (I_N \otimes D) e(t) + (I_N \otimes B) f(e(t)) \right.

\left. + (I_N \otimes F) g(e(t - \tau_1(t))) + C(t) - \sum_{k=1}^{M} h_k(L^{(k)} \otimes S_k)e(t - \tau_2(t)) \right] + \frac{1}{2} \text{trace} \left[ \sigma^T(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) \right]

\left. \sigma_{ce}(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) \right].
\]

In this paper, we make the following assumptions.

**Assumption (A1)** \([29, 30]\) *The functions \( \tilde{f}_i, \tilde{g}_i, \sigma_i, \ i = 1, 2, \cdots, N \) are Borel-measurable and satisfy the usual local Lipschitz condition and linear growth condition.*

Under this assumption, the dynamical system (5) has a unique solution \([29, 30]\) on \( t \geq 0 \) for any initial data \( \{e(t) : -\tau \leq t \leq 0\} = \eta \in C^2_2([-\tau, 0]; \mathbb{R}^{Nn}) \), and the solution is denoted by \( e(t; \eta) \) or \( e(t) \) for simplicity.

**Remark 1.** Actually, if the dynamic error system (5) has a unique solution denoted by \( e(t; \eta) \) on \( t \geq 0 \), then the synchronization theorems (Theorem 2.1 and Theorem 6.1 of [29]) are still true according to the results proved by [29]. We also employ these theorems to prove our results in this paper. Therefore, in our results, it is enough to assume that (1) \( \tilde{f}_i(x), \tilde{g}_i(x), i = 1, 2, \cdots, N \) are locally bounded in \( x \), and (2) \( \sigma_i(t, x_1, x_2, x_3), i = 1, 2, \cdots, N \) are locally bounded in \( (x_1, x_2, x_3) \) and uniformly bounded in \( t \).

**Assumption (A2)** *There exist diagonal matrices \( U_1 \) and \( U_2 \) such that*

\[
(\tilde{f}_i(y) - \tilde{f}_i(x))^T (\tilde{f}_i(y) - \tilde{f}_i(x)) \leq (y - x)^T U_1 (y - x),

(\tilde{g}_i(y) - \tilde{g}_i(x))^T (\tilde{g}_i(y) - \tilde{g}_i(x)) \leq (y - x)^T U_2 (y - x),
\]

*for all \( x, y \in \mathbb{R}^n, i = 1, 2, \cdots, N \).*
Remark 2. This assumption is a generalized Lipschitz condition. In the existing literature, these activation functions are assumed to be bounded, continuous or monotonically increasing [3, 4, 15, 55, 43]. Obviously, our assumption is weaker than those of [3, 4, 15, 55, 43].

Assumption (A3) [5, 55, 42] There exist positive constants $\tau, \rho_1$ and $\rho_2$ satisfying
\[ 0 \leq \tau_1(t) \leq \tau, \quad 0 \leq \tau_2(t) \leq \tau, \quad 0 \leq \dot{\tau}_1(t) \leq \rho_1 < 1 \quad \text{and} \quad 0 \leq \dot{\tau}_2(t) \leq \rho_2 < 1. \]

Assumption (A4) [15, 55, 42, 40] There are positive semi-definite matrices $P_1, P_2$ and $P_3$ such that
\[ \text{trace}[\sigma_i^T(t, z_1, z_2, z_3)\sigma_i(t, z_1, z_2, z_3)] \leq z_1^T P_1 z_1 + z_2^T P_2 z_2 + z_3^T P_3 z_3 \]
for all $z_1, z_2, z_3 \in \mathbb{R}^n, t \in \mathbb{R}^+$, and $i = 1, 2, \cdots, N$.

Definition 2.1. The two multi-layer complex networks (2) and (3) are (stochastically) synchronized almost surely, if
\[ \lim_{t \to \infty} e(t; \eta) = 0 \quad \text{a.s.} \]
for every $\eta \in C_{b,F}^F([-\tau, 0]; \mathbb{R}^{Nn})$.

We also provide some lemmas which may be used in the sequel.

Lemma 2.2. [2, 12] For any two vectors $x, y$ and a square matrix $P$ with compatible dimensions,
\[ 2x^T P y \leq x^T P P^T x + y^T y. \]

Lemma 2.3. [2, 12] Suppose $Q_1, Q_2, Q_3$ are matrices with appropriate dimensions such that the following operations are compatible. Then
\[ \begin{pmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{pmatrix} > 0 \]
if and only if $Q_2 > 0$ and $Q_1 - Q_3 Q_2^{-1} Q_3^T > 0$.

3. Main results and proofs. In general, the two multi-layer complex networks (2) and (3) may not achieve synchronization automatically, or may be hard to achieve the synchronization. Therefore we design two controllers in this section, namely, 1) state-feedback pinning controller; 2) adaptive pinning controller. We also assume that only part of the nodes could be controlled directly. And then we study the necessary condition such that the drive system (2) and response network (3) be stochastically synchronized under these two controllers.

3.1. Synchronization with state-feedback pinning controller. Without loss of generality, suppose that there are $l$ ($1 \leq l \leq N$) nodes could be controlled directly and the indices of these nodes be $i = 1, 2, \cdots, l$, respectively. In this situation, we set the controller [15, 55, 43] as,
\[ c_i(t) = -k_i e_i(t), \quad (7) \]
i.e.
\[ C(t) = -(K_1 \otimes I_n) e(t), \quad (8) \]
where \( k_i, i = 1, 2, \cdots, l \) are the designed control gains and \( K_1 = \text{diag}\{k_1, k_2, \cdots, k_l, 0, \cdots, 0\} \). Therefore, the error system (5) becomes
\[
d e(t) = \left( - (I_N \otimes D)e(t) + (I_N \otimes B)f(e(t)) + (I_N \otimes F)g(e(t) - \tau_1(t)) \right)
- (K_1 \otimes I_n)e(t) - \sum_{k=1}^{M} h_k (L^{(k)} \otimes S_k) e(t - \tau_2(t)) \right) dt
+ \sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) dW(t).
\]

(B2)
\[
\sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S^T_k S_k - (1 - \rho_1)Q_3 + \mu P_3 > 0.
\]

\[
\begin{pmatrix}
\Theta_{1,11} & I_N \otimes Q_1 B & I_N \otimes Q_1 F & I_N \otimes Q_1 \\
I_N \otimes B^T Q_1 & -I_N \otimes I_n & 0 & 0 \\
I_N \otimes F^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes Q_1 & 0 & 0 & -\frac{1}{\sum_{k=1}^{M} h_k} I_N \otimes I_n
\end{pmatrix}
< 0,
\]

where \( \Theta_{1,11} = I_N \otimes (-Q_1 D - D^T Q_1 + \rho_1 Q_2 + \rho_2 Q_3 + U_1 + U_2 + \mu(P_1 + P_2 + P_3) + \sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S^T_k S_k - 2K_1 \otimes Q_1 \).

(B2)
\[
\sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S^T_k S_k - (1 - \rho_2)Q_3 + \mu P_3 > 0.
\]

\[
\begin{pmatrix}
\Theta_{2,11} & I_N \otimes Q_1 B & I_N \otimes Q_1 F & I_N \otimes Q_1 \\
I_N \otimes B^T Q_1 & -I_N \otimes I_n & 0 & 0 \\
I_N \otimes F^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes Q_1 & 0 & 0 & -\frac{1}{\sum_{k=1}^{M} h_k} I_N \otimes I_n
\end{pmatrix}
< 0,
\]

where \( \Theta_{2,11} = I_N \otimes \left( Q_2 - Q_1 D - D^T Q_1 + \rho_2 Q_3 + U_1 + U_2 + \mu(P_1 + P_3) + \sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S^T_k S_k \right) - 2K_1 \otimes Q_1 \).

Proof: We choose a Lyapunov functional as follows,
\[
V(t,e(t)) = e^T(t)(I_N \otimes Q_1)e(t) + \int_{t-\tau_1(t)}^{t} e^T(s)(I_N \otimes Q_2)e(s)ds
+ \int_{t-\tau_2(t)}^{t} e^T(s)(I_N \otimes Q_3)e(s)ds.
\]

According to (6), one obtains that
\[
\mathcal{L}V(t,e(t),e(t-\tau_1(t)),e(t-\tau_2(t)))
\]
Similarly, we also have

\[
(A2)
\]

Note that, using (A2) and Lemma 2.2,

\[
2e^T(t)(I_N \otimes Q_1)f(e(t))
\leq e^T(t)(I_N \otimes Q_1)B(I_N \otimes B^TQ_1)e(t) + f^T(e(t))f(e(t))
= e^T(t)(I_N \otimes Q_1BB^TQ_1)e(t) + \sum_{i=1}^{N} f_i^T(e(t))f_i(e(t))
\leq e^T(t)(I_N \otimes Q_1BB^TQ_1)e(t) + \sum_{i=1}^{N} e_i(t)U_1e_i(t)
= e^T(t)(I_N \otimes Q_1BB^TQ_1)e(t) + e(t)(I_N \otimes U_1)e(t).
\]

Similarly, we also have

\[
2e^T(t)(I_N \otimes Q_1F)f(e(t))
\leq e^T(t)(I_N \otimes Q_1FF^TQ_1)e(t) + e(t - \tau_1(t))(I_N \otimes U_2)e(t - \tau_1(t)).
\]

Meanwhile, according to Lemma 2.2, we reach

\[
-2e^T(t)(I_N \otimes Q_1)(L^{(k)} \otimes S_k)e(t - \tau_2(t))
= e^T(t)(I_N \otimes Q_1)2e^T(t) + e^T(t - \tau_2(t))(L^{(k)})^2 \otimes S_k^T S_k)e(t - \tau_2(t))
\leq e^T(t)(I_N \otimes Q_1)2e^T(t) + (\lambda^{(k)})^2 e^T(t - \tau_2(t))(I_N \otimes S_k^T S_k)e(t - \tau_2(t)).
\]

Moreover, applying (A4),

\[
\text{trace}\left( \sigma^T(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))(I_N \otimes Q_1) \right)
= \sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))).
\]
Combining all above obtained inequalities, it follows that
\[
\begin{align*}
&= \text{trace} \left( \sum_{i=1}^{N} \sigma_i^T(t, e_i(t), e_i(t - \tau_1(t)), e_i(t - \tau_2(t)))Q_1 \right) \\
&\leq \lambda_{\text{max}}(Q_1) \sum_{i=1}^{N} \text{trace} \left( \sigma_i^T(t, e_i(t), e_i(t - \tau_1(t)), e_i(t - \tau_2(t))) \right) \\
&\leq \mu \left( e^T(t)(I_N \otimes P_1)e(t) + e^T(t - \tau_1(t))(I_N \otimes P_2)e(t - \tau_1(t)) \\
&\quad + e^T(t - \tau_2(t))(I_N \otimes P_3)e(t - \tau_2(t)) \right).
\end{align*}
\]
Combining all above obtained inequalities, it follows that
\[
\begin{align*}
&\mathcal{L}V(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) \\
&\leq -e^T(t) \left[ I_N \otimes \left( Q_1 D + D^T Q_1 - Q_2 - Q_3 - Q_1 B^T B Q_1 - Q_1 F^T F Q_1 - U_1 \right) \\
&\quad - \mu P_1 - \sum_{k=1}^{M} h_k Q_k^2 + 2K_1 \otimes Q_1 \right] e(t) \\
&\quad + e^T(t - \tau_1(t)) \left[ I_N \otimes \left( U_2 - (1 - \rho_1) Q_2 + \mu P_2 \right) \right] e(t - \tau_1(t)) \\
&\quad + e^T(t - \tau_2(t)) \left[ I_N \otimes \left( \sum_{k=1}^{M} h_k \lambda^{(k)2} S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 \right) \right] e(t - \tau_2(t)) \\
&= -e^T(t) \Pi_1 e(t) + e^T(t - \tau_1(t)) \Pi_2 e(t - \tau_1(t)) + e^T(t - \tau_2(t)) \Pi_3 e(t - \tau_2(t)),
\end{align*}
\]
where
\[
\begin{align*}
\Pi_1 &= I_N \otimes \left( Q_1 D + D^T Q_1 - Q_2 - Q_3 - Q_1 B^T B Q_1 - Q_1 F^T F Q_1 - U_1 - \mu P_1 \\
&\quad - \sum_{k=1}^{M} h_k Q_k^2 + 2K_1 \otimes Q_1, \\
\Pi_2 &= I_N \otimes \left( U_2 - (1 - \rho_1) Q_2 + \mu P_2 \right), \\
\Pi_3 &= I_N \otimes \left( \sum_{k=1}^{M} h_k \lambda^{(k)2} S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 \right).
\end{align*}
\]
**Case 1.** $U_2 - (1 - \rho_1) Q_2 + \mu P_2 > 0$. Therefore, using Schur complement lemma (Lemma 2.3),
\[
\begin{align*}
\Pi_2 + \Pi_3 - \Pi_1 &= I_N \otimes \left( - Q_1 D - D^T Q_1 + \rho_1 Q_2 + \rho_2 Q_3 + U_1 + U_2 + \mu (P_1 + P_2 + P_3) \\
&\quad + \sum_{k=1}^{M} h_k Q_k^2 + \sum_{k=1}^{M} h_k \lambda^{(k)2} S_k^T S_k + Q_1 B^T B Q_1 + Q_1 F^T F Q_1 \right) - 2K_1 \otimes Q_1.
\end{align*}
\]
is negative definite if and only if

$$\Theta_1 = \begin{pmatrix}
\Theta_{1,11} & I_N \otimes Q_1 B & I_N \otimes Q_1 F & I_N \otimes Q_1 \\
I_N \otimes B^T Q_1 & -I_N \otimes I_n & 0 & 0 \\
I_N \otimes F^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes Q_1 & 0 & 0 & -\sum_{k=1}^M h_k I_N \otimes I_n
\end{pmatrix} < 0$$

where $\Theta_{1,11} = I_N \otimes (-Q_1 D - D^T Q_1 + \rho_1 Q_2 + \rho_2 Q_3 + U_1 + U_2 + \mu(P_1 + P_2 + P_3) + \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k) - 2K_1 \otimes Q_1$. According to Theorem 6.1 of [29], the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely if the inequalities (10)-(13) are true.

**Case 2.** $U_2 - (1 - \rho_1)Q_2 + \mu P_2 < 0$. In this case,

$$\mathcal{L}V(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t))) \leq -e^T(t)\Pi_1 e(t) + e^T(t - \tau_2(t))\Pi_2 e(t - \tau_2(t))$$

Using Schur complement lemma (Lemma 2.3),

$$\Pi_1 - \Pi_1 = I_N \otimes \left(-Q_1 D - D^T Q_1 + Q_2 + \rho_2 Q_3 + U_1 + \mu(P_1 + P_3) + \sum_{k=1}^M h_k Q_k^2 + \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k + Q_1 B B^T Q_1 + Q_1 F F^T Q_1\right) - 2K_1 \otimes Q_1$$

is negative definite if and only if

$$\Theta_2 = \begin{pmatrix}
\Theta_{2,11} & I_N \otimes Q_1 B & I_N \otimes Q_1 F & I_N \otimes Q_1 \\
I_N \otimes B^T Q_1 & -I_N \otimes I_n & 0 & 0 \\
I_N \otimes F^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes Q_1 & 0 & 0 & -\sum_{k=1}^M h_k I_N \otimes I_n
\end{pmatrix} < 0,$$

where $\Theta_{2,11} = I_N \otimes (Q_2 - Q_1 D - D^T Q_1 + \rho_2 Q_3 + U_1 + \mu(P_1 + P_3) + \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k)$

$- 2K_1 \otimes Q_1$. According to Theorem 6.1 of [29], the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely if the inequalities (14)-(17) are true. Now the proof is complete.

**Remark 3.** According to (18) of the proof, this theorem is conservative. Because of this conservative inequality, if all nodes are allowed to be controlled directly, i.e., $l = N$, and if we set $k_1 = k_2 = \cdots = k_N = k^*$, then Theorem 3.1 reduce to the following corollary which can be applied much easier.

**Corollary 1.** Under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (8) with $l = N$ and $k_1 = k_2 = \cdots = k_N = k^*$, if there exist positive definite matrices $Q_1, Q_2, Q_3$ and a positive number $\mu$, such that one of the following two sets of matrix inequalities is satisfied, (B1):

$$U_2 - (1 - \rho_1)Q_2 + \mu P_2 > 0,$$

$$Q_1 \leq \mu I_n,$$

$$\sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2)Q_3 + \mu P_3 > 0,$$
where \( \Xi_{1,11} = -Q_1 D - D^T Q_1 + \rho_1 Q_2 + \rho_2 Q_3 + U_1 + U_2 + \mu (P_1 + P_2 + P_3) + \sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S_k^T S_k - 2k^* Q_1 \).

\[
\Xi_1 = \begin{pmatrix}
\Xi_{1,11} & Q_1 B & Q_1 F & Q_1 \\
B^T Q_1 & -I_n & 0 & 0 \\
F^T Q_1 & 0 & -I_n & 0 \\
Q_1 & 0 & 0 & -\frac{1}{\sum_{k=1}^{M} h_k} I_n
\end{pmatrix} < 0,
\]

under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (8), if there exist positive definite matrices \( Q_1, Q_2, Q_3 \) and a positive number \( \mu \), such that one of the following two sets of matrix inequalities is satisfied, (C1):

\[
U_2 - (1 - \rho_1) Q_2 + \mu P_2 < 0,
\]

\[
Q_1 \leq \mu I_n,
\]

\[
\sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 > 0,
\]

where \( \Xi_{2,11} = -Q_1 D - D^T Q_1 + Q_2 + \rho_2 Q_3 + U_1 + \mu (P_1 + P_3) + \sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S_k^T S_k - 2k^* Q_1 \).

**Remark 4.** The preceding theorem and corollary are quite conservative due to the conservative inequality (18) in the proof. We also provide the following theorem where we do not use such conservative inequality. In this situation, the Kronecker product in the corollary is unavoidable.

**Theorem 3.2.** Under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (8), if there exist positive definite matrices \( Q_1, Q_2, Q_3 \) and a positive number \( \mu \), such that one of the following two sets of matrix inequalities is satisfied, (C1):

\[
U_2 - (1 - \rho_1) Q_2 + \mu P_2 < 0,
\]

\[
Q_1 \leq \mu I_n,
\]

\[
\sum_{k=1}^{M} h_k (\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 > 0,
\]

where

\[
\Xi_2 = \begin{pmatrix}
\Xi_{2,11} & Q_1 B & Q_1 F & Q_1 \\
B^T Q_1 & -I_n & 0 & 0 \\
F^T Q_1 & 0 & -I_n & 0 \\
Q_1 & 0 & 0 & -\frac{1}{\sum_{k=1}^{M} h_k} I_n
\end{pmatrix} < 0,
\]

and

\[
\Theta_3 = \begin{pmatrix}
\Theta_{3,11} & -\sum_{k=1}^{M} h_k (L^{(k)} \otimes Q_1 S_k) & I_N \otimes Q_1 B & I_N \otimes Q_1 F \\
\Theta_{3,21} & -I_N \otimes ((1 - \rho_2) Q_3 - \mu P_3) & 0 & 0 \\
I_N \otimes B^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes F^T Q_1 & 0 & 0 & -I_N \otimes I_n
\end{pmatrix}
\]

and \( \Theta_{3,11} = I_N \otimes (-Q_1 D - D^T Q_1 + Q_2 + Q_3 + \mu P_1 + U_1) - 2K_1 \otimes Q_1, \Theta_{3,21} = -\sum_{k=1}^{M} h_k (L^{(k)} \otimes S_k^T S_k) Q_1 \).

\[
U_2 - (1 - \rho_1) Q_2 + \mu P_2 > 0,
\]

\[
Q_1 \leq \mu I_n,
\]

\[
\Theta_4 < 0,
\]
where
\[
\Theta_4 = \begin{pmatrix}
\Theta_{4,11} & -\sum_{k=1}^{M} h_k(L^{(k)} \otimes Q_1) & I_N \otimes Q_1 B & I_N \otimes Q_1 F \\
I_N \otimes B^T Q_1 & -I_N \otimes ((1 - \rho_2)Q_3 - \mu P_3) & 0 & 0 \\
I_N \otimes F^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
0 & 0 & 0 & -I_N \otimes I_n
\end{pmatrix},
\]
and \( \Theta_{4,11} = I_N \otimes (-Q_1D - D^TQ_1 + \rho_1Q_2 + Q_3 + \mu(P_1 + P_2) + U_1 + U_2) - 2K_1 \otimes Q_1 \),
\( \Theta_{4,21} = -\sum_{k=1}^{M} h_k((L^{(k)})^T \otimes S_k^T Q_1) \).

Proof. We choose a Lyapunov functional as that of the preceding theorem,
\[
V(t, e(t)) = e^T(t)(I_N \otimes Q_1)e(t) + \int_{t-\tau_1(t)}^{t} e^T(s)(I_N \otimes Q_2)e(s)ds + \int_{t-\tau_2(t)}^{t} e^T(s)(I_N \otimes Q_3)e(s)ds.
\]
Using (6) again,
\[
\mathcal{L}V(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t)))
\]
\[
= e^T(t) \left[ I_N \otimes (-Q_1D - D^TQ_1 + Q_2 + Q_3 - 2K_1 \otimes Q_1) \right] e(t)
+ 2e^T(t)(I_N \otimes Q_1B)f(e(t)) + 2e^T(t)(I_N \otimes Q_1F)g(e(t - \tau_1(t))
- 2e^T(t)(I_N \otimes Q_1) \sum_{k=1}^{M} h_k(L^{(k)} \otimes S_k)e(t - \tau_2(t))
- (1 - \hat{\tau}_1)e^T(t - \tau_1(t))(I_N \otimes Q_2)e(t - \tau_1(t))
- (1 - \hat{\tau}_2)e^T(t - \tau_2(t))(I_N \otimes Q_3)e(t - \tau_2(t))
+ \text{trace} \left( \sigma^T(t, e(t)), e(t - \tau_1(t)), e(t - \tau_2(t)) \right)(I_N \otimes Q_1)
\]
\[
\sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t)))
\]
And we recall that,
\[
\text{trace} \left( \sigma^T(t, e(t)), e(t - \tau_1(t)), e(t - \tau_2(t)) \right)(I_N \otimes Q_1)
\]
\[
\leq \mu \left( e^T(t)(I_N \otimes P_1)e(t) + e^T(t - \tau_1(t))(I_N \otimes P_2)e(t - \tau_1(t))
+ e^T(t - \tau_2(t))(I_N \otimes P_3)e(t - \tau_2(t)) \right).
\]
Therefore, with (A2),
\[
\mathcal{L}V(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t)))
\]
\[
\leq e^T(t) \left[ I_N \otimes (-Q_1D - D^TQ_1 + Q_2 + Q_3 + \mu P_1) - 2K_1 \otimes Q_1 \right] e(t)
+ 2e^T(t)(I_N \otimes Q_1B)f(e(t)) + 2e^T(t)(I_N \otimes Q_1F)g(e(t - \tau_1(t))
- 2e^T(t)(I_N \otimes Q_1) \sum_{k=1}^{M} h_k(L^{(k)} \otimes S_k)e(t - \tau_2(t))
+ e^T(t - \tau_1(t))(I_N \otimes (\mu P_2 - (1 - \rho_1)Q_2))e(t - \tau_1(t))
\]
where \( \xi(t) = (e^T(t), e^T(t-\tau_2(t)), f^T(e(t)), g^T(e(t-\tau_1(t))))^T, \Pi_2 = I_N \otimes (U_2 - (1-\rho_1)Q_2 + \mu P_2) \) and

\[
\Theta_3 = \begin{pmatrix}
\Theta_{3,11} & -\sum_{k=1}^{M} h_k(L^{(k)} \otimes Q_1 S_k) & I_N \otimes Q_1 B & I_N \otimes Q_1 F \\
\Theta_{3,21} & -I_N \otimes ((1-\rho_2)Q_3 - \mu P_3) & 0 & 0 \\
I_N \otimes B^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes F^T Q_1 & 0 & 0 & -I_N \otimes I_n
\end{pmatrix},
\]

\[
\Theta_{3,11} = I_N \otimes (-Q_1 D - D^T Q_1 + Q_2 + Q_3 + \mu P_1 + U_1) - 2K_1 \otimes Q_1, \quad \Theta_{3,21} = -\sum_{k=1}^{M} h_k ((L^{(k)})^T \otimes S_k^T Q_1).
\]

**Case 1.** If \( U_2 - (1-\rho_1)Q_2 + \mu P_2 < 0 \) then \( I_N \otimes (U_2 - (1-\rho)Q_2 + \mu P_2) < 0 \) and

\[
\mathcal{L}V(t,e(t),e(t-\tau_1(t)),e(t-\tau_2(t))) \leq \xi^T(t) \Theta_3 \xi(t)
\]

According to Theorem 6.1 of [29], the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely if \( \Theta_4 < 0 \).

**Case 2.** If \( U_2 - (1-\rho_1)Q_2 + \mu P_2 > 0 \), then

\[
\mathcal{L}V(t,e(t),e(t-\tau_1(t)),e(t-\tau_2(t))) \\
\leq \xi^T(t) \Theta_4 \xi(t) + e^T(t-\tau_1(t))\Pi_2 e(t-\tau_1(t)) \\
\leq \xi^T(t) \Theta_4 \xi(t) + e^T(t-\tau_1(t))\Pi_2 e(t-\tau_1(t)) + e^T(t)\Pi_2 e(t) - e^T(t)\Pi_2 e(t) \\
= \xi^T(t) \Theta_4 \xi(t) - e^T(t)\Pi_2 e(t) + e^T(t-\tau_1(t))\Pi_2 e(t-\tau_1(t))
\]

where

\[
\Theta_4 = \begin{pmatrix}
\Theta_{4,11} & -\sum_{k=1}^{M} h_k(L^{(k)} \otimes Q_1 S_k) & I_N \otimes Q_1 B & I_N \otimes Q_1 F \\
\Theta_{4,21} & -I_N \otimes ((1-\rho_2)Q_3 - \mu P_3) & 0 & 0 \\
I_N \otimes B^T Q_1 & 0 & -I_N \otimes I_n & 0 \\
I_N \otimes F^T Q_1 & 0 & 0 & -I_N \otimes I_n
\end{pmatrix},
\]

and \( \Theta_{4,11} = I_N \otimes (-Q_1 D - D^T Q_1 + \rho_1 Q_2 + Q_3 + \mu (P_1 + P_2) + U_1 + U_2) - 2K_1 \otimes Q_1, \Theta_{4,21} = -\sum_{k=1}^{M} h_k ((L^{(k)})^T \otimes S_k^T Q_1). \) Therefore

\[
\mathcal{L}V(t,e(t),e(t-\tau_1(t)),e(t-\tau_2(t))) \\
\leq -\gamma \xi^T(t) \xi(t) - e^T(t)\Pi_2 e(t) + e^T(t-\tau_1(t))\Pi_2 e(t-\tau_1(t)) \\
\leq -\gamma \xi^T(t) \xi(t) - \omega_1(e(t)) + \omega_2(e(t-\tau_1(t)))
\]

where \( \omega_1(e(t)) = e^T(t)\gamma I_N \otimes I_n + \Pi_2 e(t) \) and \( \omega_2(e(t)) = e^T(t)\Pi_2 e(t) \). Obviously, \( \omega_1(e(t)) > \omega_2(e(t)) \) for \( e(t) \neq 0 \). According to Theorem 6.1 of [29], the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely if \( \Theta_4 < 0 \).
3.2. Synchronization with adaptive pinning controller. In designing the state-feedback pinning controller, the control gains $k_i$’s in (7) or (8) are assumed to be constants, which may not be economical. Therefore, to resolve this issue, we design a pinning controller with adaptive control gains (or, adaptive pinning controller) in this subsection.

Without loss of generality, suppose that there are $l$ ($1 \leq l \leq N$) nodes could be controlled directly and the indices of these nodes be $i = 1, 2, \cdots, l$, respectively. In this situation, we set the adaptive pinning controller [15, 55, 43] as,

$$c_i(t) = -k_i(t)e_i(t)$$

where, for each $i = 1, 2, \cdots, l$, $k_i(t)$ is a scalar-valued function and is updated by the following law for a positive constant $\delta_i$,

$$\dot{k}_i(t) = \delta_i\|e_i(t)\|^2 = \delta_i e_i^T(t)e_i(t).$$

Moreover, this controller can be rewritten in the following compact form,

$$K(t) = -(K_2(t) \otimes I_n)e(t)$$

where $K_2(t) = \text{diag}\{k_1(t), k_2(t), \cdots, k_l(t), 0, \cdots, 0\}$.

Under this controller, we can also rewrite the error system as follows.

$$de(t) = \left( -(I_N \otimes D)e(t) + (I_N \otimes B)f(e(t)) + (I_N \otimes F)g(e(t - \tau_1(t)))
+ (K_2(t) \otimes I_n)e(t) - \sum_{k=1}^M h_k(L^{(k)} \otimes S_k)e(t - \tau_2(t)) \right)dt$$

$$+ \sigma(t, e(t), e(t - \tau_1(t)), e(t - \tau_2(t)))dW(t).$$

Using the similar reason as that of Theorem 3.1 and 3.2, we are able to obtain the following two theorems with such designed controllers.

**Theorem 3.3.** Under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (20), if there exist two positive numbers $q_1$ and $\mu$ ($q_1 \leq \mu$) and positive definite matrices $Q_1 = q_1 I_n$, $Q_2$, $Q_3$, such that one of the following two sets of matrix inequalities is satisfied,

**D1.**

$$U_2 - (1 - \rho_1)Q_2 + \mu P_2 > 0,$$

$$\sum_{k=1}^M h_k(\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2)Q_3 + \mu P_3 > 0,$$

$$\Theta_5 = \begin{pmatrix}
q_1 I_N \otimes B & q_1 I_N \otimes F & q_1 I_N \otimes I_n \\
q_1 I_N \otimes B^T & -I_N \otimes I_n & 0 \\
q_1 I_N \otimes F^T & 0 & -I_N \otimes I_n
\end{pmatrix} < 0,$$

where $\Theta_{5,11} = I_N \otimes (-q_1D - q_1D^T + \rho_1Q_2 + \rho_2Q_3 + U_1 + U_2 + \mu(P_1 + P_2 + P_3) + \sum_{k=1}^M h_k(\lambda^{(k)})^2 S_k^T S_k) - 2q_1K \otimes I_n$, $K = \text{diag}\{k_1, k_2, \cdots, k_l, 0, \cdots, 0\}$.

**D2.**

$$U_2 - (1 - \rho_1)Q_2 + \mu P_2 < 0,$$

$$\sum_{k=1}^M h_k(\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2)Q_3 + \mu P_3 > 0,$$
and we set \( \bar{\theta} \). We choose a Lyapunov functional similar to that of Theorem 3.1, \[ \Theta_6 = \begin{pmatrix} \Theta_{6,11} & q_1 I_N \otimes B & q_1 I_N \otimes F & q_1 I_N \otimes I_n \\ q_1 I_N \otimes B^T & -I_N \otimes I_n & 0 & 0 \\ q_1 I_N \otimes F^T & 0 & -I_N \otimes I_n & 0 \\ q_1 I_N \otimes I_n & 0 & 0 & -\frac{1}{\sum_{k=1}^M h_k} I_N \otimes I_n \end{pmatrix} < 0, \] (25)

where \( \Theta_{6,11} = I_N \otimes (-q_1 D - q_1 D^T + Q_2 + \rho_2 Q_3 + U_1 + \mu (P_1 + P_3) + \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k) \) 

\(-2q_1 \bar{K} \otimes I_n, \bar{K} = \text{diag}[\bar{k}_1, \bar{k}_2, \cdots, \bar{k}_l, 0, \cdots, 0] \}.

Proof. We choose a Lyapunov functional similar to that of Theorem 3.1,

\[ V(t,e(t)) = e^T(t)(I_N \otimes q_1 I_n) e(t) + \int_{t-\tau_1(t)}^t e^T(s)(I_N \otimes Q_2) e(s) ds \]

\[ + \int_{t-\tau_2(t)}^t e^T(s)(I_N \otimes Q_3) e(s) ds + q_1 \sum_{i=1}^l \frac{1}{\delta_i} (k_i(t) - \bar{k}_i)^2. \]

As you see, \( Q_1 \) is replaced by \( q_1 I_n \), and \( q_1 \sum_{i=1}^l \frac{1}{\delta_i} (k_i(t) - \bar{k}_i)^2 \) is added to this Lyapunov functional. Then the remaining proof is similar to that of Theorem 3.1. We omit the proof.

Remark 5. Similarly, if all nodes are allowed to be controlled directly, i.e., \( l = N \) and we set \( \bar{k}_1 = \bar{k}_2 = \cdots = \bar{k}_N = \bar{k} \), then we obtain the following corollary which does not involve Kronecker product and is easier to be applied.

Corollary 2. Under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (20), if there exist two positive numbers \( q_1 \) and \( \mu \) (\( q_1 \leq \mu \)) and positive definite matrices \( Q_1 = q_1 I_n, Q_2, Q_3, \) such that one of the following two sets of matrix inequalities is satisfied,

\[ U_2 - (1 - \rho_1) Q_2 + \mu P_2 > 0, \] (D1)

\[ \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 > 0, \] (27)

\[ \Xi_3 = \begin{pmatrix} \Xi_{3,11} & q_1 B & q_1 F & q_1 I_n \\ q_1 B^T & -I_n & 0 & 0 \\ q_1 F^T & 0 & -I_n & 0 \\ q_1 I_n & 0 & 0 & -\frac{1}{\sum_{k=1}^M h_k} I_n \end{pmatrix} < 0, \] (28)

where \( \Xi_{3,11} = q_1 D - q_1 D^T + \rho_1 Q_2 + \rho_2 Q_3 + U_1 + U_2 + \mu (P_1 + P_2 + P_3) + \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k - 2k q_1 I_n. \)

\[ U_2 - (1 - \rho_1) Q_2 + \mu P_2 < 0, \] (D2)

\[ \sum_{k=1}^M h_k (\lambda^{(k)})^2 S_k^T S_k - (1 - \rho_2) Q_3 + \mu P_3 > 0, \] (30)
Then the remaining is similar to that of Theorem 3.2. 

\[ \Xi_4 = \begin{pmatrix} \Xi_{4,11} & q_1B & q_1F & q_1I_n \\ q_1B^T & -I_n & 0 & 0 \\ q_1F^T & 0 & -I_n & 0 \\ q_1I_n & 0 & 0 & -\frac{1}{\sum_{k=1}^M h_k I_n} \end{pmatrix} < 0, \] (31)

where \( \Xi_{4,11} = -q_1D - q_1DT + Q_2 + \rho_2 Q_3 + U_1 + \mu (P_1 + P_3) + \sum_{k=1}^M h_k (\lambda(k))^2 S_k^T S_k - 2\bar{q}_1 I_n. \)

**Remark 6.** Once again, the preceding theorem and corollary are quite conservative due to the conservative inequality (18) in the proof. We also provide the following theorem where we do not use such conservative inequality. In this situation, the Kronecker product in the corollary is unavoidable.

**Theorem 3.4.** Under the assumptions (A1)-(A4), the two multi-layer complex networks (2) and (3) can be stochastically synchronized almost surely under the controller (20), if there exist two positive numbers \( q_1 \) and \( \mu (q_1 \leq \mu) \) and positive definite matrices \( Q_1 = q_1 I_n, Q_2, Q_3, \) such that one of the following two sets of matrix inequalities is satisfied,

\[ U_2 - (1 - \rho_1)Q_2 + \mu P_2 < 0, \] (E1)

\[ \Theta_7 = \begin{pmatrix} \Theta_{7,11} & -\sum_{k=1}^M h_k q_1 (L(k) \otimes S_k) & q_1 I_N \otimes B & q_1 I_N \otimes F \\ q_1 I_N \otimes B^T & -I_N \otimes ((1 - \rho_2)Q_3 - \mu P_3) & 0 & 0 \\ q_1 I_N \otimes F^T & 0 & -I_N \otimes I_n & 0 \\ q_1 I_N \otimes I_n & 0 & 0 & -I_N \otimes I_n \end{pmatrix} < 0 \]

where \( \Theta_{7,11} = I_N \otimes \left( -q_1 D - q_1 DT + Q_2 + Q_3 + \mu P_3 + U_1 \right) - 2q_1 \bar{K} \otimes I_n, \Theta_{7,21} = -\sum_{k=1}^M h_k q_1 ((L(k))^T \otimes S_k^T), \bar{K} = \text{diag}\{\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_l, 0, \ldots, 0\}. \) (E2)

\[ U_2 - (1 - \rho_1)Q_2 + \mu P_2 > 0, \]

\[ \Theta_8 = \begin{pmatrix} \Theta_{8,11} & -\sum_{k=1}^M h_k q_1 (L(k) \otimes S_k) & q_1 I_N \otimes B & q_1 I_N \otimes F \\ q_1 I_N \otimes B^T & -I_N \otimes ((1 - \rho_2)Q_3 - \mu P_3) & 0 & 0 \\ q_1 I_N \otimes F^T & 0 & -I_N \otimes I_n & 0 \\ q_1 I_N \otimes I_n & 0 & 0 & -I_N \otimes I_n \end{pmatrix} < 0, \]

where \( \Theta_{8,11} = I_N \otimes \left( -q_1 D - q_1 DT + \rho_1 Q_2 + Q_3 + \mu (P_1 + P_2) + U_1 + U_2 \right) - 2q_1 \bar{K} \otimes I_n, \Theta_{8,21} = -\sum_{k=1}^M h_k q_1 ((L(k))^T \otimes S_k^T), \bar{K} = \text{diag}\{\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_l, 0, \ldots, 0\}. \)

**Proof.** We choose the Lyapunov functional as that of Theorem 3.3,

\[ V(t, e(t)) = e^T(t)(I_N \otimes q_1 I_n) e(t) + \int_{t_{\tau_1(t)}}^t e^T(s)(I_N \otimes Q_2) e(s) ds \]

\[ + \int_{t_{\tau_2(t)}}^t e^T(s)(I_N \otimes Q_3) e(s) ds + q_1 \sum_{i=1}^l \frac{1}{\delta_i} (k_i(t) - \bar{k}_i)^2. \]

Then the remaining is similar to that of Theorem 3.2. \( \square \)
4. Numerical examples and simulation examples. The following example is given to show the effectiveness of our theoretical results.

Example 1. Firstly, we construct a multi-layer complex network with \( M = 2 \) layers and \( N = 100 \) nodes, as shown in Figure 2. The first layer is constructed by using a Watts-Strogatz small-world graph [45] with initial degree \( d = 4 \) and the rewiring probability \( p = 0.2 \), yielding the maximum eigenvalue of the Laplacian matrix \( \lambda^{(1)} = 8.9395 \). The other layer is constructed as a scale-free graph \([1, 7]\) according to the following degree distribution

\[ P(k) \sim k^{-\alpha}, \]

with the exponent \( \alpha = 2.2 \). The maximum eigenvalue of the Laplacian matrix \( \lambda^{(2)} = 8.3855 \).

\[ \text{Figure 2. Multi-layer network with two layers and 100 nodes. (a) First layer: a Watts-Strogatz small-world graph. (b) Second layer: a scale-free graph.} \]

We choose the following parameters for the drive system (2) and the response system (3),

\[
D = \begin{pmatrix} 1 & -0.8 \\ -0.8 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -0.2 \\ -6 & 3.2 \end{pmatrix}, \quad F = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix},
\]

\( h_1 = 0.1, h_2 = 0.3, S_1 = I_2, S_2 = I_2 \). Moreover, for each \( i, J_i = (0,0)^T \), \( \hat{f}_i(x_i) = \hat{g}_i(x_i) = (\tanh(x_{i1}),\tanh(x_{i2}))^T, \tau_1(t) = 0.5 \cos(t) + 0.5, \tau_2(t) = 0.4 \sin(t) + 0.6 \) and \( \sigma_i(t, \xi_i(t), \xi_i(t - \tau_1(t)), \xi_i(t - \tau_2(t))) = 0.05(\xi_i(t), \xi_i(t - \tau_1(t)) + \xi_i(t - \tau_2(t))) \). Finally, we choose initial values \( x_i(t) = (-2,8)^T \) for \( t \in [-1,0] \) and \( i = 1,2,\cdots,50 \). For \( i = 51,52,\cdots,100 \), we choose initial values \( x_i(t) = (0,0)^T \) for \( t \in [-1,0] \). For the response system, \( y_i(t) = (0,0)^T \) for \( t \in [-1,0] \) and \( i = 1,2,\cdots,100 \). To simulate the response system, the Euler-Maruyama method [11] is adopted to compute the trajectory of stochastic differential equation. Then Figure 3 shows the dynamic behavior of the node 1 in the drive system and that of the node 1 in the response system.

Under such settings, then we can obtain the following parameters in our theoretical assumptions: \( U_1 = U_2 = I_2 \) in (A2); \( \rho_1 = 0.5, \rho_2 = 0.4 \) and \( \tau = 1 \) in (A3); \( P_1 = 0.05^2 I_2, P_2 = P_3 = 2 \cdot 0.05^2 I_2 \) in (A4).
Since Theorem 3.1 and Theorem 3.3 are more conservative than Theorem 3.2 and Theorem 3.4, we only use Theorem 3.2 and Theorem 3.4 here. For simplicity, we set $l = N$ and choose $k_1 = k_2 = \cdots = k_N = k^*$ and $\bar{k}_1 = \bar{k}_2 = \cdots = \bar{k}_N = \bar{k}$ in Theorem 3.2 and Theorem 3.4 respectively.

The situation under controller (8). Solving the inequalities in Case 1 of Theorem 3.2 using the LMI toolbox in Matlab, we obtain that $k^* = 6.6392$, $\mu = 3.0365$ and $Q_1, Q_2, Q_3$ are obtained as

\[
\begin{pmatrix}
0.0864 & 0.0042 \\
0.0042 & 0.0702
\end{pmatrix}, \quad
\begin{pmatrix}
4.5583 & -0.0091 \\
-0.0091 & 4.1886
\end{pmatrix}, \quad
\begin{pmatrix}
-0.0095 & 4.9935 \\
5.0123 & -0.0095
\end{pmatrix},
\]

respectively. Solving the inequalities in Case 2 of Theorem 3.2 using the LMI toolbox in Matlab, we obtain that $k^* = 3.3592$, $\mu = 1.6404$ and $Q_1, Q_2, Q_3$ are obtained as

\[
\begin{pmatrix}
0.0445 & 0.0098 \\
0.0098 & 0.0075
\end{pmatrix}, \quad
\begin{pmatrix}
0.8945 & -0.0245 \\
-0.0245 & 0.8225
\end{pmatrix}, \quad
\begin{pmatrix}
2.7034 & -0.0082 \\
-0.0082 & 2.6796
\end{pmatrix},
\]

respectively. Hence we can choose $k_1 = k_2 = \cdots = k_N = k^* = 3.3592$, and the controller (8) becomes $C(t) = -3.3592 \epsilon(t)$. Under such controller, we draw the trajectory of the response system (3) as shown in Figure 4(a).

To show the effectiveness of our results, we also draw the time evolution (Figure 4(b)) of the total error which can be defined as

\[
\|\epsilon(t)\| = \left( \sum_{i=1}^{N} (y_i(t) - x_i(t))^T(y_i(t) - x_i(t)) \right)^{1/2}.
\]

The situation under controller (20). Here we should apply Theorem 3.4 where we set $\delta_1 = \delta_2 = \cdots = \delta_N = 0.1$. Solving the inequalities in Case 1 of Theorem 3.4 using the LMI toolbox in Matlab, we obtain that $\bar{k} = 6.0541$, $\mu = 2.5735$ and

\[
Q_1 = 0.0139I_2, \quad Q_2 = \begin{pmatrix}
4.2620 & -0.0194 \\
-0.0194 & 4.2555
\end{pmatrix}, \quad Q_3 = \begin{pmatrix}
4.2883 & -0.0008 \\
-0.0008 & 4.2881
\end{pmatrix}.
\]
SOLVING THE INEQUALITIES IN CASE 2 OF THEOREM 3.4 USING THE LMI TOOLBOX IN MATLAB, WE OBTAIN THAT $k^* = 4.5631$, $\mu = 2.5514$ AND $Q_1 = 0.0295I_2$, $Q_2 = \begin{pmatrix} 0.9311 & -0.0293 \\ -0.0293 & 0.9028 \end{pmatrix}$, $Q_3 = \begin{pmatrix} 4.1836 & -0.0175 \\ -0.0175 & 4.1698 \end{pmatrix}$.

Under controller (20), we draw the trajectory of the response system (3) as shown in Figure 5(a). To show the effectiveness of our results, we also draw the time evolution of the total error as shown in Figure 5(b).

COMPARING FIGURE 4(b) WITH FIGURE 5(b), ONE MAY MISINTERPRET THAT THE STATE-FEEDBACK PINNING CONTROLLER IS MORE EFFECTIVE THAN THE ADAPTIVE PINNING CONTROLLER.
The reason is that the control gains obtained by the state-feedback pinning controller are much bigger than those obtained by the adaptive pinning controller (Figure 6). Actually, the state-feedback pinning controller is easier to implement since the control gains are constants. The adaptive pinning controller is more economical since the control gains are adaptive and much smaller (Figure 6).

![Figure 6. The control gains under the state-feedback pinning controller versus the average controller gains under the adaptive pinning controller.](image)

5. **Conclusion.** This paper concerns the synchronization of the multi-layer dynamical networks under stochastic perturbations. Two pinning controllers are designed. One is the state-feedback pinning controller with constant control gains. The other one is the adaptive pinning controller with adaptive control gains. Several sets of sufficient conditions are obtained to ensure the synchronization of multi-layers networks under the stochastic perturbations. It is worthwhile to mention that our assumptions on the activation functions satisfy a generalized Lipschitzian condition which are weaker than those in the previous works.

Until now, there appear some related works on the distributed impulsive synchronization of Lur’e dynamical networks (e.g. [40, 41]). If we consider the distributed impulsive synchronization of multi-layer complex networks or duplex networks, it would be very interesting. Moreover, it is interesting to study the inter-layer synchronization of multi-layer complex networks or duplex networks. We leave them for further future works.

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Received August 2019; revised September 2019.

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