ON CLOSED RANGE FOR $\bar{\partial}$

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Abstract. A sufficient condition for $\bar{\partial}$ to have closed range is given for pseudoconvex domains in $\mathbb{C}^n$. Moreover, it is shown that whenever $\bar{\partial}$ has closed range on $(0, q)$-forms, then $\bar{\partial}$ on $(0, q + 1)$-forms also has closed range.

1. Introduction

Extending the Cauchy–Riemann operator, $\bar{\partial}$, initially defined pointwise, to an unbounded operator on $L^2$ allows Hilbert space methods to bear on existence and regularity questions connected to the Cauchy–Riemann equations. These methods allow one to deduce powerful results about complex function theory, especially in several complex variables. Results obtained in this manner, after the seminal work of Kohn and Hörmander in the early 1960s, are perhaps well-known enough to view extending $\bar{\partial}$ to $L^2$ as a classical part of complex analysis.

A basic question, underlying more refined existence and regularity issues, is whether the extended $\bar{\partial}$ operator has closed range in $L^2$. In this paper, for $\Omega \subset \mathbb{C}^n$ a pseudoconvex domain, we give a general sufficient condition for $\bar{\partial}$ to have closed range in $L^2_{p,q}(\Omega)$. This condition is not restricted to bounded domains; indeed, this paper primarily grew out of our interest in determining classes of unbounded domains for which $\bar{\partial}$ has closed range. A secondary interest was understanding the closed range property on non-smooth domains. The condition given is sensitive to the bi-degree $(p, q)$ where the question of closed range is posed.

We do not consider non-pseudoconvex domains in this paper until the final section. For the few results on closed range for $\bar{\partial}$ on (classes of) non-pseudoconvex domains, see [5, 12, 11, 8]. The theory needs more results on general domains, of both positive and negative type.

If $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, the fact that $\bar{\partial}$ has closed range in $L^2$, in all bi-degrees, follows from Hörmander’s estimates and the fact that $\Omega$ supports a bounded uniformly strictly plurisubharmonic function, e.g., $\phi(z) = |z|^2$. See Theorem 2.2.1 in [6] for the essential inequality; that closed range of $\bar{\partial}$ follows from this can be achieved by exhausting the domain using Theorem 5.1 and arguing as in Proposition 4.3 below. Our sufficient condition is also potential-theoretic, but more general than supporting a bounded function like $\phi$. The condition requires $\Omega$ to support two functions whose first and second derivatives combine in a certain fashion to give a uniformly positive lower bound. That this condition implies $\bar{\partial}$ has closed range follows from the refined, twisted $\bar{\partial}$ estimates in [10] rather than Hörmander’s
estimates. Two special cases of the general condition are also given, removing the interplay between two functions on $\Omega$. The simpler hypotheses in Corollaries 4.6 and 4.10 are often adequate for determining when $\bar{\partial}$ has closed range in practice. The examples discussed in Section 7 use only Corollary 6.9.

The main theorem guaranteeing closed range is stated in Section 4 for bounded domains with smooth boundary, but attention is paid to the size of the constant obtained in order to pass to the unbounded, non-smooth cases. After reviewing how an arbitrary pseudoconvex domain can be approximated by smoothly bounded pseudoconvex open sets in Section 5, it is shown in Section 6 how a uniform version of the closed range inequality (3.2) on the approximating subsets implies that $\bar{\partial}$ has closed range on the limit domain.

In Section 7, we give examples of unbounded pseudoconvex domains where $\bar{\partial}$ has closed range and others where it does not. For domains $\Omega$ in the plane, it is natural to conjecture that $\bar{\partial}$ has closed range if and only if $\Omega$ does not contain arbitrarily large complex discs. We show that this condition is necessary in general, but only establish sufficiency with an additional hypothesis (see Definition 7.1). In higher dimensions, it is reasonable to expect that if $\bar{\partial}$ on $(0,q)$-forms has closed range, then $\Omega$ cannot contain arbitrarily large $q + 1$-dimensional Euclidean balls. A sufficient condition for closed range will likely involve holomorphic images of balls, though the relation between the form level $\bar{\partial}$ acts on and the dimension of the images should remain. We hope to return to this matter in another paper.

Finally, in the last section we show that if the range of $\bar{\partial}$ acting on $(0,q)$-forms is closed, then the range of $\bar{\partial}$ on $(0,q+1)$-forms is automatically closed. This conclusion holds without assuming that the domain $\Omega$ is pseudoconvex, which is noteworthy as stronger than closed range estimates on $\bar{\partial}$, e.g., subelliptic estimates, do not enjoy this property (see Remark 8.10).

2. Preliminaries

Let $\Omega$ be a domain in $\mathbb{C}^n$. We write $\Omega \Subset \mathbb{C}^n$ to indicate $\Omega$ is bounded; more generally, write $U \Subset V$, for $U,V$ open sets, to indicate that $\overline{U}$ is a compact subset of $V$. Whether bounded or not, we shall say $\Omega$ has smooth boundary if there is a smooth, real-valued function $r$ such that $\Omega = \{ r < 0 \}$ and $dr \neq 0$ when $r = 0$; $r$ is then a defining function for $\Omega$.

For $1 \leq q \leq n$, a $(0,q)$-form $u$ can be uniquely written

$$u(z) = \sum'_{|I|=q} u_I(z) \, d\bar{z}^I,$$

where $\sum'_{|I|=q}$ denotes the sum over increasing multi-indices $I$ of length $q$, $u_I(z)$ are functions, and $d\bar{z}^I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$ when $I = (i_1, \ldots, i_q)$ is such a multi-index.

If $v = \sum'_{|I|=q} v_I \, d\bar{z}^I$ is another $(0,q)$-form, define the inner product

$$(u,v)_{L_{\bar{\partial},q}(\Omega)} = \sum'_{|I|=q} \int_{\Omega} u_I(z) \overline{v_I(z)} \, dV_E,$$

where $dV_E$ is the Euclidean volume element. Only products of components of $u$ and $v$ corresponding to the same multi-index $I$ appear in the integrand in (2.2); in
particular, forms of different bi-degree are orthogonal. Let $L^2_{0,q}(\Omega)$ denote the $(0,q)$-forms $u$ on $\Omega$ such that $\|u\|_{L^2_{0,q}(\Omega)} < \infty$, where $\|\cdot\|_{L^2_{0,q}(\Omega)}$ is the norm induced by the inner product \eqref{2.2}. The subscripts will be dropped when confusion is unlikely. To distinguish between the $L^2$-norms and the Euclidean norm on $\mathbb{C}^n$ we use the notation $|\cdot|$ for the latter.

Let $\Lambda^{0,q}(\Omega)$, $\Lambda^{0,q}(\Omega)$, and $\Lambda^{0,q}(\Omega)$ be the space of $(0,q)$-forms with coefficients in $C^\infty(\Omega)$, $C^\infty_c(\Omega)$, and $C^\infty(\Omega)$, respectively. Denote the domain, range, and null space of an operator $A$ by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively.

The Cauchy–Riemann operator, $\bar{\partial}$, on functions $f \in C^\infty(\Omega)$ is defined as

$$\bar{\partial}f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j.$$  

It is extended to $(0,q)$-forms by linearity,

$$\bar{\partial}_q u := \sum_{j=1}^{n} \sum_{|I|=q} \frac{\partial u_I}{\partial z_j} dz_j \wedge d\bar{z}^I,$$

where $u \in \Lambda^{0,q}(\Omega)$ is given by \eqref{2.1}.

$\bar{\partial}_q$ is extended to an $L^2$-operator (still called $\bar{\partial}_q$) by first letting it act on $L^2_{0,q}(\Omega)$ in the sense of distributions and then restricting its domain, $\mathcal{D}(\bar{\partial}_q)$, as follows:

$$\mathcal{D}(\bar{\partial}_q) = \{ u \in L^2_{0,q}(\Omega) : \bar{\partial}_q u \in L^2_{0,q+1}(\Omega) \}.$$  

This is the maximal extension of $\bar{\partial}_q$ to $L^2_{0,q}(\Omega)$. An equivalent description of $\mathcal{D}(\bar{\partial}_q)$ is

$$\mathcal{D}(\bar{\partial}_q) = \{ u \in L^2_{0,q}(\Omega) : \exists \{ u_j \} \subset \Lambda^{0,q}(\Omega) \text{ such that } u_j \to u \}
\in L^2_{0,q}(\Omega) \text{ and } \{ \bar{\partial}_q u_j \} \text{ is Cauchy in } L^2_{0,q+1}(\Omega).$$

Then one sets $\bar{\partial}_q u = \lim_{j \to \infty} \bar{\partial}_q u_j$ and checks easily that this is independent of the sequence $\{ u_j \}$. The extended operator $\bar{\partial}_q$ is closed and densely defined on $L^2_{0,q}(\Omega)$.

The Hilbert space adjoint, $\bar{\partial}_q^*$, of $\bar{\partial}_q$ is the operator with domain

$$\mathcal{D}(\bar{\partial}_q^*) = \{ v \in L^2_{0,q+1}(\Omega) : \exists C > 0 \text{ with } |\langle \bar{\partial}_q u, v \rangle| \leq C \|u\| \forall u \in \mathcal{D}(\bar{\partial}_q) \}$$

satisfying for $v \in \mathcal{D}(\bar{\partial}_q^*)$

$$\langle \bar{\partial}_q u, v \rangle = \langle u, \bar{\partial}_q^* v \rangle \quad \forall u \in \mathcal{D}(\bar{\partial}_q^*).$$

The subspace $\mathcal{D}^{0,q}(\Omega) := \mathcal{D}(\bar{\partial}_q^*) \cap \Lambda^{0,q}(\Omega)$ is useful for computations. The abstract conditions for $u$ to belong to $\mathcal{D}(\bar{\partial}_q^*)$ become explicit boundary conditions if $u \in \Lambda^{0,q}(\Omega)$. Also, if $\Omega$ is bounded and has smooth boundary, then $\mathcal{D}^{0,q}(\Omega)$ is dense in $\mathcal{D}(\bar{\partial}_q^*) \cap \mathcal{D}(\bar{\partial}_q)$; see [6], pages 94–98.

Finally, to use shorthand to denote the action of a complex Hessian on a $(0,q)$-form the following notation is introduced. For any $1 \leq m \leq n$ and $H$ an increasing index of length $q-1$, let $mH$ denote the multi-index $m, h_{i_1}, \ldots, h_{i_{q-1}}$ and, if $m \notin H$, $(mH)$ the increasing multi-index formed from the set $\{ m, h_{i_1}, \ldots, h_{i_{q-1}} \}$. For $u$ given by \eqref{2.1}, set

$$u_{mH} = e_{mH}^m u_{(mH)}.$$
where

\[
\varepsilon_{\langle mH \rangle}^m = \begin{cases} 
\text{sign of permutation turning } mH \text{ into } \langle mH \rangle & \text{if } m \notin H \\
0 & \text{if } m \in H
\end{cases}
\]

If \( f \) is a \( C^2 \) function, define

\[
(2.5) \quad i\partial\bar{\partial}f(u,u) := \sum_{|J|=q-1} \sum^n_{k,l=1} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_k} u_l J \bar{u}_k J, \quad u \in \Lambda^{0,q}(\Omega).
\]

When \( q = 1 \), (2.5) is standard, expressing the natural action of the \((1,1)\)-form \( i\partial\bar{\partial}f \) on the vectors \( u \) and \( \bar{u} \) associated to the forms \( u, \bar{u} \) by the Euclidean structure of \( \mathbb{C}^n \). For example, a \( C^2 \) function \( f \) is plurisubharmonic on an open \( U \subset \mathbb{C}^n \) if \( i\partial\bar{\partial}f(u,u)(p) \geq 0 \) for all \( u \in \Lambda^{0,1}(\Omega) \) and \( p \in U \).

For \( q > 1 \), the right-hand side of (2.5) is less natural. But this expression arises repeatedly when integrating by parts in the \( \bar{\partial} \)-complex and representing it by the left-hand side of (2.5) shortens many formulas, e.g., (4.2) below.

In the sequel, we shall use two equivalent notions of pseudoconvexity. If \( \Omega \subset \mathbb{C}^n \) is an arbitrary domain, we say that \( \Omega \) is pseudoconvex if there exists a plurisubharmonic exhaustion function, \( \Phi \), on \( \Omega \), i.e., a plurisubharmonic \( \Phi \) such that

\[
(2.6) \quad \{ z : \Phi(z) < c \} \Subset \Omega \quad \forall c \in \mathbb{R}.
\]

It may be assumed that this exhaustion function is in fact smooth on \( \Omega \), see, e.g., Theorem 2.6.11 in [7]. A domain \( \Omega = \{ z \in \mathbb{C}^n : r(z) < 0 \} \) with smooth boundary, is said to be Levi pseudoconvex if

\[
i\partial\bar{\partial}r(\xi,\xi)(p) \geq 0 \quad p \in \partial \Omega, \quad \xi \in \mathbb{C}^n \quad \text{with} \quad \sum^n_{j=1} \frac{\partial r}{\partial z_j}(p)\xi_j = 0.
\]

A proof that these two notions are equivalent for open, smoothly bounded sets is given in Theorem 2.6.12 of [7].

### 3. Functional Analysis

The range of \( \bar{\partial}_q \) is said to be closed if \( \mathcal{R} (\bar{\partial}_q) \subset L^2_{0,q+1}(\Omega) \) is metrically closed in \( L^2_{0,q+1}(\Omega) \). The closedness of \( \mathcal{R} (\bar{\partial}_q) \) is equivalent to \( \bar{\partial}_q \) being norm-bounded from below off its null space, \( \mathcal{N} (\bar{\partial}_q) \), and also to estimates from below on \( \bar{\partial}^*_q \). The following result summarizes these facts and is well-known (see, e.g., Theorem 1.1.1. in [6]):

**Proposition 3.1.** The following conditions are equivalent.

(i) \( \mathcal{R} (\bar{\partial}_q) \) is closed in \( L^2_{0,q+1}(\Omega) \).

(ii) There exists a constant \( C > 0 \) such that

\[
(3.2) \quad \|u\| \leq C \|\bar{\partial}_q u\| \quad \forall u \in \mathcal{D} (\bar{\partial}_q) \cap \mathcal{N} (\bar{\partial}_q)^\perp.
\]

(iii) There exists a constant \( C > 0 \) such that

\[
\|v\| \leq C \|\bar{\partial}^*_q v\| \quad \forall v \in \mathcal{D} (\bar{\partial}^*_q) \cap \mathcal{N} (\bar{\partial}^*_q)^\perp.
\]
A slightly more flexible, but equivalent, inequality will be used in Section 8

Proposition 3.3. \( R(\bar{\partial}_q) \) is closed if and only if there exists \( C > 0 \) such that

\[
\text{dist}(v, \mathcal{N}(\bar{\partial}_{q+1}^*)) \leq C \|\bar{\partial}_{q+1}^* v\| \quad \forall \ v \in \mathcal{D}(\bar{\partial}_q^*).
\]

Proof. Assume (iii) of Proposition 3.1 holds. Let \( v = a \oplus b \), where \( b \in \mathcal{N}(\bar{\partial}_{q+1}^*) \) and \( a \in \mathcal{N}(\bar{\partial}_q^*) \). Then \( \|\bar{\partial}_{q+1}^* v\| = \|\bar{\partial}_{q+1}^* a\| \) and \( \|a\| = \text{dist}(v, \mathcal{N}(\bar{\partial}_{q+1}^*)) \). Thus,

\[
\|\bar{\partial}_{q+1}^* v\| = \|\bar{\partial}_{q+1}^* a\| \geq \frac{1}{C} \|a\| \\
= \frac{1}{C} \text{dist}(v, \mathcal{N}(\bar{\partial}_{q+1}^*)) ,
\]

and so (3.4) holds. That (3.4) implies (iii) of Proposition 3.1 is trivial. \( \square \)

Closed range properties of the \( \bar{\partial} \)-operator are closely connected to the existence of the \( \bar{\partial} \)-Neumann operator. For \( 1 \leq q \leq n \), the \( \bar{\partial} \)-Neumann operator, \( N_q \), is the solution operator to the following problem: given \( \alpha \in L^2_{0,q}(\Omega) \), find \( u \in L^2_{0,q}(\Omega) \) such that

\[
\Box_q u := (\bar{\partial}_{q-1}^* \bar{\partial}_q + \bar{\partial}_q^* \bar{\partial}_q) u = \alpha, \text{ and} \\
u \in \mathcal{D}(\Box_q) := \{ u \in \mathcal{D}(\bar{\partial}_q) \cap \mathcal{D}(\bar{\partial}_q^*): \bar{\partial}_q u \in \mathcal{D}(\bar{\partial}_q^*) \}.
\]

The relationship between \( L^2 \)-boundedness of \( N_q \) and the closed range property for \( \bar{\partial} \) is:

Proposition 3.5. Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex domain with smooth boundary, \( 1 \leq q \leq n \). Then both \( \bar{\partial}_{q-1} \) and \( \bar{\partial}_q \) have closed range in \( L^2_{0,q}(\Omega) \) and \( L^2_{0,q+1}(\Omega) \), respectively, if and only if \( N_q \) is a bounded operator on \( L^2_{0,q}(\Omega) \).

Proof. This is fairly standard, so we only sketch the proof. It is straightforward to show that both \( \bar{\partial}_{q-1} \) and \( \bar{\partial}_q \) have closed range if and only if there is a constant \( C > 0 \) such that

\[
\|u\| \leq C (\|\bar{\partial}_q u\| + \|\bar{\partial}_q^* u\|) \quad \forall \ u \in \mathcal{D}(\bar{\partial}_q) \cap \mathcal{D}(\bar{\partial}_q^*) \cap (\mathcal{N}(\bar{\partial}_q) \cap \mathcal{N}(\bar{\partial}_q^*))^\perp
\]

holds, see for instance Theorem 1.1.2 in [6]. Moreover, (3.6) implies that

\[
\|u\| \leq C \|\Box_q u\| \quad \forall \ u \in \mathcal{D}(\Box_q) \cap \mathcal{N}(\Box_q)^\perp.
\]

It follows that \( \Box_q \) has closed range, since it is a closed operator, which yields the Hodge decomposition \( L^2_{0,q}(\Omega) = \mathcal{N}(\Box_q) \oplus R(\Box_q) \).

However, pseudoconvexity of \( \Omega \) forces \( \mathcal{N}(\Box_q) = \{0\} \). This follows, for instance, from (4.2) below, with \( \lambda = 0 \) and \( \tau = B - |z|^2 \) for a suitably large constant \( B > 0 \). Hence, \( \Box_q : \mathcal{D}(\Box_q) \rightarrow L^2_{0,q}(\Omega) \) is bijective and has a bounded inverse, \( N_q \).

To show that (3.6) follows if \( N_q \) is a bounded operator, one first shows that both \( \bar{\partial}_q N_q \) and \( \bar{\partial}_q^* N_q \) are bounded operators; this fact follows since \( \|\bar{\partial}_q N_q u\|^2 + \|\bar{\partial}_q^* N_q u\|^2 = (u, Nu) \). Then for \( u \in \mathcal{D}(\bar{\partial}_q) \cap \mathcal{D}(\bar{\partial}_q^*) \)

\[
\|u\|^2 = (u, u) = (\Box_q N_q u, u) = (\bar{\partial}_q N_q u, \bar{\partial}_q u) + (\bar{\partial}_q^* N_q u, \bar{\partial}_q^* u) \leq \|\bar{\partial}_q N_q\| \cdot \|\bar{\partial}_q u\| + \|\bar{\partial}_q^* N_q u\| \cdot \|\bar{\partial}_q^* u\| \leq C \|u\| (\|\bar{\partial}_q u\| + \|\bar{\partial}_q^* u\|).
\]
which yields (3.6). □

4. Smoothly bounded domains; uniform estimates

The twisted estimates derived in Proposition 3.2 in [10] (with \(g = \tau\) and \(\nu = 1\)) yield the following.

**Proposition 4.1.** Let \(\Omega \Subset \mathbb{C}^n\) be a pseudoconvex domain with smooth boundary, \(0 \leq q \leq n - 1\). Let \(\lambda, \tau \in C^2(\Omega)\) and \(\tau \geq 0\). Then

\[
\|\sqrt{\tau} \bar{\partial} u\|_{\lambda}^2 + 2\|\sqrt{\tau} \partial^* u\|_{\lambda}^2 \geq \int_{\Omega} \left( \tau i \partial \bar{\partial} \lambda(u, u) - i \partial \bar{\partial} \tau(u, u) - \frac{1}{\tau} |\langle \partial \bar{\partial}, u \rangle|^2 \right) e^{-\lambda} \, dV
\]

for all \(u \in D^{0,q+1}(\Omega)\).

Here, \(\| \cdot \|_{\lambda}\) is the norm induced by the inner product \(\langle \cdot, \cdot \rangle_{\lambda} := \int_{\Omega} \langle \cdot, \cdot \rangle e^{-\lambda} \, dV\) (on the appropriate form level). Moreover, \(\partial^*\) is the Hilbert space adjoint of \(\bar{\partial}\) with respect to \(\langle \cdot, \cdot \rangle_{\lambda}\) so that \(\mathcal{F}^* u = e^\lambda \mathcal{F}^*(e^{-\lambda} u)\) holds for all \(u \in \mathcal{D}(\partial^*)\).

For brevity, write

\[
\Theta_{\lambda, \tau}(u, u) := \tau i \partial \bar{\partial} \lambda(u, u) - i \partial \bar{\partial} \tau(u, u) - \frac{1}{\tau} |\langle \partial \bar{\partial}, u \rangle|^2.
\]

**Proposition 4.3.** Let \(\Omega \Subset \mathbb{C}^n\) be a pseudoconvex domain with smooth boundary, \(0 \leq q \leq n - 1\). Suppose \(\Omega\) admits functions \(\lambda, \tau \in C^2(\Omega)\) such that

\[
\Theta_{\lambda, \tau}(u, u) \geq c_1 |u|^2 e^{-\lambda} \quad \forall \ u \in \Lambda^{0,q+1}(\Omega)
\]

for some constant \(c_1 > 0\).

Then for each \(\alpha \in L^2_{0,q+1}(\Omega)\) with \(\bar{\partial} q \alpha = 0\) there is a \(v \in \mathcal{D}(\partial_q)\) satisfying \(\partial_q v = \alpha\) and

\[
\|v\|_{L^2_q(\Omega)} \leq C \|\alpha\|_{L^2_{0,q+1}(\Omega)},
\]

where the constant \(C\) equals \(\sqrt{2/(c_1 c_2)}\) for \(c_2 = \min\{e^{-\lambda(z)}/\tau(z) : z \in \overline{\Omega}\}\).

Moreover,

\[
\|f\| \leq C \|\partial_q f\| \quad \forall \ f \in \mathcal{D}(\partial_q) \cap \mathcal{D}(\partial_{q+1}^*)
\]

The proof of Proposition 4.3 is analogous to the proof of Theorem 4.1 in [10]. See also the proof of Theorem 4.3 in [1].

**Proof of Theorem 4.3.** Let \(\alpha \in L^2_{0,q+1}(\Omega)\) with \(\bar{\partial} \alpha = 0\) be given. Define the linear functional

\[
F : \left( \left\{ \sqrt{\tau} \partial^*_\lambda u : u \in \mathcal{D}(\partial^*) \right\}, \|\cdot\|_{\lambda} \right) \to \mathbb{C}
\]

\[
\sqrt{\tau} \partial^*_\lambda u \quad \mapsto \quad (u, \alpha)_\lambda.
\]

Write \(u = u_1 + u_2\) for \(u_1, u_2 \in L^2_{0,q+1}(\Omega)\) with \(u_1 \in \mathcal{N}(\partial_{q+1})\) and \(u_2 \perp_{\lambda} \mathcal{N}(\partial_{q+1})\) (note that \(L^2_{0,*}(\Omega) = L^2_{0,*}(\Omega, \lambda)\) as \(\lambda \in C^2(\Omega)\)). Then for \(u \in \mathcal{D}(\partial_{q+1}^*)\) it follows that

\[
F \left( \sqrt{\tau} \partial^*_\lambda u \right) = (u_1, \alpha)_\lambda.
\]
The Cauchy–Schwarz inequality, followed by (4.4), yields
\[ |F(\sqrt{\tau}u)| \leq \|u\|_2 \cdot \|\alpha\| \leq \frac{1}{\sqrt{c_1}} \left( \int_{\Omega} \Theta_{\lambda,\tau}(u_1, u_1) e^{-\lambda} dV \right)^{1/2} \cdot \|\alpha\|. \]

Note that \( u_2 \perp_{\lambda} \mathcal{N}(\bar{\partial}_{q+1}) \) implies that \( u_2 \in \mathcal{D}(\bar{\partial}_{q+1}) \) by definition of the latter space (see (2.4)). Hence \( u_1 \in \mathcal{D}(\bar{\partial}_{q+1}) \) and (4.2) holds for \( u_1 \). Therefore
\[ |F(\sqrt{\tau}u)| \leq \sqrt{\frac{2}{c_1}} \|\sqrt{\tau}u\|_\lambda \cdot \|\alpha\|. \]

The fact that \( u_2 \perp_{\lambda} \mathcal{N}(\bar{\partial}_{q+1}) \) entails
\[ \|\sqrt{\tau}u\|_\lambda = \|\sqrt{\tau}u\|_\lambda, \]
so that
\[ |F(\sqrt{\tau}u)| \leq \sqrt{\frac{2}{c_1}} \|\sqrt{\tau}u\|_\lambda \cdot \|\alpha\|. \]

That is, \( F \) is a bounded linear functional on \( \{\sqrt{\tau}u : u \in \mathcal{D}(\bar{\partial}_{q+1})\} \) which is a linear subspace of \( L^2_{0,q}(\Omega) \). The Hahn–Banach theorem says that \( F \) may be extended to a linear functional on \( L^2_{0,q}(\Omega) \), still denoted \( F \), with the same bound. The Riesz representation theorem then yields a unique \( w \in L^2_{0,q}(\Omega) \) satisfying \( F(g) = (g, w)_\lambda \) for all \( g \in L^2(\Omega) \) and
\[ \|w\|_\lambda \leq \sqrt{\frac{2}{c_1}} \|\alpha\|. \]

In particular,
\[ (\sqrt{\tau}u, w)_\lambda = (u, \alpha)_\lambda \quad \forall \ u \in \mathcal{D}(\bar{\partial}_{q+1}). \]

Since \( \mathcal{D}(\bar{\partial}_{q+1}) \) contains \( \Lambda^0_{0,q+1}(\Omega) \) which is dense in \( L^2_{0,q}(\Omega) \), \( \bar{\partial}(\sqrt{\tau}w) = \alpha \) follows. Moreover, setting \( v = \sqrt{\tau}w \) yields \( \bar{\partial}_q v = \alpha \) with the estimate
\[ c_2 \int_{\Omega} |v|^2 dV \leq \int_{\Omega} |v|^2 e^{-\lambda} dV \leq \frac{2}{c_1} \int_{\Omega} |\alpha|^2 dV. \]

It remains to show that (4.3) holds. For that, let \( f \in \mathcal{D}(\bar{\partial}) \cap \overline{\mathcal{D}(\bar{\partial}_{q+1})} \) be given. The above derivation yields a \( v \in L^2_{0,q}(\Omega) \) such that \( \bar{\partial}_q v = \bar{\partial}_q f \) and \( \|v\| \leq \sqrt{\frac{2}{c_1 c_2}} \|\bar{\partial}_q f\| \).

Since
\[ v - f \in \mathcal{N}(\bar{\partial}) = \overline{\mathcal{D}(\bar{\partial}_{q+1})} ^\perp, \]
it follows that \( f \perp (v - f) \). Hence
\[ \|f\|^2 \leq \|f\|^2 + 2Re(v - f, f) + \|v - f\|^2 = \|v\|^2 \leq \frac{2}{c_1 c_2} \|\bar{\partial}_q f\|^2, \]
which proves (4.5). \( \square \)

There are several ways that the pair of inequalities, (4.4) and \( c_2 := \min\{e^{-\lambda(z)}/\tau(z) : z \in \overline{\Omega}\} > 0 \), can be achieved. We isolate two special cases that are amenable to application. The first is related to the classical notion of hyperbolicity:
Corollary 4.6. If $\Omega \subseteq \mathbb{C}^n$ is a pseudoconvex domain with smooth boundary, $0 \leq q \leq n - 1$, and there exists a $\phi \in C^2(\Omega)$ and constants $A, B > 0$ such that

(i) $|\phi(z)| \leq A$ for all $z \in \Omega$,
(ii) $i\partial \bar{\partial} \phi(u, u) \geq B|u|^2$ for all $u \in \Lambda^{0,q+1}(\Omega)$,

then $\bar{\partial}^q$ has closed range in $L^2_{0,q+1}(\Omega)$.

Moreover, the constant $C$ in (ii) of Proposition 3.1 may be taken to be $e^A \sqrt{2/B}$.

Proof. Let $\tau = 1$ and $\lambda = \phi$ in Proposition 4.3.

For the second special case, we reformulate a definition from [10]:

Definition 4.7. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $0 \leq q \leq n - 1$. Say that $\psi \in C^2(\Omega)$ has a self-bound of $K$ on its complex gradient at level $(0,q)$ if

$$i\partial \psi \wedge \bar{\partial} \psi(u, u) \leq K^2 i\partial \bar{\partial} \psi(u, u) \quad \forall u \in \Lambda^{0,q+1}(\Omega).$$

Abbreviate (4.8) by writing $|\partial \psi|_{i\partial \bar{\partial} \psi} \leq K$ if the form level $(0,q)$ is understood.

Remark 4.9. If $q = 0$, this definition is given in [10] and $\psi$ is simply said to have self-bounded gradient $\leq K$. The abbreviated notation $|\partial \psi|_{i\partial \bar{\partial} \psi}$ in that case corresponds to the natural length measurement of the $(1,0)$-form $\partial \psi$ in the Hermitian metric associated to $i\partial \bar{\partial} \psi$. Definition 4.7 is given to avoid combinatorial constants when $q > 0$, in order to focus on the size of $C$ in Corollary 4.10 below.

Corollary 4.10. If $\Omega \subseteq \mathbb{C}^n$ is a pseudoconvex domain with smooth boundary, $0 \leq q \leq n - 1$, and there exists a $\psi \in C^2(\Omega)$ and constants $D, E > 0$ such that

(i) $|\partial \psi(z)|_{i\partial \bar{\partial} \psi} \leq D$ for all $z \in \Omega$,
(ii) $i\partial \bar{\partial} \psi(u, u) \geq E|u|^2$ for all $u \in \Lambda^{0,q+1}(\Omega)$,

then $\bar{\partial}^q$ has closed range in $L^2_{0,q+1}(\Omega)$.

Moreover, the constant $C$ in (ii) of Proposition 3.1 may be taken to be $\sqrt{2D/E}$.

Proof. Set $\tau = e^{-\alpha \psi}$ and $\lambda = \alpha \psi$ in Proposition 4.3 for a constant $\alpha > 0$ to be determined. Note that for any $\alpha$, $c_2 := \min\{e^{-\lambda(z)}/\tau(z) : z \in \Omega\} = 1$.

For these choices of $\tau, \lambda$, a straightforward computation gives

$$\Theta_{\lambda, \tau} \geq e^{-\alpha \psi} \frac{E^2}{D}|u|^2$$

Without trying to extract the sharpest lower bound, merely choose $\alpha = E/2D$. Then it follows from (4.11) that

$$\Theta_{\lambda, \tau}(u, u) \geq e^{-\alpha \psi} \frac{E^2}{D}|u|^2 \quad \forall u \in \Lambda^{0,q+1}(\Omega),$$

and Proposition 4.3 completes the proof, with the claimed constant in (ii) of Proposition 3.1.

5. Approximating Subsets

The following theorem contains the basic approximation result of pseudoconvex domains:

Theorem 5.1. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Then there exist a sequence $\{\Omega_j\}$ of open subsets of $\mathbb{C}^n$ such that
(i) $\Omega_j \subset \Omega_{j+1} \in \Omega$ for all $j \geq 1$,
(ii) $\bigcup_{j \geq 1} \Omega_j = \Omega$,
(iii) for each $j \in \mathbb{N}$ there exists an $m_j \in \mathbb{N}$ such that $\Omega_j$ is the disjoint union of smoothly bounded, Levi pseudoconvex domains $\Omega^k_j$, $1 \leq k \leq m_j$.

**Proof.** Since $\Omega$ is pseudoconvex there exists a strictly plurisubharmonic exhaustion function $\Psi \in C^\infty(\Omega)$, see (2.6) and the subsequent remark. It follows from the proof of Proposition 2.21 in Chapter II of [11] that for some $R$-linear function $\ell(z)$, the function
$$r(z) := \Psi(z) + |z|^2 + \ell(z)$$
is a smooth, strictly plurisubharmonic exhaustion function whose set of critical points on $\Omega$ is discrete. The latter fact, together with $r \in C^\infty(\Omega)$ and the boundedness of $\{z \in \mathbb{C}^n : r(z) < c\}$ for any $c \in \mathbb{R}$, implies that for any $j \in \mathbb{N}$ there is a $j^* \in (j - 1/2, j + 1/2)$ such that $\nabla r(z) \neq 0$ whenever $r(z) = j^*$. Set
$$\Omega_j = \{z \in \mathbb{C}^n : r(z) < j^*\}.$$Both (i) and (ii) then follow straightforwardly from the fact that $r$ is an exhaustion function.

Since each critical point of $r$ is isolated, it follows that any convergent sequence $\{x_n\}$ of critical points satisfies $\lim_{n \to \infty} |x_n| = \infty$. Therefore, any bounded subset of $\Omega$ contains finitely many critical points of $r$. This implies that $\Omega_j$ has finitely many connected components: $\Omega^k_j$ for $1 \leq k \leq m_j$. The fact that $\nabla r \neq 0$ on $b\Omega_j$ implies that the intersection of the closures of any two components of $\Omega_j$ are mutually disjoint. It also implies that $b\Omega^k_j$ has smooth boundary for $1 \leq k \leq m_j$. That all $\Omega^k_j$ are Levi pseudoconvex now follows from $r$ being a strictly plurisubharmonic function in $\Omega$. □

The sets $\{\Omega_j\}$ described in Theorem 5.1 will be called a sequence of approximating subsets for $\Omega$.

6. $\bar{\partial}_q$ ON GENERAL PSEUDOCONVEX DOMAINS

Proposition 4.3 is stated for bounded pseudoconvex domains with smooth boundary and functions $\lambda, \tau \in C^2(\Omega)$ because its proof hinges on Proposition 4.1; this result requires these hypotheses. Extending (4.2) to unbounded or non-smooth domains, and to $\lambda, \tau$ not necessarily smooth up to $b\Omega$, requires dealing with density issues between $D^{0,q+1}(\Omega)$ and $\mathcal{D}(\bar{\partial}_q)^* \cap \mathcal{D}(\bar{\partial})$. An extension of this kind would be delicate and would not be universally valid (it would depend on the exact lack of smoothness or unboundedness of the data).

The closed range inequality, (3.2), is less delicate. We show the conclusion of Proposition 4.3 holds on a general pseudoconvex domain $\Omega$, if $\Omega$ admits functions $\lambda, \tau \in C^2(\Omega)$ satisfying the hypotheses of Proposition 4.3 uniformly. The additional ingredient is the following

**Lemma 6.1.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Let $\{\Omega_j\}$ be a sequence of approximating subsets for $\Omega$. Suppose there exists a constant $C > 0$ such that for all
α_j \in L^2_0(q+1)(Omega_j) \cap \mathcal{N}(\bar{partial}_q+1) there exists a v_j \in L^2_0(q)(Omega_j) \cap (\mathcal{N}(\bar{partial}_q))^\perp such that \bar{partial}_q v_j = \alpha_j on Omega_j in the distributional sense and

\begin{equation}
\|v_j\|_{L^2_0(q)(Omega_j)} \leq C \|\alpha_j\|_{L^2_0(q+1)(Omega_j)}.
\end{equation}

Then \bar{partial}_q has closed range on Omega.

Recall that Omega_j = \bigcup_{\ell=1}^{m_j} Omega_j. So \alpha_j \in L^2(\Omega_j) is to mean that \alpha_j = \alpha^1_j + \cdots + \alpha^{m_j}_j for \alpha_j^e \in L^2(\Omega^e_j) and \|\alpha_j\|^2_{L^2_0(q)(Omega_j)} = \sum_{\ell=1}^{m_j} \|\alpha_j^e\|^2_{L^2_0(q)(Omega^e_j)}. Similarly, \bar{partial}_q on L^2_0(q)(Omega_j) is the direct sum of the \bar{partial}_q-operators associated to the Omega^e_j's.

**Proof.** It will be shown that

\begin{equation}
\|u\|_{L^2_0(q)(Omega)} \leq C \|ar{partial}_q u\|_{L^2_0(q+1)(Omega)} \quad \forall \ u \in \mathcal{D}(\bar{partial}_q) \cap (\mathcal{N}(\bar{partial}_q))^\perp.
\end{equation}

Suppose u \in \mathcal{D}(\bar{partial}_q) \cap (\mathcal{N}(\bar{partial}_q))^\perp. By Definition of \mathcal{D}(\bar{partial}_q), see (\ref{2.3}), there exists a sequence \{u_j\} \subset L^0_q(\Omega) such that u_j \rightharpoonup u in L^2_0(q)(Omega) and \bar{partial}_q u_j \rightharpoonup \bar{partial}_q u in L^2_0(q+1)(Omega) as j approaches \infty.

By assumption, there exists a sequence \{v_j\} \subset L^2_0(q)(Omega) \cap (\mathcal{N}(\bar{partial}_q))^\perp such that \bar{partial}_q v_j = \bar{partial}_q u_j on Omega_j and a constant C > 0 such that

\begin{equation}
\|v_j\|_{L^2_0(q)(Omega_j)} \leq C \|ar{partial}_q u_j\|_{L^2_0(q+1)(Omega_j)}.
\end{equation}

Now start estimating \|u\|^2_{L^2_0(q)(Omega)}:

\begin{align}
\|u\|^2_{L^2_0(q)(Omega)} &\leq \|u\|^2_{L^2_0(q)(Omega_j)} + \|u\|^2_{L^2_0(q)(\Omega \setminus Omega_j)} \\
&\leq \|u - u_j\|^2_{L^2_0(q)(Omega_j)} + \|u - u_j\|^2_{L^2_0(q)(\Omega \setminus Omega_j)} + \|u\|^2_{L^2_0(q)(\Omega \setminus Omega_j)} \\
&\leq \|u - u_j\|^2_{L^2_0(q)(Omega_j)} + \|u - v_j\|^2_{L^2_0(q)(Omega_j)} + \|v_j\|^2_{L^2_0(q)(Omega_j)} + \|u\|^2_{L^2_0(q)(\Omega \setminus Omega_j)}
\end{align}

Inequality (6.3) gives

\begin{equation}
\|v_j\|_{L^2_0(q)(Omega_j)} \leq C \|ar{partial}_q u_j\|_{L^2_0(q+1)(Omega_j)} \leq C \left( \|\bar{partial}_q u\|_{L^2_0(q+1)(Omega)} + \|\bar{partial}_q u - \bar{partial}_q u_j\|_{L^2_0(q+1)(Omega_j)} \right).
\end{equation}

Hence, for any \epsilon > 0 there is a j^* such that for all j \geq j^* (6.5) becomes

\begin{equation}
\|u\|_{L^2_0(q)(Omega)} \leq \epsilon + C \|ar{partial}_q u\|_{L^2_0(q+1)(Omega)} + 2 \|u - v_j\|_{L^2_0(q)(Omega_j)}.
\end{equation}

To conclude the proof, it suffices to show that \{u_j - v_j\} has a subsequence \{u_{j_k} - v_{j_k}\} such that \|u_{j_k} - v_{j_k}\|_{L^2_0(q)(\Omega_{j_k})} converges to 0 as j_k \to \infty. Notice first that v_j \perp \mathcal{N}(\bar{partial}_q) on L^2_0(q)(\Omega_j) while u_j - v_j \in \mathcal{N}(\bar{partial}_q) on L^2_0(q)(\Omega_j). Therefore,

\begin{align}
\|u_j - v_j\|^2_{L^2_0(q)(Omega_j)} &= \langle u_j - v_j, u_j \rangle_{L^2_0(q)(Omega_j)} \\
&\leq |\langle u_j - v_j, u - u_j \rangle|_{L^2_0(q)(Omega_j)} + |\langle u_j - v_j, u - u_j \rangle_{L^2_0(q)(Omega_j)}| \\
&\leq \|u - u_j\|_{L^2_0(q)(Omega_j)} \cdot \|u - u_j\|_{L^2_0(q)(Omega_j)} + \|u\|_{L^2_0(q)(Omega_j)}.
\end{align}

Since u \in L^2_0(q)(Omega) and u_j \rightharpoonup u in L^2_0(q)(Omega), it follows that \|u_j - v_j\|_{L^2_0(q)(Omega_j)} is uniformly bounded in j. Let \tilde{u}_j = u_j and \tilde{v}_j = v_j on Omega_j and \tilde{u}_j = 0 = \tilde{v}_j on Omega^e_j. Then \{\tilde{u}_j - \tilde{v}_j\}
is a bounded sequence in $L^2_{0,q}(\Omega)$. Hence, it has a subsequence $\{\tilde{u}_{j_k} - \tilde{v}_{j_k}\}_{j_k}$ which is weakly convergent. That is, for all $g \in L^2_{0,q}(\Omega)$

$$\langle \tilde{u}_{j_k} - \tilde{v}_{j_k}, g \rangle_{L^2_{0,q}(\Omega)} \to 0 \text{ as } j_k \to \infty,$$

in particular $(u_{j_k} - v_{j_k}, g)_{L^2_{0,q}(\Omega)} \to 0$ as $j_k \to \infty$. Analogous to the arguments in (6.6) we obtain

$$(6.7) \quad \|u_{j_k} - v_{j_k}\|_{L^2_{0,q}(\Omega)} \leq \|u_{j_k} - v_{j_k}\|_{L^2_{0,q}(\Omega)} \cdot \|u - u_{j_k}\|_{L^2_{0,q}(\Omega)} + \|(u_{j_k} - v_{j_k}, u)\|_{L^2_{0,q}(\Omega)}.$$ 

The last term of the right-hand side tends to 0 as $j_k \to \infty$ by the arguments preceding (6.7). The first term of the right-hand side of (6.7) also converges to 0 as $j_k \to \infty$ since its first factor is uniformly bounded while the second one goes to 0. Hence $\|u_{j_k} - v_{j_k}\|_{L^2_{0,q}(\Omega)}$ goes to 0 as $j_k \to \infty$.

Repeating the arguments, starting at (6.5), with $u_{j_k}$ and $v_{j_k}$ in place of $u_j$ and $v_j$, respectively, completes the proof.

Our main result, essentially Proposition 4.3 under relaxed hypotheses, follows easily:

**Theorem 6.8.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $0 \leq q \leq n-1$. Suppose $\Omega$ admits functions $\lambda, \tau \in C^2(\Omega)$ and constants $c_1, c_2 > 0$ such that

(i) $\Theta_{\lambda, \tau}(u, u) \geq c_1 |u|^2 e^{-\lambda} \quad \forall u \in \Lambda^0, d+1(\Omega)$

(ii) $\min \{e^{-\lambda(z)} / \tau(z) : z \in \Omega\} \geq c_2$,

then $\bar{\partial}_q$ has closed range in $L^2_{0,q+1}(\Omega)$.

**Proof.** Let $\{\Omega_j\}$ be the sequence of approximating subsets for $\Omega$ given by Theorem 5.1. Note that $\lambda, \tau \in C^2(\Omega_j)$. Proposition 4.3 applies, giving a uniform $C$ such that (6.2) holds. Lemma 6.1 completes the proof.

Generalized versions of Corollaries 4.6 and 4.10 follow as before:

**Corollary 6.9.** If $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain, $0 \leq q \leq n-1$, and there exists a $\phi \in C^2(\Omega)$ and constants $A, B > 0$ such that

(i) $|\phi(z)| \leq A$ for all $z \in \Omega$,

(ii) $i\partial \bar{\partial} \phi(u, u) \geq B |u|^2$ for all $u \in \Lambda^0, d+1(\Omega)$,

then $\bar{\partial}_q$ has closed range in $L^2_{0,q+1}(\Omega)$.

**Corollary 6.10.** If $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain, $0 \leq q \leq n-1$, and there exists a $\psi \in C^2(\Omega)$ and constants $D, E > 0$ such that

(i) $|\partial \bar{\partial} \psi(z)| \leq D$ for all $z \in \Omega$,

(ii) $i\partial \bar{\partial} \psi(u, u) \geq E |u|^2$ for all $u \in \Lambda^0, d+1(\Omega)$,

then $\bar{\partial}_q$ has closed range in $L^2_{0,q+1}(\Omega)$. 
7. Examples

7.1. Dimension 1. If \( z \in \mathbb{C} \) and \( L > 0 \), let \( \mathbb{D}(z, L) \) denote the disc centered at \( z \) of radius \( L \).

**Definition 7.1.** (i) A domain \( \Omega \subset \mathbb{C} \) is said to not contain arbitrarily large discs if there exists an \( L > 0 \) such that

\[
\mathbb{D}(z, L) \cap \overline{\Omega} \neq \emptyset \quad \forall \ z \in \Omega.
\]

(ii) A domain \( \Omega \subset \mathbb{C} \) is said to satisfy condition \( \mathcal{X} \) if there exist an \( M > 0 \) and a \( \delta > 0 \) such that for all \( z \in \Omega \) there exists a \( z^* \in \mathbb{D}(z, M) \cap \overline{\Omega} \) such that the distance of \( z^* \) to \( \overline{\Omega} \) is greater than \( \delta \).

Note that condition \( \mathcal{X} \) in (ii) above implies that (7.2) holds with \( M \) in place of \( L \). Hence, condition \( \mathcal{X} \) is only satisfied by domains which do not contain arbitrarily large discs.

**Lemma 7.3.** Let \( \Omega \subset \mathbb{C} \) be a domain. If \( \partial_0 \) has closed range on \( L^2_{0,1}(\Omega) \), then \( \Omega \) cannot contain arbitrarily large discs.

**Proof.** Suppose \( \Omega \) contains arbitrarily large discs. Then there exists a sequence \( \{z_j\}_j \subset \Omega \) such that \( \mathbb{D}(z_j, j) \subset \Omega \) for all \( j \in \mathbb{N} \). Let \( \alpha \in C_c^\infty(\mathbb{D}(0, 1)) \) be not identically zero. Set \( \alpha_j = \alpha((z - z_j)/j) \), so that \( \alpha_j \in C_c^\infty(\mathbb{D}(z_j, j)) \). Hence \( \alpha_j \in \mathcal{D} (\partial^*) \). Set \( u_j = \partial^* \alpha_j \). Then \( u_j \in \mathcal{D}(\partial) \) and

\[
\|u_j\|_{L^2(\Omega)}^2 = \|\partial^* \alpha_j\|_{L^2(\mathbb{D}(z_j, j))}^2 = \frac{1}{j^2} \int_{\mathbb{D}(z_j, j)} \left| (\partial^* \alpha) \left( \frac{z - z_j}{j} \right) \right|^2 \, dV(z) = \int_{\mathbb{D}(0, 1)} \left| (\partial^* \alpha)(z) \right|^2 \, dV(z) = \|\partial^* \alpha\|_{L^2(\mathbb{D}(0, 1))}^2.
\]

Furthermore,

\[
\|\partial u_j\|_{L^2(\Omega)}^2 = \frac{1}{j^4} \int_{\mathbb{D}(z_j, j)} \left| (\partial \partial^* \alpha) \left( \frac{z - z_j}{j} \right) \right|^2 \, dV(z) = \frac{1}{j^2} \|\partial \partial^* \alpha\|_{L^2(\mathbb{D}(0, 1))}^2.
\]

Since \( 0 < k_1 \leq \|\partial^* \alpha\|, \|\partial \partial^* \alpha\| < k_2 \) for constants independent of \( j \), it follows that there is no constant \( C > 0 \) such that

\[
\|u_j\|_{L^2(\Omega)} \leq C \|\partial u_j\|_{L^2(\Omega)}
\]

holds. Thus \( \partial \) does not have closed range on \( L^2_{0,1}(\Omega) \). \( \square \)

**Proposition 7.4.** Let \( \Omega \subset \mathbb{C} \) be a domain. If \( \Omega \) satisfies condition \( \mathcal{X} \), then \( \partial_0 \) has closed range on \( L^2_{0,1}(\Omega) \).

We prove Proposition 7.4 by constructing a function \( \phi \) in Lemma 7.6 satisfying the hypotheses of Corollary 6.9. For this construction, a perturbation of a lattice in a neighborhood of the closure of \( \Omega \) is used.

**Lemma 7.5.** Let \( \Omega \subset \mathbb{C} \) be a domain satisfying condition \( \mathcal{X} \). Then there exist constants \( M, \delta > 0 \) and \( \Lambda \subset (M\mathbb{Z})^2 \) such that
(a) \( \mathbb{D}(w,M) \cap \bar{\Omega} \neq \emptyset \neq \mathbb{D}(w,M) \cap \Omega \) for all \( w \in \Lambda \),
(b) \( \Omega \subset \bigcup_{w \in \Lambda} \mathbb{D}(w,M) \),
(c) for each \( w \in \Lambda \) there exists a \( w^* \in \bar{\Omega} \) satisfying
   (i) the distance of \( w^* \) to \( \bar{\Omega} \) is greater than \( \delta \),
   (ii) \( \mathbb{D}(w,M) \subset \mathbb{D}(w^*,3M) \).

Proof of Lemma 7.5. Let \( M, \delta > 0 \) be as in part (ii) of Definition 7.1. Set \( \Lambda = \{ w \in (M\mathbb{Z})^2 : \mathbb{D}(w,M) \cap \Omega \neq \emptyset \} \). Then (a) follows.

For each \( z \in \Omega \) there is a \( w_0 \in (M\mathbb{Z})^2 \) contained in \( \mathbb{D}(z,M) \). Hence \( z \in \mathbb{D}(w_0,M) \).
This means in particular that \( \mathbb{D}(w_0,M) \cap \bar{\Omega} \neq \emptyset \). Thus \( w_0 \in \Lambda \) and (b) follows.

Let \( w \in \Lambda \). Assume first that \( w \in \Omega \). Since \( \Omega \) satisfies condition \( \mathcal{X} \), there exists a \( w^* \in \mathbb{D}(w,M) \) such that the distance of \( w^* \) to \( \bar{\Omega} \) is greater than \( \delta \). Moreover, \( \mathbb{D}(w,M) \subset \mathbb{D}(w^*,2M) \). That is, both (i) and (ii) hold in this case. Now assume that \( w \notin \Omega \). Then, since \( w \in \Lambda \), there exists a \( z \in \Omega \) contained in \( \mathbb{D}(w,M) \). By assumption there is a \( w^* \in \mathbb{D}(z,M) \cap \bar{\Omega} \) whose distance to \( \bar{\Omega} \) is greater than \( \delta \). If \( y \in \mathbb{D}(w,M) \) then
\[
|y-w^*| \leq |y-w| + |w-z| + |z-w^*| \leq 3M,
\]
i.e., (ii) holds in this case. This concludes the proof. \( \square \)

Lemma 7.6. Let \( \Omega \subset \mathbb{C} \) be a domain which satisfies condition \( \mathcal{X} \). Then there exists a \( \phi \in \mathcal{C}^2(\Omega) \) and constants \( A, B > 0 \) such that
   (i) \( |\phi(z)| \leq A \) for all \( z \in \Omega \),
   (ii) \( i\partial\bar{\partial}\phi(u,u) \geq B|u|^2 \) for all \( u \in \Lambda_0^{q+1}(\Omega) \).

Proof. Let \( M, \delta > 0 \) be as in (ii) of Definition 7.1 and \( \Lambda \subset (M\mathbb{Z})^2 \) as in Lemma 7.5. Write \( w_{\ell,k} = \ell M + ikM \) for \( \ell, k \in \mathbb{Z} \). For each \( w_{\ell,k} \in \Lambda \) choose a \( w^*_{\ell,k} \in \bar{\Omega} \) as described in (c) of Lemma 7.5 set \( \Lambda^* = \{ w^*_{\ell,k} : w_{\ell,k} \in \Lambda \} \). It will be shown that
\[
(7.7) \quad \sum_{w^*_{\ell,k} \in \Lambda^*} |z - w^*_{\ell,k}|^{-4}
\]
converges to a function, \( \phi(z) \), of class \( \mathcal{C}^\infty \) on \( \bar{\Omega} \) satisfying conditions (i) and (ii).

Given \( \gamma \in \mathbb{N} \) and \( p = p_1 + ip_2 \in \mathbb{C} \) denote by \( Q_{\gamma M}^p \) the set bounded by the square with vertices \( (\pm \gamma M + p_1) + i(\pm \gamma M + p_2) \) and \( Q_{0M}^0 = \{ p \} \).

Fix \( z \in \bar{\Omega} \). Then there is a \( w_{\ell_0,k_0} \in \Lambda \) such that \( z \in \mathbb{D}(w_{\ell_0,k_0},M) \). To show convergence of the series (7.7) at \( z \), let us sum over the sets
\[
\mathcal{A}^w_{0M} := \Lambda^* \cap Q_{0M}^{w_{0,k_0}}, \quad \text{and} \quad \mathcal{A}^{w_{0,k_0}}_{\gamma M} := \Lambda^* \cap \left( Q_{\gamma M}^{w_{0,k_0}} \setminus Q_{(\gamma-1)M}^{w_{0,k_0}} \right)
\]
for \( \gamma \in \mathbb{N} \).

To compute the cardinality of \( \mathcal{A}^{w_{0,k_0}}_{\gamma M} \), note first that
\[
|\Lambda \cap Q_{\gamma M}^{w_{l,k}}| \leq |Q_{\gamma M}^{w_{l,k}}| = (2\gamma + 1)^2 .
\]
Moreover, it follows from (ii) of part (c) of Lemma 7.5 that
\[
|\Lambda^* \cap Q_{\gamma M}^{w_{0,k_0}}| \leq |\Lambda^* \cap Q_{(\gamma+3)M}^{w_{0,k_0}}| \leq (2\gamma + 7)^2 \quad \forall \gamma \in \mathbb{N},
\]
and
\[
\left| \Lambda^* \cap Q_{(\gamma-1)M}^{\ell,k_0} \right| \geq \left| \Lambda^* \cap Q_{(\gamma-4)M}^{\ell,k_0} \right| \geq (2\gamma - 7)^2 \quad \forall \gamma \geq 4.
\]
Therefore
\[
|A_{\gamma M}^{\ell,k_0}| \leq \begin{cases} 
(2\gamma + 7)^2 & \text{for } 0 \leq \gamma \leq 3 \\
(2\gamma + 7)^2 - (2\gamma - 7)^2 & \text{for } \gamma \geq 4
\end{cases}.
\]
Hence
\[
0 \leq \sum_{\gamma=4}^{\infty} \sum_{w_{\ell,k}^{\bullet} \in A_{\gamma M}^{\ell,k_0}} |z - w_{\ell,k}^{\bullet}|^{-4} \leq \frac{56}{M^4} \sum_{\gamma=4}^{\infty} \frac{1}{\gamma^3};
\]
which implies that the series \( \sum_{\ell,k} |z - w_{\ell,k}^{\bullet}|^{-4} \) converges absolutely to some scalar, \( \phi(z) \), at \( z \). In fact, the convergence of the series to \( \phi \) is uniform on \( \Omega \). Therefore \( \phi \) is continuous on \( \Omega \). Similarly, since any \( k \)-th derivative of \( |z - w_{\ell,k}^{\bullet}|^{-4} \) is of order \( O(|z - w_{\ell,k}^{\bullet}|^{-(4+k)}) \), it follows that \( \phi \in C^\infty(\Omega) \) and derivatives of \( \phi \) may be computed term by term. The latter implies that
\[
\phi_{zz}(z) \geq 4 |z - w_{\ell,k}^{\bullet}|^{-6} \geq \frac{4}{(3M)^6} =: B,
\]
where \( w_{\ell,k} \) is such that \( z \in \mathbb{D}(w_{\ell,k}, M) \cap \Omega \) (see (c) of Lemma 7.5). Thus (ii) is shown to hold for \( \phi \). That \( \phi \) is bounded on \( \Omega \) also follows from (7.8):
\[
\phi(z) \leq 49\delta^{-4} + \frac{1}{M^4} \left( \sum_{\gamma=1}^{3} \frac{(2\gamma + 7)^2}{\gamma^4} \sum_{\gamma=4}^{\infty} \frac{1}{\gamma^3} \right) =: A,
\]
i.e., (i) holds for \( \phi \) as well. \( \square \)

7.1.1. A class of examples. Let \( \{c_j\}_{j \in \mathbb{Z}} \) be a strictly increasing sequence of real numbers such that \( \lim_{|j| \to \infty} |c_j| = \infty \) and \( \sup_{j \in \mathbb{Z}} (c_j - c_{j-1}) \leq M \) for some \( M > 0 \). Let \( \eta_j \in C^\infty(\mathbb{R}) \) for \( j \in \mathbb{Z} \) such that
\begin{itemize}
\item[(a)] \( \eta_{2j}(x) < c_j < \eta_{2j+1}(x) \) for all \( x \in \mathbb{R} \), \( j \in \mathbb{Z} \),
\item[(b)] \( \eta_{2j-1}(x) < \eta_{2j}(x) \) for all \( x \in \mathbb{R} \), \( j \in \mathbb{Z} \).
\end{itemize}
Define \( S_j = \{ z \in \mathbb{C} : \eta_{2j-1}(\operatorname{Re}(z)) < \operatorname{Im}(z) < \eta_{2j}(\operatorname{Re}(z)) \} \) for \( j \in \mathbb{Z} \), set \( S = \bigcup_{j \in \mathbb{Z}} S_j \). It is straightforward to check that \( S \) satisfies the following properties:
\begin{itemize}
\item[(i)] \( S \) is an open set with smooth boundary – however, \( S \) is not connected.
\item[(ii)] \( S \) does not contain arbitrarily large discs.
\item[(iii)] \( S \) satisfies condition \( \mathcal{X}^* \) if and only if there is a strictly increasing subsequence \( \{j_k\}_{k \in \mathbb{Z}} \) in \( \mathbb{Z} \) such that \( \lim_{|k| \to \infty} |j_k| = \infty \), \( \sup(c_{j_{k+1}} - c_{j_k}) \leq L \) for some \( L > 0 \), and the distance between \( S_{j_k} \) and \( S_{j_{k+1}} \) is uniformly bounded from below by some positive constant \( \delta > 0 \).
\end{itemize}

Lemma 7.9. There exists a \( \varphi \in C^2(S) \) satisfying conditions (i) and (ii) of Corollary 6.4.
Proof. For each $j \in \mathbb{Z}$ choose a real-valued, smooth function $\varphi_j$ with compact support in $\{z \in \mathbb{C} : c_j-1 < \text{Im}(z) < c_j\}$ such that $\varphi_j(z) = (\text{Im}(z) - c_j)^2$ for $z \in S_j$. Then each $\phi_j$ is non-negative and bounded by $M^2$ on $S$. Moreover, $\varphi$ is subharmonic on $S$ and $(\varphi_j(z))_{j \in \mathbb{Z}} \geq 1/2$ for $z \in S_j$. Since the $\varphi_j$s have disjoint support, it follows that $\varphi := \sum_{j \in \mathbb{Z}} \varphi_j$ is a smooth, bounded function on $S$ with $\varphi_{zz} \geq 1/2$ on $S$.

Remark 7.10. Examples of domains, satisfying (i) but not (ii) of Definition 7.1, for which $\partial_0$ has closed range may be easily constructed using sets $S$ described above. Let $S$ be such a set with the additional property

$$\eta_{2j+1}(x) - \eta_{2j}(x) > \kappa_1 \quad \text{for } 2 \leq |x| \leq 3$$

for some $\kappa_1 > 0$. Let $\Omega_S$ be a smoothly bounded domain with

(a) $\Omega_S \cap \{z \in \mathbb{C} : |\text{Re}(z)| > 2\} = S \cap \{z \in \mathbb{C} : |\text{Re}(z)| > 2\}$,

(b) $\{z \in \mathbb{C} : |\text{Re}(z)| < 1\} \subset \Omega_S$.

To show that $\partial_0$ for $\Omega_S$ has closed range, a bounded function $\psi \in C^2(\Omega_S)$ with $\psi_{zz} \geq B$ on $\Omega_S$ for some $B > 0$ may be constructed as follows:

- Let $\varphi$ be the function provided by Lemma 7.9 and $\chi \in C^\infty(\mathbb{R})$ with $\chi(x) = 0$ for $|x| \leq 2$ and $\chi(x) = 1$ for $|x| > 3$. Then $\chi \cdot \varphi$ is bounded and $\chi \cdot \varphi_{zz} \geq 1/2$ on $\Omega_S \cap \{z \in \mathbb{C} : |\text{Re}(z)| > 3\}$.

- Let $\phi$ be a function as constructed in Lemma 7.6 for the set $\{z \in \mathbb{C} : |\text{Re}(z)| < 2\}$ with $w_{k}^i \in \{z \in \mathbb{C} : S : 2 < |\text{Re}(z)| < 3\}$. Then $\phi$ is bounded and strictly subharmonic on $\Omega_S$. In particular, there is a constant $b > 0$ such that $\phi_{zz} \geq b$ on $\Omega_S \cap \{z \in \mathbb{C} : |\text{Re}(z)| \leq 3\}$.

Then, for sufficiently large $K > 0$, $\psi := \chi \cdot \varphi + K \cdot \phi$ satisfies $\psi_{zz} \geq B$ for some $B > 0$ on $\Omega_S$.

7.2. Dimension $n > 1$. An argument analogous to the one given in the proof of Lemma 7.3 yields that arbitrarily large poly-discs (of dimension $n$) are an obstruction to $\partial_0$ having closed range on $L^2_{0,1}(\Omega)$ for $\Omega \subset \mathbb{C}^n$. The example given in Lemma 7.11 however shows that this is not a necessary condition.

Lemma 7.11. Let $D \subset \mathbb{C}$ be a domain which contains arbitrarily large discs. Let $m \in \mathbb{N}$ and set

$$\Omega = \{(z,w) \in D \times \mathbb{C}^m : |w|^2 = |w_1|^2 + \cdots + |w_m|^2 < 1\}.$$

Then $\partial_0$ does not have closed range on $L^2_{0,1}(\Omega)$, i.e., there is no constant $C > 0$ such that

$$\|u\| \leq C\|\partial_0 u\| \quad \forall \ u \in \mathcal{D}(\partial_0) \cap \mathcal{D}(\partial_1^*).$$

Proof. Let $r(z,w) = |w|^2 - 1$ and $\alpha_1 \in C^\infty_c(D)$. Since $r(z,w) = 0$ when $(z,w) \in b\Omega$, it then follows that the form $\alpha := \alpha_1 d\bar{z}$ belongs to the domain of $\partial_1^*$ of $\Omega$. Set

$$u(z) := \partial_1^* \alpha = -\frac{\partial \alpha_1}{\partial z}(z) = \bar{\partial}^* \alpha_1(z),$$

hence $u \in C^\infty_c(D) \cap \mathcal{D}(\partial_1^*) \cap \mathcal{D}(\partial_0)$ (for the operators attached to both $\Omega$ and $D$). If (7.12) were to hold then

$$\|u\|_{L^2(\Omega)} \leq C\|\partial_0 u\|_{L^2_{0,1}(\Omega)}$$
must hold for all functions $u$ as described above. However,

$$
\|u\|_{L^2(\Omega)}^2 = \int_D |u(z)|^2 \left( \int_{|w|^2 < 1} 1 \, dV(w) \right) \, dV(z)
= c_m \int_D |u(z)|^2 dV(z).
$$

Hence, (7.13) becomes

$$
\|u\|_{L^2(D)} \leq \tilde{C} \|\bar{\partial}_0 u\|_{L^2_{0,1}(D)} \quad \forall \ u \in C^\infty_c(D) \cap \mathcal{D}(\bar{\partial}_0) \cap \mathcal{R}(\bar{\partial}^*_q),
$$

which contradicts Lemma 7.3. □

Remark 7.14. It follows from Corollary 4.6 (with $\phi = |w|^2$) and Lemma 6.1 that $\bar{\partial}_q$ for $1 \leq q \leq m$ has closed range on $L^2_{0,q+1}(\Omega)$.

8. Percolation of closed range

In this section, we show that closed range for $\bar{\partial}$ “percolates up” the Cauchy–Riemann complex. This is an elementary fact, as it turns out, but we note that other natural estimates connected to the $\bar{\partial}$-Neumann problem do not automatically percolate up the complex (see Remark 8.10). $\Omega$ is not assumed pseudoconvex, bounded, nor smoothly bounded in this section.

To connect estimates on forms at adjacent form levels, consider a variant of interior multiplication of forms. The simple construction below was observed by the second author in [9] and recorded in [13]. We present the construction non-invariantly, using coordinates, for clarity.

Let $u \in L^2_{0,q+1}(\Omega)$ be arbitrary and written as in (2.1). Define the forms $v_1, \ldots, v_n$ by the sum

$$(8.1) \quad v_m = \sum_{|H|=q} u_m H \, d\bar{z}^H.$$ 

Notice that

$$
\begin{aligned}
\bar{\partial}^* u &= \frac{1}{q+1} \sum_{m=1}^n \bar{\partial} \bar{z}_m \wedge v_m,
\end{aligned}
$$

because, for a fixed multi-index $I^0$ of length $q + 1$, $u_{\bar{z}} \, d\bar{z}^{I^0}$ appears $q + 1$ times in the sum $\sum d\bar{z}_m \wedge v_m$, once for each $i \in I^0$.

The basic properties of the association above are collected in the next Proposition.

Proposition 8.2. The following relationships hold for $u \in L^2_{0,q+1}(\Omega)$ and the associated $v_1, \ldots, v_n \in L^2_{0,q}(U)$ given by (8.1):

(i) If $u \in \mathcal{D}(\bar{\partial}^*_{q+1})$, then each $v_m \in \mathcal{D}(\bar{\partial}^*_q)$.

(ii) $\|u\|^2 = \frac{1}{q+1} \sum_{m=1}^n \|v_m\|^2$.

(iii) $\|\bar{\partial}^* u\|^2 = \frac{1}{q} \sum_{m=1}^n \|\bar{\partial}^* v_m\|^2$. 

(iv) If \( \beta = \sum_{|J|=q} \beta_J d\bar{z}^J \), then
\[
(d\bar{z}_m \wedge \beta, u) = (\beta, v_m) \quad \forall \ m \in \{1, \ldots, n\}.
\]

Proof. Property (ii) follows easily from equation (8.1):
\[
\sum_{m=1}^{n} \|v_m\|^2 = \sum_{m=1}^{n} \sum'_{|H|=q} \|u_{mH}\|^2 = (q + 1) \sum'_{|J|=q+1} \|u_J\|^2.
\]
Focus next on property (iv). Note that if \( \beta = \sum_{|J|=q} \beta_J d\bar{z}^J \), then
\[
d\bar{z}_m \wedge \beta = \sum_{|J|=q} \beta_J d\bar{z}_m \wedge d\bar{z}^J = \sum'_{|J|=q} \beta_J \epsilon_{mJ} d\bar{z}^{(mJ)}.
\]
Thus, for \( u = \sum'_{|J|=q+1} u_J d\bar{z}^J \),
\[
(d\bar{z}_m \wedge \beta, u) = \left( \sum'_{|J|=q} \beta_J \epsilon_{mJ} d\bar{z}^{(mJ)}, \sum'_{|J|=q+1} u_J d\bar{z}^J \right) = \sum'_{|J|=q} \int_{\Omega} \beta_J \bar{u}_{mJ} dV_E
\]
by definition of the inner product, see (2.2). This last expression equals \( (\beta, v_m) \), again by (2.2), which establishes (iv).

To see properties (i) and (iii), recall the definition of \( \mathcal{D}(\bar{\partial}_{q+1}) \) (see (2.4)), and let \( u \in \mathcal{D}(\bar{\partial}_{q+1}) \). Applying (iv) to \( \beta = \bar{\partial}a \), for \( a \in \mathcal{D}(\bar{\partial}_{q-1}) \) yields
\[
(\bar{\partial}a, v_m) = (d\bar{z}_m \wedge \bar{\partial}a, u) = -\left( \bar{\partial} (d\bar{z}_m \wedge a), u \right) = -\left( d\bar{z}_m \wedge a, \bar{\partial}^* u \right).
\]
It follows from (8.3) that \( |(\bar{\partial}a, v_m)| \leq C\|a\| \), so \( v_m \in \mathcal{D}(\bar{\partial}^*_a) \) and (i) holds.

Applying (iv) to the last expression in (8.3), this time for \( \beta = a \), yields
\[
(d\bar{z}_m \wedge a, \bar{\partial}^* u) = -\left( a, \sum'_{|H|=q-1} (\bar{\partial}^* u)_{mH} d\bar{z}^H \right).
\]
Equating this with the first term in (8.3) says that
\[
\bar{\partial}^* v_m = -\sum'_{|H|=q-1} (\bar{\partial}^* u)_{mH} d\bar{z}^H.
\]
Part (iii) follows easily from this. \( \square \)

Now consider the map \( \mathcal{D} : L^2_{0,q+1}(\Omega) \rightarrow L^2_{0,q}(\Omega) \otimes \cdots \otimes L^2_{0,q}(\Omega) \) defined by \( \mathcal{D}(u) = (v_1, \ldots, v_n) \) where \( v_m \) is derived from \( u \) according to (8.1). Norm the \( n \)-fold product as usual: if \( A \in L^2_{0,q}(\Omega) \otimes \cdots \otimes L^2_{0,q}(\Omega) \),
\[
\|A\| = \left( \sum_{m=1}^{n} \|a_m\|^2 \right)^{1/2},
\]
where \( A = (a_1, \ldots, a_n) \).
Remark 8.4. It follows from (iii) of Proposition 8.2 that $\mathcal{D}$ preserves $\mathcal{N}(\bar{\partial}^*)$, i.e., $u \in \mathcal{N}(\bar{\partial}^*_{q+1})$ if and only if $v_1, \ldots, v_n \in \mathcal{N}(\bar{\partial}^*_q)$. However, we point out that $u \in \mathcal{N}(\bar{\partial}^*_{q+1})^\perp$ does not imply that each component of $\mathcal{D}(u)$ is in $\mathcal{N}(\bar{\partial}^*_q)^\perp$.

Write $\mathcal{N}^{\otimes n}(\bar{\partial}^*_q)$ for the $n$-fold product $\mathcal{N}(\bar{\partial}^*_q) \otimes \cdots \otimes \mathcal{N}(\bar{\partial}^*_q)$ of $\mathcal{N}(\bar{\partial}^*_q)$. Although $\mathcal{D}$ does not map $\mathcal{N}(\bar{\partial}^*_{q+1})^\perp$ to $\mathcal{N}^{\otimes n}(\bar{\partial}^*_q)^\perp$, elements of $\mathcal{N}(\bar{\partial}^*_{q+1})^\perp$ are not mapped close to $\mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})$. This is essentially due to the injectivity of $\mathcal{D}$ (see (ii) of Proposition 8.2).

Proposition 8.5. For all $u \in \mathcal{D}(\bar{\partial}^*_{q+1})$

$$\text{dist}(\mathcal{D}(u), \mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})) \geq \sqrt{q + 1} \cdot \text{dist}(u, \mathcal{N}(\bar{\partial}^*_{q+1})).$$

Proof. The distance functions are the usual point-set distances in a normed space $Y$; i.e.,

$$\text{dist}(p, Y) = \inf \{\|p - y\| : y \in Y\},$$

where $\|\cdot\|$ is the norm on $Y$. Thus the distance function on the right-hand side of (8.6) involves the norm on $L^2_{0,q+1}(\Omega)$, while the one on the left-hand side involves the norm associated with $L^2_{0,q}(\Omega) \otimes \cdots \otimes L^2_{0,q}(\Omega)$.

Let $\delta = \text{dist}(\mathcal{D}(u), \mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D}))$. Since $\mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})$ is a closed subspace of the Hilbert space $L^2_{0,q}(\Omega) \otimes \cdots \otimes L^2_{0,q}(\Omega)$, there exists a unique $v = (v_1, \ldots, v_n) \in \mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})$ such that

$$\delta = \|\mathcal{D}(u) - (v_1, \ldots, v_n)\|.$$ 

By the injectivity of $\mathcal{D}$ and (i) of Proposition 8.2 there exists a unique $w \in \mathcal{N}(\bar{\partial}^*_{q+1})$ such that $\mathcal{D}(w) = v$. It then follows that

$$\delta = \|\mathcal{D}(u) - \mathcal{D}(w)\| = \|\mathcal{D}(u - w)\| \geq \sqrt{q + 1} \cdot \|u - w\| \geq \sqrt{q + 1} \cdot \text{dist}(u, \mathcal{N}(\bar{\partial}^*_{q+1})).$$

because of (ii) of Proposition 8.2 and (8.7). This completes the proof. □

It follows that closed range percolates up the Cauchy–Riemann complex:

Proposition 8.8. Let $\Omega$ be a domain in $\mathbb{C}^n$. If $\mathcal{R}(\bar{\partial}^*_q)$ is closed, then $\mathcal{R}(\bar{\partial}^*_{q+1})$ is closed.

Proof. Let $u \in \mathcal{D}(\bar{\partial}^*_{q+1})$; we want to show (3.4) holds. Let $\mathcal{D}(u) = (v_1, \ldots, v_n)$ as before. The assumption that $\mathcal{R}(\bar{\partial}^*_q)$ is closed implies

$$\|\bar{\partial}^* u\| = \frac{1}{\sqrt{q}} \left( \sum_{m=1}^{n} \|\bar{\partial}^* v_m\|^2 \right)^{1/2} \geq \frac{1}{\sqrt{qC}} \left( \sum_{m=1}^{n} \left[ \text{dist}(v_m, \mathcal{N}(\bar{\partial}^*_q)) \right]^2 \right)^{1/2}$$

$$= \frac{1}{\sqrt{qC}} \text{dist}(\mathcal{D}(u), \mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})).$$

(8.9)

But Proposition 8.5 says that

$$\text{dist}(\mathcal{D}(u), \mathcal{N}^{\otimes n}(\bar{\partial}^*_q) \cap \mathcal{R}(\mathcal{D})) \geq \sqrt{q + 1} \cdot \text{dist}(u, \mathcal{N}(\bar{\partial}^*_{q+1})).$$

Proposition 8.3 then implies that $\mathcal{R}(\bar{\partial}^*_{q+1})$ is closed. □
Remark 8.10. It is significant that Proposition 8.8 holds without the assumption that $\Omega$ is pseudoconvex. Percolation up the complex does not hold for the subelliptic estimate on general non-pseudoconvex domains, nor does it seem likely to hold for the compactness estimate. For example, take $\Omega$ a smoothly bounded domain in $\mathbb{C}^5$ whose Levi form has exactly 2 strictly positive and 2 strictly negative eigenvalues at each boundary point. Then the boundary of $\Omega$ satisfies condition $Z(1)$ and $Z(3)$, but not condition $Z(2)$; see [3] for the definition of condition $Z(q)$. It is known that condition $Z(q)$ implies the existence of subelliptic estimates on $(0,q)$-forms, thus a subelliptic estimate holds on this domain for $(0,1)$ and $(0,3)$-forms. But in the non-dengenerate case of no zero eigenvalues (such as $\Omega$), it is also known that $Z(q)$ is necessary for the existence of a subelliptic estimate, see [2]. Thus, a subelliptic estimate on $(0,2)$-forms does not hold. This example shows, incidentally, that subelliptic estimates do not percolate down the complex without a pseudoconcavity hypothesis as well.

That closed range, nevertheless, always percolates up gives curious examples of “existence without regularity” in the $\overline{\partial}$-Neumann problem.

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