Statistics of Various Geometric Aspects of Number Fields

Yuval Yifrach

5th April, 2022

Abstract

It was shown by M. Bhargava and P. Harron that for \( n = 3, 4, 5 \), the shapes of rings of integers of \( S_n \)-number fields of degree \( n \) become equidistributed in a certain homogeneous space when the fields are ordered by absolute discriminant. We revisit Bhargava-Harron’s argument and find the distribution in a certain torus bundle over the aforementioned homogeneous space while extending Bhargava-Harron’s result. The main novelty of the paper lies in the formulation of a more general equidistribution question and in the use of a high dimensional equidistribution result in the flavor of Weyl’s equidistribution Theorem.

1 Introduction

Let \( F \) be a Number field. Assume that \( F \) is of type \( S_n \), meaning that its associated Galois group is isomorphic to \( S_n \) - the group of symmetries on \( \{1, \ldots, n\} \). The field \( F \) comes with \( r \) real embeddings \( \sigma_1, \ldots, \sigma_r \) and \( s \) pairs of complex embeddings: \( \tau_1, \ldots, \tau_s, \bar{\tau}_1, \ldots, \bar{\tau}_s \) where \( r + 2s = n \) (an embedding of \( F \) is simply a field embedding of \( F \) inside \( \mathbb{C} \)). We use all the embeddings together to define \( \sigma : F \to \mathbb{R}^r \oplus \mathbb{C}^s \) to be \( (\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s) \). Let \( \mathcal{O}_F \) denote the Ring of Integers of \( F \), which can be defined as the maximal order in \( F \) and in particular is a full module in \( F \) (see \([1]\) for definitions). For the simplicity of this exposition, we assume the case \( s = 0 \). The image \( \sigma(\mathcal{O}_F) \subset \mathbb{R}^n \) is a lattice, which we denote by \( \Lambda(\mathcal{O}_F) \). Since \( s = 0 \), the vector \( \mathbf{T} = (1, \ldots, 1) \in \mathbb{R}^n \) is an element of \( \Lambda(\mathcal{O}_F) \), so we can define \( P\Lambda(\mathcal{O}_F) = P_{\mathbf{T}}(\Lambda(\mathcal{O}_F)) \) to be the orthogonal projection of \( \Lambda(\mathcal{O}_F) \) on \( V_0 := \mathbf{T}^\perp = \{ x \in \mathbb{R}^n : \langle x, \mathbf{T} \rangle = 0 \} \). The orbit of \( P\Lambda(\mathcal{O}_F) \) under the natural action of the group \( \mathbb{R}_+ \times SO_{n-1}(\mathbb{R}) \) is an element of \( S_{n-1} := SL_{n-1}(\mathbb{Z}) \setminus SL_{n-1}(\mathbb{R}) / SO_{n-1}(\mathbb{R}) \), denoted \( S_{n-1}(\mathbb{F}) \) and is called ‘the shape corresponding to \( F \)’. Our starting point is the main result of Bhargava and Harron in \([4]\):

**Theorem 1.1** (\([4]\), Theorem 1). Let \( n = 3, 4, 5 \). When isomorphism classes of type \( S_n \) number fields of degree \( n \) and any given signature are ordered by discriminant, the shapes corresponding to these fields become equidistributed in \( S_{n-1} \).
We say that a sequence \( \{x_n\}_{n \geq 0} \) in \( S_{n-1} \) is equidistributed if the weak limit of the normalized counting measure exists and is equal to the natural measure on \( S_{n-1} \) defined in Subsection 2.2. In Subsection 2.2 we describe a family of 'normalizations' of \( \Lambda(F) \) parametrized by 1 co-dimensional subspaces of \( \mathbb{R}^n \). For \( F \) as above and a co-dimension 1 subspace \( E \subset \mathbb{R}^n \), this normalization is denoted by \( \Gamma_E(F) \) and called 'the E-grid corresponding to \( F \). \( \Gamma_E(F) \) is then an element of a torus bundle over \( S_{n-1} \), denoted \( Y_{n-1} \) and called 'the space of \( n-1 \) dimensional grids' (see Subsection 2.2). As a consequence, the following Theorem is more general than Theorem 1.1. We prove in Subsection 3:

**Theorem 1.2.** Let \( n = 3, 4, 5 \) and \( \bar{T} \neq u \in \mathbb{R}^n \) such that \( \bar{T} \notin u^\perp \). When isomorphism classes of type \( S_n \) number fields of degree \( n \) are ordered by discriminant, the \( u^\perp \)-grids corresponding to these fields become equidistributed in \( Y_{n-1} \).

We say that a sequence \( \{x_n\}_{n \geq 0} \) in \( Y_{n-1} \) is equidistributed if the weak limit of the normalized counting measure exists and is equal to the natural measure on \( Y_{n-1} \) defined in Subsection 2.2. As we will see in Section 3, the subspace \( V_0 \) is related to intrinsic structural properties of \( F \), implying:

**Theorem 1.3.** Let \( n = 3, 4, 5 \) and \( E := V_0 \). Order isomorphism classes of type \( S_n \) number fields of degree \( n \) are by discriminant and take \( \mu \) be any weak limit of \( E \)-grids corresponding to these fields (\( \mu \) is a probability measure on \( Y_{n-1} \)). Then there exists a torsion measures of order \( n \) (see Definition 2.7) \( \eta \) such that \( \mu \ll \eta \).

The novelty of this normalization is that it carries more information on \( \Lambda(F) \) than \( S(F) \) does. More specifically, \( S(F) \) encodes the 'shape' of the projected lattice and not its 'position' in space, while \( \Gamma_E(F) \) encodes the position and also the bundle structure of the projected lattice. Another advantage of considering the corresponding grid is the case \( n = 2 \). While \( S_1 \) is one point, \( Y_1 \) is isomorphic to the unit circle \( S^1 \).

**Theorem 1.4.** Let \( n = 2 \) and \( \bar{T} \neq u \in \mathbb{R}^n \) such that \( \bar{T} \notin u^\perp \). When isomorphism classes of type \( S_n \) number fields of degree \( n \) are ordered by discriminant, the \( u^\perp \)-grids corresponding to these fields become equidistributed in \( Y_{n-1} \). When \( u = \bar{T} \), every weak limit of \( u^\perp \)-grids corresponding to these fields is absolutely continuous with respect to some torsion measure of order \( n \) in \( Y_{n-1} \).
2 The Ingredients

2.1 Notation

Given a natural number $n$, a ring of rank $n$ over another ring $R$ is a ring which is also a free $R$-module. Given a ring $R$, we denote $GL_n(R)$ to be the space of all invertible $n \times n$ matrices with entries from $R$. The subset of $GL_n(R)$ of all matrices with determinant 1 is denoted $SL_n(R)$. $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ denotes the standard inner product structure of $\mathbb{R}^n$. Given a linear subspace $E \subset \mathbb{R}^n$, we denote $E^\perp = \{ y \in \mathbb{R}^n : \langle y, x \rangle = 0 \text{ for all } x \in E \}$ and we denote $P_E : \mathbb{R}^n \to E$ to be the orthogonal projection onto $E$. $\text{Vol}_n$ denotes the standard Lebesgue measure on $\mathbb{R}^n$. Given linear subspaces $E_1, E_2 \subset \mathbb{R}^n$, we write $\mathbb{R}^n = E_1 \oplus E_2$ if $E_1 + E_2 = \mathbb{R}^n$ and $E_1 \cap E_2 = \{0\}$. The set of all matrices $g \in GL_n(\mathbb{R})$ such that $g'g = gg' = I$ is denoted by $O(n)$, and $O(n) \cap SL_n(\mathbb{R})$ is denoted $SO(n)$. For a natural number $n$ we denote $\Sigma_n$ to be a fundamental domain for the action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{R})$. When additional properties of the fundamental domain are required, we will state so explicitly. Given a locally compact topological space $X$ and Borel measures $(\mu_n)_{n \geq 0}, \mu$ on $X$, we say that $\mu$ is a weak limit of $\mu_n$ if for any continuous and compactly supported function $f : X \to \mathbb{R}$, $\int fd\mu_n \to \int fd\mu$. Given a finite Borel measure $\mu$ on a topological space $X$, a continuous map $\pi : X \to Y$ to another topological space $Y$, $\pi_* \mu$ denotes the push-forward measure of $\mu$, which is a Borel measure on $Y$, defined by $\pi_* (\mu)(A) = \mu(\pi^{-1} A)$ for any Borel set $A \subset Y$. Given a subset $R \subset \mathbb{R}^m$, we denote $N(R)$ to be the size of $R \cap \mathbb{Z}^m$. Given $n \in \mathbb{N}$, we denote $[n] = \{1, \ldots, n\}$. A number field is a finite field extension $\mathbb{F} \supset \mathbb{Q}$. Given a number field $\mathbb{F}$ of degree $n$ and number $i = 1, \ldots, [\frac{n}{2}]$, we say that $\mathbb{F}$ is of signature $i$ if the minimal polynomial $p$ of $\mathbb{F}$ has precisely $i$ pairs of conjugate complex roots. Given a natural number $d$, a function $f : \mathbb{R}^n \to \mathbb{R}^k$ (for some natural $n, k$) is said to be homogeneous of degree $d$ if $f(\lambda v) = \lambda^d f(v)$ for every $v \in \mathbb{R}^n$ and $\lambda > 0$. We denote the (finite) set of isomorphism classes of number fields of signature $i$ with discriminant less than some $X \in \mathbb{R}_+ \cup \{\infty\}$, by $\mathcal{F}_X^{(i)}$. An $n$-dimensional Lattice is a subset of $\mathbb{R}^n$ given by $\mathbb{Z}$-span of a basis for $\mathbb{R}^n$. The covolume of a lattice in a Euclidean space $\Lambda \subset (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ denoted $\text{cov}(\Lambda)$, is defined as $\text{Vol}_n(\mathcal{F})$ and $\mathcal{F}$ is any fundamental domain for the additive action of $\Lambda$ on $\mathbb{R}^n$. Given a lattice $\Lambda \subset \mathbb{R}^n$, $\Lambda^*$ is the dual lattice defined by $\{ y \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda \}$. Given an $n$-dimensional lattice $\Lambda$, we denote $\langle \Lambda \rangle = \text{cov}(\Lambda)^{-1} \Lambda$. $S_n$ denotes the symmetric group of order $n$. 
2.2 New Lattice Normalization

In this Subsection we are going to facilitate the brief discussion carried in the Introduction. In the Introduction we noted that our main Theorems (namely Theorems 2.6 and 2.8) are more general than Bhargava-Harron’s Theorem 1.1 (see [4, Theorem 1]). In the following lines we expand on the construction carried in [4] and our new construction. At the same time, we explain why the latter is more general. We begin by describing how to construct the Shape of a Number Field.

For every totally real number field \( F \) we denote \( \sigma_1, \ldots, \sigma_n \) to be a fixed ordering of the natural embeddings of \( F \) into \( \mathbb{R} \). We will use this ordering throughout the paper. To stress this fact, we introduce:

**Notation 2.1.** For any totally real number field \( F \) of degree \( n \in \mathbb{N} \), \( \{\sigma_1, \ldots, \sigma_n\} \) denotes a fixed arbitrary ordering of the natural embeddings of \( F \) into \( \mathbb{R} \).

**Step 1:** Normalization of the co-volume of \( \Lambda_{\pi}(F) \).

For every \( \pi \in S_n \), we can find some scale \( t = t(F) > 0 \), independent on \( \pi \), such that \( \text{cov}(t\Lambda_{\pi}(F)) = 1 \). Denote \( \Lambda_{0,\pi}(F) = t\Lambda_{\pi}(F) \). One can verify that \( t(F) = \text{Disc}(F)^{-1/n} \) where \( \text{Disc}(F) \) denotes the Discriminant of \( F \).

**Step 2:** Taking an orthogonal projection of \( \Lambda_{0,\pi}(F) \) onto a subspace. To understand why such step is carried, note that \( \mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^n \) is an element of \( \Lambda_{\pi}(F) \) and thus:

\[
\text{Disc}(F)^{-1/n} \mathbf{1} \text{ is an element of } \Lambda_{0,\pi}(F). \tag{2.1}
\]

Roughly, we want to prove that when varying \( F \), \( \Lambda_{0,\pi}(F) \) distributes in the space of normalized lattices according to some natural measure. However, by Mahler criterion (see [8]) and by (2.1), \( \Lambda_{0,\pi}(F) \to \infty \) when \( t(F) \to \infty \) (in the sense that \( \Lambda_{0,\pi} \) leaves any compact set in the topological space of normalized lattices). This phenomenon thus manifests as an obstacle to distribution according to the natural measure. Bhargava-Harron eliminate the ‘small’ vector \( t(F) \mathbf{1} \) by taking the orthogonal projection of \( \Lambda_{0,\pi} \) onto \( V_0 : = \mathbf{1}^\perp \). We denote the projection operator by \( P_{V_0} \) and the projected lattice by \( P_{V_0}(\Lambda_{0,\pi}(F)) \).

**Step 3:** Another normalization. We simply denote \( PA_{0,\pi}(F) := (P_{V_0}(\Lambda_{0,\pi}(F))) \) to be a lattice of co-volume 1 lying inside the subspace \( V_0 \). Since the line \( \mathbb{R} \cdot \mathbf{1} \) is a rational line for \( \Lambda_{0,\pi}(F) \) (simply meaning that \( \mathbb{R} \mathbf{1} \cap \Lambda_{0,\pi}(F) \) is a lattice in \( \mathbb{R} \)), the resulting set with the advantage of not containing the ‘short’ vector \( t\mathbf{1} \). For the following step it is convenient to denote:

**Definition 2.1.** Given linear space \( V \), \( X(V) \) denotes the space of lattices in \( V \) up to scalar multiplication.
Step 4: Passing to Shape. For the forth (and last) step, recall that it is common to identify $X(\mathbb{R}^n)$ with $\text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z})$. It is known that $\text{SL}_n(\mathbb{Z})$ is a lattice in $\text{SL}_n(\mathbb{R})$, and so the restriction of the Haar measure on $\text{SL}_n(\mathbb{R})$ to a fundamental domain of $\text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z})$ is a finite measure on $X_n := X(\mathbb{R}^n)$. We denote the normalization of this measure to a probability measure by $m_{X_n}$. In addition, note that $SO_{n-1}(\mathbb{R})$ acts on $X(V_0)$ from the right (this action is well defined up to a choice of orthogonal basis for $(V_0, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ denotes the restriction of the standard inner product on $\mathbb{R}^n$ to the subspace $V_0$). Indeed, the action of an element $T \in SO_{n-1}(\mathbb{R})$ on $\Lambda \in X(V_0)$ is defined by applying the inverse of $T$ pointwise on every point in $\Lambda$. The space of orbits of $SO_{n-1}(\mathbb{R})$ on $X_{n-1}(V_0)$ is identified with $SO_{n-1}(\mathbb{R}) \backslash \text{SL}_{n-1}(\mathbb{R})/\text{SL}_{n-1}(\mathbb{Z})$.

**Definition 2.2.** We denote the image of $P\Lambda_0,\pi(\mathbb{F})$ inside this orbit space by $S(\mathbb{F})$ and refer it as the **Shape corresponding to $\mathbb{F}$** (Note that indeed there is no dependence on $\pi$ due to the quotient by $SO_{n-1}(\mathbb{R})$).

Existence of the natural measure $m_{X_n}$ implies the existence of a natural measure $m_{S_{n-1}}$ defined by the quotient map $\text{SL}_{n-1}(\mathbb{R})/\text{SL}_{n-1}(\mathbb{Z}) \to SO_{n-1}(\mathbb{R}) \backslash \text{SL}_{n-1}(\mathbb{R})/\text{SL}_{n-1}(\mathbb{Z})$. For convenience, we restate [4, Theorem 1]:

**Theorem.** Let $n = 3, 4, 5$. When isomorphism classes of type $S_n$ number fields of degree $n$ and any given signature are ordered by discriminant, the shapes corresponding to these fields become distributed in $S_{n-1}$ according to $m_{S_{n-1}}$.

We continue by describing our new normalization. We will 'peal off' layers from the above construction, by assigning to any number field an object in a different homogeneous space over the space of shapes. Some of the steps taken when going from the number field $\mathbb{F}$ to its shape $S(\mathbb{F})$ are a-priori redundant if all one wants is to prevent the phenomenon caused by $t[\mathbb{F}]\overline{T}$ (discussed above). The first layer we 'peal' is step 4: for every $\pi \in S_n$ we consider $P\Lambda_0,\pi(\mathbb{F}) = \text{an element of } X(V_0)$; instead of considering $S(\mathbb{F}) = \text{an element of } S_{n-1}$. As a Corollary of our main result (namely Theorem 2.6), we get:

**Theorem.** Let $n = 3, 4, 5$. When isomorphism classes of type $S_n$ number fields $\mathbb{F}$ of degree $n$ and any given signature are ordered by discriminant and $\pi \in S_n$ varies, $P\Lambda_0,\pi(\mathbb{F})$ become distributed in $X(V_0)$ according to $m_{X_{n-1}}$.

Next, we 'peal' one final layer, namely step 3, from Bhargava-Harron construction described above. To describe it, we need some more background. A lattice is a particular case of what is called a **grid** or sometimes **affine lattice**, which is simply the subset of $\mathbb{R}^n$ given by the Minkowski (i.e. pointwise) sum of a vector in $\mathbb{R}^n$ and a lattice $\Lambda \subset \mathbb{R}^n$. When $\Lambda$ is taken from $X_n$, we say that the grid $\Lambda + v$ has co-volume 1. The space of $n$-dimensional grids of co-volume 1, denoted by $Y_n$, is
then defined as \( \{v + \Lambda : v \in \mathbb{R}^n, \Lambda \in X_n \} \). Any \( M \in Y_n \) uniquely determines a lattice \( \Lambda_M \in X_n \) such that there exists \( v \in \mathbb{R}^n \), uniquely determined up to \( \Lambda_M \), such that \( M = \Lambda_M + v \). We denote the corresponding element in \( \mathbb{R}^n/\Lambda \) by \( \text{vec}(M) \) and call it the translating vector of \( M \). Note that \( X(\mathbb{R}^n) \) is embedded naturally inside \( Y_n \) by the identity map (taking \( v = 0 \) in the definition of the grid). One can also find a natural bijection from a subset of the \( n + 1 \) dimensional lattices \( X(\mathbb{R}^{n+1}) \) to \( Y_n \). Indeed, fix \( 0 \neq v \in \mathbb{R}^{n+1} \) and denote:

\[
X_{n+1}(v) = \{ \Lambda \in X_{n+1} : v^\perp \text{ is a rational subspace of } \Lambda \text{ and } \text{cov}(v^\perp \cap \Lambda) = 1 \},
\]

where we say that a \( k \)-dimensional subspace \( E \subset \mathbb{R}^{n+1} \) is a rational subspace of a lattice \( \Lambda \) if \( \Lambda \cap E \) is a \( k \)-dimensional lattice in \( E \). Thus, for any \( \Lambda \in X_{n+1}(v) \) there exists \( u \notin v^\perp \) such that \( \langle u, v \rangle > 0 \), \( u \) is unique up to \( \Lambda \cap v^\perp \) and we have the following structure for \( \Lambda \):

\[
\Lambda = \bigcup_{k \in \mathbb{Z}} \Lambda \cap v^\perp + ku.
\] (2.2)

Now, for \( \Lambda \in X_{n+1}(v) \) we correspond the grid (which is sometimes called the first layer of \( \Lambda \)) given by \( (\Lambda \cap v^\perp) + P_{v^\perp}(u) \), resulting in a bijection between \( X_{n+1}(v) \) and \( Y_n \), which is well defined up to a choice of linear isomorphism between \( v^\perp \) and \( \mathbb{R}^n \):

**Definition 2.3.** The bijection \( X_{n+1}(v) \to Y_n \) given by \( X_{n+1}(v) \ni \Lambda \mapsto (\Lambda \cap v^\perp) + P_{v^\perp}(u) \) as above is denoted by \( \rho_{n+1} \).

For our future discussions in this introduction, we arbitrarily make this choice once, and use it for any relevant \( v \). For clarity sake, let us assume that \( v = e_{n+1} \), the \( n + 1 \) standard basis element. Note that the group:

\[
G(v) := \{ g \in \text{SL}_{n+1}(\mathbb{R}) : gv^\perp = v^\perp, \text{det}(g |_{v^\perp}) = 1 \} \simeq \text{SL}_n(\mathbb{R}) \times \mathbb{R}^n
\]

acts on \( X_{n+1}(v) \) (as in the case of lattices discussed beforehand), and that the stabilizer of \( \mathbb{Z}^{n+1} \) (which is of course an element of \( X_{n+1}(v) \)) is exactly \( \text{SL}_n(\mathbb{Z}) \times \mathbb{Z}^n \) so we get:

\[
Y_n \simeq \text{SL}_n(\mathbb{R}) \times \mathbb{R}^n / \text{SL}_n(\mathbb{Z}) \times \mathbb{Z}^n.
\]

Since \( \text{SL}_n(\mathbb{Z}) \times \mathbb{Z}^n \) is a lattice in \( \text{SL}_n(\mathbb{R}) \times \mathbb{R}^n \), we can define a measure \( m_{Y_n} \) on \( Y_n \), coming from the restriction of the Haar measure of \( \text{SL}_n(\mathbb{R}) \times \mathbb{R}^n \) to a fundamental domain of \( \text{SL}_n(\mathbb{R}) \times \mathbb{R}^n / \text{SL}_n(\mathbb{Z}) \times \mathbb{Z}^n \).

**Definition 2.4.** Given subspaces \( E_1, E_2 \subset \mathbb{R}^n \) such that \( E_1 \oplus E_2 = \mathbb{R}^n \) and \( t \in \mathbb{R} \), we define a linear operator \( g = g_{E_1, E_2}^t \in \text{SL}_n(\mathbb{R}) \) by \( g(x) = e^{t/\dim(E_1)}x, \ g(y) = e^{-t/\dim(E_2)}y \) for every \( x \in E_1, y \in E_2 \). The set \( \{g_{E_1, E_2}^t\}_{t \in \mathbb{R}} \) is clearly a group, which we shall call an \( E_1, E_2 \) normalization group.
Using this normalization group we describe how to acquire an element of $(X_n(\Lambda))^*$ from a lattice $\Lambda$ in $\mathbb{R}^n$ (not necessarily of co-volume 1) such that $R \cdot \overline{T}$ is a rational line of $\Lambda$.

**Definition 2.5.** Let $\Lambda \subset \mathbb{R}^n$ be a lattice such that $R \cdot \overline{T}$ is a rational line of $\Lambda$. Let $E \subset \mathbb{R}^n$ be a subspace of co-dimension 1. Let $t \in \mathbb{R}$ be such that $e^{-t} \text{cov}(\Lambda)^{-1/n} = 1$. We define:

$$\Lambda(E) = g^t_{E, R}\overline{T}(\text{cov}(\Lambda)^{-1/n} \Lambda). \quad (2.3)$$

For $\pi \in S_n$ and $F$ a number field as above we define:

$$\Lambda^\pi_F(E) = (\Lambda^\pi(F))(E). \quad (2.4)$$

Note that by the choice of $t$ it holds that $\Lambda(E) \in X_n$ and $P_{\overline{T}} \Lambda(E) \in X(V_0)$. Equivalently, we may say that $\Lambda(E)$ is an element of $(X_n(\Lambda))^*$ and in particular so is $\Lambda^\pi_F(E)$. Denote:

$$\Gamma_E(\Lambda) = \rho_n(\Lambda(E)^*) \quad (2.5)$$

and

$$\Gamma^\pi_E(F) = \Gamma_E(\Lambda^\pi_F(E)). \quad (2.6)$$

Since $X_n(\Lambda)$ and $X_n(\overline{T})^*$ are isomorphic under the map $\Lambda \mapsto \Lambda^*$, we may consider the measure $\mu_n$ as a measure on $X_n(\overline{T})^* \simeq Y_{n-1}$ (by pushing $\mu_n$ forward with said isomorphism).

**Theorem 2.6.** Let $n = 3, 4, 5$ and $\overline{T} \neq u \in \mathbb{R}^n$ such that $\overline{T} \notin u^\perp$. When isomorphism classes of type $S_n$ number fields $F$ of degree $n$ and any given signature are ordered by discriminant and $\pi \in S_n$ varies, $\Gamma^\pi_{u^\perp}(F)$ become distributed in $Y_{n-1}$ according to $m_{Y_{n-1}}$.

To state our next result, we need the following definition:

**Definition 2.7.** Let $\Lambda \in X_n$, $m \in \mathbb{N}$ and let $u \in \mathbb{R}^n/\Lambda$ such that $mu = 0 \mod \Lambda$ but $ku \neq 0 \mod \Lambda$ for any $k = 1, \ldots, m-1$. Let $\Sigma$ be a fundamental domain for the action of $\text{stab}_{\text{SL}_n}(\mathbb{R})(\Lambda)$ on $\text{SL}_n(\mathbb{R})$.

(a) Let $f : \Sigma \to Y_{n-1}$ be defined by $g \mapsto g(\Lambda + u)$ and let $m_{\Sigma}$ be the restriction of $m_{\text{SL}_n(\mathbb{R})}$ to $\Sigma$.

A measure $\eta_u$ on $Y_n$ is called a $\Sigma$-$u$-atom if $\eta_u = f_* m_{\Sigma}$;

(b) A probability measure $\eta$ on $Y_n$ is called a $\Sigma$-torsion measure if there exist $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in \text{stab}_{\text{SL}_n(\mathbb{R})}(\Lambda)$ such that $\eta$ is the sum of $\Sigma$-$g_i u$ atoms, $i = 1, \ldots, k$;

(c) A probability measure $\eta$ on $Y_n$ is called a torsion measure of order $m$ if there exists a lattice $\Lambda, u \in \mathbb{R}^n/\Lambda$ and $\Sigma$ as above such that $\eta$ is a $\Sigma$-torsion measure.

When we take $u = \overline{T}$ a peculiar phenomenon occurs and a uniform-measure-result of the flavour of Theorem 1.1 does not hold. Instead we have:
**Theorem 2.8.** Let $n = 3, 4, 5$. Order isomorphism classes of type-$S_n$ number fields $\mathbb{F}$ of degree $n$ and any given signature by their discriminant and let $\pi \in S_n$ vary. Then there exists a torsion measure $\eta$ of order $n$ such that every weak limit $\mu$ of $\Gamma_{\nu_0}^n(\mathbb{F})$ satisfies $\mu \ll \eta$.

For convenience, we re-state Theorem 1.4 which concerns with the case $n = 2$:

**Theorem 2.9.** Let $n = 2$ and $\bar{T} \neq u \in \mathbb{R}^n$ such that $\bar{T} \notin u\perp$. When isomorphism classes of type $S_n$ number fields of degree $n$ are ordered by discriminant, the $u\perp$-grids corresponding to these fields become equidistributed in $Y_{n-1}$. When $u = \bar{T}$, every weak limit of $u\perp$-grids corresponding to these fields is absolutely continuous with respect to some torsion measure of order $n$ in $Y_{n-1}$. 


2.3 Basic Subsets

Definition 2.10. Given a lattice \( \Lambda = \text{sp}_\mathbb{Z}\{w_1, \ldots, w_n\} \subset \mathbb{R}^n \), a fundamental domain \( \Sigma \) for the action of \( \text{stab}_{\text{SL}_n(\mathbb{R})}(\Lambda) \) on \( \text{SL}_n(\mathbb{R}) \), an open and bounded subset \( S \subset \Sigma \) and \( U = I_1e_1 + \cdots + I_ne_n \subset [0,1]^n \) where \( I_i \subset [0,1] \) is an interval (such subset will be denoted by \( \text{box} \)), we define:

\[
S \times \Sigma U = \{ g^{-1}(\Lambda + I_1w_1 + \cdots + I_nw_n) : g \in S \} \subset Y_n.
\]

We refer to subsets of the above form as \( \Lambda \Sigma \)-basic subsets.

Definition 2.11. Given \( U \) as in Definition 2.10 and an ordered basis \( w = \{w_1, \ldots, w_n\} \) of \( \mathbb{R}^n \), denote:

\[
U_w = I_1w_1 + \cdots + I_nw_n.
\]

The following technical Lemma will be proved in \( \text{A.2} \).

Lemma 2.12. For any continuous \( \Sigma \)-base \( B \), the collection \( T = \{S \times_\Sigma U : S \subset \Sigma, \sigma(\partial S) = 0, U \subset [0,1]^n \text{ a box} \} \) consists of subsets of \( \mu \)-measure-0 boundary which constitute a basis for the natural topology on \( Y_n \).

2.4 Bhargava’s Correspondence

Some of the main tools we use (as do Bhargava-Harron in \( \text{[4]} \)) involve parametrizations all cubic, quartic and quintic orders which are carried in \( \text{[2, 3, 5]} \). Fix a degree \( n \in \{3, 4, 5\} \).

Definition 2.13. Let \( T \) be a ring. We define \( V_T \) to be:

(a) the space \( \text{Sym}^3T^2(\otimes T) \) of binary cubic forms over \( T \), if \( n = 3 \);

(b) the space \( \text{Sym}^2T^3 \otimes T^2 \) of pairs of ternary quadratic forms over \( T \), if \( n = 4 \);

(c) the space \( T^4 \otimes \wedge^2T^5 \) of quadruples of alternating quinary 2-forms over \( T \), if \( n = 5 \).

For \( n = 3, 4, 5 \), we set \( r = r(n) = 2, 3, 6 \) respectively. Note that \( G_T = \text{GL}_{n-1}(T) \times \text{GL}_{r-1}(T) \) acts naturally on \( V_T \). The discriminant of an element \( v \in V_T \) is defined by a polynomial of degree \( d \) in the coefficients of \( v \), where \( d = 4, 12, 40 \) when \( n = 3, 4, 5 \) respectively (see \( \text{[4]} \)).

The following Theorem can be deduced from \( \text{[5 §15]} \), \( \text{[2 Corollary 5]} \) and \( \text{[3 Corollary 3]} \) and is Theorem 2 in \( \text{[4]} \):

Theorem 2.14. The nondegenerate (i.e. with non-zero discriminant) elements of \( V_\mathbb{Z} \) are in canonical bijection with isomorphism classes of pairs \( ((R, \alpha), (S, \beta)) \), where \( R \) is a nondegenerate ring of rank \( n \) and \( S \) is a rank \( r \) resolvent ring of \( R \), and \( \alpha \) and \( \beta \) are \( \mathbb{Z} \)-bases for \( R/\mathbb{Z} \) and \( S/\mathbb{Z} \), respectively. In this bijection, the discriminant of an element of \( V_\mathbb{Z} \) is equal to the discriminant of the
corresponding ring $R$ of rank $n$. Moreover, under this bijection, the action of $G_R$ on $V_R$ results in a corresponding natural action of $G_R = GL_{n-1}(\mathbb{Z}) \times GL_{r-1}(\mathbb{Z})$ on $(\alpha, \beta)$. Finally, every isomorphism class of maximal ring $R$ of rank $n$ arises in this bijection, and the elements of $V_R$ yielding $R$ consists of exactly one $G_R$-orbit. We denote the rings corresponding to $v \in V_R$ by $R_\mathbb{Z}(v), S_\mathbb{Z}(v)$ and the bases corresponding to $v$ by $\alpha(v), \beta(v)$.

A resolvent ring of a cubic, quartic or quintic ring is some quadratic, cubic, or sextic ring respectively, which satisfies some conditions (see \cite{23} for more details). Since the precise definition is not required here, we choose not to get into the details of this matter. In fact, Theorem 2.14 holds with any field $K$ in place of $\mathbb{Z}$ with the same proofs as in \cite{23}. In particular, taking $K = \mathbb{R}$:

**Theorem 2.15.** There is a canonical bijection between the nondegenerate elements of $V_\mathbb{R}$ and isomorphism classes of pairs $((R, \alpha), (S, \beta))$, where $R$ is a nondegenerate ring of rank $n$ over $\mathbb{R}$ and $S$ is the (unique) rank $r$ resolvent ring of $R$ over $\mathbb{R}$, and $\alpha$ and $\beta$ are $\mathbb{R}$-bases for $R/\mathbb{R}$ and $S/\mathbb{R}$, respectively. Moreover, under this bijection, the action of $G_R$ on $V_\mathbb{R}$ results in the corresponding natural action of $G_R$ on $(\alpha, \beta)$. We denote the rings corresponding to $v \in V_\mathbb{R}$ by $R(v), S(v)$.

**Remark 2.16.** Theorems 2.14, 2.15 are compatible in the following sense. On the one hand $V_\mathbb{Z} \subset V_\mathbb{R}$ naturally and if $v \in V_\mathbb{Z}$ corresponds under Theorem 2.14 to the ring $R$, then it corresponds under Theorem 2.15 to $R \otimes \mathbb{R}$. Moreover, given $v \in V_\mathbb{Z}$ the multiplication tables of the algebras $R, S$ corresponding to $v$ under Theorem 2.14 with respect to the bases $\alpha, \beta$ are the same as the multiplication tables of $R', S'$ with respect to the bases $\alpha', \beta'$ which correspond to $v$ when thought of as an element of $V_\mathbb{R}$. This follows from \cite{2} §15 (1) and (2)] when $n = 3$, \cite{2} (14), (21), (22) and (23)] when $n = 4$ and by \cite{2} (16), (17), (21) and (22)] when $n = 5$.

**Remark 2.17.** Rank-$n$ rings $R$ over $\mathbb{R}$ are in particular Étale algebras over $\mathbb{R}$ of rank $n$, which implies $R \simeq \mathbb{R}^{n-2k} \times \mathbb{C}^k$ for some $k = 0, \ldots, \lfloor n/2 \rfloor$. Following the notation of \cite{4}, given $k = 0, \ldots, \lfloor n/2 \rfloor$, we denote $V_{\mathbb{R}}^{(k)}$ to be the subset of $V_{\mathbb{R}}$ of elements for which the corresponding ring $R$ (recall Theorem 2.15) has the structure $\mathbb{R}^{n-2k} \times \mathbb{C}^k$. Therefore, and as mentioned in \cite{4}, the nondegenerate orbits for the action of $G_R$ on $V_{\mathbb{R}}$ are precisely $V_{\mathbb{R}}^{(0)}, \ldots, V_{\mathbb{R}}^{\lfloor n/2 \rfloor}$. Moreover, recalling Remark 2.16 we denote $V_{\mathbb{Z}}^{(i)} = V_{\mathbb{R}}^{(i)} \cap V_{\mathbb{Z}}$ for $i = 1, \ldots, \lfloor n/2 \rfloor$.

**Definition 2.18.** Given $v \in V_\mathbb{Z}$, denote by $R_v$ the ring corresponding to it under Theorem 2.15

(a) We say that $v$ is irreducible if $R(v)$ is isomorphic to an order inside a type $S_n$ number field;
(b) we say that $v$ is maximal if $R(v)$ is a maximal ring of rank $n$ over $\mathbb{Z}$ (which means it cannot be embedded as a sub $\mathbb{Z}$ module inside any other ring of rank $n$ over $\mathbb{Z}$).

Recalling the natural identification of $V_\mathbb{Z}$ inside $\mathbb{Z}^d$ for an appropriate $d$ (depending on $n = 3, 4, 5$), we say that $v \in \mathbb{Z}^d$ is irreducible (respectively, maximal) if the corresponding element of $V_\mathbb{Z}$ is irreducible (respectively, maximal).
Corollary 2.19. Let \( n = 3, 4, 5 \) and \( X > 0 \). There is 1:1 correspondence between \( G_2 \)-orbits of irreducible maximal elements of \( V_2^{(i)} \) with discriminant less than \( X \) and rings of integers of isomorphism classes of signature \( i \) number fields with discriminant less than \( X \). The correspondence is given by \( G_2.v \leftrightarrow R_2(v) \).

In the remaining of this Subsection we assume for simplicity that we are in the totally real case.

Remark 2.20. Let \( n = 3, 4, 5 \) and fix a fundamental domain \( \Sigma \) for the action of \( SL_{n-1}(\mathbb{Z}) \) on \( SL_n(\mathbb{R}) \). Fix a base point \( v_0 \in V^{(0)}_\mathbb{R} \). In [6, p.35 for \( n = 3 \), p.39 for \( n = 4 \) and p.48 for \( n = 5 \)], Harron follows Bhargava in [2,3] as she describes the structure of the rings \( R \). Let \( \pi \) be determined by:

\[
\pi \colon R \rightarrow \mathbb{Z} \text{ satisfies (this time with pointwise product) the same multiplication table of } R \text{ (the time with pointwise product) the same multiplication table of } R \text{ we mentioned earlier. Every pair of such bases differs by applying some } \pi \in S_n \text{ (} S_n \text{ acts on } \mathbb{R}^n \text{ by permuting the coordinates). Therefore, there exist } n! \text{ choices of vectors } \{\pi_i\}_{i=1,...,n} \subset \mathbb{R}^n \text{ satisfying (this with pointwise product) the same multiplication table of } R \text{ we mentioned earlier. Every pair of such bases differs by applying some } \pi \in S_n \text{ on the basis vectors. Choose arbitrarily such a base for } v_0: \{\pi_i(v_0)\}_{i=1,...,n} \text{ and for every } v \in V^{(0)}_\mathbb{R} \text{ let } \pi_i(v) \text{ be determined by } v_0 \text{ using A.2. Then } \pi_i(v) \text{ are smooth as functions } V^{(0)}_\mathbb{R} \rightarrow \mathbb{R}^n.
\]

Definition 2.21. Given \( n = 3, 4, 5 \) we let \( v_0 \in V^{(0)}_\mathbb{R} \) and let \( u \perp = E \subset \mathbb{R}^n \) such that \( u \notin V_0 \).

(a) For any \( v \in V^{(0)}_\mathbb{R} \) let \( MT_v \) denote the multiplication table A.1 when \( n = 3 \), A.2 when \( n = 4 \) and A.3 when \( n = 5 \). Given a rank \( n \) ring \( R \) and \( x_1, \ldots, x_{n-1} \in R \) we write \( MT_v(x_1, \ldots, x_{n-1}) \) if \( (x_i \mid i = 1 \ldots, n-1) \) satisfies \( MT_v \);

(b) For \( i = 1, \ldots, n-1, \pi_i : V^{(0)}_\mathbb{R} \rightarrow \mathbb{R}^n \) denote the smooth vector functions such that:

\[
MT_v(\pi_1(v), \ldots, \pi_{n-1}(v))
\]

for any \( v \in V^{(0)}_\mathbb{R} \) (their existence is guaranteed by A.2);

(c) Given \( g \in \mathbb{R}_+ \cdot \Sigma_{n-1} \times \Sigma_{r-1} \) denote \( \Lambda_{v_0}(g) = \text{span}_\mathbb{Z}\{\overline{1}, \pi_1(gv_0), \ldots, \pi_n(gv_0)\} \).

(d) Given a basis \( w = (w_j)_{j=1}^{n-1} \) of \( V_0 \), we define a function \( f_w : \mathbb{R}_+ \cdot (\Sigma_{n-1} \times \Sigma_{r-1})v_0 \rightarrow V_0 \) by:

\[
f_w(p) = \frac{1}{\langle u, 1 \rangle} \sum_{i=1}^{n-1} \langle u, \pi_i(g_pv_0) \rangle w_i,
\]

where for \( p \in \mathbb{R}_+ \cdot \Sigma_{n-1}v_0, g_p \) is the element of \( \mathbb{R}_+ \cdot \Sigma_{n-1} \times \Sigma_{r-1} \) satisfying \( g_pv_0 = p \);
(e) Given a basis \( w = \{w_1, \ldots, w_{n-1}\} \) of \( V_0 \), we denote:

\[
\Lambda_w = \text{span}_\mathbb{Z}\{w_1, \ldots, w_{n-1}\}.
\]  

(2.8)

The main goal of this Subsection is to prove the following structural Lemma:

**Lemma 2.22.** Let \( n = 3, 4, 5 \) and let \( u^\perp = E \subset \mathbb{R}^n \) be a subspace of co-dimension 1 not containing \( \overline{T} \). Let \( \Sigma'_{n-1} \) be any fundamental domain for the action of \( \text{SL}_{n-1}(\mathbb{Z}) \) on \( \text{SL}_{n-1}(\mathbb{R}) \) and let \( v_0 \in V_0^0 \). Then there exists a fixed basis of \( V_0 \) given by \( w = (w_j)_{j=1}^{n-1} \), a function \( \Pi : F_\infty^{(0)} \to S_n \) (recall Subsection 2.1) and \( g_0 \in \text{SL}_{n-1}(\mathbb{R}) \) such that:

\[
\Sigma_{n-1} := g_0^t \Sigma'_{n-1} g_0^{-t}
\]

satisfies that for any \( \Lambda_w \cdot \Sigma_{n-1} \)-basic subset \( S \times \Sigma_{n-1} \subset Y_{n-1} \):

\[
F_X^{(0)} \cap (\Gamma_E^{(0)})^{-1}(S \times \Sigma_{n-1} \cup U) \overset{1:1}{\leftrightarrow} \{ \text{irreducible, maximal points inside } [0, X|g_0^{-1} S^{-1} g_0 \times \Sigma_{r-1})v_0 \} \cap ((f_w + g_0) \bmod \Lambda_w)^{-1}(U_w)
\]

where \( g_0 \in V_0 \) is some fixed vector (recall Definition 2.11 for definition of \( U_w \)). Note that the function \( \Gamma_E^{(0)} : F_\infty^{(0)} \to Y_{n-1} \) is defined by \( F_\infty^{(0)} \ni F \mapsto \Gamma_E^{(0)}(F) \). Moreover, for any \( \pi \in S_n \) the function \( \Pi \) can be replaced with \( \pi \circ \Pi \) (where \( (\pi \circ \Pi)(F) := \pi(\Pi(F)) \)).

**Proof.** Let \( v_0 \in V_0^0 \). Pick any fundamental domain \( \Sigma'_{n-1} \) for the action of \( \text{SL}_{n-1}(\mathbb{Z}) \) on \( \text{SL}_{n-1}(\mathbb{R}) \) and similarly pick \( \Sigma_{r-1} \) to be a fundamental domain for the action of \( \text{SL}_{r-1}(\mathbb{Z}) \) on \( \text{SL}_{r-1}(\mathbb{R}) \). For any \( v = t(g, h)v_0 \in \mathbb{R}_+ \cdot (\Sigma'_{n-1}, \Sigma_{r-1})v_0 \) recall Definiton [2.21] and denote:

\[
\Lambda_v = \Lambda_{v_0}(t(g, h)).
\]

(2.11)

Next, write:

\[
\Lambda_v = \text{span}_\mathbb{Z}\{\overline{e}_1(v), \ldots, \overline{e}_{n-1}(v), \overline{T}\}, D_v = \text{cov}(\Lambda(v))^{-1/n}.
\]

(2.12)

This is the lattice we start with. Next we follow the steps of normalization which are described in Equation (2.4) and find \( \Lambda_v(E) \) and \( \Gamma_E(\Lambda(v)) \) explicitly using \( v_0, t(g, h) \). The first step is to normalize the co-volume of \( \Lambda_v \) by defining:

\[
\Lambda_v^{0} = \text{span}_\mathbb{Z}\{D_v \overline{e}_1(v), \ldots, D_v \overline{e}_{n-1}(v), D_v \overline{T}\}.
\]

(2.13)

The next step is acting on \( \Lambda_v^{0} \) with the \( E, \mathbb{R} \cdot \overline{T} \) normalization group from Definition 2.4. To this end, write:

\[
\overline{e}_i = (\overline{e}_i)_{\perp} + c_i \overline{T} \text{ such that } (\overline{e}_i)_{\perp} \in E, i = 1, \ldots, n-1
\]

(2.14)
and obviously:

\[ D_v\alpha_i = D_v(\alpha_i)\perp + D_v e_i \perp \text{ such that } (\alpha_i)\perp \in E, i = 1, \ldots, n - 1. \]  

(2.15)

Denote \( \lambda_v = n^{-1/2} D_v^{-1} \). The appropriate element \( g_\ell \) from the \( E, \mathbb{R} \perp \) normalization group is defined uniquely by:

\[ g_\ell(\perp) = \lambda_v \perp; \]
\[ g_\ell(e) = \lambda_v^{-1} e \text{ for every } e \in E. \]

Denote also:

\[ \alpha_i^0 := \lambda_v^{-1} n^{-1/2} D_v(\alpha_i)\perp + c_i D_v \lambda_v \perp = g_\ell(D_v \alpha_i) \]  

(2.16)

Let

\[ \beta_i = \frac{\lambda_v^{-1} ((\alpha_1)\perp \times \cdots \times (\alpha_i-1)\perp \times \perp \times (\alpha_{i+1})\perp \cdots \times (\alpha_n)\perp)}{D_v(\alpha_1)\perp \times \cdots \times (\alpha_{n-1})\perp, \perp} \]  

(2.17)

and denote:

\[ \tau_v := \frac{\alpha_i^0 \times \cdots \times \alpha_n^0 \perp}{\langle \alpha_1^0 \times \cdots \times \alpha_n^{0-1}, D_v \lambda_v \perp \rangle}, P_v(\tau_v) = \frac{P_v(\alpha_1^0 \times \cdots \times \alpha_n^{0-1})}{\langle \alpha_1^0 \times \cdots \times \alpha_n^{0-1}, D_v \lambda_v \perp \rangle} = \frac{P_v(u)}{\langle u, 1 \rangle} + \sum_{i=1}^{n-1} c_i \beta_i \]  

(2.18)

Note that we have found \( \Lambda_v(E) \) and it is given by:

\[ \Lambda_v(E) = \text{span}_\mathbb{Z} \{ \lambda_v D_v \perp, \alpha_i^0, \ldots, \alpha_n^0 \perp \} \]  

(2.19)

(recall Definition 2.5). Next we find \( \Gamma_v(E) \). By Equation (A.6):

\[ P_v(\Lambda_v(E)) = g_0 g_0^{-1}(P_v \Lambda_v(E)) \]  

(2.20)

where:

\[ g_0 = \begin{pmatrix} \vert & \cdots & \vert \\ P_v \alpha_1^0(v_0) & \cdots & P_v \alpha_n^0(v_0) \\ \vert & \cdots & \vert \end{pmatrix}. \]

We claim that \( \{ \beta_i(g) \}_{i=1}^{n-1} \) is a basis for the lattice:

\[ (\Lambda_v(E))^* \cap V_0 = (P_v \Lambda_v(E))^* = (g_0 g_0^{-1}(P_v \Lambda_v(E)))^* \]  

(2.21)

where the first equality holds by known identities of the dual lattice and the second by Equation (2.20). Indeed, by definition of the dual lattice, it suffices to check that \( \langle \beta_i, \alpha_j^0 \rangle \in \{ \delta_{ij}, -\delta_{ij} \} \) and
that \( \langle \beta_i, \lambda_v D_v \mathbf{T} \rangle = 0 \) for every \( i, j = 1, \ldots, n - 1 \). Indeed, let \( i = 1, \ldots, n - 1 \). Then:

\[
\langle \beta_i, \alpha_i^0 \rangle = \left\langle \frac{\lambda_v^{-1} \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{i-1})_{\perp} \times \mathbf{T} \times (\mathbf{v}_{i+1})_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \right)}{D_v \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T} \right)}, \lambda_v^{-1} D_v (\mathbf{v}_i)_{\perp} + c_j D_v \lambda_v \mathbf{T} \right\rangle
\]

\[
= \left\langle \frac{\lambda_v^{-1} \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{i-1})_{\perp} \times \mathbf{T} \times (\mathbf{v}_{i+1})_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \right)}{D_v \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T} \right)}, (\mathbf{v}_i)_{\perp} \right\rangle
\]

\[
= \left\langle \frac{(\mathbf{v}_1)_{\perp} \cdots \times (\mathbf{v}_{i-1})_{\perp} \times \mathbf{T} \times (\mathbf{v}_{i+1})_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp}}{(\mathbf{v}_1)_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T}} (\mathbf{v}_i)_{\perp} \right\rangle
\]

\[
= \frac{(-1)^i (\mathbf{v}_1)_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T}}{(\mathbf{v}_1)_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T}} = (-1)^i
\]

where the first equality holds by definition of \( \beta_i, \alpha_i^0 \), the second holds by definition of the cross product, the third is elimination of the constants, the forth is taking out the constant and the fifth holds by two general properties of the mixed product saying that for any \( u_1, \ldots, u_n \in \mathbb{R}^n \):

\[
\langle u_1 \times \cdots \times u_{n-1}, u_n \rangle = \langle u_n \times u_1 \cdots \times u_{n-2}, u_{n-1} \rangle.
\]

and that for any \( \pi \in S_{n-1} \):

\[
u_1 \times \cdots \times u_{n-1} = \text{sgn}(\pi) u_{\pi(1)} \times \cdots \times u_{\pi(n-1)}.
\]

Let \( i \neq j \) be numbers in \( \{1, \ldots, n - 1\} \). Then:

\[
\langle \beta_i, \alpha_j^0 \rangle = \left\langle \frac{\lambda_v^{-1} \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{i-1})_{\perp} \times \mathbf{T} \times (\mathbf{v}_{i+1})_{\perp} \cdots \times (\mathbf{v}_{n-1})_{\perp} \right)}{D_v \left( (\mathbf{v}_1)_{\perp} \times \cdots \times (\mathbf{v}_{n-1})_{\perp} \mathbf{T} \right)}, \lambda_v^{-1} D_v (\mathbf{v}_j)_{\perp} + c_j D_v \lambda_v \mathbf{T} \right\rangle = 0
\]

because the element in the left side of the inner product is perpendicular to both \( \mathbf{T} \) and to \( (\mathbf{v}_j)_{\perp} \).

By the same reason, \( \langle \beta_i, \mathbf{T} \rangle = 0 \) for every \( i \).

Since for each \( i \), \( \beta_i(g) \) is a continuous function in \( g \) and by Equation \( (2.21) \) above, there exists some fixed basis \( w_1, \ldots, w_{n-1} \) of the lattice \( (P_{v_0} \Lambda_{v_0}(E))^* \) such that:

\[
\beta_i(g) = g_0^i g_0^{-1} g_0^{-i} w_i \text{ for all } i = 1, \ldots, n - 1.
\]

Define \( \Sigma_{n-1} = g_0^i \Sigma_{n-1} g_0^{-i} \) and write \( \Lambda_w = \text{span}_\mathbb{Z} \{w_1, \ldots, w_{n-1}\} \). We continue to calculate
\( \Gamma_E(\Lambda(v)) \). Recall that by Definition 2.3, \( \rho_n : X_n \to Y_{n-1} \) is an identification between lattices and grids and that by Definition 2.5: \( \Gamma_E(\Lambda(v)) = \rho_n(\Lambda_v(E)^*) \). Write by Equation (2.19) and by the definitions of \( \beta_i, \tau_v \) (recall Equations 2.17,2.18)

\[
(\Lambda_v(E))^* = \left( \begin{array}{ccc} a_1 & \cdots & a_{n-1} \\ \alpha_1 & \cdots & \alpha_{n-1} \\ \lambda_v D_v \mathbf{1} \end{array} \right) \mathbb{Z}^n = \left( \begin{array}{ccc} \beta_1 & \cdots & \beta_{n-1} \\ \cdots & \cdots & \cdots \\ \tau_v \\ \beta_n \end{array} \right) \mathbb{Z}^n
\]

which means that \( (\Lambda_v(E))^* = \text{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_{n-1}, \tau_v) \) and since \( \beta_i \in V_0 \) for every \( i = 1, \ldots, n-1 \), we can take \( u = \tau_v \) in Definition 2.3 of \( \rho_n \) and deduce that:

\[
\Gamma_E(\Lambda(v)) = \rho_n(\Lambda_v(E)^*) = \text{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_{n-1}) + P_v(\tau_v).
\]

We already know (see Corollary 2.19) that for any \( v_0 \in V_0^{(0)} \) (recall Remark 2.17) irreducible maximal points \( v = t(g,g')v_0 \in V_0 \cap R_+ \cdot (\Sigma_{n-1}^r, \Sigma_{r-1})v_0 \) correspond under \( v \mapsto R_+(v) \) to rings of integers of equivalence classes of type \( S_n \) number fields \( \mathbb{F} \). Denote the number field corresponding to \( v \) by \( \mathbb{F}_v \). Recall that in Notation 2.1 we fixed (in particular) an ordering for the natural maps of \( \mathbb{F}_v \), whom we denote by \( \sigma_1^v, \ldots, \sigma_n^v \). Let \( \pi(\mathbb{F}_v) \in S_n \) be a permutation such that:

\[
(\sigma_{\pi(\mathbb{F}_v)(1)}^v, \ldots, \sigma_{\pi(\mathbb{F}_v)(n)}) \alpha_i(t(g,g')v_0) = \pi_v(t(g,g')v_0); \forall i = 1, \ldots, n
\]

(see A.2) and define:

\[
\Pi(\mathbb{F}_v) = \pi(\mathbb{F}_v).
\]

Let \( v = t(g,g')v_0 \in V_0 \cap R_+ \cdot (\Sigma_{n-1}^r, \Sigma_{r-1})v_0 \) be irreducible maximal. Applying Equations (2.26) and (2.21) in particular to \( v \) shows that:

\[
\Gamma_E^+(\mathbb{F}_v) = \Gamma_E(\Lambda(v)) = \text{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_{n-1}) + P_v(\tau_v) = g_0 g^{-1} g_0^{-t}(\Lambda_v(E)^* \cap V_0) + P_v(\tau_v)
\]

which shows that \( \Gamma_E^+(\mathbb{F}_v) \in S \times \Sigma_{n-1}, U \) if and only if \( g_0 g^{-1} g_0^{-t} \in S \) and \( P_v(\tau_v) \in g_0 g^{-1} g_0^{-t} U \) mod \( g_0 g^{-1} g_0^{-t} \Lambda_w \) if and only if \( g^{-1} \in g_0^{-t} S g_0^t \) and:

\[
f_w(v) := \frac{P_v(u)}{\langle u, 1 \rangle} + \frac{1}{\langle u, 1 \rangle} \sum_{i=1}^{n-1} \langle u, \pi_v(gv_0) \rangle w_i \in g_0^t g^{-1} g_0^{-t} U \text{ mod } g_0 g^{-1} g_0^{-t} \Lambda_w
\]

if and only if:

\[
v \in \{ \text{irreducible, maximal points inside } 0, X \} (g_0^t S g_0^{-t} \times \Sigma_{r-1})v_0 \cap (f_w \text{ mod } \Lambda_w)^{-1}(U_w)
\]

where the second equivalence holds by Equation (2.24). Note that the only property of \( \Pi \) we used
is the continuity of $\beta_i(g) \in \mathbb{R}^n$. Since we for any $\pi \in S_n$, the functions $\pi(\beta_i(g))$ are also continuous in $g$, the same argument shows that we could have used $\pi \circ \Pi$ as claimed. ■
2.5 Weyl-Type Theorem and Verification of Analytical Properties

For Theorems 2.24 and 2.25 below, the following definition comes handy.

Definition 2.23. Let \( n, k \in \mathbb{N} \), \( F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^k \), \( \Lambda_k \subset \mathbb{R}^k \) a lattice, \( \{B_R\}_{R \geq 0} \) a monotonic family of subsets of \( \mathbb{R}^n \) and \( \Gamma \subset \mathbb{R}^n \) a discrete subset. We say that \( F \mod \Lambda_k(\Gamma) \) is \( B_R \)-equidistributed if the sequence of discrete counting measures \( \sigma_D^R \) of \( \Gamma \) points in \( B_R \cap D \) satisfies:

\[
\lim_{R \to \infty} (F \mod \Lambda_k)_* \sigma_D^R = \lambda_k
\]

in the weak sense, where \( F \mod \Lambda_k : D \rightarrow \mathbb{R}^k / \Lambda_k \) is the composition of \( F \) with the quotient map, \( (F \mod \Lambda_k)_* \sigma_D^R \) denotes the push-forward measure, and \( \lambda_k \) is the Haar probability measure on the \( k \)-dimensional torus \( \mathbb{R}^k / \Lambda_k \). When \( \Lambda_k = \mathbb{Z}^k \) we abbreviate \( F \mod \Lambda_k(\Gamma) = F \mod 1(\Gamma) \).

The following two Theorems of [11] are the main tools in the proof of Theorem 2.8.

Theorem 2.24 ([11]). Let \( n, D, k \in \mathbb{N} \). Let \( S \) be a level surface of some smooth homogeneous function \( F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^+ \) (meaning \( S = \{F = t\} \) for some \( t > 0 \)), let \( A \subset S \) be open and Jordan measurable in \( S \) and let \( f = (f_1, \ldots, f_k) : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^k \) be homogeneous of degree \( D \) and smooth inside \( \mathbb{R}^+ \cdot A \). Assume that there exists \( i \in [n] \) such that:

\[
(\frac{\partial f_i}{\partial x_i \mod 1})_{*} m_A := \left( \frac{\partial f_1}{\partial x_1 \mod 1}, \ldots, \frac{\partial f_k}{\partial x_k \mod 1} \right) \cdot m_A \ll \lambda_k
\]

where \( m_A \) is the \( n-1 \)-dimensional surface area on \( A \) and \( \lambda_k \) is the Lesbegue measure on \( (S^1)^k \). Then:

\[
f \mod 1(\mathbb{Z}^n) \text{ is } [0, R] \cdot A \text{-equidistributed.}
\]

Here, for a subset \( I \subset \mathbb{R}^+ \), \( I \cdot A \) denotes the set \( \{ta : t \in I, a \in A\} \).

Theorem 2.25 ([11], Theorem 1.2). Let \( n \in \mathbb{N} \) and let \( F : \mathbb{R}^n \rightarrow \mathbb{R} \), be a polynomial given by \( F(x) = \sum \alpha_\ell x^\ell \) where \( \ell = (\ell_1, \ldots, \ell_n) \) is a multi-index. Let \( S \) be a level surface and \( A \subset S \) as in Theorem 2.24. Assume that there exist \( \ell \) such that \( |\ell| > 0 \) and \( \alpha_\ell \notin \mathbb{Q} \), then:

\[
F \mod 1(\mathbb{Z}^n) \text{ is } [0, R] \cdot A \text{ equidistributed.}
\]

Definition 2.26. Let \( S \subset \mathbb{R}^d \) be a level surface of some smooth homogenous function.

(a) A subset \( D_\mathbb{Z} \subset \mathbb{Z}^d \) is said to be of \( S \)-density 1 in \( \mathbb{Z}^d \) if:

\[
\lim_{R \to \infty} \frac{|((0, R] \cdot S) \cap D_\mathbb{Z}|}{\text{Vol}_d([0, 1] \cdot S) R^d} = 1;
\]

17
(b) A subset $C \subset \mathbb{Z}^d$ is said to be a congruence set if it is defined by finitely many congruence conditions modulo distinct primes on the entries of $v \in C$ (namely conditions of the form $v_i \mod p = s$ where $v = (v_1, \ldots, v_n)$ and $p, s$ are integers and $p$ is prime).

We will require a modification of Theorems 2.24 and 2.25:

**Corollary 2.27.** Under the assumptions of Theorem 2.24 (of Theorem 2.25) Equation (2.33) holds (Equation (2.34) holds) for $C \cap D \mathbb{Z}$ instead of $\mathbb{Z}^d$ where $D \mathbb{Z}$ is any set of $S$-density 1 in $\mathbb{Z}^d$ and $C \mathbb{Z}$ is any congruence set.

**Proof.** Let $p_1, \ldots, p_k$ be the primes defining the congruence conditions on the first coordinate of elements in $C \mathbb{Z}$ and let $s_1, \ldots, s_k$ be the corresponding reminders. Let $P = p_1 \cdots p_k$ and for any $i$, $P_i = P/p_i$. Since the $p_i$’s are prime, $P_i, p_i$ are co-prime. By Bezout’s identity there exist integers $M_i, m_i$ such that:

$$M_iP_i + m_ip_i = 1.$$  \hspace{1cm} (2.36)

Then $x_1 := \sum_{i=1}^k s_iM_iP_i$ is a solution to the congruence system of conditions on the first entries of elements from $C \mathbb{Z}$. Since $p_i$’s are distinct primes, by the Chinese remainder theorem any other solution to the system is given by $x_1 + lP$ for some integer $l$. Repeating this argument for any other coordinate we can write:

$$C \mathbb{Z} = \{(x_1 + l_1Q_1, \ldots, x_d + l_dQ_d) : (l_1, \ldots, l_d) \in \mathbb{Z}^d\}$$  \hspace{1cm} (2.37)

for some fixed $x_i, Q_i \in \mathbb{Z}$. Note that there exists $c > 0$ such that for any $R > 0$:

$$|C \cap [0, R] \cdot S| \geq cR^d.$$  \hspace{1cm} (2.38)

Let $F$ satisfy the conditions of Theorem 2.24 (Theorem 2.25). Then the function $G(y_1, \ldots, y_d) := F(x_1 + y_1Q_1, \ldots, x_d + y_dQ_d)$ also satisfies the conditions of Theorem 2.24 (of Theorem 2.25 since $x_i, Q_i$ are integers) and therefore by Equation (2.37):

$$F \mod 1(C \mathbb{Z})$$ is $[0, R] \cdot A$ equidistributed.  \hspace{1cm} (2.39)

where $A$ is as in Theorem 2.24 (Theorem 2.25). By assumption on $D \mathbb{Z}$:

$$\lim_{R \to \infty} \frac{|([0, R] \cdot S) \cap \mathbb{Z}^d \setminus D \mathbb{Z}|}{R^d} = 0$$  \hspace{1cm} (2.40)

so by Equation (2.38):

$$\lim_{R \to \infty} \frac{|([0, R] \cdot S) \cap C \mathbb{Z} \setminus D \mathbb{Z}|}{|C \mathbb{Z} \cap [0, R]S|} = 0$$  \hspace{1cm} (2.41)
which means that $D_Z^c$ points are negligible in $C_Z$ so in fact:

\[ F \mod 1(C_Z \cap D_Z) = [0, R] \cdot A \text{ equidistributed} \quad (2.42) \]

as desired. ■

**Lemma 2.28.** The subset $\Gamma_d \subset V_Z$ of irreducible maximal lattice points in $V_Z$ (recall Definition 2.18) is the intersection of a set of $S = \Sigma_{n-1}' \times \Sigma_{r-1}v_0$ density 1 in $\mathbb{Z}^d$ and a congruence set.

**Proof.** By [4, Theorem 4] and the short discussion following it we know that the irreducible points are of $S$ density 1. By [4, Section 5], the maximal points are defined by finitely many prime congruence conditions. ■

Our main result of this Section will be stated using the function $f_{\mathbb{T}}^{E_n}$ guaranteed by Lemma 2.22. For convenience and clarity, we define it again:

**Definition 2.29.** For any hyperplane $E = u_\perp \subset \mathbb{R}^n$, $v_0 \in V^{\mathbb{T}}(0)$ and $\Sigma_{n-1}'$ be a fundamental domain for the action of $\text{SL}_{n-1}(\mathbb{Z})$ on $\text{SL}_{n-1}(\mathbb{R})$ which we will use throughout this Section, let $f_{\mathbb{T}}^{E_n}$ denote the function corresponding to the subspace $E$ under Lemma 2.22 and let $(w_i)_{i=1}^{n-1}$ be the fixed basis of $V_0$ also defined in Lemma 2.22. Explicitly, we denote:

\[
 f_{\mathbb{T}}^{E_n}(p) = \frac{1}{(u, 1)} \sum_{i=1}^{n-1} \langle u, \tau_i(g_p v_0) \rangle w_i + q_0 
 \]

for any $p \in \mathbb{R}_+ \cdot \Sigma_{n-1} \times \Sigma_{r-1}v_0$. Moreover, denote:

\[
 \Lambda_w = \text{span}_\mathbb{Z}\{w_1, \ldots, w_{n-1}\}. \quad (2.43)
 \]

The following Lemma summarizes the properties of the function $f_{\mathbb{T}}^{E_n}$ which are relevant for Section 3.

**Lemma 2.30.** Let $n = 3$ ($n = 4, n = 5$), let $u \in \mathbb{R}^n$ satisfying $u \notin (\mathbb{R}^+ \mathbb{T}) \cup \mathbb{T}^+$ and let $v_0 \in V^{(0)}_{\mathbb{R}}$. Then $f_{\mathbb{T}}^{E_n} = u_\perp$ is homogeneous of degree $D(n) = 1$ ($D(n) = 2, D(n) = 4$) and for any Jordan measurable bounded open subset $U$ of $S = \Sigma_{n-1}' \times \Sigma_{r-1}v_0$ it holds that $f_{\mathbb{T}}^{E_n}, U, S$ satisfy the conclusion of Theorem 2.24 for $k = n - 1$. Consequently:

\[
 f_{\mathbb{T}}^{E_n} \mod \Lambda_w(\Gamma_d) = [0, R] \cdot U\text{-equidistributed} \quad (2.44)
 \]

where $\Gamma_d \subset V_Z$ is the set of irreducible maximal lattice points in $V_Z$ (recall Definition 2.18). When $u = \mathbb{T}$, $n = 3, 4, 5$, $nf_{\mathbb{T}}^{E_n}(p) \in \mathbb{Z}^{n-1}$ for any $p$ in the intersection of the domain of $f_{\mathbb{T}}$ and $V_Z$.

We start by proving the case $u = \mathbb{T}$ in all dimensions:
Proof of Lemma 2.31: case $u = T$. When normalizing the lattice $\Lambda(\mathbb{F})$ of some number field $\mathbb{F}$ of degree $n$ using the group $g_{V_0, \mathbb{R}^n}$ (as in Section 1), the corresponding grid (which for now we denote $M(\mathbb{F})$) can be identified with the pair $(P_{V_0} \Lambda(\mathbb{F}), \varphi) \in X_{n-1}(V_0) \times V_0/(\Lambda(\mathbb{F})^* \cap V_0)$ where $\varphi$ is considered as a linear functional from $V_0$, such that for any $p \in P_{V_0} \Lambda(\mathbb{F})$, $p + \varphi(p) T \in \Lambda(\mathbb{F})$. In this case, $\varphi(p) = \frac{1}{n} \text{trace}(p')$ where $p'$ is the closest vector to $V_0$ on $p + \mathbb{R}^n T$ which is also in $\Lambda(\mathbb{F})$. Since $p' \in \Lambda(\mathbb{F})$, its trace is an integer and so $n \varphi(p) \in \mathbb{Z}$ for any $p \in V_0$.

We continue with the case $u \notin (\mathbb{R} \cdot T) \cup \mathbb{T}$ for $n = 3, 4, 5$ separately. Since there is an overlap in the arguments for $n = 3, 4, 5$, we write this overlap once simultaneously now. Let $u \in \mathbb{R}^n$ such that $u \notin (\mathbb{R} \cdot T) \cup \mathbb{T}$ and fix $v_0 \in V_0^{(0)}$. Let $U \subset S = \Sigma_{n-1} \times \Sigma_{r-1} v_0$ be open and Jordan measurable subset. It holds (e.g. by Weyl’s criterion for uniform distribution in [10]) that (2.44) is equivalent to the following:

For any $v_\mathbb{Z} \in \Lambda_w$, $\langle f_n^E, v_\mathbb{Z} \rangle \mod 1(\Gamma_d)$ is $[0, R] \cdot U$-equidistributed. (2.45)

Suppose for sake of contradiction that $v_\mathbb{Z} \in \mathbb{Z}^{n-1}$ is such that (2.45) does not hold. Denote $f_0 = \langle f_n^E, v_\mathbb{Z} \rangle$. Note that by Definition 2.29 there exist real constants $(\gamma_1, \ldots, \gamma_{n-1}) \neq 0$ such that:

$$f_0(v) = \sum_{i=1}^{n-1} \gamma_i \langle \alpha_i(v), u \rangle$$

(2.46)

for every $v \in \mathbb{R}_+ \cdot U \subset \mathbb{R}^d$. By Lemma 2.28, $\Gamma_d$ is the intersection of a congruence set and a set of $S$ density 1 in $\mathbb{Z}^d$. Therefore, by Corollary 2.24 we may assume that for any $v \in \mathbb{Z}^d$, the function $f_0$, $S, A$ and $v$ do not satisfy the conditions of Theorem 2.24. Indeed, otherwise the Corollary will ensure that (2.45) holds. Explicitly, it means that for any $v \in \mathbb{Z}^d$ the measure $(\frac{\partial^{D(n)} f_0}{\partial v^{D(n)}} \mod 1, m_A)$ is not absolutely continuous w.r.t $\lambda$ (recall Definition 2.28). Lemma A.1 implies that there exists a subset $B = B_v \subset U$ of positive $m_U$ measure such that $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}} = 0$ on $B$. By Subsection A.2 the smooth functions $\alpha_1, \ldots, \alpha_{n-1} : \mathbb{R}_+ \cdot U \rightarrow \mathbb{R}^n$ satisfy the multiplication table A.1 when $n = 3$, A.2 when $n = 4$ and A.3 when $n = 5$. Therefore, by Lemma A.2 for the open set $V = \mathbb{R}_+ \cdot U$ and $\alpha_1, \ldots, \alpha_{n-1}$ we deduce that $\alpha_1, \ldots, \alpha_{n-1}$ are real analytic. Since linear combinations and derivatives of real analytic functions is again real analytic, this shows that $f_0$ and $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}}$ are real analytic. By Fubini’s Theorem [1/2, 3/2], $B$ has positive $d$-dimensional Lebesgue measure so Lemma A.3 for $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}}$ and $[1/2, 3/2] \cdot B \subset V$ implies that:

$$\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}} = 0$$

(2.47)

Indeed, since $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}}$ is homogeneous of degree $-1$, $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}} |_B = 0$ implies $\nabla \frac{\partial^{D(n)} f_0}{\partial v^{D(n)}} |_{[1/2,3/2] \cdot B} = 0$. 

20
Proof of Lemma 2.30 case $n = 3$. We continue from Equation (2.47). Since $v \in \mathbb{Z}^d$ was arbitrary, we can write:

$$f_0(a_1, \ldots, a_4) = \sum_{i=1}^{4} \eta_i a_i$$

(2.48)

for some constants $\eta_i \in \mathbb{R}; i = 1, \ldots, 4$ and for every $(a_1, \ldots, a_4) \in \mathbb{R}_+ \cdot U$. In other words, for every $g = \left( \begin{array}{ccc} r & s \\ t & u \end{array} \right) \in U$:

$$\gamma_1 \langle \pi_1(gv_0), u \rangle + \gamma_2 \langle \pi_2(gv_0), u \rangle = \sum_{i=1}^{4} \eta_i (gv_0)$$

(2.49)

where for $v \in V_\mathbb{R}$ we denote $v = (a_1(v), \ldots, a_4(v))$. Since $(\pi_1(gv_0), \pi_2(gv_0))$ are proportional to $a_2(gv_0), a_3(gv_0)$ respectively (see [6] p.35), we get (possibly for different constants $\gamma_i, \eta_i$):

$$\gamma_1 \langle P_{\theta_0} \pi_1(gv_0), u \rangle + \gamma_2 \langle P_{\theta_0} \pi_2(gv_0), u \rangle = \sum_{i=1}^{4} \eta_i (gv_0).$$

(2.50)

Write $\kappa_i = \langle P_{\theta_0} \pi_i(v_0), u \rangle; i = 1, 2$, then as follows from Subsection A.2.

$$\langle P_{\theta_0} \pi_1(gv_0), u \rangle = (ru - st)(\kappa_1 r + \kappa_2 s),$$

(2.51)

By the first equation at the bottom of [6] p.38 we get:

$$\sum_{i=1}^{4} \eta_i (gv_0) = \eta_1 (a_0 r^3 + b_0 r^2 s + c_0 r s^2 + d_0 s^3) + \eta_2 (3a_0 t^2 r + 2b_0 r s t + c_0 s^2 t + b_0 r^2 u + 2c_0 r s u + 3d_0 s^2 u)$$

$$+ \eta_3 (3a_0 r t^2 + 2b_0 r t u + b_0 s^2 u + 2c_0 s t u + 3d_0 s u^2) + \eta_4 (a_0 t^3 + b_0 t^2 u + c_0 t u^2 + d_0 u^3)$$

Assume $a_0 \neq 0$. Then since neither of $r^3, r^2 t, r^2 u, t^3$ appear in Equations (2.50), (2.51), Equation (2.49) implies that $\eta_i = 0; i = 1, \ldots, 4$, a contradiction. Similarly, assuming that $d_0 \neq 0$, since $s^3, s^2 u, s t^2, u^3$ do not appear in Equations (2.50), (2.51) we get the same contradiction. Since $\text{Disc}(v_0) = b_0^2 c_0^2 b_0 - 4a_0 c_0^3 - 4b_0^2 d_0 - 27a_0^2 d_0^2 + 18a_0 b_0 c_0 d_0 \neq 0$ by assumption, we find that $b_0, d_0 \neq 0, a_0 = d_0 = 0$. The coefficients of $rst, r^2 u$ in the right hand side of (2.49) are $2\eta_2 b_0, \eta_2 b_0$ respectively. On the left hand side they are $-\gamma_1 \kappa_1 + \gamma_1 \kappa_1$ respectively. This implies that $-\gamma_1 \kappa_1 = 2\gamma_1 \kappa_1$ thus $\kappa_1 \gamma_1 = 0$. Similarly, we show that $\kappa_2 \gamma_2 = 0$. Note that by our choice of $u$, $\kappa_i = \langle P_{\theta_0} \pi_i(v_0), u \rangle \neq 0; i = 1, 2$ so that $\gamma_1 = \gamma_2 = 0$, a contradiction.

Proof of Lemma 2.30 case $n = 4$. We continue from Equation (2.47). Since $v \in \mathbb{Z}^d$ was arbitrary,
we can write:

\[ f_0(a_1, \ldots, a_4) = \sum_{i,j=1}^{6} \eta_{i,j} a_i(gv_0) a_j(gv_0) \]  

(2.52)

where \( a_i(v); i = 1, \ldots, 12 \) are the coefficients of an element \( v \in V_\mathbb{R} \). Fix \( v_0 = (A_0, B_0) \in V_\mathbb{R}^{(0)} \) where \( A_0, B_0 \) is a pair of symmetric 3 \times 3 matrices (see [6, p.46]). Write:

\[
\gamma_1 (\langle \pi_1, u \rangle) + \gamma_2 (\langle \pi_2, u \rangle) + \gamma_3 (\langle \pi_3, u \rangle) = \sum_{i,j=1}^{6} \eta_{i,j} a_i(gv_0) a_j(gv_0). 
\]

For a general \( v \in V_\mathbb{R}^{(0)} \) write \( v = (A(v), B(v)) \) where \( A(v), B(v) \) is a pair of symmetric 3 \times 3 matrices (see [6, p.46]). Equation (2.52) is equivalent to:

\[
\gamma_1 (\langle \pi_1, u \rangle) + \gamma_2 (\langle \pi_2, u \rangle) + \gamma_3 (\langle \pi_3, u \rangle) = \sum_{i,j=1}^{6} \eta_{i,j} a_i(gv_0) a_j(gv_0). 
\]

(2.54)

Write \( \kappa_i = \langle P_{v_0} \pi_i, \rangle; i = 1, 2, 3 \) and \( g = \begin{pmatrix} -v_1 & -v_2 & -v_3 \end{pmatrix} \) then it follows from Subsection A.2:

\[
\langle P_{v_0} \pi_1, u \rangle = \text{det}(g) \langle v_1, \pi \rangle, 
\]

(2.55)

\[
\langle P_{v_0} \pi_2, u \rangle = \text{det}(g) \langle v_2, \pi \rangle, 
\]

(2.56)

\[
\langle P_{v_0} \pi_3, u \rangle = \text{det}(g) \langle v_3, \pi \rangle. 
\]

(2.57)

where \( \pi = (\kappa_1, \kappa_2, \kappa_3) \). If \( g \in U \), by [5, p.46]:

\[
A(gv_0) = g A_0 g^t = \begin{pmatrix} -v_1 & -v_2 & -v_3 \end{pmatrix} \begin{pmatrix} a_0 & d_0/2 & f_0/2 \\ d_0/2 & b_0 & e_0/2 \\ f_0/2 & e_0/2 & c_0 \end{pmatrix} \begin{pmatrix} -v_1 & -v_2 & -v_3 \end{pmatrix} = \begin{pmatrix} \langle h_1, v_1 \rangle & \langle h_1, v_2 \rangle & \langle h_1, v_3 \rangle \\ \langle h_2, v_1 \rangle & \langle h_2, v_2 \rangle & \langle h_2, v_3 \rangle \\ \langle h_3, v_1 \rangle & \langle h_3, v_2 \rangle & \langle h_3, v_3 \rangle \end{pmatrix} 
\]

(2.58)

A general entry of the matrix above will be denoted by \( (i,j) := \langle v_i, \langle h_1, v_j \rangle, \langle h_2, v_j \rangle, \langle h_3, v_j \rangle \rangle \).
Therefore, by Equations (2.54), (2.55), (2.56) and (2.57) we get:

\[
\det(g)(\gamma_1 \langle v_1, \kappa \rangle + \gamma_2 \langle v_2, \kappa \rangle + \gamma_3 \langle v_3, \kappa \rangle) = \sum_{i_1, i_2, j_1, j_2=1,\ldots,6} \eta_{i_1, i_2, j_1, j_2} (i_1, j_1)(i_2, j_2). \tag{2.59}
\]

Write \( v_i = (v_{i1}, v_{i2}, v_{i3}) \) for every \( i = 1, 2, 3 \). Assume without loss of generality that \( \kappa_1 \neq 0 \) and \( \gamma_1 \neq 0 \) (it will be clear from the argument why this assumption preserves the generality). Consider the coefficients \( c_1, c_2 \) of \( v_{11}^2 v_{22} v_{33} \) and \( v_{11}^2 v_{23} v_{32} \) respectively on the left hand side of Equation (2.59). By definition of \( \det(g) \) and Equation (2.59) it holds that \( c_1 = -c_2 = \kappa_1 \gamma_1 \neq 0 \). However, direct calculation shows that the coefficient of \( v_{11}^2 v_{22} v_{33} \) on the right hand side of Equation (2.59) is \( \eta_{1,1,2,3} h_{11} h_{23} + \eta_{3,1,2,3} h_{13} h_{12} \). The same calculation shows that \( \eta_{1,1,2,3} h_{11} h_{32} + \eta_{1,3,1,2} h_{12} h_{13} \) is the coefficient of \( v_{11}^2 v_{23} v_{32} \). However, \( h \) is symmetric (recall Equation (2.53)) so these two expressions are equal. Under the assumption that Equation (2.59) poses we get that on the one hand \( c_1 = -c_2 \) and on the other \( c_1 = c_2 \). The inevitable conclusion that \( c_1 = c_2 = 0 \) contradicts our assumption that \( \kappa_1, \gamma_1 \neq 0 \). Indeed, the choice \( \kappa_1, \gamma_1 \neq 0 \) did not affect the generality: we would have otherwise considered the coefficients of \( v_{i, j}^2 v_{k, l} v_{m, p} \), \( v_{i, j}^2 v_{k, l} v_{m, p} \) for some other appropriate \( i, j, k, l, m, p \in \{1, 2, 3\} \).

**Proof of Lemma 2.69, case \( n = 5 \).** We continue from Equation (2.47). Denote:

\[
I = \{(i, j, k) : i, j = 1, \ldots, 5, k = 1, \ldots, 4\} \tag{2.60}
\]

and for \( i_0 = (i, j, k) \in I \) denote:

\[
A(i_0) = A_{i,j}^{(k)}. \tag{2.61}
\]

Since \( v \in \mathbb{Z}^d \) was arbitrary, we can write:

\[
f_0(A^{(1)}, \ldots, A^{(4)}) = \sum_{i_1, \ldots, i_5 \in I} \eta_{i_1, \ldots, i_5} A(i_1) A(i_2) A(i_3) A(i_4) A(i_5) \tag{2.62}
\]

Write \( g_i = \left( \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ u_1 & u_2 & u_3 & u_4 \end{array} \right) \) and \( g = (g_4, g_5) \). By [6, p. 52] we know that:

\[
g \cdot v_0 = \begin{bmatrix} g_4 A^{(1)} g_5^T \\ g_5 A^{(2)} g_4^T \\ g_5 A^{(3)} g_4^T \\ g_5 A^{(4)} g_4^T \end{bmatrix} \tag{2.63}
\]

where we abuse the notation \( g = (g_4, g_5) \). Now, given \( u \) as in the conditions of the Lemma we write
by \(A.2\)

\[
\langle P_v(\alpha_i(gv_0), u) = \det(g_4) \left( \sum_{j=1}^{4} v_{ij} \langle P_v(\alpha_j(v_0), u) \right) \right),
\]

\[(2.64)\]

\[
\langle P_R \mathcal{T} \alpha_i(gv_0), u \rangle = \frac{1}{n} \langle \alpha_i(gv_0), \mathcal{T} \rangle \langle u, \mathcal{T} \rangle.
\]

\[(2.65)\]

Equation (2.62) and the above two equations imply:

\[
\sum_{i_1, \ldots, i_5 \in I} \eta_{i_1, \ldots, i_5} (A(i_1)A(i_2)A(i_3)A(i_4)A(i_5))(gv_0) = f_0(A^{(1)}(gv_0), A^{(2)}(gv_0), A^{(3)}(gv_0), A^{(4)}(gv_0))
\]

\[
= \langle \alpha_i(gv_0), 1 \rangle + \frac{\det(g_4)}{\langle u, 1 \rangle} \sum_{j=1}^{4} v_{ij} \langle P_v(\alpha_j(v_0), u) \rangle + q_0
\]

\[(2.66)\]

and so:

\[
\sum_{i_1, \ldots, i_5 \in I} \eta_{i_1, \ldots, i_5} (A(i_1)A(i_2)A(i_3)A(i_4)A(i_5))(gv_0) - \langle \alpha_i(gv_0), 1 \rangle = \frac{\det(g_4)}{\langle u, 1 \rangle} \sum_{j=1}^{4} v_{ij} \langle P_v(\alpha_j(v_0), u) \rangle + q_0
\]

\[(2.67)\]

which, by Equation (2.63) is a contradiction, the left hand side of the above equation has non-trivial dependence in \(g_5\) while the right hand side doesn’t.

\[\blacksquare\]

3 Proofs

In this Section we prove Theorems 2.8 and 2.6. Assume \(n = 3, 4, 5\) and fix fundamental domains \(\Sigma_{n-1}, \Sigma_{r-1}\) for the actions of \(\text{SL}_{n-1}(\mathbb{Z})\) and \(\text{SL}_{r-1}(\mathbb{Z})\) on \(\text{SL}_{n-1}(\mathbb{R})\) and \(\text{SL}_{n-1}(\mathbb{R})\) respectively. Since we use it in the proof, we remind the reader of Definition 2.29 by repeating it once again:

**Definition.** For any hyperplane \(E = u^\perp \subset \mathbb{R}^n\), let \(f_n^E\) denote the function corresponding to the subspace \(E\) under Lemma 2.22. Explicitly, we denote:

\[
f_n^E(p) = \frac{1}{\langle u, 1 \rangle} \sum_{i=1}^{n-1} \langle u, \alpha_i(g_p) \rangle w_i + q_0
\]

for any \(p \in \mathbb{R}_+ \cdot (\Sigma_{n-1} \times \Sigma_{r-1})v_0\) (recall that in Definition 2.29 \((w_i)_{i=1}^{n-1}\) was some fixed basis of \(V_0\)).

Before proving the main Theorems we make a reduction that will serve us both in the proof of Theorem 2.6. First, for convenience of this discussion, given a choice of orderings namely \(\Pi : \mathcal{F}_\infty \rightarrow S_n\) (recall Subsection 2.1 for the definition of \(\mathcal{F}_\infty\)), write \(ED(\Pi)\) if (recall Subsection 2.2
for the definition of $\Gamma^H(F)$:

$$\Gamma^H(F)$$ is equidistributed in $Y_{n-1}$ when $F \in F_n$ are ordered by discriminant.

For $n = 3, 4, 5$, if there exists $\Pi : F^{(0)}_\infty \rightarrow S_n$ such that ED($\Pi$) and for any $\pi \in S_n$, ED($\pi \circ \Pi$) where $(\pi \circ \Pi)(F) := \pi \circ (\Pi(F))$ then averaging the counting measures for each $\pi$ proves the equidistribution statement in Theorem 2.6. Since we find such $\Pi$ using Lemma 2.22, the extra property $\text{ED}(\Pi)$ for any $\pi \in S_n$ will hold by the last part of this Lemma with the same argument. We continue with this reduction in mind.

**Proof of Theorem 2.6** Fix $n = 3, 4, 5$ and let $\mathbf{T} \neq u \in \mathbb{R}^n$ such that $u \notin V_0$ and denote $E = u^\perp$. Let $v_0 \in V^{(0)}_e$ and let $\Sigma_{n-1}$ denote a fundamental domain for the action of $\text{SL}_{n-1}(\mathbb{Z})$ on $\text{SL}_{n-1}(\mathbb{R})$ corresponding to $E$ by Lemma 2.22. Let $\Pi : F^{(0)}_X \rightarrow S_n$ be the function and let $\Sigma_{n-1}$ be the fundamental domain for the action of $\text{stab}_{\text{SL}_{n-1}(\mathbb{R})}(\Lambda_w)$ on $\text{SL}_{n-1}(\mathbb{R})$ which corresponds to $E$ and $v_0$ by Lemma 2.22. Abbreviate $f_E = f_n^E$. For an open bounded and Jordan measurable subset $S \subset \Sigma_{n-1}$ and $V = I_1w_1 + \cdots + I_{n-1}w_{n-1}$ where $I_i \subset [0, 1]$ are open intervals, let $A = S \times \Sigma_{n-1}$ $V \subset Y_{n-1}$ be a $\Lambda_w$-equidistributed subset (recall Definition 2.10). From here on, we abbreviate “open bounded and Jordan measurable” to OBJOM.

For every $\epsilon > 0$ let $\Sigma_{r-1} \subset \Sigma_{n-1}$ be OBJOM and such that:

$$\text{Vol}([0, 1] \cdot (S \times \Sigma_{r-1})v_0) \geq \text{Vol}([0, 1] \cdot (S \times \Sigma_{r-1})v_0) \geq (1 - \epsilon) \text{Vol}([0, 1] \cdot (S \times \Sigma_{r-1})v_0). \quad (3.1)$$

For every $T > 0$ denote:

$$S_T = [0, T] \cdot (S \times \Sigma_{r-1})v_0, S_T = [0, T] \cdot (S \times \Sigma_{r-1})v_0; \quad (3.2)$$

$$S_{T,Z} = S_T \cap \{\text{irreducible maximal points}\}, S_{T,Z} = S_T \cap \{\text{irreducible maximal points}\}; \quad (3.3)$$

$$L_T = [0, T] \cdot (\Sigma_{n-1} \times \Sigma_{r-1})v_0, L_{T,Z} = L_T \cap \{\text{irreducible maximal points}\}; \quad (3.4)$$

$$p_T^Z(V) = \frac{|S_{T,Z} \cap f^{-1}_E(V)|}{|S_{T,Z}|}. \quad (3.5)$$

Repeating the proof of Theorem 5 in [4] word by word but replacing the shape function (denoted as $q$ in [4, p. 7]) with the lattice function $l : V^{(0)}_\phi \rightarrow X_{n-1}$ defined for every $v = t(v)(g_1(v), g_2(v))v_0 \in \mathbb{R}_+ \cdot \Sigma_{n-1} \times \Sigma_{r-1}v_0$ by:

$$l(v) = g_0g_1(v)t g_0^{-1} \Lambda_w, \quad (3.6)$$

we get:

$$|S_{T,Z}| = \text{Vol}(S_T) + o(T^d), |L_{T,Z}| = \text{Vol}(L_T) + o(T^d). \quad (3.7)$$

25
Given a subset $B \subset Y_{n-1}$ and $T > 0$ denote $\mu_T(B)$ to be the proportion of totally real number fields $\mathbb{F}$ with discriminant less than $T$ that satisfy $\Gamma_E^3(\mathbb{F}) \in B$ (recall Subsection 2.2). Invoke Lemma 2.22 to deduce that for any $T > 0$:

$$
\mu_T(A) = \frac{|S_{T,Z} \cap f_{E}^{-1}(V)|}{|L_{T,Z}|} = \frac{|S_{T,Z} \cap f_{E}^{-1}(V)|}{|S_{T}^r \cap f_{E}^{-1}(V)|} \cdot \frac{|S_{T}^r \cap f_{E}^{-1}(V)|}{|S_{T,Z}|} \cdot \frac{|S_{T,Z}|}{|L_{T,Z}|} \quad (3.8)
$$

We estimate every element in the above product separately. First, we deal with $p_T(V) = \frac{|S_{T,Z} \cap f_{E}^{-1}(V)|}{|S_{T}^r|}$. By application of Lemma 2.30 for the OBOM set $(S \times \Sigma_{r-1}^e) v_0$:

$$
p_T(V) \to \text{Vol}_{n-1}(V) \text{ as } T \to \infty. \quad (3.9)
$$

By Equation 3.7:

$$
\frac{|S_{T,Z}|}{|L_{T,Z}|} \to \frac{\text{Vol}(S_1)}{\text{Vol}(L_1)} \text{ as } T \to \infty \quad (3.10)
$$

and by the proof of [4 Proposition 12] applied for $l^{-1}(S) \cap \{\text{Disc}(v) \leq 1\}$ instead of $R_{1,W}$:

$$
\frac{|S_{T,Z}|}{|L_{T,Z}|} \to m_{X_{n-1}}(S) \text{ as } T \to \infty \quad (3.11)
$$

(recall Subsection 2.2 for definition of $m_{X_{n-1}}$, the Haar measure on the space of lattices). Since $S_1^r$ is bounded we can use [4 Lemma 6] on it to deduce that for any $T > 0$:

$$
|S_{T,Z}| = \text{Vol}(S_1^r) + o(T^d) \quad (3.12)
$$

which implies, by Equations 3.7 and 3.1 that:

$$
\frac{|S_{T,Z}|}{|S_{T}^r|} = 1 - \epsilon + o(1) \quad (3.13)
$$

Consequently, by dividing the space into its disjoint intersections with $f_{E}^{-1}(V), f_{E}^{-1}(V^c)$:

$$
(|S_{T,Z} \cap f_{E}^{-1}(V)| - |S_{T}^r \cap f_{E}^{-1}(V)|) + (|S_{T,Z} \cap f_{E}^{-1}(V^c)| - |S_{T}^r \cap f_{E}^{-1}(V^c)|) \leq \epsilon|S_{T,Z}| + o(|S_{T,Z}|)
$$

which implies, since both terms in parenthesis in the above equation are non-negative:

$$
\frac{|S_{T,Z} \cap f_{E}^{-1}(V)|}{|S_{T}^r \cap f_{E}^{-1}(V)|} - 1 \leq \epsilon \frac{|S_{T,Z}|}{|S_{T}^r \cap f_{E}^{-1}(V)|} + o \left( \frac{|S_{T,Z}|}{|S_{T}^r \cap f_{E}^{-1}(V)|} \right) = \epsilon p_T(V) + o(p_T) \quad (3.14)
$$
but now using (3.9):
\[
\frac{|S_{T,Z} \cap f^{-1}_E(V)|}{|S'_{T,Z} \cap f^{-1}_E(V)|} = 1 + \epsilon(\text{Vol}(V) + o_T(1)).
\] (3.15)

Using (3.11), (3.13), (3.9) and (3.15) and taking \( \epsilon \to 0 \) we deduce:
\[
\mu_T(A) \to m_{X_{n-1}}(S) \text{Vol}(V) = m_{Y_{n-1}}(A)
\] (3.16)

where the last equality holds by Fubini’s Theorem and Definition 2.10. To deduce the above limit for any other Jordan measurable subset \( A \subset Y_{n-1} \) invoke Theorem B.3.

\[ \square \]

\textbf{Proof of Theorem 2.8.} The reader may find it useful to recall the definition of the corresponding shape which appears in Subsection 2.2. By Lemma 2.22, there exists a fixed basis of \( V_0 \) given by \( w = (w_j)_{j=1}^{n-1} \), a function \( \Pi : F_{\infty}^0 \to S_n \) (recall Subsection 2.1) and \( g_0 \in \text{SL}_{n-1}(\mathbb{R}) \) such that:
\[
\Sigma_{n-1} := g_0^t \Sigma_{n-1} g_0^{-t}
\] (3.17)
satisfies that for any \( \Lambda_w - \Sigma_{n-1} \)-basic subset \( S \times \Sigma_{n-1} U \subset Y_{n-1} \):
\[
\mathcal{F}_X^{(0)} \cap (\Gamma_{V_0}^{\Pi})^{-1}(S \times \Sigma_{n-1} U) \overset{1:1}{\longleftrightarrow} \{ \text{irreducible, maximal points inside } [0, X](g_0^{-t} S^{-1} g_0 \times \Sigma_{n-1})v_0 \} \cap ((f_w + g_0) \mod \Lambda_w)^{-1}(U_w)
\] (3.18)

where \( g_0 \in V_0 \) is some fixed vector (recall Definition 2.11 for definition of \( U_w \)) and in this case:
\[
f_w(p) = \frac{1}{\langle T, T \rangle} \sum_{i=1}^{n-1} \langle T, \pi_i(g_p v_0) \rangle \ w_i = \frac{1}{n} \sum_{i=1}^{n-1} \langle T, \pi_i(g_p v_0) \rangle \ w_i
\] (3.19)

by the discussion carried in [6] p.35 for \( n = 3 \), p.41 for \( n = 4 \) and p.50 for \( n = 5 \) for every \( i = 1, \ldots, n-1 \):
\[
\langle T, \pi_i(g_p v_0) \rangle \in \mathbb{Z}.
\] (3.20)

Let \( w_0 = \sum_{i=1}^{n-1} \frac{1}{n} w_i \) and let \( \Lambda_w = \text{span}_F\{w_1, \ldots, w_{n-1}\} \). Note that \( nw_0 \in \Lambda_w \) and \( lw_0 \notin \Lambda_0 \) for all \( l = 1, \ldots, n-1 \). Denote:
\[
D = \{ \sum_{i=1}^{n-1} \frac{j_i}{n} w_i \mod \Lambda_w : j_i = 1, \ldots, n \}.
\] (3.21)

By Equations (3.18)-(3.20) for every totally real \( S_n \)-number field \( F \) of degree \( n \):
\[
\Gamma_{V_0}^{\Pi}(F) \in \Sigma_{n-1} \times \Sigma_{n-1} D.
\] (3.22)

Let \( \mu \) be any weak limit of the uniform counting measures on the finite sets \( \mathcal{F}_X^{(0)} \). Let \( m_{\Sigma_{n-1}} \) be
the restriction of $m_{\text{SL}_{n-1}({\mathbb R})}$ to $\Sigma_{n-1}$.

For every $d \in D$ let $F_d : \Sigma_{n-1} \to \Omega_{n-1}$ be defined by $g \mapsto g(\Lambda w + d)$ and denote $\eta_d = (F_d)_* m_{\Sigma_{n-1}}$ to be a $d\Sigma_{n-1}$ atom (recall Definition 2.7). Denote:

$$\eta = \frac{1}{|D|} \sum_{d \in D} \eta_d$$

(3.23)

to be a $\Sigma_{n-1}$-torsion measure on $\Omega_{n-1}$.

Let $N \subset \Omega_{n-1}$ be an $\eta$-null set (namely $\eta(N) = 0$). Recall that given any grid $M = \Lambda' + w' \subset V_0$, the translating vector of $M$ is the element of $V_0/\Lambda'$ given by $w' \mod \Lambda'$. Define a map $\text{vec}_0 : \Omega_{n-1} \to V_0/\Lambda_w$ as follows. Write any grid $M$ in $\Omega_{n-1}$ as $g \Lambda w + w'$ for a unique $g \in \Sigma_{n-1}$ and define:

$$\text{vec}_0(M) = g^{-1}w' \mod \Lambda_w.$$

(3.24)

By definition of $\eta$, we can write:

$$0 = \eta(N) = \sum_{d \in D} \eta(N \cap \text{vec}_0^{-1}(d)).$$

(3.25)

By Equation (3.22) the measure $\mu$ satisfies:

$$\mu(N) = \sum_{d \in D} \mu(N \cap \text{vec}_0^{-1}(d))$$

(3.26)

so it suffices to check $\mu(N \cap \text{vec}_0^{-1}(d)) = 0$ for every $d \in D$. Fix $d \in D$ and let $A_{d,N} \subset \Sigma_{n-1}$ be such that:

$$N \cap \text{vec}_0^{-1}(d) = A_{d,N} \times \Sigma_{n-1} \{d\}.$$

(3.27)

By definition of $\eta$ and $N$:

$$0 = \eta(N \cap \text{vec}_0^{-1}(d)) = m_{\Sigma_{n-1}}(A_{d,N}).$$

(3.28)

Given a subset $B \subset \Omega_{n-1}$ and $T > 0$ denote $\mu_T(B)$ to be the proportion of totally real number fields $\mathbb F$ with discriminant less than $T$ that satisfy $\Gamma_{V_0}^\dagger(\mathbb F) \in B$ (recall Subsection 2.2). For every subspace $E \subset \mathbb R^n$ of co-dimension 1 and every $A \subset \Sigma_{n-1}$ it holds that:

$$(\Gamma_{V_0}^\dagger)^{-1}(A \times \Sigma_{n-1} [0,1]^{n-1}) = (\Gamma_{E}^\dagger)^{-1}(A \times \Sigma_{n-1} [0,1]^{n-1})$$

(3.29)

therefore by Theorem 2.6 and Equation (3.28):

$$\mu_T(A_{d,N} \times \Sigma_{n-1} \{d\}) \leq \mu_T(A_{d,N} \times \Sigma_{n-1} [0,1]^{n-1}) \to m_{\Sigma_{n-1}}(A_{d,N}) = 0$$

(3.30)
which implies:

$$\mu(N \cap \text{vec}_0^{-1}(d)) = \mu(A_{d,N} \times \Sigma_{n-1} \{d\}) = 0$$

(3.31)

and consequently that $\mu(N) = \sum_{d \in D} \mu(N \cap \text{vec}_0^{-1}(d)) = 0$ and $\mu \ll \eta$ as desired.
A Calculations

A.1 Multiplication Tables

For convenience we include explicitly the multiplication tables that define Bhargava’s correspondence in \([2, 3]\). They are copied from \([6, \text{p. 35}, 39-40]\).

A.1.1 Case \(n = 3\)

Let \(v \in V_R\) be given by \(v = (a, b, c, d)\). The multiplication table corresponding to \(v\) is:

\[
\begin{align*}
\alpha_1 \alpha_2 &= -ad \\
\alpha_1^2 &= -ac + ba_1 - a\alpha_2 \\
\alpha_2^2 &= -bd + d\alpha_1 - c\alpha_2.
\end{align*}
\] (A.1)

A.1.2 Case \(n = 4\)

Let \(v \in V_R\) be given by \(v = (a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz, b_{11}x^2 + b_{22}y^2 + b_{33}z^2 + b_{12}xy + b_{13}xz + b_{23}yz)\). The multiplication table corresponding to \(v\) is:

\[
\begin{align*}
\alpha_1^2 &= h_{11} + g_{11}\alpha_1 + f_{11}\alpha_2 + e_{11}\alpha_3 \\
\alpha_2^2 &= h_{22} + g_{22}\alpha_1 + f_{22}\alpha_2 + e_{22}\alpha_3 \\
\alpha_3^2 &= h_{33} + g_{33}\alpha_1 + f_{33}\alpha_2 + e_{33}\alpha_3 \\
\alpha_1 \alpha_2 &= h_{12} + e_{12}\alpha_3 \\
\alpha_1 \alpha_3 &= h_{13} + f_{13}\alpha_2 + e_{13}\alpha_3 \\
\alpha_2 \alpha_3 &= h_{23} + g_{23}\alpha_1 + f_{23}\alpha_2 + e_{23}\alpha_3.
\end{align*}
\] (A.2)

A.1.3 Case \(n = 5\)

Let \(v \in V_R\) be given by \(v = (A_1, \ldots, A_4)\) where \(A_i\) are \(5 \times 5\) skew symmetric matrices. The multiplication table corresponding to \(v\) is:

\[
\alpha_i \alpha_j = c_{ij}^0 + \sum_{k=1}^{4} c_{ij}^k \alpha_k
\] (A.3)
for some constants $c_{ij}^k$ which are some polynomials in the entries of $(A_1, \ldots, A_4)$ and appearing in [8, p. 51].

### A.2 Construction of a Basis

Let $n = 3, 4, 5$ and fix $v_0 \in V^{(0)}_\mathbb{R}$. Let $\pi_1(v_0), \ldots, \pi_n(v_0) \in \mathbb{R}^n, \pi_0(v_0) = \mathbf{1}$ be a set of vectors such that $MT_{v_0}(\pi_1, \ldots, \pi_n)$ with the pointwise product on $\mathbb{R}^n$.

For any $g = \begin{pmatrix} g_1 & \cdots & g_{n-1} \\ \\ \end{pmatrix}$ and $h \in SL_{r-1}(\mathbb{R})$, write $v = t(g, h)v_0; t \in \mathbb{R}$ in a unique way (it is indeed unique by [4, Section 3]) and for any $i = 1, \ldots, n - 1$, denote:

$$\pi_{i, v_0} = t \langle g_i, P_{v_0} \pi_i(v_0) \rangle.$$  \hspace{1cm} (A.4)

By the last sentence of [4, Theorem 3] we know that there exist vectors $\pi_1(v), \ldots, \pi_{n-1}(v) \in \mathbb{R}^n$ such that $MT_v(\pi_1(v), \ldots, \pi_{n-1}(v))$ and:

$$P_{v_0} \pi_i(v) = \pi_{i, v_0}$$  \hspace{1cm} (A.5)

Therefore by Equations (A.4), (A.5):

$$\begin{align*}
\text{span}_2\{P_{v_0} \pi_1(v), \ldots, P_{v_0} \pi_{n-1}(v)\} &= g_0 g_0^{-1} \text{span}_2\{P_{v_0} \pi_1(v_0), \ldots, P_{v_0} \pi_{n-1}(v_0)\} \\
\text{where:} \quad g_0 &= \begin{pmatrix} \iota & \cdots & \\ \\ P_{v_0} \pi_1(v_0) & \cdots & P_{v_0} \pi_{n-1}(v_0) \\ \iota & \cdots & \\
\end{pmatrix}.
\end{align*}$$

(A.6)

For any $i = 1, \ldots, n - 1$ write:

$$\pi_i(v) = P_{v_0} \pi_i(v) + \text{trace}(\pi_i(v)) \mathbf{1}.$$  \hspace{1cm} (A.7)

Since the action of $\mathbb{R} \times SL_{n-1} \times SL_{r-1}(\mathbb{R})$ on $V_\mathbb{R}$ is smooth and since $v \mapsto \text{trace}(\pi_i(v))$ is smooth by direct computation of the trace in the multiplication table (see [A.1]), we deduce that the functions $v \mapsto \pi_i(v)$ are smooth for $i = 1, \ldots, n - 1$.

### A.3 Additional Lemmas

In this part of the Appendix we state several supporting results for Subsection 2.5.
Lemma A.1. Let $d \in \mathbb{N}$. Let $S$ be a level surface of some smooth homogeneous function (as in Theorem 2.2) on $\mathbb{R}^d$ and let $A \subset S$ be bounded, open and Jordan measurable in $S$. Let $F : A \to \mathbb{R}$ be smooth. If $\nabla F(x) \neq 0$ for $m_A$ a.e. $x \in A$ (where $m_A$ is the surface measure on $A$) then $F_*m_A \ll \lambda_\mathbb{R}$ where $\lambda_\mathbb{R}$ is the Lebesgue measure on $\mathbb{R}$.

Proof. We first prove the claim when there exists $c_0 > 0$ such that $\|\nabla F\| > c_0$ on $A$ (and therefore on $\mathbb{R}$ because $F$ is smooth). In this case, by compactness of $\mathbb{A}$ we can find $c > 0$ and $c_0 > 0$ such that for any $\epsilon < c_0$ it holds that $\lambda_\mathbb{R}(F(B^\mathbb{R}_\epsilon(x))) \geq c\|\nabla F(x)\| m_A(B^\mathbb{R}_\epsilon(x)) \geq c\text{ccon}(A,B^\mathbb{R}_\epsilon(x))$ where $B^\mathbb{R}_\epsilon$ is an (e.g. Euclidean) open ball of radius $\epsilon$ in $\mathbb{R}$. By standard estimation argument, it will hold that $F_*m_A \ll \lambda_\mathbb{R}$. Next we prove the Lemma's claim in the general context. Define the closed subset of $A$ given by $E = \{x \in A : \nabla F(x) = 0\}$ and let $E_n \supset E$ be a sequence of open subsets of $A$ such that $m_A(E_n) \to 0$ as $n \to \infty$. For any $n$, denote $\mu_n = 1_{E_n}m_A$ to be a sequence of measures on $A$ such that $\mu_n \to m_A$ weakly as $n \to \infty$. Let $S \subset \mathbb{R}$ be a $\lambda_\mathbb{R}$-null set. By compactness of $S$, there exists $c_n > 0$ such that $\|\nabla F\| \geq c_n$ on $S \cap E_n^c$ so by what we proved before: $F_*\mu_n \ll \lambda_\mathbb{R}$ for every $n$. Since $F_*\mu_n \to F_*m_A$ weakly as $n \to \infty$ we deduce $F_*m_A \ll \lambda_\mathbb{R}$ as desired. ■

Lemma A.2. Let $n = 3, 4, 5$, $d = 4, 12, 40$, respectively and let $U \subset V^{(0)}_\mathbb{R}$ be an open subset. Assume that $\alpha_1, \ldots, \alpha_{n-1} : U \to \mathbb{R}^n$ are smooth and satisfy (A.1) if $n = 3$, (A.2) if $n = 4$ and (A.3) if $n = 5$. Then $\alpha_1, \ldots, \alpha_{n-1}$ are real analytic on $U$.

Proof. Let $v_0 \in U$ and denote $\phi : \mathbb{R} \times \text{SL}_{n-1}(\mathbb{R}) \times \text{SL}_{r-1}(\mathbb{R}) \to V^{(0)}_\mathbb{R}$ the isomorphism given by $(t,g,h) \mapsto t(g,h)v_0$. By its definition, the action of $\mathbb{R} \times \text{SL}_{n-1}(\mathbb{R}) \times \text{SL}_{r-1}(\mathbb{R})$ on $V^{(0)}_\mathbb{R}$ is by polynomials in the coefficients of $(t,g,h) \in \mathbb{R} \times \text{SL}_{n-1}(\mathbb{R}) \times \text{SL}_{r-1}(\mathbb{R})$ (see [6, p. 38, 47, 52] or [2], [3] for definition of this action) so that the map $\phi$ is real analytic. Therefore, by the inverse functions theorem for real analytic functions (see [7, Theorem 1.8.1]), so is the map $\phi^{-1}$. By Equation (A.1) it is evident that for any $i = 1, \ldots, n-1$ the function $\overline{\alpha}_i \circ \phi$ is real analytic. Indeed, the equation says:

$$
\overline{\alpha}_i(\phi(t,g,h)) = \overline{\alpha}_i(t(g,h)v_0) = P_{v_0} \overline{\alpha}_i(t(g,h)v_0) + \text{trace}(\overline{\alpha}_i(t(g,h)v_0)\overline{\alpha}_i) = t \overline{\alpha}_i + P_{v_0} \overline{\alpha}_i(v_0) + P(t,g,h)
$$

(A.8)

where $g = \begin{pmatrix} & \cdots & \cr g_1 & \cdots & g_{n-1} \cr \end{pmatrix}$ and $P$ is some polynomial in $t$ and the entries of $g, h$ by direct calculation from (A.1) to find $\text{trace}(\overline{\alpha}_i(v))$ in terms of $v$ and [6, p. 38, 47, 52] to write $v = t(g,h)v_0$ in terms of $t, g, h$. Altogether we deduce that the function:

$$
\overline{\alpha}_i = (\overline{\alpha}_i \circ \phi) \circ \phi^{-1}
$$

(A.9)

is the composition of two real analytic functions and therefore real analytic. ■
for Basic subsets, satisfying the conclusion of the Lemma. Then by Lemma B.1 and Lemma B.2 applied

\[ \lim_{\epsilon \to 0} \mu(W) = \mu(U) = 0. \]

Theorem 2.6 (see 3). Then

\[ \text{Theorem B.3.} \]

Let

\[ \text{Lemma B.2.} \]

We may assume that

\[ \text{Lemma B.1.} \]

Proof. We may assume that

\[ \text{Lemma A.3 (7), Chapter 3) } \]

For this Section, fix a lattice \( \Lambda = \text{sp}_{\mathbb{Z}}\{w_1, \ldots, w_n\} \subset \mathbb{R}^n \) and a fundamental domain \( \Sigma \) for the action of \( \text{stab}_{\text{SL}_n(\mathbb{R})}(\Lambda) \) on \( \text{SL}_n(\mathbb{R}) \). We abbreviate ‘\( \Sigma \)-\( \Lambda \) basic subsets’ to ‘basic subsets’.

**Lemma B.2.** Let \( W \subset Y_n \) be a measurable set with \( \mu(\partial W) = 0 \). Then for every \( \epsilon > 0 \) there exist disjoint basic subsets, such that

\[ \sum_{i=1}^m \mu(B_i) \geq \mu(W) - \epsilon. \]

Proof. We may assume that \( W \) is bounded, since otherwise we may intersect \( W \) with large enough bounded ball in \( Y_n \). Let \( U \supset \partial W \) be open such that \( \mu(U) \leq \epsilon \). For any \( x \in \overline{W \setminus U} \), let \( B_x \subset \text{int}(W) \) be a Basic subset. Take a finite sub-cover of \( \overline{W \setminus U} \), \( (B_i)_{i=1}^m \). Then clearly the \( B_i \)'s are included in \( W \) (they are not necessarily disjoint), and:

\[ \mu \left( \bigcup_{i=1}^m B_i \right) \geq \mu(W) - \mu(W \cap U) \geq \mu(W) - \mu(U) \geq \mu(W) - \epsilon. \]

Finally, note that the intersection of every two basic subsets can be estimated by finite union of disjoint basic subsets with arbitrarily small defect, so we may also assume that the \( B_i \)'s are disjoint.

The following Lemma is a Corollary of Lemma 2.12 and the definition of basic subsets.

**Theorem B.3.** Let \( W \subset Y_{n-1} \) be measurable with \( \mu \)-zero boundary and let \( E \) be as in the proof of Theorem 2.6 (see 3). Then

\[ \lim_{\epsilon \to 0} \mu^E_X(W) = \mu(W). \]

Proof. We may assume that \( W \) is bounded, since otherwise we can intersect \( W \) with large enough bounded ball in \( Y_{n-1} \). Let \( \epsilon > 0 \) and using the Lemma B.1 find \( B_1, \ldots, B_m \), a collection of disjoint Basic subsets, satisfying the conclusion of the Lemma. Then by Lemma B.1 and Lemma B.2 applied for \( W \setminus \bigcup_{i=1}^m B_i \) (resulting in a collection \( (D_i)_{i=1}^k \)), we get:

\[ |\mu(W) - \mu^E_X(W)| \leq |\mu(\bigcup_{i=1}^m B_i) - \mu^E_X(\bigcup_{i=1}^m B_i)| + |\mu(W \setminus \bigcup_{i=1}^m B_i) - \mu^E_X(W \setminus \bigcup_{i=1}^m B_i)| \]

\[ \leq |\mu(\bigcup_{i=1}^m B_i) - \mu^E_X(\bigcup_{i=1}^m B_i)| + \epsilon + \mu^E_X(\bigcup_{i=1}^k D_i) \leq |\mu(\bigcup_{i=1}^m B_i) - \mu^E_X(\bigcup_{i=1}^m B_i)| + \sum_{i=1}^k \mu^E_X(D_i) + \epsilon, \]

33
implying, by Lemma B.1, Equation (3.16) and since the $B_i$’s are disjoint, that for any $\epsilon > 0$, 
\begin{equation}
\limsup_{X \to \infty} |\mu(W) - \mu^E_X(W)| \leq 3\epsilon,
\end{equation}
as required.
References

[1] Frank W. Anderson and Kent R. Fuller. Rings and categories of modules, volume 13 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1992.

[2] Manjul Bhargava. Higher composition laws. III. The parametrization of quartic rings. Ann. of Math. (2), 159(3):1329–1360, 2004.

[3] Manjul Bhargava. Higher composition laws. IV. The parametrization of quintic rings. Ann. of Math. (2), 167(1):53–94, 2008.

[4] Manjul Bhargava and Piper Harron. The equidistribution of lattice shapes of rings of integers in cubic, quartic, and quintic number fields. Compos. Math., 152(6):1111–1120, 2016.

[5] B. N. Delone and D. K. Faddeev. The theory of irrationalities of the third degree. Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.

[6] Piper Harron. The Equidistribution of Lattice Shapes of Rings of Integers of Cubic, Quartic, and Quintic Number Fields: an Artist’s Rendering. 2016.

[7] Steven G. Krantz and Harold R. Parks. A primer of real analytic functions. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[8] K. Mahler. On lattice points in n-dimensional star bodies. I. Existence theorems. Proc. Roy. Soc. London Ser. A, 187:151–187, 1946.

[9] M. Sato and T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J., 65:1–155, 1977.

[10] Hermann Weyl. Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann., 77(3):313–352, 1916.

[11] Yuval Yifrach. A note about weyl equidistribution theorem, 2022.