NEW CONSERVED INTEGRALS AND INVARIANTS
OF RADIAL COMPRESSIBLE FLUID FLOW
AND GAS DYNAMICS IN $n > 1$ DIMENSIONS

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ABSTRACT. Conserved integrals and invariants (advected scalars) are studied for the
equations of radial compressible fluid flow in $n > 1$ dimensions. Apart from entropy, which is a
well-know invariant, three additional invariants are found from an explicit determination of
invariants up to first-order. One holds for a general equation of state, and the two others hold
only for entropic equations of state. A recursion operator on invariants is presented, which
produces two hierarchies of higher-order invariants. Each invariant yields a corresponding
integral invariant, describing an advected conserved integral on transported radial domains.
In addition, a direct determination of kinematic conserved densities uncovers two “hidden”
non-advected conserved integrals: one describes enthalpy-flux, holding for barotropic equa-
tions of state; the other describes entropy-weighted energy, holding for entropic equations
of state. A further explicit determination of a class of first-order conserved densities shows
that the corresponding non-kinematic conserved integrals on transported radial domains are
equivalent to integral invariants, modulo trivial densities. Since the equations of compress-
ible fluid flow are well known to be equivalent to the equations of gas dynamics, all of the
results obtained for radial fluid flow carry over to radial gas dynamics.

1. Introduction

A significant interest in conserved integrals and invariants of inviscid compressible fluid
dynamics in $n > 1$ dimensions has been on-going for several decades, [17, 27, 23, 25, 2, 3, 1, 4,
13], motivated by Hamiltonian structures and group-theoretical properties of fluid mechanics
[9, 10, 18, 29, 24, 11]. Conserved integrals are global balance equations (conservation laws)
of the form $\frac{d}{dt} \int_{V(t)} C \ dx = -\oint_{\partial V(t)} \vec{F} \cdot \vec{n} \ d^{n-1}A$ on any domain $V(t)$ transported in the flow,
where $C$ is the conserved density and $\vec{F}$ is outward flux on the moving boundary $\partial V(t)$.
These integrals describe physically conserved quantities. Invariants are quantities (scalars,
vectors, 1-forms, etc.) $J$ that are advected by the flow, $\partial_t J + \mathcal{L}_\vec{u} J = 0$, where $\mathcal{L}_\vec{u}$ is the
Lie derivative with respect to the velocity $\vec{u}$. These quantities are, physically, frozen into
the flow. There is a direct correspondence between scalar invariants and conserved integrals
whose moving flux vanishes, called integral invariants, as given by $C = \rho J$ in terms of the
density $\rho$, with $\vec{F} \equiv 0$.

Currently, a complete description is known for conserved integrals of kinematic type and
vorticity type [2, 3], and a geometric procedure is known for generating higher-order in-
variants [30, 13, 6] from the set of basic invariants. Yet some open questions remain to be
addressed: What is the complete set of all invariants up to first-order? Does every non-trivial
scalar invariant yield a non-trivial integral invariant? Are there any conserved integrals with
conserved densities of low order other than the kinematic and vorticity ones? Do conserved integrals exist having higher-order conserved densities?

The present paper fully answers these questions for $n$-dimensional compressible radial fluid dynamics and gas dynamics. There are several interesting reasons to study radial flows. A priori, since they arise from radial reduction of the $n$-dimensional governing equations, additional conserved integrals and invariants may exist which are not inherited through this reduction. The resulting radial equations have physical applications to explosive and implosive flows (see e.g. [28, 22, 31, 21, 32, 15], as well as numerous engineering applications such as radial flow turbines, pumps, impellers, and compressors (see e.g. [12, 16]). Moreover, the radial equations describe the simplest case of more general zero-vorticity flows.

For generality, radial flows with a general equation of state will be considered. In the case of fluids, the pressure will be a general function of entropy (or temperature) in addition to density; likewise in the case of gas dynamics, the sound speed will be a general function of pressure and density (or temperature). This will encompass all of the familiar equations of state — barotropic, polytropic, ideal gas — as well as the case of an entropic equation of state. Specifically, existence of conserved integrals and invariants that may arise for any special equations of state will be determined.

Several new results are obtained for invariants and conserved integrals.

Firstly, all local invariants up to first-order are derived, and a recursion operator is found which generates higher-order invariants. This operator is shown to yield all second-order invariants. The set of basic invariants is found to include a new first-order invariant which exists for a general equation of state, and two other new first-order invariants which exist in the case of an entropic equation of state. None of these invariants are inherited from $n$-dimension (non-radial) flow.

Secondly, the new invariants give rise to corresponding integral invariants. By use of the recursion operator, an infinite hierarchy of integral invariants is obtained for a general equation of state, and two additional infinite hierarchies are obtained for an entropic equation of state. All of these integral invariants represent new, advected quantities.

Thirdly, conserved integrals with local densities of at most first-order are derived using an extension of the method developed for kinematic densities in [2, 3]. In addition to kinematic ones inherited from non-radial flow, this yields two new conserved integrals, one describing enthalpy-flux in the case of a barotropic equation of state, and the other describing entropy-weighted energy in the case of an entropic equation of state. Moreover, the only non-kinematic (first-order) conserved integrals which arise are shown to be equivalent to integral invariants, modulo trivial densities.

An example of a new conserved integral with a nonlocal conserved density, and a new nonlocal invariant, are shown in remarks at the end.

The paper is organized as follows. In Section 2 the radial reduction of the governing equations is summarized, and the radial formulation of conservation laws and conserved integrals is explained. In Section 3 the radial formulation of scalar, vector, and 1-form invariants is presented, followed by the results of a classification of local invariants up to first-order. The recursion operator is constructed from a novel relationship between scalar and vector invariants. In Section 4 the results on conserved integrals are presented. Invariant integrals are summarized in 4.1 while kinematic and first-order conserved integrals are classified in 4.2.
In Section 5, some concluding remarks are made. Two appendices provide remarks on computational aspects of the main results, which have been carried out by use of Maple.

2. Governing equations and preliminaries

For fluid flow in $n$ dimensions without boundaries, the dynamical variables are functions of position $\vec{x}$ in $\mathbb{R}^n$ and time $t$: velocity $\vec{u}(\vec{x}, t)$; density $\rho(\vec{x}, t)$; pressure $p(\vec{x}, t)$. Attention is restricted to locally adiabatic flows, with entropy $S(\vec{x}, t)$.

The governing equations for compressible flows are given by the Euler equations

\begin{align*}
\vec{u}_t + \vec{u} \cdot \nabla \vec{u} &= -\frac{1}{\rho} \nabla p, \\
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\
S_t + \vec{u} \cdot \nabla S &= 0.
\end{align*}

This system is closed by specifying an equation of state, which in general is given by

\[ p = p(\rho, S). \] (2.4)

A worthwhile remark is that all thermodynamic quantities can be obtained in terms of the internal energy (per unit mass) $e(\rho, S)$ through the thermodynamic relation

\[ T \, dS = de + p \, d(1/\rho) \] (2.5)

where $T$ is the local temperature (per unit mass). In particular, from the equation of state, the internal energy is given by

\[ e(\rho, S) = \int \left( \frac{p}{\rho^2} \right) d\rho \] (2.6)

which determines the temperature

\[ T(\rho, S) = \left. \frac{\partial e}{\partial S} \right|_{\rho} = \int \left( \frac{p S}{\rho^2} \right) d\rho. \] (2.7)

As is well known [32], the Euler equations for compressible fluid flow are equivalent to the equations of gas dynamics as given by the velocity equation (2.1), the density equation (2.2), and the dynamical equation for pressure

\[ p_t + \vec{u} \cdot \nabla p + a^2 \rho \nabla \cdot \vec{u} = 0 \] (2.8)

where

\[ a = a(\rho, p) > 0 \] (2.9)

is the sound speed. The pressure equation can be derived from the equation of state (2.4) by use of the implicit function theorem to obtain $S = F(\rho, p)$, which is then substituted into the entropy equation (2.3) and simplified using the density equation (2.2), with

\[ a^2 = -F_\rho/F_p = \left. \frac{\partial p}{\partial \rho} \right|_{S=F(\rho,p)}. \] (2.10)

Conversely, the entropy equation can be recovered from the pressure equation (2.8) by solving $F_\rho + a^2(\rho, p)F_p = 0$ to obtain $S = F(\rho, p)$, which is then observed to satisfy the entropy equation as a consequence of the density and pressure equations.

It will be useful to summarize the most common equations of state (2.4) arising in physical applications for the Euler fluid equations and their counterparts for the sound speed (2.10) in gas dynamics. Here $\frac{d}{dt} = \partial_t + \vec{u} \cdot \nabla$ denotes the material derivative.
(A) non-barotropic equation of state: \( p = f(\rho, S), f_S \neq 0; a^2 = f(\rho, F(\rho, p)), F_p \neq 0. \) Physically, this describes non-isothermal flows, namely \( \frac{dT}{dt} \neq 0. \)

(B) barotropic equation of state: \( p = f(\rho), f' \neq 0; a^2 = f'(\rho). \) These flows are isothermal, namely \( \frac{dT}{dt} = 0. \)

(C) polytropic (generalized) equation of state: \( p = \kappa(S)\rho^{1+\gamma}, \kappa'(S) \neq 0 \) and \( \gamma = \text{const.}; \) \( a^2 = (1 + \gamma)p/\rho. \) In these flows, \( \frac{dT}{dt} = -\gamma T\nabla \cdot \vec{u}. \)

(D) ideal (perfect) gas law equation of state: \( p = k\rho T, \) where \( k \) is Boltzmann’s constant. This is a special case of a polytropic flow in which \( \gamma = \frac{2}{n} \) and \( \kappa(S) = \exp(\frac{2}{k}S). \)

(E) entropic equation of state: \( p = \kappa(S), \kappa'(S) \neq 0; a = 0. \) In these flows, sound does not propagate, which describes a shockless gas, with \( \frac{dT}{dt} = T\nabla \cdot \vec{u}. \)

An important consequence of the equivalence between the respective governing equations of compressible fluid flow and gas dynamics is that every conservation law and invariant admitted by the fluid equations holds for the gas dynamics equations. This correspondence can be made explicitly manifest when a conservation law or an invariant is expressed solely in terms of \( t, r, \vec{u}, \rho, p, e. \)

2.1. Radial flow. The reduction of the governing equations \( \{2.1\} - \{2.3\} \) to radial flows consists of taking \( \vec{u}, \rho, S \) to be functions of only \( r = |\vec{x}| \) and \( t, \) and requiring \( \vec{u} \) to be parallel to the radial unit vector \( \hat{r} = (1/r)\vec{x}. \) Thus,

\[
\vec{u} = \vec{u}(r, t) = U(r, t)\hat{r}, \quad \rho = \rho(r, t), \quad S = S(r, t).
\]

These variables satisfy the reduced Euler equations

\[
\begin{align*}
U_t + UU_r + (p_S S_r + p \rho_r)/\rho &= 0, \\
\rho_t + (U\rho)_r + \frac{n-1}{r} U \rho &= 0, \\
S_t + US_r &= 0.
\end{align*}
\]

Note that the vorticity of a radial flow is zero, since \( \nabla \wedge \vec{u} = -\hat{r} \wedge \nabla U = -(\hat{r} \wedge \hat{r})U_r = 0, \) due to \( \nabla \wedge \hat{r} = 0 \) and \( \nabla r = \hat{r}. \) Zero vorticity flows have the general form \( \vec{u} = \nabla \phi \) for any function \( \phi(\vec{x}, t), \) with \( \phi = \phi(r, t) \) in the particular case of radial flows.

Radial reduction \( \{2.11\} \) coincides with spherical symmetry reduction when \( n > 2 \) but it is more restrictive when \( n = 2. \) Spherical symmetry is defined by invariance of the fluid variables under the rotation group \( SO(n): \mathcal{L}_\xi \vec{u} = 0, \mathcal{L}_\xi \rho = 0, \mathcal{L}_\xi S = 0, \) for all vector fields \( \xi \) that generate rotations in \( \mathbb{R}^n. \) This invariance holds for radial reduction. In \( n = 2 \) dimensions, however, any vector field of the form \( \vec{u}(r, t) \) is spherically symmetric.

The radial fluid equations \( \{2.12\} - \{2.14\} \) are equivalent to the equations of radial gas dynamics where the radial reduction of the pressure equation \( \{2.8\} \) is given by

\[
p_t + UP_r + a^2 \rho(U_r + \frac{n-1}{r} U) = 0,
\]

while the velocity equation is formulated as

\[
U_t + UU_r + p_r/\rho = 0.
\]

Here the sound speed \( \{2.9\} \) has the role of an equation of state, through the correspondence \( \{2.10\}. \)
2.2. Radial conservation laws. A local conservation law of the fluid equations (2.1)–(2.3) is a continuity equation

\[ (D_t \Phi^t + D_\vec{x} \cdot \vec{\Phi})|_E = 0 \]  

(2.17)

holding on all solutions of the equations, where \( \Phi^t \) is the conserved density and \( \vec{\Phi} \) is the spatial flux, which are functions of the fluid variables and their spatial derivatives, in addition to \( \vec{x} \) and \( t \). In particular, note that all \( t \)-derivatives can be eliminated through use of equations (2.1)–(2.3). Here \( E \) denotes the space of solutions; \( D_t \) denotes a total time derivative, and \( D_\vec{x} \) denotes a total gradient (space derivative).

Under radial reduction, the spatial flux is required to have the form \( \vec{\Phi} = \Phi^r \hat{r} \), whereby a conservation law (2.17) becomes

\[ (D_t \Phi^t + D_r \Phi^r + \frac{n-1}{r} \Phi^r)|_E = 0 \]  

(2.18)

due to \( \nabla \cdot \hat{r} = (n-1)/r \). A radial conservation law (2.18) can be expressed in an equivalent form of a total \( t, r \)-derivative

\[ (D_t(r^{n-1}\Phi^t) + D_r(r^{n-1}\Phi^r))|_E = 0. \]  

(2.19)

The conserved density \( \Phi^t \) and radial flux \( \Phi^r \) are functions of the fluid variables \( U(r, t), \rho(t, r), S(t, r) \), and their radial derivatives, in addition to \( r \) and \( t \).

2.3. Radial conserved integrals. Every radial conservation law (2.19) can be integrated over \( 0 \leq r < \infty \) to obtain a conserved (time-independent) radial integral

\[ \frac{d}{dt} \int_0^\infty \Phi^t r^{n-1} dr |_E = 0 \]  

(2.20)

if the radial flux satisfies the conditions \( \lim_{r \to 0} r^{n-1}\Phi^r = 0 \) and \( \lim_{r \to \infty} r^{n-1}\Phi^r = 0 \). The conserved integral \( \int_0^\infty \Phi^t r^{n-1} dr |_E \) is related to the radial reduction of an \( n \)-dimensional conserved integral which satisfies \( \frac{d}{dt} \int_{\mathbb{R}^n} \Phi^t d^n x |_E = -\lim_{r \to \infty} \int_{S^n(r)} \vec{\Phi} \cdot \hat{r} d^{n-1}A \) from integration of a \( n \)-dimensional conservation law (2.17) over \( \mathbb{R}^n \) followed by use of the divergence theorem, where \( S^n(r) \) denotes the \( n \)-sphere of radius \( r \) and \( d^{n-1}A \) denotes the area element. Radial reduction leads to the relations \( \int_{\mathbb{R}^n} \Phi^t d^n x = \text{vol}(S^n) \int_0^\infty \Phi^t r^{n-1} dr \) and \( \int_{S^n(r)} \vec{\Phi} \cdot \hat{r} d^{n-1}A = \text{vol}(S^n)\Phi^r r^{n-1} \), where \( S^n \) is the unit \( n \)-sphere and \( \text{vol}(S^n) \) denotes its hypersurface area.

In general, a conservation law is locally trivial [26, 7, 8] when it holds as a differential identity that contains no local information about solutions. Namely, the conserved density is given by a total spatial divergence \( \Phi^t = D_\vec{x} \cdot \vec{\Theta} \), while the spatial flux is given by a time derivative \( \vec{\Phi} = -D_t \vec{\Theta}|_E \), where \( \vec{\Theta} \) is a vector function of the fluid variables and their spatial derivatives, in addition to \( \vec{x} \) and \( t \). Thus, in the case of a radial conservation law (2.19), it is locally trivial if and only if

\[ \Phi^t = D_r \Theta + \frac{n-1}{r} \Theta, \quad \Phi^r = -D_t \Theta|_E \]  

(2.21)

holds for some scalar function \( \Theta \) of the radial fluid variables and their radial derivatives, in addition to \( r \) and \( t \). Only non-trivial radial conservation laws are of physical interest. Two radial conservation laws that differ by a locally trivial conservation law are physically equivalent.
It is useful in fluid dynamics to consider conserved integrals on finite moving domains \( V(t) \) that are transported by the flow in \( \mathbb{R}^n \) (see e.g. \cite{19, 20, 11}). For radial flows, this type of conserved integral has the form
\[
\frac{d}{dt} \int_{V(t)} \Phi^t r^{n-1} dr |_E = - (r^{n-1} \Psi) |_{\partial V(t)} \tag{2.22}
\]
whose physical content is that the rate of change of the moving integral \( \int_{V(t)} \Phi^t r^{n-1} dr |_E \) on a transported radial domain \( V(t) \) is balanced by the net outward radial flux \( (r^{n-1} \Psi) |_{\partial V(t)} \) through the moving boundary \( \partial V(t) \). A moving conserved integral will hold for all transported radial domains if and only if the conserved density \( \Phi^t (\equiv C) \) and moving radial flux \( \Psi (\equiv \vec{F} \cdot \hat{r}) \) satisfy a local radial conservation law (2.19) in which \( \Phi^r = \Psi + U \Phi^t \). Stated equivalently, any local radial conservation law (2.19) yields a moving conserved integral (2.22) on any transported radial domain \( V(t) \), with the moving flux being given by
\[
\Psi = \Phi^r - U \Phi^t. \tag{2.23}
\]

For all transported domains, a moving conserved integral (2.22) will yield a constant of motion, \( \frac{d}{dt} \int_{V(t)} \Phi^t r^{n-1} dr |_E = 0 \), also called an integral invariant, if and only if \( \Psi |_E \equiv 0 \).

Note that the moving flux of a locally trivial radial conservation law is given by \( \Psi |_E = -(D_t \Theta + UD_r \Theta) |_E = - \frac{d}{dt} \Theta |_E \). Hence, the corresponding moving conserved integral is simply an identity \( \frac{d}{dt} \int_{V(t)} D_r (r^{n-1} \Theta) dr |_E = - (-r^{n-1} \frac{d}{dt} \Theta) |_{\partial V(t)} \). Two moving conserved integrals that differ by a locally trivial moving conserved integral are thereby physically equivalent.

### 3. Invariants

An invariant of the \( n \)-dimensional Euler equations (2.1)–(2.3) is a quantity \( J \) that is advected (frozen-in) by the flow. This property has the geometrical formulation that \( J \) is Lie dragged by the flow,
\[
(D_t J + \mathcal{L}_{\vec{u}} J) |_E = 0 \tag{3.1}
\]
where \( \mathcal{L}_{\vec{u}} \) is the Lie derivative with respect to the velocity \( \vec{u} \). The underlying operator \( D_t + \mathcal{L}_{\vec{u}} \) here is a tensorial generalization of the material derivative \( d/dt = \partial_t + \vec{u} \cdot \nabla \). In general, \( J \) is a function of the fluid variables, their spatial derivatives, and \( \vec{x}, t \). The simplest example is entropy \( J = S \), which is a scalar invariant.

For radial flow, as governed by the radial Euler equations (2.12)–(2.14), invariants can be obtained by radial reduction of invariants in \( \mathbb{R}^n \). However, apart from \( S \), all known ones \cite{11, 13, 6} involve vorticity and hence they become trivial in radial flows. Nevertheless, invariants may exist which are not inherited through reduction. Such quantities are sometimes called “hidden”.

There are three basic geometrical types of invariants: scalars, vectors, 1-forms. In radial flow, these types turn out to be mutually related to each other and to conservation laws.

**Scalars.** A scalar function \( J = J \) will be an invariant of the radial Euler equations (2.12)–(2.14) if and only if it satisfies
\[
\frac{dJ}{dt} |_E = (D_t J + UD_r J) |_E = 0 \tag{3.2}
\]
since the Lie derivative acts as \( \mathcal{L}_{\vec{u}} = \vec{u} \cdot D_{\vec{x}} = UD_r \). Thus, a scalar invariant is simply an advected (material) quantity.
Every scalar invariant can be easily shown to correspond to a local conservation law \((2.19)\) given by \(\Phi_t = r^{n-1}\rho J\) and \(\Phi^r = r^{n-1}\rho UJ\). Since \(\Psi \equiv 0\), the conservation law yields an integral invariant \(\int_{V(t)} \rho J r^{n-1} dr |_\varepsilon\).

**1-forms.** A radial 1-form can be expressed as \(J = Jdr\) in terms of a scalar function \(J\). To yield a 1-form invariant of the radial Euler equations \((2.12)-(2.14)\), \(J\) must satisfy
\[
(D_t J + D_r(UJ)) |_\varepsilon = 0. \tag{3.3}
\]
This follows from Cartan’s formula for the action of a Lie derivative on a 1-form: \(\mathcal{L}_\vec{u}J = \vec{u}dJ + d(\vec{u}[J])\). Its radial version is obtained via the relations \(dJ = J_r dr \wedge dr = 0\) and \(d(\vec{u}[J]) = d(UJ) = D_r(UJ) dr\).

Every radial 1-form invariant directly corresponds to a local conservation law \((2.19)\) with \(\Phi_t = r^{1-n}J\) and \(\Phi^r = r^{1-n}UJ\). This yields an integral invariant \(\int_{V(t)} J dr |_\varepsilon\), since the conservation law has \(\Psi \equiv 0\).

**Vectors.** A radial vector can be expressed as \(J = J\vec{r}\) in terms of a scalar function \(J\). For \(J\vec{r}\) to be a vector invariant of the radial Euler equations \((2.12)-(2.14)\), \(J\) must satisfy
\[
(D_t J + UD_r J - JD_r U)|_\varepsilon = 0. \tag{3.4}
\]
This is a consequence of the commutator formula for the action of a Lie derivative on a vector: \(\mathcal{L}_\vec{u}J = \vec{u} \cdot D_\vec{r}J - J \cdot D_\vec{r}\vec{u}\). The radial version follows from the relations \(\vec{u} \cdot D_\vec{r}J = (UD_r J)\vec{r}\) and \(J \cdot D_\vec{r}\vec{u} = (JD_r U)\vec{r}\) along with \(D_r \vec{r} = 0\).

**3.1. Properties.** These three basic types of invariants are mutually related through their determining equations \((3.2), (3.3), (3.4)\) in terms of \(J\).

**Proposition 3.1.** The following conditions are equivalent:
(i) \(J\) is a scalar invariant;
(ii) \(r^{n-1}\rho J dr\) is a 1-form invariant;
(iii) \(r^{1-n}/(\rho J)\vec{r}\) is a vector invariant;
(iv) \(\int_{V(t)} \rho J r^{n-1} dr |_\varepsilon\) is an integral invariant for all transported radial domains \(V(t)\).

The proof amounts to verifying that if \(J_{\text{scal}} = J\) satisfies equation \((3.2)\) then \(J_{\text{1-form}} = r^{n-1}\rho J\) satisfies equation \((3.3)\) and \(J_{\text{vec}} = r^{1-n}/(\rho J)\) satisfies equation \((3.4)\), and conversely.

A similar computation shows that \(J_{\text{vec}} D_r J_{\text{scal}} = J\) satisfies equation \((3.2)\). This yields the following interesting result which provides a recursion operator on invariants.

**Proposition 3.2.** Suppose \(J_1\) and \(J_2\) are scalar invariants. Then: (i) \(J_3 = (r^{1-n}/\rho)J_2 D_r J_1\) and \(J_4 = f(J_1, J_2)\) are also scalar invariants, where \(f\) is any function on \(\mathbb{R}^2\); (ii) \(I = \int_{V(t)} J_2(D_r J_1) dr |_\varepsilon\) is an integral invariant, where \(V(t)\) is any transported radial domain. This integral is non-trivial iff \(D_r(J_2/J_1) \neq 0\).

Corresponding versions can be formulated for 1-form invariants and vector invariants.

**3.2. Results.** All invariants can be found, in principle, by solving one of the determining equations \((3.2), (3.3), (3.4)\). A classification of scalar invariants up to first-order, \(J(t, r, U, \rho, S, U_r, \rho_r, S_r)\), will now be presented. Remarks on the computation are given in Appendix A.
Theorem 3.3. (i) For a general equation of state (2.4), all scalar invariants up to first-order are functions of
\[ S, \quad r^{1-n}S_r/\rho. \] (3.5)

(ii) The only special equation of state for which additional scalar invariants arise up to first-order is the entropic case \( p = \kappa(S) \), where \( \kappa \) is an arbitrary non-constant function. The additional invariants are functions of
\[ U^2 + \frac{2}{n}r\kappa'(S)S_r/\rho, \quad \int_0^r \frac{dy}{\sqrt{U^2 + \frac{2}{n}(1 - (y/r)^n)r\kappa'(S)S_r/\rho}} - t. \] (3.6)

From part (i) of Proposition 3.2, the trivial invariant \( J = 1 \) yields a recursion operator
\[ \mathcal{R} = (r^{1-n}/\rho)D_r. \] (3.7)

When this operator is applied to the basic invariant \( S \), the first-order invariant \( r^{1-n}S_r/\rho \) is obtained. Repeated application yields a hierarchy of successively higher-order invariants,
\[ J_l = \mathcal{R}^lS \] (3.8)
for \( l = 1, 2, \ldots \), with the second-order invariants being given by \( (r^{1-n}/\rho)^2(S_{rr} - S_r\rho_r/\rho + \frac{1-n}{r}S_r) \). Moreover, all second-order invariants for a general equation of state (2.4) turn out to be exhausted by an arbitrary function of \( S, J_1, J_2, \) as discussed in Appendix A.

In the case of an entropic equation of state \( p = \kappa(S) \), additional higher-order invariants can be obtained by applying \( \mathcal{R} \) to the two invariants (3.6):
\[ J_{1,l} = \mathcal{R}^{l-1}(U^2 + \frac{2}{n}r\rho_{r}/\rho), \] (3.9)
\[ J_{2,l} = \mathcal{R}^{l-1}(A(r, U, \rho_{r}/\rho) - t) = \int_0^1 \mathcal{R}^l(\frac{r}{\sqrt{U^2 + \frac{2}{n}(1 - y^n)r\rho_{r}/\rho}})dy, \] (3.10)
for \( l = 2, 3, \ldots \), where
\[ A(r, U, \rho_{r}/\rho) = \int_0^r \frac{dy}{\sqrt{U^2 + \frac{2}{n}(1 - (y/r)^n)r\rho_{r}/\rho}} = \int_0^1 \frac{r dy}{\sqrt{U^2 + \frac{2}{n}(1 - y^n)r\rho_{r}/\rho}} \] (3.11)

Thus, the following main result has been established.

Theorem 3.4. The radial Euler equations (2.12)–(2.14) possess a hierarchy of scalar invariants (3.8) of order \( l = 0, 1, 2, \ldots \) for a general equation of state, and two additional hierarchies of scalar invariants (3.6) of order 1, (3.9) and (3.10) of order \( l = 2, 3, \ldots \), for an entropic equation of state.

Finally, each hierarchy of invariants yields a corresponding hierarchy of invariant integrals, which will be discussed in the next section.

4. CONSERVED INTEGRALS

It will be useful to divide the set of non-trivial conserved integrals into two distinct types: ones whose moving flux is zero, which are integral invariants representing advected quantities; ones with non-zero moving flux, which represent non-advected conserved quantities. For each type, a classification result will be presented.

These classifications will show that the radial Euler equations (2.12)–(2.14) possess both types of conserved integrals besides those that are inherited from radial reduction of the
conserved integrals known for $n$-dimensional fluid flow [19, 20, 2, 3]. The known conserved integrals comprise kinematic conservation laws — whose conserved density and spatial flux involve only the time and space coordinates, and the fluid variables, but not their derivatives; and vorticity conservation laws — in which the conserved density involves the vorticity scalar in even dimensions and the vorticity vector in odd dimensions [11]. A complete classification is shown in Ref. [3], which encompasses all equations of state (2.4) excluding the entropic case $p = \kappa(S)$. Under radial reduction, the vorticity conserved integrals are trivial, since $\nabla \wedge \mathbf{u} = 0$.

4.1. Integral invariants. The classification of scalar invariants (up to first order) stated in Theorem 3.3 provides, through Proposition 3.1, a corresponding classification of radial integral invariants

$$\frac{d}{dt} \int_{V(t)} \Phi^t r^{n-1} dr \bigg|_\varepsilon = 0$$

with conserved densities of the form

$$\Phi^t = \rho J(t, r, U, \rho, S, U_r, \rho_r, S_r).$$

These conserved integrals are advected quantities for the radial Euler equations (2.12)–(2.14).

**Theorem 4.1.** (i) For a general equation of state (2.4), all integral invariants (4.1) up to first order (4.2) are given by

$$\frac{d}{dt} \int_{V(t)} \rho f(S, r^{1-n} S_r/\rho) r^{n-1} dr \bigg|_\varepsilon = 0$$

where $f$ is an arbitrary function. (ii) The only special equation of state for which additional integral invariants arise up to first order is the entropic case $p = \kappa(S)$, where $\kappa$ is an arbitrary non-constant function. The additional integral invariants are given by

$$\frac{d}{dt} \int_{V(t)} \rho f \left( S, r^{1-n} S_r/\rho, U^2 + 2r^2 p_r/\rho, A(r, U, p_r/\rho) - t \right) r^{n-1} dr \bigg|_\varepsilon = 0$$

where $f$ is an arbitrary function, and $A(r, U, p_r/\rho)$ is expression (3.11).

The integral invariants of zeroth order comprise mass $\int_{V(t)} \rho r^{n-1} dr$ when $f = 1$, and total entropy in the generalized form $\int_{V(t)} \rho f(S) r^{n-1} dr$ when $f$ is non-constant. These two advected quantities are inherited from radial reduction of the kinematic integral invariants known for the $n$-dimensional Euler equations (2.1)–(2.3) from the classification of kinematic conservation laws in Ref. [3].

The first-order integral invariants in both cases (i) and (ii) are new. They can be viewed as “hidden” advected quantities which exist only for radial flow.

In case (i), the integral (4.3) is non-trivial whenever $f$ is not a linear homogeneous function in its second argument, namely $f(S, J_1) \neq F(S) J_1$, since otherwise the conserved density has the locally trivial form $r^{n-1} \Phi^t = F(S) S_r$ which is a total radial derivative.

In case (ii), the integral (4.4) is non-trivial whenever $f$ is a non-constant function in at least one of its last two arguments, namely it has some dependence on $U$. As an example, the energy-like integrals $\int_{V(t)} \rho \left( \frac{1}{2} U^2 + \frac{2}{n} p_r/\rho \right) r^{n-1} dr$, $i = 1, 2, \ldots$, are advected for an entropic equation of state. For $i = 1$, note that this integral is equivalent to the energy integral (4.10) with $p = \kappa(S) = -\rho e$. The equivalence can be seen from the relation $\frac{1}{2} U^2 + \frac{1}{n} r p_r/\rho =$
$r^{n-1}\left(\frac{1}{2}pU^2\rho + D_r\left(\frac{1}{n}r^n p\right)\right)$ between the densities, which differ by a total $r$-derivative, so thus the two integrals agree modulo a trivial integral.

Another example is the integral quantity $\int_{V(t)} \rho(A(r, U, p_r/\rho) - t) r^{n-1} dr = A_{V(t)} - tM_{V(t)}$, which has explicit dependence on $t$, where $M_{V(t)} = \int_{V(t)} \rho r^{n-1} dr$ is the mass integral, and $A_{V(t)} = \int_{V(t)} \rho A(r, U, p_r/\rho) r^{n-1} dr$ is a non-conserved integral. Note, as a consequence, $\frac{d}{dt} A_{V(t)} = M_{V(t)} = \text{const.}$ implies that $A_{V(t)}$ is a non-decreasing quantity in the flow.

In both cases (i) and (ii), integral invariants of higher-order can be obtained from the hierarchies of scalar invariants (3.8), (3.9) and (3.10).

**Theorem 4.2.** For any equation of state,

$$\mathcal{I}_l = \int_{V(t)} \rho f(J_0, J_1, \ldots, J_l) r^{n-1} dr \quad (4.5)$$

is an integral invariant of order $l \geq 1$ if $f$ is non-constant in its last argument. It is non-trivial at order $l$ if and only if $f$ is nonlinear in its last argument, namely $f \neq F(J_0, J_1, \ldots, J_{l-1})J_l$.

The demonstration of non-triviality follows from expressing the conserved density as

$$r^{n-1}\Phi^r = FD_rJ_{l-1} = D_r \left( \int F dJ_{l-1} \right) - r^{n-1}\rho \int f(J_1 F_{J_0} + \cdots + J_{l-1} F_{J_{l-2}}) dJ_{l-1} \quad (4.6)$$

which is locally trivial modulo terms of order less than $l$.

A similar result holds for an entropic equation of state, where $f$ has additional dependence on the higher-order invariants (3.9), (3.10), and as well as the two first-order invariants (3.6), denoted $J_{1,1}$ and $J_{2,1}$ respectively.

**Theorem 4.3.** For an entropic equation of state,

$$\mathcal{I}_l' = \int_{V(t)} \rho f(J_0, J_1, J_{1,1}, J_{2,1}, \ldots, J_l, J_{1,l}, J_{2,l}) r^{n-1} dr \quad (4.7)$$

is an integral invariant of order $l \geq 1$ if $f$ is non-constant in at least one of its last three arguments. It is non-trivial at order $l \geq 2$ if and only if $f \neq F_1 J_{1,l} + F_2 J_{2,l}$, where $F_1 = F_{1,l-1}$ and $F_2 = F_{J_{2,l-1}}$ in terms of $F(J_0, J_{1,1}, J_{2,1}, \ldots, J_{l-1}, J_{1,l-1}, J_{2,l-1})$; at order $l = 1$, it is non-trivial if and only if $f$ is non-constant in at least one of $J_{1,1}$ and $J_{2,1}$.

The question of whether these two hierarchies of integral invariants exhaust all possible integral invariants of higher-order is much harder problem which will be considered elsewhere.

### 4.2. Non-advected conserved integrals.

Consider a conserved integral with non-zero moving flux:

$$\frac{d}{dt} \int_{V(t)} \Phi^r r^{n-1} dr = - (r^{n-1}\Psi) \bigg|_{V(t)}', \quad \Psi = \Phi^r - U\Phi^t \neq 0. \quad (4.8)$$

Here the conserved density $\Phi^r$ and radial flux $\Phi^r$ are separate functions, in contrast to the form $\Phi^r = U\Phi^t$ characterizing integral invariants.

The first aim will be to find all conserved integrals given by kinematic radial conservation laws

$$\Phi^r(t, r, U, \rho, S), \quad \Phi^t(t, r, U, \rho, S) = \Psi(t, r, U, \rho, S) + U\Phi^t \quad (4.9)$$

in which the moving flux $\Psi$ is non-zero.
The classification of kinematic conservation laws in Ref. [3] for the $n$-dimensional Euler equations (2.1)–(2.3) shows that all kinematic conserved integrals having non-zero moving flux are a linear combination given by momentum, angular momentum, Galilean momentum, and energy for a general equation of state, and also by a dilational energy and a similarity energy for a polytropic equation of state. Their radial reduction yields the following non-trivial conserved integrals for the radial Euler equations (2.12)–(2.14):

$$\text{energy } \frac{d}{dt} \int_{V(t)} \rho \left(\frac{1}{2} U^2 + e\right) r^{n-1} dr = -(r^{n-1} p U) |_{\partial V(t)} \tag{4.10}$$

in the case of a general equation of state $p = p(\rho, S)$, where $e = \int p(\rho, S)/\rho^2 d\rho$; and

$$\text{dilational energy } \frac{d}{dt} \int_{V(t)} \left(t \rho \left(\frac{1}{2} U^2 + e\right) - \frac{1}{2} \rho U r \right) r^{n-1} dr = -(r^{n-1} t U - \frac{1}{2} r p) |_{\partial V(t)}, \tag{4.11}$$

$$\text{similarity energy } \frac{d}{dt} \int_{V(t)} \left(t^2 \rho \left(\frac{1}{2} U^2 + e\right) - t r \rho U + \frac{1}{2} \rho^2 r \right) r^{n-1} dr = -(r^{n-1} t(t U - r) p) |_{\partial V(t)} \tag{4.12}$$

in the case of a polytropic equation of state $p = \kappa(S) \rho^{1+2/n}$, where $e = \frac{n}{2} \kappa(S) \rho^{2/n}$ and $\kappa$ is an arbitrary function. The radial reduction of angular momentum vanishes while momentum and Galilean momentum have no radial reduction.

Additional (new) kinematic conserved integrals (4.8) can be sought by formulating and solving determining equations for conserved densities.

Any radial conservation law (2.19) can be expressed as

$$D_t(r^{n-1} \Phi^t)|_\varepsilon = -D_r(r^{n-1} \Phi^r) \tag{4.13}$$

where the conserved density $\Phi^t$ and the radial flux $\Phi^r$ contain no $t$-derivatives of the fluid variables. Since the righthand side of this equation is a total radial derivative, a function $\Phi^t$ with no $t$-derivatives of the fluid variables will be a conserved density if and only if it satisfies the variational conditions

$$\delta(r^{n-1} D_t \Phi^t)|_\varepsilon / \delta U = 0, \quad \delta(r^{n-1} D_t \Phi^t)|_\varepsilon / \delta \rho = 0, \quad \delta(r^{n-1} D_r \Phi^r)|_\varepsilon / \delta S = 0. \tag{4.14}$$

These conditions constitute a set of linear determining equations for finding conserved densities modulo a locally trivial density $r^{n-1} \Phi^t = D_r \Theta$. This freedom in solutions disappears for kinematic conservation laws (4.9). Once a solution for $\Phi^t$ has been found, then $\Phi^r$ can be determined from equation (4.13) by inverting the total $r$-derivative.

The determining equations (4.14) for kinematic conserved densities each split with respect to $r$-derivatives of $U$, $\rho$, $S$, thereby yielding an overdetermined system of PDEs for $\Phi^t$, $p(\rho, S)$, and $n$ as unknowns. This system is computationally straightforward to solve and yields the following classification result. Remarks on the computation are given in Appendix B.

**Theorem 4.4.** (i) For a general equation of state (2.4), energy (4.10) is the only kinematic conserved integral (4.8)–(4.9) with non-zero moving flux $\Psi \neq 0$. (ii) The only special equations of state for which additional non-advected kinematic conserved integrals exist are the polytropic case $p = \kappa(S) \rho^{1+2/n}$, the barotropic case $p = p(\rho)$, and the entropic case $p = \kappa(S)$. The additional conserved integrals are respectively given by: dilational energy (4.11) and
Consequently, attention will be restricted to first-order conserved densities that are quadratic. This general problem turns out to be a very difficult computation. (2.12)–(2.14) will be undertaken by solving the determining equations (4.14) for first-order (2.3) has been carried out, beyond the results in Ref. [2, 3, 4] for vorticity conservation laws. where at least one of φ which shows how it reduces to the energy integral (4.10) when e for a barotropic equation of state, where e = \int p(ρ)/ρ^2 dρ;

entropy-weighted energy \( \frac{d}{dt} \int_{V(t)} \left( \frac{1}{2} ρ U^2 (f(S) - K(S)) \right) r^{n-1} dr \bigg| _{ε} = -\left( r^{n-1} U K(S) \right) \bigg| _{∂V(t)} \)

for an entropic equation of state, where e = −κ(S)/ρ and K(S) = \( \int f(S)κ'(S) dS \), with f being an arbitrary non-constant function.

The entropy-weighted energy (4.10) is the radial reduction of an analogous conserved integral that was first found in a classification of kinematic conserved integrals for the Euler equations formulated on n-dimensional Riemannian manifolds [4]. It can be written in the equivalent form

\[
\frac{d}{dt} \int_{V(t)} \left( \left( \frac{1}{2} U^2 + e \right) pf(S) + f'(S) p dS \right) r^{n-1} dr \bigg| _{ε} = -\left( \frac{1}{2} r^{n-1} U pf(S) - \int f'(S) p dS \right) \bigg| _{∂V(t)}
\]

which shows how it reduces to the energy integral (4.10) when f(S) = 1.

The possibility that new higher-order conserved integrals with non-zero moving flux may exist is suggested by the existence of the new first-order scalar invariants. To-date, no classification of higher-order conserved integrals for the n-dimensional Euler equations (2.1)–(2.3) has been carried out, beyond the results in Ref. [2, 3, 4] for vorticity conservation laws.

Here, a systematic search for such conserved integrals (4.8) for the radial Euler equations (2.12)–(2.14) will be undertaken by solving the determining equations (4.14) for first-order conserved densities. This general problem turns out to be a very difficult computation. Consequently, attention will be restricted to first-order conserved densities that are quadratic in \( U_r \), with no explicit dependence on t:

\[
Φ^f = U_r^2 φ_2(r, U, ρ, S, ρ_r, S_r) + U_r φ_1(r, U, ρ, S, ρ_r, S_r) + φ_0(r, U, ρ, S, ρ_r, S_r)
\]

where at least one of φ_2 and φ_1 is not identically zero.

The determining equations each split with respect to r-derivatives of \( U_r, ρ_r, S_r \), similarly to the case for kinematic conserved densities. By use of various lengthy steps, combining several integration techniques, the resulting large overdetermined system of PDEs for the unknowns Φ^f, p(ρ, S), and n can be solved, modulo locally trivial conserved densities. Computational remarks are given in Appendix B.

This yields the following classification result.

**Theorem 4.5.** (i) For a general equation of state (2.4), all first-order conserved densities of the form (4.18) are given by Φ^f = \( pf(S, J_1) \) modulo trivial densities, where f is an arbitrary function on \( \mathbb{R}^2 \). (ii) The only special equations of state for which there exist non-trivial first-order conserved densities of the form (4.18) is the entropic case \( p = κ(S) \). The conserved densities are, modulo trivial densities, a linear combination given by:

\[
Φ^f = r^{1-n} \int_0^J D_r f(S, y, U^2 + \frac{2}{n} r^n y) dy
\]
and
\[
\Phi_2^t = r^{1-n} \left( f(S, J'_1, J_{1,1})/U + \int_0^{J_1'} \int_0^r \left( (D_r(U^2 + \frac{2}{3}r^n - z^n)y)^{-1/2}) \partial_y f(S, y, U^2 + \frac{2}{3}r^n y) 
- (\partial_y(U^2 + \frac{2}{3}r^n - z^n)y)^{-1/2}) D_r f(S, y, U^2 + \frac{2}{3}r^n y) \right) dz \, dy \right)
\]
(4.20)

where \( J'_1 = r^{1-n}p_r/\rho = \kappa'(S) \) is a scalar invariant, and \( f \) is an arbitrary differentiable function on \( \mathbb{R}^3 \).

Both of these non-trivial conserved densities turn out to be locally equivalent to conserved densities with zero moving flux:
\[
\Phi_1^t = -\rho f(S, J'_1, J_{1,1}) + D_r \Theta_1
\]
(4.21)

and
\[
\Phi_2^t = r^{1-n} f(S, J'_1, J_{1,1}) D_r A(r, U, p_r/\rho) + D_r \Theta_2
\]
(4.22)

where \( A(r, U, p_r/\rho) \) is expression (3.11). This leads to the following classification result.

**Corollary 4.6.** For any equation of state, every conserved integral (4.8) given by a non-trivial first-order conserved density of the form (4.18) (modulo a trivial conserved density) is equivalent to an integral invariant.

As a consequence, unlike the situation for integral invariants, no first-order recursion operator apparently exists in general for conserved densities whose moving flux is non-zero.

5. Concluding remarks

Radial fluid flow possesses two additional kinematic conserved integrals apart from the well-known ones inherited under radial reduction of \( n \)-dimensional (non-radial) fluid flow. The inherited kinematic conserved integrals consist of total entropy, mass, and energy holding for a general equation of state, as well as dilational energy and similarity energy holding for polytropic equations of state. The new radial conserved integrals represent an enthalpy-flux quantity (4.15) which holds for barotropic equations of state, and an entropy-weighted energy (4.16) which holds for entropic equations of state.

Most interestingly, radial fluid flow with a general equation of state also possesses a hierarchy of integral invariants (4.5) which describe advected conserved integrals. These quantities arise from a corresponding hierarchy of scalar invariants, which are local quantities that are advected by the flow. The hierarchy is generated by a recursion operator (3.7) applied to the basic invariant \( S \). In the case of entropic equations of state, radial fluid flow possesses two additional hierarchies of scalar invariants (3.9) and (3.10). These yield corresponding integral invariants (4.7), which describe more advected conserved integrals.

A computational classification of non-trivial conserved integrals given by a first-order conserved density (4.18) has been carried out. The classification shows that such conserved integrals exist only for entropic equations of state and are equivalent to particular integral invariants (4.19) and (4.20) (modulo trivial conserved integrals).

All of the preceding results carry over to radial gas dynamics through the well-known equivalence between the respective governing equations of \( n \)-dimensional gas dynamics and \( n \)-dimensional compressible fluid flow. Specifically, when a conserved integral or an invariant
is expressed solely in terms of \( t, r, U, \rho, p, e \), then it manifestly holds for both radial fluid flow and radial gas dynamics.

The conservation laws underlying conserved integrals are local if the conserved density and the flux are functions of \( t, r, U, \rho, p, e \). In three dimensions, nonlocal conservation laws for compressible non-radial fluid flow have been obtained from nonlocal vorticity invariants. The method used advective potentials, derived in terms of various thermodynamic quantities. This method can be applied to radial fluid flow in \( n \) dimensions. As an example, consider a scalar potential \( \mu \) defined in terms of the temperature \( T \) by

\[
\frac{d\mu}{dt} = T. \tag{5.1}
\]

A nonlocal conservation law is now given by

\[
\Phi_t = U - \mu S_r, \quad \Phi_r = \frac{1}{2} U^2 + e + p/\rho - \mu U S_r \tag{5.2}
\]

which has non-zero moving flux \( \Psi = e + p/\rho - \frac{1}{2} U^2 \). The resulting conserved integral generalizes the enthalpy-flux (4.15) to hold for a general equation of state. Hence a new non-advected quantity is obtained. A full exploration of nonlocal conservation laws and nonlocal invariants for radial fluid flow will be considered elsewhere.

Finally, in other future work on radial fluid flow, an investigation of symmetries, Hamiltonian structure, and Casimirs is currently in progress [5].

**Appendix A. Computational remarks for scalar invariants**

The first step is substitution of \( J(t, r, U, \rho, S, U_r, \rho_r, S_r) \) into the determining equation (3.2), followed by elimination of \( t \)-derivatives through use of the radial Euler equations (2.12)–(2.14). This yields a single PDE which splits with respect to \( U_{rr}, \rho_{rr}, S_{rr} \), giving an overdetermined system of 9 PDEs. The unknowns are \( J, p(\rho, S) \), and \( n \), which are subject to the conditions \( p \neq \text{const.} \) and \( n \neq 1 \). These unknowns appear nonlinearly in the system, and hence the problem is nonlinear.

The Maple command ‘rifsimp’ is used to obtain a complete classification of all cases in which the system can be brought into a consistent involutive form. This yields two cases, distinguished by a general equation of state for \( p \), and an entropic equation of state for \( p \). In each case, the resulting simplified system is first-order and linear in \( J \), and thus it can be easily solved. The case tree of solutions leads to the results in Theorem 3.3 for scalar invariants \( J(t, r, U, \rho, S, U_r, \rho_r, S_r) \).

The same method can be used to find \( J \) with dependence on higher \( r \)-derivatives of \( U, \rho, S \), up to any specified (finite) differential order.

**Appendix B. Computational remarks for conserved densities**

**B.1. Kinematic case.** The steps for classifying kinematic conserved densities (4.9) start from

\[
(D_t \Phi^t)|_E = \Phi^t_t - \Phi^t_u (UU_r + (pS_r + p\rho_r)/\rho) - \Phi^t_p ((U\rho)_r + (n - 1)U\rho/r) - \Phi^t_S (US_r) \tag{B.1}
\]

Here \( E_v = \sum_{i \geq 0} (-D_v)^i \partial \delta v \) denotes the radial Euler operator with respect to a variable \( v \); this operator coincides with the variational derivative \( \delta / \delta v \). Substitution of expression (B.1) into the determining equations (4.14), followed by splitting each equation with respect to \( r \)-derivatives of \( U, \rho, S \), yields an overdetermined system of 14 PDEs, with \( \Phi^t, p(\rho, S) \neq \text{const.} \).
and $n \neq 1$ being the unknowns. Since the unknowns appear non-linearly in the system, the problem is nonlinear.

This system can, in principle, be solved by applying the same steps outlined for solving the overdetermined system for scalar invariants. However, ‘rifsimp’ is unable to return all cases in the classification, due to their complexity. This computational difficulty can be bypassed by first dividing the classification into the following distinct (non-overlapping) cases for the equation of state:

(a) general $p = f(\rho, S)$, $f_{\rho} \neq 0$, $f_{S} \neq 0$;
(b) entropic $p = \kappa(S)$, $\kappa \neq \text{const.}$;
(c) polytropic $p = \kappa\rho^{q}$, $q \neq 0$;
(d) barotropic $p = f(\rho)$, $p \neq \kappa\rho^{q}$.

In each case, ‘rifsimp’ is able to return a complete classification of all subcases in which the system can be brought into an involutive form. The resulting systems are second-order and linear in $\Phi_t$. They can be solved by the following integration steps.

First, solve the PDE(s) that involve only $p$, and substitute the solution for $p$ back into the system. Second, solve simplest PDEs for $\Phi_t$ (e.g. one-term or two-term equations); the solution will involve some arbitrary functions/ constants. Next, substitute back into the system; if at any step there is a variable that does not appear in all arbitrary functions, then split the system with respect to that variable. Solve simplest PDEs involving a single function, and then eliminate any redundant functions/ constants after the solution has been substituted into $\Phi_t$. Simplify the pruned system by ‘rifsimp’, and repeat the previous steps until all equations have been solved. Finally, split $\Phi_t$ with respect to any free constants/ functions.

These basic steps work for cases (a), (b), (c). In case (d), an additional step of changing variables is needed after one of the integrations; the original variables are substituted back in the final step.

The last step consists of deriving $\Phi_r$ by inverting the total $r$-derivative in the conservation law equation (4.13). The inversion can be done by a straightforward integration by parts process, starting from terms of highest-order derivatives in the righthand side and descending to terms of first-order derivatives.

This leads to the results in Theorem 4.4 for kinematic conserved integrals.

B.2. First-order quadratic case. A more refined solving process is needed for solving the determining equations (4.14) for conserved densities $\Phi^t$ with the first-order form (4.18), since there are solutions such that $\Phi^t = r^{1-n}D_r\Theta$ is a trivial density, where $\Theta(t, r, U, \rho, S)$ is an arbitrary differentiable function.

These trivial solutions can be eliminated by imposing the condition that at least one of the three expressions $E_U(r^{n-1}\Phi^t)$, $E_\rho(r^{n-1}\Phi^t)$, $E_S(r^{n-1}\Phi^t)$ is not identically zero. However, even under these conditions, there are solutions given by the sum of a kinematic conserved density and an arbitrary trivial solution. For such solutions, the expressions $E_U(r^{n-1}\Phi^t)$, $E_\rho(r^{n-1}\Phi^t)$, $E_S(r^{n-1}\Phi^t)$ will be zeroth order. Hence, all unwanted solutions can be eliminated through the condition that at least one of $E_U(r^{n-1}\Phi^t)$, $E_\rho(r^{n-1}\Phi^t)$, $E_S(r^{n-1}\Phi^t)$ has non-zero differential order:

$$\sum_{v = U, \rho, S} \left( \partial (U_r, \rho_r, S_r, U_{rr}, \rho_{rr}, S_{rr}) E_v(r^{n-1}\Phi^t) \right)^2 \neq 0$$

(B.2)

where $\partial (U_r, \rho_r, S_r, U_{rr}, \rho_{rr}, S_{rr})$ denotes the set of partial derivatives with respect to the indicated variables.
To proceed, the determining equations are split with respect to the $r$-derivatives of $U, \rho, S$ that do not appear in the unknowns $\phi_0, \phi_1, \phi_2$. This yields an overdetermined system of 96 PDEs, plus the inequation (B.2), with the additional unknowns $p(\rho, S) \neq \text{const.}$ and $n \neq 1$.

The three cases (a), (b), (c) can be handled in a combined way, with the condition $p_0 \neq 0$. In this combined case, ‘rifsimp’ finds that the system is inconsistent, and hence no solutions describing non-trivial first-order conserved densities exist.

In the remaining case (d), ‘rifsimp’ returns a consistent system. The basic steps outlined for the kinematic case enable the integration of most of this system, until a set of 6 coupled PDEs remain which are not simple to integrate. The solution is obtained through an intricate combination of changes of independent and dependent variables, direct integrations, and simplifications that ultimately reduce the PDEs into a triangular, first-order system which can be integrated by standard techniques. A crucial part of this process is the elimination of redundant functions in the $\phi$’s as well as the use of integration by parts to remove trivial terms in $\Phi_t$. Additionally, manual case splitting is invoked twice, which aids in allowing some of the steps to work.

This leads to the final result summarized in Theorem 4.5.

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