Decomposition solutions and Bäcklund transformations of the BKP and CKP equations

Xiazhi Hao\textsuperscript{1} and S. Y. Lou\textsuperscript{2}\textsuperscript{*}

\textsuperscript{1}College of Science, Zhejiang University of Technology, Hangzhou, 310014, China
\textsuperscript{2}School of Physical Science and Technology, Ningbo University, Ningbo, 315211, China

In this paper, we define the modified formal variable separation approach and show how it determines, in a remarkably simple manner, the decomposition solutions, the Bäcklund transformations, the Lax pair, and the linear superposition solution of the B-type Kadomtsev-Petviashvili equation. Also, the decomposition solutions, the Bäcklund transformation and the Lax pair relating to the C-type Kadomtsev-Petviashvili equation is obtain by the same technique. This indicates that the decomposition may provide a description of integrable behavior in nonlinear systems, while, at the same time, establishing an efficient method for determining relationships between the particular systems.

Key words: Decomposition solution, Bäcklund transformation, Lax pair, Linear superposition solution

PACS numbers: 05.45.Yv, 02.30.Ik, 47.20.Ky, 52.35.Mw, 52.35.Sb

I. INTRODUCTION

Various techniques are available for obtaining exact solutions of nonlinear partial differential equations (PDEs) such as the inverse scattering method \cite{1–3}, group theoretical method \cite{4}, the singularity approach \cite{5–8}, and some direct methods \cite{9–12}. In all these methods, one can identify a central link, Bäcklund transformation (BT for short), one analytical tool for dealing with integrability problems, which enables one to relate pairs of solutions of certain nonlinear PDEs. Indeed the number of papers produced so far on BTs is fairly large, but not surprisingly so since these transformations have attracted the attention of both mathematicians and physicists, and have been incredibly actively developed in different directions, their relevance being well established in differential geometry and algebra as well as in nonlinear science, and other fields of application \cite{13–16}.

Nevertheless, among these methods, it seems to us that the convenient one to construct BTs is the decomposition method based on the formal variable separation approach (FVSA) \cite{17–19}. We propose such a new decomposition through which the BTs may exist and show how it works for two important nonlinear PDEs: the B-type and C-type Kadomtsev-Petviashvili equations (BKP and CKP). In the authors’ opinion, this decomposition provides a very straightforward and natural way to construct BTs, in addition, yields some new results. It may be expected that these results can be carried over to other equations.

The modified FVSA consists in looking for the general decomposition solution of the (2+1)-dimensional PDE in the form

\begin{align}
 w_y & = F(x, y, t, w, w_x, w_{xx}, \ldots, w_{x^m}), \\
 w_t & = G(x, y, t, w, w_x, w_{xx}, \ldots, w_{x^n})
\end{align}

with the integrability condition

\[ w_{yt} - w_{ty} = [F, G] = 0 \]

\textsuperscript{*}Corresponding author: lousenyue@nbu.edu.cn. Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.
preserved, here, \( w_{zm} = \partial_x^m w, w_{zn} = \partial_x^n w, \) \( m \) and \( n \) are integers.  \( F \) and \( G \) depend on \( x, y, t \) and derivatives of \( w \) with respect to \( x \).

The successive practical steps of the method are the following.

(I) Determine the possible values of \( m \) and \( n \) by balancing two or more terms of the PDE and expressing that they dominate the other terms.

(II) Substitute the decompositions (1)-(2) into the PDE and integrability condition (3), then obtain the expressions for \( w \) and its derivatives in the form of an overdetermined systems.

(III) Determine \( F \) and \( G \) by solving systems obtained in step II, whose solution yields the desired decomposition solution.

Examples are always a good starting point. Consider the BKP equation

\[
ux_t + (ux_x + 15ux_{xx} + 15u^3 - 15uv - 5ux_y)_{xx} - 5uy_y = 0, \quad v_x = uy, \tag{4}
\]

where \( u_x = \partial_x u, u_{x2} = \partial_x^2 u, u_{x3} = \partial_x^3 u, \ldots \), which is a (2+1)-dimensional generalization of the Sawada-Kotera (SK) equation and exhibits many elegant integrable behavior [20, 21]. It is convenient to deal analytically with a potential function \( w \), introduced by setting \( u = w_x \), and it follows equation (4) that \( w \) can be taken to satisfy the equation

\[
w_{xt} = 5w_{yy} - (w_{x5} + 15w_xw_{x3} + 15w_x^3 - 15w_xw_y - 5w_{xxy})_x. \tag{5}
\]

The CKP equation is defined as

\[
9ux_t + (ux_x + 15ux_{xx} + 15u^3 + \frac{45}{4}u_x^2 - 15uv - 5ux_y)_{xx} - 5uy_y = 0, \quad v_x = uy, \tag{6}
\]

\[
9w_{xt} = 5w_{yy} - (w_{x5} + 15w_xw_{x3} + 15w_x^3 + \frac{45w_x^2}{4} - 15w_xw_y - 5w_{xxy})_x, \tag{7}
\]

in which \( u \) denotes the conservative field and \( w \) the potential one. When \( u_y \) tends to zero, the reduced equation of the CKP equation (6) is the well-known Kaup–Kupershmidt (KK) equation [22]. The close connection between these two completely integrable fifth order equations is well documented [23], nevertheless, despite their evident duality, equations (4) and (6) are fundamentally different. There is no scaling which reduces one to the other.

The application of the decomposition method to the potential BKP equation (5) and potential CKP equation (7) is described in sections II and III. At the same time, the BTs and the Lax pairs for the potential BKP and potential CKP equations are found in a remarkably natural manner. Finally, some lengthy and technical calculations have been moved to the Appendix.

II. THE DECOMPOSITION SOLUTIONS, BÄCKLUND TRANSFORMATION, LAX PAIR AND LINEAR SUPERPOSITION SOLUTIONS OF THE POTENTIAL BKP EQUATION

Analysis of the potential BKP equation (5) shows the leading orders \( m = 3, n = 5 \), in this case, \( w_y \) and \( w_t \) are expressible in terms of

\[
w_y = F(x, y, t, w, w_x, w_{xx}, w_{xxx}), \tag{8}
\]

\[
w_t = G(x, y, t, w, w_x, w_{xx}, w_{xxx}, w_{xxxx}, w_{xxxxx}), \tag{9}
\]

and we therefore require the equations (8) and (9) are subject to the integrability condition

\[
F_t = G_y, \tag{10}
\]

whenever equation (5) is satisfied. The derivation of \( F \) and \( G \) is elementary but somewhat lengthy and can be found in the Appendix. The result of calculation leads directly to the following theorems.
**Theorem 1.** If \( w \) is a solution of the consistent variable coefficient potential Korteweg-de Vries (KdV) decomposition system

\[
v_y = v_{xxx} - \frac{a}{2}v_x^2 + \frac{m_0}{a}x + n_y,
\]

\[
v_t = 9v_{xxxxx} - 15av_xv_{xxx} - \frac{15a}{2}v_{xx}^2 + \frac{5a^2}{2}v_x^3 + [15m_2 + 15m_1y - (5m_0^2 + \frac{1}{2}m_0)y^2 - 5m_0x]v_x
- 5m_0v + [(10m_0^2 + m_0)y - 15m_1]\frac{x}{a} + \frac{y^3}{6a}m_0(10m_0^2 + m_0a) - \frac{15y^2}{2a}m_0m_1 - \frac{15y}{a}m_0m_2
+ 5m_0n + n_t + m_3,
\]

\[
w_y = w_{xxx} + 3w_x^2 + aw_xw_x + \frac{a^2}{6}v_x^2 - \frac{1}{3}m_0(x + m_0y^2) - \frac{m_0y^2}{30} + m_1y + m_2,
\]

\[
w_t = 9w_{xxxxx} + 15(aw + 6w_x)w_{xxx} + 5a(aw + 3w)_xv_{xxx} + 15(aw + 3w)_xxw_{xx} + \frac{15}{2}w_x[(aw + 3w)^2 + 3w_x^2]
+ \frac{5a^2}{2}v_{xx}^2 + w_x(15m_2 + 15m_1y - 5m_0^2y^2 - 5m_0x - \frac{1}{2}m_0y^2) - 5m_0w - \frac{xy}{3}(m_0t + 10m_0^2) + 5m_1x
- \frac{y^3}{90}(m_0t + 30m_0m_0a + 100m_0^3) + \frac{y^2}{2}(m_1t + 10m_0m_1) + (m_2t + 10m_0m_2)y + m_4,
\]

where \( m_i, i = 0, 1, \ldots, 4 \) are arbitrary functions of \( t, a \) is an arbitrary function of \( y, t \) and \( a \) is an arbitrary constant, then \( w \) is a solution of the potential BKP equation (5).

The arbitrary parameter \( a \) determines that \( v \) defined by equations (11)-(12) may be a decomposition solution of potential BKP equation (5) as well. The reader may care to show \( v \) may indeed be transformed to a solution of potential BKP equation (5) when \( a = -3 \) and \( a = -6 \). Generalizing, we can state the following.

**Corollary 1.** Under the decomposition of Theorem 1, assume that the constant \( a \) and the function \( n \) be as

\[ a = -3, n = m_2y + \frac{1}{2}m_1y^2 - \frac{1}{90}(10m_0^2 + m_0t)y^3 + m_0(t), \]

then, both \( v \) and \( w \) given by (11)-(14) are solutions of the potential BKP equation (5).

We record another corollary of Theorem 1.

**Corollary 2.** The functions \( v \) and \( w \) fixed by (11)-(14) in the statement of Theorem 1 with the additional condition

\[ a = -6, n = m_2y + n_0(t), m_0 = m_1 = 0 \]

satisfy the potential BKP equation (5) as well.

Note that the results in Corollaries 1 and 2 indicate that potential BKP equation (5) possesses the variable coefficient KdV decomposition solutions (11)-(12) with conditions (15) and/or (16) [24]. On the other hand, the variable coefficient KdV decompositions (13)-(14) with (15) and/or (16) form BTs which connect two solutions \( v \) and \( w \) of the potential BKP equation (5).

A further consequence of decomposition is that we can establish a general BT associated with potential BKP (5). This result is formulated as the following theorem.

**Theorem 2. (Bäcklund Transformation)** The decomposition relation

\[
p_y = p_{xxx} + \frac{3}{2}(w + v)_{xx}p + \frac{3}{2}(w_x^2 - v_x^2) + \frac{1}{4}(p^3)_x, p \equiv w - v,
\]

\[
p_t = 9p_{xxxxx} + \frac{9}{16}(p^5)_x + \frac{45}{8}(w + v)_{xx}p^3 + \frac{45}{8}(3w_x^2 - 3w_{xx}^2 - 2p_{xx})p^2 + \frac{15}{4}(9v_x^2 + 9w_x^2 - 6w_xw_x
+ 4v_{xx} + 4w_{xxx} + 2v_y + 2w_y)(p' - 15w_{xxx}(v - 4w)_x - 15v_{xxx}(4v - w)_x + 45w_{xx}^2 - 45v_{xx})
+ \frac{45}{2}w_x(v_x^2 + w_x^2) + \frac{15}{2}(w + v)y_p,
\]
constitutes a BT between two solutions \( w \) and \( v \) of the potential BKP equation (5).

It is easily verified by direct calculation that this system (17)-(18) is integrable and that \( v \) satisfies equation (5). It is worth noting that the BT has the form of a conservation law

\[
p_y = \left[p_{xx} + \frac{3}{2}p(w_x + v_x) + \frac{p^3}{4}\right]_x,
\]

\[
p_t = \left[9p_{xxxx} + \frac{9p^3}{16} + \frac{45}{8}(w + v)x_p^3 + \frac{45}{4}p^2p_{xx} + (15(w + v)_{xxx} + \frac{45}{2}(w^2 + v^2) + \frac{15}{2}(w + v)y)p + \frac{45}{2}(p_x(w + v)x)_x\right]_x.
\]

Finding the BT of a PDE is equivalent to finding its Lax pair [25, 26]. Therefore, further consideration of the BT for the construction of the Lax pair seems warranted. With appeal to Theorem 2, we can obtain the well-known Lax pair [27]

\[
\psi_y = \psi_{xxx} + 3w_x\psi_x,
\]

\[
\psi_t = 9\psi_{xxxx} + 45w_x\psi_{xxx} + 45w_{xxx}\psi_x + 15(2w_{xxxx} - 3w^2_x + w_y)\psi_x
\]

of the potential BKP equation (5) by substitution of the transformation \( v = w + \ln(\psi^2) \) into the BT (17)-(18) as may be verified by direct calculation. The elimination of \( w \) is made between Lax pair, and results in the Schwarzian form of the potential BKP equation (5)

\[
S_{xxx} + C_x + 4S_x^2 - 5K_x^2 + 5(S_xK - S_yK_y) = 0
\]

with Schwarzian derivatives \( S = \frac{\psi_{xxx}}{\psi_x^3} - \frac{3\psi_{xx}^2}{\psi_x^2}, C = \frac{\psi_{xx}}{\psi_x} \) and \( K = \frac{\psi_{xx}}{\psi_x} \).

Notice that 

1. Each of two Corollaries and Theorem 2 takes the form of a system of coupled differential equations for two unknown functions, and all imply that if one of the two functions is a solution of the potential BKP (5), then also the second function solves the same equation. 2. The BT (17)-(18) implies a type of the Sharma–Tasso–Olver (STO) decomposition solution of the potential BKP equation (5) when \( v = 0 \) [24], also presents a means of finding the Lax pair, then Schwarzian equation.

**Theorem 3.** Let the function \( w \) be a solution to the consistent variable coefficient Svinolupov Sokolov (SS) system

\[
w_y = w_{xxxx} + \frac{3w_x^2}{2W} + \frac{3}{2}w_x^2 - \frac{27}{4}M^2 + \frac{3}{2}m_1x + \frac{3}{20}(m_1y + 2m_2)_t + m_3,
\]

\[
w_t = 9w_{xxxx} + 45w_xw_{xxx} + \frac{45}{2}w_x^3 + \frac{3}{4}[30m_1x - 135M^2 + 3y(m_1y + 2m_2)_t + 20m_3]w_x + \frac{45}{2}m_1w
\]

\[
- \frac{3}{2}(45m_1M - m_1y - m_2t)x + \frac{y^2}{20}(m_1y + 3m_2)_tt - \frac{27}{2}y(m_1y + 2m_2)_tM + y(m_3 - 45m_1m_3) + m_4
\]

\[
+ \frac{45}{2W}[2w_{xx}w_{xxx} + w_{xxx}^2] + \frac{180}{W^2}w_x^2w_{xxx} + \frac{315}{2W^3}w_x^4 + \frac{405}{4}M^3,
\]

where \( M \equiv m_1y + m_2, W \equiv 3M - 2w_x \) and \( m_i, i = 1, 2, 3, 4 \) are arbitrary functions of \( t \), then \( w \) is a solution of the potential BKP equation (5).

**Theorem 4.** The potential BKP equation (5) possesses the following variable coefficient KdV decomposition solution

\[
w_y = \frac{1}{2}w_{xxxx} - \frac{3}{2}w_x^2 + 6MW_x + \frac{y}{10}(m_1y + 2m_2)_t + m_1x - 6M^2 + m_3, M \equiv m_1y + m_2,
\]

\[
w_t = -\frac{9}{4}w_{xxxxx} - \frac{45}{4}(2w_xw_{xxx} + w_{xx}^2 + w_x^3) + 15m_1w + \frac{3}{2}y(m_1y + 2m_2)_tw_x + (6M^2 + m_1x + m_3)w_x
\]

\[
+ \frac{y^2}{30}(m_1y + 3m_2)_tt - 3(y^2m_1t + 10m_1x + 2ym_2t)M + M_x + (m_3 + 30m_1m_3)y - 60M^3 + m_4,
\]
where \( m_i, i = 1, 2, 3, 4 \) are arbitrary functions of \( t \).

**Theorem 5.** The variable coefficient potential SK decomposition solution of the potential BKP equation (5) possesses the form

\[
\begin{align*}
 w_y &= Mw_x + \frac{y}{30} (M + m_2)_t + \frac{1}{3} m_1 x - \frac{1}{2} M^2 + m_3, M = m_1 y + m_2, \\
 w_t &= -w_{xxxxx} + 5(M - 3w_x)w_{xxx} + 15(M - w_x)w^2_x + \left( \frac{1}{2} y^2 m_{1t} + y m_{2t} + 5m_1 x - \frac{5}{2} M^2 + 15m_3 \right) w_x \\
 & \quad + 5m_1 w + \frac{y^2}{90} (m_1 y + 3m_2)_t - \frac{y}{3} (M y - x) m_{1t} + \frac{1}{3} (x - 2M_y) m_{2t} - \frac{10}{3} m_1 (M x + 3m_3 y) \\
 & \quad + m_3 y + m_4
\end{align*}
\]

(26) with \( m_i, i = 1, 2, \ldots, 4 \) being arbitrary functions of \( t \).

Suppose that arbitrary functions \( m_i = 0, i = 1, 2, \ldots, 4 \), then by Theorems 3 to 5, three special decompositions correspond to constant coefficient SS, KdV and SK decomposition solutions of potential BKP equation (5), respectively [24].

If a BT represents one of the different aspects of the property of integrability for a PDE, the existence of a linear combination principle allows further to build explicitly some classes of solutions depending on decomposition system (11)-(12). The linear combination solution of the potential BKP equation (5) from the result of the decomposition is established.

**Theorem 6.** If \( v_1 \) and \( v_2 \) are solutions of the variable coefficient potential KdV decompositions

\[
\begin{align*}
 v_{1y} &= v_{1xxx} - \frac{1}{2} a_1 v_{1x}^2 + \frac{1}{a_1} m_0 x + n_y, M = m_1 y + m_2, \\
 v_{1t} &= 9v_{1xxxx} + \frac{5}{2} a_1^2 v_{1x}^3 - \frac{15}{2} a_1 (2v_{1x} v_{1xx} + v_{1xx}^2) + \left[ 15M - 5m_0 x - \frac{1}{2} (10m_0^2 + m_{0t}) y^2 \right] v_{1x} \\
 & \quad - 5m_0 v_1 + 5m_0 n + n_t + m_3 + \frac{1}{a_1} \left[ (10m_0^2 + m_{0t}) y - 15m_1 \right] x + \frac{5}{3} y^3 m_0^3 \\
 & \quad + \frac{1}{6} m_0 m_0 y^3 - \frac{15}{2} m_0 (m_1 y + 2m_2 y), \\
 v_{2y} &= v_{2xxx} - \frac{1}{2} a_2 v_{2x}^2 + \frac{1}{a_2} (m_0 y^2 - a_1 n_y + m_0 x + 2y^2 m_0^2 - 6M), \\
 v_{2t} &= 9v_{2xxxx} + \frac{5}{2} a_2^2 v_{2x}^3 - \frac{15}{2} a_2 (2v_{2x} v_{2xx} + v_{2xx}^2) - \frac{1}{2} y^2 v_{2x} m_{0t} + (15M - 5m_0^2 y^2 - 5m_0 x) v_{2x} - 5m_0 v_2 \\
 & \quad + \frac{1}{a_2} \left[ \frac{1}{15} m_{0tt} y^3 + \frac{1}{6} (6xy + 11m_0 y^3)m_{0t} - 3m_1 y^2 - 6m_2 y - a_1 n_t + 5(2m_0 y - 3m_1) x \right. \\
 & \quad \left. + \frac{45}{2} m_0 m_1 y^2 - 45m_0 m_2 y - 5m_0 a_1 n \right],
\end{align*}
\]

(28) and

\[
\begin{align*}
 w &= -\frac{1}{6} (a_1 v_1 + a_2 v_2)
\end{align*}
\]

(32) is a solution of the potential BKP equation (5).

Generally, the functions \( v_1 \) and \( v_2 \) in Theorem 6 are not the solutions of the potential BKP equation (5) except in certain special cases mentioned in Corollaries 1 and 2, whereas the linear combination (32) of \( v_1 \) and \( v_2 \) solves the potential BKP equation (5). The linear superposition solutions with fixed \( a_1 \) and \( a_2 \) in [24] are particular cases of Theorem 6.
III. THE DECOMPOSITION SOLUTIONS, BÄCKLUND TRANSFORMATION AND LAX PAIR OF THE POTENTIAL CKP EQUATION

As a second illustration, we repeat the entire procedure in the previous section to find the decomposition solutions and Bäcklund transformation of the potential CKP equation (7). Similarly, a straightforward calculation determines the following theorems.

Theorem 7. (Bäcklund Transformation) Let \( v \) be any solution of the potential CKP equation (7), a different solution \( w \) of equation (7) is then defined by the Bäcklund transformation

\[
p_y = [p_{xx} - \frac{3p^2}{4p} + \frac{3}{2} p(w_x + v_x) + \frac{p^3}{4}]_x, \quad p = w - v, \tag{33}
\]

\[
p_t = [p_{xxxx} - \frac{5}{2p} p_x p_{xxx} + \frac{5p}{3} (w + v)_{xxx} - \frac{5}{4p} p_{xx}^2 + \left( \frac{5p^2}{4} + \frac{5p^2}{p^3} + \frac{5}{2} (w + v)_x p_{xx} + \frac{5}{4} p_x (w + v)_{xx} \right. \nonumber
\]

\[- \frac{35p^4}{16p^3} - \frac{15(w + v)_x}{8p} p_x^2 + \frac{15}{8} p (w^2 + v^2) + \frac{p^5}{16} + \frac{5p}{6} (w + v)_y + \frac{5p^3}{8} (w + v)_x + \frac{5}{4} pw_x v_x \bigg]_x. \tag{34}
\]

This BT (33) and (34) appears to be a new result and is again in a conservation form. The Lax pair arises from the BT, which results in two equations

\[
\psi_y = \psi_{xxx} + 3w_x \psi_x + \frac{3}{2} w_{xx} \psi, \tag{35}
\]

\[
\psi_t = \psi_{xxxx} + 5w_x \psi_{xxx} + \frac{15}{2} w_{xx} \psi_{xx} + 5\left( \frac{7}{6} w_{xxx} + w_x^2 + \frac{w_y}{3} \right) \psi_x + 5\left( \frac{w_{xxx}}{3} + w_x w_{xx} + \frac{w_{xy}}{6} \right) \psi \tag{36}
\]

followed by the relation \( v = w + \ln(\partial_x^{-1} \psi)_x \).

When \( v = 0 \), the BT is reduced to the modified STO decomposition solution

\[
w_y = [w_{xx} - \frac{3w^2}{4w} + \frac{3}{2} w w_x + \frac{w^3}{4}]_x, \tag{37}
\]

\[
w_t = [w_{xxxx} - \frac{5}{2w} w_x w_{xxx} + \frac{5}{3} w w_{xx} - \frac{5}{4w} w_x^2 + 5\left( \frac{w^2}{4w} + \frac{w_x^2}{2} + \frac{3}{4} w_x \right)_x w - \frac{35w^4}{16w^3} - \frac{15w^2}{8w} \nonumber
\]

\[+ \frac{15w_x}{8} + \frac{w^5}{16} + \frac{5w_y}{6} + \frac{5w^3 w_x}{8}]_x \tag{38}
\]

of the potential CKP equation (7). This very special modified STO decomposition is linked to the STO decomposition

\[
f_y = [f_{xx} + \frac{1}{4} f^3 + \frac{3}{2} (f_x f)]_x, \tag{39}
\]

\[
f_t = [f_{xxxx} + 5f_x f_{xx} + 5\left( \frac{ff_{xxx}}{2} + \frac{f^2 f_{xx}}{2} + \frac{f^3 f_x}{4} + \frac{3f f_x^2}{4} \right) + \frac{f^5}{16}]_x \tag{40}
\]

by transformation \( w_x = f w - w^2 \), which is exactly the same as the STO decomposition of the potential BKP equation (5).

Theorem 8. The function \( w \) is a solution of the potential CKP equation (7) provided that \( w \) satisfies
the consistent variable coefficient SS system

\[
\begin{align*}
    w_y &= w_{xxx} + \frac{3w_x^2}{2W} + 3w_t^2 - \frac{9Mw_x}{2} - \frac{27M^2}{8} + \frac{3m_1x + \frac{27y}{20}(m_1y + 2m_2)t}{8} + \frac{27m_2^2}{8} + m_3, \quad (41) \\
    w_t &= w_{xxxxx} + \frac{5w_{xxx}w_{xxxx}}{W} + \frac{5w_x^2}{2W} + \frac{(20w_x^2}{2W^2} + 10w_x - \frac{15M}{2})w_{xxx} + \frac{35w_x^2}{2W^3} + \frac{5(3M + 4w_x)w_x^2}{4W} \\
         &+ 10w^3_x - \frac{45M}{2}w_x^2 + \frac{45M}{8}w_x^3 + \frac{2}{5}w_m^3 + \frac{5m_3}{3}w_x + \frac{(9w_xy^2 + 6xy - 27y^2M)m_{3t}}{4} \\
         &+ \frac{(18w_x + 6x - 27y(2m_1y + m_2))m_{2t}}{4} + \frac{9y^2(m_1y + 3m_2)t}{20} - \frac{15M}{2} \frac{m_2}{4} + \frac{45M^3}{4} - \frac{135m_1m_2^2y}{8}, \quad (42)
\end{align*}
\]

where \( M = m_1y + m_2, W = 3M - 2w_x \) and \( m_i, i = 1, 2, 3, 4 \) are arbitrary functions of \( t \).

**Theorem 9.** The potential CKP equation (7) possesses the following variable coefficient KdV decomposition solution

\[
\begin{align*}
    w_y &= \frac{1}{4}w_{xxxx} + \frac{3}{2}w_x^2 + \frac{3m_0x}{5} + \frac{27y^2(m_0 + 2m_3)}{50} + M, M = m_1y + m_2, \quad (43) \\
    w_t &= \frac{1}{16}w_{xxxxx} + \frac{5}{4}(2w_xw_{xxx} + w_x^2 + 4w_x^3) + (m_0x - \frac{9m_0y^2}{5} + \frac{5}{3}M + \frac{9m_0y^2}{10})w_x + m_0w \\
         &+ (\frac{3m_0}{5} - \frac{6m_2}{5})y + \frac{m_1}{9}x + \frac{18m_0}{25} - \frac{27m_0m_0}{50} + \frac{9m_0y}{2} - m_0m_1) \frac{y^3}{2} + \frac{m_{2t} - m_2m_2}{2} + m_3, \quad (44)
\end{align*}
\]

where \( m_i, i = 1, 2, 3, 4 \) are arbitrary functions of \( t \).

**Theorem 10.** The variable coefficient potential KK decomposition solution of the potential CKP equation (7) possesses the form

\[
\begin{align*}
    w_y &= Mw_x + \frac{3y}{10}(M + m_2)t + \frac{1}{3}m_1x - \frac{1}{2}M^2 + \frac{m_2^2}{2} + m_3, M = m_1y + m_2, \quad (45) \\
    w_t &= -\frac{w_{xxxxx}}{9} + \frac{5}{9}(M - 3w_x)w_{xxx} - \frac{w_x^2}{4} + \frac{5w_x^2}{3} + \frac{5w_x^2}{3}w_x + (\frac{5m_1x}{9} - \frac{5M}{2}) + \frac{m_2y}{6} + \frac{5m_2}{3} \\
         &+ \frac{m_1y^2}{2} + m_2y)w_x + 5m_1w + \frac{y^2}{10}(m_1y + 3m_2)t - \frac{m_1y^2}{3}M + \frac{xM}{3} + m_2y(m_2 - 2m_1y) \frac{3}{3} \\
         &+ m_3y - \frac{10m_1xM}{27} - \frac{5m_1y(m_2 + 2m_3)}{9} + m_4, \quad (46)
\end{align*}
\]

with \( m_i, i = 1, 2, \ldots, 4 \) being arbitrary functions of \( t \).

The proof of Theorems 7-10 is of the same form as that presented in the Appendix for the potential BKP equation (5). The calculation is omitted here.

We have attempted but failed to provide the linear superposition solution of the potential CKP equation (7) from the decomposition solutions.

**IV. CONCLUSIONS**

In this study, which extends the work of a previous paper [24], we have presented a modified formal variable separation approach for deriving decomposition solutions of the potential BKP and CKP equations. This method is heuristic, it gives decomposition solutions of the \((2+1)\)-dimensional PDEs in a unified way. The result makes clear that we establish the connection between the potential BKP and CKP equations with
several classic integrable systems. One might therefore hope to deduce the solutions of the potential BKP and CKP equations from solutions of the classical integrable systems.

Specifically, we are able to construct the BTs from decomposition. It is widely accepted that the existence of a BT serves as a sign to the integrability of a PDE. While the BTs for the (1+1)-dimensional SK and KK equations have been known for some time [25], the BTs for the (2+1)-dimensional SK (4) and KK (6) equations have not previously been reported. The BTs of the potential BKP and CKP equations are given here in explicit forms for the first time. Of course, other standard analytic techniques for obtaining BTs of nonlinear PDEs are available, but, the construction of the BTs from decomposition is a remarkably straightforward way.

In the case of the potential BKP and CKP equations, the system defining the BTs found has two properties. First, it is linearizable since it results in the Lax pair. Second, it has a conservation form. Observe that the solution to the potential BKP equation in Theorem 6 also has a unique property: the well-known linear superposition principle holds if the parameters are taken on some special values.

It is note that yet despite the close resemblance of the potential BKP and CKP equations, they are fundamentally different. There is no scaling which transformations one equation into the other. Whereas in view of the system (39)-(40), it is seen that the STO decomposition links the potential BKP and CKP equations.

As an elementary tool for constructing solutions of PDEs, decomposition is as straightforward to apply as we believe. We hope to expand on some of these theoretical aspects in future work.

Acknowledgement

The work was sponsored by the National Natural Science Foundations of China (Nos. 11975131, 11435005). K. C. Wong Magna Fund in Ningbo University, the Natural Science Foundation of Zhejiang Province No. LQ20A010009.

[1] A. S. Fokas, M. J. Ablowitz, On the inverse scattering transform of multidimensional nonlinear equations related to first order systems in the plane, Journal of Mathematical Physics 25 (1984) 2494–2505.
[2] R. Hirota, A new form of Bäcklund transformations and its relation to the inverse scattering problem, Prog. theor. Phys. 52 (1974) 1498–1512.
[3] V. O. Vakhnenko, E. J. Parkes, A. J. Morrison, A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation, Chaos, Solitons and Fractals 17 (2003) 683–692.
[4] A. S. Fokas, R. L. Anderson, Group theoretical nature of Bäcklund transformations, Lett. Math. Phys. 3 (1979) 117–126.
[5] J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, Journal of Mathematical Physics 24 (1983) 522–526.
[6] J. Weiss, The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pair, and the Schwarzian derivative, Journal of Mathematical Physics 24 (1983) 1405–1413.
[7] R. Conte, M. Musette, Painlevé analysis and Bäcklund transformation in the Kuramoto-Sivashinsky equation, Journal of Physics A: Mathematical and General 22 (1989) 169–177.
[8] S. Y. Lou, Painlevé test for the integrable dispersive long wave equations in two space dimensions, Physics Letters A 176 (1993) 96–100.
[9] X. N. Gao, S. Y. Lou, X. Y. Tang, Bosonization, singularity analysis, nonlocal symmetry reductions and exact solutions of supersymmetric KdV equation, Journal of High Energy Physics 05 (2013) 29.
[10] E. G. Fan, Two new applications of the homogeneous balance method, Phys. Lett. A 265 (2000) 353–357.
[11] Y. Jin, M. Jia, S. Y. Lou, Bäcklund transformations and interaction solutions of the Burgers equation, Chinese Physics Letters 30 (2013) 020203.
[12] W. X. Ma, A. Abdeljabbar, A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation, Appl. Math. Lett. 25 (2012) 1500–1504.
Appendix

We illustrate the decomposition procedure for the case of the potential BKP equation (5).

Proof: Substituting (8) and (9) with \( m > 3 \) into the potential BKP equation (5), plus the decomposition compatibility condition (10), one can find that the decomposition does not hold generally for \( m > 3 \). Therefore, we take \( m = 3 \) and then \( n = 5 \) in the decomposition relations (8) and (9). Substituting (8) and (9) into (5) gives

\[
w_{x6}(1 - 5F_{x3} - 5F_{x3} + G_{x5}) + W = 0,
\]

where \( F = F(x,y,t,w,w_x,w_{x2},w_{x3}) \equiv F(x,y,t,x_0,x_1,x_2,x_3) \), \( G = G(x,y,t,w,w_x,w_{x2},w_{x3},w_{x4},w_{x5}) \equiv G(x,y,t,x_0,x_1,x_2,x_3,x_4,x_5) \), and \( W = W(x,y,t,x_0,x_1,\ldots,x_5) \) is a complicated expression of \( x,y,t,x_0,x_1,\ldots,x_5 \). Vanishing coefficient of \( w_{x6} \), we have

\[
G = (5F_{x3} + 5F_{x3} - 1)x_5 + G_1,
\]

where \( G_1 = G_1(x,y,t,x_0,x_1,\ldots,x_4) \) is a function of \( \{x,y,t,x_0,x_1,\ldots,x_4\} \). By using the relation (48), (47) is changed to

\[
w_{x5}[G_{x4} - 5(F_{x3} + 2)(x_1F_{x0x3} + x_2F_{x1x3} + x_3F_{x2x3} + x_4F_{x3x3}) - 5F_{x2}(1 + 2F_{x3}) - 5F_{x3}(2 + F_{x3})] + W_1 = 0,
\]

(49)
where \( W_1 = W_1(x,y,t,x_0,x_1,\ldots,x_4) \) is \( w_{x5} \) independent. Eliminating the coefficient of \( w_{x5} \) yields
\[
G_1 = 5(F_{x3} + 2)(\frac{1}{2}x_4F_{xx3} + x_3F_{x2x3} + x_2F_{x1x3} + x_1F_{x0x3})x_4 + 5x_4F_{x2}(1 + 2F_{x3}) + 5x_4F_{x3}(2 + F_{x3}) + G_2,
\]
with \( G_2 = G_2(x,y,t,x_0,x_1,x_2,x_3) \) being a function of \( \{x,y,t,x_0,x_1,x_2,x_3\} \).

Similarly, substituting the decomposition (8) and (9) with \( m = 3, \ n = 5 \), the results (48) and (50) into the consistent condition (10), we have
\[
5w_{x7}(1 - F_{x3})^2(x_1F_{x0x3} + x_2F_{x1x3} + x_3F_{x2x3} + x_4F_{x3x3} + F_{xx3}) + \Gamma = 0,
\]
where \( \Gamma = \Gamma(x,y,t,x_0,\ldots,x_6) \) is a \( w_{x7} \) independent function of the lower-order differentiations of \( w \) with respect to \( x \). Vanishing the coefficient of \( w_{x7}w_{x4} \) in (51), we get
\[
F = F_1(x,y,t,x_0,x_1,x_2)x_3 + H(x,y,t,x_0,x_1,x_2).
\]
Substituting (52) into (51) and requiring the coefficient of \( w_{x7} \) being zero result \( F_1(x_0, x_1, x_2) = H_1(y,t) \). Thus, we have
\[
F = H_1(y,t)x_3 + H(x,y,t,x_0,x_1,x_2).
\]

Taking account of (53), (49) becomes
\[
w_{x4}(G_{2x3} - 5(H_1 + 2)x_3H_{x2x3} - 5(H_1 + 2)x_2H_{x1x2} + 5x_1(3 - H_1H_{x0x2} - 3H_1 - 2H_{x0x2})
- 10H_1H_{x1} - 5H_2^2 - 5H_1 - 5H_1H_{xx2} - 10H_{xx2}) + W_2 = 0
\]
with \( W_2 = W_2(x,y,t,x_0,x_1,x_2,x_3) \). Vanishing the coefficient of \( w_{x4} \) in (54) leads to
\[
G_2 = 5\left[H_1 + 2\frac{2}{x_3}H_{x2x2} + (H_1 + 2)x_2H_{x1x2} - x_1(3 - H_1H_{x0x2} - 3H_1 - 2H_{x0x2}) + 2H_1H_{x1}
+ H_2^2 + H_{x1} + H_1H_{xx2} + 2H_{xx2}\right]x_3 + \Gamma
\]
where \( \Gamma = \Gamma(x,y,t,x_0,x_1,x_2) \). Up to now, the decomposition relation is simplified to
\[
w_y = H_1w_{x3} + H, \quad w_t = (5H_1 + 5H_2^2 - 1)w_{x5} + 5H_{x2} + (1 + 2H_1)w_{x4} + 5\left[H_1 + 2\frac{2}{x_3}w_{x2}H_{x2x2} + (H_1 + 2)w_{x2}H_{x1x2}
- w_{x1}(3 - H_1H_{x0x2} - 3H_1 - 2H_{x0x2}) + 2H_1H_{x1} + H_2^2 + H_{x1} + H_1H_{xx2} + 2H_{xx2}\right]w_{x3} + \Gamma
\]
with three undetermined functions \( H_1 = H_1(y,t), H = H(x,y,t,x_0,x_1,x_2) \) and \( J = J(x,y,t,x_0,x_1,x_2) \).

Inserting (56) and (57) into the potential BKP equation (5) and the consistent condition (10), then, vanishing the coefficients of \( w_{xk} \) for \( k \geq 3 \) leaves the set of determining equations on \( \{H, H_1, J\} \). Solving this set of equations, for nontrivial solutions we shall have the following several cases for the unknowns.

**Case 1.** When \( H_1 = 0 \), further calculation then leads to the expression
\[
\begin{align*}
H &= Mw_x + \frac{y}{30}(M + m_2)t + \frac{1}{3}m_1x - \frac{1}{2}M^2 + m_3, M := m_1y + m_2, \\
J &= 15(M - w_x)w_x^2 + \left(\frac{1}{2}y^2m_1 + ym_2 + 5m_1 - \frac{5}{2}M^2 + 15m_3\right)w_x + 5m_1w + \frac{y^2}{90}(m_1y + 3m_2)t \\
- \frac{y}{3}(My - x)m_1 + \frac{1}{3}(x - 2M_y)m_2 - \frac{10}{3}m_1(Mx + 3m_3y) + m_3y + m_4
\end{align*}
\]
with \( m_i, i = 1, 2, \ldots, 4 \) being arbitrary functions of \( t \).
Case 2. Taking $H_1 = 1$ in the result of this case, then one can find three different solutions.

The first one is
\[
\begin{align*}
H &= \frac{3w_x^2}{2W} + \frac{3}{2}w_x^2 - \frac{27}{4}M^2 + \frac{3}{2}m_1x + \frac{3y}{20}(m_1y + 2m_2)x + m_3, \\
J &= \frac{45}{2}w_x^3 + \frac{3}{4}|30m_1x - 135M^2 + 3y(m_1y + 2m_2)x + 20m_3|w_x + \frac{45}{2}m_1w - \frac{3}{2}(45m_1 - m_1t)y \\
&\quad - m_2t)x + \frac{y^2}{20}(m_1y + 3m_2)t - \frac{27}{4}y(m_1y + 2m_2)tM + y(m_3 - 45m_1m_3) + m_4 \\
&\quad + \frac{45}{2W^3}[3M + w_xw_x^2] + \frac{315}{2W^3}w_x^4 + \frac{405}{4}M^3,
\end{align*}
\]
where $M \equiv m_1y + m_2, W \equiv 3M - 2w_x$ and $m_i, i = 1, 2, 3, 4$ are arbitrary functions of $t$.

The second one is
\[
\begin{align*}
H &= 3w_x^2 + aw_xw_x + \frac{a^2}{6}v_x^2 - \frac{1}{3}m_0(x + m_0y^2) - \frac{m_0y^2}{30} + m_1y + m_2, \\
J &= 5a(aw_x + 3w)_xv_{xxx} + 15(aw_x + 3w)xw_x + \frac{15}{2}w_x[(aw_x + 3w)^2]_x + \frac{5a}{2}v_x^2w_x + w_x(15m_2) \\
&\quad + 15m_1y - 5m_0g^2y - 5m_0 - \frac{1}{2}m_0g^2y - 5m_0w - \frac{xy}{3}(m_0 + 10m_2) + 5m_1x + \frac{y^3}{90}(m_0t) \\
&\quad + 30m_0m_0 + 100m_0^3 + \frac{y^2}{2}(m_1t + 10m_0m_1) + (m_2t + 10m_0m_2)y + m_4
\end{align*}
\]
with $v$ satisfying
\[
\begin{align*}
v_y &= vx - \frac{a}{2}v_x^2 + \frac{m_0}{a}x + n_y, \\
v_t &= 9vx + 15a(v + 3w)_xv_{xxx} - \frac{15}{2}v_x^2 + \frac{5a}{2}v_x^3 + [15m_2 + 15m_x - 5m_0g + \frac{1}{2}m_0g]v_x \\
&\quad - 30m_0v + [(10m_0 + m_0)t - 15m_1] + \frac{y}{6}m_0(10m_0 + m_0) - \frac{15y^2}{2a}m_0m_1 - \frac{15y}{a}m_0m_2 \\
&\quad + 5m_0n + n_t + m_3.
\end{align*}
\]

The third one is
\[
\begin{align*}
H &= -v_{xxx} + v_y + \frac{3}{2}(w + v)_x(w - v) + \frac{3}{2}(w_x - v_x)^2 + \frac{1}{4}[(w - v)^3], p \equiv w - v, \\
J &= v_t - 9v_x - 15v_x + \frac{15}{2}p(v)_x + \frac{15}{2}v_x - \frac{105}{2}v_x)w_x - 45v_x^2 - \frac{45}{2}p(v)_xw_x + 15pv_x + 45w_x^2 \\
&\quad + \frac{225}{2}p(w_x - 45p(v)_x + \frac{45}{4}p^3)w_x + \frac{45}{4}v_x^2p_x + 135\frac{p}{4}w_xp_x + 15p_xv_y + \frac{135}{4}w_xp_x + \frac{45}{16}p^4x,
\end{align*}
\]
in which case, the decomposition solution depends on another solution $v$ of the potential BKP equation (5).

Case 3. $H_1 = -\frac{1}{2}$. In this case, the functions $H$ and $J$ are fixed as
\[
\begin{align*}
H &= -\frac{3}{2}w_x^2 + 6Mw_x + \frac{y}{10}(m_1y + 2m_2)x + m_1x - 6M^2 + m_3, M \equiv m_1y + m_2, \\
J &= -\frac{45}{4}(w_x^2 + 2w_x^3) + 15m_1w + \frac{y}{2}(m_1y + 2m_2)xw_x + 15(6M^2 + m_1x + m_3)x + \frac{y^2}{30}(m_1y \\
&\quad + 3m_2)t - 3(y^2m_1t + 10m_1x + 2ym_2)t + Mtx + (m_3x - 30m_1m_3)y - 60M^3 + m_4,
\end{align*}
\]
where $m_i, i = 1, 2, 3, 4$ are arbitrary functions of $t$.

By substituting these solutions (58)–(63) into the decomposition relations (56) and (57), Theorems 1-5 are proved.