ASYMPTOTICS FOR THE EXPECTED LIFETIME OF
BROWNIAN MOTION ON THIN DOMAINS IN \( \mathbb{R}^n \)

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Abstract. We derive a three-term asymptotic expansion for the expected lifetime of Brownian motion and for the torsional rigidity on thin domains in \( \mathbb{R}^n \), and a two-term expansion for the maximum (and corresponding maximizer) of the expected lifetime. The approach is similar to that which we used previously to study the eigenvalues of the Dirichlet Laplacian and consists of scaling the domain in one direction and deriving the corresponding asymptotic expansions as the scaling parameter goes to zero. As in the case of eigenvalues, these expansions may also be used to approximate the exit time for domains where the scaling parameter is not necessarily close to zero.

1. Introduction

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) and consider the elliptic equation

\[
-\Delta u = 2, \quad x \in \Omega \\
u = 0, \quad x \in \partial \Omega
\]  

Under certain mild conditions on the regularity of the boundary, the above equation has one and only one non-negative solution in \( C(\overline{\Omega}) \). However, and except for a few domains such as ellipsoids, this is not known in closed form – see \[AFR\] for a recent result in the case of equilateral triangles.

On the other hand, solutions to equation (1.1) have a probabilistic interpretation in terms of the Brownian motion associated to the Laplacian in \( \mathbb{R}^n \). More precisely, the value of \( u \) at each point \( x \) yields the expected lifetime of a particle starting from \( x \) and it is thus of interest to be able to obtain approximations to these solutions. This is the purpose of the present paper where we apply to this problem an approach that we have used recently in the case of Dirichlet eigenvalue problems and which provides quite accurate approximations for the first eigenvalue of the Laplace operator for certain classes of domains \[BF1, BF2\]. The idea consists in scaling the domain \( \Omega \) along one direction and determining the asymptotic expansion of the solution in terms of the scaling parameter as it goes to zero. In this way we obtain an asymptotic expansion for the solution of equation (1.1) from which it is possible to derive expansions for other quantities such as...
the maximum of \( u \), the corresponding maximizer and also the integral of \( u \).

In line with the above interpretation, the second of these quantities corresponds to the point in the domain with the largest expected lifetime, while the last one is known in the elasticity literature as the torsional rigidity – see [BBC] and the references therein. As may be seen from the examples in Section 6, although our approximations are quite accurate for a fairly large range of values of the scaling parameter, the error can vary a lot as this parameter approaches one, depending on the initial domain considered – see the discussion in this section.

It should also be mentioned that there is a large number of papers devoted to the asymptotic expansions of the solutions to elliptic boundary value problems in thin domains – see [N], [NT] and the references therein. However, and to the best of our knowledge, the problem considered here has not been studied from this point of view.

The paper is organized as follows. In the next section we lay down the notation and state the main results of the paper. In Section 3 we prove the asymptotic expansion for solutions of the elliptic problem depending on a scaling parameter. Sections 4 and 5 then present the asymptotics for the maximum of this solution and for the torsional rigidity, respectively. We finish with an analysis of the error of these approximations for some specific domains.

## 2. Formulation of the problem and main results

Let \( x = (x', x_n), x' = (x_1, \ldots, x_{n-1}) \) be Cartesian coordinates in \( \mathbb{R}^n \) and \( \mathbb{R}^{n-1} \), respectively, \( \omega \) be a bounded domain in \( \mathbb{R}^{n-1} \) with \( C^1 \)-boundary. By \( h_{\pm} = h_{\pm}(x') \in C(\bar{\omega}) \cap C^1(\omega) \) we denote two arbitrary functions such that \( H(x') := h_-(x') + h_+(x') > 0 \) for \( x' \in \omega \) and \( H(x') = 0 \) on \( \partial \omega \). We also define the two functions

\[
d(x') = h_+(x') - h_-(x') \quad \text{and} \quad p(x') = h_+(x')h_-(x')
\]

We introduce a thin domain by

\[
\Omega_\varepsilon := \{ x : x' \in \omega, -\varepsilon h_-(x') < x_n < \varepsilon h_+(x') \},
\]

where \( \varepsilon \) is a small parameter, and consider the problem

\[
-\Delta u_\varepsilon = 2 \quad \text{in} \quad \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega_\varepsilon.
\]  

(2.1)

In view of the smoothness of the functions \( h_{\pm} \) and the boundary of \( \omega \) the domain \( \Omega_\varepsilon \) satisfies the exterior sphere condition at every boundary point. By Theorem 6.13 in [GT, Ch. 6, Sec. 6.3] it implies that the solution to the problem (2.1) belongs to \( C(\Omega_\varepsilon) \).

Let \( \xi_n := x_n/\varepsilon \). Assuming that the functions \( h_{\pm} \) are smooth enough, we introduce a sequence of functions

\[
\begin{align*}
u_2(x', \xi_n) &:= U_2(x', \xi_n) + \alpha_1^{(2)}(x')\xi_n + \alpha_0^{(2)}(x'), \\
U_2(x', \xi) &:= \alpha_2^{(2)}(x')\xi_n^2,
\end{align*}
\]

(2.2)
Given any \( \alpha \) and forms the basis for the remaining expansions in the paper.

The first of our results describes the uniform asymptotic expansion for \( u_\varepsilon \) and forms the basis for the remaining expansions in the paper.

**Theorem 1.** Given any \( N \geq 1 \), let the functions \( h_{\pm} \) be such that \( \alpha_i^{(2j)} \in C^2(\overline{\omega}) \), \( j \leq N \). Then the function \( u_\varepsilon \) satisfies the asymptotic formula

\[
(2.9) \quad u_\varepsilon(x) = \sum_{j=1}^{N} \varepsilon^{2j} u_{2j}(x', \frac{x_n}{\varepsilon}) + O(\varepsilon^{2N+2})
\]

in the \( C(\overline{\Omega}_\varepsilon) \)-norm. In particular,

\[
(2.10) \quad 
\begin{align*}
    u_2(x', \xi_n) &= -\xi_n^2 + \xi_n d(x') + p(x'), \\
    u_4(x', \xi_n) &= -\frac{\xi_n^4}{6} \frac{\delta}{\Delta} d(x') - \frac{\xi_n^2}{2} \Delta_p p(x') + \alpha_1^{(4)}(x') \xi_n + \alpha_0^{(4)}(x'), \\
    u_6(x', \xi_n) &= \frac{\xi_n^6}{120} \Delta^2 d(x') + \frac{\xi_n^4}{24} \Delta^2 p(x') \\
    &\quad - \frac{\xi_n^3}{36} \Delta_p \left\{ \left[ d^2(x') + p(x') \right] \Delta d(x') + 3d(x') \Delta_p p(x') \right\} \\
    &\quad - \frac{\xi_n^2}{12} \Delta_p \left[ d(x') p(x') \Delta d(x') + 3p(x') \Delta_p p(x') \right] \\
    &\quad + \alpha_1^{(6)}(x') \xi_n + \alpha_0^{(6)}(x'),
\end{align*}
\]
Then the maximum value of $u$ where 

\[ u \]

namely, the variable $x$ as above. We assume further that 

\[ \Omega \]

Theorem 2. Let the family of domains $\Omega$, and the functions $h_-, h_+$ and $H$ be as above. We assume further that $H$ satisfies the following hypothesis

- **H1** There exists a unique point $\overline{x} \in \omega$ at which $H$ achieves its global maximum which will be denoted by $H_0$;
- **H2** The Hessian matrix of $H$ at $\overline{x}$, denoted by $2H_2$, is negative definite;
- **H3** The functions $h_\pm$ are 5 times continuously differentiable in a vicinity of $\overline{x}$.

Then the maximum value of $u_\varepsilon$ in $\Omega_\varepsilon$ for sufficiently small $\varepsilon$ satisfies 

\[ \max_{x \in \Omega_\varepsilon} u_\varepsilon(x) = \frac{1}{4} H^2_0 \varepsilon^2 + \frac{1}{8} H^2_0 [d_0 \tr D_2 + 2 \tr P_2] \varepsilon^4 + O(\varepsilon^5), \]
as $\varepsilon \to 0$. Here $d_0$ is the value of $d$ at $\overline{x}$, while the matrices $2D_2$ and $2P_2$ are the Hessian matrices of $d$ and $p$ at the point $\overline{x}$, respectively.

Remark 2.1. If one drops the hypothesis that $H_2$ is nonsingular it will still be possible to obtain an expansion, but this will be much more involved and will depend on higher order terms in the expansion of $H$ around $\overline{x}$.

Remark 2.2. The hypothesis on a unique maximum of $H$ may also be dropped and provided there is only a finite number of such maxima the results still hold except that one has to construct different expansions for each maximum. The case of a continuum of maxima is also possible to handle with the techniques employed here but requires some further changes to the approach.

Remark 2.3. In the process of proving the above theorem we also obtain a two-term asymptotic expansion for the maximizer – see Theorem 4. However, this depends on higher order terms in the expansions of both $d$ and $p$ and requires the introduction of more detailed notation which we postpone till Section 4 below.

Remark 2.4. Under the hypotheses H1 and H2 of Theorem 2 and assuming that the functions $h_{\pm}$ are smooth enough in a vicinity of $\overline{x}$, it is possible to construct more terms in the asymptotic expansions for the maximum of $u_\varepsilon$ in $\Omega_\varepsilon$ and for the corresponding maximizer. In order to do it, one should follow the main lines of the proof of Theorem 2 employing Lemma 4.2 as a starting point. At the same time, it requires bulky and technical calculations which we would like to avoid. This is the reason why we provide only two-terms asymptotics in Theorem 2.

Finally, the integration of the asymptotic expansion for $u_\varepsilon$ given by Theorem 4 yields the corresponding asymptotic expansion for the torsional rigidity.

Theorem 3 (Torsional rigidity). Under the conditions of Theorem 4 we have

\[
\int_{\Omega_\varepsilon} u_\varepsilon(x) \, dx = \frac{\varepsilon^3}{6} \int_{\omega} H^3(y) \, dy \\
+ \frac{\varepsilon^5}{24} \int_{\omega} H^3(y) \left[ d(y) \Delta_y d(y) + 2 \Delta_y p(y) \right] \, dy \\
+ \varepsilon^7 \int_{\omega} \frac{1}{720} \left[ H^3(y) d_3(y) + 3d(y)p^2(y)H(y) \right] \Delta_y^3 d(y) \\
+ \frac{1}{120} \left\{ H^3(y)d^2(y) + p(y)H(y) \left[ p(y) - d^2(y) \right] \right\} \Delta_y^2 p(y) \\
- \frac{1}{144} \left[ H^3(y)d(y) - 2d(y)p(y)H(y) \right] \\
\times \Delta_y \left\{ [d^2(y) + p(y)] \Delta_y d(y) + 3d(y)\Delta y p(y) \right\}
\]
\[-\frac{1}{36} [H^3(y) - 3p(y)H(y)]
\times \Delta_y [d(y)p(y)\Delta_y d(y) + 3p(y)\Delta_y p(y)]
+ \frac{1}{2} d(y)H(y)\alpha^{(6)}_1(y) + H(y)\alpha^{(6)}_0(y) \ dy
+ O(\varepsilon^8),
\]

where \( \alpha^{(6)}_0 \) and \( \alpha^{(6)}_1 \) are as in Theorem 1.

Remark 2.5. In order to obtain the average expected lifetime it remains to divide by the volume of \( \Omega \varepsilon \) which is given by

\[|\Omega_{\varepsilon}| = \int_{\Omega_{\varepsilon}} dx = \int_\omega \int_{-\varepsilon h^- (x')} \int_{\varepsilon h^+ (x')} d\xi_n \ dx' = \varepsilon \int_\omega H(y) \ dy.\]

3. The asymptotic expansion for \( u_{\varepsilon} \)

In this section we prove Theorem 1. We begin by passing to the variables \((x', \xi_n)\) in (2.1) leading us to

\[
(3.1) \quad \left( -\varepsilon^2 \Delta_{x'} - \frac{\partial^2}{\partial \xi_n^2} \right) u_{\varepsilon} = 2\varepsilon^2 \text{ in } \Omega, \quad u_{\varepsilon} = 0 \text{ on } \partial \Omega.
\]

We construct the asymptotic expansion to the problem (3.1) as follows

\[
(3.2) \quad u_{\varepsilon}(x) = \sum_{j=0}^{\infty} \varepsilon^{2j} u_{2j}(x', \xi_n),
\]

where \( u_j(x', \xi_n) \) are functions to be determined.

We substitute the expansion (3.2) into (3.1) and equate the coefficients of like powers in \( \varepsilon \). This yields the following boundary value problems for \( u_{2j} \):

\[
(3.3) \quad -\frac{\partial^2 u_{2j}}{\partial \xi_n^2} = 2, \quad \xi \in \left( -h^-(x'), h^+(x') \right), \quad u_{2j} = 0, \quad \xi_n = \pm h_{\pm}(x'), \quad x' \in \omega,
\]

\[
(3.4) \quad -\frac{\partial^2 u_{2j}}{\partial \xi_n^2} = \Delta_{x'} u_{2j-2}, \quad \xi \in \left( -h^-(x'), h^+(x') \right), \quad u_{2j} = 0, \quad \xi_n = \pm h_{\pm}(x'), \quad x' \in \omega, \quad j \geq 2.
\]

It is easy to check that the solution to the problem (3.3) is given by the formulas (2.2), (2.3). Given \( u_{2j-2} \), the solution to the problem (3.4) is determined uniquely. This is why we can find the functions \( u_{2j} \) in a recurrent way. Moreover, at each step the solution to the problem (3.4) can be found explicitly. These explicit solutions are given by formulas (2.4), (2.5), (2.6), (2.7) and (2.8) that can be checked by direct calculations.
Given any $N \geq 1$, assume that $\alpha_{i}^{(2j)} \in C^{2}(\mathcal{M})$, $j \geq N$. Let

$$u_{\varepsilon}^{N}(x', \xi_{n}) := \sum_{j=0}^{N} \varepsilon^{2j} u_{2j}(x', \xi_{n}).$$

(3.5)

It follows from the problems (3.3), (3.4) that the function $u_{\varepsilon}^{N}$ solves the boundary value problem

$$\left( -\varepsilon^{2}\Delta_{x'} - \frac{\partial^{2}}{\partial \xi_{n}^{2}} \right) u_{\varepsilon}^{N} = 2\varepsilon^{2} + \varepsilon^{2N+2}\Delta_{x'}u_{2N} \quad \text{in} \quad \Omega, \quad u_{\varepsilon}^{N} = 0 \quad \text{on} \quad \partial\Omega.$$

Hence, the function $\tilde{u}_{\varepsilon}^{N} := u_{\varepsilon} - u_{\varepsilon}^{N}$ is the solution to

$$\left( \varepsilon^{2}\Delta_{x'} + \frac{\partial^{2}}{\partial \xi_{n}^{2}} \right) \tilde{u}_{\varepsilon}^{N} = \varepsilon^{2N+2}\Delta_{x'}u_{2N} \quad \text{in} \quad \Omega, \quad \tilde{u}_{\varepsilon}^{N} = 0 \quad \text{on} \quad \partial\Omega.$$  

(3.6)

The coefficient affecting the derivative $\frac{\partial^{2}}{\partial \xi_{n}^{2}}$ in the last equation is one. Employing this fact and applying the maximum principle in the form of inequality (1.9) in [LU, Ch. 3, Sec. 1], we obtain the estimate

$$\| \tilde{u}_{\varepsilon}^{N} \|_{C(\overline{\Omega})} \leq C\varepsilon^{2N+2}\| \Delta_{x'}u_{2N} \|_{C(\overline{\Omega})} \leq C\varepsilon^{2N+2},$$

(3.7)

where the constant $C$ is independent of $\varepsilon$. It proves the formula (2.9). The formulas (2.10) follow directly from (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8). The proof is complete.

4. Proof of Theorem 2

In the whole of this section we shall consider the function $u_{\varepsilon}(x', \xi_{n})$. This is then defined on $\Omega$ and it is clear that it is sufficient to find the maximum of $u_{\varepsilon}(x', \xi_{n})$ since after rescaling $x_{n} \rightarrow x_{n}\varepsilon$ the maximum of the function remains unaltered.

4.1. Existence of an expansion and terms of order $\varepsilon^{2}$. We begin by showing that, under the hypothesis of Theorem 2 and up to order $\varepsilon^{2}$, the maximum of $u$ has an asymptotic expansion that may be obtained directly from the expression of $u_{2}$.

**Lemma 4.1.** Under the hypothesis of Theorem 2 we have

$$\max_{x \in \Omega_{\varepsilon}} u_{\varepsilon}(x) = \max_{x \in \Omega} u_{2}(x)\varepsilon^{2} + O(\varepsilon^{4}) \quad \text{as} \quad \varepsilon \rightarrow 0.$$  

Furthermore

$$\max_{x \in \Omega} u_{2}(x) = u_{2}\left( \frac{d(x)}{2} \right) = \frac{1}{4} H_{0}^{2}$$

and is unique.

**Proof.** The first part follows directly from the asymptotics of $u_{\varepsilon}$ given by Theorem 2 since we now have

$$u_{\varepsilon}(x', \xi_{n}) = \varepsilon^{2}u_{2}(x', \xi_{n}) + O(\varepsilon^{4}).$$
For the second part, note that we may write
\[ u_2(x', \xi_n) = - \left[ \xi_n^2 + \xi_n (h_-(x') - h_+(x')) + h_-(x')h_+(x') \right] \]
\[ = - \left[ \xi_n - \frac{1}{2}d(x) \right]^2 + \frac{1}{4}H^2(x). \]
This last expression is clearly maximized when \( \xi_n = d(x)/2 \) and \( H \) is also maximized, yielding \( x' = \overline{x} \) and \( \xi = d(\overline{x})/2 \). The uniqueness follows directly from hypothesis H1.

In order to go on to obtain the next terms in the expansion for the maximum (and the corresponding maximizer), we need to show the existence of such an expansion which we do in the next lemma. This also proves that the coefficients of the terms of order one in both the expansions for \( x' \) and \( \xi_n \) vanish.

**Lemma 4.2.** Given any \( N \geq 1 \), assume that the functions \( h_{\pm} \) are \( [N/2]+2N \) times continuously differentiable in a vicinity of the point \( \overline{x} \), and the hypotheses H1 and H2 of Theorem 3 hold true. Then the function \( u_{\varepsilon,N} \) has only one stationary point which is a maximum. The corresponding maximizer has the following asymptotic expansion
\[
\begin{align*}
x_{\varepsilon,N}' &= \overline{x} + \sum_{i=1}^{[N/2]} \varepsilon^{2i} \overline{x}_{2i} + O(\varepsilon^{2[N/2]+2}), \\
\xi_{\varepsilon,N}' &= \frac{d(\overline{x})}{2} + \sum_{i=1}^{[N/2]} \varepsilon^{2i} \xi_{ni} + O(\varepsilon^{2[N/2]+2}).
\end{align*}
\]

The maximum of the function \( u_{\varepsilon} \) satisfies the identity
\[
\max_{\Omega_{\varepsilon}} u_{\varepsilon}(x) = \max_{\Omega} u_{\varepsilon}(x', \varepsilon \xi_n) = \max_{\Omega} u_{\varepsilon}^N(x', \xi_n) + O(\varepsilon^{2N+2}),
\]
and any maximizer \( (x_{\varepsilon}', \xi_{\varepsilon}) \) of this function has the asymptotic expansion
\[
\begin{align*}
x_{\varepsilon}' &= \overline{x} + \sum_{i=1}^{[N/2]} \varepsilon^{2i} \overline{x}_{2i} + O(\varepsilon^{N+1}), \\
\xi_{\varepsilon}' &= \frac{d(\overline{x})}{2} + \sum_{i=1}^{[N/2]} \varepsilon^{2i} \xi_{ni} + O(\varepsilon^{N+1}).
\end{align*}
\]

**Proof.** The identity (4.2) follows directly from the asymptotics (2.9) for \( u_{\varepsilon}^N \).

Let us find the maximum of \( u_{\varepsilon}^N \). In order to do this, we should first find the stationary points of this functions by solving the equation
\[
\nabla_{(x', \xi_n)} u_{\varepsilon}^N (x', \xi_n) = 0,
\]
which is equivalent to
\[
\sum_{i=1}^{N} \varepsilon^{2i-2} \nabla_{(x', \xi_n)} u_{2i}(x', \xi_n) = 0.
\]
It follows from Lemma 4.1 that for ε = 0 this equation has a unique solution \((\mathfrak{p}, d(\mathfrak{p})/2)\). In order to solve (4.3) for ε > 0 we apply the implicit function theorem considering \((x', \xi_n)\) as functions of ε. We first need to check that the corresponding Jacobian is non-zero. It is easy to see that this Jacobian by hypothesis H2.

The assumption for the smoothness of \(h_{\pm}\) and the formulas (2.4), (2.5), (2.6), (2.7), (2.8) for \(u_{2i}\) yield that \(u_{2i}(x', \xi_n), i = 1, \ldots, N\), are \([N/2] + 2\) times continuously differentiable in a small vicinity of \((\mathfrak{p}, d(\mathfrak{p})/2)\).

The dependence of the left hand side of (4.4) on \(\varepsilon^2\) is holomorphic and by the implicit function theorem we conclude that for \(\varepsilon\) small enough there exists a unique solution \((x_{\varepsilon,N}', \xi_{\varepsilon,N})\) to (4.4) which is \([N/2] + 1\) times continuously differentiable in \(\varepsilon^2\). Hence, we have the Taylor polynomial (4.1). The point \((x_{\varepsilon,N}', \xi_{\varepsilon,N})\) is the maximizer for \(u_{\varepsilon}^N\) since by hypothesis H2 the Hessian of \(u_{\varepsilon}^N\) at this point differs from \(2H_2\) by an error of order \(O(\varepsilon^2)\). We employ this fact and expand \(u_{\varepsilon}^N(x', \xi_n)\) in Taylor series at \((x_{\varepsilon,N}', \xi_{\varepsilon,N})\). As a result, we have the estimate

\[
(4.5) \quad u_{\varepsilon}^N(x', \xi_n) - u_{\varepsilon}^N(x_{\varepsilon,N}', \xi_{\varepsilon,N}) \leq -C_1 \left( |x' - x_{\varepsilon,N}'|^2 + |\xi_n - \xi_{\varepsilon,N}|^2 \right),
\]

where \(C_1\) is a positive constant independent of \(\varepsilon, x'\) and \(\xi_n\). This estimate is valid in a small fixed neighborhood \(Q\) of the point \((\mathfrak{p}, d(\mathfrak{p})/2)\). Since the function \(u_{\varepsilon}^N\) has the maximum at \((x_{\varepsilon,N}', \xi_{\varepsilon,N})\), we can choose the neighborhood \(Q\) so that outside it the estimate

\[
(4.6) \quad u_{\varepsilon}^N(x', \xi_n) - u_{\varepsilon}^N(x_{\varepsilon,N}', \xi_{\varepsilon,N}) \leq -C_2 < 0
\]

holds true, where the constant \(C_2\) is independent of \(\varepsilon\). Let us choose \((x', \xi_n)\) so that

\[
(4.7) \quad |x' - x_{\varepsilon,N}'|^2 + |\xi_n - \xi_{\varepsilon,N}|^2 \geq C_3 \varepsilon^{2N+2},
\]

where \(C_3\) is a positive constant independent of \(\varepsilon, x'\) and \(\xi_n\). Then it follows from (4.5), (4.6) that for such \((x', \xi_n)\) the inequality

\[
(4.8) \quad u_{\varepsilon}^N(x', \xi_n) \leq u_{\varepsilon}^N(x_{\varepsilon,N}', \xi_{\varepsilon,N}) - C_1 C_3 \varepsilon^{2N+2}
\]

is valid. Together with (4.2) it implies that a maximizer \((x_{\varepsilon}', \xi_{\varepsilon})\) of \(u_{\varepsilon}\) can not satisfy (4.7) for sufficiently small \(\varepsilon\) and sufficiently large \(C_3\) and therefore

This inequality and (4.1) prove (4.3). \(\square\)

4.2. The terms of order \(\varepsilon^4\). In order to determine \(\mathfrak{p}_2\) and \(\xi_2\) we shall need the terms of order \(\varepsilon^4\) in the asymptotics of the gradient of \(u_{\varepsilon}\), for which we need to consider \(u_4\). We must also develop \(d\) and \(p\) around \(\mathfrak{p}\). In full generality, and to obtain the full asymptotic expansion, these developments should be written in terms of homogeneous polynomials of increasing degree. However, and since in order to obtain the first two terms in the asymptotics we will only need terms up to the homogeneous polynomials of third degree,
we shall choose a form that will be more convenient for our calculations. Write thus \(d\) and \(p\) as follows.

\[
4.9 \\
d(x') = d_0 + d_1'(x' - \bar{x}) + (x' - \bar{x})^t D_2 (x' - \bar{x}) + D_3 (x' - \bar{x}) + \mathcal{O}(|x' - \bar{x}|^4),
\]

\[
p(x') = p_0 + p_1'(x' - \bar{x}) + (x' - \bar{x})^t P_2 (x' - \bar{x}) + P_3 (x' - \bar{x}) + \mathcal{O}(|x' - \bar{x}|^4),
\]

where \(d_1 = \nabla x d(\bar{x}), p_1 = \nabla x p(\bar{x}).\)

Due to the relation between \(d\) and \(p\) via the functions \(h_{\pm}\) and the fact that \(H'(\bar{x}) = h_{-}(\bar{x}) + h_{+}(\bar{x})\) must vanish, we easily obtain that

\[
p_1 = -\frac{1}{2} d_0 d_1.
\]

The homogeneous polynomials of degree three, \(D_3\) and \(P_3\), will be specified below.

In the case of \(u_{41}\), the relevant derivatives are given by

\[
\frac{\partial u_{41}}{\partial \xi_n}(x', \xi_n) = -\frac{1}{2} \xi_n^2 \Delta x d(x') - \xi_n \Delta x p(x') + \alpha_{1}^{(4)}(x')
\]

\[
\nabla_x u_{41}(x', \xi_n) = -\frac{1}{6} \xi_n^3 \nabla_x [\Delta x d(x')] - \frac{1}{2} \xi_n^2 \nabla_x [\Delta x p(x')]
\]

\[
+ \xi_n \nabla_x \alpha_{1}^{(4)}(x') + \nabla_x \alpha_{0}^{(4)}(x').
\]

We shall first obtain the term of order \(\varepsilon^4\) in the derivative of \(u_{\varepsilon}\) with respect to \(\xi_n\). This will have a component coming from the term of order \(\varepsilon^2\) in the corresponding derivative of \(u_2\), and another from the constant term in the derivative of \(u_{41}\). In the first case it is straightforward to obtain that the required coefficient is given by

\[
4.10 \\
-2 \xi_n^2 + d_1' \bar{x}.
\]

In the case of \(u_{41}\) the term coming from \(\alpha_{1}^{(4)}\) is given by

\[
\alpha_{1}^{(4)}(\bar{x} + \mathcal{O}(\varepsilon^2)) = \frac{1}{3} (d_0^2 + p_0) \text{tr}(D_2) + d_0 \text{tr}(P_2) + \mathcal{O}(\varepsilon^2).
\]

We thus obtain

\[
\frac{\partial u_{41}}{\partial \xi_n}(\bar{x} + \mathcal{O}(\varepsilon^2), \frac{1}{2} d_0 + \mathcal{O}(\varepsilon^2)) = -\frac{1}{4} d_0^2 \text{tr}(D_2) - d_0 \text{tr}(P_2)
\]

\[
+ \frac{1}{3} (d_0^2 + p_0) \text{tr}(D_2)
\]

\[
+ d_0 \text{tr}(P_2) + \mathcal{O}(\varepsilon^2)
\]

\[
= \frac{1}{12} (d_0^2 + 4p_0) \text{tr}(D_2) + \mathcal{O}(\varepsilon^2).
\]

This, together with (4.10), yields

\[
\frac{\partial u_{41}}{\partial \xi_n}(\bar{x} + \mathcal{O}(\varepsilon^2), \frac{1}{2} d_0 + \xi_n^2 \varepsilon^2 + \mathcal{O}(\varepsilon^3)) =
\]

\[
4.11 \\
= \left[ -2 \xi_n^2 + d_1' \bar{x} + \frac{1}{12} (d_0^2 + 4p_0) \text{tr}(D_2) \right] \varepsilon^4 + \mathcal{O}(\varepsilon^5).
\]
We will now proceed to compute the gradient with respect to $x'$. The case of $u_2$ is again straightforward and we obtain

\begin{equation}
\nabla_{x'} u_2(\bar{x} + \bar{x}_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3), \frac{1}{2}d_0 + \xi_n \varepsilon^2 + \mathcal{O}(\varepsilon^3)) =
\end{equation}

(4.12)

\[
= \left(\frac{1}{2}d_0 + \xi_n \varepsilon^2\right) \left[d_1 + 2D_2 \bar{x}_2 \varepsilon^2\right] - \frac{1}{2}d_0d_1 + 2P_2 \bar{x}_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3)
\]

\[
= (\xi_n d_1 + d_0 D_2 \bar{x}_2 + 2P_2 \bar{x}_2) \varepsilon^2 + \mathcal{O}(\varepsilon^3).
\]

In the case of $u_4$ we are only interested in the terms of order $\varepsilon^0$. However, there are now expressions of the form $\nabla_{x'} \Delta_{x'} d(x')$ and $\nabla_{x'} \Delta_{x'} p(x')$, this being the reason why we need the homogeneous polynomials of third degree in the expansions of $d$ and $p$. On the other hand, this implies that the only relevant terms from $D_3$ and $P_3$ are those where one of the variables appears at least twice. If we write

\[
D_3(x') = \sum_{i \neq j} d_{ijk} x_i x_j x_k
\]

\[
= \sum_{i=1}^{n-1} \left[d_{iii} x_i^3 + \sum_{j=1, j \neq i}^{n-1} (d_{iij} + d_{iji} + d_{ji}) x_i^2 x_j\right] + r_3^{d}(x'),
\]

with

\[
r_3^{d}(x') = \sum_{i \neq j \neq k} d_{ijk} x_i x_j x_k,
\]

we may assume without loss of generality that the coefficients $d_{ijk}$ are invariant under any possible permutation of the indices. If we then denote $d_{iii}$ and $d_{iij} = d_{iji} = d_{ji}$ ($i \neq j$) by $\delta_{ii}$ and $\delta_{ij}$, respectively, the expression for $D_3$ becomes

\[
D_3(x') = \sum_{i=1}^{n-1} \left(\delta_{ii} x_i^3 + 3 \sum_{j=1, j \neq i}^{n-1} \delta_{ij} x_i^2 x_j\right) + r_3^{d}(x').
\]

With this notation we get

\[
\nabla_{x'} (\Delta_{x'} D_3(x')) = 6 \nabla_{x'} \left[\sum_{i=1}^{n-1} \sum_{j=1, j \neq i}^{n-1} \delta_{ij} x_i x_j\right] = 6 \left(\sum_{j=1}^{n-1} \delta_{1j}, \ldots, \sum_{j=1}^{n-1} \delta_{n-1,j}\right).
\]
In a similar fashion, if we write
\[ P_3(x') = \sum_{i=1}^{n-1} \left( \pi_{ii} x_i^3 + 3 \sum_{j=1, j \neq i}^{n-1} \pi_{ij} x_i^2 x_j \right) + r_3^p(x'), \]
we get
\[
\nabla_{x'} (\Delta_{x'} P_3(x')) = 6 \nabla_{x'} \left[ \sum_{i=1}^{n-1} \left( \pi_{ii} x_i + \sum_{j=1, j \neq i}^{n-1} \pi_{ij} x_j \right) \right]
= 6 \left( \sum_{j=1}^{n-1} \pi_{1j}, \ldots, \sum_{j=1}^{n-1} \pi_{n-1j} \right).
\]

In this way, we obtain after some lengthy but straightforward calculations,
\[
\nabla_{x'} u_4(\bar{\xi} + \bar{\tau}_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3), \frac{1}{2} d_0 + \xi_{n2} \varepsilon^2 + \mathcal{O}(\varepsilon^3)) = -\left( \frac{d_0}{2} \right) \Sigma_{\delta} - 3 \left( \frac{d_0}{2} \right)^2 \Sigma_{\pi}
+ \frac{d_0}{2} \left[ (d_0^2 + p_0) \Sigma_{\delta} + \frac{\text{tr} D_2}{3} (2d_0 d_1 - \frac{1}{2} d_0 d_1) + 3 d_0 \Sigma_{\pi} + \text{tr} P_2 d_1 \right]
+ d_0 p_0 \Sigma_{\delta} + \frac{\text{tr} D_2}{3} (-\frac{1}{2} d_0^2 d_1 + p_0 d_1) + 3 p_0 \Sigma_{\pi} - \frac{\text{tr} P_2}{2} d_0 d_1 + \mathcal{O}(\varepsilon^2)
= \frac{1}{4} (d_0^2 + 4 p_0) \left[ 3 \Sigma_{\pi} + \frac{3}{2} d_0 \Sigma_{\delta} + \frac{\text{tr} D_2}{3} d_1 \right] + \mathcal{O}(\varepsilon^2)
\]

where
\[
\Sigma_{\delta} = \left( \sum_{j=1}^{n-1} \delta_{j1}, \ldots, \sum_{j=1}^{n-1} \delta_{j,n-1} \right) \quad \text{and} \quad \Sigma_{\pi} = \left( \sum_{j=1}^{n-1} \pi_{j1}, \ldots, \sum_{j=1}^{n-1} \pi_{j,n-1} \right).
\]

Combining this with (4.12) yields
\[
\nabla_{x'} u_4(\bar{\xi} + \bar{\tau}_2 \varepsilon^2 + \mathcal{O}(\varepsilon^3), \frac{1}{2} d_0 + \xi_{n2} \varepsilon^2 + \mathcal{O}(\varepsilon^3)) =
= \left[ \xi_{n2} d_1 + d_0 D_2 \bar{\tau}_2 + 2 P_2 \bar{\tau}_2 
+ \frac{1}{4} (d_0^2 + 4 p_0) \left[ 3 \Sigma_{\pi} + \frac{3}{2} d_0 \Sigma_{\delta} + \frac{\text{tr} D_2}{3} d_1 \right] \right] \varepsilon^4
+ \mathcal{O}(\varepsilon^5),
\]

from which we obtain the second equation for \( \bar{\tau}_2 \) and \( \xi_{n2} \) by equating the coefficient of \( \varepsilon^4 \) to zero. From equation (4.111) we get
\[
\xi_{n2} = \frac{1}{2} d_1^3 \bar{\tau}_2 + \frac{1}{24} (d_0^2 + 4 p_0) \text{tr} D_2.
\]

Substituting this into equation (4.13) yields
\[
\left[ \frac{1}{2} (d_1 d_1^3) + 2 P_2 + d_0 D_2 \right] \bar{\tau}_2 = -\frac{1}{4} (d_0^2 + 4 p_0) \left( \text{tr} D_2^2 d_1 + 3 \Sigma_{\pi} + \frac{3}{2} d_0 \Sigma_{\delta} \right).
\]

To prove that there is a unique solution, we must show that the matrix multiplying \( \bar{\tau}_2 \) is nonsingular. In order to do this, we shall relate the terms
appearing above to those in the expansions of the functions $h_\pm$ and $H$. If we write

$$h_\pm(x') = h_0^\pm + (h_1^\pm)^t (x' - \overline{x}) + (x' - \overline{x})^t H_2^\pm (x' - \overline{x}) + \mathcal{O}(|x' - \overline{x}|^3),$$

we see that

$$d_0 = h_0^+ - h_0^-$$
$$d_1 = h_1^+ - h_1^-$$
$$D_2 = H_2^+ - H_2^-$$
$$H_2 = H_2^+ + H_2^-$$

and

$$p_0 = h_0^+ h_0^-$$
$$p_1 = h_0^+ h_1^- + h_0^- h_1^+$$
$$P_2 = h_0^+ H_2^- + H_0^+ H_2^+ + h_1^- (h_1^+)^t.$$

Replacing this in the expression above yields, after some manipulation

$$d_0 D_2 + 2 P_2 + \frac{1}{2} d_1 d_1^t = H_0 H_2 + \frac{1}{2} (h_1^+ - h_1^-)(h_1^+ - h_1^-)^t + h_1^+ (h_1^-)^t$$
$$= H_0 H_2,$$

where we used the fact that $0 = H_1 = h_1^+ + h_1^-$. Since the matrix $H_0 H_2$ is negative definite by hypothesis, we may invert it to obtain

$$\overline{x}_2 = -\frac{1}{4} H_0 \left[ \text{tr} D_2 H_2^{-1} d_1 + 3 H_2^{-1} \left( \Sigma_\pi + \frac{1}{2} d_0 \Sigma_\delta \right) \right],$$

where we have used the fact that $d_0^2 + 4 p_0 = H_0^2$. Replacing this back into (4.14) yields

$$\xi_{n2} = -\frac{1}{8} H_0 \left[ \frac{1}{2} \text{tr} D_2 d_1 H_2^{-1} d_1 + 3 d_1^t H_2^{-1} \left( \Sigma_\pi + \frac{1}{2} d_0 \Sigma_\delta \right) \right]$$
$$+ \frac{1}{2} H_0^2 \text{tr} D_2.$$

If we now evaluate $u_2$ and $u_4$ at the maximizer we obtain

$$u_2 \left( \overline{x} + \overline{x} \varepsilon^2 + \mathcal{O}(\varepsilon^3), \frac{1}{2} d_0 + \xi_{n2} \varepsilon^2 + \mathcal{O}(\varepsilon^3) \right) = \frac{1}{4} H_0^2 + \mathcal{O}(\varepsilon^3)$$

and

$$u_4 \left( \overline{x} + \overline{x} \varepsilon^2 + \mathcal{O}(\varepsilon^3), \frac{1}{2} d_0 + \xi_{n2} \varepsilon^2 + \mathcal{O}(\varepsilon^3) \right)$$
$$= \frac{1}{8} H_0^2 \left[ d_0 \text{tr}(D_2) + 2 \text{tr}(P_2) \right] + \mathcal{O}(\varepsilon^2),$$

where $\xi_{n2}$ and $\overline{x}_2$ are given as above. We have thus proven the following

**Theorem 4.** Under the conditions of Theorem 2, the maximizer $(x'_*, \xi_{n*})$ of $u_\varepsilon$ given by Lemma 4.2 satisfies the asymptotic expansion

$$(x'_*, \xi_{n*}) = (\overline{x} + \overline{x} \varepsilon^2, \frac{1}{2} d_0 + \xi_{n2} \varepsilon^2 + \mathcal{O}(\varepsilon^3)) \text{ as } \varepsilon \to 0,$$
where $\overline{x}_2$ and $\xi_n$ are given by (4.15) and (4.16), respectively. The corresponding maximum satisfies the asymptotic expansion given in Theorem 2.

5. Proof of Theorem 3

To prove Theorem 3 we need only to integrate $u_\varepsilon$ in $\Omega_\varepsilon$. More precisely, and using the expression given by Theorem 1, we have to compute

$$
\int_{\Omega_\varepsilon} u_\varepsilon(x) \, dx = \int_\omega \int_{h_+(x')} \varepsilon^2 u_2 \left( x', \frac{x_n}{\varepsilon} \right) + \varepsilon^4 u_4 \left( x', \frac{x_n}{\varepsilon} \right) + \varepsilon^6 u_6 \left( x', \frac{x_n}{\varepsilon} \right) + O(\varepsilon^7) \, dx_n \, dx' = \varepsilon^3 \int_\omega \int_{-h_-(x')} u_2(x', \xi_n) \, d\xi_n \, dx' + \varepsilon^5 \int_\omega \int_{-h_-(x')} u_4(x', \xi_n) \, d\xi_n \, dx' + \varepsilon^7 \int_\omega \int_{-h_-(x')} u_6(x', \xi_n) \, d\xi_n \, dx' + O(\varepsilon^8).
$$

We shall now compute these three integrals separately. Using the expression for $u_2$ we have

$$
\int_{-h_-(x')}^{h_+(x')} u_2(x', \xi_n) \, d\xi_n = \int_{-h_-(x')}^{h_+(x')} -\xi_n^2 + \xi_n d(x') + p(x') \, d\xi_n = -\frac{1}{3} \xi_n^3 + \frac{1}{2} \xi_n^2 d(x') + p(x') \xi_n \bigg|_{-h_-(x')}^{h_+(x')} = -\frac{1}{3} \left[ h_+^3(x') - h_-^3(x') \right] + \frac{1}{2} \left[ h_+^2(x') - h_-^2(x') \right] d(x') + [h_+(x') + h_-(x')] p(x')
$$

and, taking into account the expressions for both $d$ and $p$, we get that the above equals

$$
\frac{1}{6} \left[ h_+^3(x') + h_-^3(x') + 3h_+^2(x')h_-(x') + 3h_+(x')h_-^2(x') \right] = \frac{1}{6} H_3(x')
$$
as desired.
Using now the following identities
\[ h_3^2(x') + h_5^2(x') = H^3(x') - 3p(x') H(x') \]
\[ h_4^1(x') - h_4^2(x') = H^3(x') d(x') - 2d(x') p(x') H(x') \]
\[ h_5^2(x') + h_5^2(x') = H^3(x') d^2(x') + p(x') H(x') \left[ p(x') - d^2(x') \right] \]
\[ h_5^3(x') - h_5^5(x') = H^3(x') d^3(x') + 3d(x') p^2(x') H(x') \]

For \( u_4 \) we have
\[
\int_{-h_- (x')}^{h_+ (x')} u_4 (x', \xi_n) \, d\xi_n = \int_{-h_- (x')}^{h_+ (x')} \frac{1}{6} \xi_n^3 \Delta x' d(x') - \frac{1}{2} \xi_n^2 \Delta x' p(x') + \alpha_1^{(4)} (x') \xi_n + \alpha_0^{(4)} (x') \, d\xi_n
\]
\[
= - \frac{1}{24} \xi_n^3 \Delta x' d(x') - \frac{1}{6} \xi_n^2 \Delta x' p(x') + \frac{1}{2} \alpha_1^{(4)} (x') \xi_n + \alpha_0^{(4)} (x') \xi_n \bigg|_{-h_- (x')}^{h_+ (x')}.
\]

Using the expressions for \( \alpha_0^{(4)} \) and \( \alpha_1^{(4)} \) we see that the above integral equals
\[
- \frac{1}{24} \left[ h_4^1 (x') - h_4^2 (x') \right] \Delta x' d(x') - \frac{1}{6} \left[ h_5^3 (x') + h_5^3 (x') \right] \Delta x' p(x')
+ \frac{1}{12} \left[ (d^2 (x') + p(x')) \Delta x' d(x') + 3d(x') \Delta x' p(x') \right] \left[ h_4^2 (x') - h_5^2 (x') \right]
+ \frac{1}{12} \left[ d(x') p(x') \Delta x' d(x') + 3p(x') \Delta x' p(x') \right] \left[ h_4^1 (x') + h_5^1 (x') \right]
= \frac{1}{24} \left[ h_4^1 (x') - h_4^1 (x') + 2h_4^1 (x') h_- (x') - 2h_4^1 (x') h_3^2 (x') \right] \Delta x' d(x')
+ \frac{1}{12} \left[ h_4^2 (x') + h_5^2 (x') + 3h_4^2 (x') h_- (x') + 3h_4^1 (x') h_5^2 (x') \right] \Delta x' p(x')
= \frac{1}{24} \left[ d^2 (x') H(x') + 4h_4^2 (x') h_- (x') - 4h_4^1 (x') h_5^2 (x') \right] \Delta x' d(x')
+ \frac{1}{12} H^3(x') \Delta x' p(x')
= \frac{1}{24} d(x') H(x') \left[ d^2 (x') + 4p(x') \right] \Delta x' d(x') + \frac{1}{12} H^3(x') \Delta x' p(x')
= \frac{1}{24} H^3(x') \left[ d(x') \Delta x' d(x') + 2 \Delta x' p(x') \right].
\]

Although the expression for \( u_6 \) is much more involved, the necessary computations are similar to those above. We need to compute
\[
\int_{-h_- (x')}^{h_+ (x')} u_6 (x', \xi_n) \, d\xi_n = \int_{-h_- (x')}^{h_+ (x')} \frac{\xi_n^5}{120} \Delta x' d(x') + \frac{\xi_n^4}{24} \Delta x' p(x')
- \frac{\xi_n^3}{36} \Delta x' \left\{ \left[ d^2 (x') + p(x') \right] \Delta x' d(x') + 3d(x') \Delta x' p(x') \right\}
- \frac{\xi_n^2}{12} \Delta x' \left[ d(x') p(x') \Delta x' d(x') + 3p(x') \Delta x' p(x') \right]
+ \alpha_1^{(6)} (x') \xi_n + \alpha_0^{(6)} (x') \, d\xi_n.
\]

Using now the following identities
\[
\begin{align*}
h_3^3 (x') + h_5^5 (x') & = H^3 (x') - 3p (x') H (x') \\
h_4^1 (x') - h_4^2 (x') & = H^3 (x') d (x') - 2d (x') p (x') H (x') \\
h_5^2 (x') + h_5^2 (x') & = H^3 (x') d^2 (x') + p (x') H (x') \left[ p (x') - d^2 (x') \right] \\
h_5^3 (x') - h_5^5 (x') & = H^3 (x') d^3 (x') + 3d (x') p^2 (x') H (x'),
\end{align*}
\]
after some computations we obtain
\[
\int_{-h_{-}(x')}^{h_{+}(x')} u_6(x', \xi_n) \, d\xi_n = \frac{1}{720} \left[ H^3(x') d^3(x') + 3d(x') p^2(x') H(x') \right] \Delta x' d(x')
\]
\[
+ \frac{1}{144} \left[ H^3(x') d(x') - 2d(x') p(x') H(x') \right] \times \Delta x' \left[ \left[ d^2(x') + p(x') \right] \Delta x' d(x') + 3d(x') \Delta x' p(x') \right]
\]
\[
- \frac{1}{36} \left[ H^3(x') - 3p(x') H(x') \right] \times \Delta x' \left[ d(x') p(x') \Delta x' d(x') + 3p(x') \Delta x' p(x') \right]
\]
\[
+ \frac{1}{2} d(x') H(x') \alpha_1^{(6)} (x') + H(x') \alpha_0^{(6)} (x').
\]

This, together with the expressions for the integrals of \( u_2 \) and \( u_4 \) given above yields the formula in Theorem 3.

6. Examples

Let us now apply our results to some special cases in order to test the accuracy of the approximations for concrete examples. As may be seen from Figures 6.1 and 6.2 below, although the error for either the \( L^2 \)- norm or for the torsion stays below 5\% for \( \varepsilon \) up to 0.6, this can vary substantially as the scaling parameter approaches one. For the examples considered below the error at \( \varepsilon \) equal to one for the torsion, for instance, varies between less than 2\% and 100\% in the cases of the folium and the disc, respectively. The reason for this is simply that, even in the case where a Taylor (or Laurent) series exists for the quantities under consideration, the series expansion for these quantities will have a specific radius of convergence. In the case of the disc, for instance, we have that the torsion is given by
\[
\frac{\pi}{2} \frac{\varepsilon^2}{1 + \varepsilon^2}
\]
which has a radius of convergence of one, thus explaining the large error found in this case.

6.1. Descartes’s folium. We consider the case of the domain defined by
\[
\Omega_\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + 3\varepsilon^{-2} y^2 - x^2 + \varepsilon^{-2} y^2 < 0, \ 0 < x < 1 \right\},
\]
where we have chosen coordinates it in such a way that the scaling is done along the \( y \)-axis and the corresponding height with respect to this axis is minimal. In this case we then have
\[
H(x) = 2h_+(x) = 2h_-(x) = 2x \sqrt{\frac{1 - x}{1 + 3x}}, \quad d(x) \equiv 0, \quad p(x) = \frac{(1 - x)x^2}{3x + 1},
\]
yielding
\[
\begin{align*}
\epsilon(u(x, \xi)) &= \left[\frac{(1 - x)x^2}{1 + 3x} - \xi^2\right]\epsilon^2 \\
&\quad + \left[\frac{(3x + 1)^3 - 4}{3(1 + 3x^4)}\right] \left[(x - 1)x^2 + (3x + 1)\xi^2\right]\epsilon^4 \\
&\quad - \left[\frac{(x - 1)x^2 + (3x + 1)\xi^2}{(3x + 1)^4}\right] p_1(x, \xi) \epsilon^6 + O(\epsilon^8),
\end{align*}
\]
where
\[p_1(x, \xi) := 1 - 30x + 51x^2 + 48x^3 + 135x^4 + 162x^5 + 81x^6 - 12\xi^2 - 36x\xi^2.
\]

The direct application of Theorem 2, where we only have the explicit formula for terms up to order \(\epsilon^4\), yields
\[
\max_{x \in \Omega} \epsilon u(x, y) = \frac{1}{9} (2\sqrt{3} - 3)\epsilon^2 + \frac{1}{9} (12 - 7\sqrt{3})\epsilon^4 + O(\epsilon^5).
\]
Note that in this case determining the maximum directly from (6.1) in order to obtain a better approximation implies solving an algebraic equation of degree nine. Actually, even solving explicitly for the maximizer using \(u\) up to order \(\epsilon^4\) and taking into consideration that we know beforehand using symmetry that \(\xi\) must vanish, implies having to solve an algebraic equation of degree five.

Computing the expansion for the torsion will in turn yield
\[
\varepsilon \int_{\Omega} u(x, \xi) dxd\xi = \left(\frac{16\pi}{243\sqrt{3}} - \frac{1}{9}\right)\varepsilon^3 - \left(\frac{16\pi}{243\sqrt{3}} - \frac{37}{915}\right)\varepsilon^5
\]
\[
+ \left(\frac{80\pi}{2187\sqrt{3}} - \frac{593}{9009}\right)\varepsilon^7 + O(\varepsilon^9)
\]
In Figure 6.1 we show, for various values of \(\varepsilon\), the relative errors for the \(L^2\) norm and for the torsion of the difference between the values of numerical solution determined using the method of fundamental solutions (MFS) and the asymptotic expansions given by (6.1) and (6.2). We note that the error at \(\varepsilon\) equal to one – which corresponds to the actual folium – is of the order of 3.5% and 2% respectively.

6.2. Lemniscate. We consider the domain \(\Omega_\varepsilon\) whose boundary is the lemniscate defined by
\[
(x^2 + \varepsilon^{-2}y^2)^2 = x^2 - \varepsilon^{-2}y^2, \quad x > 0.
\]
The functions \(H, h_\pm, d, p\) are given by the formulas
\[
H(x) = 2h_+(x) = 2h_-(x) = 2 \left[\frac{1}{2} - x^2 + \frac{1}{2}(1 + 8x^2)^{1/2}\right]^{1/2},
\]
\[
d(x) \equiv 0, \quad p = -\frac{1}{2} - x^2 + \frac{1}{2}(1 + 8x^2)^{1/2}, \quad \omega = (0, 1).
\]
Some straightforward calculations give

\[ u_\varepsilon(x, \xi) = \left( -\frac{1}{2} - x^2 + \frac{1}{2}(1 + 8x^2)^{1/2} - \xi^2 \right) \varepsilon^2 + \]

\[ + \left[ (1 + 8x^2)^{3/2} - 2 \right] \left( 1 + 2x^2 - \sqrt{1 + 8x^2 + 2\xi^2} \right) \varepsilon^4 \]

\[ + \left( \frac{4\xi^4}{(1 + 8x^2)^{7/2}} \right) \frac{32x^2 - 1}{(1 + 8x^2)^{7/2}} \]

\[ + \xi^2 \frac{(-512x^6 - 192x^4 - 216x^2 + 7)\sqrt{1 + 8x^2 + 256x^4 + 328x^2 - 8}}{(1 + 8x^2)^{7/2}} \]

\[ - \frac{(1 + 2x^2 - \sqrt{1 + 8x^2})p_2(x)}{2(1 + 8x^2)^{7/2}} \varepsilon^6 + \mathcal{O}(\varepsilon^8), \]

\[ p_2(x) := \sqrt{1 + 8x^2}(512x^6 + 192x^4 + 152x^2 - 5) - 128x^4 - 268x^2 + 6. \]

The maximum of \( H \) is now situated at \( \bar{x} = \sqrt{3}/(2\sqrt{2}) \) and using

\[ H_0 = \frac{\sqrt{2}}{2}, \quad d_0 = \text{tr} \, D_2 = 0, \quad 2 \, \text{tr} \, P_2 = -\frac{3}{2}, \]

in Theorem 2 then yields

\[ \max_{x \in \Omega_{\varepsilon}} u_\varepsilon(x) = \frac{\varepsilon^2}{8} - \frac{3\varepsilon^4}{32} + \mathcal{O}(\varepsilon^5). \]

By applying Theorem 3 we arrive at the asymptotics for the torsional rigidity

\[ \int_{\Omega_{\varepsilon}} u_\varepsilon(x) \, dx = \frac{3\pi - 8}{48} \varepsilon^3 + \left[ \frac{\sqrt{3}}{4} \ln(2 + \sqrt{3}) - \frac{3\pi}{16} \right] \varepsilon^5 \]

\[ + \left[ \frac{13}{18} + \frac{5\pi}{16} - \frac{20\sqrt{3}}{27} \ln(2 + \sqrt{3}) \right] \varepsilon^7 + \mathcal{O}(\varepsilon^9), \]
where the integrals appearing in the coefficients can be calculated by the Euler substitution

\[ \sqrt{1 + 8x^2} = xt + 1, \quad x = \frac{2t}{8 - t^2}, \quad t = \frac{\sqrt{1 + 8x^2} - 1}{x}. \]

As in the previous example, we show in Figure 6.2 the relative errors for the $L^2$ norm and the torsion in this case. However, comparing the two examples gives that for $\varepsilon$ larger than approximately 0.4 the errors become much larger than in the case of the folium.

![Graph of the relative errors for the $L^2$ norm and the torsion in the case of the lemniscate](image)

**Figure 2.** Graphs of the relative errors for the $L^2$ norm and the torsion in the case of the lemniscate, with the different quantities computed in a similar way to what was done in the folium case.

### 6.3. Ellipsoids.** As mentioned in the Introduction, these are one of the few examples where the explicit solution of the corresponding equation (1.1) is known. More precisely, if we consider ellipsoids defined by

\[ E = \left\{ x \in \mathbb{R}^n : \frac{x_1}{a_1^2} + \cdots + \frac{x_n}{a_n^2} = 1 \right\} \]

we have that the solution of equation (1.1) in this case is given by

\[ u(x) = \frac{1}{1/a_1^2 + \cdots + 1/a_n^2} \left[ 1 - \left( \frac{x_1}{a_1} \right)^2 - \cdots - \left( \frac{x_n}{a_n} \right)^2 \right] \]

The maximum is thus localized at the origin and is given by

\[ M = \frac{1}{1/a_1^2 + \cdots + 1/a_n^2}. \]

Let’s assume that $a_n$ is the smallest of the $a_j$’s, and we thus pick this direction to be that along which we scale the domain. We then have

\[ H(x') = 2h_+(x') = 2h_-(x') = 2a_n \left[ 1 - \left( \frac{x_1}{a_1} \right)^2 - \cdots - \left( \frac{x_{n-1}}{a_{n-1}} \right)^2 \right]^{1/2}, \]
while $d$ vanishes and

$$p(x') = a_n^2 \left[ 1 - \left( \frac{x_1}{a_1} \right)^2 - \cdots - \left( \frac{x_{n-1}}{a_{n-1}} \right)^2 \right].$$

From this it follows that the maximizer is $O(\varepsilon^3)$, while the maximum has the expansion

$$M = a_n^2 \varepsilon^2 - a_n^4 \sum_{j=1}^{n-1} \frac{1}{a_j^2} \varepsilon^4 + O(\varepsilon^6).$$

A straightforward analysis of the error

$$\mathcal{E}_M = \left[ \frac{1}{a_1^2 + \cdots + \frac{1}{\varepsilon^2 a_n^2}} - a_n^2 \varepsilon^2 + a_n^4 \sum_{j=1}^{n-1} \frac{1}{a_j^2} \varepsilon^4 \right] \left( \frac{1}{a_1^2 + \cdots + \frac{1}{\varepsilon^2 a_n^2}} \right)^{-1}$$

yields that this satisfies

$$0 \leq \mathcal{E}_M \leq \varepsilon^4 (n - 1)^2,$$

with equality on the right-hand side being achieved for the ball.

For the sake of comparison with the previous two-dimensional examples, we shall now consider the case of the planar disc (in the above notation, $n = 2, a_1 = 1, a_2 = \varepsilon$). In this case the above expressions yield that the relative error of the $L^2$ norm and of the torsion are, respectively, $\varepsilon^{12} + O(\varepsilon^{13})$ and $\varepsilon^6 + O(\varepsilon^7)$. Thus, although the approximation is very good for sufficiently small values of $\varepsilon$, the error does become quite large as $\varepsilon$ reaches one.

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