T-WEYL CALCULUS

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Abstract. Let \((W, \sigma)\) be a symplectic vector space and let \(T: W \to W\) be a linear map that satisfies a certain condition of non-degeneracy. We define the Schur multiplier \(\omega_{\sigma, T}\) on \(W\). To this multiplier we associate a \(\omega_{\sigma, T}\)-representation and we build the T-Weyl calculus, \(\text{Op}_{\sigma, T}\), whose properties are systematically studied further.

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1. Introduction

In two classical papers [5] and [19], H.O. Cordes and T. Kato develop an elegant method to deal with pseudo-differential operators. In [5], Cordes shows, among others, that if a symbol \(a(x, \xi)\) defined on \(\mathbb{R}^n \times \mathbb{R}^n\) has bounded derivatives \(D^\alpha_x D^\beta_\xi a\) for \(|\alpha|, |\beta| \leq \lfloor n/2 \rfloor + 1\), then the associated pseudo-differential operator \(A = a(X, D)\) is \(L^2\)-bounded.

This method can be extended and can be used to prove trace-class properties of pseudo-differential operators. For example, by using this method it can be proved that \(A \in \mathcal{B}_p (L^2(\mathbb{R}^n))\) if \(D^\alpha_x D^\beta_\xi a\) is in \(L^p(\mathbb{R}^n \times \mathbb{R}^n)\) for \(|\alpha|, |\beta| \leq \lfloor n/2 \rfloor + 1\) and \(1 \leq p < \infty\), where \(\mathcal{B}_p (L^2(\mathbb{R}^n))\) denote the Schatten ideal of compact operators whose singular values lie in \(l^p\). It is remarkable that this method can be used for \((\theta, \tau)\)-quantization, in particular, for both the Weyl quantization and Kohn-Nirenberg quantization.

2010 Mathematics Subject Classification. Primary 35S05, 43Axx, 46-XX, 47-XX; Secondary 42B15, 42B35.
THE \((\theta, \tau)\)-QUANTIZATION

In this section we shall briefly describe the \((\theta, \tau)\)-quantization. The definitions, the results and the notions introduced will serve both as models and as examples for the T-Weyl calculus developed in the following sections.

Let \(V\) be an \(n\) dimensional vector space over \(\mathbb{R}\) and \(V^*\) its dual. We denote by \(dx\) a Lebesgue measure in \(V\) and by \(dp\) is the dual one in \(V^*\) such that Fourier’s inversion formula holds with the usual constant. Replacing \(dx\) by \(cdx\) one must change \(dp\) to \(c^{-1}dp\) so \(dxdp\) is invariantly defined.

Let \(\theta, \tau \in \text{End}_\mathbb{R}(V)\). For \(a \in \mathcal{S}(V \times V^*)\) and \(v \in \mathcal{S}(V)\) we define

\[\text{Op}_{\theta, \tau}(a) v (x) = a_{\theta, \tau}(X, D) v (x)\]

\[= (2\pi)^{-n} \iint e^{i(x-y,p)}a(\theta x + \tau y, p) v (y) dydp.\]

If \(u, v \in \mathcal{S}(V)\), then

\[\langle \text{Op}_{\theta, \tau}(a) v, u \rangle_{\mathcal{S}(V), \mathcal{S}(V)} = (2\pi)^{-n} \iint e^{i(x-y,p)}a(\theta x + \tau y, p) u (x) v (y) dxdydp\]

\[= \left[\left(\left(1 \otimes \mathcal{F}_V^{-1}\right) a\right) \circ c_{\theta, \tau}\right] (u \otimes v)\]

\[c_{\theta, \tau}: V \times V \to V \times V, \quad c_{\theta, \tau}(x, y) = (\theta x + \tau y, x - y).\]

If \(\theta + \tau: V \to V\) is a linear isomorphism, it is possible to define \(\text{Op}_{\theta, \tau}(a)\) as an operator in \(\mathcal{B}(\mathcal{S}(V), \mathcal{S}'(V))\) for any \(a \in \mathcal{S}'(V \times V^*)\)

\[\langle \text{Op}_{\theta, \tau}(a) v, u \rangle_{\mathcal{S}'(V), \mathcal{S}(V)} = \left(\text{K}_{\text{Op}_{\theta, \tau}(a)}, u \otimes v\right)_{\mathcal{S}'(V \times V), \mathcal{S}(V \times V)},\]

\[\text{K}_{\text{Op}_{\theta, \tau}(a)} = \left(\left(1 \otimes \mathcal{F}_V^{-1}\right) a\right) \circ c_{\theta, \tau}.\]

Note that \(\det c_{\theta, \tau} = (-1)^n \det (\theta + \tau)\) and that \(\theta + \tau: V \to V\) is a linear isomorphism iff \(c_{\theta, \tau}: V \times V \to V \times V\) is a linear isomorphism.

**Definition 1.1.** (a) We denote by \(\Omega(V)\) the set

\[\Omega(V) = \{(\theta, \tau): \theta, \tau \in \text{End}_\mathbb{R}(V), \ (\theta + \tau) \text{ isomorphism}\},\]

which is symmetric i.e. \((\theta, \tau) \in \Omega(V) \iff (\tau, \theta) \in \Omega(V)\).

(b) For \((\theta, \tau) \in \Omega(V), \) the map

\[\text{Op}_{\theta, \tau}: \mathcal{S}'(V \times V^*) \to \mathcal{B}(\mathcal{S}(V), \mathcal{S}'(V)), a \to \text{Op}_{\theta, \tau}(a) \equiv a_{\theta, \tau}(X, D),\]

is called \((\theta, \tau)\)-quantization. The distribution \(a \in \mathcal{S}'(V \times V^*)\) is called \((\theta, \tau)\)-symbol of \(\text{Op}_{\theta, \tau}(a)\).

(c) For \(\tau \in \text{End}_\mathbb{R}(V), \) the map \(\text{Op}_{\tau} = \text{Op}_{\text{1} - \tau, \tau}\) is called the \(\tau\)-quantization. In particular, for \(\tau = \frac{1}{2}\) we have the Weyl quantization \(\text{Op}_{\tau} = \text{Op}_{\frac{1}{2}}\) and for \(\tau = 0\) we have the Kohn-Nirenberg quantization.

Since the equation in \(a \in \mathcal{S}'(V \times V^*), \) \(\left(\text{id} \otimes \mathcal{F}_V^{-1}\right) a \circ c_{\theta, \tau} = K,\) has a unique solution for each \(K \in \mathcal{S}'(V \times V), \) a consequence of the kernel theorem is the fact that the map

\[\text{Op}_{\theta, \tau}: \mathcal{S}'(V \times V^*) \to \mathcal{B}(\mathcal{S}(V), \mathcal{S}'(V)), a \to \text{Op}_{\theta, \tau}(a) \equiv a_{\theta, \tau}(X, D)\]

is linear, continuous and bijective. Hence for each \(A \in \mathcal{B}(\mathcal{S}, \mathcal{S}')\) there is a distribution \(a \in \mathcal{S}'(V \times V^*)\) such that \(A = \text{Op}_{\theta, \tau}(a)\). This distribution is called \((\theta, \tau)\)-symbol of \(A\).
Proposition 1.2. Let \( \omega \) be a 2-cocycle or Schur multiplier. The projective representation

\[
V \times V^* \ni (y, p) \rightarrow L_{(y, p)} \in (V \times V^*)^*,
\]

where \( \sigma : (V \times V^*) \times (V \times V^*) \rightarrow \mathbb{R} \) is the canonical symplectic form

\[
\sigma ((x, k); (y, p)) = \langle y, k \rangle_{V^*} - \langle x, p \rangle_{V^*}.
\]

For each \((\theta, \tau) \in \Omega(V)\) we consider the family \(\{W_{\theta, \tau}(z, \zeta)\}_{(z, \zeta) \in V \times V^*}\),

\[
W_{\theta, \tau}(y, p) = (e^{iL(y, p)})_{\theta, \tau}(X, D) \in B(S, S').
\]

We have

\[
W_{\theta, \tau}(y, p) = e^{i(\theta X, p)_{V^*}}e^{-i(\tau y, D)_{V^*}}e^{i(\tau X, p)_{V^*}}
\]

for every \((x, \xi), (y, \eta) \in V \times V^*\).

Let \(\omega_{\theta, \tau}\) be the function

\[
\omega_{\theta, \tau} : (V \times V^*) \times (V \times V^*) \rightarrow \mathbb{T},
\]

\[
\omega_{\theta, \tau}((x, k); (y, p)) = e^{i\sigma((x, k); (\tau y, \theta^* p))}.
\]

It follows that \(\omega_{\theta, \tau}\) satisfies the cocycle equation

\[
\omega_{\theta, \tau}((x, k); (y, p)) \omega_{\theta, \tau}((x, k) + (y, p); (z, q)) = \omega_{\theta, \tau}((x, k); (y, p) + (z, q)) \omega_{\theta, \tau}((y, p); (z, q)),
\]

\[
\omega_{\theta, \tau}((x, k); (0, 0)) = \omega_{\theta, \tau}((0, 0); (x, k)) = 1,
\]

hence \(\omega_{\theta, \tau}\) is a 2-cocycle or Schur multiplier.

Moreover, the Schur multiplier \(\omega_{\theta, \tau}\) is non-degenerate, that is

\[
\omega_{\theta, \tau}((x, k); (y, p)) \omega_{\theta, \tau}((y, p); (x, k)), \forall (y, p) \in V \times V^* \Rightarrow (x, k) = 0.
\]

Proposition 1.2. Let \((\theta, \tau) \in \Omega(V)\). Then

(a) \(\omega_{\theta, \tau}\) is a non-degenerate 2-cocycle or Schur multiplier.

(b) The couple \((L^2(V), W_{\theta, \tau})\) is a projective representation of \(V \times V^*\) with \(\omega_{\theta, \tau}\) the associated multiplier.

(c) The projective representation \((L^2(V), W_{\theta, \tau})\) is irreducible.

(d) For \(u, v \in S(V)\), the function

\[
w_{\theta, \tau, u, v} : V \times V^* \rightarrow \mathbb{R}, \quad w_{\theta, \tau, u, v}(y, p) = \langle W_{\theta, \tau}(y, p) v, u \rangle_{S(V), S(V)},
\]

is in \(S(V \times V^*)\).
For \( a \in \mathcal{S}(V \times V^*) \) we define the symplectic Fourier transform

\[
\mathcal{F}_\sigma(a)(x,k) = (2\pi)^{-n} \int \int e^{-i\langle (x,k):(y,p) \rangle} a(y,p) \, dy \, dp, \quad (x,k) \in V \times V^*
\]

and we use the same notation \( \mathcal{F}_\sigma \) for the extension to \( \mathcal{S}'(V \times V^*) \) of this Fourier transform. It follows that \( \mathcal{F}_\sigma \) is involutive (i.e. \( \mathcal{F}_\sigma^2 = 1 \)) and unitary on \( L^2(V \times V^*) \).

The family \( \{W_{\theta,\tau}(x,k)\}_{(x,k) \in V \times V^*} \) completely characterizes \((\theta,\tau)\)-Weyl calculus or \((\theta,\tau)\)-quantization. The general definition of \( \text{Op}_{\theta,\tau}(a) \) is deduced from this family and the symplectic Fourier decomposition of \( a \).

**Proposition 1.3.** Let \( (\theta,\tau) \in \Omega(V) \) and \( a \in \mathcal{S}(V \times V^*) \). Then

\[
\text{Op}_{\theta,\tau}(a) = (2\pi)^{-n} \int \int \mathcal{F}_\sigma(a)(y,p) W_{\theta,\tau}(y,p) \, dy \, dp.
\]

The integral is taken in the weak sense, i.e. for \( u,v \in \mathcal{S}(V) \)

\[
\left\langle \left( (2\pi)^{-n} \int \int \mathcal{F}_\sigma(a)(y,p) W_{\theta,\tau}(y,p) \, dy \, dp \right) v,u \right\rangle_{\mathcal{S}'(V),\mathcal{S}(V)} = \left\langle \mathcal{F}_\sigma(a), W_{\theta,\tau}(v,u) \right\rangle_{\mathcal{S}'(V \times V^*),\mathcal{S}(V \times V^*)}
\]

where \( w_{\theta,\tau,u,v} \in \mathcal{S}(V \times V^*) \) is given by

\[
w_{\theta,\tau,u,v}(y,p) = \langle W_{\theta,\tau}(y,p) v,u \rangle_{\mathcal{S}'(V),\mathcal{S}(V)}.
\]

**Proof.** It is enough to prove equality for \( a \in \mathcal{S}(V \times V^*) \). Let \( u,v \in \mathcal{S}(V) \). Then

\[
(2\pi)^{-n} \left\langle \mathcal{F}_\sigma(a), w_{\theta,\tau,u,v} \right\rangle = (2\pi)^{-n} \int \int \mathcal{F}_\sigma(a)(y,p) w_{\theta,\tau,u,v}(y,p) \, dy \, dp
\]

\[
= (2\pi)^{-n} \int \int \mathcal{F}_\sigma(a)(y,p) (W_{\theta,\tau}(y,p) u)(x) \, dx \, dy \, dp
\]

\[
= \int \left( (2\pi)^{-n} \int \mathcal{F}_\sigma(a)(y,p) (W_{\theta,\tau}(y,p) u)(x) \, dy \, dp \right) u(x) \, dx.
\]

Since

\[
(W_{\theta,\tau}(y,p) v)(x) = e^{-i\langle \tau y,p \rangle_{V,V^*}} e^{i\langle (\theta + \tau)x,p \rangle_{V,V^*}} v(x-y)
\]

\[
= e^{i\langle \theta x + \tau(x-y),p \rangle_{V,V^*}} v(x-y),
\]
we get that

\[
(2\pi)^{-n} \int \mathcal{F}_\sigma (a) (y,p) (\mathcal{W}_{\theta,\tau} (y,p) v) (x) \, dy \, dp
\]

\[
= (2\pi)^{-n} \int \mathcal{F}_\sigma (a) (y,p) e^{i(\theta x + \tau (x-y), p) v} v (x - y) \, dy \, dp
\]

\[
= (2\pi)^{-n} \int e^{i(\theta x + \tau (x-y), p) v} (F_V \otimes F_{V^*}^{-1}) a (p, y) \, dy \, dp
\]

\[
= \int v (x - y) (F_{V}^{-1} \otimes 1) \left( (F_V \otimes F_{V}^{-1}) a \right) (\theta x + \tau (x - y), y) \, dy
\]

\[
\doteq \int v (x - y) \left( (1 \otimes F_{V}^{-1}) a \right) (\theta x + \tau y, x - y) v (y) \, dy
\]

\[
= \int K_{Op_{\theta,\tau}} (a) (x, y) v (y) \, dy = Op_{\theta,\tau} (a) v (x) = a_{\theta,\tau} (X, D) v (x)
\]

Therefore

\[
(2\pi)^{-n} \left( \mathcal{F}_\sigma (a), v_{\theta,\tau, u, u} \right) = \int a_{\theta,\tau} \left( X, D \right) v (x) u (x) \, dx = \left( a_{\theta,\tau} \left( X, D \right) v, u \right)
\]

\[
\square
\]

The map \( t \to \mathcal{W}_{\theta,\tau} (ty, tp) \) is not a group representation of \( \mathbb{R} \). Instead, if we replace the family \( \{ \mathcal{W}_{\theta,\tau} (y,p) \}_{(y,p) \in V \times V^*} \) with the family \( \{ \tilde{\mathcal{W}}_{\theta,\tau} (y,p) \}_{(y,p) \in V \times V^*} \),

\[
\tilde{\mathcal{W}}_{\theta,\tau} (y,p) = e^{\frac{i}{2} (\tau y, \theta p)} \mathcal{W}_{\theta,\tau} (y,p)
\]

\[
= e^{\frac{i}{2} ((\theta - \tau) y, p)} \mathcal{W}_{\theta,\tau} (y,p)
\]

\[
= e^{\frac{i}{2} (\tau (y+\theta y), p)} e^{i (\theta + \tau) X, p) v, v^*} (y,p) \in V \times V^*,
\]

then the couple \((L^2 (V^*), \tilde{\mathcal{W}}_{\theta,\tau})\) is a projective representation of \( V \times V^* \) with \( \tilde{\mathcal{W}}_{\theta,\tau} \) the associated Schur multiplier. Here

\[
\tilde{\omega}_{\theta,\tau} ((x,k);(y,l)) = \frac{e^{\frac{i}{2} \sigma ((x,k);(\tau y, 0^* k))} e^{\frac{i}{2} \sigma ((y,l);(\tau y, 0^* l))} e^{i (\theta + \tau) x, p)}
\]

\[
= \frac{e^{\frac{i}{2} \sigma ((x,k);(\tau y, 0^* k))} e^{i (\theta + \tau) x, p)}
\]

\[
\omega_{\theta,\tau} ((x,k);(y,l)) = \frac{e^{\frac{i}{2} \sigma ((\theta + \tau) x, k);((\theta + \tau) y, l))}}{e^{\frac{i}{2} \sigma ((\theta + \tau) x, k);((\theta + \tau) y, l))}}
\]

\[
\tilde{\omega}_{\theta,\tau} ((x,k);(y,l)) = \frac{e^{\frac{i}{2} \sigma ((\theta + \tau) y, k);((\theta + \tau) x, l))}}{e^{\frac{i}{2} \sigma ((\theta + \tau) y, k);((\theta + \tau) x, l))}}
\]

The functions \( \omega_{\theta,\tau} \) and \( \tilde{\omega}_{\theta,\tau} \) are cohomologous and \( \tilde{\omega}_{\theta,\tau} \) is normalized, i.e.

\[
\tilde{\omega}_{\theta,\tau} ((x,k);(y,l)) = 1, \quad (x,k) \in V \times V^*.
\]

The map \( t \to \tilde{\mathcal{W}}_{\theta,\tau} (ty, tp) \) is a group representation of \( \mathbb{R} \) and for each \((y,p) \in V \times V^* \) there is a unique self-adjoint operator \( \phi_{\theta,\tau} (y,p) \) \((\theta, \tau)\)-field operator associated to \((y,p)\), such that

\[
\tilde{\mathcal{W}}_{\theta,\tau} (ty, tp) = e^{t \phi_{\theta,\tau} (y,p)}
\]
for all real \( t \). The map \( (y, p) \rightarrow \phi_{\theta, \tau}(y, p) \) is \( \mathbb{R} \)-linear. The \((\theta, \tau)\)-annihilation and \((\theta, \tau)\)-creation operators associated to \((y, p)\) are defined by

\[
a_{\theta, \tau}(y, p) = \frac{1}{2} \left( \phi_{\theta, \tau}(y, p) + i \phi_{\theta, \tau}(i(y, p)) \right),
\]

\[
a^{*}_{\theta, \tau}(y, p) = \frac{1}{2} \left( \phi_{\theta, \tau}(y, p) - i \phi_{\theta, \tau}(i(y, p)) \right),
\]
e.t.c.

We notice that the formula

\[
\omega_{\theta, \tau}((x, k) ; (y, p)) = e^{i \sigma((x, k); (\tau y, \theta^* p))}.
\]

highlights the symplectic structure on \( V \times V^* \) and a linear map \( \tau \times \theta^* : V \times V^* \rightarrow V \times V^* \) which satisfies a certain condition contained in the definition of the set \( \Omega(V) \). This condition is equivalent to the condition that the Schur factor \( \omega_{\theta, \tau} \) is non-degenerate. Note also that the representation

\[
\text{Op}_{\theta, \tau}(a) = (2\pi)^{-n} \int \int F_{\sigma}(a)(y, p) W_{\theta, \tau}(y, p) \, dy dp,
\]

indicates the contribution of the symplectic structure of \( V \times V^* \) in the definition of the \((\theta, \tau)\)-quantization, the operator \( \text{Op}_{\theta, \tau} \).

The \((\theta, \tau)\)-calculus built up in this section can be further generalized, and we shall do this in this paper. The paper is organized as follows. In Section 2 we summarize the most important notations and results from linear symplectic algebra. In Section 3 we define the \( \omega_{\sigma, T} \)-representation and the associated Weyl system and present some of their properties. In Section 4 we define the \( T \)-Weyl calculus and we prove one of the important results of the paper, Theorem 4.4, which is an important technical result that establishes the connection between the \( T \)-Weyl calculus and the standard Weyl calculus. In Section 5 we study modulation spaces and Schatten-class properties of operators in the \( T \)-Weyl calculus. The results in this section on Schatten-class properties of operators in the \( T \)-Weyl calculus, together with the results in Section 6 are used to prove an extension of Cordes’ lemma in Section 7. Sections 7-10 are devoted to the Cordes-Kato method for \( T \)-Weyl calculus.

As can be seen, we started from a natural definition for a pseudo-differential calculus, and we obtained a projective representation. In this paper, we shall follow the path in the opposite direction, namely using symplectic 2-form \( \sigma \) we shall associate to any linear map \( T \) on \( W \) a 2-cocycle or Schur multiplier \( \omega_{\sigma, T} \). If the Schur multiplier \( \omega_{\sigma, T} \) is non-degenerate, which may be expressed by a non-degeneracy condition of \( T \), then any two irreducible \( \omega_{\sigma, T} \)-representations are unitary equivalent. For an irreducible \( \omega_{\sigma, T} \)-representation \((\mathcal{H}, W_{\sigma, T}, \omega_{\sigma, T})\) of \( W \) there is a well defined linear, continuous and bijective map, the \( T \)-Weyl calculus,

\[
\text{Op}_{\sigma, T} : \mathcal{S}^*(W) \rightarrow \mathcal{B}(\mathcal{S}, \mathcal{S}^*), \quad a \mapsto \text{Op}_{\sigma, T}(a),
\]

where \( \mathcal{S} \) is the dense linear subspace of \( \mathcal{H} \) consisting of the \( \mathcal{C}^\infty \) vectors of the representation \( W_{\sigma, T} \), \( \mathcal{S}^* \) is the space of all continuous, anti-linear mappings \( \mathcal{S} \rightarrow \mathbb{C} \) and \( \mathcal{S}^*(W) \) is the space of all continuous, anti-linear mappings \( \mathcal{S}(W) \rightarrow \mathbb{C} \).

2. The framework

Our notations are rather standard but we recall here some of them to avoid any ambiguity. Let \((W, \sigma)\) be a symplectic vector space, that is a real finite dimensional vector space \( W \) equipped with a real antisymmetric non-degenerate bilinear form
\[ \sigma. \text{ We denote by } \sigma^b \text{ the isomorphism associated with the non-degenerate bilinear form } \sigma, \]
\[ \sigma^b : W \to W^*, \quad \sigma^b (\xi) = \sigma (\xi, \cdot), \quad \xi \in W. \]

**Symplectic adjoint**

Suppose that \((W_1, \sigma_1)\) and \((W_2, \sigma_2)\) are symplectic vector spaces and \(T : W_1 \to W_2\) is a linear map. Define the symplectic adjoint \(T^\sigma : W_2 \to W_1\) by
\[
T^\sigma : W_2 \xrightarrow{\sigma_2^\dagger} W_2^* \xrightarrow{T^*} W_1^* \xrightarrow{(\sigma_1^\dagger)^{-1}} W_1,
\]
where \(T^*\) is the usual adjoint,
\[
T^*: W_2^* \to W_1^*, \quad T^* \lambda_2 = \lambda_2 \circ T.
\]

**Lemma 2.1.** The linear map \(T^\sigma : W_2 \to W_1\) satisfies
\[
(2.1) \quad \sigma_1 (T^\sigma \xi_1, \xi_2) = \sigma_2 (\xi_1, T \xi_2), \quad \xi_1 \in W_1, \xi_2 \in W_2.
\]

**Proof.** Let \(\xi_1 \in W_1, \xi_2 \in W_2\). Then
\[
\sigma_1 (T^\sigma \xi_1, \xi_2) = \sigma_1^\dagger (T^\sigma \xi_1) (\xi_2) = \sigma_1^\dagger T^\sigma (\xi_1) \xi_2 = T^* \sigma_2^\dagger (\xi_1) \xi_2 = \sigma_2 (\xi_1, T \xi_2).
\]

**Remark 2.2.** The property (2.1) characterizes the symplectic adjoint.

**Definition 2.3.** (a) A symplectic isomorphism or a symplectomorphism \(\phi\) between symplectic vector spaces \((W_1, \sigma_1)\) and \((W_2, \sigma_2)\) is a linear isomorphism \(\phi : W_1 \to W_2\) such that \(\phi^* \sigma_2 = \sigma_1\). By definition, \(\phi^* \sigma_2 (\xi, \eta) = \sigma_2 (\phi \xi, \phi \eta), \xi, \eta \in W_1\). If a symplectomorphism exists, \((W_1, \sigma_1)\) and \((W_2, \sigma_2)\) are said to be symplectomorphic.

(b) For a symplectic vector space \((W, \sigma)\) we denote by
\[
\text{Sp} (W, \sigma) := \{ \phi \in \text{GL} (W) : \phi^* \sigma = \sigma \}
\]
the group of linear symplectomorphisms of \((W, \sigma)\).

**Remark 2.4.** We have \(\phi : W_1 \to W_2\) is a symplectic isomorphism if and only if \(\phi\) is a linear isomorphism and \(\phi^* \circ \phi = \text{Id}_{W_1}\).

**Lemma 2.5.** Let \((W, \sigma)\) be a symplectic vector space and \(S : W \to W\) a linear isomorphism. Then \(S^\sigma = S\) if and only if there is a linear isomorphism \(\phi_S : W \to W\) such that \(S = \phi_S^2 \circ \phi_S\).

**Proof.** The map
\[
\sigma_S : W \times W \to \mathbb{R}, \quad \sigma_S (\xi, \eta) = \sigma (\xi, S \eta), \quad \xi, \eta \in W,
\]
is bilinear. Let us note that \(\sigma_S\) is antisymmetric if and only if \(S^\sigma = S\) and \(\sigma_S\) is non-degenerate if and only if \(S\) is an isomorphism.
If \( S : W \to W \) is a linear isomorphism and \( S^\sigma = S \), then the 2-form \( \sigma_S \) is symplectic. Let \( \phi_S : W \to W \) a linear isomorphism that takes a symplectic basis with respect to \( \sigma_S \) to a symplectic basis with respect to \( \sigma \). Then
\[
\sigma (\xi, S\eta) = \sigma_S (\xi, \eta) = \sigma (\phi_S \xi, \phi_S \eta) = \sigma (\xi, \phi_S^* \circ \phi_S \eta), \quad \xi, \eta \in W,
\]

\[\Rightarrow \sigma (\xi, S\eta) = \sigma (\xi, \phi_S^* \circ \phi_S \eta), \quad \xi, \eta \in W,\]
hence \( S = \phi_S^* \circ \phi_S \). The converse is obvious. \( \square \)

**Corollary 2.6.** Let \((W, \sigma)\) be a symplectic vector space and \( S : W \to W \) a linear isomorphism. If \( S^\sigma = S \), then \( \det S > 0 \).

**Definition 2.7.** Let \((W, \sigma)\) be a and \( X \subset W \) be a linear subspace. The symplectic complement of \( X \) is the subspace
\[X^\sigma = \{ v \in W : \sigma(v, w) = 0 \text{ for all } w \in X \}\]
A subspace \( X \subset W \) is called isotropic if \( X \subset X^\sigma \) and involutive if \( X^\sigma \subset X \). If both are valid, i.e. \( X = X^\sigma \), then \( X \) is lagrangian. An isotropic subspace \( X \subset W \) is lagrangian if and only if \( 2 \dim X = \dim W \).

**Remark 2.8.** (a) Let \( X \) be an \( n \) dimensional vector space over \( \mathbb{R} \) and \( X^* \) its dual. Denote \( x, y, \ldots \) the elements of \( X \) and \( k, p, \ldots \) those of \( X^* \). Let \( \langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R} \) be the duality form, which is a non-degenerate bilinear form. The symplectic space is defined by \( W = T^*(X) = X \times X^* \) the symplectic form being \( \sigma ((x, k), (y, p)) = \langle y, k \rangle - \langle x, p \rangle \). Observe that \( X \) and \( X^* \) are lagrangian subspaces of \( W \).

(b) Let us mention that there is a kind of converse to this construction. Let \((X, X^*)\) be a couple of lagrangian subspaces of \( W \) such that \( X \cap X^* = 0 \) or, equivalently, \( X + X^* = W \). If for \( x \in X \) and \( p \in X^* \) we define \( \langle x, p \rangle = \sigma (p, x) \), then we get a non-degenerate bilinear form on \( X \times X^* \) which allows us to identify \( X^* \) with the dual of \( X \). A couple \((X, X^*)\) of subspaces of \( W \) with the preceding properties is called a holonomic decomposition of \( W \). Observe that if \( \xi = x + k \) and \( \eta = y + p \) are their decomposition \( s \) in \( W \), then
\[\sigma (\xi, \eta) = \langle y, k \rangle - \langle x, p \rangle .\]

**The symplectic Fourier transform**

A symplectic vector space \((W, \sigma)\) is always orientable since the 2-form \( \sigma \) is non-degenerate if and only if its \( n \)-fold exterior power is non-zero, i.e.
\[\sigma^n = \underbrace{\sigma \wedge \ldots \wedge \sigma}_{n} \neq 0,\]
where \( \dim W = 2n \). We will call the exterior power \( \sigma^n \) the symplectic volume form. When \((W, \sigma)\) is the standard symplectic space \((\mathbb{R}^{2n}, \sigma_n)\),
\[\sigma_n (z, z') = \sum (p_j x_j' - p_j' x_j),\]
\[z = (x_1, \ldots, x_n; p_1, \ldots, p_n) \text{ and } z' = (x_1', \ldots, x_n'; p_1', \ldots, p_n'),\]
then the usual volume form on \( \mathbb{R}^{2n}_z \),
\[\text{Vol}_{2n} = dp_1 \wedge \ldots \wedge dp_n \wedge dx_1 \wedge \ldots \wedge dx_n,\]
is related to the symplectic volume form by
\[\text{Vol}_{2n} = (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} \sigma^n = (-1)^{\left\lceil \frac{n}{2} \right\rceil} \frac{1}{n!} \sigma^n .\]
The form \( \frac{1}{n!} \sigma^n \) is called the Liouville volume of \((W, \sigma)\).
We define the Fourier measure $d^\sigma x$ as the unique Haar measure on $(W, \sigma)$ such that the symplectic Fourier transform or $\sigma$-Fourier transform,

$$(F_{\sigma}a) (\xi) = \int_{W} e^{-i\sigma(\xi,\eta)} a(\eta) d^\sigma \eta, \quad a \in S(W),$$

is involutive (i.e. $F_{\sigma}^2 = 1$) and unitary on $L^2(W)$. We use the same notation $F_{\sigma}$ for the extension to $S'(W)$ of this $\sigma$-Fourier transform. Let us note that

$$d^\sigma x = (2\pi)^{-\frac{\dim W}{2}} \left[ \left( \frac{\dim W}{2} \right)! \right]^{-1} |\sigma \wedge \ldots \wedge \sigma|,$$

where $|\sigma \wedge \ldots \wedge \sigma|$ is the 1-density given by the symplectic volume form $\sigma \frac{d\dim W}{2}$.

**Lemma 2.9.** Let $(W_1, \sigma_1)$ and $(W_2, \sigma_2)$ be two symplectic spaces of same dimension $2n$. If $\phi$ is a symplectic isomorphism $(W_1, \sigma_1) \rightarrow (W_2, \sigma_2)$, then

$$\phi^* |\sigma_2 \wedge \ldots \wedge \sigma_2| = |\sigma_1 \wedge \ldots \wedge \sigma_1|, \quad \phi^* (d^\sigma x_2) = d^\sigma x_1$$

and

$$F_{\sigma_1} \circ \phi^* = \phi^* \circ F_{\sigma_2} \quad \Leftrightarrow \quad F_{\sigma_2} = (\phi^*)^* \circ F_{\sigma_1} \circ \phi^*.$$

**Proof.** The first two equalities are direct consequences of the fact that $\phi$ is a symplectic isomorphism $(\phi^* \sigma_2 = \sigma_1)$. As for the third equality, it is enough to prove it equality for $b$ in $S(W)$. Let $b \in S(W)$. Then we have

$$F_{\sigma_1} (b \circ \phi) (\xi_1) = \int_{W_1} e^{-i\sigma_1(\xi_1, \eta_1)} b(\phi(\eta_1)) d^\sigma_1 \eta_1 = \int_{W_1} e^{-i\sigma_2(\phi(\xi_1), \phi(\eta_1))} b(\phi(\eta_1)) d^\sigma_1 \eta_1 = \int_{W_1} \phi^* \left( e^{-i\sigma_2(\phi(\xi_1), \cdot)} b(\cdot) \right) d\sigma_2.$$  

The equivalence is a consequence of identity $\phi^* \circ \phi = \text{id}_{W_1}$.

Let $S : W \rightarrow W$ is a linear isomorphism such that $S^\sigma = S$ and $\phi_S$ a symplectic isomorphism, $\phi_S : (W, \sigma_S) \rightarrow (W, \sigma)$, such that $S = \phi_S^* \circ \phi_S$. Then,

$$d^{\sigma_S} \eta = \phi_S^*(d^\sigma \eta) = |\det \phi_S| d^\sigma \eta = (\det S)^{\frac{1}{2}} d^\sigma \eta,$$

and for $b \in S(W)$ we have

$$F_{\sigma} (b) (\xi) = \int_{W} e^{-i\sigma(\xi, \eta)} b(\eta) d^\sigma \eta = \int_{W} e^{-i\sigma_S(\xi, S^{-1} \eta)} b(\eta) d^\sigma \eta = \det S \int_{W} e^{-i\sigma_S(\xi, \zeta)} b(S \zeta) d^\sigma \zeta = (\det S)^{\frac{1}{2}} \int_{W} e^{-i\sigma_S(\xi, \zeta)} b(S \zeta) d^\sigma \zeta = (\det S)^{\frac{1}{2}} F_{\sigma_S} (b \circ S) (\xi), \quad \xi \in W.
Lemma 2.10. Let \((W, \sigma)\) be a symplectic vector space and \(S : W \rightarrow W\) a linear isomorphism such that \(S^\sigma = S\). If \(\sigma_S\) is the symplectic form

\[\sigma_S : W \times W \rightarrow \mathbb{R}, \quad \sigma_S (\xi, \eta) = \sigma (\xi, S\eta), \quad \xi, \eta \in W,\]

then

\[\mathcal{F}_\sigma = (\det S)^{\frac{1}{2}} \mathcal{F}_{\sigma_S} \circ S^* \quad \text{on} \quad S' (W).\]

Since \(\mathrm{d}\sigma\eta\) is a multiple of the Lebesgue measure, the change of variables formula implies that if \(A : W \rightarrow W\) is a linear isomorphism, then

\[A^* \circ \mathcal{F}_\sigma = |\det A|^{-1} \mathcal{F}_\sigma \circ \left( (A^\sigma)^{-1} \right)^*.\]

Let \(b \in \mathcal{S} (W)\). Then

\[A^* \circ \mathcal{F}_\sigma (b) (\xi) = \mathcal{F}_\sigma (b) (A\xi) = \int_W e^{-i \sigma (A\xi, \eta)} b (\eta) \, \mathrm{d}\sigma\eta\]

\[= \int_W e^{-i \sigma (\xi, A^\sigma \eta)} b \circ \left( A^\sigma \eta \right) (A^\sigma) \, \mathrm{d}\sigma\eta\]

\[= |\det A|^{-1} \int_W e^{-i \sigma (\xi, \eta)} b \circ \left( A^\sigma \eta \right) \, \mathrm{d}\sigma\xi\]

\[= |\det A|^{-1} \mathcal{F}_\sigma \left( b \circ \left( A^\sigma \right)^{-1} \right) (\xi).\]

In particular, if \(S = S^\sigma : W \rightarrow W\) is a linear isomorphism, then

\[S^* \circ \mathcal{F}_\sigma = |\det S|^{-1} \mathcal{F}_\sigma \circ (S^{-1})^*\]

For \(\lambda \in C_\text{pol}^\infty (W)\), we define the operator

\[\lambda (D_\sigma) = \mathcal{F}_\sigma \circ M_{\lambda (\cdot)} \circ \mathcal{F}_\sigma,\]

where \(M_{\lambda (\cdot)}\) denotes the multiplication operator by the function \(\lambda (\cdot)\). If \(S = S^\sigma : W \rightarrow W\) is a linear isomorphism, then

\[S^* \circ \lambda (D_\sigma) = (\lambda \circ S^{-1}) (D_\sigma) \circ S^*.\]

Indeed, for \(b \in \mathcal{S} (W)\) we have

\[S^* \circ \lambda (D_\sigma) = S^* \circ \mathcal{F}_\sigma \circ M_{\lambda (\cdot)} \circ \mathcal{F}_\sigma = (\det S)^{-1} \mathcal{F}_\sigma \circ (S^{-1})^* \circ M_{\lambda (\cdot)} \circ \mathcal{F}_\sigma\]

\[= (\det S)^{-1} \mathcal{F}_\sigma \circ M_{\lambda \circ S^{-1} (\cdot)} \circ (S^{-1})^* \circ \mathcal{F}_\sigma\]

\[= (\det S)^{-1} \mathcal{F}_\sigma \circ M_{\lambda \circ S^{-1} (\cdot)} \circ \mathcal{F}_\sigma \circ S^*\]

\[= (\lambda \circ S^{-1}) (D_\sigma) \circ S^*.\]

Lemma 2.11. Let \((W, \sigma)\) be a symplectic vector space

(a) If \(A : W \rightarrow W\) is a linear isomorphism, then

\[A^* \circ \mathcal{F}_\sigma = |\det A|^{-1} \mathcal{F}_\sigma \circ \left( (A^\sigma)^{-1} \right)^*\]

(b) If \(S = S^\sigma : W \rightarrow W\) is a linear isomorphism and \(\lambda \in C_\text{pol}^\infty (W)\), then

\[S^* \circ \mathcal{F}_\sigma = (\det S)^{-1} \mathcal{F}_\sigma \circ (S^{-1})^*,\]

and

\[S^* \circ \lambda (D_\sigma) = (\lambda \circ S^{-1}) (D_\sigma) \circ S^*,\]

where

\[\lambda (D_\sigma) = \mathcal{F}_\sigma \circ M_{\lambda (\cdot)} \circ \mathcal{F}_\sigma : S' (W) \rightarrow S' (W).\]
(c) If \( S = S^\sigma : W \to W \) is a linear isomorphism and \( \lambda \in C^\infty_{pol}(W) \), then
\[
\lambda(D_\sigma) = \lambda \circ S(D_{\sigma S})
\]

Proof. (c) We use (b) and Lemma \[2.10\] twice:
\[
\begin{align*}
\lambda(D_\sigma) &= \mathcal{F}_\sigma \circ \mathcal{M}_\lambda(\cdot) \circ \mathcal{F}_\sigma = (\det S)^{\frac{1}{2}} \mathcal{F}_{\sigma S} \circ \mathcal{M}_{\lambda S(\cdot)} \circ \mathcal{F}_\sigma \\
&= (\det S)^{\frac{1}{2}} \mathcal{F}_{\sigma S} \circ \mathcal{M}_{\lambda S(\cdot)} \circ \mathcal{F}_\sigma \\
&= (\det S)^{\frac{1}{2}} \mathcal{F}_{\sigma S} \circ \mathcal{M}_{\lambda S(\cdot)} \circ \mathcal{F}_\sigma \circ (S^{-1})^* \\
&= (\det S)^{-\frac{1}{2}} \mathcal{F}_{\sigma S} \circ \mathcal{M}_{\lambda S(\cdot)} \circ \mathcal{F}_{\sigma S} \circ S^* \circ (S^{-1})^* \\
&= \mathcal{F}_{\sigma S} \circ \mathcal{M}_{\lambda S(\cdot)} \circ \mathcal{F}_{\sigma S} = \lambda \circ S(D_{\sigma S}).
\end{align*}
\]

Remark 2.12. (a) If \( a \in S^*(W) \) and \( \xi \in W \), then \( \tau_\xi a \) denote the translate by \( \xi \) of the distribution \( a \), i.e. \( (\tau_\xi a)(\cdot) = a(\cdot - \xi) \). The family \( \{\tau_\xi \}_{\xi \in W} \) is the unitary representation in \( L^2(W) \) of the additive group \( W \) by translations. The family \( \{\tau_\xi \}_{\xi \in W} \) also defines a representation of the additive group \( W \) by topological automorphisms of \( S^*(W) \) which leave \( S(W) \) invariant. Since from the definition of the symplectic Fourier transform
\[
\tau_\xi = e^{-i\sigma(D_\sigma \xi)}, \quad \xi \in W,
\]
it follows that for \( \lambda \in C^\infty_{pol}(W) \) and \( \xi \in W \)
\[
\tau_\xi \circ \lambda(D_\sigma) = \lambda(D_\sigma) \circ \tau_\xi.
\]

(b) If \( S = S^\sigma : W \to W \) is a linear isomorphism and \( \xi \in W \), then
\[
S^* \circ \tau_\xi = \tau_{S^{-1} \xi} \circ S^*.
\]
Indeed, using equality \( \tau_\xi = e^{-i\sigma(D_\sigma \xi)} \) one sees that
\[
\begin{align*}
S^* \circ \tau_\xi &= S^* \circ e^{-i\sigma(D_\sigma \xi)} = \left( e^{-i\sigma(\cdot - \xi)} \circ S^{-1} \right) \circ (D_\sigma) \circ S^* \\
&= \left( e^{-i\sigma(S^{-1} \cdot - \xi)} \circ (D_\sigma) \circ S^* = \left( e^{-i\sigma(\cdot - S^{-1} \xi)} \circ (D_\sigma) \circ S^* \\
&= \tau_{S^{-1} \xi} \circ S^*.
\end{align*}
\]

Corollary 2.13. If \( S = S^\sigma : W \to W \) is a linear isomorphism, \( \lambda \in C^\infty_{pol}(W) \) and \( \xi \in W \), then
\[
\tau_{S^{-1} \xi} \circ S^* \circ \lambda(D_\sigma) = S^* \circ \lambda(D_\sigma) \circ \tau_\xi.
\]

In many situations we need to consider additional structures such as the inner product or complex structures. We shall ask that these structures to be compatible with symplectic structure.

Recall that a complex structure on a vector space \( V \) is an automorphism \( J : V \to V \) such that \( J^2 = -\text{Id} \). We denote the space of linear complex structures on \( V \) by \( \mathcal{J}(V) \).

A complex structure \( J \) on a symplectic vector space \((W, \sigma)\) is called \( \sigma \)-compatible if
\[
(J^* \sigma)(v, w) = \sigma(Jv, Jw) = \sigma(v, w),
\]
for all \( v, w \in W \) and
\[
\sigma(v, Jv) > 0
\]
for all nonzero $v \in W$. This is equivalent to
\[ g : W \times W \to \mathbb{R}, \quad g(v, w) = \sigma(v, Jw), \quad v, w \in W, \]
is a positive definite inner product. We denote by $\mathcal{J}(W, \sigma)$ the space of $\sigma$-compatible complex structures on $(W, \sigma)$.

An inner product $g$ on a symplectic vector space $(W, \sigma)$ is called $\sigma$-compatible if there is a complex structure $J$ on $W$ such that
\[ g(u, v) = \sigma(v, Jw) \]
for all $v, w \in W$. We denote by $\mathcal{G}(W)$ the space of inner products on $W$, and by $\mathcal{G}(W, \sigma)$ the space of $\sigma$-compatible inner products on $(W, \sigma)$.

**Remark 2.14.** (a) The compatibility condition $g(u, v) = \sigma(v, Jw)$ defines a smooth diffeomorphism
\[ \mathcal{J}(W, \sigma) \to \mathcal{G}(W, \sigma). \]
In fact, the same formula $g(u, v) = \sigma(v, Jw)$ defines a linear isomorphism $J \to g$ from the ambient vector space of linear maps $V \to V$ to the ambient vector space of bilinear forms $V \times V \to \mathbb{R}$. The bijection $\mathcal{J}(W, \sigma) \to \mathcal{G}(W, \sigma)$ is the restriction of this linear isomorphism, so it is a diffeomorphism.

(b) There is a canonical retraction $\mathcal{G}(W) \to \mathcal{G}(W, \sigma)$ (see for instance Proposition 2.5.6 in [20]), so we can associate to any inner product in a canonical manner a $\sigma$-compatible one.

3. $\omega_{\sigma,T}$-REPRESENTATION AND THE ASSOCIATED WELLY SYSTEM

**Lemma 3.1.** Let $T : W \to W$ be a linear map, and let $\omega_{\sigma,T}$ be the function
\[ \omega_{\sigma,T} : W \times W \to \mathbb{T}, \]
\[ \omega_{\sigma,T}(\xi, \eta) = e^{i\sigma(\xi, T\eta)}, \quad \xi, \eta \in W. \]

Then $\omega_{\sigma,T}$ is a 2-cocycle or Schur multiplier. Moreover, the Schur multiplier $\omega_{\sigma,T}$ is non-degenerate, that is
\[ \omega_{\sigma,T}(\xi, \eta) = \omega_{\sigma,T}(\eta, \xi), \quad \forall \eta \in W \]
\[ \Rightarrow \xi = 0, \]
if and only if $T + T^{\sigma}$ is an isomorphism.

**Proof.** We have to show that $\omega_{\sigma,T}$ satisfies the cocycle equation
\[ \omega_{\sigma,T}(\xi, \eta) \omega_{\sigma,T}(\xi + \eta, \zeta) = \omega_{\sigma,T}(\xi, \eta + \xi)\omega_{\sigma,T}(\eta, \zeta), \quad \xi, \eta, \zeta \in W. \]

By definition this equality is equivalent to
\[ \sigma(\xi, T\eta) + \sigma(\xi + \eta, T\zeta) = \sigma(\xi, T(\eta + \zeta)) + \sigma(\eta, T\zeta) \]
\[ = \sigma(\xi, T\eta) + \sigma(\xi + \eta, T\zeta), \quad \xi, \eta, \zeta \in W. \]

Obviously we have $\omega_{\sigma,T}(\xi, 0) = \omega_{\sigma,T}(0, \xi) = 1, \xi \in W$. Hence $\omega_{\sigma,T}$ is a 2-cocycle or Schur multiplier.

Let $\xi \in W$. Then
\[ \omega_{\sigma,T}(\xi, \eta) = \omega_{\sigma,T}(\eta, \xi), \quad \forall \eta \in W \]
\[ \Leftrightarrow \sigma(\xi, T\eta) = \sigma(\eta, T\xi), \quad \forall \eta \in W \]
\[ \Leftrightarrow \sigma((T + T^{\sigma})\xi, \eta) = 0 \quad \forall \eta \in W \]
\[ (\forall \eta \in W) \ (\omega_{\sigma,T}(\xi,\eta) = \omega_{\sigma,T}(\eta,\zeta)) \Leftrightarrow (\xi \in \ker(T + T^\sigma)). \]

So we deduce that
\[ \omega_{\sigma,T}(\xi,\eta) \equiv (\xi \in \ker(T + T^\sigma)), \]
and this clearly implies that \( \omega_{\sigma,T} \) is non-degenerate if and only if \( T + T^\sigma \) is an isomorphism.

\[ \textbf{Remark 3.2.} \]
(a) We know that for any continuous multiplier \( \omega \) on \( W \), there is a projective representation \( \{ W(\xi) \}_{\xi \in W} \equiv (\mathcal{H}, W, \omega) \) whose multiplier is \( \omega \), that is a strongly continuous map
\[ W : W \rightarrow U(\mathcal{H}) \]
which satisfies
\[ W(\xi)W(\eta) = \omega(\xi,\eta)W(\xi + \eta), \quad \xi, \eta \in W. \]

\( W \) is called a \( \omega \)-representation (or, less precisely, a multiplier or ray, or cocycle representation).

(b) For instance, \((L^2(W), \mathcal{R}_\omega, \omega)\) is a projective representation of \( W \) with \( \omega \) the associated multiplier, where
\[ \mathcal{R}_\omega : W \rightarrow U(L^2(W)), \quad \mathcal{R}_\omega(\xi)f = \omega(\cdot,\xi)f(\cdot + \xi), \quad \xi \in W. \]

This representation is called the regular \( \omega \)-representation of \( W \).

Let \( \{ W_{\sigma,T}(\xi) \}_{\xi \in W} \equiv (\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T}) \) be a \( \omega_{\sigma,T} \)-representation of \( W \). For fixed \( \xi \) in \( W \), the map
\[ \mathbb{R} \ni t \rightarrow W_{\sigma,T}(t\xi) \in \mathcal{B}(\mathcal{H}) \]
satisfies
\[ W_{\sigma,T}(t\xi)W_{\sigma,T}(s\xi) = W_{\sigma,T}((t+s)\xi), \quad s,t \in \mathbb{R}, \]
\[ W_{\sigma,T}(t\xi)W_{\sigma,T}(s\xi) = e^{it\sigma(\xi,T\xi)}W_{\sigma,T}((t+s)\xi), \quad s,t \in \mathbb{R}. \]

Here \( t \rightarrow W_{\sigma,T}(t\xi) \) it is not in general a group representation of \( \mathbb{R} \). Instead, by using equality
\[ ts = \frac{1}{2} \left( (t+s)^2 - t^2 - s^2 \right), \]
we find that the map \( t \rightarrow \tilde{W}_{\sigma,T}(t\xi) \) is a group representation of \( \mathbb{R} \), where
\[ \{ \tilde{W}_{\sigma,T}(\xi) \}_{\xi \in W} \equiv (\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T}) \]
is the \( \tilde{\omega}_{\sigma,T} \)-representation of \( W \) given by
\[ \tilde{W}_{\sigma,T}(\xi) = e^{\frac{1}{2}\sigma(\xi,T\xi)}W_{\sigma,T}(\xi), \quad \xi \in W, \]
and
\[ \tilde{\omega}_{\sigma,T}(\xi,\eta) = \frac{e^{\frac{1}{2}\sigma(\xi,T\xi)}e^{\frac{1}{2}\sigma(\eta,T\eta)}}{e^{\frac{1}{2}\sigma((\xi+\eta,T(\xi+\eta)))}}\omega_{\sigma,T}(\xi,\eta) \]
\[ = e^{-\frac{1}{2}\sigma(\xi,T\eta) - \frac{1}{2}\sigma(\eta,T\xi)}e^{\sigma(\xi,T\eta)} \]
\[ = e^{\frac{1}{2}(\sigma(\xi,T\eta) - \sigma(\eta,T\xi))} \]
\[ = e^{\frac{1}{2}\sigma((\xi-(T+T^\sigma))\eta)} = e^{\sigma(\xi,\frac{1}{2}(T+T^\sigma))} \]
\[ = \omega_{\sigma,\frac{1}{2}(T+T^\sigma)}(\xi,\eta), \quad \xi, \eta \in W. \]
We recall that the set of all possible multipliers on $W$ can be given an abelian group structure by defining the product of two multipliers as their pointwise product. The resulting group we denote by $Z^2(W; T)$. The set of all multipliers satisfying

$$\alpha(\xi, \eta) = \frac{\mu(\xi + \eta)}{\mu(\xi) \mu(\eta)}, \quad \xi, \eta \in G,$$

for an arbitrary function $\mu : G \to \mathbb{T}$ such that $\mu(0) = 1$, forms an invariant subgroup $B^2(W; T)$ of $Z^2(W; T)$. Thus we may form the quotient group

$$H^2(W; T) = \frac{Z^2(W; T)}{B^2(W; T)}.$$

Two multipliers $\omega_1$ and are $\omega_2$ equivalent (or cohomologous) if $\sigma_{\omega_1} \in B^2(W; T)$.

The functions $\omega_{\sigma, T}$ and $\tilde{\omega}_{\sigma, T}$ are cohomologous and $\sigma_{\omega, T}$ is normalized, i.e.

$$\tilde{\omega}_{\sigma, T}(\xi, -\xi) = 1, \quad \xi \in W.$$

The map $t \to \tilde{\mathcal{W}}_{\sigma, T}(t\xi)$ is a group representation of $\mathbb{R}$ and for each $\xi \in W$ there is a unique self-adjoint operator $\phi_T(\xi)$, $T$-field operator associated to $\xi$, such that

$$\tilde{\mathcal{W}}_{\sigma, T}(t\xi) = e^{it\phi_T(\xi)}$$

for all real $t$.

Since

$$\tilde{\mathcal{W}}_{\sigma, T}(\xi) = e^{i\tilde{\sigma}(\xi, T\xi)}\mathcal{W}_{\sigma, T}(\xi), \quad \xi \in W,$$

and

$$\mathcal{W}_{\sigma, T}(\xi) = e^{-i\tilde{\sigma}(\xi, T\xi)}\tilde{\mathcal{W}}_{\sigma, T}(\xi) = e^{i\tilde{\sigma}(\xi, T^*\xi)}\mathcal{W}_{\sigma, T}(\xi) = e^{i\sigma_T(\xi, T^*\xi)}\mathcal{W}_{\sigma, T}(\xi) \quad \xi \in W,$$

we get

$$\{\mathcal{W}_{\sigma, T}(\xi) : \xi \in W\}' = \left\{\tilde{\mathcal{W}}_{\sigma, T}(\xi) : \xi \in W\right\}'.$$

where $S'$ is the commutant of the subset $S \subset \mathcal{B}(\mathcal{H})$. Thus, we have partially proved the following result.

**Lemma 3.3.** (a) The functions $\omega_{\sigma, T}$ and $\tilde{\omega}_{\sigma, T}$ are cohomologous and $\tilde{\omega}_{\sigma, T}$ is normalized.

(b) The map $t \to \tilde{\mathcal{W}}_{\sigma, T}(t\xi)$ is a group representation of $\mathbb{R}$ and for each $\xi \in W$ there is a unique self-adjoint operator $\phi_T(\xi)$, $T$-field operator associated to $\xi$, such that

$$\tilde{\mathcal{W}}_{\sigma, T}(t\xi) = e^{it\phi_T(\xi)}$$

for all real $t$.

(c) The map

$$\sigma_{T+T^*} : W \times W \to \mathbb{R},$$

$$\sigma_{T+T^*}(\xi, \eta) = \sigma(\xi, T\eta) + \sigma(\eta, T^*\xi) = \sigma(\xi, (T + T^*)\eta), \quad \xi, \eta \in W,$$

is a bilinear antisymmetric 2-form. $\sigma_{T+T^*}$ is symplectic (i.e. $\sigma_{T+T^*}$ is non-degenerate) if and only if $T + T^*$ is an isomorphism ($\Leftrightarrow \omega_{\sigma, T}$ is a non-degenerate Schur multiplier).

(d) If $T + T^*$ is an isomorphism (i.e. $\omega_{\sigma, T}$ is a non-degenerate Schur multiplier), then $\sigma_{T+T^*}$ is a symplectic form and in this case

$$\left\{\tilde{\mathcal{W}}_{\sigma, T}(\xi) : \xi \in W\right\} \equiv \left\{\mathcal{H}, \tilde{\mathcal{W}}_{\sigma, T}, \tilde{\omega}_{\sigma, T}\right\}.$$
is a Weyl system for the symplectic space $(W, \sigma_T + T^\sigma)$, i.e.,

$$\tilde{W}_{\sigma,T}(\xi) \tilde{W}_{\sigma,T}(\eta) = e^{\frac{i}{2} \sigma_T + T^\sigma(\xi,\eta)} \tilde{W}_{\sigma,T}(\xi + \eta), \quad \xi, \eta \in W.$$  

Also, in this case, the map $\xi \rightarrow \phi_T(\xi)$ is $\mathbb{R}$-linear.

(c) The projective representations $(\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T})$ and $(\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T})$ satisfy

$$\tilde{W}_{\sigma,T}(\xi) = e^{\frac{i}{2} \sigma(\xi,T\xi)} \tilde{W}_{\sigma,T}(\xi), \quad \xi \in W,$$

$$W_{\sigma,T}(\xi) = e^{-\frac{i}{2} \sigma(\xi,T\xi)} \tilde{W}_{\sigma,T}(\xi) = e^{\frac{i}{2} \sigma(\xi,T^\sigma \xi)} \tilde{W}_{\sigma,T}(\xi), \quad \xi \in W.$$  

In particular,

$$\{W_{\sigma,T}(\xi) : \xi \in W\}' = \left\{\tilde{W}_{\sigma,T}(\xi) : \xi \in W\right\}' \quad \triangleright$$

**Proof.** As noted before, almost all the statements have been proven. The rest is a simple interpretation of the definitions except the $\mathbb{R}$-linearity of the map $\xi \rightarrow \phi_T(\xi)$, which is a consequence of the fact that $(\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T})$ is a Weyl system. $\square$

From now on, we shall always assume that $T + T^\sigma$ is an isomorphism.

**Corollary 3.4.** $(\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T})$ is an irreducible $\omega_{\sigma,T}$-representation if and only if $(\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T}, (W, \sigma_T + T^\sigma))$ is an irreducible projective representation.

**Corollary 3.5.** Any two irreducible $\omega_{\sigma,T}$-representations are unitary equivalent.

**Corollary 3.6.** Suppose that $(\mathcal{H}, \tilde{W}, \tilde{\omega}_{\sigma,T})$ is an irreducible $\omega_{\sigma,T}$-representation and $S : W \rightarrow W$ is a linear map.

(a) If $S$ satisfies $S^\sigma (T + T^\sigma) S = (T + T^\sigma)$, then there is a unitary transformation $U$ in $\mathcal{H}$, uniquely determined apart from a constant factor of modulus 1, such that

$$W \circ S(\xi) = \mu(\xi) U^{-1} W(\xi) U, \quad \xi \in W,$$

where

$$\mu(\xi) = e^{\frac{i}{2} [\sigma(\xi, T \xi) - \sigma(S \xi, T S \xi)]}, \quad \xi \in W.$$

(b) If $S$ satisfies $S^\sigma S T S = T$, then there is a unitary transformation $U$ in $\mathcal{H}$, uniquely determined apart from a constant factor of modulus 1, such that

$$W \circ S(\xi) = U^{-1} W(\xi) U, \quad \xi \in W.$$

**Proof.** The hypothesis $S^\sigma (T + T^\sigma) S = (T + T^\sigma)$ is equivalent to the fact that $S$ is a symplectic transformation in $(W, \sigma_T + T^\sigma)$. Now (a) follows from Segal’s theorem, Theorem 18.5.9 in [17], and the previous lemma, and (b) is a consequence of (a). $\square$

**Example 1.** In $(\theta, \tau)$-quantization we have $W = V \times V^*$ with $\sigma : (V \times V^*) \times (V \times V^*) \rightarrow \mathbb{R}$ is the canonical symplectic form

$$\sigma : W \times W \rightarrow \mathbb{R}, \quad \sigma((x,p);(x',p')) = \langle x',p \rangle_{V',V^*} - \langle x,p' \rangle_{V,V^*}.$$  

In this case

$$T = \begin{pmatrix} \tau & 0 \\ 0 & \theta^* \end{pmatrix} : W = V \times V^* \rightarrow W = V \times V^*,$$

and the condition $T + T^\sigma$ is an isomorphism ($\Leftrightarrow T + T^\sigma$ is invertible) is equivalent to $\tau + \theta$ is invertible.
More generally, if
\[
T = \begin{pmatrix} \tau & \alpha \\ \beta & \theta^* \end{pmatrix} : W = V \times V^* \to W = V \times V^*,
\]
then
\[
T^* = \begin{pmatrix} \theta & -\alpha^* \\ -\beta^* & \tau^* \end{pmatrix} : W = V \times V^* \to W = V \times V^*,
\]
and the condition \( T + T^* \) is invertible is equivalent to
\[
\begin{pmatrix} \tau + \theta & \alpha - \alpha^* \\ \beta - \beta^* & (\tau + \theta)^* \end{pmatrix} \text{ is invertible.}
\]

4. T-Weyl Calculus

Let \( \mathcal{S} \) be the dense linear subspace of \( \mathcal{H} \) consisting of the \( C^\infty \) vectors of the representation \( \mathcal{W}_{\sigma,T} \)
\[
\mathcal{S} = \mathcal{S}(\mathcal{H}, \mathcal{W}_{\sigma,T}) = \left\{ \varphi \in \mathcal{H} : W \ni \xi \rightarrow \mathcal{W}_{\sigma,T}(\xi) \varphi \in \mathcal{H} \text{ is a } C^\infty \text{ map} \right\}.
\]
Since
\[
\mathcal{W}_{\sigma,T}(\xi) = e^{i\pi(\xi, T\xi)} \mathcal{W}_{\sigma,T}(\xi), \ \xi \in W,
\]
it follows that \( \mathcal{S} \) is also the space of \( C^\infty \) vectors of the representation \( \mathcal{W}_{\sigma,T} \)
\[
\mathcal{S} = \mathcal{S}(\mathcal{H}, \mathcal{W}_{\sigma,T}) = \left\{ \varphi \in \mathcal{H} : W \ni \xi \rightarrow \mathcal{W}_{\sigma,T}(\xi) \varphi \in \mathcal{H} \text{ is a } C^\infty \text{ map} \right\}.
\]
The space \( \mathcal{S} \) can be described in terms of the subspaces \( D(\phi_T(\xi)) \), where \( \phi_T(\xi) \) is \( T \)-field operator associated to \( \xi \),
\[
\mathcal{S} = \bigcap_{k \in \mathbb{N}} \bigcap_{\xi_1, \ldots, \xi_k \in W} D(\phi_T(\xi_1) \ldots \phi_T(\xi_k))
\]
\[
= \bigcap_{k \in \mathbb{N}} \bigcap_{\xi_1, \ldots, \xi_k \in \mathcal{B}} D(\phi_T(\xi_1) \ldots \phi_T(\xi_k)),
\]
where \( \mathcal{B} \) is a (symplectic) basis. The topology in \( \mathcal{S} \) defined by the family of seminorms \( \{\|\cdot\|_{k, \xi_1, \ldots, \xi_k}\}_{k \in \mathbb{N}, \xi_1, \ldots, \xi_k \in W} \)
\[
\|\varphi\|_{k, \xi_1, \ldots, \xi_k} = \|\phi_T(\xi_1) \ldots \phi_T(\xi_k) \varphi\|_H, \ \varphi \in \mathcal{S}
\]
makes \( \mathcal{S} \) a Fréchet space. We denote by \( \mathcal{S}^* = \mathcal{S}(\mathcal{H}, \mathcal{W}_{\sigma,T})^* \) the space of all continuous, antilinear (semilinear) mappings \( \mathcal{S} \to \mathbb{C} \) equipped with the weak topology \( \sigma(\mathcal{S}^*, \mathcal{S}) \). Since \( \mathcal{S} \hookrightarrow \mathcal{H} \) continuously and densely, and since \( \mathcal{H} \) is always identified with its adjoint \( \mathcal{H}^* \), we obtain a scale of dense inclusions
\[
\mathcal{S} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}^*
\]
such that, if \( \langle \cdot, \cdot \rangle : \mathcal{S} \times \mathcal{S}^* \to \mathbb{C} \) is the antiduality between \( \mathcal{S} \) and \( \mathcal{S}^* \) (antilinear in the first and linear in the second argument), then for \( \varphi \in \mathcal{S} \) and \( u \in \mathcal{H} \), if \( u \) is considered as an element of \( \mathcal{S}^* \), the number \( \langle \varphi, u \rangle \) is just the scalar product in \( \mathcal{H} \). For this reason we do not distinguish between the the scalar product in \( \mathcal{H} \) and the antiduality between \( \mathcal{S} \) and \( \mathcal{S}^* \). See p. 83-85 in [1].

**Lemma 4.1.** If \( \varphi, \psi \in \mathcal{S} \), then the map \( \mathcal{W} \ni \xi \rightarrow \langle \psi, \mathcal{W}_{\sigma,T}(\xi) \varphi \rangle_{\mathcal{S}, \mathcal{S}^*} \in \mathbb{C} \) belongs to \( \mathcal{S}(\mathcal{W}) \). Moreover, for each continuous seminorm \( p \) on \( \mathcal{S}(\mathcal{W}) \) there are continuous seminorms \( q \) and \( q' \) on \( \mathcal{S} \) such that
\[
p\left( \langle \psi, \mathcal{W}_{\sigma,T}(\cdot) \varphi \rangle_{\mathcal{S}, \mathcal{S}^*} \right) \leq q(\psi)q'(\varphi).
\]
Proof. For a proof see Lemma 1.1 in [1]. \[\square\]

We use the symplectic Fourier decomposition of \(a\),

\[
a = \int e^{-i\sigma(\cdot,\xi)}\mathcal{F}_\sigma(a)(\xi)\,d\sigma\xi
\]

to introduce the following operator in \(B(\mathcal{H})\)

\[
\text{Op}_{\sigma,T}(a) = \int \mathcal{F}_\sigma(a)(\xi)\mathcal{W}_{\sigma,T}(\xi)\,d\xi, \quad a \in S(W).
\]

Lemma 4.1 allows us to use the same formula to define the \(T\)-Weyl calculus.

**Definition 4.2 \((T\text{-Weyl calculus})\).** Let \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})\) be a \(\omega_{\sigma,T}\)-representation of \(W\), \(S = S(\mathcal{H}, \mathcal{W}_{\sigma,T})\) and \(S^* = S(\mathcal{H}, \mathcal{W}_{\sigma,T})^*\). Then for each \(\mu, a \in S^*(W)\) we can define the operators

\[
\mathcal{W}_{\sigma,T}(\mu) : S \to S^*, \quad \text{Op}_{\sigma,T}(a) : S \to S^*,
\]

by

\[
\mathcal{W}_{\sigma,T}(\mu) = \int_{\mathcal{W}} \mathcal{W}_{\sigma,T}(\xi)\mu(\xi)\,d\xi, \quad \text{Op}_{\sigma,T}(a) = \int_{\mathcal{W}} \mathcal{W}_{\sigma,T}(\xi)\mathcal{F}_\sigma(a)(\xi)\,d\xi.
\]

The above integrals make sense if they are taken in the weak sense, i.e. for \(\varphi, \psi \in S\)

\[
\langle \varphi, \mathcal{W}_{\sigma,T}(\mu)\psi \rangle_{S,S^*} = \left\langle \langle \varphi, \mathcal{W}_{\sigma,T}(\cdot)\psi \rangle_{S,S^*}, \mu \rangle_{S(W),S^*(W)},
\]

\[
\langle \varphi, \text{Op}_{\sigma,T}(a)\psi \rangle_{S,S^*} = \left\langle \langle \varphi, \mathcal{W}_{\sigma,T}(\cdot)\psi \rangle_{S,S^*}, \mathcal{F}_\sigma(a) \rangle_{S(W),S^*(W)}.
\]

Moreover, from Lemma 4.1 one obtains that

\[
\left| \langle \varphi, \mathcal{W}_{\sigma,T}(\mu)\psi \rangle_{S,S^*} \right| + \left| \langle \varphi, \text{Op}_{\sigma,T}(a)\psi \rangle_{S,S^*} \right| \leq p \left( \langle \varphi, \mathcal{W}_{\sigma,T}(\cdot)\psi \rangle_{S,S^*} \right) \leq q(\varphi)q'(\psi),
\]

where \(p\) is a continuous seminorm on \(S(W)\) and \(q\) and \(q'\) are continuous seminorms on \(S\).

If we consider on \(S^*(W)\) the weak* topology \(\sigma(S^*(W), S(W))\) and on \(B(S,S^*)\), the topology defined by the seminorms \(\{p_{\varphi,\psi}\}_{\varphi,\psi \in S}\),

\[
p_{\varphi,\psi}(A) = \left| \langle \varphi, A\psi \rangle \right|, \quad A \in B(S,S^*),
\]

the mappings

\[
\mathcal{W}_{\sigma,T} : S^*(W) \to B(S,S^*), \quad \mu \mapsto \mathcal{W}_{\sigma,T}(\mu),
\]

\[
\text{Op}_{\sigma,T} : S^*(W) \to B(S,S^*), \quad a \mapsto \text{Op}_{\sigma,T}(a),
\]

are well defined linear and continuous.

On \(W\) we have two symplectic structures, the first, the initial one, is given by the 2-form \(\sigma\), and the second one obtained in the normalization process of the factor \(\omega_{\sigma,T} (\omega_{\sigma,T} \to \tilde{\omega}_{\sigma,T})\), namely the structure given by the 2-form form \(\sigma_{T+T^*}\). Accordingly, we have two symplectic Fourier transformations \(\mathcal{F}_\sigma\) and \(\mathcal{F}_{\sigma_{T+T^*}}\). Their connection will be established below.
Lemma 4.3. If $b \in S'(W)$, then

$$F_\sigma (b) = (\det (T + T^\sigma))^\frac{1}{2} F_{\sigma T + T^\sigma} (b \circ (T + T^\sigma)),$$

or in operator form,

$$F_\sigma = (\det (T + T^\sigma))^\frac{1}{2} F_{\sigma T + T^\sigma} \circ (T + T^\sigma)^* \text{ on } S'(W).$$

Proof. We take $S = T + T^\sigma$ in Lemma 2.10.

We have two projective representations $(\mathcal{H}, W_\sigma, \omega_\sigma, T, \omega_\sigma, T)$ and $(\mathcal{H}, \tilde{W}_\sigma, \tilde{\omega}_\sigma, T, \tilde{\omega}_\sigma, T)$ with $\omega_\sigma, T$ and $\tilde{\omega}_\sigma, T$ cohomologous associated multipliers, two symplectic structures $(W, \sigma)$ and $(W, \sigma_{T+T^\sigma})$ with two associated symplectic Fourier transformations $F_\sigma$ and $F_{\sigma T + T^\sigma}$. Accordingly, we have the $T$-Weyl calculus, $\text{Op}_{\omega_\sigma, T}$, corresponding to the $\omega_\sigma, T$-representation $(\mathcal{H}, W_\sigma, \omega_\sigma, T)$ and the standard Weyl calculus, $\text{Op}^{\tilde{\omega}}$, corresponding to the Weyl system $(\mathcal{H}, \tilde{W}_\sigma, \tilde{\omega}_\sigma, T)$ (for the symplectic space $(W, \sigma_{T+T^\sigma})$). The connection between these two calculus will be established hereafter.

We write $\theta_\sigma, T$ for the quadratic form on $W$ given by

$$\theta_\sigma, T (\xi) = \sigma (\xi, T\xi), \quad \xi \in W,$$

with the associated symmetric bilinear form $\beta_\sigma, T : W \times W \rightarrow \mathbb{R}$ defined by

$$\beta_\sigma, T (\xi, \eta) = \frac{1}{2} [\sigma (\xi, T\eta) + \sigma (\eta, T\xi)] = \frac{1}{2} [\sigma (\xi, (T - T^\sigma) \eta)], \quad \xi, \eta \in W.$$

We write $\lambda_\sigma, T$ for the function in $C^\infty_{\text{pol}}(W)$ given by

$$\lambda_\sigma, T (\xi) = e^{-\frac{i}{2} \theta_\sigma, T(\xi)} = e^{-\frac{i}{2} \sigma (\xi, T\xi)}, \quad \xi \in W.$$

To this function, we associate the convolution operator $\lambda_\sigma, T(D_\sigma)$ defined by using the symplectic Fourier transformation, i.e.

$$\lambda_\sigma, T(D_\sigma) = F_\sigma \circ M_{\lambda_\sigma, T}(\cdot) \circ F_\sigma : S'(W) \rightarrow S'(W),$$

where $M_{\lambda_\sigma, T}(\cdot)$ denotes the multiplication operator by the function $\lambda_\sigma, T(\cdot)$. For this operator we shall also use the notation $e^{-\frac{i}{2} \theta_\sigma, T(D_\sigma)}$, i.e.

$$\lambda_\sigma, T(D_\sigma) = e^{-\frac{i}{2} \theta_\sigma, T(D_\sigma)}.$$

For $a$ in $S'(W)$, we shall denote by $a^w_\sigma, T$ the temperate distribution in $S'(W)$ given by

$$a^w_\sigma, T = \lambda_\sigma, T(D_\sigma)(a) \circ (T + T^\sigma) = \left(\lambda_\sigma, T \circ (T + T^\sigma)^{-1}\right)(D_\sigma)(a \circ (T + T^\sigma)).$$
If \( a \in \mathcal{S}(W) \), then we have

\[
\text{Op}_{\sigma,T}(a) = \int \mathcal{F}_\sigma(a)(\xi) \mathcal{W}_{\sigma,T}(\xi) \, d^\sigma \xi = \int e^{-\frac{i}{2} \theta_{\sigma,T}(D_\sigma)}(a) \mathcal{F}_\sigma(a)(\xi) \mathcal{W}_{\sigma,T}(\xi) \, d^\sigma \xi
\]

\[
= \int \lambda_{\sigma,T}(\xi) \mathcal{F}_\sigma(a)(\xi) \tilde{\mathcal{W}}_{\sigma,T}(\xi) \, d^\sigma \xi = \int \mathcal{F}_\sigma(\lambda_{\sigma,T}(D_\sigma)(a))(\xi) \tilde{\mathcal{W}}_{\sigma,T}(\xi) \, d^\sigma \xi
\]

\[
= \int (\det(T + T^\sigma))^2 \mathcal{F}_{\sigma,T+T^\sigma}(\lambda_{\sigma,T}(D_\sigma)(a) \circ (T + T^\sigma))(\xi) \tilde{\mathcal{W}}_{\sigma,T}(\xi) \, d^\sigma \xi
\]

\[
= \int (\det(T + T^\sigma))^2 \mathcal{F}_{\sigma,T+T^\sigma}(a_{\sigma,T}^w)(\xi) \tilde{\mathcal{W}}_{\sigma,T}(\xi) \, d^\sigma \xi
\]

\[
= \int \mathcal{F}_{\sigma,T+T^\sigma}(a_{\sigma,T}^w)(\xi) \tilde{\mathcal{W}}_{\sigma,T}(\xi) \, d^\sigma \xi = \text{Op}^\sim(a_{\sigma,T}^w).
\]

Hence

\[
\text{Op}_{\sigma,T}(a) = \text{Op}^\sim(a_{\sigma,T}^w), \quad a \in \mathcal{S}(W).
\]

Here we used the equality

\[
d^\sigma \xi = (\det(T + T^\sigma))^\frac{1}{2} d^\sigma \eta.
\]

By continuity and density arguments we find that

\[
\text{Op}_{\sigma,T}(a) = \text{Op}^\sim(a_{\sigma,T}^w), \quad a \in \mathcal{S}'(W),
\]

with

\[
a_{\sigma,T}^w = \lambda_{\sigma,T}(D_\sigma)(a) \circ (T + T^\sigma) = e^{-\frac{i}{2} \theta_{\sigma,T}(D_\sigma)}(a) \circ (T + T^\sigma)
\]

\[
= \left( \lambda_{\sigma,T} \circ (T + T^\sigma)^{-1} \right) (D_\sigma)(a) \circ (T + T^\sigma).
\]

By Lemma \text{2.11},

\[
\lambda_{\sigma,T}(D_{\sigma,T+T^\sigma})(a) \circ (T + T^\sigma) = \lambda_{\sigma,T}(D_{\sigma,T+T^\sigma})(a) \circ (T + T^\sigma).
\]

Thus we have proved one of the important results of the paper.

**Theorem 4.4.** Let \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})\) be a \(\omega_{\sigma,T}\)-representation of \(W\), the T-Weyl calculus, \(\text{Op}_{\sigma,T}\), corresponding to this representation and the standard Weyl calculus, \(\text{Op}^\sim\), corresponding to the associated Weyl system \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \mathcal{W}_{\sigma,T})\) (for the symplectic space \((W, \sigma_{T+T^\sigma})\)). Then for every \(a \in \mathcal{S}'(W)\)

\[
\text{Op}_{\sigma,T}(a) = \text{Op}^\sim(a_{\sigma,T}^w),
\]

where

\[
a_{\sigma,T}^w = (T + T^\sigma)^* \left( \lambda_{\sigma,T}(D_\sigma)(a) \right) = (T + T^\sigma)^* \left( e^{-\frac{i}{2} \theta_{\sigma,T}(D_\sigma)}(a) \right)
\]

\[
= \left( \lambda_{\sigma,T} \circ (T + T^\sigma)^{-1} \right) (D_\sigma) \left( \left( T + T^\sigma \right)^*(a) \right)
\]

\[
= \lambda_{\sigma,T}(D_{\sigma,T+T^\sigma})(a) \circ (T + T^\sigma).
\]

The equation in \(a \in \mathcal{S}'(W)\),

\[
a_{\sigma,T}^w = \lambda_{\sigma,T}(D_\sigma)(a) \circ (T + T^\sigma) = e^{-\frac{i}{2} \theta_{\sigma,T}(D_\sigma)}(a) \circ (T + T^\sigma),
\]
has a unique solution in $S'(W)$,
\[ a = \overline{\lambda_{\sigma,T}} (D_{\sigma}) \left( a_{\sigma,T}^w \circ (T + T^\sigma)^{-1} \right) = e^{i \theta_{\sigma,T}(D_{\sigma})} \left( a_{\sigma,T}^w \circ (T + T^\sigma)^{-1} \right), \]
where $\overline{\lambda_{\sigma,T}}(\cdot) = 1/\lambda_{\sigma,T}(\cdot) = e^{i \theta_{\sigma,T}(\cdot)}$. It follows that the correspondence at the symbol level $S'(W) \ni a \rightarrow a_{\sigma,T}^w \in S'(W)$ is bijective and obviously continuous.

Next, we assume that the $\omega_{\sigma,T}$-representation $(H, W_{\sigma,T}, \omega_{\sigma,T})$ is irreducible, which is equivalent to $(H, \dot{W}_{\sigma,T}, \dot{\omega}_{\sigma,T})$ is an irreducible Weyl system. In this case, $(H, \dot{W}_{\sigma,T}, \dot{\omega}_{\sigma,T})$ is unitary equivalent to the Schrödinger representation. It follows that the map

\[ \text{Op}_{\sigma,T}^w : S^*(W) \rightarrow B(S, S^*), \quad a \rightarrow \text{Op}_{\sigma,T}^w(a), \]

it is linear, continuous and bijective, so the same is true for the map $\text{Op}_{\sigma,T}$.

**Proposition 4.5.** Let $(H, W_{\sigma,T}, \omega_{\sigma,T})$ be an irreducible $\omega_{\sigma,T}$-representation of $W$. Then the map

\[ \text{Op}_{\sigma,T} : S^*(W) \rightarrow B(S, S^*), \quad a \rightarrow \text{Op}_{\sigma,T}(a), \]

it is linear, continuous and bijective.

From now on, we shall always assume that $(H, W_{\sigma,T}, \omega_{\sigma,T})$ is an irreducible $\omega_{\sigma,T}$-representation.

**Definition 4.6.** For each $A \in B(S, S^*)$ there is a unique distribution $a \in S^*(W)$ such that $A = \text{Op}_{\sigma,T}(a)$. This distribution is called $T$-symbol of $A$. Likewise, for each $A \in B(S, S^*)$ there is a unique distribution $b \in S^*(W)$ such that $A = \text{Op}_{\sigma,T}^w(b)$. This distribution is called the Weyl symbol of $A$.

If $\text{Op}_{\sigma,T}(a) = A = \text{Op}_{\sigma,T}^w(b)$, then we have

\[ b = a_{\sigma,T}^w = \left( \text{Op}_{\sigma,T}^w \right)^{-1} \text{Op}_{\sigma,T}(a) = \lambda_{\sigma,T}(D_{\sigma})(a) \circ (T + T^\sigma), \]

and

\[ a = (\text{Op}_{\sigma,T})^{-1} \text{Op}_{\sigma,T}^w(b) = \overline{\lambda_{\sigma,T}}(D_{\sigma}) \left( b \circ (T + T^\sigma)^{-1} \right). \]

### 5. Modulation spaces and Schatten-class properties of operators in the T-Weyl calculus

The importance of Theorem 4.4 lies in the fact that it establishes both the connection between the $T$-Weyl calculus and the standard Weyl calculus, as well as the connection that exists between the symbols used in them. The maps,

\[ \Lambda_{\sigma,T}^w = \left( \text{Op}_{\sigma,T}^w \right)^{-1} \text{Op}_{\sigma,T} : S^*(W) \rightarrow S^*(W), \quad \Lambda_{\sigma,T}^w(a) = \lambda_{\sigma,T}(D_{\sigma})(a) \circ (T + T^\sigma), \]

\[ \left( \Lambda_{\sigma,T}^w \right)^{-1} : S^*(W) \rightarrow S^*(W), \quad \left( \Phi_{\sigma,T}^w \right)^{-1}(b) = \overline{\lambda_{\sigma,T}}(D_{\sigma}) \left( b \circ (T + T^\sigma)^{-1} \right), \]

are continuous linear isomorphisms. Clearly $S(W)$ is an invariant subspace for both maps. Other invariant subspaces for these maps are particular cases of modulation spaces.

Now we shall recall the definition of the classical modulation space $M^{p,q}(\mathbb{R}^n)$ with parameters $1 \leq p, q \leq \infty$. 

Definition 5.1. Let $1 \leq p, q \leq \infty$. We say that a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $M^{p,q}(\mathbb{R}^n)$ if there is $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$ such that the measurable function
\[
U_{\chi,p} : \mathbb{R}^n \to [0, +\infty],
\]
\[
U_{\chi,p}(\xi) = \begin{cases}
\sup_{y \in \mathbb{R}^n} |\hat{u}_y \chi(\xi)| & \text{if } p = \infty \\
\left( \int |\hat{u}_y \chi(\xi)|^p dy \right)^{1/p} & \text{if } 1 \leq p < \infty 
\end{cases}
\]
then \[
\hat{u}_y \chi(\xi) = \left\langle u, e^{-i(\cdot \xi)} \chi(\cdot - y) \right\rangle.
\]
belongs to $L^q(\mathbb{R}^n)$.

These spaces are special cases of modulation spaces which were introduced by Hans Georg Feichtinger \cite{Feichtinger1983} in 1983 (see also \cite{Toft1990}). They were used by many authors (Boulkhemair, Gröchenig, Heil, Sjöstrand, Toft ...) in the analysis of pseudo-differential operators defined by symbols more general than usual.

Now we give some properties of these spaces.

Proposition 5.2. (a) Let $u \in M^{p,q}(\mathbb{R}^n)$ and let $\chi \in C_0^\infty(\mathbb{R}^n)$. Then the measurable function
\[
U_{\chi,p} : \mathbb{R}^n \to [0, +\infty],
\]
\[
U_{\chi,p}(\xi) = \begin{cases}
\sup_{y \in \mathbb{R}^n} |\hat{u}_y \chi(\xi)| & \text{if } p = \infty \\
\left( \int |\hat{u}_y \chi(\xi)|^p dy \right)^{1/p} & \text{if } 1 \leq p < \infty 
\end{cases}
\]
belongs to $L^q(\mathbb{R}^n)$.

(b) If we fix $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$ and if we put
\[
\|u\|_{M^{p,q},\chi} = \|U_{\chi,p}\|_{L^q}, \quad u \in M^{p,q}(\mathbb{R}^n),
\]
then $\|\cdot\|_{M^{p,q},\chi}$ is a norm on $M^{p,q}(\mathbb{R}^n)$ and the topology that defines does not depend on the choice of the function $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$.

(c) $M^{p,q}(\mathbb{R}^n)$ is a Banach space.

(d) If $1 \leq p_0 \leq p_1 \leq \infty$ and $1 \leq q_0 \leq q_1 \leq \infty$, then
\[
\mathcal{S}(\mathbb{R}^n) \subset M^{1,1}(\mathbb{R}^n) \subset M^{p_0,q_0}(\mathbb{R}^n) \subset M^{p_1,q_1}(\mathbb{R}^n) \subset M^{\infty,\infty}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).
\]

To go further, we need a more convenient way to describe $M^{p,q}(\mathbb{R}^n)$’s topology.

Lemma 5.3. (a) Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $\chi(D)u \in \mathcal{S}'(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$. In fact we have
\[
\chi(D)u(x) = (2\pi)^{-n} \left\langle \hat{\chi}, e^{i(x \cdot \cdot)} \right\rangle, \quad x \in \mathbb{R}^n.
\]

(b) Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in C_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$). Then
\[
\mathbb{R}^n \times \mathbb{R}^n \ni (x, \xi) \mapsto \chi(D - \xi)u(x) = (2\pi)^{-n} \left\langle \hat{\chi}, e^{i(x \cdot \cdot)} \chi(\cdot - \xi) \right\rangle \in \mathbb{C}
\]
is a $C^\infty$-function.
Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\tilde{\varphi}(x) = \varphi(-x)$. We have
\[
\langle \chi(D)u, \varphi \rangle = \langle \chi^{-1}(\chi \tilde{u}), \varphi \rangle = \langle \chi \tilde{u}, \chi^{-1} \varphi \rangle = (2\pi)^{-n} \int \langle \tilde{u}, e^{-i(x,\cdot)} \chi \rangle \varphi(x) \, dx
\]

Hence
\[
\chi(D)u(x) = (2\pi)^{-n} \langle \tilde{u}, e^{i(x,\cdot)} \chi \rangle, \quad x \in \mathbb{R}^n.
\]

Lemma 5.4. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then
\[
\chi(D - \xi)u(x) = (2\pi)^{-n} e^{i(x,\xi)} \hat{u}(\xi), \quad x, \xi \in \mathbb{R}^n.
\]

Proof. Suppose that $u, \chi \in \mathcal{S}(\mathbb{R}^n)$. Then using
\[
\mathcal{F} \left( e^{i(x,\cdot)} \chi (-\xi) \right)(y) = \int \hat{\chi}(\eta) e^{i(x,\eta)} \chi(y - \xi) \, d\eta = \int \hat{\chi}(\eta) e^{i(x,\eta + y,\xi)} \chi(\eta) \, d\eta
\]
we obtain
\[
\chi(D - \xi)u(x) = (2\pi)^{-n} \int \hat{\chi}(\eta) e^{i(x,\eta)} \hat{u}(\eta) \, d\eta
\]

and
\[
\mathcal{F} \left( e^{i(x,\cdot)} \chi (-\xi) \right)(y) = \int \hat{\chi}(\eta) e^{i(x,\eta + y,\xi)} \hat{u}(\eta) \, d\eta = (2\pi)^{-n} e^{i(x,\xi)} \hat{u}(\xi).
\]
The general case is obtained from the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ noticing that both,
\[
\chi(D - \xi)u(x) = (2\pi)^{-n} \langle \tilde{u}, e^{i(x,\cdot)} \chi (-\xi) \rangle
\]
and
\[
\hat{u}(\xi) = \langle u, e^{-i(\cdot,\xi)} \chi(-x) \rangle,
\]
depend continuously on $u$. \hfill $\square$

Corollary 5.5. Let $1 \leq p \leq \infty$, $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then
\[
\| \chi(D - \xi)u \|_{L^p} = (2\pi)^{-n} U_{\xi,p}(\chi), \quad \xi \in \mathbb{R}^n.
\]

Corollary 5.6. Let $1 \leq p, q \leq \infty$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then the following statements are equivalent:
\begin{enumerate}
\item[(a)] $u \in M^{p,q}(\mathbb{R}^n)$;
\item[(b)] There is $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$ so that $\xi \mapsto \| \chi(D - \xi)u \|_{L^p}$ is in $L^q(\mathbb{R}^n)$.
\end{enumerate}

Corollary 5.7. Let $1 \leq p, q \leq \infty$ and $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$. Then
\[
M^{p,q}(\mathbb{R}^n) \ni u \mapsto \left( \int \| \chi(D - \xi)u \|_{L^p}^q \, d\xi \right)^{1/q} \equiv \| u \|_{M^{p,q},\chi}
\]
\[
u \mapsto \| \chi(D - \cdot)u \|_{L^q} = \| u \|_{M^{p,q},\chi}
\]
is a norm on $M^{p,q}(\mathbb{R}^n)$. The topology defined by this norm coincides with the topology of $M^{p,q}(\mathbb{R}^n)$. 

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Let $A$ be a real, symmetric and non-singular matrix, $\Phi_A$ the quadratic form in $\mathbb{R}^n$ defined by $\Phi_A(x) = -\langle Ax, x \rangle / 2$, $x \in \mathbb{R}^n$. Then
\[
\hat{e}^{i\Phi_A}(\xi) = (2\pi)^{n/2} |\det A|^{\frac{1}{2}} e^{\frac{i\text{sgn}A}{4}\pi} e^{-i(A^{-1}x,\xi)/2}
\]
This formula suggests the introduction of the operator
\[
T_A : S'(\mathbb{R}^m \times \mathbb{R}^n) \rightarrow S(\mathbb{R}^m \times \mathbb{R}^n)
\]
defined by
\[
T_A u = (2\pi)^{-n/2} |\det A|^{\frac{1}{2}} e^{-\frac{i\text{sgn}A}{4}(\delta \otimes e^{i\Phi_A}) \cdot u}.
\]
Then
\[
\widetilde{T_A} u = (2\pi)^{-n/2} |\det A|^{\frac{1}{2}} e^{-\frac{i\text{sgn}A}{4}(1 \otimes e^{i\Phi_A}) \cdot \hat{u}} = (1 \otimes e^{i\Phi_A^{-1}}) \cdot \hat{u},
\]
so $T_A : S(\mathbb{R}^m \times \mathbb{R}^n) \rightarrow S'(\mathbb{R}^m \times \mathbb{R}^n)$ is invertible and $T_A^{-1} = T_A^{-1}$.

**Remark 5.8.** Let $\xi = (\xi', \xi'')$ for an element in $\mathbb{R}^m \times \mathbb{R}^n$ and accordingly
\[
\rho \in \mathbb{R}^n \setminus 0, \psi \in S(\mathbb{R}^n) \setminus 0, H = F_m^{-1}(h), \Psi = F_n^{-1}(\psi) \text{ and } \chi = h \otimes \psi.
\]
For $u \in M^{p,q}(\mathbb{R}^m \times \mathbb{R}^n)$ we shall evaluate
\[
\| \chi^2 (D - \xi) T_A u \|_{L^p}.
\]
Using the equality $F^{-1} \circ \tau_\xi = \hat{e}^{i(\cdot,\xi)} F^{-1}$, if we set
\[
C_{A,n} = (2\pi)^{-n/2} |\det A|^{\frac{1}{2}} e^{-\frac{i\text{sgn}A}{4}},
\]
then
\[
\chi^2 (D - \xi) T_A u
\]
\[
= C_{A,n} \left[ \hat{e}^{i(\cdot,\xi')} F_m^{-1}(h) \otimes \left( \left( \hat{e}^{i(\cdot,\xi'')} F_n^{-1}(\psi) \right) \ast e^{i\Phi_A} \right) \right] \ast \chi(D - \xi) u
\]
\[
= C_{A,n} \left[ \left( \hat{e}^{i(\cdot,\xi')} H \right) \otimes \left( \left( \hat{e}^{i(\cdot,\xi'')} \Psi \right) \ast u \ast e^{i\Phi_A} \right) \right] \ast \chi(D - \xi) u.
\]
Since
\[
\left( \hat{e}^{i(\cdot,\xi'')} \Psi \right) \ast u \ast e^{i\Phi_A} (x'') = \int e^{i(\langle A(x'' - y''), x'' - y'' \rangle / 2 + \langle y'', \xi'' \rangle)} \Psi(y'') \, dy''
\]
\[
= e^{i\Phi_A(x'')} \int e^{i(y''(Ax'' + \xi''))} e^{i\Phi_A(y'')} \Psi(y'') \, dy
\]
\[
= e^{i\Phi_A(x'')}(\Psi_A(Ax'' + \xi''))
\]
where \( \Psi_A = \mathcal{F} (e^{i\Phi_A} \Psi) \) it follows that
\[
\|\chi^2 (D - \xi) T_A u\|_{L^p}^2 
\leq (2\pi)^{-n/2} |\det A|^{2/3} \left\| \left( e^{-i\xi \cdot \xi^*} H \right) \odot \left( \left( e^{i\Phi_A} \right)^* e^{i\Phi_A} \right) \right\|_{L^1} \|\chi (D - \xi) u\|_{L^p} 
\]
\[
= (2\pi)^{-n/2} |\det A|^{-1} \|H\|_{L^1} \|\Psi_A\|_{L^1} \|\chi (D - \xi) u\|_{L^p}, \quad \xi \in \mathbb{R}^n \times \mathbb{R}^n.
\]

This estimate implies (a) and (b). For part (c), we use this estimate and Lebesgue’s dominated convergence theorem. \( \Box \)

For \( \lambda \in \text{End}_\mathbb{R} (\mathbb{R}^n) \) and \( u \in S' (\mathbb{R}^n) \) put \( u_\lambda = u \circ \lambda \) whenever it makes sense.

**Theorem 5.10.** If \( \lambda \in \text{End}_\mathbb{R} (\mathbb{R}^n) \) is invertible and \( u \in M^{p,q} (\mathbb{R}^n) \), then \( u_\lambda \in M^{p,q} (\mathbb{R}^n) \) and there is \( C \in (0, +\infty) \) independent of \( u \) and \( \lambda \) such that
\[
\|u_\lambda\|_{M^{p,q}} \leq C |\det \lambda|^{-1/p - 1/q'} (1 + |\lambda|)^n \|u\|_{M^{p,q}}
\]
where \( 1/q + 1/q' = 1 \).

**Proof.** Let \( \chi \in C_0^\infty (\mathbb{R}^n) \) be such that \( \int \chi (x) \, dx = 1 \). We shall use the notation \( \|\cdot\|_{M^{p,q}} \) for \( \|\cdot\|_{M^{p,q},\chi} \). Let \( r > 0 \) be such that \( \text{supp} \chi \subset \{ x : |x| \leq r \} \). We denote by \( \chi_1 \) the characteristic function of the unit ball in \( \mathbb{R}^n \). We evaluate
\[
\overline{u_\lambda \tau_y \chi} (\xi) = \int e^{-i(x,\xi)} u (\lambda x) \chi (x - y) \, dx 
\]
\[
= \int e^{-i(x,\xi)} u (\lambda x) \chi (x - y) \chi (\lambda x - z) \, dx \, dz.
\]
Since \( \chi (x - y) \chi (\lambda x - z) \neq 0 \) implies \( |x - y| \leq r \) and \( |\lambda x - z| \leq r \), we get that \( |z - \lambda y| \leq r (1 + |\lambda|) \) on the support of the integrand. We can write
\[
\chi (x - y) \chi (\lambda x - z) = \chi (x - y) \chi (\lambda x - z) \chi_1 \left( \frac{z - \lambda y}{r (1 + |\lambda|)} \right).
\]
It follows that
\[
\overline{u_\lambda \tau_y \chi} (\xi) = \int \int e^{-i(x,\xi)} (u_\tau \chi) (\lambda x) \chi (x - y) \chi_1 \left( \frac{z - \lambda y}{r (1 + |\lambda|)} \right) \, dx \, dz
\]
\[
= (2\pi)^{-n} \int \chi_1 \left( \frac{z - \lambda y}{r (1 + |\lambda|)} \right) \left( \int e^{-i(x,\xi)} e^{i(x - y, \eta)} (u_\tau \chi) (\lambda x) \, \tilde{\chi} (\eta) \, d\eta \right) \, dz
\]
and
\[
\int e^{-i(x,\xi)} e^{i(x - y, \eta)} (u_\tau \chi) (\lambda x) \, \tilde{\chi} (\eta) \, d\eta
\]
\[
= \int e^{-i(x,\xi) + i(x - \lambda y, \eta)} (u_\tau \chi) (x) \, \tilde{\chi} (t \lambda \eta) \, d\eta
\]
\[
= \int e^{-i(x,\lambda^{-1} \xi - \eta) - i(y, \lambda \eta)} (u_\tau \chi) (x) \, \tilde{\chi} (t \lambda \eta) \, d\eta
\]
\[
= \int e^{-i(y, \lambda \eta)} \overline{u_\tau \chi} (t \lambda^{-1} \xi - \eta) \, \tilde{\chi} (t \lambda \eta) \, d\eta
\]
\[
= \int e^{-i(y, \xi - t \lambda \eta)} \overline{u_\tau \chi} (\eta) \, \tilde{\chi} (\xi - t \lambda \eta) \, d\eta.
\]
We obtain that
\[
\tilde{u}_\lambda \tau_y \tilde{\chi} (\xi) = (2\pi)^{-n} \int \int e^{-i(y,x-\xi \cdot \lambda \eta)} \chi_1 \left( \frac{z - \lambda \eta}{r (1 + \|\lambda\|)} \right) \tilde{u}_\lambda \tau_y \tilde{\chi} (\eta) \tilde{\chi} (\xi - t \lambda \eta) \, d\eta \, dz
\]
\[
= (2\pi)^{-n} \int \int e^{-i(y,x-\xi \cdot \lambda \eta)} \chi_1 \left( \frac{x}{r (1 + \|\lambda\|)} \right) \tilde{u}_\lambda \tau_y \tilde{\chi} (\eta) \tilde{\chi} (\xi - t \lambda \eta) \, d\eta \, dx,
\]
and
\[
|\tilde{u}_\lambda \tau_y \tilde{\chi} (\xi)| \leq (2\pi)^{-n} \int \int \chi_1 \left( \frac{x}{r (1 + \|\lambda\|)} \right) \left| \tilde{u}_\lambda \tau_y \tilde{\chi} (\eta) \right| \left| \tilde{\chi} (\xi - t \lambda \eta) \right| \, d\eta \, dx.
\]
This estimate implies
\[
U_{\lambda, x, p} (\xi) = \left( \int \left| \tilde{u}_\lambda \tau_y \tilde{\chi} (\xi) \right|^p \, dy \right)^{1/p}
\]
\[
\leq (2\pi)^{-n} \int \int \chi_1 \left( \frac{x}{r (1 + \|\lambda\|)} \right) \left| \tilde{\chi} (\xi - t \lambda \eta) \right| \left( \int \left| \tilde{u}_\lambda \tau_y \tilde{\chi} (\eta) \right|^p \, dy \right)^{1/p} \, d\eta \, dx
\]
\[
= (2\pi)^{-n} |\text{det} \lambda|^{-1/p} \int \int \chi_1 \left( \frac{x}{r (1 + \|\lambda\|)} \right) \left| \tilde{\chi} (\xi - t \lambda \eta) \right| U_{\lambda, x, p} (\eta) \, d\eta \, dx
\]
\[
= (2\pi)^{-n} r^n |\text{det} \lambda|^{-1/p} (1 + \|\lambda\|)^n \text{vol} \{ |x| \leq 1 \} \int U_{\lambda, x, p} (\eta) \left| \tilde{\chi} (\xi - t \lambda \eta) \right| \, d\eta
\]
The integral in the last row can be estimated using Hölder’s inequality:
\[
\int U_{\lambda, x, p} (\eta) \left| \tilde{\chi} (\xi - t \lambda \eta) \right| \, d\eta
\]
\[
\leq \left( \int U_{\lambda, x, p}^q (\eta) \left| \tilde{\chi} (\xi - t \lambda \eta) \right| \, d\eta \right)^{1/q} \left( \int \left| \tilde{\chi} (\xi - t \lambda \eta) \right|^q \, d\eta \right)^{1/q'}
\]
\[
= |\text{det} \lambda|^{-1/q' \|\tilde{\chi}\|_{L^q}} \left( \int U_{\lambda, x, p}^q (\eta) \left| \tilde{\chi} (\xi - t \lambda \eta) \right|^q \, d\eta \right)^{1/q}
\]
If \( c = (2\pi)^{-n} r^n \text{vol} \{ |x| \leq 1 \} \), then
\[
U_{\lambda, x, p}^q (\xi) \leq c^q |\text{det} \lambda|^{-q/p} (1 + \|\lambda\|)^{q n} |\text{det} \lambda|^{-q/q'} \|\tilde{\chi}\|_{L^q}^{q/q'} \left( \int U_{\lambda, x, p}^q (\eta) \left| \tilde{\chi} (\xi - t \lambda \eta) \right|^q \, d\eta \right)^{1/q}
\]
which by integration with respect to \( \xi \) gives us
\[
\|U_{\lambda, x, p}\|_{L^q}^q \leq c^q |\text{det} \lambda|^{-q/p} (1 + \|\lambda\|)^{q n} |\text{det} \lambda|^{-q/q'} \|\tilde{\chi}\|_{L^q}^{q/q'} + 1 \|U_{\lambda, x, p}\|_{L^q}^q
\]
\[
\|U_{\lambda, x, p}\|_{L^q} \leq c |\text{det} \lambda|^{-1/p - 1/q'} (1 + \|\lambda\|)^n \|\tilde{\chi}\|_{L^q} \|U_{\lambda, x, p}\|_{L^q}
\]
\[
\|u_\lambda\|_{M^{p, q}} \leq C |\text{det} \lambda|^{-1/p - 1/q'} (1 + \|\lambda\|)^n \|u\|_{M^{p, q}}
\]
with \( C = c \|\tilde{\chi}\|_{L^q} = (2\pi)^{-n} r^n \text{vol} \{ |x| \leq 1 \} \|\tilde{\chi}\|_{L^q} \).

From Theorems \([5,9]\) and \([5,10]\) we deduce another important result of the present paper.
Theorem 5.11. Let $1 \leq p, q \leq \infty$ and $u \in S'(W)$. Then
(a) $u \in M^{p,q}(W)$ if and only if $\Lambda^w_{\sigma,T} u \in M^{p,q}(W)$.
(b) The operator
$$\Lambda^w_{\sigma,T} : M^{p,q}(W) \to M^{p,q}(W)$$
is a bounded isomorphism.

This theorem together with Theorem 4.4 and the corresponding results from the standard Weyl calculus implies the first results on Schatten-class properties for operators in the $T$-Weyl calculus.

Theorem 5.12. Let $(\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})$ be an irreducible $\omega_{\sigma,T}$-representation of $W$ and $1 \leq p < \infty$.
(a) If $a \in M^{p,1}(W)$, then
$$\Omega_{\sigma,T}(a) \in \mathcal{B}_p(\mathcal{H}),$$
where $\mathcal{B}_p(\mathcal{H})$ denote the Schatten ideal of compact operators whose singular values lie in $l^p$. We have
$$\|\Omega_{\sigma,T}(a)\|_{\mathcal{B}_p(\mathcal{H})} \leq Cst \|a\|_{M^{p,1}(W)}.$$
(b) If $a$ is in the Sjöstrand algebra $M^{\infty,1}(W)$, then
$$\Omega_{\sigma,T}(a) \in \mathcal{B}(\mathcal{H}),$$
and
$$\|\Omega_{\sigma,T}(a)\|_{\mathcal{B}(\mathcal{H})} \leq Cst \|a\|_{M^{\infty,1}(W)}.$$

Proof. This theorem is a consequence of the previous theorem and the fact that it is true for pseudo-differential operators with symbols in $M^{p,1}(W)$ (see for instance [3] or [23] for $1 \leq p < \infty$ and [4] for $p = \infty$).

6. AN EMBEDDING THEOREM

Results such as those on pseudo-differential operators from [1] and [2], can be obtained using Theorem 5.12 and an embedding theorem. To formulate the result we define some Sobolev type spaces ($L^p$-style). These spaces are defined by means of weight functions.

Definition 6.1. (a) A positive measurable function $k$ defined in $\mathbb{R}^n$ will be called a weight function of polynomial growth if there are positive constants $C$ and $N$ such that
$$k(\xi + \eta) \leq C \langle \xi \rangle^N k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$  
The set of all such functions $k$ will be denoted by $K_{pol}(\mathbb{R}^n)$.
(b) For a weight function of polynomial growth $k$, we shall write $M_k(\xi) = C \langle \xi \rangle^N$, where $C$, $N$ are the positive constants that define $k$.

Remark 6.2. (a) An immediate consequence of Peetre’s inequality is that $M_k$ is weakly submultiplicative, i.e.
$$M_k(\xi + \eta) \leq C_k M_k(\xi) M_k(\eta), \quad \xi, \eta \in \mathbb{R}^n,$$where $C_k = 2^{N/2}C^{-1}$ and that $k$ is moderate with respect to the function $M_k$ or simply $M_k$-moderate, i.e.
$$k(\xi + \eta) \leq M_k(\xi) k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$
Lemma 6.5. Let \( k \in K_{pol} (\mathbb{R}^n) \). From definition we deduce that
\[
\frac{1}{M_k (\xi)} = C^{-1} (\xi)^{-N} \leq \frac{k (\xi + \eta)}{k (\eta)} \leq C (\xi)^N = M_k (\xi), \quad \xi, \eta \in \mathbb{R}^n.
\]
In fact, the left-hand inequality is obtained if \( \xi \) is replaced by \(-\xi\) and \( \eta \) is replaced by \( \xi + \eta \) in \((6.1)\). If we take \( \eta = 0 \) we obtain the useful estimates
\[
C^{-1} k (0) (\xi)^{-N} \leq k (\xi) \leq C k (0) (\xi)^N, \quad \xi \in \mathbb{R}^n.
\]

The following lemma is an easy consequence of the definition and the above estimates.

Lemma 6.3. Let \( k, k_1, k_2 \in K_{pol} (\mathbb{R}^n) \). Then:
(a) \( k_1 + k_2, k_1 \cdot k_2, \sup (k_1, k_2), \inf (k_1, k_2) \in K_{pol} (\mathbb{R}^n) \).
(b) \( k^n \in K_{pol} (\mathbb{R}^n) \) for every real \( s \).
(c) If \( \hat{k} (\xi) = k (-\xi), \xi \in \mathbb{R}^n \), then \( \hat{k} \) is \( M_k \)-moderate hence \( \hat{k} \in K_{pol} (\mathbb{R}^n) \).
(d) \( 0 < \inf_{\xi \in K} k (\xi) \leq \sup_{\xi \in K} k (\xi) < \infty \) for every compact subset \( K \subset \mathbb{R}^n \).

Definition 6.4. If \( k \in K_{pol} (\mathbb{R}^n) \) and \( 1 \leq p \leq \infty \), we denote by \( H_p^k (\mathbb{R}^n) \) the set of all distributions \( u \in \mathcal{S}' \) such that \( k (D) u \in L^p \). For \( u \in H_p^k (\mathbb{R}^n) \) we define
\[
\| u \|_{k,p} = \| k (D) u \|_{L^p} < \infty.
\]
\( H_p^k (\mathbb{R}^n) \) is a Banach space with the norm \( \| \cdot \|_{k,p} \). We have
\[
\mathcal{S} (\mathbb{R}^n) \subset H_p^k (\mathbb{R}^n) \subset \mathcal{S}' (\mathbb{R}^n)
\]
continuously and densely.

Lemma 6.5. Let \( g : (0, +\infty) \to \mathbb{C} \) and \( k : \mathbb{R}^n \to (0, +\infty) \) be two differentiable maps of class \( \geq r \) satisfying the following conditions:
(a) For any \( j \leq r \), there is \( C_{g,j} > 0 \) such that
\[
t^j \left| g^{(j)} (t) \right| \leq C_{g,j} |g (t)|, \quad t > 0.
\]
(b) For any multi-index \( \alpha = (\alpha_1, ..., \alpha_n) \) of length \( |\alpha| \leq r \), there is \( C_{k,\alpha} > 0 \) such that
\[
|\partial^\alpha k (x)| \leq C_{k,\alpha} k (x), \quad x \in \mathbb{R}^n.
\]
Then, for any multi-index \( \alpha = (\alpha_1, ..., \alpha_n) \), \( |\alpha| \leq r \), there is \( C_{g,k,\alpha} > 0 \) such that
\[
|\partial^\alpha (g \circ k) (x)| \leq C_{g,k,\alpha} |g \circ k (x)|, \quad x \in \mathbb{R}^n.
\]

Proof. By using the formula
\[
\left( h^{(k)} (\cdot) (v_1, ..., v_k) \right)' (v_0) = h^{(k+1)} (v) (v_0, v_1, ..., v_k)
\]
to recursively construct the coefficients \( \{ a_{n,j,\alpha,\sigma} \} \), it is shown by induction that
\[
(g \circ k)^{(m)} (x) = \sum_{j=1}^m \sum_{|\ell| = m} \sum_{\sigma \in S_n} a_{n,j,\alpha,\sigma} g^{(j)} (k (x)) \sigma \cdot (k^{(\ell_1)} (x) \otimes ... \otimes k^{(\ell_j)} (x)).
\]
This formula together with the assumptions on \( g \) and \( k \) imply the conclusion of the lemma.

Lemma 6.6. Let \( 1 \leq p \leq \infty \), \( \chi \in \mathcal{S} (\mathbb{R}^n) \) and \( v \in L^p (\mathbb{R}^n) \). Then \( \chi (D) v \in L^p (\mathbb{R}^n) \) and
\[
\| \chi (D) v \|_{L^p} \leq \| \mathcal{F}^{-1} \chi \|_{L^1} \| v \|_{L^p}.
\]
Proof. We have
\[
\chi(D)v = \mathcal{F}^{-1}(\chi \cdot \hat{v}) = (2\pi)^{-n} \mathcal{F}(\chi \cdot \hat{v}) = (2\pi)^{-n}(2\pi)^{-n} (\mathcal{F}\chi) \ast (\mathcal{F}\hat{v}) = \mathcal{F}^{-1}\chi \ast v.
\]

Now Schur’s lemma implies the result. \hfill \square

**Theorem 6.7.** Let \( 1 \leq p, q \leq \infty, r = \left[ \frac{n}{2} \right] + 1 \) and \( k \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \).
Assume that \( k \) satisfies the conditions:
(a) For any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| \leq 2r \), there is \( C_\alpha > 0 \) such that
\[
|\partial^\alpha k(\xi)| \leq C_\alpha k(\xi), \quad \xi \in \mathbb{R}^n.
\]
(b) \( 1/k \in L^q(\mathbb{R}^n) \).

Then
\[
H^k_p(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n).
\]

Proof. Let \( \chi \in C_0^\infty(\mathbb{R}^n) \setminus 0 \). For \( u \in H^k_p(\mathbb{R}^n) \), we shall show that
\[
\xi \to \|\chi(D - \xi)u\|_{L^p}
\]
is in \( L^q(\mathbb{R}^n) \). Using previous lemma we get
\[
\|1/k(D)\chi(D - \xi)k(D)u\|_{L^p} \leq \left\| \mathcal{F}^{-1}\left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1} \|k(D)u\|_{L^p} = \left\| \mathcal{F}^{-1}\left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1} \|u\|_{K,p}.
\]

It remains to evaluate \( \left\| \mathcal{F}^{-1}\left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1} \). We have
\[
\mathcal{F}^{-1}\left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right)(x) = (2\pi)^{-n} \int e^{i(x,\eta)} \frac{\chi(\eta - \xi)}{k(\eta)} \, d\eta = (2\pi)^{-n} \langle x \rangle^{-2r} \int (1 - \triangle)^r \left( e^{i(x,\eta)} \right) (\eta) \frac{\chi(\eta - \xi)}{k(\eta)} \, d\eta = (2\pi)^{-n} \langle x \rangle^{-2r} \int e^{i(x,\eta)} (1 - \triangle)^r \left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right)(\eta) \, d\eta.
\]

Consequently,
\[
\left\| \mathcal{F}^{-1}\left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1} \leq (2\pi)^{-n} \left\| \langle \cdot \rangle^{-2r} \right\|_{L^1} \left\| (1 - \triangle)^r \left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1}
\]
with
\[
\left\| (1 - \triangle)^r \left( \frac{\chi(\cdot - \xi)}{k(\cdot)} \right) \right\|_{L^1} \leq \sum_{|\alpha| \leq 2r} C_\alpha \left\| \frac{1}{k(\cdot)} \partial^\alpha \chi (\cdot - \xi) \right\|_{L^1}
\]
by Lemma 6.6 above. Further we use the inequality (6.1) to obtain
\[
k(\xi) \left\| \frac{1}{k(\cdot)} \partial^\alpha \chi (\cdot - \xi) \right\|_{L^1} = \int \frac{k(\xi)}{k(\eta)} |\partial^\alpha \chi (\eta - \xi)| \, d\eta \leq \int C |\eta - \xi|^N |\partial^\alpha \chi (\eta - \xi)| \, d\eta = \|M_k(\cdot)\partial^\alpha \chi (\cdot)\|_{L^1}.
\]
Hence
\[
\left\| (1 - \triangle)^r \frac{\chi (\cdot - \xi)}{k (\cdot)} \right\|_{L^1} \leq \left( \sum_{|\alpha| \leq 2r} C_\alpha \| M_k (\cdot) \partial^\alpha \chi (\cdot) \|_{L^1} \right) \frac{1}{k (\xi)}.
\]
Consequently we have
\[
\| u \|_{M^{p,q}} = \| \chi (D - \cdot) u \|_{L^p}
\leq (2\pi)^{-n} \| \langle \cdot \rangle^{-2r} \|_{L^1} \left( \sum_{|\alpha| \leq 2r} C_\alpha \| M_k (\cdot) \partial^\alpha \chi (\cdot) \|_{L^1} \right) \frac{1}{k (\cdot)} \| u \|_{k,p}.
\]
This completes the proof. \[\square\]

Let \( \ell \in \{1, ..., n\} \), \( 1 \leq p \leq \infty \) and \( t = (t_1, ..., t_\ell) \in \mathbb{R}^\ell \). Let \( \mathbb{V} = (V_1, ..., V_\ell) \) denote an orthogonal decomposition, i.e.
\[
\mathbb{R}^n = V_1 \oplus \cdots \oplus V_\ell.
\]
We introduce the Banach space \( H^{t_1, ..., t_\ell}_{p,\mathbb{V}} (\mathbb{R}^n) = H^{t_1, ..., t_\ell}_{p,V_1, ..., V_\ell} (\mathbb{R}^n) \) defined by
\[
H^{t_1, ..., t_\ell}_{p,\mathbb{V}} (\mathbb{R}^n) = \left\{ u \in S' (\mathbb{R}^n) : (1 - \triangle_{V_j})^{t_j/2} \cdots (1 - \triangle_{V_\ell})^{t_\ell/2} u \in L^p (\mathbb{R}^n) \right\},
\]
\[
\| u \|_{H^{t_1, ..., t_\ell}_{p,\mathbb{V}}} = \left\| (1 - \triangle_{V_1})^{t_1/2} \cdots (1 - \triangle_{V_\ell})^{t_\ell/2} u \right\|_{L^p}, \quad u \in H^{t_1, ..., t_\ell}_{p,\mathbb{V}}.
\]
Then
\[
H^{t_1, ..., t_\ell}_{p,\mathbb{V}} (\mathbb{R}^n) = H^k_p (\mathbb{R}^n)
\]
where
\[
k (\cdot) = \langle \cdot \rangle^{t_1}_1 \cdots \langle \cdot \rangle^{t_\ell}_\ell,
\]
\[
\langle \xi \rangle_j = \langle \text{Pr}_{V_j} (\xi) \rangle = \left( 1 + |\text{Pr}_{V_j} (\xi)|^2 \right)^{1/2}, \quad \xi \in \mathbb{R}^n, 1 \leq j \leq \ell.
\]
For \( 1 \leq q < \infty \), it is clear that if \( qt_1 > \dim V_1, ..., qt_\ell > \dim V_\ell \), then the weight function \( k \) satisfies the hypotheses of the previous theorem.

**Corollary 6.8.** Suppose that \( \mathbb{R}^n = V_1 \oplus \cdots \oplus V_\ell \). If \( 1 \leq p \leq \infty \), \( 1 \leq q < \infty \), and \( qt_1 > \dim V_1, ..., qt_\ell > \dim V_\ell \), then \( H^{t_1, ..., t_\ell}_{p,V_1, ..., V_\ell} (\mathbb{R}^n) \hookrightarrow M^{p,q} (\mathbb{R}^n) \).

**Theorem 6.9.** Let \( (\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T}) \) be an irreducible \( \omega_{\sigma,T} \)-representation of \( W \) and \( k \in K_{\text{pol}} (W) \cap C^\infty (W) \). Assume that \( k \) satisfies the conditions:

(a) For any \( j \leq 2 (n + 1) \), there is \( C_j > 0 \) such that
\[
|k^{(j)} (\xi)| \leq C_j k (\xi), \quad \xi \in W.
\]
(b) \( 1/k \in L^1 (\mathbb{R}^n) \).

Then

(i) \( 1 \leq p < \infty \) and \( a \in H^k_p (W) \) imply \( \text{Op}_{\sigma,T} (a) \in B_p (\mathcal{H}) \) and
\[
\| \text{Op}_{\sigma,T} (a) \|_{B_p (\mathcal{H})} \leq C_s t \| a \|_{H^k_p (W)}.
\]

(ii) \( a \in H^\infty_k (W) \) implies \( \text{Op}_{\sigma,T} (a) \in B (\mathcal{H}) \) and
\[
\| \text{Op}_{\sigma,T} (a) \|_{B (\mathcal{H})} \leq C_s t \| a \|_{H^\infty_k (W)}.
\]

Here \( B_p (\mathcal{H}) \) denote the Schatten ideal of compact operators whose singular values lie in \( l^p \).
7. Cordes’ lemma

Recall the definition of generalized Sobolev spaces. Let \( s \in \mathbb{R}, 1 \leq p \leq \infty \). Define
\[
H^s_p (\mathbb{R}^n) = \{ u \in \mathcal{S}' (\mathbb{R}^n) : \langle D \rangle^s u \in L^p (\mathbb{R}^n) \},
\]
where
\[
\| u \|_{H^s_p} = \| \langle D \rangle^s u \|_{L^p}, \quad u \in H^s_p.
\]

**Lemma 7.1.** If \( s > n \) and \( 1 \leq p \leq \infty \), then \( H^s_p (\mathbb{R}^n) \hookrightarrow M^{p,1} (\mathbb{R}^n) \).

**Proof.** This is a particular case of theorem \([6,7]\).

To state and prove Cordes’ lemma we shall work with a very restrictive class of symbols. We shall say that \( a : \mathbb{R}^n \rightarrow \mathbb{C} \) is a symbol of order \( m \) if \( a \in C^\infty (\mathbb{R}^n) \) and for any \( \alpha \in \mathbb{N}^n \), there is \( C_\alpha > 0 \) such that
\[
|\partial^\alpha a (x)| \leq C_\alpha \langle x \rangle^{-m-|\alpha|}, \quad x \in \mathbb{R}^n.
\]
We denote by \( \mathcal{S}^m (\mathbb{R}^n) \) the vector space of all symbols of order \( m \) and observe that
\[
m_1 \leq m_2 \Rightarrow \mathcal{S}^{m_1} (\mathbb{R}^n) \subset \mathcal{S}^{m_2} (\mathbb{R}^n), \quad \mathcal{S}^{m_1} (\mathbb{R}^n) : \mathcal{S}^{m_2} (\mathbb{R}^n) \subset \mathcal{S}^{m_1+m_2} (\mathbb{R}^n).
\]
Observe also that \( a \in \mathcal{S}^m (\mathbb{R}^n) \Rightarrow \partial^\alpha a \in \mathcal{S}^{m-|\alpha|} (\mathbb{R}^n) \) for each \( \alpha \in \mathbb{N}^n \). The function \( \langle x \rangle^m \) clearly belongs to \( \mathcal{S}^m (\mathbb{R}^n) \) for any \( m \in \mathbb{R} \). We denote by \( \mathcal{S}^\infty (\mathbb{R}^n) \) the union of all the spaces \( \mathcal{S}^m (\mathbb{R}^n) \) for any \( m \in \mathbb{R} \). We note that \( \mathcal{S} (\mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} \mathcal{S}^m (\mathbb{R}^n) \) the space of tempered test functions. It is clear that \( \mathcal{S}^m (\mathbb{R}^n) \) is a Fréchet space with the semi-norms given by
\[
|a|_{m,\alpha} = \sup_{x \in \mathbb{R}^n} \langle x \rangle^{-m-|\alpha|} |\partial^\alpha a (x)|, \quad a \in \mathcal{S}^m (\mathbb{R}^n), \alpha \in \mathbb{N}^n.
\]
In addition to the previous lemma, we use part (iii) of Proposition 2.4 from \([2]\) which state that
\[
\mathcal{F}^{-1} \left( \bigcup_{m < 0} \mathcal{S}^m (\mathbb{R}^n) \right) \subset L^1 (\mathbb{R}^n).
\]

**Corollary 7.2.** \( \mathcal{F}^{-1} \left( \bigcup_{m > n} \mathcal{S}^{-m} (\mathbb{R}^n) \right) \cup \mathcal{F} \left( \bigcup_{m > n} \mathcal{S}^{-m} (\mathbb{R}^n) \right) \subset M^{1,1} (\mathbb{R}^n) \).

**Proof.** It is sufficient to prove the inclusion
\[
\mathcal{F}^{-1} \left( \bigcup_{m > n} \mathcal{S}^{-m} (\mathbb{R}^n) \right) \subset M^{1,1} (\mathbb{R}^n).
\]

Let \( m > n \) and \( s \in (n, m) \). We shall show that if \( a \in \mathcal{S}^{-m} (\mathbb{R}^n) \), then \( \mathcal{F}^{-1} a \in H^s_p \).

Indeed, if \( u = \mathcal{F}^{-1} a \), then
\[
\langle D \rangle^s u = \mathcal{F}^{-1} (\langle \cdot \rangle^s a) \in L^1 (\mathbb{R}^n),
\]
since \( \langle \cdot \rangle^s a \in \mathcal{S}^{s-m} (\mathbb{R}^n) \) and \( s - m < 0 \). Using previous lemma we obtain that \( \mathcal{F}^{-1} a \) is in the Feichtinger algebra \( M^{1,1} (\mathbb{R}^n) \).

Using the above lemma, the fact that the map
\[
M^{p,1} (\mathbb{R}^n) \times M^{p,1} (\mathbb{R}^n) \ni (u, v) \rightarrow u \otimes v \in M^{p,1} (\mathbb{R}^n \times \mathbb{R}^n)
\]
is well defined and continuous and Theorem \([5,12]\) we get the following extension of Cordes’ lemma.
Corollary 7.3 (Cordes’ lemma). Suppose that $W = V_1 \oplus ... \oplus V_\ell$ is an orthogonal decomposition with respect to a $\sigma$-compatible inner product on $(W, \sigma)$. Let $t_1 > n_1 = \dim V_1, ..., t_\ell > n_\ell = \dim V_\ell$. Let $a_1 \in S^{-t_1} (V_1), ..., a_k \in S^{-t_\ell} (V_\ell)$ and $g : W \to \mathbb{C}$,

$$g = \mathcal{F}_V^1 (a_1) \otimes ... \otimes \mathcal{F}_V^1 (a_\ell) ,$$

where $\mathcal{F}_V^1$ is either $\mathcal{F}_W$ or $\mathcal{F}_W^{-1}$. If the representation $(\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T})$ is irreducible, then $\text{Op}_{\sigma,T} (g) \in B_1 (\mathcal{H})$.

8. Kato’s identity

In this section we shall describe an extension of a formula due to T. Kato [19]. On the symplectic vector space $(W, \sigma)$ both, duality $\langle \cdot, \cdot \rangle_{\sigma,\sigma'}$ and antiduality $\langle \cdot, \cdot \rangle_{\sigma,S}$ are defined taking into account the symplectic structure of $W$, i.e. if $\varphi, u \in S (W)$, then

$$\langle \varphi, u \rangle_{\sigma,S'} = \int_W \varphi (\eta) u (\eta) d^\sigma \eta, \quad \langle \varphi, u \rangle_{\sigma,S} = \int_W \overline{\varphi (\eta)} u (\eta) d^\sigma \eta.$$

Likewise, the convolution, $\ast_\sigma$, is defined by means of the Fourier measure, $d^\sigma \xi$, by

$$\langle \varphi \ast_\sigma u \rangle (\xi) = \int_W \varphi (\xi - \eta) u (\eta) d^\sigma \eta, \quad \varphi, u \in S (W).$$

Let $\varphi \in S (W)$ and $u \in S' (W)$ (or $u \in S^* (W)$). Then the formal integral

$$\langle \varphi \ast_\sigma u \rangle (\xi) = \int_W \varphi (\xi - \eta) u (\eta) d^\sigma \eta$$

has a rigorous meaning when interpreted as the action of the distribution $u$ on the test function $\eta \to \varphi (\xi - \eta)$ (or $\eta \to \overline{\varphi (\xi - \eta)}$), i.e.

$$\langle \varphi \ast_\sigma u \rangle (\xi) = \langle \varphi (\xi - \cdot), u \rangle_{\sigma,S'}, \quad \langle \varphi \ast_\sigma u \rangle (\xi) = \langle \overline{\varphi (\xi - \cdot)}, u \rangle_{\sigma,S'} .$$

If $S : W \to W$ is a linear isomorphism such that $S^\sigma = S$ then $d^\sigma \eta = (\det S)^{\frac{1}{2}} d^\sigma \eta$,

$$\langle \varphi, u \rangle_{\sigma,S} = (\det S)^{\frac{1}{2}} \langle \varphi, u \rangle_{\sigma,S'}, \quad \langle \varphi, u \rangle_{\sigma,S'} = (\det S)^{\frac{1}{2}} \langle \varphi, u \rangle,$$

and

$$\varphi \ast_{\sigma,S} u = (\det S)^{\frac{1}{2}} (\varphi \ast_\sigma u) .$$

For $b, c \in S (W)$,

$$((b \ast_\sigma c) \circ S) (\xi) = \int_W b (S (\xi) - \eta) c (\eta) d^\sigma \eta$$

$$= \int_W (b \circ S) (\xi - S^{-1} \eta) (c \circ S) (S^{-1} \eta) d^\sigma \eta$$

$$= \det S \int_W (b \circ S) (\xi - \zeta) (c \circ S) (\zeta) d^\sigma \eta$$

$$= (\det S)^{\frac{1}{2}} \int_W (b \circ S) (\xi - \zeta) (c \circ S) (\zeta) d^\sigma \eta$$

$$= (\det S)^{\frac{1}{2}} ((b \circ S) \ast_{\sigma,S} (c \circ S)) (\xi) ,$$

hence

$$(b \ast_\sigma c) \circ S = (\det S)^{\frac{1}{2}} ((b \circ S) \ast_{\sigma,S} (c \circ S)) ,$$

and this formula remains true in all cases where the operations of convolution and composition with a linear bijection make sense.
If \( b \in \mathcal{S}(W), \ c \in \mathcal{S}^*(W) \), then \( b *_{\sigma} c \in \mathcal{S}^*(W) \) and from Theorem 4.4 one obtains that

\[
\text{Op}_{\sigma,T} (b *_{\sigma} c) = \text{Op}^{\tilde{w}} \left( (b *_{\sigma} c)^{w}_{\sigma,T} \right),
\]

where

\[
(b *_{\sigma} c)^{w}_{\sigma,T} = \lambda_{\sigma,T} (D_{\sigma^*_T + T^*}) ((b *_{\sigma} c) \circ (T + T^*))
\]

\[
= (\det (T + T^*))^{\frac{1}{2}} \lambda_{\sigma,T} (D_{\sigma^*_T + T^*}) (b \circ (T + T^*))_{\sigma^*_T + T^*} (c \circ (T + T^*))
\]

\[
= (\det (T + T^*))^{\frac{1}{2}} \lambda_{\sigma,T} (D_{\sigma^*_T + T^*}) (b \circ (T + T^*))_{\sigma^*_T + T^*} (c \circ (T + T^*))
\]

\[
= (\det (T + T^*))^{\frac{1}{2}} ((b \circ (T + T^*))_{\sigma^*_T + T^*} \lambda_{\sigma,T} (D_{\sigma^*_T + T^*}) (c \circ (T + T^*))),
\]

i.e.

\[
(b *_{\sigma} c)^{w}_{\sigma,T} = (\det (T + T^*))^{\frac{1}{2}} ((b \circ (T + T^*))_{\sigma^*_T + T^*} \lambda_{\sigma,T} (D_{\sigma^*_T + T^*}) (c \circ (T + T^*))).
\]

Recalling that \( \text{Op}^{\tilde{w}} \) is the standard Weyl calculus corresponding to the Weyl system \((\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{w}_{\sigma,T})\), for the symplectic space \((W, \sigma_{T + T^*})\), this can be used together with Lemma 2.1 and Lemma 2.2 from [1] to represent \( \text{Op}_{\sigma,T} (b *_{\sigma} c) \).

\[
\text{Op}_{\sigma,T} (b *_{\sigma} c)
\]

\[
= (\det (T + T^*))^{\frac{1}{2}} \int_W (b \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}^{\tilde{w}} (\xi) \tilde{W}_{\sigma,T} (-\xi) d^{\sigma^*_T + T^*} \xi
\]

\[
= \det (T + T^*) \int_W (b \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (c) \tilde{W}_{\sigma,T} (-\xi) d^\sigma \xi
\]

\[
= \det (T + T^*) \int_W (c \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (b) \tilde{W}_{\sigma,T} (-\xi) d^\sigma \xi
\]

i.e.

\[
\text{Op}_{\sigma,T} (b *_{\sigma} c)
\]

\[
= \det (T + T^*) \int_W (b \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (c) \tilde{W}_{\sigma,T} (-\xi) d^\sigma \xi
\]

\[
= \det (T + T^*) \int_W (c \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (b) \tilde{W}_{\sigma,T} (-\xi) d^\sigma \xi,
\]

where the first integral is weakly absolutely convergent while the second one must be interpreted in the sense of distributions and represents the operator defined by

\[
\langle \varphi, \int_W (b \circ (T + T^*)) (\xi) \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (b) \tilde{W}_{\sigma,T} (-\xi) d^\sigma \xi \psi \rangle_{S^*,S^*},
\]

\[
= \langle \varphi, \tilde{W}_{\sigma,T} (\xi) \text{Op}_{\sigma,T} (b) \tilde{W}_{\sigma,T} (-\xi) \psi \rangle_{S^*(W),S^*(W)}.
\]
for all $\varphi, \psi \in S$.

It may be appropriate here to introduce $\{U_{\sigma,T}(\xi)\}_{\xi \in W}$, defined by

$$U_{\sigma,T}(\xi) = \tilde{W}_{\sigma,T}\left((T + T^\sigma)^{-1}\xi\right)$$

$$= e^{\frac{i}{2}\sigma((T+T^\sigma)^{-1}\xi(T+T^\sigma)^{-1}\xi)}W_{\sigma,T}\left((T + T^\sigma)^{-1}\xi\right)$$

$$= e^{\frac{i}{2}\sigma(\xi(T+T^\sigma)^{-1}T(T+T^\sigma)^{-1}\xi)}W_{\sigma,T}\left((T + T^\sigma)^{-1}\xi\right), \quad \xi \in W.$$

Then a change of variables gives

$$\text{Op}_{\sigma,T}(b \ast_{\sigma} c)$$

$$= \int_W b(\xi)\tilde{W}_{\sigma,T}\left((T + T^\sigma)^{-1}\xi\right)\text{Op}_{\sigma,T}(c)\tilde{W}_{\sigma,T}\left(-(T + T^\sigma)^{-1}\xi\right)d^\sigma\xi$$

$$= \int_W b(\xi)U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(c)U_{\sigma,T}(-\xi)d^\sigma\xi$$

$$= \int_W c(\xi)U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(b)U_{\sigma,T}(-\xi)d^\sigma\xi,$$

again with the first and second integrals weakly absolutely convergent and the third interpreted in the sense of distributions.

The family $\{U_{\sigma,T}(\xi)\}_{\xi \in W}$ has very nice properties that can be deduced from Lemma 2.1 in [1] by means of Theorem 4.4.

**Lemma 8.1.** Let $(H, W_{\sigma,T}, \omega_{\sigma,T})$ be a $\omega_{\sigma,T}$-representation of the symplectic space $(W, \sigma)$, $S = S(H, W_{\sigma,T}) = S(H, \tilde{W}_{\sigma,T})$ the space of $C^\infty$ vectors and $\text{Op}_{\sigma,T}$ the T-Weyl calculus. Let $\varphi, \psi \in S$.

(a) If $a \in S^*(W)$ and $\xi \in W$, then

$$U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(a)U_{\sigma,T}(\xi) = \text{Op}_{\sigma,T}(\tau_\xi a),$$

where $\tau_\xi a$ denote the translate by $\xi$ of the distribution $a$, i.e. $(\tau_\xi a)(\cdot) = a(\cdot - \xi)$.

(b) If $a \in S^*(W)$, then the map

$$W \ni \xi \to \langle \varphi, U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(a)U_{\sigma,T}(\xi)\psi \rangle_{S, S,*} \in \mathbb{C}$$

belongs to $C^\infty_{pol}(W)$.

(c) If $a \in S(W)$, then the map

$$W \ni \xi \to \langle \varphi, U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(a)U_{\sigma,T}(\xi)\psi \rangle_{S, S,*} \in \mathbb{C}$$

belongs to $S(W)$.

**Proof.** Part (a) of the lemma follows immediately from Theorem 4.3 Lemma 2.1 in [1] and Corollary 2.13

$$U_{\sigma,T}(\xi)\text{Op}_{\sigma,T}(a)U_{\sigma,T}(\xi)$$

$$= \tilde{W}_{\sigma,T}\left((T + T^\sigma)^{-1}\xi\right)\text{Op}_{\sigma,T}(a_{\sigma,T}^w)\tilde{W}_{\sigma,T}\left(-(T + T^\sigma)^{-1}\xi\right)$$

by Theorem 4.3

$$= \text{Op}_{\sigma,T}(\tau_\xi a)_{\sigma,T}^w$$

by Lemma 2.1 in [1]

$$= \text{Op}_{\sigma,T}(\tau_\xi a)$$

by Corollary 2.13

$$= \text{Op}_{\sigma,T}(\tau_\xi a)$$

by Theorem 4.4.
Here we applied Corollary 2.13 for \( S = T + T^\sigma \) and \( \lambda = \lambda_{\sigma,T} (\cdot) = e^{-\frac{i}{\lambda} \theta_{\sigma,T}(\cdot)} \) as follows:

\[
\begin{align*}
  a_{\sigma,T}^w &= \tau_{(T+T^\sigma)\tau} (T + T^\sigma)^* (\lambda_{\sigma,T} (D_\sigma) (a)) \\
  &= (T + T^\sigma)^* (\lambda_{\sigma,T} (D_\sigma) (\tau_{\xi}a)) \\
  &= (\tau_{\xi}a)_{\sigma,T}^w.
\end{align*}
\]

For (b) and (c) we use (a) and Lemma 3.1. By this lemma we know that \( w = w_{\varphi,\psi} = \langle \varphi, W_{\sigma,T}(\cdot) \psi \rangle_{S,S^*} \in \mathcal{S}(W) \). Assume that \( a \in S^* (W) \). Then, writing \( \tilde{a} \) for \( \mathcal{F}_{\sigma}a \), we have

\[
\langle \varphi, \mathcal{U}_{\sigma,T} (\xi) \mathcal{O}_{\sigma,T} (a) \mathcal{U}_{\sigma,T} (-\xi) \psi \rangle_{S,S^*} = \langle \varphi, \mathcal{O}_{\sigma,T} (\tau_{\xi}a) \psi \rangle_{S,S^*},
\]

and

\[
\langle \varphi, \mathcal{O}_{\sigma,T} (\tau_{\xi}a) \psi \rangle_{S,S^*} = \langle \langle \varphi, W_{\sigma,T}(\cdot) \psi \rangle_{S,S^*}, \tau_{\xi}a \rangle_{\mathcal{S}(W),\mathcal{S}^*(W)} = \langle \langle \varphi, \tau_{\xi}a \rangle_{\mathcal{S}(W),\mathcal{S}^*(W)} = \langle \tau_{-\xi} \tilde{W}(\cdot - -\xi), a \rangle_{\mathcal{S}(W),\mathcal{S}^*(W)}.
\]

Hence

\[
(8.1) \quad \langle \varphi, \mathcal{U}_{\sigma,T} (\xi) \mathcal{O}_{\sigma,T} (a) \mathcal{U}_{\sigma,T} (-\xi) \psi \rangle_{S,S^*} = a *_{\sigma} \tilde{w} (-\xi), \quad \xi \in W,
\]

and (b) and (c) follows at once from this equality. \( \square \)

**Theorem 8.2** (Kato’s identity). Let \( (\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T}) \) be a \( \omega_{\sigma,T} \)-representation of the symplectic space \( (W, \sigma) \), \( S = \mathcal{S}(\mathcal{H}, W_{\sigma,T}) = \mathcal{S}(\mathcal{H}, \tilde{W}_{\sigma,T}) \) the space of \( \mathcal{C}^\infty \) vectors and \( \mathcal{O}_{\sigma,T} \) the \( T \)-Weyl calculus.

(a) If \( b \in \mathcal{S}(W), c \in S^* (W) \), then \( b *_{\sigma} c \in S^* (W) \) and

\[
\mathcal{O}_{\sigma,T} (b *_{\sigma} c) = \int_W b (\xi) \mathcal{U}_{\sigma,T} (\xi) \mathcal{O}_{\sigma,T} (c) \mathcal{U}_{\sigma,T} (-\xi) d^\sigma \xi,
\]

where the first integral is weakly absolutely convergent while the second one must be interpreted in the sense of distributions and represents the operator defined by

\[
\langle \varphi, \left( \int_W c (\xi) \mathcal{U}_{\sigma,T} (\xi) \mathcal{O}_{\sigma,T} (b) \mathcal{U}_{\sigma,T} (-\xi) d^\sigma \xi \right) \psi \rangle_{S,S^*} = \langle \langle \varphi, \mathcal{U}_{\sigma,T} (\cdot) \mathcal{O}_{\sigma,T} (b) \mathcal{U}_{\sigma,T} (-\cdot) \psi \rangle_{S(SW),S^*(SW)}, c \rangle_{S(SW),S^*(SW)},
\]

for all \( \varphi, \psi \in \mathcal{S} \).

(b) Let \( h \in \mathcal{C}^\infty_{pol}(W) \). If \( b \in L^p (W) \) and \( c \in L^q (W) \), where \( 1 \leq p, q \leq \infty \) and \( p^{-1} + q^{-1} \geq 1 \), then \( b *_{\sigma} c \in L^r (W) \), \( r^{-1} = p^{-1} + q^{-1} - 1 \) and

\[
(8.2) \quad \mathcal{O}_{\sigma,T} (h (D_\sigma) (b *_{\sigma} c)) = \int_W b (\xi) \mathcal{U}_{\sigma,T} (\xi) \mathcal{O}_{\sigma,T} (h (D_\sigma) c) \mathcal{U}_{\sigma,T} (-\xi) d^\sigma \xi,
\]

where the integral is weakly absolutely convergent.
Proof. (a) Let \( \varphi, \psi \in S \). Then \( w = w_{\varphi, \psi} = \langle \varphi, W_{\sigma, T}(\eta) \psi \rangle_{S, S^*} \in S(W) \).

First we consider the case when \( b, c \in S(W) \). Then, writing \( \hat{a} \) for \( F_{\sigma} a \), we have

\[
\langle \varphi, \Op_{\sigma, T}(b \ast_{\sigma} c) \psi \rangle_{S, S^*} = \int_{w} \frac{b(\xi)}{c(\eta)} \langle \varphi, W_{\sigma, T}(\eta) \psi \rangle_{S, S^*} d^\sigma \eta
\]

where in the last equality we used the formula

\[\Op_{\sigma, T}(\tau c) = U_{\sigma, T}(\xi) \Op_{\sigma, T}(c) \Op_{\sigma, T}(\tau)(-\xi) \].

Let \( c \in S^*(W) \). Then

\[
\langle \varphi, U_{\sigma, T}(\xi) \Op_{\sigma, T}(c) U_{\sigma, T}(\tau)(-\xi) \psi \rangle_{S, S^*} = \langle \varphi, \Op_{\sigma, T}(\tau c) \psi \rangle_{S, S^*}
\]

\[= \langle \varphi, (\tau c)^\sigma \rangle_{S(W), S^*(W)}\]

\[= \langle \varphi, \xi \rangle_{S(W), S^*(W)} \]

\[= \langle \varphi, \xi \rangle_{S(W), S^*(W)} \].

If \( \{c_j\} \subset S(W) \) is such that \( c_j \to c \) weakly in \( S^*(W) \), then

\[
\langle \varphi, U_{\sigma, T}(\xi) \Op_{\sigma, T}(c_j) U_{\sigma, T}(\tau)(-\xi) \psi \rangle_{S, S^*} \to \langle \varphi, U_{\sigma, T}(\xi) \Op_{\sigma, T}(c) U_{\sigma, T}(\tau)(-\xi) \psi \rangle_{S, S^*},
\]

for every \( \xi \in W \), and the uniform boundedness principle implies that there are \( M \in \mathbb{N}, C = C(M, w) > 0 \) such that

\[
|\langle \varphi, U_{\sigma, T}(\xi) \Op_{\sigma, T}(c_j) U_{\sigma, T}(\tau)(-\xi) \psi \rangle_{S, S^*}| \leq C \langle \xi \rangle^M, \quad \xi \in W,
\]

where \( \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}} \) and \( |\cdot| \) is an euclidean norm on \( W \).

The general case can be deduced from the above case if we observe that

\[
\langle \varphi, \Op_{\sigma, T}(b \ast_{\sigma} c_j) \psi \rangle_{S, S^*} \to \langle \varphi, \Op_{\sigma, T}(b \ast_{\sigma} c) \psi \rangle_{S, S^*},
\]
Hence we proved (b) in the case when either $p, q \leq \infty$, $p^{-1} + q^{-1} \geq 1$, $r^{-1} = p^{-1} + q^{-1} - 1$, $b \in L^p(W)$ and $c \in L^q(W)$, then $b * c \in L^r(W)$ and

$$
\|b * c\|_{L^r(W)} \leq \|b\|_{L^p(W)} \|c\|_{L^q(W)}.
$$

Let $p', r' \geq 1$ such that $p^{-1} + p'^{-1} = r^{-1} + r'^{-1} = 1$. Then $r'^{-1} + q^{-1} \geq 1$ and $p'^{-1} = r'^{-1} + q^{-1} - 1$.

If $b \in S(W)$ and $c \in S(W)$, then $g = h(D_\sigma) c \sigma \in S(W)$ and $h(D_\sigma)(b * \sigma c) = b * \sigma h(D_\sigma) c = b * \sigma g$. Using (a) it follows that

$$
\langle \varphi, \Omega_{\sigma,T} (b * \sigma g) \rangle_{S,S^*} = \int_W b(\xi) \langle \varphi, \Omega_{\sigma,T} (\xi) \Omega_{\sigma,T} (g) \Omega_{\sigma,T} (-\xi) \rangle_{S,S^*} d^\sigma \xi.
$$

Similarly, if $b \in L^p(W) \subset S(W)$ and $c \in S(W)$, then $g = h(D_\sigma) c \sigma \in S(W)$ and $h(D_\sigma)(b * \sigma c) = b * \sigma h(D_\sigma) c = b * \sigma g$. Using (a) again, it follows that

$$
\langle \varphi, \Omega_{\sigma,T} (b * \sigma g) \rangle_{S,S^*} = \int_W b(\xi) \langle \varphi, \Omega_{\sigma,T} (\xi) \Omega_{\sigma,T} (g) \Omega_{\sigma,T} (-\xi) \rangle_{S,S^*} d^\sigma \xi.
$$

Hence we proved (b) in the case when either $b \in S(W)$ or $c \in S(W)$.

The general case can be obtained from these particular cases by an approximation argument. Observe that condition $p^{-1} + q^{-1} \geq 1$ implies that $p < \infty$ or $q < \infty$. For the convergence of the left-hand side of (8.2) we use the Young inequality and the continuity of the map $\Omega_{\sigma,T}$. To estimate right-hand side of (8.2) we use (8.1) and the Hölder and Young inequalities. We have

$$
\langle \varphi, \Omega_{\sigma,T} (\xi) \Omega_{\sigma,T} (h(D_\sigma) c) \Omega_{\sigma,T} (-\xi) \rangle_{S,S^*} = \langle h(D_\sigma) c * \hat{w}, -\xi \rangle = \left(c * \hat{h}w\right)(-\xi)
$$

and

$$
\left| \left\langle \varphi, \left( \int_W b(\xi) \Omega_{\sigma,T} (\xi) \Omega_{\sigma,T} (h(D_\sigma) c) \Omega_{\sigma,T} (-\xi) d^\sigma \xi \right) \rangle_{S,S^*} \right| = \left| \left( \int_W b(\xi) \left(c * \hat{h}w\right)(-\xi) d^\sigma \xi \right) \right| \leq \|b\|_{L^p(W)} \|c * \hat{h}w\|_{L^{p'}(W)} \leq \|b\|_{L^p(W)} \|c\|_{L^q(W)} \|\hat{h}w\|_{L^{p'}(W)},
$$

where $w = w_{\varphi, \psi} = \langle \varphi, \Omega_{\sigma,T} (\xi) \rangle_{S,S^*} \in S(W)$.

\begin{remark}
The last two results are true for $\omega_{\sigma,T}$-representations that are not necessarily irreducible.
\end{remark}
9. Kato’s operator calculus

In [5], H.O. Cordes noticed that the $L^2$-boundedness of an operator $a(x,D)$ in $OPS^0_{0,0}$ could be deduced by a synthesis of $a(x,D)$ from trace-class operators. In [19], T. Kato extended this argument to the general case $OPS^0_{\rho,\rho}, 0 < \rho < 1$, and abstracted the functional analysis involved in Cordes’ argument. This operator calculus can be extended further to investigate the Schatten-class properties of operators in the $T$-Weyl calculus for an irreducible $\omega_{\sigma,T}$-representation $(\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T})$ of $W$.

Let $\mathcal{H}$ be a separable Hilbert space. For $1 \leq p < \infty$, we denote by $B_p(\mathcal{H})$ the Schatten ideal of compact operators on $\mathcal{H}$ whose singular values lie in $l^p$ with the associated norm $\| \cdot \|_p$. For $p = \infty$, $B_\infty(\mathcal{H})$ is the ideal of compact operators on $\mathcal{H}$ with $\| \cdot \|_\infty = \| \cdot \|_1$.

**Definition 9.1.** Let $T, A, B \in B(\mathcal{H}), A \geq 0, B \geq 0$. We write

$$T \ll (A; B) \quad \text{def} \quad \| (u, Tv) \|^2 \leq (u, Au) (v, Bv), \quad \text{for } u, v \in \mathcal{H}.$$  

**Lemma 9.2.** Let $S, T, A, B \in B(\mathcal{H}), A \geq 0, B \geq 0$. Then

(i) $T \ll (|T^*|; |T|)$.

(ii) $T \ll (A; B) \Rightarrow T^* \ll (B^*; A)$.

(iii) $T \ll (A; B) \Rightarrow S^*TS \ll (S^*AS; S^*BS)$.

**Proof.** For a proof see Lemma 3.2 in [1]. □

**Lemma 9.3.** Let $Y$ be a measure space and $Y \ni y \to U(y) \in B(\mathcal{H})$ a weakly measurable map.

(a) Assume that there is $C > 0$ such that

$$\int_Y |(\varphi, U(y) \psi)|^2 \, dy \leq C \| \varphi \|^2 \| \psi \|^2, \quad \varphi, \psi \in \mathcal{H}.$$  

If $b \in L^1(Y)$ and $G \in B_1(\mathcal{H})$, then the integral

$$b \{G\} = \int_Y b(y) U(y)^* GU(y) \, dy$$  

is weakly absolutely convergent and defines a bounded operator such that

$$\| b \{G\} \| \leq C \| b \|_{L^\infty} \| G \|_1.$$  

(b) Assume that there is $C > 0$ such that

$$\| U(y) \| \leq C_1^2 \quad \text{a.e. } y \in Y.$$  

If $b \in L^1(Y)$ and $G \in B_1(\mathcal{H})$, then the integral

$$b \{G\} = \int_Y b(y) U(y)^* GU(y) \, dy$$  

is absolutely convergent and defines a trace class operator such that

$$\| b \{G\} \|_1 \leq C \| b \|_{L^1} \| G \|_1.$$  

(c) Assume that there is $C > 0$ such that

$$\| U(y) \| \leq C_1^2 \quad \text{a.e. } y \in Y,$$

and

$$\int_Y |(\varphi, U(y) \psi)|^2 \, dy \leq C_\infty \| \varphi \|^2 \| \psi \|^2, \quad \varphi, \psi \in \mathcal{H}.$$
If \( b \in L^p(Y) \) with \( 1 \leq p < \infty \) and \( G \in B_1(\mathcal{H}) \), then the integral
\[
b \{ G \} = \int_Y b(y) U(y)^* G U(y) \, dy
\]
is weakly absolutely convergent and defines an operator \( b \{ G \} \) in \( B_p(\mathcal{H}) \) and
\[
\|b \{ G\}\|_p \leq C_1\|b\|_p \|G\|_1.
\]

**Proof.** For a proof see Lemma 3.3 in [1]. \( \square \)

**Lemma 9.4.** Let \((\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T})\) be an irreducible \(\omega_{\sigma,T}\)-representation of \(W\) and let \(\{U_{\sigma,T}(\xi)\}_{\xi \in \mathcal{W}}\) be the family of unitary operators
\[
U_{\sigma,T}(\xi) = \tilde{W}_{\sigma,T}
\end{align*}
\(\left((T + T^*)^{-1} \xi\right), \quad \xi \in \mathcal{W}.
\]
If \(\phi, \psi \in \mathcal{H}\), then the map \(\xi \mapsto (\phi, U_{\sigma,T}(\xi) \psi)_{\mathcal{H}}\) belongs to \(L^2(W) \cap C_\infty(W)\) and
\[
\int_W |(\phi, U_{\sigma,T}(\xi) \psi)|^2 \, d^\sigma \xi = (\det(T + T^*))^{\frac{1}{2}} \|\phi\|^2 \|\psi\|^2,
\]
and
\[
\|(\phi, U_{\sigma,T}(\xi) \psi)\|_\infty \leq \|\phi\| \|\psi\|.
\]
If \(\phi, \psi, \phi', \psi' \in \mathcal{H}\), then
\[
\int_W \overline{(\phi', U_{\sigma,T}(\xi) \psi')} (\phi, U_{\sigma,T}(\xi) \psi) \, d^\sigma \xi = (\det(T + T^*))^{\frac{1}{2}} (\phi, \phi')(\psi', \psi).
\]

**Proof.** Since \((\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T})\) is an irreducible \(\omega_{\sigma,T}\)-representation if and only if \((\mathcal{H}, \tilde{W}_{\sigma,T}, \tilde{\omega}_{\sigma,T}, (W, \sigma_{T+T^*}))\) is an irreducible Weyl system, the lemma can be deduced from Lemma 3.4 in [1] by noticing that
\[
\int_W |(\phi, U_{\sigma,T}(\xi) \psi)|^2 \, d^\sigma \xi = \int_W \left| (\phi, \tilde{W}_{\sigma,T}(T + T^*)^{-1} \xi) \psi \right|^2 \, d^\sigma \xi
\]
\[
= \det (T + T^*) \int_W \left| (\phi, \tilde{W}_{\sigma,T}(\eta) \psi) \right|^2 \, d^\sigma \eta
\]
\[
= (\det(T + T^*))^{\frac{1}{2}} \int_W \left| (\phi, \tilde{W}_{\sigma,T}(\eta) \psi) \right|^2 \, d^{\sigma_{T+T^*}} \eta
\]
\[
= (\det(T + T^*))^{\frac{1}{2}} \|\phi\|^2 \|\psi\|^2 \quad \text{by Lemma 3.4 in [1].}
\]
The last formula is a consequence of polarization identity. \( \square \)

**Theorem 9.5** (Kato’s operator calculus). Let \((\mathcal{H}, W_{\sigma,T}, \omega_{\sigma,T})\) be an irreducible \(\omega_{\sigma,T}\)-representation of \(W\).

(a) If \(b \in L^\infty(W)\) and \(G \in B_1(\mathcal{H})\), then the integral
\[
b \{ G \} = \int_W b(\xi) U_{\sigma,T}(\xi) G U_{\sigma,T}(-\xi) \, d^\sigma \xi
\]
is weakly absolutely convergent and defines a bounded operator such that
\[
\|b \{ G\}\| \leq (\det(T + T^*))^{\frac{1}{2}} \|b\|_{L^\infty} \|G\|_1.
\]
Moreover, if \(b\) vanishes at \(\infty\) in the sense that for any \(\varepsilon > 0\) there is a compact subset \(K\) of \(W\) such that
\[
\|b\|_{L^\infty(W \setminus K)} \leq \varepsilon,
\]
then \(b \{ G\}\) is a compact operator.
The mapping \( (b, G) \rightarrow b \{ G \} \) has the following properties.

(i) \( b \geq 0, G \geq 0 \Rightarrow b \{ G \} \geq 0 \).

(ii) \( 1 \{ G \} = (\det (T + T^\sigma))^\frac{1}{2} \Tr (G) \text{id}_H \).

(iii) \( \langle b_1 b_2 \{ G \} \rangle < \left\langle \left( b_1 \right)^2 \{ G^* \}; \left\{ b_2 \right\} \{ G \} \right\rangle \).

(b) If \( b \in L^p (W) \) with \( 1 \leq p < \infty \) and \( G \in B_1 (H) \), then the integral
\[
b \{ G \} = \int_W b (\xi) U_{\sigma, T} (\xi) \sigma b U_{\sigma, T} (-\xi) \, d^\sigma \xi
\]
is weakly absolutely convergent and defines an operator \( b \{ G \} \) in \( B_p (H) \) and
\[
\| b_{\sigma, T} \{ G \} \|_p \leq (\det (T + T^\sigma))^\frac{1}{2} (1 - \frac{\xi}{2}) \| b \|_{L^p} \| G \|_1 .
\]

Proof. (ii) Let \( G = |\varphi (\psi)| = (\psi, \cdot ) \varphi, \varphi, \psi \in H \). Then
\[
U_{\sigma, T} (\xi) G U_{\sigma, T} (-\xi) = U_{\sigma, T} (\xi) \varphi (\xi) U_{\sigma, T} (\xi) \psi = (U_{\sigma, T} (\xi) \psi, \cdot ) U_{\sigma, T} (\xi) \varphi
\]
and
\[
(u, 1 \{ \varphi (\psi ) \} v) = \int_W (u, U_{\sigma, T} (\xi) \varphi (\xi) U_{\sigma, T} (\xi) \psi, v) \, d^\sigma \xi
\]
\[
= (\det (T + T^\sigma))^\frac{1}{2} (\psi, \varphi) (u, v)
\]
\[
= (u, (\det (T + T^\sigma))^\frac{1}{2} \Tr (|\varphi (\psi)|) v) .
\]
So the equality holds for operators of rank 1. Next we extend this equality by linearity and continuity.

(iii) We have
\[
U_{\sigma, T} (\xi) G U_{\sigma, T} (-\xi) \ll (U_{\sigma, T} (\xi) |G^*| U_{\sigma, T} (-\xi) ; U_{\sigma, T} (\xi) |G| U_{\sigma, T} (-\xi))
\]
which gives
\[
|b_1 (\xi) b_2 (\xi) (\varphi, U_{\sigma, T} (\xi) G U_{\sigma, T} (-\xi) \psi)| \leq \left( |b_1 (\xi)|^2 (\varphi, U_{\sigma, T} (\xi) |G^*| U_{\sigma, T} (-\xi) \varphi) \right)^\frac{1}{2}
\]
\[
\cdot \left( |b_2 (\xi)|^2 (\psi, U_{\sigma, T} (\xi) |G| U_{\sigma, T} (-\xi) \psi) \right)^\frac{1}{2} .
\]
Next, all that remains is to use Schwarz inequality to conclude that (iii) is true. \( \square \)

10. Schatten-class properties of operators in the T-Weyl calculus.

The Cordes-Kato method.

Now, if \( (H, W_{\sigma, T}, \omega_{\sigma, T}) \) is an irreducible \( \omega_{\sigma, T} \)-representation of \( W \), then we are able to study boundedness and Schatten-class properties of certain operators in the T-Weyl calculus using the Cordes-Kato method.

**Theorem 10.1.** Let \( (H, W_{\sigma, T}, \omega_{\sigma, T}) \) be an irreducible \( \omega_{\sigma, T} \)-representation of \( W \), and assume that \( W = V_1 \oplus \ldots \oplus V_t \) is an orthogonal decomposition with respect to a \( \sigma \)-compatible inner product on \( (W, \sigma) \). Let \( a \in S'(W) \).

(a) If there are \( t_1 > n_1 = \dim V_1, \ldots, t_t > n_t = \dim V_t \) such that
\[
b = (1 - \Delta V_1)^{t_1/2} \otimes \ldots \otimes (1 - \Delta V_t)^{t_t/2} a \in L^\infty (W),
\]
then \( \text{Op}_{\sigma, T} (a) \in B (H) \) and
\[
\| \text{Op}_{\sigma, T} (a) \|_{B (H)} \leq C st \| (1 - \Delta V_1)^{1/2} \otimes \ldots \otimes (1 - \Delta V_t)^{1/2} a \|_{L^\infty (W)} .
\]
(b) Let \( 1 \leq p < \infty \). If there are \( t_1 > n_1 = \dim V_1, \ldots, t_\ell > n_\ell = \dim V_\ell \) such that

\[
b = (1 - \triangle V_1)^{t_1/2} \otimes \cdots \otimes (1 - \triangle V_\ell)^{t_\ell/2} a \in L^p(W),
\]
then \( \text{Op}_{\sigma,T} (a) \in \mathcal{B}_p (\mathcal{H}) \) and

\[
\|\text{Op}_{\sigma,T} (a)\|_{\mathcal{B}_p (\mathcal{H})} \leq \text{Cst} \left\| (1 - \triangle V_1)^{t_1/2} \otimes \cdots \otimes (1 - \triangle V_\ell)^{t_\ell/2} a \right\|_{L^p(W)}.
\]

**Proof.** For \( j \in \{1, \ldots, \ell\} \), let \( \psi_{t_j} \) be the unique solution within \( S' (V_j) \) for

\[
(1 - \triangle V_j)^{t_j/2} \psi_{t_j} = \delta.
\]
Then \( g = \psi_{t_1} \otimes \cdots \otimes \psi_{t_\ell} \in S' (W) \).

Recall that the Sobolev space \( H^s_p (W) \), \( s \in \mathbb{R}, 1 \leq p \leq \infty \), consists of all \( a \in S' (W) \) such that \( (1 - \triangle W)^{s/2} a \in L^p(W) \), and we set

\[
\|a\|_{H^s_p(W)} = \left\| (1 - \triangle W)^{s/2} a \right\|_{L^p(W)}.
\]

**Theorem 10.2.** Let \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})\) be an irreducible \( \omega_{\sigma,T}\)-representation of \( W \).

(a) If \( s > 2n \) and \( a \in H^s_\infty (W) \), then \( \text{Op}_{\sigma,T} (a) \in \mathcal{B} (\mathcal{H}) \) and

\[
\|\text{Op}_{\sigma,T} (a)\|_{\mathcal{B} (\mathcal{H})} \leq \text{Cst} \|a\|_{H^s_\infty(W)}.
\]

(b) If \( 1 \leq p < \infty \), \( s > 2n \) and \( a \in H^s_p (W) \), then \( \text{Op}_{\sigma,T} (a) \in \mathcal{B}_p (\mathcal{H}) \) and

\[
\|\text{Op}_{\sigma,T} (a)\|_{\mathcal{B}_p (\mathcal{H})} \leq \text{Cst} \|a\|_{H^s_p(W)}.
\]

If we note that \( \text{Op}_{\sigma,T} (a) \in \mathcal{B}_2 (\mathcal{H}) \) whenever \( a \in L^2 (W) = H^0_\infty (W) \) for any irreducible \( \omega_{\sigma,T}\)-representation \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})\) of \( W \), then standard interpolation results in Sobolev spaces give us the following result.

**Theorem 10.3.** Let \((\mathcal{H}, \mathcal{W}_{\sigma,T}, \omega_{\sigma,T})\) be an irreducible \( \omega_{\sigma,T}\)-representation of \( W \). Let \( \mu > 1 \) and \( 1 \leq p < \infty \). If \( a \in H^{2\mu n - (1 - 2/p)}_p (W) \), then \( \text{Op}_{\sigma,T} (a) \in \mathcal{B}_p (\mathcal{H}) \) and

\[
\|\text{Op}_{\sigma,T} (a)\|_{\mathcal{B}_p (\mathcal{H})} \leq \text{Cst} \|a\|_{H^{2\mu n - (1 - 2/p)}_p(W)}.
\]
References

[1] G. Arsu, On Schatten-von Neumann class properties of pseudo-differential operators. The Cordes-Kato method, *J. Operator Theory*, **59** (2008), 81-114.
[2] G. Arsu, On Schatten-von Neumann class properties of pseudo-differential operators. Cordes’ lemma, [https://arxiv.org/abs/math/0610397](https://arxiv.org/abs/math/0610397).
[3] G. Arsu, On Schatten-von Neumann class properties of pseudo-differential operators. Boulkhemair’s method, [http://arxiv.org/abs/0910.5316](http://arxiv.org/abs/0910.5316).
[4] A. Boulkhemair, Remarks on a Wiener type pseudodifferential algebra and Fourier integral operators, *Math. Res. Lett.*, **4**(1997), 53–67.
[5] H.O. Cordes, On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators, J. Funct. Anal. **18** (1975), 115-131.
[6] N. Dinculeanu, *Integration on locally compact spaces*, Noordhoff International Publishing, 1974.
[7] Evans, David Emrys and Lewis, John T., Dilations of Irreversible Evolutions in Algebraic Quantum Theory. Communications of the Dublin Institute for Advanced Studies, Series A: Theoretical Physics, vol. 24. Dublin: Dublin Institute for Advanced Studies, 1977.
[8] H. G. Feichtinger, Modulation spaces on locally compact abelian groups, Technical report, University of Vienna, Vienna, 1983.
[9] H.G. Feichtinger, Banach convolution algebras of Wiener type, In *Function Series, Operators*, Vol.II (Budapest 1980), pages 509-524. North-Holland, Amsterdam 1983.
[10] H. G. Feichtinger, Modulation spaces of locally compact Abelian groups, In R. Radha M. Krishna and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56, Chennai, January 2002, 2003, Allied Publishers, New Delhi.
[11] Gerald B. Folland, *Real Analysis Modern Techniques and Their Applications* (2nd ed.), John Wiley, 1999, ISBN 978-0-471-31716-6.
[12] K. Gröchenig, Foundations of time frequency analysis, Birkhäuser Boston Inc., Boston MA, 2001.
[13] K. Gröchenig, A Pedestrian Approach to Pseudodifferential Operators, In C. Heil, editor, *Harmonic Analysis and Applications: In honour of John J. Benedetto*, Birkhäuser, Boston, 2006.
[14] K. Gröchenig and C. Heil, Modulation spaces and pseudodifferential operators, *Integral Equations Operator Theory*, **34** (1999), 439-457.
[15] L. Hörmander, The Weyl calculus of pseudo-differential operators, *Comm. Pure Appl. Math.*, **32**(1979), 359-443.
[16] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. I, III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983, 1985.
[17] Lars Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudodifferential Operators*, Springer 2007.
[18] D. Kastler, The $C^*$-algebras of a free boson field. I. Discussion of basic facts, *Comm. Math. Phys.*, **9**(1965), 14-48.
[19] T. Kato, Boundedness of some pseudo-differential operators, Osaka J. Math. 13 (1976), 1-9.
[20] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford University Press; 2nd Edition, 1999.
[21] M.A. Naimark, *Normed Algebras*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1972.
[22] Segal, I. E., Transforms for Operators and Symplectic Automorphisms over a Locally Compact Abelian Group, Mathematica Scandinavica, **13** (1963), 31-43 (1963).
[23] J. Toft, Subalgebras to a Wiener type algebra of pseudo-differential operators, *Ann. Inst. Fourier*, **51**(2001), 1347-1383.
[24] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.

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