LARGE SIEVE INEQUALITY WITH CHARACTERS FOR POWERFUL MODULI

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Abstract. In this paper we aim to generalize the results in [1], [2], [19] and develop a general formula for large sieve with characters to powerful moduli that will be an improvement to the result in [19].

keywords: large sieve inequality; power moduli.

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1. Introduction

Throughout this paper, we reserve the symbols $c_i$ ($i = 1, 2, ...$) for absolute positive constants. Large sieve was an idea originated by J. V. Linnik [10] in 1941 while studying the distribution of quadratic non-residues. Refinements of this idea were made by many. In this paper, we develop a large sieve inequality for powerful moduli. More in particular, we aim to have an estimate for the following sum

$$
\sum_{q \leq Q} \sum_{a=1 \atop (a,q)=1}^{q^k} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q^k n} \right) \right|^2,
$$

where $k \geq 2$ is a natural number. In the sequel, let

$$Z := \sum_{n=M+1}^{M+N} |a_n|^2.
$$

With $k = 1$ in (1.1), it is

$$\ll (Q^2 + N) Z.
$$

This is in fact the consequence of a more general result first introduced by H. Davenport and H. Halberstam [7] in which the Farey fractions in the outer sums of (1.1) can be replaced by any set of well-spaced points. Applying the said more general result, (1.1) is bounded above by

$$\ll (Q^{k+1} + QN) Z, \text{ and } \ll (Q^{2k} + N) Z
$$

(see [19]). Literature abound on the subject of the classical large sieve. See [3], [6], [7], [8], [10], [12], [13] and [14]. In [19] it was proved that the sum (1.1) can be estimated by

$$\ll \left( Q^{k+1} + \left( N Q^{1-1/\kappa} + N^{1-1/\kappa} Q^{1+k/\kappa} \right) N^{\varepsilon} \right) Z,
$$

where $\kappa := 2^{k-1}$ and the implied constant depends on $\varepsilon$ and $k$. Furthermore, when appropriate, some of the constants $c_i$‘s and the implied constants in $\ll$ in the remainder of this paper will depend on $\varepsilon$ or both $\varepsilon$ and $k$. In [1] and [2] this bound was improved for $k = 2$. Extending the elementary method in [1] to higher power moduli, we here establish the following bound for (1.1).
Theorem 1: We have
\[
\sum_{q \leq Q} \sum_{a=1}^{q^k} \left| \sum_{n=M+1}^{M+N} a_ne\left(\frac{a}{q^n}\right) \right|^2 \ll \left( \log \log 10NQ \right)^{k+1}(Q^{k+1} + N + N^{1/2+\varepsilon}Q^k)Z.
\]

For \( k \geq 3 \) Theorem 1 improves the classical bounds \([1,3]\) as well as Zhao’s bound \([1,4]\) in the range \( N^{1/(2k)+\varepsilon} \ll Q \ll N^{(k-2)/(2(k-1)\kappa-2k)-\varepsilon} \). In particular, for \( k = 3 \) we obtain an improvement in the range \( N^{1/6+\varepsilon} \ll Q \ll N^{1/5-\varepsilon} \). We note that for a large \( k \) the exponent \( (\kappa - 2)/(2(k-1)\kappa - 2k) \) is close to \( 1/(2(k-1)) \).

Extending the Fourier analytic methods in \([2,19]\), we establish another bound for cubic moduli which improves the bounds \([1,3], [1,4]\) in the range \( N^{7/23+\varepsilon} \ll Q \ll N^{1/3-\varepsilon} \).

Theorem 2: Suppose that \( 1 \leq Q \leq N^{1/2} \). Then we have
\[
\sum_{q \leq Q} \sum_{a=1}^{q^3} \left| \sum_{n=M+1}^{M+N} a_ne\left(\frac{a}{q^n}\right) \right|^2 \ll \begin{cases} 
N^\varepsilon(Q^4 + N^{9/10}Q^{6/5})Z, & \text{if } N^{7/24} \leq Q \leq N^{1/2}, \\
NQ^{6/7+\varepsilon}Z, & \text{if } 1 \leq Q < N^{7/24}.
\end{cases}
\]

Unfortunately, our Fourier analytic method does not yield any improvement if \( k \geq 4 \).

2. Proof of Theorem 1

Let \( S \) be the set of \( k \)-th powers of natural numbers. Let \( Q_0 \geq \sqrt{N} \). Set
\[
S(Q_0) = S \cap (Q_0, 2Q_0].
\]

We first note, by classical large sieve, setting \( Q = \sqrt{N} \) in \([1,2]\),
\[
\sum_{q \leq \sqrt{N}} \sum_{a=1}^{q^3} \left| \sum_{n=M+1}^{M+N} a_ne\left(\frac{a}{q^n}\right) \right|^2 \leq 2NZ.
\]

Let
\[
S_t(Q_0) = \{ q \in S : tq \in S(Q_0) \}.
\]

Let \( t = p_1^{v_1} \cdots p_n^{v_n} \) be the prime decomposition of \( t \). Furthermore, let
\[
u_i := \left\lceil \frac{v_i}{k} \right\rceil,
\]
where for \( x \in \mathbb{R}, \left\lceil x \right\rceil = \min\{k \in \mathbb{Z} : k \geq x \} \) is the ceiling of \( x \). Moreover, set
\[f_t = p_1^{u_1} \cdots p_n^{u_n}.
\]

Therefore, for all \( q_0^k = q \in S \), \( q \) is divisible by \( t \) if and only if \( q_0 \) is divisible by \( f_t \). Therefore, we have
\[
S_t(Q_0) = \{ q_0^{1/k} / f_t < q_0^k / f_t \leq (2Q_0)^{1/k} / f_t \},
\]

where
\[
g_t := \frac{f_t}{t}.
\]

Moreover we note that
\[
S_t(Q_0) \subset (Q_0/t, 2Q_0/t]
\]
and that
\[
|S_t(Q_0)| \leq \frac{(2Q_0)^{1/k}}{f_t}.
\]

We set for \( m \in \mathbb{N}, l \in \mathbb{Z} \) with \( (m, l) = 1 \)
\[
A_t(u, m, l) = \max_{q_0^k/t \leq y \leq 2q_0^k/t} \{|q \in S_t(Q_0) \cap (y, y+u] : q \equiv l \mod m|\}.
\]
Let \( \delta_t(m,l) \) be the number of solutions \( x \) to the congruence 

\[ x^k g_t \equiv l \mod m. \]

We now use Theorem 2 in [1] with \( Q_0 \geq \sqrt{N} \):

**Theorem 3:** Assume that for all \( t \in \mathbb{N} \), \( m \in \mathbb{N} \), \( l \in \mathbb{Z} \), \( u \in \mathbb{R} \) with \( t \leq \sqrt{N} \), \( m \leq \sqrt{N}/t \), \( (m,l) = 1 \), \( mQ_0/\sqrt{N} \leq u \leq Q_0/t \) the conditions

\[
A_t(u,m,l) \leq C \left( 1 + \frac{|S_t(Q_0)|/m}{Q_0/t} \cdot u \right) \delta_t(m,l),
\]

(2.4)

\[
\sum_{l=1}^{m} \delta_t(m,l) \leq m,
\]

(2.5)

\[
\delta_t(m,l) \leq X
\]

hold for some suitable positive numbers \( C \) and \( X \). Then,

(2.6)

\[
\sum_{q \in S(Q_0)} \frac{q}{\phi(q)} \sum_{a=1}^{\varphi(q)} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q} \right) \right|^2 \leq c_0 C\left( \min \{Q_0X,N\} + Q_0 \right) \left( \sqrt{N} \log \log 10N + \max_{r \leq \sqrt{N}} \sum_{t|r} |S_t(Q_0)| \right) Z.
\]

First, we have to check the validity of the conditions (2.4), (2.5) and (2.6). Conditions (2.4) and (2.5) are obviously satisfied with \( C \) absolute. We further suppose that \( (g_t,m) = 1 \) for otherwise \( \delta_t(m,l) = 0 \) since \( (m,l) = 1 \). Therefore, we must estimate the number of solutions to

(2.8)

\[ x^k \equiv \overline{g_t l} \mod m, \]

denotes the multiplicative inverse of \( g_t \) modulo \( m \). By the virtue of the Chinese remainder theorem, it suffices to estimate the number of solutions to (2.8) with \( m \) as a prime power, say \( m = p^e \), for \( p \in \mathbb{P} \) and \( e \in \mathbb{N} \). Note that the function

\[ \sigma_k : (\mathbb{Z}/p^e \mathbb{Z})^* \rightarrow (\mathbb{Z}/p^e \mathbb{Z})^* : x \rightarrow x^k \]

is an endomorphism. Hence it is enough to estimate the size of its kernel \( \ker(\sigma_k) \). If \( k = \pi_1^{a_1} \cdots \pi_h^{a_h} \) is the prime decomposition of \( k \), then

\[ \sigma_k = \prod_{i=1}^{h} \sigma_{\pi_i}^{a_i}. \]

Thus,

(2.9)

\[ |\ker \sigma_k| \leq \prod_{i=1}^{h} |\ker \sigma_{\pi_i}|^{a_i}. \]

Hence, it suffices to estimate the size of \( |\ker \sigma_{\pi_i}| \) for prime numbers \( \pi \).

For \( p \in \mathbb{P} \),

\[ x^\pi - 1 \equiv 0 \mod p \]

has at most \( \pi \) solutions. By elementary result (see [15], for example), a solution, \( a \) mod \( p^e \) with \( e \geq 1 \), of the congruence

(2.10)

\[ x^\pi - 1 \equiv 0 \mod p^e \]

lifts to more than one solution to

\[ x^\pi - 1 \equiv 0 \mod p^{e+1} \]

only when \( p|\pi a^{\pi - 1} \) and \( p^{e+1}|a^{\pi} - 1 \). If \( p \neq \pi \), \( p|\pi a^{\pi - 1} \) implies \( p|a \), but it is not possible that \( p^{e+1}|a^{\pi} - 1 \) as \( (a^{\pi} - 1,a) = 1 \). Thus, in this case (2.10) has at most \( \pi \) solutions for all \( e \geq 1 \). In the following, we consider the case \( p = \pi \).
By Fermat’s little theorem, there exists only one solution of the congruence
\[ x^\pi - 1 \equiv 0 \mod \pi, \]
namely \( 1 \mod \pi \). This solution lifts to exactly \( \pi \) solutions to
\[ x^\pi - 1 \equiv 0 \mod \pi^2, \]
namely
\[ 1, 1 + \pi, 1 + 2\pi, \ldots, 1 + (\pi - 1)\pi \mod \pi^2. \]
More generally, if \( a \mod \pi^e \) is a solution to
\[ (2.11) \]
then, if \( a \) lifts to solutions to
\[ x^\pi - 1 \equiv 0 \mod \pi^e, \]
they are of the form
\[ (2.12) \]
\[ a, a + \pi^e, a + 2\pi^e, \ldots, a + (\pi - 1)\pi^e \mod \pi^e + 1. \]
Assume there are \( j_1, j_2 \in \{0, \ldots, \pi - 1\} \), \( j_1 \neq j_2 \) such that both \( a + j_1\pi^e \) and \( a + j_2\pi^e \) lift to solutions modulo \( \pi^{e+2} \). Then \( \pi^{e+2}|(a + j_1\pi^e)^{\pi} - 1 \) and \( \pi^{e+2}|(a + j_2\pi^e)^{\pi} - 1 \), hence
\[ (a + j_1\pi^e)^{\pi} - (a + j_2\pi^e)^{\pi} = (j_1 - j_2)\pi^e \sum_{i=0}^{\pi-1} (a + j_1\pi^e)^{\pi-1-i}(a + j_2\pi^e)^i \]
is divisible by \( \pi^{e+2} \). If \( e \geq 2 \), this implies \( a \equiv 0 \mod \pi \), but then \( a \) cannot be a solution to \( (2.11) \). Therefore, if \( e \geq 2 \), only one of the solutions \( (2.12) \) lifts to a solution modulo \( \pi^{e+2} \). From this we infer that the number of solutions to \( (2.11) \) never exceeds \( \pi^2 \), i.e.
\[ |\ker \sigma_{\pi}| \leq \pi^2. \]
Combining this with \( (2.9) \), we get
\[ |\ker \sigma_{k}| \leq k^2. \]
Therefore, by the Chinese remainder theorem, we obtain
\[ \delta_t(m, l) \leq k^{2\omega(m)}, \]
where \( \omega(m) \) is the number of distinct prime divisors of \( m \). Since \( 2^{\omega(m)} \) is the number of square-free divisors of \( m \), we have
\[ k^{2\omega(m)} \leq \tau(m)^{2\log_2 k} \ll m^\varepsilon, \]
where \( \tau(m) \) is the number of divisors of \( m \). Thus, if \( m \leq \sqrt{N} \), \( (2.6) \) holds with
\[ X \ll N^\varepsilon. \]
Now, by Theorem 3,
\[ (2.13) \]
\[ \sum_{q \in S(Q_0)} \sum_{a=1 \atop (a,q)=1}^q \left| \sum_{n=M+1}^{M+N} a_n \left( \frac{a}{n} \right) \right|^2 \]
is majorized by
\[ \ll (\min\{Q_0N^\varepsilon, N\} + Q_0) \left( \sqrt{N} \log \log(10N) + \max_{r \leq \sqrt{N}} \sum_{t|r} Q_0^{1/k} f_t^{-1} \right) \]
\[ \cdot Z. \]
The function
\[ G(r) = \sum_{t|r} \frac{1}{f_t} \]
is clearly multiplicative. If \( r \) is a prime power \( p^v \), then
\[ G(r) \leq 1 + k \left( \frac{1}{p} + \frac{1}{p^2} + \ldots \right) = 1 + \frac{k}{p-1} \leq \left( 1 + \frac{1}{p-1} \right)^k = \left( \frac{p^v}{\varphi(p^v)} \right)^k. \]
Hence, for all \( r \in \mathbb{N} \) we have

\[
G(r) \leq \left( \frac{r}{\varphi(r)} \right)^k \ll (\log \log r)^k.
\]

Hence (2.14) is

\[
\ll (\log \log 10NQ_0)^{k+1} (\sqrt{N} + Q_0^{1/k}) (\min\{Q_0N^\epsilon, N\} + Q_0).
\]

The above is always majorized by

\[
\ll (\log \log 10NQ_0)^{k+1} (Q_0^{1/k} + N^{1/2+\epsilon}Q_0^\epsilon).
\]

Summing over all relevant dyadic intervals and combining with (2.1), we see that (1.1) is majorized by

\[
\ll (\log \log 10NQ_0)^{k+1} (Q_0^{k+1} + N + N^{1/2+\epsilon}Q_0^\epsilon).
\]

Therefore, our result follows.

\[\blacksquare\]

3. Proof of Theorem 2

3.1. Reduction to Farey fractions in short intervals. As in [1], [2], our starting point is the following general large sieve inequality.

**Lemma 1**: Let \((\alpha_r)_{r \in \mathbb{N}}\) be a sequence of real numbers. Suppose that \(0 < \Delta \leq 1/2\) and \(R \in \mathbb{N}\). Put

\[
K(\Delta) := \max_{\alpha \in \mathbb{R}} \sum_{r=1}^R 1, \quad \text{subject to } ||\alpha_r - \alpha|| \leq \Delta,
\]

where \(||x||\) denotes the distance of a real \(x\) to its closest integer. Then

\[
\sum_{r=1}^R |S(\alpha_r)|^2 \leq c_1 K(\Delta) (N + \Delta^{-1}) Z.
\]

In the sequel, we suppose that \(S\) is the set of cubes of natural numbers and that \(\alpha_1, \ldots, \alpha_R\) is the sequence of Farey fractions \(a/q\) with \(q \in S(Q_0), 1 \leq a \leq q\) and \((a, q) = 1\), where \(Q_0 \geq 1\). We further suppose that \(\alpha \in \mathbb{R}\) and \(0 < \Delta \leq 1/2\). Put

\[
I(\alpha) := [\alpha - \Delta, \alpha + \Delta] \quad \text{and} \quad P(\alpha) := \sum_{q \in S \cap \{Q_0, 2Q_0\}} \sum_{\substack{(a, q) = 1 \\text{ and } a/q \notin I(\alpha) \\text{ and } \alpha_{a/q}}} 1.
\]

Then we have

\[
K(\Delta) = \max_{\alpha \in \mathbb{R}} P(\alpha).
\]

Therefore, the proof of Theorem 2 reduces to estimating \(P(\alpha)\).

As in [1] and [2], we begin with an idea of D. Wolke [13]. Let \(\tau\) be a positive number with

\[
1 \leq \tau \leq \frac{1}{\sqrt{\Delta}}.
\]

In [1] and [2] we put \(\tau := 1/\sqrt{\Delta}\), but in fact our method works for all \(\tau\) satisfying (3.2). We will later fix \(\tau\) in an optimal manner. In the said earlier papers, \(\tau = 1/\sqrt{\Delta}\) was the optimal choice.

By Dirichlet’s approximation theorem, \(\alpha\) can be written in the form

\[
\alpha = \frac{b}{r} + z,
\]

where

\[
r \leq \tau, \quad (b, r) = 1, \quad |z| \leq \frac{1}{r\tau}.
\]

Thus, it suffices to estimate \(P(b/r + z)\) for all \(b, r, z\) satisfying (3.3).
We further note that we can restrict ourselves to the case when
\[ z \geq \Delta. \]
If \( |z| < \Delta \), then
\[ P(\alpha) \leq P\left(\frac{b}{r} - \Delta\right) + P\left(\frac{b}{r} + \Delta\right). \]
Furthermore, we have
\[ \Delta \leq \frac{1}{r^2} \leq \frac{1}{r\tau}. \]
Therefore this case can be reduced to the case \( |z| = \Delta \). Moreover, as \( P(\alpha) = P(-\alpha) \), we can choose \( z \) positive. So we can assume (3.4), without any loss of generality.

Summarizing the above observations, we deduce

**Lemma 2:** We have
\[ K(\Delta) \leq 2 \max_{r \in \mathbb{N}} \max_{b \in \mathbb{Z}} \max_{\Delta \leq z \leq 1/(r\tau)} P\left(\frac{b}{r} + z\right). \]

3.2. **Estimation of** \( P(b/r + z) - \text{first way} \). We now prove a first estimate for \( P(b/r + z) \) by using some results in [1]. In the sequel, we suppose that the conditions (3.2), (3.3) and (3.4) are satisfied.

By inequality (41) in [1], we have
\[ P\left(\frac{b}{r} + z\right) \leq 1 + 6 \sum_{t|r} \sum_{0 < m \leq 4rzQ_0/t} \max_{(m,r/t)=1} A_t\left(\frac{\Delta Q_0}{t}, \frac{r}{t}, -bm\right), \]
where \( A_t(u,m,l) \) is defined as in (2.3) and \( b \) is the multiplicative inverse of \( b \) modulo \( r \). By the results of section 2, for the set of cubes, the conditions (2.4), (2.5) and (2.6) with \( X = \Delta^{-\varepsilon} \) are satisfied for all \( t \in \mathbb{N}, m \in \mathbb{N}, l \in \mathbb{Z}, u \in \mathbb{R} \) with \( t \leq \tau, m \leq \tau/t, (m,l) = 1, mQ_0/\tau \leq u \leq Q_0/t \). Conditions (2.4) and (2.6) imply
\[ \sum_{0 < m \leq 4rzQ_0/t} \max_{(m,r/t)=1} A_t\left(\frac{\Delta Q_0}{t}, \frac{r}{t}, -bm\right) \leq C \left(1 + \frac{\Delta t|S_t(Q_0)|}{rzr^2}\right) \frac{4rzQ_0X}{t}. \]

From (3.6) and (3.7) and
\[ \sum_{t|r} \frac{1}{l} \leq \prod_{p|r} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \prod_{p|r} \frac{p}{p - 1} \leq c_2 \log \log 10r, \]
we derive
\[ P\left(\frac{b}{r} + z\right) \leq 1 + c_3 Q_0 X \left(rz \log \log 10r + \Delta \sum_{t|r} |S_t(Q_0)|\right). \]
Furthermore, by (2.2) and (2.14), we have
\[ \sum_{t|r} |S_t(Q_0)| \ll (\log \log 10r)^3 Q_0^{1/3}. \]
Thus, from (3.8) and the fact that \( r \leq \tau = \Delta^{-1/2} \), we obtain

**Proposition 1:** Let \( S \) be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have
\[ P\left(\frac{b}{r} + z\right) \leq 1 + c_4 \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta + Q_0rz\right). \]
3.3. Estimation of $P(b/r + z)$ - second way. We now prove a second estimate for $P(b/r + z)$ by extending the Fourier analytic methods in [2], [19] to cubic moduli. The following bound for $P(b/r + z)$ can be proved in the same manner as Lemma 2 in [2].

**Lemma 3:** Let $S$ be the set of cubes of natural numbers. Suppose that

$$\frac{Q_0 \Delta}{z} \leq \delta \leq Q_0. \tag{3.10}$$

Then,

$$P\left(\frac{b}{r} + z\right) \leq c_5 \left(1 + \frac{1}{\delta} \frac{2Q_0}{\zeta} \Pi(\delta, y) \, dy\right), \tag{3.11}$$

where $I(\delta, y) = [y^{1/3} - c_6 \delta/Q_0^{2/3}, y^{1/3} + c_6 \delta/Q_0^{2/3}]$, $J(\delta, y) = [(y - 4\delta)rz, (y + 4\delta)rz]$ and

$$\Pi(\delta, y) = \sum_{q \in I(\delta, y)} \sum_{m \in J(\delta, y)} \frac{1}{m - bq \mod r}. \tag{3.12}$$

We shall prove the following

**Proposition 2:** Let $S$ be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have

$$P\left(\frac{b}{r} + z\right) \leq c_7 \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta + Q_0^{1/3} \Delta^{-1/3}z^{-1} + \Delta^{-1/2}(rz)^{1/2}\right). \tag{3.13}$$

To derive Proposition 2 from Lemma 3, we need the following standard results from Fourier analysis.

**Lemma 4:** (Poisson summation formula, [5]) Let $f(x)$ be a complex-valued function on the real numbers that is piecewise continuous with only finitely many discontinuities and for all real numbers $a$ satisfies

$$f(a) = \frac{1}{2} \left(\lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x)\right).$$

Moreover, suppose that $f(x) \leq c_8(1 + |x|)^{-c}$ for some $c > 1$. Then,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \text{ where } \hat{f}(x) := \int_{-\infty}^{\infty} f(y)e(xy)\,dy,$n$$

the Fourier transform of $f(x)$.

**Lemma 5:** (see [19], for example) For $x \in \mathbb{R} \setminus \{0\}$ define

$$\phi(x) := \left(\sin \frac{\pi x}{2x}\right)^2, \text{ and } \phi(0) := \lim_{x \to 0} \phi(x) = \frac{\pi^2}{4}.$$n

Then $\phi(x) \geq 1$ for $|x| \leq 1/2$, and the Fourier transform of the function $\phi(x)$ is

$$\hat{\phi}(s) = \frac{\pi^2}{4} \max\{1 - |s|, 0\}.$$n

**Lemma 6:** (see Lemma 3.1. in [9]) Let $F : [a, b] \to \mathbb{R}$ be twice differentiable. Assume that $|F'(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_{a}^{b} e^{iF(x)} \, dx \right| \leq \frac{c_9}{u}. $$
Lemma 7: (see Lemma 4.3.1. in [4]) Let $F : [a, b] \to \mathbb{R}$ be twice continuously differentiable. Assume that $|F''(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_a^b e^{iF(x)}dx \right| \leq \frac{c_{10}}{\sqrt{u}}.$$ 

We shall also need the following estimates for cubic exponential sums.

Lemma 8: (see [11], [17]) Let $c \in \mathbb{C}$, $k, l \in \mathbb{Z}$ with $(k, c) = 1$. Then,

$$\sum_{d=1}^c e\left(\frac{kd^3 + ld}{c}\right) \leq c_{11}c^{1/2+\varepsilon}(l, c).$$

Furthermore,

$$\sum_{d=1}^c e\left(\frac{kd^3}{c}\right) \leq c_{11}c^{2/3}.$$ 

Proof of Proposition 2: We put

$$\delta := \frac{Q_0\Delta}{z}.$$ 

By Lemma 5, (3.12) can be estimated by

$$\Pi(\delta, y) \leq \sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_6\delta/Q_0^{2/3}}\right) \sum_{m \equiv -bq^3 \mod r} \phi\left(\frac{m - yrz}{8\delta rz}\right).$$

Using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.15) into

$$\sum_{m \equiv -bq^3 \mod r} \phi\left(\frac{m - yrz}{8\delta rz}\right) = 8\delta z \sum_{j \in \mathbb{Z}} e\left(\frac{jbq^3}{r} + jyz\right) \tilde{\phi}(8\delta z).$$

Therefore, we get for the double sum on the right-hand side of (3.15)

$$\sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_6\delta/Q_0^{2/3}}\right) \sum_{m \equiv -bq^3 \mod r} \phi\left(\frac{m - yrz}{8\delta rz}\right)$$

$$= 8\delta z \sum_{j \in \mathbb{Z}} e(jyz) \tilde{\phi}(8\delta z) \sum_{d=1}^{\tilde{r}} e\left(\frac{jbq^3}{r}\right) \sum_{k \equiv d \mod \tilde{r}} \phi\left(\frac{k - y^{1/3}}{2c_6\delta/Q_0^{2/3}}\right),$$

where $\tilde{r} := r/(r, j)$ and $\tilde{j} := j/(r, j)$. Again using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.16) into

$$\sum_{k \in \mathbb{Z}} \phi\left(\frac{k - y^{1/3}}{2c_6\delta/Q_0^{2/3}}\right) = \frac{2c_6\delta}{\tilde{r}Q_0^{2/3}} \sum_{l \in \mathbb{Z}} e\left(l \cdot \frac{d - y^{1/3}}{\tilde{r}}\right) \tilde{\phi}\left(\frac{2c_6d}{\tilde{r}Q_0^{2/3}}\right).$$
Therefore, the right-hand side of (3.19) is majorized by

\[ \sum_{m \equiv -bq^r \mod r} \phi \left( \frac{m - yrz}{8drz} \right) \]

(3.18)

If \( r \) is accounted for, we deduce

From (3.16) and (3.17), we obtain

Applying the Lemmas 5 and 8 to the right-hand side of (3.18), and taking (3.21) into account, we deduce

(3.19)

\[ \sum_{|j| \leq 1/(8dz) \atop |l| \leq (\tilde{r}Q_0^{1/3})/(2cn\delta)} \sum_{l \neq 0} \frac{1}{|l|} \left( \tilde{r} \right) \int_{Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{\tilde{r}} \right) dy \]

\[ + \sum_{|j| \leq 1/(8dz)} \frac{1}{|j|} \left( \tilde{r} \right) \int_{Q_0} e \left( jyz \right) dy \]

If \( j \neq 0 \), then

\[ \left| \int_{Q_0} e \left( jyz \right) dy \right| \leq \frac{1}{|j|z}. \]

If \( j = 0 \) and \( l \neq 0 \), then

\[ \left| \int_{Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{\tilde{r}} \right) dy \right| \leq \frac{c_13Q_0^{2/3}}{|l|} \]

by Lemma 6 (take into account that \( \tilde{r} = 1 \) if \( j = 0 \)). If \( j \neq 0 \) and \( l \neq 0 \), then Lemma 7 yields

\[ \left| \int_{Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{\tilde{r}} \right) dy \right| \leq \frac{c_{14}\sqrt{r}Q_0^{5/6}}{|l|}. \]

Therefore, the right-hand side of (3.19) is majorized by

(3.20)

\[ \leq c_{15}\Delta^{-\varepsilon} \left( zQ_0^{1/3} + \frac{1}{Q_0^{2/3}} \sum_{1 \leq l \leq 1/(8dz)} \frac{1}{l} + z \sum_{1 \leq l \leq Q_0^{1/3}/(2cn\delta)} \frac{1}{l} + zQ_0^{1/6} \sum_{1 \leq l \leq 1/(8dz)} \sum_{1 \leq l \leq \tilde{r}Q_0^{2/3}/(2cn\delta)} \frac{(l, \tilde{r})}{\sqrt{l}} \right). \]

Now, we estimate the sums in the last line of (3.20). Using (3.2), (3.3) and (3.4), we obtain

(3.21)

\[ \sum_{1 \leq l \leq Q_0^{1/3}/(2cn\delta)} \frac{1}{l} \leq c_{16}\Delta^{-\varepsilon}. \]

Using the definition of \( \tilde{r} \), (3.22), (3.23) and (3.24), we obtain

(3.22)

\[ \sum_{1 \leq j \leq 1/(8dz)} \frac{1}{j^{1/3}} = \frac{1}{\sqrt{r}} \sum_{t | r} \sqrt{t} \sum_{1 \leq j \leq 1/(8dz) \atop (r, j) = t} \frac{1}{j} \leq \frac{c_{17}\Delta^{-\varepsilon}}{\sqrt{r}} \sum_{t | r} t^{-2/3} \leq c_{18}\Delta^{-\varepsilon} r^{-1/3}. \]
For $A \geq 1$, we have
\[
\sum_{1 \leq i \leq A} \frac{(l, \bar{r})}{\sqrt{l}} \leq \sum_{t \mid i} t \sum_{1 \leq i \leq A/t} \frac{1}{\sqrt{lt}} \ll \sqrt{A} \sum_{t \mid \bar{r}} 1 \ll \bar{r}^\varepsilon \sqrt{A}.
\]
Therefore,
\[
(3.23) \quad \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq i \leq rQ_{i}^{2/3}/(2c_{0} \delta)} \frac{(l, \bar{r})}{\sqrt{l}} \leq c_{19} \Delta^{-\varepsilon} Q_{0}^{4/3} \sqrt{r} \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{r}.
\]
Using the definition of $\bar{r}$, we obtain
\[
(3.24) \quad \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{r} \sum_{1 \leq i \leq 1/(8\delta z)} \sum_{(r, t) = 1} \frac{1}{\sqrt{t}} 1 \leq \frac{\sqrt{r}}{8\delta z} \sum_{1 \leq i \leq r} \frac{1}{t^{3/2}} \leq c_{20} \sqrt{r}.
\]
Combining Lemma 3 and (3.24), we obtain
\[
(3.25) \quad P \left( \frac{b}{r} + z \right) \leq c_{7} \Delta^{-3\varepsilon} \left( 1 + \delta z Q_{0}^{1/3} + \delta Q_{0}^{-2/3} r^{-1/3} + \delta^{-1/2} Q_{0}^{1/2} \sqrt{r} \right).
\]
From (3.24) and (3.25), we infer the desired estimate. Note that the first term in the right-hand side of (3.25) can be absorbed into the last term on the right-hand side of (3.24) by (3.3). □

3.4. Final proof of Theorem 2. Combining Propositions 1,2 and (3.3), we obtain
\[
(3.26) \quad P \left( \frac{b}{r} + z \right) \leq c_{21} \Delta^{-\varepsilon} \left( Q_{0}^{1/3} \Delta + \min \left\{ Q_{0} r z, Q_{0}^{1/3} \Delta r^{-1/3} z^{-1} \right\} + \Delta^{-1/2} r^{-1/2} \right).
\]
If
\[
\Delta \leq Q_{0}^{-1/3} r^{-2/3},
\]
then
\[
\min \left\{ Q_{0} r z, Q_{0}^{1/3} \Delta r^{-1/3} z^{-1} \right\} = Q_{0} r z \leq Q_{0}^{2/3} \Delta^{1/2} r^{1/3}.
\]
If
\[
\Delta > Q_{0}^{-1/3} r^{-2/3},
\]
then
\[
\min \left\{ Q_{0} r z, Q_{0}^{1/3} \Delta r^{-1/3} z^{-1} \right\} = Q_{0}^{1/3} \Delta r^{-1/3} z^{-1} \leq Q_{0}^{2/3} \Delta^{1/2} r^{1/3}.
\]
From the above inequalities and (3.3), we deduce
\[
(3.27) \quad \min \left\{ Q_{0} r z, Q_{0}^{1/3} \Delta r^{-1/2} z^{-1} \right\} \leq Q_{0}^{2/3} \Delta^{1/2} r^{1/3} \leq Q_{0}^{2/3} \Delta^{1/2} r^{1/3}.
\]
Combining (3.26) and (3.27), we get
\[
(3.28) \quad P \left( \frac{b}{r} + z \right) \leq c_{22} \Delta^{-\varepsilon} \left( Q_{0}^{4/3} \Delta r^{\varepsilon} + Q_{0}^{2/3} \Delta^{1/2} r^{1/3} + \Delta^{-1/2} r^{-1/2} \right).
\]
Now we choose
\[
\tau := \begin{cases} N^{6/5} Q_{0}^{-4/5}, & \text{if } N^{7/8} \leq Q_{0} \leq N^{3/2}, \\ Q_{0}^{1/7}, & \text{if } 1 \leq Q_{0} < N^{7/8}, \end{cases} \quad \text{and} \quad \Delta := \begin{cases} N^{-1}, & \text{if } N^{7/8} \leq Q_{0} \leq N^{3/2}, \\ Q_{0}^{-8/7}, & \text{if } 1 \leq Q_{0} < N^{7/8}. \end{cases}
\]
Then the condition (3.22) is satisfied in each case, and from (3.28) and Lemmas 1,2, we obtain
\[
(3.29) \quad \sum_{Q_{0}^{1/3} \leq q \leq (2Q_{0})^{1/3}} \sum_{a=1}^{\frac{3}{(a,q)=1}} \left| S \left( \frac{a}{q^{3}} \right) \right|^{2} \leq \begin{cases} N^{\tau} \left( Q_{0}^{1/3} + N^{9/10} Q_{0}^{2/5} \right) Z, & \text{if } N^{7/8} \leq Q_{0} \leq N^{3/2}, \\ N Q_{0}^{2/7+\varepsilon} Z, & \text{if } 1 \leq Q_{0} < N^{7/8}. \end{cases}
\]
We can divide the interval $[1, Q]$ into $O(\log Q)$ subintervals of the form $\left[ Q_{0}^{1/3}, (2Q_{0})^{1/3} \right]$, where $1 \leq Q_{0} \leq Q^{3}$. Hence, the result of Theorem 2 follows from (3.29). □
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