CLOSEDNESS AND INVERTIBILITY FOR THE SUM OF TWO CLOSED OPERATORS

NIKOLAOS ROIDOS

Abstract. We show a Kalton-Weis type theorem for the general case of non-commuting operators. More precisely, we consider sums of two possibly non-commuting linear operators defined in a Banach space such that one of the operators admits bounded $H^\infty$-calculus, the resolvent of the other one satisfies some weaker boundedness condition and the commutator of their resolvents has certain decay behavior with respect to the spectral parameters. Under this consideration, we show that the sum is closed and that after a sufficiently large positive shift it becomes invertible, and moreover sectorial. As an application, we employ this result in combination with a resolvent construction technique, and recover a classical result on the existence, uniqueness and maximal $L^p$-regularity of solution for the abstract non-autonomous linear parabolic problem.

1. Introduction

Let $E$ be a complex Banach space and $A : \mathcal{D}(A) \rightarrow E$, $B : \mathcal{D}(B) \rightarrow E$ are two closed possibly non-commuting linear operators in $E$. We consider the question of whether the sum $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is also closed. Further, we ask under which assumptions the sum can be invertible. The last is connected to the existence, uniqueness and maximal regularity of solution for the following abstract linear equation in the Banach space $E$, namely

$$(A + B)x = y, \quad y \in E.$$ 

We can distinguish between two cases according to whether the operators commute or not, where by commuting we mean resolvent commuting (see e.g. in [1] (III.4.9.1)).

For a quick review on the above two problems, we start with the classical result of Da Prato and Grisvard [4], where they introduced for first time the inverse of the sum formula. In the case of resolvent commuting operators they showed that if both operators are sectorial and the strong parabolicity condition on their sectoriality angles is fulfilled (i.e. their sum is greater than $\pi$), then the sum is closable and the closure is invertible. If the operators do not commute, then the same result still holds if some further commutation condition [4, (6.5)] is satisfied. They applied their result to the first and the second order abstract linear Cauchy problems and provided existence, uniqueness and regularity results of solution.

If we restrict to the commuting case, then we mention two classical results that further provide closedness. Dore and Venni in [6] employed a different formula for the inverse of the sum and used the underline properties of the UMD Banach spaces. In such a space they showed that if two resolvent commuting operators have bounded imaginary powers and the strong parabolicity condition for their power angles is satisfied (i.e. their sum is less that $\pi$), then their sum is closed and invertible. As an application, a criterion to the question of
maximal $L^p$-regularity for the abstract linear parabolic problem was also given. Next, Kalton and Weis in [3] treated the problem as a special case of operator valued holomorphic functional calculus. Without assumptions on the geometry of the Banach space, they showed that if one of the operators admits bounded $H^\infty$-calculus, the other one is $R$-sectorial (or $U$-sectorial) and the standard strong parabolicity condition on the corresponding angles is fulfilled (i.e. their sum is greater than $\pi$), then the sum is closed and invertible. They further provided an application to the problem of maximal $L^p$-regularity for the abstract linear parabolic equation that led to characterization of this property in UMD spaces.

For the general case of non-commuting operators we mention the following two remarkable generalizations. First, Monniaux and Prüss in [10] showed that the Dore-Venni theorem can be extended to the non-commuting case if further the Labbas-Terreni condition [10, (2.6)] is satisfied. As an application they showed maximal regularity results to a general evolutionary integral equation. Then, Prüss and Simonett, extended the Kalton-Weis theorem to the general case of non-commuting operators provided that either the Da Prato-Grisvard condition [11, (3.1)] or the Labbas-Terreni condition [11, (3.2)] is satisfied. Their application gave maximal regularity results for the abstract linear parabolic problem for the Laplacian on wedge domains.

In the current paper we give an answer to the problems of the closedness and invertibility of the sum of two operators simultaneously (Theorem 3.3) by generalizing the Kalton-Weis theorem in two aspects. Firstly, we extend it to the case of possibly non-commuting operators under certain decay assumptions to the commutator of the two resolvents with respect to the spectral parameters. This condition (Condition 2.12) that breaks down into three parts of equivalent strength, ensures that the resolvent commutator decays faster than the composition of the resolvents themselves. Secondly, instead of asking for one of the summands to be $R$-sectorial we introduce a weaker boundedness condition, namely the $\mathcal{ON}$-sectoriality, that is a boundedness property for the resolvent based on Bochner-norm estimates over an arbitrary measure space (Definition 2.6).

We approach the problem by applying first the Da Prato-Grisvard formula to certain complex powers, or respectively by applying certain complex powers to the formula, and then by perturbing the result in order to construct some unbounded approximations for the left and the right inverse of the sum by using only the commutation condition. Then, we employ the extra assumptions of the boundedness of the $H^\infty$-calculus and the $\mathcal{ON}$-boundedness to our operators and make the above approximations to serve as the full inverse of the sum. For the final estimates we use dyadic decomposition of our integral formulas together with the essential unconditionality result of Kalton and Weis [8, Lemma 4.1].

As an application of our main theorem, we recover a standard result on the existence, uniqueness and maximal $L^p$-regularity of solution for the abstract non-autonomous linear parabolic problem in UMD spaces (Theorem 4.2). We regard the problem as an inverse of the sum question and the non-autonomous term as a parametric family of $R$-sectorial operators with the parameter over a compact topological space. Further, by regarding the time derivative as a second summand, our previous theory will provide us a set of local sectorial approximation of the full sum. Then, an old method for the construction of the full resolvent of such a family by using only the sectoriality of each local approximation combined with standard perturbation results will give us the full inverse.
2. Notation and preliminaries

We start with the notion of a sectorial operator of positive type, which allows us for the development of the functional calculus. We denote by $\rho$ and $\sigma$ the resolvent and the spectrum respectively.

**Definition 2.1.** Let $E$ be a complex Banach space, $\theta \in [0, \pi)$ and $\kappa \geq 1$. Let $\mathcal{P}_\kappa(\theta)$ be the class of closed densely defined linear operators in $E$ such that if $A \in \mathcal{P}_\kappa(\theta)$, then

$$S_\theta = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| \leq \theta \} \cup \{0\} \subset \rho(-A) \quad \text{and} \quad (1 + |\lambda|)(A + \lambda)^{-1}\|_{\mathcal{L}(E)} \leq \kappa, \forall \lambda \in S_\theta.$$  

Also, let $\mathcal{P}(\theta) = \cup_\kappa \mathcal{P}_\kappa(\theta)$. The elements in $\mathcal{P}(\theta)$ are called (invertible) sectorial operators of angle $\theta$ and if $A \in \mathcal{P}_\kappa(\theta)$, $\kappa$ is called sectorial bound of $A$.

If $A \in \mathcal{P}_\kappa(\theta)$, then a sectoriality extension argument (see e.g. [1, (III.4.7.11)]) implies that

$$\Omega_{\kappa, \theta} = \cup_{z \in \mathbb{C}} \{ \lambda \in \mathbb{C} \mid |\lambda - z| \leq \frac{1 + |z|}{2\kappa} \} \subset \rho(-A)$$

and

$$(1 + |\lambda|)(A + \lambda)^{-1}\|_{\mathcal{L}(E)} \leq 2\kappa + 1, \forall \lambda \in \Omega_{\kappa, \theta}.$$  

Therefore, whenever $A \in \mathcal{P}(\theta)$ we can assume that $\theta > 0$ (see e.g. [1, (III.4.6.4)] and [1, (III.4.6.5)]). For any $\rho \geq 0$ and $\theta \in (0, \pi)$, let the positively oriented path

$$\Gamma_{\rho, \theta} = \{ re^{i\theta} \in \mathbb{C} \mid r \geq \rho \} \cup \{ re^{i \phi} \in \mathbb{C} \mid \theta \leq \phi \leq 2\pi - \theta \} \cup \{ re^{i \phi} \in \mathbb{C} \mid r \geq \rho \},$$

where we denote $\Gamma_{0, \theta}$ simply by $\Gamma_\theta$. We can define the holomorphic functional calculus for sectorial operators by the Dunford integral. Then, the following basic property can be satisfied.

**Definition 2.2.** Let $E$ be a complex Banach space and $A \in \mathcal{P}(\theta), \theta \in (0, \pi)$ and $\phi \in [0, \theta)$. Let $H_0^\infty(\phi)$ be the space of all bounded holomorphic functions $f : \mathbb{C} \setminus S_\phi \to \mathbb{C}$ such that

$$|f(\lambda)| \leq c(\frac{|\lambda|}{1 + |\lambda|^2})^\eta, \quad \text{for any} \quad \lambda \in \mathbb{C} \setminus S_\phi,$$

with some $c > 0$ and $\eta > 0$ depending on $f$. Any $f \in H_0^\infty(\phi)$ defines an element $f(-A) \in \mathcal{L}(E)$ by

$$f(-A) = \frac{1}{2\pi i} \int_{\Gamma_\phi} f(\lambda)(A + \lambda)^{-1} d\lambda.$$  

We say that the operator $A$ admits a bounded $H^\infty$-calculus of angle $\phi$, and we denote by $A \in \mathcal{H}^\infty(\phi)$, if

$$\|f(-A)\|_{\mathcal{L}(E)} \leq C_{A, \phi} \sup_{\lambda \in \mathbb{C} \setminus S_\phi} |f(\lambda)|, \quad \text{for any} \quad f \in H_0^\infty(\phi),$$

where the constant $C_{A, \phi} > 0$ depends only on $A$ and $\phi$.

Let $A : \mathcal{D}(A) \to E$ be a linear operator in a complex Banach space $E$ such that $A \in \mathcal{P}(\theta) \cap \mathcal{H}^\infty(\phi), 0 \leq \phi < \theta < \pi$. Denote by $A^* : \mathcal{D}(A^*) \to E^*$ the adjoint of $A$ defined in the continuous dual space $E^*$ of $E$. Then, $A^* \in \mathcal{P}(\theta) \cap \mathcal{H}^\infty(\phi)$ provided that $\mathcal{D}(A^*)$ is dense in $E^*$, which we will always assume in the sequel. This is [5, Proposition 1.3 (v)] and [5, Proposition 2.11 (v)]. We also recall the following boundedness property for operators having bounded $H^\infty$-functional calculus, which will be of particular importance in the later estimates. Denote by $\mathbb{D}$ the closed unit disk in $\mathbb{C}$.  


Lemma 2.3. Let $E$ be a complex Banach space, $A \in \mathcal{H}^{\infty}(\phi)$ and $h \in H_0^{\infty}(\phi)$. For any $t > 0$ and any finite sequence $\{a_k\}_{k \in \{0, \ldots, n\}}$, $n \in \mathbb{N}$, with $a_k \in \mathbb{D}$ for each $k$, we have that

$$
\| \sum_{k=0}^{n} a_k h(-t^{2^{-k}} A) \|_{L(E)} \leq C_{A,h},
$$

with the constant $C_{A,h}$ depending only on $A$ and $h$.

Proof. This is Lemma 4.1 in [3].

A typical example of the functional calculus for a sectorial operator $A \in \mathcal{P}(\theta)$ are the complex powers. For $\text{Re}(z) < 0$ they are defined by

$$
(2.1) \quad A^z = \frac{1}{2\pi i} \int_{\Gamma_{\rho,\theta}} (-\lambda)^z (A + \lambda)^{-1} d\lambda,
$$

where $\rho > 0$ is sufficiently small. The above family together with $A^0 = I$ is a strongly continuous holomorphic semigroup on $E$ (see e.g. [1] Theorem III.4.6.2 ] and [1] Theorem III.4.6.5)). Further, by a sectoriality extension argument, we can replace $\Gamma_{\rho,\theta}$ in (2.1) by $-\delta + \Gamma_{\theta}$ with $\delta > 0$ sufficiently small. Each operator $A^z$, $\text{Re}(z) < 0$, is an injection and therefore, the complex powers for positive real part are defined by the inverses of the above ones. The imaginary powers are defined by a deformation of the formula (2.1) and a closability argument. We refer to [1, Section III.4.6] for a detailed description. For technical purposes, we also recall the following elementary decay property of the resolvent of a sectorial operator involving the fractional powers.

Lemma 2.4. Let $E$ be a Banach, $A \in \mathcal{P}_{\kappa}(\theta)$, $\theta > 0$, and $\rho \in (0,1)$. Then, for any $\phi \in [0,\theta)$ and $\eta \in [0,1-\rho)$ we have that

$$
\| A^\phi(A + z)^{-1} \|_{L(E)} \leq \frac{\gamma}{1 + |z|^{\eta}}, \quad z \in S_{\phi},
$$

for some constant $\gamma > 0$ depending on the data $\kappa, \theta, \rho, \phi$ and $\eta$.

Proof. For some $\delta > 0$ sufficiently small (due to a sectoriality extension argument) and for any $z \in S_{\phi}$, by Cauchy’s theorem we have that

$$
A^\phi(A + z)^{-1} = \frac{1}{2\pi i} A \int_{-\delta + \Gamma_{\theta}} (-\lambda)^{\phi-1} (A + \lambda)^{-1} (A + z)^{-1} d\lambda
$$

$$
= \frac{1}{2\pi i} A \int_{-\delta + \Gamma_{\theta}} \frac{(-\lambda)^{\phi-1}}{z - \lambda} (A + \lambda)^{-1} (A + z)^{-1} d\lambda
$$

$$
= \frac{1}{2\pi i} A \int_{-\delta + \Gamma_{\theta}} \frac{(-\lambda)^{\phi-1}}{z - \lambda} (A + \lambda)^{-1} d\lambda - \frac{1}{2\pi i} A (A + z)^{-1} \int_{-\delta + \Gamma_{\theta}} \frac{(-\lambda)^{\phi-1}}{z - \lambda} d\lambda
$$

$$
= \frac{1}{2\pi i} A \int_{-\delta + \Gamma_{\theta}} \frac{(-\lambda)^{\phi-1}}{z - \lambda} (A + \lambda - \lambda)(A + \lambda)^{-1} d\lambda
$$

$$
= \frac{1}{2\pi i} A \int_{-\delta + \Gamma_{\theta}} \frac{(-\lambda)^{\phi-1}}{z - \lambda} \lambda d\lambda + \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta}} \frac{\lambda(-\lambda)^{\phi-1}}{\lambda - z} (A + \lambda)^{-1} d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta}} \frac{\lambda(-\lambda)^{\phi-1}}{\lambda - z} (A + \lambda)^{-1} d\lambda.
$$
Therefore, we obtain

\[ z^\eta A^\rho (A + z)^{-1} = \frac{1}{2\pi i} \int_{\delta - \Gamma} \frac{(\lambda)^\eta}{1 + \lambda} \lambda^{\rho + \eta - 1} (A - \lambda)^{-1} d\lambda, \]

and the estimate follows. \[ \Box \]

We introduce next a boundedness property for families of bounded operators with respect to orthonormal sets on an arbitrary measure spaces.

**Notation 2.5.** Denote by \( S = (\Omega, \Sigma, \mu) \) an arbitrary finite measure space and by \( \mathcal{E}_n = \{e_1, \ldots, e_n\}, n \in \mathbb{N} \), a finite sequence of vectors in \( L^\infty (\Omega; \mathbb{C}, d\mu) \) with \( \|e_k\|_{L^\infty (\Omega; \mathbb{C}, d\mu)} \leq 1 \), \( k \in \{1, \ldots, n\} \), such that all vectors in \( \mathcal{E}_n \) are orthonormal in \( L^2 (\Omega; \mathbb{C}, d\mu) \). Further, let \( E \) be a complex Banach space and denote by \( X_n = \{x_1, \ldots, x_n\} \) a finite sequence of vectors in \( E \). Also, if \( F \subset \mathcal{L}(E) \) is a family of bounded operators in \( E \), denote by \( T_n = \{T_1, \ldots, T_n\} \) a finite sequence of vectors in \( F \). Finally, we denote by \( L^2 (\Omega; E, d\mu) \) the Bochner space.

**Definition 2.6.** Let \( E \) be a complex Banach space and \( F \subset \mathcal{L}(E) \) be a family of bounded operators in \( E \). According to the previous notation, \( F \) is called orthonormally bounded with respect to the measure space \( S \) if for any triple \( \tau = (n, X_n, T_n) \) there exists some \( \mathcal{E}_n \) that depends on \( \tau \), such that

\[ \| \sum_{k=1}^n e_k T_k x_k \|_{L^2 (\Omega; E, d\mu)} \leq C_{F,S} \left( \sup_{a_k \in \mathbb{D}} \| \sum_{k=1}^n a_k x_k \|_E \right), \]

for some constant \( C_{F,S} \geq 1 \) which is called orthonormal bound or ON-bound and depends only on \( F \) and \( S \). If for some family \( F \) there exists some finite measure space \( S \) such that \( F \) is orthonormally bounded with respect to \( S \), then we say that \( F \) is orthonormally bounded or ON-bounded.

Actually in our estimates we will require a weaker boundedness condition than the ON-boundedness. This is described in the following lemma.

**Lemma 2.7.** Let \( E \) be a complex Banach space and \( F \subset \mathcal{L}(E) \) be ON-bounded with ON-bound equal to \( C_{F,S} \) with respect to some measure space \( S = (\Omega, \Sigma, \mu) \). Then, for any \( x_1, \ldots, x_n \in E \), \( x_1^*, \ldots, x_n^* \in E^* \) and \( T_1, \ldots, T_n \in F \), \( n \in \mathbb{N} \), we have that

\[ \left| \sum_{k=1}^n \langle T_k x_k, x_k^* \rangle \right| \leq \tilde{C}_{F,S} \left( \sup_{a_k \in \mathbb{D}} \| \sum_{k=1}^n a_k x_k \|_E \right) \left( \sup_{b_k \in \mathbb{D}} \| \sum_{k=1}^n b_k x_k^* \|_{E^*} \right), \]

where \( \tilde{C}_{F,S} = C_{F,S} (\text{Vol}(\Omega))^{\frac{1}{2}} \).
Proof. By Cauchy-Schwarz inequality we have that
\[
\left| \sum_{k=1}^{n} (T_k x_k, x_k^*) \right| \\
= \left| \int_{\Omega} \left( \sum_{i=1}^{n} e_i T_i x_i, \sum_{j=1}^{n} \bar{e}_j x_j^* \right) d\mu \right| \\
\leq \int_{\Omega} \left\| \sum_{i=1}^{n} e_i T_i x_i \right\| E \left\| \sum_{j=1}^{n} \bar{e}_j x_j^* \right\| E d\mu \\
\leq \left( \int_{\Omega} \left\| \sum_{i=1}^{n} e_i T_i x_i \right\|^2_E d\mu \right)^{\frac{1}{2}} \left( \int_{\Omega} \left\| \sum_{j=1}^{n} \bar{e}_j x_j^* \right\|^2_E d\mu \right)^{\frac{1}{2}},
\]
for certain vectors \( e_1, \ldots, e_n \) in \( L^\infty(\Omega; \mathbb{C}, d\mu) \) with \( \| e_k \|_{L^\infty(\Omega; \mathbb{C}, d\mu)} \leq 1, k \in \{1, \ldots, n\} \), such that \( e_1, \ldots, e_n \) are orthonormal in \( L^2(\Omega; \mathbb{C}, d\mu) \). Then, the estimate follows. \( \square \)

If we restrict the operator family to the case of the resolvent of an operator, we can generalize the notion of sectoriality as follows.

Definition 2.8. Let \( E \) be a Banach and \( A \in \mathcal{P}(\theta) \). We say that \( A \) is ON-sectorial of angle \( \theta \), and denote by \( A \in \text{ON}(\theta) \), if the family \( \{ \lambda(A + \lambda)^{-1} \mid \lambda \in S_\theta \setminus \{0\} \} \) is ON-bounded. In this case, we call the ON-bound as ON-sectorial bound.

Similarly to the sectoriality, the ON-sectoriality of an operator is preserved under appropriate shifts and the resulting ON-sectorial bound remains uniformly bounded.

Lemma 2.9. Let \( A : \mathcal{D}(A) \to E \) be ON-sectorial of angle \( \theta > 0 \), and let \( C_A \) be the ON-sectorial bound. If \( \omega \in [0, \min\{\theta, \pi - \theta\}] \), then for any \( c \in S_\omega \), \( A + c \) is ON-sectorial of angle \( \theta \) with ON-sectorial bound \( \leq \frac{C_A}{\sin(\theta + \omega)} \).

Proof. Let \( S = (\Omega, \Sigma, \mu) \) the finite measure space subject to the ON-sectoriality of \( A \). For any \( \lambda_1, \ldots, \lambda_n \in S_0 \setminus \{0\} \) and \( x_1, \ldots, x_n \in E \), \( n \in \mathbb{N} \), we have that
\[
\| \sum_{k=1}^{n} e_k \lambda_k (A + c + \lambda_k)^{-1} x_k \|_{L^2(\Omega; E, d\mu)} \\
= \| \sum_{k=1}^{n} e_k (c + \lambda_k) (A + c + \lambda_k)^{-1} \lambda_k \bar{e}_k x_k \|_{L^2(\Omega; E, d\mu)} \\
\leq C_A \sup_{a_k \in \mathcal{D}} \| \sum_{k=1}^{n} a_k \frac{\lambda_k}{c + \lambda_k} x_k \|_E,
\]
for some vectors \( e_1, \ldots, e_n \) in \( L^\infty(\Omega; \mathbb{C}, d\mu) \) with \( \| e_k \|_{L^\infty(\Omega; \mathbb{C}, d\mu)} \leq 1, k \in \{1, \ldots, n\} \), such that \( e_1, \ldots, e_n \) are orthonormal in \( L^2(\Omega; \mathbb{C}, d\mu) \). The result now follows by the estimate
\[
\sup_{\lambda \in S_\omega} | \frac{\lambda}{c + \lambda} | \leq \frac{1}{\sin(\theta + \omega)}.
\]
\( \square \)

We can further restrict the notion of ON-boundedness to the more flexible case where the vectors involving the estimate are taken from a fixed orthonormal set.
Definition 2.10. Let $E$ be a complex Banach space, $F \subset L(E)$ be a family of bounded operators in $E$, $S = (\Omega, \Sigma, \mu)$ be a finite measure space and $\mathcal{E} = \{e_k\}_{k=1}^\infty$ be a fixed orthonormal set in $L^2(\Omega; \mathbb{C}, d\mu)$ such that $e_k \in L^\infty(\Omega; \mathbb{C}, d\mu)$ with $\|e_k\|_{L^\infty(\Omega; \mathbb{C}, d\mu)} \leq 1$ for each $k \in \mathbb{N}$. Let $X_n = \{x_1, \ldots, x_n\}$ a finite sequence of vectors in $E$ and $T_n = \{T_1, \ldots, T_n\}$ be a finite sequence of vectors in $F$, $n \in \mathbb{N}$. We say that $F$ is $\mathcal{E}$-bounded if for any triple $\tau = (n, X_n, T_n)$ there exists a finite sequence $a_1, \ldots, a_n \in \mathbb{D}$ that depends on $\tau$, such that

$$\left\| \sum_{k=1}^n a_k e_k x_k \right\|_{L^2(\Omega; E, d\mu)} \leq C_{F, \mathcal{E}} \left\| \sum_{k=1}^n a_k e_k x_k \right\|_{L^2(\Omega; E, d\mu)},$$

for some constant $C_{F, \mathcal{E}} \geq 1$ which is called $\mathcal{E}$-bound and depends only on $F$ and $\mathcal{E}$. Further, an operator $A \in \mathcal{P}(\theta)$ in $E$ is called $\mathcal{E}$-sectorial of angle $\theta$, and we denote by $A \in \mathcal{E}(\theta)$, if the family $\{\lambda(A+\lambda)^{-1} \mid \lambda \in S_\theta \setminus \{0\}\}$ is $\mathcal{E}$-bounded. In this case, we call the $\mathcal{E}$-bound as $\mathcal{E}$-sectorial bound.

A typical example of an $\mathcal{E}$-sectorial operator is any $R$-sectorial operator, i.e. a sectorial operator $A$ where the family $\{\lambda(A+\lambda)^{-1} \mid \lambda \in S_\theta \setminus \{0\}\}$ is Rademacher bounded (see e.g. [S] Section 5). Due to the nice properties of the Rademacher functions, i.e. the Kahane’s contraction principle (see e.g. [9] Proposition 2.5), in this case the numbers $a_k$ in the above definition can be taken equal to one. Further, in [12] Theorem 2.8, it has been shown that if an operator $A$ defined in a $UMD$ (unconditionality of martingale differences property) space has bounded imaginary powers with power angle $\phi < \pi$, then the family $F_\theta = \{(1+re^{\theta+k+1\omega})A^{-1} \mid r \in [0,1], k \in \mathbb{N}\}$, for any $\theta \in (\phi-\pi, \pi-\phi)$, is $\mathcal{E}$-bounded with respect to the orthonormal set $\mathcal{E} = \{e^{ikt}/\sqrt{2\pi}\}_{k=0}^\infty$ in $L^2(0,2\pi)$. This together with [12] Theorem 2.6 showed that the sum of two resolvent commuting operators in a $UMD$ space such that one admits bounded $H^\infty$-calculus and the other one has bounded imaginary powers is closed and invertible, provided that the standard parabolic condition between the corresponding angles is satisfied.

The class of $\mathcal{E}$-sectorial operators behaves nicely in relatively small perturbations as we can see in the following elementary statement.

Proposition 2.11. Let $E$ be a complex Banach space, $A \in \mathcal{E}(\theta)$ with $\mathcal{E}$-sectorial bound equal to $C_A$, and $B$ be a linear operator in $E$ such that $D(A) \subseteq D(B)$ and $\|BA^{-1}\|_{L(E)} < 1/(1+C_A)$. Then $A + B \in \mathcal{E}(\theta)$ and the $\mathcal{E}$-sectorial bound of $A + B$ is $\leq C_A/(1 - (1+C_A)\|BA^{-1}\|_{L(E)})$.

Proof. Clearly, the family $\{A(A+\lambda)^{-1} \mid \lambda \in S_\theta \setminus \{0\}\}$ is also $\mathcal{E}$-bounded with $\mathcal{E}$-bound $\leq 1 + C_A$. Therefore, any $\lambda \in S_\theta$ belongs to $\rho(-(A + B))$ and the resolvent is given by the following absolutely convergent Neumann series

$$(A + B + \lambda)^{-1} = (A + \lambda)^{-1} \sum_{k=0}^\infty (-1)^k (B(A + \lambda)^{-1})^k.$$

Using successively the $\mathcal{E}$-boundedness of $A(A+\lambda)^{-1}$ we obtain the result. $\square$

In the sequel, we are going to consider sums of possibly non-commuting operators. In order to extend the existing theory from the case of commuting operators, we need to introduce some extra commutation condition. The condition we impose is certain decay property for the commutator of the resolvents of two sectorial operators that breaks down into three similar estimates. Namely, we consider the following.
**Condition 2.12.** Let $A : \mathcal{D}(A) \to E$, $B : \mathcal{D}(B) \to E$ are linear operators in a complex Banach space $E$ such that $A \in \mathcal{P}(\theta_A)$, $B \in \mathcal{P}(\theta_B)$ and $(B + \mu)^{-1} : \mathcal{D}(A) \to \mathcal{D}(A)$ for all $\mu \in S_{\theta_B}$. Assume that there exists some $C > 0$ and $\alpha_j, \beta_j \geq 0$, $j \in \{1, 2, 3\}$, such that

$$[(A + \lambda)^{-1}, (B + \mu)^{-1}] : S_{\theta_A} \times S_{\theta_B} \ni (\lambda, \mu) \mapsto \mathcal{L}(X_j, Y_j)$$

is a well defined Lebesgue measurable map and

$$\|[(A + \lambda)^{-1}, (B + \mu)^{-1}]\|_{\mathcal{L}(X_j, Y_j)} \leq \frac{C}{(1 + |\lambda|^{\alpha_j})(1 + |\mu|^{\beta_j})} \quad \text{when} \quad (\lambda, \mu) \in S_{\theta_A} \times S_{\theta_B},$$

where

1. $X_1 = Y_1 = E$, $\alpha_1 + \beta_1 > 2$, $\alpha_1 > 0$, $\beta_1 > 0$.
2. $X_2 = E$, $Y_2 = \mathcal{D}(A)$, $\alpha_2 + \beta_2 > 1$, $\alpha_2 \geq 0$, $\beta_2 > 0$.
3. $X_3 = Y_3 = \mathcal{D}(A)$, $\alpha_3 + \beta_3 > 2$, $\alpha_3 \geq 0$, $\beta_3 > 0$.

In view of the following formally written equalities

\begin{equation}
[(A + \lambda)^{-1}, (B + \mu)^{-1}]
= (A + \lambda)^{-1}(B + \mu)^{-1}[A, B]B^{-1(1+\nu)}B(B + \mu)^{-1}B^{-1}A^{-\rho}A^{-v}(A + \lambda)^{-1}
\end{equation}

and

\begin{equation}
[(A + \lambda)^{-1}, (B + \mu)^{-1}]
= A(A + \lambda)^{-1}(B + \mu)^{-1}[A, B]B^{-1(1+\gamma)}B(B + \mu)^{-1}B^{-1}B^{-\tau}B^{\gamma + \gamma}A^{-1}(A + \lambda)^{-1}
\end{equation}

for certain $\nu, \rho, \gamma, \tau \in (0, 1)$, by Lemma 2.4, we see that Condition 2.12 is fulfilled when the commutator $[A, B]$ is of lower order e.g. this can happen in the case of differential operators. Let us now consider one further condition in terms of real interpolation spaces that will turn out to be stronger than Condition 2.12.

**Condition 2.13.** Let $A : \mathcal{D}(A) \to E$, $B : \mathcal{D}(B) \to E$ are linear operators in a complex Banach space $E$ such that $A \in \mathcal{P}(\theta_A)$, $B \in \mathcal{P}(\theta_B)$ and $(B + \mu)^{-1} : \mathcal{D}(A) \to \mathcal{D}(A)$ for all $\mu \in S_{\theta_B}$. Assume that there exists some $\eta \in (0, 1)$, $p \in (1, \infty)$ and $M > 0$ such that $[A, (B + \mu)^{-1}] : S_{\theta_B} \ni \mu \mapsto \mathcal{L}(\mathcal{D}(A), (E, \mathcal{D}(A))_{\eta, p})$ is a well defined Lebesgue measurable map and

$$\|[(A, (B + \mu)^{-1}]]_{\mathcal{L}(\mathcal{D}(A), (E, \mathcal{D}(A))_{\eta, p})} \leq \frac{M}{1 + |\mu|}, \quad \mu \in S_{\theta_B}.$$ 

Further, assume that there exists some $\xi \in (0, 1)$ and $q \in (1, \infty)$ such that for each $\mu \in S_{\theta_B}$, $[A, (B + \mu)^{-1}] : \mathcal{D}(A) \to E$ admits a bounded extension $[A, (B + \mu)^{-1}] \in \mathcal{L}((E, \mathcal{D}(A))_{\xi, q}, E)$ such that $[A, (B + \mu)^{-1}] : S_{\theta_B} \ni \mu \mapsto \mathcal{L}((E, \mathcal{D}(A))_{\xi, q}, E)$ is Lebesgue measurable and

$$\|[(A, (B + \mu)^{-1}]_{\mathcal{L}((E, \mathcal{D}(A))_{\xi, q}, E)} \leq \frac{M}{1 + |\mu|}, \quad \mu \in S_{\theta_B}.$$ 

By \[1\] (I.2.5.2)], \[1\] (I.2.9.6)], the equalities

$$[(A + \lambda)^{-1}, (B + \mu)^{-1}] = (A + \lambda)^{-1}A^{-\nu}A^{-v}[(B + \mu)^{-1}, A](A + \lambda)^{-1}$$

and

$$[(A + \lambda)^{-1}, (B + \mu)^{-1}] = (A + \lambda)^{-1}[(B + \mu)^{-1}, A]A^{-\rho}A^{-\rho}(A + \lambda)^{-1},$$

with $\nu \in (0, \eta)$ and $\rho \in (\xi, 1)$, together with Lemma 2.4 imply the following.

**Remark 2.14.** Condition 2.13 implies Condition 2.12.
In the sequel, we will employ the standard Da Prato and Grisvard formula for the inverse of the closure of the sum of two resolvent commuting sectorial operators.

**Notation 2.15.** Let \( E \) be a complex Banach space and \( A, B \) are linear operators in \( E \) such that \( A \in \mathcal{P}(\theta_A) \) and \( B \in \mathcal{P}(\theta_B) \) with \( \theta_A + \theta_B > \pi \). Let \( c \in \mathbb{S}_\omega, \omega \in [0, \pi - \max\{\theta_A, \theta_B\}] \), and consider the bounded operators \( K_c, L_c \in \mathcal{L}(E) \) defined by

\[
K_c = \frac{1}{2\pi i} \int_{\Gamma_{\theta_B}} (A - z)^{-1} (B_c + z)^{-1} dz \quad \text{and} \quad L_c = \frac{1}{2\pi i} \int_{\Gamma_{\theta_B}} (B_c + z)^{-1} (A - z)^{-1} dz,
\]

where \( B_c = B + c \). By a sectoriality extension argument we can replace the path \( \Gamma_{\theta_B} \) in the above formulas by \( \Gamma_{\rho, \theta_B} \) or by \( \{\rho - \varepsilon, \rho + \varepsilon\} \), for sufficiently small \( \rho, \varepsilon > 0 \).

**Remark 2.16.** In the definition of \( K_c \), if we keep \( \theta_B \) fixed and replace \( B \) by a family of operators \( B(\xi) \in \mathcal{P}(\theta_B) \), \( \xi \in \Xi \), indexed by a set \( \Xi \), such that the sectorial bounds of \( B(\xi) \) are uniformly bounded in \( \xi \in \Xi \), then we can still replace \( \Gamma_{\theta_B} \) by \( \Gamma_{\rho, \theta_B} \) for some fixed \( \rho > 0 \) independent of \( \xi \).

Next, we record the following basic mapping property of the operator \( K_c \).

**Lemma 2.17.** Let \( E \) be a complex Banach space and \( A, B \) are linear operators in \( E \) such that \( A \in \mathcal{P}(\theta_A) \) and \( B \in \mathcal{P}(\theta_B) \) with \( \theta_A + \theta_B > \pi \). Then, the operator \( K_c \) maps \( D(B) \) to \( D(A) \).

**Proof.** If \( w \in \mathbb{C} \) such that \( \text{Re}(w) < 0 \), then

\[
K_c B^w = \frac{1}{2\pi i} \int_{\Gamma_{\theta_B} - \rho} (A - z)^{-1} \left( \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta_B}} (-\lambda)^w (B_c + z)^{-1} (B + \lambda)^{-1} d\lambda \right) dz
\]

where we have used Fubini’s theorem. By Cauchy’s theorem the first term in the right hand side of the above equation is zero. Therefore,

\[
K_c B^w = \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta_B}} (-\lambda)^w (A - z)^{-1} (B + \lambda)^{-1} d\lambda,
\]

which converges absolutely. By (2.3), we have that \( K_c B^w \) maps \( E \) to \( D(A) \). \( \square \)

Finally, for later use, we recall the following elementary commutation formula.

**Lemma 2.18.** Let \( E \) be a complex Banach space and \( A, B \) are linear operators in \( E \) with \( A \in \mathcal{P}(\theta_A) \). Then, for any \( f \in H^\infty_c(\phi), \phi \in [0, \theta_A) \), and \( \lambda \in \rho(-B) \neq \emptyset \) we have that

\[
[f(-A), (B + \lambda)^{-1}] = Q_f(\lambda),
\]
Proof. By the integral formula for the functional calculus of $A$ we estimate
\[
\begin{align*}
\quad f(-A)(B + \lambda)^{-1} \\
= \frac{1}{2\pi i} \int_{\Gamma_{\theta_A}} f(z) (A + z)^{-1}(B + \lambda)^{-1}dz \\
= (B + \lambda)^{-1} f(-A) + \frac{1}{2\pi i} \int_{\Gamma_{\theta_A}} f(z) ((A + z)^{-1}(B + \lambda)^{-1} - (B + \lambda)^{-1}(A + z)^{-1})dz.
\end{align*}
\]

\[\square\]

3. THE SUM OF NON-COMMUTING OPERATORS

In this section we consider sums of possibly non-commuting operators and show, under certain assumptions, closedness and invertibility. We follow three steps. First, under our commutation assumption, we perturb certain assumptions, closedness and invertibility. We follow three steps. First, under our commutation assumption, we perturb certain assumptions, closedness and invertibility. We follow three steps. First, under our commutation assumption, we perturb certain assumptions, closedness and invertibility. We follow three steps.

Proposition 3.1. Let $E$ be a complex Banach space and $A$, $B$ are linear operators in $E$ such that $A \in \mathcal{P}(\theta_A)$ and $B \in \mathcal{P}(\theta_B)$ with $\theta_A + \theta_B > \pi$. If Condition \[\text{2.12}\] is satisfied, then the operator $K_c$ maps the range $\text{Ran}(A + B_c)$ to $\mathcal{D}(A)$ and there exists some $P_c \in \mathcal{L}(E)$ such that
\[
AK_c(A + B_c) = (I + P_c)A \quad \text{in} \quad \mathcal{D}(A + B).
\]

Further, $\|P_c\|_{\mathcal{L}(E)} \to 0$ when $|c| \to \infty$.

Proof. If $\text{Re}(w) < 0$, we have that
\[
\begin{align*}
A^wK_c &= \frac{1}{2\pi i} \int_{\Gamma_{\theta_B}} A^w(A - z)^{-1}(B_c + z)^{-1}dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_{\theta_B}} \left( \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta_A}} (-\lambda)^w(A + \lambda)^{-1}(A - z)^{-1}d\lambda \right)(B_c + z)^{-1}dz \\
&= \frac{1}{(2\pi i)^2} \int_{\Gamma_{\theta_B}} \left( \int_{-\delta + \Gamma_{\theta_A}} (-\lambda)^w(\lambda + z)^{-1}(A - z)^{-1} - (A + \lambda)^{-1}d\lambda \right)(B_c + z)^{-1}dz \\
&= \frac{1}{(2\pi i)^2} \int_{\Gamma_{\theta_B}} \left( \int_{-\delta + \Gamma_{\theta_A}} (-\lambda)^w(\lambda + z)^{-1}(A - z)^{-1}(B_c + z)^{-1}d\lambda dz \right) \\
&\quad - \frac{1}{(2\pi i)^2} \int_{-\delta + \Gamma_{\theta_A}} \left( \int_{\Gamma_{\theta_B}} (-\lambda)^w(\lambda + z)^{-1}(A + \lambda)^{-1}(B_c + z)^{-1}dz d\lambda,\right.
\end{align*}
\]
where at the last step we have used Fubini’s theorem. By Cauchy’s theorem, the first term on the right hand side of the above equation is zero. Therefore we find that
\[ A^w K_c = \frac{1}{2\pi i} \int_{-\delta+\Gamma_{sp}} (-\lambda)^w (A + \lambda)^{-1} (B_c - \lambda)^{-1} d\lambda, \]
and hence if \( \theta \in (0,1) \) by Cauchy’s theorem we further obtain
\[ (3.5) \quad A^{-\theta} K_c = \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} (A - \lambda)^{-1} (B_c + \lambda)^{-1} d\lambda. \]
Then, if \( x \in D(A + B) \) we have that
\[
A^{-\theta} K_c (A + B_c)x
= \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} (A - \lambda)^{-1} B_c (B_c + \lambda)^{-1} x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} (B_c + \lambda)^{-1} A (A - \lambda)^{-1} x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] A x d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} (A - \lambda)^{-1} x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{1-\theta} (A - \lambda)^{-1} (B_c + \lambda)^{-1} x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda^{-\theta} [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] A x d\lambda
\]
By taking the limit in the above equation when \( \theta \to 0 \), since \( A^{-\theta} \to I \) strongly, by the dominated convergence theorem we obtain
\[ K_c (A + B_c)x \]
\[ (3.6) \quad x = \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] A x d\lambda, \]
where we have employed Condition 2.12 for the existence of the dominant and for the absolute convergence of the above integrals. Since by Condition 2.12 the integrals
\[ \int_{\Gamma_{sp}} \lambda A [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] x d\lambda \quad \text{and} \quad \int_{\Gamma_{sp}} A [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] A x d\lambda \]
converge absolutely, [3.10] implies that \( K_c (A + B_c) \) maps \( D(A + B) \) to \( D(A) \) and
\[ AK_c (A + B_c)x \]
\[ (3.7) \quad = (I - \frac{1}{2\pi i} \int_{\Gamma_{sp}} \lambda A [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] x d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma_{sp}} A [(A - \lambda)^{-1}, (B_c + \lambda)^{-1}] x d\lambda) A x, \]
for all $x \in \mathcal{D}(A + B)$. Further, by Condition $\text{2.12}$ the operator norm of

$$P_c = -\frac{1}{2\pi i} \int_{\Gamma_{\theta B}} \lambda A[[A - \lambda]^{-1}, (B_c + \lambda)^{-1}]A^{-1}d\lambda$$

(3.8)

$$+ \frac{1}{2\pi i} \int_{\Gamma_{\theta B}} A[[A - \lambda]^{-1}, (B_c + \lambda)^{-1}]d\lambda \in \mathcal{L}(E)$$

can become arbitrary small by taking $|c|$ sufficiently large.

In the same manner, we can build an approximation of the right inverse of the sum by applying the Da Prato and Grisvard formula to certain fractional powers as follows.

**Proposition 3.2.** Let $E$ be a complex Banach space and $A$, $B$ are linear operators in $E$ such that $A \in \mathcal{P}(\theta A)$ and $B \in \mathcal{P}(\theta B)$ with $\theta A + \theta B > \pi$. If Condition $\text{2.12}$ is satisfied, then the operator $L_c$ maps $\mathcal{D}(A)$ to $\mathcal{D}(A + B)$ and there exists some $T_c \in \mathcal{L}(E)$ such that

$$(A + B_c)L_c = I + T_c \quad \text{in} \quad \mathcal{D}(A).$$

Further, $||T_c||_{\mathcal{L}(E)} \to 0$ when $|c| \to \infty$.

**Proof.** We need an analogue of formula (3.7). If $\Re(w) < 0$, then

$$L_c A^w = \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta}} (B_c + z)^{-1} \left( \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta A}} (-\lambda)^w (A - z)^{-1} (A + \lambda)^{-1} d\lambda \right) dz$$

$$= \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta}} (B_c + z)^{-1} \left( \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta A}} (-\lambda)^w (\lambda + z)^{-1} ((A - z)^{-1} - (A + \lambda)^{-1}) d\lambda \right) dz$$

$$= \left( \frac{1}{2\pi i} \right)^2 \int_{-\delta + \Gamma_{\theta}} \int_{-\delta + \Gamma_{\theta A}} (-\lambda)^w (\lambda + z)^{-1} (B_c + z)^{-1} (A - z)^{-1} d\lambda dz$$

$$- \left( \frac{1}{2\pi i} \right)^2 \int_{-\delta + \Gamma_{\theta}} \int_{-\delta + \Gamma_{\theta A}} (-\lambda)^w (\lambda + z)^{-1} (B_c + z)^{-1} (A + \lambda)^{-1} dz d\lambda,$$

where we have used Fubini’s theorem. By Cauchy’s theorem the first term in the right hand side of the above equation is zero. Therefore,

(3.9) \quad $L_c A^w = \frac{1}{2\pi i} \int_{-\delta + \Gamma_{\theta A}} (-\lambda)^w (B_c - \lambda)^{-1} (A + \lambda)^{-1} d\lambda$.

Let $\theta \in (0, 1)$. Since the integral

$$\int_{\Gamma_{\theta}} (-\lambda)^{-\theta} B_c (B_c - \lambda)^{-1} (A + \lambda)^{-1} d\lambda$$

covers absolutely, by (3.9) we have that $L_c A^{-\theta}$ maps $E$ to $\mathcal{D}(B)$ and

$$B_c L_c A^{-\theta} = \frac{1}{2\pi i} \int_{\Gamma_{\theta}} (-\lambda)^{-\theta} (B_c - \lambda + \lambda) (B_c - \lambda)^{-1} (A + \lambda)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\theta}} (-\lambda)^{-\theta} (A + \lambda)^{-1} d\lambda - \frac{1}{2\pi i} \int_{\Gamma_{\theta}} (-\lambda)^{1-\theta} (B_c - \lambda)^{-1} (A + \lambda)^{-1} d\lambda$$

(3.10) \quad $A^{-\theta} = \frac{1}{2\pi i} \int_{\Gamma_{\theta}} (-\lambda)^{1-\theta} (B_c - \lambda)^{-1} (A + \lambda)^{-1} d\lambda.$
Moreover, by Condition \(2.12\) the integrals
\[
\int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A(A + \lambda)^{-1}(B_c - \lambda)^{-1}d\lambda \quad \text{and} \quad \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda
\]
converge absolutely. Hence, by (3.9), we find that \(L_c A^{-\theta}\) maps \(E\) to \(\mathcal{D}(A)\) and
\[
AL_c A^{-\theta} = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta}(A + \lambda - \lambda)(A + \lambda)(B_c - \lambda)^{-1}d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta}(A + \lambda)^{-1}(B_c - \lambda)^{-1}d\lambda
\]

(3.11)

\[
- \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda.
\]

Therefore, by (3.10) and (3.11) we obtain that
\[
(A + B_c) L_c A^{-\theta} = A^{-\theta} + \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{1-\theta}(A + \lambda)^{-1}(B_c - \lambda)^{-1}d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda.
\]

Hence, if \(x \in \mathcal{D}(A)\), we have that
\[
(A + B_c) L_c x = x + \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{1-\theta}(A + \lambda)^{-1}(B_c - \lambda)^{-1}A^\theta x d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} (-\lambda)^{-\theta} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]A^\theta x d\lambda.
\]

By the Dunford integral for the complex powers and the dominated convergence theorem, for any \(x \in \mathcal{D}(A)\) we have that \(A^\theta x = A^\theta x \to x\) when \(\theta \to 0\). Hence, by taking the pointwise limit in the above equation, by the dominated convergence theorem we obtain that
\[
(A + B_c) L_c x
\]

\[
= (I - \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} \lambda[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda - \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda) x,
\]

for all \(x \in \mathcal{D}(A)\), where we have used Condition \(2.12\) for the existence of the dominant and for the absolute convergence of the last integrals in the operator norm. Finally, by Condition \(2.12\) the operator norm of
\[
T_c = - \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} \lambda[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda
\]

(3.12)

\[- \frac{1}{2\pi i} \int_{\Gamma_{\epsilon A}} A[(A + \lambda)^{-1}, (B_c - \lambda)^{-1}]d\lambda \in \mathcal{L}(E),\]

can become arbitrary small by taking \(|c|\) large enough.
We are now in the position to impose further assumptions to our operators and make the above constructed perturbations to serve as the left and the right inverse of the sum. The main task is to show that the left inverse approximation, which is given by Proposition 3.1, becomes a bounded operator. We manage this by decomposing dyadically the integral representation of the unbounded part and then using the consequences of $ON$-boundedness and the boundedness of the $H^\infty$-calculus.

**Theorem 3.3.** Let $E$ be a complex Banach space and $A, B$ are linear operators in $E$ such that $A \in H^\infty(\theta_A)$ and $B \in ON(\theta_B)$ with $\theta_A + \theta_B > \pi$. If Condition 2.12 is satisfied, then $A + B$ is closed and there exists some $c_0 \geq 0$ such that $\sigma(A + B + c_0) \in S_{\max\{\theta_A, \theta_B\}}$. Further, for any $\omega \in [0, \pi - \max\{\theta_A, \theta_B\})$, we have that $A + B + c_0 \in \mathcal{P}(\omega)$.

**Proof.** Let $\psi_A = \theta_A - \varepsilon$, with $\varepsilon > 0$ sufficiently small. Since the integral

$$\int_{-\Gamma_{\psi_A}} \lambda^{-\theta} A(A - \lambda)^{-1}(B_c + \lambda)^{-1} d\lambda$$

converges absolutely when $\theta \in (0, 1)$, by (3.3) we have that $A^{-\theta} K_c$ maps $E$ to $\mathcal{D}(A)$ and

$$AA^{-\theta} K_c = U_\theta + G_\theta,$$

with

$$U_\theta = \frac{1}{2\pi i} \int_{-\Gamma_{\psi_A} \cap \mathbb{D}} \lambda^{1-(\theta + \phi)} A^\phi (A - \lambda)^{-1}(B_c + \lambda)^{-1} d\lambda$$

and

$$G_\theta = \frac{1}{2\pi i} \int_{-\Gamma_{\psi_A} \cap (\mathbb{C} \setminus \mathbb{D})} \lambda^{1-(\theta + \phi)} A^\phi (A - \lambda)^{-1}(B_c + \lambda)^{-1} d\lambda,$$
where we have used Lemma 2.4 for the absolute convergence of the above integral. For any \( m \in \mathbb{N}, m \geq 1 \), define
\[
G_m^\pm = \frac{e^{\pm i(\pi - \psi_A)(2 - \theta + \phi)}}{2\pi i} \int_1^{2^m} A^\phi(A - re^{\pm i(\pi - \psi_A)})^{-1}(B_c + re^{\pm i(\pi - \psi_A)})^{-1}r^{1 - (\theta + \phi)}dr
\]
\[= \frac{e^{\pm i(\pi - \psi_A)(2 - \theta + \phi)}}{2\pi i} \sum_{k=0}^{m-1} \int_k^{2^k + 1} A^\phi(A + re^{-\pi i\psi_A})^{-1}(B_c - re^{-\pi i\psi_A})^{-1}r^{1 - (\theta + \phi)}dr
\]
\[= \frac{e^{\pm i(\pi - \psi_A)(2 - \theta + \phi)}}{2\pi i} \sum_{k=0}^{m-1} \int_1^{2^k} A^\phi(A + t2^ke^{-\pi i\psi_A})^{-1}(B_c - t2^ke^{-\pi i\psi_A})^{-1}t^{1 - (\theta + \phi)}2^k(2 - \theta + \phi)dt
\]
(3.14) \[= \frac{e^{\pm i(\psi_A - \pi)(\theta + \phi)}}{2\pi i} \int_1^2 W_m^\pm(t)\frac{dt}{t},
\]
where
\[
W_m^\pm(t) = \sum_{k=0}^{m-1} A^\phi(A + t2^ke^{-\pi i\psi_A})^{-1}(B_c - t2^ke^{-\pi i\psi_A})^{-1}t^{2 - (\theta + \phi)}2^k(2 - (\theta + \phi))e^{\mp 2\pi i\psi_A},
\]
and take any \( x \in E, x^* \in E^* \). We proceed now as in [8] in order to estimate uniform bounds for the above operator families. More precisely we have that
\[
|\langle W_m^\pm(t)x, x^* \rangle|
\]
\[= |\sum_{k=0}^{m-1} \left( t\frac{1 - (\theta + \phi)}{2}\right)^k e^{-\pi i\psi_A} A^\phi(A + t2^ke^{-\pi i\psi_A})^{-1}(B_c - t2^ke^{-\pi i\psi_A})^{-1}(A^*)^\phi(A^* + t2^ke^{-\pi i\psi_A})^{-1}\frac{2\pi i}{q} x^*|,
\]
with \( p, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Denote
\[
f_{k,p}^\pm(z) = t^{-\frac{2 - k}{2}} \frac{1 - (\theta + \phi)}{2} e^{-\pi i\psi_A} h_j^\pm(z^{1 - 2^-k}),
\]
where
\[
h_j^\pm(w) = ((-w)^\phi(-w + e^{\pi i\psi_A})^{-1})\frac{1}{2} \in H_0^\phi(\theta A - \frac{\phi}{2}), j \in \{p, q\}.
\]
Then, Lemma 2.18 implies
\[
|\langle W_m^\pm(t)x, x^* \rangle|
\]
\[= |\sum_{k=0}^{m-1} f_{k,p}^\pm(-A)(B_c - t2^ke^{-\pi i\psi_A})^{-1}t2^ke^{-\pi i\psi_A}x, f_{k,q}^\pm(-A^*)x^*|\]
\[\leq |\sum_{k=0}^{m-1} \langle (B_c - t2^ke^{-\pi i\psi_A})^{-1}t2^ke^{-\pi i\psi_A}f_{k,p}^\pm(-A)x, f_{k,q}^\pm(-A^*)x^*|\]
\[+ |\sum_{k=0}^{m-1} (Q f_{k,p}^\pm(c - t2^ke^{-\pi i\psi_A})t2^ke^{-\pi i\psi_A}x, f_{k,q}^\pm(-A^*)x^*)|,
\]
By Lemma 2.14, $B_r$ is again in $\mathcal{ON}(\theta B)$ and its $\mathcal{ON}$-sectorial bound is uniformly bounded in $c$. Hence, by Lemma 2.27 we obtain

\[
|\langle W_{m}^{\pm}(t)x,x^{*}\rangle| 
\leq C \left( \sup_{a_{k}\in\mathbb{D}} \| \sum_{k=0}^{m-1} a_{k} f_{k}^{\pm}(\cdot-A)x \|_{E} \right) \left( \sup_{b_{k}\in\mathbb{D}} \| \sum_{k=0}^{m-1} b_{k} f_{k-1}^{\pm}(\cdot-A^{*})x^{*} \|_{E^{*}} \right)
\]

(3.15) \quad + C \left( \sum_{k=0}^{m-1} a_{k} Q f_{k}^{\pm}(c-t^{2k}e^{\mp i\psi A})t^{2k}e^{\mp i\psi A} \right) \left( \sum_{k=0}^{m-1} b_{k} f_{k-1}^{\pm}(\cdot-A^{*})x^{*} \|_{E^{*}} \right),

for some constant $C_B > 0$ that depends only on the $\mathcal{ON}$-sectorial bound of $B$. Concerning the family $Q f_{k}^{\pm}(c-t^{2k}e^{\mp i\psi A})t^{2k}e^{\mp i\psi A}$ of bounded operators, by Condition 2.12 we estimate

\[
\| Q f_{k}^{\pm}(c-t^{2k}e^{\mp i\psi A})t^{2k}e^{\mp i\psi A} \|_{E^{(E)}} 
\leq C \frac{t^{2k}\|f_{k}^{\pm}(z)\|}{(1+|z|^{\alpha_{1}})(1+|c-t^{2k}e^{\mp i\psi A}|^{\beta_{1}})}dz
\]

\[
\leq C \frac{t^{-\frac{n}{2}}2^{-\frac{k}{2}}}{2\pi} \int_{\Gamma_{\theta A}} \frac{t^{2k}|zt^{-1/2-i\theta}|^{\frac{\alpha_{1}}{2}}}{(1+|z|^{\alpha_{1}})(1+|c-t^{2k}e^{\mp i\psi A}|^{\beta_{1}})|zt^{-1/2-i\theta}e^{\mp i\psi A}|^{\frac{\beta_{1}}{2}}})dz.
\]

By changing variables and taking appropriate values of $p$ and $\phi$, the last integral in the above inequality converges absolutely and it is uniformly bounded in $t$ and $\theta$ by $2^{(2-\alpha_{1}-\beta_{1})k}$, for each $k$. More precisely, by possibly increasing $C$ we can assume that $\alpha_{1} < 2$, and then, by taking $p$ close to 1 and $\phi$ closed to 0 when $\alpha_{1} < 1$ and $p$, $\phi$ both closed to 1 when $\alpha_{1} \geq 1$, we have that

\[
\| Q f_{k}^{\pm}(c-t^{2k}e^{\mp i\psi A})t^{2k}e^{\mp i\psi A} \|_{E^{(E)}} \leq C_{Q} 2^{(2-\alpha_{1}-\beta_{1})k}
\]

for some constant $C_Q$ independent of $t$, $\theta$ and $k$. Therefore, (3.13) and Lemma 2.25 imply that

\[
|\langle W_{m}^{\pm}(t)x,x^{*}\rangle| \leq C_{A,B}\|x\|_{E}\|x^{*}\|_{E^{*}},
\]

with some constant $C_{A,B}$ independent of $m$, $t$ and $\theta$. Hence, by (3.14) we have

\[
\| G_{m}^{\pm}x\|_{E} \leq \frac{C_{A,B}}{2\pi}\|x\|_{E},
\]

and by taking the limit as $m \to \infty$ we obtain

(3.16) \quad \| G_{\theta}x\|_{E} \leq \frac{C_{A,B}}{\pi}\|x\|_{E}.

Clearly,

(3.17) \quad \| U_{\theta}x\|_{E} \leq C'_{A,B}\|x\|_{E},

for some constant $C'_{A,B}$ independent of $\theta$. Therefore, (3.13), (3.16) and (3.17) imply that

\[
\| AA^{-\theta}K_{x}\|_{E} \leq \tilde{C}_{A,B}\|x\|_{E},
\]

with some constant $\tilde{C}_{A,B}$ independent of $\theta$. By Lemma 2.17 if $y \in D(B)$, we have that

\[
\| A^{-\theta}K_{x}y\|_{E} \leq \tilde{C}_{A,B}\|y\|_{E},
\]
that

By combining the above equations we find that

and

the above integral converges absolutely and it is uniformly bounded in

AK

Hence, (3.20) and (3.21) can be improved to

the closedness of

for some sufficiently small fixed

ρ>

later on.

(3.21)

θ

and by taking the limit as \( \theta \to 0 \) we obtain

(3.18)

\[ \|AKc\|_E \leq \tilde{C}_{A,B}\|y\|_E. \]

By the closedness of \( A \) and a density argument, we conclude that \( K_c \) maps \( E \) to \( \mathcal{D}(A) \) and \( AK_c \in \mathcal{L}(E) \). By taking \( |c| \) sufficiently large, from Proposition 3.1 and Proposition 3.2 we have that

(3.19)

\[ A^{-1}(I + P_c)^{-1}AK_c(A + B_c) = I \quad \text{in} \quad \mathcal{D}(A + B). \]

and

(3.20)

\[ (A + B_c)L_c = I + T_c \quad \text{in} \quad \mathcal{D}(A). \]

By combining the above equations we find that

(3.21)

\[ A^{-1}(I + P_c)^{-1}AK_c(I + T_c) = L_c \quad \text{in} \quad \mathcal{D}(A). \]

By the boundedness of \( AK_c \) the above formula also holds in \( E \). Therefore, we conclude that \( L_c \) maps \( E \) to \( \mathcal{D}(A) \) and \( AL_c \in \mathcal{L}(E) \). Then, by (3.20), i.e. by

\[ B_cL_c = I + T_c - AL_c \quad \text{in} \quad \mathcal{D}(A), \]

the closedness of \( B \) and a density argument we conclude also that \( L_c \) maps \( E \) to \( \mathcal{D}(B) \) and \( B_cL_c \in \mathcal{L}(E) \). The right inverse of \( A + B_c \) then follows by (3.20) and the invertibility of \( I + T_c \).

Hence, (3.20) and (3.21) can be improved to

\[ (A + B_c)L_c = I + T_c \quad \text{in} \quad E, \]

and

\[ A^{-1}(I + P_c)^{-1}AK_c(I + T_c) = L_c \quad \text{in} \quad E. \]

By the invertibility of \( I + T_c \), the last equality implies that the left inverse of \( A + B_c \), which is given by (3.19), maps to \( \mathcal{D}(A + B) \), and therefore closedness follows.

Now for the sectoriality of \( A + B + c_0 \), for sufficiently large \( c_0 > 0 \), we first note by Proposition 3.2 that the norm \( \| (I + T_c)^{-1} \|_{\mathcal{L}(E)} \) is uniformly bounded in \( |c| \in [c_0, \infty). \) By changing \( z = |c|\mu \) in the formula for \( L_c \), we find that

\[ L_c = \frac{1}{2\pi i} \int_{\Gamma_{\rho}} (B + c + |c|\mu)^{-1}(A - |c|\mu)^{-1}|c|d\mu \]

\[ = \frac{1}{2\pi i} \int_{\Gamma_{\rho}} (B + c + |c|\mu)^{-1}(A - |c|\mu)^{-1}|c|d\mu, \]

for some sufficiently small fixed \( \rho > 0 \). Therefore, by standard sectoriality we estimate

(3.22)

\[ \|L_c\|_{\mathcal{L}(E)} \leq \frac{\kappa_{AKB}}{2\pi|c|} \int_{\Gamma_{\rho}} \frac{\kappa_B}{(|c|^{-1} + |\mu + c|^{-1})(|c|^{-1} + |\mu|)}d\mu, \]

where \( \kappa_A \) and \( \kappa_B \) are the sectorial bounds of \( A \) and \( B \) respectively. If \( c = c_0 + \nu, \nu \in S_\omega, \) then the above integral converges absolutely and it is uniformly bounded in \( |\nu| \). The result now follows by

\[ (A + B_c)^{-1} = L_c(I + T_c)^{-1}. \]

\[ \square \]

From the proof of the above theorem we can make the following observation, which we use later on.
Lemma 3.4. Suppose that in the assumptions of Theorem 3.3 we keep \( \theta_B \) fixed and replace \( B \) by a family of operators \( B(\xi) \in \text{ON}(\theta_B) \), \( \xi \in \Xi \), indexed by a set \( \Xi \), such that \( A \) and \( B(\xi) \) are resolvent commuting for each \( \xi \). Then,

(i) The shifts \( c_0 \) can be chosen to be equal to zero for each \( \xi \in \Xi \).

(ii) If the sectorial bounds and the ON-sectorial bounds of \( B(\xi) \) are uniformly bounded in \( \xi \in \Xi \), then the sectorial bounds of \( A + B(\xi) \in \mathcal{P}(0) \) can be chosen to be uniformly bounded in \( \xi \). Further, the \( L(E) \) norms of \( (A + B(\xi) + \nu)^{-1} \) are uniformly bounded in \( (\xi, \nu) \in \Xi \times [0, \infty) \).

Proof. In the proof of Theorem 3.3 \( c_0 \) was taken large enough in order to make the \( L(E) \) norms of the operators \( P_c \) and \( T_c \) from Proposition 3.1 and Proposition 3.2 respectively, sufficiently small. Since \( A \) and \( B(\xi) \) are resolvent commuting for each \( \xi \), by [11, III.4.9.1 (ii)], (3.8) and (3.12) we clearly have that \( P_c = T_c = 0 \). Therefore we can take \( c_0 = 0 \) for all \( \xi \) (see also [1, Theorem 3.7]).

Further, only the sectorial bounds of the two summands contribute to the estimate (3.22),

where \( p \) can be chosen to be fixed due to Remark 2.16. Hence, the sectorial bounds of \( A + B(\xi) \) can be chosen to be uniformly bounded in \( \xi \). Finally, the \( L(E) \) norm of \( B(\xi)(A + B(\xi) + \nu)^{-1} \) can be estimated by the sectorial bound of \( A + B(\xi) \) and the \( L(E) \) norm of \( A(A + B(\xi) + \nu)^{-1} \). By 3.18, \( \xi \) and \( \nu \) contribute to the estimate of the last norm only by the ON-sectorial bound of \( B(\xi) \). Here we have noted that the operator \( Q_f \) from Lemma 2.18 is zero in our case. Thus, the \( L(E) \) norm of \( B(\xi)(A + B(\xi) + \nu)^{-1} \) is uniformly bounded in \( (\xi, \nu) \in \Xi \times [0, \infty) \).

\[ \Box \]

4. An Application to the Abstract Linear Non-Autonomous Parabolic Problem

In this section, we apply the previous result on the closedness and invertibility of the sum of two closed operators in order to recover a classical result on the existence, uniqueness and maximal \( L^p \)-regularity of solution for the abstract non-autonomous linear parabolic equation.

We will require the natural extensions of certain operators from the original space to the Bochner space to be ON-sectorial. Therefore, we restrict to an ideal case of \( R \)-sectorial operators, namely to the case of \( R \)-sectorial operators. We start by recalling the definition of this property.

Definition 4.1. An operator \( A \in \mathcal{P}(\theta) \) in a complex Banach space \( E \) is called \( R \)-sectorial of angle \( \theta \), if for any choice of \( \lambda_1, \ldots, \lambda_n \in S_\theta \setminus \{0\} \) and \( x_1, \ldots, x_n \in E \), \( n \in \mathbb{N} \), we have that

\[
\left\| \sum_{k=1}^{n} \epsilon_k \lambda_k (A + \lambda_k)^{-1} x_k \right\|_{L^2(\theta; E)} \leq C_{A, R} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|_{L^2(\theta; E)}^1,
\]

for some constant \( C_{A, R} \geq 1 \), called \( R \)-bound, and the sequence \( \{\epsilon_k\}_{k=1}^\infty \) of the Rademacher functions.

Let \( T > 0 \), \( E_1 \overset{d}{\rightarrow} E_0 \) be a densely and continuously injected Banach couple and let some continuous family of linear operators \( A(t) \in C([0, T]; \mathcal{L}(E_1, E_0)) \). Consider the Cauchy problem

\begin{align*}
(4.23) & \quad u'(t) + A(t) u(t) = g(t), \quad t \in (0, T], \\
(4.24) & \quad u(0) = 0,
\end{align*}

in the space \( L^p(0, T; E_0) \), where \( p \in (1, \infty) \). We need to combine Theorem 3.3 together with an old method for the construction of the full resolvent of a parametric family of sectorial operators, with the parameter over a topological space, based on standard sectoriality perturbation...
and a Neumann series representation argument (see e.g. [9], Theorem 5.7), in order to show well-posedness for the above problem. Hence, we wish to recover the result of [3, Theorem 2.7] for the case of continuously dependent family $A(t)$ over the non-autonomous parameter. For an alternative approach to the problem, we also refer to [2].

**Theorem 4.2.** Assume that $E_0$ is UMD and that there exists some $\theta > \frac{\omega}{2}$ such that for each $t \in [0,T]$, $A(t)$ is $R$-sectorial of angle $\theta$. Then, the problem (4.23)-(4.24) is well-posed, i.e. for any $g \in L^p(0,T;E_0)$ there exists a unique solution $u \in W^{1,p}(0,T;E_0) \cap L^p(0,T;E_1)$ that depends continuously on $g$.

**Proof.** Let the Banach spaces $X_0 = L^p(0,T;E_0)$, $X_1 = L^p(0,T;E_1)$ and $X_2 = \{u \in W^{1,p}(0,T;E_0) \mid u(0) = 0\}$.

Let the operators in $X_0$ defined by

$$A(t) : u(t) \mapsto (Au)(t) = A(t)u(t) \quad \text{with} \quad \mathcal{D}(A(t)) = X_1$$

and

$$B : u(t) \mapsto \partial_t u(t) \quad \text{with} \quad \mathcal{D}(B) = X_2.$$ 

For any fixed $\xi \in [0,T]$, the operator $A(\xi) : E_1 \to E_0$ is $R$-sectorial of angle $\theta$. Further, the sectorial bounds and the $R$-bounds of $A(\xi)$ can be chosen to be uniformly bounded in $\xi$ when $\xi \in [0,T]$. This is easy to see by the continuity of $A(t)$. For convenience, for e.g. the $R$-bounds we argue by contradiction as follows. For each $\xi$, let $\inf C_\xi$ be the infimum of all $R$-bounds of $A(\xi)$. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence in $[0,T]$ such that $\inf C_{\xi_k} \to \infty$ when $k \to \infty$. By passing to a subsequence we can assume that $\xi_k \to \bar{\xi}$ as $k \to \infty$. We have that $A(\bar{\xi})$ is $R$-sectorial with $R$-bound equal to $C_{\bar{\xi}}$. Let $r > 0$ be sufficiently small such that $t \in [\bar{\xi} - r, \bar{\xi} + r]$ (with $[0, \bar{\xi} + r]$ instead of $[\bar{\xi} - r, \bar{\xi} + r]$ when $\bar{\xi} = 0$, and similarly when $\bar{\xi} = T$) ensures that $\|A(\bar{\xi}) - A(t)\|_{\mathcal{L}(E_1,E_0)} < \kappa$, where

$$0 < \kappa < \frac{1}{(1 + C_{\bar{\xi}})\|A(\bar{\xi})^{-1}\|_{\mathcal{L}(E_0,E_1)}}.$$ 

Then, Proposition 2.11 implies that when $t \in [\bar{\xi} - r, \bar{\xi} + r]$, we also have that $A(t)$ is $R$-sectorial and its $R$-bound is uniformly bounded by

$$\frac{C_{\bar{\xi}}}{1 - \kappa(1 + C_{\bar{\xi}})\|A(\bar{\xi})^{-1}\|_{\mathcal{L}(E_0,E_1)},}$$

which gives us the contrary. Therefore, each extension of $A(\xi)$ from $X_1$ to $X_0$ given by $(A(\xi)u)(t) = A(\xi)u(t)$, which we denote again by $A(\xi)$, is also $R$-sectorial of angle $\theta$ and the sectorial bounds together with the $R$-bounds are uniformly bounded in $\xi$.

Since $E_0$ is UMD, by Theorem 8.5.8 in [7], for any $\omega < \frac{\pi}{\theta}$, the operator $B$ admits bounded $H^\infty$-calculus of angle $\omega$. For any $\xi \in [0,T]$, the operators $A(\xi) \text{ and } B$ are resolvent commuting and hence by Theorem 3.3 for each $\xi$ there exists come $c_0(\xi) \geq 0$ such that $A(\xi) + B + c_0(\xi)$ with $\mathcal{D}(A(\xi)+B+c_0(\xi)) = X_1 \cap X_2$ in $X_0$ is closed and belongs to the class $\mathcal{P}(0)$. Further, by Lemma 1.7 all $c_0(\xi)$ can be chosen equal to zero and the $\mathcal{L}(X_0)$ norms of $A(\xi)(A(\xi)+B+c)^{-1} \text{ are uniformly bounded in } (\xi,c) \in [0,T] \times [0,\infty).$

Take $t_1,\ldots,t_n \in [0,T]$, $r_1,\ldots,r_n \in (0,1)$, $n \in \mathbb{N}$, and let $\{\chi_i\}_{i=1}^{n}$ be a collection of smooth positive functions bounded by one such that $\chi_i = 1$ in $[t_i - r_i, t_i + r_i]$ and $\chi_i = 0$ outside of $[t_i - 2r_i, t_i + 2r_i]$, for each $i$. Let $\{\phi_i\}_{i=1}^{n}$ and $\{\psi_i\}_{i=1}^{n}$ be two further collections of smooth positive functions such that $\text{supp } \phi_i \subset \text{supp } \chi_i$, $\text{supp } \psi_i \subset \text{supp } \chi_i$, $\chi_i = 1$
on $\text{supp} \psi_i$ and $\psi_i = 1$ on $\text{supp} \phi_i$, for each $i$. Increase $n$ and take $\{t_1, ..., t_n\}$, $\{r_1, ..., r_n\}$ in such a way that $\{\phi_i\}_{i \in \{1, ..., n\}}$ is a partition of unity in $[0, T]$. By the argument in the previous paragraph, each $A(t_i) + B$ belongs to $\mathcal{P}(0)$. Define

$$A_i = \chi_i(A(t_i) + B) + (1 - \chi_i)(A(t_i) + B) = A(t_i) + B + \chi_i(A(t) - A(t_i)),$$

with $\mathcal{D}(A_i) = X_1 \cap X_2$ in $X_0$ for each $i \in \{1, ..., n\}$. By taking $\max |r_i|$ sufficiently small (and possibly $n$ large enough), from the continuity of $A(t)$ and the uniform boundedness in $(\xi, c) \in [0, T] \times [0, \infty)$ of the $\mathcal{L}(X_0)$ norm of $A(\xi)(A(\xi) + B + c)^{-1}$, we have that each $A_i$ belongs again to $\mathcal{P}(0)$. More precisely, we have that

$$(A_i + c)^{-1} = (A(t_i) + B + c)^{-1} \sum_{k=0}^{\infty} (-1)^k \left(\chi_i(A(t) - A(t_i))(A(t_i) + B + c)^{-1}\right)^k,$$

for any $c \geq 0$, provided that we have taken $\|\chi_i(A(t) - A(t_i))(A(t_i) + B + c)^{-1}\|_{X_0} \leq \nu$, for some fixed $\nu \in (0, 1)$.

Take $u \in X_1 \cap X_2$, $g \in X_0$ and $c \geq 0$. By multiplying

$$(A(t) + B + c)u = g$$

with $\phi_i$ we get that

$$(A(t) + B + c)\phi_i u = \phi_i g + [A(t) + B + c, \phi_i]u = \phi_i g + \phi_i' u,$$

where we have noted that the commutator $[A(t) + B + c, \phi_i]$ acts by multiplication with the derivative $\phi_i'$ of $\phi_i$. By applying the inverse of $A_i + c$, we obtain

$$\phi_i u = (A_i + c)^{-1}(\phi_i g + \phi_i' u),$$

and therefore

$$\phi_i u = \psi_i(A_i + c)^{-1}(\phi_i g + \phi_i' u).$$

By summing up we finally find that

$$u = \sum_{i=1}^{n} \psi_i(A_i + c)^{-1}\phi_i g + \sum_{i=1}^{n} \psi_i(A_i + c)^{-1}\phi_i' u. \tag{4.25}$$

The $\mathcal{L}(X_1 \cap X_2)$ norm of $(A_i + c)^{-1}$ decays like $c^{-1}$ when $c \to \infty$, due to the sectoriality of $A_i$ in $X_0$. Further, multiplication by $\phi_i'$ induces a bounded map in $X_1 \cap X_2$. Hence, by taking $c$ sufficiently large, from (4.25) we obtain a left inverse of $A(t) + B + c$ that maps to $X_1 \cap X_2$.

Denote this left inverse by $L$. Then,

$$(A(t) + B + c)L = (A(t) + B + c)\sum_{i=1}^{n} \psi_i(A_i + c)^{-1}(\phi_i + \phi_i' L)$$

$$= \sum_{i=1}^{n} \psi_i(A(t) + B + c)(A_i + c)^{-1}(\phi_i + \phi_i' L)$$

$$+ \sum_{i=1}^{n} [A(t) + B + c, \psi_i](A_i + c)^{-1}(\phi_i + \phi_i' L)$$

$$= \sum_{i=1}^{n} \psi_i \phi_i + \sum_{i=1}^{n} \psi_i \phi_i' L + \sum_{i=1}^{n} \psi_i(A_i + c)^{-1}(\phi_i + \phi_i' L). \tag{4.26}$$
Note that $\sum_{i=1}^{n} \psi_i \phi_i = 1$ and $\sum_{i=1}^{n} \psi_i \phi'_i = 0$. Also, by the sectoriality of $A_i$, we have that the $\mathcal{L}(X_0)$ norm of $(A_i + c)^{-1}$ tends to zero when $c \to \infty$. Therefore, (4.26) provides us, by possibly increasing $c$, a right inverse of $B + A(t) + c$. The result now follows by changing of variables $u(t) \to e^t v(t)$ in (4.23). \hfill \Box

References

[1] H. Amann. Linear and quasilinear parabolic problems Vol. I. Monographs in Mathematics Vol. 89, Birkhäuser Verlag (1995).
[2] H. Amann. Maximal regularity for nonautonomous evolution equations. Advanced Nonlinear Studies 4, 417–430 (2004).
[3] W. Arendt, R. Chill, S. Fornaro and C. Poupaud. $L^p$-maximal regularity for non-autonomous evolution equations. J. Differential Equations 237, no. 1, 1–26 (2007).
[4] G. Da Prato and P. Grisvard. Sommes d’opérateurs linéaires et équations différentielles opérationnelles. J. Math. Pures Appl. (9) 54, no. 3, 305–387 (1975).
[5] R. Denk, M. Hieber and J. Prüss. $R$-boundedness, Fourier multipliers, and problems of elliptic and parabolic type. Memoirs of the American Mathematical Society vol. 166, no. 788, Oxford University Press (2003).
[6] G. Dore and A. Venni. On the closedness of the sum of two closed operators. Math. Z. 196, 189–201 (1987).
[7] M. Haase. The functional calculus for sectorial operators. Operator theory: Advances and applications. Vol. 169, Birkhäuser Verlag (2006).
[8] N. Kalton and L. Weis. The $H^\infty$-calculus and sums of closed operators. Math. Ann. 321, 319–345 (2001).
[9] P. C. Kunstmann and L. Weis. Maximal $L^p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus. Functional Analytic Methods for Evolution Equations, Lecture Notes in Mathematics vol. 1855, Springer Verlag, 65–311 (2004).
[10] S. Monniaux and J. Prüss. A theorem of the Dore-Venni type for non-commuting operators. Transactions Amer. Math. Soc. 349, 4787–4814 (1997).
[11] J. Prüss and G. Simonett. $H^\infty$-calculus for the sum of non-commuting operators. Transactions Amer. Math. Soc. 359, 3549–3565 (2007).
[12] N. Roidos. Preserving closedness of operators under summation. J. Funct. Anal. 266, no. 12, 6938–6953 (2014).

Institut für Analysis, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
E-mail address: nikolaosroidos@gmail.com