CONSTRUCTING IRREDUCIBLE REPRESENTATIONS OF QUANTUM GROUPS $U_q(f_m(K))$

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Abstract: In this paper, we construct families of irreducible representations for a class of quantum groups $U_q(f_m(K))$. First, we give a natural construction of irreducible weight representations for $U_q(f_m(K))$ using methods in spectral theory developed by Rosenberg. Second, we study the Whittaker model for the center of $U_q(f_m(K))$. As a result, the structure of Whittaker representations is determined, and all irreducible Whittaker representations are explicitly constructed. Finally, we prove that the annihilator of a Whittaker representation is centrally generated.

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0. Introduction

Note that the quantized enveloping algebra $U_q(sl_2)$ has played a fundamental role in the study of the quantized enveloping algebra $U_q(g)$ of any semisimple Lie algebra $g$. In order to study the deformations of $U_q(g)$, it is natural to first study deformations of $U_q(sl_2)$.

In [7], a class of algebras $U_q(f(K))$ parameterized by Laurent polynomials $f(K) \in \mathbb{C}[K, K^{-1}]$ was introduced as generalizations of $U_q(sl_2)$. The condition for the existence of a Hopf algebra structure on $U_q(f(K))$ was determined and finite dimensional irreducible representations were explicitly constructed.

Such generalizations yield a family of quantum groups in the sense of Drinfeld [3]. For some special parameters $f(K) = a(K^m - K^{-m})$, $a \neq 0$, $m \in \mathbb{N}$, $U_q(f(K))$ are quantum groups, and all finite dimensional representations of $U_q(f(K))$ are completely reducible [7]. In particular, $U_q(f_m(K))$ with $f_m = \frac{K^m - K^{-m}}{q - q^{-1}}$ are quantum groups. In this paper, we study the irreducible representations of these quantum groups $U_q(f_m(K))$.

As a matter of fact, $U_q(f(K))$ can also be realized as Hyperbolic algebras [12] (or under the name of Generalized Weyl algebras [11]). For Hyperbolic algebras, the spectral theory developed in [12] is convenient for the explicit construction of irreducible weight representations. As an application, we construct all irreducible weight representations for the quantum groups $U_q(f_m(K))$ via methods developed in [12].

The Whittaker model for the center $Z(g)$ of the universal enveloping algebra $U(g)$ for any semisimple Lie algebra $g$ was first studied by Kostant in the seminal paper [8]. Whittaker model is closely related to Whittaker equations, and has a nice application in the theory of Toda lattice. Via the Whittaker model for $Z(g)$, the structure of Whittaker representations was determined, and all irreducible Whittaker representations were classified in [8].
The quantum analogue of the Whittaker model for $U_q(sl_2)$ was obtained in [11]. For the semisimple Lie algebras of higher ranks, the quantum Whittaker model was constructed in [14] for the topological version of quantized enveloping algebras via their Coxeter realizations. In this paper, we try to generalize Kostant’s results to the quantum groups $U_q(f_m(K))$. We construct the Whittaker model for the center of $U_q(f_m(K))$. As a result, we determine the structure of any Whittaker representation and construct all irreducible Whittaker representations explicitly. In addition, we prove that the annihilator of any Whittaker representation is centrally generated.

The above two constructions of irreducible representations for $U_q(f_m(K))$ are perpendicular in the sense that Whittaker representations are not weight representations. Though the constructions may work for general parameters $f(K)$, the calculations are more involved.

The paper is organized as follows. In Section 1, we recall the definition and some basic facts about $U_q(sl_2)$. In Section 2, we recall some background about spectral theory and Hyperbolic algebras from [12]. Then we illustrate how to realize $U_q(sl_2)$ as Hyperbolic algebras, and construct irreducible weight representations for $U_q(f_m(K))$. In Section 3, we describe the center of $U_q(sl_2)$ and construct the Whittaker model for the center. We study the properties of Whittaker representations.

1. The algebras $U_q(f(K))$

Let $\mathbb{C}$ be the field of complex numbers and $0 \neq q \in \mathbb{C}$ such that $q^2 \neq 1$. The quantized enveloping algebra $U_q(sl_2)$ corresponding to the simple Lie algebra $sl_2$ is the associative $\mathbb{C}-$algebra generated by $K^\pm 1, E, F$ subject to the following relations:

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad KK^{-1} = K^{-1}K = 1;$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

It is well-known that $U_q(sl_2)$ is a Hopf algebra with the following Hopf algebra structure:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\epsilon(E) = 0 = \epsilon(F), \quad \epsilon(K) = 1 = \epsilon(K^{-1}),$$

$$s(E) = -K^{-1}E, \quad s(F) = -FK, \quad s(K) = K^{-1}.$$

As generalizations of $U_q(sl_2)$, a class of algebras $U_q(f(K))$ parameterized by Laurent polynomials $f(K) \in \mathbb{C}[K, K^{-1}]$ was introduced in [7]. We recall their definition here.

**Definition 1.1.** (See [7]) For any Laurent polynomial $f(K) \in \mathbb{C}[K, K^{-1}], U_q(f(K))$ is defined to be the $\mathbb{C}-$algebra generated by $E, F, K^\pm 1$ subject to the following relations:

$$KE = q^2EK, \quad KF = q^{-2}FK,$$

$$KK^{-1} = K^{-1}K = 1;$$

$$EF - FE = f(K).$$

The ring theoretic properties and finite dimensional representations were studied in detail in [7]. We recall some basic results without proof.
Proposition 1.1. (Prop 3.3 in [7]) Assume $f(K)$ is a non-zero Laurent polynomial in $\mathbb{C}[K, K^{-1}]$. Then the non-commutative algebra $U_q(f(K))$ is a Hopf algebra such that $K, K^{-1}$ are group-like elements, and $E, F$ are skew primitive elements if and only if $f(K) = a(K^m - K^{-m})$ with $m = t - s$ and the following conditions are satisfied:

\[
\begin{align*}
\Delta(K) &= K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}; \\
\Delta(E) &= E^s \otimes E + E \otimes K^t, \quad \Delta(F) = K^{-t} \otimes F + F \otimes K^{-s}; \\
\epsilon(K) &= \epsilon(K^{-1}) = 1, \quad \epsilon(E) = \epsilon(F) = 0; \\
S(K) &= K^{-1}, \quad S(K^{-1}) = K; \\
S(E) &= -K^{-s}KE^{-t}, \quad S(F) = -K^tFK^s.
\end{align*}
\]

\[\blacksquare\]

In particular, $U_q(f_m(K))$ are quantum groups for $f_m(K) = \frac{K^m - K^{-m}}{q^2 - 1}$, $m \in \mathbb{N}$. When $q$ is not a root of unity, the finite dimensional irreducible representations were proved to be highest weight and were constructed explicitly in [7]. Furthermore, we have the following:

Theorem 1.1. (Thm 4.17 in [7]) If $q$ is not a root of unity, then any finite dimensional representation $V$ of $U_q(f_m(K))$ is completely reducible.

\[\blacksquare\]

2. HYPERBOLIC ALGEBRAS AND THEIR REPRESENTATIONS

In this section, we realize $U_q(f(K))$ as Hyperbolic algebras and use the methods in spectral theory developed in [12] to construct irreducible weight representations for $U_q(f_m(K))$. For the reader’s convenience, we recall some background about spectral theory from [12].

2.1. Preliminaries on spectral theory. Spectral theory of abelian categories was started by Gabriel in [4]. Gabriel defined the injective spectrum of any noetherian Grothendieck category with enough injectives. This spectrum consists of isomorphism classes of indecomposable injective objects of the category. Let $R$ be a commutative noetherian ring, then the spectrum of the category of all $R$–modules is isomorphic to the prime spectrum $Spec(R)$ of $R$ as a scheme. Furthermore, one can reconstruct any noetherian commutative scheme $(X, O_X)$ using the spectrum of the category of quasi-coherent sheaves of modules on $X$. The spectrum of any abelian category was later on defined by Rosenberg in [12]. This spectrum works for any abelian category. Via this spectrum, one can reconstruct any quasi-separated and quasi-compact commutative scheme $(X, O_X)$ via the spectrum of the category of quasi-coherent sheaves of modules on $X$.

Though spectral theory is more important for the purpose of non-commutative algebraic geometry, it has nice applications to representation theory. Note that the spectrum has a natural analogue of the Zariski topology and its closed points are in a one-to-one correspondence with the irreducible objects of the category under certain mild finiteness condition. This is the case for the category of representations over an algebra. To study irreducible representations, one can study the spectrum of the category of all representations, then single out closed points of the spectrum with respect to the associated topology.
Let $C_X$ be an abelian category and $M, N \in C_X$ be any two objects; We say that $M \succ N$ if and only if $N$ is a sub-quotient of the direct sum of finitely many copies of $M$. It is easy to verify that $\succ$ is a pre-order. We say $M \approx N$ if and only if $M \succ N$ and $N \succ M$. It is obvious that $\approx$ is an equivalence. Let $\text{Spec}(X)$ be the family of all nonzero objects $M \in C_X$ such that for any non-zero sub-object $N$ of $M$, we have $N \succ M$.

**Definition 2.1.** (See [12]) The spectrum of any abelian category is defined to be:

$$\text{Spec}(X) = \text{Spec}(X)/\approx.$$ 

2.2. **The left spectrum of a ring.** If $C_X$ is the category $A-\text{mod}$ of left modules over a ring $A$, then it is sometimes convenient to express the points of $\text{Spec}(X)$ in terms of left ideals of the ring $A$. In order to do so, the *left spectrum* $\text{Spec}_l(A)$ was defined in [12], which is by definition the set of all left ideals $p$ of $A$ such that $A/p$ is an object of $\text{Spec}(X)$. The relation $\succ$ on $A-\text{mod}$ induces a specialization relation among left ideals, in particular, the specialization relation on $\text{Spec}_l(A)$. Namely, $A/m \succ A/n$ iff there exists a finite subset $x$ of elements of $A$ such that such that the ideal $(n : x) = \{a \in A \mid ax \subset n\}$ is contained in $m$. Following [12], we denote this by $n \preceq m$. Note that the relation $\preceq$ is just the inclusion if $n$ is a two-sided ideal. In particular, it is the inclusion if the ring $A$ is commutative. The map which assigns to an element of $\text{Spec}_l(A)$ induces a bijection of the quotient $\text{Spec}_l(A)/\preceq$ of $\text{Spec}_l(A)$ by the equivalence relation associated with $\preceq$ onto $\text{Spec}(X)$. From now on, we will not distinguish $\text{Spec}_l(A)/\preceq$ from $\text{Spec}(X)$ and will express results in terms of the left spectrum.

2.3. **Hyperbolic algebra $R\{\xi, \theta\}$ and its spectrum.** Hyperbolic algebras are studied by Rosenberg in [12] and by Bavula under the name of Generalized Weyl algebras in [1]. Hyperbolic algebra structure is very convenient for the construction of points of the spectrum. Many interesting algebras such as the first Weyl algebra $A_1, U(sl_2)$ and their quantized versions have a Hyperbolic algebra structure. Points of the spectrum of the category of modules over Hyperbolic algebras are constructed in [12]. We recall some basic facts about Hyperbolic algebras and two important construction theorems from [12].

Let $\theta$ be an automorphism of a commutative algebra $R$; and let $\xi$ be an element of $R$.

**Definition 2.2.** The Hyperbolic algebra $R\{\theta, \xi\}$ is defined to be the $R$–algebra generated by $x, y$ subject to the following relations:

$$xy = \xi, \quad yx = \theta^{-1}(\xi)$$

and

$$xa = \theta(a)x, \quad ya = \theta^{-1}(a)y$$

for any $a \in R$. And $R\{\theta, \xi\}$ is called a Hyperbolic algebra over $R$.

From [12], we have the following construction theorems:

**Theorem 2.1.** (Thm 3.2.2.in [12])

(1) Let $P \in \text{Spec}(R)$, and assume the orbit of $P$ under the action of the automorphism $\theta$ is infinite.
(a) If \( \theta^{-1}(\xi) \in P \), and \( \xi \in P \), then the left ideal
\[
P_{1,1} = P + R\{\theta, \xi\}x + R\{\theta, \xi\}y
\]
is a two-sided ideal from \( \text{Spec}_l(R\{\theta, \xi\}) \).

(b) If \( \theta^{-1}(\xi) \in P \), \( \theta^i(\xi) \notin P \) for \( 0 \leq i \leq n - 1 \), and \( \theta^n(\xi) \in P \), then the left ideal
\[
P_{1,n+1} = R\{\theta, \xi\}P + R\{\theta, \xi\}x + R\{\theta, \xi\}y^{n+1}
\]
belongs to \( \text{Spec}_l(R\{\theta, \xi\}) \).

(c) If \( \theta^i(\xi) \notin P \) for \( i \geq 0 \) and \( \theta^{-1}(\xi) \in P \), then
\[
P_{1,\infty} = R\{\theta, \xi\}P + R\{\theta, \xi\}x
\]
belongs to \( \text{Spec}_l(R\{\theta, \xi\}) \).

(d) If \( \xi \in P \) and \( \theta^{-1}(\xi) \notin P \) for all \( i \geq 1 \), then the left ideal
\[
P_{\infty,1} = R\{\theta, \xi\}P + R\{\theta, \xi\}y
\]
belongs to \( \text{Spec}_l(R\{\theta, \xi\}) \).

(2) If the ideal \( P \) in (b), (c), or (d) is maximal, then the corresponding left ideal of \( \text{Spec}_l(R\{\theta, \xi\}) \) is maximal.

(3) Every left ideal \( Q \in \text{Spec}_l(R\{\theta, \xi\}) \) such that \( \theta^\nu(\xi) \in Q \) for a \( \nu \in \mathbb{Z} \) is equivalent to one left ideal as defined above uniquely from a prime ideal \( P \in \text{Spec}(R) \). The latter means that if \( P \) and \( P' \) are two prime ideals of \( R \) and \( (\alpha, \beta) \) and \( (\nu, \mu) \) take values \( (1, \infty), (\infty, 1), (\infty, \infty) \) or \( (1, n) \), then \( P_{\alpha, \beta} \) is equivalent to \( P'_{\nu, \mu} \) if and only if \( \alpha = \nu, \beta = \mu \) and \( P = P' \).

\[\square\]

**Theorem 2.2.** (Prop 3.2.3. in [12])

(1) Let \( P \in \text{Spec}(R) \) be a prime ideal of \( R \) such that \( \theta^i(\xi) \notin P \) for \( i \in \mathbb{Z} \) and \( \theta^i(P) - P \neq \emptyset \) for \( i \neq 0 \), then \( P_{\infty, \infty} = R\{\xi, \theta\}P \in \text{Spec}_l(R\{\xi, \theta\}) \).

(2) Moreover, if \( P \) is a left ideal of \( R\{\theta, \xi\} \) such that \( P \cap R = P \), then \( P = P_{\infty, \infty} \). In particular, if \( P \) is a maximal ideal, then \( P_{\infty, \infty} \) is a maximal left ideal.

(3) If a prime ideal \( P' \subset R \) is such that \( P_{\infty, \infty} = P'_{\infty, \infty} \), then \( P' = \theta^n(P) \) for some integer \( n \). Conversely, \( \theta^n(P)_{\infty, \infty} = P_{\infty, \infty} \) for any \( n \in \mathbb{Z} \).

\[\square\]

2.4. **Realize \( U_q(f(K)) \) as Hyperbolic algebras.** Let \( R \) be the sub-algebra of \( U_q(f(K)) \) generated by \( EF, K^{\pm 1} \), then \( R \) is a commutative algebra. We define an algebra automorphism of \( R \) as follows:

\[\theta : R \rightarrow R\]

by

\[\theta(EF) = EF + f(\theta(K)), \quad \theta(K^{\pm 1}) = q^{\mp 2}K^{\pm 1} \]

It is obvious that \( \theta \) extends to an algebra automorphism of \( R \). We also have the following lemma:
Lemma 2.1. The following identities hold:
\[ E(EF) = \theta(EF)E, \]
\[ F(EF) = \theta^{-1}(EF)F, \]
\[ EK = \theta(K)E, \]
\[ FK = \theta^{-1}(K)F. \]

Proof: The Proof is straightforward. \( \square \)

From Lemma 2.1, we have the following:

Proposition 2.1. \( U_q(f(K)) = R\{\xi = EF, \theta\} \) is a Hyperbolic algebra with \( R \) and \( \theta \) defined as above. \( \square \)

We have a corollary:

Corollary 2.1. The Gelfand-Kirillov dimension of \( U_q(f(K)) \) is 3.

Proof: This follows from the fact that \( R \) has Gelfand-Kirillov dimension 2 and \( U_q(f(K)) \) is a Hyperbolic algebra over \( R \). \( \square \)

2.5. Irreducible weight representations of \( U_q(f_m(K)) \).

Now we apply the above construction theorems to the case of \( U_q(f_m(K)) \) and construct families of irreducible weight representations for \( U_q(f_m(K)) \).

Let \( \alpha, 0 \neq \beta \in \mathbb{C} \). We denote by \( P = M_{\alpha, \beta} = (\xi - \alpha, K - \beta) \subset R \) the maximal ideal of \( R \) generated by \( \xi - \alpha, K - \beta \). We need a lemma:

Lemma 2.2. \( \theta^n(M_{\alpha, \beta}) \neq M_{\alpha, \beta} \) for any \( n \geq 1 \). In particular, \( M_{\alpha, \beta} \) has infinite orbit under the action of \( \theta \).

Proof: We have
\[ \theta^n(K - \beta) = (q^{-2n}K - \beta) = q^{-2n}(K - q^{2n}\beta). \]
Since \( q \) is not a root of unity, \( q^{2n} \neq 1 \) for any \( n \neq 0 \). Thus \( \theta^n(M_{\alpha, \beta}) \neq M_{\alpha, \beta} \) for any \( n \geq 1 \). \( \square \)

Another lemma is in order:

Lemma 2.3. (1) For \( n \geq 0 \), we have the following:
\[ \theta^n(EF) = EF + \frac{1}{q - q^{-1}}\left(q^{-2m(1 - q^{-2nm})}K^m - q^{2n(1 - q^{2nm})}K^{-m}\right). \]

(2) For \( n \geq 1 \), we have the following:
\[ \theta^{-n}(EF) = EF - \frac{1}{q - q^{-1}}\left(1 - q^{2nm}\right)K^m - \frac{1 - q^{-2nm}}{1 - q^{-2m}}K^{-m}. \]

Proof: For \( n \geq 1 \), we have
\[ \theta^n(EF) = EF + \frac{1}{q - q^{-1}}\left((q^{2m} + \cdots + q^{2nm})K\right) - (q^{2m} + \cdots + q^{2nm})K^{-m}) \]
\[ = EF + \frac{1}{q - q^{-1}}\left(q^{-2m(1 - q^{-2nm})}K^m - q^{2m(1 - q^{2nm})}K^{-m}\right). \]
The second statement can be verified similarly. \( \square \)
Theorem 2.3.\hspace{1em} (1) If $\alpha = \frac{\beta_m - \beta_m}{q^{-n}}$, $\beta_m = \pm q^{m}$ for some $n \geq 0$, then $\theta^m(\xi) \in M_{\alpha, \beta}$, and $\theta^{-1}(\xi) \in M_{\alpha, \beta}$, thus $U_q(f_m(K))/P_{1, n+1}$ is a finite dimensional irreducible representation of $U_q(f(K))$.

(2) If $\alpha = \frac{\beta_m - \beta_m}{q^{-n}}$ and $\beta_m \neq \pm q^{m}$ for all $n \geq 0$, then $U_q(f_m(K))/P_{1, \infty}$ is an infinite dimensional irreducible representation of $U_q(f_m(K))$.

(3) If $\alpha = 0$ and $0 \neq \frac{1}{q^{-n}}\left(\frac{1 - q^{2m}}{1 - q^{-2m}}\beta_m - \frac{1 - q^{-2m}}{1 - q^{2m}}\beta^{-m}\right)$ for any $n \geq 1$, then $U_q(f_m(K))/P_{\infty, 1}$ is an infinite dimensional irreducible representation of $U_q(f_m(K))$.

Proof: Since $\theta^{-1}(\xi) = \xi - \frac{K_m - K_{-m}}{q^{-1}}$, thus $\theta^{-1}(\xi) \in M_{\alpha, \beta}$ if and only if $\alpha = \frac{\beta_m - \beta_m}{q^{-n}}$. Now by the proof of Lemma 2.3, we have

$$\theta^n(\xi) = \xi + \frac{1}{q - q^{-1}}((q^{-2m} + \cdots + q^{-2nm})K^m - (q^{2m} + \cdots + q^{2nm})\beta^{-m})$$

Hence $\theta^n(\xi) \in M_{\alpha, \beta}$ if and only if

$$0 = \alpha + \frac{1}{q - q^{-1}}((q^{-2m} + \cdots + q^{-2nm})\beta_m - (q^{2m} + \cdots + q^{2nm})\beta^{-m})$$

$$= \alpha + \frac{1}{q - q^{-1}}(\frac{q^{-2m}(1 - q^{-2nm})}{1 - q^{2m}}\beta_m - \frac{q^{2m}(1 - q^{2nm})}{1 - q^{-2m}}\beta^{-m}).$$

Hence when $\alpha = \frac{\beta_m - \beta_m}{q^{-n}}$, $\beta_m = \pm q^{m}$ for some $n \geq 0$, we have

$$\theta^n(\xi) \in M_{\alpha, \beta}, \theta^{-1}(\xi) \in M_{\alpha, \beta}.$$ 

By Theorem 2.1, $U_q(f_m(K))/P_{1, n+1}$ is a finite dimensional irreducible representation of $U_q(f_m(K))$. We have already proved the first statement, the rest of the statements can be similarly verified. 

Remark 2.1. The representations we constructed in Theorem 2.3 exhaust all finite dimensional irreducible representations, highest weight irreducible representations and lowest weight irreducible representations of $U_q(f_m(K))$. For each $n \geq 1$, there are exactly $2m$ irreducible representations of dimension $n$.

In addition, we have the following:

Theorem 2.4. If $\alpha \neq -\frac{1}{q - q^{-1}}(\frac{q^{-2m}(1 - q^{-2nm})}{1 - q^{2m}}\beta_m - \frac{q^{2m}(1 - q^{2nm})}{1 - q^{-2m}}\beta^{-m})$ for any $n \geq 0$ and $\alpha \neq -\frac{1}{q - q^{-1}}(\frac{1 - q^{-2nm}}{1 - q^{2m}}\beta_m - \frac{1 - q^{2nm}}{1 - q^{-2m}}\beta^{-m})$ for any $n \geq 1$, then $U_q(f_m(K))/P_{\infty, \infty}$ is an infinite dimensional irreducible weight representation of $U_q(f_m(K))$.

Proof: The proof is similar to the one of Theorem 2.3, we omit it here. 

Corollary 2.2. The representations constructed in Theorem 2.3 and Theorem 2.4 exhaust all irreducible weight representations of $U_q(f_m(K))$.

Proof: It follows directly from Theorems 2.1, 2.2, 2.3 and 2.4. 

Remark 2.2. When $m = 1$, the above results recover the weight representations of $U_q(sl_2)$. So our results are just a natural generalization of those for $U_q(sl_2)$. 

3. The Whittaker model of the center $Z(U_q(f_m(K)))$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. The Whittaker model for the center of $U(\mathfrak{g})$ was studied by Kostant in [8]. The Whittaker model for the center $Z(U(\mathfrak{g}))$ is defined by a non-singular character of the nilpotent Lie subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$. Using the Whittaker model, Kostant studied the structure of Whittaker modules of $U(\mathfrak{g})$ and several important results about Whittaker modules were obtained in [8]. Later on, Kostant’s idea was further generalized by Lynch in [9] and by Macdowell in [10] to the case of singular characters of $\mathfrak{n}^+$ and similar results were proved to hold.

The obstacle of generalizing the Whittaker model to the quantized enveloping algebra $U_q(\mathfrak{g})$ with $\mathfrak{g}$ of higher ranks is that there is no non-singular character existing for the positive part $(U_q(\mathfrak{g}))^>0$ of $U_q(\mathfrak{g})$ because of the quantum Serre relations. In order to overcome this difficulty, it was Sevostyanov who first realized to use the topological version $U_h(\mathfrak{g})$ over $\mathbb{C}[[h]]$ of quantum groups. Using a family of Coxeter realizations $U_h(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ indexed by the Coxeter elements $s_\pi$, he was able to prove Kostant’s results for $U_h(\mathfrak{g})$ in [14]. While in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$, the quantum Serre relations are trivial, and a direct generalization has been found in [11].

In addition, it is reasonable to expect that the Whittaker model exists for most of the deformations of $U_q(\mathfrak{sl}_2)$. In this section, we show that there is such a Whittaker model for the center of $U_q(f_m(K))$ and study their Whittaker modules. We prove several similar results as in [8] and [11]. We follow the approach in [8] and [11]. For the reader’s convenience, we will work out all the details here. For the simplicity of notations, we denote $f_m(K) = \frac{K^m - K^{-m}}{q - q^{-1}}$ by $f(K)$ from now on.

### 3.1. The center $Z(U_q(f(K)))$ of $U_q(f(K))$

We first give a description of the center of $U_q(f(K))$. To proceed, we define a Casimir element $\Omega$ by:

$$\Omega = FE + \frac{q^{2m}K^m + K^{-m}}{(q^{2m} - 1)(q - q^{-1})}.$$

We have the following:

**Proposition 3.1.**

$$\Omega = FE + \frac{q^{2m}K^m + K^{-m}}{(q^{2m} - 1)(q - q^{-1})} = EF + \frac{K^m + q^{2m}K^{-m}}{(q^{2m} - 1)(q - q^{-1})}.$$

**Proof:** The proof is easy.

In addition, we have the following lemma:

**Lemma 3.1.** $\Omega$ is in the center of $U_q(f(K))$.

**Proof:** The proof is easy.

We have the following description of the center $Z(U_q(f(K)))$ of $U_q(f(K))$:

**Proposition 3.2.** (See also [7]) $Z(U_q(f(K)))$ is the subalgebra of $U_q(f(K))$ generated by $\Omega$. In particular, $Z(U_q(f(K)))$ is isomorphic to the polynomial algebra in one variable.
So we are done.

**Proposition 3.4.**

Definition 3.2. An algebra homomorphism \( \eta: U_q(E) \to \mathbb{C} \) is called a non-singular character of \( U_q(E) \) if \( \eta(E) \neq 0 \).

From now on, we always fix such a non-singular character of \( U_q(E) \) and denote it by \( \eta \). As in [8], we have the following:

**Definition 3.2.** Let \( V \) be a \( U_q(f(K)) \)-module, a vector \( 0 \neq v \in V \) is called a Whittaker vector if \( E \) acts on \( v \) through the non-singular character \( \eta \), i.e., \( Ev = \eta(E)v \). If \( V = U_q(f(K))v \), then we call \( V \) a Whittaker module of type \( \eta \) and \( v \) is called a cyclic Whittaker vector of type \( \eta \) for \( V \).

By the definition of \( U_q(f(K)) \), the following decomposition of \( U_q(f(K)) \) is obvious:

**Proposition 3.3.** \( U_q(f(K)) \) is isomorphic to \( U_q(F, K^{\pm 1}) \otimes_{\mathbb{C}} U_q(E) \) as vector spaces and \( U_q(f(K)) \) is a free module over the subalgebra \( U_q(E) \).

Let us denote the kernel of \( \eta: U_q(E) \to \mathbb{C} \) by \( U_{q, \eta}(E) \), and we have the following decompositions of \( U_q(E) \) and \( U_q(f(K)) \).

**Proposition 3.4.** We have \( U_q(E) = \mathbb{C} \oplus U_{q, \eta}(E) \). Furthermore,

\[
U_q(f(K)) \simeq U_q(F, K^{\pm 1}) \oplus U_q(f(K))U_{q, \eta}(E).
\]

**Proof:** It is obvious that \( U_q(E) = \mathbb{C} \oplus U_{q, \eta}(E) \). And we have

\[
U_q(f(K)) = U_q(F, K^{\pm 1}) \otimes (\mathbb{C} \oplus U_{q, \eta}(E)),
\]

thus

\[
U_q(f(K)) \simeq U_q(F, K^{\pm 1}) \oplus U_q(f(K))U_{q, \eta}(E).
\]

So we are done. \( \square \)

Now we define a projection

\[
\pi: U_q(f(K)) \to U_q(F, K^{\pm 1})
\]

from \( U_q(f(K)) \) onto \( U_q(F, K^{\pm 1}) \) by taking the \( U_q(F, K^{\pm 1}) \)-component of any \( u \in U_q(f(K)) \). We denote the image \( \pi(u) \) of \( u \in U_q(f(K)) \) by \( u^\eta \) for short.
Lemma 3.2. If $v \in Z(U_q(f(K)))$, then we have $u^iv^n = (uv)^n$ for any $u \in U_q(f(K))$.

Proof: Let $u \in U_q(f(K))$, $v \in Z(U_q(f(K)))$, then we have
\[
uv - u^iv^n = (u - u^i)v + u^i(v - v^n)
\]
which is in $U_q(f(K))U_{q,n}(E)$. Hence $(uv)^n = u^iv^n$.

From the definition of $\Omega$, we have the following description of $\pi(\Omega)$:

Lemma 3.3.
\[
\pi(\Omega) = \eta(E)F + \frac{q^{2m}K^m + K^{-m}}{(q^{2m} - 1)(q - q^{-1})}.
\]

Proposition 3.5. The map
\[
\pi: Z(U_q(f(K))) \longrightarrow U_q(F, K^{\pm 1})
\]
is an algebra isomorphism of $Z(U_q(f(K)))$ onto its image $W(F, K^{\pm 1})$ in $U_q(F, K^{\pm 1})$.

Proof: It follows from Lemma 3.2. that $\pi$ is a homomorphism of algebras. It remains to show that $\pi$ is injective. Suppose $\pi(u) = 0$ for some $0 \neq u \in Z(U_q(f(K)))$. Since $Z(U_q(f(K)))$ is a polynomial algebra in $\Omega$, then $u = \sum_{i=0}^Nu_i\Omega^i$. By Lemma 3.4., $\pi(\Omega) = \eta(E)F + \frac{q^{2m}K^m + K^{-m}}{(q^{2m} - 1)(q - q^{-1})}i = 0$.

By direct computations, we have $u_N(\eta(E))^NF^N = 0$. So $u_N(\eta(E))^N = 0$, which is a contradiction. So $\pi$ is an injection. Thus $\pi$ is an algebra isomorphism from $Z(U_q(f(K)))$ onto its image $W(F, K^{\pm 1})$ in $U_q(F, K^{\pm 1})$.

Lemma 3.4. If $u^i = u$, then we have
\[
u^0v^n = (uv)^n
\]
for any $v \in U_q(f(K))$.

Proof: We have
\[
uv - u^iv^n = (u - u^i)v + u^i(v - v^n)
\]
which is in $U_q(f(K))U_{q,n}(E)$. So we have
\[
u^0v^n = (uv)^n
\]
for any $v \in U_q(f(K))$.

Let $\hat{A}$ be the subalgebra of $U_q(f(K))$ generated by $K^{\pm 1}$. Then $\hat{A}$ is a graded vector space with $A_0 = \mathbb{C}$, and
\[
\hat{A}_{[n]} = \mathbb{C}K^n \oplus \mathbb{C}K^{-n}
\]
for $n \geq 1$, and
\[
\hat{A}_{[n]} = 0
\]
for $n \leq -1$. 
As in [8] and [11], we define a filtration of $U_q(F, K^{±1})$ as follows:

$$U_q(F, K^{±1})_{[n]} = \bigoplus_{im+j \leq nm} U_q(F, K^{±1})_{i,j}$$

where $U_q(F, K^{±1})_{i,j}$ is the vector space spanned by $F^i K^j$. We denote by

$$W(F, K^{±1})_{[p]} = \mathbb{C} - span\{1, \Omega^0, \cdots, (\Omega^p)^3\}$$

for $p \geq 0$. It is easy to see that

$$W(F, K^{±1})_{[p]} \subset W(F, K^{±1})_{[p+1]}, \quad W(F, K^{±1}) = \sum_{p \geq 0} W(F, K^{±1})_{[p]}$$

And $W(F, K^{±1})_{[p]}$ give a filtration of $W(F, K^{±1})$, which is compatible with the filtration of $U_q(F, K^{±1})$. In particular,

$$W(F, K^{±1})_{[p]} = W(F, K^{±1}) \cap U_q(F, K^{±1})_{[p]}$$

for $q \geq 0$.

Now we have the following decomposition of $U_q(F, K^{±1})$.

**Theorem 3.1.** $U_q(F, K^{±1})$ is free (as a right module) over $W(F, K^{±1})$. And the multiplication induces an isomorphism

$$\Phi: \hat{A} \otimes W(F, K^{±1}) \rightarrow U_q(F, K^{±1})$$

as right $W(F, K^{±1})$-modules. In particular, we have the following

$$\bigoplus_{l+p=nm} \hat{A}_{[l]} \otimes W(F, K^{±1})_{[p]} \cong U_q(F, K^{±1})_{[n]}$$

**Proof:** First of all, the map $\hat{A} \times W(F, K^{±1}) \rightarrow U_q(F, K^{±1})$ is bilinear. So by the universal property of the tensor product, there is a map from $\hat{A} \otimes W(F, K^{±1})$ into $U_q(F, K^{±1})$ defined by the multiplication. It is easy to check this map is a homomorphism of right $W(F, K^{±1})$-modules and is surjective as well. Now we show that it is injective. Let $0 \neq u \in \hat{A} \otimes W(F, K^{±1})$ such that $\Phi(u) = 0$. We write $u = \sum_{i=0}^L a_i(K, K^{-1}) \otimes b_i(\pi(\Omega))^i$, where $a_i(K, K^{-1})$ are non-zero Laurent polynomials in $\mathbb{C}[K, K^{-1}]$ and $b_i \in \mathbb{C}^*$. Then $0 = \Phi(u) = \sum_{i=0}^L a_i(K, K^{-1})b_i(\eta(E)^L F + \frac{q^{2mK^m} + K^{-m}}{q^{m+1}(1-q^{-m})})^i).$ By direct computations, we have so $a_L(K, K^{-1})b_L(\eta(E)) F^L = 0.$ Thus $a_L(K, K^{-1})b_L(\eta(E)) F^L = 0$, which is a contradiction. So we have proved that $\Phi$ is a isomorphism of vector spaces. In addition, by counting the degrees of both sides, we also have

$$\bigoplus_{l+p=nm} \hat{A}_{[l]} \otimes W(F, K^{±1})_{[p]} \cong U_q(F, K^{±1})_{[n]}$$

Thus we have proved the theorem.

Let $Y_\eta$ be the left $U_q(f(K))$-module defined by

$$Y_\eta = U_q(f(K)) \otimes_{U_q(E)} \mathbb{C}_\eta$$

where $\mathbb{C}_\eta$ is one dimensional $U_q(E)$-module defined by the character $\eta$. It is easy to see that

$$Y_\eta \cong U_q(f(K))/U_q(f(K))U_{q, \eta}(E)$$

is a Whittaker module with a cyclic vector denoted by $1_\eta$. Now we have a quotient map from $U_q(f(K))$ to $Y_\eta$

$$U_q(f(K)) \rightarrow Y_\eta.$$
If \( u \in U_q(f(K)) \), then there is a \( u^n \) which is the unique element in \( U_q(F, K^{\pm 1}) \) such that 
\( u1_q = u^n1_q \). As in \([8]\), we define the \( \eta \)-reduced action of \( U_q(E) \) on \( U(F, K^{\pm 1}) \) as follows:

\[
x \cdot v = (xv)^n - \eta(x)v
\]

where \( x \in U_q(E) \) and \( v \in U_q(F, K^{\pm 1}) \).

**Lemma 3.5.** Let \( u \in U_q(F, K^{\pm 1}) \) and \( x \in U_q(E) \), we have

\[
x \cdot u^n = [x, u]^{\eta}
\]

**Proof:** \([x, u]1_q = (xu - ux)1_q = (xu - \eta(x)u)1_q \). Hence

\[
[x, u]^{\eta} = (xu)^{\eta} - \eta(x)u^{\eta} = (xu)^{\eta} - \eta(x)u^{\eta} = x \cdot u^{\eta}
\]

\( \square \)

**Lemma 3.6.** Let \( x \in U_q(E) \), \( u \in U_q(F, K^{\pm 1}) \), and \( v \in W(E, K^{\pm 1}) \), then we have

\[
x \cdot (uv) = (x \cdot u)v.
\]

**Proof:** Let \( v = w^q \) for some \( w \in Z(U_q(f(K))) \), then \( uv = uw^q = u^qw^q = (uw)^q \). Thus

\[
x \cdot (uv) = x \cdot (uw)^q = [x, uw]^q
\]

\[
= ([x, u]^q)^q = [x, u]^q w^q
\]

\[
= [x, u]^q v
\]

\[
= (x \cdot u^q)v
\]

\[
= (x \cdot u)v.
\]

So we are done. \( \square \)

Let \( V \) be an \( U_q(f(K)) \)-module and let \( U_{q, V}(f(K)) \) be the annihilator of \( V \) in \( U_q(f(K)) \). Then \( U_{q, V}(f(K)) \) defines a central ideal \( V \subset Z(U_q(f(K))) \) by setting \( Z_V = U_{q, V}(f(K)) \cap Z(U_q(f(K))) \). Suppose that \( V \) is a Whittaker module with a cyclic Whittaker vector \( w \), we denote by \( U_{q, w}(f(K)) \) the annihilator of \( w \) in \( U_q(f(K)) \). It is obvious that

\[
U_q(f(K))U_{q, \eta}(E) + U_q(f(K))Z_V \subset U_{q, w}(f(K)).
\]

In the next theorem we show that the reverse inclusion holds. First of all, we need an auxiliary Lemma:

**Lemma 3.7.** Let \( X = \{ v \in U_q(F, K^{\pm 1}) \mid (x \cdot v)w = 0, x \in U_q(E) \} \). Then

\[
X = \hat{A} \odot W_V(F, K^{\pm 1}) + W(F, K^{\pm 1})
\]

where \( W_V(F, K^{\pm 1}) = (Z_V)^q \). In fact, \( U_{q, V}(F, K^{\pm 1}) \subset X \) and

\[
U_{q, w}(F, K^{\pm 1}) = \hat{A} \odot W_w(F, K^{\pm 1})
\]

where \( U_{q, w}(F, K^{\pm 1}) = U_{q, w}(f(K)) \cap U_q(F, K^{\pm 1}) \)

**Proof:** Let us denote by \( Y = \hat{A} \odot W_V(F, K^{\pm 1}) + W(F, K^{\pm 1}) \) where \( W(F, K^{\pm 1}) = (Z(U_q(f(K))))^q \). Thus we need to verify \( X = Y \). Let \( v \in W(F, K^{\pm 1}) \), then \( v = u^n \).
for some $u \in Z(U_q(f(K)))$. So we have
\[
x \cdot v = x \cdot u^n = [x, u]^n = (xu)^n - \eta(x)u^n = x^0u^n - \eta(x)u^n = 0.
\]

So we have $W(F, K^{\pm 1}) \subset X$. Let $u \in Z_V$ and $v \in U_q(F, K^{\pm 1})$. Then for any $x \in U_q(F)$, we have
\[
x \cdot (uv^n) = (x \cdot v)u^n
\]

Since $u \in Z_V$, then $uv^n \in U_q(w(f(K)))$. Thus we have $uv^n \in X$, hence
\[
\tilde{A} \otimes W_V(F, K^{\pm 1}) \subset X
\]

which proves $Y \subset X$. Note that $\tilde{A}_{[i]}$ is the two dimensional subspace of $\mathbb{C}[K^{\pm 1}]$ spanned by $K^{\pm i}$ and $\bar{W}_V(F, K^{\pm 1})$ is the complement of $W_V(F, K^{\pm 1})$ in $W(F, K^{\pm 1})$. Let us set
\[
M_i = \tilde{A}_{[i]} \otimes \bar{W}_V(F, K^{\pm 1})
\]

thus we have the following:
\[
U_q(F, K^{\pm 1}) = M \oplus Y
\]

where $M = \sum_{i \geq 1} M_i$. We show that $M \cap X \neq 0$. Let $M_{[k]} = \sum_{1 \leq i \leq k} M_i$, then $M_{[k]}$ are a filtration of $M$. Suppose $n$ is the smallest $n$ such that $X \cap M_{[n]} \neq 0$ and $0 \neq y \in X \cap M_{[n]}$. Then we have $y = \sum_{1 \leq i \leq n} y_i$ where $y_i \in \tilde{A}_{i} \otimes \bar{W}_V(F, K^{\pm 1})$. Suppose we have chosen $y$ in such a way that $y$ has the fewest terms. Then by direct computations, we have $0 \neq y - \frac{1}{n!}E \cdot y \in X \cap M_{[n]}$ with fewer terms than $y$. This is a contradiction. So we have $X \cap M = 0$.

Now we prove that $U_{q,w}(F, K^{\pm 1}) \subset X$. Let $u \in U_{q,w}(F, K^{\pm q})$ and $x \in U_q(E)$, then we have $xuw = 0$ and $uxw = \eta(x)uw = 0$. Thus $[x, u] \in U_{q,w}(F, K^{\pm 1})$, hence $[x, u]^n \in U_{q,w}(F, K^{\pm 1})$. Since $u \in U_{q,w}(F, K^{\pm 1}) \subset U_{q,w}(E, F, K^{\pm 1})$, then $x \cdot u = [x, u]^n$. Thus $x \cdot u \in U_{q,w}(F, K^{\pm 1})$. So $u \in X$ by the definition of $X$. Now we are going to prove the following:

\[
W(F, K^{\pm 1}) \cap U_{q,w}(F, K^{\pm 1}) = W_V(F, K^{\pm 1})
\]

In fact, $W_V(F, K^{\pm 1}) = (Z_V^\eta)$ and $W_V(F, K^{\pm 1}) \subset U_{q,w}(F, K^{\pm 1})$. So if $v \in W_w(F, K^{\pm 1}) \cap U_{q,w}(F, K^{\pm 1})$, then we can uniquely write $v = u^n$ for $uZ(U_q(f(K)))$. Then $uv^n = 0$ implies $uw = 0$ and hence $u \in Z(U_q(f(K))) \cap U_{q,w}(F, K^{\pm 1})$. Since $V$ is cyclically generated by $w$, we have proved the above statement. Obviously, we have $U_q(f(K))Z_V \subset U_{q,w}(f(K))$. Thus we have $\tilde{A} \otimes W_V(F, K^{\pm 1}) \subset U_{q,w}(F, K^{\pm 1})$, hence we have $U_{q,w}(F, K^{\pm 1}) = \tilde{A} \otimes W_V(F, K^{\pm 1})$. So we have finished the proof.$\square$

**Theorem 3.2.** Let $V$ be a Whittaker module admitting a cyclic Whittaker vector $w$, then we have

\[
U_{q,w}(f(K)) = U_q(f(K))Z_V + U_q(f(K))U_{q,\eta}(E).
\]

**Proof:** It is obvious that

\[
U_q(f(K))Z_V + U_q(f(K))U_{\eta}(E) \subset U_{q,w}(f(K))
\]

Thus we have

\[
U_{q,w}(f(K)) = U_q(f(K))Z_V + U_q(f(K))U_q(f(K))U_{q,\eta}(E).
\]

This completes the proof. $\square$
Let $u \in U_q, w(f(K))$, we show that $u \in U_q(f(K))Z_V + U_q(f(K))U_{q, \eta}(E)$. Let $v = u^q$, then it suffices to show that $v \in \hat{A} \otimes W_V(F, K^{\pm 1})$. But $v \in U_{q, w}(F, K^{\pm 1}) = \hat{A} \otimes W_V(F, K^{\pm 1})$. So we have proved the theorem.

**Theorem 3.3.** Let $V$ be any Whittaker module for $U_q(f(K))$, then the correspondence

$$V \rightarrow Z_V$$

sets up a bijection between the set of all equivalence classes of Whittaker modules and the set of all ideals of $Z(U_q(f(K)))$.

**Proof:** Let $V_i, i = 1, 2$ be two Whittaker modules. If $Z_V = Z_{V_2}$, then clearly $V_1$ is equivalent to $V_2$ by the above Theorem. Now let $Z_*$ be an ideal of $Z(U_q(f(K)))$ and let $L = U_q(f(K))Z_* + U_q(f(K))U_{q, \eta}(E)$. Then $V = U_q(f(K))/L$ is a Whittaker module with a cyclic Whittaker vector $w = 1$. Obviously we have $U_{q, w}(f(K)) = L$. So that $L = U_{q, w}(f(K)) = U_q(f(K))Z_V + U_q(f(K))U_{q, \eta}(E)$. This implies that

$$\eta(L) = \pi(Z_*) = \pi(R_w) = \pi(Z_V).$$

Since $\pi$ is injective, thus $Z_V = Z_*$. Thus we have finished the proof.

**Theorem 3.4.** Let $V$ be an $U_q(f(K))$–module. Then $V$ is a Whittaker module if and only if

$$V \cong U_q(f(K)) \otimes_{L(U_q(f(K))) \otimes U_q(F)} (Z(U_q(f(K)))Z_*).$$

In particular, in such a case the ideal $Z_*$ is uniquely determined to be $Z_V$.

**Proof:** If $1_*$ is the image of $1$ in $Z(U_q(f(K)))Z_*$, then

$$Ann_{Z(U_q(f(K))) \otimes U_q(F)}(1_*) = U_q(E)Z_* + Z(U_q(f(K)))U_{q, \eta}(E).$$

Thus the annihilator of $w = 1 \otimes 1_*$ is

$$U_{q, w}(f(K)) = U_q(f(K))Z_* + U_q(f(K))U_{q, \eta}(E).$$

Then the result follows from the previous theorem.

**Theorem 3.5.** Let $V$ be an $U_q(f(K))$–module with a cyclic Whittaker vector $w \in V$. Then any $v \in V$ is a Whittaker vector if and only if $v = uw$ for some $u \in Z(U_q(f(K)))$.

**Proof:** If $v = uw$ for some $u \in Z(U_q(f(K)))$, then it is obvious that $v$ is a Whittaker vector. Conversely, let $v = uw$ for some $u \in U_q(f(K))$ be a Whittaker vector of $V$. Then $v = u^qw$ by the definition of Whittaker module. So we may assume that $u \in U_q(F, K^{\pm 1})$. If $x \in U_q(E)$, then we have $xuw = \eta(x)uw$ and $uxw = \eta(x)uw$. Thus $[x, u]w = 0$ and hence $[x, u]^qw = 0$. But we have $x \cdot u = [x, u]^q$. Thus we have $w \in X$. We can now write $u = u_1 + u_2$ with $u_1 \in U_q(F, K^{\pm 1})$ and $u_2 \in W(F, K^{\pm 1})$. Then $u_1w = 0$. Hence $u_2w = v$. But $u_2 = u_3w$, which proves the theorem.

Now let $V$ be a Whittaker module and $End_{U_q(f(K))}(V)$ be the endomorphism ring of $V$ as a $U_q(f(K))$–module. Then we can define the following homomorphism of algebras using the action of $Z(U_q(f(K)))$ on $V$:

$$\pi_V: Z(U_q(f(K))) \rightarrow End_{U_q(f(K))}(V)$$

It is clear that

$$Z(U_q(f(K)))/Z_V(U_q(f(K))) \cong \pi_V(Z(U_q(f(K)))) \subset End_{U_q(f(K))}(V).$$

In fact, the next theorem says that this inclusion is an equality as well.
Theorem 3.6. Let $V$ be a Whittaker $U_q(f(K))$-module. Then $\text{End}_{U_q(f(K))}(V) \cong Z(U_q(f(K)))/Z_V$. In particular, $\text{End}_{U_q(f(K))}(V)$ is commutative.

Proof: Let $v \in V$ be a cyclic Whittaker vector. If $\alpha \in \text{End}_{U_q(f(K))}(V)$, then $\alpha(v) = vw$ for some $u \in Z(U_q(f(K)))$ by Theorem 3.5. Thus we have $\alpha(vw) = vvw = uvw$. Hence $\alpha = \pi_u$, which proves the theorem.

We have the following description about the basis of an irreducible Whittaker module $(V, w)$ where $w \in V$ is a cyclic Whittaker vector.

Theorem 3.7. Let $(V, w)$ be an irreducible Whittaker module with a Whittaker vector $w$, then $V$ has a $\mathbb{C}$-basis consisting of elements $\{K^i w \mid i \in \mathbb{Z}\}$.

Proof: The proof is straightforward.

Now we are going to construct explicitly some Whittaker modules. Let

$$\xi : Z(U_q(f(K))) \longrightarrow \mathbb{C}$$

be a central character of the center $Z(U_q(f(K)))$. For any given central character $\xi$, let $Z_\xi = \text{Ker}(\xi) \subset Z(U_q(f(K)))$ and $Z_\xi$ is a maximal ideal of $Z(U_q(f(K)))$.

Since $\mathbb{C}$ is algebraically closed, then $Z_\xi = (\Omega - a_\xi)$ for some $a_\xi \in \mathbb{C}$. For any given central character $\xi$, let $\mathbb{C}_{\xi, \eta}$ be the one dimensional $Z(U_q(f(K))) \otimes U_q(E)$-module defined by $uvy = \xi(u)\eta(v)y$ for any $u \in Z(U_q(f(K)))$ and any $v \in U_q(E)$. We set

$$Y_{\xi, \eta} = U_q(f(K)) \otimes Z(U_q(f(K))) \otimes U_q(E) \mathbb{C}_{\xi, \eta}.$$

It is easy to see that $Y_{\xi, \eta}$ is a Whittaker module of type $\eta$ and admits a central character $\xi$. By Schur’s lemma, we know every irreducible representation has a central character. As studied in [7], we know $U_q(f(K))$ has a similar theory for Verma modules. In fact, Verma modules also fall into the category of Whittaker modules if we take the trivial character of $U_q(E)$. Namely we have the following

$$M_\lambda = U_q(f(K)) \otimes_{U_q(E, K^{\pm 1})} \mathbb{C}_\lambda$$

where $K$ acts on $\mathbb{C}_\lambda$ through the character $\lambda$ of $\mathbb{C}[K^{\pm 1}]$ and $U_q(E)$ acts trivially on $\mathbb{C}_\lambda$. It is obvious that $M_\lambda$ admits a central character. It is well-known that Verma modules may not be necessarily irreducible, even though they have central characters. While Whittaker modules are in the other extreme as shown in the next theorem:

Theorem 3.8. Let $V$ be a Whittaker module for $U_q(f(K))$. Then the following statements are equivalent.

(1) $V$ is irreducible.

(2) $V$ admits a central character.

(3) $Z_V$ is a maximal ideal.

(4) The space of Whittaker vectors of $V$ is one-dimensional.

(5) All nonzero Whittaker vectors of $V$ are cyclic.

(6) The centralizer $\text{End}_{U_q(f(K))}(V)$ is reduced to $\mathbb{C}$. 
(7) $V$ is isomorphic to $Y_{\xi,\eta}$ for some central character $\xi$.

**Proof:** It is easy to see that (2) – (7) are equivalent to each other by using the previous Theorems we have just proved. Since $\mathbb{C}$ is algebraically closed and uncountable, we also know (1) implies (2) by using a theorem due to Dixmier [2]. To complete the proof, it suffices to show that (2) implies (1), namely if $V$ has a central character, then $V$ is irreducible. Let $\omega \in V$ be a cyclic Whittaker vector, then $V = U_q(f(K))\omega$. Since $V$ has a central character, then it is easy to see from the description of the center that $V$ has a $\mathbb{C}$-basis consisting of elements $\{K^i \omega \mid i \in \mathbb{Z}\}$. Let $0 \neq v = \sum a_i K^i \omega \in V$, then $E \sum a_i K^i \omega = \sum q^{-2i} a_i K^i E \omega = \eta(E) \sum q^{-2i} a_i K^i \omega$. Thus we have $0 \neq \eta(E) q^{-2n} v - Ev \in V$, in which the top degree of $K$ is $n - 1$. By repeating this operation finitely many times, we will finally get an element $0 \neq a \omega$ with $a \in \mathbb{C}^*$. This means that $V = U_q(f(K))v$ for any $0 \neq v \in V$. So $V$ is irreducible. Therefore, we are done with the proof. 

**Theorem 3.9.** Let $V$ be a $U_q(f(K))$–module which admits a central character. Assume that $w \in V$ is a Whittaker vector. Then the submodule $U_q(f(K))w \subset V$ is irreducible.

**Proof:** First of all, $U_q(f(K))w$ is a Whittaker module. Since $V$ has a central character, then $U_q(f(K))w$ has a central character. Thus $U_q(f(K))w$ is an irreducible Whittaker module. 

**Theorem 3.10.** Let $V_1, V_2$ be any two irreducible $U_q(f(K))$–modules with the same central character. If $V_1$ and $V_2$ contain Whittaker vectors, then these vectors are unique up to scalars. And furthermore $V_1$ and $V_2$ are isomorphic to each other as $U_q(f(K))$–modules.

**Proof:** Since $V_i$ are irreducible and have Whittaker vectors, then they are irreducible Whittaker modules. In addition, they have a central character, so the subspace of Whittaker vectors is one dimensional, hence the Whittaker vectors are unique up to scalars. In this case, it is obvious that they are isomorphic to each other. 

3.3. The submodule structure of a Whittaker module $(V, w)$. In this section, we spell out the details about the structure of submodules of a Whittaker module $(V, w)$. We have a clean description about all submodules through the algebraic geometry of the affine line $\mathbb{A}^1$. Throughout this section, we fix a Whittaker module $(V, w)$ of type $\eta$ and a cyclic vector $w$ of $V$. Our argument is more or less the same as the one in [11].

**Lemma 3.8.** Let $Z(U_q(f(K))) = \mathbb{C}[\Omega]$ be the center of $U_q(f(K))$, then any maximal ideal of $Z(U_q(f(K)))$ is of the form $(\Omega - a)$ for some $a \in \mathbb{C}$.

**Proof:** This fact follows directly from the fact that $\mathbb{C}$ is an algebraically closed field and Hilbert Nullstellenlantz Theorem.

Let $Z_V$ be the annihilator of $V$ in $Z(U_q(f(K)))$, then $Z_V = (g(\Omega))$ is the two sided ideal of $\mathbb{C}[\Omega]$ generated by some polynomial $g(\Omega) \in Z(U_q(f(K)))$. Suppose that we have a decomposition for $g$ as follows:

$$g = \prod_{i=1,2,\ldots, m} g_i^{n_i},$$

where $g_i$ are irreducible. Then we have the following:
Proposition 3.6. \( V_i = U_q(f(K)) \prod_{j \neq i} g_j^{n_j} w \) are indecomposable submodules of \( V \). In particular we have
\[
V = V_1 \oplus \cdots \oplus V_m
\]
as a direct sum of submodules.

**Proof:** It is easy to verify that \( V_i \) are submodules. Now we show each \( V_i \) is indecomposable. Suppose not, we can assume without loss of generality that \( V_1 = W_1 \oplus W_2 \). Now \( Z_V = Z_{W_1} \cap Z_{W_2} \). Since \( Z(U_q(f(K))) \) is a Principal Ideal Domain, hence \( Z_{W_i} = (g_i(\Omega)) \). Thus we have \( g_i | f_{n_1} \). This implies that the decomposition is not a direct sum. Therefore \( V_i \) are all indecomposable. Finally, the decomposition follows from the Chinese Reminder Theorem.

Proposition 3.7. Let \( (V, w) \) be a Whittaker module and \( Z_V = \langle g^n \rangle \) where \( g \) is an irreducible polynomial in \( \mathbb{C}[\Omega] \). Let \( V_i = U_q(f(K))g_i w, i = 0, \cdots, n \) and \( S_i = V_i/V_{i+1}, i = 0, \cdots, n-1 \). Then \( S_i, i = 0, \cdots, n-1 \) are irreducible Whittaker modules of the same type \( \eta \) and form a composition series of \( V \). In particular \( V \) is of finite length.

**Proof:** The proof follows from the fact that \( Z_{S_i} = \langle g \rangle \) for all \( i \). □

Corollary 3.1. \( V \) has a unique maximal submodule \( V_1 \).

**Proof:** This is obvious because the only maximal ideal of \( Z_V \) is \( \langle g \rangle \). □

Based on the above propositions, the irreducibility and indecomposability are reduced to the structure of \( Z_V \). \( V \) is irreducible if and only if \( Z_V \) is maximal in \( Z(U_q(f(K))) \). And \( V \) is indecomposable if and only if \( Z_V \) is a primary ideal of \( Z(U_q(f(K))) \). The following proposition is a refinement of the submodule structure of \( (V, w) \).

Proposition 3.8. Suppose \( (V, w) \) is an indecomposable Whittaker module with \( Z_V = \langle g^n \rangle \), then any submodule \( T \subset V \) is of the form
\[
T = U_q(f(K))g_i w
\]
for some \( i \in \{0, \cdots, n\} \).

**Proof:** The proof is obvious. □

Now we investigate the submodule structure of any Whittaker module \( (V, w) \) with a nontrivial central annihilator \( Z_V \). First of all, we recall some notations from [8]. Let \( T \subset V \) be any submodule of \( V \), we define an ideal of \( Z \) as follows:
\[
Z(T) = \{ x \in Z \mid xT \subset T \}
\]
We may call \( Z(T) \) the normalizer of \( T \) in \( Z \). Conversely for any ideal \( J \subset Z \) containing \( Z_V \), \( JV \subset V \) is a submodule of \( V \). We have the following Theorem:

Theorem 3.11. Let \( (V, w) \) be a Whittaker module with \( Z_V \neq 0 \). Then there is a one-to-one correspondence between the set of all submodules of \( V \) and the set of all ideals of \( Z \) containing \( Z_V \) given by the
\[
T \mapsto Z(T)
\]
and
\[
J \mapsto JV
\]
which are inverse to each other.
Proof: The proof is a straightforward. 

Now we give a description of the basis of any Whittaker module \((V, w)\).

**Proposition 3.9.** Let \((V, w)\) be a Whittaker module and suppose that \(Z_V = \langle g(\Omega) \rangle \) where \(g \neq 0\) is monic polynomial of degree \(n\). Then

\[
B = \{F^i K^j \mid 0 \neq i \leq n - 1, j \in \mathbb{Z}\}
\]

is a \(C\)-basis of \(V\). If \(g = 0\), then

\[
B = \{F^i K^j \mid i \geq 0, j \in \mathbb{Z}\}
\]

is a \(C\)-basis of \(V\). 

3.4. The annihilator of a Whittaker module. In [11], it was proved the annihilator of any Whittaker module of \(U_q(sl_2)\) is centrally generated. In this section we generalize this result to our situation. We closely follow the approach in [11].

First of all, we need some lemmas:

**Lemma 3.9.** Let \((V, w)\) be a Whittaker module of type \(\eta\) with a fixed Whittaker vector \(w\). Suppose there is a \(u \in U_q(f(K))\) such that \(u K^i w = 0\) for all \(i > 0\). Then \(uw = 0\).

**Proof:** (We will adopt the proof of Lemma 6.1 in [11]). We can write \(u = \sum_{i \in \mathbb{Z}} x_i\) where \(K x_i = q^i x_i K\). Suppose the statement is false, then there exists a minimal such \(u\) with respect to the length of the above expression of \(u\). We may assume \(u\) has more than one summand, otherwise \(0 = x_r K w = q^r K x_r w\) implies that \(uw = 0\). Since we have \(u K^i w = 0\) for all \(i > 0\). In particular, we have

\[
0 = (\sum_{r} x_r)Kw
\]

and

\[
0 = (\sum_{r} x_r)K^2 w = K(\sum_{r} q^{-r} x_r)Kw.
\]

Thus we have \(0 = (\sum_{r} q^{-r} x_r)Kw\). Hence both \(u = \sum_{r} x_r\) and \(u' = \sum_{r} q^{-r} x_r\) annihilate \(Kw\). Note that

\[
K u' K^i w = K(\sum_{r} x_r)K^i w = (\sum_{r} x_r)K^{i+1} w = u K^{i+1} w = 0
\]

for any \(i > 0\). Thus \(0 = K^{-1} K u' K^i w = u' K^i w\). Now let \(m = \max\{r \mid x_r \neq 0\}\), and note that \(u - q^m u'\) annihilates \(K^i w\) for \(i > 0\). But

\[
u - q^m u' = u - q^m(\sum_{r} q^{-r} x_r) = \sum_{r \neq m} (1 - q^{m-r}) x_r
\]

has fewer nonzero terms than \(u\). This is a contradiction. 

The following corollary can be proved the same as in [11]:

**Corollary 3.2.** Let \(V\) be a Whittaker module with \(Z_V = 0\), then \(U_V = U Z_V = 0\). 

In addition, the following similar lemma holds:

**Lemma 3.10.** Assume that for any simple Whittaker module \(V\), we have \(U_V = U Z_V\). Then for any Whittaker module \(V\) of finite length, \(U_V = U Z_V\).
As remarked in [11], the above lemma reduces the problem to the case where $V$ is an irreducible Whittaker module with a nonzero central annihilator $Z_V = \langle \Omega - a \rangle$ for some $a \in \mathbb{C}$. To summarize the result, we have the following:

**Theorem 3.12.** Let $V$ be a Whittaker module, then $\text{Ann}(V)$ is centrally generated, i.e.

$$\text{Ann}_{U_q(f(K))}(V) = U_q(f(K))\text{Ann}_{Z(U_q(f(K)))}(V)$$

**Proof:** (This argument is essentially due to Smith [13]). First of all, the primitive ideal $U_V$ has infinite codimension since $U_V \subset U_w$ and $U/U_w = V$ is infinite-dimensional. Since $U_q(f(K))$ is a domain, then $(0)$ is a prime ideal of $U_q(f(K))$. $U_q(f(K))Z_V$ is also prime. Since $U_q(f(K))_V$ is primitive, it is prime. Thus we have a chain of prime ideals: $(0) \subset U_q(f(K))Z_V \subset U_q(f(K))_V$. Let $R = U/U_V$, then $R$ is a primitive ring. If $R$ is artinian, then $R$ is finite dimensional which is contradicting to the fact that $U_V$ has infinite codimension. So $R$ is not artinian. Now $U$ has GK-dimension 3. Suppose that the prime ideals $(0) \subset UZ_V \subset U_V$ are different, then GK-dimension of $U/U_V$ is at most 1. But there are no non-artinian finitely generated noetherian primitive $\mathbb{C}$–algebras with GK-dimension 1. Thus $R$ has to be finite dimensional, which is a contradiction. So we are done with proof.\[\square\]

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