THE BORSUK-ULAM-PROPERTY, TUCKER-PROPERTY 
AND CONSTRUCTIVE PROOFS IN COMBINATORICS

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Abstract. This article is concerned with a general scheme on how to obtain constructive proofs for combinatorial theorems that have topological proofs so far. To this end the combinatorial concept of Tucker-property of a finite group \( G \) is introduced and its relation to the topological Borsuk-Ulam-property is discussed. Applications of the Tucker-property in combinatorics are demonstrated.

1. Introduction

Topological combinatorialists prove combinatorial theorems by means of topological tools. For many combinatorialists there seems to remain an unsatisfactory feeling arising from the expectation that a purely combinatorial question also deserves a combinatorial proof. In many cases this is substantiated by the quest to find algorithms that provide solutions \([3, 20, 21]\).

An outstanding application of topological methods and the proof that offered a completely new perspective on topological combinatorics was Lovász’ proof \([12]\) of the Kneser conjecture in 1978. It has an interesting history of simplifications and generalizations, which culminated in a combinatorial proof by Matoušek \([13]\) in 2000. Other questions to be mentioned are various fair division problems \([21, 20]\), one closely related to the necklace problem \([2]\), and further coloring problems of graphs and hypergraphs \([15, 24]\).

At the heart of Matoušek’s combinatorial proof of Lovász’ theorem is the deeper understanding of the relation of the Borsuk-Ulam-theorem to its combinatorial counterpart, a lemma by Tucker \([22, 9]\).

In this article we will investigate and generalize such a correspondence and show its applicability towards the necklace problem and the related fair division problem, or more generally towards any problem solvable by the relatives of the Borsuk-Ulam-theorem. This might eventually lead to a general recipe to produce constructive, combinatorial proofs for theorems in topological combinatorics.

Outline of the paper. In order to introduce the setup and to motivate the subsequent generalizations, we start with the correspondence of the Borsuk-Ulam-theorem and the lemma by Tucker, followed by a short discussion of Matoušek’s proof of Lovász’ theorem. This provides a motivation for the introduction of Borsuk-Ulam pairs and their combinatorial counterpart: Tucker triples. In turn this leads to the Borsuk-Ulam-property for a group \( G \), a concept originally introduced by Sarkaria \([19]\), and the corresponding combinatorial property referred to as the Tucker-property for \( G \). We investigate the connection between these concepts and discuss its applications towards combinatorial problems.
2. THE BORSUK-ULAM-THEOREM AND THE TUCKER-LEMMA

Let us recall the Borsuk-Ulam-theorem, a topological result which is often illustrated to a layman by the claim that at any moment there is a pair of antipodal points on the surface of the earth with the same temperature and air pressure. This theorem has many nice proofs usually applying some argument from homology theory [8].

**Theorem** (Borsuk–Ulam). Let $f : S^n \to \mathbb{R}^n$ be a continuous, antipodal map, i.e., a map such that $f(-x) = -f(x)$ for each $x \in S^n$. Then there exists an $x$ with $f(x) = 0$.

As it turns out the Borsuk-Ulam-theorem can easily be derived from the following combinatorial lemma, and vice versa. An antipodally symmetric triangulation of the sphere is a triangulation $K$ with the property that $\sigma \in K$ implies $-\sigma \in K$.

**Lemma** (Tucker [22]). Let $K$ be an antipodally symmetric triangulation of the $n$-sphere $S^n$ refining the triangulation of the sphere induced by the coordinate hyperplanes. Let $\lambda : \text{vert}(K) \to \{\pm 1, \ldots, \pm n\}$ be an antipodal labeling of the vertices of $K$, i.e., $\lambda(-x) = -\lambda(x)$ for all $x \in \text{vert}(K)$. Then there exists $i \in \{1, \ldots, n\}$, and an edge in $K$ whose vertices are labeled by complementary labels $-i$ and $+i$.

An elementary, constructive proof of the lemma was given by Freund and Todd [9]. For the proof and its relation to the Borsuk-Ulam-theorem we refer to Matoušek’s book [14]. The requirement that the triangulation “refines the triangulation of the sphere induced by the coordinate hyperplanes” can be weakened [17]. It actually can be removed, but then the proof proceeds by a detour using the continuous Borsuk-Ulam-theorem, for details see [14]. But in this article we do not want to consider such detours.

We want to comment on the word “constructive”: The proof of Tucker’s lemma by Freund and Todd is based on the construction of a particular graph of degree at most two. Following a path in this graph starting at a known vertex of degree one, the inclined mathematician will end up in a vertex corresponding to the desired edge. In order to do this one actually will only need to construct the graph along this path. In general this will be much quicker then to search all edges of the triangulation. The reader might be reminded of the proof of the Sperner lemma which indeed is very similar, and we point out that Sperner’s lemma is the combinatorial counterpart to Brouwer’s fixed point theorem [1], a connection that has proved to be very fruitful in finding combinatorial proofs and algorithms as well (see e.g., [21]).

3. MATOUŠEK’S PROOF OF THE KNESEK CONJECTURE AND A GENERAL SCHEME

For self-containment, we recall Kneser’s original conjecture [11] from 1955 first proved by Lovász in 1978: For every partition of $\binom{[n]}{k}$ into $n - 2k + 1$ sets $C_1, \ldots, C_{n-2k+1}$ there exists a set $C_i$ containing a pair of disjoint $k$-sets. Here and in the sequel $[n]$ denotes the set $\{1, \ldots, n\}$, $\binom{[n]}{k}$ denotes the $k$-subsets of $[n]$.

Jiří Matoušek found an ingenious combinatorial proof [13] of the Kneser conjecture in 2000. It is based on the insight that the combinatorial counterpart to the Borsuk-Ulam-theorem, namely Tucker’s lemma, is only needed in a very mild form. He presents a two-step procedure in order to obtain the direct proof. First he shows how a special case of Tucker’s lemma suffices, i.e., the octahedral Tucker lemma.
And in a second step he eliminates all intermediate steps and provides a compact elegant constructive proof of Lovász’ theorem. Although Matoušek presents this as a proof by contradiction, his proof in fact finds a pair of disjoint $k$-sets constructively.

Matoušek’s proof suggests a general scheme to obtain constructive proofs for theorems with topological proofs. Note that step iii) might be easier to achieve if it is known how the combinatorial result implies the topological one.

i) Find a combinatorial counterpart to the topological theorem used in the proof.

ii) Identify a special case of the combinatorial statement as needed, and prove it directly.

iii) Replace the topological argument and the accompanying spaces etc. by the combinatorial counterparts.

4. Borsuk-Ulam pairs and Tucker triples

The relationship between the Borsuk-Ulam theorem and Tucker lemma, reviewed in Section 2, deserves a closer analysis and motivates the introduction of more general concepts. For all topological and combinatorial concepts that appear we refer to [14], see also [26].

Let $K$ be a finite simplicial complex with a simplicial action of a finite group $G$. Let $V$ be a real representation of $G$ and $Q \subset V$ a $G$-invariant convex polytope such that $0 \in \text{int}(Q)$.

**Example 1.** As a special case of such a complex $K$ the following is playing a special role. Let $G$ be a non-trivial finite group of order $k$, let $n \geq 1$, and let $N := n(k-1)$. Consider $E_N G = G \ast \ldots \ast G$, the $(N+1)$-fold join of $G$ with itself, where $G$ is regarded as a 0-dimensional simplicial complex. We denote the vertex set of this complex by $G \times [N+1]$. We will denote the elements of the (geometric realization of) $E_N G$ by $(t_0 \cdot g_0, \ldots, t_N \cdot g_N)$ where $t_i \geq 0$, $\sum t_i = 1$, $g_i \in G$. $E_N G$ is a compact $N$-dimensional, $(N-1)$-connected space with a free $G$-action, given by the diagonal action $g \cdot (t_0 \cdot g_0, \ldots, t_N \cdot g_N) = (t_0 \cdot g_0, \ldots, t_N \cdot g g_N)$.

**Definition 1.** A pair $(K, V)$ is called a Borsuk-Ulam pair for the group $G$, or just a Borsuk-Ulam pair if $G$ is fixed in advance, if each $G$-equivariant map $f : K \to V$ has a zero, that is if $0 \in \text{Image}(f)$.

**Definition 2.** A triple $(K, V, Q)$ is called a Tucker triple for a group $G$, or just a Tucker triple for short, if for each $G$-equivariant map (labelling) $\phi : \text{vert}(K) \to \text{vert}(Q)$, there exists a simplex $\sigma \in K$ such that $0 \in \text{conv}(\phi(\text{vert}(\sigma)))$.

The Tucker lemma can be rephrased as the statement that $(K, \mathbb{R}^n, \hat{Q}^n)$ is a Tucker triple for the group $\mathbb{Z}_2$ where $\hat{Q}^n := \text{conv}\{+e_i, -e_i\}_{i=1}^n$ is the crosspolytope in $\mathbb{R}^n$, and $K$ is a $\mathbb{Z}_2$-complex homeomorphic to $\mathbb{S}^n$ with the symmetric triangulation. The following example shows that such a “crosspolytope” exists for each finite group $G$.

**Example 2.** Suppose that $G$ is a group of order $k$. Let $\mathbb{R}^k \cong \text{span}\{e_g \mid g \in G\} \cong \mathbb{R}[G]$ be the real regular representation [10] of $G$ and $W_G := \{ x \in \mathbb{R}^k \mid x_1 + \ldots + x_k = 0 \}$ the representation obtained by taking orthogonal complement of the diagonal, i.e., the trivial 1-dimensional representation. The generalized crosspolytope $\hat{Q}_G^k = \hat{Q}_G^n$ is defined as the convex hull of the union $\bigcup_{i=1}^n \Delta_{(i)} \subset (W_G)^n$ where $\Delta_{(i)}$ is the simplex in the $i$-th copy $(W_G)_{(i)} \cong W_G$, spanned by the projections (of $\mathbb{R}^k$ onto $(W_G)_{(i)}$) of the orthonormal basis vectors $e_1, \ldots, e_k$. The polytope $\hat{Q}_G^n$ is clearly a
$G$-invariant subspace of $(W_G)^n$ such that $0 \in \text{int}(\hat{\mathcal{O}}^n_G)$. Note that $\hat{\mathcal{O}}^n_G$ depends only on the order $k$ of the group $G$ which justifies the notation $\hat{\mathcal{O}}^n_G = \mathcal{O}^n_G$.

Tucker triples are easily generated from the Borsuk-Ulam pairs. A moment of reflection shows that each Borsuk-Ulam pair $(K, V)$ can be upgraded to a Tucker triple $(K, V, Q)$ where $Q$ is an arbitrary $G$-invariant polytope in $V$ such that $0 \in \text{int}(Q)$. Indeed, a $G$-equivariant labelling $\phi : \text{vert}(K) \rightarrow \text{vert}(Q)$ is linearly (simplicially) extended to a $G$-equivariant map $f : K \rightarrow V$ and a zero of $f$ is inside a simplex $\sigma$ such that $0 \in \text{conv}(\phi(\text{vert}(\sigma)))$.

The converse is not true. The following proposition shows that there may be striking differences between the two notions.

**Proposition 1.** Assume that $G$ is a group of order $k$ and let $V$ be a real $G$-representation of dimension $N = n(k - 1)$ such that $V^G = \{0\}$, i.e., such that $g \cdot x = 0$ for all $g \in G$ if and only if $x = 0$. Let $Q \subset V$ be a simplicial, $G$-invariant polytope such that $0 \in \text{int}(Q)$. Then $(E_N G, V, Q)$ is always a Tucker triple for the group $G$.

An example for a space $V$ as in the proposition is the representation $(W_G)^n$ where $W_G$ is the $(k - 1)$-dimensional, real $G$-representation obtained from the regular representation by factoring out the 1-dimensional trivial representation.

**Proof.** The result is an easy consequence of the following remarkable result from convex geometry due to I. Bárány [4, 5].

Suppose that

\[
\Omega = \begin{bmatrix}
v_{1,1} & v_{1,2} & \ldots & v_{1,d+1} \\
v_{2,1} & v_{2,2} & \ldots & v_{2,d+1} \\
\vdots & \vdots & \ddots & \vdots \\
v_{m,1} & v_{m,2} & \ldots & v_{m,d+1}
\end{bmatrix}
\]

is a matrix where the entries $v_{i,j}$ are vectors in a vector space $\mathbb{R}^d$. Moreover, we assume that $0 \in \text{conv}\{v_{i,\nu}\}_{i=1}^m$ for each $\nu$, i.e., that the origin is in the convex hull of each column of the matrix $\Omega$. Then there exists a function $\alpha : [d + 1] \rightarrow [m]$ such that $0 \in \text{conv}\{v_{\alpha(1),1}, v_{\alpha(2),2}, \ldots, v_{\alpha(d+1),d+1}\}$.

Suppose that $\phi : \text{vert}(E_N G) \rightarrow \text{vert}(Q)$ is a $G$-equivariant labelling. Assume $m := k$ and $d := n(k - 1) = N$. Let $\Omega = [v_{g,j}]$ be the vector-valued matrix defined by $v_{g,j} := \phi((g, j))$ for each $(g, j) \in \text{vert}(E_N G) = G \times [N + 1]$. For each $\nu$,

\[x_{\nu} := \sum_{g \in G} v_{g,\nu} = \sum_{g \in G} \phi(g(e, \nu)) = \sum_{g \in G} g\phi(e, \nu)\]

is a $G$-invariant element in $V$. By the assumption $V^G = \{0\}$, hence $x_{\nu} = 0$ for each $\nu$, and we conclude that the matrix $\Omega$ satisfies the conditions of Bárány’s theorem. Consequently, there exists a function $\alpha : [N + 1] \rightarrow G$ such that $0 \in \text{conv}\{v_{\alpha(i),i}\}_{i=1}^{N+1}$, or equivalently,

$0 \in \text{conv}\{\phi((\alpha(i), i))\}_{i=1}^{N+1} = \text{conv}(\phi(\text{vert}(\sigma)))$

where $\sigma \in E_N G$ is the simplex determined by the function $\alpha$. \hfill \qed
There exist examples of groups $G$ of order $k$ such that $(E_N G, (W_G)^n)$ is not a Borsuk-Ulam pair. Such an example is provided already by the group $\mathbb{Z}_6$, see part (II) where it was shown that there exists a $\mathbb{Z}_6$-equivariant map $f : S^5 \to S^4 \subset \mathbb{R}^5 \cong W_{\mathbb{Z}_6}$. Let $G = \mathbb{Z}_6$, i.e., $k = 6$, and $n := 1$, hence $N = 5$. In this case the map $f$ also implies the existence of a $\mathbb{Z}_6$-equivariant map $E_N G \to W_G \cong \mathbb{R}^5$ without zeroes. So among the consequences of Proposition 1 is the following observation.

**Corollary 1.** There exists a finite group $G$ and a Tucker triple $(K, V, Q)$ such that $(K, V)$ is not a Borsuk-Ulam pair. For example one can take $G = \mathbb{Z}_6$, define $V := W_{\mathbb{Z}_6} \cong \mathbb{R}^5$ as the real representation of $\mathbb{Z}_6$ obtained from the regular representation modulo the 1-dimensional trivial representation, and choose $Q$ to be a $\mathbb{Z}_6$-invariant, 5-dimensional simplex in $W_{\mathbb{Z}_6}$.

5. The Borsuk-Ulam-property of $G$

In order to pursue the scheme outlined in Section 3, we first state a family of generalizations of the Borsuk-Ulam-theorem that has proven to be useful in topological combinatorics.

Consider the space $E_{n,k}$ of all $(n \times k)$-matrices with real entries and the property that all row sums are zero. In other words $E_{n,k}$ is the space of all matrices orthogonal to the space of all matrices with entries in each row being identical. In particular, $E_{n,k}$ has dimension $n(k-1)$. By labeling the columns with elements of $G$, $G$ acts on $E_{n,k}$ by column permutations, and the only fixed point of this action is the zero matrix.

The reader can easily convince herself that $E_{n,k} \cong (W_G)^n$, where $W_G$ is the representation described in Example 2. In other words $E_{n,k}$ is just a more concrete presentation of the representation $W_G$. Note that in particular, the simplex $\Delta_i \subset (W_G)_{(i)}$ corresponds in $E_{n,k}$ up to scaling to the convex hull of the set of matrices

\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
(e_g - \frac{1}{n} \sum_{h \in G} e_h)^{t} \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad g \in G,
\]

where the non-zero entries are in row $i$.

In the spirit of Sarkaria [19] we introduce the following definition.

**Definition 3.** A group $G$ of order $k$ has the Borsuk-Ulam-property if $(E_N G, E_{n,k})$ is a Borsuk-Ulam pair for each $n \geq 1$. In other words for each $n \geq 1$, every $G$-equivariant continuous map $f : E_N G \to E_{n,k}$ must have a zero.

Let us briefly review the case of the group $G = \mathbb{Z}_2$. In this case $E_N G = G^{* (n+1)} \cong \mathbb{S}^n$, and the $G$-action is given by the antipodal map. The space $E_{n,2}$ can be identified with $\mathbb{R}^n$ together with the action $x \mapsto -x$. We conclude that $G = \mathbb{Z}_2$ has the Borsuk-Ulam-property which is just a restatement of the Borsuk-Ulam-theorem.

The following theorem is very important tool in topological combinatorics with numerous and diverse applications.
Theorem 1 (Özaydin [16], Sarkaria [19], Volovikov [23]). For $p$ prime, $r \geq 1$, the group $G = (\mathbb{Z}_p)^r$ has the Borsuk-Ulam-property.

The groups $G = (\mathbb{Z}_p)^r$ are the only groups for which we know that the Borsuk-Ulam-property holds. For more information about this and related problems we refer the reader to [9].

The previous theorem has been used in the proofs of numerous combinatorial theorems, most notably the topological Tverberg theorem and its relatives [14, 19, 25, 26]. In order to demonstrate its strength and as an overture to the proof of Theorem 3, we present a short proof of a theorem by Alon on simultaneous equipartitions (splitting) of a set of probability measures (necklaces). Compared to Alon’s original approach and other existing proofs, see [14] and [26] for the references, the proof doesn’t offer new ideas. However the exposition is smooth and short providing an excellent example of how a zero of a continuous map encodes all the information needed for the solution (equipartition) of a geometric problem.

Theorem (Alon [2]). Let $\mu_1, \ldots, \mu_n$ be continuous probability measures on the unit interval and $k \geq 2$. Then it is possible to cut the interval in $n(k - 1)$ places and to partition the $n(k - 1) + 1$ resulting intervals into $k$ families $F_1, \ldots, F_k$ such that $\mu_i(\cup F_j) = \frac{1}{k}$ for all $i$ and $j$.

It is easy to see that the number of cuts is best possible in general.

Proof. It is a straightforward combinatorial exercise to reduce the problem first to the case $k = p$ a prime number. Then the elements of $E_N G$, with $G = \mathbb{Z}_p$ and $N := n(p - 1)$, define $n(p - 1)$ cuts of the unit interval together with a partition $F_1, \ldots, F_p$ of the resulting intervals: consider $(t_0 \cdot g_0, \ldots, t_N \cdot g_N) \in E_N G$ and define $x_{-1} := 0$ and $x_j := \sum_{i=0}^{j} t_i$ for $j = 0, \ldots, N$. Then the cuts are given by $x_0, \ldots, x_{N-1}$ and the resulting intervals are partitioned by setting $F_i := \{[x_{j-1}, x_j] : j \in \{0, \ldots, N\}, g_j = i + p\mathbb{Z}\}$. (Degenerate intervals with $x_{j-1} = x_j$ may be put into any of the $F_i$ since they have measure zero.) Next we define a map

$$E_N G \rightarrow E_{n,p}$$

$$(t_0 \cdot g_0, \ldots, t_N \cdot g_N) \mapsto (E_{ij})_{i=1,\ldots,n \atop j=1,\ldots,p}$$

where $E_{ij} := \mu_i(\cup F_j) - \mu_i(\cup F_{j-1})$ with the $j$-indices considered modulo $p$. Note that $(E_{ij})$ has row sums equal to zero by construction. With the columns of the matrix labeled appropriately by the elements of $G$ this map is continuous and $G$-equivariant. Hence by the previous theorem there exists a zero, which yields the desired cuts and the partition. \hfill $\square$

6. The Tucker-property for $G$

The combinatorial counterpart to a Borsuk-Ulam pair is a Tucker triple, as discussed in Section 4. Similarly, the Borsuk-Ulam-property for $G$ has a combinatorial counterpart, referred to as the Tucker property for $G$.

Another motivation for introducing this concept comes from the conjecture of Simmons and Su [20], discussed in Section 8. Since a Borsuk-Ulam pair can be upgraded to a Tucker triple in many ways, depending on the choice of a $G$-invariant polytope $Q$, it is clear that there does not exist a unique way of defining the “Tucker property” for $G$. For example a different generalization of Tucker’s lemma has been...
used by Ziegler in \[24\]. Our definition is based on the choice of the generalized crosspolytope \(\Diamond^n_k = \Diamond^n_G\) introduced in Example \[2\].

As before, let \(G\) be a non-trivial finite group of order \(k\), and \(N = n(k - 1)\) where \(n \geq 1\). Consider a \(G\)-invariant triangulation \(K\) of \(E_N G\), i.e., if \(g \in G\) and \(\sigma \in K\) then \(g \cdot \sigma \in K\). Furthermore, we assume that \(K\) refines the natural triangulation of \(E_N G\) induced from the \((N + 1)\)-fold join operation of the 0-dimensional complex \(G\).

**Definition 4.** The group \(G\) has the Tucker property if \((K, (W_G)^n, \Diamond^n_k)\) is a Tucker triple for \(G\) for each \(n \geq 1\) and each \(G\)-invariant subdivision \(K\) of the complex \(E_N G\), \(N = n(k - 1)\).

A slightly more combinatorial reformulation of Definition \[4\] is the following. A group \(G\) is said to have the Tucker property if for all \(n \geq 1\) and all \(K\) as above, every \(G\)-equivariant labelling \(\lambda : \text{vert}(K) \to G \times [n]\), i.e., a labelling such that \(\lambda(g \cdot v) = g \cdot \lambda(v)\), has the property that there exists an \(i \in [n]\) and a \((k - 1)\)-simplex in \(K\) whose vertices are labelled by \(\{(g, i) : g \in G\}\). The equivalence of the two formulations follows from the observation that if \(0 \in \text{conv}(S)\) for some \(S \subset \text{vert}(\Diamond^n_k)\), then \(\text{vert}(\Delta(\sigma)) \subset S\) for some \(i\).

Let us consider the case of \(G = \mathbb{Z}_2\) again. In this case, \(K\) turns out to be an antipodally symmetric triangulation of the \(n\)-sphere refining the triangulation induced by the coordinate hyperplanes. Hence \(G = \mathbb{Z}_2\) has the Tucker property by Tucker’s lemma.

More generally, a consequence of Theorem \[1\] and Proposition \[2\] is that the group \(G = (\mathbb{Z}_p)^r\) has the Tucker-property for each prime \(p\) and arbitrary \(r \geq 1\).

7. BORSUK-ULAUM VS. TUCKER-PROPERTY

As already observed in Section \[4\] if \((K, V)\) is a Borsuk-Ulam pair then \((K, V, Q)\) is a Tucker triple for any \(G\)-invariant convex polytope in \(V\). The following proposition is an easy consequence.

**Proposition 2.** If a group \(G\) has the Borsuk-Ulam-property than it also has the Tucker-property.

**Proof.** Let \(k = |G| \geq 2\), \(n \geq 1\), \(N = n(k - 1)\) and let \(K\) be a \(G\)-invariant triangulation of \(E_N G\) refining the natural triangulation. Furthermore, let \(\lambda : \text{vert}(K) \to \text{vert}(\Diamond^n_k)\) be a \(G\)-equivariant map. Let \(\Lambda : K \to \Diamond^n_k \subset E_{n,k}\) be the linear (affine) extension of this map. Since by assumption \(G\) has the Borsuk-Ulam property, there exists a simplex \(\sigma \in K\) and \(x \in \sigma\), such that \(\Lambda(x) = 0\). It follows that \(0 \in \text{conv}(\Lambda(\text{vert}(\sigma)))\).

The following theorem shows that the converse to Proposition \[2\] is also true.

**Theorem 2.** If a group \(G\) has the Tucker property than it also has the Borsuk-Ulam property.

**Proof.** For the sake of contradiction assume that \((E_N G, (W_G)^n)\) is not a Borsuk-Ulam pair for some \(n \geq 1\). In other words we assume that there exists a \(G\)-equivariant map \(f : E_N G \to (W_G)^n\) such that \(0 \notin \text{Image}(f)\). By compactness we can assume that \(\text{Image}(f) \subset \mathbb{R}^k \setminus U\) for some neighborhood of \(0\) and by rescaling we can assume that \(U = \Diamond^n_k\). The radial projection \(R : \mathbb{R}^k \setminus \Diamond^n_k \to \partial(\Diamond^n_k)\) to the boundary of the crosspolytope is \(G\)-equivariant, so we can assume that \(\text{Image}(f) \subset \partial(\Diamond^n_k)\). By the (equivariant) simplicial-approximation theorem, there is
a $G$-invariant subdivision $K$ of the complex $E_N G$ and a simplicial, $G$-equivariant map $g : K \to \partial(\mathcal{N}_k^c)$ approximating $f$ in a suitable sense. Then the restriction of $g$ on the 0-skeleton $K^{(0)} = \text{vert}(K)$ defines a $G$-equivariant labelling $\phi : \text{vert}(K) \to \text{vert}(\mathcal{N}_k^c)$ which contradicts the assumption that $(K,(W_G)^n,\mathcal{N}_k^c)$ is a Tucker triple. Indeed, $0 \notin \text{conv}(\phi(\text{vert}(\sigma)))$ for each $\sigma \in K$ is a consequence of the fact that $\text{conv}(\phi(\text{vert}(\sigma))) \subset g(\sigma) \subset \partial(\mathcal{N}_k^c)$. □

8. $G$-Tucker-property and combinatorial proofs

In this section we discuss a possible application of the $G$-Tucker-property towards a combinatorial problem: finding approximate solutions for the consensus-$\frac{1}{k}$-division problem. This also relates to a conjecture of Simmons and Su, which we will discuss as well.

The necklace problem and consensus-$\frac{1}{k}$-division. In [2], Alon investigates the theft of a necklace with beads of $n$ different colors by $k$ thieves. Under the assumption that there are a multiple of $k$ beads of each color and that the necklace is opened at the clasp, the question is whether it is possible to always cut the necklace at $n(k-1)$ places and to distribute the resulting pieces among the thieves in such a way that each of them gets the same number of beads of each color. In order to show that this is indeed the fact, Alon proved the equipartition of measures theorem that we discussed in Section 5 and showed how it applies to the discrete situation.

In [20], Simmons and Su consider the question of subdividing an object into two portions in such a way that $n$ given people believe that the two portions are equal in value. If the problem is modeled in terms of simultaneously equipartitioning a set of $n$ measures, the existence of such a partition is given by Alon’s theorem for $k = 2$, but there is no algorithm on how to obtain such a solution. Using Tucker’s lemma Simmons and Su describe an algorithm to obtain an $\varepsilon$-approximate solution to the problem for any given $\varepsilon > 0$. Here we want to address the generalization of this problem already mentioned in [20]: Subdividing an object into $k$ portions such that according to the $n$ individual measures all the portions have value $\frac{1}{k^2}$. Again the existence of such a subdivision is guaranteed by Alon’s theorem. But what about algorithmic $\varepsilon$-approximations?

This problem can be divided into two steps: Finding a constructive proof of the Tucker-property for $G$, and showing how the Tucker-property can be applied to yield approximate solutions. So far we were only able to provide the second step, which we will demonstrate here. As in the proof of Alon’s theorem the approximation problem easily reduces to the case $k = p$, $p$ prime.

Theorem 3. Let $k = p$ be a prime, $\mu_1,\ldots,\mu_n$ be continuous probability measures on the unit interval, and $\varepsilon > 0$ be given. Then a single application of the Tucker-property for $G = \mathbb{Z}_p$ yields $n(p-1)$ cuts of the unit interval together with a partition $F_1,\ldots,F_p$ of the resulting intervals, such that $|\mu_i(\cup F_j) - \frac{1}{k^2}| < \varepsilon$ for all $i$ and $j$.

The following proof relies on ideas from [20], but has to deal with some technical problems that do not occur in the case $k = 2$.

Proof. As in the proof of Alon’s theorem in Section 5 the elements $v$ of $E_N G$, $N := n(p-1)$, encode $n(p-1)$ cuts of the unit interval together with a partition $F(v) = \{F_1,\ldots,F_p\}$ of the resulting intervals. By continuity of the $\mu_i$, let $K$ be a $G$-invariant triangulation of $E_N G$ refining the natural triangulation with the
property that for all pairs of neighboring vertices \(v, w\) of \(K\) with corresponding partitions \(\mathcal{F}(v) = \{F_1, \ldots, F_p\}\) and \(\mathcal{F}(w) = \{F'_1, \ldots, F'_p\}\) the inequality
\[
|\mu_i(\cup F_j) - \mu_i(\cup F'_j)| < \frac{\varepsilon}{(p-1)^2}
\]
holds for all \(i\) and \(j\). Again, such a triangulation can be obtained by iterated barycentric subdivision. We will now define a labeling
\[
\lambda: \text{vert}(K) \rightarrow G \times [n]
\]
\[
v \mapsto (\lambda_1(v), \lambda_2(v)),
\]
where \(\lambda_1\) and \(\lambda_2\) are defined as follows. Let \(v \in \text{vert}(K)\) with corresponding partition \(\mathcal{F}(v) = \{F_1, \ldots, F_p\}\). Consider \(m(v) := \min_{i,j}\{\mu_i(\cup F_j)\}\), and let \(\lambda_2(v)\) be the smallest \(i\) such that there exists a \(j\) with \(\mu_i(\cup F_j) = m(v)\). In order to define \(\lambda_1(v)\) consider the sign vector \((\varepsilon_1, \ldots, \varepsilon_p) \in \{+, -\}^p\) defined by
\[
\varepsilon_j := \begin{cases} 
+1, & \text{if } \mu_{\lambda_2(v)}(\cup F_{j+1}) > \mu_{\lambda_2(v)}(\cup F_j), \\
-1, & \text{if } \mu_{\lambda_2(v)}(\cup F_{j+1}) < \mu_{\lambda_2(v)}(\cup F_j), \\
0, & \text{if } \mu_{\lambda_2(v)}(\cup F_{j+1}) = \mu_{\lambda_2(v)}(\cup F_j),
\end{cases}
\]
with \(j\)-indices considered modulo \(p\). If \((\varepsilon_1, \ldots, \varepsilon_p) = (0, \ldots, 0)\) then \(\mu_{\lambda_2(v)}(\cup F_1) = \mu_{\lambda_2(v)}(\cup F_2) = \cdots = \mu_{\lambda_2(v)}(\cup F_p) = \frac{1}{p}\) and hence, by definition of \(\lambda_2(v)\), we have \(\mu_i(\cup F_j) = \frac{1}{p}\) for all \(i\) and \(j\). In this lucky event we found what we were looking for.

If \((\varepsilon_1, \ldots, \varepsilon_p) \neq (0, \ldots, 0)\), then in particular the sign vector is not constant, and we can define \(\lambda_1(v) := [j]\), where \(j \in \{1, \ldots, p\}\) is such that
\[
(\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_p, \varepsilon_1, \ldots, \varepsilon_{j-1})
\]
is the lexicographic smallest vector among all cyclic permutations of the \(\varepsilon\)-vector, with respect to the linear order \(< 0 < +\) of \(\{+, -, 0\}\). Thus we have defined a \(G\)-equivariant labeling. We can think of this labeling as saying which person \(\lambda_2(v)\) is distressed most by the fact that according to its measure the portion \(\cup F_{\lambda_1(v)}\) is the smallest with respect to all portions and measures. By the Tucker-property for \(G\), we obtain an \(i_0 \in [n]\) and a \((p-1)\)-simplex \(\{v_{i_1}, \ldots, v_{i_p}\}\) such that \(\lambda(v_{i_r}) = ([r], i_0)\) for \(r = 1, \ldots, p\). In other words, for person \(i_0\) there exist \(p\) different partitions very close to each other, such that the person is distressed about a different portion every time. But this means that they all must have similar size close to \(\frac{1}{p}\). Since person \(i_0\) was most distressed, the portions must be similar in size for all other people as well. From here on we will just carry out the filthy details of this consideration.

Let \(\mathcal{F}(v_r) := \{F'_1, \ldots, F'_p\}\), and define \(x^i_{rj} := \mu_i(\cup F'_r)\). We have the following properties.

1. For all \(i, j\) and \(r \neq r'\): \(|x^i_{rj} - x^i_{r'j}| < \frac{\varepsilon}{(p-1)^2}\) by definition of the triangulation,
2. for all \(i, r\): \(\sum_{j=1}^{p} x^i_{rj} = 1\) since the \(\mu_i\) are probability measures,
3. and we have for all \(j\): \(m(v_r) = x^i_{0r} \leq m(v_r) = (\varepsilon, i_0)\).

First, we will be concerned with the numbers \(x^i_{0j}\). Properties (2) and (3) yield for all \(r\): \(x^i_{0r} \leq \frac{1}{p}\). Hence by (1), we obtain for all \(r\) and \(j\): \(x^i_{0r} < \frac{1}{p} + \frac{\varepsilon}{(p-1)^2}\), which together with (2) yields \(x^i_{0r} > \frac{1}{p} - \frac{\varepsilon}{(p-1)^2}\). Now let \(i, j\), and \(r\) be arbitrary. Then by the definition of the labeling \(x^i_{rj} \geq m(v_r) = x^i_{0r} > \frac{1}{p} - \frac{\varepsilon}{(p-1)^2}\). By (2) we therefore obtain \(x^i_{rj} < \frac{1}{p} + \varepsilon\). The last two inequalities yield the result. \(\square\)
Note that the previous theorem also yields a proof of Alon’s theorem by compactness of the space $E_N$. 

**A conjecture of Simmons and Su.** In [20] Simmons and Su consider the following space

$$S^n_k = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} |z_i| = 1, z_i^k = |z_i|^k \right\}. $$

Let $\omega := e^{2\pi i/k}$, then the elements of $S^n_k$ have the form $(t_0\omega^{j_0}, \ldots, t_n\omega^{j_n})$ for some $j_i \in \{1, \ldots, k\}$ and $t_i \geq 0$ with $\sum t_i = 1$. The symmetric group $\text{Sym}(k)$ acts on $S^n_k$ by

$$\pi \cdot (t_0\omega^{j_0}, \ldots, t_n\omega^{j_n}) = (t_0\omega^{\pi(j_0)}, \ldots, t_n\omega^{\pi(j_n)}).$$

**Conjecture (Simmons & Su [20]).** Suppose that $S^n_k$ is triangulated invariantly with respect to the action of the symmetric group $\text{Sym}(k)$, and suppose that the vertices $V$ of the triangulation are labeled by a function

$$\ell : V \rightarrow \{\omega^j m : 1 \leq j \leq k, 1 \leq m \leq n\},$$

such that for $\pi \in \text{Sym}(k)$ the condition $\ell(\pi(v)) = \pi(\ell(v))$ holds for all $v \in V$. Then there must exist $k$ adjacent vertices in the triangulation with labels $\{\omega^j m : 1 \leq j \leq k\}$ for a fixed $m$.

We will see that this conjecture holds in the case where $k$ is a prime power $p^r$, in which case the requirement that everything is symmetric with respect to the whole symmetric group can be weakened to symmetry with respect to the subgroup $G = (\mathbb{Z}_p)^r$.

**Proposition 3.** Let $G$ be any group of order $k$. Then $G$ considered as a subgroup of $\text{Sym}(k)$ via an enumeration $\{g_1, \ldots, g_k\}$ of $G$ acts on $S^n_k$ and there is an equivariant homeomorphism from $S^n_k$ to $E_n G$.

**Proof.** The homeomorphism is given by $(t_0\omega^{j_0}, \ldots, t_n\omega^{j_n}) \mapsto (t_0g_{j_0}, \ldots, t_ng_{j_n})$. \(\square\)

**Corollary 2.** The conjecture by Simmons and Su holds in the case $k = p^r$ a prime power, in which case the requirement that everything is symmetric with respect to the whole symmetric group $\text{Sym}(k)$ can be weakened to symmetry with respect to the subgroup $G = (\mathbb{Z}_p)^r$.

**Proof.** As stated in Section 7, $G$ has the Tucker-property. Now apply the previous proposition. \(\square\)

9. **Towards constructive proofs in topological combinatorics**

The progress in further pursuing the scheme discussed in Section 3 heavily depends on the question whether there is a constructive proof of the Tucker-property for $G$ at least in the case where $G = \mathbb{Z}_p$ for $p > 2$ a prime. Such a proof would dramatically increase the chances for obtaining combinatorial proofs for many theorems from topological combinatorics.

The scheme for discovering constructive proofs for combinatorial statements, originally deduced by topological arguments, outlined in Section 3 is certainly not unique. There ought to exist other approaches which may be more suitable...
for some applications. For example the Tucker lemma and its generalizations can be incorporated into a general problem of finding combinatorial formulas for Stiefel-Whitney and other characteristic cohomological classes. More generally, the topological methods used in combinatorial applications are often naturally seen as part of topological obstruction theory. Consequently, there ought to be a close relationship between the problem of finding constructive proofs with the program of developing effective obstruction theory which has been recognized as one of the problems paradigmatic for computational topology.

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