EQUATIONS DEFINING TORIC VARIETIES

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This article is based on my lecture at the AMS Summer Institute in Algebraic Geometry at Santa Cruz, July 1995. The topic is toric ideals, by which I mean the defining ideals of subvarieties of affine or projective space which are parametrized by monomials. With one exception there will be no proofs given in this paper. For most assertions which are stated without a reference, the reader can find a proof and combinatorial details in my monograph “Gröbner Bases and Convex Polytopes” [Stu]. Connections to recent developments in algebraic geometry and theoretical physics are discussed in the beautiful survey by David Cox [Cox].

1. Toric ideals and their polytopes

In many branches of mathematics and its applications one encounters algebraic varieties which are parametrized by monomials. Such varieties are called toric varieties in this article. This stands in contrast to common practise in algebraic geometry (see [Cox]), where toric varieties are assumed to be normal. From the point of view taken by the author it is more natural to start out with the following definition. Let \( A = (a_1, \ldots, a_n) \) be any integer \( d \times n \)-matrix. Each column vector \( a_i = (a_{i1}, \ldots, a_{id})^T \) is identified with a Laurent monomial \( t^{a_i} = t_{a_{i1}} \cdots t_{a_{id}} \). The toric ideal \( I_A \) associated with \( A \) is the kernel of the \( k \)-algebra homomorphism

\[
\mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}], \quad x_i \mapsto t^{a_i}.
\]

Every vector \( u \) in \( \mathbb{Z}^n \) can be written uniquely as \( u = u_+ - u_- \), where \( u_+ \) and \( u_- \) are non-negative and have disjoint support. The difference of two monomials is called a binomial. An ideal generated by binomials is called a binomial ideal.

Lemma 1.1.
(a) The toric ideal \( I_A \) is generated by the binomials \( x^{u_+} - x^{u_-} \), where \( u \) runs over all integer vectors in the kernel of the matrix \( A \).
(b) An ideal in \( \mathbb{C}[x_1, \ldots, x_n] \) is toric if and only if it is prime and binomial.

Here is an easy method for computing generators of \( I_A \). Assume for simplicity that \( A \subset \mathbb{N}^d \). Then our toric ideal equals the elimination ideal

\[
I_A = \langle x_1 - t^{a_1}, x_2 - t^{a_2}, \ldots, x_n - t^{a_n} \rangle \cap \mathbb{C}[x_1, x_2, \ldots, x_n],
\]

which can be computed by lexicographic Gröbner bases in \( \mathbb{C}[x_1, \ldots, x_n, t_1, \ldots, t_d] \). More efficient algorithms for the same task are described in Section 12.1 of [Stu].

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The zero set of $I_A$ in affine $n$-space is denoted $X_A$ and called the affine toric variety defined by $A$. The dimension of $X_A$ equals the rank of the matrix $A$. If all columns of $A$ have the same coordinate sum, then the ideal $I_A$ is homogeneous and defines a projective toric variety $Y_A$ in $\mathbb{P}^{n-1}$. In what follows we identify $A$ with the point configuration given by its columns.

**Examples 1.2.** Here are some familiar examples of projective toric varieties.

(a) The twisted cubic curve in $\mathbb{P}^3$ is defined by four equidistant points on a line:

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$  

The corresponding toric ideal is generated by three quadratic binomials:

$$I_A = \langle x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2 \rangle.$$  

More generally, the $r$-th Veronese embedding of $\mathbb{P}^{n-1}$ equals $Y_A$ for

$$A = \left\{ (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n : i_1 + i_2 + \cdots + i_n = r \right\}.$$  

(b) All rational normal scrolls are toric. For instance, the cubic scroll $S_{2,1}$ in $\mathbb{P}^4$ is defined by the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$  

If we were to add a column vector $(0, 0, 2)$ to this matrix then we would get the quadratic Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$.

(c) The Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ into $\mathbb{P}^{r+s+1}$ is toric. Here $I_A$ is the ideal of $2 \times 2$-minors of an $(r+1) \times (s+1)$-matrix of indeterminates, and the configuration $A$ consists of the $(r+1)(s+1)$ points of the product of two regular simplices $\Delta_r \times \Delta_s$.

(d) Consider a generic point in the Grassmannian of lines in $\mathbb{P}^3$. The closure of its orbit under the natural $(\mathbb{C}^*)^4$-action is the toric variety $Y_A$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$  

The closure of any torus orbit in a flag variety arises from a configuration $A$ of weights in a $GL_n(\mathbb{C})$-module. That module is $\wedge^2 \mathbb{C}^4$ in the above case. □

The $d$-dimensional algebraic torus $(\mathbb{C}^*)^d$ acts on affine $n$-space $\mathbb{C}^n$ via

$$(x_1, \ldots, x_n) \mapsto (x_1 t^{a_1}, \ldots, x_n t^{a_n}).$$

The affine toric variety $X_A$ is the closure of the $(\mathbb{C}^*)^d$-orbit of the point $(1, 1, \ldots, 1)$. A basic invariant of $X_A$ is the convex polyhedral cone $\text{pos}(A)$ consisting of all non-negative linear combinations of column vectors in $A$. For a projective toric variety $Y_A$ we also consider the convex hull $\text{conv}(A)$ of the points in $A$. This is a convex polytope of dimension $\text{rank}(A) - 1$. Note that $\text{pos}(A)$ equals the cone over $\text{conv}(A)$. 


Proposition 1.3. 
(a) The \((\mathbb{C}^*)^d\)-orbits on the affine toric variety \(X_A\) are in order-preserving bijection with the faces of the cone \(\text{pos}(A)\).
(b) The \((\mathbb{C}^*)^d\)-orbits on the projective toric variety \(Y_A\) are in order-preserving bijection with the faces of the polytope \(\text{conv}(A)\).

We next determine the degree of a projective toric variety \(Y_A\) in \(\mathbb{P}^{n-1}\). Let \(L\) be the sublattice of \(\mathbb{Z}^d\) affinely generated by \(A\). We normalize the volume form on \(L \otimes \mathbb{R}\) in such a way that each primitive lattice simplex has unit volume. The normalized volume of the polytope \(\text{conv}(A)\) is a positive integer denoted \(\text{Vol}(A)\).

Theorem 1.4. The degree of \(Y_A\) in \(\mathbb{P}^{n-1}\) equals the normalized volume \(\text{Vol}(A)\).

It is instructive to verify Theorem 1.4 for the varieties in Examples 1.2.

(a) For the twisted cubic, \(\text{conv}(A)\) is a line segment of normalized length 3.
(b) For the cubic scroll \(S_{2,1}\), \(\text{conv}(A)\) is a quadrangle with three edges of unit length and one edge of length two. Its normalized area equals \(\text{Vol}(A) = 3\).
(c) The product of simplices \(\Delta_r \times \Delta_s\) can be triangulated into \(\binom{r+s}{r}\) simplices of unit volume. Hence the degree of the Segre variety equals \(\binom{r+s}{r}\).
(d) For the generic torus orbit on the Grassmannian of lines in \(\mathbb{P}^3\), the polytope \(\text{conv}(A)\) is a regular octahedron. It can be triangulated into four tetrahedra of unit volume. Therefore \(Y_A\) is a toric threefold of degree four in \(\mathbb{P}^5\).

The toric ideals \(I_A\) in Example 1.2 are generated by quadratic binomials, and their varieties \(Y_A\) are projectively normal (see Section 2). The quadratic generators are easy to find, in view of the special structure of the matrices \(A\). The simplicity of these geometric examples is misleading: for a general matrix \(A\) it is difficult to identify generators for \(I_A\). One objective of this article is to describe methods for studying and solving this problem. We illustrate this issue for two families of toric ideals which arise from an application in computational statistics.

Example 1.5. (The toric variety of the Birkhoff polytope)
Let \(A\) be the set of \(p \times p\)-permutation matrices. Here \(d = p^2\), \(n = p!\), and \(\text{conv}(A)\) is the Birkhoff polytope consisting of all doubly-stochastic matrices. The dimension of \(\text{conv}(A)\) and hence of the toric variety \(Y_A\) is \((p-1)^2\). The variables in the toric ideal \(I_A\) are indexed by permutations. For instance, for \(p = 3\) we have

\[ I_A = \langle x_{123}x_{231}x_{312} - x_{132}x_{213}x_{321} \rangle. \]

In general, the toric ideal \(I_A\) is generated by forms of degree \(p\). It is a challenging combinatorial problem to determine \(\text{Vol}(A) = \text{degree}(Y_A)\). The known values are:

\[
\begin{array}{cccccc}
\text{dim}(Y_A) & 4 & 9 & 16 & 25 & 36 \\
\text{degree}(Y_A) & 3 & 352 & 4718075 & 14666561365176 & 17832560768358341943028 \\
p & 3 & 4 & 5 & 6 & 7
\end{array}
\]

Another open problem is to bound the degree of the universal Gröbner basis of \(I_A\).

Example 1.6. (Looks like Segre but isn’t)
Fix integers \(r \leq s \leq t\). Let \(n = rst\), \(d = rs + rt + st\) and identify \(\mathbb{Z}^d\) with the
direct sum $\mathbb{Z}^r \times s \oplus \mathbb{Z}^s \times t$. We denote the standard basis vectors in the three components as $e_{ij}$, $e'_{ik}$ and $e''_{jk}$ respectively. Our configuration in this example is

$$\mathcal{A} = \{ e_{ij} \oplus e'_{ik} \oplus e''_{jk} : i = 1, \ldots, r, j = 1, \ldots, s, k = 1, \ldots, t \}.$$ 

The toric ideal $I_{\mathcal{A}}$ is the kernel of the ring map

$$\mathbb{C}[x_{ijk}] \to \mathbb{C}[u_{ij}, v_{ik}, w_{jk}], \quad x_{ijk} \mapsto u_{ij} \cdot v_{ik} \cdot w_{jk},$$

where the indices $i, j, k$ run as in $\mathcal{A}$. The smallest example is $r = s = t = 2$, where

$$I_{\mathcal{A}} = \langle x_{111}x_{122}x_{212}x_{221} - x_{112}x_{121}x_{211}x_{222} \rangle.$$ 

At first glance the projective toric variety $Y_{\mathcal{A}}$ looks similar to a Segre variety. But this is a deception. The following questions are widely open for general $r, s, t \geq 3$:

- Characterize the faces of $\text{conv}(\mathcal{A})$, i.e. the torus orbits on $Y_{\mathcal{A}}$.
- Determine the normalized volume of $\text{conv}(\mathcal{A})$, i.e. the degree of $Y_{\mathcal{A}}$.
- Find minimal generators for $I_{\mathcal{A}}$, or at least bound their degree.

2. Normal toric varieties

In algebraic geometry one often assumes that $X_{\mathcal{A}}$ is normal and that $Y_{\mathcal{A}}$ is projectively normal. This imposes strong combinatorial restrictions on the configuration $\mathcal{A}$. We first discuss these restrictions and then we present known results and open problems concerning the generators of the corresponding toric ideals $I_{\mathcal{A}}$.

Let $\mathbb{N}_{\mathcal{A}}$ denote the semigroup spanned by $\mathcal{A}$. Throughout this paper we assume that $\mathbb{N}_{\mathcal{A}} \cap -\mathbb{N}_{\mathcal{A}} = \{0\}$. This condition means that the semigroup algebra $\mathbb{C}[\mathcal{A}] := \mathbb{C}[x_1, \ldots, x_n]/I_{\mathcal{A}}$ has no non-trivial units. The semigroup $\mathbb{N}_{\mathcal{A}}$ lies in the intersection of the abelian group $\mathbb{Z}_{\mathcal{A}}$ and the convex polyhedral cone $\text{pos}(\mathcal{A})$:

$$\mathbb{N}_{\mathcal{A}} \subseteq \text{pos}(\mathcal{A}) \cap \mathbb{Z}_{\mathcal{A}}. \quad (2.1)$$

We say that the configuration $\mathcal{A}$ is normal if equality holds in (2.1).

**Lemma 2.1.** The affine variety $X_{\mathcal{A}}$ is normal if and only if $\mathcal{A}$ is normal.

If $\mathcal{A}$ is not normal, then one can replace it (e.g. using Algorithm 13.2 in [Stu]) by the unique minimal finite subset $\mathcal{B}$ of $\mathbb{Z}^d$ such that

$$\mathbb{N}_{\mathcal{B}} = \text{pos}(\mathcal{A}) \cap \mathbb{Z}_{\mathcal{A}}.$$ 

In this case $X_{\mathcal{B}}$ is the normalization of $X_{\mathcal{A}}$.

Singularities of toric varieties are characterized as follows:

**Lemma 2.2.** The affine toric variety $X_{\mathcal{A}}$ is smooth if and only if the semigroup $\mathbb{N}_{\mathcal{A}}$ is isomorphic to the free semigroup $\mathbb{N}^r$ for some $r$. 

Example 2.3. The affine toric surface $X_A$ defined by $A = \{(2,0), (1,1), (0,2)\}$ is normal but not smooth. It is the cone over a smooth quadric curve in $\mathbb{P}^2$.

From now on we assume that the ideal $I_A$ is homogeneous. It defines a projective toric variety $Y_A$ in $\mathbb{P}^{n-1}$. The intersection of $Y_A$ with the affine chart $\{x_i \neq 0\}$ equals the affine toric variety $X_{A-a_i}$ defined by the configuration

$$A - a_i = \{a_1 - a_i, \ldots, a_{i-1} - a_i, a_{i+1} - a_i, \ldots, a_n - a_i\}.$$ 

Thus $Y_A$ has an open cover consisting of $n$ affine toric varieties. In general the number $n$ can be lowered, by the following proposition.

Proposition 2.4. The projective toric variety $Y_A$ is covered irredundently by the affine varieties $X_{A-a_i}$, where $a_i$ runs over the vertices of the polytope $\operatorname{conv}(A)$.

Thus $Y_A$ is normal (resp. smooth) if and only if each affine chart $X_{A-a_i}$ in the above cover is normal (resp. smooth). In particular, $Y_A$ is normal if and only if

$$N(A - a_i) = \operatorname{pos}(A - a_i) \cap \mathbb{Z}(A - a_i) \quad \text{for} \quad i = 1, \ldots, n.$$ 

Let $H_A(s)$ denote the Hilbert polynomial of $Y_A$, that is, $H_A(s) = \dim_k(\mathbb{C}[A]_s)$ for $s \gg 0$. The Ehrhart polynomial $E_A$ of the lattice polytope $\operatorname{conv}(A)$ is defined by

$$E_A(s) = \#(s \cdot \operatorname{conv}(A) \cap \mathbb{Z}^d) \quad \text{for all} \quad s \in \mathbb{N}.$$ 

The relationship between the Hilbert polynomial and the Ehrhart polynomial was studied by A. Khovanskii in [Kho].

Proposition 2.5. A projective toric variety $Y_A$ is normal if and only if its Hilbert polynomial $H_A$ is equal to its Ehrhart polynomial $E_A$.

A much stronger requirement is to ask that $Y_A$ be projectively normal, which means that the affine cone $X_A$ over $Y_A$ is normal, i.e., $N_A = \operatorname{pos}(A) \cap \mathbb{Z}A$.

Example 2.6. (Normal versus projectively normal)

Let $d = 2, n = 4, r \geq 4$ and $A = \{(r,0), (r-1,1), (1,r-1), (0,r)\}$. Here $Y_A$ equals the projective line $\mathbb{P}^1$. It is smooth (hence normal) but not projectively normal. The toric ideal $I_A \subset \mathbb{C}[x_1, x_2, x_3, x_4]$ is minimally generated by one quadric and $r - 1$ binomials of degree $r - 1$. In this example $X_A$ is not Cohen-Macaulay. □

A well-known result due to M. Hochster states that normal affine toric varieties are Cohen-Macaulay. A consequence of this fact is the following degree bound.

Theorem 2.7. If $Y_A$ is a projectively normal $r$-dimensional toric variety then the homogeneous toric ideal $I_A$ is generated by binomials of degree at most $r$.

This degree bound is sharp since the toric hypersurface $x_0^r = x_1 x_2 \cdots x_r$ is projectively normal. It is unknown whether Theorem 2.7 extends to Gröbner bases. In the following conjecture we do not allow any linear changes of coordinates.

Conjecture 2.8. If $Y_A$ is projectively normal $r$-dimensional toric variety, then $I_A$ has a Gröbner basis consisting of binomials of degree at most $r$.

To appreciate the distinction between generators and Gröbner bases consider the following example: If $d = 3, n = 8$ and $A = \{(3,0,0), (0,3,0), (0,0,3), (2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,2,1), (0,1,2)\}$, then $I_A$ is generated by quadrics but has no quadratic Gröbner basis. It is unknown whether $\mathbb{C}[A]$ is a Koszul algebra.

Another problem is to better understand the effect of smoothness on the degrees of the defining equations.
Conjecture 2.9. If $Y_A$ is a smooth and projectively normal toric variety, then the toric ideal $I_A$ is generated by quadratic binomials.

It may even be conjectured that in this case $I_A$ possesses a quadratic Gröbner basis. It was briefly believed in January '95 that Conjecture 2.9 had been proven, but that proof was withdrawn. The answer is affirmative for scrolls by a result of Ewald and Schmeinck [EwS]. Conjecture 2.9 is also known to be true for toric surfaces. This is a consequence of the following theorem due to Bruns, Gubeladze and Trung [BGT].

Theorem 2.10. Let $Y_A$ be a projectively normal toric surface and suppose that the polygon $\text{conv}(A)$ has at least four lattice points on its boundary. Then $I_A$ possesses a quadratic lexicographic Gröbner basis.

There is an important sufficient condition for $Y_A$ to be projectively normal. (It is not necessary; see Example 13.17 in [Stu]). A triangulation $\Delta$ of the configuration $A$ is a triangulation of the polytope $\text{conv}(A)$ whose vertices lie in $A$. We call the triangulation $\Delta$ unimodular if each simplex in $\Delta$ is a primitive lattice simplex.

Theorem 2.11. There exists a unimodular triangulation of $A$ if and only if some initial monomial ideal of $I_A$ is radical. In this case $Y_A$ is projectively normal.

The following theorem was established by Knudsen and Mumford in the early days of toric geometry. It is a key ingredient in Mumford’s proof of the semi-stable reduction theorem. See [KKMS] for details and the proof of Theorem 2.12.

Theorem 2.12. Let $Q$ be any lattice polytope in $\mathbb{Z}^d$. There exists an integer $m \gg 0$ such that the configuration $mQ \cap \mathbb{Z}^d$ possesses a unimodular triangulation.

It would be interesting to find an effective version of this theorem.

Problem 2.13. Does there exist a bound $M(d)$ such that, for every $m \geq M(d)$ and every lattice polytope $Q$ in $\mathbb{Z}^d$, the set $mQ \cap \mathbb{Z}^d$ has a unimodular triangulation?

In our discussion so far $Y_A$ was given as an explicit subvariety of some projective space $\mathbb{P}^{n-1}$. What can be said about $Y_A$ as an abstract scheme, independently of any choice of a very ample line bundle? This is where polyhedral fans enter the picture. Let $Q$ be any polytope in a real vector space $V$. For a face $F$ of $Q$ we consider the set of linear functionals on $Q$ which attain their maximum at $F$. This is a convex polyhedral cone in the dual space $V^*$. The collection of these cones, as $F$ runs over all faces of $Q$, is a polyhedral fan. It is called the normal fan of $Q$.

Proposition 2.14. Two projective toric varieties $Y_A$ and $Y_B$ have isomorphic normalizations if and only if their polytopes $\text{conv}(A)$ and $\text{conv}(B)$ have the same normal fan.

Here “isomorphic” refers to an isomorphism of equivariant torus embeddings. If $Y_A$ is normal, then the normal fan of $\text{conv}(A)$ retains just enough information to remember $Y_A$ as an abstract torus embedding. But it forgets the specific line bundle which was used to map $Y_A$ into $\mathbb{P}^{n-1}$. In the synthetic approach to toric varieties, as presented in the books [Ful] and [Oda], the normal fan comes before the polytope. Starting with any complete fan, one constructs an abstract complete normal toric variety by gluing affine pieces as in Proposition 2.4. See also [Cox].
Example 2.15. Fix positive integers \( i < j < k \) and consider the configuration
\[
\mathcal{A} := \text{conv} \{ (i, j, k), (i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i) \} \cap \mathbb{Z}^3.
\]
The normal fan of the hexagon \( \text{conv}(\mathcal{A}) \) is independent of the choice of \( i, j, k \). All toric surfaces \( Y_{\mathcal{A}} \) arising for different choices of \( i, j, k \) are normal and isomorphic to one another. The abstract scheme \( Y_{\mathcal{A}} \) equals \( \mathbb{P}^2 \) blown up at three points. All ideals \( I_{\mathcal{A}} \) possess quadratic lexicographic Gröbner bases, by Theorem 2.10.

In many applications of toric geometry one encounters toric varieties \( Y_{\mathcal{A}} \) which are not normal. Or sometimes they are normal but this is difficult to verify. We describe two instances of the latter kind arising from representation theory.

Let \( G \) be a connected semi-simple algebraic group over \( \mathbb{C} \). Fix a maximal torus \( (\mathbb{C}^*)^d \) in \( G \), let \( P \) be a parabolic subgroup containing \( (\mathbb{C}^*)^d \) and consider the flag variety \( G/P \). The following result is due to R. Dabrowski [Dab].

**Theorem 2.16.** The closure in \( G/P \) of a generic \( (\mathbb{C}^*)^d \)-orbit is normal.

It is open whether the generic torus orbit closures are projectively normal for all very ample line bundles on \( G/P \). It is also open whether the closures of all (non-generic) \( (\mathbb{C}^*)^d \)-orbits are normal, or even projectively normal. This latter conjecture is known to be true in the case when \( G/P \) is the classical Grassmannian \( Gr(r, \mathbb{C}^d) \) in its Plücker embedding. In this case \( \mathcal{A} \) is a subset of the **hypersimplex**
\[
\{(i_1, \ldots, i_d) \in \{0, 1\}^d : i_1 + \cdots + i_d = r\}
\]
such that \( \mathcal{A} \) consists of the incidence vectors of the bases of a realizable matroid. For instance, in Example 1.2 (d) that matroid is the uniform rank 2 matroid on 4 elements. We refer to [GGMS] for an introduction to matroids from an algebro-geometric point of view. The following result is due to Neil White [Wh1].

**Theorem 2.17.** Let \( \mathcal{A} \) be the set of incidence vectors of the bases of a matroid. Then \( Y_{\mathcal{A}} \) is projectively normal.

We close with a reformulation of a combinatorial conjecture in [Wh2].

**Conjecture 2.18.** Let \( \mathcal{A} \) be the set of incidence vectors of the bases of a matroid. Then the homogeneous toric ideal \( I_{\mathcal{A}} \) is generated by quadratic binomials.

### 3. Binomial Zoo

Our objective is to understand the minimal generators and Gröbner bases of the toric ideal \( I_{\mathcal{A}} \). To this end we introduce the following three definitions. A binomial \( x^{u^+} - x^{u^-} \) in \( I_{\mathcal{A}} \) is called a **circuit** if its support (i.e. the set of variables appearing in that binomial) is minimal with respect to inclusion. We write \( \mathcal{C}_{\mathcal{A}} \) for the set of all circuits in \( I_{\mathcal{A}} \). Geometrically speaking, we consider all images of \( X_{\mathcal{A}} \) under projection into coordinate subspaces of \( \mathbb{P}^{n-1} \); such an image is called a circuit of \( X_{\mathcal{A}} \) if it has codimension 1 in its coordinate subspace. We define the **universal Gröbner basis** \( \mathcal{U}_{\mathcal{A}} \) to be the union of all reduced Gröbner bases of \( I_{\mathcal{A}} \). We say that a binomial \( x^{y^+} - x^{y^-} \) in \( I_{\mathcal{A}} \) lies in the **Graver basis** \( \mathcal{G}_{\mathcal{A}} \) if there exists no other binomial \( x^{y^+} - x^{y^-} \in I_{\mathcal{A}} \) such that \( x^{y^+} \) divides \( x^{u^+} \) and \( x^{y^-} \) divides \( x^{u^-} \).
Proposition 3.1. For any toric ideal $I_A$ we have the inclusions

\[(3.1) \quad C_A \subseteq U_A \subseteq \text{Gr}_A.\]

Each of the four combinations of strict or non-strict inclusions is possible.

Examples 3.2.

(1) For the twisted cubic curve in Example 1.2 (a) we have

\[
C_A = \{x_1 x_3 - x_2^2, x_2 x_4 - x_3^2, x_1^2 x_4 - x_3^3, x_1 x_1^2 - x_3^3\} \quad \text{and} \quad U_A = \text{Gr}_A = C_A \cup \{x_1 x_4 - x_2 x_3\}.
\]

In this example the circuits do not generate the ideal $I_A$ although they do define the twisted cubic as a subscheme of $\mathbb{P}^3$.

(2) For the Veronese surface in $\mathbb{P}^5$, the set of circuits $C_A$ equals the universal Gröbner basis $U_A$, but the Graver basis $\text{Gr}_A$ properly contains $U_A$.

In general, the circuits of a projective toric variety $Y_A$ do not define $Y_A$ scheme-theoretically. But they do set-theoretically. We state this result in the affine case.

Theorem 3.3. The toric variety $X_A$ is cut out by its circuits: $\text{Rad}(C_A) = I_A$. The embedded components of $(C_A)$ are supported on torus orbits of $X_A$, i.e. every associated prime of $(C_A)$ has the form $I_{F_G \cap A}$ for some face $F$ of the cone $\text{pos}(A)$.

Theorem 3.3 was proved jointly with David Eisenbud in [EiSt]. The main result in [EiSt] concerns the decomposition of arbitrary binomial schemes. It underlines the importance of toric ideals as combinatorial building blocks in algebraic geometry:

Theorem 3.4. Let $I$ be any binomial ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Then $I$ possesses a primary decomposition into primary binomial ideals. In particular, every associated prime of $I$ has the form $I_A + (x_{i_1}, \ldots, x_{i_r})$ for some configuration $A$.

There is an important class of toric varieties for which both inclusions in (3.1) are equalities. They are called unimodular and defined by the following theorem.

Theorem 3.5. For a configuration $A$ the following conditions are equivalent:

1. Every triangulation of $A$ is unimodular.
2. For every subset $B$ of $A$, the quotient group $\mathbb{Z}A/\mathbb{Z}B$ is free abelian.
3. Every initial monomial ideal of $I_A$ is a radical ideal.
4. In every circuit of $I_A$ both monomials are square-free.

If these four equivalent conditions hold, then we call $A$ unimodular.

Proposition 3.6. If $A$ is unimodular then $X_A$ is normal.

Example 3.7. The term “circuits” originates from the following class of unimodular toric varieties. Let $G$ be a finite directed graph. We label the vertices of $G$ by $1, 2, \ldots, d$ and we encode the edges of $G$ as differences of unit vectors in $\mathbb{Z}^d$:

\[
A_G = \{e_i - e_j \in \mathbb{Z}^d : (i, j) \text{ is a directed edge of } G\}.
\]
The unimodularity of $A_G$ is a basic result in matroid theory. The circuits in the toric ideal $I_{A_G}$ correspond to the directed circuits in $G$. The following examples of circuits of length five illustrate this correspondence:

| Directed circuit in $G$ | Circuit in $I_{A_G}$ |
|------------------------|----------------------|
| $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ | $x_{12}x_{23}x_{34}x_{45}x_{51} - 1$ |
| $1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \leftarrow 1$ | $x_{12}x_{23}x_{45}x_{51} - x_{43}$ |

If $G$ is the complete directed bipartite graph $K_{r,s}$ then $I_{A_G}$ is the ideal of $2 \times 2$-minors of an $r \times s$-matrix of indeterminates and $Y_{A_G}$ is the Segre variety $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$. The degrees of the circuits in $I_{A_G}$ range from 2 to $\max\{r, s\}$.

If $G$ is a complete graph, then $A_G$ equals the root system of type $A_n-1$. The toric variety $X_{A_G}$ is the closure of a generic $(\mathbb{C}^+)^n$-orbit in the adjoint representation of $GL_n(\mathbb{C})$. Passing to subgraphs corresponds to passing to non-generic orbit closures in the adjoint representation. Thus the affine toric varieties defined by directed graphs are precisely the closures of $(\mathbb{C}^+)^n$-orbit in the adjoint representation of $GL_n(\mathbb{C})$. These toric varieties are all normal by Proposition 3.6. □

A configuration $X_A$ is called hereditarily normal if all closures of torus orbits on $X_A$ are normal. This is equivalent to the requirement that $X_B$ is normal for every subset $B$ of $A$. Unimodular configurations are hereditarily normal, but not conversely. Normality of torus orbits for arbitrary semisimple algebraic groups was studied by J. Morand [Mor]. She showed that the root systems of types $A_n$, $B_2$, $C_2$, $D_4$ are hereditarily normal and that all other root systems are not hereditarily normal.

The following criterion underlines the geometric significance of the circuits.

**Theorem 3.8.** An affine toric variety $X_A$ is hereditarily normal if and only if every circuit in $I_A$ has at least one square-free monomial.

The Graver basis $Gr_A$ has the following geometric interpretation. Consider the closure of the affine variety $X_A \subset \mathbb{C}^n$ in the $n$-fold product of projective lines $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. It is defined by a toric ideal $I_{\Lambda(A)}$ in $\mathbb{C}[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]$, which is homogeneous with respect to each pair of variables $(x_i, y_i)$. Here $\Lambda(A)$ is a certain configuration of $2n$ vectors in $\mathbb{Z}^{n+d}$, which is called the Lawrence lifting of $A$. A key property of such configurations $\Lambda(A)$ is the following.

**Theorem 3.9.** The defining ideal $I_{\Lambda(A)}$ of a toric subvariety of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is minimally generated by its Graver basis $Gr_{\Lambda(A)}$.

The original ideal $I_A$ is recovered from $I_{\Lambda(A)}$ by dehomogenizing, that is, by replacing the variables $y_1, \ldots, y_n$ by 1. This induces a bijection between the Graver basis of $\Lambda(A)$ and the Graver basis of $A$. Under this bijection we have the following geometric interpretation of the Graver basis.

**Corollary 3.10.** The Graver basis of an affine toric variety consists of the minimal generators of the ideal of its closure in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

For instance, if $A$ is the configuration in Example 1.2 (a), then

$$I_{\Lambda(A)} = \langle x_1x_3y_2^2 - x_2y_1y_3, x_1x_4y_2y_3 - x_2x_3y_1y_4, x_2x_4y_1^2 - x_3^2y_2y_4, x_1x_3y_2^3 - x_2^3y_1y_4, x_1x_4y_2^3 - x_2^3y_1y_4, x_1x_3y_2^3 - x_3^3y_1y_4 \rangle.$$
This ideal defines a toric surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It is the closure of the affine cone over the twisted cubic curve. We get the Graver basis for $I_A$, the ideal of the twisted cubic, by setting $y_1, y_2, y_3, y_4$ to $1$ in these five binomials.

4. Degree Bounds

The following problem is of great interest in computational algebraic geometry.

**Conjecture 4.1.** Let $Y$ be an irreducible projective variety of degree $d$. Then the homogeneous prime ideal of $Y$ is generated by forms of degree at most $d$.

Many authors have concentrated on proving stronger inequalities involving the (Castelnuovo-Mumford) regularity $\text{reg}(Y)$ of $Y$. Let $I_Y$ denote the ideal of $Y$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then $\text{reg}(Y)$ is the maximum of the numbers $\deg(\sigma) - i$ where $\sigma$ is a minimal $i$-th syzygy of $I_Y$. Here $i = 0$ is for ideal generators, $i = 1$ for syzygies among them, etc... Following [BS], two equivalent definitions of regularity are:

1. $\text{reg}(Y)$ is the maximum degree of an element in the reduced Gröbner basis of $I_\phi(Y)$ with respect to the reverse lexicographic term order, where $I_\phi(Y)$ is the image of $I_Y$ under a generic linear automorphism $\phi \in \text{GL}_n(\mathbb{C})$;
2. $\text{reg}(Y)$ is the smallest integer such that $H^i(\mathbb{P}^{n-1}, I_Y(s)) = 0$ for all $i \geq 0$ and $s \geq \text{reg}(Y) - i$, where $I_Y$ is the ideal sheaf of $Y$.

The following conjecture due to Eisenbud and Goto [EG] implies Conjecture 4.1.

**Conjecture 4.2.** Let $Y$ be an irreducible projective variety, not contained in any hyperplane. Then $\text{reg}(Y) \leq \text{degree}(Y) - \text{codim}(Y) + 1$.

The Eisenbud-Goto inequality holds when $Y$ is arithmetically Cohen-Macaulay. Hence, by Hochster’s Theorem, it holds for projectively normal toric varieties. Conjecture 4.2 was proved for curves by Gruson-Lazarsfeld-Peskine [GLP], and for irreducible smooth surfaces and 3-folds by Lazarsfeld [Laz] and Ran [R].

Conjectures 4.1 and 4.2 are widely open in general, even for toric varieties $Y_A$. Recall from Theorem 1.4 that the degree of $Y_A$ equals the volume of the polytope $\text{conv}(A)$. Also the regularity of $Y_A$ can be expressed combinatorially, using the simplicial representation of Koszul homology (see e.g. Theorem 12.12 in [Stu]). A class of toric varieties for which the Eisenbud-Goto inequality is valid was identified by Irena Peeva and the author in [PS]. For toric varieties $Y_A$ in codimension 2 we explicitly construct the minimal free resolution of $I_A$. Our construction implies:

**Theorem 4.3.** Let $Y_A$ be a projective toric variety of codimension 2 in $\mathbb{P}^{n-1}$, not contained in any hyperplane. Then $\text{reg}(Y_A) \leq \text{degree}(Y_A) - 1$.

**Example 4.4.** The following family of toric surfaces in $\mathbb{P}^4$ shows that the inequality in Theorem 4.3 is tight. For any integer $d \geq 3$ let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & d \end{pmatrix}.$$

The surface $Y_A$ is not arithmetically Cohen-Macaulay, its toric ideal $I_A$ is generated by one quadric and $d$ forms of degree $d$, and $\text{degree}(Y_A) = \text{Vol}(\text{conv}(A)) = d + 1$.

Given an arbitrary toric variety in higher codimension, the following is the currently best general bound for its regularity in terms of its degree.
Theorem 4.5. Let \( Y_A \) be a projective toric variety in \( \mathbb{P}^{n-1} \). Then
\[
\text{reg}(Y_A) \leq n \cdot \text{degree}(Y_A) \cdot \text{codim}(Y_A).
\]

We shall outline the proof of Theorem 4.5. First, we make use of the fact that regularity is upper semi-continuous with respect to flat families. This implies
\[
\text{reg}(I_A) \leq \text{reg}(\text{in}_{\succeq}(I_A))
\]
for any term order \( \succeq \) on \( \mathbb{C}[x_1, \ldots, x_n] \). The Taylor resolution (see page 439 in [Eis]) for the monomial ideal \( \text{in}_{\succeq}(I_A) \) implies the inequality
\[
\text{reg}(\text{in}_{\succeq}(I_A)) \leq n \cdot \text{maxgen}(\text{in}_{\succeq}(I_A)),
\]
where \( \text{maxgen} \) denotes the maximal degree of any minimal generator. In view of the inclusion \( \mathcal{U}_A \subseteq \text{Gr}_A \) in Proposition 3.1, it suffices to prove the following:

Lemma 4.6. Let \( Y_A \) be a projective toric variety. Every binomial in the Graver basis \( \text{Gr}_A \) of the homogeneous ideal \( I_A \) has degree at most \( \text{degree}(Y_A) \cdot \text{codim}(Y_A) \).

For a set \( S \) of polynomials let \( \text{maxdeg}(S) \) denote the maximum degree of any element in \( S \). The proof of Lemma 4.6 is derived from the next two inequalities:
\[
\text{maxdeg}(C_A) \leq \text{degree}(Y_A)
\]
This holds because the hypersurface defined by any circuit is a projection of \( Y_A \). The second inequality is
\[
\text{maxdeg}(\text{Gr}_A) \leq \text{codim}(Y_A) \cdot \text{maxdeg}(C_A).
\]
This follows from a standard inequality for Hilbert bases of integer monoids. See Chapter 4 in [Stu] for details.

While the inequality (4.4) is tight, there seems to be some room for improvement left in (4.5). In my lecture at Santa Cruz I asked whether even the equality \( \text{maxdeg}(\text{Gr}_A) = \text{maxdeg}(C_A) \) might be true. This would have implied Conjecture 4.1 for all toric varieties. Unfortunately this equality was too good to be true. Serkan Hosten and Rekha Thomas found the following counterexample.

Example 4.7. We shall present a projective toric variety in \( \mathbb{P}^9 \) which satisfies
\[
16 = \text{maxdeg}(\text{Gr}_A) = \text{maxgen}(I_A) > \text{maxdeg}(C_A) = 15.
\]
Our starting point is the affine toric surface \( X_B \subset \mathbb{C}^5 \) defined by
\[
B = \begin{pmatrix} 1 & 3 & 4 & 6 & 0 \\ 0 & 0 & 0 & -5 & 1 \end{pmatrix}.
\]
We then construct their Lawrence lifting \( \mathcal{A} := \Lambda(B) \). In other words, \( I_A \) is the toric ideal defining the closure of \( X_B \) in \( \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). This ideal equals
\[
I_A = \langle x_2y_1^3 - x_3y_2, x_3y_1^4 - x_1y_3, x_3y_2^4 - x_2y_3, x_4y_1^2y_2^2 - x_2y_4y_5, x_4y_2y_5^2 - x_1y_4y_5, \\
x_4y_3y_4^2 - x_3y_4y_5, x_5y_2y_4^2 - x_3y_1y_5, x_5y_3y_5^2 - x_2y_1y_5, x_5y_4y_5^2 - x_1y_2y_5, \\
x_1x_2x_3y_2y_3 - x_2x_3y_1y_4y_5, x_1x_2x_4y_2y_3 - x_3y_1y_2y_5, x_1x_2x_5y_2y_3 - x_4y_1y_2y_5, \\
x_2x_3x_4y_2y_3 - x_1x_3y_1y_4y_5, x_2x_3x_5y_2y_3 - x_1x_4y_1y_4y_5, x_2x_4x_5y_2y_3 - x_1x_5y_1y_4y_5 \rangle.
\]
By Theorem 3.9, these 16 minimal generators of $I_A$ coincide with the Graver basis $Gr_A$. The first six binomials in this list are the circuits. Thus (4.6) holds for this example. We remark that the projective variety $Y_A \subset \mathbb{P}^9$ defined by $I_A$ satisfies $\text{codim}(Y_A) = 3$, $\text{degree}(Y_A) = 54$, and $\text{reg}(Y_A) = 17$. □

In light of this counterexample, I wish to propose the following improvement of (4.5). Consider any circuit $C \in C_A$ and regard its support $\text{supp}(C)$ as a subset of $A$. The lattice $\mathbb{Z}(\text{supp}(C))$ has finite index in the (possibly bigger) lattice $\mathbb{R}(\text{supp}(C)) \cap \mathbb{Z}A$. We call this index the index of the circuit $C$. We define the true degree of the circuit $C$ to be the product $\text{degree}(C) \cdot \text{index}(C)$. It can be shown that the true degree of any circuit is bounded above by $\text{degree}(Y_A)$. The following conjecture would thus imply Conjecture 4.1 for projective toric varieties.

**Conjecture 4.8.** The degree of any element in the Graver basis $Gr_A$ of a toric ideal $I_A$ is bounded above by the maximal true degree of any circuit $C \in C_A$.

Note that Conjecture 4.8 is consistent with Example 4.7 because the underlined circuit $x_2^2 x_5^{10} y_3^3 - x_3^3 y_4^2 y_5^{10}$ has index two and hence its true degree is 30.
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