MASSIVE SCALING LIMIT OF THE ISING MODEL: SUBCRITICAL ANALYSIS AND ISOMONODROMY

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ABSTRACT. We study the spin n-point functions of the planar Ising model on a simply connected domain Ω discretised by the square lattice δZ² under near-critical scaling limit. While the scaling limit on the full-plane C has been analysed in terms of a fermionic field theory, the limit in general Ω has not been studied. We will show that, in a massive scaling limit wherein the inverse temperature is scaled β ∼ βc − m₀δ for a constant m₀ < 0, the renormalised spin correlations converge to a continuous quantity determined by a boundary value problem set in Ω. In the case of Ω = C and n = 2, this result reproduces the celebrated formula of [WMTB76] involving the Painlevé III transcendent. To this end, we generalise the comprehensive discrete complex analytic framework used in the critical setting to the massive setting, which results in a perturbation of the usual notions of analyticity and harmonicity.

CONTENTS

1. Introduction 1
  1.1. Main Results 2
  1.2. Notation 3
  1.3. Proof of the Main Theorems 4
  1.4. Structure of the Paper 5

Acknowledgements

2. Massive Fermions
  2.1. Bounded Domain Fermions and Discrete Analysis 5
  2.2. Full-Plane Fermions 6
  2.3. Massive Complex Analysis: Continuous Fermions and their Uniqueness 10

3. Discrete Analysis: Scaling Limit
  3.1. Bulk Convergence 13
  3.2. Analysis near the Singularity 14

4. Continuum Analysis: Isomonodromy and Painlevé III

Appendix A. Harmonicity Estimates 20

References 23

1. INTRODUCTION

The Ising model is a classical model of ferromagnetism first introduced by Lenz [Len20], whose simplicity and rich emergent structure have allowed for applications in various areas of science. In two dimensions, it famously exhibits a continuous phase transition [Oms44, Yan52], where the characteristic length of the model diverges and the model becomes scale invariant. Consequently the model at the critical temperature βc is expected to exhibit conformal symmetry under scaling limit, a prediction which has been formalised in terms of the Conformal Field Theory [BPZ84].

Given the infinite dimensional array of 2D local conformal symmetry [DMS97], it is natural to study the scaling limit of the model not only on the full plane C but on an arbitrary simply connected domain Ω. Accordingly, recent research has focused on giving a rigorous description of the interaction between various physical quantities of the model under critical scaling limit and the conformal geometry of Ω, which results in explicit formulae for the limit of spin correlations at the microscopic scale [HoSm13, GHP18], at the macroscopic scale [CHI15], or in some mixture of the two [Hon10, CHI18] in terms of quantities such as the conformal radius and its derivatives.

Central to such analyses of the critical regime are discrete fermion correlations, which manifest themselves as discrete functions capable of encoding relevant physical quantities. They are discrete counterparts of the massless free fermion correlations in the continuous CFT, which turn out to be explicit holomorphic functions thanks to conformal symmetry. Discrete fermions instead enjoy a strong notion of discrete analyticity [Smi10], which, unlike some of its weaker variants, readily lends itself to precompactness estimates that ultimately yield convergence to the continuous fermion.

However, the free fermion is not an object unique to the massless (conformal) field theory; indeed, the general field theory of the free fermion specifies a mass parameter m, or equivalently a length scale ξ ∝ 1/m². In general, the corresponding regime in the Ising model is the near-critical scaling limit, where the deviation from criticality βc − β scales proportionally
to the lattice spacing $\delta$. Such scaling keeps the physical correlation length $\xi$ asymptotically constant, allowing the limit to be physically described by the massive fermion.

The discrete fermion survives in this near-critical setup, and discrete analyticity persists albeit in a perturbed sense [DGPT14, HKZ15, ST18]. While the continuous massive fermion is usually described by the two-dimensional massive Dirac equation, our strong discrete analyticity in fact features twice as many relations, resulting in (the discrete counterpart of) a perturbation of the ordinary Cauchy-Riemann equations in 1D. Since there are as many lattice equations in the near-critical limit as in the critical limit, it is natural to attempt to carry out in the former the analogues of analyses from the latter. We note here that another direction of research has recently focused on universality with respect to general lattice, see [Che18].

In this paper, we undertake the analysis of macroscopic Ising spin correlations on a simply connected domain $\Omega$ in the near-critical scaling limit where $\beta_c - \beta$ is held equal to $m_0\delta$ for a fixed $m_0 < 0$ with + boundary conditions. We establish the existence of scaling functions to which renormalised spin correlations converge, and show that their logarithmic derivatives are determined by an explicit boundary value problem set in $\Omega$. This extends the results of [CH115] to the massive regime (save for the conformal covariance, which should not hold), and our proof combines the strategies of that paper with a massive perturbation of analytic function theory, both in the discrete and the continuous settings. In the former, massive harmonic and holomorphic functions can be studied via their relation to massive (extinguished) random walk; in the latter, the perturbed Cauchy-Riemann equation is dubbed Vekua equation and extensively treated in a theory established by Carleman, Bers, and Vekua, among others [Ber56, Vek62].

In the full plane, the massive scaling limit of the spin correlations was revealed to exhibit a surprising integrability property. Wu, McCoy, Tracy, and Barouch [WMTB76] first demonstrated that the 2-point function on the plane can be described in terms of the Painlevé III transcendent. Subsequently, Sato, Miwa, and Jimbo [SMJ77] recast the continuous analysis in terms of isomonodromic deformation theory, where Painlevé equations are known to arise, and obtained a closed set of differential equations for the $n$-point function. Letting $\Omega = \mathbb{C}$, we reproduce the scaling limit in the case of full-plane (whose classical treatment is given in, e.g., [PalTr83, Pal07], setting up the continuous analysis. We explicitly carry out the 2-point case following the formulation of [KaKo80].

1.1. Main Results. Let $\Omega$ be a bounded simply connected domain with smooth boundary. We will treat the unbounded cases $\Omega = \mathbb{C}, \mathbb{H}$ as well. Define the rotated square lattice $\Omega_{\delta} := \Omega \cap \delta(1+i)\mathbb{Z}^2 = \Omega \cap \mathbb{C}_\delta$. We define the Ising probability measure $P = P_{\Omega_{\delta},\beta}$ with + boundary conditions at inverse temperature $\beta > 0$ on the space of spin configurations $\{\pm 1\}^{\Omega_{\delta}}$ by

$$P_{\Omega_{\delta},\beta}[\sigma : \Omega_{\delta} \to \{\pm 1\}] \propto \exp \sum_{i > j} \beta \sigma_i \sigma_j,$$

where the sum is over pairs $\{i, j\} \subset \mathbb{C}_\delta$, $i \in \Omega_\delta$ such that $|i - j| = \sqrt{2}\delta$ and we define $\sigma_j = 1$ for $j \notin \Omega_\delta$ ($j \in \mathcal{F}[\Omega_{\delta}]$ in terms of detailed notation in Section 1.2). If $a \in \Omega$, we understand by $\sigma_a$ the spin at a closest point in $\Omega_\delta$ to $a$.

The planar Ising model on the square lattice undergoes a phase transition at the critical temperature $\beta_c = \frac{1}{2} \ln (1 + \sqrt{2})$. Henceforth we will fix a negative parameter $m$ and set $\beta = \beta(\delta) = \beta_c - \frac{m_0}{1 + \sqrt{2}}$. This is a subcritical massive limit, where the spins stay in the ordered phase while approaching criticality.

**Theorem 1.** Let $\Omega$ be a bounded simply connected domain and suppose $a_1, \ldots, a_n \in \Omega$. Under $\delta \downarrow 0$, $\beta = \beta_c - \frac{m_0}{1 + \sqrt{2}}$, the spin $n$-point function converges to a continuous function of $a_1, \ldots, a_n$,

$$\delta^{-n} E_{\Omega_{\delta},\beta(\delta)}^- [\sigma_{a_1} \cdots \sigma_{a_n}] \to (a_1, \ldots, a_n)^+_{\Omega,m},$$

and its logarithmic derivative $2\partial_{a_1} \ln \langle a_1, \ldots, a_n \rangle_{\Omega,m}^+ = A_1^+ + iA_1^-$, where $\partial_{a_1} = \frac{1}{2} (\partial_{x_1} - i\partial_{y_1})$ is determined by the solution to the boundary value problem of Proposition 17 set on the domain $\Omega$.

The derivation of the following result in our analytical setup may be of independent interest.

**Corollary 2** ([WMTB76, SMJ77, KaKo80]). The 2-point function in the full-plane is given by

$$(a, a)^{\pm}_{\mathbb{C},\mathbb{C}} = \text{cst} \cdot \cosh h_0(\eta) \cdot \exp \left[ \int_{-\infty}^{\eta_{am}} r \left( (h_0'(r))^2 - 4 \sinh^2 2h_0(r) \right) dr \right],$$

$$(a, a)^{\pm}_{\mathbb{C},-\mathbb{C}} = \text{cst} \cdot \sinh h_0(\eta) \cdot \exp \left[ \int_{-\infty}^{\eta_{am}} r \left( (h_0'(r))^2 - 4 \sinh^2 2h_0(r) \right) dr \right],$$

where $a > 0$ and $\eta_{0} = -\frac{1}{2} \ln h_0$ is a solution to the Painlevé III equation

$$r\eta\eta' = r(\eta')^2 - \eta\eta_0 - 4r + 4r\eta_0^4.$$
1.2. Notation. Following signs will be used throughout the paper:

\[ \lambda := e^{\frac{i\pi}{2}}, \beta_c = \frac{1}{2} \ln \left( 1 + \sqrt{2} \right), \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}z > 0 \}, A \oplus B := (A \cup B) \setminus (A \cap B). \]

We will write partial derivatives in contracted form, i.e. \( \partial_x = \frac{\partial}{\partial x} \), etc. And denote by \( \partial_z, \partial_{\bar{z}} \) the Wirtinger derivatives:

\[ \partial_z := \frac{\partial_x - i\partial_y}{2}, \quad \partial_{\bar{z}} := \frac{\partial_x + i\partial_y}{2} = e^{i\theta} \left( \partial_x + iv^{-1}\partial_y \right). \]

We will also use \( \partial \) to denote directional derivatives; i.e. \( \partial_x = \frac{\partial}{\partial x} \), \( \partial_y = \frac{\partial}{\partial y} \), and so on. If \( z \in \partial \Omega \), denote by \( \nu_{\text{out}}(z) \in \mathbb{C} \) as the unit normal at \( x \), i.e. the unit complex number which points to the direction of outer normal vector at \( z \). Then \( \partial_{\nu_{\text{out}}} \) is the outer normal derivative in the direction of \( \nu_{\text{out}} \).

We denote by \( \partial^6, \Delta^3 \) the following discrete operators:

\[ \partial^6 f(z) := f(z + \sqrt{2}\lambda\delta) - f(z), \]

\[ \Delta^3 f(z) := f(z + \sqrt{2}\lambda\delta) + f(z - \sqrt{2}\lambda\delta) + f(z - \sqrt{2}\lambda\delta) - 4f(z), \]

wherever they make sense, if \( z \notin \mathcal{V}[\Omega_3] \). On \( \mathcal{V}[\Omega_3] \), we make a small modification in the coefficients in \( \Delta^3 \); see (2.10).

**Note that** \( (\sqrt{2}\lambda)^{-2}\Delta^3 \to \Delta \).

**Mass Parametrisation.** There are various equivalent ways of parametrising the deviation \( \beta_c - \beta \), and we summarise the relation amongst them here at once.

- **Discrete mass** \( M := \beta_c - \beta \) is scaled \( M = \frac{m\delta}{\pi} \) with the continuous mass \( m \) being a constant.

- Pure phase factor \( e^{2i\Theta} := e^{-\frac{\lambda^2 \pi^2}{2\pi \delta}} \) with \( \Theta \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \). Equivalently \( e^{2\beta} = \cot \left( \frac{\pi}{4} + \Theta \right) \). \( \Theta \) is scaled \( \Theta \sim \frac{m\delta}{\pi} \).

Also define \( M_H := 2\sin 2\Theta \sqrt{2 \cos \frac{\pi\delta}{2}} \) which shows up as the mass coefficient in massive harmonicity.

**Graph Notation.** Recall that we work with the rotated square lattice \( \mathbb{C}_\delta := \delta(1+i)\mathbb{Z}^2 \). Our graph \( \Omega_3 \) comprises the following components (see Figure 1):  

- **faces** \( \mathcal{F} [\Omega_3] := \Omega \cap \delta(1+i)\mathbb{Z}^2 \),
- **vertices** \( \mathcal{V} [\Omega_3] := \{ f + \delta, f + i\delta : f \in \mathcal{F} [\Omega_3] \} \subset \mathbb{C}_\delta \),
- **edge** \( \mathcal{E} [\Omega_3] := \{ (ij) : i,j \in \mathcal{V} [\Omega_3], |i-j| = \sqrt{2}\delta \} \), and
- **corners** \( \mathcal{C} [\Omega_3] := \{ (vf) : v \in \mathcal{V} [\Omega_3], f \in \mathcal{F} [\Omega_3], |v-f| = \delta \} \).

For consistency with the low-temperature expansion of the model, we prefer to visualise the lattice in its dual form. Note that just as the faces are represented by their midpoints above, an edge \( (ij) \) and a corner \( (vf) \) will be identified with their midpoints \( \frac{i+j}{2} \) and \( \frac{v+f}{2} \), respectively. Additionally, we draw a half-edge between either an edge midpoint or a corner to a nearest vertex.

For \( \tau = 1, i, \lambda, \bar{\lambda} \), a corner \( c \in \mathcal{C}[\Omega_3] \) is in \( C^\tau [\Omega_3] \) if the nearest vertex is in the direction \( -\tau^{-2} \). The edges in \( C^1[\Omega_3], C^0[\Omega_3] \) are respectively called real and imaginary corners.

We will frequently denote union of various sets by concatenation, e.g. \( \mathcal{E} \mathcal{C} [\Omega_3] := \mathcal{E} [\Omega_3] \cup \mathcal{C} [\Omega_3] \).

**Graph Boundary.**

- **faces** \( \mathcal{F} [\Omega_3] := \{ (ij) : i,j \in \mathcal{V} [\Omega_3], |i-j| = \sqrt{2}\delta \} \), \( \partial \mathcal{E} [\Omega_3] := \mathcal{F} [\Omega_3] \setminus \mathcal{E} [\Omega_3] \),
- **boundary faces** \( \partial \mathcal{F} [\Omega_3] \) are faces in \( \mathcal{F} [\Omega_3] \setminus \mathcal{E} [\Omega_3] \) which do not point to any corner of \( \mathcal{V} [\Omega_3] \), and
- **boundary corners** \( \partial \mathcal{C} [\Omega_3] := \{ (vf) : v \in \mathcal{V} [\Omega_3], f \in \partial \mathcal{F} [\Omega_3], |v-f| = \delta \} \),
- \( \nu_{\text{out}}(z) \) for \( z \in \partial \mathcal{E} [\Omega_3] \) is the unit complex number corresponding to the orientation of \( z \) pointing outwards from \( \Omega \).

**Double Cover.** The fermions we introduce in forthcoming sections are functions defined on the double cover of the continuous and discrete domains \( \Omega, \Omega_3 \). Define \( [\Omega, a_1, \ldots, a_n] \) as the double cover of \( \Omega \) ramified at distinct interior points \( a_1, \ldots, a_n \in \Omega \); in particular, it is a Riemann surface where \( \sqrt{(z-a_1) \cdots (z-a_n)} \) is well-defined, smooth and single-valued. In the case where \( \{ a_1, \ldots, a_n \} = \{ a_1, \ldots, a_n \} \), conjugation on the double cover is defined by requiring that

\[ \sqrt{(z-a_1) \cdots (z-a_n)} \cdot \sqrt{(\bar{z}-a_1) \cdots (\bar{z}-a_n)} = \sqrt{(z-a_1) \cdots (\bar{z}-a_n)}. \]

On \( \mathbb{C}, a_1 \), we will refer to the slit domains \( X^+ := \{ \text{Re}\sqrt{z} > 0 \} \) and \( Y^+ := \{ \text{Im}\sqrt{z} > 0 \} \).

When the choice of the lift of \( z \in \Omega \) is clear, we will write \( z \) to denote the lift in \( [\Omega, a_1, \ldots, a_n] \). Conversely, if \( z \in [\Omega, a_1, \ldots, a_n], \pi(z) \in \Omega \setminus \{ a_1, \ldots, a_n \} \) is the projection onto the planar domain; \( z \in [\Omega, a_1, \ldots, a_n] \) is the lift of \( \pi(z) \) which is not \( z \). A function which switches sign under switching \( z \) and \( z' \) is called a spinor.

We say that two points \( z, w \in [\Omega, a_1, \ldots, a_n] \) are on the same sheet if we can draw a straight line segment between them; i.e. the straight line segment on \( \Omega \) which connects \( \pi(z), \pi(w) \) can be lifted to connect \( z, w \) on \( [\Omega, a_1, \ldots, a_n] \).
For the discrete double cover $[\Omega_\delta, a_1, \ldots, a_n]$, we will take closest faces in $\Omega_\delta$ to $a_1, \ldots, a_n \in \Omega$, and then lift components of $\Omega_\delta$ minus those $n$ faces. Clearly, $[\Omega_\delta, a_1, \ldots, a_n]$ is a lattice which is locally isomorphic to the planar lattice $\Omega_\delta$. Given the first monodromy face $a_1$, we will then lift $a_1 + \frac{\delta}{2}$ to $\Omega_\delta$ and then lift components

1.3. Proof of the Main Theorems.

Proof of Theorem 4. Given the fermion convergence of Theorem 25 and identification of Ising quantities in terms of the fermions of Proposition 5, we can integrate the discrete logarithmic derivative which converges in the scaling limit (see also CHI15 Theorem 1.5, Remark 2.23)

\[
\frac{1}{2\delta} \left[ \frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1+2\delta \sigma_{a_2} \cdots \sigma_{a_n}} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]} - 1 \right] \rightarrow A_0^1(a_1, \ldots, a_n),
\]

to get that scales to a continuous limit for $a_1, \ldots, a_n, b_1, \ldots, b_n \in \Omega$. The analogue for $A_0$ is clear by considering the result in a $-90^\circ$ rotated domain.

Now it remains to relate the massive convergence rate to the massless convergence rate:

\[
\frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]} \rightarrow \frac{\langle a_1, \ldots, a_n \rangle_\Omega}{\langle a_1, \ldots, a_n \rangle_\Omega^0}
\]

for some $a_1, \ldots, a_n \in \Omega$. Given the convergence of $\delta^{-\frac{1}{2}} \frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]}$ to a continuous limit $\langle a_1, \ldots, a_n \rangle_\Omega$, we have the desired result.

The procedure is largely the same as the process in the massless case of relating the bounded domain correlations to full-plane correlations from CHI15, since their argument only refers to finite distributions, without relying on the criticality of $\beta_c$, we will give an explanation of their argument step by step, deferring to Sections 2.7, 2.8, and 2.9 of said reference for details.

Note that $\beta > \beta_c$, and denote the dual temperature by $\beta^* < \beta_c$. We always assume a scaling $\beta_c - \beta = \frac{m^2}{2}$ for $m < 0$.

(1) Relating two point functions:

\[
\frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]} \rightarrow \text{a continuous limit, since}
\]

\[
1 \leq \frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]} \leq \frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]} \leq \frac{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]}{E_{\Omega_\delta}^{\beta_+} \left[ \sigma_{a_1} \sigma_{a_2} \right]} \rightarrow |B_\Omega(a_1, a_2|m)|^{-1},
\]

where we successively used the monotonicity of spin correlation in inverse temperature (e.g. by coupling with FK-Ising) and in boundary condition (FKG inequality). We also used the convergence of Theorem 25 of the Ising ratio to $|B_\Omega(a_1, a_2|m)|$. We conclude by noting that $|B_\Omega(a_1, a_2|m)|$ can be made arbitrarily close to 1 (Lemma 20) by merging $a_1, a_2$.
(2) One point functions: first note, again by monotonicity in $\beta$,
\[ 1 \leq \frac{\mathbb{E}^{\beta +}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]}{\mathbb{E}^{\beta +}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]} \leq \frac{\mathbb{E}^{\beta, \text{free}}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]}{\mathbb{E}^{\beta, \text{free}}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]} \rightarrow |\mathcal{B}_{\Omega(a_1, a_2|m)|^{-1},}
so \frac{\mathbb{E}^{\beta, \text{free}}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]}{\mathbb{E}^{\beta, \text{free}}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]} \text{can be made arbitrarily close to 1. FKG and GHS inequalities imply}
\[ 1 - \frac{\mathbb{E}^{\beta, \text{free}}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]}{\mathbb{E}^{\beta +}_{\Omega \setminus a_m} [\sigma_a, \sigma_{a_2}]} \leq \frac{\mathbb{E}^{\beta +}_{\Omega +} [\sigma_a, \sigma_{a_2}]}{\mathbb{E}^{\beta +}_{\Omega +} [\sigma_a, \sigma_{a_2}]} 
\leq 1,
so one-point functions scale as square root of two-point functions: this is mirrored in the massless case, thus ratio of one-point functions converge to a continuous limit.

(3) More points: we proceed by induction, where we rely on the uniform decorrelation result
\[ \frac{\mathbb{E}^{\beta +} [\sigma_{a_1} \ldots \sigma_{a_n}]}{\mathbb{E}^{\beta +} [\sigma_{a_1} \ldots \sigma_{a_n}]} \rightarrow 1,
uniformly in $\delta$ as $a_1$ approaches the boundary. So the $n$-point functions scale as the product of a 1-point function and $(n - 1)$-point functions. See [CHI13, Lemma 2.27] for details.

\textbf{Proof of Corollary 3} From Theorem 25 we need to integrate $A_C^k(-a, a)$. Recall $r = am$, and define $\langle r \rangle^+_C := (-a, a)^+_C.$

By (4.10), $-m^{-1} \partial_a \ln \langle r \rangle^+_C$$_{\Omega\setminus a_m} = -\frac{1}{2} \left( \ln \cosh h_0(r) \right)' + \left( \frac{1}{2} \left( h_0(r) \right)^2 - 2 \sinh^2 2h_0(r) \right)_{r=am},$ which can be rephrased as
\[ \partial_r \ln \langle r \rangle^+_C = m^{-1} \partial_r \langle -a, a \rangle^+_{\Omega, m} = \left( \ln \cosh h_0(r) \right)' + \left( \frac{1}{2} \left( h_0(r) \right)^2 - 4 \sinh^2 2h_0(r) \right)_{r=am},
Then
\[ \langle -a, a \rangle^+_{\Omega, m} = \langle r \rangle^+_C = \exp \left( \frac{1}{2} \int_{-\infty}^{am} r \left( \frac{1}{2} \left( h_0(r) \right)^2 - 4 \sinh^2 2h_0(r) \right) dr \right).
Then the identification $B_0 = \tanh h_0$ gives the other case.

\textbf{1.4. Structure of the Paper.} This paper contains four sections and one appendix to which technical calculations and estimates are deferred. Section 2 defines our main analytical tool, the discrete fermions. The combinatorial definition in 2.1 involves contours on the discrete bounded domain and is seen to naturally encode the logarithmic derivative of the spin correlation. Its discrete complex analytic properties are then established, which are exploited in Section 2.2 to give a definition on the full plane by an infinite volume limit.

Since analysis of the continuous fermions is needed for the scaling limit process (for a unique characterisation of the continuous limit), we carry out the continuum analysis first in Section 2.3. We formulate the boundary value problem on $\Omega$ for our continuous fermions, which will be a massive perturbation of holomorphic functions treated in [Ber56, Vek62]. We verify various properties we will use: such as the expansion in terms of formal ‘powers’.

Convergence of the discrete fermion under scaling limit is done Section 3. The analysis is divided into two parts: bulk convergence (Section 3.1), where the discrete fermion evaluated at a fixed $z \in [\Omega, a_1, \ldots, a_n] \text{ shows to converge to the}$
continuum fermion, and analysis near the singularity (Section 3.2), where the discrete fermion evaluated at a point in $[\Omega, a_1, \ldots, a_n]$ microscopically away from a monodromy face is identified from the coefficients of a massive analytic version of power series expansion of the continuous fermion. Bulk convergence is done in the standard manner, by first showing that the set of discrete fermion correlations are precompact and then uniquely identifying the limit. Analysis near the singularity mainly uses ideas from [CHI13], where the continuous power series expansion is modelled in the discrete setting then the coefficients carefully matched.

Then in Section 4 we carry out the isomonodromic analysis and obtain the Painlevé III transcendent, which can be identified in the logarithmic derivative of spin correlations in $\mathbb{C}$ given the convergence results.

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\textbf{2. Massive Fermions}
them in the complex plane in Section 2.2. Then, we carry out analysis of the continuous spinors, to which the discrete
spinors presented in the previous section is shown to converge in Section 3. Since the proof of scaling limit requires unique
identification of the continuous limit, we first give necessary analytic background in Section 2.3.

2.1. Bounded Domain Fermions and Discrete Analysis. We introduce here the main object of our analysis, the
discrete fermion $F$. Note that this function is essentially the same object as in [CH15] Definition 2.1 albeit at general $\beta$, and
we try to keep the same normalisation and notation where appropriate. The contents of this subsection are valid for
any $\beta > 0$.

In order to use the low-temperature expansion of the Ising model, we first define $\Gamma_{\Omega_3} \subset 2^{\mathcal{E}[\Omega_3]}$ as the collection of
closed contours, i.e. set $\omega$ of edges in $\Omega_3$ such that an even number of edges in $\omega$ meet at any given vertex. Given +
boundary condition, any $\omega$ is clearly in one-to-one correspondence with a spin configuration $\sigma$ (where $\omega$ delineates clusters of identical spins), and we can compute the partition function of the model and the correlation

$$
Z_{\Omega_3}^{\pm, \beta} := \sum_{\omega \in \Gamma_{\Omega_3}} e^{-2|\beta| |\omega|},
$$

$$
E_{\Omega_3}^{\pm, \beta} [\sigma_{a_1} \cdots \sigma_{a_n}] = \left| Z_{\Omega_3}^{\pm, \beta} \right|^{-1} \sum_{\omega \in \Gamma_{\Omega_3}} e^{-2|\beta| |\omega|} (-1)^{\# \text{loops}_{a_1 \cdots a_n}(\omega)},
$$

where $\# \text{loops}_{a_1 \cdots a_n}$ denotes the parity of loops in $\omega$ which separate the boundary spins from an odd number of $a_1, \ldots, a_n$.
The unnormalised correlation $Z_{\Omega_3}^{\pm, \beta} [\sigma_{a_1} \cdots \sigma_{a_n}] := Z_{\Omega_3}^{\pm, \beta} \cdot E_{\Omega_3}^{\pm, \beta} [\sigma_{a_1} \cdots \sigma_{a_n}]$ will be used for normalisation below.

**Definition 3.** For a bounded simply connected domain $\Omega \subset \mathbb{C}$ with $n$ distinct interior points $a_1, \ldots, a_n$ and inverse
temperature $\beta > 0$, define for $z \in \mathcal{E}[\Omega]$, $(a_1, \ldots, a_n)$ which is not a lift of $a_1 + \frac{\delta}{2}$ the discrete massive fermion,
or simply the discrete fermion

$$
F_{[\Omega_3, a_1, \ldots, a_n]} (z | \beta) = \frac{1}{Z_{\Omega_3}^{\pm, \beta} [\sigma_{a_1} \cdots \sigma_{a_n}]} \sum_{\gamma \in \Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, z)} c_{z} e^{-2|\beta| |\gamma|} \cdot \phi_{a_1, \ldots, a_n} (\gamma, z)
$$

where

- $\Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, z)$ is the collection of $\gamma = \omega \oplus \gamma_0$, where $\omega$ runs over elements over $\Gamma_{\Omega_3}$, $\gamma_0$ is a fixed simple lattice path from $a_1 + \frac{\delta}{2}$ to $\pi(z) \in \mathcal{E}[\Omega_3]$, and $\oplus$ refers to the XOR (symmetric difference) operation. $|\gamma|$ is the number of full edges in $\gamma$. $c_z := \cos \left( \frac{\pi}{2} + \Theta(\beta) \right)$ if $z$ is an edge, and 1 if it is a corner. Note that none of these definitions refer to the double cover.

- $\phi_{a_1, \ldots, a_n} (\gamma, z)$ is a pure phase factor, independent of $\beta$, defined by

$$
\phi_{a_1, \ldots, a_n} (\gamma, z) = e^{-\frac{\nu}{2} \text{wind}(p(\gamma))(\gamma \beta)} (-1)^{\# \text{loops}_{a_1 \cdots a_n}(\gamma \beta) \cdot \text{sheet}_{a_1 \cdots a_n}(p(\gamma))},
$$

where $p(\gamma)$ is a simple path (we allow for self-touching, as long as there is no self-crossing) from $a_1 + \frac{\delta}{2}$ to $\pi(z)$ chosen in $\gamma$, $\text{wind}(p(\gamma))$ is the total turning angle of the tangent of $p(\gamma)$, and $\text{sheet}_{a_1 \cdots a_n}(p(\gamma)) \in \{\pm 1\}$ is defined to be $+1$ if the lift of $p(\gamma)$ to the double cover starting from the fixed lift of $a_1 + \frac{\delta}{2}$ (fixed once forever in Section 1.2) ends at $z$ and $-1$ if it ends at $z^\ast$. $\phi_{a_1, \ldots, a_n}$ is well-defined; see e.g. [Chiz13] [CH15].

Note that $F_{[\Omega_3, a_1, \ldots, a_n]}$ is naturally a spinor, i.e. $F_{[\Omega_3, a_1, \ldots, a_n]} (z | \beta) = -F_{[\Omega_3, a_1, \ldots, a_n]} (z | \beta)$.

The massive fermion satisfies a perturbed notion of discrete holomorphicity, called massive s-holomorphicity. First, define the projection operator $\text{Proj}_{e^{\pm i \Phi} x} := \frac{e^{\mp i \Phi} x}{2}$ to be the projection of the complex number $x$ to the line $e^{i \beta} \mathbb{R}$.

**Proposition 4.** The discrete massive fermion $F_{[\Omega_3, a_1, \ldots, a_n]} (\gamma | \beta)$ is massive s-holomorphic, i.e. it satisfies

$$
e^{\mp i \beta} \text{Proj}_{e^{\pm i \Phi} \tau(c) \mathbb{R}} F_{[\Omega_3, a_1, \ldots, a_n]} \left( c + \frac{\tau(c)^{-2} i \delta}{2} \beta \right) = F_{[\Omega_3, a_1, \ldots, a_n]} (c | \beta),
$$

between $c + \frac{\tau(c)^{-2} i \delta}{2} \in \mathcal{E}[\Omega_3, a_1, \ldots, a_n]$ and $c \in \mathcal{C}[\Omega_3, a_1, \ldots, a_n]$ which is not a lift of $a_1 + \frac{\delta}{2}$. At $a_1 + \frac{\delta}{2}$, we have instead

$$
e^{\mp i \beta} \text{Proj}_{e^{\pm i \Phi} \mathbb{R}} F_{[\Omega_3, a_1, \ldots, a_n]} \left( a + \frac{\delta \pm \delta i}{2} \beta \right) = \mp i \beta.
$$

**Proof.** The proof of massive s-holomorphicity, essentially identical to the massless case [CH15] Subsection 3.1., uses
the bijection between $\Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, c + \frac{\tau(c)^{-2} i \delta}{2})$ and $\Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, c)$ by the $\oplus$ (symmetric difference) operation. Without loss of generality, notice that $\gamma \mapsto \gamma \oplus t$ with the fixed path $t := \left\{ (c - \frac{\tau(c)^{-2} i \delta}{2}, c - \frac{\tau(c)^{-2} \delta}{2}), (c - \frac{\tau(c)^{-2} \delta}{2}, c) \right\}$ maps $\Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, c - \frac{\tau(c)^{-2} i \delta}{2})$ to $\Gamma_{\Omega_3}(a_1 + \frac{\delta}{2}, c)$ bijectively (see Figure 1.1).
We now only need to show that the summand in the definition of $F_{[\Omega_\delta, a_1, \ldots, a_n]}(c + \frac{\tau(c)\delta}{2})$ for a given $\gamma$ transforms to the summand in $F_{[\Omega_\delta, a_1, \ldots, a_n]}(c|\beta)$ for $\gamma \oplus t$. Given $p(\gamma) \subset \gamma$, we may take $p(\gamma \oplus t) := p(\gamma) \oplus t$. Clearly $(-1)^{\#\text{loops}}$, sheet remain the same for $\gamma$ and $\gamma \oplus t$, while

$$\text{wind } p(\gamma \oplus t) = \begin{cases} \text{wind } p(\gamma) + \frac{\pi}{4} & \text{if } \gamma \cap t \neq \emptyset \\ \text{wind } p(\gamma) - \frac{3\pi}{4} & \text{if } \gamma \cap t = \emptyset \in \tau(c)\mathbb{R}. \end{cases}$$

In the case where $\gamma \cap t \neq \emptyset$, $|\gamma \oplus t| = |\gamma|$ and

$$e^{-i\Theta \text{Proj}_{\mathcal{H}}e^{i\Theta \tau(c)\mathbb{R}}}e^{-\frac{i}{2}\text{wind}(p(\gamma))} = e^{-i\Theta \text{Proj}_{\mathcal{H}}e^{i\Theta \tau(c)\mathbb{R}}}e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} e^{-\frac{\pi i}{4} - i\Theta}$$

$$= e^{-i\Theta} \left[ e^{i\Theta} e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} \right] \text{Re } e^{\frac{3\pi i}{8} - i\Theta}$$

$$= e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} \cos \left( \frac{\pi}{8} + \Theta \right),$$

while if $\gamma \cap t = \emptyset$, $|\gamma \oplus t| = |\gamma| + 1$ and similarly

$$e^{-i\Theta \text{Proj}_{\mathcal{H}}e^{i\Theta \tau(c)\mathbb{R}}}e^{-\frac{i}{2}\text{wind}(p(\gamma))} = e^{-i\Theta} \left[ e^{i\Theta} e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} \right] \text{Re } e^{\frac{3\pi i}{8} - i\Theta}$$

$$= e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} \sin \left( \frac{\pi}{8} + \Theta \right)$$

$$= e^{-\frac{i}{2}\text{wind}(p(\gamma \oplus t))} e^{2\beta} \cos \left( \frac{\pi}{8} + \Theta \right),$$

and the result follows.

At $a_1 + \frac{\delta}{2}$, (say) $\gamma \in \Gamma_{\Omega_\delta}(a_1 + \frac{\delta}{2}, a_1 + \frac{\delta}{2} + \frac{i\delta}{2})$ is mapped bijectively to $\gamma \oplus t \in \Gamma_{\Omega_\delta}$. It is easy to see that

$$\text{wind } p(\gamma) = \begin{cases} -\frac{5\pi}{4} \pm 2\pi & \text{if } t \subset \gamma, \\ -\frac{\pi}{4} \pm 2\pi & \text{if } \gamma \cap t = \emptyset, \end{cases}$$

and $(-1)^{\#\text{loops}_{a_1, \ldots, a_n}(\gamma)p(\gamma)} \text{sheet}_{a_1, \ldots, a_n}(p(\gamma)) = (-1)^{\#\text{loops}_{a_1, \ldots, a_n}(\gamma \oplus t)}$. Now a simple computation similar to above yields the result.

We will take $2.1$ as the definition of $(M, \beta, \Theta)$-massive $s$-holomorphicity (or just $s$-holomorphicity when the mass is clear) at $c$ (or between $c + \frac{\tau(c)\delta}{2}$). We will see later that massive $s$-holomorphicity corresponds to the continuous notion of perturbed analyticity $\partial \bar{z} f = m \bar{f}$.

The fermion defined above is a deterministic function without explicit connection to the Ising model—we now record how it encodes probabilistic information.

**Proposition 5.** For $\nu = -1, 0, 1$, we have

$$F_{[\Omega_\delta, a_1, \ldots, a_n]}(a_1 + \delta + \frac{e^{i\nu \pi}}{2} \delta) = e^{-i\nu \frac{\pi}{4}} \left[ \frac{E^\beta + \left[ \sigma_{a_1 + \delta + e^{i\nu \pi} \delta} \sigma_{a_2} \cdots \sigma_{a_n} \right]}{E^\beta + \left[ \sigma_{a_1} \cdots \sigma_{a_n} \right]} \right].$$

Figure 2.1. The proof of massive s-holomorphicity by bijection.
where \( a_1 + \delta + \frac{e^{i\pi/2} \delta}{2} \) is lifted to the same sheet as the fixed lift of \( a_1 + \frac{\delta}{2} \), and thus for any \( \beta_1 < \beta_2 \) there exist constants \( C_1(\beta_1, \beta_2), C_2(\beta_1, \beta_2) > 0 \), uniform in \( \beta \in [\beta_1, \beta_2] \), \( \Omega, \delta \), such that

\[
C_1 < \left| F_{\Omega, a_1, \ldots, a_n} \left( a_1 + \delta + \frac{e^{i\pi/2} \delta}{2} \beta \right) \right| < C_2.
\]

In the case \( n = 2 \) we have also

\[
(2.4) \quad \left| F_{\Omega_1, a_1, a_2} \left( a_2 + \frac{\delta}{2} \beta \right) \right| = \frac{E_{\Omega_1}^{\beta+, free} [\sigma_{a_1} + \delta \sigma_{a_2 + \delta}]}{E_{\Omega_1}^{\beta+, free} [\sigma_{a_1}, \sigma_{a_2}]}.
\]

Proof. \([2.3]\) and \((2.4)\) were proved for the massless case in \([\text{CHIL}3\text{a}, \text{Lemma }2.6]\) and the proof is easily seen to not depend on a specific value of \( \beta_2 \).

We carry out the \( \nu = 1 \) case here: the path \( t := \{(a_1 + \frac{\delta}{2}, a_1 + \delta) \} \) is admissible (i.e. without a crossing) as \( p(\gamma) \) with \( \text{wepd}(p(\gamma)) = \frac{\pi}{2} \), sheet \( a_{1, \ldots, n} \) \( (p(\gamma)) = 1 \) in \( \gamma \in \Gamma_{\Omega_1}(a_1 + \frac{\delta}{2}, a_1 + \delta) \) if and only if \( \gamma \) does not contain a loop separating \( a_1 \) from \( a_1 + \delta + i \delta \) and \(( -1)^{\#\text{loops}_{a_1, \ldots, n} (\gamma \setminus t)} = ( -1)^{\#\text{loops}_{a_1 + \delta + i \delta, \ldots, n} (\gamma \setminus t)} \). If \( \gamma \) does contain such a loop \( L \), we need to choose \( p(\gamma) = t \cup L \) with \( \text{wepd}(p(\gamma)) = \frac{\pi}{2} \pm 2\pi \). In this case, sheet \( a_{1, \ldots, n} \) \( (p(\gamma)) = 1 \) if and only if \( L \) encloses an even number of \( a_1 \). This means that

\[
( -1)^{\#\text{loops}_{a_1, \ldots, n} (\gamma \setminus p(\gamma))} \text{sheet}_{a_1, \ldots, n} (p(\gamma)) = ( -1)^{\#\text{loops}_{a_1, \ldots, n} (\gamma \setminus t)} = ( -1)^{\#\text{loops}_{a_1 + \delta + i \delta, \ldots, n} (\gamma \setminus t)},
\]

so in all cases the summand in the definition of \( F_{\Omega_1, a_1, \ldots, n} \) is \( \lambda^{-1} \) \( ( -1)^{\#\text{loops}_{a_1 + \delta + i \delta, \ldots, n} (\gamma \setminus t)} \) and the result follows. The uniform bound comes from the finite energy property of the Ising model (such expectations are uniformly bounded from 0 and \( \infty \) in any finite distribution). \( \square \)

A crucial feature of (discrete) massive s-holomorphicity, shared with its continuous counterpart, is that the line integral \( \text{Re} \int [F_{\Omega_1, a_1, \ldots, n}] (z | \beta) \frac{dz}{z} \) can be defined path-independently.

Proposition 6. There is a single valued function \( H_{\Omega_1, a_1, \ldots, n} (x | \beta) \) up to a global constant on \( \mathcal{V} F \Omega_1 \) constructed by

\[
(2.5) \quad H_{\Omega_1, a_1, \ldots, n}^0 (f | \beta) - H_{\Omega_1, a_1, \ldots, n}^0 (v | \beta) := 2\delta \left| F_{\Omega_1, a_1, \ldots, n} \left( \frac{v + f}{2} | \beta \right) \right|^2,
\]

where \( f \in \mathcal{F} \Omega_1 \cup \partial \mathcal{F} \Omega_1 \) and \( v \in \mathcal{V} \Omega_1 \) are \( \delta \) away from each other so that \( \frac{v + f}{2} \) is the corner between them. Put

\[
\left| F_{\Omega_1, a_1, \ldots, n} (a_1 + \frac{\delta}{2} | \beta) \right|^2 = 1.
\]

At the boundary faces \( H^0 \) is constant, so we may put

\[
(2.6) \quad H_{\Omega_1, a_1, \ldots, n}^0 (f | \beta) := 0 \text{if} f \in \partial \mathcal{F} \Omega_1,
\]

and further define \( H_{\Omega_1, a_1, \ldots, n}^0 (v | \beta) := 0 \) if \( v \in \partial \mathcal{V} \Omega_1 \), then across a boundary edge \( e = (a_{int} a) \) with \( a_{int} \in \mathcal{V} \Omega_1 \), \( a \in \partial \mathcal{V} \Omega_1 \), we have

\[
(2.7) \quad \partial_{a_{int}} H_{\Omega_1, a_1, \ldots, n}^0 (e | \beta) := -H_{\Omega_1, a_1, \ldots, n}^0 (a_{int} a) \beta := 2 \cos^2 \left( \frac{\pi}{8} + \Theta \right) \beta \left| F_{\Omega_1, a_1, \ldots, n} (e | \beta) \right|^2 \geq 0.
\]

Proof. \([2.5]\) gives rise to a single valued function because doing a loop around any edge will give zero: if \( e = (v_1 v_2) \) is an edge which is incident to faces \( f_1, f_2, H_{\Omega_1, a_1, \ldots, n}^0 (f_1) - H_{\Omega_1, a_1, \ldots, n}^0 (v_1) + H_{\Omega_1, a_1, \ldots, n}^0 (f_2) - H_{\Omega_1, a_1, \ldots, n}^0 (v_2) \) according to \([2.5]\), \( f_1 \) is simply equal to \( 2\delta \left| F_{\Omega_1, a_1, \ldots, n} \left( \frac{f_1 + v_1}{2} \right) \right|^2 + 2\delta \left| F_{\Omega_1, a_1, \ldots, n} \left( \frac{f_2 - v_2}{2} \right) \right|^2 = 2\delta \left| F_{\Omega_1, a_1, \ldots, n} (e | \beta) \right|^2 \), since the two corner values at \( \frac{f_1 + v_1}{2}, \frac{f_2 - v_2}{2} \) are simply projections of \( F_{\Omega_1, a_1, \ldots, n} (e) \) onto orthogonal lines by s-holomorphicity. 

\( H_{\Omega_1, a_1, \ldots, n}^0 (f_1) - H_{\Omega_1, a_1, \ldots, n}^0 (v_1) + H_{\Omega_1, a_1, \ldots, n}^0 (f_1) - H_{\Omega_1, a_1, \ldots, n}^0 (v_2) \) is also equal to \( 2\delta \left| F_{\Omega_1, a_1, \ldots, n} (e | \beta) \right|^2 \), and increments along the loop \( f_1 \sim v_1 \sim f_2 \sim v_2 \sim f_1 \) sum to zero.

Then the boundary behaviour can be easily verified noting that \( F_{\Omega_1, a_1, \ldots, n} (e | \beta) \in \nu_{out} (e)^{-1/2} \Re \) if \( e \) is a boundary edge. See \([\text{CHIL}3\text{a}, \text{Proposition }3.6]\) for a massless counterpart. \( \square \)

Remark. Writing corner values in terms of edge projections, we obtain that \( H_{\Omega_1, a_1, \ldots, n} \) is a discrete version of the integral \( \text{Re} \int [F_{\Omega_1, a_1, \ldots, n}] (z | \beta) \frac{dz}{z} \) in that

\[
(2.8) \quad H_{\Omega_1, a_1, \ldots, n}^0 (f_1 | \beta) - H_{\Omega_1, a_1, \ldots, n}^0 (f_2 | \beta) = \sqrt{2} \sin \left( \frac{\pi}{4} + 2\Theta \right) \Re \left[ F_{\Omega_1, a_1, \ldots, n} (z | \beta)^2 \left( f_1 - f_2 \right) \right],
\]
for all \( f_1, f_2 \) of distance \( \sqrt{2}\delta \) from each other, and
\[
(2.9) \quad H_{[\Omega],a_1,\ldots,a_n}^\bullet (v_1|\beta) - H_{[\Omega],a_1,\ldots,a_n}^\bullet (v_2|\beta) = \sqrt{2} \sin \left( \frac{\pi}{4} - 2\Theta \right) \Re \left[ F_{[\Omega],a_1,\ldots,a_n} (z|\beta)^2 (v_1 - v_2) \right],
\]
for \( v_1, v_2 \) of distance \( \sqrt{2}\delta \) from each other.

The functions \( H_{\bullet}^{\cdot} \) constructed in the previous proposition satisfy a discrete version of a second-order partial differential equation. We first recall the standard discrete operators \( \partial^2_e, \Delta^\delta \); as alluded to in the notation section, we make a small modification to the conventional definition for the laplacian in \( V[\Omega] \). We make this boundary modification specifically for \( V[\Omega] \), which lets us define \( H_{[\Omega],a_1,\ldots,a_n}^\bullet = 0 \) on boundary vertices; see [ChSm12] for a motivation.

**Definition 8.** Suppose \( A \) is a function defined on \( E[\Omega] \) and \( B \) on \( V[\Omega] \) (or \( F[\Omega] \), or any locally isomorphic graph). We define the discrete Wirtinger derivative and laplacian by
\[
\partial^\delta_e A(x) := \sum_{n=0}^3 i^m e^{i\pi/4} A \left( x + i^m e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right) \text{ if, e.g., } x \in V F[\Omega],
\]
\[
(2.10) \quad \Delta^\delta B(x) := \sum_{m=0}^3 c_m \left[ B \left( x + i^m e^{i\pi/4} \sqrt{2}\delta \right) - B(x) \right] \text{ if, e.g., } x \in V[\Omega],
\]
where \( c_m = 1 \) if \( x + i^m e^{i\pi/4} \sqrt{2}\delta \in V[\Omega] \), and \( c_m = \frac{\sin \left( \frac{\pi}{4} - 2\Theta \right)}{\cos \left( \frac{\pi}{4} + \Theta \right)} \) if \( x + i^m e^{i\pi/4} \sqrt{2}\delta \in \partial V[\Omega] \). For any other lattice, \( c_m \equiv 1 \).

If \( \Delta^\delta B(x) = M^2_H B(x) \) for \( M_H^2 = \frac{8\sin^2 2\Theta}{\cos 4\Theta} \), we call \( B(\Theta) \)-massive harmonic at \( x \).

We note that the spinor \( F_{[\Omega],a_1,\ldots,a_n} \) is massive harmonic [BeDC12, DGPT14, HKZ15], at least away from monodromy; see Proposition 26 and also the proof of Proposition 13 for the behaviour near monodromy.

**Proposition 9.** For \( x = a_1 + \delta, a_2, \ldots, a_n \), we have
\[
\Delta^\delta H_{[\Omega],a_1,\ldots,a_n}^\bullet (x) = 2 \sin \left( \frac{\pi}{4} + 2\Theta \right) \delta.
\]
\[
(2.11) \quad \left[ A\Theta \sum_{n=0}^3 F \left( x + i^m e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right)^2 + B\Theta \left| \partial^\delta_e F \right|^2 (x) \right] - \left[ A\Theta \sum_{n=0}^3 F \left( x + i^m e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right)^2 - B\Theta \left| \partial^\delta_e F \right|^2 (x) \right] = 0
\]
where \( A\Theta, B\Theta \) are explicit constants (see (A.3)). \( A \) is odd and \( B \) is even in \( \Theta \), \( A\Theta \sim 4\sqrt{2}\Theta \) and \( B\Theta \sim \frac{1}{2\sqrt{2}} \) as \( \Theta \to 0 \).

**Proof.** We calculate \( \partial^\delta_e F_{[\Omega],a_1,\ldots,a_n}^2 \) in Proposition 26. The laplacians follow straightforwardly by noting that \( \Delta^\delta H_{[\Omega],a_1,\ldots,a_n}^\bullet (x) = \Re \left[ 2 \sin \left( \frac{\pi}{4} \pm \Theta \right) \delta \partial^\delta_e F_{[\Omega],a_1,\ldots,a_n}^2 \right] \), which is true if \( x \in V[\Omega] \) is adjacent to \( \partial V[\Omega] \) as well (the boundary conductances are defined precisely to preserve this relation). See also Remark 10.

**Remark 10.** Note that Proposition 9 applies for \( x = a_1 \) as well, since, thanks to the singularity (2.22), projections of \( F_{[\Omega],a_1,\ldots,a_n} \left( a_1 + e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right) \) on the line \( i \mathbb{R} \) are equal to \( -i \); \( F_{[\Omega],a_1,\ldots,a_n} \) does show s-holomorphicity between 4 edges surrounding \( a_1 \), as long as we take it on the sheet which is cut along \( \mathbb{R}_{\geq 0} \). The singularity effectively transfers the monodromy at face \( a_1 \) to the vertex \( a_1 + \delta \).

2.2. Full-Plane Fermions. We now define the fermion \( F_{[\Omega],a_1,\ldots,a_n} \) for \( \Omega = \mathbb{C} \) by taking increasingly bigger balls \( B_R = B_R(0) \), allowing us to extract informations about the Ising measure on the corresponding discretised domains.

**Lemma 11.** Fix \( \delta > 0, \Theta < 0 \). There exists a constant \( C = C(\delta, \Theta) \) such that \( \left| F_{[(B_R)_\delta],a_1,\ldots,a_n} (c) \right| \leq C \) for any \( c \in C( [(B_R)_\delta], a_1, \ldots, a_n) \).

**Proof.** Discrete Green’s formula implies that
\[
\sum_{v \in V[(B_R)_\delta]} \Delta^\delta H_{(B_R)_\delta,a_1,\ldots,a_n}^\bullet (v) = \frac{\sin \left( \frac{\pi}{4} - 2\Theta \right)}{\cos^2 \left( \frac{\pi}{4} + \Theta \right)} \sum_{e \in \partial E[(B_R)_\delta]} \partial^\delta_e H_{(B_R)_\delta,a_1,\ldots,a_n}^\bullet (e),
\]
and we can use (2.11) on vertices other than $a_1 + \delta$ for the laplacian and (2.7) for the boundary outer derivatives, so leaving just the laplacian at $(a_1 + \delta)$ on the left hand side

$$
\Delta^{\delta} H^\bullet_{\{B_R\}_{a_1, \ldots, a_n}}(a_1 + \delta) = 
\sum_{v \in \mathcal{V}\{B_R\}_{a_1, \ldots, a_n}} 2 \sin \left(\frac{\pi}{4} - \Theta\right) \delta \cdot \left| A_{-\Theta} \sum_{n=0}^{3} F \left( v + i^n \frac{e^{i\pi}}{\sqrt{2}} \right) \right|^2 + B_{-\Theta} \left| \partial_{\delta} F \right|^2(v) 
+ \sum_{e \in \partial C\{B_R\}_{a_1, \ldots, a_n}} 2 \delta \sin \left(\frac{\pi}{4} - 2\Theta\right) \left| F_{\{B_R\}_{a_1, \ldots, a_n}}(e) \right|^2
$$

(2.12)

Moreover, there is only one such function $F_{\{C, a_1, \ldots, a_n\}}$ on the whole of $[C, a_1, \ldots, a_n]$. It suffices to show that such a limit must be unique.

**Proposition 12.** Any subsequential limit $F_{\{C, a_1, \ldots, a_n\}}$ of $F_{\{B_R\}_{a_1, \ldots, a_n}}$ as $R \to \infty$

1. shows s-holomorphicity and the singularity at $a_1 + \frac{\delta}{2}$ as in Proposition 4.
2. $F_{\{C, a_1, \ldots, a_n\}} \to 0$ uniformly and its square integral $\int_{\mathcal{C}} \left| F_{\{C, a_1, \ldots, a_n\}} \right|^2$ is finite and constant at infinity, (3)

(2.13)

$$
\sum_{v \in \mathcal{V}\{C\}_s} \Delta^{\delta} H^\bullet_{\{C, a_1, \ldots, a_n\}}(v) < \infty,
$$

Moreover, there is only one such function $F_{\{C, a_1, \ldots, a_n\}}$.

**Proof.** The first entry immediate from the corresponding properties of $F_{\{B_R\}_{a_1, \ldots, a_n}}$.

The inequality (2.13) can also be deduced from the uniform bound for the sum of $\Delta^{\delta} H^\bullet_{\{B_R\}_{a_1, \ldots, a_n}}$. Since the sums are uniformly bounded by (2.12) and (2.13), monotone convergence gives the desired inequality for the limit.

The infinity behaviour is immediate from the fact that the sum of $\left| F_{\{\Omega, a_1, \ldots, a_n\}} \right|^2$ along any line in $\Omega \setminus B_R$ vanishes uniformly as $R \to \infty$.

If there are two such $F_{\{C, a_1, \ldots, a_n\}}$, their difference $\tilde{F}_{\{C, a_1, \ldots, a_n\}}$ is everywhere s-holomorphic, and since the sum of $\left| \tilde{F}_{\{C, a_1, \ldots, a_n\}} \right|^2$ along any line in $\mathcal{C} \setminus B_R$ is finite and decays to zero, the square integral $\int_{\mathcal{C}} \left| \tilde{F}_{\{C, a_1, \ldots, a_n\}} \right|^2$ is finite and constant at infinity. $\tilde{H}^\bullet_{\{C, a_1, \ldots, a_n\}}$ is everywhere superharmonic and is finite at infinity, so $H^\bullet_{\{C, a_1, \ldots, a_n\}}$ is constant and $F_{\{\Omega, a_1, \ldots, a_n\}} \equiv 0$.

On the full plane, we have an explicit characterisation of the one point spinor in terms of the *massive harmonic measure* of the slit plane $\text{hm}^0_{(1+i)\mathbb{Z} \setminus \mathbb{R}_{<0}}: \text{hm}^0_{(1+i)\mathbb{Z} \setminus \mathbb{R}_{<0}}(\cdot|\Theta)$ for $z \in (1+i)\mathbb{Z}^2$ is the probability of a simple random walk started at $z$ extinguished at each step with probability $\frac{2\sin^2 2\Theta}{\cos 2\Theta}$ to successfully reach 0 before hitting $(1+i)\mathbb{Z}^2 \cap \mathbb{R}_{<0}$. $\text{hm}^0_{(1+i)\mathbb{Z} \setminus \mathbb{R}_{<0}}(\cdot|\Theta)$ is the unique $\Theta$-massive harmonic function (in the sense of (A.2)) on $(1+i)\mathbb{Z}^2 \setminus \mathbb{R}_{<0}$ which has the boundary values 1 at 0 and 0 on $(1+i)\mathbb{Z}^2 \cap \mathbb{R}_{<0}$ and infinity.

**Proposition 13.** Denote the slit planes $\mathbb{X}^+ := \{z \in [C, 0]: \text{Re}\sqrt{z} > 0\} \equiv \mathbb{C} \cap \mathbb{R}_{>0}$ and $\mathbb{Y}^+ := \{z \in [C, 0]: \text{Im}\sqrt{z} > 0\} \equiv \mathbb{C} \cap \mathbb{R}_{>0}$. Then for $\Theta < 0$

$$
F_{\{C, 0\}}(c\delta|\Theta) = \begin{cases} 
\text{hm}^0_{(1+i)\mathbb{Z} \setminus \mathbb{R}_{<0}}(c - \frac{3\delta}{2}|\Theta) & c\delta \in \mathbb{X}^+ \cap \mathcal{C}^1_{[C, 0]} \\
-\text{ihm}^0_{(1+i)\mathbb{Z} \setminus \mathbb{R}_{<0}}(c - \frac{1}{2}|\Theta) & c\delta \in \mathbb{Y}^+ \cap \mathcal{C}^1_{[C, 0]}.
\end{cases}
$$

**Proof.** By Proposition 20 $F_{\{C, 0\}}(c\delta)$ restricted to $\mathcal{C}^1_{[C, 0]}$ or $\mathcal{C}^1_{[C, 0]}$ is $\Theta$-massive harmonic, except possibly at the lifts of $\pm \frac{\delta}{2}$ (because there is no planar neighbourhood $G_\delta$ around these points in $[C, 0]$, see Figure A.1) and $\pm \frac{\delta}{2}$ (because $F_{\{C, 0\}}(c\delta)$ is not s-holomorphic at the lifts of $\frac{\delta}{2}$).
From Definition \[3\] it is clear that \( F_{|c_{\delta}| \cap B_{\rho_0}}(\zeta) = 0 \) for any \( e \in C^1 |c_{\delta}, 0| \) on the (lift of) negative real line and \( e \in C^1 |C_{\delta}, 0| \) on the (lift of) positive real line, since the complex phase \( \phi_{a_1, \ldots, a_n}(\gamma, e) \) for any \( \gamma \in \Gamma_{C_{\delta} \cap B_{\rho_0}}(\frac{\delta}{2}, \pi(e)) \) switches sign for the reflection across the real line \( \gamma_r := \{ \bar{e} : e \in \gamma \} \), and \( \Gamma_{C_{\delta} \cap B_{\rho_0}}(\frac{\delta}{2}, \pi(e)) \) is invariant under the reflection. We conclude that \( F_{|c_{\delta}, 0|}(\zeta) = 0 \) for \( e \in C^1 |C_{\delta}, 0| \).

Also note that \( F_{|c_{\delta}, 0|}(\frac{\delta}{2}) = 1 \), where \( \frac{\delta}{2} \) is precisely the lift at which the ratio is on the same sheet as the fixed lift of \( \frac{\delta}{2} \), since the corresponding Ising quantity in \[2, \frac{3}{2}\] is precisely the ratio of two adjacent magnetisations (spin 1-point functions); in the infinite volume limit with \( \beta > \beta_c \), the ratio tends to 1.

In all, \( F_{|c_{\delta}, 0|}(\cdot) \) restricted to \( X^+ \cap C^1 |C_{\delta}, 0| \) is the massive harmonic function which takes the boundary values 1 on the lift of \( \frac{\delta}{2} \) in \( X^+ \cap C^1 |C_{\delta}, 0| \) and 0 on the lift of the negative real line and infinity, whence the identification with \( \lim_{n \to \infty} Z_\nu \), since \( \frac{\delta}{2} \) is precisely the ratio of two adjacent magnetisations (spin 1-point functions); in the infinite volume limit with \( \beta > \beta_c \), the ratio tends to 1.

Proof. See [Ber56 Section 10] or [Vek62 Section III.4]: basically, we can define \( s(z) := -\frac{m}{\pi} \int_U \frac{f}{z-w} \, dw \) and show holomorphicity of \( e^{-zf} \). Then there is a holomorphic function on \( U \) whose imaginary part equals that of \( s \), unique up to a global real constant, which can be used to cancel out its imaginary part.

\[ \text{Corollary 15.} \quad \text{There exists a family of m-massive holomorphic functions} \quad Z^1_n(z), Z^s_n(z) \quad \text{for each} \quad n \in \mathbb{Z} \quad \text{such that} \quad z \to 0 \quad Z^1_n(z) \sim z^n, Z^s_n(z) \sim iz^n, \]

and any function \( f \) that is \( m \)-massive holomorphic in a punctured neighbourhood of \( a \) can be expressed near \( a \) as a formal power series in \( Z_n \), i.e.

\[ f(z) = \sum_n \left[ A_n^1 Z^1_n(z - a) + A_n^s Z^s_n(z - a) \right] \]

for real numbers \( A_n^1, A_n^s \). If \( f(z) \) is a spinor defined on a double cover ramified at \( a \), \( f \) admits formal power series in analogous functions indexed by half-integers \( Z_{n+\frac{1}{2}} \).

Proof. See [Ber56 Section 5].

Remark 16. There is no canonical choice of the ‘local formal powers’ \( Z^1_n \). We will need explicit functions to expand continuous fermions around their singularities and analyse them further to derive Painlevé III in Section 4; we hereby fix the following radially symmetric functions for half-integers \( \nu \):

\[
Z^1_\nu(re^{i\theta}) := \frac{\Gamma(\nu + 1)}{|m|^\nu} \left[ e^{i\nu \theta} I_\nu(2|m|r) + (\text{sgn} m) \cdot e^{-i(\nu + 1)\theta} I_{\nu+1}(2|m|r)^\ast \right],
\]

\[
Z^s_\nu(re^{i\theta}) := \frac{\Gamma(\nu + 1)}{|m|^\nu} \left[ e^{i\nu \theta} I_\nu(2|m|r) - i(\text{sgn} m) \cdot e^{-i(\nu + 1)\theta} I_{\nu+1}(2|m|r)^\ast \right],
\]

where \( I_n \) is the modified Bessel function of the first kind. One can easily verify the desired asymptotics and massive holomorphicity from the corresponding facts about \( I_\nu \), namely that \( I_\nu(r) \sim 0 \Gamma(\nu + 1) \left( \frac{r}{2} \right)^\nu \) and \( I'_\nu(r) = I_{\nu+1}(r) \pm \frac{\nu}{r} I_\nu(r) \).

As a special case, we have \( Z^1_{\frac{1}{2}}(z) = \frac{z^m r^{\nu/2}}{\sqrt{\pi}} \).

\[ \text{Proposition 17.} \quad \text{Fix} \quad m < 0. \quad \text{The following boundary value problem on a bounded simply connected domain} \quad \Omega \quad \text{has at most one solution}:
\]

\[ f : [\Omega, a_1, \ldots, a_n] \to \mathbb{C} \quad \text{satisfies}
\]
Proof. If there are any two such functions \( f_1, f_2 \), applying Green-Riemann’s formula to their difference \( \hat{f} \) on \( \Omega \setminus \cup_j B_r(a_j) \) for small \( r > 0 \) yields
\[
\hat{f} \sim \text{cst} \cdot (z - a_i)^{1/2} \quad \text{near } a_i \quad \text{in view of Corollary \[15\].}
\]
so \( \int_{\partial \Omega} f^2 \, dz \) yields zero as \( r \to 0 \), and \( \hat{f} \to i \hat{B}_j(z - a_j)^{-1/2} \) for some \( \hat{B}_j \in \mathbb{R} \) as \( z \) tends to any other \( a_j \). On the inner circles, we have
\[
\int_{\partial B_r(a_j)} f^2 \, dz \xrightarrow{r \to 0} - \frac{\hat{B}_j^2}{2\pi i} \in i\mathbb{R} \quad \text{for } j = 2, \ldots, n,
\]
whereas the boundary condition readily gives \( i \int_{\partial \Omega} |f|^2 \, dz = \int_{\partial \Omega} |f|^2 \, ds \geq 0 \). Therefore
\[
0 \leq i \int_{\partial \Omega} f^2 \, dz = -2 \int_{\partial \Omega} 2m|\hat{f}|^2 \, dz - \sum_j \frac{\hat{B}_j^2}{2\pi i} \leq 0,
\]
so \( \hat{f} \equiv 0 \).

Remark 18. As in the proof of the Proposition \[17\] it is easy to show that \( h := \text{Re} \int f^2 \, dz \) is globally well-defined in \( \Omega \setminus \{a_1, \ldots, a_n\} \). In terms of \( h \), the boundary condition \( f(z) \in \bar{\Omega} \) and the asymptotics around \( a_i \) is equivalent to
1. \( h \) is constant on \( \partial \Omega \) and there is no \( x_0 \in \partial \Omega \) such that \( h > h(\partial \Omega) \) in a neighbourhood of \( x_0 \), and
2. \( h \) is bounded below near \( a_2, \ldots, a_n \) and \( h^{1/2} := \text{Re} \int w \left( f(z) - Z_{-1/2}^- (z - a_1) \right)^2 \, dz \) is single valued and bounded near \( a_1 \).

We note that this boundary problem can easily be extended to the case where \( \Omega = \mathbb{C} \), by requiring that \( h \) be finite and constant at infinity (we will in fact require that \( f \) decays exponentially fast, see also Lemma \[22\]).

We now define quantities which reflect the geometry of \( \Omega \) exploiting the boundary value problem above, which will turn out to be directly related to the Ising correlations through the connection which is precisely our main convergence result in Section \[3.2\]. Determination of these quantities through isomonodromic deformation is the main subject of Section \[4\].

**Definition 19.** Given a solution \( f_{[\Omega,a_1,\ldots,a_n]} \) of the boundary value problem presented in Proposition \[17\] (the continuous massive fermion), define \( A^{(1)}_{1i} = A^{(1)}_{1i} (a_1, \ldots, a_n, m) \) as the real coefficients in the expansion
\[
f_{[\Omega,a_1,\ldots,a_n]}(z) = \sum_{j=1}^n A^{(1)}_{1j} \left( z - a_j \right) + 2A^{(1)}_{1j} Z_{1/2}^- (z - a_j) + 2A^{(1)}_{1j} Z_1^- (z - a_j) + O \left( (z - a)^{3/2} \right).
\]

In addition, in the case where \( n = 2 \), define \( B_\Omega = B_\Omega (a_1, a_2, m) \) as the coefficient
\[
f_{[\Omega,a_1,a_2]}(z) = B_\Omega Z_{-1/2}^-(z - a_j) + O \left( (z - a)^{1/2} \right).
\]

For notational convenience, we do not assume \( B_\Omega > 0 \) as in \[15\]; instead, its sign depend on the sheet choice of \( Z_{-1/2}^- (z - a_i) \), which we will explicitly fix whenever needed.

The following lemma, which is useful in the proof of the main theorem, is a direct consequence of the similarity principle.

**Lemma 20.** Let \( \Omega \) be bounded simply connected domain. For \( m < 0 \), \( |B_\Omega (a_1, a_2, m)| \to 1 \) as \( |a_1 - a_2| \to 0 \) with \( a_1, a_2 \) uniformly away from the boundary.

**Proof.** By \[15\] Remark 2.24, the result holds for \( m = 0 \).

Now let \( \Omega \) be bounded. Using Lemma \[14\] write \( f_{[\Omega,a_1,a_2]}(z|m) = \phi(z)e^{s(z)} \), where \( s \) is Hölder, real on boundary, and \( \text{Res}_1(a_1) = 0 \). \( \phi \) is a holomorphic function with monodromies at \( a_1, a_2 \), where it has singularities of order at most \( 1/2 \). Denote the coefficients of \( \frac{1}{\sqrt{z-a_i}} \) respectively by \( B_1(a_1, a_2), B_2(a_1, a_2) \). By \[15\] Lemma 2.9 (i.e. uniqueness of solutions to the boundary value problem in the massless case),
\[
\phi(z) = (\text{Re}B_1) f_{[\Omega,a_1,a_2]}(z|0) + (\text{Re}B_2) f_{[\Omega,a_2,a_1]}(z|0).
\]

Since the Hölder constant of \( s \) is uniform and \( |B_1(a_1, a_2)| = 1 \), as \( |a_1 - a_2| \to 0 \), \( |B_2|/|B_\Omega (a_1, a_2, m)| \to 1 \). But \( B_2 = (\text{Re}B_1) \cdot iB_\Omega (a_1, a_2, 0) + \text{Re}B_2 \) and \( B_1 = \text{Re}B_1 + (\text{Re}B_2) \cdot iB_\Omega (a_2, a_1, 0) \). Given \( |B_\Omega (a_1, a_2, 0)|, |B_\Omega (a_2, a_1, 0)| \to 1 \), we deduce \( |B_2| \to 1 \), and thus \( |B_\Omega (a_1, a_2, m)| \to 1 \). \( \square \)
3. Discrete Analysis: Scaling Limit

In this section, we show the convergence of the discrete fermions introduced in Section 2 to their continuous counterparts. Then we show that the family of discrete fermions as $\delta \downarrow 0$ is precompact in Section 3.1, whose limit satisfies a unique characterisation as laid out in Proposition 17. These suffice to show convergence to the desired limit.

3.1. Bulk Convergence. We finally get to the convergence of the discrete fermion $f_{[\Omega, a_1, \ldots, a_n]}$ to its continuous counterpart $f_{[\Omega, a_1, \ldots, a_n]}$ in scaling limit. First, we need to interpolate the discrete function defined on $[\Omega, a_1, \ldots, a_n]$ on the continuous domain $[\Omega, a_1, \ldots, a_n]$. While any reasonable interpolation (e.g., linear interpolation used in many papers dealing with massless case) should converge to the unique continuous limit, we will assume an interpolation scheme with a continuously differentiable $F_{[\Omega, a_1, \ldots, a_n]}$ as an easy way to show that the limit itself is continuous differentiable. While we do not explicitly carry it out, we could show the limit is smooth by using arbitrarily more regular interpolation scheme.

Proposition 21. Suppose $m < 0$ and $\Omega$ is simply connected with smooth boundary or $\mathbb{C}$. For any compact subset $K \subset \Omega$, any infinite collection $(\frac{2}{\pi} \delta_k)^{-1/2} F_{[\Omega, a_1, \ldots, a_n]} =: f_{[\Omega, a_1, \ldots, a_n]}$ with $\delta_k \downarrow 0$ has a subsequence that (suitably interpolated as above) converges in $C^1(K)$-topology to a continuously differentiable limit.

Proof. By Arzelà-Ascoli, it suffices to show that the discrete derivatives $\delta^{-1} \partial^2 f_{[\Omega, a_1, \ldots, a_n]}(z) := \delta^{-1} \left[ f_{[\Omega, a_1, \ldots, a_n]}(z + 2\delta) - f_{[\Omega, a_1, \ldots, a_n]}(z) \right]$ are equicontinuous on $K$. In view of Proposition 20, it suffices to show that the 'discrete $L^2$ norm' $\sum_{c \in C(K)} \left| f_{[\Omega, a_1, \ldots, a_n]}(c) \right|^2 \delta^2$ is uniformly bounded.

We recall (2.12) and (2.13), dividing both sides by $\delta$,

\begin{equation}
\text{cst} \geq \delta^{-1} \Delta^1 H^\bullet_{[\Omega, \delta]\cap B_R(a_1, \ldots, a_n)}(a_1 + \delta) \geq 2 \sin \left( \frac{\pi}{4} - \Theta \right) \sum_{c \in C[\delta]} \delta A_{-\Theta} \left| f_{[\Omega, a_1, \ldots, a_n]}(c) \right|^2 
+ \sum_{e \in \partial C[\delta]} 2\delta \sin \left( \frac{\pi}{4} - 2\Theta \right) \left| f_{[\Omega, a_1, \ldots, a_n]}(e) \right|^2,
\end{equation}

and since $A_{-\Theta} \sim -2\sqrt{2m}\delta$, we have the desired $L^2$ bound from the sum of $\delta A_{-\Theta} \left| f_{[\Omega, a_1, \ldots, a_n]}(c) \right|^2$. \hfill \square

With a sequence $K_m$ of increasing compact subsets such that $\bigcup_m K_m = \Omega \setminus \{a_1, \ldots, a_n\}$ and using diagonalisation, we can find a global subsequential limit $f_{[\Omega, a_1, \ldots, a_n]}$ with uniform convergence in compact subsets of $\Omega$. We finish the proof of convergence by showing that a limit must satisfy the boundary value problem of Proposition 17 and thus is unique. We note that continuous differentiability, square integrability and the condition $\partial_2 f_{[\Omega, a_1, \ldots, a_n]} = m f_{[\Omega, a_1, \ldots, a_n]}$ follows straightforwardly from Proposition 20 and Remark 27, so we are left to verify the boundary conditions, as laid out in Remark 18.

We first treat the following explicit case:

Lemma 22. For $m < 0$, the one point spinor $f_{[\Omega, a_1]}$ converges to $Z^1_{\frac{1}{2}}(z = re^{i\theta}) = \frac{e^{2m\pi r}}{\sqrt{2}}$ uniformly in compact subsets of $[\Omega, a_1]$.

Proof. To use the unique characterisation of Remark 18, we need to show that $f_{[\Omega, a_1]}$ vanishes at infinity sufficiently fast to yield that $h_{[\Omega, a_1]} := \text{Re} \int f_{[\Omega, a_1]}^2 dz$ is constant at infinity. But this is obvious given the identification of the discrete spinor with the massive harmonic measure (the hitting probability of an extinguished random walk) of the tip of a slit in Proposition 13 and the exponential estimates of Proposition 20.

It now remains to identify the singularity at $a_1$ as $f_{[\Omega, a_1]} \sim (z - a_1)^{-1/2}$. Note that the massless harmonic measure $K_{\Omega_7}$ and thus the spinor $[\Omega, a_1]$ (Lemma 2.14) for the identification $\partial(\delta) = (\frac{z}{\sqrt{\delta}})^{1/2}$, see [GHPT13 Lemma 5.14] show the same behaviour. Since the hitting probability of a massive random walk is dominated by the hitting probability of a simple random walk, $f_{[\Omega, a_1]} \sim r \cdot Z^1_{\frac{1}{2}}$ for some $r \in [0, 1]$. But the probability that the massive random walk is extinguished can be made arbitrarily close to 0 as length scale becomes negligible compared to $\frac{1}{|a_1|}$, so we conclude $r = 1$.

Proposition 23. For $m < 0$, a subsequential limit $f_{[\Omega, a_1, \ldots, a_n]}$ of $f_{[\Omega, a_1, \ldots, a_n]}$ satisfies the conditions of Remark 18 and thus is unique.

Proof. We first suppose $\Omega \neq \mathbb{C}$. Consider the renormalised discrete square integral $h_{[\Omega, a_1, \ldots, a_n]} := \left( \frac{2\delta}{\pi} \right)^{-1} H_{[\Omega, a_1, \ldots, a_n]}$. By Propositions 9 and 12, $h_{[\Omega, a_1, \ldots, a_n]} \sim m f_{[\Omega, a_1, \ldots, a_n]}$ is discrete superharmonic on $\mathcal{V}[\Omega_{a_1}]$ and takes the boundary value 0 on $\partial\mathcal{V}[\Omega_{a_1}] \cup \{\infty\}$. Moreover, by Proposition 21, $\Delta^1 h_{[\Omega, \delta]\cap B_R(a_1, \ldots, a_n)}(a_1 + \delta) \leq \text{cst}$. Thus, by superharmonicity, $\text{cst}^{-1} h_{[\Omega, a_1, \ldots, a_n]}$ is lower bounded by the domain Green’s function $K := K_{\mathcal{V}[\Omega_{a_1}]}, a_1 + \delta$. \hfill \square
Since $g$ is $O(\delta)$ on any vertex $a_{\text{int}} \in \mathcal{V}[\Omega_\delta]$ adjacent to $a \in \mathcal{V}[\Omega_\delta]$ (Lemma 28), we have that $O(\delta) = K(a_{\text{int}}) \leq h^\bullet_{\Omega_\delta} (a_{\text{int}}) \leq \|h\|_{\partial \Omega_\delta}$ (Lemma 28). Since $F \to \infty$ at infinity and $|F|^2$ is subharmonic by (3.5), $f|_{\delta B_r(a_1)}$ is uniformly bounded on $\Omega_\delta \setminus \bigcup_r B_r(a_1)$ for small $r > 0$. Therefore by equicontinuity $h^\bullet_{\Omega_\delta} \to h_{\Omega_\delta}$ uniformly on $\overline{\Omega_\delta}$, and $h_{\Omega_\delta}$ is continuous up to the boundary.

(1) $h_{\Omega_\delta}$ is the solution of a Poisson equation with bounded data and smooth boundary condition; $h_{\Omega_\delta}$ is continuously differentiable up to the boundary thanks to standard Green's function estimates (e.g. [ChTr13, Theorem 4.3]). From [ChSm12, Remark 6.3], which only uses the superharmonicity (in their normalisation, subharmonicity) of $h^\bullet_{\Omega_\delta}$ and does not depend on a particular property of $h^\bullet_{\Omega_\delta}$, we see that there is no neighbourhood of $x_0 \in \partial \Omega$ where $h > 0$.

(2) We see also that $h_{\Omega_\delta}$ is bounded below by $K$, which is bounded from negative infinity away from $a_1$ in the scaling limit. Near $a_2, \ldots, a_n$, $h_{\partial \Omega_\delta}$ is bounded below. For the asymptotic near $a_1$, note that $f_{\partial \Omega_\delta} := f_{\Omega_\delta} - f_{\Gamma}$ is $h$-holomorphic near $a_1$ with $f_{\partial \Omega_\delta}(a_1 + \frac{x}{\delta}) = 0$, so by Proposition 26 it is everywhere massive harmonic (unlike in the proof of Proposition 13, the zero prevents a singularity near monodromy; see also [CHPT15, Remark 2.6]). But both $f_{\partial \Omega_\delta}$, $f_{\Gamma}$ are uniformly bounded, say, in a discrete circle $S_r := B_{r+\delta}(a_1) \setminus B_r(a_1)$ for small $r > 0$. As above, we conclude that $f_{\partial \Omega_\delta}$ is uniformly bounded in $B_r(a_1)$. Given Lemma 22 this suffices to show boundedness and well-definedness for the continuous limits $f_\Gamma, h_\Gamma$.

If $\Omega = \mathbb{C}$, we first show that $f_{\Omega_\delta} \to 0$ at infinity uniformly in $\delta$ and $h_{\Omega_\delta}$ is constant and finite at infinity using Proposition 29 as in the proof of Lemma 22 then simply take the Green’s function $K$ on a big enough ball $B_H(0)$ to carry out the rest of the above analysis in $B_H(0)$. □

3.2. Analysis near the Singularity. To show convergence of $F_{\partial \Omega_\delta}$ in the discrete setting, and carefully analyse the magnitude of the difference. The candidate for discrete $Z^1_{-\frac{1}{2}}$ is clear: $F_{\Gamma} := \Gamma F_{\partial \Omega_\delta}$ is able to cancel out the singularity of $F_{\partial \Omega_\delta}$, and converges to $Z^1_{-\frac{1}{2}}$ by Lemma 22.

Then, we need to build a discrete analogue of $Z^1_{-\frac{1}{2}}$. Following [CHI15, (3.12)], we define a discrete function $G_{\Gamma}$ by discrete integrating the $F_{\Gamma}$.

Proposition 24. Construct the spinor $G_{\Gamma} : \mathcal{C}^1[\Omega_\delta, a_1] \to \mathbb{R}$ by

$$g_{\Gamma}(z) := \delta \sum_{j=0}^{\infty} \Gamma^j F_{\Gamma}(z - 2j\delta),$$

where $\Gamma(\Theta) := \tan^2 \left( \frac{\pi}{4} + 2\Theta \right)$, and $z - 2j\delta$ is taken on the same sheet as $z$ if $\pi(z) \notin a_1 + \mathbb{R}_{>0}$ or if $\pi(z), \pi(z - 2j\delta) \in a_1 + \mathbb{R}_{>0}$, while $F_{\Gamma}(z - 2j\delta) = 0$ naturally as soon as $\pi(z - 2j\delta) \in \mathbb{R}_{<0}$ (cf. Proposition 13).

It is massive harmonic with coefficient $M_\delta^2$ on $\mathbb{R}^+ \cap \mathcal{C}^1[\Gamma_\delta, 0]$, and $(\frac{\delta}{\pi})^{-1/2} G_{\Gamma}$ converges uniformly in compact subsets of $[\Gamma_\delta, a_1]$ to $g_{\Gamma}$ which has the asymptotic behaviour $g_{\Gamma}(z) \sim \sqrt{z - a_1}$ near $a_1$.

Proof. We know that $(\frac{\delta}{\pi})^{-1/2} F_{\Gamma}(z)$ converges to $\Re e^{2i\pi x}$ on $\mathcal{C}^1[\Omega_\delta, a_1]$ uniformly in compact subsets away from $a_1$. Also note that in the scaling limit $(x + iy) - 2j\delta = (x_0 + iy_0) + \Gamma^j$ converges to $e^{-2\pi i(x - x_0)}$. The scaling limit shows $x^{-3/2}$ decay as $x \to -\infty$, and this is true in the discrete as well; massive random walk–simple random walk coupled with a geometric clock–satisfies the same exponential bound (Thm 14).

If $z$ is in a compact subset, the discrete integrand $\Gamma F_{\Gamma}(z)$ decays to zero as $x_0 \to -\infty$ uniformly in $z$, so we conclude that $g_{\Gamma}(z) \sim (\frac{\delta}{\pi})^{-1/2} G_{\Gamma}(z)$ converges

$$g_{\Gamma}(z) := \sum_{j=0}^{\infty} \frac{1}{2} \frac{e^{2m(x_0 - x)}}{\sqrt{x_0 + iy_0}} dx_0.$$
Massive harmonicity is clear unless $\pi(z) \in \mathbb{R}_{>0}$. If $\pi(z) \in a_1 + \mathbb{R}_{>0}$, $(\Delta^s - M_H^2) F_{[C_s,a]}(z - 2j\delta) = 0$ if $\pi(z - 2j\delta) \in a_1 + \frac{3\delta}{2} + \mathbb{R}_{\geq 0}$ while $F_{[C_s,a]}(z - 2j\delta) = 0$ if $\pi(z - 2j\delta) \in a_1 + \mathbb{R}_{<0}$, thus

$$(\Delta^s - M_H^2) G_{[C_s,a]}(z) = \delta \sum_{j=[(z-a_1)/2\delta]}^{\infty} \Gamma^j (\Delta^s - M_H^2) F_{[C_s,a]}(z - 2j\delta)$$

$$= \delta \Gamma^j [(z-a_1)/2\delta] \sum_{j=0}^{\infty} \Gamma^j (\Delta^s - M_H^2) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta),$$

where the laplacian is taken on the planar slit domain $X^+ \cap C^1 [C_s,0]$ with zero boundary values on the slit.

We need to show that the last sum vanishes.

$${\sum_{j=0}^{N}} \Gamma^j (\Delta^s - M_H^2) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta)$$

$$= \sum_{j=0}^{N} \sum_{s=\pm 1} \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + s)\delta + i\delta) - \tan \left( \frac{\pi}{4} - s2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right]$$

$$+ \sum_{j=0}^{N} \sum_{s=\pm 1} \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + s)\delta - i\delta) - \tan \left( \frac{\pi}{4} + s2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right],$$

where two sums are done respective above and below the slit. By massive discrete holomorphicity, we can convert the differences of real corner values into differences of imaginary corner values: we need to be careful, since the points in $X^+$ directly above and below the cut are on opposite sheets. We can in fact think of $a_1 + \frac{\delta}{2}$ as lying on the slit, since the singularity (2.2) shows two values at $a_1 + \frac{\delta}{2}$ above and below $a_1 + \mathbb{R}$. Above the slit, (2.1) implies

$${\sum_{s=\pm 1}} \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + s)\delta + i\delta) - \tan \left( \frac{\pi}{4} - s2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right]$$

$$= -i \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta + i\delta) - \tan \left( \frac{\pi}{4} + 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j - 1)\delta) \right]$$

$$+ i \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta + i\delta) - \tan \left( \frac{\pi}{4} - 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + 1)\delta) \right]$$

$$= i \tan \left( \frac{\pi}{4} + 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j - 1)\delta) - i \tan \left( \frac{\pi}{4} - 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + 1)\delta).$$

Thanks to the factor of $\Gamma^j$, the sum telescopes (see Figure A.1)

$${\sum_{j=0}^{N}} \sum_{s=\pm 1} \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + s)\delta + i\delta) - \tan \left( \frac{\pi}{4} - s2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right]$$

$$= i \tan \left( \frac{\pi}{4} + 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{5\delta}{2}) - i \Gamma^j \tan \left( \frac{\pi}{4} - 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2N + 1)\delta).$$

The sum below the slit analogously gives

$${\sum_{j=0}^{N}} \sum_{s=\pm 1} \left[ F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2j + s)\delta - i\delta) - \tan \left( \frac{\pi}{4} + s2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) \right]$$

$$= -i \tan \left( \frac{\pi}{4} + 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{5\delta}{2}) + i \Gamma^j \tan \left( \frac{\pi}{4} - 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2N + 1)\delta),$$

where $a_1 + \frac{3\delta}{2} - (2N + 1)\delta$ is approached from below the sheet. Summing the two and taking $a_1 + \frac{3\delta}{2} - (2N + 1)\delta$ from above the slit,

$${\sum_{j=0}^{N}} \Gamma^j (\Delta^s - M_H^2) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - 2j\delta) = -2i \Gamma^j \tan \left( \frac{\pi}{4} - 2\Theta \right) F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2N + 1)\delta).$$

Now, by Proposition 13, $-i \cdot F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2N + 1)\delta)$ is the massive harmonic measure of the point $a_1 + \frac{\delta}{2}$ in the discrete plane $Y^+ \cap C^1 [C_s,0]$ slit by $a_1 + \mathbb{R}_{>0}$. $-i \cdot \Gamma^j F_{[C_s,a]}(a_1 + \frac{3\delta}{2} - (2N + 1)\delta) = O(N^{-1/2})$, and the sum decays to zero, as desired.

\[\square\]
Theorem 25. As $\delta \to 0$,

\begin{equation}
F_{[\Omega_\delta,a_1,...,a_n]} \left( a_1 + \frac{3\delta}{2} \right) = 1 + 2\delta\Lambda \left( a_1, \ldots, a_n \right) + o(\delta),
\end{equation}

\begin{equation}
\left| F_{[\Omega_\delta,a_1,a_2]} \left( a_2 + \frac{\delta}{2} \right) \right| = |B_\Omega(a_1, a_2)| + o(1).
\end{equation}

Proof. We use the strategy of [CHI15 Subsection 3.5]: the massive harmonic nature of our functions are hardly visible since at short length scale massive hitting probabilities approach simple random walk hitting probabilities. While we try to use the same notation for corresponding notions where appropriate, the functions we work with are all massive harmonic.

Note that, due to explicit constructions in Propositions 13 and 24, (3.4) can be written as

\[ F_{[\Omega_\delta,a_1,...,a_n]} \left( a_1 + \frac{3\delta}{2} \right) = F_{[\Omega_\delta,a_1]} \left( a_1 + \frac{3\delta}{2} \right) + 2\Lambda \left( a_1, \ldots, a_n \right) G_{[\Omega_\delta,a_1]} \left( a_1 + \frac{3\delta}{2} \right) + o(\delta). \]

Define the reflection $\mathcal{R}(\Omega)$ of $\Omega$ across the line $a_1 + \mathbb{R}$. Then there is a ball $B(a_1)$ around $a_1$ which belongs to $\Omega \cap \mathcal{R}(\Omega)$. Denote $\Lambda$ to be the lift of the slit neighbourhood $B(a_1) \setminus \{a_1 + \mathbb{R}_{>0}\}$ such that both $F_{[\Omega_\delta,a_1,...,a_n]} \mathcal{R}(\Omega_\delta)$, $a_1,...,a_n$ have their fixed lift of the path origin $a_1 + \frac{\delta}{2}$ on $\Lambda$. By symmetry arguments about the path set $\Gamma(a_1 + \frac{\delta}{2}, z)$ similar to the proof of Proposition 13 we have $F_{[\Omega_\delta,a_1,...,a_n]} \left( a_1 + \frac{3\delta}{2} \right) = F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]} \left( a_1 + \frac{3\delta}{2} \right)$, whereas the boundary value $F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]}(z)F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]}(z)$ on the slit $\pi(z) \in a_1 + \mathbb{R}, z > 0$ cancel out to give $F_{[\Omega_\delta,a_1,...,a_n]}(z)F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]}(z) = 0$ for $z \in C^1[\Omega_\delta,a_1] \cap \partial \Lambda$ (see also [CHI15 Subsection 3.5]).

Then, restricted to $C^1[\Omega_\delta,a_1] \cap \partial \Lambda$,

\[ K_\delta := \frac{1}{2} \left[ f_{[\Omega_\delta,a_1,...,a_n]} + f_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]} \right] - f_{[\Omega_\delta,a_1]} - 2\Lambda \left( a_1, \ldots, a_n \right) g_{[\Omega_\delta,a_1]} \]

is everywhere massive harmonic (as seen before in the proof of Proposition 23) with zero boundary values on the slit $C^1[\Omega_\delta,a_1] \cap \partial \Lambda$. We have to show

\[ K_\delta(a_1 + \frac{3\delta}{2}) = f_{[\Omega_\delta,a_1,...,a_n]}(a_1 + \frac{3\delta}{2}) - 1 - 2\Lambda \left( a_1, \ldots, a_n \right) \delta = o(\delta^{1/2}). \]

Noting the expansion of Definition 19 we see that, on $\Lambda$, away from $a_1$,

\[ K_\delta \to \frac{1}{2} \left[ f_{[\Omega_\delta,a_1,...,a_n]} + f_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]} \right] - Z \left( a_1, \ldots, a_n \right) \text{Re} \sqrt{z - a_1} = o \left( (z - a_1)^{1/2} \right), \]

so on the discrete circle $S_r = B_{r+5\delta}(a_1) \setminus B_r(a_1)$ for $r > 0$, max $S_r$, $|K_\delta| = o(r^{1/2})$ as $r \to 0$. Sharp discrete Beurling estimate (see [CHI15 (3.4)]) for the form used here) for harmonic functions may be used to dominate the value of the massive harmonic function $S_r$ at $a_1 + \frac{3\delta}{2}$, and we have

\[ \left| K_\delta(a_1 + \frac{3\delta}{2}) \right| \leq cst \cdot \delta^{1/2} r^{-1/2} \max_{S_r} |K_\delta| = cst \cdot \delta^{1/2} o(1), \]

where $o(1)$ holds for $r \to 0$. We conclude the right hand side is $o(\delta^{1/2})$ as $\delta \to 0$.

For (3.5), without loss of generality assume $B_\Omega(a_1,a_2) > 0$. Define this time $\mathcal{R}(\Omega)$ as the reflection of $\Omega$ across $a_2 + \mathbb{R}$. As above, write $\Lambda$ for the lift of the slit disc $B(a_2) \setminus \{a_2 + \mathbb{R}_{>0}\}$ such that we simultaneously define $F_{[\Omega_\delta,a_1,a_2]}(a_2 + \frac{\delta}{2}) = F_{[\mathcal{R}(\Omega_\delta),a_1,a_2]}(a_2 + \frac{\delta}{2}) = \Lambda(a_1,a_2)$ with $\Lambda > 0$. Again, by symmetry, $f_{[\Omega_\delta,a_1,a_2]} + f_{[\mathcal{R}(\Omega_\delta),a_1,a_2]}$ is zero on the boundary $\partial \Lambda \cap C^i[\Omega_\delta,a_1,a_2]$, i.e. the lift of $a_2 + \mathbb{R}_{>0}$. Define the massive harmonic measure of the point $a_1 + \frac{\delta}{2}$ in the lattice $\Lambda \cap C^i[\Omega_\delta,a_1,a_2]$ as $W_\delta$. Then define

\[ T_\delta := \frac{1}{2} \left[ F_{[\Omega_\delta,a_1,...,a_n]} + F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]} \right] - iB_\delta W_\delta, \]

which is massive harmonic on $\Lambda \cap C^i[\Omega_\delta,a_1,a_2] \setminus \{a_2 + \frac{\delta}{2}\}$, takes the value 0 on $\partial \Lambda \cap C^i[\Omega_\delta,a_1,a_2]$ and $\{a_2 + \frac{\delta}{2}\}$. Since $\frac{1}{2} \left[ F_{[\Omega_\delta,a_1,...,a_n]} + F_{[\mathcal{R}(\Omega_\delta),a_1,...,a_n]} \right]$ restricted to the imaginary corners converges to a continuous limit with asymptotic $i \text{Re} \frac{B_\Omega(a_1,a_2)}{\sqrt{z - a_2}} + o((z - a_2)^{-1/2})$ on the discrete circle $S_r = B_{r+5\delta}(a_2) \setminus B_r(a_2)$, it is easy to see that unless $B_\delta \to B_\Omega(a_1,a_2)$ and $T_\delta(a_2 + \frac{\delta}{2}) = o(1)$, we can find a point in the bulk which is greater than any value on the boundary $S_r$, contradicting the maximum principle; see [CHI15 (3.23)].

4. Continuum Analysis: Isomonodromy and Painlevé III

In this section, we take the convergence results established in Section 3 and derive established correlation results in the full plane, first shown in [Ward17] and reformulated in terms of isomonodromic deformation in [SMJ17]. We will explicitly carry out the basic 2-point case following the presentation of [Kato80 Sections III, IV], using the continuous limit of our discrete massive fermions which has been characterised in terms of a boundary value problem in Definition
We cannot directly cite their formulae, since instead of considering a complex space of functions which solve a two-dimensional Dirac equation, we cast them in terms of a real space of massive holomorphic functions because massive holomorphicity is an $\mathbb{R}$-linear notion. The resulting analysis is equivalent.

**Useful Formulae.** Recall $\partial_z = \frac{1}{2} e^{-i\theta} (\partial_r - i r^{-1} \partial_\theta), \partial_\theta = \frac{1}{2} e^{i\theta} (\partial_r + i r^{-1} \partial_\theta)$ and $I_{\nu} (r) = I_{\nu \pm 1}(r) \pm \frac{\nu}{2} I_{\nu}(r)$. Given that we work with negative indices $\nu$, we note also that $I_{\nu,\nu}(r) = I_{\nu}(r) + \frac{\nu}{2} \sin \nu \pi K_{\nu}(r)$, where $K_{\nu}$ is the modified Bessel function of the second kind.

Define $W_\nu(re^{i\theta}) := e^{i\nu\theta} I_\nu \left(2 \log r\right)$. The formal powers $Z_{\nu}^{1,1}$ defined in can be written as

$$Z_{\nu}^{1} = \frac{\Gamma(\nu+1)}{|m|^\nu} \left(W_\nu + (\text{sgn} m) W_{\nu+1}\right),$$

$$Z_{\nu}^{1} = \frac{\Gamma(\nu+1)}{|m|^\nu} \left(iW_\nu - i (\text{sgn} m) W_{\nu+1}\right).$$

Then $\partial_r W_\nu(re^{i\theta}) = e^{i\nu\theta} \cdot 2|m| \cdot \left(I_{\nu \pm 1}(2 |m| r) \pm \frac{\nu}{2|m|} I_{\nu}(2 |m| r)\right)$ and $\partial_\theta W_\nu(re^{i\theta}) = i e^{i\nu\theta} I_{\nu} \left(2 |m| r\right)$, and we see that $\partial_2 W_\nu = |m| W_{\nu-1}, \partial_3 W_\nu = |m| W_{\nu+1}$, and the corresponding identities for $Z_{\nu}^{1,1}$ follow. In fact, we will record, noting $\partial_x = \partial_\nu + \partial_\theta, \partial_y = i (\partial_\nu - \partial_\theta)$,

$$\partial_x Z_{\nu}^{1} = i Z_{\nu-1}^{1} + (\nu + 1)^{-1} m Z_{\nu+1}^{1}, \partial_y Z_{\nu}^{1} = \nu Z_{\nu-1}^{1} + (\nu + 1)^{-1} m Z_{\nu+1}^{1},$$

$$\partial_x Z_{\nu}^{1} = -i Z_{\nu-1}^{1} - (\nu + 1)^{-1} m Z_{\nu+1}^{1}, \partial_y Z_{\nu}^{1} = -\nu Z_{\nu-1}^{1} - (\nu + 1)^{-1} m Z_{\nu+1}^{1}.$$

We would also like to show how the functions behave under rotation around the origin. We will compose rotation of the coordinate system with multiplication by a phase factor and denote it by $R_\theta W_\nu(z) := W_\nu(e^{-i\psi z}) e^{-i\phi/2}$ and so on: first we see that $R_\theta W_\nu(z) = e^{-i(\nu+\frac{1}{2})\phi} W_\nu(z)$, and similarly

$$R_\phi Z_{\nu}^{1} = \frac{\Gamma(\nu+1)}{|m|^\nu} \left(e^{-i(\nu+\frac{1}{2})\phi} W_\nu + (\text{sgn} m) e^{i(\nu+\frac{1}{2})\phi} W_{\nu+1}\right)$$

$$= \cos \left(\nu + \frac{1}{2}\right) \phi \right] Z_{\nu}^{1} + \sin \left(\nu + \frac{1}{2}\right) \phi \right] Z_{\nu}^{1},$$

$$R_\phi Z_{\nu}^{1} = \cos \left(\nu + \frac{1}{2}\right) \phi \right] Z_{\nu}^{1} - \sin \left(\nu + \frac{1}{2}\right) \phi \right] Z_{\nu}^{1}. $$

**Expansion.** Fix $m < 0$. Suppose $a > 0$ is a positive real number, and consider the double cover $[C, -a, a]$. Consider the real vector space of $m$-massive holomorphic functions on the double cover which have singularity of order at most $3/2$ at each monodromy and decay at infinity. Around each monodromy, we can expand the singular part of a function in $Z_{\nu}^{1,1}$, and from Proposition [17] we see in fact fixing the coefficients of $Z_{\nu}^{1,1}, Z_{\nu}^{1,1}$ at each monodromy fixes the function. 6 basis functions are given by the two fermions $f_1 := f_{[C,-a, a]}, f_2 := f_{[C, a, -a]}$ and their derivatives $\partial_x f_1, \partial_y f_1, \partial_x f_2, \partial_y f_2$. The idea is to express the variation of $f_1$ under movement of the monodromies $\pm a$ as a linear combination of these six functions, and to get a nontrivial equality by looking at the dependent coefficient of $Z_{\nu}^{1,1}$. First, we augment the expansions (2.18), (2.19): $f_{[C, -a, a]}$ is equal to $(C^{1,1} \text{ are real constants, unrelated to the discrete notation } C^{1,1}[\Omega])$

$$Z_{1/2}^{1/2}(z + a) + 2A^1 Z_{1/2}^{1/2}(z + a) + 2A^1 Z_{1/2}^{1/2}(z + a) + 2D^1 Z_{1/2}^{1/2}(z + a)$$

$$+ 2D^1 Z_{1/2}^{1/2}(z + a) + O \left((z + a)^{5/2}\right) \text{ near } z = -a,$$

$$B_C Z_{1/2}^{2/2}(z - a) + 2C^1 Z_{1/2}^{2/2}(z - a) + 2C^1 Z_{1/2}^{2/2}(z - a)$$

$$+ 2E^1 Z_{1/2}^{2/2}(z - a) + O \left((z - a)^{5/2}\right) \text{ near } z = a.$$
\[ f_{[\varepsilon,-ae^{i\phi},ae^{i\phi}]}(z) = R_\phi f_1(z) = Z_{-\frac{1}{2}}^1(z + ae^{i\phi}) + 2 \cos \phi A^1 Z_{-\frac{1}{2}}^1(z + ae^{i\phi}) + 2 \sin \phi A^1 Z_{-\frac{1}{2}}^1(z + ae^{i\phi}) + 2 \cos 2\phi D^1 Z_{-\frac{1}{2}}^1(z + ae^{i\phi}) + 2 \sin 2\phi D^1 Z_{-\frac{1}{2}}^1(z + ae^{i\phi}) + O \left( (z + ae^{i\phi})^{5/2} \right) \text{ near } z = -a, \text{ and} \]

\[ f_{[\varepsilon,-ae^{i\phi},ae^{i\phi}]}(z) = B\Omega Z_{-\frac{1}{2}}^1(z - ae^{i\phi}) + 2 \cos \phi C^i Z_{-\frac{1}{2}}^1(z - ae^{i\phi}) - 2 \sin \phi C^i Z_{-\frac{1}{2}}^1(z - ae^{i\phi}) + 2 \cos 2\phi D^i Z_{-\frac{1}{2}}^1(z - ae^{i\phi}) + O \left( (z - ae^{i\phi})^{5/2} \right) \text{ near } z = a. \]

We are now ready to analyse the variation of \( R_\phi f_1 \) under both \( \partial_a \) and \( \partial_\phi \) at \( \phi = 0 \). Around \( z = -a \),

\[
\begin{align*}
    f_1(z) &= Z_{-\frac{1}{2}}^1(z + a) + 2A^1 Z_{-\frac{1}{2}}^1(z + a) + 2D^1 Z_{-\frac{1}{2}}^1(z + a) + O \left( (z + a)^{5/2} \right), \\
    \partial_a f_1(z) &= -\frac{1}{2} Z_{-\frac{1}{2}}^1(z + a) + A^1 Z_{-\frac{1}{2}}^1(z + a) + (3D^1 + 2m^2) Z_{-\frac{1}{2}}^1(z + a) + O \left( (z + a)^{3/2} \right), \\
    \partial_\phi f_1(z) &= -\frac{1}{2} Z_{-\frac{1}{2}}^1(z + a) + A^1 Z_{-\frac{1}{2}}^1(z + a) + (3D^1 - 2m^2) Z_{-\frac{1}{2}}^1(z + a) + O \left( (z + a)^{3/2} \right),
\end{align*}
\]

while around \( a \),

\[
\begin{align*}
    f_1(z) &= B Z_{-\frac{1}{2}}^i(z - a) + 2C^i Z_{-\frac{1}{2}}^1(z - a) + 2E^i Z_{-\frac{1}{2}}^1(z - a) + O \left( (z - a)^{5/2} \right), \\
    \partial_a f_1(z) &= -\frac{B}{2} Z_{-\frac{1}{2}}^i(z - a) + C^i Z_{-\frac{1}{2}}^1(z - a) + (3E^i + 2m^2B) Z_{-\frac{1}{2}}^1(z - a) + O \left( (z - a)^{3/2} \right), \\
    \partial_\phi f_1(z) &= \frac{B}{2} Z_{-\frac{1}{2}}^1(z - a) - C^i Z_{-\frac{1}{2}}^1(z - a) + (2m^2B - 3E^i) Z_{-\frac{1}{2}}^1(z - a) + O \left( (z - a)^{3/2} \right),
\end{align*}
\]

and similar formulae hold for \( f_2 \) with \( -a \) and \( a \) interchanged and the signs in front of \( A, B, E \) reversed. As for the varied functions, we have

\[
\begin{align*}
    \partial_a f_1(z) &= -\frac{1}{2} Z_{-\frac{1}{2}}^1(z + a) + A^1 Z_{-\frac{1}{2}}^1(z + a) + \left( 2 \partial_a A^1 + 3D^1 + 2m^2 \right) Z_{-\frac{1}{2}}^1(z + a) + O \left( (z + a)^{3/2} \right), \\
    \partial_\phi R_\phi f_1(z) &= a Z_{-\frac{1}{2}}^i(z + a) - a A^1 Z_{-\frac{1}{2}}^1(z + a) + \left( 2A^1 - 3aD^1 + 2am^2 \right) Z_{-\frac{1}{2}}^1(z + a) + O \left( (z + a)^{3/2} \right), \\
    \partial_a R_\phi f_1(z) &= \frac{aB}{2} Z_{-\frac{1}{2}}^1(z - a) - a C^i Z_{-\frac{1}{2}}^1(z - a) + \left( 2am^2B - 3aE^i - 2C^i \right) Z_{-\frac{1}{2}}^1(z - a) + O \left( (z - a)^{3/2} \right), \\
    \partial_\phi R_\phi f_1(z) &= \frac{aB}{2} Z_{-\frac{1}{2}}^1(z - a) - a C^i Z_{-\frac{1}{2}}^1(z - a) + \left( 2am^2B - 3aE^i - 2C^i \right) Z_{-\frac{1}{2}}^1(z - a) + O \left( (z - a)^{3/2} \right).
\end{align*}
\]

The resulting expansions are

\[
\begin{align*}
    \partial_a f_1 &= \frac{2(A^1 B + C^i)B}{1 - B^2} f_1 + \frac{1 + B^2}{1 - B^2} \partial_a f_1 - \frac{2B}{1 - B^2} \partial_\phi f_2, \\
    \partial_\phi R_\phi f_1 &= \frac{-2a(A^1 B + C^i)B}{1 - B^2} f_2 - \frac{2aB}{1 - B^2} \partial_a f_2 - \frac{a + B^2}{1 - B^2} \partial_\phi f_1.
\end{align*}
\]

Derivation. Comparing the coefficients of \( Z_{-\frac{1}{2}}^1, Z_{-\frac{1}{2}}^1, Z_{-\frac{1}{2}}^i \), we get from (4.2)

\[
\begin{align*}
    \partial_a B - C^i &= \frac{-2(A^1 B + C^i)B}{1 - B^2} B + \frac{1 + B^2}{1 - B^2} C^i + \frac{2B}{1 - B^2} A^1, \\
    2 \left( \partial_a A^1 \right) + 3D^1 + \frac{2m^2}{3} &= \frac{-2(A^1 B + C^i)B}{1 - B^2} 2A^1 + \frac{1 + B^2}{1 - B^2} (3D^1 + 2m^2) - \frac{2B}{1 - B^2} (-2m^2B + 3E^i), \\
    2 \left( \partial_a C^i \right) - 3E^i - \frac{2m^2B}{3} &= \frac{-2(A^1 B + C^i)B}{1 - B^2} 2C^i + \frac{1 + B^2}{1 - B^2} (3E^i + 2m^2B) - \frac{2B}{1 - B^2} (3D^1 - 2m^2),
\end{align*}
\]
while for (4.3) we get
\[
2A^4 - 3aA^3 + \frac{2am^2}{3} = -2a(\frac{A^3}{1 - B^2})2C^4 - \frac{2aB}{1 - B^2}(-3E^4 - 2m^2B) - a \frac{1 + B^2}{1 - B^2}(3D^3 - 2m^2),
\]
and
\[
\frac{2am^2B}{3} - 3aE^4 - 2C^4 = 2a(\frac{A^3}{1 - B^2})2A^4 - \frac{2aB}{1 - B^2}(3D^3 + 2m^2) - a \frac{1 + B^2}{1 - B^2}(2m^2B - 3E^4).
\]

We now make the dependence in \(m\) explicit. Similarly to above, for any \(k > 0\), \(f_{[\mathbb{C}, -ak^{-1}, -ak^{-1}]}(z|mk) = f_{[\mathbb{C}, -a]}(kz|m)k^{1/2}\). Analysing the effect of this dilation, which leaves \(r := am\) fixed, on the individual coefficients, we can write \(A^1(a, m) =: mA_0(r), B(a, m) =: B_0(r), C^1(a, m) =: mC_0(r)\). Then we have \(\partial_amA^4 = m^2A_0', \partial_mB = mB_0', \partial_mC^1 = m^2C_0'\). In terms of these functions, we have

\[
E_0' = \frac{2(A_0B_0 + C_0)}{1 - B_0^2}B_0 + \frac{2}{1 - B_0^2}C_0 + \frac{2B_0}{1 - B_0^2}A_0 = 2A_0B_0 + 2C_0,
\]
\[
A_0' = -\frac{2B_0(A_0B_0 + C_0)}{1 - B_0^2}A_0 + \frac{4B_0^2}{(1 - B_0^2)} + \frac{3m^{-2}B}{1 - B^2}(BD^1 - E^1),
\]
\[
C_0' = -\frac{2B_0(A_0B_0 + C_0)}{1 - B_0^2}C_0 + \frac{4B_0}{(1 - B_0^2)} - \frac{3m^{-2}}{1 - B^2}(BD^1 - E^1),
\]
\[
A_0 = -\frac{2r(A_0B_0 + C_0)}{1 - B_0^2}A_0 + \frac{4rB_0^2}{(1 - B_0^2)} + \frac{3am^{-1}B}{1 - B^2}(BD^1 - E^1),
\]
\[
C_0 = -\frac{2r(A_0B_0 + C_0)}{1 - B_0^2}C_0 + \frac{4rB_0}{(1 - B_0^2)} + \frac{3am^{-1}}{1 - B^2}(BD^1 - E^1).
\]

Define \(B_0 := \tanh h_0\). Then \(\frac{E_0'}{1 - B_0^2} = h_0'\) and \(\frac{4B_0^2}{(1 - B_0^2)} = \sinh^2 2h_0\). From (4.7), (4.8),

\[
A_0 + B_0 \frac{A_0B_0 + C_0}{1 - B_0^2} = A_0 + B_0C_0 = -r \left[ \frac{2(A_0B_0 + C_0)^2}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} \right],
\]
and noting (4.4),

\[
A_0 = -\frac{1}{2} \left( \ln \cosh h_0 \right)' - r \left[ \frac{1}{2} (h_0')^2 - 2 \sinh^2 2h_0 \right].
\]

To characterise \(h\), first combine (4.5), (4.6) to get

\[
A_0' + C_0'B_0 = \frac{2(A_0B_0 + C_0)(A_0B_0 + B_0^2C_0)}{(1 - B_0^2)^2} + \frac{8B_0^2}{(1 - B_0^2)^2},
\]
\[
= \left[ \frac{2(A_0B_0 + C_0)^2}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} \right] + B_0'C_0 \frac{B_0}{1 - B_0^2},
\]
then differentiate (4.9) to get

\[
\frac{A_0' + C_0'B_0 + B_0'C_0}{1 - B_0^2} + \frac{2B_0B_0'(A_0 + B_0C_0)}{(1 - B_0^2)^2} \frac{1 - B_0^2}{(1 - B_0^2)^2} - r \left[ \frac{2(A_0B_0 + C_0)^2}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} \right]' = \frac{2A_0B_0 + C_0}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} - r \left[ \frac{2(A_0B_0 + C_0)^2}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} \right]'.
\]

Then combining the two we finally have

\[
\frac{2B_0'(A_0B_0 + C_0)}{(1 - B_0^2)^2} = -r \left[ \frac{2(A_0B_0 + C_0)^2}{(1 - B_0^2)^2} - \frac{8B_0^2}{(1 - B_0^2)^2} \right]',
\]
or

\[
(h_0')^2 = -r \left[ \frac{1}{2} (h_0')^2 - 2 \sinh^2 2h_0 \right]'.
\]

Simplifying, we have \(h_0'' + \frac{h_0'}{r} = 4 \sinh 4h_0(r)\). This is equivalent to the Painlevé III equation \(r\eta_0\eta_0'' = r (\eta_0')^2 - \eta_0\eta_0' - 4r + 4r\eta_0^4\) by a change of variables \(\eta_0 = e^{-2h_0}\) [KaKo80] (4.12)).
APPENDIX A. Harmonicity Estimates

In this Appendix, we collect together the discrete analytic calculations and estimates used in the paper. Fix a discrete simply connected planar graph $G_\delta$, which can be thought of as a subgraph of $\Omega_\delta$ or $[\Omega_\delta, a_1, \ldots, a_n]$.

**Proposition 26.** A massive $s$-holomorphic function $F : \mathcal{E}[G_\delta] \rightarrow \mathbb{C}$ is massive discrete holomorphic, that is to say

$$\cos\left(\frac{\pi}{4} + 2\Theta\right) F(r_+) - \cos\left(\frac{\pi}{4} - 2\Theta\right) F(r_-) = -i \left(\cos\left(\frac{\pi}{4} + 2\Theta\right) F(i_+) - \cos\left(\frac{\pi}{4} - 2\Theta\right) F(i_-)\right),$$

(A.1)

$$\cos\left(\frac{\pi}{4} - 2\Theta\right) F(i_+) - \cos\left(\frac{\pi}{4} + 2\Theta\right) F(i_-) = -i \left(\cos\left(\frac{\pi}{4} - 2\Theta\right) F(r_+) - \cos\left(\frac{\pi}{4} + 2\Theta\right) F(r_-)\right),$$

(A.2)

if there is a $\lambda$-corner $c$ such that $r_\pm = c \pm \frac{\delta + 3i}{2}$ (real corners) and $i_\pm = c \pm \frac{-\delta + 3i}{2}$ (imaginary corners), or a $\bar{\lambda}$-corner $c'$ such that $r_\mp = c \pm \frac{\delta - i}{2}$ and $i_\mp = c \pm \frac{-\delta - i}{2}$ (resp. imaginary and real corners).

It is massive harmonic, i.e.

$$\Delta^\delta F(c) = 2 \frac{\cos \left(\frac{\pi}{4} - 2\Theta\right)}{\cos \left(\frac{\pi}{4} + 2\Theta\right)} - 2 \frac{\cos \left(\frac{\pi}{4} + 2\Theta\right)}{\cos \left(\frac{\pi}{4} - 2\Theta\right)} F(c) = \left(8 \sin^2 \frac{2\Theta}{cos 4\Theta}\right) F(c) = : M^2_H F(c) \text{ for } c \in C^{1, \eta}[G_\delta].$$

In addition, its square satisfies

$$\partial^\delta \partial^\delta F^2(x) \left\{ \begin{array}{ll} A_\Theta \sum_{n=0}^{3} |F(x + i^n \frac{\delta}{2})|^2 + B_\Theta |\partial_x F|^2(x) & x \in \mathcal{F}[\Omega_\delta] \setminus \{a_1, \ldots, a_n\} \\
-A_\Theta \sum_{n=0}^{3} |F(x + i^n \frac{\delta}{2})|^2 - B_\Theta |\partial_x F|^2(x) & x \in \mathcal{V}[\Omega_\delta] \setminus \{a_1 + \delta\} \end{array} \right.$$  

(A.3)

where $A_\Theta = \frac{2(\sqrt{2} \cos (\frac{\pi}{4} - 2\Theta))^{-1}}{\sqrt{2} \cos (\frac{\pi}{4} - 2\Theta)}$, $B_\Theta = \frac{1}{\sqrt{2} \cos (\frac{\pi}{4} - 2\Theta)}$.

**Proof.** For the first line in (A.1), note that by massive s-holomorphicity we have the edge values $F\left(\frac{r_1+\delta}{2}\right) = e^{-i\Theta} [F(r_+) + F(i_-)]$ and $F\left(\frac{r_1+\delta}{2}\right) = e^{i\Theta} [F(i_+) + F(r_-)]$. Since $F$ is $s$-holomorphic at the $\lambda$-corner $c$, which is adjacent to both of them, writing $e^{-i\Theta} \text{Pro}_{c,\alpha,\beta} F\left(\frac{r_1+\delta}{2}\right) = e^{i\Theta} \text{Pro}_{c,\alpha,\beta} F\left(\frac{r_1+\delta}{2}\right)$, equivalent to $\frac{1}{2} \left(\left[ e^{-2i\Theta} (F(r_+) + F(i_-)) + ie^{2i\Theta} (F(r_-) - F(i_+)) \right] = \frac{1}{2} \left[ e^{2i\Theta} (F(i_+) + F(r_-)) + ie^{-2i\Theta} (F(i_+) + F(r_-)) \right]$, and rearranging gives the result. For the second line, notice that $iF$ is $(-\Theta)$-massive $s$-holomorphic if we move to the dual graph $G^*_\delta$ (i.e. $\mathcal{V}(G^*_\delta) := \mathcal{F}(G_\delta)$). Since this duality transformation converts $\lambda$-corners into $\lambda$-corners, we can use the previous calculation.

For (A.2), suppose $c$ is a real corner. Take four copies of the previous result (A.1) (see Figure A.1) around each of the four middle corners $c \pm \frac{\delta + i\delta}{2}$. Each of them involve $c$ and one of the four neighbouring real corners $c \pm (\delta \pm i\delta)$; summing the four equations with scalar factors so that the coefficients of $F(c \pm (\delta \pm i\delta))$ in each equation is 1, the result is straightforward. The case where $c$ is imaginary is immediate from duality as above.

For (A.3), take $x \in \mathcal{F}[G_\delta]$ and note that the value at each of the neighbouring edges $x + i^n \frac{\delta}{\sqrt{2}}$ can be reconstructed from two of the four corners $x \pm i^n \frac{\delta}{\sqrt{2}}$. Explicitly, inverting s-holomorphicity projections give

$$\cos \left(\frac{\pi}{4} + 2\Theta\right) \lambda^{n+1} F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right) = e^{i\Theta} F \left( x + i^n \frac{\delta}{2} \right) - e^{-i\Theta} F \left( x + i^{n+1} \frac{\delta}{2} \right)$$

$$= -i^n \left[ e^{i\Theta} F \left( x + i^n \frac{\delta}{2} \right) - e^{-i\Theta} F \left( x + i^{n+1} \frac{\delta}{2} \right) \right].$$

noting that $F \left( x + i^n \frac{\delta}{2} \right) = -i^n F \left( x + i^n \frac{\delta}{2} \right)$.
So multiplying the two lines
\[
\cos^2 \left( \frac{\pi}{4} + 2\Theta \right) i^{2n+2} \lambda^2 F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right) =
- i^n \left[ e^{2i\Theta} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2 + i e^{-2i\Theta} \left| F \left( x + i^{n+1} \frac{\delta}{2} \right) \right|^2 
\right. 

- F \left( x + i^n \frac{\delta}{2} \right) \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right) - i F \left( x + i^n \frac{\delta}{2} \right) \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right) \bigg]
\]
\[
= - i^n \left[ e^{2i\Theta} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2 + i e^{-2i\Theta} \left| F \left( x + i^{n+1} \frac{\delta}{2} \right) \right|^2 
\right. 

+ 2 i^{n+1} F \left( x + i^n \frac{\delta}{2} \right) \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right) \bigg].
\]

So
\[
\cos^2 \left( \frac{\pi}{4} + 2\Theta \right) \partial_x^2 F (x)^2 = \cos^2 \left( \frac{\pi}{4} + 2\Theta \right) \sum_{n=0}^{3} i^n \lambda F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right)^2
\]
\[
= \lambda^{-1} \left[ e^{2i\Theta} + i e^{-2i\Theta} \right] \sum_{n=0}^{3} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2
\]
\[
+ 2 \lambda^{-1} \sum_{n=0}^{3} i^{n+1} F \left( x + i^n \frac{\delta}{2} \right) \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right)
\]
\[
= 2 \cos \left( \frac{\pi}{4} - 2\Theta \right) \sum_{n=0}^{3} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2
\]
\[
+ 2 \lambda^{-1} \sum_{n=0}^{3} i^{n+1} F \left( x + i^n \frac{\delta}{2} \right) \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right).
\]

Now reuse the first relation
\[
\cos \left( \frac{\pi}{4} + 2\Theta \right) \lambda^{-1} i^{-n} F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right) = - i^{-2n} \left[ e^{i\Theta} \bar{F} \left( x + i^n \frac{\delta}{2} \right) - e^{-i\Theta} \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right) \right]
\]
\[
\cos \left( \frac{\pi}{4} + 2\Theta \right) \lambda i^n F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right) = (-1)^{n+1} \left[ e^{-i\Theta} \bar{F} \left( x + i^n \frac{\delta}{2} \right) - e^{i\Theta} \bar{F} \left( x + i^{n+1} \frac{\delta}{2} \right) \right]
\]
\[
\cos \left( \frac{\pi}{4} + 2\Theta \right) \partial_x \bar{F} (x) = (e^{i\Theta} + e^{-i\Theta}) \sum_{n=0}^{3} (-1)^{n+1} F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right)
\]
\[
= 2 \cos \Theta \sum_{n=0}^{3} (-1)^{n+1} F \left( x + i^n \frac{\lambda \delta}{\sqrt{2}} \right).
\]

Taking squares
\[
\cos^2 \left( \frac{\pi}{4} + 2\Theta \right) \left| \partial_x \bar{F} (x) \right|^2 = 4 \cos^2 \Theta \left[ \sum_{n=0}^{3} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2 \right.
\]
\[
+ \Re \sum_{n \neq n'} (-1)^{n+1} F \left( x + i^{n'} \frac{\delta}{2} \right) (-1)^{n'+1} \bar{F} \left( x + i^{n'} \frac{\delta}{2} \right)
\]
\[
= 4 \cos^2 \Theta \left[ \sum_{n=0}^{3} \left| F \left( x + i^n \frac{\delta}{2} \right) \right|^2 \right.
\]
\[
- \Re \sum_{n \neq n'} (-1)^{n+n'} \bar{F} \left( x + i^n \frac{\delta}{2} \right) F \left( x + i^{n'} \frac{\delta}{2} \right) \bigg],
\]
but \(i^n F(x + \frac{i^n \delta}{2}) F(x + \frac{i^n \delta}{2}) \in i\mathbb{R} \) if \(|n - n'| = 2\). The remaining 8 combinations of \(n, n'\) all give rise to purely real terms, and resumming gives

\[
\text{Re} \sum_{n \neq n'} (-1)^{n+n'} i^n F(x + \frac{i^n \delta}{2}) F(x + \frac{i^n \delta}{2}) = -\sum_{n=0}^{3} (i^n + i^{n+1}) F(x + \frac{i^n \delta}{2}) F(x + \frac{i^{n+1} \delta}{2})
\]

\[
\cos^2 \left( \frac{\pi}{4} + 2\Theta \right) |\partial \bar{F}(x)|^2 = 4 \cos^2 \Theta \left[ \sum_{n=0}^{3} |F(x + \frac{i^n \delta}{2})|^2 \right]
\]

\[
+ 3 \sum_{n=0}^{3} \sqrt{2} i^{n+1} \lambda^{-1} F(x + \frac{i^n \delta}{2}) F(x + \frac{i^{n+1} \delta}{2})
\].

Comparing the two expressions

\[
\cos^2 \left( \frac{\pi}{4} + 2\Theta \right) \left[ 2 \sqrt{2} \cos^2 \Theta \cdot \partial_z \bar{F}(x)^2 - |\partial \bar{F}(x)|^2 \right]
\]

\[
= 4 \left( \sqrt{2} \cos \left( \frac{\pi}{4} - 2\Theta \right) - 1 \right) \sum_{n=0}^{3} |F(x + \frac{i^n \delta}{2})|^2,
\]

we have the full result given duality. \(\square\)

**Remark 27.** \(\{A,1\}\) is equivalent to massive \(s\)-holomorphicity in the sense that if we have such values of \(F\) on \(C^{1,1}[G_\delta]\) then it is easy to see from the proof that we have enough data to extend the values \(s\)-holomorphically first to \(E[G_\delta]\) and then the \(\lambda, \lambda\)-corners. In other words, bound on \(C^{1,1}\) is equivalent to a global bound in an \(s\)-holomorphic function. On \(E[G_\delta]\), \(\{A,1\}\) becomes

\[
\partial^A \bar{F}(x) := \sum_{n=0}^{3} i^n e^{i\pi/4} F \left( x + i^n e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right)
\]

\[
= \sin \Theta \sec \left( \frac{\pi}{4} + 2\Theta \right) \sum_{n=0}^{3} \frac{F \left( x + i^n e^{i\pi/4} \frac{\delta}{\sqrt{2}} \right)}{\partial \bar{F}(x)},
\]

i.e. a discretised version of \(\partial \bar{f} = m \bar{f}\) given \(\Theta \sim \frac{m \delta}{\sqrt{2}}\).

**Lemma 28.** Suppose \(\Omega' \subset \Omega\) are smooth simply connected domains. Any function \(H_0\) on \(V[(\Omega \setminus \Omega')_\delta]\) which is harmonic and takes the boundary value 0 on \(\partial V[\Omega_\delta]\) and 1 on \(\partial V[\Omega'_\delta]\) satisfies \(0 \leq H_0(a) \leq C(\Omega, \Omega')\delta\) on any \(a \in V[(\Omega \setminus \Omega')_\delta]\) adjacent to \(\partial V[\Omega_\delta]\) for a constant \(C(\Omega, \Omega')\).

**Proof.** We believe this lemma is standard. One possible proof would proceed by mapping \(\Omega \setminus \Omega'\) to the annulus \(B_1 \setminus B_{r_0}\) for some \(r_0 > 0\) using a Riemann map which smoothly extends to \(B_1 \setminus B_{r_0}\). The radial function \(\frac{1-r}{1-r_0}\) on \(B_1 \setminus B_{r_0}\) is superharmonic, so its composition with the Riemann map is (continuous) superharmonic on \(\Omega \setminus \Omega'\); the restriction to \(V[(\Omega \setminus \Omega')_\delta]\) is discrete superharmonic for small enough \(\delta\) since the discrete laplacian (suitably renormalised) and continuous laplacian are uniformly close on smooth functions, and we can use it to upper bound \(H_0\). \(\square\)

We frequently have local \(L^2\)-bounds for our function \(F\); it turns out that thanks to massive harmonicity, this is sufficient for equicontinuity.

Recall that the massive harmonic measure \(hm^\delta_A(z|\Theta)\) for a discrete domain \(A, z \in A,\) and \(a \in \partial A \cup A\) is the probability of a simple random walk, which is started at \(z\) and extinguished after each step with probability \(1 - \frac{2 \sin^2 \Theta}{\cos \Theta}\), hitting \(a\) before \(\partial A \setminus \{a\}\). It is the unique \(\Theta\)-massive harmonic function on \(A\) which takes the boundary value 1 at \(a\) and 0 on \(\partial A \setminus \{a\}\). In the scaling limit \(\delta \downarrow 0\) and \(\Theta \sim \frac{m \delta}{\sqrt{2}}\), the massive random walk is extinguished after an exponential step of mean \(\frac{1}{2\cos \Theta}\). Taking into account the square-root scaling for the random walk, this corresponds to a scaling of order \(\sqrt{2} \cdot \frac{1}{\sqrt{2}m \delta} = -\frac{1}{m}\). For a more precise asymptotic, see [BdTR17].

**Proposition 29.** There are constants \(C, C', c > 0\) such that, for a real massive harmonic function \(F : C^{1,1}(B_{R\delta}) \rightarrow \mathbb{C}\) with \(\Delta F = M^\Theta_H F\),

\[
\delta^{-1} \left| F \left( \frac{\delta}{2} + i \delta \right) - F \left( -\frac{\delta}{2} \right) \right| \leq C e^{c m R} \frac{L}{R^3},
\]

where \(L = \sum_{c \in C^{1,1}(B_{R\delta})} |F(c)|^2 \delta^2\).
Proof. For the first bound, note that $F^2 \geq 0$ is subharmonic:

$$\Delta^\delta F^2(c) = (2M_r^2 + \frac{M_h^2}{4})F^2(c) + \frac{1}{2} \sum_{x,y} |F(x) - F(y)|^2.$$

So we can use the mean value property for harmonic functions: for $0 < r < R$, write the discrete circle $S_r = C^1(B_R) \cap (B_{R + \delta} \setminus B_r)$

$$\left| F\left(-\frac{\delta}{2}\right) \right|^2 \leq \frac{cst}{r} \sum_{c \in S_r} |F(c)|^2 \delta,$$

multiplying by $\delta$ and summing over the $O(\delta)$ discrete circles $S_r$ such that their union equals $B_R \setminus B_{R/2}$

$$\left| F\left(-\frac{\delta}{2}\right) \right|^2 \leq \frac{cst}{R} \sum_{c \in B_R \setminus B_{R/2}} \left| F\left(-\frac{\delta}{2}\right) \right|^2 \delta^2 \leq \frac{cst \cdot L}{R}.$$

For the desired bounds, note that by first applying (A.6) to smaller balls of radii $R/2$ we can opt for a bound of the form

$$\left| F\left(-\frac{\delta}{2}\right) \right| \leq \frac{cst \cdot c^m R}{R/2} \max_{B_{R/2}} |F|,$$

$$\delta^{-1} \left| F\left(-\frac{\delta}{2}\right) \right| \leq \frac{cst \cdot c^m R \max_{B_{R/2}} |F|}{R/2}.$$

By writing $F(c) = \sum_{c' \in S_{R/2}} \text{hm}_{B_{R/2}}(c)F(c')$ for $c \in B_{R/2}$, where $\text{hm}_{B_{R/2}}$ is the massive harmonic function on $B_{R/2}$ whose boundary value is zero on $S_{R/2} \setminus \{c'\}$ and one at $c'$, it suffices to have such estimates for each $\text{hm}_{B_{R/2}}$.

$\text{hm}_{B_{R/2}}(c)$ is the hitting probability of $c'$ of the massive random walk started at $c$. For the bound at $-\frac{\delta}{2}$, simply note that the probability of a massive random walk reaching a distance $R$ decays exponentially. For the second, we know that the hitting probability for the simple random walk (i.e. the harmonic measure of the point $c'$, with $m = 0$) satisfies the desired estimate (e.g. CheSm11 Proposition 2.7): as $\delta \to 0$, the difference of the probabilities of simple random walk started at neighbouring points reaching a macroscopic distance is of order $\delta$. For massive random walk, these instances need to survive to contribute to the difference; therefore the difference also decays by an exponential factor. \hfill \Box

Remark 30. The second bound in (A.4) is valid for differences in other directions as well, since massive harmonicity and the bound are rotationally invariant. By considering smaller balls within $B_{R/2}$, we in fact deduce uniform bounds for $F$ and its discrete derivative in, say, $B_{R/2}$. Then, defining $D_\delta^2 F(c) := F(c + \delta + i \delta) - F(c), D_\delta^4 F(c) := F(c + \delta - i \delta) - F(c)$, which are massive harmonic functions uniformly bounded in $B_{R/2}$, and using the bound (A.7) on them, we actually have bound on discrete derivatives of second order in, say, $B_{R/4}$. Recursively, we see that derivatives of any order can be locally bounded.

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