Efficient Computation by Three Counter Machines

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Abstract
We show that multiplication can be done in polynomial time on a three counter machine that receives its input as the contents of two counters. The technique is generalized to functions of two variables computable by deterministic Turing machines in linear space.

1 Preliminaries

In an investigation of the power of simple counter machines [1], Schroeppe1 described a four counter machine that multiplies two numbers in quadratic time and posed as a hard problem to multiply two numbers using only three counters. He presented a solution based on an exponential encoding of numbers and asked whether there is a way that takes less time.

The purpose of this note is to first present a solution for multiplication that runs in time polynomial in the maximum input value and then generalize the technique to all two variable functions that can be computed by a deterministic Turing machines operating in linear space (linear bounded automaton) to which the arguments are passed in binary representation on its work-tape. This class of functions includes integer division.

A $k$ counter machine is equipped with $k$ counters each storing a nonnegative integer. Its operation is controlled by a deterministic sequential program consisting of four types of instructions:

Increment: Add 1 to the specified counter.

Conditional Decrement: If the specified counter stores a value greater than 0, subtract 1. Otherwise leave the counter unchanged and jump to a specified instruction out of the normal flow of control.

Unconditional Jump: Flow of control is transferred to the specified instruction.

Halt: Stop execution.
Notice that unlike many computational models studied in Complexity Theory, a counter machine has no separate input-tape or output-tape and receives input values as the contents of its counters, which corresponds to a unary encoding.

Instead of breaking down an algorithm into these four basic instruction, we will use additional notation borrowed from higher programming languages. In the following we sketch how to simulate these constructs with macros on counter machines. By $R$, $R_1$, $R_2$ we denote any of the first $k-1$ counters, while counter $k$ is reserved as auxiliary scratch memory. Counter $k$ is assumed to have 0 as its initial value and all macros will restore this value.

**if $R > 0$ then begin...end:** Decrement $R$ and jump to the instruction after end if $R$ was 0 before the instruction. Otherwise increment $R$ (restoring its value before the if) and continue with the instructions between begin and end.

**while $R > 0$ do begin...end:** Like if $R > 0$ then... described above, but jump back to the decrement instruction before the end.

**if odd($R$) then begin...end:** In a loop decrement $R$ twice while incrementing counter $k$. If $R$ has the value 0 in the first decrement instruction, its initial value was even. Then restore its value by incrementing $R$ twice while decrementing counter $k$. Skip the begin...end block. If $R$ has the value 0 in the second decrement instruction of the loop, its initial value was odd. Restore its value as in the case when its value was even, additionally adding 1. Execute the instructions between begin and end.

**while even($R$) do begin...end:** Like if odd($R$) then... described above (roles of odd/even interchanged), but jump back to the test before the end.

**$R_1 := R_2$:** In a loop decrement $R_1$ until it is 0. In another loop decrement $R_2$ while incrementing $R_1$ and counter $k$. Finally in a loop decrement counter $k$ and increment $R_2$, restoring its previous value.

**$R_1 := R_1 + R_2$:** In a loop decrement $R_2$ until it is 0 while incrementing $R_1$ and counter $k$. In another loop decrement counter $k$ and increment $R_2$, thus restoring its previous value.

**$R_1 := R_1 - R_2$:** In a loop decrement $R_2$ until it is 0 while decrementing $R_1$ (if possible) and incrementing counter $k$. In another loop decrement counter $k$ and increment $R_2$, thus restoring its previous value. Note that if $R_1 < R_2$, the resulting value of $R_1$ will be 0. This operation is sometimes called modified minus.

**$R := m * R$:** In a loop decrement $R$ while incrementing counter $k$. In another loop decrement counter $k$ and increment $R$ in every iteration $m$ times.

**$R := R \div m$:** In a loop decrement $R$ while incrementing counter $k$. In another loop decrement counter $k$ $m$ times and increment $R$ for every full iteration.
2 Results

Proposition 1 A three counter machine can compute $X \times Y$ for nonnegative integers $X$ and $Y$ in polynomial time as the contents of a counter when $X$ and $Y$ are initially stored in two counters.

Proof. The algorithm will be presented using the macros introduced above:

procedure mult; (* input: X in A, Y in B; output: B *)
begin
    if B > 0 then (* special case Y = 0, output Y *)
    begin
        A := 2 * A + 1; (* flag in lowest bit *)
    (* Loop I *)
    while B > 0 do
        begin
            A := 2 * A;
            if odd(B) then begin A := A + 1 end;
            A := 2 * A;
            B := B div 2
        end;
        B := A;
        A := A + 1; (* flag in lowest bit *)
    (* Loop II *)
    while even(B) do
        begin
            A := 2 * A;
            B := B div 4
        end;
        B := B - 1; (* remove flag *)
    (* Loop III *)
    while even(A) do
        begin
            B := 8 * B;
            A := A div 2
        end;
        A := A - 1; (* remove flag *)
    (* Loop IV *)
    while even(A) do
        begin
            A := A div 2;
            B := B div 4;
            if odd(A) then begin A := A + B end;
            A := A div 2;
            B := B div 2
        end;
        A := A - B; (* adjust initial value of A *)
        B := A;
        B := B div 2 (* remove flag *)
end
In the following the purpose of the four loops is outlined:

**Loop I:** For each bit $d$ of $B$ (input $Y$) shift two bits $d0$ into $A$, thus reversing the order of bits.

**Loop II:** Each two-bit group generated in loop I is translated into 0 and shifted into $A$. At the end of loop II counter $A$ contains (starting from lowest bits) $\log_2(Y+1)$ 0-bits, $2\log_2(Y+1)$ groups of two bits each containing a single bit of $Y$ in the higher order bit, a single 1 and the input $X$ shifted by $3\log_2(Y + 1) + 1$ positions. At the end of loop II counter $B$ contains $2Y + 1$.

**Loop III:** $B$ is shifted by $3\log_2(Y + 1)$ positions, while the trailing 0-bits are removed from $A$.

**Loop IV:** Add the multiples of $X$ stored in $B$ to an ‘accumulator’ in $A$ with initial value $Y$. In addition, $A$ stores the two-bit representation of $X$. It is decoded and controls the additions.

Notice that the maximum value handled by the algorithm is of order $X \cdot Y^3$. The macros introduced are time-bounded by this value and the loops of the main algorithm are executed $\log_2(Y + 1)$ times. Thus the running time of algorithm is polynomial in the input values.

**Theorem 1** The class of functions of two variables computable by three counter machines in polynomial time coincides with the class of functions of two variables computable by deterministic Turing machines in linear space, where input and output of the Turing machines are encoded in binary.

**Proof.** We first show how to simulate a Turing machine efficiently with the help of a three counter machine. Given a concise encoding of the input it is well known how to do this (see, e.g., the Theorem on p. 2 of [1]). Thus we will only sketch this part and focus on the input- and output-problem.

Let Turing machine $M$ computing function $f(x, y)$ have the tape alphabet $\Sigma$ that includes a blank symbol, symbols 0, 1 and a separator #. We assume that the input is encoded as $\text{bin}(x)\#\text{bin}(y)^R$ (where $w^R$ is the reversal of string $w$). $M$ starts its operation with its tape head before the first symbol of $\text{bin}(x)$, and $M$ stops with $\text{bin}(f(x, y))$ as its tape contents with its head again before the first symbol (the simulation can easily be adopted to other input-/output-conventions).

A three counter machine $C$ simulating $M$ has $x$ on counter 1 and $y$ on counter 2. For encoding $M$’s tape alphabet $C$ reserves $k = \lceil \log_2|\Sigma| \rceil$ bits per symbol and uses the following codes for $M$’s symbols:

- **blank:** A sequence of $k$ zeroes (this in mandatory, since the infinite number of blank symbols is represented by counter value 0).
- **#:** The sequence $0^{k-1}$.
- **0, 1, and further symbols:** Sequences $0^{k-2}10$, $0^{k-2}11$ and so on.
First $C$ computes $2^k x + 1$ (thus encoding a separator) and then (using counter 3 as scratch memory) repeatedly divides counter 2 by 2 and multiplies counter 1 by $2^k$ adding the appropriate constants encoding 0 and 1. This process continues until counter 2 is 0.

In the next loop $C$ decodes $k$ bits from counter counter 1 and puts the encoding on counter 2 until the encoding of separator # has been transferred. Notice that the least significant bits of $y$ are encoded in the least significant $k$-bit blocks.

Finally the analogous process is carried out for $x$, putting the blocks onto counter 2.

After this preparation, $C$ carries out the standard simulation of $M$ treating blocks of $k$ bits as an encoding of symbols from $\Sigma$.

When $M$ stops, $C$ translates the encoding back to a number stored on counter 1 by reversing the encoding outlined above. Since all numbers can be encoded in $O(\log x + y)$ bits, the simulation is polynomial in the input values.

For the simulation of a polynomial time $k$ counter machine by a Turing machine observe that due to the limited arithmetic the numbers generated on the counters are polynomial and can be encoded in linear space. Therefore a Turing machine can simulate the counter machine by updating $k$ binary strings representing the counter contents.

References

[1] R. Schroeppe1. A two-counter machine cannot calculate $2^N$. Technical Report 257, Massachusetts Institute of Technology, A. I. Laboratory, 1973.