Ascending Disk: Theoretical and Numerical Study

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In this work we analyzed a classical mechanical problem: a disc with a circular hole rotating and ascending on a inclined plane. The principal idea is to show the basic equations in both Lagrangian and Newtonian formalisms and to present numerical solutions. The equations are not solvable analytically, then we proceed to integrate numerically using the standardized Runge-Kutta method of order 4. We present several phase space showing the time evolution of the system. Additionally we presented an extension of the system considering a damping factor that decays exponentially in time.

1. Introduction

In his research on cognitive development Jerome Bruner [1] proposes three modes of representation: enactive (action based), iconic (image based) and symbolic (language based). In a natural learning process these modes occur sequentially in the order mentioned. In the context of learning Physics, the enactive representation is directly related to the phenomenon, in other words, is the experience in the lab, iconic representation is such that is related to the graphs, and symbolic representation is necessarily related to the theoretical formalism.

The traditional physics courses at university level generally are focused on the development and learning theory, to enable the student to solve theoretical problems, thereby covering mainly the iconic representations. The symbolic representations sometimes are linked to the activities developed in the laboratory, but nevertheless this happens not together with the corresponding courses. So in this way the student engages in an abstract reasoning without reference to a particular physical system which relate their cognitive development in the given representations. Actually to have a deeper understanding of physics, it is necessary to access to the concepts through: 1) experimental work (enactive), 2) theory (symbolic) 3) numerical developments, simulation (iconic and symbolic). It is benefit for the student to show the connections between these three ways of solving a physics problem, with the intention that the student to assimilate precisely the concepts involved.

In this work we analyzed a classical mechanical problem: a disc with a circular hole rotating and ascending on a inclined plane. There are several similar studies in the literature [2,3], then in this brief report the principal idea is to show two of the three modes of representation previously mentioned. First, we present the basic equations in both Lagrangian and Newtonian formalisms
(symbolic). As the equations are not solvable analytically, then we proceed to integrate numerically using the standard Runge-Kutta method of order 4 (iconic and symbolic). We present several phase space showing the time evolution of the system. Eventually, this system could be analysed experimentally completing the three modes of representation.

2. Ascending disc with a circular hole

2.1. The model

We considered a uniform disc with radius $R$ with a circular hole with radius $a < R$ ascending on a inclined plane according with Figure 1. The ratio between the radii is defined as $\gamma$.

$$\gamma = \frac{a}{R}, \quad (1)$$

It is possible to prove that the inertial moment of the disc with a circular hole is given by

$$I_R = \frac{1}{2} M_R R^2 \kappa, \quad (2)$$

where $\kappa$ is an adimensional number given by

$$\kappa = 1 - \gamma^2 + (2 \eta^2 - 1) \left( \frac{\gamma^2}{1-\gamma^2} \right) \gamma^2, \quad (3)$$

2.2. Newtonian Formulation

Using the non-slipping condition we have $\dot{x} = R \dot{\theta}$, it is means $= R(\theta - \theta_0)$.

As a first approximation, we considered the no presence of dissipative forces, in other words, the case when the mechanical energy is conserved. The potential energy is given by

$$U = M_R g (H + h), \quad (4)$$

where $H$ and $h$ are written in terms of the high climbed on the plane and the distance to the geometrical center of the disc of radius $R$ according with Figure 1. Another parametrized form of the potential energy is

Figure 1. Schematic representation of a disc with a circular hole ascending on an inclined plane.
\[ U(\theta) = M_R g R \left[ (\theta - \theta_0) \sin \varphi + \gamma \cos \theta \right] \] (5)

For another hand the expression for the kinetic energy is given by
\[ K = \frac{2+\kappa}{4} M_R R^2 \dot{\theta}^2 \] (6)

Finally, the mechanical energy of the system is
\[ E = \frac{2+\kappa}{4} M_R R^2 \dot{\theta}^2 + M_R g R \left[ (\theta - \theta_0) \sin \varphi + \gamma \cos \theta \right] \] (7)

The initial mechanical energy, when the angular velocity of the disc is zero, is
\[ E_0 = M_R g R \left[ \theta_0 \sin \varphi + \gamma \cos \theta \right] \] (8)

Then, to find the solution for \( \theta \) we must to solve the following non-linear differential equation
\[ \dot{\theta} = \frac{2}{R} \left( \frac{E_0 - M_R g R \left[ (\theta - \theta_0) \sin \varphi + \gamma \cos \theta \right]}{[2+\kappa]M_R} \right)^{1/2} \] (9)

and using the non slipping condition is possible to obtain the solution for \( \dot{x} \).

### 2.3. Lagrangian formulation

The Lagrangian function of the system is
\[ L = \frac{2+\kappa}{4} M_R R^2 \dot{\theta}^2 - M_R g R \left[ (\theta - \theta_0) \sin \varphi + \gamma \cos \theta \right] \] (10)

And using the Euler-Lagrange equation [4, 5], we can obtain the following movement equation
\[ \ddot{\theta} + \omega^2 \sin \theta + \psi \sin \varphi = 0 \] (11)

where
\[ \omega^2 = \frac{2 g \gamma}{(2+\kappa)R}, \quad \psi = \frac{\omega^2}{\gamma} \] (12)

### 3. Numerical Study

The numerical solution was obtained using the regular and well known Runge-Kutta method of order 4. In order to obtain the numerical solutions of the equation (11). The initial conditions were \( \theta(0) = \theta_0 \) and \( \dot{\theta}(0) = 0 \). It is very useful to rewrite the original problem as two coupled differential equations of first order solved simultaneously
\[ u(t) = \theta(t) \]
\[ \dot{u}(t) = \dot{\theta}(t) \]
\[ \dot{u}(t) = \frac{d}{dt} u(t) = \dot{\theta}(t) = -\psi \sin \varphi - \omega^2 \sin \theta \] (13)
The numerical results were obtained using in the previous theoretical expressions and using the data presented in Table I. The value of the initial angle was generated randomly in the interval [0, 0.4] for \( \theta \) and \( \dot{\theta} \) in order to obtain several results to show in the phase space. The interval of integration was since 0 to 5, and this interval was split in 975 equal parts, then the time step was \( \delta h = 5.12 \times 10^{-3} \), the error to the simulation given in the numerical method was \( \epsilon = 6.87 \times 10^{-10} \). The value of the acceleration of the gravity used in the simulation was the reported for the Mexico City.

Additionally with these numerical results in the following Figures, 3 and 4, the behavior of the system in the presence of dissipative forces is showed.

**Table I. Parameters used in the numerical simulation**

| \( \varphi \) | \( \theta_0 \) | \( \gamma \) | \( \beta \) | \( N \) |
|----------------|----------------|-------------|------------|------|
| 1.0 rad | 0.0 rad | 1.291 | 0 | 975 |
| \( \dot{\theta}_0 \) | 0.01 rad/s | M | 0.6 kg | \( \delta h \) |
| 0.126 m | \( 0.365 \) s | 0 | 0 | 500 |

\( N \) is the number of parts of the interval of integration, \( n \) is the number of initial conditions simulated. The interval from the initial conditions for \( \theta_0 \) and \( \dot{\theta}_0 \) were taken was [0, 0.4].

4. Obtained Results.

Figure 2 show two graphics for the lagrangian function, there is a function of the generalized coordinate: \( \theta \), \( \dot{\theta} \), and time, the time is evaluate in two different value \( t_1 = 0 \) s and \( t_2 = 16 \) s. To this surface show in the figure, is possible obtain the level curves, which correspond to the phase diagrams of system, this surface show are to the first case, when the damped is not present.

**Figure 2.** Lagrangian surfaces obtained with the constant value of \( t \), \( \theta \) and \( \dot{\theta} \), for two different times, \( t = 0 \) s and \( t = 16 \) s.
Figure 3. Behavior of the distance $x$ as a function of time a) for the isoenergetic hypothesis and b) system with damping. In both situations one can see an oscillating behavior. The initial condition were obtained in a random way, with the amplitude of the oscillation is constant always. Figure 3b) show when the dissipative force is present and the hypothesis isoenergetic fail, in this figure we show that the amplitude have an exponential decay given by the damped factor.

Figure 4a show plane phase in $\theta$ coordinate for the ideal case, in this figure the curves are closed, in the background are the level curves to lagrangian surface, and this comparison between numerical results and theory model is very well. Figure 4b show the system when the dissipative force is present, here the curves are open because the hypothesis isoenergetic is not right. But the comparison with the theory is adequate.

Conclusion

In this work we show the model for the problem of a uniform disc with a circular hole ascending in an inclined plane in order to make a correspondence between the modes of representations in education [1]. Both formulations, Newtonian and Lagrangian were presented. The expressions obtained were simulated using the regular Runge-Kutta numerical method of order 4 and were presented results considering the presence or not of the dissipative forces.
This results with the incorporation of dissipative force we think that are better to reproduce the experimental results.

References

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