HIGHER FANO MANIFOLDS AND RATIONAL SURFACES

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Abstract. Let $X$ be a Fano manifold of pseudo-index $\geq 3$ such that $c_1(X)^2 - 2c_2(X)$ is nef. Irreducibility of some spaces of rational curves on $X$ (in fact, a weaker hypothesis) implies a general point of $X$ is contained in a rational surface.

1. Introduction

One consequence of the bend-and-break lemma is uniruledness of Fano manifolds, [MM86]. In fact, in characteristic 0, Fano manifolds are rationally connected, [KMM92], [Cam92]. We prove an analogous theorem with rational curves replaced by rational surfaces for Fano manifolds satisfying positivity of the second graded piece of the Chern character.

Definition 1.1. A Fano manifold is 2-Fano if $\text{ch}_2(T_X)$ is nef, where $\text{ch}_2(T_X)$ is the second graded piece of the Chern character, $\frac{1}{2}(c_1(T_X)^2 - 2c_2(T_X))$. In other words, $\deg(\text{ch}_2(T_X)|_S)$ is nonnegative for every surface $S$ in $X$.

Let $M$ be a positive-dimensional, irreducible component of the Artin stack $\overline{M}^{0,0}(X)$ of genus 0 stable maps to $X$ whose general point of $M$ parametrizes a stable map with irreducible domain. Denote by $\overline{M}$ the coarse moduli space of $M$. Denote by $\Delta$ the locally principal closed substack of $\overline{M}^{0,0}(X)$ parametrizing stable maps with reducible domain. The closed substack $M \cap \Delta$ is a Cartier divisor. The question we consider is uniruledness of $M$.

Theorem 1.2. If $X$ is 2-Fano, every point of $M$ parametrizing a free curve and contained in a proper curve in $M - M \cap \Delta$ is contained in a rational curve in $M$.

If a general point of $M$ parametrizes a birational, free curve and is contained in a proper curve in $M - M \cap \Delta$, then a general point of $X$ is contained in a rational surface.

The proof uses the bend-and-break approach of [MM86]. Given a general curve $C$ in $M$, we need to bound the dimension of $\text{Hom}(C, M)$ from below. Although $M$ and $\overline{M}$ may be very singular, the deformation theory of stable maps nonetheless gives a useful lower bound. This uses Grothendieck-Riemann-Roch computations from [dJS05a]. Unfortunately, the formula has a negative term coming from intersection points of $C$ and $\Delta$. This is the reason for the hypothesis that $M - M \cap \Delta$ contains a proper curve. Luckily, there are nice sufficient conditions for $M - M \cap \Delta$ to contain many proper curves.

Proposition 1.3. If the pseudo-index of $X$ is $\geq 3$ and every irreducible component of $M \cap \Delta$ is an irreducible component of $\Delta$, then $M - M \cap \Delta$ is a union of proper curves.
This uses a contraction of the locally principal closed subspace $\Delta$ in $\overline{\mathcal{M}}_{0,0}(X)$ discovered in [CHS05] and independently by Adam Parker [Par05].

Section 3 gives some examples of 2-Fano manifolds and makes some observations about classification. Section 4 shows Theorem 1.2 is sharp in 2 ways. First, there are Fano manifolds that are not 2-Fano where the components $\mathcal{M}$ are not uniruled. Second, there are 2-Fano manifolds where the components $\mathcal{M}$ are uniruled but not rationally connected. Finally Section 5 speculates on sufficient conditions for the components $\mathcal{M}$ to be rationally connected.

2. Proof of the theorem

For every point $x$, denote by $\text{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ the open subscheme of $\text{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ parametrizing nonconstant morphisms.

Lemma 2.1. The dimension of every irreducible component of $\text{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ is at least as large as the pseudo-index of $X$.

Proof. This follows from [Ko96] Theorem II.1.2, Corollary II.1.6].

Proof of Proposition 1.3. Let $f : X \rightarrow \mathbb{P}^N$ be a plurianticanonical embedding. Denote by $\overline{\mathcal{M}}_{0,0}(f) : \overline{\mathcal{M}}_{0,0}(X) \rightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N)$ the associated embedding. Denote by $\phi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N) \rightarrow Y$ the contraction of the boundary constructed in [CHS05]. Denote by $N$ the image of $\mathcal{M}$ in $Y$.

Since the restriction of $\phi$ to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N) - \Delta$ is an open immersion, the restriction of $\phi \circ \overline{\mathcal{M}}_{0,0}(f)$ to $\mathcal{M} - \Delta$ is and immersion. Since $\mathcal{M}_{\text{free}}$ is dense in $\mathcal{M}$, $\mathcal{M}$ has pure dimension equal to the expected dimension, and $\mathcal{M} \cap \Delta$ is a Cartier divisor. Therefore $\text{dim}(N)$ equals $\text{dim}(\mathcal{M})$ and $\text{dim}(\mathcal{M} \cap \Delta)$ equals $\text{dim}(\mathcal{M}) - 1$.

If $i \leq j$, the restriction of $\phi$ to the boundary divisor $\Delta_{i,j}$ factors through the projection $\pi_j : \Delta_{i,j} \rightarrow \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, j)$. Denote $\Delta_i \cap \overline{\mathcal{M}}_{0,0}(X)$ by $\Delta_{X,i,j}$. Denote the restriction of $\pi_j$ by $\pi_{X,j} : \Delta_{X,i,j} \rightarrow \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, j)$. By Lemma 2.1, every irreducible component of every fiber of $\pi_{X,j}$ has dimension $\geq 1$, i.e., the difference of the pseudo-index and $\text{dim}(\text{Aut}(\mathbb{P}^1, 0))$. Therefore, for every irreducible component $\Delta'$ of $\Delta$, the dimension of $\phi(\overline{\mathcal{M}}_{0,0}(f)(\Delta'))$ is strictly less than the dimension of $\Delta'$. By hypothesis, every irreducible component $\Delta'$ of $\mathcal{M} \cap \Delta$ is an irreducible component of $\Delta$. Since $\text{dim}(\Delta')$ equals $\text{dim}(\mathcal{M}) - 1$, the image of $\Delta'$ in $N$ has dimension $\leq \text{dim}(N) - 2$.

Since every connected component of $Y$ is projective, also $N$ is projective. Because $\text{dim}(\text{Image}(\Delta')) \leq \text{dim}(N) - 2$, a general intersection of $N$ with $\text{dim}(N) - 1$ hyperplanes containing a point of $N - \text{Image}(\Delta')$ is a complete curve that does not intersect $\text{Image}(\Delta')$. Because there are only finitely many irreducible components of $\mathcal{M} \cap \Delta$, a general intersection of $N$ with $\text{dim}(N) - 1$ hyperplanes containing a point of $N - \text{Image}(\mathcal{M} \cap \Delta)$ is a complete curve that does not intersect $\text{Image}(\mathcal{M} \cap \Delta)$. The inverse image of this curve in $\mathcal{M} - \mathcal{M} \cap \Delta$ is a complete curve containing a given point of $\mathcal{M} - \mathcal{M} \cap \Delta$.

Let $C$ be a smooth, proper, connected curve and let $\zeta : C \rightarrow \overline{\mathcal{M}}_{0,0}(X) - \Delta$ be a nonconstant $1$-morphism whose general point parametrizes a free curve of $(-K_X)$-degree $e$. Let $B$ be a finite set of closed points of $C$. Denote by $(\pi : \Sigma \rightarrow C, F : \Sigma \rightarrow X)$ the associated family of stable maps.
Lemma 2.2. The dimension at $[\zeta]$ of $\text{Hom}(C, \mathcal{M}_{0,0}(X), \zeta|_B)$ is at least,

$$\deg(ch_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} \deg(c_1(T_X)^2|_{F(\Sigma)}) + (e + \dim(X) - 3)(1 - g(C) - \#(B)).$$

Proof. Consider the finite morphism $(\pi, g): \Sigma \to C \times X$. Denote by $N$ the cokernel of the map,

$$d(\pi, g): T_\Sigma \to \pi^*T_C \oplus g^*T_X.$$ 

By a natural generalization of [Kol96, Theorem II.5.8], the dimension of $\text{Hom}(C, \mathcal{M}_{0,0}(X), \zeta|_B)$ at $\zeta$ is at least,

$$h^0(\Sigma, N) - h^1(\Sigma, N).$$

By the Leray spectral sequence, $h^2(\Sigma, N)$ equals $h^1(C, R^1\pi_*N)$. Because a general point of $C$ parametrizes a free curve, the restriction of $N$ to a general fiber of $\pi$ is generated by global sections, thus has no higher cohomology. Thus $R^1\pi_*N$ is a torsion sheaf so that $h^1(C, R^1\pi_*N)$ is 0. Therefore, the lower bound actually equals $\chi(\Sigma, N)$.

Finally, by the Grothendieck-Riemann-Roch computations from [dJS05a], $\chi(\Sigma, N)$ equals,

$$\deg(ch_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} \deg(c_1(T_X)^2|_{F(\Sigma)}) + (e + \dim(X) - 3)(1 - g(C) - \#(B)).$$

Proof of Theorem 1.2. Every proper curve in $M - M \cap \Delta$ is the image of a non-constant 1-morphism $\zeta: C \to \mathcal{M} - \mathcal{M} \cap \Delta$ from a smooth curve $C$. The induced morphism $\text{Hom}(C, \mathcal{M} - \mathcal{M} \cap \Delta) \to \text{Hom}(C, M)$ is finite. By Lemma 2.2, $\dim(\text{Hom}(C, \mathcal{M}; \zeta|_B))$ behaves as if $M$ is smooth along the image of $\zeta$ and the anticanonical degree of $\zeta(C)$ equals

$$\deg(ch_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} \deg(c_1(T_X)^2|_{F(\Sigma)}).$$

Because $X$ is 2-Fano, this degree is positive. Therefore the usual bend-and-break argument applies, cf. [Kol96] Theorem II.5.8. \qed

3. Examples of 2-Fano Manifolds

All the results of this section, and more, are discussed and proved in the note [dJS05b]. There are two families of 2-Fano manifolds. The first family comes from complete intersections. Let $\mathbb{P}$ be a weighted projective space of dimension $n$. Let $X \subset \mathbb{P}$ be a smooth complete intersection of type $(d_1, \ldots, d_r)$. Then $X$ is Fano if and only if $d_1 + \cdots + d_r \leq n$. It is 2-Fano if and only if $d_1^2 + \cdots + d_r^2 \leq n$.

The second family comes from Grassmannians. Let $G$ be a Grassmannian $\text{Grass}(k, n)$ of $k$-dimensional subspaces of a fixed $n$-dimensional vector space. Without loss of generality, assume $n \geq 2k$. This is Fano. It is 2-Fano if and only if either $k = 1$, $n = 2k$ or $n = 2k + 1$.

There are two operations for producing new 2-Fano manifolds. First, if $X$ and $Y$ are each 2-Fano, then the product $X \times Y$ is 2-Fano. The second operation is more interesting. Let $X$ be a smooth Fano manifold and let $L$ be a nef invertible sheaf. The $\mathbb{P}^1$-bundle $\mathbb{P}(O_X \oplus L')$ is Fano if and only if $c_1(T_X') - c_1(L)$ is ample. Assuming it is Fano, it is 2-Fano if and only if $\deg(ch_2(T_X) + \frac{1}{2}c_1(L)^2)$ is nef. Notice, it is not necessary that $\deg(ch_2(T_X)$ is nef, i.e., $X$ need not be 2-Fano.
There are other operations on Fano manifolds. It is reasonable to ask which of these produce 2-Fano manifolds. For instance, a projective bundle $\mathbb{P}(E)$ of fiber dimension $\geq 2$ over a Fano manifold is also Fano if $E$ satisfies a weak version of stability. However, if $P(E)$ is 2-Fano then the pullback of $E$ to every curve is a semistable bundle. If $X$ is $\mathbb{P}^n$, for instance, this implies $P(E)$ is simply $\mathbb{P}^m \times \mathbb{P}^n$. This, and other examples, suggest the following principle: an operation on Fano manifolds produces a 2-Fano manifold only if some vector bundle associated to the operation is semistable.

4. The theorem is sharp

The theorem is sharp in 2 ways. First, let $X$ be a general cubic hypersurface in $\mathbb{P}^5$. This is Fano, but it is not 2-Fano. By the main theorem of [dJS04], there are infinitely many non-uniruled irreducible components $\mathcal{M}$ of $\overline{M}_{0,0}(X)$ satisfying the hypotheses of Theorem 1.2. Second, let $Y$ be the $\mathbb{P}^1$-bundle over $X$, $Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_{\mathbb{P}^5}(-2)|_X)$. By the construction in the last section, $Y$ is 2-Fano. Associated to the projection $\pi : Y \to X$, there is a 1-morphism $\overline{\mathcal{M}}_{0,0}(\pi) : \overline{\mathcal{M}}_{0,0}(Y) \to \overline{\mathcal{M}}_{0,0}(X)$. For an irreducible component $\mathcal{N}$ of $\overline{\mathcal{M}}_{0,0}(Y)$ containing a free curve, it is easy to prove the boundary $\mathcal{N} \cap \Delta$ is contracted. (However it is not true that every component of $\mathcal{N} \cap \Delta$ is a component of $\Delta$.) Thus Theorem 1.2 implies $N$ is uniruled. In fact, the restriction of $\overline{\mathcal{M}}_{0,0}(\pi)$ to $\mathcal{N}$ is birational to a projective bundle over the image component $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(X)$. Choosing $N$ appropriately, $\mathcal{M}$ is one of the infinitely many non-uniruled irreducible components of $\overline{\mathcal{M}}_{0,0}(X)$. Therefore $N$ is not rationally connected, and the MRC quotient of $N$ is precisely $\mathcal{M}$.

5. Speculation

For the counterexample $Y$ in the previous section, $\text{ch}_2(T_Y)$ is nef. But it is not “positive”. It has intersection number 0 with the surface $\pi^{-1}(B)$ for every curve $B$ in $X$. If $X$ is a Fano manifold such that $\text{ch}_2(T_X)$ has positive intersection number with every surface, is $\mathcal{M}$ rationally connected? We know no counterexample.

References

[Cam92] F. Campana. Connexité rationnelle des variétés de Fano. Ann. Sci. École Norm. Sup. (4), 25(5):539–545, 1992.

[CHS05] Izzet Coskun, Joe Harris, and Jason Starr. The ample cone of the Kontsevich moduli space. preprint submitted Trans. Amer. Math. Soc., 2005.

[dJS04] A. J. de Jong and Jason Starr. Cubic fourfolds and spaces of rational curves. Illinois J. Math., 48(2):415–450, 2004.

[dJS05a] A. J. de Jong and J. Starr. Divisor classes and the virtual canonical bundle. preprint, 2005.

[dJS05b] A. J. de Jong and J. Starr. A note on Fano manifolds whose second Chern character is positive. in preparation, 2005.

[KJM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. J. Differential Geom., 36(3):765–779, 1992.

[Kol96] János Kollár. Rational curves on algebraic varieties, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996.

[MM86] Yoichi Miyaoka and Shigefumi Mori. A numerical criterion for uniruledness. Ann. of Math. (2), 124(1):65–69, 1986.
[Par05] Adam Parker. *An elementary GIT construction of the moduli space of stable maps.*
PhD thesis, University of Texas at Austin, 2005.