ON THE $n$-TH DERIVATIVE AND THE FRACTIONAL INTEGRATION OF BESSEL FUNCTIONS WITH RESPECT TO THE ORDER

J. L. GONZÁLEZ-SANTANDER

(Communicated by )

Abstract. We obtain integral representations of the $n$-th derivatives of the Bessel functions with respect to the order. The numerical evaluation of these expressions is very efficient using a double exponential integration strategy. Also, from the integral representation corresponding to the Macdonald function, we have calculated a new integral. Finally, we calculate integral expressions for the fractional derivatives of the Bessel functions with respect to the order. Simple proofs for some particular cases given in the literature are provided as well.

1. INTRODUCTION

Bessel functions are the canonical solutions $y(t)$ of Bessel’s differential equation:

\begin{equation}
    t^2 y'' + t y' + (t^2 - \nu^2) y = 0,
\end{equation}

where $\nu$ denotes the order of the Bessel function. This equation arises when finding separable solutions of Laplace equation in cylindrical coordinates, and Helmholtz equation in spherical coordinates [8, Chap. 6]. The general solution of (1.1) is a linear combination of the Bessel functions of the first and second kind, i.e. $J_\nu(t)$ and $Y_\nu(t)$ respectively. These functions are usually defined as [9, Eqns. 10.2.2&3]

\begin{equation}
    J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k}}{k! \Gamma(\nu + k + 1)},
\end{equation}

and

\begin{equation}
    Y_\nu(t) = \frac{J_\nu(t) \cos \pi \nu - J_{-\nu}(t)}{\sin \pi \nu}, \quad \nu \notin \mathbb{Z}.
\end{equation}

In the case of pure imaginary argument, the solutions to the Bessel equations are called modified Bessel functions of the first and second kind, $I_\nu(t)$ and $K_\nu(t)$ respectively, where [9, Eqns. 10.25.2&27.4]

\begin{equation}
    I_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(t/2)^{2k}}{k! \Gamma(\nu + k + 1)},
\end{equation}

and

\begin{equation}
    K_\nu(t) = \frac{\pi}{2} \frac{I_{-\nu}(t) - I_\nu(t)}{\sin \pi \nu}, \quad \nu \notin \mathbb{Z}.
\end{equation}
Despite the fact, the properties of the Bessel functions has been studied extensively in the literature [12, 4], studies about successive derivatives and repeated integrals of the Bessel functions with respect to the order $\nu$ are relatively scarce. In the literature, we find the following series representations [9, Eqs. 10.15.1&38.1] for the derivative with respect to the order $\nu \notin \mathbb{Z}$:

$$\frac{\partial J_\nu(t)}{\partial \nu} = J_\nu(t) \log \left(\frac{t}{2}\right) - \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu+k+1)(-1)^k(t/2)^{2k}}{k!\Gamma(\nu+k+1)}, \tag{1.6}$$

and

$$\frac{\partial I_\nu(t)}{\partial \nu} = I_\nu(t) \log \left(\frac{t}{2}\right) - \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu+k+1)(t/2)^{2k}}{k!\Gamma(\nu+k+1)}, \tag{1.7}$$

which are obtained directly from (1.2) and (1.4). For the $n$-th derivative of the modified Bessel functions $I_\nu$ and $J_\nu$, we find in [10] a more complex expression in series form.

Regarding integral representations of the derivative of $I_\nu(t)$ and $J_\nu(t)$ with respect to the order, we find in [2] $\forall \Re \nu > 0$,

$$\frac{\partial J_\nu(t)}{\partial \nu} = \pi \nu \int_0^{\pi/2} \tan \theta Y_0(t \sin^2 \theta) J_\nu(t \cos^2 \theta) d\theta, \tag{1.8}$$

and

$$\frac{\partial I_\nu(t)}{\partial \nu} = -2\nu \int_0^{\pi/2} \tan \theta K_0(t \sin^2 \theta) I_\nu(t \cos^2 \theta) d\theta. \tag{1.9}$$

Other integral representations of the order derivative of $J_\nu(z)$ and $Y_\nu(z)$ are given in [5] for $\nu > 0$ and $t \neq 0$, $|\arg t| \leq \pi$, which read as,

$$\frac{\partial J_\nu(t)}{\partial \nu} = \pi \nu \left[ Y_\nu(t) \int_0^t \frac{J_\nu^2(z)}{t} dz + J_\nu(t) \int_t^\infty \frac{J_\nu(z) Y_\nu(z)}{z} dz \right], \tag{1.10}$$

and

$$\frac{\partial Y_\nu(t)}{\partial \nu} = \pi \nu \left[ J_\nu(t) \left( \int_t^\infty \frac{Y_\nu^2(z)}{z} dz - \frac{1}{2\nu} \right) - Y_\nu(t) \int_t^\infty \frac{J_\nu(z) Y_\nu(z)}{z} dz \right]. \tag{1.11}$$

Recently, in [6], we find the following integral representations of the derivatives of the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$ with respect to the order for $\nu > 0$ and $t \neq 0, |\arg t| \leq \pi$,

$$\frac{\partial I_\nu(t)}{\partial \nu} = -2\nu \left[ I_\nu(t) \int_t^\infty \frac{K_\nu(z) I_\nu(z)}{z} dz + K_\nu(t) \int_0^t \frac{I_\nu^2(z)}{z} dz \right], \tag{1.12}$$

and

$$\frac{\partial K_\nu(t)}{\partial \nu} = 2\nu \left[ K_\nu(t) \int_t^\infty \frac{I_\nu(z) K_\nu(z)}{z} dz - I_\nu(t) \int_t^\infty \frac{K_\nu^2(z)}{z} dz \right]. \tag{1.13}$$

The great advantage of the integral expressions (1.10)-(1.13) is that the integrals involved in them can be calculated in closed-form [6]. Also, expressions in closed-form for the second and third derivatives with respect to the order are found in [4], but these expressions are extraordinarily complex, above all for the third derivative.
Regarding integration of Bessel functions with respect to the order, the results found in the literature are even more scarce. For instance, in [2], we find

\[
\int_{\nu}^{\infty} J_\mu(t) \, d\mu = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\pi} \sin(t \sin x - \nu x) \frac{dx}{x} \\
- \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t \sinh x - \nu x}}{\pi^2 + x^2} (x \sin \pi \nu + \cos \pi \nu) \, dx,
\]

and

\[
\int_{\nu}^{\infty} I_\mu(t) \, d\mu = \frac{e^{t}}{2} - \frac{1}{\pi} \int_{0}^{\pi} e^{t \cos x} \sin \nu x \frac{dx}{x} \\
- \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t \cosh x - \nu x}}{\pi^2 + x^2} (x \sin \pi \nu + \cos \pi \nu) \, dx,
\]

which are calculated using the complex contour integration of a particular inverse Laplace transform [3, Sect. 88]. In the Appendix, we provide simples proofs of (1.14) and (1.15) by direct integration. Notice that a second integration with respect to the order in (1.14) or (1.15), taking again as integration interval \((\nu, \infty)\), would be divergent. However, repeated integration of Bessel functions with respect to the order is possible if we take a finite integration interval. It is worth noting that the latter seems to be absent in the most common literature.

Therefore, the goal of this article is two-folded. On the one hand, in Section 2, we obtain simple integral representations for the \(n\)-th derivatives of the Bessel functions with respect to the order. The great advantage of these expressions is that its numerical evaluation is quite rapid and straightforward. As a by-product, we obtain the calculation of an integral which does not seem to be reported in the literature. Also, the values of the integral representations obtained at argument \(t = 0\) are calculated.

On the other hand, in Section 3, we calculate the iterated integrals of Bessel functions, using fractional integration. Also, values at argument \(t = 0\) are calculated. It is worth noting that the latter is not trivial from the integral representations obtained.

Finally, we collect our conclusions in Section 4.

2. INTEGRAL REPRESENTATIONS OF \(n\)-TH ORDER DERIVATIVES

In order to perform the \(n\)-th derivatives of Bessel and modified Bessel functions with respect to the order, first we state the following \(n\)-th derivatives, that can be proved easily by induction.

**Lemma 2.1.** The \(n\)-th derivative of the functions

\[
\begin{align*}
f_1(\nu) &= \cos(t \sin x - \nu x), \\
f_2(\nu) &= \sin(t \sin x - \nu x), \\
f_3(\nu) &= e^{-\nu x} \sin \pi \nu = \Im \left( e^{(i\pi-x)\nu} \right), \\
f_4(\nu) &= e^{-\nu x} \cos \pi \nu = \Re \left( e^{(i\pi-x)\nu} \right),
\end{align*}
\]
with respect to the order $\nu$ are given by

\begin{align}
(2.1) & \quad f_1^{(n)}(\nu) = x^n \cos \left(t \sin x - \nu x - n\pi/2\right), \\
(2.2) & \quad f_2^{(n)}(\nu) = x^n \sin \left(t \sin x - \nu x - n\pi/2\right), \\
(2.3) & \quad f_3^{(n)}(\nu) = e^{-\nu x} \Im \left[(i\pi - x)^n e^{i\nu\pi}\right], \\
(2.4) & \quad f_4^{(n)}(\nu) = e^{-\nu x} \Re \left[(i\pi - x)^n e^{i\nu\pi}\right].
\end{align}

To obtain the $n$-th derivative of the Bessel function of the first kind with respect to the order, we depart from the Schläfli integral representation of $J_\nu(t)$ \cite{9} Eqn. 10.9.6, wherein we have $\forall \Re t > 0$,

\begin{equation}
J_\nu(t) = \frac{1}{\pi} \int_0^\pi \cos \left(t \sin x - \nu x\right) dx - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-t \sinh x - \nu x} dx.
\end{equation}

Therefore, applying (2.1) and (2.4), the $n$-th derivative of $J_\nu(t)$ with respect to the order is

\begin{equation}
\frac{\partial^n}{\partial \nu^n} J_\nu(t) = \frac{1}{\pi} \int_0^\pi x^n \cos \left(t \sin x - \nu x - \pi/2 n\right) dx - \frac{1}{\pi} \int_0^\infty e^{-t \sinh x - \nu x} \Im \left[(i\pi - x)^n e^{i\nu\pi}\right] dx.
\end{equation}

Similar calculations can be carried out for the Bessel function of the second kind, whose integral representation is \cite{4} Eqn. 10.9.7, $\forall \Re t > 0$,

\begin{equation}
Y_\nu(t) = \frac{1}{\pi} \int_0^\pi \sin \left(t \sin x - \nu x\right) dx - \frac{1}{\pi} \int_0^\infty e^{-t \sinh x} \left(e^{\nu x} + e^{-\nu x} \cos \nu \pi\right) dx.
\end{equation}

Therefore, according to (2.2) and (2.4), we obtain

\begin{equation}
\frac{\partial^n}{\partial \nu^n} Y_\nu(t) = \frac{1}{\pi} \int_0^\pi x^n \sin \left(t \sin x - \nu x - \pi/2 n\right) dx - \frac{1}{\pi} \int_0^\infty e^{-t \sinh x} \left(x^n e^{\nu x} - e^{-\nu x} \Re \left[(i\pi - x)^n e^{i\nu\pi}\right]\right) dx.
\end{equation}

For the modified Bessel function, we find in the literature the following integral representation \cite{11} Eqn. 10.32.4], $\forall \Re t > 0$,

\begin{equation}
I_\nu(t) = \frac{1}{\pi} \int_0^\pi e^{t \cos x} \cos \nu x dx - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-t \cosh x - \nu x} dx.
\end{equation}

Therefore, according to (2.1) with $t = 0$ and (2.3), we have

\begin{equation}
\frac{\partial^n}{\partial \nu^n} I_\nu(t) = \frac{1}{\pi} \int_0^\pi x^n e^{t \cos x} \cos \left(\nu x + \pi/2 n\right) dx - \frac{1}{\pi} \int_0^\infty e^{-t \cosh x - \nu x} \Im \left[(i\pi - x)^n e^{i\nu\pi}\right] dx.
\end{equation}
Also, from the integral representation of the Macdonald function [8 Eqn. 5.10.23], \( \forall \Re t > 0 \),

\[
(2.11) \quad K_\nu(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\nu x - t \cosh x} \, dx,
\]

we have

\[
(2.12) \quad \frac{\partial^n}{\partial \nu^n} K_\nu(t) = \frac{1}{2} \int_{-\infty}^{\infty} x^n e^{\nu x - t \cosh x} \, dx.
\]

For \( n = 1 \), the above integral (2.12) is calculated in [6] in closed-form, thus \( \forall \nu > 0, \Re t > 0 \), we have

\[
(2.13) \quad \int_{-\infty}^{\infty} x e^{\nu x - t \cosh x} \, dx = \nu \left[ \frac{K_\nu(z)}{\sqrt{\pi}} G_{2,4}^{5,1} \left( \frac{z^2}{2}; 1/2,1 \atop 0,0,\nu,-\nu \right) - \sqrt{\pi} I_\nu(z) G_{2,4}^{4,0} \left( \frac{z^2}{2}; 1/2,1 \atop 0,0,\nu,-\nu \right) \right],
\]

where \( G_{p,q}^{m,n} \) denotes the Meijer-G function [9 Eqn. 16.17.1]. If \( \nu \notin \mathbb{Z} \), the above expression is reduced in terms of generalized hypergeometric functions \( \, _pF_q \) [9 Eqn. 16.2.1] as [4],

\[
(2.14) \quad \int_{-\infty}^{\infty} x e^{\nu x - t \cosh x} \, dx = \pi \csc \pi \nu \left\{ \pi \cot \pi \nu I_\nu(z) - \left[ I_\nu(z) + I_{-\nu}(z) \right] \right\}
\]

\[
\left[ \frac{z^2}{4(1-\nu^2)} \right] _3F_4 \left( \frac{1}{2}, \frac{3}{2}, 2 \atop 2,2,2-\nu,2+\nu \right| z^2 \right) + \log \left( \frac{z}{2} \right) - \psi(\nu) - \frac{1}{2\nu} \right\}
\]

\[
+ \frac{1}{2} \left\{ I_{-\nu}(z) \Gamma^2(-\nu) \left( \frac{z}{2} \right)^{2\nu} _2F_3 \left( \frac{\nu}{2}, \nu, \frac{1}{2} \atop 1+\nu,1+\nu,1+2\nu \right| z^2 \right) - I_\nu(z) \Gamma^2(\nu) \left( \frac{z}{2} \right)^{-2\nu} _2F_3 \left( -\nu, \frac{1}{2} - \nu \atop 1-\nu,1-\nu,1-2\nu \right| z^2 \right) \right\}.
\]

It is worth noting that the numerical evaluation of the integral representations for the \( n \)-th derivatives of the Bessel functions, i.e. (2.6), (2.8), (2.10), and (2.12), is very efficient if we use a double exponential strategy [11]. This is especially significant when \( n \geq 3 \). In order to have an idea of how quick is the performance of the formulas presented here, define \( t_{\text{int}} \) as the timing of the numerical evaluation of integral representations (2.6), (2.8), (2.10), and (2.12), and \( t_{\text{num}} \) as the timing of the numerical method provided by MATHEMATICA to compute derivatives, thus the timing ratio of both methods is \( \chi = t_{\text{num}}/t_{\text{int}} \). For instance, Fig. 4 shows the plot of \( \partial^3/\partial \nu^3 I_\nu(t) \) in the domain \( (\nu,t) \in (0,10) \times (0,10) \), and the timing ratio in this case is \( \chi = t_{\text{num}}/t_{\text{int}} \approx 35 \). Similar ratios are obtained for the other \( n \)-th derivatives integral expressions, and the higher is \( n \), the higher is the timing ratio \( \chi \).

2.1. Values at \( t = 0 \). On the one hand, the derivatives of the irregular solutions of the Bessel’s equation are infinite \( \forall t = 0 \). Indeed, from (2.8), we have \( \forall \nu \geq 0, n = 0,1,\ldots \)

\[
\lim_{t \to 0^+} \frac{\partial^n}{\partial \nu^n} Y_\nu(t) = -\infty,
\]
and from (2.12),
\[
\lim_{t \to 0^+ \atop \nu} \frac{\partial^n}{\partial \nu^n} K_{\nu}(t) = \infty.
\]

On the other hand, for the regular solutions, the derivatives with respect to order \( \forall t = 0 \) are null. Indeed, according to the series representations (1.2) and (1.4), we have
\[
(2.15) \quad J_{\nu}(0) = I_{\nu}(0) = \begin{cases} 
1, & \nu = 0 \\
0, & \nu > 0 
\end{cases}
\]

Therefore, \( \forall \nu \geq 0, n = 1, 2, \ldots \)
\[
(2.16) \quad \frac{\partial^n}{\partial \nu^n} J_{\nu}(0) = \frac{\partial^n}{\partial \nu^n} I_{\nu}(0) = 0.
\]

However, (2.16) is not obvious from the integral representations (2.5) and (2.9).

Next, we provide a simple proof of (2.16) from this point of view.

\textbf{Proof.} Rewrite (2.6) and (2.10) as
\[
\frac{\partial^n}{\partial \nu^n} J_{\nu}(t) = \frac{1}{\pi} \Re \int_0^\pi e^{it \sin x} (ix)^n e^{i \nu x} dx \\
- \frac{1}{\pi} \Im \int_0^\infty e^{-t \sinh x} (i\pi - x)^n e^{(i\pi - x)\nu} dx,
\]
and
\[
\frac{\partial^n}{\partial \nu^n} I_{\nu}(t) = \frac{1}{\pi} \Re \int_0^\pi e^{t \cos x} (ix)^n e^{i \nu x} dx \\
- \frac{1}{\pi} \Im \int_0^\infty e^{-t \cosh x} (i\pi - x)^n e^{(i\pi - x)\nu} dx.
\]
Therefore,
\[
\frac{\partial^n}{\partial \nu^n} J_\nu (0) = \frac{\partial^n}{\partial \nu^n} I_\nu (0) = \frac{1}{\pi} \Re \int_0^\pi (ix)^n e^{ix} dx
\]
\[= \frac{1}{\pi} \Re \int_0^\infty (i\pi - x)^n e^{i\pi(x-x)} dx. \quad (2.17)\]

Note that, performing the substitution \( z = i\pi - x \) in (2.17) and splitting the integration path in real and pure imaginary parts, we have \( \forall \nu > 0 \)
\[
\Re \int_0^\infty (i\pi - x)^n e^{i\pi(x-x)} dx = \Re \int_{i\pi}^{-\infty} z^n e^{z\nu} dz
\]
\[= \Re \left[ \int_0^\pi z^n e^{z\nu} dz + \int_0^i \pi z^n e^{z\nu} dz \right] \]
\[= \Re \int_0^\pi z^n e^{z\nu} dz. \]

Now, perform the substitution \( z = ix \) and apply the property \( \Re (iz) = \Re (z) \) to arrive at
\[
\Re \int_0^\infty (i\pi - x)^n e^{i\pi(x-x)} dx = \Re \int_0^\pi (ix)^n e^{ix} dx. \quad (2.18)
\]
Inserting (2.18) in (2.17), the proof is completed. \( \square \)

3. Repeated integration with respect to the order

Here, we will adopt the following notation for the repeated integration,
\[
D_{x-x_0}^{-\alpha} f (x) = \int_x^{x_0} \int_{x_0}^{t_{n-1}} \cdots \int_{x_0}^{t_1} \int_{x_0}^{t_0} f (t_0) dt_0 \cdots dt_{n-1}.
\]

According to "Cauchy's iterated integral" \[7\], we have
\[
D_{x-x_0}^{-\alpha} f (x) = \frac{1}{(n-1)!} \int_x^{x_0} (x-t)^{n-1} f (t) dt.
\]

We can generalize the above result replacing \( n \) by \( \alpha \),
\[
D_{x-x_0}^{-\alpha} f (x) = \frac{1}{\Gamma (\alpha)} \int_x^{x_0} (x-t)^{\alpha-1} f (t) dt, \quad \Re \alpha > 0. \quad (3.1)
\]

When \( x_0 = 0 \), (3.1) is called the Riemann-Liouville integral. Thereby, \( D_{x-x_0}^{-\alpha} \) denotes the fractional integration of order \( \alpha \). In order to perform the fractional integration of the Bessel functions with respect to the order, we need the following result.

**Lemma 3.1.** The fractional integration of the exponential function is given by
\[
D_{x-x_0}^{-\alpha} e^{sx} = \frac{e^{sx}}{s^\alpha} P (\alpha, s (x-x_0)), \quad (3.2)
\]
where
\[
P (\alpha, z) = \frac{1}{\Gamma (\alpha)} \int_0^z t^{\alpha-1} e^{-t} dt, \quad \Re \alpha > 0, \quad (3.3)
\]
denotes the regularized lower incomplete gamma function [9, Eqn. 8.2.4].

Proof. Taking in (3.1) \( f(x) = e^{sx} \), and performing the change of variables \( v = s(x-t) \), we arrive at

\[
D^{-\alpha}_{x-x_0} e^{sx} = \frac{e^{sx}}{\Gamma(\alpha) s^\alpha} \int_{0}^{s(x-x_0)} v^{\alpha-1} e^{-v} dv,
\]

which, according to (3.3), is the desired result given in (3.2).

According to the integral representation (2.5), and taking into account (3.2), the fractional integration of the Bessel function of the first kind with respect to the order is,

\[
\forall \Re t > 0, \quad D^{-\alpha\nu}_{\nu-\nu_0} J_{\nu}(t) = \frac{1}{\pi} \left\{ \int_{0}^{\infty} e^{-\alpha \Re x} x^{-\alpha} e^{i(t \sin x - \nu x + \pi \alpha/2)} P(\alpha, -ix(\nu - \nu_0)) \right\} \ dx
\]

Similarly, from (2.9) and taking into account (3.2), the fractional integration of the modified Bessel function is,

\[
\forall \Re t > 0, \quad D^{-\alpha\nu}_{\nu-\nu_0} I_{\nu}(t) = \frac{1}{\pi} \left\{ \int_{0}^{\infty} e^{-\alpha \Im x} x^{-\alpha} e^{i(t \cos x - \nu x)} P(\alpha, ix(\nu - \nu_0)) \right\} \ dx
\]

Also, from (2.7), \( \forall t > 0, \)

\[
\forall \Re t > 0, \quad D^{-\alpha\nu}_{\nu-\nu_0} Y_{\nu}(t) = \frac{1}{\pi} \left\{ \int_{0}^{\infty} e^{-\alpha \Im x} x^{-\alpha} e^{i(t \sin x - \nu x + \pi \alpha/2)} P(\alpha, -ix(\nu - \nu_0)) \right\} \ dx
\]

\[
\forall \Re t > 0, \quad D^{-\alpha\nu}_{\nu-\nu_0} K_{\nu}(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\alpha \Re x} x^{-\alpha} e^{-t \cosh x} P(\alpha, x(\nu - \nu_0)) \right\} \ dx,
\]

wherein we have taken the real part of the integral because, according to (3.4), \( D^{-\alpha\nu}_{\nu-\nu_0} K_{\nu}(t) \) is real \( \forall t > 0 \), thus the imaginary part of (3.7) must vanish.

It is worth noting that the finite integrals given in (3.4)-(3.6) are efficiently evaluated using the numerical double exponential strategy [11].
3.1. Values at $t = 0$. Note that from (2.7), we have that $\lim_{t \to 0^+} Y_\nu(t) = -\infty$, thus, according to (3.1), we have

$$\lim_{t \to 0^+} D_{\nu-\nu_0}^{-\alpha} Y_\nu(t) = -\infty.$$ 

Also, from (2.11), we have that $\lim_{t \to 0^+} K_{\nu}(t) = \infty$, thus

$$\lim_{t \to 0^+} D_{\nu-\nu_0}^{-\alpha} K_{\nu}(t) = \infty.$$ 

Also, according to (2.15) and (3.1), we have

$$D_{\nu-\nu_0}^{-\alpha} J_\nu(0) = D_{\nu-\nu_0}^{-\alpha} I_\nu(0) = 0.$$ 

However, (3.8) is not obvious from the integral representations given in (3.4) and (3.5). Next, we provide a simple proof of the latter.

**Proof.** Substitute $t = 0$ in (3.4) and (3.5), and rewrite the result as follows,

$$D_{\nu-\nu_0}^{-\alpha} J_\nu(0) = D_{\nu-\nu_0}^{-\alpha} I_\nu(0) = 0.$$ 

Notice that the first integral on the RHS of (3.9) is independent of the ‘±’ sign since $\forall a \in \mathbb{R}, P(a, z) = P(a, \bar{z})$ and $\Re(z) = \Re(\bar{z})$. Perform the change of variables $z = i\pi - x$ in the second integral on the RHS of (3.9) to obtain

$$\Re \int_0^\pi e^{ix\nu} P(\alpha, \pm i\pi (\nu - \nu_0)) dx.$$ 

Taking into account that $\forall a \in \mathbb{R}, P(a, z) = P(a, \bar{z})$, and the property $\Im(i\bar{z}) = \Re(z) = \Re(\bar{z})$, we finally get

$$\Im \int_0^\pi e^{ix\nu} P(\alpha, \pm i\pi (\nu - \nu_0)) dx.$$ 

Insert (3.11) in (3.9) to complete the proof. \qed
4. Conclusions

On the one hand, we have obtained integral expressions for the $n$-th derivatives of the Bessel functions with respect to the order in (2.6), (2.8), (2.10), and (2.12). Numerically, these integral expressions are much more efficient than the numerical evaluation of the corresponding derivatives for $n \geq 3$ using a double exponential integration strategy. As a by-product, we have calculated the integral given in (2.13), which does not seem to be reported in the literature. Also, since the null value of $\partial^n/\partial \nu^n J_\nu(0)$ and $\partial^n/\partial \nu^n I_\nu(0)$ is not trivial from the integral representations obtained, a simple proof is included.

On the other hand, we have calculated the fractional integration of the Bessel functions with respect to the order in (3.4)-(3.7). Also, we have included a simple proof to calculate the value of $D^{-\alpha}_\nu t - \nu J_\nu(0)$ and $D^{-\alpha}_\nu t - \nu I_\nu(0)$ from the corresponding integral representations obtained.

Finally, in the Appendix, we provide simple proofs of the infinite integrals of $J_\nu(t)$ and $I_\nu(t)$ with respect to the order, i.e. (1.14) and (1.15). In the literature, these integrals are derived calculating the complex contour integral of a particular inverse Laplace transform, but here we have derived them by direct integration.

**Appendix A. Derivation of (1.14)**

Integrating with respect to the order in (2.5) and exchanging the order of integration, we have

\[
\int_{\nu}^{\infty} J_\mu(t) \, d\mu = \frac{1}{\pi} \int_{0}^{\pi} \cos(t \sin x - \mu x) \, d\mu \, dx
\]

\[
-\frac{1}{\pi} \int_{0}^{\infty} e^{-\sinh x} \int_{\nu}^{\infty} e^{-\mu x} \sin \mu \pi \, d\mu \, dx.
\]

Notice that

\[
\int_{\nu}^{\infty} \cos(z \sin x - \mu x) \, d\mu = \lim_{b \to \infty} \Re \left[ e^{it \sin x} \int_{\nu}^{b} e^{-i\mu x} \, d\mu \right]
\]

\[
= \frac{1}{x} \lim_{b \to \infty} \Re \left[ i e^{i(t \sin x - \mu x)} \right]_{\mu=\nu}.
\]

Since $\Re(iz) = -\Im(z)$ and $\lim_{b \to \infty} \sin(t \sin x - bx) = -\lim_{b \to \infty} \sin bx$, we have

\[
\int_{\nu}^{\infty} \cos(z \sin x - \mu x) \, d\mu = \frac{-1}{x} \lim_{b \to \infty} \Im \left[ e^{i(t \sin x - \nu x)} \right]_{\mu=\nu}.
\]

Also, considering that $x > 0$,

\[
\int_{\nu}^{\infty} e^{-\mu x} \sin \mu \pi \, d\mu = \Im \int_{\nu}^{\infty} e^{i(\pi - x)\mu} \, d\mu
\]

\[
= -\Im \left[ e^{i(\pi - x)\mu} \right] \mu = \frac{e^{-\nu x}}{\pi^2 + x^2} (\pi \cos \pi \nu + x \sin \pi \nu).
\]
Substituting in (A.1) the results (A.2) and (A.3), we obtain

\[ \int_{\nu}^{\infty} J_{\mu} (z) \, d\mu = \frac{1}{\pi} \lim_{b \to \infty} \int_{0}^{\pi} \frac{\sin bx}{x} \, dx + \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin (t \sin x - \nu x)}{x} \, dx \]

\[ - \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t \sinh z - \nu x}}{\pi^2 + x^2} (\pi \cos \pi \nu + x \sin \pi \nu) \, dx. \]

Considering the definition of the sine integral [9, Eqn. 6.2.9], we have

\[ \frac{1}{\pi} \lim_{b \to \infty} \int_{0}^{\pi} \frac{\sin bx}{x} \, dx = \lim_{b \to \infty} \frac{\text{Si} (b \pi)}{\pi} = \frac{1}{2}, \]

where we have applied [9, Eqn. 6.2.14],

\[ \lim_{x \to \infty} \text{Si} (x) = \frac{\pi}{2}. \]

Therefore, substituting (A.5) in (A.4), we obtain the integral representation given in (1.14).

**APPENDIX B. DERIVATION OF (1.15)**

Integrating with respect to the order in (2.9) and exchanging the order of integration, we have

\[ \int_{\nu}^{\infty} I_{\mu} (t) \, d\mu = \frac{1}{\pi} \int_{0}^{\pi} e^{t \cos x} \int_{\nu}^{\infty} \cos \mu x \, d\mu \, dx \]

\[ - \frac{1}{\pi} \int_{0}^{\infty} e^{-t \cosh x} \int_{\nu}^{\infty} \sin \pi \mu e^{-\mu x} \, d\mu \, dx. \]

Note that

\[ \int_{\nu}^{\infty} \cos \mu x \, d\mu = \lim_{b \to \infty} \frac{\sin bx}{x} - \frac{\sin \nu x}{x}. \]

Therefore, substituting in (B.1) the results (A.5) and (B.2), we obtain

\[ \int_{\nu}^{\infty} I_{\mu} (t) \, d\mu = \frac{1}{\pi} \lim_{b \to \infty} \int_{0}^{\pi} e^{t \cos x} \frac{\sin bx}{x} \, dx - \frac{1}{\pi} \int_{0}^{\pi} e^{t \cos x} \frac{\sin \nu x}{x} \, dx \]

\[ - \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t \cosh x - \nu x}}{\pi^2 + x^2} (\pi \cos \pi \nu + x \sin \pi \nu) \, dx. \]

We can calculate the above limit performing the substitution \( u = bx \),

\[ \frac{1}{\pi} \lim_{b \to \infty} \int_{0}^{\pi} e^{t \cos x} \frac{\sin bx}{x} \, dx = \frac{1}{\pi} \lim_{b \to \infty} \int_{0}^{b \pi} e^{t \cos (u/b)} \frac{\sin u}{u} \, du \]

\[ = \frac{e^{t}}{\pi} \lim_{b \to \infty} \int_{0}^{b \pi} \frac{\sin u}{u} \, du \]

\[ = \frac{e^{t}}{\pi} \lim_{b \to \infty} \text{Si} (b \pi) = \frac{e^{t}}{2}, \]

where we have used (A.6). Therefore, inserting in (B.3) the result (B.4), we arrive at (1.15).
References

1. G.E. Andrews, R. Askey, R. Roy, *Special functions: Encyclopedia of Mathematics and its Applications*, vol. 71, Cambridge University Press, NY, 2004.
2. A. Apelblat, N. Kravitsky, *Integral representations of derivatives and integrals with respect to the order of the Bessel functions* Jν(t), Iν(t), the Anger function Jν(t), and the integral Bessel function Jν(t). IMA J. Appl. Math. 34 (1985) 187–210.
3. J.W. Brown, R.V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, NY, 2009.
4. Y.A. Brychkov, *Higher derivatives of the Bessel functions with respect to the order*. Integr. Transf. Spec. Funct. 27 (2016) 566–577.
5. T.M. Dunster, *On the order derivatives of Bessel functions*. Constr. Approx. 46 (2017) 47–68.
6. J.L. González-Santander, *Closed-form expressions for derivatives of Bessel functions with respect to the order*. J. Math. Anal. Appl. 466 (2018) 1060–1081.
7. J.L. Lavoie, T.J. Osler, R. Tremblay, *Fractional Derivatives and Special Functions*. SIAM Rev. 18(2) (1976) 240–268.
8. N.N. Lebedev, *Special Functions and their Applications*, Prentice-Hall Inc, NJ, 1965.
9. F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, NY, 2010.
10. J. Sesma, *Derivatives with respect to the order of the Bessel function of the first kind*. (2014) arXiv:1401.4850
11. H. Takahasi, M. Mori, *Double exponential formulas for numerical integration*. Publ. Res. I. Math. Sci. 9(3) (1974) 721–741.
12. G.N. Watson *A Treatise on the theory of Bessel functions*. Second Edition. Cambridge University Press, Cambridge, 2006.