Isoclinism of skew braces

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Abstract
We define isoclinism of skew braces and present several applications. We study some properties of skew braces that are invariant under isoclinism. For example, we prove that right nilpotency is an isoclinism invariant. This result has application in the theory of set-theoretic solutions to the Yang–Baxter equation. We define isoclinic solutions and study multipermutation solutions under isoclinism.

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1 | INTRODUCTION

The fundamental problem of constructing combinatorial solutions of the Yang–Baxter equation (YBE) is nowadays based on the use of specific (associative and non-associative) algebraic structures associated with the solution. One particular algebraic structure stands out: skew braces. Such structures were introduced in [15] and [11]. The theory originates in Jacobson radical rings, and it is now considered a hot topic, as skew braces appear in several different areas of mathematics, see, for example, [8, 17, 22].

Skew braces classify solutions [1, 2]. This justifies the search for classification results on skew braces. In this vein, several results are known; see, for example, [4, 11]. However, the classification of skew braces of order $p^n$ ($p$ prime) is still hard to achieve.

To classify $p$-groups, in [12], Hall introduced a specific equivalence relation that simplifies the problem. Without technicalities, one defines isoclinic groups as groups that have “essentially the same” commutator functions. Recall that the commutator function for the group $G$ is the map

$$G/Z(G) \times G/Z(G) \to G, \quad (xZ(G), yZ(G)) \mapsto [x, y] = xyy^{-1}x^{-1}.$$  

Isoclinism is an equivalence relation that generalizes isomorphisms.

There are several different motivations to consider this notion in the context of skew braces. With isoclinism of skew braces, we will have a new tool to classify finite skew braces of prime-
power order, something that ultimately will have applications in other branches of mathematics, for example, in the theory of pre-Lie algebras [19–21].

In [14], the authors defined Schur covers of finite skew braces and proved that any two Schur covers are isoclinic.

In this work, we define isoclinism in the context of skew braces. We show that the following properties are preserved under isoclinism: triviality (Theorem 3.1), two-sidedness (Theorem 3.12), right nilpotence (Theorem 3.4), the lattice of sub skew braces containing the annihilator (Proposition 2.9), and the size of \( \lambda \)- and \( \rho \)-orbits up to a constant factor (Theorem 3.19). As an application, we define a notion of isoclinism of set-theoretic solutions to the YBE. This new equivalence relation defined on the space of solutions suggests a way of attacking the classification problem of set-theoretic solutions to the YBE. Isoclinism classes of solutions could be relevant in the study of multipermutation solutions [3, 6, 9, 10, 13]. We prove in Section 4 that if \((X, r)\) and \((Y, s)\) are isoclinic set-theoretic solutions and \((X, r)\) is multipermutation, then \((Y, s)\) is multipermutation (see Theorem 4.3).

2 | ISOCLINISM OF SKEW BRACES

Recall that a skew brace is a triple \((A, +, \circ)\), where \((A, +)\) and \((A, \circ)\) are groups such that the compatibility condition \(a \circ (b + c) = a \circ b - a + a \circ c\) holds for all \(a, b, c \in A\). The inverse of an element \(a \in A\) with respect to the circle operation \(\circ\) will be denoted by \(a'\). We will denote the groups \((A, +)\) and \((A, \circ)\), respectively, by \(A_+\) and \(A_\circ\).

Let \(A\) be a skew brace. There are two canonical actions by automorphism:

\[
\lambda : A_\circ \to \text{Aut}(A_+), \quad \lambda_a(b) = -a + a \circ b,
\]
\[
\rho : A_\circ \to \text{Aut}(A_+), \quad \rho_a(b) = a \circ b - a.
\]

By definition, a sub skew brace \(B\) of \(A\) is an ideal of \(A\) if \(B_+\) is normal in \(A_+\), \(B_\circ\) is normal in \(A_\circ\), and \(\lambda_a(B) \subseteq B\) for all \(a \in A\).

We write \(A^{(2)} = A * A\) to denote the additive subgroup of \(A\) generated by \(\{a * b : a, b \in A\}\), where \(a * b = -a + a \circ b - b\) for all \(a, b \in A\). One proves that \(A^{(2)}\) is an ideal of \(A\). If \(a, b \in A\), we write \([a, b]_+ = a + b - a - b\) and \([a, b]_\circ = a \circ b \circ a' \circ b'\). We write \([A, A]_+\) to denote the commutator subgroup of the additive group of \(A\). The socle of \(A\) is the ideal \(\text{Soc}(A) = \ker \lambda \cap Z(A, +)\) and the annihilator of \(A\) is the ideal \(\text{Ann}(A) = \text{Soc}(A) \cap Z(A, \circ)\); see [5].

**Definition 2.1.** Let \(A\) be a skew brace. The commutator \(A'\) of \(A\) is the additive subgroup of \(A\) generated by \([A, A]_+\) and \(A^{(2)}\).

**Proposition 2.2.** Let \(A\) be a skew brace. Then, \(A'\) is an ideal of \(A\).

**Proof.** It follows from the fact that for all \(a \in A\) and \(x \in A'\lambda_a(x) = a * x + x \in A', a + x - a = x + [-x, a]_+ \in A'\), and

\[
a \circ x \circ a' = \lambda_a(x) + [-a \circ x, a]_+ \]
\[
- a' * a + (a \circ x) * a' + [a', -a \circ x \circ a' + a \circ x]_+ \in A'
\]

for all \(a \in A\) and \(x \in A'\). \(\square\)
A skew brace $A$ is said to be trivial if $a + b = a \circ b$ for all $a, b \in A$.

**Example 2.3.** In the case of a trivial brace $A$, $A'$ is the commutator of the underlying group.

Notation 2.4. In general, if $A/B$ is a quotient of a skew brace, we will denote the equivalence class of $a \in A$ in $A/B$ by $\bar{a}$.

**Remark 2.5.** $A'$ is the smallest ideal of $A$ such that $A^{ab} = A/A'$ is an abelian group.

**Lemma 2.6.** We define two collections of maps $\phi_+$ and $\phi_*$ that associate, respectively, to every skew brace $B$ the maps

$$
\phi_+^B : (B / Ann B)^2 \to B', \quad (\bar{a}, \bar{b}) \mapsto [a, b]_+,
$$

$$
\phi_*^B : (B / Ann B)^2 \to B', \quad (\bar{a}, \bar{b}) \mapsto a * b.
$$

The maps (2.1) and (2.2) are well defined.

**Proof.** A direct calculation shows that (2.1) is well defined. To prove that (2.2) is well defined, let $(\bar{a}, \bar{b}) = (\bar{c}, \bar{d}) \in (B / Ann B)^2$. We have

$$
-a + a \circ b - b - (-c + c \circ d - d) = -a + a \circ d - c \circ d + c.
$$

After conjugating by $c$, we get

$$
c - a + a \circ d - c \circ d = (c - a) \circ a \circ d - c \circ d
$$

$$
= ((c - a) + a) \circ d - c \circ d
$$

$$
= c \circ d - c \circ d
$$

$$
= 0.
$$

Since the conjugation by $c$ is an automorphism, we have $a * b = c * d$. \qed

**Definition 2.7.** We say that the braces $A$ and $B$ are isoclinic if there are two isomorphisms $\xi : A / Ann A \to B / Ann B$ and $\vartheta : A' \to B'$ such that

$$
A' \xleftarrow{\phi^A_*} (A / Ann A)^2 \xrightarrow{\phi^A_+} A' \xrightarrow{\vartheta} B' \xleftarrow{\phi^B_*} (B / Ann B)^2 \xrightarrow{\phi^B_+} B'
$$

(2.3)

commutes. We call the pair $(\xi, \vartheta)$ a skew brace isoclinism.

**Notation 2.8.** For $A$ a skew brace and $B$ and $C$ sub skew braces, let

$$
B + C = \{ b + c : b \in B, c \in C \}.
$$

Note that, in general, if $B$ and $C$ are sub skew braces of $A$, then $B + C$ is not a skew brace. However, $B + C$ is always a skew brace if $C$ is an ideal.
Proposition 2.9. Let $A$ and $B$ be isoclinic skew braces. Then, sub skew braces of $A$ containing $\text{Ann}(A)$ are in bijective correspondence with sub skew braces of $B$ containing $\text{Ann}(B)$. Furthermore, these corresponding sub skew braces are isoclinic.

Proof. Let $(\xi, \theta)$ be an isoclinism between $A$ and $B$. The correspondence is given by $A_1 \mapsto \pi^{-1}(\xi(A_1/\text{Ann}(A)))$ for all skew braces $\text{Ann}(A) \subseteq A_1 \subseteq A$, where the map $\pi : B \rightarrow B/\text{Ann}(B)$ is the canonical map. Let $\text{Ann}(A) \subseteq A_1 \subseteq A$ and $\text{Ann}(B) \subseteq B_1 \subseteq B$ be corresponding skew braces. Then, by the commutativity of the diagram (2.3), $\theta$ restricts to an isomorphism $A_1' \rightarrow B_1'$ and $\xi$ factors through an isomorphism $\overline{A_1} \rightarrow \overline{B_1}$. These two maps form an isoclinism from $A_1$ to $B_1$. □

Proposition 2.10. Let $A$ be a skew brace and $K$ be a sub skew brace of $A$. Then, $K$ is isoclinic to $K + \text{Ann}(A)$.

Proof. It is a direct consequence of the fact that $K' = (K + \text{Ann}(A))'$ and that $K/\text{Ann}(K) \cong (K + \text{Ann}(A))/\text{Ann}(K + \text{Ann}(A))$. □

Proposition 2.11. Let $A$ be a skew brace and $K$ be a sub skew brace of $A$. If $A/\text{Ann}(A)$ is finite, then $A$ is isoclinic to $K$ if and only if $K + \text{Ann}(A) = A$.

Proof. Assume $A$ is isoclinic to $K$. Then,

$$|K/\text{Ann}(K)| = |A/\text{Ann}(A)| \geq |K/K \cap \text{Ann}(A)| \geq |K/\text{Ann}(K)|.$$ 

Therefore, $(K + \text{Ann}(A))/\text{Ann}(A) = A/\text{Ann}(A)$. Thus, $K + \text{Ann}(A) = A$. □

Proposition 2.12. Let $A$ and $B$ be isoclinic skew braces with isoclinism $(\xi, \theta)$. If $K \subseteq A'$ is an ideal of $A$, then $\theta(K) \subseteq B'$ is an ideal of $B$. In addition, $A/K$ is isoclinic to $B/\theta(K)$.

Proof. First notice that for all elements $x \in A'$, we have $\xi(x) = \overline{\theta(x)}$. It is enough to verify it on the generators of $A'$. The commutativity of (2.3) implies that for all $a, a_1 \in A$,

$$\overline{\theta([a, a_1]_+)} = [\xi(a), \xi(a_1)]_+, \quad \overline{\theta(a * a_1)} = \xi(a) * \xi(a_1).$$

We show that $\theta(K)$ is an ideal of $B$. Clearly $\theta(K)_+$ is a subgroup of $B_+$ and $\theta(K)_O$ is a subgroup of $B_O$. Let $k \in K$. Let $b \in B$ and $a \in A$ be such that $\xi(a) = \overline{b}$. The commutativity of the diagram (2.3) implies that $\theta([a, k]_+) = [b, \theta(k)]_+$ and $\theta(a * k) = b * \theta(k)$. Thus, $b + \theta(a - k) = \theta(a + k - a) \in \theta(K)$ and $\lambda_b(\theta(k)) = \xi(k)$. Therefore, $\theta(K)$ is a left ideal of $B$ that is normal in the additive group of $B$. Similarly, one shows that $K_\theta$ is a normal subgroup of $B_\theta$ using the commutativity of the diagram (3.3). It is left to see that $A/K$ is isoclinic to $B/\theta(K)$. First, $(A/K)' = A'/K$ and $(B/\theta(K))' = B'/\theta(K)$. Thus, $\theta$ factors through an isomorphism $\overline{\theta} : (A/K)' \rightarrow (B/\theta(K))'$. The annihilator of $A/K$ corresponds by the canonical homomorphism to an ideal $Q$ of $A$ such that

$$\text{Ann}(A) + K \subseteq Q \subseteq A.$$ 

Similarly, the annihilator of $B/\theta(K)$ corresponds to an ideal $Q_1$ of $B$ such that

$$\text{Ann}(B) + \theta(K) \subseteq Q_1 \subseteq B.$$
Thus,

\[
\frac{A}{K} \simeq \frac{A}{Q} \simeq \frac{A}{Q/\text{Ann}(A)}, \quad \frac{B}{\vartheta(K)} \simeq \frac{B}{Q_1} \simeq \frac{B}{Q_1/\text{Ann}(B)}.
\]

It is left to see that \(\xi(Q/\text{Ann}(A)) = Q_1/\text{Ann}(B)\). Let \(q \in Q\), then \([q, x]_+ \in K\), \(q \ast x \in K\), and \([q, x]_0 \in K\) for all \(x \in A\). Let \(q_1 \in B\) be a representative of \(\xi(q)\). By the commutativity of the diagrams (2.3) and (3.3), \([q_1, x]_+ \in \vartheta(K)\), \(q_1 \ast x \in \vartheta(K)\) and \([q_1, x]_0 \in \vartheta(K)\) for all \(x \in B\). By a symmetric argument, \(Q_1/\text{Ann}(B) \subseteq \xi(Q/\text{Ann}(A))\). The commutativity of (2.3) is straightforward.

**Proposition 2.13.** Let \(A\) be a skew brace and \(K\) an ideal of \(A\). Then, \(A/K\) is isoclinic to \(A/(K \cap A')\).

**Proof.** We have \((A/K)' = (A' + K)/K \simeq A'/((K \cap A')')\). The annihilator of \(A/K\) corresponds to an ideal \(Q\) of \(A\) such that \(\text{Ann}(A) + K \subseteq Q \subseteq A\). Similarly, the annihilator of \(A/(K \cap A')\) corresponds to an ideal \(Q_1\) of \(A\) such that \(\text{Ann}(A) + K \subseteq Q_1 \subseteq A\). Thus \(A/K \simeq A/Q\) and \(A/K \cap A' \simeq A/Q_1\). It is left to see that \(Q = Q_1\). It is clear that \(Q_1 \subseteq Q\). Let \(q \in Q\). Then, \([q, x]_+ \in K\), \(q \ast x \in K\), and \([q, x]_0 \in K\) for all \(x \in A\). In addition, by the definition of \(A'\), \([q, x]_+ \in A'\), \(q \ast x \in A'\), and \([q, x]_0 \in A'\) for all \(x \in A\). Thus, \(q \in Q_1\). The commutativity of (2.3) is straightforward.

**Corollary 2.14.** Let \(A\) be a finite skew brace and \(K\) an ideal of \(A\). Then, \(A\) is isoclinic to \(A/K\) if and only if \(K \cap A' = \{0\}\).

**Proof.** If \(K \cap A' \neq \{0\}\), then

\[
|\frac{A}{K}'| = \frac{|A'|}{(K \cap A')} = \frac{|A'|}{|K \cap A'|} < |A'|.
\]

Thus, \(A\) cannot be isoclinic to \(A/K\). The other implication is a direct consequence of Proposition 2.13.

**Remark 2.15.** Let \(A, A_1, B, B_1\) be skew braces. If \(A\) is isoclinic to \(A_1\) and \(B\) is isoclinic to \(B_1\), then \(A \times B\) is isoclinic to \(A_1 \times B_1\).

The following definition is motivated by its group-theoretic analog.

**Definition 2.16.** A skew brace \(A\) such that \(\text{Ann}(A) \subseteq A'\) will be called a *stem skew brace*.

**Proposition 2.17.** Two isoclinic stem skew braces have the same order.

**Proof.** Let \(A\) and \(B\) be isoclinic stem skew braces with isoclinism \((\xi, \vartheta)\). As \(\text{Ann}(A) \subseteq A'\), \(\vartheta(x) = \xi(x)\) for all \(x \in \text{Ann}(A)\). Thus, \(\vartheta(\text{Ann}(A)) \subseteq \text{Ann}(B)\). A symmetric argument shows that \(\vartheta\) restricts to an isomorphism of \(\text{Ann}(A)\) and \(\text{Ann}(B)\). The statement is then a consequence of \(A/\text{Ann}(A) \simeq B/\text{Ann}(B)\).

**Theorem 2.18.** Every skew brace is isoclinic to a stem skew brace.
Proof. Let \( A \) be a skew brace and \( I \) a set of indices (possibly uncountable). Let \( \{\xi_i : i \in I\} \) be a set of generators of \( A \) (as a skew brace). Let \( G \) be the free abelian group generated by \( \{\eta_i : i \in I\} \). Then, \( A \) is isoclinic to \( A \times G \). Let \( A_1 \subseteq A \times G \) be the skew brace generated by \( \{(\xi_i, \eta_i) : i \in I\} \). Then, clearly \( A_1 + G \) coincides with \( A \times G \). Since \( G \) is contained in the annihilator of \( A \times G \), it follows that \( A_1 \) is isoclinic to \( A \times G \). Thus, \( A_1 \) is isoclinic to \( A \). Let \( \pi : A_1 \rightarrow G \) denote the projection on the second coordinate. Then, \( \pi \) factors through a morphism of abelian groups \( \overline{\pi} : A_1/A_1' \rightarrow G \). By the universal property of the free abelian group \( G \), there is a homomorphism \( \phi : G \rightarrow A_1/A_1' \) that maps \( \eta_i \) to \( (\xi_i, \eta_i) \) for all \( i \in I \). In fact, \( \phi \) is the inverse map of \( \overline{\pi} \). Therefore, \( A_1/A_1' \cong G \). Moreover,
\[
\text{Ann}(A_1)/(\text{Ann}(A_1) \cap A_1') \cong (\text{Ann}(A_1) + A_1')/A_1',
\]
that is, \( \text{Ann}(A_1)/(\text{Ann}(A_1) \cap A_1') \) is isomorphic to a subgroup of a free abelian group. Thus, \( \text{Ann}(A_1) \) is the direct product of \( \text{Ann}(A_1) \cap A_1' \) with another group \( K \). Since \( K \cap A_1' = \{0\} \), the skew brace \( A_1/K \) is isoclinic to \( A_1 \). The annihilator of \( A_1/K \) is \( \text{Ann}(A_1)/K \) and is contained in the commutator, which is \( (A_1' + K)/K \). \( \square \)

For \( n \geq 2 \), let \( C_n \) be the cyclic group of order \( n \).

Notation 2.19. Let \( n \geq 2 \) be an integer. Let \( d \) be an integer such that \( d \mid n \) and every prime divisor of \( n \) divides \( d \). We will denote by \( C(n, d) \) the skew brace \( C_n \) with multiplication defined as \( x \circ y = x + dx y + y \).

The skew braces \( C(n, d) \) appear in the work of Rump [16].

Example 2.20. Let \( m, n > 2 \) be integers, We have \( a \ast b = m^{n-1}ab \) for all \( a, b \in C(m^n, m^{n-1}) \). The commutator is the set of multiples of \( m^{n-1} \) and the annihilator is the set of multiples of \( m \). They coincide if and only if \( n = 2 \). Thus, \( C(m^2, m) \) is a stem skew brace. It is straightforward to see that \( C(m^2, m) \) is isoclinic to \( C(m^n, m^{n-1}) \) for all \( n > 1 \).

3 ISOCLINISM INVARIANTS

Theorem 3.1. Let \( A \) and \( B \) be skew braces. If \( A \) and \( B \) are isoclinic, then \( A \) is trivial if and only if \( B \) is trivial.

Proof. It is a direct consequence of the commutativity of the right part of the diagram (2.3). \( \square \)

Remark 3.2. Two trivial skew braces are isoclinic if and only if their underlying groups are isoclinic. Therefore, we will not distinguish between the two notions when dealing with trivial skew braces.

Remark 3.3. Let \( A \) and \( B \) be isoclinic skew braces. The quotients \( A/\text{Ann}(A) \) and \( B/\text{Ann}(B) \) are isomorphic. It follows that \( A \) is annihilator nilpotent if and only if \( B \) is annihilator nilpotent.

A skew brace \( A \) is said to be right nilpotent if \( A^{(n)} = \{0\} \) for some \( n \), where \( A^{(1)} = A \) and \( A^{(k+1)} = A^{(k)} \ast A \) for all \( k \).
**Theorem 3.4.** Let $A$ and $B$ be isoclinic skew braces. Then, $A$ is right nilpotent if and only if $B$ is right nilpotent.

**Proof.** Let $(\xi, \theta)$ be an isoclinism between $A$ and $B$. Assume that $A$ is right nilpotent. By [7, Lemma 2.5], $A / \text{Soc}(A)$ is right nilpotent.

We claim that

$$\xi(\text{Soc}(A) / \text{Ann}(A)) = \text{Soc}(B) / \text{Ann}(B).$$  \hspace{1cm} (3.1)

Let $a \in \text{Soc}(A)$ be such that $\xi(a + \text{Ann}(A)) = b + \text{Ann}(B)$ for some $b \in B$. Let $a_1 \in A$ and $b_1 \in B$ such that $\xi(a_1 + \text{Ann}(A)) = b_1 + \text{Ann}(B)$. By the commutativity of (2.3),

$$b * b_1 = \phi_B^a(b + \text{Ann}(B), b_1 + \text{Ann}(B))$$

$$= \theta(\phi_A^a(a + \text{Ann}(A), a_1 + \text{Ann}(A)))$$

$$= \delta(a * a_1) = 0.$$  

Similarly, $b$ is central in $(B, +)$. Now (3.1) follows.

It follows from the first isomorphism theorem that

$$(A / \text{Ann}(A))/ (\text{Soc}(A) / \text{Ann}(A)) \simeq (B / \text{Ann}(B))/ (\text{Soc}(B) / \text{Ann}(B)).$$

Hence, $A / \text{Soc}(A) \simeq B / \text{Soc}(B)$. Thus, $B / \text{Soc}(B)$ is right nilpotent. By [7, Proposition 2.17], $B$ is right nilpotent.  \hfill $\square$

We now present the notions of terms and term functions from universal algebra in the context of skew braces. A more general approach can be found in section 10, and section 11 of the book [18].

Let $X$ be a set of objects called variables. The set of skew brace terms over $X$ is the smallest set $T(X)$ such that

1. $X \cup \{0\} \subseteq T(X)$, where 0 is a formal element, and
2. if $p_1, p_2 \in T(X)$, then the “strings” $p_1 \circ p_2, p_1 + p_2, -p_1, p'_2 \in T(X)$.

For $p \in T(X)$, we write $p$ as $p(x_1, \ldots, x_n)$ to indicate that the variables occurring in $p$ are among $x_1, \ldots, x_n$. We say that a skew brace term $p$ is $n$-ary if the number of variables appearing explicitly in $p$ is $\leq n$.

Given a skew brace term $p(x_1, \ldots, x_n)$ over some set of variables $X$ and a skew brace $A$, we define a map $p^A : A^n \to A$ inductively as follows:

1. if $p$ is a variable $x_i \in X$, then $p^A(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A$;
2. if $p$ is of the form $p_1(x_1, \ldots, x_n) \circ p_2(x_1, \ldots, x_n)$, then

$$p^A(a_1, \ldots, a_n) = p_1^A(a_1, \ldots, a_n) \circ p_2^A(a_1, \ldots, a_n),$$

where $\circ$ denotes either $\circ$ or $+$;
3. if $p$ is of the form $p_1(x_1, \ldots, x_n)'$, then $p^A(a_1, \ldots, a_n) = p_1^A(a_1, \ldots, a_n)'$;
4. if $p$ is of the form $-p_1(x_1, \ldots, x_n)$, then $p^A(a_1, \ldots, a_n) = -p_1^A(a_1, \ldots, a_n)$.
**Example 3.5.** Let \( X = \{x, y\} \), then \( p(x, y) = -x + x \circ y - y \) is a skew brace term over \( X \). Given a skew brace \( A \), its term function over \( A \) is the map \( p^A : A^2 \to A, (a, b) \mapsto a \ast b \).

**Notation 3.6.** Let \( A \) be a skew brace. From now on, we denote by \( \overline{A} \) the quotient \( A/ \text{Ann} A \).

**Lemma 3.7.** Let \( n, m \) be integers, \( \eta_1, \ldots, \eta_{2m} \) be \( n \)-ary skew brace terms and \( p \) be an \( m \)-ary skew brace term. Let \( \phi_1, \ldots, \phi_m \) be collections of maps where each \( \phi_i \) is either \( \phi_+ \) or \( \phi_- \). Then, one can construct a collection of well-defined maps \( \phi \) that associate to every skew brace \( B \) a map \( \phi^B : \overline{B}^n \to \overline{B}' \) such that

\[
\phi^B(\overline{b_1}, \ldots, \overline{b_n}) = p^B(a_1(\overline{b_1}, \ldots, \overline{b_n}), \ldots, a_m(\overline{b_1}, \ldots, \overline{b_n})),
\]

where \( a_i : \overline{B} \to B \) is the map

\[
(b_1, \ldots, b_n) \mapsto \phi^B_i \left( \eta^B_{2i-1}(b_1, \ldots, b_n), \eta^B_{2i}(b_1, \ldots, b_n) \right)
\]

for all \( 1 \leq i \leq m \). In addition, if \( A \) and \( B \) are two isoclinic skew braces with isoclinism \( (\xi, \theta) \), then the following diagram

\[
\begin{array}{ccc}
\overline{A}^n & \xrightarrow{\phi^A} & A' \\
\downarrow \xi^n & & \downarrow \theta \\
\overline{B}^n & \xrightarrow{\phi^B} & B'
\end{array}
\]

commutes.

**Proof.** Let \( B \) be a skew brace. One has that \( \phi^B \) is the composition of the maps

\[
\overline{B}^n \xrightarrow{\theta} \overline{B}^{2m} \xrightarrow{\gamma} B^m \xrightarrow{p^B} B
\]

where

\[
\theta(\overline{b_1}, \ldots, \overline{b_n}) = \left( \eta^B_1(\overline{b_1}, \ldots, \overline{b_n}), \ldots, \eta^B_{2m}(\overline{b_1}, \ldots, \overline{b_n}) \right),
\]

\[
\gamma(\overline{f_1}, \ldots, \overline{f_{2m}}) = \left( \phi^B_1(\overline{f_1}, \overline{f_2}), \ldots, \phi^B_m(\overline{f_{2m-1}}, \overline{f_{2m}}) \right).
\]

Suppose that \( A \) and \( B \) are isoclinic skew braces with isoclinism \( (\xi, \theta) \). Since \( (\xi, \theta) \) is an isoclinism, [18, Theorem 10.3] implies that for all \( a_1, \ldots, a_n \in A \) and \( 1 \leq i \leq m \),

\[
\phi^B_i \eta^B_{2i-1} \left( \xi(a_1), \ldots, \xi(a_n) \right) = \theta \left( \phi^A_i \left( \eta^A_{2i-1}(\overline{a_1}, \ldots, \overline{a_n}), \eta^A_{2i}(\overline{a_1}, \ldots, \overline{a_n}) \right) \right).
\]

Therefore,

\[
\phi^B \left( \xi(\overline{a_1}, \ldots, \overline{a_n}) \right) = \theta \left( \phi^A(\overline{a_1}, \ldots, \overline{a_n}) \right).
\]

This means that diagram (3.3) is commutative. \(\square\)
Notation 3.8. For $X$ a skew brace and $x, y \in X$, let

$$
\begin{align*}
r(x, y) &= -y - x + x \circ y = [-y, -x + x \circ y]_+ + x \star y, \\
l(x, y) &= x \circ y - y - x = [x \circ y - y, -x]_+ + x \star y.
\end{align*}
$$

Proposition 3.9. For all skew braces $X$, the map

$$
\phi^X : X^2 \to X', \quad (\bar{a}, \bar{b}) \mapsto [a, b],
$$

is well defined. In addition, if $A$ and $B$ are isoclinic skew braces with isoclinism $(\xi, \vartheta)$, then

$$
\begin{array}{ccc}
A^2 & \xrightarrow{\phi^A} & A' \\
\downarrow \xi \times \xi & & \downarrow \vartheta \\
B^2 & \xrightarrow{\phi^B} & B'
\end{array}
$$

commutes.

Proof. Let $X$ be a skew brace. For all $a, b \in X$,

$$
[a, b]_o = [a, b]_+ - l(b, a) - r(b \circ a, a' \circ b') + r(b, a' \circ b') + r(a, b \circ a' \circ b').
$$

Lemma 3.7 concludes the proof. \[\square\]

Proposition 3.10. Let $A$ and $B$ be two skew braces. If $A$ and $B$ are isoclinic, then $A_+$ is isoclinic to $B_+$ and $A_\circ$ is isoclinic to $B_\circ$.

Proof. Assume $A$ is isoclinic to $B$ with isoclinism $(\xi, \vartheta)$. Because of the commutativity of (2.3), the map $\vartheta$ restricts to an isomorphism $\vartheta_1 : [A, A]_+ \to [B, B]_+$ and $\xi(Z(A_+)/\text{Ann } A) = Z(B_+)/\text{Ann } B$. Therefore, $\xi$ induces a group isomorphism $\xi_1 : A_+/Z(A_+) \to B_+/Z(B_+)$ such that

$$
\begin{array}{ccc}
(A_+/Z(A_+))^2 & \xrightarrow{\xi \times \xi} & A'_+ \\
\downarrow \xi \times \xi & & \downarrow \vartheta_1 \\
(B_+/Z(B_+))^2 & \xrightarrow{\vartheta_1} & B'_+
\end{array}
$$

commutes, where the horizontal maps are the classical commutator maps for groups. Proposition 3.9 and a similar argument shows that $A_\circ$ is isoclinic to $B_\circ$. \[\square\]

Remark 3.11. If $A$ and $B$ are isoclinic skew braces, then $A$ is of nilpotent type if and only if $B$ is of nilpotent type.

Recall that a skew brace $A$ is said to be two-sided if $(a + b) \circ c = a \circ c - c + b \circ c$ holds for all $a, b, c \in A$. In [15], Rump proved that radical rings are exactly two-sided skew braces with an abelian additive group.
| Size | Radical rings | Abelian type | Two-sided | All |
|------|--------------|--------------|-----------|-----|
| 8    | 8            | 12           | 16        | 20  |
| 27   | 10           | 13           | 25        | 38  |

**Theorem 3.12.** Let $A$ and $B$ be skew braces. If $A$ and $B$ are isoclinic, then $A$ is two-sided if and only if $B$ is two-sided.

**Proof.** We claim that for all skew braces $X$,

$$\phi_X : (\overline{X})^3 \rightarrow X', (\overline{a}, \overline{b}, \overline{c}) \mapsto (a + b)oc - boc + c - aoc,$$

is well defined. In addition, if $A$ and $B$ be isoclinic skew braces with isoclinism $(\xi, \theta)$, the following diagram

$$\begin{array}{ccc}
\overline{A}^3 & \xrightarrow{\phi_A} & A' \\
\varepsilon^3 \downarrow & & \downarrow \phi \\
\overline{B}^3 & \xrightarrow{\phi_B} & B'
\end{array}$$

(3.4)

commutes. This is a direct consequence of Lemma 3.7 and the fact that for any elements $a, b, c$ of any brace, the following equation holds:

$$(a + b)oc - boc + c - aoc = l(a + b, c) - l(b, c) - [a, boc - c - b]_+ - l(a, c).$$

This concludes the proof. □

**Example 3.13.** Computer calculations using the database of [11] show that among the 47 skew braces of size 8, there are 20 isoclinism classes. Moreover, there are eight isoclinism classes of radical rings of size 8, 12 isoclinism classes of skew braces of abelian type of size 8, and 16 isoclinism classes of two-sided skew braces of size 8. There are 101 skew braces of size 27 and there are 38 isoclinism classes. See Table 3.1 for other numbers.

**Notation 3.14.** Let $B$ be a skew brace. There is a canonical group homomorphism

$$B_+ \rtimes_{\rho} B_o \rightarrow \text{Aut}(B_+), \quad (a, b) \mapsto (c \mapsto a + \rho_b(c) - a).$$

We write $\rtimes(a, b)c = a + \rho_b(c) - a$.

**Notation 3.15.** Let $A$ be a skew brace, denote, respectively, by $\overline{A}_+$ and $\overline{A}_o$ the groups $A_+ / \text{Ann}(A)$ and $A_o / \text{Ann}(A)$.

**Remark 3.16.** The group homomorphism $\rho : A_o \rightarrow \text{Aut}(A_+)$ induces a group homomorphism $\overline{\rho} : \overline{A}_o \rightarrow \text{Aut}(\overline{A}_+)$. One can check that the group $(\overline{A}_+ \rtimes_{\overline{\rho}} \overline{A}_o)$ is isomorphic to $(\overline{A}_+ \rtimes_{\overline{\rho}} \overline{A}_o) / (\text{Ann}(A) \times \text{Ann}(A))$. Thus, $\overline{A}_+ \rtimes_{\overline{\rho}} \overline{A}_o$ acts canonically on $\overline{A}_+$. 

**Remark 3.17.** If $A$ and $B$ are isoclinic skew braces, then $\overline{A}_+ \rtimes_{\overline{\rho}} \overline{A}_o \simeq \overline{B}_+ \rtimes_{\overline{\sigma}} \overline{B}_o$. 

---

**TABLE 3.1** Number of isoclinism classes of skew braces

| Size | Radical rings | Abelian type | Two-sided | All |
|------|--------------|--------------|-----------|-----|
| 8    | 8            | 12           | 16        | 20  |
| 27   | 10           | 13           | 25        | 38  |
Notation 3.18. Let $A$ be a skew brace. Let $H$ be a subgroup of $\overline{A}_+ \rtimes \overline{A}_o$. We call the orbit of an element $a \in A$ under the induced action of $H$ an $H$-orbit.

**Theorem 3.19.** Let $A$ and $B$ be isoclinic skew braces. Let $H$ be a subgroup of $\overline{A}_+ \rtimes \overline{A}_o$ and $K$ be the corresponding subgroup of $\overline{B}_+ \rtimes \overline{B}_o$. For $c \in \mathbb{Z}_{\geq 1}$, let $m_1$ (resp. $m_2$) be the number of $H$-orbits (resp. $K$-orbits) of size $c$. Then,

$$m_1 = m_2 |A|/|B|.$$

**Proof.** Lemma 3.7 and the fact that

$$\varpi(a, b)c - c = [a, b \circ c - b]_+ + [b \circ c, -b]_+ + b \ast c$$

imply that the map

$$\phi^X : \overline{X}^3 \to X', \quad (\overline{a}, \overline{b}, \overline{c}) \mapsto \varpi(a, b)c - c$$

is well defined for all skew brace $X$. In addition, the diagram

$$\begin{array}{ccc}
\overline{A}^3 & \xrightarrow{\phi^A} & A' \\
\downarrow{\xi^3} & & \downarrow{\theta} \\
\overline{B}^3 & \xrightarrow{\phi^B} & B'
\end{array}
$$

(3.5)

commutes.

An element $a \in A$ has an $H$-orbit of size $c$ if and only if the index of the subgroup $C(\overline{a}) = \{w \in H : \phi^A(w, \overline{a}) = 0\}$ in $H$ is $c$. Let $S$ be the subset of $\overline{A}$ that consists of elements $w$ such that $C(w)$ has index $c$ in $H$. If $\pi : A \to \overline{A}$ is the canonical homomorphism, $\pi^{-1}(S)$ is the set of elements of $A$ that have an $H$-orbit of size $c$. Hence, $m_1 c = |\text{Ann } A||S|$. Because of the commutativity of (3.5), one also has that $m_2 c = |\text{Ann } B||S|$. Hence, the claim follows.

**Remark 3.20.** We use the notations of Theorem 3.19.

1. Let

$$H = \{(\overline{-a}, \overline{a}) : a \in A\} \subseteq \overline{A}_+ \rtimes \overline{A}_o,$$

$$K = \{(\overline{-b}, \overline{b}) : b \in B\} \subseteq \overline{B}_+ \rtimes \overline{B}_o.$$

Note that $K$ is the subgroup of $\overline{B}_+ \rtimes \overline{B}_o$ corresponding to $H$ by the isomorphism induced by isoclinism. Then, the $H$-orbits and the $K$-orbits are, respectively, the orbits of the canonical actions $\lambda : A_0 \to \text{Aut}(A_+)$ and $\overline{\lambda} : B_0 \to \text{Aut}(B_+)$. 

2. Similarly, Theorem 3.19 applies to the pair $H = \{(\overline{0}, \overline{a}) : a \in A\}$ and $K = \{(\overline{0}, \overline{b}) : b \in B\}$. In this case, the $H$-orbits and the $K$-orbits are, respectively, the orbits of the canonical actions $\rho : A_0 \to \text{Aut}(A_+)$ and $\rho : B_0 \to \text{Aut}(B_+)$. 

**Example 3.21.** Let $A$ be the skew brace $C_2 \times C_4$ with multiplication given by

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 2x_1y_2)$$
and $B$ the skew brace $C_2 \times C_4$ with multiplication given by
\[(x_1, y_1) \circ (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 2(x_1 + y_2)x_2 + 2y_1y_2).\]

In the skew brace $A$,
\[(x_1, y_1) \ast (x_2, y_2) = (0, 2x_1y_2)\]

and in $B$
\[(x_1, y_1) \ast (x_2, y_2) = (0, 2(x_1 + y_2)x_2 + 2y_1y_2).\]

Both $A$ and $B$ have commutator $C_2$ and annihilator quotient $C_2 \times C_2$. In addition, $A_+ \cong B_+$ and $A_o \cong D_8 \cong B_o$ where $D_8$ is the dihedral group of order 8. However, $A$ and $B$ are not isoclinic as $A$ has four $\lambda$-orbits of size 1 and two of size 2 and $B$ has two $\lambda$-orbits of size 1 and three of size 2.

In group theory, the notion of isoclinism is very convenient in the study of finite $p$-groups as these groups have non-trivial center and their commutator is a proper subgroup. This implies that isoclinism only depends on relations between groups of smaller order. However, there exist skew braces of prime-power size that have trivial annihilator. This is not the case for two-sided skew braces. An interesting property of two-sided skew braces is that the multiplicative conjugation is an action by automorphism of the multiplicative group over the additive one. Using this, one can extend the action defined earlier (see Notation 3.14) for two-sided skew braces. Let $B$ be a two-sided skew brace. Since the underlying multiplicative group acts on itself by conjugation, we can consider the semi-direct product $B_o \rtimes B_o$. The map
\[B_o \rtimes B_o \to \text{Aut}(B_+), \quad (a, b) \mapsto (c \mapsto \rho_a(b \circ c \circ b')),\]
defines an action by automorphisms of $B_o \rtimes B_o$ over $B_+$. The latter comes from the fact that $\rho_{boaob'}(b \circ c \circ b') = b \circ \rho_a(c) \circ b'$ for all $a, b, c \in B$. Thus, we can consider the semi-direct product $B_+ \rtimes (B_o \rtimes B_o)$. Finally, straightforward computations show that the map
\[B_+ \rtimes (B_o \rtimes B_o) \to \text{Aut}(B_+), \quad (a, b, c) \mapsto (d \mapsto a + \rho_b(c d \circ c') - a),\]
defines an action by automorphisms of $B_+ \rtimes (B_o \rtimes B_o)$ over $B_+$. It is straightforward to see that the elements of $B$ whose orbits have size 1 are exactly the elements of the annihilator. Moreover, if $B$ is a two-sided skew brace of size $p^n$, the group $B_+ \rtimes (B_o \rtimes B_o)$ has size $p^{3n}$. Thus, the non-trivial orbits of the action have size a power of $p$. Let $n_1, \ldots, n_m$ denote the sizes of the $m$ non-trivial orbits of $B$, then we have the following class equation:
\[p^n = |\text{Ann}(B)| + \sum_{i=1}^{m} n_i.\]

Therefore, $p$ divides $|\text{Ann}(B)|$. We have proved the following result:

**Proposition 3.22.** Let $p$ be a prime number and $B$ be a two-sided skew brace of size $p^n$ for some integer $n \geq 1$. Then, $\text{Ann}(B)$ is non-trivial.
**Proposition 3.23.** Let $p$ be a prime number and $B$ be a two-sided skew brace of size $p^n$ for some integer $n \geq 1$. Then, $B'$ is a proper ideal of $B$.

**Proof.** We proceed by induction on $n$. If $n = 1$, then $B$ is the trivial skew brace $C_p$. Assume now $n \geq 2$. Since $|B/\text{Ann}(B)| < p^n$, it follows by the induction hypothesis that $\overline{B} = (B' + \text{Ann}(B))/\text{Ann}(B)$ is a proper sub skew brace of $\overline{B}$. Thus, $B' + \text{Ann}(B) \subsetneq B$. □

4 | AN APPLICATION TO THE YBE

A set-theoretic solution to the YBE is a pair $(X, r)$, where $X$ is a set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

By convention, we will consider finite non-degenerate solutions, that is solutions $(X, r)$, where $X$ is a finite set and

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

where the maps $\sigma_x : X \to X$ and $\tau_x : X \to X$ are bijective for every $x \in X$.

If $(X, r)$ is a solution, there is an equivalence relation on $X$ given by

$$x \sim y \iff \sigma_x = \sigma_y \text{ and } \tau_x = \tau_y.$$  

This equivalence relation induces a solution $\text{Ret}(X, r)$ on the set of equivalence classes. The solution $\text{Ret}(X, r)$ is called the retraction of $(X, r)$.

A solution $(X, r)$ is said to be multipermutation if there exists an integer $m \geq 1$ such that $|\text{Ret}^m(X, r)| = 1$, where $\text{Ret}^1(X, r) = \text{Ret}(X, r)$ and

$$\text{Ret}^k(X, r) = \text{Ret}(\text{Ret}^{k-1}(X, r))$$

for $k \geq 2$.

The permutation group of $(X, r)$ is the group $G(X, r) = \langle \sigma_x, \tau_x : x \in X \rangle$. The permutation group of $(X, r)$ is a skew brace [1].

**Definition 4.1.** Let $(X, r)$ and $(Y, s)$ be solutions to the YBE. We say that $(X, r)$ and $(Y, s)$ are permutation isoclinic if the skew braces $G(X, r)$ and $G(Y, s)$ are isoclinic.

The following result follows from [7, Proposition 2.17, Theorem 2.20] and [6, Theorem 4.13].

**Lemma 4.2.** Let $(X, r)$ be a finite non-degenerate solution to the YBE. The following statements are equivalent:

1. $(X, r)$ is multipermutation.
2. $G(X, r)$ is right nilpotent of nilpotent type.
3. $G(X, r)$ is right nilpotent of nilpotent type.
Theorem 4.3. Let \((X, r)\) and \((Y, s)\) be permutation isoclinic solutions to the YBE. Then, \((X, r)\) is multipermutation if and only if \((Y, s)\) is multipermutation.

Proof. If \((X, r)\) is multipermutation, then \(\mathcal{G}(X, r)\) is right nilpotent. Thus, \(\mathcal{G}(Y, s)\) is right nilpotent and hence \((Y, s)\) is multipermutation.

We conclude the paper with concrete examples of involutive solutions up to permutation isoclinism. Recall that solution \((X, r)\) is said to be involutive if \(r^2 = \text{id}\). If \((X, r)\) is involutive, then

\[ \tau_y(x) = \sigma_{\sigma_y(x)}^{-1}(x) \]

for all \(x, y \in X\).

Example 4.4. There are four permutation isoclinism classes of involutive solutions of size 4. Let \(X = \{1, 2, 3, 4\}\). The following list provides a complete set of representatives over the set \(X\):

1. The flip \((x, y) \mapsto (y, x)\).
2. \(\sigma_1 = \sigma_2 = \text{id}, \sigma_3 = (34), \) and \(\sigma_4 = (12)(34)\).
3. \(\sigma_1 = (34), \sigma_2 = (1324), \sigma_3 = (1423), \) and \(\sigma_4 = (1, 2)\).
4. \(\sigma_1 = (12), \sigma_2 = (1324), \sigma_3 = (34), \) and \(\sigma_4 = (1423)\).

Remark 4.5. Permutation isoclinism of solutions does not preserve indecomposability. For example, let \(X = \{1, 2, 3, 4\}\) and \(\sigma = (1234)\). Then \((X, r)\), where

\[ r(x, y) = (\sigma(y), \sigma^{-1}(x)) \]

is indecomposable and \(\mathcal{G}(X, r)\) is the trivial skew brace over the cyclic group \(C_4\). It follows that \((X, r)\) is isoclinic to the flip over \(X\), as both solutions have isoclinic permutation braces (note that the permutation group of the flip is the trivial group).

Remark 4.6. Permutation isoclinism of solutions does not preserve the multipermutation level. For example, let \(X = \{1, 2, 3, 4\}\) and \(\sigma_1 = \sigma_2 = \sigma_3 = \text{id}, \sigma_4 = (23)\). Then, \((X, r)\) has multipermutation level 2. Moreover, \(\mathcal{G}(X, r)\) is the trivial skew brace over the cyclic group \(C_2\). Hence, the solution \((X, r)\) is permutation isoclinic to the flip over \(X\).

Example 4.7. There are six permutation isoclinism classes of involutive solutions of size 5. Let \(X = \{1, 2, 3, 4, 5\}\). The following list provides a complete set of representatives over \(X\):

1. The flip \((x, y) \mapsto (y, x)\).
2. \(\sigma_1 = \sigma_2 = \sigma_3 = \text{id}, \sigma_4 = (45), \sigma_5 = (23)(45)\).
3. \(\sigma_1 = \sigma_2 = \sigma_3 = \text{id}, \sigma_4 = (23)(45), \sigma_5 = (12)(45)\).
4. \(\sigma_1 = \text{id}, \sigma_2 = (45), \sigma_3 = (2435), \sigma_4 = (2534), \) and \(\sigma_5 = (23)\).
5. \(\sigma_1 = \text{id}, \sigma_2 = (23), \sigma_3 = (2435), \sigma_4 = (45), \) and \(\sigma_5 = (2534)\).
6. \(\sigma_1 = \sigma_2 = (45), \sigma_3 = (14)(25), \) and \(\sigma_4 = \sigma_5 = (12)\).

Note that flips of size 4 and 5 are permutation isoclinic. The second solution is permutation isoclinic to the second solution of Example 4.4. The fourth solution is permutation isoclinic to the third solution of Example 4.4. The fifth solution is permutation isoclinic to the fourth
solution of Example 4.4. This is a complete set of permutation isoclinisms between solutions of sizes 4 and 5.

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