SOME SUBMANIFOLDS OF ALMOST CONTACT MANIFOLDS WITH NORDEN METRIC *

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In this paper we study submanifolds of almost contact manifolds with Norden metric of codimension two with totally real normal spaces. Examples of such submanifolds as a Lie subgroups are constructed.

Keywords: Norden metric, almost contact manifold, submanifold, Lie group

1. Introduction

Let \((M, \varphi, \xi, \eta, g)\) be a \((2n+1)\)-dimensional almost contact manifold with Norden metric, i.e. \((\varphi, \xi, \eta)\) is an almost contact structure\(^1\) and \(g\) is a metric\(^3\) on \(M\) such that

\[
\varphi^2 X = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \\
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

where \(id\) denotes the identity transformation and \(X, Y\) are differentiable vector fields on \(M\), i.e. \(X, Y \in \mathfrak{X}(M)\). The tensor \(\tilde{g}\) given by

\(\tilde{g}(X, Y) = g(X, \varphi Y) + \eta(X)\eta(Y)\) is a Norden metric, too. Both metrics \(g\) and \(\tilde{g}\) are indefinite of signature \((n+1, n)\).

Let \(\nabla\) be the Levi-Civita connection of the metric \(g\). The tensor field \(F\) of type \((0, 3)\) on \(M\) is defined by

\[
F(X, Y, Z) = g((\nabla_X \varphi)Y, Z).
\]

A classification of the almost contact manifolds with Norden metric with respect to the tensor \(F\) is given in\(^3\) and eleven basic classes \(\mathcal{F}_i (i = 1, 2, \ldots, 11)\) are obtained.

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Let $R$ be the curvature tensor field of $\nabla$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

The corresponding tensor field of type $(0, 4)$ is determined as follows

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Let $(\overline{M}, \varphi, \xi, \eta, g)$ $(\dim M = 2n + 3)$ be an almost contact manifold with Norden metric and let $M$ be a submanifold of $\overline{M}$. Then for each point $p \in \overline{M}$ we have

$$T_p \overline{M} = T_p M \oplus (T_p M)^\perp,$$

where $T_p M$ and $(T_p M)^\perp$ are the tangent space and the normal space of $\overline{M}$ at $p$ respectively. When the submanifold $M$ of $\overline{M}$ is of codimension 2 we denote $(T_p M)^\perp$ by $\alpha = \{N_1, N_2\}$, i.e. $\alpha$ is a normal section of $M$.

Let $\alpha$ be a 2-dimensional section in $T_p \overline{M}$. Let us recall a section $\alpha$ is said to be

- **non-degenerate, weakly isotropic or strongly isotropic** if the rank of the restriction of the metric $g$ on $\alpha$ is 2, 1 or 0 respectively;
- **of pure or hybrid type** if the restriction of $g$ on $\alpha$ has a signature $(2, 0)$, $(0, 2)$ or $(1, 1)$ respectively;
- **holomorphic** if $\varphi \alpha = \alpha$;
- **$\xi$- section** if $\xi \in \alpha$;
- **totally real** if $\varphi \alpha \perp \alpha$.

Submanifolds $M$ of $\overline{M}$ of codimension 2 with a non-degenerate of hybrid type normal section $\alpha$ are studied. In two basic types of such submanifolds are considered: $\alpha$ is a holomorphic section and $\alpha$ is a $\xi$- section. In the normal section $\alpha = \{N_1, N_2\}$ is such that $\varphi N_1 \notin \alpha, \varphi N_2 \in \alpha$. In this paper we consider submanifolds $M$ of $\overline{M}$ of codimension 2 in the case when the normal section $\alpha$ is a non-degenerate of hybrid type and $\alpha$ is a totally real. The totally real sections $\alpha$ are two types: $\alpha$ is non-orthogonal to $\overline{\xi}$ and $\alpha$ is orthogonal to $\overline{\xi}$.

2. **Submanifolds of codimension 2 of almost contact manifolds with Norden metric with totally real non-orthogonal to $\overline{\xi}$ normal spaces**

Let $(\overline{M}, \varphi, \xi, \eta, g)$ $(\dim M = 2n + 3)$ be an almost contact manifold with Norden metric and let $M$ be a submanifold of codimension 2 of $\overline{M}$. We assume that there exists a normal section $\alpha = \{N_1, N_2\}$ defined globally over the submanifold $M$ such that
• $\alpha$ is a non-degenerate of hybrid type, i.e.

$$g(N_1, N_1) = -g(N_2, N_2) = 1, \quad g(N_1, N_2) = 0; \quad (2)$$

• $\alpha$ is a totally real, i.e.

$$g(N_1, \nabla N_1) = g(N_2, \nabla N_2) = g(N_1, \nabla N_2) = g(N_2, \nabla N_1) = 0; \quad (3)$$

• $\alpha$ is a non-orthogonal to $\xi \ (\xi / \in T_pM)$ and $\xi / \in \alpha$.

Then we obtain the following decomposition for $\xi, \varphi X, \varphi N_1, \varphi N_2$ with respect to $\{N_1, N_2\}$ and $T_pM$

$$\xi = \xi_0 + aN_1 + bN_2;$$
$$\varphi X = \varphi X + \eta_1^1(X)N_1 + \eta_2^2(X)N_2, \ X \in \chi(M);$$
$$\varphi N_1 = \xi_1;$$
$$\varphi N_2 = -\xi_2; \quad (4)$$

where $\varphi$ denotes a tensor field of type $(1, 1)$ on $M$; $\xi_0, \xi_1, \xi_2 \in \chi(M)$; $\eta_1^1$ and $\eta_2^2$ are 1-forms on $M$; $a, b$ are functions on $M$ such that $(a, b) \neq (0, 0)$. We denote the restriction of $g$ on $M$ by the same letter.

Let $a \neq 0$, $|a| > b$ and $a^2 - b^2 = k^2$. Taking into account the equalities (1) $\div$ (4) we compute

$$\eta_i^i(X) = g(X, \xi_i), \ (i = 0, 1, 2); \quad (5)$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta_0^0(X)\eta_0^0(Y) - \eta_1^1(X)\eta_1^1(Y) + \eta_2^2(X)\eta_2^2(Y); \quad (6)$$

$$\varphi^2 X = -X + \eta_0^0(X)\xi_0 - \eta_1^1(X)\xi_1 + \eta_2^2(X)\xi_2;$$
$$\eta_0^0(\varphi X) = -a\eta_0^0(X) + b\eta_0^0(X);$$
$$\eta_1^1(\varphi X) = a\eta_0^0(X); \quad \eta_2^2(\varphi X) = b\eta_0^0(X); \quad (7)$$

$$\varphi \xi_0 = -a\xi_1 + b\xi_2; \quad \varphi \xi_1 = a\xi_0; \quad \varphi \xi_2 = b\xi_0; \quad (8)$$

$$g(\xi_0, \xi_0) = 1 - a^2 + b^2; \quad g(\xi_1, \xi_1) = a^2 - 1;$$
$$g(\xi_2, \xi_2) = 1 + b^2; \quad g(\xi_0, \xi_1) = g(\xi_0, \xi_2) = 0;$$
$$g(\xi_1, \xi_2) = ab; \quad (9)$$

for arbitrary $X, Y \in \chi(M)$. 

Now we define a vector field $\xi$, an 1-form $\eta$ and a tensor field $\phi$ of type $(1,1)$ on $M$ by

$$\xi = -\frac{b}{k} \xi_1 + \frac{a}{k} \xi_2;$$

$$\eta(X) = -\frac{b}{k} \eta^1(X) + \frac{a}{k} \eta^2(X), \quad X \in \chi(M);$$

$$\phi X = \lambda \varphi^3 X + \mu \varphi X, \quad X \in \chi(M);$$

where

$$\lambda_1 = \frac{\epsilon}{k(k+1)}, \quad \mu_1 = \frac{\epsilon(1 + k^2 + k)}{k(k+1)};$$

$$\lambda_2 = \frac{\epsilon}{k(k-1)}, \quad \mu_2 = \frac{\epsilon(1 + k^2 - k)}{k(k-1)}; \quad \epsilon = \pm 1.$$

Further we consider the following cases for $k$:

1) $k^2 \neq 1 \iff k \neq \pm 1$. In this case $\phi, \xi, \eta$ are given by (10) and $\lambda = \lambda_1, \mu = \mu_1$ or $\lambda = \lambda_2, \mu = \mu_2$.

2) $k = -1$. We obtain $\phi, \xi, \eta$ from (10) by $k = -1$ and $\lambda = \lambda_2 = \frac{\epsilon}{2}, \mu = \mu_2 = \frac{3\epsilon}{2}$.

3) $k = 1$. We obtain $\phi, \xi, \eta$ from (10) by $k = 1$ and $\lambda = \lambda_1 = \frac{\epsilon}{2}, \mu = \mu_1 = \frac{3\epsilon}{2}$.

Using (5) ÷ (10) we verify that $(\phi, \xi, \eta)$ is an almost contact structure on $M$ and the restriction of $g$ on $M$ is Norden metric. Thus, the submanifolds $(M, \phi, \xi, \eta, g)$ of $\overline{M}$ considered in 1), 2), 3) are $(2n+1)$-dimensional almost contact manifolds with Norden metric.

Denoting by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of the metric $g$ in $\overline{M}$ and $M$ respectively, the formulas of Gauss and Weingarten are

$$\overline{\nabla}_X Y = \nabla_X Y + g(A_{N_1} X, Y) N_1 - g(A_{N_2} X, Y) N_2;$$

$$\overline{\nabla}_X N_1 = -A_{N_1} X + \gamma(X) N_2;$$

$$\overline{\nabla}_X N_2 = -A_{N_2} X + \gamma(X) N_1, \quad X, Y \in \chi(M);$$

where $A_{N_i}$ $(i = 1, 2)$ are the second fundamental tensors and $\gamma$ is an 1-form on $M$. 
3. Submanifolds of codimension 2 of almost contact manifolds with Norden metric with totally real orthogonal to $\xi$ normal spaces

Let $(\overline{M}, \varphi, \xi, \eta, g) \ (\dim \overline{M} = 2n + 3)$ be an almost contact manifold with Norden metric and let $\overline{M}$ be a submanifold of codimension 2 of $\overline{M}$. We assume that there exists a normal section $\alpha = \{N_1, N_2\}$ defined globally over the submanifold $\overline{M}$ such that

- $\alpha$ is a non-degenerate of hybrid type, i.e. the equality (2) holds;
- $\alpha$ is a totally real;
- $\alpha$ is orthogonal to $\xi$, i.e. $\xi \in T_p \overline{M}$.

Then from (4) by $a = b = 0$ we obtain the following decomposition with respect to $\{N_1, N_2\}$ and $T_p \overline{M}$

$$
\begin{align*}
\varphi &= \varphi_0; \\
\varphi X &= \varphi X + \eta^1(X)N_1 + \eta^2(X)N_2, \quad X \in \chi(\overline{M}); \\
\varphi N_1 &= \xi_1; \\
\varphi N_2 &= -\xi_2.
\end{align*}
$$

Substituting $a = b = 0$ in (7), (8), (9) we have

$$
\begin{align*}
\eta^0(\varphi X) &= \eta^1(\varphi X) = \eta^2(\varphi X) = 0; \\
\varphi \xi_0 &= \varphi \xi_1 = \varphi \xi_2 = 0; \\
g(\xi_0, \xi_0) &= g(\xi_2, \xi_2) = 1; \quad g(\xi_1, \xi_1) = -1; \\
g(\xi_0, \xi_1) &= g(\xi_0, \xi_2) = g(\xi_1, \xi_2) = 0.
\end{align*}
$$

Now we define a vector field $\xi$, an 1-form $\eta$ and a tensor field $\phi$ of type $(1, 1)$ on $\overline{M}$ by

$$
\begin{align*}
\xi &= t_0 \xi_0 - t_2 \xi_2; \\
\eta(X) &= t_0 \eta^0(X) - t_2 \eta^2(X), \quad X \in \chi(\overline{M}); \\
\phi X &= \varphi X + t_0 \{\eta^1(X) \xi_2 + \eta^2(X) \xi_1\} + \\
t_2 \{\eta^0(X) \xi_1 + \eta^1(X) \xi_0\};
\end{align*}
$$

where $t_0, t_2$ are functions on $\overline{M}$ and $t_0^2 + t_2^2 = 1$.

Using (5), (6), (13), (14) we verify that $(\phi, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and the restriction of $g$ on $M$ is Norden metric. So, the submanifolds $(M, \phi, \xi, \eta, g)$ of $\overline{M}$ are $(2n + 1)$-dimensional almost contact manifolds with Norden metric. The formulas of Gauss and Weingarten are the same as those in section 2.
4. Examples of submanifolds of codimension 2 of almost contact manifolds with Norden metric with totally real normal spaces

In a Lie group as a 5-dimensional almost contact manifold with Norden metric of the class \( F_9 \) is constructed. We will use this Lie group to obtain examples of submanifolds considered in sections 2 and 3.

First we recall some facts from which we need. Let \( g \) be a real Lie algebra with a global basis of left invariant vector fields \( \{X_1, X_2, X_3, X_4, X_5\} \) and \( G \) be the associated with \( g \) real connected Lie group. The almost contact structure \( (\varphi, \xi, \eta) \) and the Norden metric \( g \) on \( G \) are defined by:

\[
\begin{align*}
\varphi X_i &= X_{2+i}, \quad \varphi X_{2+i} = -X_i, \quad \varphi X_5 = 0, \quad (i = 1, 2); \\
g(X_i, X_i) &= -g(X_{2+i}, X_{2+i}) = g(X_5, X_5) = 1, \quad (i = 1, 2); \\
g(X_j, X_k) &= 0, \quad (j \neq k, \ j, k = 1, 2, 3, 4, 5); \\
\xi &= X_5, \quad \eta(X_i) = g(X_i, X_5), \quad (i = 1, 2, 3, 4, 5).
\end{align*}
\]

(15)

The commutators of the basis vector fields are given by:

\[
\begin{align*}
[X_1, X_2] &= -[X_1, X_3] = aX_4, \quad [X_2, X_3] = aX_2 + aX_3, \\
[X_3, X_4] &= -[X_2, X_4] = aX_1, \quad [X_2, \xi] = 2mX_1, \\
[X_3, \xi] &= -2mX_4, \quad [X_1, X_4] = [X_1, \xi] = [X_4, \xi] = 0, \\
a, m &\in \mathbb{R}.
\end{align*}
\]

(16)

So, the manifold \((G, \varphi, \xi, \eta, g)\) is an almost contact manifold with Norden metric in the class \( F_9 \).

Theorem 4.1. Let \( G \) be a Lie group with a Lie algebra \( g \) and \( \tilde{b} \) be a subalgebra of \( g \). There exists an unique connected Lie subgroup \( H \) of \( G \) such that the Lie algebra \( b \) of \( H \) coincides with \( \tilde{b} \).

From the equalities for the commutators of the basis vector fields \( \{X_1, X_2, X_3, X_4, X_5\} \) it follows that the 3-dimensional subspaces of \( g \) \( b_1 \) with a basis \( \{X_1, X_2, X_3\} \), \( b_2 \) with a basis \( \{X_1, X_3, X_4\} \) and \( b_3 \) with a basis \( \{X_1, X_4, \xi\} \) are closed under the bracket operation. Hence \( b_i \) \( (i = 1, 2, 3) \) are real subalgebras of \( g \). Taking into account Theorem 4.1 we have there exist Lie subgroups \( H_i \) \( (i = 1, 2, 3) \) of the Lie group \( G \) with Lie algebras \( b_i \) \( (i = 1, 2, 3) \) respectively. The normal spaces \( \alpha_i \) \( (i = 1, 2, 3) \) of the submanifolds \( H_i \) \( (i = 1, 2, 3) \) of \( G \) are: \( \alpha_1 = \{X_4, \xi\}, \alpha_2 = \{X_2, \xi\}, \alpha_3 = \{X_2, X_3\} \). Because of (15) we have \( \alpha_1 \) is \( \xi \)-section of hybrid type, \( \alpha_2 \) is \( \xi \)-section of pure type and \( \alpha_3 \) is a totally real orthogonal to \( \xi \)-section of hybrid type.
So, the submanifold $H_3$ of $G$ is of the same type submanifolds considered in section 3.

We choose the unit normal fields of $H_3 N_1 = X_2$ and $N_2 = X_3$. For an arbitrary $X \in \chi(H_3)$ we have $X = x^1X_1 + x^4X_4 + \overline{\eta}(X)\xi$. Taking into account (15) we compute

\[
\xi = \xi_0; \\
\varphi X = -x^4X_2 + x^1X_3; \\
\varphi X_2 = X_4; \\
\varphi X_3 = -X_1.
\]  

From (12), (17) it follows

\[
\eta^0(X) = \overline{\eta}(X); \\
\varphi X = 0; \quad \eta^1(X) = -x^4; \quad \eta^2(X) = x^1; \\
\xi_1 = X_4; \quad \xi_2 = X_1.
\]  

Substituting (18) in (14) for the almost contact structure on $H_3$ we obtain

\[
\xi = t_0\xi - t_2X_1; \\
\eta(X) = t_0\overline{\eta}(X) - t_2x^1; \\
\varphi X = t_0\{-x^4X_1 + x^1X_4\} + t_2\{\overline{\eta}(X)X_4 - x^4\xi\};
\]  

where $t_0$, $t_2 \in \mathbb{R}$ and $t_0^2 + t_2^2 = 1$.

Using the well known condition for the Levi-Civita connection $\nabla$ of $g$

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)
\]  

we get the following equation for the tensor $F$ of $H_3$

\[
F(X, Y, Z) = \frac{1}{2} \{g([X, \phi Y] - \phi[X, Y], Z) + g(\phi[Z, X] - [\phi Z, X], Y) + g([Z, \phi Y] - [\phi Z, Y], X)\}, \quad X, Y, Z \in \chi(H_3).
\]  

From (16) we have $[X_1, X_4] = [X_1, \xi] = [X_4, \xi] = 0$. Having in mind the last equalities, (19) and (21) for the tensor $F$ of $H_3$ we obtain $F = 0$. Thus, the submanifold $(H_3, \phi, \xi, \eta, g)$ of $G$, where $(\phi, \xi, \eta)$ is defined by (19) is an almost contact manifold with Norden metric in the class $\mathcal{F}_0$.

In order to construct an example for a submanifold from section 2 we
make the following change of the basis of $g$

\[
\begin{pmatrix}
    E_1 \\
    E_2 \\
    E_3 \\
    E_4 \\
    E_5
\end{pmatrix}
= \begin{pmatrix}
    X_1 \\
    X_2 \\
    \xi \\
    X_3 \\
    X_4
\end{pmatrix}, \quad T = \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\
    0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix} \in O(3, 2). \quad (22)
\]

Taking into account (16) and (22) we compute the commutators of the basis vector fields $\{E_1, E_2, E_3, E_4, E_5\}$ of $g$

\[
[E_1, E_2] = \frac{\sqrt{3}}{2} a E_5, \quad [E_1, E_3] = \frac{1}{2} a E_5, \quad [E_3, E_5] = -\frac{1}{2} a E_1,
\]

\[
[E_2, E_5] = -\frac{\sqrt{3}}{2} a E_1, \quad [E_1, E_4] = -a E_5,
\]

\[
[E_2, E_4] = \frac{3}{4} a E_2 + \frac{\sqrt{3}}{4} a E_3 + \frac{\sqrt{3}}{2} a E_4 - m E_5,
\]

\[
[E_3, E_4] = \frac{\sqrt{3}}{4} a E_2 + \frac{1}{4} a E_3 + \frac{1}{2} a E_4 + \sqrt{3} m E_5,
\]

\[
[E_4, E_5] = a E_1, \quad [E_2, E_3] = 2m E_1, \quad [E_1, E_3] = 0.
\]

Because of the elements of the matrix $T$ are constants the Jacobi identity for the vector fields $\{E_1, E_2, E_3, E_4, E_5\}$ is valid. Now, we compute the matrix $B$ of $\varphi$ and the coordinates of $\xi$ with respect to the basis $\{E_1, E_2, E_3, E_4, E_5\}$

\[
B = \begin{pmatrix}
    0 & 0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
    0 & 0 & 0 & 0 & -\frac{1}{2} \\
    1 & 0 & 0 & 0 & 0 \\
    0 & \frac{\sqrt{3}}{2} & 1 & 0 & 0
\end{pmatrix} ; \quad \xi = \begin{pmatrix}
    0 \\
    -\frac{1}{2} \\
    \frac{\sqrt{3}}{2} \\
    0 \\
    0
\end{pmatrix} . \quad (24)
\]

From $T \in O(3, 2)$ it follows the matrix of the metric $g$ with respect to the basis $\{E_1, E_2, E_3, E_4, E_5\}$ is the same as the matrix

\[
C = \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 0 & -1
\end{pmatrix} . \quad (25)
\]
of \( g \) with respect to the basis \( \{X_1, X_2, \xi, X_3, X_4\} \).

Using (23) we have that the 3-dimensional subspace \( b \) of \( g \) with a basis \( \{E_1, E_2, E_5\} \) is a subalgebra of \( g \). Let \( H \) be the Lie subgroup of \( G \) with a Lie algebra \( b \). Having in mind (24), (25) we obtain that the section \( \alpha = \{E_3, E_4\} \) is a normal to the submanifold \( H \), \( \alpha \) is a totally real non-orthogonal to \( \xi \) section of hybrid type and \( \xi \notin \alpha \), i.e. \( H \) is of the same type submanifolds considered in section 2. We have the following decomposition of \( \xi, \varphi X, \varphi E_3, \varphi E_4 \) with respect to \( \{E_3, E_4\} \) and \( T_pH \)

\[
\begin{align*}
\xi & = -\frac{1}{2}E_2 + \frac{\sqrt{3}}{2}E_3; \\
\varphi X & = -\frac{\sqrt{3}}{2}x^5E_2 + \frac{\sqrt{3}}{2}x^2E_5 - \frac{1}{2}x^5E_3 + x^1E_4; \\
\varphi E_3 & = \frac{1}{2}E_5; \\
\varphi E_4 & = -E_1;
\end{align*}
\]

(26)

where \( X \in \chi(H) \) and \( X = x^1E_1 + x^2E_2 + x^5E_5 \). We substitute

\[
a = \frac{\sqrt{3}}{2}, \quad b = 0, \quad k = \frac{\sqrt{3}}{2}, \quad \lambda = \frac{4\sqrt{3}(2 - \sqrt{3})}{3}, \quad \mu = \lambda + 1,
\]

\[
\xi_2 = E_1, \quad \eta^2(X) = x^1, \quad \varphi X = -\frac{\sqrt{3}}{2}x^5E_2 + \frac{\sqrt{3}}{2}x^2E_5
\]

in (10) and obtain an almost contact structure \((\phi, \xi, \eta)\)

\[
\begin{align*}
\xi & = E_1; \\
\eta^1(X) & = x^1; \\
\phi X & = \frac{2\sqrt{3}}{3} \varphi X = -x^5E_2 + x^2E_5;
\end{align*}
\]

(27)

on the submanifold \( H \).

Using (11), (16) and (20) we get

\[
A_{E_3}X = -m\overline{x}^2E_1 - m\overline{x}^5E_2; \quad A_{E_4}X = -\frac{1}{2} \left( \frac{3}{2} m\overline{x}^2 + m\overline{x}^5 \right) E_2 + \frac{1}{2} m\overline{x}^2E_5;
\]

\[
\gamma(X) = \frac{\sqrt{3}}{2} \left( m\overline{x}^2 - m\overline{x}^5 \right).
\]
Then the formulas of Gauss and Weingarten (11) become
\[ \nabla_X Y = \nabla_X Y - m (x^2 y^1 + x^1 y^2) E_3 + \frac{1}{2} \left( \left( \frac{3}{2} \alpha x^2 + m x^5 \right) y^2 + m x^2 y^5 \right) E_4; \]
\[ \nabla_X E_3 = m x^2 E_1 + m x^1 E_2 + \frac{\sqrt{3}}{2} (\alpha x^2 - m x^5) E_4; \]
\[ \nabla_X E_4 = \frac{1}{2} \left( \frac{3}{2} \alpha x^2 + m x^5 \right) E_2 - \frac{1}{2} m x^2 E_5 + \frac{\sqrt{3}}{2} (\alpha x^2 - m x^5) E_3. \]

Having in mind the last formulas, (26) and (27) we compute the tensor \( F \) of \( H \)
\[ F(X, Y, Z) = -\frac{\sqrt{3}}{2} (\alpha x^2 (y^1 z^2 + y^2 z^1)), \quad X, Y, Z \in \chi(H) \]
and verify that the submanifold \((H, \phi, \xi, \eta, g)\) of \( G \), where \((\phi, \xi, \eta)\) is defined by (27) is an almost contact manifold with Norden metric in the class \( \mathcal{F}_4 \oplus \mathcal{F}_8 \).

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