REMARKS ON ESSENTIAL CODIMENSION

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Abstract. We look for generalizations of the Brown–Douglas–Fillmore essential codimension result, leading to interesting local uniqueness theorems in KK theory. We also study the structure of Paschke dual algebras.

1. Introduction

The notion of essential codimension was introduced by Brown–Douglas–Fillmore (BDF) in their groundbreaking paper [5] where they classified all essentially normal operators using Fredholm indices. Since then, this notion has had manifold applications (e.g., [1, 7]). This includes, among other things, an explanation for the mysterious integers appearing in Kadison’s Pythagorean theorem ([14, 15, 16, 22]) as well as other Schur–Horn type results ([4, 12]).

Here is the BDF definition of essential codimension:

Definition 1.1. (BDF) Let $P, Q \in B(l^2)$ be projections such that $P - Q \in K$. The essential codimension of $P$ and $Q$ is given by

$$[P : Q] = \begin{cases} Tr(P) - Tr(Q) & \text{if } Tr(P) + Tr(Q) < \infty \\ Ind(V^*W) & \text{if } Tr(P) = Tr(Q) = \infty, \text{ where } V^*V = W^*W = 1, \\ WW^* = P, \ VV^* = Q. & \\
\end{cases}$$

In the above, “Ind” means Fredholm index.

It is not hard to show that, if $Q \leq P$, then essential codimension reduces to the usual codimension. Basic properties of essential codimension and their proofs can be found in [6]. We note that, given that $P - Q \in K$, the essential codimension essentially measures “local differences”.

A fundamental result on essential codimension which was stated in [5] (a proof can be found in [6]) is the following:

Theorem 1.2. Let $P, Q \in B(l^2)$ be projections such that $P - Q \in K$.

Then there exists a unitary $U \in C1 + K$ such that $UPU^* = Q$ if and only if $[P : Q] = 0$.

The main goal of this paper is to find generalizations of this result. We are following the path first travelled on by [6, 19, and 20] (see also, [21] and [8]). Lee ([19]) observed that essential codimension is a basic example of $KK^0$, and thus the BDF essential codimension result (Theorem 1.2) is connected to powerful

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uniqueness theorems, and our goal is to work out some of the operator theoretic consequences.

In Section 2 we undertake a study of the Paschke dual algebra $A_d^B$ of $A$ relative to $B$ in the context of when $A$ is a unital separable nuclear C*-algebra and $B$ is a separable stable C*-algebra. In this setting we prove a number of results. We first establish that the Paschke dual algebra is $K_1$-injective (Theorem 2.5 and Theorem 2.9) under certain restrictions on the canonical ideal, which is essential for proving our theorems in Section 3. We note that the Paschke dual algebra is a unital properly infinite C*-algebra, and it is an open problem whether every properly infinite unital C*-algebra is $K_1$ injective$^1$. We then prove that the Paschke dual algebra is dual in the sense that $A$ and $A_d^B$ are each other’s relative commutants in the corona algebra $C(B)$, where $A$ is identified with its image under the Busby map (Theorem 2.10). This generalizes a remark of Valette (26). The key technique throughout this section is the Elliott–Kucerovsky theory of absorbing extensions [9].

In Section 3 we prove a few theorems (Theorems 3.3 and 3.4) which can be considered as generalizations of BDF’s Theorem 1.2 to the realm of $KK$-theory where the essential codimension is interpreted as an element of $KK_0$, and the unitary which is a compact perturbation of the identity is replaced by the notion of proper asymptotic unitary equivalence due to Dadarlat and Eilers [8]. In order to make this abstract notion of essential codimension more concrete, we simply take $A = C$ and, with a few modest hypotheses, arrive at a generalization of Theorem 1.2 that bears true resemblance to it (see Theorem 3.6).

In Section 4, we prove a technical lemma which is used in one of the main results in a previous section.

In a separate paper$^2$ we study the connection between essential codimension and projection lifting.

2. The Paschke dual algebra

For a nonunital C*-algebra $B$, $M(B)$ and $C(B)$ denote the multiplier and corona algebras of $B$ respectively. $\pi : M(B) \to C(B)$ denotes the natural quotient map.

Definition 2.1. Let $A$ be a unital separable C*-algebra, let $B$ be a separable stable C*-algebra, and let $\phi : A \to M(B)$ be a unital absorbing trivial extension. The Paschke dual algebra of $A$ relative to $B$ is defined to be $A_d^B = df(\pi \circ \phi(A))' \in C(B)$. Sometimes, to emphasize the map $\phi$, we will use the notation $D_\phi = df A_d^B$.

We note that $A_d^B$ is, up to *-isomorphism, independent of $\phi$. However, the map $\phi$ is quite important, and in many treatments of Paschke duality, one has “$\phi$” in the notation. Hence, we also use the alternate notation “$D_\phi$”. There is also a definition for nonunital $A$, but we focus on the unital case where the definition is simpler (essentially Paschke’s and Valette’s original definition). We so name the Paschke dual algebra because of Paschke duality, which asserts the existence of group isomorphisms $K_j(A_d^B) \cong KK^{j+1}(A, B)$ for $j = 0, 1$. (See 11, 21, 25, 26.) We will show below (Theorem 2.10) that the Paschke dual algebra is also dual in another sense, thus generalizing a remark of Valette (26).

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$^1$See, for example, [9].
$^2$J. Loreaux and P. W. Ng, Remarks on essential codimension: Lifting projections. Preprint.
Paschke ([24]) focused on the case where $B = K$. However, many of his assertions and arguments remain true in general. Sometimes the modifications are straightforward and other times they are quite nontrivial.

We fix a notation from extension theory. Let $A, B$ be C*-algebras with $B$ nonunital, and let $\phi, \psi : A \to C(B)$ be *-homomorphisms. We say that $\phi$ and $\psi$ are unitarily equivalent and write

$$\phi \sim \psi$$

(2.1)

if there exists a unitary $u \in M(B)$ such that

$$\pi(u)\phi(a)\pi(u)^* = \psi(a)$$

for all $a \in A$.

The argument of the first result is very similar to that of [24] Lemma 1, but every occurrence of Voiculescu’s noncommutative Weyl–von Neumann theorem ([27]) is replaced with the Elliott–Kucerovsky theory of absorbing extensions ([9]). We go through the proof for the convenience of the reader, expanding some details.

Lemma 2.2. Let $A$ be a unital separable nuclear C*-algebra, and let $B$ be a separable stable C*-algebra.

Then we have the following:

(a) The unit of $A_B^d$ is properly infinite. In fact, $1 \sim 1 \oplus 1$ in $M_2 \otimes A_B^d$.

(b) The stable equivalence classes of projections in $A_B^d$ constitute all of $K_0(A_B^d)$.

Proof. (a): Let $\phi : A \to M(B)$ be a unital trivial absorbing extension. Hence, we may identify $A = \pi \circ \phi(A) \subset C(B)$, and we may thus view $A$ as a unital C*-subalgebra of $C(B)$. And by [9], the inclusion map $\iota : A \hookrightarrow C(B)$ is a unital trivial absorbing extension. (For triviality, note that the map $\phi(A) \to M(B) : \pi \circ \phi(a) \mapsto \phi(a)$ is a *-homomorphism, and note that we are identifying $A = \pi \circ \phi(A)$.)

We may also identify $A_B^d = (\pi \circ \phi(A))' \subseteq C(B)$.

Since $\iota$ is trivial and absorbing

$$\iota \oplus \iota \sim \iota.$$

Therefore, there exists an isometry $\tilde{v} \in M_2 \otimes M(B)$ such that

$$v(\iota \oplus \iota)v^* = \iota \oplus 0$$

where $v =_{df} \pi(\tilde{v})$. In particular, we have that

$$v(x \oplus x)v^* = x \oplus 0$$

for all $x \in A$. Hence, since $A$ is unital,

(2.2) \hspace{1cm} v^*v = 1 \oplus 1 \text{ and } vv^* = 1 \oplus 0.$$

From the above, we have that for all $x \in A$,

$$v(x \oplus x) = (x \oplus 0)v$$

$$= (x \oplus x)v \text{ (since } vv^* = 1 \oplus 0).$$

Hence, $v \in M_2 \otimes A_B^d$. From this and (2.2), the unit of $A_B^d$ is Murray–von Neumann equivalent to two copies of itself.

(b): This follows immediately from (a). \qed
We note that it is an open problem whether every unital properly infinite C*-algebra is $K_1$ injective [3], and the Paschke dual algebra is an interesting and important case of this. We now move towards proving $K_1$ injectivity under additional hypotheses.

The next lemma ensures that under appropriate conditions, given any unitary $u$ in the commutant of $\mathcal{A}$ (relative to some larger unital algebra), and given a unital trivial absorbing extension, the image of $u$ in the Paschke dual of $\mathcal{A}$ lies in the connected component of the identity in the unitary group.

**Lemma 2.3.** Let $\mathcal{C}$ be a unital C*-algebra and $\mathcal{A} \subseteq \mathcal{C}$ a separable nuclear unital C*-subalgebra. Say that $u \in \mathcal{A}' (\subseteq \mathcal{C})$ is a unitary. Let $\mathcal{B}$ be a separable simple stable C*-algebra. Let $\phi : C^*(\mathcal{A}, u) \to \mathcal{M}(\mathcal{B})$ be a unital trivial absorbing extension.

Then there exists a norm-continuous path of unitaries $\{v_t\}_{t \in [0, 1]}$ in $(\pi \circ \phi(\mathcal{A}))'$ such that $v_0 = \pi \circ \phi(u)$ and $v_1 = 1$.

**Proof.** Since $\mathcal{B}$ is stable, we may work with $\mathcal{B} \otimes \mathcal{K}$ instead of $\mathcal{B}$.

By the universal property of the maximal tensor product, $C^*(\mathcal{A}, u)$ is a quotient of $\mathcal{A} \otimes_{\max} C(S^1)$, which is nuclear since $\mathcal{A}$ and $C(S^1)$ are nuclear. Hence, $C^*(\mathcal{A}, u)$ is a nuclear C*-algebra.

Since $C^*(\mathcal{A}, u)$ is separable, let $\{\sigma_n\}_{n=1}^\infty$ be a dense sequence in $C^*(\mathcal{A}, u)$ (the space of irreducible *-representations of $C^*(\mathcal{A}, u)$) such that every term in $\{\sigma_n\}$ reoccurs infinitely many times. Let $\sigma' : C^*(\mathcal{A}, u) \to \mathcal{B}(l_2)$ be the unital essential *-representation given by

$$\sigma' = \bigoplus_{n=1}^\infty \sigma_n.$$ 

Then by [17] Theorem 6 (see also [2] Theorem 15.12.4 and [9] Theorem 17), the map

$$\sigma : C^*(\mathcal{A}, u) \to \mathcal{M}(\mathcal{B} \otimes \mathcal{K}) : x \mapsto 1_{\mathcal{M}(\mathcal{B})} \otimes \sigma'(x)$$

is a unital trivial absorbing extension. Hence, since $\phi$ is also a unital trivial absorbing extension, there exists a unitary $w \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K})$ such that

$$\phi(x) - w\sigma(x)w^* \in \mathcal{B} \otimes \mathcal{K}$$

for all $x \in C^*(\mathcal{A}, u)$.

Note that for all $n$, since $\sigma_n$ is an irreducible *-representation of $C^*(\mathcal{A}, u)$, and since $u$ commutes with every element of $C^*(\mathcal{A}, u)$, $\sigma_n(u) \in S^1$. So let $\theta_n \in [0, 2\pi)$ such that $\sigma_n(u) = e^{i\theta_n}$.

Now for all $t \in [0, 1]$, let

$$v_t' = d_f w(1_{\mathcal{M}(\mathcal{B})} \otimes \bigoplus_{n=1}^\infty e^{i(1-t)\theta_n})w^*.$$ 

And let

$$v_t = d_f \pi(v_t').$$

Then $\{v_t'\}_{t \in [0, 1]}$ is a norm continuous path of unitaries in $w\sigma(\mathcal{A})'w^*$ ($\subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K})$), and so $\{v_t\}_{t \in [0, 1]}$ is a norm continuous path of unitaries such that

$$v_0 = \pi \circ \phi(u), \ v_1 = 1$$

and $v_t \in (\pi \circ \phi(\mathcal{A}))'$ for all $t \in [0, 1]$. \qed
Recall that for a unital C*-algebra \( D \), \( U(D) \) denotes the unitary group of \( D \), and \( U(D)_0 \) denotes the elements of \( U(D) \) that are in the connected component of the identity.

We first focus on the case where the canonical ideal is either \( K \) or simple purely infinite. It is well-known that this is exactly the case with “nicest” extension theory, since, among other things, a BDF–Voiculescu type absorption result holds. In fact, in this context, under a nuclearity hypothesis, Kasparov’s \( KK^1 \) classifies all essential extensions.

The next result generalizes [24] Lemma 3(2).

**Lemma 2.4.** Let \( A \) be a unital separable nuclear C*-algebra, and \( B \) a separable stable simple C*-algebra such that either \( B \cong K \) or \( B \) is purely infinite. Then the map

\[
U(A_d) / U(A_d)_0 \to U(M_2 \otimes A_d) / U(M_2 \otimes A_d)_0
\]

given by

\[
[u] \to [u \oplus 1]
\]

is injective.

**Proof.** Let \( \phi : A \to \mathcal{M}(B) \) be a unital trivial absorbing extension. We may identify

\[
A_d = (\pi \circ \phi(A))'.
\]

Let \( u \in A_d \) be a unitary such that

\[
u \oplus 1 \sim h \oplus 1
\]

in \( M_2 \otimes A_d \).

Let \( \sigma : C^*(\pi \circ \phi(A), u) \to \mathcal{M}(B) \) be a unital trivial absorbing extension.

Since \( \sigma|_{\pi \circ \phi(A)} \) is a unital trivial absorbing extension, conjugating \( \sigma \) by an appropriate unitary if necessary, we may assume that \( \pi \circ \sigma(x) = x \) for all \( x \in \pi \circ \phi(A) \).

(After all, by [9], the map \( \pi \circ \phi(A) \to \mathcal{M}(B) : \pi \circ \phi(a) \to \phi(a) \) is also a unital trivial absorbing extension.)

By Lemma [23] we have that

\[
\pi \circ \sigma(u) \sim h
\]

in \( (\pi \circ \phi(A))' = (\pi \circ \phi(A))' = A_d \).

Since either \( B \cong K \) or \( B \) is simple purely infinite, it follows, by [9] Theorem 17, that the inclusion map \( \iota : C^*(\pi \circ \phi(A), u) \to C(B) \) is a unital trivial absorbing extension. Hence,

\[
\iota \oplus (\pi \circ \sigma) \sim \iota.
\]

Hence, there exists an isometry \( W \in M_2 \otimes \mathcal{M}(B) \) such that \( W^*W = 1 \oplus 1 = 1_{M_2 \otimes \mathcal{M}(B)} \), \( WW^* = 1 \oplus 0 \) and if \( w = df \pi(W) \), then

\[
w(\iota \oplus (\pi \circ \sigma))w^* = \iota \oplus 0.
\]

As a consequence, we have that

\[
w(u \oplus (\pi \circ \sigma(u)))w^* = u \oplus 0,
\]

and

\[
w(x \oplus x)w^* = x \oplus 0
\]

for all \( x \in \pi \circ \sigma(A) \).
Note that by (2.5), for all \( x \in \pi \circ \sigma(A) \),
\[
w(x \oplus x) = (x \oplus 0)w = (x \oplus x)w \text{ (since } ww^* = 1 \oplus 0 )\]

Hence,
\[w \in M_2 \otimes A_{B}^d.\]

Now by (2.3),
\[u \oplus (\pi \circ \sigma(u)) \sim_h u \oplus 1\]
in \( M_2 \otimes A_{B}^d \). Also, by the hypothesis on \( u \),
\[u \oplus 1 \sim_h 1 \oplus 1\]
in \( M_2 \otimes A_{B}^d \). So
\[u \oplus (\pi \circ \sigma(u)) \sim_h 1 \oplus 1\]
in \( M_2 \otimes A_{B}^d \). Conjugating the continuous path of unitaries by \( w \) and applying (2.4),
we have that
\[u \sim_h 1\]
in \( A_{B}^d \).

\[\square\]

**Theorem 2.5.** Let \( A \) be a unital separable nuclear \( C^* \)-algebra and \( B \) a separable simple stable \( C^* \)-algebra such that either \( B \cong K \) or \( B \) is purely infinite. Then \( A_{B}^d \) is \( K_1 \)-injective. Moreover, for all \( n \geq 1 \), the map
\[U(M_n \otimes A_{B}^d)/U(M_n \otimes A_{B}^d)_0 \to U(M_2n \otimes A_{B}^d)/U(M_2n \otimes A_{B}^d)_0\]
given by
\[[u] \mapsto [u \oplus 1]\]
is injective.

**Proof.** By Lemma 2.2, we have that the unit of the Paschke algebra \( A_{B}^d \) satisfies
\[1 \oplus 1 \sim 1.\] Hence, for all \( n \), \( A_{B}^d \cong M_n \otimes A_{B}^d \). Thus, the result follows from Lemma 2.4. \[\square\]

We now move towards understanding \( K_1 \) injectivity of the Paschke dual algebra, when the canonical ideal is no longer elementary nor simple purely infinite. Outside of these small number of cases, our knowledge of extension theory is highly incomplete and the questions that arise are much more challenging.

Let \( D \) be a \( C^* \)-algebra and \( C \subseteq D \) a \( C^* \)-subalgebra. We say that \( C \) is **strongly full** in \( D \) if every nonzero element of \( C \) is full in \( D \). For every nonzero \( x \in D \), we say that \( x \) is **strongly full** in \( D \) if \( C^*(x) \) is a strongly full \( C^* \)-subalgebra of \( D \).

**Lemma 2.6.** Let \( D \) be a unital \( C^* \)-algebra and \( A \subseteq D \) a unital simple \( C^* \)-subalgebra. Suppose that \( u \in A' \) is a strongly full unitary element of \( D \).

Then \( C^*(u, A) \) is strongly full in \( D \).

**Proof.** It suffices to prove that every nonzero positive element of \( C^*(u, A) \) is full in \( D \).

Let \( c \in C^*(u, A) \) be a nonzero positive element. Hence, there exists a continuous function \( g : S^1 \to [0, 1] \), and an element \( a \in A_+ \) such that \( g(u)a \neq 0 \) and \( 0 \leq g(u)a \leq c \).
Since $A$ is unital and simple, let $x_1, x_2, \ldots, x_n \in A$ be such that
\[
\sum_{j=1}^{n} x_j ax_j^* = 1.
\]
Hence,
\[
\sum_{j=1}^{n} x_j g(u)ax_j^* = \sum_{n=1}^{n} g(u)x_j ax_j^* = g(u).
\]
Since $g(u)$ is a full element of $D$, it follows that $g(u)a$ is a full element of $D$. Hence, $c$ is a full element of $D$. Since $c$ was arbitrary, $C^*(u, A)$ is a strongly full C*-subalgebra of $D$. □

Recall that a separable stable C*-algebra $B$ is said to have the corona factorization property (CFP) if every norm-full projection in $\mathcal{M}(B)$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(B)}$ [18].

Many C*-algebras have the CFP. For example, all separable simple C*-algebras that are either purely infinite or have strict comparison of positive elements, including all simple C*-algebras classified in the Elliott program, have the CFP. In fact, it is quite difficult to construct a simple separable C*-algebra without CFP.

Recall also, that a map $\phi : A \to C$ between C*-algebras is said to be norm full or full if for every $a \in A - \{0\}$, $\phi(a)$ is a full element of $C$, i.e., Ideal($\phi(a)$) $= C$.

We say that a *-homomorphism $\phi : A \to \mathcal{M}(B)$ absorbs 0 if $\pi \circ \phi \oplus 0 = \pi \circ \phi$.

In [18], the following result was proven:

**Theorem 2.7.** Let $B$ be a separable stable C*-algebra with the CFP, $A$ a separable C*-algebra, and $\phi : A \to \mathcal{M}(B)/B$ an essential extension such that either $\phi$ is unital or $\phi$ absorbs 0.

Then $\phi$ is nuclearly absorbing if and only if $\phi$ is norm-full.

As a consequence, if, in addition, $A$ is nuclear, then $\phi$ is absorbing if and only if $\phi$ is norm-full.

In the above, when $\phi(1) = 1$ and we say that $\phi$ is absorbing, we mean that $\phi$ is absorbing in the unital sense.

Let $B$ be a nonunital separable stable simple C*-algebra with a nonzero projection $e \in B$. We let $T_e(B)$ denote the set of all tracial states on $eB_e$. It is well known that $T_e(B)$, with the weak* topology, is a Choquet simplex. Moreover, it is also well known that $B \cong eB_e \otimes \mathcal{K}$ and that every $\tau \in T_e(B)$ extends to a trace (which can take the value $\infty$) on $\mathcal{M}(B)_+$. If $e' \in B$ is another nonzero projection, then $T_e(B)$ and $T_{e'}(B)$ are homeomorphic, and $T_e(B)$ has finitely many extreme points if and only if $T_{e'}(B)$ has finitely many extreme points. Our results will be independent of the choice of nonzero projection in $B$, and hence, we will write $T(B)$ to mean $T_e(B)$ for some $e \in \text{Proj}(B) - \{0\}$.

Recall that for all $a \in B_+$ and for all $\tau \in T(B)$,
\[
d_{\tau}(a) = \lim_{n \to \infty} \tau(a^{1/n}).
\]

Recall that $B$ is said to have strict comparison for positive elements if for all $a, b \in B_+$,
\[
d_{\tau}(a) < d_{\tau}(b) \text{ whenever } d_{\tau}(b) < \infty \forall \tau \in T(B) \text{ if and only if } a \preceq b.
\]

In the above, $a \preceq b$ means that there exists $\{x_k\}$ in $B$ such that $x_kbx_k^* \to a$. 
In the next proof, we use a key technical lemma, Lemma 4.4, whose proof we provide in the later Section 4.

**Lemma 2.8.** Let $A$ be a unital separable simple nuclear C*-algebra, and $B$ a separable stable simple C*-algebra with a nonzero projection, strict comparison of positive elements and for which $T(B)$ has finitely many extreme points.

Suppose that there exists a *-embedding $A \hookrightarrow B$.

Then the map

$$U(A^d_B)/U(A^d_B)_0 \to U(M_2 \otimes A^d_B)/U(M_2 \otimes A^d_B)_0$$

given by

$$[u] \mapsto [u \oplus 1]$$

is injective.

**Proof.** By the hypotheses, there exist a sequence $\{p_n\}_{n=1}^\infty$ of pairwise orthogonal projections in $B$, a sequence $\{\phi_n\}_{n=1}^\infty$ of *-embeddings from $A$ to $B$, and a sequence $\{v_{n,1}\}_{n=1}^\infty$ of partial isometries in $B$ such that the following statements are true:

1. $p_m \sim p_n$ for all $m,n$.
   
   In fact, $v_{n,1}^*v_{n,1} = p_1$ and $v_{n,1}v_{n,1}^* = p_n$ for all $n$.

2. $\sum_{n=1}^\infty p_n = 1_{\mathcal{M}(B)}$, where the sum converges strictly.

3. $\phi_n(1) = p_n$ for all $n$.

4. $v_{n,1}\phi_n(x)v_{n,1}^* = \phi_n(x)$, for all $x \in A$ and for all $n$.

Let $\phi : A \to \mathcal{M}(B)$ be the unital *-homomorphism given by

$$\phi = \sum_{n=1}^\infty \phi_n.$$

Then by [21] (see also [9] Theorem 17), $\pi \circ \phi$ is a unital trivial absorbing extension. (In the literature, $\phi$ is often called the “Lin extension”.)

We may identify $A^d_B = (\pi \circ \phi(A))'$.

Let $u \in A^d_B$ be a unitary such that

$$u \oplus 1 \sim_h 1 \oplus 1$$

in $M_2 \otimes A^d_B$.

By Lemma 4.4 there exists a unitary $v \in A^d_B$ such that

$$u \sim_h v$$

in $A^d_B$, and $v$ is strongly full in $C(B)$. Hence, we may assume that $u$ is a strongly full element of $C(B)$. Hence, by Lemma 2.6 $C^*(u, \pi \circ \phi(A))$ is a strongly full unital C*-subalgebra of $C(B)$.

Hence, by Theorem 2.7 the inclusion map

$$\iota : C^*(u, \pi \circ \phi(A)) \to C(B)$$

is a unital absorbing extension.

The rest of the proof is exactly the same as that of Lemma 4.4.

**Theorem 2.9.** Let $A$ be a unital separable simple nuclear C*-algebra, and $B$ a separable stable simple C*-algebra with a nonzero projection, strict comparison of positive elements, and for which $T(B)$ has finitely many extreme points.

Then $A^d_B$ is $K_1$-injective. Moreover, for all $n \geq 1$, the map

$$U(M_n \otimes A^d_B)/U(M_n \otimes A^d_B)_0 \to U(M_{2n} \otimes A^d_B)/U(M_{2n} \otimes A^d_B)_0$$

is injective.

□
given by

\[ [u] \mapsto [u \oplus 1] \]

is injective.

**Proof.** The proof is exactly the same as that of Theorem 2.5, except that Lemma 2.4 is replaced with Lemma 2.8. □

We fix a terminology that will only be used in the next theorem. Let \( A \) be a unital separable nuclear C*-algebra, and let \( B \) be a separable stable C*-algebra. Let \( \phi : A \to M(B) \) be a unital trivial absorbing extension. Recall that we can identify \( A = (\pi \circ \phi(A))' \subseteq C(B) \). Since \( \pi \circ \phi \) is injective, we may identify \( A \) with \( \pi \circ \phi(A) \). When \( A \) and \( A \oplus B \) sit in \( C(B) \) in the above manner, we say that \( A \) and \( A \oplus B \) are in standard position in \( C(B) \).

**Theorem 2.10.** Let \( A \) be a separable simple unital nuclear C*-algebra, and let \( B \) be a separable stable simple C*-algebra. Suppose that \( A \) and \( A \oplus B \) are in standard position in \( C(B) \). Then

\[
A' = A \oplus B \text{ and } (A \oplus B)' = A.
\]

**Proof.** The first equality follows trivially from the definition of \( A \oplus B \).

The proof of the second equality is exactly the same as that of 23 Theorem 1. We note that, in our context, the inclusion map \( \iota : A \to C(B) \) is a unital trivial absorbing extension. Hence, the hypothesis, that \( [\iota] \in T \) (notation as in 23 Theorem 1) in 23 Theorem 1 is satisfied. Also, since \( \iota \) is absorbing, the hypothesis that \( B \) satisfies the CFP in 23 Theorem 1 is unnecessary. □

Thus, the Paschke dual algebra is “dual” in still another sense.

3. Essential codimension

In what follows, we will let \( KK \) denote the generalized homomorphism picture of KK theory (see, for example, 13 Chapter 4).

In 19, Lee observed that the BDF notion of essential codimension (Definition 1.1) is a special case of an element of \( KK^0 \). He thus gave the following definition:

**Definition 3.1.** Let \( B \) be a separable stable C*-algebra, and let \( P, Q \in M(B) \) be projections such that \( P - Q \in B \). Let \( \phi, \psi : C \to M(B) \) be *-homomorphisms for which \( \phi(1) = P \) and \( \psi(1) = Q \). The essential codimension of \( P \) and \( Q \) is given by

\[
[P : Q] = \{[\phi, \psi] \in KK(C, B) \mid K_0(B)\}.
\]

Here, \([\phi, \psi]\) is the class of the generalized homomorphism \((\phi, \psi)\) in \( KK(C, B) \).

It is not hard to see (e.g., 20 Remark 2.2) that in the case where \( B = K \), Definition 3.1 coincides with the original BDF essential codimension (Definition 1.1). Thus, \( KK^0 \) concerns the local aspects of operator theory, as opposed to \( KK^1 \) which deals with the asymptotic aspects (e.g., classifying essentially normal operators up to unitary equivalence modulo the compacts).

Towards generalizing the BDF essential codimension result (Theorem 1.2), we recall the notion of proper asymptotic unitary equivalence (see 8).
Definition 3.2. Let \( \mathcal{A}, \mathcal{B} \) be \( \mathcal{C}^* \)-algebras, with \( \mathcal{B} \) nonunital. Let \( \phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) be two \( \ast \)-homomorphisms.

(1) \( \phi \) and \( \psi \) are said to be asymptotically unitarily equivalent (\( \phi \sim_{\text{asymp}} \psi \)) if there exists a (norm-) continuous path \( \{ u_t \}_{t \in [0, \infty)} \) of unitaries in \( \mathcal{M}(\mathcal{B}) \) such that for all \( a \in \mathcal{A} \),
   i. \( \phi(a) - u_t \psi(a) u_t^* \in \mathcal{B} \), for all \( t \), and
   ii. \( \| \phi(a) - u_t \psi(a) u_t^* \| \to 0 \) as \( t \to \infty \).

(2) \( \phi \) and \( \psi \) are said to be properly asymptotically unitarily equivalent (\( \phi \approx \psi \)) if \( \phi \) and \( \psi \) are asymptotically unitarily equivalent where the path of unitaries satisfy that \( u_t \in \mathcal{C}1 + \mathcal{B} \) for all \( t \).

We note that proper asymptotic unitary equivalence is a local notion. This is in fitting with the BDF essential codimension theorem.

In [8], the following generalization of Theorem 1.2 was given: Let \( \mathcal{A}, \mathcal{B} \) be separable \( \mathcal{C}^* \)-algebras with \( \mathcal{B} \) stable, and let \( \phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) be \( \ast \)-homomorphisms such that \( \phi(a) - \psi(a) \in \mathcal{B} \) for all \( a \in \mathcal{A} \). Then \( [\phi, \psi] = 0 \) in \( \text{KK}(\mathcal{A}, \mathcal{B}) \) if and only if there exists a \( \ast \)-homomorphism \( \sigma : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) such that \( \phi \oplus \sigma \approx \psi \oplus \sigma \).

We note that [8] was inspired by and extensively used ideas from the earlier stable uniqueness paper [21]. We also note that results of the above type can be used to produce (unbounded) stable uniqueness theorems. This idea is essentially due to Lin [21].

We now introduce and prove our generalization of Theorem 1.2. The proof essentially follows that of [19] Theorem 2.11 which follows that of [8] Theorem 3.12. As noted above, [8] used extensively the ideas of [21]. In fact, the argument is essentially that of [21]: A proper asymptotic unitary equivalence induces a continuous path of automorphisms on \( \phi(\mathcal{A}) + \mathcal{B} \). Then, following [21], we prove innerness of the automorphisms. We sketch the proof for the convenience of the reader.

Recall that \( \text{KK} \) denotes the generalized homomorphism picture of KK theory (e.g., see [13] Chapter 4). In the next proof, we will let \( \text{KK}_{\text{Higson}} \) denote Higson’s definition of KK theory (e.g., see [10] Section 2).

Recall that a trivial extension \( \phi \) is said to absorb the zero extension if \( \pi \circ \phi \oplus 0 \sim \pi \circ \phi \).

Theorem 3.3. Let \( \mathcal{A}, \mathcal{B} \) be separable \( \mathcal{C}^* \)-algebras with \( \mathcal{A} \) nuclear and \( \mathcal{B} \) stable and simple purely infinite. Let \( \phi, \psi : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) be essential extensions such that \( \phi(a) - \psi(a) \in \mathcal{B} \) for all \( a \in \mathcal{A} \).

Suppose also that either both \( \phi \) and \( \psi \) are unital, or both \( \phi \) and \( \psi \) absorb the zero extension.

Then \( [\phi, \psi] = 0 \) in \( \text{KK}(\mathcal{A}, \mathcal{B}) \) if and only if \( \phi \approx \psi \).

Proof. The “if” direction is trivial.

We now prove the “only if” direction. Note that by [9] Theorem 17, both \( \phi \) and \( \psi \) are absorbing extensions.

Let \( \tilde{\mathcal{A}} \) denote the unitization of \( \mathcal{A} \) if \( \mathcal{A} \) is nonunital, and \( \mathcal{A} \oplus \mathcal{C} \) if \( \mathcal{A} \) is unital. By [9], if \( \phi : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) is an absorbing extension, then the map \( \tilde{\phi} : \tilde{\mathcal{A}} \to \mathcal{M}(\mathcal{B}) \) given by \( \tilde{\phi}|{\mathcal{A}} = \phi \) and \( \tilde{\phi}(1) = 1 \) is a unital absorbing extension. Thus, we may assume that \( \mathcal{A} \) is unital and \( \phi \) and \( \psi \) are unital absorbing trivial extensions.

\[3\] Of course, when both are unital, we mean that they are unitally absorbing.
As in the previous section, we may identify the Paschke dual algebra $A_B^d = (\pi \circ \phi(A))' \in C(B)$.

By [19] Theorem 2.5, $\phi \sim_{a\text{sym}} \psi$. I.e., there exists a norm continuous path 
\[ \{u_t\}_{t \in [0, \infty)} \] of unitaries in $\mathcal{M}(B)$ such that 
\[ u_t \phi(a) u_t^* - \psi(a) \in B \]
for all $t$ and for all $a \in A$, and 
\[ \|u_t \phi(a) u_t^* - \psi(a)\| \to 0 \]
as $t \to \infty$, for all $a \in A$.

It is trivial to see that this implies that 
\[ [\phi, u_0 u_0^*] = [\phi, \psi] = 0, \]
and that $\pi(u_t) \in (\pi \circ \phi(A))' = A_B^d$ for all $t$.

It is well-known that we have a group isomorphism $KK(A, B) \to KK_{Higson}(A, B) : [\phi, \psi] \to [\phi, \psi, 1]$. Hence, $[\phi, u_0 u_0^*, 1] = 0$ in $KK_{Higson}(A, B)$. Hence, by [10] Lemma 2.3, $[\phi, \phi, u_0^*] = 0$ in $KK_{Higson}(A, B)$.

By Thomsen’s Paschke duality theorem ([25] Theorem 3.2), there is a group isomorphism $K_1(A_B^d) \to KK_{Higson}(A, B)$ which sends $[\pi(u_0)]$ to $[\phi, \phi, u_0^*]$. Hence, $[\pi(u_0)] = 0$ in $K_1(A_B^d)$. Hence, by Theorem [29] $\pi(u_0) \sim_h 1$ in $A_B^d = (\pi \circ \phi(A))'$. Hence, there exists a unitary $v \in C(1 + B)$ such that $v^* u_0 \sim_h 1$ in $\pi^{-1}(A_B^d)$.

Hence, modifying an initial segment of $\{v^* u_t\}_{t \in [0, \infty)}$ if necessary, we may assume that $\{v^* u_t\}_{t \in [0, \infty)}$ is a norm continuous path of unitaries in $\pi^{-1}(A_B^d)$ such that $v^* u_0 = 1$.

Now for all $t \in [0, \infty)$, let $\alpha_t \in Aut(\phi(A) + B)$ be given by $\alpha_t(x) = v^* u_t x u_t^* v$ for all $x \in \phi(A) + B$. Thus, $\{\alpha_t\}_{t \in [0, \infty)}$ is a norm continuous path of automorphisms of $\phi(A) + B$ such that $\alpha_0 = id$. Hence, by [8] Proposition 2.15 (see also [21] Theorem 3.2 and 3.4), there exist a continuous path $\{v_t\}_{t \in [0, \infty)}$ of unitaries in $\phi(A) + B$ such that $v_0 = 1$ and $\|v_t x u_t^* - v^* u_t x u_t^* v\| \to 0$ as $t \to \infty$ for all $x \in \phi(A) + B$. Thus, $\|v_0 x u_0^* - v^* u_0 x u_0^* v\| \to 0$ as $t \to \infty$ for all $x \in \phi(A) + B$.

We now proceed as in the last part of the proof of [8] Proposition 3.6 Step 1 (see also the proof of [21] Theorem 3.4). For all $t \in [0, \infty)$, let $a_t \in A$ and $b_t \in B$ such that $v v_t = \phi(a_t) + b_t$. Since $\pi \circ \phi$ is injective, we have that for all $t$, $a_t$ is a unitary in $A$, and hence, $\phi(a_t)$ is a unitary in $\phi(A) + B$. Note also that since $\pi \circ \phi = \pi \circ \psi$ and both maps are injective, $\|a_t a_t^* - a\| \to 0$ as $t \to \infty$ for all $a \in A$. For all $t$, let $w_t = v v_t \phi(a_t)^* + 1 + B$. Then $\{w_t\}_{t \in [0, 1)}$ is a norm continuous path of unitaries in $1 + B$, and for all $a \in A$,
\[ \|u_t \phi(a) u_t^* - \psi(a)\| \]
\[ \leq \|u_t \phi(a) u_t^* - v v_t \phi(a) v_t^* v^*\| + \|v v_t \phi(a) v_t^* v^* - u_t \phi(a) u_t^*\| + \|u_t \phi(a) u_t^* - \psi(a)\| \]
\[ = \|v v_t \phi(a) a_t a_t^* - a\| v_t^* v^* + \|v v_t \phi(a) v_t^* v^* - u_t \phi(a) u_t^*\| + \|u_t \phi(a) u_t^* - \psi(a)\| \]
\[ \to 0. \]

We have another generalization of the BDF essential codimension theorem:

**Theorem 3.4.** Let $A$ be a unital separable simple nuclear C*-algebra, and $B$ a separable simple stable C*-algebra with a nonzero projection, strict comparison of positive elements and for which $T(B)$ has finitely many extreme points.

Suppose that there exists a *-embedding $A \to B$. 

Let $\phi, \psi : A \to \mathcal{M}(B)$ be unital extensions such that $\phi(a) - \psi(a) \in B$ for all $a \in A$.

Then $[\phi, \psi] = 0$ in $KK(A, B)$ if and only if $\phi \equiv \psi$.

Proof. Note that since $A$ is simple, $\phi$ and $\psi$ are both norm full extension. Hence, since $B$ has the CFP, it follows, by Theorem 2.7 that $\phi$ and $\psi$ are both unitally absorbing extensions.

The rest of the proof is exactly the same as that of Theorem 3.3 except that Theorem 2.5 is replaced with Theorem 2.9. □

We note once more, that, as in Theorem 1.2 Theorems 3.5 and 3.6 are essentially about local phenomena.

Towards more concrete generalizations, we first need a technical result.

Lemma 3.5. If $B$ is a nonunital $C^*$-algebra and $P, Q \in \mathcal{M}(B)$ are projections with $P - Q \in B$ and $\|P - Q\| < 1$, then there exists a unitary $U \in 1 + B$ such that $P = UQU^*$. Moreover, we can choose $U$ as above so that $\|U - 1\| \leq 4\|P - Q\|$.

Proof. Brief sketch of standard argument: $Z = PQ + (1 - P)(1 - Q)$ satisfies $Z - 1 = (1 - 2P)(P - Q)$, and thus $\|Z - 1\| < 1$. Hence, $Z$ is invertible and if $U$ is the unitary in the polar decomposition of $Z$, then $UQU^* = P$. Moreover, since $P - Q \in B$, $Z \in 1 + B$ and hence, $U \in 1 + B$.

Also, $\|Z^**Z - 1\| \leq \|Z^**Z - Z\| + Z - 1\| = \|Z^** - 1\| 1\| + 1\| = 3\|P - Q\|. So \|\|Z - 1\| \leq \|Z^** - 1\| 1\| + 1\| = 3\|P - Q\|. So \|U - 1\| \leq \|U - U|Z|| + \|Z - 1\| = \|1 - |Z|| + \|P - Q\| \leq 4\|P - Q\|$. □

We now move towards a more concrete generalization of the BDF essential codimension theorem. We will be using the notion of generalized essential codimension in Definition 3.1.

Theorem 3.6. Let $B$ be a separable stable simple purely infinite $C^*$-algebra, and $P, Q \in \mathcal{M}(B)$ projections such that $P, Q, 1 - P, 1 - Q \notin B$, and $P - Q \in B$.

Then $[P : Q] = 0$ in $K_0(B)$ if and only if there exists a unitary $U \in 1 + B$ such that $UPU^* = Q$.

Proof. Since $B$ is simple purely infinite, it follows, from the hypotheses, that $P \sim 1 - P \sim Q \sim 1 - Q \sim 1$. Let $\phi, \psi : C \to \mathcal{M}(B)$ be $*-$homomorphisms such that $\phi(1) = P$ and $\psi(1) = Q$. Then $\phi$ and $\psi$ are absorbing trivial extensions. (And both absorb the zero extension.)

The “if” direction then follows immediately from Theorem 3.3 (See also [20] Lemma 2.4.)

We now prove the “only if” direction. We have that $[\phi, \psi] = [P : Q] = 0$. Hence, by Theorem 3.3 there exists a norm continuous path $\{u_t\}_{t \in [0, 1]}$ of unitaries in $\mathbb{C}1 + B$ such that $\|u_t Pu_t^* - Q\| \to 0$ as $t \to \infty$.

Choose $s \in [0, \infty)$ such that $\|u_s Pu_s^* - Q\| < 1$. We may assume that $u_s \in 1 + B$. Then, by Lemma 3.5 there exists a unitary $V \in 1 + B$ such that $Vu_s Pu_s^* V^* = Q$. Take $U =_{\text{def}} V u_s$. □

We note that there is a mistake in [19] Theorem 2.14. It is not true that if $B$ is a separable simple stable purely infinite $C^*$-algebra for which $\mathcal{M}(B)$ has real rank zero, and if $P, Q \in \mathcal{M}(B)$ are projections with $P - Q \in B$, $P \notin B$, for which
Now use the concrete structure of $p$ (Sketch of argument for choosing $\delta$). Find a polynomial $p$ then choose $\delta > 0$, if $\|p\| > 0$. Let $P =_{df} 1_{\mathcal{M}(O_2 \otimes K)}$ and $Q = P - r$. Then $P - Q = r \in O_2 \otimes K$, $P \notin O_2 \otimes K$, and $|P : Q| = 0$ in $K_0(O_2 \otimes K)$. But it is not true that $P$ is unitarily equivalent to $Q$.

The mistake in the argument of [19] Theorem 2.14 is essentially a mistake about absorbing extensions. If $\phi : A \to \mathcal{M}(B)$ is an absorbing extension then $\phi \otimes 0 \sim \phi$, i.e., $\phi$ must absorb the 0 extension, and thus $\text{ran}(\phi)^\perp$ must be big. (Of course, this must be separated from the unital case where $\phi(1) = 1$ and $\phi$ is unitally absorbing — meaning absorbing all strongly unital trivial extensions.)

Finally, we note that in a separate paper, where we also investigate the relationship between essential codimension and projection lifting, we will look more extensively at concrete generalizations of the BDF essential codimension result, as in the above.

4. TECHNICAL LEMMA

For $\delta > 0$, let $f_\delta : [0, \infty) \to [0, 1]$ be the unique continuous function for which

$$f_\delta(t) = \begin{cases} 1 & t \in [\delta, \infty) \\ 0 & t = 0 \\ \text{linear on } [0, \delta]. \end{cases}$$

If $\mathcal{C}$ is a unital $C^*$-algebra and $p \in \mathcal{C}$ is a projection, we follow standard convention by letting $p^\perp =_{df} 1 - p$.

In what follows, for elements $a, b$ in a $C^*$-algebra, we use $a \approx_\epsilon b$ to denote $\|a - b\| < \epsilon$.

**Lemma 4.1.** Let $\mathcal{B}$ be a separable stable $C^*$-algebra with an approximate unit $\{e_n\}$ consisting of increasing projections. (We define $e_0 =_{df} 0$.)

Suppose that $A, A', A'' \in \mathcal{C}(\mathcal{B})_+$ are contractive elements and $\delta > 0$ such that

$$AA' = A'$$

and

$$A'' \in \text{her}((A' - \delta)^+) \text{.}$$

Let $A_0 \in \mathcal{M}(\mathcal{B})$ be any contractive lift of $A$, and let $\epsilon > 0$ be given.

Then for every $M \geq 0$, there exists an $A''_0 \in e_M^{+\mathcal{M}(\mathcal{B})} e_M^+$ which is a contractive positive lift of $A''$ such that for all $l \geq 1$,

$$A_0 (A''_0)^{1/l} \approx_\epsilon A''_0 \approx_\epsilon (A''_0)^{1/l} A_0.$$ 

**Proof.** Choose $\delta_1 > 0$ such that for any contractive operators $B, C$, with $C \geq 0$, if

$$BC \approx_{\delta_1} C \approx_{\delta_1} CB$$

then

$$Bf_\delta(C) \approx_\epsilon f_\delta(C) \approx_\epsilon f_\delta(C)B.$$ 

(Sketch of argument for choosing $\delta_1$: By the Weierstrass approximation theorem, find a polynomial $p(t)$, with $p(0) = 0$, such that $|f_\delta(t) - p(t)| < \epsilon/2$ for all $t \in [0, 1]$. Now use the concrete structure of $p(t)$ to determine $\delta_1$.)
Let $A'_0 \in e_{\frac{1}{M}}^+ \mathcal{M}(\mathcal{B}) e_{\frac{1}{M}}^+$ be any contractive positive lift of $A'$. Note that we can restrict to the corner $e_{\frac{1}{M}}^+$ because the image of $\pi(e_{\frac{1}{M}}^+)$ equals 1 since $e_M \in \mathcal{M}(\mathcal{B})$. Because $AA' = A'$ (and since they are positive, we also have $A' = A'A$) it follows that $AA'^{1/2} = A'^{1/2}$ and so also $A'^{1/2}A = A'^{1/2}$. Therefore, there exist $c, c' \in \mathcal{B}$ for which

$$A_0 A_0'^{1/2} = A_0'^{1/2} + c,$$

and

$$A_0'^{1/2} A_0 = A_0'^{1/2} + c'.$$

Since $\{e_n\}$ is an approximate identity for $\mathcal{B}$, we can choose $N \geq 1$ so that $c e_N^\perp \approx_{\delta_1} 0 \approx_{\delta_1} e_N^\perp c'$.

Then, combining the above displays yields

$$A_0 A_0'^{1/2} e_N^\perp \approx_{\delta_1} A_0'^{1/2} e_N^\perp,$$

and

$$e_N^\perp A_0'^{1/2} A_0 \approx_{\delta_1} e_N^\perp A_0'^{1/2}.$$

Hence, if we define

$$D = \delta f A_0'^{1/2} e_N^\perp A_0'^{1/2}$$

then

$$A_0 D \approx_{\delta_1} D \approx_{\delta_1} DA_0.$$

Hence, by the definition of $\delta_1$,

\begin{equation}
(4.1) \quad A_0 f_\delta(D) \approx_{\epsilon} f_\delta(D) \approx_{\epsilon} f_\delta(D) A_0.
\end{equation}

Note that $\pi(D) = A'$, which follows since $\pi(e_N^\perp) = 1$. Because the algebra $\pi((D - \delta)_+^+ \mathcal{M}(\mathcal{B})(D - \delta)_+^+)$ equals $\text{her}((A' - \delta)_+^+)$, we can find a contractive positive lift $A''_0 \in (D - \delta)_+^+ \mathcal{M}(\mathcal{B})(D - \delta)_+^+$ of $A''$. Note that $A''_0 \in e_{\frac{1}{M}}^+ \mathcal{M}(\mathcal{B}) e_{\frac{1}{M}}^+$ since $A'_0$, and consequently, $D$ and $(D - \delta)_+$ are.

We remark that $f_\delta(D)(D - \delta)_+ = (D - \delta)_+$, and for all $l \geq 1$, $A_0'^{1/l}$ is a contraction. Combining these facts with \((4.1)\) we obtain

$$A_0(A''_0)^{1/l} = A_0 f_\delta(D)(A''_0)^{1/l} \approx_{\epsilon} f_\delta(D)(A''_0)^{1/l} = (A''_0)^{1/l}.$$

Similarly,

$$(A''_0)^{1/l} A_0 \approx_{\epsilon} (A''_0)^{1/l}. \quad \Box$$

We now fix some notation which will be used for the rest of this section.

Let $\mathcal{B}$ be a separable simple stable $C^*$-algebra with a nonzero projection. Let $\{p_k\}_{k=1}^\infty$ be a sequence of pairwise orthogonal projections of $\mathcal{B}$ such that

$$\sum_{k=1}^\infty p_k = 1_{\mathcal{M}(\mathcal{B})},$$

where the series converges strictly.

For all $m \leq n$, let

$$p_{m,n} = df \sum_{k=m}^n p_n$$
and let
\[ e_n = \sum_{k=1}^{n} p_k. \]

(Hence, \( \{e_n\} \) is an approximate unit for \( B \).)

Let \( U \in C(B) \) be a unitary and let \( V \in M(B) \) be a partial isometry such that
\[ \pi(V) = U. \]

Also, we let \( B(0,1) \) denote the closed unit ball of the complex plane, i.e.,
\[ B(0,1) = \{ \alpha \in \mathbb{C} : |\alpha| \leq 1 \}. \]

Recall also that for a C*-algebra \( C \), for a \( \tau \in T(C) \) and for any \( a \in C_+ \),
\[ d_{\tau}(a) = \lim_{n \to \infty} \tau(a^{1/n}). \]

\textbf{Lemma 4.2.} Let \( h_1, h_2, h_3 : S^1 \to [0,1] \) be continuous functions and let \( \delta_1 > 0 \) be such that
\[ h_1 h_2 = h_2 \]
and
\[ \text{supp}(h_3) \subset \text{supp}((h_2 - \delta_1)_+). \]

Let \( \delta_2 > 0 \) and \( \hat{h} \) be a complex polynomial such that
\[ |\hat{h}(\lambda) - h_1(\lambda)| < \frac{\delta_2}{10} \]
for all \( \lambda \in S^1 \).

Then for every \( L, L' \geq 1 \), there exist \( L_1 > L' \), there exist contractive \( A \in e_{L_1}^+, M(B)_+ e_{L_1}^+ \) which is a lift of \( h_3(U) \) such that for every contractive \( a \in (AB\hat{A})_+ \) there exist \( M > L, M_1 > L_1 \) and \( x \in p_{L_1+1,M_1} B p_{L_1+1,M_1} \) for which
\[ \hat{x} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) \approx_{\delta_2} a \]
for every sequence \( \{\alpha_j\} \) in \( B(0,1) \) (closed unit ball of the complex plane) such that \( \alpha_j = 1 \) for all \( L \leq j \leq M \).

\textbf{Proof.} Let \( A_0 \in M(B)_+ \) be a contractive lift of \( h_1(U) \).

Since \( U \) is unitary and because of the conditions on \( \hat{h} \), we know \( \hat{h}(U) \approx_{\delta_2} h_1(U) \).
Moreover, since \( \hat{h} \) is a polynomial, \( \hat{h}(U) = \hat{h}(\pi(V)) = \pi(\hat{h}(V)) \) and also \( h_1(V) = \pi(A_0) \). Using these facts along with the fact that \( \{e_n\} \) is an approximate identity for \( B \), we can choose \( L_1 > L' \) so that
\[ e_{L_1}^+ \hat{h}(V) e_{L_1}^+ \approx_{\delta_2} e_{L_1}^+ A_0 e_{L_1}^+ \]
and
\[ \hat{x} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) e_{L_1}^+ \approx_{\delta_2} \hat{x} \left( \sum_{j=1}^{\infty} \alpha'_j p_j V \right) e_{L_1}^+ \]
for all sequences \( \{\alpha_j\} \) and \( \{\alpha'_j\} \) in \( B(0,1) \) such that \( \alpha_j = \alpha'_j \) for all \( j \geq L \).
By Lemma 4.1 (instantiated with $A, A', A'', \delta, \epsilon, M$ chosen to be $h_1(U), h_2(U), h_3(U), \delta_1, \delta_2, L_1$), there exists $A \in \mathcal{E}(B)e_{L_1}^{+}$ which is a contractive positive lift of $h_3(U)$ for which
\[ e_{L_1}^{+}A_{0}e_{L_1}^{+}A^{1/2} \approx \frac{\delta_1}{10} A^{1/2} \]
for all $l \geq 1$.

Hence, if we let $a \in \overline{AB\bar{A}}$ be an arbitrary contractive positive element, then because the previous display holds for all $l \geq 1$,
\[ e_{L_1}^{+}A_{0}e_{L_1}^{+}a^{1/2} \approx \frac{\delta_1}{10} a^{1/2}. \]
Chaining this with (4.2) yields
\[ e_{L_1}^{+}\hat{h}(V)e_{L_1}^{+}a^{1/2} \approx \frac{\delta_1}{10} a^{1/2}. \]
Therefore, if we let $y = df a^{1/2}$ then
\[ ye_{L_1}^{+}\hat{h}(V)e_{L_1}^{+}y^{*} \approx \frac{\delta_1}{10} a. \]

By the definition of $L_1$ and since $y, a \in B$, we can choose $M > L$ and $M_1 > L_1$ such that if we define
\[ x = df p_{L_1+1,M_1}y_{L_1+1,M_1} \]
then
\[ x\hat{h}(\sum_{j=1}^{\infty} \alpha_j p_{j}V) x^{*} \approx \frac{\delta_2}{10} a \]
for every sequence $\{\alpha_j\}$ in $B(0,1)$ for which $\alpha_j = 1$ for all $L \leq j \leq M$. \hfill \Box

**Lemma 4.3.** Suppose that, in addition, $B$ has strict comparison of positive elements and $T(B)$ has finitely many extreme points.

Let $p \in B$ be a nonzero projection and let $\epsilon > 0$ be given.

Let $h_1, h_2, h_3 : S^1 \to [0,1]$ be a continuous functions $\delta_1 > 0$ and $\lambda_1, ..., \lambda_m \in S^1$ such that
\[ h_1 h_2 = h_2 \]
\[ \text{supp}(h_3) \subset \text{supp}((h_2 - \delta_1)_+) \]
and the function
\[ \lambda \mapsto \sum_{j=1}^{m} h_3(\lambda_j \lambda) \]
is a full element in $C(S^1)$.

There exists $\delta_2 > 0$ such that if $\hat{h}$ a complex polynomial for which
\[ |\hat{h}(\lambda) - h_1(\lambda)| < \frac{\delta_2}{10} \]
for all $\lambda \in S^1$ then the following holds:

For every $L, L' \geq 1$, there exist $L < L_1 < L_2 < ... < L_m, L' < M < M'$, a projection $q \in B$ for which $q \sim p$, and contractive $x \in p_{M+1,M'}Bp_{M+1,M'}$ such that
\[ x\hat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_{j}V\right) x^{*} \approx \epsilon q \]
where $\{\alpha_j\}$ is any sequence in $B(0,1)$ such that $\alpha_j = \lambda_k$ for all $L_{k-1} < j \leq L_k$ and all $1 \leq k \leq m$. (Here, $L_0 = df L_1$.)
Proof. Let \( \mathcal{F} \) be the finitely many extreme points of \( T(\mathcal{B}) \).

Let \( L, L' \geq 1 \) be arbitrary.

Let \( \epsilon > 0 \) and let \( \delta_2 > 0 \) be any constant for which \( \delta_2 < \frac{1}{m} \).

We construct a elements \( L_j, A_j, b_j, \epsilon_j, M, M_j', M_j'', M_j''' \) and \( x_j \) for \( 1 \leq j \leq m \).

The construction is by induction on \( j \).

**Basis step:** \( j = 1 \).

By Lemma 1.2 choose \( M > L' \) and contractive positive \( A_1 \in e_{\frac{1}{M}} \mathcal{M}(\mathcal{B}) e_{\frac{1}{M}} \) such that

\[
\pi(A_1) = h_3(\lambda_1 U).
\]

We let

\[
M_1' = \text{df } M.
\]

Let \( \mathcal{F}_1 = \{ \tau \in \mathcal{F} : \tau(A_1) = \infty \} \).

Let \( a_1 \in A_1 \mathcal{B} A_1 \) be a strictly positive element. Choose \( \epsilon_1 > 0 \) so that

\[
d_\tau(a_1 - 2\epsilon_1) > \tau(p)
\]

for all \( \tau \in \mathcal{F}_1 \).

By Lemma 1.2 choose \( L > L, M_1'' > M_1' = M \) and a contractive element

\[
x_1 \in p_{M+1,M_1''} \mathcal{B} p_{M+1,M_1''}
\]

so that

\[
x_1 \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) x_1^* \approx_{\delta_2} \mathcal{F}_1(a_1)
\]

for every sequence \( \{ \alpha_j \} \) in \( B(0,1) \) for which \( \alpha_j = \lambda_1 \) for all \( L < j \leq L_1 \).

We let \( M_1'' > M_1' \) be a number that is big enough so that if we define

\[
b_1 = \text{df } p_{M+1,M_1''} a_1 p_{M+1,M_1''}
\]

then

\[
d_\tau((b_1 - 2\epsilon_1) < \tau(p)
\]

for all \( \tau \in \mathcal{F}_1 \) and

\[
x_1 \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) x_1^* \approx_{\delta_2} \mathcal{F}_1(b_1)
\]

for every sequence \( \{ \alpha_j \} \) in \( B(0,1) \) for which \( \alpha_j = \lambda_1 \) for all \( L < j \leq L_1 \).

**Induction step.** Suppose that \( L_k, A_k, b_k, \epsilon_k, M_k', M_k'', M_k''' \) and \( x_k \) have been chosen. We now construct the constants with \( k \) replaced with \( k + 1 \).

Choose \( N > M_k''' \) big enough so that

\[
\left\| e_{M_k''} \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) e_{\frac{1}{M_k''}} \right\|, \left\| e_{\frac{1}{N}} \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) e_{M_k'} \right\| < \delta_2,
\]

for every sequence \( \{ \alpha_j \} \) in \( B(0,1) \).

By Lemma 1.2 choose \( M_{k+1}' > N \) and a contractive positive element \( A_{k+1} \in e_{\frac{1}{M_{k+1}} \mathcal{B} e_{\frac{1}{M_{k+1}}}} \) such that

\[
\pi(A_{k+1}) = h_3(\lambda_{k+1} U).
\]
Let $F_{k+1} = \{ \tau \in F : \tau(A_{k+1}) = \infty \}$. Let $a_{k+1} \in \mathbb{A}_{k+1}B \mathbb{A}_{k+1}$ be a strictly positive element. Choose $\epsilon_{k+1} > 0$ so that
\[
d_\tau((a_{k+1} - 2\epsilon_{k+1})_+) > \tau(p)
\]
for all $\tau \in F_{k+1}$.

By Lemma 4.2, choose $L_{k+1} > L_k$, $M_{k+1}'' > M_{k+1}'$ and contractive $x_{k+1} \in pM_{k+1}'+1.M_{k+1}'' B pM_{k+1}'+1.M_{k+1}''$ so that
\[
x_{k+1} \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) \approx_{\delta_2} f_{\epsilon_{k+1}}(a_{k+1})
\]
for every sequence $\{\alpha_j\}$ in $B(0,1)$ for which $\alpha_j = \lambda_{k+1}$ for all $L_k < j \leq L_{k+1}$.

Find $M_{k+1}'' > M_{k+1}'$ big enough so that if we define
\[
b_{k+1} = \mathcal{A} M_{k+1}'+1.M_{k+1}'' a_{k+1} pM_{k+1}'+1.M_{k+1}''
\]
then
\[
d_\tau((b_{k+1} - 2\epsilon_{k+1})_+) > \tau(p)
\]
for all $\tau \in F_{k+1}$.

and
\[
x_{k+1} \hat{h} \left( \sum_{j=1}^{\infty} \alpha_j p_j V \right) \approx_{\delta_2} f_{\epsilon_{k+1}}(b_{k+1})
\]
for every sequence $\{\alpha_j\}$ in $B(0,1)$ for which $\alpha_j = \lambda_{k+1}$ for all $L_k < j \leq L_{k+1}$.

This completes the inductive construction.

Now let $x \in p_{M+1},M_{m+1}'' B p_{M+1},M_{m+1}''$ be the contractive element defined by
\[
x = \mathcal{A} \sum_{j=1}^{m} x_j.
\]

Let $\{\alpha_j\}$ be any sequence in $B(0,1)$ such that $\alpha_j = \lambda_k$ for all $L_{k-1} < \alpha_j \leq L_k$ and for all $1 \leq k \leq m$. (Here $L_0 = \mathcal{A} L$.)

Then
\[
x \hat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V)x^*
\]
\[
\approx_{(m^2-m)\delta_2} \sum_{k=1}^{m} x_k \hat{h}(\sum_{j=1}^{\infty} \alpha_j p_j V)x_k^*
\]
\[
\approx_{m \delta_2} \sum_{k=1}^{m} f_{\epsilon_k}(b_k).
\]

Since
\[
d_\tau \left( \sum_{k=1}^{m} (b_k - 2\epsilon_k)_+ \right) > \tau(p)
\]
for all $\tau \in T(B)$ and since $B$ has strict comparison for positive elements, there exists a projection $q \in (\sum_{k=1}^{m}(b_k - 2\epsilon_k)_+)B(\sum_{k=1}^{m}(b_k - 2\epsilon_k)_+)B(\sum_{k=1}^{m}(b_k - 2\epsilon_k)_+)$ such that $q \sim p$.

Hence,

$$qx\hat{h}\left(\sum_{j=1}^{\infty} \alpha_j p_j V\right)x^*q \approx m^2 \delta_2 q$$

for every sequence $\{\alpha_j\}$ in $B(0,1)$ for which $\alpha_j = \lambda_k$ for every $L_{k-1} < j \leq L_k$ and all $1 \leq k \leq m$. Since $\delta_2 < \frac{\epsilon_m}{m^2}$ we are done. □

Let $D$ be a unital C*-algebra. Recall that a nonzero element $x \in D$ is said to be strongly full in $D$ if every nonzero element of $C^*(x)$ is a full element of $D$.

**Lemma 4.4.** Say that, in addition, $B$ has strict comparison for positive elements and $T(B)$ has finitely many extreme points.

Then there exists a sequence $\{\alpha_j\}$ in $S^1$ such that the unitary $\pi(\sum_{j=1}^{\infty} \alpha_j p_j)U$ is a strongly full element of $C(B)$.

**Proof.** For every $k \geq 1$, let $h_{k,1,j}, h_{k,2,j}, h_{k,3,j} : S^1 \to [0,1]$ be continuous functions, $\lambda_{k,j} \in S^1$ (for $1 \leq j \leq k$), and $\delta_k > 0$ be such that

$$\sum_{j=1}^{k} h_{k,3,j}$$

is a full element of $C(S^1)$,

$$\operatorname{supp}(h_{k,3,j}) \subset \operatorname{supp}((h_{k,2,j} - \delta_k)_+),$$

$$h_{k,1,j}h_{k,2,j} = h_{k,2,j},$$

$$h_{k,3,j}(\lambda) = h_{k,3,1}(\lambda_{k,j})$$

for all $1 \leq j \leq k$, and

$$\max_{1 \leq j \leq k} \operatorname{diam}(\operatorname{supp}(h_{k,1,j})) \to 0$$

as $k \to \infty$.

Let $\{h_l\}_{l=1}^{\infty}$ be a sequence of continuous functions from $S^1$ to $[0,1]$ such that for all $l$, there exists $k,j$ such that $h_l = h_{k,1,j}$ and for all $k,j$, $h_{k,1,j}$ occurs infinitely many times as a term in the sequence $\{h_l\}_{l=1}^{\infty}$.

Let $\{r_n\}$ be a sequence of pairwise orthogonal projections in $B$ such $r_m \sim r_n$ for all $m,n$, and

$$\sum_{n=1}^{\infty} r_n = 1_{M(B)}$$

where the sum converges strictly.

Let $\{\epsilon_k\}$ be a decreasing sequence in $(0,1)$ and $\{\epsilon_{k,l}\}$ a (decreasing in $k+l$) biinfinite sequence in $(0,1)$ such that

$$\sum_{k=1}^{\infty} \epsilon_k < \infty$$

and

$$\sum_{1 \leq k,l < \infty} \epsilon_{k,l} < \infty.$$
Note that for every $\gamma > 0$, $Y \in \mathcal{M}(\mathcal{B})$ and $y \in \mathcal{B}$, there exists $N \geq 1$ such that
$$\|yY e^+_N\| < \gamma.$$ 
Also, for every $\gamma > 0$, complex polynomial $\hat{h}$ and $L \geq 1$, there exists $N \geq 1$ so that
$$\hat{h}\left(\sum_{j=1}^{\infty} \beta_j p_n V\right) e^+_N \approx \gamma \hat{h}\left(\sum_{j=1}^{\infty} \beta'_j p_n V\right) e^+_N$$
for all sequences $\{\beta_j\}$ and $\{\beta'_j\}$ in $B(0,1)$ for which $\beta_j = \beta'_j$ for all $j \geq L$.

By using the above two principles and by repeatedly applying Lemma 4.3 (first to $h_1$; then to $h_2$; then to $h_3$; and so forth), we can find a sequence $\{x_k\}$ of pairwise orthogonal contractive elements of $\mathcal{B}$, a sequence $\{\alpha_k\}$ in $S^1$, and a sequence $\{\hat{h}_k\}$ of complex polynomials such that the following statements hold:

1. $\sum_{k=1}^{\infty} x_k$ converges strictly in $\mathcal{M}(\mathcal{B})$.
2. For all $k \geq 1$,
$$\max_{n \in \mathbb{Z}} |h_k(\lambda) - \hat{h}_k(\lambda)| < \epsilon_k.$$ 
3. For all $k \geq 1$, there exists a subsequence $\{x_{j_k}\}$ of $\{x_j\}$ such that
   (a) $\left\|x_{j_k}\hat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V\right) x_{j_k}^*\right\| < \epsilon_{l,s}$ for all $l \neq s$, and
   (b) $x_{j_k}\hat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V\right) x_{j_k}^* \approx \epsilon_{l,s}$ for all $l$.

We denote the above statements by “(*)”.

(Sketch of argument on how to choose the subsequence in (*) (3) above: Firstly, from the construction of the sequence, we already have part (3)(b). Next, note that, from Lemma 4.3 there is a sequence of pairwise orthogonal projections $\{s_j\}$ in $\mathcal{B}$ such that $\sum_{j=1}^{\infty} s_j$ converges strictly in $\mathcal{M}(\mathcal{B})$ and $x_j = r_j x_j s_j$ for all $j$. Now fix a $k$. The subsequence is constructed in two steps (a subsequence of a subsequence). 
Step 1: Let $\{j_i\}$ be a subsequence of the positive integers for which $h_{j_i} = h_k$ for all $i$. Step 2: Extract the subsequence of $\{j_i\}$ by observing that for all $\delta$, for all $Y \in \mathcal{M}(\mathcal{B})$, for all $i_1$, there exists $i_2$ such that for all $i \geq i_2$, $\|x_{j_i} Y x_{j_i}^*\| < \delta$.)

Let $m \geq 1$ be given. We will now show that $h_m(\pi(\sum_{n=1}^{\infty} \alpha_n p_n U))$ is full in $\mathcal{C}(\mathcal{B})$. Let $\epsilon > 0$. Since each term of the sequence $\{h_i\}_{i=1}^{\infty}$ is repeated infinitely many times there is some $k$ for which $\epsilon_k < \frac{\epsilon}{2}$ and $h_k = h_m$.

Choose a subsequence $\{x_{j_i}\}$ of $\{x_j\}$ as in (3) of (*), corresponding to $\hat{h}_k$. Let $A \in \mathcal{M}(\mathcal{B})_+$ be a contractive element so that
$$\pi(A) = h_k(\pi(\sum_{n=1}^{\infty} \alpha_n p_n U)).$$

We can choose $L \geq 1$ great enough so that if we define
$$X = \sum_{l=L}^{\infty} x_{j_l},$$
then
$$X \hat{h}_k \left(\sum_{n=1}^{\infty} \alpha_n p_n V\right) X^* \approx \epsilon_k X A X^*. $$
Increasing $L$ if necessary, we may assume that
\[
\sum_{l \geq L} \varepsilon_l + \sum_{m,n \geq L} \varepsilon_m,n < \frac{\varepsilon}{2}.
\]

Consider the projection $R = \sum_{l \geq 1} r_l \in \mathcal{M}(\mathcal{B})$ and note that $R \sim 1_{\mathcal{M}(\mathcal{B})}$ since $\sum_{n=1}^{\infty} r_n \sim 1$, and because all the projections $r_n$ are equivalent. From (3) of (\ref{eq:sequence}),
\[
X \hat{h}_k \left( \sum_{n=1}^{\infty} \alpha_n p_n V \right) X^* \sim_{\delta} R
\]
where
\[
\delta = \sum_{l=L}^{\infty} \varepsilon_l + \sum_{L \leq l,s < \infty} \varepsilon_{l,s}.
\]
Therefore,
\[
\|XAX^* - R\| < \delta + \varepsilon_k < \varepsilon.
\]

Since $R \sim 1_{\mathcal{M}(\mathcal{B})}$, there is some partial isometry $W$ implementing the equivalence so that $WW^* = 1_{\mathcal{M}(\mathcal{B})}$ and $W^*W = R$. Then
\[
\|WXAX^*W^* - 1_{\mathcal{M}(\mathcal{B})}\| \leq \|XAX^* - R\| < \varepsilon.
\]

Applying $\pi$, we obtain
\[
\|\pi(WX)\pi(A)\pi(X^*W^*) - 1_{\mathcal{C}(\mathcal{B})}\| < \varepsilon.
\]
Therefore, $\pi(A) = h_k(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U) = h_m(\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U)$ is full in $\mathcal{C}(\mathcal{B})$.

Since $m \geq 1$ was arbitrary, and by the definition of the sequence $\{h_k\}$, we claim that $\pi(\sum_{n=1}^{\infty} \alpha_n p_n)U$ is a strongly full element of $\mathcal{C}(\mathcal{B})$.

To see this, note that every nonnegative continuous function $f \in C(S^1)$ has some $h_1$ which is in the ideal generated by $f$. Indeed, there is some arc of positive width $\eta$ centered at $s \in S^1$ on which $f$ is greater than some $\zeta > 0$. Since $\max_{1 \leq j \leq k} \text{diam}(\text{supp}(h_{k,1,j})) \to 0$, there is some $k$ such that the maximum of these diameters is less than $\frac{\eta}{3}$. Moreover, since $\sum_{j=1}^{k} h_{k,3,j}$ is full in $C(S^1)$, there is some $1 \leq j \leq k$ such that $h_{k,1,j}(s) \neq 0$. Then, because $\text{diam}(\text{supp}(h_{k,1,j})) < \frac{\eta}{3}$, the support of $h_{k,1,j}$ is entirely contained within the arc on which $f \geq \zeta > 0$. Therefore $h_{k,1,j}$ is in the ideal generated by $f$. Finally, by the definition of $\{h_l\}_{l=1}^{\infty}$, there is some $l$ for which $h_l = h_{k,1,j}$ (in fact, there are infinitely many such $l$).

\[\Box\]

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