SHUFFLE ALGEBRAS, LATTICE PATHS
AND THE COMMUTING SCHEME

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Abstract. The commutative trigonometric shuffle algebra $A$ is a space of symmetric rational functions satisfying certain wheel conditions [FO97, FHH+09]. We describe a ring isomorphism between $A$ and the center of the Hecke algebra using a realization of the elements of $A$ as partition functions of coloured lattice paths associated to the $R$-matrix of $U_{q, t}(\hat{gl}_\infty)$. As an application, we compute under certain conditions the Hilbert series of the commuting scheme and identify it with a particular element of the shuffle algebra $A$, thus providing a combinatorial formula for it as a “domain wall” type partition function of coloured lattice paths.

Dedicated to Jasper Stokman on the occasion of his 50th birthday

1. Introduction

1.1. Center of Hecke and shuffle products. Consider the Hecke algebra $H_n$ ($n \in \mathbb{Z}_{\geq 0}$), which we define for our purposes to be the algebra (over a field extension $\mathbb{F}$ of $\mathbb{Q}(t)$ to be specified below) with generators $T_1, \ldots, T_{n-1}$ and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i^2 = (t-1)T_i + t, \quad T_i T_j = T_j T_i \ (|i-j| \geq 2)$$

The standard basis $(T_w)_{w \in S_n}$ of $H_n$ is obtained by writing $T_w := T_{i_1} \ldots T_{i_k}$ if $w = s_{i_1} \ldots s_{i_k}$ is a reduced word in the elementary transpositions $s_i$. Denote by $|w| = k$ the length of $w$.

Combine the centers $Z(H_n)$ into a graded algebra

$$Z := \bigoplus_{n \geq 0} Z(H_n)$$

by defining a shuffle product

$$(1) \quad \ast : Z(H_k) \otimes Z(H_\ell) \to Z(H_{k+\ell})$$

$$x \mapsto \sum_{w \in S_{k,\ell}} t^{-|w|} T_w x T_{w^{-1}}$$

where $H_k \otimes H_\ell$ is embedded inside $H_{k+\ell}$ in the obvious way, and $S_{k,\ell}$ is the set of shortest representatives of cosets in $S_{k+\ell}/(S_k \times S_\ell)$.

Let us also consider $\Lambda$, the algebra of symmetric functions with coefficients in $\mathbb{F}$. One can for example define it as a polynomial ring in countably many variables: $\Lambda := \mathbb{F}[p_1, p_2, \ldots]$ where $\deg p_r = r$.

There is a graded algebra isomorphism $\Phi$ between $Z$ and $\Lambda$ [WW15], which will be reviewed in §2; it is a deformation of the classical Frobenius map in the context of the symmetric group.

In what follows we fix the base field to be

$$\mathbb{F} = \mathbb{Q}(q, t)$$

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Define $F[x_1^\pm, \ldots, x_n^\pm]_0$ to be the space of Laurent polynomials with specific degree bounds:

\[
F[x_1^\pm, \ldots, x_n^\pm]_0 = \mathbb{F}\text{-span of } \left\{ x_1^{i_1} \ldots x_n^{i_n}, \ i_1, \ldots, i_n \in \mathbb{Z} : \sum_{j=1}^r |i_j| \leq r(n-r), \ 0 \leq r \leq n \right\}
\]

(in particular, these are homogeneous polynomials of degree 0) and $F[x_1^\pm, \ldots, x_n^\pm]_{S^n}$ to be its subspace of symmetric Laurent polynomials.

Consider $A_n = \{ P \in F[x_1^\pm, \ldots, x_n^\pm]_{S^n} : P(x, qx, tx, x_4, \ldots, x_n) = P(tqx, qx, tx, x_4, \ldots, x_n) = 0 \}$

One can again introduce a shuffle product

\[
* : A_k \otimes A_\ell \to A_{k+\ell}
\]

\[
P \otimes Q \mapsto \sum_{w \in S^{k,\ell}} P(x_{w(1)}, \ldots, x_{w(k)})Q(x_{w(k+1)}, \ldots, x_{w(k+\ell)}) \prod_{1 \leq i \leq k, \ k+1 \leq j \leq k+\ell} \tilde{\omega}(x_{w(i)}/x_{w(j)})
\]

where $\tilde{\omega}(x) = \frac{(1-q^{-1}x)(1-q^{-1}x)(1-tx)}{-x(1-x)}$ making

\[
A = \bigoplus_{n \geq 0} A_n
\]

a graded algebra. There is another graded algebra isomorphism $\Upsilon$ from $A$ to $\Lambda$ \footnote{A_n is not quite the shuffle algebra as normally defined, but is more convenient for us; the standard shuffle algebra, denoted by $\mathcal{A}_n$, will be given in \S 3}, which will be reviewed in \S 3.3.

It is natural to ask if there is an interpretation of the isomorphism $\Upsilon^{-1} \circ \Phi$ from $\mathbb{Z}$ to $A$. Note however that $\Lambda$ possesses graded algebra automorphisms, and in particular we shall allow ourselves to work modulo the automorphisms that rescale the variables $p_r$.

1.2. Lattice paths. Consider a $n \times n$ square grid, and a set of $n$ paths entering from the left external edges of the grid and exiting at the top. We label incoming paths on the left $1, \ldots, n$ from top to bottom. We say that a set of paths $P$ has connectivity $\text{conn}(P) = w \in S_n$ if the path labelled $i$ (i.e., the $i$th incoming path on the left counted from the top) exits at the $w^{-1}(i)$th location at the top (counted from the left).

Example 1. Here are all the paths with connectivity 2431:
around that vertex:

\[
\text{wt}(P) = \prod_{i,j=1}^{n} \begin{cases} 
1 - t & \text{,} \\
q^{-1}x_ix_j^{-1}(1 - t) & \text{,} \\
t(1 - q^{-1}x_ix_j^{-1}) & \text{,} \\
1 - q^{-1}x_ix_j^{-1} & \text{,} \\
1 - tq^{-1}x_ix_j^{-1} & \text{.}
\end{cases}
\]

where the convention is that colours can be substituted as long as one preserves the ordering; that is, if two paths are present at a vertex, red (resp. green) stands for the one with the lower (resp. greater) label.

To any element \( c = \sum_{v \in S_n} c_v T_v \in Z(\mathcal{H}_n) \) (\( c_v \in \mathbb{F} \)) of the center of the Hecke algebra we associate the \textit{partition function}

\[
f(c) := \alpha_n \sum_{\text{lattice paths } P \text{ on the } n \times n \text{ grid}} \text{wt}(P) c_{\text{conn}(P)}
\]

where \( \alpha_n = (q/t)^{n(n-1)/2}(1 - t)^{-n} \) is a constant which we introduce for convenience.

We can now formulate our first theorem:

\textbf{Theorem 1}. There is a commuting square of graded algebra morphisms

\[
\begin{array}{ccc}
Z & \xrightarrow{\Phi} & \Lambda \\
\downarrow f & & \downarrow \sigma_{q^{-1}\sigma_t^{-1}} \\
\Lambda & \xrightarrow{\gamma} & \Lambda \\
\end{array}
\]

where \( \sigma_{q^{-1}\sigma_t^{-1}} \) is the automorphism that sends \( p_r \) to \( (-1)^{r-1}(1-q^r(1-t))p_r/((1-q^r)(1-t)^r) \).

1.3. \textbf{Partition function with connectivity 12...n}. We consider here the special case of the unit \( 1_n \in Z(\mathcal{H}_n) \) of \( \mathcal{H}_n \) which corresponds to the partition function with connectivity 12...n

\[
f(1_n) = \alpha_n \sum_{\text{lattice paths } P \text{ on the } n \times n \text{ grid}} \text{wt}(P)
\]

\textbf{Example 2}. Here are the identity paths at \( n = 3 \):
Theorem 2. The partition function \( f(1_n) \), as an element of \( \mathbb{F}[x_1^\pm, \ldots, x_n^\pm]^{S_n} \), is determined by \( f(1_0) = 1 \) and the recurrence relation

\[
(5) \quad (x_1 + \cdots + x_n)f(1_n) = \sum_{i=1}^{n} x_i f(1_{n-1})[\hat{x}_i] \prod_{j=1, j \neq i}^{n} \frac{(1 - q^{-1}x_j/x_i)(1 - qt^{-1}x_j/x_i)(1 - tx_i/x_j)}{-x_j/x_i(1 - x_j/x_i)}
\]

where \( f(1_{n-1})[\hat{x}_i] \in \mathbb{F}[x_1^\pm, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n^\pm]^{S_n} \).

In §2.4 we will show that the partition functions \( f(c) \) can be interpreted using the R-matrices of \( \mathcal{U}_{\text{gl}}(gl_{\infty}) \). This means that one can study them using the quantum inverse scattering method (QISM) \( \text{KBI97} \) (see also \( \text{Lam18} \)), and indeed the partition function \( f(1_n) \) was already considered in \( \text{BW18} \). We note, however, that the recurrence relation \( 5 \) has an unusual form from the point of view of the QISM due to the division by \( q \) at every \( j \)th step of the recursive computation of \( f(1_n) \). It is not clear to us if one can prove \( 5 \) or find an alternative recursion relation for \( f(1_n) \) using the QISM. Our proof of \( 5 \) is indirect; it follows as a corollary of our proof of Theorem 1 given in §1.

1.4. The commuting scheme. We now give a geometric interpretation of the partition function \( f(1_n) \) of the previous section in terms of the commuting scheme. (The geometric interpretation of other \( f(c) \) is left for future work.)

Given \( n \in \mathbb{Z}_{\geq 0} \), define the commuting scheme \( \mathfrak{C}_n \) to be the affine scheme in \( \mathfrak{gl}_n(\mathbb{C})^2 \) consisting of pairs of commuting \( n \times n \) matrices:

\[
(6) \quad \mathfrak{C}_n := \{(A, B) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) : [A, B] = 0\}
\]

\( \mathfrak{C}_n \) is known to be irreducible, of dimension \( n(n+1) \) \( \text{MT55} \), smooth in codimension 1 \( \text{Pop08} \), and we have the conjecture, generally attributed to Artin and Hochster in 1982:

Conjecture 1. \( \mathfrak{C}_n \) is Cohen–Macaulay.

which would imply that \( \mathfrak{C}_n \) is reduced and normal. This conjecture will not be addressed in the present paper.

The group \( GL_2(\mathbb{C}) \times GL_n(\mathbb{C}) \) acts on \( \mathfrak{gl}_n(\mathbb{C})^2 \) by

\[
(f, g) \in GL_2(\mathbb{C}) \times GL_n(\mathbb{C}) : \begin{pmatrix} A \\ B \end{pmatrix} \mapsto \begin{pmatrix} gAg^{-1} \\ gBg^{-1} \end{pmatrix}
\]

and leaves \( \mathfrak{C}_n \) invariant. In particular, the maximal torus \( T = (\mathbb{C}^\times)^{2+n} \) induces a \( \mathbb{Z}^{2+n} \)-grading of the corresponding coordinate rings; explicitly,

\[
\deg A_{ij} = \varepsilon_1 + \varepsilon_{i+2} - \varepsilon_{j+2} \quad \deg B_{ij} = \varepsilon_2 + \varepsilon_{i+2} - \varepsilon_{j+2} \quad i, j = 1, \ldots, n
\]

where \( \varepsilon_i \) is the unit vector with nonzero coordinate \( i \).

We consider the Hilbert series \( \chi(\mathfrak{C}_n) \) of the commuting scheme with the multigrading above (i.e., the \( T \)-character of its coordinate ring); in fact, it is convenient to consider instead the \( K \)-polynomial (or Poincaré polynomial) of \( \mathfrak{C}_n \), defined by

\[
K_n = \frac{\chi(\mathfrak{C}_n)}{\chi(\mathfrak{gl}_n(\mathbb{C})^2)}
\]

Explicitly, introducing formal variables \( q_1, q_2, x_1, \ldots, x_n \) corresponding to \( (\mathbb{C}^\times)^{2+n} \), one has

\[
\chi(\mathfrak{gl}_n(\mathbb{C})^2) = \prod_{i,j=1}^{n} \frac{1}{(1 - q_1 x_i x_j^{-1})(1 - q_2 x_i x_j^{-1})}
\]
$K_n$ is known to be a Laurent polynomial in $\mathbb{Z}[q_1^\pm, q_2^\pm, x_1^\pm, \ldots, x_n^\pm]$. Furthermore, because of the $GL_2(\mathbb{C}) \times GL_n(\mathbb{C})$-invariance of $C_n$, $K_n$ is invariant under the action of the corresponding Weyl group, i.e., it is a symmetric Laurent polynomial in the variables $q_1, q_2$ and $x_1, \ldots, x_n$ separately.

Finally, one can perform the substitution $q_i \mapsto 1 - q_i, x_i \mapsto 1 - x_i$ in $K_n$, expand in power series and keep only the lowest degree terms in those variables. We obtain this way a homogeneous polynomial of degree $\text{codim} \, C_n = n(n-1)$ in $\mathbb{Z}[q_1, q_2, x_1, \ldots, x_n]$ (with its ordinary grading), called the multidegree of $C_n$ and denoted by

$$D_n := \text{mdeg} \, C_n$$

Our first result on the commuting scheme is the following:

**Theorem 3.** Assuming Conjecture [1], the following formula holds for the $K$-polynomial of the commuting scheme:

$$K_n = (q_1q_2)^{\frac{n(n-1)}{2}} f(1_n)$$

with the identification $q = q_1^{-1}, t = (q_1q_2)^{-1}$.

Performing the substitution of variables above, it is convenient to slightly rearrange the Boltzmann weights of §1.2 to:

$$\text{wt}_K(P) = \prod_{i,j=1}^n \begin{cases} 
1 - q_1q_2 & \leftarrow, \\
q_1(1 - q_1q_2) & \leftarrow \\
1 - q_1x_ix_j^{-1} & \leftarrow \\
q_1q_2(1 - q_1x_ix_j^{-1}) & \leftarrow \\
q_2(1 - q_1x_ix_j^{-1}) & \leftarrow \\
x_ix_j^{-1}(1 - q_2x_i^{-1}x_j) & \leftarrow 
\end{cases}$$

Then (7) can be formulated equivalently as

$$K_n = (1 - q_1q_2)^{-n} \sum_{\text{lattice paths } P \atop \text{on the } n \times n \text{ grid}} \text{wt}_K(P)$$

with identity connectivity.
Example 3. At $n = 2$, the weights are

$$\text{wt}_K \begin{pmatrix} \; & \; & \; \\ \; & \; & \; \\ \hline \; & \; & \; \end{pmatrix} = (1 - q_1 q_2)^3(1 - q_2)$$

$$\text{wt}_K \begin{pmatrix} \; & \; & \; \\ \; & \; & \; \\ \hline \; & \; & \; \end{pmatrix} = q_2(1 - q_1 q_2)^2(1 - q_1 x_1 x_2^{-1})(1 - q_1 x_2 x_1^{-1})$$

and one can check that their sum divided by the common factor $(1 - q_1 q_2)^2$ indeed reproduces the $K$-polynomial of $C_2$

$$K_2 = 1 + q_1^2 q_2 - q_1 q_2 x_1 x_2^{-1} - q_1 q_2 - q_1 q_2 x_1^{-1} x_2 + q_1 q_2^2$$

Define another (Boltzmann) weight in a similar fashion:

$$\text{wt}_H(P) = \prod_{i,j=1}^{n} \begin{cases} q_1 + q_2, & \text{if } i = j \\ q_1 + x_i - x_j, & \text{if } |i - j| = 1 \\ q_2 - x_i + x_j, & \text{if } |i - j| = 0 \end{cases}$$

(9)

where colours of paths can be substituted freely.

We can state our second result:

**Theorem 4.** The following formula holds for the multidegree of the commuting scheme:

$$D_n := (q_1 + q_2)^{-n} \sum_{\text{lattice paths } P \text{ on the } n \times n \text{ grid} \atop \text{with identity connectivity}} \text{wt}_H(P)$$

(10)

Note that Conjecture 1 and Theorem 3 imply Theorem 4, as can be easily checked. However Theorem 4 can be proven independently of Conjecture 1.

1.5. Plan of the paper. The rest of the paper is organized as follows. In §2 we provide details on the Hecke algebra construction of lattice paths partition functions, and on the map $\Phi$ from the center of Hecke to symmetric functions. In §3 we give details on the shuffle algebra, the map $\Upsilon$ and derive the recurrence relations which are required for proving Theorems 1-3. In §4 we prove our Theorem 1 by an explicit comparison of $\Phi$ and $\Upsilon \circ f$, and obtain Theorem 2 as a corollary of all that precedes. In §5 we turn to the geometry of the commuting scheme and prove Theorems 3 and 4 ending with some comments.

2. Hecke algebra and lattice paths

2.1. The Hecke algebra and its center. As in §1, we consider the Hecke algebra $H_n$, an $F$-algebra with generators $T_1, \ldots, T_{n-1}$ and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i^2 = (t - 1)T_i + t, \quad T_i T_j = T_j T_i \; (|i - j| \geq 2)$$

In this paper we choose $F$ to be $\mathbb{Q}(q,t)$, though we shall not need the variable $q$ until §2.3. Note $H_0 \cong H_1 \cong F$. 
We have the simple lemma:

**Lemma 1.**

\[
T_i T_w = \begin{cases} 
T_{s_i w} & |s_i w| > |w| \\
(t-1)T_w + t T_{s_i w} & |s_i w| < |w| 
\end{cases}
\]

\[
T_w T_i = \begin{cases} 
T_{w s_i} & |ws_i| > |w| \\
(t-1)T_w + t T_{w s_i} & |ws_i| < |w| 
\end{cases}
\]

expressing the action of \( T_i \) on the standard basis \( (T_w)_{w \in S_n} \), where \( T_w := T_{i_1} \cdots T_{i_k} \) if \( w = s_{i_1} \cdots s_{i_k} \) is a reduced word in the elementary transpositions \( s_i \), and \( |w| = k \).

We shall often need two particular elements of the center \( Z(\mathcal{H}_n) \), namely the complete symmetrizer

\[ S_n = \sum_{w \in S_n} T_w \]  

and the complete antisymmetrizer

\[ A_n = \sum_{w \in S_n} (-t)^{-|w|} T_w \]

Both are primitive idempotents up to normalization.

Next, we consider the shuffle product structure:

**Proposition 1.** The product \( * \) in (11) is a well-defined map, making \( Z = \bigoplus_{n \geq 0} Z(\mathcal{H}_n) \) a graded (associative) algebra.

The nontrivial part of the proof can be formulated by repeatedly applying Lemma 1; instead, we shall illustrate it by using the standard diagrammatic way to depict the Hecke algebra, as acting on \( n \) strands:

\[ T_i = \begin{array}{cccc} 
\cdots & \cdots & \times \times \cdots & \cdots \\
\hline
i-1 & n-i-1 
\end{array} \]

The arrows keep track of the ordering of operators: moving forward w.r.t. the orientation of a line is reading an expression right to left. Sometimes we will omit arrows; then all lines are implicitly oriented downwards. (The diagrammatic calculus that follows can be viewed as warming up for the considerably more difficult proof of our Theorem which will also be diagrammatic.)

**Proof.** Associativity of the product is obvious: an \( m \)-fold product can be expressed as

\[ x_1 \cdots x_m = \sum_{w \in S_{\lambda_1} \cdots \lambda_m} t^{-|w|} T_w (x_1 \otimes \cdots \otimes x_m) T_{w^{-1}} \]

where \( x_i \in \mathcal{H}_{\lambda_i} \) and \( S_{\lambda_1, \ldots, \lambda_m} \) is the set of shortest representatives in \( S_n / S_{\lambda_1, \ldots, \lambda_m} \); \( S_{\lambda_1, \ldots, \lambda_m} = S_{\lambda_1} \times \cdots \times S_{\lambda_m} \).

The only nontrivial part is to show that \( * \) is well-defined as a map from \( Z(\mathcal{H}_k) \otimes Z(\mathcal{H}_\ell) \) to \( Z(\mathcal{H}_{k+\ell}) \), i.e., we need to show that the right hand side of (11) is central, or equivalently that it commutes with all \( T_i \)’s.
We recall that $S^{k,\ell}$ is the set of shortest representatives in $S_{k+\ell}/(S_k \times S_\ell)$; equivalently, it is the set of (Grassmannian) permutations of size $k + \ell$ with a unique possible descent at $(k, k + 1)$.

Fixing $i = 1, \ldots, n - 1$, we distinguish four subsets of $S^{k,\ell}$ according to the preimages of $i$ and $i + 1$:

$$S_{\leq \leq} = \{ w \in S^{k,\ell} : w^{-1}(i) \leq k + 1/2, \ w^{-1}(i + 1) \leq k + 1/2 \}$$

Note that $w \mapsto s_i w$ induces a bijection between $S_{<,>}$ and $S_{>,<}$. We now discuss the corresponding terms in (1) separately:

- If $w \in S_{<,>}$, then $T_w(a \otimes b)T_{w^{-1}}$ commutes with $T_i$ provided $a \in Z(\mathcal{H}_k)$, as shown diagrammatically:

$$T_w(a \otimes b)T_{w^{-1}}T_i = T_iT_w(a \otimes b)T_{w^{-1}}$$

- Similarly, if $w \in S_{>,>}$, then $T_w(a \otimes b)T_{w^{-1}}$ commutes with $T_i$ provided $b \in Z(\mathcal{H}_\ell)$.

- Finally, we group together the two terms corresponding to $w \in S_{<,>}$ and $s_i w \in S_{>,<}$ to obtain:

$$(T_w(a \otimes b)T_{w^{-1}} + t^{-1}T_{s_i w}(a \otimes b)T_{w^{-1} s_i})T_i$$

Clearly, the same result is obtained by multiplying by $T_i$ on the left.
By multiplying by $t^{-|w|}$ and summing over $w$, we obtain the desired commutation property for the right hand side of (1). □

We define an algebra morphism $\Psi$ from $\Lambda = \mathbb{F}[p_1, p_2, \ldots]$ to $\mathbb{Z}$ as follows. Define Jucys–Murphy elements in $\mathcal{H}_n$ (see e.g. [Ram97]) to be

$$J_{j,n} = \sum_{i=1}^{j-1} t^{i-j+1} T_i T_{i+1} \ldots T_{j-1} T_{j-2} \ldots T_i, \quad j = 2, \ldots, n$$

They form a commutative subalgebra of $\mathcal{H}_n$, and it is well-known that symmetric polynomials in the $J_{j,n}$ are central. Then $\Psi$ sends $p_n$ to the element

$$\Psi(p_n) = [n]_{t-1} \prod_{j=2}^{n} J_{j,n}$$

where $[n]_{t-1} = 1 + \cdots + t^{-(n-1)}$.

**Proposition 2** ([WW15]). $\Psi$ is a graded algebra isomorphism from $\Lambda$ to $\mathbb{Z}$.

Its inverse is the $\Phi$ of §1.4 As a corollary, $\mathbb{Z}$ is commutative.

Denote by

\begin{align*}
   \tilde{h}_n &= \Phi(1_n) \\
   \tilde{e}_n &= \Phi(t^{-\frac{n(n-1)}{2}} T_{w_0}) \\
   h_n &= \Phi(S_n) \\
   e_n &= \Phi(A_n)
\end{align*}

where $1_n$ is the identity of $\mathcal{H}_n$, and $w_0$ the longest element of $S_n$. We can identify these symmetric functions. Given $u \in \mathbb{F}$, define $\sigma_u$ to be the automorphism of $\Lambda$

$$\sigma_u : \ p_r \mapsto \frac{(1 - u)^r}{1 - u^r} p_r$$

**Proposition 3** ([Las06]). $h_n$ (resp. $e_n$) is the complete (resp. elementary) symmetric function of degree $n$.

Furthermore, $\tilde{h}_n = \sigma_{i_2}(h_n)$ and $\tilde{e}_n = \sigma_{i_2}(e_n)$.

We also denote by $\sigma_{\infty} : \ p_r \mapsto (-1)^{r-1} p_r$. Note $\sigma_{u-1} = \sigma_{\infty} \sigma_u$. It is well-known that $\sigma_{\infty}$ exchanges $h_n$ and $e_n$ (and similarly for $\tilde{h}_n$ and $\tilde{e}_n$).

2.2. Partition functions from the Hecke algebra. We now work inside the algebra $\mathcal{H}_{2n}$. There are two obvious embeddings of $\mathcal{H}_n$ into $\mathcal{H}_{2n}$, given by $T_i \mapsto T_i$ and $T_i \mapsto T_{i+n}$ respectively, and we denote the corresponding subalgebras by $\mathcal{H}_n^{(1)}$ and $\mathcal{H}_n^{(2)}$. Similarly, denote by $S_2^{(1)}$ and $S_2^{(2)}$ the two subgroups of $S_{2n}$ (both isomorphic to $S_n$) acting nontrivially on $\{1, \ldots, n\}$ (resp. $\{n+1, \ldots, 2n\}$) only.

Also define the $R$-matrix

$$\hat{R}_i(u) = 1 - t + (1 - u) T_i \quad i = 1, \ldots, 2n - 1$$

as an element of $\mathcal{H}_{2n}[u^\pm]$. 

Extending the diagrammatic calculus introduced in the proof of Proposition 1, we depict $R$-matrices as (flat) crossings of two lines carrying variables (so-called spectral parameters), the parameter of the $R$-matrix being the ratio of spectral parameters:

$$\begin{array}{c}
\cdots \\
\begin{array}{c}
\cdots \\
i - 1 \\
2n - i - 1
\end{array}
\end{array}$$

so that (19) can be expressed as:

$$\begin{array}{c}
\cdots \\
\begin{array}{c}
\cdots \\
u''/u'
\end{array}
\end{array}$$

The Yang–Baxter equation is known to hold:

$$\begin{array}{c}
\cdots \\
\begin{array}{c}
\cdots \\
v
\end{array}
\end{array}$$

as an identity in $H_{2n}[u^\pm, v^\pm]$, as well as the unitarity equation

$$\begin{array}{c}
\cdots \\
\begin{array}{c}
\cdots \\
u''/u'
\end{array}
\end{array}$$

The central object of study of this section is the “formal partition function”, which is the following element of $H_{2n}[x^\pm_1, \ldots, x^\pm_n, y^\pm_1, \ldots, y^\pm_n]$:

$$\begin{array}{c}
\cdots \\
\begin{array}{c}
\cdots \\
y_1 \\
y_n \ldots \\
x_1 \\
x_n \\
\vdots \\
\vdots
\end{array}
\end{array}$$

We also write $Z = Z(x_1, \ldots, x_n, y_1, \ldots, y_n)$ when there is no need to emphasize the dependence on the variables.

$Z$ has an expansion of the form

$$Z = \sum_{w \in S_{2n}} Z_w T_w$$

where each $Z_w \in \mathbb{Z}[t^\pm, x^\pm_1, \ldots, x^\pm_n, y^\pm_1, \ldots, y^\pm_n]$. 


2.3. Exchange relation and symmetry. Consider the operator $\tau_i$ (resp. $\tau'_j$) that permutes variables $x_i$ and $x_{i+1}$ (resp. $y_j$ and $y_{j+1}$), $i, j = 1, \ldots, n - 1$.

We have the following fundamental lemma:

**Lemma 2.** $Z$ satisfies the exchange relations:

$$
(\tau_i Z) \tilde{R}_{n+i}(x_{i+1}/x_i) = \tilde{R}_i(x_{i+1}/x_i)Z \quad i = 1, \ldots, n - 1
$$

**Proof.** Both equalities result from repeated application of the Yang–Baxter equation \((20)\). \qed

We now consider the specialization $y_i = qx_i$, $i = 1, \ldots, n$. We first note

**Lemma 3.** $Z_{\mid y_i = qx_i} \in \mathbb{F}[x_1^{\pm}, \ldots, x_n^{\pm}]_0$.

**Proof.** Recall that each of the $n^2$ factors of $Z$ corresponds to some vertex $(i, j)$ on the diagram of \((22)\), and is a polynomial of degree 1 in the parameter $x_i/y_j$, so after specialization of the Lemma, in $x_i/(qx_j)$. Now fix $r \in \{0, \ldots, n\}$ and consider the degree $i_1 + \cdots + i_r$ in the first $r$ variables $x_1, \ldots, x_r$. Among the factors of $Z$, the ones corresponding in the diagram to the top-left $r \times r$ square have degree zero, and similarly for the bottom $(n - r) \times (n - r)$ square. The ones for the top-right $r \times (n - r)$ square can have monomials of degree 0 or 1, whereas for the bottom-left $(n - r) \times r$ square they can be of degree 0 or $-1$. Summing these degrees, we find the desired inequality $|i_1 + \cdots + i_r| \leq r(n - r)$.

Of course the same proof would work with any $r$-subset of variables, suggesting that we should study the behaviour of $Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n)$ under the interchange of the variables $x_i$. We have the following condition:
Proposition 4. Let \( \chi \) be a linear form on \( \mathcal{H}_{2n} \). \( \chi(Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n)) \) is a symmetric Laurent polynomial in the \( x_i \)'s provided \( \chi \) satisfies
\[
\chi(ab) = \chi(ba) \quad \forall a \in \mathcal{H}_{2n}, \; b \in \mathcal{H}_{n}^{(1)} \otimes \mathcal{H}_{n}^{(2)}
\]

Proof. We assume \( n \geq 2 \), otherwise the Proposition is trivial. By definition exchanging \( x_i \) and \( x_{i+1} \) in \( \chi(Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n)) \) amounts to considering \( \chi(\tau_i \tau'_i Z) \) and then substituting \( y_i = qx_i \). One has
\[
\chi(\tau_i \tau'_i Z) = \chi(\hat{R}_i(x_{i+1}/x_i) \hat{R}_{n+i}(y_i/y_{i+1})^{-1} Z \hat{R}_i(y_i/y_{i+1}) \hat{R}_{n+i}(x_{i+1}/x_i)^{-1})
\]
Assuming the property (26) of the Proposition, one can rewrite this
\[
\chi(\tau_i \tau'_i Z) = \chi(\hat{R}_i(y_i/y_{i+1}) \hat{R}_i(x_{i+1}/x_i) Z \hat{R}_{n+i}(y_i/y_{i+1})^{-1} \hat{R}_{n+i}(x_{i+1}/x_i)^{-1})
\]
Substituting \( y_i = qx_i \) and simplifying using unitarity equation (21), we obtain the desired symmetry of \( \chi(Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n)) \).

In what follows, we restrict ourselves to the following class of linear forms. Denote by \( \langle \bullet \rangle \) the linear form that extracts the coefficient of 1 in a linear combination of \( T_w \), i.e., \( \langle T_w \rangle := \delta_{w,1} \). One checks that
\[
\langle ab \rangle = \langle ba \rangle = \sum_{w \in S_{2n}} t_{[w]} a_w b_{w^{-1}}, \quad a = \sum_{w \in S_{2n}} a_w T_w, \; b = \sum_{w \in S_{2n}} b_w T_w
\]
Given \( c \in Z(\mathcal{H}_n) \), we then define
\[
f(c) := \alpha_n \langle (c \otimes S_n)Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n) \rangle
\]
where \( S_n \) was defined in (11), and \( \alpha_n = (q/t)^{n(n-1)/2} (1-t)^{-n} \) is the same constant as in the introduction. Clearly, \( \chi = \langle (c \otimes S_n)\bullet \rangle \) fulfills the condition of Proposition 4.

Example 4. Three important examples are:
- the identity: \( f(1_n) \) is the object of (11, 12) and [6]
- the complete antisymmetrizer: \( f(A_n) \) will be computed in (4)
- the complete symmetrizer: \( f(S_n) \) is also interesting, since the symmetrization makes the model "color-blind", and thus equal to the famous Izergin determinant [Ize87, Kor82] for the partition function of the six-vertex model with domain wall boundary conditions, as can be shown by the usual symmetry and recurrence arguments [Kor82]. We leave the proof to the interested reader.

We finally introduce a lattice path representation of \( f(c) \), reconnecting it to the content of §12.

2.4. Lattice path representation. Consider the following representation of \( \mathcal{H}_{2n} \). As a vector space, \( \mathcal{M} = (\mathbb{F}^{n+1})^{\otimes 2n} \). Letting \( (e_i)_{i=1,\ldots,n+1} \) be the standard basis of \( \mathbb{F}^{n+1} \), define the action of \( \mathcal{H}_{2n} \) by
\[
T_i(e_{k_1} \otimes \cdots \otimes e_{k_{2n}}) = \sum_{k_i', k_{i+1}'} \left[ \begin{array}{c} k_i \\ k_{i+1} \end{array} \right] \left[ \begin{array}{c} k_i' \\ k_{i+1}' \end{array} \right] e_{k_1} \otimes \cdots \otimes e_{k_i-1} \otimes e_{k_i'} \otimes e_{k_i+1} \otimes e_{k_{i+2}} \otimes \cdots \otimes e_{k_{2n}}
\]
where

\[
\begin{bmatrix}
  i & j & k \\
  \ell & & \\
\end{bmatrix} =
\begin{cases}
  t - 1 & i = j > k = \ell \\
  1 & j = k > i = \ell \\
  t & j = k \leq i = \ell \\
  0 & \text{else}
\end{cases}
\]

(28)

Given \( v \in \mathcal{S}_{2n} \), define

\[
|v| = e_{\omega(v(1))} \otimes \cdots \otimes e_{\omega(v(2n))}
\]

where \( \omega(i) = i \) for \( 1 \leq i \leq n \) and \( \omega(i) = n + 1 \) for \( n + 1 \leq i \leq 2n \). Note that \(|v|\) only depends on the class of \( v \) inside \( \mathcal{S}_n^{(2)} \backslash \mathcal{S}_{2n} \); we define this way a basis indexed by \( \mathcal{S}_n^{(2)} \backslash \mathcal{S}_{2n} \) of the submodule of \( \mathcal{M} \) of interest to us.

Similarly, define dual basis elements

\[
\langle v \rangle = e_{\omega(v(1))}^* \otimes \cdots \otimes e_{\omega(v(2n))}^*
\]

in \( \mathcal{M}^* \).

We have the easy lemma

**Lemma 4.** For all \( w \in \mathcal{S}_{2n} \),

(a) \( \langle 1 | T_w = t^{|w|} | w \rangle \).

(b) \( T_w | 1 \rangle = t^{|w_2|} | w^{-1} \rangle \) where \( w_2 \) is defined by \( w^{-1} = w_1 w_2 \), \( w_1 \in \mathcal{S}_n^{(2)} \) and \( w_2 \) a minimal representative in \( \mathcal{S}_n^{(2)} \backslash \mathcal{S}_{2n} \).

**Proof.** (a) Write \( w = w_1 w_2 \) where \( w_1 \in \mathcal{S}_n^{(2)} \) and \( w_2 \) minimal representative in \( \mathcal{S}_n^{(2)} \backslash \mathcal{S}_{2n} \). One has directly \( \langle 1 | T_w = t^{|w_1|} \langle 1 | \). Then use a reduced decomposition of \( w_2 \) and note that in (28), one only ever uses the case \( j < \ell \) which leads to \( \langle 1 | T_{w_2} = t^{|w_2|} \langle w_2 | \) inductively. At both stages, the only nonzero contributions come from the third line of (28). Case (b) is similar except the first (resp. second) stage involves the second (resp. third) line of (28).

Define, noting that \( \mathcal{S}_n^{(1)} \) naturally sits inside \( \mathcal{S}_n^{(2)} \backslash \mathcal{S}_{2n} \),

\[
F_v(x_1, \ldots, x_n, y_1, \ldots, y_n) = \langle 1 | Z(x_1, \ldots, x_n, y_1, \ldots, y_n) | v \rangle \quad v \in \mathcal{S}_n^{(1)}
\]

(29)

This is a slightly more general definition than we need for the interpretation of the right hand side of (22); indeed, we have:

**Proposition 5.**

\[
\sum_{\text{lattice paths } P \text{ on the } n \times n \text{ grid } \text{ cons}(P)=v} \text{wt}(P) = F_v(x_1, \ldots, x_n, qx_1, \ldots, qx_n)
\]

**Proof.** Consider the diagrammatic representation (22) of \( Z \) and apply the representation described at the start of this section; this amounts to assigning to each edge a label in \( \{1, \ldots, n + 1\} \). \( F_v \) corresponds, in accordance with (29), to further imposing that labels of external edges be fixed: \( 1, \ldots, n \) on the left side, \( v(1), \ldots, v(n) \) on the top side, and \( n + 1 \) on the right and bottom sides. Now identify labels \( 1, \ldots, n \) with the corresponding lattice paths, and \( n + 1 \) with “empty”. Then this produces precisely a lattice path with connectivity \( P \). Finally, what needs to be checked is that the weight assigned to a lattice path coincides with the entry of \( Z \). This can be done locally at the level of each individual crossing (\( R \)-matrix); and indeed, combining (28) with (19) and setting \( y_i = qx_i \) leads to the weights in (22). □
Lemma 5. If \( c = \sum_{v \in S_{n}} c_{v} T_{v} \in Z(\mathcal{H}_{n}) \), then \( c_{v} = c_{v^{-1}} \) for all \( v \in S_{n} \) and
\[
f(c) = \alpha_{n} \sum_{v \in S_{n}} c_{v} F_{v}(x_{1}, \ldots, x_{n}, qx_{1}, \ldots, qx_{n})
\]
Proof. First we observe that
\[
\langle x (1 \otimes S_{n}) \rangle = \langle 1 | x | 1 \rangle \quad \forall x \in \mathcal{H}_{2n}
\]
by testing this on \( x = T_{w}, w \in S_{2n} \), and noting that both sides of the equality are 0 unless \( w \in S_{2n}^{(2)} \), in which case they are equal to \( l^{\|w\|} \) (using the definition (11) for the left hand side and Lemma 4 for the right hand side). We now start from the definition (27) of \( f(c) \) and apply the equality above, as well as Lemma 4 (b):
\[
f(c) = \alpha_{n} \langle 1 | Zc | 1 \rangle = \alpha_{n} \sum_{v \in S_{n}} c_{v^{-1}} F_{v}(x_{1}, \ldots, x_{n}, qx_{1}, \ldots, qx_{n})
\]
This is almost (4), except a small simplification was made: any central element satisfies \( c_{v} = c_{v^{-1}} \) (short proof: it is obviously true for the generators \( S_{n} \) of \( Z \), and \( * \) preserves this property).

We conclude this section by reformulating more explicitly the exchange relations of Lemma 2 in terms of the \( F_{v} \). The next proposition will not be used in what follows, but is included to reconnect to the existing literature [BW13].

Proposition 6. For \( v \in S_{n}^{(1)} \) and \( i, j = 1, \ldots, n - 1 \),
\[
(1 - tx_{i+1}/x_{i}) \tau_{i} F_{v} = \begin{cases} 
(1 - t)F_{v} + (1 - x_{i+1}/x_{i})F_{s_{i}v} & s_{i}v > v \\
(1 - t)(x_{i+1}/x_{i})F_{v} + t(1 - x_{i+1}/x_{i})F_{s_{i}v} & s_{i}v < v 
\end{cases}
\]
\[
(1 - ty_{j}/y_{j+1}) \tau_{j} F_{v} = \begin{cases} 
(1 - t)F_{v} + (1 - y_{j}/y_{j+1})F_{v s_{j}} & v s_{j} > v \\
(1 - t)(y_{j}/y_{j+1})F_{v} + t(1 - y_{j}/y_{j+1})F_{v s_{j}} & v s_{j} < v 
\end{cases}
\]
Proof. First, note that (23), (29), and Lemma 4 (a) imply
\[
F_{v} = \sum_{w \in S_{n}^{(2)}} t^{\|w\|} Z_{w}
\]
where the summation is over the coset of \( v \) in \( S_{n}^{(2)} \setminus S_{2n} \).

We then start from the identity (21) of Lemma 2.

Using (23) and Lemma 1 we obtain for the right hand side:
\[
\tilde{R}_{i}(x_{i+1}/x_{i}) Z = \sum_{w \in S_{2n}} Z_{w} \begin{cases} 
(1 - t)T_{w} + (1 - x_{i+1}/x_{i})T_{s_{i}w} & s_{i}w > w \\
(1 - t)(x_{i+1}/x_{i})T_{w} + t(1 - x_{i+1}/x_{i})T_{s_{i}w} & s_{i}w < w 
\end{cases}
\]
We now take the bra-ket \( \langle 1 | \cdot | v \rangle \) and note that only \( w \) such that either \( w \in S_{n}^{(2)} v \) or \( w \in S_{n}^{(2)} s_{i}v \) contribute. We get for \( s_{i}v > v \)
\[
\langle 1 | \tilde{R}_{i}(x_{i+1}/x_{i}) Z | v \rangle = (1 - t) \sum_{w \in S_{n}^{(2)}} t^{\|w\|} Z_{w} + t(1 - x_{i+1}/x_{i}) \sum_{w \in S_{n}^{(2)} s_{i}v} t^{\|s_{i}w\|} Z_{w}
\]
\[
= (1 - t)F_{v} + (1 - x_{i+1}/x_{i})F_{s_{i}v}
\]
and for $s_iv < v$

$$\langle 1 | \tilde{R}_i(x_{i+1}/x_i)Z | v \rangle = (1 - t)(x_{i+1}/x_i) \sum_{w \in S_n^{(2)}v} t^{|w|}Z_w + (1 - x_{i+1}/x_i) \sum_{w \in S_n^{(2)}s_iv} t^{|w|}Z_w$$

$$= (1 - t)(x_{i+1}/x_i)F_v + t(1 - x_{i+1}/x_i)F_{s_iv}$$

The bra-ket $\langle 1 | v \rangle$ of the left hand side is simple to evaluate: we note that $\tilde{R}_{n+i}(x_{i+1}/x_i) | v \rangle = (1 - tx_{i+1}/x_i) | v \rangle$ because the $t$th and $(i + 1)$th factors of $| v \rangle$ are $e_{n+1} \otimes e_{n+1}$; therefore

$$\langle 1 | (\tau_iZ)\tilde{R}_{n+i}(x_{i+1}/x_i) | v \rangle = (1 - tx_{i+1}/x_i)\tau_iF_v$$

This proves the first identity of the Proposition.

The second identity is actually simpler to prove (there isn’t a full symmetry between rows and columns). Start analogously from (25) of Lemma 2. Now apply the bra-ket $\langle 1 | \cdot | v \rangle$ directly. One has $\langle 1 | \tilde{R}_{n+j}(y_j/y_{j+1}) = (1 - ty_j/y_{j+1}) | 1 \rangle$ by the same argument as above; but one also has from (25)

$$\tilde{R}_j(y_j/y_{j+1}) | v \rangle = \begin{cases} (1 - t) | v \rangle + (1 - y_j/y_{j+1}) | vs_j \rangle & v s_j > v \\ (y_j/y_{j+1})(1 - t) | v \rangle + t(1 - y_j/y_{j+1}) | vs_j \rangle & v s_j < v \end{cases}$$

which leads to the desired identity.

These relations can be used to determine the $F_v$ inductively. More precisely, consider first the computation of $F_{w_0}$ where $w_0$ is the longest element of $S^{(1)}_n$. There is a unique lattice path with such connectivity, of the form

```
  1  2  3  4  5
  |
  |
  |
  |
  |
```

from which we conclude

$$F_{w_0} = (1 - t)^n t^{n(n-1)/2} \prod_{1 \leq i,j \leq n, i+j < n+1} (1 - x_i/y_j) \prod_{1 \leq i,j \leq n, i+j > n+1} (1 - tx_i/y_j)$$

One can then apply either the second relation or the fourth relation of Proposition 6 to define the $F_w$ inductively.

3. The shuffle algebra

This section is devoted to the trigonometric Feigin–Odesskii shuffle algebra $[FO97]$. It was studied notably in $[FHH+09, FTI11, SV13, Neg14]$. We will review some known results and derive several formulas which are required for proving Theorems 11. We will follow the conventions of $[FHH+09]$ and define the shuffle algebra $A$ which is slightly different from the algebra $A$ in §1. The precise connection between $A$ and $A$ is given in Remark 11. For the treatment of the shuffle algebra $A$ it is convenient to use three parameters $q_1, q_2, q_3$, expressed via $q$ and $t$ as

$$q_1 = q^{-1} \quad q_2 = qt^{-1} \quad q_3 = (q_1q_2)^{-1} = t$$

Let us outline the goals of this section:
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• Discuss the shuffle algebra $A^+$, its commutative subalgebra $A$ and the isomorphism $\Upsilon$ of Theorem 1
• Give a characterization of the distinguished basis elements $\epsilon_\lambda$ of the shuffle algebra $A$ via specializations which will be used in \S 4 to prove Theorem 1
• Introduce a special shuffle element $\kappa_n$ which will be used in \S 4 to prove Theorem 2 and in \S 5.1 to prove Theorem 3

3.1. The shuffle algebra $A^+$. The shuffle algebra $A^+$ is a vector space whose elements are symmetric rational functions of the form

\[ P(x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2} \quad p(x_1, \ldots, x_n) \in F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^S \]

where $p(x_1, \ldots, x_n)$ satisfies the wheel condition

\[ p(x_1, \ldots, x_n) = 0 \quad \text{if} \quad (x_i, x_j, x_k) = (x, q_1 x, q_2 x) \quad \text{or} \quad (x_i, x_j, x_k) = (x, q_2 x, q_1 x) \]

The number of arguments $(x_1, \ldots, x_n)$ gives a grading

\[ A^+ = \bigoplus_{n \geq 0} A^+_n \]

We set $A^+_0 = F$ and $A^+_1 = F[x^{\pm 1}]$. The shuffle algebra $A^+$ is closed under the shuffle product $*$, defined for two elements $F \in A^+_k$ and $G \in A^+_l$ by

\[ (F * G)(x_1, \ldots, x_{k+l}) = \frac{1}{k! l!} \text{Sym} \left[ F(x_1, \ldots, x_k)G(x_{k+1}, \ldots, x_{k+l}) \prod_{1 \leq i \leq k \leq l} \omega(x_{k+j}, x_i) \right] \]

where $\text{Sym}$ is the symmetrization over all arguments $x_i$

\[ \text{Sym} [f(x_1, \ldots, x_n)] = \sum_{w \in S_n} f(x_{w(1)}, \ldots, x_{w(n)}) \]

and $\omega(x, y)$ is defined as follows

\[ \omega(x, y) = \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3} \]

It was shown in [Neg14] that the algebra $A^+$ is generated by the elements $x^i \in A^+_1$ for $i \in \mathbb{Z}$.

3.2. The commutative subalgebra $A$. Consider a subalgebra $A \subset A^+$ of the elements $P \in A^+_n$ for which the two limits

\[ \lim_{\xi \to 0} P(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n), \]

\[ \lim_{\xi \to \infty} P(\xi x_1, \ldots, \xi x_r, x_{r+1}, \ldots, x_n) \]

exist and coincide for all fixed $r = 1, \ldots, n$. This subalgebra splits into components of fixed degree in the same way as $A^+$

\[ A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_n \]
Remark 1. The shuffle algebras $\mathcal{A}_n$ and $\mathcal{A}_n$ are isomorphic. For $P(x_1, \ldots, x_n) \in \mathcal{A}_n$ and $Q(x_1, \ldots, x_n) \in \mathcal{A}_n$, s.t. $Q \mapsto P$, we have
\begin{equation}
P(x_1, \ldots, x_n) = \frac{1}{V_n(x_1, \ldots, x_n)} Q(x_1, \ldots, x_n)
\end{equation}
where $V_n = V_n(x_1, \ldots, x_n)$ is defined by
\begin{equation}
V_n := \prod_{1 \leq i < j \leq n} (1 - x_i/x_j)
\end{equation}

Proof. The product $V_n^{-1}Q(x_1, \ldots, x_n)$ is of the form (34) and the wheel conditions of $\mathcal{A}_n^+$ and $\mathcal{A}_n$ coincide. The fact that the limiting conditions (39) and (40) of $\mathcal{A}_n$ are equivalent to the degree constraints (2) of $\mathcal{A}_n$ was shown in [FHH+09]. \[\square\]

Another important result of [FHH+09] is the following proposition.

Proposition 7. The algebra $(\mathcal{A}, *)$ is commutative and the dimension of the graded subspace $\mathcal{A}_n$ is equal to the number of partitions of $n$.

There are several distinguished bases in the space $\mathcal{A}$. These bases are built using the shuffle elements
\begin{equation}
\epsilon_n(x_1, \ldots, x_n; q_k) = \prod_{1 \leq i < j \leq n} \frac{(x_i - q_k x_j)(x_i - q_k^{-1} x_j)}{(x_i - x_j)^2}
\end{equation}
In the following we will abbreviate them as $\epsilon_n(q_k) = \epsilon_n(x_1, \ldots, x_n; q_k)$. Due to commutativity the products $\epsilon_i(q_k) * \epsilon_j(q_k)$ can be ordered such that $i > j$. Thus for a partition $\lambda$ the elements
\begin{equation}
\epsilon_\lambda(q_k) = \epsilon_{\lambda_1}(q_k) * \epsilon_{\lambda_2}(q_k) * \cdots * \epsilon_{\lambda_{|\lambda|}}(q_k)
\end{equation}
give three bases, one for each $k = 1, 2, 3$.

3.3. Macdonald operators and the isomorphism $\Upsilon$. Define the Heisenberg algebra $\mathfrak{h}$ with generators $a_{\pm n}$, $n > 0$, satisfying the relations
\begin{equation}
[a_m, a_n] = \delta_{m,-n} m \frac{1 - q^{|m|}}{1 - q^{|m|}}
\end{equation}
We have two commutative subalgebras generated by $\{a_{\pm n}\}_{n>0}$ and containing the elements $a_{\pm \lambda} = a_{\pm \lambda_1} \cdots a_{\pm \lambda_{|\lambda|}}$. Define the Fock space $\mathcal{F}$ by introducing the vacuum vector $|\emptyset\rangle$ which is annihilated by the positive modes $a_n |\emptyset\rangle = 0$, for $n > 0$. The action of the negative modes creates a basis
\begin{equation}
|a_\lambda\rangle = a_{-\lambda} |\emptyset\rangle
\end{equation}
for all partitions $\lambda$.

Let $x = (x_1, x_2, \ldots)$ be the alphabet for the ring of symmetric functions $\Lambda$ with the base field $\mathbb{F}$. Recall the power sum basis of $\Lambda$
\begin{equation}
p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{|\lambda|}} \quad p_n = x_1^n + x_2^n + \ldots
\end{equation}
The Fock space $\mathcal{F}$ is isomorphic to $\Lambda$ where the isomorphism is explicitly given by
\begin{equation}
|a_\lambda\rangle \mapsto p_\lambda
\end{equation}
In $\Lambda$ a symmetric function is considered in the power sum basis and the above isomorphism replaces it with a vector in the Fock space. A distinguished basis in $\Lambda$ is given by the Macdonald symmetric functions $P_\lambda$ (see [Mac98]). We view it as a basis in the Fock space denoted by $|P_\lambda\rangle$. 


The Macdonald operators are defined as the operators which act diagonally on $P_\lambda$. The first Macdonald operator $E$ [Mac98, Ch. VI] acts as follows
\begin{equation}
E |P_\lambda\rangle = \sum_{i \geq 1} (q^{\lambda_i} - 1)t^{-i} |P_\lambda\rangle
\end{equation}
This operator is realized on the Fock space with the help of the vertex operator [Shi06]
\begin{equation}
\eta(z) := \exp \left( \sum_{n > 0} \frac{1 - t^{-n}a_n z^n}{n} \right) \exp \left( - \sum_{n > 0} \frac{1 - t^{n}a_n z^{-n}}{n} \right)
\end{equation}
The Fourier modes of this operator $\eta_n (n \in \mathbb{Z})$ are given by
\begin{equation}
\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n}
\end{equation}
and have a well defined action on $\mathcal{F}$. The operator $E$ is given by the zero mode [Shi06]
\begin{equation}
\eta_0 = (t - 1)E + 1
\end{equation}
We can write the eigenvalue equation as follows
\begin{equation}
\frac{t^{-1}}{1 - t^{-1}} \eta_0 |P_\lambda\rangle = u_\lambda(e_1) |P_\lambda\rangle
\end{equation}
where
\begin{equation}
u_\lambda(x_i) = q^{\lambda_i}t^{-i}
\end{equation}
and $e_1$ is the first elementary symmetric function in the infinite alphabet $e_1 = x_1 + x_2 + \ldots$. One can build higher order operators in a systematic way using the algebra $\mathcal{A}$ and the operators $\eta$.

The product of Heisenberg operators is normally ordered if all positive modes are on the right and negative modes on the left. The product $\eta(z)\eta(w)$ can be ordered at the expense of a rational function
\begin{equation}
\eta(z)\eta(w) = \frac{(1 - w/z)(1 - qt^{-1}w/z)}{(1 - qw/z)(1 - t^{-1}w/z)} : \eta(z)\eta(w) :
\end{equation}
where the operators inside :: are normally ordered. For $f \in \mathcal{A}_n$ define the mapping $\mathcal{O} : \mathcal{A} \to \text{End}_F(\mathcal{F})$ by
\begin{equation}
\mathcal{O}(f) = \frac{1}{(t - 1)^n n!} \oint \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} f(z_1, \ldots, z_n) e_n(z_1, \ldots, z_n; q_1)e_n(z_1, \ldots, z_n; q_3) : \eta(z_1) \cdots \eta(z_n) :
\end{equation}
where the integration contours are given by $|z_1| = \cdots = |z_n| = 1$ and $|t|, |q| < 1$. It was shown in [FHH+09] that $\mathcal{O}$ is compatible with the shuffle product, i.e.
\begin{equation}
\mathcal{O}(f * g) = \mathcal{O}(f)\mathcal{O}(g)
\end{equation}
Since $\mathcal{A}$ is commutative this implies the commutativity of the operators $[\mathcal{O}(f), \mathcal{O}(g)] = 0$ for any two elements $f, g \in \mathcal{A}$. The map $\mathcal{O}$ was used in [FHH+09] to define an isomorphism $\Upsilon' : \mathcal{A} \to \Lambda$:

**Proposition 8.** Given $f \in \mathcal{A}_n$, there exists an $S \in \Lambda_n$ such that for any partition $\lambda$,
\begin{equation}
\mathcal{O}(f) |P_\lambda\rangle = u_\lambda(S) |P_\lambda\rangle
\end{equation}
and $\Upsilon' : f \mapsto S$ thus defined is a ring isomorphism from $\mathcal{A}$ to $\Lambda$. 
By inserting a $V_n^{-1}$ in (54), cf Remark [1], we obtain the isomorphism $\Upsilon$ from $A$ to $\Lambda$ that is stated in the introduction and used in Theorem [1]. The symmetric functions in $\Lambda$ which correspond to the shuffle elements $\epsilon_n(q_k)$, $k = 1, 2, 3$ are all given by a plethystic substitution applied to the elementary symmetric functions $e_n$. Recalling [18] we set

$$e_n^{(1)} = e_n \quad e_n^{(2)} = \sigma_{q^{-1}}\sigma_{q^{-1}}^{-1}(e_n) \quad e_n^{(3)} = \sigma_{q^{-1}}\sigma_{q^{-1}}^{-1}(e_n)$$

Note that the plethystic substitution in $e_n^{(3)}$ appears in the statement of Theorem [1]. This element will play a special role. We have [FHH+09]

$$\Upsilon'(\epsilon_n(q_k)) = e_n^{(k)} \quad k = 1, 2, 3$$

or equivalently

$$\mathcal{O}(\epsilon_n(q_k)) |P_\lambda\rangle = u_\lambda(e_n^{(k)}) |P_\lambda\rangle \quad k = 1, 2, 3$$

The eigenvalue equation (57) for $k = 1$ shows that the operator $\mathcal{O}(\epsilon_n(q_1))$ coincides with the $n^{th}$ Macdonald operator [Mac98]. Thus one shows (57) by matching the Macdonald difference operators $E_\nu$ with $\mathcal{O}(\epsilon_n(q_1))$. Based on this result one can show the other two eigenvalue equations for $k = 2, 3$ using the Wronski relations

$$\sum_{j=0}^{n}(q_k^j - q_l^{-n-j})\frac{(1 - q_l^j)}{(1 - q_k^j)}\epsilon_{n-j}(q_k) \ast \epsilon_j(q_l) = 0 \quad k, l = 1, 2, 3$$

In [FHH+09] the authors proved these Wronski relations. Applying $\mathcal{O}$ to (58) with $l = 1, k = 2, 3$ gives a relation between the eigenvalues of $\mathcal{O}(\epsilon_n(q_1))$ and $\mathcal{O}(\epsilon_n(q_k))$ which determines the eigenvalue in (57) for $k = 2, 3$.

We finish this section by giving a lemma which will be used at a later stage.

**Lemma 6.** The following identities hold

$$\sum_{i=1}^{n} x_i \epsilon_{n-1}(q_1)[\hat{x}_i] = \frac{1}{1 - q_1} \sum_{i=1}^{n} x_i \epsilon_{n-1}(q_1)[\hat{x}_i]$$

$$\prod_{j=1}^{n} (x_j - q_1 x_i)(x_j - q_2 x_i)(x_j - q_3 x_i) - q_1 \prod_{j=1}^{n} (x_i - q_1 x_j)(x_i - q_2 x_j)(x_i - q_3 x_j)$$

where $\epsilon_{n-1}(q_1)[\hat{x}_i]$ depends on the $n - 1$ arguments $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Noting that

$$\epsilon_{n-1}(q_1)[\hat{x}_i] \prod_{j=1}^{n} (x_j - q_1 x_i) / (x_j - x_i) = \epsilon_n(q_1) \prod_{j=1}^{n} (x_j - x_i) / (x_j - q_1^{-1} x_i)$$

$$\epsilon_{n-1}(q_1)[\hat{x}_i] \prod_{j=1}^{n} (x_i - q_1 x_j) / (x_i - x_j) = \epsilon_n(q_1) \prod_{j=1}^{n} (x_i - x_j) / (x_i - q_1^{-1} x_j)$$
leads to
\[ C_n = \epsilon_n(q_1) \frac{1}{1 - q_1} \sum_{i=1}^{n} x_i \left( \prod_{j=1 \atop j \neq i}^{n} \frac{(x_j - q_2 x_j)(x_j - q_3 x_i)}{(x_j - q_1 x_j)(x_j - x_i)} - q_1 \prod_{j=1 \atop j \neq i}^{n} \frac{(x_i - q_2 x_j)(x_i - q_3 x_j)}{(x_i - q_1 x_j)(x_i - x_j)} \right) \]

The factor next to \( \epsilon_n(q_1) \) does not have poles at \( x_i = x_j \) or \( x_i = q_1^{-1} x_j \) because of the symmetry in \( x_i \) and vanishing residues at \( x_i = q_1^{-1} x_j \) which can be easily verified. It means that this factor must be a symmetric polynomial of degree 1. Thus it is proportional to the elementary symmetric polynomial \( e_1 \) in \( n \) arguments
\[ \frac{1}{1 - q_1} \sum_{i=1}^{n} x_i \left( \prod_{j=1 \atop j \neq i}^{n} \frac{(x_i - q_2 x_j)(x_i - q_3 x_j)}{(x_i - q_1 x_j)(x_i - x_j)} - q_1 \prod_{j=1 \atop j \neq i}^{n} \frac{(x_i - q_2 x_i)(x_i - q_3 x_i)}{(x_i - q_1 x_i)(x_i - x_i)} \right) = C \sum_{i=1}^{n} x_i \]

We need to show that the proportionality factor \( C = 1 \). This can be verified by setting \( x_i = 0 \) for \( i > 1 \) in the above equation. This completes the proof of (59).

3.4. Shuffle elements of \( \mathcal{A} \) and specializations. In this section we look at the problem of determining the shuffle algebra elements of \( \mathcal{A} \) recursively via specializations of the arguments \( x_i \). This problem was investigated in \([\text{Neg}14, \text{Neg}16]\) using a coproduct map of \( \mathcal{A}^+ \). Our treatment of this problem is appropriate for the goal of matching shuffle elements with partition functions in §4.

Recall that the shuffle elements of \( \mathcal{A} \) have the form
\[ P(x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2} \]
where \( p(x_1, \ldots, x_n) \) is a symmetric polynomial. We start with a lemma which gives us information about the degree of \( p(x_1, \ldots, x_n) \).

**Lemma 7.** Let \( P(x_1, \ldots, x_n) \) be an element of \( \mathcal{A}_n \) as in (60). Let \( x \) be an indeterminate and \( c_1, \ldots, c_r \in \mathbb{F} \) then \( p(c_1 x, \ldots, c_r x, x_{r+1}, \ldots, x_n) \in \mathbb{F}[x, x_{r+1}, \ldots, x_n] \) is of the form
\[ x^{r(r-1)} G(x, x_{r+1}, \ldots, x_n; c_1, \ldots, c_r) \]
and \( G(x, x_{r+1}, \ldots, x_n; c_1, \ldots, c_r) \) is a polynomial of degree at most \( 2r(n-r) \) in \( x \).

**Proof.** Consider the specialization \( x_k = c_k x \) \( (k = 1, \ldots, r) \) of (60)
\[ P(c_1 x, \ldots, c_r x, x_{r+1}, \ldots, x_n) = \frac{p(c_1 x, \ldots, c_r x, x_{r+1}, \ldots, x_n)}{x^{r(r-1)} \prod_{1 \leq i < j \leq n} (c_i x - x_j)^2 \prod_{1 \leq i < j \leq r} (c_i - c_j)^2 \prod_{r+1 \leq i < j \leq n} (x_i - x_j)^2} \]
The existence of the limit \((39)\) implies that, for \( x \to 0 \), the polynomial in the numerator \( p(c_1 x, \ldots, c_r x, x_{r+1}, \ldots, x_n) \) contains an overall factor \( x^{r(r-1)} \) to compensate the same factor in the denominator. Thus we established the form \((61)\) and
\[ P(c_1 x, \ldots, c_r x, x_{r+1}, \ldots, x_n) = \frac{G(x, x_{r+1}, \ldots, x_n; c_1, \ldots, c_r)}{\prod_{1 \leq i < j \leq r} (c_i - c_j)^2 \prod_{r+1 \leq i < j \leq n} (x_i - x_j)^2} \]
The existence of the limit \((40)\) implies that, for \( x \to \infty \), the degree in \( x \) of the polynomial \( G(x, x_{r+1}, \ldots, x_n) \) is controlled by \( \prod_{i=1}^{r} \prod_{j=r+1}^{n} (c_i x - x_j)^2 \), i.e. bounded by \( 2r(n-r) \). \( \square \)
In the following we will only be interested in the situations when \( c_i = q_k^{-1} \). The next statement gives a recipe for a recursive computation of shuffle elements of \( A \).

**Proposition 9.** Let \( r \in \{1, \ldots, n\} \), and \( P(x_1, \ldots, x_n) \) be an element of \( A_n \) as in (60). Fix \( k \in \{1, 2, 3\} \), then \( P(x_1, \ldots, x_n) \) is determined by the following specializations:

1. \( \lim_{x_i \to 0} \cdots \lim_{x_{i+1} \to 0} P(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, r - 1. \)
2. \( P(x, q_k x, \ldots, q_k^{-1} x, x_{r+1}, \ldots, x_n). \)

**Proof.** We fix \( k', k'' \in \{1, 2, 3\} \) such that \( k, k', k'' \) are all distinct. The proof is by induction on \( r \). The case \( r = 1 \) is trivial. Now assume the property true at \( r \geq 1 \), i.e. knowing (1.r) and (2.r) allows us to compute \( P(x_1, \ldots, x_n) \). Let us show that the specializations (1.r + 1) and (2.r + 1) allow us to compute the specializations (1.r) and (2.r). Since the specializations (1.r) are a subset of the specializations (1.r + 1) we only need to focus on computing (2.r)

\[
P(x, q_k x, \ldots, q_k^{-1} x, x_{r+1}, \ldots, x_n)
\]

According to Lemma 7 with \( c_i = q_k^{-1} \) for \( 1 \leq i \leq r \) we have

\[
P(x, q_k x, \ldots, q_k^{-1} x, x_{r+1}, \ldots, x_n) = \frac{G(x, x_{r+1}, \ldots, x_n)}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} (q_k^{-1} x - x_j)^2 \prod_{1 \leq i < j \leq r} (q_k^{-1} - q_k^{-1})^2 \prod_{r+1 \leq i < j \leq n} (x_i - x_j)^2}
\]

where \( G \) is a polynomial of degree \( 2r(n - r) \) in \( x \). According to the wheel conditions (55), it has the following factorization:

\[
G(x, x_{r+1}, \ldots, x_n) = \prod_{i=0}^{r-2} \prod_{j=r+1}^{n} (q_k x_j - q_k^{-1} x)(q_k^{-1} x_j - q_k x) H(x, x_r, \ldots, x_n)
\]

with the degree of \( H \) in \( x \) being \( 2r(n - r) - 2(r - 1)(n - r) = 2(n - r) \). To determine \( H \), we therefore need to know its values at \( 2(n - r) + 1 \) distinct specializations of \( x \). The value of \( H \) at \( x = 0 \) gives one specialization and can be computed from the knowledge of

\[
\lim_{x_i \to 0} \cdots \lim_{x_{r+1} \to 0} P(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)
\]

The specialization (2), using the fact that \( P \) is a symmetric polynomial, is equivalent to computing \( H \) at \( x = q_k^{-r} x_j \) and \( x = q_k x_j \), \( j = r + 1, \ldots, n \). This is the required number of specializations, which means \( G \) is determined and with it \( P(x, q_k x, \ldots, q_k^{-1} x, x_{r+1}, \ldots, x_n) \).

Let us apply this result to the computation of the basis elements \( \epsilon_{\lambda}(q_k) \) with \( \ell(\lambda) = m \) and \( \lambda_1 + \cdots + \lambda_m = n \). For convenience we only consider the case \( k = 3 \) and denote by

\[
\epsilon_{\lambda} = \epsilon_{\lambda}(q_3) = \epsilon_{\lambda}(t)
\]

We apply Proposition 9 with \( r = n \) and \( k = 3 \) to the problem of computing \( \epsilon_{\lambda} \). This means that in order to determine \( \epsilon_{\lambda} \) we need to know the values of

\[
(62) \quad \lim_{x_i \to 0} \cdots \lim_{x_{i+1} \to 0} \epsilon_{\lambda}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, n - 1
\]

\[
(63) \quad \epsilon_{\lambda}(x, tx, \ldots, t^{n-1} x)
\]

One can compute the limits in (62) using the following lemma.
Lemma 8. The elements $\epsilon_{\lambda_1,\ldots,\lambda_m}$, with $\lambda_1 + \cdots + \lambda_m = n$, satisfy the recurrence relation

$$
\lim_{x_n \to 0} \epsilon_{\lambda_1,\ldots,\lambda_m}(x_1, \ldots, x_{n-1}, x_n) = \sum_{i=1}^{m} \epsilon_{\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_m}(x_1, \ldots, x_{n-1})$

Proof. We start by writing $\epsilon_{\lambda_1,\ldots,\lambda_m}$ explicitly as a shuffle product of $\epsilon_{\lambda_i}$

$$
\epsilon_{\lambda_1,\ldots,\lambda_m} = \sum_{w \in S_n/S_{\lambda_1,\ldots,\lambda_m}} \prod_{i=1}^{m} \epsilon_{\lambda_i}(x_{w(i+\sum_{j=1}^{i-1} \lambda_j)}, \ldots, x_{w(\sum_{j=1}^{i} \lambda_j)})
$$

(65)

$$
\prod_{1 \leq a < b \leq m} \prod_{i \in I_a, j \in I_b} \omega(x_{w(j)}, x_{w(i)})
$$

where we defined the subsets $I_a$ by

$$
I_a := \left\{ \sum_{s=1}^{a-1} \lambda_s + 1, \ldots, \sum_{s=1}^{a} \lambda_s \right\}
$$

From definitions (63) and (68) we have two obvious identities

$$
\epsilon_k(x_1, \ldots, x_{k-1}, 0) = \epsilon_{k-1}(x_1, \ldots, x_{k-1}) \quad \omega(x, 0) = \omega(0, x) = 1
$$

Fix $r \in \{1, \ldots, m\}$ and consider the summands of (65) with $w$ such that $n \in w(I_r)$. The factor $\epsilon_{\lambda_r}$ will be replaced by $\epsilon_{\lambda_r-1}$ upon setting $x_n = 0$ while the factors of $\omega$ containing $x_n$ will disappear. The summation over $w \in S_n/S_{\lambda_1,\ldots,\lambda_m}$ (with $n \in w(I_r)$) reduces to the summation over $w' \in S_{n-1}/S_{\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_m}$ which is equal to the shuffle product of $\epsilon_{\lambda_1,\ldots,\epsilon_{\lambda_r-1,\ldots,\lambda_m}}$. Treating in this way every summand with $n \in w(I_r)$ for $r \in \{1, \ldots, m\}$ gives rise to the formula (64).

It is clear that computing (62) can be done inductively using (64). The specializations in (63) are not difficult to compute and in fact follow from more general specializations considered in [FHH+09]. We have

$$
\epsilon_\lambda(x, tx, \ldots, t^{n-1}x) = 0, \quad \lambda \neq (1, 1, \ldots, 1)
$$

(67)

$$
\epsilon_{(1,1,\ldots,1)}(x, tx, \ldots, t^{n-1}x) = \prod_{1 \leq i < j \leq n} \frac{(qt^i - t^j)(qt^i - qt^j)(t^i - t^j+1)}{qt(t^i - t^j)^3}
$$

(68)

Let us explain (68). We use the explicit expression (65) with $\lambda_i = 1$ in which case all $\epsilon_{\lambda_i} = 1$ and all sets $I_a$ have a single element. Due to the vanishing of $\omega(tx, x) = 0$ only one term in the summation is non-vanishing. This term must have $x$’s with fully ordered indices as follows

$$
\prod_{1 \leq i < j \leq n} \omega(x_i, x_j) \big|_{x_k = t^{k-1}x}
$$

which is equal to the product on the right hand side of (68).

3.5. The shuffle element $\kappa_n$. The results of this section are gathered in the following proposition:
Proposition 10. Let \( \kappa_0 = 1 \in A_0 \). Each of the following formulas uniquely determines the same sequence of shuffle algebra elements \( \kappa_n \in A_n \):

\[
(69) \quad \tau'(\kappa_n) = \sigma_q(\epsilon_n)
\]

\[
(70) \quad \kappa_n = \sum_{j=0}^{n} q_k^{-n}(q_k - 1)^{n-j} \kappa_j \ast \epsilon_{n-j}(q_k) \quad k = 1, 2, 3
\]

\[
(71) \quad (x_1 + \cdots + x_n) \kappa_n = \kappa_{n-1} \ast x_1
\]

where in (70) any one choice \( k \in \{1, 2, 3\} \) suffices.

**Proof.** It is clear that each of these equations uniquely determines the element \( \kappa_n \), we only need to show their consistency. We can take (71) together with \( \kappa_0 = 1 \) as the definition of \( \kappa_n \) and prove that it implies (70) and (69).

The derivation of (70) is recursive. The case \( n = 0 \) of (70) obviously holds. Using (71) we will derive (70) for \( \kappa_n \) assuming that (70) holds for \( n - 1 \). We act with \( \ast x_1 \) on both sides of (70) where \( n \) is replaced by \( n - 1 \)

\[
(72) \quad \kappa_{n-1} \ast x_1 = \sum_{j=0}^{n-1} q_k^{-n+1}(q_k - 1)^{n-1-j} \kappa_j \ast \epsilon_{n-j-1}(q_k) \ast x_1
\]

On the left hand side we apply (71) and on the right hand side (69)

\[
\sum_{i=1}^{n} x_i \kappa_n = \sum_{j=0}^{n-1} q_k^{-n}(q_k - 1)^{n-j} \kappa_j \ast \left( \sum_{i=1}^{n-j} x_i \epsilon_{n-j}(q_k) \right)
+ \sum_{j=0}^{n-1} q_k^{-n}(q_k - 1)^{n-j-1} \kappa_j \ast x_1 \ast \epsilon_{n-j-1}(q_k)
\]

On the right hand side we apply (71) to \( \kappa_j \ast x_1 \)

\[
\sum_{i=1}^{n} x_i \kappa_n = \sum_{j=0}^{n-1} q_k^{-n}(q_k - 1)^{n-j} \kappa_j \ast \left( \sum_{i=1}^{n-j} x_i \epsilon_{n-j}(q_k) \right)
+ \sum_{j=0}^{n-1} q_k^{-n}(q_k - 1)^{n-j-1} \left( \sum_{i=1}^{j+1} x_i \kappa_{j+1} \right) \ast \epsilon_{n-j-1}(q_k)
\]

In the last summation we can shift the index \( j \)

\[
(73) \quad \sum_{i=1}^{n} x_i \kappa_n = \sum_{j=0}^{n-1} q_k^{-n}(q_k - 1)^{n-j} \kappa_j \ast \left( \sum_{i=1}^{n-j} x_i \epsilon_{n-j}(q_k) \right)
+ \sum_{j=1}^{n} q_k^{-n}(q_k - 1)^{n-j} \left( \sum_{i=1}^{j} x_i \kappa_{j} \right) \ast \epsilon_{n-j}(q_k)
\]

From the definition of the shuffle product (66) it is easy to check that

\[
\kappa_j \ast \left( \sum_{i=1}^{n-j} x_i \epsilon_{n-j}(q_k) \right) + \left( \sum_{i=1}^{j} x_i \kappa_{j} \right) \ast \epsilon_{n-j}(q_k) = \left( \sum_{i=1}^{n} x_i \right) \kappa_j \ast \epsilon_{n-j}(q_k)
\]
Using this identity in (73) for every pair of summands with \( j = 1, \ldots, n - 1 \) leads to
\[
\sum_{i=1}^{n} x_{i} \kappa_{n} = \sum_{i=1}^{n} x_{i} q_{k}^{-n}(q_{k} - 1)^{n} \kappa_{0} \ast \epsilon_{n}(q_{k}) + \sum_{i=1}^{n} x_{i} q_{k}^{-n} \kappa_{n} \ast \epsilon_{0}(q_{k}) + \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n-1} q_{k}^{-n}(q_{k} - 1)^{n-j} \kappa_{j} \ast \epsilon_{n-j}(q_{k})
\]

After removing the common factor \( \sum_{i=1}^{n} x_{i} \) and putting all terms together on the right hand side we obtain (70) for \( \kappa_{n} \).

Next we show that (69) holds. For that we will use (70). It is sufficient to consider \( k = 1 \) in (70). We apply the map \( \Upsilon' \) to this equation
\[
\Upsilon' (\kappa_{n}) = \sum_{j=0}^{n} q^{n}(q^{-1} - 1)^{n-j} \Upsilon'(\kappa_{j}) \Upsilon'(\epsilon_{n-j}(q^{-1})) \tag{74}
\]
From (56) we have \( \Upsilon'(\epsilon_{n-j}(q^{-1})) = \epsilon_{n-j} \). Let us denote by \( e'_{n} = \Upsilon'(\kappa_{n}) \), then (74) becomes
\[
e'_{n} = \sum_{j=0}^{n} q^{j}(1 - q)^{n-j} e'_{j} \epsilon_{n-j} \tag{75}
\]
We need to show that \( e'_{n} = \sigma_{q}(\epsilon_{n}) \). Note that, together with the initial condition, (75) uniquely determines \( e'_{n} \). Let us solve (75) using generating functions, set
\[
E'(z) := \sum_{n=0}^{\infty} e'_{n} z^{n} \quad E(z) := \sum_{n=0}^{\infty} \epsilon_{n} z^{n} = \exp \left( - \sum_{j>0} \frac{(-1)^{j}}{j} p_{j} z^{j} \right) \tag{76}
\]
Multiply both sides of (75) by \( z^{n} \) and sum over \( n \geq 0 \)
\[
E'(z) = E'(qz) E((1 - q)z) \tag{77}
\]
By iterating this equation we find
\[
E'(z) = \prod_{k=0}^{\infty} E((1 - q)q^{k}z) = \exp \left( - \sum_{k \geq 0} \sum_{j>0} \frac{(-1)^{j}(1 - q)^{j}}{j} p_{j} q^{j} k z^{j} \right)
\]
\[
= \exp \left( - \sum_{j>0} \frac{(-1)^{j}(1 - q)^{j}}{j(1 - q^{j})} p_{j} z^{j} \right) = \exp \left( - \sum_{j>0} \frac{(-1)^{j}}{j} \sigma_{q}(p_{j}) z^{j} \right) = \sum_{n=0}^{\infty} \sigma_{q}(\epsilon_{n}) z^{n}
\]
Comparing the last result with (76) shows that \( e'_{n} = \sigma_{q}(\epsilon_{n}) \).

4. Matching the Partition Functions with Shuffle Elements

We now investigate the partition functions associated to the elements \( A_{\lambda_{1}} \ast \cdots \ast A_{\lambda_{m}} \) (where \( A_{n} \) was defined in (12) that were studied in the previous section on the shuffle algebra side.

**Proposition 11.** Given \( \lambda_{1}, \ldots, \lambda_{m} \in \mathbb{Z}_{>0} \) with \( n = \sum_{i=1}^{m} \lambda_{i} \), one has the limit
\[
\lim_{x_{n} \to 0} x_{n}^{n-1} f(A_{\lambda_{1}} \ast \cdots \ast A_{\lambda_{m}}) = (-1)^{n-1} \prod_{i=1}^{m} x_{i} \sum_{i=1}^{m} f(A_{\lambda_{1}} \ast \cdots \ast A_{\lambda_{i-1}} \ast A_{\lambda_{i}}) \tag{77}
\]
Note that by commutativity, \( \lambda \) can be chosen to be a partition.
Proof. We start from the definition of $A_{\lambda_1} \ast \cdots \ast A_{\lambda_m}$ (cf. (13)):

$$A_{\lambda_1} \ast \cdots \ast A_{\lambda_m} = \sum_{v \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} \sum_{w \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} (-1)^{|w|} t^{-(|v|+|w|)} T_w T_v T_{w^{-1}}$$

By definition of the groups involved, one has $T_w T_v = T_{wv}$ (and similarly $T_v T_{w^{-1}} = T_{wv^{-1}}$, but we choose to use the first identity).

Plugging this into (27), we have

$$\alpha_n^{-1} f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m}) = \sum_{v \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} \sum_{w \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} (-1)^{|w|} t^{-(|v|+|w|)} \langle Z \rangle_{y=qx} (T_w T_v T_{w^{-1}} \otimes S_n)$$

$$= \sum_{v \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} \sum_{w \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} (-1)^{|w|} t^{-(|v|+|w|)} \langle T_{w^{-1}} Z \rangle_{y=qx} (T_{wv} 1 \otimes S_n)$$

Using (30), we can rewrite this as

$$\alpha_n^{-1} f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m}) = \sum_{v \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_n}} \sum_{w \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}} (-1)^{|w|} t^{-(|v|+|w|)} \langle 1 \rangle T_{w^{-1}} Z \langle y=qx \rangle T_{wv} 1$$

(78)

This has the following interpretation: $\alpha_n^{-1} f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m})$ is the sum over $v$ and $w$ of partition functions of lattice paths, where the paths are labelled $w^{-1}(1), \ldots, w^{-1}(n)$ on the left side and $v^{-1}(w^{-1}(1)), \ldots, v^{-1}(w^{-1}(n))$ on the top side (with an extra weight $(-t)^{|v|}$).

We can now take the limit $x_n \to 0$. In this paragraph we denote by $a = w^{-1}$, $b = v^{-1} w^{-1}$, which are two arbitrary permutations, and consider $\langle a \rangle Z \langle y=qx \rangle b$ as $x_n \to 0$. We are only interested for the purposes of deriving (77) in the terms of order $x_n^{-1}$; from Lemma 3 we know that this is the minimum order in $x_n$ and in fact, as in the proof of the lemma, we know that the only dependence on $x_n$ comes from the last row and column of the diagram of $Z$. Let us examine these in more detail with our boundary conditions $\langle a \rangle$ and $\langle b \rangle$. We look at the destiny of the path $b(n)$ which ends at the top right: to maximize the degree in $x_n^{-1}$, we must pick up a $x_n^{-1}$ at each vertex $(i, n)$, $i = 1, \ldots, n - 1$; this means, according to (3) that the first type of vertex (a path making a bend when the right edge is either empty or has a higher labelled path) must be excluded. Since all right edges are empty in the last column, one can prove inductively that the paths must go straight through every vertex $(i, n)$, $i = 1, \ldots, n - 1$, incurring a weight of $-q^{-1} x_i x_n^{-1}$. At vertex $(n, n)$, because both bottom and right edges are empty, we must have the first type of vertex in (3), incurring a $1 - t$. From there, the path can only go straight left to its destination with label $a(n)$, incurring a weight $t^{n-1}$ and imposing $a(n) = b(n)$. Combining all this, we find

(79) $\langle a \rangle Z(x_1, \ldots, x_n, qx_1, \ldots, qx_n) \langle b \rangle = \delta_{a(n), b(n)} (1 - t) t^{n-1} x_n^{-1} \prod_{i=1}^{n-1} (-q^{-1} x_i)$

$$\times \langle a(1), \ldots, a(n-1) \rangle Z(x_1, \ldots, x_{n-1}, qx_1, \ldots, qx_{n-1}) \langle b(1), \ldots, b(n-1) \rangle + O(x_n^{n+2}) a, b \in S_n^{(1)}$$

We apply this formula to $f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m})$. We fix $w \in S_{\lambda_1} \ast \cdots \ast S_{\lambda_m}$ and denote by $k = w^{-1}(n)$. Because of the form of $w$, $k$ must be the last element of its block $I_a$, $1 \leq a \leq m$ ($I_a$ was defined in (66)). According to (79), the summation can be restricted to $v$ such that $v(k) = k$. Consider now the permutation $w'$ with $w'(i) = w(i)$, $i = 1, \ldots, k - 1$, $w'(i) = w(i + 1)$, $i =
\( k, \ldots, n - 1 \), and the permutation \( v' \) with \( v'(i) = i \) for \( i = 1, \ldots, k - 1 \) and \( v'(i) = v(i + 1) - 1 \) for \( i = k, \ldots, n - 1 \). Because \( k \) is last of its block, \( |v'| = |v| \) and we have

\[
\alpha_n^{-1} f(A_{\lambda_1} \cdots A_{\lambda_m}) \xrightarrow{x \sim 0} (1 - t) t^{n-1} x_n^{-n+1} \sum_{i=1}^{n-1} (-q^{-1} x_i) \sum_{v' \in S_{\lambda_1, \ldots, \lambda_{n-1}, \ldots, \lambda_m}} (-t)^{-|v'|} \sum_{w' \in S_{\lambda_1, \ldots, \lambda_{n-1}, \ldots, \lambda_m}} (-t)^{-|v'|} \langle w'^{-1}(1), \ldots, w'^{-1}(n - 1) | Z(x_1, \ldots, x_{n-1}, qx_1, \ldots, qx_{n-1}) | v^{-1}(w'^{-1}(1)), \ldots, v^{-1}(w'^{-1}(n - 1)) \rangle
\]

Comparing with (78) and using \( \alpha_n^{-1} \alpha_n = (1 - t)(t/q)^{n-1} \), we finally find the desired limit (77).

In what follows, we shall use the diagrammatic language. We shall repeatedly use the following simple identities:

**Lemma 9.** One has for \( k \leq \ell, m \geq 2 \):

\[
\begin{align*}
A_k A_{\ell} & = [k]_{t-1}! \quad \text{and} \quad S_k S_{\ell} = [k]_{t-1}! \quad \text{and} \quad \text{where } [k]_t = \prod_{i=1}^{k} [i]_t \text{ and } [i]_t = 1 + \cdots + t^{i-1}.
\end{align*}
\]

For general \( c \in Z(\mathcal{H}_n) \), \( f(c) \) can be described as

\[
f(c) = \alpha_n \langle \quad \rangle
\]

This immediately implies

**Lemma 10.** For any \( c \in Z \), \( f(c) \) satisfies the wheel conditions

\[
f(c)(x, tx, qx, x_4, \ldots, x_n) = f(c)(tqx, tx, qx, x_4, \ldots, x_n) = 0
\]
Proof. \( f(c) \) being symmetric in its arguments according to Proposition \([4]\) we can pick any three variables and specialize them. Let us then consider the diagram above and set \( x_{n-2} = x, \ x_{n-1} = tx, \ x_n = qx \). Zooming in on the bottom right corner, we have, using \( \hat{R}(t^{-1}) = (1 - t)A_2, \)

\[
\begin{align*}
\cdots & \downarrow qx \ 
\cdots \ 
\cdots \\
\cdots & \downarrow tx \ 
\cdots \ 
\cdots \\
S_n & \downarrow q^2x \\
\cdots & \\
\cdots & \\
\end{align*}
\]

\( = (1 - t) \)

\[
\begin{align*}
\cdots & \downarrow qx \ 
\cdots \ 
\cdots \\
\cdots & \downarrow txq \ 
\cdots \ 
\cdots \\
S_n & \downarrow q^2x \\
\cdots & \\
\cdots & \\
\end{align*}
\]

\( = 0 \)

Switching rows and columns results similarly in the second wheel condition. \( \square \)

We now go back to \( A_{\lambda_1} \ast \cdots \ast A_{\lambda_m} \). Starting from the expression

\[
f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m}) = \alpha_n \sum_{w \in S^{\lambda_1, \ldots, \lambda_m}} t^{-|w|} \langle (T_w(A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_m})T_{w^{-1}} \otimes S_n)Z \rangle |_{y_i = qx_i, \ i = 1, \ldots, n}
\]

and using the cyclicity of \( \langle \cdot \rangle \), we can describe the expression inside it as the diagram: (compare with \([83]\))
As a side remark, using the idempotency of $A_n$ and $S_n$, we can introduce the more symmetric diagram

\[
\tilde{X}_w := y_n \cdots y_2 y_1 x_1 x_2 \cdots x_n.
\]

with $\tilde{X}_w = (X_w) A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_m} \otimes S_n$. We shall not use $\tilde{X}_w$ in what follows.

**Proposition 12.**

- For $m < n$, one has the vanishing condition:

\[
f(A_{\lambda_1} \ast \cdots \ast A_{\lambda_m})(x, tx, \ldots, t^m x, x_{m+2}, \ldots, x_n) = 0
\]

- The partition function $f(A_n)$ has the fully factorized form

\[
f(A_n) = t^{-\frac{n(n-1)}{2}} \prod_{i,j=1}^{n} \left(1 - tx_i/x_j\right)
\]

- One has the specialization

\[
f(A_1 \ast \cdots \ast A_1)(1, t, \ldots, t^{n-1}) = [n]_{t-1}! \prod_{1 \leq i < j \leq n} (q - t^{j-i})(q - t^{i-j+1})
\]

**Proof.** We first consider $m = 1$, and investigate the effect of setting $x_1 = tx$, $x_2 = x$ in $X_w$, where here $w = 1$. Noting that $R_1(t) = (1 - t)(1 + T_1) = (1 - t)S_2$, and using (81), we can insert a crossing at the right of the top two horizontal lines and then apply the Yang–Baxter equation repeatedly to move the crossing to the left, resulting in zero according to (82):

\[
X_1|_{x_2 = x, x_2 = x} = \frac{1}{1 + t}
\]
We conclude that \( x_1 - tx_2 \) divides \( f(A_n) \), and by symmetry, \( f(A_n) \) is a multiple of \( \prod_{i \neq j} (1 - tx_i/x_j) \). The latter has degree range \([- (n-1), (n-1)]\) in each \( x_i \), which is the maximum allowed according to (2), so only a constant in \( \mathbb{F} \) remains to be determined. The latter is determined inductively by using Proposition 11.

We proceed identically for \( 1 < m < n \). We set \( x_1 = t^m x, \ldots, x_m = tx, x_{m+1} = x \). We note that the “\( R \)-matrix associated to the longest element \( w_0 \) of \( S_{m+1} \),” i.e., the product of \( R \)-matrices whose diagram reproduces (any) wiring diagram of \( w_0 \), with spectral parameters set to \( t^m, \ldots, 1 \), is nothing but \((1-t)^{m(m+1)/2} \prod_{i=1}^{m} [i]_t! S_{m+1} \) (see e.g. [Las06, Lem. 3] or [IO09, 3.1]), and move it across to the left using the Yang–Baxter equation:

\[
X_w|_{x_1=t^m x, \ldots, x_{m+1}=x} = \frac{1}{[m+1]_t!}
\]

The pigeonhole principle tells us there will be two numbers between 1 and \( m+1 \) whose preimages under \( w \) lie in the same block (i.e., are associated to the same part of \( \lambda \)). Since \( w \in S_{\lambda_1, \ldots, \lambda_m} \), all numbers whose preimages are in the same block are consecutive, and so we can apply (22) to conclude that the result is zero. By summing over \( w \) and using symmetry in the exchange of variables, we obtain the result (85) as stated in the Proposition.

The only case left is \( m = n \), i.e., \( A_1 \ast \cdots \ast A_1 \). We set \( x_1 = t^{n-1} x, \ldots, x_{n-1} = tx, x_n = x \) but this time the result does not vanish. However, noting once again that \( S_n \) is up to normalization the product of \( R \)-matrices associated to the longest permutation \( w_0 \) of \( S_n \), we can move it to the left and absorb any crossing from \( T_w \).
where in the last line we find it convenient to reintroduce the longest element $w_0$, before moving back $S_n$ to the right. Performing the summation over $w$ (one could instead absorb the top crossings as well, but the result would be the same), we find

$$
\sum_{w \in S_n} t^{-|w|} X_w = t^{-\frac{n(n-1)}{2}}
$$
We then impose $y_i = qx_i$ and note that up to normalization, the $S_n$ at the top is nothing but the $R$-matrix associated to the longest element of $S_n^{(2)}$; we can pull it to the bottom:

$$
\sum_{w \in S_n} t^{-|w|} X_w[y_i = qx_i] = t^{-\frac{n(n-1)}{2}}
$$

Finally, the bracket is

$$
f(A_1 \ast \cdots \ast A_1) = \alpha_n t^{-\frac{n(n-1)}{2}} [n]_t!
$$

This is essentially the same situation that was considered at the end of §2. First we expand each $R$-matrix in the upper-left triangle as $\tilde{R}_i(u) = u(1-t) + t(1-u)T_i^{-1}$ and note that the only term that has a nonzero identity coefficient is

$$
f(A_1 \ast \cdots \ast A_1) = \alpha_n [n]_t! (1-t)^n \prod_{1 \leq i, j \leq n} (1-q^{-1}t^n-i-j-1)
$$

Secondly, we absorb all the crossings of the lower-right triangle into $S_n$, resulting in a weight of $\prod_{1 \leq i, j \leq n} (1-q^{-1}t^n-i-j+2)$. Recombining and simplifying, we obtain (87). □

We are now in a position to prove Theorem 1. First, combining Lemma 3, Proposition 4 and Lemma 10, we find that $f(c)$, from its definition (27), is $\alpha_n$ times an element of $A_n$. $\alpha_n$ contains a factor of $(1-t)^{-n}$, but it is compensated by the fact that any lattice path entering from the left and exiting from the top must have a bend, resulting in a factor of
The recurrence relation for Proposition 13.

Next, we identify \( f(A_{\lambda_1} \ldots A_{\lambda_m}) \). We use the characterization of Proposition 8 so that \( \epsilon_{\lambda_1, \ldots, \lambda_m} \) is entirely determined by Lemma 8 and (42). On the other hand, it is easy to check that Proposition 12 implies the exact same relations for \( V_n^{-1} f(A_{\lambda_1} \ldots A_{\lambda_m}) \) (where \( V_n \) is defined in (12), and implements the isomorphism \( V_n \times \) from \( A_n \) to \( A_n \)). Therefore,

\[
f(A_{\lambda_1} \ldots A_{\lambda_m}) = V_n \epsilon_{\lambda_1, \ldots, \lambda_m}
\]

We finally compare \( \Upsilon(f(A_{\lambda_1} \ldots A_{\lambda_m})) \) and \( \Phi(A_{\lambda_1} \ldots A_{\lambda_m}) \).

According to the ring map property of \( \Upsilon \) and (36) applied to \( \epsilon_r = \epsilon_r^{(3)} \),

\[
\Upsilon(f(A_{\lambda_1} \ldots A_{\lambda_m})) = \Upsilon(V_n \epsilon_{\lambda_1}^{(3)} \ldots \epsilon_{\lambda_m}^{(3)}) = \prod_{i=1}^{m} \Upsilon(V_n \epsilon_{\lambda_i}^{(3)}) = \prod_{i=1}^{m} \sigma_{q^{-1}} \sigma_{t^{-1}}(e_{\lambda_i})
\]

On the other hand, from (17) and the ring map property of \( \Phi \),

\[
\Phi(A_{\lambda_1} \ldots A_{\lambda_m}) = \prod_{i=1}^{m} e_{\lambda_i}
\]

We have therefore checked the statement of Theorem 1 on a basis of \( Z \).

**Corollary 1.** We have the following identity

(88) \( f(1_n) = V_n \kappa_n \)

**Proof.** According to (69) and Proposition 8 \( \Upsilon(V_n \kappa_n) = \Upsilon'(\kappa_n) = \sigma_q(e_n) \). On the other hand, according to Proposition 8 \( \Phi(1_n) = \hat{h}_n = \sigma_t(h_n) = \sigma_t \sigma_{\infty}(e_n) \). Applying Theorem 1 we conclude that (88) holds. \( \square \)

Using the definition of the shuffle product (36) we can write

(89) \(
(x_1 + \cdots + x_n) \kappa_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \kappa_{n-1}(1, \ldots, \hat{x}_i, \ldots, x_n) \prod_{j=1 \atop j \neq i}^{n} \frac{(x_i - q_1 x_j)(x_i - q_2 x_j)(x_i - q_3 x_j)}{(x_i - x_j)^3}
\)

The recurrence relation for \( f(1_n) \) stated in (5) follows from (88), (89) and (12) thus proving Theorem 2.

5. Application to the Commuting Scheme

5.1. Proof of Theorem 3. In this section, all schemes are over \( \mathbb{C} \), as in the introduction.

**Proposition 13.** Assuming Conjecture 7, \( K_n \) satisfies the following recurrence relation:

(90) \( (x_1 + \cdots + x_n)K_n = \sum_{i=1}^{n} x_i K_{n-1}[\hat{x}_i] \prod_{j=1 \atop j \neq i}^{n} \frac{(1 - q_1 x_j/x_i)(1 - q_2 x_j/x_i)(1 - q_1 q_2 x_j/x_i)}{1 - x_j/x_i} \)

where \( K_{n-1}[\hat{x}_i] = K_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \).
Proof. For the purposes of this proof, we need more geometry than in the rest of the paper. We think of the multi-graded Hilbert series as the pushforward to a point in localized equivariant $K$-theory. Equivalently, the $K$-polynomial of $\mathcal{C}_n$ is viewed as (the pushforward of) the $K_T$-class $[O_{\xi_n}]$ in the ambient space $\mathfrak{gl}_n \times \mathfrak{gl}_n$. Note that $K_T(\mathfrak{gl}_n \times \mathfrak{gl}_n) \cong K_T(pt) \cong \mathbb{Z}[q_1, q_2, x_1^\pm, \ldots, x_n^\pm]$ where the variables $q_1, q_2, x_1, \ldots, x_n$ are the inverses of the equivariant parameters attached to the torus $T$.

Consider

$$\mathfrak{S}_n = \{(v, A, B) \in \mathbb{P}^{n-1} \times \mathfrak{gl}_n \times \mathfrak{gl}_n : [A, B] = 0 \text{ and } v \text{ eigenvector of } A \text{ and } B\}$$

There is a natural projective morphism

$$\mathfrak{S}_n \xrightarrow{p} \mathcal{C}_n$$

$$(v, A, B) \mapsto (A, B)$$

which is generically $n$-to-1 (choice of one eigenvector among $n$). There is also the affine morphism $f : \mathfrak{S}_n \to \mathbb{P}^{n-1}$. Since all fibers are isomorphic by $GL_n$-action, the dimension of a fiber is $\dim \mathfrak{S}_n - \dim \mathbb{P}^{n-1} = n^2 + 1$.

The computation is in two parts:

- Computing $\pi_*[O_{\xi_n}(1)]$ by equivariant localization (which relates it to $\pi_*[O_{\xi_n}^{-1}]$ using $(v, A, B) \mapsto (A|_{C^n/v}, B|_{C^n/v}))$.
- Comparing $p_*[O_{\xi_n}(1)]$ and $[O_{\xi_n}]$.

5.1.1. Equivariant localization. Torus fixed points in $\mathbb{P}^{n-1}$ are basis vectors $e_i$, $i = 1, \ldots, n$. Consider such a fixed point, which up to $S_n$ action, we can choose to be $v = e_n$. The fiber is

$$f^{-1}(v) = \{v\} \times X, \quad X = \left\{ \left( A = \begin{pmatrix} A' & 0 \\ \alpha & a \end{pmatrix}, B = \begin{pmatrix} B' & 0 \\ \beta & b \end{pmatrix} \right) : AB = BA \right\}$$

More explicitly, the equations are

$$A'B' = B'A'$$

$$a\beta + \alpha B' = b\alpha + \beta A'$$

In particular note that $(A', B') \in \mathcal{C}_{n-1}$. There are $2n$ extra variables $\alpha, \beta, a, b$ and $n-1$ extra equations, and $\dim \mathcal{C}_{n-1} = 2n - (n - 1) = n^2 + 1 = \dim f^{-1}(v)$, so assuming Conjecture 1 (Cohen–Macaulay property for $\mathcal{C}_{n-1}$), which implies it for $\mathcal{C}_{n-1} \times A^{2n+2}$, the unmixedness theorem asserts that the extra equations (92) form a regular sequence.

We then conclude by standard exact sequence arguments that

$$[O_X] = [O_{\xi_{n-1}}] \prod_{j=1}^{n-1} (1 - q_1 x_j/x_n)(1 - q_2 x_j/x_n)(1 - q_1 q_2 x_n/x_j)$$

where $[O_X]$ is the $K_T$ class of $O_X$ in $\mathfrak{gl}_n \times \mathfrak{gl}_n$, whereas $[O_{\xi_{n-1}}]$ is the $K_T$ class of $\mathcal{C}_{n-1}$ in $\mathfrak{gl}_{n-1} \times \mathfrak{gl}_{n-1}$ (where the $T$-action on $\mathfrak{gl}_{n-1} \times \mathfrak{gl}_{n-1}$ has a one-dimensional kernel leading to the obvious inclusion $K_T/\mathbb{C}^\times \subset K_T$, i.e., as a polynomial $[O_{\xi_{n-1}}] = K_{n-1}$ is independent of $x_n$). The factors $1 - q_1 x_j/x_n$ (resp. $1 - q_2 x_j/x_n$) come from the zero column of $A$ (resp. $B$) whereas the factors $1 - q_1 q_2 x_n/x_j$ come from the equations (92).

We have thus managed to compute $[f^{-1}(v)] = [v][O_X]$. 
The weights at the tangent space of $e_i$ are $x_i/x_j$, $j \neq i$ (recalling that the $x_i$ are the inverses of the equivariant parameters). Pulling back the relation

$$1 = \sum_{i=1}^n \frac{[e_i]}{\prod_{j=1}^n (1 - x_j/x_i)}$$

in localized equivariant $K$-theory of $\mathbb{P}^{n-1}$ to $\mathfrak{F}_n$ and multiplying by $[O_{\mathfrak{F}_n}(1)]$, we obtain

$$[O_{\mathfrak{F}_n}(1)] = \sum_{i=1}^n x_i [O_{\mathfrak{F}_{n-1}}(1)](\hat{x}_i) \prod_{j=1}^n (1 - q_2 x_j/x_i)(1 - q_1 x_j/x_i)(1 - q_1 q_2 x_i/x_j)/(1 - x_j/x_i)$$

Finally, pushing forward to $\mathfrak{gl}_n \times \mathfrak{gl}_n$ leads to:

$$p_n[O_{\mathfrak{F}_n}(1)] = \sum_{i=1}^n x_i [O_{\mathfrak{F}_{n-1}}(1)](\hat{x}_i) \prod_{j=1}^n (1 - q_2 x_j/x_i)(1 - q_1 x_j/x_i)(1 - q_1 q_2 x_i/x_j)/(1 - x_j/x_i)$$

5.1.2. **Pushforward.** Now let us try to compute directly the pushforward of $[O_{\mathfrak{F}_n}(1)]$ under $p : \mathfrak{F}_n \to \mathfrak{C}_n$. Write $\mathfrak{C}_n = \text{Spec} R$ and $\mathfrak{F}_n = \text{Proj} S$ where $S_0 = R$. Since $\mathfrak{C}_n$ is affine, the pushforward is simply given in terms of the sheaf cohomology groups of $O_{\mathfrak{F}_n}(1)$ (viewed as modules over $R$). We claim the following

$$H^i(\mathfrak{F}_n, O_{\mathfrak{F}_n}(1)) \cong \begin{cases} S_1 & i = 0 \\ 0 & i > 0 \end{cases}$$

**Proof.** Let us first write the equations of $\mathfrak{F}_n$ explicitly:

$$AB = BA \quad \text{minors of size 2 of } (v, Av) \text{ and } (v, Bv) = 0$$

Now define the patches $U_i = \{v_i \neq 0\}$. Let us consider $U_1 \cap U_2$ inside $U_1$, $U_2$. Denote by $z = v_1/v_2$, $r_i = v_i/v_2$, $i > 2$ and collectively $r = (r_i)$. A section on $U_1$ is given by a polynomial $P(z, r, A, B)$ modulo the relations (95) rewritten in terms of $z, r, A, B$. Let us call these equations $q_i(z)$ where dependence on $r, A, B$ is suppressed. Note that the $q_i(z)$ are of degree less or equal to 2 in $z$. Similarly a section on $U_2$ is given by a polynomial $Q(z^{-1}, rz^{-1}, A, B)$ modulo the same equations expressed in terms of $z^{-1}, rz^{-1}, A, B$. The $O(1)$ matching condition on $U_1 \cap U_2$ is

$$P(z) - zQ(z^{-1}) = \sum c_i(z)q_i(z)$$

where the $c_i(z)$ are Laurent polynomials in $z$. Now assume that $P$ is of degree greater than 1 in $z$. Then its top coefficient is equal to the top coefficient of $Q := \sum c_i^{\text{pol}}(z)q_i(z)$ where $c_i^{\text{pol}}(z)$ is the part of $c_i(z)$ with nonnegative powers of $z$. By subtracting $Q$ from $P$, we can reduce the degree of $P$ until it is at most 1.

This way, one can show that the restriction of a global section to $U_i$ is of degree in $v_i/v_j$ at most 1 for all $i \neq j$; this immediately implies that the $v_i$ generate the global sections of $O(1)$. Linear independence is obvious (it can be tested at the generic point).

(96) also implies that the map $(P, Q) \mapsto P(z) - zQ(z^{-1})$ is surjective, so that $H^1(O_{\mathfrak{F}_n}(1)) = 0$. Vanishing of higher cohomology follows similarly. \qed
Since $H^0(\mathcal{O}_{\tilde{\mathcal{S}}_n}(1)) \cong S_1$ is a free $R$-module with basis $(v_1, \ldots, v_n)$ and higher sheaf cohomology vanishes, we conclude that

\[(97)\quad p^*_n[\mathcal{O}_{\tilde{\mathcal{S}}_n}(1)] = \left[\mathcal{O}_{\mathcal{C}_n}\right] \sum_{j=1}^{n} x_j\]

where $x_j$ is the weight of the basis element $v_j$.

Combining (94) and (97) leads to the recurrence relation (90). \hfill \Box

We are now in a position to prove Theorem 3 by relating the shuffle element $\kappa_n$ and the $K$-polynomial $K_n$. $K_n$ is uniquely determined by (90) and the initial condition $K_0 = 1$. Similarly $\kappa_n$ can be computed recursively with (89). By comparing (89) with (90) (and checking the trivial initial condition $\kappa_0 = K_0 = 1$) we conclude that

\[(98)\quad \kappa_n(x_1, \ldots, x_n) = \frac{t^{n(n-1)/2}}{V_n(x)} K_n(x_1, \ldots, x_n)\]

Combining (88) with (98) gives us the statement of Theorem 3.

5.2. Proof of Theorem 4. Theorem 4 involves computing the multidegree of $\mathcal{C}_n$, which means one needs to work in equivariant cohomology. The situation is much simpler than in equivariant $K$-theory (which was needed for Theorem 3), and we only sketch the corresponding geometric argument. By abuse of notation, we use the same symbols $q_1, q_2, x_1, \ldots, x_n$ for the cohomology equivariant parameters. This is consistent with our passage from $K$-polynomial to multidegree, which was defined in the introduction as the substitution $A^i B^j$ for the cohomology equivariant parameters. This is consistent with our passage from $K$-polynomial to multidegree, which was defined in the introduction as the substitution $A^i B^j$ for the cohomology equivariant parameters. This is consistent with our passage from $K$-polynomial to multidegree, which was defined in the introduction as the substitution $A^i B^j$ for the cohomology equivariant parameters. This is consistent with our passage from $K$-polynomial to multidegree, which was defined in the introduction as the substitution $A^i B^j$ for the cohomology equivariant parameters.

View the flag variety $\mathcal{B}_n$ as the variety of Borel subalgebras of $\mathfrak{gl}_n$ and consider $\tilde{\mathcal{S}}_n = \{(b, A, B) \in \mathcal{B}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n : [A, B] = 0, A, B \in b\}$. There is a projection map $p$ from $\tilde{\mathcal{S}}_n$ to $\mathcal{C}_n$ which is generically $n!$-to-1 (generic matrices are diagonalizable); one then obtains by equivariant localization techniques (similar to §5.1.1) the formula

\[(99)\quad D_n = \frac{1}{n!} \prod_{i,j=1 \atop i \neq j}^{n} (q_1 + x_i - x_j)(q_2 + x_i - x_j) \sum_{w \in S_n} w \left( \prod_{1 \leq i < j \leq n} \frac{q_1 + q_2 + x_i - x_j}{(x_j - x_i)(q_1 + x_i - x_j)(q_2 + x_i - x_j)} \right)\]

where the symmetric group acts by permuting variables, and Sym denotes symmetrization w.r.t. to it.

One can think of this expression as the GRR limit of the shuffle element $\frac{1}{n!} f(A_1 \ast \cdots \ast A_1)$ (studied in §4).\footnote{It should be noted that the element $f(A_1 \ast \cdots \ast A_1)$ itself has a geometric interpretation [Gin11] as the Hilbert series of the coordinate ring of the normalization of the isospectral commuting variety $X_{\text{norm}}$. What we are finding here is that to go from $K$-theory to cohomology, instead of using the structure sheaf of $\mathcal{C}_n$, one can use the combinatorially simpler structure sheaf of $X_{\text{norm}}$.}

At $t = 1$, from (11), one has the identity in the center of the symmetric group algebras $A_1 \ast \cdots \ast A_1 = n! 1_n$, so that in the GRR limit, $\frac{1}{n!} f(A_1 \ast \cdots \ast A_1)$ and $f(1_n)$ coincide. As
was already remarked in §1.4, expanding the weights (8), which correspond to \( f(1_n) \), in the GRR limit, result in the weights (9). This leads to the formula of Theorem [4]

A few comments are in order. In [DFZJ06, KZJ07] another expression for the multidegree of the commuting variety was obtained, using apparently unrelated methods. There is actually a connection: consider Proposition [5] and take the GRR limit. One recovers one of the exchange relations of the work above (cf [DFZJ06, Eq. (3.10)] or [KZJ07, Prop. 6]). The expression for the multidegree of \( C_n \) there is essentially a repeated application of these identities; it is not obvious if it is more or less useful than our (new) formula (99). (The extension of these ideas to \( K \)-theory will be investigated elsewhere [KZJ].)

The expression of Theorem 4, in contrast, has many desirable properties; for instance, it is division- and subtraction-free. Regarding positivity, let us only consider the bidegree with respect to scaling of the two matrices \( A \) and \( B \), i.e., set the \( x_i \) to zero. On the one hand, (99) specializes to the bidegree being the constant term of the Laurent polynomial
\[
\frac{1}{n!} \prod_{1 \leq i < j \leq n} (u_i + u_j)(q_1 - u_i + u_j)(q_2 - u_i + u_j)(q_1 + q_2 + u_i - u_j)
\]
which clearly involves subtractions. On the other hand, the expression in (10) is a polynomial in \( q_1 \) and \( q_2 \) with manifestly positive coefficients. If we further restrict to the ordinary degree, i.e., set \( q_1 = q_2 = 1 \), then we have the simple formula
\[
\deg C_n = \sum_{\text{lattice paths } P \text{ on the } n \times n \text{ grid} \atop \text{with identity connectivity}} 2^{\#(i,j): \text{bends}} - n
\]
(100) is also extremely effective computationally; here is the degree of \( C_{12} \) (computed in a few seconds on a laptop):
\[
\deg C_{12} = 1862632561783036151478238040096092649
\]
(note that this is beyond the currently known entries of [OEIS A029729].)

5.3. The \( S_3 \) symmetry. As an obvious corollary of (71) of Proposition [10] one has:

**Corollary 2.** \( \kappa_n \) possesses the \( S_3 \) symmetry of permutations of \( q_1, q_2, q_3 \).

In view of (88), the same holds for \( f(1_n) \); this symmetry is however not obvious at the level of the partition function. In a similar vein, assuming Conjecture [1] this also implies the symmetry property
\[
K_n(q_1, q_2) = K_n(q_2, q_1) = (q_1 q_2)^{n(n-1)/2} K_n(q_1, (q_1 q_2)^{-1})
\]

**Example 5.** Here are the coefficients of \( K_3 | x_i = 1 \):

\[
\begin{array}{cccccc}
1 & 1 & 1 & -8 \\
q_1 & 1 & -16 & 29 & 1 \\
1 & -8 & 29 & -27 & -16 & 1 \\
q_2 & 1 & -16 & 29 & 1 \\
1 & 1 & -8 \\
1 & & & & & \\
\end{array}
\]
It is not a priori obvious why the $K$-polynomial of the commuting scheme $\mathfrak{C}_n$ should possess the second symmetry property. Here we propose an “explanation”:

**Conjecture 2.** Consider the triple commuting scheme

$$\hat{\mathfrak{C}}_n := \{(A, B, C) \in \mathfrak{gl}_3^n : [A, B] = [A, C] = [B, C] = 0\}$$

and its grading w.r.t. scaling $(A, B, C) \mapsto (q_1 A, q_2 B, q_3 C)$ and conjugating by diagonal matrices. Then its $K$-polynomial $\hat{K}_n$ satisfies

$$\hat{K}_n(q_1, q_2, (q_1 q_2)^{-1}) = K_n(q_1, q_2) K_n(q_1^{-1}, q_2^{-1})$$

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3This is to be compared with the following two toroidal algebra representations: the one coming from the Hilbert scheme of points in $\mathbb{C}^2$, which does not have manifest $S_3$ symmetry, and the one coming from the Hilbert scheme of points in $\mathbb{C}^3$, which does.
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