Development of a unified tensor calculus for the exceptional Lie algebras

A. J. Macfarlane and Hendryk Pfeiffer

1 Centre for Mathematical Sciences, DAMTP, Wilberforce Road, Cambridge CB3 0WA, UK
2 Perimeter Institute for Theoretical Physics, 35 King Street N, Waterloo ON N2J 2W9, Canada
3 Emmanuel College, St. Andrew's Street, Cambridge CB2 3AP, UK

16 December 2002

Abstract

The uniformity of the decomposition law, for a family $F$ of Lie algebras which includes the exceptional Lie algebras, of the tensor powers $ad^\otimes n$ of their adjoint representations $ad$ is now well-known. This paper uses it to embark on the development of a unified tensor calculus for the exceptional Lie algebras. It deals explicitly with all the tensors that arise at the $n = 2$ stage, obtaining a large body of systematic information about their properties and identities satisfied by them. Some results at the $n = 3$ level are obtained, including a simple derivation of the the dimension and Casimir eigenvalue data for all the constituents of $ad^\otimes 3$. This is vital input data for treating the set of all tensors that enter the picture at the $n = 3$ level, following a path already known to be viable for $a_1 \in F$.

The special way in which the Lie algebra $\mathfrak{d}_4$ conforms to its place in the family $F$ alongside the exceptional Lie algebras is described.

1 Introduction

1.1 Notation and conventions

Let $\mathfrak{g}$ be a simple complex Lie algebra with generators $X_k$ such that

$$[X_i, X_j] = ic_{ijk}X_k.$$ (1)

Here and in the following summation over repeated indices is understood. The adjoint representation $ad$ of $\mathfrak{g}$ is defined by $X_k \mapsto ad_k := [X_k, \cdot]$ with matrix elements

$$(ad_i)_{jk} = ic_{ikj},$$ (2)

and our normalisations are fixed by requiring that the Cartan-Killing form of $\mathfrak{g}$ satisfies

$$\kappa_{jk} = \operatorname{Tr}(ad_j ad_k) = \delta_{jk},$$ (3)
so that the structure constants are totally antisymmetric and satisfy

\[ c_{ijp} c_{kpq} = \delta_{jk}. \] (4)

It follows that the quadratic Casimir operator of \( g \)

\[ C^{(2)} = X_k X_k, \] (5)

has, for each \( g \), the eigenvalue \( c_2(ad) = 1 \), since

\[ C^{(2)}(ad)_{ij} = c_{pqi} c_{pqj} = \delta_{ij}. \] (6)

We use the notation \( \mathcal{E} \) to indicate the set of all exceptional Lie algebras

\[ \mathcal{E} = \{ g_2, f_4, e_6, e_7, e_8 \}. \] (7)

They are a subset of the set

\[ \mathcal{F} = \{ a_1, a_2, g_2, f_4, e_6, e_7, e_8 \} \] (8)

of Lie algebras in the last line of the extended Freudenthal magic square \([1, 2]\).

We note that we use the informal abbreviation irrep for irreducible representation. Here the term is understood as irreducible over the field of complex numbers, but only up to diagram automorphisms. The groups of diagram automorphisms of the algebras \( g \in \mathcal{F} \) are \( \mathbb{Z}_2 \) for \( a_2 \) and \( e_6 \), \( S_3 \) for \( \mathfrak{d}_4 \), and the trivial group for all the others. As the adjoint irrep is always mapped to itself under diagram automorphisms, we find in the complete decomposition of its tensor products over the complex numbers either irreps that are self-conjugate or pairs of conjugate irreps for \( a_2 \) and \( e_6 \). For \( \mathfrak{d}_4 \), the constituents are either single irreps that are stable under triality or triples and sextuples of irreps that are related by triality. We call the direct sum of all irreps belonging to such a pair, triple or sextuple, an irrep in the sense of this article.

We refer informally to a "Clebsch" as an abbreviation for a Clebsch-Gordan coefficient. We thought the abbreviations were marginally preferable to the acronyms IR and CGC. We refer to irreps usually by their dimensions because our studies give a central role to dimension formulas for families of irreps, one for each of \( g \in \mathcal{F} \), as a function of \( D = \dim g \). When we wish to use the Dynkin co-ordinate or highest weight designation of an irrep, we follow the conventions that stem from the choice of Cartan matrix used in \([3, 4]\).

1.2 On the context and the content of this paper

The work of Meyberg \([5]\) for \( j = 2 \), and its extension \([4, 6, 7]\) to \( j = 3 \) and 4, demonstrates the uniformity for \( g \in \mathcal{F} \) of the decomposition into irreps of the \( j \)-th tensor power

\[ ad \otimes^j \] (9)

of the adjoint irrep \( ad \) of \( g \in \mathcal{F} \). This striking property opens the way towards the main purpose of the present paper: the development of a comprehensive tensor calculus for the exceptional Lie algebras \( \mathcal{E} \), in a form uniform over \( \mathcal{E} \). The first phase of this programme is implemented here, mainly but not exclusively, at the \( j = 2 \) level.
We have in an earlier paper \[9\] succeeded in developing quite far a tensor calculus for \(g_2\), one that is based, not on \(ad\) as here, but on the seven dimensional defining irrep of \(g_2\). This can no doubt serve satisfactorily applications to areas like spin-chains, Gaudin models or to integrable quantal or supersymmetric models with \(g_2\) invariance. However it promises no discernible path to a comparable treatment of the larger exceptional groups, nor is it as amenable, as are the methods of the present paper, to extension beyond the context of the two-fold direct product. The uniform tensor calculus that this paper displays underlines the view that the tensor powers (9) of \(ad\) for \(g\in E\) have very special properties that deserve to be, and here will be, exploited as fully as possible.

The complete reduction of

\[
ad\otimes ad = (ad\otimes ad)_A + (ad\otimes ad)_S,
\]

provides a result for the antisymmetric part

\[
(ad\otimes ad)_A \equiv ad + X_2,
\]

in a universal form, one that is valid for each simple complex \(g\), and a result

\[
(ad\otimes ad)_S \equiv R_1 + R_2 + R_3,
\]

valid for \(g\in F\) but not for other \(g\). It is a simple matter to show that \(X_2\) is an irrep of \(g\) in the sense of this article with the universal properties

\[
\Delta_2 \equiv \dim X_2 = \frac{1}{2}D(D-3)
\]

\[
c_2(X_2) = 2,
\]

where \(D = \dim g\) and \(c_2(R)\) denotes the eigenvalue of the quadratic Casimir operator \(C^{(2)}\) for the irrep \(R\) of \(g\). (we use the notations \(X_2\) and \(X_3\) below, see [16], following [8], but sometimes write \(X_2 = R_0\) and \(X_3 = R_7\) to enable generic reference to representations \(R_r\).) The membership for \(g\in F\) of the family of irreps \(X_2\) is shown in Table [1], along with the membership of the three families \(R_1, R_2\) and \(R_3\) arising in (12).

It is known [3] that there are formulas for

\[
D_2 = \dim R_2, \quad D_3 = \dim R_3, \quad c_2(R_2) = \frac{1}{2}(1 + \ell_2), \quad c_2(R_3) = \frac{1}{2}(1 + \ell_3),
\]

as functions of \(D = \dim g\), valid in each case for each member of the family in question. Eq. [14] also defines the useful variables \(\ell_{2,3}\); \(D_{2,3}\) and \(\ell_{2,3}\) are essential as input into many important formulas derived below.

While we have favoured the use of \(D\) as a parameter in formulas valid across a family of irreps of \(F\), other parameters are in common use elsewhere. To facilitate comparisons, we have collected some information about them in the Appendix.

We review the analysis of \(ad\otimes ad\) carefully in Sec. [2], which makes various additions to results in [3], and employs an elementary method that lends itself to generalisation beyond the case of \(ad\otimes ad\). One matter of interest that arises here is the absence [10] for \(g\in E\) of a primitive quartic Casimir operator. We review this too, giving some extra results of later use, and explain how the case of \(d_4\), which has two independent primitive quartic Casimir operators, nevertheless conforms fully to the family picture for \(F\). Apart from this topic we are concerned almost exclusively with the exceptionals.
Table 1: Irreps of $g$ for $ad \otimes ad$.
The remaining sections of this paper take up the establishment, first at the \( j = 2 \) level, of a tensor calculus applicable uniformly across \( \mathcal{E} \), and the demonstration that our methods are sufficiently general as to permit extension to the case of \( \text{ad} \otimes \text{ad} \otimes \text{ad} \), although a systematic treatment of this is left to a future publication.

In Sec. 3 we embark on the tensor calculus accessible using tensor products such as \( v_i v_j \) and \( \phi_i \phi_j \), where \( v_i \) is an adjoint vector, i.e. one that transforms according to the adjoint representation of \( \mathfrak{g} \in \mathcal{E} \), and \( \phi_i \) is an adjoint vector with anticommuting components, e.g. fermionic creation operators. This brings into focus many of the most important third rank isotropic tensors, these being isotropic under adjoint action. We deduce basic identities involving them, and consider the interpretation of them as Clebsches. Further, we note the occurrence of results like those which in the quantum theory of angular momentum see the appearance of Racah coefficients, or what is the same thing to within a phase of no significance here, Wigner \( 6j \) symbols.

Just as for full mastery of \( \mathfrak{a}_n = su(n + 1) \) tensor and related algebraic methods, stems from development of the properties both of the Gell-mann \( \lambda \)-matrices and the tensors that enter their product law

\[
\lambda_i \lambda_j = \frac{2}{n} \delta_{ij} + (d + if)ijk \lambda_k, \tag{15}
\]

so also is there a best approach to the tensor calculus for \( \mathfrak{g} \in \mathcal{E} \).

Thus in Sec. 4, we introduce a basis of matrices for \( \text{hom}_\mathbb{H}(\mathcal{V}_{\text{ad}}, \mathcal{V}_{\text{ad}}) \), where \( \mathcal{V}_{\text{ad}} \) is the vector space in which \( \text{ad} \) acts. It is not surprising that we meet a much more complicated situation when we attempt to generalise from \( \mathfrak{a}_n \) to the set of exceptionals \( \mathcal{E} \) treated uniformly. We write and confront fully the set of all product laws involving matrices of the basis, thereby identifying all the isotropic third rank tensors arising at the \( j = 2 \) level of our study. Their interpretation as Clebsches is discussed. Amongst the large body of identities, applicable uniformly across \( \mathcal{E} \), that are proved in Sec 4. are again some that have an interpretation in terms of Racah coefficients. Further we identify explicitly (and evaluate quadratic Casimir operators for) the matrices that transform under \( \mathfrak{g} \in \mathcal{E} \) according to the irreps \( R_2, R_3 \) and \( X_2 \). This is essential input into Sec. 5.

In Sec. 5, we study \( \text{ad} \otimes \text{ad} \otimes \text{ad} \), following a straightforward method for deducing, for all \( \mathfrak{g} \in \mathcal{E} \), formulas for the dimensions and the quadratic Casimir eigenvalues for all the (new) families of irreps of \( \mathfrak{g} \in \mathcal{E} \) that enter \( \text{ad} \otimes R \) for \( R = X_2, R_2, R_3 \).

This is vital input data for a study of all the tensors that enter the picture for \( \text{ad} \otimes^3 \). It is hoped to describe progress in this direction in a future publication. That a viable approach is available is known from a preliminary study for the simplest case of \( \mathfrak{a}_1 \in \mathcal{F} \), but details of this are not given here.

In Sec. 6, we make some remarks regarding the status of \( \mathfrak{d}_4 \) within \( \mathcal{F} \). We show explicitly how the fact that \( \mathfrak{d}_4 \) has two primitive quartic Casimir operators, whereas all other members of \( \mathcal{F} \) have none, is fully compatible with this status.

The paper concludes with three appendices. The first describes the parametrisation of family formulas by \( m \) instead of \( D = \dim \mathfrak{g} \), the second gives a listing of dimension formulas in terms of \( m \), whilst the third compares the definition of Racah coefficients, or \( 6j \) symbols, in quantum theory of angular momentum with various formulas derived in the text expressing products of three trilinear tensors in terms of one. The relevance of this follows from a view described in Sec. 4 of such tensors as Clebsch-Gordan coefficients.
Development of a unified tensor calculus

\[ a_2 \quad g_2 \quad f_4 \quad e_6 \quad e_7 \quad e_8 \]

| \( \ell_2 \) | \(-\frac{1}{2}\) | \(-\frac{5}{12}\) | \(-\frac{5}{18}\) | \(-\frac{1}{4}\) | \(-\frac{3}{8}\) | \(-\frac{1}{5}\) |
| \( \ell_3 \) | \(\frac{1}{3}\) | \(\frac{1}{4}\) | \(\frac{1}{6}\) | \(\frac{1}{12}\) | \(\frac{1}{18}\) | \(\frac{1}{36}\) |

Table 2: Eigenvalues of \( L \)

2 Analysis of \( ad \otimes ad \)

2.1 The \( L \)-operator

Our approach here employs the \( L \)-operator for \( g \). Writing \( X_1 = X_i \otimes I \) and \( X_2 = I \otimes X_2 \), this is defined, for \( X_i = X_{1i} + X_{2i} \), with \( X_{1i} \mapsto ad_{1i}, X_{2i} \mapsto ad_{2i} \), by writing

\[
C^{(2)}(ad \otimes ad) = (X_{1i} + X_{2i})(X_{1i} + X_{2i}) = C^{(2)}(ad) + 2L + C^{(2)}(ad)
\]

in agreement with (15). Clearly \( L = ad_{1i}ad_{2i} \) has the same eigenspaces as \( C^{(2)}(ad \otimes ad) \) so that its eigenvalues are given by

\[
\ell = \frac{1}{2}c_1(ad \otimes ad) - 1.
\]

We have \( \ell = -1, -\frac{1}{2}, 0 \) for \( R_1, ad, X_2 \) in all cases, and for \( R_2 \) and \( R_3 \) we have the eigenvalues given in Table 2. From these data one can see empirically the result

\[
\ell_2 + \ell_3 + \frac{1}{6} = 0,
\]

which we derive in Sec. 4.

2.2 Trace results

From (15) and (16), we get

\[
L_{ij,pq} = -c_{ipk}c_{jqk}.
\]

Thus we have the following trace results

\[
\begin{align*}
\text{Tr } I &= D^2 \\
\text{Tr } L &= 0 \\
\text{Tr } L^2 &= D \\
\text{Tr } L^3 &= -\frac{1}{4}D
\end{align*}
\]

The third result here comes from (15), while the fourth one depends on the consequence

\[
c_{ipj}c_{jqk}c_{kri} = -\frac{1}{5}c_{pqr}
\]

of the Jacobi identity. Such a result is valid for all \( g \), but the actual number on the right side depends on our conventions, (14) and (15).

Alongside (21) we note the result

\[
c_{ipj}c_{jqk}c_{kri}c_{pqrs} = -\frac{1}{2}\delta_{rs}.
\]
2.3 Projectors

We begin by stating a well-known result. If a hermitian operator $A$ has distinct eigenvalues $a_i$, $1 \leq i \leq p$, then the projector onto its $i$-th eigenspace is given by

$$P_i = \prod_{k \neq i} \frac{A - a_k I}{a_i - a_k}$$

(23)

where $I$ is the unit operator. Further the result

$$(A - a_i I)P_i = 0,$$

(24)

with no sum implied on $i$, is, for each $i$, a possibly reduced version of the characteristic equation of $A$. Rather than employ the full $L$-operator of (16), with unit operator $I$ such that

$$(I)_{ij,pq} = \delta_{ip}\delta_{jq},$$

(25)

we treat separately the symmetric and antisymmetric parts of $L$

$$L_S = LI_S \quad , \quad L_A = LI_A,$$

(26)

the corresponding unit operators being

$$L_A = -\frac{1}{2}P_{ad}, \quad P_{ad} + P_0 = I_A \quad , \quad P_1 + P_2 + P_3 = I_S.$$

(28)

We turn now first to $L_A$ and second to $L_S$.

2.4 The antisymmetric subspace

We wish to note the universal features of the results for $L_A = LI_A$. The result $\ell_{ad} = -\frac{1}{2}$ for $ad$ is obvious for any $g$. From the Jacobi identity (see, for example, [11]) we obtain,

$$L_A = -\frac{1}{2}P_{ad},$$

(29)

and also the reduced characteristic equation of $L_A$,

$$L_A(L_A + \frac{1}{2}) = 0,$$

(30)

which can be rewritten as

$$L(L + \frac{1}{2})I_A = 0.$$ 

(31)

This implies that $L$ has got two distinct eigenvalues $-1/2$ and $0$ on the antisymmetric subspace, therefore $\ell_0 = 0$.

Although much of the most important information about $L$ for our purposes resides in $L_S$, (31) helps us simplify our work.

We note one other result. Let $\Delta_2 = \dim X_2$, then (11) leads to

$$\text{Tr} \ I_A = \frac{1}{2}D(D - 1) = D + \Delta_2,$$

(32)
so that
\[ \Delta_2 = \frac{1}{2} D(D - 3), \] (33)
holds for all simple \( g \). For \( a_2 = su(3) \) this means \( \Delta_2 = 20 \), where \( 20 \equiv 10 + \overline{10} \) is irreducible in the sense of our paper, namely a pair of conjugate irreps.

Since easily
\[ \text{Tr } L I_A = - \frac{1}{2} D, \] (34)
equation (31) yields
\[ \text{Tr } L^2 I_A = \frac{1}{2} D, \quad \text{Tr } L^3 I_A = - \frac{1}{8} D. \] (35)
This and (20) provide all the trace results for \( L \) needed below.

2.5 The symmetric subspace

We recall the formula in (21) for \( L \), and the result (27) for \( I_S \). We know trivially
\[(P_1)_{ij,pq} = \frac{1}{D} \delta_{ij} \delta_{pq}.\] (36)
To derive, for each of the exceptional Lie algebras, which are governed by (12), an identity quartic in the structure constants, we must eliminate \( P_2 \) and \( P_3 \) from the equations
\[(L - \ell_i)P_i = 0, \quad \ell_i = -1, \quad i = 1, 2, 3.\] (37)
There is no sum on \( i \) in (37). Also \( L \) can therein be replaced by \( LI_S \) because \( I_S P_i = P_i \) for each of the three symmetric projectors \( P_i \). Since \( P_2 + P_3 = I_S - P_1 \), it is easy to find the result
\[(L - \ell_2)(L - \ell_3)(I_S - P_1) = 0.\] (38)
To reach the sought after identities, we use \( \ell_1 = -1 \) and expand (38)
\[L^2 I_S - (\ell_2 + \ell_3)LI_S + \ell_2 \ell_3 I_S - (1 + \ell_2)(1 + \ell_3)P_1 = 0,\] (39)
add \( L^2 I_A \) to this using (31), and obtain
\[L^2 = (\ell_2 + \ell_3)LI_S - \frac{1}{4} LI_A = \ell_2 \ell_3 I_S + (1 + \ell_2)(1 + \ell_3)P_1.\] (40)
Now the \( ij,pq \) matrix element of (40) yields
\[c_{ij}c_{pq} = \text{Tr } (ad_i ad_p ad_q ad_j) \]
\[= -\frac{1}{6}(\ell_2 + \ell_3 - \frac{1}{2})c_{ij}c_{pq} + \frac{1}{6}(\ell_2 + \ell_3 + \frac{1}{2})c_{ip}c_{jq} - \frac{1}{2}\ell_2 \ell_3 (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \frac{1}{6}(1 + \ell_2)(1 + \ell_3)\delta_{ij}\delta_{pq}.\] (41)
We have already put \( \ell_1 = -1 \) into this result. If we make use also of (38), we may simplify the final expression on the right side of (41) obtaining
\[\frac{1}{6}c_{ij}c_{jq} + \frac{1}{6}c_{iq}c_{pj} - \frac{1}{4}\ell_2 \ell_3 (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \frac{5 + 6\ell_2 \ell_3}{6D} \delta_{ij}\delta_{pq}.\] (42)
From this we may deduce the result
\[\text{Tr } (ad_i ad_p ad_q ad_j) = \frac{1}{6D} [5 + 6(1 - D)\ell_2 \ell_3] \delta_{ij}\delta_{pq},\] (43)
where the enclosure of a set of suffices by round brackets indicate symmetrisation at unit weight. This result is used below in the discussion of the non-primitivity of the quartic Casimir invariants of exceptional algebras.
2.6 Derivation of formulas for $\ell_2$ and $\ell_3$

The results of Sec. 4 and 5 can be given in a nice form, applicable uniformly to all exceptional $g$, by deriving formulas for $\ell_2 + \ell_3$ and $\ell_2 \ell_3$ in terms of $D = \dim g$.

Thus we examine the equations

\[
\begin{array}{ll}
\text{Tr } I_S &= \frac{1}{2} D(D + 1) = 1 + D_2 + D_3 \\
\text{Tr } L_S &= \frac{1}{2} D = -1 + \ell_2 D_2 + \ell_3 D_3 \\
\text{Tr } L^2 I_S &= \frac{3}{4} D = 1 + \ell_2^2 D_2 + \ell_3^2 D_3 \\
\text{Tr } L^3 I_S &= -\frac{1}{8} D = -1 + \ell_2^3 D_2 + \ell_3^3 D_3,
\end{array}
\]

(44)

in which $D_i = \dim R_i$, $i = 1, 2, 3$ with $D_1 = 1$. Also we have set $\ell_1 = -1$. We write the first two and the last two of these equations in a matrix form

\[
\begin{pmatrix}
D + 2 \\
\frac{D - 1}{D + 2} \\
\frac{(3D - 4)/4}{D + 2}
\end{pmatrix}
\begin{pmatrix}
D_2 \\
D_3
\end{pmatrix}
= \begin{pmatrix}
\ell_2 \\
\ell_3
\end{pmatrix}
\begin{pmatrix}
\ell_2^2 \\
\ell_3^2
\end{pmatrix}
\begin{pmatrix}
\ell_2^3 \\
\ell_3^3
\end{pmatrix}
\begin{pmatrix}
D_2 \\
D_3
\end{pmatrix}.
\]

(45)

Elimination of $D_2$ and $D_3$ matrixwise leads to

\[
\begin{align*}
\frac{3D - 4}{2(D + 2)} &= -x(D - 1) + y \\
\frac{8 - D}{4(D + 2)} &= y\left[-x(D - 1) + y\right] - x,
\end{align*}
\]

(46)

where $y = \ell_2 + \ell_3$ and $x = \ell_2 \ell_3$, and hence to $y = -1/6$, as noted empirically. The result

\[
\ell_2 \ell_3 = -\frac{5}{3(D + 2)},
\]

(47)

follows easily, and hence expressions for $\ell_2, \ell_3, D_2, D_3$ as explicit functions of $D$. These are displayed in Sec. 2.7.

The results given in Table 2 are all in agreement with (47). The important trace result (41) can now be given in a form in which all numerical coefficients determined solely by $D$. Also (43) now reads

\[
\text{Tr } (ad_ia_ad_p ad_q ad_j) = \frac{5}{2(D + 2)} \delta_{(ij} \delta_{pq)}.
\]

(48)

2.7 Explicit results

To make explicit as functions of $D = \dim g$ some results given in previous subsections, we define

\[
\Delta = \left[\frac{242 + D}{2 + D}\right]^{\frac{1}{2}},
\]

(49)

denoted by $w$ in [3]. This takes these values for $g \in F$:

\[
7, 5, 4, 3, \frac{7}{3}, 2, \frac{5}{3}, \frac{7}{5}, \frac{7}{6}.
\]

(50)
Development of a unified tensor calculus...

Then, from Sec. 2.6, we have the following

\[ \ell_2 = \frac{(-1 - \Delta)}{12}, \]
\[ \ell_3 = \frac{(-1 + \Delta)}{12}, \]
\[ D_2 = \frac{(D + 2)}{4\Delta}(-D - 11 + \Delta(D - 1)) \]
\[ D_3 = \frac{(D + 2)}{4\Delta}(D + 11 + \Delta(D - 1)) \]
\[ c_2(R_2) = \frac{1}{6}(11 - \Delta) \]
\[ c_2(R_3) = \frac{1}{6}(11 + \Delta). \]  

(51)

Since \( \Delta = \frac{m^2}{m + 2} \) follows (169) and (49), all the formulas listed here are rational functions of \( m \). Thus it will often be true that simplifications are easier to find by working in terms of \( m \) rather than \( D \).

2.8 Quartic Casimir operators for the exceptionals

We deal here with \( g \in F \) excluding \( d_4 \) which requires separate treatment, provided in Sec. 3.

The simplest thing to do in this context is to define the general adjoint matrix \( A = b_i \, ad_i \), \( b_i \in \mathbb{R} \), and look at \( \text{Tr} \, A^4 \). We may use (47) to deduce

\[ \text{Tr} \, A^4 = b_i b_j b_p b_q \text{Tr}(ad_i ad_j ad_p ad_q)) = \frac{5}{2(D + 2)}(b_k b_k)^2, \]

(52)

which exhibits explicitly the failure of \( \text{Tr} \, A^4 \) to be primitive.

More generally, we must define a quartic Casimir operator. Set \( M = ad_i X_i \), where the \( X_i \) denote the hermitian Lie algebra generators themselves, and define

\[ C^{(4)} = \text{Tr} \, M^4 \]
\[ = \text{Tr} \, (ad_i ad_j ad_p ad_q)X_i X_j X_p X_q, \]  

(53)

We employ (3), with \( \ell_2 + \ell_3 \) and \( \ell_2\ell_3 \) given by (13) and (10), to evaluate the right side (53). We can complete the evaluation with the aid of (1), (6) and (21), obtaining

\[ C^{(4)} = \frac{5}{2(D + 2)}C^{(2)} + \frac{D - 3}{12(D + 2)}C^{(2)}, \]

(54)

valid for all exceptional \( g \).

2.9 \( C^{(4)} \) for irreps of \( g \in E \) and \( \text{Tr} \, L^4 \)

Let \( X_i \mapsto M_i \) define the matrices of an irrep \( R \) of \( g \in E \) with \( L \)-operator \( L = ad_i X_i \) represented by

\[ L_M = ad_i M_i \]
\[ (L_M)_{ja,kb} = (ad_i)_{jk}(M_i)_{ab} \]
\[ = -c_{ijk}m_{ab}, \]  

(55)
where \((M_t)_{ab} = -im_{tab}, \ a,b \in \{1,2,\ldots, \dim R\}\).

The properties (4) and (21) allow us to evaluate traces
\[
\begin{align*}
\Tr L_M &= 0 \\
\Tr L_M^2 &= m_{tab}m_{tab} = \Tr M_t M_t = c_2(R) \dim R \\
\Tr L_M^3 &= -\frac{1}{4} \Tr M_t M_t.
\end{align*}
\]
Also
\[
\Tr L_M^4 = \Tr (ad_j ad_k ad_q ad_p) \Tr (M_j M_k M_q M_p) = \left\{ \frac{5}{(D+2)} c_2(R)^2 + \frac{D-3}{12(D+2)} c_2(R) \right\} (\dim R),
\]
where we have used (54). Note also
\[(M_t M_t)_{ab} = c_2(R) \delta_{ab}, \quad (58)\]
compatibly with the second in result (56).

In the special case of \(M = ad\), so that \(L_M = L\), these results reproduce those of Sec. 2.1, while (57) gives rise to
\[
\Tr L^4 = \frac{D(D+27)}{12(D+2)},
\]
since \(c_2(ad) = 1\). This can be confirmed correct using (40) and results from Sec. 2.1.

More specific applications of (57) arise in Sec. 5, upon identification of explicit expressions for the matrices of \(R_2, R_3, X_2\) etc.

3 Simple tensor methods for \(g \in \mathcal{E}\)

3.1 Second rank tensor decomposition

Given a vector \(v_i\) which transforms under \(g \in \mathcal{E}\) according to its adjoint representation, i.e. an adjoint vector, we have, for \(v_i v_j\) which transforms according to \((ad \otimes ad)_S\), the decomposition into tensors irreducible under \(g\):
\[
v_i v_j = \frac{1}{D} v_k v_k \delta_{ij} + d_{ija} x_a + d_{ij\alpha} y_\alpha,
\]
where
\[x_a = d_{ija} v_i v_j, \quad y_\alpha = d_{ij\alpha} v_i v_j,\]
Here, to within normalisation, \(d_{ija}, d_{ij\alpha}\) are Clebsch-Gordan coefficients, referred to often here as Clebsches for short, for

\[
\begin{align*}
ad \otimes ad &\rightarrow R_2 \\
ad \otimes ad &\rightarrow R_3.
\end{align*}
\]
They are distinguished by virtue of having index sets of different natures. Our index conventions here are
\[
\begin{align*}
i,j,k,\ldots &\text{ for } ad \in \{1,2,\ldots, \dim g\} \\
a,b,c,\ldots &\text{ for } R_2 \in \{1,2,\ldots, D_2 = \dim R_2\} \\
\alpha,\beta,\gamma,\ldots &\text{ for } R_3 \in \{1,2,\ldots, D_3 = \dim R_3\},
\end{align*}
\]

\(\Box\)
as well as
\[ \mu, \nu, \rho, \ldots \quad \text{for} \quad X_2 \in \{1, 2, \ldots, \Delta_2 = \dim X_2\}, \quad (64) \]
needed soon.

Eqs. (60) and (61) reflect the normalisations
\[ d_{ija}d_{ijb} = \delta_{ab}, \]
\[ d_{ija}d_{ij\alpha} = \delta_{\alpha\beta}, \quad (65) \]
and the traceless properties
\[ d_{iia} = 0, \quad d_{iia} = 0. \quad (66) \]
Also orthogonality of different sets of Clebsches gives
\[ d_{ija}d_{ij\alpha} = 0. \quad (67) \]

To bring \((ad \otimes ad)_A\) into the picture, let \(\phi_i\) be a fermionic adjoint vector, *i.e.* one with anticommuting components, *e.g.* fermionic creation operators, as for \(a_2\) in [13]. Then the analogue of (60) is
\[ \phi_i\phi_j = X_{ij} + c_{ijk}\psi_k, \quad (68) \]

where
\[ \psi_i = c_{ijk}\phi_j\phi_k \quad (69) \]
is the adjoint vector expected from
\[ \quad (ad \otimes ad)_A \equiv ad + X_2, \quad (70) \]
and \(X_{ij}\) is the tensor, clearly of dimension
\[ \Delta_2 = \frac{1}{2}D(D - 3) = \dim X_2, \quad (71) \]
associated with the second term of (70). Also we may view the \(c_{ijk}\) as Clebsches for \(ad \otimes ad \rightarrow ad\).

### 3.2 A projector view

We obtain another useful view of (60) by applying to \(v_i v_j\) the result, from (28),
\[ I_S = P_1 + P_2 + P_3, \quad (72) \]
where we write now
\[ (P_1)_{ij,kl} = \frac{1}{2}\delta_{ij}\delta_{kl} \]
\[ (P_2)_{ij,kl} = d_{ija}d_{kla} \]
\[ (P_3)_{ij,kl} = d_{ija}d_{kla}. \quad (73) \]

All the usual properties of projectors are satisfied: \(\text{Tr } P_1 = 1\) is trivial, \(\text{Tr } P_2 = D_2\) and \(\text{Tr } P_3 = D_3\) follow (65), while
\[ P_2^2 = P_2, \quad P_3^2 = P_3, \quad P_2P_3 = 0, \quad P_1P_3 = 0, \quad P_1P_2 = 0 \quad (74) \]
follow (65), (66) and (67). Similarly we may apply
\[ I_A = P_{ad} + P_0, \] to \( \phi_i \phi_j \), where
\[ (P_{ad})_{ij,kl} = c_{ijt}c_{ikt} \]
\[ (P_0)_{ij,kl} = g_{ij\mu}g_{k\mu}, \] (76)
where the Clebsches for \( ad \otimes ad \rightarrow X_2 \) are normalised, like all the other Clebsches introduced so far, so that
\[ g_{ij\mu}g_{ij\nu} = \delta_{\mu\nu}. \] (77)
The application simply reproduces (68) with
\[ X_{ij} = (P_0)_{ij,kl}\phi_k\phi_l. \] (78)
We note also \( \text{Tr } P_0 = \delta_{\mu\mu} = \Delta_2 = \text{dim } X_2 \), and the orthogonality relation
\[ c_{ijt}g_{ij\mu} = 0. \] (79)
Also alongside (64) and (77) we note the identities
\[ d_{ijadkja} = \frac{D_2}{D}\delta_{ik}, \]
\[ d_{ijadkja} = \frac{D_3}{D}\delta_{ik}, \]
\[ g_{ij\mu}g_{kj\mu} = \frac{1}{2}(D - 3)\delta_{ik}. \] (80)

### 3.3 Two basic tensor identities

From the equations
\[ I_S = P_1 + P_2 + P_3 \]
\[ LI_S = -P_1 + \ell_2P_2 + \ell_3P_3, \] (81)
we may eliminate \( P_2 \) and \( P_3 \) in turn, getting
\[ (L - \ell_2)I_S = -(1 + \ell_2)P_1 + (\ell_3 - \ell_2)P_3 \]
\[ (L - \ell_3)I_S = -(1 + \ell_3)P_1 + (\ell_2 - \ell_3)P_2. \] (82)

Taking matrix elements with the aid of (73) gives
\[ (\ell_2 - \ell_3)d_{ijadpqa} = -\frac{1}{2}(c_{ipt}c_{jq} + c_{iqt}c_{jp}) - \frac{1}{2}\ell_3(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \frac{1 + \ell_3}{D}\delta_{ij}\delta_{pq}, \] (83)
plus a result for \( d_{ijadpqa} \) obtained by interchange of \( \ell_2 \) and \( \ell_3 \) in (83).

It is a non-trivial but instructive task to verify that various contractions of (83) are identically satisfied; results from Sec. 2.8 are needed.
3.4 Trilinear tensor identities

Since we regard $c_{ijk}$ as defining the set of Clebsches for

$$ad \otimes ad \to ad,$$    

eq. (21) can be regarded as an analogue of a result in the quantum theory of angular momentum that defines a Racah coefficient. There are many more identities of this sort. Appendix C provides a little background from the quantum theory of angular momentum.

It is easy to contract (83) with $d_{ija}$ and get

$$c_{ikt}c_{tlj}d_{ija} = \ell_2d_{kla}.$$    

There is a similar result for $d_{ija}$ obtained by replacing $\ell_2$ on the right of (85) by $\ell_3$. There are many other pairs of identities related in this fashion; they should not need to be indicated explicitly again.

To contract (83) with a $c$-tensor and get

$$d_{ija}d_{pqa}c_{ipt} = -\ell_2D_2D_c_{pqs},$$    

is harder, and requires the use of (21) and results from Sec. 2.8.

Also, using (75), we get

$$g_{ij\mu}g_{k\ell\mu}c_{iks} = -c_{jls}.$$    

Further, equipped with (83), we can deduce

$$d_{ija}d_{pqa}d_{ipb} = -\frac{11 + D + \Delta}{2D\Delta}d_{jpb},$$    

where $\Delta$ is defined by (49).

A further consequence of (83) is

$$d_{(ij}^ad_{pq)a} = \frac{2D_2}{D(D + 2)}\delta_{(ij}\delta_{pq)},$$    

in which the round brackets denote symmetrisation over the enclosed at unit weight. The first index $a$ is raised, without any metric significance, just to take it outside the round brackets. This result, (89), can be used to give an independent derivation of (88), with the aid of the result in (51) for $D_2$.

Obviously there are more results of the type here treated, of increasing complication. We have shown how one might work towards them if and when the need to do so arises.

4 Matrices and associated tensors

4.1 The basis set

To gain full control of the formalism, systematically identifying all the tensors of importance as they arise and determining their essential properties, it is useful to introduce a complete set of basis matrices for hom$_R(\mathcal{V}_{ad}, \mathcal{V}_{ad})$, where $\mathcal{V}_{ad}$ is the vector space in which the irrep $ad$ of $\mathfrak{g} \in \mathcal{E}$ acts.
The basis

\[ M_A, \quad A \in \{1, 2, \ldots, D^2\}, \quad D = \text{dim } \mathfrak{g}, \]  

consists of matrices

\[ \left( \frac{1}{D} I, \; D_a, \; Y_\alpha \right), (F_i, \; G_\mu). \]  

The first parenthesis contains a total of \(1 + D_2 + D_3 = \frac{1}{2} D(D + 1)\) symmetric matrices defined by

\[ I_{ij} = \delta_{ij}, \quad (D_a)_{ij} = -d_{ija}, \quad (Y_\alpha)_{ij} = -d_{ija}. \]  

The second parenthesis in (91) contains \(D + \Delta_2 = \frac{1}{2} D(D - 1)\) antisymmetric matrices defined by

\[ (F_i)_{jk} = -ic_{ijk}, \quad (G_\mu)_{ij} = -ig_{ij\mu}. \]  

These definitions are all given in terms of tensors already introduced in Sec. 3.

The matrices \(M_A\) all hermitian, and possess the trace properties

\[ \text{Tr } M_A = 0, \quad \text{Tr } (M_A M_B) = \delta_{AB}. \]  

By expanding symmetric \(A \in \text{hom}_\mathbb{R}(\mathcal{V}_\text{ad}, \mathcal{V}_\text{ad})\) with respect to our basis

\[ A = aI + u_a D_a + v_a Y_\alpha \]  

we obtain a completeness relation

\[ d_{ija} d_{kla} + d_{ija} d_{kla} + \frac{1}{D} \delta_{ij} \delta_{kl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]  

the compatibility of which with (89) and its analogue involving \(D_3\) can be checked.

Similarly we have

\[ c_{ikt} c_{jlt} + g_{ik\mu} g_{jlt\mu} = \frac{1}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}), \]  

which contains the same information as (75),

### 4.2 The product \(F_i F_j\)

By considering the action of \(I = I_A + I_S\) on \(F_i F_j\) we find

\[ F_i F_j = \frac{1}{D} \delta_{ij} + \frac{1}{2} i c_{ijk} F_k + \ell_2 d_{ija} D_a + \ell_3 d_{ija} Y_\alpha, \]  

with no term in \(G_\mu\) because of the closure of the Lie algebra \(\mathfrak{g}\).

Here we have used facts like

\[ \text{Tr } (F_i F_j D_a) = c_{piq} c_{qij} d_{rpa} = \ell_2 d_{ija}, \]  

evaluated using (85). The factors \(i\) needed for the hermiticity of the \(F\)-matrices accounts for the minus sign in the definition (92).
4.3 The full set of product laws

It is necessary to be prepared to contemplate all the product laws within \( M_A M_B \) and all the tensors that arise in them. The full list is

\[
\begin{align*}
F_i F_j &= \frac{1}{D} \delta_{ij} + \frac{1}{2} i c_{ijk} F_k + \ell_2 d_{ij a} D_a + \ell_3 d_{ij a} Y_\alpha \\
F_i D_a &= \frac{1}{2} i m_{i a b} D_b + \ell_2 d_{ij a} F_j + d_{i a b} G_\mu \\
F_i Y_\alpha &= \frac{1}{2} i m_{i a b} Y_\beta + \ell_3 d_{ij a} F_j + d_{i a b} G_\mu \\
F_i G_\mu &= \frac{1}{2} i g_{i a b} G_\nu + d_{i a b} D_a + d_{i a b} D_a \\
D_a D_b &= \frac{1}{D} \delta_{a b} I + \frac{1}{2} i m_{i a b} F_i + \frac{1}{2} i m_{a b} G_\mu + d_{a b c} D_c + d_{a b a} Y_\alpha \\
D_a Y_\alpha &= \frac{1}{2} i m_{a b c} G_\mu + d_{a b a} D_b + d_{a a b} Y_\beta \\
G_\mu D_a &= \frac{1}{2} i m_{a b c} D_b + \frac{1}{2} i m_{a b c} Y_\alpha + d_{a b a} F_i + d_{a b a} G_\nu \\
Y_\alpha Y_\beta &= \frac{1}{D} \delta_{a b} + \frac{1}{2} i m_{i a b} F_i + \frac{1}{2} i m_{a b c} G_\mu + d_{a a b} Y_\gamma + d_{a a b} Y_\alpha \\
Y_\alpha G_\mu &= \frac{1}{2} i m_{a b c} D_a + \frac{1}{2} i m_{a b c} Y_\beta + d_{a b a} F_i + d_{a b a} G_\nu \\
G_\mu G_\nu &= \frac{1}{D} \delta_{a b} + \frac{1}{2} i g_{i a b} F_i + \frac{1}{2} i g_{i a b} G_\nu + d_{a b a} D_a + d_{a b a} D_a.
\end{align*}
\]

Thus we need to consider 10 products with 39 terms, involving 4 Kronecker deltas and 18 isotropic third rank tensors. Again, tensors named by the same letter, but carrying distinct types of index sets, are to be regarded as distinct tensors. To understand fully the results, all of which are relevant at this point even though we may not make this completely explicit, bring in six families of irreps which have not so far been mentioned. These are defined by their dimensions for \( g_2, f_1, e_6 \ldots \) in that order, as follows

\[
\begin{align*}
R_4 &= 7, 273, 650, 1463, - \\
R_5 &= 64, 4096, 11648, 40755, 147250 \\
R_6 &= 189, 10829, 34749, 152152, 779247 \\
X_3 ( = R_7 ) &= 182, 19448, 70070, 365750, 2450240 \\
R_8 &= 273, 12376, 43758, 238602, 1763125 \\
R_9 &= 448, 29172, 105600, 573440, 4096000.
\end{align*}
\]

The point here is that the result (111) relevant to (101), contains only three terms relevant to our basis matrices \( M_A \) namely the first three, which correspond to the \( F, G, D \) terms of
There is no \( Y \) term in (101) because \( R_3 \) does not occur in (111). Tracing with \( F_k \) accounts for the coefficient \( \ell_2d_{ij\alpha} \), while the other two allowed terms necessitate the introduction of two new tensors. Eqs. (110–112) relate similarly to (103–101). And so on one proceeds. The repetition in later products of tensors introduced in earlier products is accounted for by requiring consistency under tracing. Various tensors have obvious symmetry or antisymmetry properties. Also the order of various matrices of different types within trace definitions should be irrelevant, to within a sign; e.g. it can be proved that the three traces
\[
\text{Tr} (F_iD_aG_\mu) = \text{Tr} (F_\mu G_iD_a) = \text{Tr} (G_\mu F_iD_a) = d_{i\alpha\mu},
\]
are mutually consistent.

We draw attention to the absence of an \( F \)-term in (103). This follows from Lie algebra closure. Another view of this states that
\[
\text{Tr} (F_iF_jG_\mu) = 0.
\]
This in turn gives an identity that can be proved, using the Jacobi identity for the structure constants, in exactly the same way as (21) was proved
\[
c_{piq}c_{qjr}g_{r\mu} = 0.  \quad (121)
\]

4.4 Tensors as Clebsches

There are in the product laws (100–109) four distinct Kronecker deltas associated trivially with
\[
R_1 = 1 \in \text{ad} \otimes \text{ad}, \quad R_2 \otimes R_2, \quad R_3 \otimes R_3, \quad X_2 \otimes X_2. \quad (122)
\]

For the third rank isotropic tensors we have drawn up Table 3. It indicates symmetry properties with respect to interchange of indices of the same type, with the letter \( T \) standing for totally. The table also specifies the triad of irreps for which each tensors provides a set of Clebsches. We emphasise that only the terms \( \text{ad}, R_2, R_3, X_2 \) on the right sides of (110–112) are relevant at present, but see Sec. 5.

4.5 Jacobi identities and matrix irreps of \( \mathfrak{g} \in \mathcal{E} \)

The results, from (101–103),
\[
[F_i, D_a] = im_{iab}D_b, \quad [F_i, Y_\alpha] = im_{i\alpha\beta}Y_\beta, \quad [F_i, G_\mu] = ig_{i\mu\nu}G_\nu, \quad (123)
\]

imply that \( D_a, Y_\alpha, G_\mu \) transform under \( \mathfrak{g} \in \mathcal{E} \) according to the irreps \( R_2, R_3, X_2 \) of \( \mathfrak{g} \in \mathcal{E} \). The tensors that appear on the right side of (123) are very important ones. To see this, we use Jacobi identities of the sort \( F, F, X \) for \( X = D, Y, G \) in turn. The first one translates into
\[
[M_i, M_j] = ic_{ijk}M_k, \quad (124)
\]
where the matrices \( M_i \) are defined by
\[
(M_i)_{ab} = -im_{iab}. \quad (125)
\]
| Notation | Symmetries | Triad of irreps |
|----------|------------|----------------|
| $e_{ijk}$ | TA         | $ad \otimes ad \rightarrow ad$ |
| $d_{ija}$ | S          | $ad \otimes ad \rightarrow R_2$ |
| $d_{ija}$ | S          | $ad \otimes ad \rightarrow R_3$ |
| $m_{iab}$ | A          | $ad \otimes R_2 \rightarrow R_2$ |
| $m_{i\alpha\beta}$ | A | $ad \otimes R_3 \rightarrow R_3$ |
| $d_{ia\mu}$ | A | $ad \otimes R_2 \rightarrow X_2$ |
| $d_{ia\mu}$ | A | $ad \otimes R_3 \rightarrow X_2$ |
| $m_{\mu ab}$ | A | $R_2 \otimes R_2 \rightarrow X_2$ |
| $d_{abc}$ | TS         | $R_2 \otimes R_2 \rightarrow R_2$ |
| $d_{a\beta\alpha}$ | S | $R_2 \otimes R_2 \rightarrow R_3$ |
| $m_{\mu\alpha\beta}$ | A | $R_3 \otimes R_3 \rightarrow X_2$ |
| $d_{\alpha\beta\gamma}$ | TS | $R_3 \otimes R_3 \rightarrow R_3$ |
| $d_{\alpha\beta\alpha}$ | S | $R_3 \otimes R_3 \rightarrow R_2$ |
| $g_{i\mu\nu}$ | A | $X_2 \otimes X_2 \rightarrow ad$ |
| $m_{a\alpha\mu}$ | A | $R_2 \otimes R_3 \rightarrow X_2$ |
| $d_{a\mu\nu}$ | S | $X_2 \otimes X_2 \rightarrow R_2$ |
| $d_{a\mu\nu}$ | S | $X_2 \otimes X_2 \rightarrow R_3$ |
| $g_{\mu\nu\rho}$ | TA | $X_2 \otimes X_2 \rightarrow X_2$ |

Table 3: Third rank isotropic tensors
Thus $X_i \mapsto M_i$ defines the $D_2 \times D_2$ matrices of the irrep $R_2$ of $\mathfrak{g}$.

Similarly
\begin{align*}
(N_1)_{\alpha\beta} &= -i m_{\alpha\beta} \\
(G_1)_{\mu\nu} &= -i g_{\mu\nu},
\end{align*}
(126)
defines the matrix irreps $X_i \mapsto N_i$ and to $G_i$ for the irreps $R_2$ and $X_2$ of $\mathfrak{g} \in \mathcal{E}$.

### 4.6 Eigenvalues of $C^{(2)}$ for $X_i \mapsto M_i, N_i, G_i$

It is easiest in the case of $G_i$ to show that the definition of Sec. 4.5 is consistent with the knowledge, already to hand, that
\begin{equation}
G_i G_i = 2I.
\end{equation}
(127)
Thus we note the results
\begin{align*}
(G_i G_i)_{\mu\rho} &= g_{\mu\nu} g_{\nu\rho} \\
\frac{1}{2} i g_{m\mu\nu} &= \text{Tr} (G_{\mu} G_{\nu} F_m) = -g_{jk\mu} g_{kl\nu} c_{jlm}.
\end{align*}
(128)
We insert the second one, not directly into the first, but rather into
\begin{equation}
g_{m\mu\nu} g_{n\mu\nu}.
\end{equation}
(129)
Then use of (21, 121, 4) and (77) enables the proof that
\begin{equation}
g_{m\mu\nu} g_{n\mu\nu} = (D - 3) \delta_{mn}.
\end{equation}
(130)
This is tantamount to proving
\begin{equation}
g_{m\mu\nu} g_{m\nu\rho} = 2 \delta_{\mu\rho},
\end{equation}
(131)
which is as required.

The same method can be applied to showing that
\begin{align*}
(M_i M_i)_{ac} &= m_{ib\delta} m_{\delta c} = 2(1 + \ell_2) \delta_{ac} \\
(N_i N_i)_{\alpha\gamma} &= m_{\alpha\beta} m_{\beta\gamma} = 2(1 + \ell_3) \delta_{\alpha\gamma}.
\end{align*}
(132)
The first of these requires \textbf{[54], [56] etc.}, and emerges upon use of formulas form Sec. 2.8.

In (131) and (132), the required eigenvalues of $C^{(2)}$ are seen explicitly on the right sides.

### 5 Towards $ad \otimes ad \otimes ad$

#### 5.1 dim $X_3$ and $c_2(X_3)$

The result
\begin{equation}
(ad \otimes ad \otimes ad)_A \equiv R_1 + R_2 + R_3 + X_2 + X_3,
\end{equation}
(133)
is known from \textbf{[1]} on the basis of \textbf{[3]}. It can be proved, as in \textbf{[15]}, using methods based on the Molien function \textbf{[16, 17]}, a method capable \textbf{[15]} of treating higher $ad^{\otimes \tau}$.

Its importance resides in the fact that all but the last family of irreps of $\mathfrak{g} \in \mathcal{E}$ have been treated fully already at the level of $ad \otimes ad$. Accordingly \textbf{[133]} gives us an easy passage to the treatment of the family $X_3$. 

We define the operator $M$ via
\[
C^{(2)} = (X_1 + X_2 + X_3)^2 = 3 + 2M \\
M = X_1.X_2 + X_2.X_3 + X_3.X_1,
\]
in which all the $X_i = ad_i$. We define projectors onto the eigenspaces of $C^{(2)}$, and hence of $M$, so that
\[
I_A = P_1 + P_2 + P_3 + P_0 + P_7 \\
\frac{1}{6} D(D-1)(D-2) = (1 + D_2 + D_3) + \Delta_2 + \Delta_3,
\]
where $\Delta_{2,3} = \text{dim } X_{2,3}$. Hence
\[
\Delta_3 = \frac{1}{6} D(D-1)(D-8).
\]
Next we apply $M = \frac{1}{2}(C^{(2)} - 3)$ to the first entry of (135). This gives
\[
\text{Tr } (MI_A) = -\frac{3}{2} + (\ell_2 - \frac{1}{2})D_2 + (\ell_3 - \frac{1}{2})D_3 - \frac{1}{2}\Delta_2 + m\Delta_3,
\]
where $m$ is the eigenvalue of $M$ for $X_3$. One calculates the left side directly getting $D - \frac{1}{2}D^2$. All the other quantities in (137) are known as functions of $D = \text{dim } g$. Hence, with the aid of (144), we find that $m = 0$, so that $C^{(2)}$ has eigenvalue
\[
c_2(X_3) = 3,
\]
completing algebraic derivation of the expected result.

5.2 **Trace equations for $ad \otimes ad \otimes ad$**

The approach here is based on the results (110 – 112), and depends on the fact that formulas for $c_2(R)$ and $\text{dim } R$ are known for
\[
R \in \{ad, R_2, R_3, X_2, X_3(\equiv R_7)\}.
\]
It will be seen soon that the fact that $X_3$ has been treated (in Sec. 5.1) is crucial, enabling us to deduce corresponding results for
\[
R \in \{R_4, R_5, R_6, R_8, R_9\}.
\]
we begin by calculating
\[
\text{Tr } C^r, \quad r = 0, 1, 2, 3,
\]
where
\[
C = C^{(2)}_{ad \otimes R}, \quad R = X_2, R_2, R_3.
\]
We have
\[
C^{(2)}_{ad \otimes R} = (ad_t + M_t)(ad_t + M_t) = 1 + c_2(R) + L_M,
\]
where $X_i \mapsto M_i$ defines the matrices of $R$, and $L_M$ is as defined in (53) by $L_M = ad_t M_t$, so that $\text{Tr} \,(L_M)^r$ is known, from the work of Sec. 2.9, for $r = 0, 1, 2, 3$. It is known for $m = 4$ also but this is not needed now. Thus we obtain

\[
\begin{align*}
\text{Tr} \, I &= DD(R) \\
\text{Tr} \, C &= DD(R)(1 + c_2(R)) \\
\text{Tr} \, C^2 &= DD(R)(1 + c_2(R))^2 + 4c_2(R)D(R) \\
\text{Tr} \, C^3 &= DD(R)(1 + c_2(R))^3 + 12c_2(R)D(R)(1 + c_2(R)) - 2c_2(R)D(R).
\end{align*}
\tag{144}
\]

We now outline how, by reference to (110) and (111), we can evaluate $c_2(R)$ and $D(R)$ for $R_9$ and $R_4$. A similar method applied to (110) and (112) can be used to treat $R_5$ and $R_8$, leaving the easy final step of handling $R_6$ to complete the job.

From (110) and (111) we get

\[
I = P_{ad} + P_0 + P_2 + P_3 + P_5 + P_7 + P_9
\]

\[
I = P_{ad} + P_0 + P_2 + P_4 + P_5 + P_6.
\tag{145}
\]

We have not distinguished the different unit operators. Taking the traces of these equations and subtracting allows much cancellation and gives

\[
D_9 - D_4 = D(\Delta_2 - D_2) - (D_3 + D_7) = f_0(D),
\tag{146}
\]

where $D_9 = \dim R_9, D_4 = \dim R_4$. Next acting on (145) with the appropriate Casimirs $C$, taking traces using (144), and subtracting, gives

\[
c_9 D_9 - c_4 D_4 = f_1(D),
\tag{147}
\]

where

\[
f_1(D) = 3D\Delta_2 - DD_2(1 + c_2) - (c_3 D_3 + 3D_7),
\tag{148}
\]

and $c_r = c_2(R_r), r \in \{9, 4, 2, 3, 7\}$ with $c_7 = 3$. Similarly using the square and the cube of $C$ we complete the derivation of the set of four equations

\[
c_9^r D_9 - c_4^r = f_r(D), \quad r = 0, 1, 2, 3,
\tag{149}
\]

where we have not displayed expressions for $f_2(D)$ or $f_3(D)$. The method of Sec. 2.4 (matrixwise elimination of $D_9$ and $D_4$) now immediately yields

\[
\begin{align*}
c_9 + c_4 &= \frac{f_0 f_3 - f_1 f_2}{f_0 f_2 - f_1^2} \\
c_9 c_4 &= \frac{f_1 f_3 - f_2^2}{f_0 f_2 - f_1^2}.
\end{align*}
\tag{150}
\]

It is obvious how to assign the two solutions of these equations appropriately to the correct families, $R_4, \, R_9$. The explicit evaluation of the right sides of (150) is a task best left to MAPLE. Because of non-rational dependence on $D$, it is better to work in terms of $m$, related to $D$ by (169) of Appendix A. However the results are already known: [7], where [8]
was employed. Since, in confirming them, we have used a different parametrisation from that of \[7\], we quote

\[ c_9 = c_2(R_9) = \frac{3m + 7}{m + 2}, \quad c_4 = c_2(R_4) = \frac{2(m + 1)}{m + 2} \]
\[ c_5 = c_2(R_5) = \frac{5m + 8}{2(m + 2)}, \quad c_8 = c_2(R_8) = \frac{3m + 8}{m + 2} \]
\[ c_6 = c_2(R_6) = \frac{8}{3}. \tag{151} \]

Further, for convenience of readers, we have listed the expressions in terms of \(m\) for the dimensions of \(X_2 = R_0, R_2, \ldots, R_6, X_3 = R_7, R_8, R_9\) in Appendix B. Once \(c_9\) and \(c_4\) have been found it is a simple matter to use (146–147) to reach \(D_9\) and \(D_4\), etc.

6 The case of \(d_4\), and of its quartic Casimirs

6.1 \(ad \otimes ad\)

The versions of (11) and (12) that apply to \(d_4\) read as

\[ (ad \otimes ad)_A = ad + 350 \tag{153} \]
\[ (ad \otimes ad)_S = 1 + \{35 + 35 + 35\} + 300 \]
\[ = (0, 0, 0, 0) + \{(2, 0, 0, 0) + (0, 0, 2, 0) + (0, 0, 0, 2)\} + (0, 2, 0, 0). \tag{154} \]

The irrep 350 here agrees with \([13]\) for \(\text{dim } d_4 = D = 28\), but, in the role of \(R_2\) in \([12]\), \([154]\) suggests the direct sum of three inequivalent irreps of \(d_4\). These three irreps, whose Dynkin labels are given explicitly above, are a set of three related by triality, all of which share the eigenvalue \(\frac{4}{3}\) of \(C^2(2)\) for \(d_4\). It is the latter fact that enables their direct sum, viewed as a single entity, to fulfill exactly the role of \(R_2\) in the general discussion that applies to other members of \(g \in F\).

Since the parameter \(\Delta\) of \([19]\) has the value 3 for \(d_4\), we get \(D_2 = 105\), correctly, and the expected values of \(D_3, \ell_2, \ell_3\) for \(d_4\) by inserting \(\Delta = 3\) into the results of Sec. 2.8.

The discussion of the situation surrounding irreps of \(d_4\) related by triality, such as the 35’s in \([154]\), can be refined by consideration of irreps of the group obtained by extending the group \(SO(8)\) by the group of automorphisms of its Dynkin diagram. Here we merely refer to \([7]\) for this and similar considerations for \(a_2\) and \(e_6\).

6.2 The quartic Casimirs of \(d_4\)

Sec. 2.9 explains why the exceptional Lie algebras \(E \subset F\) do not possess a primitive quartic Casimir operator. Since \(d_4 = so(8) \subset F\) has two independent primitive quartic Casimir operators, it might seem that \(d_4\) fails to conform fully to its implied status within \(F\). We show next that is not the case, showing explicitly exactly how it conforms.

The projector \(P_2\) that projects onto the \(R_2\) subspace of \((ad \otimes ad)_S\) is given by \([73]\) for all \(g \in F\) in the form

\[ (P_2)_{ij,pq} = d_{ij}d_{pqa}, \tag{155} \]
and a view of the $d_{ij a}$ as a set of Clebsches for $ad \otimes ad \rightarrow R_2$ was indicated in Sec. 4.4. Since

$$P_2 = \frac{(L + 1)(L - \ell_3)I_S}{(L + 1)(L - \ell_3)},$$

(156)

we find, with a temporary abbreviation $g(D)$ for the denominator of the right side of (156),

$$g(D)d_{ija}d_{pqa} = \frac{1}{2}c_{ikt}c_{ilt}(c_{kps}c_{tqs} + c_{kqs}c_{tps}) - \frac{1}{2}(1 - \ell_3)(c_{ipt}c_{jqt} + c_{iqt}c_{jpt}) + \frac{1}{2}\ell_3(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}).$$

(157)

If we now define a quartic Casimir invariant for the vector $v_i$ according to

$$Q^{(4)} = d_{ija}d_{pqa}v_iv_jv_pv_q,$$

(158)

we get

$$g(D)Q^{(4)} = c_{ikt}c_{jtl}c_{spk}v_i v_j v_p v_q - \ell_3 v_kv_kv_l,$$

(159)

which shows the definition (158) is a satisfactory alternative to that of $C^{(4)}$ used in (52). It further reduces, as $C^{(4)}$ itself reduced using (41), to a multiple of the square of the quadratic invariant $v_kv_k$. We do not exhibit the result as the multiple does not simplify into a nice enough form.

Putting $D = 28$ and $\ell_3 = \frac{1}{6}$ naively into our result for $Q^{(4)}$, we find

$$Q^{(4)} = \frac{1}{2}v_kv_kv_l.$$

(160)

Using the notation $x_a = d_{ija}v_iv_j$ of (61), we write this as

$$2Q^{(4)} = x_ax_a.$$

(161)

As noted for the exceptionals, this is the whole story; there is one irreducible vector $x_a$, and one equation, e.g. (161) which means that the square $x_ax_a$ does not define a primitive quartic Casimir. For $\mathfrak{so}_4$ the difference from other $g \in \mathcal{F}$ lies in the reducibility of the representation $R_2$ for $\mathfrak{so}_4$. In fact, the projector $P_3$ is the sum of three orthogonal projectors. Put otherwise, there are three orthogonal sets of Clebsch-Gordan coefficients for 35’s belonging to $(ad \otimes ad)_S$ and three pairwise orthogonal 35 component entities

$$w^{(r)}_a = k_rd^{(r)}_{ija}v_iv_j,$$

(162)

in which the $k_r$ are constants, such that

$$x_a = w^{(1)}_a + w^{(2)}_a + w^{(3)}_a.$$

(163)

Now $x_ax_a$ itself is not itself primitive. But, since the $w^{(r)}$ are orthogonal,

$$x_ax_a = w^{(1)}_aw^{(1)}_a + w^{(2)}_aw^{(2)}_a + w^{(3)}_aw^{(3)}_a,$$

(164)

and this leaves two linear combinations of the three squares, which can serve as independent and primitive quartic invariants. This places the known situation for $\mathfrak{so}_4$ correctly within, and not superficially outside, the family context.

Acknowledgements
The research of AJM is supported in part by PPARC. HP is grateful to Emmanuel College, Cambridge, for a Research Fellowship. We thank Bruce Westbury for stimulating discussions, and for generously providing us with copies of manuscripts of his research work, including a preliminary version of [19].

**Appendix A: Other parametrisations**

In our work we have chosen to use the $D = \dim \mathfrak{g}$ as the parameter in formulas such as (12) for dimensions $\dim R$, or eigenvalues $c_2(R)$ of $\mathcal{O}(2)$ with

$$D = 3, 8, 14, 28, 52, 78, 133, 248,$$

for $\mathfrak{g} \in \mathcal{F}$. Other workers in the general area have made different choices. In [3] and [7] one finds

$$\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{18}, \frac{1}{30}.$$  

(166)

This has the significance that $\alpha$ is the inverse of the dual Coxeter number $h^\vee$ for each $\mathfrak{g}$, [3] p37. The choice has a natural interpretation also in terms of our work:

$$\alpha = \ell_3 = \frac{1}{2}c_2(R_3) - 1.$$  

(167)

Here $\ell_3$ denotes the eigenvalue for $R_3$ of the $L$-operator used in Sec. 2 in the analysis of $ad \otimes ad$.

In recent studies [18, 19], one meets the parameter $m$ with values

$$m = -\frac{4}{3}, -1, -\frac{2}{3}, 0, 1, 2, 4, 8$$

(168)

related to $D$ via

$$D = \frac{2(3m + 7)(5m + 8)}{m + 4},$$

(169)

and to $\alpha = \ell_3$ via

$$\alpha = \ell_3 = \frac{1}{3(m + 2)},$$

(170)

so that $h^\vee = 3(m + 2)$.

For the Lie algebras of the last line of the Freudenthal magic square itself there is the further observation that $m$ is equal to the dimension of the division algebra involved in the Freudenthal construction of each one.

One other thought: suppose one solves (169) for $m$ in terms of $D$. Of the two roots of the quadratic equation in question here, one is $m$ and has values related to $\ell_3$ by (170). The other root has different values, $m'$, say, such that

$$\ell_2 = \frac{1}{3(m' + 2)},$$

(171)

where $\ell_2$ denotes the eigenvalue for $R_2$ of the $L$-operator used in Sec. 3. Comparison of (170) and (171) reveals a close relationship to the involution $\ast$ used in [7].

We also note the parameter $\Delta = \Delta(D)$ of (49), and its role, see Sec. 2.7, in formulas not dependent linearly upon $D$. Also $\Delta \mapsto -\Delta$ corresponds to the star involution of [7].

**Appendix B: Dimension formulas in terms of $m$**

As mentioned above, many formulas are in essentially their simplest form when written in terms of $m$ rather than $D = \dim \mathfrak{g}$, especially ones which involve the quantity $\Delta$ of (49), a rational function of $m$ but not of $D$. This applies to many dimension formulas. We have
\[ D = \dim g = \frac{2(5m + 8)(3m + 7)}{(m + 4)}, \quad (172) \]

\[ \dim R_0 \equiv \dim X_2 = \frac{5(5m + 8)(3m + 7)(3m + 4)(2m + 5)}{(m + 4)^2}, \quad (173) \]

\[ \dim R_2 = \frac{90(3m + 7)(3m + 4)(m + 2)^2}{(m + 6)(m + 4)^2}, \quad (174) \]

\[ \dim R_3 = \frac{45(5m + 8)(2m + 5)(m + 2)^2}{(m + 6)(m + 4)}, \quad (175) \]

\[ \dim R_4 = \frac{5(5m + 8)(3m + 7)(3m + 4)(2m + 3)(m + 2)(8 - m)}{(m + 6)(m + 5)(m + 4)^3}, \quad (176) \]

\[ \dim R_5 = \frac{5120(3m + 7)(2m + 5)(2m + 3)(m + 2)^2(m + 1)}{(m + 8)(m + 6)(m + 4)^3}, \quad (177) \]

\[ \dim R_6 = \frac{27(5m + 12)(5m + 8)(3m + 7)(3m + 4)(2m + 5)(2m + 3)}{(m + 9)(m + 5)(m + 4)^2}, \quad (178) \]

\[ \dim R_7 = \frac{10(5m + 12)(5m + 8)(3m + 8)(3m + 7)(2m + 3)(m + 1)}{(m + 4)^3}, \quad (179) \]

\[ \dim R_8 = \frac{10(5m + 12)(5m + 8)(3m + 11)(3m + 7)(2m + 5)(m + 2)^2}{(m + 8)(m + 6)(m + 4)^2}, \quad (180) \]

\[ \dim R_9 = \frac{40(5m + 12)(5m + 8)(3m + 8)(3m + 4)(m + 2)^2}{(m + 6)(m + 5)(m + 4)}. \quad (181) \]
Appendix C:  Racah coefficients

For more details the reader may refer to textbooks devoted to the quantum theory of angular momentum, or to the valuable reprint volume [14].

Racah coefficients arise in the comparison of different ways of coupling three angular momenta to define the total angular momentum. One way of presenting the definition in terms of angular momentum Clebsches is

$$\sum_m \langle j_1 j_6 m_1 m_6 | j_5 m_5 \rangle \langle j_2 j_4 m_2 m_4 | j_6 m_6 \rangle \langle j_3 j_4 m_3 m_4 | j_5 m_5 \rangle = \sqrt{(2j_3 + 1)(2j_6 + 1)} W(j_1 j_2 j_5 j_4 j_3 j_6),$$  \(182\)

in which \(m_1, m_2\) and \(m_3\) take on fixed values. Thus in (182), the sum over \(m\) denotes a single sum, over \(m_6\) for example. The Racah coefficient \(W\) involves four triad of angular momenta

\[(j_1, j_6, j_5), \quad (j_2, j_4, j_6), \quad (j_3, j_4, j_5), \quad (j_1, j_2, j_3).\]  \(183\)

We intend to pursue the analogy of results like (21) in a somewhat loose or qualitative way. Thus we consider the square root factors in (182) as being absorbed into the Racah coefficient \(W\), and ignore signs.

We begin by comparing (182) and (21). We have already mentioned the view of \(c_{ijk}\) as a set of Clebsches for \(ad \otimes ad \rightarrow ad\) in a basis of Cartesian rather than angular momentum type. Now we regard the the numerical factor \(\frac{1}{2}\) on the right side of (21) as a Racah coefficient with all six arguments equal to \(ad\).

Likewise, (85) suggests that the Racah coefficients with five arguments \(ad\) and its fifth argument \(R\), in the place corresponding to \(j_3\) in (182), takes the value \(\ell_2\), whilst (86) suggests that the Racah coefficients with five arguments \(ad\) and its sixth argument \(R\) takes the value \(\ell_2 D_2 / D\).

We wish here to make the point that, if one were to define Racah coefficients systematically for \(g \in E\), then it would be expected that they would display full uniformity across \(E\). We have indicated a few simple examples in justification of this. Also

$$W(ad ad ad, R_2 R_3) = \frac{(11 + D + \Delta)}{2D\Delta}.\quad (184)$$

References

[1] H. Freudenthal, *Lie groups in the foundations of geometry*, Adv. Math. **1** 145-196 (1964).

[2] J. R. Faulkner and J. C. Ferrar, *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. Lond. Math. Soc. **9** 1-35 (1977).

[3] L. Frappat, A. Sciarrino and P. Sorba, *A dictionary on Lie algebras and superalgebras*, Academic Press, London, (2001).

[4] J. F. Cornwell, *Group Theory in Physics*, Vol. 2, Academic Press, London, 1984.

[5] K. Meyberg, *Spurformeln in einfachen Lie-algebren*, Abh. Math. Sem. Univ. Hamburg **54** 177-189 (1984).
[6] P. Deligne, *La série exceptionnelle de groupes de Lie*, C. R. Acad. Sci. Paris 322, Série I, 321-326 (1996).

[7] A. M. Cohen and R. de Man, *Computational evidence for Deligne’s conjecture regarding the exceptional groups*, C. R. Acad. Sci. Paris 322, Série I, 427-432 (1996).

[8] M. A. van Leeuwen and A. M. Cohen, *LiE, a package for Lie group computation*, CAN, Amsterdam, (1992).

[9] A. J. Macfarlane, *Lie algebra and invariant tensor technology for \( g_2 \)*, Intern. J. Mod. Phys. 16 3067-3097 (2001).

[10] S. Okubo, *Quartic trace identities for exceptional Lie algebras*, J. Math. Phys. 20 586-593 (1979).

[11] A. J. Macfarlane and H. Pfeiffer, *On characteristic equations, trace identities and Casimir operators of simple Lie algebras*, J. Math. Phys. 41 No. 5 3192-3225 (2000), 42 No. 2 977 (2001).

[12] K. Meyberg, *Okubo’s quartic trace formula for exceptional Lie algebras*, J. Alg. 84 279-284 (1984).

[13] C. Chryssomalakos, J.A. de Azcárraga, A.J. Macfarlane and J.C. Pérez Bueno, *Higher order BRST and anti-BRST operators and Lie algebra cohomology for compact lie groups*, J. Math. Phys. 40 6009-6032 (1999).

[14] L. C. Beidenharn and H. van Dam, *Quantum Theory of Angular Momentum*, Academic Press, London, (1984).

[15] A. J. Macfarlane and H. Pfeiffer, *Representations of the exceptional and other Lie algebras with integral eigenvalues of the Casimir operator*, DAMTP-2002-90, math-ph/0208014.

[16] T. Molien, Sitzungsber. König. Preuss. Akad. Wiss. 1152-1158 (1897).

[17] D. H. Sattinger and O. L. Weaver, *Lie groups and Lie algebras and applications to physics, geometry and mechanics*, Springer-Verlag, Berlin, (1987).

[18] J. M. Landsberg and L. Manivel, *Triality, exceptional Lie algebras and the Deligne dimension formulas*, math.AG/0107032. *Series of Lie groups*, math.AG/0203247.

[19] B. W. Westbury, *R-matrices and the magic square*. 