ON SELF-ADJOINTNESS OF A SCHRODINGER OPERATOR ON DIFFERENTIAL FORMS

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Abstract. Let $M$ be a complete Riemannian manifold and let $\Omega^\bullet(M)$ denote the space of differential forms on $M$. Let $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ be the exterior differential operator and let $\Delta = dd^* + d^*d$ be the Laplacian. We establish a sufficient condition for the Schrödinger operator $H = \Delta + V(x)$ (where the potential $V(x) : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is a zero order differential operator) to be self-adjoint. Our result generalizes a theorem by I. Oleinik about self-adjointness of a Schrödinger operator which acts on the space of scalar valued functions.

1. Introduction. Suppose $M$ is a complete Riemannian non-compact manifold. We will assume that $M$ is oriented and connected. Let $T^*M$ denote the cotangent bundle to $M$ and let $\bigwedge^\bullet(T^*M) = \bigoplus_i \bigwedge^i(T^*M)$ denote the exterior algebra of $T^*M$. We denote by $L^2\Omega^\bullet(M)$ the space of square integrable complex valued differential forms on $M$, i.e. the space of sections of $\bigwedge^\bullet(T^*M) \otimes \mathbb{C}$ which are square integrable with respect to the scalar product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta, \quad \alpha, \beta \in L^2\Omega^\bullet(M).$$

Here $\star$ denotes the Hodge operator associated to the Riemannian metric on $M$. Note that $L^2\Omega^0(M)$ is just the space of square integrable complex valued functions on $M$.

Let $d : L^2\Omega^\bullet(M) \to L^2\Omega^{\bullet+1}(M)$ denote the exterior differential and let $d^*$ be the operator formally adjoint to $d$ with respect to the scalar product (1).

Let $\Delta = dd^* + d^*d$ be the Laplacian and consider the Schrödinger operator

$$H = \Delta + V(x) : L^2\Omega^\bullet(M) \to L^2\Omega^\bullet(M)$$

where the potential $V(x)$ is a measurable section of the bundle $\text{End} \left( \bigwedge^\bullet(T^*M) \right)$ of endomorphisms of $\bigwedge^\bullet(T^*M)$ which belongs to the class $L^\infty_{\text{loc}}$ (i.e. such that for any compact set $K \subset M$ there exists a constant $C_K > 0$ such that $|V(x)| \leq C_K$ for almost all $x \in K$).

We denote by $H_0$ the restriction of $H$ on the space $\Omega^\bullet_c(M)$ of smooth differential forms with compact support. The purpose of this paper is to introduce a sufficient condition on the potential $V(x)$ for operator $H_0$ to be self-adjoint.

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2. Statement of results. For \(x, y \in M\) let \(\text{dist}(x, y)\) denote the Riemannian distance between \(x\) and \(y\). Fix a point \(p \in M\) and set \(r(x) = \text{dist}(x, p)\).

Fix \(x \in M\). The Riemannian metric on \(M\) defines a scalar product \(\langle \cdot, \cdot \rangle_x\) on the fiber \(\bigwedge^\bullet (T^*_x M) \otimes \mathbb{C}\) of the vector bundle \(\bigwedge^\bullet (T^* M) \otimes \mathbb{C}\). As usual, we write \(V(x) \geq C\) if

\[
\langle V(x) \xi, \xi \rangle_x \geq C \langle \xi, \xi \rangle_x
\]

for any \(\xi \in \bigwedge^\bullet (T^*_x M) \otimes \mathbb{C}\). Note that it follows from (3) that \(V(x)\) is a self-adjoint endomorphism of \(\bigwedge^\bullet (T^*_x M)\).

**Theorem A.** Assume that for almost all \(x \in M\) the potential \(V(x)\) of the operator (2) satisfies the estimate

\[
V(x) \geq -Q(x),
\]

where \(1 \leq Q(x) \leq \infty\) and \(Q^{-1/2}(x)\) is a Lipschitz function on \(M\) such that

\[
|Q^{-1/2}(x) - Q^{-1/2}(y)| \leq K \text{dist}(x, y) \quad \text{for any } x, y \in M.
\]

If for any piecewise smooth curve \(\gamma : [0, \infty) \to M\) such that \(\lim_{t \to \infty} r(\gamma(t)) = \infty\) the integral

\[
\int_{\gamma} Q^{-1/2}(x) \, d\gamma = \infty
\]

then the operator \(H_0\) is essentially self-adjoint.

For the case of a Schrödinger operator acting on scalar valued functions this theorem was established by I. Oleinik [O2]. Note that \(Q(x)\) may be equal to infinity on a set of positive measure.

As a simple consequence of Theorem A we obtain the following

**Theorem B.** Suppose that for almost all \(x \in M\) the potential \(V(x)\) satisfies the estimate \(V(x) \geq -q(r(x))\), where \(1 \leq q \leq \infty\) and \(q^{-1/2}(t)\) is a Lipschitz function on \(\mathbb{R}\) such that \(\int_0^\infty q^{-1/2}(t) \, dt = \infty\). Then the operator \(H_0\) is essentially self-adjoint.

In particular, if \(M = \mathbb{R}^n\) and \(V(x) \geq -C|x|^2\) then the operator \(H_0\) is essentially self-adjoint.

**Remark.** Theorem A remains true if we replace \(L^2\Omega^\bullet(M)\) by the space of square integrable forms on \(M\) with values in a flat Hermitian vector bundle \(\mathcal{F}\) over \(M\), provided that the Hermitian structure on \(\mathcal{F}\) is flat. In this case the differential \(d\) should be replaced by the covariant differential associated to the flat structure on \(\mathcal{F}\). The proof is a verbatim repetition of the proof for the scalar case, cf. below. However the notation in the vector valued case is more complicated.

3. Historical remarks. An analogue of Theorem B for the case \(M = \mathbb{R}^1\) was established by Sears [Se]. B. Levitan [Le] proved the Sears theorem for the Schrödinger operator acting on scalar valued functions on \(M = \mathbb{R}^n\). F. Rofe-Beketov [RB] extended these results to the case where the potential \(V(x)\) can not be estimate by a function depending only on \(\text{dist}(x, p)\). Many results and references
about the essential self-adjointness of Schrödinger operators on $\mathbb{R}^n$ may be found in [RS].

I. Oleinik [O1,O2] established Theorem A for the Schrödinger operator acting on scalar valued functions on a complete Riemannian manifold.

Essential self-adjointness of a pure Laplacian (without lower order terms) on differential forms on a complete Riemannian manifold was first stated and proved by M. P. Gaffney [Ga1,Ga2]. A number of related results may be found in [Sh].

In [BFS], Theorem B is established for the case where $M$ is a manifold with cylindrical ends and the potential $V(x) \geq 0$. The result is used there to study Witten deformation of the Laplacian on a non-compact manifold.

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5. The domain of $D(H_0^*)$. Let $H_0^*$ denote the operator adjoint to $H_0$. The domain $D(H_0^*)$ of $H_0^*$ consists of forms $\alpha \in L^2\Omega^*(M)$ such that $H\alpha$ understood in the sense of distributions also belongs to $L^2\Omega^*(M)$.

The operator $H_0$ is symmetric. Hence, to show that its closure is self-adjoint it is enough to show that the adjoint operator $H_0^*$ is symmetric. In other words we have to prove that

$$
\int_M (H\alpha \wedge \ast \beta - \alpha \wedge \ast H\beta) = 0
$$

for any $\alpha, \beta \in D(H_0^*)$.

To prove (7) we need some information about the behavior of differential forms from $D(H_0^*)$. The main result of this section is the following lemma, which provides us with this information.

**Lemma 1.** If $\alpha \in D(H_0^*)$ then the forms $Q^{-1/2}d\alpha$, $Q^{-1/2}d^*\alpha$ are square integrable.

**Remark.** 1. By the standard theory of elliptic operators any $\alpha \in D(H_0^*)$ belongs to the Sobolev space $H^2_{loc}$. Hence, $d\alpha$, $d^*\alpha$ are locally square integrable. Thus the lemma provides us with an information about the behavior of the forms from $D(H_0^*)$ at infinity.

2. For the Schrödinger operator on scalar valued functions on $\mathbb{R}^n$ an analogous lemma was established in [RB]. The proof was adopted in [O1,O2] to the case of a Riemannian manifold. In our proof we follow rather closely the lines of [O2]. However, the fact that we deal with differential forms rather than with scalar valued functions demands a more careful analysis.

**Proof.** Recall that we fixed a point $p \in M$ and that for any $x \in M$ we denoted by $r(x)$ the Riemannian distance between $x$ and $p$.

It is shown in [O2, Proof of Lemma 1] that for any $R > 0, \varepsilon > 0$ there exist smooth functions $r_{R,\varepsilon}(x), F_{R,\varepsilon}(x)$ on $M$ which approximate the Lipschitz functions $r(x), Q^{-1/2}(x)$ in the sense that

$$
|r_{R,\varepsilon}(x) - r(x)| < \varepsilon, \quad Q^{-1/2}(x) - \varepsilon < F_{R,\varepsilon}(x) < (1 + \varepsilon)Q^{-1/2}(x),
$$

$$
\lim_{\varepsilon \to 0} |dr_{R,\varepsilon}| < 1, \quad \lim_{\varepsilon \to 0} |dF_{R,\varepsilon}| < K,
$$

where $K$ is a constant dependent only on $R$. Then

$$
\int_M |Q^{-1/2}d\alpha|^2 \leq K \int_M |\alpha|^2,
$$

for any $\alpha \in D(H_0^*)$, where $K$ is a constant depending only on $R$. The second inequality follows from the same arguments used in the proof of Lemma 1 in [O2].
for any \( x \in r_{R, \varepsilon}^{-1}([0, R + 1]) \). Here \( K \) is the same constant as in (5).

Let \( \Psi : [0, +\infty) \to [0, 1] \) be a smooth function which is equal to one when \( t \leq 1/2 \) and which is equal to zero when \( t \geq 1 \). Set

\[
\psi_{R, \varepsilon}(x) = \begin{cases} 
\Psi \left( \frac{r_{R, \varepsilon}(x)}{R} \right) F_{R, \varepsilon}(x) & \text{if } r_{R, \varepsilon}(x) \leq R; \\
0 & \text{outside of the set } r_{R, \varepsilon}(x) \leq R.
\end{cases}
\]

For any \( R > 0 \) the functions \( \psi_{R, \varepsilon}, \varepsilon < 1 \) vanish outside of the compact set \( r_{R}^{-1}([0, R + 1]) \). Hence, it follows from (8) and (4) that there exist a constant \( K_{1} > 0 \) not depending on \( R \) and a number \( \varepsilon_{R} > 0 \) (which does depend on \( R \)) such that

\[
\|d\psi_{R, \varepsilon}(x)\| \leq K_{1}, \quad \psi_{R, \varepsilon}^{2}(x) \leq 2, \quad \left| \int_{M} \psi_{R, \varepsilon}^{2}(x) \wedge \ast V \alpha \right| \leq 2 \|\alpha\|^{2},
\]

for any \( x \in M, R > 1, 0 < \varepsilon < \varepsilon_{R}, \alpha \in L^{2}\Omega^{*}(M) \). Here \( \|\alpha\| = \langle \alpha, \alpha \rangle^{1/2} \) denotes the \( L^{2} \)-norm of the form \( \alpha \).

Functions \( \psi_{R, \varepsilon} \) have compact support. Hence, in view of the remark 1 after the statement of the lemma, the forms \( \psi_{R, \varepsilon}d\alpha \) and \( \psi_{R, \varepsilon}d^{*}\alpha \) are square integrable. Assume that \( \alpha \in D(H_{0}^{*}) \) is a real valued form and set

\[
J_{R, \varepsilon}^{2} = \left\| \psi_{R, \varepsilon}d\alpha \right\|^{2} + \left\| \psi_{R, \varepsilon}d^{*}\alpha \right\|^{2} = \int_{M} \psi_{R, \varepsilon}(x)^{2} (d\alpha \wedge \ast d\alpha + d^{*}\alpha \wedge \ast d^{*}\alpha).
\]

It follows from (8), (9) that to prove the lemma it is enough to show that

\[
\lim_{R \to \infty} \lim_{\varepsilon \to 0} J_{R, \varepsilon} < \infty.
\]

Let us first rewrite the integrand in (11) in a more convenient form. In the calculations below we use the equality (cf. [Wa, §6.1]) \( d^{*}\alpha = (-1)^{|\alpha|} \ast^{-1} d \ast \alpha \) where \( |\alpha| \) denotes the degree of the differential form \( \alpha \).

\[
\psi_{R, \varepsilon}^{2} \, d\alpha \wedge \ast d\alpha = d(\psi_{R, \varepsilon}^{2} \, d\alpha \wedge \ast d\alpha) - 2\psi_{R, \varepsilon}d\psi_{R, \varepsilon} \wedge \alpha \wedge \ast d\alpha + \psi_{R, \varepsilon}^{2} \, d\alpha \wedge \ast d^{*}d\alpha,
\]

\[
\psi_{R, \varepsilon}^{2} \, d^{*}\alpha \wedge \ast d^{*}\alpha = (-1)^{|\alpha|} \psi_{R, \varepsilon}^{2} \, d^{*}\alpha \wedge \ast d\alpha = -d(\psi_{R, \varepsilon}^{2} \, d^{*}\alpha \wedge \ast \alpha) + 2\psi_{R, \varepsilon}d\psi_{R, \varepsilon} \wedge \alpha \wedge \ast \alpha + \psi_{R, \varepsilon}^{2} \, dd^{*}\alpha \wedge \ast \alpha.
\]

It follows now from (13), (10) and from the Stokes theorem that, if \( R > 1, \varepsilon < \varepsilon_{R}, \) then

\[
\left\| \psi_{R, \varepsilon}d\alpha \right\|^{2} = \int_{M} \psi_{R, \varepsilon}^{2} \, d\alpha \wedge \ast d\alpha = -2\langle d\psi_{R, \varepsilon} \wedge \alpha, \psi_{R, \varepsilon}d\alpha \rangle + \langle \alpha, \psi_{R, \varepsilon}^{2} \, d^{*}d\alpha \rangle \\
\leq 2K_{1}\|\alpha\| \left\| \psi_{R, \varepsilon}d\alpha \right\| + \langle \alpha, \psi_{R, \varepsilon}^{2} \, d^{*}d\alpha \rangle,
\]

\[
\left\| \psi_{R, \varepsilon}d^{*}\alpha \right\|^{2} = \int_{M} \psi_{R, \varepsilon}^{2} \, d^{*}\alpha \wedge \ast d^{*}\alpha = 2\langle d\psi_{R, \varepsilon} \wedge \psi_{R, \varepsilon}d^{*}\alpha, \alpha \rangle + \langle \psi_{R, \varepsilon}^{2} \, dd^{*}\alpha, \alpha \rangle \\
\leq 2K_{1}\left\| \psi_{R, \varepsilon}d^{*}d\alpha \right\| + \langle \alpha, \psi_{R, \varepsilon}^{2} \, dd^{*}\alpha \rangle.
\]
Summing these two equations we obtain

\[
J_{R,\varepsilon}^2 \leq 2K_1 \| \alpha \| (\| \psi_{R,\varepsilon} d\alpha \| + \| \psi_{R,\varepsilon}^* \alpha \|) + \langle \alpha, \psi_{R,\varepsilon}^2 \Delta \alpha \rangle \\
\leq 4K_1 \| \alpha \| J_{R,\varepsilon} + \int_M \psi_{R,\varepsilon}^2 (\alpha \land \ast H\alpha - \alpha \land \ast V\alpha) \\
\leq 4K_1 \| \alpha \| J_{R,\varepsilon} + 2 \| \alpha \| \| H\alpha \| + 2 \| \alpha \|^2.
\]

Here the last inequality follows from (10).

It follows from (14) that the set \( \{ J_{R,\varepsilon} : R > 1, \varepsilon < \varepsilon_R \} \) is bounded from above. Hence (12) holds. The proof of the lemma is completed. □

6. Proof of Theorem A. We apply a modification of the method used in [RB] suggested by I. Oleinik [O2].

The quantity

\[
\tilde{\rho}(x, y) = \inf_{\gamma} \int_\gamma Q^{-1/2}(x) \, d\gamma,
\]

where the infimum is taken over all piecewise smooth curves connecting the points \( x, y \in M \), is called generalized distance between \( x \) and \( y \). It is a symmetric function in \( x, y \) which satisfies the triangular inequality. The first metric axiom is not valid in general. Note, however, that (6) implies, that the sets \( P^{-1}([0, R]) \) are compact for any \( R > 0 \).

Recall that in Section 2 we have fixed a point \( p \in M \). Set \( P(x) = \tilde{\rho}(x, p) \). Then (cf. [O2, Lemma 2])

\[
|P(x) - P(y)| \leq Q^{-1/2}(x) \text{dist}(x, y) + \frac{K}{2}(\text{dist}(x, y))^2
\]

for any \( x, y \in M \). It follows (cf. [O2]) that for any \( R > 0, \varepsilon > 0 \) there exists a smooth function \( \tilde{P}_{R,\varepsilon}(x) \) which approximates \( P(x) \) in the sense that

\[
|\tilde{P}_{R,\varepsilon}(x) - P(x)| \leq \varepsilon, \quad \lim_{\varepsilon \to 0} |d\tilde{P}_{R,\varepsilon}(x)| \leq Q^{-1/2}(x),
\]

for any \( x \in P^{-1}([0, R + 1]) \).

Assume that \( \varepsilon < 1 \) so that \( \tilde{P}_{R,\varepsilon}^{-1}([0, R]) \subset P^{-1}([0, R + 1]) \). Let us define a piecewise smooth function \( P_{R,\varepsilon}(x) \) on \( M \) by the formula

\[
P_{R,\varepsilon}(x) = \begin{cases} 
\tilde{P}_{R,\varepsilon}(x) & \text{if } \tilde{P}_{R,\varepsilon}(x) \leq R; \\
R & \text{outside the set } \tilde{P}_{R,\varepsilon}(x) \leq R.
\end{cases}
\]

By (17), the inequality

\[
\lim_{\varepsilon \to 0} |dP_{R,\varepsilon}(x)| \leq Q^{-1/2}(x)
\]

holds almost everywhere on \( M \).
Recall from Section 5, that the statement of Theorem A is equivalent to equality (7). Fix $\alpha, \beta \in D(H_0^*)$ and consider the following approximation of the integral (7)

$$ I_{R,\varepsilon} = \int_M \left(1 - \frac{P_{R,\varepsilon}}{R}\right) (H\alpha \wedge \ast \beta - \alpha \wedge \ast H\beta) $$

$$ = \int_M \left(1 - \frac{P_{R,\varepsilon}}{R}\right) (\Delta\alpha \wedge \ast \beta - \alpha \wedge \ast \Delta\beta). $$

By the Fatou theorem ([RS, Theorem I.17]), it is enough to show that

$$ \lim_{R \to \infty} \lim_{\varepsilon \to 0} I_{R,\varepsilon} = 0. $$

We will need the following “integration by parts” lemma

**Lemma 2.** Let $\phi : M \to \mathbb{R}$ be a smooth function with compact support. Then

$$ \int_M \phi \Delta\alpha \wedge \ast \beta $$

$$ = \int_M \phi (d\alpha \wedge \ast d\beta + d^*\alpha \wedge \ast d^*\beta) + \int_M d\phi \wedge (\beta \wedge \ast d\alpha - d^*\alpha \wedge \ast \beta) $$

for any $\alpha, \beta \in D(H_0^*).$

Note that, by remark 1 after the statement of Lemma 1, all the integrals in (22) have sense.

**Proof.** Recall that $d^*u = (-1)^{|u| - 1} d^* u$ where $|u|$ denotes the degree of the differential form $u.$ Hence, if $|u| = |v| - 1,$ then

$$ \phi du \wedge \ast v = \phi u \wedge \ast d^* v - d\phi \wedge u \wedge \ast w + d(\phi u \wedge \ast v) $$

Substituting into (23) first $u = d^*\alpha,$ $v = \beta$ and then $u = \beta,$ $v = d\alpha$ we obtain

$$ \phi dd^*\alpha \wedge \ast \beta = -d\phi \wedge d^*\alpha \wedge \ast \beta + \phi d^*\alpha \wedge \ast d^*\beta + d(\phi d^*\alpha \wedge \ast \beta), $$

$$ \phi d^*\alpha \wedge \ast \beta = \phi \beta \wedge \ast d^*\alpha = d\phi \wedge \beta \wedge \ast d\alpha + \phi d\alpha \wedge \ast d\beta - d(\phi \beta \wedge \ast d\alpha). $$

In the last equality we used that $u \wedge \ast v = v \wedge \ast u$ for any differential forms $u,v$ of the same degree. Summing the above equations, integrating over $M$ and using the Stokes theorem we get (22). $\square$

Using definition (20) of $I_{R,\varepsilon}$ and Lemma 2 we obtain

$$ I_{R,\varepsilon} = \frac{1}{R} \int_M dP_{R,\varepsilon} \wedge (\beta \wedge \ast d\alpha - d^*\alpha \wedge \ast \beta - \alpha \wedge \ast d\beta + d^*\beta \wedge \ast d\alpha). $$

Let $d\mu(x)$ denote the Riemannian density on $M.$ For any $\xi \in \bigwedge^k(T^*M) \otimes \mathbb{C}$ we denote by $|\xi|$ its norm with respect to the scalar product on $\bigwedge^k(T^*M) \otimes \mathbb{C}$ induced by the Riemannian structure on $M.$ Then

$$ |(\alpha, \beta)| \leq \int_M |\alpha \wedge \ast \beta| d\mu(x) \leq \int_M |\alpha| |\beta| d\mu(x) \leq ||\alpha|| ||\beta|| $$
for any $\alpha, \beta \in L^2 \Omega^*(\mathcal{M})$.

Let us estimate the behavior of the right hand side of (24) as $\varepsilon \to 0$. For the first term we obtain

\[
\lim_{\varepsilon \to 0} \left| \frac{1}{R} \int_M dP_{R,\varepsilon} \wedge \beta \wedge *d\alpha \right| \leq \frac{1}{R} \lim_{\varepsilon \to 0} \int_M |dP_{R,\varepsilon}| |d^*\alpha| |\beta| d\mu(x)
\leq \frac{1}{R} \int_M |Q^{-1/2}d^*\alpha| |\beta| d\mu(x) \leq \frac{\|Q^{-1/2}d^*\alpha\| \|\beta\|}{R}.
\]

In the second inequality in (26) we used the estimate (19). The last inequality in (26) follows from Lemma 1.

Analogously, one can estimate the other terms in the right hand side of (24). That proves (21) and Theorem A. □

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