Sequential Effect Systems with Control Operators

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Abstract—Sequential effect systems are a class of effect system that exploits information about program order, rather than discarding it as traditional commutative effect systems do. This extra expressive power allows effect systems to reason about behavior over time, capturing properties such as atomicity, unstructured lock ownership, or even general safety properties. We now understand the essential denotational (categorical) models fairly well, application of these ideas to real software is hampered by the sheer variety of source level control flow constructs in real languages. Denotational approaches are general enough to accommodate any particular control flow construct, but provide no guidance on the details, let alone applications.

We address this new problem by appeal to a classic idea: macro-expression of commonly-used programming constructs in terms of control operators. We give an effect system for a subset of Racket’s tagged delimited constructs, as a lifting of an effect system for a language without direct control operators. This gives the first account of sequential effects in the presence of general control operators. Using this system, we re-derive the sequential effect system rules for control flow constructs previously shown sound directly, and derive sequential effect rules for new constructs not previously studied in the context of source-level sequential effect systems. This offers a way to directly extend source-level support for sequential effect systems to real programming languages.

I. Introduction

Effect systems extend type systems to reason about not only the shape of data, and available operations — roughly, what a computation produces given certain inputs — but to also reason about how the computation produces its result. Examples include ensuring data race freedom by reasoning about what locks a computation assumes held during its execution [1, 8, 9, 19], restricting sensitive actions (like UI updates) to dedicated threads [28], ensuring deadlock freedom [20, 27], checking safe arena-based memory management [41, 56], or most commonly checking that a computation handles (or at least indicates) all errors it may encounter — Java’s checked exceptions [29] are the quintessential example of a static effect system.

Most effect systems discard information about program order — the same join operation on a join semilattice of effects is used to overapproximate different branches of a conditional or different subexpressions executed in sequence. Despite this simplicity, these traditional commutative effect systems (where the combination of effects is always a commutative operation) are powerful. Still, many program properties of interest are sensitive to evaluation order. For example, commutative effect systems handle scoped synchronized blocks as in Java with ease: the effect of (the set of locks required by) the synchronized’s body is permitted to contain the synchronized lock, in addition to the locks required by the overall construct. But to support explicit lock acquisition and release operations that are not block-structured, an effect system must track whether a given expression acquires and/or releases locks, and must distinguish their ordering: releasing and then acquiring a given lock is not the same as acquiring before releasing. To this end, sequential effect systems (so named by Tate [55]) reason about effects with knowledge of the program’s evaluation order.

Sequential effect systems are much more powerful than commutative effect systems, with examples extending through generic reasoning about program traces [52, 53] and even propagation of liveness properties from oracles [37] — well beyond what most type systems support. Yet for all the power of this approach, for years each of the many examples of sequential effect systems in the literature — including effect systems for deadlock freedom [27, 54], atomicity [21, 22], trace-based security properties [52, 53], safety of concurrent communication [2, 44], and general linear temporal properties with a liveness oracle [57] — individually rederived much structure common to all sequential effect systems. We would like to provide a general characterization of control for sequential effect systems.

Recent years have seen efforts to unify understanding of these sequential effect systems, first denotationally [35, 43, 55], and recently as an extension to the join semilattice model. Gordon [25] proposed effect quantales, which capture most of the structure common to core languages used to study sequential effect systems — including imperative looping constructs — but still omits critical features of real languages and concrete effect systems that clearly interact with evaluation order. Chief among these are various control flow constructs, as well as control operators like exceptions and generators.

Control operators effectively reorder, drop, or duplicate portions of a program’s execution at runtime, changing evaluation order. In order to reason precisely about flexible rearrangement of evaluation order, a sequential effect system must reason about control operators. The classic example is again Java’s try-catch: if the body of a try block both acquires and releases a lock this is good, but if an exception is thrown mid-block the release may need to be handled in the corresponding catch. Clearly, applying sequential effect systems to real software requires support for exceptions in a sequential...
effect system. Working out just those rules is tempting, but exceptions interact with loops. The effect before a throw inside a loop — which a catch block may need to “complete” (e.g., by releasing a lock) — depends on whether the throw occurs on the first or nth iteration. Many languages now go beyond try-catch, for example with the generators (a form of coroutine) now found in C# [1], Python, and JavaScript. These interact with exceptions and loops. Incrementally enriching the theory of sequential effect systems for each new control operator seems at best inefficient, and at worst ineffective.

An alternative to studying the panoply of control flow constructs in real languages individually is to study more general constructs, such as the powerful control operators present in Racket, which are useful in their own right, and can also express all of the more specialized constructs of interest. Then general principles can be derived for the general constructs, which can then be applied to or specialized for the constructs of interest, including considering interactions between various control flow constructs and control operators. This both solves the open question of how to treat general control operators with sequential effect systems, and leads to a basis for more compositional treatment of loops, exceptions, generators, and future additions to real languages. This is the avenue we pursue in this paper.

Delimited continuations [16, 18] are known to macro-express most useful control structures, including loops, exceptions [23], and coroutines [31, 32]. They are also directly available to user programs in Racket [23], Scala [48], and Haskell [36], among others. So directly treating delimited continuations in a sequential effect system is useful directly, as well as via embedding.

Delimited continuations solve the generality challenge, but introduce new challenges of their own since sequential effect systems can track evaluation order [26, 37, 52, 53]. The effect of an expression that aborts out of a prompt depends on what was executed before the abort, but not after. At the same time, the body of a continuation capture (callcc) must be typed knowing the effect of the enclosing context — the code executing after, but not before (up to the enclosing prompt).

We lay the groundwork for handling modern control operators in a sequential effect system. Specifically:

- We give the first generic characterization of sequential effects for continuations, by giving a generic characterization of how to lift a control-unaware sequential effect system into one that can support tagged delimited continuations. As a consequence, we can transfer prior sequential effect systems designed without control operators to a setting with control operators.
- We give sequential effect system type rules for while loops, try-catch, and generators by deriving them from their macro-expression [17] in terms of more primitive operators. The loop characterization was previously known (and technically a control flow construct, not a general control operator), but was given as primitive. The other characterizations are new to our work.
- We demonstrate how the notion of an iteration operator derived from a closure operator on the underlying effect lattice is not specific to the loops it was used for in prior work, but rather provides a general tool for solving recursive partial order constraints in sequential effect systems.
- We prove syntactic type soundness for our sequential control effect transformation when applied to any effect quantale.

Section III gives general background useful for the rest of the paper, in particular explaining some prior work on sequential effect systems and control operators. Sections III and IV progressively explain the concepts behind our type system, starting with just exceptions, then full control operators subject to some nesting restrictions, and finally unrestricted control operators.

II. Background

We briefly recall the details of standard type-and-effect systems, sequential type-and-effect systems, and tagged delimited continuations. We emphasize the view of effect systems in terms of a control flow algebra [43] — an algebraic structure with operations corresponding to the different ways an effect system might combine the effects from subexpressions in a program.

A. Effect Systems

Traditional type-and-effect systems extend the typing judgment $\Gamma \vdash e : \tau$ for an additional component. The extended judgment form $\Gamma \vdash e : \tau | \chi$ is read “under local variable assumptions $\Gamma$, the expression $e$ evaluates to a value of type $\tau$ (or diverges), with effect $\chi$ during evaluation.” The last clause of that reading is vague, but carries specific meanings for specific effect systems. For checked exceptions, it could be replaced by “possibly throwing exceptions $\chi$ during evaluation” where $\chi$ would be a set of checked exceptions. For a data race freedom type system reasoning about lock ownership, it could be replaced by “and is data race free if executed while locks $\chi$ are held.”

In traditional effect systems the set of effects tracked forms a join semilattice: a partial order with a (binary) least-upper bound operation (join, written $\sqcup$), and a least element $\bot$. As is standard for join semilattices, $\sqcup$ is commutative and associative. Any time effects of subexpressions must be combined, they are mixed with this join. Functions introduce an additional complication that requires modifying function types: the effect of a function’s body does not occur when a function (e.g., a lambda expression) is evaluated, but only when it is applied. So the effect of a function expression itself (like other values) may simply be bottom. A function type then carries the latent effect of the body — the effect that does not “happen” until the function is actually invoked. For example, consider checked exceptions in Java. The allocation of a class instance (such as what a lambda allocation there

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1 C# calls them iterators. We mostly avoid this terminology, which is based on the generators’ consumption through a more traditional iterator interface.
translates to) does not actually run any method(s) of the class—invocation does. So allocating a class instance throws no exceptions (assuming the constructor throws no exceptions). But invoking a method with a throws clause may—the throws clause is the latent effect of the method for Java’s checked exceptions. To make this more explicit, let us consider the standard type rules for lambda expressions and function application in a generic effect system:

\[
\begin{array}{c}
\text{T-LAMBDA} \quad \frac{\Gamma, x : \tau \vdash e : \sigma \mid \chi}{\Gamma \vdash \lambda x. e : \tau \xrightarrow{\chi} \sigma} \\
\Gamma \vdash e_1 : \tau \xrightarrow{\chi} \sigma \mid \chi_1 \quad \frac{\Gamma \vdash e_2 : \tau \mid \chi_2}{\Gamma \vdash e_1 e_2 : \sigma \mid \chi_1 \sqcup \chi_2 \sqcup \chi}
\end{array}
\]

Key to note here are that the lambda expression’s type carries the latent effect of the function body, but itself has only the bottom effect; and that when a function is applied, the overall effect of the expression is the combination (via join) of all subexpressions’ effects and the latent effect of the function. We call these effect systems commutative not only to distinguish them from the broader class of systems we study in this paper, but also because all combinations of effects in such systems are commutative, and disregard evaluation order—the only means to combine an effect in a commutative effect system is with the (commutative) join operation. Other rules with multiple subexpressions, such as while loops, conditions, and more, similar join effects without regard to program order or repetition.

By contrast, many effect systems use a richer structure to reason about cases where evaluation order is important. This includes effect systems for atomicity [21, 22], deadlock freedom [3, 27, 54], race freedom with explicit lock acquisition and release [23, 54], security checks [53], and (with the aid of an oracle for liveness properties) general linear-time properties [37]. Tate labels this class of systems sequential effect systems [55], as their distinguishing feature is the use of an additional sequencing operator to join effects where one is known to be evaluated before another. Consider the sequential rules for function application and while loops:

\[
\begin{array}{c}
\text{T-APP} \quad \frac{\Gamma \vdash e_1 : \tau \xrightarrow{\chi} \sigma \mid \chi_1}{\Gamma \vdash \lambda x. e_1 : \tau \xrightarrow{\chi} \sigma} \\
\Gamma \vdash e_1 : \tau \xrightarrow{\chi} \sigma \mid \chi_1 \quad \frac{\Gamma \vdash e_2 : \tau \mid \chi_2}{\Gamma \vdash e_1 e_2 : \sigma \mid \chi_1 \sqcup \chi_2 \sqcup \chi}
\end{array}
\]

\[
\Gamma \vdash e : \text{boolean} \mid \chi_e \quad \Gamma \vdash e_h : \tau \mid \chi_h
\]

\[
\Gamma \vdash \text{while } e_c \ e_h : \text{unit} \mid \chi_e \triangleright (\chi_h \triangleright \chi_e)^* 
\]

The sequencing operator \( \triangleright \) is associative but not (necessarily) commutative. Thus the effect in the new T-APP reflects left-to-right evaluation order: first the function position is reduced to a value, then the argument, and then the function body is executed. The while loop uses an iteration operator \((-)^*\) to represent 0 or more repetitions of its argument; we will return to its details later. The effect of T-WHILE reflects the fact that while the condition will always be executed, followed by 0 or more repetitions of the loop body and checking the loop condition again.

Gordon [25] proposed a source-level algebraic characterization of sequential effect systems—the effect quantale—and showed that rather than capturing strictly necessary aspects of sequential effect systems (as some semantic work has done [35, 43, 55]), it captured sufficient structure to model and extend some prior effect systems from the literature [22, 54]. This was demonstrated in part by giving a core language with effect system, parameterized by a choice of effect quantale, and giving a type-and-effect-preserving translation between Flanagan and Qadeer’s early atomicity effect system [22] and an instantiation of that core language. The structure extends a join semilattice with a sequencing operator, a designated error element to model possibly-undefined combinations, and laws specifying how the operators interact.

**Definition 1 (Effect Quantale).** An effect quantale is a join-semilattice-ordered monoid with nilpotent top. That is, it is a structure \((E, \sqcup, \sqcap, \triangleright, I)\) where:

- \((E, \sqcup, \sqcap)\) is an upper-bounded join semilattice,
- \((E, \triangleright, I)\) is a monoid,
- \(\sqcap\) is nilpotent for sequencing (\(\forall x, y, z \sqcap = \sqcap = \sqcap \triangleright x\)), and
- \(\triangleright\) distributes over \(\sqcap\) on both sides: \(a \triangleright (b \sqcup c) = (a \triangleright b) \sqcup (a \triangleright c)\) and \(a \sqcup b \triangleright c = (a \triangleright c) \sqcup (b \triangleright c)\)

\(\sqcap\) is used as an indication of an error, for modeling partial join or sequence operators, \(\sqcup\) is used to model nondeterministic joins (e.g., for branches) as in the commutative systems, and \(\triangleright\) is used for sequencing. The default effect of “uninteresting” program actions becomes the unit \(I\) rather than a bottom element (note that effect quantales need not possess a bottom element). As a consequence of the distributivity laws, it follows that \(\triangleright\) is also monotone in both arguments, for the standard partial order derived from a join semilattice:

\[
x \sqsubseteq y \equiv x \sqcup y = y
\]

Gordon [25] also showed how to exploit closure operators [6, 7, 24] to impose a well-behaved notion of iteration (the \((-)^*\) operator from T-WHILE) that coincides with manually-derived versions for the effect quantales modeling prior work. The construction was only defined for effect quantales satisfying certain criteria. The original definitions turned out to be more restrictive than necessary, which prevented the iteration operator from applying to behavioral effect systems; applying the same construction without the unnecessary restriction works for large classes of known effect systems [26]. All known effect quantales satisfy the (generalized [26]) criteria for iteration to be defined. The effect quantales for which the generalized iteration is defined are called laxly iterable. An effect quantale is laxly iterable if for every element \(x\), the set of subidempotent elements greater than both \(x\) and \(I\) has a least element; an element \(s\) is subidempotent if \(s \triangleright s \sqsubseteq s\). This is true of all known effect quantales corresponding to systems in the literature.

The iteration operator for an iterable effect quantale takes each effect \(x\) to the least subidempotent effect greater than or

\[2\text{In contrast the original proposal, which was unnecessarily strict.}\]
equal to \(x \sqcup I\) (which exists, by the definition of laxly iterable). This iteration operator satisfies 5 essential properties for any notion of iteration, which we will find useful when deriving rules for loop\(^4\).

- **Extensive**: \(\forall e. e \subseteq e^*\)
- **Idempotent**: \(\forall e. (e^*)^* = e^*\)
- **Monotone**: \(\forall e, f. e \subseteq f \Rightarrow e^* \subseteq f^*\)
- **Foldable**: \(\forall e. e \triangleright e^* \subseteq e^*\) and \(e^* \triangleright e \subseteq e^*\)
- **Possibly-Empty**: \(\forall e. I \subseteq e^*\)

Another useful property of iteration that we will sometimes use without remarking on it, is that \(\forall x, y. x^* \sqcup y^* \subseteq (x \sqcup y)^*\) \(\square\).

Gordon’s work \(25, 26\) gives more details on closure operators and the derivation of iteration. We merely require its existence and properties.

Gordon \(25\) gives an effect quantale for enforcing data race freedom in the presence of unstructured locking, where the elements are pairs of multisets of locks — a multiset counting (recursive) lock acquisitions assumed before an expression, and a multiset counting (recursive) lock claims after an expression. Using pre- and post-multisets rather than simply tracking acquisitions and releases makes it possible to enforce data race freedom: an atomic action (e.g., field read) accessing data guarded by a lock \(\ell\) can have effect \(\{(\ell), (\ell)\}\), which indicates the guarding lock must be held before and remains held. The idempotent elements for this effect quantale are the error element, plus those where the pre- and post-multisets are the same — so iterating an expression where the lock claims are loop-invariant does nothing, while iterating an action that acquires and/or releases locks (in aggregate, not internally) yields an error.

Another example from Gordon \(26\) is a simplification of various trace or history effect systems \(37, 52, 53\). For a set (alphabet) of events \(\Sigma\), consider the non-empty subsets of \(\Sigma^*\) — the set of possibly-empty strings of letters drawn from \(\Sigma\) (the strings, not the subsets, may be empty). This gives rise to an effect quantale \(T(\Sigma)\) whose elements are these subsets or an additional top-most error element \(Err\). Join is simply set union lifted to propagate \(Err\). Sequencing is the double-lifting of concatenation, first to sets \((A \cdot B) = \{xy \mid x \in A \land y \in B\}\), then again to propagate \(Err\). The unit for sequencing is the singleton set containing the empty string, \(\{\epsilon\}\). If \(\Sigma\) is a set of events of interest — e.g., security events — then effects drawn from this effect quantale represent sets of possible finite event sequences.

We will often refer to the examples of tracking locks or event sequences for motivation or intuition throughout the paper, as continuations (below) duplicate, drop, and/or reorder program fragments. The key challenge we face in this paper is, viewed through the lense of these examples, to ensure that when continuations are used in those ways, we want to ensure the effect system does not lose track of events of interest.

\(^3\)The original proposal gave a slightly different and stronger set of 5 properties, including an extra distributivity property.

\(^4\)\(x^* \sqcup y^* \subseteq (x \sqcup y)^* \sqcup y^*\) by monotonicity and \(x \subseteq x \sqcup y\), so for \(y\) similarly \(\ldots \subseteq (x \sqcup y)^* \sqcup (x \sqcup y)^* = (x \sqcup y)^*\)

B. Tagged Delimited Continuations

Control operators have a long and rich history, reaching far beyond what we discuss here. Many different control operators exist, and many are macro-expressible \(17\) in terms of each other (i.e., can be translated by direct syntactic transformation into another operator), though some of these translations require the assumption of mutable state, for example. But a priori there is no single most general construct to study which obviously yields insight on the source-level effect typing of other constructs. A suitable starting place, then, is to target a highly expressive set of operators that see use in a real language. If the operators are sufficiently expressive, this provides not only a sequential type system for an expressive source language directly, but also supports deriving type rules for other languages' control constructs, based on their macro-expression in terms of the studied control operators.

We study a subset of the tagged delimited control operators \(16, 18, 49, 50, 51\) present in Racket \(23\), which includes both composable and non-composable continuations. We briefly recall this subset here; Racket aficionados familiar with Flatt et al.’s work may skip ahead while noting we omit continuation marks, dynamic-wind, and composable continuations, deferring these to future work. Continuation marks are little-used outside Racket. dynamic-wind is the heart of constructs like Java’s finally block or the unconditional lock release of a synchronized block, but we leave them to future work. Composable continuations extend their context, rather than replacing it. We give their semantics below for comparison and because they are necessary for completeness in some models \(50, 51\), but leave their treatment in a sequential effect system to future work as well.

All continuations in Racket are delimited, and tagged. That is, there is a form of prompt that limits the scope of any continuation capture: \((\& \text{tag } e e2)\) is a tagged prompt with tag \(\text{tag}\), body \(e\), and abort handler \(e2\). Without tagging, any continuation captured within \(e\) would only extend as far as the closest enclosing prompt, and restoring that (non-composable) continuation would only replace the context within that prompt. With tagging, continuation capture specifies the prompt tag delimiting the capture. If the nearest enclosing prompt with that tag lays outside prompts with other tags, those inner prompts are captured as part of the continuation.

Without tags, different uses of continuations — e.g., error handling or concurrency abstractions — can interfere with each other \(49\). Thus prompts, the continuation-capturing primitives \texttt{callcc} and \texttt{callcomp}, and the \texttt{abort} primitives all specify a tag, and only prompts with the specified tag are used to interpret continuation and abort boundaries. In most presentations of delimited continuations, tags are ignored (equivalently, all tags are equal), while most implementations retain them for the reasons above. In this paper, the tags are essential to explain the theory as well: an abort that “skips” a different prompt must be handled differently by our type-and-effect system to preserve precision.
callcc tag f is the standard (delimited) call-with-current-continuation: f is invoked with a delimited continuation representing the current continuation up to the nearest prompt with tag tag. Invoking that continuation replaces the context up to the nearest dynamically enclosing prompt with the same tag, leaving outer parts of the continuation alone.

callcomp tag f captures the same continuation, but in a composable form. f is invoked with the corresponding composable continuation, which when invoked extends the current evaluation context similarly to typical function application, rather than replacing it up to the enclosing appropriately-tagged prompt. This is seen most clearly in the operational semantics for capturing and invoking tagged delimited continuations in Figure 1. For now, ignore the type annotations in the terms (σ, χ). Both capture and replacement are bounded by the nearest enclosing prompt for the specified tag. The surrounding captured or replaced context (E′ in both rules) may contain prompts for other tags, but not the one specified by the operator.

Racket, and the semantics in Figure 1, include one additional construct, abort. (abort t e) evaluates e to a value, then replaces the enclosing prompt (of the specified tag) with an invocation of the specified handler.

This set of primitives is among the most expressive control operator sets. Sitaram showed that this set of operators is complete with respect to several denotational models [50, 51], and it is known to macro-express another popular form of delimited continuations, the combination of shift and reset [23]. In this paper we do not consider callcomp and composable continuations, but only the non-composable (and better known) varieties. We leave composable continuations to future work. The non-composable operators we study can still express loops, exceptions, coroutines [31, 32], and generators [11]; we study the latter later in this paper.

This set of primitives is preferable for our purposes (over shift/reset) because they are implemented as primitives in a real, mature language implementation (Racket), used in real software [38], and general enough to use for deriving rules for higher-level constructs like generators from their macro-expansion.

III. Warming Up: Observing to the Left and Right

To build intuition for our eventual technical solution, we first present informal discussion of how a sequential effect system might be extended to reason about control operations that either skip or duplicate parts of a program. For the former, we will discuss how to handle abort, which discards unexecuted code up to a prompt, by appealing to our intuitions about exceptions — which can be macro-expressed using abort. For the latter, we will examine a very simple control operator that runs provided code twice; solving this offers a key insight that we later show scales up to encodings of non-trivial control operators, and by extension non-trivial direct uses of control operators.

In general we would like to derive type rules in a sequential effect system based on the expression of control flow constructs in terms of continuations. This necessitates macro-expressing [17] the control flow constructs, and using the continuation-aware type rules to derive rules stated directly in terms of the control flow construct.

We would also like these derived rules to make sense in any sequential effect system, rather than one particular sequential effect system. To this end, we will work with regard to an unspecified underlying sequential effect system given as an effect quantale, and specify the effects for a language with control operators as a construction over that effect quantale — a construction which, when applied to an effect quantale, yields a new continuation-aware version of that effect quantale. Throughout, we will use the term underlying effect to refer to an effect drawn from this original effect quantale.

They key idea we employ, in two forms, is that of an accumulator effect, which collects effects sequenced on one side (the left or right), but ignores effects sequenced on the other.

A. Aborting Computations

We can macro-express try-catch with a number of handlers for disjoint exception types C as

$$\lambda e \cdot \text{catch}_{C_1} \ldots \text{catch}_{C_n}$$

Here we assume types are valid prompt tags (or at least in some known 1:1 correspondence with such tags) and we tag the throw with the expected exceptional type.

A key requirement for the static effect of a try-catch with n possible checked exceptions to be sound is that the overall effect must over-approximate not only the non-exceptional executions (which throw no exceptions), but also the exceptional executions. The exceptional executions include some “prefix” of the body, as well as the execution of the corresponding handler. This means we must track for each exception type the sequential effect prior to throwing that exception. Then when typing the overall try-catch, the overall effect must be greater than (or equal to) the sequencing of that body effect before the exception was thrown and the effect of the corresponding handler. In addition, this tracking must support multiple throws of the same exception in different locations.

We can track the effect prior to a throw by augmenting effects with a set of non-local control transfers (uses of abort). We can ensure that this “prefix” of the execution before a throw is tracked correctly by having the sequence operator $\triangleright$ prefix these effects with underlying effects from the left (the prefix), and ignore effects to the right (for the control transfers). We call these control transfer effects left accumulators, because they accumulate approximations.

3Technically this encoding differs from the standard behavior of exceptions in Java or C#: if a handler throws an exception caught by another handler in the same try-catch block, in Java or C# this will always propagate to the next enclosing catch (if one exists). In this encoding, if the handler for $C_n$ throws exception $C_m$ for $n > m$, it will be caught locally due to nesting. In practice, such scenarios are uncommon. In the type rules we give, we restrict the catch handlers to not throw exceptions.
E ::= • | (E e) | (v E) | (% t E v) | (call/cc t E) | (call/comp t E) | (abort t e)

\[\begin{align*}
\sigma; v \rightarrow_\delta \sigma; e\\
\sigma; ((\lambda x. e) v) \rightarrow_\delta \sigma; e[v/x]\\
\sigma; (p \tau) \rightarrow_\delta \sigma', v\\
\sigma; e \rightarrow_\delta \sigma; e\\
\sigma; e \rightarrow_\delta \sigma; e\\
\sigma; e \rightarrow_\delta \sigma; e\\
\end{align*}\]

E-App

E-PRIMAPP

\[\begin{align*}
\delta(\sigma, p \tau) &= (\sigma', v, \chi) \\
Values(\tau) &= \sigma; e \rightarrow_\delta \sigma', v
\end{align*}\]

E-Context

E-Abort

E-CALLCC

E-INVOKECC

E-CALLCOMP

\[\begin{align*}
E' \text{ contains no prompts for } \ell & \quad \sigma; E[\% \ell E'[\text{(abort } \ell \text{ v)}] \, h]) \rightarrow_\delta \sigma; E[h \, v] \\
\sigma; E' \text{ contains no prompts for } \ell & \quad \sigma; E[\% \ell E'[\text{(abort } \ell \text{ v)}] \, h]) \rightarrow_\delta \sigma; E[h \, v] \\
\sigma; E' \text{ contains no prompts for } \ell & \quad \sigma; E[\% \ell E'[\text{(abort } \ell \text{ v)}] \, h]) \rightarrow_\delta \sigma; E[h \, v] \\
\sigma; E' \text{ contains no prompts for } \ell & \quad \sigma; E[\% \ell E'[\text{(abort } \ell \text{ v)}] \, h]) \rightarrow_\delta \sigma; E[h \, v]
\end{align*}\]

Fig. 1. Formal semantics

of the (linear) past execution from the left. Consider an effect aborts \(\chi\), which describes a computation that first performs computation with effect \(\chi\), then aborts. If this is executed after code with effect \(\gamma\), then the resulting effect should be \(\gamma \triangleright (\text{aborts } \chi) = \text{aborts } (\gamma \triangleright \chi)\). By contrast, if this is run before code with effect \(\gamma\), we would expect the later code’s effect to be discarded since it will not execute: \((\text{aborts } \chi) \triangleright \gamma = (\text{aborts } \chi)\). Returning to tracking lock and unlock effects with exceptions, this correctly gives the body of the following \texttt{try-catch} its expected effect of only acquiring the lock:

\texttt{try (lock(\ell): if (cond ...) (throw e) else (unlock(\ell))) catch ...}

Giving the \texttt{throw} a left accumulator effect ensures the effect “before” the throw includes the lock acquisition and omits the release that is not run in the same executions as the throw.

B. Duplicating Computations

Left accumulators alone are insufficient to deal with the full range of control operators. Consider the following use of \texttt{callcc} (with prompt handlers ellided):

\texttt{(\% t ((callcc t (lambda k. (\% t (k () ...)))) (foo ()) ...))}

This program awkwardly executes \texttt{(foo ())} exactly twice. The \texttt{callcc} captures the continuation \((\bullet; (\texttt{foo} ()))\). The body of the \texttt{callcc} then nests a new prompt whose body replaces the (inner) prompt body with a use of that continuation — introducing a second invocation of \texttt{foo}. Immediately after the reduction step that applies the continuation to \(\ell\), the overall expression is \((\% t ((\% t (\ell)); (\texttt{foo} ())) ...); (\texttt{foo} ())) ...\). The outer call to \texttt{foo} comes from the original call outside the \texttt{callcc}, and the inner call (inside the inner prompt) comes from invoking the continuation.

There’s an important subtlety in reasoning about this kind of duplication. For left-accumulation, we can simply give a closed effect for \texttt{abort} and have that effect collect to its left. But the body of the \texttt{callcc} itself does not contain the information we require about the captured continuation — and in fact, syntactically identical \texttt{callccs} in different contexts can capture continuations with very different effects.

Somehow we must both use some effect to characterize the behaviors from invoking the continuation \(k\) in subexpressions, but we must also relate this to effects that occur outside the \texttt{callcc}. We will take a “guess-and-check” approach. The type rule for \texttt{callcc} will choose some assumed latent effect for the continuation, and check the body under that assumption. But the overall effect will produce a \textit{right-accumulator}, which is initially empty (the identity effect), but accumulates the effects sequenced to the right. In the example above, the \texttt{callcc} typing will guess an effect \(K\), and use that. It will also produce an accumulator that collects effects outside, including the effect of the call to \texttt{foo} in the original program. When this accumulator has “observed” all code between the \texttt{callcc} and the enclosing prompt, we can check that the observed effects are a subeffect of what was guessed. This results in splitting the work between the rule for \texttt{callcc}, which posits a continuation effect and includes the guess (and accumulator) in its own effect, and the rule for \texttt{prompt}, which checks the prediction against the observation.

Extending these intuitions further requires dealing with a few additional difficulties. First, the guess made when locally checking a \texttt{callcc} body might need to predict the existence of other uses of \texttt{callcc}, since guesses are themselves an effect. Second, the effects must track which prompt they refer to, since when multiple control constructs or control operators are modeled, they will use different tags. Finally, accumulation must sometimes pause, skip some effects, and then resume. This becomes necessary when mixing continuations from mul-

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6While we are the first to describe effects as accumulating on the left or right, we are not the first to use accumulators. Koskinen and Terauchi [13] use a similar idea when handling the sets of infinite executions in their effect system.
multiple prompts, and is the key reason we cannot simply ignore tags in our formalism and track them in an implementation — the interaction itself is subtle and non-obvious.

IV. EFFECTS FOR FULL CONTROL OPERATORS

This section gives type rules for the non-compositional control operators recalled in Section III. The core idea for the type system remains the use of accumulators. The effect quantale construction presented in this section adds to an existing, underlying effect quantale support for tagged delimited continuations with abort handlers and (non-compositional) continuation invocation. Beyond tracking an underlying effect, the system must track two other kinds of effects, each accumulating in a different direction.

a) Control Effects: These characterize non-local control transfer from both aborts and continuation invocation, which each discard the continuation enclosing their use. Abort effects must track a prefix effect over-approximating any linear past of the expression at some point in execution, as well as the type of value thrown, and the prompt tag that selects the handler. Continuation replacements track the same notion of prefix and target tag, and a type representing the result type of the current continuation, checked at the appropriate prompt boundary. Because both variants of control effects track “prefixes” of execution before a use of non-local control, both are left-acumulators.

b) Prophecies: These address the non-local nature of reasoning about continuations, connecting the use of a \texttt{callcc} deep within a context (which “generates” a prophecy) to the prompt delimiting the capture (whose type rule checks validity of relevant prophecies within its body). Most prophecies are right accumulators, because they track what effects may occur in their continuation. What they accumulate is a bit complex, and we come back to that after some more development. Because the accumulation follows evaluation order, the control and underlying effects accumulated “between” a \texttt{callcc} and the respective prompt is tantamount to building a parallel typing derivation for the context, inside the main typing judgment (our soundness proof makes this precise in Section VII). Some prophecies do not accumulate: if a continuation generates prophecies, then invoking that continuation should not accumulate behaviors from the surrounding context that the invocation would discard — they should only accumulate starting from the boundary of the \textit{resumed} context. For example, in \((\% t \{ (c 3); (unlock l) \} \ldots)\), if \(c\) is a continuation (for \(t\) or any other tag), the \texttt{unlock} will never be executed, so prophecies in the context restored by \(c\)'s invocation should not observe it. While they are not accumulating, we call these prophecies \textit{frozen}. At the appropriate prompt boundary, they are transformed to permit further accumulation.

Figure 2 gives the formal definition of the transformed effect quantale, for a variant of Gordon's core language, a simply-typed lambda calculus parameterized by a choice of state and primitive operations, with equirecursive types, a subsumption rule (for both types and effects), and the control operators we study. Most type rules are standard for a sequential effect system (e.g., along the lines of those in Section III) so we do not repeat them inline.

Figure 2's \texttt{ContinuationEffects} \(\chi\) are triples of a prophecy set, control effect set, and underlying effect (from the underlying effect quantale). As suggested above, a control effect tracks either an abort or replacement (continuation invocation) with the target prompt tag and prefix effect. Prophecies, in addition to tracking the anticipated type of the enclosing prompt, track (1) the actual effect predicted for the captured continuation (this is used to type the continuation passed to the body of \texttt{callcc}), and (2) an accumulator for effects sequence on the right.

Rather than accumulating only underlying effects, prophecies accumulate full effects — including other prophecies and control effects. Recall that the purpose of the prophecy is to capture the unevaluated effect that would occur in the continuation captured by a particular \texttt{callcc}. If that continuation would include further uses of control operators, those must be predicted, and uses of the resulting context must checked for correctness with respect to those control operators. Because of how sequencing (\(\triangleright\)) is used by the type system, that effect corresponds by construction to the effects sequenced to the right of a prophecy raised by a \texttt{callcc} (see Figure 2 shortly). This capture must include not only the underlying effect and “exceptional exits” (control effects) of the continuation, but also other prophecies arising from unevaluated \texttt{callcc}s captured in the continuation — including their prophecies, particularly for captures of continuations for different tags. We will see later that this is essential for interactions between different control constructs.

Sequencing continuation effects requires sequencing the components appropriately:

- Underlying effects are sequenced using the underlying effect quantale's \(\triangleright\) operator.
- Control effects represent the set of non-local control transfers (and execution prefixes) of an expression. The non-local control transfers for two sequenced subexpressions include the transfers of the first along with the transfers of the second, though the latter — if they occur — would occur after the regular (underlying) effect of the first expression.
- Prophecies on the left of \(\triangleright\) accumulate full effects from the right, via a per-prophecy accumulation \(\triangleright\), lifted to operate on sets of prophecies. This operator \(\triangleright\) mimics

\footnote{For simplicity, our type system does not support answer type modification. We foresee no major barriers to supporting answer type modification since our core language uses explicit, rather than implicitly-inferred polymorphism, but presenting such an integration would add significant technical detail that is orthogonal to the key ideas of this paper.}

\footnote{We drop parametric type and effect polymorphism, and singleton types. We add subsumption and equirecursive types (not effects). We foresee no difficulties adding back the dropped features, we simply have not proven soundness for those features.}

\footnote{\(\triangleright\) is well-founded, recurring through sets, much like a rose tree's recursion scheme.
the effect of sequencing each prophecy’s observation “thus far” with the the effects to the right in the current context. Prophecies from the right of $\gg$ are also unioned into the result, as those are also top-level prophecies to be validated. Note that in $\triangleright$, frozen prophecies — those arising from continuation invocations, which therefore summarize the prophecy as far as the enclosing prompt — do not accumulate. Intuitively, the accumulation in prophecies builds a parallel effect derivation for the continuation a $\text{call/cc}$ would capture inside the effect captured in the source-level derivation. Our soundness proof makes this precise, with a lemma proving that prophecies accumulate exactly the effects of their enclosing context (Section VII and Appendix B:B).

For brevity and clarity, we consider the ContinuationEffect definitions in Figure 2, quotiented by the least reflexive relation that also relates any full effect containing a $\top_Q$ (error element) from the underlying effect quantale to any other such effect. This avoids explicitly dealing with error elements in the definitions below. When needed, we use $(\emptyset, \emptyset, \top_Q)$ as $\top$, which we take as the representative element of that equivalence class.

The figure also gives a definition of a partial order on sets of control effects. We treat control effect sets as equivalent if the greatest underlying effect (with respect to $\subseteq$ in the underlying effect quantale) is equal for each tag-and-type for each variety (abort or replace) of control effect. We treat the definition of $C$ as quotiented by this relation, which leads to the more intuitive partial order on control effect sets $C_1 \subseteq C_2$. Joins of prophecy sets are treated similarly, considering $P_1 \subseteq P_2$ when every prophecy in $P_1$ has its observation (only) overapproximated with respect to subeffecting.

Our transformation yields a new effect quantale, and moreover, the exception quantale over an effect quantale is a subset of the corresponding continuation quantale by the natural embedding.

**Theorem 1 (Continuation Quantale).** For any effect quantale $Q$, the continuation (pre-)quantale $(\mathcal{C}(Q), \sqcup, \gg, \top, (\emptyset, \emptyset, I))$ is in fact an effect quantale (a join-semilattice-ordered monoid with top nilpotent for sequencing, where $\gg$ distributes over $\sqcup$ on both sides).

**Proof.** Follows from the definitions in Figure 2 and the effect quantale axioms for $Q$; each property is tedious but straightforward to verify, except nilpotency of $\top$, which follows trivially from nilpotency in the underlying effect quantale and our quotienting by a relation equating all continuation effects containing an error from the underlying effect quantale.

**A. Iterating Continuation Effects**

Another critical property we will need later is the fact that this construction preserves lax iterability [26], and therefore has a well-defined iteration operator which we give here. The construction is somewhat detailed and only necessary in this paper for our discussion of generators in Section VI-F, so a reader inclined to see applications earlier could skip this section on a first read, and revisit it before Section VI-F.

Lax iterability requires that for any element $x \in \mathcal{C}(Q)$, there is a unique least element $y \in \mathcal{C}(Q)$ such that $x \subseteq y$, $I \subseteq y$, and $y \gg y \subseteq y$ (i.e., $y$ is subidempotent). The induced iteration operator takes each effect $x$ to the corresponding $y$ above, so the proof also defines the iteration operator. We will first describe some constraints to build intuition for why the operator is appropriate, then define it, then highlight some special cases of interest.

Since we are interested in subidempotent elements of $\mathcal{C}(Q)$, we can expand the definition of subidempotence for non-$\top$ elements ($\top$ is trivially subidempotent). If an effect $(P_y, C_y, Q_y)$ is subidempotent, then by expanding the meaning of $\subseteq$ in $(P_y, C_y, Q_y) \gg (P_y, C_y, Q_y) \subseteq (P_y, C_y, Q_y)$:

- $Q_y \gg Q_y \subseteq Q_y$
- $\forall y : Q_y \subseteq C_y$
- $(P_y \gg (P_y, C_y, Q_y)) \subseteq P_y \subseteq P_y$

The last two constraints may be simplified to:

- $(Q_y \gg C_y) \subseteq C_y$
- $(P_y \gg (P_y, C_y, Q_y)) \subseteq P_y$

The lower bound on $C_y$ implies that each underlying effect in $C_y$ inside an $\text{abort or replace}$ takes the form $Q_y \gg Q'$, where the $Q'$ may vary between different escape effects. The point is that this ensures that prefixing any of the escape effects by $Q_y$ again yields an effect already over-approximated by $C_y$.

The lower bound on $P_y$ is slightly less obvious. The definition of $\gg$ suggests every prophecy in $P_y$ must either be frozen (thus, not accumulating observations), or predict an infinite set of prophecies, each individually of finite depth —
Theorem 2 (Lax Iterability with Continuations). For a laxly iterable underlying effect quantale $Q$, the continuation quantale $C(Q)$ is also laxly iterable, with the closure operator given by:

$$(P, C, Q)^* = \left( \bigcup_{i \in \mathbb{N}} P \triangleright (P, C, Q)^i, Q^* \triangleright C, Q^* \right)$$

lifted to propagate $\top$.

Proof. To prove this is the closure operator, we must prove that the right hand side is the minimum subidempotent element greater than both $I$ and $(P, C, Q)$. Subidempotence follows directly from the infinite union of prophecies, and the properties of the underlying effect quantale’s iteration. Being greater than the original input and identity is straightforward. So it remains to prove minimality. By contradiction. Assume there is a lesser such element than the above, $(P', C', Q')$. The fact that this is supposedly less than the result of $(P, C, Q)^*$ defined above requires that each component be ordered less, and at least one of these component-wise inequalities must be strict:

- $Q' \subseteq Q^*$
- $C' \subseteq Q^* \triangleright C$
- $P' \subseteq \bigcup_{i \in \mathbb{N}} P \triangleright (P, C, Q)^i$

The first is only possible if $Q' = Q^*$, since $Q^*$ is minimal in the same way within the underlying effect quantale. The second degenerates to an equality requirement for the same reason, so the final constraint must be a strict $\sqsubseteq$. The final component-wise constraint requires any prophecy in $P'$ to be over-approximated by some prophecy in the infinite union term. So for the assumed $(P', C', Q')$ to be strictly less than $(P, C, Q)^*$, either at least one prophecy in $P'$ is strictly over-approximated wherever it is over-approximated in the infinite union, or at least one prophecy in the infinite union is not necessary to over-approximate an element of $P'$. The latter case is straightforwardly not possible: since the infinite union contains exactly all prophecies obtainable by finite iteration of $(P, C, Q)$, omitting any such prophecy from $P'$ would mean that for some $n$, $(P', C', Q') \triangleright (P, C, Q)^m$ would contain a prophecy not contained in $P'$, even though because $(P', C', Q')$ is an iteration result and subidempotent, it should be the case $(P', C', Q') \triangleright (P, C, Q)^m \subseteq (P', C', Q')$; so this is not possible. The former case is similar: it is not possible for a prophecy $p \in P'$ to be only strictly over-approximated by any larger elements of the infinite union, because that would require $p$ to be a prophecy that could not be generated by finite iteration of $(P, C, Q)$.

The requirements for lax iterability dictate exactly the operator above for iteration, but we can consider the relationship between this operator and various representative cases that may arise in typechecking to understand why this is not only mathematically necessary, but actually corresponds in a sensible way to the runtime semantics.

We can build some intuition for the operator above by considering two special cases, then discussing the general case.

Example 1 (Control-Free Iteration). In the case where an iterated effect has no (escaping) prophecies or control effects, it behaves exactly as the iteration from the underlying effect quantale: $(\emptyset, \emptyset, Q)^* = (\emptyset, \emptyset, Q^*)$.

Example 2 (Prophecy-Free Iteration). In the case where the prophecies are empty — where there are no unresolved continuation captures — the results correspond to the intuitive idea that the control effects would occur after 0 or more non-exceptional runs of the underlying effect — that any exceptional control action in $C$ would occur only after repeating $Q$ some (possibly-zero) number of times: $(\emptyset, C, Q)^* = (\emptyset, Q^* \triangleright C, Q^*)$.

Control-free iteration appears any time the code that is repeated has no escaping control effects (e.g., it is straight-line code, or uses only closed control flow constructs like loops). Prophecy-free iteration covers cases like throwing exceptions from inside a loop (or invoking another continuation inside a loop, which requires use of call/cc elsewhere outside the loop).

While these examples “merely” drop certain components of Theorem 2, it helps to work from the simples case up to the more complex versions, since the examples above correspond intuitively to various execution paths. The infinite union in the prophecy set is the most subtle part of the operation to explain. Consider an expression with the structure:

$$\text{while } c \ldots (\text{call/cc } t \ldots) \ldots$$

Assume the tag $t$ for the continuation captured inside the loop does not occur elsewhere inside the loop — in particular, that the captured continuation would extend outside the loop. Considering the runtime execution, in some sense the prophecy captured by the first loop iteration must predict not only the regular execution and exceptional executions of future iterations, but even the need for more prophecies to be generated by the callcc’s in future iterations as well! This is why the set of prophecies must still be sequenced with some form of themselves, rather than just some subset. During static typechecking, we must therefore conservatively overapproximate the number of iterations following a prophecy. It may be 0, 1, 2, … or any number. So the approximation must consider all of those possibilities, hence the infinite union of finite repetitions following the prophecies.

V. Typing with Continuation Effects

Figure 5 gives type rules for prompts, aborts, continuation capture, and continuation invocation. Rules for closures and closure application were given in Section II (T-LAMBDA and T-APP). We also assume typing for a family of primitive operations (for E-PRIMAPP in Figure 1) that introduce non-trivial effects from the underlying effect quantale (roughly,
an underlying effect for each primitive operation reduction, following prior work \cite{22,23}. We explain the rules specific to tagged delimited (non-composable) continuations bottom-up in Figure 3’s rendering, finishing with the type rule for prompts, and its supporting metafunctions (above the horizontal line).

T-ABORT is most straightforward: it checks the type and effect of the expression to throw to the enclosing handler, and its effect is the effect of that expression followed by (\(\triangleright\)) an effect indicating no prophecies or underlying effect, but that there is non-local control: an abort to the enclosing prompt for \(\ell\), throwing some value of type \(\tau\), after the trivial prefix \(I\). Note that the sequencing this after the evaluation expression thrown to the handler, this both models evaluation order and results in the control effects for the overall expression indicating that when the abort in question occurs, it does so after a non-exceptional execution of \(e\). This is the rule used to derive T-THROW in Section~\ref{III}

T-APPCont is similar with respect to typing its subexpressions and matching use of \(\triangleright\) to evaluation order. However the final piece of the effect is different because it is a different use of non-local control. T-APPCont also has no underlying effect (invoking a continuation has no local behavior, by definition). Its prophecy set is the continuation’s latent prophecy set, but \(\text{frozen}\) until the next prompt for \(\ell\). This corresponds to the fact that if the restored continuation makes prophecies, it is already known what those prophecies accumulate up to the prompt for \(\ell\) when the continuation was captured. Moreover, those prophecies should not accumulate further observations between the site of the invocation and the enclosing prompt, because that fragment of the evaluation context would be discarded when the applied continuation is restored. The control effect extends the restored continuation’s latent control effects (the restored context may itself exit non-locally) with an indication that the expression invokes a continuation whose remaining local effect is \(Q\), and whose result type is \(\gamma\).

T-CALLCont types uses of \texttt{callcc}. The overall effect reflects that evaluation of this expression first reduces the function to invoke to a value (effect \(\chi_v\)) and subsequently invokes it (latent effect \(\chi\) from \(e\)’s type). The final effect sequenced by T-CALLCont is a prophecy, which will accumulate the “latent” effect of the continuation captured by that particular \texttt{callcc}. Even though it is conceptually tempting to consider the “act of prophecy” occurring before the body is invoked, it is essential to place this right-acumulating prophecy \textit{after} the effects of the subexpression, to ensure only effects that would occur outside the \texttt{callcc} are accumulated.

T-PROMPT is the most complex, and most subtle rule of the type system. Naturally, it requires a well-typed body \(e\) and handler \(h\). The complexity in T-PROMPT arises from its two essential purposes: to validate optimistic assumptions made in the body (\texttt{validEffects}), and to “resolve” the control effects that would be scoped to the body of the prompt (\texttt{validEffects} and the non-trivial transformations in the conclusion’s effect).

The \texttt{validEffects} rule V-Effects validates the types carried by all control effects or prophecies:

\begin{itemize}
  \item that the values thrown according to \texttt{abort} effects are subtypes of the handler’s input, and
  \item that context result types assumed by continuation capture (prophecies) and invocation (\texttt{replace}) are valid replacements for the prompt body.
\end{itemize}

It also performs more thorough validation for continuation capture. It \textit{nearly} checks that the observations for a given prophecy are a subeffect of what was predicted, which would be \((P, C', Q_p) \subseteq \chi_{prop}$. Instead, the comparison on the predicted and observed the prophecy sets is slightly different. We will revisit it with concrete examples in hand in Section
After validating prophecies and control effect types, TPROMPT gives the types for the overall prompt. The prophecy set escaping the prompt is the body’s set with two changes: removing now-validated (and locally-scoped) prophecies for the current prompt tag, and unfreezing (and “compressing”) prophecies for other tags that were not accumulating within the current prompt (i.e., those prophecies arising from invoking a continuation with its own prophecies). This occurs via an operation we call masking, \( P \setminus Q \ell \). The control effects escaping are simply those of the body, less the ones limited to the current prompt (aborts and continuation invocations for \( \ell \)): \( C \setminus \ell \). And the underlying effect joins (1) the underlying effect of the body, (2) all underlying effects from replacing the body by invoking a continuation, and (3) the sequencing via \( \triangleright \) of all pre-abort effects with the underlying effect of the handler. The join of those three over-approximates the execution trace effects from any execution of the body which completes by reducing the prompt expression to a value\(^{[1]}\).

VI. DERIVING SEQUENTIAL EFFECT RULES FOR CONTROL FLOW

Section [IV] developed the core rules, which give sequential effects to programs making direct use of tagged delimited control. As we have discussed, most programs do not use the full power of delimited control, and instead use only control flow constructs or weaker control operators. This section uses the type-and-effect rules of Section [IV] to derive sequential effect rules for a range of control flow constructs and weaker control operators macro-expressed in terms of prompts.

Our examples fall into two groups. First, we consider checking consistency of derived rules for typical control flow constructs with those hand-designed in prior work, for infinite loops (Section [VI-A]) and while loops (Section [VI-B]). Second, we consider derived rules for common constructs never addressed in prior work on sequential effect systems, namely exceptions (Section [VI-E]). Finally, we consider expressing a weaker control operator, a form of generator close to an \( \lambda \), namely \( \lambda k. k \) (underlying effect \( \chi \)), which we take to be \( \chi_e \), is repeated arbitrarily many times. We take this expansion as the body of a macro \([\text{loop } e\]]\).

A. Infinite Loops

Consider a simple definition of an infinite loop using the constructs we have derived here:

\[
[\text{loop } e] = (\% \ell \ (\text{let } cc = ((\text{call/cc } \ell \ (\lambda k. k)) \ (e; cc cc)) \ (\_\_ \ tt))
\]

The term above executes \( e \) repeatedly, forever (assuming \( e \) does not abort). Thus, its effect ought to indicate that \( e \)'s effect, which we take to be \( \chi_e \), is repeated arbitrarily many times. We take this expansion as the body of a macro \([\text{loop } e]\).

This program can be well-typed in our system, with an appropriate effect (assuming the underlying effect quantale is iterable per Section [II]). The body of the call/cc is pure, but for the expression to be well-typed, the call/cc’s own effect must prophesize some effect \((\emptyset, C_p, Q_p)\) of the enclosing continuation up to the prompt for \( \ell \) (because no call/cc occurs in the continuation of another, the prophecy set can be empty).

The right-accumulator of the prophecy effect, initially \((0, 0, \_\_\_\_\_\_\_)\), eventually accumulates a control set \((Q_e \triangleright C_p) \cup \{\text{replace } \ell : Q_e \triangleright Q_p \sim \text{unit}\}\) and underlying effect \(Q_e\), because between capturing the continuation and the prompt, the program evaluates \( e \) (underlying effect \( Q_e \)) and then invokes the captured continuation (prophesized effect \((\emptyset, C_p, Q_p)\)). This is also the resulting control effect set for the body; we will refer to it as \( C \). The type rule for the prompt itself removes all \( \ell \)-related prophecies and control effects, leaving both empty (since we assume no control effects escape \( e \), \( C_p \) should only contain \( \ell \)-related effects, while the prophecy set contains the single prophecy from the callcc). For the underlying effect, TPROMPT joins the immediate underlying effect \( Q_e \) (from the overall judgment, not the prophecy) with all \( \ell \)-related behaviors in \( C - e \) has no escaping control effects, and the macro-expanded loop contains no aborts, so \( C_p \) ought to have only \text{replace} effects, meaning \( C \) contains only replace effects, and \( Q_e \cup \bigcup C_p \bigcup C \) will join the underlying effect of all continuations invoked by the body. TPROMPT also performs some checking of result types (which all hold trivially since all types involved are unit), and prophecy validity checks that yield constrains we can solve to derive a closed-form type rule for the loop.

Completing a typing derivation with final underlying effect \( Q_\ell = Q_e \cup \bigcup C_p \bigcup C \) is possible given the solutions to the effect-related constraints imposed by \text{validEffects}: \( \Gamma \vdash Q_e \subseteq Q_p \), and \( \Gamma \vdash (Q_e \triangleright C_p) \cup \{\text{replace } \ell : Q_e \triangleright Q_p \sim \text{unit}\} \subseteq C_p \). These could be read off a hypothetical derivation (for example, see Figure [14] in Appendix [A] that would yield the derived rule

\[
\begin{align*}
\Gamma \vdash Q_e & \subseteq Q_p \\
\Gamma \vdash (Q_e \triangleright C_p) & \cup \{\text{replace } \ell : Q_e \triangleright Q_p \sim \text{unit}\} \subseteq C_p \\
\Gamma & \vdash e : \tau | (\emptyset, 0, Q_e) \\
\Gamma & \vdash [\text{loop } e] : \text{unit} | Q_e \cup \bigcup C_p
\end{align*}
\]
However, this rule is more complex than we would like for a simple infinite loop (note we have not expanded $C = (Q_e \triangleright C_p) \cup \{\text{replace } \ell : Q_e \triangleright Q_p \rightsquigarrow \text{unit}\}$), and also exposes details of the continuation-aware effects — which is undesirable if the goal is to derive closed rules for using the loop by itself, without developer access to full continuations. These constraints can be satisfied by $Q_p = Q_e^*$, with $C_p = \{\text{replace } \ell : Q_e^* \rightsquigarrow \text{unit}\}$. The choice for $C_p$ ensures that any “unrolling” of the loop to include any number of $Q_p$ prefixes (as generated by the left operand of the union in the last constraint) is in fact less than the replacement effect $(Q_e \triangleright Q_e^* \subseteq Q_e^*)$. This then implies that $Q_e \cup [\chi e]_I^\ell = Q_e \cup (Q_e^*) = Q_e^*$, by properties of Gordon’s iteration operator [23, 24] (Section III). Assuming $c \not\in \Gamma$ (or the use of hygienic macros), this leads us to the pleasingly simple derived rule:

$$\text{D-INFLOOP} \quad \Gamma \vdash e : \tau \mid (\emptyset, \emptyset, Q_e) \quad \quad \Gamma \vdash [\text{loop } e] : \text{unit} \mid (\emptyset, \emptyset, Q_e)$$

### B. While Loops

While loops can similarly be macro-expressed via continuation [23]:

$$[\text{while } c e] = (\% \ell (\text{if}(c) (\text{call/cc } \text{id}) \in (e; \text{if}(c) (\text{cc } cc) (tt))) \in (\lambda_{\ell, tt}))$$

Assume no other control effects escape $e$ or $c$ (i.e., $\Gamma \vdash e : \tau \mid (\emptyset, \emptyset, Q_e)$ and $\Gamma \vdash c : \text{bool} \mid (\emptyset, \emptyset, Q_e)$).

Writing only the underlying effects as shorthand for the case where no control behaviors appear, the effect of the prompt’s body is detailed and simplified in Figure 4.

As in the infinite loop case, this body effect along with T-PROMPT imposes a set of constraints which, if satisfied, allows the while loop to be well-typed in our type system.

In short, a derived type rule requires some underlying effect $Q_e = (Q_e \triangleright Q_e \triangleright Q_e) \cup Q_e^* \cup \bigcup \chi \Gamma Q_e^* e]^\ell$, and a choice for the prophecized control effect set $C_p$ and underlying $Q_p$ where:

- $\Gamma \vdash Q_e \triangleright Q_e \subseteq Q_p$
- $\Gamma \vdash Q_e \triangleright Q_e \triangleright C_p \cup \{\text{replace } \ell : (Q_e \triangleright Q_e \triangleright Q_p) \rightsquigarrow \text{unit}\} \subseteq C_p$

This leads to another complex derived rule, which can be further simplified:

$$\Gamma \vdash e : \tau \mid (\emptyset, \emptyset, Q_e) \quad \quad \Gamma \vdash c : \text{bool} \mid (\emptyset, \emptyset, Q_e)$$

As in the infinite loop case, a simpler solution is available as long as the underlying effect quantale has an iteration operator, and because the callcc does not capture other callccs, we start from the assumption that the prophecy predicts no other prophecies. In this case, we set $Q_p = (Q_e \triangleright Q_e)^*$, and $C_p = \{\text{replace } \ell : (Q_e \triangleright Q_e)^* \rightsquigarrow \text{unit}\}$. Then $Q_e$ simplifies using properties of effect quantales and the iteration operator:

$$Q_e = ((Q_e \triangleright Q_e \triangleright Q_e) \cup Q_e^*) \cup \bigcup \chi \Gamma (Q_e \triangleright Q_e)^* e]^\ell$$

This justifies the following derived rule:

$$\text{D-WHILE} \quad \quad \Gamma \vdash e : \tau \mid (\emptyset, \emptyset, Q_e) \quad \quad \Gamma \vdash c : (Q_e \triangleright Q_e)^* \mid (\emptyset, \emptyset, Q_e)$$

This derived rule is an important consistency check against prior work. Setting aside the additional enforcement that no other control effects escape $e$ or $c$ (as they would not in languages where control operators were used only for loops), this is nearly identical to the prior rule for typing while loops with sequential effects recalled in Section II, as in prior work [21, 22, 25] where the rule’s soundness was proven directly. The only difference is the presence of the (empty) behavior sets for other control effects and prophecies from working in a continuation-aware effect quantale.

### C. While Loops Without Subexpression Prophecies

Thus far we have only shown derived rules for simple loops. The infinite and while loops are limited in ways beyond simply being expected based on prior work that addressed them directly: they also ignore the potential for “improper” nesting of control operators — the cases studied thus far assume subexpressions that are not part of the macro expansion do not involve further unresolved control effects — we have not seen the interaction of loops with aborts, invoking continuations for prompts outside a loop, or prophecies from a loop body that need to observe the presence of iteration. Here we remedy the first two limitations, and in the next subsection address iteration of loop bodies with arbitrary control effects.

We first study iteration under the assumption that loop components may have aborts or continuation invocations that would exit the loop. While this stops short of the full generality of our system, it still encompasses many languages whose control flow constructs and operators — when expressed in terms of tagged delimited continuations — do not nest the capture of continuations inside macro arguments. This includes loops and exceptions [5]. In these cases, subexpressions of $c$ and $e$ that capture continuations occur under prompts that are themselves within $c$ or $e$, so $P_c$ and $P_e$ above would be $\emptyset$. In this case, $P_c \setminus \ell = \emptyset$. Figure 5 simplifies $C_e \setminus \ell$. The closed derived rule under these assumptions then becomes:

$$\text{D-ABORTINGWHILE} \quad \quad \Gamma \vdash e : \text{unit} \mid (\emptyset, \emptyset, Q_e) \quad \quad \Gamma \vdash c : (Q_e \triangleright Q_e)^* \mid (\emptyset, \emptyset, Q_e)$$

$C_{\emptyset} \setminus \ell \neq \emptyset$. Figure 5 simplifies $C_e \setminus \ell$. The closed derived rule under these assumptions then becomes:

$$\text{D-ABORTINGWHILE} \quad \quad \Gamma \vdash e : \text{unit} \mid (\emptyset, \emptyset, Q_e) \quad \quad \Gamma \vdash c : (Q_e \triangleright Q_e)^* \mid (\emptyset, \emptyset, Q_e)$$
D. Infinite Loops with Control

We have shown a general solution for while loops with control effects in the condition and expression is possible, but omitted the closed form because it is verbose and difficult to decipher. A more manageable and instructive example is possible for plain infinite loops. While it is possible to derive a fully general rule for while loops that admit the full flexibility of our effect system — without restricting the effects of subexpressions — the details are somewhat verbose. We instead demonstrate the principles on the slightly simpler example of the infinite loop; following the same process for the while loop yields a similar but correspondingly more complex result (in particular, the control effect set is the same as in Section VI-C). Showing the example of the infinite loop also demonstrates quite clearly that the effect of our transformation preserving lax iterability (Section IV-A) is not only a theoretical nicety, but useful.

The effect \( \chi \) of the prompt body in this case, for \( \chi_e = (P_e, C_e, Q_e) \) and \( \chi_{\text{in}} = \llbracket P_e \rrbracket, C_p \cup \{ \text{replace } \ell : (Q_p) \leadsto \text{unit}, I \} \), is given in Figure 6. This requires a choice of prophecy, satisfying (after simplifying \( \chi_p \) above):

- \( Q_e \subseteq Q_p \)
- \( C_e \cup (Q_e \triangleright C_e) \cup \{ \text{replace } \ell : (Q_p) \leadsto \text{unit} \} \subseteq C_p \)
- \( \llbracket P_e \rrbracket \text{ s.t. } \llbracket P_e \rrbracket, C_p \cup \{ \text{replace } \ell : (Q_p) \leadsto \text{unit}, I \} \subseteq \llbracket P_e \rrbracket \).

Unsurprisingly, the underlying and control constraints suggest a choice of \( Q_p = Q_e^* \) and \( C_p = (Q_e^* \triangleright C_e) \cup \{ \text{replace } \ell : (Q_e^*) \leadsto \text{unit} \} \). So a solution to the prophecy constraint would be given by a solution to the (slightly stricter)

\[
\llbracket P_e \rrbracket, (Q_p \triangleright C_e) \cup \{ \text{replace } \ell : (Q_p) \leadsto \text{unit}, I \} \subseteq \llbracket P_e \rrbracket
\]

which simplifies to

\[
P_e \triangleright (P_p, (Q_e^* \triangleright C_e) \cup \{ \text{replace } \ell : (Q_e^*) \leadsto \text{unit}, I \}) \subseteq \llbracket P_p \rrbracket
\]

We would like a solution without prophecies frozen for \( \ell \), in which case we may solve

\[
P_e \triangleright (P_p, (Q_e^* \triangleright C_e) \cup \{ \text{replace } \ell : (Q_e^*) \leadsto \text{unit}, I \}) \subseteq P_p
\]

This is solved for

\[
P_p = \bigcup_{i \in \mathbb{N}} P_e \triangleright ((P_e, (Q_e^* \triangleright C_e) \cup \{ \text{replace } \ell : (Q_e^*) \leadsto \text{unit}, I \})^i
\]

which is also a solution to the original constraint.

Solving for the final effect of the prompt expression itself:

- \( Q = Q_e \cup (Q_e^* \triangleright C_e) = (Q_e^* \triangleright C_e) = Q_e^{14} \)
- \( C = C_e \)
- \( P = P_e \)

A thorough reader will notice that for these results, \( P, C, Q \subseteq (P_e, C_e, Q_e) \) using Section IV-A's notion of iteration, licensing the following derived rule that accounts for arbitrary body effects despite its superficial simplicity:

\[
\text{D-FULLINLOOP } \Gamma \vdash e : \tau_e \mid \chi_e
\]

When the prophecy set is empty, this rule simplifies (by unfolding the definition of \((-)^*\)) to D-INFLUOOP from Section VI-A. If the process above is followed for the while loop expansion used in Sections VI-B and VI-C the resulting rule simplifies to D-ABORTINGWHILE and D-WHILE when assuming the same constraints as in those sections.

a) Unfreezing Prophecies: This is a good time to pause and reflect on a particularly subtle aspect of T-PROMPT, where the observations and predictions of a prophecy at a prompt boundary are unfrozen before comparison. This is the first example we have considered where the difference matters, because it is the first we have shown a prophecy that predicts further prophecies. Concretely: why is \( P_e \subseteq \llbracket P_p \rrbracket \), the correct constraint? A more obvious choice for the constraint

\[14\text{Last step follows because } Q_e \subseteq Q_e^* \text{ for iteration, which is defined as the join being the larger element.}\]
imposed by validEffects (specialized to this example) would be \( P \subseteq P_p \). This is in fact sound for the type system, but breaks this example where the condition and body may have unresolved prophecies. \( P_{P_b} \subseteq P \subseteq P_p \), which would imply that any solution required that all prophecies in \( P_p \) be frozen! Clearly, any unresolved prophecies from subexpressions would not be frozen at the “time” they were accumulated by the loop’s prophecy. Such a constraint would in general then have no solution — sound, but problematic for our goals (this problem arises only when a continuation captures its own invocation, but that includes most loop constructs). This requires the use of freezing in prophecy validation.

E. Exceptions

In Section III we informally considered typing a macro expansion of basic exception handling facilities:

\[
\begin{align*}
\text{try } e \text{ catch } C_i \Rightarrow e_i & = (\% C_1 \ldots (\% C_n e e_n) \ldots e_1) \\
\text{throw}_{C_C} & = (\text{abort } C)
\end{align*}
\]

The earlier discussion focused on the need to track what effects occurred before a throw vs. after. Now that we have discussed the type rules for prompts and aborts, this mapping is clear, and derived rules for the simple case (no escaping control effects) follow easily from the rules for prompt and abort. Assuming there is a designated prompt label \( \ell_C \) corresponding to every thrown type \( C \):

\[
\begin{align*}
\text{D-TryCatch} & \quad \Gamma \vdash e : \tau \mid (\emptyset, \{\text{abort } C \ C \ @ Q \}, Q) \\
& \quad \Gamma \vdash \text{catch } C \Rightarrow h : \tau \mid (\emptyset, \emptyset, I) \\
& \quad \Gamma \vdash \text{try } e \text{ catch } C \Rightarrow h : \tau \mid (\emptyset, \emptyset, Q) \cup (Q \Rightarrow Q_h) \\
\text{D-Throw} & \quad \Gamma \vdash \text{throw}_{C_C} : \tau \mid (\emptyset, \{\text{abort } C \ C \ @ Q \}, I)
\end{align*}
\]

Iterating the construction for D-TryCatch while permitting other aborts in the body effect and still preventing control effects in each handler gives a simple rule for an arbitrary set of exceptions. Mimicking the exact semantics of Java- or C#-style exceptions with multiple catch blocks per try — specifically, that a throw within one catch block is not handled by catch blocks for the same try — requires sum types and rethrowing, which is possible but does not illuminate the details of our continuation effects.

F. Generalized Iterators

Here we consider a simple encoding of generators in terms of delimited continuations. Our encoding is similar to Coyle and Crogono [11], but written independently (we first gave an encoding ourselves, then figuring it was unlikely to be new, located a reference with a similar approach).

The design of our encoding is as follows: the code that traverses some data structure (or lazily enumerates a sequence) is given as a function taking two arguments: a function to pass a value to a consumer (often a primitive named \texttt{yield} in many implementations, like C#), and a function to indicate that iteration is complete (we will call it \texttt{done}, but it is sometimes given other names such as \texttt{yield break} in C#). Given such code, the function \texttt{iterate} that we define returns a stateful procedure, each invocation of which returns either the next value from the iterator, or a value indicating completion (via an option type). Only one invocation of \texttt{iterate} is required; then each time client code is ready for the next value, it invokes the same function returned from the one call to \texttt{iterate}.

Figure 7 gives untyped Racket code for a simple generator.

Compared to our core language, Racket names several primitives slightly differently, moves the tag to the last argument position (it is optional in Racket), and uses a keyword argument \#tag to specify the prompt tag. \texttt{iterate} takes as an argument a two-parameter function \( \ell \). It allocates a fresh prompt tag \texttt{tag}, and a placeholder for the resumption continuation — initially \( * \) — which will be used to store the continuation that will produce the remaining items to be generated. \texttt{get-next} assumes that placeholder has been initialized: when invoked, it creates a new prompt, and invokes the resumption context. \texttt{yield} captures the enclosing context up to the nearest prompt for \texttt{tag}, stores it into \texttt{resumption}, and then aborts with (an option of) the generated value. The intent is that \texttt{yield} is invoked inside the prompt created by
define "a" argument, as-
form of stream).

get-next, by f. The abort/cc throws the value\textsuperscript{15} to the
for that particular iterator construction should predict. (The

get-next). Before we discuss typing uses of iterate, let us consider
put the code that knows how to generate

get-next. The main body after the let-bindings opens a new prompt,

get-next. The body of that
call/cc then aborts, yielding the function get-next to the
caller of iterate (by way of applying the identity-function

We can express nearly the same code in our core language.

We must explicitly use sum types for resumption and

In general rather than foo, which is not very interesting
by itself, iterate would be used with routines that yield
successive elements of a data structure (list, tree, etc.), or
perform some non-trivial computation only on demand (i.e., a
form of stream).

Moreover, the assumed argument types for iterate reflect the declared types

In addition, our core language cannot give iterate its own
type, but must instead define it as a macro: our core language
lacks new-prompt to declare fresh prompt tags, and even
with that, we would require type-level abstraction over tags
to give f a type. A full language implementation would need to
resolve these limitations, but for our current purposes the
macro approach is adequate.

\textsuperscript{15}The use of back-tick and comma here is how Racket (like Scheme)
exposes a shorthand for quasiquotation.

Figure 9 gives a version in our core language, assuming
an instantiation with mutable references (with the identity effect
for all uses) and a sum type, with type annotations. It
makes the distinctions mentioned above, aborting from inside
the inner initial prompt (the gen prompt) to the outer initial
prompt (the init prompt) to separate the result types, but
still return get-next to iterate’s client only after the
the resumption is initialized. As in Figure 7 the initially-
captured continuation used for the first call to get-next is
still (begin • (f yield finish) (finish)).

Before we discuss typing uses of iterate, let us consider
what a desirable typing would entail. First, note that the result of
a “call” to iterate is a closure (specifically get-next), and
assuming the underlying effect quantale ignores the reference
manipulation during initialization, the immediate effect of evaluating a use of iterate should be \((\emptyset, 0, I)\).

The assumed argument types for f reflect the declared types
for yield and finish. The main point to justify above is the
latent effect assumed for f. Consider E to be an upper bound
on the underlying effect of f’s body between two successive calls to yield\textsuperscript{16}.

Following our approach above, we can give the derived rule in
Figure 10 assuming all prophecies and control effects are related to the generator tag gen. Key to this derived rule above
is the fact that f’s body is constrained to have prophecies and
control effects related only to gen. Because all invocations of
f or continuations containing parts of f’s body occur under
prompts for gen, all of those effects are resolved inside calls
to get-next, leaving only the underlying effect. (This rule
does assume no other effects — such as aborts from exceptions — escape the body of the generator.) D-ITERATE also
permits the prophecy “emitted” by the continuation capture
for yield to vary between uses of D-ITERATE: \(P_p\) and \(C_p\)
are the prophecies and control effects that the uses of yield
for that particular iterator construction should predict. (The
GenProphs antecedent in D-ITERATE ensures the prophecy
is validated.)

With the derived rule in hand, we would hope that it is
precise enough to validate the intuitive effect for common
constructions.

Example 3 (Iterating a Pure Function). Consider the example
of a generator that always immediately yields the boolean
true. Under the assumptions made earlier (that reference
mutations are ignored by the effect system), we may derive:
\[
\Gamma \vdash \begin{array}{ll}
\text{iterate init gen } (\lambda y. f. [\text{loop (y true)])] & : \text{unit} \leftarrow \text{unit} \rightarrow \text{option bool} \mid (\emptyset, 0, I)
\end{array}
\]

This follows from the rule above because the loop body
(y true) has effect \(\chi\) as defined in Figure 7 which is essentially
the assumed latent effect for the yield argument, assuming
the presence of (eqi-irreversibly-defined prophecies. By the derived rule from Section 7.1.4 the overall loop then has
effect \(\chi^*\), which is then the latent effect of the function.
The conditions on \(C\) and \(P\) in D-ITERATE are then clearly satisfied with \(E = I\): the infinite union in the iteration above

\textsuperscript{16}Or between a yield and a final call to finish, or between finish and any “regular” return by f.
Fig. 9. A typed generator, parameterized (here implicitly) by two tags, init and get.

\[
\text{GenProphs}(P, \ell, E) = \\
\forall \ell', P', C', Q', P'', C'', Q'', \tau, \text{prophecy } \ell' (P', C', Q') \rightsquigarrow \tau \text{ obs } (P'', C'', Q'') \in P \Rightarrow \\
\ell' = \ell \land \tau = \text{bool } \land \text{GenProphs}(P', \ell, E) \land C'' \subseteq \{ \text{abort } \ell (\text{option } \tau)@E'' \} \land Q' \subseteq E''
\]

D-ITERATE

\[
\Gamma \vdash f : \\
(\text{iterate init gen f}) : \text{unit } \Rightarrow \\
(\text{iterate init gen f}) : \text{unit } \Rightarrow \\
(\text{iterate init gen f}) : \text{unit } \Rightarrow \\
C \subseteq \{ \text{abort } \ell (\text{option } \tau)@E'' \} \\
\text{GenProphs}(P, \text{gen}, E)
\]

Fig. 10. Derived rule for generators.

only creates prophecies satisfying GenProphs, in particular because all finite iterations of the effect produce prophecies less than the recursive prophecy (after the unfreezing).

To provide a bit more intuition for this, note that the prophecy set component \( P_\chi \) of \( \chi \) is itself less than the recursive prophecy \( P_{fix} \) predicted by \( P_\chi \subseteq P_{fix} \). This is enough to show that the prophecy component of \( \chi^2 \) (as shown in Figure 11) remains valid (in the sense of validEffects’ prophecy validation). This extends to any of the finite iterations introduced by the iteration operator of Section IV-A. Because \( P_{fix} \)’s recursively-defined observations are again \( P_{fix} \), then sequencing any finite iteration of \( \chi \) with itself yields a finite approximation of \( P_{fix} \). The approximation is \( C \subseteq P_{fix} \) in every case because at the point the approximation drops off with the “base case” observation \( (0,0,\ell, I) \), this is less than the observation in \( P_{fix} \) because the prophecy and control components are merely empty sets, and \( I \subseteq I \).

So the overall latent effect of the iterator produced by iterate is \( (0,0,\ell, I) \).

This example assumes recursive prophecies, not present in our initial presentation, and which up to now we have not required. While we can construct the semantics of any finite prefix of an execution by taking the union over all finite iterations per Section IV-A that construction ignores whether or not the observations in the resulting prophecies could actually be consistent with the predictions. In order to write an actual prediction that over-approximates these observations, we require recursive prophecies.

Example 4 (Iterating an Impure Looping Generator). One style of use for generators is to implement on-demand (pull) streams. In a concurrent setting this may be implemented in a way where on each request the generator takes a lock in order to find the next element to produce. To avoid unnecessary serialization, the lock must be released before yielding a new element, then reacquired. There are two ways to implement this. The first approach acquires, searches, and releases the lock on each call to get-next. In this case the body effect is pure as above. An alternative is to assume the caller holds the lock initially, and the iterator should release and reacquire the lock.

Consider iterating the following generator function, which presumes the existence of some auxiliary state in the environ-
let \( P_{\text{fix}} \equiv \{ \mu \mathcal{P}, \text{prophecy} \ \mathcal{G} \ \{ P \}, \{ \text{abort gen bool@I} \}, I \} \text{obs} \{ \{ P \}, \{ \text{abort gen bool@I} \}, I \} \rightsquigarrow \text{unit} \)

\[
P_{\text{fix}} \equiv \{ \text{prophecy} \ \mathcal{G} \ \{ P_{\text{fix}}, \{ \text{abort gen bool@I} \}, I \} \text{obs} \{ P_{\text{fix}}, \{ \text{abort gen bool@I} \}, I \} \rightsquigarrow \text{unit} \]

\[
\chi \equiv \{ \text{prophecy} \ \mathcal{G} \ \{ P_{\text{fix}}, \{ \text{abort gen bool@I} \}, I \} \text{obs} \{ 0, 0, I \}, \{ \text{abort gen option bool@I} \}, I \}
\]

\[
P_{\chi}^2 \equiv \{ \text{prophecy} \ \mathcal{G} \ \{ P_{\text{fix}}, \{ \text{abort gen bool@I} \}, I \} \text{obs} \{ 0, 0, I \} \} \triangleright \chi
\]

\[
\chi \equiv \{ \text{prophecy} \ \mathcal{G} \ \{ P_{\text{fix}}, \{ \text{abort gen bool@I} \}, I \} \text{obs} \{ P_{\chi}, C_{\chi}, Q_{\chi} \} \}
\]

Fig. 11. The effect of the infinite loop body that always immediately yields a value.

\[
(\lambda \ (\text{yield finish}) \ (\begin{align*}
\text{begin} & (\text{while} \ (\text{not-done?}) \\
& \text{begin} \ (\text{cond \ wait \ l}) \ \\
& \text{obs} \ (\text{cond \ wait \ l})
\end{align*}))
\]

The underlying effect of the body would be \( \{(l, l)\}^\ast = \{(l, l)\} \) in the underlying locking effect quantale (it begins assuming \( l \) is held, and finishes assuming \( l \) is held). This effect is observed in the execution prefix preceding both the call to \( \text{finish} \) and prior to “each” call to \( \text{yield} \). The loop body has nearly the same effect as the body in the previous example, but with some uses of \( I \) replaced by \( \{(l, l)\} \), per Figure 12. The same argument for the prophecies always remaining valid extends to this case, taking advantage of the fact that \( \{(l, l)\}^\ast = \{(l, l)\} \). This makes the latent effect of the generator function for the above also \( \{(l, l)\} \).

Because \( \text{D-ITERATE} \) is a derived rule in a sound type-and-effect system, we know it is sound. We could also go beyond the rule \( \text{D-ITERATE} \) above, which assumes the only control effects and prophecies escaping the body of \( \varepsilon \) are related to the generator infrastructure. This is naturally not always the case — in a language like \#t, the body of a generator can throw exceptions. We do not explore the details of deriving rules for such combinations here, but the same methodology employed thus far still applies to that case.

VII. Soundness

We have proven syntactic type soundness for the type system presented in Section 17. We follow Gordon [25, 20] in giving soundness for a language parameterized by abstract states \( \sigma \) with state types \( \Sigma \), along with primitives \( p_i \) that may manipulate the state, subject to some consistency assumptions on the interactions of the primitives, states, and the assumed types of primitives. Progress is uninteresting (standard as long as primitives satisfy progress), but preservation is slightly non-standard:

Type Preservation: If \( \vdash \sigma : \Sigma \) and \( \Sigma ; \Gamma \vdash e : \tau \mid \chi \), then either \( e \) is a value or \( \sigma ; e \vdash q_1, \ldots, q_n \vdash \sigma' ; e' \) and there exists a \( \Sigma' \), \( \tau' \), \( \chi' \) such that:

- \( \Sigma \leq \Sigma' \)
- \( \vdash \sigma' : \Sigma' \)

\( \text{17} \)Technically this code should contain an inner loop because condition variable implementations permit spurious wake-ups, for a variety of reasons, including that even if the intended condition was true when the sleeping thread was signalled, it may have been falsified again before the thread has a chance to execute again. This is a simplified example to focus on the effects.

\[
\sum_{\tau} ; \Gamma \vdash e : \tau' \mid \chi'
\]

\[
\tau' < : \tau
\]

\[
(0, 0, q_1) \triangleright \ldots \triangleright (0, 0, q_n) \triangleright \chi' \subseteq \chi
\]

\[\text{Proof. See Lemma 6 in Appendix C} \]

This is mostly standard, except for the last conclusion (which is standard for syntactic type soundness for sequential effect systems). Essentially, this says that the static effect \( \chi \) of an expression over-approximates the sequencing of the effect of any finite execution with the static effect of the remaining expression. In the special case where \( e' \) is a value, \( \chi \) over-approximates the effect of the entire execution.

The general structure is mostly standard for semantics based on evaluation contexts, except for the addition of an explicit context typing judgment \( \sum, \Gamma \vdash E :: \tau/\chi \rightsquigarrow \tau'/\chi' \) used both for manipulating contexts in proofs, and to give types to continuation values — which contain contexts with holes. A key step in the proof, used for ensuring that prophecy validation is sound, is a separate lemma validating prophecies:

Valid Prophecies: For any \( \Sigma, \Gamma, E, \tau, \chi, \chi', \sigma, \ell, \chi_\ell, \gamma \), if

- \( \sum, \Gamma \vdash E :: \tau/\chi \rightsquigarrow \sigma/\chi' \) and
- \( \text{prophecy} \ \ell \ \chi_\ell \rightsquigarrow \gamma \text{ obs} \ (0, 0, I) \in P_\chi \)
- \( E \) contains no prompts for \( \ell \)

then there exists a \( P, C, Q \) such that:

- \( \text{prophecy} \ \ell \ \chi_\ell \rightsquigarrow \gamma \text{ obs} \ (P, C, Q) \in P_\chi' \)
- \( \sum, \Gamma \vdash E :: \tau/I \rightsquigarrow \sigma/(P, C, Q) \)

\[\text{Proof. By induction on } E. \text{ See Lemma 2 in Appendix B.3} \]

A. Syntactic vs. Semantic Soundness

While this is a type safety result, note that due to the syntactic nature and agnosticism over the particular underlying effect quantale, this is effectively a coherence result for sequential effect systems. No part of this imparts \textit{semantic} meaning to effects, beyond the way in which the use of \( \subseteq \) in the last conclusion suggests the effect is a form of predicate on effect traces. This is adequate for some properties (e.g., finite trace effects [37, 53]), but insufficient for others which require further proofs of semantic properties. For example, Flanagan and Qader [21] prove a further lemma about atomicity to show their effect system actually enforces atomicity, and Gordon et al. [27] require a separate proof beyond type safety to show their deadlock freedom effect system ensures deadlock freedom.

Ignoring the syntactic nature of soundness leads to counter-intuitive misunderstandings. Consider an effect quantale with
3 elements — Total ⊆ Partial ⊆ T — intended to model total or partial computations. If sequencing simply takes least-upper bound with respect to the partial order (≥ = ⊔), this is a valid effect quantale with Total as the identity. But Gordon’s iteration operator will set Total = Total, suggesting that infinite loops of “Total” actions are Total. This is because the soundness proof does not account for what each effect should mean, and the syntactic effect Total is not semantically tied to termination. Notice that this effect quantale is isomorphic to one that simply expresses whether or not a computation uses termination. Notice that this effect quantale is isomorphic to one that simply expresses whether or not a computation uses reflection: NoReflection ⊆ Reflection ⊆ T.

Rather than being a limitation of syntactic type soundness, this is a result of working with respect to an arbitrary sequential effect system, and is a caveat common to all work on abstract effect systems, including related sequential models (joinads [43], graded monads [35], and productors [55]) and abstract models of commutative effect systems [42]. Incorporating appropriate effect-system-specific soundness into any of these abstract approaches in a general way is an important area of future work.

VIII. RELATED WORK

Here we briefly recall other related work not covered in Section III.

a) Sequential Effect Systems: The past few years have seen great progress on semantic models for sequential effect systems [35, 43, 55], centering on what are now known as graded monads. These are monads indexed by some kind of monoid (to model sequential composition), commonly a partially-ordered monoid following Katsumata [35]. Most of this work, however, emphasizes various necessary features of sequential effect systems, rather than models sufficient for capturing source-level sequential effect systems. Gordon [25] shifted focus to capturing sufficient detail to model prior work, leading to the first abstract characterization of sequential effect systems that included singleton effects, effect polymorphism, and iteration of sequential effects. Gordon’s work is syntactic — like ours — though the semantic approaches to singletons and polymorphism have been well-explored separately from effect systems.

Another smaller piece of the puzzle is the use of accumulators. To the best of our knowledge we are the first to use the term “accumulator” as we do, and the first to use right-accumulators, but left-accumulators have appeared at least once before. Koskinen and Terauchi [37] use a form of left-accumulator in their effect system for safety and liveness properties (requiring an oracle for liveness). Effects in their system are a pair of sets, one a set of finite traces (for terminating executions) and the other a set of infinite traces (for non-terminating executions. The set of infinite traces acts as a left-accumulator: code that comes after a non-terminating expression in program-order never runs. On the other hand, finite executions from code before an infinite execution extend the prefix of the infinite executions.

b) Effects and Continuations: Effect systems treating continuations are nearly as old as effect systems themselves [35]. To the best of our knowledge, we are the first to consider the integration of sequential effects with exceptions, generators, or continuations — any control flow construct beyond while loops. We are also the first we know of to consider the combination of any effect system with tagged delimited continuations. The primary motivation for tagging was to prevent encodings of separate control operators from interfering with each other [49]; without tagging, for example, a throw inside a loop would abort to the loop boundary, rather than a try-catch enclosing the loop. Without treating multiple tags/prompts, we could not have typed the example programs from Section VI-F that combined generators and loops. For most purposes, ignoring tags in the theory (formalizing only untagged prompts) and treating them in the implementation is a workable solution. In our case, the tags are indispensable: tracking execution order with multiple tags together is non-trivial so we needed to work out the multiple-tag handling in detail; this is the reason we require a general notion of prophecy freezing in the system, rather than a simpler solution to only address continuations capturing their own invocation. Tagged delimited continuations can macro-express untagged undelimited continuations, but not vice versa [49], making untagged delimited continuations unsuitable for our purpose: deriving type rules for new control operators macro-expressed in terms of more primitive operators. This choice of more expressive control operators is also a key distinction between our work and some of the most closely related work.

Tov and Pucella [57] examined the interaction of untagged delimited continuations with substructural types (a co-effect [45]). Delbianco and Nanevski adapted Hoare Type Theory for untagged algebraic continuations [13]. The particular form of prompt and abort they study places handlers at the site of an abort, rather than at the prompt, in order to satisfy some useful computational equalities (see below). As a consequence, encoding non-trivial control flow constructs in their system becomes significantly more complex: for example, simulating the standard semantics of throwing exceptions to the nearest enclosing catch block for the exception type would require catching, dispatching, and re-throwing at every prompt. This adds another way, beyond lack of tags, that their approach would not support compositional study of multiple control flow constructs.

c) Algebraic Effects: Algebraic effects with handlers [46] are a means to describe the semantics of effects in terms of a set of operations (the effectful operations) along with handlers that interpret those operations as actions on some resource. The combination yields an algebra characterizing equality of different effectful program expressions, hence the
term “algebraic”. Languages with algebraic effects include an effect system to reason about which effects a computation uses. Some implementations even use row polymorphism [58] to support effect inference [39]. Handlers for algebraic effects receive both the action to interpret and the continuation of the program following the effectful action. Thus they can implement many control operators, including generators and cooperative multithreading [40], as with the delimited continuations we study. Computationally, they are quite powerful, but the general form of continuations we study is more expressive [33]. The effect systems considered for algebraic effects thus far have only limited support for reasoning about sequential effects.

The types given for individual algebraic effects do support reasoning about the existence of a certain type of resource before and after the computation [5, 10]. However, the way this is done corresponds to a parameterized monad [4], which Tate [55] showed crisply do not include all meaningful sequential effect systems. Every parameterized monad’s algebra of sequencing and lifting can be described as an effect quantale (they are quite similar to the recursive monad acquisition effect quantale given by Gordon [25]). However, there are effect quantales which cannot be represented as parameterized monads, and therefore express finer-grained distinctions on program order than those imposed by effect systems for algebraic effects. Examples include effect quantales for atomicity (see Gordon [25]) and trace effects [37, 52, 53].

There is no reason to believe this is a fundamental limitation of the algebraic effects approach. Rather that general considerations of sequential effect systems have just not yet been explored for algebraic effects. When it is considered, it seems likely ideas from our development (particularly prophecies) will be useful. For example, Dolan et al. [14] offer two reasons for dynamically enforcing linearity of continuations in their handlers: performance, but also avoiding the sorts of errors prevented by sequential effect systems, such as closing a file twice by reusing a continuation.

\(d\) Derived Typing Rules: The approach we take to deriving type rules for control flow constructs and control operators is reminiscent of work done in parallel with ours by Pombrio and Krishnamurthi [43]. They address the problem of producing useful type rules when a language semantics and type rules are defined directly for a simpler core language, and a full source language is defined using syntactic sugar (i.e., macros) that expand into core language expressions with the intended semantics, such as the approach taken by \(\lambda_{IE} [30]\). There the issue is that type errors given in terms of the elaborated core terms are difficult to understand for developers writing in the unelaborated source language. Pombrio and Krishnamurthi offer an approach to automatically lift core language type rules through the desugaring process to the source language, providing sensible source-level type errors. Their work focuses on type systems without effects, but including such notions as subtyping and existential types. They do not consider control operators (delimited continuations) or effects (neither commutative nor sequential). Extending their approach to support the language features and types (effects) we consider would make our approach more useful to effect system designers, though this is non-trivial due to the many ways to combine sequential effects.

IX. CONCLUSIONS

We have given the first characterization of how to integrate sequential effect systems with tagged delimited control operators. We have used this characterization to derive sequential effect system rules for standard control flow structures macro-expressed via continuations, including deriving known forms (loops) and giving the first characterization of exceptions and generators in sequential effect systems.

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APPENDIX A
HYPOTHETICAL TYING DERIVATION FOR INFINITE LOOPS

Figure 13 gives a hypothetical tying derivation for an infinite loop, assuming no control effects escape the loop’s body. Choosing as in Section VI-A, \( Q_p = Q_e^p \) and \( C_p = \{ \text{replace } \ell : Q_e^p \rightarrow \text{unit} \} \) makes this derivation valid. In that case, the final underlying effect in the derivation is equal to \( Q_e \cup (Q_e^p) = Q_e^p \).

Note that the derivation is already partly specialized, based on the assumption that no prophecies need to be predicted by the callcc’s effect (i.e., the prophecy set of the continuation prophecy is already assumed to be \( \emptyset \) in Figure 13). This is satisfactory because if we hypothesized some \( P_p \) as an unknown prophecy set for the continuation’s latent effect, the validEffects derivation would check, as part of validating the prophecy’s observations against its predictions, that \( \emptyset \subseteq P_p \) (because the prophecy observes the invocation of the continuation). Clearly choosing \( P_p = \emptyset \) satisfies this additional constraint.

APPENDIX B
CONTEXT TYING AND SUBSTITUTIONS

For syntactic type safety, we must give types to terms that exist only at runtime, which include reified continuations. For this we introduce the evaluation context type \( \tau / \chi \rightarrow \tau' / \chi' \), which characterizes an evaluation context with a hole of type \( \tau \), which when filled by an expression of appropriate type and effect at most \( \chi \) yields an expression producing \( \tau' \) with overall effect \( \chi' \). This is only used by the context-tying judgment \( \Sigma; \Gamma \vdash E : \tau / \chi \rightarrow \tau' / \chi' \), which has no effect of its own, because evaluation contexts do not appear in expression positions during evaluation. This judgment plays both a convenient administrative role in the soundness proof, and a role in tying the runtime form of continuations.

Note that we have avoided explicitly tracking a notion of latent effect for a continuation while typing the main program expression. There are two reasons for this. First, doing this explicitly would make the type rules significantly more complex due to the need to identify various contexts and associate latent effects to them. Second, it is unnecessary in the presence of prophecies: the observations made by a prophecy capture the latent effect between the point of the continuation capture and the enclosing prompt of the same tag. And these are the only continuations for which a latent effect is useful. Our use of prophecies permits the inference of these latent effects from the characterization above, in cases where they are required (see Lemma 2).

The context typing judgment is defined in parallel with typing derivations, one case for each possible way of typing an evaluation context. For example, there are two rules for typing the function-hole contexts:

\[
\begin{align*}
\text{T-CtxFunApp} & : & \Sigma; \Gamma \vdash E' : \tau / \chi \rightsquigarrow (\sigma \triangleleft \sigma') / \chi' & \quad \Sigma; \Gamma \vdash e : \sigma | \chi_e \\
\end{align*}
\]

The other context typing rules are defined similarly, each effectively exchanging one inductive hypothesis for a recursive context typing hypothesis with the same type and effect for the hole. The base case is the natural rule for the hole, requiring that the type and effect of the value plugged into the hole are subtype or subeffect of the of the “plugged” context’s type (since plugging an expression into the empty context is simply that expression):

\[
\begin{align*}
\text{T-CtxHole} & : & \tau \rightarrow \tau' & \quad \chi \subseteq \chi' \\
\end{align*}
\]

The context typing judgment is defined mutually with the term typing, as the type rule for continuation values refers back to the context typing judgment:

\[
\begin{align*}
\text{T-ContC} & : & \Sigma; \Gamma \vdash E : \tau' & \quad C_0 \subseteq C \quad Q_0 \subseteq Q \\
\end{align*}
\]

A. Context Decomposition

Because the preservation proof will destruct full expressions into evaluation contexts and redexes, and we will require both local typing information about the redex and information about replacing the redex within the context, we must be able to decompose the typing derivation of a (filled) evaluation context in parallel with the operational semantics.

Lemma 1 (Context Typing Decomposition). If

\[
\begin{align*}
\Sigma; \Gamma \vdash E[e] : \tau | \chi \\
then there exists a \( \tau', \chi' \), such that:
\end{align*}
\]

\[
\begin{align*}
\Sigma; \Gamma \vdash e : \tau' | \chi' \\
\Sigma; \Gamma \vdash E : \tau' / \chi' \rightarrow \tau | \chi \\
\end{align*}
\]

Proof. By induction on \( E \) (with other variables universally quantified in the inductive hypothesis).

- Case \( E = \cdot \): Here \( E[e] = e \), so \( \tau' = \tau \), \( \chi' = \chi \), and \( \chi'' = \cdot \).

- Case \( E = (E'[e] : e') \): Here we have two cases, for application of functions or application of continuations.

  We present the function application case; the continuation application is similar. Given:

  \[
  - \Sigma; \Gamma \vdash E'[e] : \tau' | \chi' \\
  \]

  By inversion on the typing derivation, for the function application case:

  \[
  - \Sigma; \Gamma \vdash E'[e] : \tau' \quad \chi_e \rightarrow \tau | \chi_f \\
  \]
\[ C = (Q_e \triangleright C_p) \cup \{ \text{replace } \ell : Q_e \triangleright Q_p \rightsquigarrow \text{unit} \} \]
\[ \mathcal{J} = \{ \mathcal{J}_\ell \} \]
\[ \mathcal{J}_\ell = \{ \ell \text{ contains no prompts for } \ell \} \]

**Prophecy and Implicit Effect**

- \( \Sigma; \Gamma \vdash \ell : \tau \rightarrow \chi_e \)
- \( \ell \) is a prophecy.
- Notice that post-composing an effect after one with the next expression to reduce up to an enclosing prompt. In evaluation contexts, in a non-local manner. Intuitively, a use of prophecies to capture the residual effect of various B. Valid Prophecies

**Case**:
- \( \Sigma; \Gamma \vdash E : \tau' \rightarrow \chi' \)
- \( \Sigma; \Gamma \vdash E' : \tau' \rightsquigarrow (\tau_e, \chi_{\text{latent}}) \rightarrow \tau) / \chi_f \)

We can then invent on context typing and use T-APP to prove

\[ \Sigma; \Gamma \vdash E : \tau' / \chi' \rightsquigarrow \tau / \chi \]

The inversion on typing produces a second case, for applying continuations, which proceeds similarly.

- **Case** \( E = \{ v \ E'[e] \} \): Similar to the other function application context.
- **Case** \( E = (\% \ E'[e]) \): Similar to previous cases.
- **Case** \( E = (\text{call/cc } \ell E'[e]) \): Similar to previous cases.

**B. Valid Prophecies**

One of the most subtle parts of the effect system is the use of prophecies to capture the residual effect of various evaluation contexts, in a non-local manner. Intuitively, a prophecy captures all possible effects from the point of the prophecy (the point where the continuation capture would be the next expression to reduce) up to an enclosing prompt. In particular, notice that post-composing an effect after one with a prophecy performs the same "transformations" on the C and Q components of the prophecy as on those components of the actual effect, effectively type-checking a context twice simultaneously. The lemma below makes the intuition precise, and shows that the type system in fact matches that intuition.

**Lemma 2** (Valid Prophecies). For any \( \Sigma, \Gamma, E, \tau, \chi, \chi', \sigma, \ell, \chi_e, \gamma, \) if

- \( \Sigma; \Gamma \vdash E : \tau / \chi \rightsquigarrow \sigma / \chi' \) and
- prophecy \( \ell \chi_e \rightsquigarrow \gamma \) obs \((0, 0, I) \in \mathcal{P}_x \)

- \( E \) contains no prompts for \( \ell \)


then there exists a \( P, C, \) and \( Q \) such that:

- prophecy \( \ell \chi_e \rightsquigarrow \gamma \) obs \((P, C, Q) \in \mathcal{P}_x \)
- \( \Sigma; \Gamma \vdash E : \tau / \chi \rightsquigarrow \sigma / (P, C, Q) \)

**Proof.** By induction on \( E \). We present a demonstrative inductive case and the one interesting case.

- **Case** \( E = (E' e) \): Inversion on the context typing produces a case for function application, and a case for continuation application. We show the former; the latter is similar. By that inversion and the inductive hypothesis:

- prophecy \( \ell \chi_e \rightsquigarrow \gamma \) obs \((P', C', Q') \in \mathcal{P}_x \)
- \( \Sigma; \Gamma \vdash E' : \tau' / \chi_e \rightsquigarrow (\tau_e, \chi_{\text{latent}}) \rightarrow \tau) / \chi_f \)

Applying T-CtxtxtFunApp with the "plugged" type for \( E' \) produces the expected result type (\( \sigma \)) and a result effect

\[ (P, C, Q) \triangleq (P', C', Q') \triangleright \chi_e \triangleright \chi_f \]

And given the prophecy observing \((P', C', Q') \) in \( \mathcal{P}_x \), we know the \((P, C, Q) \) above is present in \( P_E, P_E' \triangleright \chi_e \triangleright \chi_f \) will contain

- prophecy \( \ell \chi_e \rightsquigarrow \gamma \) obs \((P', C', Q') \triangleright \chi_e \triangleright \chi_f \)

- **Case** \( E = (\% \ E'[h]) \): By assumption, \( \ell \neq \ell' \). By the inversion on context typing and the inductive hypothesis:

- prophecy \( \ell \chi_e \rightsquigarrow \gamma \) obs \((P', C', Q') \in \mathcal{P}_x \)
- \( \Sigma; \Gamma \vdash E' : \tau' / \chi_e \rightsquigarrow (\tau_e, \chi_{\text{latent}}) \rightarrow \tau) / \chi_f \)
- \( P_E = P_E' \setminus \ell' \)
- \( C_E = C_E' \setminus \ell' \)
- \( Q_E = Q_E' \setminus \bigcup C[Q_i] \)
- validEffects\((P_{E'}, C_{E'}, Q_{E'}, \ell', \sigma, \sigma') \) (\( \sigma' \) is the argument type of the handler)

Note that the changes from the body effect to prompt effect imposed by T-CtxtxtPrompt preserve the prophecy of interest (suitably modified itself). For choices:

- \( P = P' \setminus Q_k \ell' \)
- \( C = C' \setminus Q_k \ell' \)
- \( Q = Q' \setminus \bigcup (C')[Q_i] \)
Therefore, let $\chi'' = \chi_f \triangleright \chi_e \triangleright \gamma$. Then the typing result for $E[e]$ holds by T-APP (note we’ve suppressed the details of handling the choice of $e'$ being a value or the function having non-dependent type). For the subeffect obligation:

- $\Sigma; \Gamma \vdash q \triangleright \chi'' = q \triangleright \chi_f \triangleright \chi_e \triangleright \gamma \subseteq \chi_f \triangleright \chi_e \triangleright \gamma = \chi' \triangleright \gamma$

follows from the definition of $\chi''$, the definition of $\chi'$ from inversion, and the inductive result.

- Case $E = (\nu E')$: Similar to other cases.
- Case $E = (\% \ell E' \ h)$: By inversion on context typing

\[
\begin{align*}
\Sigma; \Gamma & \vdash E' :: \tau / \chi \triangleright \tau' / \chi_b \\
\chi_b & = (P_b, C_b, Q_b) \\
\chi' & = (P_b \setminus \ell, C_b \setminus \ell, Q_b \sqcup \bigcup C_b|_{Q_b}) \\
\Sigma; \Gamma & \vdash b : \sigma \triangleright (\emptyset, b, Q_b) \triangleright \tau' \setminus I \\
\Sigma; \Gamma & \vdash \text{validEffects}(P_b, C_b, Q_b, \ell, \tau', \sigma)
\end{align*}
\]

By the inductive hypothesis:

- $\Sigma; \Gamma \vdash E'[e] : \tau' | \chi'_b$
- $\Sigma; \Gamma \vdash q \triangleright \chi''_b \subseteq \chi_b$

Destructure $\chi''_b$ as $(P'_b, C'_b, Q'_b)$. Let $\chi'' = (P'_b \setminus \ell, C'_b \setminus \ell, Q'_b \sqcup \bigcup C_b|_{Q_b})$. Because $\Gamma \vdash q \triangleright (P'_b, C'_b, Q'_b) \subseteq (P_b \setminus \ell, C_b \setminus \ell, Q_b \sqcup \bigcup C_b|_{Q_b})$.

- Case $E = (\text{call/cc } \ell E')$: Similar to other cases.

\section*{Appendix C

\section*{Soundness}

We prove syntactic type soundness for a lightly type-annotated source language. We require some type annotations because the dynamic semantics form a labelled transition system where the label is the effect of the reduction. Because some of the control effects contain types, we must add those to the term language. Specifically:

- \textit{abort}$^\sigma$ is the abort operator labelled with the type of the value passed to the handler.
- \textit{call/cc}$^\chi$ "predicts" a continuation result type (the type of the enclosing prompt) of $\sigma$ and latent continuation effect of $\chi$ (as in the continuation type, this is a flattened underlying effect)

In each case, the runtime typing rule is modified to enforce the correct relationship between the term and the type. These type-annotations are straightforward to produce from a valid source typing derivation.

The structure of our proof borrows heavily from Gordon’s modular proof of type safety for a lambda calculus defined with respect to an unspecified effect quantale, where the language is parameterized by a selection of primitives, new values, new types, a notion of state, state types, and operations and relations giving types to the new primitives, values, and states. The proof is also parameterized by a lemma that amounts to one-step type preservation for executing fully-applied primitive operations — these are the only parts of the language that may modify the state (the rest of the framework...
is defined without knowledge of the state’s internals). Sufficient restrictions are placed on these parameters to ensure various traditional lemmas continue to hold (for example, ensuring that the primitives do not add a third boolean, so the typical Canonical Forms lemma stands).

We impose an additional requirement on the parameters for primitive typing, beyond what Gordon requires. Whereas Gordon requires that for any primitive, only its final effect is non-I, we also require that all control behaviors and freeze sets for any primitive added are empty. This ensures that all primitives are in fact local operations. We retain Gordon’s model of dynamic primitive behavior, which reduces fully-applied primitives to a new value, transforms the (pluggable) state, and produces a dynamic effect — in the underlying effect quantale, rather than the control-enhanced one.

**Lemma 4 (Redex Preservation).** If
- \( \vdash \Gamma \)
- \( \vdash \sigma : \Sigma \)
- \( \Sigma; \Gamma \vdash e : \tau \mid \chi \)
- \( \sigma; e \not\Rightarrow \sigma'; e' \)
then there exists \( \Sigma' \) and \( \chi' \) such that
- \( \Sigma \leq \Sigma' \)
- \( \vdash \sigma' : \Sigma' \)
- \( \Sigma'; \Gamma \vdash e' : \tau \mid \chi' \)
- \( q \triangleright \chi' \subseteq \chi \)

**Proof.** By induction on the reduction step, followed by inversion on the typing derivation in each case. The proof is nearly identical to Gordon’s proofs beyond the addition of equivariant recursive types, non-dependent products and sum types, and subsumption. □

**Lemma 5 (Context Substitution).** If
- \( \Sigma; \Gamma \vdash E : \tau/\chi \rightsquigarrow \sigma/\chi' \)
- \( \Sigma; \Gamma \vdash e : \tau \mid \chi \)
then \( \Sigma; \Gamma \vdash E[e] : \tau \mid \chi' \)

**Proof.** By straightforward induction on the context typing. □

Finally, we are ready to tackle the central preservation lemma.

**Lemma 6 (Context Reduction).** If
- \( \vdash \Gamma \)
- \( \vdash \sigma : \Sigma \)
- \( \Sigma; \Gamma \vdash e : \tau \mid \chi \)
- \( \sigma; e \not\Rightarrow \sigma'; e' \)
then there exists a \( \Sigma' \) such that
- \( \Sigma \leq \Sigma' \)
- \( \vdash \sigma' : \Sigma' \)
- \( \Sigma'; \Gamma \vdash e' : \tau \mid \chi' \)
- \( q \triangleright \chi' \subseteq \chi \)

**Proof.** By induction on the derivation of \( \sigma; e \not\Rightarrow \sigma'; e' \). In each case below other than the case for the context reduction (E-CONTEXT), nothing will change the state. So in all cases other than E-CONTEXT, we choose \( \Sigma \overset{def}{=} \Sigma \), and the semantics ensure \( \sigma = \sigma' \), and so in all cases \( \vdash \sigma' : \Sigma' \equiv \vdash \sigma : \Sigma \). Thus we present only the expression- and effect-related details below. Moreover, we continue to use \( \sigma \) as an additional meta-variable for states, to minimize the number of prime marks that must be counted by the reader (or author) of the proof.

- **Case E-CONTEXT:** This case follows from context decomposition (Lemma 1), redex reduction (Lemma 4), and the reduct effects lemma (Lemma 3). In this case,
  - \( \sigma; e \not\Rightarrow \sigma'; e' \)
  - \( \Sigma; \Gamma \vdash E[e] : \tau \mid \chi \)
  - \( \vdash \Gamma \)
  - \( \vdash \sigma : \Sigma \)

By context decomposition (Lemma 1):
- \( \Sigma; \Gamma \vdash e : \tau_e \mid \chi_e \)
- \( \Sigma; \Gamma \vdash E : \tau_e/\chi_e \rightsquigarrow \tau/\chi \)

By redex reduction (Lemma 4), there exist \( \Sigma' \) and \( \chi_e \) such that:
- \( \Sigma \leq \Sigma' \)
- \( \vdash \sigma' : \Sigma' \)
- \( \Sigma' ; \Gamma \vdash e' : \tau_e \mid \chi' \)
- \( \Sigma' ; \Gamma \vdash q \triangleright \chi' \subseteq \chi_e \)

By weakening and repetition of context decomposition:
- \( \Sigma' ; \Gamma \vdash E : \tau_e/\chi_e \rightsquigarrow \tau/\chi \)

Then by the reduct effects lemma (Lemma 3), we can derive the appropriate result for some \( \chi' \):
- \( \Sigma' ; \Gamma \vdash E[e'] : \tau \mid \chi' \)
- \( \Sigma' ; \Gamma \vdash q \triangleright \chi' \subseteq \chi_e \)

- **Case E-ABORT:** In this case,
  - \( e = E[\langle \ell, h \rangle] \wedge (\ell \neq \langle \ell, h \rangle) \),
- \( q = I \)
- \( e' = E[h \ell] \)
- \( E' \) contains no prompts for \( \ell \)

By context decomposition (Lemma 1), there exist \( \tau_{prompt}, \chi_{prompt} \) where:
- \( \Sigma; \Gamma \vdash \langle \ell, h \rangle : \tau_{prompt} \mid \chi_{prompt} \)
- \( \Sigma; \Gamma \vdash E : \tau_{prompt}/\chi_{prompt} \rightsquigarrow \tau/\chi \)

By inversion on the prompt’s typing:
- \( \Sigma; \Gamma \vdash E[\langle \ell, h \rangle] : \tau_{prompt} \mid (P_{\text{body}}, C_{\text{body}}, Q_{\text{body}}, h) \rightarrow \tau_{prompt} \mid (\emptyset, \emptyset, I) \)
- \( \Sigma; \Gamma \vdash h \mid \sigma_h \rightarrow \tau_{prompt} \mid (\emptyset, \emptyset, I) \)
- \( \Sigma; \Gamma \vdash \text{validEffects}(P_{\text{body}}, C_{\text{body}}, Q_{\text{body}}, \ell, \tau_{prompt}, \sigma_h) \)
- \( \chi_{prompt} = (P_{\text{body}} \setminus Q_h_\ell, C_{\text{body}} \setminus Q_h_\ell, Q_{\text{body}} \cup C_{\text{body}} \ell \setminus Q_h_\ell) \)

Applying context decomposition again to \( E' \) and its contents:
- \( \Sigma; \Gamma \vdash \langle \ell \rangle : \tau_{abort} \mid \chi_{abort} \)
- \( \Sigma; \Gamma \vdash E' : \tau_{abort}/\chi_{abort} \rightsquigarrow \tau_{prompt}/\chi_{prompt} \)

By inversion on the typing of \( \text{abort} \) and value typing:
- \( \Sigma; \Gamma \vdash v : \sigma \mid I \)
- \( \Sigma; \Gamma \vdash \langle \{0, \lambda \ell. \sigma @I \}, I \rangle \subseteq \chi_{abort} \)
Because abort effects for \( \ell \) are propagated through \( E' \) (which contains no \( \ell \)-prompts) without accumulating new prefixes (since \( E' \) is an evaluation context):
- \( \text{abort } \ell \sigma \emptyset I \in C_{body} \)

By inversion on the validEffects conclusion above:
- \( \Sigma; \Gamma \vdash \sigma <: \sigma_h \)

Then by T-APP:
- \( \Sigma; \Gamma \vdash h \vdash: \tau_{prompt} | (\emptyset, \emptyset, Q_h) \)

This is a subeffect of \( \chi_{prompt} \), because \( (I \triangleright Q_h) \in C_{body}|_{Q_h} \) as a result of the abort control effect. This allows the use of context substitution (Lemma 5) to place this result back in \( E \).

- **Case E-CALLCC:** In this case:
  - \( e = E'[\% \ell \ E'[\langle \text{call/cc}_\chi \ell k \rangle] h)] \)
  - \( q = I \)
  - \( e' = E'[\% \ell \ E'[\langle \text{call/cc}_\chi \ell k \rangle] h)] \)

By context decomposition (Lemma 1):
- \( \Sigma; \Gamma \vdash (\% \ell \ E'[\langle \text{call/cc}_\chi \ell k \rangle] h) : \tau_{prompt} \mid \chi_{prompt} \)
- \( \Sigma; \Gamma \vdash E : \tau_{prompt} / \chi_{prompt} \nRightarrow \tau / \chi \)

By inversion on the prompt typing:
- \( \Sigma; \Gamma \vdash E'[\% \ell \ E'[\langle \text{call/cc}_\chi \ell k \rangle] h) : \tau_{prompt} \mid \chi_{prompt} \)
- \( \Sigma; \Gamma \vdash \text{validEffects}(P_{\text{body}}, C_{\text{body}}, Q_{\text{body}}, \ell, \tau_{prompt}, \sigma_h) \)
- \( \chi_{prompt} = (P_{\text{body}} \setminus \ell, C_{\text{body}} \setminus \ell, Q_{\text{body}} \cup \bigcup C_{\text{body}}|_{Q_h}) \)

Applying context decomposition again to \( E' \) and its contents:
- \( \Sigma; \Gamma \vdash (\text{call/cc}_\chi \ell k) : \tau_{hole} \mid \chi_{hole} \)
- \( \Sigma; \Gamma \vdash E' : \tau_{hole} / \chi_{hole} \nRightarrow \tau_{prompt} / \chi_{prompt} \)

where \( k \) is a value. By inversion on the call/cc typing and value typing:
- \( \Sigma; \Gamma \vdash k : (\text{cont } \ell \tau_{hole} \chi_{proph} \sigma) \nRightarrow (\emptyset, \emptyset, I) \)
- \( \chi_{hole} = \chi k \nRightarrow (\{ \text{prophecy } \ell \chi_{proph} \nRightarrow \sigma \} \emptyset, \emptyset, I) \)

By the valid prophecies lemma (Lemma 2):
- \( \text{prophecy } \ell \chi_{proph} \nRightarrow \sigma \text{ obs } (P_{\text{obs}}, C_{\text{obs}}, Q_{\text{obs}}) \in P_{\text{body}} \)
- \( \forall v, \Sigma; \Gamma \vdash v : \tau_{hole} | (\emptyset, \emptyset, I) \rightarrow \Sigma; \Gamma \vdash E[v] : \tau_{prompt} | (P_{\text{obs}}, C_{\text{obs}}, Q_{\text{obs}}) \)

From the validEffects assumption above, conclude:
- \( P_{\text{obs}} \subseteq P_{\text{proph}} \)
- \( C_{\text{obs}} \subseteq C_{\text{proph}} \)
- \( Q_{\text{obs}} \subseteq Q_{\text{proph}} \)
- \( \Sigma; \Gamma \vdash \tau_{prompt} \nRightarrow \sigma \)

These are exactly the requirements for the typing of continuation values via T-CONTC, thus:
- \( \Sigma; \Gamma \vdash (\text{cont } \ell \ E') : \text{cont } \ell \chi_{proph} \sigma | I \)

The use of T-APP to apply \( k \) to the continuation at its assumed type, along with Context Substitution (Lemma 5) completes the case.

- **Case E-INVOKEC:** In this case:
  - \( e = E'[\% \ell \ E'[\langle \text{cont } \ell \ E'' \rangle] h)] \)
  - \( e' = E'[\% \ell \ E'[\langle \text{cont } \ell \ E'' \rangle] h)] \)
  - \( q = (\ell \rightarrow \text{replaced } I \nRightarrow \sigma, I) \)

By Context Decomposition (Lemma 1):
- \( \Sigma; \Gamma \vdash (\% \ell \ E'[\langle \text{cont } \ell \ E'' \rangle] h) : \tau' \mid \chi_{prompt} \)
- \( \Sigma; \Gamma \vdash E : \tau_{prompt} / \chi_{prompt} \nRightarrow \tau / \chi \)

By inversion on the prompt typing:
- \( \Sigma; \Gamma \vdash E'[\% \ell \ E'[\langle \text{cont } \ell \ E'' \rangle] h) : \tau_{prompt} \mid \chi_{prompt} \)
- \( \Sigma; \Gamma \vdash \text{validEffects}(P_{\text{body}}, C_{\text{body}}, Q_{\text{body}}, \ell, \tau_{prompt}, \sigma_h) \)
- \( \chi_{prompt} = (P_{\text{body}} \setminus \ell, C_{\text{body}} \setminus \ell, Q_{\text{body}} \cup \bigcup C_{\text{body}}|_{Q_h}) \)

By context decomposition for \( E' \):
- \( \Sigma; \Gamma \vdash (\% \ell \ E'' \rangle) : \tau_{hole} | \chi_{hole} \)
- \( \Sigma; \Gamma \vdash E' : \tau_{hole} / \chi_{hole} \nRightarrow \tau_{prompt} / \chi_{prompt} \)

By inversion on the continuation application, and value typing:
- \( \Sigma; \Gamma \vdash (\% \ell \ E'') : \text{cont } \ell \gamma (P_{E''}, C_{E''}, Q_{E''}) \sigma \mid I \)
- \( \Sigma; \Gamma \vdash v : \gamma \mid I \)
- \( \chi_{hole} = \chi \gamma \setminus P_{E''} \subseteq C_{E''} \cup \{ \text{replace } \ell : Q_{E''} \nRightarrow \sigma \}, I \)

By inversion on typing of the continuation (T-CONTC):
- \( \Sigma; \Gamma \vdash E'' : \gamma / (\emptyset, \emptyset, I) \nRightarrow \sigma / (P_{E''}, C_{E''}, Q_{E''}) \)
- \( P_{E''} \subseteq P_{E''} \subseteq C_{E''} \)
- \( Q_{E''} \subseteq Q_{E''} \)

Thus we conclude by Lemma 5 for the \( v \) at hand:
- \( \Sigma; \Gamma \vdash E'[v] : \sigma | (P_{E''}, C_{E''}, Q_{E''}) \)

Because control effects for \( \ell \) are carried through \( E' \) without change (since it is an evaluation context containing no \( \ell \)-prompts)
- \( C_{E''} \cup \{ \text{replace } \ell : Q_{E''} \nRightarrow \sigma \} \subseteq C_{\text{body}} \)

Likewise, \( P_{E''} \subseteq P_{\text{body}} \) (originating from \( \chi_{hole} \)). To apply T-PROMPT, we require that \( P_{E''} \setminus Q_h \subseteq P_{\text{body}} \setminus Q_h \ell \).

Fortunately, if the unmasked \( P_{E''} \) is already less than a set masked by \( Q_h \ell \), then masking it by \( Q_h \ell \) again will have no effect. Thus \( P_{E''} \setminus Q_h \ell \subseteq P_{\text{body}} \setminus Q_h \ell \). And because of the replacement present in \( C_{\text{body}} \), \( Q_{E''} \subseteq Q_{\text{body}} \cup \bigcup C_{\text{body}}|_{Q_h} \). Thus, applying T-PROMPT to \( \% \ell \ E'[v] \) it has effect \( (P_{E''} \setminus Q_h \ell, C_{E''} \setminus Q_h \ell, Q_{E''} \cup \bigcup C_{\text{body}}|_{Q_h}) \), which is a smaller effect than \( \chi_{prompt} \). The rest of the case (plugging into \( E \)) follows from subsumption and Context Substitution (Lemma 5).