I. INTRODUCTION

The collective behaviour of many elementary single particles combined in an interacting system can be very rich and differ greatly from the physics governing each individual particle. One of the most remarkable phenomena in nature arising from collective behaviour is the phase transition, whereby tuning a physical field the system may undergo a singular change in its static properties. This phenomenon is ubiquitous in nature and so the theory of phase transitions naturally extends into quantum systems with the paradigmatic model of a 2nd order quantum phase transition being the 1d anisotropic XY model in a transverse field. By tuning the transverse field the ground state undergoes a singular change from a ferromagnetic state to a paramagnetic state. However focusing on equilibrium properties does not necessarily capture the full dynamical behaviour of a complex many-body system. Recent studies have found that consideration of purely dynamical observables, more specifically time-integrated observables, provide insights into the dynamics of many-body systems. Using full counting statistics (FCS) the moment generating function (MGF) of these observables can be calculated and is treated analogously to a partition sum. Moreover the counting field ‘s’, that is the field conjugate to the time-integrated observable, is treated like a full thermodynamic variable. Within this formalism singular features in the long-time limit of the cumulant generating function (CGF) mark phase transitions in the FCS.

In a recent paper we focussed on the time-integrated transverse magnetization in the ground state of an associated model, the transverse field Ising model (TFIM), using this approach. We uncovered a whole curve of 2nd order FCS phase transitions of which the static quantum critical points were the endpoints. We also found that a special class of states exist which capture these FCS singular points in an analogous manner to the ground state. By considering this critical curve as a quantum critical line in this extended class of states these FCS points corresponded to a 3rd order static phase transition which was probed by tuning the field s. These special states were identified as eigenstates of a non-Hermitian operator which forms the MGF. Relating this non-Hermitian operator to the effective Hamiltonian of an appropriate open quantum system it was shown that the state could be prepared via the no jump evolution of this open system. A second approach to dynamics and quantum nonequilibrium focusses on the formal link between the boundary function and the Loschmidt echo associated with a quantum quench. This formal link allows one to extend the Lee-Yang theory of phase transition to nonequilibrium quantum dynamics. Recently we studied the connection between the singular features in the FCS captured by these states and the emergence of temporal nonanalyticities in the return amplitude on quenching these states across the FCS critical line. These temporal nonanalyticities dubbed dynamical phase transitions (DPTs) often only appear for quenches in certain areas of parameter space and in the previous studies quenching across either a quantum or FCS critical point results in there emergence. A recent work by Vajna et al. highlighted that the presence of equilibrium phase transitions does not imply DPTs will emerge, and so a natural question to consider is whether this result applies to FCS transitions also.

The focus of this work is twofold, in the first part we focus on analytic properties of the generating functions of the time-integrated transverse magnetization and anisotropy in the 1d XY model. Examining the properties of the special states associated with these observables, which we call the s-states, we find the 2nd order FCS phase transitions associated with these observables correspond to 3rd order static phase transitions in these states. Furthermore when considering the latter observable without any anisotropy we find for every point in the ferromagnetic regime there exists a 3rd order static phase transition. However contrary to expectation DPTs...
do not emerge when quenching across the FCS critical points of this observable.

In the second part of this study we explore the preparation of these states in detail. We study the likelihood of reaching the $s$-states using appropriate Markovian baths and the timescales required to reach these states. We find that in the thermodynamic limit dissipation prevents one from preparing such a state and one must consider an alternative method using an ancillary system.

We begin in Sec. II A with a description of the theoretical framework of time-integrated observables. We then discuss the proposed method of preparing such special states using Markovian baths in Sec. II B. Following this we present a theoretical primer on DPTs and the quench protocol in Sec. II C. Then in Sec. III we present our results on 3rd order phase transitions and DPTs in the $XY$ model. We discuss the timescales required to prepare the $s$-state in Sec. IV along with a discussion on the use of an ancilla system in preparing these states. To finish we present our conclusions in Sec. V.

II. THEORETICAL BACKGROUND

A. Time-integrated observables

A closed quantum system is described by its Hamiltonian, $H$, which defines the time evolution of this system under an associated unitary operator. To capture the dynamics of this system we wish to examine the moments of a general dynamical observable

$$Q_t = \int_0^t q(t')dT', \quad (1)$$

here $q(t)$ is the operator of interest in the Heisenberg representation. To construct the MGF of this time-integrated observable we deform $H$ to a non-Hermitian operator $H_s$. Associated with this new operator is a non-unitary evolution operator $T_t(s)$, both of these operators are defined as

$$T_t(s) \equiv e^{-itH_s}, \quad H_s \equiv H - \frac{is}{2}q. \quad (2)$$

From these definitions it is easy to see that the generating function of $Q_t$ is given by

$$\Theta_t(s) \equiv \langle T_t(s)T_t(s) \rangle. \quad (3)$$

With the MGF the moments of $Q_t$ are generated simply via differentiation, $\langle Q_t^n \rangle = (-1)^n \frac{\partial^n}{\partial s^n} \Theta_t(s)|_{s=0}$, and the CGF is simply the logarithm of this object, $\Theta_t(s) \equiv \log Z_t(s)$. These objects define the FCS but compared to the usual approach where the FCS focuses on the characteristic function here we consider the generating function with real $s$. In the study of FCS phase transitions it is useful to study that analytic properties of a scaled version of the CGF in the long-time limit

$$\theta(s) = \lim_{N,t \to \infty} \frac{\Theta_t(s)}{Nt}. \quad (4)$$

Making a connection with equilibrium thermodynamics we consider this CGF a full dynamical “free energy” and define an order parameter, $\kappa_s = -\partial_s \theta(s)$, and corresponding FCS susceptibility, $X_s = (\partial^2_s \theta(s)$. Similar to their counterparts in equilibrium statistical physics we use these quantities to characterize the FCS phases of the system and a diverging $X_s$ is indicative of a 2nd order FCS critical point. Moreover for each point in parameter space we can define a state $|s\rangle$ which are right eigenstates of $H_s$ and play a role analogous to the ground state of the $XY$ model. Starting with an initial state $|i\rangle$ this state is defined as $|s\rangle \equiv \lim_{t \to \infty} T_t(s)|i\rangle$ with an appropriate normalization. With this state we define an $s$-biased expectation value of an observable

$$(\mathcal{O})_s \equiv \lim_{t \to \infty} \frac{\langle 0| T_t^\dagger(s) \mathcal{O} T_t(s) |0 \rangle}{Z_t(s)}, \quad (5)$$

where for the initial state $|i\rangle$ we use the ground state of the $XY$ model $|0\rangle$. Taking $\mathcal{O}$ to be the operator of interest $q$, in the long-time limit it is easy to see that

$$\frac{\langle q \rangle_s}{N} = \frac{-\theta(s)}{s}. \quad (6)$$

From this equation it becomes apparent that the 2nd derivative of the $s$-biased expectation of $q$ with respect to $s$ will diverge at a FCS critical point. Therefore a 2nd order FCS transition is equivalent to a a 3rd order phase transition existing in this extended $s$-state regime.

B. Preparation of $s$-state

The $s$-state is the result of long-time evolution under a non-unitary evolution operator, this operator evolves the density matrix $\rho(t)$ according to $\dot{\rho}(t) = -i[H, \rho(t)] - \frac{s}{2}\{q, \rho(s)\}$. This evolution is similar in appearance to a Lindblad master equation without recycling terms, $\dot{\rho}(s) = -i[H, \rho(s)] + \sum_i L_i \rho L_i - \frac{1}{2} \{L_i L_i, \rho\}$, this full evolution describes a system connected to a Markovian bath. Noticing that if the observable of interest $q$ is related to the jump operators $L_i$ via $\sum_i L_i^\dagger L_i = sq$ then $T_t(s)$ defines this associated open quantum system in between quantum jumps. From this the MGF is simply the probability of no jumps occurring $P_0(t)$ up to a time $t$ in the associated open quantum system and $s$ plays the role of the decay rate.

For the case of the TFIM and the total transverse magnetization, if we make a trivial shift so that $q = \sum_i (\sigma^z_i + 1)$, we can define the jump operators as $L_i = \sqrt{2s} \{|-\rangle_i \langle +|_i \}$ where $\sigma^z_i |\pm\rangle = |\pm\rangle$. It was shown that using cold ions one could simulate this associated open quantum system and from the jump statistics extract features of the critical curve for finite sizes and short times. The question we pose now is this, if we have access to such baths can we prepare the $|s\rangle$ state directly from the no jump evolution and tune the decay rate to probe this critical line? Furthermore, if this is possible what is the timescale required to prepare these $s$-states?
C. Return Amplitude and Quantum Quenches

One of the central quantities in equilibrium statistical physics is the boundary partition function

\[ Z(L) = \langle \psi_a | e^{-LH} | \psi_b \rangle, \]  

(7)

where \( L \) is the length of the boundary, \( H \) is the Hamiltonian and \( |\psi_a/b\rangle \) are the boundary states. This quantity may be connected to the nonequilibrium protocol known as the quantum quench by analytically continuing to the complex plane, \( L \to \beta \) where \( \beta \in \mathbb{C} \). Taking identical boundaries \( a = b = 0 \) one can readily show that if \( \beta = it \) the analytically continued boundary partition function is the Loschmidt amplitude

\[ G(t) = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle. \]  

(8)

In the context of the quantum quench \( |\psi_0\rangle \) is the initial state and \( H \) is the post quench Hamiltonian. The boundary partition function has zeros in the complex \( L \) plane and in the thermodynamic limit these zeros may coalesce to form a critical line. If this line intersects the real \( L \) axis for some parameter values then the system will undergo a phase transition. However this critical line may also intersect the imaginary \( L \) axis and result in so-called dynamical phase transitions. Analogous to equilibrium statistical physics these DPTs manifest as temporal nonanalyticities in the large deviation function associated with the return amplitude

\[ l(t) = \lim_{N \to \infty} \frac{-1}{N} \log |G(t)|^2, \]  

(9)

where here \( N \) represents the system size. It is important to note that although this mapping is formally exact the return amplitude does not provide information on the equilibrium statistics in this problem.

In this work we consider a specific type of quantum quench protocol which we refer to as the s-quench. The protocol is as follows: the system is initially prepared in the relevant \( |s\rangle \) state, then it is “quenched” to \( s = 0 \) by allowing the state to evolve under the original \( s = 0 \) XY model Hamiltonian \( H \),

\[ |s_i\rangle = e^{-itH} |s\rangle. \]  

(10)

In the next Section we will examine the analytic properties of the return amplitude associated with this “quench” protocol.

III. 3RD ORDER PHASE TRANSITIONS IN THE XY MODEL

We now focus on the 1d XY model with periodic boundary conditions, this is a paradigmatic model of a quantum phase transition and is defined by the Hamiltonian

\[ H = -\sum_i \left[ \frac{1}{2} \sigma_i^x \sigma_{i+1}^x + \sum_i \frac{1}{2} \sigma_i^y \sigma_{i+1}^y - \lambda \sum_i \sigma_i^z \right] \]

\[ = \sum_k \epsilon_k(\lambda)(A_k^\dagger A_k - 1/2), \]  

(11)

where \( \sigma^x,y \) are pauli spin operators, \( \gamma \) is the anisotropy parameter and \( \lambda \) denotes the strength of the transverse field. Using a Jordan-Wigner transformation followed by a Bogoliubov rotation this Hamiltonian may be diagonalized and has an energy spectrum

\[ \epsilon_k(\lambda) = 2\sqrt{(\lambda - \cos k)^2 + \gamma^2 \sin^2 k}. \]  

(12)

This model has critical points at \( \lambda = \pm 1 \), where the ground state changes from a ferromagnetic state to being paramagnetic in a singular fashion. Within the ferromagnetic regime there is also a critical line along \( \gamma = 0 \) either side of which the ground state is ferromagnetic with the spins aligned along either the \( x \) or \( y \) direction.

We consider time integrals of the transverse magnetization and the total anisotropy in the ground state of this model. Beginning with the former we set \( q = \sum_i \sigma_i^z \), the deformed \( H_\gamma \) is still diagonalizable via free fermion methods and the analytic form of the CGF is accessible,

\[ \theta(s) = \frac{2}{\pi} \text{Im} \left( \int_0^\pi dk |\sqrt{(\lambda + is/2 - \cos k)^2 + \gamma^2 \sin^2 k}| \right). \]  

(13)

The dynamic susceptibility associated with this diverges along a critical curve in the \( s, \lambda \) plane, of which the end points are the static quantum phase transitions. This curve obeys the equation

\[ \lambda^2 + (s/2\gamma)^2 = 1, \ |\lambda| \leq 1, \]  

(14)

where for each value of \( \lambda \) there is a critical \( k \) mode whose associated energy spectrum becomes 0 at the critical line.
Switching focus to the time-integrals of the anisotropy parameter  
$q = \sum_l \sigma_l^z\sigma_{l+1}^z - \sigma_l^z\sigma_{l+1}^x$, we denote the relevant 
quantities of interest with a superscript $A$. The relevant $H^A_s$ is once again diagonalizable using free fermion techniques\cite{38–40} and we find the FCS “free” energy is

$$\theta^A(s) = \frac{2}{\pi} \text{Im} \left( \int_0^\pi dk |\sqrt{(\lambda - \cos k)^2 + (\gamma + is)^2 \sin^2 k}| \right).$$

(15)

When $\gamma \neq 0$ there are no FCS critical points but along the anisotropy coexistence line $\gamma = 0$ the susceptibility $X^A_s$ diverges with a square root singularity when

$$k^* = \cos^{-1} \lambda. \text{ We note that taking } \gamma = 0 \text{ we obtain the critical curve of Ref.}\cite{19}. \text{ This curve marks a 3rd order phase transition in the magnetization of the } |s\rangle\text{ states, where the static susceptibility } \chi_s = \partial_s \langle m_x \rangle \text{ has a “kink” and its derivative } \chi_s^\prime \text{ diverges, is shown in Fig. 1.}$

![Graph](image)

**FIG. 2.** (Color Online) (a) The dynamical activity of the anisotropy parameter for $N = 2000$ spins is plotted, tuning $s$ many discontinuities associated with FCS singularities can be seen. (b) The discontinuities in $\kappa^A_s$, see above, manifest as divergences in the dynamical susceptibility and mark a 3rd order static phase transition in the associated $|s\rangle$-states.

$\lambda = \cos k \pm s \sin k$. Examining this equation we find the critical $k$ mode which satisfies this equality is given by

$$k^* = |\cos^{-1} \left[ \frac{\lambda \pm s\sqrt{1 + s^2 - \lambda^2}}{1 + s^2} \right]|.$$

(16)

with the constraint that the $k$ modes lie in the range $[0, \pi]$. Despite this restriction on the range of $k$ modes we find that in ferromagnetic regime $|\lambda| < 1$ there are two solutions to Eq. (16). Therefore this region is critical as there are 3rd order phase transitions in the $|s\rangle$ states for every point in the $s-\lambda$ region. Moving to the paramagnetic regime $-1 > \lambda > 1$ solutions to Eq. (16) only exist for this equation when $1 + s^2 \geq \lambda^2$. This results in the emergence of 1st order FCS transition at $s_c = \pm \sqrt{\lambda^2 - 1}$, beyond this critical point there once again exists solutions to Eq. (16) and 3rd order FCS phase transitions at each point in parameter space.

When there exists no $k^*$ the spectrum of $H^A_s$ is real and there exists an associated unbroken $\mathcal{PT}$-symmetry\cite{38,19}, for a regular Hermitian operator this symmetry is simply Hermitian conjugation. We do not discuss the technical details of this symmetry but it has implications on the temporal scaling of the cumulants. This is easy to see from Eqs. (3) and (15), in the ferromagnetic regime the spectrum of $H_s$ has complex eigenvalues which contribute to $\theta^A(s)$. From Eq. (4) this implies the cumulants at $s = 0$ scale at least linearly in time, however in the paramagnetic regime $\theta^A(s)$ is strictly 0 in the vicinity of $s = 0$. This is because for $|s| < |\lambda|$ the eigenvalues of $H^A_s$ are real, the form of the CGF in this parameter regime implies the physical cumulants are oscillatory or scale sublinearly with time. These results are interesting and highlight the importance of the spectrum of $H^A_s$ when considering time-integrated observables, as discussed in Ref.\cite{38–40}.
Starting from the critical regime ($|\lambda| < 1$), the analytic form of $|s\rangle$ is analytically accessible (see Appendix). Quenching this state to $s = 0$ we find the return amplitude rate function takes the form

$$l(t) = -2\text{Re}\left(\int_{k_1}^{k_2} \frac{\sin^2 \alpha_k^s}{\cosh(2\text{Im}(\alpha_k^s))} e^{-2i\tau t_k(\lambda)}\right),$$  \hspace{1cm} (17)

where $\alpha_k^s$ is the difference between Bogoliubov angles which arise in diagonalizing $H_A$ and $k_{1,2}$ are the upper and lower solutions to Eq. (16) respectively. Quenching from both within the critical regime ($|\lambda| < 1$) and across the 1st order FCS critical line ($|\lambda| > 1$) does not result in the emergence of DPTs, this is shown in Fig. 3. Although for finite $N$ there exists two families of solutions, $k = k_{1,2}$, where $|\cos \alpha_k^s| = |\sin \alpha_k^s|$, in the thermodynamic limit these zeros do not result in DPTs emerging in Eq. (17). The lack of DPTs on quenching across these static critical points in $|s\rangle$ space is surprising and highlights that the presence of such critical points is not sufficient for these DPTs to emerge upon quenching.

IV. TIMESCALES TO REACH S-STATE

Having discussed the 3rd phase transitions that may occur in this model we now focus on the timescales required to prepare the $|s\rangle$ states associated with the time-integrated transverse magnetization in the TFIM, therefore $\gamma = 1$ throughout this section. Specifically we are interested in the $|s\rangle$ states which undergo 3rd order phase transitions by tuning $s$ and so we take $\lambda = 0.5$ in this section. We begin by examining the timescale required for $Z_t(s)$ to converge to $e^{iN\theta(s)}$ as a function of the number of spins $N$, and the decay rate $s$. We measure this convergence by examining the scaled “distance” to the long-time regime

$$\Delta_t = \frac{\log Z_t(s)/Nt}{\theta^N(s)},$$  \hspace{1cm} (18)

where $\theta^N(s)$ is the CGF of Eq. (1) for finite $N$. When $\Delta_t = 0$ the ground state has converged to $|s\rangle$ under the no jump evolution of the bath. Taking $N = 500$ we find to strictly reach the $|s\rangle$ state requires an infinitely long-time for all values of $s$. However the timescale required to produce a state close to $|s\rangle$, $|\Delta_t| < 0.05$, decreases with increasing $s$. This is captured by the timescale $\tau_{1/2}$ defined by $\Delta_{\tau_{1/2}} = 0.5$, where this equation may have multiple solutions so we take the largest $\tau_{1/2}$. This timescale is plotted in Fig. 4(c) and shows that as $s \to 0$ the timescale required to prepare a state very close to the $|s\rangle$ state diverges. This timescale decreases with increasing $s$ but the rate of decrease with increasing decay rate tails off for $s \gg 1$. Focussing on fixed $s$ and varying system size we find the timescale required to reach the final state is independent of the system size, this is shown in Fig. 4(b).

The evolution of the system when connected to these Markovian environments is stochastic in nature and so there is only a finite probability that the system will evolve under the no jump evolution defined by $H_s$. This probability is known as the survival probability and is analytically calculable for this model. In fact due to the shift in observable to $q = \sum_i (\sigma_i^z + 1)$ the survival probability is trivially related to the MGF of the time-integrated transverse magnetization by the equation

$$P_0(t) = e^{-s N \lambda}(Z_t(s)).$$  \hspace{1cm} (19)

This quantity is highly dependent on system size and in the thermodynamic limit is zero for all finite $t$. This result is shown in Fig. 4. Combined with the system size independence of $\Delta_t$ we conclude that to prepare a state close to $|s\rangle$ it is better to use a few body system. Such few body open quantum systems may be experimentally probed via digital simulation with ultracold atoms. Although the finite size effects for such small system sizes are large, it was demonstrated in Ref. [23] that some signatures of the transition are still extractable.

If one wishes to probe the 3rd order phase transition in the thermodynamic limit it is necessary to evolve the system under the no jump evolution directly and avoid the prospect of emissions which prevent us from preparing the system close to $|s\rangle$ state. This may be done via introducing an ancillary two level system in conjunction with the Markovian baths. Consider the evolution of the full system+ancilla qubit in a Markovian environment, its density matrix $\varrho$ evolves under a master equation

$$\dot{\varrho} = -i[H, \varrho] + \sum_i J_i \varrho J_i^\dagger - \frac{1}{2} \{J_i^\dagger J_i, \varrho\}. \hspace{1cm} (20)$$

Choosing $H = H \otimes |0\rangle\langle 0|$, $J_i = L_i \otimes |1\rangle\langle 0|$, where $H$ is the TFIM Hamiltonian and the projectors $|0\rangle\langle 0|$ and $|1\rangle\langle 0|$ act on the ancilla subspace, one can readily show that the no jump evolution defined by $H_s$ is readily given by

$$\dot{\varrho} = \text{Tr}_{\text{Anc.}}(\mathbb{I} \otimes |0\rangle\langle 0| \dot{\varrho})$$

$$= -i[H, \varrho] - \frac{1}{2} \{L_i^\dagger L_i, \varrho\}, \hspace{1cm} (21)$$

where the trace is performed over the ancilla subspace. Using this simple two level ancillary system one may implement the no jump evolution required to prepare a state very close to $|s\rangle$ in the thermodynamic limit and thus probe the 3rd order static phase transitions in this extended state space.

V. CONCLUSIONS

In this paper we examined the singularities of the generating functions of the time-integrated transverse magnetization and anisotropy in the 1d $XY$ model. In the former case there exists an elliptical curve of FCS critical points in the $s$-$\lambda$ plane where the eccentricity of the
ellipse is set by the anisotropy $\gamma$. In the latter case when $\gamma = 0$ we uncovered a critical regime where every point in parameter space has an FCS singularity but the FCS singularities of the anisotropy are not marked by the emergence of DPTs in the return amplitude upon quenching. The FCS singularities in both cases manifest themselves as 3rd order phase transitions in the $|s\rangle$ states. We also examined the timescales required to prepare the $s$-states which exhibit these novel phase transitions. We focussed on the example of the TFIM and the $s$-states associated with the time-integrated transverse magnetization. We found that, starting in the ground state, the timescale to reach the $s$-state is independent of the system size and is weakly dependent on the decay rate when $s \gg 1$. However the probability of evolving without emission becomes zero in the limit of large system size due to dissipation. Thus to prepare $|s\rangle$ and find features of the 3rd order phase transitions few body systems are preferable such as cold ion systems used in digital simulation. To reach the thermodynamic limit one needs to use an ancillary system where the desired evolution is contained as a block of the joint system+ancilla density matrix. Beyond their properties in capturing FCS singularities as static quantum critical points and DPTs, many questions remain about the nature of these $|s\rangle$ states. It would be interesting to study the entanglement properties of the $|s\rangle$ states and their relationship to scaling theories in non-unitary conformal field theories. Finally there is the interesting question of how static observables in the ground state relate to the properties of these $|s\rangle$ states.

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Appendix A: Diagonalizing $H_A^s$ and finding $|s\rangle$

The non-Hermitian operator associated with the time-integrated anisotropy is given by

$$H_A^s = -\sum_i \frac{1 + \gamma + is}{2\lambda} \sigma_i^x \sigma_{i+1}^x - \sum_i \frac{1 - \gamma - is}{2\lambda} \sigma_i^y \sigma_{i+1}^y - \lambda \sum_i \sigma_i^z. \quad (A1)$$

To diagonalize this operator we first perform a Jordan-Wigner transformation followed by Bogoliubov rotation. The first transformation maps the spin operators at each site $\sigma_i^x, \sigma_i^y$, and $\sigma_i^z$ to fermionic creation and annihilation operators.
operators $c_i$ and $c_i^\dagger$ with $\{c_i, c_j\} = \delta_{i,j}$ via
\[
\sigma_i^+ = 1 - 2c_i c_i^\dagger, \\
\sigma_i^- = \prod_{j<i} (1 - 2c_j^\dagger c_j), \\
\sigma_i = \prod_{j<i} (1 - 2c_j^\dagger c_j^\dagger).
\] (A2)

Fourier transforming the transformed Hamiltonian results in an operator which may be diagonalized by Bogoliubov rotation
\[
c_k = \cos \frac{\phi_k}{2} B_k + i \sin \frac{\phi_k}{2} B_{-k}, \\
c_k^\dagger = \cos \frac{\phi_k}{2} B_k - i \sin \frac{\phi_k}{2} B_{-k}.
\] (A3)

It is worth noting here we restrict ourselves to an even number of spins $N$ and assume periodic boundary conditions. With this rotation $H^A_s$ is diagonal in the complex fermionic pair $\{B_k, B_k^\dagger\} = \delta_{k,k'}$, where $B_k \neq B_k^\dagger$ provided $\phi_k = -\phi_k^*$ and
\[
\tan \phi_k = \frac{\gamma + is}{\lambda - \cos k}. \\
\] (A4)

In the case of no anisotropy ($\gamma = 0$) the free fermion dispersion of $H^A_s$ is then found to be
\[
\epsilon_k(\lambda, s) = 2\sqrt{(\lambda - \cos k)^2 - s^2 \sin^2 k}.
\] (A5)

The key property in determining the evolution of the ground state $|0\rangle$ of $H^A_s$ is that the fermionic states at $s = 0$ may be expressed in terms of the fermionic states at finite $s$. In fact the ground state is simply a BCS state of $H^A_s$
\[
|0\rangle = \frac{1}{N} \exp \left( \sum_{k>0} \hat{B}(k) \hat{B}_k \hat{B}_{-k} \right) |0\rangle_s \\
= \frac{1}{\sqrt{\cosh(2\text{Im}(\alpha_k^s))}} \bigotimes_{k>0} |\cos \alpha_k^s |0_k, 0_{-k}\rangle_s - i \sin \alpha_k^s |1_k, 1_{-k}\rangle_s.
\] (A6)

Here the $k$th mode of the $s$-vacuum $|0_k, 0_{-k}\rangle_s$ is defined such that $B_{\pm k} |0_k, 0_{-k}\rangle_s = 0$ and the complex angles in the coefficients is simply $\alpha_k^s = \frac{\phi_k^0 - \phi_k^s}{2}$. While $|1_k, 1_{-k}\rangle_s = B_k B_{-k} |0_k, 0_{-k}\rangle_s$ signifies the occupied fermionic state with the wavevector $|k\rangle$ that diagonalizes $H^A_s$. With Eq. (A6) we can evolve $|0\rangle$ under the non-unitary evolution defined by $H^A_s$ to obtain the $|s\rangle$.

There is however a subtlety here, as often there exists some $k$ modes for which $\epsilon_k(\lambda, s)$ is real. The $|s\rangle$ state is then chosen such that Eq. (5) holds and from (4) it is clear in the long-time limit these $k$ modes do not contribute to $\theta^A(s)$ and hence $|s\rangle$. Thus we find
\[
|s\rangle = \prod_{k>0, \text{Im} \epsilon_k^s \neq 0} \frac{|1_k, 1_{-k}\rangle_s}{\sqrt{\cosh(2\text{Im}(\alpha_k^s))}}.
\] (A7)

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