Finite One-Loop Corrections and Perturbative Gauge Invariance in Quantum Gravity Coupled to Photon Fields

Nicola Grillo*

Institut für Theoretische Physik, Universität Zürich
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

August 8, 2018

Abstract

One-loop calculations in quantum gravity coupled to $U(1)$-Abelian fields (photon fields) are ultraviolet finite and cutoff-free in the framework of causal perturbation theory. We compute the photon loop graviton self-energy and the photon self-energy in second order perturbation theory. The condition of perturbative gauge invariance to second order is shown and generates the gravitational Slavnov–Ward identities. Quantum corrections to the Newtonian potential through the photon loop graviton self-energy are also derived.

PACS numbers: 0460, 1110

Keywords: Quantum Gravity

Preprint: ZU-TH 39/1999

*grillo@physik.unizh.ch
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1 Introduction

The standard field theoretical perturbative approach to quantum gravity (see the introduction to this subject in [1] and references therein), considered as a flat space-time relativistic quantum field theory of gravitons, massless rank-2 tensor fields, coupled to photons, massless vector fields, leads to non-renormalizable ultraviolet (UV) divergences. These were found by means of dimensional regularization and background field method [2], [3], [4] and [5].

A closer investigation of loop calculations in quantum field theory shows that the reason for the UV divergences lies basically in the fact that one performs mathematically ill-defined operations, when using Feynman rules for closed loop graphs, because one multiplies Feynman propagators as if they were ordinary functions.

We show how it is possible to overcome these discouraging outcomes by applying an improved perturbation scheme which has as central objects the time-ordered products and as constructing principle causality. The $S$-matrix is constructed inductively as a sum of smeared operator-valued $n$-point distributions avoiding UV divergences.

UV finiteness of $S$-matrix elements is then a consequence of a deeper mathematical understanding of how loop graph contributions have to be calculated.

This idea goes back to Stückelberg, Bogoliubov and Shirkov and the program (causal perturbation theory) was carried out successfully by Epstein and Glaser [6] for scalar field theories and subsequently applied to QED by Scharf [7], to non-Abelian gauge theories by Dütsch et al. [8] and to quantum gravity [9].

In this work we carry out the analysis of the coupled quantized Einstein–Maxwell system, Sec. 2.2. Two main topics will be investigated in this paper: the gauge structure of the theory and the UV finiteness of loop graphs in second order perturbation theory.

The first is formulated by means of a ‘gauge charge’ that generates infinitesimal gauge variations of the fundamental free quantum fields. Gauge invariance of the $S$-matrix implies then a set of identities between the C-number part of the 2-point distributions, Sec. 2.4, which implies the gravitational Slavnov–Ward identities (SWI) [10].

The second is obtained by applying the ‘causal perturbation’ scheme, Sec. 3, to the calculations of the photon loop graviton self-energy, Sec. 4, and of the photon self-energy, Sec. 5, in second order perturbation theory. In both cases the results are UV finite and cutoff-free and the inherent ambiguity in the normalization of the 2-point distribution is settled by requiring appropriate normalization conditions for mass and coupling constant.

In addition, the causal scheme preserves the gauge symmetries of the theory.

The trace of the graviton self-energy tensor vanishes. This property was manifestly broken using dimensional regularization in order to extract finite results from UV divergent quantities [11].

From the photon loop graviton self-energy we sketch the derivation of quan-
tum corrections to the Newtonian potential between two spinless massive bodies in the static non-relativistic limit, Sec. 4.5.

Gauge structure and one-loop calculations with gravitational self-coupling are not considered here, see [9], [12], [13], and [14]. These references contain also the notations and the conventions used here.

We use the unit convention: $\hbar = c = 1$, Greek indices $\alpha, \beta, \ldots$ run from 0 to 3, whereas Latin indices $i, j, \ldots$ run from 1 to 3.

2 Quantized Einstein–Maxwell System and Perturbative Gauge Invariance

2.1 Inductive $S$-Matrix Construction

In causal perturbation theory [7], [15], one makes an ansatz for the $S$-matrix as a formal power series in the coupling constant, namely as a sum of smeared operator-valued distributions:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \cdot \ldots \cdot g(x_n),$$

(2.1)

where the Schwartz test function $g \in \mathcal{S}(\mathbb{R}^4)$ switches adiabatically the interaction and provides a natural infrared cutoff in the long range part of the interaction. The $S$-matrix maps the asymptotically incoming free fields on the outgoing ones and it is possible to express the $T_n$’s by means of free fields. Interacting quantum fields are never used in the causal scheme.

The $n$-point distribution $T_n$ is a well-defined ‘renormalized’ time-ordered product expressed in terms of Wick monomials of free fields $\mathcal{O}(x_1, \ldots, x_n)$:

$$T_n(x_1, \ldots, x_n) = \sum_{\mathcal{O}} \mathcal{O}(x_1, \ldots, x_n) : t_n^\mathcal{O}(x_1 - x_n, \ldots, x_{n-1} - x_n).$$

(2.2)

The $t_n$’s are C-number distributions. $T_n$ is constructed inductively from the first order $T_1(x)$, which plays the rôles of the interaction Lagrangian in terms of free fields, and from the lower orders $T_j$, $j = 2, \ldots, n - 1$ by means of Poincaré covariance and causality. The latter leads directly to UV finite and cutoff-free $T_n$-distributions.

2.2 Quantized Einstein–Maxwell System

In the context of linearized gravity (without graviton self-interactions [9], [13], [14]) coupled to photon field [3], the interaction between the quantized symmetric tensor field $h^{\alpha\beta}(x)$, the graviton, and the quantized vector field $A^\mu(x)$, the photon, is considered in the background of Minkowski space-time.

The free graviton field satisfies the wave equation

$$\Box h^{\mu\nu}(x) = 0,$$

(2.3)
and is quantized according to
\[ [h^{\mu\nu}(x), h^{\alpha\beta}(y)] = -i b^{\mu\nu\alpha\beta} D_0(x - y), \] (2.4)
where the \( b \)-tensor is constructed from the Minkowski metric
\[ b^{\mu\nu\alpha\beta} := \frac{1}{2} \left( \eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta} \right), \] (2.5)
and \( D_0(x) \) is the mass-zero causal Jordan–Pauli distribution:
\[ D_0(x) = D^{(+)}_0(x) + D^{(-)}_0(x) = \frac{1}{2} \pi \delta(x^2) \text{sgn}(x^0) = \frac{i}{(2\pi)^3} \int d^4p \delta(p^2) \text{sgn}(p^0) e^{-i p \cdot x}. \] (2.6)
(see [12] and [13] for the details of this procedure). The additional degrees of freedom present in the symmetric tensor field \( h^{\mu\nu} \) (gravitons are massless spin-2 particles) could be eliminated by imposing appropriate gauge and trace conditions, but these are disregarded in the following and only considered later as a characterization of the physical states [12].

From the point of view of Lagrangian field theory [3], the Hilbert–Einstein Lagrangian
\[ L_{HE} = -2\kappa^{-2} \sqrt{-g} \, \eta^{\mu\nu} R_{\mu\nu}, \] (2.3)
\( (\kappa^2 = 32\pi G) \), is written by means of the Goldberg variable \( \tilde{g}^{\mu\nu} := \sqrt{-g} \, g^{\mu\nu} \) and expanded as \( \tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} \) around the flat space-time \( \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1) \). From the zeroth order, choosing the gauge \( h^{\mu\nu},_\nu = 0 \), one obtains (2.3).

The free photon field fulfills also the wave equation
\[ \Box A^\mu(x) = 0, \] (2.7)
and is quantized as
\[ [A^\mu(x), A^\nu(y)] = +i \eta^{\mu\nu} D_0(x - y). \] (2.8)
Also the field strength \( F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \) will be used in the following.

### 2.3 Gauge Invariance and First Order Coupling \( T^A_1(x) \)

The \( U(1)_{EM} \) gauge content [7] of the photon field is formulated by means of the gauge charge \( Q_A \)
\[ Q_A := \int_{x^0=\text{const}} \, d^3x \, A^\nu(x),_\nu \, \partial^0_x \, v(x), \] (2.9)
where \( v(x) \) is a C-number scalar field satisfying \( \Box v(x) = 0 \). Actually, due to the generality of the gravitational coupling, one should also consider the coupling between \( h^{\mu\nu} \) and the quantized electromagnetic ghost \( v \) and anti-ghost \( \tilde{v} \) scalar fields, but this would have the form \( \kappa \eta^{\mu\nu} h^{\mu\nu} \tilde{v} \Box v \) from \( \sqrt{-g} \, \tilde{v} \Box v \) and therefore
can be written as a total divergence in the sense of vector analysis because of
the presence of two equal derivatives on three massless fields. For this reason
the electromagnetic ghosts will not be considered here.

On the other side, the gauge content of the graviton field (which is related
to the general covariance of $g_{\mu
u}(x)$ under coordinate transformations [1]) is
formulated by the gauge charge [9], [14]
\[
Q := \int_{x^0=\text{const}} d^3 x \, h^{\mu
u}(x)_{,\nu} \partial_0^\mu u_\mu(x).
\] (2.10)

In order for gauge invariance to first order in pure quantum gravity to hold
(see Eq. (2.15)) [9], for the construction of the physical subspace [12] and in
order to prove unitarity of the $S$-matrix on the physical subspace [12], we need
to quantize the ghost field $u_\mu(x)$, together with the anti-ghost field $\tilde{u}^\nu(x)$, as
fermionic vector fields
\[
\Box u_\mu(x) = 0, \quad \Box \tilde{u}^\nu(x) = 0, \quad \{u_\mu(x), \tilde{u}^\nu(y)\} = i \eta^{\mu\nu} D_0(x - y),
\] (2.11)
whereas all other anti-commutators vanish. The gauge charges generate the
following infinitesimal gauge variations of the fundamental quantum fields:
\[
d_Q h^{\mu\nu}(x) := [Q, h^{\mu\nu}(x)] = -i b^{\mu\rho\sigma} u_\rho(x)_{,\sigma}, \\
d_Q u^\alpha(x) := [Q, u^\alpha(x)] = 0, \quad d_Q \tilde{u}^\alpha(x) := [Q, \tilde{u}^\alpha(x)] = i h^{\alpha\beta(x)_{,\beta}}, \\
d_Q A^\alpha(x) := [Q_A, A^\alpha(x)] = i \partial_0^\alpha v(x), \quad d_Q F^{\alpha\beta}(x) = 0.
\] (2.12)

The operator $d_Q$ obeys also the Leibniz rule
\[
d_Q(A B) = (d_Q A) B + (-1)^{n_Q(A)} A d_Q B,
\] (2.13)
for arbitrary operators $A$ and $B$, where $n_Q(A)$ is the number of ghost fields
minus the number of anti-ghost fields in the Wick monomial $A$.

For convenience of notation, the trace of the graviton field is written as $h = h^\gamma_\gamma$ and all Lorentz indices of the fields are written as superscripts whereas the
derivatives acting on the fields are written as subscripts. All indices occurring
twice are contracted by the Minkowski metric $\eta^{\mu\nu}$. We skip the space-time
dependence if the meaning is clear.

The first order interaction between graviton field and photon fields is dictated
by gauge invariance and by the assumption that the graviton interacts with the
traceless and conserved electromagnetic energy-momentum tensor, see below.

Gauge invariance of the $S$-matrix means formally
\[
\lim_{g \to 1} d_Q S(g) = 0, \quad \text{and} \quad \lim_{g \to 1} d_Q A S(g) = 0.
\] (2.14)

Since the existence of the adiabatic limit $g \to 1$ in massless theories can be
problematic, we consider the condition of perturbative gauge invariance to $n$-th order
\[
d_Q T_n(x_1, \ldots, x_n) = d_Q A T_n(x_1, \ldots, x_n) = \text{sum of divergences},
\] (2.15)
which implies $S$-matrix gauge invariance, because divergences do not contribute in the adiabatic limit $g \to 1$ due to partial integration and Gauss’ theorem.

We define $T_A^1(x)$ as

$$T_A^1(x) := i \frac{K}{2} : h^{\mu \nu}(x) b_{\mu \alpha \beta} T_{EM}^{\alpha \beta}(x) :,$$

(2.16)

where $T_{EM}^{\alpha \beta}(x)$ corresponds to the electromagnetic energy-momentum tensor and reads

$$T_{EM}^{\alpha \beta}(x) := - F^{\alpha \gamma}(x) F^{\beta}_{\gamma}(x) + \frac{1}{4} \eta^{\alpha \beta} F^{\rho \sigma}(x) F_{\rho \sigma}(x).$$

(2.17)

Actually, the $b$-tensor is not essential, because $T_{EM}^{\alpha \beta}$ is traceless. Its origin lies in the Goldberg variable expansion. $T_A^1(x)$ corresponds to the term of order $\kappa$ in the expansion of the generally covariant electromagnetic Lagrangian density

$$L_A = \frac{1}{4} \sqrt{-g} F_{\mu \nu} F^{\alpha \beta} g_{\mu \alpha} g_{\nu \beta},$$

(2.18)

as a function of $\tilde{g}^{\mu \nu}$ around the flat space-time.

Perturbative gauge invariance to first order is readily established. $T_A^1(x)$ is $U(1)_{EM}$ gauge invariant

$$d_Q T_A^1(x) = 0,$$

(2.19)

due to $d_Q F^{\mu \nu}(x) = 0$. The $U(1)_{EM}$ gauge structure of the theory is rather trivial and will not be considered.

Gauge invariance with respect to the gauge charge $Q$

$$d_Q T_A^1(x) = \frac{\kappa}{2} : u^\rho(x) : T_{EM}^{\rho \sigma}(x) := \partial_\sigma \left( \frac{K}{2} : u^\rho(x) T_{EM}^\sigma : \right) =: \partial_\sigma T_{1/1}^\sigma(x)$$

(2.20)

holds because of $T_{EM}^{\alpha \beta}(x, \beta) = 0$. $T_{1/1}^\sigma(x)$ is the so-called $Q$-vertex $\mathbb{Q}$, $\mathbb{Q}$. It allows us to formulate the condition of perturbative gauge invariance to the $n$-th order in a precise way:

$$d_Q T_n(x_1, \ldots, x_n) = \sum_{l=1}^n \frac{\partial}{\partial x_l^\nu} T_{n/l}^\nu(x_1, \ldots, x_l, \ldots, x_n),$$

(2.21)

where $T_{n/l}^\nu$ is the ‘renormalized’ time-ordered product, obtained according to the inductive causal scheme, with one $Q$-vertex at $x_l$, while all other $n-1$ vertices are ordinary $T_1$-vertices.

### 2.4 Consequences of Perturbative Gauge Invariance to Second Order

Without performing any calculation, from the structure of $T_A^1$ we can anticipate that the two-point distribution describing loop graphs has the form

$$T_2(x, y)^{\text{loops}} := h^{\alpha \beta}(x) h^{\mu \nu}(y) : t_{hh}(x, y)_{\alpha \beta \mu \nu} + : F^{\alpha \beta}(x) F^{\mu \nu}(y) : t_{FF}(x, y)_{\alpha \beta \mu \nu}.$$
Here, the first term represents the photon loop graviton self-energy and the second term the photon self-energy. The C-number distributions $t_{hh}$ and $t_{FF}$ will be explicitly calculated in Sec. 4 and in Sec. 5, respectively.

Perturbative gauge invariance to second order, Eq. (2.21) with $n = 2$, enables us to derive a set of identities for these distributions by comparing the distributions attached to the same external operators on both sides of Eq. (2.21), as in [17]. Calculating $d_Q T_2(x, y)$ loops we obtain

$$d_Q T_2(x, y)_{\text{loops}} = + : u^\rho(x),_\sigma h^{\mu\nu}(y) : \left( - i b^{\alpha\beta\rho\sigma} t_{hh}(x, y)_{\alpha\beta\mu\nu} \right) + : h^{\alpha\beta}(x) u^\rho(y),_\sigma : \left( - i b^{\mu\nu\rho\sigma} t_{hh}(x, y)_{\alpha\beta\mu\nu} \right). \quad (2.22)$$

On the other side, the loop contributions coming from $T_2^\sigma(x, y)$ can only be of the form

$$T_2^\sigma(x, y) = : u^\rho(x) h^{\mu\nu}(y) : t^\sigma_{uh}(x, y)_{\rho\mu\nu} + : u^\alpha(x) h^{\mu\nu}(y) : t_{uh}(x, y)_{\mu\nu}. \quad (2.23)$$

Applying $\partial_x^\sigma$ to the expression above we find

$$\partial^\sigma_x T_2^\sigma_{/1}(x, y) = + : u^\rho(x),_\sigma h^{\mu\nu}(y) : \left\{ t^\sigma_{uh}(x, y)_{\rho\mu\nu} + \eta^\sigma_{\rho\mu\nu} t_{uh}(x, y)_{\mu\nu} \right\} + : u^\alpha(x) h^{\mu\nu}(y) : \partial^\sigma_x \left\{ t^\alpha_{uh}(x, y)_{\rho\mu\nu} + \eta^\alpha_{\rho\mu\nu} t_{uh}(x, y)_{\mu\nu} \right\}. \quad (2.24)$$

We compare the C-number distributions in (2.22) and in (2.24) attached to the external operators

$$: u^\rho(x),_\sigma h^{\mu\nu}(y) : \quad \text{and} \quad : u^\alpha(x) h^{\mu\nu}(y) :. \quad (2.25)$$

No such terms come from $T_2^\sigma_{/2}(x, y)$. Therefore, we obtain the identities

$$-i b^{\rho\alpha\sigma\beta} t_{hh}(x, y)_{\alpha\beta\mu\nu} = \left\{ t^\sigma_{uh}(x, y)_{\rho\mu\nu} + \eta^\sigma_{\rho\mu\nu} t_{uh}(x, y)_{\mu\nu} \right\}, \quad 0 = \partial^\sigma_x \left\{ t^\alpha_{uh}(x, y)_{\rho\mu\nu} + \eta^\alpha_{\rho\mu\nu} t_{uh}(x, y)_{\mu\nu} \right\}. \quad (2.26)$$

By applying $\partial^\sigma_x$ to the first identity and inserting the second one, we obtain

$$b^{\alpha\beta\rho\sigma} \partial^\sigma_x t_{hh}(x, y)_{\alpha\beta\mu\nu} = 0 \quad (2.27)$$

This identity has been explicitly checked in Eq. (4.20) and implies the gravitational Slavnov–Ward identities [3], [10] for the two-point connected Green function (see Sec. 4.3)

### 3 Causal Construction of the Two-Point Distribution

We outline briefly the main steps in the inductive causal construction of $T_2(x, y)$ from the first order interaction $T_1^A(x)$. Following the inductive scheme [8], we first calculate

$$R_2'(x, y) := -T_1^A(y) T_1^A(x), \quad A_2'(x, y) := -T_1^A(x) T_1^A(y). \quad (3.1)$$
From these two quantities, we form the causal distribution
\[ D_2(x, y) := R'_2(x, y) - A'_2(x, y) = [T_1^a(x), T_1^a(y)]. \] (3.2)

In order to obtain \( D_2(x, y) \), one has to carry out all possible contractions between the normally ordered \( T_1 \) using Wick’s lemma, so that \( D_2(x, y) \) has the following structure (see (2.2))
\[ D_2(x, y) = \sum_{\mathcal{O}} : \mathcal{O}(x, y) : d_O^2(x - y). \] (3.3)

\( d_O^2(x - y) \) is a numerical distribution that depends only on the relative coordinate \( x - y \), because of translation invariance.

\( D_2(x, y) \) contains tree (one contraction), loop (two contractions) and vacuum graph (three contractions) contributions. Due to the presence of normal ordering, tadpole diagrams do not appear. \( D_2(x, y) \) is causal, i.e. \( \text{supp}(d_O^2(z)) \subseteq V^+(z) \cup V^-(z) \), with \( z := x - y \).

In order to obtain \( T_2(x, y) \) we have to split \( D_2(x, y) \) into a retarded part, \( R_2(x, y) \), and an advanced part, \( A_2(x, y) \), with respect to the coincident point \( z = 0 \), so that \( \text{supp}(R_2(z)) \subseteq V^+(z) \) and \( \text{supp}(A_2(z)) \subseteq V^-(z) \). This splitting, or ‘time-ordering’, has to be carried out in the distributional sense with great care so that the retarded and advanced part are well-defined and UV finite [6], [7].

The splitting affects only the numerical distribution \( d_O^2(x - y) \) and must be accomplished according to the correct ‘singular order’ \( \omega(d_O^2) \) which describes roughly speaking the behaviour of \( d_O^2(x - y) \) near \( x - y = 0 \), or that of \( \hat{d}_O^2(p) \) in the limit \( p \to \infty \), respectively. If \( \omega < 0 \), then the splitting is trivial and agrees with the standard time-ordering. If \( \omega \geq 0 \), then the splitting is non-trivial and non-unique:
\[ d_O^2(x - y) \longrightarrow r_O^2(x - y) + \sum_{|a|=0} C_{a,\mathcal{O}} D^a \delta^{(4)}(x - y) , \] (3.4)
and a retarded part \( r_O^2(x - y) \) is obtained in momentum space by means of a dispersion-like integral of the type
\[ r_O^2(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{\hat{d}_O^2(tp)}{(t - i0)^{\omega+1} (1 - t + i0)} , \quad p \in V^+, \] (3.5)
which requires modifications in the case of massless theories [8].

Eq. (3.4) contains a local normalization ambiguity: the \( C_{a,\mathcal{O}} \)’s are undetermined finite normalization constants, which multiply terms with point support \( D^a\delta^{(4)}(x - y) \) (\( D^a \) is a partial differential operator). This freedom in the normalization has to be restricted by physical conditions. For example, Lorentz covariance and gauge invariance will be used in our discussions.
Finally, \( T_2(x, y) \) is obtained by subtracting \( R'_2(x, y) \) from \( R_2(x, y) \) and can be written as

\[
T_2(x, y) + N_2(x, y) = \sum_\mathcal{O}(x, y) :\mathcal{O}(x, y) : \left\{ t_2^\mathcal{O}(x - y) + \sum_{|a| = 0} \omega^\mathcal{O} \cdot \delta^a \mathcal{O}(x - y) \right\},
\]

(3.6)

where \( N_2(x, y) \) represents the normalization terms given by the second term on the right side.

### 4 Photon One-Loop Contribution to the Graviton Self-Energy

#### 4.1 Causal \( D_2(x, y) \)-Distribution for the Photon Loop

We now apply the causal scheme described in Sec. 3 in order to calculate the 2-point distribution \( T_2(x, y)^{gSE} \) that describes the photon loop graviton self-energy.

First of all, from the commutation rules (2.4) and (2.8) we derive the contractions between two field operators

\[
C\{ h^{\alpha\beta}(x) h^{\mu\nu}(y) \} := \left[ h^{\alpha\beta}(x) \cdot h^{\mu\nu}(y) \right] = -i \delta^{\alpha\beta\mu\nu} D_0^{(+)(x - y)},
\]

\[
C\{ A^\mu(x) A^\nu(y) \} := \left[ A^\mu(x) \cdot A^\nu(y) \right] = +i \eta^\mu\nu D_0^{(+)(x - y)},
\]

(4.1)

where \((\pm)\) refers to the positive/negative frequency part of the corresponding quantity.

The distributions \( R'_2(x, y)^{gSE} \) and \( A'_2(x, y)^{gSE} \), defined in Eq. (3.1), are obtained by performing two photon contractions between the two first order interactions \( T_1^A(x) \) and \( T_1^A(y) \). After this operation, we obtain

\[
A'_2(x, y)^{gSE} = + : h^{\alpha\beta}(x) h^{\mu\nu}(y) : a'_2(x - y)^{gSE}_{\alpha\beta\mu\nu},
\]

\[
R'_2(x, y)^{gSE} = + : h^{\alpha\beta}(x) h^{\mu\nu}(y) : r'_2(x - y)^{gSE}_{\alpha\beta\mu\nu}.
\]

(4.2)

The C-number tensorial distributions \( a'_2(x - y)^{gSE}_{\alpha\beta\mu\nu} \) and \( r'_2(x - y)^{gSE}_{\alpha\beta\mu\nu} \) are given by linear combinations of products between positive/negative frequency parts of the Jordan–Pauli distributions (2.6) and carry four derivatives (each photon field in \( T_1^A \) carries one derivative). Therefore, the basic distribution appearing here is

\[
D^{(+)}_{\alpha\beta\mu\nu}(x - y) := \partial^\alpha \partial^\beta \partial^\mu D_0^{(+)}(x - y) \cdot \partial^\nu D_0^{(+)}(x - y).
\]

(4.3)
Summing up all the contributions coming from \(-T_1^4(x)T_1^4(y)\) after performing two photon contractions, we have

\[
a_2^2(x-y)^{g\text{SE}}_{\alpha\beta\mu\nu} = \frac{\kappa^2}{4} \left\{ -2\eta_{\mu\nu} D^{(+)\alpha\rho\beta\beta}_{\rho\rho\rho\rho} - 2\eta_{\alpha\beta} D^{(+)\alpha\beta\rho\rho}_{\rho\rho\rho\rho} - 4D^{(+)\alpha\beta\mu\nu}_{\rho\rho\rho\rho} + 4\eta_{\alpha\mu} D^{(+)\alpha\beta\rho\rho}_{\rho\rho\rho\rho} + + 2\eta_{\alpha\nu} D^{(+)\beta\beta\rho\rho}_{\rho\rho\rho\rho} + 2\eta_{\beta\mu} D^{(+)\alpha\beta\rho\rho}_{\rho\rho\rho\rho} + - (\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu}) D^{(+)\gamma\rho\gamma\rho}_{\gamma\rho\gamma\rho} \right\}, \tag{4.4}
\]

as a function of the \(D^{(+)\gamma\rho\gamma\rho}_{\gamma\rho\gamma\rho}\)-distributions. The distribution \(r_2^2(x-y)^{g\text{SE}}_{\alpha\beta\mu\nu}\) has the same form as above, but with \(D^{(-)}_{\gamma\rho\gamma\rho}\) instead of \(D^{(+)\gamma\rho\gamma\rho}\) (because of the relation \(D^{(+)\gamma\rho\gamma\rho}(y-x) = -D^{(-)}_{\gamma\rho\gamma\rho}(x-y)\)). The \(D^{(\pm)}_{\gamma\rho\gamma\rho}\)-distribution are easily evaluated in momentum space by means of Eq. (2.6):

\[
\hat{D}^{(\pm\alpha\beta\mu\nu)}_{\gamma\rho\gamma\rho}(p) = \frac{-1}{(2\pi)^4} \int d^4k \delta((p - k)^2) \Theta(\pm (p^0 - k^0)) \delta(k^0) \Theta(\pm k^0) 
\times \left[ + p_\alpha p_\beta k_\mu k_\nu - p_\alpha k_\beta k_\mu k_\nu - p_\beta k_\alpha k_\mu k_\nu + k_\alpha k_\beta k_\mu k_\nu \right], \tag{4.5}
\]

(see App. 1 of [13]), because products of Jordan–Pauli distributions go over into convolutions of the corresponding Fourier transforms. Therefore, we see that we have to deal with integrals of the type

\[
I^{(\pm)}_{\alpha\beta\mu\nu} := \int d^4k \delta((p - k)^2) \Theta(\pm (p^0 - k^0)) \delta(k^0) \Theta(\pm k^0) 
\times \left[ 1, k_\alpha, k_\alpha k_\beta, k_\alpha k_\beta k_\mu, k_\alpha k_\beta k_\mu k_\nu \right]. \tag{4.6}
\]

These integrals have been calculated in detail in App. 2 of [13] and partially in [8]. We give here only the final results:

\[
I^{(\pm)}(p) = \frac{\pi}{2} \Theta(p^2) \Theta(\pm p^0),
\]

\[
I^{(\pm)}(p)_{\alpha\beta} = \frac{\pi}{6} \left( p_\alpha p_\beta - \frac{p^2}{4\eta_{\alpha\beta}} \right) \Theta(p^2) \Theta(\pm p^0), \tag{4.7}
\]

\[
I^{(\pm)}(p)_{\alpha\beta\mu} = \frac{\pi}{8} \left( p_\alpha p_\beta p_\mu - \frac{p^2}{6} \right) \left( p_\alpha \eta_{\beta\mu} + p_\beta \eta_{\alpha\mu} + p_\mu \eta_{\alpha\beta} \right) \Theta(p^2) \Theta(\pm p^0),
\]

\[
I^{(\pm)}(p)_{\alpha\beta\mu\nu} = \frac{\pi}{10} \left( p_\alpha p_\beta p_\mu p_\nu - \frac{p^2}{8} \right) \left( + p_\alpha p_\beta \eta_{\mu\nu} + p_\alpha p_\mu \eta_{\beta\nu} + p_\alpha p_\nu \eta_{\beta\mu} + + p_\beta p_\mu \eta_{\alpha\nu} + p_\beta p_\nu \eta_{\alpha\mu} + p_\mu p_\nu \eta_{\alpha\beta} \right) + + \frac{p^4}{48} \left( + \eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu} + \eta_{\alpha\beta} \eta_{\mu\nu} \right) \Theta(p^2) \Theta(\pm p^0).
\]
From (4.5), (4.6) and (4.7), we obtain the relations for the \( \hat{D}^{(\pm)} \)-distributions

\[
\hat{D}^{(\pm)}_{\alpha\rho|\beta\rho}(p) = \frac{-1}{(2\pi)^4} \left[ p_\alpha p_\rho I^{(\pm)}(p)_{\rho\beta} - p_\rho I^{(\pm)}(p)_{\rho\alpha} \right] \\
= \frac{-\pi}{48(2\pi)^4} \left[ 2p^2 p_\alpha p_\beta + p^4 \eta_{\alpha\beta} \right] \Theta(p^2) \Theta(\pm p^0),
\]

(4.8)

\[
\hat{D}^{(\pm)}_{\gamma\rho|\gamma\rho}(p) = \frac{-1}{(2\pi)^4} \left[ p_\gamma p_\rho I^{(\pm)}(p)_{\gamma\gamma} \right] \\
= \frac{-\pi}{8(2\pi)^4} \left[ 2p^2 \right] \Theta(p^2) \Theta(\pm p^0).
\]

Therefore, from (4.2) with \( \Theta(p^2) \Theta(-p^0) - \Theta(p^2) \Theta(+p^0) = -\Theta(p^2) \text{sgn}(p^0) \) we find

\[
D_2(x, y)^{gSE} = h^{\alpha\beta}(x) \varepsilon^{\mu\nu}(y) \cdot d_2(x - y)^{gSE}_{\alpha\beta\mu\nu}.
\]

(4.9)

With Eqs. (4.4), (4.8) and after symmetrization in \((\alpha\beta) \leftrightarrow (\mu\nu)\) the C-number tensorial \( d_2 \)-distribution reads in momentum space

\[
\hat{d}_2(p)^{gSE}_{\alpha\beta\mu\nu} = \frac{\kappa^2 \pi}{960(2\pi)^4} \left[ -16 p_\alpha p_\beta p_\mu p_\nu - 8 p^2 \left( p_\alpha p_\beta \eta_{\mu\nu} + p_\mu p_\nu \eta_{\alpha\beta} \right) + \\
+ 12 p^2 \left( p_\alpha p_\mu \eta_{\beta\nu} + p_\beta p_\nu \eta_{\alpha\mu} + p_\beta p_\mu \eta_{\alpha\nu} + p_\alpha p_\nu \eta_{\beta\mu} \right) + \\
-12 p^4 \left( \eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu} \right) + \\
+ 8 p^4 \eta_{\alpha\beta} \eta_{\mu\nu} \right] \Theta(p^2) \text{sgn}(p^0).
\]

(4.10)

For simplicity, we use the shorthand notation

\[
\hat{d}_2(p)^{gSE}_{\alpha\beta\mu\nu} = \Upsilon \hat{P}(p)^{gSE}_{\alpha\beta\mu\nu} \hat{d}(p),
\]

\[
\hat{P}(p)^{gSE}_{\alpha\beta\mu\nu} := \left[ -16, -8, +12, -12, +8 \right],
\]

(4.11)

where \( \Upsilon := \kappa^2 \pi/(960(2\pi)^4) \), and \( \hat{d}(p) := \Theta(p^2) \text{sgn}(p^0) \) (this scalar distribution is typical for a loop with massless particles) and the numerical coefficients always refer to the polynomial structure as in Eq. (4.10). The polynomial \( \hat{P}(p)^{gSE}_{\alpha\beta\mu\nu} \) has degree four in \( p \), because of the four derivatives present on the two contracted photon lines.

4.2 Singular Order, Distribution Splitting and Graviton Self-Energy Tensor

According to the inductive construction of \( T_2(x, y) \), Sec. 3, the next step is the splitting of \( D_2(x, y) \) into a retarded part \( R_2(x, y) \) and an advanced part \( A_2(x, y) \). In this procedure the singular order of the distribution \( D_2(x, y) \) plays
an essential rôle. From (4.10) it follows that $\omega(\hat{d}_2) = 4$, because of the presence of the polynomial of degree four in $p$.

A general formula for the singular order of any distribution in linearized gravity coupled to photon fields can be found by considering in the $n$-th order of perturbation theory an arbitrary $n$-point distribution

$$T_n^G(x_1, \ldots, x_n) = \prod_{j=1}^{n_h} h(x_{k_j}) \prod_{i=1}^{n_A} A(x_{m_i}) : t_n^G(x_1, \ldots, x_n).$$  \hfill (4.12)

This $T_n^G$ corresponds to a graph $G$ with $n_h$ external graviton lines and $n_A$ external photon lines. Then we state that the singular order of $G$ is given by

$$\omega(G) \leq 4 - n_h - n_A - d + n.$$ \hfill (4.13)

Here, $d$ is the number of derivatives on the external field operators in (4.12). The explicit presence of the order of perturbation theory renders the theory ‘non-normalizable’: the number of free undetermined, but finite normalization terms in Eq. (3.6) increases with the order of perturbation theory, that is the theory has a weaker predictive power but it is still well-defined in the sense of UV finiteness. The proof of (4.13) has the same structure as in QED, Yang–Mills theories and pure quantum gravity, see therefore [7], [8] and [13], respectively.

In the case of the graviton self-energy contribution, Eq. (4.9), we obtain from (4.13) the correct result

$$\omega(\hat{d}_2) = 4,$$

being $n_h = 2$, $n_A = 0$, $d = 0$ and $n = 2$. The singular order of a distribution remains unchanged after distribution splitting.

Because of the decomposition (4.11), it suffices to split $\hat{d}(p)$, which has $\omega(\hat{d}) = 0$, in order to obtain a retarded part $\hat{r}(p)$. The full retarded distribution is then given by $\hat{r}_2(p)^{g_{SE}}_{\alpha\beta\mu\nu} = \hat{\gamma}(p)^{g_{SE}}_{\alpha\beta\mu\nu} \hat{r}(p)$. The ambiguity in the normalization appearing in the splitting formula (3.4) will be discussed in Sec. 4.4.

The splitting of the scalar distribution $\hat{d}(p)$ was already carried out in [8], using a modification of the formula (3.5) for the retarded part, see also [13]. We quote only the result for the analytic continuation to $p \in \mathbb{R}^4$ of the retarded part:

$$\hat{r}(p)^{an} = \frac{i}{2\pi} \log \left( \frac{-p^2 + i p^0}{M^2} \right),$$ \hfill (4.14)

where $M > 0$ is a scale invariance breaking mass. Then

$$R_2(x, y)^{g_{SE}} = : h^{\alpha\beta}(x) h^{\mu\nu}(y) : r_2(x - y)^{g_{SE}}_{\alpha\beta\mu\nu},$$

$$\hat{r}_2(p)^{g_{SE}}_{\alpha\beta\mu\nu} = i \gamma \frac{\hat{P}(p)^{g_{SE}}_{\alpha\beta\mu\nu}}{2\pi} \log \left( \frac{-p^2 + i p^0}{M^2} \right).$$ \hfill (4.15)

The distribution $T_2(x, y)$ is obtained from $R_2(x, y)$ by subtracting $R_2'(x, y)$, Eq. (4.2). This subtraction affects only the scalar distribution. Since $\hat{r}'(p) = \hat{r}_2(p)^{an}$.

\[ \text{Page 13} \]
\[ (-\Theta(p^2) \Theta(-p^0), \text{ we obtain} \]
\[ \hat{t}(p) = \hat{r}(p)^{an} - \hat{r}'(p) = \frac{i}{2\pi} \log \left( \frac{-p^2 - i0}{M^2} \right). \]  

(4.16)

Therefore, the 2-point distribution for the photon loop graviton self-energy reads
\[ T_2(x, y)^{gSE} = \Pi(x - y)^{gSE}, \]
\[ i\hat{\Pi}(p)_{\alpha\beta\mu\nu} = i\Xi \left[ -16, -8, +12, -12, +8 \right] \log \left( \frac{-p^2 - i0}{M^2} \right), \]  

(4.17)

with \( \Xi := \frac{1}{2\pi} \Upsilon. \) The most important aspect of this result lies in the fact that the obtained \( T_2(x, y)^{gSE} \)-distribution is UV finite without the introduction of counterterms [2], [3] and cutoff-free. Note that \( M \) is a normalization constant and not a cutoff.

### 4.3 Slavnov–Ward Identities from Perturbative Gauge Invariance

We investigate the gauge properties of the graviton self-energy tensor. For simplicity let us denote by \( [A, B, C, E, F] \) the five numerical coefficients of \( \Pi(x - y)_{\alpha\beta\mu\nu} \) given as in Eq. (4.10).

First of all, \( T_2(x, y)^{gSE} \) satisfies the condition of perturbative gauge invariance to second order (2.21). Using the infinitesimal operator gauge variation (2.12), we obtain
\[ d_Q T_2(x, y)^{gSE} = + \partial_\sigma^x \left( :u^\rho(x)h^{\mu\nu}(y): \left[ + b^{\alpha\beta\rho\sigma} \Pi(x - y)_{\alpha\beta\mu\nu} \right] \right) + \partial_\sigma^y \left( :u^\alpha(x)u^\beta(y): \left[ + b^{\mu\nu\rho\sigma} \Pi(x - y)_{\alpha\beta\mu\nu} \right] \right), \]  

(4.18)

because, after working out explicitly in momentum space, the necessary condition for (4.18) to hold,
\[ b^{\alpha\beta\rho\sigma} p_\sigma \hat{\Pi}(p)_{\alpha\beta\mu\nu} = 0, \]  

(4.19)

corresponds to the identity (2.27) and is satisfied by \( \hat{\Pi}(p)_{\alpha\beta\mu\nu} \) of Eq. (4.17). In terms of the \( A, \ldots, F \) coefficients, (4.19) implies
\[ C + E = 0, \quad A - 2B = 0, \quad B - 2E - 2F = 0. \]  

(4.20)

and these relations are satisfied by the polynomial in (4.17).

We show now that these relations implies the gravitational Slavnov–Ward identities (SWI) [3], [10]. We construct the 2-point connected Green function with one photon loop
\[ \hat{G}(p)_{\alpha\beta\mu\nu}^{[2]} := b^{\alpha\beta\gamma\delta} \hat{D}_0^\gamma(p) \hat{\Pi}(p)^{\gamma\delta\rho\sigma} b_{\rho\sigma\mu
u} \hat{D}_0^\delta(p), \]  

(4.21)
with the Feynman propagator \( \hat{D}_0(p) = (2\pi)^{-2}(-p^2 - i0)^{-1} \). The gravitational SWI reads

\[
p^\alpha \hat{G}(p)^{[2]}_{\alpha\beta\mu\nu} = 0,
\]

namely the 2-point connected Green function is transversal. Eq. (4.22) implies the relations

\[
C + E = 0, \quad A - 2B = 0, \quad \frac{A}{4} + C - F = 0.
\]

Analysis of (4.20) and (4.23) shows that the condition of perturbative gauge invariance is equivalent to the gravitational SWI. In addition, the photon loop is transversal also without the \( b \)-tensors that come from the graviton propagators:

\[
p^\alpha \hat{\Pi}(p)_{\alpha\beta\mu\nu} = 0,
\]

because the equivalent relations

\[
C + E = 0, \quad B + F = 0, \quad A + B + 2C = 0.
\]

are fulfilled by (4.17). This property follows directly from our perturbative invariance condition (2.27), because the trace of \( \hat{\Pi}(p)_{\alpha\beta\mu\nu} \) vanishes (see below).

The last two properties of \( \hat{\Pi}(p)_{\alpha\beta\mu\nu} \) concern its trace and are related to conformal transformations. Conformal invariance is manifest in the vanishing of the trace of the Maxwell energy-momentum tensor in four dimension. Basically, the photon loop graviton self-energy consists of a ‘time-ordered’ product of two such traceless photon energy-momentum tensors (2.17). Therefore, in addition to the identities (4.19), it is expected that \( \hat{\Pi}(p)_{\alpha\beta\mu\nu} \) and \( \hat{G}(p)^{[2]}_{\alpha\beta\mu\nu} \) are traceless. The condition

\[
\eta^{\alpha\beta} \hat{\Pi}(p)_{\alpha\beta\mu\nu} = 0,
\]

namely that the photon loop graviton self-energy tensor is traceless, implies the relations

\[
B + 2E + 4F = 0, \quad A + 4B + 4C = 0.
\]

These are satisfied by (4.17). Note that (4.27) does not follow from the perturbative gauge invariance conditions (4.20). In addition, in our approach, also the trace of the 2-point connected Green function vanishes:

\[
\eta^{\alpha\beta} \hat{G}(p)^{[2]}_{\alpha\beta\mu\nu} = 0,
\]

because the equivalent relations

\[
\frac{A}{2} + 3B + 2C + 2E + 4F = 0, \quad A + 4B + 4C = 0.
\]
are satisfied by the coefficients of the photon loop self-energy tensor in Eq. (4.17). This is in sharp contrast to the calculation in \cite{3,11}, performed within the dimensional regularization scheme. This scheme generates conformal trace anomalies \cite{18}, because the trace operation is not dimensional invariant. As a consequence, the extraction of the finite part from UV divergent quantities breaks this symmetry.

Causal perturbation theory not only provides us with UV finite results but preserves the underlying symmetries of the theory such as the SWI and the vanishing of the trace of the self-energy tensor. Invariance under conformal transformations is broken by the presence of the mass scale $M$ in the logarithm of Eq. (4.17). If necessary, we could exploit the ambiguity in the normalization to restore it, see Sec. 4.4.

### 4.4 Freedom in the Normalization of the Graviton Self-Energy

We still have to discuss the ambiguity in the splitting procedure, namely the appearance of undetermined local normalization terms.

Having singular order four, the 2-point photon loop graviton self-energy contribution $T_2(x, y)^{gsE}$ admits a general normalization of the form

$$N_2(x, y)^{gsE} =: h^{\alpha \beta}(x) h^{\mu \nu}(y) : i N(\partial_x, \partial_y)_{\alpha \beta \mu \nu} \delta^{(4)}(x - y),$$

$$\tilde{N}(p)_{\alpha \beta \mu \nu} = \tilde{N}(p)_{\alpha \beta \mu \nu}^{(0)} + \tilde{N}(p)_{\alpha \beta \mu \nu}^{(2)} + \tilde{N}(p)_{\alpha \beta \mu \nu}^{(4)},$$

where the odd terms are excluded by parity. $\tilde{N}(p)_{\alpha \beta \mu \nu}^{(i)}$ is a polynomial in $p$ of degree $i$ with $i = 0, 2, 4$. We assume in addition that only scalar constants should be considered, because vector-valued or tensor-valued constants may endanger Lorentz covariance. Taking the symmetries of $\Pi(x - y)_{\alpha \beta \mu \nu}$ into account, we make the following ansatz

$$\tilde{N}(p)_{\alpha \beta \mu \nu}^{(0)} = \Xi \left[ + c_1 (\eta_{\alpha \mu} \eta_{\beta \nu} + \eta_{\alpha \nu} \eta_{\beta \mu}) + c_2 \eta_{\alpha \beta} \eta_{\mu \nu} \right],$$

$$\tilde{N}(p)_{\alpha \beta \mu \nu}^{(2)} = \Xi \left[ + c_3 (p_\alpha \eta_{\beta \mu} + p_\beta \eta_{\mu \alpha} + p_\alpha \eta_{\nu \beta} + p_\beta \eta_{\nu \alpha}) + c_4 (p_\alpha \eta_{\beta \nu} + p_\beta \eta_{\mu \alpha} + p_\beta \eta_{\nu \beta} + p_\beta \eta_{\nu \alpha}) + c_5 p^2 (\eta_{\alpha \mu} \eta_{\beta \nu} + \eta_{\alpha \nu} \eta_{\beta \mu}) + c_6 p^2 \eta_{\alpha \beta} \eta_{\mu \nu} \right],$$

$$\tilde{N}(p)_{\alpha \beta \mu \nu}^{(4)} = \Xi \left[ c_7, c_8, c_9, c_{10}, c_{11} \right].$$

c_1, \ldots, c_{11}$ are undetermined real numbers. The normalization polynomials have to fulfil the same symmetries as $\tilde{\Pi}_{\alpha \beta \mu \nu}$, namely: (4.20), (4.25), (4.27) and (4.29). These reduce the choice to

$$\tilde{N}(p)_{\alpha \beta \mu \nu}^{(0)} = 0, \quad \tilde{N}(p)_{\alpha \beta \mu \nu}^{(2)} = 0, \quad \tilde{N}(p)_{\alpha \beta \mu \nu}^{(4)} = \Xi \frac{c_{11}}{4} \left[ -16, -8, +12, -12, +8 \right].$$

Since $\tilde{N}(p)_{\alpha \beta \mu \nu}^{(4)}$ has the same structure as $\tilde{\Pi}_{\alpha \beta \mu \nu}$, we absorb the undetermined parameter $c_{11}$ in the mass scale $M$ appearing in Eq. (4.17) through the rescaling
\( c_{11} = 4 \log(M^2/M_0^2) \). The whole freedom in the normalization is reduced to the mass parameter \( M_0 \) in the logarithm owing to the symmetries that the self-energy tensor and its normalization have to fulfil.

This normalization automatically preserves graviton mass- and coupling constant-normalizations. If one sums up the infinite series with an increasing number of photon loop graviton self-energy insertions, the vanishing of \( \hat{N}(p)^{(0)}_{\alpha\beta\mu\nu} \) implies that the graviton mass is not shifted by quantum corrections and the vanishing of \( \hat{N}(p)^{(2)}_{\alpha\beta\mu\nu} \) implies that the coupling constant \( \kappa \) is not changed by quantum corrections.

As pointed out at the end of Sec. 4.3, we could exploit this normalization to restore conformal invariance. Through a rescaling of \( M_0 \) we can compensate the variation of \( p^2 \) under such a transformation. The drawback lies in the arbitrariness of this operation.

### 4.5 Corrections to the Newtonian Potential

As pointed out in [19], [20], [21] and [22], massless particle loop corrections to the graviton propagator leads to quantum corrections of the Newtonian potential between massive spinless bodies in the static non-relativistic limit. These can be appropriately defined by considering the whole set of diagrams in the scattering \( \varphi_1 \varphi_2 \to \varphi_1 \varphi_2 \), where \( \varphi_i \) represents a scalar field of mass \( m_i \), of the order \( \kappa^4 \) (i.e. \( \sim G^2 \)). Then one isolates the non-local contributions. These lead to \( r^{-2} \) and \( r^{-3} \) corrections to the Newtonian potential \( V(r) = -Gm_1 m_2 r^{-1} \) in the static non-relativistic limit. For this purpose, the logarithm-dependent result (4.17) generates the correction

\[
V(r) = -G \frac{m_1 m_2}{r} \left(1 + \frac{Gh}{c^3 \pi} \frac{8}{15} \frac{1}{r^2}\right),
\]

as calculated in [13], where \( \hbar \) and \( c \) are put back in the expression and the relevant length scale appears to be the Planck length \( l_p = \sqrt{G\hbar/c^3} \).

Note that the mass scale \( M \) in Eq. (4.17) is irrelevant, because it contributes only to local terms \( \sim \delta^{(3)}(x) \). Therefore, the still remaining freedom in the normalization, namely the choice of \( M \), is irrelevant to physical quantum corrections of the Newtonian potential.

The correction in Eq. (4.33) is only a partial one, because we have taken into account only the photon loop graviton self-energy contribution and not the complete set of diagrams of order \( \kappa^4 \) contributing to these corrections, as, for example, the vertex correction or the double scattering. Therefore we cannot make any statement on the absolute sign and magnitude of the numerical factor in Eq. (4.33). To our knowledge, photon loop corrections to the exchanged graviton have not been considered yet.
5 Photon Self-Energy

We now turn to the calculation of the photon self-energy. Since the main features of the causal scheme have already been presented, we can go on more speedily. Moreover this graph is not interesting from the point of view of gauge invariance: it is trivially gauge invariant due to $dQ_AF^{\alpha\beta} = 0$.

In order to obtain the relevant $A'_2(x, y)$ and $R'_2(x, y)$, we carry out one graviton and one field-strength contraction between $T_1^A(x)$ and $T_1^A(y)$ and obtain

$$A'_2(x, y) =: F^{\alpha\gamma}(x)F^\mu\rho(y) : a'_2(x - y)_{\alpha\gamma|\mu\rho},$$

where

$$a'_2(x - y)_{\alpha\gamma|\mu\rho} := \frac{\kappa^2}{4} \left( - \eta_{\gamma\rho} D^{(+)}_{\gamma\mu} - \eta_{\gamma\mu} D^{(+)}_{\gamma\rho} - \eta_{\alpha\rho} D^{(+)}_{\alpha\mu} + 5\eta_{\alpha\mu} D^{(+)}_{\gamma\rho} + 2\eta_{\alpha\gamma} D^{(+)}_{\gamma\mu} + 2\eta_{\gamma\mu} D^{(+)}_{\alpha\rho} \right)(x - y),$$

with

$$d_p(\alpha\gamma)D^{(+)}_{\alpha\gamma} = \frac{1}{(2\pi)^4} \int d^4k \delta((p - k)^2) \Theta(p^0 - k^0) \delta(k^0) \Theta(k^0) k_\mu k_\rho.$$ 

The result for $R'_2(x, y)$ is analogous to that of $A'_2(x, y)$: the C-number tensor distribution $r'_2(x - y)_{\alpha\gamma|\mu\rho}$ depends on the $D^{(\pm)}_{\...}$ distributions so that in momentum space the factor $\Theta(-p^0)$ is present instead of $\Theta(p^0)$. The causal $D_2$-distribution is then given by

$$D_2(x, y)_{\alpha\gamma|\mu\rho} =: F^{\alpha\gamma}(x)F^\mu\rho(y) : d_2(x - y)_{\alpha\gamma|\mu\rho},$$

where

$$d_2(p)_{\alpha\gamma|\mu\rho} = \frac{\kappa^2}{48(2\pi)^4} \left[ 2p_\alpha p_\rho \eta_{\gamma\mu} - 2p_\gamma p_\rho \eta_{\alpha\mu} - 2p_\alpha p_\mu \eta_{\gamma\rho} + 2p_\gamma p_\mu \eta_{\alpha\rho} + p^2 \eta_{\gamma\mu} \eta_{\alpha\rho} + p^2 \eta_{\alpha\mu} \eta_{\gamma\rho} \right] \Theta(p^0) \Theta(p^0).$$

The result for $R_2(x, y)$ is analogous to that of $A_2(x, y)$: the C-number tensor distribution $r_2(x - y)_{\alpha\gamma|\mu\rho}$ depends on the $D^{(\pm)}_{\...}$-distributions so that in momentum space the factor $\Theta(-p^0)$ is present instead of $\Theta(p^0)$. The causal $D_2$-distribution is then given by

$$D_2(x, y)_{\alpha\gamma|\mu\rho} =: F^{\alpha\gamma}(x)F^\mu\rho(y) : d_2(x - y)_{\alpha\gamma|\mu\rho},$$

where

$$d_2(p)_{\alpha\gamma|\mu\rho} = \frac{\kappa^2}{48(2\pi)^4} \left[ \text{the same as in (5.4)} \right] \Theta(p^0) \Theta(p^0).$$

From Eq. (1.13) or from direct inspection, we ascertain that the singular order of the distribution is two. The distribution splitting is the same as in Sec. 4.2, Eq. (4.10), because the loop consists of massless particles. Therefore, the $T_2(x, y)$-distribution corresponding to the photon self-energy reads

$$T_2(x, y)_{\alpha\gamma|\mu\rho} =: F^{\alpha\gamma}(x)F^\mu\rho(y) : \left( - i \Pi(x - y)_{\alpha\gamma|\mu\rho} \right),$$

where

$$\Pi_2(p)_{\alpha\gamma|\mu\rho} = \frac{\kappa^2}{48(2\pi)^4} \left[ \text{the same as in (5.4)} \right] \log \left( \frac{-p^2 - i0}{M^2} \right).$$
$U(1)_{EM}$-perturbative gauge invariance to second order holds trivially:
\[ dQ_A T_2(x, y)^{PSE} = 0. \]
(5.7)

The photon self-energy 2-point distribution can be written also in terms of the photon field $A^\mu$:
\[
T_2(x, y)^{PSE} = A^\gamma(x) A^\rho(y) : \left(-4i \Pi(x - y)^{PSE}_{\gamma\rho}\right) = A^\gamma(x) A^\rho(y) : \left(-i \Pi(x - y)\right),
\]
(5.8)

by disregarding non-contributing divergences (in the adiabatic limit $g \to 1$ of Eq. (2.1)). Then $T_2(x, y)^{PSE}$ has the same structure as the photon self-energy in QED [7] and the ‘reduced’ photon self-energy tensor appearing Eq. (5.8) is
\[
\hat{\Pi}(p)_{\gamma\rho} = 4 p^\mu p^\alpha \hat{\Pi}(p)_{\alpha\gamma|\mu\rho}
\]
\[
= \frac{\kappa^2 \pi}{12(2\pi)^5} \left[p^2 p_{\gamma\rho} - p^4 \eta_{\gamma\rho}\right] \log\left(-\frac{p^2 - i0}{M^2}\right)
\]
(5.9)

and satisfies the Ward identity
\[ p^\gamma \hat{\Pi}(p)_{\gamma\rho} = 0 \]
(5.10)
as in QED.

As a last point, we discuss the normalization of $T_2(x, y)^{PSE}$ in the second line of Eq. (5.8) with singular order four from (5.9). Actually, we should consider the normalization of Eq. (5.6) with singular order two, but owing to the relation (5.9), the two analyses lead to the same conclusion. The normalization terms have the form
\[
N_2(x, y)^{PSE} = A^\gamma(x) A^\rho(y) : \left(-i N(\partial_x, \partial_y) \delta^{(4)}(x - y)\right) ,
\]
\[
\hat{N}(p)_{\gamma\rho} = \hat{N}(p)_{\gamma\rho}^{(0)} + \hat{N}(p)_{\gamma\rho}^{(2)} + \hat{N}(p)_{\gamma\rho}^{(4)}. \]
(5.11)

Due to the same reasons as pointed out in Sec. 4.4, the local normalization can be expressed in momentum space through the polynomials
\[
\hat{N}(p)_{\gamma\rho}^{(i)} = \Psi \left[c_i \eta_{\gamma\rho}\right],
\]
\[
\hat{N}(p)_{\gamma\rho}^{(2)} = \Psi \left[c_2 p_{\gamma\rho} + c_3 \eta_{\gamma\rho} p^2\right],
\]
\[
\hat{N}(p)_{\gamma\rho}^{(4)} = \Psi \left[c_4 p_{\gamma\rho}^2 + c_5 \eta_{\gamma\rho} p^4\right].
\]
(5.12)

Here, $c_1, \ldots, c_5$ are undetermined real numbers and $\Psi := \kappa^2 \pi/12(2\pi)^5$. Requiring (5.10) for $\hat{N}(p)_{\gamma\rho}^{(i)}$, $i = 0, 2, 4$, we reduce the freedom in the choice of the normalization parameters in the polynomials to
\[
\hat{N}(p)_{\gamma\rho}^{(0)} = 0, \quad \hat{N}(p)_{\gamma\rho}^{(2)} = c_2 \Psi \left[p_{\gamma\rho} - \eta_{\gamma\rho} p^2\right],
\]
\[
\hat{N}(p)_{\gamma\rho}^{(4)} = c_4 \Psi \left[p_{\gamma\rho}^2 + \eta_{\gamma\rho} p^4\right].
\]
(5.13)
The normalized photon self-energy tensor then reads

\[ \hat{\Pi}(p^2) = \left( \frac{p^\gamma p^\rho}{p^2} - \eta_{\gamma\rho} \right) \hat{\Pi}(p^2), \quad \text{with} \]

\[ \hat{\Pi}(p^2) := \Psi \left[ p^4 \log \left( -\frac{p^2 - i0}{M^2} \right) + c_2 p^2 + c_4 p^4 \right]. \quad (5.15) \]

In order to fix the remaining free parameters \( c_2 \) and \( c_4 \), we consider the total photon propagator in momentum space, defined as the sum of the free photon Feynman propagator and an increasing number of self-energy insertions \[ \hat{D}(p) \]

\[ \hat{D}(p)^{\mu\nu}_{\text{tot}} = +\eta_{\mu\nu} \hat{D}_0^\mu(p) + \hat{D}_0^\lambda(p) \hat{\Pi}(p)_{\lambda\mu} \hat{D}_0^\rho(p) + \]

\[ + \hat{D}_0^\rho(p) \hat{\Pi}(p)_{\mu\lambda} \hat{D}_0^\mu(p) \hat{\Pi}(p)_{\lambda\nu} \hat{D}_0^\lambda(p) + \ldots \quad (5.16) \]

with \( \hat{\Pi}(p)^{\mu\nu}_{\lambda\lambda} := (2\pi)^4 \hat{\Pi}(p)^{\mu\nu}_{\lambda\lambda} \). Multiplying \( \hat{D}_0^\mu(p)^{-1} \) and with \( \eta_{\rho\nu} \) we obtain

\[ \left[ \eta_{\mu\nu} \hat{D}_0^\lambda(p)^{-1} - \hat{\Pi}(p)^{\mu\lambda}_{\mu\rho} \right] \hat{D}_0^\rho(p) = \eta_{\rho\nu}. \quad (5.17) \]

The inverse of the total photon propagator is then

\[ (\hat{D}(p)_{\text{tot}})^{-1})_{\mu\nu} = (2\pi)^2 \left[ \eta_{\mu\nu}(-p^2 - i0) - (2\pi)^2 \hat{\Pi}(p)^{\mu\nu}_{\mu\nu} \right]. \quad (5.18) \]

Therefore, inverting the expression above, the total propagator reads

\[ \hat{D}(p)^{\mu\nu}_{\text{tot}} = \frac{1}{(2\pi)^2} \left[ \eta^{\mu\nu} \frac{1}{-p^2 - i0 - \hat{\Pi}(p^2)} + \frac{p^{\mu} p^{\nu}}{p^2} F(p^2) \right]. \quad (5.19) \]

The explicit form of the function \( F(p^2) \) is not important, because the last term vanishes between transversal photon operators. Photon mass- and coupling constant normalization

\[ \hat{\Pi}(p^2) \bigg|_{p^2=0} = 0 \quad \text{and} \quad \frac{\hat{\Pi}(p^2)^N}{p^2} \bigg|_{p^2=0} = 0, \quad (5.20) \]

yield \( c_2 = 0 \). The last parameter \( c_4 \) is non-essential, because it only shifts the mass scale \( M \). Therefore, all local finite ambiguities in \( (5.11) \) can be reduced to a single unknown normalization parameter.

This concludes our discussion of the loop graphs to second order in causal perturbation theory.

**Acknowledgements**

I would like to thank Prof. G. Scharf, Adrian Müller and Mark Wellmann for discussions and comments regarding these topics.
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