Normal Power Series Class of Distributions: Model, Properties and Applications

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Abstract: A new class of distributions, called as normal power series (NPS), which contains the normal one as a particular case, is introduced in this paper. This new class which is obtained by compounding the normal and power series distributions, is presented as an alternative to the class of skew-normal and Balakrishnan skew-normal distributions, among others. The density and distribution functions of this new class of distributions, are given by a closed expression which allows us to easily compute probabilities, moments and related measurements. Estimation of the parameters of this new model by maximum likelihood method via an EM-algorithm is given. Finally, some applications are shown as examples.

Keywords: Normal distribution; Power series distributions; EM algorithm; Maximum likelihood estimation.

Subject Classifications:

1. INTRODUCTION

Recently, many distributions to model lifetime data have been studied and generalized by compounding of some discrete and important lifetime distributions. Adamidis and Loukas (1998), introduced exponential-geometric (EG) distribution by compounding the exponential and geometric distributions. In the similar manner exponential-Poisson (EP), exponential-logarithmic (EL), exponential-power series (EPS), Weibull-geometric (WG) and Weibull-power series (WPS) distributions were introduced by Tahmasbi and Rezaei (2008), Chahkandi and Ganjali (2009), Barreto-Souza et al. (2011) and Morais and Barreto-Souza (2011), respectively.

In recent years, techniques for extending the family of normal distributions have been proposed. The method applied here can be considered an alternative to the well-known skew-normal distribution (Azzalini (1985)), whose properties (Azzalini (1986); Azzalini and Chiogna (2004)), estimation (Gupta and Gupta (2008)), diagnostics (Xie et al. (2009)), generalization (Gupta and Gupta (2004) and multivariate extension (Azzalini and Valle, 1996; Azzalini and Capitanio, 1999; Arnold and Beaver, 2002) have been widely developed. Other ways of obtaining skewed normal distributions have also been introduced, such as the Balakrishnan skew-normal density in Sharafi and Behboodian (2008), the variance-gamma process in Fung and Seneta (2007) and the generalized normal distribution in Nadarajah (2005), among others. Whenever the Fisher information matrix of this skew-normal model is singular for values of the added parameter λ, and the maximal likelihood estimate of this parameter can be infinite with a positive probability, an alternative model would be desirable.

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In this paper, we introduce a new generalization of the normal distribution which is called the NPS class of distributions and is denoted by $NPS(\mu, \sigma, \theta)$. The method used to insert the new parameter is described in Marshall and Olkin (1997) for the first time, where it was applied to the exponential and Weibull families. This method enables us to obtain explicit expressions for the probability density functions and survival functions and allow us to MLE-estimate via an EM-algorithm of the model parameters with the help of appropriate R software.

Note that all of these distribution have ranges on $(0, \infty)$, but in this paper we introduced the new generalization of normal distribution, which has range in $(-\infty, \infty)$. To begin with, we shall use the following notation throughout this paper: $\phi(\cdot)$ for the standard normal probability density function (pdf), $\phi_n(\cdot; \mu, \Sigma)$ for the pdf of $N_n(\mu, \Sigma)$ ($n$-variate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$), $\Phi_n(\cdot; \mu, \Sigma)$ for the cdf of $N_n(\mu, \Sigma)$ (in both singular and non-singular cases), simply $\Phi_n(\cdot; \Sigma)$ for the case when $\mu = 0$.

The rest of this paper is organized as follows. In Section 2, we define the class of NPS distributions. The density, hazard rate and survival functions and some of their properties are given in this section. In Section 3, we derive moments of NPS distributions by two method. In Section 4, we present some special distributions which are studied in details. Some properties of sub model of NPS distribution are studied in section 5. Estimation of the parameters by maximum likelihood method and inference for large sample are presented in Section 6. The EM-algorithm with a method evaluating the standard errors from the EM-algorithm is presented in Section 7. Simulation study is given in Section 8. Applications to two real data sets are given in Section 9. Finally, Section 10 concludes the paper.

### 2. The NPS class

Let $X_1, \ldots, X_N$ be a random sample from normal distribution with mean $\mu$ and variance $\sigma^2$ and let the random variable $N$ has a power series distributions (truncated at zero) with the probability mass function

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)},$$

where $a_n > 0$ depends only on $n$, $C(\theta) = \sum_{n=0}^{\infty} a_n \theta^n$ and $\theta \in (0, s)$ ($s$ can be $\infty$) is such that $C(\theta)$ is finite. Table 1 lists some particular cases of the truncated (at zero) power series distributions (geometric, Poisson, logarithmic, binomial and negative binomial). Detailed properties of power series distributions can be found in Noack (1950). Here, $C'(\theta)$, $C''(\theta)$ and $C'''(\theta)$ denote the first, second and third derivatives of $C(\theta)$ with respect to $\theta$, respectively.

| Distribution      | $a_n$ | $C(\theta)$ | $C'(\theta)$ | $C''(\theta)$ | $C'''(\theta)$ | $s$ |
|-------------------|-------|-------------|--------------|---------------|---------------|-----|
| Geometric         | 1     | $\theta(1-\theta)^{-1}$ | $(1-\theta)^{-2}$ | $2(1-\theta)^{-3}$ | $6(1-\theta)^{-4}$ | 1   |
| Poisson           | $n!^{-1}$ | $e^\theta - 1$ | $e^\theta$ | $e^\theta$ | $e^\theta$ | $\infty$ |
| Logarithmic       | $n^{-1}$ | $-\log(1-\theta)$ | $(1-\theta)^{-1}$ | $(1-\theta)^{-2}$ | $2(1-\theta)^{-3}$ | 1   |
| Binomial          | $\binom{k}{n}$ | $(1+\theta)^k - 1$ | $\frac{k}{(1+\theta)^1-k}$ | $\frac{k(k-1)}{(1+\theta)^2-k}$ | $\frac{k(k-1)(k-2)}{(1+\theta)^3-k}$ | $\infty$ |
| Neg. Binomial     | $\binom{n-1}{k-1}$ | $\frac{\mu^k}{(1-\theta)^k}$ | $\frac{k\mu^{k-1}}{(1-\theta)^{k+1}}$ | $\frac{k(k+2\theta-1)}{\theta^2-k(1-\theta)^{k+2}}$ | $\frac{k(k^2+6k\theta+6\theta^2-3k-6\theta+2)}{\theta^3-k(1-\theta)^{k+3}}$ | 1   |

Table 1. Quantities for power series distributions
Let \( Y = X_{(N)} = \max (X_1, \ldots, X_N) \), then the conditional cdf of \( Y|N = n \) is given by
\[
G_{Y|N=n}(y) = \left( \Phi \left( \frac{y - \mu}{\sigma} \right) \right)^n,
\]
where \( \Phi(\cdot) \) denotes the cdf of standard normal distribution.

The cdf of normal power series (NPS) class of distributions is defined by the marginal cdf of \( Y \), i.e.,
\[
F(y; \mu, \sigma, \theta) = \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C'(\theta)} \left( \Phi \left( \frac{y - \mu}{\sigma} \right) \right)^n = \frac{C \left( \theta \Phi \left( \frac{y - \mu}{\sigma} \right) \right)}{C(\theta)}, \quad y \in \mathbb{R}, \; \mu \in \mathbb{R}, \; \sigma > 0. \tag{2.1}
\]

We denote a random variable \( Y \) follows NPS distributions by \( NPS(\mu, \sigma, \theta) \). The density function of NPS follows immediately as:
\[
f(y; \mu, \sigma, \theta) = \frac{\theta}{\sigma} \phi \left( \frac{y - \mu}{\sigma} \right) \frac{C'(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{C(\theta)}.
\]

The corresponding survival and hazard rate functions are
\[
S(y; \mu, \sigma, \theta) = 1 - \frac{C(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{C(\theta)},
\]
and
\[
h(y; \mu, \sigma, \theta) = \frac{\theta}{\sigma} \phi \left( \frac{y - \mu}{\sigma} \right) \frac{C'(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{C(\theta) - C(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))},
\]
respectively.

One can put \( \mu = 0 \) and \( \sigma = 1 \) and obtains the standard version on NPS class of distributions denote by \( NPS(0, 1, \theta) \). Fig. 1 shows the probability density function and hazard rate function of the classical normal distribution and the NPS distributions proposed in this paper for some choices of \( C(\theta) \). It can be seen that the new model is very versatile and that the value of \( \theta \) has a substantial effect on the skewness of the probability density function.

**Proposition 2.1.** For the pdf and cdf of NPS class of distributions, we have
\[
\lim_{y \to \pm \infty} f(y; \mu, \sigma, \theta) = 0, \quad \lim_{y \to \pm \infty} h(y; \mu, \sigma, \theta) = \pm \infty. \tag{2.3}
\]

**Proposition 2.2.** Let \( c = \min\{n \in \mathbb{N} : a_n > 0\} \). As \( \theta \to 0^+ \) we have
\[
\lim_{\theta \to 0^+} F(y; \mu, \sigma, \theta) = \lim_{\theta \to 0^+} \frac{C(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{C(\theta)} = \lim_{\theta \to 0^+} \frac{\sum_{n=1}^{\infty} a_n\theta^n(\Phi \left( \frac{y - \mu}{\sigma} \right))^n}{\sum_{n=1}^{\infty} a_n\theta^n} = \frac{\sum_{n=1}^{c-1} a_n\theta^n(\Phi \left( \frac{y - \mu}{\sigma} \right))^n + a_c\theta^c(\Phi \left( \frac{y - \mu}{\sigma} \right))^c + \sum_{n=c+1}^{\infty} a_n\theta^n(\Phi \left( \frac{y - \mu}{\sigma} \right))^n}{\sum_{n=1}^{\infty} a_n\theta^n + a_c\theta^c + \sum_{n=c+1}^{\infty} a_n\theta^n} = \frac{\Phi \left( \frac{y - \mu}{\sigma} \right)^c + a_c^{-1} \sum_{n=c+1}^{\infty} a_n\theta^n(\Phi \left( \frac{y - \mu}{\sigma} \right))^n}{1 + a_c^{-1} \sum_{n=c+1}^{\infty} a_n\theta^n} = \left( \Phi \left( \frac{y - \mu}{\sigma} \right) \right)^c.
\]

**Proposition 2.3.** The densities of NPS class of distributions can be written as infinite number of linear combination of density of order statistics. We know that \( C'(\theta) = \sum_{n=1}^{\infty} na_n\theta^{n-1} \), therefore
\[
f(y; \mu, \sigma, \theta) = \frac{\theta}{\sigma} \phi \left( \frac{y - \mu}{\sigma} \right) \frac{C'(\theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{C(\theta)} = \sum_{n=1}^{\infty} g_{X(n)}(y; n)P(N = n),
\]
where \( g_{X(n)}(y; n) \) denotes the density function of \( X(n) = \max (X_1, \ldots, X_n) \).
Proposition 2.4. The \( \gamma \)th quantile of the NPS class of distributions is given by

\[
y_{\gamma} = \sigma \Phi^{-1} \left( \frac{C^{-1}(\gamma C(\theta))}{\theta} \right) + \mu.
\]

One can use this expression for generating a random sample from NPS distributions with generating data from uniform distribution.

3. Moments of the NPS distributions

In this section we give two methods to obtain the moments of the NPS distributions.

(i) First method

Jamalizadeh and Balakrishnan (2010) considered unified skew-elliptical distributions which contain unified skew-normal distribution as a special case. The univariate random variable \( Z_{k,\theta} \sim SN (k, \theta) \), if its pdf is

\[
\phi_{SN} (z; k, \theta) = \frac{\phi (z) \Phi_k (\lambda z + \gamma; \Omega)}{\Phi_k (\gamma; \Omega + \lambda \lambda^T)}.
\]

Furthermore, the moment generating function of \( Z_{k,\theta} \sim SN (k, \theta) \) is, for \( s \in \mathbb{R} \),

\[
M_{SN} (s; k, \theta) = \exp \left( \frac{1}{2} s^2 \right) \Phi_k (\lambda s + \gamma; \Omega + \lambda \lambda^T)
\]

The mean of \( Z_{k,\theta} \) can be obtained as the following lemma.

Lemma 3.1. If \( Z_{k,\theta} \sim SN (k, \theta) \), then

\[
E \left( Z_{k,\theta} \right) = \frac{1}{\Phi_k (\gamma; \Omega + \lambda \lambda^T)} \sum_{i=1}^{k} \frac{\lambda_i}{\omega_{ii} + \lambda_i^2} \phi \left( \frac{\gamma_i}{\sqrt{\omega_{ii} + \lambda_i^2}} \right)
\]

\[
\times \Phi_{k-1} \left( \gamma - \frac{\gamma_i}{\omega_{ii} + \lambda_i^2} (\omega_{-ii} + \lambda_i \lambda_{-i}) ; (\Omega + \lambda \lambda^T)_{-ii} \right),
\]

where, for some \( i \),

\[
\lambda = \left( \begin{array}{c} \lambda_i \\ \lambda_{-i} \end{array} \right), \quad \gamma = \left( \begin{array}{c} \gamma_i \\ \gamma_{-i} \end{array} \right), \quad \Omega = \left( \begin{array}{cc} \omega_{ii} & \omega_{-ii} \\ \omega_{-ii} & \Omega_{-ii} \end{array} \right),
\]

with \((\Omega + \lambda \lambda^T)_{-ii} = \Omega_{-ii}^{-1} + \lambda_{-i} \lambda_{-i}^T - (\omega_{-ii} + \lambda_i \lambda_{-i}) (\omega_{-ii} + \lambda_i \lambda_{-i})^T / \omega_{ii} + \lambda_i^2 \).

In the special case when \( \gamma = 0 \), the moments can be determined rather easily. If in this case the generalized skew-normal is denoted by \( Z_{k,\lambda,\Omega} \), we then have

\[
E \left( Z_{k,\lambda,\Omega} \right) = \frac{1}{\Phi_k (0; \Omega + \lambda \lambda^T) \sqrt{2\pi}} \sum_{i=1}^{k} \frac{\lambda_i}{\omega_{ii} + \lambda_i^2} \Phi_{k-1} \left( 0 ; (\Omega + \lambda \lambda^T)_{-ii} \right),
\]

where \((\Omega + \lambda \lambda^T)_{-ii} \) is as given above. In this case, was obtained a recurrence formula for the moments of \( Z_{k,\lambda,\Omega} \sim SN (k, \lambda, \Omega) \). For simplicity, they presented in the following lemma this recurrence formula for the case when \( \Omega \) is the correlation matrix.
Lemma 3.2. We have, for \( m = 1, 2, \cdots, \)
\[
E \left( Z_{k, \lambda, \Omega}^{m+1} \right) = mE \left( Z_{k, \lambda, \Omega}^{m-1} \right) + \frac{1}{\sqrt{2\pi} \Phi_k(0; \Omega + \lambda^T)} \sum_{i=1}^{k} \frac{\lambda_i \Phi_{k-1} \left( 0; \Omega - i\lambda + \lambda_i^* \lambda_i^T \right)}{(1 + \lambda_i^* \lambda_i^T)^{m+1/2}} E \left( Z_{k-1, \lambda_i^*, \Omega - i\lambda}^{m} \right),
\]
where \( \lambda_i^* = \frac{\lambda_i}{\sqrt{1 + \lambda_i^* \lambda_i^T}}. \)

In addition when \( X \sim N_n \left( \mu I_n, \sigma^2 \left( (1 - \rho)I_n + \rho I_n \right) \right), \mu \in \mathbb{R}, \sigma > 0, -\frac{1}{\sqrt{n}} < \rho < 1, \) they proved that
\[
\frac{X(n) - \mu}{\sigma} \sim SN(n-1, \theta),
\]
where
\[
\theta = \left( \sigma (1 - \rho) I_{n-1}, 0, \sigma^2 (1 - \rho) \{I_{n-1} + \rho J_{n-1} J_{n-1}^T \} \right).
\]

Proposition 3.1. If \( Y \sim NPS(0, 1, \theta), \) then the moment generating function, \( k \)th moment and mean of \( Y \) are given by
\[
M_Y(t) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} M_{X(n)}(t) = \exp \left( \frac{1}{2} t^2 \right) \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times n \Phi_{n-1} \left( 1_{n-1} ; I_{n-1} + 1_{n-1} \right),
\]
\[
E(Y^{k+1}) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} k E(Y^{k-1}) + \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times \frac{(n-1) \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2} 1_{n-2} \right)}{2 \sqrt{\pi} \Phi_{n-1} \left( 0; I_{n-1} + 1_{n-1} \right)} \times E \left( Z_{n-2, \sqrt{\frac{1}{2}} 1_{n-2}, I_{n-2}}^k \right),
\]
and
\[
E(Y) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times n(n-1) \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2} 1_{n-2} \right),
\]
respectively.

One can derive the second moment of NPS distributions as
\[
E(Y^2) = 1 + \frac{1}{4} \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times n(n-1) \Phi_{n-3} \left( 0; I_{n-3} + \frac{1}{2} 1_{n-3} \right).
\]

(ii) Second method

In the following proposition we present another formulation for calculate the \( k \)th moment around the origin
of the random variable $Y \sim NPS(\mu, \sigma, \theta)$. First, we give two well-known relationship, which are necessary in the following proposition. If $\Phi(x; \mu, \sigma)$ denotes the cdf of $N(\mu, \sigma^2)$ distribution, then we have

$$\Phi(x; \mu, \sigma) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x - \mu}{\sigma \sqrt{2}} \right) \right], \quad (3.2)$$

and

$$\Phi^{-1}(t; \mu, \sigma) = \mu + \sigma \sqrt{2} \text{erf}^{-1}(2t - 1). \quad (3.3)$$

**Proposition 3.2.** We have

$$E(Y^k) = \int_0^1 \left\{ \mu + \sqrt{2}\sigma \text{erf}^{-1} \left( \frac{2C^{-1}(C(\theta)u)}{\theta} - 1 \right) \right\}^k du. \quad (3.4)$$

**Proof.** We know that $E(Y^k) = \int_{-\infty}^{\infty} y^k f(y; \mu, \sigma, \theta) dy$. Then substituting $f(y; \mu, \sigma, \theta)$ from (2.2) and change of variable $\Phi(y; \mu, \sigma) = t$ gives

$$E(Y^k) = \frac{\theta}{C(\theta)} \int_0^1 \left[ \Phi^{-1}(t; \mu, \sigma) \right]^k C'(\theta t) dt.$$ 

Now, by changing the variable to $u = \frac{C(\theta)}{C(\theta)}$, we have

$$E(Y^k) = \int_0^1 \left\{ \Phi^{-1} \left( \frac{C^{-1}(C(\theta)u)}{\theta}; \mu, \sigma \right) \right\}^k du,$$

thus, the result follows from the Eq. (3.3). \qed

Table 1 contains values for $\mu_k$ ($k = 1, 2, 3, 4$) in a $NPS(0, 1, \theta)$ distributions for some values of $\theta$. Also, the variance and the skewness coefficient $Sk$ given by $Sk = \mu_3/\sigma^3$, is included in Table 1.

### 4. Special cases of NPS class of distributions

In this section four important sub-models of NPS class of distributions are studied in details. These models are normal geometric (NG), normal Poisson (NP), normal logarithmic (NL) and normal binomial (NB) distributions.

#### 4.1. Normal geometric distribution

Using Table II, the NPS distributions contain NG distribution when $a_n = 1$ and $C(\theta) = \frac{\theta}{1-\theta}$ (0 < $\theta$ < 1). Using Eq. (2.1), the cdf of NG is given by

$$F(y; \mu, \sigma, \theta) = \frac{(1 - \theta)\Phi \left( \frac{y - \mu}{\sigma} \right)}{1 - \theta\Phi \left( \frac{y - \mu}{\sigma} \right)}, \quad y \in \mathbb{R}, \quad (4.1)$$

The pdf and hazard rate functions of NG distribution are

$$f(y; \mu, \sigma, \theta) = \frac{(1 - \theta)\phi \left( \frac{y - \mu}{\sigma} \right)}{\sigma(1 - \theta\Phi \left( \frac{y - \mu}{\sigma} \right))^2}, \quad (4.2)$$

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Figure 1. Plots of density and hazard rate functions of NG distribution for selected parameter values \( \theta < 1 \) with \( \mu = 0, \sigma = 1 \).

and

\[
h(y; \mu, \sigma, \theta) = \frac{(1 - \theta)\phi\left(\frac{y-\mu}{\sigma}\right)}{\sigma(1 - \Phi\left(\frac{y-\mu}{\sigma}\right))(1 - \theta\Phi\left(\frac{y-\mu}{\sigma}\right))},
\]

respectively, where \( y \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0 \) and \( 0 < \theta < 1 \). We use the notation \( Y \sim NG(\mu, \sigma, \theta) \) when the random variable \( Y \) has NG distribution with location \( \mu \), scale \( \sigma \) and shape parameter \( \theta \).

Remark 4.1. Even when \( \theta \leq 0 \), Equation (4.2) is also a density function. We can then define the NG distribution by Equation (4.2) for any \( \theta < 1 \). Some special sub-models of the NG distribution are obtained as follows. If \( \theta = 0 \), we have the normal distribution. When \( \theta \to 1^- \), the NG distribution tends to a distribution degenerate in zero. Hence, the parameter \( \theta \) can be interpreted as a concentration parameter.

Figure 3 shows the NG density and hazard rate functions for selected values \( \theta \) where \( \mu = 0 \) and \( \sigma = 1 \).

Theorem 4.1. If \( Y_1 \sim NG(0,1,\theta_1) \) and \( Y_2 \sim NG(0,1,\theta_2) \), and \( \theta_1 > \theta_2 \), then \( Y_2 <_{LR} Y_1 \).

Proof. The logarithm of the likelihood ratio

\[
v(y) = \log \frac{f(y;0,1,\theta_1)}{f(y;0,1,\theta_2)} = \log \left(\frac{1 - \theta_1}{1 - \theta_2}\right) + 2\log(1 - \theta_2\Phi(y)) - 2\log(1 - \theta_1\Phi(y)),
\]

is an increasing function of \( y \) if \( \theta_1 > \theta_2 \), since

\[
v'(y) = \frac{2(\theta_1 - \theta_2)\phi(y)}{(1 - \theta_2\Phi(y))(1 - \theta_1\Phi(y))} > 0,
\]

for all \( y \). Therefore, the NG has the likelihood ratio ordering, which implies it has the failure rate ordering as well as the stochastic ordering and the mean residual life ordering.

Proposition 4.1. The moment generating function, mean and second central moment of \( NG \) are given by

\[
M_Y(t) = \exp\left(\frac{1}{2}t^2\right) \sum_{n=1}^{\infty} n(1 - \theta)n^{-1}\Phi_{n-1}\left(I_{n-1}t; I_{n-1} + 1_{n-1}\right),
\]
Table 2. The first four moments, variance, skewness and kurtosis of NG distribution for \( \mu = 0, \sigma = 1 \) and different values \( \theta \)

| \( \theta \)  | -5  | -2  | -0.5 | 0   | 0.3 | 0.5 | 0.8 | 0.9 |
|----------------|-----|-----|------|-----|-----|-----|-----|-----|
| \( E(X) \)     | -0.9841 | -0.6134 | -0.2284 | 0   | 0.2010 | 0.3894 | 0.8884 | 1.2445 |
| \( E(X^2) \)   | 1.8465 | 1.3270 | 1.0452 | 1   | 1.0350 | 1.1315 | 1.6887 | 2.3609 |
| \( E(X^3) \)   | -3.1487 | -1.6981 | -0.5795 | 0   | 0.5083 | 1.0155 | 2.7254 | 4.5206 |
| \( E(X^4) \)   | 7.2110 | 4.4974 | 3.1974 | 3   | 3.1526 | 3.5829 | 6.3424 | 10.313 |
| \( Var \)      | 0.8781 | 0.9508 | 0.9930 | 1   | 0.9946 | 0.9799 | 0.8995 | 0.8123 |
| \( Sk \)       | 0.4821 | 0.3046 | 0.1141 | 0   | -0.1004 | -0.1942 | -0.4371 | -0.5999 |
| \( Kur \)      | 3.5440 | 3.2104 | 3.0291 | 3   | 3.0225 | 3.0846 | 3.4429 | 3.8702 |

Figure 2. Plots of skewness and kurtosis of NG distribution for selected parameter values \( \theta < 1 \).

\[
E(Y) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} n(n-1)(1-\theta)\theta^{n-1} \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2}1_{n-2}^T 1_{n-2} \right),
\]

and

\[
E(Y^2) = 1 + \frac{1}{4\sqrt{3\pi}} \sum_{n=1}^{\infty} n(n-1)(n-3)(1-\theta)\theta^{n-1} \Phi_{n-3} \left( 0; I_{n-3} + \frac{1}{3}1_{n-3}^T 1_{n-3} \right),
\]

respectively.

Table 2 gives the first four moments, variance, skewness and kurtosis of the \( NG(0, 1, \theta) \) for different values \( \theta < 1 \). One can see from this table that.

Figure 2 shows the skewness and kurtosis plot of the \( NG(0, 1, \theta) \) for different values \( \theta < 1 \) with \( \mu = 0, \sigma = 1 \).
Figure 3. Plots of density and hazard rate functions of NP distribution for selected parameter values \( \theta < 1 \) with \( \mu = 0 \) and \( \sigma = 1 \).

4.2. Normal Poisson distribution

The normal Poisson distribution is obtained when \( a_n = \frac{1}{n!} \) and \( C(\theta) = e^\theta - 1 \). The cdf, pdf and hazard rate function of NP distribution are given by

\[
F(y; \mu, \sigma, \theta) = \frac{e^{\theta \Phi(y - \mu / \sigma)} - 1}{e^\theta - 1}, y \in \mathbb{R},
\]

\[
f(y; \mu, \sigma, \theta) = \frac{\theta \phi(y / \sigma) e^{\theta \Phi(y / \sigma)}}{e^\theta - 1},
\]

and

\[
h(y; \mu, \sigma, \theta) = \frac{\theta \phi(y / \sigma) e^{\theta \Phi(y / \sigma)}}{e^\theta - e^{\theta \Phi(y / \sigma)}},
\]

respectively, where \( y \in \mathbb{R} \) and \( \theta \in (0, \infty) \). We use the notation \( Y \sim NP(\mu, \sigma, \theta) \) when the random variable \( Y \) has NP distribution with location \( \mu \), scale \( \sigma \) and shape parameter \( \theta \).

Remark 4.2. Even when \( \theta < 0 \), Equation (4.3) is also a density function. We can then define the NG distribution by Equation (4.3) for any \( \theta \in \mathbb{R} - \{0\} \).

Figure ?? shows the NP density and hazard rate functions for selected values \( \theta \) where \( \mu = 0 \) and \( \sigma = 1 \).

Theorem 4.2. If \( Y_1 \sim NP(\mu, \sigma, \theta_1) \) and \( Y_2 \sim NP(\mu, \sigma, \theta_2) \), and \( \theta_1 > \theta_2 \), then \( Y_2 <_{LR} Y_1 \).

Proof. The proof is similar to the proof of Theorem 4.1 and is omitted.

Proposition 4.2. The moment generating function, mean and second central moment of NP are given by

\[
M_Y(t) = \exp \left( \frac{1}{2} t^2 \sum_{n=1}^{\infty} \frac{\theta^n}{n!(e^\theta - 1)} \right) \times n \Phi_{n-1}(1_{n-1}t; I_{n-1} + 1_{n-1}1_{n-1}^T),
\]
Table 3. The first four moments, variance, skewness and kurtosis of NP distribution for $\mu = 0$, $\sigma = 1$ and different values $\theta$.

| $\theta$ | $E(X)$ | $E(X^2)$ | $E(X^3)$ | $E(X^4)$ | $V ar$ | $Sk$ | $Kur$ |
|----------|--------|----------|----------|----------|-------|------|-------|
| 0.01     | 0.0028 | 1.0000   | 0.0071   | 3.0000   | 1.0000 | -0.0014 | 3.0000 |
| 0.3      | 0.0845 | 1.0041   | 0.2114   | 3.0179   | 0.9970 | -0.0421 | 3.0074 |
| 0.5      | 0.1405 | 1.0114   | 0.3520   | 3.0495   | 0.9917 | -0.0697 | 3.0204 |
| 0.8      | 0.2236 | 1.0290   | 0.5617   | 3.1259   | 0.9790 | -0.1097 | 3.0515 |
| 1        | 0.2781 | 1.0450   | 0.7003   | 3.1954   | 0.9677 | -0.1349 | 3.0792 |
| 3        | 0.7541 | 1.3477   | 2.0013   | 4.5372   | 0.7790 | -0.2764 | 3.5076 |
| 6        | 1.1997 | 1.9673   | 3.5904   | 7.4821   | 0.5279 | -0.0956 | 3.6846 |
| 10       | 1.5045 | 2.6533   | 5.2127   | 11.2262  | 0.3898 | 0.1973  | 3.4236 |

Figure 4. Plots of skewness and kurtosis of NP distribution for selected parameter values $\theta$.

$$E(Y) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{n(n-1)\theta^n}{n!(e^\theta - 1)} \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2} I_{n-2}^T \right),$$

and

$$E(Y^2) = 1 + \frac{1}{4\sqrt{3\pi}} \sum_{n=1}^{\infty} \frac{\theta^n}{n!(e^\theta - 1)} \times n(n-1)(n-3) \Phi_{n-3} \left( 0; I_{n-3} + \frac{1}{3} I_{n-3}^T \right).$$

Table 3 gives the first four moments, variance, skewness and kurtosis of the $NP(0, 1, \theta)$ for different values $\theta > 0$. One can see from this table that.

Figure 4 shows the skewness and kurtosis plot of the $NP(0, 1, \theta)$ for different values $\theta$. 
Figure 5. Plots of density and hazard rate functions of NB distribution for selected parameter values $\theta$ with $\mu = 0, \sigma = 1$.

### 4.3. Normal-binomial distributions

When $a_n = \binom{m}{n}$ and $C(\theta) = (\theta + 1)^m - 1 (\theta > 0)$, where $m (n \leq m)$ is the number of replicas, we obtain the Normal-binomial distribution ($NB$) with cdf

$$F(y; \mu, \sigma, \theta) = \frac{(\theta \Phi(\frac{y-\mu}{\sigma}) + 1)^m - 1}{(\theta + 1)^m - 1}, \; y \in \mathbb{R},$$

(4.4)

the pdf and hazard rate function are

$$f(y; \mu, \sigma, \theta) = \frac{\theta}{\sigma} \phi \left( \frac{y - \mu}{\sigma} \right) \frac{m(\theta \Phi(\frac{y-\mu}{\sigma}) + 1)^{m-1}}{(\theta + 1)^m - 1},$$

(4.5)

and

$$h(y; \mu, \sigma, \theta) = \frac{\theta}{\sigma} \phi \left( \frac{x - \mu}{\sigma} \right) \frac{m(\theta \Phi(\frac{y-\mu}{\sigma}) + 1)^{m-1}}{(\theta + 1)^m - (\theta \Phi(\frac{y-\mu}{\sigma}) + 1)^m},$$

respectively, where $y \in \mathbb{R}$ and $\theta \in (0, \infty)$. We use the notation $Y \sim NB(\mu, \sigma, \theta)$ when the random variable $Y$ has NB distribution with location $\mu$, scale $\sigma$ and shape parameter $\theta$.

**Remark 4.3.** Even when $\theta < 0$, Equation (4.5) is also a density function. We can then define the NB distribution by Equation (4.5) for any $\theta \in \mathbb{R} \setminus \{0\}$.

Figure 5 shows the NB density and hazard rate functions for selected values $\theta$ where $\mu = 0$ and $\sigma = 1$.

**Proposition 4.3.** The moment generating function, mean and second central moment of NB are given by

$$M_X(t) = \exp \left( \frac{1}{2} t^2 \right) \sum_{n=1}^{\infty} \frac{\theta^n}{(\theta + 1)^m - 1} \times n \Phi_{n-1}(1_{n-1}; I_{n-1} + 1_{n-1} I_{n-1}),$$

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\[ E(Y) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \binom{m}{n} \frac{\theta^n}{(\theta + 1)^m - 1} n(n-1) \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2} I_{n-2}^T \right), \]

and

\[ E(Y^2) = 1 + \frac{1}{4\sqrt{3\pi}} \sum_{n=1}^{\infty} \binom{m}{n} \frac{n(n-2)(n-3)\theta^n}{(\theta + 1)^m - 1} \Phi_{n-3} \left( 0; I_{n-3} + \frac{1}{3} I_{n-3}^T \right), \]

### 4.4. Normal-logarithmic distributions

When \( a_n = \frac{1}{n} \) and \( c(\theta) = -\log(1 - \theta)(0 < \theta < 1) \) we obtain the Normal-logarithmic distribution (\( NL \)) with cdf

\[ F(y; \mu, \sigma, \theta) = \frac{\log(1 - \theta \Phi \left( \frac{y - \mu}{\sigma} \right))}{\log(1 - \theta)}, \quad y \in \mathbb{R}. \]

The pdf and hazard rate function are

\[ f(y; \mu, \sigma, \theta) = \frac{\theta \phi \left( \frac{y - \mu}{\sigma} \right)}{(\theta \Phi \left( \frac{y - \mu}{\sigma} \right) - 1) \log(1 - \theta)}, \]

and

\[ h(y; \mu, \sigma, \theta) = \frac{\theta \phi \left( \frac{y - \mu}{\sigma} \right)}{(\theta \Phi \left( \frac{y - \mu}{\sigma} \right) - 1) \log \frac{1 - \theta}{1 - \theta \Phi \left( \frac{y - \mu}{\sigma} \right)}}, \]

respectively, where \( y \in \mathbb{R} \) and \( \theta \in (0, 1) \). We use the notation \( Y \sim NL(\mu, \sigma, \theta) \) when the random variable \( Y \) has NL distribution with location \( \mu \), scale \( \sigma \) and shape parameter \( \theta \).

**Remark 4.4.** Even when \( \theta < 0 \), Equation (4.6) is also a density function. We can then define the NL distribution by Equation (4.6) for any \( \theta \in (-\infty, 0) \cup (0, 1) \).

Figure 6 shows the NL density and hazard rate functions for selected values \( \theta \) where \( \mu = 0 \) and \( \sigma = 1 \).

**Proposition 4.4.** The moment generating function, mean and second central moment of NL are given by

\[ M_Y(t) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} M_Y^{(n)}(t) = \exp \left( \frac{1}{2} t^2 \right) \sum_{n=1}^{\infty} \frac{\theta^n}{n \log(1 - \theta)} \times \Phi_{n-1} \left( 1_{n-1} t; I_{n-1} + 1_{n-1} I_{n-1}^T \right), \]

\[ E(Y) = -\frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\theta^n}{n \log(1 - \theta)} (n-1) \Phi_{n-2} \left( 0; I_{n-2} + \frac{1}{2} I_{n-2}^T \right), \]

and

\[ E(Y^2) = 1 - \frac{1}{4\sqrt{3\pi}} \sum_{n=1}^{\infty} \frac{n(n-2)(n-3)\theta^n}{n \log(1 - \theta)} \Phi_{n-3} \left( 0; I_{n-3} + \frac{1}{3} I_{n-3}^T \right), \]

respectively.
Figure 6. Plots of density and hazard rate functions of NL distribution for selected parameter values $\theta$ with $\mu = 0$, $\sigma = 1$.

5. Properties of sub model of NPS distribution

In this section we present some additional and useful properties of the sub models of NPS distribution.

Proposition 5.1. For NG, NB and NL distributions we have

$$F(y; 0, 1, \theta) = 1 - F \left( -y; 0, 1, \frac{\theta}{\theta - 1} \right),$$

and for NP distribution we have

$$F(y; 0, 1, \theta) = 1 - F (-y; 0, 1, -\theta),$$

Proof. We shall prove for NG distribution. Proofs of other distributions are similar. From 4.1 it can be found that

$$1 - F(-y; 0, 1, \frac{\theta}{\theta - 1}) = 1 + \frac{\frac{1}{\sqrt{\frac{\theta - 1}{\theta}}} \Phi(-y)}{1 - \frac{\theta - 1}{\theta} \Phi(-y)} = 1 + \frac{1 - \Phi(y)}{\theta \Phi(y) - 1}$$

and hence the proof is completed.

In the following proposition we give approximations for first and second moments around the origin of NG, NP and NB distributions.

Proposition 5.2. We have

(i) If $Y \sim NG(\mu, \sigma, \theta)$, then $E(Y)$ and $E(Y^2)$ are approximated as:

$$E(Y) \simeq \frac{1}{2\theta^2} \left\{ 2\theta^2 \mu + \sqrt{2\pi}\sigma \left( 2 \log (1 - \theta)(1 - \theta) + 2\theta - \theta^2 \right) \right\},$$
\[ E(Y^2) \simeq \frac{1}{4\theta(\theta - 1)} \left\{ \theta^3 \left( \pi \sigma^2 + 2\mu^2 - 2\sqrt{2/\pi} \frac{\pi \sigma}{2} \right) + 16\pi \theta \sigma^2 + 28\theta^2 \left( \sqrt{2\pi} \mu - 2\pi \sigma \right) + 8\sigma \log(1 - \theta) \left( \sqrt{2\pi} \theta \mu (1 - \theta) + (\theta (\theta - 3) + 2) \pi \sigma \right) \right\}, \]

(ii) If \( Y \sim NP(\mu, \sigma, \theta) \), then \( E(Y) \) and \( E(Y^2) \) are approximated as:

\[ E(Y) \simeq \frac{1}{2\theta \sigma (e^\theta - 1)} \left\{ (2 + \theta) \sigma \sqrt{2\pi} - 2\theta \mu + \left( 2\theta \mu + (\theta - 2) \sigma \sqrt{2\pi} \right) e^\theta \right\}, \]

\[ E(Y^2) \simeq \frac{1}{2\theta^2 (e^\theta - 1)} \left\{ \sigma \theta \left( 4\sqrt{2\pi} + 2\sqrt{2\pi \theta} \right) - 8\pi \sigma^2 - \theta^2 (\pi \sigma^2 + 2\mu^2) - 4\pi \theta^2 \right. \]
\[ \left. \quad + \left( 2\theta^2 \mu^2 + 8\pi \sigma^2 - 4\pi \theta^2 \sigma^2 + \pi \theta^2 \sigma^2 + 2\sqrt{2\pi \theta^2} \mu \sigma - 4\sqrt{2\pi \theta} \sigma \mu \right) e^\theta \right\}. \]

(iii) If \( Y \sim NB(\mu, \sigma, \theta) \), then \( E(Y) \) and \( E(Y^2) \) are approximated as:

\[ E(Y) \simeq \frac{1}{2\theta \mu (\theta + 1)^m - 1} \left\{ 2\sqrt{2\pi} \sigma + \left( \sqrt{2\pi} \sigma (1 + m) - 2\mu (1 + m) \right) \theta \right. \]
\[ \left. \quad - \left[ 2\sqrt{2\pi} \sigma (1 + m) + \left( \sqrt{2\pi} \sigma (1 + m) - 2\mu (1 + m) \right) \theta - 2\sqrt{2\pi} \mu \sigma (\theta + 1) \right] (\theta + 1)^m \right\}. \]

\[ E(Y^2) \simeq \frac{1}{\theta^2 (\theta + 1)^m - 1 (m + 2)(m + 1)} \left\{ (-2\mu^2 + \sqrt{2\pi} \mu) \sigma - \frac{3}{2} \pi m \sigma^2 - \pi \sigma^2 - 3m \mu^2 \right. \]
\[ + \left. \left( 3\sqrt{2\pi} \mu \sigma \mu - m \mu^2 - \frac{1}{2} \pi m \sigma^2 + 2\sqrt{2\pi \sigma \mu} \theta + (4\sqrt{2\pi} \mu + 2\sqrt{2\pi} \sigma \mu - 2\pi m \sigma^2 - 4\pi \sigma^2) \theta \right. \right. \]
\[ \left. \quad + \left[ (2\mu^2 + 3m \mu^2 + \pi \sigma^2 + \sqrt{2\pi} \mu \sigma + 2\sqrt{2\pi} \sigma \mu - 2\sqrt{2\pi} \mu \sigma - \frac{1}{2} \pi m \sigma^2 + \frac{1}{2} \pi m^2 \sigma^2 + m^2 \mu^2) \theta^2 \right. \right. \]
\[ \left. \quad + \left. 4\pi \sigma^2 + (4\pi \sigma^2 - 2\pi m \sigma^2 - 4\sqrt{2\pi} \sigma \mu - 2\sqrt{2\pi} \sigma \mu \theta] (\theta + 1)^m \right\}. \]

**Proof.** For (i), note first that

\[ E(Y) = \int_{-\infty}^{\infty} \frac{(1 - \theta) y \phi(\mu, \sigma, \theta)}{(1 - \theta \Phi(\mu, \sigma, \theta))^2}. \]

After the change of variable \( u = 1 - \theta \Phi(\mu, \sigma, \theta) \), we obtain \( y = \Phi^{-1} \left( \frac{1 - u}{\sigma} ; \mu, \sigma \right) = \mu + \sqrt{2\sigma} \text{erf}^{-1} \left( 2 \left( \frac{1 - u}{\sigma} \right) - 1 \right) \).

Now because \( \text{erf}^{-1}(z) = z \frac{\sqrt{\pi}}{2} + O(z^3) \), we can write \( \text{erf}^{-1}(z) \simeq z \frac{\sqrt{\pi}}{2} \). Therefore we have

\[ \mu_1 = -\frac{1 - \theta}{\theta} \int_1^{1 - \theta} \frac{\mu + \sqrt{2\sigma} \left( \frac{1 - u}{\sigma} \right) - 1}{u^2} du. \]

the result is obtained by solving the integral. Finally, \( E(Y^2) \) is derived in the same manner after simple computation. Parts (ii) and (iii) follow in a same way.

6. Estimation and inference

In this section, we discuss the estimation of the parameters of \( NPS \) distribution. Let \( Y_1, Y_2, ..., Y_n \) be a random sample with observed values \( y_1, y_2, ..., y_n \) from a \( NPS(\mu, \sigma, \theta) \) and \( \Psi = (\mu, \sigma, \theta)^T \) be a parameter vector. The total log-likelihood function is given by

\[ l_n = l_n(\Theta; y) = n \log(\theta) - n \log(\sigma) - \frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{n} z_i^2 + \sum_{i=1}^{n} \log(C' (\theta \Phi(z_i)) - n \log(C(\theta)), \]
where \( z_i = \frac{y_i - \mu}{\sigma} \). The maximum likelihood estimation (MLE) of \( \Psi \), say \( \hat{\Psi} \), is obtained by solving the nonlinear system \( \left( \frac{\partial l_n}{\partial \mu}, \frac{\partial l_n}{\partial \sigma}, \frac{\partial l_n}{\partial \theta} \right)^T = 0 \), where

\[
\begin{align*}
\frac{\partial l_n}{\partial \mu} &= \frac{1}{\sigma} \sum_{i=1}^{n} z_i - \frac{\theta}{\sigma} \sum_{i=1}^{n} \frac{\phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))}, \\
\frac{\partial l_n}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} z_i^2 - \frac{\theta}{\sigma} \sum_{i=1}^{n} \frac{z_i \phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))}, \\
\frac{\partial l_n}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \frac{\Phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} - \frac{nC'(\theta)}{C(\theta)}.
\end{align*}
\]

The solution of this nonlinear system of equation has not a closed form. The observed information matrix is obtained for approximate confidence intervals and hypothesis tests of the vector. The \( 3 \times 3 \) observed information matrix is

\[
I_n(\Psi) = -\begin{bmatrix}
I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\theta} \\
I_{\mu\sigma} & I_{\sigma\sigma} & I_{\sigma\theta} \\
I_{\mu\theta} & I_{\sigma\theta} & I_{\theta\theta}
\end{bmatrix},
\]

where

\[
\begin{align*}
I_{\mu\mu} &= -\frac{n}{\sigma^2} - \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \left[ \frac{z_i \phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} - \theta C''(\theta\Phi(z_i)) \phi^2(z_i) \right] \frac{C'(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} \left( C'(\theta\Phi(z_i)) \right)^2, \\
I_{\mu\sigma} &= -\frac{2}{\sigma^2} \sum_{i=1}^{n} z_i + \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \frac{\phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} \\
&+ \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \left[ \frac{z_i^2 \phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} - \theta z_i \phi^2(z_i) \right] \frac{C'(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} \left( C'(\theta\Phi(z_i)) \right)^2, \\
I_{\mu\theta} &= -\frac{1}{\sigma} \sum_{i=1}^{n} \frac{\phi(z_i)C''(\theta\Phi(z_i))}{C'(\theta\Phi(z_i))} \\
&- \frac{\theta}{\sigma} \sum_{i=1}^{n} \frac{\Phi(z_i) \phi(z_i) C''(\theta\Phi(z_i)) C'(\theta\Phi(z_i)) - \Phi(z_i) \phi(z_i) \left( C'(\theta\Phi(z_i)) \right)^2}{(C'(\theta\Phi(z_i)))^2},
\end{align*}
\]
\[
I_{\sigma\sigma} = \frac{n}{\sigma^2} - \frac{3}{\sigma^2} \sum_{i=1}^{n} z_i^2 + \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \frac{z_i \phi(z_i) C''(\theta \Phi(z_i))}{C'(\theta \Phi(z_i))} \\
+ \frac{\theta}{\sigma^2} \sum_{i=1}^{n} \left[ (z_i^2 \phi(z_i) - z_i \phi(z_i)) C''(\theta \Phi(z_i)) - \theta z_i^2 \phi^2(z_i) C''(\theta \Phi(z_i)) \right] C'(\theta \Phi(z_i)) \\
- \frac{\theta^2}{\sigma^2} \sum_{i=1}^{n} \frac{z_i^2 \phi^2(z_i) (C''(\theta \Phi(z_i)))^2}{(C'(\theta \Phi(z_i)))^2},
\]

\[
I_{\sigma\theta} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} \frac{z_i \phi(z_i) C''(\theta \Phi(z_i))}{C'(\theta \Phi(z_i))} \frac{C'(\theta \Phi(z_i)) - \theta z_i \phi(z_i) \Phi(z_i) \left( C''(\theta \Phi(z_i)) \right)^2}{(C'(\theta \Phi(z_i)))^2},
\]

\[
I_{\theta\theta} = -\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{\Phi^2(z_i) C''(\theta \Phi(z_i)) C'(\theta \Phi(z_i)) - \Phi^2(z_i) \left( C''(\theta \Phi(z_i)) \right)^2}{(C'(\theta \Phi(z_i)))^2} \\
- \frac{n C''(\theta)}{C(\theta)} + n \frac{(C'(\theta))^2}{(C(\theta))^2}.
\]

The asymptotic distribution of \( \sqrt{n} \left( \hat{\Psi} - \Psi \right) \) is \( N_3(0, I_n(\Psi)^{-1}) \), that can be used to create approximate intervals and confidence regions for the parameters and for the hazard and the survival functions. An 100(1 - \( \gamma \)) asymptotic confidence interval for each parameter \( \Psi_r \) is given by

\[
ACI_r = \left( \hat{\Psi}_r - Z_{\gamma/2} \sqrt{I_{\hat{\Psi}rr}}, \hat{\Psi}_r + Z_{\gamma/2} \sqrt{I_{\hat{\Psi}rr}} \right),
\]

where \( \hat{\Psi}_{rr} \) is the \((r,r)\) diagonal element of \( I_n(\hat{\Psi})^{-1} \) for \( r = 1, 2, 3 \) and \( Z_{\gamma/2} \) is the quantile \( 1 - \gamma/2 \) of the standard normal distribution.

7. EM-algorithm

The EM algorithm is one such elaborate technique. The EM algorithm is a general method of finding the maximum likelihood estimate of the parameters of an underlying distribution from a given data set when the data is incomplete or has missing values. There are two main applications of the EM algorithm. The first occurs when the data indeed has missing values, due to problems with or limitations of the observation process. The second occurs when optimizing the likelihood function is analytically intractable but when the likelihood function can be simplified by assuming the existence of and values for additional but missing (or hidden) parameters.

We define a hypothetical complete-data distribution with a joint probability density function in the form

\[
g(z, y; \Psi) = \frac{a z^\theta}{\sigma C(\theta)} \Phi \left( \frac{y_i - \mu}{\sigma} \right) \Phi^{z-1} \left( \frac{y_i - \mu}{\sigma} \right),
\]

where \( \sigma > 0, \theta, y \in R \) and \( z \in N \). The probability density function of \( Z \) given \( Y = y \) is given by

\[
g(z | y) = \frac{g(z, y; \Psi)}{f(y)} = \frac{a z^{\theta-1} \Phi^{z-1} \left( \frac{y_i - \mu}{\sigma} \right)}{C'(\theta \Phi \left( \frac{y_i - \mu}{\sigma} \right))},
\]

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After some simple calculation we have
\[
E(Z \mid Y = y) = 1 + \frac{\theta \Phi \left( \frac{y_i - \mu}{\sigma} \right) C'' \left( \theta \Phi \left( \frac{y_i - \mu}{\sigma} \right) \right)}{C' \left( \theta \Phi \left( \frac{y_i - \mu}{\sigma} \right) \right)}.
\]

The complete-data log-likelihood has the form
\[
l_n^* (y, z; \mu, \sigma, \theta) \propto \sum_{i=1}^{n} z_i \log \theta - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2 + \sum_{i=1}^{n} (z_i - 1) \log \Phi \left( \frac{y_i - \mu}{\sigma} \right) - n \log C(\theta).
\]

The components of the score function \(U_c(y, z; \Psi) = \left( \frac{\partial l_n^*}{\partial \mu}, \frac{\partial l_n^*}{\partial \sigma}, \frac{\partial l_n^*}{\partial \theta} \right)\), are
\[
\begin{align*}
\frac{\partial l_n^*}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( y_i - \mu \right) - \frac{1}{\sigma} \sum_{i=1}^{n} (z_i - 1) \frac{\phi \left( \frac{y_i - \hat{\mu}^{(h+1)}}{\hat{\sigma}^{(h+1)}} \right)}{\Phi \left( \frac{y_i - \hat{\mu}^{(h+1)}}{\hat{\sigma}^{(h+1)}} \right)}, \\
\frac{\partial l_n^*}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu)^2 - \frac{1}{\sigma^2} \sum_{i=1}^{n} (z_i - 1) \left( y_i - \hat{\mu}^{(h)} \right) \frac{\phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h)}} \right)}{\Phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h)}} \right)}, \\
\frac{\partial l_n^*}{\partial \theta} &= \frac{1}{\theta} \sum_{i=1}^{n} \left( y_i - \hat{\mu}^{(h+1)} \right) - n C'(\theta) \frac{C''(\theta)}{C(\theta)}.
\end{align*}
\]

The maximum likelihood estimates can be obtained from the iterative algorithm given by
\[
\begin{align*}
\frac{1}{\hat{\sigma}^{(h)}} \sum_{i=1}^{n} \left( y_i - \hat{\mu}^{(h+1)} \right) - \sum_{i=1}^{n} \left( \hat{z}_i^{(h)} - 1 \right) \frac{\phi \left( \frac{y_i - \hat{\mu}^{(h+1)}}{\hat{\sigma}^{(h+1)}} \right)}{\Phi \left( \frac{y_i - \hat{\mu}^{(h+1)}}{\hat{\sigma}^{(h+1)}} \right)} &= 0, \\
\frac{1}{\left( \hat{\sigma}^{(h+1)} \right)^2} \sum_{i=1}^{n} \left( y_i - \hat{\mu}^{(h)} \right)^2 - \frac{1}{\hat{\sigma}^{(h+1)}} \sum_{i=1}^{n} \left( \hat{z}_i^{(h)} - 1 \right) \left( y_i - \hat{\mu}^{(h)} \right) \frac{\phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h+1)}} \right)}{\Phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h+1)}} \right)} - n &= 0, \\
\hat{\mu}^{(h+1)} &= \frac{C(\hat{\mu}^{(h+1)})}{n C'(\hat{\theta}^{(h+1)})} \sum_{i=1}^{n} \hat{z}_i^{(h)},
\end{align*}
\]

where \(\hat{\mu}^{(h+1)}, \hat{\sigma}^{(h+1)}\) and \(\hat{\theta}^{(h+1)}\) are found numerically. Here, for \(i = 1, \ldots, n\), we have that
\[
\hat{z}_i^{(h+1)} = 1 + \frac{\hat{\theta}^{(h)} \Phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h)}} \right) C'' \left( \hat{\theta}^{(h)} \Phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h)}} \right) \right)}{C' \left( \hat{\theta}^{(h)} \Phi \left( \frac{y_i - \hat{\mu}^{(h)}}{\hat{\sigma}^{(h)}} \right) \right)}.
\]

**7.1. Evaluation of the standard errors from the EM-algorithm**

By using the results of Louis (1982) we obtain the standard errors of the estimators from the EM-algorithm. The elements of the \(3 \times 3\) observed information matrix \(I_c(\Psi; y, z) = -\left[ \frac{\partial U_c(y, z; \Psi)}{\partial \psi} \right] \) are given by
\[ \frac{\partial^2 l_n}{\partial \mu^2} = \frac{n}{\sigma^2} \sum_{i=1}^{n} (z_i - 1) \frac{(y_i - \mu)}{\sigma} \frac{\phi \left( \frac{y_i - \mu}{\sigma} \right) \Phi \left( \frac{y_i - \mu}{\sigma} \right) + \phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ \frac{\partial^2 l_n}{\partial \mu \partial \sigma} = \frac{\partial}{\partial \sigma} \frac{\partial^2 l_n}{\partial \mu^2} = \frac{2}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu) \frac{1}{\sigma^2} \sum_{i=1}^{n} (z_i - 1) \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ \frac{\partial^2 l_n}{\partial \sigma^2} = -\frac{n}{\sigma^2} + 3 \frac{n}{\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2 + \frac{1}{\sigma^2} \sum_{i=1}^{n} (z_i - 1) \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ \frac{\partial^2 l_n}{\partial \theta^2} = \frac{1}{\theta^2} \sum_{i=1}^{n} z_i + n \frac{C''(\theta) C(\theta) - (C'(\theta))^2}{C^2(\theta)}, \]

Taking the conditional expectation of \( I_c (\Psi; y, z) \) given \( y \), we obtain the 3 \times 3 matrix

\[ l_c (\Psi; y, z) = E \left( I_c (\Psi; y, z) \mid y \right) \right] = [c_{ij}] \]

\[ \text{(7.1)} \]

where

\[ c_{11} = \frac{n}{\sigma^2} + \frac{n}{\sigma^2} \sum_{i=1}^{n} \left( E(Z_i \mid y) - 1 \right) \frac{(y_i - \mu)}{\sigma} \frac{\phi \left( \frac{y_i - \mu}{\sigma} \right) \Phi \left( \frac{y_i - \mu}{\sigma} \right) + \phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ c_{21} = \frac{2}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu) \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( E(Z_i \mid y) - 1 \right) \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ + \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( E(Z_i \mid y) - 1 \right) \frac{(y_i - \mu)^2 \phi \left( \frac{y_i - \mu}{\sigma} \right) \Phi \left( \frac{y_i - \mu}{\sigma} \right) - (y_i - \mu) \phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ c_{22} = -\frac{n}{\sigma^2} + \frac{3}{\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2 + \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( E(Z_i \mid y) - 1 \right) \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ + \frac{1}{\sigma^3} \sum_{i=1}^{n} \left( E(Z_i \mid y) - 1 \right) (y_i - \mu) \frac{(y_i - \mu)^2 \phi \left( \frac{y_i - \mu}{\sigma} \right) \Phi \left( \frac{y_i - \mu}{\sigma} \right) - (y_i - \mu) \phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi^2 \left( \frac{y_i - \mu}{\sigma} \right)}, \]

\[ c_{33} = \frac{1}{\theta^2} \sum_{i=1}^{n} E(Z_i \mid y) + n \frac{C''(\theta) C(\theta) - (C'(\theta))^2}{C^2(\theta)}, \]

\[ c_{13} = c_{31} = c_{23} = c_{32} = 0, \]

\[ \text{and} \]

\[ E(Z_i \mid y) = 1 + \frac{\theta \Phi \left( \frac{y_i - \mu}{\sigma} \right) C''(\theta \Phi \left( \frac{y_i - \mu}{\sigma} \right))}{C'(\theta \Phi \left( \frac{y_i - \mu}{\sigma} \right))}. \]

Moving now to the computation of \( l_m (\Psi; y) \) as

\[ l_m (\Psi; y) = Var[U_C (y, z; \Psi) \mid y] = [v_{ij}], \]

\[ \text{(7.2)} \]
where

\[ v_{11} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \phi^2 \left( \frac{y_i - \mu}{\sigma} \right) \text{Var}[Z_i \mid y] , \quad v_{22} = \sum_{i=1}^{n} \left( \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\sigma^2 \Phi \left( \frac{y_i - \mu}{\sigma} \right)} \right)^2 \text{Var}[Z_i \mid y] , \]

\[ v_{33} = \frac{1}{\theta^2} \sum_{i=1}^{n} \text{Var}[Z_i \mid y] , \quad v_{21} = v_{12} = \frac{1}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu) \left( \frac{\phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)} \right)^2 \text{Var}[Z_i \mid y] , \]

\[ v_{13} = v_{31} = -\frac{1}{\sigma \theta} \sum_{i=1}^{n} \frac{\phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)} \text{Var}[Z_i \mid y] , \]

\[ v_{23} = v_{32} = -\frac{1}{\theta \sigma^2} \sum_{i=1}^{n} \frac{(y_i - \mu) \phi \left( \frac{y_i - \mu}{\sigma} \right)}{\Phi \left( \frac{y_i - \mu}{\sigma} \right)} \text{Var}[Z_i \mid y] . \]

And

\[ \text{Var}[Z_i \mid y] = E \left( Z_i^2 \mid y \right) - (E(Z_i \mid y))^2 \]

\[ = \frac{1}{C'(\theta_*)} \sum_{i=1}^{n} a_{zz}^2 \theta_*^{z-1} \left( C'(\theta_*) + \theta_* C''(\theta_*) \right)^2 \]

\[ = \frac{1}{C'(\theta_*)} \left[ \theta_* C'''(\theta_*) + C'(\theta_*) + 3\theta_* C''(\theta_*) \right] - \frac{[C'(\theta_*) + \theta_* C''(\theta_*)]^2}{(C'(\theta_*)^2)} , \]

in which \( \theta_* = \theta \phi \left( \frac{\mu - \mu}{\sigma} \right) \).

And applying (7.1) and (7.2), we obtain the observed information as

\[ I \left( \hat{\Psi}; y \right) = l_c \left( \hat{\Psi}; y \right) - l_m \left( \hat{\Psi}; y \right) . \]

The standard errors of the MLEs of the EM-algorithm are the square root of the diagonal elements of the \( I \left( \hat{\Psi}; y \right) \).

8. Simulation

This section provides the results of simulation study. Simulations have been performed in order to investigate the proposed estimator of \( \mu, \sigma, \theta \) of the proposed MLE method. We simulate 1000 times under the NG distribution with three different sets of parameters and sample sizes \( n = 50, 100, 200 \) and 300. For each sample size, we compute the MLEs by EM-method. We also compute the standard error of the MLEs of the EM-algorithm determined though the Fisher information matrix. The simulated values of \( se(\hat{\mu}) \), \( se(\hat{\sigma}) \), \( se(\hat{\theta}) \), \( Cov(\hat{\mu}, \hat{\sigma}) \), \( Cov(\hat{\mu}, \hat{\theta}) \) and \( Cov(\hat{\sigma}, \hat{\theta}) \) obtained by averaging the corresponding values of the observed information matrices, are computed. The results for the NP distribution are reported in Tables 4. Some of the points are quite clear from the simulation results: (i) Convergence has been achieved in all cases and this emphasizes the numerical stability of the EM-algorithm. (ii) The differences between the average estimates and the true values are almost small. (iii) These results suggest that the EM estimates have performed consistently. (iv) As the sample size increases, the standard errors of the MLEs decrease. (v) Additionally, the standard errors of MLEs of the EM-algorithm obtained from the observed information matrix are quite close to the simulated ones for large values of \( n \).
Table 4. The averages of the 1000 MLE’s, mean of the simulated standard errors and mean of the standard errors of EM estimators obtained using observed information matrix of the NP distribution.

| n   | (μ, σ, θ) | Average estimators | S.t.d | Cov                        |
|-----|-----------|--------------------|------|----------------------------|
| 30  | (0,1.0,1.0) | 0.39 1.06 -8.92   | 1.05 0.17 100.67 | 0.10 -40.46 -5.90 |
|     | (0,1.0,0.5) | 0.17 1.02 -1.95   | 0.9 0.14 21.66    | 0.00 -7.17 -0.59  |
|     | (0,1.0,0.8) | -0.17 1.03 0.57   | 1.02 0.17 0.81    | -0.14 -0.46 0.03  |
| 100 | (0,1.0,1.0) | 0.16 1.02 -1.74   | 0.7 0.1 10.78     | 0.04 -4.34 -0.60  |
|     | (0,1.0,0.5) | 0.02 1.01 0.11    | 0.62 0.1 1.75     | -0.02 -0.69 -0.03 |
|     | (0,1.0,0.8) | -0.12 1.03 0.69   | 0.79 0.14 0.46    | -0.09 -0.19 0.02  |
| 200 | (0,1.0,1.0) | 0.07 1.01 -0.52   | 0.47 0.00 2.90     | 0.46 -1.41 -0.14  |
|     | (0,1.0,0.5) | 0.00 1.00 0.39    | 0.37 0.00 0.56     | -0.01 -0.15 0.01  |
|     | (0,1.0,0.8) | -0.08 1.02 0.75   | 0.59 0.1 0.17     | -0.05 -0.08 0.01  |
| 300 | (0,1.0,1.0) | 0.03 1.00 -0.27   | 0.39 0.00 2.33     | 0.01 -0.73 -0.09  |
|     | (0,1.0,0.5) | 0.00 1.00 0.40    | 0.3 0.00 0.9      | 0.00 -0.16 0.00   |
|     | (0,1.0,0.8) | -0.06 1.01 0.77   | 0.48 0.1 0.17     | -0.03 -0.06 0.01  |

Table 5. Parameter estimates, AIC and BIC for AIS data.

| Dist. | Parameter estimates | log(L) | AIC   | BIC   |
|-------|---------------------|--------|-------|-------|
| NG    | ³=136.001, ³=13.642, ³=0.998 | 348.376 | 702.752 | 710.567 |
| NP    | ³=167.106, ³=9.208, ³=3.398 | 349.145 | 704.291 | 712.106 |
| NL    | ³=169.353, ³=7.947, ³=0.897 | 350.872 | 707.745 | 715.560 |
| Normal| ³=174.594, ³=8.209 | 352.319 | 708.635 | 713.846 |
| ASN   | ³=170.320, ³=8.002, ³=0.0016 | 352.032 | 710.636 | 718.451 |

9. Application

In this section, we try to illustrate the better performance of the proposed model. For this end, we fit NG, NP and NL models to two real data sets. We also fit the Azzalini’s skew-normal (ASN) and normal distributions to make a comparison with the NPS models. The first data concerning the heights (in centimeters) of 100 Australian athletes. The data have been previously analyzed in Cook and Weisberg and are available for download at [http://azzalini.stat.unipd.it/SN/index.html](http://azzalini.stat.unipd.it/SN/index.html). We estimate parameters by numerically maximizing the likelihood function. The MLEs of the parameters, the maximized loglikelihood, the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) for the NG, NP, NL, normal and azzalini’s skew-normal models are given in Table 5.

As is well known, a model with a minimum AIC value is to be preferred. Therefore NG distribution provides a better fit to this data set than the other distributions and hence could be chosen as the best distribution. Also this conclusion is confirmed from the plots of the densities functions in Figure 7.

The second data represent the Oits IQ Scores for 52 non-White males hired by a large insurance company in 1971 given in Roberts (1988). Table 5 gives the MLEs of the parameters, the maximized log-likelihood, the AIC and BIC for the NG, NP, NL, normal and ASN models for the second data set.

The results for this data set show that the NG distributions yield the best fit among the NG, NL, normal and ASN distributions. Also the plots of the densities function in Figure 8 confirmed this conclusion.
Figure 7. Histogram of heights of 100 Australian athletes. The lines represent distributions fitted using maximum likelihood estimation: NG (Black), NP (Red), NL (Green) and ASN (Blue).

Table 6. Parameter estimates, AIC and BIC for OTIS IQ scores data.

| Dist. | Parameter estimates | log(L) | AIC    | BIC    |
|-------|---------------------|--------|--------|--------|
| NG    | \([\hat{\mu}=112.875, \hat{\sigma}=182.313, \hat{\theta}=-2.989]\) | \(182.313\) | \(370.628\) | \(376.479\) |
| NP    | \([\hat{\mu}=106.263, \hat{\sigma}=8.227, \hat{\theta}=0.0000002]\) | \(182.313\) | \(372.850\) | \(376.479\) |
| NL    | \([\hat{\mu}=106.308, \hat{\sigma}=7.947, \hat{\theta}=0.0000002]\) | \(183.433\) | \(372.867\) | \(378.719\) |
| Normal| \([\hat{\mu}=106.654, \hat{\sigma}=8.230]\) | \(183.387\) | \(370.774\) | \(374.676\) |
| ASN   | \([\hat{\mu}=98.790, \hat{\sigma}=11.380, \hat{\theta}=1.710]\) | \(182.436\) | \(370.872\) | \(376.726\) |

Figure 8. Histogram of heights of 100 Australian athletes. The lines represent distributions fitted using maximum likelihood estimation: NG (Black), NP (Red), NL (Green), normal (pink) and ASN (Blue).
10. Conclusion

In this paper we introduce a new three-parameter class of distributions called the normal power series distributions (NPS), which is an alternative to the Azzalini skew-normal distribution for fitting skewed data. The NPS distributions contain the NG, NP, NB and NL distributions as special cases. We obtain closed form expressions for the moments. The estimation of the unknown parameters of the proposed distribution is approached by the EM-algorithm. Finally, we fitted NPS models to two real data sets to show the potential of the new proposed class.

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NB Density

\[ \theta = -10 \]
\[ \theta = -5 \]
\[ \theta = 0 \]
\[ \theta = 5 \]
\[ \theta = 10 \]