Non-Hermitian Noncommutative Quantum Mechanics

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Abstract

In this work we present a general formalism to treat non-Hermitian and noncommutative Hamiltonians. This is done employing the quantum mechanics phase-space formalism, which allows to write a set of robust maps connecting the Hamiltonians and the associated Wigner functions to the different Hilbert space structures, namely, those describing the non-Hermitian and noncommutative, Hermitian and noncommutative, and Hermitian and commutative systems. A general recipe is provided to obtain the expected values of the more general Hamiltonian. Finally, we apply our method to the harmonic oscillator under linear amplification and discuss the implications of both non-Hermitian and noncommutative effects.
I. INTRODUCTION

In the last two decades two distinct formalisms have been separately developed for the treatments of non-Hermitian [1] or non-commutative [2] Hamiltonians. These formalisms extend significantly the scope of traditional quantum mechanics, whose observables comprise only Hermitian operators and whose components of a given operator (position and momentum) always commute with each other. The replacement of the (predominantly mathematical) requirement of hermiticity by that (of greater physical bias) of invariance by spatial reflection and time reversal, introduces into $\mathcal{PT}$-symmetric quantum mechanics a wide range of physical systems not previously contemplated [3]. The same occurs when the non-commutation between the components of a given physical observable is admitted, and new phenomena emerge both from non-Hermitian [4] and non-commutative quantum mechanics [5, 6].

There is a strong research activity in the field of non-Hermitian quantum mechanics, both as regards its foundation [7, 8] and its application to a wide variety of phenomena, such as for metrology [9], chaos in optomechanics [10, 11], and for stimulating the fluctuation superconductivity [12]. The finding of $\mathcal{PT}$-symmetry breaking leading to the coalescence of the energy levels has also been reported in distinct systems as microwave billiard [13], tunneling heterostructures [14], lattice model [15], a ferromagnetic superfluid [16], etc., considerably broadening the range of interest in the physics of non-Hermitian Hamiltonians.

Non-commutative Hamiltonians have also been useful to study new fundamental properties emerging when considering a deformed Heisenberg-Weyl algebra [17]. As examples of this effort, the noncommutative quantum mechanics has been studied in the harmonic oscillator Hamiltonian [18, 20], in the context of the thermodynamical limit and quantum information theory [21] and in the well known gravitational quantum well for ultra-cold neutrons, where corrections in the eigenenergies of the system have been obtained [22, 23], and in quantum heat engines [24].

In what follows we briefly revisit the foundations of non-Hermitian and noncommutative quantum mechanics, discussing their key ingredients: the Dyson map and the associated metric operator for non-Hermitian Hamiltonians, and the Seiberg-Witten transformation for the noncommutative ones. The Dyson map enable us to construct the Hermitian counterpart of a non-Hermitian Hamiltonian, both being isospectral partners, and to define the metric
operator which ensures the unitarity of the time evolution of the system. The Seiberg-Witten transformation, by its turn, allow us to describe a noncommutative operator in terms of its commutative counterparts, converting all noncommutative structure to the standard Hilbert space. Although the Seiberg-Witten map is not unique, once it depends of arbitrary parameters, it could be proved that the eigenvalues of a noncommutative system, such as the eigenenergies, does not depend on the choice of the map.

The rest of this work is organized as follows. The section II is devoted to review the main structures of the non-Hermitian and noncommutative quantum mechanics separately, in order to establish the basic formalism to be used. In section III the general non-Hermitian and noncommutative formalism is presented in detail and the main aspects are discussed. We apply our method in an example in section IV. The conclusion and final remarks are presented in section V.

II. REVIEW OF NON-HERMITIAN AND NONCOMMUTATIVE QUANTUM MECHANICS

In this section we consider the main features of the two extension of quantum mechanics separately, i.e., the non-Hermitian and the noncommutative quantum mechanics. Concerning the first one, the more important aspect is the Dyson map, whereas the second one is based on the Seiberg-Witten maps. Both are described in details in the following.

Non-Hermitian Quantum Mechanics

Let us begin by revisiting the non-Hermitian quantum mechanics when time independent Hamiltonians and Dyson maps are considered. As anticipated above, the key ingredients here are the positive definite Dyson map $\eta$ and its associated metric operator $\Theta = \eta^\dagger \eta$, which ensure, respectively, the essential features of quantum mechanics: real energy spectrum and unitary (probability-preserving) time evolution. Let us consider a non-Hermitian Hamiltonian, $\mathcal{H}_{NH}(q, p)$, which is $\mathcal{PT}$-symmetric and thus remains invariant under the transformation

$$q \rightarrow -q, \; p \rightarrow p, \; i \rightarrow -i,$$  \hspace{1cm} (1)
Under the quasi-Hermiticity relation
\[ \Theta(q, p) H_{NH}(q, p) = H^\dagger_{NH}(q, p) \Theta(q, p), \]  
(2)
the non-Hermitian \( H_{NH}(q, p) \) is mapped to its Hermitian counterpart
\[ H_H(q, p) = \eta(q, p) H_{NH}(q, p) \eta^{-1}(q, p), \]  
(3)
where the Dyson map as well as the metric operator are defined by the same set of variable operators as the Hamiltonians.

From the solutions of the Schrödinger equations for \( H_{NH} \) and \( H_H \), given by \( \{ |\psi_n\rangle \} \) and \( \{ |\phi_n\rangle \} \), respectively, with \( |\varphi_n\rangle = \eta |\psi_n\rangle \), we immediately verify that both Hamiltonians are isospectral partners, sharing the same eigenvalues \( \{ \varepsilon_n \} \). Moreover, it is straightforward to see that besides having real energy spectrum, the non-Hermitian \( H_{NH} \) generates unitary time evolution under the redefined metric
\[ \left\langle \psi | \tilde{\psi} \right\rangle_\Theta \equiv \left\langle \psi | \Theta \tilde{\psi} \right\rangle = \left\langle \varphi | \tilde{\varphi} \right\rangle, \]  
(4)
\( \psi \) and \( \tilde{\psi} \), (as well as \( \varphi \) and \( \tilde{\varphi} \)) being generic superpositions of the basis states \( \{ |\psi_n\rangle \} \) \( \{ |\phi_n\rangle \} \). The matrix elements of the observables,
\[ O(q, p) = \eta^{-1}(q, p) O(q, p) \eta(q, p) \]  
(5)
associated with the non-Hermitian \( H \) are accordingly computed in the new metric as
\[ \left\langle \psi_m | O | \psi_n \right\rangle_\Theta = \left\langle \psi_m | \Theta O | \psi_n \right\rangle = \left\langle \varphi_m | O | \varphi_n \right\rangle, \]  
(6)
the calligraphic (italic) capitals referring to the non-Hermitian (Hermitian) system, as it has become clear by now. Using the density operator formalism, we have \( \rho_\varphi = \eta \rho_\psi \eta^\dagger \) and the expected values\[34],
\[ \left\langle O \right\rangle_\Theta = (\text{Tr}[\rho_\psi O])_\Theta = \text{Tr}[\rho_\psi \Theta O] = \text{Tr}[\rho_\varphi O] = \left\langle O \right\rangle. \]  
(7)

The expected value from an observable obtained from the density operator formalism will be very useful when dealing with the Wigner function and its map from the non-Hermitian and noncommutative Hamiltonian to the Hermitian and commutative one.
Noncommutative Quantum Mechanics

Regarding noncommutative quantum mechanics, deformed commutation relations are defined under the assumption that a different space-time structure emerges in the Planck scale, \( l_P = 10^{-33}\text{cm} \) \cite{2, 25–27}:

\[
[q_k, q_\ell] = i\theta_{k\ell}, \quad [q_k, p_\ell] = i\hbar\delta_{k\ell}, \quad [p_k, p_\ell] = i\zeta_{k\ell},
\]  
\[ \text{(8)} \]

with \( k \) and \( \ell \) labeling the components of the position and momentum operators, whereas the quantities \( \theta_{k\ell} = \theta\epsilon_{k\ell} \) and \( \zeta_{k\ell} = \zeta\epsilon_{k\ell} \) introduce two new fundamental constants in the theory, \( \theta \) and \( \zeta \), with units \([\theta] = L^2\) and \([\zeta] = M^2 L^2 / T^2\). Here, \( \epsilon_{kk} = 0 \) and \( \epsilon_{k\ell} = -\epsilon_{\ell k} \), in contrast with the standard commutation relations,

\[
[Q_k, Q_\ell] = 0, \quad [Q_k, P_\ell] = i\hbar\delta_{k\ell}, \quad [P_k, P_\ell] = 0.
\]  
\[ \text{(9)} \]

The connection between the noncommutative algebra and the standard one is performed through the Seiberg-Witten map \cite{2}, which is a linear transformation written in general as

\[
q_k = A_{k\ell} Q_\ell + B_{k\ell} P_\ell, \quad p_k = C_{k\ell} Q_\ell + D_{k\ell} P_\ell,
\]

where \( A_{k\ell}, B_{k\ell}, C_{k\ell}, \) and \( D_{k\ell} \) are elements of the matrices \( A, B, C, \) and \( D \), assumed reals, constants and invertibles. Using the relations (8) and (9) we derive Seiberg-Witten map,

\[
q_k = \nu Q_k - \theta_{k\ell} \zeta_{k\ell} \frac{1}{2\nu\hbar^2} P_\ell, \quad p_k = \mu P_k + \theta_{k\ell} \zeta_{k\ell} \frac{1}{2\mu\hbar^2} Q_\ell,
\]  
\[ \text{(10)} \]

\( \mu \) and \( \nu \) being arbitrary parameters constrained by

\[
\frac{\theta \zeta}{4\hbar^2} = \nu\mu(1 - \nu\mu).
\]  
\[ \text{(11)} \]

The inverse Seiberg-Witten map reads,

\[
Q_k = \mu \left(1 - \frac{\theta \zeta}{\hbar^2}\right)^{-1/2} \left(q_k + \frac{\theta_{k\ell}}{2\nu\mu\hbar^2} p_\ell\right),
\]  
\[ \text{(12a)} \]

\[
P_k = \nu \left(1 - \frac{\theta \zeta}{\hbar^2}\right)^{-1/2} \left(p_k - \frac{\zeta_{k\ell}}{2\nu\mu\hbar^2} q_\ell\right).
\]  
\[ \text{(12b)} \]

From equations (12a) and (12b) one notices that the commutative variables are recovered when \((\theta, \zeta) \to (0,0)\). Besides, the way how the commutative variables are written as function of the noncommutative ones is responsible for the NC effects quantified by the noncommutative constants once the Hamiltonian of the system is defined.
III. NON-HERMITIAN NON-COMMUTATIVE QUANTUM MECHANICS

In order to put together the non-Hermitian and the non-commutative formalisms, we first observe that the Seiberg-Witten is not a $\mathcal{PT}$-symmetric map. Therefore, when starting from a non-Hermitian non-commutative $\mathcal{PT}$-symmetric Hamiltonian $H^{\mathcal{PT}}_{NH,NC}(q,p)$, we can not first employ the Seiberg-Witten map to transform this Hamiltonian to a commutative non-Hermitian one, since in this way we lose its fundamental $\mathcal{PT}$-symmetric character, and we are left with a non-physical (non-Hermitian non-$\mathcal{PT}$-symmetric) Hamiltonian. Therefore, the only way to bring both formalisms together is to first transform the Hamiltonian $H^{\mathcal{PT}}_{NH,NC}(q,p)$, through the Dyson map, into the Hermitian non-commutative one $H_{H,NC}(q,p)$, and then to use the Seiberg-Witten map to reach the Hermitian commutative form $H_{H,C}(Q,P)$, i.e.,

$$H^{\mathcal{PT}}_{NH,NC}(q,p) \xrightarrow{\eta(q,p)} H_{H,NC}(q,p) \xrightarrow{\text{Seiberg-Witten}} H_{H,C}(Q,P), \quad (13)$$

where, referring to the non-hermitian noncommutative, hermitian non-commutative and hermitan commutative systems, respectively, we use calligraphic, italic and roman capitals letters.

As a next step, we will introduce another well known way to describe the state of a quantum system, the so called Wigner function [28], which has been extensively used to describe noncommutative quantum systems. This is done because, unlike the wave function, the Wigner function contains information about both position and momentum and then it is possible to describe noncommutative states. To obtain the Wigner function, we first introduce the Weyl transform which converts an operator in a c-number, defined as [6],

$$A^W(Q,P) = \int d\mathbf{y} e^{-i\mathbf{P} \cdot \mathbf{y}/\hbar} \langle Q + \mathbf{y}/2 | A(Q,P) | Q - \mathbf{y}/2 \rangle, \quad (14)$$

where $A(Q,P)$ is an arbitrary operator and the superscript “W” stands for the Weyl transform. The Wigner function is the Weyl transform of the density operator and, given the Hamiltonian $H_{H,C}(Q,P)$ and the associated density operator $\rho$, it follows that

$$W(Q,P) = \int d\mathbf{y} e^{-i\mathbf{P} \cdot \mathbf{y}/\hbar} \langle Q + \mathbf{y}/2 | \rho | Q - \mathbf{y}/2 \rangle, \quad (15)$$

with the probability density in position (momentum) representation being obtained by integrating the Wigner function over the momentum (position) variable. The Wigner-Weyl
formalism of quantum mechanics also encompasses the so called Moyal product \[6, 29\], which allow us to obtain all features of the quantum mechanics formalism based in operator structure and is defined as,

\[\ast = \ast_h \ast_\theta \ast_\zeta\]  

(16)

where each term, given by

\[\ast_h = \sum_k \exp \left[ \frac{ih}{2} \left( \frac{\partial}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial}{\partial q_k} \right) \right],\]

\[\ast_\theta = \sum_{k,\ell} \exp \left[ \frac{i\theta}{2} \left( \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_\ell} - \frac{\partial}{\partial q_\ell} \frac{\partial}{\partial q_k} \right) \right],\]

\[\ast_\zeta = \sum_{k,\ell} \exp \left[ \frac{i\zeta}{2} \left( \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_\ell} - \frac{\partial}{\partial p_\ell} \frac{\partial}{\partial p_k} \right) \right],\]

(17a, 17b, 17c)

refers to an associated commutation relation in Eq. (8), with the arrows, pointing to the right and left, indicating the directions of operation of the derivatives.

An alternative form to compute the Wigner function \(W(Q, P)\) given in Eq. (15) follows by applying the Weyl transform on the eigenvalue equation for \(H_{H,C}(Q, P)\) in the density matrix formulation, i.e., \(H_{H,C}(Q, P) \langle E \rangle \langle E \rangle = E \langle E \rangle \langle E \rangle\), what leads to the \(\ast\)-value equation \[18\],

\[H_{H,C}^W(Q, P) \ast W(Q, P) = E W(Q, P),\]

(18)

where \(H_{H,C}\) represents the Weyl transform of the associated Hamiltonian, \(W(Q, P)\) describes the state of the system and \(E\) are the eigenenergies.

In order to unify the notation, we refer to \(\rho, \rho_\varphi\), and \(\rho_\psi\) as being the density operators related to the Hamiltonians \(H_{H,C}, H_{H,NC}\), and \(H_{PT,NH,NC}\), respectively, as well as the definition introduced in eq. \[13\] applied to the Wigner functions \([W(Q, P), W(q, p)\) and \(W(q, p)]\) and the observables \([O(Q, P), O(q, p)\) and \(O(q, p)]\). Using the inverse Seiberg-Witten map, we straightforwardly obtain the Wigner function \(W(q, p)\) from \(W(Q, P)\) \[6\], and by Weyl-transforming the expression \(\rho_\varphi = \eta \rho_\psi \eta^\dagger\), we automatically derive a relation between \(W(q, p)\) and \(W(q, p)\); from \(\rho_\varphi^W = (\eta \rho_\psi \eta^\dagger)^W = \eta^W \ast \rho_\psi^W \ast (\eta^\dagger)^W\) we obtain

\(W(q, p) = \eta^W(q, p) \ast W(q, p) \ast (\eta^\dagger)^W(q, p),\)

(19)

or equivalently

\[W(q, p) = (\eta^{-1})^W(q, p) \ast W(q, p) \ast \left[(\eta^\dagger)^{-1}\right]^W(q, p).\]

(20)
Note that the star product given by eq. (16) was essential to obtain a phase-space representation of the map connecting the non-Hermitian to the Hermitian counterparts. Just for stress the results above, \( \mathcal{W}(q,p) \) is the Wigner function associated with the non-Hermitian noncommutative Hamiltonian whereas \( W(Q,P) \) is the counterpart associated to the Hermitian and commutative case. In the following we elucidate how the description of the map links the observables from NHNC Hamiltonians to the HC ones.

**Expected Values**

Starting again form the Hermitian commutative Hamiltonian, the expectation value of an observable \( O(Q,P) \) associated with \( H_{H,C}^W(Q,P) \), in the phase-space formalism of quantum mechanics, is given by [30]:

\[
\langle O(Q,P) \rangle = \int \int dQ dP \, W(Q,P) O^W(Q,P).
\]  

Using the inverse Seiberg-Witten map, we verify that the expectation value \( \langle O(Q,P) \rangle \) is related to that associated with the Hamiltonian \( H_{H,NC}(q,p) \) in the form

\[
\langle O(Q,P) \rangle = \int \int dq dp \, \frac{\partial (Q,P)}{\partial (q,p)} W(q,p) O^W(q,p) = \langle O(q,p) \rangle
\]  

\[
\partial (Q,P)/\partial (q,p) \text{ being the Jacobian of the transformation from } (Q,P) \text{ to } (q,p), \text{ and the Wigner function } W(q,p) \text{ associated with the Hermitian non-commutative Hamiltonian } H_{H,NC}(q,p) \text{ is defined as } [6]
\]

\[
W(q,p) = \frac{\partial (Q,P)}{\partial (q,p)} W(q,p). \tag{23}
\]

Now, to compute the expectation value of the observable \( O(Q,P) \) we redefine the metric to obtain

\[
\langle O(q,p) \rangle_\Theta = \int \int dq dp \, W(q,p) \Theta^W(q,p) O^W(q,p), \tag{24}
\]

which, together with Eq. (20), the Weyl transform of the metric \( \Theta^W = (\eta^i)^W \star \eta^W \), and the relation between the Weyl transforms of the observables \( O \) and \( O \) (derived using the property \( (ABC)^W = A^W \star B^W \star C^W \):

\[
O^W(q,p) = (\eta^{-1})^W(q,p) \star O^W(q,p) \star \eta^W(q,p). \tag{25}
\]
enable us to verify, as required, that the expectation value \( \langle \mathcal{O}(q,p) \rangle_\Theta \) equals \( \langle \mathcal{O}(q,p) \rangle \) and \( \langle \mathcal{O}(Q,P) \rangle \):

\[
\langle \mathcal{O}(q,p) \rangle_\Theta = \int \int dq dp W(q,p) O^W(q,p) = \langle \mathcal{O}(q,p) \rangle = \langle \mathcal{O}(Q,P) \rangle,
\]

where, inside the integrals, the \( \ast \)-product reduces to the regular operator product [29]. The expression above is our last important result and it completes the formalism to describe non-Hermitian noncommutative Hamiltonians by establishing maps connecting the different Hilbert structures in a robust way.

IV. ILLUSTRATIVE EXAMPLE: THE LINEARLY AMPLIFIED HARMONIC OSCILLATOR

After the NHNC QM formalism has been constructed, we would like to present an example. Let us consider the following non-Hermitian and non-commutative \( \mathcal{P}\mathcal{T}\)-symmetric Hamiltonian,

\[
\mathcal{H}_{NH,NC}^{PT}(q,p) = \sum_{i=1}^{2} \frac{p_i^2}{2m} + \frac{1}{2} m \omega_i^2 q_i^2 + \gamma_i p_i + i \delta_i q_i,
\]

where \( m, \omega_i, \gamma_i, \) and \( \delta_i \) are all real and constant coefficients. Considering the anzats for the Dyson map

\[
\mathcal{\eta}(q,p) = \sum_{i=1}^{2} e^{A_i \dot{q}_i + B_i \dot{p}_i},
\]

\( A_i \) and \( B_i \) being complex coefficients, so we can rewrite this coefficients in the polar form \( A_i = |A_i| e^{i \theta_{A_i}} \) and \( B_i = |B_i| e^{i \theta_{B_i}} \). Using the relation \( \mathcal{H}_{H,NC}(q,p) = \eta(q,p) \mathcal{H}_{NH,NC}^{PT}(q,p) \eta^{-1}(q,p) \), we derive the Hermitian counterpart of Hamiltonian [27]:

\[
\mathcal{H}_{H,NC}(q,p) = \sum_{i=1}^{2} \left\{ \frac{p_i^2}{2m} + \frac{1}{2} m \omega_i^2 q_i^2 + p_i V_i + q_i T_i + \frac{1}{2m \omega_i} \left[ m^2 \omega_i^2 (V_i^2 - \gamma_i^2) + (\delta_i^2 + T_i^2) \right] \right\},
\]

where

\[
V_i = -\frac{\varsigma_{ji} \hbar (\delta_j + \delta_j^*)(\tan[\theta_{B_j}] - \tan[\theta_{A_i}])}{2m^2 \omega_i^2 (\varsigma_{ji} \theta_{ij} + \hbar^2)} + \gamma_i,
\]

\[
T_i = \frac{i(\delta_i - \delta_j^*)}{2} + \frac{(\delta_i + \delta_j^*)(\tan[\theta_{B_j}] \theta_{ij} \varsigma_{ij} + \tan[\theta_{B_j}] \hbar^2)}{2(\theta_{ij} \varsigma_{ij} + \hbar^2)},
\]

(30a)
with $j = i + 1 \mod 2$. Note that the coefficients $\delta$ is real in eq. (27) in order to have a non-Hermitian Hamiltonian. However, we will consider the general case to analyze how this can affect our results.

For the Hamiltonian $H_{H,NC}(q, p)$ to be hermitian, we need to impose that the real part of the coefficients in Dyson map are

\begin{equation}
\Re [A_i] = |A_i| \cos [\theta_{A_i}] = -\frac{(\delta_j + \delta_j^*)\varsigma_{ji}}{2m\omega_j^2(\varsigma_{ji}\theta_{ij} + \hbar^2)},
\end{equation}

\begin{equation}
\Re [B_i] = |B_i| \cos [\theta_{B_i}] = \frac{(\delta_i + \delta_i^*)\hbar}{2m\omega_i^2(\theta_{ji}\varsigma_{ij} + \hbar^2)},
\end{equation}

on the other hand there is no restriction on the imaginary part of the coefficients in Dyson map. It is evident that the Hamiltonian (27) is $\mathcal{PT}$-symmetric if $\delta_j$ is real, resulting in $(\delta_j + \delta_j^*) = 2\delta_j$. However, if we assume the Hamiltonian (27) is Hermitian, this implies $\delta$ must be purely imaginary, $\delta_j = i\tilde{\delta}_j$, leading the coefficients of the Dyson map to zero, consequently the Dyson map drive to identity and the Hamiltonian (29) becomes the Hermitian form of then Hamiltonian (27), as expected. If we assume coefficients of the Dyson map purely real, and $\delta_i \in \mathbb{R}$, the Hamiltonian $H_{H,NC}(q, p)$ reduces to

\begin{equation}
H_{H,NC}(q, p) = \sum_{i=1}^{2} \left[ \frac{p_i^2}{2m} + \frac{1}{2}m\omega_i^2q_i^2 + \gamma_ip_i + \frac{\delta_i^2}{2m\omega_i^2} \right].
\end{equation}

Comparing with the Hamiltonian (29) is easy to see that imaginary part of Dyson map are responsible for the position linear term in the Hamiltonian (29). As we note from Eq. (32), this simple case leads to a shift in the eigenenergies of the system, which depend exclusively on the parameters of the Hamiltonian, while in the Hamiltonian (29) these term also depend on the Dyson map parameters. Next, following the formalism developed here, we have to apply the Seiberg-Witten map, Eq. (10), in Eq. (29) resulting in

\begin{equation}
H_H(Q, P) = \sum_{i=1}^{2} \left[ \left( \frac{\mu^2}{2m} + \frac{m\omega_i^2\theta_{ji}}{8\nu^2\hbar^2} \right) P_i^2 + \left( \frac{1}{2}m\omega_i^2\nu^2 + \frac{\varsigma_{ji}^2}{8m\mu^2\hbar^2} \right) Q_i^2 + \left( \frac{\varsigma_{ij} - m^2\omega_i^2\theta_{ji}}{4m\hbar} \right) \right] \times \{P_i, Q_j\} + \Xi_iP_i + \Lambda_iQ_i + \Omega_i,
\end{equation}

where $\Lambda_i = \left( \frac{\mu\nu_i}{\nu_{ih}} + \nu T_i \right)$, $\Xi_i = \left( \mu\nu_i - \frac{\theta_{ji}\nu_{ih}}{2h} \right)$, $\Omega_i = \frac{m^2\omega_i^2(\nu_i^2 - \nu_j^2) + (\nu_i^2 + \delta_j^2)}{2m\omega_i^2}$ and $\{P_i, Q_j\} = P_iQ_j + Q_jP_i$. Unlike the Hamiltonian (29), imposing the Dyson’s map coefficients real, and $\delta \in \mathbb{R}$, none of the terms of the Hamiltonian goes to zero. However, as the Hamiltonian (29) when $\{\omega_i, \gamma_i\} \in \mathbb{R}$, $\delta = -i\tilde{\delta}$, $\theta_{ij} = \varsigma_{ij} \rightarrow 0$ and $\mu = \nu \rightarrow 1$, the Hamiltonian (33)
where \( n_i \) are the number states, the unitary operators can be written in terms of position and moment as follows: 
\[ U(\Upsilon) = e^{i\frac{\hbar}{2}(m\sqrt{\omega_1}\omega_2 Q_1 Q_2 + \frac{n_i \rho_i}{\sqrt{\rho_i^2}})}, \quad K(\Gamma) = e^{i\frac{\hbar}{2}(m\sqrt{\omega_1}\omega_2 Q_1 - \frac{n_i \rho_i}{\sqrt{\rho_i^2}})}, \]
\[ S_i(r_i, \phi_i) = e^{-i\frac{\hbar}{2}\sin[\phi_i]\left(\frac{\rho_i^2}{\omega_i} - m\omega_i Q_i^2 + \cos[\phi_i](Q_i, P_i)\right)}, \quad D_i(\chi_i) = e^{(\chi_i - \chi_i')\sqrt{\frac{\omega_i}{\hbar}} Q_i - (\chi_i + \chi_i')\sqrt{\frac{1}{2m\omega_i} P_i}} \]
with \( \phi_i = k_i \pi \), where \( k_i = 0, 1, 2 \ldots, i = 1, 2 \) and,
\[ \Upsilon = \tan^{-1}\left[\frac{2\hbar[2F_- l_+ + G_+ l_+]}{4(f_1^2 - f_2^2) - (g_1^2 - g_2^2)}\right], \quad \Gamma = \tan^{-1}\left[\frac{G_- \tan[\Upsilon] + 2\hbar}{2L_+ \sqrt{1 + \tan^2[\Upsilon]}\right], \]
\[ \tanh[2r_1] = \frac{(\tanh[\Upsilon](F_- - 2\cos[\Upsilon] F_- + l_+ \hbar \sinh[\Gamma]) - \sin[\Upsilon] \sinh[\Gamma] G_-)}{2 \sinh[\Gamma] \sinh[\Gamma] F_- + \hbar + \cosh[\Gamma] G_+ + \cos[\Upsilon] (G_- + 2L_+ \hbar \sinh[\Gamma])}, \]
\[ \tanh[2r_2] = \frac{-\tanh[\Upsilon](F_- - 2\cos[\Upsilon] F_- + l_+ \hbar \sinh[\Gamma]) - \sin[\Upsilon] \sinh[\Gamma] G_- + 2L_+ \hbar \sinh[\Gamma] - \sinh[\Gamma] F_- + \hbar \sin[\Upsilon]}{2 \sinh[\Gamma] \sinh[\Gamma] F_- + \hbar + \cosh[\Gamma] (G_- + 2L_+ \hbar \sinh[\Gamma])}, \]
\[ \chi_1 = \frac{\sqrt{2} \cosh[2r_1] \left(i \Xi_1\kappa_1, \sqrt{m\omega_1} \hbar + \Xi_2\lambda_1, \sqrt{m\omega_2} \hbar + \Lambda_1\kappa_1, \sqrt{\frac{\hbar}{m\omega_1}} - i\Lambda_2\lambda_1, \sqrt{\frac{\hbar}{m\omega_2}}\right)}{-2 \sin[\Upsilon] \sinh[\Gamma] F_- + \hbar + \cosh[\Gamma] G_+ + \cos[\Upsilon] (G_- + 2L_+ \hbar \sinh[\Gamma])}, \]
\[ \chi_2 = \frac{\sqrt{2} \cosh[2r_2] \left(\Xi_1\lambda_2, \sqrt{m\omega_1} \hbar + i\Xi_2\kappa_2, \sqrt{m\omega_2} \hbar + \Lambda_2\kappa_2, \sqrt{\frac{\hbar}{m\omega_1}} - i\Lambda_1\lambda_2, \sqrt{\frac{\hbar}{m\omega_2}}\right)}{2 \sinh[\Gamma] (L_+ \hbar \cos[\Upsilon] - F_- \sin[\Upsilon]) + \cosh[\Gamma] (2L_+ \hbar \sin[\Upsilon] - G_- \cos[\Upsilon]) + G_+}, \]
\[ \lambda_{2 \pm} = (\cosh[r_2] \pm e^{i\phi_2} \sinh[r_2]) \left(\sin\left[\frac{\Upsilon}{2}\right] \cosh\left[\frac{\Gamma}{2}\right] \pm \cos\left[\frac{\Upsilon}{2}\right] \sinh\left[\frac{\Gamma}{2}\right]\right), \]
\[ \kappa_{2 \pm} = (\cosh[r_2] \pm e^{i\phi_2} \sinh[r_2]) \left(\cos\left[\frac{\Upsilon}{2}\right] \cosh\left[\frac{\Gamma}{2}\right] \pm \sin\left[\frac{\Upsilon}{2}\right] \sinh\left[\frac{\Gamma}{2}\right]\right), \]
and $F_- = f_1 - f_2$, $F_+ = f_1 + f_2$, $G_- = g_1 - g_2$, and $G_+ = g_1 + g_2$. With the eigenvalues given by

$$
\varepsilon_{n_1,n_2} = C \left( n_1 + \frac{1}{2} \right) + D \left( n_2 + \frac{1}{2} \right) - \left( C|\chi_1|^2 + D|\chi_2|^2 - E \right),
$$

(45)

where

$$
C = \frac{1}{2} \text{sech}[2r_1][G_+ \cosh[\Gamma] - 2 \sin[\Upsilon](l_+ h + F_+ \sinh[\Gamma])] + \cos[\Upsilon](G_- + 2l_- h \sinh[\Gamma])],
$$

$$
D = \frac{1}{2} \text{sech}[2r_2][(G_+ + 2 \sin[\Upsilon](l_+ h \cosh[\Gamma] - F_+ \sinh[\Gamma])) - \cos[\Upsilon](G_- \cosh[\Gamma] - 2l_- h \sinh[\Gamma])],
$$

$$
E = \Omega_1 + \Omega_2 - \frac{1}{2} \left[ g_1 \cos^2 \left( \frac{\Upsilon}{2} \right) + g_2 \sin^2 \left( \frac{\Upsilon}{2} \right) - hl_+ \sin[\Upsilon] \right] (\cosh[\Gamma] - 1).
$$

(46)

The analytical solution allows us to analyze the effects of the non-Hermiticity and non-commutativity in the Hamiltonian eigenstates and eigenenergies. Through the equations (45-46) we can easily see that the functions $\Upsilon$, $\Gamma$ and $r_i$ are dependent on the non-commutative parameters and independent on the non-Hermitian parameters, this implies that the rotation and the compression in the states are independent on the non-Hermiticity of the Hamiltonian. In fact we can analyze only the effects of the non-commutativity, by doing $\{\omega_i, \gamma_i\} \in \mathbb{R}$, $\delta = -i \tilde{\delta}$, this leads $V_i = \gamma_i$ and $T_i = \delta_i$, the functions $\Lambda_i$, $\Xi_i$ and $\Omega_i$ become $\Lambda_i = \left( \frac{\mu_i \gamma_i}{2\nu \hbar} + \nu \tilde{\delta}_i \right)$, $\Xi_i = \left( \mu \gamma_i - \frac{\theta_i \delta_i}{2\nu \hbar} \right)$, $\Omega_i = \frac{\delta_i^2}{m \omega_i^2}$. The only change in relation of the eigenstates (34) is a change in the displacement operator $\chi_i$, eqs. (38-39). Clearly, the deformation in the space given by non-commutativity, results in the rotation of the state, in the compression of the modes and contributes to the displacement. The only change in the eigenenergies (45), is in the quadratic terms dependent on the displacement parameter. This shows that most of the contribution in the evolution of this state and in the eigenenergies comes from non-commutativity. To see the contribution of non-hermiticity in this Hamiltonian we take $\{\omega_i, \gamma_i, \delta_i\} \in \mathbb{R}$, $\theta_{ij} = \zeta_{ij} \to 0$ and $\mu = \nu \to 1$, meaning that the Hamiltonian (27) goes to be “commutative” (described by conventional quantum mechanics) and non-Hermitian with the eigenvectors given by

$$
|\psi\rangle = D_1(-\chi_1)D_2(-\chi_2)|n_1,n_2\rangle,
$$

(47)

where

$$
\chi_i = \sqrt{\frac{1}{2m \hbar \omega^2}} \delta_i \tan[\theta_{Bi}] + i \sqrt{\frac{m}{2\omega_i \hbar}} \gamma_i,
$$

(48)
with the eigenvalues
\[
\varepsilon_{n_1,n_2} = \left( n_1 + \frac{1}{2} \right) \omega_1 \hbar + \left( n_2 + \frac{1}{2} \right) \omega_2 \hbar + \frac{1}{2m} \left( \frac{\delta_1^2}{\omega_1^2} + \frac{\delta_2^2}{\omega_2^2} \right).
\] (49)

Exactly as we had predicted, the contribution of non-hemiticity occurs in the displacement parameter proportional to \( \gamma_i \), adding an energy to the system proportional to the non-Hermitian parameter \( \delta_i \). We can compare the energies, (45) and (49), with the energy of the Hermitian linearly amplified harmonic oscillator. For that we just have to take \( \{ \omega_i, \gamma_i \} \in \mathbb{R} \), \( \delta = -i \tilde{\delta} \), \( \theta_{ij} = \zeta_{ij} \to 0 \) and \( \mu = \nu \to 1 \), meaning that the Hamiltonian (29) goes to be “commutative” (described by standard quantum mechanics) and Hermitian with the eigenvectors given by
\[
|\psi\rangle = D_1(-\chi_1)D_2(-\chi_2)|n_1, n_2\rangle,
\] where
\[
\chi_i = -\sqrt{\frac{1}{2m \omega_i^3} \tilde{\delta}_i} + i \sqrt{\frac{m}{2 \omega_i} \gamma_i},
\] (51)
and the eigenvalues
\[
\varepsilon_{n_1,n_2} = \left( n_1 + \frac{1}{2} \right) \omega_1 + \left( n_2 + \frac{1}{2} \right) \omega_2 - \frac{1}{2m} \left( \frac{\tilde{\delta}_1^2}{\omega_1^2} + \frac{\tilde{\delta}_2^2}{\omega_2^2} \right).
\] (52)

As can be seen from the equation (52), the linear amplification in the Hermitian Hamiltonian draws energy from the system in proportion to the square of the amplification parameters, \( \gamma_i \) and \( \delta_i \), and the state of the system is a displacement of the number states, with that displacement proportional to \( \gamma_i \) and to \( -\tilde{\delta}_i \). We can choose the free parameter on the Dyson map, \( \theta_{B_i} \), such that the displacement parameter in the state whose evolution is given by a non-Hermitian Hamiltonian (48) stays equal to the displacement parameter of the state of their Hermitian Hamiltonian (51). Thus, both states are the same. Although we choose \( \theta_{B_i} \), so that both have the same states, the Hamiltonians remain different and this results in different eigenenergies. Moreover, even with the isospectral partner of the non-Hermitian Hamiltonian obtaining the same states of the Hermitian Hamiltonian, the latter has a smaller energy, eq.(52), compared to the isospectral partner of the non-Hermitian Hamiltonian, eq.(49).
V. CONCLUSIONS

In this work we have formulated a unified formalism to treat non-Hermitian and non-commutative Hamiltonians. By considering the Dyson and Seiberg-Witten maps for the non-Hermitian and non-commutative operators respectively, it was possible in the phase-space formalism of the quantum mechanics to develop a formal and correct way to employ both generalizations in the canonical quantum mechanics.

Since the non-Hermitian aspects of quantum mechanics had been elucidated in many quantum systems as mentioned in the introduction, we ask how these features will appear in a scenario where the non-commutative of the phase-space becomes relevant. For this, the formalism developed in this work presents the correct way to deal simultaneously with both aspects of the quantum generalizations involved. Our example, even in the particular case, allows to show the applicability of the method and its practical consequences. Further, in a future work, we intend to analyze the complete Hamiltonian in appendix in details, on the aspects of the Wigner function and the eigenenergies and its full properties.

We would like to compare our method to that presented in [31]. In that work, the authors address the problem from a different point of view. The main point of analyze is to impose $\mathcal{PT}$-symmetry on the general commutation relations of the non-commutative quantum mechanics, Eq. (8) in our text. The authors then verify that this realization does not preserve the commutation relations $\mathcal{PT}$-symmetric. To solve the problem, they propose another form for the relations, for instance, they write $[q_k, q_\ell] = \theta_{k\ell}$, in our notation. Although the authors argues that this procedure solve the problem, it is direct to note that in doing so the relations are no longer invariant under self-adjoint transformation. In order to avoid this apparent problem we follow another direction in our work, explicitly showing that a robustness way is, given a non-Hermitian and non-commutative operator, first obtain the Hermitian counterpart of the operator and then perform the Siberg-Witten map, not being necessary change the commutation relations. Another different approach has been taken in [32], where is argued about the possibility of the $\mathcal{PT}$-symmetry change the sign of the new non-commutative constants in the theory. Again, our method does not assume this option, mainly based on the fact that these constants are invariant such that the Plank constant.

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In this expression we use $\rho_\psi = \sum_n p_n |\psi_n\rangle \langle \psi_n|$, where $|\psi_n\rangle$ are the eigenstates of $H_{NH}$ and consequently the $\Theta$-trace in Eq. (7) represents $\sum_n p_n \langle \psi_n| \eta^\dagger \eta O |\psi_n\rangle$. 

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[34] In this expression we use $\rho_\psi = \sum_n p_n |\psi_n\rangle \langle \psi_n|$, where $|\psi_n\rangle$ are the eigenstates of $H_{NH}$ and consequently the $\Theta$-trace in Eq. (7) represents $\sum_n p_n \langle \psi_n| \eta^\dagger \eta O |\psi_n\rangle$.
Appendix

Here it will be presented the general case of the Hamiltonian in example above. After we use the Dyson map in the Hamiltonian $H_{NH,NC}^{PT}(q,p)$ we get,

$$H_{H,NC}(q, p) = \sum_{i=1}^{\text{mod}2} \left[ \alpha_i p_i^2 + \beta_i q_i^2 + p_i (2i\alpha_i (hA_i + \zeta_{i+1}B_{i+1,i}) + \gamma_i) + iq_i (2\beta_i (\theta_{i+1,i}A_{i+1} - hB_i) + \delta_i) \right. \\
- \alpha_i (hA_i + \zeta_{i+1,i}B_{i+1})^2 - \beta_i (\theta_{i+1,i}A_{i+1} - hB_i)^2 \\
+ \left. i\gamma_i(hA_i + \zeta_{i+1,i}B_{i+1}) + \delta_i(hB_i - \theta_{i+1,i}A_{i+1}) \right], \quad (53)$$

where mod2 implies that $2 + 1 = 1$.

After the transformation, the Hamiltonian $H_{H,NC}(q, p)$ must be Hermitian, that is, $H_{H,NC}(q, p) = H_{H,NC}(q, p)^\dagger$. Using this condition, the polar notation for complex quantities and after some mathematical manipulations, one has,

$$H_{H,NC}(q, p) = \sum_{i=1}^{2} \left[ \alpha_i p_i^2 + \beta_i q_i^2 + p_i \left( -\frac{\alpha_i \zeta_i h\delta_j (\tan[\theta_{B_i}] - \tan[\theta_{A_i}])}{\beta_j (\zeta_i \theta_{ij} + h^2)} + \gamma_i \right) + q_i \left( \frac{\delta_i (\tan[\theta_{A_i}] \theta_{ji} \zeta_j \theta_{ij} + \tan[\theta_{B_i}] h^2)}{(\theta_{ji} \zeta_{ij} + h^2)} \right) \right. \\
- \alpha_i \left( \frac{i\delta_j (\tan[\theta_{B_i}] - \tan[\theta_{A_i}]) \zeta_j h}{2\beta_j (\zeta_i \theta_{ij} + h^2)} \right)^2 - \frac{i^2}{4\beta_i} \left( -1 - \frac{i(\tan[\theta_{A_i}] \theta_{ji} \zeta_j \theta_{ij} + \tan[\theta_{B_i}] h^2)}{(\theta_{ji} \zeta_{ij} + h^2)} \right)^2 \\
- \gamma_i \delta_j \left( \frac{\tan[\theta_{B_i}] - \tan[\theta_{A_i}]}{2\beta_j (\zeta_i \theta_{ij} + h^2)} \right) + \frac{\delta^2_i}{2\beta_i} \left[ 1 + \frac{i(\tan[\theta_{B_i}] h^2 + \tan[\theta_{A_i}] \theta_{ji} \zeta_j)}{(\theta_{ji} \zeta_{ij} + h^2)} \right] \right), \quad (54)$$

where $j = 1 \text{ mod } 2$. If we considering the parameters of the Dyson map real we have the $\theta_{B_i} = \theta_{A_i} = 0$, this leads the equation above

$$H_{H,NC}(q, p) = \sum_{i=1}^{2} \left[ \alpha_i p_i^2 + \beta_i q_i^2 + p_i \gamma_i + \frac{\delta^2_i}{4\beta_i} \right]. \quad (55)$$

Now, using the Seiberg-Witten in Hamiltonian $H_{H,NC}(q, p)$ one has, after mathematical manipulations,

$$H_{H,C}(Q, P) = \sum_{i=2}^{2} F_i P_i^2 + G_i Q_i^2 + H_i\{P_i, Q_i\} + I_i P_i + K_i Q_i + L_i, \quad (56)$$
where the constants $F_i$, $G_i$, $H_i$, $I_i$, $K_i$ and $L_i$ are given by

$$
F_i = \alpha_i \mu^2 + \frac{\beta_i \theta_j^2}{4 \nu^2 \hbar^2},
$$
$$
G_i = \beta_i \nu^2 + \frac{\alpha_j \varsigma_{ji}^2}{4 \mu^2 \hbar^2},
$$
$$
H_i = \frac{\alpha_j \varsigma_{ji}}{2 \hbar} - \frac{\beta_j \theta_{ji}}{2 \hbar},
$$
$$
I_i = \gamma_i - \frac{\mu \delta_j}{(\varsigma_{ji} \theta_{ji} + \hbar^2)} \left( \frac{\alpha_j \varsigma_{ji} \hbar (\tan[\theta_{Bi}] - \tan[\theta_{Ai}]) + \frac{\theta_j}{2 \nu \hbar} (\tan[\theta_{Ai}] \varsigma_{ji} \theta_{ji} + \tan[\theta_{Bi}] \hbar^2)}{2 \mu \hbar \beta_j} \right),
$$
$$
K_i = \frac{\varsigma_{ji}}{2 \mu \hbar} \gamma_j + \frac{\delta_i}{(\theta_{ji} \varsigma_{ij} + \hbar^2)} \left( - \frac{\alpha_j \varsigma_{ji} \varsigma_{ij} \hbar (\tan[\theta_{Bi}] - \tan[\theta_{Ai}])}{2 \mu \hbar \beta_j} + \nu (\tan[\theta_{Ai}] \theta_{ji} \varsigma_{ij} + \tan[\theta_{Bi}] \hbar^2) \right),
$$
$$
L_i = -\alpha_i \left( \frac{i \delta_j (\tan[\theta_{Bi}] - \tan[\theta_{Ai}]) \varsigma_{ji} \hbar}{2 \beta_j (\varsigma_{ji} \theta_{ji} + \hbar^2)} \right)^2 - \beta_i \left( \frac{\delta_i}{2 \beta_i} - \frac{i (\tan[\theta_{Ai}] \theta_{ji} \varsigma_{ij} + \tan[\theta_{Bi}] \hbar^2) \delta_j}{2 \beta_i (\varsigma_{ji} \theta_{ij} + \hbar^2)} \right)^2

+ i \gamma_i \left( \frac{i (\tan[\theta_{Bi}] - \tan[\theta_{Ai}]) \delta_j \varsigma_{ji} \hbar}{2 \beta_j (\varsigma_{ji} \theta_{ji} + \hbar^2)} \right) + \delta_i \left( \frac{\delta_i}{2 \beta_i} + i \frac{\delta_i (\tan[\theta_{Bi}] \hbar^2 + \tan[\theta_{Ai}] \theta_{ji} \varsigma_{ij})}{2 \beta_i (\varsigma_{ji} \theta_{ij} + \hbar^2)} \right),
$$

It is direct to see that this Hamiltonian naturally differs from the one just with non-commutative effects.

In a next work we intend to treat mathematically and physically this complex Hamiltonian in details, analyzing its eigenstates and eigenenergies.