An averaging principle for fractional stochastic differential equations with Lévy noise

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This paper is devoted to the study of an averaging principle for fractional stochastic differential equations in $\mathbb{R}^n$ with Lévy motion, using an integral transform method. We obtain a time-averaged equation under suitable assumptions. Furthermore, we show that the solutions of averaged equation approach the solutions of the original equation. Our results in this paper provide better understanding for effective approximation of fractional dynamical systems with non-Gaussian Lévy noise.

Keywords: Stochastic averaging principle, Fractional order systems, Lévy noise

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Fractional stochastic differential equations are alternative models for anomalous dynamics in various complex systems under non-Gaussian random fluctuations. Although stochastic averaging methods are widely used to gain macroscopic dynamics for multiscale stochastic differential equations, it is still a great challenge to derive effective models to approximate fractional stochastic differential equations, letting alone the systems with non-Gaussian Lévy motion. We take up the challenge to study a fractional averaging principle for a dynamical system with Lévy motion and provide a theoretical foundation of fractional stochastic averaging methods. This offers a reduced yet effective way to accurately predict the solution paths of fractional stochastic systems with Lévy motion, under suitable conditions.

I. INTRODUCTION

This paper is devoted to the averaging theory of Caputo type fractional stochastic equations with Lévy motion, looking for an averaged equation in the mean square convergence sense. The theory of averaging are indispensable that they provide evidences or justifications for the averaging procedures of complex equations arising from mathematics, control, engineering mechanics and several problems. These theoretical results play a crucial role in investigating perturbation theory and nonlinear dynamical systems during their long history.

Stochastic averaging principles, as a kind of effective analysis tool, are presented to help us approach stochastic differential equations (SDEs) with various different noises, such as multiplicative noise, Poisson noise, fractional Brownian motion, general stochastic measure and the like. Lévy noise is an important non-Gaussian noise, and the enthusiasm of researchers are growing in this field. Recently, Xu in a literature gave the integer averaging principle with Lévy noise. However, until now, the fractional averaging behavior with Lévy motion is not well understood.

Fractional derivative, which can characterize the memory and hereditary properties of various practical dynamical systems, has been widespread concerned. However, most scholars prefer integer order derivative to fractional derivative in the concrete research process, because there is no rigorous mathematic tool. With the in-depth research, they grad-
ually find that fractional averaging methods can more easily help us simplify and obtain approximate solutions to fractional nonclassical dynamical systems. Roughly speaking, the theory of fractional averaging provides us ample opportunity to accurately reveal the essence of real life.\(^{14}\)

Fractional averaging principle is in the innovation phase. In our previous papers\(^{15,16}\), by analyses of solutions before and after averaging, we have proved that averaging principles are satisfied for both Caputo fractional stochastic differential equations with Brownian motion and fractional neutral equations with Poisson jumps. Here, we consider the following fractional stochastic equations,

\[
\begin{aligned}
D^\beta_t X(t) &= b(t, X(t)) + \sigma(t, X(t)) \frac{dL(t)}{dt}, \\
X(0) &= X_0,
\end{aligned}
\]

where \(D^\beta_t\) is the Caputo fractional derivative, \(\beta \in (\frac{1}{2}, 1)\), initial value \(\mathbb{E}|X_0|^2 < \infty\), functions \(b: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n\) and \(\sigma: [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times r}\) are measurable, \(L(t)\) is a \(r\)-dimensional Lévy motion. The above type of equations are of significant importance in applications\(^{17}\), which appear in many problems, such as high variability and long memory signal models\(^{18}\), subsurface solute transport\(^{19,20}\), infrared remote sensing\(^{21}\), among others. They are the ones which attract so much attention since the beginning. We consider in this paper is the averaging character of these substantial interesting equations.

We would like to highlight the fact that our work here is motivated by Xu et al\(^{10}\), who studied the averaging principle with Lévy noise. In this article, we shall generalize the classical Khasminskii averaging approach to Caputo type fractional stochastic equations with Lévy motion.

We first recall some essential definitions and existing results (Section 2). Then we present the fractional averaging results, establishing a stochastic averaging principle for Caputo type fractional equations with Lévy motion (Section 3). Finally, the example is discussed to illustrate the main results (Section 4), and conclusion is given (Section 5).

**II. DEFINITIONS AND EXISTING RESULTS**

Before researching our averaging results, let us give some basic concepts about fractional calculus and Lévy motion.
**Definition II.1.** Let \( f \) be a Lebesgue integrable function, \( \frac{1}{2} < \beta < 1 \), the \( \beta \)-order integral is defined by

\[
I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s) \, ds, \quad t \in [t_0, \infty),
\]

where \( \Gamma \) is the gamma function.

**Definition II.2.** Let \( \frac{1}{2} < \beta < 1 \), the \( \beta \)-order Caputo derivative for the function \( f \) is

\[
D^\beta_t f(t) = \frac{1}{\Gamma(1-\beta)} \int_{t_0}^{t} \frac{f'(s)}{(t-s)^\beta} \, ds, \quad t \in [t_0, \infty).
\]

**Theorem II.1.** If Lévy motion \( L(t) \) is in \( \mathbb{R}^r \), then the expression

\[
L(t) = mt + B(t) + \int_{|x|<c} x \tilde{N}(t, dx) + \int_{|x|\geq c} x N(t, dx)
\]

is called the Lévy-Itô decomposition, where vector \( m \in \mathbb{R}^r \), constant \( c > 0 \), \( r \)-dimensional Brownian motion \( B(t) \) has the covariance matrix \( A \), \( N(t, dx) : \mathbb{R}^+ \times \{\mathbb{R}^r - 0\} \) controlling small jumps, is Poisson random measure, \( \tilde{N}(t, dx) = N(t, dx) - t\nu(dx) \) controlling large jumps, is compensated Poisson random measure, \( \nu \) is the jump measure.

Using the above theorem, let’s rewrite equation (1) and give a more general representation

\[
\begin{cases}
D_t^\beta X(t) = f(t, X(t-)) + G(t, X(t-)) \frac{dB(t)}{dt} + \frac{1}{dt} \int_{|x|<c} H(t, X(t-), x) \tilde{N}(dt, dx) \\
+ \frac{1}{dt} \int_{|x|\geq c} Q(t, X(t-), x) N(dt, dx), \\
X(0) = X_0,
\end{cases}
\]

where functions \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, G : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times r}, H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( Q : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) are measurable. According to the technique presented in the literature, we set our sights on the Lévy motion without large jumps,

\[
\begin{cases}
D_t^\beta X(t) = f(t, X(t-)) + G(t, X(t-)) \frac{dB(t)}{dt} + \\
+ \frac{1}{dt} \int_{|x|<c} H(t, X(t-), x) \tilde{N}(dt, dx), \\
X(0) = X_0.
\end{cases}
\]

Let us make two assumptions on functions \( f, G \) and \( H \),
Let $x_1, x_2 \in \mathbb{R}^n$, $t \in [0, T]$ and constant $C_1 > 0$. Then

\[
|f(t, x_1) - f(t, x_2)|^2 \vee \|a(t, x_1, x_1) - 2a(t, x_1, x_2) + a(t, x_2, x_2)\| \vee \\
\int_{|x| < c} |H(t, x_1, x) - H(t, x_2, x)|^2 \nu(dx) \leq C_1 |x_1 - x_2|^2,
\]

where $| \cdot |$ is $\mathbb{R}^n$-form, $\| \cdot \|$ is matrix form, $x_1 \vee x_2 = \max\{x_1, x_2\}$ and $a(t, x_1, x_2) = G(t, x_1)G(t, x_2)'$ is a $n \times n$ matrix.

Let $x_1 \in \mathbb{R}^n$, $t \in [0, T]$ and constant $C_2 > 0$. Then

\[
|f(t, x_1)|^2 \vee \|a(t, x_1, x_1)\| \vee \int_{|x| < c} |H(t, x_1, x)|^2 \nu(dx) \leq C_2 (1 + |x_1|^2).
\]

Conduct assumptions $(H_1)$ and $(H_2)$, equation (2) have the unique, adapted and cadlag mild solution

\[
X(t) = X_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, X(s-)) ds + \\
\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} G(s, X(s-)) dB(s) + \\
\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \int_{|x| < c} H(s, X(s-), x) \tilde{N}(ds, dx),
\]

where $\mathbb{E}(\int_0^T |X(t)|^2 dt) < \infty$. See \cite{10, 22, 25} for more details.

III. AVERAGING PRINCIPLE FOR FRACTIONAL STOCHASTIC EQUATIONS

In previous sections, all tools needed for the averaging problem of Caputo type fractional stochastic equations with Lévy noise have been prepared. Start this section by giving the standard form of equation (2):

\[
\begin{cases}
D_t^\beta X_\epsilon(t) = \epsilon f(t, X_\epsilon(t-)) + \sqrt{\epsilon} G(t, X_\epsilon(t-)) \frac{dB(t)}{dt} + \\
\frac{\sqrt{\epsilon}}{dt} \int_{|x| < c} H(t, X_\epsilon(t-), x) \tilde{N}(dt, dx), \\
X_\epsilon(0) = X_0,
\end{cases}
\]
which is obtained by some time scale transformations. Here, $\epsilon << 1$ is a small positive parameter in $(0, \epsilon_0]$.

Taking the average of functions $f, G, H$ with respect to $t$, we are going to show that the solutions of equation (4) can be approached by the solutions of time-averaged equation

$$
\begin{align*}
\left\{ \begin{array}{l}
D_\beta Z_\epsilon (t) = & \epsilon \bar{f} (Z_\epsilon (t-)) + \sqrt{\epsilon} \bar{\mathcal{G}} (Z_\epsilon (t-)) \frac{dB (t)}{dt} + \\
& \frac{\sqrt{\epsilon}}{dt} \int_{|x| < c} \bar{H} (Z_\epsilon (t-), x) \tilde{N} (dt, dx), \\
Z_\epsilon (0) = & X_0,
\end{array} \right.
\end{align*}
$$

(5)

where functions $\bar{f}, \bar{G}, \bar{H}$ satisfying

$(H_3)$ Let $x_1 \in \mathbb{R}^n$, $T_1 \in [0, T]$ and $\alpha_i (T_1) > 0$, $i = 1, 2, 3$. Then

$$
|f (T_1, x_1) - \bar{f} (x_1)| \leq \alpha_1 (T_1) (1 + |x_1|),
$$

$$
\|a (T_1, x_1, x_1) - \bar{a} (x_1, x_1)\| \leq \alpha_2 (T_1) (1 + |x_1|^2),
$$

and

$$
\int_{|x| < c} |H (T_1, x_1, x) - H (x_1, x)|^2 \nu (dx) \leq \alpha_3 (T_1) (1 + |x_1|^2),
$$

where $\lim_{T_1 \rightarrow \infty} \alpha_i (T_1) = 0$.

**Theorem III.1.** Let $\delta_1 > 0$ and suppose assumptions $(H_1)-(H_3)$ hold for functions $f, G$ and $H$. Then there exist three constants $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\lambda \in (0, 1)$ such that

$$
\mathbb{E} (\sup_{t \in [0, L \epsilon^{-\lambda}]} |X_\epsilon (t) - Z_\epsilon (t)|^2) \leq \delta_1
$$

for all $\epsilon \in (0, \epsilon_1]$.

**Proof.** Take a value from $(0, T]$ and define as $u$. Then for any $t \in [0, u]$

$$
X_\epsilon (t) - Z_\epsilon (t) = \frac{\epsilon}{\Gamma (\beta)} \int_0^t (t-s)^{\beta-1} \left[ f (s, X_\epsilon (s-)) - \bar{f} (Z_\epsilon (s-)) \right] ds +
$$

$$
\frac{\sqrt{\epsilon}}{\Gamma (\beta)} \int_0^t (t-s)^{\beta-1} \left[ G (s, X_\epsilon (s-)) - \bar{G} (Z_\epsilon (s-)) \right] dB(s) +
$$

$$
\frac{\sqrt{\epsilon}}{\Gamma (\beta)} \int_0^t (t-s)^{\beta-1} \int_{|x| < c} \left[ H (s, Z_\epsilon (s-), x) - \bar{H} (Z_\epsilon (s-), x) \right] \tilde{N} (ds, dx).
$$

6
Computing the mathematical expectation, we have

\[
\mathbb{E}(\sup_{0 \leq t \leq u} |X_\epsilon(t) - Z_\epsilon(t)|^2) \leq \frac{3\epsilon^2}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ f(s, X_\epsilon(s)) - f(s, Z_\epsilon(s)) \right] ds \right|^2 + \\
\frac{3\epsilon}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ G(s, X_\epsilon(s)) - \overline{G}(Z_\epsilon(s)) \right] dB(s) \right|^2 + \\
\frac{3\epsilon}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \int_{|x| < \epsilon} \left[ H(s, X_\epsilon(s), x) - \overline{H}(Z_\epsilon(s), x) \right] \widetilde{N}(ds, dx) \right|^2
\]

(6)

\[= J_1 + J_2 + J_3.\]

Evaluating \(J_1\) with the technic of integration by parts and conditions \((H_1)-(H_3)\), produces

\[
J_1 \leq \frac{6\epsilon^2}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ f(s, X_\epsilon(s)) - f(s, Z_\epsilon(s)) \right] ds \right|^2 + \\
\frac{6\epsilon^2}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ f(s, Z_\epsilon(s)) - \overline{f}(Z_\epsilon(s)) \right] ds \right|^2 + \\
\frac{6\epsilon^2}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 2} \left[ f(s, X_\epsilon(s)) - f(s, Z_\epsilon(s)) \right] ds \right|^2 \leq K_1 \epsilon^2 u \int_0^u (u - s)^{2\beta - 2} \mathbb{E}(\sup_{0 \leq t \leq s} |X_\epsilon(t) - Z_\epsilon(t)|^2) ds + K_{12} \epsilon^2 u^{2\beta},
\]

where \(K_1 = \frac{6\epsilon^2}{\Gamma(\beta)^2}\) and \(K_{12} = \frac{12 \beta^2}{\beta^2 \Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \alpha_1(t)^2 [1 + \mathbb{E}(\sup_{0 \leq \tau \leq u} |Z_\epsilon(\tau)|^2)].\)

Similarly, for \(J_2\),

\[
J_2 \leq \frac{6\epsilon}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ G(s, X_\epsilon(s)) - G(s, Z_\epsilon(s)) \right] dB(s) \right|^2 + \\
\frac{6\epsilon}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t (t - s)^{\beta - 1} \left[ G(s, Z_\epsilon(s)) - \overline{G}(Z_\epsilon(s)) \right] dB(s) \right|^2,
\]

adding Doob’s martingale inequality and Itô isometry,

\[
J_2 \leq K_{21} \epsilon \int_0^u (u - s)^{2\beta - 2} \mathbb{E}(\sup_{0 \leq s_1 \leq s} |X_\epsilon(s_1) - Z_\epsilon(s_1)|^2) ds + \\
\frac{6\epsilon}{\Gamma(\beta)^2} \mathbb{E} \sup_{0 \leq t \leq u} \left| \int_0^t \|a(s, Z_\epsilon(s)) - \overline{a}(Z_\epsilon(s))\| d \left[ \frac{-(t - s)^{2\beta - 1}}{2\beta - 1} \right] \right| + \\
K_{21} \epsilon \int_0^u (u - s)^{2\beta - 2} \mathbb{E}(\sup_{0 \leq s_1 \leq s} |X_\epsilon(s_1) - Z_\epsilon(s_1)|^2) ds + K_{22} \epsilon u^{2\beta - 1},
\]

(8)
where $K_{21} = \frac{6c_2^2}{\Gamma(\beta)^2}$ and $K_{22} = \frac{6}{(2\beta-1)\Gamma(\beta)^2} \sup_{0 \leq t \leq u} \alpha_2(t) [1 + \mathbb{E}( \sup_{0 \leq \tau \leq u} |Z_\epsilon(\tau)|^2)]$.

In the sequel, for $J_3$,

$$J_3 \leq \frac{6\epsilon}{\Gamma(\beta)^2} \mathbb{E} \left( \int_0^u (u-s)^{2\beta-2} \int_{[x] < c} |H(s, X_{\epsilon}(s^-), x) - H(s, Z_{\epsilon}(s^-), x)|^2 \nu(dx)ds \right)$$

$$+ \frac{6\epsilon}{\Gamma(\beta)^2} \mathbb{E} \left( \int_0^u (u-s)^{2\beta-2} \int_{[x] < c} |H(s, X_{\epsilon}(s^-), x) - \mathcal{H}(Z_{\epsilon}(s^-), x)|^2 \nu(dx)ds \right),$$

noting the assumptions and formula used above,

$$J_3 \leq K_{31}\epsilon \int_0^u (u-s)^{2\beta-2} \mathbb{E} \left( \sup_{0 \leq s_1 \leq s} |X_{\epsilon}(s_1) - Z_{\epsilon}(s_1)|^2 \right) ds +$$

$$\frac{6\epsilon}{\Gamma(\beta)^2} \int_0^u \int_{[x] < c} \left| H(s, X_{\epsilon}(s^-), x) - \mathcal{H}(Z_{\epsilon}(s^-), x) \right|^2 \nu(dx)d \left( \frac{(u-s)^{2\beta-1}}{2\beta - 1} \right)$$

$$\leq K_{31}\epsilon \int_0^u (u-s)^{2\beta-2} \mathbb{E} \left( \sup_{0 \leq s_1 \leq s} |X_{\epsilon}(s_1) - Z_{\epsilon}(s_1)|^2 \right) ds + K_{32}\epsilon u^{2\beta-1},$$

where $K_{31} = \frac{6c_2^2}{\Gamma(\beta)^2}$ and $K_{32} = \frac{6}{(2\beta-1)\Gamma(\beta)^2} \sup_{0 \leq t \leq u} \alpha_3(t) [1 + \mathbb{E}( \sup_{0 \leq \tau \leq u} |Z_\epsilon(\tau)|^2)]$.

Hence, we get

$$\mathbb{E} \left( \sup_{0 \leq t \leq u} |X_{\epsilon}(t) - Z_{\epsilon}(t)|^2 \right)$$

$$\leq K_{12}\epsilon^2 u^{2\beta} + (K_{22} + K_{32})\epsilon u^{2\beta-1} + (K_{11}\epsilon^2 u + K_{21}\epsilon + K_{31}\epsilon)$$

$$\int_0^u (u-s)^{(2\beta-1)-1} \mathbb{E} \left( \sup_{0 \leq s_1 \leq s} |X_{\epsilon}(s_1) - Z_{\epsilon}(s_1)|^2 \right) ds,$$

moreover \cite{26, 27},

$$\mathbb{E} \left( \sup_{0 \leq t \leq u} |X_{\epsilon}(t) - Z_{\epsilon}(t)|^2 \right)$$

$$\leq \left( K_{12}\epsilon^2 u^{2\beta} + (K_{22} + K_{32})\epsilon u^{2\beta-1} \right)$$

$$\sum_{k=0}^{\infty} \frac{\left[ (K_{11}\epsilon^2 u^{1+\beta} + (K_{21} + K_{31})\epsilon u^\beta) \Gamma(\beta) \right]^k}{\Gamma(k\beta + 1)}.$$

Thus, we can find $L > 0$ and $\lambda \in (0, 1)$ such that for every $t \in (0, L\epsilon^{-\lambda}] \subseteq [0, T]$ having

$$\mathbb{E} \left( \sup_{0 \leq t \leq L\epsilon^{-\lambda}} |X_{\epsilon}(t) - Z_{\epsilon}(t)|^2 \right) \leq C\epsilon^{1-\lambda},$$

where

$$C = \left( K_{12}L^{2\beta}\epsilon^{1+\lambda-2\lambda\beta} + (K_{22} + K_{32})L^{2\beta-1}\epsilon^{2\lambda(1-\beta)} \right)$$

$$\sum_{k=0}^{\infty} \frac{\left[ (K_{11}L^{1+\beta}\epsilon^{2-\lambda-\lambda\beta} + (K_{21} + K_{31})L^\beta \epsilon^{-\beta\lambda}) \Gamma(\beta) \right]^k}{\Gamma(k\beta + 1)}.$$
is a constant.

IV. EXAMPLE

In this section, we reduce a Caputo type fractional stochastic equations into simpler form to illustrate our main result.

Consider

\[
\begin{cases}
D^\beta_t X_\epsilon(t) = 2\epsilon X_\epsilon \cos(t) + \sqrt{\epsilon} \frac{dB(t)}{dt} + \sqrt{\epsilon} \int_{|x|<\epsilon^c} 2x^4 \sin(t)^2 X_\epsilon(t) \nu_\alpha(dx), \\
X_\epsilon(0) = 0.1,
\end{cases}
\]

(10)

where \(\beta \in (\frac{1}{2}, 1)\), \(\alpha\)-stable Lévy jump measure \(\nu_\alpha(dx) = \frac{\gamma}{x^{1+\alpha}} dx\), constants \(\gamma > 0\), \(\alpha \in (0, 2)\).

Together with the averaged system

\[
D_t^{\gamma} Z_\epsilon(t) = \epsilon Z_\epsilon(1 + \gamma_1) + \sqrt{\epsilon} \frac{dB(t)}{dt}, \quad Z_\epsilon(0) = 0.1, \quad \gamma_1 = \frac{\gamma c^{4-\alpha}}{\sqrt{\epsilon}(4-\alpha)},
\]

(11)

define

\[
Er = \left[ |X_\epsilon(t) - Z_\epsilon(t)|^2 \right]^{\frac{1}{2}},
\]

we numerically compare solution paths for equations (10) and (11) in Fig. 1.
Fig. 1. Solution paths of equations (10) and (11) when $\epsilon = 0.001$, $c = 0.5$: (a) $\beta = 0.6, \alpha = 0.3, \gamma = 3$, (b) $\beta = 0.6, \alpha = 1.1, \gamma = 0.6$, (c) $\beta = 0.85, \alpha = 0.3, \gamma = 0.6$, (d) $\beta = 0.85, \alpha = 1.9, \gamma = 3$.

Note that a good agreement is demonstrated.

V. CONCLUSION

In this paper, we have proved an averaging principle for the Caputo type fractional stochastic equations with non-Gaussian Lévy motion. It provides an effective stochastic approximation of solutions of fractional stochastic dynamical systems.

As fractional stochastic differential equations arise as models for a variety of complex systems under non-Gaussian random influences, our method for fractional averaging will be beneficial in extracting effective dynamical behaviors of such systems.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in GitHub.

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