Abstract
In this note, we establish a functional central limit theorem for the capacity of the range for a class of $\alpha$-stable random walks on the integer lattice $\mathbb{Z}^d$ with $d > 5\alpha/2$. Using similar methods, we also prove an analogous result for the cardinality of the range when $d > 3\alpha/2$.

Keywords
The range of a random walk · Capacity · Functional central limit theorem

Mathematics Subject Classification 60F17 · 60F05 · 60G50 · 60G52

1 Introduction
Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{\xi_i\}_{i \geq 1}$ be a sequence of i.i.d. $\mathbb{Z}^d$-valued random variables defined on $(\Omega, \mathcal{F}, P)$, where $\mathbb{Z}^d$ denotes the $d$-dimensional integer lattice. Further, let $S_0 = x$, and $S_n = S_{n-1} + \xi_n, n \geq 1$, be a $\mathbb{Z}^d$-valued random walk starting from $x \in \mathbb{Z}^d$. The range of the random walk $\{S_n\}_{n \geq 0}$ is defined as the random
set
\[ \mathcal{R}_n = \{S_0, \ldots, S_n\}, \quad n \geq 0. \]

Throughout the paper, we use the notation \(|\mathcal{R}_n|\) to denote the cardinality of \(\mathcal{R}_n\).

In addition to the cardinality of the range, we also consider its capacity. Let \(\mathbb{P}_x\) be the probability measure (on the space \((\Omega, \mathcal{F})\)) which corresponds to \(\{S_n\}_{n \geq 0}\) starting at \(x \in \mathbb{Z}^d\). We write \(\mathbb{P}\) instead of \(\mathbb{P}_0\). For any \(A \subseteq \mathbb{Z}^d\), we denote by \(T_A^+\) the first hitting time of the set \(A\) by \(\{S_n\}_{n \geq 0}\), that is,
\[ T_A^+ = \inf\{n \geq 1 : S_n \in A\}. \]

Also, when \(A = \{x\}\) for \(x \in \mathbb{Z}^d\), we write \(T_x^+\) instead of \(T_{\{x\}}^+\). Recall \(\{S_n\}_{n \geq 0}\) is said to be transient if \(\mathbb{P}(T_0^+ = \infty) > 0\); otherwise, it is said to be recurrent, which implies that every random walk is either transient or recurrent. In the case when \(\{S_n\}_{n \geq 0}\) is transient, the capacity of a set \(A \subseteq \mathbb{Z}^d\) is defined as
\[ \text{Cap}(A) = \sum_{x \in A} \mathbb{P}_x(T_A^+ = \infty). \]

For \(n \geq 0\), we denote \(C_n = \text{Cap}(\mathcal{R}_n)\). Observe that \(C_n\) is a random variable. The aim of this article is to prove a functional central limit theorem (FCLT) for the capacity and the cardinality of the range of the random walk \(\{S_n\}_{n \geq 0}\), that is, for the stochastic processes \(\{C_{\lfloor nt\rfloor}\}_{t \geq 0}\) and \(\{|\mathcal{R}_{\lfloor nt\rfloor}|\}_{t \geq 0}\).

The study on the range of random walks in \(\mathbb{Z}^d\) has a long history. A pioneering work is due to Dvoretzky and Erdős [8] where they obtained the strong law of large numbers for \(\{|\mathcal{R}_n|\}_{n \geq 0}\) of a simple random walk in \(d \geq 2\). This result was later extended by Spitzer [23] to all random walks in \(d \geq 1\). A central limit theorem for \(\{|\mathcal{R}_n|\}_{n \geq 0}\) was first obtained by Jain and Orey [10] for strongly transient random walks (see below for the definition of strong transience). Jain and Pruitt [13] later extended this result to all random walks in \(d \geq 3\). Le Gall [18] proved a version of a central limit theorem for \(\{|\mathcal{R}_n|\}_{n \geq 0}\) of all two-dimensional random walks with zero mean and finite second moment. It is remarkable that in this case the limit law is not normal. For \(d = 1\), Jain and Pruitt [15, Theorem 6.1] proved that \(\mathbb{E}[|\mathcal{R}_n|] \asymp \sqrt{n}\), where the symbol \(\asymp\) means that the ratio of the two expressions is bounded from below and above by some positive constants. Le Gall and Rosen [19] established the strong law of large numbers and central limit theorem for \(\{|\mathcal{R}_n|\}_{n \geq 0}\) of a class of \(\alpha\)-stable random walks.

Studies on the long-time behavior of \(\{C_n\}_{n \geq 0}\) were initiated by Jain and Orey [10] where they obtained a version of the strong law of large numbers for any transient random walk. Asselah et al. [1] proved a central limit theorem for \(\{C_n\}_{n \geq 0}\) of a simple random walk in \(d \geq 6\). Versions of the law of large numbers and central limit theorem in \(d = 4\) were proved by the same authors in [2], see also [6]. Recently, Schapira [22] proved a central limit theorem for \(\{C_n\}_{n \geq 0}\) of a class of symmetric random walks in \(\mathbb{Z}^5\) which satisfy appropriate moment conditions. In [7], the present authors established a central limit theorem for \(\{C_n\}_{n \geq 0}\) of a class of \(\alpha\)-stable random walks in \(d > 5\alpha/2\).
A FCLT for $|\mathcal{R}_n|_{n \geq 0}$ was proved by Jain and Pruitt [16] for all random walks in $d \geq 3$ satisfying $\mathbb{P}(T_0^+ = \infty) < 1$ (note that if $\mathbb{P}(T_0^+ = \infty) = 1$, then $|\mathcal{R}_n| = n + 1$ a.s.). This result is a version of Donsker’s invariance principle, and it states that suitably normalized and linearly interpolated process $\{|\mathcal{R}_n|\}_{n \geq 0}$ converges weakly in the space of continuous functions endowed with the locally uniform topology to a standard one-dimensional Brownian motion. The purpose of the present article is to prove an analogous result for $\{C_{\lfloor nt\rfloor}\}_{t \geq 0}$ and $\{|\mathcal{R}_{\lfloor nt\rfloor}|\}_{t \geq 0}$ of a class of $\alpha$-stable random walks.

We remark that there is a correspondence between the limit behavior of the cardinality of the range when the ratio of the dimension to the index of stability is $\rho$ and the capacity of the range when this ratio is $\rho + 1$ (as already indicated in [7]). Basing upon the results for the range when $d/\alpha \leq 3/2$ from [19], we expect that for $d/\alpha \leq 5/2$ the capacity of the range behaves in a different manner than presented in this article. If $d/\alpha = 5/2$, we conjecture that the limit law is again Gaussian but the scaling sequence should be of the form $\sqrt{n g(n)}$, where $g(n)$ is a slowly varying function. This corresponds to the scaling sequence for the range process in the case $d/\alpha = 3/2$, as established in [19, Section 4.5]. We remark that the case $\alpha = 2$ (and $d = 5$) for the capacity process has been recently partially solved by Schapira [22] where he studied a class of symmetric random walks which satisfy some moment condition and for this class he obtained the normal law in the limit while the scaling sequence was $\sqrt{n \log n}$. For $2 \leq d/\alpha < 5/2$, we conjecture that the limit law is nonnormal and it is given in terms of the self-intersection local time of the limiting stable process (see [2, Theorem 1.2] for the case $\alpha = 2$) and with the scaling sequence of the form $n^{3-d/\alpha} g(n)$, where $g(n)$ is again a slowly varying function. In the context of the range process, we expect the same asymptotic behavior but the corresponding scaling sequence should be of the form $n^{2-d/\alpha} g(n)$, see [19, Result 2].

We remark that there are further interesting results pertaining to the limit behavior of the process $\{|\mathcal{R}_n|\}_{n \geq 0}$, including the law of the iterated logarithm and an almost sure invariance principle, see [3,4,9,11,12,14,16].

Before we state the main result of this article, we formulate and briefly discuss assumptions which we impose on the random walk $\{S_n\}_{n \geq 0}$.

(A1) $\{S_n\}_{n \geq 0}$ is aperiodic, that is, the set $\{x \in \mathbb{Z}^d : \mathbb{P}(S_1 = x) > 0\}$ generates (as an additive subgroup) the whole of $\mathbb{Z}^d$.

(A2) $\{S_n\}_{n \geq 0}$ belongs to the domain of attraction of a nondegenerate $\alpha$-stable law with index $0 < \alpha \leq 2$. This means there exists a regularly varying function $b(x)$ with index $1/\alpha$ such that

$$\frac{S_n}{b(n)} \xrightarrow{(d) \quad n \to \infty} X_\alpha,$$

where $X_\alpha$ is an $\alpha$-stable random variable in $\mathbb{R}^d$ and $(d)$ stands for the convergence in distribution.

(A3) $\{S_n\}_{n \geq 0}$ is symmetric and strongly transient.

(A4) $\{S_n\}_{n \geq 0}$ admits one-step loops, that is, $\mathbb{P}(S_1 = 0) > 0$. 

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Theorem 1.1 Assume \((A1)\) is not restrictive. If \(\{S_n\}_{n \geq 0}\) were not aperiodic, we could perform our analysis (and obtain the same results) on the (smallest) additive subgroup generated by the set \(\{x \in \mathbb{Z}^d : \mathbb{P}(S_1 = x) > 0\}\).

Assumption \((A2)\) is of fundamental importance for our analysis. It allows us to apply error estimates in the capacity decomposition which we use in the proof of a FCLT for \(\{C_{\lfloor nt \rfloor}\}_{t \geq 0}\). Similarly, in view of \((A2)\) we can apply results from [19] to estimate the number of intersection points of two independent copies of our random walk which are necessary to prove a FCLT for \(\{|R_{\lfloor nt \rfloor}|\}_{t \geq 0}\).

To discuss assumption \((A3)\), recall that \(\{S_n\}_{n \geq 0}\) is transient if \(\mathbb{P}(T_0^+ = \infty) > 0\). Transience is equivalent to the convergence of the series \(\sum_{n \geq 1} \mathbb{P}(S_n = 0)\). The random walk \(\{S_n\}_{n \geq 0}\) is called strongly transient if the series \(\sum_{n \geq 1} \sum_{k \geq n} \mathbb{P}(S_k = 0)\) converges. We remark that in \(d \geq 3\) every random walk is transient, and in \(d \geq 5\) it is strongly transient. However, such random walks can also appear in lower dimensions. In particular, under assumption \((A2)\), \(\{S_n\}_{n \geq 0}\) is transient if \(d > \alpha\) and strongly transient if \(d > 2\alpha\), see [21, Theorem 3.4] and [24, Theorem 7]). The notion of strong transience was first introduced in [20] for Markov chains and was later used in [10] in the context of the limit behavior of the range of random walks.

Assumption \((A4)\) is purely technical. Accompanied by assumptions \((A1)–(A3)\), it has recently enabled the present authors in [7] to conclude that the appropriately centered and normalized stochastic process \(\{C_n\}_{n \geq 0}\) converges weakly to a normal law. Notice that \((A4)\) excludes a simple random walk. We show, however, that a FCLT for \(\{C_{\lfloor nt \rfloor}\}_{t \geq 0}\) and \(\{|R_{\lfloor nt \rfloor}|\}_{t \geq 0}\) of this process holds true as well.

We now state the main results of the article.

**Theorem 1.1** Assume \((A1)–(A4)\), \(0 < \alpha \leq 2\), and \(d/\alpha > 5/2\). Then, there is a constant \(\sigma_d > 0\) such that

\[
\left\{ C_{\lfloor nt \rfloor} - \mathbb{E}[C_{\lfloor nt \rfloor}] \right\}_{t \geq 0} \xrightarrow{n \to \infty} \{B_t\}_{t \geq 0},
\]

where \([x]\) denotes the integer part of \(x \in \mathbb{R}\), \(\xrightarrow{n \to \infty} \) stands for the weak convergence in the Skorohod space \(D([0, \infty), \mathbb{R})\) endowed with the \(J_1\) topology, and \(\{B_t\}_{t \geq 0}\) denotes a standard one-dimensional Brownian motion.

Our proof is based on the central limit theorem from [7] and a tightness argument. To establish tightness, we utilize the capacity decomposition from [1, Corollary 2.1] and combine it with the error term estimates from [7, Lemma 3.1] which hold under crucial assumptions \((A1)–(A3)\). Using a similar reasoning and estimates of the number of intersection points (extracted from [19]), we prove an analogous result for \(\{|R_{\lfloor nt \rfloor}|\}_{t \geq 0}\).

**Theorem 1.2** Assume \((A1), (A2)\), \(0 < \alpha \leq 2\), \(d/\alpha > 3/2\) and \(\mathbb{P}(T_0^+ = \infty) < 1\). Then, there is a constant \(\sigma_d > 0\) such that

\[
\left\{ |R_{\lfloor nt \rfloor}| - \mathbb{E}[|R_{\lfloor nt \rfloor}|] \right\}_{t \geq 0} \xrightarrow{n \to \infty} \{B_t\}_{t \geq 0},
\]

where we use the same notation as in Theorem 1.1.
This result has to be compared with the Donsker’s invariance principle for \(|R_n|_{n \geq 0}\) which was found by Jain and Pruitt \([16]\). They proved that performing an appropriate linearization of \(|R_n|_{n \geq 0}\), the uniform convergence towards a Brownian motion holds in two cases: (i) for all random walks in dimensions \(d \geq 3\) which satisfy \(\mathbb{P}(T_0^+ = \infty) < 1\), and (ii) for all strongly transient random walks for which a certain moment condition is valid. We notice that this condition is usually not easy to check, see the closing Remark in \([16]\). We establish a slightly more general result, as we prove convergence in the Skorohod space, but for stable random walks only. Notice, however, that we may well have \(d < 3\). We also mention that we could easily deduce the locally uniform convergence in the space of continuous functions by employing a suitable linearization.

2 FCLT for the Process \(\{C_{\lfloor nt \rfloor}\}_{t \geq 0}\)

Throughout this section, we assume that \(\{S_n\}_{n \geq 0}\) satisfies \((A1)–(A4)\). We denote by \(G(x, y)\) the corresponding Green function, that is,

\[
G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(S_n = y), \quad x, y \in \mathbb{Z}^d.
\]

Also, for \(A, B \subseteq \mathbb{Z}^d\) we denote

\[
G(A, B) = \sum_{x \in A} \sum_{y \in B} G(x, y).
\]

In order to prove a FCLT for \(\{C_{\lfloor nt \rfloor}\}_{t \geq 0}\), we employ the following classical two-step scheme, see \([17, \text{Theorems 16.10 and 16.11}]\). Let \(\{X^n\}_{n \geq 0}\) be a sequence of random elements in the Skorohod space \(\mathcal{D}([0, \infty), \mathbb{R})\) endowed with the \(J_1\) topology. The sequence \(\{X^n\}_{n \geq 0}\) converges weakly to a random element \(X\) (in \(\mathcal{D}([0, \infty), \mathbb{R})\)) if the following two conditions are satisfied:

1. The finite-dimensional distributions of \(\{X^n\}_{n \geq 0}\) converge weakly to the finite-dimensional distributions of \(X\).
2. For any bounded sequence \(\{T_n\}_{n \geq 1}\) of \(\{X^n\}_{n \geq 0}\)-stopping times, any sequence \(\{h_n\}_{n \geq 1} \subset [0, \infty)\) converging to zero, and any \(\varepsilon > 0\), it holds that

\[
\lim_{n \to \infty} \mathbb{P}( |X^n_{T_n + h_n} - X^n_{T_n}| \geq \varepsilon ) = 0.
\]

**Proof of Theorem 1.1** We consider the following sequence of random elements which are defined in the space \(\mathcal{D}([0, \infty), \mathbb{R})\),

\[
X^n_t = \frac{C_{\lfloor nt \rfloor} - \mathbb{E}[C_{\lfloor nt \rfloor}]}{\sigma_d \sqrt{n}}, \quad n \geq 1, \tag{2.1}
\]
where $\sigma_d$ is a positive constant. We prove validity of conditions (i) and (ii) under assumptions (A1)–(A4). Let us start by showing condition (i).

**Condition (i).** By [7, Theorem 1.1.], we have that for any $t > 0$

$$X^n_t = \frac{C_{[nt]} - \mathbb{E}[C_{[nt]}]}{\sigma_d\sqrt{nt}} \cdot \frac{\sqrt{nt}}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0, t),$$

(2.2)

where $\xrightarrow{(d)}$ stands for the convergence in distribution, the constant $\sigma_d > 0$ is determined in [7, Theorem 1.1.] and $\mathcal{N}(0, t)$ stands for a Gaussian random variable with mean zero and variance $t$. Let $k \geq 1$ be an arbitrary integer and choose $0 = t_0 < t_1 < t_2 < \cdots < t_k$. We need to prove that

$$(X^n_{t_1}, X^n_{t_2}, \ldots, X^n_{t_k}) \xrightarrow{(d)} (B_{t_1}, B_{t_2}, \ldots, B_{t_k}).$$

In view of the Cramér–Wold theorem [17, Corollary 5.5], it suffices to show that

$$\sum_{j=1}^{k} v_j X^n_{t_j} \xrightarrow{(d)} \sum_{j=1}^{k} v_j B_{t_j}, \quad (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k. \quad (2.3)$$

To prove (2.3), we first find lower and upper bounds for $C_{[nt_j]}, \ j = 1, \ldots, k$. We follow closely the arguments from the capacity decomposition [1, Corollary 2.1]. For $i = 1, \ldots, k$, we have that

$$C_{[nt_i]} = \text{Cap} \left( \mathcal{R}_{[nt_i]} \cup \mathcal{R}([nt_1], [nt_i]) \right) \leq \text{Cap} \left( \left( \mathcal{R}_{[nt_i]} - S_{[nt_i]} \right) \cup \left( \mathcal{R}([nt_1], [nt_i]) - S_{[nt_i]} \right) \right),$$

where for $1 \leq m \leq n$, $\mathcal{R}[m,n] := \{S_m, \ldots, S_n\}$. By the Markov property, the two random variables

$$\mathcal{R}^{(1)}_{[nt_1]} := \mathcal{R}_{[nt_1]} - S_{[nt_1]} \quad \text{and} \quad \mathcal{R}^{(2)}_{[nt_i]-[nt_1]} := \mathcal{R}([nt_1], [nt_i]) - S_{[nt_1]}$$

are independent, and $\mathcal{R}^{(2)}_{[nt_i]-[nt_1]}$ has the same law as $\mathcal{R}([nt_i]-[nt_1])$. The symmetry of the random walk $\{S_n\}_{n \geq 0}$ implies that $\mathcal{R}^{(1)}_{[nt_1]}$ is equal in law to $\mathcal{R}^{(2)}_{[nt_i]-[nt_1]}$. Hence, [1, Proposition 1.2] implies

$$C_{[nt_i]} \geq \text{Cap} \left( \mathcal{R}^{(1)}_{[nt_1]} \right) + \text{Cap} \left( \mathcal{R}^{(2)}_{[nt_i]-[nt_1]} \right) - 2G \left( \mathcal{R}^{(1)}_{[nt_1]}, \mathcal{R}^{(2)}_{[nt_i]-[nt_1]} \right).$$
We now present how to deal with the next step of the decomposition. We have

\[
\text{Cap}\left(\mathcal{R}_{[nt_2]-[nt_1]}^{(2)}\right) = \text{Cap}\left(\mathcal{R}_{[nt_1], [nt_2]} - S_{[nt_1]}\right) \\
= \text{Cap}\left(\mathcal{R}_{[nt_1], [nt_2]}\right) \\
= \text{Cap}\left((\mathcal{R}_{[nt_1], [nt_2]} - S_{[nt_2]}) \cup (\mathcal{R}_{[nt_2], [nt_1]} - S_{[nt_2]})\right).
\]

Similarly as before, the two random variables

\[
\mathcal{R}_{[nt_2]-[nt_1]}^{(2)} := \mathcal{R}_{[nt_1], [nt_2]} - S_{[nt_2]} \quad \text{and} \quad \mathcal{R}_{[nt_2]-[nt_1]}^{(3)} := \mathcal{R}_{[nt_2], [nt_1]} - S_{[nt_2]}
\]

are independent. Also, the random variable \(\mathcal{R}_{[nt_2]-[nt_1]}^{(2)}\) has the same law as \(\mathcal{R}_{[nt_2]-[nt_1]}\), and \(\mathcal{R}_{[nt_2]-[nt_1]}^{(3)}\) has the same law as \(\mathcal{R}_{[nt_2]-[nt_1]}\). If we continue with this procedure and use the same arguments as above, together with subadditivity property of the capacity (see [23, Proposition 25.11]) for the upper bound, we obtain the following estimates

\[
\sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)} - 2 \sum_{i=1}^{j-1} E_{[nt_j]}^{(i)} \leq C_{[nt_j]} \leq \sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)}, \quad j = 1, \ldots, k,
\]

where

\[
C_{[nt_1]-[nt_{i-1}]}^{(i)} := \text{Cap}\left(\mathcal{R}_{[nt_1]-[nt_{i-1}]}^{(i)}\right) \quad \text{and} \quad E_{[nt_j]}^{(i)} := G(\mathcal{R}_{[nt_j]}^{(i)}, \mathcal{R}_{[nt_j]-[nt_i]}^{(i+1)}).
\]

The random variables \(C_{[nt_1]-[nt_{i-1}]}^{(i)}\), \(i = 1, \ldots, k\), are independent, \(\mathcal{R}_{[nt_1]-[nt_{i-1}]}^{(i)}\) has the same law as \(\mathcal{R}_{[nt_1]-[nt_{i-1}]}\) and \(E_{[nt_j]}^{(i)}\) has the same law as \(G(\mathcal{R}_{[nt_j]}^{(i)}, \mathcal{R}_{[nt_j]-[nt_i]}^{(i+1)})\), with \(\mathcal{R}_{[nt_j]-[nt_i]}^{(i)}\) being an independent copy of \(\mathcal{R}_{[nt_j]-[nt_i]}\).

We now find lower and upper bounds for the left-hand side expression in (2.3). For \(v_j \geq 0\), we have

\[
v_j \sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)} - 2v_j \sum_{i=1}^{j-1} E_{[nt_j]}^{(i)} \leq v_j C_{[nt_j]} \leq v_j \sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)},
\]

and for \(v_j < 0\),

\[
v_j \sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)} - 2v_j \sum_{i=1}^{j-1} E_{[nt_j]}^{(i)} \geq v_j C_{[nt_j]} \geq v_j \sum_{i=1}^{j} C_{[nt_1]-[nt_{i-1}]}^{(i)}.
\]

Thus, by splitting the sum into two parts, we obtain the following lower bound

\[
\sum_{j=1}^{k} v_j X^{\ast}_{j} \geq \sum_{1 \leq j \leq k} \frac{1}{\sigma_d \sqrt{n}} \left( v_j \sum_{i=1}^{j} \left( C_{[nt_1]-[nt_{i-1}]}^{(i)} - \mathbb{E}[C_{[nt_1]-[nt_{i-1}]}^{(i)}]\right) - 2v_j \sum_{i=1}^{j-1} E_{[nt_j]}^{(i)} \right).
\]
\[ + \sum_{1 \leq j < k} \frac{1}{\sigma_d \sqrt{n}} \left( v_j \sum_{i=1}^{j} \left( c^{(i)}_{\lfloor nt_j \rfloor - \lfloor nt_{i-1} \rfloor} - \mathbb{E}\left[ c^{(i)}_{\lfloor nt_j \rfloor - \lfloor nt_{i-1} \rfloor} \right] \right) + 2v_j \sum_{i=1}^{j-1} \mathbb{E}\left[ \mathcal{E}_{\lfloor nt_j \rfloor}^{(i)} \right] \right) \]

\[ = \sum_{i=1}^{k} \left( \sum_{j=i}^{k} v_j \right) J_n^{(i)} - \frac{2}{\sigma_d \sqrt{n}} \sum_{1 \leq j < k} v_j \sum_{i=1}^{j-1} \mathcal{E}_{\lfloor nt_j \rfloor}^{(i)} \]

\[ + \frac{2}{\sigma_d \sqrt{n}} \sum_{1 \leq j < k} v_j \sum_{i=1}^{j-1} \mathbb{E}\left[ \mathcal{E}_{\lfloor nt_j \rfloor}^{(i)} \right], \quad (2.4) \]

where

\[ J_n^{(i)} = \frac{c^{(i)}_{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor} - \mathbb{E}\left[ c^{(i)}_{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor} \right]}{\sigma_d \sqrt{n}}, \quad i = 1, \ldots, k. \]

We now study weak convergence of \( \{J_n^{(i)}\}_{n \geq 1} \), and for this, we can replace \( \{c^{(i)}_n\}_{n \geq 0} \) by \( \{c^{(i)}_n\}_{n \geq 0} \). We clearly have

\[ |x - y| \leq |x| - |y| \leq |x - y| + 1, \quad x \geq y \geq 0. \quad (2.5) \]

Also, the map \( n \mapsto c_n \) is monotone, and it holds that

\[ c_{n+1} = \text{Cap} (\mathcal{R}_{n+1}) = \text{Cap} (\mathcal{R}_n \cup \{S_{n+1}\}) \leq \text{Cap} (\mathcal{R}_n) + \text{Cap} (S_{n+1}) \leq c_n + 1. \quad (2.6) \]

Combining (2.5) and (2.6) with (2.2), we easily conclude that

\[ J_n^{(i)} \xrightarrow{(d)} n \rightarrow \infty \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, 2, \ldots, k. \]

Next, we show that the two last terms in (2.4) are negligible. Indeed, the Markov inequality together with [7, Lemma 3.2] implies that there is a constant \( C > 0 \) such that for every \( \varepsilon > 0 \),

\[ \mathbb{P}\left( n^{-1/2} \mathcal{E}_{\lfloor nt_j \rfloor}^{(i)} > \varepsilon \right) \leq \frac{\mathbb{E}\left[ \mathcal{E}_{\lfloor nt_j \rfloor}^{(i)} \right]}{\varepsilon \sqrt{n}} \leq \frac{\mathbb{E}\left[ G(\mathcal{R}_{\lfloor nt_j \rfloor}, \widetilde{\mathcal{R}}_{\lfloor nt_j \rfloor}) \right]}{\varepsilon \sqrt{n}} \leq \frac{CH_d(\lfloor nt_j \rfloor)}{\varepsilon \sqrt{n}}, \quad (2.7) \]

where

\[ H_d(n) = \begin{cases} 1, & d/\alpha > 3, \\ \sum_{k=1}^{n} k^{-1} \ell(k)^{-d}, & d/\alpha = 3, \\ n^3(b(n))^{-d}, & 2 < d/\alpha < 3, \end{cases} \quad (2.8) \]

and the function \( b(x) \) is necessarily of the form

\[ b(x) = x^{1/\alpha} \ell(x), \quad x \geq 0, \quad (2.9) \]
where \( \ell(x) \) is a slowly varying function, see [5]. Moreover, by [19, Lemma 2.2], the function \( n \mapsto \sum_{k=1}^{n} k^{-d} \ell(k)^{-d} \) is slowly varying. Hence, \( H_d(n) \) is a slowly varying function in the case \( d/\alpha \geq 3 \) and regularly varying function with index strictly smaller than \( 1/2 \) when \( d/\alpha \in (5/2, 3) \). This implies that the last term in (2.7) tends to zero as \( n \to \infty \). Recall that the random variables \( C_{[n_i]} - [n_{i-1}] \), \( i = 1, \ldots, k \), are independent. Therefore, after performing the same analysis as in (2.4) for the upper bound, we conclude that

\[
\sum_{j=1}^{k} \sum_{i=1}^{k} v_j X_{t_j}^{n} \xrightarrow{(d)} \mathcal{N} \left( 0, \sum_{i=1}^{k} \left( \sum_{j=i}^{k} v_j \right)^2 \left( t_i - t_{i-1} \right) \right).
\]

We finally notice that

\[
\sum_{i=1}^{k} \left( \sum_{j=i}^{k} v_j \right)^2 \left( t_i - t_{i-1} \right) = \sum_{i=1}^{k} \left( v_i t_i + 2 \sum_{j>i} v_i v_j t_i \right).
\]

Hence, the finite-dimensional distributions of \( \{X^n\}_{n \geq 1} \) converge weakly to the finite-dimensional distributions of a one-dimensional standard Brownian motion.

**Condition (ii).** Let \( \{T_n\}_{n \geq 1} \) be a bounded sequence of \( \{X^n\}_{n \geq 1} \)-stopping times and \( \{h_n\}_{n \geq 1} \subset [0, \infty) \) an arbitrary sequence which converges to zero. We want to prove that

\[
X^n_{T_n+h_n} - X^n_{T_n} \xrightarrow{\mathbb{P}} 0, \quad \text{(2.10)}
\]

where \( \xrightarrow{\mathbb{P}} \) stands for the convergence in probability. By (2.1), we have

\[
X^n_{T_n+h_n} - X^n_{T_n} = \frac{C_{[nT_n+h_n]} - \mathbb{E}[C_{[nT_n+h_n]}]}{\sigma_d \sqrt{n}} - \frac{C_{[nT_n]} - \mathbb{E}[C_{[nT_n]}]}{\sigma_d \sqrt{n}}.
\]

Proceeding as in the capacity decomposition from [1, Corollary 2.1] and combining the strong Markov property with the subadditivity and monotonicity of the capacity yield

\[
C_{[n(T_n+h_n)]} \leq C_{[nT_n]+[n h_n]} + 1 \leq C_{[nT_n]} + C_{[n h_n]} + 1,
\]

and

\[
C_{[n(T_n+h_n)]} \geq C_{[nT_n]+[n h_n]} \geq C_{[nT_n]} + C_{[n h_n]} - 2\mathcal{E}([nT_n], [n h_n]),
\]

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where $C_{[nT_n]}^{(1)}$ and $C_{[nh_n]}^{(2)}$ are independent and have the same law as $C_{[nT_n]}$ and $C_{[nh_n]}$, respectively. Moreover, the random variable $E([nT_n], [nh_n])$ has the same law as $G(R_{[nT_n]}, \tilde{R}_{[nh_n]})$, with $\tilde{R}_{[nh_n]}$ being an independent copy of $R_{[nh_n]}$.

Using these inequalities, we now bound the expression $X^n_{T_n+h_n} - X^n_{T_n}$ from below and above with quantities converging to zero in probability. We start with the lower bound

$$X^n_{T_n+h_n} - X^n_{T_n} \geq \frac{C_{[nh_n]}^{(2)} - \mathbb{E}[C_{[nh_n]}^{(2)} + 1]}{\sigma_d \sqrt{n}} - \frac{2\mathbb{E}( [nT_n], [nh_n] )}{\sigma_d \sqrt{n}} + \frac{\mathbb{E}[C_{[nh_n]}^{(2)}] - \mathbb{E}[C_{[nh_n]}^{(2)} + 1]}{\sigma_d \sqrt{n}} \geq \frac{C_{[nh_n]}^{(2)} - \mathbb{E}[C_{[nh_n]}^{(2)}]}{\sigma_d \sqrt{n}} - \frac{2\mathbb{E}( [nT_n], [nh_n] )}{\sigma_d \sqrt{n}} - \frac{1}{\sigma_d \sqrt{n}},$$

where in the last line we used (2.6). It suffices to show that

$$\frac{C_{[nh_n]}^{(2)} - \mathbb{E}[C_{[nh_n]}^{(2)}]}{\sigma_d \sqrt{n}} - \frac{2\mathbb{E}( [nT_n], [nh_n] )}{\sigma_d \sqrt{n}} \overset{\mathbb{P}}{\longrightarrow} 0. \quad (2.11)$$

Take arbitrary $\varepsilon > 0$. The Markov inequality together with the fact that there is a constant $C > 0$ such that $\sup_{n \geq 1} \max[T_n, h_n] \leq C$ implies

$$\mathbb{P} \left( n^{-1/2} \left| (C_{[nh_n]}^{(2)} - \mathbb{E}[C_{[nh_n]}^{(2)}]) - 2\mathbb{E}( [nT_n], [nh_n] ) \right| > \sigma_d \varepsilon \right) \leq \mathbb{P} \left( 2 |C_{[nh_n]}^{(2)} - \mathbb{E}[C_{[nh_n]}^{(2)}]| > \varepsilon \sigma_d \sqrt{n} \right) + \mathbb{P} \left( 4\mathbb{E}( [nT_n], [nh_n] ) > \varepsilon \sigma_d \sqrt{n} \right) \leq \frac{4 \text{Var}(C_{[nh_n]}^{(2)})}{\varepsilon^2 \sigma_d^2 n} + \frac{4 \mathbb{E}[G(R_{[nT_n]}, \tilde{R}_{[nh_n]})]}{\sigma_d \varepsilon \sqrt{n}} \leq \frac{4C_1 nh_n}{\varepsilon^2 \sigma_d^2 n} + \frac{4C_2 H_d(Cn)}{\sigma_d \varepsilon \sqrt{n}} \overset{\mathbb{P}}{\longrightarrow} 0,$$

where in the last line we applied [7, Lemma 4.3] to conclude that there is a constant $C_1 > 0$ such that $\text{Var}(C_n) \leq C_1 n$ for all $n \geq 1$, and [7, Lemma 3.2] to find a constant $C_2 > 0$ such that $\mathbb{E}[G(R_{[nT_n]}, \tilde{R}_{[nh_n]})] \leq C_2 H_d(Cn)$ for all $n \geq 1$, where $H_d(n)$ is defined in (2.8). Its index of regular variation is strictly smaller than $1/2$ for all $d/\alpha > 5/2$. This gives us (2.11).

To obtain the upper bound, we write

$$X^n_{T_n+h_n} - X^n_{T_n} \leq \frac{C_{[nh_n]}^{(2)} + 1 - \mathbb{E}[C_{[nh_n]}]}{\sigma_d \sqrt{n}} + \frac{2\mathbb{E}[G(R_{[nT_n]}, \tilde{R}_{[nh_n]})]}{\sigma_d \sqrt{n}}$$
One can easily show that in view of (2.6) the last term converges to zero in law (and whence in probability). Finally, for the remaining terms, we use the same arguments as before, which allow us to conclude (2.10) and the proof is finished. □

The next theorem is the corresponding FCLT for a simple random walk. Recall that a simple random walk does not satisfy (A4).

**Theorem 2.1** Let \( \{S_n\}_{n \geq 0} \) be a symmetric simple random walk in \( \mathbb{Z}^d \) with \( d \geq 6 \). Then, the following convergence holds

\[
\left\{ \frac{C_{\lfloor nt \rfloor} - \mathbb{E}[C_{\lfloor nt \rfloor}]}{\sigma_d \sqrt{n}} \right\}_{t \geq 0} \xrightarrow{n \to \infty} \{B_t\}_{t \geq 0},
\]

where \( \sigma_d > 0 \) is the constant from [1, Theorem 1.1].

**Proof** To prove the theorem, we use the same reasoning as in the proof of Theorem 1.1 but we use results from [1] instead of results from [7]. More precisely, instead of [7, Lemma 3.2], we use [1, Lemma 3.2], [7, Lemma 4.3] has to be replaced with [1, Lemma 3.5] and, finally, [7, Theorem 1.1] is replaced with [1, Theorem 1.1]. □

### 3 FCLT for the Process \(|R_{\lfloor nt \rfloor}|\)_{t \geq 0}

In this section, we show how to adapt the methods from the previous section and prove Theorem 1.2. Recall that we assume that \( \{S_n\}_{n \geq 0} \) satisfies (A1), (A2), \( d/\alpha > 3/2 \) and \( \mathbb{P}(T_0^+ = \infty) < 1 \) (if \( \mathbb{P}(T_0^+ = \infty) = 1 \), then \( |\mathcal{R}_n| = n + 1 \) a.s.). Also, notice that by [19, Proposition 2.4] if \( \alpha > 1 \), then \( \mathbb{P}(T_0^+ = \infty) < 1 \) holds true.

Before we prove Theorem 1.2, we formulate the following lemma which shows how the range of a random walk up to time \( n \) can be decomposed into two independent ranges. This idea was first used by Le Gall in [18] to establish a central limit theorem for \(|R_n|\)_{n \geq 0}. Let \( \{S_n\}_{n \geq 0} \) and \( \{\tilde{S}_n\}_{n \geq 0} \) be two independent and identically distributed random walks (defined on the same probability space). By \( I_n \), we denote the number of intersection points up to time \( n \) of the paths of \( \{S_n\}_{n \geq 0} \) and \( \{\tilde{S}_n\}_{n \geq 0} \), that is,

\[
I_n = |\mathcal{R}_n \cap \tilde{\mathcal{R}}_n|,
\]

where \( \{\mathcal{R}_n\}_{n \geq 0} \) is the range of \( \{S_n\}_{n \geq 0} \), and \( \{\tilde{\mathcal{R}}_n\}_{n \geq 0} \) is the range of \( \{\tilde{S}_n\}_{n \geq 0} \).

**Lemma 3.1** For all \( m, n \in \mathbb{N} \), we have

\[
|\mathcal{R}_{m+n}| = |\mathcal{R}_m^{(1)}| + |\mathcal{R}_n^{(2)}| - \mathcal{E}(m, n),
\]

where the random variables \( \mathcal{R}_m^{(1)} \) and \( \mathcal{R}_n^{(2)} \) are independent and have the same law as \( \mathcal{R}_m \) and \( \mathcal{R}_n \), respectively. Moreover, the random variable \( \mathcal{E}(m, n) \) has the same law as \(|\mathcal{R}_m \cap \tilde{\mathcal{R}}_n|\). In particular, \( |\mathcal{R}_m \cap \tilde{\mathcal{R}}_n| \leq I_{m+n} \).
Proof We clearly have

\[ |\mathcal{R}_{m+n}| = |\mathcal{R}_m \cup \mathcal{R}[m, m+n]| = |(\mathcal{R}_m - S_m) \cup (\mathcal{R}[m, m+n] - S_m)|
\]

\[ = |\mathcal{R}_m^{(1)} \cup \mathcal{R}_n^{(2)}|
\]

\[ = |\mathcal{R}_m^{(1)}| + |\mathcal{R}_n^{(2)}| - |\mathcal{R}_m^{(1)} \cap \mathcal{R}_n^{(2)}|.
\]

The Markov property implies that the random variables \( \mathcal{R}_m^{(1)} \) and \( \mathcal{R}_n^{(2)} \) are independent and that the law of \( \mathcal{R}_n^{(2)} \) is equal to the law of \( \mathcal{R}_m \). By symmetry, \( \mathcal{R}_m^{(1)} \) has the same law as \( \mathcal{R}_m \). Evidently, the random variable \( |\mathcal{R}_m^{(1)} \cap \mathcal{R}_n^{(2)}| \) is equal in law to \( |\mathcal{R}_m \cap \mathcal{R}_n| \). The last inequality \( |\mathcal{R}_m \cap \mathcal{R}_n| \leq l_{m+n} \) follows by a monotonicity argument, and the proof is finished. \( \square \)

Proof of Theorem 1.2 To establish the desired result, we again show validity of conditions (i) and (ii) discussed before the proof of Theorem 1.1. We consider the following sequence of random elements in the space \( D([0, \infty), \mathbb{R}) \),

\[ X_t^n = \frac{|\mathcal{R}_{nt}| - \mathbb{E}[|\mathcal{R}_{nt}|]}{\sigma_d \sqrt{n}}, \quad n \geq 1,
\]

for a constant \( \sigma_d > 0 \). By \cite[Theorem 4.5]{19},

\[ X_t^n = \frac{|\mathcal{R}_{nt}| - \mathbb{E}[|\mathcal{R}_{nt}|]}{\sigma_d \sqrt{nt}} \cdot \frac{\sqrt{nt}}{\sqrt{n}} \cdot \frac{\sigma_d}{\sqrt{n}} \xrightarrow{(d) \; n \to \infty} \mathcal{N}(0, t). \tag{3.1}
\]

Condition (i). We choose arbitrary integer \( k \geq 1 \) and fix \( 0 = t_0 < t_1 < t_2 < \cdots < t_k \). By Cramér–Wold theorem \cite[Corollary 5.5]{17}, it suffices to prove that

\[ \sum_{j=1}^{k} v_j X^n_{t_j} \xrightarrow{(d) \; n \to \infty} \sum_{j=1}^{k} v_j B_{t_j}, \quad (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k. \tag{3.2}
\]

Proceeding in an analogous way as in the proof of Theorem 1.1, and applying Lemma 3.1, we obtain

\[ |\mathcal{R}_{nt_j}| = \sum_{i=1}^{j} |\mathcal{R}_{[nt_i] - [nt_{i-1}]}^{(i)}| - \sum_{i=1}^{j-1} \mathcal{E}_{[nt_i]}^{(i)}, \quad j = 1, \ldots, k,
\]

where the random variables \( \mathcal{R}_{[nt_i] - [nt_{i-1}]}^{(i)} \), \( i = 1, \ldots, k \), are independent, and the random variable \( \mathcal{R}_{[nt_j] - [nt_{j-1}]}^{(i)} \) has the same law as \( \mathcal{R}_{[nt_j] - [nt_{j-1}]} \). Moreover,

\[ \mathcal{E}_{[nt_j]}^{(i)} = |\mathcal{R}_{[nt_j] - [nt_{j-1}]}^{(i)} \cap \mathcal{R}_{[nt_j] - [nt_{j-1}]}^{(i+1)}|,
\]

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and it has the same law as $|\mathcal{R}_{[nt_j]} - [nt_{i-1}]| \cap \widetilde{\mathcal{R}}_{[nt_j]} - [nt_{i-1}]|$. Then, we can adapt all the arguments from the proof of Theorem 1.1, and thus we only show how to establish the desired weak convergence. We have the following equality

$$\sum_{j=1}^{k} v_j X_{t_j}^n = \sum_{i=1}^{k} \left( \sum_{j=i}^{k} v_j \right) J_n^{(i)} - \frac{1}{\sigma_d \sqrt{n}} \sum_{j=1}^{k} v_j \sum_{i=1}^{j-1} \mathcal{E}_{[nt_j]}^{(i)} + \frac{1}{\sigma_d \sqrt{n}} \sum_{j=1}^{k} v_j \sum_{i=1}^{j-1} \mathbb{E}[\mathcal{E}_{[nt_j]}^{(i)}],$$

where

$$J_n^{(i)} = \frac{|\mathcal{R}_{[nt_i]} - [nt_{i-1}]| - \mathbb{E}[|\mathcal{R}_{[nt_i]} - [nt_{i-1}]|]}{\sigma_d \sqrt{n}}.$$

Using monotonicity argument together with (2.5), one can apply (3.1) to arrive at

$$J_n^{(i)} \xrightarrow{(d)} \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, 2, \ldots, k.$$

We now investigate the convergence of the two last (error) terms in (3.3). Recall that $\{S_n\}_{n \geq 0}$ and $\{\widetilde{S}_n\}_{n \geq 0}$ are two independent and identically distributed random walks (defined on the same probability space), satisfying (A1) and (A2). Then, in view of [19, Remark after Corollary 3.2], the expectation of the number of their intersection points up to time $n$ admits the following bound

$$\mathbb{E}[I_n] \leq C F_d(n),$$

where $C > 0$ is a constant and $F_d(n)$ is given by

$$F_d(n) = \begin{cases} 1, & d/\alpha > 2, \\ \sum_{k=1}^{n} k^{-1} \ell(k)^{-d}, & d/\alpha = 2, \\ n^2 (b(n))^{-d}, & 1 < d/\alpha < 2, \end{cases}$$

where $b(x)$ and $\ell(x)$ are as in (2.9). Combining the Markov inequality with Lemma 3.1, we obtain

$$\mathbb{P}\left(n^{-1/2} \mathcal{E}_{[nt_j]}^{(i)} > \varepsilon\right) \leq \frac{\mathbb{E}[\mathcal{E}_{[nt_j]}^{(i)}]}{\varepsilon \sqrt{n}} \leq \frac{\mathbb{E}[I_{[nt_j]}]}{\varepsilon \sqrt{n}} \leq \frac{C F_d([nt_j])}{\varepsilon \sqrt{n}}.$$

For $d/\alpha \geq 2$, using [19, Lemma 2.2], we clearly have

$$\frac{F_d([nt_j])}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0.$$
If $3/2 < d/\alpha < 2$, we proceed as follows

$$F_d(\lfloor nt_j \rfloor) = \frac{|\lfloor nt_j \rfloor|^{2-d/\alpha}(\ell(\lfloor nt_j \rfloor))^{-d}}{\sqrt{n}} \leq \frac{(nt_j)^{2-d/\alpha}(\ell(\lfloor nt_j \rfloor))^{-d}}{\sqrt{n}} \leq \frac{t_j^{2-d/\alpha}(\ell(\lfloor nt_j \rfloor))^{-d}}{n^{d/\alpha-3/2}} \to 0.$$  

These relations imply that the first error term in (3.3) converges in probability (and whence in distribution) to zero and the second term is negligible. Using the fact that $R_{\lfloor nt_i \rfloor-\lfloor nt_{i-1} \rfloor}$, $i = 1, \ldots, k$, are independent, we obtain that the quantity in (3.3) converges in law to a normal random variable with mean zero and variance

$$\sum_{i=1}^k (\sum_{j=i}^k v_j)^2 (t_i - t_{i-1}).$$

We finally conclude that the finite-dimensional distributions of $\{X_n\}_{n \geq 1}$ converge weakly to the finite-dimensional distributions of a one-dimensional standard Brownian motion, which means that condition (i) is satisfied.

**Condition (ii).** Let $\{T_n\}_{n \geq 1}$ be a bounded sequence of $\{X_n\}_{n \geq 1}$-stopping times and $\{h_n\}_{n \geq 1} \subset [0, \infty)$ a sequence converging to zero. We prove that

$$X_{T_n+h_n}^n - X_{T_n}^n \xrightarrow{\mathbb{P}} 0.$$  

By the definition,

$$X_{T_n+h_n}^n - X_{T_n}^n = \frac{|R_{\lfloor nT_n+h_n \rfloor} - \mathbb{E}[|R_{\lfloor nT_n+h_n \rfloor}]]}{\sigma_d \sqrt{n}} - \frac{|R_{\lfloor nT_n \rfloor} - \mathbb{E}[|R_{\lfloor nT_n \rfloor}]]}{\sigma_d \sqrt{n}}.$$  

Combining Lemma 3.1 with the following trivial inequality

$$|R_{n+1}| = |R_n \cup \{S_{n+1}\}| \leq |R_n| + 1,$$  

and with the strong Markov property, we obtain

$$|R_{nT_n+h_n}| \leq |R_{nT_n} + [nh_n]| \leq |R_{nT_n}| + |R_{[nh_n]}|,$$

and

$$|R_{nT_n+h_n}| \geq |R_{nT_n} + [nh_n]| \geq |R_{nT_n}| + |R_{[nh_n]}| - \mathcal{E}(\lfloor nT_n \rfloor, [nh_n]),$$

where $R_{nT_n}$ and $R_{[nh_n]}$ are independent and have the same law as $R_{\lfloor nT_n \rfloor}$ and $R_{\lfloor nh_n \rfloor}$, respectively. Moreover, the random variable $\mathcal{E}(\lfloor nT_n \rfloor, [nh_n])$ has the same law as $|R_{\lfloor nT_n \rfloor} \cap \tilde{R}_{[nh_n]}|$. 

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We now show how to bound the sequence \( \{X_{T_n+h_n}^n - X_{T_n}^n\}_{n \geq 1} \) from below and above with quantities that converge to zero in probability. We only sketch the argument for the lower bound, as the second case is similar. We have

\[
X_{T_n+h_n}^n - X_{T_n}^n \geq \frac{|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor+1}|]}{\sigma_d \sqrt{n}} - \frac{\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor)}{\sigma_d \sqrt{n}}
\]

\[
= \frac{|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|]}{\sigma_d \sqrt{n}} - \frac{\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor)}{\sigma_d \sqrt{n}}
\]

\[
+ \frac{\mathbb{E}[|R_{\lfloor nh_n \rfloor}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor+1}|]}{\sigma_d \sqrt{n}}
\]

\[
\geq \frac{|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|]}{\sigma_d \sqrt{n}} - \frac{\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor)}{\sigma_d \sqrt{n}} - \frac{1}{\sigma_d \sqrt{n}}.
\]

where in the last line we used (3.5). It remains to prove that

\[
\frac{|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|]}{\sigma_d \sqrt{n}} - \frac{\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor)}{\sigma_d \sqrt{n}} \xrightarrow{\mathbb{P}} 0. \quad (3.6)
\]

We take an arbitrary \( \varepsilon > 0 \) and apply Markov’s inequality to arrive at

\[
\mathbb{P}\left(n^{-1/2}\left(|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|]\right) - \mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor) > \sigma_d \varepsilon \right)
\]

\[
\leq \frac{\mathbb{P}\left(2|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|] > \varepsilon \sigma_d \sqrt{n}\right) + \mathbb{P}\left(2\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor) > \varepsilon \sigma_d \sqrt{n}\right)}{\varepsilon^2 \sigma_d^2 n}.
\]

By [19, Theorem 4.4], we know that there exists a constant \( C_1 > 0 \) such that \( \text{Var}(|R_n|) \leq C_1 n \), for all \( n \geq 1 \). By our assumptions, there is a constant \( C \) such that \( \sup_{n \geq 1} \max \{T_n, h_n\} \leq C \). Moreover, by Lemma 3.1, \( \mathbb{E}[\mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor)] \leq \mathbb{E}[I_{\lfloor Cn \rfloor}] \). Hence, by (3.4), we obtain

\[
\mathbb{P}\left(n^{-1/2}\left(|R_{\lfloor nh_n \rfloor}^{(2)}| - \mathbb{E}[|R_{\lfloor nh_n \rfloor}|]\right) - \mathcal{E}(\lfloor nT_n \rfloor, \lfloor nh_n \rfloor) > \sigma_d \varepsilon \right)
\]

\[
\leq \frac{4C_1 n h_n}{\varepsilon^2 \sigma_d^2 n} + \frac{2C_2 F_d(\lfloor Cn \rfloor)}{\varepsilon \sigma_d \sqrt{n}}.
\]

Since \( h_n \to 0 \) and \( F_d(\lfloor Cn \rfloor) / \sqrt{n} \to 0 \) as \( n \nearrow \infty \), we conclude (3.6), and the proof is finished. \( \square \)

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