Overlapping Resonances in the Resistance of Superposition States to Decoherence

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(Dated: January 23, 2010)

Abstract

Overlapping resonances are shown to provide new insights into the extent of decoherence experienced by a system superposition state in the regime of strong system-environment coupling. As an example of this general approach, a generic system comprising spin-half particles interacting with a thermalized oscillator environment is considered. We find that (a) amongst the collection of parametrized Hamiltonians, the larger the overlapping resonances contribution, the greater the maximum possible purity, and (b) for a fixed Hamiltonian, the larger the overlapping resonances contribution, the larger the range of possible values of the purity as one varies the phases in the system superposition states. Systems displaying decoherence free subspaces show that largest overlapping resonances contribution.

PACS numbers: 03.65.Yz, 03.67.Pp
I. INTRODUCTION

Coherent control of atomic and molecular processes is a subject of considerable ongoing interest [1, 2]. Specifically, a number of successful control scenarios have been theoretically proposed and experimentally implemented. The vast majority of well characterized scenarios are, however, based upon molecules treated in isolation, with the understanding that systems in an external environment are subject to decoherence effects [3, 4] that can be deleterious to control [5]. Decoherence effects of this kind are also detrimental to “quantum technologies”, such as quantum information processing and quantum computing [6] insofar as they cause the loss of the essential quantum characteristics that make these research areas interesting.

As a consequence, efforts are ongoing to find suitable mechanisms to reduce or compensate for decoherence effects. For example, several techniques, such as quantum dynamical decoupling (QDD) [7] and the use of decoherence free subspace (DFS) [8], have been proposed in the quantum information community. However, the QDD method is quite challenging to implement using the present technology, and the DFS requires explicit symmetries in the Hamiltonian, which are not generic. Hence, methods for identifying conditions for states that are resistant to decoherence is of great interest.

Indeed, considerable effort has gone into identifying characteristic features of particular system states, interacting with different environments, that lead to stability against decoherence. Sample approaches include the general predictability sieve approach of Zurek et al. [9], and predictions of the unusual stability of coherent matter states, low lying eigenstates, and localized states [10, 11] for typical environmental coupling.

In this paper we introduce a new perspective, based on overlapping resonances, to understanding and optimize the resistance of system superposition states to decoherence. The approach applies in the important regime where system-environment coupling is large. Specifically, we show that interference between the overlapping resonances that exist in this regime can be used to build system superposition states that are relatively resistant to decoherence. The extent to which such states are stable against decoherence is shown to directly correlate with the degree to which the system-environment Hamiltonian displays overlapping resonances. Thus, even if the system does not possess symmetry properties such as decoherence free subspaces (DFS), states can be constructed which are more resistant to decoherence than are others.
From a formal viewpoint, the approach described herein constitutes a major extension, to product Hilbert spaces characteristic of open quantum systems, of our work on controlling radiationless transitions in large molecules \[12\]. There, population transfer between a Q-subspace and the remainder of the Hilbert space, the P-subspace, was minimized by preparing initial superpositions in the Q-subspace, relying on the presence of overlapping resonances. This approach was applicable to the case where the full Hilbert space was partitioned into the sum of two subspaces. By contrast, decoherence involves the effect of an environment on the system, i.e. a product of two Hilbert spaces. The difference is substantial and significant.

The inter-relationship between overlapping resonances and the ability to design states that are resistant to decoherence is general. It proves useful, however, to introduce the approach via a specific generic example. For this reason we choose a particular class of spin-boson problems. Note that system decoherence is a manifestation of the loss of quantum information when one neglects the environment with which the system is entangled\[4\]. As such, decoherence can occur effectively even with a small environment\[13\] as is manifest in the example below.

The structure of the paper is as follows. In Sec. II, we introduce a generic system that describes a spin system interacting with a environment, providing a specific example for this general approach. Section III, considers the nature of decoherence in this system, and its relationship to overlapping resonance. Some specific numerical examples are provided in Sect. IV to demonstrate correlations between overlapping resonances and reduced state decoherence. Sect. V provides a summary.

II. A GENERIC CLASS OF HAMILTONIANS

The spin-boson problem has been extensively studied in Chemistry in various scenarios and for a wide variety of applications\[14\]. Here we consider, as a specific example, the control of decoherence in a simple spin system interacting with a small thermal environment, i.e. a single spin-half system and an environment comprising a single-mode oscillator. The most general system-environment Hamiltonian for this case is given by

\[
H/\hbar = \omega a^\dagger a + \frac{\omega_0}{2} S_z + g_r(S_+ a + S_- a^\dagger) + g_{nr}(S_+ a^\dagger S_- a) + g_{ph}S_z(a + a^\dagger),
\]  

(1)
where \( a \) and \( a\dagger \) are the environment annihilation and creation operators, \( \omega \) is the frequency of the environment mode, \( \hbar \omega_0 \) is the energy difference between the two spin-half states \(|\pm\rangle\), and the spin operators are defined as \( S_\pm = |\pm\rangle\langle \mp| \), \( S_z = |+\rangle\langle +| - |-\rangle\langle -| \). Note that the rotating wave approximation (RWA) has not been invoked: the \( g_r \) term in the Hamiltonian is the spin-environment interaction that is retained under the RWA, whereas the \( g_{nr} \) term is that which would be neglected in the RWA. The \( g_{ph} \) term corresponds to decoherence of the spin due to the operator \( S_z \), which induces a relative phase between the spin states \(|+\rangle\) and \(|-\rangle\). The \( g_r \) and \( g_{nr} \) terms mediate energy exchange between the spin and the environment.

Our focus below is in the region of strong coupling, where \( g_{nr} \) and \( g_r \) are both greater than \( \omega_0 \). It is important to emphasize that, although the environment is small (a single oscillator) its coupling to the system does induce significant system decoherence.

The class of Hamiltonians [Eq. (1)] possesses a plane defined by \( g_r = g_{nr} \) in the three dimensional \( g_{nr}, g_r, g_{ph} \) parameter space upon which the system displays a decoherence free subspace (DFS). That is, in these special cases the Hamiltonian Eq. (1) leads to a DFS due to a Lie algebraic symmetry. In these cases, the interaction part of the total Hamiltonian can be written as \( \mathcal{E}_I \otimes B_I \), where \( B_I = (a + a\dagger) \) is the environment operator, and the spin operator \( \mathcal{E}_I \) can be written in a closed matrix form in the \((|+\rangle, |-\rangle)\) spin basis:

\[
\mathcal{E}_I = \begin{pmatrix}
g_{ph} & g \\
g & -g_{ph}
\end{pmatrix}, \quad g = g_r = g_{nr}.
\]  

(2)

The DFS corresponds to the eigenstates of the \( \mathcal{E}_I \) matrix. For other values of \( g_{nr} \) and \( g_r \), of primary interest here, the system does not have a DFS.

### III. DECOHERENCE OF A SPIN SUPERPOSITION

We consider the stability against decoherence of an initial spin superposition state \(|\psi\rangle = c_+|+\rangle + c_-|-\rangle\), where \( c_\pm \) are complex amplitudes and the environment is in thermal equilibrium at temperature \( T \). The latter is described in terms of the oscillator eigenstates \(|n\rangle\) by the environment density matrix \( \rho_B = \sum_n p_n |n\rangle\langle n| \), \( p_n = e^{-E_n/k_B T} / (\sum_n e^{-E_n/k_B T}) \) and \( E_n = (n + 1/2) \hbar \omega \). In order to follow the temporal evolution of the system we expand the total (system+environment) evolution operator \( U \) in terms of \(|\gamma\rangle\), the eigenstates of the full Hamiltonian [i.e., \( (E_\gamma - H)|\gamma\rangle = 0 \)] with \( U = \sum_\gamma e^{-iE_\gamma t/\hbar} |\gamma\rangle\langle \gamma| \). Expanding the eigenstates \(|\gamma\rangle\) in the zeroth order basis of “bare” spin+environment states \(|\pm, n\rangle \equiv |\pm\rangle \otimes |n\rangle\), (i.e.
the eigenstates of $H$ without the spin-environment interaction) gives, for the time-evolved density matrix of the total spin+environment system:

$$\rho(t) = \sum_n p_n U [\langle \psi | \otimes | n \rangle \langle n |] U^\dagger = \sum_{\gamma',\gamma,n} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} |\gamma'\rangle \langle \gamma' | \langle \gamma | \psi, n \rangle \langle \psi, n | \gamma' \rangle \ .$$  \hspace{1cm} (3)

Further, tracing over the environment states, gives the following reduced system density matrix of the total spin+environment system:

$$(\rho_s)_{kl} = \sum_{m,n,\gamma,\gamma'} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} \langle k, m | \gamma \rangle \langle \gamma | l, m \rangle \langle \gamma | \psi, n \rangle \langle \psi, n | \gamma' \rangle \langle \gamma' | l, m \rangle \ .$$  \hspace{1cm} (4)

Expanding $|\psi\rangle$ as $c_+|+\rangle + c_-|--\rangle$, we can rewrite the above expression as

$$(\rho_s)_{kl} = |c_+|^2 Q_{kl}(t) + |c_-|^2 R_{kl}(t) + c_+ c_+^* P_{kl}(t) + c_- c_-^* T_{kl}(t) \ ,$$  \hspace{1cm} (5)

where $P,Q,R,T$ are complex matrices with elements

$$P_{kl}(t) = \sum_{m,n,\gamma,\gamma'} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} \langle k, m | \gamma \rangle \langle \gamma | l, m \rangle \langle \gamma | +, n \rangle \langle +, n | \gamma' \rangle \ ,$$  \hspace{1cm} (6)

$$Q_{kl}(t) = \sum_{m,n,\gamma,\gamma'} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} \langle k, m | \gamma \rangle \langle \gamma | l, m \rangle \langle \gamma | +, n \rangle \langle +, n | \gamma' \rangle \ ,$$  \hspace{1cm} (6)

$$R_{kl}(t) = \sum_{m,n,\gamma,\gamma'} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} \langle k, m | \gamma \rangle \langle \gamma | l, m \rangle \langle \gamma | -, n \rangle \langle -, n | \gamma' \rangle \ ,$$  \hspace{1cm} (6)

$$T_{kl}(t) = \sum_{m,n,\gamma,\gamma'} p_n e^{-i(E_{\gamma'} - E_{\gamma})t} \langle k, m | \gamma \rangle \langle \gamma | l, m \rangle \langle \gamma | -, n \rangle \langle +, n | \gamma' \rangle \ .$$  \hspace{1cm} (6)

Consider the decoherence of the spin state, quantified in terms of the purity $S$ of the spin, where $S = \text{Tr}(\rho_s^2)$. Given the above results, $S$ can be evaluated via

$$S = \sum_{k,l \in +, -} \langle k | \rho_s | l \rangle \langle l | \rho_s | k \rangle = \sum_{k,l \in +, -} |(\rho_s)_{kl}|^2$$  \hspace{1cm} (7)

so that, using $P_{kl}^* = T_{lk}$:

$$S = |c_+|^4 \sum_{k,l \in +, -} |Q_{kl}|^2 + |c_-|^4 \sum_{k,l \in +, -} |R_{kl}|^2$$

$$+ |c_+|^2 |c_-|^2 \sum_{k,l \in +, -} \left[ |P_{kl}|^2 + |T_{kl}|^2 + \{Q_{kl}R_{kl}^* + Q_{kl}^* R_{kl}\} \right]$$

$$+ 2 \text{Re} \left[ \sum_{k,l \in +, -} \left[ c_+^2 c_-^2 P_{kl} P_{lk}^* + c_+^2 c_-^2 Q_{kl} P_{kl}^* + P_{lk}^* Q_{kl}^* \right] + |c_-|^2 c_+^2 R_{kl} P_{kl}^* + P_{lk}^* R_{kl} \right]$$  \hspace{1cm} (8)
The first three of these terms depend solely on the magnitude of the \( c_\pm \) coefficients, whereas the remainder of the terms depend upon the phases of the coefficients as well. The latter are clearly contributions to \( S \) that rely upon quantum interference between the spin states.

Note, for comparison later below, that \( S = 1 \) denotes a pure state. If the state is completely decohered in the spin basis then \( (\rho_s)_{kl} = \frac{1}{2} \delta_{kl} \) so that the purity, by Eq. (7), is \( S = 2(1/2)^2 = 1/2 \).

With the \( P, Q, R, T \) matrices known, one can optimize \( S \) with respect to \( c_+ \) and \( c_- \) to determine the state with maximum purity \( S \) at a fixed time. In particular, the relative phase of \( c_+ \) and \( c_- \) is important if either \( Q_{kl} \) or \( R_{kl} \) is non-zero, and \( P_{kl} \) is non-zero. Although not evident from Eq. (8), numerical results below clearly demonstrate a reliance on this relative phase to maximize \( S \).

IV. OVERLAPPING RESONANCES AND SYSTEM PURITY

A. Overlapping Resonances

As noted above, the bare states \( |\pm, n \rangle \) are not eigenstates of \( H \), and will evolve in the presence of the system-environment coupling. Insight into the nature of this time evolution is afforded by expanding the zeroth order state in the exact eigenstates \( |\gamma \rangle \). For example, a bare state such as \( |+, j \rangle \) expands as

\[
|+, j \rangle = \sum_\gamma |\gamma \rangle \langle \gamma |+, j \rangle,
\]

and evolves as

\[
|+, j \rangle_t = \sum_\gamma |\gamma \rangle \langle \gamma |+, j \rangle e^{-iE_\gamma t/\hbar}.
\]

The \( E_\gamma \) dependence of the square of the expansion coefficient, i.e. \( |D_\gamma|^2 \equiv |\langle \gamma |+, j \rangle|^2 \) provides the energy width over which the zeroth order state \( |+, j \rangle \) is spread due to the system-environment interaction. In the simplest cases the inverse of this width provides a qualitative measure of the time scale for the evolution of \( |+, j \rangle \). Hence, the zero order states are indeed resonances with a characteristic width \( |D_\gamma|^2 \). This is the analog, in a bound state spectrum, of the well known resonance in the continuum.

Similarly, by analogy to the continuum case, we can define overlapping resonances, as resonances that overlap in energy space, i.e. resonances that share a common \( |\gamma \rangle \) in their
respective eigenstate expansions. These are the precise bound state analogues to overlapping resonances in the continuum [e.g. Fig 1, Ref. [16]]. Examples of bound state overlapping resonances in molecular systems have also been previously reported [12].

In the case of the spin-boson Hamiltonians given by Eq. (1), overlapping resonances were clearly evident in, but not limited to, the strong coupling regime characterized by $g_r$ and $g_{nr}$ greater than $\omega_0$. They occur as two states $|k, m\rangle$ and $|l, n\rangle$ with $k \neq l$ and/or $m \neq n$ ($k, l \in +, -$) that have non-zero overlap with the same eigenstate $|\gamma\rangle$. Examples of such overlapping resonances are shown in Fig. 1. Quite clearly the overlap between the resonances can be significant, enhanced by the fact that the overall width of each resonances can be significantly larger than the energy spacing between the zeroth order states. The resonances are also seen to be rather highly structured.

**B. Contributions to $\text{Tr} \rho_s^2$**

Consider then the role of these overlapping resonances in the dynamical evolution of $\rho_s(t)$ given in Eq. (5). Specifically, note that all the matrices $P, Q, R, T$ depend on overlapping resonance contributions. Their dependences differ, however, in the important case where $k \neq l, k, l \in +, -$. In that case overlapping resonances contribute to $T_{kl}$ and $P_{kl}$, but not to $Q_{kl}$ and $R_{kl}$. That is, the terms in $(\rho_s)_{kl}$ that are sensitive to the relative phase of $c_+$ and $c_-$ [see Eq. (5)], and are hence associated with phase control over the dynamics, are directly dependent on overlapping resonances contributions.

\[ A_{+-} = \sum_{m \neq n} A_+^{m,n} A_-^{m,n}, \quad A_{\pm}^{m,n} = \left[ \sum_\gamma |\langle \pm, m|\gamma\rangle\langle\gamma|\pm, n\rangle| \right]. \quad (11) \]

For a pair of the oscillator states $|m\rangle$ and $|n\rangle$, $A_+^{m,n}$ determines the overlap (summed over all eigenstates $|\gamma\rangle$) between the bare states $|+, m\rangle$ and $|+, n\rangle$, when the spin state is $|+\rangle$. If, for the same pair of $|m\rangle$ and $|n\rangle$, there exists non-zero overlap $A_-^{m,n}$ between the bare state $|-, m\rangle$ and $|-, n\rangle$, then $A_{+-}$ provides a measure of the overlap between states $|+\rangle$ and $|\pm, n\rangle$. In the case of the spin-boson Hamiltonians given by Eq. (1), overlapping resonances were clearly evident in, but not limited to, the strong coupling regime characterized by $g_r$ and $g_{nr}$ greater than $\omega_0$. They occur as two states $|k, m\rangle$ and $|l, n\rangle$ with $k \neq l$ and/or $m \neq n$ ($k, l \in +, -$) that have non-zero overlap with the same eigenstate $|\gamma\rangle$. Examples of such overlapping resonances are shown in Fig. 1. Quite clearly the overlap between the resonances can be significant, enhanced by the fact that the overall width of each resonances can be significantly larger than the energy spacing between the zeroth order states. The resonances are also seen to be rather highly structured.
that arises through the overlap between states \(|\pm, m\rangle\) and \(|\pm, n\rangle\). Note that Eq. (11) is somewhat similar to the off diagonal \(T_{+-}\) and \(P_{+-}\) terms at \(t = 0\).

Note that the number and character of the overlapping resonances are functions of the parameters in the total Hamiltonian, e.g., here \(g_r, g_{nr}, g_{ph}, \omega,\) and \(\omega_0\). For example, in the extreme case where there is no system-environment coupling (i.e. \(g_r = g_{nr} = g_{ph} = 0\)), the total Hamiltonian is diagonal in the spin-boson bare state basis. The overlap between the spin-boson bare states then vanishes, and in turn, so does the overlap between the states \(|\pm\rangle\). That is, as expected, in absence the system of coupling to the environment, the spin states are not broadened and thus do not overlap.

C. Computational Results

Consider then the relationship between overlapping resonances as reflected in \(A_{+-}\) and decoherence as embodied in the system purity \(S\). As an example, we examine dynamics where the environment temperature \(T = 25\) mK and choose to vary \(g_r\), keeping the other Hamiltonian parameters fixed.

Numerical results are obtained by diagonalizing the Hamiltonian matrix in the zeroth order basis of bare states \(|\pm, n\rangle\). At the temperature of \(T = 25\) mK, a total of 20 oscillator states contribute, allowing us to diagonalize a \(40 \times 40\) dimensional Hamiltonian matrix to obtain the desired \(E_\gamma, |\gamma\rangle\).

To assess the decoherence we find, for each value of \(g_r\), the maximum possible purity \(S_{\text{max}}\) in the system at a fixed time \(t\) by optimizing the complex coefficients \(c_\pm\). Results are shown in Fig. 2 where both \(S_{\text{max}}\) and the overlapping resonance measure \(A_{+-}\) are plotted as a function of \(g_r\) at \(t = 0.1\) ns. The correspondence between the behavior of these two functions is evident, with larger \(A_{+-}\) correlating well with larger \(S_{\text{max}}\). That is, the greater the overlapping resonances contribution, the more resistant is the optimal superposition to decoherence. Note, significantly, that the optimally resistant state need not be, and is often not, an eigenstate of the spin Hamiltonian, but is rather a superposition of \(|+\rangle\) and \(|-\rangle\) eigenstates.

Prominent in the graph (and some presented below) is the DFS point \(g_r = g_{nr}\), where the \(A_{+-}\) curve displays a sharp peak, commensurate with \(S_{\text{max}}\) reaching its maximum value of unity. That is, maximal purity clearly correlates with the maximum \(A_{+-}\). The deviation
of $S_{\text{max}}$ from unity as one moves away from this point is significant, particularly in scenarios such as quantum computation, where extremely high degrees of coherence are necessary.

The results clearly show that states take advantage of overlapping resonances (and hence of quantum interference) to increase the state purity. This is the case even if the system-environment coupling is very large, as is clear from Fig. 2(b), where, for example, $g_r \sim 2\pi \times 1.5 \text{ GHz} \ (g_r \gg \omega_0)$, but where the purity remains as large as 0.92 for $g_{ph} = 2\pi \text{ GHz}$.

The presence of overlapping resonances, as seen above, influences the degree of decoherence when preparing initial states with different initial coefficients $c_{\pm}$. Sample dependences of $S$ on $c_{\pm}$ are shown in Fig. 3 for the case of $g_r = 2\pi \times 400 \text{ MHz}$, $g_{nr} = 2\pi \text{ GHz}$, and $g_{ph} = 2\pi \times 500 \text{ MHz}$ [i.e. parameters associated with Fig. 2(a)]. Shown are the time evolution of $S$ for three states, one being the best case superposition $c_+ = -0.539998$ and $c_- = \sqrt{1 - |c_+|^2}$ which shows $S_{\text{max}} = 0.96$ at the time $t = 0.1 \text{ ns}$ where $S$ is optimized. Comparison is made with the time evolution of $S$ for two other sample states with different $c_{\pm}$ values: $c_+ = 1$ and $c_- = 1/\sqrt{2}$. The range of $S$ as a function of initial state is large, with the $c_+ = 1$ case showing almost fully mixed behavior at long time. Interestingly, the curves shown do not cross as a function of time. Hence the optimal superposition, determined by coefficients obtained for the $t = 0.1 \text{ ns}$ case is, in fact, the optimal superposition for the times preceding that target time as well. This was also confirmed by optimizations independently carried out at shorter times.

Additional results showing how overlapping resonances between spin states is manifest in decoherence is shown in Fig. 4. Here, for a given value of $g_r$, we obtain the $|c_+|$ and $|c_-|$ that maximize $S$ and then vary the relative phase $\theta$ between $c_+$ and $c_-$ to obtain $S_{\text{diff}}$, defined as the difference between the maximum and minimum purity so attained. This difference is plotted in Fig. 4 as a function of $g_r$, along with the corresponding value of the overlapping resonance contribution $A_{+-}$. The extent to which $S_{\text{diff}}$ varies is seen to reflect the variation of $A_{+-}$. Specifically, as $A_{+-}$ increases, so does the extent to which the purity varies with $\theta$. Hence, in the presence of large overlapping resonance, one can actively control the purity of the system by taking advantage of the phase dependent quantum interference contribution to $S$. Indeed, in accord with Eq. (8), an observed variation of the decoherence with the phase of a superposition state provides evidence of the presence of overlapping resonances.

The extent to which $S_{\text{diff}}$ varies with $A_{+-}$ depends upon the Hamiltonian parameters. Consider, for example, an interesting case of the nanoscale Hamiltonian system in Eq. (1),

the interaction of a Cooper pair box (CPB) with a nanomechanical resonator [17]. This system is given by:

\[ H_{\text{CPB}} = 4E_C \delta n S_z - (E_J/2)S_x + \hbar \omega a^\dagger a + \hbar g S_z (a + a^\dagger), \]  

(12)

where \( E_C \) and \( E_J \) are the charging energy and the Josephson energy of the CPB, \( \omega \) is the fundamental frequency of the resonator, and \( \delta n \) lies between \(-1/2\) and \(1/2\). At the degeneracy point \( \delta n = 0 \) this Hamiltonian can be written in the form of that of Eq. (1) under a similarity transformation, where we identify \( \omega_0 = E_J, g_{ph} = 0, \) and \( g_r = g_{nr} = g \). The similarity transformation is given by the operator \( e^{iS_y} \), with \( S_y \) being the \( y \)-projection of the spin operator.

\[ \]  

From the symmetry of the Hamiltonian (12), one finds a DFS corresponding to the state \( c_+ = c_- = 1/\sqrt{2} \) and \( c_+ = -c_- = 1/\sqrt{2} \). Hence, the value of \( S_{\text{max}} \) will always be unity for all values of \( g \), irrespective of the values of \( A_{\pm} \). Nonetheless, overlapping resonances affect the decoherence in this system. Specifically, as \( A_{\pm} \) increases with \( g \), the variation of \( S \) with changes in the relative phase \( \theta \) between \( c_{\pm} \) also increases. This is shown in Fig. 5 where the difference \( S_{\text{diff}} \) between the maximum and minimum values of \( S \) and the values of \( A_{\pm} \) with \( g \) are seen to be strongly correlated.

On the other hand, if \( g_r \neq g_{nr} \), there is no DFS in the system. In that case, the results (not shown) behave similarly to, e.g., Fig. 2. That is, a plot of \( S_{\text{max}} \) and \( A_{\pm} \) as a function of \( g_r \), with \( g_r \neq g_{nr} \) and \( g_{ph} = 0 \), shows a clear correlation between the maximum achievable value of \( S \) and the overlapping resonance contribution.

V. SUMMARY

We have introduced a new unifying approach to considering the resistance to decoherence of system superposition states. Specifically, using the generic class of spin-boson systems as an example we have shown that overlapping resonances of the system bare states, and the associated interference between these states, provide insight into the dependence of the decoherence of system superposition states in the important regime of strong system-environment coupling. For example, within a given class of Hamiltonians, increasing overlapping resonances contributions allow for states of higher purity. The approach is completely general, although it has been applied here to a spin 1/2 particle interacting with a single thermal
oscillator. Extensions to other cases are underway. In the first instance we anticipate that increasing the environment size will increase the participation and effectiveness of overlapping resonances in decoherence control.

Acknowledgments: This work was supported by NSERC and by the Centre for Quantum Information and Quantum Control, University of Toronto.

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FIG. 1: Plot of the overlap $|D_\gamma|^2$ between various bare states and the eigenstates $|\gamma\rangle$ as a function of the energy $E_\gamma$: (a) $|+,0\rangle$, $|-,0\rangle$, (b) $|+,0\rangle$, $|-,2\rangle$, (c) $|+,2\rangle$, $|-,2\rangle$. The data points for the first of each pair are connected by a solid line whereas the data points for the second of each pair are connected by solid lines. Data are for the parameters $\omega = 2\pi$ GHz, $\omega_0 = 2\pi \times 100$ MHz, $g_r = g_{nr} = 2\pi$ GHz and $g_{ph} = 0$. 
FIG. 2: The dependence of the maximum achievable purity $S_{\text{max}}$ of the system (solid line) and the overlapping resonance $A_{+-}$ (dashed line; right y-axis) on the coupling strength $g_r/(2 \pi)$. Here (a) $g_{ph} = 2 \pi \times 500 \text{ MHz}$, $g_{nr} = 2 \pi \text{ GHz}$ and (b) $g_{ph} = 2 \pi \text{ GHz}$, $g_{nr} = 2 \pi \times 500 \text{ MHz}$. The remaining parameters are $\omega = 2 \pi \text{ GHz}$, $\omega_0 = 2 \pi \times 100 \text{ MHz}$, and the temperature $T = 25 \text{ mK}$. The purity is calculated at a time $t = 0.1 \text{ ns}$.

FIG. 3: Variation of purity with time for the initial states $(|-0.539998|0.841665\rangle) / \sqrt{2}$ (dashed line), and $|+\rangle$ (dotted line). Here the interaction strengths $g_r = 2 \pi \times 400 \text{ MHz}$, $g_{nr} = 2 \pi \text{ GHz}$, and $g_{ph} = 2 \pi \times 500 \text{ MHz}$. The other parameters are as in Fig. 2.
FIG. 4: Variation of $S_{\text{diff}}$ (solid line) and the overlap $A_{+-}$ (dashed line; right y-axis) with $g_r$ for $g_{nr} = 2\pi$ GHz and $g_{ph} = 2\pi \times 500$ MHz. The remaining parameters are as in Fig. 2.

FIG. 5: Variation of $S_{\text{diff}}$ (solid line) and the overlap $A_{+-}$ (dashed line; right y-axis) with $g_r = g_{nr}$ for $g_{ph} = 0$. The other parameters are as in Figs. 2.