ON THE EXISTENCE OF MULTIPLE SOLUTIONS FOR FRACTIONAL BREZIS NIRENBERG TYPE EQUATIONS

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Abstract. The present paper studies the non-local fractional analogue of the famous paper of Brezis and Nirenberg in [4]. Namely, we focus on the following model,

\[
(P) \begin{cases}
(-\Delta)^s u - \lambda u = \alpha |u|^{p-2}u + \beta |u|^{2^* - 2}u \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \((-\Delta)^s\) is the fractional Laplace operator, \(s \in (0, 1)\), with \(N \geq 3s\), \(2 < p < 2^*\), \(\beta > 0\), \(\lambda, \alpha \in \mathbb{R}\) and establish the existence of nontrivial solutions and sign-changing solutions for the problem \((P)\).

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1. Introduction

In the famous paper of Brezis and Nirenberg [4], they have researched on the following nonlinear critical elliptic partial differential equation:

\[
(1.1) \begin{cases}
-\Delta u - \lambda u = \alpha |u|^{p-2}u + |u|^{2^* - 2}u \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), \(2 < p < 2^*\) where \(2^* = \frac{2N}{N-2}(N \geq 3)\) is the Sobolev critical exponent, \(\lambda, \alpha\) are parameters. They have proved the following (see Corollaries 2.1-2.4 in [4]):

(I) For \(N \geq 4\), problem \((1.1)\) has a positive nontrivial solution for \(\alpha > 0\) and \(\lambda \in (0, \lambda_1)\) where \(\lambda_1\) is the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition in \(\Omega\).

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For $N = 3$, $\alpha > 0$, $\lambda \in (0, \lambda_1)$, problem (II) has a solution provided $4 < p < 6$ and problem (I) has a solution for each $\alpha \geq \alpha_0$ for some $\alpha_0 > 0$ if $2 < p \leq 4$.

In the interest of the pioneering work of H.R. Brezis and L. Nirenberg [1], a massive study is doing the rounds about the results dealing with semilinear problems involving critical Sobolev exponents, we mention some of the well-celebrated papers [3, 4, 5, 8] and the references therein. In [2], Bandle and Benguria have studied the classical Brezis-Nirenberg problem [1] on $\mathbb{S}^3$. In [4], the authors have worked on the Brezis-Nirenberg problem on the hyperbolic space. In [23], the authors have researched about the existence of non-trivial solutions to semilinear Brezis-Nirenberg problems involving Hardy potential and singular coefficients.

In the present paper, we are interested in the following nonlocal elliptic equation with Sobolev-critical exponent:

\[
(P) \begin{cases}
(\Delta)^s u - \lambda u = \alpha |u|^{p-2} u + \beta |u|^{2^*-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where $s \in (0, 1)$ is fixed, $N \in \mathbb{N}$, $N > 2s$, $2^* = \frac{2N}{N-2s}$ (the fractional critical Sobolev exponent), $2 < q < 2^*$, $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $(\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as:

\[
-(\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N.
\]

In the nonlocal setting, Brezis-Nirenberg type problems are in the pipeline and have been widely studied by many researchers. To name a few, we cite [9, 14, 15, 17, 20, 19]. In [19], Servadei and Valdinoci have studied the following model:

\[
(\text{II}) \begin{cases}
(\Delta)^s u - \lambda u = |u|^{2^*-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where $\Omega$ is an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 4s$, $s \in (0, 1)$. They have proved that for any $\lambda \in (0, \lambda_1)$, problem (II) admits a nontrivial solution $u \in H^s(\mathbb{R}^N)$ such that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, where $\lambda_1$ is the first eigenvalue of the nonlocal operator $(\Delta)^s$ with homogeneous Dirichlet boundary condition. In [21], the author has dealt with the existence and nonexistence of positive solutions to Brezis-Nirenberg type problems which involve the square root of the Laplace operator in a smooth bounded domain in $\mathbb{R}^N$. In [14], the authors have worked on the existence, nonexistence, and regularity of weak solutions for a non-local system involving fractional Laplacian. Very recently, in [19], Colorado and Ortega have studied existence of solutions to the nonlocal critical Brezis-Nirenberg problem involving mixed Dirichlet-Neumann boundary conditions. In [11], Cora and Iacopetti have studied the asymptotic behavior and qualitative properties of least energy radial sign-changing solutions of the problem (II) in a ball of $\mathbb{R}^N$.

**Functional Setting.** We mean by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm

\[
\|g\|_{H^s(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}^N)} + \left( \int_{\mathbb{R}^{2N}} \frac{|g(x) - g(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}.
\]
Let us signify 

\[ Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c), \Omega^c = \mathbb{R}^N \setminus \Omega \] 

and we define 

\[ X_0 := \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}, \]

with the norm 

\[ \| u \|_{X_0} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}. \]

With this norm, \( X_0 \) is a Hilbert space with the scalar product 

\[ \langle u, v \rangle_{X_0} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy, \]

(see [16, lemma 7]). For further details on \( X_0 \) and for their properties, we refer to [13] and the references therein.

Define 

\[ D^{s,2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\}, \]

and 

\[ S := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{(\int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx)^{2^*/2}}, \]

is the best Sobolev constant for the fractional Sobolev embedding.

In this article, we will prove the following main results:

**Theorem 1.1.** Under the assumptions \( N \in \mathbb{N} \) with \( N \geq 4s \), \( s \in (0,1) \), \( \lambda, \alpha, \beta \in (0, \infty) \), (P) has a non-trivial solution and has a sign changing solution provided \( \lambda \geq \lambda_1 \), where \( \lambda_1 \) is the first eigen value of \( (-\Delta)^s \) in \( X_0 \).

**Theorem 1.2.** Let \( N \geq 3s, \lambda \in \mathbb{R}, \alpha, \beta > 0 \). Then for all \( m \in \mathbb{N} \), there exists \( \beta_m \geq 0 \) such that (P) has \( m \) nontrivial solutions for all \( \beta \in (0, \beta_m) \). Moreover, for \( \lambda \geq \lambda_1 \), (P) has \( m \) sign changing solutions.

**Theorem 1.3.** Let \( N \geq 4s \) with \( s \in (0,1), \lambda > 0, \alpha < 0, \beta > 0 \). Then, problem (P) has a non-trivial solution for all \( \alpha \in (-\alpha_0,0) \) for some \( \alpha_0 > 0 \).

In the classical case, this present problem is addressed in [24]. The nonlocal framework has made this problem challenging. As far as we know, such result for existence of multiple and sign-changing solutions in the non-local framework, is not available in the literature.

**Organization of the paper.** The present manuscript consists of the following sections. Section 2 is devoted to recall some preliminary results. In section 3, we prove Theorem 1.1. Section 4 consists of the proof of Theorem 1.2. In section 5, we establish the proof of Theorem 1.3.

2. Preliminaries

2.1. Eigenvalue problem. In this section, we focus on the following eigenvalue problem:

\[ \begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \]
where \( s \in (0, 1) \), \( N > 2s \) and \( \Omega \subset \mathbb{R}^N \) be an open, smooth bounded domain. We say \( \lambda \in \mathbb{R} \) is an eigenvalue of \(( -\Delta )\) if there exists a nontrivial solution of \((\mathcal{E})\) In \(X_0\). We recall the following properties of eigenvalues of \(-\Delta\) in \(X_0\). We refer [Proposition 9, [18]] for details.

(i) The problem \((\mathcal{E})\) has an eigenvalue \( \lambda_1 > 0 \) which can be characterised as:

\[
\lambda_1 := \inf_{u \in \mathcal{X}_0 \setminus \{0\}} \frac{\|u\|_{X_0}}{|u|_2},
\]

where \(| \cdot |_p\) denotes the \(L^p\) norm in \(\Omega\) for \(1 \leq p \leq \infty\). Moreover, \(\lambda_1\) is simple and the eigen function corresponding to \(\lambda_1\) is not sign-changing.

(ii) The set of eigen values of \((\mathcal{E})\) consist of a sequence \(\{\lambda_k\}_{k \geq 1}\) such that

\[0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \text{ and } \lim_{k \to \infty} \lambda_k = \infty.\]

(iii) The sequence \(\{\epsilon_k\}_{k \in \mathbb{N}}\) of eigen-functions in \(X_0\) corresponding to \(\{\lambda_k\}_{k \in \mathbb{N}}\) forms an orthonormal basis of \(L^2(\Omega)\) and an orthogonal basis of \(X_0\).

(iv) For each \(k \in \mathbb{N}\), the eigen value \(\lambda_k\) has finite multiplicity.

For \(k \geq 1\), let us denote \(X_k := \text{span}\{e_1, e_2, \ldots, e_k\}\). For \(\lambda > 0\), let us denote

\[
\lambda^+ = \min_{k \in \mathbb{N}} \{\lambda_k : \lambda_k > \lambda\}.
\]

We note that

\[
\lambda^+ = \lambda_l \text{ for some } l \in \mathbb{N} \text{ and } \lambda < \lambda_l.
\]

For \(\varepsilon > 0\), let us consider the functions \(v_\varepsilon : \mathbb{R}^N \to \mathbb{R}\) defined by:

\[
v_\varepsilon(x) = \frac{[N(N-2s)\varepsilon]^{\frac{N-2s}{2}}}{(\varepsilon + |\phi|^2)^{\frac{N-2s}{2}}} \text{ for all } x \in \mathbb{R}^N.
\]

We know that (see [12]) \(S\), defined in (1.3), is achieved by the family \(\{v_\varepsilon\}_{\varepsilon > 0}\). Let us consider a smooth function \(\psi\) of \(B_{1/2}(0)\) w.r.t. \(B_1(0)\), such that \(\psi \in C_\infty(\mathbb{R}^N)\), \(0 \leq \psi \leq 1\), \(\psi(x) \equiv 1\) on \(B_{1/2}(0)\) and \(\text{supp}(\psi) \subset B_1(0)\).

Let us define \(\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}\) by

\[
\phi_\varepsilon(x) = \psi(x)v_\varepsilon(x), \text{ } x \in \mathbb{R}^N.
\]

For \(k \in \mathbb{N}\), let us denote the eigen space corresponding to the eigen value \(\lambda_k\) by \(Y_k\), that is,

\[
Y_k := \{u \in X_0 : (-\Delta)^s u = \lambda_k u \text{ in } \Omega\}.
\]

By property (iv) above, we note that \(\dim Y_k < \infty\) for each \(k \in \mathbb{N}\). Let us consider the orthogonal projection \(P_k : X_0 \to Y_k\) of \(X_0\) onto \(Y_k\). Set \(\eta_k := (I - P_{l-1})\phi_\varepsilon = \phi_\varepsilon - P_{l-1}\phi_\varepsilon\) where \(l\) is given in (2.2) and define the set \(V_\varepsilon\) by

\[
V_\varepsilon := \{u \in X_0 : u = v + t\psi_\varepsilon, v \in X_{l-1}, t \in \mathbb{R}\},
\]

where

\[
\psi_\varepsilon = \begin{cases} 
\phi_\varepsilon & \text{ if } \lambda \neq \lambda_{l-1}, \\
\eta_\varepsilon & \text{ if } \lambda = \lambda_{l-1}.
\end{cases}
\]

We observe that \(V_\varepsilon = X_{l-1} + \text{span}\{\psi_\varepsilon\}\).

We need the following result from [Proposition 12, [17]] to establish our main result Theorem 1.1.
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**Proposition 2.1.** For \( \lambda \in (0, \infty) \), let us define the functional \( Q_\lambda : H^s(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R} \) by
\[
Q_\lambda(u) := \frac{\|u\|^2_{X_0} - \lambda |u|_2^2}{|u|_2^2}, \quad \text{for all } u \in H^s(\mathbb{R}^N) \setminus \{0\}.
\]
Then, there exists \( \varepsilon_0 > 0 \) (small enough) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and for all \( \lambda \in (0, \infty) \),
\[
\sup_{u \in V_\varepsilon} Q_\lambda(u) < S,
\]
where \( S \) is defined in (1.3).

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 which consists of several steps.

**Step-1.** Let us consider the energy functional \( E \in C^1(X_0, \mathbb{R}) \) corresponding to the problem \( (P) \) given by:
\[
E(u) = \frac{1}{2} \|u\|^2_{X_0} - \frac{\lambda}{2} |u|_2^2 - \frac{\alpha}{q} |u|^q - \frac{\beta}{2^*} |u|_2^{2^*}, \quad u \in X_0,
\]
for \( \lambda, \alpha, \beta, c, d \in \mathbb{R}, c \leq d \), let us define
\[
E_c := \left\{ u \in X_0 : E(u) \geq c \right\}, \quad E^d := \left\{ u \in X_0 : E(u) \leq d \right\}.
\]
Then note that \( E^{-1}[c, d] = E_c \cap E^d \). For \( t > 0 \), let us define \( B_t := \left\{ u \in X_0 : \|u\|_{X_0} < t \right\} \). We now claim the following:

**Claim 1:** Let \( \lambda > 0 \). Then, there exists \( t \in \mathbb{N}, t, \mu > 0 \) such that:
\[
X_{t-1}^\perp \cap \partial B_t \subset E_\mu \text{ and } X_t^\perp \cap (B_t \setminus \{0\}) \subset (E_0 \setminus E^{-1}(0)).
\]

**Proof of Claim 1.** By the definition of \( \lambda^+ \), we note that, there exists \( t \in \mathbb{N} \) such that \( \lambda^+ = \lambda_t \). Using the Raleigh quotient characterization of \( \lambda_t \), Sobolev embedding and interpolation inequality, we have,
\[
E(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) \|u\|^2_{X_0} - c_1 \|u\|_{X_0}^p - c_2 \|u\|_{X_0}^{2^*}, \quad \text{for all } u \in X_{t-1}^\perp,
\]
for some \( c_1, c_2 > 0 \). Now, the result follows from elementary analysis of the function
\[
f(x) = \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) x^2 - c_1 x^p - c_2 x^{2^*}, \quad x \geq 0.
\]
Therefore, for \( 0 < \|u\|_{X_0} < r, E(u) \geq f(\|u\|_{X_0}) > 0 \) and we have the claim.

**Step-2.** Let \( \mu, l \) be given as in Step-1 and \( \beta > 0 \). Let us denote
\[
c := \sup_{h \in \Gamma} \inf_{u \in \partial(B_t) \cap X_{t-1}^\perp} E(h(u)), \quad b := \inf_{h \in \Gamma^*} \sup_{u \in K} E(u) \quad \text{and} \quad c^* := \frac{1}{N \beta^{N/2s}} S^{N/2s},
\]
where \( \Gamma := \left\{ h \in C(X_0, X_0) : h(B_1) \subset E_0 \cup B_r, h \text{ is an odd homeomorphism of } X_0 \right\}, \) and
\[
\Gamma^* := \left\{ K \subset X_0 : K \text{ is compact, symmetric and } \gamma(K \cap h(\partial(B_1))) \geq l \text{ for all } h \in \Gamma \right\},
\]
where \( \gamma \) is the genus. We notice that \( \gamma(K \cap h(\partial(B_1))) \) in \( \Gamma^* \) is well-defined.

**Claim 2.** The following holds true:
\[
0 < \mu \leq c < b < c^*.
\]
Proof of Claim 2. As \( h = r \text{Id} \in \Gamma \), using Step-1, we have,

\[
c \geq \inf_{u \in \partial(B_1) \cap \mathcal{X}_{l-1}^+} E(ru) \geq \mu > 0.
\]

We observe that for any \( K \in \Gamma^* \) and \( h \in \Gamma \), we have \( K \cap h(\partial(B_1) \cap \mathcal{X}_{l-1}^+) \neq \emptyset \). See [1] for details. Thus, for \( w_{h,k} \in K \cap h(\partial(B_1) \cap \mathcal{X}_{l-1}^+) \), there exists \( u_{h,k} \in \partial(B_1) \cap \mathcal{X}_{l-1}^+ \) such that \( h(u_{h,k}) = w_{h,k} \). Therefore,

\[
\inf_{u \in \partial(B_1) \cap \mathcal{X}_{l-1}^+} E(h(u)) \leq E(h(u_{h,k})) = E(w_{h,k}) \quad \text{for all} \quad h \in \Gamma.
\]

This implies,

\[
c \leq \sup_{h \in \Gamma^*} E(w_{h,k}) \leq \sup_{u \in K} E(u).
\]

Hence, \( c \leq b \).

For any finite dimensional subspace \( \tilde{X} \subset X_0 \) we have \( E_0 \cap \tilde{X} \) is bounded in \( X_0 \). Suppose not. Then, as in finite dimensional subspace \( \tilde{X} \), all norms are equivalent, so there exists sequence \( \{ u_k \}_{k \geq 1} \subset E_0 \cap \tilde{X} \) such that \( \| u_k \|_{X_0} \to \infty, k \to \infty \). Now, as \( u_k \in E_0 \), we have

\[
0 \leq E(u_k) \leq \frac{1}{2} \| u_k \|_{X_0}^2 - c_1 \| u_k \|_{X_0}^2 - c_2 \| u_k \|_{X_0}^2 - c_3 \| u_k \|_{X_0}^{2*}.
\]

Letting \( k \to \infty \), we have the contradiction. In particular, we take \( \tilde{X} = V_\varepsilon \). Then \( E_0 \cap V_\varepsilon \) is bounded. So there exists \( R > r > 0 \) sufficiently large such that \( E_0 \cap V_\varepsilon \subset K_R \) where \( K_R := V_\varepsilon \cap \overline{B}_R \). We note that \( K_R \) is compact and symmetric. Then for any \( h \in \Gamma \), we have,

\[
h(B_1) \cap V_\varepsilon \subset (E_0 \cup \text{overline}{B}_R) \cap V_\varepsilon = (E_0 \cap V_\varepsilon) \cup K_R \subset K_R.
\]

Since \( h(B_1) \cap V_\varepsilon \) is a bounded neighbourhood of 0 in the \( l \) dimensional subspace \( V_\varepsilon \), then \( \gamma(\partial(h(B_1) \cap V_\varepsilon)) = l \). As \( h \) is a homeomorphism, we have

\[
l \leq \gamma(h(B_1) \cap V_\varepsilon)) \leq \gamma(h(B_1) \cap V_\varepsilon).
\]

Also \( h \) being a homeomorphism, \( h(B_1) \cap V_\varepsilon \subset K_R \). This implies, \( h(\partial B_1) \cap V_\varepsilon \subset K_R \) and \( K_R \cap h(\partial B_1) = h(\partial B_1) \cap V_\varepsilon \). Therefore, \( \gamma(K_R \cap h(\partial B_1)) \geq l \). So, \( K_R \in \Gamma^* \).

Thus

\[
b \leq \sup_{u \in V_\varepsilon} E(u) \leq \sup_{u \in V_\varepsilon} E_{\beta}(u),
\]

where \( E_{\beta}(u) = \frac{1}{2} \| u \|_{X_0}^2 - \frac{\lambda}{2} \| u \|_{2}^2 - \frac{\delta}{2} \| u \|_{2*}^2 \). Then we have,

\[
E_{\beta}(u) \leq \max_{t} E_{\beta}(tu) \leq \frac{1}{sN^{1/2s}} \left( \frac{\| u \|_{X_0}^2 - \lambda \| u \|_{2}^2}{\| u \|_{2*}^2} \right)^{N/2s},
\]

\( u \in X_0 \setminus \{0\} \). Indeed, by a rudimentary analysis follows by constructing the function

\[
N(t) = \frac{1}{2} \| u \|_{X_0}^2 - \frac{\lambda}{2} \| u \|_{2}^2 - \frac{\delta}{2} \| u \|_{2*}^2, \quad t \geq 0. \]

So, \( N'(t_0) = 0 \) implies \( t_0 = 0 \) or

\[
t_0 = \left( \frac{\| u \|_{X_0}^2 - \lambda \| u \|_{2}^2}{\| u \|_{2*}^2} \right)^{\frac{1}{1-2}}.
\]
Hence, \( N''(t_0) = -(2^* - 2) \left( \|u\|_{X_0}^2 - \lambda|u|_2^2 \right) \leq 0 \). So, \( N(t) \) attains maximum at \( t = t_0 \). Therefore, an easy computation yields,

\[
N(t_0) = \frac{t_k^2}{2} \left( \|u\|_{X_0}^2 - \lambda|u|_2^2 - \frac{t_k^{(2^*-2)} - 2}{2^{2^*-2}} |u|_2^{2^*} \right) = \frac{s}{N} \beta \left( \frac{\|u\|_{X_0}^2 - \lambda|u|_2^2}{|u|_2^{2^*}} \right) \frac{2}{\gamma - 2^*}.
\]

This implies,

\[
E_{0,\beta}(u) : \max_t E_{0,\beta}(tu) \leq \frac{s}{N} \beta \left( \frac{\|u\|_{X_0}^2 - \lambda|u|_2^2}{|u|_2^{2^*}} \right) \frac{2}{\gamma - 2^*}.
\]

So by (2.10) we have,

\[
\sup_{u \in V_{c^*}} E_{0,\beta}(u) \leq \frac{s}{N} \beta \frac{2}{\gamma - 2^*} S^{N/2s} = c^*.
\]

This finishes the proof of Claim 2.

**Step-3.** In this step, we show \( E \) satisfies \( (PS)_c \) condition for any \( c < c^* \). To this goal, let \( \{u_k\}_{k \geq 1} \) be a sequence in \( X_0(\Omega) \) such that \( E(u_k) \to c \) and \( E'(u_k) \to 0 \) in \( (X_0(\Omega))', k \to \infty \). It is easy to check that \( \{u_k\}_{k \geq 1} \) is bounded in \( X_0(\Omega) \). Hence, going to a subsequence, if necessary, we can assume that as \( k \to \infty \), there exists \( u \in X_0(\Omega) \) such that

\[
\begin{align*}
\text{Step-3.} & 
\end{align*}
\]

\[ u_k \rightharpoonup u \quad \text{in} \quad X_0(\Omega), \]
\[ u_k \to u \quad \text{strongly in} \quad L^q(\mathbb{R}^N) \quad \text{for} \quad q \in [1, 2^*), \]
\[ u_k \to u \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{for} \quad 1 \leq \gamma < 2^*, \]

and there exists \( l \in L^q(\mathbb{R}^N) \) such that

\[
|u_k(x)| \leq l(x) \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{for all} \quad k \geq 1.
\]

Using Vitali Convergence Theorem, we have,

\[
\int_{\Omega} u_k \phi \, dx \to \int_{\Omega} u \phi \, dx, k \to \infty \quad \text{for all} \quad \phi \in X_0(\Omega),
\]
\[
\int_{\Omega} |u_k|^{p-2} u_k \phi \, dx \to \int_{\Omega} |u|^{p-2} u \phi \, dx, k \to \infty \quad \text{for all} \quad \phi \in X_0(\Omega),
\]
and

\[
\int_{\Omega} |u_k|^{2^*-2} u_k \phi \, dx \to \int_{\Omega} |u|^{2^*-2} u \phi \, dx, k \to \infty \quad \text{for all} \quad \phi \in X_0(\Omega).
\]

Hence, \( u \) is the solution of : \( (-\Delta)^\gamma u - \lambda u = \alpha |u|^{p-2} u + \beta |u|^{2^*-2} u \). This gives,

\[
\|u\|_{X_0}^2 - \lambda|u|_2^2 - \alpha|u|_p^p - \beta|u|_2^{2^*} = 0.
\]

Using this, we have

\[
E(u) = \alpha \left( \frac{1}{2} - \frac{1}{p} \right) |u|_p^p + \beta \left( \frac{1}{2} - \frac{1}{2^*} \right) |u|_2^{2^*}.
\]

Let \( v_n = u_n - u \). Brezis-Lieb lemma leads to:

\[
\|v_n\|_{X_0}^2 = |u_n|_2^{2^*} + |v_n|_2^{2^*} + o(1).
\]
So we have,

\[ E(u) + \frac{1}{2} \| v_n \|_{X_0}^2 - \frac{\beta}{2^*} \| v_n \|_{2^*}^2 \]

\[ = \frac{1}{2} (\| u \|_{X_0}^2 - \| u_n \|_{X_0}^2) - \frac{\lambda}{2} (\| u \|_{2}^2 - \| u_n \|_{2}^2) - \frac{\alpha}{p} (\| u \|_{p}^p - \| u_n \|_{p}^p) \]

\[ - \frac{\beta}{2^*} (\| u \|_{2^*}^2 - \| u_n \|_{2^*}^2) + \| v_n \|_{X_0}^2 - \| u \|_{X_0}^2) - \langle u_n, u \rangle + E(u_n) \]

\[ = \| u \|_{X_0}^2 - \langle u_n, u \rangle - \frac{\alpha}{p} (\| u \|_{p}^p - \| u_n \|_{p}^p) - \frac{\lambda}{2} (\| u \|_{2}^2 - \| u_n \|_{2}^2) + o(1) + E(u_n) \]

= \( o(1) + E(u_n) \rightarrow c, n \rightarrow \infty \).

This implies,

\[ (3.5) \quad E(u) + \frac{1}{2} \| v_n \|_{X_0}^2 - \frac{\beta}{2^*} \| v_n \|_{2^*}^2 \rightarrow c, n \rightarrow \infty. \]

Since \( \langle E'(u_n), u_n \rangle \rightarrow 0, n \rightarrow \infty \) so we have,

\[ ||v_n||_{X_0}^2 - \beta ||v_n||_{2^*}^2 = ||u||_{X_0}^2 - \lambda ||u||_{2}^2 - \alpha ||u||_{p}^p - \beta ||u_n||_{2^*}^2 + \lambda ||u_n||_{2}^2 + \alpha ||u||_{p}^p \]

\[ + \beta ||u||_{2^*}^2 - ||u||^2 - 2 \langle u_n, u \rangle \]

\[ = \langle E'(u_n), u_n \rangle + \lambda ||u||_{2}^2 + \alpha ||u||_{p}^p + \beta ||u||_{2^*}^2 - 2 \langle u_n, u \rangle \]

\[ \rightarrow - \lambda ||u||_{2}^2 + \alpha ||u||_{p}^p + \beta ||u||_{2^*}^2 - ||u||_{X_0}^2 \]

= - \langle E'(u), u \rangle = 0. \]

This gives us,

\[ ||v_n||_{X_0}^2 - \beta ||v_n||_{2^*}^2 \rightarrow 0. \]

Since \((u_n)_{n \geq 1}\) is bounded in \(X_0(\Omega)\), so \(||v_n||_{X_0}\) is bounded. Hence, up to a subsequence we may assume that \(||v_n||_{X_0}^2 \rightarrow b\). Using \(3.4\), we have,

\[ \beta ||v_n||_{2^*}^2 \rightarrow b, \]

and by Sobolev embedding,

\[ ||v_n||_{X_0}^2 \geq S ||v_n||_{2^*}^2. \]

These two estimates imply

\[ (3.7) \quad b \geq S \left( \frac{b}{\beta} \right)^{2/2^*}. \]

If \(b = 0\), then we are done. Suppose not. Then we have \(b \geq \frac{s}{N} \left( \frac{2^*}{\beta s} \right)^{1-2/s} \). From \(3.4\), we have, \(E(u) + \frac{1}{2}b - \frac{1}{2}b = c\). This implies,

\[ c = \left( \frac{1}{2} - \frac{1}{2^*} \right)b + E(u) \]

\[ = \frac{s}{N} (b + \beta ||u||_{2^*}^2) + \left( \frac{1}{2} - \frac{1}{p} \right) ||u||_{p}^p \]

\[ \geq \frac{s}{N} S \left( \frac{2^*}{\beta s} \right)^{1-2/s} + \frac{s}{N} \beta ||u||_{2^*}^2 + \alpha \left( \frac{1}{2} - \frac{1}{p} \right) ||u||_{p}^p \]

\[ \geq c^* + \frac{s}{N} \beta ||u||_{2^*}^2 + \alpha \left( \frac{1}{2} - \frac{1}{p} \right) ||u||_{p}^p. \]
As $p \geq 2$, $\alpha, \beta > 0$, we have $c \geq c^*$, which is a contradiction to Claim 2 in Step-2. Hence, $b = 0$ and the conclusion follows. This finishes Step-3.

**Step-4.** In this Step, we conclude the proof of Theorem 1.1. Before that, we recall the following lemma from [Lemma 3.1, [22]]. (Also, see [Lemma 2.3, [24]]).

**Lemma 3.1.** Let $\mu, \epsilon$ as in Step-1. Let $\epsilon \in (0, \frac{1}{2}(c - \frac{\rho}{2}))$, $\delta > 0$. Let $h \in \Gamma$ be such that
\[
\inf_{u \in \partial B \cap E_{i-1}} E(h(u)) \geq c - \epsilon.
\]
Then, there exists $v_\epsilon \in X_0$ such that
\[
\begin{align*}
(1) & \quad c - 2\epsilon \leq E(v_\epsilon) \leq c + 2\epsilon, \\
(2) & \quad \text{dist}(v_\epsilon, h(\partial B \cap Y_{i-1})) \leq 2\delta, \\
(3) & \quad \|E'(v_\epsilon)\| \leq \frac{8\delta}{\epsilon}.
\end{align*}
\]
Using Lemma 3.1, we can say there exists a sequence $(v_n)_{n \geq 1} \in X_0$ such that $E(v_n) \to c$ and $E'(v_n) \to 0$, as $n \to \infty$. So by Step-3, there exists a subsequence, still denoted by $v_n$ and $u \in X_0$ such that $v_n \to u$ in $X_0$ and $E(u) = c$, $E'(u) = 0$, that is, $(E'(u), \phi) = 0$ for all $\phi \in X_0$. In particular, for $\phi = \phi_1 > 0$ where $\phi_1$ is the first eigenfunction of $(-\Delta)^s$, we have,
\[
\langle E'(u), \phi_1 \rangle = 0.
\]
This implies,
\[
\langle u, \phi_1 \rangle_{X_0} - \lambda \int \Omega u \phi_1 = \alpha \int \Omega |u|^{p-2} u \phi_1 + \beta \int \Omega |u|^{2s} u \phi_1.
\]
Note that $\langle u, \phi_1 \rangle_{X_0} = \langle u, (-\Delta)^s \phi_1 \rangle_2 = \lambda_1 \langle u, \phi_1 \rangle_2 = \lambda_1 \int \Omega u \phi_1$. These two estimates together imply,
\[(3.8) \quad (\lambda_1 - \lambda) \int \Omega u \phi_1 = \alpha \int \Omega |u|^{p-2} u \phi_1 + \beta \int \Omega |u|^{2s} u \phi_1.
\]
Let $\lambda_1 \leq \lambda, \alpha, \beta > 0$. Then $u^+ \neq 0, u^- \neq 0$. If not, that is, if $u^+ \equiv 0$, then R.H.S of (3.8) > 0 but L.H.S. < 0, and if $u^- \equiv 0$, then R.H.S of (3.8) < 0 while L.H.S. > 0. Hence, $u$ is sign changing and this completes the proof.

4. Proof of Theorem 1.2

In this section, we consider the case $N \geq 3s$, $s \in (0, 1)$, $\lambda \in \mathbb{R}, \alpha, \beta > 0$. We rewrite the energy functional taking care of $\beta$ as a perturbation term,
\[E_{\alpha,\beta}(u) \equiv E(u) = \left\{ \frac{1}{2} \|u\|_{X_0}^2 - \frac{\lambda}{2} |u|^2 - \frac{\alpha}{p} |u|^p \right\} + \left\{ - \frac{\beta}{2s} |u|^{2s} \right\} \equiv K_\alpha(u) + J_\beta(u),
\]
where $K_\alpha(u) = \frac{1}{2} \|u\|_{X_0}^2 - \frac{\lambda}{2} |u|^2 - \frac{\alpha}{p} |u|^p$, $J_\beta(u) = - \frac{\beta}{2s} |u|^{2s}$. Let $A \subset X_0(\Omega)$ be a subset. We denote the neighbourhood of $A$ by $A^d$ where
\[A^d := \bigcup_{u \in A} B_d(u), \quad B_d(u) := \left\{ v \in X_0(\Omega) : \|u - v\|_{X_0} \leq d \right\}.
\]
The proof is also divided into several steps.

**Step-1.** In the first step, we recall the following results from [Theorem 3.7, [22]].

**Lemma 4.1.** For each $k \in \mathbb{N}$, there exists $R_k > 0$ such that $K_\alpha(u) < 0$ with $u \in X_k, \|u\|_{X_0} = R_k$. 

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Lemma 4.2. For each \( k \in \mathbb{N} \), there exists \( 0 < r_k < R_k \) such that for \( u \in X^\perp_{k-1} \) and \( \|u\|_{X_0} = r_k, K_\alpha(u) \to \infty, k \to \infty \).

We also list some notations which we will use in this section:

\[
B_k := \left\{ u \in X_k : \|u\|_{X_0} \leq R_k \right\}, \quad \Gamma_k := \left\{ h \in C(B_k, X_0(\Omega)) : h \text{ is odd}, h|_{\partial B_k} = \text{id} \right\},
\]

\[
c_{ak} := \inf_{h \in \Gamma_k} \sup_{u \in B_k} K_\alpha(h(u)), \quad a_{ak} := \sup_{u \in \partial B_k} K_\alpha(u), \quad b_{ak} := \inf_{u \in \partial B_k} K_\alpha(u),
\]

where

\[
Z_k := \left\{ u \in X^\perp_{k-1} : \|u\|_{X_0} \leq r_k \right\}.
\]

By Lemma 4.1, there exists \( k_1 \) such that for all \( k \geq k_1 \), there exists \( 0 < r_k < R_k \) such that \( u \in X^\perp_{k-1}, \|u\|_{X_0} = r_k, K_\alpha(u) \geq 1 \). Hence, for all \( k \geq k_1 \), we have

\[
\inf_{u \in X^\perp_{k-1} : \|u\|_{X_0} = r_k} K_\alpha(u) \geq 1 > 0 > K_\alpha(u) \text{ for all } u \in X_k \text{ with } \|u\|_{X_0} = R_k.
\]

So, we have for \( k \) large enough \( b_{ak} > a_{ak} \) and \( c_{ak} \geq b_{ak} \). We refer [Theorem 3.5, [22]] for proof. For all \( k \in \mathbb{N} \) large enough the functional \( K_\alpha \) satisfies \((PS)_c\) condition. We know that for any \( c > 0 \), \( K_\alpha \) satisfies \((PS)_c\) condition and thus, \( K_\alpha \) has infinitely many critical values. For each \( k \in \mathbb{N} \), let us define:

\[
V_{ak} := \left\{ u \in X_0(\Omega) \setminus \{0\} : K'_\alpha(u) = 0, K_\alpha(u) = c_{ak} \right\}.
\]

Therefore, \( V_{ak} \) is non-empty and compact. Let \( S_{1k} := \sup_{u \in V_{ak}} \|u\|_{X_0} \).

Step-2. In this step, we prove the following claim.

Claim 1. For \( c > 0 \), the following holds true:

\[
\lim_{\beta \to 0} \sup_{\|u\|_{X_0} \leq c} |J_\beta(u)| = \lim_{\beta \to 0} \sup_{\|u\|_{X_0} \leq c} |J'_\beta(u)| = 0.
\]

Proof of Claim 1. We have,

\[
|J_\beta(u)| = \frac{\beta}{2} \|u\|_2^2 \leq \frac{\beta}{2} S^{-2^*/2} \|u\|_{X_0}^{2^*}.
\]

This implies,

\[
\sup_{\|u\|_{X_0} \leq c} |J_\beta(u)| \leq \frac{\beta}{2} S^{-2^*/2} c^{2^*}.
\]

and so \( \lim_{\beta \to 0} \sup_{\|u\|_{X_0} \leq c} |J_\beta(u)| = 0 \). Let \( v \in X_0(\Omega) \). Then, we have,

\[
|\langle J'_\beta(u), v \rangle| \leq \beta \|u\|_{X_0}^{2^*-1} \|v\|_{X_0} S^{-2^*/2}.
\]

This evidently implies,

\[
\frac{|\langle J'_\beta(u), v \rangle|}{\|v\|_{X_0}} \leq \beta \|u\|_{X_0}^{2^*-1} S^{-2^*/2}.
\]

Therefore, \( \sup_{\|u\|_{X_0} \leq c} |J'_\beta(u)|\|v\|_{X_0} \leq \beta c^{2^*-1} S^{-2^*/2} \) and Claim 1 follows.

Step-3. In this section, we again set some notations:

\[
c_{\beta k} := \inf_{u \in \Gamma_k} \sup_{u \in B_k} E_{\alpha \beta}(h(u)), \quad a_{\beta k} := \sup_{u \in \partial B_k} E_{\alpha \beta}(u), \quad b_{\beta k} := \inf_{u \in \partial B_k} E_{\alpha \beta}(u).
\]
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Note that \( b_{\beta k} > a_{\beta k} \) for \( \beta \) sufficiently small and \( k \in \mathbb{N} \) large enough. Indeed, 
\[
E_{\alpha \beta}(u) = K_{\alpha}(u) + J_{\beta}(u) \leq K_{\alpha}(u) \text{ for all } u \in X_{0}(\Omega). \]
So, \( a_{\beta k} \leq a_{\alpha k} < 0 \). For \( \beta > 0 \) small enough, we have
\[
b_{\beta k} = \inf_{u \in \partial \mathbb{Z}_{k}} (K_{\alpha}(u) + J_{\beta}(u)) > a_{\beta k}.
\]
Also, for any \( h \in \Gamma_{k} \) we have, \( h(B_{k}) \cap \partial \mathbb{Z}_{k} \neq \emptyset \). So, \( c_{\beta k} \geq b_{\beta k} \) by [Theorem 3.5, [22]] and we get a \((PS)_{c_{\beta k}}\) sequence of \( I_{\alpha \beta} \).

**Step-4.** In this section, we claim the following.  
Claim 2. For \( \lambda \in \mathbb{R}, \beta, \alpha > 0 \), \( \{u_{\beta kn}\}_{n \geq 1} \) is a \((PS)_{c_{\beta k}}\) sequence of \( E_{\alpha \beta} \), then there exists \( S_{2k} \) independent of \( \beta \) such that \( \|u_{\beta kn}\|_{X_{0}} \leq S_{2k} \) for all \( n \) large enough.

**Proof of Claim 2.** We note that, \( c_{\beta k} \leq c_{\alpha k} \), since \( E_{\alpha \beta}(u) \geq K_{\alpha}(u) \text{ for all } u \in X_{0}(\Omega) \). By choosing \( \rho \in \left( \frac{1}{p}, \frac{1}{2} \right) \) for \( n \) large enough we have,
\[
c_{\alpha k} + 1 + \|u_{\beta kn}\|_{X_{0}} \geq c_{\beta k} + 1 + \|u_{\beta kn}\|_{X_{0}} \geq E_{\alpha \beta}(u_{\beta kn}) - \rho \left( E'_{\alpha \beta}(u_{\beta kn}) \cdot u_{\beta kn} \right) \geq \left( \frac{1}{2} - \rho \right)\|u_{\beta kn}\|_{X_{0}}^{2} + \alpha \left( \rho - \frac{1}{\beta} \right) - \varepsilon\|u_{\beta kn}\|_{X_{0}}^{p} - c_{\varepsilon},
\]
where \( 0 < \varepsilon < \alpha(\rho - \frac{1}{\beta}) \) and \( c_{\varepsilon} \) independent of \( \beta \). This completes the proof of the Claim.

**Step-5.** We signify \( S_{k}^{\varepsilon} := 2\max\{S_{1k}, S_{2k}, R_{k}\} \). In this step, we prove the following Claim.

Claim 3. We have \( \lim_{\beta \to 0} c_{\beta k} = c_{\alpha k} \).

**Proof of Claim 3.** Let \( \varepsilon > 0 \). From the definition of \( c_{\alpha k} \) we have \( h_{\varepsilon} \in \Gamma_{k} \) such that
\[
\sup_{v \in B_{\varepsilon}} K_{\alpha}(h_{\varepsilon}(v)) \leq c_{\alpha k} + \varepsilon,
\]
that is,
\[
\sup_{u \in h_{\varepsilon}(B_{k})} K_{\alpha}(u) - \varepsilon \leq c_{\alpha k}.
\]
Let \( \Gamma_{k, \varepsilon, \alpha} := \{h_{\varepsilon} \in \Gamma_{k} : \sup_{u \in h_{\varepsilon}(B_{k})} K_{\alpha}(u) - \varepsilon \leq c_{\alpha k}\} \). Then,
\[
\lim_{\varepsilon \to 0} \inf_{h \in \Gamma_{k, \varepsilon, \alpha}} \sup_{u \in h(B_{k})} K_{\alpha}(u) = c_{\alpha k}.
\]
We notice that, for each \( \varepsilon > 0 \), there exists \( v_{\varepsilon} \in h_{\varepsilon}(B_{k}) \) such that
\[
\sup_{u \in h_{\varepsilon}(B_{k})} K_{\alpha}(u) = K_{\alpha}(v_{\varepsilon}),
\]
and \( \{v_{\varepsilon}\}_{\varepsilon \geq 0} \) is a \((PS)_{c_{\alpha k}}\) sequence of \( K_{\alpha} \) as \( \varepsilon \to 0 \).
Therefore, for \( \varepsilon > 0 \) small enough we have \( v_{\varepsilon} \in B_{S_{k}}(0) \). Thus,
\[
c_{\alpha k} \leq K_{\alpha}(v_{\varepsilon}) \leq \sup_{u \in h_{\varepsilon}(B_{k}) \cap B_{S_{k}}(0)} K_{\alpha}(u).
\]
So, we have, \( c_{\alpha k} = \lim_{\varepsilon \to 0} \inf_{h \in \Gamma_{k, \varepsilon, \alpha}} \sup_{u \in h(B_{k}) \cap B_{S_{k}}(0)} K_{\alpha}(u) \). Similarly, we can show that
\[
c_{\beta k} = \lim_{\varepsilon \to 0} \inf_{h \in \Gamma_{k, \varepsilon, \beta}} \sup_{u \in h(B_{k}) \cap B_{S_{k}}(0)} J_{\beta}(u),
\]
where
\[
\Gamma_{k, \varepsilon, \beta} := \{h \in \Gamma_{k} : \sup_{u \in h(B_{k})} E_{\alpha \beta}(u) - \varepsilon \leq c_{\beta k}\}.
\]
By Claim 2, for any \( h \in \Gamma_k \) and \( u \in h(B_k) \cap B_{\delta_k}(0) \) we get, \( K_\alpha(u) = \lim_{\beta \to 0} E_{\alpha\beta}(u) \).

As \( c_{\alpha k} \geq c_{\beta k} \) for any \( h \in \Gamma_{k,\varepsilon,\beta} \) such that

\[
\sup_{u \in h(B_k) \cap B_{\delta_k}(0)} E_{\alpha\beta}(u) - \varepsilon \leq c_{\beta k} \leq c_{\alpha k}.
\]

So we have \( h \in \Gamma_{k,\varepsilon,\alpha} \) as \( \beta \to 0 \). Hence, \( \lim_{\beta \to 0} \inf c_{\beta k} \geq c_{\alpha k} \). Therefore, \( c_{\alpha k} = \lim_{\beta \to 0} c_{\beta k} \). This completes the proof of Claim-3.

**Step-6.** In this step, we claim the following, thereby completing the proof.

Claim 4. For any \( d > 0 \), there exists \( \beta_{0k} > 0 \) such that for any \( \beta \in (0, \beta_{0k}) \) and any \((PS)_{c_{\beta k}}\) sequence \( \{u_{\beta kn}\}_{n=1}^\infty \) of \( E_{\alpha\beta} \), there exists \( n_0 \in \mathbb{N} \) such that \( u_{\beta kn} \in V_{\alpha k}^d \) for all \( n \geq n_0 \).

Proof of Claim 4. We will prove the claim by the method of contradiction. Suppose not, there exists \( \beta_{0k} > 0 \) as \( n_j \to \infty \), such that

\[
\lim_{n_j \to \infty} E_{\alpha\beta_j}(u_{\beta_j kn_j}) = c_{\beta_j k},
\]

but \( u_{\beta_j kn_j} \notin V_{\alpha k} \). Using Claim 2, there exists \( N_{2k} > 0 \) such that \( \|u_{\beta_j kn_j}\|_{X_0} \leq N_{2k} \).

Using Claim 1 and Claim 3, we have \( \lim_{n_j \to \infty} K'_{\alpha}(u_{\beta_j kn_j}) = c_{\alpha k} \). Indeed,

\[
|K_\alpha(u_{\beta_j kn_j}) - c_{\alpha k}| \leq |E_{\alpha\beta_j}(u_{\beta_j kn_j}) - c_{\beta_j k}| + |c_{\beta_j k} - c_{\alpha k}| + |J_{\beta_j}(u_{\beta_j kn_j})| \to 0, j \to \infty,
\]

and

\[
\|K'_{\alpha}(u_{\beta_j kn_j})\| \leq \|K'_{\alpha}(u_{\beta_j kn_j}) + J'_{\beta_j}(u_{\beta_j kn_j})\| + \|J'_{\beta_j}(u_{\beta_j kn_j})\| = \|E'_{\alpha}(u_{\beta_j kn_j})\| + \|J'_{\beta_j}(u_{\beta_j kn_j})\| \to 0, j \to \infty.
\]

Therefore, \( \{u_{\beta_j kn_j}\} \) is a \((PS)_{c_{\alpha k}}\) sequence of \( K_\alpha \). As \( K_\alpha \) satisfies \((PS)_{c_{\alpha k}}\) condition, so up to a subsequence, there exists \( u_{0k} \in V_{\alpha k} \) such that \( u_{\beta_j kn_j} \to u_{0k} \). Therefore, \( u_{\beta_j kn_j} \in B_d(u_{0k}) \subset V_{\alpha k}^d \) for \( j \) large, which is a contradiction. This proves Claim 4.

By the above Claim 4, we consider \( d_k > 0 \) small so that for any \( u \in V_{\alpha k}^{d_k} \), we have \( u \neq 0 \). For any \( \beta \in (0, \beta_{0k}) \), let \( \{u_{\beta kn}\} \) be \((PS)_{c_{\beta k}}\) sequence for \( E_{\alpha\beta} \), then \( \{u_{\beta kn}\} \in V_{\alpha k}^{d_k} \). Therefore, there exists \( u_{0kn} \in V_{\alpha k} \) such that \( u_{\beta kn} \in B_{d_k}(u_{0kn}) \). As \( V_{\alpha k} \) is compact, there exists \( u_{0k} \in V_{\alpha k} \) such that \( u_{0kn} \to u_{0k} \). We have \( u_{\beta kn} \in B_{2d_k}(u_{0kn}) \) for large \( n \). Also, there exists \( u_{\beta kn} \) such that \( u_{\beta kn} \to u_{\beta k} \) and \( E'_{\alpha\beta}(u_{\beta k}) = 0 \). As \( \beta_{2d_k}(u_{0k}) \) is closed and convex, so it is weakly closed. Therefore, \( u_{\beta kn} \in B_{2d_k}(u_{0k}) \). Hence, \( u_{\beta k} \) is a nontrivial critical point of \( E_{\alpha\beta} \) and we have for \( m \in \mathbb{N} \), every \( \beta \in (0, \beta_m) \) the problem \((\text{P})\) has \( m \) nontrivial solutions where \( \beta_m := \min_{1 \leq k \leq n} \{\beta_{0k}\} \). This completes the proof.

5. **Proof of Theorem 1.3**

Let us denote the energy functional as in Step-1 of the proof of Theorem 1.1.

\[
E_{\alpha}(u) = E(u) = \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|_2^2 - \frac{\beta}{2^*}\|u\|_{2^*}^2 - \frac{\alpha}{q}\|u\|_q^q := E_0(u) + G_{\alpha}(u),
\]

where

\[
E_0(u) = \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|_2^2 - \frac{\beta}{2^*}\|u\|_{2^*}^2
\]

and \( G_{\alpha}(u) = -\frac{\alpha}{q}\|u\|_q^q \),

with \( \lambda > 0, \alpha < 0, \beta > 0, N \geq 4s \) with \( s \in (0, 1) \).
We prove the result in the following steps.

**Step-1.** Approaching similarly as in the proof of Claim 1 of Step-1 in the proof of Theorem 1.2 [4], using Raleigh quotient characterization of \( \lambda_0 \), Sobolev embedding that,

\[
E_0(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) ||u||_{X_0}^2 - C||u||_{X_0}^2,
\]

for all \( u \in X_{l-1}^+ \) and for some \( C > 0 \). This yields that, there exists \( \mu_0, r_0 > 0 \) such that,

\[
E_0(u) > \mu_0 \text{ for all } u \text{ with } ||u||_{X_0} = r_0,
\]

and

\[
E_0(u) > 0 \text{ for all } u \text{ with } 0 < ||u||_{X_0} < r_0.
\]

Let \( \Gamma_0 = \left\{ h \in C(X_0, X_0) : h(B_1) \subset (E_0)_0 \cup \overline{B}_{r_0}, h \text{ is an odd homeomorphism of } X_0 \right\} \),

where \( E_0 = \{ u \in X_0 : E_0(u) \geq 0 \} \) and \( B_{r_0} = \{ u \in X_0 : ||u||_{X_0} < r_0 \} \), and

\[
\Gamma_0 = \left\{ K \subset X_0 : K \text{ is compact, symmetric and } \gamma(K \cap h(\partial B_1)) \geq 1 \text{ for all } h \in \Gamma \right\},
\]

where \( \Gamma \) is the genus. We set the following notations:

\[
b_0 := \inf_{K \in \Gamma_0} \sup_{u \in K} E_0(u), \quad c_0 := \sup_{h \in \Gamma_0} \inf_{u \in \partial B_1 \cap X_{l-1}^+} E_0(h(u)).
\]

With the above notations, proceeding as in Claim 2 of Step-2 in the proof of Theorem 1.2, that,

\[
0 < \mu_0 \leq b_0 \leq c_0 < c^*,
\]

and similarly by Step-3 in the proof of Theorem 1.2, we get, \( E_0 \) has a \((PS)_{c_0}\) sequence and \( c_0 \) is a critical value of \( E_0 \). Let us consider the set of critical points of \( F \) with critical value \( c_0 \), that is,

\[
Z_{E_0} := \left\{ u \in X_0 \setminus \{ 0 \} : E_0(u) = c_0, E'_0(u) = 0 \right\}.
\]

Hence, the set \( Z_{E_0} \neq \emptyset \) and compact. We now signify: \( L_1 := \sup_{u \in Z_{E_0}} ||u||_{X_0} \).

**Step-2.** As in Claim 2 of Step-2 in the proof of Theorem 1.2, we can show that for any \( C > 0 \),

\[
\lim_{\alpha \to 0} \sup_{||u||_{X_0} \leq C} |G_\alpha(u)| = 0 = \lim_{\alpha \to 0} \sup_{||u||_{X_0} \leq C} |G'_\alpha(u)|.
\]

Let us set \( \Gamma_\alpha = \left\{ h \in C(X_0, X_0) : h(B_1) \subset (E_\alpha)_0 \cup \overline{B}_{r_0}, h \text{ is an odd homeomorphism of } X_0 \right\} \).

We notify: \( c_\alpha = \sup_{h \in \Gamma_\alpha} \inf_{u \in \partial B_1 \cap X_{l-1}^+} E_\alpha(h(u)) \). Then, one can observe that,

\[
E_\alpha(u) \geq E_0(u) \geq \mu_0 \text{ for } u \in X_{l-1}^+, ||u||_{X_0} = r_0.
\]

Hence, \( c_\alpha \geq \mu_0 \) and \( E_\alpha \) satisfies \((PS)_{c_\alpha}\).

**Step-3.** In this step, we first claim the following.

*Claim 1.* Let \( \lambda > 0, \alpha < 0 \) with \( |\alpha| \) small, \( \beta > 0 \). Assume \( \{ (u_\alpha) \}_n \) be a \((PS)_{c_\alpha}\) sequence of \( E_\alpha \). Then, there exists \( L_2 > 0 \) (independent of \( \alpha \)) such that,

\[
||(u_\alpha)_n||_{X_0} \leq L_2 \text{ for all } n \in \mathbb{N}.
\]
Proof of Claim 1. For \( \alpha > 0 \), let us denote the set
\[
\Gamma_\alpha = \{ K \subset X_0 : K \text{ is compact, symmetric and } \gamma(K \cap h(B_1)) \geq \lambda \text{ for all } h \in \Gamma_\alpha \}.
\]
We set \( b_\alpha = \inf_{K \in \Gamma_\alpha} \sup_{u \in K} E_\alpha(u) \). We observe that, \( c_\alpha \leq b_\alpha \). There exists \( R_0 > r_0 > 0 \) large enough such that for \( |\alpha| \) small, we have, \( (E_\alpha)_0 \cap V_\varepsilon \subset \overline{B}_{R_0} \). Let \( K_{R_0} := V_\varepsilon \cap \overline{B}_{R_0} \). Then, \( K_{R_0} \subset \Gamma_\alpha^* \).

Thus, we obtain,
\[
b_\alpha \leq \sup_{u \in K_{R_0}} E_\alpha(u) \leq \sup_{V_\varepsilon} E_0(u) + c_0 < c^* + c_0 = c^*.
\]
where \( c_0 > 0 \) (independent of \( \alpha \)). By choosing \( \rho \in (\frac{1}{2} - \frac{1}{\lambda}) \), as in the proof of Claim 2 of Step-2 in the proof of Theorem 1.2 we note that,
\[
c^* + \|u_{\alpha_n}\|_{X_0} \geq E_\alpha(u_{\alpha_n}) - \rho \langle E'_\alpha(u_{\alpha_n}), u_{\alpha_n} \rangle
\]
\[
\geq \left( \frac{1}{2} - \rho \right) \|u_{\alpha_n}\|_{X_0}^2 + \left( \beta(\rho - \frac{1}{2\lambda}) - \varepsilon \right) \|u_{\alpha_n}\|_{L^2}^2 - c\varepsilon
\]
\[
\geq \left( \frac{1}{2} - \rho \right) \|u_{\alpha_n}\|_{X_0}^2 - c\varepsilon,
\]
where \( \varepsilon \in (0, \beta(\rho - \frac{1}{2\lambda})) \), \( c\varepsilon \) is independent of \( \alpha \). This yields, there exists \( L_2 > 0 \) such that
\[
\|u_{\alpha_n}\|_{X_0} \leq L_2 \text{ for all } n \in \mathbb{N}.
\]

Step-4. We note that, \( \Gamma_0 \subset \Gamma_\alpha \) for \( \alpha < 0 \). We also observe that, for \( u \in X_0 \), we have,
\[
\lim_{\alpha \to 0} E_\alpha(u) = E_0(u).
\]
Therefore, for any \( h \in \Gamma_\alpha \) with \( E_\alpha(h(B_1)) \geq 0 \), we have, \( E_0(h(B_1)) \geq 0 \). In a similar fashion, as in Claim 3 of Step-5 in the proof of Theorem 1.2 we obtain,
\[
\lim_{\alpha \to 0} c_\alpha = c_0.
\]
Using similar arguments as in Step-6 in the proof of Theorem 1.2 for any \( d > 0 \), there exists \( \alpha_0 > 0 \) such that for any \( \alpha \in (-\alpha_0, 0) \), there exists \( (PS)_{c_\alpha} \) sequence of \( E_\alpha \), denoted by \( \{u_{\alpha_n}\} \subset Z_{E_\alpha} \). Taking \( d > 0 \) sufficiently small, we obtain a nontrivial solution \( u_\alpha \) in a neighborhood of a solution \( u_0 \) of \( E_\alpha \) with \( E_\alpha(u_0) = c_0 \). This finishes the proof.

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References

[1] Ambrosetti, A., and Rabinowitz, P. H. Dual variational methods in critical point theory and applications. J. Functional Analysis 14 (1973), 349–381.
[2] Bandle, C., and Benguria, R. The Brézis-Nirenberg problem on \( S^3 \). J. Differential Equations 178, 1 (2002), 264–279.
[3] Benci, V., and Cerami, G. Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. Calc. Var. Partial Differential Equations 2, 1 (1994), 29–48.
[4] Brézis, H., and Nirenberg, L. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. 36, 4 (1983), 437–477.
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[5] Capozzi, A., Fortunato, D., and Palmieri, G. An existence result for nonlinear elliptic problems involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 2, 6 (1985), 463–470.

[6] Carriaño, P. C., Lehrer, R., Miyagaki, O. H., and Vícente, A. A Brezis-Nirenberg problem on hyperbolic spaces. *Electron. J. Differential Equations* (2019), Paper No. 67, 15.

[7] Cerami, G., Fortunato, D., and Struwe, M. Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1, 5 (1984), 341–350.

[8] Cerami, G., Solimini, S., and Struwe, M. Some existence results for superlinear elliptic boundary value problems involving critical exponents. *J. Funct. Anal.* 69, 3 (1986), 289–306.

[9] Chen, K.-S., Montenegro, M., and Yan, X. The Brezis-Nirenberg problem for fractional elliptic operators. *Math. Nachr.* 290, 10 (2017), 1491–1511.

[10] Colorado, E., and Ortega, A. The Brezis-Nirenberg problem for the fractional Laplacian with mixed Dirichlet-Neumann boundary conditions. *J. Math. Anal. Appl.* 473, 2 (2019), 1002–1025.

[11] Cora, G., and Iacopetti, A. On the structure of the nodal set and asymptotics of least energy sign-changing radial solutions of the fractional Brezis-Nirenberg problem. *Nonlinear Anal.* 176 (2018), 226–271.

[12] Coti Zoli, A., and Tavoularis, N. K. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.* 295, 1 (2004), 225–236.

[13] Di Nezza, E., Palatucci, G., and Valdinoci, E. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* 136, 5 (2012), 521–573.

[14] Farina, L. F. O., Miyagaki, O. H., Pereira, F. R., Squassina, M., and Zhang, C. The Brezis-Nirenberg problem for nonlocal systems. *Adv. Nonlinear Anal.* 5, 1 (2016), 85–103.

[15] Mosconi, S., Pereira, K., Squassina, M., and Yang, Y. The Brezis-Nirenberg problem for the fractional p-Laplacian. *Calc. Var. Partial Differential Equations* 55, 4 (2016), Art. 105, 25.

[16] Servadei, R., and Valdinoci, E. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.* 389, 2 (2012), 887–898.

[17] Servadei, R., and Valdinoci, E. A Brezis-Nirenberg result for non-local critical equations in low dimension. *Commun. Pure Appl. Anal.* 12, 6 (2013), 2445–2464.

[18] Servadei, R., and Valdinoci, E. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.* 33, 5 (2013), 2105–2137.

[19] Servadei, R., and Valdinoci, E. The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.* 367, 1 (2015), 67–102.

[20] Servadei, R., and Valdinoci, E. Fractional Laplacian equations with critical Sobolev exponent. *Rev. Mat. Complut.* 28, 3 (2015), 655–676.

[21] Tan, J. The Brezis-Nirenberg type problem involving the square root of the Laplacian. *Calc. Var. Partial Differential Equations* 42, 1-2 (2011), 21–41.

[22] Willem, M. *Minimax theorems*, vol. 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.

[23] Xuan, B., Su, S., and Yan, Y. Existence results for Brezis-Nirenberg problems with Hardy potential and singular coefficients. *Nonlinear Anal.* 67, 7 (2007), 2091–2106.

[24] Yue, X., and Zou, W. Remarks on a Brezis-Nirenberg’s result. *J. Math. Anal. Appl.* 425, 2 (2015), 900–910.

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