Towards Scalable Spectral Sparsification of Directed Graphs

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Abstract—Recent spectral graph sparsification research allows constructing nearly-linear-sized subgraphs that can well preserve the spectral (structural) properties of the original graph, such as the first few eigenvalues and eigenvectors of the graph Laplacian, leading to the development of a variety of nearly-linear time numerical and graph algorithms. However, there is not a unified approach that allows for truly-scalable spectral sparsification of both directed and undirected graphs. For the first time, we prove the existence of linear-sized spectral sparsifiers for general directed graphs, and introduce a practically-efficient yet unified spectral graph sparsification approach that allows sparsifying real-world, large-scale directed and undirected graphs with guaranteed preservation of the original graph spectra. By exploiting a highly-scalable (nearly-linear complexity) spectral matrix perturbation analysis framework for constructing nearly-linear sized (directed) subgraphs, it enables to well preserve the key eigenvalues and eigenvectors of the original (directed) graph Laplacians.

I. INTRODUCTION

There is not a unified approach that allows for truly-scalable spectral sparsification of both directed and undirected graphs. For example, the state-of-the-art sampling-based methods for spectral sparsification are only applicable to undirected graphs [10], [13], [15]; the latest algorithmic breakthrough in spectral sparsification of directed graphs [4], [5] can only handle strongly-connected directed graphs \(^1\), which inevitably limits its applications when confronting real-world graphs, since many directed graphs may not be strongly connected, such as the graphs used in chip design automation (e.g. timing analysis) tasks as well as the graphs used in machine learning and data mining tasks. Consequently, there is still a pressing need for the development of highly-robust (theoretically-rigorous) and truly-scalable (nearly-linear complexity) algorithms for reducing real-world large-scale (undirected and directed) graphs while preserving key graph spectral (structural) properties.

For the first time we prove the existence of linear-sized spectral sparsifiers for general directed graphs, and introduce a practically-efficient yet unified spectral sparsification approach that allows simplifying real-world, large-scale directed and undirected graphs with guaranteed preservation of the original graph spectra [17]. More specifically, we exploit a highly-scalable (nearly-linear complexity) spectral matrix perturbation analysis framework for constructing ultra-sparse (directed) subgraphs that can well preserve the key eigenvalues and eigenvectors of the original graph Laplacians. Unlike the prior state-of-the-art methods that are only suitable for handling specific types of graphs (e.g. undirected or strongly-connected directed graphs [5], [13]), the proposed approach is more general and thus will allow for truly-scalable spectral sparsification of a much wider range of real-world complex graphs that may involve billions of elements. The spectrally-sparse directed graphs constructed by the proposed approach will potentially lead to the development of much faster numerical and graph-related algorithms. For example, spectrally-sparsefied social (data) networks allow for more efficient modeling and analysis of large social (data) networks; spectrally-sparsefied neural networks allow for more scalable model training and processing in emerging machine learning tasks; spectrally-sparsefied web-graphs allow for much faster computations of personalized PageRank vectors; spectrally-sparsefied integrated circuit networks will lead to more efficient partitioning, modeling, simulation, optimization and verification of large chip designs, etc.

II. PRELIMINARIES

A. Laplacians for directed and undirected graphs

Consider a directed graph \(G = (V, E_G, w_G)\) with \(V\) denoting the set of vertices, \(E_G\) representing the set of directed edges, and \(w_G\) denoting the associated edge weights. In the following, we denote the diagonal matrix by \(D_G\) with \(D_G(i, i) = \sum_{j \in V} w_G(i, j)\) being equal to the (weighted) outdegree of node \(i\), as well as the adjacency matrix of \(G\) by \(A_G\):

\[ A_G(i, j) = \begin{cases} w_G(i, j) & \text{if } (i, j) \in E_G \\ 0 & \text{otherwise} \end{cases}. \] (1)

Then the directed Laplacian matrix can be constructed as follows [5]:

\[ L_G = D_G - A_G^T. \] (2)

Undirected graphs can be converted into equivalent directed graphs by replacing each undirected edge with two opposite directed edges. While for most directed graphs \(L_G\) may not be a symmetric matrix.

It can be shown that any directed (undirected) graph Laplacian constructed using (2) will satisfy the following properties: 1) Each column (and row) sum is equal to zero; 2) All off-diagonal elements are non-positive; 3) The Laplacian matrix is asymmetric (symmetric) and indefinite (positive semidefinite).

B. Spectral sparsification of undirected graphs

Graph sparsification aims to find a subgraph (sparsifier) \(S = (V, E_S, w_S)\) that has the same set of vertices but much fewer edges than the original graph \(G\). There are two types of sparsification methods: the cut sparsification methods preserve cuts in the original graph through random sampling of edges [2], whereas spectral sparsification methods preserve the graph spectral (structural) properties, such as distances between vertices, effective resistances, cuts in the graph, as well as the stationary distributions of Markov chains [4], [5], [14]. Therefore, spectral graph sparsification is a much stronger notion than cut sparsification.

For undirected graphs, spectral sparsification aims to find an ultra-sparse subgraph proxy that is spectrally-similar to the original one. \(G\) and \(S\) are said to be \(\sigma\)-spectrally similar if the following condition holds for all real vectors \(x \in \mathbb{R}^V\): \(\frac{\|L_G x\|_2}{\|L_S x\|_2} \leq \sigma \) \(\|L_G x\|_2 \leq \sigma \|L_S x\|_2\). When \(L_G\) and \(L_S\) denote the symmetric diagonally dominant (SDD) Laplacian matrices of graphs \(G \) and \(S\), respectively. By defining the relative condition number to be \(\kappa(L_G, L_S) = \lambda_{\text{max}}(L_G) / \lambda_{\text{min}}(L_S)\), where \(\lambda_{\text{max}}(\lambda_{\text{min}})\) denotes the largest (smallest) eigenvalues of \(L_G\) and \(L_S\), \(\kappa\) denotes the Moore-Penrose pseudoinverse of \(L_S\), it can be further shown that \(\kappa(L_G, L_S) \leq \sigma^2\), implying that a smaller relative condition number or \(\sigma^2\) corresponds to a higher (better) spectral similarity between two graphs.

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1\(^A\) A strongly connected directed graph is a directed graph in which any node can be reached from any other node in the graph.
C. Spectral sparsification of directed graphs

A significant progress has been made for spectral analysis of directed graphs in [3], which for the first time has proved the Cheeger inequality for directed graphs and shown the connection between directed graph partitioning and the smallest (nontrivial) eigenvalue of directed Laplacian. More specifically, the transition probability matrix and the stationary distributions of Markov chains have been exploited for constructing the undirected Laplacians for strongly-connected directed graphs. The latest algorithmic breakthrough in spectral sparsification for strongly-connected directed graphs has been introduced based on the results in [3], which proposes to first convert strongly connected graphs into Eulerian graphs via Eulerian scaling, and subsequently sparsify the undirected graphs obtained via Laplacian symmetrization [3] by leveraging existing spectral graph theory for undirected graphs [5]. It has been shown that such an approach for directed graphs can potentially lead to the development of almost-linear-time algorithms for solving asymmetric linear systems, computing the stationary distribution of a Markov chain, computing expected commute times in a directed graph, etc [5].

For directed graphs, the subgraph $S$ can be considered spectrally similar to the original graph $G$ if the condition number or the ratio between the largest and smallest singular values of $L_S G$ is close to 1 [4], [5]. Since the singular values of $L_S G$ correspond to the square roots of eigenvalues of $(L_S G) \top (L_S G)$, spectral sparsification of directed graphs is equivalent to finding an ultra-sparse subgraph $S$ such that the condition number of $(L_S G) \top (L_S G)$ is small enough.

III. A Unified Spectral Sparsification Framework

We introduced a unified spectral graph sparsification framework [17] that allows handling both directed and undirected graphs in nearly-linear time. The core idea of our approach is to leverage a novel spectrum-preserving Laplacian symmetrization procedure to convert directed graphs into undirected ones (as shown in Figure 1). Then existing spectral sparsification methods for undirected graphs [1], [6], [7], [11], [14] can be exploited for directed graph spectral sparsification tasks.

Our approach for symmetrizing directed graph Laplacians is motivated by the following fact: the eigenvalues of $(L_S G) \top (L_S G)$ will always correspond to the eigenvalues of $(L_S L_S \top) \top (L_S L_S)$ under the condition that $L_S$ and $L_G$ are diagonalizable. It can be shown that $L_G L_S$ and $L_S L_S$ can be considered as special graph Laplacian matrices corresponding to undirected graphs that may contain negative edge weights. Consequently, as long as a directed subgraph $S$ can be found such that the undirected graphs corresponding to $L_S L_S$ and $L_G L_S$ are spectrally similar to each other, the subgraph $S$ can be considered spectrally similar to the original directed graph $G$. Unlike the recent theoretical breakthrough in directed graph sparsification [4], [5], our approach does not require the underlying directed graphs to be strongly connected, and thus can be applied to a much wider range of large-scale real-world problems, such as the neural networks adopted in many machine learning and data mining applications [8], [16], the directed graphs (e.g. timing graphs) used in various circuit analysis and optimization tasks [9], [12], [18], etc. Detailed proofs, algorithm flows and experimental results can be found in [17].

IV. Conclusions

For the first time we prove the existence of linear-sized spectral sparsifiers for general directed graphs, and propose a practically-efficient yet unified spectral graph sparsification framework. Such a novel spectral sparsification approach allows sparsifying real-world, large-scale directed and undirected graphs with guaranteed preservation of the original graph spectral properties. The proposed method has been validated using various kinds of directed graphs obtained from public domain sparse matrix collections, showing promising spectral sparsification and partitioning results for general directed graphs.

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