Herein, the Hidden Markov Model is expanded to allow for Markov chain observations. In particular, the observations are assumed to be a Markov chain whose one step transition probabilities depend upon the hidden Markov chain. An Expectation-Maximization analog to the Baum-Welch algorithm is developed for this more general model to estimate the transition probabilities for both the hidden state and for the observations as well as to estimate the probabilities for the initial joint hidden-state-observation distribution. A belief state or filter recursion to track the hidden state then arises from the calculations of this Expectation-Maximization algorithm. A dynamic-programming analog to the Viterbi algorithm is also developed to estimate the most likely sequence of hidden states given the sequence of observations.

1. Introduction. Hidden Markov models (HMMs) were introduced in a series of papers by Baum and collaborators [1], [2]. Traditional HMMs have enjoyed tremendous success in applications like computational finance [28], single-molecule kinetic analysis [27], animal tracking [33], forecasting commodity futures [12] and protein folding [35]. In HMMs the unobservable hidden states \( X \) are a discrete-time Markov chain and the observations process \( Y \) is some distorted, corrupted partial information or measurement of the current state of \( X \) satisfying the condition

\[
P(Y_n \in A | X_n, X_{n-1}, ..., X_1) = P(Y_n \in A | X_n).
\]

These probabilities, \( P(Y_n \in A | X_n) \), are called the emission probabilities.

This type of observation modeling can be limiting. Consider observations \( Y \) consisting of daily stock price and volume that are based upon a hidden (bullish/bearish type) market state \( X \). If there was really just an emission probability, the prior day’s price and volume would be completely forgotten and a new one would be chosen randomly only depending solely upon the market bull/bear state. Clearly, this is not what happens. The next day’s price and volume is related to the prior day’s in some way. Perhaps, prices are held in a range by recent earnings or volume is elevated for several days due to some company news. Indeed, the autoregressive HMM (AR-HMM) was been introduced because the (original) HMM does not allow for an observation to depend upon a past observation. For the AR-HMM the observations take the structure:

\[
Y_n = \beta_0(X_n) + \beta_1(X_n)Y_{n-1} + \cdots + \beta_p(X_n)Y_{n-p} + \varepsilon_n,
\]

where \( \{\varepsilon_n\}_{n=1}^{\infty} \) are a (usually zero-mean Gaussian) i.i.d. sequence of random variables and the autoregressive coefficients are functions of the current hidden state \( X_n \). Most critically, one might view this AR-HMM as a linear, Gaussian partial patch to the HMM deficiency and expect a more general, useful theory. Still, the AR-HMM has experienced strong success in applications like speech recognition (see [5]), diagnosing blood infections (see [34]) and...
the study of climate patterns (see [40]). Finally, there are general models that truly incorporate (possibly non-linear) dependencies of $Y_n$ on past values of $Y$ referred to as Markov-switching models or sometimes Markov jump systems. These are very general models that are particularly important in financial applications. However, as mentioned in [6] the analyses of Markov-switching models can be far more intricate than those of HMM due to the fact that the properties of the observed process are not directly controlled by those of the hidden chain.

It is perhaps easiest to explain our work in the context of the most general Pairwise Markov Chain (PMC) model from [29]. In [29], it was only assumed that $(X, Y)$ was jointly Markov and important formula for Bayesian Maximal Posterior Mode restoration were still derived. However, when it came to parameter estimation from incomplete data it was realized that the Baum-Welch algorithm could not be generalized to this most general PMC setting and, instead, the general Iterative Conditional Estimation was resorted to. Likewise no Viterbi-like algorithm exists for finding the most likely sequence from an observed data sequence of a PMC. Our goal is to narrow the gap between the limited HMM and AR-HMM where Baum-Welch and Viterbi algorithms are known and the practically-important PMC which has no such algorithms by introducing a model that falls between the two that still has these algorithms. In particular, we will establish Baum-Welch-like and Viterbi-like algorithms for estimating (initial and transition) probabilities and the most likely sequence from the observed data for a new, still practically important model in the discrete setting. We refer to our models as Markov Observation Models (MOM).

Perhaps, the most important goals of HMM are calibrating the model, real-time belief state propagation, i.e. filtering, and decoding the whole hidden sequence from the observation sequence. The first problem is solved mathematically in the HMM setting by the Baum-Welch re-estimation algorithm, which is an application of the Expectation-Maximization (EM) algorithm, predating the EM algorithm. The filtering problem is also solved effectively using a recursive algorithm that is similar to part of the Baum-Welch algorithm. In practice, there can be numeric problems like a multitude of local maxima to trap the Baum-Welch algorithm or inefficient matrix operations when the state size is large but the hidden state resides in a small subset most of the time. In these cases, it can be advisable to use particle filters or other alternative methods, which are not the subject of this note (see instead [6] for more information). The forward and backward propagation probabilities of the Baum-Welch algorithm also tend to get very small over time. While satisfactory results can sometimes be obtained by (often logarithmic) rescaling, this is still a severe problem limiting the use of the Baum-Welch algorithm (see more explanation within). Our raw algorithms for the more general Markov observation models will also share these difficulties but as a secondary contribution we will explain how to avoid this small number problem when we give our final pseudocode so our EM algorithm will truly apply to many big data problems.

The optimal complete-observation sequence decoding problem in the HMM case is solved by the Viterbi algorithm (see [38], [31]), which is a dynamic programming type algorithm. Given the sequence of observations $\{Y_i\}_{i=1}^N$ and the model probabilities, the Viterbi algorithm returns the most likely hidden state sequence $\{X_i^{*}\}_{i=1}^N$. The Viterbi algorithm is a forward-backward algorithm like the Baum-Welch algorithm and hence computer efficient but not real time. The most natural applications of the Viterbi algorithm are perhaps speech recognition [31] and text recognition [32]. We develop a Markov observation model generalization to the Viterbi algorithm and explain how to handle the small number problem in this algorithm as well.

The HMM can be thought of as a nonlinear generalization of the earlier Kalman filter (see [17], [18]). Nonlinear filtering theory is another related generalization of the Kalman filter and has many celebrated successes like the Fujisaki-Kallianpur-Kunita and the Duncan-Mortensen-Zakai equations (see e.g. [41], [15], [23] for some of the original work and [22],
MARKOV OBSERVATION MODELS

3

[24] for some of the more recent general results). The hidden state, called signal in nonlinear filtering theory, can be a general Markov process model and live in a general state space but there is no universal EM algorithm for identifying the model like the Baum-Welch algorithm nor dynamic programming algorithm for identifying a most likely hidden state path like the Viterbi algorithm. Rather the goals are usually to compute filters, predictors and smoothers, for which there are no exact closed form solutions, except in isolated cases (see [20]), and approximations have to be used. Like HMM, nonlinear filtering has enjoyed widespread application. For instance, the subfield of nonlinear particle filtering, also known as sequential Monte Carlo, has a number of powerful algorithms (see [30], [13], [21], [8]) and has been applied to numerous problems in areas like bioinformatics [16], economics and mathematical finance [9], intracellular movement [26], fault detection [11], pharmacokinetics [4] and many other fields. Still, like HMM, the observations in nonlinear filter models are largely limited to distorted, corrupted, partial observations of the signal with very few limited exceptions like [10].

The purpose of this note is to promote a class of Markov Observation Models (MOM) that will be shown to subsume the HMM and AR-HMM models in the next section. MOM is also very different than the models considered in non-linear filtering. Hence, to the author’s knowledge, MOM represents a practically important class of models to analyze and apply to real world problems. Both the Baum-Welch and the Viterbi algorithms will be extended to these MOM models as, together with the model itself, the main contributions. A real-time filtering recursion is also extended. It should be noted that our EM and dynamic programming generalizations of the Baum-Welch and Viterbi algorithms include new methods for handling an unseen first observation that is not even part of the HMM model. Finally, the small number problem encountered in HMM and the raw MOM algorithms is resolved.

The layout of this note is as follows. In the next section, we give our model as well as our main notation. In Section 3, we apply EM techniques to derive an analog to the Baum-Welch algorithm for identifying the system (probability) parameters. In particular, joint recursive formulas for the hidden state transition probabilities, observation transition probabilities and the initial joint hidden-observation state distribution are derived. Section 4 translates these formula into a pseudocode implementation of our EM algorithm. More calculations and explanations are included to explain how we avoid the small number problem often encountered in HMM. Section 5 is devoted to connecting the limit points of the EM type algorithm to the maxima of the conditional likelihood given the observations. Section 6 contains our real-time filter process recursion and our forward-backward most likely hidden sequence detection. Specifically, it contains our dynamic programming analog to the Viterbi algorithm for MOM as well as its derivation and pseudocode implementation. Finally, Section 7 features a application of our (Baum-Welch-like) EM and our (Viterbi-like) dynamic programming algorithms on real bitcoin data to detect uptrends.

2. Model. Suppose $N$ is some positive integer (representing the final time) and $O$ is some discrete observation space. In our model, like HMM, the hidden state is a homogeneous Markov chain $X$ on some discrete (finite or countable) state space $E$ with one step transition probabilities denoted by $p_{x \rightarrow x'}$ for $x, x' \in E$. However, in contrast to HMM, we allow self dependence in the observations. (This is illustrated by right arrows between the $Y$’s in Figure 1 below.) In particular, given the hidden state $\{X_i\}_{i=0}^N$, we take the observations to be a (conditional) Markov chain $Y$ with transitions probabilities

$$P(Y_{n+1} = y | \{X_i = x_i\}_{i=0}^{n+1}, \{Y_j = y_j\}_{j=0}^n) = q_{y_n \rightarrow y}(x_{n+1}) \forall x_0, \ldots, x_N \in E; y, y_n \in O$$

that do not affect the hidden state transitions in the sense

$$P(X_{n+1} = x' | X_n = x, \{X_i\}_{i<n}, \{Y_j\}_{j \leq n}) = p_{x \rightarrow x'}, \forall x, x' \in E, n \in \mathbb{N}_0$$
which, given the hidden state parameters, we need means to estimate this initial distribution
observations to be had prior to this time. Further, since we generally do not know the model
is a MOM.

The joint Markov property then implies that
\[
P \left( X_{n+1} = x, Y_{n+1} = y \mid X_n = x_n, Y_n = y_n \right) = p_{x_n \rightarrow x} q_{y_n \rightarrow y} (x) \quad \forall x, x_n \in E; \ y, y_n \in O.
\]

Notice that this generalizes the emission probability to
\[
P (Y_n \in A \mid X_n, X_{n-1}, \ldots, X_1; Y_{n-1}, \ldots, Y_1) = P (Y_n \in A \mid Y_{n-1}, X_n) = \sum_{y \in A} q_{Y_{n-1} \rightarrow Y} (X_n)
\]
so MOM generalizes HMM by just taking \( q_{Y_{n-1} \rightarrow Y} (X_n) = b_{X_n} (y) \), a state dependent probability mass function. To see that MOM generalizes AR-HMM, we re-write (1) as
\[
\begin{bmatrix}
Y_n \\
Y_{n-1} \\
Y_{n-2} \\
\vdots \\
Y_{n-p+1} \\
\end{bmatrix}
\begin{bmatrix}
\beta_1^{(X_n)} \\
\beta_2^{(X_n)} \\
\beta_3^{(X_n)} \\
\vdots \\
\beta_p^{(X_n)}
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
Y_{n-1} \\
Y_{n-2} \\
Y_{n-3} \\
\vdots \\
Y_{n-p}
\end{bmatrix}
+ \begin{bmatrix}
\beta_0^{(X_n)} + e_n \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]
which, given the hidden state \( X_n \), gives an explicit formula for \( Y_n \) in terms of only \( Y_{n-1} \) and some independent noise \( e_n \). Hence, \( \{Y_n\} \) is obviously conditionally Markov and \( \{(X_n, Y_n)\} \) is a MOM.

A subtly that arises with our Markov Observation Model (MOM) over HMM is that we need an enlarged initial distribution since we have a \( Y_0 \) that is not observed (see Figure 1). Rather, we think of starting up the observation process at time 1 even though there were observations to be had prior to this time. Further, since we generally do not know the model parameters, we need means to estimate this initial distribution
\[ P(X_0 \in dx_0, Y_0 \in dy_0) = \mu(dx_0, dy_0). \]

It is worth noting that our model resembles the stationary PMC under Condition (H) in [29], which forces the Hidden state to be Markov by Proposition 2.2 of [29].

### 2.1. Key Notation.

- We will use the shorthand notation \( P(Y_1, \ldots, Y_n) \) for \( P(Y_1 = y_1, \ldots, Y_n = y_n) \) when \( y_1 = \ldots = y_n \).
- \( \alpha^k_n(x) = P^k(X_n = x, Y_1, \ldots, Y_n) \) and \( \beta^k_n(x) = P^k(Y_{n+1}, \ldots, Y_N | X_n = x, Y_n) \) (both defined differently when \( n = 0 \) below) are probabilities computed using the current estimates \( p^k_{x \rightarrow x'}, q^k_{y \rightarrow y'}(x) \) and \( \mu(x, y) \) of the transition and initial probabilities. \( \alpha^k_n(x) \) and \( \beta^k_n(x) \) will be key variables in the forward respectively backward propagation step of our raw Baum-Welch-like EM algorithm for estimating the transition and initial probabilities. For notational ease, we will drop the fact \( P \) depends on \( k \) hereafter.
- The filter \( \pi^k_n(x) = P(X_n = x | Y_1, \ldots, Y_n) \) and \( \chi^k_n(x) = \beta^k_n(x) P(Y_1, \ldots, Y_n) \) are used in our refined Baum-Welch-like algorithm to replace \( \alpha^k_n \) respectively \( \beta^k_n \) of the raw algorithm in order to solve the small number problem discussed below. Whereas \( \alpha^k_n(x) \beta^k_n(\xi) \) is often the product of two tiny, unqiely sized factors, \( \pi^k_n \) and \( \chi^k_n \) are scaled to always be manageable factors. Yet, \( \pi^k_n(x) \chi^k_n(\xi) = \frac{\alpha^k_n(x) \beta^k_n(\xi)}{P(Y_1, \ldots, Y_{n-1})} \) and both \( \pi^k_n \) and \( \chi^k_n \) satisfy nice forward and backward recursions so they are efficient to compute and our refined EM algorithm for MOM is efficient and avoids the small number problem.
- \( \delta_n(x) = \max_{y_0;X_0,x_0,\ldots,x_{n-1}} P(Y_0 = y_0; X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}; X_n = x; Y_1, \ldots, Y_n) \) is the key internal function in our Viterbi-like dynamic programming algorithm for determining the most likely sequence of hidden states. \( \delta_n \) also suffers from the small number problem as it tends to get ridiculously small as \( n \) increases. However, since there is only one factor it is easy to scale and scaling each \( \delta_n \) does not affect Viterbi-like algorithm, we can replace \( \delta_n \) with a properly scaled version \( \gamma_n \) below.

### 3. Probability Estimation via EM algorithm.

In this section, we develop a recursive expectation-maximum algorithm that can be used to create convergent estimates for the transition and initial probabilities of our MOM models. We leave the theoretical justification of convergence to Section 5.

The main goal of developing an EM algorithm would be to find \( p_{x \rightarrow x'} \) for all \( x, x' \in E \), \( q_{y \rightarrow y'}(x) \) for all \( y, y' \in O, x \in E \) and \( \mu(x, y) \) for all \( x \in E, y \in O \). Noting every time step is considered to be a transition in a discrete-time Markov chain, we would ideally set:

\[
\begin{align*}
(6) \quad p_{x \rightarrow x'} &= \frac{\text{transitions } x \text{ to } x'}{\text{occurrences of } x} \\
(7) \quad q_{y \rightarrow y'}(x) &= \frac{\text{transitions } y \text{ to } y' \text{ when } x \text{ is true}}{\text{occurrences of } y \text{ when } x \text{ is true}}.
\end{align*}
\]

Here, ‘when \( x \) is true’ means when the hidden state is in state \( x \). However, we can never see \( x \) nor \( x' \) in MOM from our data so we must estimate when they are true. Hence, we replace the above with

\[
(8) \quad p_{x \rightarrow x'} = \frac{\text{Expected transitions } x \text{ to } x'}{\text{Expected occurrences of } x} = \frac{\sum_{n=1}^{N} P(X_{n-1} = x, X_n = x' | Y_1, \ldots, Y_N)}{\sum_{n=1}^{N} P(X_{n-1} = x | Y_1, \ldots, Y_N)}
\]
(9) \[ q_{y \rightarrow y'}(x) = \frac{\text{Expected transitions } y \text{ to } y' \text{ when } x \text{ is true}}{\text{Expected occurrences of } y \text{ when } x \text{ is true}} \]

\[
1_{Y_1 = y'} P(Y_0 = y, X_1 = x \mid Y_1, \ldots, Y_N) + \sum_{n=2}^{N} 1_{Y_{n-1} = y, Y_n = y} P(X_n = x \mid Y_1, \ldots, Y_N)
\]

\[
P(Y_0 = y, X_1 = x \mid Y_1, \ldots, Y_N) + \sum_{n=2}^{N} 1_{Y_{n-1} = y} P(X_n = x \mid Y_1, \ldots, Y_N)
\]

which means we must compute \( P(Y_0 = y, X_1 = x \mid Y_1, \ldots, Y_N) \), \( P(X_n = x \mid Y_1, \ldots, Y_N) \) for all \( 0 \leq n \leq N \) and \( P(X_{n-1} = x, X_n = x' \mid Y_1, \ldots, Y_N) \) for all \( 1 \leq n \leq N \) to get these two transition probability estimates. However, let

(10) \[
\begin{align*}
\alpha_0(x, y) &= P(Y_0 = y, X_0 = x) \\
\alpha_n(x) &= P(Y_1, \ldots, Y_n, X_n = x), 1 \leq n \leq N
\end{align*}
\]

and

(11) \[
\left\{ \begin{array}{ll}
\beta_0(x_1, y) &= P(Y_1, \ldots, Y_N \mid X_1 = x_1, Y_0 = y) \\
\beta_n(x_{n+1}) &= P(Y_{n+1}, \ldots, Y_N \mid X_{n+1} = x_{n+1}, Y_n) \quad \forall 0 < n < N - 1 \\
\beta_{N-1}(x_N) &= P(Y_N \mid X_N = x_N, Y_{N-1}) = q_{Y_{N-1} \rightarrow Y_N}(x_N)
\end{array} \right.
\]

Notice we include an extra variable \( y \) in \( \alpha_0, \beta_0 \). This is because we do not see the first observation \( Y_0 \) so we have to consider all possibilities and treat it like another hidden state. Now, by Bayes’ rule, (11) and (10)

(12) \[
P(Y_0 = y, X_1 = x \mid Y_1, \ldots, Y_N)
\]

\[
= P(Y_1, \ldots, Y_N \mid X_1 = x, Y_0 = y) P(X_1 = x, Y_0 = y)
\]

\[
= \frac{\beta_0(x, y) \sum_{x_0} p_{x_0 \rightarrow x} \alpha_0(x_0, y)}{\sum_{\xi} \alpha_N(\xi)}
\]

Next, by the Markov property and (10)

(13) \[
P(X_{n-1} = x, X_n = x' \mid Y_1, \ldots, Y_N)
\]

\[
= P(X_{n-1} = x, X_n = x', Y_1, \ldots, Y_N)
\]

\[
= \frac{\alpha_{n-1}(x) P(X_n = x', Y_n, \ldots, Y_N \mid X_{n-1} = x, Y_1, \ldots, Y_{n-1})}{P(Y_1, \ldots, Y_N)}
\]

\[
= \frac{\alpha_{n-1}(x) P(X_n = x', Y_n, \ldots, Y_N \mid X_{n-1} = x, Y_{n-1})}{P(Y_1, \ldots, Y_N)}
\]

so by (3, 4, 11, 10)

(14) \[
P(X_{n-1} = x, X_n = x' \mid Y_1, \ldots, Y_N)
\]

\[
= \frac{\alpha_{n-1}(x) P(X_n = x', Y_n, \ldots, Y_N, X_{n-1} = x, Y_{n-1}) P(X_n = x', X_{n-1} = x, Y_{n-1})}{P(Y_1, \ldots, Y_N) P(X_n = x', X_{n-1} = x, Y_{n-1}) P(X_{n-1} = x, Y_{n-1})}
\]
Our new strategy is to define
\[ P(X_{n-1} = x, X_n = x', Y_1, \ldots, Y_N) = \frac{\alpha_{n-1}(x) p_{x \rightarrow x' \mid X_n = x', Y_n-1} \alpha_{n-1}(x')}{\sum_{\xi} \alpha_N(\xi)} \]
for \( n = 2, 3, \ldots, N \).

**Remark 3.1.** The Baum-Welch algorithm for regular HMM also constructs the joint conditional probability in (14). In the HMM case, the numerator in (14) looks like
\[ P(X_{n-1} = x, X_n = x', Y_1, \ldots, Y_N) = \frac{\alpha_{n-1}(x) \beta_{n-1}(x')}{\sum_{\xi} \alpha_N(\xi)} \]
which works well when the observations are conditionally independent. However, this multiplication rule does not apply in our more general Markov observations case. Moreover, there is no conditional independence so
\[ P(X_n = x', Y_n, \ldots, Y_N \mid X_{n-1} = x, Y_{n-1}) \neq P(X_n = x' \mid X_{n-1} = x, Y_{n-1}) P(Y_n, \ldots, Y_N \mid X_{n-1} = x, Y_{n-1}). \]

Our new strategy is to define
\[ \beta_{n-1}(x') = P(Y_n, \ldots, Y_N \mid X_n = x', Y_{n-1}) \]
and note that
\[ \beta_{n-1}(x') = P(Y_n, \ldots, Y_N \mid X_n = x', X_{n-1} = x, Y_{n-1}) \]
for all \( x \). Surprisingly, with such modest changes, the algorithms making HMM such a powerful tool translate to the more general MOM models.

It follows from (14) that
\[ P(X_n = x \mid Y_1, \ldots, Y_N) = \alpha_n(x) \sum_{x_{n+1}} \frac{\beta_n(x_{n+1} \mid x, Y_n) p_{x \rightarrow x_{n+1}}}{\sum_{\xi} \alpha_N(\xi)} \]
for \( n = 1, 2, \ldots, N - 1 \) and
\[ P(X_n = x \mid Y_1, \ldots, Y_N) = \beta_{n-1}(x) \sum_{x_{n-1}} \frac{p_{x_{n-1} \rightarrow x \mid x, Y_{n-1}} \alpha_{n-1}(x_{n-1})}{\sum_{\xi} \alpha_N(\xi)} \]
for \( n = 2, 3, \ldots, N \). Similarly to (13,14), one has that
\[ P(X_0 = x, X_1 = x' \mid Y_1, \ldots, Y_N) \]
\[ = \frac{\sum_y P(X_0 = x, Y_0 = y) P(X_1 = x'; Y_1, \ldots, Y_N \mid X_0 = x, Y_0 = y)}{P(Y_1, \ldots, Y_N)} \]
\[ = \frac{\sum_y \alpha_0(x, y) p_{x \rightarrow x'} \beta_0(x', y)}{\sum_{\xi} \alpha_N(\xi)} \]
and so

\begin{equation}
\sum_{y} \alpha_0(x, y)p_{y \rightarrow x} \beta_0(x', y) / \sum_{\xi} \alpha_N(\xi).
\end{equation}

\alpha_n \text{ and } \beta_n \text{ are computed recursively below using the prior estimates of } p_{x \rightarrow x'}, q_{y \rightarrow y'}(x) \text{ and } \mu.

Recalling that there are prior observations that we do not see, we must also estimate an initial joint distribution for an initial hidden state and observation. An expectation-maximization argument for the initial distribution leads one to the assignment

\begin{equation}
(19) \quad \mu(x, y) = P(X_0 = x, Y_0 = y | Y_1, ..., Y_N)
\end{equation}

\begin{equation}
= \frac{P(Y_1, ..., Y_N | X_0 = x, Y_0 = y)P(X_0 = x, Y_0 = y)}{P(Y_1, ..., Y_N)}
\end{equation}

for all \( x \in E, y \in O \), which is Bayes’ rule.

Expectation-maximization algorithms use these types of formula and prior estimates to produce better estimates. We take estimates for \( p_{x \rightarrow x'}, q_{y \rightarrow y'}(x) \) and \( \mu(x, y) \) and get new estimates for these quantities iteratively using (8), (17), (14), (18) and (15):

\begin{equation}
(20) \quad p_{x \rightarrow x'} = \frac{\sum_{y} \alpha_0(x, y)p_{x \rightarrow x} \beta_0(x', y) + \sum_{n=1}^{N-1} \alpha_n(x)p_{x \rightarrow x} \beta_n(x')}{\sum_{y} \sum_{x_1} \alpha_0(x, y)p_{x \rightarrow x_1} \beta_0(x_1, y) + \sum_{n=1}^{N-1} \sum_{x_{n+1}} p_{x \rightarrow x_{n+1}} \beta_n(x_{n+1}) \alpha_n(x)}
\end{equation}

then using (9), (12,16)

\begin{equation}
(21) \quad q_{y \rightarrow y'}(x) = \frac{1_{\xi = y'} \beta_0(x, y) \sum_{\xi} p_{\xi \rightarrow x} \alpha_0(\xi, y) + \sum_{n=1}^{N-1} 1_{\xi = y, y_{n+1} = y'} \beta_n(x) \sum_{\xi} \alpha_n(\xi)p_{\xi \rightarrow x}}{\beta_0(x, y) \sum_{\xi} p_{\xi \rightarrow x} \alpha_0(\xi, y) + \sum_{n=1}^{N-1} \beta_n(x) \sum_{\xi} \alpha_n(\xi)p_{\xi \rightarrow x}}
\end{equation}

and using (19)

\begin{equation}
(22) \mu'(x, y) = \frac{\sum_{x_1} P(Y_1, ..., Y_N | X_1 = x_1, X_0 = x, Y_0 = y)P(X_1 = x_1 | X_0 = x, Y_0 = y)\mu(x, y)}{P(Y_1, ..., Y_N)}
\end{equation}

\begin{equation}
= \frac{\sum_{x_1} \beta_0(x_1, y)p_{x \rightarrow x_1} \mu(x, y)}{\sum_{\xi} \alpha_N(\xi)}.
\end{equation}

\textbf{Remark 3.2.} 1) Different iterations of \( p_{x \rightarrow \xi}, \mu(x, y) \) will be used on the left and right hand sides of (20,22). The new estimates on the left are denoted \( p'_{x \rightarrow \xi}, \mu'(x, y) \). Moreover, \( \alpha_N \) also depends on (the earlier iteration of) \( \mu \) so the equation is not linear. It should be thought of as a Bayes’ rule with the \( \mu \) on the right being a prior (to incorporating the observations with the current set of parameters) and the one on the left being a posterior.

2) Setting a \( p_{x \rightarrow \xi} = 0 \) or \( \mu(x, y) = 0 \) will result in it staying zero for all updates. This effectively removes this parameter from the EM optimization update and should be avoided unless it is known that one of these should be 0.
3) If there is no successive observations with \(Y_n = y\) and \(Y_{n+1} = y'\) in the actual observation sequence, then all new estimates \(q'_{y \rightarrow y'}(x)\) will either be set to 0 or close to it. They might not be exactly zero due to the first term in the numerator of (21) where we could have an estimate of \(Y_0 = y\) and an observed \(Y_1 = y'\).

Naturally, our solution degenerates to the Baum-Welch algorithm in the HMM case. However, the extra Markov component of MOM complicates this algorithm and its derivation. We start with \(\alpha\), which is the most similar to HMM. Here, we have by the joint Markov property and (10) that:

\[
\alpha_n(x) = P(Y_1, \ldots, Y_n, X_n = x) = \sum_{x_{n-1}} P(Y_1, \ldots, Y_{n-1}, X_{n-1} = x_{n-1}, X_n = x) = \sum_{x_{n-1}} P(Y_1, \ldots, Y_{n-1}, X_{n-1} = x_{n-1}) P\left(X_n = x, Y_n \mid Y_1, \ldots, Y_{n-1}, X_{n-1} = x_{n-1}\right) = q_{Y_{n-1} \rightarrow Y_n}(x) \sum_{x_{n-1}} \alpha_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x},
\]

which can be solved forward for \(n = 2, 3, \ldots, N - 1, N\), starting at

\[
\alpha_1(x_1) = \sum_{x_0} \sum_{y_0} \mu(x_0, y_0) p_{x_0 \rightarrow x_1} q_{y_0 \rightarrow Y_1}(x_1).
\]

Recall \(\alpha_0 = \mu\) is assigned differently.

Our iterative estimates for \(p_{x \rightarrow x'}, q_{y \rightarrow y'}(x)\) and \(\mu(x, y)\) also rely on the second (backward) recursion for \(\beta_n\). It also follows from the Markov property, our transition probabilities and (3, 4) that:

\[
\beta_n(x) = P\left(Y_{n+1}, \ldots, Y_N \mid X_{n+1} = x, Y_n\right) = P\left(Y_{n+2}, \ldots, Y_N \mid X_{n+1} = x, Y_{n+1}, Y_n\right) = \sum_{x' \in E} P\left(Y_{n+2}, \ldots, Y_N \mid X_{n+2} = x', X_{n+1} = x, Y_{n+1}\right) \beta_{n+1}(x') = \sum_{x'} \beta_{n+1}(x') p_{x \rightarrow x'} q_{y \rightarrow Y_{n+1}}(x),
\]

which can be solved backward for \(n = N - 2, N - 3, \ldots, 3, 2, 1, 0\), starting from

\[
\beta_{N-1}(x) = P(Y_N \mid X_N = x, Y_{N-1}) = q_{Y_{N-1} \rightarrow Y_N}(x).
\]

It is worth noting that when we use \(\beta_n\) with \(n = 0\) we will have \(Y_0 = y\) is some fixed value of interest not the missed observation that we never see and we use the notation \(\beta_0(x_0, y_0)\). We only see \(Y_1, Y_2, \ldots, Y_N\).

We now have everything required for our algorithm, which is given in Algorithm 1 in Section 4.
To be able to show convergence in Section 5, we need to track when parameters could become 0. The following lemma follows immediately from (23), (24), induction and the fact that \( \sum_{x'} p_{x \to x'} = 1 \). Any sensible initialization of our EM algorithm would ensure the condition \( q_{Y_n \to Y_{n+1}}(x) > 0 \) holds.

**Lemma 3.3.** Suppose \( q_{Y_n \to Y_{n+1}}(x) > 0 \) for all \( x \in E \) and \( n \in \{1, ..., N-1\} \). Then,

1. \( \beta_m(x) > 0 \) for all \( x \in E \) and \( m \in \{1, ..., N-1\} \).
2. \( \beta_0(x, y) > 0 \) for any \( x \in E, y \in O \) such that \( q_{y \to Y_1}(x) > 0 \).
3. \( \alpha_m(x) > 0 \) for all \( x \in E \) and \( m \in \{1, ..., N\} \) if both \( \sum_{x'} p_{x' \to x} > 0 \) and \( \sum_{y_0} \mu(x_0, y_0) q_{y_0 \to Y_1}(x) > 0 \) for all \( x, x_0 \in E \).
4. \( \alpha_0(x, y) > 0 \) if \( \mu(x, y) > 0 \).

Notice the condition \( \sum_{x'} p_{x' \to x} > 0 \) for all \( x \) says that any hidden state can be reached from at least one other state while \( \sum_{y_0} \mu(x_0, y_0) q_{y_0 \to Y_1}(x) > 0 \) for all \( x, x_0 \) ensures that all the initial hidden states are meaningful. The following result is the key to ensuring that our non-zero parameters stay non-zero. It follows from the prior lemma as well as (20, 21, 22, 24).

**Lemma 3.4.** Suppose \( N \geq 2 \), \( q_{Y_n \to Y_{n+1}}(x) > 0 \) for all \( x \in E \) and \( n \in \{1, ..., N-1\} \), \( \sum_{x'} p_{x' \to x} > 0 \) for all \( x \in E \) and \( \sum_{y_0} \mu(x_0, y_0) q_{y_0 \to Y_1}(x) > 0 \) for all \( x, x_0 \in E \). Then,

1. \( p'_{x \to x'} > 0 \) if and only if \( p_{x \to x'} > 0 \) for any \( x, x' \).
2. \( q_{y \to y'}(x) > 0 \) for all \( x \in E \) if either \( \sum_{n=1}^{N-1} 1_{Y_n=y, Y_{n+1}=y'} > 0 \) or \( \mu(\xi, y_1)y_{1=n} q_{y \to Y_1}(\xi) > 0 \) for all \( \xi \in E \).
3. \( \mu'(x, y) > 0 \) if \( \mu(x, y) > 0 \) and \( q_{y \to Y_1}(\xi) > 0 \) for all \( \xi \in E \). \( \mu'(x, y) = 0 \) if \( q_{y \to Y_1}(\xi) = 0 \) for all \( \xi \in E \).

The algorithm; given explicitly in Section 4; starts with initial estimates of all \( p_{x \to x'}, q_{y \to y'}(x), \mu(x, y); \) say \( p_{x \to x'}, q_{y \to y'}(x), \mu^1(x, y); \) and uses the formula for \( p_{x \to x'}, q_{y \to y'}(x), \mu^1(x, y) \) to refine these estimates successively to the next estimates \( p_{x \to x'}, q_{y \to y'}(x), \mu^2(x, y); \) \( p_{x \to x'}, q_{y \to y'}(x), \mu^3(x, y); \) etc. It is important to know that our estimates \( \{p_{x \to x'}, q_{y \to y'}(x), \mu^k(x, y)\} \) are getting better as \( k \to \infty \). Lemma 3.4 will be used in some cases to ensure that an initially positive parameter stays positive as \( k \) increases, which important in our proofs to follow.

### 4. EM Algorithm and Small Number Problem.

The raw algorithm that we have considered hitherto computes \( \alpha_n \) and \( \beta_n \) recursively. By their definitions,

\[
\alpha_n(x) = P(Y_1, ..., Y_n, X_n = x)
\]

\[
\beta_n(x) = P(Y_{n+1}, ..., Y_N | X_{n+1} = x, Y_n)
\]

both can get extremely small when \( N \) is large. In this case, \( \alpha_1(x) \) would be a reasonable number as it is just a probability of the event \( \{Y_1 = Y_1, X_0 = x\} \). However, \( \beta_1(x) \) would be a conditional probability of an exact occurrence of \( Y_2, ..., Y_N \), which would usually be extraordinarily small. Conversely, \( \alpha_N(x) \) would usually be extraordinarily small and \( \beta_N(x) \) may be a reasonable number. In between, the product \( \alpha_n(x) \beta_n(x) \) would usually be extraordinarily small. The unfortunate side-effect of this is that our \( p_{x \to x'} \) (and \( q \) calculations are basically
going to result in zero over zero most of the time when a computer is employed. We need a fix.

This small number problem is resolved by using the filter instead of $\alpha$. Observe that the filter

$$\pi_n(x) = P(X_n = x|Y_1, \ldots, Y_n) = \frac{\alpha_n(x)}{\sum_{\xi} \alpha_n(\xi)}$$

is a (conditional) probability of a single event regardless of $n$. Hence, it does not necessarily get extraordinarily small. However, scaling $\alpha_n$ in a manner depending upon $n$ means we will have to scale $\beta_n$ as well in a counteracting way. The idea is to note that $\alpha_n(x)\beta_n(x)$ appear together in computing the $p'_{x \rightarrow x'}$ and $q'_{y \rightarrow y'}(x)$ in such a way that we can divide every $\alpha_n(x)\beta_n(x)$ by the same small number without changing the values of the $p$’s and $q$’s. Specifically, we replace

$$\alpha_n(x)\beta_n(x) \Rightarrow \frac{\alpha_n(x)\beta_n(x)}{a_1a_2 \cdots a_{N-1}} = \pi_n(x) \chi_n(x), \forall n \in \{1, \ldots, N - 1\},$$

where $\pi_n(x)$ is the filter and $\chi_n(x) = \frac{\beta_n(x)}{a_{n+1} \cdots a_{N-1}}, a_1, \ldots, a_N$ are normalizing constants and $\alpha_n(x, y)\beta_n(x, y)$ is scaled similarly. Using (23,24), one finds the recursions for $\pi$ and $\chi$ are:

$$\rho_n(x) = q_{Y_{n-1} \rightarrow Y_n}(x) \sum_{x_{n-1}} \pi_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x},$$

$$\pi_n(x) = \frac{\rho_n(x)}{a_n}, a_n = \sum_x \rho_n(x_n),$$

which can be solved forward for $n = 2, 3, \ldots, N - 1, N$, starting at

$$\pi_1(x) = \frac{\sum \sum \mu(x_0, y_0) p_{x_0 \rightarrow x} q_{y_0 \rightarrow Y_1}(x_1)}{a_1}, a_1 = \sum_x \sum_{y_0} \mu(x_0, y_0) p_{x_0 \rightarrow x_1} q_{y_0 \rightarrow Y_1}(x_1).$$

Like $\beta$, $\chi$ is a backward recursion starting from

$$\chi_{N-1}(x) = P(Y_N|x_N = x, Y_{N-1}) = q_{Y_{N-1} \rightarrow Y_N}(x)$$

and then continuing as

$$\chi_n(x) = \frac{q_{Y_n \rightarrow Y_{n+1}}(x)}{a_{n+1}} \sum_{x' \rightarrow x} \chi_{n+1}(x') p_{x \rightarrow x'},$$

which can be solved backward for $n = N - 2, N - 3, \ldots, 3, 2, 1$.

Finally, the $n = 0$ value for $\pi$ and $\chi$ become

$$\chi_0(x, y) = \sum_{x'} \frac{\chi_1(x')}{a_1} p_{x \rightarrow x'} q_{y \rightarrow Y_1}(x'),$$

$$\pi_0(x, y) = \alpha_0(x, y) = \mu(x, y).$$

The adjusted, non-raw algorithm is given in Algorithm 1

**Note:** In the three probability ($p, q, \mu$) update steps of Algorithm 1, it usually better from numeric and performance perspectives to compute the numerators and then use the facts that they must be probability mass functions to properly normalize rather than use the full equation as given.
Algorithm 1: EM algorithm for MOM

Data: Observation sequence: $Y_1, ..., Y_N$
Input: Initial Estimates: $\{p_{x \rightarrow x'}\}, \{q_{y \rightarrow y'}(x)\}, \{\mu(x, y)\}$
Output: Final Estimates: $\{p_{x \rightarrow x'}\}, \{q_{y \rightarrow y'}(x)\}, \{\mu(x, y)\}$ // Characterize MOM models

/* Initialization. */

while $p, q$, and $\mu$ have not converged do
  /* Forward propagation. */
  $p_0(x, y) = \mu(x, y) \forall x \in E; y \in O;$
  $p_1(x) = \sum_{x_0 \in E, y_0 \in O} \mu(x_0, y_0) p_{x_0 \rightarrow x} q_{y_0 \rightarrow Y_1} (x) \forall x \in E;$
  $a_1 = \sum_x p_1(x)$
  $p_1(x) = p_1(x) / a_1.$

for $n = 2, 3, ..., N$ do
  $\rho_n(x) = q_{Y_n \rightarrow Y_n}(x) \sum_{x_{n-1} \in E} \pi_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x} \forall x \in E.$
  $a_n = \sum_x \rho_n(x).$
  $\pi_n(x) = \rho_n(x) / a_n.$

/* Backward propagation. */

$\chi_{N-1}(x) = q_{Y_{N-1} \rightarrow Y_{N-1}}(x) \forall x \in E.$

for $n = N - 2, N - 3, ..., 1$ do
  $\chi_n(x) = \chi_{n+1}(x) \sum_{x' \in E} \chi_{n+1}(x') p_{x' \rightarrow x} \forall x \in E.$
  $\chi_0(x, y) = q_{y \rightarrow Y_1(x)} / a_1 \sum_{x' \in E} \chi_1(x') p_{x' \rightarrow x} \forall x \in E, y \in O.$

/* Probability Update. */

$q_{y \rightarrow y'}(x) = \sum_\xi p_{\xi \rightarrow x} \left[ 1_{Y_1= y'} \chi_0(x, y) \pi_0(\xi, y) + \sum_{n=1}^{N-1} 1_{Y_n= y, Y_{n+1}= y'} \chi_n(x) \pi_n(\xi) \right]$

$\forall x \in E; y, y' \in O.$

$\mu(x, y) = \sum_\xi \sum_\theta \mu(\xi, \theta) \sum_{x_1} \chi_0(x_1, y) p_{x_1 \rightarrow x_1} \forall x \in E; y \in O.$

$p_{x \rightarrow x'} = \sum_{y} p_{x, y} \chi_0(x, y) p_{x \rightarrow x'} \forall x, x' \in E.$

/* Initial Estimates */

5. Convergence of Probabilities. In this section, we establish the convergence properties of the transition probabilities and initial distribution $\{p_{x \rightarrow x'}, q_{y \rightarrow y'}(x), \mu(x, y)\}$ that we derived in Section 3. Our method adapts the ideas of Baum et. al. [3], Liporace [25] and Wu [39] to our setting.

We think of the transition probabilities and initial distribution as parameters, and let $\Theta$ denote all of the non-zero transition and initial distribution probabilities in $p, q, \mu$. Let $e = |E|$ and $o = |O|$ be the cardinalities of the hidden and observation spaces. Then, the whole parameter space has cardinality $d' = e^2 + e \cdot o^2 + e \cdot o$ for the $p_{x \rightarrow x'}$ plus $q_{y \rightarrow y'}(x)$ plus $\mu(x, y)$ and lives on $[0, 1]^{d'}$. However, we are removing the values that will be set to zero and adding sum to one constraints to consider a constrained optimization problem on $(0, \infty)^d$.
for some \(d \leq d'\). Removing these zero possibilities gives us necessary regularity for our re-estimation procedure. However, it was not enough to just remove them at the beginning. We had to ensure that zero parameters did not creep in during our iterations or else we will be doing such things as taking logarithms of 0. Lemma 3.4 suggests a strategy for initially assigning estimates so zeros will not occur in later estimates in the case that the value of \(Y_1\) also appears later in the observation sequence.

1. Pick initial estimate \(\{p^1_{x \rightarrow x'}\} \) such that \( \sum_x p^1_{x \rightarrow x'} > 0 \) for all \(x'\). This says that any hidden state can be reached from somewhere. From above we know \( p^1_{x \rightarrow x'} \rightarrow p^1_{x' \rightarrow x'} \) for all \(k\) so \( \sum_x p^k_{x \rightarrow x'} > 0 \) for all \(x'\).

2. Pick \( q^1_{y \rightarrow y'}(x) > 0 \) for all \(x \in E\) if and only if \( \sum_{n=1}^{N-1} 1_{Y_n=y, Y_{n+1}=y'} > 0 \).

3. Pick \( \mu^1(x,y) > 0 \) if and only if \( q^1_{y \rightarrow y'}(x) > 0 \). Here you are using the values you just picked in the previous step to make this decision.

This will produce an example of a zero separating sequence in the case the value of \(Y_1\) is repeated as at least one \(Y_n\) with \(n > 1\).

**Definition 5.1.** A sequence of estimates \(\{p^k, q^k, \mu^k\}\) is zero separating if:

1. \( p^1_{x \rightarrow x'} > 0 \) iff \( p^1_{x' \rightarrow x'} > 0 \) for all \(k = 1, 2, 3, \ldots\)
2. \( q^1_{y \rightarrow y'}(x) > 0 \) iff \( q^1_{y \rightarrow y'}(x) > 0 \) for all \(k = 1, 2, 3, \ldots\), and
3. \( \mu^1(x,y) > 0 \) iff \( \mu^1(x,y) > 0 \) for all \(k = 1, 2, 3, \ldots\)

Here, iff stands for if and only if.

This means that we can potentially optimize over \(p, q, \mu\) that we initially do not set to zero. Henceforth, we factor the zero \(p, \mu, q\) out of \(\Theta\), consider \(\Theta \subset (0, \infty)^d\) with \(d \leq d'\) and define the parameterized mass functions

\[
(31) \quad p_{y_0, y_1 \ldots y_N}(x; \Theta) = p_{x_0 \rightarrow x_1} q_{y_0 \rightarrow y_1}(x_1) p_{x_1 \rightarrow x_2} q_{y_1 \rightarrow y_2}(x_2) \cdots p_{x_{N-1} \rightarrow x_N} q_{y_{N-1} \rightarrow y_N}(x_N) \mu(x_0, y_0)
\]

in terms of the non-zero values only. The observable likelihood

\[
(32) \quad P_{Y_1 \ldots Y_N}(\Theta) = \sum_{x_0, x_1 \ldots x_N} \sum_{y_0} p_{y_0, Y_1 \ldots Y_N}(x_0, x_1, \ldots, x_N; \Theta)
\]

is not changed by removing the zero values of \(p, \mu, q\) and this removal allows us to define the re-estimation function

\[
(33) \quad Q_{Y_1 \ldots Y_N}(\Theta', \Theta) = \sum_{x_0, \ldots, x_N} \sum_{y_0} p_{y_0, Y_1 \ldots Y_N}(x_0, \ldots, x_N; \Theta) \ln p_{y_0, Y_1 \ldots Y_N}(x_0, \ldots, x_N; \Theta').
\]

**Note:** Here and in the sequel, the summation in \(P, Q\) above are only over the non-zero combinations. We would not include an \(x_i, x_{i+1}\) pair where \(p_{x_i \rightarrow x_{i+1}} = 0\) nor an \(x_0, y_0\) pair where \(\mu(x_0, y_0) = 0\). Hence, our parameter space is

\[
\Gamma = \{\Theta \in (0, \infty)^d : \sum_{x' \rightarrow x''} p_{x \rightarrow x'} = 1, \sum_{y \rightarrow y'} q_{y \rightarrow y'}(x) = 1 \forall x, \sum_{x,y} \mu(x, y) = 1\}.
\]

Later, we will consider the extended parameter space

\[
K = \{\Theta \in [0, 1]^d : \sum_{x' \rightarrow x''} p_{x \rightarrow x'} = 1, \sum_{y \rightarrow y'} q_{y \rightarrow y'}(x) = 1 \forall x, \sum_{x,y} \mu(x, y) = 1\}
\]
as limit points. Note: In both \( \Gamma \) and \( \Theta \), \( \Theta \) is only over the \( p_{x \rightarrow x'} \), \( q_{y \rightarrow y'}(x) \) and \( \mu(x, y) \) that are not just set to 0 (before limits).

Then, equating \( Y_0 \) with \( y_0 \) to ease notation, one has that

\[
Q(\Theta, \Theta') = \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow Y_n}(x_n) \right] \mu(x_0, y_0)
\]

\[
\left[ \sum_{m=1}^{N} \{ \ln p'_{x_{m-1} \rightarrow x_m} + \ln q'_{y_{m-1} \rightarrow Y_m}(x_m) \} + \ln \mu'(x_0, y_0) \right].
\]

The re-estimation function will be used to interpret the EM algorithm we derived earlier. We impose the following condition to ensure everything is well defined.

(Zero) The EM estimates are zero separating.

The following result that is motivated by Theorem 3 of Liporace [25].

**Theorem 5.2.** Suppose (Zero) holds. The expectation-maximization solutions (20, 21, 22) derived in Section 3 are the unique critical point of the re-estimation function \( \Theta' \rightarrow Q(\Theta, \Theta') \), subject to \( \Theta' \) forming probability mass functions. This critical point is a maximum taking value in \((0, 1]^d\) for \( d \) explained above.

We consider it as an optimization problem over the open set \((0, \infty)^d\) but with the constraint that we have mass functions so the values have to be in the set \((0, 1]^d\).

**Proof.** One has by (34) as well as the constraint \( \sum_{x'} p'_{x \rightarrow x'} = 1 \) that the maximum must satisfy

\[
0 = \frac{\partial}{\partial p'_{x \rightarrow x'}} \left\{ Q(\Theta, \Theta') - \lambda \left( \sum_{\xi} p'_{x \rightarrow \xi} - 1 \right) \right\}
\]

\[
= \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow Y_n}(x_n) \right] \sum_{m=1}^{N} \frac{1_{x_{m-1} = x} 1_{x_m = x'}}{p'_{x \rightarrow x'}} \mu(x_0, y_0) - \lambda
\]

where \( \lambda \) is a Lagrange multiplier. Multiplying by \( p'_{x \rightarrow x'} \), summing over \( x' \) and then using the Markov property as well as the argument in (14,15), one has that

\[
\lambda = \sum_{m=1}^{N} \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow Y_n}(x_n) \right] 1_{x_{m-1} = x} \mu(x_0, y_0)
\]

\[
= \sum_{m=1}^{N} P(X_{m-1} = x, Y_1, \ldots, Y_N)
\]

\[
= \sum_{y} \sum_{x_1} \beta_0(y_1, y)p_{x \rightarrow x_1} \alpha_0(x, y) + \sum_{m=2}^{N} \sum_{x_m} \beta_{m-1}(x_m)p_{x \rightarrow x_m} \alpha_{m-1}(x).
\]

Substituting (36) into (35), one has by the Markov property that

\[
p'_{x \rightarrow x'} = \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow Y_n}(x_n) \right] \sum_{m=1}^{N} \frac{1_{x_{m-1} = x} 1_{x_m = x'}}{\lambda} \mu(x_0, y_0)
\]
Clearly, the value on the far right of (37) is in \((0,1]\) (since we assumed \(p_{x\to x'} > 0\)). Similarly,

\[
(38) 0 = \frac{\partial}{\partial q_{y\to y'}(x)} \left\{ Q(\Theta, \Theta') - \lambda \left( \sum_{\theta \in O} q_{y\to \theta}(x) - 1 \right) \right\}
\]

\[
= \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \to x_n} q_{Y_{n-1} \to Y_{n}}(x_n) \right] \sum_{m=1}^{N} 1_{Y_{m-1} = y} 1_{Y_{m} = y'} 1_{X_{m} = x} \frac{\mu(x_0, y_0)}{q_{y\to y'}(x)} - \lambda,
\]

where \(\lambda\) is a Lagrange multiplier. Multiplying by \(q_{y\to y'}(x)\), summing over \(y'\) and then using the Markov property as well as the argument in (14,16), one has that

\[
(39) \quad \lambda = \sum_{m=1}^{N} \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \to x_n} q_{Y_{n-1} \to Y_{n}}(x_n) \right] 1_{Y_{m-1} = y} 1_{x_m = x} \mu(x_0, y_0)
\]

\[
= P(Y_0 = y, X_1 = x, Y_1, \ldots, Y_N) + \sum_{m=2}^{N} 1_{Y_{m-1} = y} P(X_m = x, Y_1, \ldots, Y_N)
\]

\[
= \beta_0(x, y) \sum_{x_0} p_{x_0 \to x} \mu(x_0, y) + \sum_{n=1}^{N-1} 1_{Y_{n} = y} \beta_n(x) \sum_{x_n} p_{x_n \to x} \alpha_n(x_n).
\]

Substituting (39) into (38), one has that

\[
(40) \quad q_{y\to y'}(x)
\]

\[
P(Y_0 = y, X_1 = x, Y_1 = y', Y_2, \ldots, Y_N) + \sum_{m=2}^{N} 1_{Y_{m-1} = y, Y_{m} = y'} P(X_m = x, Y_1, \ldots, Y_N)
\]

\[
= \frac{\beta_0(x, y) \sum_{x_0} p_{x_0 \to x} \mu(x_0, y) + \sum_{n=1}^{N-1} 1_{Y_{n} = y} \beta_n(x) \sum_{x_n} p_{x_n \to x} \alpha_n(x_n)}{\beta_0(x, y) \sum_{x_0} p_{x_0 \to x} \mu(x_0, y) + \sum_{n=1}^{N-1} 1_{Y_{n} = y} \beta_n(x) \sum_{x_n} p_{x_n \to x} \alpha_n(x_n)}
\]

Finally, for a maximum one also requires

\[
(41) \quad 0 = \frac{\partial}{\partial \mu'(x, y)} \left\{ Q(\Theta, \Theta') - \lambda \left( \sum_{\xi \in E, \theta \in O} \mu'(\xi, \theta) - 1 \right) \right\}
\]

\[
= \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \to x_n} q_{Y_{n-1} \to Y_{n}}(x_n) \right] \frac{1_{x_0 = x} 1_{y_0 = y}}{\mu'(x, y)} \mu(x_0, y_0) - \lambda,
\]
where $\lambda$ is a Lagrange multiplier. Multiplying by $\mu'(x, y)$ and summing over $x, y$, one has that
\begin{equation}
\lambda = \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{Y_{n-1} \rightarrow Y_n}(x_n) \right] \mu(x_0, y_0) \\
= P(Y_1, \ldots, Y_N) \\
= \sum_{\xi} \alpha_N(\xi).
\end{equation}

Substituting (42) into (41), one has by (3,4) that
\begin{equation}
\begin{aligned}
\mu'(x, y) &= \sum_{x_{0}, \ldots, x_N, y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{Y_{n-1} \rightarrow Y_n}(x_n) \right] 1_{x_0=x_1} 1_{y_0=y} \mu(x_0, y_0) \\
&= \frac{P(X_0 = x, Y_0 = y, Y_1, \ldots, Y_N)}{\sum_{\xi} \alpha_N(\xi)} \\
&= \frac{\sum_{x_1} P(Y_1, \ldots, Y_N | X_1 = x_1, X_0 = x, Y_0 = y) P(X_1 = x_1 | X_0 = x, Y_0 = y) \mu(x, y)}{\sum_{\xi} \alpha_N(\xi)} \\
&= \frac{\sum_{x_1} \beta_0(x_1, y) p_{x \rightarrow x_1} \mu(x, y)}{\sum_{\xi} \alpha_N(\xi)}.
\end{aligned}
\end{equation}

If we were to sum the numerator on the far right of (43), then upon substitution of $\beta$ we would get $P(Y_1, \ldots, Y_N)$, which matches the denominator. Hence, $\mu'(x, y) \in [0, 1]$ like the other new estimates. Now, we have established that the EM algorithm of Section 3 corresponds to the unique critical point of $\Theta' \rightarrow Q(\Theta, \Theta')$. Moreover, all mixed partial derivative of $Q$ in the components of $\Theta'$ are 0, while
\begin{equation}
\begin{aligned}
\frac{\partial^2 Q_{Y_1, Y_2, \ldots, Y_N}(\Theta, \Theta')}{\partial p'_{x \rightarrow x'}} &= -\sum_{y_0: x_0, \ldots, x_N} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{Y_{n-1} \rightarrow Y_n}(x_n) \right] \sum_{m=1}^{N} 1_{x_{m-1}=x, x_{m}=x'} \frac{\mu(x_0, y_0)}{p'_{x \rightarrow x'}} \\
\frac{\partial^2 Q_{Y_1, Y_2, \ldots, Y_N}(\Theta, \Theta')}{\partial q'_{y \rightarrow y'}(x)^2} &= -\sum_{y_0: x_0, \ldots, x_N} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{Y_{n-1} \rightarrow Y_n}(x_n) \right] \sum_{m=1}^{N} 1_{y_{m-1}=y, y_{m}=y'} \frac{\mu(x_0, y_0)}{q'_{y \rightarrow y'}(x)^2} \\
\text{and} \\
\frac{\partial^2 Q_{Y_1, Y_2, \ldots, Y_N}(\Theta, \Theta')}{\partial \mu'(x, y)^2} &= -\sum_{y_0: x_0, \ldots, x_N} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n} q_{Y_{n-1} \rightarrow Y_n}(x_n) \right] \sum_{m=1}^{N} 1_{y_0=y_0, x_0=x} \frac{\mu'(x, y)^2}{\mu'(x, y)} \mu(x_0, y_0).
\end{aligned}
\end{equation}
Hence, the Hessian matrix is diagonal with negative values along its axis and the critical point is a maximum.

The upshot of this result is that, if the EM algorithm produces parameters \( \{ \Theta^k \} \subset \Gamma \), then \( Q(\Theta^k, \Theta^{k+1}) \geq Q(\Theta^k, \Theta^k) \). Now, we have the following result, based upon Theorem 2.1 of Baum et. al. [3], that establishes the observable likelihood is also increasing i.e. \( P(\Theta^{k+1}) \geq P(\Theta^k) \).

**Lemma 5.3.** Suppose (Zero) holds. \( Q(\Theta, \Theta') \geq Q(\Theta, \Theta) \) implies \( P(\Theta') \geq P(\Theta) \). Moreover, \( Q(\Theta, \Theta') > Q(\Theta, \Theta) \) implies \( P(\Theta') > P(\Theta) \).

**Proof.** \( \ln(t) \) for \( t > 0 \) has convex inverse \( \exp(t) \). Hence, by Jensen’s inequality

\[
(47) \quad \frac{Q(\Theta, \Theta') - Q(\Theta, \Theta)}{P(\Theta)}
\]

\[
= \ln \left[ \sum_{x_0, x_1, \ldots, x_N} \sum_{y_0} \ln \left( \frac{p_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta')}{p_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta)} \right) \right] \frac{P_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta)}{P(\Theta)}
\]

\[
\leq \ln \left( \frac{\sum_{x_0, x_1, \ldots, x_N} \sum_{y_0} P_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta) \frac{p_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta)}{P_{y_0, x_0, \ldots, x_N}(x_0, x_1, \ldots, x_N; \Theta')}}{P(\Theta)} \right)
\]

\[
= \ln \left( \frac{P(\Theta')}{P(\Theta)} \right)
\]

and the result follows.

The stationary points of \( P \) and \( Q \) are also related.

**Lemma 5.4.** Suppose (Zero) holds. A point \( \Theta \in \Gamma \) is a critical point of \( P(\Theta) \) if and only if it is a fixed point of the re-estimation function, i.e. \( Q(\Theta; \Theta') = \max_{\Theta'} Q(\Theta; \Theta') \) since \( Q \) is differentiable on \((0, \infty)^d \) in \( \Theta' \).

**Proof.** The following derivatives are equal:

\[
(48) \quad \frac{\partial P_{y_1, \ldots, y_N}(\Theta)}{\partial p_{x \rightarrow x'}} = \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^N p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow y_n}(x_n) \right] \sum_{m=1}^N \frac{1_{x_{n-1}=x_{m-1},x_m=x'}}{p_{x_{m-1} \rightarrow x_m}} \mu(x_0, y_0)
\]

\[
= \left. \frac{\partial Q_{y_1, y_2, \ldots, y_N}(\Theta, \Theta')}{\partial p_{x \rightarrow x'}} \right|_{\Theta=\Theta'}
\]

which are defined since \( p_{x \rightarrow x'} \neq 0 \). Similarly,

\[
(49) \quad \frac{\partial P_{y_1, \ldots, y_N}(\Theta)}{\partial q_{y \rightarrow y'}} = \sum_{x_0, \ldots, x_N} \sum_{y_0} \left[ \prod_{n=1}^N p_{x_{n-1} \rightarrow x_n} q_{y_{n-1} \rightarrow y_n}(x_n) \right] \sum_{m=1}^N \frac{1_{y_{n-1}=y_{m-1},y_m=y'}}{q_{y_{m-1} \rightarrow y_m}(x)} \mu(x_0, y_0)
\]

\[
= \left. \frac{\partial Q_{y_1, y_2, \ldots, y_N}(\Theta, \Theta')}{\partial q_{y \rightarrow y'}} \right|_{\Theta=\Theta'}
\]
and
\[ \frac{\partial P_{Y_i,...,Y_N}(\Theta)}{\partial \mu(x,y)} = \sum_{x_0,...,x_N} \sum_{y_0} \left[ \prod_{n=1}^{N} p_{x_{n-1} \rightarrow x_n, q_{Y_{n-1} \rightarrow Y_n}}(x_n) \right] 1_{(x_0,y_0)=(x,y)} = \frac{\partial Q_{Y_1,Y_2,...,Y_N}(\Theta,\Theta')}{\partial \mu'(x,y)} \bigg|_{\Theta'=\Theta}. \]

We can rewrite (37,40,43) in recursive form with the values of \(\alpha\) and \(\beta\) substituted in to find that
\[ \Theta^{k+1} = M(\Theta^k), \]
where \(M\) is a continuous function. Moreover, \(P : K \rightarrow [0,1]\) is continuous and satisfies \(P(\Theta^k) \leq P(M(\Theta^k))\) from above. Now, we have established everything we need for the following result, which follows from the proof of Theorem 1 of [39].

**Theorem 5.5.** Suppose (Zero) holds. Then, \(\{\Theta^k\}_{k=1}^{\infty}\) is relatively compact, all its limit points (in \(K\)) are stationary points of \(P\), producing the same likelihood \(P(\Theta^*)\) say, and \(P(\Theta^k)\) converges monotonically to \(P(\Theta^*)\).

[39] has several interesting results in the context of general EM algorithms to guarantee convergence to local or global maxima under certain conditions. However, the point of this note is to introduce a new model and algorithms with just enough theory to justify the algorithms. Hence, we do not consider theory under any special cases here but rather refer the reader to Wu [39].

6. Viterbi algorithm. Like for HMM, the filter for MOM can be computed in real time (once the parameters are known)
\[ \pi_n(x) = P \left( X_n = x \bigg| Y_1,...,Y_n \right), \forall x \in E. \]
So, we can just compute \(\alpha_n(x)\) and then just normalize i.e.
\[ \pi_n(x) = \frac{\alpha_n(x)}{\sum_{\xi} \alpha_n(\xi)} \quad \text{and} \quad \pi(A) = \sum_{x \in A} \pi(x). \]
This provides our tracking estimate of the hidden state given the observations. Prediction can then be done by running the Kolmogorov forward equation starting from this estimate.

We can compute the most likely single hidden state values \(x^*_n\) of the hidden state \(X_n\), given the back observations \(Y_1,...,Y_n\) by finding the values that maximize \(x \rightarrow P(X_n = x \bigg| Y_1,...,Y_n)\). However, the Viterbi algorithm is used in HMM to find the most likely whole sequence of hidden state given the complete sequence of observations. This is particularly important in problems like decoding or recognition but is still useful in a widerange of application. It is a dynamic programming type algorithm.

As there is a EM analog to the Baum-Welch algorithm for our MOM models, it is natural to wonder if there is a dynamic programming analog to the Viterbi algorithm for our MOM models. The answer is yes and it is more similar to the Viterbi algorithm than our MOM EM algorithm is to the Baum-Welch algorithm. There are three small variants that one can
We define a sequence of functions \( \delta_n(y, x_0, x_1) \), \( \{ \delta_n(x) \}_{n=2}^N \), the maximum functions, and a sequence of estimates \( \{ y_0^*, x_0^*, x_1^*, ..., x_N^* \} \), the most likely sequence, within our Viterbi algorithm, Algorithm 2 below. Then, we show the algorithm works by noting

\[
\delta_n(x) = \max_{y_0:x_0,x_1,...,x_{n-1}} P(Y_0 = y_0; X_0 = x_0, ..., X_{n-1} = x_{n-1}; X_n = x; Y_1, ..., Y_n),
\]

for all \( n, x \) and establishing that \( \{ y_0^*, x_0^*, ..., x_N^* \} \) satisfies \( x_N^* = \text{arg max}_x \delta_N(x) \) and

\[
\delta_N(x_N^*) = P(Y_0 = y_0^*; X_0 = x_0^*, X_1 = x_1^*, ..., X_{N-1} = x_{N-1}^*; X_N = x_N^*; Y_1, ..., Y_N).
\]

**Algorithm 2:** Viterbi algorithm for MOM

**Input:** Observation sequence: \( Y_1, ..., Y_N \)

**Output:** Most likely Hidden state sequence: \( P^*; y_0^*, x_0^*, x_1^*, ..., x_N^* \)

**Data:** Probabilities \( \{ p_{x \rightarrow x'}, \{ q_{y \rightarrow y'}(x) \}, \{ \mu(x, y) \} \} \) \hspace{1em} // Distinguish MOM models

1. \( \delta_{0,1}(y_0, x_0, x_1) = \mu(x_0, y_0) p_{x_0 \rightarrow x_1} q_{y_0 \rightarrow Y_1}(x_1), \forall y_0, x_0, x_1 \) // Initialize joint distribution.

2. \( \delta_2(x_2) = \max_{y_0 \in O: x_0, x_1 \in E} \left[ \delta_{0,1}(y_0, x_0, x_1) p_{x_1 \rightarrow x_2} q_{Y_1 \rightarrow Y_2}(x_2) \right] \)

3. \( \psi_2(x_2) = \text{arg max}_{y_0 \in O: x_0, x_1 \in E} \left[ \delta_{0,1}(y_0, x_0, x_1) p_{x_1 \rightarrow x_2} \right] \)

4. for \( n=3 \) to \( N \) do

5. \( \delta_n(x_n) = \max_{x_{n-1} \in E} \left[ \delta_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x_n} \right] q_{Y_{n-1} \rightarrow Y_n}(x_n) \) // Maximums

6. \( \psi_n(x_n) = \text{arg max}_{x_{n-1} \in E} \left[ \delta_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x_n} \right] \) // Maximum Locations

7. \( P^* = \max_{x_N \in E} \left[ \delta_N(x_N) \right] \)

8. \( x_N^* = \text{arg max}_{x_N \in E} \left[ \delta_N(x_N) \right] \)

9. for \( n=N \) down to \( 2 \) do

10. \( x_n^* = \psi_{n+1}(x_{n+1}^*) \)

11. \( (y_0^*, x_0^*, x_1^*) = \psi_2(x_2^*) \)

6.1. **Dynamic Programming Explanation.** \( \delta_2(x) \) as defined in Algorithm 2 verifiably satisfies (52) by simple substitution. Next, assume \( \delta_n(x) \) satisfies (52) for some \( n \geq 3 \). Then, by the algorithm and the Markov property:

\[
\delta_n(x_n) = \max_{x_{n-1} \in E} \left[ \delta_{n-1}(x_{n-1}) p_{x_{n-1} \rightarrow x_n} \right] q_{Y_{n-1} \rightarrow Y_n}(x_n)
\]
\[
\begin{align*}
&= \max_{x_{n-1}} \left[ \max_{y_0, x_0, \ldots, x_{n-2}} P(Y_0 = y_0; X_0 = x_0, \ldots, X_{n-1} = x_{n-1}; Y_1, \ldots, Y_{n-1}) p_{x_{n-1} \to x_n} q_{y_{n-1} \to Y_n}(x_n) \right] \\
&= \max_{y_0, x_0, \ldots, x_{n-2}, x_{n-1}} P(Y_0 = y_0; X_0 = x_0, \ldots, X_{n-2} = x_{n-2}, X_{n-1} = x_{n-1}; X_n = x_n; Y_1, \ldots, Y_n),
\end{align*}
\]
and (52) follows for all \( n \) by induction. Next, it follows from the algorithm that \( x_N^* = \arg \max \delta_N(x) \). Finally, we have by the Path Back Tracking part of the algorithm as well as induction that
\[
(54) \quad \delta_N(x_N^*) = \delta_{N-1}(x_{N-1}^*) p_{x_{N-1}^* \to x_N^*} q_{Y_{N-1} \to Y_N}(x_N^*) \\
= \delta_{0,1}(y_0^*, x_0^*, x_1^*) \prod_{n=2}^{N} p_{x_{n-1}^* \to x_n^*} q_{Y_{n-1} \to Y_n}(x_n^*) \\
= P(Y_0 = y_0^*; X_0 = x_0^*, X_1 = x_1^*, \ldots, X_{N-1} = x_{N-1}^*; X_N = x_N^*; Y_1, \ldots, Y_N)
\]
and the most likely sequence is established.

**Remark 6.1.** Our Viterbi dynamic programming algorithm can be thought of as a direct generalization of the orginal Viterbi algorithm for HMM. Indeed, we need only let \( y \to \gamma(x) = b_x(y') \) for some probability mass function (depending upon \( x \) \( b_x \)) to recover the normal HMM and the normal Viterbi algorithm. Then, we would drop the consideration of the most likely starting point \( (y_0^*, x_0^*) \) and be back to the original setting.

**6.2. Small Number Problem.** The Viterbi-type algorithm also suffers from extraordinarily small, shrinking numbers. Indeed, since we have multiple events in both \( X \) and \( Y \), the numbers will shrink faster than our EM algorithm in \( n \). On the other hand, we are not taking ratios and it is easier to scale this algorithm than the EM algorithm. Still, one might wonder if we can handle the small number problem for our dynamic programming algorithm in a similar manner as we did for our EM algorithm.

While we did not adjust Algorithm 2, this algorithm can be adjusted for small numbers. The idea is similar to that used in the EM algorithm. Simply replace \( \delta_2 \) with
\[
v_2(x_2) = \max_{y_0 \in O; x_0, x_1 \in E} \left[ \delta_{0,1}(y_0, x_0, x_1)p_{x_1 \to x_2} \right] q_{Y_1 \to Y_2}(x_2)
\]
and \( \delta_n, n \geq 3 \) with \( \gamma_n \), where
\[
v_n(x_n) = \max_{x_{n-1} \in E} \left[ \gamma_{n-1}(x_{n-1}) p_{x_{n-1} \to x_n} \right] q_{Y_{n-1} \to Y_n}(x_n)
\]
and
\[
\gamma_n(x_n) = \frac{v_n(x_n)}{a_n}, \quad a_n = \sum_{\xi} v_n(\xi).
\]
Then, replace \( \delta_n \) with \( \gamma_n \) everywhere else in the algorithm. Of course, the maximum sequence likelihood \( P^* \) would have be scaled down by multiplying by the product of the \( a_n \)'s. Otherwise, the algorithm would remain the same.
7. Bitcoin Example. To establish the applicability of our model and algorithms to real-world big data problems, we include an illustrative MOM model example application. In particular, no Gaussian approximation is imposed. We work with discrete data and rely on our solution to the small number problem. Bitcoin is a highly volatile digital currency that can be traded by various means. Further, holding Bitcoin during uprends has proven to be a superlative investment, while holding it during other periods has been extremely risky and painful. Therefore, it is of interest to see if our MOM model algorithms might be able to isolate uprend periods and provide a two-hidden-state Markov Observation Model that matches historical data reasonably well. Accordingly, we applied our (Baum-Welch-like) EM and (Viterbi-like) dynamic programming algorithms to identify and demonstrate a (hidden) regime-change model for daily Bitcoin closing prices from Sept 1, 2018 until Sept 1, 2022.

The price varied (rather dramatically) from a low of $3235.76 USD on Dec. 15, 2018 to a high of $67566.83 USD on Nov. 8, 2021 over our four year period of interest. Instead of raw prices, we took our observations $Y$ be the natural logarithm of prices, which ranged from 8.0823 to 11.12 (see the continuous orange line in Figure 2), and divided those into $b = 25$ equal-sized bins. For example, bin 0 consisted of log prices with the range 8.08 to 8.2016 corresponding to actual prices $3230 to $3646.79 while the last bin, bin 24, consisted of log prices with the range 11.0005 to 11.122 corresponding to actual prices $59904.09 to $67643.06 all in US dollars. For ease of assimilation we just chose to have $s = 2$ hidden states 0 and 1.

7.1. Initialization. It is well known that the Baum-Welch algorithm for HMM will get stuck at a local maximum. This should be even more true for our EM algorithm of our MOM model as we have even more to estimate. (MOM has a larger initial distribution and more complex Markov transitions probabilities compared to HMM’s single state initial distribution and emission probabilities.) Therefore, it makes sense to start the algorithm with an idea of the solution that we seek. Most importantly, we want to differentiate the hidden states so we plan that state 1 will represent an uprend and state 0 will represent everything else and initialize accordingly. However, since we want our algorithm to find a variety of uprends, we will allow some inconsistencies in our initial set up that will force the algorithm to make significant changes. Also, we recognize that it is the algorithm that decides what the hidden states are. While we suggesting state 1 will be an uprend, the algorithm, by the time it has finished, may have decided 1 represents something completely different like high volatility say.

Our first step was to use the data to come up with initial $q_{y \rightarrow y'}(1)$ for uprend observation transitions and $q_{y \rightarrow y'}(0)$ other observation transitions. Accordingly, we made a somewhat arbitrarily decision about when Bitcoin might be in an uprend. In particular, we decided, based on a brief glimpse at the graph, to say it was in an uprend from the low on December 15, 2018 until the high on July 3, 2019, then again from the low on March 12, 2020 until the high on April 15, 2021, and finally from the low on July 20, 2021 until the high on November 8, 2021. Otherwise, it was not in an uprend. This amounts to six changes over the 1461 days in these four years.

**Remark 7.1.** Naturally, there were down and up days for both hidden states. Also, the very first price was excluded as this is our $Y_0$ price that we would not see in practice. Finally, we need to emphasize that we expect that the bin size effect was rather huge. We used 25 bins, which is extremely crude, and an arbitrary 4 year period with no sign of numerical issues. Also, the amount of data was very uneven over the bins, which we simply ignored, but it certainly hampered algorithm performance. A larger, finer study by more experienced computer programmers is definitely recommended.
To create our initial observation transitions, we initialized our $b \times b$ matrices $\{q_{y \rightarrow y'}(0)\}$ and $\{q_{y \rightarrow y'}(1)\}$ to zero. (Here, $y$ and $y'$ refer to either bin number or rounded log price through a one-to-one mapping.) Starting from $n = 1$ and going through to $n = N - 1$ we added 1 to $q_{Y_n \rightarrow Y_{n+1}}(1)$ if we were in an uptrend and otherwise 1 to $q_{Y_n \rightarrow Y_{n+1}}(0)$. Then, we normalized both matrices so that the non-zero rows added to one.

To initialize $\mu(x, y)$, we first set it all to zero. Next, we went through $n \in \{1, ..., N - 1\}$ if $Y_{n+1}$ was equal to $Y_1$ then we added $q_{Y_n \rightarrow Y_1}(x)$ to $\mu(x, Y_n)$ for $x = 0, 1$. Finally, we normalized $\mu(x, y)$ so it summed to 1.

We set the stopping criterion to be extremely tight, making sure that the $p$’s and $\mu$’s were essentially done changing. (The $q$’s will also be done in this case so there is little need to check this bulky matrix.) The initialization of the $p$ will be varied and explained in the results.

7.2. Results. Our first goal was to see if the algorithms would return the three uptrends that were supplied. To do this, we initialized $p$ as follows:

\begin{equation}
\begin{bmatrix}
p_{0 \rightarrow 0} & p_{0 \rightarrow 1} \\
p_{1 \rightarrow 0} & p_{1 \rightarrow 1}
\end{bmatrix} = \begin{bmatrix}0.997 & 0.003 \\
0.003 & 0.997\end{bmatrix}.
\end{equation}

This means that it should switch states every 333 days on average, which is roughly consistent with my initial take of three uptrends given above.

After $k = 11$ iterations, the EM algorithm converged and the combined result of both algorithms is displayed in Figure 2. It reduced the number of uptrends from what I supplied from three to two. I believe that the algorithm’s uptrends are at least as good as my initial ones. The final $p$ matrix in this case was

\begin{equation}
\begin{bmatrix}
p_{0 \rightarrow 0} & p_{0 \rightarrow 1} \\
p_{1 \rightarrow 0} & p_{1 \rightarrow 1}
\end{bmatrix} = \begin{bmatrix}0.99643132 & 0.00356868 \\
0.00302665 & 0.99697335\end{bmatrix}.
\end{equation}
Blue High: Most likely uptrend; Blue Low: Anything but an uptrend.

**Figure 3.** BitCoin Uptrend Detection - Short Uptrends

From an investor’s perspective, shorter, steeper uptrends might be more desirable. Hence, we investigated the possibility of finding more, shorter uptrends without retraining the $q$ matrices. Instead, we merely changed the initial $p$ matrix to

\[
\begin{bmatrix}
    p_{0 \rightarrow 0} & p_{0 \rightarrow 1} \\
    p_{1 \rightarrow 0} & p_{1 \rightarrow 1}
\end{bmatrix} = \begin{bmatrix}
    0.90 & 0.10 \\
    0.01 & 0.99
\end{bmatrix},
\]

which initially makes all changes more likely. However, it sets the initial expected time in an uptrend to just ten days initially. After $k = 43$ iterations, the EM algorithm converged and the combined result of both algorithms is displayed in Figure 3. Compared to the earlier result the uptrends were split and shrunk. In addition, a new uptrend was added in the later part of the data stream. It is very interesting that it did a decent job of finding a different type of uptrend without any new training, but rather just a different $p$ matrix initialization. The final $p$ matrix was:

\[
\begin{bmatrix}
    p_{0 \rightarrow 0} & p_{0 \rightarrow 1} \\
    p_{1 \rightarrow 0} & p_{1 \rightarrow 1}
\end{bmatrix} = \begin{bmatrix}
    0.98329893 & 0.01670107 \\
    0.0127778 & 0.9872222
\end{bmatrix}.
\]

The EM algorithm spent those 43 iterations making significant changes. In particular, the final $p$ matrix and graph suggests a near equal time in uptrends as not. However, this is somewhat out of our control. We supply the data, the number of hidden states and some initial estimates and then tell the EM algorithm to give us the locally optimal model, whatever that may be. In both cases the result exceeded our expectations.

**REFERENCES**

[1] Baum, L. E. and Petrie, T. (1966). Statistical Inference for Probabilistic Functions of Finite State Markov Chains. *The Annals of Mathematical Statistics*, 37 (6): 1554-1563. doi:10.1214/aoms/1177699147.
[2] Baum, L. E. and Eagon, J. A. (1967). An inequality with applications to statistical estimation for probabilistic functions of Markov processes and to a model for ecology. *Bulletin of the American Mathematical Society*, 73 (3): 360. doi:10.1090/S0002-9904-1967-11751-8. Zbl 0157.11101.
[32] Shinghal, R. and Toussaint, G.T. (1979). "Experiments in text recognition with the modified Viterbi algorithm," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-1 184-193.

[33] Sidrow, E., Heckman, N., Fortune, S. M., Trits, A. W., Murphy, I., and Auger-Méthé, M. (2022). Modelling multi-scale, state-switching functional data with hidden Markov models. *Canadian Journal of Statistics*, 50(1), 327-356.

[34] Stanculescu, I., Williams, C. K. I., and Freer, Y. (2014). Autoregressive Hidden Markov Models for the Early Detection of Neonatal Sepsis. *IEEE Journal of Biomedical and Health Informatics* 18(5):1560-1570. DOI: 10.1109/JBHI.2013.2294692

[35] Stigler, J., Ziegler, F., Gieseke, A., Gebhardt, J. C. M. and Rief, M. (2011). The Complex Folding Network of Single Calmodulin Molecules. *Science*. 334 (6055): 512-516. Bibcode:2011Sci...334..512S. doi:10.1126/science.1207598

[36] van Dijk, H. K.; Kloek, T. (1984). Experiments with some alternatives for simple importance sampling in Monte Carlo integration. In Bernardo, J. M.; DeGroot, M. H.; Lindley, D. V.; Smith, A. F. M. (eds.). *Bayesian Statistics. Vol. II*. Amsterdam: North Holland. ISBN 0-444-87746-0.

[37] Van Leeuwen, P.J., Künsch, H.R., Nerger, L., Potthast, R., Reich, S. (2019). Particle filters for high-dimensional geoscientific applications: A review. *Q. J. R. Meteorol Soc.* 145: 2335–2365. doi: 10.1002/qj.3551.

[38] Viterbi, A. J. (1967). "Error bounds for convolutional codes and an asymptotically optimum decoding algorithm". *IEEE Transactions on Information Theory*. 13 (2): 260-269. doi:10.1109/TIT.1967.1054010.

[39] Wu, C.F.J. (1983). "On the Convergence Properties of the EM Algorithm," *Ann. Statist.* 11(1): 95-103.

[40] Xuan, T. (2004) Autoregressive Hidden Markov Model with Application in an El Nino Study. MSc. Thesis, University of Saskatchewan, Saskatoon.

[41] Zakai, M. (1969). On the optimal filtering of diffusion processes. *Z. Wahrsch. Verw. Gebiete* 11, 230–243.