Deformed Heisenberg algebra with reflection

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Abstract

A universality of deformed Heisenberg algebra involving the reflection operator is revealed. It is shown that in addition to the well-known infinite-dimensional representations related to parabosons, the algebra has also finite-dimensional representations of the parafermionic nature. We demonstrate that finite-dimensional representations are representations of deformed parafermionic algebra with internal $Z_2$-grading structure. On the other hand, any finite- or infinite-dimensional representation of the algebra supply us with irreducible representation of $osp(1|2)$ superalgebra. We show that the normalized form of deformed Heisenberg algebra with reflection has the structure of guon algebra related to the generalized statistics.

Key words: deformed Heisenberg algebra, parabosons, deformed parafermionic algebra, $osp(1|2)$ superalgebra, generalized statistics

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1 Introduction

The deformed Heisenberg algebra involving the reflection operator $R$ appeared in the context of quantization schemes generalizing bosonic commutation relations. Such generalized schemes lead naturally to the concept of parafields and parastatistics [1, 2, 3]. The same $R$-deformed Heisenberg algebra (RDHA) was also used for solving quantum mechanical Calogero model [4, 5, 6]. Recently this algebra has been employed for bosonization of supersymmetric quantum mechanics [7, 8] and for describing anyons in (2+1) [9] and (1+1) dimensions [10]. All the applications as well as the parabosonic constructions [2, 3] use infinite-dimensional unitary representations of RDHA.

In this paper we shall reveal a universality of the $R$-deformed Heisenberg algebra. We shall show that in addition to the infinite-dimensional representations related to parabosons, the algebra has also finite-dimensional representations of the parafermionic nature. Finite-dimensional representations will be investigated in different aspects. This includes their interpretation as representations of some non-degenerate paragrassmann algebra [11, 12] as well as treating them as representations of generalized deformed parafermionic [13] and deformed $su(2)$ algebras [14]. We shall find that any irreducible representation of RDHA supplies us with corresponding finite- or infinite-dimensional irreducible representation of $osp(1|2)$ superalgebra. The $osp(1|2)$ generators turn out to be realizable in a universal form in terms of generators of the underlying deformed Heisenberg algebra. Besides, we shall show that the normalized form of RDHA has the structure of guon algebra [15] related to generalized statistics [16, 17].

The paper is organized as follows. In Section 2 we consider Fock space representations of the $R$-deformed Heisenberg algebra. Here the relationship of the algebra to parabosonic trilinear (anti)commutation relations is discussed and finite-dimensional representations of parafermionic type are found. Finite-dimensional representations are analyzed in detail in Section 3, where we discuss also a universal $osp(1|2)$ superalgebraic aspect of RDHA. Section 4 is devoted to the construction of the normalized form of the algebra. We get the standard fermionic algebra, presented in the form of $R$-algebra, as a limit case of the normalized RDHA and generalize it into the algebra with phase operator. We show that such a generalization is related to the $q$-deformed Heisenberg algebra [15, 16] with $q$ being a primitive root of unity. In Section 5 we discuss possible applications of the results.

Appendix A concerns the realization of finite-dimensional representations on the $q$-paragrassmann algebra [11, 12].

Appendix B illustrates a relationship of two lower-dimensional representations of RDHA to the spinor and vector representations of (2+1)-dimensional Lorentz group.

The paper is devoted to the memory of Dmitrij Vasilievich Volkov.

2 Representaions of $R$-deformed Heisenberg algebra

The $R$-deformed Heisenberg algebra is given by the generators $1$, $a^-$, $a^+$, $R$, which satisfy the (anti)commutation relations

$$[a^-, a^+] = 1 + \nu R, \quad \{a^\pm, R\} = 0, \quad R^2 = 1, \quad (2.1)$$
and \([a^+, 1] = [R, 1] = 0\). Here \(\nu \in \mathbb{R}\) is a deformation parameter and hermitian operator \(R\) is the reflection operator [2]. The relationship of \(a^+\) and \(a^-\) under hermitian conjugation will be specified lately.

As a consequence of eq. (2.1), operators \(a^\pm\) obey trilinear (anti)commutation relations not containing either operator \(R\) or deformation parameter:

\[
\{a^-, a^+\}, a^- = -2a^-, \quad \{a^-, a^+\}, a^+ = 2a^+.
\]

(2.2)

Trilinear relations (2.2) characterize parabosons [1, 2]. They can be represented equivalently as \([a^+, a^-] = -2a^-\) and \([a^-, a^+] = 2a^+,\) or as

\[
\{a^-, [a^-, a^+]\} = 2a^-, \quad \{a^+, [a^-, a^+]\} = 2a^+.
\]

(2.3)

These trilinear relations themselves lead to \(R\)-deformed Heisenberg algebra (2.1). To show this, let us define operator \(G = [a^-, a^+] - 1\). Relations (2.3) mean that \([G, a^\pm] = 0\) and as a consequence, \([G^2, a^\pm] = 0\). Hence, for irreducible representation we have \(G^2 = \text{const}\). If in such representation \(G\) is hermitian operator, then \(G = \nu R\) with \([R, a^\pm] = 0, R^2 = 1\) and \(\nu \in \mathbb{R}\).

Now we proceed to consideration of irreducible representations of algebra (2.1). One introduces the vacuum state \(|0\rangle, a^-|0\rangle = 0, \langle 0|0\rangle = 1, R|0\rangle = |0\rangle\), and defines the states \(|n\rangle = C_n(a^+)^n|0\rangle, n = 0, 1, \ldots\), with \(C_n\) being some normalization constants. From the relation

\[
[a^-, (a^+)^n] = \left(n + \frac{1}{2}(1 - (-1)^n)\nu R\right)(a^+)^{n-1}
\]

(2.4)

we conclude that algebra (2.1) has infinite-dimensional unitary representations when \(\nu > -1\). Only in this case the states \(|n\rangle\) with \(C_n = (|n\rangle \nu !)^{-1/2}, |n\rangle \nu ! = \prod_{l=1}^{n} [l]_{\nu}, [l]_{\nu} = l + \frac{1}{2}(1 - (-1)^l)\nu\), form the complete orthonormal basis of Fock representation and operators \(a^+\) and \(a^-\) are hermitian conjugate. The reflection operator can be realized in terms of creation and annihilation operators via the number operator \(N\),

\[
R = (-1)^N = \cos \pi N,
\]

(2.5)

\[
N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1), \quad N|n\rangle = n|n\rangle.
\]

(2.6)

Trilinear commutation relations (2.2) characterize the parabosonic system of order \(p = 1, 2, \ldots\) in the case when \(\nu = p - 1 = 0, 1, \ldots\) [1, 2]. The existence of infinite-dimensional unitary representations of algebra (2.1) on the half-line \(\nu > -1\) and the relationship between trilinear commutation relations and \(R\)-deformed Heisenberg algebra (2.1) mean that the latter can be considered as the algebra supplying us with some generalization of parabosons for the case of non-integer statistical parameter \(p = \nu + 1 > 0\). Such generalized parabosonic aspect of algebra (2.1) was discussed recently by Macfarlane [3].

Now we shall show that algebra (2.1) has finite-dimensional representations, and, as a consequence, it is also related to parafermionic systems. To this end we note that eq. (2.4) gives some special values of the deformation parameter,

\[
\nu = -(2p + 1), \quad p = 1, 2, \ldots,
\]
for which the relation \( \langle m|n \rangle = 0, |n \rangle \equiv (a^+)^n|0 \rangle \), takes place for \( n \geq 2p + 1 \) and arbitrary \( m \). This means that there are \((2p + 1)\)-dimensional irreducible representations of algebra (\ref{2.1}), with \( \nu = -2p + 1 \), in which the relations \((a^+)^{2p+1} = (a^-)^{2p+1} = 0\) are valid. The latter relations are a characteristic property of parafermions of order \( 2p \) \cite{2}. The relationship of these \((2p + 1)\)-dimensional representations of the \( R \)-deformed Heisenberg algebra to parafermions will be discussed in detail in the next section.

One concludes that the \( R \)-deformed Heisenberg algebra turns out to be related to parabosons and to parafermions. In this sense it has some properties of universality.

3 Finite-dimensional representations

We shall investigate finite-dimensional representations in different aspects. First we shall discuss them as representations of some non-degenerate paragrassmann algebra. Then we shall obtain the corresponding matrix representation. In particular, we shall show that operators \( a^+ \) and \( a^- \) are mutually conjugate with respect to indefinite scalar product. We shall find another set of operators, \( f^+ \) and \( f^- \), which are related to \( a^+ \) and \( a^- \) in a simple way. These new operators are hermitian and give a possibility to work with a Hilbert space of states. Operators \( f^\pm \), unlike \( a^\pm \), will satisfy anticommutation relations involving the reflection operator. We shall show that \( f^\pm \) are the creation-annihilation operators of some deformed parafermionic algebra, which at \( p = 1 \) is reduced to the standard parafermionic algebra of order 2. For \( p > 1 \), operators \( f^\pm \) generate the related special deformation of \( su(2) \) algebra. The realization of deformed \( su(2) \) generators in terms of the standard \( su(2) \) generators reflects a nontrivial relationship of our deformed parafermionic algebra to the standard parafermionic algebra of the same order. We shall show that there is also a simple realization of the standard \( su(2) \) generators in terms of operators \( f^\pm \). However, the price necessary to pay will be reducible action of these \( su(2) \) generators. The related reducible quadratic realization of \( so(2, 1) \) will be obtained in terms of initial operators \( a^\pm \). It will give the standard (non-unitary) finite-dimensional representations of \((2 + 1)\)-dimensional Lorentz group. The above-mentioned indefinite scalar product turns out to be the standard indefinite scalar product which is necessary to have Lorentz generators as self-conjugate operators. We shall show that operators \( a^+ \) and \( a^- \) together with quadratic Lorentz generators form the set of generators of \( osp(1|2) \) superalgebra and give us its \((2p + 1)\)-dimensional irreducible representations.

3.1 \( R \)-paragrassmann algebra

Let us consider finite-dimensional representations in more detail. We have arrived at the nilpotent algebra

\[
[a^-, a^+] = 1 - (2p + 1)R, \quad \{a^\pm, R\} = 0, \quad R^2 = 1, \tag{3.1}
\]

\[
(a^\pm)^{2p+1} = 0, \quad p = 1, 2, \ldots \tag{3.2}
\]

As a consequence of relations (\ref{3.1}), (\ref{3.2}), we have the relations

\[
(1 - R)a^{+2p} = (1 - R)a^{-2p} = 0. \tag{3.3}
\]
These relations and eqs. (3.1), in turn, lead to conditions (3.2). This means that here the nilpotency conditions (3.2) are equivalent to relations (3.3).

One can interpret $a^+$ as a paragrassmann variable $\theta$, $\theta^{2p+1} = 0$. Then operator $a^-$ can be considered as corresponding differentiation operator $\partial$ defined by relation (2.4) (see refs. [11, 12]). Therefore, algebra (3.1), (3.2) is a paragrassmann algebra of order $2p$ with special differentiation operator. One can call it the $R$-paragrassmann algebra.

In addition to the universal realization (2.7), (2.6), for the $R$-paragrassmann algebra we have the normal ordered representation

$$ R = \sum_{n=0}^{2p} f_n a^+ n a^- n. \quad (3.4) $$

The c-number factors $f_n$ are given by finite recursive relations

$$ 2f_{n-1} + [n]_\nu f_n - (2p + 1) \sum_{i=0}^{[n/2]-1} f_{2i+1} f_{n-(2i+1)} = 0, \quad n = 1, \ldots, 2p, $$

where $[n/2]$ is an integer part of $n/2$, and as it follows just from eq. (3.3), $f_0 = 1$. In simplest cases this gives

$$ R = 1 + a^+ a^- + \frac{1}{2} a^+ a^- 2, \quad p = 1, $$

$$ R = 1 + \frac{1}{2} a^+ a^- + \frac{1}{8} a^+ a^- 2 - \frac{1}{32} a^+ a^- 3 - \frac{3}{128} a^+ a^- 4, \quad p = 2. $$

The explicit form of normal ordered representation (3.4) and the form of commutation relations (2.4) say that the $R$-paragrassmann algebra is non-degenerate paragrassmann algebra characterized by the properties $\partial \theta^n = \alpha_n \theta^{n-1} + (\ldots) \theta_n$, $\alpha_n \neq 0$, $n = 1, 2, \ldots, p$, $\theta^{2p+1} = 0$ [12]. Due to isomorphism between non-degenerate paragrassmann algebras of the same order [12], it is possible to realize algebra (3.1), (3.2) on the $q$-paragrassmann algebra with $q$ being a corresponding primitive root of unity. Such a realization is given in Appendix A.

### 3.2 Matrix realization

Finite-dimensional Fock space representations of $R$-deformed Heisenberg algebra (2.1) contain states with negative norm. One can introduce the normalized states as $\langle n \rangle = |\langle n | n \rangle|^{-1/2} \cdot |n\rangle$. They give the indefinite metric operator $\eta = \eta^1$, $\eta^2 = 1$, with matrix elements

$$ ||\eta||_{mn} = \langle m \cdot n \rangle = \text{diag}(1, -1, -1, +1, +1, -1, -1, \ldots, (-1)^{p-1}, (-1)^{p-1}, (-1)^{p}, (-1)^{p}). $$

Then the indefinite scalar product can be defined as

$$ (\Psi_1, \Psi_2) = \langle \Psi_1 | \eta \Psi_2 \rangle = \Psi^*_1, \eta_{mn} \Psi_{2m}, \quad (3.5) $$

where $\Psi_n = \langle n | \Psi \rangle$. Operators $a^+$ and $a^-$ are given by the matrices

$$ (a^+)_{mn} = A_n \delta_{m-1,n}, \quad (a^-)_{mn} = B_m \delta_{m+1,n}. \quad (3.6) $$
They satisfy relations (2.1) with diagonal operator $R$ with respect to indefinite scalar product (3.5), $\Psi^* = \Psi$ instead of operators $\Psi$. Considered in this case as not basic operators. However, we shall see further that the usage of positive-definite scalar product (3.8), $\langle \Psi_1, a^- \Psi_2 \rangle = \langle \Psi_1, a^+ \Psi_2 \rangle$, from now on one could work in terms of hermitian conjugate operators (3.8) using ordinary product turns out also to be useful for some physical applications.

One can introduce hermitian conjugate operators

\[ f^+ = a^+, \quad f^- = a^- R \]  

instead of operators $a^\pm$. Then finite-dimensional representations of the $R$-deformed Heisenberg algebra can be specified by the relations

\[
\begin{align*}
\{f^+, f^-\} &= (2p + 1) - R, \\
\{R, f^\pm\} &= 0, \quad R^2 = 1, \quad (f^\pm)^{2p+1} = 0.
\end{align*}
\]

From now on one could work in terms of hermitian conjugate operators (1.8) using ordinary positive-definite scalar product $\langle \Psi_1, \Psi_2 \rangle = \Psi_1^* \Psi_2$. Operators $a^+ = f^+$, $a^- = f^- R$ can be considered in this case as not basic operators. However, we shall see further that the usage of operators $a^+$, $a^-$ as basic conjugate operators together with corresponding indefinite scalar product turns out also to be useful for some physical applications.

The present situation with two sets of operators, $a^+$, $a^-$ and $f^+$, $f^-$, is similar to that taking place for finite-dimensional representations of the $q$-deformed Heisenberg algebra [18, 19]. That algebra can be given by commutation relations in the Biedenharn-Macfarlane form $b^+ b^- - q b^+ b^- = q^{-N}$ with $N$ being a number operator. In the case $q^{k+1} = -1$, $k = 1, 2, \ldots$, the algebra has $(k + 1)$-dimensional representations with $(b^\pm)^{k+1} = 0$, in which operators $b^+$ and $b^-$ are hermitian conjugate. In terms of operators $c^+ = b^q N/2$ and $c^- = q^{N/2} b^-$, the algebra is given in more simple Arik-Coon-Kuryshkin form, $c^- c^+ - q^2 c^+ c^- = 1$, but here operators $c^\pm$ are not hermitian conjugate, $(c^+)^\dagger = q^{-N} c^-.$

### 3.3 Parafermions and deformed $su(2)$

Here we shall show that finite-dimensional representations of the $R$-deformed Heisenberg algebra are representations of deformed parafermionic algebra which is related to some non-linear deformation of $su(2)$. In the case of lowest representation ($p = 1$), the deformed parafermionic algebra is reduced to the standard parafermionic algebra of order 2 giving us a vector representation of the standard $su(2)$.

First one notes that the operators $a^\pm$ satisfying eqs. (3.1), (3.2) obey the relations

\[
\begin{align*}
a^+ a^{-2p} + a^- a^+ a^{-(2p-1)} + \ldots + a^{-(2p-1)} a^+ a^- + a^{-2p} a^+ &= 0, \\
a^- a^{+2p} + a^+ a^- a^{(2p-1)} + \ldots + a^{+(2p-1)} a^- a^+ + a^{+2p} a^- &= 0.
\end{align*}
\]

Such relations together with relations $a^{(2p+1)} = 0$ take place for $N = 2$ parasupercharge operators in the case of trivial parasupersymmetry characterized by zero parasuperhamiltonian [20]. The alternating sum

\[ f^+ f^{+2p} - f^- f^{-(2p-1)} f^+ f^- - f^{-2p} f^+ = 0 \]

(3.12)
and its hermitian conjugate analog are equivalent to relations (3.11). The following relations are also valid:

\[
f^+ f^- f^+ p = f^{+(2p-1)} C_p, \quad f^- p f^+ f^- p = f^{-(2p-1)} C_p,
\]

where \( C_p = -p \) for even \( p \) and \( C_p = p + 1 \) for odd \( p \). At \( p = 1 \) they are reduced to

\[
f^- f^+ f^- = 2f^-, \quad f^+ f^- f^+ = 2f^+.
\]

(3.13)

Taking into account eq. (3.13), we reduce equation (3.12) and its conjugate to the relations

\[
f^ - 2 f^+ f^- + f^+ f^- 2 f^+ = 2 f^+, \quad f^- f^+ f^- 2 f^+ f^- = 2 f^+.
\]

(3.14)

As a consequence of eqs. (3.13), (3.14), at \( p = 1 \) one has

\[
[[f^+, f^-], f^+] = 2 f^+; \quad [[f^+, f^-], f^-] = -2 f^-.
\]

(3.15)

Relations (3.13), (3.14) and \( f^\pm = 0 \) are the relations characterizing the parafermions of order 2. Spin-1 \( su(2) \) generators are realized in terms of these parafermionic operators as \( I_+ = f^+, \quad I_- = f^-, \quad I_3 = \frac{1}{2}[f^+, f^-] \). For \( p > 1 \), operators \( I_+ = f^+ \), \( I_- = f^- \) generate a nonlinear deformation of \( su(2) \) algebra of the form

\[
[I_+, I_+] = 2I_3(-1)^{I_3+p},
\]

(3.16)

\[
[I_3, I_\pm] = \pm I_\pm,
\]

(3.17)

involving the reflection operator

\[
R = (-1)^{I_3+p}.
\]

(3.18)

At \( p = 1 \), one has the relation \( I_3(-1)^{I_3+1} = I_3 \), and deformed \( su(2) \) algebra (3.16), (3.17) turns into the standard \( su(2) \). Relations (3.16), (3.17) can be represented in the form

\[
[[f^+, f^-], f^\mp] = 2(2I_3 \mp 1)(-1)^{I_3+p} f^\mp
\]

(3.19)

with operator \( I_3 \) given by

\[
I_3 = \frac{1}{2}[f^+, f^-] \cdot (-1)^{\frac{1}{2}[f^+, f^-] + p}.
\]

(3.20)

Relations (3.19) generalize trilinear parafermionic relations (3.13) for the case \( p > 1 \) and mean that we have deformed parafermionic algebra.

The (anti)commutation relations (3.9) and (3.16) can be represented as

\[
\{f^+, f^-\} = F(N + 1) + F(N), \quad [f^-, f^+] = F(N + 1) - F(N),
\]

(3.21)

with function \( F(N) = N(-1)^{N} + (p+\frac{1}{2})(1-(-1)^N) \) being a function of the number operator \( N = I_3 + p \). This function is characterized by the properties

\[
F(n) > 0, \quad n = 1, 2, \ldots, p', \quad F(p' + 1) = 0,
\]

(3.22)

and here \( p' = 2p \). The algebra with defining relations (3.21), (3.22) is the generalized deformed parafermionic algebra introduced by Quesne [13]. For \( F(x) = x(p' + 1 - x) \) it is reduced to the standard parafermionic algebra of order \( p' \). It is interesting to note that
algebra (3.21), (3.22) contains also $q$-deformed parafermionic algebra [21], which for $p' = 2$ is reduced to the standard parafermionic algebra as it happens in our case.

Generators of deformed $su(2)$ algebra (3.16), (3.17) can be related to the generators of the standard $su(2)$ algebra,

$$[J_+, J_-] = 2J_3, \quad [J_3, J_{\pm}] = \pm J_{\pm},$$

(3.23)
as

$$I_3 = J_3, \quad I_- = J_- \Phi(J_3), \quad I_+ = \Phi(J_3)J_+.$$  \hspace{1cm} (3.24)

Here function $\Phi(J_3)$ is given by

$$\Phi(J_3) = \sqrt{\frac{2p + 1 + R(2J_3 - 1)}{2(p + J_3)(p - J_3 + 1)}}, \quad R = (-1)^{J_3 + p}.$$  \hspace{1cm} (3.25)

We assume that generators $J_\pm$, $J_3$ realize $(2p + 1)$-dimensional representation of $su(2)$ which are characterized by the corresponding value of the Casimir operator, $J_3^2 + \frac{1}{2} \{J_+, J_-\} = p(p + 1)$. Definition (3.25) gives indefinite relation of the form $0/0$ at $J_3 = -p$, where it should be supplemented by any finite nonzero value. This supplementing a definition is formally necessary to have a well defined action of operator $I_-(I_+)$ on the state $| - p \rangle$ ($\langle - p |$) being the eigenstate of $J_3$, $J_3| - p \rangle = -p| - p \rangle$. Then, for $p > 1$, the transformation inverse to (3.24) and eq. (3.20) give essentially nonlinear realization of the standard $su(2)$ in terms of operators $f^\pm$. Further we shall show that the special nature of our deformed parafermionic algebra admits, nevertheless, a simple (but reducible) realization of the standard $su(2)$. Note here that the deformations of $su(2)$ of the form (3.24) were considered by Polychronakos and Roček [14].

### 3.4 Finite-dimensional representations of $osp(1|2)$

Our deformed parafermionic algebra admits also a simple realization of the standard $su(2)$. This turns out to be possible due to a special $\mathbb{Z}_2$-grading structure of deformed $su(2)$ algebra (3.10), (3.17), which consists in the presence of reflection operator (3.18) in deformed commutation relation (3.16). The related construction in terms of initial operators $a^\pm$ will supply us with quadratic realization of generators of $(2+1)$-dimensional Lorentz group and will give a self-conjugate set of $osp(1|2)$ generators. As we shall see, the indefinite scalar product introduced in section 3.2 will be recognized as the standard scalar product necessary to have spin-$j$ Lorentz generators as self-conjugate operators.

One introduces hermitian conjugate quadratic operators

$$J_\pm = \frac{1}{2} f^{\pm 2}, \quad J_- = \frac{1}{2} f^{-2}.$$  \hspace{1cm} (3.26)

Direct calculation gives $[J_+, J_-] = 2J_3$, with

$$J_3 = \frac{1}{2} I_3,$$

(3.27)

where $I_3$ is the diagonal operator introduced before, $I_3 = N - p$. Eq. (3.17) leads to the relations $[J_3, J_\pm] = \pm J_\pm$. Therefore, operators (3.20) and (3.27) form, unlike the generators
of the superalgebra. As a consequence, operators \((3.26)\) satisfy standard trilinear parafermionic commutation relations of the form \((3.15)\). But since \((f^\pm)^{p+1} = 0\), our \((2p+1)\)-dimensional representation is reducible with respect to the action of operators \((3.26)\) as parafermionic generators. It is a direct sum of \((p+1)\)-dimensional odd subspaces spanned by even, \(|+\rangle\), and odd, \(|-\rangle\), eigenstates of reflection operator, \(R|\pm\rangle = \pm|\pm\rangle\). These subspaces carry unitary spin-\(j_+\) and spin-\(j_-\) \(su(2)\)-representations: \(C|\pm\rangle = j_+(j_++1)|\pm\rangle\), where \(C\) is the \(su(2)\) quadratic Casimir operator, \(C = J_3^2 + \frac{1}{2}\{J_+, J_-\}\), and \(j_+ = p/2, j_- = (p-1)/2\).

Let us consider operators \(J_+ = J_+, \ J_- = -J_, \ J_0 = J_3 = \frac{1}{4}[J^+, J^-]R\). In terms of operators \(a^\pm\), they are represented as quadratic operators:

\[
J_0 = \frac{1}{4}\{a^+, a^-\}, \quad J_+ = \frac{1}{2}a^{+2}, \quad J_- = \frac{1}{2}a^{-2}.
\] (3.28)

These generators form \(so(2, 1)\) algebra,

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_-, J_+] = 2J_0.
\] (3.29)

Such realization gives reducible representation of this algebra being the direct sum of spin-\(j_+\) and spin-\(j_-\) representations: \(C|\pm\rangle = -j_+(j_++1)|\pm\rangle\), where \(C = -J_0^2 + \frac{1}{2}\{J_+, J_-\}\) is \(so(2, 1)\) Casimir operator, whereas subspaces \(|\pm\rangle\) and numbers \(j_\pm\) have been specified above. Operators \(a^+\) and \(a^-\), being ‘square root’ operators of \(so(2, 1)\) generators, have with them the following nontrivial commutation relations:

\[
[J_+, a^-] = -a^+, \quad [J_-, a^+] = a^-, \quad [J_0, a^\pm] = \pm\frac{1}{2}a^\pm.
\] (3.30)

These relations mean that operators \(a^+\) and \(a^-\) are components of \((2+1)\)-dimensional spinor. Reading relations \((3.28)\) in inverse order, from the right to the left hand side, and taking into account eqs. \((3.24), (3.30)\), one concludes that operators \(a^+, a^-, J_0, J_+\) and \(J_-\) form \(osp(1|2)\) superalgebra \([22]\) with \(a^\pm\) being odd generators and \(J_0, J_\pm\) being even generators of the superalgebra. The corresponding Casimir operator of the superalgebra is \(C_{osp} = -J_0^2 + \frac{1}{2}\{J_+, J_-\} - \frac{1}{8}[a^-, a^+]\), and it takes here the value \(C_{osp} = -\frac{1}{4}p(p+1)\). Therefore, every \((2p+1)\)-dimensional representation of the \(R\)-deformed Heisenberg algebra supplies us with reducible representation of the standard (non-deformed) \(osp(1|2)\) superalgebra. To have operators \(a^+, a^-\) and \(J_+\), \(J_-\) as mutually conjugate operators, it is necessary to use indefinite scalar product \((3.5)\). There is nothing surprising in this fact since it is well known that finite-dimensional representations of \(so(2, 1)\) are non-unitary (see, e.g., ref. [23]). Dirac conjugate field \(\hat{\psi}\) containing \(\gamma^0\)-factor in the case of spinor representation and indefinite metric operator of the form \(\eta_{\mu\nu} = diag(-1, +1, +1)\) in the case of vector representation reflect this non-unitarity. The relationship of indefinite scalar product \((3.5)\) to the above mentioned standard simplest physical manifestations of non-unitarity of finite-dimensional \(so(2, 1)\) representations is illustrated in Appendix B.

Therefore, we have shown that \((2p+1)\)-dimensional representations of the \(R\)-deformed Heisenberg algebra supply us with irreducible (non-unitary) representations of \(osp(1|2)\) superalgebra. They are analogs of the well known infinite-dimensional half-bounded unitary representations of this superalgebra, which are realized on corresponding infinite-dimensional
representations of the $R$-deformed Heisenberg algebra. In the case of infinite-dimensional representations the superalgebra generators are realized via operators $a^\pm$ in the same form as here (see refs. [3, 4, 5]). Such universality of the construction comprising finite-dimensional and infinite-dimensional representations of $osp(1|2)$ can find interesting application in $(2+1)$-dimensional physics. This point will be discussed in the last section.

4 Normalized $R$-deformed Heisenberg algebra

We have analyzed representations of the $R$-deformed Heisenberg algebra proceeding from its standard non-normalized form (2.1). Here we shall find the normalized form of the algebra. Using such a terminology we have in mind the normalization of the right hand side of the corresponding commutation relations. As we shall see, the normalized form represents by itself a guon-like algebra [13] which is similar to a normalized form of $q$-deformed Heisenberg algebra, $c^-c^+ -qc^+c^- = 1$ [18], but unlike the latter, it contains some special $g$-operator factor instead of $c$-number $q$-factor. The initial form of our algebra (2.1) corresponds to the non-normalized variant of the $q$-deformed Heisenberg algebra, $b^-b^+ -qb^+b^- = q^{-N}$ [19].

4.1 Guons

Let us suppose that $\nu \neq 1$, and define the operators

$$c^- = a^-G_\nu^{-1/2}(R), \quad c^+ = G_\nu^{-1/2}(R)a^+, \quad G_\nu(R) = |1 - \nu R|,$$

where for the moment we suppose that $R = (-1)^N$ with $N$ given by eq. (2.6). These operators anticommute with the reflection operator, $\{ R, c^\pm \} = 0$, and satisfy the commutation relation $c^-c^+ - G_\nu(R)G_\nu^{-1}(R)c^+c^- = sign(1 + \nu R)$, where $sign x$ is +1 for $x > 0$ and −1 for $x < 0$. The operator $G_\nu(R)$ in transformation (4.1) is reduced to $G_\nu(R) = 1 - \nu R$ for $-1 < \nu < 1$; for two other cases we have $G_\nu(R) = \nu - R$, $\nu > 1$, and $G_\nu(R) = R - \nu$, $\nu = -(2p + 1)$. As a result, commutation relation is represented in first case in a normalized form

$$c^-c^+ - g_\nu c^+c^- = 1, \quad g_\nu = (1 - \nu)^R(1 + \nu)^{-R}, \quad -1 < \nu < 1, \quad (4.2)$$

whereas in two other cases it is reduced to

$$c^-c^+ - g_\nu c^+c^- = R, \quad (4.3)$$

where $g_\nu = (\nu - 1)^R(1 + \nu)^{-R}$ for $\nu > 1$, and $g_\nu = p^R(p + 1)^{-R}$ for $\nu = -(2p + 1)$. In the case corresponding to finite-dimensional representations, the final form (1.3) has been obtained via additional changing $R \rightarrow -R$. In all three cases operator-valued function $g_\nu$ satisfies the relation $g_\nu c^\pm = c^\pm g_\nu^{-1}$. The normalized form of algebra (4.2) is similar to the Scipioni guon algebra [13] which was introduced in the context of generalized statistics [16, 17]. Algebra (4.3) gives some modification of (4.2).

The corresponding number operator $N = N(c^+, c^-)$ is given by

$$N = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{(1 - \nu^2)(2c^+c^- - \beta)(2c^-c^+ - \beta) + 1},$$

10
where $\alpha = -\nu$, $\beta = 1$ in the case $-1 < \nu < 1$, and $\alpha = -\nu + \nu^2 - 1$, $\beta = |\nu|$ for two other cases. Implying in relations (4.2), (4.3) that $R = (-1)^{N(c^+, c^-)}$, one can represent them in a closed form containing only creation-annihilation operators $c^\pm$.

### 4.2 Fermions and $P$-algebra

Let us consider a limit $|\nu| \to \infty$ proceeding from relation (4.3). Both cases, $\nu > -1$ and $\nu = -(2p + 1)$, lead to the algebra

$$
c^-c^+ - c^+c^- = R, \quad \{R, c^\pm\} = 0, \quad R^2 = 1.
$$

(Anti)commutation relations (4.4) have irreducible two-dimensional Fock space representation in which $c^-|0\rangle = 0$, $R|0\rangle = |0\rangle$ and $(c^\pm)^2 = 0$. This corresponds to fermionic representation. Indeed, here reflection operator is realized as $R = 1 - 2c^+c^-$, and $R$-commutation relations (4.4) are reduced to the fermionic relations $c^+c^- + c^+c^- = 1$, $c^2 = c^2 = 0$. Therefore, fermionic algebra can be obtained from normalized $R$-deformed Heisenberg algebra (4.3) in the limit $|\nu| \to +\infty$.

Algebra (4.4) has a natural generalization related to the $q$-deformed Heisenberg algebra. To show this, we note that $R$ is a phase operator, $R^2 = 1$. Then commutation relation (4.4) can be generalized to

$$
[a, \bar{a}] = P,
$$

where $P$ is a phase operator with the properties

$$
P^{p'} = 1, \quad Pa = qaP, \quad P\bar{a} = q^{-1}\bar{a}P, \quad q = e^{-\frac{2\pi i}{p'}}, \quad p' = 2, 3, \ldots.
$$

At $p' = 2$, relations (1.3), (4.6) reproduce equations (4.4). Using these relations, one finds that operators $a^{p'}$ and $\bar{a}^{p'}$ commute with operators $a$, $\bar{a}$ and $P$. Hence, in irreducible representation they are reduced to some constants. Assuming the existence of the vacuum state $|0\rangle$, $a|0\rangle = 0$, one finds that in Fock representation of algebra (4.3), (4.4), we have the relations $a^{p'} = \bar{a}^{p'} = 0$. Multiplying the relation (1.3) from the left by the operator $P^{-1} = P^{p'-1}$, we reduce it to the form of Lie-admissible algebra (13): $aT\bar{a} - \bar{a}Sa = 1$, $T = q^{-1}P^{-1}$, $S = qP^{-1}$. Defining new creation-annihilation operators as $c = q^{-1/2}aP^{-1/2}$, $\bar{c} = q^{-1/2}P^{-1/2}\bar{a}$, one gets finally the normalized $q$-deformed Heisenberg algebra (18, 19), $c\bar{c} - q\bar{c}c = 1$. Let us note that the $q$-deformed Heisenberg algebra in this form with $q$ being a primitive root of unity, $q^{p'} = 1$, was used by Chou for introducing the so called genon statistics (17).

Thus, the normalized form of $R$-deformed Heisenberg algebra has the structure of guon algebra (13). In the limit case $|\nu| \to \infty$ such guon-like algebra turns into the standard fermionic algebra represented in the form of $R$-algebra (4.4). The natural generalization of the latter into the algebra (1.3), (4.6) involving the phase operator $P$ results in $q$-deformed Heisenberg algebra with $q$ being a primitive root of unity.

### 5 Outlook

Let us discuss briefly possible physical applications of the obtained results.
Recently it was shown [24] that finite-dimensional representations of the \( q \)-deformed Heisenberg algebra give a possibility to construct new variants of parasupersymmetry. Their physical properties can be essentially different from the properties of the parasupersymmetries realized in terms of the standard parafermionic algebra [20, 25]. It seems to be interesting to investigate analogous possibility of constructing parasupersymmetric systems with the help of deformed parafermionic algebra obtained here. Perhaps, the intrinsic \( \mathbb{Z}_2 \)-grading structure peculiar to the deformation could find physically interesting consequences.

We have established a universality of quadratic realization of \( so(2, 1) \) generators in terms of \( R \)-deformed Heisenberg algebra operators \( a^+, a^- \). The generators have the same operator form in the cases of infinite-dimensional half-bounded unitary representations and finite-dimensional non-unitary representations of (2+1)-dimensional Lorentz group. This universality can be used for the construction of the set of linear differential equations which will describe either fractional spin fields (anyons) or ordinary integer and half-integer spin fields. The equations and corresponding field action will be governed by the choice of the appropriate value of the deformation parameter \( \nu \) in the underlying \( R \)-deformed Heisenberg algebra [26]. Moreover, the related \( osp(1|2) \) superalgebraic structure can supply us with a natural basis for realizing (2+1)-dimensional supersymmetry [26]. This possibility is based on the fact that every described \( osp(1|2) \) representation contains a direct sum of two corresponding \( so(2, 1) \) representations with spin shifted in one-half.

The constructed guon-like algebra of the form (4.2) or (4.3) contains the operator-valued function \( g \). Unlike the original guon algebra [15], here \( g^2 \neq 1 \) and \( [g, c^\pm] \neq 0 \). The condition of the form \( [g, c^\pm] = 0 \) appeared in [15] from the requirement of micro causality under assumption that observables should be bilinear in fields or in creation-annihilation operators. On the other hand, it is known that in the field-theoretical anyonic constructions involving the Chern-Simons gauge field, there are observables (e.g., total angular momentum operator) which are not bilinear in creation-annihilation operators [27]. Moreover, the gauge-invariant fields carrying fractional spin and statistics themselves turn out to be nonlocal operators [28] being decomposable in some infinite series in degrees of creation-annihilation operators of the initial matter field. It seems that guon-like algebra appeared here could also find some applications in the theory of anyons. In particular, it could be useful for establishing spin-statistics relation for fractional spin fields within the framework of the group-theoretical approach to anyons [7].

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A Realization of \( R \)-paragrassmann algebra on \( q \)-paragrassmann algebra

Due to the isomorphism between non-degenerate paragrassmann algebras of the same order [12], \( R \)-paragrassmann algebra (6.1), (6.2) can be realized with the help of \( q \)-paragrassmann
algebra of order $2p + 1$. The latter is given by the relations
\[ \partial \theta - q \theta \partial = 1, \quad \theta^{2p+1} = \partial^{2p+1} = 0, \quad q^{2p+1} = 1, \quad \partial 1 = 0. \]

For this algebra we have the relations
\[ \partial^n \theta = (n-1)_q \theta^{n-1} + q^n \theta^n, \quad \partial^n \theta = (n-1)_q \theta^{n-1} + q^n \theta^n \partial, \]
where $n = 1, \ldots, 2p$, and we have used the notation $(n)_q = \sum_{k=0}^{n} q^k$. With the help of this algebra, the operators $a^+$ and $R$ can be realized as $a^+ = \theta$ and
\[ R = \sum_{n=0}^{2p} f_n \theta^n \partial^n, \tag{A.1} \]
where
\[ f_0 = 1, \quad f_{n+1} = (-1)^n (1 + q^2)^{n+1} \prod_{k=0}^{n} (1 + q^k), \quad n = 0, \ldots, 2p - 1, \]
and so, $f_1 = -2$ and $f_2 = 2$ for any $p$. The operator $R$ anticommutes with $\theta$ and $q$-derivative $\partial$. Our differentiation operator $a^-$ is realized in the form
\[ a^- = \sum_{n=0}^{2p-1} g_n \theta^n \partial^{n+1} \tag{A.2} \]
with coefficients given by the recursive relations
\[ g_n(n)_q + g_{n-1}(q^n - 1) = -(2p + 1)f_n, \quad n = 1, \ldots, 2p - 1, \quad g_0 = -2p. \]

Here the highest coefficient $g_{2p-1}$ has the form $g_{2p-1} = (2p - 1)f_{2p}/(1 - q^{2p})$.

In the simplest case of $p = 1$, one has
\[ a^- = -2\partial + 6(1 - q^2)^{-1}\theta \partial^2, \quad R = 1 - 2\theta \partial + 2\theta^2 \partial^2. \]

For $p = 2$, the operators $R$ and $a^-$ are given by relations (A.1) and (A.2) with nontrivial coefficients
\[ f_0 = 1, \quad f_1 = f_2 = 2, \quad f_3 = 2q^2(1 + q^2) \cdot (1 + q)^{-1}, \quad f_4 = 2(3 + 2q^2 + 2q^3), \]
\[ g_0 = -2, \quad g_1 = (6 + 4q)(1)^{-1}, \quad g_2 = -4 + 2q(2)^{-1}, \quad g_3 = 10(2q^3 + q^2 + q + 2)(1 - q^2)^{-1}. \]

### B Finite-dimensional matrix representations: $p = 1, 2$

Here we give the explicit expressions for matrix finite-dimensional representations of the $R$-deformed Heisenberg algebra and realization of $\text{so}(2,1)$ generators for two simplest cases, $p = 1$ and $p = 2$.

For $p = 1$, the corresponding operators are realized as
\[ a^+ = \begin{pmatrix} \sqrt{2} & & \\ & \ddots & \\ \sqrt{2} & & \end{pmatrix}, \quad a^- = \begin{pmatrix} -\sqrt{2} & & \\ & \ddots & \\ & & \sqrt{2} \end{pmatrix}, \]
\[ R = \sum_{n=0}^{2} f_n \theta^n \partial^n, \tag{A.1} \]
where
\[ f_0 = 1, \quad f_{n+1} = (-1)^n (1 + q^2)^{n+1} \prod_{k=0}^{n} (1 + q^k), \quad n = 0, \ldots, 2 - 1, \]
and so, $f_1 = -2$ and $f_2 = 2$ for any $p$. The operator $R$ anticommutes with $\theta$ and $q$-derivative $\partial$. Our differentiation operator $a^-$ is realized in the form
\[ a^- = \sum_{n=0}^{2-1} g_n \theta^n \partial^{n+1} \tag{A.2} \]
with coefficients given by the recursive relations
\[ g_n(n)_q + g_{n-1}(q^n - 1) = -(2p + 1)f_n, \quad n = 1, \ldots, 2p - 1, \quad g_0 = -2p. \]

Here the highest coefficient $g_{2p-1}$ has the form $g_{2p-1} = (2p - 1)f_{2p}/(1 - q^{2p})$.

In the simplest case of $p = 1$, one has
\[ a^- = -2\partial + 6(1 - q^2)^{-1}\theta \partial^2, \quad R = 1 - 2\theta \partial + 2\theta^2 \partial^2. \]

For $p = 2$, the operators $R$ and $a^-$ are given by relations (A.1) and (A.2) with nontrivial coefficients
\[ f_0 = 1, \quad f_1 = f_2 = 2, \quad f_3 = 2q^2(1 + q^2) \cdot (1 + q)^{-1}, \quad f_4 = 2(3 + 2q^2 + 2q^3), \]
\[ g_0 = -2, \quad g_1 = (6 + 4q)(1)^{-1}, \quad g_2 = -4 + 2q(2)^{-1}, \quad g_3 = 10(2q^3 + q^2 + q + 2)(1 - q^2)^{-1}. \]

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For $p = 1$, the corresponding operators are realized as
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\[ R = \sum_{n=0}^{2} f_n \theta^n \partial^n, \tag{A.1} \]
where
\[ f_0 = 1, \quad f_{n+1} = (-1)^n (1 + q^2)^{n+1} \prod_{k=0}^{n} (1 + q^k), \quad n = 0, \ldots, 2 - 1, \]
and so, $f_1 = -2$ and $f_2 = 2$ for any $p$. The operator $R$ anticommutes with $\theta$ and $q$-derivative $\partial$. Our differentiation operator $a^-$ is realized in the form
\[ a^- = \sum_{n=0}^{2-1} g_n \theta^n \partial^{n+1} \tag{A.2} \]
with coefficients given by the recursive relations
\[ g_n(n)_q + g_{n-1}(q^n - 1) = -(2p + 1)f_n, \quad n = 1, \ldots, 2p - 1, \quad g_0 = -2p. \]

Here the highest coefficient $g_{2p-1}$ has the form $g_{2p-1} = (2p - 1)f_{2p}/(1 - q^{2p})$.

In the simplest case of $p = 1$, one has
\[ a^- = -2\partial + 6(1 - q^2)^{-1}\theta \partial^2, \quad R = 1 - 2\theta \partial + 2\theta^2 \partial^2. \]

For $p = 2$, the operators $R$ and $a^-$ are given by relations (A.1) and (A.2) with nontrivial coefficients
\[ f_0 = 1, \quad f_1 = f_2 = 2, \quad f_3 = 2q^2(1 + q^2) \cdot (1 + q)^{-1}, \quad f_4 = 2(3 + 2q^2 + 2q^3), \]
\[ g_0 = -2, \quad g_1 = (6 + 4q)(1)^{-1}, \quad g_2 = -4 + 2q(2)^{-1}, \quad g_3 = 10(2q^3 + q^2 + q + 2)(1 - q^2)^{-1}. \]
where dots correspond to zero elements, and \( R = \text{diag}(1, -1, 1) \), \( \eta = \text{diag}(1, -1, -1) \), \( J_0 = \text{diag}(-\frac{1}{2}, 0, \frac{1}{2}) \),

\[
J_1 = \frac{1}{2} \begin{pmatrix} \cdots & \cdot & -1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots \end{pmatrix}, \quad J_2 = \frac{i}{2} \begin{pmatrix} \cdots & \cdot & -1 \\ \cdots & \cdots & \cdots \\ -1 & \cdots & \cdots \end{pmatrix}.
\]

Here generators \( J_{1,2} \) are related to \( J_{\pm} \) as \( J_{\pm} = J_1 \pm iJ_2 \). In the even subspace, where the operator \( R \) takes the value +1, the metric operator and generators of \( \text{so}(2,1) \) are reduced to the following matrices: \( \eta = \sigma_3 \), \( J_0 = -\frac{1}{2}\sigma_3 \), \( J_1 = -\frac{i}{2}\sigma_2 \), \( J_2 = -\frac{i}{2}\sigma_1 \). Therefore, in this subspace the indefinite scalar product is the Dirac scalar product: \((\Psi_1, \Psi_2) = \bar{\Psi}_1 \Psi_2 \), where \( \bar{\Psi} = \Psi^{\dagger} \gamma_0 \) is the Dirac conjugate wave function and \( \gamma_0 = -2J_0 = \sigma_3 \).

For \( p = 2 \) we have

\[
a^+ = \begin{pmatrix} 2 & \cdots & \cdots & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdot & \sqrt{2} & \cdots & \cdots \\ \cdot & \cdot & \sqrt{2} & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad a^- = \begin{pmatrix} -\frac{2}{\sqrt{2}} & \cdots & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} \\ \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},
\]

and \( R = \text{diag}(1, -1, 1, -1, 1) \), \( \eta = \text{diag}(1, -1, -1, 1, 1) \), \( J_0 = \text{diag}(-1, -\frac{1}{2}, 0, \frac{1}{2}, 1) \),

\[
J_1 = \frac{1}{2} \begin{pmatrix} \cdots & \cdot & \sqrt{2} & \cdots & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & 1 & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} \end{pmatrix}, \quad J_2 = -\frac{i}{2} \begin{pmatrix} \cdots & \cdot & \sqrt{2} & \cdots & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & 1 & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} \end{pmatrix}.
\]

On the even subspace the metric operator is reduced to \( \eta = \text{diag}(1, -1, 1) \). This operator and corresponding \( \text{so}(2,1) \) generators being reduced to this subspace are unitary equivalent to the standard vector realization of \((2 + 1)\)-dimensional Lorentz group. The latter is given by \( \tilde{\eta}_{\alpha\beta} = \text{diag}(-1, 1, 1) \) and \((\tilde{J}_\mu)^\alpha_\beta = -i\epsilon^{\alpha}_{\mu\beta} \), \( \alpha, \beta, \mu = 0, 1, 2 \), with totally antisymmetric tensor normalized as \( \epsilon^{012} = 1 \) \cite{23}. The equivalence is established by the relations \( \tilde{\eta} = U\eta U^\dagger \), \( \tilde{J}_\mu = UJ_\mu U^\dagger \), where \( U \) is the unitary matrix,

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & -\sqrt{2} \\ \cdot & 1 & \cdot \\ \cdot & \cdot & i \end{pmatrix}.
\]
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