Fully nonlinear Jeans instabilities for expanding Newtonian universes under homogeneous and isotropic perturbations

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Abstract

Based on mathematically rigorous analysis of nonlinear differential equations studied in our companion article [1], we construct a model which describes the nonlinear gravitational instability on a local portion of the universe characterized by the expanding Newtonian universe. In this portion, the perturbations are homogeneous and isotropic. This result, to some extent, can be viewed as a nonlinear version of the Jeans instability. The growth rate of the relative density due to the nonlinear effects is much faster (at least $\sim \exp(t^2)$ or blowup at a finite time according to the data) than the one predicted by the classical linear version of the Jeans instability ($\sim t^2$), and it leads to a better, or potentially substantial impacts on, understanding of the formation of the nonlinear structures in the universe and stellar systems. This article associated with [1] provides a new window into the rigorously mathematical and robust method, instead of the most used approximations and numerical calculations, of the fully nonlinear analysis of the Jeans instability for general cases.

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I. INTRODUCTION

The gravitational instability characterizes the mass accretions of self-gravitating systems and helps us understand the formations of stellar systems and the nonlinear structures in the universe. The first gravitational instability was studied by Jeans [2] for Newtonian gravity in 1902 (thus it is also known as Jeans instability). However, it is worth noting that Jeans’ work is only in the linear regime since he linearized the Euler–Poisson system. Then it was generalized to general relativity by Lifshitz [3] in 1946. Due to the expanding universe, the Jeans instability was extended to the expanding background universe by Bonnor [4] (also see [5, 6]). After these works, most studies of Jeans instability refer to the solution with the growing density of the linearized Euler–Poisson system (see [5, 6]). However, the linear Jeans instability has some inconveniences. The first inconvenience comes from the linearization of the Euler–Poisson system. Due to this linearizations, the linear Jeans instability can be only applied to the case with small perturbations of the uniform density distribution (i.e., the relative density \( \varrho := (\rho - \bar{\rho})/\bar{\rho} < 1 \)) and only for a short time before the perturbations growing large, since the larger perturbations will lead to the larger deviations from the linearized scheme. With the accretions of the mass, the derivations of the linear Jeans instability will be completely spoiled since the increasing density leads to the derivations significantly deviate the linear regime. On the other hand, the second inconvenience is the growth rate of the density predicted by the classical linearized version of the Jeans instability can not yield the observed large inhomogeneities of the universe nowadays and formations of galaxies, because this growth rate (\( \sim t^{\frac{3}{2}} \), see [4–7]) is too slow and thus is much less efficient. Therefore, it is urgent to study the fully nonlinear Jeans instability and as pointed out by Rendall [8] in 2002, there are no results on Jeans instability available for the fully nonlinear case, and it becomes a long-standing open problem.

In fact, the key result of this article is that, by the mathematical tools developed in our companion paper [1], the rigorous derivations confirm, by taking the nonlinear effects of certain cases into account, the growth rates of the relative density are at least of order \( \sim \exp(t^{\frac{3}{2}}) \) (see later (19) for details) or blow up at the finite time (see (20)). This is much faster than the one given by the classical Jeans instability of the linearized Euler–Poisson system (\( \sim t^{\frac{3}{2}} \)).

Some nonlinear strategies have been discussed in several references, for example, [5, 6,
However, most of these works involve approximations and numerical methods, for example, the famous Zel’dovich solutions (see [5, 6]). Another famous exact solution describing the evolutions and collapses of the inhomogeneity is the Tolman solution (see [6, §6.4.1], [11]). Although the Tolman solution gives an exact spherical symmetric dust solution, it can not include the fluids with the nonvanishing pressure or non-spherical symmetric distributions, and the inconvenience of the parametric form of the solution is too complex to visualize the actual behaviors of it.

The final goal of our previous works [1, 7] and the current one is to build and develop a robust and systematic method for the fully nonlinear analysis of the Jeans instability which eventually relies on the mathematically rigorous analysis of the nonlinear differential equations (without approximations when analyzing the solutions). In this article, as a simplified testing ground, by neglecting the spatial inhomogeneities and considering a local homogeneous perturbation, we mostly focus on the nonlinearities of the Euler–Poisson system. This local homogeneous perturbation can be viewed as an inhomogeneous perturbation on a larger scale of the universe but it fails in capturing the behaviors near the boundary of the local region that we are focusing on (see details in the following model).

The method in this article associated with the framework in [1] provides a new window into the rigorously theoretical analysis of the fully nonlinear Jeans instability for general cases (e.g., cases with the nonspherical symmetry or pressure) instead of the most used approximations and numerical methods or for some special cases such as Tolman solutions. These general cases are being prepared and will be presented in near future.

II. MODELS

We imagine the whole universe is modeled by the FLRW metric in general relativity and there is a relativistic coordinate \{\bar{x}^{\mu}\} associating with it. For every small enough local region of the universe, we use Newtonian gravity to approximate it. To be specific, let us assume\(^1\) \(B(0, 2R)\) is a domain such that in the exterior of \(B(0, 2R)\), the universe has FLRW metric and assume there is a region \(B(0, R)\), such that the portion in \(B(0, R)\) of the universe can be approximated by the homogeneous, isotropic and expanding Newtonian universe. According to the fact (see, for example, [12, p. 75]) that in the exterior region of \(B(0, R)\),

\(^1\) We use notation \(B(0, R) := \{\bar{x}^{i} \mid ||\bar{x}^{i}|| < R\}\) to denote a ball centered at the origin 0 with the radius \(R\), in the relativistic coordinate \(\bar{x}^{i}\).
the FLRW metric has no influence on the interior of $B(0, R)$. Noting the relation between the Newtonian coordinate $x^i$ and the relativistic coordinate $\bar{x}^i = \epsilon x^i$ (this expression of mathematically rigorous Newtonian limits can be found in [13–15]), the Newtonian gravity dominates in the domain for $x^i \in B(0, R/\epsilon)$ (in the Newtonian coordinate). Let us take $\epsilon \searrow 0$, the Newtonian gravity holds almost for all $x^i \in \mathbb{R}^3$ in the Newtonian coordinate since $R/\epsilon \to \infty$ as $\epsilon \searrow 0$. In this article, we consider a homogeneous and isotropic perturbation for the relativistic coordinate $\bar{x}^i \in B(0, R)$ (i.e., $x^i \in B(0, R/\epsilon)$ in the Newtonian coordinate) of this homogeneous and isotropic Newtonian expanding local universe. In this model, if we can prove in the local expanding Newtonian universe, the growth rate (at least $\sim \exp(t^{3/2})$, see (19)) of relative density due to the nonlinear Jeans instability is way faster than the one $\sim t^{3/2}$ predicted by the classical linear Jeans instability, then we can, to some extent, conclude the way faster growth rates of the local inhomogeneity yielded by the nonlinear effects of the Jeans instability. Note although the perturbation in $B(0, R)$ is homogeneous, in the large scales, there is only a perturbation for $\bar{x}^i \in B(0, R)$. For all $\bar{x}^i \in \mathbb{R}^3$, it can be viewed as an inhomogeneous perturbation, but this model fails to describe the behaviors outside $B(0, R)$. This defect would be compensated by our preparing works in near future, and this idea with the framework in [1] should work.

For the nonlinear version of the Jeans instability, we can not use Fourier analysis to solve the nonlinear differential equations derived from the Euler–Poisson system (see [7] for alternative non-Fourier based proof of the linear Jeans instability). In [1], we developed some preparing techniques for a class of the nonlinear ordinary differential equations (ODE) and hyperbolic equations for the nonlinear Jeans instability. We intend to apply it to conclude the result of this article.

Let us focus on the local expanding Newtonian universe by using the Newtonian coordinate $x^i$. This can be described by the Euler–Poisson system (we use the Einstein summation convention), i.e.,

$$\partial_t \rho + \partial_i (\rho v^i) = 0, \quad (1)$$

$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0, \quad (2)$$

$$\Delta \phi = \delta^{ij} \partial_i \partial_j \phi = 4\pi G \rho, \quad (3)$$

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with initial data at $t = t_0$,

$$\dot{\rho}(t_0) = \frac{1}{6\pi G t_0^2}, \quad \dot{\rho}(t_0) = p(\dot{\rho}(t_0)),$$

(4)

$$\dot{v}^i(t_0, x^k) = \frac{2}{3t_0} x^i, \quad \dot{\phi}(t_0, x^k) = \frac{2}{3} \pi G \dot{\rho}(t_0) \delta_{ij} x^i x^j,$$

(5)

where $\rho(t, x^i)$, $v^i(t, x^k)$, $p(\rho)$, and $\phi(t, x^{\cdot})$ are the energy density, 3-velocities, pressure of the fluids and gravitational potential, respectively.

An expanding, homogeneous and isotropic Newtonian universe according to the assumptions in [6, §6.3] can be given below. Due to the homogeneity and isotropy, the velocities obey the Hubble law:

$$\rho(t, x^k) = \dot{\rho}(t), \quad v^i(t, x^k) = \dot{v}^i(t, x^k) = H(t) x^i.$$

(6)

Then, substituting $^2$ (6) into the Euler–Poisson system (1)–(3), we arrive at the conservation of the total mass and Friedmann equation (see [6, §6.3] for details),

$$\partial_t \dot{\rho} + 3H \dot{\rho} = 0,$$

(7)

$$\partial_t H + H^2 = -\frac{4\pi G}{3} \dot{\rho}. $$

(8)

In this case, the data (4)–(5) become

$$\dot{\rho}(t_0) = \frac{1}{6\pi G t_0^2} \quad \text{and} \quad H(t_0) = \frac{2}{3t_0}.$$

There is a well known exact solution to this homogeneous and isotropic Euler–Poisson (8) (see [6, §1.2.3] or [5, §10.2]),

$$\dot{\rho}(t) = \frac{1}{6\pi G t^2} \quad \text{and} \quad H(t) = \frac{2}{3t}.$$

(9)

Furthermore, a solution to the Euler–Poisson system (1)–(3) is obtained by

$$\dot{\rho}(t) = \frac{1}{6\pi G t^2}, \quad \hat{\rho}(t) = p(\dot{\rho}(t)),$$

(10)

$^2$ Take the divergence of the momentum equation in the Euler equation and use the Poisson equation to obtain the second equation in (8).
\[ \dot{v}^i(t, x^k) = \frac{2}{3t} x^i \quad \text{and} \quad \ddot{\phi}(t, x^k) = \frac{2}{3} \pi G \dot{\rho} \delta_{ij} x^i x^j, \quad (11) \]

where \( p \) is a smooth, positive and increasing function, are the exact homogeneous solution to the Euler–Poisson system (1)–(3) in \((t, x^k) \in [t_0, \infty) \times \mathbb{R}^3\). In order to simplify the expressions in this article, in the rest we normalize \( t_0 = 1 \), while for general \( t_0 \neq 1 \), the results can be obtained by taking the time transform \( \hat{t} := t_0 t \).

### III. MAIN RESULTS AND IDEAS

This article intends to conclude the nonlinear behavior of the homogeneous and isotropic perturbations of the relative density \( \dot{\rho} := (\rho - \dot{\rho})/\rho \) by the following two steps and the main results are given by the following estimates (19)–(20) of the lower bounds of the growth rate of the relative density.

Step 1: Let us assume \( \beta \) and \( \gamma \) are two given positive constants, the initial data (at \( t = 1 \)) of (1)–(3) have homogeneous and isotropic initial perturbations comparing with the data (4)–(5) which characterized by the given positive constants \( \beta \) and \( \gamma \) in the following ways,

\[ \rho(1) = (1 + \beta) \dot{\rho}(1), \quad p(1) = p(\rho(1)), \quad (12) \]
\[ v^i(1, x^i) = \left( \frac{2}{3} - \gamma \right) x^i, \quad (13) \]

Then the solution of the Euler–Poisson system (1)–(3) will become (we use notation \( (\cdot)' := d(\cdot)/dt \))

\[ \rho(t) = (1 + f(t)) \dot{\rho}(t) = \frac{1 + f(t)}{6\pi G t^2}, \quad p(t) = p(\rho(t)), \quad (14) \]
\[ v^i(t, x^i) = \frac{2}{3t} x^i - \frac{f'(t)}{3(1 + f(t))} x^i, \quad (15) \]
\[ \dot{\phi}(t, x^i) = \frac{2}{3} \pi G \dot{\rho} (1 + f(t)) |x|^2 = \frac{(1 + f(t)) |x|^2}{9t^2} \quad (16) \]

and the relative density \( g(t) = f(t) \) where \( |x|^2 := \delta_{ij} x^i x^j \) and \( f(t) \) is a solution of a nonlinear ODE,

\[ f''(t) + \frac{4}{3t} f'(t) - \frac{2}{3t^2} f(t)(1 + f(t)) - \frac{4 (f'(t))^2}{3(1 + f(t))} = 0, \quad (17) \]
\[ f|_{t=t_0} = \beta \quad \text{and} \quad f'|_{t=t_0} = 3(1 + \beta)\gamma. \]  \text{(18)}

**Step 2:** In Step 1, we have represented the perturbation solution in terms of functions \( f(t) \) and its derivative \( f_0(t) := f'(t) \). In order to understand the behaviors of the perturbation solution, especially the growth rates of the relative density \( \varrho \), we have to know the detailed behaviors of the functions \( f \) and \( f_0 \). In fact, the behaviors of \( f \) and \( f_0 \) can be acquired by solving the ODE (17)–(18). This ODE (17)–(18), in fact, has been studied in our companion article [1] which includes this ODE and a class of the second order nonlinear hyperbolic equations which may work for the nonlinear Jeans instability for general cases. We list the conclusions of the solutions to the ODE (17)–(18) in Theorem 1 (see the last section on the Mathematical appendix) and we use these results to conclude the behaviors of the homogeneous perturbations. In fact, eventually, the relative density has the lower bound estimate according to Theorem 1, for \( t \in (1, t_m) \),

\[ \varrho(t) = f(t) > \exp\left(\frac{3(\ln(1 + \beta) + 3\gamma)t^\frac{2}{3} + 2(\ln(1 + \beta) - \frac{3}{2}\gamma)t^{-\frac{1}{2}}}{5}\right) - 1. \] \text{(19)}

In addition, by Theorem 1, if further the initial data satisfies \( \gamma > 1/3 \), we have an improved lower bound estimate on the growth rate of the relative density, i.e., for \( t \in (1, t_m) \),

\[ \varrho(t) = f(t) > \frac{1 + \beta}{(1 - 3\gamma + 3\gamma t^{\frac{1}{3}})^3} - 1. \] \text{(20)}

The lower bound of \( \varrho \) blows up at \( t^* = \left(1 - \frac{1}{3\gamma}\right)^{-3} > t_0 = 1 \). These lower bounds give an estimate of the growth rates of the relative density.

In the rest of this article, we will only need to elaborate Step 1, i.e., solving the Euler–Poisson system (1)–(3) under the perturbed data (12)–(13) and further the perturbation equations.
IV. EQUATIONS OF PERTURBATIONS

Let us first decompose the variables ($\rho, v^i, p, \phi$) to the exact background solution (10)–(11) and the perturbed parts, along with a relative density $\rho$,

$$\rho = \hat{\rho} + \tilde{\rho}, \quad v^i = \hat{v}^i + \tilde{v}^i, \quad \phi = \hat{\phi} + \tilde{\phi}, \quad \text{(21)}$$

$$p = \hat{p} + \tilde{p}, \quad \text{and} \quad \varrho := \frac{\tilde{\rho}}{\hat{\rho}}. \quad \text{(22)}$$

Let us introduce the Lagrangian coordinates $q^k$ defined by $x^k = a(t)q^k$ where $a(t)$ := 1, and the time derivatives are obtained at $q^k$ (i.e., the material derivatives). We also denote

$$D_t := \partial_t|_{q^k} = \partial_t|_{x^k} + \hat{v}^i \partial_i = \partial_t|_{x^k} + Hx^j \partial_j, \quad \text{(23)}$$

$$D_i := a(t)\partial_i. \quad \text{(24)}$$

Note in the next, we will slightly abuse the notations and do not distinguish the variables in terms of the Eulerian $x^i$ and Lagrangian coordinate $q^i$, that is, we abuse, for example, $\tilde{v}^k(t, x^i)$ and $\tilde{v}^k(t, q^i)$ for the simplicity of the statements and readers should be clear according to the contexts.

Let us review how to reexpress the Euler–Poisson system (1)–(3) in terms of the perturbation variables ($\varrho, \tilde{v}^i, \tilde{\phi}$) given by (21)–(22). Firstly, we focus on the conservation of mass. Substituting the decomposition (21)–(22) into the equation (1), and subtracting the background (7), the conservation of mass (1) becomes

$$\partial_t \hat{\rho} + \partial_i (\hat{\rho} \hat{v}^i) + \partial_i (\hat{\rho} \tilde{v}^i) + \partial_i (\hat{\rho} \tilde{v}^i) = 0.$$

Using the Hubble laws (6) and applying the Lagrangian coordinates (23)–(24), the above equation becomes

$$D_t \hat{\rho} + 3H \hat{\rho} + \hat{\rho} a^{-1} D_i \tilde{v}^i + \hat{\rho} a^{-1} D_i \hat{v}^i + \tilde{v}^i a^{-1} D_i \tilde{\rho} = 0.$$

Using (22), with the help of (7), we replace $\partial_i \tilde{\rho}$ by $-3H \hat{\rho}$ and after straightforward calcula-

\[^3\text{In fact, } a(t) = a(1)t^\frac{1}{2} = t^\frac{1}{2} \text{ provided } a(1) = 1, \text{ since by the Hubble law (6) and the Lagrangian coordinates } x^k = a(t)q^k, \text{ we obtain } H(t) := \frac{\hat{a}(t)}{a(t)}. \text{ Then by (9), we can solve } a(t).\]
tions, we obtain
\[ D_t \varrho + (1 + \varrho) a^{-1} D_i \tilde{v}^i + \tilde{v}^i a^{-1} D_i \varrho = 0. \] (25)

Secondly, by (22) and (24), the Poisson equation in terms of the Lagrangian coordinate \( q^i \) becomes
\[ \delta^{ij} D_i D_j \tilde{\phi} = 4\pi a^2 G \tilde{\rho} = 4\pi a^2 G \dot{\rho} \varrho = \frac{2}{3 t^3} \varrho. \] (26)

In the end, we turn to the balance of momentum (2). Note (8), with the help of (11), implies
\[ x^i \partial_t H + H^2 x^i + \partial^i \tilde{\phi} = 0. \] (27)

By subtracting the background (27) from (2), and in terms of Lagrangian coordinates, using (23)–(24), with the help of (22), the balance of momentum (2) becomes
\[ D_t \tilde{v}^i + H \tilde{v}^i + \tilde{v}^i a^{-1} D_j \tilde{v}^j + \frac{a^{-1} D_i (\tilde{p}/\dot{\rho})}{1 + \varrho} + a^{-1} \tilde{D}^i \tilde{\phi} = 0. \] (28)

Gathering above equations (25), (26) and (28) together, the Euler–Poisson system (1)–(3), in terms of the Lagrangian coordinates, become
\[ D_t \varrho + \frac{(1 + \varrho)}{a} D_i \tilde{v}^i + \frac{\tilde{v}^i}{a} D_i \varrho = 0, \] (29)
\[ D_t \tilde{v}^i + H \tilde{v}^i + \frac{\tilde{v}^j}{a} D_j \tilde{v}^i + \frac{D_i (\tilde{p}/\dot{\rho})}{a(1 + \varrho)} + \frac{1}{a} D^i \tilde{\phi} = 0, \] (30)
\[ \delta^{ij} D_i D_j \tilde{\phi} = 4\pi a^2 G \dot{\rho} \varrho = \frac{2}{3 t^3} \varrho. \] (31)

V. HOMOGENEOUS AND ISOTROPIC PERTURBATIONS

In this section, let us focus on solving the equations (29)–(31) if there is a homogeneous and isotropic initial perturbation (12)–(13). In this assumption, we have the relative density is independent of the spatial variables and the velocity satisfies the Hubble’s law, that is, we can assume the forms of \( \varrho \) and \( \tilde{v}^i \) by
\[ \varrho(t, q^k) = \varrho(t) \quad \text{and} \quad \tilde{v}^i(t, q^k) = \tilde{H}(t) q^i \] (32)
where the function $\tilde{H}(t)$ is to be determined. In this case, by (31), direct calculations imply

$$
\tilde{\phi} = \frac{2}{3} \pi a^2 G \tilde{\rho} \dot{\varrho}(t) |q|^2 = \frac{f|q|^2}{9t^{\frac{5}{2}}} \quad \text{and} \quad \delta^{ij} D_i \tilde{\phi} = \frac{2 \dot{\varrho}(t)}{9t^{\frac{5}{2}}} q^j
$$

(33)

where $|q|^2 := \delta_{ij} q^i q^j$.

Now our task becomes to solve $\varrho(t)$ and $H(t)$ from the equations (29) and (30), while substituting (33) into (30). Taking the divergence of $\tilde{\nu}^i$ leads to $D_i \tilde{\nu}^i = 3 \tilde{H}(t)$ and substituting this into (29), we obtain, with the help of (32),

$$
\tilde{H}(t) = -\frac{t^{\frac{5}{2}}}{3} (\ln(1 + \varrho(t)))' \Rightarrow \tilde{\nu}^i = -\frac{t^{\frac{5}{2}}}{3} q^i (\ln(1 + \varrho(t)))'.
$$

(34)

Next, let us solve $\varrho(t)$. Taking (33) and (34) into (30), straightforward calculations imply

$$
\dot{\varrho}''(t) + \frac{4}{3t} \dot{\varrho}'(t) - \frac{2}{3t^2} \varrho(t)(1 + \varrho(t)) - \frac{4}{3} \left( \frac{\varrho'(t)}{1 + \varrho(t)} \right)^2 = 0.
$$

(35)

By the data (12)–(13) and the definition of the perturbed variables (21)–(22), we obtain the data of $\varrho(t_0)$,

$$
\varrho|_{t=1} = \beta.
$$

(36)

Since (35) is a second order ODE, to solve this equation, we have to know the initial data of $\varrho'|_{t=1}$. In order to find this data and solve the Euler–Poisson system (29)–(31), we note (29) must hold at the initial time $t = 1$. Thus, with the help of (12)–(13) and the data $\tilde{\nu}^i|_{t=1} = (v^i - \tilde{v}^i)|_{t=1} = -\gamma q^i$ (by (5) and (13), and noting $a(1) = 1$), a necessary condition of (29) is

$$
\varrho'|_{t=1} = -\left( \frac{1 + \varrho}{a} D_i \tilde{\nu}^i \right)|_{t=1} - \left( \frac{\tilde{\nu}^i}{a} D_i \varrho \right)|_{t=1} = 3(1 + \beta) \gamma.
$$

(37)

Using the ODE (35) with the data (36) and (37), with the help of Theorem 1 (see the last section on the Mathematical appendix), we arrive at

$$
\varrho(t) = f(t).
$$

(38)
Using (34), we obtain
\[
\tilde{v}^j = -\frac{t^2}{3} q^j \partial_t \ln(1 + f(t)) = -\frac{t^2 f'(t)}{3(1 + f(t))} q^j.
\] (39)

By direct substituting these solutions (38)–(39) and (33) back into (29)–(31), we are able to verify (38)–(39) and (33) indeed solve the system (29)–(31). In the end, taking (38)–(39) and (33) to (21) and (22), we conclude the solutions (14)–(16). This completes Step 1.

VI. CONCLUSIONS AND DISCUSSIONS

Let us firstly compare the nonlinear growth rates (19) and (20) of the relative density with the one (∼ \(t^\frac{2}{3}\)) predicted by the classical linear version of the Jeans instability. We, by using \(t^{-1} \leq \tilde{t}^\frac{2}{3}\) for \(\ln(1 + \beta) - \frac{2}{5} \gamma < 0\) and \(t^{-1} > 0\) for \(\ln(1 + \beta) - \frac{2}{5} \gamma \geq 0\), note (19) implies
\[
\rho(t) > \exp(At^\frac{2}{3}) - 1 = At^\frac{2}{3} + O(t^\frac{4}{3})
\] (40)

where \(A := \min\{\ln(1 + \beta), \frac{2}{5}(\ln(1 + \beta) + 3\gamma)\}\) is a constant and \(O(t^\frac{4}{3})\) means the remainder terms are at least of order \(t^\frac{4}{3}\). We note the growth rate (∼ \(t^\frac{2}{3}\)) by the classical Jeans instability is just the first order approximation of the lower bound estimate (40) of the relative density \(\rho\) if expanding \(\exp(At^\frac{2}{3})\) with respect to \(t^\frac{2}{3}\). The nonlinear effects indeed significantly boost the growth of the relative density as \(\rho\) grows larger. According to the Taylor expansion (40), we see when \(t\) is small enough, it consists with the result of the classical linear Jeans instability. From these improved faster growth rates due to the nonlinear effects, we can see that indeed the classical Jeans instability with the linearized Euler–Poisson system can only be applied to the small initial perturbation of the density and only work for a short time before the density grows large enough. However, the nonlinear method proposed by this article does not require the initial perturbations small and it works for a long time before the Euler–Poisson system breaks down.

The method of this article relies on the nonlinear analysis of a type of differential equations, which is mathematically rigorous without approximations and numerical calculations. In addition, this method is robust and systematic, and if we also use the ideas and method (the Cauchy problem of the Fuchsian formulation of a second order hyperbolic equation
which allows certain pressure) from our previous paper [1, §3], it may be possible to study
the general cases of the nonlinear Jeans instability, at least for the case with nonvanishing
pressure and small inhomogeneous perturbations. The inaccuracy of this article comes from
the simplified physical model, but this will be complemented when we can take the pressure
into account. These works are progressing and will be presented in near future.

Although this model of the universe in the present paper is a simplified model, it can
still capture some of the main nonlinear effects on the locally homogeneous and isotropic
perturbations. Thus this result helps us have a better, or potentially substantial impacts
on, understanding of the formation of the nonlinear structures in the universe and stellar
systems. The lower bound estimates (19) and (20) of the growth rates of the relative den-
sity are far more efficient than the one predicted by the classical linear Jeans instability.
Therefore, it may be possible that these nonlinear results of the growth rates can contribute
to the explanations of the observed large inhomogeneities of the universe nowadays and the
formations of galaxies. Due to these much larger and more efficient growth rates of the
relative density, we may not require substantial initial perturbations of the relative density
at the early times of the universe and weaken the constraints on the initial spectrum of
perturbations.

In summary, this model gives mathematically rigorous and physically decent nonlinear
estimates on the growth rates of the relative density $\rho$ on the local portion of the universe
characterized by the Newtonian expanding universe. The mathematical tools and methods
have the potential for the more general cases of the nonlinear version of the Jeans instability.
Based on these tools and methods developed by our companion article [1], the fully nonlinear
Jeans instabilities both for the Newtonian universe and general relativity are in progress.

VII. MATHEMATICAL APPENDIX OF A NONLINEAR ODE

In [1, §2], we have proven a mathematical theorem on the solutions $f(t)$ to the following
type ODE,

\[ f''(t) + \frac{a}{t} f'(t) - \frac{6}{t^2} f(t)(1 + f(t)) - \frac{c(f'(t))^2}{1 + f(t)} = 0, \]

\[ f(t_0) = \beta > 0 \quad \text{and} \quad f'(t_0) = \beta_0 > 0. \]
where \( \beta, \beta_0 > 0 \) are positive constants and

\[
a > 1, \quad \delta > 0 \quad \text{and} \quad 1 < c < 3/2. \tag{43}
\]

From now on, in order to simplify the notations, we denote

\[
\Delta := \sqrt{(1-a)^2 + 4\delta} > -\bar{a}, \quad \bar{a} = 1 - a < 0, \quad \bar{c} = 1 - c < 0
\]

and we introduce constants \( A, B, C, D \) and \( E \) depending on the initial data \( \beta \) and \( \beta_0 \) to (41)–(42) and parameters \( a, \delta \) and \( c \),

\[
A := \frac{t_0^{\frac{\bar{a}-\Delta}{\Delta}}}{\Delta} \left( \frac{t_0\beta_0}{(1+\beta)^2} - \frac{\bar{a} + \Delta}{2} \frac{\beta}{1+\beta} \right),
\]

\[
B := t_0^{\frac{\bar{a}+\Delta}{\Delta}} \left( \frac{\bar{a} - \Delta}{2} \frac{\beta}{1+\beta} - \frac{t_0\beta_0}{(1+\beta)^2} \right) < 0,
\]

\[
C := \frac{2}{2 + \bar{a} + \Delta} \left( \ln(1+\beta) + \frac{\bar{a} + \Delta}{2\delta} \frac{t_0\beta_0}{1+\beta} \right) t_0^{\frac{\bar{a}+\Delta}{2}} > 0,
\]

\[
D := \frac{\bar{a} + \Delta}{2 + \bar{a} + \Delta} \left( \ln(1+\beta) - \frac{1}{\delta} \frac{t_0\beta_0}{1+\beta} \right) t_0,
\]

\[
E := \frac{\bar{c}\beta_0 t_0^{1-\bar{a}}}{\bar{a} (1+\beta)} > 0.
\]

We define the following two critical times \( t_* \) and \( t^* \).

1. Let \( \mathcal{R} := \{ t_r > t_0 \mid A t_r^{\frac{\bar{a}-\Delta}{\Delta}} + B t_r^{\frac{\bar{a}+\Delta}{\Delta}} + 1 = 0 \} \) and define \( t_* := \min \mathcal{R} \).

2. If \( t_0^{\bar{a}} > E^{-1} \), we define \( t^* := (t_0^{\bar{a}} - E^{-1})^{1/\bar{a}} \in (0, \infty) \), i.e., \( t = t^* \) solves \( 1 - Et_0^{\bar{a}} + Et^{\bar{a}} = 0 \).

We are now in a position to state the main theorem on ODE (41)–(42) and the proof can be found in [1, §2].

**Theorem 1.** Suppose constants \( a, \delta \) and \( c \) are defined by (43), \( t_* \) and \( t^* \) are defined above and the initial data \( \beta, \beta_0 > 0 \), then

1. \( t_* \in [0, \infty) \) exists and \( t_* > t_0 \);

2. there is a constant \( t_m \in [t_*, \infty] \), such that there is a unique solution \( f \in C^2([t_0, t_m]) \)
to the equation (41)-(42), and

\[ \lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty. \]

3. \( f \) satisfies upper and lower bound estimates,

\[
\begin{align*}
1 + f(t) &> \exp\left( Ct \frac{\hat{a}}{\bar{c}^2} + Dt^{-1}\right) \quad \text{for} \quad t \in (t_0, t_m); \\
1 + f(t) &< \left( At \frac{\hat{a}}{\bar{c}^2} + Bt \frac{\hat{a}}{\bar{c}^2} + 1\right)^{-1} \quad \text{for} \quad t \in (t_0, t_*) .
\end{align*}
\]

Furthermore, if the initial data satisfies \( \beta_0 > \bar{a}(1 + \beta)/(\hat{c}t_0) \), then

4. \( t_* \) and \( t^* \) exist and finite, and \( t_0 < t_* < t^* < \infty \);

5. there is a finite time \( t_m \in [t_*, t^*) \), such that there is a solution \( f \in C^2([t_0, t_m]) \) to the equation (41) with the initial data (42), and

\[ \lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty. \]

6. the solution \( f \) has improved lower bound estimates, for \( t \in (t_0, t_m) \),

\[
(1 + \beta) \left( 1 - E t_0^a + E t^a \right)^{1/\xi} < 1 + f(t) .
\]

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