Exact solution of damped harmonic oscillator with a magnetic field in a time dependent noncommutative space

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Abstract

In this paper we have obtained the exact eigenstates of a two dimensional damped harmonic oscillator in the presence of an external magnetic field varying with respect to time in time dependent noncommutative space. It has been observed that for some specific choices of the damping factor, the time dependent frequency of the oscillator and the time dependent external magnetic field, there exists interesting solutions of the time dependent noncommutative parameters following from the solutions of the Ermakov-Pinney equation. Further, these solutions enable us to get exact analytic forms for the phase which relates the eigenstates of the Hamiltonian with the eigenstates of the Lewis invariant. Then we compute the expectation value of the Hamiltonian. The expectation values of the energy are found to vary with time for different solutions of the Ermakov-Pinney equation corresponding to different choices of the damping factor, the time dependent frequency of the oscillator and the time dependent applied magnetic field. We also compare our results with those in the absence of the magnetic field obtained earlier.
1 Introduction

The Landau problem of a charged particle moving in a two dimensional plane under the influence of a magnetic field acting perpendicular to the plane has been looked upon by physicists over the years, not only from the pedagogical interest of formation of discrete energy levels known as Landau levels, but also due to the multifaceted applications of this problem. In this context, an extremely intriguing problem is that of a charged oscillator placed in a magnetic field, acting perpendicular to the plane in which it is oscillating. In addition, one may also consider an electric field lying along the plane of oscillation. The eigenfunction and the eigenvalues of a charged particle in a magnetic field in the presence of a time-dependent background electric field with time-dependent mass and frequency have been determined in [1]. The problem becomes even more fascinating when one places the Landau oscillator in a noncommutative phase space. This problem has been looked upon by an earlier study [2] in the presence of a time dependent magnetic field. The system there was considered in a noncommutative space. The simplest setting of noncommutative [NC] space is a two dimensional quantum mechanical space in which one replaces the standard set of commutation relations between the canonical coordinates by NC commutation relations $[X, Y] = i\theta$, where $\theta$ is a positive real constant. Study of quantum mechanical systems in such NC space has allured theoretical physicists since the work by Synder [3]. Since then the necessity for NC spaces has been established to ensure the attainment of gravitational stability [4] in the present theories of quantum gravity, namely, string theory [5, 6] and loop quantum gravity [7]. This has triggered several studies on quantum mechanical systems in such spaces in the literature [8]-[18].

However, the mentioned study [2] on charged oscillators in the presence of magnetic field, considers only noncommutativity amongst position variables. So, we in our present communication extend the study to a space where noncommutativity exists not only amongst spatial variables but also amongst momentum variables. Also unlike the previous study we have considered the NC space to be time dependent. Moreover, our oscillator is considered to be damped by an explicit damping factor in order to model a realistic situation. A damped oscillator in two dimensional NC space has been studied by us in an earlier communication [19]. Before our communication one of the very few works which had studied damped quantum harmonic oscillators in two dimensional space was the work by Lawson et.al. [20]. We extended their model to a two dimensional NC space. But at present our objective is to study how the interplay of damping and an external time dependent magnetic field modulates the energetics of a charged oscillator in a time dependent NC space where spatial as well as momentum noncommutativity is present.

Our present study is one of the very first to study a damped quantum harmonic oscillator with time varying frequency in two dimensional NC space in the presence of a time-dependent magnetic field. In our earlier study [19] we had seen that the expectation value of energy of an oscillator decays due to damping even in NC space. In the present study we intend to investigate the change in energetics of the damped quantum oscillator in NC space under the influence of magnetic field having various kinds of time dependence. For this purpose we first set up the Hamiltonian of the damped quantum harmonic oscillator in two dimensional NC space under the presence of a time varying magnetic field and then express it in terms of commutative variables. This is done in Section 2. After that we solve the Hamiltonian using the method of invariants [21, 22, 23] in Section 3. It must be noted that the eigenfunction of the said Hamiltonian is a product of the eigenfunction of the invariant and a phase factor. Both the eigenfunction and phase factor are expressed in terms of time dependent parameters which obey the non-linear differential equation known as Ermakov-Pinney (EP) equation [24, 25]. Next in Section 4 we choose the parameters of the damped system in the presence of magnetic field in such a way that they satisfy all the equations representing the system as well as provide us exact closed form solutions for the system for various choices of time variation of the applied field. In Section 5 we calculate the expectation value of energy of the damped oscillator.
explicitly and graphically explore how the time dependence of the magnetic field alters the time evolution of the energetics of the damped quantum system having a time varying frequency of oscillation in two dimensional NC space.

2 Model of the two-dimensional harmonic oscillator in magnetic field

In our study the system that we consider is a combination of two non-interacting damped harmonic oscillators affected by a time dependent magnetic field in two dimensional NC space. Both the oscillators have equal time dependent frequencies, coefficients of friction, equal mass and equal charge in NC space. Such a model of two-dimensional Landau problem of harmonic oscillator in a time dependent magnetic field was considered in an earlier communication [2] in spatially noncommutative configuration space. In this work, however, we extend the model by considering the system in time dependent NC space. Also it must be noted that the noncommutativity we consider is not restricted to spatial variables like the earlier studies but also extends to momentum noncommutativity.

The Hamiltonian of the two dimensional oscillator in magnetic field has the following form,

$$H(t) = \frac{f(t)}{2M} \left[ (P_1 - qA_1)^2 + (P_2 - qA_2)^2 \right] + \frac{M\omega^2(t)}{2f(t)} (X_1^2 + X_2^2) ;$$  

where the damping factor \( f(t) \) is given by,

$$f(t) = e^{-\int_0^t \eta(s) ds}$$  

with \( \eta(s) \) being the coefficient of friction and \( A_i \), the vector potential of a time dependent magnetic field \( B(t) \) is chosen in Coulomb gauge as,

$$A_i = -\frac{B(t)}{2} \epsilon_{ij} X_j ;$$

where \( i, j = 1, 2 \) and \( \epsilon_{ij} = -\epsilon_{ji} \) with \( \epsilon_{12} = 1 \). Here \( \omega(t) \) is the time dependent angular frequency of the oscillators , \( M \) and \( q \) are their mass and charge respectively. The position and momentum coordinates \((X_i, P_i)\) are noncommuting variables in NC space, that is, their commutators are \([X_1, X_2] \neq 0 \) and \([P_1, P_2] \neq 0 \). The corresponding canonical variables \((x_i, p_i)\) in commutative space are such that the commutator \([x_i, p_j] = i\hbar \delta_{i,j} \), \([x_i, x_j] = 0 = [p_i, p_j] ; (i, j = 1, 2)\).

In order to express the NC Hamiltonian in terms of the standard commutative variables explicitly, we apply the standard Bopp-shift relations [26] (\( \hbar = 1 \)):

$$X_1 = x_1 - \frac{\theta(t)}{2} p_2 ; \quad X_2 = x_2 + \frac{\theta(t)}{2} p_1$$

$$P_1 = p_1 + \frac{\Omega(t)}{2} x_2 ; \quad P_2 = p_2 - \frac{\Omega(t)}{2} x_1 .$$

Here \( \theta(t) \) and \( \Omega(t) \) are the NC parameters for space and momentum respectively, such that \([X_1, X_2] = i\theta(t) \), \([P_1, P_2] = i\Omega(t) \) and \([X_1, P_1] = i[1 + \frac{\theta(t)\Omega(t)}{4}] = [X_2, P_2] ; (X_1 \equiv X, X_2 \equiv Y, P_1 \equiv P_x, P_2 \equiv P_y)\).

The Hamiltonian in terms of \((x_i, p_i)\) coordinates is therefore given by the following relation,

$$H = a(t) (p_1^2 + p_2^2) + \frac{b(t)}{2} (x_1^2 + x_2^2) + c(t)(p_1 x_2 - p_2 x_1) .$$

\(^1\)We shall be considering NC phase space in our work. However, we shall be generically referring to this as NC space.
The time dependent coefficients in the above Hamiltonian are given as,

\[ a(t) = \frac{f(t)}{M} + \frac{qB(t)f(t)\theta(t)}{2M} + \frac{1}{4} \left[ \frac{q^2B^2(t)f(t)}{4M} + \frac{M\omega^2(t)}{f(t)} \right] \theta^2(t) \]

\[ b(t) = \frac{q^2B^2(t)f(t)}{4M} + \frac{M\omega^2(t)}{f(t)} + \frac{qB(t)f(t)\Omega(t)}{2M} + \frac{f(t)\Omega^2(t)}{4M} \]

\[ c(t) = \frac{1}{2} \left[ \frac{qB(t)f(t)}{M} \left( 1 + \frac{\theta(t)\Omega(t)}{4} \right) + \frac{\Omega(t)f(t)}{M} + \left( \frac{q^2B^2(t)f(t)}{4M} + \frac{M\omega^2(t)}{f(t)} \right) \theta(t) \right]. \]

Here it must be noted that although our Hamiltonian given by Eqn. (6) has the same form as that in [27] and [19] to study a system of a two dimensional harmonic oscillator and damped harmonic oscillator in NC space, the time dependent Hamiltonian coefficients (given by Eqn(s). (7), (8), (9)) have a modified form. This is because our present system of damped harmonic oscillator is studied in the presence of a time dependent magnetic field in two-dimensional NC space. Thus, both the damping factor \( f(t) \) and the magnetic field \( B(t) \) modulate and alter the Hamiltonian coefficients from the form considered in earlier study [27] and [19] respectively. It is relevant to mention that those coefficients also differ from those obtained in [2] as the considered noncommutativity in that study is time independent and exists only in the configuration space.

### 3 Solution of the model Hamiltonian

In order to find the solutions of the model Hamiltonian \( H(t) \) (Eqn.(6)) representing the two-dimensional damped harmonic oscillator with magnetic field in NC space, we follow the route suggested by Lewis et al. [21] in their work. First we construct the time-dependent Hermitian invariant operator \( I(t) \) corresponding to our Hamiltonian operator \( H(t) \) (given by Eqn. (6)). This is because if one can solve for the eigenfunctions of \( I(t) \), \( \phi(x_1, x_2) \), such that,

\[ I(t)\phi(x_1, x_2) = \epsilon \phi(x_1, x_2) \]

where \( \epsilon \) is an eigenvalue of \( I(t) \) corresponding to eigenstate \( \phi(x_1, x_2) \), one can obtain the eigenstates of \( H(t) \), \( \psi(x_1, x_2, t) \), using the relation given by Lewis et al. [21] which is as follows,

\[ \psi(x_1, x_2, t) = e^{i\Theta(t)}\phi(x_1, x_2) \]

where the real function \( \Theta(t) \) which acts as the phase factor will be discussed in details later.

#### 3.1 The Time Dependent Invariant

Next, following the approach taken by Lewis et al. [21], we need to construct the operator \( I(t) \) which is an invariant with respect to time, corresponding to the Hamiltonian \( H(t) \), as mentioned earlier, such that \( I(t) \) satisfies the condition,

\[ \frac{dI}{dt} = \partial_t I + \frac{1}{i}[I, H] = 0. \]

The procedure is to choose the Hermitian invariant \( I(t) \) to be of the same homogeneous quadratic form defined by Lewis et al. [21] for time-dependent harmonic oscillators. However, since we are dealing with a two-dimensional system in the present study, \( I(t) \) takes on the following form,

\[ I(t) = \alpha(t)(p_1^2 + p_2^2) + \beta(t)(x_1^2 + x_2^2) + \gamma(t)(x_1p_1 + p_2x_2). \]
Here we will consider $\hbar = 1$ since we choose to work in natural units. Now, using the form of $I(t)$ defined by Eqn.\(^\textit{13}\) in Eqn.\(^\textit{12}\) and equating the coefficients of the canonical variables, we get the following relations,

\[
\dot{\alpha}(t) = -a(t)\gamma(t) \tag{14}
\]

\[
\dot{\beta}(t) = b(t)\gamma(t) \tag{15}
\]

\[
\dot{\gamma}(t) = 2 \left[ b(t)a(t) - \beta(t)a(t) \right] \tag{16}
\]

where dot denotes derivative with respect to time $t$.

To express the above three time dependent parameters $\alpha$, $\beta$ and $\gamma$ in terms of a single time dependent parameter, we parametrize $\alpha(t) = \rho^2(t)$. Substituting this in Eqn(s).\(^\textit{14, 16}\), we get the other two parameters in terms of $\rho(t)$ as,

\[
\gamma(t) = -\frac{2\rho\dot{\rho}}{a(t)} , \quad \beta(t) = \frac{1}{a(t)} \left[ \frac{\dot{\rho}^2}{a(t)} + \rho^2b + \frac{\rho\dot{\rho}}{a(t)} - \frac{\rho\ddot{\rho}}{a^2} \right] . \tag{17}
\]

Now, substituting the value of $\beta$ in Eqn.\(^\textit{15}\), we get a non-linear equation in $\rho(t)$ which has the form of the non-linear Ermakov-Pinney (EP) equation with a dissipative term \cite{24, 25}. The form of the non-linear equation is as follows,

\[
\ddot{\rho} - \frac{a}{\rho^2} \dot{\rho} + ab\rho = \xi^2 a^2 \frac{\dot{\rho}^2}{\rho^4} \tag{18}
\]

where $\xi^2$ is a constant of integration. This equation has similar form to the EP equation obtained in \cite{27}, which is expected since our $H(t)$ has the same form as theirs. However, once again it should be mentioned that the explicit form of the time-dependent coefficients are different due to the presence of the external magnetic field as well as the fact that the oscillator is damped.

Now, using the EP equation we get a simpler form of $\beta$ as,

\[
\beta(t) = \frac{1}{a(t)} \left[ \frac{\dot{\rho}^2}{a(t)} + \xi^2 a(t) \rho^2 \right] . \tag{19}
\]

Next, substituting the expressions of $\alpha$, $\beta$ and $\gamma$ in Eqn.\(^\textit{13}\), we get the following expression for $I(t)$,

\[
I(t) = \rho^2(p_1^2 + p_2^2) + \left( \frac{\dot{\rho}^2}{a^2} + \frac{\xi^2}{\rho^2} \right) (x_1^2 + x_2^2) - \frac{2\rho\dot{\rho}}{a}(x_1p_1 + p_2x_2) . \tag{20}
\]

This form of the Lewis invariant in Cartesian coordinate is converted to polar coordinate using the same procedure as followed in our previous communication \cite{19}. The invariant in polar coordinate takes the following form,

\[
I(t) = \frac{\xi^2}{\rho^2} r^2 + \left( \rho p_r - \frac{\dot{\rho}}{r} \right)^2 + \left( \frac{\rho p_\theta}{r} \right)^2 - \left( \frac{\rho h}{2r} \right)^2 \tag{21}
\]

where the canonical coordinates in polar representation takes the following form,

\[
p_r = -i \left( \partial_r + \frac{1}{2r} \right) , \quad p_\theta = -i \partial_\theta . \tag{22}
\]

Now we note from Eqn.\(^\textit{21}\) that the invariant $I(t)$ has the same form as that used in \cite{27} to study the undamped harmonic oscillator in NC space. The time-dependent coefficients involved in the present study however differ due to the presence of external magnetic field and damping in our system. Thus, we can just borrow the expression of eigenfunction and the phase factors from \cite{27} for our present system.
3.2 Eigenfunction and phase factor

We depict the set of eigenstates of the invariant operator $I(t)$ as $|n,l\rangle$, following the convention in [27]. Here, $n$ and $l$ are integers such that $n + l \geq 0$. So we have the condition $l \geq -n$. Thus, if $l = -n + m$, then $m$ is a positive integer; and the corresponding eigenfunction in polar coordinate system has the following form (restoring $\hbar$),

\[
\phi_{n,m}(r,\theta) = \langle r,\theta|n,m-n \rangle \tag{23}
\]

\[
= \lambda_n \left( i\sqrt{\hbar} \right)^m \frac{r^{n-m}}{\sqrt{m!}} e^{i(m-n)\theta - \frac{a(t) - i\rho}{2\hbar \rho^2} r^2} U \left( -m, 1 - m + n, \frac{r^2}{\hbar \rho^2} \right) \tag{24}
\]

where $\lambda_n$ is given by

\[
\lambda_n^2 = \frac{1}{\pi n!(\hbar \rho^2)^{1+n}}. \tag{25}
\]

Here, $U \left( -m, 1 - m + n, \frac{r^2}{\hbar \rho^2} \right)$ is Tricomi’s confluent hypergeometric function [28] [29] and the eigenfunction $\phi_{n,m-n}(r,\theta)$ satisfies the following orthonormality relation,

\[
\int_0^{2\pi} d\theta \int_0^\infty r dr \phi^*_{n,m-n}(r,\theta) \phi_{n',m'-n'}(r,\theta) = \delta_{nn'} \delta_{mm'}. \tag{26}
\]

Again following [27], the expression of the phase factor $\Theta(t)$ is given by,

\[
\Theta_{n,l}(t) = (n + l) \int_0^t \left( c(T) - \frac{a(T)}{\rho^2(T)} \right) dT. \tag{27}
\]

For a given value of $l = -n + m$, it would be given by [27],

\[
\Theta_{n,m-n}(t) = m \int_0^t \left( c(T) - \frac{a(T)}{\rho^2(T)} \right) dT. \tag{28}
\]

We shall use this expression to compute the phase explicitly as a function of time for various physical cases in the subsequent discussion.

The eigenfunction of the Hamiltonian therefore reads (using Eqn(s). [11] [24] [28])

\[
\psi_{n,m-n}(r,\theta,t) = e^{i\Theta_{n,m-n}(t)} \phi_{n,m-n}(r,\theta) \tag{29}
\]

\[
= \lambda_n \left( i\sqrt{\hbar} \right)^m \frac{r^{n-m}}{\sqrt{m!}} \exp \left[ im \int_0^t \left( c(T) - \frac{a(T)}{\rho^2(T)} \right) dT \right] \times r^{n-m} e^{i(m-n)\theta - \frac{a(t) - i\rho}{2\hbar \rho^2} r^2} U \left( -m, 1 - m + n, \frac{r^2}{\hbar \rho^2} \right).
\]

4 Solutions for the noncommutative damped oscillator in magnetic field

In this communication we are primarily concerned about the evolution of the solution due to the inclusion of a time dependent magnetic field in the system. For this purpose we want to find the eigenfunctions of the corresponding Hamiltonian due to interplay of damping and magnetic field. The various kinds of damping in the presence of the applied magnetic field are represented by various forms of the time dependent coefficients of the Hamiltonian, namely, $a(t)$, $b(t)$ and $c(t)$. However, the various forms must be constructed in such a way that they satisfy the
non-linear EP equation given by Eqn. (18). The procedure of this construction of exact analytical solutions is based on the Chiellini integrability condition [30] and this formalism was followed in [27]. So, in this communication for various forms of \(a(t)\) and \(b(t)\), we get the corresponding form of \(\rho(t)\) using the EP equation together with the Chiellini integrability condition. In the subsequent discussion we shall proceed to obtain solutions of the EP equation for the damped oscillator in a magnetic field considered in NC space.

4.1 Solution Set-I for Ermakov-Pinney equation : Exponentially decaying solutions

4.1.1 The Solution Set

The simplest kind of solution set of the EP equation under damping is the exponentially decaying set used in [27]. The solution set is given by the following relations,

\[
a(t) = \sigma e^{-\vartheta t}, \quad b(t) = \Delta e^{\vartheta t}, \quad \rho(t) = \mu e^{-\vartheta t/2}
\]

where \(\sigma, \Delta\) and \(\mu\) are constants. Here, \(\vartheta\) is any positive real number. Substituting the expressions for \(a(t)\), \(b(t)\) and \(\rho(t)\) in the EP equation, we can easily verify the relation between these constants to be as follows,

\[
\mu^4 = \frac{4\xi^2\sigma^2}{4\sigma\Delta - \vartheta^2}.
\]

4.1.2 Study of the corresponding eigenfunctions

We now write down the eigenfunctions of the Hamiltonian for the chosen set of time-dependent coefficients. For this purpose we need to choose explicit forms of the damping factor \(f(t)\), angular frequency of the oscillator \(\omega(t)\) and the applied magnetic field \(B(t)\). The eigenfunction of the invariant \(I(t)\) (which is given by Eqn.(24)) takes on the following form for the solution Set-I:

\[
\phi_{n,m-n}(r,\theta) = \lambda_n \frac{(i\mu e^{-\vartheta t/2})^m}{\sqrt{m!}} r^{-m} e^{i(m-n)\vartheta/4} \mu^{-2} \mu^{2/\mu^2} e^{-\vartheta t/2} U\left(-m, 1 - m + n, \frac{r^2 e^{\vartheta t}}{\mu^2}\right)
\]

where \(\lambda_n\) is given by

\[
\lambda_n^2 = \frac{1}{\pi n! [\mu^2 e^{-\vartheta t}]^{1+n}}.
\]

Next, we proceed to obtain explicit expressions of the phase factors for various forms of the damping factor \(f(t)\) and the angular frequency \(\omega(t)\) of the oscillator. The value of the applied magnetic field \(B(t)\) is also tuned accordingly.

At first, we consider the most general form of damping factors and the applied magnetic field which are given as follows,

\[
f(t) = e^{-\Gamma t}; \quad \omega(t) = \omega_0 e^{-\delta t/2}; \quad B(t) = B_0 e^{\Lambda t};
\]

where \(\Gamma\) and \(\delta\) are non-negative real constants and \(\Lambda\) is an arbitrary real constant. Substituting these relations in Eqn(s). (34), we get the most general form of the time dependent NC parameters as,

\[
\theta(t) = \frac{8M e^{-\Gamma t}}{q^2 B_0^2 e^{2(\Lambda-\Gamma)t} + 4M^2 \omega_0^2 e^{-\delta t}} \left[\frac{(\xi^2 B_0^2 e^{(2\Lambda-\Gamma-\vartheta)t})}{4M} + \omega_0^2 e^{-\delta t} (M \sigma e^{(\vartheta-\vartheta)t} - 1) - \frac{q B_0 e^{(\Lambda-\Gamma)t}}{2M}\right]
\]

\[
\Omega(t) = -q B_0 e^{\Lambda t} + 2 e^{\Gamma t} \sqrt{M \Delta e^{(\vartheta-\vartheta)t} - M^2 \omega_0^2 e^{-\delta t}}.
\]
In order to get the exact analytical form of the phase factor we choose some suitable special forms of the constants \(\theta\), \(\Gamma\), \(\delta\) and \(\Lambda\).

(a) **Set-I, Case I**

Here we set the constants

\[
\theta = \Gamma, \quad \delta = 0, \quad \Lambda = 0.
\]

So, the parameters can be depicted by the following relations,

\[
f(t) = e^{-\Gamma t} ; \quad \omega(t) = \omega_0 ; \quad B(t) = B_0.
\]

Therefore, substituting Eqn.(38) in Eqn(s). (35,36), the reduced form of the NC parameters for this case are as follows,

\[
\theta(t) = \frac{8M e^{-\Gamma t}}{q^2 B_0^2 e^{-2\Gamma t} + 4M^2 \omega_0^2} \left[ \sqrt{\frac{q^2 B_0^2 \sigma e^{-2\Gamma t}}{4M} + \frac{\omega_0^2}{4M} (M\sigma - 1) - \frac{q B_0 e^{-\Gamma t}}{2M}} \right]
\]

\[
\Omega(t) = -q B_0 + 2 e^{\Gamma t} \sqrt{M \Delta - M^2 \omega_0^2}.
\]

It can be checked that in the limit \(B \rightarrow 0\), the expressions for \(\theta(t)\) and \(\Omega(t)\) reduce to those in [19]. Substituting these relations in the expression for \(c(t)\) in Eqn.(39), we get,

\[
c(t) = \frac{1}{q^2 B_0^2 e^{-2\Gamma t} + 4M^2 \omega_0^2} \left[ (4M^2 \omega_0^2 + 2q B_0 e^{-\Gamma t}) \sqrt{M \Delta - M^2 \omega_0^2} \right] \sqrt{\frac{q^2 B_0^2 \sigma e^{-2\Gamma t}}{4M} + \omega_0^2 (M\sigma - 1)}
\]

\[
-2q B_0 M \omega_0^2 e^{-\Gamma t} - \frac{q^2 B_0^2 \sigma}{M} e^{-2\Gamma t} \sqrt{M \Delta - M^2 \omega_0^2} + \sqrt{\frac{\Delta}{M} - \omega_0^2}.
\]

Substituting the expressions of \(a(t)\), \(\rho(t)\) and \(c(t)\) in Eqn.(39), we can get an expression for the phase in a closed form as,

\[
\Theta_{n,t}(t) = (n+l) \int_0^t \left[ c(t) - \frac{a}{\rho^2} \right] dT = (n+l) \left[ \sqrt{\frac{\Delta}{M} - \omega_0^2} - \frac{\sigma}{\mu^2} \right] t
\]

\[
+ \frac{(n+l)\omega_0 \sqrt{(M\sigma - 1)}}{\Gamma} \log \frac{\omega_0 \sqrt{M\sigma - 1} e^{\Gamma t} + \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1) e^{2\Gamma t}}}{\omega_0 \sqrt{M\sigma - 1} + \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)}}
\]

\[
+ \frac{(n+l)\omega_0}{\Gamma} \left[ \tan^{-1} \frac{\omega_0 e^{\Gamma t}}{\sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1) e^{2\Gamma t}}} - \tan^{-1} \frac{\omega_0}{\sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)}} \right]
\]

\[
+ \frac{(n+l)\sqrt{M \Delta - M^2 \omega_0^2}}{\Gamma M} \left[ \tan^{-1} \frac{2M \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1) e^{2\Gamma t}}}{q B_0} - \tan^{-1} \frac{2M \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)}}{q B_0} \right]
\]

\[
- \frac{(n+l)\sqrt{(\Delta - M\omega_0^2)\sigma}}{\Gamma} \left[ \tan^{-1} \frac{q^2 B_0^2 \sigma + 4M \omega_0^2 (M\sigma - 1) e^{2\Gamma t}}{2q B_0^2 \sigma} - \frac{q^2 B_0^2 \sigma + 4M \omega_0^2 (M\sigma - 1)}{2q B_0^2 \sigma} \right]
\]

\[
- \frac{(n+l)\omega_0}{\Gamma} \left[ \tan^{-1} \frac{2M \omega_0 e^{\Gamma t}}{q B_0} - \tan^{-1} \frac{2M \omega_0}{q B_0} \right] + (n+l) \sqrt{M \Delta - M^2 \omega_0^2} \log \frac{q^2 B_0^2 e^{-2\Gamma t} + 4M^2 \omega_0^2}{2q B_0^2 + 4M^2 \omega_0^2}.
\]

(b) **Set-I, Case II**
Here we set the constants

\[ \vartheta = \Gamma , \ \delta = 0 , \ \Lambda = \Gamma . \]  

(42)

So, the situation can be depicted by the following relations,

\[ f(t) = e^{-\Gamma t} ; \ \omega(t) = \omega_0 ; \ B(t) = B_0 e^{\Gamma t} . \]  

(43)

Therefore, substituting Eqn.(42) in Eqn(s). (35,36), the reduced form of the NC parameters for this case are as follows,

\[ \theta(t) = \frac{8Me^{-\Gamma t}}{q^2B_0^2 + 4M^2\omega_0^2} \left[ \left( 4M^2\omega_0^2 + 2qB_0\sqrt{M\Delta - M^2\omega_0^2} \right) \sqrt{\frac{q^2B_0^2\sigma}{4M} + \omega_0^2(M\sigma - 1)} \right] \]  

\[ -2qB_0M\omega_0^2 - \frac{q^2B_0^2}{M}\sqrt{M\Delta - M^2\omega_0^2} \]  

\[ + \sqrt{\frac{\Delta}{M} - \omega_0^2} . \]  

(44)

The point that is to be noted is that the multiplication of the two time dependent NC parameters obtained for this case reduces to a constant value and later we observe that the constant value is equal to the same found for another case discussed in Eqn.(70). It can be checked that in the limit \( B \to 0 \), the expressions for \( \theta(t) \) and \( \Omega(t) \) reduce to those obtained in [19]. Substituting these relations in the expression for \( c(t) \) in Eqn.(49), we get,

\[ c(t) = \frac{1}{q^2B_0^2 + 4M^2\omega_0^2} \left[ \left( 4M^2\omega_0^2 + 2qB_0\sqrt{M\Delta - M^2\omega_0^2} \right) \sqrt{\frac{q^2B_0^2\sigma}{4M} + \omega_0^2(M\sigma - 1)} \right] \]  

\[ -2qB_0M\omega_0^2 - \frac{q^2B_0^2}{M}\sqrt{M\Delta - M^2\omega_0^2} \]  

\[ + \sqrt{\frac{\Delta}{M} - \omega_0^2} . \]  

(45)

Substituting the expressions of \( a(t) , \rho(t) \) and \( c(t) \) in Eqn.(28), we can get an expression for the phase in a closed form in the following way.

\[ \Theta_{n,t}(t) = \frac{(n + l)}{q^2B_0^2 + 4M^2\omega_0^2} \left[ \left( 4M^2\omega_0^2 + 2qB_0\sqrt{M\Delta - M^2\omega_0^2} \right) \sqrt{\frac{q^2B_0^2\sigma}{4M} + \omega_0^2(M\sigma - 1)} \right] \]  

\[ -2qB_0M\omega_0^2 - \frac{q^2B_0^2}{M}\sqrt{M\Delta - M^2\omega_0^2} \]  

\[ t + (n + l) \left[ \sqrt{\frac{\Delta}{M} - \omega_0^2} - \frac{\sigma}{\mu^2} \right] t . \]  

(46)

In this case the phase is varying linearly with respect to time. In the limit \( B_0 \to 0 \), we can easily recover the same form of the phase factor corresponding to the solution set Ib obtained in [19].

(c) Set-I , Case III

Here we set the constants

\[ \vartheta = \Gamma , \ \delta = 0 , \ \Lambda = -\Gamma . \]  

(47)

So, the situation can be depicted by the following relations,

\[ f(t) = e^{-\Gamma t} ; \ \omega(t) = \omega_0 ; \ B(t) = B_0 e^{-\Gamma t} . \]  

(48)

Therefore, substituting Eqn.(47) in Eqn(s). (35,36), the reduced form of the NC parameters for this case are as follows,

\[ \theta(t) = \frac{8Me^{-\Gamma t}}{q^2B_0^2e^{-4\Gamma t} + 4M^2\omega_0^2} \left[ \sqrt{\frac{q^2B_0^2\sigma e^{-4\Gamma t}}{4M} + \omega_0^2(M\sigma - 1)} - \frac{qB_0e^{-2\Gamma t}}{2M} \right] \]  

\[ \Omega(t) = -qB_0e^{-\Gamma t} + 2e^{\Gamma t}\sqrt{M\Delta - M^2\omega_0^2} . \]  

(49)
Substituting these relations in the expression for \(c(t)\) in Eqn. \(50\), we get,

\[
c(t) = \frac{1}{q^2 B_0^2 e^{-4\Gamma t} + 4M^2 \omega_0^2} \left[ 4M^2 \omega_0^2 + 2qB_0 e^{-2\Gamma t} \sqrt{M \Delta - M^2 \omega_0^2} \right] \frac{q^2 B_0^2 e^{-4\Gamma t}}{4M} + \omega_0^2 (M \sigma - 1) \\
-2qB_0 M \omega_0^2 e^{-2\Gamma t} - \frac{q^2 B_0^2 e^{-4\Gamma t}}{M} \sqrt{M \Delta - M^2 \omega_0^2} + \frac{\Delta}{M} - \omega_0^2.
\] (50)

Substituting the expressions of \(a(t)\), \(\rho(t)\) and \(c(t)\) in Eqn. \(28\), we can get an expression for the phase factor in a closed form as,

\[
\Theta_{n, t}(t) = (n + l) \int_0^t \left[ c(t) - \frac{a}{\rho^2} \right] dT = (n + l) \left[ \frac{\Delta}{M} - \omega_0^2 - \frac{\sigma}{\mu^2} \right] t \\
+ \frac{(n + l) \omega_0 \sqrt{(M \sigma - 1)}}{2 \Gamma} \log \frac{\omega_0 \sqrt{M \sigma - 1} e^{2\Gamma t} + \sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1) e^{4\Gamma t}}}{\omega_0 \sqrt{M \sigma - 1} + \sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1)}} \\
+ \frac{(n + l) \omega_0}{2 \Gamma} \left[ \tan^{-1} \frac{\omega_0 e^{2\Gamma t}}{\sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1) e^{4\Gamma t}}} - \frac{\omega_0}{\sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1)}} \right] \\
- \frac{(n + l) \sqrt{M \Delta - M^2 \omega_0^2} \sigma}{2 \Gamma} \left[ \tanh^{-1} \frac{2M \sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1) e^{4\Gamma t}}}{qB_0} - \frac{2M \sqrt{q^2 B_0^2 \sigma + \omega_0^2 (M \sigma - 1)}}{qB_0} \right] \\
- \frac{(n + l) \omega_0}{2 \Gamma} \left[ \tan^{-1} \frac{2M \omega_0 e^{2\Gamma t}}{qB_0} - \frac{2M \omega_0}{qB_0} \right] + (n + l) \sqrt{M \Delta - M^2 \omega_0^2} \log \frac{q^2 B_0^2 e^{-4\Gamma t} + 4M^2 \omega_0^2}{q^2 B_0^2 + 4M^2 \omega_0^2}.
\] (51)

(d) **Set-I, Case IV**

Here we set the constants

\[
\vartheta = \delta = \Lambda = \Gamma.
\] (52)

So, the situation can be depicted by the following relations,

\[
f(t) = e^{-\Gamma t} ; \quad \omega(t) = \omega_0 e^{-\Gamma t/2} ; \quad B(t) = B_0 e^{\Gamma t}.
\] (53)

Substituting Eqn. \(52\) in Eqn. \(35, 36\), the reduced form of the NC parameters for this case are as follows,

\[
\theta(t) = \frac{8M e^{-\Gamma t}}{q^2 B_0^2 + 4M^2 \omega_0^2 e^{-\Gamma t}} \left[ \sqrt{q^2 B_0^2 \sigma + \omega_0^2 e^{-\Gamma t} (M \sigma - 1) - \frac{qB_0}{2M}} \right], \quad \Omega(t) = -qB_0 e^{\Gamma t} + 2e^{\Gamma t} \sqrt{M \Delta - M^2 \omega_0^2 e^{-\Gamma t}}.
\] (54)

Substituting these relations in the expression for \(c(t)\) in Eqn. \(50\), we get,

\[
c(t) = \frac{1}{q^2 B_0^2 + 4M^2 \omega_0^2 e^{-\Gamma t}} \left[ \left( 4M^2 \omega_0^2 e^{-\Gamma t} + 2qB_0 \sqrt{M \Delta - M^2 \omega_0^2 e^{-\Gamma t}} \right) \sqrt{q^2 B_0^2 \sigma + \omega_0^2 e^{-\Gamma t} (M \sigma - 1)} \\
-2qB_0 M \omega_0^2 e^{-\Gamma t} - \frac{q^2 B_0^2}{M} \sqrt{M \Delta - M^2 \omega_0^2 e^{-\Gamma t}} \right] + \frac{\Delta}{M} - \omega_0^2 e^{-\Gamma t}.
\] (55)
We are able to obtain the exact form of the phase factor and it has been shown in the Appendix.

4.2 Solution Set-II for Ermakov-Pinney equation: Rationally decaying solutions

4.2.1 The Solution Set

We now consider rationally decaying solutions of the EP equation similar to that used in [27] which is of the form,

\[
a(t) = \frac{\sigma \left( \frac{1 + 2}{k} \right)^{(k+2)/k}}{(\Gamma t + \chi)^{(k+2)/k}}, \quad b(t) = \frac{\Delta \left( \frac{k}{k+2} \right)^{(2-k)/k}}{(\Gamma t + \chi)^{(k-2)/k}}, \quad \rho(t) = \frac{\mu \left( \frac{1 + 2}{k} \right)^{1/k}}{(\Gamma t + \chi)^{1/k}} ;
\]

(56)

where \(\sigma, \Delta, \mu, \Gamma\) and \(\chi\) are constants such that \((\Gamma t + \chi) \neq 0\), and \(k\) is an integer. Substituting the expressions of \(a(t)\), \(b(t)\), and \(\rho(t)\) in the EP equation, we can easily verify the relation between these constants to be as follows,

\[
\Gamma^2 \mu = (k+2)^2 \left( \sigma \Delta \mu - \frac{\xi^2 \sigma^2}{\mu^3} \right).
\]

(57)

4.2.2 Study of the corresponding eigenfunctions

The eigenfunction of the invariant operator \(I(t)\) [given by Eqn. (24)] for this solution Set-II is of the following form,

\[
\phi_{n,m-n}(r,\theta) = \lambda_n \left( \frac{i\mu}{\sqrt{m!}} \right)^m \left( \frac{k + 2}{k(\Gamma t + \chi)} \right)^{m/k} e^{i(m-n)\theta} \frac{\left[ \sigma (k+2) + i\mu^2 \Gamma \right]}{2\sigma (k+2)^{k+2/k}\mu^2} \frac{r^{k+2/k}}{r^2}
\]

\[
\times U \left(-m, 1 - m + n, \frac{r^2[k(\Gamma t + \chi)]^{2/k}}{\mu^2 (k+2)^{2/k}} \right)
\]

(58)

where \(\lambda_n\) is given by

\[
\lambda_n^2 = \frac{1}{\pi n! \mu^{2n+2}} \left[ \frac{k(\Gamma t + \chi)}{k+2} \right]^{2(1+n)/k}.
\]

(59)

In order to get the eigenfunction of the Hamiltonian \(H(t)\), we need to calculate the associated phase factor. Once again for this we need to fix up the forms of the damping factor \(f(t)\), angular frequency \(\omega(t)\) of the oscillator and the applied magnetic field \(B(t)\). In order to explore the solution of \(H(t)\) for rationally decaying coefficients, we choose a rationally decaying form for \(\omega(t), B(t)\) and set \(f(t) = 1\). Thus, we have the following relations,

\[
\eta(t) = 0 \Rightarrow f(t) = 1, \quad \omega(t) = \frac{\omega_0}{(\Gamma t + \chi)} , \quad B(t) = \frac{B_0}{(\Gamma t + \chi)}.
\]

(60)

(c) Set-II, Case I

As we want to study how the nature of the rationally decaying solution gets altered when the system is placed in a magnetic field, we set \(k = 2\) in Eqn. (56). The system has been studied without applying any external field for this particular \(k\) parameter in an earlier communication [19].

When we set \(k = 2\), the set \(a(t), b(t)\) and \(\rho(t)\) takes the following simplified form,

\[
a(t) = \frac{4\sigma}{(\Gamma t + \chi)^2}, \quad b(t) = \Delta, \quad \rho(t) = \left[ \frac{2\mu^2}{\Gamma t + \chi} \right]^{1/2}.
\]

(61)
Substituting the expressions for \(a(t), b(t), \omega(t), f(t)\) and \(B(t)\) in the Eqn(s). \([7,8]\), we get the time dependent NC parameters as,

\[
\theta(t) = \frac{8M}{q^2 B_0^2 + 4M^2 \omega_0^2} \left[ \sqrt{\frac{q^2 B_0^2 \sigma}{M} + \omega_0^2 4 \sigma M - \omega_0^2 (\Gamma t + \chi)^2} - \frac{q B_0}{2M} (\Gamma t + \chi) \right]
\]

\[
\Omega(t) = -\frac{q B_0}{(\Gamma t + \chi)} + 2 \sqrt{M \Delta - \frac{M^2 \omega_0^2}{(\Gamma t + \chi)^2}}.
\] (62)

Substituting these relations in the expression for \(c(t)\) in Eqn. \([9]\) gives,

\[
c(t) = \frac{1}{4M^2 \omega_0^2 + q^2 B_0^2} \left[ \frac{4M^2 \omega_0^2}{(\Gamma t + \chi)^2} + \frac{2q B_0 \sqrt{M \Delta (\Gamma t + \chi)^2 - M^2 \omega_0^2}}{(\Gamma t + \chi)^2} \right] \sqrt{\frac{q^2 B_0^2 \sigma}{M} - \omega_0^2 (\Gamma t + \chi)^2} + \frac{\Delta}{M} - \frac{\omega_0^2}{(\Gamma t + \chi)^2}.
\] (63)

The additional terms that appear in the expression due to the presence of the magnetic field are mostly seen to be decaying functions of time. Their contribution becomes more evident when we study the evolution of expectation value of energy with time in a later section. We are able to obtain the exact form of the phase factor and it has been shown in the Appendix.

\(\langle f \rangle\) Set-II , Case II

It is observed from the solution set given by Eqn.\([56]\), that the time dependent parameters \(a(t)\) and \(\rho(t)\) vanish, while \(b(t)\) diverges if we set \(k = -2\). In order to avoid this and study the system at this critical value of \(k\), we choose the corresponding solution set to be,

\[
a(t) = \frac{\sigma}{(\Gamma t + \chi)^{(k+2)/k}}, \quad b(t) = \frac{\Delta}{(\Gamma t + \chi)^{(k-2)/k}}, \quad \rho(t) = \frac{\mu}{(\Gamma t + \chi)^{1/k}}.
\] (64)

We now set \(k = -2\) in Eqn. \([63]\) to obtain the following set,

\[
a = \sigma, \quad b = \frac{\Delta}{(\Gamma t + \chi)^2}, \quad \rho = \mu \sqrt{\Gamma t + \chi}.
\] (65)

Substituting these relations in the EP equation the constraint condition is found to be,

\[-\mu^4 \Gamma^2 + 4 \sigma \Delta \mu^4 = 4 \xi^2 \sigma^2.
\] (66)

The above relation matches with that found in Eqn.\([31]\) while considering \(\vartheta = \Gamma\).

As the Eqn.\([55]\) vanish at \(k = -2\), the eigenfunction of the invariant for this solution is needed to be calculated separately and it is the form of

\[
\phi_{n, m-n}(r, \vartheta) = \lambda_n \left( \frac{i \mu \sqrt{(\Gamma t + \chi)^m}}{\sqrt{m!}} \right) r^{n-m} e^{-2 \sigma - i \mu^2 \Gamma} U \left( -m, 1 - m + n, \frac{r^2}{\mu^2 (\Gamma t + \chi)} \right)
\] (67)

where \(\lambda_n\) is given by

\[
\lambda_n^2 = \frac{1}{\pi n! [\mu^2 (\Gamma t + \chi)]^{1+n}}.
\] (68)

Again we consider the same explicit forms of the damping factor \(f(t)\), angular frequency \(\omega(t)\) and applied magnetic field \(B(t)\) as in Eqn.\([41]\),

\[
f(t) = 1, \quad \omega(t) = \frac{\omega_0}{(\Gamma t + \chi)}, \quad B(t) = \frac{B_0}{(\Gamma t + \chi)}.
\] (69)
Substituting the expressions for $a(t)$, $b(t)$, $\omega(t)$, $f(t)$ and $B(t)$ in the Eqn(s).\textsuperscript{7,8}, we get the time dependent NC parameters as,

$$\theta(t) = \frac{8 M (\Gamma t + \chi)}{q^2 B_0^2 + 4 M^2 \omega_0^2} \left[ \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)} - \frac{q B_0}{2M} \right], \quad \Omega(t) = 2 \frac{\sqrt{M \Delta - M^2 \omega_0^2}}{(\Gamma t + \chi)} - \frac{q B_0}{(\Gamma t + \chi)}. \quad (70)$$

Once again it is interesting to note that a constant value is found after multiplication of the two time dependent NC parameters obtained above. Here we recall Eqn.\textsuperscript{11} where we discussed that the multiplication of two time dependent NC parameters reduces to a constant value which is equal to the same obtained for this case.

Substituting these relations in the expression for $c(t)$ in Eqn.\textsuperscript{10} gives,

$$c(t) = \frac{1}{(4 M^2 \omega_0^2 + q^2 B_0^2)(\Gamma t + \chi)} \left[ (4 M^2 \omega_0^2 + 2 q B_0 \sqrt{M \Delta - M^2 \omega_0^2}) \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)} \right.$$

$$- 2 q B_0 M \omega_0^2 - \frac{q^2 B_0^2}{M} \sqrt{M \Delta - M^2 \omega_0^2} \left. + \frac{1}{(\Gamma t + \chi)} \sqrt{\frac{\Delta}{M} - \omega_0^2} \right]. \quad (71)$$

Substituting these expressions for $a(t)$, $\rho(t)$ and $c(t)$ in Eqn.\textsuperscript{28}, we get the following expression for the phase factor in a closed form as,

$$\Theta_{n,l}(t) = \frac{(n + l)}{(4 M^2 \omega_0^2 + q^2 B_0^2)} \left[ (4 M^2 \omega_0^2 + 2 q B_0 \sqrt{M \Delta - M^2 \omega_0^2}) \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)} \right.$$

$$- 2 q B_0 M \omega_0^2 - \frac{q^2 B_0^2}{M} \sqrt{M \Delta - M^2 \omega_0^2} \left. \ln \left( \frac{\Gamma t + \chi}{\chi} \right) + \frac{(n + l)}{M} \left[ \sqrt{\frac{\Delta}{M} - \omega_0^2} - \frac{\sigma}{\mu^2} \right] \ln \left( \frac{\Gamma t + \chi}{\chi} \right) \right]. \quad (72)$$

### 5 Analysis of the expectation value of energy

In this section, we intend to calculate the expectation value of energy. It is shown in \textsuperscript{19} that the expectation value of energy $\langle E_{n,m-n}(t) \rangle$ with respect to energy eigenstate $\psi_{n,m-n}(r, \theta, t)$ can be expressed as,

$$\langle E_{n,m-n}(t) \rangle = \frac{1}{2} (n + m + 1) \left[ b(t) \rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\rho^2(t)}{a(t)} + c(t) (n - m) \right]. \quad (73)$$

Substituting the expression of $c(t)$ in the above equation, the expectation value of energy for our model takes the following form,

$$\langle E_{n,m-n}(t) \rangle = \frac{1}{2} (n + m + 1) \left[ b(t) \rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\rho^2(t)}{a(t)} \right]$$

$$+ \frac{(n - m)}{2} \left[ q B(t) f(t) \left( 1 + \frac{\theta(t)}{4} \right) + \frac{\Omega(t) f(t)}{M} + \frac{q^2 B^2(t) f(t)}{4M} + \frac{M \omega^2(t)}{f(t)} \right] \theta(t) \right]; \quad (74)$$

which reduces to the same obtained in \textsuperscript{19} in the limit $B \to 0$.

The energy expression depends on the charge explicitly. It contains both terms having linear and quadratic dependence on charge. So the energy does not remain invariant when the charge of the particle changes its sign. Another notable point is that even when the frequency of oscillation $\omega \to 0$, and the applied field $B \to 0$; the expectation value of energy is non-zero. This is because all the three parameters of the Hamiltonian $a(t)$, $b(t)$ and $c(t)$ are finite even as $\omega \to 0, B \to 0$, as is clear from the Eqn(s).\textsuperscript{7,8}. Now we will proceed to study the time-dependent behaviour of $\langle E_{n,m-n}(t) \rangle$ for various types of damping and applied magnetic field.
5.1 Solution Set-I: Exponentially decaying solution

For the exponentially decaying solution given by Eqn. (30), the energy expectation value takes the following form,

$$\langle E_{n,m-n}(t) \rangle = (n + m + 1)\mu^2\Delta + c(t) (n - m)$$

(75)

where we have set the constant $\xi^2$ to unity and used the constraint relation given by Eqn. (31).

(A) Set-I, Case I

Here we set $f(t) = e^{-\Gamma t}$, $\omega(t) = \omega_0$ and $B(t) = B_0$. With this the energy expression for the ground state takes the form,

$$\langle E_{n,-n}(t) \rangle = (n+1)\mu^2\Delta + \frac{n}{q^2B_0^2e^{-2\Gamma t} + 4M^2\omega_0^2} \left[-2qB_0M\omega_0^2e^{-\Gamma t} - \frac{q^2B_0^2}{M}e^{-2\Gamma t}\sqrt{M\Delta - M^2\omega_0^2} \right]$$

$$+ \left(4M^2\omega_0^2 + 2qB_0e^{-\Gamma t}\sqrt{M\Delta - M^2\omega_0^2} \right) \sqrt{\frac{q^2B_0^2\sigma e^{-2\Gamma t}}{4M} + \omega_0^2(M\sigma - 1)} + n\sqrt{\frac{\Delta}{M} - \omega_0^2}.$$ 

(76)

The nature of this energy expectation value depends on the value of the constants. Specifically, the sign of the charge plays a crucial role for determining the nature of this expectation value. It is noteworthy that in both the limits $t \to \infty$ and $B \to 0$ this energy expression gains the same constant value which is already found in [19] for a damped oscillator in time dependent NC space. Apart from it, an inclusion of constant magnetic field in the system considered in [19] also makes the Hamiltonian non-hermitian after a certain limit of time beyond which the energy becomes imaginary. The condition for getting the expectation value of energy to be real is as follows,

$$t \leq \frac{1}{2\Gamma} \ln \frac{q^2B_0^2\sigma}{4M\omega_0^2(1-M\sigma)}.$$ 

(77)

It is interesting to note that the upper bound of time below which the energy expectation value remains real does not depend on the sign of charge of the oscillator, although the value of energy itself does. The nature of variation of energy expectation value with time is shown in Fig. [I]. Though the energy shows an initial decrease, eventually it tends to be a constant over time.

(B) Set-I, Case II

Here we set $f(t) = e^{-\Gamma t}$, $\omega(t) = \omega_0$ and $B(t) = B_0e^{-\Gamma t}$. With this the energy expression for the ground state takes the form,

$$\langle E_{n,-n}(t) \rangle = (n+1)\mu^2\Delta + \frac{n}{q^2B_0^2 + 4M^2\omega_0^2} \left[4M^2\omega_0^2 + 2qB_0\sqrt{M\Delta - M^2\omega_0^2} \right] \sqrt{\frac{q^2B_0^2\sigma e^{-2\Gamma t}}{4M} + \omega_0^2(M\sigma - 1)}$$

$$-2qB_0M\omega_0^2 - \frac{q^2B_0^2}{M}\sqrt{M\Delta - M^2\omega_0^2} + n\sqrt{\frac{\Delta}{M} - \omega_0^2} = \text{constant}.$$ 

(78)

As expected we observe from Fig. [I] the energy is a constant over time. In the limit $B \to 0$, the constant value of energy reduces to the same obtained in [19] for a damped oscillator in time dependent NC space.

(C) Set-I, Case III

Here we set $f(t) = e^{-\Gamma t}$, $\omega(t) = \omega_0$ and $B(t) = B_0e^{-\Gamma t}$. With this the energy expression for the ground state takes the form,

$$\langle E_{n,-n}(t) \rangle = (n+1)\mu^2\Delta + \frac{n}{q^2B_0^2e^{-2\Gamma t} + 4M^2\omega_0^2} \left[-2qB_0M\omega_0^2e^{-2\Gamma t} - \frac{q^2B_0^2}{M}e^{-4\Gamma t}\sqrt{M\Delta - M^2\omega_0^2} \right]$$

$$+ \left(4M^2\omega_0^2 + 2qB_0e^{-2\Gamma t}\sqrt{M\Delta - M^2\omega_0^2} \right) \sqrt{\frac{q^2B_0^2\sigma e^{-4\Gamma t}}{4M} + \omega_0^2(M\sigma - 1)} + n\sqrt{\frac{\Delta}{M} - \omega_0^2}.$$ 

(79)
The nature of variation of the energy expectation value with time has almost the same characteristics as obtained in Case I. However, a closer observation tells us that in Case III the rate of decay is faster than in Case I for the same set of parameters. It must be because in Case III, unlike in Case I, the applied field is decaying as well with respect to time. The upper bound of time beyond which the system becomes non-physical (since the energy ceases to be real after this time) is also half of that obtained in Eqn.(77). Here the bound is found to be

\[ t \leq \frac{1}{4\Gamma} \ln \left( \frac{q^2 B_0^2 \sigma}{4M \omega_0^2 (1 - M \sigma)} \right). \]  

(80)

An important inference that can be drawn from the above study is that, in the presence of the external magnetic field, the energy of a damped oscillator decays with time if the field either decays with time or is at least a constant. If the field tends to grow as fast as the damping factor decays, then the energy of the oscillator tends to be a constant with time. In Fig. 1 we have also plotted the evolution of energy with time in this situation if the applied field is turned off. We see that if \( B = 0 \), then the energy is a constant over time. This is expected from the energy expectation value expressions given by Eqns. 76, 78 and 79. Even a constant magnetic field is able to bring about time variation in this energy value.

\( \langle D \rangle \)

Set-I, Case IV

Here we set \( f(t) = e^{-\Gamma t} \), \( \omega(t) = \omega_0 e^{-\Gamma t/2} \) and \( B(t) = B_0 e^{\Gamma t} \). With this the energy expression for the ground state takes the form,

\[ \langle E_{n, -n}(t) \rangle = (n + 1)\mu^2 \Delta + \frac{n}{q^2 B_0^2 + 4M^2 \omega_0^2} \left[ -2qB_0 M \omega_0^2 e^{-\Gamma t} - \frac{q^2 B_0^2}{M} \sqrt{M \Delta - M^2 \omega_0^2 e^{-\Gamma t}} + (4M^2 \omega_0^2 e^{-\Gamma t} + 2qB_0 \sqrt{M \Delta - M^2 \omega_0^2 e^{-\Gamma t}}) \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 e^{-\Gamma t} (M \sigma - 1)} \right] + n \sqrt{\frac{\Delta}{M} - \omega_0^2 e^{-\Gamma t}}. \]

(81)

It can be verified that the above energy expectation value reduces to a decaying form as obtained in [19] in the absence of magnetic field. However, it is interesting to note that for certain choice of parameters (as shown in Fig. 1) the energy may also exhibit an initial growth with time. In fact for the given set of parameters in Fig. 1, it initially decays and then increases. Eventually, as \( t \to \infty \), the energy tends to be a constant. In this situation, if we turn off the applied magnetic field, then we see from Fig 1 the energy decays off with time. However, due to the absence of the exponentially growing field, the energy of the oscillator is seen to be much lower. Also the exponential growth in energy seen in the presence of the field is absent within the time range studied. The system also possess two lower bounds of time below which it becomes non-physical due to the imaginary value of energy. The conditions are as follows,

\[ t \geq \frac{1}{\Gamma} \ln \left( \frac{M \omega_0^2}{\Delta} \right); \quad t \geq \frac{1}{\Gamma} \ln \left( \frac{4M \omega_0^2 (1 - M \sigma)}{q^2 B_0^2 \sigma} \right). \]

(82)

The greater of the two bounds serves as the actual lower bound.

5.2 Solution Set-II: Rationally decaying solution

In previous section we considered two different solution set generated from Eqn.(56). The special form of the solution shown in Eqn.(61) is directly produced by substituting \( k = 2 \) in the Eqn.(56) and the special form of the solution shown in Eqn.(65) is obtained by substituting \( k = -2 \) in the modified form of Eqn.(56).
For the rationally decaying solution given by Eqn.(61), the energy expectation value takes the following form

\[ \langle E \rangle = \frac{1}{\omega_0} \left\langle \frac{\langle E \rangle}{\omega_0} \right\rangle \]

in order to make it dimensionless, as we vary \( \Gamma t \) (again a dimensionless quantity). Here we consider mass \( M=1 \), charge \( q=1 \), magnetic field \( B_0=10^2 \), \( \mu=1 \), \( \Delta=10^7 \), \( \sigma=10^7 \), \( \omega_0=10^3 \) and \( \Gamma=1 \) in natural units. The constants \( n=1 \) and \( m=0 \). The expectation value of energy \( \langle E \rangle \) is calculated for exponentially decaying solution set when Case I: \( f(t) = e^{-\Gamma t} \), \( \omega(t) = \omega_0 \) and \( B(t) = B_0 \); Case II: \( f(t) = e^{-\Gamma t} \), \( \omega(t) = \omega_0 \) and \( B(t) = B_0 e^{\Gamma t} \); Case III: \( f(t) = e^{-\Gamma t} \), \( \omega(t) = \omega_0 \) and \( B(t) = B_0 e^{-\Gamma t} \) and Case IV: \( f(t) = e^{-\Gamma t} \), \( \omega(t) = \omega_0 e^{-\Gamma t/2} \) and \( B(t) = B_0 e^{\Gamma t} \). While for Case I and Case III the energy first decreases, then becomes constant with time, for Case II the energy remains constant as we vary time. For Case IV the behaviour of energy with time is seen to be extremely interesting. It first decreases and then increases with time. Along with these, we have also plotted what happens in the absence of magnetic field for comparison. When the angular frequency of oscillation is a constant (Case I, Case II and Case III), if the magnetic field is set to zero, then the energy of the oscillator is a constant with time. So, the magnetic field, even when it is a constant brings about time variation in energy for an exponentially damped oscillator having a constant frequency. However, when the angular frequency is decaying exponentially too (Case IV), then even when \( B=0 \), the energy decays with time. Nevertheless, the variation of energy is remarkably different from that seen in the presence of the field for Case IV.

### 5.2.1 Set-II, Case I

For the rationally decaying solution given by Eqn.\[\text{[61]}\], the energy expectation value takes the following form

\[ \langle E_{n,m-n}(t) \rangle = \frac{(n + m + 1)}{2(\Gamma t + \chi)} \left[ 2 \left( \frac{\sigma}{\mu^2} + \Delta \mu^2 \right) + \frac{\mu^2 \Gamma^2}{8\sigma} \right] + (n - m)c(t) \]  

where we have set the constant \( \xi^2 \) to unity and used the constraint relation given by Eqn.\[\text{[67]}\]. Here we set \( f(t) = 1 \), \( \omega(t) = \omega_0/(\Gamma t + \chi) \) and \( B(t) = B_0/(\Gamma t + \chi) \). With this the energy expression for the ground state takes the form,

\[ \langle E_{n,-m}(t) \rangle = \frac{(n + 1)}{2(\Gamma t + \chi)} \left[ 2 \left( \frac{\sigma}{\mu^2} + \Delta \mu^2 \right) + \frac{\mu^2 \Gamma^2}{8\sigma} \right] + \frac{n}{4M^2 \omega_0^2 + q^2 B_0^2} \left[ -\frac{2qB_0M \omega_0^2}{\Gamma t + \chi} - \frac{q^2 B_0^2 \sqrt{M \Delta(\Gamma t + \chi)^2 - M^2 \omega_0^4}}{M(\Gamma t + \chi)} \right] \]

\[ + \left( \frac{4M^2 \omega_0^2}{(\Gamma t + \chi)^2} + \frac{2qB_0 \sqrt{M \Delta(\Gamma t + \chi)^2 - M^2 \omega_0^4}}{(\Gamma t + \chi)^2} \right) \sqrt{\frac{q^2 B_0^2 \sigma}{M} - \omega_0^2(\Gamma t + \chi)^2 + 4\omega_0^2 \sigma M} \]  

\[ + n \sqrt{\frac{\Delta}{M} - \frac{\omega_0^2}{(\Gamma t + \chi)^2}} \]  

\[ \text{[84]} \]
Figure 2: A study of the variation of expectation value of energy, scaled by $\frac{1}{\omega_0} \langle \frac{E}{\omega_0} \rangle$ in order to make it dimensionless, as we vary $\Gamma t$ (again a dimensionless quantity). Here we consider mass $M=1$, charge $q=1$, magnetic field $B_0=10^{20}$, $\mu=1$, $\Delta=10^3$, $\sigma=10^7$, $\omega_0=10^3$ and $\Gamma=1$ in natural units. The constants $n=1$ and $m=0$. The expectation value of energy $\langle E \rangle$ is calculated for rationally decaying solution set when $\langle A \rangle$ Case I: $a(t) = \frac{4\sigma}{(\Gamma t + \chi)^2}$, $b(t) = \Delta$, $\rho(t) = \left[\frac{2\mu^2}{\Gamma t + \chi}\right]^{1/2}$ and $\langle B \rangle$ Case II: $a = \sigma$, $b(t) = \frac{\Delta}{(\Gamma t + \chi)^2}$, $\rho = \mu\sqrt{\Gamma t + \chi}$. For both Case I and Case II the energy decays with time. However, the rate of decay is higher for Case I as compared to Case II. This is expected since two of the Hamiltonian parameters $a(t)$ and $\rho(t)$ are decaying with time for Case I, whereas only one parameter $b(t)$ is decaying while another one $\rho(t)$ is increasing with time for Case II. For comparison, we have also plotted Case I and II, in the absence of the magnetic field in the same plot. It is noteworthy, that although the energy of the oscillator decays even in the absence of the field, when we apply the field the energy is enhanced. This is due to the presence of magnetic energy in the system in this situation.

which is a decaying function of time and is seen to reduce to the same obtained in [19] in the limit $B \to 0$. The time range beyond which the system becomes non-physical due to imaginary energy expectation value is as follows,

$$\frac{1}{\Gamma} \left( \omega_0 \sqrt{\frac{M}{\Delta} - \chi} \right) \leq t \leq \frac{1}{\Gamma} \left[ \sqrt{\frac{q^2 B_0^2 \sigma}{M \omega_0^2} + 4\sigma M - \chi} \right].$$

(85)

In Fig. 2 we have done a comparative study of the time variation of energy in this situation, when the applied magnetic field is present and when it is turned off. We see from the Figure that in both these cases the energy decays with time. However the energy of the oscillator is higher in the presence of the field due to the presence of the magnetic energy in the system.

5.2.2 Set-II, Case II

For the rationally decaying solution given by Eqn.(65), the energy expectation value takes the following form,

$$\langle E_{n,m-n}(t) \rangle = \frac{(m+n+1)\mu^2 \Delta}{\Gamma t + \chi} + c(t)(n-m).$$

(86)

where we have set the constant $\xi^2$ to unity and used the constraint relation given by Eqn.(66). Here we set $f(t) = 1$, $\omega(t) = \omega_0/(\Gamma t + \chi)$ and $B(t) = B_0/(\Gamma t + \chi)$. With this the energy expression for the ground state takes the form,
\[\langle E_{n,-n}(t) \rangle = \frac{(n+1)\mu^2 \Delta}{\Gamma t + \chi} + \frac{n}{(4M^2 \omega_0^2 + q^2 B_0^2)(\Gamma t + \chi)} \left[ -2qB_0 M \omega_0^2 \right] - \frac{q^2 B_0^2 \sqrt{M \Delta - M^2 \omega_0^2}}{M} \\
+ \left( 4M^2 \omega_0^2 + 2qB_0 \sqrt{M \Delta - M^2 \omega_0^2} \right) \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M\sigma - 1)} \right] + \frac{n}{(\Gamma t + \chi)} \sqrt{\frac{\Delta}{M} - \omega_0^2}. \quad (87)\]

As we see from Fig. 2, the corresponding energy expectation value decays off with time. Infact, it approaches zero at large time. However, the rate of decay is lower than Case I. This is expected since in Case II, unlike in Case I, while one of the Hamiltonian parameters \(b(t)\) is decreasing with time, another parameter \(\rho(t)\) is growing with time. The resultant energy is decreasing with time since the rate at which \(b(t)\) is decaying (rate \(\sim t^{-2}\)) is higher than the rate at which \(\rho(t)\) is growing (rate \(\sim t^{1/2}\)). When we turn off the field in this situation, we see from Fig. 2 the energy still decays off due to the decaying angular frequency of the oscillator. But in this situation the energy of the oscillator is reduced due to the absence of magnetic energy in the system.

6 Conclusion

In conclusion, our primary objective through this study has been to investigate the effect that an external time-dependent magnetic field has on a two dimensional damped harmonic oscillator in noncommutative space. The behaviour of the system under the influence of this time varying field is seen to be dependent on the nature of the field. For this purpose we have first set up the Hamiltonian for our system in the presence of a general magnetic field in noncommutative space. Then we map this Hamiltonian in terms of commutative variables by using a shift of variables connecting the noncommutative and commutative space, known in the literature as Bopp-shift. We have then obtained the exact solution of this time dependent system in the presence of an applied time varying magnetic field by using the well known Lewis invariant which in turn leads to a non-linear differential equation known as the Ermakov-Pinney equation. Then we make various choices of the parameters of the system and study the solutions depending on these choices as we tune the applied magnetic field. In this study we have primarily considered two different sets of parameters for our damped system, namely, exponentially decaying solutions and rationally decaying solutions. Interestingly, the solutions obtained make it possible to integrate the expression of the phase factor exactly thereby giving an exact solution for the eigenstates of the Hamiltonian. Then we compute the expectation value of the Hamiltonian. Expectedly, the expectation value of the energy varies with time. For the exponentially decaying system, the nature of the time dependent magnetic field crucially determines the nature of evolution of the energy with time. There is basically an interplay between the damping factor, applied time varying magnetic field and time dependent angular frequency of the harmonic oscillator in determining the time evolution. While an exponentially growing magnetic field is able to maintain the energy to a constant value inspite of damping, a constant or an exponentially decaying field makes the energy fall off faster with damping. Remarkably, the presence of an exponentially decaying frequency along with the damping factor makes the behaviour of the system under the influence of an exponentially growing field even more interesting. While initially the energy decays off with time due to the damping present in the system, later the energy starts growing under the influence of the growing field. For the rationally decaying situation, even when damping is not present, a rationally decaying magnetic field in combination with a rationally decaying angular frequency is able to eventually damp out the energy of the oscillator. While the decaying oscillation corresponding to the Case I of rational EP solution cannot remain physical at the zero value of energy due to the existance of upper bound of time, the same corresponding to the Case II of rational EP solution is physically damped out to be zero with time. However, we observe that at a given instant of time, the expectation value of energy is greater in the presence of
the magnetic field than when the field is turned off. This is because magnetic energy is absent in the system in the latter case.

**Appendix: Explicit forms of some phases**

We discussed about the eigenfunctions corresponding to the exponentially decaying EP solution set in section 4.1. We mentioned that the exact form of the phase factor to construct the eigenfunction of a damped oscillator having exponentially decaying frequency in the presence of a magnetic field increasing exponentially with respect to time in NC space can be found [Set I, Case IV]. The phase factor is as follows,

$$
\Theta_{n,l}(t) = \frac{2(n + l)}{\Gamma} \left[ \sqrt{\frac{\Delta}{M}} - \omega_0^2 - \sqrt{\frac{\Delta_{\sigma}}{M} - \omega_0^2 e^{-\Gamma t}} \right] + \frac{2(n + l)}{\Gamma} \left\{ \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M \sigma - 1)} - \sqrt{\frac{q^2 B_0^2 \sigma}{4M} + \omega_0^2 (M \sigma - 1) e^{-\Gamma t}} \right\} + \frac{n + l}{\mu^2} t + \frac{i(n + l)}{M \Gamma} \sqrt{q^2 B_0^2 + 4M \Delta} \left[ \tan^{-1} \frac{\sqrt{q^2 B_0^2 + 4M \Delta}}{2 \sqrt{M^2 \omega_0^2 - M \Delta}} - \tan^{-1} \frac{\sqrt{q^2 B_0^2 + 4M \Delta}}{2 \sqrt{M^2 \omega_0^2 e^{-\Gamma t} - M \Delta}} \right] + \frac{q B_0 (n + l)}{2M \Gamma} \log \left( \frac{4 \omega_0^2 q^2 B_0^2}{2M \omega_0^2 + q^2 B_0^2} \right) \left\{ \left( \omega_0^2 (M \sigma - 1) + \frac{q^2 B_0^2 \sigma}{4M} + \frac{q B_0 \sigma}{2} \right)^2 + \omega_0^2 (M \sigma - 1)^2 \right\} + \frac{(n + l) \sqrt{q^2 B_0^2 + 4M \Delta}}{2M \Gamma} \log \left( \frac{4 \omega_0^2 q^2 B_0^2}{2M \omega_0^2 + q^2 B_0^2} \right) \left\{ \frac{q^2 B_0^2 \sigma^2 M \omega_0^2}{4M} + \frac{q^2 B_0^2 \sigma \Delta}{4 M \omega_0^2} \right\} + \frac{(n + l) \sqrt{q^2 B_0^2 + 4M \Delta}}{\Gamma} \omega_0^2 \left( M \omega_0^2 - M \Delta - \frac{q^2 B_0^2 \sigma}{4M} + \frac{q^2 B_0^2 \sigma}{4 M \omega_0^2} \right) \right] \left( \omega_0^2 (M \sigma - 1) + \frac{q^2 B_0^2 \sigma}{4M} + \frac{q B_0 \sigma}{2} \right)^2 + \omega_0^2 (M \sigma - 1)^2 \right\} + \frac{q B_0 (n + l)}{2M \Gamma} \log \frac{q^2 B_0^2 + 4M^2 \omega_0^2 e^{-\Gamma t}}{q^2 B_0^2 + 4M^2 \omega_0^2} \left\{ \frac{q^2 B_0^2 \sigma^2 M \omega_0^2}{4M} + \frac{q^2 B_0^2 \sigma \Delta}{4 M \omega_0^2} \right\} + \frac{(n + l) q B_0 \sqrt{M \sigma - 1}}{2M \Gamma} \left\{ \frac{M \sigma \Delta - M \Delta - \frac{q^2 B_0^2 \sigma}{4}}{2M \omega_0^2 (M \sigma - 1) e^{-\Gamma t} - 2i \sqrt{(M \sigma - 1) \left( \omega_0^2 (M \sigma - 1) e^{-\Gamma t} + \frac{q^2 B_0^2 \sigma}{4M} \right) (M \Delta - M^2 \omega_0^2 e^{-\Gamma t})}} \right\} + \frac{(n + l) q B_0 \sqrt{M \sigma - 1}}{2M \Gamma} \left\{ \frac{M \sigma \Delta - M \Delta - \frac{q^2 B_0^2 \sigma}{4}}{2M \omega_0^2 (M \sigma - 1) - 2i \sqrt{(M \sigma - 1) \left( \omega_0^2 (M \sigma - 1) + \frac{q^2 B_0^2 \sigma}{4M} \right) (M \Delta - M^2 \omega_0^2)}} \right\} \right\} \right\}

(88)
having rationally decaying frequency in the presence of a magnetic field decaying rationally with respect to time in NC space [Set II, Case I] can be found. The phase factor is as follows,

\[
\Theta_{n,t}(t) = -\frac{2(n+l)}{\Gamma} \left( \frac{\sigma^2}{\mu^2} + \frac{qB_0 M \omega_0^2}{4M^2 \omega_0^2 + q^2 B_0^2} \right) \log \frac{\Gamma t + \chi}{\chi} + \frac{4(n+l)M^2 \omega_0^2}{(q^2 B_0^2 + 4M^2 \omega_0^2)\Gamma} \left[ \sqrt{\frac{q^2 B_0^2 \sigma}{M} + 4 \omega_0^2 \sigma M} \right] \frac{1}{\chi^2 - \omega_0^2}
\]

\[-\sqrt{\frac{q^2 B_0^2 \sigma}{M} + 4 \omega_0^2 \sigma M} \left( \frac{1}{\Gamma (t + \chi)^2 - \omega_0^2 + \omega_0} \left\{ \tan^{-1} \frac{\sqrt{M \omega_0^2 \chi}}{\sqrt{(q^2 B_0^2 \sigma + 4 \omega_0^2 \sigma M^2) - \omega_0^2 \chi^2 M}} \right\} \right] - \frac{2(n+l)qB_0}{q^2 B_0^2 + 4M^2 \omega_0^2} \left[ \frac{1}{\Gamma} \left\{ \frac{\Delta - \omega_0^2}{\chi^2} \right\} \left( \frac{4M^2 \omega_0^2 \sigma}{q^2 B_0^2 \sigma - M \omega_0^2 \chi^2} \right) \right] \]

\[-\left( \frac{M^2 \omega_0^2}{(t + \chi)^2} \right) \left( \frac{q^2 B_0^2 \sigma}{M} + 4 \omega_0^2 \sigma M - \omega_0^2 (\Gamma t + \chi)^2 \right) \left\{ \frac{2tM \omega_0^2}{\Gamma} \left( i \frac{i \omega_0 \sqrt{M \chi}}{\sqrt{q^2 B_0^2 \sigma + 4 \omega_0^2 \sigma M^2}} \right) \right\} \]

\[-\frac{i(q^2 B_0^2 \sigma M^2 \Delta)}{\Gamma M \omega_0^4} \left\{ \frac{i \omega_0 \sqrt{M \chi}}{\sqrt{q^2 B_0^2 \sigma + 4 \omega_0^2 \sigma M^2}} \right\} \left( \frac{q^2 B_0^2 \sigma M^2 \Delta}{M^2 \omega_0^4} \right) \]

\[-(n+l) \frac{(t + \chi)^2 - \omega_0^2 - \sqrt{\frac{M \Delta}{M} (\Gamma t + \chi)^2 - M^2 \omega_0^2}}{\Delta (t + \chi)^2 - M^2 \omega_0^2} - \frac{(n+l)q^2 B_0}{M (4M^2 \omega_0^2 + q^2 B_0^2)\Gamma} \left[ \sqrt{M \Delta (\Gamma t + \chi)^2 - M^2 \omega_0^2} \right] \frac{M \omega_0}{\sqrt{M \Delta (\Gamma t + \chi)^2 - M^2 \omega_0^2} - \tan^{-1} \frac{M \omega_0}{\sqrt{M \Delta \chi^2 - M^2 \omega_0^2}}} \right]. \tag{89}
\]

Here \textit{EllipticF} and \textit{EllipticE} are the incomplete elliptic integrals of the first and second kinds respectively.

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