RANDOMIZED MULTIVARIATE CENTRAL LIMIT THEOREMS
FOR ERGODIC HOMOGENEOUS RANDOM FIELDS II.
REDUCTION OF THE MOMENT CONDITION.

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Abstract
In the previous work by the second author (A. Tempelman, Randomized multivariate central limit theorems for ergodic homogeneous random fields, Stochastic Processes and their Applications, 143 (2022), 89-105) CLTs for homogeneous random fields on $\mathbb{R}^m$ and $\mathbb{Z}^m$ ($m \geq 1$), satisfying the condition: $E[|X(0)|^{2+\delta}] < \infty$ have been proved. In the present article the authors prove that these theorems are valid under the weaker condition $E[(X(0))^2] < \infty$.

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1 Preliminaries

1.1 On the CLT

The Central Limit Theorem (CLT) is one of the remarkable statements of Probability. Originally proved for sequences of independent random variables, in many works this theorem was generalized to stationary random processes and homogeneous random fields under some additional assumptions: Markov, strong mixing and/or rather strong moment conditions, etc. (see, e.g., [3, 6, 7, 9, 10, 11, 12, 14, 15, 17, 18, 26, 27, 29, 38, 41, 42, 45, 46, 48, 53-56]).

However, the CLT may fail even when the sequence $X_1, X_2, \ldots$, is stationary, quite strongly mixing, orthogonal and with $\lim \frac{1}{n} \text{Var}(\sum_{i=1}^{n} X_i) = \sigma^2 > 0$ [24]; two other interesting counterexamples are provided in [7]. Earlier Chung [12] and Davydov [15, 16] proved that the CLT may fail for strictly stationary irreducible aperiodic Markov chains with a countable set of states (these and other counterexamples can be found in vol. 3, Chapters 30 and 31 in the book [7]. Ibragimov and Linnik [28] have constructed a strongly mixing stationary sequence of random variables $X_1, X_2, \ldots$, with finite variances such that the self-normalized sums $(\text{Var}[\sum_{i=1}^{n} X_i])^{-\frac{1}{2}} \sum_{i=1}^{n} X_i$ converge in distribution to a non-normal random variable (see Ch. 19, §5 therein). A rather common case when the CLT fails is when $\lim \frac{1}{n} \text{Var}[\sum_{i=1}^{n} X_i] = 0$ (see, e.g., [18], p.153).

A well-known source of counterexamples for the classical CLT for strict sense stationary random sequences is the class of coboundaries with respect to measure preserving transformations. Let $\gamma : \omega \mapsto \gamma \omega$ be an ergodic measure preserving transformation of a probability space $(\Omega, F, P)$; let $\alpha \geq 1$; then for each $f \in L^\alpha(\Omega, F, P)$, $f \neq 0$ a.s., the random sequence $X_i^{(f)} = f(\gamma^i \omega)$, $i = 1, 2, \ldots$, is an ergodic strict sense stationary sequence and $E[|X_i^{(f)}|^\alpha] < \infty$ (see, e.g., [36], §1.4). Let $g \in L^\alpha(\Omega, F, P)$ and let $f$ be the coboundary of $g$: $f(\omega) = g(\gamma \omega) - g(\omega)$. It is clear that $E(X_i^{(f)}) = 0$, $i = 1, 2, \ldots$. We have also: $\sum_{i=1}^{n} X_i^{(f)} = \sum_{i=1}^{n} f(\gamma^i \omega) = g(\gamma^{n+1} \omega) - g(\omega)$, hence $E[|\sum_{i=1}^{n} X_i^{(f)}|^\alpha] \leq 2^\alpha ||g||_{L^\alpha}^\alpha$, and $E[|n^{-\beta} \sum_{i=1}^{n} X_i^{(f)}|^\alpha] \to 0$ if $\beta > 0$ (when $\alpha > 1$, the converse is true: if $E[f] = 0$ and $\sup_n E[|\sum_{i=1}^{n} X_i^{(f)}|^\alpha] < \infty$, it is a coboundary of some $g \in L^\alpha$; see Lemma 5 in [8]); the set of coboundaries is dense in the space $L^\alpha(\Omega, F, P) \ominus \mathbb{R}$, by the ergodic decomposition. It is evident that, if $f$ is a coboundary of a function $g \in L^\alpha$, then $n^{-\beta} \sum_{i=1}^{n} X_i^{(f)} \to 0$ in probability for any $\beta > 0$, and, if $g$ is
bounded, then this is true in the sense of a.s. convergence.

There is an extensive literature related to limit theorems for “long memory” (or “long-range dependent”) stationary sequences, where the slowly decreasing dependence between receding terms is characterized by the rate of decrease of the correlation function or by properties of the spectrum near 0) - see [1], [22], [49] and the references therein.

1.2 Main results

In the previous work by the second author [52], CLTs for homogeneous random fields on $\mathbb{R}^m$ and $\mathbb{Z}^m$ ($m \geq 1$), satisfying the condition: $E[|X(0)|^{2+\delta}] < \infty$ have been proved. In the present article the authors prove that these theorems are valid under the weaker condition $E[(X(0))^2] < \infty$.

In this paper we present randomized versions of the CLT, which are valid for each ergodic homogeneous measurable random field $X(\cdot)$ on $\mathbb{R}^m$ or $\mathbb{Z}^m$ ($m \geq 1$), (in particular, for each ergodic stationary random process and each ergodic stationary random sequence) with a finite second moment (in some versions ergodicity may be omitted). Specifically, observations of the random field at randomly chosen points are used.

If the field $X$ is multivariate and each of its components satisfies the above conditions, another feature is obtained by the randomization: in Theorems [14] and Corollaries 1, 2 the components of the limit normal vector are independent whatever are the components of $X$. When $E[|X(0)|^2] < \infty$, these theorems are valid also in all cases, mentioned in Subsection 1.1, when the conventional CLT fails.

For example, consider the stationary random sequences $\{X^{(f)}_i\}$, which have been introduced in Subsection [14] (each sequence is specified by some coboundary $f \in L^2(\Omega, \mathcal{F}, P)$); we have: $E[|X^{(f)}_i|^2] < \infty$; as mentioned above, the set of coboundaries is dense in $L^2(\Omega, \mathcal{F}, P) \ominus \mathbb{R}$, and the conventional CLT fails for $\{X^{(f)}_i\}$, if $f$ a coboundary: the limit is degenerate ($\delta_0$) and not normal. However, the randomized CLTs are valid for all sequences $\{X^{(f)}_i\}$ with $E[|X^{(f)}_i|^2] < \infty$.

In [26] (see also Theorem 18.5.3 in [28]) I.A. Ibragimov proved the CLT for strongly mixing stationary processes under the condition: $E[|X(0)|^{2+\delta}] < \infty$ for some $\delta > 0$, and later Yu. Davydov [16], Bradley [4, 5] and Häggström [25] proved that this moment condition is essential.

Randomization allowed us to assume that the field is only homogeneous (in some statements it is assumed that the field is also ergodic) and its second moment is finite.
The main tools, used in the proofs, are the Lindeberg CLT and the Pointwise and Mean Ergodic Theorems.

To illustrate our results, we state a simple corollary of Theorem 3.

Let \( X(t), t \in \mathbb{R} \), be an ergodic stationary measurable random process and \( E[(X(0))^2] < \infty \); denote \( \sigma^2 = \text{Var}[X(0)] \), \( M_n = \frac{1}{n} \int_0^n X(t) dt \). Let \( \tau_{n,i} \) (\( n = 1, 2, \ldots \) and \( i = 1, \ldots, n \)) be random variables, independent of \( X \), and, for each \( n \), independent of each other and uniformly distributed on \([0, n]\).

Then, if \( n \to \infty \),
\[
\frac{\sum_{i=1}^n (X(\tau_{n,i}) - M_n)}{\sigma \sqrt{n}} \Rightarrow N(0, 1)
\]

A slightly more complex corollary with \( \mu = E[X(0)] \) instead of \( M_n \) can be derived from Theorem 4; as usual, if \( \sigma \) is known, this statement may be used for consistent statistical inference on the expectation \( \mu \).

1.3 The "time" set \( T \)

In this paper we study random fields defined on a set \( T \), which is the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) or the \( m \)-dimensional integer lattice \( \mathbb{Z}^m \), \( m \geq 1 \) (when \( m = 1 \), the random fields turn into random processes or random sequences). We denote by \( \mathcal{B} \) the Borel \( \sigma \)-field on \( T \) (if \( T = \mathbb{Z}^m \), then \( \mathcal{B} \) coincides with the collection of all subsets of \( \mathbb{Z}^m \), and each function \( f \) on \( \mathbb{Z}^m \) is \( \mathcal{B} \)-measurable); \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^m \) and the counting measure on \( \mathbb{Z}^m \) (in the latter case \( \lambda(A) \) is the cardinality of \( A \subset \mathbb{Z}^m \), and \( \int_A f(t) \lambda(dt) = \sum_{t \in A} f(t) \) if \( \lambda(A) < \infty \)).

1.4 Random fields

We consider a \( d \)-dimensional random field \( X(t) = (X^1(t), \ldots, X^d(t)), t \in T \), over a probability space \((\Omega_X, \mathcal{F}_X, P_X)\).

Let us recall several definitions. The field \( X \) is (strict-sense) homogeneous if all finite dimensional distributions of \( X \) are shift-invariant:

\[
P_X(\{\omega : X(t_1 + t, \omega) \in A_1, \ldots, X(t_k + t, \omega) \in A_k}\}) = \\
P_X(\{\omega : X(t_1, \omega) \in A_1, \ldots, X(t_k, \omega) \in A_k\})
\]

for all \( t, t_i \in T, \ i, k \in \mathbb{N} \) and for all sets \( A_i \), belonging to the Borel \( \sigma \)-field \( \mathcal{B}(\mathbb{R}^d) \).
A family of invertible transformations $\gamma = \{\gamma_t, t \in T\}$ of $\Omega_X$ is said to be a **group** if $\gamma_0 \equiv \omega, \gamma_{s+t} = \gamma_s \gamma_t, \gamma_t^{-1} = \gamma_{-t}$, where $s,t \in T, \omega \in \Omega_X$. The field $X$ is generated by a group $\gamma$, if $X(t, \omega) = X(0, \gamma_t \omega), t \in T, \omega \in \Omega_X$; since $X(0, \gamma_{s+t} \omega) = X(0, \gamma_s \gamma_t \omega) = X(s, \gamma_t \omega)$, this implies:

$$X(s + t, \omega) = X(s, \gamma_t \omega), s, t \in T. \quad (1)$$

A family $\gamma$ is **measure preserving**, if the transformations $\gamma_t$ are $\mathcal{F}_X$-measurable and $P_X(\gamma_t \Lambda) = P_X(\Lambda), t \in T, \Lambda \in \mathcal{F}_X$. If the random field $X$ is generated by a measure preserving group of transformations of $\Omega_X$, then, by Property (1), it is homogeneous. Denote by $\mathcal{I}_X$ the $\sigma$-field of all events $\Lambda \in \mathcal{F}_X$, which are invariant mod $P_X$ with respect to all transformations $\gamma_t$, i.e. $P_X(\Lambda \triangle \gamma_t \Lambda) = 0, t \in T$. The group $\gamma$ is said to be **metrically transitive** if $\mathcal{I}_X = \{\Lambda : \Lambda \in \mathcal{F}_X, P_X(\Lambda) = 0 \text{ or } 1\}$. The field is **ergodic** if it is generated by a metrically transitive measure preserving group $\gamma$.

Without loss of generality, we assume that $(\Omega_X, \mathcal{F}_X, P_X)$ is the probability space of function type, i.e., $\Omega_X$ is the space of $\mathbb{R}^d$-valued "sample functions" $x(\cdot)$ on $T$, $\mathcal{F}_X$ is the $\sigma$-field generated by the sets

$$\{x(\cdot) : x(t) \in A\}, \quad t \in T, A \in \mathcal{B}(\mathbb{R}^d),$$

and $P_X$ is a probability measure on $\mathcal{F}_X$. $X$ is the coordinate random field: $X(t, x(\cdot)) = x(t)$, where $t \in T, x(\cdot) \in \Omega_X$ (sometimes, we write $\omega$ instead of $x(\cdot)$, when we refer to an element of $\Omega_X$).

Consider the invertible measurable "shift" transformations $\gamma_t, t \in T$, of $\Omega_X$, defined as follows: $\gamma_t x(\cdot) = x(\cdot + t)$; it is clear that the family $\{\gamma_t\}$ is a group; moreover, $X$ is generated by this group:

$$X(0, \gamma_t x(\cdot)) = X(0, x(\cdot + t)) = x(0 + t) = x(t) = X(t, \cdot).$$

The field is homogeneous, if and only if the shift transformations preserve the measure $P_X$.

Each component field $X^l$ may be considered over the probability space $(\Omega^l_X, \mathcal{F}^l_X, P^l_X)$ where $\Omega^l_X$ is the set of scalar functions $x^l(\cdot)$ on $T$, $\mathcal{F}^l_X$ is the $\sigma$-field generated by the events $\{x^l(\cdot) : x^l(t) \in D\}, \quad t \in T, \quad D \in \mathcal{B}(\mathbb{R})$. Let $A^l$ be an event in $\mathcal{F}^l_X$; the event $\Lambda_{A^l} := \{x(\cdot) : x^l(\cdot) \in A^l\} \in \mathcal{F}_X$, and $P^l_X$ is the projection of the measure $P_X$ onto $\mathcal{F}^l$: for each $A^l \in \mathcal{F}^l_X$, $P^l_X(A^l) = P_X(\Lambda_{A^l})$. $X^l$ is the coordinate field: $X^l(t, x^l(\cdot)) = x^l(t)$, where $t \in T, x^l(\cdot) \in \Omega^l_X$.

In what follows we assume that the fields $X^l$ are homogeneous, i.e., for each $l$ all transformations $\gamma^l_t$ preserve the measure $P^l_X$. It is also assumed that $E_X[|X^l(0)|^2] < \infty$ for some $\delta > 0$ ($l = 1, \ldots, d$). Let

$$\mu^l := E_X[X^l(0)]; \quad \sigma^l := Var_X[X^l(0)]. \quad If \ T = \mathbb{R}_m, \ we \ always \ assume \ that \ the \ random \ fields \ X^l(t) = X^l(t, x^l) \ are \ \mathcal{B} \times \mathcal{F}^l\text{-measurable}; \ by \ the \ Fubini$
theorem, these assumptions imply that the sample functions $x^l(\cdot)$ are Borel measurable with $P_X^l$-probability 1 and for each finite Borel measure $Q$ the integral $\int_T x^l(t)Q(dt)$ exists with $P_X^l$-probability 1.

Denote by $\mathcal{I}_X^l$ the $\sigma$-field of all $\gamma$-invariant mod ($P_X^l$) events in $\mathcal{F}_X^l$ (we remind that $X^l$ is said to be ergodic if $\mathcal{I}_X^l$ is trivial). $E_X[X^l(0)|\mathcal{I}_X^l]$, $\text{Var}_X[X^l(0)|\mathcal{I}_X^l]$ are the conditional expectation and variance; in the sequel it is assumed that for each $l$ $\text{Var}_X[X^l(0)|\mathcal{I}_X^l] > 0$ with $P_X^l$-probability 1.

In some cases it is assumed that the $\mathbb{R}^d$-valued random field $X$ is homogeneous, that is the transformations $\gamma_t$ preserve the measure $P_X$; then the transformations $\gamma_t^l$ preserve the measures $P_X^l$, and the components $X^l$ are also homogeneous; if the random field $X$ is ergodic, then all its component fields $X^l$ are ergodic, too.

### 1.5 Randomizing random vectors

Let $\{q^l_n\}$ be sequences of probability measures on $\mathcal{B}$, $(l = 1, \ldots, d)$, $d \in \mathbb{N}$ (these sequences may coincide for some or even all $l$). For each natural $n$ we consider $d$ mutually independent $k_n$-dimensional random vectors $\tau_n^l = (\tau_{n,1}^l, \ldots, \tau_{n,k_n}^l)$, $l = 1, \ldots, d$, over a probability space $(\Omega_\tau, \mathcal{F}_\tau, P_\tau)$ possessing the following properties:

a) the vectors $\tau_n^l$ do not depend on the random field $X$,

and

b) the components of each vector $\tau^l$ are i.i.d. $T$-valued random vectors with the distribution $q^l_n$.

(Of course, if $T = \mathbb{Z}^m$ and if for some $l$ the support of $q^l_n$ is finite, then some points may appear in the sample $\tau_{n,1}^l, \ldots, \tau_{n,k_n}^l$ several times). It is assumed that $k_n \uparrow \infty$ as $n \to \infty$ (the sequence $\{k_n\}$ may also depend on $l$; to simplify the notation, we always drop the index $l$ in $k_n^l$). Since each scalar field $X^l = X^l(t, \omega)$ is a $\mathcal{B} \times \mathcal{F}_X$-measurable function, $X(\tau_{n,i}^l, \omega)$ are random variables over the probability space

$$(\Omega_{X,\tau}, \mathcal{F}_{X,\tau}, P_{X,\tau}) := (\Omega_X \times \Omega_\tau, \mathcal{F}_X \times \mathcal{F}_\tau, P_X \times P_\tau)$$

### 1.6 Notation

In the sequel we use the following notation:

- $E_X$, $E_\tau$ and $E_{X,\tau}$ denote the expectation with respect to the measure $P_X$, $P_\tau$ and $P_{X,\tau}$.
• $Y_n \rightarrow Y$ P-a.s., $Y_n \overset{P}{\rightarrow} Y$ mean convergence almost sure, respectively in probability.
• $Y_n \overset{Q}{\Rightarrow} Y$ means convergence in distribution with respect to the measure $Q$.
• $a := b$ means that the quantity $a$ is defined by the expression $b$.
• $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote the sets of real numbers, of integers and of natural numbers.
• $T = \mathbb{R}^m$ or $\mathbb{Z}^m$ ($m \in \mathbb{N}$).
• $1_A$ denotes the indicator of the set $A$.

1.7 Terminology

All results are stated in the terms of homogeneous random fields (of course, they are valid for stationary random sequences and processes, too). If $m = 1$, the words "homogeneous random field" mean "stationary random process" or "stationary random sequence"; the words "ball", "cube" and "parallelepiped" mean "interval" and, if $m = 2$, these words mean "sphere" "square" and "parallelogram", respectively.

2 Randomization using uniform distributions on subsets $T^l_n$ of $T$

Let $X(t) = (X^l(t), ..., X^d(t)), t \in T,$ be a homogeneous random field on $T$, 
\{T^l_n\} be sequences of bounded Borel sets of positive measure in $T$
($l = 1, ..., d), \ d \in \mathbb{N}$ (these sequences may coincide for some or even all $l$).
It is supposed that $\lambda(T^l_n) \to \infty$ as $n \to \infty$.

We consider the $T_n$-valued random vectors $\tau^{l}_{n,1}, ..., \tau^{l}_{n,k_n}, \ l = 1, ..., d, \ n \in \mathbb{N},$ introduced in Subsect. 1.5.

**Lemma 1.** If $f$ is a measurable function on $\mathbb{R}$ such that $E_X[|f(X(0))|] < \infty$, then $E_X,\tau[|f(X(\tau^{l}_{n,i}))|] < \infty$ and with $P_X$-probability 1

$E_{\tau}[|f(X(\tau^{l}_{n,i}))|] < \infty$.

**Proof.** It follows immediately from the Fubini - Tonelli Theorem. □

Assume that for each $l \in \{1, ..., d\}$ and for each natural number $n$ the random vectors $\tau^{l}_{n,i}, \ i = 1, ..., k_n$, introduced in Subsect. 1.5 are uniformly distributed (with respect to $\lambda$) on the set $T^l_n$. In this section we use the
following estimators of \( \mu^l \) and \((\sigma^l)^2\):

\[
M_n^l := \frac{1}{\lambda(T_n^l)} \int_{T_n^l} X^l(t) \lambda(dt), \\
V_n^l := \frac{1}{\lambda(T_n^l)} \int_{T_n^l} (X^l(t) - M_n^l)^2 \lambda(dt) = \frac{1}{\lambda(T_n^l)} \int_{T_n^l} (X^l(t))^2 \lambda(dt) - (M_n^l)^2.
\]

If \( \omega \in \Omega_X \) is fixed, the random variables \( X^l(\tau_{n,i}^l, \omega) \) form \( d \) triangular arrays and for each \( n \) they are independent and identically distributed. For each measurable function \( f \) on \( \mathbb{R} \), such that \( f[X^l(t)] \) is integrable on \( T_n^l \),

\[
E_\tau[f(X^l(\tau_{n,i}^l))] = \frac{1}{\lambda(T_n^l)} \int_{T_n^l} f(X^l(t)) \lambda(dt), \quad i = 1, \ldots, k_n, \quad l = 1, \ldots, d.
\]

In particular,

\[
E_\tau[X^l(\tau_{n,i}^l)] = M_n^l; \quad Var_\tau[X^l(\tau_{n,i}^l)] = V_n^l.
\]

The above relation implies:

\[
E_\tau[\sum_{i=1}^{k_n} f(X^l(\tau_{n,i}^l))] = k_n \frac{1}{\lambda(T_n^l)} \int_{T_n^l} f(X^l(t)) \lambda(dt) \quad (l = 1, \ldots, d). \tag{2}
\]

**Definition 1.** We say that a sequence of Borel sets \( \{T_n\} \) is **pointwise averaging** if the following **Pointwise Ergodic Theorem** (PET) holds with this sequence:

*Let \( f(\cdot) \) be a measurable function on \( \mathbb{R} \). If \( X \) is a scalar homogeneous random field on \( T \) over \((\Omega_X, \mathcal{F}_X, P_X)\) with \( E_X[f(X(0))] < \infty \) and \( I_X \) is the \( \sigma \)-field of shift-invariant \( \mod (P_X) \) events in \( \mathcal{F}_X \), then with \( P_X \)-probability 1

\[
\lim_{n \to \infty} \frac{1}{\lambda(T_n)} \int_{T_n} f(X(t)) \lambda(dt) = E_X[f(X(0))|I_X];
\]

(if \( X \) is ergodic, then \( E_X[f(X(0))|I_X] = E_X[f(X(0))] \) with \( P_X \)-probability 1).

**Example 1.** In \( \mathbb{R}^m \) each increasing sequence of bounded convex sets \( T_n^l \subset \mathbb{R}^m \), containing balls \( B_n \) of radii \( r(B_n) \to \infty \), is pointwise averaging (see Corollary 3.3 in Ch. 6 in [50]). In particular, any sequence of concentric balls \( B_n \) with \( r(B_n) \to \infty \) and any increasing sequence of parallelepipeds \( T_n \subset \mathbb{R}^m \) with infinitely growing edges is pointwise averaging (in particular, the sequence of cubes \([0,n]^m \) has this property). The intersections of the mentioned sets with \( \mathbb{Z}^m \) form pointwise averaging sequences in \( \mathbb{Z}^m \) (see Subsect. 5.2 in [51]).
Example 2. Let $A$ be a compact set in $\mathbb{R}^m$ containing 0 with $\lambda(A) > 0$. If $s_n \uparrow \infty$, the sequence of homothetic sets $\{s_n A\}$ is pointwise averaging (see Example 2.9 in Ch. 5 in [50] or Subsection 5.4.1 in [51]). Note: if $0 \notin A$, this sequence is not increasing.

Remark 1. Along with the pointwise averaging sequences of sets, mean averaging sequences may be considered, i.e., sequences $T_n$ for which The Mean Ergodic Theorem (MET) for each homogeneous random field with $E[(X(0))^2] < \infty$:

$$E_X \left[ \frac{1}{\lambda(T_n)} \int_{T_n} X(t) \lambda(dt) - E_X[X(0)|I_X]\right]^2 \to 0.$$ 

Under our condition $E[(X(0))^2] < \infty$, each pointwise averaging sequence is mean averaging (see Corollary 2 in Subsect 9.4 in [11]). Consider the Hilbert subspace $H_X$ of $L^2(\Omega_X, \mathcal{F}_X, P_X)$, spanned by the random variables $X(t), t \in T$; it is invariant with respect to the shift transformations: $W(\gamma_t \omega) \in H_X$ if $W(\omega) \in H_X$; denote by $I$ the subspace of all random variables invariant with respect to all $\gamma_t$. The conditional expectation $E_X[X(0)|I_X] = \tilde{E}_X[X(0)|I_X]$, the orthogonal projection of $X(0)$ onto $I$. This random variable equals $E[X(0)]$, if $X$ is "wide-sense ergodic", i.e., if $I = \mathbb{R}$; of course this condition is much weaker than ergodicity, which assures that the subspace of all $\gamma$-invariant random variables in $L^2(\Omega_X, \mathcal{F}_X, P_X)$ coincides with $\mathbb{R}$. The class of the mean averaging sequences is rather wide; in particular, the monotonicity condition in Example 1 is redundant. The MET is essentially used in the sequel (see also Remark 3 where the speed of convergence in the MET is discussed).

Let us remind that the main probability space has the following form $(\Omega_{X,\tau}, \mathcal{F}_{X,\tau}, P_{X,\tau}) = (\Omega_X \times \Omega_\tau, \mathcal{F}_X \times \mathcal{F}_\tau, P_X \times P_\tau)$.

Lemma 2. Let $X$ be a homogeneous random field, $E[(X(0))^2] < \infty$, s and let $\{T_n\}$ be a pointwise averaging sequence of sets. Then for $P_X$-almost all $\omega \in \Omega_X$ the Lindeberg condition is fulfilled:

for each $\epsilon$ as $n \to \infty$

$$L_n(\omega) := \frac{\sum_{i=1}^{k_n} E_\tau[[X(\tau_{n,i},\omega) - M_n(\omega)]^2 1_{B_n}]}{\sum_{i=1}^{k_n} V_n(\omega)} \to 0,$$

where $B_n = \{ |X(\tau_{n,i},\omega) - M_n(\omega)| > \epsilon k_n^{1/2} (V_n)^{1/2} \}$.

Proof. For a fixed ”good” $\omega$, we will apply the Lindeberg theorem (see, e.g., Theorem 27.2 in [2]) to the random variables over the space $(\Omega_\tau, \mathcal{F}_\tau, P_\tau)$. 

9
By relation (2), for each $\omega \in \Omega_X$ and for each $\epsilon > 0$, the Lindeberg fraction for the random variables $X(\tau_{n,i}(w), \omega)$, $i = 1, \ldots, k_n$,

$$L_n(\omega) = \frac{1}{V_n(\omega)} \frac{1}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - M_n(\omega))^2 1_{C_n(t)} \lambda(dt),$$

where $C_n(t) = \{|X(t, \omega) - M_n(\omega)| > \epsilon k_n(V_n)^{\frac{1}{2}}\}$.

We have to prove that with $P_X$-probability 1 $L_n(\omega) \to 0$. Since the sequence $\{T_n\}$ is pointwise averaging, by the PET, with $P_X$-probability 1

$$M_n(\omega) \to z(\omega), \quad (3)$$

$$V_n(\omega) = \frac{1}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - M_n(\omega))^2 \lambda(dt) \to v(\omega), \quad (4)$$

where

$$z(\omega) = E_X[X(0, \omega)|I_{X}](\omega), \quad (5)$$

$$v(\omega) = Var_X[(X(0), \omega)^2|I_{X}](\omega) > 0. \quad (6)$$

It is clear that

$$L_n(\omega) \leq \Sigma_n^{(1)} + \Sigma_n^{(2)},$$

where

$$\Sigma_n^{(1)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - z(\omega))^2 1_{C_n(t)} \lambda(dt),$$

$$\Sigma_n^{(2)} = M_n^2 \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (z(\omega) - M_n)^2 1_{C_n(t)} \lambda(dt).$$

Due to (3) with $P_X$-probability 1

$$\Sigma_n^{(2)} \leq (z(\omega) - M_n)^2 \frac{2}{V_n(\omega)} \to 0. \quad (7)$$

Consider $\Sigma_n^{(1)}$. Denote by $\Lambda \in \mathcal{F}_X$ the set of $P_X$-measure 1 in $\Omega$ where the limits (3) and (4) exist, and fix some $\omega \in \Lambda$. It is clear that $k_nV_n(\omega) \to \infty$ as $n \to \infty$. For each $C > 0$, let us also fix some $m_C(\omega) \in \mathbb{N}$ such that, if $n > m_C(\omega)$, then $|M_n(\omega) - z(\omega)| < \frac{1}{2} \epsilon C$ and $(k_nV_n(\omega))^{\frac{2}{3}} > C > 0$. 

10
As for \( n > m_C(\omega) \)
\[
\{ |X(t, \omega) - M_n| > \varepsilon k_n^\frac{1}{2} (V_n)^{\frac{1}{2}} \} \subset \{ |X(t, \omega) - z(\omega)| > \frac{\varepsilon C}{2} \} \cup \{ |z(\omega) - M_n| > \frac{\varepsilon C}{2} \},
\]
we have:
\[
\Sigma_{n}^{(1)} \leq \delta_{n}^{(1)} + \delta_{n}^{(2)}, \tag{8}
\]
where
\[
\delta_{n}^{(1)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - z(\omega))^2 1_{\{|X(t, \omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \lambda(dt),
\]
\[
\delta_{n}^{(2)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - z(\omega))^2 1_{\{|z(\omega) - M_n| > \frac{\varepsilon C}{2}\}} \lambda(dt).
\]
By the PET, with \( P_X \)-probability 1
\[
\lim_{n} \delta_{n}^{(1)} = \frac{2}{v(\omega)} E \left[ (X(0, \omega) - z(\omega))^2 1_{\{|X(0, \omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \right], \tag{9}
\]
and due to (3)
\[
\lim_{n} \delta_{n}^{(2)} = \frac{2}{v(\omega)} \lim_{n} \frac{1}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - z(\omega))^2 \lambda(dt) \cdot \lim_{n} 1_{\{|z(\omega) - M_n| > \frac{\varepsilon C}{2}\}} = 0. \tag{10}
\]
It follows from (7)-(10) that
\[
\limsup_{n} L_n(\omega) \leq 2 E \left[ (X(0, \omega) - z(\omega))^2 1_{\{|X(0, \omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \right].
\]
Letting \( C \) tend to infinity, we finally obtain with \( P_X \)-probability 1
\[
\lim_{n} L_n(\omega) = 0.
\]
\[\square\]

**Assumptions 1.** In Theorems 1-4, all \( X^l, l = 1, \ldots, d, \) are homogeneous random fields and, for all \( l, \)
\[
E_X[|X^l(0)|^2] < \infty. \tag{11}
\]
If \( X^l \) is not ergodic, we assume \( \text{Var}_X[X^l(0)|T^l_X] > 0 \) with \( P^l_X \)-probability 1, \( l = 1, \ldots, d. \)
We remind that \( k_n \in \mathbb{N}, k_n \uparrow \infty. \)
For each \( l, \) \( \{T^l_n\} \) is a pointwise averaging sequence of sets.
Remark 2. We will prove that, together with the conditions mentioned above, in the non-ergodic case, the condition \( \text{Var}_X[X^l(0)|I^l_X] > 0 \) \( P_X \)-a.s., \( l = 1, \ldots, d \), is sufficient for the relation (12) to hold with \( P_X \)-probability 1. And, for each fixed \( l \), it is also necessary for this statement to hold for the component \( X^l \). Indeed, if the variance of the conditional distribution \( \text{Var}_X[X^l(0)|I^l_X] = 0 \), then \( X^l(0) = E_X[X^l(0)|I^l_X] \) \( P_X \)-a.s., hence \( X^l(0) \) is \( I^l_X \)-measurable and \( X^l(t, \omega^l) = X^l(0, \gamma_l^l \omega^l) = X^l(0, \omega^l) \) \( P_X \)-a.s. so, for each \( t \in T \), \( X^l(t) = X^l(0) \) \( P_X \)-a.s. It is clear, that this dependence is too strong for any version of the CLT to hold for \( X^l \). In this case, it becomes senseless: note that \( M^l_n = X^l(0), V^l_n = 0 \); therefore, the expression

\[
\sum_{i=1}^{k_n} \left( X^l(\tau^l_{n,i}) - M^l_n \right) \frac{1}{k_n^n{V^l_n}^{\frac{1}{2}}} = \frac{k_n^n(X^l(0) - M^l_n)}{k_n^n{V^l_n}^{\frac{1}{2}}}
\]

(see (12)) is of type \( 0 \); conditions (15) and (16) for \( X^l \) do not hold, and the expressions (13), (14) for \( X^l \) are of type \( 1 \).

We denote: \( Z = (Z^1, \ldots, Z^d) \) where \( Z^1, \ldots, Z^d \) are independent standard normal random variables.

**Theorem 1.** With \( P_X \)-probability 1

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - M^1_n), \ldots, \sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - M^d_n) \right) \xrightarrow{P_X} Z;
\]

if all \( X^l \) are ergodic, then with \( P_X \)-probability 1

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - M^1_n), \ldots, \sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - M^d_n) \right) \xrightarrow{P_X} Z.
\]

**Proof.** By Lemma 2 we may apply the Lindeberg theorem and obtain: with \( P_X \)-probability 1 for each \( l \)

\[
\sum_{i=1}^{k_n} (X^l(\tau^l_{n,i}) - M^l_n) \xrightarrow{k_n}{Z^l}.
\]

If \( X \) is ergodic, then \( (V^l_n)^{\frac{1}{2}} \rightarrow \sigma^l \) with \( P_X \)-probability 1, and, by the Slutsky theorem ([37], Theorem 23.3), with \( P_X \)-probability 1

\[
\sum_{i=1}^{k_n} (X^l(\tau^l_{n,i}) - M^l_n) \xrightarrow{k_n}{Z^l}.
\]

12
Since for each $n$ the $d$ vectors $(\tau_{n,i}^l, i = 1, \ldots, k_n), \ l = 1, \ldots, d$, are mutually independent, using (13) and (14) we obtain the convergence (12), respectively (1), of vectors.

Since the limiting distributions in (12) and (1) with $P_X$-probability 1 are the same, we immediately deduce from the previous theorem the nonconditional weak convergence:

**Theorem 2.** Under Assumptions 1,

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - M_n^1) \frac{1}{k_n^\frac{1}{2} (V_n^1)^{\frac{1}{2}}}, \ldots, \sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - M_n^d) \frac{1}{k_n^\frac{1}{2} (V_n^d)^{\frac{1}{2}}} \right) \xrightarrow{P_X} Z;
\]

if all $X^l$ are ergodic, then with $P_X$-probability 1

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - M_n^1) \frac{1}{k_n^\frac{1}{2} \sigma^1}, \ldots, \sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - M_n^d) \frac{1}{k_n^\frac{1}{2} \sigma^d} \right) \xrightarrow{P_X} Z.
\]

The following Theorems 3-4 can be deduced subsequently from Theorems 1-2 literally as Th. 3 and Th. 4 in [52].

**Theorem 3.** In addition to the above Assumptions 1, assume that

\[
k_n^\frac{1}{2} (M_n^l - \mu^l) \to 0 \text{ with } P_X\text{-probability } 1, \ l = 1, \ldots, d. \quad (15)
\]

Then with $P_X$-probability 1

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - \mu^1) \frac{1}{k_n^\frac{1}{2} (V_n^1)^{\frac{1}{2}}}, \ldots, \sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - \mu^d) \frac{1}{k_n^\frac{1}{2} (V_n^d)^{\frac{1}{2}}} \right) \xrightarrow{P_X} Z;
\]

if all $X^l$ are ergodic, then with $P_X$-probability 1

\[
\left( \sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - \mu^1) \frac{1}{k_n^\frac{1}{2} \sigma^1}, \ldots, \sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - \mu^d) \frac{1}{k_n^\frac{1}{2} \sigma^d} \right) \xrightarrow{P_X} Z.
\]

**Theorem 4.** Let the conditions stated in Assumptions 1 be fulfilled and

\[
k_n^\frac{1}{2} (M_n^l - \mu^l) \xrightarrow{P_X} 0. \quad (16)
\]
Then
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{k_n^{1/2} (V^1_n)^{1/2}}, \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{k_n^{1/2} (V^d_n)^{1/2}} \right) \Rightarrow Z;
\]
if all \(X^l\) are ergodic, then
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{k_n^{1/2} \sigma^1}, \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{k_n^{1/2} \sigma^d} \right) \Rightarrow Z.
\]
Condition (16) puts some restrictions to the growth of the number \(k_n\) of observations of the values of the field \(X\). It is often of the form \(k_n = o(\lambda(T_n))\) (see Remark 3). On the other hand, the accuracy of the approximation of the normal distribution by the CLT is higher when \(k_n\) is large. An obvious way to increase \(k_n\) without violation of this condition is to increase \(\lambda(T_n)\) proportionally. Another way to increase the number of observations without breaking this condition puts more work for the statistician; it is based on a special case of Theorem 4 when, instead of \(X\), an auxiliary \(R^{wd}\)-valued random field \(Y\) is considered, in which each component \(X^l\) participates \(w\) times.

Corollary 1. Let the conditions of Theorem 4 be fulfilled. Consider the family of independent random variables \(\{\tau^l_{n,i}, n \in \mathbb{N}, 1 \leq i \leq k_n, 1 \leq l \leq d, 1 \leq u \leq w\}\) which, for each \(n, l, u, i\), are distributed uniformly on \(T^l_n\). Then
\[
\frac{\sum_{i=1}^{k_n} (X^l(\tau^l_{n,i}) - \mu^l)}{k_n^{1/2} (V^l_n)^{1/2}} \Rightarrow Z^{l,u}, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w.
\]
where all random variables \(Z^{l,u}, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w\), are standard normal and independent.

If the field \(X\) is ergodic, the "empirical" standard deviations \(V^l_n\) can be replaced by the "theoretical" standard deviations \(\sigma^l\).

Proof. Apply Theorem 4 to the random fields \(Y^{l,u}(t) := X^l(t), \quad t \in T, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w\).

We consider a generalization of this corollary to various parameters of the marginal or multidimensional distributions of the field \(X\); for simplicity we assume that \(X\) is \(R\)-valued. Let \(\theta^l, l = 1, \ldots, d\), be parameters of the
distribution of a vector \((X(t_1),...,X(t_k))\). We assume that there exists a \(B(\mathbb{R}^k)\)-measurable functions \(f^l(x_1,...,x_k)\) such that
\[
E_X[|f^l(X(t_1),...,X(t_k))|^2] < \infty \text{ for some } \delta > 0,
\]
and \(\theta^l = E_X[f^l(X(t_1),...,X(t_k))]\). Denote:
\[
\sigma^l_f := \sqrt{\text{Var}_X[f^l(X(t_1),...,X(t_k))]},
\]
\[
M^l_{f,n} := \frac{1}{\lambda(T_n)} \int_{T_n} f^l(X(t_1 + t),...,X(t_k + t)) \lambda(dt),
\]
\[
V^l_{f,n} := \frac{1}{\lambda(T_n)} \int_{T_n} (f^l(X(t_1 + t),...,X(t_k + t)) - M^l_{f,n})^2 \lambda(dt).
\]
For each \(l\) we consider the random field
\[
Y^l_f(t) := f^l(X(t_1 + t),...,X(t_k + t)), \ t \in T,
\]
over \((\Omega_X, \mathcal{F}_X, P_X)\). Note that \(Y^l_f(0) := f^l(X(t_1),...,X(t_k + t))\), so the condition \(E_X[Y^l_f(0)]^2 < \infty\) is fulfilled; the finite-dimensional distributions of the field \(Y^l_f\) coincide with finite-dimensional distributions of \(X^l\), and \(Y^l_f\) is generated by the shift transformations: by Property (1),
\[
Y^l_f(0, \gamma \omega) = f^l(X(t_1, \gamma \omega),...,X(t_k, \gamma \omega)) = f^l(X(t_1 + t), \omega),...,X(t_k + t), \omega)) = Y^l_f(t, \omega);
\]
therefore, the field \(Y^l_f\) is homogeneous, and, if the field \(X^l\) is ergodic, the field \(Y^l_f\), is ergodic, too. Note that \(Y^l_f(0) := f^l(X(t_1),...,X(t_k))\), so the condition \(E_X[Y^l_f(0)]^2 < \infty\) is fulfilled.

For example, if \(\theta\) is a mixed moment, i.e.
\[
\theta = E[(X(t_1))^{v_1}...(X(t_k))^{v_k}], \ (v_1,...,v_k \in \mathbb{N}),
\]
we have: \(f^l(x_1,...,x_k) = (x(t_1))^{v_1}...(x(t_k))^{v_k}\),
\[
Y^l_f(t) = (X(t_1 + t))^{v_1}...(X(t_k + t))^{v_k}.
\]

Application of Corollary 1 to the fields \(Y^l_f(t)\) brings us to the following statement.
Corollary 2. Let $X$ be a homogeneous scalar random field. Under the conditions of Corollary 1 (with $Y, M_{l,n}, V_{l,n}, \theta^l$ instead of $X, M_{n}, V_{n}, \mu^l$, respectively), for $l = 1, ..., d$, $u = 1, ..., w$,

$$\frac{\sum_{i=1}^{k_n} f^l(X(t_1 + \tau_{n,i}^l), ..., X(t_k + \tau_{n,i}^l)) - \theta^l}{k_n^{\frac{1}{2}} (V_{l,n})^{\frac{1}{2}}} \overset{P}{\Rightarrow} Z^l,u.$$ 

where all random variables $Z^l,u$ are standard normal and independent.

If $X$ is ergodic, the "empirical" standard deviations $(V_{l,n})^{\frac{1}{2}}$ can be replaced by the "theoretical" ones $\sigma^l$. 

Now we present a randomized version of the CLT when the limit normal distribution is not standard and its covariance matrix coincides with the marginal covariance matrix of the field $X$. This statement is a generalization of the classical CLT for i.i.d. random vectors (see Theorem 29 in [2]) and can be readily deduced from our Theorem 4 (with $d = 1$) by using the Cramér-Wold theorem (compare with the proof of the mentioned Theorem 29).

Theorem 5. Let the conditions of Theorem 4 be fulfilled and, moreover, let the field $X$ be strict sense stationary and ergodic. Let the pointwise averaging sequence $\{T_n\}$ and the randomizing sequence $(\tau_{n,i})$ be the same for all components $X^l$. Denote by $\Sigma$ the (nonsingular) covariance matrix of the vector $X(0)$. Then

$$\left( k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^1(\tau_{n,i}) - \mu^1, ..., k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^d(\tau_{n,i}) - \mu^d) \right) \overset{P}{\Rightarrow} V \sim N(0, \Sigma).$$

Our theorems and Examples 1 and 2 imply the following statement.

Corollary 3. 1. Let $T = \mathbb{R}^m$. Then Theorems 1-5 and Corollaries 1 and 2 hold if $\{T_n\}$ is

- an increasing sequence of bounded convex sets, containing balls of radii $r_n \to \infty$, or
- a sequence of homothetic sets $\{s_n A\}$ considered in Example 2

2. The same is true if $T = \mathbb{Z}^m$ and $T_n^l$ are restrictions onto $\mathbb{Z}^m$ of convex sets in $\mathbb{R}^m$ considered above.

In what follows we put $d = 1$ for simplicity.

Remark 3. The proofs of Theorems 1 and 2 suggest that, in these theorems, it is reasonable to use sequences $\{k_n\}$ growing to $\infty$ rather fast, i.e., to
chose the “size of randomization” as large as possible. But in Theorems 2 and 4 conditions (15) and (16) put some restrictions to this growth. These conditions are related to the speed of convergence in the Pointwise and Mean Ergodic Theorems, respectively. In the following notes we provide an idea of these conditions. We start with well known results related to condition (15) in the case $T = \mathbb{Z}$, $T^l_n = \{1, \ldots, n\}$.

1. There are no universal sequences $k_n$ satisfying condition (15): for each non-decreasing sequence $k_n \to \infty$ there is a bounded ergodic stationary sequence $\{X_n\}$ such that $\{k_n\}$ does not satisfy this condition [36].

2. The sequence $k_n = n^2$ does not satisfy condition (15) [30].

3. For each ergodic stationary random field there is a non-decreasing sequence of integers $k_n \to \infty$ that satisfies condition (15) [34].

4. Note 1 shows that the rate of convergence in the PET depends on the properties of the field $X$; about this relation see [30, 33, 34] and references therein.

5. If $\sup_n (n^\gamma \text{Var}_X(M_n))) < \infty$, where $\gamma > 0$, then (15) holds with $k_n = n^\alpha$, where $\alpha < \frac{\gamma}{2}$ ([13], Proposition 1). See Notes 6 and 9 - 11 below for examples.

Now we turn to condition (16).

6. By the Chebyshev inequality, condition (16) holds if $k_n \text{Var}_X(M_n) \to 0$. If the field $X$ is “wide-sense ergodic”, then, by the Mean Ergodic Theorem, $\text{Var}_X(M_n) \to 0$ (see Remark 1); therefore, in this case (16) is always fulfilled if $k_n$ grows sufficiently slow:

$$k_n = o((\text{Var}_X(M_n))^{-1}).$$

7. If $\{X(k)\}$ is a mixing stationary random sequence, $E_X[(X(0))^2] < \infty$ and $T^l_n = \{1, \ldots, n\}$, then the sequence $k_n = n^2$ does not satisfy condition (17).

8. Let $T = \mathbb{R}^m$ or $\mathbb{Z}^m, m \geq 1$. Assume that the CLT holds in its classical form: $(\lambda(T_n))^{\frac{1}{2}} \sigma^{-1}(M_n - \mu) \overset{D}{\to} Z$, where $Z$ is the standard normal random variable. We have: $k_n^{\frac{1}{2}}(M_n - \mu) = (\frac{k_n \sigma^2}{\lambda(T_n)})^{\frac{1}{2}}(\lambda(T_n))^{\frac{1}{2}} \sigma^{-1}(M_n - \mu)$; therefore $k_n^{\frac{1}{2}}(M_n - \mu) \overset{D}{\to} 0$, if and only if $k_n = o((\lambda(T_n)))$. A rather often case when the classical CLT fails is when $(\lambda(T_n))^{\frac{1}{2}}(M_n - \mu) \overset{D}{\to} 0$, hence condition (16) is fulfilled with $k_n = O((\lambda(T_n)))$.

9. Let $\{\xi_k\}$ be an ergodic stationary Markov chain with the probability state space $(X, A, m)$ ($m$ the initial probability measure). Let $Q(x, A)$ be the transition function, and let $F$ be the (dense) set in $L^2(X, A, m)$ consisting of all functions $f : f(x) = g(x) - \int_X g(y)Q(x, dy), g \in L^2(X, A, m)$. Then, for
all $f \in F$ the following alternative holds: the stationary random sequence $X_k = f(\xi_k)$ either satisfies the CLT in the classical form or $\text{Var}[M_n] = O(n^{-1})$ [23]; this alternative holds also for $f$ in a larger class of functions, which contains $F$ (see [44]). According to the previous note, in both cases each sequence $k_n = o(n)$ satisfies condition (16).

10. Let $T = \mathbb{R}^m$ or $\mathbb{Z}^m, m \geq 1$, and let $T_n = (0, n]^m, n \in \mathbb{N}$; if $X$ is wide-sense stationary and possesses a spectral density $\psi$, which is continuous at 0, then condition (16) holds if $k_n = o(n^m)$. This follows from the relation

$$n^m \text{Var}_X \left( \int_{T_n} X(t) \lambda(dt) \right) = (2\pi)^m \psi(0) + o(1)$$

(in the cases $T = \mathbb{Z}$ and $T = \mathbb{R}$ these are, respectively, Theorems 18.2.1 and 18.3.2 in [28]; if $T = \mathbb{Z}^m, m \geq 2$, this is Theorem 3 in [31]; the case $T = \mathbb{R}^m$, $m \geq 2$, can be treated similarly).

11. In the case $m = 1$ various restrictions on the spectrum and the correlation function implying a given rate of convergence in the Mean Ergodic Theorem, hence on restrictions on the choice of $k_n$ in (16) are discussed in [30, 32, 33, 35].

3 Randomization using non-uniform distributions

Let $X(t) = (X^1(t), \ldots, X^d(t)), t \in T$, be a $\mathcal{B} \times \mathcal{F}$-measurable random field with homogeneous components ($d \in \mathbb{N}$) and $E[|X(0)|^2] < \infty$ for some $\delta > 0$. For each $l = 1, \ldots, d, n \in \mathbb{N}$, consider a probability Borel measure $\{q^l_n\}$ on $T^l_n$, which is the distribution of the $T^l_n$-valued i.i.d. randomizing random vectors $\tau^l_{n,i}, i = 1, \ldots, k_n$, introduced in Subsection 1.5. By the Fubini theorem, for each measurable function $f$ on $\mathbb{R}$ such that $E_P[f(X(0))] < \infty$ we have:

$$\int_{T^l_n} |f(X(t))|q^l_n(dt) < \infty \text{ a.s.}$$

and

$$E_r[f(X(\tau_{n,i}))] = \int_{T^l_n} f(X(t))q^l_n(dt), i = 1, \ldots, k_n.$$

In particular,

$$M^l_n := E_r[X(\tau^l_{n,i})] = \int_{T^l_n} X(t)q^l_n(dt);$$

$$V^l_n := \text{Var}_r[X(\tau^l_{n,i})] = \int_{T^l_n} (X(t) - M^l_n)^2q^l_n(dt).$$
Hence
\begin{equation*}
E_T \left[ \sum_{i=1}^{k_n} f(X(\tau_{n,i})) \right] = k_n \int_{T^m} f(X(t)) q_n^t(dt).
\end{equation*}

**Definition 2.** We say that the sequence of probability Borel measures \( \{q_n\} \) is **pointwise averaging** if for each homogeneous random field \( X \) the following PET with the "weights" \( q_n \) holds: if \( E_X[|f(X(0))|] < \infty \) then 
\begin{equation*}
\lim_{n \to \infty} \int_{T^m} f(X(t)) q_n^t(dt) = E_X[f(X(0)|X] \text{ with probability 1; in the case, when the measures } q_n \text{ possess densities } \varphi_n \text{ with respect to } \lambda, \text{ we say that the sequence } \{\varphi_n\} \text{ is pointwise averaging.}
\end{equation*}

We provide a simple lemma that helps to construct pointwise averaging sequences in \( \mathbb{R}^m \).

**Lemma 3.** Let \( \{\varphi_n\} \) be a sequence of densities on \( \mathbb{R}^m \) with compact supports \( S_n \) in \( T \); if there are positive numbers \( a_n \) such that the sets \( T_n^m := \{(x,y): x \in S_n, 0 \leq y \leq a_n \varphi_n(x)\} \) form a pointwise averaging sequence in \( \mathbb{R}^{m+1} \), then \( \{\varphi_n\} \) is pointwise averaging.

**Proof.** Apply Definition 1 to the random field \( Y(t,s) \equiv X(t), t \in \mathbb{R}^m, s \in \mathbb{R}, \) (note that \( \lambda(T_n^m) = a_n \), hence \( \frac{1}{\lambda(T_n^m)} \int_{T_n^m} Y(t,s) \lambda(dt,ds) = \int_{S_n} X(t) \varphi_n(t) dt; \) here \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^{m+1} \)).

The following two examples are implied by Lemma 3 and Examples 1 and 2 (see also Corollaries 4.2 and 4.3 in Ch. 6 in [50]).

**Example 3.** Let \( \{\varphi_n\} \) be a sequence of densities on \( \mathbb{R}^m \) which are concave on their supports \( S_n \) (the sets \( S_n \) are compact and convex). If the sequence \( \{S_n\} \) is increasing and the sets \( S_n \) contain balls of radii \( r_n \to \infty \), then the sequence \( \{\varphi_n\} \) is pointwise averaging; see §5.3 in [51].

**Example 4.** Let \( \{c_n\} \) be a sequence of positive numbers tending to \( \infty \). If \( \varphi \) is a bounded density on \( \mathbb{R}^m \) with a compact support \( S \) containing 0, then the sequence of rescaled densities \( \varphi_n(x) = c_n^{-m} \varphi(c_n^{-1}x) \) is pointwise averaging.

**Remark 4.** When \( \varphi(t) > 0 \) for all \( t \in \mathbb{R}^m \), some more conditions on \( \varphi \) are needed, but the class of "good" rescaled densities is still rather wide (see Proposition 5.3 in [51]); for example, if \( \varphi \) is the density of a nondegenerate symmetric normal distribution, then the sequence of rescaled densities is pointwise averaging.

For more examples, related to averages by convolutions of probability measures, see the next section.
It is easy to see that Theorems 1 - 5 hold for all pointwise averaging sequences \( \{q_n^l\} \) (of course, in the proofs \( \int_{T_n^l} f(X(t)) q_n^l (dt) \) has to be considered instead of \( \frac{1}{\lambda(T_n^l)} \int_{T_n^l} f(X(t)) \lambda(dt) \)).

4 Randomized CLT on general groups

In the above study the simplest groups \( \mathbb{R}^m \) and \( \mathbb{Z}^m \) were considered. But the results are valid for all groups \( T \) on which the PET holds with some sequence of sets \( \{T_n^l\} \) or "weights" \( \{q_n^l\} \). Then the proof of the generalized versions of Theorems 2 and 4 may be repeated almost word for word. There is a rather rich literature related to PETs on groups; see [39], [47], [50], [51] and the bibliographical survey therein; the PET with exponential rates of convergence on semisimple Lie groups is presented in [43]. In locally compact topological groups it is natural to consider pointwise averaging sequences of sets or densities with respect to the Haar measure. If the group is not locally compact, there is no Haar measure on it, and one has to consider pointwise averaging with sequences of probability Borel measures \( \{q_n\} \) as "weights" (see §3). If \( T \) is a second countable topological group and the smallest closed group containing the support of a probability Borel measure \( p \) is \( T \), then the sequence \( \{q_n := \frac{1}{n} \sum_{k=1}^n p^{*k}\} \) is pointwise averaging on \( T \) (see Corollary 5.3 in Ch. 3, Proposition 1.1 in Ch. 5 and Theorem 6.1 in Chapter 6 in [50]). In all these cases the analogs of our CLTs for homogeneous random fields with \( E[(X(0)^2) < \infty \) are valid.

If, in addition, \( p \) is symmetric (i.e. \( p(A^{-1}) = p(A) \) for each Borel set \( A \) in \( T \)) we may put \( q_n := p^{*n} \) (see Note 6.4 in Chapter 6 in [50] and the references therein); the PET with these weights is valid for a homogeneous field \( X \), if \( EX[|X(0)|^{1+\delta}] < \infty \) for some \( \delta > 0 \); note that condition

\[
EX[|X(0)|^{2+\delta}] < \infty \text{ for some } \delta > 0
\]

guarantees that the PET holds also for the random field \( (X(t))^2 \); this, in its turn, implies the analog of Lemma 2 and, hence, the analogs of the rest results. Therefore, under this condition, the CLTs hold with the "weights" \( p^{*n} \), too.

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