The dynamics of universe for exponential decaying dark energy

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(Dated: March 26, 2015)

In this study we consider an exponential decaying form for dark energy as EoS parameter in order to discuss the dynamics of the universe. Firstly, assuming that universe is filled with an ideal fluid which consists of exponential decaying dark energy we obtain time dependent behavior of several physical quantities such as energy density, pressure and others for dark energy, dark energy-matter coupling and non-coupling cases. Secondly, using scalar field instead of an ideal fluid we obtain these physical quantities in terms of scalar potential and kinetic term for the same cases in scalar-tensor formalism. Finally we show that ideal fluid and scalar-tensor description of dark energy give mathematically equivalent results for this EoS parameter.

PACS numbers: 95.36.+x; 98.80.-k

I. INTRODUCTION

Recent cosmological observations show that the current expansion of the universe is accelerating [1–7]. The discovery of cosmic acceleration of the Universe is one of the most significant discoveries over last decade and it challenges to physicists. So far, many models have been proposed to explain accelerated expansion of the universe. The current procedure to explain the cosmic acceleration consists in introducing a dark energy with negative pressure and negative entropy [8–11]. According to dark energy scenario, the Universe is filled by dark energy of unknown form. The simplest model being the cosmological constant Λ with constant dark energy denoted by DE. The EoS parameter w = p/ρ of DE is −1 in Ref. [15].

In this study we consider, as an example, exponential decaying EoS parameter

\[ w(t) = -1 + w_1 e^{-t/\tau} \]  

(1)

where \( w_1 \) and \( \tau \) are positive and real parameters of time-dependent w(t) function which responsible for accelerating expansion or decaying of the universe. According to presented model, the additional term \( w_1 e^{-t/\tau} \) in Eq. (1) corresponds to perturbation of dark energy. The \( \tau \) also indicates macroscopic relaxation parameter of dark energy. Using this EoS parameter we will discuss several physical quantities such as energy density, pressure and others for dark energy, dark energy-matter coupling and non-coupling cases. We also will present the time dependent behavior of these quantities for dark energy for EoS parameter. Additionally using scalar-tensor description of dark energy we will obtain these physical quantities in terms of scalar potential and kinetic term for the same cases. Finally we will show that ideal fluid and scalar-tensor description of dark energy give mathematically equivalent results for this EoS parameter.

The organization of this paper is structured as follows: In Sec. II, theoretical framework of the FRW equations is briefly presented. In Sec. III, the some physical parameters and dynamics of universe for exponential decaying...
dark energy form are computed and discussed. In Sec. IV, the analytical results in the scalar-tensor description are presented. Finally conclusion is given in Sec. V.

II. FRW MODEL

In standard cosmology, spatially flat and homogeneous universe’s FRW metric is presented in the form

\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 \]  

(2)

FRW equations in this metric are given by

\[ H^2 = \frac{\kappa^2}{3}\rho, \quad \dot{H} = -\frac{\kappa^2}{2}(\rho + p) \]  

(3)

where \( \kappa^2 = 8\pi G \) is the gravitational constant, \( H = \frac{\dot{a}}{a} \) is Hubble rate and \( \rho \) is the energy density.

These FRW equations satisfy the energy conservation law

\[ \dot{\rho} + 3H(\rho + p) = 0 \]  

(4)

where \( \dot{} \) indicates the time derivative. \( p \) is the pressure of the FRW universe. The relation between \( \rho \) and \( p \) is given as

\[ p = w\rho \]  

(5)

where \( w \) is EoS parameter of FRW equation which is constant. It takes \(-1 \) for FRW framework. However, EoS parameter \( w \) may be chosen depending on time \( t \). Especially, in order to explain time evolution of the universe for different time epoch, it is supposed that \( w \) changes with time \( t \) \[15, 37\]. In this case time dependent EoS is given by

\[ p = w(t)\rho \]  

(6)

which also satisfies energy conservation

\[ \dot{\rho} + \kappa\sqrt{3}(1 + w(t))\rho \dot{t}^2 = 0 . \]  

(7)

Integrating of Eq. (7) we can obtain energy density in terms of time dependent EoS as

\[ \rho = \frac{4}{3\kappa^2 \left( \int dt(1 + w(t)) \right)^2} . \]  

(8)

If Eq. (8) is put into in Eq. (3)’s first equation, time dependent Hubble parameter can be obtained as

\[ H = h(t) = \frac{2}{3 \left( \int dt(1 + w(t)) \right)} . \]  

(9)

However, when \( w \) is chosen as a constant, the standard expression of Hubble is recovered

\[ H = \frac{2}{3(1 + w)(t - t_s)} . \]  

(10)

The dynamics of universe depends on time and EoS parameter \( w \). For instance, the cosmological time \( t \) should be restricted to be \( t_s \) and if \( \int dt(1 + w(t)) = 0 \) which occurs at \( t = t_s \), \( H \) diverges which behavior of corresponds to the Big Rip singularity. On the other hand, in the case of \( w < -1 \), Hubble parameter takes \( H > 0 \) which corresponds to phantom era and \( w > -1 \) corresponds to quintessence where the Hubble takes \( H < 0 \).

III. IDEAL FLUID DESCRIPTION

A. Dark Energy Case

In this section let us consider the universe filled with dark energy whose EoS is \( w(t) = -1 + w_1 e^{-t/\tau} \) defined in Eq. (1). We will obtain some physical quantities using by this EoS and give numerical solutions. The time dependence behavior of EoS parameter \( w(t) \) is plotted in Fig. 1 for \( w_1 = \tau^{-1} = 1 \). As it can be seen from figure that the EoS is \( w(t) = 0 \) at \( t = 0 \) which is max value of the EoS and it exponentially decreases with time and reach to \(-1 \) at relatively large time. In Fig. (1) EoS \( w \) changes

![FIG. 1. This figure reflects the time dependence of EoS presented in this work for \( w_1 = \tau^{-1} = 1 \).](image)

with time between \(-1 < w < 0 \), it has been shown in previous studies \[13, 39–42\] that this range fits current cosmological observations best. However, we note that time evolution of the dark energy is still debate. Present EoS models may not enough to carry out the dynamics of universe, therefore, alternative EoS models can be used. Our model is consistent with previous studies \[13, 39–42\].

Time dependent dark energy density for EoS parameter in Eqs. (1) can be obtained as

\[ \rho = \frac{4\tau^{-2}}{3\kappa^2 (w_0 - w_1 e^{-t/\tau})^2} \]  

(11)

where \( w_0 \) is an integral constant, \( w_1 \) and \( \tau \) are other constants mentioned above. Dark energy density (11) satisfies conservation laws (4) and (7). Eq. (11) has an singularity at \( t = 0 \) for all \( w_0 = w_1 \). The time dependence behavior of dark energy density \( \rho \) is given in Fig. (2).
As it can be seen from Fig. (2) that energy density has maximum value at $t = 0$. However, dark energy density also exponentially decreases with time and it reaches to constant value at relatively large time. The behavior of the energy density is natural consequences of EoS parameter. Dramatically decreasing of energy density can be interpreted that universe is rapidly expanding, as unexpectedly, or dark energy may transform from one of kind to another form due to interactions energy-energy or energy-matter so on. On the other hand, we note that $\dot{\rho}$ has a physical meaning. For instance, it is assumed that the universe in phantom phase era if $\rho > 0$. In this regime, energy density of universe exponentially grows. On the contrary, universe is in non-phantom phase if $\rho < 0$. Then, in this case, energy density exponentially decreases. In our model, $\dot{\rho}$ is less than zero. This provides that EoS model consistent with non-phantom phase of universe.

Hubble parameter and its time derivative for EoS in Eq. (1) are respectively given as

$$H = \frac{2\tau^{-1}}{3(w_0 - w_1 e^{-t/\tau})}$$  \hspace{1cm} (12)$$

and

$$\dot{H} = -\frac{2\tau^{-2}w_1 e^{-t/\tau}}{3(w_0 - w_1 e^{-t/\tau})^2}.$$  \hspace{1cm} (13)$$

Hubble parameter and its time derivative have singularities for all $w_0 = w_1$ at $t = 0$. These behaviors correspond to Big Rip singularity. Time dependence of Hubble parameter and its time derivative are respectively plotted in Fig. (3) and (4) for constant parameters $w_0 = w_1 = \tau^{-1} = 1$. As it can be seen from Fig. (3) while the value of Hubble parameter is maximum at $t = 0$. It exponentially decreases with time and reaches to constant value at relatively large time. On the other hand, the time derivative of Hubble parameter shows different behavior from Hubble parameter. As it can be seen from $\dot{H}$ takes negative values and its value increases with time up to reach to approximately zero. In FRW model, it is assumed that Hubble parameter is a constant. However it changes with time for time dependent EoS parameter as in present study.

The acceleration parameter is given by

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{2\tau^{-2}}{3(w_0 - w_1 e^{-t/\tau})^2} \left(\frac{2}{3} - w_1 e^{-t/\tau}\right)$$ (14)$$

which defines acceleration of the universe in cosmological time. This parameter also has singularity for all $w_0 = w_1$ at $t = 0$. The acceleration parameter is plotted versus time in Fig. (5) for constant parameters $w_0 = w_1 = \tau^{-1} = 1$. As it can be seen from Fig. (5), acceleration parameter for constant values of parameters rapidly maximum value in sort time range and later it smoothly decreases to a constant value at large time. It may be considered that the sort time regime where rapidly increasing of acceleration parameter corresponds to inflation era of the universe. The large time behavior of the acceleration parameter in Fig. (5) also show that universe reaches to constant accelerating parameter.
On the other hand, the pressure can be obtained from

\[ p = -\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2\right) \]  

(15)

which is known as second FRW equation. With help of Eqs. (12) and (13), pressure of the dark energy for exponential decaying EoS can be obtained as

\[ p = -\frac{4\tau^{-2}(1 - w_1e^{-t/\tau})}{3\kappa^2(w_0 - w_1e^{-t/\tau})^2}. \]  

(16)

The Eq. (16) also indicates that the pressure has singularity for all \( w_0 = w_1 \) at \( t = 0 \). The pressure in

\[ p = -\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2\right) \]  

(17)

where \( \rho_0 \) is a constant and an \( w_m \) EoS parameter for matter. Hence, in the presence matter dark energy interaction, the dark energy density \( \rho \) is described as

\[ \rho = \frac{4\tau^{-2}(w_0 - w_1e^{-t/\tau})^2 - \rho_0a^{-3(1+w_m)}}{3\kappa^2}. \]  

(18)

Additionally, the total pressure for matter and dark energy combination is given by \( p_{\text{tot}} = p + p_m \) where \( p \) indicates dark energy pressure for EoS in Eq. (1) and \( p_m \) corresponds to matter pressure which is defined as

\[ p_m = w_m\rho_m \]  

(19)

with EoS parameter \( w_m \) is constant. From the second FRW equation, dark energy pressure in the presence of dark energy and matter coupling can be written as

\[ p = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) - w_m\rho_0a^{-3(1+w_m)}. \]  

(20)

Using (15) and (16), we can write pressure as

\[ p = -\frac{4\tau^{-2}(1 - e^{-t/\tau})}{3\kappa^2(w_0 - w_1e^{-t/\tau})^2} - w_m\rho_0a^{-3(1+w_m)} \]  

(21)

which also satisfies conservation relation (4). Now we can define a new EoS parameter for the coupling dark energy and matter as

\[ w(t) = -1 - \frac{2}{\kappa^2} \frac{\dot{H} + (1 + w_m)\rho_0a^{-3(1+w_m)}}{\dot{H}^2 - \rho_0a^{-3(1+w_m)}}. \]  

(22)

Here we note that all physical quantities can be redefined for a suitable scale factor \( a \).

\section*{C. Non-Coupling Case of Dark Energy and Matter}

On the other hand in the case of non-coupling between dark energy and matter, the pressure and density cannot be defined separately. However, EoS parameter can be described by

\[ p = \omega \rho + \frac{2}{\kappa^2} Hg(t) \]  

(23)

where \( w = -1 \) for FRW and \( g(t) \) is an arbitrary function of time \( t \). Hence EoS for dark energy fluid for no-coupling case is written as in Ref. [38]
\[ p = -\rho + \frac{4(1 + w(t))}{3 \kappa^2 \int dt(1 + w(t))} - (1 + w_m) \rho_m \exp \left\{ -3(1 + w_m) \frac{2}{3 \int dt(1 + w(t))} \right\}. \quad (24) \]

Finally using EoS in Eq. (1), Eq. (24) can be arranged as

\[ p = -\rho + \frac{4}{3 \kappa^2} \frac{\tau^{-1} w_1 e^{-t/\tau}}{(w_0 - w_1 e^{-t/\tau})} - (1 + w_m) \rho_m \exp \left\{ -\frac{2(1 + w_m) \tau^{-1}}{(w_0 - w_1 e^{-t/\tau})} \right\}. \quad (25) \]

which also satisfies energy conservation (4).

**IV. SCALAR-TENSOR DESCRIPTION**

**A. Scalar Field Case**

In this section, we will obtain the same physical quantities discussed above by using scalar-tensor description following method in Refs. [43–48]. Let us start with following action:

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2 \kappa^2} R - \frac{1}{2} \Omega(\phi) \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right\}. \quad (26) \]

where \( \Omega(\phi) \) is the kinetic term and \( V(\phi) \) is the scalar potential in terms of scalar field \( \phi \) which only depends on the time \( t \).

In the scalar-tensor formalism, energy density \( \rho \) and pressure \( p \) caused from in the presence of scalar field \( \phi \) are respectively given by

\[ \rho = \frac{1}{2} \Omega(\phi) \dot{\phi}^2 + V(\phi) \quad (27) \]

and

\[ p = \frac{1}{2} \Omega(\phi) \dot{\phi}^2 - V(\phi). \quad (28) \]

Combining (3) with (27) and (28) we can obtain kinetic function and scalar potential as

\[ \Omega(\phi) = -\frac{2}{\kappa^2} \dot{H} \quad (29) \]

and

\[ V(\phi) = \frac{1}{\kappa^2} (3H^2 + \dot{H}) \quad (30) \]

where \( H \) is the Hubble parameter. In case of \( \phi = t \) and \( H = f(t) \), equations (29) and (30) can be presented as

\[ \Omega(\phi) = -\frac{2}{\kappa^2} f'(\phi) \quad (31) \]

\[ V(\phi) = \frac{1}{\kappa^2} \left( 3f(\phi)^2 + f'(\phi) \right) \quad (32) \]

where \( f(\phi) \) is described as

\[ f(\phi) = \frac{2\tau^{-1}}{3(w_0 - w_1 e^{-\phi/\tau})}. \quad (33) \]

These conditions in (31) and (32) satisfy scalar-field equation

\[ 0 = \Omega(\phi) \ddot{\phi} + \frac{1}{2} \Omega'(\phi) \dot{\phi}^2 + 3H \Omega(\phi) \dot{\phi} + V'(\phi) \quad (34) \]

where \( \dot{\phi} \) indicates time derivation and \( \prime \) denotes derivation according to scalar field. We note that the scalar field \( \phi \) can be redefined as \( \phi \rightarrow F(\phi) \) where \( F(\phi) \) is the arbitrary function. Moreover the scalar field \( \phi \) can be used as time coordinate such as \( \phi = t \). Having in mind these details and using Eqs. (29) and (30), we obtain energy density and pressure in the form

\[ \rho = \frac{3}{\kappa^2} f(\phi)^2 \quad (35) \]

and

\[ p = -\frac{3}{\kappa^2} f(\phi)^2 - \frac{2}{\kappa^2} f'(\phi). \quad (36) \]

For the action, (26) when \( \phi = f^{-1}(\kappa \sqrt{\frac{\rho}{3}}) \), we can re-obtain inhomogeneous EoS for dark energy in the form

\[ p = -\rho - \frac{2}{\kappa^2} f'(f^{-1}(\kappa \sqrt{\frac{\rho}{3}})). \quad (37) \]

On the other hand, in the scalar-tensor description, using definitions (12) and (13), the kinetic term \( \Omega(\phi) \) and the scalar potential \( V(\phi) \) for EoS (1) can be written as

\[ \Omega(\phi) = \frac{4\tau^{-2} w_1 e^{-\phi/\tau}}{3 \kappa^2 (w_0 - w_1 e^{-\phi/\tau})^2} \quad (38) \]

and

\[ V(\phi) = \frac{\tau^{-2}(4 - 2w_1 e^{-\phi/\tau})}{3 \kappa^2 (w_0 - w_1 e^{-\phi/\tau})^2} \quad (39) \]

respectively. If we put Eqs. (38) and (39) into Eqs. (27) and (28), we get energy density and pressure relations

\[ \rho = \frac{4\tau^{-2}}{3 \kappa^2 (w_0 - w_1 e^{-\phi/\tau})^2} \quad (11) \]
and
\[ p = -\frac{4\tau^{-2}(1 - w_1 e^{-t/\tau})}{3\kappa^2 (w_0 - w_1 e^{-t/\tau})^2} \]  
which clearly confirms that EoS and scalar-tensor analysis are equivalent.

### B. Coupling Case of Scalar Field and Matter

Let us consider a universe filled with scalar field \( \phi \) and matter. EoS of matter is \( p_m = w_m \rho_m \) (\( w_m \) is constant) and scalar field \( \phi \) depends on the time \( t \) for coupling case. In this case, action is given by
\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} \Omega(\phi) \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) + L_m \right) \]
where \( L_m \) is the matter Lagrangian density.

For metric (2), corresponding FRW equations are given by
\[ H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_\phi) \]
and
\[ \dot{H} = -\frac{\kappa^2}{2}(\rho_m + p_m + \rho_\phi + p_\phi) \]
where \( \rho_\phi \) and \( p_\phi \) are energy density and pressure for scalar field \( \phi \), on the other hand, \( \rho_m \) and \( p_m \) are energy density and pressure for matter.

Combining (41) and (42) with (29) and (30) we can obtain kinetic function and scalar potential for coupling case between scalar field and matter, respectively
\[ \Omega(\phi) = -\frac{2}{\kappa^2} \dot{H} - (\rho_m + p_m) \]
\[ V(\phi) = \frac{1}{\kappa^2} (3H^2 + \dot{H}) - \frac{\rho_m - p_m}{2} . \]

By using Eqs. (12) and (13), kinetic function and scalar potential for coupling case to the exponential dark energy are respectively rewritten as
\[ \Omega(\phi) = \frac{4\tau^{-2}w_1 e^{-\phi/\tau}}{3\kappa^2(w_0 - w_1 e^{-\phi/\tau})^2} - (\rho_m + p_m) \]
\[ V(\phi) = \frac{\tau^{-2}(4 - 2w_1 e^{-\phi/\tau})}{3\kappa^2(w_0 - w_1 e^{-\phi/\tau})^2} - \frac{(\rho_m - p_m)}{2} . \]
Finally, if we put Eqs. (45) and (46) into Eqs. (27) and (28), we get energy density and pressure relations
\[ \rho = \frac{4\tau^{-2}}{3\kappa^2(w_0 - w_1 e^{-t/\tau})^2} - \rho_0 a^{-3(1+w_m)} \]
and
\[ p = -\frac{4\tau^{-2}(1 - e^{-t/\tau})}{3\kappa^2(w_0 - w_1 e^{-t/\tau})^2} - w_m \rho_0 a^{-3(1+w_m)} . \]
These results clearly confirm that EoS and scalar-tensor analysis are mathematically equivalent.

### C. Non-Coupling Case of Scalar Field and Matter

Now, let us consider non-coupling between dark energy and matter as a final case. In the case of no-coupling, matter and scalar field satisfy energy conservation separately. Hence kinetic function and scalar potential can be obtained as
\[ \Omega(\phi) = -\frac{2}{\kappa^2} f'(\phi) - (w_m + 1)F_0 e^{-3(1+w_m)}F(\phi) \]
\[ V(\phi) = \frac{1}{\kappa^2} [3f(\phi)^2 + f'(\phi)] + \frac{w_m - 1}{2} F_0 e^{-3(1+w_m)} \]
Under assumptions \( \phi = t \) and \( H = f(t) \), by using Eqs. (12) and (13), kinetic function and scalar potential for non-coupling case are respectively rewritten as
\[ \Omega(\phi) = \frac{4\tau^{-2}w_1 e^{-\phi/\tau}}{3\kappa^2(w_0 - w_1 e^{-\phi/\tau})^2} - (w_m + 1)F_0 e^{-3(1+w_m)} \]
\[ V(\phi) = \frac{\tau^{-2}(4 - 2w_1 e^{-\phi/\tau})}{3\kappa^2(w_0 - w_1 e^{-\phi/\tau})^2} + \frac{w_m - 1}{2} F_0 e^{-3(1+w_m)} \]
where \( F(\phi) = \int d\phi f(\phi) \) and \( F_0 \) is an integration constant. Using the relation between scale factor and arbitrary scalar function \( F(t) \)
\[ a(t) = a_0 e^{F(t)}, \quad a_0 = \left( \frac{\rho_{m0}}{F_0} \right)^{1/(3(1+w_m))} \]
inhomogeneous EoS for non-coupling case can be obtained as
\[ p = -\rho + \frac{4}{3\kappa^2} \tau^{-1}w_1 e^{-t/\tau} - (1 + w_m) \rho_{m0} \exp \left\{ -\frac{2}{1 + w_m} \frac{1}{\tau^{-1} \left( w_0 - w_1 e^{-t/\tau} \right)} \right\} \]
for exponential dark energy.
V. CONCLUSION

In summary, considering an exponential decaying form for dark energy as an example we compute several physical quantities such as energy density, pressure and others for dark energy, dark energy-matter coupling and dark energy-matter uncoupling cases in ideal fluid and scalar-tensor description of dark energy. Finally we show that these approximations are mathematically equivalent. We state that the exponential decaying EoS parameter may be used to dynamics of universe as well other EoS parameters.

ACKNOWLEDGMENTS

Authors would like to thank Istanbul University for financial support (Grant No. 48081). This work has been completed at Istanbul University, Graduate School of Natural and Applied Sciences and is the subject of the forthcoming M.Sc. Thesis of Nilay Bostan.