Mean-field description of multicomponent exciton-polariton superfluids

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Abstract. This is a review of spin-dependent (polarization) properties of multicomponent exciton-polariton condensates in conditions when quasi-equilibrium mean-field Gross-Pitaevskii description can be applied. Mainly two-component (spin states $\pm 1$) polariton condensates are addressed, but some properties of four-component exciton condensates, having both the bright (spin $\pm 1$) and the dark (spin $\pm 2$) components, are discussed. Change of polarization state of the condensate and phase transitions in applied Zeeman field are described. The properties of fractional vortices are given, in particular, I present recent results on the warping of the field around half-vortices in the presence of longitudinal-transverse splitting of bare polariton bands, and discuss the geometrical features of warped half-vortices (in the framework of the lemon, monstar, and star classification).

1.1 The Gross-Pitaevskii equation

Currently, most theoretical descriptions of exciton-polariton condensates observed in incoherently excited semiconductor microcavities are based on the Gross-Pitaevskii equation (GPE). When the polarization of the condensate is of interest, this equation can be generically written as

$$i\hbar \frac{\partial}{\partial t} \psi(r, t) = \frac{\delta H}{\delta \psi^*(r, t)}, \quad H = \int \mathcal{H}(\psi^*, \psi) d^2 r,$$

(1.1)

where the order parameter $\psi(r, t)$ of the condensate is a complex 2D vector function of the 2D position in the microcavity plane $r$ and time $t$. Alternatively, one can expand $\psi$ on the circular polarization basis

$$\psi = \frac{\hat{x} + i\hat{y}}{\sqrt{2}} \psi_{+1} + \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \psi_{-1},$$

(1.2)

and obtain the coupled GPEs for two circular components $\psi_{\pm 1}$ (see Eqs. (1.32a,b) in Section 1.4).
GPE is used in two main flavors, strongly non-equilibrium GPE and quasi-equilibrium GPE, that treat the energy relaxation in two extreme ways. It is completely neglected in the former, and it is considered to be essential in the later. Mathematically, these approaches differ in Hamiltonian density $\mathcal{H}(\psi, \psi^*)$: it is complex in the former and it is real in the later. Each approach has its benefits and drawbacks.

The imaginary part of the Hamiltonian for non-equilibrium GPE is given by the difference of income and escape rates of exciton-polaritons into and out of the condensate. While the escape rate is given by the reciprocal radiative life-time of exciton-polaritons and is independent of the particle density, the income rate is a non-linear function of it. The nonlinearity is essential to stabilize the solution and it appears due to the depletion of an incoherently pumped reservoir. This approach resulted to be quite successful in modeling the experimental data on condensate density profiles for spatially nonuniform exciton-polariton condensates. On the other hand, it cannot describe the spontaneous formation of linear polarization of the condensate—the fact that is quite unfortunate since the observation of spontaneous linear polarization is one of the direct experimental evidences of Bose-Einstein condensation of exciton-polaritons. A workaround is to add the Landau-Khalatnikov relaxation into (1.1), which is equivalent to adding an imaginary part to time $t$. But this relaxation is rather artificial because it changes the number of particles in the condensate.

In what follows, we consider the opposite limit assuming a fast relaxation of exciton-polaritons, so that they reach quasi-equilibrium, even with the temperature that can be different from the lattice one. The balance of income and outcome rates produces some steady-state concentration of exciton-polaritons, that can be defined, as usual, by introducing the chemical potential $\mu$. The coherent fraction of condensed particles can be described by the traditional GPE with real $\mathcal{H}(\psi, \psi^*)$,

$$\mathcal{H} = \mathcal{T} - \mu n + \mathcal{H}_{\text{int}} + \mathcal{H}'.$$  \hspace{1cm} (1.3)

Here $\mathcal{T}$ is the density of the kinetic energy, $\mathcal{H}_{\text{int}}$ describes interaction between the particles, $\mathcal{H}'$ stands for some possible perturbations, and $n = \psi^* \cdot \psi$ is the exciton-polariton density.

The kinetic energy of exciton-polaritons in planar microcavities depends on the orientation of vector $\psi$ with respect to the direction of motion. Near the bottom of lower polariton branch one has

$$\mathcal{T} = \frac{\hbar^2}{2m_l} |\nabla \cdot \psi|^2 + \frac{\hbar^2}{2m_t} |\nabla \times \psi|^2$$  \hspace{1cm} (1.4a)

$$= \frac{\hbar^2}{m_l} \left| \frac{\partial \psi_{+1}}{\partial z^*} + \frac{\partial \psi_{-1}}{\partial z} \right|^2 + \frac{\hbar^2}{m_t} \left| \frac{\partial \psi_{+1}}{\partial z^*} - \frac{\partial \psi_{-1}}{\partial z} \right|^2,$$  \hspace{1cm} (1.4b)

where $m_l$ and $m_t$ are the longitudinal and transverse effective masses of polaritons, and the complex derivatives
are used. The vector $\psi$ is proportional to the in-plane electric field vector of exciton-polariton mode. According to (1.4), the frequency of transverse electric (TE) mode with in-plane wave vector $k \perp \psi$ is $\hbar k^2/2m_t$, while for the transverse magnetic (TM) mode with $k \parallel \psi$ the frequency is $\hbar k^2/2m_l$. (The same bare frequencies of both modes at $k = 0$ are removed from (1.4).)

The polariton-polariton interaction is also anisotropic: it depends on mutual orientation of $\psi$ and $\psi^*$. One can construct two quartic invariants from these two vectors and $\mathcal{H}_{\text{int}}$ is given by

$$\mathcal{H}_{\text{int}} = \frac{1}{2} (U_0 - U_1) (\psi^* \cdot \psi)^2 + \frac{1}{2} U_1 |\psi^* \times \psi|^2$$  \hspace{1cm} (1.6a)

$$= \frac{1}{2} U_0 (|\psi_{+1}|^4 + |\psi_{-1}|^4) + (U_0 - 2U_1) |\psi_{+1}|^2 |\psi_{-1}|^2.$$  \hspace{1cm} (1.6b)

It is seen that $U_0$ is the amplitude of interaction of polaritons with the same circular polarization (with the same spin), and $U_0 - 2U_1$ is the amplitude of interaction of polaritons with opposite circular polarizations (opposite spins). These quantities are denoted by $\alpha_1$ and $\alpha_2$ in some papers. The constant $U_0$ is positive and can be estimated as $\sim \mathcal{E}_b a_B^2$, where $\mathcal{E}_b$ is the exciton binding energy and $a_B$ is the exciton Bohr radius. The interaction of exciton-polaritons with opposite spins depends substantially on the electron-electron and hole-hole exchange processes and is defined by the electron and hole confinement within quantum wells and by the number of quantum wells in the microcavity. As a result, the value of $U_1$ is sensitive to the microcavity geometry.

To end this section it is important to mention the limitations of any GPE in application to the condensates of exciton-polaritons in microcavities, or to condensates of any other bosonic excitations that have a finite radiative lifetime. Due to interference of light emitted from different parts of condensate there appears dissipative long-range coupling in the system. Most importantly, the escape rate becomes dependent on the symmetry of the condensate wavefunction and this favors the formation of particular long-living many-particle states [10]. These effects cannot be properly treated in the framework of Gross-Pitaevskii equation (1.1).

1.2 Polarization and effects of Zeeman field

The interaction energy (1.6) of the polariton condensate is polarization dependent. While the first term in (1.6a) does not depend on polarization and is simply proportional to the square of the polariton concentration $n = (\psi^* \cdot \psi)$, the second term in (1.6b) is sensitive to the degree of the circular polarization of the condensate. For $U_1 > 0$ the interaction energy is minimized when the second term in (1.6b) is annulled, which is achieved for polarization satisfying $\psi^* \times \psi = 0$, i.e., for the linear polarization. On the other hand, in the case
$U_1 < 0$ the minimum is reached for the circular polarization of the condensate, when $\psi^* \times \psi = \pm in$.

So, there is qualitative change in the ground state of the condensate when $U_1$ changes sign [1].

(i) $U_1 > 0$. The ground state is characterized by two angles, the total phase angle $\theta$ and the polarization angle $\eta$. These angles are defined from the Descartes components of the order parameter $\psi_x = \sqrt{n} e^{i\theta} \cos \eta$ and $\psi_y = \sqrt{n} e^{i\theta} \sin \eta$. The circular components are then $\psi_{\pm 1} = \sqrt{n/2} e^{i(\theta \mp \eta)}$. There are two broken continuous symmetries and, consequently, the excitation spectrum consists of two Bogoliubov branches. The sound velocities for these branches at $m_l = m_t = m^*$ are $v_0 = \sqrt{\mu/m^*}$ and $v_1 = \sqrt{nU_1/m^*}$, where $\mu = (U_0 - U_1)n$ is the chemical potential. The presence of TE-TM splitting leads to the anisotropy of sound velocities (see [1] for details).

(ii) $U_1 < 0$. In this case one of the circular components is zero and the other is $\sqrt{n} e^{i\theta}$. Since there is only one broken continuous symmetry, the excitation spectrum consists of only one Bogoliubov branch, and the other branch is gaped parabolic with the gap $2|U_1|n$. The chemical potential is $\mu = U_0 n$ in this domain, so that the sound velocity for the Bogoliubov excitations is $\sqrt{\mu/m^*}$.

The mean-field theory predicts an arbitrary polarization for $U_1 = 0$ since in this case the energy of the condensate is polarization independent. In reality, fluctuations destroy the order in this case at any finite temperature $T$. It can be already understood from the excitation spectrum, because, apart from the Bogoliubov branch, there is the gapless parabolic branch with dispersion $\hbar^2 k^2/2m^*$ and the condensate would evaporate completely due to excitation of these quasiparticles. One can also map this case to the O(4) nonlinear sigma model, where the order is proven to be absent for $T > 0$ [1].

Note the similarity between the two-component condensates of exciton-polaritons and three-component condensates of spin-1 cold atoms [12,13]. Due to the 3D rotational symmetry, there are also only two interaction constants in the latter case. These constants are defined by the cross-sections of scattering of two atoms with the total spin 0 and 2. Two different atomic condensates can also be found depending on the sign of the scattering length with the total spin 2: ferromagnetic and anti-ferromagnetic (or polar), which are analogs of circularly and linearly polarized exciton-polariton condensates, respectively.

It is the first case, $U_1 > 0$, that is realized in the exciton-polariton condensates observed so far. The linearly polarized condensate can be seen as

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1. The concentration of quasiparticles with the energy $\epsilon(k)$ is given by $\int \frac{d^2k}{(2\pi)^2} [\exp(\epsilon(k)/T) - 1]^{-1} d^2k$ and the integral diverges logarithmically for small $k$ when $\epsilon(k) \propto k^2$.

2. The case of the total spin 1 is irrelevant since the orbital wave function of colliding bosons is antisymmetric and it cannot be realized within the condensate.
composition of equal numbers mutually coherent spin-up and spin-down polaritons. Therefore, it is interesting to study the effect of applied magnetic field to this state \([15]\). Considering only weak fields, when the magnetic length is much greater than the exciton Bohr radius, one can study only the effects of Zeeman field, that is described by adding

\[ \mathcal{H}' = \Omega \left( |\psi_{-1}|^2 - |\psi_{+1}|^2 \right) \]  

(1.7)

into Hamiltonian \((1.3)\). Here the Zeeman field \(\Omega\) is given by the half of the Zeeman splitting energy for a single polariton.

To find the order parameter for the uniform condensate in the presence of Zeeman field it is convenient to introduce the concentrations of the components \(n_{\pm 1} = |\psi_{\pm 1}|^2\) satisfying \(n_{+1} + n_{-1} = n\). Assuming both \(n_{+1}\) and \(n_{-1}\) to be nonzero, one can take variations of the Hamiltonian

\[ \mathcal{H}_{\text{int}} + \mathcal{H}' - \mu n \]

\[ = \frac{1}{2} U_0 (n_{+1} + n_{-1})^2 - 2U_1n_{+1}n_{-1} - (\mu + \Omega)n_{+1} - (\mu - \Omega)n_{-1} \]  

(1.8)

over \(n_{\pm 1}\) to obtain

\[ -2U_1n_{\pm 1} = (\mu - U_0 n \mp \Omega) \]  

(1.9)

The sum and the difference of Eqs. (1.9) results in

\[ \mu = (U_0 - U_1)n, \quad n_{\pm 1} = \frac{1}{2} \left( n \pm \frac{\Omega}{U_1} \right), \quad \text{for } |\Omega| < \Omega_c \equiv nU_1, \]  

(1.10)

For higher Zeeman fields, \(|\Omega| > \Omega_c\), one of the components becomes empty, \(n_{-1}\) for \(\Omega > \Omega_c\) and \(n_{+1}\) for \(\Omega < -\Omega_c\), and in this case \(\mu = U_0 n - |\Omega|\).

Remarkably, for subcritical fields the chemical potential does not change at all, so that there is no change in the position of the emission line. The only effect of applied Zeeman field is the change of circular polarization degree \(\varrho_c = \frac{(n_{+1} - n_{-1})/(n_{+1} + n_{-1})}{\Omega/nU_1} = \Omega/nU_1\), that increases linearly with the field. The elliptical polarization of the condensate for subcritical fields is characterized by two angles, and in the the same way as for the linearly polarized condensate, there are two Goldstone modes; only the sound velocities change with the Zeeman field. This implies the full suppression of the Zeeman splitting by polariton-polariton interactions within the condensate \([15]\). Note also that for subcritical fields there are two phase transitions in the left and in the right circular component of the condensate, respectively \([16]\). The Zeeman splitting (the gap in the exciton spectrum) appears only for supercritical fields \(|\Omega| > \Omega_c\) where the condensate becomes circularly polarized. This effect, observed experimentally by Larionov et al. \([17]\), allows to measure the spin-dependent interaction constant \(U_1\).
1.3 Vortices in exciton-polariton condensates

Vortices play a key role in various physical phenomena both on macroscopic and microscopic level. While the vortex formation is very important for description of different effects in fluid mechanics, in particular, in aerodynamics and turbulent flow motion, the understanding of properties of quantized vortices is crucial for description of phase transitions in condensed matter. The well known examples are the phase transitions in type II superconductors in applied magnetic field, which are related to the formation and melting for vortex lattices [18], and the Berezinskii-Kosterlitz-Thouless (BKT) phase transition [19, 20, 21].

As it was discussed above, the exciton-polariton condensates possess two-component order parameter (1.2) and these condensates allow half-quantum vortices (half-vortices) [22]. Moreover, the half-vortices are basic topological excitations in this case (see [23] for a review on basic properties of half-quantum vortices). In spite of recent observation of both integer [24] and half-integer [25] vortices in exciton-polariton condensates, the presence of half-vortices was recently questioned [26] for the case of two-band dispersion with TE-TM splitting of polariton band given by Eq. (1.4). In this section I present the details on how the vortex solutions should be found in this case (a short summary of this theory has been given in [27]). In what follows only the case of zero Zeeman field will be considered.

For a 2D system of radius $R$ the energy of a vortex is finite but logarithmically large,

$$E_{\text{vor}} = E_c + E_s \ln(R/a), \quad (1.11)$$

where $a = \hbar/\sqrt{2m^*\mu}$ is the characteristic radius of vortex core (the effective mass $m^*$ is defined below in Eq. (1.17)). The fact that $E_{\text{vor}}$ diverges logarithmically at $R \to \infty$ is good: it prevents the single vortices to be excited at low temperatures and thus protects the long-range order of the condensate. Knowledge of prefactor $E_s$ allows to estimate [24] the BKT transition temperature $T_c$. The proliferation of single vortices appears when the free energy $E_{\text{vor}} - TS$ crosses zero. The vortex core area is $a^2$ and it can be appear in $R^2/a^2$ places, so that the entropy $S = \ln(R/a)^2$ and this gives $T_c = E_s/2$ if one neglects the energy of the core $E_c$ in (1.11). The energy $E_s \ln(R/a)$ is elaborated on large distances from the vortex core $r \gg a$, which we will refer to as the elastic region, and the study of vortices should begin with establishing the behavior of the order parameter in this region. When TE-TM splitting is present this behavior is, in general, nontrivial.

1.3.1 The order parameter on large distances

In the elastic region the order parameter changes within the order parameter manyfold, i.e., the polarization of the condensate is linear everywhere in this domain. The circular-polarization components $\psi_{\pm1}$ defined in (1.2) can be written in cylindrical coordinates $(r, \phi)$ as
\[
\psi_{\pm 1}(r \gg a, \phi) = \sqrt{n} e^{i[\theta(\phi) \mp \eta(\phi)]},
\]  

(1.12)

where \(n\) is the constant concentration of the condensate at large distances, and the phases are written in terms of total phase angle \(\theta\) and polarization angle \(\eta\). These angles do not depend on the radius \(r\) (such dependence would only increase the vortex energy), but they are functions of the azimuthal angle \(\phi\). Since the order parameters should be uniquely defined in the whole space, one has

\[
\eta(\phi + 2\pi) - \eta(\phi) = 2\pi k, \quad \theta(\phi + 2\pi) - \theta(\phi) = 2\pi m.
\]  

(1.13)

These conditions divide all possible solutions of GPE into topological sectors. Each sector is defined by two topological charges (or winding numbers), \(k\) and \(m\). The state from one sector cannot be continuously transformed into another sector, or, in other words, any state of the condensate evolves within its own topological sector. The sector \(k = m = 0\) is the ground state sector; the minimum energy here is reached for position-independent order parameter. By definition, the vortex is the state that minimizes the energy in a topological sector with at least one non-zero winding number. The energy of the \((k, m)\)-vortex \(1.11\) is counted from the ground state energy, i.e., it is the difference between the minimal energy in the \((k, m)\) sector and the minimal energy of \((0, 0)\) sector (the ground state energy). Since only the sum and the difference, \(\eta \pm \eta\), enter Eq. (1.12), the winding numbers can be either both integer or both half-integer, and the corresponding vortices are referred accordingly. Note also that the vortex corresponds to a minimum of Hamiltonian \(H\) for specific boundary conditions: \(\delta H/\delta \psi^* = 0\) for the vortex solution and, therefore, it is a static solution of GPE \((1.1)3\).

According to \((1.13)\) one can add any periodic functions of \(\phi\) to \(\eta(\phi)\) and \(\theta(\phi)\) without changing the topological sector. The proper functions \(\eta(\phi)\) and \(\theta(\phi)\) for the \((k, m)\)-vortex should be found from minimization of Hamiltonian in elastic region. The corresponding part of Hamiltonian is related solely to the kinetic energy term \(\int T d^2r\). After substitution of \((1.12)\) into \((1.11)\) and use of the asymptotic behavior of the complex derivative for \(r \to \infty\),

\[
\frac{\partial}{\partial z} \to -\frac{i}{2r} e^{-i\phi} \frac{\partial}{\partial \phi},
\]  

(1.14)

one obtains the product of integrals over \(r\) and \(\phi\) that results in the second term of Eq. \((1.11)\). The integral over \(r\) diverges logarithmically and should be cut by the core size \(a\) at small \(r\), and by system radius \(R\) at large \(r\). This gives the factor \(\ln(R/a)\). The prefactor is then given by

\(^3\)Note, however, that this does not imply that a single vortex gives an absolute minimum of the \(H\) in the corresponding topological sector. For example, the integer vortex \((1, 0)\) can be unstable with respect to decay into the pair of \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{-1}{2}, \frac{1}{2})\) half-vortices for \(m_l < m_t\) (see subsection \((1.3)\)2 below).
\[ E_s = \frac{\hbar^2 n}{2m^*} \int_0^{2\pi} \left\{ \left[ 1 + \gamma \cos(2u) \right] (1 + u')^2 + \left[ 1 - \gamma \cos(2u) \right] \theta'^2 \right\} d\phi \tag{1.15} \]

where the prime denotes the derivative over \( \phi \) and

\[ u(\phi) = \eta(\phi) - \phi. \tag{1.16} \]

The effective mass \( m^* \) and the TE-TM splitting parameter \( \gamma \) are defined in (1.15) by

\[ \frac{1}{m^*} = \frac{1}{2} \left( \frac{1}{m_t} + \frac{1}{m_l} \right), \quad \gamma = \frac{m_t - m_l}{m_t + m_l}. \tag{1.17} \]

Variations of the functional (1.15) over \( \theta \) and \( u \) lead to the equations

\[ [1 - \gamma \cos(2u)] \theta'' + 2\gamma \sin(2u)u'\theta' = 0, \tag{1.18a} \]

\[ [1 + \gamma \cos(2u)] u'' + \gamma \sin(2u) \left( 1 - u'^2 - \theta'^2 \right) = 0. \tag{1.18b} \]

In general, the polarization will be radial at least at one specific direction and it is convenient to count the azimuthal angle from this direction and set the total phase to be zero at this direction as well. Then, the solutions of Eqs. (1.18a,b) for \((k,m)\)-vortex should satisfy the boundary conditions

\[ u(0) = 0, \quad \theta(0) = 0, \tag{1.19a} \]

\[ u(2\pi) = 2(k - 1)\pi, \quad \theta(2\pi) = 2m\pi. \tag{1.19b} \]

The solutions in question are trivial for some particular vortices.

(i) **Hedgehog vortices.** These are \((1,m)\)-vortices having \( \theta = m\phi \) and \( u \equiv 0 \), so that the polarization angle \( \eta = \phi \). Polarization points into the radial direction everywhere and these vortices look like hedgehogs. These solutions are similar to magnetic monopoles \([28]\).

(ii) **Double-quantized polarization vortex** \((2,0)\). In this special case \( \theta \equiv 0 \), but \( u = \phi \), resulting in \( \eta = 2\phi \). Polarization rotates twice when one encircles the vortex core. These vortices and experimental possibilities of their excitation in exciton-polariton fields have been studied by Liew et al. \([29]\).

In other cases the solutions should be found numerically. Both Eqs. (1.18a) and (1.18b) can be integrated once to give

\[ [1 - \gamma \cos(2u)] \theta' = C_1, \tag{1.20a} \]

\[ [1 + \gamma \cos(2u)] u'^2 + [1 - \gamma \cos(2u)] \theta'^2 - \gamma \cos(2u) = C_2, \tag{1.20b} \]

and the solutions can be written as integrals of elementary functions. The constants \( C_{1,2} \) should then be found, e.g., by shooting, to satisfy the boundary conditions (1.19). The functions \( \theta(\phi) \) and \( \eta(\phi) \) are shown in Fig. 1.1 for
Fig. 1.1. The dependence of polarization angle $\eta$ (solid lines) and phase angle $\theta$ (dashed lines) on the azimuthal angle $\phi$ for two values of TE-TM splitting parameter: $\gamma = -0.4$ (thin lines) and $\gamma = -0.9$ (thick lines). The panels show the behavior of the angles for the $(\frac{1}{2}, \frac{1}{2})$ half-vortex (a), the $(\frac{1}{2}, \frac{1}{2})$ half-vortex (b), the $(−1, 0)$ polarization vortex (c), and the $(0, 1)$ phase vortex (d). In the last case the periodic function $\eta(\phi)$ has been upscaled for clarity.

elementary half-vortices and for two integer vortices $(-1, 0)$ and $(0, 1)$, that also exhibit nonlinear dependencies of polarization and phase angles.

Fig. 1.1 demonstrates the behavior of angles for negative values of the TE-TM splitting parameter $\gamma$. The functions $\theta(\phi)$ and $\eta(\phi)$ for positive $\gamma$ can be found by the shift. Indeed, the change $u \to u + (\pi/2)$ in Eqs. (1.18a,b) results in the change of the sign of $\gamma$. More precisely, to satisfy the boundary conditions (1.19) the transformations can be written as

$$\gamma \to -\gamma,$$

$$u(\phi) \to u \left( \phi + \frac{\pi}{2(k-1)} \right) + \frac{\pi}{2},$$

$$\theta(\phi) \to \left( \phi + \frac{\pi}{2(k-1)} \right) - \theta \left( \frac{\pi}{2(k-1)} \right),$$

and they can be applied to all vortices except the hedgehogs with $k = 1$ (and where they are not necessary, of course, since $u(\phi) \equiv 0$).
The nonlinear change of angles seen in Fig. 1.1 becomes especially evident when $\gamma$ approaches $\pm 1$. This limit correspond to a strong inequality between effective masses, e.g., $m_l \gg m_t$ for $\gamma \to -1$. Qualitatively the strong non-linearities can be understood if one introduces the effective masses for the phase $m_\theta$ and for the polarization $m_\eta$,

$$
\frac{1}{m_\theta} = \frac{\cos^2 u}{m_t} + \frac{\sin^2 u}{m_l}, \quad \frac{1}{m_\eta} = \frac{\sin^2 u}{m_t} + \frac{\cos^2 u}{m_l},
$$

and writes the energy (1.15) as

$$
E_s = \frac{\hbar^2 n^2}{2} \int_0^{2\pi} \left\{ \frac{\eta'^2}{m_\eta} + \frac{\theta'^2}{m_\theta} \right\} d\phi.
$$

The effective masses (1.22) depend on the orientation of polarization. Since $u(\phi)$ changes between the values specified by (1.19), there are sectors where $m_\theta \approx m_t$ and $m_\eta \approx m_t$, and there are sectors where $m_\theta \approx m_t$ and $m_\eta \approx m_l$. To minimize the energy (1.23) in the case $m_l \gg m_t$, the phase angle changes rapidly and the polarization angle stays approximately constant in the former, while there is the opposite behavior in the latter.

### 1.3.2 The energies and interactions of vortices

In the absence of TE-TM splitting the energies $E_s$ of vortices are

$$
E_{s0}^{(k,m)} = E_0(k^2 + m^2), \quad E_0 = \frac{\pi \hbar^2 n}{m^*}, \quad (\text{for } \gamma = 0).
$$

It is seen that in this case the energy of an elementary half-vortex is exactly half of the energy of an elementary integer vortex. Important consequences can be drawn from this relation concerning the interactions between half-vortices. The four elementary half-vortices can be divided in two kinds, right half-vortices with $k + m = \pm 1$, and left ones with $k - m = \pm 1$. One can see from (1.12) that the right half-vortices possess the vorticity of the left-circular component of the order parameter, the amplitude of this component goes to zero and the phase of this component becomes singular in the vortex core, and, as a result, the polarization becomes right-circular in the core center. For left half-vortices the picture is opposite. It follows from (1.24) that the left and the right half-vortices do not interact with each other. Consider, for example, the $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$ half-vortices. The elastic energy of this pair is $E_0 \ln(R/a)$ both when they are far away from each other and when they are in the same place forming the phase vortex $(0, 1)$. So, the only possible coupling between the left and the right half-vortices is of short range, related to the overlap of their cores and resulting change of the core energy term $E_c$ in (1.11). So, the only long-range coupling is present between the half-vortices of the same kind. It can be shown that identical half-vortices
repel each other logarithmically, while the half-vortices and anti-half-vortices, 
\((k, m)\) and \((-k, -m)\), attract each other logarithmically, as it is in the case of
vortices and antivortices in one-component condensates [30]. This simple
picture is changed in the presence of TE-TM splitting that leads to the long-
range interaction between half-vortices of different kind.

The logarithmic prefactors \(E_s^{(k,m)}\) for elementary half-vortices and ele-
mentary integer vortices are shown in Fig. 1.2. For all of them, expe
ct the hedgehog \((1,0)\)-vortex, these energies are even functions of
\(\gamma\), which can be
proven using the transformations (1.21a). These energies decre
ase with in-
creasing \(\gamma^2\). The case of hedgehog is special, as it has been discussed above.
The hedgehog polarization is radial everywhere and \(E_s^{(1,0)}\) is defined purely
by the longitudinal effective mass,

\[
E_s^{(1,0)} = \pi \hbar^2 n/m = E_0 (1 + \gamma). \tag{1.25}
\]

It can be shown that when two vortices with winding numbers \((k_1, m_1)\) and
\((k_2, m_2)\) are injected in the condensate and they are separated by distance \(r\),
such that \(a \ll r \ll R\), the energy of the condensate is increased in logarithmic
approximation (i.e., omitting the core energies) by

\[
E_s^{(k_1+k_2,m_1+m_2)} \ln(R/a)
+ \left[ E_s^{(k_1,m_1)} + E_s^{(k_2,m_2)} - E_s^{(k_1+k_2,m_1+m_2)} \right] \ln(r/a). \tag{1.26}
\]

The second term in (1.26) gives the interaction energy of two vortic
es. The
coupling between vortices arising due to TE-TM splitting can be analyzed
analytically in the limit of small \(\gamma\).

The solutions of Eqs. (1.18a,b) for \(k \neq 1\) are written as series in \(\gamma\),

\[
\theta(\phi) = m\phi + \frac{m}{2(k-1)} \gamma \sin[2(k-1)\phi]
+ \frac{m[(k-1)^2 - (1-m^2)]}{16(k-1)^3} \gamma^2 \sin[4(k-1)\phi] + \ldots, \tag{1.27a}
\]

\[
u(\phi) = (k-1)\phi - \frac{[(k-1)^2 + (m^2-1)]}{4(k-1)^2} \gamma \sin[2(k-1)\phi]
+ \frac{[5(k-1)^4 + 2(k-1)^2(m^2-3) + (m^2-1)^2]}{64(k-1)^4} \gamma^2 \sin[4(k-1)\phi] + \ldots. \tag{1.27b}
\]

Substitution of these expression into (1.15) gives

\[
\frac{E_s^{(k,m)}}{E_0} = (k^2+m^2) - \frac{[k^2(k-2)^2 + 2(2 + 3k(k-2))m^2 + m^4]}{8(k-1)^2} \gamma^2 - \ldots. \tag{1.28}
\]
Fig. 1.2. The logarithmic prefactor of vortex energies $E_s$ (see Eqs. (1.11) and (1.15)) for half-vortices and integer vortices as functions of TE-TM splitting parameter $\gamma$ (1.17). The curves are labeled by the winding numbers $(k, m)$ of the vortices, and the energies are given in the units of $E_0 = \pi \hbar^2 n/m^*$

and, in particular,

$$E_s^{(-1,0)} = E_0 \left[ 1 - \frac{9}{32} \gamma^2 - \ldots \right], \quad E_s^{(0,\pm 1)} = E_0 \left[ 1 - \frac{5}{8} \gamma^2 - \ldots \right].$$  \hspace{1cm} (1.29)

There is no difference between the energies of half-vortices at this order of $\gamma$-series. The difference, however, appears in the next order. The series for the angles up to $\gamma^4$ are rather cumbersome to be presented, but they result in

$$E_s^{\left(\frac{1}{2}, \pm \frac{1}{2}\right)} = \frac{E_0}{2} \left[ 1 - \frac{\gamma^2}{2} - \frac{3\gamma^4}{16} - \ldots \right],$$  \hspace{1cm} (1.30a)

$$E_s^{\left(-\frac{1}{2}, \pm \frac{1}{2}\right)} = \frac{E_0}{2} \left[ 1 - \frac{\gamma^2}{2} - \frac{11\gamma^4}{144} - \ldots \right].$$  \hspace{1cm} (1.30b)

Eqs. (1.25), (1.29), and (1.30) can be used to find the interactions between half-vortices according to Eq. (1.26). Most important interaction that appears due to TE-TM splitting is between the $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, -\frac{1}{2}\right)$ half-vortices. For small $\gamma$ their coupling constant is linear in $\gamma$ and the interaction energy is [31]

$$V_{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)} \simeq -\gamma E_0 \ln(r/a).$$  \hspace{1cm} (1.31)

It should be noted that the interrelation between $m_l$ and $m_l$, i.e., the sign of $\gamma$, depends on the detuning of the frequency of the cavity photon mode from the center of the stop-band of the distributed Bragg mirror [32]. So, one can
have both attraction and repulsion of the \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{2}, -\frac{1}{2})\) half-vortices. The coupling of the other left and right half-vortices is quadratic in \(\gamma\). The \((-\frac{1}{2}, \frac{1}{2})\) and \((-\frac{1}{2}, -\frac{1}{2})\) half-vortices repel each other with the interaction energy being \(-\frac{7}{32}\gamma^2 E_0 \ln(r/a)\). The \((-\frac{1}{2}, \pm \frac{1}{2})\) and \((\frac{1}{2}, \pm \frac{1}{2})\) half-vortices attract each other with the interaction energy being \((1/8)\gamma^2 E_0 \ln(r/a)\).

In the absence of TE-TM splitting there is no coupling between the right half-vortices (with \(km > 0\)) and the left ones (with \(km < 0\)) and there are two decoupled BKT transitions, corresponding to the dissociation of pairs of left and right half-vortices \([22, 16]\). The transition temperature is then estimated from the energy of single half-vortex as \(E_0/4\). The TE-TM splitting of polariton bands changes this picture substantially. First, because all four half-vortices become coupled and, secondly, because the energies of a vortex and its antivortex become different, so it is not clear with one should be used in the estimation of critical temperature.

One expects qualitative modifications of the BKT transition in the region of \(\gamma\) close to \(-1\). In this region the attraction of the \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{2}, -\frac{1}{2})\) half-vortices becomes very strong and, as a result, the hedgehog is the vortex with the smallest energy in the system for \(\gamma < \gamma_c \approx -0.6\) (see Fig. 1.2). It does not mean, however, that the transition temperature can be estimated from the energy of the hedgehog in this region. In fact, the phase transition occurs due to dissociation of vortex-antivortex pairs, and the energy of the \((1, 0)\) and \((-1, 0)\) pair is still bigger than the energy of the pair of two half-vortices. One expects that when pairs of half-vortices are thermally excited in the system they will tend to form molecules consisting of the hedgehog (formed by merging of the \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{2}, -\frac{1}{2})\) half-vortices) with the \((-\frac{1}{2}, -\frac{1}{2})\) and \((-\frac{1}{2}, \frac{1}{2})\) half-vortices being attached to it. The proliferation of these \((-\frac{1}{2}, -\frac{1}{2}) - (1, 0) - (-\frac{1}{2}, \frac{1}{2})\) molecules defines the phase transition for \(ml \gg m_t\).

### 1.4 Geometry of the half-vortex fields

In general, two coupled Gross-Pitaevskii equations for the circular components of the order parameter

\[
\begin{align*}
    i\hbar \frac{\partial \psi_{+1}}{\partial t} &= -\frac{\hbar^2}{2m^*} \left( \Delta \psi_{+1} + 4\gamma \frac{\partial^2 \psi_{-1}}{\partial z^2} \right) - \mu \psi_{+1} + U_0(|\psi_{+1}|^2 + |\psi_{-1}|^2) - 2U_1|\psi_{-1}|^2 \psi_{+1}, \\
    i\hbar \frac{\partial \psi_{-1}}{\partial t} &= -\frac{\hbar^2}{2m^*} \left( \Delta \psi_{-1} + 4\gamma \frac{\partial^2 \psi_{+1}}{\partial z^2} \right) - \mu \psi_{-1} + U_0(|\psi_{-1}|^2 + |\psi_{+1}|^2) - 2U_1|\psi_{+1}|^2 \psi_{-1},
\end{align*}
\]

are not separated in cylindrical coordinates \((r, \phi)\). The variables are separated only in special cases of hedgehog vortices and the double-quantized
Fig. 1.3. Showing the geometry of half-vortices for different values of TE-TM splitting parameter $\gamma$ (1.17). The plots are obtained from numerical solutions of GPEs (1.9). The interaction constants are related as $U_1 = 0.55U_0$. The local polarization ellipses are drawn with the thick (red) lines. The streamlines of the current are shown by thin (green) lines. The panels demonstrate the following cases: (a) the half-vortices ($\frac{1}{2}, \pm \frac{1}{2}$) for $\gamma = -0.5$ (the lemon morphology); (b) the half-vortices ($\frac{3}{2}, \pm \frac{1}{2}$) for $\gamma = 0.5$ (the lemon morphology); (c) the half-vortices ($-\frac{1}{2}, \pm \frac{1}{2}$) for $\gamma = -0.5$ (the star morphology); (d) the half-vortices ($-\frac{3}{2}, \pm \frac{1}{2}$) for $\gamma = 0.5$ (the star morphology).

polarization vortex discussed in previous section after Eqs. (1.19). For other vortices one needs to solve GPEs numerically in two spacial dimensions with the boundary conditions at large distances defined by Eqs. (1.12) and (1.18).

Each vortex with nonzero phase winding number $m$ is characterized by a finite superfluid current circulating around its core center. Performing numerical solutions it is important to take into account the fact that streamlines of the current are deformed with respect to perfect circles in the presence of TE-TM splitting. The circular components of the current $J$ are given by
\[ J_{+1} = \frac{i \hbar}{\sqrt{2} m^*} \left\{ \left( \psi_{+1} \frac{\partial \psi_{+1}^*}{\partial z} + \psi_{-1} \frac{\partial \psi_{-1}^*}{\partial z} - \psi_{+1}^* \frac{\partial \psi_{+1}}{\partial z} - \psi_{-1}^* \frac{\partial \psi_{-1}}{\partial z} \right) \right. \]
\[ + 2\gamma \left( \psi_{+1} \frac{\partial \psi_{-1}^*}{\partial z^*} - \psi_{-1}^* \frac{\partial \psi_{+1}}{\partial z^*} \right) \left\} \right., \quad (1.33) \]

and \( J_{-1} = J_{+1}^* \). They are related to the radial \( J_r \) and the azimuthal \( J_\phi \) components of the current by
\[ J_r = \frac{1}{\sqrt{2}} \left( e^{i\phi} J_{+1} + e^{-i\phi} J_{-1} \right), \quad J_\phi = i \sqrt{2} \left( e^{i\phi} J_{+1} - e^{-i\phi} J_{-1} \right). \quad (1.34) \]

At large distances \( r \gg a = \hbar / \sqrt{2 m^* \mu} \) one can use Eq. (1.12) to find
\[ J_r = \frac{\hbar n}{m^* r} \gamma \sin(2\theta) d\theta d\phi, \quad (1.35a) \]
\[ J_\phi = \frac{\hbar n}{m^* r} \left[ 1 - \gamma \cos(2\theta) \right] d\theta d\phi. \quad (1.35b) \]

Note that the condition of conservation of the total number of polaritons for the static vortex solution of Eqs. (1.32),
\[ \text{div} \mathbf{J} = 1 \frac{\partial}{\partial r} \left[ r J_r \right] + \frac{\partial J_\phi}{\partial \phi} = 0, \quad (1.36) \]
implies \( \partial J_\phi / \partial \phi = 0 \). So, Eq. (1.18b) obtained in the previous section is in fact the condition of conservation of the azimuthal current.

The warping of streamlines of current is shown in Fig. 1.3. The order parameter has been found numerically \[33\] for different values of TE-TM splitting parameter. To find the static solutions of GPE (1.32), we have been choosing an initial order parameter \( \psi(r, t = 0) \) satisfying the boundary conditions that follow from Eqs. (1.18) and that are shown in Fig. 1.1 for a given topological sector \((k, m)\). Apart from this the initial functions were continuous but arbitrary. Then the functions were evolved according to (1.32) in imaginary time. As a result, the order parameter relaxed to corresponding static half-vortex solution. The resulting half-vortex solutions are found to be independent of the initial shape of \( \psi(r, t = 0) \).

In Fig. 1.3 one can see two distinct morphologies of basic half-vortices. The geometry of half-vortex solutions can be discussed in terms of singular optics \[34, 35\], where the polarization singularity related to a half-vortex is referred as C-point, to indicate that the polarization is circular at the vortex center and, therefore, the direction of the main axis of polarization ellipse is not defined. The morphologies of the field around C-points are classified by the index of associated real tensor field, and, additionally, by the number of straight polarization lines\[36\] that terminate at C-point \[30\]. The tensor index
\[ 4 \] The tangents of polarization lines define by the direction of the main axis of polarization ellipse in each point.
coincides with the polarization winding number $k$, and the number of lines could be either one or three. As a result, three different morphologies can be found. Following Berry and Hannay [37,38], these morphologies are referred as lemon, star, and monstar.

The lemon configuration is characterized by $k = \frac{1}{2}$ and by only one straight polarization line terminating in the vortex center. This is the morphology of vortices in Fig. 13a,b with the straight polarization line being defined by $\phi = 0$. The star configuration is characterized by $k = -\frac{1}{2}$. In this case there are always three straight lines terminating in the vortex center. The stars are realized in Fig. 13c,d and three straight polarization lines are defined by $\phi = 0, \pm 2\pi/3$.

The change of parameter $\gamma$ leads to deformation of polarization texture and to deformation of streamlines of the current, but it does not result in the change of morphologies of half-vortices. In principle, one could expect the transformation of lemon into monstar, since these morphologies possess the same topological index $k = \frac{1}{2}$. Contrary to the lemon case, however, the monstar is characterized by three straight polarization lines terminating in the vortex center, similar to the star configuration. So, from geometrical point of view the monstar has got intermediate structure between the lemon and the star, and this is why its name is constructed from “(le)mon-star”.

To have the monstar configuration one needs a special behavior of polarization angle $\eta(\phi)$. Namely, it is necessary to have

$$\left. \frac{d\eta}{d\phi} \right|_{\phi=0} > 1 \quad \text{for} \quad k = \frac{1}{2}. \quad (1.37)$$

In this case the polarization angle initially rotates faster than the azimuthal angle $\phi$, but since the total rotation of $\eta$ should be still $\pi$ when $\phi$ is changing up to $2\pi$, as it is dictated by the winding number $k = \frac{1}{2}$, there will be three roots of the equation $\eta(\phi) = \phi$. These roots, 0 and $\pm \phi_m$, define three straight polarization lines terminating in the vortex center for the monstar geometry.

One can see from Fig. 1.1a and 1.3a that when $\gamma$ approaches $-1$ the derivative becomes very close to unity, but it never becomes bigger than 1, so that the monstar is not formed. The reason preventing the appearance of the monstar is that it is not energetically favorable to satisfy the condition (1.37). In fact, it is the most energetically favorable to have $\eta = \phi$, as for the hedgehog—the vortex having the smallest energy when $\gamma \to -1$ (see Fig. 1.2). The rotation of polarization of the half-vortex is also synchronous with the azimuth in rather wide sector, but the polarization never overruns the azimuth. The monstar half-vortices, however, are expected to be found in the exciton-polariton condensates out of equilibrium [39,9,40], where their appearance is not restricted by energetics.

\[5\] Note that for the monstar all polarization lines residing within the sector $-\phi_m < \phi < \phi_m$ terminate in the vortex center, but only three of them are straight, i.e., are having nonzero inclination at $r \to 0$ (see [36] for the details).
1.5 Four-component exciton condensates

Excitons formed by an electron and a heavy hole in the semiconductor quantum wells can be in four spin states. The states with the total spin projection ±1 are optically active. These bright excitons are formed by the heavy hole with the spin $\pm \frac{3}{2}$ and the electron with the spin $\pm \frac{1}{2}$, or by the heavy hole with the spin $\pm \frac{1}{2}$ and the electron with the spin $\pm \frac{3}{2}$. The other two states are hidden from the observer and are usually referred to as the dark excitons. The total momentum of these states is ±2 and they are formed either by the $\pm \frac{3}{2}$ hole and the $\pm \frac{1}{2}$ electron, or by the $\mp \frac{1}{2}$ hole and the $\pm \frac{3}{2}$ electron.

The exciton-polaritons discussed in the previous sections are coupled states of quantum-well excitons and microcavity photons. Only the bright excitons are involved in this coupling, and the resulting condensates are two-component. Since the frequency of a single exciton-polariton is shifted down with respect to the single exciton frequency by a half of the Rabi frequency, the presence of dark excitons is irrelevant in this case provided the exciton-photon coupling is strong enough. Contrary, when pure exciton condensates are of interest, all four exciton spin states should be, in general, taken into account. The formation of exciton condensates is possible for cold indirect excitons in coupled quantum wells. The life-time of these excitons is long enough, the excitons can travel coherently over long distances, and the condensates can be formed in quasi-equilibrium conditions. The presence of four-component exciton condensates has also been experimentally demonstrated recently.

The indirect excitons are dipoles oriented along the growth axis of the semiconductor structure, and their main interaction is spin-independent dipole-dipole repulsion. The condensate state, however, is defined by weak spin-dependent interactions arising from electron-electron, hole-hole, and exciton-exciton exchanges. In what follows, I will assume the signs of these interactions to be such that they favor the distribution of excitons over all four spin states, populating both bright and dark components. This state is similar to the linearly polarized two-component condensates described above, but there is one important qualitative difference between them. The exchange scattering of two excitons can result in transformation of their spin states. Namely, two bright excitons can turn into two dark ones after collision and vice versa. These processes are described microscopically by the Hamiltonian

$$\hat{H}_{\text{mix}} = W \left[ \hat{\psi}_{+2}^\dagger \hat{\psi}_{-2} + \hat{\psi}_{+1}^\dagger \hat{\psi}_{-1} + \hat{\psi}_{+1}^\dagger \hat{\psi}_{-2} + \hat{\psi}_{+2}^\dagger \hat{\psi}_{-1} \right].$$ (1.38)

In mean-field approximation, the creation $\hat{\psi}_{\sigma}^\dagger$ and annihilation $\hat{\psi}_{\sigma}$ exciton operators ($\sigma = \pm 1, \pm 2$) are replaced by the order parameter components, $\hat{\psi}_{\sigma}^*$ and $\hat{\psi}_{\sigma}$, respectively. The contribution of the resulting exciton-mixing term $\hat{H}_{\text{mix}}$ into the total energy of the exciton condensate depends on the relative phases of the components. The term of this type is absent in the two-component exciton-polariton case. Remarkably, the mixing of excitons always
leads to the decrease of the condensate energy, which is achieved by fixing the proper interrelation between the phases. Denoting by $\theta_{\sigma}$ the phase of $\psi_{\sigma}$, one can see that the following relation holds within the order parameter manifold

$$\theta_{+2} + \theta_{-2} - \theta_{+1} - \theta_{-1} = \begin{cases} 0 \mod 2\pi, & \text{if } W < 0, \\ \pi \mod 2\pi, & \text{if } W > 0. \end{cases} \quad (1.39)$$

The mixing term $H_{\text{mix}}$ additionally favors the formation of the four-component exciton condensate with equal occupations of the components. This fact can be seen from different perspective. $H_{\text{mix}}$ describes the transformation of pairs of excitons, and, in the same way as in the BCS theory of the superconductivity, this term leads to the pairing of particles. This pairing leads to a decrease in the energy of the system and results in appearance of the gap in the excitation spectrum for one excitation branch. The other three excitation branches are Bogoliubov-like. This follows from the fact that the phase locking condition (1.39) leaves there angles to be undefined, so that there are three Goldstone modes apart from the gaped mode induced by the mixing.

The effect of applied Zeeman field on four-component exciton condensate is expected to be very spectacular [48]. The Zeeman splitting is different for dark and bright excitons: the $g$-factor is given by the sum of the electron and hole $g$-factors for the former, and by their difference for the latter. When such a field is applied to the exciton condensate its action is two-fold. On the one hand, it polarizes the bright and dark components with different degrees of circular polarization, and thus reduces the Zeeman energy of the condensate. On the other hand, the induced imbalance in the occupation of the components increases the energy of the mixing term $H_{\text{mix}}$ and suppresses the gap in the spectrum discussed above. The interplay between these two effects can lead to the first-order transitions from the four-component exciton condensate to the two- or the one-component condensates. Note also that due to the presence of $H_{\text{mix}}$ the system of equations defining the concentrations of the components of the exciton condensate in the Zeeman field is nonlinear, contrary to the case of two component exciton-polariton condensate (see Eq. (1.39)).

Finally, it is important to note that vortices in the four-component exciton condensate in the presence of the mixing of the component are composite: the vorticity of one component should be accompanied by the vorticity of another component to satisfy the the phase-locking condition (1.39). As a result one expect twelve elementary vortices. These are four polarization vortices (two in each components), and the eight paired half-vortices in the bright and dark components.

1.6 Conclusions and perspectives

The mean-field approximation provides simple and reliable method to study the polarization properties and topological excitations of exciton-polariton...
and exciton condensates that possess two and four components of the order parameter, respectively. This includes, in particular, the description of the polarization of the ground state and elementary excitations of the condensates and their change in applied Zeeman field, as well as the description of the texture of vortices and vortex interactions.

The elementary topological excitations in two-component exciton-polariton condensates are four half-vortices \((k, m)\) with \(k, m = \pm \frac{1}{2}\), characterized by half-quantum changes of polarization and phase angles. In the absence of transverse-electric-transverse-magnetic (TE-TM) splitting of the lower polariton band there is no coupling between the left half-vortices (with \(km < 0\)) and the right ones (with \(km > 0\)), and one expects two decoupled Berezinskii-Kosterlitz-Thouless (BKT) superfluid transitions happening at the same temperature in the system. The TE-TM splitting results in two qualitative effects. First, the cylindrical symmetry of the half-vortex field is spontaneously broken that leads to warping of the polarization field around a half-vortex and to deviation of the streamlines of the supercurrent from the perfect circles. Secondly, there appears long-range interactions between left and right half-vortices. These interactions are particularly important in the case of large longitudinal polariton mass \(m_l\) when it favors the formation of hedgehog (monopole) vortices \((1, 0)\) from the \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{2}, -\frac{1}{2})\) half-vortices. The peculiarities of the superfluid transition in this case and related features of polarization textures of the exciton-polariton condensates are subjects of further studies. In what concerns the geometry of the half-vortex field it is shown that only two configurations, lemon and star, are realized. The monstar configuration is not energetically favorable for any value and sign of TE-TM splitting.

The essential feature of four-component exciton condensates is the presence of mixing and related phase locking between dark and bright excitons. One expects a nontrivial Zeeman-field effect resulting in a discontinuous change of the polarization state of the condensate in course of the first-order transition. The presence of composite vortices in different components should lead to the formation of interesting polarization patterns in driven exciton condensates that provide an important topic for investigation, both experimental and theoretical.

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