Single-photon generation of entangled triplet state in an atomic spin dimer

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Abstract. We show that the entanglement between two atomic spins, coupled via $XY$ interaction, can be achieved with the aid of a single photon in a controllable manner. Assuming a constant spin-photon coupling, two distinct cases are considered — excitation with a constant and with a linear time-dependent photon frequency. We demonstrate that these problems reduce to the study of the well known Jaynes-Cummings and Jaynes-Cummings – Landau-Zener models, respectively, where the two-level system is formed by the zero-momentum mode states of the dimer. Owing to the exact solubility of both models, by tuning the relevant parameters, one can design and coherently control the dynamics of the excitation process. For instance, one can adiabatically switch from an atomic spin system initialized in a state with both spins down to the maximally entangled triplet state.

1. Introduction

The production and manipulation of entanglement in quantum systems is currently a major issue of research interest. Apart from the fundamental tests of quantum mechanics, the entangled states are used as a resource for quantum information science. Controllable creation and manipulation of entanglement in two-qubit systems, in particular, is the simplest and hence most studied problem, mainly theoretically and especially with respect to potential applications, see for instance [1, 2]. The bipartite entanglement is used as a means for quantum computations, for realization of quantum key distribution and quantum teleportation, etc. [3]. Moreover, multipartite entanglement, needed for certain quantum algorithms [3], can be created via iteration procedures with creation of a bipartite entanglement at each step.

In this paper we propose a method for coherent control of a pair of interacting qubits. More precisely, a method for a single-photon generation of maximally entangled triplet state ($S = 1, S_z = 0$) in a pair of two-level atoms interacting according to the $XY$ model. The latter will be referred to as atomic spin dimer. We will assume that there is a single excitation in the composite system consisting of a photon field and a dimer. The case of a single excitation is of peculiar interest since its solution can be obtained in a closed form for some selected values of the atom-photon coupling and detuning [4]. We point out that the semi-classical counterpart of this model (subjected to a classical field) has already been used to propose adiabatic state preparation of interacting two-level systems [5].

Half-spin Hamiltonians describe not only spin systems but also serve as effective models to describe interacting two-qubit systems with different physical natures including semiconductor quantum dots [6], neutral cold atoms [7] and so forth. In this sense, our method is applicable to any physical implementations of two $XY$-interacting qubits.
2. Atomic dimer coupled to a photon mode

In this section we write down the explicit expression of the Hamiltonian describing our system and explain its main components. The sketch of the physical setup of the model is plotted in figure 1.

Figure 1. Schematic setup of Hamiltonian (3). The atomic spin dimer is represented by two two-level atoms within an ellipse. The dimer is placed in a cavity with left-circular polarization of the electric component of the field mode.

The interaction between atoms constituting the dimer is chosen to be of the $XY$ type [8] and the associated Hamiltonian reads

$$\hat{H}_{a-a} = 2J (\hat{S}_1^+ \hat{S}_2^- + \hat{S}_1^- \hat{S}_2^+),$$

(1)

where $J$ is the coupling constant and $\hat{S}_{i}^{\pm}, i = 1, 2,$ stand for the half-spin operators. These latter operators satisfy the commutation relations $[\hat{S}_{i}^{+}, \hat{S}_{j}^{-}] = 2\delta_{ij}\hat{S}_{i}^{z}, [\hat{S}_{i}^{z}, \hat{S}_{j}^{\pm}] = \pm \delta_{ij}\hat{S}_{i}^{\pm}, i, j = 1, 2$. Observe that Hamiltonian (1) is formally equivalent to the Hamiltonian of the periodic $XY$ model with spin-spin coupling $J$, in the nearest-neighbor approximation. This property is exploited in the next section since it allows us to write down directly the quasi-momentum -- spin representation for the full Hamiltonian.

Further, we assume that the atom-spin – photon coupling mechanism is described by the Dicke model. Within the rotating wave approximation the Dicke Hamiltonian [9] turns into the Tavis-Cummings Hamiltonian [10]

$$\hat{H}_{a-p} = \hbar\Delta \sum_{i=1}^{2} \hat{S}_{i}^{z} - \frac{\hbar\Omega}{\sqrt{2}} \sum_{i=1}^{2} \left( \hat{a} \hat{S}_{i}^{+} + \hat{a}^{\dagger} \hat{S}_{i}^{-} \right).$$

(2)

Here $\Omega$ is the atom-photon coupling, $\Delta$ is the detuning, and $\hat{a} (\hat{a}^{\dagger})$ is the photon annihilation (creation) operator for the left-circularly polarized photons.

Eventually, the full Hamiltonian describing the problem of a dimer interacting with a photon mode along $z$-axis is

$$\hat{H} = \hbar\omega \hat{N} + \hat{H}_{a-a} + \hat{H}_{a-p},$$

(3)

where $\hat{N} = (\hat{n} + 1 + \sum_{i=1}^{2} \hat{S}_{i}^{z})$ is the number of excitation operator and $\hat{n}$ is the number of photons in the mode. The number of excitations $N$ is a conserved quantity, and hence, for a given $N$ the dynamics of the system is constrained to the corresponding Hilbert subspace. Furthermore, since $N$ enters the global phase of the Schrödinger’s equation solution we will omit it hereafter. The selection rules for the total spin of the atomic dimer are given by $\Delta S_z = \pm 1, \Delta S = 0$.

3. Quasi-momentum–spin representation

We consider the case of a single excitation, $N = 1$, with the excitation contained in the field mode at the beginning of the dynamical process. It has been recently shown [4] that Hamiltonian (3) can be expressed
in terms of half-spin operators \( \hat{\Sigma}_{q, z} \) as

\[
\hat{H} = P_{-}\hat{\Sigma}_{-\pi, z} + \hat{H}_{JC},
\]

where

\[
\hat{H}_{JC} = p_{0}\hat{\Sigma}_{0, z} + \frac{Q}{2}\left( a\hat{\Sigma}_{0} + a^{\dagger}\hat{\Sigma}_{0} \right),
\]

with \( Q = -2\hbar\Omega, P_{0, -\pi} = \pm 2f + \hbar\Delta \). Operators \( \hat{\Sigma}_{q, z} \) (\( \{ \hat{\Sigma}_{q, z}, \hat{\Sigma}_{q', z} \} = 2\delta_{q, q'}\hat{\Sigma}_{q, z}, \{ \hat{\Sigma}_{q, z}, \hat{\Sigma}_{q, \pm} \} = \pm \delta_{q, q'}\hat{\Sigma}_{q, z}, q, q' = -\pi, 0 \)) act on the two points with \( q = -\pi, 0 \), that form the Brillouin zone of the periodic \( XY \) chain. Note that the first term in (4) describes a free two-level system uncoupled from the field-induced dynamics. The essential dynamical information is derivable from \( \hat{H}_{JC} \) (5). Hamiltonian \( \hat{H}_{JC} \) is formally equivalent to the ubiquitous Janes-Cummings model [11], that describes a two-level atom interacting with a quantized mode of electromagnetic field. The dynamics induced by \( \hat{H}_{JC} \) is confined to the two-dimensional Hilbert subspace spanned by \( \{ |1\rangle_{ph}| -\frac{1}{2}k_{-\pi}, -\frac{1}{2}k=0 \rangle \equiv |1; -\frac{1}{2}\rangle, |0\rangle_{ph}| -\frac{1}{2}k_{-\pi}, \frac{1}{2}k=0 \rangle \equiv |0; \frac{1}{2}\rangle \} \).

### 4. Generation of entangled states

As we have already shown the dynamics of the atomic dimer interacting with a single photon is confined in the two-dimensional subspace spanned by \( \{ |1\rangle_{ph}| -\frac{1}{2}\rangle, |0\rangle_{ph}| \frac{1}{2}\rangle \} \) where the spin component is given in \( \Sigma \)-representation. Then, the state vector \(|\Phi(t)\rangle\), that is solution of the Schrödinger equation with Hamiltonian \( \hat{H}_{JC} \) reads

\[
|\Phi(t)\rangle = a_{+}(t)|0; \frac{1}{2}\rangle + a_{-}(t)|1; -\frac{1}{2}\rangle,
\]

where \( a_{\pm}(t) \) are the time-dependent amplitudes for the respective states.

Correspondingly, in \( S \)-representation, we have

\[
|1; -\frac{1}{2}\rangle = |1\rangle_{ph}| -\frac{1}{2}\rangle - \frac{1}{\sqrt{2}}|\frac{1}{2}\rangle,
\]

\[
|0; \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|0\rangle_{ph}\left( | -\frac{1}{2}\rangle + |\frac{1}{2}\rangle \right).
\]

It is then clear that in \( S \)-representation their associated spin-dimer components are triplet states – the total spin of the dimer is \( S = \Sigma = 1 \) and its projection in these states is \( S_{z} = \Sigma_{z} = -1 \) and \( S_{z} = \Sigma_{z} = 0 \), respectively.

In our setup, by assumption, the system initially resides in the eigenstate \(|\Phi(t_{i})\rangle = |1; -\frac{1}{2}\rangle \) (7). Our goal is to propagate this pure state in time, so to obtain the maximally entangled spin-triplet state (8). In other words we need to achieve complete population transfer from state (7) to state (8) under the evolution governed by \( \hat{H}_{JC} \) (5), i.e. we require that

\[
P_{+}(t_{f}) = |a_{+}(t_{f})|^{2} = 1,
\]

at the final time of the process \( t_{f} \). An important remark is that the probability to find the atomic system in the entangled state is conditional in the sense that it is photon-state dependent. To this end, below we consider two well-known exactly solvable models.

To study the time-dependence of the amount of entanglement between the constituent atomic spins of the dimer we make use of the quantity dubbed concurrence [12]. One can easily show, however, that in the case considered here the concurrence coincides with the probability to find the dimer in the excited state (9).
4.1. Jaynes-Cummings model
The on-resonance Jaynes-Cummings model [11], corresponds to the case $P_0 = 0$ and $Q = \text{const}$, with the initial condition $|\Phi(t_i = 0)\rangle = |1; -\frac{1}{2}\rangle_\Sigma$. It can be shown that the respective time-dependent amplitudes read

$$a_-(t) = \cos\left(\frac{Qt}{2\hbar}\right), \quad a_+(t) = -i\sin\left(\frac{Qt}{2\hbar}\right).$$

(10)

Evidently, the transition from the state $|1; -\frac{1}{2}\rangle$ to the state $|0; \frac{1}{2}\rangle$, with unit probability

$$P_+(t_f) = \sin^2\left(\frac{Qt_f}{2\hbar}\right) = 1,$$

(11)

can be achieved with the use of $\pi$-pulse. That is, when the coupling is a rectangular pulse, starting at time $t_i = 0$ and ending at $t_f = T$, with the duration of the pulse $T$ defined by the $\pi$-area condition

$$\frac{QT}{\hbar} = \pi.$$  

(12)

4.2. Jaynes-Cummings – Landau-Zener model
Let us now assume that the coupling $Q$ is constant, the detuning is linearly dependent on time, i.e. $P_0 = 2\lambda t$ ($\lambda > 0$), and the process initiates at $t_i = -\infty$ and ends at $t_f = +\infty$. This two-level model is the Jaynes-Cummings – Landau-Zener model [13]. The amplitude to find the system in the initial state $|\Phi(t_i)\rangle = |1; -\frac{1}{2}\rangle_\Sigma$ at $t_f$ is

$$a_-(t_f) = \exp\left(\frac{-\pi Q^2}{8\hbar\lambda}\right),$$

(13)

that implies

$$P_+(t_f) = 1 - \exp\left(-\frac{\pi Q^2}{4\hbar\lambda}\right).$$

(14)

for the transition probability. With the aim to ensure the nearly unit transition probability one needs to work in the adiabatic regime, i.e.

$$\frac{Q^2}{4\hbar\lambda} \gg 1.$$  

(15)

An illustration of the time-dependence of the concurrence for specific values of the experimental parameters is shown in figure 2 for both cases of interest. Note the difference in the time scales between the Jaynes-Cummings model in panel a) and the Jaynes-Cummings – Landau-Zener model in panel b) for the same values of $\Omega$ and $J$, and $\lambda$ such that (15) is fulfilled. On one hand, the latter evolution is adiabatic and hence the transition is robust against small fluctuations of the experimental parameters. On the other, stronger adiabaticity is related to transitions over longer times and in turn to vulnerability of the process to decoherence, thus decreasing the accuracy of our model (3) with respect to real physical systems. Furthermore, the infinite time of the process needed for implementation of the Landau-Zener model is unphysical but it is known that the major part of the transition is concentrated in the vicinity of the crossing of diabatic states (seen also in figure 2), and thence, it is sufficient to take small/large enough values for the start/end time of the process in order to have the above results valid. That is, in practice one would implement the finite Landau-Zener model [14].
5. Conclusions
We propose a method for generation of a maximally entangled triplet state between two atomic spins coupled via $XY$ interaction. Initially, both atoms are prepared in their ground states, and then they are let to interact with a single photon. The key observation is that the whole system, photon and dimer, is described by a two-level system Hamiltonian, with levels defined by the zero-momentum mode states of the dimer. This allows one to coherently control the state of the atomic dimer using a variety of techniques developed for the two-level system problem. We demonstrate, in particular, the creation of the maximally entangled triplet state by means of two exactly solvable models, namely the Jaynes-Cummings and the Jaynes-Cummings–Landau-Zener models. In the latter the entangled state is created in an adiabatic manner. Such entangled states can find different applications, especially in the fields of quantum computations and quantum communications.

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