Fluctuation identities for omega-killed Markov additive processes and dividend problem

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Abstract

In this paper we solve the exit problems for an one-sided Markov additive process (MAP) which is exponentially killed with a bivariate killing intensity $\omega(\cdot, \cdot)$ dependent on the present level of the process and the present state of the environment. Moreover, we analyze respective resolvents. All identities are given in terms of new generalizations of classical scale matrices for the MAP. We also remark on a number of applications of the obtained identities to (controlled) insurance risk processes. In particular, we show that our results can be applied to the so-called Omega model, where bankruptcy occurs at rate $\omega(\cdot, \cdot)$ when the surplus process becomes negative. Finally, we consider the Markov modulated Brownian motion (MMBM) and present the results for the particular choice of piecewise intensity function $\omega(\cdot, \cdot)$.

Keywords: Markov modulation, Omega model, Potential measures, Fluctuation theory, Dividends.

1 Introduction

In the fields of risk theory, financial mathematics, environmental problems, queueing and so forth, there are various applications of a Markov additive process (MAP) which in continuous time is a natural generalization of a Lévy process (see, e.g., \cite{1 4 5 6 12}). Furthermore, MAP can be seen as a Lévy process in Markov environment, which provides rich modeling possibilities. This paper solves exit problems for spectrally negative MAP which is exponentially killed with a bivariate killing intensity $\omega(\cdot, \cdot)$ dependent on the present states of the process and the environment. Moreover, we analyze respective $\omega$-killed resolvents. Recently, Li and Palmowski \cite{17} investigated $\omega$-killed exit identities and resolvents for a general

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(reflected) spectrally negative Lévy process. This paper generalized their results to Markov additive framework.

Before entering our discussion of this subject, we shall begin by defining the class of processes we intend to work with. Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a filtrated probability space, with filtration \(\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}\) which satisfies usual conditions. Throughout this article, we will consider a bivariate process \((X, J) = \{(X_t, J_t)\}_{t \geq 0}\) such that \(X\) is a real-valued càdlàg (right-continuous with left limits) process and \(J\) is a right-continuous jump process with a finite state space \(E = \{1, 2, ..., N\}\). We say that \((X, J)\) is a MAP if, given \(\{J_t = i\}\), the vector \((X_{t+s} - X_t, J_{t+s})\) is independent of \(\mathcal{F}_t\) and has the same law as \((X_s - X_0, J_s)\) given \(\{J_0 = i\}\) for all \(s, t \geq 0\) and \(i \in E\). Usually \(X\) is called an additive component and \(J\) is a background process representing the environment. Moreover, we find the following representation of every MAP important. Straightforward conclusion from the definition gives that \(J\) is a Markov chain. Furthermore, one can observe that process \(X\) evolves as some Lévy process \(X^i\) when \(J\) is in state \(i\). In addition, when \(J\) transits to state \(j \neq i\), process \(X\) jumps according to the distribution of the random variable \(U_{ij}\), where \(i, j \in E\). All above components are assumed to be independent. The above structure explains why the another name for MAP is “Markov-modulated Lévy process”. Furthermore, let us remark that when \(J\) lives on a single state, \(X\) reduces to a Lévy process. Throughout this paper we assume that process \(X\) has no positive jumps, thus \(X^i\) is a spectrally negative Lévy process and \(U_{ij} \leq 0\) a.s. (for every \(i, j \in E\)). We exclude the case when \(X\) has monotone paths. We assume that \(J\) is an irreducible Markov chain, with \(Q\) being its transition probability matrix and \(\pi\) being its unique stationary vector.

One of the main contributions of this paper is the identification of the so-called \(\omega\)-scale matrices \(W^{(\omega)}\) and \(Z^{(\omega)}\), which appear as the solutions of exit problems for MAPs with \(\omega\)-killing. Moreover, it is shown that these new generalizations of scale matrices are solutions to some integral equations. In the case where the killing intensity \(\omega\) is constant for every \(i \in E\) and \(x \in \mathbb{R}\), our results are consistent with the classical exit identities and resolvents obtained in [15] and [14] in terms of the so-called scale matrices.

The paper is organized as follows. Section 2 recalls some basic definitions and properties of MAPs and also introduces the bivariate \(\omega\)-function and \(\omega\)-killed exit problems. In Section 3, we formally define \(\omega\)-scale matrices and present our main results. In Section 4, we apply our results to find the value function for dividends paid until ruin in the so-called Omega model. Section 5 is dedicated to the analysis of some particular examples of \(\omega\). Further numerical computations are provided. Finally, we postpone the existence of \(\omega\)-killed scale matrices as well as the proofs of the main results in Appendixes A and B for conciseness.

## 2 Preliminaries

In this section, we present some basic definitions and properties of MAPs. Let \(F(\alpha)\) be the matrix analogue of the Laplace exponent of the spectrally negative Lévy process, namely

\[
\mathbb{E}(e^{\alpha X_t}, J_t = j | J_0 = i) = (e^{F(\alpha) t})_{ij}, \quad \text{for } \alpha \geq 0,
\]

which has an explicit representation

\[
F(\alpha) = \text{diag}(\psi_1(\alpha), ..., \psi_N(\alpha)) + Q \circ \mathbb{E}(e^{\alpha U_{ij}}).
\]
Recall that $Q$ is the $N \times N$ transition rate matrix of $J$. Further, we denote by $\psi_i$ the Laplace exponent of Lévy process $X^i$ when $J_t = i$ (i.e., $\mathbb{E}(e^{\alpha X^i_t}) = e^{\psi_i(\alpha)}$), and $A \circ B = (a_{ij}b_{ij})$ stands for entry-wise (Hadamard) matrix product. Note that $F(0)$ is the transition rate matrix of $J$, and hence our MAP is non-defective if and only if $F(0)\mathbf{1} = \mathbf{0}$, where $\mathbf{0}$ and $\mathbf{1}$ denote the (column) vectors of $0$s and $1$s respectively (whereas the identity and the zero matrices are denoted by $I$ and $\mathbf{0}$ respectively.) Throughout this article, the law of $(X, J)$ such that $X_0 = x$ and $J_0 = i$ is denoted by $\mathbb{P}_{x,i}$ and its expectation by $\mathbb{E}_{x,i}$. We will also use equivalently $\mathbb{E}_x[\cdot | J_0 = i]$ for $\mathbb{E}_{x,i}[\cdot]$ to emphasis the starting state. When $x = 0$, we will write $\mathbb{P}(\cdot | J_0 = i)$ and $\mathbb{E}[\cdot | J_0 = i]$ respectively. For a stopping time $\tau$, the notation $\mathbb{E}_x[\cdot, J_\kappa | J_0 = i]$ is used to denote a $N \times N$ matrix whose $(i,j)$ entry equals to $\mathbb{E}_x[\cdot, J_\kappa = j | J_0 = i]$.

In the study of exit problems of spectrally negative MAPs, the so-called scale matrices play an essential role, which can be defined analogously as the scale functions of spectrally negative Lévy processes. First, let us define the first passage times:

$$\tau^+_x = \inf\{t > 0 : X_t \geq x\}, \quad \tau^-_x = \inf\{t > 0 : X_t < x\}.$$ 

From Kyprianou and Palmowski [16], for $q \geq 0$, there exists a continuous, invertible matrix function $W^{(q)} : [0, \infty) \rightarrow \mathbb{R}^{N \times N}$ such that for all $0 \leq x \leq a$,

$$\mathbb{E}_x \left[ e^{-q \tau^+_x}, \tau^+_x < \tau^0, J_{\tau^+_x} | J_0 \right] = W^{(q)}(x)W^{(q)}(a)^{-1}. \tag{2.1}$$

Moreover, Ivanovs [13] and Ivanovs and Palmowski [15] showed that $W^{(q)}$ can be characterized by

$$\tilde{W}^{(q)}(\alpha) = (F(\alpha) - qI)^{-1}, \quad \text{for large enough } \alpha, \tag{2.2}$$

where $\tilde{f}(\alpha) = \int_0^\infty e^{-\alpha x} f(x)dx$ denotes the Laplace transform of the matrix function $f$. Furthermore, the domain of $W^{(q)}$ can extended to the negative half line by taking $W^{(q)}(x) = \mathbf{0}$ for $x < 0$. The basis of the above transform lies a probabilistic construction of the scale matrix $W^{(q)}$ which involves the first hitting time at level $x$ and can be written as

$$W^{(q)}(x) = e^{-\Lambda^q x}L^q(x),$$

where $\Lambda^q$ is the transition rate matrix of Markov chain $\{J^q_t\}_{t \geq 0}$, i.e., $\mathbb{P}(\tau^+ < \kappa, J_{\tau^+} = e_q) = e^{\Lambda^q x}$ with $e_q$ being an independent exponential random variable of rate $q > 0$. Moreover, $L^q(x)$ is a matrix of expected occupation times at $0$ up to the first passage time over $x$. In addition, the matrix $L^q := L^q(\infty)$ is the expected occupation density at $0$ and it is known that $L^q$ has finite entries and is invertible unless the process is non-defective and $\pi \mathbb{E}[X_1, J_1 | J_0] \mathbf{1} = 0$ (see [15]). Hence, we have

$$\lim_{x \to \infty} e^{\Lambda^q x}W^{(q)}(x) = \lim_{x \to \infty} W^{(q)}(x)e^{\Lambda^q x} = L^q, \tag{2.3}$$

where the matrix $R^q := (L^q)^{-1} \Lambda^q L^q$. Moreover, it is easy to see that $\lim_{a \to \infty} W^{(q)}(a)^{-1} = \mathbf{0}$, since the Expectation (2.1) tends to $\mathbf{0}$ when $a \to \infty$, therefore, from the above argument,

$$\lim_{x \to \infty} e^{\Lambda^q x} = \lim_{x \to \infty} W^{(q)}(x)^{-1}L^q(x) = \mathbf{0}.$$
The second scale matrix $Z^{(q)}(x)$ is then defined through the $W^{(q)}(x)$ matrix function:

$$Z^{(q)}(x) = I - \int_0^x W^{(q)}(y) dy \left( F(0) - qI \right).$$

Note that $Z^{(q)}(x)$ is continuous in $x$ with $Z^{(q)}(0) = I$. Furthermore,

$$\lim_{x \to \infty} e^{\Lambda x} Z^{(q)}(x) = \int_0^\infty e^{\Lambda z} dz \left( qI - F(0) \right).$$

**Remark 2.1** For the case without exponential killing $(q = 0)$, the upper subscript $q$ will be omitted in all above quantities, which write as $W(x), Z(x), L(x), \Lambda$, etc.

For more details of the scale matrices, we refer the reader to [14, 15].

**Definition 2.1** Let $\omega : E \times R \to R^+$ be a function defined as $\omega(i, x) = \omega_i(x)$, where for a fixed $i \in E$, $\omega_i : R \to R^+$ is a bounded, nonnegative measurable function and its value formulates the matrix $\omega(x) := \text{diag}(\omega_1(x), ..., \omega_N(x))$. Let $\lambda > 0$ be the upper bound of $|\omega_i(x)|$ on $[0, \infty)$ for all $i \in E$.

Our main interest of this paper is deriving closed-form formulas for the occupation times (up to some exit times), weighted by the $\omega$ function defined above. More specifically, for $d \leq x \leq c$ and $1 \leq i, j \leq N$, we are interested in the expectation matrices whose $(i, j)$-th elements are, respectively,

$$E_x \left[ e^{-\int_0^{\tau_d+} \omega_j(X_s) ds} \right. \left. \tau_d^+ < \tau_d^-, J_{\tau_d^+} = j, J_0 = i \right] \quad \text{and} \quad E_x \left[ e^{-\int_0^{\tau_d-} \omega_j(X_s) ds} \right. \left. \tau_d^- < \tau_d^+, J_{\tau_d^-} = j, J_0 = i \right].$$

Further discussions about applications with some particular $\omega$ will be presented in Section 5.

### 3 Main results

#### 3.1 Omega scale matrices

Before presenting our main results, we shall devote a little time to establishing some further notations. Our main aim is to represent fluctuation identities for MAPs with $\omega$-killing in the terms of new $\omega$-scale matrices defined as the unique solutions of the following equations:

$$\begin{align*}
W^{(\omega)}(x) &= W(x) + W \ast (\omega W^{(\omega)}) (x), \\
Z^{(\omega)}(x) &= I + W \ast (\omega Z^{(\omega)}) (x),
\end{align*}$$

where $f \ast g(x) = \int_0^x f(x-y)g(y)dy$ denotes the convolution of two matrix functions $f$ and $g$. The following lemma shows that the above $\omega$-scale matrices $W^{(\omega)}$ and $Z^{(\omega)}$ are well-defined and exist uniquely (see Appendix A for the proof).
Lemma 3.1 For every $i,j \in E$, let us assume that $h_{ij}$ is a locally bounded function and $\omega_i$ is a bounded function on $\mathbb{R}$. There exists an unique solution to the following equation:

\[(3.2) \quad H(x) = h(x) + W \ast (\omega H)(x), \]

where $H(x) = h(x)$ for $x < 0$. Furthermore, for any fixed $\delta > 0$, $H$ satisfies (3.2) if and only if $H$ satisfies:

\[(3.3) \quad H(x) = h_s(x) + W^s \ast ((\omega - \delta I)H)(x), \]

where $h_s(x) = h(x) + \delta W^s \ast h(x)$.

We further introduce more general scale matrices $W^{(\omega)}(x,y)$ and $Z^{(\omega)}(x,y)$ to allow shifting:

\[(3.4) \quad W^{(\omega)}(x,y) = W(x - y) + \int_y^x W(x - z)\omega(z)W^{(\omega)}(z,y)dz, \]

\[(3.5) \quad Z^{(\omega)}(x,y) = I + \int_y^x W(x - z)\omega(z)Z^{(\omega)}(z,y)dz. \]

Also note that $W^{(\omega)}(x,0) = W^{(\omega)}(x)$, $Z^{(\omega)}(x,0) = Z^{(\omega)}(x)$, as well as

\[(3.6) \quad W^{(\omega^*)}(x - y) = W^{(\omega)}(x,y), \quad \text{and} \quad Z^{(\omega^*)}(x - y) = Z^{(\omega)}(x,y), \]

with $\omega^*(\cdot, z) = \omega(\cdot, z + y)$, and

Since $W^{(\delta)} - W = \delta W^{(\delta)} \ast W$ and $Z^{(\delta)} - Z = \delta W^{(\delta)} \ast Z$, it is easy to check that

\[(3.7) \quad W^{(\omega)}(x,y) = W^{(\delta)}(x - y) + \int_y^x W^{(\delta)}(x - z)(\omega(z) - \delta I)W^{(\omega)}(z,y)dz, \]

\[(3.8) \quad Z^{(\omega)}(x,y) = Z^{(\delta)}(x - y) + \int_y^x W^{(\delta)}(x - z)(\omega(z) - \delta I)Z^{(\omega)}(z,y)dz. \]

To solve the one-sided upward problem (i.e., to get Corollary 3.1 (i)) we have to assume additionally that

\[(3.9) \quad \omega_i(x) \equiv \beta \geq 0, \quad \text{for all } x \leq 0 \text{ and } i \in E. \]

Hence we define a matrix function $\mathcal{H}^{(\omega)}(x)$ which satisfies the following integral equation

\[(3.10) \quad \mathcal{H}^{(\omega)}(x) = e^{-R^\delta x} + \int_0^x W^{(\beta)}(x - z)(\omega(z) - \beta I)\mathcal{H}^{(\omega)}(z)dz. \]

3.2 Exit problems and resolvents

In this section, we establish our main results of fluctuation identities and resolvents for spectrally negative $\omega$-killed MAPs. The proofs of the below theorems and corollary are postponed to Appendix B since the arguments tend to be technical, and the results intuitively hold in a similar manner as presented in [17].
Theorem 3.1 (Two-sided exit problem)
For invertible matrix function \( W^{(\omega)} \) and for \( Z^{(\omega)} \) given in (3.4) and (3.5) respectively, the following hold:

(i) For \( d \leq x \leq c \),
\[
A^{(\omega)}_d (x, c) := E_x \left[ e^{-\int_{\tau_0^-}^{\tau_0^+} \omega_s(X_s) ds} , \tau_0^+ < \tau_0^- , J_{\tau_0^+} | J_0 \right] = W^{(\omega)}(x, d) W^{(\omega)}(c, d)^{-1}.
\]

(ii) For \( d \leq x \leq c \),
\[
B^{(\omega)}_d (x, c) := E_x \left[ e^{-\int_{\tau_0^-}^{\tau_0^+} \omega_s(X_s) ds} , \tau_0^- < \tau_0^+ , J_{\tau_0^-} | J_0 \right] = Z^{(\omega)}(x, d) - W^{(\omega)}(x, d) W^{(\omega)}(c, d)^{-1} Z^{(\omega)}(c, d).
\]

Remark 3.1 When \( d = 0 \), we use simplified notations: \( A^{(\omega)}(x, c) := A^{(\omega)}_0 (x, c) \) and \( B^{(\omega)}(x, c) := B^{(\omega)}_0 (x, c) \).

Now, taking the limits \( d \to -\infty \) and \( c \to \infty \) (as well as \( d = 0 \)) in Theorem 3.1 (i) and (ii) respectively, we obtain the following corollary.

Corollary 3.1 (One-sided exit problem)

(i) Under the assumption (3.9), for \( x \leq c \),
\[
E_x \left[ e^{-\int_{\tau_0^-}^{\tau_0^+} \omega_s(X_s) ds} , \tau_0^+ < \infty , J_{\tau_0^+} | J_0 \right] = H^{(\omega)}(x) H^{(\omega)}(c)^{-1},
\]
for invertible matrix function \( H^{(\omega)} \) given in (3.10).

(ii) For \( x \geq 0 \) and \( \lambda > 0 \),
\[
E_x \left[ e^{-\int_{\tau_0^-}^{\tau_0^+} \omega_s(X_s) ds} , \tau_0^- < \infty , J_{\tau_0^-} | J_0 \right] = Z^{(\omega)}(x) - W^{(\omega)}(x) C W^{(\infty)}(c) Z^{(\omega)}(c),
\]
where matrix
\[
C W^{(\infty)}(c) Z^{(\omega)}(c) := \lim_{c \to \infty} W^{(\omega)}(c)^{-1} Z^{(\omega)}(c)
\]
exists and has finite entries.

Next, we present the representation of \( \omega \)-type resolvents.

Theorem 3.2 (Resolvents)

(i) For \( d \leq x \leq c \),
\[
U^{(\omega)}_{(d, c)} (x, dy) := \int_{0}^{\infty} E_x \left[ \exp \left( -\int_{0}^{t} \omega_s(X_s) ds \right) , X_t \in dy , t < \tau_d^- \land \tau_c^+ , J_t | J_0 \right] dt = \left( W^{(\omega)}(x, d) W^{(\omega)}(c, d)^{-1} W^{(\omega)}(c, y) - W^{(\omega)}(x, y) \right) dy.
\]
(ii) For $x \geq 0$ and $\lambda > 0$ ,

$$U^{(\omega)}_{(0,\infty)}(x,dy) := \int_{0}^{\infty} \mathbb{E}_x \left[ \exp \left( - \int_{0}^{t} \omega_{J_s}(X_s) ds \right) , X_t \in dy, t < \tau_0^- , J_t | J_0 \right] dt$$

$$= (W^{(\omega)}(x)C_{W(\infty)^{-1}W(\infty)}(y) - W^{(\omega)}(x,y)) dy,$$

where

$$C_{W(\infty)^{-1}W(\infty)}(y) := \lim_{c \to \infty} W^{(\omega)}(c)^{-1}W^{(\omega)}(c,y)$$

is well defined and finite matrix.

(iii) For $x,y \leq c$,

$$U^{(\omega)}_{(-\infty,c)}(x,dy) := \int_{0}^{\infty} \mathbb{E}_x \left[ \exp \left( - \int_{0}^{t} \omega_{J_s}(X_s) ds \right) , X_t \in dy, t < \tau_c^+ , J_t | J_0 \right] dt$$

$$= (H^{(\omega)}(x)H^{(\omega)}(c)^{-1}W^{(\omega)}(c,y) - W^{(\omega)}(x,y)) dy.$$

(iv) For $x \in \mathbb{R}$,

$$U^{(\omega)}_{(-\infty,\infty)}(x,dy) := \int_{0}^{\infty} \mathbb{E}_x \left[ \exp \left( - \int_{0}^{t} \omega_{J_s}(X_s) ds \right) , X_t \in dy, J_t | J_0 \right] dt$$

$$= (H^{(\omega)}(x)C_{H(\infty)^{-1}W(\infty)}(y) - W^{(\omega)}(x,y)) dy,$$

where matrix $C_{H(\infty)^{-1}W(\infty)} = \lim_{c \to \infty} H^{(\omega)}(c)^{-1}W^{(\omega)}(c,y)$ exists and has finite entries.

4 Dividends in the Omega ruin model

In this section, we present one application of the previously obtained results on dividend problem. The optimal dividend problem is very popular in the field of applied mathematics. De Finetti [9] was the first who introduced the dividend model in risk theory. He proposed the model in which company’s surplus is described by random walk with increments $\pm 1$. In his work, it was proved that, under the rule of maximization of expected discounted dividends before the classical ruin occurs (the surplus reaches below level 0), the optimal strategy is the so-called barrier strategy which is described as follows. For a fixed level $c > 0$, whenever the surplus process reaches this level, one reflects the process and pays all funds above $c$ as dividends. In the literature, there is a rich set of articles in which this problem was studied in the continuous time; see, e.g., Loeffen [18], Loeffen and Renaud [19] and Avram et al. [2] where the value function of the barrier strategy and the optimal barrier level was described in the terms of the scale functions.

In this paper, we assume that the company’s reserve process is governed by a Markov additive process $(X,J)$. Moreover, we assume that this company pays dividends according to the barrier strategy until omega ruin time defined in the following way. Fix an exponential random variable $e_1$ (with mean 1) and level $-d \leq 0$, and then omega ruin time is defined as

$$\tau_{\omega}^d = \inf\{t \geq 0 : \int_{0}^{t} \omega_{J_s}(X_s) dx > e_1 \text{ or } X_t < -d \},$$
where, for all \( i \in E, \omega_i(x) \geq 0 \) for \( x \geq -d \) and \( \omega_i(x) = 0 \) when \( x < -d \). Thus ruin can occur in two situations. The first is the situation in which the process crosses a fixed level \(-d \leq 0\) (for \( d = 0 \) we have a case of classical ruin time). The second possibility is when bankruptcy happens in the so-called red zone and the intensity of this bankruptcy is a function of current level of the additive component \( X \) and the Markov chain \( J \). For more details related to this omega ruin time, we refer to [10] and [17].

Immediately from the definition of \( \tau^d_\omega \), one can conclude that

\[
P_x(\tau^d_\omega > t) = \mathbb{E}_x \left[ e^{-\int_0^{\tau^d_\omega} \omega_j(X_s) ds}, \tau^d_\omega > t \right].
\]

We denote dividend barrier strategy (at \( c \)) \( \pi^c \) as follows

\[
\pi^c = \{ L^c_s : t \geq 0 \},
\]

which is a non-decreasing, left-continuous \( \mathcal{F} \)-adapted process starting at zero. Random variable \( L^c_t \) can be interpreted as the cumulative dividends paid up to time \( t \). In the case of the barrier strategy, we have

\[
L^c_\tau = \sup_{s \leq \tau} [X_s - c] \vee 0.
\]

In the following theorem, we set \( d = 0 \) and then consider the general \( d \) in the corollary.

**Theorem 4.1** Assume that dividends are discounted at a constant force of interest \( \delta > 0 \) and \( d = 0 \). The expected discounted present value of the dividends paid before omega ruin \((\tau_\omega := \tau^0_\omega) \) under a constant dividend barrier \( c \) is given by

\[
v_c(x) := \mathbb{E}_x \left[ \int_0^{\tau_\omega} e^{-\delta t} dL_t, J_\tau, J_0 \right] = \begin{cases} W^{(\delta+\omega)}(x)W^{(\delta+\omega)}(c)^{-1}, & \text{for } 0 < x \leq c, \\ (x - c) + W^{(\delta+\omega)}(c)W^{(\delta+\omega)}(c)^{-1}, & \text{for } x > c, \end{cases}
\]

for the invertible matrix function:

\[
W^{(\delta+\omega)}(c) = W'(c) + \int_0^c W'(c - y)(\omega(y) + \delta I)W^{(\delta+\omega)}(y)dy + W(0)(\omega(c) + \delta I)W^{(\delta+\omega)}(c).
\]

**Proof.** At the beginning we will treat the case of \( 0 < x \leq c \). Conditioning on reaching the level \( c \) first, we have

\[
v_c(x) = A^{(\omega)}(x, c)v_c(c) = W^{(\delta+\omega)}(x)W^{(\delta+\omega)}(c)^{-1}v_c(c).
\]

As a first step we will find a lower bound for \( v_c(c) \). For \( m \in \mathbb{N}, \) consider that the dividend is not paid until reaching the level \( c + \frac{1}{m} \):

\[
v_c(c) \geq \mathbb{E}_c \left[ e^{-\int_0^{\tau^c_\omega} (\delta+\omega)ds}, \tau^c_\omega < \tau_0, J_{\tau^c_\omega}, J_0 \right] v_c \left( c + \frac{1}{m} \right)
\]

\[
= \mathbb{E}_c \left[ e^{-\int_0^{\tau^c_\omega} (\delta+\omega)ds}, \tau^c_\omega < \tau_0, J_{\tau^c_\omega}, J_0 \right] \left( v_c(c) + \frac{1}{m} \right),
\]

\[
= \mathbb{E}_c \left[ e^{-\int_0^{\tau^c_\omega} (\delta+\omega)ds}, \tau^c_\omega < \tau_0, J_{\tau^c_\omega}, J_0 \right] v_c \left( c + \frac{1}{m} \right) + \frac{1}{m}.
\]

8
where the last equality is due to the dividend of $\frac{1}{m}$ paid immediately and the fact that the drop in surplus will not cause the state transition.

On the other hand, an upper bound can be found as

\[
\mathbf{v}_c(c) \leq \mathbb{E}_c \left[ e^{-\int_0^{\tau^+_{c+1/m}} (\delta + \omega J_\omega(X_s)) ds}, \tau^+_{c+1/m} < \tau_0, J_{\tau_0^+} | J_0 \right] \left( \mathbf{v}_c(c) + \frac{1}{m} \mathbf{I} \right)
+ \frac{1}{m} \mathbb{E}_c \left[ \int_0^{\tau^+_{c+1/m}} e^{-\delta t} dt \ e^{-\int_0^{\tau^+_{c+1/m}} \omega J_\omega(X_s) ds}, \tau^+_{c+1/m} < \tau_0, J_{\tau_0^+} | J_0 \right] 
+ \mathbb{E}_c \left[ \int_0^{\tau_\omega} e^{-\delta t} dL_t^c, \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega} | J_0 \right],
\]

where $L_t^c$ will be bounded by $\frac{1}{m}$ for the process starting from level $c$ to level $c + \frac{1}{m}$, i.e.,

\[
\mathbb{E}_c \left[ \int_0^{\tau_\omega} e^{-\delta t} dL_t^c, \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega} | J_0 \right] \leq \frac{1}{m} \mathbb{P}_c \left( \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega} | J_0 \right).
\]

Note that as $m \to \infty$, the following two limits approach to 0:

\[
\lim_{m \to \infty} \mathbb{E}_c \left[ \int_0^{\tau^+_{c+1/m}} e^{-\delta t} dt \ e^{-\int_0^{\tau^+_{c+1/m}} \omega J_\omega(X_s) ds}, \tau^+_{c+1/m} < \tau_0, J_{\tau_0^+} | J_0 \right] = 0,
\]

and

\[
\lim_{m \to \infty} \mathbb{P}_c \left( \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega} | J_0 \right) = 0.
\]

See Renaud and Zhou [20] and Czarna et al. [8] for more details.

Therefore, by the upper and lower bounds,

\[
\mathbf{v}_c(c) = \mathbb{E}_c \left[ e^{-\int_0^{\tau^+_{c+1/m}} (\delta + \omega J_\omega(X_s)) ds}, \tau^+_{c+1/m} < \tau_0, J_{\tau_0^+} | J_0 \right] \left( \mathbf{v}_c(c) + \frac{1}{m} \mathbf{I} \right) + o \left( \frac{1}{m} \right)
= \mathcal{W}^{(\delta + \omega)}(c) \mathcal{W}^{(\delta + \omega)}(c + \frac{1}{m})^{-1} \left( \mathbf{v}_c(c) + \frac{1}{m} \mathbf{I} \right) + o \left( \frac{1}{m} \right),
\]

and hence

\[
\left( \mathbf{I} - \mathcal{W}^{(\delta + \omega)}(c) \mathcal{W}^{(\delta + \omega)}(c + \frac{1}{m})^{-1} \right) \mathbf{v}_c(c) = \frac{1}{m} \mathcal{W}^{(\delta + \omega)}(c) \mathcal{W}^{(\delta + \omega)}(c + \frac{1}{m})^{-1} + o \left( \frac{1}{m} \right),
\]

\[
\frac{1}{m} \left( \mathcal{W}^{(\delta + \omega)}(c + \frac{1}{m}) \mathcal{W}^{(\delta + \omega)}(c) - \mathbf{I} \right) \mathbf{v}_c(c) = \mathbf{I} + o \left( \frac{1}{m} \right),
\]

\[
\left( \mathcal{W}^{(\delta + \omega)}(c + \frac{1}{m}) - \mathcal{W}^{(\delta + \omega)}(c) \right) \mathcal{W}^{(\delta + \omega)}(c)^{-1} \mathbf{v}_c(c) = \mathbf{I} + o \left( \frac{1}{m} \right).
\]

Letting $m \to \infty$, it turns out

\[
\mathcal{W}^{(\delta + \omega)}(c) \mathcal{W}^{(\delta + \omega)}(c)^{-1} \mathbf{v}_c(c) = \mathbf{I},
\]
where matrix
\[
W^{(\delta+\omega)'}(c) = W'(c) + \int_0^c W'(c - y)(\omega(y) + \delta I)W^{(\delta+\omega)}(y)dy + W(0)(\omega(c) + \delta I)W^{(\delta+\omega)}(c)
\]
is well-defined since the scale matrix $W$ is almost everywhere differentiable, see [10]. Furthermore, one can observe that, from representation (B.4), the above matrix is invertible for any $c > 0$ and then $v_c(c) = W^{(\delta+\omega)}(c)W^{(\delta+\omega)'}(c)^{-1}$.

To end this proof, note that for $x > c$, one is immediately paying dividend of size $x - c$ (and this will not cause the state transition), therefore
\[
v_c(x) = (x - c) + v_c(c) = (x - c) + W^{(\delta+\omega)}(c)W^{(\delta+\omega)'}(c)^{-1}.
\]

□

Making use of Theorem 4.1 and the shifting argument, we can state the representation for value function for a general $d \geq 0$.

**Corollary 4.1** For $\delta > 0$, the expected present value of the dividend paid before omega ruin $(\tau_d^\omega)$ under a constant dividend barrier $c$ is
\[
v_c^d(x) := \mathbb{E}_x \left[ \int_0^{\tau_d^\omega} e^{-\delta t} dL_t, J_{\tau_d^\omega} \right] = \begin{cases} W^{(\delta+\omega)}(x, -d)W^{(\delta+\omega)'}(c, -d)^{-1} & \text{for } -d < x \leq c, \\
(x - c) + W^{(\delta+\omega)}(c, -d)W^{(\delta+\omega)'}(c, -d)^{-1} & \text{for } x > c.
\end{cases}
\]

for invertible matrix:
\[
W^{(\delta+\omega)'}(c, -d) = W'(c + d) + \int_{-d}^c W'(c - y)(\omega(y) + \delta I)W^{(\delta+\omega)}(y, -d)dy
+ W(0)(\omega(c) + \delta I)W^{(\delta+\omega)}(c, -d).
\]

### 5 Examples

The aim of this section is to give examples of $\omega$-scale matrices when the $\omega$ function is specified. We would like to present relations between $W^{(\omega)}$ and $W^{(q)}$, for some $q \geq 0$, as well as numerical examples which help to understand better the nature of explored matrix-valued functions. We will start with short analyse of $W^{(q)}$ for Markov modulated Brownian motion, since this model will be a base for more complicated scale matrices.

#### 5.1 Markov modulated Brownian motion

In this part, we will consider a special case when $(X, J)$ is a Markov modulated Brownian motion. Our aim is to derive some relations which will be useful in the subsequent examples. Let $X_i$ be a Brownian motion with variance $\sigma_i^2 > 0$ and drift $\mu_i$ for all $i \in E$. Further denote $\sigma$ and $\mu$ as the (column) vectors of $\sigma_i$ and $\mu_i$, and $\Delta_v$ as the diagonal matrix with $v$ on the diagonal. Therefore, the matrix Laplace exponent $F(s)$ is given by
\[
F(s) = \frac{1}{2} \Delta_v^2 s^2 + \Delta_\mu s + Q.
\]
Despite the case when $\kappa := \pi \mu = 0$ and $q = 0$, Ivanovs [11] gives the representation of the $q$-scale matrix

\begin{equation}
W^{(q)}(x) = \left( e^{-\Lambda^+_q x} - e^{\Lambda^-_q x} \right) \Xi_q,
\end{equation}

where $\Xi^{-1}_q = -\frac{1}{2} \Delta^2_q (\Lambda^+_q + \Lambda^-_q)$ and $\Lambda^\pm_q$ are the (unique) right solutions to the matrix integral equation $F(\mp \Lambda^\pm_q) = 0$, that is,

\begin{equation}
\Delta_{x^2} (\Lambda^\pm_q)^2 \mp \Delta \mu \Lambda^\pm_q + (Q - qI) = 0.
\end{equation}

In the next lemma, we present relations between $\Lambda^+_q$ and $\Lambda^-_q$.

**Lemma 5.1** For $q \geq 0$, we have

\begin{equation}
\Delta_{x^2} \left((\Lambda^+_q)^2 - (\Lambda^-_q)^2\right) = \Delta \mu \left(\Lambda^+_q + \Lambda^-_q\right),
\end{equation}

and

\begin{equation}
\Delta_{x^2} \left((\Lambda^+_q)^2 - (\Lambda^-_q)^2\right) = \Delta \mu \left(\Lambda^+_q + \Lambda^-_q\right) \left(\Lambda^+_q - \Lambda^-_q\right)^{-1},
\end{equation}

where

\begin{equation}
C_q = (\Lambda^+_q + \Lambda^-_q) \Lambda^-_q (\Lambda^+_q + \Lambda^-_q)^{-1}, \quad D_q = \left(\Lambda^+_q + \Lambda^-_q\right) \Lambda^+_q \left(\Lambda^+_q + \Lambda^-_q\right)^{-1}.
\end{equation}

**Proof.**

Using equations (5.2) altogether, one can obtain

\[
\Delta_{x^2} \left((\Lambda^+_q)^2 - (\Lambda^-_q)^2\right) = \Delta \mu \left(\Lambda^+_q + \Lambda^-_q\right),
\]

hence,

\[
\Delta_{x^2} = \left((\Lambda^+_q)^2 - (\Lambda^-_q)^2\right) \left(\Lambda^+_q + \Lambda^-_q\right)^{-1}
= \left(\Lambda^+_q (\Lambda^+_q + \Lambda^-_q) - (\Lambda^+_q + \Lambda^-_q) \Lambda^-_q\right) \left(\Lambda^+_q + \Lambda^-_q\right)^{-1}
= \Lambda^+_q - C_q.
\]

Now, the above relationship together with (5.2) gives that:

\[
C_q \Lambda^+_q = \Delta_{x^2} \left[-Q + qI\right].
\]

The remaining part of the proof can be done in a similar way by using

\[
(\Lambda^+_q)^2 - (\Lambda^-_q)^2 = (\Lambda^+_q + \Lambda^-_q) \Lambda^+_q - \Lambda^-_q (\Lambda^+_q + \Lambda^-_q).
\]

In the special case of $q = 0$ we will write $\Lambda^+, \Lambda^-, C$ and $D$ for $\Lambda^+_0, \Lambda^-_0, C_0$ and $D_0$, respectively.\[\square\]
Note that if \((X, J)\) is a MMBM with one single state (i.e., one dimensional Brownian motion), we have, for \(q \geq 0\),

\[
\Lambda^+_q = -\rho_2, \quad \Lambda^-_q = -\rho_1,
\]

where \(\rho_1 - \rho_2 = \frac{2\mu}{\sigma^2}\) and \(\rho_1 + \rho_2 = \frac{2\sqrt{\mu^2 + 2q^2}}{\sigma^2}\). In general, for the MMBM, we can only calculate explicit analytical formulas for \(W(q)(x)\), \(\Lambda^+_q\), and \(\Lambda^-_q\) for some special cases. For instance, consider the following parameters

\[
\begin{align*}
\Delta_\sigma &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\
\Delta_\mu &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
Q &= \begin{pmatrix} -q_{11} & q_{11} \\ q_{22} & -q_{22} \end{pmatrix}
\end{align*}
\]

and \(q > 0\), for \(\sigma_1, \sigma_2, q_{11}, q_{22} \in \mathbb{R}_+\). Then the matrix \(F(s) - qI\) is of the form

\[
F(s) - qI = \begin{pmatrix} \frac{\sigma_1^2}{2} s^2 - q_{11} - q & q_{11} \\ q_{22} & \frac{\sigma_2^2}{2} s^2 - q_{22} - q \end{pmatrix}.
\]

Thus,

\[
(F(s) - qI)^{-1} = \frac{1}{\left(\frac{\sigma_1^2}{2} s^2 - q_{11} - q\right)\left(\frac{\sigma_2^2}{2} s^2 - q_{22} - q\right) - q_{11}q_{22}} \begin{pmatrix} \frac{\sigma_1^2}{2} s^2 - q_{22} - q & -q_{11} \\ -q_{22} & \frac{\sigma_2^2}{2} s^2 - q_{11} - q \end{pmatrix}.
\]

Inversion of the Laplace transform \((2.2)\) with respect to \(s\) gives:

\[
W(q)(x) = \left(\begin{array}{cc} 2(q_{22} + q) - \alpha_2^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + q) - \alpha_2^2 \sigma_1^2 \end{array}\right) \frac{e^{\alpha_2 x} - e^{-\alpha_2 x}}{(\alpha_1^2 - \alpha_2^2)\alpha_2 \sigma_1^2 \sigma_2^2} - \left(\begin{array}{cc} 2(q_{22} + q) - \alpha_1^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + q) - \alpha_2^2 \sigma_1^2 \end{array}\right) \frac{e^{\alpha_1 x} - e^{-\alpha_1 x}}{(\alpha_1^2 - \alpha_2^2)\alpha_1 \sigma_1^2 \sigma_2^2},
\]

where

\[
\begin{align*}
\alpha_1 &= \sqrt{M_q + \sqrt{\left(M_q\right)^2 - 4\sigma_1^2 \sigma_2^2 K_q}}, \\
\alpha_2 &= \sqrt{M_q - \sqrt{\left(M_q\right)^2 - 4\sigma_1^2 \sigma_2^2 K_q}}, \\
M_q &= \sigma_1^2 (q_{22} + q) + \sigma_2^2 (q_{11} + q), \\
K_q &= (q_{11} + q_{22} + q) q.
\end{align*}
\]

It is straightforward that

\[
W(q)'(x) = \left(\begin{array}{cc} 2(q_{22} + q) - \alpha_2^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + q) - \alpha_2^2 \sigma_1^2 \end{array}\right) \frac{e^{\alpha_2 x} + e^{-\alpha_2 x}}{(\alpha_1^2 - \alpha_2^2)\sigma_1^2 \sigma_2^2} - \left(\begin{array}{cc} 2(q_{22} + q) - \alpha_1^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + q) - \alpha_2^2 \sigma_1^2 \end{array}\right) \frac{e^{\alpha_1 x} + e^{-\alpha_1 x}}{(\alpha_1^2 - \alpha_2^2)\sigma_1^2 \sigma_2^2}.
\]
Our last step is to derive the formulas for $\Lambda^+_q$ and $\Lambda^-_q$. First, note that $\Lambda^+_q = \Lambda^-_q$ due to the assumption of $\mu_1 = \mu_2 = 0$ and equation (5.2). Then (5.3) becomes

$$(\Lambda^+_q)^2 = \Delta_2 \sigma^2 \left[-Q + qI\right].$$

Since $-\alpha_1$ and $-\alpha_2$ are eigenvalues of $\Lambda^+_q$, thus after some basic algebra, we get that

$$\Lambda^+_q = \Lambda^-_q = \begin{pmatrix}
-\sqrt{2\sigma^2(\alpha_1+\alpha_2)}(q_{11}+q) - 4q_{11}q_{22} & \frac{2q_{11}}{\sigma_1} \\
\frac{2q_{22}}{\sigma_2} & -\sqrt{2\sigma^2(\alpha_1+\alpha_2)}(q_{22}+q) - 4q_{11}q_{22}
\end{pmatrix} \frac{1}{\alpha_1 + \alpha_2}.$$

Finally, we will provide a graphical example of the scale matrix. Consider the following setting of the parameters

$$\Delta_\sigma = \begin{pmatrix} 1 & 0 \\
0 & 1.2
\end{pmatrix}, \quad \Delta_\mu = \begin{pmatrix} 0 & 0 \\
0 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix} -0.05 & 0.05 \\
0.1 & -0.1
\end{pmatrix}, \quad \text{and} \quad q = 0.05.$$

Using the formula (5.5), the scale matrix $W^{(q)}$ is plotted in Figure 1. We can see that the diagonal cells of this matrix have the same shape as the one dimensional scale functions, where off-diagonal ones are reflected in shape. In the subsequent examples, we will provide plots of omega-matrices to compare them to these traditional ones.
5.2 Constant state-dependent discount rates

Consider the special case where \( \omega_i(x) \equiv \omega_i \) is a constant for all \( x \in \mathbb{R} \) and \( i \in E \). Therefore, the discounting structure depends on the state of the chain \( J \) only. Before calculating \( \omega \)-scale matrix, let us state the following proposition.

**Proposition 5.1** Let \( \omega_i(x) \equiv \omega_i \) for all \( x \in \mathbb{R} \) and \( i \in E \). The \( \omega \)-scale matrix has the Laplace transform

\[
\tilde{W}^{(\omega)}(s) = (F(s) - \omega)^{-1}.
\]

**Proof.** Taking the Laplace transform on both sides of (3.1), we have

\[
\tilde{W}^{(\omega)}(s) = \tilde{W}(s) + \tilde{W}(s)\omega\tilde{W}^{(\omega)}(s),
\]

which gives

\[
\tilde{W}^{(\omega)}(s) = \left( I - \tilde{W}(s)\omega \right)^{-1}\tilde{W}(s) = (F(s) - \omega)^{-1}.
\]

\[\Box\]

As an example of such \( \omega \)-scale matrix, we take again the model of Markov modulated Brownian motion with the following parameters: \( \omega_1(x) = \omega_1 \), \( \omega_2(x) = \omega_2 \),

\[
\Delta_\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Delta_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} -q_{11} & q_{11} \\ q_{22} & -q_{22} \end{pmatrix}.
\]

Using the same method as in the previous subsection, we will obtain analytical formula for the \( \omega \)-killed matrix. Taking the inverse of \( F(s) - \omega \), one has

\[
(F(s) - \omega)^{-1} = \frac{1}{(\sigma_1^2 s^2 - q_{11} - \omega_1)(\sigma_2^2 s^2 - q_{22} - \omega_2)} \begin{pmatrix} \sigma_1^2 s^2 - q_{22} - \omega_2 & -q_{11} \\ -q_{22} & \sigma_2^2 s^2 - q_{11} - \omega_1 \end{pmatrix},
\]

whose Laplace inversion gives

\[
W^{(\omega)}(x) = \begin{pmatrix} 2(q_{22} + \omega_2) - \alpha_2^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + \omega_1) - \alpha_2^2 \sigma_1^2 \end{pmatrix} \frac{e^{\alpha_2 x} - e^{-\alpha_2 x}}{(\alpha_1^2 - \alpha_2^2) \alpha_2 \sigma_1^2 \sigma_2^2} - \begin{pmatrix} 2(q_{22} + q_2) - \alpha_1^2 \sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11} + \delta_1) - \alpha_1^2 \sigma_1^2 \end{pmatrix} \frac{e^{\alpha_1 x} - e^{-\alpha_1 x}}{(\alpha_1^2 - \alpha_2^2) \alpha_1 \sigma_1^2 \sigma_2^2},
\]

where

\[
\alpha_1 = \sqrt{M_\omega + \sqrt{(M_\omega)^2 - 4\sigma_1^2 \sigma_2^2 K_\omega}}, \quad \alpha_2 = \sqrt{M_\omega - \sqrt{(M_\omega)^2 - 4\sigma_1^2 \sigma_2^2 K_\omega}},
\]

\[
M_\omega = \sigma_1^2 (q_{22} + \omega_2) + \sigma_2^2 (q_{11} + \omega_1), \quad K_\omega = q_{11} \omega_2 + \omega_1 q_{22} + \omega_1 \omega_2.
\]

Note that, for \( \omega_1 = \omega_2 = q \), the result is consistent with the previous result for the \( (q) \)-scale matrix \( W^{(q)} \) in (5.5). Now, consider the following setting of the parameters

\[
\Delta_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1.2 \end{pmatrix}, \quad \Delta_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} -0.05 & 0.05 \\ 0.1 & -0.1 \end{pmatrix}, \quad \omega_1(x) = 0.05, \quad \omega_2(x) = 0.25,
\]

\[\omega \]
which results in the plots of $\omega$-scale matrix in Figure 2. In Figure 2 one can observe that $\omega$-scale matrix has similar shape as $W^{(q)}$.

![Figure 2: Entries of $\omega$-scale matrix function $W^{(\omega)}$ for state-dependent $\omega$ function](image)

### 5.3 Step $\omega$-scale matrix

In this example, we consider omega function as a positive step function which depends only on the position of the process $X$. Such an assumption is motivated by the situation where the company has the discounting structure depending on its current financial status. Li and Palmowski [17] showed that, in the case of spectrally negative Lévy processes, such $\omega$-scale functions have recurrent nature. Same observation holds true for MAPs.

**Proposition 5.2** Assume that omega function is of the form

$$\omega(i, x) := \omega(x) = p_0 + \sum_{j=1}^{n} (p_j - p_{j-1})1_{\{x > x_j\}}, \quad \text{for all } i \in E,$$

where $n \in \mathbb{N}$, $\{p_j\}_{j=0}^{n}$ is a fixed sequence and $\{x_j\}_{j=1}^{n}$ is an increasing sequence dividing $\mathbb{R}$ into $(n + 1)$ parts. Then the omega matrix $W^{(\omega)}(x, y)$ satisfies

$$W^{(\omega)}(x, y) = W^{(\omega)}_n(x, y),$$

for $x > y$, where $W^{(\omega)}_n(x, y)$ is defined recursively as follows:

$$W^{(\omega)}_0(x, y) = W^{(p_0)}(x - y),$$
and

\[ W^{(\omega)}_{k+1}(x, y) = W^{(\omega)}_k(x, y) + (p_{k+1} - p_k) \int_{x_{k+1}}^x W^{(p_{k+1})}(x - z) W^{(\omega)}_k(z, y) dz, \]

for \( x > x_{k+1} \) and \( k = 0, 1, \ldots, n - 1 \).

**Proof.**
Denote \( \omega^{(k)}(x) := p_0 + \sum_{j=1}^k (p_j - p_{j-1}) 1_{\{x > x_j\}} \) with \( \omega^{(0)}(x) = p_0 \). From Equation (3.7), we get that

\begin{align*}
W^{(\omega)}_k(x, y) &= W^{(p_{k+1})}(x - y) + \int_y^x (\omega^{(k)}(z) - p_{k+1}) W^{(p_{k+1})}(x - z) W^{(\omega)}_k(z, y) dz, \\
W^{(\omega)}_{k+1}(x, y) &= W^{(p_{k+1})}(x - y) + \int_y^x (\omega^{(k+1)}(z) - p_{k+1}) W^{(p_{k+1})}(x - z) W^{(\omega)}_{k+1}(z, y) dz.
\end{align*}

Note that \( \omega^{(k+1)}(z) - p_{k+1} = 0 \) for \( z > x_{k+1} \) and \( \omega^{(k+1)}(z) = \omega^{(k)}(z) \) for \( z \leq x_{k+1} \). Thus from Lemma 3.1, we have

\[ W^{(\omega)}_{k+1}(x, y) = W^{(\omega)}_k(x, y), \]

for \( x \leq x_{k+1} \). Equation (5.7) could be rewritten as

\[ W^{(\omega)}_{k+1}(x, y) = W^{(p_{k+1})}(x - y) + \int_y^x (\omega^{(k+1)}(z) - p_{k+1}) W^{(p_{k+1})}(x - z) W^{(\omega)}_{k+1}(z, y) dz = W^{(p_{k+1})}(x - y) + \int_y^x (\omega^{(k)}(z) - p_{k+1}) W^{(p_{k+1})}(x - z) W^{(\omega)}_{k+1}(z, y) dz = W^{(\omega)}_k(x, y) - \int_{x_k}^x (\omega^{(k)}(z) - p_{k+1}) W^{(p_{k+1})}(x - z) W^{(\omega)}_k(z, y) dz, \]

where the last step uses (5.6). The proof is completed by noticing that \( \omega^{(k)}(z) - p_{k+1} = p_k - p_{k+1} \) for \( z > x_{k+1} \).

Note also that the similar considerations will lead to the same result for the second \( \omega \)-scale matrix \( Z^{(\omega)} \).

In the next proposition, we will compute the matrix \( W^{(\omega)} \) for one particular case.

**Proposition 5.3** Let \( (X, J) \) be a Markov modulated Brownian motion with \( \mu_i \in \mathbb{R} \) and \( \sigma_i^2 > 0 \) for all \( i \in E \). Assume that \( n = 1 \) \( \{p_j\}_{j=0}^n = \{p_0, p_1\} \) and \( \{x_j\}_{j=1}^n = \{x_1\} \) with \( p_0, p_1, x_1 \) being positive numbers. Then for \( x \leq x_1 \),

\[ W^{(\omega)}(x, y) = W^{(p_0)}(x - y), \]

and for \( x > x_1 \),

\[ W^{(\omega)}(x, y) = \left( e^{-\Lambda^+_1(x-x_1)} (\Lambda^+_{p_1} + \Lambda^-_{p_1})^{-1} \Lambda^-_{p_1} + e^{\Lambda^-_1(x-x_1)} (\Lambda^+_{p_1} + \Lambda^-_{p_1})^{-1} \Lambda^+_{p_1} \right) W^{(p_0)}(x_1 - y) - W^{(p_1)}(x - x_1) \Delta \sigma^2 \cdot W^{(p_0)}(x_1 - y). \]
Proof. Note that the case when \( x \leq x_1 \) is a straightforward conclusion from Proposition 5.2. For \( x > x_1 \), from previous proposition and (5.1), we have

\[
\mathcal{W}_1(x, y) = \mathcal{W}_0(x, y) + (p_1 - p_0) \int_{x_1}^{x} \mathcal{W}^{(p_1)}(x - z) \mathcal{W}_0(z, y) dz
\]

(5.8) \[
= \mathcal{W}^{(p_0)}(x - y) + (p_1 - p_0) \int_{x_1}^{x} \left( e^{-\Lambda^+_{p_1}(x-z)} \Xi_{p_1} e^{-\Lambda^-_{p_0}(z-y)} - e^{-\Lambda^+_{p_1}(x-z)} \Xi_{p_0} e^{-\Lambda^+_{p_0}(z-y)} + e^{\Lambda^-_{p_1}(x-z)} \Xi_{p_1} e^{\Lambda^-_{p_0}(z-y)} \right) dz \Xi_{p_0}.
\]

We start from identifying the following integral appearing in Equation (5.8):

\[
\int_{x_1}^{x} \left( e^{-\Lambda^+_{p_1}(x-z)} \Xi_{p_1} e^{-\Lambda^-_{p_0}(z-y)} \right) dz.
\]

(5.9) Consider (5.9) as a function \( M_1 : A \rightarrow \mathbb{R}^{N \times N} \), where

\[
A = \{(x, y) : x \geq x_1, x > y\},
\]

and \( N \) is the size of the matrix \( \mathcal{W}^{(p_0)} \). Then

\[
M_1(x, y) = \int_{x_1}^{x} \left( e^{-\Lambda^+_{p_1}(x-z)} \Xi_{p_1} e^{-\Lambda^-_{p_0}(z-y)} \right) dz = e^{-\Lambda^+_{p_1}x} \int_{x_1}^{x} \left( e^{\Lambda^-_{p_1}z} \Xi_{p_1} e^{-\Lambda^-_{p_0}z} \right) dz e^{\Lambda^-_{p_0}y}.
\]

Let

\[
K_1(x) := M_1(x, x_1) = \int_{x_1}^{x} \left( e^{-\Lambda^+_{p_1}(x-z)} \Xi_{p_1} e^{-\Lambda^-_{p_0}(z-x_1)} \right) dz.
\]

The derivative of \( K_1(x) \) equals

\[
(5.10) \quad K'_1(x) = -\Lambda^+_{p_1} K_1(x) + \Xi_{p_1} e^{-\Lambda^-_{p_0}(x-x_1)} \quad \text{with the boundary condition} \quad K_1(x_1) = 0.
\]

We will prove that the solution of above differential equation is of the form

\[
(5.11) \quad K_1(x) = C e^{-\Lambda^-_{p_0}(x-x_1)} - e^{-\Lambda^+_{p_1}(x-x_1)} C,
\]

where \( C \) is some constant matrix. To do this, we need to put our proposition for \( K_1(x) \) into (5.10) and after some calculation we get that (5.11) is indeed our solution if the following equation holds true:

\[
(5.12) \quad \Lambda^+_{p_1} C - C \Lambda^+_{p_0} = \Xi_{p_1}.
\]

The above equality is an example of well known Sylvester equation. Usually to solve equations of this type we must use numerical methods, however in this case we can guess the formula for \( C \):

\[
C = -\left( \Lambda^+_{p_1} + \Lambda^-_{p_1} \right)^{-1} \left( \Lambda^+_{p_0} + \Lambda^-_{p_0} \right)^{-1} \cdot \frac{1}{p_1 - p_0}.
\]
We need to check if such formula for $C$ is indeed correct. Therefore, from equation (5.12) one can get that

$$-\Lambda_{p_i}^+ \left( \Lambda_{p_i}^+ + \Lambda_{p_i}^- \right)^{-1} \left( \Lambda_{p_0}^+ + \Lambda_{p_1}^- \right) \cdot \frac{1}{p_1 - p_0} + \left( \Lambda_{p_1}^+ + \Lambda_{p_1}^- \right)^{-1} \left( \Lambda_{p_0}^+ + \Lambda_{p_1}^- \right) \cdot \frac{1}{p_1 - p_0} \Lambda_{p_0}^+ = \Xi_{p_1}$$

$\left[ \left( \Lambda_{p_1}^+ + \Lambda_{p_1}^- \right) \Lambda_{p_1}^+ \left( \Lambda_{p_1}^+ + \Lambda_{p_1}^- \right)^{-1} \left( \Lambda_{p_0}^+ + \Lambda_{p_1}^- \right) - \left( \Lambda_{p_0}^+ + \Lambda_{p_1}^- \right) \Lambda_{p_0}^+ \right] \cdot \frac{1}{p_1 - p_0} = \Delta \frac{I}{\sigma^2}$

$\left[ \Delta \frac{2 \sigma}{\sigma^2} \Lambda_{p_0}^+ + \Delta \frac{2 \sigma}{\sigma^2} \Lambda_{p_0}^- + (\Lambda_{p_0}^-)^2 - (\Lambda_{p_0}^+)^2 \right] \cdot \frac{1}{p_1 - p_0} = \Delta \frac{I}{\sigma^2}$

$$\left[ (Q - p_0 \mathbf{1}) - (Q - p_1 \mathbf{1}) \right] \cdot \frac{1}{p_1 - p_0} = \mathbf{I}$$

$I = \mathbf{I}$.

In the second line of above calculations we used the definition of $\Xi_{p_1}$. Third equality follows from the second by the relation [5.4]. Finally, to get the fifth equation we used $[5.3]$ and again [5.4].

Therefore, $K_1(x)$ is a solution to differential equation [5.10]. Returning to $M_1(x, y)$ it is now straightforward to guess and check the formula for $M_1$, namely

$$M_1(x, y) = C e^{-\Lambda_{p_0}^+ (x-y)} - e^{-\Lambda_{p_1}^+ (x-x_1)} C e^{-\Lambda_{p_0}^- (x_1-y)}.$$

Now, using similar reasoning as for deriving $M_1$ we can identify other integrals appearing in Equation [5.8]:

$$M_2(x, y) = \int_{x_1}^{x} e^{-\Lambda_{p_1}^+ (x-z)} \Xi_{p_1} e^{\Lambda_{p_0}^- (z-y)} dz,$$

$$M_3(x, y) = \int_{x_1}^{x} e^{\Lambda_{p_1}^- (x-z)} \Xi_{p_1} e^{-\Lambda_{p_0}^+ (z-y)} dz,$$

$$M_4(x, y) = \int_{x_1}^{x} e^{\Lambda_{p_1}^- (x-z)} \Xi_{p_1} e^{\Lambda_{p_0}^- (z-y)} dz.$$

Precisely,

$$M_1(x, y) = C e^{-\Lambda_{p_0}^+ (x-y)} - e^{-\Lambda_{p_1}^+ (x-x_1)} C e^{-\Lambda_{p_0}^+ (x_1-y)},$$

$$M_2(x, y) = D e^{\Lambda_{p_0}^- (x-y)} - e^{-\Lambda_{p_1}^+ (x-x_1)} D e^{\Lambda_{p_0}^- (x_1-y)},$$

$$M_3(x, y) = E e^{-\Lambda_{p_0}^+ (x-y)} - e^{\Lambda_{p_1}^- (x-x_1)} E e^{-\Lambda_{p_0}^+ (x_1-y)},$$

$$M_4(x, y) = F e^{\Lambda_{p_0}^- (x-y)} - e^{\Lambda_{p_1}^- (x-x_1)} F e^{\Lambda_{p_0}^- (x_1-y)}.$$
where matrices $C, D, E, F$ are given by

\[
C = -\left(\Lambda_{p_1}^+ + \Lambda_{p_1}^-\right)^{-1} \left(\Lambda_{p_0}^+ + \Lambda_{p_1}^-\right) \cdot \frac{1}{p_1 - p_0}, \\
D = \left(\Lambda_{p_1}^+ + \Lambda_{p_1}^-\right)^{-1} \left(\Lambda_{p_0}^- - \Lambda_{p_1}^-\right) \cdot \frac{1}{p_1 - p_0}, \\
E = -\left(\Lambda_{p_1}^+ + \Lambda_{p_1}^-\right)^{-1} \left(\Lambda_{p_0}^+ - \Lambda_{p_1}^+\right) \cdot \frac{1}{p_1 - p_0}, \\
F = \left(\Lambda_{p_1}^+ + \Lambda_{p_1}^-\right)^{-1} \left(\Lambda_{p_0}^- + \Lambda_{p_1}^+\right) \cdot \frac{1}{p_1 - p_0}.
\]

Thus from (5.8), for $x > x_1$,

\[
W^{(\omega)}_1(x, y) = \left(e^{-\Lambda_{p_0}^+(x-y)} - e^{\Lambda_{p_0}^-(x-y)}\right)\Xi_{p_0}
+ (p_1 - p_0)\left(M_1(x, y) - M_2(x, y) - M_3(x, y) + M_4(x, y)\right)\Xi_{p_0}
= \left[I - (p_1 - p_0)(E - C)\right]e^{-\Lambda_{p_0}^+(x-y)} - \left[I - (p_1 - p_0)(D + F)\right]e^{\Lambda_{p_0}^-(x-y)}
+ (p_1 - p_0)\left(e^{\Lambda_{p_1}^-(x-x_1)} - e^{\Lambda_{p_0}^-(x-1-y)}\right)
- e^{-\Lambda_{p_1}^+(x-x_1)}\left(C e^{-\Lambda_{p_0}^+(x-1-y) - D e^{\Lambda_{p_0}^-(x-1-y)}}\right)\Xi_{p_0}
= \left[e^{-\Lambda_{p_1}^+(x-x_1)}\left(\Lambda_{p_1}^+ + \Lambda_{p_1}^+\right)^{-1} \Lambda_{p_1}^- + e^{\Lambda_{p_1}^-(x-x_1)}\left(\Lambda_{p_1}^+ + \Lambda_{p_1}^-\right)^{-1} \Lambda_{p_1}^+\right]W^{(p_0)}(x_1 - y)
- W^{(p_1)}(x - x_1)\Delta_x^2 W^{(p_0)}(x_1 - y),
\]

where we notice the facts that

\[(p_1 - p_0)(E - C) = I, \quad (p_1 - p_0)(D + F) = I.\]

This completes the proof of this proposition. Note that the uniqueness of this result is straightforward conclusion from Lemma 3.1. \(\square\)

Remark 5.1 In general, if we choose to divide $\mathbb{R}$ into more intervals, similar idea could be used for the computations of $\omega$-scale matrix.

We take the following parameters for the numerical analysis:

\[
\Delta_\sigma = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.85 \end{pmatrix}, \quad \Delta_\mu = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, \quad Q = \begin{pmatrix} -0.1 & 0.1 \\ 0.3 & -0.3 \end{pmatrix},
\]

$p_0 = 0.25, \quad p_1 = 0.03, \quad x_1 = 4.$

Note that we do not assume that $\Delta_\mu \neq 0$ and thus we cannot use the formula (5.5). Therefore for the computations, we used numerical package [13] instead.

From Figure 3 one can see that in every cell we have interesting relation that $W^{(\omega)}$ lies between $W^{(p_0)}$ and $W^{(p_1)}$ and this functions are similar in shape.
In Section 4, we considered dividend problem in the general Markov additive model and we derived the formula for the value function in the terms of \( \omega \)-scale matrix. In this subsection, we will analyze it for the specific choice of \( \omega \) function:

\[
\omega(i, x) := \omega(x) = (\gamma_0 + \gamma_1(x + d))1_{\{-d \leq x \leq 0\}}, \quad \text{for all } i \in E,
\]

and for MAP being a Markov modulated Brownian motion. Similar model for the Lévy-risk process was analyzed in Li and Palmowski [17].

Fix a constant force of interest \( \delta \geq 0 \). Using (3.4) we obtain that \( \mathcal{W}^{(\omega+\delta)} \) satisfy following equation: for \( x \in [-d, 0] \),

\[
\mathcal{W}^{(\omega+\delta)}(x, -d) = \mathcal{W}(x + d) + \int_{-d}^{x} (\omega(z) + \delta) \mathcal{W}(x - z) \mathcal{W}^{(\omega)}(z, -d) dz
\]

\[
= \mathcal{W}(x + d) + \int_{0}^{x+d} (\omega(y - d) + \delta) \mathcal{W}(x + d - y) \mathcal{W}^{(\omega)}(y - d, -d) dy
\]

\[
= \mathcal{W}^{(\gamma_0+\delta)}(x + d) + \gamma_1 \int_{0}^{x+d} y \mathcal{W}^{(\gamma_0+\delta)}(x + d - y) \mathcal{W}^{(\omega)}(y - d, -d) dy.
\]

Now, let \( z = x + d \geq 0 \), \( \omega_1(z) := \omega(x + d) = (\gamma_0 + \gamma_1z)1_{\{0 \leq z \leq d\}} \) and

\[
G(z) := \mathcal{W}^{(\omega+\delta)}(z - d, -d) = \mathcal{W}^{(\omega+\delta)}(x, -d),
\]

(5.13)
Then we can rewrite equation for $\mathcal{W}^{(\omega+\delta)}$ as
\[
G(z) = W^{(\gamma_0+\delta)}(z) + \gamma_1 \int_0^z y W^{(\gamma_0+\delta)}(z-y) G(y) dy.
\]
From equation (5.1) for $W^{(\gamma_0+\delta)}$ we obtain that
\[
(\frac{d}{dz} - C_{\gamma_0+\delta}) \left( \frac{d}{dz} + \Lambda_{\gamma_0+\delta}^+ \right) W^{(\gamma_0+\delta)}(z) = 0,
\]
where $C_{\gamma_0+\delta} = (\Lambda_{\gamma_0+\delta}^+ + \Lambda_{\gamma_0+\delta}^-) \Lambda_{\gamma_0+\delta}^-(\Lambda_{\gamma_0+\delta}^+ + \Lambda_{\gamma_0+\delta}^-)^{-1}$.

Starting from (5.13) for $z \in [0, d]$ (or equivalently for $x \in [-d, 0]$) we have
\[
(\frac{d}{dz} - C_{\gamma_0+\delta}) \left( \frac{d}{dz} + \Lambda_{\gamma_0+\delta}^+ \right) G(z) = \gamma_1 z \Delta_2 \frac{\partial}{\partial z} G(z)
\]
with the boundary conditions
\[
G(0) = 0, \quad G'(0) = \Delta_2 \frac{\partial}{\partial z}.
\]

For better understanding of the nature of the above differential matrix equation we rewrite it into the following form
\[
G''(z) + \left( \Lambda_{\gamma_0+\delta}^+ - C_{\gamma_0+\delta} \right) G'(z) - \left( C_{\gamma_0+\delta} \Lambda_{\gamma_0+\delta}^+ + 2 \gamma_1 z \Delta_1 \sigma^2 \right) G(z) = 0,
\]
which could be simplified to, by (5.3),
\[
\Delta_2 \sigma^2 G''(z) + \Delta_\mu G'(z) + QG(z) - (\omega_1(z) + \delta) G(z) = 0, \quad \text{for } z \in [0, d].
\]

Now, we will treat the case of $z \geq d$ (or equivalently for $x \geq 0$). We first rewrite formula
\[
\mathcal{W}^{(\omega+\delta)}(x; -d) = W(x + d) + \int_0^{x+d} \omega(y - d) W(x + d - y) \mathcal{W}^{(\omega+\delta)}(y - d; -d) dy, \quad \text{for } x \geq 0,
\]
in the terms of matrix $G(z)$ with respect to $z \geq d$:
\[
G(z) = W(z) + \int_0^d (\delta + (\gamma_0 + \gamma_1 y)) W(z-y) G(y) dy + \delta \int_d^z W(z-y) G(y) dy.
\]

Similar to (5.14) and (5.15), we have, respectively
\[
(\frac{d}{dz} - C) \left( \frac{d}{dz} + \Lambda^+ \right) W(z) = 0,
\]
and
\[
(\frac{d}{dz} - C) \left( \frac{d}{dz} + \Lambda^+ \right) G(z) = \delta \Delta_2 \frac{\partial}{\partial z} G(z), \quad \text{for } z \geq d,
\]
where $C = (\Lambda^+ + \Lambda^-) \Lambda^- (\Lambda^+ + \Lambda^-)^{-1}$. Using (5.3) for $q = 0$, we get that
\[
\Delta_2 \sigma^2 G''(z) + \Delta_\mu G'(z) + QG(z) - \delta G(z) = 0, \quad \text{for } z > d.
\]
Then the system has a unique solution $y$.

Figures 4 and 5 present entries of the numerical approximations of the matrices $W$.

Set the following parameters

for $z \in [0, d]$

for $z > d$

with the boundary conditions $G(0) = 0$, and $G'(0) = \frac{\partial G}{\partial x}$. Therefore from (5.13) for $x \in [-d, 0]$ we obtain:

and for $x > 0$,

with the boundary conditions $W(-d, -d) = 0$ and $W(-d, -d) = \frac{\partial W}{\partial x}$.

Before we proceed to the numerical example, we recall that $N$ is the cardinality of the state space $E$ and $W^{(\omega+\delta)}$ maps $\mathbb{R}$ into $\mathbb{R}^{N \times N}$. Thus, we can see that differential equations for $W^{(\omega+\delta)}$ can be treat as $(2N)$th-order system of second-order initial-value problems. For second-order initial-value problems we can introduce new unknown functions being derivative of remaining functions. Then we get $(4N)$th-order system of first-order initial-value problems for which there exist rich collection of iterative algorithms. Let us focus on the uniqueness and existence in the general case. Namely, recall that every $m$th-order system of first-order initial-value problems can be written in the form of

where for all $i \in \{1, 2, ..., m\}$, $g_i$ is assumed to be defined on some set

Then the system has a unique solution $y_1(t), y_2(t), ..., y_m(t)$, for $a \leq t \leq b$ if all $g_i$’s are continuous on $D_i$ and satisfy a Lipschitz condition with respect to $(y_1, y_2, ..., y_m)$.

In the framework of this section, we choose $a = -d$ and $b = t_{max}$ as a upper limit of our approximation. It is also clear that if we choose $\omega$ to be continuous then above sufficient condition holds true. Set the following parameters

where $\gamma_0 = 0.5$, $\gamma_1 = -0.1$, $d = 5$, $t_{max} = 10$ and $\delta = 0.04$. Figures 4 and 5 present entries of the numerical approximations of the matrices $W^{(\omega+\delta)}$ and $v_1^d(x)$ respectively.
Figure 4: Entries of $\omega$-scale matrix function $W^{(\omega+\delta)}$

Figure 5: Entries of the value matrix function $v^d_c(x)$
A Proof of Lemma 3.1

To prove the uniqueness of the solution, we will show that \( H(x) = 0 \) is the only solution to

\[
H(x) = \int_0^x W(x - y) \omega(y) H(y) dy.
\]

Taking the Laplace transform on both sides of \( (A.1) \) (with an argument \( s_0 \)), we get

\[
\tilde{H}(s_0) = \tilde{W}(s_0) \tilde{\omega} \tilde{H}(s_0).
\]

Recall that \( \lambda \) is the upper bound of \( |\omega_i(y)| \) on \([0, \infty)\) for all \( i \in E \). Using \( (2.2) \), we obtain that the matrix norm of \( \tilde{H}(s_0) \) fulfills the inequality

\[
(A.2) \quad \| \tilde{H}(s_0) \| \leq \lambda \| \tilde{W}(s_0) \| \| \tilde{\omega} \| \| \tilde{H}(s_0) \|.
\]

Next we will show that there exists \( s_0 \) such that

\[
(A.3) \quad \| F^{-1}(s_0) \| < \frac{1}{2\lambda}, \quad \text{for all } s \geq s_0.
\]

To do so, we recall the expression for \( F(\alpha) \):

\[
F(\alpha) = \text{diag}(\psi_1(\alpha), \ldots, \psi_N(\alpha)) + Q \circ \mathbb{E}(e^{\alpha U_{ij}}).
\]

Observe that its diagonal goes to infinity, as \( \alpha \) goes to infinity, and each element (entry-wise) other than the diagonal is bounded by the (fixed) \( q_{ij} \).

We now prove that (using the induction argument with respect to the dimension of \( F(\alpha) \))

\[
F^{-1}(\alpha) \to 0, \quad \text{as } \alpha \to \infty.
\]

Define a series sub-matrices of \( F(\alpha) \), for \( m = 1, 2, \ldots, N \),

\[
F_m(\alpha)^{-1} := F(\alpha)^{-1}_{m \times m} = \left( \{ F_{ij}(\alpha) \}_{i,j=1}^m \right)^{-1},
\]

and in what follows, we will show that

\[
(A.4) \quad F_m^{-1}(\alpha) \to 0_{m \times m}, \quad \text{as } \alpha \to \infty.
\]

Clearly, \( F_N(\alpha)^{-1} = F(\alpha)^{-1} \).

When \( m = 1 \), \( F_1(\alpha)^{-1} = \frac{1}{\psi_1(s_0) + q_{11}} \), which makes \( (A.4) \) hold obviously, and \( s_0 \) in \( (A.3) \) is chosen such that \( \frac{1}{\psi_1(s_0) + q_{11}} < \frac{1}{2\lambda} \). Assume \( (A.4) \) holds for the dimension \( m = k - 1 \). Then in the dimension \( m = k \), we have

\[
F_k(\alpha)^{-1} = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1},
\]

where

\[
A_{(k-1) \times (k-1)} = F_{k-1}(\alpha),
\]
\[
B_{(k-1)\times 1} = (q_{1k}E(e^{\alpha t_{1k}}), \ldots, q_{(k-1)k}E(e^{\alpha t_{(k-1)k}}))^T,
\]
\[
C_{1\times (k-1)} = (q_{1k}E(e^{\alpha t_{1k}}), \ldots, q_{k(k-1)}E(e^{\alpha t_{k(k-1)}})),
\]
and
\[
D_{1\times 1} = \psi_k(\alpha) - q_{kk}.
\]

Using the property for the inverse of the block matrix
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1}
\end{pmatrix},
\]
it is easy to see that each block for \( F_k(\alpha)^{-1} \) goes to 0 as \( \alpha \to \infty \), since
\[
A^{-1} = F_{k-1}(\alpha)^{-1} \to 0_{(k-1)\times (k-1)},
\]
\[
(D - CA^{-1}B)^{-1} = \frac{1}{\psi_k(\alpha) - q_{kk} - CA^{-1}B} \to 0,
\]
and \( B, C \) have bounded (non-negative) elements.

This completes the proof of (A.3).

Plugging (A.3) into (A.2) gives:
\[
\tilde{H}(s_0) = 0, \text{ i.e., } H(x) = 0,
\]
which completes the proof of uniqueness of the solution of Equation (3.2).

To prove the existence of solution of of Equation (3.2), we construct a series of matrices \( \{\tilde{H}_m\} \) which converge to the unique solution. Define the operator \( \mathcal{G} \) on a matrix: for \( z > 0 \),
\[
\mathcal{G}\tilde{K}(z) = \int_0^\infty e^{-zx} \int_0^x e^{-s_0(x-y)}W(x-y)\omega(y)\tilde{K}(y)dydx = \tilde{W}(s_0 + z) \tilde{\omega}\tilde{K}(z),
\]
\[
\mathcal{G}^{(m+1)}\tilde{K}(z) = \mathcal{G}(\mathcal{G}^{(m)})\tilde{K}(z),
\]
\[
\tilde{H}_0(z) = \int_0^\infty e^{-zx} e^{-s_0x} h(x)dx = \tilde{h}_0(s_0 + z), \quad \tilde{H}_{m+1}(z) = \tilde{H}_0(z) + \mathcal{G}\tilde{H}_m(z).
\]

Then we have \( \mathcal{G} \) is a linear operator such that \( \|\mathcal{G}\tilde{K}(z)\| < \frac{1}{2}\|\tilde{K}(z)\| \) for \( z > 0 \). Therefore, for \( m > l \), we have
\[
\|\tilde{H}_m(z) - \tilde{H}_l(z)\| = \|\sum_{k=l+1}^m \mathcal{G}^{(k)}\tilde{H}_0(z)\| < 2^{-l}\|\tilde{H}_0(z)\|,
\]
which means \( \{\tilde{H}_m(z), z > 0\}_{m \geq 0} \) forms a Cauchy sequence (entry-wise) that admits a limit \( \tilde{\omega}\tilde{\omega}(z) \) for any \( z > 0 \) satisfying:
\[
\tilde{\omega}(z) = \tilde{H}_0(z) + \mathcal{G}\tilde{\omega}(z) = \tilde{h}_0(s_0 + z) + \tilde{W}(s_0 + z) \tilde{\omega}\tilde{\omega}(z).
\]

Using the uniqueness of Laplace transform, we have
\[
\tilde{\omega}(x) = e^{-s_0x}h_0(x) + \int_0^x e^{-s_0(x-y)}W(x-y)\omega(y)\tilde{\omega}(y)dy,
\]
which shows that $H(x) = e^{\omega x} \delta_t(x)$ is the solution to (3.2).

As for the second statement in this lemma, we see that if $H$ satisfies (3.3), by letting $\delta = 0$, we obtain (3.2) immediately. Now we only need to show that if $H$ is the solution to (3.2), it is also the solution to (3.3). We convolute both sides of (3.2) with $\delta W^{(\delta)}$ (on the left),

$$
\delta W^{(\delta)} \ast H(x) = \delta W^{(\delta)} \ast h(x) + \delta W^{(\delta)} \ast W \ast (\omega H)(x)
$$

$$
= \delta W^{(\delta)} \ast h(x) + (W^{(\delta)} - W) \ast (\omega H)(x)
$$

where the last step using the identity $W^{(\delta)} - W = \delta W^{(\delta)} \ast W$ (which can be easily seen from the Laplace transform). Therefore,

$$
H(x) = h(x) + \delta W^{(\delta)} \ast h(x) + W^{(\delta)} \ast ((\omega - \delta I)H)(x),
$$

which completes the proof. \qed

\section{B Proofs of main results}

\subsection*{B.1 Proof of Theorem 3.1}

\subsubsection*{B.1.1 Proof of the case \((i)\)}

In what follows, we prove the case of $d = 0$, and then the general result holds true using the shifting argument as well as the identity (3.6).

First, applying the strong Markov property of $X$ at $\tau_{\omega} + y$ and using the fact that $X$ has no positive jumps, we get that:

(B.1) $$A^{(\omega)}(x, z) = A^{(\omega)}(x, y)A^{(\omega)}(y, z),$$

for all $0 \leq x \leq y \leq z$.

Following the similar argument as in Li and Palmowski [17], we recall that $\lambda > 0$ is the arbitrary upper bound of $\omega_i(x)$ (for all $x \in \mathbb{R}$ and $1 \leq i \leq N$). Let $\Psi = \{\Psi_t, t \geq 0\}$ be a Poisson point process with a characteristic measure $\mu(dt, dy) = \lambda dt \frac{1}{\lambda} 1_{\{0, \lambda\}}(y)dy$. Hence $\Psi = \{(T_k, M_k), k = 1, 2, \ldots \}$ is a doubly stochastic marked Poisson process with jump intensity $\lambda$, jumps epochs $T_k$ and marks $M_k$ being uniformly distributed on $[0, \lambda]$. Moreover, we construct $\Psi$ to be independent of $X$. Therefore, for $T^\omega := \inf \{T_k > 0 : M_k < \omega_{J_{T_k}}(X_{T_k}); \text{ for } k \geq 1\}$, we have

$$A^{(\omega)}_{ij}(x, c) = \mathbb{P}_{x, i} \left( \tau_c^+ < \tau_0^- \land T^\omega, J_{\tau_c^+} = j \right)$$

$$= \mathbb{P}_{x, i} \left( \#_k \{M_k < \omega_{J_{T_k}}(X_{T_k}) \text{ for } T_k < \tau_c^+, \tau_c^- < \tau_0^-, J_{\tau_c^+} = j \} = 0 \right).$$

In this case, there are two scenarios following: either there is no $T_k$ which occurs before reaching level $c$ or the first jump time $T_1$ occurs in state $m$ and the process renews from
state m. Hence:

\[
A_{ij}^{(\omega)}(x, c) = \mathbb{P}_{x,i}(T_1 > \tau_c^+, \tau_c^+ < \tau^-_0, J_{c^+} = j) \\
+ \sum_{m=1}^{N} \mathbb{E}_{x,i} \left[ A_{m,j}^{(\omega)}(X_{T_1}, c), T_1 < \tau_c^+ \wedge \tau^-_0, M_1 > \omega_m(X_{T_1}), J_{T_1} = m \right] \\
= \mathbb{E}_{x,i} \left[ e^{-\lambda \tau_c^+}; \tau_c^+ < \tau^-_0, J_{c^+} = j \right] \\
+ \int_0^\infty \sum_{m=1}^{N} \mathbb{E}_{x,i} \left[ X_{T_1} \in dy, T_1 < \tau_c^+ \wedge \tau^-_0, J_{T_1} = m \right] \frac{\lambda - \omega_m(y)}{\lambda} A_{m,j}^{(\omega)}(y, c),
\]

which is equivalent to

\[
\mathbf{A}^{(\omega)}(x, c) = \mathbb{E}_x \left[ e^{-\lambda \tau_c^+}; \tau_c^+ < \tau^-_0, J_{c^+} \right] \\
+ \int_0^c \mathbb{E}_x \left[ X_{T_1} \in dy, T_1 < \tau_c^+ \wedge \tau^-_0, J_{T_1} \right] \frac{1}{\lambda} (\lambda \mathbf{I} - \mathbf{\omega}(y)) \mathbf{A}^{(\omega)}(y, c),
\]

where

\[
\mathbb{E}_x \left[ e^{-\lambda \tau_c^+}; \tau_c^+ < \tau^-_0, J_{c^+} \right] = \mathbf{W}^{(\lambda)}(x) \mathbf{W}^{(\lambda)}(c)^{-1}
\]

and

\[
\frac{1}{\lambda} \mathbb{E}_x \left[ X_{T_1} \in dy, T_1 < \tau_c^+ \wedge \tau^-_0, J_{T_1} \right] = \left( \mathbf{W}^{(\lambda)}(x) \mathbf{W}^{(\lambda)}(c)^{-1} \mathbf{W}^{(\lambda)}(c-y) - \mathbf{W}^{(\lambda)}(x-y) \right) dy,
\]

are given in Ivanovs and Palmowski [15] and Ivanovs [14], respectively.

Taking the last increment to the other side of the above equality and applying relation (B.1) gives

\[
\mathbf{W}^{(\omega)}(x)^{-1} := \mathbf{W}^{(\lambda)}(x)^{-1} \left( \mathbf{I} + \int_0^x \mathbf{W}^{(\lambda)}(x-y) (\lambda \mathbf{I} - \mathbf{\omega}(y)) \mathbf{A}^{(\omega)}(y, x) dy \right)
\]

Define

\[
\mathbf{W}^{(\omega)}(x)^{-1} := \mathbf{W}^{(\lambda)}(x)^{-1} \left( \mathbf{I} + \int_0^x \mathbf{W}^{(\lambda)}(x-y) (\lambda \mathbf{I} - \mathbf{\omega}(y)) \mathbf{A}^{(\omega)}(y, x) dy \right)
\]

and then we obtain the required identity

\[
\mathbf{A}^{(\omega)}(x, c) = \mathbf{W}^{(\omega)}(x) \mathbf{W}^{(\omega)}(c)^{-1}.
\]

The proof of the invertibility of matrix \( \mathbf{W}^{(\omega)}(x)^{-1} \) is deferred to Proposition B.1. After replacing \( \mathbf{A}^{(\omega)}(y, x) = \mathbf{W}^{(\omega)}(y) \mathbf{W}^{(\omega)}(x)^{-1} \) in (B.3), we have

\[
\mathbf{W}^{(\lambda)}(x) = \left( \mathbf{I} + \int_0^x \mathbf{W}^{(\lambda)}(x-y) (\lambda \mathbf{I} - \mathbf{\omega}(y)) \mathbf{A}^{(\omega)}(y, x) dy \right) \mathbf{W}^{(\omega)}(x)
\]

\[
= \mathbf{W}^{(\lambda)}(x) + \int_0^x \mathbf{W}^{(\lambda)}(x-y) (\lambda \mathbf{I} - \mathbf{\omega}(y)) \mathbf{W}^{(\omega)}(y) dy
\]
Now using the identity
\[ W^{(\delta)} - W = \delta W \ast W^{(\delta)}, \]
it is easy to show
\[ W^{(\omega)}(x) = W(x) + \int_0^x W(x - y)\omega(y)W^{(\omega)}(y) \, dy. \]

**Proposition B.1** The matrix \( W^{(\omega)}(x)^{-1} \) is invertible for any \( x > 0 \).

**Proof.** From (B.3), one can see that it is enough to prove that the matrix
\[ P(x) := I + \int_0^x W^{(\lambda)}(x - y) (\lambda I - \omega(y)) A^{(\omega)}(y, x) \, dy \]
is invertible for every \( x \geq 0 \). Using similar argument as in [16], note that for all \( y > 0 \) there exists some \( N \times N \) sub-stochastic invertible intensity matrix \( \Lambda_{\omega, *}(y) \) such that
\[ P_x (\tau_c^+ < \tau_0^- \wedge T^\omega, J_{\tau_c^+} | J_0) = \exp \left( \int_x^c \Lambda_{\omega, *}(y) \, dy \right). \]
This observation implies that the matrix \( A^{(\omega)}(x, c) \) is invertible for any \( x, c \in \mathbb{R}_+ \) such that \( 0 < x \leq c \). The matrix \( A^{(\omega)}(x, c) \) is also continuous (entry wise) with respect of \( c \). Now, assume that there exists \( c > 0 \) such that matrix \( P(x) \) is invertible for some \( 0 < x < c \) and is singular for \( x = c \). Then from relation (B.2) we get contradiction, because the left-hand side of it is invertible (as a product of invertible matrices) and the right-hand side is singular from the assumption. Hence, only two scenarios are possible: the matrix \( P(x) \) is invertible for all \( x > 0 \) or it is singular for all \( x > 0 \). Finally, since \( P(0) = I \) and \( P(x) \) is continuous in \( x \geq 0 \) we obtain that \( P(x) \) must be invertible for all \( x \geq 0 \). \( \square \)

**B.1.2 Proof of the case (ii)**
Let \( \{ (X_t, J_t) \}_{t \geq 0} \) be a MAP with the lifetime \( \xi \), transition probabilities and \( q \)-resolvent measures, given, respectively by
\[ Q_{t,ij} f_j(x) = \mathbb{E}_{x,i} [f_j(X_t), t < \xi, J_t = j] \]
and
\[ K^{(q)}_{ij} f_j(x) = \int_0^\infty e^{-qt} Q_{t,ij} f_j(x) \, dt, \]
where \( \{ f_j \}_{j=1}^N \) is a set of nonnegative, bounded, continuous functions on \( \mathbb{R} \) such that \( \sup_{i,j} K^{(0)}_{ij} f_j(x) < \infty \). Then the \( \omega \)-type resolvent \( K^{(\omega)}_{ij} \) is defined by
\[ K^{(\omega)}_{ij} f_j(x) := \int_0^\infty Q^{(\omega)}_{t,ij} f_j(x) \, dt, \]
Using the resolvent identity
\[ \lambda K(x, t) = \int_0^t \lambda K(x, s) ds \]
where by matrix compounding, we mean \( A \) matrix form, we have
Note that the superscript \( 0 \) for all \( \lambda > 0 \) we have

**Lemma B.1** The matrix \( K(\omega f(x) = \{ K_{ij}(\omega f_j(x) \}_i,j=1^N \) satisfies the following equality:

\[ K(\omega f(x) = K(0) (f - \omega K(\omega f(x)), \]
where \( f = diag(f_1, \ldots, f_N) \).

**Proof.** As before without loss of generality, we assume that \( \omega_i(x) \) is bounded by some \( \lambda > 0 \) for all \( x \in \mathbb{R} \) and \( i \in E \). The finiteness of \( K_{ij}(\omega f_j(x) \) comes from the fact that \( K(\omega f_j(x) < K(0)(\omega f_j(x) \) for all \( 1 \leq i \leq N \). Using similar arguments as in the proof in [B.1.1], we have

\[
Q_{t,i,j}(\omega f_j(x) := \mathbb{E}_{x,i} \left[ f_{i,t}(X_t); t < \xi \text{ and } M_k > \omega,J_k(X_T) \text{ for all } T_k < t, J_t = j \right] \\
= \mathbb{E}_{x,i} \left[ f_{i,t}(X_t); t < \xi, T_1 > t, J_t = j \right] \\
+ \sum_{l=1}^N \int_0^t \mathbb{E}_{x,i} \left[ Q_{t-s,l,j}(\omega f_j(X_s), M_1 > \omega,J(X_s), J_s = l \right] \mathbb{P}(T_1 \in ds) \\
= \mathbb{E}_{x,i} \left[ e^{-\lambda t} f_{i,t}(X_t); t < \xi, J_t = j \right] \\
+ \sum_{l=1}^N \int_0^t \mathbb{E}_{x,i} \left[ (\lambda - \omega l)(X_s))Q_{t-s,l,j}(\omega f_j(X_s), J_s = l \right] e^{-\lambda s} ds \\
= Q_{t,i,j}(\omega f_j(x) + \sum_{l=1}^N \int_0^t Q_{s,l}(\omega f_j(x) ds.
\]

Note that the superscript \( \lambda \) denotes a counterpart for fixed \( \omega_i(x) \equiv \lambda \). Equivalently, in a matrix form, we have

\[
Q_t(\omega f(x) = Q_t(\omega f(x) + \int_0^t Q_s(\omega (\lambda I - \omega)Q_{t-s}(\omega f(x)) ds.
\]
where by matrix compounding, we mean \( (A(B(x))_{ij} = \sum_{m=1}^N A_{im}B_{mj}(x) \).

Thus,

\[
K(\omega f(x) = \int_0^\infty Q_t(\omega f(x) dt = K(0) f(x) + K(\lambda) (\lambda I - \omega)K(\omega f(x)).
\]

Using the resolvent identity \( \lambda K(0) K(\lambda) = K(0) - K(\lambda) \), we have

\[
\lambda K(0)(\lambda) f(x) = \lambda K(0)(\lambda) f(x) + \lambda K(0)(\lambda) (\lambda I - \omega)K(\omega f(x)) \\
= (K(0) - K(\lambda)) f(x) + (K(0) - K(\lambda)) (\lambda I - \omega)K(\omega f(x)).
\]
Comparing \([\text{B.5}]\) with \([\text{B.6}]\) completes the proof. □

Now we can proceed the proof of the case \((ii)\). Again we prove the case of \(d = 0\), and then the general result holds true using the shifting argument as well as the identity \((3.6)\).

For \(i, j \in E\), define

\[
\text{B}^{(\omega)}_{ij}(x) := \lim_{c \to \infty} \text{B}^{(\omega)}_{ij}(x, c) = \mathbb{E}_{x,i} \left[ e^{-\int_{0}^{\tau_{0}^{-}} \omega_{ij}(X_{s})ds}, \tau_{0}^{-} < \infty, J_{\tau_{0}^{-}} = j \right].
\]

Note that for any \(i, j \in E\) and \(x, c \in \mathbb{R}\) such that \(x < c\) matrix function \(\text{B}^{(\omega)}_{ij}(x, c)\) is monotone in \(c\), and it is bounded by \(0 \leq \text{B}^{(\omega)}_{ij}(x, c) \leq \mathbb{P}_{x,i} \left( \tau_{0}^{-} < \tau_{c}^{+}, J_{\tau_{0}^{-}} = j \right) \leq 1\), so the limit in \((\text{B.7})\) exists and is finite. The strong Markov property and spectrally negativity of \(X\) give that

\[
\text{B}^{(\omega)}(x, c) = \text{B}^{(\omega)}(x) - \text{A}^{(\omega)}(x, c)\text{B}^{(\omega)}(c).
\]

To identify \(\text{B}^{(\omega)}(x)\), we use lemma \([\text{B.1}]\) with \(\xi = \tau_{0}^{-}\) and \(f(\cdot) = \omega(\cdot)\). Hence

\[
\text{I}(x) - \text{B}^{(\omega)}(x) = \mathbb{E}_{x} \left[ \int_{0}^{\tau_{0}^{-}} \omega_{ij}(X_{t}) \exp \left( - \int_{0}^{t} \omega_{ij}(X_{s})ds \right) dt, t < \tau_{0}^{-}, J_{t} \right] = \int_{0}^{\infty} \mathbb{E}_{x} \left[ \omega_{ij}(X_{t}) \exp \left( - \int_{0}^{t} \omega_{ij}(X_{s})ds \right), t < \tau_{0}^{-}, J_{t} \right] dt
\]

\[
= \int_{0}^{\infty} \left( \mathbf{W}(x)e^{Ry} - \mathbf{W}(x-y) \right) \left[ \omega(y) - \omega(I - \text{B}^{(\omega)})(y) \right] dy,
\]

where the potential measure

\[
K^{(0)}(1_{(0,\infty)}(X_{t} \in dy))(x) = U_{(0,\infty)}(x, dy) = \left( \mathbf{W}(x)e^{Ry} - \mathbf{W}(x-y) \right) dy,
\]

was obtained in \([14]\) with \(R = R^{0}\). We may rewrite it as

\[
\text{B}^{(\omega)}(x) = \text{I}(x) - \mathbf{W}(x)\mathbf{C}_{\text{B}^{(\omega)}} + \int_{0}^{x} \mathbf{W}(x-y)\omega(y)\text{B}^{(\omega)}(y)dy,
\]

where

\[
\mathbf{C}_{\text{B}^{(\omega)}} = \int_{0}^{\infty} e^{Ry}\omega(y)\text{B}^{(\omega)}(y)dy.
\]

Note that \(0 \leq \text{B}^{(\omega)}_{ij}(y) \leq 1\) and recall that \(0 \leq \omega_{ij}(x) \leq \lambda\). Hence last increment on the right hand side of equation \([\text{B.10}]\) is finite and then matrix \(\mathbf{C}_{\text{B}^{(\omega)}}\) is well defined and finite.

From the definitions of \(\omega\)-scale matrices we have

\[
\text{B}^{(\omega)}(x) = \mathbf{Z}^{(\omega)}(x) - \mathbf{W}^{(\omega)}(x)\mathbf{C}_{\text{B}^{(\omega)}}.
\]

Equation \([\text{B.8}]\) completes the proof. □
B.2 Proof of Corollary 3.1

B.2.1 Proof of the case (i)

First we will prove that

\[(B.13) \quad \lim_{d \to -\infty} \mathcal{W}^{(\omega)}(x, d) \mathcal{W}^{(\omega)}(c, d)^{-1} = \mathcal{H}^{(\omega)}(x) \mathcal{H}^{(\omega)}(c)^{-1}.\]

Then the result will follow from Theorem 3.1(i). Recall that for \(x \geq d\) and any fixed \(\beta \geq 0\) we have:

\[
\mathcal{W}^{(\omega)}(x, d) = \mathcal{W}^{(\beta)}(x - d) + \int_0^x \mathcal{W}^{(\beta)}(x - z) \left(\mathcal{W}^{(\omega)}(z, d) - \beta I\right) \mathcal{W}^{(\omega)}(z, d) \, dz.
\]

Moreover, for \(x = 0\),

\[
\mathcal{W}^{(\omega)}(0, d) e^{-R\beta d} = \mathcal{W}^{(\beta)}(-d) e^{-R\beta d}.
\]

Hence from (2.3) we have

\[
\lim_{d \to -\infty} \mathcal{W}^{(\omega)}(0, d) e^{-R\beta d} = \lim_{d \to -\infty} \mathcal{W}^{(\beta)}(-d) e^{-R\beta d} = L^\beta.
\]

From Theorem 3.1(i), for \(x > 0\),

\[
\mathbb{E} \left[ e^{-\int_0^{\tau_+} \omega_s(X_s) \, ds} \mid \tau_+ < \tau_d, J_{\tau_+} \right] \mathcal{W}^{(\omega)}(x, d) = \mathcal{W}^{(\omega)}(0, d).
\]

Since the above expectation is increasing with respect to \(d\) the following limit is well-defined and finite for every \(x > d\):

\[
\lim_{d \to -\infty} \mathbb{E} \left[ e^{-\int_0^{\tau_+} \omega_s(X_s) \, ds} \mid \tau_+ < \tau_d, J_{\tau_+} \right] \mathcal{W}^{(\omega)}(x, d) e^{-R\beta d} = \mathcal{L}^\beta.
\]

Note also that, since matrix \(L^\beta\) is invertible as it was noted above Equation (2.3), from above equation it follows that the matrix \(\lim_{d \to -\infty} \mathcal{W}^{(\omega)}(x, d) e^{-R\beta d}\) is also invertible. Taking

\[
\mathcal{H}^{(\omega)}(x) := \lim_{d \to -\infty} \mathcal{W}^{(\omega)}(x, d) e^{-R\beta d} \left( L^\beta \right)^{-1}.
\]

completes the proof of the first part of the corollary. To show that the above form of \(\mathcal{H}^{(\omega)}(x)\) satisfies (3.10), note that

\[
\mathcal{W}^{(\omega)}(x, d) e^{-R\beta d} = \left(\mathcal{W}^{(\beta)}(x - d) + \int_0^x \mathcal{W}^{(\beta)}(x - z) \left(\omega(z) - \beta I\right) \mathcal{W}^{(\omega)}(z, d) \, dz\right) e^{-R\beta d}.
\]

Then by taking the limit \(d \to -\infty\) and applying the dominated convergence theorem the result follows. \(\square\)
B.2.2 Proof of the case (ii)

The proof follows by taking the limit (B.7), which exists and is finite. Moreover, the limit

$$\lim_{c \to \infty} W^{(\omega)}(c)^{-1} Z^{(\omega)}(c) = C_{W(\infty)^{-1} Z(\infty)} = C_{B^{(\omega)}}$$

is by (B.11) finite. This completes the proof. □

B.3 Proof of Theorem 3.2

B.3.1 Proof of the case (i)

Using Lemma B.1, we have

$$U^{(\omega)}(\omega) f(x) = \int_0^{\infty} \mathbb{E}_x \left[ \int_0^t \omega_j(X_s) \exp \left( -\int_0^t \omega_j(X_s) ds \right), t < \tau^+_\omega, J_t \right] dt$$

(B.14)

$$ = \int_d^c U^{(\omega)}(d,c)(x,dy) \left( f(y) - \omega(y) U^{(\omega)}(d,c) f(y) \right),$$

where $U^{(\omega)}(d,c)(x,dy)$ is the potential measure of the MAP without $\omega$-killing, given in Theorem 1 of Ivanovs [14]:

$$U^{(\omega)}(d,c)(x,dy) = \left( W(x-d) W(c-d)^{-1} W(c-y) - W(x-y) \right) dy.$$

Hence, we can rewrite Equation (B.14) as

$$U^{(\omega)}(d,c)(x,dy) = (W(x-d) W(c-d)^{-1} W(c-y) - W(x-y)) dy.$$

Multiplying Equation (3.4) by $C_U$ gives that

$$W^{(\omega)}(x, d) C_U = W(x-d) C_U + \int_d^x W(x-y) \omega(y) U^{(\omega)}(d,c)(x,dy),$$

where

$$C_U = \int_d^c W(c-d)^{-1} W(c-y) \left( f(y) - \omega(y) U^{(\omega)}(d,c) f(y) \right) dy.$$

Multiplying Equation (3.4) by $C_U$ gives that

$$W^{(\omega)}(x, d) C_U = W(x-d) C_U + \int_d^x W(x-y) \omega(y) W^{(\omega)}(y, d) C_U dy,$$

and define the operator $R^{(\omega)} f(x) := \int_d^x W^{(\omega)}(x, y) f(y) dy$, which leads to

$$R^{(\omega)} f(x) = \int_d^x W(x-y) f(y) dy + \int_d^x W(x-y) \omega(y) R^{(\omega)} f(y) dy.$$

Therefore, by the uniqueness property in Lemma 3.1, we have

$$U^{(\omega)}(d,c)(x,dy) = W^{(\omega)}(x, d) C_U - R^{(\omega)} f(x).$$

To find the constant matrix $C_U$, we use the boundary condition $U^{(\omega)}(d,c)(c) = 0$. One completes the proof by denoting the density of $U^{(\omega)}(d,c)(x,dy)$ as $U^{(\omega)}(d,c)(x,dy)$. □
B.3.2 Proof of the case (ii)

This identity follows directly from Theorem 3.2 (i) by taking the limit and using (2.3) together with the dominated convergence theorem.

□

B.3.3 Proof of the case (iii)

The formula follows by taking the limit lim_{d \to -\infty} in Theorem 3.2 (i) and then using (B.13).

□

B.3.4 Proof of the case (ii)

This identity follows from Theorem 3.2 (iii) by taking the limit c \to \infty. Since \mathcal{H}^{(\omega)}(c)^{-1}\mathcal{W}^{(\omega)}(c, y) is monotonic of c then the result holds.

□

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