On the $k$-free divisor problem

Jun FURUYA and Wenguang ZHAI

Jun Furuya,
Department of Integrated Arts and Science,
Okinawa National College of Technology,
Nago, Okinawa, 905-2192, Japan
E-mail: jfuruya@okinawa-ct.ac.jp

Wenguang Zhai,
School of Mathematical Sciences,
Shandong Normal University,
Jinan, Shandong, 250014, P.R.China
E-mail: zhaiwg@hotmail.com

Acta Arith.123 (2006), 267-287

Abstract. Let $\Delta^{(k)}(x)$ denote the error term of the $k$-free divisor problem for $k \geq 2$. In this paper we establish an asymptotic formula of the integral $\int_{T}^{T} |\Delta^{(k)}(x)|^2 dx$ for each $k \geq 4$.

1 Introduction

Let $d(n)$ denote the divisor function. Dirichlet first proved that the error term

$$\Delta(x) := \sum_{n \leq x} d(n) - x \log x - (2\gamma - 1)x, \quad x \geq 2$$

satisfies $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result is due to Huxley [4], who proved that

$$\Delta(x) = \left( x^{131/416} (\log x)^{26957/8320} \right).$$

It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \quad (1.1)$$

2000 Mathematics Subject Classification: 11N37.
Key Words: Dirichlet divisor problem, $k$-free divisor problem.

The second-named author is supported by National Natural Science Foundation of China(Grant No. 10301018).
which is supported by the classical mean-square result
\[
\int_1^T \Delta^2(x)dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + O(T\log^5 T)
\] (1.2)
proved by Tong [15].

Let \( k \geq 2 \) denote a fixed integer. An integer \( n \) is called \( k \)-free if \( p^k \) does not divide \( n \) for any prime \( p \). Let \( d^{(k)}(n) \) denote the number of \( k \)-free divisors of the positive integer \( n \) and define
\[
D^{(k)}(x) := \sum_{n \leq x} d^{(k)}(n).
\]

Then the expected asymptotic formula of \( D^{(k)}(x) \) is
\[
D^{(k)}(x) = C_1^{(k)} x \log x + C_2^{(k)} x + \Delta^{(k)}(x),
\]
where \( C_1^{(k)}, C_2^{(k)} \) are two constants, \( \Delta^{(k)}(x) \) is the error term. In 1874 Mertens [9] proved that \( \Delta^{(2)}(x) \ll x^{1/2} \log x \). In 1932 Hölder [4] proved that
\[
\Delta^{(k)}(x) \ll \begin{cases} 
 x^{1/2}, & \text{if } k = 2, \\
 x^{1/3}, & \text{if } k = 3, \\
 x^{33/100}, & \text{if } k \geq 4.
\end{cases}
\]

For \( k = 2, 3 \), it is very difficult to improve the exponent \( 1/k \) in the bound \( \Delta^{(k)}(x) \ll x^{1/k} \), unless we have substantial progress in the study of the zero-free region of \( \zeta(s) \). Therefore it is reasonable to get better improvements by assuming the truth of the Riemann Hypothesis (RH). Such results were given in [1, 2, 9, 12, 13, 14]. Especially in [2] R. C. Baker proved \( \Delta^{(2)}(x) \ll x^{4/11+\varepsilon} \) and in [9] Kumchev proved \( \Delta^{(3)}(x) \ll x^{27/85+\varepsilon} \) under RH. For \( k \geq 4 \), it is easy to show that if \( \Delta(x) \ll x^\alpha \) is true, then the estimate \( \Delta^{(k)}(x) \ll x^\alpha \log x \) follows.

We believe that the estimate
\[
\Delta^{(k)}(x) = O(x^{1/4+\varepsilon})
\] (1.3)
would be true for any \( k \geq 2 \), which is an analogue of (1.1). For \( k \geq 4 \) it is easily seen that if the conjecture (1.3) is true, then so is (1.3). For \( k = 2, 3 \), we cannot deduce the conjecture (1.3) from (1.1) directly; in this case we don’t know the truth of (1.3) even if both (1.1) and RH are true. However for any \( k \geq 2 \), the conjecture (1.3) cannot be proved by the present method.

In this paper we shall study the mean square of \( \Delta^{(k)}(x) \) for \( k \geq 4 \), from which the truth of the conjecture (1.3) \( (k \geq 4) \) is supported partly. Our result is an analogue of (1.2).

**Theorem 1.** We have the asymptotic formula
\[
\int_1^T |\Delta^{(k)}(x)|^2dx = \frac{B_k}{6\pi^2}T^{3/2} + \begin{cases} 
 O(T^{3/2}e^{-c\delta(T)}), & \text{for } k = 4, \\
 O(T^{\delta_k+\varepsilon}), & \text{for } k \geq 5,
\end{cases}
\]

where
\[ B_k := \sum_{m=1}^{\infty} g_k^2(m)m^{-3/2}, \quad g_k(m) := \sum_{m=nd^k} \mu(d)d(n)d^{k/2}, \]
\[ \delta(u) := (\log u)^{3/5}(\log \log u)^{-1/5}, \]
\[ \delta_5 := 75/52, \quad \delta_k := 3/2 - 1/2k + 1/k^2 \quad (k \geq 6), \]
and where \( c > 0 \) is an absolute constant.

**Corollary 1.** If \( k \geq 4 \), then we have
\[ \Delta^{(k)}(x) = \Omega(x^{1/4}). \]

By the same method we can study the mean square of \( \Delta(1, 1, k; x) \), which is defined by
\[ \Delta(1, 1, k; x) := \sum_{n \leq x} d(1, 1, k; n) - x \{ \zeta(k) \log x + k\zeta'(k) + (2\gamma - 1)\zeta(k) \} - \zeta^2\left(\frac{1}{k}\right)x^{1/k}, \]
where \( d(1, 1, k; n) = \sum_{n=m_1m_2d^k} 1 \) and \( \gamma \) is the Euler constant. This is a special three-dimensional divisor problem. From the formula (5.3) of Ivić [7] we have
\[ \int_1^T \Delta^2(1, 1, k; x)dx \ll T^{3/2+\varepsilon}. \quad (1.4) \]

From Krätzel [8] we know that
\[ \Delta(1, 1, k; x) = \Omega(x^{1/4}) \quad (1.5) \]
if \( k \geq 5 \).

Now we prove the following Theorem 2, which improves (1.4).

**Theorem 2.** Suppose \( k \geq 3 \) is a fixed integer. Then we have
\[ \int_1^T \Delta^2(1, 1, k; x)dx = \frac{C_k}{6\pi^2} T^{3/2} + \left\{ \begin{array}{ll}
O(T^{53/36} \log^3 T), & \text{if } k = 3, \\
O(T^{29/20} \log^{503} T), & \text{if } k = 4, \\
O(T^{75/52} \log^{1000} T), & \text{if } k = 5, \\
O(T^{3/2-1/2k+1/k^2+\varepsilon}), & \text{if } k \geq 6,
\end{array} \right. \]
where
\[ C_k := \sum_{m=1}^{\infty} f_k^2(m)m^{-3/2}, \quad f_k(m) := \sum_{m=nd^k} d(n)d^{k/2}. \]

**Corollary 2.** The formula (1.5) holds for \( k = 3, 4 \).

**Notations.** For a real number \( u \), \([u]\) denotes the integer part of \( u \), \( \{u\} \) denotes the fractional part of \( u \), \( \psi(u) = \{u\} - 1/2 \), \( \|u\| \) denotes the distance from \( u \) to the
integer nearest to \( u \). \( \mu(d) \) is the Möbius function. Let \((m, n)\) denote the greatest common divisor of natural numbers \( m \) and \( n \). \( n \sim N \) means \( N < n \leq 2N \). \( \varepsilon \) always denotes a sufficiently small positive constant which may be different at different places. \( SC(\Sigma) \) denotes the summation condition of the sum \( \Sigma \).

2 The expression of \( \Delta^{(k)}(x) \)

In order to prove Theorem 1, we shall give a simple expression of \( \Delta^{(k)}(x) \) in this section.

Lemma 2.1. There exists an absolute constant \( c_1 > 0 \) such that the estimate

\[
M(u) := \sum_{n \leq u} \mu(n) \ll u e^{-c_1 \delta(u)}
\]

holds for \( u \geq 2 \).

This is Theorem 12.7 of Ivić [6]. Now we prove the following

Lemma 2.2. Suppose \( 10 \leq y \ll x^{1/k} \), then we have

\[
\Delta^{(k)}(x) = \sum_{d \leq y} \mu(d) \Delta \left( \frac{x}{d^k} \right) + O \left( x y^{1-k} e^{-c_1 \delta(y)} \log x \right).
\]

Proof. We have

\[
D^{(k)}(x) = \sum_{d \leq y} \sum_{n \leq x} \mu(d) = \sum_{d \leq y} \mu(d) \sum_{n \leq x} \mu(n)
\]

\[
= \sum_{d \leq y} \mu(d) D \left( \frac{x}{d^k} \right) + \sum_{n \leq x/y^k} d(n) M \left( \frac{x}{n} \right)^{1/k} - D \left( \frac{x}{y^k} \right) M(y)
\]

\[
= \sum_1 + \sum_2 - \sum_3,
\]

say. From Lemma 2.1 and the estimate \( D(u) \ll u \log u \) directly we have

\[
\sum_3 \ll x y^{1-k} e^{-c_1 \delta(y)} \log x.
\]

From Lemma 2.1, the estimate \( D(u) \ll u \log u \) and partial summation we have

\[
\sum_2 \ll x y^{1-k} e^{-c_1 \delta(y)} \log x
\]

if we note that \( e^{-c_1 \delta((x/n)^{1/k})} \leq e^{-c_1 \delta(y)} \) for all \( n \leq x/y^k \). By Lemma 2.1 and simple calculations we have

\[
\sum_1 = \sum_{d \leq y} \mu(d) \left\{ \frac{x}{d^k} \log \frac{x}{d^k} + (2\gamma - 1) \frac{x}{d^k} \right\} + \sum_{d \leq y} \mu(d) \Delta \left( \frac{x}{d^k} \right)
\]

\[
= \text{(Main term)} + \sum_{d \leq y} \mu(d) \Delta \left( \frac{x}{d^k} \right) + O \left( x y^{1-k} e^{-c_1 \delta(y)} \log x \right).
\]

Whence Lemma 2.2 follows. \( \square \)
3 Proof of Theorem 1(Beginning)

Suppose $T \geq 10$ is large. It suffices for us to evaluate the integral $\int_{T}^{2T} |\Delta^{(k)}(x)|^2 dx$.

Let $T^\varepsilon \ll y \ll T^{1/k - \varepsilon}$, $T^\varepsilon \ll z \ll T^{1 - \varepsilon}$ be two parameters to be determined later. Let

$$
\Delta_1(u) := \frac{u^{1/4}}{\pi \sqrt{2}} \sum_{n \leq z} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n u} - \frac{\pi}{4} \right), \quad \Delta_2(u; z) := \Delta(u) - \Delta_1(u).
$$

Then by Lemma 2.2 we can write

$$
\Delta^{(k)}(x) = R_1^{(k)}(x) + R_2^{(k)}(x) + R_3^{(k)}(x), \tag{3.1}
$$

where

$$
R_1^{(k)}(x) := \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{d \leq y} \frac{\mu(d)}{d^{k/4}} \sum_{n \leq z} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n x d} - \frac{\pi}{4} \right),
$$

$$
R_2^{(k)}(x) := \sum_{d \leq y} \mu(d) \Delta_2 \left( \frac{x}{d^k}; z \right) \quad \text{and} \quad R_3^{(k)}(x) := O \left( xy^{1-k} e^{-c_1 \delta(y) \log x} \right).
$$

Lemma 3.1. Suppose $A > 0$ is any fixed constant, $T^\varepsilon \ll V \ll T^A$. Then we have

$$
\int_{V}^{2V} \Delta_2^2(u; z) du \ll V^{3/2} z^{-1/2} \log^2 V + V \log^5 V.
$$

Proof. Suppose $\min(z, V^{11}) < N \ll V^B$ is a large parameter, where $B > 0$ is a constant suitably large. By Lemma 3 of Meurman [11] we have

$$
\Delta_2(u; z) = \frac{u^{1/4}}{\pi \sqrt{2}} \sum_{z < n \leq N} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n u} - \frac{\pi}{4} \right) + \Delta_2(u; N),
$$

where $\Delta_2(u; N) \ll u^{-1/4}$ if $\|u\| \gg u^{5/2} N^{-1/2}$, and $\Delta_2(u; N) \ll u^\varepsilon$ otherwise. Thus we have

$$
\int_{V}^{2V} \Delta_2^2(u; z) du \ll \int_1 + \int_2,
$$

where

$$
\int_1 = \int_{V}^{2V} \left| \frac{u^{1/4}}{\pi \sqrt{2}} \sum_{z < n \leq N} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n u} - \frac{\pi}{4} \right) \right|^2 du, \quad \int_2 = \int_{V}^{2V} \Delta_2^2(u; N) du
$$
For \( f_1 \) we have
\[
\int_1 \ll \int_V^{2V} |u^{\frac{1}{3}} \sum_{z < n \leq N} \frac{d(n)}{n^{\frac{3}{4}}} e(2\sqrt{nu})|^2 du
\]
\[
\ll T^{3/2} \sum_{z < n \leq N} \frac{d^2(n)}{n^{3/2}} + V \sum_{z < m < n \leq N} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})}
\]
\[
\ll \frac{V^{3/2} \log^3 V}{z^{1/2}} + V \log^5 V,
\]
where we used the well-known estimates
\[
\sum_{n \leq u} d^2(n) \ll u \log^3 u, \tag{3.2}
\]
\[
\sum_{z < m < n \leq N} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})} \ll \log^5 N \ll \log^5 V.
\]

For \( f_2 \) we have
\[
\int_2 \ll V(V^{5/2+\varepsilon} N^{-1/2} + V^{-1/4}) \ll V^{7/2+\varepsilon} N^{-1/2} + V^{3/4} \ll V.
\]

Now Lemma 3.1 follows from the above estimates. \( \square \)

By Cauchy’s inequality and Lemma 3.1 we get
\[
\int_T^{2T} |R_2^{(k)}(x)|^2 dx = \int_T^{2T} \left| \sum_{d \leq y} \mu(d)d^{-1/2}d^{1/2} \Delta_2 \left( \frac{x}{d^k}; z \right) \right|^2 dx \tag{3.3}
\]
\[
\ll \int_T^{2T} \left( \sum_{d \leq y} d^{-1} \right) \left( \sum_{d \leq y} d \Delta_2 \left( \frac{x}{d^k}; z \right) \right)^2 dx
\]
\[
\ll \sum_{d \leq y} d \int_T^{2T} \left| \Delta_2 \left( \frac{x}{d^k}; z \right) \right|^2 dx \log y
\]
\[
\ll \sum_{d \leq y} d^{k+1} \int_{T/d^k}^{2T/d^k} \left| \Delta_2(u; z) \right|^2 du \log y
\]
\[
\ll \sum_{d \leq y} d^{k+1} \left( \left( \frac{T}{d^k} \right)^{3/2} z^{-1/2} \log^3 T + T d^{-k} \log^5 T \right) \log y
\]
\[
\ll T^{3/2} z^{-1/2} \sum_{d \leq y} d^{1-k/2} \log^4 T + Ty^2 \log^6 T
\]
\[
\ll \begin{cases} 
T^{3/2} z^{-1/2} y^{1/2} \log^4 T + Ty^2 \log^6 T, & \text{if } k = 3,
T^{3/2} z^{-1/2} \log^5 T + Ty^2 \log^6 T, & \text{if } k \geq 4.
\end{cases}
\]
If $k = 4$, we take $y = T^{1/4}e^{-c_2\delta(T)}$, where $c_2 = c_1/4^{8/5}$. It is easy to see that $R^{(k)}_3(x) \ll T^{1/4}e^{-c_3\delta(T)}$ holds for all $T \leq x \leq 2T$, where $0 < c_3 < c_1/4^{8/5}$ is an absolute constant. Hence

$$\int_T^{2T} |R^{(4)}_3(x)|^2 dx \ll T^{3/2}e^{-2c_3\delta(T)}.$$  \hfill (3.4)

If $k \geq 5$, then we have

$$\int_T^{2T} |R^{(k)}_3(x)|^2 dx \ll T^3y^{2-2k}.$$  \hfill (3.5)

Now we consider the mean square of $R^{(k)}_1(x)$. By the elementary formula

$$\cos u \cos v = \frac{1}{2}(\cos(u - v) + \cos(u + v))$$

we may write

$$|R^{(k)}_1(x)|^2 = \frac{x^{1/2}}{2\pi^2} \sum_{d_1,d_2 \leq y} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \sum_{n_1,n_2 \leq z} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}$$

$$\times \cos \left(4\pi \sqrt{\frac{n_1x}{d_1} - \frac{n_2x}{d_2}} - \frac{\pi}{4}\right) \cos \left(4\pi \sqrt{\frac{n_1x}{d_1} - \frac{n_2x}{d_2}} - \frac{\pi}{4}\right),$$

$$= S_1(x) + S_2(x) + S_3(x),$$

where

$$S_1(x) = \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq y; n_1,n_2 \leq z} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}},$$

$$S_2(x) = \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq y; n_1,n_2 \leq z} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \cos \left(4\pi \sqrt{x \left(\sqrt{\frac{n_1x}{d_1}} - \sqrt{\frac{n_2x}{d_2}}\right)}\right),$$

$$S_3(x) = \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq y; n_1,n_2 \leq z} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \sin \left(4\pi \sqrt{x \left(\sqrt{\frac{n_1x}{d_1}} + \sqrt{\frac{n_2x}{d_2}}\right)}\right).$$

We have

$$\int_T^{2T} S_1(x) dx = \frac{B_k(y,z)}{4\pi^2} \int_T^{2T} x^{1/2} dx,$$

$$B_k(y,z) := \sum_{d_1,d_2 \leq y; n_1,n_2 \leq z} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}.$$
By the first derivative test we get

\[
\int_0^T S_2(x) \, dx \ll T E_k(y, z),
\]

(3.8)

where

\[
E_k(y, z) = \sum_{d_1, d_2 \leq y; n_1, n_2 \leq z \atop n_1 d_2 \neq n_2 d_1} \frac{d(n_1)d(n_2)}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}} \min \left( T^{1/2}, \frac{1}{\sqrt{\frac{n_1}{d_1^2} - \frac{n_2}{d_2^2}}} \right).
\]

By the first derivative test again we get

\[
\int_0^T S_3(x) \, dx \ll \sum_{d_1, d_2 \leq y; n_1, n_2 \leq z} \frac{d(n_1)d(n_2)}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}} \frac{1}{\sqrt{\frac{n_1}{d_1^2} + \frac{n_2}{d_2^2}}}^{1/2}
\]

\[
\ll \sum_{d_1, d_2 \leq y; n_1, n_2 \leq z} \frac{d(n_1)d(n_2)}{n_1n_2} \ll y^2 \log^4 z,
\]

where the inequality \(ab \geq 2\sqrt{ab}\) and the estimate \(D(u) \ll u \log u\) were used.

Now the problem is reduced to evaluating \(B_k(y, z)\) and estimating \(E_k(y, z)\).

### 4 Evaluation of \(B_k(y, z)\)

In this section we shall evaluate \(B_k(y, z)\). We have

\[
B_k(y, z) = \sum_{d_1, d_2 \leq y; n_1, n_2 \leq z \atop n_1 d_2 = n_2 d_1} \frac{\mu(d_1)\mu(d_2)d(n_1)d(n_2)(d_1d_2)^{k/2}}{(n_1d_2^k n_2 d_1^k)^{3/4}}
\]

\[
= \sum_{m \leq zy^k} g^2(m; y, z)m^{-3/2},
\]

where

\[
g_k(m; y, z) := \sum_{\substack{m = nd^k \atop n \leq z; d \leq y}} \mu(d) n^{d^{k/2}}.
\]

Let

\[
g_k(m) = \sum_{\substack{m = nd^k \atop n \leq z}} \mu(d) n^{d^{k/2}}, \quad g_0(m) = f_k(m) = \sum_{\substack{m = nd^k \atop n = 1}} d(n)n^{d^{k/2}}.
\]
Let $z_0 := \min(y, z)$. Obviously,
\[ g_k(m; y, z) = g_k(m), \quad m \leq z_0, \]
\[ |g_k(m; y, z)| \leq g_0(m), |g_k(m)| \leq g_0(m), m \geq 1. \]

Thus
\begin{equation}
B_k(y, z) = \sum_{m \leq z_0} g_k^2(m) m^{-3/2} + \sum_{z_0 < m \leq zy^k} g_k^2(m; y, z) m^{-3/2} \tag{4.1}
\end{equation}

\[ = \sum_{m \leq z_0} g_k^2(m) m^{-3/2} + O \left( \sum_{z_0 < m \leq zy^k} |g_0^2(m)| m^{-3/2} \right). \]

For any $1 < U < V < \infty$, we shall estimate the sum
\[ W_k(U, V) := \sum_{U < m \leq V} |g_0^2(m)| m^{-3/2}. \]

Obviously $g_0(m)$ is a multiplicative function. So for $m > 1$, we have
\[ g_0(m) = \prod_{p^\alpha \| m} g_0(p^\alpha). \]

If $1 \leq \alpha \leq k - 1$, then $g_0(p^\alpha) = \alpha + 1$, which implies that if $n$ is $k$-free then $g_0(n) = d(n)$.

Now suppose $ek \leq \alpha < (e + 1)k$ for some integer $e \geq 1$. It can be easily seen that if we write $p^\alpha$ in the form $p^\alpha = nd^k$, then $n = p^{\alpha - jk}$, $d = p^j$, $j = 0, 1, 2, \ldots, e$.

Then we have
\[ g_0(p^\alpha) = \sum_{j=0}^e (\alpha - jk + 1)p^{jk/2} = p^{ek/2} \sum_{j=0}^e (\alpha - jk + 1)p^{-(e-j)k/2} \]
\[ \leq (\alpha + 1)p^{ek/2} \sum_{j=0}^e p^{-(e-j)k/2} = (\alpha + 1)p^{ek/2} \sum_{j=0}^e p^{-jk/2} \]
\[ \leq (\alpha + 1)p^{ek/2} \sum_{j=0}^\infty 2^{-jk/2} \leq 2(\alpha + 1)p^{\alpha/2}, \]

which implies that if $l$ is $k$-full, then
\[ g_0(l) \leq \prod_{p^\alpha \| l} 2(\alpha + 1)p^{\alpha/2} = 2^{\omega(l)}d(l)l^{1/2} \leq d^2(l)l^{1/2}. \]

Let $\delta_k(n)$, $\delta^{(k)}(n)$ denote the characteristic function of $k$-free and $k$-full numbers, respectively. Each integer $m$ can be uniquely written as $m = nl$, $(n, l) = 1$, $\delta_k(n) =$
Thus we have
\[
W_k(U, V) = \sum_{\substack{u \leq n \leq V \\ (n, l) = 1}} g_0^2(n) g_0^2(l) \delta(k)(n) \delta(k)(l) (nl)^{-3/2}
\]
\[
\ll \sum_4 + \sum_5,
\]
where
\[
\sum_4 := \sum_{l \leq U/3, U < nl \leq V} g_0^2(n) g_0^2(l) \delta(k)(n) \delta(k)(l) (nl)^{-3/2},
\]
\[
\sum_5 := \sum_{l > U/3, U < nl \leq V} g_0^2(n) g_0^2(l) \delta(k)(n) \delta(k)(l) (nl)^{-3/2}.
\]

**Lemma 4.1.** We have the estimate
\[
\sum_{n \leq u} d^4(n) \delta(k)(n) \ll u^{1/k} \log^{(k+1)4} u, u \geq 2. \tag{4.2}
\]

**Proof.** For \( \Re s > 1/k \), it is easy to show that
\[
\sum_{n=1}^{\infty} d^4(n) \delta(k)(n) n^{-s} = \zeta(k+1)^4 (ks) G_k(s),
\]
where \( G_k(s) \) is absolutely convergent for \( \Re s > 1/(1 + k) \). And whence (4.2) follows. \( \square \)

By (3.2), partial summation and Lemma 4.1 we have
\[
\sum_4 \ll \sum_{l \leq U/3} g_0^2(l) \delta(k)(l) l^{-3/2} \sum_{U/l < n \leq V/l} g_0^2(n) n^{-3/2}
\]
\[
\ll \sum_{l \leq U/3} g_0^2(l) \delta(k)(l) l^{-3/2} (U/l)^{-1/2} \log^3 U
\]
\[
\ll U^{-1/2} \log^3 U \sum_{l \leq U/3} d^4(l) \delta(k)(l)
\]
\[
\ll U^{-1/2+1/k} \log^{(k+1)4+2} U
\]
and
\[
\sum_5 \ll \sum_{l > U/3} g_0^2(l) \delta(k)(l) l^{-3/2} \sum_m g_0^2(n) n^{-3/2}
\]
\[
\ll \sum_{l > U/3} d^4(l) \delta(k)(l) l^{-1/2}
\]
\[
\ll U^{-1/2+1/k} \log^{(k+1)4+2} U.
\]
Thus
\[
W_k(U, V) \ll U^{-1/2+1/k} \log^{(k+1)^4+2} U.
\] (4.3)

From (4.1) and (4.3) we immediately get
\[
B_k(y, z) = \sum_{m=1}^{\infty} g_k^2(m)m^{-3/2} + O \left( z_0^{-1/2+1/k} \log^{(k+1)^4+2} z_0 \right).
\] (4.4)

5 Estimation of \( E_k(y, z) \)

In this section we shall estimate \( E_k(y, z) \). By a splitting argument, we have
\[
E_k(y, z) \ll E_k(D_1, D_2, N_1, N_2) z^\varepsilon \log^2 y
\] (5.1)
for some \((D_1, D_2, N_1, N_2)\) with \(1 \ll D_j \ll y, 1 \ll N_j \ll z, j = 1, 2\), where
\[
E_k(D_1, D_2, N_1, N_2) = \sum \frac{1}{(d_1d_1)^{k/4}(n_1n_2)^{3/4}} \min \left( T^{1/2}, \frac{1}{|\sqrt{d_1} - \sqrt{d_2}|} \right).
\]

We write
\[
E_k(D_1, D_2, N_1, N_2) = \sum_6 \frac{1}{(d_1d_1)^{k/4}(n_1n_2)^{3/4}} \min \left( T^{1/2}, \frac{1}{|\sqrt{d_1} - \sqrt{d_2}|} \right)
+ \sum_7 \frac{1}{(d_1d_1)^{k/4}(n_1n_2)^{3/4}} \min \left( T^{1/2}, \frac{1}{|\sqrt{d_1} - \sqrt{d_2}|} \right),
\]
where

\[
SC(\sum_6) : d_1 \sim D_1, d_2 \sim D_2, n_1 \sim N_1, n_2 \sim N_2,
\]

\[
\left| \sqrt{n_1} - \sqrt{n_2} \right| \geq \left( \sqrt{n_1} \sqrt{d_1} \sqrt{n_2} \right)^{1/2} / 10,
\]

\[
SC(\sum_7) : d_1 \sim D_1, d_2 \sim D_2, n_1 \sim N_1, n_2 \sim N_2,
\]

\[
\left| \sqrt{n_1} - \sqrt{n_2} \right| < \left( \sqrt{n_1} \sqrt{d_1} \sqrt{n_2} \right)^{1/2} / 10.
\]

Trivially we have
\[
\sum_6 \ll \sum_{d_j \sim D_j, n_j \sim N_j} \frac{1}{(d_1d_1)^{k/4}(n_1n_2)^{3/4}} \left( \sqrt{n_1} \sqrt{d_1} \sqrt{n_2} \right)^{-1/2} \ll D_1D_2 \ll y^2.
\] (5.2)
Suppose $\delta > 0$, and let $\mathcal{A}(D_1, D_2, N_1, N_2; \delta)$ denote the number of the solutions of the inequality
\[
\left| \sqrt{\frac{n_1}{d_1^2}} - \sqrt{\frac{n_2}{d_2^2}} \right| \leq \delta, d_1 \sim D_1, d_2 \sim D_2, n_1 \sim N_1, n_2 \sim N_2.
\]

In order to estimate $\sum_{i}$, we need an upper bound of $\mathcal{A}(D_1, D_2, N_1, N_2; \delta)$.

**Lemma 5.1.** We have
\[
\mathcal{A}(D_1, D_2, N_1, N_2; \delta) \ll \delta(D_1 D_2)^{1 + k/4} (N_1 N_2)^{3/4} + (D_1 D_2 N_1 N_2)^{1/2} \log 2D_1 D_2 N_1 N_2,
\]
where the implied constant is absolute.

**Proof.** We shall use an idea of Fouvry and Iwaniec [3]. Suppose $u$ and $v$ are two positive integers and let $\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta)$ denote the number of the solutions of the inequality (5.3) with $(n_1, n_2) = (u, d_1, d_2) = v$. Set $n_j = m_j u, d_j = l_j v (j = 1, 2)$, then $(m_1, m_2, l_1, l_2)$ satisfies
\[
\left| \sqrt{\frac{m_1}{m_2}} - \sqrt{\frac{k_1}{k_2}} \right| \leq 2^{k/2} \delta D_1^{k/2} N_2^{-1/2}
\]
and
\[
\left| \sqrt{\frac{m_2}{m_1}} - \sqrt{\frac{k_2}{k_1}} \right| \leq 2^{k/2} \delta D_2^{k/2} N_1^{-1/2}.
\]

It is easy to show that $\sqrt{\frac{m_1}{m_2}}$ is $u^2 N_2^{-3/2} N_1^{-1/2}$-spaced, so from (5.4) we get
\[
\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{D_1 D_2}{v^2} \left( 1 + \frac{\delta D_1^{k/2} N_2^{1/2}}{u^2} \right) + \frac{D_1 D_2}{u^2} \frac{\delta D_1^{k/2} N_2^{1/2}}{v^2 u^2}.
\]

Similarly, since $\sqrt{\frac{m_2}{m_1}}$ is $u^2 N_1^{-3/2} N_2^{-1/2}$-spaced, from (5.5) we get
\[
\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{D_1 D_2}{v^2} + \frac{\delta D_1 D_2^{k/2} N_1^{1/2}}{u^2 v^2}.
\]

From the above two estimates we get
\[
\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{D_1 D_2}{v^2} + \frac{\delta D_1 D_2}{u^2 v^2} \min(D_1^{k/2} N_2^{1/2}, D_2^{k/2} N_1^{1/2}) + \frac{\delta (D_1 D_2)^{1 + k/4} (N_1 N_2)^{3/4}}{u^2 v^2}
\]

(5.6)
if we note that \( \min(a, b) \leq a^{1/2}b^{1/2} \).

It is easy to show that \((l_1/l_2)^{k/2}\) is \(v^2D_2^{-2}(D_1/D_2)^{k/2-1}\)-spaced, from (5.4) we get

\[
A_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{N_1N_2}{u^2}(1 + \delta D_1^{k/2}N_2^{-1/2}v^{-2}D_2^2(D_1/D_2)^{-k/2+1}) \\
\ll \frac{N_1N_2}{u^2} + \frac{\delta D_1D_2D_2^{k/2}N_2^{1/2}}{u^2v^2}.
\]

Similarly from (5.5) we get

\[
A_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{N_1N_2}{u^2} + \frac{\delta D_1D_2D_1^{k/2}N_2^{1/2}}{u^2v^2}.
\]

From the above two estimates we have

\[
A_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{N_1N_2}{u^2} + \frac{\delta(D_1D_2)^{1+k/4}(N_1N_2)^{3/4}}{u^2v^2},
\]

which combining (5.6) gives

\[
A_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{\delta(D_1D_2)^{1+k/4}(N_1N_2)^{3/4}}{u^2v^2} + \min\left(\frac{N_1N_2}{u^2}, \frac{D_1D_2}{v^2}\right).
\]

Summing over \(u\) and \(v\) completes the proof of Lemma 5.1. \(\square\)

Now we estimate \(\Sigma_7\). Let \(\Omega = \sqrt{\frac{m_1}{d_1}} - \sqrt{\frac{m_2}{d_2}}\). By Lemma 5.1 the contribution of \(T^{1/2}\) is (note that \(|\Omega| \leq T^{-1/2}\)

\[
\ll \frac{T^{1/2}}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}} A_{u,v}(D_1, D_2, N_1, N_2; T^{-1/2}) \\
\ll \frac{T^{1/2} \log T}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} + D_1D_2.
\]

Divide the remaining range into \(O(\log T)\) intervals of the form \(T^{-1/2} < \delta < |\Omega| \leq 2\delta\). By Lemma 5.1 again we find that the contribution of \(1/|\Omega|\) is

\[
\ll \log T \max_{\delta > T^{-1/2}} \frac{A_{u,v}(D_1, D_2, N_1, N_2; 2\delta)}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}\delta} \\
\ll \frac{T^{1/2} \log^2 T}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} + D_1D_2 \log T.
\]

From the above two estimates we get

\[
\sum_7 \ll \frac{T^{1/2} \log^2 T}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} + y^2 \log T. \tag{5.7}
\]
Now we give another estimate of $\sum_7$. By noting that $\sqrt{n_1\delta_1} \approx \sqrt{n_2\delta_2}$ we get

$$\frac{1}{|\Omega|} = \frac{\sqrt{n_1\delta_1} + \sqrt{n_2\delta_2}}{|n_1\delta_1 - n_2\delta_2|} \ll \frac{(d_1d_2)^k(\sqrt{n_1\delta_1} + \sqrt{n_2\delta_2})}{|n_1d_2^k - n_2d_1^k|} \ll (d_1d_2)^k(\sqrt{n_2\delta_2})^{1/2} \ll (d_1d_2)^{3k/4}(n_1n_2)^{1/4} \ll (D_1D_2)^{3k/4}(N_1N_2)^{1/4}.$$ 

The range of $\Omega$ can be divided into $O(\log T)$ intervals of the form

$$(D_1D_2)^{-3k/4}(N_1N_2)^{-1/4} \ll \delta \leq |\Omega| \leq 2\delta.$$ 

By Lemma 5.1 we have

$$\sum_7 \ll \frac{1}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}} \sum_\Omega \frac{1}{|\Omega|} \ll \sum_\Omega \frac{\log T}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}} \max_\delta \frac{A_{a,v}(D_1, D_2, N_1, N_2; \delta)}{\delta} \ll (D_1D_2)^{(k+1)/2} \log^2 T,$$ 

if we note that $\delta \gg (D_1D_2)^{-3k/4}(N_1N_2)^{-1/4}$.

From (5.7) and (5.8) we get

$$\sum_7 \ll y^2 \log T + \min \left( T^{1/2} \frac{T^{1/2}}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}}, (D_1D_2)^{(k+1)/2} \log^2 T \right) \ll y^2 \log T + \left( T^{1/2} \frac{T^{(2k+2)/3k}}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} \right) ((D_1D_2)^{(k+1)/2})^{(k-2)/3k} \log^2 T \ll y^2 \log T + T^{(k+1)/3k} \log^2 T.$$ 

Finally, from (5.1), (5.2) and (5.9) we have

$$E_k(y, z) \ll y^2 z^\varepsilon \log^4 T + T^{(k+1)/3k} z^\varepsilon \log^4 T.$$ 

(5.10)

6 Proof of Theorem 1(Completion)

First consider the case $k = 4$. Take $z = e^{10c_3\delta(T)}$, where $c_3$ was the constant in (3.4).

From (3.3) and (3.4) we get

$$\int_T^{2T} |R_2^{(4)}(x) + R_3^{(4)}(x)|^2 dx \ll T^{3/2} e^{-2c_3\delta(T)}.$$
From (3.6)-(3.9), (4.4) and (5.10) we get
\[
\int_T^{2T} |R_1^{(4)}(x)|^2 \, dx = \frac{B_4}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{3/2} z_0^{-1/4} \log^{627} T) + O(T y^2 \varepsilon \log^{5} T + T^{17/12} \varepsilon \log^{6} T)
\]
\[
= \frac{B_4}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{3/2} e^{-2c_3\delta(T)}).
\]

From the above two estimates and Cauchy’s inequality we get
\[
\int_T^{2T} R_1^{(4)}(x)(R_2^{(4)}(x) + R_3^{(4)}(x)) \, dx \ll T^{3/2} e^{-c_3\delta(T)}.
\]

From the above three estimates we get
\[
\int_T^{2T} |\Delta^{(4)}(x)|^2 \, dx = \int_T^{2T} |R_1^{(4)}(x)|^2 \, dx + 2 \int_T^{2T} R_1^{(4)}(x)(R_2^{(4)}(x) + R_3^{(4)}(x)) \, dx \quad (6.1)
\]
\[
= \int_T^{2T} |R_2^{(4)}(x) + R_3^{(4)}(x)|^2 \, dx
\]
\[
= \frac{B_4}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{3/2} e^{-c_3\delta(T)}),
\]

which implies the case \( k = 4 \) of Theorem 1.

Now suppose \( k \geq 5 \). Take \( z = T^{1-\varepsilon} \). From (3.3) and (3.5) we get
\[
\int_T^{2T} |R_2^{(k)}(x) + R_3^{(k)}(x)|^2 \, dx \ll T^{1+\varepsilon} y^2 + T^{3} y^{2-2k}.
\]

From (3.6)-(3.9), (4.4) and (5.10) we get
\[
\int_T^{2T} |R_1^{(k)}(x)|^2 \, dx = \frac{B_k}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{3/2+\varepsilon} y^{1/k-1/2})
\]
\[
+ O(T^{1+\varepsilon} y^2 + T^{1+(k+1)/3k+\varepsilon}).
\]

The above two estimates imply
\[
\int_T^{2T} R_1^{(k)}(x)(R_2^{(k)}(x) + R_3^{(k)}(x)) \, dx \ll T^{5/4+\varepsilon} y + T^{9/4} y^{1-k}.
\]

From the above three estimates we get
\[
\int_T^{2T} |\Delta^{(k)}(x)|^2 \, dx = \frac{B_k}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{1+(k+1)/3k+\varepsilon})
\]
\[
+ O(T^{5/4+\varepsilon} y + T^{9/4} y^{1-k} + T^{3/2+\varepsilon} y^{1/k-1/2}).
\]

Now on taking \( y = T^{5/26} \) if \( k = 5 \) and \( y = T^{1/k-\varepsilon} \) if \( k \geq 6 \), we get
\[
\int_T^{2T} |\Delta^{(k)}(x)|^2 \, dx = \frac{B_k}{4\pi^2} \int_T^{2T} x^{1/2} \, dx + O(T^{\delta_k+\varepsilon}), \quad (6.2)
\]
where \( \delta_k \) was defined in Section 1. The case \( k \geq 5 \) of Theorem 1 now follows from (6.2).
7 An expression of $\Delta(1, 1, k; x)$

In order to prove Theorem 2, we shall give an expression of $\Delta(1, 1, k; x)$ in this section. We write

$$D(1, 1, k; x) = \sum_{nd^k \leq x} d(n)$$

$$= \sum_{d \leq y} D\left(\frac{x}{d^k}\right) + \sum_{n \leq \frac{x}{yk}} d(n) \left[\left(\frac{x}{n}\right)^{1/k}\right] - D\left(\frac{x}{y^k}\right) [y]$$

$$= \sum_8 + \sum_9 - \sum_{10}$$

say, where $x^\varepsilon \ll y \ll x^{1/k-\varepsilon}$ is a parameter.

We write $\sum_8$ as

$$\sum_8 = \sum_{d \leq y} \left(\frac{x}{d^k} \log \frac{x}{d^k} + (2\gamma - 1)\frac{x}{d^k} + \Delta\left(\frac{x}{d^k}\right)\right)$$

$$= x \log x \sum_{d \leq y} \frac{1}{d^k} - kx \sum_{d \leq y} \frac{\log d}{d^k} + (2\gamma - 1)x \sum_{d \leq y} \frac{1}{d^k} + \sum_{d \leq y} \Delta\left(\frac{x}{d^k}\right).$$

By the well-known Euler-Maclaurin’s formula we have

$$\sum_{d \leq y} \frac{1}{d^k} = \zeta(k) - \sum_{d > y} \frac{1}{d^k} = \zeta(k) - \frac{y^{1-k}}{k-1} - \psi(y)y^{-k} + O(y^{-k-1})$$

and

$$\sum_{d \leq y} \frac{\log d}{d^k} = -\zeta'(k) - \sum_{d > y} \frac{\log d}{d^k}$$

$$= -\zeta'(k) + \frac{y^{1-k} \log y}{1-k} - \frac{y^{1-k}}{(k-1)^2} - \frac{\psi(y) \log y}{y^k} + O(y^{-k-1} \log y).$$

From the above three formulas we get

$$\sum_8 = \zeta(k)x \log x - \frac{xy^{1-k} \log x}{k-1} - \psi(y)xy^{-k} \log x$$

$$+ k\zeta'(k)x - \frac{kxy^{1-k} \log y}{1-k} + \frac{xy^{1-k}}{(k-1)^2} + \frac{kx\psi(y) \log y}{y^k}$$

$$+ (2\gamma - 1)\zeta(k)x - (2\gamma - 1)\frac{xy^{1-k}}{k-1} - (2\gamma - 1)\psi(y)xy^{-k}$$

$$+ \sum_{d \leq y} \Delta\left(\frac{x}{d^k}\right) + O(xy^{-k-1} \log x).$$

(7.2)
We write
\[ \sum_9 = \sum_{n \leq \frac{x}{y^k}} d(n) \left( (x/n)^{1/k} - 1/2 - \psi((x/n)^{1/k}) \right) \]
\[ = x^{1/k} \sum_{n \leq \frac{x}{y^k}} d(n)n^{-1/k} - \frac{1}{2}D(x^{-k}) - \sum_{n \leq \frac{x}{y^k}} d(n)\psi((x/n)^{1/k}). \]

By partial summation we get \( M = xy^{-k} \)
\[ \sum_{n \leq M} d(n)n^{-1/k} = \int_{1-}^M \frac{dD(u)}{u^{1/k}} = \int_{1-}^M \frac{d(u \log u + (2\gamma - 1)u)}{u^{1/k}} + \int_{1-}^M \frac{d\Delta(u)}{u^{1/k}} \]
\[ = \int_{1-}^M \frac{\log u + 1 + 2\gamma - 1}{u^{1/k}} du + \frac{\Delta(M)}{M^{1/k}} + \frac{1}{k} \int_{1-}^M \frac{\Delta(u)}{u^{1+1/k}} du \]
\[ = \zeta^2(1/k) + \frac{M^{1-1/k} \log M}{1 - 1/k} - \frac{M^{1-1/k}}{(1 - 1/k)^2} + \frac{M^{1-1/k}}{1 - 1/k} + (2\gamma - 1) \frac{M^{1-1/k}}{1 - 1/k} \]
\[ + \Delta(M)M^{-1/k} + O(M^{-1/k}), \]
where we used the estimate
\[ \int_M^\infty \frac{\Delta(u)}{u^{1+1/k}} du \ll M^{-1/k}, \]
which follows from the well-known estimate \( \int_1^t \Delta(u)du \ll t. \)

From the above two formulas we get
\[ \sum_9 = \zeta^2(1/k)x^{1/k} + \frac{xy^{1-k} \log xy^{-k}}{1 - 1/k} - \frac{xy^{1-k}}{(1 - 1/k)^2} + \frac{xy^{1-k}}{1 - 1/k} \]
\[ + (2\gamma - 1) \frac{xy^{1-k}}{1 - 1/k} + y\Delta(xy^{-k}) - \frac{1}{2}D(xy^{-k}) \]
\[ - \sum_{n \leq \frac{x}{y^k}} d(n)\psi((x/n)^{1/k}) + O(y). \] (7.3)

For \( \sum_{10} \) we have
\[ - \sum_{10} = \psi(y)xy^{-k} \log xy^{-k} + (2\gamma - 1)\psi(y)xy^{-k} + \psi(y)\Delta(xy^{-k}) \]
\[ + \frac{1}{2}D(xy^{-k}) - xy^{1-k} \log xy^{-k} - (2\gamma - 1)xy^{1-k} - y\Delta(xy^{-k}). \] (7.4)

From (7.1)–(7.4) we get
\[ \Delta(1, 1, k; x) = \sum_{d \leq y} \Delta\left(\frac{x}{y^k}\right) - \sum_{n \leq \frac{x}{y^k}} d(n)\psi((x/n)^{1/k}) + O(y) \]
\[ + O(xy^{-k-1} \log x) + O(|\Delta(xy^{-k})|). \]
From $\Delta(u) \ll u^{1/3}$ we get

$$|\Delta(xy^{-k})| \ll x^{1/3}y^{-k/3} \ll y + xy^{-k-1}.$$}

Thus we get the following Lemma.

**Lemma 7.1.** Suppose $x^\varepsilon \ll y \ll x^{1/k-\varepsilon}$. Then

$$\Delta(1, 1, k; x) = \sum_{d \leq y} \Delta(\frac{x}{y^k}) - \sum_{n \leq \frac{x}{y}} d(n)\psi\left(\left(\frac{x}{n}\right)^{1/k}\right) + O(x^{y-1}\log x) + O(y).$$

### 8 Proof of Theorem 2

It suffices for us to evaluate $\int_T^{2T} \Delta^2(1, 1, k; x)dx$ for large $T$. Suppose $T^\varepsilon \ll y \ll T^{1/k-\varepsilon}$ is a parameter to be determined later and $z = T^{1-\varepsilon}$. For simplicity, we write $L = \log T$ in this section. Similar to (3.1), by Lemma 7.1 we may write

$$\Delta(1, 1, k; x) = R_{1,k}(x) + R_{2,k}(x) - R_{3,k}(x), \quad (8.1)$$

where

$$R_{1,k}(x) := \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{d \leq y} \frac{1}{d^{k/4}} \sum_{n \leq z} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{\frac{nx}{d}} - \frac{\pi}{4}\right),$$

$$R_{2,k}(x) := \sum_{d \leq y} \Delta_2(\frac{x}{d^k}; z),$$

$$R_{3,k}(x) := \sum_{n \leq \frac{x}{y^k}} d(n)\psi((x/n)^{1/k}) + O(x^{y-1}\log x) + O(y).$$

Similar to the mean square of $R_1^{(k)}(x)$, we can prove that

$$\int_T^{2T} |R_{1,k}(x)|^2dx = \frac{C_k}{4\pi^2} \int_T^{2T} x^{1/2}dx + O(T^{3/2+\varepsilon}y^{1/k-1/2})$$

$$+ O(T^{1+\varepsilon}y^2 + T^{1+(k+1)/3k+\varepsilon}). \quad (8.2)$$

From (3.3) we have

$$\int_T^{2T} |R_{2,k}(x)|^2dx \ll Ty^2L^6. \quad (8.3)$$

Now we study the mean square of

$$S(x) = \sum_{n \leq \frac{x}{y^k}} d(n)\psi((x/n)^{1/k}).$$
Let $J = [\log^{-1} 2 \log(Ty^{-k}L^{-1})]$, then $J \ll \mathcal{L}$ and we may write
\[ S(x) = \sum_{j=0}^{J} S_j(x) + O(\mathcal{L}^2), \]
\[ S_j(x) := \sum_{x2^{-j-1}y^{-k} < n \leq x2^{-j}y^{-k}} d(n)\psi((x/n)^{1/k}). \]

Let $1/T \ll \eta < 1/10$ is a real number and let $\eta T = N$. Let
\[ M(x, \eta) := \sum_{\eta x < n \leq 2\eta x} d(n)\psi((x/n)^{1/k}). \]
Then $S_j(x) = M(x, 2^{-j-1}y^{-k})$, $j = 0, 1, \ldots, J$. We shall study $\int_T^{2T} M^2(x, \eta)dx$.

According to Vaaler [16], we may write
\[ \psi(t) = \sum_{|h| \leq N} a(h)e(ht) + O\left(\sum_{|h| \leq N} b(h)e(ht)\right) \]
with $a(h) \ll 1/|h|$, $b(h) \ll 1/N$. Thus
\begin{align*}
M(x, \eta) &= \sum_{1 \leq |h| \leq N} a(h) \sum_{\eta x < n \leq 2\eta x} d(n)e(h(x/n)^{1/k}) \\
&\quad + O\left(\sum_{|h| \leq N} b(h) \sum_{\eta x < n \leq 2\eta x} d(n)e(h(x/n)^{1/k})\right) \\
&\ll 1 + \sum_{1 \leq h \leq N} h^{-1/2}h^{-1/2} \left|\sum_{\eta x < n \leq 2\eta x} d(n)e(h(x/n)^{1/k})\right|^2.
\end{align*}

By Cauchy’s inequality we get
\[ M^2(x, \eta) \ll 1 + \sum_{1 \leq h \leq N} \mathcal{L} \left|\sum_{\eta x < n \leq 2\eta x} d(n)e(h(x/n)^{1/k})\right|^2. \]
Integrating, squaring out and then by the first derivative test we get

\[
\int_T^{2T} M^2(x, \eta) dx \ll T + \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \int_T^{2T} \left| \sum_{\eta x < n \leq 2 \eta x} d(n) e(h(x/n)^{1/k}) \right|^2 dx
\]

\[
= T + \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \int_T^{2T} \sum_{\eta x < n \leq 2 \eta x} d^2(n) dx
\]

\[
+ \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \int_T^{2T} \sum_{\eta x < n, \eta x \leq 2 \eta x} d(m) d(n) e(h x^{1/k} (m^{-1/k} - n^{-1/k})) dx
\]

\[
= O(TN\mathcal{L}^5) + \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \sum_{N<n,m \leq 4N \atop m \neq n} d(m) d(n) \int_{I(m,n)} e(h x^{1/k} (m^{-1/k} - n^{-1/k})) dx
\]

\[
\ll TN\mathcal{L}^5 + \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \sum_{N<n,m \leq 4N \atop m \neq n} T^{1-1/k} d(n) d(m) \frac{T^{1-1/k} d(n) d(m)}{h |m^{-1/k} - n^{-1/k}|}
\]

\[
\ll TN\mathcal{L}^5 + \sum_{1 \leq h \leq N} \frac{\mathcal{L}}{h} \sum_{N<n,m \leq 4N \atop m \neq n} T^{1-1/k} N^{1+1/k} d(n) d(m) \frac{T^{1-1/k} N^{1+1/k} d(n) d(m)}{h |m - n|}
\]

\[
\ll TN\mathcal{L}^5 + T^{1-1/k} N^{2+1/k} \mathcal{L}^5,
\]

where \(I(m, n)\) is a subinterval of \([T, 2T]\).

From Cauchy’s inequality and the above estimate we get

\[
\int_T^{2T} S^2(x) dx \ll \int_T^{2T} \left| \sum_{j=0}^{J} S_j(x) \right|^2 dx + T \mathcal{L}^2
\]

\[
\ll \mathcal{L} \sum_{j=0}^{J} \int_T^{2T} |S_j(x)|^2 dx + T \mathcal{L}^2
\]

\[
\ll (T^2 y^{-k} + T^3 y^{-2k-1}) \mathcal{L}^6,
\]

which implies that

\[
\int_T^{2T} R_{3k}^2(x) dx \ll (T^2 y^{-k} + T^3 y^{-2k-1}) \mathcal{L}^6 + T y^2. \quad (8.4)
\]

From (8.2)-(8.4) and Cauchy’s inequality we get

\[
\int_T^{2T} R_{1k}(x) (R_{2k}(x) + R_{3k}(x)) dx \ll T^{5/4} y \mathcal{L}^3 + T^{7/4} y^{-k/2} \mathcal{L}^3 + T^{9/4} y^{-k-1/2} \mathcal{L}^3. \quad (8.5)
\]
From (8.1)-(8.5) we get

\[
\int_T^{2T} \Delta^2(1,1,k;x)dx = \frac{C_k}{4\pi^2} \int_T^{2T} x^{1/2}dx
\]

\[
+ O(T^{5/4}y\mathcal{L}^3 + T^{7/4}y^{-k/2}\mathcal{L}^3 + T^{9/4}y^{-k-1/2}\mathcal{L}^3)
\]

\[
+ O(T^{3/2}y^{1-k-1/2}\mathcal{L}^{(k+1)^4+2} + T^{1+(k+1)/3k+\varepsilon}).
\]

Now on taking \(y = T^{2/9}\) if \(k = 3\), \(y = T^{1/5}\mathcal{L}^{2496/5}\) if \(k = 4\), \(y = T^{5/26}\mathcal{L}^{10(6^3-1)/13}\) if \(k = 5\) and \(y = T^{1/k-\varepsilon}\) if \(k \geq 6\) we get

\[
\int_T^{2T} \Delta^2(1,1,k;x)dx = \frac{C_k}{4\pi^2} \int_T^{2T} x^{1/2}dx + \begin{cases}
O(T^{53/36}\mathcal{L}^3), & \text{if } k = 3, \\
O(T^{29/20}\mathcal{L}^{503}), & \text{if } k = 4, \\
O(T^{75/52}\mathcal{L}^{1000}), & \text{if } k = 5, \\
O(T^{3/2-1/2k+1/k^2+\varepsilon}), & \text{if } k \geq 6.
\end{cases}
\]

(8.6)

Theorem 2 follows from (8.6) immediately.

References

[1] R. C. Baker, The square-free divisor problem, Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 179, 269–277.

[2] R. C. Baker, The square-free divisor problem. II, Quart. J. Math. Oxford Ser. (2) 47 (1996), no. 186, 133–146.

[3] E. Fouvry and H. Iwaniec, Exponential sums with monomials, J. Number Theory 33 (1989), no. 3, 311–333.

[4] O. Hölder, Über einen asymptotischen Ausdruck, Acta Math. 59 (1932), 89–97.

[5] M. N. Huxley, Exponential sums and lattice points III, Proc. London Math. Soc. 87 (3) (2003), 591–609.

[6] A. Ivić, The Riemann-zeta function, John Wiley & Sons, New York, 1985.

[7] A. Ivić, The general divisor problem, J. Number Theory 27 (1987), no. 1, 73–91.

[8] E. Krätzel, Teilerprobleme in drei dimensionen, Math. Nachr. 42 (1969), 275–288.

[9] A. Kumchev, The \(k\)-free divisor problem, Monatsh. Math. 129 (2000), 321–327.

[10] F. Mertens, Über einige asymptotische Gesetze der Zahlentheorie, J. Reine. Angew. Math. 77 (1874), 289–338.
[11] T. Meurman, On the mean square of the Riemann zeta-function, Quart. J. Math. Oxford Ser. (2) 38 (1987), no. 151, 337–343.

[12] W. G. Nowak, M. Schmeier, Conditional asymptotic formulae for a class of arithmetical functions, Proc. Amer. Math. Soc. 103 (1988), 713–717.

[13] B. Saffari, Sur le nombre de diviseurs ”r-libres” d’un entier, et sur les points à coordonnées entières dans certaines régions du plan, C. R. Acad. Sci. Paris. Sér. A-B 266 (1968), A601–A603.

[14] V. Siva Rama Prasad, D. Suryanarayana, The number of k-free divisors of an integer, Acta Arith. 17 (1970/71), 345–354.

[15] K. C. Tong, On divisor problem III, Acta Math. Sinica 6 (1956), 515-541.

[16] J. D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. 12 (1985), 183–216.