Classical Effective Field Theory and Caged black holes

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Abstract: Matched asymptotic expansion is a useful technique in General Relativity and other fields whenever interaction takes place between physics at two different length scales. Here matched asymptotic expansion is argued to be equivalent quite generally to Classical Effective Field Theory (ClEFT) where one (or more) of the zones is replaced by an effective theory whose terms are organized in order of increasing irrelevancy, as demonstrated by Goldberger and Rothstein in a certain gravitational context. The ClEFT perspective has advantages as the procedure is clearer, it allows a representation via Feynman diagrams, and divergences can be regularized and renormalized in standard field theoretic methods. As a side product we obtain a wide class of classical examples of regularization and renormalization, concepts which are usually associated with Quantum Field Theories. We demonstrate these ideas through the thermodynamics of caged black holes, both simplifying the non-rotating case, and computing the rotating case. In particular we are able to replace the computation of six two-loop diagrams by a single factorizable two-loop diagram, as well as compute certain new three-loop diagrams. The results generalize to arbitrary compactification manifolds. For caged rotating black holes we obtain the leading correction for all thermodynamic quantities. The angular momentum is found to non-renormalize at leading order.
1. Introduction and Summary

Matched Asymptotic Expansion (MAE) is an analytical tool which applies to problems containing two (or more) separate scales. In mathematical physics this idea goes back as far as Laplace who used it to find the shape of a drop of liquid on a surface - see [1] and references therein for a historical review. In [2, 3] Gorbonos and one of the authors (BK) applied MAE to the problem of small caged black holes, namely black holes which are much smaller than their background compactification manifold. On p.7 of [3] it was recognized that the divergences which appear at higher order of MAE and their regularization are “reminiscent of renormalization in Quantum Field Theory”. In [4] Chu, Goldberger and Rothstein applied an effective field theory approach, rather than MAE, to the same problem of small black holes, thereby simplifying the derivation of their thermodynamics and
extending it to a higher order. That work built on the ideas of Goldberger and Rothstein regarding an effective field theory of gravity for extended objects \[5\]; see also \[6\] and a pedagogical introduction in \[7\].

In this paper we further develop these ideas. The paper is composed of three parts. In the first part, subsection 2.1, we argue for a quite general equivalence of MAE and effective field theory. In the other parts we proceed to apply and illustrate these ideas in the context of the thermodynamics of caged black holes. In the second part, we start in subsections 2.2–2.6 by describing several improvements to the effective field theory analysis of caged black holes, and we continue in section 3 to significantly economize the derivation of the thermodynamics of static caged black holes, and to perform a new computation. In the third part, section 4, we apply the method to obtain new results for rotating caged black holes. We end this introduction with a summary of results.

General Equivalence of MAE and EFT

In subsection 2.1 we claim that Matched Asymptotic Expansion (MAE) is equivalent quite generally to an effective field theory. We observe the phenomena of regularization and renormalization in this classical set-up and as a way of stressing it we refer to the method as Classical Effective Field Theory (ClEFT). Even though ClEFT is formally equivalent to MAE we indicate the advantages of the ClEFT perspective: a clear representation via Feynman diagrams, the effective action as a way of studying a zone once and for all, easy power counting, and the use of dimensional regularization. Finally we characterize quite generally the domain of validity of ClEFT to be whenever an extended object (such as a soliton) moves in a background whose length scale is much larger than the object’s size.

Caged black holes

As a concrete realization of the ideas regarding the equivalence of Matched Asymptotic Expansion and Effective Field Theory we apply them to the problem of caged black holes, namely black holes in the background \( \mathbb{R}^d \times X \) where the compactification manifold will be taken to be \( X = S^1 \) throughout most of the paper.

This problem was motivated by the effort to establish the phase diagram of the black-hole black-string transition \[3\]; see the reviews \[3, 10\] and references therein. The problem was engaged with a combination of analytic and numeric methods. It was studied analytically in \[11\] using adapted coordinates in a single patch; in \[4\] with a two-zone MAE; in \[12\] the asymptotic thermodynamics properties were computed to \( \mathcal{O}(m_0^3) \) in 5d MAE; in \[3\] the \( \mathcal{O}(m_0^2) \) correction was found for all \( d \) together with a systematic discussion of regularization; Finally in \[4\] effective field theory was used to compute to order \( \mathcal{O}(m_0^3) \) for all \( d \). Numerical studies include a 5d simulation \[13\]; a 6d simulation \[14\] relying on an earlier brane-world simulation \[15\] and finally \[16\] which produced significantly larger 5d black holes. For another perspective see a review of “phenomenological” work on black holes in theories with large extra dimensions \[17\].

In section 3 (except for subsection 2.1) we study static caged black holes. We present several improvements to the ClEFT method which allow us to reproduce the results of Chu, Goldberger and Rothstein \[3\] (and to perform a new computation). Actually we believe
that we have finally discovered the shortest route to these results. The main improvements to the method are

- We perform a change of variables through a dimensional reduction over $t$. It has the advantage that the propagator is diagonal with respect to the field $\phi \sim h_{00}/2$ which couples to the world-line at lowest order (through the interaction $(-)m_0\phi$).

- It is shown that the mass renormalization $\delta m$ can be read off a zero-point function, rather than a 1-point function. Moreover, this zero-point function serves as a partition function and thermodynamic potential. From it we are able to derive the tension, temperature and area (completing all equations of state) at the price of computing another quantity, the red-shift, up to a similar order.

- We note several points where the classical nature of our problem allows simplification through the elimination of certain quantum features which appear in the approach of Rothstein and Goldberger [5, 7] which is based on a background in Quantum Field Theory (QFT). These features are: Planck’s constant $\hbar$ (implicit in the definition of the Planck mass), the complex number $i$ and Feynman path integrals.

Through these improvements we obtain the following results in section 3

- Our method replaces the 6 diagrams required for the 2-loop computation of $\delta m$ in [4] (fig.2) by a single diagram (fig.3(b))! Moreover, it does not require the quartic vertex of GR, nor the cubic one as it happens. This diagram happens to manifestly factorize and hence is simple to compute, thereby explaining the factorization which was observed in [4].

- We reproduce other thermodynamic quantities: the temperature (3.18), tension and entropy (3.24).

- Ignoring finite-size effects it is now possible to proceed to higher orders. We perform part of the calculation of $\delta m$ to order $O(m_0^4)$ (3.15). Neglecting finite-size effects is justified for $d < 7$, which happen to be the numerically studied cases.

- We point out that the results immediately generalize to any (Ricci-flat) compactification manifold.

**Rotating black holes**

In addition to economizing the computations for static black holes (and performing a certain extension thereof), we apply in section 4 the CIETF method to compute for the first time the caging effect on the thermodynamics of the rotating Myers-Perry black holes [18].

The thermodynamics of caged rotating black holes could be useful for determining a black-hole black-string phase diagram in the presence of angular momentum. Non-uniform rotating black string solutions were studied in the case of equal angular momenta in 6d [19]. In the rotating case the isometry group is much reduced and accordingly a MAE analysis
would require a much larger number of metric components. In ClEFT thermodynamics, on the other hand, the reduced symmetry hardly manifests itself and up to the relevant order all we need to add are several new world-line vertices. Actually determining these vertices is one of our motivations as a step towards the full effective action of moving and spinning black holes.

We proceed to describe our main results. We determine the two leading world-line vertices which involve the local angular momentum $j_0$ (figure 11) and confirm that they agree with the existing literature [20, 21, 22]. We compute the leading expressions for all thermodynamic quantities: mass (4.12), angular momentum (4.10), temperature and angular velocity (4.13), and finally tension and area (4.22). In the computation of the mass we proceed to compute $\delta m$ to order $O(j_0^2) \simeq O(m r_0^2)$. Somewhat unexpectedly we find that although the mass renormalizes the angular momentum does not renormalize at leading order (4.10). It would be interesting to know whether this non-renormalization holds to all orders and if so to prove it. It could be especially interesting if insight from the mechanism behind this non-renormalization could apply to non-renormalization in QFT as well.

Note added (v2). Minor changes upon publication. These include: a global change in sign $J \rightarrow -J$ to conform with standard conventions, appearing in (4.3, 4.5, 4.9) and fig. 11 which does not affect the final results; a factor of 2 in (4.6); and at the top of p.3 in the first item we corrected that our change of variables should not be considered a change of gauge.

Address. In presenting this paper we hope that it would be especially enjoyed by Gorbonos, Chu, Goldberger and Rothstein upon whose work we build here.

2. Main Ingredients

We start with a general discussion of the equivalence of Matched Asymptotic Expansion (MAE) with Classical Effective Field Theory (ClEFT). Then we proceed to set-up the problem of caged black holes, and discuss several more specific improvements to the method in that case.

2.1 From MAE to ClEFT

Matched Asymptotic Expansion (MAE) entails the use of two zones (or more) at widely separated scales. In each zone one of the scales is fixed while the other is infinitely small or infinitely large. The interaction dialogue between the scales occurs through supplying each other with boundary conditions. For instance, applying MAE to the problem of caged black holes [4, 5] requires two zones: the near zone where the black hole has fixed size $r_0$ but the compactification scale is infinitely far, and an asymptotic zone where the compactification size $L$ is fixed and the black hole is point-like and fixed to the origin.

Divergences and the associated need for regularization were observed to appear at higher orders in the small parameter $r_0/L$ [4]. The first instance was while solving for the next to leading correction in the asymptotic zone. In that zone the leading correction
is simply the Newtonian potential of the point-like object, which solves a Laplace-like equation. At the next order one needs to solve a similar equation, only the non-linear nature of General Relativity (GR) introduces a source term quadratic in the Newtonian potential (and its derivatives). Since the Newtonian potential diverges near the origin (the location of the object) this source term has an even worse divergence, and the Green’s function integral diverges. Quantitatively, the first correction to the metric \( h^{(1)} \) is determined by the Newtonian potential \( \Phi \) which solves \( \Delta \Phi \propto \delta(\vec{r}) \); it behaves as \( \Phi \sim 1/r^{d-3} \) where \( r \) is the distance from the black hole, and \( d \) is the total space-time dimension; the equation for the second order perturbation to the metric \( h^{(2)} \) is schematically \( \Delta h^{(2)} = Src \sim (\partial \Phi)^2 \sim 1/r^{2(d-2)} \); hence the Green function integral behaves as \( h^{(2)}(x') = \int dx G(x', x) Src(x) \sim \int r^{d-2} dr / r^{2(d-2)} \sim 1/\epsilon^{d-3} \), and certainly diverges for all relevant dimensions \( d \geq 5 \).

The concept of renormalization can also be seen to arise in the context of caged black holes. The simplest example is the mass of the black hole: while an observer at a distance \( r_0 \ll r \ll L \) measures the local mass \( m_0 \), a distant asymptotic observer at \( r \gg L \) measures a different mass \( m \), which is slightly smaller, the leading effect being the Newtonian binding energy between the black hole and its images. This can be interpreted as a dependence of the mass on the length scale at which it is measured, exactly in the spirit of renormalization.

Recall that historically, divergences quite similar to these obstructed the development of Quantum Field Theory (QFT) for about two decades from soon after the discovery of quantum mechanics in 1926-7 till their treatment with counter-terms and the completion of the theory of Quantum Electro-Dynamics (QED) in 1948. It took even longer, till the early 1970’s to reveal the renormalization significance of regularization. Actually several Nobel prizes were awarded for these achievements: to Feynman, Schwinger and Tomonaga for QED and to Wilson for the theory of second order phase transitions which is intimately connected with renormalization.

It is quite obvious that at the time physicists were not familiar with any examples of regularization and renormalization, definitely not in classical physics. Even today we are not familiar with too many such examples (the authors would appreciate correspondence on this issue). A notable exception is the classical regularization and renormalization near the boundary of Anti-de-Sitter space [23].

Here we claim that Matched Asymptotic Expansion is equivalent quite generally to an effective field theory. The equivalence is achieved by replacing one (or more) of the zones by a point-like effective action (usually it is the near zone but we may also consider replacing an asymptotic zone by effective boundary conditions at the asymptotic region of the near zone). The physics of the eliminated zone is coded in various interaction terms in the effective action. A more precise statement of the equivalence is that the ClEFT is equivalent not to all the observables of the MAE but rather to those which do not reside in the replaced zone.

Let us present a general argument for this claim. Whenever we have two (or more) widely separated length scales, we may cleanly decompose the fields into corresponding components by performing a spatial Fourier transform and dividing the field according to the scale of the spatial frequencies. This is equivalent to the decomposition into zones. Then we may integrate out the field component in one of the zones. Integrating out a near-
zone with its high spatial frequencies is analogous to a Born-Oppenheimer approximation which integrates out fast degrees of freedom. The definition of classically integrating out a field and its eligibility are discussed in subsection 2.6. This integration replaces by definition the integrated field or zone with interaction terms in the effective action. The opposite direction poses an interesting question, namely to what extent can the action in the integrated-out zone be reconstructed given the effective action.

The concepts of regularization and renormalization appear in these equivalent methods quite generally. We stress the fact that this happens in a completely classical set-up by referring to the theory as a Classical Effective Field Theory, acronym ClEFT. This perspective provides a large class of new classical examples of renormalization and regularization (some features will appear only in non-linear theories), a class which includes boundary layer phenomena in hydrodynamics, waves in a background with defects and black holes moving in a slowly varying background. These examples may be useful in building classical insight into the appearance of the same concepts in QFT.

Despite ClEFT being equivalent to MAE it does offer several advantages in perspective, as well as in practical computations

- Feynman diagrams provide as a clear representation of the computation.

- The near zone needs to be considered only once.

  In MAE one alternates between zones. In ClEFT we need to go once through the process of replacing the near zone by an effective action (at least up to a prescribed order), and then we can forget the near zone altogether.

- Easy power counting.

  In ClEFT the effective interaction terms are ordered by powers of the small parameter which determine an order of relevancy. Accordingly the power counting of each Feynman diagram is easily recognized in terms of the vertices which appear in it.

- Dimensional regularization.

  While in GR one of the standard regulators is Hadamard’s (Partie finie), which requires certain care and attention to case by case details, the field theory perspective suggests dimensional regularization which proves to be symmetry preserving, straight-forward and efficient in ClEFT just as it is in more standard QFT. We find dimensional regularization to be equivalent to the regularization used in [3] – it is equivalent to Hadamard’s method as both are essentially an analytic continuation, and it is seen to realize the no-self-interaction feature (see subsection 3.2).

Some of these points appeared already in [4].

We proceed to stress and clarify two general points which appear in [5].

The first issue involves the domain of validity. The central application of [5] is to the Post-Newtonian expansion for the radiation from an inspiraling binary. In that expansion the small parameter is the velocity \( v \ll 1 \). However, the natural domain of ClEFT is wider and simple to state: ClEFT is valid whenever an extended object (such as a soliton)
moves in a background whose length scale is much larger than the object’s size. Similar statements for the gravitational context appear in [6]. Note that CI-EFT applies not only to gravity and would be equally useful to say a monopole placed in a non-trivial background in a Yang-Mills theory. Returning to the binary inspiral problem the system contains two independent dimensionless parameters: the velocity $v$ and the mass ratio $m_1/m_2$. CI-EFT applies not only when $v \ll 1$ but also when $m_1/m_2 \ll 1$ and while the first condition always fails at the last stage of the inspiral, the second condition allows in principle to compute the radiation throughout the whole evolution in a controlled way.

The second issue is the classical nature of problem. While [5] is rooted in a QFT background the problem at hand is classical and as such it allows for certain adjustments in the theory (more precisely certain quantum issues can be avoided and left out of the theory). The ingredients which can be avoided include Planck’s constant $\hbar$, the complex number $i$ and the Feynman path integral, as we discuss in detail in subsection 2.6.

2.2 Caged black holes – set-up

For concreteness we turn to consider static caged black holes, and we start by setting-up the problem. Consider a compactification background of the form $\mathbb{R}^\hat{d} \times X$ where $X$ is a compact manifold. For simplicity we take the theory to be pure gravity (though additional fields could be accommodated) and hence $X$ is assumed to be Ricci-flat. The total space-time dimension is $d = \hat{d} + \text{dim} X$. We make another simplifying assumption by considering mostly $X = S^1$, a circle of size $L$ parameterized by the coordinate $z$ (see however subsection 3.4), and accordingly $d = \hat{d} + 1$.

Next we consider placing a small static black hole at a point in $\mathbb{R}^\hat{d} \times X$. As long as a certain no-self-force is obeyed the black hole will remain at rest. Here we do not need the explicit form of the no-force condition and it suffices to observe that certain symmetries are enough to guarantee it. If the black hole position $p \in X$ is a fixed point of an isometry\footnote{More generally an isolated member of the fixed-set of a subgroup of isometries.} then the force vector must vanish. Moreover, assuming there is at least one equilibrium point in $X$ (this must be true because there is no external energy source), then if $X$ is homogeneous any point in it would be an equilibrium point. Since $X = S^1$ is both homogeneous and enjoys the discrete symmetry of inversion ($z \to 2z_0 - z$ for arbitrary $z_0$), any point on $S^1$ is an equilibrium point.

Our aim is to compute the thermodynamics of this system, as encoded by the fundamental thermodynamic relation $G = G(\beta, \Omega_i, L)$ where $\beta$ is the inverse temperature of the black hole and $\Omega_i$ are the angular velocities in the rotating case.

The basic feature of the problem of small caged black holes is that we have two widely separated length scales

$$r_0 \ll L \quad (2.1)$$

where $r_0$ is the Schwarzschild radius. Accordingly the metric (and any other field) can be decomposed

$$g_f \supset g \supset \bar{g} \quad (2.2)$$
where \( g_f \) is the full metric including all length scales, \( g \) includes only length scales of order \( L \) or larger, and finally \( \bar{g} \) includes only length scales much larger than \( L \) and can be thought to live at the asymptotic region. We sometimes write \( g_f = g_S + g \) where \( g_S \) is the component of the metric field with short length scales of order \( r_0 \), and \( g = g_L + \bar{g} \) where \( g_L \) is the \( L \)-scale component of the metric.

The original action is purely gravitational, without any source terms

\[
S = \int R[g_f] .
\] (2.3)

Our basic tool is to integrate out the short degrees of freedom around the black hole and replace them by an effective world-line action

\[
S_{\text{eff}}[\bar{g}] = I[S,g_S] = \int R[\bar{g}] + S_{\text{BH}}[g, x, e_I^\mu] \tag{2.4}
\]

where we denote by \( I[S,g_S] \) “integrating \( g_S \) out of \( S \)” as defined below in subsection 2.6. The black hole effective action depends on \( x, e_I^\mu, g \). The first two are black hole degrees of freedom: \( x = x(\tau) \) is its location while “the frame” \( e_I^\mu = e_I^\mu(\tau) \) is a rotational degree of freedom. \( g \) represents here the local background at the location of the black hole.

The black hole effective action, \( S_{\text{BH}} \), needs to be evaluated only once (up to the required order) and then it can be used to study BH motion through any background (whose typical length scale is much larger than the black hole). Naturally \( S_{\text{BH}} \) must be invariant under world-line reparameterization as well as the more general background diffeomorphisms. Its leading term is the point-particle action \( S_p \) characterized by the (local) mass \( m_0 \)

\[
S_{\text{BH}} = S_p + \ldots
\]

\[
S_p = -m_0 \int d\tau
\] (2.5)

where \( d\tau \equiv \sqrt{g_{\mu\nu}(x) dx^\mu dx^\nu} \) is the proper time interval along the world-line, and the ellipsis denote terms which depend on gradients of the background.

### 2.3 Dimensional reduction and the Newtonian potential

Caged black holes have a time translation symmetry which we now turn to exploit. Given this symmetry there exists a natural change of variables, namely the outcome of a Dimensional reduction over \( t \). The new variables will be especially useful to simplify the computations since \( g_{00} \) which appears in the leading (mass) term of the world-line effective action (2.5) will be separated from the other metric components and mapped onto a scalar \( \phi \).

Dimensional reduction is commonly used to reduce over a compact spatial dimension which the fields do not depend upon, but it can be used equally well for reducing over the non-compact temporal direction, as long as all the fields are \( t \)-independent. The standard Kaluza-Klein ansatz is given by

\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2\phi} \left( dt - A_i dx^i \right)^2 - e^{-2\phi/(d-3)} \gamma_{ij} dx^i dx^j ,
\] (2.6)
which defines a change of variables $g_{\mu\nu} \rightarrow (\gamma_{ij}, A_i, \phi)$. We let Greek indices run over all coordinates while Latin indices are spatial, namely $\mu \rightarrow (t \equiv 0, i)$. Note that our signature convention for $g$ is mostly minus, $(+ - \cdots -)$, as in field theory, while for the purely spatial metric $\gamma_{ij}$ we change the signature to be all $(+)$. In particular the scalar field $\phi$ is defined through

$$e^{2\phi} = g_{00}.$$ \hfill (2.7)

Since in the stationary case the action is proportional to $\int dt$ we may factor it out and define a reduced action

$$S_R := \frac{S}{\int dt}$$ \hfill (2.8)

where from hereon we shall suppress the subscript ‘R’. The resulting bulk action is

$$S = \frac{1}{16\pi G} \int R[g] \rightarrow \frac{1}{16\pi G} \int dx^{d-1}\sqrt{\gamma} \left[ R + \frac{d-2}{d-3} (\partial \phi)^2 - \frac{1}{4} e^{2(\phi-2)/d} F^2 \right] ,$$ \hfill (2.9)

where the second line displays the reduced action (2.8) and in which only the metric $\gamma$ is being used: $R = R[\gamma]$, $(\partial \phi)^2 = \gamma^{ij} \partial_i \phi \partial_j \phi$ including the standard definitions $F^2 = F_{ij} F^{ij}$, $F_{ij} = \partial_i A_j - \partial_j A_i$. The action describes a metric $\gamma_{ij}$ with a standard Einstein-Hilbert action (this is achieved through the Weyl rescaling factor in front of $\gamma_{ij}$ in the ansatz) a negative-kinetic-term vector field $A_i$ with a $\phi$ dependent pre-factor, and a minimally coupled scalar field $\phi$ which is related to $g_{00}$. The negative kinetic term for $A_i$ is directly related to the fact that the spin-spin force in gravity has an opposite sign relative to electro-dynamics, namely “north poles attract” [24]. Finally the constant pre-factor $(d-2)/(d-3)$ which appears in the kinetic term for $\phi$ is related to the polarization dependence of the $g$ propagator (the original graviton), see footnote\(^7\).

Given the time-translation symmetry we can be more specific regarding the black hole effective action (2.3). The BH stands at a spatial point which we denote as the origin $O$. The BH degrees of freedom are frozen (more precisely the velocity $\dot{x}$ and the angular velocity $\Omega^\mu_\nu$ are frozen) and $S_{BH} = S_{BH}[g]$. This action must be supplemented by certain no-force and no-torque constraints,\(^2\) which originate from the equations of motion for $x, e^\mu_I$. The (reduced) point-particle action becomes

$$S_p = -m_0 \sqrt{g_{00}} = -m_0 e^\phi$$ \hfill (2.10)

where all field are to be evaluated at $O$ and in the second equality we used the change of variables (2.7).

There is a nice physical interpretation for the scalar field $\phi$ which appears in the dimensional reduction. It is a free scalar field which couples to the mass and as such it is quite similar to the Newtonian potential in Newtonian gravity. Its equation of motion to leading order (first in $\phi$ and zeroth in the other fields) is

$$\Delta \phi = 8\pi G \frac{d-3}{d-2} m_0 \delta(x)$$ \hfill (2.11)

\(^2\)In our applications these constraints will be satisfied automatically due to symmetry. See the discussion of the no-force constraint in the previous subsection.
\[ -\frac{im}{2} \int \bar{h}_0 dt = \begin{array}{c} \text{top line} \\ \text{bottom line} \end{array} = \begin{array}{c} \text{asymp} \\ \text{phys} \end{array} + \begin{array}{c} \text{grav} \\ \text{phys} \end{array} + \ldots \]

\[ -m \bar{\phi}(0) = \begin{array}{c} \text{asymp} \\ \text{phys} \end{array} + \begin{array}{c} \text{prop} \end{array} + \ldots \]

**Figure 1:** A definition of the (asymptotic, renormalized) ADM mass \( m \) in terms of a 1-point function as in [4]. The top line is pre-dimensional reduction as in [4] and the wavy lines represent gravitational perturbations (of \( g \)). The external leg is the asymptotic component \( \bar{h}_{00} \) and there is no propagator associated with it. The double solid lines denote the black hole world-line. The bottom line is the translation of the top line into the dimensionally reduced fields which we use. \( \bar{\phi} \) is the longest scale (\( \gg L \)) component of the metric field related to \( g_{00} \) through 2.7. The solid internal lines denote the propagator (3.6) for the scalar field \( \phi \). More details about the Feynman rules will be given later in figure 5.

where \( \triangle \) is the flat-space spatial Laplacian.

In our system there are additional symmetries beyond time translation, namely azimuthal symmetries. In principle one could dimensionally reduce over the corresponding angles, but in this paper we reduce only over time which is special both because the associated scalar field \( \phi \) appears in the leading world-line interaction and because a reduction over time is more generic and applies to all stationary sources (neutron stars, ordinary stars etc.) which unlike black holes do not necessarily possess an azimuthal symmetry. Actually in higher dimensions the Myers-Perry black holes are symmetric with respect to \( [(d-1)/2] \) angles while so far a proof guarantees a single angular symmetry for a general higher-dimensional black hole [25].

### 2.4 Vacuum diagrams

A central objective is to calculate the (asymptotic) ADM mass \( m \) given \( m_0 \), the local mass of the caged black hole. As remarked already this can be considered to be a renormalization of the mass from the scale \( r_0 \ll r \ll L \) where \( m_0 \) is defined to the scale \( r \gg L \) where \( m \) is defined. In [4] \( m \) was calculated from a 1-pt function relation, which when translated to the language of dimensional reduction, is represented by the Feynman diagrams in figure 4.

We suggest an improved calculational definition for \( m \) represented by the Feynman diagrams in figure 2. This definition avoids the need for an external leg and along with it
reduces the maximum “connectivity index” of the required vertices (for example, unlike [4] we avoid using the 4-graviton vertex at two loop, as well as the 3-vertex, as it happens).

We explain fig.2 first intuitively and then more formally. Intuitively, the classical vacuum diagrams describe the vacuum energy at the location of the black hole, namely its mass $m$, while $m_0$ is the tree level value. Technically one needs to prove that the two definitions are identical. To see that we integrate out the scale $L$ component of the metric $g_L$. By residual diffeo invariance the action must look like $S_{eff}[\bar{g}] = - \int d\tau m$ where the proper time element is $d\tau = \sqrt{g_{\mu\nu} dX^\mu dX^\nu} = \sqrt{\bar{g}_{00}} dt = e^{\phi} dt = (1 + \bar{\phi} + \ldots) dt$ (or equivalently $d\tau = (1 + \bar{h}_{00}/2 + \ldots) dt$) and $m$ is a scalar. Thus the coefficient of the two terms in the effective action must coincide and we can indeed identify the two diagrammatic definitions (fig.1,fig.2) of the mass $m$.

2.5 The thermodynamic potential

In the previous subsection we defined the black hole mass in terms of “vacuum” diagrams

$$m = m(m_0, a_i, L)$$

(2.12)

where $m$ is considered as a function of the local black hole parameters, namely the local mass $m_0$ and the local rotation parameters $a_i$ (in higher dimensions there are several independent rotation parameters), and the parameters of the compactification such as $L$. As such it is analogous to the “vacuum energy” or “partition function” of translationally invariant quantum field theories, which is known to encode all properties of the QFT (more precisely, the partition function in the presence of arbitrary sources encodes in its derivatives all the correlation functions of the QFT). On the other hand in thermodynamics, and in particular in black hole thermodynamics, all the thermodynamic information is known to be contained in a “fundamental thermodynamic relation” specifying a thermodynamic potential in terms of its natural variables. In this subsection we will build from the mass function such a thermodynamic relation.

It is convenient to choose horizon parameters as our basic variables, namely the temperature $T$ (or equivalently the surface gravity) and the angular velocities $\Omega_i$. Correspondingly
the relevant ensemble is the grand canonical one and the potential is the Gibbs potential

\[ G = G(T, \Omega_i, L) . \]  

(2.13)

From derivatives of \( G \) one can infer all the equations of state, namely the expressions for the entropy, tension and angular momenta

\[ S = -\partial G / \partial T, \quad \hat{\tau} = \partial G / \partial L, \quad J_i = -\partial G / \partial \Omega_i \]  

(2.14)

In the static (non-rotating) case where \( \Omega_i \) do not appear the ensemble coincides with the canonical ensemble and the potential becomes the free energy.

The natural parameters which appear in the computations are \( m_0, a_i, L \) while the natural parameters for \( G \) are \( T, \Omega_i, L \) and therefore we need to find the relation between these two sets of parameters. This is done through an intermediate set \( T_0, \Omega_0 \), the local temperature and angular velocities (we omit the index \( i \) from the angular velocities for clarity of notation) which are defined to be the quantities measured by an “intermediate” observer at a distance \( r_0 \ll r \ll L \) from the black hole. Schematically the transformation is

\[ (m_0, a) \rightarrow (T_0, \Omega_0) \rightarrow (T, \Omega) \]  

(2.15)

The first transformation is carried out considering the black hole to be embedded in Minkowski space-time, namely by using the standard relations of the Myers-Perry black hole \(^3\).

In order to perform the second transformation, from local horizon quantities to their values for an asymptotic observer another ingredient is needed. While the intermediate and asymptotic observer agree on their definition of proper distances, their notion of time differs due to the red-shift factor

\[ R := \sqrt{g_{00}(O)} \equiv e^{\phi(O)} = \frac{t_0}{t} \]  

(2.16)

where \( g_{00}(O) \) is the metric at the black hole location (after the scale \( r_0 \) component of the metric was integrated out), and \( t, t_0 \) are the asymptotic and intermediate times. In terms of the red-shift the asymptotic angular velocity is given by

\[ \Omega = dl / dt = R dl / dt_0 = R \Omega_0, \]  

where \( dl \) is an element of proper distance on the horizon (more formally one should obtain the angular velocity from the coefficients in the decomposition of the Killing generator of the horizon into the time translation and angular shift Killing vectors). The asymptotic temperature (and surface gravity) is canonically conjugate to time and hence transforms inversely to \( t \), namely

\[ T / T_0 = t_0 / t = R \]  

(2.17)

Alternatively, the Hawking temperature is red-shifted exactly according to \( R \). Altogether the asymptotic quantities are given by

\[ T = RT_0 \]

\[ \Omega = R \Omega_0 . \]  

(2.18)

The red-shift itself

\[ R = R(m_0, a_i, L, \ldots) \]  

(2.19)

may be defined in terms of Feynman diagrams as in fig.3.

---

\(^3\)We denote the tension by \( \hat{\tau} \) to prevent any possible confusion with the proper time \( \tau \).
We note that our formulae for the transformation $(m_0, a_i) \rightarrow (T, \Omega_i)$ are stated and were tested up to the order which we need, but not necessarily beyond.

Having found the relation between the natural parameters of the problem and the natural parameters for thermodynamics we need to transform the mass function into the Gibbs potential. They are related through the standard thermodynamical relation

$$ G := m - T S - \Omega J. $$

(2.18)

More details on method of computation will be given in the next sections.

We can now compare our method with that of CGR [4]. There the mass and tension are computed separately in terms of 1-point functions. Additional thermodynamic variables of interest such as the angular momentum would need to be computed separately as well. In the current method on the other hand we compute a single thermodynamic potential $m$ and it has no external legs thereby requiring simpler vertices and simplifying the computation. The price to pay is the computation of the red-shift (which is roughly of the same computational difficulty as $m$) and the introduction of possibly lengthy thermodynamic manipulations.

Another difference is that while in CGR $m_0, a_i$ are interpreted as bare couplings which have no physical meaning here they are assigned meaning by relating them to horizon quantities.

2.6 Non-quantum field theory

In this subsection we stress several issues where the classical nature of our problem allows simplification in comparison with quantum field theory (QFT). The approach of Rothstein and Goldberger [5, 7] is based on a background in QFT and as such factors of $\hbar$ (implicit
in the definition of the Planck mass) and the complex number $i$ are commonplace as well as Feynman path integrals. However, from the classical point of view it is clear that all factors of $\hbar$ must cancel and hence should not appear in the first place. Similarly, since the action and all quantities are real there is no reason for $i$’s to appear. Finally Feynman path integrals in QFT are notoriously difficult to define rigorously despite their long-time usage. On the other hand in classical physics we do not have any such apparent uncertainty in the definition of the theory and accordingly we would expect to be able to do away with this notion in a classical set-up. Here we believe we clarify these points, which are perhaps not very deep, but still quite useful.

The discussion of $\hbar$ factors is related to the issue of units in classical GR vs. QFT, which we proceed to review. In classical physics we have three fundamental dimensions time, length and mass denoted $T$, $L$ and $M$ respectively. Special relativity introduces the speed of light $c$ as a universal constant, and setting it to 1 identifies $T \equiv L$, leaving us with two fundamental dimensions $L, M$ which continue to be the fundamental dimensions in classical field theory (ClFT). In QFT we introduce $\hbar$ as a second fundamental constant. Setting it to 1 allows us to identify $L \equiv 1/M$, and it is standard custom to measure all QFT quantities in units of mass. In GR on the other hand $\hbar$ is absent but rather $G$ is introduced as a second fundamental constant identifying $M \equiv L^{d-3}$ and it is useful to keep $L$ as the fundamental dimension. Accordingly the dimensions of various field theory quantities vary between ClFT, QFT and GR, see table 1. Note especially that in GR the Lagrangian density has dimensions $1/L^2$, canonical scalar fields are dimensionless and the elements of the Fourier space dual to space-time is better described as wavenumbers rather than momenta. The notion of the Planck mass merits another comment. In quantum gravity we have both $G, \hbar$ and hence we have a natural unit for each dimension. Up to numerical constants of convention one defines the Planck length $l_P^{d-2} = G \hbar$ and accordingly the Planck mass $m_P = \hbar/l_P = \hbar^{(d-3)/(d-2)} / G^{1/(d-2)}$. Since $m_P$ requires $\hbar$ for its definition it has no place in a classical theory.

A second inheritance from QFT is to include factors of $i$ in the Feynman rules for every vertex and propagator. The reason for that is that in the functional integration (sum over histories) the weight factor is $\exp iS$. In ClFT on the other hand all quantities are real and there is no reason to have any $i$’s in the formulae. Indeed by reviewing the origin of the Feynman rules (for the computation of the effective action or otherwise) one finds that all

|                     | ClFT   | QFT    | GR     |
|---------------------|--------|--------|--------|
| action              | $ML$   | 1      | $L^{d-2}$ |
| Lagrangian density  | $M/L^{d-1}$ | $M^d$   | $1/L^2$  |
| wavenumber          | $L^{-1}$ | $M$    | $L^{-1}$   |
| canonical scalar field | $(M/L^{d-3})^{1/2}$ | $M^{(d-2)/2}$ | 1 |

**Table 1:** Dimensions of various physical quantities compared between three types of field theories: classical (ClFT), quantum (QFT) and General Relativity (GR). $M$ denotes the mass dimension, and $L$ length.
Figure 4: The Feynman rules for CIEFT are naturally real. As an example we display them for the $\phi^4$ scalar theory.

Factors of $i$ can be omitted: interaction terms come with the same sign as in the action, while the propagator gets an additional minus sign multiplying the inverse of the kinetic term (from “moving it to the other side of the equation”). For example given the Lagrangian of the $\phi^4$ theory $\mathcal{L} = (\partial \phi)^2/2 - (m^2 \phi^2/2 + \lambda \phi^4/4!) = -\phi(\partial^2 + m^2)\phi/2 - \lambda \phi^4/4!$ the Feynman rules are given by fig.4, namely the 4-vertex is given by $(-\lambda)$ while the propagator is $+1/(\partial^2 + m^2) = -1/(k^2 - m^2)$. Actually from this discussion it is clear that factors of $i$ can be omitted not only from CIEFT but also from tree-level QFT computations (at least of $S_{\text{eff}}$). One could be motivated to generalize this to all QFT computations. When loops are added an amendment is required which is seen (by comparison with the standard rules) to be a factor of $(-i)$ for each (bosonic) loop. As usual a fermionic loop adds another negative sign so its contribution would be $4^4 (+i)$.

Our last issue concerns the Feynman path integrals which are a standard tool in QFT. In particular they are used in the definition of an effective action resulting from “integrating out” a field, for example

$$iS_{\text{eff}}[g]/\hbar = \int Dg_S \exp(iS[g_S,g]/\hbar)$$

(2.19)

where $g_S$, $g$ are the short and long wavelength components of a certain metric $g$. However, as intuitive and as useful as path integrals are, they are also notoriously difficult to define rigorously. In classical physics there clearly should not be any reason for such uncertainties nor for the loop diagrams which appear in (2.13) at higher orders of $\hbar$. So we would

---

4One may wonder whether additional phases are required under some circumstances in order to agree with the standard prescription. In all cases which we checked this is not necessary, and the key point is the assumption that $V - P + L = 1$ where $V$, $P$ and $L$ are the number of vertices, propagators and loops, respectively, in the diagram. This relation holds for Lorentz invariant field theories and may hold even more generally.
expect there exists a purely classical definition of the effective action. Indeed it is known that in the classical limit the path integral reduces in the saddle point approximation to a computation around a classical solution. Accordingly we should define “integrating out” a field $g_S$ classically out of the action $S$, denoted $I[S,g_S]$, as follows

$$S_{eff}[g] \equiv I[S,g_S] := S[g,g_S(g)]$$

where the right hand side means that one should first solve for $g_S$ given the prescribed long wavelength $g$ and then evaluate the action. This definition is natural and does not produce any uncertainty. Moreover, it stresses that one is allowed to integrate out only when the remaining fields can specify a solution, for instance, fields on the boundary (or asymptotic fields in the case of an unbounded space).

3. Caged black hole: improved calculation

3.1 Action and Feynman rules

Let us consider an isolated static black hole in a background with a single compact dimension $\mathbb{R}^{d-2,1} \times S^1$, where $d$ is the total space-time dimension. Coordinates on $\mathbb{R}^{d-2,1}$ are denoted by $(x^0, \mathbf{x})$ and $z$ labels the coordinate along $S^1$. The asymptotic period of the $S^1$ is $L$ (as measured by an observer at $|\mathbf{x}| \to \infty$). In addition the black hole is static, thus one can take $x^\mu = (t, \mathbf{x} = 0, z = 0)$ without loss of generality.

As a first step towards the action and the corresponding derivation of Feynman rules we integrate out the short degrees of freedom $g_S$ and replace the space-time in the vicinity of the horizon with an effective Lagrangian for the black hole world-line coupled to gravity. The resulting effective action takes the form

$$S_{eff}(g, \phi) = -\frac{1}{16\pi G} \int d^{d-1}x \sqrt{\gamma} \left[ R[\gamma] + \frac{d-2}{d-3}(\partial \phi)^2 \right] - m_0 e^{\phi(O)} + \ldots ,$$

where the ellipsis denote finite-size higher order terms which we shall not require. Such terms depend on the values of the fields at the origin and respect diffeomorphism and world-line reparameterization invariance. The leading finite-size term is

$$O := \partial_i \partial_j \phi(O) \partial^i \partial^j \phi(O) .$$

Starting from the effective action (3.1) we decompose the metric tensor and the scalar field into a long wavelength non-dynamical background fields $\bar{\phi}$, $\bar{\gamma}_{ij}$ which live at the asymptotics and the scale $L$ fields $\phi_L$, $\gamma_{Lij}$

$$\phi = \phi_L + \bar{\phi}$$

$$\gamma_{ij} = \gamma_{Lij} + \bar{\gamma}_{ij} .$$

5Due to the symmetry $t \to -t$ the vector field $A_t$ vanishes in the static case.

6Note that the operator $\partial_i \phi(O) \partial^i \phi(O)$ is redundant (can be removed by field re-definitions, see for instance [7]) since $\partial_t \phi = 0$ for a stationary black hole, whereas operators involving the Ricci tensor are redundant since the black hole is placed in a Ricci flat background.
\[ D(x_\perp - x'_\perp; z - z') = D(\mathbf{x}_\perp - \mathbf{x}'_\perp; z - z') \]

\[ = \chi \chi = -m_0 \]

**Figure 5:** Feynman rules obtained from the expansion of (3.1)

Since we assume the perturbative regime (2.1) all the fields are weak and therefore it is consistent to linearize about flat space

\[ \gamma_{ij} = \eta_{ij} + \delta\gamma_{ij} \quad (3.4) \]

Integrating out in the sense of (2.20) or equivalently (2.19) the scale \( L \) fields \( \gamma_{\ell ij}, \phi_L \) while holding the black hole world-line fixed leads to an effective action \( \Gamma_{\text{eff}}[\bar{\phi}, \delta\gamma_{ij}] \) valid on a scale much larger than \( L \). According to (fig.1, fig.2) the relation between the ADM mass \( m \) for a caged black hole and the local mass \( m_0 \) can be read off from either the constant or the linear term in \( \Gamma_{\text{eff}}[\bar{\phi}, \delta\gamma_{ij}] \)

\[ \Gamma_{\text{eff}}[\bar{\phi}, \delta\gamma_{ij}] = -m - m\bar{\phi}(O) + \ldots \quad (3.5) \]

The constant term is represented by the Feynman diagrams of fig.2, whereas the linear term is represented by the Feynman diagrams of fig.1.

We turn to construct the Feynman rules, summarized in fig.5. Solid internal lines denote the propagator for the scalar field \( \phi \) on flat \( \mathbb{R}^{d-2,1} \times S^1 \)

\[ D(x - x'; z - z') = \frac{8\pi G}{L} \frac{d - 3}{d - 2} \sum_{n = -\infty}^{\infty} \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2}} \frac{1}{k^2_\perp + (2\pi n/L)^2} e^{ik_\perp(x - x')_\perp + 2\pi n(z - z')/L}, \quad (3.6) \]

where \( k_\perp \equiv k \). The double solid line denotes the black hole world-line. There are no propagators associated with this line. Finally the vertices are constructed from the expansion of (3.1) about flat space. Those relevant for our computations are listed on fig.5. The Feynman rules for the fields in the decomposition (3.3) are directly related to those of fig.5. As usual, diagrams that become disconnected by the removal of the particle world-line, such as fig.6(b), do not contribute to the terms in \( \Gamma_{\text{eff}}[\bar{\phi}, \delta\gamma_{ij}] \).

**Power counting.** Each Feynman diagram in the CIEFT contributes a definite power of \( \lambda \sim (r_0/L)^{d-3} \) to the terms in (3.3) and we now explain how to evaluate this power for each diagram in a straightforward manner (similar to [1]). Our problem contains two dimensionful parameters \( m_0 = r_0^{d-3} \) and \( L \). Powers of \( r_0 \) can come only from world-line
vertices and not from the bulk. Actually since we neglect finite-size effects each diagram is simply proportional to $m_0^{n_V}$ where $n_V$ is the number of world-line vertices. Powers of $L$ must arrange themselves automatically by dimensional analysis.

In more detail, since the only scale in the propagator is $L$ (we set $G = c = 1$) we assign $k \sim \partial_t \sim L^{-1}$ and thus

$$D \sim \int \frac{dk^{d-1}}{k^2} \sim L^{3-d}. \quad (3.7)$$

Based on this reasoning the propagators of vector and metric fields $A_i, \delta\gamma_{ij}$ are assigned the same scaling, $L^{3-d}$. No scaling factors are assigned to the asymptotic fields.

Altogether the diagrams of fig. 2 scale like

$$\text{fig. 2(a) } \sim m_0 \lambda$$
$$\text{fig. 2(b) } \sim m_0 \lambda^2 \quad (3.8)$$

and thus their contribution to $m$ is suppressed by a single and quadratic power of $\lambda$ respectively.

In order to count powers of finite-size higher-order terms in the effective world-line action (3.1) we note that by dimensional analysis the dimension of a term sets the dimension of its coefficient and $r_0$ is the only dimensionful parameter which can enter into the expression for such a coefficients. For example, the coefficient of $\mathcal{O}$ defined in (3.2) must be proportional to $m_0 r_0^4 \propto r_0^{d+1}$ and the proportionality constant is fixed by matching the effective Lagrangian of equation (3.1) to the full black hole theory, so that observables calculated in the CIEFT agree with those of the full theory.

The first finite-size correction to $m$ (through the constant term in $\Gamma_{\text{eff}}[\bar{\phi}, \delta\gamma_{ij}]$) is due to an insertion of $\mathcal{O}$ as in fig. 3(a). According to the power counting rules

$$\mathcal{O} \sim m_0 \lambda^{\frac{d-3}{d-3}}$$
$$\text{fig. 3(a) } \sim m_0 \lambda^{\frac{2(d-1)}{d-3}} \ll m_0 \lambda^2 \quad (3.9)$$

Therefore, the contribution of $\mathcal{O}$ along with other finite-size higher derivative terms is always beyond second order in $\lambda$, whereas for $d = 5, 6$ finite-size effects are beyond third order.

3.2 The renormalized mass at one-loop and higher

According to the definition of the perturbative regime (2.1), the small parameter $\lambda$ should be proportional to a (positive) power of $r_0/L$. We set the normalization of the small parameter to be

$$\lambda := \left(\frac{r_0}{L}\right)^{d-3} \zeta(d-3) = \frac{16\pi G m_0}{(d-2)\Omega_{d-2} L^{d-3}} \zeta(d-3) \quad (3.10)$$

where the equality relates the Schwarzschild radius $r_0$ in the first expression to the mass $m_0$ in the second expression and $\Omega_{d-2} := (d-1)\pi^{(d-1)/2}/\Gamma[(d + 1)/2]$ is the area if $S^{d-2}$. This normalization is such that $\phi(O)- = -\lambda + \ldots$ as can be seen either from of the
Figure 6: (a) The leading finite-size contribution to $m$, which is due to the term $O$. The thick square vertex denotes an insertion of $O$. (b) A diagram that becomes disconnected by the removal of the particle world-line. Therefore it does not contribute to the computation of $m$ through the effective action.

The first correction to the mass of the system arises from the 1-loop diagram of fig.2(a). Using the Feynman rules of the ClEFT (see fig.3) this diagram is evaluated to be

$$f i g . 2(a) = \frac{\lambda}{2} m_0 .$$

This reproduces the results of [11],[3],[4]. It can be understood in Newtonian terms by comparing to the expression of the total Newtonian gravitational energy $E = \int \phi dm/2$ (like in electro-statics).

Appendix A contains details of the derivation or (3.11). Basically the loop gives the factor of $\lambda$ while the $1/2$ is a symmetry factor. An interesting point is that from the perspective of the wave-number space the sum over the Kaluza-Klein harmonics gives a factor of $\zeta(4 - d)$ while from the configuration space perspective we expect the Newtonian potential to be proportional to the sum $\sum_n 1/(nL)^{d-3}$ which is proportional to $\zeta(d - 3)$. It turns out that the two can be traded according to an identity involving the functions zeta and gamma.

7This definition contains an extra factor of $(d - 3)/(d - 2)$ relative to propagator of a canonically normalized scalar field which originates from the pre-factor of the kinetic term for $\phi$ in the action (2.9).

If we were to compute the Newtonian potential in the original action, prior to dimensional reduction, this same factor would have emerged from the graviton propagator in the standard Feynman gauge. In this sense we get insight to this pre-factor in the action which is somewhat curious at first sight.
Figure 7: The six 2-loops diagrams which were computed in [4] to determine \( m \) to order \( \mathcal{O}(\lambda^2) \). Compare this with the single diagram fig.2(b) which is required by our improved method.

The regularization is a second noteworthy point about the derivation. In appendix A we use dimensional regularization, while within the method of MAE [3] advocated Hadamard’s regularization which was claimed to be equivalent to omitting self-interaction terms (“no-SI”). Since both dimensional regularization and Hadamard’s are essentially analytic continuations they are guaranteed to agree, but in this case we can moreover see explicitly the equivalence with no-SI. Considering the sum over \( n \), the quantized KK wavenumber, the only divergent term is the one with \( n = 0 \), while \( n \neq 0 \) can be thought to arise from the images of the black hole (in the covering space). Dimensional regularization puts the \( n = 0 \) term to zero which indeed amounts to omitting self-interaction, keeping only the interaction with the images.

2-loop. The next contribution to \( m \) is suppressed by a factor of \( \lambda \) relative to the 1-loop result and is given by a 2-loop diagram in fig.2(b). Using the same Feynman rules as before we obtain

\[
\text{fig.2(b)} = -\frac{m_0}{2} \lambda^2. \tag{3.12}
\]

Again, the factor 1/2 is a symmetry factor. Adding up, we reproduce the result of [4] up to second order in \( \lambda \)

\[
m = m_0 \left( 1 - \frac{1}{2} \lambda + \frac{1}{2} \lambda^2 + \ldots \right) \tag{3.13}
\]

Note that, whereas [4] computed six 2-loop diagrams (fig.7) each with one external leg, including a diagram with the quartic coupling of GR, we compute a single 2-loop diagram with no external legs. Moreover our diagram happens to factorize into two integrals, which explains the factorization observed by [4] for the sum of their diagrams.

High order corrections for \( d = 5, 6 \). According to the power counting rules established in
subsection 3.1 finite-size effects do not contribute to $m$ at order $O(\lambda^3)$ for $d = 5, 6$. The relevant diagrams are those of fig. 8. Using the Feynman rules of fig.8 yields

$$
\begin{align*}
fig 8(a) &= \frac{m_0}{2} \lambda^3 \\
fig 8(b) &= \frac{m_0}{6} \lambda^3. 
\end{align*}
$$

(3.14)

Combining altogether we obtain for the ADM mass in $d = 5, 6$ up to the evaluation of the non-factorizable diagram fig.8(c)

$$
m = m_0 \left(1 - \frac{1}{2} \lambda + \frac{1}{2} \lambda^2 - \left(\frac{2}{3} + \frac{1}{m_0 \lambda^2} fig 8(c)\right) \lambda^3 + \ldots\right)
$$

(3.15)

3.3 Thermodynamics

In order to calculate other thermodynamic quantities including the tension and the entropy, we find it convenient to use the free energy potential which, as explained in section 2.5, plays a fundamental role in the system under consideration.

We start by calculating the red-shift factor (2.15). For this purpose we use the effective action (3.1) in order to calculate the value of $\phi(O)$ at the black hole location. Up to the second order in $\lambda$ the diagrams contributing to the red-shift appear on fig.8.
Their value is given by
\[
\begin{align*}
\text{fig. 3}(a) &= -\lambda \\
\text{fig. 3}(b) &= \lambda^2
\end{align*}
\]
(3.16)

As a result we obtain
\[
\phi(O) = -\lambda + \lambda^2 + \ldots \\
R = \sqrt{g_{00}(O)} = e^\phi = 1 - \lambda + \frac{3}{2}\lambda^2 + \ldots
\]
(3.17)

Altogether the asymptotic temperature is given by
\[
T = RT_0 = T_0 \left( 1 - \lambda + \frac{3}{2}\lambda^2 + \ldots \right)
\]
(3.18)

where \(T_0 = \frac{d-3}{4\pi r_0}\) is the local temperature of the black hole.

Next we relate the free energy \(F = m - TS\) to the asymptotic charges \(m, \hat{\tau}\) using Smarr’s relation \((d-3)m = (d-2)TS + \hat{\tau}L\), where \(S, \hat{\tau}\) are the entropy and the tension of the black hole respectively.\(^3\) to eliminate the term \(TS\)

\[
m = (d-2)F - \hat{\tau}L.
\]
(3.19)

In this equation \(m\) is known from (3.13) while \(F\) is unknown. Considering the tension to be a derivative of \(F\) via \(\hat{\tau} = \partial F / \partial L\) we get a differential equation which we can solve for \(F\). Since we use \((m_0, L)\) as our basic variables, we need to express \(\hat{\tau}\) accordingly

\[
\hat{\tau} = \left( \frac{\partial F}{\partial L} \right)_T = \frac{\partial (F, T)}{\partial (m_0, L)} \frac{\partial (m_0, L)}{\partial (L, T)} = -\frac{\partial (F, T)}{\partial (m_0, L)} \left( \frac{\partial T}{\partial m_0} \right)_L^{-1}
\]
(3.21)

Substituting in the expression for \(T\) (3.18) and solving the resulting differential equation order by order in \(\lambda\) we obtain

\[
F(m_0, L) = \frac{m_0}{d-2} \left( 1 + \frac{d-4}{2}\lambda + \frac{7-2d}{2}\lambda^2 + \ldots \right)
\]
(3.22)

Having the expression for the free energy at hand one can compute all the thermodynamic quantities of interest. We list them below

\[
\hat{\tau}L = \frac{1}{2}(d-3)\lambda - (d-3)\lambda^2 + \ldots \\
S = -\left( \frac{\partial F}{\partial T} \right)_L = -\left( \frac{\partial F}{\partial m} \right)_L \left( \frac{\partial T}{\partial m} \right)_L^{-1} = S_0 (1 + 0 \cdot \lambda + 0 \cdot \lambda^2 + \ldots)
\]
(3.23)

\(^8\)These are nothing but the standard relations for changing variables, in thermodynamics or otherwise, stated concisely à la Landau-Lifshitz \([26]\) in terms of Jacobians. We use \(\frac{\partial (u, v)}{\partial (x, y)}\) to denote the Jacobian determinant

\[
\frac{\partial (u, v)}{\partial (x, y)} = \det \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right)
\]
(3.20)
where $S_0 = \Omega_d - 2 r_0^d / (4G)$ is the entropy of an uncompactified black hole. The expression for the tension is identical to the corresponding one in [4], whereas the entropy at first sight looks different, but turns out to agree. To see that one needs to express the local mass $m_0$ in terms of the asymptotic one $m$ through the relation (3.13), substitute it in (3.23) and expand the result in powers of $\lambda$. Our result merely states that entropy gets no corrections up to a second order in $\lambda$, though we may expect it to change when finite-size effects are taken into account and the black hole is seen to deform from spherical symmetry.

3.4 Generalization to all $X$

Consider generalizing the previous analysis from an $S^1$ compactification to a general compactification manifold $X$. One observes that although some Feynman rules change the diagrams to be computed are the same. Actually, there is no change in the vertices for the effective world-line action of the black hole, and the only change enters through the propagator.

Since all the results up to this order depend on a single quantity $\lambda$ it is sufficient to generalize the definition of $\lambda$ (3.10), and to define it to be the value of the Newtonian potential at the location of the black hole, or in formulae

$$\lambda := |\phi(O)|$$

where $\phi$ solves the linearized $\phi$ equation of motion (2.11) this time on $\mathbb{R}^d \times X$ and $\phi(O)$ is the constant term in the Laurent series for $\phi$ around the origin.

This is a definition of $\lambda$ through a linear partial differential equation that in general may be solved through numerical relaxation. In some cases an analytic solution may be available such as in our case $X = S^1$ where the method of images serves, as well as in the more general case of the $n$-dimensional torus $X = T^n$.

Summarizing, our results (3.15,3.18,3.23) generalize to an arbitrary compactification manifold $X$ once the definition of $\lambda$ (3.10) is generalized to (3.24).

4. Application: rotating caged black hole

In this section we propose an extension of the CIEFT approach to black hole thermodynamics which includes spin. We obtain the leading spin vertices in the world-line action. We compute the leading $O(\lambda)$ corrections to the thermodynamic quantities $m, \hat{\tau}, S$ and angular-momentum $J$. We supplement the power counting rules of the previous section with the scaling of the angular momentum, and proceed to compute $m$ and $J$ to the next to leading order. Finite-size effects do not contribute at this order.

4.1 Action and Feynman rules

We consider a stationary spinning black hole in the same background as in the static case — $\mathbb{R}^{d-2,1} \times S^1$. The local angular momentum tensor which is measured by an “intermediate” observer at a distance $r_0 \ll r \ll L$ from the black hole is denoted by $J_\mu^\nu_0 = -J_\nu^\mu_0$ (conventionally normalized such that in 4d $|J_{12}| = |J_3|$), whereas we denote the asymptotic angular-momentum by $J_\mu^\nu$. The rest of the notation is left unchanged.
After compactifying the extra dimension one loses the rotational symmetry between the compact and extended dimensions, therefore the angular momenta associated with these rotations are no longer conserved and should be set to zero in a stationary phase. Actually any rotation in such a plane would ultimately dissipate into gravitational waves due to the compactification-induced quadrupole moment of the black hole which would create a varying quadrupole moment once rotation starts. Temporal components of the angular momentum tensor vanish as well by its definition as the momentum conjugate to rotations. Combining altogether yields

\[ J_0^{\mu} = J_0^{\nu} = 0 \] (4.1)

Therefore in general our system is characterized by \( \left[ \frac{d-2}{2} \right] \) parameters \( J_0^A \), where \( \left[ \frac{d-2}{2} \right] \) is the rank of \( SO(d-2) \) (the dimension of the Cartan subalgebra). \( J_0^A \) are then the angular momenta associated with commuting rotations in the corresponding planes.

Let us discuss the terms that need to be added to the black hole effective action. Integrating out short degrees of freedom \( g_S \) replaces the space-time in the vicinity of the horizon with an effective Lagrangian for the black hole world-line coupled to gravity. In general, such an effective action includes an infinite set of possible non-minimal couplings of the point object to the space-time metric. The mass term, a universal part of the action which is independent of the object’s structure, is given by (2.10) for a spinless particle of the point object to the space-time metric. The mass term, a universal part of the action generally, such an effective action includes an infinite set of possible non-minimal couplings which is independent of the object’s structure, is given by (2.10) for a spinless particle (static black hole) and needs to be supplemented in our case by including the spin degrees of freedom. The procedure for constructing the action for the spinning point particle can be found in [20, 21, 22], and here we are satisfied with mentioning some key points. The rotation degrees of freedom (of a rigid body) are represented on the world-line by a frame variable \( e_I^\mu(t) \), where \( I \) is a “body” index while \( \mu \) is a space-time index. The angular velocity is defined to be \( \Omega^\mu := e_I^\mu D_\tau e_I^\mu \) where \( D_\tau \equiv \dot{X}^\rho D_\rho \) is a covariant derivative in the direction tangent to the world-line. Due to the isotropy of the object, the action depends on \( e_I^\mu \) only through \( \Omega^\mu \) and actually the terms of interest to us can be obtained from the term \( \frac{1}{2} I \Omega^2 \subset S \), where the inertia tensor \( I \) is related to the angular momentum via \( J_{\mu\nu} = I \Omega_{\mu\nu} \).

The leading order terms in \( S_{BH} \) which involve \( J_0 \) are

\[ S_{SG} = \frac{1}{2} \int J_0^{\alpha\beta} h_{\alpha\gamma,\beta} \dot{x}^\gamma dt + \frac{1}{4} \int J_0^{\beta\gamma} \left( \frac{1}{2} h_{\beta\lambda,\mu} + h_{\mu\lambda,\beta} - h_{\mu\beta,\lambda} \right) h_{\gamma}^\lambda \dot{x}^{\mu} dt + \ldots \] (4.2)

where “SG” stands for spin-gravity interaction; the metric perturbation \( h_{\mu\nu} \) is defined by \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) where \( g_{\mu\nu} \) is the metric prior to the dimensional reduction (2.6); in the perturbative regime (2.11) \( h_{\mu\nu} \) can be considered to be small; and the ellipsis denote terms which are of higher order in \( h \) (and proportional to \( J_0 \)). Combining with (2.9) and (2.10) yields

\[ S_{\text{eff}}[\phi, A_i, \gamma_{ij}] = -\frac{1}{16 \pi G} \int d^{d\!-\!1} \sqrt{\gamma} \left[ R[\gamma] + \frac{d}{d-3} \left( \frac{d-2}{2} \right) (\partial \phi)^2 - \frac{1}{4} e^{2(d-2)\phi/(d-3)} F^2 \right] \]

\[ - m_0 - m_0 \phi(O) - \frac{m_0}{2} \phi(O)^2 + \frac{J_0^{ij}}{2} F_{ij}(O) \left( \frac{1}{2} + \frac{d-2}{d-3} \phi(O) \right) \]

\[ - \frac{J_0^{ij}}{2} A_i(O) \partial_j \phi(O) - \frac{J_0^{ij}}{4} \delta_{ij}^k (O) F_{ik}(O) + \ldots \] (4.3)
Figure 9: A diagram which represents the spin-spin contribution to the renormalized mass $m$ according to the effective action (4.5). It is of order $J_0^2 \sim m_0 \lambda^{d-1}$. 

In this action we decompose the metric tensor, the vector field and the scalar field into a long wavelength non-dynamical background fields $\bar{\phi}$, $\bar{A}_i$ and $\bar{\gamma}_{ij}$ which live at the asymptotic region and the short wavelength fields $\phi_L$, $A_{Li}$, $\gamma_{Li j}$ which include the scales of order $L$

$$\phi = \phi_L + \bar{\phi}$$
$$A_i = A_{Li} + \bar{A}_i$$
$$\gamma_{ij} = \gamma_{Li j} + \bar{\gamma}_{ij} = \gamma_{Li j} + \eta_{ij} + \delta\gamma_{ij}$$ \hspace{1cm} (4.4)

We now define the renormalized mass $m$ and angular momentum $J$. Integrating out in the sense of (2.20) or equivalently (2.19) the short wavelength fields $\phi_L$, $A_{Li}$, $\gamma_{Li j}$, while holding the black hole world-line fixed leads to an effective action $\Gamma_{\text{eff}}[\bar{\phi}, \bar{A}, \delta\gamma]$ valid on a scale much larger than $L$. The relation between the ADM mass $m$ for a rotating caged black hole and the local mass $m_0$ along with the relation between the local angular-momentum tensor $J_{0ij}$ and the asymptotic one $J_{ij}$ can be read off

$$\Gamma_{\text{eff}}[\bar{\phi}, \bar{A}, \delta\gamma] = -m - m \phi(O) + \frac{J_{ij}}{4} \bar{F}_{ij}(O) + \ldots$$ \hspace{1cm} (4.5)

The mass $m$ is the sum of Feynman diagrams like those of fig.9 and fig.9, whereas $J$ is given by the sum of tadpole diagrams like those of fig.10. As always, diagrams that become disconnected by the removal of the particle world-line do not contribute to the effective action $\Gamma_{\text{eff}}[\bar{\phi}, \bar{A}, \delta\gamma]$.

The additional Feynman rules beyond those of fig.8 are summarized in fig.11. The dashed lines denote the propagator for the vector field $A_i$ on flat $\mathbb{R}^{d-2,1} \times \mathbb{S}^1$

$$D_{ij}(x - x'; z - z') = -\frac{16\pi G}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{\delta_{ij}}{k_\perp^2 + (2\pi n/L)^2} e^{ik_\perp \cdot (x-x')_\perp + 2\pi n(z-z')/L},$$ \hspace{1cm} (4.6)

where we used the Feynman gauge defined by adding to the action (4.3) the following gauge fixing term

$$S_{GF} = \frac{1}{32\pi G} \int d^{d-1} x (\bar{\phi}' A_i)^2$$ \hspace{1cm} (4.7)
Figure 10: A diagrammatic representation of the definition of the renormalized angular momentum $J$ according to the effective action (4.5). Both (a) and (b) represent corrections of order $\lambda$.

The vertices in the bulk are constructed from the expansion of (4.3) about flat space. Those relevant for our computations are summarized in fig.5 and fig.11.

Counting powers of $r_0$ needs to be supplemented by a scaling of $J_0$ and we assign $J_0 \sim m_0 r_0$.

4.2 The renormalized mass and angular momentum

Here we calculate the renormalized mass according to fig.2. The leading $O(\lambda)$ correction to $m$ is still nothing but the 1-loop diagram in fig.2(a). Therefore the leading order contribution to the mass of the rotating black hole is identical to the static case and is given by (3.11)

$$m = m_0 \left( 1 - \frac{1}{2} \lambda + \ldots \right)$$

(4.8)

In order to calculate the leading contribution (of order $m_0 J_0$) to the angular-momentum $J_{ij}$ one needs to compute the 1-loop tadpole diagrams of fig.10(a),(b). Using the Feynman rules listed on fig.5 and fig.11 we obtain

$$fig.10(a) = \frac{-d-2}{d-3} \lambda \frac{J_0^{ij}}{2} \bar{F}_{ij}(O)$$

$$fig.10(b) = \frac{d-2}{d-3} \lambda \frac{J_0^{ij}}{2} \bar{F}_{ij}(O)$$

(4.9)

As a result, the overall contribution to $J_{ij}$ vanishes at linear order in $\lambda$

$$J_{ij} = J_0^{ij} \left( 1 + 0 \cdot \lambda + \ldots \right) .$$

(4.10)
OFJ
ij
4
0
= ( ) ( )
OOFd
ij
ij
φ
3
2
2
2
0
−
−
= \int_{−}^{−} \phi \pi
2
3
2
32
1
F dx d
G
( )
';'zzxxD
ij
−−
⊥⊥

Figure 11: Feynman rules obtained from the expansion of (4.3)

Actually, there can be no contribution to \( J \) of order \( J^2 \) either due to the absence of a cubic vertex for the vector field.

Higher order correction. Fig. uses a Feynman diagram which contributes to the mass \( m \) at the next to leading order \( O(\lambda^{1+\frac{2}{d-3}}) \). Applying the Feynman rules of fig. it is evaluated to be

\[
fig.11 = -\frac{d-2}{4} \frac{J^i_j J^j_i}{m_0 r_0^2} (\frac{r_0}{L})^{d-1} \zeta(d-1)
\]  

(see Appendix A for details). As a result, the mass of the rotating caged black hole (4.8) is modified

\[
m = m_0 \left(1 - \frac{1}{2} \lambda + \frac{d-2}{4} \frac{J^i_j J^j_i}{(m_0 r_0)^2} (\frac{r_0}{L})^{d-1} \zeta(d-1) + \ldots \right)
\]  

We note that this result is consistent with the 4d spin-spin interaction

\[
V_{SS} = +G \left(3(\vec{S}_1 \cdot \hat{r}) (\vec{S}_2 \cdot \hat{r}) - \vec{S}_1 \cdot \vec{S}_2 \right) / r^3
\]  

\[24\] see also \[22\].
4.3 Thermodynamics

In this section we derive additional thermodynamic quantities through the use of the Gibbs potential. We consider only the leading order corrections to the thermodynamic quantities. For simplicity and without loss of generality we assume that only one of the local spin parameters $J^A_0$ is non-zero and we denote it by $J_0$.

The calculation of the red-shift factor up to linear order in $\lambda$ does not differ from the static case. Thus the temperature and the angular velocity possess the same form as (3.18)

$$T = RT_0 = T_0 (1 - \lambda + \ldots)$$
$$\Omega = R\Omega_0 = \Omega_0 (1 - \lambda + \ldots)$$

where $T_0, \Omega_0$ are the local temperature and angular velocity of the rotating Myers-Perry black hole [18]

$$T_0 = \frac{d - 5}{4\pi r_+} + \frac{1}{2\pi} \frac{r_+^{d-4}}{r_0^{d-3}}$$
$$\Omega_0 = \frac{a}{r_+^2 + a^2}$$

(4.14)
a denotes the rotation parameter in terms of which the hole’s angular momentum is

$$J_0 = \frac{2m_0a}{(d - 2)}$$

(4.15)
and $r_+$ is the location of the horizon given implicitly by

$$r_0^{d-3} = r_+^{d-5}(r_+^2 + a^2) .$$

(4.16)

For later use we list also the hole’s entropy

$$S_0 = \frac{\pi^{d/2} r_+^{d-3}}{4GT_0\Gamma(\frac{d-3}{2})} r_0^{d-3} \left( 1 - \frac{2}{d - 3} \frac{a^2}{r_+^2 + a^2} \right) = \frac{4\pi r_+ m_0}{d - 2} ,$$

(4.17)

where only the first expression appears in [18].

We now wish to translate our knowledge of the thermodynamic potential $m$ into the Gibbs potential $G$ which is more appropriate for the natural variables of the problem. Using the definition of the Gibbs potential (2.18) and the Smarr formula $(d - 3)m = (d - 2)(\Omega J + TS) + \hat{\tau}L$ yields the simple relation

$$m = (d - 2)G - \hat{\tau}L .$$

(4.18)

In this equation $m$ is known from (4.8), while $G$ is unknown. Considering the tension to be a derivative of $G$ via $\hat{\tau} = \partial G/\partial L$ we get a differential equation which we can solve for $G$.

As mentioned in subsection 2.5, the natural parameters for the computation are $a$ and $m_0$ rather than $\Omega$ and $T$, therefore one needs to establish the relations between the
derivatives of the Gibbs potential expressed in terms of these two sets. We list some useful relations

\[ \hat{\tau} = \left( \frac{\partial G}{\partial L} \right)_{T, \Omega} = \frac{\partial(G, T, \Omega)}{\partial(L, T, \Omega)} = \frac{\partial(G, T, \Omega)}{\partial(L, m_0, a)} \left( \frac{\partial(L, T, \Omega)}{\partial(L, m_0, a)} \right)^{-1} \]

\[ J = - \left( \frac{\partial G}{\partial \Omega} \right)_{T, L} = -\frac{\partial(G, T, L)}{\partial(\Omega, T, L)} = -\frac{\partial(G, T, L)}{\partial(m_0, a, L)} \left( \frac{\partial(\Omega, T, L)}{\partial(m_0, a, L)} \right)^{-1} \]

\[ S = - \left( \frac{\partial G}{\partial T} \right)_{\Omega, L} = -\frac{\partial(G, \Omega, L)}{\partial(T, \Omega, L)} = -\frac{\partial(G, \Omega, L)}{\partial(m_0, a, L)} \left( \frac{\partial(T, \Omega, L)}{\partial(m_0, a, L)} \right)^{-1} \] (4.19)

The expression for \( \hat{\tau} \) simplifies at the leading order in \( \lambda \) and we obtain

\[ \hat{\tau} = \left( \frac{\partial G}{\partial L} \right)_{T, \Omega} = \left( \frac{\partial G}{\partial L} \right)_{m_0, a} + \frac{m_0 (d-3)^2}{L (d-2)} \lambda + \ldots \] (4.20)

In appendix B we present useful identities for the derivation of the above relation for \( \hat{\tau} \) and those which follow.

Substituting (4.20) and (4.8) in the relation between \( m \) and \( G \) (4.18) and solving the resulting differential equation for \( G(m_0, a, L) \) up to linear order in \( \lambda \) yields

\[ G(m_0, L) = \frac{m_0}{d-2} \left( 1 + \frac{d-4}{2} \lambda + \ldots \right) \] (4.21)

Substituting this expression into (4.19,4.20) we finally obtain

\[ \frac{\hat{\tau} L}{m_0} = \frac{d-3}{2} \lambda + \ldots \]

\[ J = J_0 \left( 1 + 0 \cdot \lambda + \ldots \right) \]

\[ S = S_0 \left( 1 + 0 \cdot \lambda + \ldots \right) \] (4.22)

where \( J_0 \), \( S_0 \) are defined by (4.13,4.17). This result for \( J \), obtained through a 0-point function, reproduces (4.10) obtained through a 1-point function. It states that the angular-momentum is left “unrenormalized” in the leading order. This agreement can be considered a consistency check for the Feynman rule of the \( J_0 F \phi \) vertex in fig.11. The other results are consistent with the static case 3.23.

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A. Calculations for Feynman diagrams

In this appendix we calculate certain integrals denoted by \( I_0, I_1 \) and defined below, which are useful for evaluating the Feynman diagrams (3.11,3.11), respectively. Both integrals are ultraviolet divergent and we use dimensional regularization.
We start from
\[ I_0(L) := \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}k_{\perp}}{(2\pi)^{d-2} k_{\perp}^2 + (2\pi n/L)^2} \] (A.1)

Let us use the dimensional regularization result
\[ \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + \Delta)^m} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2})}{\Gamma(m)} \Delta^{\frac{D}{2} - m} \] (A.2)

with \( D = d - 2, m = 1 \) and \( \Delta = \left( \frac{2\pi n}{L} \right)^2 \), then the \( n = 0 \) term in \( I_0(L) \) vanishes and the rest yields
\[ I_0(L) = \frac{\pi^{d-6}}{4L^{d-3}} \zeta(4-d) \Gamma \left( \frac{4-d}{2} \right) \] (A.3)

Note that \( \Gamma[(4 - d)/2] \) has a pole for \( d = 4, 6, 8, \ldots \), while \( \zeta(4-d) \) has a zero for exactly the same values of \( d \). We can avoid this feature by using a relation between the Gamma function and the Riemann zeta function
\[ \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} \zeta(s) = \Gamma \left( \frac{1-s}{2} \right) \pi^{-(1-s)/2} \zeta(1-s) \] (A.4)

from which we get
\[ \Gamma \left( \frac{4-d}{2} \right) \zeta(4-d) = \pi^{7/2-d} \Gamma \left( \frac{d-3}{2} \right) \zeta(d-3) = \frac{4}{d-3} \frac{\pi^{3-d/2}}{\Omega_{d-2}} \zeta(d-3). \] (A.5)

Substituting back into (A.3) we finally obtain
\[ I_0(L) = \frac{\Gamma \left( \frac{d-3}{2} \right)}{(4\pi)^{d/2}} \left( \frac{2}{L} \right)^{d-3} \zeta(d-3). \] (A.6)

We now turn to \( I_1 \) defined by
\[ I_1(L) := \frac{2}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-2}k_{\perp}}{(2\pi)^{d-2} k_{\perp}^2 + (2\pi n/L)^2} \] (A.7)

First we use the following dimensional regularization result
\[ \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 + \Delta)^m} = \frac{D/2}{(4\pi)^{D/2}} \frac{\Gamma(m - \frac{D}{2} - 1)}{\Gamma(m)} \Delta^{\frac{D}{2} - m+1} \] (A.8)

with the same \( D = d - 2, m = 1 \) and \( \Delta = \left( \frac{2\pi n}{L} \right)^2 \) as before. The \( n = 0 \) term in \( I_1(L) \) vanishes and the rest yields
\[ I_1(L) = \frac{2\pi^{d-4}}{L^{d-1}} (d-2) \zeta(2-d) \Gamma \left( \frac{2-d}{2} \right) \] (A.9)

Finally, applying relation (A.4) gives
\[ I_1(L) = (d-2)(d-3) \frac{\Gamma \left( \frac{d-3}{2} \right)}{\pi^{d-1}} \zeta(d-1) \frac{\Omega_{d-2}}{L^{d-1}}. \] (A.10)
B. Useful thermodynamic identities

In this appendix we consider a spinning black hole imbedded in an uncompactified $d$-dimensional space-time \[18\]. We assume that only one of the spin parameters is non-zero and present different identities valid in this case. These identities are found to be useful for the derivation of the thermodynamics of a rotating caged black hole considered in the text. The notation is explained in the text.

We start from

$$
\frac{\partial(T_0, \Omega_0)}{\partial(m_0, a)} = \frac{1}{2m_0 a} \frac{\partial T_0}{\partial a}
$$

(B.1)

This equation restates the thermodynamic identity $J_0 = -\partial G/\partial \Omega$ in the $(m_0, a)$ set of variables

$$
J_0 = \frac{2m_0 a}{d-2}
$$

$$
\left(\frac{\partial G_0}{\partial \Omega_0}\right)_{T_0} = \frac{\partial (G_0, T_0)}{\partial (\Omega_0, T_0)} = \frac{\partial (G_0, T_0)}{\partial (m_0, a)} \left(\frac{\partial (\Omega_0, T_0)}{\partial (m_0, a)}\right)^{-1}
$$

$$
= \frac{1}{d-2} \frac{\partial T_0}{\partial a} \left(\frac{\partial (\Omega_0, T_0)}{\partial (m_0, a)}\right)^{-1}
$$

(B.2)

where in the last equality we used that $G_0 = m_0/(d-2)$ according to (4.18) with no $\hat{\tau}$ in the uncompactified case.

Another set of useful identities can be obtained after taking account of scaling dimensions. Indeed, performing a scaling transformation $L \rightarrow (1 + \epsilon)L$ and recalling that $m_0, a, T_0, \Omega_0$ have length dimensions $d-3, 1, -1, -1$ respectively, we get by expanding

$$
\begin{pmatrix}
\frac{dT_0}{d\Omega_0}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial T_0}{\partial m_0} & \frac{\partial T_0}{\partial a} \\
\frac{\partial \Omega_0}{\partial m_0} & \frac{\partial \Omega_0}{\partial a}
\end{pmatrix} \begin{pmatrix}
\frac{dm_0}{da}
\end{pmatrix}
$$

(B.3)

to first order in $\epsilon$

$$
\begin{pmatrix}
-T_0 \\
-\Omega_0
\end{pmatrix} = \begin{pmatrix}
\frac{\partial T_0}{\partial m_0} & \frac{\partial T_0}{\partial a} \\
\frac{\partial \Omega_0}{\partial m_0} & \frac{\partial \Omega_0}{\partial a}
\end{pmatrix} \begin{pmatrix}
(d - 3) m_0 \\
a
\end{pmatrix}
$$

(B.4)

This expression can be inverted and rewritten as follows

$$
\begin{pmatrix}
(3-d)m_0 \\
am
\end{pmatrix} = \begin{pmatrix}
\partial(T_0, \Omega_0)
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial T_0}{\partial a} & -\frac{\partial T_0}{\partial a} \\
-\frac{\partial \Omega_0}{\partial m_0} & \frac{\partial \Omega_0}{\partial m_0}
\end{pmatrix} \begin{pmatrix}
T_0 \\
\Omega_0
\end{pmatrix}
$$

(B.5)

Combining this result with (B.1), we finally obtain

$$
\begin{align*}
\frac{3-d}{2a} \frac{\partial T_0}{\partial a} &= T_0 \frac{\partial \Omega_0}{\partial a} - \Omega_0 \frac{\partial T_0}{\partial a} \\
\frac{1}{2m_0} \frac{\partial T_0}{\partial a} &= T_0 \frac{\partial \Omega_0}{\partial m_0} - \Omega_0 \frac{\partial T_0}{\partial m_0}
\end{align*}
$$

(B.6)
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