A Note on Contractible Edges in Chordal graphs

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Abstract. Contraction of an edge merges its end points into a new vertex which is adjacent to each neighbor of the end points of the edge. An edge in a $k$-connected graph is contractible if its contraction does not result in a graph of lower connectivity. We characterize contractible edges in chordal graphs using properties of tree decompositions with respect to minimal vertex separators.

1 Introduction

Chordal graphs are also known as triangulated graphs [3] and have applications in the study of linear sparse systems, scheduling and relational database systems. In this paper, we focus on $k$-connected chordal graphs. We study the impact of contraction on connectivity in $k$-connected chordal graphs. In a graph $G$, contraction of an edge $e$ with endpoints $u, v$ is the replacement of $u$ and $v$ with a single vertex $z$. In the resulting graph, the edges incident on $u$ and $v$ are incident on $z$. Edge contraction and in general clique contraction plays a significant role in the proof of the Perfect Graph Theorem, see [8]. Edge contraction also plays a significant role in min-cut algorithms by using the basic property that the contraction of an edge can only increase the size of the min-cut. The basic idea exploited in randomized algorithms for min-cut is that contracting a randomly chosen edge does not increase the size of the min-cut [9]. This leads to expected polynomial time algorithms for min-cut, and these algorithms are fundamentally different from the classical max-flow based techniques.

1.1 Past Results on Contractible Edges

As with many a problem in Graph Theory, the study of contractible edges was initiated by Tutte in [7] where a constructive characterization of 3-connected graphs was presented. One consequence of this characterization was that in any 3-connected graph with at least five vertices, there is at least one contractible edge. In the work by Saito et. al [6], this lower bound was improved to $\frac{|V(G)|}{2}$, and the structure of graphs that have exactly so many contractible edges is studied. For $k$-connected graph with $k \geq 4$, it is still ongoing research to find necessary and sufficient conditions for the presence of contractible edges. For example, Thomassen [2] has shown that there is a contractible edge in triangle-free $k$-connected graphs in which the minimum degree is more than $\frac{2k-3}{2}$. Kriesell’s
survey of contractible edges [4] is an excellent source for many results in this area, and is also the motivation point of our work.

1.2 Definitions

We have, to a large extent, followed the notation and definitions as in the Graph Theory text by West[1]. Let $G = (V, E)$ be an undirected non weighted graph where $V(G)$ is the set of vertices and $E(G) \subseteq \{\{u, v\}|u, v \in V(G), u \neq v\}$. Order of $G$ and size of $G$ are $|V(G)|$ and $|E(G)|$, respectively. The neighborhood of a vertex $v$ in a graph $G$ is the set $\{u|\{u, v\} \in E(G)\}$ and is denoted by $N_G(v)$. A separating set or cut set of a graph $G$ is a set $S \subseteq V(G)$ such that the induced subgraph, denoted by $G - S$, on the vertex set $V(G) \setminus S$ has more than one connected component. The vertex connectivity of a graph $G$, written $\kappa(G)$, is the minimum cardinality of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex. $\gamma_G$ is the set of all minimum order cut sets. We let $G.e$ denote the graph obtained by contracting an edge $e = \{u, v\}$ in $G$ such that $V(G.e) = V(G) \setminus \{u, v\} \cup \{z_{uv}\}$ and $E(G.e) = \{\{z_{uv}, x\}|\{u, x\} \in E(G)\} \cup \{\{x, y\}|x \neq u, y \neq v \in E(G)\}$. An edge $e \in E(G)$ is contractible if the connectivity of $G.e$ is same as the connectivity of $G$. $E_c(G)$ denotes the set of contractible edges in $G$. A $k$-connected graph $G$ is said to be contraction critical, if for each edge $e$, connectivity of $G.e$ is smaller than the connectivity of $G$. The following lemma relates cut sets and contractible edges [4].

**Lemma 1.** An edge $e = \{u, v\}$ of $G$ is non contractible if and only if there is a minimum cut set $T \in \gamma_G$ such that $u \in T$ and $v \in T$.

A tree decomposition of a graph $G = (V, E)$ is a tree $T$, where each node $x$ has a label $l(x) \subseteq V(G)$ such that:

- $\bigcup_{x \in V(T)} l(x) = V(G).$(We say that "all vertices are covered.")
- For any edge $\{v, w\} \in E(G)$, there exists a node $x$ in $T$ such that $v, w \in l(x).$(We say that "all edges are covered.")
- For any $v \in V(G)$, the set of all nodes of $T$ whose label contains $v$ form a connected subtree in $T.$(We call this the "connectivity condition")

**Chordal Graph Preliminaries**

A **chord** of a cycle $C$ is an edge not in $C$ whose endpoints lie in $C$. A **chordless cycle** in $G$ is a cycle of length at least 4 in $G$ that has no chord. A graph $G$ is **chordal** if it is simple and has no chordless cycle. We can represent a chordal graph $G$ using a tree decomposition $T$ as follows: for each vertex $x \in V(T)$ the associated label $l(x) \subseteq V(G)$ induces a maximal clique in $G$, and for each $v \in V(G)$, $T_v$, the subgraph of $T$ induced by the set $\{x \in V(T)|v \in l(x)\}$, is a tree. We use $M$ to denote the set of minimal vertex separators of $G$, and the graph to which the symbol $M$ applies is always clear from the context. A stable (or independent) set is a set of pairwise nonadjacent vertices of the graph $G$. A **split** graph $G$ is a graph with two partitions, a stable set $I$ and a clique $K$.
such that $V(G) = I \cup K$. $E(G) \subseteq \{\{u, v\} | u \in I, v \in K\}$. For a chordal graph $G$ and its tree decomposition $T$, a minimal vertex separator $S$, and an edge $e \in E(G)$, we consider fixed tree decompositions of $G \setminus S$ and $G, e$, denoted by $T \setminus S$ and $T, e$, respectively. $T \setminus S$ and $T, e$ are defined as follows: The vertex set of both $T \setminus S$ and $T, e$ are same as the vertex set of $T$. The removal of $S$ and the contraction of $e$ only changes the labels associated with the vertices. In $T \setminus S$, for each $x \in V(T \setminus S)$, we write $l(x) = l(x) \setminus S$, if $S \cap l(x) \neq \phi$, otherwise $l(x)$ is the same set as in $T$. In $T, e$, for each $x \in V(T, e)$, $l(x) = l(x) \setminus \{u, v\} \cup \{z_{uv}\}$, if $l(x) \cap \{u, v\} \neq \phi$. Otherwise, $l(x)$ is the same set as in $T$. Clearly, $T \setminus S$ and $T, e$ are tree decompositions of $G \setminus S$ and $G, e$, respectively.

2 The Structure of Contractible edges in $k$-connected Chordal Graphs

We first prove a theorem which characterizes the set of minimal vertex separators of a chordal graph. This result is used subsequently to prove our characterisation of contractible edges in chordal graphs.

Lemma 2. Let $G$ be a chordal graph and $T$ be its tree decomposition. $G$ is connected iff for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.

Proof. Necessity: If $G$ is connected then we need to show that for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$. We prove this by contradiction. Suppose there exists an edge $\{x, y\} \in E(T)$ and $l(x) \cap l(y) = \phi$. Consider the two components $C_1$ and $C_2$ obtained by removing the edge $\{x, y\}$. Assume that $x \in C_1$ and $y \in C_2$. Let $A = \bigcup_{z \in C_1} l(z)$, $B = \bigcup_{z \in C_2} l(z)$. Since $T$ is a tree decomposition and $l(x) \cap l(y) = \phi$, it follows that $A \cap B = \phi$. Further, each edge $e \in E(G)$ is contained in the graph induced by $A$ or $B$ but not both. Hence $G$ is disconnected. However, by our hypothesis $G$ is connected. Hence our assumption is wrong. Therefore, if $G$ is connected then for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.

Sufficiency: Given that for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$, we now show that $G$ is connected. We show that $\forall u, v \in V(G), u \neq v$, there exists a path between $u$ and $v$ in $G$. Let $x, y$ be any two vertices in $V(T)$ such that $u \in l(x)$ and $v \in l(y)$. Consider the path $x = z_1, z_2, ..., z_j = y$ in the tree $T$. Further, $l(z_i) \cap l(z_{i+1})$ is non empty in $T$. This implies that there exists a vertex $r_i \in l(z_i) \cap l(z_{i+1})$. Hence the sequence of edges $\{u, r_1\} \{r_1, r_2\} ... \{r_{j-1}, v\}$ is a uv path in $G$. The reason this is true in $G$ is because $G$ is a chordal graph and label of each node in $T$ is a maximal clique. Therefore $u$ and $v$ are connected in $G$. Hence $G$ is connected.

Note: For a simple graph $G$ and any tree decomposition $T$, if $G$ is connected then for each edge $\{x, y\} \in E(T)$, $l(x) \cap l(y) \neq \phi$.

Theorem 1. Let $G$ be a $k$-connected chordal graph and let $T$ be its tree decomposition. Let $M' = \{X | X = l(x) \cap l(y) \text{ where } \{x, y\} \in E(T)\}$ and $M'' = \{Y | Y \in M' \text{ and for all } Z \in M', Z \not\subset Y\}$. $M = M''$. In other words, $M''$ is the set of minimal vertex separators of $G$. 
Theorem 2. Let $G$ be a $k$-connected chordal graph with $|V(G)| \geq (k + 2)$. An edge $e = \{u, v\} \in E(G)$ is contractible if and only if one of the following holds:

(i) $e$ is in a unique maximal clique in $G$

(ii) For $x, y \in V(T)$, $\{u, v\} \subseteq l(x) \cap l(y)$ and $\{x, y\} \in E(T)$, $|l(x) \cap l(y)| > k$.

Proof. Necessity:
(i): Given that $e$ is contractible implies that $G.e$ is $k$-connected. If $e$ is in a unique maximal clique in $G$, then we are done. In the case when $e$ is not in a unique maximal clique, let $e \in l(x) \cap l(y)$ for some $\{x, y\} \in E(T)$. We now show that $|l(x) \cap l(y)| > k$. We prove this claim by contradiction. Let us assume that $|l(x) \cap l(y)| \leq k$. On contraction of $e$ in $G$, the tree decomposition of $G.e$ is $T.e$. In $T.e$, the $|l(x) \cap l(y)| \leq k - 1$. From lemma 2 it follows that that $l(x) \cap l(y)$ is a vertex separator of $G.e$, and since $|l(x) \cap l(y)| \leq k - 1$, it follows that $G.e$ is $k - 1$-connected. This is a contradiction to the hypothesis that $G.e$ is $k$-connected. Therefore, our assumption that $|l(x) \cap l(y)| \leq k$ is wrong. It follows that $|l(x) \cap l(y)| > k$.

Sufficiency: First, we consider the case when $e$ is in a unique maximal clique and show that $e$ is contractible. If $e$ is in a unique maximal clique in $G$ implies that $e$ is contained in the label of a unique node in $T$. Therefore, for each $x, y \in T$, $|l(x) \cap l(y)|$ remains unchanged in $T.e$. From theorem 1 the connectivity of $G.e$ is at least as much as the connectivity of $G$. Therefore, $e$ is contractible. In the case when $|l(x) \cap l(y)| > k$ for all $\{x, y\} \in E(T)$, after contracting $e$, in $T.e$ $|l(x) \cap l(y)|$ is at least $k$ and hence the connectivity of $G.e$ is at least $k$, by theorem 1. Hence $G.e$ is $k$-connected. Therefore, $e$ is contractible in $G$.

As a consequence of this lemma, it follows that each edge incident on a simplicial vertex in a $k$-connected chordal graph is contractible. Therefore, a $k$-connected
chordal graph has at least $2k$ contractible edges. We now apply the main lemma to understand contractible edges in split graphs. Let $G$ be a non regular split graph. An edge $e = \{u, v\}$ such that $u \in K$ and $v \in I$ is contractible. Clearly such an edge $e$ is in a unique maximal clique in $G$. By theorem 2 $e$ is contractible.

For the case when $G$ is a regular $k$-connected split graph with at least $k + 2$ vertices, it follows that $G$ is contraction critical, that is none of the edges of $G$ are contractible. The reason is that, given that $G$ is regular implies that there is exactly one vertex in $I$. Thus the resulting graph is a complete graph and each edge in every complete graph is non contractible. Therefore, $G$ is contraction critical.

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