Controllable Conditional Quantum Oscillations and Quantum Gate Operations in Superconducting Flux Qubits

Ai Min Chen and Sam Young Cho*

Centre for Modern Physics and Department of Physics, Chongqing University, Chongqing 400044, China

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1. Introduction

Quantum gates lie at the heart of realization of quantum computing.1-3 Two-qubit gates as well as single-qubit gates have been demonstrated in various types of quantum systems such as cavity quantum electrodynamics (QED),2,3) ion traps,4,5) nuclear magnetic resonance (NMR),6-9) quantum dots,10) and superconducting charge11) and flux12,13) qubits.

To perform qubit operations, normally, an electromagnetic field is applied such as microwave fields, laser pulses, and oscillating voltages, which can induce quantum oscillations between qubit states. Especially, quantum Rabi oscillations can be achievable to manipulate a controlled-gate operation because it implies a complete transition between two quantum states at the half-period time of the oscillation. Of particular interests are to manipulate such quantum oscillations between qubit states by applying an external field in association with controlled-gate operations.

A gate operation depending on a control qubit state can be performed, which is called conditional gate operation, where the target states can be flipped for a control-NOT (CNOT) gate operation. It has been experimentally demonstrated for a pair of superconducting qubits in refs. 11 and 13. In the experiments, an individual conditional operation has been applied to observe a CNOT gate operation in the superconducting charge and flux qubits. Also, in ref. 14, a similar scheme has been theoretically discussed via time evolution of a two qubit system for an applied pulsed-bias field.

In our study, a conditional quantum oscillation rather than a conditional gate operation is introduced by considering an effective interacting two-qubit Hamiltonian, which can be achievable in adjusting system parameter ranges. Compared to the conventional proposals that are based on a combination of several steps of gate operations, manipulating conditional quantum oscillations enable us to perform an aimed gate operation in a single step. Thus, we investigate how conditional quantum oscillations can be simultaneously manipulated to perform a controlled-gate operation in a controllable and accurate manner. To clearly discuss analytically an implementation of conditional quantum oscillations to quantum gate operations, at first, we restrict ourselves to a rotating wave approximation (RWA) in the presence of applied time-dependent fields, which allows us to capture an essential physics for controlled quantum gate operations based on conditional quantum oscillations.

Within the approximation, at the resonant frequencies, the conditional Rabi and non-Rabi oscillations are shown to characterize the time-dependent dynamics of the two qubit system. A frequency matching condition is discussed for achievable controlled-gate operations. By synchronizing the two oscillation frequencies on the matching condition, the CNOT gate performance and operation time are shown to be controllable for a very accurate gate operation. Also, we perform a numerical calculation without any approximation. The numerical results show a well agreement with the analytic results. Further, conditional quantum oscillations and their frequency synchronization can be applicable to various quantum gate operations in solid-state multi-qubit systems such as Toffoli and Fredkin gates.

2. Model

Any two-state system can play a role as a qubit. In solid-state systems, qubit parameters are controllable. In terms of the pseudo-spin-1/2 language, a single qubit in the two states of the basis \(\{|\uparrow\rangle, |\downarrow\rangle\}\) can be described by the Hamiltonian \(H_0 = \sum_\sigma \epsilon_\sigma \sigma^\sigma_\sigma\), where \(\sigma^\sigma_\sigma\) is the Pauli matrices with the identity matrix \(\sigma^\sigma\), \(\epsilon_\sigma\), and \(\Delta_\sigma\) correspond to the energy difference and transition amplitude between the two states of the qubit \(\sigma\), respectively. \(V(t)\) is responsible for an interplay between the states of the qubit \(\sigma\) and a time-dependent applied field. In the basis \(\{|\uparrow\rangle, |\downarrow\rangle\}\), let us consider the Hamiltonian\(^{13,15}\) of interacting two qubits \((\sigma \in \{A, B\})\) written by

\[
H = H_0 \otimes \sigma^\sigma_\sigma + \sigma^\sigma_\sigma \otimes H_B + J \sigma^\sigma_A \otimes \sigma^\sigma_B,
\]

where \(J\) is the interaction strength characterizing the interaction between the two qubits. If the transition amplitudes \(\Delta_A\) and \(\Delta_B\) become negligible by means of controlling system parameters, the Hamiltonian has a form of block-diagonal matrix. Each block of the matrix Hamiltonian corresponds to a conditional Hamiltonian. Suppose that the qubit \(A\) is the control qubit and the transition amplitude \(\Delta_A\) is negligible compared to other energies. Then, for the states \(|\uparrow\rangle\) and \(|\downarrow\rangle\) of qubit \(A\), one can rewrite the Hamiltonian in eq. (1), i.e., the two effective Hamiltonians of the qubit \(B\) given by

\[E-mail address: sycho@cqu.edu.cn]
where $\sigma^B_0 = -\Delta_B \sigma^B_1$.

Equations (2) and (3) show that the time-dependent dynamics of the system can be understood in association with a combination of two conditional quantum oscillations of the qubit $B$. Away from the qubit degeneracy point, where the qubit energy $\epsilon_A$ (transition amplitude $\Delta_A$) becomes relatively bigger (smaller) than other parameters, such conditional quantum oscillations can be achievable.\(^\text{13}\)

### 3. Conditional Quantum Oscillations

Let us discuss about conditional quantum oscillations of two qubits in the absence of time-dependent applied fields before we consider conditional quantum oscillations controlled by external fields in the next section. In the absence of time-dependent applied fields $V_i(t) = 0$ ($i \in \{A, B\}$), the time-independent conditional Hamiltonians generate the time evolution of the states of qubit $B$ through the Schrödinger equation $i\hbar \dot{\psi}_{B,i}(t) = H_{B,i}(t)\psi_{B,i}(t)$. For the two conditional time evolutions of the qubit states, at time $t$, the states are written by $|\psi_{B,i}(t)| = G_{B,i}(t)|\psi_{B,i}(0)|$, where $G_{B,i}(t) = U^{-1}(\eta_{B,i}(0)) \exp(-i H_{B,i}(0) t) U(\eta_{B,i}(0))$ with $H_{B,i} = U(\eta_{B,i}(0)) H_B(0) U^{-1}(\eta_{B,i}(0))$. Here, $U(\eta)$ is a unitary transformation matrix which makes the conditional Hamiltonians $H_{B,i}$ diagonal respectively with $\eta_{B,i}(0)|_i = \tan^{-1}\Delta_B/2\epsilon_B \pm 2J_i$.

The unitary transformation matrix is written by

$$U(\eta) = \begin{pmatrix}
\cos \frac{\eta}{2} & -\sin \frac{\eta}{2} \\
\sin \frac{\eta}{2} & \cos \frac{\eta}{2}
\end{pmatrix}.$$  

These unitary transformations give the eigenvalues of the conditional Hamiltonians, i.e., the eigenvalues of the system, without time-dependent external fields:

$$\epsilon_{\pm}^{(0)}_{\pm} = \frac{1}{2}(\epsilon_B \pm 2J^2 + \Delta_B^2/2),$$

$$\epsilon_{\pm}^{(0)}_{\mp} = -\frac{1}{2}(\epsilon_B \pm 2J^2 + \Delta_B^2/2).$$

In fact, the two conditional time evolutions of the one qubit can characterize the dynamics of two qubit system. If the initial state of qubit $B$ is chosen as an arbitrary state $|\psi_{B,i}(0)| = |a\rangle\langle b|$, the occupation probabilities of the states $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ at time $t$ are obtained as

$$P_{\pm}(t) = 1 - P_{\mp}(t) = a^2 + [b^2 - (a \sin n_{B,i} + b \cos n_{B,i})^2] \sin^2[\Omega_B(t)/2]$$

for the $|i\rangle$ states of qubit $A$, where the conditional oscillation frequencies are respectively denoted by the qubit Larmor frequencies $\Omega_B(t)|_i = [\epsilon_B \pm 2J^2 + \Delta_B^2]/2$ corresponding to the resultant energy levels. (i) For $a = 1$ and $b = 0$, i.e., $|\psi_{B,i}(0)| = |\uparrow\rangle$ or (ii) for $a = 0$ and $b = 1$, i.e., $|\psi_{B,i}(0)| = |\downarrow\rangle$, if $\epsilon_B = -2J$ ($\epsilon_B = 2J$) then the conditional Larmor oscillation between the states $|\uparrow\uparrow\rangle$ and $\pm |\downarrow\rangle$ occurs with the characteristic frequency $\Omega_B(t) = \Delta_B$. These conditional Larmor oscillations show that an initial state of the qubit $B$ can be in the other flipped state of the qubit $B$ at the periodic time $t = 2\pi(m - 1/2)/\Delta_B$ with a positive integer $m$.

In Fig. 1, we plot the probabilities to find the system in the states $|ss\rangle$ ($ss \in \{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$) for an initial state chosen as $|\psi(0)| = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. Here, $\tau_s = 2\pi/\Omega_B^{(0)}$ is the period of the conditional Larmor oscillation. The parameters are chosen as $\epsilon_A = 5J$, $\epsilon_B = -2J$, $\Delta_A = 0.01J$, and $\Delta_B = 4\sqrt{3}J$. At the half periodic times of the complete conditional oscillations, the CNOT gate operations can be realized.

### 4. Controlled Operations with Controllable Conditional Quantum Oscillations

Let us consider a time-dependent applied field $V_i(t) = V \cos wt$, where $V$ and $\omega$ are its amplitude and frequency, respectively. The time-dependent applied field can give rise to an external field-driven conditional Rabi oscillation. Recall the unitary transformation $U(\eta_{B,i}(0))$, where $\eta_{B,i}(0) = \tan^{-1}\Delta_B/2\epsilon_B$. Actually, any other unitary transformation can be used for the qubit $B$ such as $\eta_{B,i}(0)$.
tional Hamiltonians are then transformed as $\hat{H}_{B,t} = U(\phi_B^{(0)}) U_B^{-1}(\phi_B^{(0)})$. The basis $\{|i\}, |j\rangle \}$ of qubit $B$ are transformed into the basis $\{|0\}, |1\rangle \}$ by $\langle \Psi_{B,s} | = U(\phi_B^{(0)}) |\Psi_{B,s} \rangle$.

In order to show clearly a possible controlled-gate operation, we will employ a rotating wave approximation (RWA) which implies that an applied time-dependent field can be a static field in a rotating frame. The usual experimental condition, i.e., $\epsilon_B V_B / \Omega_B^{(0)} < \Omega_B^{(0)}$ (10) will be considered to limit our analytic calculation for the parameter regime. We will also assume that the applied fields satisfy the regime $V_A \ll \epsilon_A$ and $|2J| \ll V_B$, where $\Omega_B^{(0)} = [\epsilon_B^2 + \Delta_B^2]^{1/2}$. Within the approximations, then, the conditional Hamiltonians in the rotating frame are obtained through $\hat{H}_{B,s}^{(0)} = -iU(t)\partial_t U^{-1}(t) + U(t)\hat{H}_B U^{-1}(t)$ with $U(t) = \exp[\text{not} \sigma_B^z/2]$ as

$$
\hat{H}_{B,t}^{(0)} = \frac{1}{2} \left( \frac{\Omega_B^{(0)} + 2J \epsilon_B}{\Omega_B^{(0)}} - \omega \right) \sigma_B^z + \epsilon_A \sigma_B^y + \frac{V_B}{2} \left( \frac{\Delta_B}{\Omega_B^{(0)}} \right) \sigma_B^x.
$$

$$
\hat{H}_{B,\bar{t}}^{(1)} = \frac{1}{2} \left( \frac{\Omega_B^{(0)} - 2J \epsilon_B}{\Omega_B^{(0)}} - \omega \right) \sigma_B^z - \epsilon_A \sigma_B^y + \frac{V_B}{2} \left( \frac{\Delta_B}{\Omega_B^{(0)}} \right) \sigma_B^x.
$$

To see a quantum gate operation, one can employ a probability amplitude table $U(t)$ at time $t$. For the conditional quantum oscillations in two qubit systems, then, a two-qubit gate operation can be seen in the probability amplitude table $U(t)$ expressed as

$$
U(t) = \begin{pmatrix}
P_{\{1\} \rightarrow \{0\}}(t) & P_{\{1\} \rightarrow \{1\}}(t) & 0 & 0 \\
P_{\{0\} \rightarrow \{1\}}(t) & P_{\{0\} \rightarrow \{0\}}(t) & 0 & 0 \\
0 & 0 & P_{\{0\} \rightarrow \{1\}}(t) & P_{\{1\} \rightarrow \{0\}}(t) \\
0 & 0 & P_{\{1\} \rightarrow \{1\}}(t) & P_{\{1\} \rightarrow \{1\}}(t)
\end{pmatrix},
$$

where $P_{\beta \rightarrow \alpha}$ denotes the probability that an $\alpha$ input state becomes a $\beta$ output state with $\alpha, \beta \in \{0\}, \{1\}, \{0\}, \{1\}\}$. Within the approximations, by solving the Schrödinger equations of the effective Hamiltonians $i\partial_t |\Psi_{B,t}(t)\rangle = \hat{H}_{B,s}^{(0)} |\Psi_{B,s}(t)\rangle$ in eqs. (4) and (5), we obtain the probabilities as $P_{\{1\} \rightarrow \{0\}}(t) = \sin^2 \eta_B \sin^2 \frac{\Omega_B t}{2}$

$$
P_{\{1\} \rightarrow \{1\}}(t) = \cos^2 \eta_B \sin^2 \frac{\Omega_B t}{2},
$$

with the relations $P_{\{0\} \rightarrow \{0\}} = P_{\{0\} \rightarrow \{0\}} = 1 - P_{\{1\} \rightarrow \{0\}} = 1 - P_{\{0\} \rightarrow \{1\}}$, where the conditional oscillation frequencies are

$$
\Omega_{B,s} = \left[ \left( \frac{\omega - \Omega_B^{(0)} - 2J \epsilon_B}{\Omega_B^{(0)}} \right)^2 + \left( \frac{\Delta_B V_B}{\Omega_B^{(0)}} \right)^2 \right]^{1/2},
$$

for $|\uparrow\rangle/|\downarrow\rangle = \pm$. The transformation angles are denoted by $\eta_B = \tan^{-1} \left( \frac{\Delta_B V_B}{2 \left( \omega - \Omega_B^{(0)} - 2J \epsilon_B/\Omega_B^{(0)} \right) \Omega_B^{(0)}} \right)$.

There are two resonant frequencies (i) $\omega = \Omega_B^{(0)} + 2J \epsilon_B/\Omega_B^{(0)}$ and (ii) $\omega = \Omega_B^{(0)} - 2J \epsilon_B/\Omega_B^{(0)}$, each of which can induce a conditional Rabi oscillation. (i) For $\omega = \Omega_B^{(0)} + 2J \epsilon_B/\Omega_B^{(0)}$, $P_{\{1\} \rightarrow \{0\}}(t) = \sin^2 \eta_B t/2$ undergoes a Rabi oscillation while $P_{\{1\} \rightarrow \{1\}}(t) = \sin^2 \eta_B t/2$ can do a non-Rabi oscillation, where the conditional Rabi and non-Rabi oscillation frequencies are respectively given by

$$
\Omega_R = \frac{V_B}{2} \left( \frac{\Delta_B}{\Omega_B^{(0)}} \right),
$$

$$
\Omega_{n,R} = \left[ 16J^2 \left( \frac{\epsilon_B}{\Omega_B^{(0)}} \right)^2 + \frac{V_B^2}{4} \left( \frac{\Delta_B}{\Omega_B^{(0)}} \right)^2 \right]^{1/2}.
$$

The amplitude of the non-Rabi oscillation is determined by $\eta_{n,R} = \tan^{-1} \left( \frac{\Delta_B V_B}{8J \epsilon_B} \right)$.

The Rabi oscillation has a longer period than the non-Rabi oscillation because the non-Rabi frequency is larger than the Rabi oscillation frequency, $\Omega_R < \Omega_{n,R}$. It should be noted that, in the chosen basis, the Rabi frequency does not depend on the interaction strength $J$ within our approximations while the non-Rabi frequency depends on the interaction. This shows that if no interaction exists between the qubits, in fact, the conditional quantum oscillations are not realizable in the basis.

If one chooses other basis, however, a Rabi frequency can be dependent of the interaction strength $J$. (ii) For the other resonant frequency $\omega = \Omega_B^{(0)} - 2J \epsilon_B/\Omega_B^{(0)}$, the two conditional quantum oscillations exchange their roles each other, i.e., $P_{\{1\} \rightarrow \{1\}}(t) = \sin^2 \eta_B t/2$ undergoes a Rabi oscillation while $P_{\{1\} \rightarrow \{0\}}(t) = \sin^2 \eta_{n,R} t/2$ can do a non-Rabi oscillation.

To see clearly a role of the two conditional quantum oscillations for CNOT gate operations, let us introduce the fidelity $F$ of the probability amplitude table for the truth table of CNOT gate as

$$
F(t) = \frac{1}{4} \text{Tr}[U(t)U_{\text{CNOT}}].
$$

The fidelity $F$ and its error $\delta F(t) = 1 - F(t)$ give the estimations of CNOT gate performance and its reliability. In terms of the transition probability amplitudes, with $U_{\text{CNOT}} = \text{diag}([0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1])$, the fidelity is generally given by

$$
F = \frac{1}{4} \left( P_{\{1\} \rightarrow \{1\}} + P_{\{1\} \rightarrow \{1\}} \right)
+ \left( P_{\{1\} \rightarrow \{0\}} + P_{\{1\} \rightarrow \{1\}} \right).
$$

For the conditional quantum oscillations, from eq. (6), the fidelity becomes

$$
F(t) = \frac{1}{2} \left( P_{\{1\} \rightarrow \{1\}}(t) + P_{\{1\} \rightarrow \{1\}}(t) \right)
$$

because $P_{\{1\} \rightarrow \{1\}}(t) = P_{\{1\} \rightarrow \{1\}}(t)$ and $P_{\{1\} \rightarrow \{1\}}(t) = P_{\{1\} \rightarrow \{1\}}(t)$. This shows that the two conditional quantum oscillations with their characteristic frequencies $\Omega_R$ and $\Omega_{n,R}$ determine a CNOT operation performance and its reliability. For the resonant frequency $\omega = \Omega_B^{(0)} + 2J \epsilon_B/\Omega_B^{(0)}$, the Rabi oscillation $P_{\{1\} \rightarrow \{1\}}(t)$ shows that the initial state is in its flipped state at a periodic time $t = (m - 1/2) \tau_R$ with a positive integer $m$, where $\tau_R = 2\pi/\Omega_R$ is the period of Rabi oscillation. The non-Rabi oscillation
$P_{|10\rangle\rightarrow|10\rangle}(t)$ shows that the initial state is in its original state at a periodic time $t = m\tau_R$, where $\tau_R = 2\pi/\Omega_{n-R}$ is the period of non-Rabi oscillation. Performing a CNOT gate operation ($F = 1$) is, in fact, to synchronize the periods $\tau_R$ and $\tau_{n-R}$ of the Rabi and non-Rabi oscillations by varying the system parameters because, according to the control qubit states, the target state should be in a flipped state (Rabi oscillation: $P_{|10\rangle\rightarrow|+\rangle} = 1$) or the original state (non-Rabi oscillation: $P_{|10\rangle\rightarrow|1\rangle} = 0$) at a certain operation time $t_{\text{OP}}$.

5. Synchronization of Two Conditional Oscillations for Controlled Gate Operation

To synchronize the two conditional quantum oscillations for a CNOT gate operation, we discuss a frequency matching condition between the quantum oscillations. If the operation time of CNOT gate is $t_{\text{OP}}^c = \tau_R/2$ for the first flipped state in the Rabi oscillation, a CNOT gate operation can be performed with a positive multiple-integer period of non-Rabi oscillation $n\tau_{n-R} = \tau_R/2$, where $n$ is a positive integer. Once the frequencies are matched with the condition, the CNOT gate is operated periodicaly at the periodic operation time $t_{\text{OP}}^c = (m - 1/2)\tau_R$ with the $m$-th flipped state of the Rabi oscillation. More generally, if a CNOT gate operation is performed at the $l$-th flipped states in the Rabi oscillation, i.e., the operation time becomes $t_{\text{OP}} = (l - 1/2)\tau_R$ with a positive integer $l$, then the matching frequencies are given by the relation $(n + l - 1)\tau_{n-R} = (l - 1/2)\tau_R$ because of $\tau_{n-R} < \tau_R$ ($\Omega_R < \Omega_{n-R}$). In other words, a $(n + l - 1)$ multiple-integer period of non-Rabi oscillation matches with the period $(l - 1/2)\tau_R$ of Rabi oscillation for the CNOT gate operation. For $l = 1$, these matching frequencies are reduced to the relation $n\tau_{n-R} = \tau_R/2$ at the first flipped state. Consequently, the matching frequencies between the conditional non-Rabi and Rabi oscillations for a CNOT gate operation are given by the relation

$$\Omega_{n-R}(V_B) = 2\left(\frac{n + l - 1}{2l - 1}\right)\Omega_R(V_B).$$

(11)

Both the Rabi and non-Rabi oscillations can be tuned by varying the amplitude of applied fields $V_B$ for a CNOT gate operation. By using the conditional quantum oscillations, therefore, a CNOT gate operation can be achievable by synchronizing the periods of conditional Rabi and non-Rabi oscillations. Note that, for the other resonant frequency $\omega = \Omega_B^{(0)} = 2J_{B}/\Delta_B^{(0)}$, the matching frequencies in eq. (11) leads to another CNOT gate operation with the ideal truth table $U_{\text{CNOT}} = \text{diag}(1, 0, 0, 0, 1, 0, 0, 0)$ because $P_{|1+\rangle\rightarrow|+\rangle} = 0$ and $P_{|1-\rangle\rightarrow|-\rangle} = 1$ at the operation time $t_{\text{OP}} = (l - 1/2)\tau_R$.

When the CNOT gate operation has carried out, from eq. (11), the amplitudes of the applied time-dependent field $V_B$ are given by

$$V_B^{\text{CNOT}}(n, l) = \frac{2l - 1}{\sqrt{2(n - 1)(2n + 4l - 3)}}\left(\frac{8J_{B}}{2} \right)^{n-R}.$$

(12)

Then as $V_B$ varies a CNOT gate operation is executed for $V_B = V_B^{\text{CNOT}}(n, l)$ consecutively. This implies that the synchronization of the operation time $t_{\text{OP}} = (l - 1/2)\tau_R$ can be achievable by tuning the applied time-dependent field $V_B$. Also, it is shown that from eqs. (8), (9), and (12) the CNOT gate operation can be performed with possible Rabi and non-Rabi frequencies given as

$$\Omega_{\text{CNOT}}(n, l) = \frac{2l - 1}{\sqrt{2(n - 1)(2n + 4l - 3)}}\left(\frac{8J_{B}}{2} \right)^{n-R}.$$

(13)

$$\Omega_{\text{CNOT}}(n, l) = \frac{2l - 1}{\sqrt{2(n - 1)(2n + 4l - 3)}}\left(\frac{8J_{B}}{2} \right)^{n-R}.$$

(14)

As a result, synchronizing well conditional quantum oscillations by varying system parameters makes it possible to achieve a CNOT gate operation with a very accurate performance rate, which can be applied to various types of qubit systems. In addition, the operation time can be controlled by means of the matching frequencies.

A CNOT gate is a special case of the controlled-$U$ gate. If the conditional non-Rabi oscillation is suppressed to make the states of target qubit staying in their original states during the conditional Rabi oscillation, the two qubit system can be a controlled-$U$ gate. Actually, if $\eta_{n-R} \ll 1$, i.e., $\Delta_B^{(0)} \approx 8J_{B}$, the amplitude of the non-Rabi oscillation becomes negligible $\sin^2 \eta_{n-R} \approx 0$. Without the matching frequencies, a controlled-$U$ gate operation can then be obtained. Another possible way for a controlled-$U$ gate is also to be the matching frequencies. From eq. (11), the amplitude of the non-Rabi oscillation is given by

$$\sin^2 \eta_{\text{CNOT}} = \frac{1}{4} \left(\frac{2l - 1}{n + l - 1}\right)^2.$$

(15)

As $n$ increases, i.e., the amplitude of time-dependent field decreases in eq. (12), the amplitude of the non-Rabi oscillation will become significantly suppressed, $\sin^2 \eta_{n-R} \approx 0$. This results in a realization of a controlled-$U$ gate rather than a controlled-CNOT gate. In the case of $\omega = \Omega_B^{(0)} = 2J_{B}/\Delta_B^{(0)}$, then the probability amplitude tables become a truth table of controlled-$U$ gate respectively given as

$$U_{\text{CU}}^{(+)}(t) \simeq \begin{pmatrix} \cos^2 \Omega_{B}/2 & \sin^2 \Omega_{B}/2 & 0 & 0 \\ \sin^2 \Omega_{B}/2 & \cos^2 \Omega_{B}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(16)

$$U_{\text{CU}}^{(-)}(t) \simeq \begin{pmatrix} \cos^2 \Omega_{B}/2 & \sin^2 \Omega_{B}/2 & 0 & 0 \\ 0 & 0 & \cos^2 \Omega_{B}/2 & \sin^2 \Omega_{B}/2 \\ 0 & \cos^2 \Omega_{B}/2 & 0 & \sin^2 \Omega_{B}/2 \\ 0 & 0 & \cos^2 \Omega_{B}/2 & \sin^2 \Omega_{B}/2 \end{pmatrix}.$$

(17)

As a consequence, if one of conditional quantum oscillations is suppressed by controllable system parameters, the two qubit system can provide a controlled-$U$ gate.

6. Numerical Results and Discussion

Based on the conditional quantum oscillations, so far, a possibility of a CNOT gate operation has been discussed with the analytic equations within a RWA. To give a clear understanding on the CNOT gate operation by means of conditional quantum oscillations, in this section, we will discuss our numerical results. As discussed, in order to realize the conditional quantum oscillations, there is only one constraint condition in eq. (1), i.e., $\Delta_A$ is negligibly...
smaller than other energy parameters. Then, all numerical calculations are performed in the parameter regimes of conditional quantum oscillations. Especially, we focus on the simplest case for \( l = 1 \) in the numerical calculation.

In Fig. 2, we plot the time evolutions of fidelities for different synchronization conditions (\( n = 1, 2, 3, 4 \)) in eq. (11). Here, the chosen other parameters are \( \varepsilon_A = 500J, \varepsilon_B = 300J, \Delta_A = 0.01J, \Delta_B = 100J, \) and \( V_A = 5J. \) Also, the amplitudes of applied fields are chosen from eq. (12) as \( V_B(1,1) = 13.856J, V_B(2,1) = 6.197J, V_B(3,1) = 4.057J, \) and \( V_B(4,1) = 3.024J. \) It is shown that the fidelities can reach their maximum value \( F \simeq 1 \) periodically in the period time of \( \tau_B(n,1) \) for different system parameters and a CNOT gate operation is realizable at the time of the maximum fidelities. The gate operation times are to be \( t = (m - 1/2)\tau_B(n,1). \) Also, the relation \( \tau_B(n,1) = 2\pi/\Omega_R(n,1) \) indicates that the gate operations are performed in the period of the Rabi oscillations. This numerical result agrees well with our analytic results. Hence, this shows that the conditional Rabi oscillations play a significant role to realize a CNOT gate operation. Further, it enables us to obtain a CNOT gate with a very accurate performance rate.

In Fig. 3, we display the fidelity as a function of the applied field at the time \( t = \tau_R/2. \) Here, the other parameters are chosen as \( \varepsilon_A = 500J, \varepsilon_B = 300J, \Delta_A = 0.01J, \Delta_B = 100J, \) and \( V_A = 5J. \) Other parameters are chosen as \( \varepsilon_A = 500J, \varepsilon_B = 300J, \Delta_A = 0.01J, \Delta_B = 100J, \) and \( V_A = 5J. \) The peaks of the fidelity correspond to the first maximum fidelities for \( n = 1, 2, 3, \) and \( 4 \) in Fig. 2. In the inset, the fidelity error is displayed. For \( n = 1, \) as an example, near the maximum fidelity, the vertical green shade indicates an amplitude window \( \Delta V_B = 1.043J \) within the error value \( 10^{-4} \) for a CNOT gate operation.

Also be very useful for an experimental achievement of CNOT gate operation.

If the frequency of the applied field does not tune to a perfect resonant frequency, i.e., \( \omega = \omega_0 + \Delta \omega \) with the resonant frequency \( \omega_0 = \Omega_B^{(0)} + 2J\varepsilon_B/\Omega_B^{(0)} \) and its deviation \( \Delta \omega, \) the deviation of the resonant frequency may spoil the performance rate of the CNOT gate operation. Then, in Fig. 4, we plot the fidelity error \( \delta F \) as a function of \( \Delta \omega \) at time \( t = \tau_R/2 \) for \( n = 1. \) To satisfy the relation \( \Omega_{\omega - R} = 2\omega_R, \) other parameters are chosen as \( \varepsilon_A = 500J, \varepsilon_B = 300J, \Delta_A = 0.01J, \Delta_B = 100J, \) and \( V_A = 5J. \) As the \( \Delta \omega \) increases, the fidelity error \( \delta F(\Delta \omega) \) increases monotonically. It is shown that, within \( \Delta \omega \simeq 1 \times 10^{-4}\omega_0, \) the fidelity error occurs around \( \delta F \simeq 5.0 \times 10^{-4} \) close for the fault-tolerant quantum computing.

### 7. Multi-Qubit System and Conditional Quantum Oscillation

For multi-qubit systems, conditional quantum oscillation can be realizable if a similar adjustment is made in the system parameters. Once conditional quantum oscillations are achieved in multi-qubit systems, one may synchronize their...
characteristic frequencies to perform a controlled multi-qubit gate operation such as Toffoli and Fredkin gates which are an extension of two-qubit gates to multi-qubit gates.

8. Summary

We investigated conditional quantum oscillations in interacting solid-state qubit systems. It was shown that a conditional quantum oscillation can be achievable in a way of tuning a range of system parameters. Synchronizing conditional quantum oscillations by varying applied time-dependent fields as well as system parameters enables to perform quantum gate operations such as controlled-NOT and $-U$ gate operations with a very accurate performance rate and adjustable operation time. Controlled multiple-qubit gate operations such as Toffoli and Fredkin gates can be implemented with conditional quantum oscillations and their synchronization.

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