A $C^1$ PETROV-GALERKIN METHOD AND GAUSS COLLOCATION METHOD FOR 1D GENERAL ELLIPTIC PROBLEMS AND SUPERCONVERGENCE

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ABSTRACT. In this paper, we present and study $C^1$ Petrov-Galerkin and Gauss collocation methods with arbitrary polynomial degree $k$ ($\geq 3$) for one-dimensional elliptic equations. We prove that, the solution and its derivative approximations converge with rate $2k-2$ at all grid points; and the solution approximation is superconvergent at all interior roots of a special Jacobi polynomial of degree $k+1$ in each element, the first-order derivative approximation is superconvergent at all interior $k - 2$ Lobatto points, and the second-order derivative approximation is superconvergent at $k - 1$ Gauss points, with an order of $k + 2$, $k + 1$, and $k$, respectively. As a by-product, we prove that both the Petrov-Galerkin solution and the Gauss collocation solution are superconvergent towards a particular Jacobi projection of the exact solution in $H^2$, $H^1$, and $L^2$ norms. All theoretical findings are confirmed by numerical experiments.

1. Introduction. Superconvergence phenomenon means that the convergent rate exceeds the best possible global rate at some special points. Those points are called superconvergent points. During the past several decades, the subject has attracted much attention from the scientific and engineering computing community, and it is well understood for the $C^0$ finite element method (see, e.g., [4, 7, 16, 15, 20, 21, 22, 23, 24, 26, 27, 33]), the $C^0$ finite volume method (see, e.g., [8, 11, 13, 18, 29]), the discontinuous Galerkin method (see, e.g., [1, 2, 3, 9, 10, 12, 17, 18, 28, 30]), and the spectral Galerkin method (see, e.g., [31, 32]). Here by $C^0$ element methods we mean that the approximation space is continuous while its derivative function space is not continuous. As comparison, the relevant study for $C^1$ element methods (i.e., both the approximation space and its derivative function space are continuous) is lacking. Only very special and simple cases have been discussed (see, e.g., [27, 6, 5]).
Comparing with continuous Galerkin (or $C^0$ element) and discontinuous Galerkin (DG) methods, the most attractive feature of $C^1$ element methods is the continuity of the derivative approximation across the element interface. As early as 1995, Wahlbin investigated the superconvergence of $C^1$ Galerkin (not Petrov Galerkin) and spline Galerkin methods in [27] for two-point boundary value problems and established a mathematical theory to find superconvergence points for the $C^1$ finite element solution under the locally uniform mesh assumption. It was proved in [27] that the function value approximation of the $k$-th $C^1$ Galerkin method is superconvergent with order $k + 2$ at zeros of a special polynomial, and the derivative error is $k + 1$-th order superconvergent at grid points as well as element mid-point when $k$ is odd. While for even $k$, the superconvergence behavior changes: the function value approximation is superconvergent at interior Lobatto points, mesh points, and element mid-points, and the derivative is superconvergent at the Gauss points. All those superconvergence rates are one order higher than the counterpart optimal convergence rates and the superconvergence results are valid in case that the mesh is locally uniform. However, the generalization of the superconvergence analysis to quasi-uniform meshes is not straightforward. In 1999, Bialecki [6] studied piecewise Hermite bi-cubic orthogonal spline collocation solution of the Poisson equation on rectangular mesh and proved a fourth-order accuracy of the first order partial derivatives of the collocation solution at the partition nodes. Only recently, Bhal and Danumjaya in [5] presented a cubic spline collocation method for the one dimensional Helmholtz equation with discontinuous coefficients, and proved a fourth-order accuracy for the function value approximation and for the first-order derivative value approximation at the grid points.

In this paper, we present and study a $C^1$ Petrov-Galerkin method and Gauss collocation method for elliptic equations in 1D. The trial space is taken as the $C^1$ polynomial space of degree not more than $k$, while the test space of the $C^1$ Petrov-Galerkin method is chosen as the $L^2$ polynomial space of degree not more than $k - 2$. As the reader may recall, the total degrees of freedom for the $C^1$ Petrov-Galerkin method is the same as that for the counterpart $C^0$ element method. The main purpose of our current work is to provide a unified mathematical approach to establish the superconvergence theory of $C^1$ element methods. We prove that, for general 1D elliptic equations, the solution of the $C^1$ Petrov-Galerkin method is superclose to a particular Jacobi projection of the exact solution and thus establish the following superconvergence results at some special points: 1) both the function value and the first-order derivative approximations are superconvergent with order $2k - 2$ at mesh nodes; 2) the function value approximation is superconvergent with order $k + 2$ at roots of a generalized Jacobi polynomial; 3) the first-order derivative approximation is superconvergent with order $k + 1$ at interior Lobatto points; 4) the second-order derivative approximation is superconvergent with order $k$ at interior Gauss points. By interpreting the Gauss collocation method as a Petrov-Galerkin method up to some higher-order numerical integration errors, we also prove that the Gauss-collocation solution inherits almost all the superconvergence properties from the counterpart Petrov-Galerkin solution.

The main contribution of this paper lies in that: in one hand, we provide a unified approach to establish the superconvergence theory of $C^1$ element methods and discover some new superconvergence phenomena, especially the $(2k - 2)$-th convergence rate of the derivative approximation at grid points and the superconvergence for the second order derivative approximation, which is greatly different from the
C\(^0\) element method and DG method, even the C\(^1\) finite element method in [27]; on the other hand, all our superconvergence results are valid for non-uniform meshes. In other words, we improve the mesh condition from locally uniform meshes in [27] to quasi-uniform meshes. Furthermore, the superconvergence results for the C\(^1\) Gauss collocation method can be viewed as the generalization of the one presented in [5]. Actually, the cubic spline collocation method in [5] is a special case of our current C\(^1\) Gauss collocation method in case of \(k = 3\).

The rest of the paper is organized as follows. In section 2, we present a C\(^1\) Petrov-Galerkin method and Gauss collocation method for elliptic equations under the one-dimensional setting. In section 3, we investigate approximation properties and superconvergence properties of a special Jacobi projection of the exact solution, which is the basis to establish the superconvergence theory for C\(^1\) element methods. In section 4 and section 5, we separately study the superconvergence behavior of C\(^1\) Petrov-Galerkin and Gauss collocation methods, where superconvergence at the grid points (function and first order derivative value approximations), at interior roots of Jacobi polynomials (function value approximation), at interior Lobatto points (first order derivative value approximation) and Gauss points (the second order derivative value approximation) are investigated. Numerical experiments supporting our theory are presented in section 6. Some concluding remarks are provided in section 7.

Throughout this paper, we adopt standard notations for Sobolev spaces such as \(W^{m,p}(D)\) on sub-domain \(D \subset \Omega\) equipped with the norm \(\| \cdot \|_{m,p,D}\) and semi-norm \(| \cdot |_{m,p,D}\). When \(D = \Omega\), we omit the index \(D\); and if \(p = 2\), we set \(W^{m,p}(D) = H^m(D), \| \cdot \|_{m,p,D} = \| \cdot \|_{m,D}\), and \(| \cdot |_{m,p,D} = | \cdot |_{m,D}\). Notation \(A \lesssim B\) implies that \(A\) can be bounded by \(B\) multiplied by a constant independent of the mesh size \(h\). \(A \sim B\) stands for \(A \lesssim B\) and \(B \lesssim A\).

2. C\(^1\) Petrov-Galerkin methods and Gauss collocation methods. We consider the following two-point boundary value problem

\[-(\alpha u')' + \beta u' + \gamma u = f, \quad x \in \Omega = (a, b),\]
\[u(a) = u(b) = 0,\]

where \(\alpha \geq \alpha_0 > 0, \gamma - \frac{\alpha'}{2} \geq 0, \gamma \geq 0, \alpha, \beta, \gamma \in L^\infty(\Omega),\) and \(f\) is real-valued function defined on \(\Omega\). For simplicity, we assume that \(\alpha, \beta, \gamma\) are all constants. Other than technical complexity, there is no essential difficulty in analysis for variable coefficients as long as the above conditions are satisfied.

Let \(a = x_0 < x_1 < \ldots < x_N = b\) be \(N + 1\) distinct points on the interval \(\Omega\). For all positive integers \(r\), we define \(\mathbb{Z}_r = \{1, \ldots, r\}\) and denote by

\[\tau_j = (x_{j-1}, x_j), \quad j \in \mathbb{Z}_N.\]

Let \(h_j = x_j - x_{j-1}\), and \(h = \max \limits_{j} h_j\). We assume that the mesh is quasi-uniform, i.e., there exists a constant \(c\) such that

\[h \leq ch_j, \quad j \in \mathbb{Z}_N.\]

Define

\[V_h := \{ v \in C^1(\Omega) : v|_{\tau_j} \in P_k(\tau_j), \quad j \in \mathbb{Z}_N \}\]

to be the \(C^1\) finite element space, where \(P_k, k \geq 3\) denotes the space of polynomials of degree not more than \(k\). Let

\[V_h^0 := \{ v \in V_h : v(a) = v(b) = 0 \}.\]
We adopt two numerical methods to solve the problem (1), i.e., the Petrov-Galerkin method and the Gauss collocation method. To establish the Petrov-Galerkin method, we choose \( V^0_h \) as our trial space and the piecewise polynomial space of degree \( k - 2 \) as the test space, which is defined as follows:

\[
W_h := \{ w \in L^2(\Omega) : w|_{\tau_i} \in P_{k-2}(\tau_i), \; j \in \mathbb{Z}_N \}.
\]

**Petrov-Galerkin method:** The Petrov-Galerkin method for solving (1) is to find a \( u_h \in V^0_h \) such that

\[
(-\alpha u''_h, v_h) + (\beta u'_h + \gamma u_h, v_h) = (f, v_h), \; \forall v_h \in W_h.
\]

**Gauss collocation method:** Given any \( i \in \mathbb{Z}_N \), we denote by \( g_{im}, m \in \mathbb{Z}_{k-1} \) the \( k-1 \) Gauss points in the interval \( \tau_i \). That is, \( \{g_{im}\}_{m=1}^{k-1} \) are zeros of the Legendre polynomial of degree \( k - 1 \). Then the Gauss collocation method to (1) is: Find a \( \bar{u}_h \in V^0_h \) such that

\[
(-\alpha \bar{u}''_h + \beta \bar{u}'_h + \gamma \bar{u}_h)(g_{im}) = f(g_{im}), \; (i, m) \in \mathbb{Z}_N \times \mathbb{Z}_{k-1}.
\]

3. **Approximation and superconvergence properties of the truncated Jacobi projection.** In this section, we define a \( C^1 \) Jacobi projection of the exact solution and study its approximation and superconvergence properties, which is of great importance to establish superconvergence results for the \( C^1 \) numerical solution, especially the discovery of superconvergence points.

We begin with some preliminaries. We first introduce the Jacobi polynomials. The Jacobi polynomials, denote by \( J^r_s(n, l) \), \( r, l > -1 \), are orthogonal with respect to the Jacobi weight function \( \omega_{r,s}(s) := (1-s)^r(1+s)^l \) over \( I := (-1,1) \). That is,

\[
\int_{-1}^{1} J^r_s(n, l)(s)J^r_m(n, l)(s)\omega_{r,s}(s)ds = \kappa_{n,m}^r \delta_{mn},
\]

where \( \delta \) denotes the Kronecker symbol and

\[
\kappa_{n,m}^r = \|J^r_s(n, l)\|^2_{\omega_{r,s}} := \frac{2^{r+l+1} \Gamma(n+r+1)\Gamma(n+l+1)}{(2n+r+l+1)\Gamma(n+1)\Gamma(n+r+l+1)}.
\]

Here \( \Gamma(n) \) denotes the Gamma function. Note that when \( r = l = 0 \), the Jacobi polynomial \( J^0_s(n, l) \) is reduced to the standard Legendre polynomial. That is, \( J^0_s(n, l) = L_n(s) \) with \( L_n(s) \) being the Legendre polynomial of degree \( n \) over \([-1,1]\). We extend the definition of the classical Jacobi polynomials to the cases where both parameters \( (r, l) \) are negative, i.e.,

\[
J^r_s(n, l) := (1-s)^r(1+s)^l J^{-r-l}_{n+r+l}(s), \; r, l \leq -1, \; n \geq -r-l.
\]

The above defined polynomial is referred to as the generalized Jacobi polynomial with index \( (r, l) \). It is clear that the so-defined generalized Jacobi polynomial \( J^r_s(n, l) \) is a polynomial of degree \( n \).

It was proved in [25] (see Lemma 6.2) that \( J^r_s(n, l) \) satisfies the following derivative recurrence relation

\[
\partial_s J^r_s(n, l)(s) = C^r_s(n, l) J^{r+1,l+1}_s(n+1, l+1)(s),
\]

where

\[
C^r_s(n, l) = \begin{cases} 
-2(n+r+l+1), & \text{if } r, l \leq -1, \\
-n, & \text{if } r \leq -1, l > -1, \text{ or } r > -1, l \leq -1, \\
\frac{1}{2}(n+r+l+1), & \text{if } r, l > -1.
\end{cases}
\]
By taking \( r = l = -2 \) in (4) and using the derivative recurrence relation (5), we obtain
\[
J_{n}^{2,-2}(s) = (1 - s)^2(1 + s)^2J_{n-4}^{2,2}(s) = \frac{2}{n}(1 - s)^2(1 + s)^2\partial_s J_{n-3}^{1,1}(s)
\]
\[
= \frac{4}{n(n-1)}(1 - s)^2(1 + s)^2\partial^2_s J_{n-2}^0(s)
\]
\[
= \frac{4}{n(n-1)}(1 - s)^2(1 + s)^2\partial^2_s L_{n-2}(s).
\]

On the other hand, we have, from (5)
\[
\partial^2_s J_{n}^{2,-2}(s) = -(n-3)\partial_s J_{n-1}^{1,-1}(s) = c_n J_{n-2}^0(s), \quad c_n = 4(n-3)(n-2).
\]

The above generalized Jacobi polynomial plays an important role in our later superconvergence analysis.

Given any function \( u \in C^1(\Omega) \), suppose \( u(x) \) has the following Jacobi expansion in each element \( \tau_i, i \in \mathbb{Z}_N \)
\[
u(x)|_{\tau_i} = H_3u(x) + \sum_{n=1}^{\infty} u_n \hat{J}_{n}^{2,-2}(x),
\]
where \( \hat{J}_{n}^{2,-2}(x)|_{\tau_i} = J_{n}^{2,-2}\left(\frac{2x-x_i-x_{i-1}}{h_i}\right) = J_{n}^{2,-2}(s), \quad s \in [-1,1] \) is the generalized Jacobi polynomial of degree \( n \) over \( \tau_i \), and \( H_3u \in P_3 \) denotes the Hermite interpolation of \( u \), i.e.,
\[
\partial^2 m H_3u(x_i) = \partial^2 x u(x_i), \quad \partial^2 m H_3u(x_{i-1}) = \partial^2 x u(x_{i-1}), \quad m = 0, 1.
\]

For \( n \geq 4 \), noticing that \( \partial^2 m H_3u \in P_1 \subset P_{n-3} \), we take the second derivative on both sides of (8) and then use (7), the orthogonality properties of Legendre polynomials, and the fact that \( L_{n-2} \perp P_{n-3} \) to obtain
\[
u_n = \frac{h_i^2}{4c_n} \int_{\tau_i} (\partial^2 x L_{i,n-2})^2(x)dx / \int_{\tau_i} L_{i,n-2}(x)L_{i,n-2}(x)dx, \quad n \geq 4.
\]

Here \( c_n \) is the same as that in (7) and \( L_{i,n}(x) \) denotes the Legendre polynomial of degree \( n \) over \( \tau_i \), that is,
\[
L_{i,n}(x) = L_n\left(\frac{2x-x_i-x_{i-1}}{h_i}\right) = L_n(s), \quad s \in [-1,1].
\]

Now we define a truncated Jacobi projection \( u_I \in V_h \) of \( u \) as follows:
\[
u_I(x)|_{\tau_i} := \begin{cases} 
H_3u(x) + \sum_{n=4}^{k} u_n \hat{J}_{n}^{2,-2}(x), & \text{if } k \geq 4, \\
H_3u(x), & \text{if } k = 3.
\end{cases}
\]

We have the following orthogonal and approximation properties for \( u_I \).

**Proposition 1.** Assume that \( u \in W^{k+2,\infty}(\Omega) \) is the solution of (1), and \( u_I \) is the Jacobi truncation projection of \( u \) defined by (10). Then the following orthogonality and approximation properties hold true.

1. \( u_I \) is exactly the same at mesh nodes for both function value and derivative value approximations, i.e.,
\[
(u - u_I)(x_i) = 0, \quad (u - u_I)'(x_i) = 0, \quad \forall i \in \mathbb{Z}_N.
\]
2. Orthogonality:

\[
\int_{\tau_i} (u - u_I)''v dx = 0, \quad \int_{\tau_i} (u - u_I)'v' dx = 0, \quad \int_{\tau_i} (u - u_I)v'' dx = 0, \quad \forall v \in P_{k-2}(\tau_i). \tag{12}
\]

3. Optimal error estimates:

\[
\|u - u_I\|_{0,\infty,\tau} + h(k-1) + h^2 k^{-2} \|u - u_I\|_{2,\infty,\tau} \lesssim \frac{h^{k+1}}{2^{k+1}} \|u\|_{k+1,\infty,\tau}. \tag{13}
\]

4. Superconvergence of function value approximation on interior roots of the generalized Jacobi polynomial:

\[
|(u - u_I)(l_{im})| \lesssim \frac{h^{k+2}}{2^k(k+1)!} |u|_{k+2,\infty,\tau}, \tag{14}
\]

where for \( k \geq 4 \), \( l_{im} \neq x_i, x_{i-1}, m \in \mathbb{Z}_{k-3} \) are interior roots of \( \hat{J}_{k+1}^{2,-2}(x) \) in \( \tau_i \).

5. Superconvergence of first order derivative value approximation on Gauss-Lobatto points:

\[
|(u - u_I)'(g_{lm})| \lesssim \frac{h^{k+1}}{2^k k!} |u|_{k+2,\infty,\tau}, \tag{15}
\]

where \( g_{lm} \neq x_i, x_{i-1}, n \in \mathbb{Z}_{k-2} \) are interior roots of \( \partial_x \hat{J}_{k+1}^{2,-2}(x) = c_k \hat{J}_{k}^{1,-1}(x) \) on \( \tau_i \). That is, \( g_{lm}, i \leq k - 2 \) are interior Gauss-Lobatto points of degree \( k - 2 \).

6. Superconvergence of second order derivative value approximation on Gauss points:

\[
|(u - u_I)''(g_{lm})| \lesssim \frac{h^k}{2^k(k-1)!} |u|_{k+2,\infty,\tau}, \tag{16}
\]

where \( g_{lm}, n \leq k - 1 \) are interior roots of \( L_{i,k-1}(x) \), i.e., the \( k - 1 \) Gauss points.

**Proof.** First, subtracting \( 10 \) from \( 8 \) yields that

\[
\partial_x^m (u - u_I)(x) = \sum_{n=k+1}^{\infty} u_n \partial_x^n \hat{J}_{n}^{2,-2}(x), \quad m = 0, 1, 2. \tag{17}
\]

Since

\[
\partial_x^p \hat{J}_{n}^{2,-2}(x_i) = \partial_x^p \hat{J}_{n}^{2,-2}(x_{i-1}) = 0, \quad p = 0, 1,
\]

we easily get

\[
\partial_x^p u_I(x_i) = \partial_x^p u(x_i), \quad \partial_x^p u_I(x_{i-1}) = \partial_x^p u(x_{i-1}), \quad p = 0, 1. \tag{18}
\]

Then \( 11 \) follows.

On the other hand, by using \( 7 \) and the orthogonal properties of Legendre polynomials, we derive

\[
\int_{\tau_i} (u - u_I)''v dx = 0, \quad \forall v \in P_{k-2}(\tau_i). \tag{19}
\]

Consequently, a simple integration by parts and \( 19 \) lead to

\[
\int_{\tau_i} (u - u_I)'v' dx = 0, \quad \int_{\tau_i} (u - u_I)v'' dx = 0, \quad \forall v \in P_{k-2}(\tau_i). \tag{20}
\]

That is,

\[
(u - u_I) \perp P_{k-4}, \quad (u - u_I)' \perp P_{k-3}, \quad (u - u_I)'' \perp P_{k-2}.
\]
Then (12) follows.

We now prove the approximation and superconvergence properties (13)-(16). By a scaling from $\tau_i$ to $[-1,1]$ and a simple integration by parts for (9), we have

$$u_n = \frac{(2n-3)}{2c_n} \int_{-1}^{1} \partial_s^n u(s)L_{n-2}(s)ds = \gamma_n \int_{-1}^{1} \partial_s^n u(s) \frac{d^{n-2}(1-s^2)^{n-2}}{ds^{n-2}} ds$$

$$= (-1)^m \gamma_n \int_{-1}^{1} \partial_s^n u(s) \frac{d^{n-m}(1-s^2)^{n-2}}{ds^{n-m}} ds, \forall m \leq n,$$

where

$$u(s) = u\left(\frac{2x - x_i - x_{i-1}}{h_i}\right) = u(x), \ s \in [-1,1], x \in \tau_i, \ \gamma_n = \frac{(2n-3)}{c_n2^{n-1}(n-2)!}.$$ 

Noticing that

$$\partial_s^n u(s) = \left(\frac{h_i}{2}\right)^m \partial_s^n u(x), \ \forall m \geq 1,$$

we have for all $n \geq 4$,

$$|u_n| \leq C|\gamma_n|\|\partial_s^n u\|_{0,\infty,\tau_i} \leq C h^n \frac{(2n-3)}{2^{n+m}(n-3)(n-2)!} \|u\|_{m,\infty,\tau_i}, \ m \leq n. \quad (21)$$

Here and in the rest of this paper, $C$ is a positive constant independent of $m, n$ and the mesh size $h$, which is not necessary the same at every appearance. By choosing $m = k + 1$ in (21) and using $\tilde{J}_n^{-2,-2}(x) \leq C$, we derive

$$|(u - u_f)(x)| = \left| \sum_{n=k+1}^{\infty} u_n \tilde{J}_n^{-2,-2}(x) \right| \leq C \sum_{n=k+1}^{\infty} |u_n|$$

$$\leq Ch^{k+1} \|u\|_{k+1,\infty,\tau_i} \sum_{n=k+1}^{\infty} 2^{n-3} \frac{2n-3}{2^{n}(n-3)(n-2)(n-2)!}$$

$$\leq C h^{k+1} \|u\|_{k+1,\infty,\tau_i} \quad (22).$$

Similarly, by (7) and a scaling from $(x_{i-1}, x_i)$ to $(-1,1)$, we have

$$|\partial_x \tilde{J}_n^{-2,-2}(x)| \leq C h^{-1} (n-2), \ |\partial_x^2 \tilde{J}_n^{-2,-2}(x)| \leq Ch^{-2} (n-2)(n-3),$$

and thus

$$|(u - u_f)'(x)| \leq \frac{Ch^k}{2^k(k-1)!} \|u\|_{k+1,\infty,\tau_i}, \quad |(u - u_f)''(x)| \leq \frac{Ch^{k-1}}{2^k(k-2)!} \|u\|_{k+1,\infty,\tau_i}.$$ 

Then (13) follows.

At roots of $\partial_x \tilde{J}_{k+1}^{-2,-2}(x), p = 0, 1, 2$, there hold

$$|(u - u_f)(l_{im})| = \left| \sum_{n=k+2}^{\infty} u_n \tilde{J}_n^{-2,-2}(l_{im}) \right| \leq \sum_{n=k+2}^{\infty} |u_n|, \ m \in \mathbb{Z}_{k-3},$$

$$|(u - u_f)'(g_{il})| = \left| \sum_{n=k+2}^{\infty} u_n \partial_x \tilde{J}_n^{-2,-2}(g_{il}) \right| \leq h^{-1} \sum_{n=k+2}^{\infty} n |u_n|, \ n \in \mathbb{Z}_{k-2},$$

$$|(u - u_f)''(g_{ir})| = \left| \sum_{n=k+2}^{\infty} u_n \partial_x^2 \tilde{J}_n^{-2,-2}(g_{il}) \right| \leq h^{-2} \sum_{n=k+2}^{\infty} n^2 |u_n|, \ r \in \mathbb{Z}_{k-1}.$$ 

By choosing \( m = k + 2 \) in (21), we derive
\[
|u - u_I|\|_{(l_{im})} + h^k|u - u_I'|\|_{(g_{im})} + h^2h^k|u - u_I''|\|_{(g_{ir})} \\
\leq h^{k+2}\|u\|_{k+2,\infty,\tau} \sum_{n=k+2}^{\infty} \frac{2n-3}{2^{n-3}(n-2)(n-2)!} \leq \frac{h^{k+2}}{2^k(k+1)!}\|u\|_{k+2,\infty,\tau}.
\]
Here the hidden constant is independent of \( k \). Then (14)-(16) follow. The proof is complete.

4. Superconvergence for \( C^1 \) Petrov-Galerkin methods. In this section, we study superconvergence properties of the \( C^1 \) Petrov-Galerkin method for (1). To this end, we begin with the introduction of the bilinear form of the finite element method and some Green functions.

First, we denote by \( a(\cdot,\cdot) \) the bilinear form of the finite element method, which is defined as
\[
a(u,v) = (au',v') - (bu,v) + (cu,v), \quad \forall u,v \in H^1(\Omega).
\]
Second, given any \( x \in \Omega \), let \( G(x,\cdot) \) be the Green function for the problem (1). Then for any \( v \in H^1(\Omega) \),
\[
v(x) = a(v,G(x,\cdot)), \quad \forall x \in \Omega.
\]
Especially, if \( v(x) \in H^1_0(\Omega) \), then the Green function \( G(x,\cdot) \) satisfies \( G(x,a) = G(x,b) = 0 \).

Let \( S_h \) be the \( C^0 \) finite element space, i.e.,
\[
S_h = \{ v \in C^0(\Omega) : v|_{\tau_i} \in P_k, v(a) = v(b) = 0, i \in Z_N \}.
\]
Denote by \( G_h \in S_h \) the Galerkin approximation of \( G(x,\cdot) \), that is,
\[
v_h(x) = a(v_h,G_h) = a(v_h,G(x,\cdot)), \quad \forall v_h \in S_h.
\]
Finally, we use the following notations in the rest of this paper
\[
ed_h := u - u_h = \xi + \eta, \quad \xi := u_I - u_h, \quad \eta := u - u_I.
\]
We have the following optimal error estimates for the \( C^1 \) Petrov-Galerkin method.

**Lemma 4.1.** Assume that \( u \in W^{k+1,\infty}(\Omega) \) is the solution of (1), and \( u_h \) is the solution of (2). Then
\[
\|u - u_h\|_{0,\infty} + h\|u - u_h\|_{1,\infty} \lesssim \frac{h^{k+1}}{2^k(k-2)!}\|u\|_{k+1,\infty}, \quad \|u - u_h\|_2 \lesssim \frac{h^{k-1}}{2^k(k-2)!}\|u\|_{k+1,\infty},
\]
where the hidden constant is independent of \( h \) and \( k \).

**Proof.** First, noticing that the exact solution \( u \) also satisfy (2), we have
\[
(-\alpha e''_h, v_h) + (\beta e'_h + \gamma e_h, v_h) = 0, \quad \forall v_h \in W_h.
\]
Especially, we choose \( v_h = -\xi'' \) in the above equation and using the orthogonal property of \( \eta \) in (12) and (13) to get
\[
(\alpha \xi'', \xi'') + (\gamma \xi', \xi') - \frac{\beta}{2}(\xi'(b)^2 - \xi'(a)^2) \\
= (-\alpha \eta'' + \beta \eta' + \gamma \eta, \xi'') \\
\leq \left( \frac{|\beta| h^{k}}{2^k(k-1)!} + \frac{\gamma h^{k+1}}{2^k k!} \right) \|\xi''\|_0|u|_{k+1,\infty}.
\]

On the other hand, noticing that \( e'_h(x) \in C^0(\Omega) \subset H^1(\Omega) \), we take \( v = e'_h \) in (22) and use the integration by parts to obtain

\[
e'_h(x_i) = a(e'_h, G(x_i, \cdot)) = (\alpha e''_h - \beta e'_h, G'(x_i, \cdot)) + (\gamma e'_h, G(x_i, \cdot))
\]

\[
= (\alpha e''_h - \beta e'_h - \gamma e_h, G'(x_i, \cdot))
\]

\[
= (\alpha e''_h - \beta e'_h - \gamma e_h, G'(x_i, \cdot) - I_{k-2}G'(x_i, \cdot)).
\]

Here \( I_{k-2}v \) denotes the \( L^2 \) projection of \( v \) onto \( P_{k-2} \). Since the Green function \( G(x_i, \cdot) \in C^k(\tau_j), j \in \mathbb{Z}_N \) is bounded, we have for all \( i = 0, \ldots, N \)

\[
|\xi'(x_i)| = |e'_h(x_i)| \lesssim h^{k-1} \|e_h\|_2 \lesssim h^{k-1}(\|\eta\|_2 + \|\xi\|_2)
\]

\[
\lesssim h^{2(k-1)}| u | _{k+1, \infty} + h^{k-1}| \xi\|_2.
\]

(27)

Substituting (27) into (26) and using the Cauchy-Schwarz inequality yields

\[
|\xi|^2 + |\xi|^2 \leq C(\frac{|\beta| h}{2^k(k-1)!} + \frac{\gamma h^{k+1}}{2^k k!} + \frac{h^{2(k-1)}}{2^k (k-2)!})^2 |u|_{k+1, \infty}^2 + (\frac{1}{2} + C_1 h^{2(k-1)})|\xi|^2,
\]

where \( C, C_1 \) are some positive constants independent of \( h \). Consequently, when \( h \) is sufficiently small, there holds

\[
|\xi|_2 + |\xi|_1 \lesssim (\frac{|\beta| h}{2^k(k-1)!} + \frac{\gamma h^{k+1}}{2^k k!} + \frac{h^{2(k-1)}}{2^k (k-2)!})|u|_{k+1, \infty}.
\]

(28)

Similarly, we choose \( v_h = \xi \in H^1_0(\Omega) \) in (23) and again use the integration by parts to obtain

\[
|\xi(x)| = |a(\xi, G_h)| = |(-\alpha \xi'' + \beta \xi' + \gamma \xi, G_h - \bar{G}_h) + (-\alpha \xi'' + \beta \xi' + \gamma \xi, \bar{G}_h)|
\]

\[
\lesssim h|\xi||G_h||1 + ||(-\alpha \eta'' + \beta \eta' + \gamma \eta, \bar{G}_h)|
\]

\[
\lesssim h|\xi||G_h||1 + \frac{h^{k+1}}{2^k k!}|u|_{k+1, \infty}||\bar{G}_h||1,1.
\]

Here \( \bar{G}_h|_{\tau_j} \in P_0(\tau_j) \) denotes the cell average of \( G_h \). It has been proved in [14] that

\[
\|G_h\|_{2,1} \lesssim 1,
\]

which yields, together with the embedding theory

\[
\|G_h\|_1 \lesssim \|G_h\|_{2,1} \lesssim 1.
\]

Consequently,

\[
|\xi(x)| \lesssim \frac{h^{k+1}}{2^k k!}|u|_{k+1, \infty} + h|\xi|_2 \lesssim \frac{h^{k+1}}{2^k (k-2)!}|u|_{k+1, \infty},
\]

where in the last step, we have used (28). Then

\[
|\xi|_{0, \infty} \lesssim \frac{h^{k+1}}{2^k (k-2)!}|u|_{k+1, \infty}.
\]

Similarly, by (27) and (28), we have

\[
|\xi'(x)| = |\xi'(x_{i-1}) + \int_{x_{i-1}}^{x} \xi''(x)dx| \lesssim \frac{h^k}{2^k (k-2)!}|u|_{k+1, \infty}, \ \forall x \in \tau_i,
\]

and thus

\[
|\xi|_{1, \infty} \lesssim \frac{h^k}{2^k (k-2)!}|u|_{k+1, \infty}.
\]
Then (24) follows from the triangle inequality and the standard approximation theory. This finishes our proof.

Now we are ready to present the superconvergence of the solution for the $C^1$ Petrov-Galerkin method. As we may observe from Lemma 4.1, the convergence of the error $u - u_h$ can be achieved by increasing the polynomial degree $k$ or decreasing the mesh size $h$. In this paper, we focus our attention on the superconvergence of $h$-version Petrov-Galerkin methods, i.e., convergence is achieved by decreasing the mesh size, while the polynomial degree $k$ is fixed. Therefore, in the rest of this paper, the polynomial degree $k$ is omitted in all our error estimates since it is treated as a bounded constant. As for the $p$-version Petrov-Galerkin method (i.e., $h$ is fixed and polynomial degree is increasing), the superconvergence is much more sophisticated and needs to be discussed separately.

**Theorem 4.2.** Assume that $u \in W^{k+2,\infty}(\Omega)$ is the solution of (1), and $u_h$ is the solution of (2). The following superconvergence properties hold true.

1. **Super closeness between the numerical solution and truncation projection in the $H^2$ norm:**
   \[
   \|u_h - u_I\|_2 \lesssim h^k|u|_{k+1,\infty}, \text{ if } \beta \neq 0, \quad \|u_h - u_I\|_2 \lesssim h^{k+1}|u|_{k+1,\infty}, \text{ if } \beta = 0, \gamma \neq 0. \tag{29}
   \]

2. If $\beta = \gamma = 0$, then
   \[
   u_h(x) = u_I(x), \quad (u - u_h)(x_i) = 0, \quad (u - u_h)'(x_i) = 0. \tag{30}
   \]

3. **Superconvergence for both function value and derivative value approximations at nodes:**
   \[
   |(u - u_h)(x_i)| \lesssim h^{2(k-1)}|u|_{k+1,\infty}, \quad |(u - u_h)'(x_i)| \lesssim h^{2(k-1)}|u|_{k+1,\infty}, \quad i \in \mathbb{Z}_N. \tag{31}
   \]

4. **Superconvergence of function value approximation on interior roots of the generalized Jacobi polynomial for $k \geq 4$:**
   \[
   |(u - u_h)(l_m)| \lesssim h^{k+2}\|u\|_{k+2,\infty}, \tag{32}
   \]
   where $l_m, m = 1, \cdots, k - 3$ are $k - 3$ interior roots of $j_{k-1}^{2,-2}(x)$ in $\tau_i$.

5. **Superconvergence of first order derivative value approximation on Gauss-Lobatto points:**
   \[
   |(u - u_h)'(g_{in})| \lesssim h^{k+1}\|u\|_{k+2,\infty}, \tag{33}
   \]
   where $g_{in}, n \leq k - 2$ are interior Gauss-Lobatto points of degree $k - 2$.

6. **Superconvergence of second order derivative value approximation on Gauss points:**
   \[
   |(u - u_h)''(g_{in})| \lesssim h^{k}\|u\|_{k+2,\infty}, \tag{34}
   \]
   where $g_{in}, n \leq k - 1$ are $k - 1$ roots of $L_{i,k-1}(x)$.

**Proof.** First, (29) follows directly from (28). Furthermore, there holds from (26),

\[
|u_h - u_I|_2 + |u_h - u_I|_1 = 0, \quad \text{if } \beta = \gamma = 0,
\]

which indicates that $u_h - u_I$ is a constant. Noticing that $(u_h - u_I)(a) = 0$, we have $u_I = u_h$.

Then (30) follows.
Now we consider the superconvergence at nodes. In light of (22) and (25), we obtain
\[ |e_h(x_i)| = |a(e_h, G(x_i, \cdot))| = |(-\alpha e_h'' + \beta e_h' + \gamma e_h, G(x_i, \cdot) - \mathcal{I}_{k-2}G(x_i, \cdot))| \]
\[ \lesssim h^{2(k-1)}|u|_{k+1, \infty}. \]
where in the last step, we have used the fact the Green function $G(x_i, \cdot) \in C^k(\tau_j)$, $j \in \mathbb{Z}_N$ is bounded. Similarly, we have from (27) and (24)
\[ |e_h'(x_i)| \lesssim h^{k-1}\|\phi_h\|_2 \lesssim h^{2(k-1)}|u|_{k+1, \infty}. \]
Then (31) follows.

We next prove (32)-(34). We consider two cases, i.e., $k \geq 4$ and $k = 3$.

**Case 1.** $k \geq 4$

For any function $v \in L^2(\Omega)$, we denote by $\mathcal{I}_{k-2}v$ the $L^2$ projection of $v$ onto $P_{k-2}$ and define
\[ \partial_x^{-1}v(x) := \int_a^x v(t)dt. \]
In light of (22), we have from the integration by parts, the orthogonality (12) and (25),
\[ \xi(x) = a(\xi, G_h) = (-\alpha \xi'' + \beta \xi' + \gamma \xi, G_h) = (-\alpha \xi'' + \partial_{\xi}G_h + (\beta \xi' + \gamma \xi, G_h) \]
\[ = (-\alpha \xi'' + \partial_{x}G_h + (\beta \xi' + \gamma \xi, G_h) \]
\[ = -(\beta e_h' + \gamma e_h, \mathcal{I}_{k-2}G_h) + (\beta \xi' + \gamma \xi, G_h) \]
\[ = (\beta e_h' + \gamma e_h, G_h - \mathcal{I}_{k-2}G_h) - (\beta \eta' + \gamma \eta, G_h) = I_1 - I_2. \]
Now we estimate the two terms $I_1, I_2$, respectively. In light of (24) and the fact that $\|G_h\|_{2,1} \lesssim 1$ (see, e.g., [14]), we have
\[ |I_1| = |(\beta e_h' + \gamma e_h, G_h - \mathcal{I}_{k-2}G_h)| \lesssim h^2(\|\epsilon_h\|_{1, \infty} + \|\epsilon_h\|_{0, \infty})\|G_h\|_{2,1} \lesssim h^{k+2}|u|_{k+1, \infty}. \]
On the other hand, by (12), there holds for $k \geq 4$,
\[ \int_{\tau_i}(u - u_I)(x)dx = 0, \]
and thus
\[ (\partial_x^{-1}\eta)(x_i) = 0, \quad i \in \mathbb{Z}_N, \quad (\partial_x^{-1}\eta)(x) = \int_{x_{i-1}}^x \eta(t)dt, \forall x \in \tau_i. \]
Then a direct integration by parts yields
\[ |I_2| = |(\beta \eta' + \gamma \eta, G_h)| = |(\beta \partial_x^{-1}\eta, G_h'') - (\gamma \partial_x^{-1}\eta, G_h')| \]
\[ \lesssim \|\partial_x^{-1}\eta\|_{0, \infty}\|G_h\|_{2,1} \lesssim h\|\eta\|_{0, \infty}. \]
Consequently,
\[ |\xi(x)| \leq |I_1| + |I_2| \lesssim h^{k+2}|u|_{k+1, \infty}, \forall x \in \Omega, \]
which yields, together with the inverse inequality,
\[ \|\xi\|_{0, \infty} \lesssim h^{k+2}|u|_{k+1, \infty}, \quad \|\xi\|_{1, \infty} \lesssim h^{k+2}|u|_{k+1, \infty}, \quad \|\xi\|_{2, \infty} \lesssim h^{k}|u|_{k+1, \infty}. \]
Then the desired results (32)-(34) follow from the triangle inequality and the approximation properties of $u_I$ in Theorem 4.2.
Case 2. $k = 3$

To prove (33) and (34) for $k = 3$, we first construct a special function $w_h \in P_3 \cap C^1(\Omega)$ satisfying the following condition:

$$
(\alpha w''_h, v) = (\beta \eta' + \gamma \eta, v) \quad \forall v \in P_1(\tau_j) \setminus P_0(\tau_j),
$$

$$
w''_h(x_i) = 0, \quad w_h(a) = 0 \quad \forall i = 0, \ldots, N. 
$$

We can prove that the function $w_h$ is uniquely defined. Actually, if the right hand side of (35) equals to zero, we can easily obtain that $w''_h = 0$. Then the boundary condition (36) indicates that $w_h = 0$. We next estimate the function $w_h$. We suppose

$$
w_h(x) = \sum_{i=1}^{N} c_i \phi_i(x)
$$

with $\phi_i(x) \in P_3 \cap C^1(\Omega)$ being the basis function associated with the node $x_i$, that is,

$$
\phi_i(x) = \begin{cases} 
\frac{1}{h_{i+1}^2} (x_{i+1} - x)^2 (2x + x_{i+1} - 3x_i), & \text{if } x \in \tau_{i+1}, \\
\frac{1}{h_{i}^2} (x - x_{i-1})^2 (3x_i - 2x - x_{i-1}), & \text{if } x \in \tau_i, \\
0, & \text{else}.
\end{cases}
$$

We choose $v = x$ in (35) to obtain

$$
\frac{12(c_j - c_{j-1})}{h_j^3} (\bar{x}_j - x, x)_j = (\beta \eta' + \gamma \eta, x)_j,
$$

where $(w_h, v)_j = \int_{\tau_j} w_h v dx, \bar{x}_j = \frac{x_j + x_{j+1}}{2}$. Consequently,

$$
|c_j - c_{j-1}| \lesssim h_j^{k+3} |u|_{k+1, \infty},
$$

and thus,

$$
\|w''_h\|_{0, \infty, \tau_j} \lesssim \frac{|c_j - c_{j-1}|}{h_j^3} \lesssim h^k |u|_{k+1, \infty}.
$$

Moreover, there holds for all $x \in \tau_j$

$$
|w''_h(x)| = \left| w''_h(x_{j-1}) + \int_{x_{j-1}}^{x} w''_h(x) dx \right| \leq h_j \|w''_h\|_{0, \infty, \tau_j} \lesssim h^{k+1} |u|_{k+1, \infty}.
$$

Then

$$
\|w'_h\|_{0, \infty} \lesssim h^{k+1} |u|_{k+1, \infty}, \quad \|w_h\|_{0, \infty} \lesssim \|w'_h\|_{0, \infty} \lesssim h^{k+1} |u|_{k+1, \infty}.
$$

Now we are ready to prove (33) and (34) for $k = 3$. Let

$$
e_h = u - u_h = \tilde{\xi} + \tilde{\eta}, \quad \tilde{\xi} := u_I - u_h - w_h, \quad \tilde{\eta} := u - u_I + w_h.
$$

Choosing $v_h = -\tilde{\xi}''$ in (25) and following the same argument as that in (26), we obtain

$$
(\alpha \tilde{\xi}'' + (\gamma \tilde{\xi}', \tilde{\xi}'') - \frac{\beta}{2} (|\tilde{\xi}'(b)|^2 - |\tilde{\xi}'(a)|^2) = (-\alpha \tilde{\epsilon}' + \beta \eta' + \gamma \tilde{\eta}', \tilde{\xi}'') \quad = (\beta \eta' + \gamma \tilde{\eta} + \tilde{\xi}'') + (-\alpha w''_h + \beta w'_h + \gamma w_h, \tilde{\xi}'') = I.
$$

We now estimate the term $I$. Since $\tilde{\xi}''|_{\tau_j} \in P^1(\tau_j)$, we have the following decomposition

$$
\tilde{\xi}'' = \xi_0 + \xi_1, \quad \xi_0 \in P^0(\tau_j), \quad \xi_1 \in P^1(\tau_j) \setminus P_0(\tau_j).$$
By (35) and the integration by parts, we get
\[
|I| = |(\beta\eta' + \gamma\xi_0) + (\alpha w''_h + \beta w'_h + \gamma w_h, \xi_0) + (\beta w'_h + \gamma w_h, \xi_1) - (\eta, \xi_0) + (\beta w' + \gamma w_h, \xi_0) + (\beta w'_h + \gamma w_h, \xi_1)| \\
= |(\eta, \xi_0) + (\beta w' + \gamma w_h, \xi_0) + (\beta w'_h + \gamma w_h, \xi_1)| \lesssim (\|\eta\|_0 + \|w_h\|_1)\|\xi''\|_0.
\]
In light of the estimates for \(w_h\) and \(\eta\), we get
\[
(\alpha\xi'', \tilde{\xi}''') + (\gamma\xi', \tilde{\xi}'') - \frac{\beta}{2}(|\xi''(b)|^2 - |\xi'(a)|^2) \lesssim h^{k+1}|u|_{k+1,\infty}|\xi|_2. \tag{37}
\]
By (31), there holds
\[
|\tilde{\xi}'(b)| = |\xi'(b)| \lesssim h^{2(k-1)}|u|_{k+1,\infty}.
\]
Substituting the above estimate into (37) and using the Cauchy-Schwarz inequality yields
\[
\|\tilde{\xi}''\|_0 \lesssim h^{k+1}|u|_{k+1,\infty}.
\]
By the triangle inequality and the inverse inequality,
\[
\|\xi''\|_{0,\infty} \leq \|\tilde{\xi}''\|_{0,\infty} + \|w''_h\|_{0,\infty} \lesssim h^{-\frac{1}{2}}\|\xi''\|_0 + h^k|u|_{k+1,\infty} \lesssim h^k|u|_{k+1,\infty}.
\]
Furthermore, there holds for all \(x \in \tau_j\)
\[
|\xi'(x)| = \left|\xi'(x_{j-1}) + \int_{x_{j-1}}^x \xi''(x)dx\right| \lesssim h^{2(k-1)}|u|_{k+1,\infty} + h\|\xi''\|_{0,\infty} \lesssim h^{2(k-1)}|u|_{k+1,\infty} + h^{k+1}|u|_{k+1,\infty}.
\]
Then (33) and (34) follows from the triangle inequality and the approximation properties of \(u_I\) for \(k = 3\). This finishes our proof.

**Remark 1.** As we may observe from the above theorem, for problems with constant coefficients, the convergence rate of the error \(\|u_h - u_I\|_2\) is two order higher than the optimal convergence rate \(k - 1\) in case of \(\beta = 0, \gamma \neq 0\). However, this superconvergence result may not hold true for problems with variable coefficients. Actually, in case of \(\beta = 0, \alpha \neq 0, \gamma \neq 0\) with \(\alpha\) a variable function, we have from (26)
\[
(\alpha \xi'', \xi''') + (\gamma \xi', \tilde{\xi}'') - \frac{\beta}{2}(|\xi''(b)|^2 - |\xi'(a)|^2) = (-\alpha\eta' + \beta\xi' + \gamma \eta, \xi'') = ((\bar{\alpha} - \alpha)\eta'' + \beta\eta' + \gamma \eta, \xi'') \lesssim h^k\|\xi''\|_0|u|_{k+1,\infty},
\]
where \(\bar{\alpha}\) denotes the cell average of \(\alpha\), i.e., \(\bar{\alpha}|_{\tau_j} = h_j^{-1}\int_{\tau_j} \alpha(x)dx\). Then we follow the same argument as that in Lemma 4.1 to obtain
\[
\|\xi''\| \lesssim h^k|u|_{k+1,\infty}.
\]
In other words, the convergence rate of \(\|u_h - u_I\|_2\) for problems with variable coefficients is always \(k\), only one order higher than the optimal convergence rate. This is the difference between the constant coefficients and variable coefficients. Our numerical examples will demonstrate this point.

**Remark 2.** We would like to point out that similar superconvergence results still hold true for variable coefficients. Actually, as we may observe from Theorem 4.2, the only difference between the variable coefficients and constant coefficients lies...
in the estimates for the error bound \(a(\eta, \theta)\) for \(\theta \in W_h\). For variable coefficients \(\alpha, \beta, \gamma\), we use the orthogonality of \(\eta''\) and the estimates of \(\eta\) to derive that
\[
|a(\eta, \theta)| = |(\alpha v'' + \beta \eta' + \gamma \eta, \theta)| = |(\bar{\alpha} \eta'' + \beta \eta' + \gamma \eta, \theta)| \lesssim h^k ||\eta||_0 ||w_{k+1, \infty}||.
\]
Here again \(\bar{\alpha}\) denotes the cell average of \(\alpha\). This error bound is exactly the same as that for the constant coefficients. In other words, by following the same arguments as what we did in Theorem 4.2, we can obtain similar superconvergence results for the variable coefficients.

5. Superconvergence for \(C^1\) Gauss collocation methods. This section is dedicated to the superconvergence analysis of the \(C^1\) Gauss collocation method. Our analysis is along this line: we first prove that the Gauss collocation solution \(\bar{u}_h\) is superclose to the Petrov-Galerkin solution \(u_h\); then due to the supercloseness between \(\bar{u}_h\) and \(u_h\), the numerical solution \(\bar{u}_h\) shares the same superconvergence results with those of \(u_h\), and finally we establish all superconvergence results for the solution of the Gauss collocation method.

We begin with some preliminaries. We first denote by \(\omega_{im}, (i, m) \in \mathbb{Z}_{N} \times \mathbb{Z}_{k-1}\) the wight of Gauss quadrature. For any function \(u, v\), we define the following discrete \(L^2\) inner product \((\cdot, \cdot)_s\) as
\[
(u, v)_s := \sum_{i=1}^{N} \sum_{m=1}^{k-1} (uv)(g_{im})\omega_{im}.
\]
For any \(v_h \in W_h\), we multiply \(v_h(g_{im})\omega_{im}, (i, m) \in \mathbb{Z}_{N} \times \mathbb{Z}_{k-1}\) on both sides of (3) and sum up all \(m\) from 1 to \(k - 1\) to derive
\[
\sum_{m=1}^{k-1} (-\alpha \bar{u}'' + \beta \bar{u}' + \gamma \bar{u}_h)(g_{im})v_h(g_{im})\omega_{im} = \sum_{m=1}^{k-1} f(g_{im})v_h(g_{im})\omega_{im}.
\]
As we may observe, the \(C^1\) Gauss collocation method can be viewed as the counterpart Petrov-Galerkin method up to a Gauss numerical quadrature error. Note that the \((k - 1)\)-point Gauss quadrature is exact for all polynomials of degree not less than \(2k - 3\). Then
\[
a_s(\bar{u}_h, v_h) := (-\alpha \bar{u}'' + \beta \bar{u}' + \gamma \bar{u}_h)(f, v_h)_s + (f, v_h)_s, \quad \forall v_h \in W_h.
\]
Denote
\[
\bar{e}_h = u_h - \bar{u}_h.
\]
Subtracting (39) from (2), we have
\[
(-\alpha \bar{e}'' + \beta \bar{e}' + \gamma \bar{e}_h, v_h) + (\gamma u_h, v_h) = (f, v_h)_s + (f, v_h)_s, \quad \forall v_h \in W_h,
\]
or equivalently,
\[
(-\alpha \bar{e}'' + \beta \bar{e}' + \gamma \bar{e}_h, v_h) = (\gamma u_h, v_h)_s + (\gamma u_h, v_h) + (f, v_h)_s - (f, v_h)_s, \quad \forall v_h \in W_h.
\]
We note that up to a Gauss numerical quadrature error, the right hand side of the above equation equals to zero.

We have the following supercloseness result for the error \(\bar{e}_h\).

**Theorem 5.1.** Assume that \(u \in W^{2k, \infty}(\Omega)\) is the solution of (1), and \(u_h\) and \(\bar{u}_h\) is the solution of (2) and (3), respectively. Then
\[
||u_h - \bar{u}_h||_{0, \infty} + h||u_h - \bar{u}_h||_{1, \infty} + h^2||u_h - \bar{u}_h||_2 \lesssim h^{k+2}||u||_{2k, \infty}.
\]
Proof. Noticing that \( \bar{e}_h \in V_0^0 \), we choose \( \nu_h = -\bar{e}_h'' \in W_h \) in (40) and use the integration by parts to obtain

\[
(\alpha \bar{e}_h'', \bar{e}_h') + (\gamma \bar{e}_h', \bar{e}_h) - \frac{\beta}{2}(\bar{e}_h'(b))^2 - (\bar{e}_h'(a))^2 = (f - \gamma u_h, \bar{e}_h') - (f - \gamma u_h, \bar{e}_h'').
\]

(43)

For any function \( w \), we denote \( I_h w \in P_{k-1} \) the Gauss interpolation function of \( w \) satisfying

\[
I_h w(x_i) = w(x_i), \quad I_h w(g_{im}) = w(g_{im}), \quad m \in \mathbb{Z}_{k-1}.
\]

Since \( v_h I_h w \in P_{2k-3} \) for all \( v_h \in W_h \), we have

\[
|(w, v_h) - (w, v_h)_*| = |(w - I_h w, v_h)| \lesssim h^k ||v||_k ||v_h||_0, \quad \forall v_h \in W_h.
\]

(44)

Plugging the above estimate into (43) gives

\[
(\alpha \bar{e}_h'', \bar{e}_h') + (\gamma \bar{e}_h', \bar{e}_h) \lesssim h^k ||f||_k ||u||_{k} + ||\bar{e}_h'(b)||^2 \lesssim h^k ||f||_k + ||u||_{k+1, \infty} ||\bar{e}_h''||_0 + ||\bar{e}_h'(b)||^2,
\]

(45)

where in the last step, we have used (24), the inverse inequality and the triangle inequality to get

\[
||u_h||_k \lesssim ||u||_k + h^{-k} ||u||_0
\]

\[
\lesssim ||u||_k + ||u - u_I||_k + h^{-k} (||u_I - u||_0 + ||u_h - u||_0) \lesssim ||u||_{k+1, \infty}.
\]

(46)

To estimate \( \bar{e}_h'(b) \), we choose \( v = \bar{e}_h' \) in (22) and use the integration by parts again to obtain

\[
\bar{e}_h'(x_i) = a(\bar{e}_h', G(x_i, \cdot)) = (\alpha \bar{e}_h'' - \beta \bar{e}_h, G'(x_i, \cdot)) + (\gamma \bar{e}_h, G(x_i, \cdot))
\]

\[
= (\alpha \bar{e}_h'' - \beta \bar{e}_h' - \gamma \bar{e}_h, G'(x_i, \cdot) - \bar{G}') + (\alpha \bar{e}_h'' - \beta \bar{e}_h' - \gamma \bar{e}_h, \bar{G}')
\]

\[
= (\alpha \bar{e}_h'' - \beta \bar{e}_h' - \gamma \bar{e}_h, G'(x_i, \cdot) - \bar{G}') - (f, \bar{G}') + (f, G')_*,
\]

where \( \bar{G}' \in P_0 \) denotes the cell average of \( G'(x_i, \cdot) \) and in the last step, we have used (41) and the fact that

\[
(\bar{u}_h, v) - (\bar{u}_h, v)_* = 0, \quad \forall v \in P_{k-3}.
\]

Using the fact that \( G(x_i, \cdot) \in C^k(\tau_j) \) is bounded, we get

\[
(\alpha \bar{e}_h'' - \beta \bar{e}_h' - \gamma \bar{e}_h, G'(x_i, \cdot) - \bar{G}') \lesssim h ||\bar{e}_h||_2.
\]

On the other hand, by (44), we have

\[
|(f, \bar{G}') - (f, G')_*| \lesssim h^k ||f||_k.
\]

Consequently,

\[
||\bar{e}_h'(x_i)|| \lesssim h ||\bar{e}_h||_2 + h^k ||f||_k.
\]

Substituting the above inequality into (45) and using the Cauchy-Schwarz inequality yields

\[
(\alpha \bar{e}_h'', \bar{e}_h') + (\gamma \bar{e}_h', \bar{e}_h) \lesssim \left( \frac{\alpha}{4} + Ch^2 \right) ||\bar{e}_h''||_0^2 + C_1 h^{2k} (||f||_k + ||u||_{k+1, \infty})^2
\]

for some positive \( C, C_1 \). Therefore, when \( h \) is sufficient small, there holds

\[
||\bar{e}_h''||_0 \lesssim h^k (||u||_{k+1, \infty} + ||f||_k) \lesssim h^k ||u||_{k+2, \infty}.
\]
We next estimate \( \| \bar{e}_h \|_{0, \infty} \). Choosing \( v = \bar{e}_h \) in (23) and using (41), we get
\[
\bar{e}_h(x) = a(\bar{e}_h, G_h) = (-\alpha \bar{e}_h'' + \beta \bar{e}_h' + \gamma \bar{e}_h, G_h) \\
= (-\alpha \bar{e}_h'' + \beta \bar{e}_h' + \gamma \bar{e}_h, G_h - I_{k-2}G_h) + (-\alpha \bar{e}_h'' + \beta \bar{e}_h' + \gamma \bar{e}_h, I_{k-2}G_h) \\
= (\beta \bar{e}_h' + \gamma \bar{e}_h, G_h - I_{k-2}G_h) + (f - \gamma \bar{u}_h, I_{k-2}G_h) - (f - \gamma \bar{u}_h, I_{k-2}G_h)...
\]
Here again \( I_{k-2}G_h \) denotes the \( L^2 \) projection of \( G_h \) onto \( P_{k-2} \). By using the error of Gauss quadrature (see, e.g., [19], P.98 (2.7.12)), there exists some \( \theta_j \in \tau_j \) such that
\[
\begin{align*}
(f, I_{k-2}G_h) - (f, I_{k-2}G_h)_* &= \sum_{j=1}^N h^{2k-1}[(k-1)!]^4 \left( f I_{k-2}G_h \right)^{(2k-2)}(\theta_j) \\
&\leq h^{2k-1} \| f \|_{2k-2, \infty} \sum_{j=1}^N \| I_{k-2}G_h \|_{k-2, \infty, \tau_j} \\
&\leq h^{k+2} \| f \|_{2k-2, \infty} \| G_h \|_{2,1}.
\end{align*}
\]
Here in the last step, we have used the inverse inequality
\[
\| v_h \|_{m,p} \leq h^{n-m+\frac{1}{p} - \frac{1}{q}} \| v_h \|_{n,q}, \ \forall n < m.
\]
Similarly, there holds
\[
\begin{align*}
(\gamma \bar{u}_h, I_{k-2}G_h) - (\gamma \bar{u}_h, I_{k-2}G_h)_* &= \sum_{j=1}^N h^{2k-1}[(k-1)!]^4 \left( \bar{u}_h I_{k-2}G_h \right)^{(2k-2)}(\theta_j) \\
&\leq h^{2k-1} \| \bar{u}_h \|_{k, \infty} \| G_h \|_{k-2, \infty} \lesssim h^{k+2} \| \bar{u}_h \|_{k, \infty} \| G_h \|_{2,1} \\
&\lesssim h^{k+2} (\| \bar{e}_h \|_{k, \infty} + \| u_h \|_{k, \infty}) \| G_h \|_{2,1}.
\end{align*}
\]
On the other hand,
\[
\begin{align*}
\| (\beta \bar{e}_h' + \gamma \bar{e}_h, G_h - I_{k-2}G_h) \|_{2,1} &\lesssim h^2 \| G_h \|_{2,1} \| \bar{e}_h \|_{1, \infty} \lesssim h \| \bar{e}_h \|_{0, \infty} \| G_h \|_{2,1}.
\end{align*}
\]
Since \( \| G_h \|_{2,1} \) is bounded, we have
\[
\begin{align*}
| \bar{e}_h(x) | &\lesssim h^{k+2} \| f \|_{2k-2, \infty} \| \bar{e}_h \|_{0, \infty} + h^{k+2} (\| \bar{e}_h \|_{k, \infty} + \| u_h \|_{k, \infty}) \\
&\lesssim h^{k+2} \| u \|_{2k, \infty} + h \| \bar{e}_h \|_{0, \infty}.
\end{align*}
\]
Here in the last step, we have used (46) and the inverse inequality \( \| \bar{e}_h \|_{k, \infty} \lesssim h^{-k} \| \bar{e}_h \|_{0, \infty} \). Consequently,
\[
\begin{align*}
\| \bar{e}_h \|_{0, \infty} &\lesssim h^{k+2} \| u \|_{2k, \infty}, \quad \| \bar{e}_h \|_{1, \infty} \lesssim h^{-1} \| \bar{e}_h \|_{0, \infty} \lesssim h^{k+1} \| u \|_{2k, \infty}.
\end{align*}
\]
This finishes our proof. \( \square \)

Using the conclusions in the above theorem and the superconvergence results for the \( C^1 \) Petrov-Galerkin method, we have the following superconvergence properties for the solution of Gauss collocation methods.

**Theorem 5.2.** Assume that \( u \in W^{2k, \infty}(\Omega) \) is the solution of (1), and \( \bar{u}_h \) is the solution of (3). The following superconvergence properties hold true.
1. **Superconvergence of function value approximation on interior roots of the generalized Jacobi polynomial for** $k \geq 4$:

\[
|(u - \bar{u}_h)(l_{im})| \lesssim h^{k+2}\|u\|_{2k,\infty},
\]

where $l_{im}$, $m = 1, \cdots, k - 3$ are interior roots of $\tilde{J}_{k+1}^{-2,-2}(x)$ in $\tau_i$.

2. **Superconvergence of first order derivative value approximation on Gauss-Lobatto points**:

\[
|(u - \bar{u}_h)'(gl_{in})| \lesssim h^{k+1}\|u\|_{2k,\infty},
\]

where $gl_{in}$, $i \leq k - 2$ are interior Gauss-Lobatto points of degree $k - 2$.

3. **Superconvergence of second order derivative value approximation on Gauss points**:

\[
|(u - \bar{u}_h)''(g_{in})| \lesssim h^k\|u\|_{2k,\infty},
\]

where $g_{in}$, $n \leq k - 1$ are roots of $L_{i,k-1}$.

4. **Supercloseness between the numerical solution and the truncation projection of the exact solution in the $H^2$ norm**:

\[
\|\bar{u}_h - u_I\|_2 \lesssim h^k\|u\|_{2k,\infty}.
\]

5. **Superconvergence for both function value and derivative value approximations at nodes**:

\[
|(u - \bar{u}_h)(x_i)| \lesssim h^{2(k-1)}\|u\|_{2k,\infty}, \quad |(u - \bar{u}_h)'(x_i)| \lesssim h^{2(k-1)}\|u\|_{2k,\infty}, \quad i \in \mathbb{Z}_N.
\]

**Proof.** We only prove (52) since (48)-(51) follow directly from Theorem 4.2 and Theorem 5.1. In light of (47), we have

\[
|\bar{e}_h(x_i)| = |(\gamma \bar{u}_h - f, I_{k-2}G_h(x_i, \cdot)) + (\gamma \bar{u}_h - f, I_{k-2}G_h(x_i, \cdot))|
\lesssim h^{k-1}\|e_h\|_2 + |I|,
\]

where we used that $G_h \in C^k(\tau_j)$ and

\[
I = (\gamma \bar{u}_h - f, I_{k-2}G_h(x_i, \cdot)) + (\gamma \bar{u}_h - f, I_{k-2}G_h(x_i, \cdot)).
\]

Again we use the error of Gauss numerical quadrature and the fact \(|I_{k-2}G_h|_{k-2,\infty} \lesssim 1| to get

\[
|I| \lesssim h^{2k-2}(\|f\|_{2k-2,\infty} + \|\bar{u}_h\|_{k,\infty})|I_{k-2}G_h|_{k-2,\infty} \lesssim h^{2k-2}\|u\|_{2k,\infty},
\]

and thus

\[
|\bar{e}_h(x_i)| \lesssim h^{2k-2}\|u\|_{2k,\infty}.
\]

Similarly, we can prove

\[
|\bar{e}_h'(x_i)| \lesssim h^{2(k-1)}(\|u\|_{k+1,\infty} + \|f\|_{2k-2,\infty}) \lesssim h^{2k-2}\|u\|_{2k,\infty}.
\]

Then the proof is complete.

**Remark 3.** As we may observe from Theorem 4.2 and Theorem 5.2, to achieve the same superconvergence result, the regularity assumption of the exact solution $u$ for the Gauss collocation method is much more stronger than that for the counterpart Petrov-Galerkin method.
6. Numerical experiments. In this section, we present some numerical examples to demonstrate the method and to verify the theoretical findings established in previous sections.

In our numerical experiments, we solve the model problem (1) by the $C^1$ Petrov-Galerkin method (2) and the Gauss collocation method (3) with $k = 3$ and $k = 4$. We test various errors in our examples, including the $H^2$ error of $u_h - u_I$ denoted as $\|u_h - u_I\|_2$, the maximum errors of $u - u_h$ and $(u - u_h)'$ at mesh points, the maximum errors of $u - u_h$ at interior roots of $\hat{J}_{k+1}^{-2, -2}(x)$, $(u - u_h)'$ and $(u - u_h)''$ at interior Gauss-Lobatto and Gauss points, respectively. They are defined by

\[
e_{un} = \max_i |(u - u_h)(x_i)|, \quad e_{un}' = \max_i |(u - u_h)'(x_i)|,
\]

\[
e_u = \max_{i,m} |(u - u_h)(l_{im})|, \quad e_{u'} = \max_{i,m} |(u - u_h)'(gl_{im})|, \quad e_{u''} = \max_{i,m} |(u - u_h)''(g_{im})|.
\]

Here $l_{im}, 1 \leq m \leq k - 3$ are interior roots of $\hat{J}_{k+1}^{-2, -2}(x)$, and $gl_{im}, 1 \leq n \leq k - 2$ are interior Lobatto points, and $g_{im}, 1 \leq n \leq k - 1$ are interior Gauss points in $\tau$. For simplicity, we do not distinguish the error symbols when the method is clearly stated in the following tables.

**Example 1.** We consider the following equation with Dirichlet boundary condition:

\[
\begin{align*}
-\alpha u''(x) + \beta u'(x) + \gamma u(x) &= f(x), \quad x \in [0, 1], \\
u(0) &= u(1) = 0.
\end{align*}
\]

We take the constant coefficients as

\[
\alpha = \beta = \gamma = 1,
\]

and choose the right-hand side function $f$ such that the exact solution to this problem is

\[
u(x) = \sin(\pi x).
\]

Non-uniform meshes of $N$ elements are used in our numerical experiments with $N = 2, \ldots, 32$, which are obtained by randomly and independently perturbing each node of a uniform mesh by up to some percentage. To be more precise,

\[
x_j = \frac{j}{N} + 0.01 \frac{1}{N} \sin\left(\frac{j\pi}{N}\right) \text{randn}, \quad 0 \leq j \leq N,
\]

where $\text{randn}()$ returns a uniformly distributed random number in $(0, 1)$.

We list in Table 1 various approximation errors calculated by the $C^1$ Petrov-Galerkin method for $k = 3, 4$. As we may observe, both the convergence rates of the error $e_{un}$ and $e_{u'n}$ are $2k - 2$, and the convergence rate of $e_u, e_{u'}, e_{u''}$ is $k + 2, k + 1, k$, respectively. All these results are consistent with our theoretical findings in Theorem 4.2.

We next test the superconvergence behavior of the $C^1$ Gauss collocation method. We present in Table 2 the numerical data for various errors and the corresponding convergence rates calculated by the $C^1$ Gauss collocation method. We observe a convergence rate of $2k - 2$ for $e_{un}$ and $e_{u'n}$, $k + 2$ for $e_u$, $k + 1$ for $e_{u'}$, and $k$ for $e_{u''}$, which confirms the theory established in Theorem 5.2.

Furthermore, we also test the supercloseness result between the $C^1$ numerical solution and the Jacobi truncation projection of the exact solution under the $H^2$ norm for two different choices of parameters: i.e., $\beta = 0, \gamma \neq 0$ and $\beta \neq 0, \gamma \neq 0$. Listed in Table 3 are the approximation errors of $\|u_h - u_I\|_2$ and their corresponding convergence rates. From Table 3 we observe that, for both Petrov-Galerkin and Gauss collocation methods, the convergence rate of $\|u_h - u_I\|_2$ is $k$ in case of
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Table 1. Errors, corresponding convergence rates for $C^1$ Petrov-Galerkin method, $\alpha = \beta = \gamma = 1$.

| $k$ | $N$ | $e_u$ | order | $e_u'$ | order | $e_u''$ | order |
|-----|-----|-------|-------|-------|-------|-------|-------|
| 2   | 8.03e-04 | - | 6.11e-03 | - | 7.59e-03 | - | 1.31e-01 | - |
| 4   | 7.02e-05 | 3.49 | 4.92e-04 | 3.61 | - | 5.61e-04 | 3.73 | 1.66e-02 | 2.98 |
| 8   | 4.59e-06 | 3.95 | 3.02e-05 | 4.04 | - | 3.73e-05 | 3.92 | 1.66e-03 | 2.93 |
| 16  | 2.94e-07 | 4.00 | 1.90e-06 | 4.05 | - | 2.66e-06 | 3.87 | 1.66e-04 | 3.03 |
| 32  | 1.90e-08 | 4.00 | 1.18e-07 | 3.99 | - | 1.77e-07 | 3.90 | 1.38e-05 | 2.98 |

Table 2. Errors, corresponding convergence rates for $C^1$ Gauss collocation method, $\alpha = \beta = \gamma = 1$.

| $k$ | $N$ | $e_u$ | order | $e_u'$ | order | $e_u''$ | order |
|-----|-----|-------|-------|-------|-------|-------|-------|
| 2   | 5.25e-03 | - | 1.36e-02 | - | 1.44e-02 | - | 8.32e-03 | - |
| 4   | 2.88e-04 | 4.13 | 7.26e-04 | 4.18 | - | 8.35e-04 | 4.06 | 5.17 | 8.88e-04 | 3.88 |
| 8   | 1.16e-05 | 3.94 | 4.66e-05 | 3.91 | - | 5.89e-05 | 3.84 | 1.61e-03 | 2.85 |
| 16  | 5.18e-06 | 3.90 | 2.91e-06 | 3.84 | - | 4.01e-06 | 3.79 | 1.91e-04 | 2.88 |
| 32  | 2.92e-07 | 3.86 | 1.81e-07 | 3.87 | - | 2.65e-07 | 3.82 | 4.83e-05 | 2.95 |

$\beta \neq 0, \gamma \neq 0$. However, when $\beta = 0, \gamma \neq 0$, the convergence rate is $k$ for the Gauss collocation method, and $k + 1$ for the Petrov-Galerkin method, which is one order higher than that for the Gauss collocation method. Note that the $(k+1)$-th super-convergence rate for the Petrov-Galerkin method in case $\beta = 0$ is two order higher than the counterpart optimal convergence rate. All those results are consistent with our theoretical results predicted in (29) and (51).

Example 2. We consider the following equation with Dirichlet boundary condition:

$$\begin{align*}
-((\alpha(x)u'(x))' + \beta(x)u'(x) + \gamma(x)u(x)) &= f(x), & x \in [0, 1],
\end{align*}$$

(54)

In our experiments, we test the problems of variable coefficients and consider the following three cases:

- Case 1: $\alpha(x) = e^x, \beta(x) = \cos(x), \gamma(x) = x$;
- Case 2: $\alpha(x) = e^x, \beta(x) = 0, \gamma(x) = x$;
- Case 3: $\alpha(x) = e^x, \beta(x) = 0, \gamma(x) = 0$.

The right-hand side function $f$ is chosen such that exact solution is $u(x) = \sin x(x^{12} - x^{11})$. 
Table 3. $\|u_h - u_I\|_2$ and the corresponding convergence rates, constant coefficients.

| $k$ | $N$ | $\|u_h - u_I\|_2$ | $C^1$ Petrov-Galerkin | $C^1$ Gauss collocation |
|-----|-----|-------------------|-----------------------|------------------------|
|     |     | $\alpha = \beta = \gamma = 1$ | $\alpha = \beta = \gamma = 1$, $\alpha = \gamma = 1$, $\beta = 0$ | $\alpha = \gamma = 1$, $\beta = 0$ |
| 2   | 3.93e-02 | 5.57e-03 | 1.12e-01 | 8.32e-02 |
| 4   | 5.15e-03 | 3.59e-04 | 3.94e-02 | 3.00e-02 | 1.07e-02 | 2.93 |
| 8   | 6.47e-04 | 2.23e-05 | 3.99e-02 | 3.04e-02 | 1.34e-03 | 3.03 |
| 16  | 8.12e-05 | 1.40e-06 | 4.02e-02 | 2.96e-02 | 1.70e-04 | 2.96 |
| 32  | 1.01e-05 | 2.99e-07 | 8.75e-08 | 3.01e-05 | 2.12e-05 | 3.03 |

We use the piecewise uniform meshes, which are constructed by equally dividing each interval $[0, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ into $N/2$ subintervals with $N = 4, ..., 64$.

We present various approximation errors and the corresponding convergence rates in Tables 4-5 for the $C^1$ Petrov-Galerkin method, and in Tables 6-7 for the $C^1$ Gauss collocation method, for three different cases with $k = 3, 4$, respectively. Again, we observe the same superconvergence results as those for the constant coefficients in Example 1, i.e., both errors $e_{u_n}$ and $e_{u_n'}$ converge with a rate of $2k - 2$, and the convergence rates of $e_u$, $e_{u'}$, $e_{u''}$ are $k + 2$, $k + 1$, $k$, respectively. In other words, superconvergence results in Theorem 4.2 and Theorem 5.2 are still valid for the case of variable coefficients.

Table 4. Errors and corresponding convergence rates for $C^1$ Petrov-Galerkin method, variable coefficients, $k = 3$.

| $k$ | $N$ | $e_{u_n}$ | $e_{u_n'}$ | $e_{u_n''}$ | $e_{u_n'''}$ |
|-----|-----|---------|---------|---------|---------|
|     |     | error   | order   | error   | order   | error   | order   | error   | order   |
|     |     |        |         |         |         |         |         |         |         |
| 4   | 4.24e-04 | -   | 4.42e-04 | -   | 1.45e-02 | -   | 4.98e-01 | -   |
| 8   | 2.75e-05 | 3.94 | 2.94e-05 | 3.91 | 1.38e-02 | 3.39 | 8.75e-04 | 2.51 |
| 16  | 1.75e-06 | 3.97 | 1.87e-06 | 3.98 | 1.03e-04 | 3.74 | 1.25e-04 | 2.81 |
| 32  | 1.10e-07 | 4.00 | 1.17e-07 | 3.99 | 6.85e-06 | 3.91 | 1.63e-04 | 2.94 |
| 64  | 6.86e-09 | 4.00 | 7.34e-09 | 4.00 | 4.36e-07 | 3.97 | 2.05e-04 | 2.99 |

Case 2

| $k$ | $N$ | $e_{u_n}$ | $e_{u_n'}$ | $e_{u_n''}$ | $e_{u_n'''}$ |
|-----|-----|---------|---------|---------|---------|
|     |     | error   | order   | error   | order   | error   | order   | error   | order   |
| 4   | 3.36e-04 | -   | 3.34e-04 | -   | 3.37e-02 | -   | 4.88e-01 | -   |
| 8   | 2.26e-05 | 3.89 | 2.82e-05 | 3.65 | 1.30e-03 | 3.39 | 8.56e-02 | 2.51 |
| 16  | 1.44e-06 | 3.97 | 1.97e-06 | 3.84 | 9.74e-05 | 3.74 | 1.22e-02 | 2.81 |
| 32  | 9.03e-08 | 4.00 | 1.23e-07 | 3.93 | 6.45e-06 | 3.91 | 1.58e-03 | 2.95 |
| 64  | 5.66e-09 | 4.00 | 8.26e-09 | 3.97 | 4.09e-07 | 3.98 | 2.00e-04 | 2.99 |

Case 3

| $k$ | $N$ | $e_{u_n}$ | $e_{u_n'}$ | $e_{u_n''}$ | $e_{u_n'''}$ |
|-----|-----|---------|---------|---------|---------|
|     |     | error   | order   | error   | order   | error   | order   | error   | order   |
| 4   | 3.36e-04 | -   | 3.36e-04 | -   | 3.37e-02 | -   | 4.88e-01 | -   |
| 8   | 2.26e-05 | 3.89 | 2.82e-05 | 3.65 | 1.30e-03 | 3.39 | 8.56e-02 | 2.51 |
| 16  | 1.44e-06 | 3.97 | 1.97e-06 | 3.84 | 9.74e-05 | 3.74 | 1.22e-02 | 2.81 |
| 32  | 9.05e-08 | 4.00 | 1.29e-07 | 3.93 | 6.45e-06 | 3.91 | 1.59e-03 | 2.95 |
| 64  | 5.66e-09 | 4.00 | 8.26e-09 | 3.97 | 4.09e-07 | 3.98 | 2.00e-04 | 2.99 |
### Table 5. Errors and corresponding convergence rates for \( C^1 \) Petrov-Galerkin method, variable coefficients, \( k = 4 \).

| \( k \) | \( N \) | \( e_{u_n} \) | error | order | \( e_{u_n'} \) | error | order | \( e_u \) | error | order | \( e_u' \) | error | order | \( e_u'' \) | error | order |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 2.56e-06 | - | 1.71e-06 | - | 1.19e-05 | - | 6.21e-02 | - |
| 8 | 4.06e-08 | 5.98 | 3.46e-08 | 5.62 | 8.78e-07 | 5.55 | 4.83e-05 | 4.63 | 4.91e-03 | 3.66 |
| 16 | 6.25e-10 | 6.02 | 6.31e-10 | 5.78 | 1.35e-08 | 6.03 | 1.38e-06 | 5.13 | 2.90e-04 | 4.08 |
| 32 | 1.05e-11 | 5.89 | 1.03e-11 | 5.94 | 2.25e-10 | 5.90 | 4.78e-08 | 4.85 | 1.94e-05 | 3.90 |
| 64 | 1.63e-13 | 6.01 | 1.65e-13 | 5.96 | 4.02e-12 | 5.81 | 1.52e-09 | 4.98 | 1.24e-06 | 3.97 |

### Table 6. Errors and corresponding convergence rates for \( C^1 \) Gauss collocation method, variable coefficients, \( k = 3 \).

| \( k \) | \( N \) | \( e_{u_n} \) | error | order | \( e_{u_n'} \) | error | order | \( e_u \) | error | order | \( e_u' \) | error | order | \( e_u'' \) | error | order |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1.25e-06 | - | 8.07e-07 | - | 3.91e-05 | - | 1.16e-03 | - | 6.07e-02 | - |
| 8 | 2.01e-08 | 5.96 | 2.10e-08 | 5.56 | 3.48e-07 | 4.95 | 1.13e-05 | 4.27 | 4.25e-03 | 3.66 |
| 16 | 3.10e-10 | 6.02 | 3.99e-10 | 5.78 | 1.35e-08 | 6.03 | 1.38e-06 | 5.13 | 2.84e-04 | 4.08 |
| 32 | 5.12e-12 | 5.92 | 6.70e-12 | 5.90 | 1.26e-08 | 6.05 | 1.33e-06 | 5.12 | 2.84e-04 | 4.08 |
| 64 | 8.02e-14 | 6.00 | 1.05e-13 | 5.75 | 3.89e-12 | 5.77 | 1.46e-09 | 4.98 | 1.21e-06 | 3.97 |

We also test the error \( \| u_I - u_h \|_2 \) in the above three cases of variable coefficients. We list in Table 8 and Table 9 the numerical data \( \| u_I - u_h \|_2 \) and the convergence rate for \( C^1 \) Petrov-Galerkin approximation and Gauss collocation approximation with \( k = 3, 4 \). We observe that the convergence rate is always \( k \) in different choices.
Table 7. Errors and corresponding convergence rates for $C^1$ Gauss collocation method, variable coefficients, $k = 4$.

| $k$ | $N$ | $e_{u_{nn}}$ | $e_{u_{nn}}$ | $e_u$ | $e_u^\prime$ | $e_u^\prime\prime$ |
|-----|-----|-------------|-------------|-------|-------------|--------------|
|     |     | error order | error order | error | error order | error order |
| 4   | 1.45e-05 | 1.16e-04 | 8.66e-05 | 1.00e-03 | 8.32e-03 |
| 8   | 4.69e-07 | 4.95 | 3.01e-06 | 5.27 | 1.53e-06 | 3.87e-05 | 4.69 | 7.68e-04 | 3.44 |
| 16  | 1.25e-08 | 5.23 | 4.84e-08 | 5.96 | 1.64e-08 | 6.55 | 1.14e-06 | 5.09 | 5.07e-05 | 3.75 |
| 32  | 2.23e-10 | 5.81 | 7.53e-10 | 6.01 | 4.01e-10 | 5.35 | 3.76e-08 | 4.92 | 3.73e-06 | 3.94 |
| 64  | 3.61e-12 | 5.95 | 1.18e-11 | 6.00 | 7.73e-12 | 5.70 | 1.18e-09 | 4.99 | 2.34e-07 | 4.00 |

Case 2

| $k$ | $N$ | $e_{u_{nn}}$ | $e_{u_{nn}}$ | $e_u$ | $e_u^\prime$ | $e_u^\prime\prime$ |
|-----|-----|-------------|-------------|-------|-------------|--------------|
|     |     | error order | error order | error | error order | error order |
| 4   | 1.60e-05 | 1.15e-04 | 9.17e-05 | 1.09e-03 | 1.06e-02 |
| 8   | 4.82e-07 | 5.05 | 3.09e-06 | 5.22 | 1.63e-06 | 4.81e-05 | 4.70 | 9.85e-04 | 3.42 |
| 16  | 1.31e-08 | 5.20 | 5.06e-08 | 5.93 | 1.78e-08 | 6.52 | 1.25e-06 | 5.06 | 7.21e-05 | 3.77 |
| 32  | 2.35e-10 | 5.80 | 7.92e-10 | 6.00 | 4.08e-10 | 5.45 | 4.10e-08 | 4.93 | 4.68e-06 | 3.94 |
| 64  | 3.80e-12 | 5.95 | 1.24e-11 | 6.00 | 7.92e-12 | 5.69 | 1.28e-09 | 5.00 | 2.93e-07 | 4.00 |

Case 3

| $k$ | $N$ | $e_{u_{nn}}$ | $e_{u_{nn}}$ | $e_u$ | $e_u^\prime$ | $e_u^\prime\prime$ |
|-----|-----|-------------|-------------|-------|-------------|--------------|
|     |     | error order | error order | error | error order | error order |
| 4   | 1.61e-05 | 1.15e-04 | 9.18e-05 | 1.09e-03 | 1.05e-02 |
| 8   | 4.84e-07 | 5.06 | 3.08e-06 | 5.22 | 1.63e-06 | 4.81e-05 | 4.70 | 9.86e-04 | 3.42 |
| 16  | 1.32e-08 | 5.20 | 5.07e-08 | 5.92 | 1.78e-08 | 6.52 | 1.25e-06 | 5.06 | 7.21e-05 | 3.77 |
| 32  | 2.37e-10 | 5.80 | 7.96e-10 | 5.99 | 4.06e-10 | 5.45 | 4.10e-08 | 4.93 | 4.68e-06 | 3.95 |
| 64  | 3.84e-12 | 5.95 | 1.25e-11 | 6.00 | 7.89e-12 | 5.69 | 1.28e-09 | 5.00 | 2.93e-07 | 4.00 |

Table 8. $\|u_h - u_I\|_2$ and corresponding convergence rates, variable coefficients, $k = 3$.

| $k$ | $N$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ |
|-----|-----|-------------|-------------|-------------|-------------|
|     |     | error order | error order | error order | error order |
| 4   | 2.35e-02 | - | 1.89e-02 | - | 1.89e-02 |
| 8   | 3.46e-03 | 2.77 | 2.82e-03 | 2.75 | 2.82e-03 | 2.75 |
| 16  | 4.49e-04 | 2.94 | 3.67e-04 | 2.94 | 3.67e-04 | 2.94 |
| 32  | 5.66e-05 | 2.99 | 4.64e-05 | 2.99 | 4.64e-05 | 2.99 |
| 64  | 7.10e-06 | 3.00 | 5.81e-06 | 3.00 | 5.81e-06 | 3.00 |
| $C^1$ Petrov-Galerkin | 3 | 16 | 4.49e-04 | 2.94 | 3.67e-04 | 2.94 |
| 32  | 5.66e-05 | 2.99 | 4.64e-05 | 2.99 | 4.64e-05 | 2.99 |
| 64  | 7.10e-06 | 3.00 | 5.81e-06 | 3.00 | 5.81e-06 | 3.00 |

| $k$ | $N$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ | $\|u_h - u_I\|_2$ |
|-----|-----|-------------|-------------|-------------|-------------|
|     |     | error order | error order | error order | error order |
| 4   | 1.86e-01 | - | 1.93e-01 | - | 1.93e-01 |
| 8   | 2.65e-02 | 2.81 | 2.75e-02 | 2.81 | 2.75e-02 | 2.81 |
| 16  | 3.37e-03 | 2.97 | 3.51e-03 | 2.97 | 3.51e-03 | 2.97 |
| 32  | 4.22e-04 | 3.00 | 4.39e-04 | 3.00 | 4.39e-04 | 3.00 |
| 64  | 5.27e-05 | 3.00 | 5.49e-05 | 3.00 | 5.49e-05 | 3.00 |
| $C^1$ Gauss collocation | 3 | 16 | 3.37e-03 | 2.97 | 3.51e-03 | 2.97 |
| 32  | 4.22e-04 | 3.00 | 4.39e-04 | 3.00 | 4.39e-04 | 3.00 |
| 64  | 5.27e-05 | 3.00 | 5.49e-05 | 3.00 | 5.49e-05 | 3.00 |

To sum up, superconvergence phenomena for problems with variable coefficients under non-uniform meshes still exist, and the superconvergence behavior for variable coefficients problems is similar with that for the constant coefficients problems.

7. Conclusion. In this work, we have presented a unified approach to study superconvergence properties of $C^1$ Petrov-Galerkin and Gauss collocation methods for one-dimensional elliptic equations. Our main theoretical results include the proof of the $2k - 2$ superconvergence rate for both solution and its first order derivative approximations at grid points, the $k + 2$-th order function value approximation at...
Table 9. $\|u_h - u_I\|_2$ and corresponding convergence rates, variable coefficients, $k = 4$.

| $k$ | $N$ | $\|u_h - u_I\|_2$ | Error order | Error order | Error order |
|-----|-----|-----------------|-------------|-------------|-------------|
|     |     | Case 1 | Case 2 | Case 3 | Case 1 | Case 2 | Case 3 |
| 4   | 5.03e-03 | - | 4.48e-03 | - | 4.48e-03 | - |
| 8   | 3.53e-04 | 3.83 | 3.17e-04 | 3.82 | 3.17e-04 | 3.82 |
| 16  | 2.24e-05 | 3.98 | 2.01e-05 | 3.98 | 2.01e-05 | 3.98 |
| 32  | 1.40e-06 | 4.00 | 1.26e-06 | 4.00 | 1.26e-06 | 4.00 |
| 64  | 8.75e-08 | 4.00 | 7.86e-08 | 4.00 | 7.86e-08 | 4.00 |

Our analysis indicates that for constant coefficients, the Gauss collocation method is essentially equivalent to the Petrov-Galerkin method up to practically neglectable numerical quadrature errors, see (39). Indeed, we always use numerical quadrature instead of exact integration in practice.

Comparing with the traditional $C^0$ Galerkin method, the major gain of the $C^1$ Petrov-Galerkin method discussed in this work is the $2k - 2$ convergence rate of the derivative approximation at nodes, with the sacrifice of function value convergence rate at nodes dropping from $2k$ to $2k - 2$.

Comparing with the $C^1$ Galerkin method studied in [27], the $C^1$ Petrov-Galerkin method discussed in this work has equal or better convergence rates in all respect. It seems that that the $L^2$ test function is superior to the $C^1$ test function. Therefore, the $C^1$ Petrov-Galerkin method is a method to recommend if one is also interested in derivative approximations.

Based on the analysis, extension of our results to the higher dimensional tensor-product space is feasible.

REFERENCES

[1] S. Adjerid and T. C. Massey, Superconvergence of discontinuous Galerkin solutions for a nonlinear scalar hyperbolic problem, *Comput. Methods Appl. Mech. Engrg.*, 195 (2006), 3331–3346.
[2] S. Adjerid and T. Weinhart, Discontinuous Galerkin error estimation for linear symmetric hyperbolic systems, *Comput. Methods Appl. Mech. Engrg.*, 198 (2009), 3113–3129.
[3] S. Adjerid and T. Weinhart, Discontinuous Galerkin error estimation for linear symmetrizable hyperbolic systems, *Math. Comp.*, 80 (2011), 1335–1367.
[4] I. Babuška, T. Strouboulis, C. S. Upadhyay and S. K. Gangaraj, Computer-based proof of the existence of superconvergence points in the finite element method: Superconvergence of the derivatives in finite element solutions of Laplace's, Poisson’s, and the elasticity equations, *Numer. Meth. PDEs*, 12 (1996), 347–392.
[5] S. K. Bhal and P. Danumjaya, A Fourth-order orthogonal spline collocation solution to 1D-Helmholtz equation with discontinuity, *J. Anal.*, 27 (2019), 377–390.

[6] B. Bialecki, Superconvergence of the orthogonal spline collocation solution of Poisson’s equation, *Numerical Methods for Partial Differential Equations*, 15 (1999), 285–303.

[7] J. H. Bramble and A. H. Schatz, High order local accuracy by averaging in the finite element method, *Math. Comp.*, 31 (1977), 94–111.

[8] Z. Q. Cai, On the finite volume element method, *Numer. Math.*, 58 (1991), 713–735.

[9] W. Cao, C.-W. Shu, Y. Yang and Z. Zhang, Superconvergence of Discontinuous Galerkin method for nonlinear hyperbolic equations, *SIAM. J. Numer. Anal.*, 56 (2018), 732–765.

[10] W. Cao and Z. Zhang, Superconvergence of Local Discontinuous Galerkin method for one-dimensional linear parabolic equations, *Math. Comp.*, 85 (2016), 63–84.

[11] W. Cao, Z. Zhang and Q. Zou, Superconvergence of any order finite volume schemes for 1D general elliptic equations, *J. Sci. Comput.*, 56 (2013), 566–590.

[12] W. Cao, Z. Zhang and Q. Zou, Superconvergence of Discontinuous Galerkin method for linear hyperbolic equations, *SIAM. J. Numer. Anal.*, 52 (2014), 2555–2573.

[13] W. Cao, Z. Zhang and Q. Zou, Is 2k-conjecture valid for finite volume methods?, *SIAM. J. Numer. Anal.*, 53 (2015), 942–962.

[14] C. Chen, *Structure Theory of Superconvergence of Finite Elements*, Hunan Science and Technology Press, Hunan, China, 2001.

[15] C. Chen and S. Hu, The highest order superconvergence for bi-k degree rectangular elements at nodes- a proof of 2k-conjecture, *Math. Comp.*, 82 (2013), 1337–1355.

[16] C. Chen and Y. Huang, *High Accuracy Theory of Finite Elements*, Hunan Science and Technology Press, Hunan, China, 1995.

[17] Y. Cheng and C.-W. Shu, Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension, *SIAM J. Numer. Anal.*, 47 (2010), 4044–4072.

[18] S.-H. Chou and X. Ye, Superconvergence of finite volume methods for the second order elliptic problem, *Comput. Methods Appl. Mech. Eng.*, 196 (2007), 3706–3712.

[19] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Second edition, Computer Science and Applied Mathematics, Academic Press, Inc., Orlando, FL, 1984.

[20] R. E. Ewing, R. D. Lazarov and J. Wang, Superconvergence of the velocity along the Gauss lines in mixed finite element methods, *SIAM J. Numer. Anal.*, 28 (1991), 1015–1029.

[21] M. Krížek and P. Neittaanmäki, On superconvergence techniques, *Acta Appl. Math.*, 9 (1987), 175–198.

[22] M. Krížek, P. Neittaanmäki and R. Stenberg (Eds.), *Finite Element Methods: Superconvergence, Post-processing, and A Posteriori Estimates*, Lecture Notes in Pure and Applied Mathematics Series Vol. 196, Marcel Dekker, Inc., New York, 1997.

[23] Q. Lin and N. Yan, *Construction and Analysis of High Efficient Finite Elements*, Hebei University Press, P.R. China, 1996.

[24] A. H. Schatz, I. H. Sloan and L. B. Wahlbin, Superconvergence in finite element methods and meshes which are symmetric with respect to a point, *SIAM J. Numer. Anal.*, 33 (1996), 505–521.

[25] J. Shen, T. Tang and L.-L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Series in Computational Mathematics, 41. Springer, Heidelberg, 2011.

[26] V. Thomée, High order local approximation to derivatives in the finite element method, *Math. Comp.*, 31 (1977), 652–660.

[27] L. B. Wahlbin, *Superconvergence In Galerkin Finite Element Methods*, Lecture Notes in Mathematics, 1605. Spring, Berlin, 1995.

[28] Z. Xie and Z. Zhang, Uniform superconvergence analysis of the discontinuous Galerkin method for a singularly perturbed problem in 1-D, *Math. Comp.*, 79 (2010), 35–45.

[29] J. Xu and Q. Zou, Analysis of linear and quadratic simplitical finite volume methods for elliptic equations, *Numer. Math.*, 111 (2009), 469–492.

[30] Y. Yang and C.-W. Shu, Analysis of optimal superconvergence of discontinuous Galerkin method for linear hyperbolic equations, *SIAM J. Numer. Anal.*, 50 (2012), 3110–3133.

[31] Z. Zhang, Superconvergence points of polynomial spectral interpolation, *SIAM J. Numer. Anal.*, 50 (2012), 2966–2985.

[32] Z. Zhang, Superconvergence of a Chebyshev spectral collocation method, *J. Sci. Comput.*, 34 (2008), 237–246.
[33] Q. Zhu and Q. Lin, *Superconvergence Theory of the Finite Element Method*, Hunan Science and Technology Press, Hunan, China, 1989.

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