On a Jordan-algebraic formulation of quantum mechanics : Hilbert space construction

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Abstract

In this note I discuss some aspects of a formulation of quantum mechanics based entirely on the Jordan algebra of observables. After reviewing some facts of the formulation in the $C^*$-approach I present a Jordan-algebraic Hilbert space construction (inspired by the usual GNS-construction), thereby obtaining a real Hilbert space and a (Jordan-) representation of the algebra of observables on this space. Taking the usual case as a guideline I subsequently derive a Schrödinger equation on this Hilbert space.
1 Introduction

The algebraic approach to quantum theory takes the observables to play the central role in the formulation. This viewpoint was initiated by Segal [16] and taken later by Haag and Kastler [8] to formulate quantum field theory. The case of ordinary quantum mechanics is studied by Roberts and Roepstorff in [15].

One of the main ideas of this approach is to base the formulation of quantum theory on "essential" ingredients: observables, states, expectation values and their time evolution. The observables are taken to be the self-adjoint part of a $C^*$-algebra $A$, the states are positive, linear functionals on $A$ [16], and the (real) number a state assigns to an self-adjoint element is interpreted as the expectation value of the corresponding observable. A normal state $\phi$ can be represented by a positive trace-class element (density matrix) $\rho$ of $A$ such that

$$\phi(A) = \text{tr}\rho A .$$

The time-evolution equation (in the Schrödinger picture) is given by the familiar

$$\dot{\rho} = -i[H, \rho]$$

where the Hamiltonian $H$, a self-adjoint element of $A$ (here assumed to be time-independent), is the generator of the one-parameter group of time translations of the system under consideration. The solution is

$$\rho_t = e^{-itH} \rho_0 e^{itH} .$$

Contact with the Hilbert space formulation is made via the GNS-construction: One takes a pure state $\phi$ on $A$ which gives a hermitean form $\langle A|B \rangle = \phi(A^*B)$, divides out the set of "zero-elements" $\mathcal{I}_\phi = \{ A \in A \mid \phi(A^*A) = 0 \}$ which is an ideal in $A$, and obtains with $\phi$ a positive-definite, hermitean scalar product on $A/\mathcal{I}_\phi$ and, subsequently, a norm $\| A \|_\phi = \sqrt{\langle A|A \rangle}$. Closure with respect to this norm gives a (complex since $A$ is complex) Hilbert space

$$\mathcal{H}_\phi = A/\mathcal{I}_\phi .$$

$A$ can be represented in the algebra of bounded, linear operators on $A_\phi$, i.e. there is a *-homomorphism $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$ that acts simply by multiplication on $A/\mathcal{I}_\phi$

$$\pi_\phi(A)B = AB .$$

Pure states are now given by rays in $\mathcal{H}_\phi$, and one has for the expectation value

$$\phi_B(A) = \langle B|A|B \rangle = \phi(B^*AB) .$$

In Hilbert space the time evolution equation for (pure) state vectors $v \in \mathcal{H}$ is given by the Schrödinger equation

$$\dot{v} = Hv ,$$

and its solution takes the form $v_t = e^{-itH}v_0$.
Now, the $C^*$-algebra one starts with contains a lot of non-self-adjoint elements and can therefore hardly be considered the most essential structure for the observables. Since the $C^*$-product of two self-adjoint elements in general is not self-adjoint, the observables do not even form a $C^*$-subalgebra but only a real subspace. But they do form an algebra under the product

$$A \cdot B = \frac{1}{2}(AB + BA)$$

which is commutative but not fully associative anymore. Instead, one has

$$(A \cdot B) \cdot A^2 = A \cdot (B \cdot A^2)$$

which is equivalent to the power-associativity of $A$ \[10, 14\]. A real, commutative algebra $A$ satisfying this latter identity is called a Jordan algebra \[13\]. Jordan algebras that can be embedded in an associative algebra (with the above anticommutator as a product) are called special. They are formally real if $\sum_{i=1}^n A_i^2 = 0$ already implies $A_i = 0$. The classification of finite-dimensional, formally real Jordan algebras \[14\] gives two classes of special algebras: hermitean matrix algebras over the reals, the complexes or the quaternions and spin factors \[9\]. In addition, there is one exceptional (i.e. non-special) Jordan algebra, the hermitean $3 \times 3$-matrices over the octonions, denoted by $H_3(O)$ \[7, 9\].

The infinite-dimensional case was investigated from the late seventies on, initiated by \[1\] (see also \[9\]) where the direct analogues of $C^*$-algebras, the so-called JB-algebras are considered. These are Jordan algebras that are also a Banach space satisfying

$$\|A^2\| = \|A\|^2,$$

$$\|A^2 - B^2\| \leq max\{\|A^2\|, \|B^2\|\}.$$

See \[1\] for this version of the JB-axioms. A consequence of these two axioms is $\|A \cdot B\| \leq \|A\|\|B\|$ which is the analog of the familiar Banach-space axiom. All JB-algebras are formally real; in finite dimensions there is even equivalence of the two concepts.

Since with exception of $H_3(O)$ all other JB-algebras can be seen as the self-adjoint part of a $C^*$-algebra ($JC$-algebras) and can thus be represented on a complex Hilbert space \[1\], and since, so far, there seems to be no physical applications for $H_3(O)$ (see on the other hand e.g. \[3, 4, 18\]), one could come to the conclusion that there is no need to discuss Jordan algebras any further. Contrary to that, I would like to argue that

1. since one agrees that the observables do form a JB-algebra ,

2. since one wants to incorporate all algebras of that kind (an approach placing the algebra of observables in the centre should be able to handle all admissible algebras)

3. if one really wants to base the formulation of quantum theory on essential ingredients only,
then one must look for a possibility to formulate quantum theory in terms of the Jordan algebra of observables alone.

Over the past, there has already been some work on a Jordan-algebraic formulation of quantum mechanics \[1, 2, 3, 18\], often concentrating on the exceptional character of \(H_3(O)\) and the presumed absence of a Hilbert space formulation for it. In this work, however, I present a unified frame for all algebras of observables. In particular, I construct a representation of a Jordan algebra on a Hilbert space of states and, finally, give a Schrödinger-like time evolution equation for these state vectors.

Of course, the results of ordinary quantum theory will be recovered in this Jordan-algebraic version. The associative case will be taken as a guideline and connection to it will be made whenever possible. The starting point is the same as in the associative case \[15\]: One starts with the JB-algebra of observables \(A\), takes the states to be positive, normed, linear functionals on \(A\), and interprets their values on algebra elements as the expectation value of an observable in that state.

One of the next steps would certainly be to imitate the usual GNS-construction for \(A\) \[9\]. In this case, a state \(\phi\) gives a real, symmetric, bilinear form on \(A\): \(\langle A|B\rangle = \phi(A\circ B)\). Again, one looks at \(I_\phi = \{A \in A \mid \phi(A^2) = 0\}\) which is an orthogonal subspace of \(A\). So on \(A/I_\phi\) we have a positive-definite (real) scalar product and therefore a norm. So one has a real Hilbert space
\[
\mathcal{H}_J = A/I_\phi
\]

The problem is that \(I_\phi\) is not an ideal in \(A\), i.e. \(A\) does not act on \(\mathcal{H}_J\). So this Hilbert space is, at least for our purposes, without use.

The aim of the following section is to enlarge the associative formulation of quantum theory (in the sense of \[3, 19\]) to allow for the transition to a (then more general) Jordan version.
2 The Hilbert space of Hilbert-Schmidt operators

In the introduction we have seen that, given a pure state φ of a $C^*$-algebra $\mathcal{A}$, one can construct a Hilbert space $\mathcal{H}$ and represent $\mathcal{A}$ on $\mathcal{H}$ in such a way that, on $\mathcal{A}/\mathcal{I}_\phi$, the algebra acts simply as multiplication operators.

There is another Hilbert space construction based on traces instead of states: Let us assume that $\mathcal{A}$ is a von Neumann algebra of type I (or II). Then there always exists a semi-finite, faithful, normal trace $tr$ on $\mathcal{A}$ [2, 17]. I denote the positive elements $A \in \mathcal{A}$ that have finite trace by $A^+_1$ and those $A \in \mathcal{A}$ for which $tr A^* A < \infty$ by $A_2$. $A_2$ is an ideal in $\mathcal{A}$.

We now parallel the GNS-construction. $tr$ gives a hermitean form $\langle A | B \rangle = tr A^* B$ on $A_2$. Since $tr$ is faithful, this form is already positive-definite and therefore defines a norm

$$\|A\|_{tr} := \sqrt{tr A^* A} .$$

$A_2$ can be closed with respect to this norm, i.e. we get a Hilbert space $\hat{\mathcal{H}} = \overline{A_2}^{tr}$

Due to the associativity of $\mathcal{A}$ and the fact that $A_2$ is an ideal, $\mathcal{A}$ can be represented on $A_2$ as multiplication operators

$$\pi(A) B = AB$$

for any $A \in \mathcal{A}, B \in A_2$. $\hat{\mathcal{H}}$ is an $\mathcal{A}$-bimodule and $\pi$ can be extended to a faithful representation of $\mathcal{A}$ on $\hat{\mathcal{H}}$ [17].

The difference to the usual Hilbert space of pure states is that now every (pure and mixed) state is being represented by at least one vector in $\hat{\mathcal{H}}$. This is easily seen as follows: Take a state $\phi$ represented by a density matrix $\rho$, i.e. $\rho \in A^+_1$, then there is at least one $B \in A_2$ with $\rho = BB^*$. So we have for the expectation value of an observable $A \in \mathcal{A}$:

$$\phi(A) = tr \rho A = tr BB^* A = tr B^* AB = \langle B | A | B \rangle .$$

On the other hand every vector $B \in A_2$ describes a state since $\frac{1}{\|B\|^2} BB^*$ is positive and trace-class and therefore a density matrix. The representation of states in $\hat{\mathcal{H}}$ is by no means unique. Take any vector $B \in A_2$ and any unitary element $U$ in $\mathcal{A}$, then $BU$ is a vector in $\hat{\mathcal{H}}$ describing the same state as $B$ does

$$BUU^* B^* = BB^* = \rho .$$

Again, this transfers to all of $\hat{\mathcal{H}}_J$. This means that instead of a single ray, as in the usual Hilbert space, the states are being represented in $\hat{\mathcal{H}}$ by “right-unitary orbits”.

The connection between the two Hilbert spaces is as follows: Start with $\mathcal{H}$ and take $\mathcal{A} = B(\mathcal{H})$, then every usual GNS-Hilbert space is isomorphic to $\mathcal{H}$. The construction
based on traces gives the Hilbert space of Hilbert-Schmidt-operators with norm given by \(
\| \cdot \|_{tr}\).

This norm is a cross-norm for the tensor product \( \mathcal{H} \otimes \mathcal{H}^* \) and the completion with respect to it just gives the Hilbert space of Hilbert-Schmidt-operators \[ ^{17} \]

\[ \hat{\mathcal{H}} = \overline{\mathcal{H} \otimes \mathcal{H}^*_{tr}} . \]

This Hilbert space \( \hat{\mathcal{H}} \) is investigated by Uhlmann \[ ^{19} \] and also Dabrowski and Grosse \[ ^{3} \] to study the Berry phase for mixed states. They use the “right-unitary” ambiguity of the states as a generalized phase. They also consider possible time evolution equations which turn out to be, in their most general form

\[ \dot{\psi} = \hat{H} \psi \]

where \( \hat{H} = L_H - R_{\tilde{H}} \) such that

\[ \dot{\psi} = H \psi - \psi \tilde{H} . \]

\( H \) is the “usual” Hamiltonian and \( \tilde{H} \) is an additional generator of “right-unitary translations”; it does not change the states:

\[ \psi_t = e^{-itH} \psi_0 e^{it\tilde{H}} . \]

To see this we take any density matrix \( \rho_t \) and get

\[ \rho_t = B_t B_t^* = e^{-itH} B_0 B_0^* e^{itH} = e^{-itH} \rho_0 e^{itH} , \]

i.e. we regain the familiar form of the time evolution of a state.

3 Jordan-GNS-Construction

It is our goal now to repeat the constructions of the last chapter but base them on the JB-Algebra of observables instead of an associative \( C^* \)-algebra. As already stated in the introduction, the trinity of observables, states and expectation values is not touched by the restriction to the actual algebra of observables. Yet, a Jordan version of the usual GNS-construction did not lead to a representation of \( \mathcal{A} \) on a Hilbert space. (In fact it did not even lead to an action of \( \mathcal{A} \) on \( \mathcal{H}_J \).)

A different situation is met in the construction based on traces. A trace on a Jordan algebra is defined to be a weight on \( \mathcal{A} \) (i.e. for positive elements \( A, B \in \mathcal{A} \) and positive real numbers \( \lambda \) we have \( tr(A + B) = trA + trB \), and \( trA = \lambda trA \)) with the additional property

\[ trA \circ (B \circ C) = tr(A \circ B) \circ C , \]

replacing the cyclicity condition for traces on associative algebras. There also is an analogue of abstract von Neumann algebras, the JBW-algebras \[ ^{4} \], i.e. JB-algebras.
with Banach predual. One can work out the same machinery for normal, faithful, semi-
finite traces on JBW-algebras as one did in the $W^*$-case [11]. In particular, we have
the same definitions of $A_1^+, A_2$. $A_2$ is a Jordan ideal.
Based on this we construct a Hilbert space as follows: $tr$ induces a bilinear, symmetric,
real scalar product on $A_2$
\[
\langle A|B \rangle = tr A \circ B
\]
which is positive-definite and thus yields a norm
\[
\|A\|_{tr} = \sqrt{tr A^2}
\]
As in the associative case closure with respect to this norm yields a (this time real)
Hilbert space
\[
\hat{H}_J = \overline{A_2}^r
\]
We now turn to the problem of representing $\mathcal{A}$ on this Hilbert space. For this we
need the notion of a Jordan-module [12]: Let $\mathcal{A}$ be a Jordan algebra and $V$ a (real)
vector space. Furthermore, let there be two bilinear mappings (both denoted by “.”)
$(\mathcal{A}, V) \rightarrow V : (A, v) \mapsto A.v$ and $(\mathcal{A}, V) \rightarrow V : (A, v) \mapsto v.A$. $V$ is called a Jordan
module, if for any $v \in V$ and $A, B \in \mathcal{A}$
\[
A.v = v.A,
A^2.(A.v) = A.(A^2.v),
2A.(B.(A.v)) + (B \circ A^2).v = 2(A \circ B).(A.v) + A^2.(B.v).
\]
Based on this we can define: A Jordan representation is a linear mapping
\[
\pi_J : \mathcal{A} \rightarrow Hom_R(V, V)
\]
with the following properties:
\[
\pi_J(A^2)\pi_J(A) = \pi_J(A)\pi_J(A^2),
2\pi_J(A)\pi_J(B)\pi_J(A) + \pi_J(B \circ A^2) = 2\pi_J(A \circ B)\pi_J(A) + \pi_J(A^2)\pi_J(B).
\]
Let now be $B \in A_2$. We have for any $A \in \mathcal{A}$ a multiplication operator, i.e. a linear
mapping $T_A : A_2 \rightarrow A_2$
\[
T_A B = A \circ B.
\]
$T_A$ maps into $A_2$ since $A_2$ is an ideal. One sees that $A_2$ with the multiplication by
elements of $\mathcal{A}$ is a Jordan module: The first two relations are just algebra relations.
The third is easily verified in the case of a special Jordan algebra and therefore, by
Macdonald’s theorem [9], valid in every Jordan algebra.
Due to the continuity of multiplication the operators $T_A$ can be extended to the whole
of $\hat{H}_J$, and we have that, with the mapping
\[
T : \mathcal{A} \rightarrow Hom_R(\hat{H}_J, \hat{H}_J) : A \mapsto T_A
\]
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as a module mapping, also $\mathcal{H}_J$ is a Jordan module. Finally, one can state the

**Proposition:** The mapping $T$ is a Jordan representation of $A$ on $\mathcal{H}_J$.

The expectation value is still given by

$$\omega_v(A) = \frac{\langle v | A | v \rangle}{\|v\|^2_{tr}}.$$  

For a normal state vector $B \in \mathcal{A}_2 \subseteq \mathcal{H}_J$ this can be brought into density matrix form by using the associativity of the trace (let $\|B\|_{tr} = 1$)

$$\omega_B(A) = \langle B | A | B \rangle = tr(B \circ (A \circ B)) = trB^2 \circ A = tr\rho \circ A.$$  

At this point it seems appropriate to compare the spaces so far constructed. Let $\mathcal{B}$ be a $C^*$-algebra with self-adjoint part $\mathcal{A}$, the latter being considered as a JB-algebra. Then $\mathcal{H}$ is the space of Hilbert-Schmidt operators and $\mathcal{H}_J$ is the real subspace of self-adjoint Hilbert-Schmidt operators. We have to clarify some of the ambiguities of the state representation in $\mathcal{H}_J$.

In $\mathcal{H}_J$ the “right-unitary orbit” does not appear anymore but is replaced by a real ray. As in the associative case it may happen that a state is being described by more than one real ray (formerly right-unitary orbit). An example: Take $\mathcal{A}$ to be $H_2(\mathbb{C})$, and the density matrix

$$\rho = \frac{1}{5} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$  

Then two self-adjoint and unitarily inequivalent matrices $v_{1,2}$ (i.e. vectors in $\mathcal{H}_J$) with $\rho = (v_{1,2})^2$ are given by

$$v_{1,2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & \pm 2 \end{pmatrix}.$$  

Yet, we have the following

**Proposition:** A pure state is being represented in $\mathcal{H}_J$ by a single real ray.

To see this consider a pure state, given by an idempotent $P_o$. We can take $P_o$ also as the first basis vector of $\mathcal{H}_J$, with the other basis vectors denoted by $P_i$. Then any “root” $B$ of $P$ is of the form $B = \sum_{i=1}^n \lambda_i P_i$ with $\sum \lambda_i^2 = 1$. So we have for the state $P_o = B^2 = \sum_{i=o}^n \lambda_i^2 P_i$, i.e. we get $\lambda_o = \pm 1$ and $P_{i>o} = 0$. 

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4 Jordan-Schrödinger Equation

In order to write down a time evolution equation for state vectors (i.e. a Schrödinger equation) we take the associative case as a guideline. The time evolution equation for density matrices in a $C^*$-algebra is given by

$$\dot{\rho} = -i[H, \rho] = -K_{iH} \rho$$

where $K_{iH} = i[H, .]$ is the (inner) derivation determined by the Hamiltonian $H$, a self-adjoint element of the algebra. In the case of Jordan algebras inner derivations are given by the associator \[4, 20\]

$$[R, A, S] = (R \circ A) \circ S - R \circ (A \circ S)$$

which acts as a derivation on its second argument. In the case of special Jordan algebras this can be expressed by a double commutator

$$[R, A, S] = [A, [R, S]]$$

and the connection with the associative case is made by the fact that in a $W^*$-algebra every (self-adjoint) element $H$ can be expressed by a finite sum of commutators \[20\]

$$H = i \sum_{j=1}^{N} [R_j, S_j]$$

So we have as a (Jordan-algebraic) evolution equation for density matrices (for the simple case of only one associator)

$$\dot{\rho} = [R, \rho, S] = [T_S, T_R] \rho$$

the latter term being the representation of the associator in $\Omega(A)$, the multiplication envelope of $A$ \[12\]. The solution of this is given by

$$\rho_t = e^{i[T_S, T_R]} \rho_0$$

The next step is to derive a Schrödinger-like equation for the state vectors. We recall again that we can write any density matrix as $\rho = B^2$ for some $B \in A_2$. This means

$$\dot{\rho} = 2B \circ \dot{B}$$

On the other hand

$$[R, \rho, S] = [R, B^2, S] = 2B \circ [R, B, S]$$

Comparison yields

$$\dot{B} = [R, B, S] = [T_S, T_R]B$$

which can be, again, extended to all of $\hat{H}_J$ to yield

$$\dot{v} = [R, v, S] = [T_S, T_R]v$$
This Jordan-Schrödinger equation has the same form as the evolution equation for density matrices. Its solution is similarly given by

$$v_t = e^{t[T_S,T_R]} v_0 .$$

Contact with the usual (associative) form is made if we take the Jordan algebra to be the self-adjoint part of a $C^*$-algebra. For the state vectors we get

$$\dot{v} = [T_S, T_R] v = -i K_H v = -i Hv + \nu H .$$

The solution is

$$v_t = e^{t[T_S,T_R]} v_0 = e^{-itK_H} v_0 = e^{-itK_H} v_0 e^{itH} .$$

We saw earlier that unitary transformations from the right do not change the state. Therefore, this solution describes, in the case of a JC-algebra, the same time evolution of a state as the usual, associative formulation, i.e. we have for the density matrix $\rho = B^2$:

$$\rho_t = B^2_t = e^{-itH} B_o e^{itH} e^{-itH} B_o e^{itH} = e^{-itH} B^2_o e^{itH}$$

which again gives

$$\rho_t = e^{-itH} \rho_0 e^{itH} .$$

We see, therefore, that it is indeed possible to formulate a Hilbert space version of Jordan algebraic quantum mechanics. The approach not only has, in the case of JC-algebras, the same content as the usual, associative formulation but also, without additional effort, incorporates the exceptional Jordan algebra $H_3(O)$.

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