On the Nonuniform Quantum Turbulence in Superfluids

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The problem of quantum turbulence in a channel with an inhomogeneous counterflow of superfluid turbulent helium is studied. The counterflow velocity $V_{ns}^+(y)$ along the channel is supposed to have a parabolic profile in the transverse direction $y$. Such statement corresponds to the recent numerical simulation by Khomenko et al. [Phys. Rev. B 91, 180504 (2015)]. The authors reported about a sophisticated behavior of the vortex line density (VLD) $L(r,t)$, different from $L \propto V_{ns}^+(y)^2$, which follows from the naive, straightforward application of the conventional Vinen theory. It is clear, that Vinen theory should be refined by taking into account transverse effects and the way it ought to be done is the subject of active discussion in the literature. In the work we discuss several possible mechanisms of the transverse flux of VLD $L(r,t)$ which should be incorporated in the standard Vinen equation to describe adequately the inhomogeneous quantum turbulence (QT). It is shown that the most effective among these mechanisms is the one that is related to the phase slippage phenomenon. The use of this flux in the modernized Vinen equation corrects the situation with an unusual distribution of the vortex line density, and satisfactory describes the behavior $L(r,t)$ both in stationary and nonstationary situations. The general problem of the phenomenological Vinen theory in the case of nonuniform and nonstationary quantum turbulence is thoroughly discussed.

I. INTRODUCTION.

The question of evolution of the vortex line density (VLD) $L(r,t)$ of the vortex tangle (VT) is the key issue in the macroscopic theory of quantum turbulence (QT). Although the VLD is a rough characteristic of the QT, it is responsible for many (mainly hydrodynamic) phenomena in superfluids and the knowledge of its exact dynamics is very important for an adequate interpretation of various experiments.

Long ago Vinen [2] suggested that the rate of change of VLD $\partial L(t)/\partial t$ can be described in terms of only the quantity $L(t)$ itself (and also other, external parameters, such as the counterflow velocity $V_{ns}$ and the temperature). He called this statement as a self-preservation assumption. The corresponding balance equation for the quantity $L(r,t)$, the so called Vinen equation, reads:

$$\frac{\partial L}{\partial t} = \alpha_V |V_{ns}| L^{3/2} - \beta_V L^2.$$  (1)

Here $\alpha_V$ and $\beta_V$ are the parameters of the theory, $\alpha_V$ is close to the mutual friction coefficient $\alpha$, $\beta_V$ is of the order of the quantum of circulation $\kappa$. Throughout its long history, the Vinen equation has undergone various improvements and modifications (see e.g. [3], [4], [5], [6], [7]) although at present the form (1) is mainly used.

One of serious problems, is the application of the Vinen theory to complicated situations, in particular to inhomogeneous flows (for recent papers see e.g. [8], [9], [10], [11], [12]). In the cited papers the authors, analyzing numerically the steady counterflowing helium in an inhomogeneous channel flow, obtained a very specific behavior of the VLD $L(r,t)$, which cannot be interpreted in terms of equation (1). Thus, Khomenko et al. [8] observed that the VLD field is concentrated near the side walls. Quite similar behavior was observed in the work by Yui et al. [10].

Analyzing the obtained results, the authors of the paper [8] proposed, that the first term on the right hand side of the Vinen equation (the so called production term) has the structure $\propto |V_{ns}|^3 L^{1/2}$, a combination that has never been discussed before. This conclusion was the subject of a polemics between the authors of the article [8] and the author of this paper (see [13] and [14]).

In the present paper, I would like to digress from the content of the mentioned polemics, and to present our view on the macroscopic behavior of the VLD in inhomogeneous flows (referring to numerical results of the work [8]). In the study I retain the conventional form of the production term in the Vinen equation (1).

In short, the results of work [8] can be formulated as follows. In a rectangular channel $2 \times 0.05$ cm wide, a parabolic counterflow $V_{ns}^+(y) = V_0(1 - (y/0.05)^2)$ is applied in $x$ direction. The periodic conditions were assumed in all directions. The resulting distributions of the dimensional VLD, the normal and counterflow velocities $L(y), V_n(y), V_0(y)$ are presented in Fig. 1.

If one applies straightforwardly the well-known relation $L = \gamma^2 V_{ns}^+(y)^2 \approx 2 \times 10^{-2} V_{ns}^+(y)^2$ (here $\gamma = \alpha_V/\beta_V$), which immediately arises from equation (1), then the dimensionless $L$ should be about $2 \times 10^{-2} V_{ns}^+(y)^2$, which essentially exceeds the value obtained in [8]. Another striking feature is that the profile $L(y)$ is radically different from the quadratic velocity profile $L \propto (V_{ns}^+(y))^2$.

In the paper we develop an approach explaining this unusual (from the point of view of the naive use of the Vinen theory) behavior of the VLD $L(y)$. In the inhomogeneous situation the Vinen equation should be corrected to include the transverse spatial effects. In particular we offer to incorporate into classic Vinen theory an addi-
tional space flux $\mathbf{J}(r,t)$ of the VLD, which redistributes the quantity $\mathcal{L}(y)$ in the $y$ direction. It is clear that a transverse gradient of the flux $\partial J_y(r,t)/\partial y$ should be added into the balance equation (1).

In the next Sec. II we discuss several mechanisms of these possible fluxes, derive mathematical expressions and compare contributions from them. In Sec. III we present numerical solutions for stationary and nonstationary cases and compare the results with the numerical data of paper [8]. In Sec. IV we discuss the problem of nonuniform and unsteady quantum turbulence and the Vinen phenomenological theory. The Conclusion is devoted to a discussion of the results and probable generalizations of the presented approach.

II. VORTEX-LINE DENSITY FLUX

Let’s describe various ideas on the transverse vortex-line density flux $\mathbf{J}(r,t)$ in inhomogeneous flows/counterflows of superfluid helium. As it was mentioned above, the first remark in this respect had been made by Vinen himself in the context of the possible influence of the channel width [2]. Unfortunately, no advanced theory had been supplemented. It is clear that the most general expression for the flux of quantity $\mathcal{L}$ is $\mathbf{J}(r,t) = \mathbf{L} \nabla \mathcal{L}$, where $\mathbf{V}_L$ is the macroscopic local velocity of the vortex tangle (see explanations in papers [15,16,4]). However, unless we don’t have a general expression for $\mathbf{V}_L$ as a function (functional) of quantity $\mathcal{L}$, we can not ascertain a closure procedure, i.e. obtain a description of the vortex tangle dynamics in terms of the VLD itself. This procedure is not uniquely defined and admits different approaches.

Thus, in the cited paper [8] the authors proceeded from the following microscopic expression for the transverse flux $J_{\text{micro}}$

$$J_{\text{micro}} = \frac{1}{\Omega} \int |\mathbf{V}_n(y)| s_x' d\xi = \frac{\alpha}{\Omega} \int |\mathbf{V}_n(y)| s_x' d\xi. \quad (2)$$

Here the integration is performed over the whole vortex line configuration, so it should be understood as an integration along each vortex loops constituting the vortex tangle and summation over all loops, i.e.

$$\int d\xi \to \sum_j \int_0^{L_j} d\xi_j.$$

The quantity $\Omega$ is the total volume, $\alpha$ is the mutual friction coefficient. The authors of work [8] calculated the quantity $\alpha$ in numerical simulation and concluded that the macroscopic expression

$$J_{Vb}(r,t) = \frac{\alpha}{2k} C_{\text{flux}} \frac{\partial \mathcal{V}_n^2}{\partial y}, \quad (3)$$

best corresponds to the microscopic flux (2). The quantity $C_{\text{flux}}$ is a constant, determined from numerical simulations. Another mechanism, frequently discussed in the problems of nonuniform flow, is related to the diffusion flux [16,17]. That mechanism is not connected with mutual friction, and realized by the emission of vortex loops, (see, e.g., [18,19]). The diffusion flux can be written as follows

$$\mathbf{J}_{\text{diff}}(r,t) = D \nabla \mathcal{L}, \quad (4)$$

where the diffusion coefficient is estimated as $D = 2 \times 10^{-3} \text{ cm}^2/\text{s}$.

The next contribution, which we consider here, is related to the so called phase slippage phenomenon. This phenomenon implies appearance of additional the chemical potential $\nabla \mu$, and accordingly the mutual friction when the crossing by the vortices of the main flow. This effect is especially important for monitoring the quantization of vortices. We will use the corresponding technique to describe the transverse flux of VLD $J_y(y,t)$. To find an analytical expression for $J_y(y,t)$, consider the following equation (see [20,21,22])

$$\mathcal{A} = \int (\hat{s}(\xi) \times s'(\xi)) d\xi. \quad (5)$$

The right-hand side of (5) is a net area, swept out by the motion of the line elements. Therefore, the $x$ -component of vector $\mathcal{A}$ is simply the rate of phase slippage (without the factor $k$) caused by the transverse motion of the vortex lines (see [22]). It is important, however, that the sign of the $x$ -component of the vector $\mathcal{A}$ does not depend on the direction of motion of vortex line segments (either in the positive or in the negative directions along axis $y$). It makes no differences in the calculation of the phase slippage, and accordingly the

\[ FIG. 1: \] (Color online) Prescribed parabolic normal velocity profile $V_n$ (- -), the resulting counterflow profile $V_{ns}$ (---) and the resulting profile of $\mathcal{L}(y)$ (-) in dimensionless unites, $T = 1.6 \text{ K}$ (from paper by Khomenko et al. [8])
additional drop in the chemical potential $\nabla \mu$, but it is essential for our purposes to determine flux $J_{ps}(y, t)$ of the VLD $L$ to the side wall. To overcome this problem we assume that all the vortex filaments are closed loops, so the averaged fluxes in both directions are equal. Therefore, the required transverse flux $J_{ps}(y, t)$ of the VLD $L$ is just half of the $x$-component of the vector $A$. Taking velocity of elements $\dot{s}(\xi)$ in the form of the local induction approximation (see e.g. [2]), we arrive at the following expression

$$J_{ps}(r, t) = \frac{1}{2} \int \left( [\alpha s' \times (V_{ns} - \beta (s' \times s'')] \right) \times s'(\xi) \, d\xi.$$

(6)

Here the combination $\dot{s}_I = \beta (s' \times s'')$ is the self-induced velocity of the line elements in the local induction approximation.

To move further we have to introduce the closure procedure and to express the right hand side of Eq. (6) via quantities $L$ and $V_{ns}$. It corresponds to the self-preservation assumption expressed by Vinen, that the macroscopic dynamics of the vortex tangle depends only on the VLD $L(t)$. The other, more subtle characteristics of the vortex structure, different from $L$, must adjust to it. In particular, the first contribution, containing the external counterflow velocity can be written as $\alpha I_I L |V_{ns}|$, where $I_I$ is the structure parameter of the vortex tangle, introduced by Schwarz [8]. The last term in Eq. (6) with the self-induced velocity can be expressed as $\alpha I I \beta L L^{-1/2}$, where $I_I$ is another structure parameter. Usually at this point the substitution $L^{1/2} = \gamma |V_{ns}|$ is used, and both contributions are reduced to a combination

$$J_{ps,1}(r, t) = \frac{1}{2} \alpha (I_I - \gamma \beta I_I) L |V_{ns}|.$$

(7)

Being multiplied by $\rho_{h, v}$ this expression (up to a factor 1/2) coincides with the formula for mutual friction. This is not surprising, because it is well known from the vortex dynamics that a vortex crossing the channel transfers the momentum to the main flow (see [23]). Therefore the final expression should be proportional to $V_{ns}$ and the whole scheme becomes self-consistent. But this above consideration concerns only homogeneous or near - homogeneous cases. In the highly inhomogeneous situation, which we are interested in here, the simple relations such as $L^{1/2} = \gamma |V_{ns}|$, do not work and the question of determining the transverse flux remains open. A very similar problem of using the structure parameters of the vortex tangle also arises for nonstationary situations (see a related discussion in the review article [4]). This problem is very intriguing, and we decided to explore yet another version of the closure procedure, which leads to the following formula for the transverse flux

$$J_{ps}(y, t) = \alpha I_I L |V_{ns}| - \alpha I_I L^{3/2}.$$

(8)

Thus, we have obtained two forms for the transverse flux associated with the phase slippage mechanism. They are identical in case of an uniform flow, when $L^{1/2} = \gamma |V_{ns}|$, however, in inhomogeneous flow they differ and can result in different results.

Our further goal is to analyze the results on the nonequilibrium quantum turbulence obtained in the numerical work by Khomenko et al. [8], basing on supposition of the transverse flux of VLD $L(y)$. Using the conditions of their modeling and taking that $|V_{ns}| \sim 1 \text{ cm/s}$, $L \sim 10^4$ $1/\text{cm}^2$, $\alpha \sim 0.1$, $\partial / \partial y \sim 1/0.05$, we conclude that the most effective mechanism among those considered above, is the one related to the phase slippage mechanism. It exceeds other contributions almost by the order and further we will concentrate on the only this effect.

Beside the usual estimation and comparison of various fluxes written above we can appeal to the fact that neither Khomenko et al. flux $J_{Kb, v}$ nor the diffusion flux $J_{diff}$ are effective enough to produce the complicated spatial distribution of the vortex line density which was observed in paper [8] and is shown in Fig 1. As far as the Khomenko et al. flux $J_{Kb, vl}$ this problem was discussed in details in the paper [13] (Sec. IV).

The impact of the diffusion flux was studied in a recent work by Saluto et al. [11]. The authors observed that the influence of the vortex diffusion is focused on the local values of $L(y)$ rather than on the form of the spatial distribution VLD. Thus the diffusion term (4) is also small for this particular problem, although, being a second-order derivative, it would be essential for other situations. In this paper we will not consider this term.

### III. SOLUTIONS

Thus we introduced and discussed several mechanisms for the transverse flux of VLD and concluded that the most effective of them is associated with the phase slippage mechanism. A microscopic equation for this flux is given by Eq. (6), its macroscopic closure variants are given by the formulas (7), (8). Our goal now is to incorporate these terms into the Vinen equation (11)

$$\frac{\partial L}{\partial t} + \frac{\partial J_{ps}(y, t)}{\partial y} = \alpha_V |V_{ns}| L^{3/2} - \beta_V L^2,$$

(9)

and to study its solutions under the conditions that identical to those studied in the work by Khomenko et al. [8]. Namely, we have selected the temperature of system, the geometry and size of the of the channel, parabolic counterflow velocity $V_{ns}(y)$ coinciding with the ones accepted in their work. We study two cases, a stationary situation and a completely unsteady problem.

#### A. Stationary case, profile of VLD $L(y)$.

In Fig. 2 we displayed the VLD $L(y)$ profiles obtained through the numerical solution of equation (9) without the term $\partial L/\partial t$. The upper and lower images
correspond to different expressions for the transverse flux \( \mathcal{L} \). We have chosen the system temperature \( T = 1.6 \) K, the channel size \( 2 \times 0.05 \) cm, the parabolic counterflow velocity \( \mathbf{V}_{ns}(y) = 1.2(1 - (y/0.05)^2) \) cm/s, coinciding with the conditions adopted in the work \([8]\). Additionally, only half of the channel width is considered, namely \( 0 \leq y \leq 0.05 \) cm. The boundary condition \( \mathcal{L}(y = 0) = 1000 \) 1/cm\(^2\) had been taken from the result of paper \([8]\) and from the solution of the fully nonstationary problem (see below). It is noteworthy that they are very close to each other.

The most important (albeit expected) result is that the VLD profile does not really satisfy the standard Vinen relation \( \mathcal{L}(y) = \gamma^2 |\mathbf{V}_{ns}|^2 \). On the contrary, the vortex tangle is concentrated in the region closer to the side wall, (but not directly on the wall). This behavior can be understood qualitatively from the following considerations. The structure of flux expressed by the formula \([7]\) is that its maximal value is at the central parts \( y = 0 \) of the channel (due to the large value of the counterflow velocity \( \mathbf{V}_{ns} \)) and the VLD \( \mathcal{L} \) is intensively removed from this region. On the contrary, because of the vanishing of the counterflow velocity \( \mathbf{V}_{ns} \) on the side walls \( y = 0.05 \), the flux is almost extinguished, and \( \mathcal{L} \) does not penetrate into this region. Clearly, to support a stationary solution in the regions where \( \mathcal{L}(y) \neq \gamma^2 |\mathbf{V}_{ns}|^2 \), either the production or the decay (second) term on the right hand side of equation \([9]\) should prevail. Another remarkable result is that there is a very good agreement, both qualitative and quantitative, with the data of the paper \([8]\) depicted in Fig. \([1]\).

One more important result concerns the fundamental question of the use of the Schwarz’s relations for the structure parameters of the nonuniform quantum turbulence. In the lower picture of Fig. \([2]\) we presented the quantity \( \mathcal{L}(y) \) obtained in numerical solution of the equation \([9]\) with the transverse flux expressed by Eq. \([8]\), which includes an alternative variant of the structure parameter. It is easy to see that qualitatively solutions are very similar, although they are a bit different. This fact confirms the widespread view that the Vinen equation can be a good tool for studying rough engineering problems, although relevant approaches may require some fitting parameters. At the same time the whole Vinen macroscopic theory is not suitable for the investigation of the fine structure of the vortex tangle.

**B. Nonstationary case, development of quantum turbulence in the inhomogeneous counterflow.**

The rather elegant results are obtained when solving the full equation \([9]\), with the term \( \partial \mathcal{L}/\partial t \). This procedure faces the standard problem of initial conditions, typical for the Vinen theory. Equation \([9]\) is a balance relation between the growth and the disappearance of vortex lines. The mechanism of spontaneous appearance of vortices in the helium flow has not been built into this equation.

At present, there are various theories of the initial appearance of vortex filament, which can be divided into two groups. The first group offers the different mechanisms (tunnelling, fluctuation growth, etc.) of initial generation of vortices. Another group is based on the idea that in the helium permanently exists a background of remnant vortices. From the point of view of the phenomenological theory the former group can be taken into account by introducing the initiating term into the Vinen equation. In turn, the latter group should lead to some initial value of VLD \( \mathcal{L}(t = 0) = \mathcal{L}_{back} \) in the Vinen equation. The better agreement between experimental data on the propagation of intense heat pulses (generating vortices and interacting with these “own” vortices) and the corresponding numerical solution, was obtained when assuming the existence of an initial level of VLD \( \mathcal{L}_{back} \), whereas the introduction of the initiating term led to an unsatisfactory correlation with the experimental observations (see e.g. \([24]\)). Thus it may be surmised that this is an argument in favour of the theory of remnant vortices. Usually, the level of the remnant vorticity \( \mathcal{L}_{back} \) is estimated approximately as \( 10^2 - 10^3 \) 1/cm\(^2\).

The spatio-temporal behavior of VLD \( \mathcal{L}(y,t) \) obtained in the numerical solution of the equation \([9]\) with the nonstationary term \( \partial \mathcal{L}/\partial t \) is shown in Fig. \([3]\). The upper and lower images correspond to the different expressions for the flux \([7],[8]\). We again have chosen all conditions of work \([8]\). As for initial conditions we assume that the background vorticity \( \mathcal{L}_{back} = 1000 \) 1/cm\(^2\).
FIG. 3: (Color online) The spatio-temporal behavior of VLD $L(t,y)$ obtained in numerical solution of the equation $\square$. The upper and lower pictures correspond to different expressions for transverse flux $\square$. The obtained picture confirms all the conclusions on the behavior of the VLD $L(y,t)$, made in the previous paragraph, and demonstrates how the according scenario is developing in time. On a time slice of $2 \, \text{s}$ (It is probable graph, and demonstrates how the according scenario is upper and lower pictures correspond to different expression s behavior of the VLD $\square$ describe the macroscopic dynamics of statistical systems macroscopic vortex dynamics can be described in terms homogenious situations. theory as applied to the complex nonstationary and in-bilities we have chosen two variants, leading to different parameters. Bearing in mind to compare various possi-

is, in general, a difficult and delicate step. For instance, the usual gas dynamics variables, such as density, momentum and energy (per unit volume) are just the first moments of the distribution function of the Boltzmann’s kinetic theory. Higher moments relax to approach equilibirium much faster than do the first listed variables. This circumstance allows one to truncate an infinite hierarchy of the moment equations and obtain a closed description using the listed quantities.

Unfortunately in case quantum turbulence, the assumption of self-preservation is not motivated, the restriction to the only variable $L(t)$ is not justified, and, in general, the Vinen equation is not valid. Indeed, let us consider a very simple counterexample. Assume that the velocity $V_{ns}(s,t)$ changes instantly to the opposite. Since the Vinen-type equation include the absolute value of relative velocity $|V_{ns}(s,t)|$ magnitude, then formally the system remains unaffected by the change. This is wrong, of course. The structure of the VT, mean curvature, anisotropy and polarization parameters will become reorganized. That implies the violation of the self-preservation assumption, and dynamics of the VLD $L(t)$ depends on other, more subtle characteristics of the vortex structure, different from $L(t)$.

To clarify the situation, let us consider a way of derivation of VE from the dynamics of vortex filaments in the local induction approximation (see, e.g. [23]). It will suffice for the illustration sake. Integrating an equation for the change of the length of line element over $\xi$ inside a volume $\Omega$, Schwarz concluded that in the counterflowing helium II the quantity $L(t)$ obeys the equation (see [3])

$$\frac{\partial L}{\partial t} = \frac{\alpha V_{ns}}{\Omega} \int \langle s' \times s'' \rangle \, d\xi - \frac{\alpha^3}{\Omega} \int \langle |s''|^2 \rangle \, d\xi . \quad (10)$$

The quantity $L(t)$ is related to the first derivative $s'$ of the function $s(\xi)$, since $L(t) \propto \int |s'| d\xi$. The rate of change of $L(t)$ includes quantities involving the higher-order derivative $s''$, namely $\langle s' \times s'' \rangle$ and $\langle |s''|^2 \rangle$. In a steady-state, these higher-order quantities are are directly expressed via the VLD $L$ as $\langle s' \times s'' \rangle \propto I_1 L^{1/2}$ and $\langle |s''|^2 \rangle \propto c_2^2(T) L$. Here the $I_1, c_2(T)$ are temperature dependent parameters introduced by Schwarz [3]. But in the nonstationary situation $s''$ is a new independent variable, and one needs a new independent equation for it and for other quantities, related to curvature of line. This new equation, in turn, will involve higher derivatives $s''', s^{IV}$ and so on. This infinite hierarchy can be truncated if, for some reasons, the higher-order derivatives relax faster, than the low-order derivatives, and take their ”equilibrium” values (with respect to the moments of low order).

Strictly speaking, there are no theoretical grounds for assuming that the relaxation of higher moments is faster than that of the quantity $L(t)$. Thus, in general, no equation of the type $\partial L(t)/\partial t = F(L)$ exists! At the same time, in some (unclear) conditions, and with the use of additional arguments (see, [2]), the required equation can be written down. The attempt was successful, this theory

IV. NONUNIFORM QUANTUM TURBULENCE AND THE VIVEN PHENOMENOLOGICAL THEORY

In Sec. II we described the problems of the closure procedure for the microscopic equation for the flux $\square$ and questions of the choice of the form for the structure parameters. Bearing in mind to compare various possibilities we have chosen two variants, leading to different expressions $\square, \square$. In this regard, it seems appropriate to return to the basics of Vinen’s phenomenological theory as applied to the complex nonstationary and in-homogenous situations.

The main idea of the Vinen approach was the assumption of self-preservation , i.e. the suggestion that the macroscopic vortex dynamics can be described in terms of the quantity $L(t)$ only. Selecting a set of variables to describe the macroscopic dynamics of statistical systems

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explained a large number of hydrodynamic experiments, including the main experiment by Gorter and Mellink [20] (see, for details, the review by [27]). It concerned, however, only stationary or near-stationary situations. In a strongly unsteady case, the region of applicability of this equation is unclear, see the above counterexample with a sudden inversion of the counterflow velocity.

Meanwhile, it seems intuitively plausible that for slow changes (both in space and time) the assumption of self-preservation is valid. That was the starting point in the construction of the so-called Hydrodynamics of Superfluid Turbulence (HST), which was the unification of the Vinen equation and the classical two-fluid hydrodynamics (see, e.g., [15, 28, 29]). The HST equations have been applied to study a large number of hydrodynamic and thermal problems, including heat transfer and boiling in He II (see, e.g., [30, 31, 32, 33, 34, 35, 24]). The numerical and analytic results were in very good agreement with numerous experimental data. This fact pointed out that the Vinen equation is robust and is, in general, quite suitable for the unsteady hydrodynamic problems.

It follows from the results of this work that the situation with inhomogeneous flow is quite similar. This is confirmed by the curves depicted on the upper and lower images in Figures 2 and 3. In these images we display the results obtained from solutions of the Vinen equation (9) with different expressions (7, 8) for the transverse flux. The qualitative similarity and closeness of the quantitative solutions indicates again that the Vinen equation is rather insensitive to a particular choice of the transverse flux. This construction forces the vortex filaments to escape from the central part, at the same time does not allow them to touch the walls.

One of our results, important for the macroscopic theory of quantum turbulence concerns the structure functions of the vortex tangle, such as the parameters of anisotropy and polarization. Just like in the unsteady situation, the use of such parameters in the usual form, introduced by Schwarz, can only be done approximately and with reservations. This fact confirms the widespread view that the Vinen equation can be used to explore the rough, engineering problems (although the corresponding studies may require some fitting parameters), but it’s not suitable for the description of the fine structure of the vortex tangle.

We conclude by saying that the study of the inhomogeneous flow/counterflow of superfluids in the channel on the basis of the Vinen equation [11] requires the introduction of additional terms describing the transverse flux of the VLD $\mathcal{L}$ towards the side walls. The analysis demonstrated that the most efficient mechanism is related to the phase slippage mechanism. The corresponding solutions of the Vinen equation with the additional term in both stationary and nonstationary cases agree with observations obtained earlier in numerical simulations. They showed that the VLD $\mathcal{L}(y,t)$, as function of $y$ is concentrated in the domain near the side walls. The reason for this behavior is the special structure of the transverse flux. This construction forces the vortex filaments to escape from the central part, at the same time does not allow them to touch the walls.

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[20] M. Rasetti and T. Regge, Physica A: Statistical Mechanics and its Applications 80, 217 (1975).
[21] S. K. Nemirovskii, Phys. Rev. B 57, 5972 (1998).
[22] C. Swanson and R. Donnelly, J. Low Temp. Phys. 61, 363 (1985).
[23] S. K. Nemirovskii and M. Tsubota, Journal of Low Temperature Physics 113, 591 (1998).
[24] L. Kondaurova, V. Efimov, and A. Tsoi, Journal of Low Temperature Physics 187, 80 (2017), ISSN 1573-7357.
[25] K. W. Schwarz, Phys. Rev. B 18, 245 (1978).
[26] C. J. Gorter and J. H. Mellink, Physica 15, 285 (1949).
[27] J. Tough, Progress in Low Temperature Physics, Vol. 8 (North-Holland, Amsterdam, 1982).
[28] K. Yamada, S. Kashiwamura, and K. Miyake, Physica B: Condensed Matter 154, 318 (1989), ISSN 0921-4526.
[29] J. Geurst, Physica A: Statistical Mechanics and its Applications 183, 279 (1992), ISSN 0378-4371.
[30] W. Fiszdon, M. v. Schwerdtner, G. Stamm, and W. Poppe, Journal of Fluid Mechanics 212, 663 (1990).
[31] M. Murakami, S. Gotoh, N. Koshizuka, S. Tanaka, T. Matsushita, S. Kambe, and K. Kitazawa, Cryogenics 30, 390 (1990).
[32] M. Murakami and K. Iwashita, Computers & Fluids 19, 443 (1991).
[33] W. Poppe, G. Stamm, and J. Pakleza, Physica B: Condensed Matter 176, 247 (1992).
[34] A. N. Tsoi and M. O. Lutset, Inzh. Fiz. Zh. 51, 5 (1986).
[35] U. Ruppert, W. Z. Yang, and K. Luders, Jpn. J. Appl. Phys. 26, Suppl. 26 (1987).