A $W$-String Realization of the Bosonic String

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Abstract

It has recently been shown that the ordinary bosonic string can be represented by a special background of N=1 or N=2 strings. In this paper, it will be shown that the bosonic string can also be represented by a special background of $W$-strings.
By adding fermionic gauge fields to the bosonic string, it was recently shown that an N=1 superconformal algebra could be constructed [1], and that the cohomology of the corresponding N=1 BRST charge coincides with the cohomology of the original bosonic string [2,3]. Furthermore, it was shown that the N=1 prescription for calculating scattering amplitudes with this special choice of matter fields produces the usual bosonic string amplitudes [1,3], allowing one to view the bosonic string as a background of N=1 strings. It was speculated first by Vafa [1], and later by others [2,4], that the bosonic string might similarly be represented by a special background of $W$-strings.

In this paper, it will be shown that by adding bosonic gauge fields to the bosonic string, a $W_3$ algebra can be constructed, and the cohomology of the corresponding $W_3$ BRST charge is precisely the cohomology of the ordinary bosonic string. If it is assumed that $W$-moduli can be safely ignored in this special background, the resulting scattering amplitudes are just those of the ordinary bosonic string. These results easily generalize to the case of $W_2,a$ strings which contain one spin 2 generator and one spin s generator.

Over the past few years, it was shown that for a particular representation of the $W_3$ algebra, the corresponding $W_3$ string theory has a cohomology containing Ising model primary fields and has tree-level scattering amplitudes which contain Ising model correlators in their integrands [5]. These results suggest that this $W_3$ string can be thought of as a non-critical bosonic string coupled to a $c = \frac{1}{2}$ Ising model. The matter fields in this particular representation of the $W_3$ string consist of a scalar boson $\varphi$ with background charge $\sqrt{\frac{49}{8}}$ and $d$ scalar bosons $x^\mu$ ($\mu = 0$ to $d - 1$) whose background charge $\alpha^\mu$ satisfies $\alpha^\mu \alpha_\mu = (d - 25\frac{1}{2})/12$.

It will be shown in this paper that for the choice $d = 27$ and $\alpha^\mu = (0, ..., 0, 1/\sqrt{8})$, the cohomology of the BRST charge for this $W_3$ string, when expressed in terms of a new set of fields, corresponds to the cohomology of an ordinary bosonic string. This will be proven by showing that the effects of $x^{26}$ and $\varphi$ are precisely canceled by the spin $(3, -2)$ $W_3$ ghosts, $d$ and $e$. In other words, the $x^{26}$ and $\varphi$ fields can be completely gauged away using the $W_3$-transformations, leaving only the 26 $x^\mu$'s of the ordinary bosonic string. By
replacing the first 26 $x^\mu$'s by any other $c = 26$ system, this same proof can be used to show that any $c = 26$ matter system can be represented by a $W_3$ string.

The proof will consist of finding a field redefinition such that the BRST charge for this representation of the $W_3$ string takes the simple form:

$$Q = Q_B + \int dz fG$$

where $Q_B = \int dz(cT_B + c\partial cb)$ is the BRST charge for the ordinary bosonic string (or any other matter system whose central charge is 26) and $fG$ is a non-minimal term constructed out of $x^{26}, \varphi, d,$ and $e$ in such a way that $(f, g)$ are a conjugate pair of free fermions and $(F, G)$ are a conjugate pair of free bosons. Since $(f, g, F, G)$ appear only in this non-minimal term, the “quartet” argument of Kugo and Ojima can be used to prove that the cohomology of $Q$ is equivalent to the cohomology of $Q_B$.

The BRST charge for this particular representation of the $W_3$ string is given by

$$Q = \int dz[c(T^m + \frac{1}{2}T^{gh}) + e(W^m + \frac{1}{2}W^{gh})],$$

where

$$T^m = T_\varphi + T_x, \quad T_\varphi = -\frac{1}{2}(\partial \varphi)^2 - \sqrt{\frac{49}{8}}\partial^2 \varphi, \quad T_x = T_B - \frac{1}{2}(\partial x^{26})(\partial x^{26}) - \frac{i}{\sqrt{8}}\partial^2 x^{26},$$

$$T^{gh} = -2b\partial c - \partial b c - 3d\partial e - 2\partial d e,$$

$$W^m = -\frac{2i}{\sqrt{261}} \left[ \frac{1}{3}(\partial \varphi)^3 + \sqrt{\frac{49}{8}}\partial \varphi \partial^2 \varphi + \frac{1}{3} \frac{49}{8} \partial^3 \varphi + 2\partial \varphi T_x + \sqrt{\frac{49}{8}}\partial T_x \right],$$

$$W^{gh} = -\partial d c - 3d\partial c - \frac{8}{261}[\partial(b e T^m) + b \partial e T^m]$$

$$+ \frac{25}{1566}(2e\partial^3 b + 9de \partial^2 b + 15\partial^2 e \partial b + 10\partial^3 e b).$$

The first step in simplifying $Q$ is to use the field redefinition of reference [6]:

$$\tilde{c} = c + \frac{7\sqrt{58}i}{174}\partial e - \frac{8}{261}b\partial ee - \frac{4\sqrt{29}i}{87}\partial \varphi e,$$

$$\tilde{d} = d + \frac{7\sqrt{58}i}{174}\partial b - \frac{8}{261}b b e e + \frac{4\sqrt{29}i}{87}\partial \varphi e, \quad \tilde{\varphi} = \varphi - \frac{4\sqrt{29}i}{87}b,$$
which allows $Q$ to be written in the simpler form:

$$Q = Q_0 + Q_1$$

where

$$Q_0 = \int dz [\tilde{c}(T_B - \frac{1}{2}(\partial x^{26})^2 - \frac{i}{\sqrt{8}} \partial^2 x^{26} + T_{\phi} + T_{\bar{d}}) + \tilde{c}\partial\bar{c}],$$

$$Q_1 = \int dz \left[-\frac{4\sqrt{25i}}{261}e \left(2(\partial \tilde{\phi})^3 + \frac{42}{\sqrt{8}} \partial^2 \tilde{\phi}\partial\tilde{\phi} + \frac{19}{4} \partial^3 \tilde{\phi} + 9\partial\tilde{\phi}\tilde{d}\partial e + \frac{21}{\sqrt{8}} \partial\tilde{d}\partial e \right)e\right].$$

In $Q_0$, the stress tensors for the redefined fields are the same as those of the original fields.

The next step in simplifying $Q$ is to define new fields [7]

$$\phi_1 = -3\rho - i\sqrt{8}\tilde{\phi}, \quad \phi_2 = -i\sqrt{8}\rho + 3\tilde{\phi}$$

where $\rho$ comes from bosonizing the $W_3$ ghosts in the standard way:

$$\tilde{d} = e^{-i\rho}, \quad e = e^{i\rho}.$$  

Since $\phi_1$ and $\phi_2$ have free-field OPE’s, it is easy to show that

$$Q_0 = \int dz [\tilde{c}(T_B - \frac{1}{2}(\partial x^{26})^2 - \frac{i}{\sqrt{8}} \partial^2 x^{26} + T_{\phi_1} + T_{\phi_2}) + \tilde{c}\partial\bar{c}],$$

where

$$T_{\phi_1} = -\frac{1}{2}(\partial\phi_1)^2 - \frac{i}{2} \partial^2 \phi_1, \quad T_{\phi_2} = -\frac{1}{2}(\partial\phi_2)^2 - \frac{1}{\sqrt{8}} \partial^2 \phi_2.$$  

$Q_1$ can be checked to take the following form when expressed in terms of $\phi_1$ and $\phi_2$ [7]:

$$Q_1 = \int dz e^{\sqrt{8}\phi_2} e^{-4i\phi_1} \partial^3 e^{i\phi_1}$$

where we have dropped an overall irrelevant factor of $-i\sqrt{58}/1044$ from the expression for $Q_1$. This can be simplified further by defining a new field $\hat{\phi}_1$ in terms of $\phi_1$ by means of the relations [7]

$$e^{i\phi_1} = e^{-i\phi_1}$$

$$i\partial e^{-i\phi_1} = -e^{i\phi_1}.$$
By taking repeated operator products of $e^{i\hat{\phi}_1}$ with itself it can be shown that, in terms of $\hat{\phi}_1$ and $\phi_2$, $Q_1$ takes the simple form [7]

$$ Q_1 = \int dz e^{3i\hat{\phi}_1} e^{\sqrt{8}\phi_2}. \quad (15) $$

Furthermore, because all fields satisfy free-field OPE’s, $Q_0$ remains of the form

$$ Q_0 = \int dz [\hat{c}(T_B - \frac{1}{2}(\partial x^{26})^2 - \frac{i}{\sqrt{8}}\partial^2 x^{26} + T_{\hat{\phi}_1} + T_{\phi_2}) + \hat{c}\partial\hat{c}b], \quad (16) $$

where $T_{\hat{\phi}_1}$ is the same stress tensor as $T_{\hat{\phi}_1}$.

The last step in simplifying $Q$ is to combine $x^{26}$, $\hat{\phi}_1$, and $\phi_2$ into a conjugate pair of free fermions ($f, g$) and a conjugate pair of free bosons ($F, G$). This is done by defining

$$ f = e^{iw}, \quad g = e^{-iw}, \quad F = \partial(e^{-iy})e^{-\sigma} \quad G = e^{iy}e^{\sigma}, \quad (17) $$

where

$$ w = \hat{\phi}_1 - \frac{i}{\sqrt{2}}\phi_2 - \frac{1}{\sqrt{2}}x^{26}, \quad y = \hat{\phi}_1 - \frac{i}{\sqrt{2}}\phi_2 + \frac{1}{\sqrt{2}}x^{26}, \quad \sigma = i\hat{\phi}_1 + \sqrt{2}\phi_2. \quad (18) $$

The coefficients in equation (18) have been chosen such that

$$ Q_1 = \int dz fG. \quad (19) $$

Furthermore, since $f, g, F$, and $G$ are bosonized in the standard way for free fermions and free bosons, it is easy to show that

$$ Q_0 = \int dz \left[ \hat{c}\left(T_B + \frac{1}{2}(\partial fg - f\partial g) + \frac{1}{2}(\partial FG - F\partial G)\right) + \hat{c}\partial\hat{c}b\right]. \quad (20) $$

Finally, by defining

$$ \hat{f} = f + \hat{c}\partial F + \frac{1}{2}\partial\hat{c}F, \quad \hat{G} = G + \hat{c}\partial g + \frac{1}{2}\partial\hat{c}g, \quad \hat{b} = b + \frac{1}{2}F\partial g - \frac{1}{2}g\partial F, \quad (21) $$

the BRST charge for this particular representation of the $W_3$ string takes the form

$$ Q = Q_0 + Q_1 = \int dz [(\hat{c}T_B - \hat{c}\partial\hat{c}b) + \hat{f}\hat{G}]. \quad (22) $$
Since \((\hat{f}, g, F, \hat{G})\) only appear in the last term of equation (22), the “quartet” argument of Kugo and Ojima can be used to prove that all dependence of physical states on these extra fields can be gauged away, and therefore the cohomology of \(Q\) is simply the cohomology of the BRST charge for the bosonic string, \(Q_B = \int dz (\partial \tilde{b} + \tilde{c} \partial \tilde{c})\). Furthermore, since the field redefinitions of equations (6), (9), (14) and (21) preserve all OPE’s, correlation functions of physical vertex operators of this \(W_3\) string are the same as correlation functions of the corresponding physical vertex operators of the bosonic string. Therefore, assuming that the role of \(W\)-moduli can be ignored, the resulting scattering amplitudes are just those of the ordinary bosonic string.

Note that the “quartet” argument of Kugo and Ojima depends on the assumption that all \(W_3\)-string states can be constructed from the vacuum using creation modes of the \((\hat{f}, g, F, \hat{G})\) fields and creation modes of the bosonic string matter and ghost fields. Although this includes all states which are normalizable using the standard free-field norm of the \((\hat{f}, g, F, \hat{G})\) fields, it does not include all states which are normalizable using the original norm defined for the \((x^{26}, \phi, d, e)\) fields. For example, there are many states which are normalizable in terms of the original norm which can not be constructed from a vacuum using only creation modes of the \((\hat{f}, g, F, \hat{G})\) fields and creation modes of the bosonic string matter and ghost fields. In fact of all the physical vertex operators described in reference [6], only the operator \(cV\) for \(k^{26} = 0\) can be constructed in this way. For this reason, the cohomology of \(Q\) which was found in references [6,8] using normalizable states built out of the \((x^{26}, \phi, d, e)\) fields is much bigger than the cohomology found in this paper using normalizable states built out of the \((\hat{f}, g, F, \hat{G})\) fields.

This \(W_3\) string realization of the bosonic string is easily generalized to the \(W_{2,s}\) string. In reference [7] it was explained how, starting from the bosonic field \(\tilde{\phi}\) and bosonized ghost \(\rho\) of the \(W_{2,s}\) string, we can define new fields

\[
\begin{align*}
\phi_1 &= -s \rho - i \sqrt{s^2 - 1} \tilde{\phi} \\
\phi_2 &= -i \sqrt{s^2 - 1} \rho + s \tilde{\phi}
\end{align*}
\]

in terms of which the non-trivial part \(Q_1\) of the BRST charge becomes

\[
Q_1 = \int dz e^{\sqrt{s^2 - 1} \phi_2} e^{-i(s+1)\phi_1} \partial \phi e^{i\phi_1}.
\]
We can rewrite this expression in terms of a field $\hat{\phi}_1$ defined by equation (14), to obtain

$$Q_1 = \int dz \exp\{\sqrt{s^2 - 1}\phi_2 - is\hat{\phi}_1\},$$

and then the quartet mechanism can be used to show that the cohomology of the BRST charge is the same as that of the bosonic string.

In this paper, we have found a representation of the $W_3$ string that reduces to the bosonic string; namely, its physical states and scattering amplitudes are those of the bosonic string. Thus the bosonic string can be viewed as a special background of the $W_3$ string, as well as the N=1 and N=2 strings. It would seem likely in view of these results that the bosonic string can be embedded into any string which is based on a symmetry which includes the Virasoro algebra.

Although this might appear surprising, it is possible that the strong consistency conditions of string theory inevitably force a string to be the bosonic string once it has $c=26$ matter tensored with an appropriate choice of gauge fields. A potential analogy worth bearing in mind is the situation that occurs in the theory of non-linear realizations; given a classical theory invariant under a rigid symmetry group, we can promote it to be invariant under any larger symmetry group which contains the original group by introducing the appropriate Goldstone bosons [9].

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