Convergence rate of solutions toward stationary solutions to the isentropic micropolar fluid model in a half line

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Abstract

In this paper, we study the asymptotic behavior of solutions to the initial boundary value problem for the one-dimensional compressible isentropic micropolar fluid model in a half line \( \mathbb{R}_+ := (0, \infty) \). We mainly investigate the unique existence, the asymptotic stability and convergence rates of stationary solutions to the outflow problem for this model. We obtain the convergence rates of global solutions towards corresponding stationary solutions if the initial perturbation belongs to the weighted Sobolev space. The proof is based on the weighted energy method by taking into account the effect of the microrotational velocity on the viscous compressible fluid.

Key words. isentropic micropolar fluid, stationary solutions, convergence rate, weighted energy method.

AMS subject classifications. 34K21, 35B35, 35Q35.

1 Introduction

The 1-D compressible viscous micropolar fluid model in the half line \( \mathbb{R}_+ := (0, +\infty) \) reads in Eulerian coordinates:

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + p(\rho)_x = \lambda u_{xx}, \\
(\rho \omega)_t + (\rho u \omega)_x + \mu \omega = \nu \omega_{xx}.
\end{cases}
\]

(1.1)

Here the unknown functions \( \rho, u \) and \( \omega \) represent the density, the velocity and microrotational velocity, respectively. The pressure \( p(\rho) = K \rho^\gamma \), with the adiabatic exponent \( \gamma \geq 1 \) and the gas constant \( K > 0 \). The positive constants \( \lambda, \mu \) and \( \nu \) are the viscosities. The model of micropolar fluid was first introduced by Eringen [13] in 1966. This model can be used to describe the motions of a large variety of complex fluids consisting of dipole elements such as the suspensions, animal blood, liquid crystal, etc. For more physical background on this model, we refer to [14, 24].

Much attention has been paid to the compressible micropolar fluid model by many mathematicians in the last several decades. For the isentropic case, Chen in [1] investigated the global existence of strong solutions to the one-dimensional compressible micropolar fluid model with initial vacuum. Later, Chen and his collaborators in [3] further studied the global weak solutions to the compressible micropolar fluid model with discontinuous initial data and vacuum. For three-dimensional compressible micropolar fluid model, the optimal decay rate in \( L^2 \) norm was studied by Liu and Zhang in

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with or without an external force. Later, Wu and Wang \cite{43} derived the pointwise estimates of the solution to the compressible micropolar fluid model and extended the optimal $L^p$ decay rate in \cite{22} to the $L^p$ optimal decay rate with $p > 1$. We also mention that there have been many results on the incompressible micropolar fluid system, see \cite{5} \cite{11} \cite{8} and the references therein.

For the non-isentropic case, Mujaković first investigated the one-dimensional compressible micropolar fluid model and obtained a series of results concerning the local-in-time existence, the global existence and the regularity of solutions to an initial-boundary value problem with homogeneous \cite{28} \cite{29} \cite{30} and non-homogeneous \cite{33} \cite{35} \cite{34} \cite{37} boundary conditions. The authors in \cite{16} \cite{31} \cite{32} \cite{36} \cite{42} studied the large time behavior of the solutions and the regularity of solutions to initial-boundary value problem and the Cauchy problem of the one-dimensional compressible micropolar fluid model. Duan in \cite{11} \cite{12} investigated the global existence of strong solutions for the one-dimensional compressible micropolar fluids. Chen and his collaborators in \cite{2} \cite{3} studied the blow up criterion of strong solutions to the three-dimensional compressible micropolar fluid model. Dražić and Mujaković in \cite{10} \cite{9} studied the regularity and large-time behavior of the spherical symmetry solutions for the three-dimensional compressible micropolar fluid model.

Recently, there have been a series of mathematical results in the literature to study of the stability of wave patterns for the compressible nonisentropic micropolar fluid model: Liu and Yin \cite{21} for stability of contact discontinuity for the Cauchy problem; Jin and Duan \cite{17} for the stability of rarefaction waves for the Cauchy problem; Yin \cite{14} for stability of stationary solutions for the inflow problem; Cui and Yin \cite{6} for stability of composite waves for the inflow problem; Cui and Yin \cite{7} for the stability and convergence rate of stationary solutions for the outflow problem. However, the articles mentioned above about the stability of waves are all based on the assumption of microrotation velocity $\omega = 0$ for the large time behavior. In this paper, we expect to study the asymptotic stability of stationary solutions to the one-dimensional compressible isentropic micropolar fluid model for the outflow problem without the assumption of $\omega = 0$ for the large time behavior.

Initial data for system \ref{1.1} is given by

$$
(\rho, u, \omega)(x, 0) = (\rho_0, u_0, \omega_0)(x), 
\inf_{x \in \mathbb{R}^+} \rho_0(x) > 0.
$$

We assume that the initial data at the far field $x = +\infty$ is constant, namely

$$
\lim_{x \to +\infty} (\rho_0, u_0, \omega_0)(x) = (\rho_+, u_+, \omega_+),
$$

In particular, to construct a classical solution of the micropolar fluid model \ref{1.1}$_3$, it is necessary to require that

$$
\omega_+ = 0.
$$

The boundary data for $u$ and $\omega$ at $x = 0$ is given by

$$
(u, \omega)(0, t) = (u_b, \omega_b), \quad \forall t \geq 0,
$$

where $u_b < 0, \omega_b \neq 0$ are constants and the following compatibility conditions hold

$$
u_0(0) = u_b, \quad \omega_0(0) = \omega_b.
$$

The assumption $u_b < 0$ means that fluid blows out from the boundary $x = 0$ with the velocity $u_b$. Thus this problem is called an outflow problem (see \cite{25}). The outflow boundary condition implies that the characteristic of the hyperbolic equation \ref{1.1}$_1$ for the density $\rho$ is negative around the boundary so that boundary conditions on $u$ and $\omega$ to parabolic equations \ref{1.1}$_2$ and \ref{1.1}$_3$ are necessary and sufficient for the wellposedness of this problem.
We guess that the large time behavior of solutions to the initial boundary value problem (1.1), (1.2), (1.3), (1.6) are the stationary solutions to (1.1) independent of a time variable $t$

$$
\begin{align*}
\begin{cases}
(\rho \overline{u})_x = 0, \\
(\rho \overline{u}^2)_x + p(\rho)\overline{u} = \lambda \overline{\omega}_{xx}, \\
(\rho \overline{\omega})_x + \mu \overline{\omega} = \nu \overline{\omega}_{xx},
\end{cases}
\end{align*}
$$

(1.7)

with the boundary data

$$
\inf_{x \in \mathbb{R}^+} \rho(x) > 0, \quad \lim_{x \to \infty} (\rho, \overline{u}, \overline{\omega})(x) = (\rho_+, u_+), \quad \overline{u}(0) = u_b < 0, \quad \overline{\omega}(0) = \omega_b \neq 0.
$$

(1.8)

Integrating (1.7) over $(x, \infty)$ for $x > 0$ yields

$$
\rho(x)\overline{u}(x) = \rho_+ u_+,
$$

(1.9)

which implies by letting $x \to 0^+$,

$$
u_+ = \frac{\rho(0)\overline{u}(0)}{\rho_+} = \frac{\rho(0)u_b}{\rho_+} < 0.
$$

(1.10)

Now define

$$
\tilde{\delta} = \max \{|\omega_b|, |u_b - u_+|\}, \quad \sigma = \min \{|r_1|, \xi_0\},
$$

(1.11)

where constants $r_1$ and $\xi_0$ are defined in Section 2.

Main results of the present paper are stated in the following theorems.

**Theorem 1.1.** The boundary value problem (1.7), (1.8) has a unique smooth solution $(\rho, \overline{u}, \overline{\omega})$ if and only if $M_+ \geq 1$ and $\chi_+ u_+ > u_b$.

(i) If $M_+ > 1$, then the solution $(\rho, \overline{u}, \overline{\omega})$ satisfies the estimates

$$
|\partial^k_x (\rho - \rho_+, \overline{u} - u_+, \overline{\omega} - \omega_+)| \leq C\tilde{\delta}e^{-\sigma x}, \quad \text{for } k = 0, 1, 2, \cdots,
$$

(1.12)

where $C$ and $\sigma$ are positive constants.

(ii) If $M_+ = 1$, then the solution $(\rho, \overline{u}, \overline{\omega})$ satisfies the estimates

$$
|\partial^k_x (\rho - \rho_+, \overline{u} - u_+)| \leq C\frac{(k+1)!\tilde{\delta}^{k+1}}{(k+1)!}, \quad |\partial^k_x (\overline{\omega} - \omega_+)| \leq C\tilde{\delta}e^{-\sigma x} \quad \text{for } k = 0, 1, 2, \cdots
$$

(1.13)

where $C$ and $\sigma$ are positive constants.

**Theorem 1.2.** Suppose that stationary solution $(\rho, \overline{u}, \overline{\omega})$ exists.

(i) Assume that $M_+ > 1$ holds. For an arbitrary positive constant $\vartheta$, there exist positive constants $\beta$ and $\varepsilon_1$ such that if $(1 + \beta x)^{\frac{1}{2}}(\rho_0 - \rho), (1 + \beta x)^{\frac{1}{2}}(u_0 - \overline{u}), (1 + \beta x)^{\frac{1}{2}}(\omega_0 - \overline{\omega})$ respectively belongs to the Lebesgue space $L^2(\mathbb{R}^+)$ and $\|\rho_0 - \rho, u_0 - \overline{u}, \omega_0 - \overline{\omega}\|_{1 + \beta + \tilde{\delta} \leq \varepsilon_1}$, then the initial boundary value problem (1.1), (1.2), (1.3), (1.6) has a unique solution $[\rho, u, \omega]$ verifying the decay estimate

$$
\|\rho - \rho_0, u - \overline{u}, \omega - \overline{\omega}\|_\infty \leq C(1 + t)^{-\frac{\vartheta}{\beta}}.
$$

(1.14)

(ii) Assume that $M_+ = 1$ holds. There exists a positive constant $\varepsilon_2$ such that if the initial data satisfies $\|[(1 + B x)^{\frac{1}{2}}(\rho_0 - \rho), (1 + B x)^{\frac{1}{2}}(u_0 - \overline{u}), (1 + B x)^{\frac{1}{2}}(\omega_0 - \overline{\omega})]\|_1 \leq \varepsilon_2$ for positive constants $B$ and $\theta$ satisfying $\theta \in [2, \theta^*]$ where $B$ and $\theta^*$ is respectively defined by (3.30) and

$$
\theta^*(\theta^* - 2) = \frac{4}{\gamma + 1}, \quad \text{and } \theta^* > 0,
$$

(1.15)

then the initial boundary value problem (1.1), (1.2), (1.3), (1.5), (1.6) has a unique solution $(\rho, u, \omega)$ verifying the decay estimate

$$
\|\rho - \rho_0, u - \overline{u}, \omega - \overline{\omega}\|_\infty \leq C(1 + t)^{-\frac{\theta^*}{\beta}}.
$$

(1.16)
If the microstructure of the fluid is not taken into account, that is to say the effect of the micro-
rotational velocity is omitted, i.e., $\omega = 0$, then equations (1.1) reduce to the classical Navier-Stokes
equations. So far, there have been a great number of mathematical studies about the outflow prob-
lem, impermeable wall problem and inflow problem for Navier-Stokes equations, please referring to
[15, 20, 27, 38, 39, 19] and the references therein. Under the assumption of microrotation velocity
$\omega = 0$ for the large time behavior, there also have been a series of mathematical results about the
above mentioned problems for the compressible nonsentropic micropolar fluid model, see [44, 6, 7]
and the references therein. Theorem 1.1 shows the unique existence of stationary solutions without
the assumption of $\omega = 0$ for the large time behavior. We point out that the stationary equa-
tion (1.7) can be divided into two independent stationary equations. One is the equation (2.1) which
is also the stationary equation of the isentropic Navier-Stokes equation. The other is the equation (2.3)
about the microrotation velocity $\omega$ which is a second order homogeneous linear ordinary differential
equation with constant coefficients. The detailed process about stationary solutions can be seen in
Section 2. Theorem 1.2 shows that the time asymptotic stability and convergence rates of stationary
solutions for the compressible isentropic micropolar fluid model without the assumption of $\omega = 0$
for the large time behavior.

The rest of the paper is arranged as follows. In Section 2, we prove the existence of the stationary
solution. In the main part Section 3, we give the a priori estimates on the solutions of the perturbative
equations for the supersonic case $M_+ > 1$ and transonic case $M_+ = 1$, respectively. The proof of
Theorem 1.2 is concluded in Section 4.

Notation: Throughout the paper, we denote positive constants (generally large) and (generally
small) independent of $t$ by $C$ and $c$, respectively. And the character “$C$” and “$c$” may take different
values in different places. $L^p = L^p(\mathbb{R}_+)$ ($1 \leq p \leq \infty$) denotes the usual Lebesgue space on $[0, \infty)$
with its norm $\| \cdot \|_{L^p}$, and when $p = 2$, we write $\| \cdot \|_{L^2(\mathbb{R}_+)} = \| \cdot \|$. $H^s(\mathbb{R}_+)$ denotes the usual $s$-th
order Sobolev space with its norm $\| f \|_{H^s(\mathbb{R}_+)} = \| f \|_s = (\sum_{i=0}^s \| \partial^i f \|_2^2)^{\frac{1}{2}}$. A norm with algebraic weight
is defined as follows:

$$
\| f \|_{\alpha,\beta,i} := \left( \int W_{\alpha,\beta} \sum_{j \leq i} (\partial^j f)^2 dx \right)^{\frac{1}{2}}, \quad i, j \in \mathbb{Z}, \quad i, j \geq 0,
$$

$$
W_{\alpha,\beta} := (1 + \beta x)^\alpha, \quad \alpha > 0.
$$

Note that this norm is equivalent to the norm defined by $\| (1 + \beta x)^{\frac{1}{2}} f \|_i$. The last subscript $i$ is often
dropped for the case of $i = 0$, i.e. $\| f \|_{\alpha,\beta} := \| f \|_{\alpha,\beta,0}$.

2 Existence of the stationary solution

In this section, we will prove the existence of the stationary solution to the boundary value problem
(1.7)-(1.8), which is stated in Theorem 1.1. Notice that the stationary problem (1.7)-(1.8) can be
divided into the following two independent stationary equations

$$
\begin{cases}
(\hat{\rho}u)_x = 0, \\
(\hat{\rho}u^2)_x + p(\hat{\rho})_x = \lambda \hat{u}_{xx},
\end{cases}
$$

(2.1)
with the boundary data

\[
\inf_{x \in \mathbb{R}^+} \hat{\rho}(x) > 0, \quad \lim_{x \to \infty} (\hat{\rho}, \hat{u})(x) = (\rho_+, u_+), \quad \hat{u}(0) = u_b < 0,
\]

and

\[
(\hat{\rho}\hat{\omega})_x + \mu \hat{\omega} = \nu \hat{\omega}_{xx}.
\]

with the boundary data

\[
\lim_{x \to \infty} \hat{\omega}(x) = 0, \quad \hat{\omega}(0) = \omega_b \neq 0.
\]

Now we firstly prove the existence of the stationary solution to the stationary problem (2.3)-(2.4). From (2.3) and (1.9), we have

\[
\rho_+ + u_+ + \hat{\omega}_x + \mu \hat{\omega} = \nu \hat{\omega}_{xx},
\]

which is a second order homogeneous linear ordinary differential equation with constant coefficients. The characteristic equation of (2.5) is given by

\[
\nu r^2 - \rho_+ + u_+ + r - \mu = 0.
\]

Then the roots of characteristic equation are

\[
r_1 = \frac{\rho_+ + u_+ - \sqrt{\rho_+^2 + u_+^2 + 4\nu\mu}}{2\nu}, \quad r_2 = \frac{\rho_+ + u_+ + \sqrt{\rho_+^2 + u_+^2 + 4\nu\mu}}{2\nu}.
\]

From (1.10), we deduce that \( r_1 < 0 \) and \( r_2 > 0 \). Then from the theory of the homogeneous linear ordinary differential equation ([1]), we get the general solution of (2.5) as follows

\[
\hat{\omega}(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants. Notice that \( r_1 < 0 \), \( r_2 > 0 \), and the boundary data (2.4), arbitrary constants \( C_1 \) and \( C_2 \) can be uniquely determined as follows \( C_1 = \omega_b \) and \( C_2 = 0 \). Hence the particular solution of (2.5) with boundary condition (2.4) is given by

\[
\hat{\omega}(x) = \omega_b e^{r_1 x}.
\]

Here we should note that the existence of the stationary problem (2.1)-(2.2) has been proved by Kawashima, Nishibata and Zhu in [20]. Here we summarize the results in [20]. We firstly define

\[
\hat{\chi}(x) := \frac{\hat{u}(x)}{u_+} = \frac{\rho_+}{\hat{\rho}(x)} > 0.
\]

Then integrating (2.1) over \((x, +\infty)\) and substituting (2.10) in the resulting equality, we get

\[
\lambda u_+ \hat{\chi}_x = F(\hat{\chi}),
\]

where

\[
F(\hat{\chi}) := K \rho_+^\gamma (\hat{\chi}^{-\gamma} - 1) + \rho_+ u_+^2 (\hat{\chi} - 1).
\]

\( \hat{\chi} \) satisfies boundary conditions

\[
\hat{\chi}(0) = \frac{u_b}{u_+} > 0, \quad \lim_{x \to +\infty} \hat{\chi}(x) = 1
\]

which can be derived by (2.2) and (1.10). We introduce Mach number \( M_+ \) at the far field \( x = +\infty \):

\[
M_+ = \frac{u_+}{c_+}, \quad \text{where} \quad c_+ = \sqrt{\rho_+^{-1}} = \sqrt{K \gamma \rho_+^{-1}}, \quad \text{is sound speed}.
\]

If \( M_+ > 1 \), the equation \( F(\hat{\chi}) = 0 \) has the distinct two roots \( \hat{\chi} = 1 \) and \( \hat{\chi} = \chi_c \) satisfying \( \chi_c < \hat{\chi}(0) \). If \( M_+ = 1 \), the equation \( F(\hat{\chi}) = 0 \) admits only one root \( \hat{\chi} = \chi_c = 1 \).
Lemma 3.1. (See [24]) The boundary value problem (2.11)−(2.13) has a unique smooth solution \( \tilde{\chi} \) if and only if \( M_+ \geq 1 \) and \( \chi_x < \tilde{\chi}(0) \). Moreover, if \( \tilde{\chi}(0) \leq 1 \), then \( \chi_x \geq 0 \).

(i) If \( M_+ > 1 \), then the solution \( \tilde{\chi}(x) \) satisfies the estimates

\[
|\partial_x^k(\tilde{\chi}(x) - 1)| \leq C|\tilde{\chi}(0) - 1|e^{-\xi_0 x}, \quad \text{for } k = 0, 1, 2, \cdots.
\]

where \( C \) and \( \xi_0 \) are positive constants.

(ii) If \( M_+ = 1 \), then the solution \( \tilde{\chi} \) is monotonically decreasing and satisfies the estimates

\[
|\partial_x^k(\tilde{\chi}(x) - 1)| \leq C\frac{|\tilde{\chi}(0) - 1|^{k+1}}{(1 + |\tilde{\chi}(0) - 1|x)^{k+1}}, \quad \text{for } k = 0, 1, 2, \cdots.
\]

Then we complete the proof of Theorem 1.1 from (2.9), (2.10) and Lemma 2.1.

3 Energy estimates

To prove Theorem 1.2, we use the energy method. Define the perturbation as

\[
\partial_t\varphi + u\partial_x\varphi + \rho\partial_x\psi = -\partial_x\tilde{\varphi} - \partial_x\tilde{\psi},
\]

\[
\rho(\partial_t\psi + u\partial_x\psi) + \partial_x[p(\rho) - p(\tilde{\rho})] = \lambda\partial_x^2\varphi - \partial_x^2(\tilde{u}\varphi + \rho\psi),
\]

\[
\rho(\partial_t\zeta + u\partial_x\zeta) + \mu\zeta = \nu\partial_x^2\zeta - \partial_x\omega(\tilde{u}\varphi + \rho\psi),
\]

with initial data

\[
[\varphi, \psi, \zeta](x, 0) = [\varphi_0, \psi_0, \zeta_0](x) = [\rho_0(x) - \tilde{\rho}(x), u_0(x) - \tilde{u}(x), \omega_0(x) - \tilde{\omega}(x)]
\]

and boundary condition

\[
\psi(0, t) = \zeta(0, t) = 0.
\]

In the paper, to prove Theorem 1.2 for brevity we only devote ourselves to obtaining the global-in-time \textit{a priori} estimates in the following. We look for the solution \([\varphi, \psi, \zeta](x, t)\) in the solution space \( X([0, +\infty)) \) which is defined as follows:

\[
X([0, T]) = \left\{ [\varphi, \psi, \zeta]|\varphi, \psi, \zeta \in C([0, T]; H^1(\mathbb{R}^+)), [\partial_x\varphi, \zeta] \in L^2([0, T]; L^2(\mathbb{R}^+)), \partial_x[\psi, \zeta] \in L^2([0, T]; H^1(\mathbb{R}^+)) \right\}
\]

for some \( 0 < T \leq +\infty \). Lemma 3.1 plays an important role in the proof of the \textit{a priori} estimates for supersonic case \( M_+ > 1 \) and transonic case \( M_+ = 1 \), respectively.

Lemma 3.1. (i) For any function \( h(\cdot, t) \in H^1(\mathbb{R}^+) \), there is a positive constant \( C \) such that

\[
\int_{\mathbb{R}^+} e^{-\sigma x}|h|^2 \, dx \leq C \left( h^2(0, t) + \|\partial_x h(t)\|^2 \right).
\]

(ii) Let \( k > 1 \). For any function \( h(\cdot, t) \in H^1(\mathbb{R}^+) \), there is a positive constant \( C \) such that

\[
\int_{\mathbb{R}^+} \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} h^2 \, dx \leq C\delta^k h^2(0, t) + C\delta^{k-1} \|\partial_x h\|^2.
\]
Proof. (i) (3.4) can be derived from the following Poincaré type inequality:

\[ |h(x,t)| \leq |h(0,t)| + x^\frac{1}{2} \|\partial_x h\|. \quad (3.6) \]

(ii) Letting \( k > 1 \) and using Poincaré type inequality (3.6), we compute as

\[
\int_{\mathbb{R}^+} \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} h^2 dx \leq \int_{\mathbb{R}^+} \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} (h^2(0,t) + x\|\partial_x h\|^2) dx \leq C\delta^k h^2(0,t) + C\delta^{k-1}\|\partial_x h\|^2. 
\]

\[ \square \]

### 3.1 The a priori estimates for \( M_+ > 1 \)

The key to the proof of our main Theorem 1.2 (i) is to derive the uniform a priori estimates of solutions to the initial boundary value problem \( (3.1), (3.2) \) and \( (3.3) \). For the convenience of stating the a priori assumption, we use the notations

\[ N_1(T) := \sup_{0 \leq t \leq T} \|(\varphi, \psi, \zeta)(t)\|_1 \]

**Proposition 3.1.** Assume the same conditions as in Theorem 1.2 (i) hold. Let \( \alpha, \beta \) and \( \kappa \) be positive constants. Suppose \( (\varphi, \psi, \zeta) \in X([0,T)) \) is a solution to \( (3.1), (3.2) \) and \( (3.3) \) which satisfies \( (1 + \beta x)\varphi, (1 + \beta x)^2 \psi, (1 + \beta x)^2 \zeta \in C([0,T];L^2(\mathbb{R}^+)) \) for a certain positive constant \( T \). For arbitrary \( \alpha \in [0, \vartheta] \), there exist positive constants \( C \) and \( \varepsilon_1 \) independent of \( T \) such that if \( N_1(T) + \delta + \beta \leq \varepsilon_1 \) is satisfied, it holds for an arbitrary \( t \in [0, T] \) that

\[
(1 + t)^{\alpha + \kappa} \||\varphi, \psi, \zeta(t)\|^2_2 + \int_0^t (1 + \tau)^{\alpha + \kappa} \||\varphi, \psi, \zeta(\tau)\|^2_2 d\tau \leq C(1 + t)^{\kappa} \left( \||\varphi_0, \psi_0, \zeta_0\|^2_2 + \||\varphi_0, \psi_0, \zeta_0\|^2_{\alpha, \beta} \right). \quad (3.8)
\]

**Lemma 3.2.** There exists a positive constant \( \varepsilon_1 \) such that if \( N_1(T) + \delta + \beta \leq \varepsilon_1 \), then

\[
(1 + t)^{\xi} \||\varphi, \psi, \zeta(t)\|^2_{\alpha, \beta} + \beta \int_0^t (1 + \tau)^{\xi} \||\varphi, \psi, \zeta(\tau)\|^2_{\alpha-1, \beta} d\tau + \xi \int_0^t (1 + \tau)^{\xi - 1} \||\varphi, \psi, \zeta(\tau)\|^2_{\alpha, \beta} d\tau + C\delta \int_0^t (1 + \tau)^{\xi} \||\partial_x \varphi\|^2(\tau) d\tau \leq C \||\varphi_0, \psi_0, \zeta_0\||^2_{\alpha, \beta} + \xi \int_0^t (1 + \tau)^{\xi - 1} \||\varphi, \psi, \zeta(\tau)\|^2_{\alpha, \beta} d\tau + C\delta \int_0^t (1 + \tau)^{\xi} \||\partial_x \varphi\|^2(\tau) d\tau \quad (3.9)
\]

holds for \( \alpha \in [0, \vartheta] \) and \( \xi \geq 0 \).

**Proof.** A direct computation by using \( (3.1) \) and \( (3.3)_1 \), we have the following identity

\[
[p\Phi(\rho, \hat{\rho})]_1 + [p\mu \Phi(\rho, \hat{\rho})]_x + (p(\rho) - p(\hat{\rho}))\psi x + \hat{u}_x[p(\rho) - p(\hat{\rho})] - p'(\hat{\rho})\varphi + \frac{p'(\hat{\rho})x}{\hat{\rho}} \varphi \psi = 0, \quad (3.10)
\]

where \( \Phi(\rho, \hat{\rho}) = \int_{\hat{\rho}}^\rho \frac{p(s) - p(\hat{\rho})}{s^2} ds \).

It is easy to see that \( \Phi(\rho, \hat{\rho}) \) is equivalent to \( |\varphi|^2 \), i.e.,

\[
c|\varphi|^2 \leq \Phi(\rho, \hat{\rho}) \leq C|\varphi|^2, \quad (3.11)
\]

since there exist positive constants \( c \) and \( C \) such that \( \rho \) and \( \hat{\rho} \) satisfying

\[
0 < c \leq \rho, \quad \hat{\rho} \leq C. \quad (3.12)
\]
Multiply (3.1) by \( \psi \) and (3.1) by \( \zeta \), we have the following identity
\[
\left( \frac{\rho}{2} \psi^2 + \frac{\rho}{2} \zeta^2 \right)_t + \left( \frac{\rho u}{2} \psi^2 + \frac{\rho u}{2} \zeta^2 \right)_x + (p(\rho) - p(\hat{\rho})) \psi - (\lambda \psi \psi_x + \nu \zeta \zeta_x)_x + \mu \zeta^2 + \lambda \psi^2 + \nu \zeta^2
\]
\[
= - \partial_x \hat{u}[\hat{u} \varphi \psi + \rho \psi^2] - \partial_x \hat{\omega}(\hat{u} \varphi \zeta + \rho \psi \zeta).
\]
(3.13)

Combining (3.3) and (3.10), we have
\[
\left[ \rho \Phi(\rho, \hat{\rho}) + \frac{\rho}{2} \psi^2 + \frac{\rho}{2} \zeta^2 \right]_t + \left[ \rho u \Phi(\rho, \hat{\rho}) + \frac{\rho u}{2} \psi^2 + \frac{\rho u}{2} \zeta^2 + (p(\rho) - p(\hat{\rho})) \psi - \lambda \psi \psi_x - \nu \zeta \zeta_x \right]_x
\]
\[
+ \mu \zeta^2 + \lambda \psi^2 + \nu \zeta^2
\]
\[
= - \partial_x \hat{u}[\hat{u} \varphi \psi + \rho \psi^2] + (p(\rho) - p(\hat{\rho})) \varphi - \partial_x \hat{\omega}(\hat{u} \varphi \zeta + \rho \psi \zeta) - \frac{p(\hat{\rho}) x \varphi \psi}{\hat{\rho}}.
\]
(3.14)

Multiplying (3.14) by \( W_{\alpha, \beta} = (1 + \beta x) \) defined in Notation, then we integrate the resulting equality over \( \mathbb{R}_+ \) to get
\[
\frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha, \beta} \left[ \rho \Phi(\rho, \hat{\rho}) + \frac{\rho}{2} \psi^2 + \frac{\rho}{2} \zeta^2 \right] dx + \int_{\mathbb{R}_+} W_{\alpha, \beta} \left[ \mu \zeta^2 + \lambda \psi^2 + \nu \zeta^2 \right] dx
\]
\[
- \alpha \beta \int_{\mathbb{R}_+} W_{\alpha-1, \beta} \left[ \rho u \Phi(\rho, \hat{\rho}) + \frac{\rho u}{2} \psi^2 + \frac{\rho u}{2} \zeta^2 + (p(\rho) - p(\hat{\rho})) \psi \right] dx - \rho(0, t) u_0 \Phi(\rho(0, t), \hat{\rho}(0))
\]
\[
= - \int_{\mathbb{R}_+} \underbrace{W_{\alpha, \beta} \left[ \partial_x \hat{u}[\hat{u} \varphi \psi + \rho \psi^2 + (p(\rho) - p(\hat{\rho})) \varphi - \partial_x \hat{\omega}(\hat{u} \varphi \zeta + \rho \psi \zeta) + \frac{p(\hat{\rho}) x \varphi \psi}{\hat{\rho}} \right]}_{J_1, \, J_2} dx.
\]
(3.15)

where we have used boundary condition \( \psi(0, t) = 0 \) and \( \zeta(0, t) = 0 \).

Now we estimate each term in (3.15). We decompose \( \rho \) as \( \rho = \varphi + (\hat{\rho} - \rho_+) + \rho_+ \), \( u \) as \( u = \psi + (\hat{u} - u_+) + u_+ \) and \( \omega = \omega = \zeta + \hat{\omega} \). Then we see, under the condition \( M_+ > 1 \) and \( u_+ < 0 \), that
\[
- \left[ \rho u \Phi(\rho, \hat{\rho}) + \frac{\rho u}{2} \psi^2 + \frac{\rho u}{2} \zeta^2 + (p(\rho) - p(\hat{\rho})) \psi \right]
\]
\[
\geq \left[ -\frac{\gamma K u_+}{2} \varphi^2 - K \gamma \rho_+^{-1} \varphi \psi - \frac{\rho_+ u_+}{2} \psi^2 \right] - \frac{\rho_+ u_+}{2} \zeta^2
\]
\[
- C(N_1(T) + \hat{\delta})(\varphi^2 + \psi^2 + \zeta^2)
\]
\[
= \| \varphi, \psi \|_{M_1[\varphi, \psi]}^T + \frac{\rho_+ |u_+|}{2} \zeta^2 - C(N_1(T) + \hat{\delta})(\varphi^2 + \psi^2 + \zeta^2),
\]
where \( [\cdot]^T \) denotes the transpose of a row vector, and the \( 2 \times 2 \) real symmetric matrix \( M_1 \) is given by
\[
\begin{pmatrix}
-\frac{\gamma K u_+}{2} \rho_+^{-2} & -\frac{K \gamma \rho_+^{-1}}{2} \\
-\frac{K \rho_+^{-1}}{2} & -\frac{\rho_+ u_+}{2}
\end{pmatrix}.
\]

One can compute all the leading principal minors \( \Delta_{ll} \) \((1 \leq l \leq 2)\) of \( M_1 \) as follows:
\[
\Delta_{11} = -\frac{\gamma K u_+}{2} \rho_+^{-2} > 0, \quad \Delta_{22} = \frac{\gamma K}{4} \rho_+^{-1}(u_+^2 - \gamma K \rho_+^{-1}) > 0,
\]
where we have used the condition \( M_+ > 1 \) and \( u_+ < 0 \).

Thus we have
\[
-\alpha \beta \int_{\mathbb{R}_+} W_{\alpha-1, \beta} \left[ \rho u \Phi(\rho, \hat{\rho}) + \frac{\rho u}{2} \psi^2 + \frac{\rho u}{2} \zeta^2 + (p(\rho) - p(\hat{\rho})) \psi \right] dx \geq c_\beta \| \varphi, \psi, \zeta \|_{\alpha-1, \beta}^2,
\]
8
where we take $N_1(T)$ and $\tilde{\delta}$ small enough.

It is easy to obtain that
\[-\rho(0,t) u_t \Phi(\rho(0,t), \hat{\rho}(0)) \geq c \varphi(0,t)^2.\]

Using Theorem 14.1(i), Lemma 3.1(i), (3.3) and Cauchy-Schwarz's inequality with $0 < \eta < 1$, we have
\[
|J_1| \leq C\tilde{\delta}\varphi(0,t)^2 + C\tilde{\delta}\|\partial_x \varphi, \psi, \zeta\|^2,
\]
\[
|J_2| \leq \eta\|\partial_x \varphi, \psi, \zeta\|_{-1,\beta}^2 + C\eta^2\|\varphi, \psi, \zeta\|_{-1,\beta}^2.
\]

Inserting the above estimations into (3.14) and then choosing $\eta, N_1(T), \tilde{\delta}$ and $\beta$ suitably small, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha,\beta} \left[ \rho \Phi(\rho, \hat{\rho}) + \frac{\rho}{2} \psi^2 + \frac{\rho}{2} \zeta^2 \right] dx + c \varphi(0,t)^2
\]
\[
+ c\beta \|\partial_x \varphi, \psi, \zeta\|_{-1,\beta}^2 + c\|\partial_t \varphi, \psi, \zeta\|_{-1,\beta}^2 \leq C\tilde{\delta}\|\partial_x \varphi\|^2.
\]
(3.16)

Multiplying (3.14) by $(1 + t)^\xi$ and integrating in $\tau$ over $[0, t]$ for any $0 \leq t \leq T$, we have the desired estimate (3.9) for $\alpha \in (0, \nu]$. Next, we prove (3.9) holds for $\alpha = 0$. Multiplying (3.14) by $(1 + t)^\xi$ and integrating the resulting equality over $\mathbb{R}_+ \times (0, t)$, then (3.9) holds for $\alpha = 0$ with the aid of Theorem 14.1(i), Lemma 3.1(i) and boundary condition (3.3).

Lemma 3.3. There exists a positive constant $\varepsilon_1$ such that if $N_1(T) + \tilde{\delta} + \beta \leq \varepsilon_1$, then
\[
(1 + t)^\xi \|\partial_x \varphi\|^2 + \int_0^t (1 + \tau)^\xi \|\partial_x \varphi\|^2 d\tau + \int_0^t (1 + \tau)^\xi (\partial_x \varphi)^2(0, \tau) d\tau
\]
\[
\leq C \left( \|\varphi_0, \psi_0, \zeta_0\|^2 \|\partial_x \varphi_0\|^2 + \|\partial_x \varphi_0\|^2 \right) + \xi \int_0^t (1 + \tau)^\xi \left( \|\varphi, \psi, \zeta\|_{-1,\beta}^2 + \|\partial_x \varphi\|^2 \right) d\tau
\]
\[
+ CN_1(T) \int_0^t (1 + \tau)^\xi \left\| \partial_x^2 \varphi \right\|^2 d\tau
\]
(3.17)
holds for $\xi \geq 0$.

Proof. We first differentiate (3.1) with respect to $x$, multiplying the resulting equations and (3.2) by $\frac{\lambda \partial_x \varphi}{\rho^2}$ and $\frac{\partial_x \varphi}{\rho}$ respectively to obtain
\[
\lambda \frac{\lambda \partial_x \varphi}{\rho^2} \partial_t \partial_x \varphi + \lambda \partial_x u \frac{(\partial_x \varphi)^2}{\rho^2} + \lambda u \frac{\partial_x \varphi \partial_x^2 \varphi}{\rho^2}
\]
\[
+ \lambda \frac{\partial_x \varphi}{\rho^2} \partial_t \partial_x \varphi + \lambda \partial_x \rho \frac{\partial_x \varphi}{\rho^2}
\]
\[
= -\lambda \frac{\partial_x^2 \varphi}{\rho^2} \frac{\partial_x \varphi}{\rho} - \lambda \partial_x \varphi \varphi \frac{\partial_x \varphi}{\rho^2} - \lambda \partial_x \rho \varphi \frac{\partial_x \varphi}{\rho^2} + \partial_t \varphi - \partial_x \varphi.
\]
(3.18)
\[
\partial_t \psi \partial_x \varphi + u \partial_x \psi \partial_x \varphi + \partial_x [p(\rho) - p(\hat{\rho})] \frac{\partial_x \varphi}{\rho} = \lambda \partial_x \varphi \varphi \frac{\partial_x \varphi}{\rho} - \frac{\tilde{u}_\tau \tilde{u}}{\rho} \varphi \partial_x \varphi - \partial_x \tilde{u}_\tau \psi \partial_x \varphi.
\]
(3.19)
The summation of (3.18) and (3.19), and then taking integration over $\mathbb{R}_+$ further imply
\[
\frac{d}{dt} \int_{\mathbb{R}_+} \left[ \psi \partial_x \varphi + \frac{\lambda (\partial_x \varphi)^2}{2\rho^2} \right] dx + \int_{\mathbb{R}_+} \frac{p'(\rho)}{\rho} (\partial_x \varphi)^2 dx
\]
\[
= \int_{\mathbb{R}_+} \psi \partial_t \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} (\partial_x \varphi)^2 \partial_x \rho dx - \lambda \int_{\mathbb{R}_+} \partial_x [p(\rho) - p(\hat{\rho})] \frac{\partial_x \varphi}{\rho} dx
\]
\[
- \int_{\mathbb{R}_+} \frac{\partial_x p'(\rho)}{\rho} \partial_x \varphi dx - \int_{\mathbb{R}_+} u \partial_x \psi \partial_x \varphi dx - \int_{\mathbb{R}_+} \tilde{u}_\tau \partial_x \psi \partial_x \varphi dx - \int_{\mathbb{R}_+} \partial_x \tilde{u}_\tau \psi \partial_x \varphi dx
\]
\[
- \lambda \int_{\mathbb{R}_+} \partial_x u \frac{(\partial_x \varphi)^2}{\rho^2} dx - \lambda \int_{\mathbb{R}_+} \frac{\partial_x \varphi \partial_x^2 \varphi}{\rho^2} dx - \lambda \int_{\mathbb{R}_+} \frac{\partial_x \varphi}{\rho^2} \partial_x \rho \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} \partial_x \tilde{u}_\tau \psi \partial_x \varphi dx
\]
\[
- \lambda \int_{\mathbb{R}_+} \partial_x \rho \partial_x \psi \partial_x^2 \varphi dx - \lambda \int_{\mathbb{R}_+} \partial_x \tilde{u}_\tau \psi \partial_x \varphi dx - \int_{\mathbb{R}_+} \lambda \partial_x \tilde{u}_\tau \frac{(\partial_x \varphi)^2}{\rho^2} dx = \sum_{l=1}^{16} J_l,
\]
(3.20)
where $J_l$ ($3 \leq l \leq 15$) denote the corresponding terms on the left of (3.20).

Applying Sobolev’s inequality, Young’s inequality and Cauchy-Schwarz’s inequality with $0 < \eta < 1$ and using Theorem 1.1(i), Lemma 3.1(i), one has (3.17).

Proof.

Lemma 3.4.

There exists a positive constant $\varepsilon_1$ such that if $N_1(T) + \tilde{\delta} + \beta \leq \varepsilon_1$, then

\begin{align*}
(1 + t)^{\xi} &\left[\|\partial_x \psi, \zeta(t)\|^2 + \int_0^t (1 + \tau)^{\xi} \|\partial_x^2 \psi, \zeta(t)\|^2 d\tau\right] \\
&\leq C \left(\|\varphi_0, \psi_0, \zeta_0\|_{1, \beta}^2 + \|\partial_x [\varphi, \psi_0, \zeta_0]\|^2\right)
\end{align*}

and

\begin{align*}
\int_0^t (1 + \tau)^{\xi-1} \left(\|\varphi, \psi, \zeta(\tau)\|^2_{0, \beta} + \|\partial_x [\varphi, \psi, \zeta]\|^2\right) d\tau
\end{align*}

holds for $\xi \geq 0$.

Proof. Multiplying (3.31) by $-\frac{\partial^2 \psi}{\rho}$, and then integrating the resulting equations over $\mathbb{R}_+$, one has

\begin{align*}
\frac{d}{dt} &\int_{\mathbb{R}_+} \frac{(\partial_x \psi)^2}{2} dx + \lambda \int_{\mathbb{R}_+} \frac{(\partial_x^2 \psi)^2}{\rho} dx \\
= &\int_{\mathbb{R}_+} \frac{\partial_x [p(\rho) - p(\tilde{\rho})]}{\rho} \partial_x^2 \psi dx + \int_{\mathbb{R}_+} u \partial_x^2 \psi \partial_x \psi dx + \int_{\mathbb{R}_+} \frac{\tilde{u} \partial_x \tilde{u} \partial_x^2 \psi}{\rho} dx + \int_{\mathbb{R}_+} \partial_x \tilde{u} \partial_x^2 \psi \partial_x \psi dx. \tag{3.23}
\end{align*}

We utilize integration by parts, Cauchy-Schwarz’s inequality and Lemma 3.1 to address the following estimates:

\begin{align*}
|J_{17}| &\leq \eta \|\partial_x^2 \psi\|^2 + C_\eta \|\partial_x \psi, \zeta\|^2,
\end{align*}
and

\[ |J_{17}| + |J_{19}| + |J_{20}| \leq \eta \| \partial_x^2 \psi \|^2 + C_\eta \partial_x [\varphi, \psi] \|^2 + C_\delta \varphi^2 (0, t). \]

Substituting the above estimates for \( J_l \) (17 \( \leq l \leq 20 \)) into \( 3.23 \) and taking \( \eta \) small enough, one has

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{\partial_x \psi}{2} dx + \| \partial_x^2 \psi \|^2 \leq C \| \partial_x [\varphi, \psi] \|^2 + C \varphi^2 (0, t). \tag{3.24}
\]

Multiplying \( 3.13 \) by \( -\frac{\partial_x \zeta}{\rho} \) and integrating the resulting equality over \( \mathbb{R}^+ \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^+} (\partial_x \zeta)^2 dx + \nu \int_{\mathbb{R}^+} \frac{\partial_x^2 \zeta}{\rho} dx = \int_{J_{21}} \mu \frac{\partial_x \zeta}{\rho} dx + \int_{J_{22}} \frac{\partial_x \zeta}{\rho} \frac{\partial_x \varphi}{\rho^2} dx + \int_{J_{23}} \frac{\partial_x \zeta}{\rho} \left( \frac{\partial_x \varphi}{\rho} + \rho \varphi \right)^2 dx. \tag{3.25}
\]

To obtain the estimates for \( J_{21} - J_{23} \), we use Cauchy-Schwarz’s inequality with \( 0 < \eta < 1 \) to get

\[
|J_{21}| + |J_{22}| \leq \eta \| \partial_x^2 \zeta \|^2 + C_\eta \| [\zeta, \partial_x \zeta] \|^2,
\]

\[
|J_{23}| \leq \eta \| \partial_x^2 \zeta \|^2 + C_\eta \partial_x [\varphi, \psi] \|^2 + C_\delta \varphi^2 (0, t).
\]

Then we have

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \frac{\partial_x \zeta}{2} dx + \| \partial_x^2 \zeta \|^2 \leq C \| [\zeta, \partial_x \zeta] \|^2 + C_\tilde{\delta} \| \partial_x [\varphi, \psi] \|^2 + C_\delta \varphi^2 (0, t) \tag{3.26}
\]

if \( \eta \) is small enough.

The summation of \( 3.24 \) and \( 3.26 \), and multiplying the resulting inequality by \( (1 + t) \xi \), then integrating the resulting inequality in \( \tau \) over \( [0, t] \) for any \( 0 \leq t \leq T \), using \( 3.9 \), \( 3.17 \) and Cauchy-Schwarz’s inequality, one has \( 3.22 \).

**Proof of Proposition 3.1** Now, following the three steps above, we are ready to prove Proposition 3.1. Summing up the estimates \( 3.9 \), \( 3.17 \) and \( 3.22 \), and taking \( \tilde{\delta} \) and \( N_3 (T) \) suitably small, we have

\[
(1 + t) \xi \left( \| [\varphi, \psi, \zeta(t)] \|_{\alpha, \beta}^2 + \| \partial_x [\varphi, \psi, \zeta(t)] \|^2 \right) + \int_0^t (1 + \tau) \xi \| \partial_x [\varphi, \partial_x [\varphi, \psi, \zeta(t)] \|^2 d\tau
\]

\[
+ \int_0^t (1 + \tau) \xi (\tilde{\delta} \| [\varphi, \psi, \zeta(t)] \|_{1- \alpha, \beta}^2 + \| [\zeta, \partial_x [\varphi, \psi, \zeta(t)] \|^2_{\alpha, \beta}) d\tau
\]

\[
\leq C \left( \| [\varphi, \psi, \zeta_0(t)] \|_{\alpha, \beta}^2 + \| \partial_x [\varphi, \psi, \zeta_0(t)] \|^2 \right)
\]

\[
+ \xi \int_0^t (1 + \tau) \xi^{-1} \left( \| [\varphi, \psi, \zeta(t)] \|_{\alpha, \beta}^2 + \| \partial_x [\varphi, \psi, \zeta(t)] \|^2 \right) d\tau, \tag{3.27}
\]

where \( C \) is a positive constant independent of \( T, \alpha, \beta, N_3 (T) \) and \( \tilde{\delta} \). Hence, similarly as in \( 18 \), \( 40 \), applying an induction to \( 3.27 \) gives desired estimate \( 3.28 \).

**3.2 The a priori estimates for \( M_+ = 1 \)**

This subsection is devoted to prove the algebraic decay estimate for the transonic case \( M_+ = 1 \) in Theorem 1.2. For the convenience of stating the a priori assumption, we use the notations

\[
N_2 (T) := sup_{0 \leq t \leq T} \| [\varphi, \psi, \zeta(t)] \|_{\alpha, \beta, 1}. \tag{3.28}
\]
Proposition 3.2. Assume the same conditions as in Theorem 1.2 (ii) hold. Suppose \([\varphi, \psi, \zeta] \in X((0, T))\) is a solution to (3.1), (3.2) and (3.3) which satisfies \((1 + Bx)^{\frac{\gamma}{2}} \varphi, (1 + Bx)^{\frac{\gamma}{2}} \psi, (1 + Bx)^{\frac{\gamma}{2}} \zeta \in C([0, T]; H^1(\mathbb{R}_+))\) for certain positive constants \(T, B\) and \(\vartheta \in [2, \vartheta^*]\), where \(B\) and \(\vartheta^*\) is respectively defined in (3.20) and (1.1b). For arbitrary \(\alpha \in [0, \vartheta]\), there exist positive constants \(C\) and \(\varepsilon_2\) independent of \(T\) such that if \(N_2(T) + \tilde{\delta} \leq \varepsilon_2\) is satisfied, it holds for an arbitrary \(t \in [0, T]\) that

\[
(1 + t)^{\frac{\gamma}{2} + \kappa}\|\varphi, \psi, \zeta(t)\|^2_{\alpha,B,1} + \int_0^t (1 + \tau)^{\frac{\gamma}{2} + \kappa}\|\zeta, \partial_x \varphi, \psi, \zeta, \partial^2_x \psi, \zeta\|_{\alpha,B,1}(\tau) \, d\tau \\
\leq C(1 + t)^{\kappa}\|\varphi_0, \psi_0, \zeta_0\|^2_{\alpha,B,1},
\]

where \(\kappa\) is positive constant.

In order to prove Proposition 3.2, we need to get a lower estimate for \(\tilde{u}(x)\) and the Mach number \(\tilde{M}\) on the stationary solution \((\tilde{\rho}(x), \tilde{u}(x))\) defined by \(\tilde{M} := \frac{|\tilde{u}(x)|}{\sqrt{\tilde{\rho}(\tilde{\rho}(x))}}\).

Lemma 3.5. (See [20]) The stationary solution \(\tilde{u}(x)\) satisfies

\[
\tilde{u}_x(x) \geq A \left( \frac{u_+}{u_b} \right)^{\gamma + 2} \frac{\tilde{\delta}^2}{(1 + Bx)^2} > 0, \quad A := \frac{(\gamma + 1)\rho_+}{2\lambda}, \quad B = \tilde{\delta} A
\]

for \(x \in (0, \infty)\). Moreover, there exists a positive constant \(C\) such that

\[
\frac{(\gamma + 1)\tilde{\delta}}{2|u_+|(1 + Bx)} - C \frac{\tilde{\delta}^2}{(1 + Bx)^2} \leq \tilde{M} - 1 \leq C \frac{\tilde{\delta}}{1 + Bx}.
\]

Proof. Please refer to [20] for the detailed proof.

Lemma 3.6. There exists a positive constant \(\varepsilon_2\) such that if \(N_2(T) + \tilde{\delta} \leq \varepsilon_2\), then

\[
(1 + t)^{\xi}\|\varphi, \psi, \zeta(t)\|^2_{\alpha,B} + \tilde{\delta}^2 \int_0^t (1 + \tau)^{\xi}\|\varphi, \psi, \zeta(\tau)\|^2_{\alpha-2,B} \, d\tau + \tilde{\delta} \int_0^t (1 + \tau)^{\xi}\|\zeta(\tau)\|^2_{\alpha-1,B} \, d\tau \\
+ \int_0^t (1 + \tau)^{\xi}\|\zeta, \partial_x \varphi, \psi, \zeta(\tau)\|^2_{\alpha,B} \, d\tau + \int_0^t (1 + \tau)^{\xi}\varphi^2(0, \tau) \, d\tau \\
\leq C\|\varphi_0, \psi_0, \zeta_0\|^2_{\alpha,B} + \xi \int_0^t (1 + \tau)^{\xi-1}\|\varphi, \psi, \zeta(\tau)\|^2_{\alpha,B} \, d\tau + C\tilde{\delta} \int_0^t (1 + \tau)^{\xi}\|\partial_x \varphi\|^2_{\alpha,B}(\tau) \, d\tau
\]

holds for \(\alpha \in [0, \vartheta]\) and \(\xi \geq 0\).

Proof. Using (1.7), the equation (3.14) is rewritten to

\[
\left[ \rho\Phi(\rho, \tilde{\rho}) + \frac{\rho}{2}\psi^2 + \frac{\rho}{2}\zeta^2 \right] + \left[ \rho u \Phi(\rho, \tilde{\rho}) + \frac{\rho u}{2}\psi^2 + \frac{\rho u}{2}\zeta^2 + (p(\rho) - p(\tilde{\rho}))\psi - \lambda \psi \zeta - \nu \zeta \zeta \right] \]

\[
+ \mu \zeta^2 + \nu \zeta^2 + \partial_x \tilde{\omega}(\partial_x \psi) + (p(\rho) - p(\tilde{\rho})) = -\partial_x \tilde{\omega}(\tilde{u} \varphi \zeta + \rho \psi \zeta) - \frac{\lambda \partial_x^2 \tilde{u}}{\tilde{\rho}} \varphi \psi.
\]

Multiplying (3.33) by \(\tilde{W}_{\alpha,B} := (1 + Bx)^{\alpha}\) defined in Notation, then we integrate the resulting equality
over $\mathbb{R}_+$ to get

\[
\frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha,B} \left[ \rho \Phi(\rho, \tilde{\rho}) + \frac{\rho}{2} \psi^2 + \frac{\rho}{2} \rho^2 \right] dx \quad \text{with}
\]

\[
\int_{\mathbb{R}_+} W_{\alpha,B} \left[ \mu \zeta^2 + \lambda \psi^2 + \nu \zeta^2 \right] dx
\]

\[
- \rho(0,t)u_0 \Phi(\rho(0,t), \tilde{\rho}(0)) + \alpha B \int_{\mathbb{R}_+} W_{\alpha-1,B} \left[ -\rho \Phi(\rho, \tilde{\rho}) - \frac{\rho}{2} \psi^2 - (p(\rho) - p(\tilde{\rho})) \psi \right] dx
\]

\[
+ \int_{\mathbb{R}_+} W_{\alpha,B} \partial_t \tilde{u} \left[ \rho \psi^2 + p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho}) \varphi \right] dx - \frac{\lambda \alpha}{2} B^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \psi^2 dx
\]

\[
- \frac{\alpha B}{2} \int_{\mathbb{R}_+} W_{\alpha-1,B} \rho \zeta^2 dx - \frac{\mu \alpha}{2} B^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \zeta^2 dx
\]

\[
= - \int_{\mathbb{R}_+} W_{\alpha,B} \partial_t \tilde{w}(\tilde{u} \varphi \zeta + \rho \psi \zeta) dx - \int_{\mathbb{R}_+} W_{\alpha,B} \frac{\lambda \partial_t \tilde{u}}{\rho} \varphi \psi dx,
\]

(3.34)

where we have used integration by parts and boundary condition \[(3.3)\].

It is easy to obtain that

\[-\rho(0,t)u_0 \Phi(\rho(0,t), \tilde{\rho}(0)) \geq c_\varphi(0,t)^2.\]

Using Lemma \[(3.5)\] and decomposing $\rho$ as $\rho = \varphi + (\tilde{\rho} - \rho_\pm) + \rho_\pm$, $u$ as $u = \psi + (\tilde{u} - u_\pm) + u_\pm$, we have the following estimates:

\[
G_1 \geq \alpha B \int_{\mathbb{R}_+} W_{\alpha-1,B} \left[ \frac{\rho'(\rho_\pm)^2}{2\rho_\pm} \varphi^2 + \rho + \frac{\rho'}{2} \psi^2 \right] (\tilde{M} - 1) + \frac{\rho'(\tilde{\rho})}{2\tilde{\rho}} \left( \sqrt{\rho'(\tilde{\rho})} \varphi - \tilde{\varphi} \right) \right] dx
\]

\[
- C_\alpha B \| (1 + Bx) \|_\infty \| \varphi, \psi \|_2^2 \| \varphi, \psi \|_{\alpha - 2,B} - C\tilde{\alpha} B \int_{\mathbb{R}_+} W_{\alpha-1,B} (\varphi^2 + \psi^2)(\tilde{M} - 1) dx
\]

\[
\geq \alpha B \int_{\mathbb{R}_+} W_{\alpha-1,B} \left[ \frac{\rho'(\rho_\pm)^2}{2\rho_\pm} \varphi^2 + \rho + \frac{\rho'}{2} \psi^2 \right] \left( \frac{(\gamma + 1)\delta}{2|u_\pm|(1 + Bx)} - C \frac{\tilde{\delta}^2}{(1 + Bx)^2} \right) dx
\]

\[
- C_\alpha B \| (1 + Bx) \|_\infty \| \varphi, \psi \|_2^2 \| \varphi, \psi \|_{\alpha - 2,B}
\]

\[
\geq \frac{(\gamma + 1)\alpha A}{2|u_\pm|} \delta^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \left[ \frac{\rho'(\rho_\pm)^2}{2\rho_\pm} \varphi^2 + \rho + \frac{\rho'}{2} \psi^2 \right] dx
\]

\[
- C\delta (N_2(T) + \delta^2) \| \varphi, \psi \|_{\alpha - 2,B}^2,
\]

(3.35)

\[
G_2 \geq A \left( \frac{u_\pm}{u_b} \right) \frac{(\gamma + 1)\alpha A}{2|u_\pm|} \delta^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \left[ \rho \psi^2 + p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho}) \varphi \right] dx
\]

\[
\geq A \left( \frac{u_\pm}{u_b} \right) \frac{(\gamma + 1)\alpha A}{2|u_\pm|} \delta^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \left( \rho + \psi^2 + \frac{\rho'(\rho_\pm)^2}{2} \psi^2 \right) dx - C\delta^2 (\| \varphi \|_2 \delta + \tilde{\delta}) \| \varphi, \psi \|_{\alpha - 2,B}^2
\]

\[
\geq A \left( \frac{u_\pm}{u_b} \right) \delta^2 \int_{\mathbb{R}_+} W_{\alpha-2,B} \left( \rho + \psi^2 + \frac{\rho'(\rho_\pm)^2}{2} \psi^2 \right) dx - C\delta^2 (N_2(T) + \tilde{\delta}) \| \varphi, \psi \|_{\alpha - 2,B}^2.
\]

(3.36)
For $\alpha \in (0, \vartheta]$, we obtain the lower estimate of $G_1 + G_2 + G_3$ as

$$G_1 + G_2 + G_3 \geq A \delta^2 \int_{\mathbb{R}^+} W_{\alpha - 2, B} \left[ \left( \gamma + 1 \right) \alpha + \frac{\|u\|}{u_b} \right]^{\gamma + 2} \varphi^2 \psi^2 \, dx + A \delta^2 \int_{\mathbb{R}^+} W_{\alpha - 2, B} \left[ \left( \gamma + 1 \right) \alpha + \frac{\|u\|}{u_b} \right]^{\gamma + 2} \varphi^2 \psi^2 \, dx \geq A \delta^2 \int_{\mathbb{R}^+} \alpha \alpha - 2 \gamma + 1 \alpha + \left( 3.38 \right) \alpha \alpha - 2 \alpha - 1, B.$$

where we have used Lemma 3.5, $\alpha < \vartheta^*$ and $M_+ = 1$.

Similarly, we have

$$G_4 + G_5 \geq \alpha A \delta^2 \int_{\mathbb{R}^+} W_{\alpha - 2, B} \left[ \left( \gamma + 1 \right) \alpha + \frac{\|u\|}{u_b} \right]^{\gamma + 2} \varphi^2 \psi^2 \, dx \geq \alpha A \delta^2 \int_{\mathbb{R}^+} \delta \|\psi\|^2 \alpha - 2, B - C \delta (N_2 (T) + \delta) \|\psi\|^2 \alpha - 1, B.$$

Using Poincaré type inequality \(3.36\) and \(1.13\), we have

$$K_1 \leq C \delta \int_{\mathbb{R}^+} W_{\alpha, B} e^{-\sigma_2} (\varphi^2 + \psi^2 + \zeta^2) \, dx \leq C \delta \int_{\mathbb{R}^+} W_{\alpha, B} e^{-\sigma_2} \left[ \varphi (0, t)^2 + x \|\varphi, \psi, \zeta\|^2 \right] \, dx \leq C \delta \varphi (0, t)^2 + C \delta \|\varphi, \psi, \zeta\|^2 \leq C \delta \varphi (0, t)^2 + C \delta \|\varphi, \psi, \zeta\|^2 \alpha, B,$$

\(3.39\)

Inserting the above estimates into \(3.34\) and then taking $N_2 (T)$ and $\delta$ suitably small to satisfy $N_2 (T) \ll \delta^2$ and $\delta \ll 1$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^+} W_{\alpha, B} \left[ \rho \Phi (\rho, \varphi) + \frac{\rho}{2} \varphi^2 + \frac{\rho}{2} \zeta^2 \right] \, dx + c \delta \varphi (0, t)^2 + C \delta \|\varphi, \psi\|^2 \alpha, B.$$

\(3.41\)

Multiplying \(3.41\) by $(1 + t)^{\xi}$ and integrating in $\tau$ over $[0, t]$ for any $0 \leq t \leq T$, we have the desired estimate \(3.32\) for $\alpha \in (0, \vartheta]$. Next, we prove \(3.32\) holds for $\alpha = 0$. Multiplying \(3.33\) by $(1 + t)^{\xi}$
and integrating the resulting equality over $\mathbb{R}_+ \times (0, t)$, we have

$$(1 + t)^\xi ||\varphi, \psi, \zeta(t)||^2 + \int_0^t (1 + \tau)^\xi ||\partial_x \varphi, \zeta||^2 d\tau + \int_0^t (1 + \tau)^\xi \varphi^2(0, \tau)d\tau 
\leq C||\varphi_0, \psi_0, \zeta_0||^2 + \xi \int_0^t (1 + \tau)^{\xi - 1} ||\varphi, \psi, \zeta(\tau)||^2 d\tau 
+ C \int_0^t (1 + \tau)^\xi \int_{\mathbb{R}_+} |\partial_x \tilde{u}|(|\varphi\zeta| + |\psi\zeta|)d\tau + C \int_0^t (1 + \tau)^\xi \int_{\mathbb{R}_+} |\partial_x^2 \tilde{u}|\varphi\psi|d\tau, \quad (3.42)$$

where we have used the fact that $\partial_x \tilde{u} |\rho\psi|^2 + p(\rho) - p(\hat{\rho}) - p'(\hat{\rho})\varphi \geq 0$ holds. Applying Lemma 3.1 to the third term and fourth term on the right-hand side of (3.42) with the aid of (3.13), we obtain the estimate (3.32) for the case of $\alpha = 0$.

**Lemma 3.7.** There exists a positive constant $\varepsilon_2$ such that if $N_2(T) + \delta \leq \varepsilon_2$, then

$$(1 + t)^\xi ||\partial_x \varphi||^2_{\alpha, B} + \int_0^t (1 + \tau)^\xi ||\partial_x \varphi||^2_{\alpha, B} d\tau + \int_0^t (1 + \tau)^\xi (\partial_x \varphi)^2(0, \tau)d\tau 
\leq C( ||\varphi_0, \psi_0, \zeta_0||^2_{B, \alpha} + ||\partial_x \varphi_0||^2_{\delta, B}) + \xi \int_0^t (1 + \tau)^{\xi - 1} ( ||\varphi, \psi, \zeta(\tau)||^2_{\alpha, B} + ||\partial_x \varphi||^2_{\alpha, B}) d\tau 
+ C N_2(T) \int_0^t (1 + \tau)^\xi ||\partial_x^2 \psi||^2_{\alpha, B} d\tau \quad (3.43)$$

holds for $\alpha \in [0, \delta]$ and $\xi \geq 0$.

**Proof.** Multiplying the summation of (3.18) and (3.19) by $W_{\alpha, B}$, and then taking integration over $\mathbb{R}_+$ further imply

$$\frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha, B} \left[ \psi \partial_x \varphi + \lambda \left( \frac{\partial_x \varphi}{\rho} \right)^2 \right] dx + \int_{\mathbb{R}_+} W_{\alpha, B} \frac{\partial_p(\varphi)}{\rho} (\partial_x \varphi)^2 dx 
= \int_{\mathbb{R}_+} W_{\alpha, B} \psi \partial_t \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} W_{\alpha, B} (\partial_x \varphi)^2 \rho^{-3} \partial_p dx - \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x (p(\rho) - p(\hat{\rho}) - p'(\hat{\rho})\varphi) \frac{\partial_x \varphi}{\rho} dx 
- \int_{\mathbb{R}_+} W_{\alpha, B} \frac{\partial_p(\varphi)}{\rho} \varphi \partial_x \varphi dx - \int_{\mathbb{R}_+} W_{\alpha, B} \rho \partial_x \psi \partial_x \varphi dx - \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \varphi \partial_x \varphi dx 
- \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \varphi \rho \rho \partial_x \psi \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \rho \partial_x \varphi \partial_x \psi \partial_x \varphi dx 
- \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \varphi \rho \rho \partial_x \psi \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x^2 \varphi \rho \rho \partial_x \psi \partial_x \varphi dx 
- \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \varphi \rho \rho \partial_x \psi \partial_x \varphi dx - \lambda \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \varphi \rho \rho \partial_x \psi \partial_x \varphi dx 
= \sum_{i=3}^{16} K_i, \quad (3.44)$$

where $K_1$ ($3 \leq l \leq 16$) denote the corresponding terms on the left of (3.44).

Applying Sobolev’s inequality, Young’s inequality and Cauchy-Schwarz’s inequality with $0 < \eta < 1$ and using Theorem 1.1(ii), one has

$$K_3 = \int_{\mathbb{R}_+} W_{\alpha, B} \partial_x \psi \partial_x (\rho u - \hat{\rho} \tilde{u}) dx + \int_{\mathbb{R}_+} \partial_x (W_{\alpha, B}) \psi \partial_x (\rho u - \hat{\rho} \tilde{u}) dx 
\leq (N_2(T) + \eta ||\partial_x \varphi||^2_{\alpha, B} + (C + C_\eta) \delta ||\varphi, \psi||^2_{\alpha, B} + (C + C_\eta + N_2(T)) ||\partial_x \psi||^2_{\alpha, B}.$$
\[ K_4 + K_{10} + K_{11} \]
\[ = - \frac{\lambda |u_b|}{2 \rho(0,t)^2} (\partial_x \varphi)^2(0,t) + \frac{\lambda}{2} \int_{\mathbb{R}} W_{\alpha,B} \partial_x \hat{u} (\partial_x \varphi)^2 \rho^{-2} dx \]
\[ + \frac{\lambda}{2} \int_{\mathbb{R}} W_{\alpha,B} \partial_x \psi (\partial_x \varphi)^2 \rho^{-2} dx + \frac{\lambda}{2} \int_{\mathbb{R}} \partial_x(W_{\alpha,B}) u(\partial_x \varphi)^2 \rho^{-2} dx \]
\[ \leq - \frac{\lambda |u_b|}{2 \rho(0,t)^2} (\partial_x \varphi)^2(0,t) + C \delta \| \partial_x \varphi \|_{a,B}^2 + C \| \partial_x \psi \|_{H^1} \| \partial_x \varphi \|_{a,B}^2 \]
\[ \leq - \frac{\lambda |u_b|}{2 \rho(0,t)^2} (\partial_x \varphi)^2(0,t) + C (\delta + N_2(T)) \| \partial_x \varphi \|_{a,B}^2 + C N_2(T) \| \partial_x \psi \|_{a,B}^2. \]

Inserting the above estimates for \( |K_l| \) \((3 \leq l \leq 16)\) into (3.44) and then choosing \( N_2(T), \delta \) and \( \eta \) suitably small, we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}} W_{\alpha,B} \left[ \psi \partial_x \varphi + \lambda \frac{(\partial_x \varphi)^2}{2\rho} \right] dx + \| \partial_x \varphi \|_{a,B}^2 + (\partial_x \varphi)^2(0,t) \]
\[ \leq C \| \partial_x \psi \|_{a,B}^2 + C N_2(T) \| \partial_x \varphi \|_{a,B}^2 + C \delta^2 \| \partial_x \varphi \|_{a,B}^2 + C N_2(T) \| \partial_x \psi \|_{a,B}^2 + C \delta^2 \| \partial_x \varphi \|_{a,B}^2. \quad (3.45) \]

Multiplying (3.45) by \((1 + t)^\xi\) and integrating in \( \tau \) over \([0,t]\) for any \( 0 \leq t \leq T \), using (3.32) and Cauchy-Schwarz’s inequality, we have the desired estimate (3.43) for \( \alpha \in (0, \theta) \). Then we can prove (3.43) holds for \( \alpha = 0 \) with the help of Lemma 3.1 and Sobolev’s inequality.

**Lemma 3.8.** There exists a positive constant \( \varepsilon_2 \) such that if \( N_2(T) + \delta \leq \varepsilon_2 \), then
\[ (1 + t)^\xi \| \partial_x \psi \|_{a,B}^2 + \int_0^t (1 + \tau)^\xi \| \partial_x \psi \|_{a,B}^2 d\tau \leq C \left( \| \varphi_0 \varphi_0, \psi_0, \zeta_0 \|_{a,B}^2 + \| \partial_x \varphi_0, \psi_0, \zeta_0 \|_{a,B}^2 \right) \]
\[ + \xi \int_0^t (1 + \tau)^{\xi - 1} \left( \| \varphi, \psi, \zeta \|_{a,B}^2 + \| \partial_x [\varphi, \psi, \zeta] \|_{a,B}^2 \right) d\tau \quad (3.46) \]
holds for \( \alpha \in (0, \theta) \) and \( \xi \geq 0 \).

**Proof.** Multiplying (3.32) by \(-W_{\alpha,B} \frac{\partial_x^2 \psi}{\rho}\), and then integrating the resulting equations over \( \mathbb{R} \), one has
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} W_{\alpha,B} \partial_x \psi^2 dx + \lambda \int_{\mathbb{R}} W_{\alpha,B} \frac{(\partial_x^2 \psi)^2}{\rho} dx \]
\[ = - \int_{\mathbb{R}} \partial_x(W_{\alpha,B}) \partial_x \psi \partial_x \psi dx + \int_{\mathbb{R}} W_{\alpha,B} \frac{\partial_x [\rho(p) - p(\rho)]}{\rho} \partial_x^2 \psi dx + \int_{\mathbb{R}} W_{\alpha,B} u \partial_x \psi \partial_x^2 \psi dx \]
\[ + \int_{\mathbb{R}} W_{\alpha,B} \frac{u \partial_x^2 \psi}{\rho} dx + \int_{\mathbb{R}} W_{\alpha,B} \partial_x \psi \partial_x \psi^2 dx. \quad (3.47) \]

We utilize Cauchy-Schwarz’s inequality with \( 0 < \eta < 1 \) to address the following estimates:
\[ |K_{17}| \leq C \delta \| \partial_x [\varphi, \psi, \partial_x \psi] \|_{a,B}^2 + C \delta^2 \| \partial_x \psi \|_{a,B}^2. \]
\[ |K_{18}| + |K_{19}| \leq \eta \| \partial_x^2 \psi \|^2_{\alpha,B} + C_\eta \| \partial_x [\varphi, \psi] \|^2_{\alpha,B}, \]
\[ |K_{20}| + |K_{21}| \leq \eta \| \partial_x^2 \psi \|^2_{\alpha,B} + C_\eta \tilde{\delta}^2 \| [\varphi, \psi] \|^2_{\alpha-2,B}. \]

Substituting the above estimates for \( K_i \) \((17 \leq i \leq 21)\) into (3.47) and taking \( \eta \) small enough, one has
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha,B} (\partial_x \psi)^2 \, dx + \| \partial_x^2 \psi \|^2_{\alpha,B} \leq C \| \partial_x [\varphi, \psi] \|^2_{\alpha,B} + C \tilde{\delta}^2 \| [\varphi, \psi] \|^2_{\alpha-2,B}. \tag{3.48}
\]

Multiplying (3.13) by \(-W_{\alpha,B} \frac{\partial_x^2 \psi}{\rho}\), and integrating the resulting equality over \( \mathbb{R}_+ \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha,B} (\partial_x \zeta)^2 \, dx + \nu \int_{\mathbb{R}_+} W_{\alpha,B} \left( \frac{\partial_x^2 \zeta}{\rho} \right)^2 \, dx = - \int_{\mathbb{R}_+} \partial_x (W_{\alpha,B}) \partial_t \zeta \partial_x \zeta \, dx \int_{\mathbb{R}_+} W_{\alpha,B} u \partial_x^2 \zeta^2 \, dx + \mu \int_{\mathbb{R}_+} W_{\alpha,B} \frac{\zeta}{\rho} \partial_x^2 \zeta^2 \, dx + \int_{\mathbb{R}_+} W_{\alpha,B} \frac{\partial_x \omega}{\rho} (\tilde{u} \varphi + \rho \psi) \partial_x^2 \zeta \, dx. \tag{3.49}
\]

To obtain the estimates for \( K_{22} - K_{25} \), we use Cauchy-Schwarz’s inequality with \( 0 < \eta < 1 \) to get
\[
|K_{22}| \leq C \tilde{\delta} \| \partial_x [\zeta, \partial_x \zeta] \|^2_{\alpha,B} + C \tilde{\delta} \| \zeta \|^2_{\alpha,B} + C \tilde{\delta}^2 \| [\varphi, \psi] \|^2_{\alpha-2,B},
\]
\[
|K_{21}| + |K_{24}| \leq \eta \| \partial_x^2 \zeta \|^2_{\alpha,B} + C_\eta \| \zeta, \partial_x \zeta \|^2_{\alpha,B},
\]
\[
|K_{25}| \leq \eta \| \partial_x^2 \zeta \|^2_{\alpha,B} + C_\eta \tilde{\delta}^2 \| [\varphi, \psi] \|^2_{\alpha-2,B}. \]

Then we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+} W_{\alpha,B} (\partial_x \zeta)^2 \, dx + \| \partial_x^2 \zeta \|^2_{\alpha,B} \leq C \| [\zeta, \partial_x \zeta] \|^2_{\alpha,B} + C \tilde{\delta} \| \zeta \|^2_{\alpha,B} + C \tilde{\delta}^2 \| [\varphi, \psi] \|^2_{\alpha-2,B}, \tag{3.50}
\]
if \( \eta \) is small enough.

The summation of (3.48) and (3.50), and multiplying the resulting inequality by \((1 + t)\xi\), then integrating the resulting inequality in \( \tau \) over \([0, t] \) for any \( 0 \leq t \leq T \), using (3.32) and (3.43), then (3.46) holds for \( \alpha \in (0, \tilde{\delta}) \). Then we can prove (3.46) holds for \( \alpha = 0 \) with the help of Lemma 3.1 and Sobolev’s inequality. \( \square \)

**Proof of Proposition 3.2** Now, following the three steps above, we are ready to prove Proposition 3.2. Summing up the estimates (3.32), (3.43) and (3.46), and taking \( \tilde{\delta} \) and \( N_1(T) \) suitably small, we have
\[
(1 + t)^\xi \| [\varphi, \psi, \zeta] (t) \|^2_{\alpha,B} + \int_0^t (1 + \tau)^\xi \left( \tilde{\delta}^2 \| [\varphi, \psi, \zeta] (\tau) \|^2_{\alpha-2,B} + \| [\zeta, \partial_x [\varphi, \psi, \zeta], \partial_x^2 [\psi, \zeta]] (\tau) \|^2_{\alpha,B} \right) d\tau \leq C \| [\varphi_0, \psi_0, \zeta_0] \|^2_{\alpha,B} + \xi \int_0^t (1 + \tau)^{\xi-1} \| [\varphi, \psi, \zeta] (\tau) \|^2_{\alpha,B} d\tau, \tag{3.51}
\]
where \( C \) is a positive constant independent of \( T, \alpha, \beta \), \( N_1(T) \) and \( \tilde{\delta} \). Hence, similarly as in [18, 40], applying an induction to (3.51) gives desired estimate (3.24).

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References

[1] M.T. Chen, Global strong solutions for the viscous, micropolar, compressible flow, J. Partial Differ. Equ., 24(2011), 158-164.

[2] M.T. Chen, Blowup criterion for viscous, compressible micropolar fluids with vacuum, Nonlinear Anal., Real World Appl., 13(2012), 850-859.

[3] M.T. Chen, B. Huang, J.W. Zhang, Blowup criterion for the three-dimensional equations of compressible viscous micropolar fluids with vacuum, Nonlinear Anal., 79(2013), 1-11.

[4] M.T. Chen, X.Y. Xu, J.W. Zhang, Global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum, Commun. Math. Sci., 13(2015), 225-247.

[5] Q. L. Chen, C. X. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, J. Differential Equations, 252(2012), 2698-2724.

[6] H.B. Cui, H.Y. Yin, Stability of the composite wave for the inflow problem on the micropolar fluid model, Commun. Pure Appl. Anal., 16(2017), 1265-1292.

[7] H.B. Cui, H.Y. Yin, Stationary solutions to the one-dimensional micropolar fluid model in a half line: existence, stability and convergence rate, J. Math. Anal. Appl., 449(2017), 464-489.

[8] B.Q. Dong, J.N. Li, J.H. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, J. Differential Equations, 262(2017), 2488-3523.

[9] I. Dražić, N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: large time behavior of the solution, J. Math. Anal. Appl., 431(2015), 545-568.

[10] I. Dražić, L. Simčić, N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: regularity of the solution, J. Math. Anal. Appl., 438(2016), 162-183.

[11] R. Duan, Global solutions for a one-dimensional compressible micropolar fluid model with zero heat conductivity, J. Math. Anal. Appl., 463(2018), 477-495.

[12] R. Duan, Global strong solution for initial-boundary value problem of one-dimensional compressible micropolar fluids with density dependent viscosity and temperature dependent heat conductivity, Nonlinear Anal. Real World Appl., 42(2018), 71-92.

[13] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech., 16(1966), 1-18.

[14] A.C. Erigen, Microcontinuum Field Theories: I. Foundations and solids, Springer. New York., 1999.

[15] F.M. Huang, X.H. Qin, Stability of boundary layer and rarefaction wave to an outflow problem for compressible Navier-Stokes equations under large perturbation, J. Differential Equations, 246(2009), 4077-4096.

[16] L. Huang, D.Y. Nie, Exponential stability for a one-dimensional compressible viscous micropolar fluid, Math. Methods Appl. Sci., 38(2015), 5197-5206.

[17] J. Jin, R. Duan, Stability of rarefaction waves for 1-D compressible viscous micropolar fluid model, J. Math. Anal. Appl., 450(2017), 1123-1143.

[18] S. Kawashima, A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, Comm. Math. Phys., 101(1985), 97-127.
[19] S. Kawashima, T. Nakamura, S. Nishibata, P.C. Zhu, Stationary waves to viscous heat-conductive gases in half space: existence, stability and convergence rate, Math. Models Methods Appl. Sci., 20(2010), 2201-2035.

[20] S. Kawashima, S. Nishibata, P.C. Zhu, Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space, Comm. Math. Phys., 240(2003), 483-500.

[21] Q.Q. Liu, H.Y. Yin, Stability of contact discontinuity for 1-D compressible viscous micropolar fluid model, Nonlinear Anal.: Theory, Methods Appl., 149(2017), 41-55.

[22] Q.Q. Liu, P.X. Zhang, Optimal time decay of the compressible micropolar fluids, J. Differential Equations, 260(2016), 7634-7661.

[23] Q.Q. Liu, P.X. Zhang, Long-time behavior of solution to the compressible micropolar fluids with external force, Nonlinear Anal. Real World Appl., 40(2018), 361-376.

[24] G. Lukaszewicz, Micropolar Fluids. Theory and Applications, Modeling and Simulation in Science, Engineering and Technology. Birkhäuser, Baston, 1999.

[25] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas, Methods Appl. Anal., 8(2001) 645-666.

[26] A. Matsumura, M. Mei, Convergence to travelling fronts of solutions of the p-system with viscosity in the presence of a boundary, Arch. Ration. Mech. Anal., 146(1999), 1-22.

[27] A. Matsumura, K. Nishihara, Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas, Comm. Math. Phys., 222(2001), 449-474.

[28] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: a local existence theorem, Glas. Mat., 33(1998), 71-91.

[29] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: a global existence theorem, Glas. Mat., 33(1998), 199-208.

[30] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: regularity of the solution, Rad. Mat., 10(2001), 181-193.

[31] N. Mujaković, Global in time estimates for one-dimensional compressible viscous micropolar fluid model, Glas. Mat. Ser.III, 40(2005), 103-120.

[32] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: stabilization of the solution, in: Z. Drmač, M. Marušić, Z. Tutek (Eds.), Proceedings of the Conference on Applied Mathematics and Scientific Computing, Springer, Netherlands, 2005, pp. 253-262.

[33] N. Mujaković, Nonhomogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: a local existence theorem, Ann. Univ. Ferrara Sez. VII Sci. Mat., 53(2007), 361-379.

[34] N. Mujaković, Nonhomogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: regularity of the solution, Bound. Value Probl., 2008, Article ID 189748 (2008).

[35] N. Mujaković, Nonhomogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: a global existence theorem, Math. Inequal. Appl., 12(2009), 651-662.
[36] N. Mujaković, One-dimensional compressible viscous micropolar fluid model: stabilization of the solution for the Cauchy problem, Bound. Value Probl., 2010, Article ID 796065 (2010).

[37] N. Mujaković, The existence of a global solution for one dimensional compressible viscous micropolar fluid with non-homogeneous boundary conditions for temperature, Nonlinear Anal. Real World Appl., 19(2014), 19-30.

[38] T. Nakamura, S. Nishibata, T. Yuge, Convergence rate of solutions toward stationary solutions to the compressible Navier-Stokes equation in a half line, J. Differential Equations, 241(2007), 94-111.

[39] T. Nakamura, S. Nishibata, Stationary wave associated with an inflow problem in the half line for viscous heat-conductive gas, J. Hyperbolic Differ. Equ., 8(2011), 651-670.

[40] M. Nishikawa, Convergence rate to the traveling wave for viscous conservation laws, Funkcial. Ekvac., 41(1998), 107-132.

[41] B. Nowakowski, Large time existence of strong solutions to micropolar equations in cylindrical domains, Nonlinear Anal. Real World Appl., 14(2013), 635-660.

[42] Y. Qin, T. Wang, G. Hu, The Cauchy problem for a 1D compressible viscous micropolar fluid model: analysis of the stabilization and the regularity, Nonlinear Anal., Real World Appl., 13(2012), 1010-1029.

[43] Z.G. Wu, W.K. Wang, The pointwise estimates of diffusion wave of the compressible micropolar fluids, J. Differential Equations, 265(2018), 2544-2576.

[44] H.Y. Yin, Stability of stationary solutions for inflow problem on the micropolar fluid model, Z. Angew. Math. Phys., 68(2017), 44.