Stationary Velocity Distributions in Traffic Flows

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We introduce a traffic flow model that incorporates clustering and passing. We obtain analytically the steady state characteristics of the flow from a Boltzmann-like equation. A single dimensionless parameter, \( R = c_0 v_0 t_0 \) with \( c_0 \) the concentration, \( v_0 \) the velocity range, and \( t_0^{-1} \) the passing rate, determines the nature of the steady state. When \( R \ll 1 \), uninterrupted flow with single cars occurs. When \( R \gg 1 \), large clusters with average mass \( \langle m \rangle \sim R^\alpha \) form, and the flux is \( J \sim R^{-\gamma} \). The initial distribution of slow cars governs the statistics. When \( P_0(v) \sim v^{\alpha} \) as \( v \to 0 \), the scaling exponents are \( \gamma = 1/(\mu + 2) \), \( \alpha = 1/2 \) when \( \mu > 0 \), and \( \alpha = (\mu + 1)/(\mu + 2) \) when \( \mu < 0 \).

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I. INTRODUCTION

Traffic flows are strongly interacting many-body systems. They also present a natural testbed for theories and techniques developed for physical systems such as kinetic theory and hydrodynamics. Traffic systems have been receiving much attention recently [1], and a number of approaches were suggested including fluid mechanics [2,3], cellular automata [4–11], particle hopping [12–15], and ballistic motion [16–18]. Traffic networks can be viewed as low dimensional systems. For example, rural traffic is intrinsically one-dimensional and urban grid traffic is two-dimensional. Despite this important simplifying feature, most studies in this area are numeric in nature.

Ballistic models are harder to simulate than cellular automata and particle hopping models. However, they are more realistic since time and space are treated as continuous variables. They can also prove useful for analytical treatment. An exactly solvable clustering process shows that extremal properties of the velocity distribution determine the kinetic behavior [13]. However, it results in ever-growing and ever-slowing jams with a trivial steady state in a finite system. In this study, we investigate more realistic situations where fast cars can pass slow cars. This is motivated by and should be applicable to passing zones of one lane roadways as well as multilane highways. Our goal is to determine analytically statistical properties of the flow such as the flux, and characterize their dependence on the intrinsic velocity distribution.

We start by formulating the model. Consider a one-dimensional traffic flow with sizeless cars (“particles”) moving with a constant velocity. We assume that cars have intrinsic velocities by which they would drive on an empty road. Initially, cars are randomly distributed in space and they drive with their intrinsic velocities. However, the presence of slower cars forces some cars to drive behind a slower car and therefore leads to the formation of clusters. Simple collision and escape mechanisms are implemented. When a cluster overtakes a slower cluster, a larger cluster forms. It moves with the smaller of the two velocities. Meanwhile, all cars in a given cluster may escape their respective cluster and resume driving with their intrinsic velocity. We assume a constant escape rate \( t_0^{-1} \). The actual collision and escape times are proportional to the car size and thus set to zero (these time scales should become important in heavy traffic).

A heuristic argument suggests that a single dimensionless parameter underlies the steady state. Consider a state where the car concentration is \( c_0 \), and the typical intrinsic velocity range is \( v_0 \). Let the steady state cluster density be \( c < c_0 \), which implies the typical cluster size \( \langle m \rangle = c_0/c \). If large clusters form, \( \langle m \rangle \gg 1 \), then the overall escape rate can be estimated by \( \langle m \rangle t_0^{-1} \). Assuming that most collisions involve fast cars and slow clusters, the typical collision rate is \( cv_0 \). In the steady state, the number of cars joining and leaving clusters should balance and thus, \( c_0/(ct_0) = v_0c \) or \( c = (c_0/v_0 t_0)^{1/2} \). This heuristic argument gives the leading behavior of the average cluster size

\[
\langle m \rangle \sim R^{1/2} \quad \text{when} \quad R \gg 1,
\]

where \( R \) is the ratio of the two elementary time scales, the escape time \( t_{\text{esc}} = t_0 \) and the collision time \( t_{\text{col}} = (c_0 v_0)^{-1} \):

\[
R = \frac{t_{\text{esc}}}{t_{\text{col}}} = c_0 v_0 t_0.
\]

We term this dimensionless quantity the “collision number”. For large collision numbers, large clusters occur according to Eq. (1), while for small collision numbers

Fig.1 Space time diagram of the traffic model. Formation of a cluster with two fast cars is shown to the left and formation of a one car cluster and its breakup due to escape is shown to the right.
the effect of collisions is small \((m) \approx 1 + aR\). Analysis of the master equations detailed below confirms this heuristic picture under quite general conditions.

The rest of this paper is organized as follows. In Sec. II, the master equations are used to derive analytical expressions for various velocity distributions in the steady state. The leading behavior in the limiting cases of light and heavy traffic are highlighted in Sec. III. Explicit expressions are written for the special cases of uniform initial and final velocity distributions as well as discrete distributions in Sec. IV. We close with some open problems, a discussion, and possible applications.

II. THEORY

In the following, it is convenient to introduce dimensionless velocity \(v/v_0 \rightarrow v\), space \(x_0 \rightarrow x\), and time \(\epsilon_0 v_0 t \rightarrow t\) variables. This rescales the escape rate \(t^{-1}_0\) to the inverse collision number \(R^{-1}\). Let \(P(v, t)\) be the density of clusters moving with velocity \(v\) at time \(t\). Initially, isolated single cars drive with their intrinsic velocities drawn from the distribution \(P_0(v) \equiv P(v, t = 0)\). This intrinsic velocity distribution is normalized to unity, \(\int dv P_0(v) = 1\). The flow is invariant under a velocity translation, and the minimal velocity is set to zero.

Initially, the velocities and the positions of the particles are uncorrelated. Escape effectively mixes the positions and the velocities. Assuming that no spatial correlations develop, a closed master equation for the velocity distribution of clusters \(P(v, t)\) can be written

\[
\frac{\partial P(v, t)}{\partial t} = R^{-1} [P_0(v) - P(v, t)]
- P(v, t) \int_0^v dv' (v - v') P(v', t).
\]

The density of slowed down cars with intrinsic velocity \(v\) is \(P_0(v) - P(v, t)\). Such cars escape their clusters with rate \(R^{-1}\), and thus the escape term. Collisions occur with rate proportional to the velocity difference as well as the product of the velocity distributions. The integration limits ensure that only collisions with slower cars are taken into account.

Steady state is obtained by taking the long time limit \(t \rightarrow \infty\) or \(\partial/\partial t = 0\). Since we are primarily interested in the steady state, we omit the time variable \(P(v) \equiv P(v, t = \infty)\). Equating the right-hand side of the master equation to zero, a relation between the intrinsic car distribution and steady state cluster distribution emerges

\[
P(v) \left[1 + R \int_0^v dv' (v - v') P(v') \right] = P_0(v).
\]

Given the intrinsic velocity distribution this relation gives the final cluster velocity distribution only implicitly. In contrast, the inverse problem is simpler as knowledge of

the final distribution, the observed quantity in real traffic flows, gives explicitly the intrinsic distribution. We confirm that in the limit \(R \rightarrow \infty\), all clusters move with the minimal velocity \(P(v) \rightarrow \delta(v)\), while in the limit \(R \rightarrow 0\), all cars move with their intrinsic velocity \(P(v) \rightarrow P_0(v)\).

It is convenient to transform the integral equation \(4\) into a differential one. Consider the auxiliary function

\[
Q(v) = R^{-1} + \int_0^v dv' (v - v') P(v'),
\]

which gives the cluster distribution by second differentiation

\[
P(v) = Q''(v).
\]

Thence, the steady state condition \(4\) reduces to the second order nonlinear differential equation

\[
Q(v)Q''(v) = R^{-1} P_0(v).
\]

The boundary conditions are \(Q(0) = R^{-1}\) and \(Q'(0) = 0\). The cluster concentration is found from the cluster velocity distribution using

\[
c = \int_0^\infty dv P(v),
\]

and the average cluster mass is simply \(\langle m \rangle = c^{-1}\). Furthermore, the average cluster velocity is obtained from

\[
\langle v \rangle = c^{-1} \int_0^\infty dv v P(v).
\]

Cars may drive with a velocity smaller than their intrinsic one, and it is natural to consider the joint velocity distribution \(P(v, v')\), the density of cars of intrinsic velocity \(v\) driving with velocity \(v'\). The master equation for the joint distribution reads

\[
\frac{\partial P(v, v')}{\partial t} = - R^{-1} P(v, v') + (v - v') P(v) P(v')
- P(v, v') \int_0^{v'} dv'' (v' - v'') P(v'')
+ P(v') \int_{-v'}^{v'} dv'' (v'' - v') P(v, v'').
\]

The first term accounts for loss due to escape, while the rest of the terms represent changes due to collisions. For instance, the last term describes events where a \(v\)-car driving with velocity \(v''\) is further slowed down after a collision with a \(v'\)-cluster. One can verify that the total number of \(v\)-cars,

\[
P_0(v) = P(v) + \int_0^v dv' P(v, v'),
\]

is conserved by the evolution Eqs. \(3\) and \(11\).

At the steady state, the joint distribution satisfies
\[ P(v, v')Q(v') = (v - v')P(v)P(v') + Q(v, v')P(v'), \quad (12) \]

obtained using the definition of \( Q(v) \) and the joint auxiliary function

\[ Q(v, v') = \int_v^w dw(w - v')P(w, v). \quad (13) \]

Although the collision number \( R \) does not appear in Eq. (12) explicitly, it enters through \( Q(v) \) and \( P(v) \).

Combining (12) with Eqs. (13), (10), and using the relationship \( P(v, v') = \partial Q(v, v')/\partial v' \) yields

\[ \frac{\partial}{\partial v'} \left[ \frac{Q^2(v')}{Q(v')} \right] = (v - v')P(v)P(v'). \quad (14) \]

Integrating twice over \( v' \) gives the joint auxiliary function in terms of the single variable functions

\[ Q(v, v') = P(v)Q(v') \int_v^w \frac{du}{Q^2(u)} \int_v^u dw(w - v')P(w). \quad (15) \]

The boundary conditions \( Q(v, v) = \frac{\partial}{\partial v}Q(v, v)|_{v'=v} = 0 \) were used to obtain this expression. Furthermore, integration by parts of \( \int_v^w dw(w - v')P(w) = \int_v^w dw - \int_v^w \frac{dv}{vP(v)} \) gives

\[ Q(v, v') = P(v) \left[ Q(v)Q(v') \int_v^w \frac{du}{Q^2(u)} - (v - v') \right]. \quad (16) \]

Substituting Eq. (16) into (12) and then replacing \( PQ \) with \( R^{-1}P_0 \) we find a relatively simple expression for the joint velocity distribution

\[ P(v, v') = \frac{P_0(v)P_0(v')}{Q(v')} \int_v^w \frac{du}{[RQ(u)]^2}. \quad (17) \]

Another interesting quantity is the flux or the average velocity given by \( J = \int dv \ vP(v) + \int_0^v dw wP(w, v) \).

From the definition of the joint auxiliary function, the second integral is identified with \( Q(v, 0) \), implying

\[ J = \int_0^\infty dv \ [vP(v) + Q(v, 0)]. \quad (18) \]

The integrand can be considerably simplified using Eq. (10), \( Q(0) = R^{-1} \), and Eq. (11). The term \( vP(v) \) cancels and we find a useful expression for the flux

\[ J = \int_0^\infty dvP_0(v) \int_0^v \frac{du}{[RQ(u)]^2}. \quad (19) \]

One can also ask for the actual velocity distribution of cars defined via

\[ G(v) = P(v) + \int_v^\infty dw \ P(w, v). \quad (20) \]

Substituting the joint velocity distribution allows us to express the car velocity distribution via single variable distributions

\[ G(v) = P(v) \left[ 1 + R \int_v^\infty dw \ P_0(w \int_v^w \frac{du}{[RQ(\mu)]^2}) \right]. \quad (21) \]

The car velocity distribution satisfies the normalization conditions \( 1 = \int dv \ G(v) \) and \( J = \int dv \ vG(v) \).

In summary, for arbitrary intrinsic velocity distributions, the entire steady state problem is reduced to the nonlinear second order differential equation (1). Given \( Q(v) \), steady state characteristics such as \( P(v) \), \( P(v, v') \), \( J \), and \( G(v) \) can be calculated using the explicit formulae (3), (7), (19), and (21), respectively.

### III. LIMITING CASES

Although one cannot solve Eq. (7) analytically in general, it is still possible to obtain the leading behavior in the limits of \( R \to 0 \) and \( R \to \infty \).

#### A. Low Collision Numbers

To analyze the flow characteristics in the collision-controlled regime, \( R \ll 1 \), we use Eq. (4) to write \( P(v) \) as a perturbation expansion in \( R \):

\[ P(v) \cong P_0(v) \left[ 1 - R \int_0^v dv' (v - v')P_0(v') \right]. \quad (22) \]

In this limit, the auxiliary function is roughly constant \( RQ(v) \cong 1 \), and Eq. (17) gives the joint distribution to first order in \( R \)

\[ P(v, v') \cong R(v - v')P_0(v)P_0(v'). \quad (23) \]

The final density and flux are

\[ c \cong 1 - c_1R, \quad J \cong J_0 - J_1R, \quad (24) \]

with \( c_1 = \int dv \ vP_0(v) \int_0^v dv' (v - v')P_0(v') \), \( J_0 = M_1 \), \( J_1 = M_2 - M_1^2 \) (\( M_n \) are the moments of the intrinsic velocity distribution \( M_n = \int dv v^n P_0(v) \)). The coefficient \( J_1 \geq 0 \) equals the width of the initial velocity distribution. This gives a simple intuitive picture: the larger the initial velocity fluctuations, the smaller the flux. By either substituting the joint velocity distribution into the definition of \( G(v) \), or from Eq. (21), the car velocity distribution is

\[ G(v) \cong P_0(v) \left[ 1 + R \int_0^\infty dv' (v' - v)P_0(v') \right]. \quad (25) \]

As the integral is over the entire velocity range, the order \( R \) correction is positive for small \( v \) and negative for large \( v \). In other words \( G(v) > P_0(v) \) when \( v < v_c \). The crossover velocity equals the average intrinsic velocity \( v_c = J_0 = M_1 \), as seen from Eq. (23).

We conclude that the collision-controlled limit is weakly interacting, explicit expressions for the leading corrections of the steady state properties are possible.
B. Large Collision Numbers

The analysis in the complementary escape-controlled regime, $R \gg 1$, is more subtle since the condition $R \int_0^\infty dv (v-v') P_0(v') \ll 1$ is satisfied only for small velocities. No matter how large $R$ is, sufficiently slow cars are not affected by collisions, and $P(v)$ is given by Eq. (22) when $v \ll v^*$. The threshold velocity $v^* \equiv v^*(R)$ is estimated from $R \int_0^{v^*} dv (v-v') P_0(v') \sim 1$.

It is useful to consider algebraic intrinsic distributions

$$P_0(v) = (\mu + 1) v^\mu, \quad \mu > -1, \tag{26}$$

in the velocity range [0:1] with the prefactor ensuring unit normalization. For such distributions, the threshold velocity decreases with growing $R$ according to $v^* \sim R^{-\frac{\mu}{\mu+2}}$. For $v \gg v^*$, the integral in Eq. (i) dominates over the constant factor and $R P(v) \int_0^v dv (v-v') P_0(v') \sim v^\mu$. Anticipating an algebraic behavior for the cluster velocity distribution, $P(v) \sim R^\mu v^\mu$ when $v \gg v^*$, gives different answers for positive and negative $\mu$. The leading behavior for $v \gg v^*$ can be summarized as follows

$$P(v) \sim \begin{cases} R^{-1/(\mu+2)} v^{\mu-1} & \mu < 0; \\ R^{-2} v^{-1} \ln(v/v^*)^{-1} & \mu = 0; \\ R^{-2} v^{-\mu-1} & \mu > 0. \end{cases} \tag{27}$$

The small and large velocity components of $P(v)$ match at the threshold velocity, $P(v^*) \sim P_0(v^*)$. Careful analysis, detailed in the following section, is needed to get the logarithmic corrections in the borderline case $\mu = 0$. Substituting the leading asymptotic behavior of Eq. (27) into Eq. (i), the average cluster size is found

$$\langle m \rangle \sim \begin{cases} R^{(\mu+1)/(\mu+2)} & \mu < 0; \\ (R/\ln R)^{1/2} & \mu = 0; \\ R^{1/2} & \mu > 0. \end{cases} \tag{28}$$

Similarly, the average cluster velocity defined in Eq. (i) is evaluated

$$\langle v \rangle \sim \begin{cases} R^{\mu/(\mu+2)} & \mu < 0; \\ 1/|\ln R| & \mu = 0; \\ \text{const} & \mu > 0. \end{cases} \tag{29}$$

Two distinct regimes of behavior emerge. For $\mu > 0$, car-cluster collisions dominate while for $\mu < 0$ cluster-cluster collisions dominate. The scaling argument given in the introduction assumes the former picture, and thus it does not hold in general. A postiori, one can extend the scaling argument to the $\mu < 0$ regime. The argument becomes involved, and we do not present it here. Interestingly, in the cluster-cluster dominated regime, the scaling behavior for the average cluster size, $\langle m \rangle \sim R^\alpha$ with $\alpha = (\mu + 1)/(\mu + 2)$, is identical to the kinetic scaling, $\langle m \rangle \sim (c_0 v_0 t)^\alpha$ with the same $\alpha$, found in the no passing limit [16]. This suggests an analogy between the dimensionless collision number $R = c_0 v_0 t_0$ and the dimensionless time $c_0 v_0 t$. The flux can be evaluated in a similar fashion using Eq. (i)

$$J \sim v^* \sim R^{-\frac{\mu}{\mu+2}}. \tag{30}$$

Interestingly, the flux is proportional to the threshold velocity $v^*$. As a result, the flux exponent $\gamma = 1/(\mu + 2)$ is a regular function of $\mu$ unlike the cluster size exponent $\alpha$. Eq. (i) is also consistent with identification of the crossover velocity $v_\psi$ with the marginal velocity $v^*$. No flux reduction occurs when the intrinsic distribution is dominated by fast cars, i.e., in the limit $\mu \to -1$. In the other extreme, the maximal flux reduction $J \sim R^{-1}$ is realized when $\mu \to -1$.

The car velocity distribution is strongly enhanced in the low velocity limit, as seen by evaluating Eq. (i)

$$G(v) \sim R^{\frac{\mu-1}{\mu+2}} v^\mu (1 - \text{const} \times v^{\mu+1}), \quad v \ll v^*. \tag{31}$$

As a check of self-consistency, one can verify that $1 \sim \int_0^{v^*} dv G(v)$, and $J \sim v^* \sim \int_0^{v^*} dv v G(v)$. Near the maximal velocity, the car velocity distribution approaches the cluster distribution $G(v) \approx P(v)$.

In summary, as $R \to \infty$ the solution to the differential equation (i) exhibits a boundary layer structure. Inside the boundary layer, $v < v^*$, the cluster velocity distribution is only slightly affected by collisions, while in the outer region $v > v^*$, the cluster velocity distribution is much smaller than the intrinsic velocity distribution. The threshold velocity $v^*$ is determined by the small velocity behavior of the intrinsic velocity distribution, and for the algebraic distributions (26) we have found $v^* \sim R^{-\frac{\mu}{\mu+2}} \to 0$. The behavior detailed above in the escape controlled limit is not restricted to purely algebraic distributions but is quite general. We conclude that a single parameter

$$\mu = \lim_{v \to 0} \frac{\partial}{\partial v} \ln P_0(v) \tag{32}$$

determines the behavior as $R \to \infty$. In short, extreme statistics underly the escape-limited flow properties. Additionally, an interesting transition between a slow and a fast velocity dominated flow occurs at $\mu = 0$.

IV. EXAMPLES

Although the above analysis is quite general, it applies only to the limiting values of $R$. To examine intermediate behavior, it is also useful to obtain explicit solutions for some special cases. Below, we consider two relevant cases: uniform $P_0(v)$ and $P(v)$. We also obtain explicit expressions in the case of discrete velocity distributions.
A. Uniform Intrinsic Distribution

We now consider the case of a uniform intrinsic distribution, \( P_0(v) = 1 \) for \( 0 < v < 1 \). This case appears to be the most relevant to real traffic flows since the intrinsic velocity distribution should be regular near the the minimal velocity. Integrating \( QQ'' = -R^{-1} \) subject to the boundary conditions \( Q(0) = R^{-1} \) and \( Q'(0) = 0 \) gives \( Q' = \sqrt{2R^{-1}\ln(RQ)} \). Second integration gives

\[
\int_1^{RQ} \frac{dq}{\sqrt{2\ln q}} = v\sqrt{R},
\]

and thus implicitly determines \( Q(v) \). Evaluating the leading behavior when \( R \gg 1 \), we find

\[
\langle m \rangle \simeq \sqrt{\frac{R}{\ln R}}, \quad \langle v \rangle \simeq \frac{1}{\ln R}, \quad J \sim \sqrt{\frac{\pi}{2R}}.
\]

Fig. 2 shows the velocity distribution obtained numerically using Eqs. (33) and (21) for \( R = 10 \). For \( v \ll v^* \), \( G(v) \gg P_0(v) \), and for \( v \gg v^* \), \( G(v) \simeq P(v) \ll P_0(v) \). The calculated distributions are consistent with the predictions, \( G(0) \sim R^{1/2} \) and \( v^* \sim R^{-1/2} \). The car velocity distribution is linear near the origin in agreement with Eq. (21).

\[
P_0(v) = c \left[ 1 + \frac{1}{2}Rcv^2 \right].
\]

The overall initial concentration is unity, thereby relating \( R \) and \( c \) via \( 1 = c + \frac{1}{6}Rc^2 \). The flux is calculated from Eq. (19),

\[
J = \frac{(3 + \lambda)\sqrt{\lambda} \tan^{-1} \sqrt{\lambda} + \lambda - \ln(1 + \lambda)}{3R},
\]

with \( \lambda = \frac{1}{2}Rc = (3/2) \left[ \sqrt{1 + 2R/3} - 1 \right] \). These explicit solutions agree with our low and high \( R \) predictions. For instance, when \( R \gg 1 \) we find \( \langle m \rangle \sim R^{1/2} \) and \( J \sim R^{-1/4} \). If we look at the initial distribution, \( P_0(v) \simeq (6/R)^{1/2} + 3v^2 \), then the constant part is negligible and the distribution corresponds to the \( \mu = 2 \) case of the power-law distribution (24). For this case the size exponent is \( \alpha = 1/2 \) and the flux exponent is \( \gamma = 1/4 \), see (28) and (30), in agreement with our findings.

Substituting \( P_0(v) \) and \( Q(v) \) in Eq. (27) and performing the integration gives the joint distribution

\[
P(v, v') = 2R^{-1}\lambda^2 \frac{(v - v')(1 - \lambda vv')}{1 + \lambda vv'} + 2R^{-1}\lambda^3/2(1 + \lambda v^2) \left[ \tan^{-1}(\lambda^{1/2}v) - \tan^{-1}(\lambda^{1/2}v') \right].
\]

A direct integration of the joint distribution confirms the conservation law (17), thus providing a useful check of self-consistency. The joint velocity distribution is linear in the velocity difference for small \( v \) and \( v' \). This is reminiscent of the small collision number behavior of Eq. (23). As the velocity difference increases, significant curvature develops (see Fig. 3).

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\[
J = \frac{(3 + \lambda)\sqrt{\lambda} \tan^{-1} \sqrt{\lambda} + \lambda - \ln(1 + \lambda)}{3R},
\]
C. Discrete Velocity Distribution

The results formulated for continuous distributions can be used to study the special case of discrete velocity distribution as well. Here we quote the results in terms of the original (non-dimensionless) quantities. Consider the intrinsic velocity distribution

\[ P_0(v) = \sum_{i=1}^{n} c_i \delta(v - v_i), \]  

(38)

with \( v_1 < v_2 < \cdots < v_n \). We denote by \( p_i \) the discrete counterpart of the cluster velocity distribution, e.g., \( P(v) = \sum_{i=1}^{n} p_i \delta(v - v_i) \). The steady state condition of Eq. (3) reads

\[ p_i \left[ 1 + t_0 \sum_{j=1}^{i-1} (v_i - v_j) p_j \right] = c_i. \]  

(39)

Substituting the intrinsic velocity distribution and solving iteratively, we get

\[ p_1 = c_1 \]
\[ p_2 = \frac{c_2}{1 + c_1 (v_2 - v_1) t_0} \]
\[ p_3 = \frac{c_3}{1 + c_1 (v_3 - v_1) t_0 + \frac{c_2 (v_2 - v_3) t_0}{1 + c_1 (v_2 - v_1) t_0}} \]

etc. Rather than a solution to a differential equation, the steady state solution is in the form of an explicit continued fraction. This expression involves the initial distribution and the velocity differences, and can be useful to analyze data in a histogram form. In a similar way, explicit expressions can be obtained for the rest of the steady state properties.

V. DISCUSSION

An important property, the cluster size distribution is absent from our treatment so far. Naturally, the size and the velocity of a cluster are strongly correlated and one must consider \( P_m(v) \), the distribution of clusters of size \( m \) and velocity \( v \). The joint cluster size-velocity distribution obeys the master equation

\[ \frac{\partial P_m(v)}{\partial t} = R^{-1} \left[ m P_{m+1}(v) - (m - 1) P_m(v) \right] + R^{-1} \delta_{m,1} [P_0(v) - P(v)] - F(v) P_m(v) \]
\[ + \int_0^v dv' \left( v' - v \right) \sum_{j=1}^{m} P_j(v') P_{m-j}(v) \]

(41)

which applies for all \( m \geq 1 \). Terms proportional to \( R^{-1} \) account for escape, while the rest represent collisions. The factor \( F(v) = \int_0^v dv' |v - v'| P(v') \) measures the overall collision rate experienced by a \( v \)-cluster, and is reminiscent of kinetic theory. Summing Eqs. (41), one recovers the rate equation (3) for \( P(v) = \sum_m P_m(v) \). On the other hand, integration over the entire velocity range does not reduce Eqs. (41) to a closed system of rate equations for the cluster size distribution \( P_m = \int dv P_m(v) \). Therefore, the entire joint distribution is needed to determine \( P_m \). Additionally, we note that in Eqs. (3) and (41), the integration limits include only slower velocities, a feature that considerably simplifies the analysis. This property is lost for Eqs. (41), thereby putting analytical solution out of reach.

Nevertheless, a leading order analysis is still possible for low collision numbers. In the limit \( R \ll 1 \), we find

\[ P_1(v) \approx P_0(v) \left[ 1 - R \int_0^\infty dv' |v - v'| P_0(v') \right], \]
\[ P_2(v) \approx R P_0(v) \int_0^\infty dv' (v' - v) P_0(v'), \]
\[ P_m(v) \approx R^{m-1} P_m(v). \]

(42)

Heuristically, clusters with \( m \) cars are created by \( m - 1 \) collisions and a factor \( R \) is generated in each collision. Although the functions \( P_m(v) \) are quite complicated, the overall prefactor \( R^{m-1} \) suggests an exponential cluster size distribution in the dilute limit.

In the special case of a bimodal velocity distribution, a solution is possible. The structure of clusters here is simple: A cluster of size \( m \) consists of a leading slow car and \( m - 1 \) fast cars behind it. The rate equation (41) simplifies considerably, and a Poisson size distribution is found \( P_m \propto e^{-f} f^{m-1}/(m-1)! \). The collision rate \( f \) is equal to the product of the escape time, the velocity difference, and the fast car concentration. This steady state distribution satisfies a detailed balance condition as the escape rate and the collision rate are equal microscopically, \((m - 1)P_m = f P_{m-1}\). Thus, an equilibrium steady state is reached. However, in general, a nonequilibrium steady state is approached with the collision rate and the escape rate balancing only macroscopically. This is seen by noting that the cluster size may increase by an arbitrary number due to collisions, but can decrease only by one due to escape.

Further investigation of the collision term in the rate equation will be useful as well. In the no escape case \( R^{-1} = 0 \), the exact Boltzmann equation

\[ \frac{\partial P(v, t)}{\partial t} = -P(v, t) \int_0^v dv' (v - v') P_0(v') \]

(43)
The model and the results presented above can be generalized to study other traffic situations. First, a multilane flow can be treated as a system of coupled one lane flows. Escape naturally couples neighboring lanes. Second, a natural generalization is to heterogeneous situations where passing is allowed only in a fraction $r$ of the road. We expect that for regular distribution of these passing segments the problem should reduce to the homogeneous case with a renormalized collision number $R/r$.

The most challenging question appears to be the role played by the escape mechanism. We considered the case where all cars are equally likely to escape. This assumption simplified the master equation considerably as the escape term is linear in $P(v)$, and thus, is exact. The complementary case where only the first car in the cluster can escape is interesting as well. For low collision numbers, large clusters are unlikely, and the behavior is independent of the escape mechanism. However, for high collision numbers the escape mechanism becomes weaker and larger clusters form. Indeed, a scaling argument along the lines of Eq. (1) gives $\langle m \rangle \sim R$ in the car-cluster dominated regime.

In conclusion, despite the simplifying assumptions made, the suggested model results in quite realistic behavior. The overall picture is both familiar and intuitive: due to the presence of slower cars, clusters form and the overall flux is reduced. For heavy traffic, the characteristics of the flow are solely determined by the distribution of slow cars. A single dimensionless parameter, the collision number $R$ ultimately determines the nature of the steady state. The stationary distributions obtained analytically provide a simple practical recipe for calculating the flow properties for arbitrary intrinsic distributions. It will be interesting to analyze observed traffic data using these theoretical tools.

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