A statistical physics of stationary and metastable states

A Cabo\(^1\), S Curilef\(^2\), A González\(^1\), N G Cabo-Bizet\(^3,4\) and C A Vera\(^5\)

\(^1\) Departamento de Física Teórica, Instituto de Cibernética, Matemática y Física, Calle E, No. 309, Vedado, La Habana, Cuba
\(^2\) Facultad de Física, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta, Chile
\(^3\) Departamento de Física, Centro de Aplicaciones Tecnológicas y Desarrollo Nuclear (CEADEN), Calle 30, esq. a 5ta Ave, Miramar, La Habana, Cuba
\(^4\) Physikaliches Institut der Universität Bonn, Nussallee 12, Bonn, Germany
\(^5\) Instituto de Física, Universidad de Antioquia, Medellín, Colombia

E-mail: cabo@icmf.inf.cu, scurilef@ucn.cl, namuye@gmail.com and caverac@gmail.com

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Abstract. We present a generalization of Gibbs statistical mechanics designed to describe a general class of stationary and metastable equilibrium states. It is assumed that the physical system maximizes the entropy functional \(S\) subject to the standard conditions plus an extra conserved constraint function \(F\), imposed to force the system to remain in the metastable configuration. After requiring additivity for two quasi-independent subsystems, and the commutation of the new constraint with the density matrix \(ρ\), it is argued that \(F\) should be a homogeneous function of \(ρ\), at least for systems in which the spectrum is sufficiently dense to be considered as continuous. Therefore, surprisingly, the analytic form of \(F\) turns out to be of the kind \(F(ρ_i) = ρ_i^q\), where the \(ρ_i\) are the eigenvalues of the density matrix and \(q\) is a real number to be determined. Thus, the discussion identifies the physical relevance of Lagrange multiplier constraints of the Tsallis kind and their \(q\) parameter, as enforced by the additivity of the constraint \(F\) which fixes the metastable state. An approximate analytic solution for the probability density is found for \(q\) close to unity. The procedure is applied to describe the results from the plasma experiment of Huang and Driscoll. For small and medium values of the radial distance, the measured density is predicted with a precision similar to that achieved by minimal enstrophy and Tsallis procedures. Also, the particle density...
is predicted at all the radial positions. Thus, the discussion gives a solution to the conceptual difficulties of the two above mentioned approaches as applied to this problem, which both predict a non-analytic abrupt vanishing of the density above a critical radial distance.

**Keywords:** rigorous results in statistical mechanics, spin glasses (theory), metastable states

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1. **Introduction**

Nowadays, a relevant topic of research in statistical physics is the characterization of metastable and non-equilibrium states of physical systems [1]–[6]. Multiple efforts to explain this kind of behavior have been made, but the discussion continues intensively because of the large variety and complexity of the physical processes involved [5, 10]–[12].

In this work we intend to give a complete formulation of a statistical description of metastable states, for which only the basic formal ideas have been presented before, in [13]. We expect to describe the statistical physics of a large class of metastable and stationary states. Our main suggestion for dealing with these situations is that the statistical properties of the metastable or stationary non-equilibrium states depart from the Gibbs thermal states in equilibrium in the following sense: the metastable or non-equilibrium configuration may be the result of an additional dynamical constraint, keeping the system for a while in the metastable state if the restriction is only an approximate one, and keeping the system in a time independent configuration in the case of non-equilibrium stationary states. After the removal of the condition, the system is allowed to evolve in a thermal state described by the Bloch density matrix. Hence, the time independence of the statistical properties of the systems leads to commutation of the density matrix and the Hamiltonian during the time for which the metastable or stationary state exists. As was mentioned, we will assume that an extra constraint $F$ is conserved in time; then $F$ will also commute with the Hamiltonian. In the large class of systems having a non-degenerate
spectrum of the energy, the Hamiltonian $H$, the density matrix $\rho$ and the extra constraint $F$ can be diagonalized in the same common basis of eigenstates. The same property is not necessarily valid when the energy spectrum is degenerate. From now on, we will simply restrict the discussion to systems exhibiting the aforementioned commutativity among $H$, $\rho$ and $F$.

Before continuing, we should remark that the relevance of incorporating constraints (in addition to the usually imposed energy conservation one) in the search for statistical descriptions of metastable states has been underlined in [6]. The discussion presented in the present work, then, extends the investigation of the role of such constraints in understanding the statistical mechanics of stationary non-equilibrium and metastable states. It should also be stated that the discussion in [6] is in fact more general than the present one, when considered as a search for allowed entropy formulas in constructing valid statistical descriptions of metastable and stationary states. However, the basic research aim of the present study is not determining possible entropy definitions. On the contrary, our purpose is to (after assuming the standard form for the entropy) construct a procedure in which the metastable and stationary states are established thanks to the action of constraints additional to the usual one: the energy conservation. The independent nature of the contributions of the two works is established by the fact that the quantity $F[\rho] = f\rho^q$ appearing in our paper is linked not with an alternative form of the entropy, but with a structure required for the added constraint imposed, in order to assure the additive property for the statistical procedure.

Let us continue now, by assuming that the extra constraint defines the metastable state. Then, it is possible to note the existence of some related conserved constraints. A particularly interesting one is given by the product operator $HF$. Note that the assumed commutativity of $\rho$ with $F$ implies the possibility of expressing $F$ as a function of $\rho$. The above remark motivated our idea of constructing a special analytical form of the condition to be incorporated in the Lagrange multiplier scheme for the maximization of the entropy. This condition will be constructed as a modified expectation value of the energy, in which in the density matrix in the usual mean value is substituted by the constraint function $F$. This quantity turns out to be conserved as a direct consequence of the validity of the constraint $F$.

As a result of the above construction, it follows that, when the additivity condition of the resulting statistical description is assumed to be valid for two approximately independent subsystems, the constraint function $F$ should have the Tsallis homogeneous structure $F(\rho) = C_q\rho^q$ where $q$ is a real number to be determined. It should be underlined that this conclusion follows for systems in which the energy spectrum is sufficiently dense to be considered continuous. For such systems, the analysis seems to indicate an interpretation of the Tsallis $q$ parameter as corresponding to the degree of homogeneity of the constraint $F$ after being represented as a function of the density matrix $\rho$.

We consider that this is the central conclusion of this work. Under some general assumptions, including a wide class of physical systems (specifically a non-degenerate and continuous energy spectrum), it had been argued that the constraint (which is assumed to define the metastable or stationary non-equilibrium state) should have a strict power law form if additivity should be obeyed. Surprisingly, this analytic form is independently shown by the Tsallis $q$ energy constraint. In our view, this is a strong conclusion. It suggests the interesting possibility of alternatively justifying the successful
fitting description which the Tsallis statistical physics furnishes for the measured data in many physical systems, but by means of the proposed scheme, in which the entropy is taken as the usual one. In other words, it is not ruled out that the coincidence in form of the constraints in the two descriptions will allow them to determine similar approximate results for the large number of systems for which the Tsallis approach gives very approximate fitting data. The need for that strict form of the constraint after additivity is assumed has not, to our knowledge, been reported in the literature.

The above remarks, and also the present study, might be thought of as in some way related to an issue discussed in [7]–[9]. In these works the question is investigated of whether the so-called $q$ Gaussian distributions (which are linked with the Tsallis approach) could analytically describe the statistical properties of systems for which the Tsallis statistics is expected to work exactly. But, it can be stated that our analysis has only an indirect link with the cited works. This is because, in fact, our discussion did not make use of a $q$ dependent entropy function, since the formula employed is the standard one. The appearance of the parameter $q$ in our work is related to the form of the added constraint, which turned out to coincide with the Tsallis one. In these circumstances, we have no a priori reasons to expect the $q$ Gaussian distributions to emerge as exact solutions in the proposed scheme. However, the analysis suggests a possibility for understanding the very approximate reproduction of the exact results (derived for a few models in [7]–[9]) using the ones obtained by fitting them with $q$ Gaussian distributions, in spite of the lack of a rigorous coincidence. At least for values of $q$ close to unity, it seems to be feasible that the optimization of the standard entropy formula, rather than the Tsallis one, but including the $q$ dependent new constraint, could allow us to explain the small differences between the exact statistical properties and the ones given by fitted $q$ Gaussian distributions. This possibility is expected to be considered elsewhere.

In order to start investigating its implications, the procedure was applied in this work to describe the experimental data obtained from the plasma experiment performed by Huang and Driscoll [10]. The measurements of the electron density were chosen as an input of an iterative procedure for solving the equations for the probability density $\rho$. The results predicted for the densities at radial distances at which the their values are not small describe the experimental measurements with similar quality to the ones following from the two theoretical schemes existing for this problem. They are the minimization of the enstrophy on one hand and the maximization of the Tsallis entropy on the other one [10,12]. It is known that these procedures are equivalent and their applications to the problem considered are recognized as the main studies existing in recent literature [12]. In connection with the zone of small radial distances, the two analyses describe this region with similar precision and their results are close to the experimental data. However, at large radial distances, where the density is small enough, their results predict a non-analytic behavior [10]. Our conclusions improve the outcome of the aforementioned theoretical procedures in this large distance region, by predicting a smooth vanishing tail of the density profile, as has been experimentally measured. However, since the density values at large radial distances are comparable with the experimental errors, the tails could also be the result of the inaccuracy of the measurements. In spite of the possibility just signaled, each one of the aforementioned alternative procedures needs to justify the presence of a non-analytical abrupt vanishing of the density distributions at a radial distance value, as coming from the requirement for non-negative character of the
electron density. In the discussion given here, such a procedure is not necessary and, as noted before, the dependence of the density as a function of the radial distance is predicted to smoothly decay.

It should also be remarked that the iterative solution found shows a value of the parameter \( q \) which is very close to unity and also the resulting values of the Lagrange multipliers assume very high values. These results led to the idea that the proposed statistical description for metastable states could be associated with an analytic dependence that is singular around the value \( q = 1 \), which corresponds to the Gibbs distribution. In an appendix, we address this problem by searching for solutions showing singular single-pole structures within a Laurent expansion in \( \epsilon = q - 1 \). Solutions for the probability distribution showing a pole singularity for the Lagrange multipliers were found up to the first order in the series expansion in \( \epsilon \). This result permits us to understand the properties of the iterative solution found in section 4. An interesting outcome is that the probability distribution of zero order in \( \epsilon \) takes an exponential form similar to the Gibbs one, but with a ‘distorted’ energy spectrum.

It can be remarked that a less elaborate form of the scheme (whose formal elements were advanced before, in [13]) was applied to describe the quasi-equilibrium state of excitonic polaritons [14]. These polaritons are quasibosonic quasiparticles arising from the strong interaction of excitons and confined light modes in a microcavity. A finite model was studied in [14] with the help of a master equation for the density matrix. The result is that the density matrix approximately commutes with the Hamiltonian. We assumed that the quasi-equilibrium is the result of a constraint in phase space and showed that an analytical expression for the density matrix resulting from our procedure gives a reasonable fit to the numerical results.

The paper proceeds as follows. In section 2 we present the basic elements of the proposed statistical description. Section 3 then continues by arguing that the assumption of additivity in the statistical description implies that the constraint function should be a simple \( q \) power of the density matrix. In section 4 the procedure is applied to describe the results of the plasma experiment of Huang and Driscoll. Further, section 5 summarizes the results of the work and an appendix describes the finding of an explicit solution of the equation for the distribution for \( q \) close to 1.

2. The statistical mechanics for a class of metastable and stationary states

Let us consider a physical system having a quantum dynamics described by a Hamiltonian \( H \). In the Gibbs approach, the properties of the system in thermal equilibrium are contained in the Bloch density matrix

\[
\rho = \exp \left( -\frac{H}{kT} \right),
\]

which satisfies \([H, \rho] = 0\) and determines the conditional maximum of the entropy functional

\[
S = -\text{Tr}[\rho \log(\rho)] + \alpha(\text{Tr}[\rho H] - E) + \beta(\text{Tr}[\rho] - 1),
\]

where \( \alpha \) and \( \beta \) are Lagrange multipliers corresponding to imposing two conditions: the conservation of the energy \( E \) and the normalization of the trace of the density operator.
\( \rho \). The latter condition is imposed in order to furnish a probability interpretation for the diagonal elements of \( \rho \), this is \( \text{Tr}[\rho] = 1 \). Eventually, additional conserved quantities can be added by also introducing their corresponding multipliers.

Our main assumption in this work is that during a large relaxation time \( \tau \), the physical systems considered are forbidden to reach the thermal equilibrium state, as a consequence of the action of extra constraints. These restrictions are assumed to be dynamically generated and able to obstruct during macroscopic time intervals the standard evolution which normally drops the density matrix to the Bloch form. In other words, we propose the existence of a conserved quantity \( F \), whose effect on the motion of the system is to delay the usual evolution, then leading the system to an intermediate equilibrium state differing from the Gibbs thermal one: the metastable or non-equilibrium stationary state. Therefore, the system is kept in those states during the time interval in which the extra constraint is being satisfied. Thus, as a result of the preceding assumptions we will have

\[
[H, F] = 0.
\]

Therefore, the constraint \( F \) and the density matrix \( \rho \) both commute with the Hamiltonian. As mentioned before, in the large class of systems in which \( H \) has a non-degenerate spectrum, it is possible to simultaneously diagonalize \( H, \rho \) and \( F \), within a common basis of eigenfunctions. However, in a more general situation, we will simply restrict the discussion to the cases where all three quantities commute among themselves. Thus, the conserved constraint \( F \) can be expressed as a certain function of the density matrix \( F = F(\rho) \). From the aforementioned assumptions if follows that the quantity \( F(\rho)H \) is also conserved:

\[
[F(\rho)H, H] = 0. \tag{2}
\]

Further, the evolution of the system in the metastable or stationary states considered also implies the time invariance of the specially constructed quantity

\[
\frac{\text{Tr}[F(\rho)H]}{\text{Tr}[F(\rho)]}.
\]

Now, let us precisely state our main dynamical principle: the metastable and stationary states are determined by the maximization of the entropy \( S \) subject to the conditions of conservation of normalization, energy and the aforementioned constraint. Then, the usual constraints plus the extra one, after being multiplied by their corresponding Lagrange multipliers, are added to the entropy functional to construct the modified form of the Gibbs Lagrange multiplier scheme. The functional takes the explicit form

\[
S = -\text{Tr}[\rho \log(\rho)] + \alpha(\text{Tr}[\rho H] - E) + \beta(\text{Tr}[\rho] - 1) + \gamma \left( \frac{\text{Tr}[F(\rho)H]}{\text{Tr}[F(\rho)]} - E_C \right)
\]

\[
= -\sum_i p_i \log(p_i) + \alpha \left( \sum_i p_i \epsilon_i - E \right) + \beta \left( \sum_i p_i - 1 \right)
\]

\[
+ \gamma \left( \frac{\sum_i F(p_i) \epsilon_i}{\sum_i F(p_i)} - E_F \right), \tag{3}
\]

where \( p_i, i = 1, 2, 3 \ldots \), are the diagonal elements of the density matrix \( \rho = \sum_i p_i |i \rangle \langle i | \).
3. The Tsallis $q$ parameter from the additivity of the description

Let us now consider implications of the modified statistical physics being introduced. They follow after assuming the additivity in the statistical description for the combination of two quasi-independent systems. That is, each one of the subsystems will be assumed to be in the same kind of metastable state, showing statistical properties analogous to those of the whole body. Then, let us consider a pair of such systems that mutually weakly interact. Any one of the bodies is considered in the same kind of metastable state. The entropy and the constraint functions of the first system can be written as

$$S^{(1)} = -\sum_i p_i^{(1)} \log(p_i^{(1)}),$$  
$$E^{(1)} = \sum_i p_i^{(1)} \epsilon_i^{(1)}, \quad 1 = \sum_i p_i^{(1)},$$  
$$E_F^{(1)} = \frac{\sum_i F(p_i^{(1)}) \epsilon_i^{(1)}}{\sum_i F(p_i^{(1)})}.$$  

Analogously, the same quantities for the second system may be expressed as

$$S^{(2)} = -\sum_i p_i^{(2)} \log(p_i^{(2)}),$$  
$$E^{(2)} = \sum_i p_i^{(2)} \epsilon_i^{(2)}, \quad 1 = \sum_i p_i^{(2)},$$  
$$E_F^{(2)} = \frac{\sum_i F(p_i^{(2)}) \epsilon_i^{(2)}}{\sum_i F(p_i^{(2)})}.$$  

Now, let us consider that the statistical description is also valid for the combination of the two systems. Then, the entropy and constraints can be written as follows:

$$S^{(1,2)} = -\sum_{i,j} p_{i,j}^{(1,2)} \log(p_{i,j}^{(1,2)}),$$  
$$E^{(1,2)} = \sum_{i,j} p_{i,j}^{(1,2)} \epsilon_{i,j}^{(1,2)}, \quad 1 = \sum_{i,j} p_{i,j}^{(1,2)},$$  
$$E_F^{(1,2)} = \frac{\sum_{i,j} F(p_{i,j}^{(1,2)}) \epsilon_{i,j}^{(1,2)}}{\sum_{i,j} F(p_{i,j}^{(1,2)})},$$  

where the pair $(i, j)$ indicates the state of the composite system in terms of the indices of the states of the single systems as $|i, j\rangle = |i\rangle \times |j\rangle$.

Henceforth, the assumed separability of the two bodies permits us to write a relation for the probability of the combined states $|i, j\rangle$ in the following way:

$$p_{i,j}^{(1,2)} = p_i^{(1)} p_j^{(2)},$$  
$$\sum_{i,j} p_{i,j}^{(1)} p_{j}^{(2)} = \sum_i p_i^{(1)} \sum_j p_j^{(2)} = 1.$$
Thus, the probability constraint for the individual subsystems is valid for the probability constraint of the composite system.

The additivity of entropies follows in a similar way:

$$ S^{(1,2)} = -\sum_{(i,j)} p_{(i,j)}^{(1,2)} \log(p_{(i,j)}^{(1,2)}) $$

$$ = -\sum_{(i,j)} p_i^{(1)} p_j^{(2)} \log(p_i^{(1)} p_j^{(2)}) $$

$$ = -\sum_i p_i^{(1)} \log(p_i^{(1)}) - \sum_j p_j^{(2)} \log(p_j^{(2)}) $$

$$ = S^{(1)} + S^{(2)}. \quad (15) $$

The same outcome follows for the addition of energies:

$$ E^{(1,2)} = \sum_{(i,j)} p_{(i,j)}^{(1,2)} \epsilon_{(i,j)}^{(1,2)} $$

$$ = \sum_{(i,j)} p_i^{(1)} p_j^{(2)} \left( \epsilon_i^{(1)} + \epsilon_j^{(2)} \right) $$

$$ = \sum_{(i,j)} p_i^{(1)} p_j^{(2)} \epsilon_i^{(1)} + \sum_{(i,j)} p_i^{(1)} p_j^{(2)} \epsilon_j^{(2)} $$

$$ = \sum_i p_i^{(1)} \epsilon_i^{(1)} + \sum_j p_j^{(2)} \epsilon_j^{(2)} $$

$$ = E^{(1)} + E^{(2)}. \quad (16) $$

Let us come to the main new element considered in this section. Assume that the $q$ expectation value of the energy also satisfies the statistical independence condition for the combined system:

$$ F(\rho_{(i,j)}^{(1,2)}) = F(\rho_i^{(1)}) F(\rho_j^{(2)}). \quad (17) $$

Therefore, the $q$ expectation value also satisfies the additivity properties, as follows:

$$ E^{(1,2)}_F = \sum_{(i,j)} F^{(1,2)}_{(i,j)} \epsilon_{(i,j)}^{(1,2)} $$

$$ = \sum_{(i,j)} F_i^{(1)} F_j^{(2)} \left( \epsilon_i^{(1)} + \epsilon_j^{(2)} \right) $$

$$ = \sum_{(i,j)} F_i^{(1)} F_j^{(2)} \epsilon_i^{(1)} + \sum_{(i,j)} F_i^{(1)} F_j^{(2)} \epsilon_j^{(2)} $$

$$ = \frac{\sum_i F_i^{(1)} \epsilon_i^{(1)}}{\sum_i F_i^{(1)}} + \frac{\sum_j F_j^{(2)} \epsilon_j^{(2)}}{\sum_i F_i^{(1)}} $$

$$ = E^{(1)}_F + E^{(2)}_F. \quad (18) $$
However, this condition also imposes a strong restriction on the possible forms of the function $F$ defining the newly introduced expectation value. In order to see this central property of the presentation, the statistical independence condition (17) will be rewritten. Consider $F$ as expanded in powers of its argument $x$, in the general form

$$F(x) = x^\nu \sum_{n=0}^{\infty} f_n x^n.$$  \hfill (19)

Combining equations (17) and (19), it is possible to write

$$(p_i^{(1)} p_j^{(2)})^\nu \sum_{n=0}^{\infty} f_n (p_i^{(1)})^n (p_j^{(2)})^n = (p_i^{(1)})^\nu (p_j^{(2)})^\nu \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_n f_m (p_i^{(1)})^n (p_j^{(2)})^m.$$  \hfill (20)

After a suitable rearrangement of the variables, it follows that

$$0 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_n (p_i^{(1)})^n (p_j^{(2)})^m (\delta_{nm} - f_m).$$  \hfill (21)

Next, let us assume that the domain of the probability $p_i$ for all the values of the indices $i$ is a continuous set. Then, the completeness of the basis formed by the powers of a variable implies

$$f_n (\delta_{nm} - f_m) = 0,$$  \hfill (22)

for all $m$ and $n$.

At this point, let us suppose that $f_m \neq 0$, for a particular value of $m = m_o$. Then, the validity of the above relation directly implies that $f_m = 0$ for all the values of $m \neq m_o$. Therefore, the explicit form of the operator $F$ as a function of $\rho$ can be written in the following simple way:

$$F(\rho) = f_{m_o} \rho^{m_o+\nu} = f_q \rho^q = \sum_i p_i^q |i⟩⟨i|. $$  \hfill (22)

The parameter $q$ is a real number whose value should be dynamically determined. The above conclusion completes the argument indicating the similarity of the Tsallis condition structure with the dynamical constraint $F$ considered here.

### 3.1. Extremal entropy equations

Let us examine in this subsection the $F$ dependent constraint. As noted before, its most interesting property is the full coincidence with the Tsallis $q$ condition

$$E_q \equiv E_F = \frac{\sum_i p_i^q \epsilon_i}{\sum_i p_i^q},$$  \hfill (23)

where the index $q$ is the power of the probability in (22). The modified entropy functional takes the form

$$S = -\sum_i p_i \log(p_i) + \alpha \left( \sum_i p_i \epsilon_i - E \right) + \beta \left( \sum_i p_i - 1 \right) + \gamma \left( \frac{\sum_i p_i^q \epsilon_i}{\sum_i p_i^q} - E_q \right).$$  \hfill (24)
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and the Lagrange extremum equations following from the functional are

$$\frac{\partial S}{\partial p_i} = 0, \quad i = 1, 2, \ldots, \quad \frac{\partial S}{\partial \alpha} = 0,$$

$$\frac{\partial S}{\partial \beta} = 0, \quad \frac{\partial S}{\partial \gamma} = 0.$$  \hspace{1cm} (25)

Their explicit calculation leads to the following set of coupled equations for the eigenvalues of the density matrix $\rho$:

$$1 - \alpha \epsilon_i - \beta = -\log(p_i) + \gamma \sum_{i=1}^{q-1} p_i^{q_i} (\epsilon_i - E_q), \quad i = 1, 2, \ldots,$$  \hspace{1cm} (26)

$$E = \sum_i p_i \epsilon_i,$$  \hspace{1cm} (27)

$$1 = \sum_i p_i,$$  \hspace{1cm} (28)

$$E_q = \frac{\sum_i p_i^{q_i} \epsilon_i}{\sum_i p_i^{q_i}}.$$  \hspace{1cm} (29)

After some algebra, we obtain the integral relation

$$0 = -\sum_i p_i \log(p_i) + \gamma \sum_i \frac{q_{p_i}^{q_i-1}}{p_i^{q_i}} (\epsilon_i - E_q) - 1 + \alpha E + \beta$$

$$= -\sum_i p_i \log(p_i) - 1 + \alpha E + \beta$$

$$= S + \alpha E - 1 + \beta.$$  \hspace{1cm} (30)

The last expression establishes a connection among the entropy, the energy and the Lagrange multipliers. As usual, it seems likely to be employed in constructing generalizations of the free energy and other thermodynamical potentials and relations. However, we will not deal with this discussion here.

4. Metaequilibrium states in electron plasma columns

In this section, we apply the statistical procedure being investigated to an important example: the relaxation of a 2D turbulence to a metaequilibrium state. A relevant experiment related to this problem was performed and discussed by Huang and Driscoll [10]. They compared results from several theoretical approaches to their measured data. The theoretical scheme which furnishes the best approximation is the minimum enstrophy model. The experiment considered the 2D dynamics of a electron plasma fluid. The authors were able to identify and measure properties of a metaequilibrium state (MES) for a plasma sample in which a turbulence initially occurs within a rotating magnetized cylindrical electron column. The turbulence relaxes to a long-lasting metaequilibrium state with axial symmetry. The relevant conserved quantities of the electron column in the MES are the following: energy $H$, entropy $S$, angular momentum $P_\theta$ and number of particles $N_L$. 

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We start from the Poisson equation in order to relate the potential \( \Psi = \phi(1/eN_L) \) to the particle density \( \rho_e = n(R_w^2/N_L) \), where \( \phi \) and \( n \) are the potential and the particle density (in the international system of units). The parameter \( e \) is the electron charge and \( R_w \) is the radius of the rotating drum. In agreement with [10], all magnitudes are expressed in a special set of units, in which the cylinder radius is equal to 1. The Poisson equation for \( \Psi \) is

\[
\nabla^2 \Psi = 4\pi \rho_e(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\Psi(r)}{dr} \right).
\]

(32)

The solution of this equation can be obtained in the form

\[
\Psi(r) = \int_0^1 2\pi r' \frac{d}{dr} G^S_0(r, r') \rho_e(r'),
\]

(33)

\[
G^S_0(r, r') = 2\ln(r)\Theta(r-r') + 2\ln(r')\Theta(r'-r),
\]

where the gauge freedom of the potential has been employed to obtain a symmetric expression for the Green function \( G^S_0(r, r') \). Note that this fact may introduce a change in the value of the total electrostatic energy with respect to the treatment in [10].

The entropy \( S \), normalization \( N \), energy \( H \), angular momentum \( P_\theta \) and constraint \( E_q \) are given by the expressions

\[
H = -\frac{1}{2} \int_0^1 2\pi r dr \Psi(r) \rho_e(r),
\]

(34)

\[
P_\theta = \int_0^1 d^2r (1-r^2) \rho_e(r),
\]

(35)

\[
N = 1 = \int_0^1 2\pi r dr \rho_e(r),
\]

(36)

\[
S = -\int_0^1 2\pi r dr \rho_e(r) \ln[\rho_e(r)],
\]

(37)

\[
E_q = \int \frac{\rho^g_e(r) \left(-\frac{\Psi(r)}{2}\right)}{\int \rho^g_e(r) d^2r} d^2r.
\]

(38)

We should then find the extremum of the functional

\[
A = -\int \rho_e(r) \ln[\rho_e(r)] d^2r + \alpha_E \left( H - H^0 \right) + \alpha_N \left( N - 1 \right) + \alpha_P \left( P_\theta - P^{0}_\theta \right) + \gamma \left( E_q - E^{0}_q \right).
\]

(39)

The Lagrange multipliers are given by \( \alpha_E, \alpha_P, \alpha_N \) and \( \gamma \). The corresponding Euler equations are

\[
\ln[\rho_e(\omega)] = (\alpha_N - 1) + \alpha_P (1-\omega^2) - \alpha_E \int d^2r G^S_0(r, \omega) \rho_e(r)
\]

\[
+ \frac{\gamma}{\int d^2r \rho^g_e(r)} \left(-\frac{1}{2} q \rho^{g-1}_e(\omega) \Psi(\omega)\right).
\]
The integrated version of the above relation is

\[
\ln[\rho_{\omega}(\omega_j)] = (\alpha_N - 1) + \alpha_P (1 - \omega_j^2) - \alpha_E \sum_i 2\pi r_i \Delta r_i G_{0}^{S}(r_i, \omega_j) \rho_e(r_i) + \sum_i 2\pi r_i \Delta r_i \rho_e^0(r_i) G_{0}^{S}(r_i, \omega_j) \times \left(-\frac{1}{2} \gamma q \rho_e^{q-1}(\omega_j) \right) + \sum_i 2\pi r_i \Delta r_i \rho_e^0(r_i) (-E_q \rho_e^{q-1}(\omega_j)),
\]

(42)
to obtain consecutive approximations to the density distribution by exponentiating the logarithm in the left-hand side (lhs) of the relation (42). Now, the indices \(i, j = 1, 2, \ldots\) indicate the increasing values of all the radial positions \(r_i\) (or \(w_i\)) at which the electron density measurements were reported in [10]. The initial density function \(\rho_e^0(\omega_j)\) was constructed by assigning the density value measured at each point \(w_j\) in [10]. The distance elements \(\Delta r_k\) are defined as the differences \(\Delta r_k = (r_{k+1} - r_{k-1})/2, k = 2, \ldots, N - 1\). But \(\Delta r_1 = r_2 - r_1\) and \(\Delta r_N = r_N - r_{N-1}\). Let us denote the lhs of equation (42) evaluated in the density values obtained in [10] as \(L_j^0\), and the right-hand side (rhs) as \(R_j^0\). The procedure starts by evaluating the vector \(L_j^0 - R_j^0\), for all its components \(j\), in the initial density data \(\rho_e^0(\omega_j)\). Then, this vector becomes a function of the Lagrange multipliers and the value of \(q\). Next, the values of those parameters \(\alpha_E^0, \alpha_P^0, \alpha_N^0, \gamma^0\) and \(q^0\) that minimize the quadratic difference \(\sum_j (L_j^0 - R_j^0)^2\) were determined. Finalizing this first step, the set of multipliers obtained were substituted in the expression for \(R_j^0\) together with the initial density \(\rho_e^0(\omega_j)\), defining in this way \(L_j^1\) as the lhs of (42). The new density data to be employed in the next iteration \(\rho_e^1(\omega_j)\) was determined from \(\rho_e^1(\omega_j) = \exp[L_j^1]/(\sum_k 2\pi r_k \Delta r_k \exp[L_k^1])\), where the denominator shows that the normalization is imposed. The new step continues by writing a new vector \(L_j^1 - R_j^1\) in which the density \(\rho_e^1(\omega_j)\) is substituted in \(R_j^1\) and the parameters are made free again. Then, the expression \(\sum_j (L_j^1 - R_j^1)^2\) is again minimized to get the new optimal multipliers and \(q\). This iterative procedure was carried out thirty times and the resulting degree of convergence is expressed by the value \(\sum_j (L_j^{30} - R_j^{30})^2 = 2.58 \times 10^{-7}\), in comparison with the initial evaluation giving a result \(\sum_j (L_j^0 - R_j^0)^2 = 0.33\). Another indicator of the convergence is equation (41). In this case the lhs goes from \(-6742\) in the first step, down to 0.1 in the last one. Thus, the numerical solution of the system was taken as the 30th
Figure 1. Density of particles $\rho_e(r)$ versus the radial position. The squares show the experimental result $\rho_0^0(r_j)$ of [10]. The continuous curve illustrates the density obtained from the minimal enstrophy model. The diamonds indicate the density evaluated from the iterative solution of the maximum entropy principle subject to the constraint $E_q$: $\rho_{30}^e(r_j)$.

iterative result for the density values $\rho_{30}^e(\omega_j)$ and the set of parameters $a_{30}^{E}, a_{30}^{P}, \alpha_{N}^{30}, \gamma_{30}$ and $\hat{q}_{30}$.

In figure 1 the data of Huang and Driscoll [10] are depicted in comparison with the ones predicted by the restricted minimum enstrophy model and our iterative solution. The values of the parameters obtained from the iterative procedure are given by

$$a_{30}^{E} = (3.9 \pm 0.3)10^5,$$

$$a_{30}^{N} = (1.8 \pm 0.1)10^5,$$

$$a_{30}^{P} = -4.627 \pm 0.001,$$

$$\gamma_{30} = (3.9 \pm 0.3)10^5,$$

$$\hat{q}_{30} = 0.999993 \pm 0.000001.$$

Both for the parameters and the physical magnitudes, the errors are given by the difference between the value of the quantity in the last iteration, the 30th, and in the previous one, the 29th. The values obtained for the physical magnitudes are

$$S = 0.0716522 \pm 0.000001,$$

$$H = 0.476445 \pm 0.000003,$$

$$P_{\theta} = 0.866057 \pm 0.0000001,$$

which are reasonably close to the experimental values [10] given the natural presence of experimental errors and our discrete approximation of the integrals. The experimental values are $S^{exp} = 0.087$ and $P_{\theta}^{exp} = 0.861$.

It can be noted that the results for the Lagrange multipliers associated with the energy constraint, the Tsallis-constraint-like condition and the number of particles requirement
are relatively high quantities. This curious outcome turns out to be compatible with the result of the value of $q$ which is very close to unity. In this case, the only way in which the finite differences between the Gibbs and the present result for the density could arise is due to those large values for the multipliers. The natural occurrence of this behavior will be argued in the appendix as resulting from the singular dependence of the Lagrange multipliers around the value $q = 1$. Further, in order to rule out the existence of a possible instability of the results for the multipliers, as a function of the step in the iterative process, in figure 2 we plotted the multipliers for the energy and Tsallis constraints versus the iteration number. As can be seen, the results continuously vary and decrease in moduli with the increase of the iterative step.

In order to search for the sensitivity of the results to the manner of evaluating the integrals, the iterative method described was also carried out in a different way, to approximate the integrals—they were calculated by taking a trapezoidal approach: $\int_0^R g(r) \, dr \approx \sum_{i=1}^{n-1} \Delta r_i ((g(r_i) + g(r_{i+1}))/2)$, with $\Delta r_k = r_{k+1} - r_k$. The results obtained are basically the same as the ones shown here for the density solution of equation (42), the magnitudes of the Lagrange multipliers and the convergence of the lhs from equation (41) to zero.

Figure 1 indicates that our solution agrees reasonably well with the experimental data presented in [10]. At the points with bigger values of the density the results are close to the ones predicted from the restricted enstrophy calculation. However, comparing the curves, one can appreciate that the minimum enstrophy model has an abrupt ending for a given radius $r_0$, which is a limitation of this approach. In this case, the density is written as an analytical expression given by $\rho_\nu(r) = \alpha(J_0(\beta r) - J_0(\beta r_0))$ for $r \leq r_0$ and vanishing otherwise. In contrast, the present discussion correctly predicts the experimental smooth decay of the density to zero at large distances. Thus, in that region, the proposed procedure is able to avoid the singularity predicted by the minimal enstrophy analysis. It can be concluded that the numerical study performed shows that the statistical mechanics proposal investigated in this work describes the metaequilibrium state found in [10] reasonably well.
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Figure 3. Comparison of the results for the density obtained by disregarding or not disregarding the Tsallis-like constraint. The open squares show the Gibbs results, that is the solution of the iterative procedure obtained by only retaining the fixed energy constraint. The diamonds again indicate the density evaluated from the iterative solution of the maximum entropy principle subject to both the fixed energy condition and the new constraint $E_q$.

In figure 3 and for comparison purposes, the density obtained is plotted together with the density evaluated from the same procedure but in which the Tsallis-like constraint $E_q$ is not considered: $\gamma = 0$. That is, the conserved magnitudes are energy, angular momentum and normalization. The two results drastically deviate from each other at small radial distances. A recent work studying the behavior of a similar system [4] establishes that in certain ranges of the parameters, it could dominate an evolution guided by a near maximum entropy principle, but for others, an alternative minimum enstrophy describes experience better. The Huang–Driscoll experiment furnishes a primer for the second case, since it is approximately well described by a minimum enstrophy state. Therefore, as we are able to describe the data [10] reasonably well, our results deviate from the Gibbs behavior, and furnish support for the additional constraint argued here to be relevant in determining metastable and stationary states.

Let us now comment about some initially intriguing aspects of the iterative solution found. They were the very small deviations from unity of the resulting $q$ number and its associated large values for the Lagrange multipliers. These outcomes led to suspicion of the presence of an analytic dependence on $\epsilon = q - 1$ showing a singularity at $q = 1$. Therefore, in an appendix we present a study of this problem. Specifically, we present a search for solutions expanded in a singular Laurent expansion around $q = 1$. It emerged that a solution for the first-order coefficients of the Laurent expansion of the probability distribution and the Lagrange multipliers can be found. It is shown that large values of the multipliers arise for small values of $\epsilon = q - 1$. This behavior is similar to the one exhibited by the iterative solution revealed in this section. A side result of this discussion consists in the distribution having an exponential Gibbs-like appearance within the approximation considered, but in terms of a ‘deformed’ energy spectrum. We expect to make use of this simple expression for the probability distribution for $q$ being close to 1, for inspecting the applicability of the scheme. This study can help in checking whether the procedure proposed here can or cannot improve the Tsallis description for the numerous non-thermal distributions which have been investigated up to now within this analysis [5].

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5. Conclusions

In this work, we present a new statistical mechanics devoted to describing stationary and metastable equilibrium states of physical systems. As usual, the time evolution is assumed to maximize the value of the entropy $S$ by satisfying the standard constraints, like constant energy and normalization. However, the description of the non-equilibrium states is searched for by considering that those configurations are stabilized by the presence of an extra constraint $F$, which is (approximately) conserved in the motion. The constraint, being restricted to being time invariant, is assumed to play the central role in stopping the evolution of the entropy to its maximal value in the Gibbs thermal equilibrium state. Afterward, we also assume that the description which is valid for a composite system formed by two quasi-independent subsystems is also applicable to each of them. This supposition surprisingly implies that, if the constraint $F$ commutes with density matrix, then $F(p_i)$ has a simplified dependence on $p_i$ of the form $F(p_i) = p_i^q$. This can be seen as the main result of the present work. An interesting idea arises from this property: the interpretation of the Tsallis $q$ parameter as the order of homogeneity of the constraint $F$ when it is expressed as a function of the density matrix.

It can be recalled that the form of the constraint being given by a power of order $q$ of the density matrix is a direct consequence of the assumed validity of the additive property for the statistical description of weakly interacting subsystems. This is almost a kinematical form, which shows the only freedom in the value of the real number $q$, whose selection defines different particular statistical states of the system. Therefore, it can be interpreted that the present work finds the unique forms of additive constraints commuting with the density matrices, which can be imposed in defining a Lagrange multiplier description of the statistical mechanics associated with the considered metastable (or stationary) physical systems. Thus, this conclusion indicates that a class of metastable or stationary states should be necessarily described by generic constraint functions showing a form similar to the Tsallis one. Henceforth, one of the motivating extensions of the work is to search for the possibility that the proposed scheme could approximate the known capacity of the Tsallis statistical mechanics for accurately fitting the statistical data for a large number of physical systems. It is our guess that perhaps the essence of the described ability of the Tsallis scheme could be essentially resting on the imposition of the modified $q$ dependent constraint and not so much on the modified formula for the entropy. In fact, this property can be qualitatively argued as follows. First, it can be noted that the form of the $q$ dependent constraint appears to be almost a kinematic result of the assumed additivity of the physical systems. Then, the statistical mechanics constructed here can be considered as being imposed by the maximal entropy principle, which is a very natural dynamical assumption. Henceforth, assuming the entropy maximization principle and additivity can be expected to explain the descriptive power of the Tsallis procedure, having its origin in the imposition of the modified $q$ dependent constraint, after assuming that the also reformulated entropy expression is not affecting the results very much.

It should be underlined that although the new constraint introduced here has an expression similar to the one defined in the Tsallis scheme, in our discussion it is not defined to be employed for evaluating mean values for all the physical quantities. The mean values are defined in this work so as to be calculated by using the same quantum mechanical method: as the trace of the corresponding Hermitian operator after being
multiplied by the density matrix. This represents a basic difference from the Tsallis procedure. In particular, this implies that the recent debate directed to clarifying the stability of the Tsallis mean value does not translate to our discussion [15]–[19].

For checking its predictions, the procedure is applied to describe the experimental data obtained from the plasma experiment performed by Huang and Driscoll [10]. The results evaluated for the densities at small and medium radial distances, at which their values are not small, describe the experimental results with similar quality to the ones following from the best two theoretical schemes existing for this problem. They are the minimization of the enstrophy in one hand, and the maximization of the Tsallis entropy on the other hand [10, 12]. It is known that these schemes are equivalent and their applications to the problem considered are recognized as the best studies existing in the recent literature [12]. This outcome furnishes a first positive check of the construction.

A new result obtained from this application, is related to the ability of predicting the density values at arbitrary radial distances. In the large radial distance zone, where the density is small, the results of the enstrophy minimization and Tsallis entropy maximization both predict a non-analytic abrupt vanishing of the density for distances larger than a fixed length value. However, our results radically differ from the outcome of the aforementioned theoretical procedures by predicting the density at all the radial positions. In addition, the smooth decreasing tail shown by the experimental density distribution is well matched by the density profile evaluated here [10]. However, it should be noted that the measurement errors in these large radial distances, for which the density is small, could also explain the presence of the smooth tail. In any case the analysis proposed here avoids the non-analyticity of the density profile at large distances arising in the aforementioned approaches, which is recognized as one of their main drawbacks [10, 12].

Further, the results for the values of $q$ very close to 1 and large values of the multiplier led us to consider a study of the analytic dependence on $q$ of the solutions of the equations. Approximate solutions are determined for values of $q$ close to 1 within an appendix. They show a singular expansion (Laurent series) around $q = 1$, resulting in large values of the multipliers for small values of $q - 1$. This discussion also furnishes a Gibbs-like exponential form to the probability density in terms of a ‘deformed’ energy spectrum.

In concluding, it can again be underlined that the proposed statistical mechanics could give a qualitative justification of the success of the Tsallis statistical mean values, by starting from the conditional maximization of the entropy, subject to a constraint, which fixes the system to the metastable state. It can also be noted that the description has also been applied, in [14], to a gas of polaritons contained within a quantum dot pumped by an optical resonator. Like in the plasma problem discussed here, the scheme furnished a fine prediction for the density profiles associated with the system.

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Appendix. Singular solutions at \( q = 1 \)

The study presented in this appendix is motivated by the results of the iterative solutions of the equations for the statistical distribution of the magnetized plasma equations in section 4. These results evidenced a remarkably small deviation of the optimal value of the parameter \( q \) from 1, combined with very high values of the Lagrange multipliers \( \alpha, \beta \) and \( \gamma \). However, in spite of this value of \( q \) close to unity, the evaluated distribution function turned out to be radically different from the outcome in the Gibbs case, \( \gamma = 0 \). The aforementioned properties led to the suspicion of singular dependence of the class of solutions found on the parameter \( q \) around its value \( q = 1 \). Therefore, in this appendix, we investigate the existence of solutions of the equations for the probability distribution in which the solutions can show a singular dependence of \( q \) around \( q = 1 \), which will be assumed to be defined by a Laurent expansion in \( \epsilon = q - 1 \). This form of singular dependence is clearly not the most general one, but it furnishes a pole kind of singularity structure around \( q = 1 \) which has a chance of catching the behavior of the solutions which were obtained iteratively. We will see that a closed set equations for the first coefficients in the series expansion of the probability distribution and the Lagrange multipliers can be determined and solved.

Let us recall for bookkeeping purposes the expression for the Lagrange multiplier entropy functional appropriate for imposing the constraints under consideration: the unit value of the total probability, the conserved mean value of the energy \( E \) and the also conserved special \( q \) dependent energy mean value \( E_T \). The functional has the form

\[
S[p_i, \alpha, \beta, \gamma, q] = - \sum_i p_i \log(p_i) + \alpha \left( \sum_j p_j \epsilon_j - E \right) + \beta \left( \sum_j p_j - 1 \right) + \gamma \left( \sum_j p_j q \right) \left( \sum_j p_j \epsilon_j - E_T \right).
\] (A.1)

The Lagrange equations for the \( N \) values of the probability distribution \( p_i, i = 1, 2, 3, \ldots, N \), and the three values of the Lagrange multipliers \( \alpha, \beta, \gamma \) are

\[
\frac{\partial S}{\partial p_i} = \frac{\partial S}{\partial \alpha} = \frac{\partial S}{\partial \gamma} = \frac{\partial S}{\partial \beta} = 0, \quad i = 1, 2, 3, \ldots, N.
\] (A.2)

These equations take the explicit forms

\[
1 - \epsilon_i \alpha - \beta + \log(p_i) = \frac{q \gamma (p_i)^{q-1}}{\sum_i p_i^q} \left( \epsilon_i - \frac{\sum_j \epsilon_j p_j^q}{\sum_j p_j^q} \right),
\] (A.3)

\[
\sum_j p_j - 1 = 0,
\] (A.4)
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\[ \sum_j p_j^0 \epsilon_j - E = 0, \]  
(A.5)

\[ \frac{\sum_j p_j^1 \epsilon_j}{\sum_j p_j^0} - E_T = 0, \]  
(A.6)

\[ S_G + \alpha E + \beta - 1 = 0, \]  
(A.7)

\[ S_G = - \sum_i p_i \log(p_i). \]  
(A.8)

The relation \( S_G + \alpha E + \beta - 1 = 0 \) directly follows after multiplying equation (A.3) by \( p_i \) and summing over \( i \).

Let us start now the search for solutions showing a singular dependence on the parameter \( \epsilon = q - 1 \) around \( \epsilon = 0 \). The probability distribution and the Lagrange multipliers will be assumed to have the following Laurent expansions:

\[ p_i = p_i(\epsilon) = \sum_{n=0}^{\infty} p^n_i \epsilon^n, \quad i = 1, 2, 3, \ldots, N \]  
(A.9)

\[ \alpha(\epsilon) = \sum_{n=-1}^{\infty} \alpha^n \epsilon^n, \]  
(A.10)

\[ \gamma(\epsilon) = \sum_{n=-1}^{\infty} \gamma^n \epsilon^n, \]  
(A.11)

\[ \beta(\epsilon) = \sum_{n=-1}^{\infty} \beta^n \epsilon^n, \]  
(A.12)

where the probabilities are assumed to have a regular Taylor representation. The three multipliers, on the other hand, can show a behavior with a single pole in \( \epsilon \). It might be helpful to underline that the parameters \((E, E_T)\), being fixed from the start, are in fact \( \epsilon \) independent. That is, they only show the null power of \( \epsilon \) in their expansion.

Expanding the constraint fixing the total probability as equal to 1, it follows that

\[ \sum_j \left( \sum_{n=0}^{\infty} p^n_j \epsilon^n \right) = \sum_{n=0}^{\infty} \left( \sum_j p^n_j \right) \epsilon^n = 1, \]
which for the first two powers, \( n = 0, 1 \), leads to the following particular relations which are useful for the next steps:

\[ \sum_j p^0_j = 1, \quad \sum_j p^1_j = 0. \]  
(A.13)

In the case of the expansion of the conserved energy constraint, we have

\[ \sum_j \left( \sum_{n=0}^{\infty} p^n_j \epsilon^n \right) \epsilon_j = \sum_{n=0}^{\infty} \left( \sum_j \epsilon_j p^n_j \right) \epsilon^n = E, \]
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which implies in zero order in $\epsilon$ that the conserved value of the energy is also given by the mean value over the component of zero order in $\epsilon$ of the probability distribution:

$$E = \sum_j \epsilon_j p_j^0. \quad (A.14)$$

Further, the expansion in $\epsilon$ of the constraint fixing the $q$ dependent mean value to $E_T$ can be evaluated up to the first order in $\epsilon$ as follows:

$$\sum_j p_j^0 \epsilon_j \sum_j p_j^0 = E_T$$

$$= \sum_j p_j^0 \epsilon_j \sum_j p_j^0 + \frac{\epsilon}{\sum_j p_j^0} \left[ \sum_j \epsilon_j (p_j^0 + p_j^0 \log(p_j^0)) - \sum_j p_j^0 \epsilon_j \sum_j (p_j^1 + p_j^0 \log(p_j^0)) \right]$$

$$= E + \epsilon \left[ \sum_j \epsilon_j (p_j^0 + p_j^0 \log(p_j^0)) - E \sum_j (p_j^1 + p_j^0 \log(p_j^0)) \right]$$

$$= E + \epsilon M, \quad (A.15)$$

where the coefficient of the linear term in $\epsilon$ takes the form

$$M = \sum_j \epsilon_j (p_j^1 + p_j^0 \log(p_j^0)) - E \sum_j (p_j^1 + p_j^0 \log(p_j^0)). \quad (A.16)$$

in which (A.13) has been used. Let us consider now the expansion of the set of $N$ equations for the probability distribution. Substituting the series for the probabilities and the Lagrange multipliers in equation (A.3) gives

$$1 - \epsilon_i \sum_{n=-1}^{\infty} \alpha^n \epsilon^n - \sum_{n=-1}^{\infty} \beta^n \epsilon^n = -\log \left( \sum_{n=0}^{\infty} p_i^n \epsilon^n \right)$$

$$+ \frac{(1 + \epsilon) \left( \sum_{n=-1}^{\infty} \gamma^n \epsilon^n \right) \left( \sum_{n=0}^{\infty} p_i^n \epsilon^n \right)^\epsilon}{\sum_i \left( \sum_{n=0}^{\infty} p_i^n \epsilon^n \right)^{1+\epsilon}} \left( \epsilon_i - \sum_j \epsilon_j (\sum_{n=0}^{\infty} p_j^n \epsilon^n)^{1+\epsilon} \right). \quad (A.17)$$

Collecting the single-pole terms in $\epsilon$ and employing equations (A.13) and (A.14) leads to the relation

$$-\beta^{-1} - \alpha^{-1} \epsilon_i = \gamma^{-1} \left( \epsilon_i - \sum_j \epsilon_j \sum_j p_j^0 \epsilon_j \right)$$

$$= \gamma^{-1} (\epsilon_i - E). \quad (A.18)$$

Then, considering that the energy spectrum of the system is not degenerate, this forces the following equalities:

$$\alpha^{-1} = -\gamma^{-1}, \quad (A.19)$$

$$\beta^{-1} = \gamma^1 E. \quad (A.20)$$

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Substituting in the general formula (A.17) the series expansions of the probabilities and Lagrange multipliers taken in their first two orders, up to the first-two-order precision, we can write

\[ 1 - \epsilon_i \alpha^0 - \epsilon_i \frac{\alpha^{-1}}{\epsilon} - \beta^0 - \frac{\beta^{-1}}{\epsilon} = -\log(p_i^0) + \frac{(1 + \epsilon)(\gamma^{-1}\epsilon^{-1} + \gamma^0)(1 + \log(p_i^0)\epsilon)}{\sum_j(\sum_{n=0}^{\infty} p_j^n \epsilon^n)^{1+\epsilon}}(\epsilon_i - E - \epsilon M), \quad (A.21) \]

which after expanding the products allows us to write

\[ 1 = -\log(p_i^0) + \epsilon_i \alpha^0 + \beta^0 - \gamma^{-1}M - \sum_i p_i^0 \]

\[ - \left( \epsilon_i - \frac{\sum_j \epsilon_i p_j^0}{\sum_j p_j^0} \right) \left( \frac{\gamma^0}{\sum_i p_i^0} + \frac{\gamma^{-1}}{\sum_i p_i^0}(1 + \log(p_i^0)) \right) \]

\[ - \frac{\gamma^{-1}}{\sum_i p_i^0} \left( \sum_j (p_j^0 + p_j^0 \log(p_j^0)) \right) \sum_i p_i^0 \]

\[ \sum_i p_i^0 \sum_i p_i^0 \]

\[ (A.22) \]

The use of the relations already found permits us to simplify the above formula to

\[ \log(p_i^0)(1 - \gamma^{-1}(\epsilon_i - E)) = \epsilon_i \alpha^0 + \beta^0 - 1 - \gamma^{-1}M \]

\[ - (\epsilon_i - E) \left( \gamma^0 + \gamma^{-1} - \gamma^{-1} \sum_j p_j^0 \log(p_j^0) \right), \quad (A.23) \]

where the order 1 probabilities \( p_j^1 \) are now absent thanks to the relation (A.13) following from the expansion of the normalization of probabilities constraint.

Let us expand now the general relation (A.7) in powers of \( \epsilon \) up to the first order. It follows that

\[ S_G + \alpha E + \beta - 1 = S_G^0 + \left( \frac{\alpha^{-1}}{\epsilon} + \alpha^0 \right) E + \frac{\beta^{-1}}{\epsilon} + \beta^0 - 1 + \cdots \]

\[ = S_G^0 + \alpha^0 E - 1 + \beta^0 + \frac{1}{\epsilon}(\alpha^{-1} E + \beta^{-1}) + \cdots, \quad (A.24) \]

from which two new relations emerge, expressing \( \beta^0 \) and \( \beta^{-1} \) in terms of \( \alpha^0 \) and \( \alpha^{-1} \):

\[ \beta^{-1} = -\alpha^{-1} E, \quad (A.26) \]

\[ \beta^0 = -S_G^0 - \alpha^0 E + 1, \quad S_G^0 = -\sum_i p_i^0 \log(p_i^0). \quad (A.27) \]

The above derived expressions directly allow us to write the following formulas:

\[ p_i^0 = \frac{1}{Z_{\text{eff}}} \exp[\alpha^0 \epsilon_i^{\text{eff}}], \quad i = 1, 2, \ldots, N, \quad (A.28) \]

\[ \frac{1}{Z_{\text{eff}}} = \sum_i \exp[\alpha^0 \epsilon_i^{\text{eff}}], \quad (A.29) \]

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They implicitly define the zeroth-order components of the probability densities and Lagrange multipliers, as well as the single-pole parts of the three Lagrange multipliers for a fixed value of \( \epsilon = q - 1 \). This is a set of \( N + 3 \) equations for the equivalent number of parameters \( p_i, i = 1, 2, \ldots, N \), plus the three independent constants \( \alpha^0, \gamma^0, \gamma^{-1} \). The rest of the constants appearing are functions of \( E \) and the \( p_i, i = 1, 2, \ldots, N \), and \( \alpha^0, \gamma^0, \gamma^{-1} \) though the relations

\[
\beta^0 = -S_G^0 - \alpha^0 E + 1, \tag{A.34}
\]

\[
\beta^{-1} = \gamma^{-1} E, \tag{A.35}
\]

\[
\alpha^{-1} = -\gamma^{-1}, \tag{A.36}
\]

\[
S_G^0 = -\sum_j p_j^0 \log(p_j^0), \tag{A.37}
\]

\[
M = \sum_j \epsilon_j p_j^0 \log(p_j^0) - E \sum_j p_j^0 \log(p_j^0). \tag{A.38}
\]

### A.1. The extremum of the entropy with respect to \( q \)

In the previous discussion in this appendix, we had considered \( q \) as an arbitrary free parameter. However, having determined the equations for \( q \) arbitrary (but close to 1), we will also consider the physical condition that the most stable state of the system should have a maximum value of the entropy under variations of \( q \) around the value characterizing this state. Therefore, this extremum condition for \( S \) with respect to \( q \) leads to

\[
\frac{dS}{dq} = \frac{\partial S}{\partial q} + \frac{\partial S}{\partial p_i} \frac{dp_i}{dq} + \frac{\partial S}{\partial \alpha} \frac{d\alpha}{dq} + \frac{\partial S}{\partial \beta} \frac{d\beta}{dq} + \frac{\partial S}{\partial \gamma} \frac{d\gamma}{dq}
\]

\[
= \frac{\partial S}{\partial q} + \frac{\partial}{\partial q} \left( \gamma \left( \frac{\sum_j p_j^0 \epsilon_j}{\sum_j p_j^0} - E_T \right) \right)
\]

\[
= \gamma \frac{\partial}{\partial q} \left( \frac{\sum_j p_j^0 \epsilon_j}{\sum_j p_j^0} - E_T \right)
\]

\[
= 0 \tag{A.39}
\]
where the satisfaction of all the Lagrange equations was assumed and \( \gamma \) was extracted from the derivative thanks to the assumed satisfaction of all the constraints. Then, the condition for determining the value of \( q \) at the extremum is

\[
0 = \frac{\partial}{\partial q} \left( \frac{\sum_j p^0_j \epsilon_j}{\sum_j p^0_j} - E_T \right) = \left( \frac{\sum_j \log(p^0_j) p^0_j \epsilon_j}{\sum_j p^0_j} - \frac{\sum_j \log(p^0_j) \sum_j p^0_j \epsilon_j}{\sum_j p^0_j} \right).
\]

(A.40)

Expanding this relation up to linear order in \( \epsilon \) leads to

\[
0 = \sum_j \log(p^0_j) p^0_j \epsilon_j - \sum_j p^0_j \epsilon_j \sum_j \log(p^0_j) p^0_j + 2 \epsilon \left( \sum_j \log^2(p^0_j) p^0_j \epsilon_j - \sum_j \log(p^0_j) p^0_j \sum_j \log(p^0_j) p^0_j \right) - E \left( \sum_j \log(p^0_j) p^0_j \right)^2 - E \sum_j \log^2(p^0_j) p^0_j + \cdots O(\epsilon^2),
\]

(A.41)

which after assuming that \( \epsilon \) is a small number leads to the following definition of the value of \( q \):

\[
q = 1 - \frac{\left( \sum_j \log(p^0_j) p^0_j \epsilon_j - \sum_j p^0_j \epsilon_j \sum_j \log(p^0_j) p^0_j \right)}{D},
\]

(A.42)

\[
D = 2 \left( \sum_j \log^2(p^0_j) p^0_j \epsilon_j - \sum_j \log(p^0_j) p^0_j \epsilon_j \sum_j \log(p^0_j) p^0_j \right) - E \left( \sum_j \log(p^0_j) p^0_j \right)^2 - E \sum_j \log^2(p^0_j) p^0_j).
\]

(A.43)

Therefore, in this appendix we have argued for the existence of solutions of the set of equations for the probability distributions associated with metastable and stationary states, which show singular dependence on the \( q \) parameter around its value \( q = 1 \). This result gives an explanation for the very small value of the parameter \( q \), and the large values of the Lagrange multipliers, found after iteratively solving the equations derived here for the magnetized plasma metastable states, in section 4.

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