Existence of solutions for some nonlinear problems with boundary value conditions

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Abstract
In this paper we study the existence of solutions for nonlinear boundary value problems
\[
\begin{align*}
(\varphi(u'))' &= f(t, u, u') \\
l(u, u') &= 0,
\end{align*}
\]
where \(l(u, u') = 0\) denotes the Dirichlet or mixed conditions on \([0, T]\), \(\varphi\) is a bounded, singular or classic homeomorphism such that \(\varphi(0) = 0\), \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a continuous function, and \(T\) a positive real number. All the contemplated boundary value problems are reduced to finding a fixed point for one operator defined on a space of functions, and Schauder fixed point theorem or Leray-Schauder degree are used.

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Key words: Dirichlet problem, Schauder fixed point theorem, Leray-Schauder degree, mixed boundary value problems.

1 Introduction

The purpose of this article is to obtain some existence results for nonlinear boundary value problems of the form
\[
\begin{align*}
(\varphi(u'))' &= f(t, u, u') \\
l(u, u') &= 0,
\end{align*}
\]
where \(l(u, u') = 0\) denotes the Dirichlet or mixed boundary conditions on the interval \([0, T]\), \(\varphi\) is a bounded, singular or classic homeomorphism such that \(\varphi(0) = 0\), \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a continuous function, and \(T\) a positive real number.

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Recently, the problem (1.1) in special cases, when \( \varphi \) is an increasing homeomorphism from \((-a, a)\) to \(\mathbb{R}\) such that \(\varphi(0) = 0\) and \(l(u, u') = 0\) denotes the periodic, Neumann or Dirichlet boundary conditions, has been investigated by C. Bereanu and J. Mawhin in [4].

In [3], the authors have studied the problem (1.1), where \(\varphi : \mathbb{R} \to (-a, a)\) \((0 < a \leq \infty)\) and periodic boundary conditions. They obtained the existence of solutions by means of the Leray-Schauder degree theory.

The paper is organized as follows. In Section 2, we introduce some notations and preliminaries, which will be crucial in the proofs of our results. Section 3 is devoted to the study of existence of solutions for the Dirichlet problems with bounded homomorphisms

\[
\begin{aligned}
(\varphi(u'))' &= f(t, u, u') \\
u(0) &= 0 = u(T).
\end{aligned}
\]

In particular, C. Bereanu and J. Mawhin in [5] proved the existence of at least one solution by means of the Leray-Schauder degree.

**Theorem 1.1** (Bereanu and Mawhin). *If the function \( f \) satisfies the condition

\[\exists c > 0 \text{ such that } |f(t, x, y)| \leq c < \frac{a}{2T}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},\]

the Dirichlet problem has at least one solution.*

The main purpose of this section is an extension of the results obtained in the previous theorem. For this, we use topological methods based upon Leray-Schauder degree [10] and more general properties of the function \( f \). In Section 4, we use the fixed point theorem of Schauder to show the existence of at least one solution for boundary value problems of the type

\[
\begin{aligned}
(\varphi(u'))' &= f(t, u, u') \\
u(T) &= u(0) = u'(T).
\end{aligned}
\]

where \( \varphi : (-a, a) \to \mathbb{R} \) (we call it singular). We call solution of this problem any function \( u : [0, T] \to \mathbb{R} \) of class \( C^1 \) such that \( \max_{[0, T]} |u'(t)| < a \), satisfying the boundary conditions and the function \( \varphi(u') \) is continuously differentiable and \((\varphi(u'(t)))' = f(t, u(t), u'(t))\) for all \( t \in [0, T] \). In Section 5, for \( u(T) = u'(0) = u'(T) \) boundary conditions and classic homeomorphisms \( (\varphi : \mathbb{R} \to \mathbb{R}) \), we investigate the existence of at least one solution using Leray-Schauder degree, where a solution of this problem is any function \( u : [0, T] \to \mathbb{R} \) of class \( C^1 \) such that \( \varphi(u') \) is continuously differentiable, which satisfies the boundary conditions and \((\varphi(u'(t)))' = f(t, u(t), u'(t))\) for all \( t \in [0, T] \). Such problems do not seem to have been studied in the literature. In the present paper generally we follow the ideas of Bereanu and Mawhin [1, 2, 3, 4, 5, 6].
2 Notation and preliminaries

We first introduce some notation. For fixed $T$, we denote the usual norm in $L^1 = L^1([0,T], \mathbb{R})$ for $\| \cdot \|_{L^1}$. For $C = C([0,T], \mathbb{R})$ we indicate the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\| \cdot \|_\infty$, $C^1 = C^1([0,T], \mathbb{R})$ denote the Banach space of continuously differentiable functions from $[0,T]$ into $\mathbb{R}$ endowed with the usual norm $\| u \|_1 = \| u \|_\infty + \| u' \|_\infty$ and for $C^1_0$ we designate the closed subspace of $C^1$ defined by $C^1_0 = \{ u \in C^1 : u(T) = 0 = u(0) \}$.

We introduce the following applications:

- the Nemytskii operator $N_T : C^1 \to C$, $N_T(u)(t) = f(t, u(t), u'(t))$,
- the integration operator $H : C \to C^1$, $H(u)(t) = \int_0^t u(s)ds$,
- the following continuous linear applications:
  - $K : C \to C^1$, $K(u)(t) = -\int_t^T u(s)ds$,
  - $Q : C \to C$, $Q(u)(t) = \frac{1}{T} \int_0^T u(s)ds$,
  - $S : C \to C$, $S(u)(t) = u(T)$,
  - $P : C \to C$, $P(u)(t) = u(0)$.

For $u \in C$, we write

$$ u_m = \min_{[0,T]} u, \quad u_M = \max_{[0,T]} u, \quad u^+ = \max \{ u, 0 \}, \quad u^- = \max \{ -u, 0 \}. $$

The following lemma is an adaptation of a result of [4] to the case of a homeomorphism which is not defined everywhere. We present here the demonstration for better understanding of the development of our research.

**Lemma 2.1.** Let $B = \{ h \in C : \| h \|_\infty < a/2 \}$. For each $h \in B$, there exists a unique $Q_\varphi = Q_\varphi(h) \in \text{Im}(h)$ (where $\text{Im}(h)$ denotes the range of $h$) such that

$$ \int_0^T \varphi^{-1}(h(t) - Q_\varphi(h))dt = 0. $$

Moreover, the function $Q_\varphi : B \to \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

**Proof.** Let $h \in B$. We define the continuous application $G_h : [h_m, h_M] \to \mathbb{R}$ for
\[ G_h(s) = \int_0^T \varphi^{-1}(h(t) - s)dt. \]

We now show that the equation

\[ G_h(s) = 0 \] (2.2)

has a unique solution \( Q_\varphi(h) \). Let \( r, s \in [h_m, h_M] \) be such that

\[ \int_0^T \varphi^{-1}(h(t) - r)dt = 0, \quad \int_0^T \varphi^{-1}(h(t) - s)dt = 0, \]

that is

\[ \int_0^T \varphi^{-1}(h(t) - r)dt = \int_0^T \varphi^{-1}(h(t) - s)dt. \]

It follows that there exist \( \tau \in [0, T] \) such that

\[ \varphi^{-1}(h(\tau) - r) = \varphi^{-1}(h(\tau) - s). \]

Using the injectivity of \( \varphi^{-1} \) we deduce that \( r = s \). Let us now show the existence. Because \( \varphi^{-1} \) is strictly monotone and \( \varphi^{-1}(0) = 0 \), we have that

\[ G_h(h_m)G_h(h_M) \leq 0. \]

It follows that there exists \( s \in [h_m, h_M] \) such that \( G_h(s) = 0 \). Consequently for each \( h \in B \), the equation (2.2) has a unique solution. Thus, we define the function \( Q_\varphi : B \to \mathbb{R} \) such that

\[ \int_0^T \varphi^{-1}(h(t) - Q_\varphi(h))dt = 0. \]

On the other hand, because \( h \in B \), we have that

\[ |Q_\varphi(h)| \leq \|h\|_\infty < a/2. \]

Therefore, the function \( Q_\varphi \) sends bounded sets into bounded sets.

Finally, we show that \( Q_\varphi \) is continuous on \( B \). Let \( (h_n)_n \subset C \) be a sequence such that \( h_n \to h \) in \( C \). Since the function \( Q_\varphi \) sends bounded sets into bounded sets, then \( (Q_\varphi(h_n))_n \) is bounded. Hence, \( (Q_\varphi(h_n))_n \) is relatively compact. Without loss of generality, passing if necessary to a subsequence, we can assume that

\[ \lim_{n \to \infty} Q_\varphi(h_n) = \tilde{a}, \]

where for each \( n \in \mathbb{N} \) we obtain

\[ \lim_{n \to \infty} \int_0^T \varphi^{-1}(h_n(t) - Q_\varphi(h_n))dt = 0. \]

Using the dominated convergence theorem, we deduce that

\[ \int_0^T \varphi^{-1}(h(t) - \tilde{a})dt = 0, \]

so we have that \( Q_\varphi(h) = \tilde{a} \). Hence, the function \( Q_\varphi \) is continuous. \( \square \)
The following extended homotopy invariance property of the Leray-Schauder degree, can be found in [9].

**Proposition 2.2.** Let $X$ be a real Banach space, $V \subset [0,1] \times X$ be an open, bounded set and $M$ be a completely continuous operator on $V$ such that $x \neq M(\lambda, x)$ for each $(\lambda, x) \in \partial V$. Then the Leray-Shauder degree
\[
\text{deg}_{LS}(I - M(\lambda, \cdot), V_\lambda, 0)
\]
is well defined and independent of $\lambda$ in $[0, 1]$, where $V_\lambda$ is the open, bounded (possibly empty) set defined by $V_\lambda = \{x \in X : (\lambda, x) \in V\}$.

3 Dirichlet problems with bounded homeomorphisms

In this section we are interested in Dirichlet boundary value problems of the type
\[
\begin{cases}
(\varphi(u'))' = f(t, u, u') \\
u(0) = 0 = u(T)
\end{cases}
\] (3.3)
where $\varphi : \mathbb{R} \to (-a, a)$ is a homeomorphism such that $\varphi(0) = 0$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In order to apply Leray-Schauder degree theory to show the existence of at least one solution of (3.3), we introduce, for $\lambda \in [0, 1]$, the family of Dirichlet boundary value problems
\[
\begin{cases}
(\varphi(u'))' = \lambda f(t, u, u') \\
u(0) = 0 = u(T)
\end{cases}
\] (3.4)
Let
\[
\Omega = \{(\lambda, u) \in [0, 1] \times C^1_0 : \|\lambda H(N_f(u))\|_\infty < a/2\}.
\]
Clearly $\Omega$ is an open set in $[0, 1] \times C^1_0$, and is nonempty because $\{0\} \times C^1_0 \subset \Omega$. Using Lemma 2.1 we can define the operator $M : \Omega \to C^1_0$ by
\[
M(\lambda, u) = H(\varphi^{-1}[\lambda H(N_f(u)) - Q_\varphi(\lambda H(N_f(u)))]).
\] (3.5)
Here $\varphi^{-1}$ with an abuse of notation is understood as the operator $\varphi^{-1} : B_a(0) \subset C \to C$ defined by $\varphi^{-1}(v)(t) = \varphi^{-1}(v(t))$. It is clear that $\varphi^{-1}$ is continuous and sends bounded sets into bounded sets.

When the boundary conditions are periodic or Neumann, an operator has been considered by Bereanu and Mawhin [5].

The following lemma plays a pivotal role to study the solutions of the problem (3.4).
Lemma 3.1. The operator $M$ is well defined and continuous. Moreover, if $(\lambda, u) \in \Omega$ is such that $M(\lambda, u) = u$, then $u$ is solution of (3.4).

Proof. Let $(\lambda, u) \in \Omega$. We show that in fact $M(\lambda, u) \in C_0^1$. It is clear that

$$(M(\lambda, u))' = \varphi^{-1} [\lambda H(N_f(u)) - Q_\varphi(\lambda H(N_f(u)))]$$

where the continuity of $M(\lambda, u)$ and $(M(\lambda, u))'$ follows from the continuity of the applications $H$ and $N_f$.

On the other hand using Lemma 2.1, we have

$$M(\lambda, u)(0) = 0 = M(\lambda, u)(T).$$

Therefore $M(\Omega) \subset C_0^1$ and $M$ is well defined. The continuity of $M$ follows by the continuity of the operators which compose it $M$.

Now suppose that $(\lambda, u) \in \Omega$ is such that $M(\lambda, u) = u$. It follows from (3.5) that

$$u(t) = M(\lambda, u)(t) = H(\varphi^{-1} [\lambda H(N_f(u)) - Q_\varphi(\lambda H(N_f(u)))])(t)$$

for all $t \in [0, T]$. Differentiating, we obtain

$$u'(t) = \varphi^{-1} [\lambda H(N_f(u)) - Q_\varphi(\lambda H(N_f(u)))](t).$$

Applying $\varphi$ to both of its members and differentiating again, we deduce that

$$(\varphi(u'(t)))' = \lambda N_f(u)(t)$$

for all $t \in [0, T]$. Thus, $u$ satisfies problem (3.4). This completes the proof.

Remark 3.2. Note that the reciprocal of Lemma 3.1 is not true because we can not guarantee that $\|\lambda H(N_f(u))\|_\infty < a/2$ for $u$ solution of (3.4).

In our main result, we need the following lemma to obtain the required a priori bounds for the possibles fixed points of $M$.

Lemma 3.3. Assume that there exist $h \in C([0, T], \mathbb{R}^+)$ and $n \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\|h\|_{L^1} < a/2, \quad \varphi(y)n'(x)y \geq 0, \quad n(0) = 0,$$

and

$$|f(t, x, y)| \leq f(t, x, y)n(x) + h(t)$$

for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. If $(\lambda, u) \in \Omega$ is such that $M(\lambda, u) = u$, then

$$\|\lambda H(N_f(u))\|_\infty \leq \|h\|_{L^1}, \quad \|u\|_\infty \leq L \quad \text{and} \quad \|u\|_1 \leq L + LT,$$

where $L = \max \{ |\varphi^{-1}(-2 \|h\|_{L^1})|, |\varphi^{-1}(2 \|h\|_{L^1})| \}$.  

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Proof. Let $\lambda \neq 0$ and $(\lambda, u) \in \Omega$ be such that $M(\lambda, u) = u$. Using Lemma 3.1, we have that $u$ is solution of (3.4), which implies that

$$\varphi(u') = \lambda H(N_f(u)) - Q_\varphi(\lambda H(N_f(u))), \quad u(0) = 0 = u(T),$$

where for all $t \in [0, T]$, we obtain

$$|\lambda H(N_f(u))(t)| \leq \int_0^T |f(s, u(s), u'(s))| \, ds$$

$$\leq \int_0^T f(s, u(s), u'(s))n(u(s)) \, ds + \int_0^T h(s) \, ds.$$

On the other hand, because $\varphi$ is a homeomorphism such that $\varphi(y)n'(x)y \geq 0$ for all $x, y \in \mathbb{R}$, then

$$-\int_0^T \varphi(u'(t))n'(u(t))u'(t) \, dt \leq 0.$$

Using the integration by parts formula, the boundary conditions and the fact that $n(0) = 0$, we deduce that

$$\int_0^T (\varphi(u'(t)))'n(u(t)) \, dt = -\int_0^T \varphi(u'(t))n'(u(t))u'(t) \, dt \leq 0.$$

Since $\lambda \in (0, 1]$ and $u$ is solution of (3.4), we have that

$$\int_0^T f(t, u(t), u'(t))n(u(t)) \, dt \leq 0,$$

and hence

$$|\lambda H(N_f(u))(t)| \leq \|h\|_{L^1}.$$

On the other hand, since that $Q_\varphi(\lambda H(N_f(u))) \in \text{Im}(\lambda H(N_f(u)))$, we get

$$|\varphi(u'(t))| \leq 2 \|h\|_{L^1}$$

for all $t \in [0, T]$. It follows that

$$\|\varphi(u')\|_{\infty} \leq 2 \|h\|_{L^1},$$

which implies that $\|u'\|_{\infty} \leq L$, where $L = \max\{\varphi^{-1}(-2 \|h\|_{L^1}), \varphi^{-1}(2 \|h\|_{L^1})\}$. Using again the boundary conditions, we have that

$$|u(t)| \leq \int_0^t |u'(s)| \, ds \leq \int_0^T |u'(s)| \, ds \leq LT \quad (t \in [0, T]),$$

and hence

$$\|u\|_1 \leq L + LT.$$

Finally, if $u = M(0, u)$, then $u = 0$, so the proof is complete.  \qed
Let $\rho, \kappa \in \mathbb{R}$ be such that $\|h\|_{L^1} < \kappa < a/2$, $\rho > L + LT$ and consider the set

$$
V = \{ (\lambda, u) \in [0,1] \times C^1_0 : \|\lambda H(N_f(u))\|_\infty < \kappa, \|u\|_1 < \rho \}.
$$

Since the set $\{0\} \times \{u \in C^1_0 : \|u\|_1 < \rho\} \subset V$, then we deduce that $V$ is nonempty. Moreover, it is clear that $V$ is open and bounded in $[0,1] \times C^1_0$ and $\overline{V} \subset \Omega$. On the other hand using an argument similar to the one introduced in the proof of Lemma 6 in \cite{5}, it is not difficult to see that $M : \overline{V} \to C^1_0$ is well defined, completely continuous and $u \neq M(\lambda, u)$ for all $(\lambda, u) \in \partial V$.

### 3.1 Existence results

In this subsection, we present and prove our main result.

**Theorem 3.4.** If $f$ satisfies conditions of Lemma 3.3, then problem (3.3) has at least one solution.

**Proof.** Let $M$ be the operator given by (3.5). Using Proposition 2.2, we deduce that

$$
deg_{LS}(I - M(0, .), V_0, 0) = deg_{LS}(I - M(1, .), V_1, 0),
$$

where $deg_{LS}(I - M(0, .), V_0, 0) = deg_{LS}(I, B_\rho(0), 0) = 1$. Thus, there exists $u \in V_1$ such that $M(1, u) = u$, which is a solution for (3.3).

**Remark 3.5.** Note that Theorem 3.4 is a generalization of Theorem 1.1.

**Corollary 3.6.** Assume that $\varphi$ is an increasing homomorphism. Let $h \in C([0, T], \mathbb{R}^+)$ be such that

$$
\|h\|_{L^1} < a/2, \ |f(t, x, y)| \leq f(t, x, y)x + h(t)
$$

for all $x, y \in \mathbb{R}$ and $t \in [0, T]$. If $(\lambda, u) \in \Omega$ is such that $M(\lambda, u) = u$, then

$$
\|\lambda H(N_f(u))\|_\infty \leq \|h\|_{L^1}, \ |u'|_\infty \leq L \quad \text{and} \quad \|u\|_1 \leq L + LT,
$$

where $L = \max \{|\varphi^{-1}(-2 \|h\|_{L^1})|, |\varphi^{-1}(2 \|h\|_{L^1})|\}$.

**Proof.** Since $\varphi$ is an increasing homomorphism we have that $\varphi(y)y \geq 0$ for all $y \in \mathbb{R}$. Using Lemma 3.3 with $n(x) = x$ for all $x \in \mathbb{R}$, we can obtain the conclusion of Corollary 3.6. The proof is achieved.

**Theorem 3.7.** If $f$ satisfies conditions of Corollary 3.6, then problem (3.3) has at least one solution.
Let us give now an application of Theorem 3.7 when \( f \) is unbounded.

**Example 3.8.** Let \( \phi : \mathbb{R} \to (-1,1) \) with \( \phi(y) = \frac{y}{\sqrt{1+y^2}} \), \( f(t,x,y) = x - 2 \) and \( h(t) = 4 \). Using Theorem 3.7, we obtain that the problem
\[
\left( \frac{u'}{\sqrt{1+u'^2}} \right)' = u - 2, \quad u(0) = u(T) = 0,
\]
has at least one solution if \( T < 1/8 \).

4 Problems with singular homeomorphisms and tree-point boundary conditions

In this section we study the existence of at least one solution for boundary value problems of the type
\[
\begin{cases}
(\phi(u'))' = f(t,u,u') \\
u(T) = u(0) = u'(T),
\end{cases}
\tag{4.6}
\]
where \( \phi : (-a,a) \to \mathbb{R} \) is a homeomorphism such that \( \phi(0) = 0 \) and \( f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function.

In order to transform problem \((4.6)\) to a fixed point problem we use a similar argument introduced in Lemma 2.1 for \( h \in C \).

**Lemma 4.1.** \( u \in C^1 \) is a solution of \((4.6)\) if and only if \( u \) is a fixed point of the operator \( M \) defined on \( C^1 \) by
\[
M(u) = \phi^{-1}(-Q\phi(K(N_f(u)))) + H \left( \phi^{-1} \left[ K(N_f(u)) - Q\phi(K(N_f(u))) \right] \right).
\]

**Proof.** If \( u \in C^1 \) is solution of \((4.6)\), then
\[
(\phi(u'(t)))' = N_f(u)(t) = f(t,u(t),u'(t)), \quad u(0) = u(T), u(0) = u'(T)
\]
for all \( t \in [0,T] \). Applying \( K \) to both members and using the fact that \( u(0) = u'(T) \), we deduce that
\[
\phi(u'(t)) = \phi(u(0)) + K(N_f(u))(t).
\]
Composing with the function \( \phi^{-1} \), we obtain
\[
u'(t) = \phi^{-1}[K(N_f(u))(t) + c],
\]
where \( c = \phi(u(0)) \). Integrating from 0 to \( t \in [0,T] \), we have that
\[
u(t) = u(0) + H \left( \phi^{-1} \left[ K(N_f(u)) + c \right] \right) (t).
\]
Because \( u(0) = u(T) \), then
\[ \int_0^T \varphi^{-1} [K(N_f(u))(t) + c] \, dt = 0. \]

Using an argument similar to the introduced in Lemma 2.1, it follows that \( c = -Q\varphi(K(N_f(u))) \). Hence,
\[
 u = \varphi^{-1}(-Q\varphi(K(N_f(u)))) + H \left( \varphi^{-1} [K(N_f(u)) - Q\varphi(K(N_f(u)))] \right).
\]

Let \( u \in C^1 \) be such that \( u = M(u) \). Then
\[
 u(t) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) + H \left( \varphi^{-1} [K(N_f(u)) - Q\varphi(K(N_f(u)))] \right)(t) \quad (4.7)
\]
for all \( t \in [0, T] \). Since \( \int_0^T \varphi^{-1} [K(N_f(u))(t) - Q\varphi(K(N_f(u)))] \, dt = 0 \), therefore, we have that \( u(0) = u(T) \). Differentiating (4.7), we obtain that
\[
 u'(t) = \varphi^{-1} [K(N_f(u)) - Q\varphi(K(N_f(u)))](t).
\]

In particular,
\[
 u'(T) = \varphi^{-1}(0 - Q\varphi(K(N_f(u)))) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) = u(0).
\]

Applying \( \varphi \) to both members and differentiating again, we deduce that
\[
 (\varphi(u'(t)))' = N_f(u)(t), \quad u(0) = u(T), \quad u(0) = u'(T)
\]
for all \( t \in [0, T] \). This completes the proof. \( \square \)

**Lemma 4.2.** The operator \( M : C^1 \to C^1 \) is completely continuous.

**Proof.** Let \( \Lambda \subset C^1 \) be a bounded set. Then, if \( u \in \Lambda \), there exists a constant \( \rho > 0 \) such that
\[
 ||u||_1 \leq \rho. \quad (4.8)
\]

Next, we show that \( \overline{M(\Lambda)} \subset C^1 \) is a compact set. Let \( (v_n)_n \) be a sequence in \( M(\Lambda) \), and let \( (u_n)_n \) be a sequence in \( \Lambda \) such that \( v_n = M(u_n) \). Using (4.8), we have that there exists a constant \( L > 0 \) such that, for all \( n \in \mathbb{N} \),
\[
 ||N_f(u_n)||_\infty \leq L,
\]
which implies that
\[
 ||K(N_f(u_n)) - Q\varphi(K(N_f(u_n)))||_\infty \leq 2LT.
\]
Hence the sequence \((K(N_f(u_n)) - Q\varphi(K(N_f(u_n))))_n\) is bounded in \(C\). Moreover, for \(t, t_1 \in [0, T]\) and for all \(n \in \mathbb{N}\), we have that

\[
\begin{align*}
|K(N_f(u_n))(t) - Q\varphi(K(N_f(u_n))) - K(N_f(u_n))(t_1) + Q\varphi(K(N_f(u_n)))| \\
\leq \left| -\int_t^{t_1} f(s, u_n(s), u_n'(s))ds + \int_{t_1}^{T} f(s, u_n(s), u_n'(s))ds \right| \\
\leq \left| \int_t^{t_1} f(s, u_n(s), u_n'(s))ds \right| \\
\leq L |t - t_1|,
\end{align*}
\]

which implies that \((K(N_f(u_n)) - Q\varphi(K(N_f(u_n))))_n\) is equicontinuous. Thus, by the Arzelà-Ascoli theorem there is a subsequence of \((K(N_f(u_n)) - Q\varphi(K(N_f(u_n))))_n\), which we call \((K(N_f(u_{n_j})) - Q\varphi(K(N_f(u_{n_j}))))_j\), which is convergent in \(C\). Using that \(\varphi^{-1} : C \to B_0(0) \subset C\) is continuous it follows from

\[
M(u_{n_j})' = \varphi^{-1} \left[ K(N_f(u_{n_j})) - Q\varphi(K(N_f(u_{n_j}))) \right]
\]

that the sequence \((M(u_{n_j}))_j\) is convergent in \(C\). Then, passing to a subsequence if necessary, we obtain that \((v_{n_j})_j = (M(u_{n_j}))_j\) is convergent in \(C^1\). Finally, let \((v_n)_n\) be a sequence in \(M(\Lambda)\). Let \((z_n)_n \subseteq M(\Lambda)\) be such that

\[
\lim_{n \to \infty} \|z_n - v_n\|_1 = 0.
\]

Let \((z_{n_j})_j\) be a subsequence of \((z_n)_n\) such that converge to \(z\). It follows that \(z \in M(\Lambda)\) and \((z_{n_j})_j\) converge to \(z\). This concludes the proof.

The next result is based on Schauder’s fixed point theorem.

**Theorem 4.3.** Let \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) be continuous. Then \([4.6]\) has at least one solution.

**Proof.** Let \(u \in C^1\). Then

\[
M(u) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) + H \left( \varphi^{-1} \left[ K(N_f(u)) - Q\varphi(K(N_f(u))) \right] \right),
\]

where

\[
M(u)(0) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) = M(u)(T),
\]

\[
M(u)'(T) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) = M(u)(0).
\]

Moreover,

\[
\|M(u)\|_\infty = \|\varphi^{-1} \left[ K(N_f(u)) - Q\varphi(K(N_f(u))) \right]\|_\infty < a
\]

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and
\[ \|M(u)\|_\infty < a + aT. \]
Hence,
\[ \|M(u)\|_1 = \|M(u)\|_\infty + \|M(u)'\|_\infty < a + aT + a = 2a + aT. \]
Because the operator \( M \) is completely continuous and bounded, we can use Schauder’s fixed point theorem to deduce the existence of at least one fixed point. This, in turn, implies that problem (4.6) has at least one solution. The proof is complete.

5 Problems with classic homeomorphisms and tree-point boundary conditions

We finally consider boundary value problems of the form
\[
\begin{cases}
(\varphi(u'))' = f(t, u, u') \\
u(T) = u'(0) = u'(T),
\end{cases}
\tag{5.9}
\]
where \( \varphi : \mathbb{R} \to \mathbb{R} \) is a homeomorphism such that \( \varphi(0) = 0 \) and \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function. We remember that an \textit{solution} of this problem is any function \( u : [0, T] \to \mathbb{R} \) of class \( C^1 \) such that \( \varphi(u'(t))' = f(t, u(t), u'(t)) \) for all \( t \in [0, T] \).

Let us consider the operator
\[
M_1 : C^1 \to C^1, \\
u \mapsto S(u) + Q(N_f(u)) + K(\varphi^{-1}[H(N_f(u)) - Q(N_f(u))] + \varphi(S(u))].
\]
Analogously to the section 3, here \( \varphi^{-1} \) is understood as the operator \( \varphi^{-1} : C \to C \) defined for \( \varphi^{-1}(v)(t) = \varphi^{-1}(v(t)) \). It is clear that \( \varphi^{-1} \) is continuous and sends bounded sets into bounded sets.

**Lemma 5.1.** \( u \in C^1 \) is a solution of (5.9) if and only if \( u \) is a fixed point of the operator \( M_1 \).

**Proof.** Let \( u \in C^1 \), we have the following equivalences:
\[
(\varphi(u'))' = N_f(u), \quad u'(T) = u'(0), \quad u'(0) = u(T)
\]
\[\iff (\varphi(u'))' = N_f(u) - Q(N_f(u)), \quad Q(N_f(u)) = 0, \quad u'(0) = u(T)\]
\[\iff \varphi(u') = H(N_f(u) - Q(N_f(u))) + \varphi(u'(0)),\]

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\[ Q(N_f(u)) = 0, \ u'(0) = u(T) \]
\[ \Leftrightarrow u' = \varphi^{-1} [H(N_f(u) - Q(N_f(u)) + \varphi(u'(0))], \]
\[ Q(N_f(u)) = 0, \ u'(0) = u(T) \]
\[ \Leftrightarrow u = u(T) + K(\varphi^{-1} [H(N_f(u) - Q(N_f(u)) + \varphi(u(T))]), \]
\[ Q(N_f(u)) = 0, \ u'(0) = u(T) \]
\[ \Leftrightarrow u = u(T) + Q(N_f(u)) + K(\varphi^{-1} [H(N_f(u) - Q(N_f(u)) + \varphi(u'(T))]) \]
\[ \Leftrightarrow u = S(u) + Q(N_f(u)) + K(\varphi^{-1} [H(N_f(u) - Q(N_f(u)) + \varphi(S(u))]). \]  

**Remark 5.2.** Note that \( u'(T) = u'(0) \Leftrightarrow Q(N_f(u)) = 0. \)

Using an argument similar to the introduced in Lemma 4.2, it is easy to see that, \( M_1 : C^1 \rightarrow C^1 \) is completely continuous.

In order to apply Leray-Schauder degree to the operator \( M_1 \), we introduced a family of problems depending on a parameter \( \lambda \). We remember that to each continuous function \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) we associate its Nemytskii operator \( N_f : C^1 \rightarrow C \) defined by
\[ N_f(u)(t) = f(t, u(t), u'(t)). \]

For \( \lambda \in [0, 1] \), we consider the family of boundary value problems
\[
\left\{ \begin{array}{l}
(\varphi(u'))' = \lambda N_f(u) + (1 - \lambda) Q(N_f(u)) \\
u(T) = u'(0) = u'(T).
\end{array} \right. \tag{5.10}
\]

Notice that (5.10) coincide with (5.9) for \( \lambda = 1 \). So, for each \( \lambda \in [0, 1] \), the operator associated to (5.10) by Lemma 5.1 is the operator \( M(\lambda, \cdot) \), where \( M \) is defined on \([0, 1] \times C^1 \) by
\[ M(\lambda, u) = S(u) + Q(N_f(u)) + K(\varphi^{-1} [\lambda H(N_f(u) - Q(N_f(u)) + \varphi(S(u))]). \]

Using the same arguments as in the proof of Lemma 4.2, we show that the operator \( M \) is completely continuous. Moreover, using the same reasoning as above, the system (5.10) (see Lemma 5.1) is equivalent to the problem
\[ u = M(\lambda, u). \]

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5.1 Existence results

In this subsection, we present and prove our main results. These results are inspired on works by Bereanu and Mawhin [5] and Manásevich and Mawhin [8]. We denote by $\text{deg}_B$ the Brouwer degree and for $\text{deg}_{LS}$ the Leray-Schauder degree, and define the mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G : \mathbb{R}^2 \to \mathbb{R}^2, \quad (a, b) \mapsto (aT + bT^2 - bT - \frac{1}{T} \int_0^T f(t, a + bt, b)dt, b - a - bT).$$

**Theorem 5.3.** Assume that $\Omega$ is an open bounded set in $C^1$ such that the following conditions hold.

1. For each $\lambda \in (0, 1)$ the problem

\[
\begin{aligned}
(\varphi(u'))' &= \lambda N_f(u) \\
u(T) &= u'(0) = u'(T),
\end{aligned}
\]  

(5.11)

has no solution on $\partial \Omega$.

2. The equation

$$G(a, b) = (0, 0),$$

has no solution on $\partial \Omega \cap \mathbb{R}^2$, where we consider the natural identification $(a, b) \approx a + bt$ of $\mathbb{R}^2$ with related functions in $C^1$.

3. The Brouwer degree

$$\text{deg}_B(G, \Omega \cap \mathbb{R}^2, 0) \neq 0.$$

Then problem (5.9) has a solution.

**Proof.** Let $\lambda \in (0, 1]$. If $u$ is a solution of (5.11), then $Q(N_f(u)) = 0$, hence $u$ is a solution of problem (5.10). On the other hand, for $\lambda \in (0, 1]$, if $u$ is a solution of (5.10) and because

$$Q(\lambda N_f(u) + (1 - \lambda)Q(N_f(u))) = Q(N_f(u)),$$

we have $Q(N_f(u)) = 0$, then $u$ is a solution of (5.11). It follows that, for $\lambda \in (0, 1]$, problems (5.10) and (5.11) have the same solutions. We assume that for $\lambda = 1$, (5.10) does not have a solution on $\partial \Omega$ since otherwise we are done with proof. It follows that (5.10) has no solutions for $(\lambda, u) \in (0, 1] \times \partial \Omega$. If $\lambda = 0$, then (5.10) is equivalent to the problem

\[
\begin{aligned}
(\varphi(u'))' &= Q(N_f(u)) \\
u(T) &= u'(0) = u'(T),
\end{aligned}
\]  

(5.12)
and thus, if $u$ is a solution of (5.12), we must have
\[ \int_0^T f(t, u(t), u'(t))dt = 0. \] (5.13)

Moreover, $u$ is a function of the form $u(t) = a + bt$, $a = b - bT$. Thus, by (5.13)
\[ \int_0^T f(t, a + bt, b)dt = 0, \]
which, together with hypothesis 2, implies that $u = b - bT + tb \notin \partial \Omega$. Thus we have proved that [5.10] has no solution in $\partial \Omega$ for all $\lambda \in [0, 1]$. Then we have that for each $\lambda \in [0, 1]$, the Leray-Schauder degree $\text{deg}_{LS}(I - M(\lambda, \cdot), \Omega, 0)$ is well defined, and by the homotopy invariance imply that
\[ \text{deg}_{LS}(I - M(0, \cdot), \Omega, 0) = \text{deg}_{LS}(I - M(1, \cdot), \Omega, 0). \]

On the other hand, we have that
\[ \text{deg}_{LS}(I - M(0, \cdot), \Omega, 0) = \text{deg}_{LS}(I - (S + QN_f + KS), \Omega, 0). \]

But the range of the mapping
\[ u \mapsto S(u) + Q(N_f(u)) + K(S(u)) \]
is contained in the subspace of related functions, isomorphic to $\mathbb{R}^2$. Thus, using a reduction property of Leray-Schauder degree [7, 10]
\[ \text{deg}_{LS}(I - (S + QN_f + KS), \Omega, 0) = \text{deg}_{B}(I - (S + QN_f + KS)|_{\Omega \cap \mathbb{R}^2}, \Omega \cap \mathbb{R}^2, 0) = 0. \]

Then, $\text{deg}_{LS}(I - M(1, \cdot), \Omega, 0) \neq 0$, where $I$ denotes the unit operator. Hence, there exists $u \in \Omega$ such that $M_1(u) = u$, which is a solution for (5.9). \qed

The following result gives a priori bounds for the possible solutions of (5.11), adapts a technique introduced by Ward [11].

**Theorem 5.4.** Assume that $f$ satisfies the following conditions.

1. There exists $c \in C$ such that
\[ f(t, x, y) \geq c(t) \]
for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

2. There exists $M_1 < M_2$ such that for all $u \in C^1$,
\[ \int_0^T f(t, u(t), u'(t))dt \neq 0 \text{ if } u'_m \geq M_2, \]
\[ \int_0^T f(t, u(t), u'(t))dt \neq 0 \text{ if } u'_M \leq M_1. \]

If \((\lambda, u) \in (0, 1) \times C^1\) is such that \(u\) is solution of (5.11), then
\[ \|u\|_1 < r(2 + T), \]
where
\[ r = \max \{ |\varphi^{-1}(L + 2 \|c^-\|_{L^1})|, |\varphi^{-1}(-L - 2 \|c^-\|_{L^1})| \}, \]
\[ L = \max \{ |\varphi(M_2)|, |\varphi(M_1)| \}. \]

**Proof.** Let \((\lambda, u) \in (0, 1) \times C^1\) be such that \(u\) is a solution of (5.11). Then for all \(t \in [0, T]\),
\[ (\varphi(u'(t)))' = \lambda N_f(u)(t), \quad u'(0) = u'(T) = u(T) \]
and
\[ \int_0^T f(t, u(t), u'(t))dt = 0. \]

Using hypothesis 2, we have that \(u'_m < M_2\) and \(u'_M > M_1\).

It follows that there exists \(\omega \in [0, T]\) such that \(M_1 < u'(\omega) < M_2\) and
\[ \int_\omega^t (\varphi(u'(s)))'ds = \lambda \int_\omega^t N_f(u)(s)ds, \]
which implies that
\[ |\varphi(u'(t))| \leq |\varphi(u'(\omega))| + \int_0^T |f(s, u(s), u'(s))|ds, \]
where
\[ \int_0^T |f(s, u(s), u'(s))|ds \leq \int_0^T f(s, u(s), u'(s))ds + 2 \int_0^T c^-(s)ds. \]

Hence,
\[ |\varphi(u'(t))| < L + 2 \|c^-\|_{L^1}, \]
where \(L = \max \{ |\varphi(M_2)|, |\varphi(M_1)| \}\) and \(t \in [0, T]\). It follows that
\[ \|u'\|_{\infty} < r, \]
where \(r = \max \{ |\varphi^{-1}(L + 2 \|c^-\|_{L^1})|, |\varphi^{-1}(-L - 2 \|c^-\|_{L^1})| \}. \) Because \(u \in C^1\) is such that \(u'(0) = u(T)\) we have that
\[ |u(t)| \leq |u(T)| + \int_0^T |u'(s)|ds < r + rT \quad (t \in [0, T]), \]

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and hence $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty < r + rT + r = r(2 + T)$. This proves the theorem.

Now we show the existence of at least one solution for problem (5.9) by means of Leray-Schauder degree.

**Theorem 5.5.** Let $f$ be continuous and satisfy condition (1) and (2) of Theorem 5.4. Assume that the following conditions hold for some $\rho \geq r(2 + T)$.

1. The equation
   
   $$G(a, b) = (0, 0),$$
   
   has no solution on $\partial B_\rho(0) \cap \mathbb{R}^2$, where we consider the natural identification $(a, b) \approx a + bt$ of $\mathbb{R}^2$ with related functions in $C^1$.

2. The Brouwer degree
   
   $$\text{deg}_B(G, B_\rho(0) \cap \mathbb{R}^2, 0) \neq 0.$$

Then problem (5.9) has a solution.

**Proof.** Let $(\lambda, u) \in (0, 1) \times C^1$ be such that $u$ is a solution of (5.11). Using Theorem 5.4 we have

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty < r + rT + r = r(2 + T),$$

where $r = \max \{|\varphi^{-1}(L + 2\|c\|_{L^1})|, |\varphi^{-1}(-L - 2\|c\|_{L^1})|\}$. Thus the conditions of Theorem 5.3 are satisfied with $\Omega = B_\rho(0)$, where $B_\rho(0)$ is the open ball in $C^1$ center 0 and radius $\rho$. This concludes the proof.

Let us give now an application of Theorem 5.5.

**Example 5.6.** Let us consider the problem

$$\left((u')^3\right)' = \frac{e^{u'}}{2} - 1, \quad u(T) = u'(0) = u'(T), \quad (5.14)$$

for $M_1 = -1$, $M_2 = 1$, $\rho \geq (1 + 2T)^{1/3}(2 + T)$ and $c(t) = -1$ for all $t \in [0, T]$. So, problem (5.14) has at least one solution.

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References

[1] C. Bereanu and J. Mawhin, *Nonhomogeneous boundary value problems for some nonlinear equations with singular $\varphi$-laplacian*, J. Math. Anal. Appl. 352 (2009), 218-233.

[2] C. Bereanu and J. Mawhin, *Boundary value problems for some nonlinear systems with singular $\varphi$-laplacian*, J. Fixed Point Theory Appl. 4 (2008), 57-75.

[3] C. Bereanu and J. Mawhin, *Periodic solutions of nonlinear perturbations of $\varphi$-laplacians with possibly bounded $\varphi$*, Nonlinear Anal. 68 (2008), 1668-1681.

[4] C. Bereanu and J. Mawhin, *Existence and multiplicity results for some nonlinear problems with singular $\varphi$-laplacian*, J. Differential Equations. 243 (2007), 536-557.

[5] C. Bereanu and J. Mawhin, *Boundary-value problems with non-surjective $\varphi$-laplacian and one-sided bounded nonlinearity*, Advances Differential Equations. 11 (2006), 35-60.

[6] C. Bereanu and J. Mawhin, *Nonlinear Neumann boundary value problems with $\varphi$-laplacian operators*, An. Stiint. Univ. Ovidius Constanta. 12 (2004), 73-92.

[7] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[8] R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p-laplacian-like operators*, Differential Equations. 145 (1997), 367-393.

[9] J. Mawhin, *Leray-Schauder degree: A half century of extensions and applications*, Topological Methods Nonlinear Anal. 14 (1999), 195-228.

[10] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS series No. 40, American Math. Soc., Providence RI, 1979.

[11] J.R. Ward Jr, *Asymptotic conditions for periodic solutions of ordinary differential equations*, Proc. Amer. Math. Soc. 81 (1981), 415-420.