ESTIMATES FOR INITIAL COEFFICIENTS OF CERTAIN BI–UNIVALENT FUNCTIONS

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Abstract. Estimates are obtained for the initial coefficients of a normalized analytic function \( f \) in the unit disk \( D := \{ z : |z| < 1 \} \) such that \( f \) and the analytic extension of \( f^{-1} \) to \( D \) belong to certain subclasses of univalent functions. The bounds obtained improve some existing known bounds.

1. Introduction and Preliminaries

Let \( A \) be the class of analytic functions defined on the unit disk \( D := \{ z : |z| < 1 \} \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]  

(1.1)

Suppose that \( S \) is the subclass of \( A \) consisting of univalent functions. Being univalent, the functions in class \( S \) are invertible; however, the inverse need not be defined on the entire unit disk. The Koebe one-quarter theorem ensures that the image of the unit disk under every univalent function contains a disk of radius \( 1/4 \). Thus, a function \( f \in S \) has an inverse defined on a disk containing disk \( |z| < 1/4 \). It can be easily seen that

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + \cdots
\]

in some disk of radius at least \( 1/4 \). A function \( f \in A \) is said to be bi–univalent in \( D \) if both \( f \) and analytic extension of \( f^{-1} \) to \( D \) are univalent in \( D \). The class of bi–univalent functions, denoted by \( \sigma \), was introduced by Lewin [15] in 1967, who also showed that the second coefficient of a bi–univalent function satisfies the inequality \( |a_2| \leq 1.51 \). Let \( \sigma_1 \) be the class of the functions \( f = \phi \circ \psi^{-1} \), where \( \phi \) and \( \psi \) are univalent analytic functions mapping \( D \) onto a domain containing \( D \) and satisfy \( \phi'(0) = \psi'(0) \). Clearly, \( \sigma_1 \subset \sigma \), though \( \sigma_1 \neq \sigma \) (see [6]). In 1969, Suffridge [25] gave a function in class \( \sigma_1 \) with \( a_2 = 4/3 \) and conjectured that \( |a_2| \leq 4/3 \) for the functions in the class \( \sigma \). Netanyahu [20] proved the conjecture for the subclass \( \sigma_1 \). The conjecture was later disproved by Styler and Wright [24] in 1981, who showed that \( a_2 > 4/3 \) for some function in \( \sigma \). Brannan and Clunie [6] conjectured that \( |a_2| \leq \sqrt{2} \) for a function \( f \in \sigma \). Kedziewski [11] proved this for a special case when the functions \( f \) and \( f^{-1} \) are starlike functions.

For analytic functions \( f \) and \( g \) in \( D \), the function \( f \) is subordinate to the function \( g \), written as \( f(z) \prec g(z) \), if there is a Schwarz function \( w \) such that \( f = g \circ w \). If \( g \) is univalent, then \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \). The

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method of subordination is quite useful for establishing relations in terms of inequalities in the complex plane. Padmanabhan and Parvatham [21] gave a unified representation of various classes of starlike and convex functions using convolution with the function \( z/(1-z)^{\alpha} \), for \( \alpha \in \mathbb{R} \). Later in 1989, for a convex function \( h \) and a fixed function \( g \), Shanmugam [22] introduced a class \( S^*_g(h) \) which consists of functions \( f \in \mathcal{A} \) satisfying \( z(f * g)'(z)/(f * g)(z) < h(z) \). Further, if \( g(z) = z/(1-z) \) and \( h = \varphi \) is an analytic function with positive real part in \( \mathbb{D} \) such that \( \varphi(0) = 1, \varphi'(0) > 0 \) and \( \varphi(\mathbb{D}) \) is symmetric about the real axis and starlike with respect to 1, then the class \( S^*_g(h) \) reduces to the class \( S^*(\varphi) \) which was introduced by Ma and Minda [16]. The growth, distortion and covering theorems for the class \( S^*(\varphi) \) are also proved in [16]. For particular choices of \( \varphi \), we have the following subclasses of the univalent functions. If \( \varphi(z) = (1 + Az)/(1 + Bz) \), where \(-1 \leq B < A \leq 1\), then the class \( S^*(\varphi) \) is termed as the class of Janowski starlike functions [10], denoted by \( S^*[A, B] \). For \( 0 \leq \beta < 1 \), the class \( S^*[1 - 2\beta, -1] =: S^*(\beta) \) is the class of starlike functions of order \( \beta \) and for \( \beta = 0 \), the class \( S^* := S^*(0) \) is simply the class of starlike functions. If \( 0 < \alpha \leq 1 \), then the class \( SS^*\alpha := S^*((1 + z)/(1 - z))^{\alpha} \) is the class of strongly starlike functions of order \( \alpha \). Similarly, the class \( \mathcal{K}(\varphi) \) of convex functions consists of the univalent functions satisfying \( 1 + zf''(z)/f'(z) < \varphi(z) \). Let \( \mathcal{R}(\varphi) \) be the class of univalent functions satisfying \( f'(z) < \varphi(z) \). For \( b \in \mathbb{C} \setminus \{0\} \) and \( p \in \mathbb{N} \), the classes \( \mathcal{R}_{b,p}(\varphi) \) and \( S^*_{b,p}(\varphi) \) consist of the functions of the form \( f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \) satisfying

\[
1 + \frac{1}{b} \left( \frac{f'(z)}{p z^{p-1}} - 1 \right) < \varphi(z) \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{1}{z} f'(z) - 1 \right) < \varphi(z),
\]

respectively. Ali et al. [4] obtained Fekete-Szegö inequalities and bound on the coefficient \( a_{p+3} \) for the functions in these classes. On coefficient estimates for the functions that belong to certain subclasses of univalent functions, one can refer [12, 14].

Analogous to the class of starlike (and convex) functions of order \( \beta \) (with \( 0 \leq \beta < 1 \)), the class of bi–starlike (and bi–convex) functions of order \( \beta \), denoted by \( \mathcal{S}^*_\sigma(\beta) \) (and \( \mathcal{K}_\sigma(\beta) \)), is the class of bi–univalent functions \( f \) such that \( f \) and analytic extension of \( f^{-1} \) to \( \mathbb{D} \) are both starlike (and convex) of order \( \beta \) in \( \mathbb{D} \). For \( 0 < \alpha \leq 1 \), a bi–univalent function \( f \) is in class \( \mathcal{S}^*_\sigma(\alpha) \) of strongly bi–starlike functions of order \( \alpha \) if \( f \) and analytic extension of \( f^{-1} \) to \( \mathbb{D} \) are strongly starlike functions of order \( \alpha \) in \( \mathbb{D} \). Brannan and Taha [7] introduced these classes and gave bound on initial coefficients of the functions in these classes. Also, for a function \( f \in \mathcal{K}_\sigma(0) \) given by \( \{1\} \), they showed \( |a_2| \leq 1 \) and \( |a_3| \leq 1 \) with extremal function given by \( z/(1-z) \) and its rotations. Particularly if \( \beta = 0 \), then the class \( \mathcal{S}^*_\sigma(\beta) \) reduces to the class of bi–starlike functions. Kedzierski [14] proved that for a bi–starlike function \( f \) of the form \( \{1\} \), \( |a_2| \leq \sqrt{2} \). Further, [17] and [18] improved the estimates for coefficients \( a_2 \) and \( a_3 \) and also found estimates for the fourth coefficient for the functions in classes \( \mathcal{S}^*_\sigma(\beta) \) and \( \mathcal{S}^*_g(\alpha) \). For coefficient estimates for the functions in some particular subclasses of bi–univalent functions, one may see [3, 9, 13, 19, 23, 26, 27, 28].

Let the function \( \varphi \) be an analytic function in \( \mathbb{D} \) of the form

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,
\]

where \( B_1 > 0 \). For the function \( \varphi \) and \( \lambda \geq 0 \), Kumar et al. [13] introduced the following subclass \( \mathcal{R}_\sigma(\lambda, \varphi) \) of bi–univalent functions.
Definition 1.1. Let \( \lambda \geq 0 \). A bi-univalent function \( f \) given by (1.1) is in class \( \mathcal{R}_\sigma(\lambda, \varphi) \), if it satisfies
\[
(1 - \lambda)\frac{f'(z)}{z} + \lambda f'(z) < \varphi(z) \quad \text{and} \quad (1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) < \varphi(w),
\]
where \( g \) denotes the univalent extension of \( f^{-1} \) to the unit disk.

With the particular values of \( \lambda \) and \( \varphi \), the class \( \mathcal{R}_\sigma(\lambda, \varphi) \) reduces to many earlier classes as mentioned below:

(i) \( \mathcal{R}_\sigma(\lambda, (1 + (1 - 2\beta)z)/(1 - z)) = \mathcal{R}_\sigma(\lambda, \beta) \quad (\lambda \geq 1; 0 \leq \beta < 1) \) [9 Definition 3.1]
(ii) \( \mathcal{R}_\sigma(\lambda, ((1 + z)/(1 - z))^\alpha) = \mathcal{R}_{\sigma, \alpha}(\lambda) \quad (\lambda \geq 1; 0 < \alpha \leq 1) \) [9 Definition 2.1]
(iii) \( \mathcal{R}_\sigma(1, \varphi) = \mathcal{R}_\sigma(\varphi) \) [3, p. 345].
(iv) \( \mathcal{R}_\sigma(1, (1 + (1 - 2\beta)z)/(1 - z)) = \mathcal{R}_\sigma(\beta) \quad (0 \leq \beta < 1) \) [23 Definition 2].
(v) \( \mathcal{R}_\sigma(1, ((1 + z)/(1 - z))^\alpha) = \mathcal{R}_{\sigma, \alpha} \quad (0 < \alpha \leq 1) \) [23 Definition 1]

The class of bi-starlike functions of Ma-Minda type was given by Ali et al. [3].

Definition 1.2. A function \( f \in \sigma \) of the form (1.1), is said to be in the class of Ma-Minda bi-starlike functions, denoted by \( \mathcal{S}^*_\sigma(\varphi) \), if the following subordinations hold:
\[
\frac{zf'(z)}{f(z)} < \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} < \varphi(w),
\]
where \( g \) denotes the univalent extension of \( f^{-1} \) to \( \mathbb{D} \) and \( \varphi \) is the function of the form (1.2) satisfying the conditions as in the definition of the class \( \mathcal{S}^*(\varphi) \) as mentioned earlier.

The class \( \mathcal{S}^*_\sigma(\varphi) \) includes some well-known classes of the bi-univalent functions. For example:

(i) \( \mathcal{S}^*_\sigma((1 + (1 - 2\beta)z)/(1 - z)) =: \mathcal{S}^*_\sigma(\beta), \ 0 \leq \beta < 1 \).
(ii) \( \mathcal{S}^*_\sigma(((1 + z)/(1 - z))^\alpha) =: \mathcal{S}^*_\sigma[\alpha], \ 0 < \alpha \leq 1 \).

Using the Fekete-Szegő inequalities and principles of subordination, in this paper, the estimates for the coefficients \( a_2 \) and \( a_3 \) of the functions of the form (1.1) in the classes \( \mathcal{R}_\sigma(\lambda, \varphi) \) and \( \mathcal{S}^*_\sigma(\varphi) \) have been obtained. Moreover, the estimates so obtained are observed to be an improvement over the ones derived in [3] [5] [13]. For some particular choices of \( \lambda \) and \( \varphi \), the bounds determined are smaller than those mentioned in [7] [9] [17] [18] [23] for the coefficients of the functions in the respective classes.

More precisely, the following theorem derives the estimates for the coefficients \( a_2 \) and \( a_3 \) for the functions given by (1.1) that belong to the class \( \mathcal{R}_\sigma(\lambda, \varphi) \).

Theorem 1.3. Let \( \varphi \) be an analytic function given by the series (1.2) such that \( B_2 \in \mathbb{R} \). For \( \lambda \geq 0 \), let the function \( f \in \mathcal{R}_\sigma(\lambda, \varphi) \) and \( \tau := (1 + \lambda)^2/(1 + 2\lambda) \).

(a) If \( \tau B_2 \leq B_1^2 \), then
\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1 + 2\lambda)(B_1^2 - \tau B_2 + \tau B_1)}} \quad \text{and} \quad |a_3| \leq \frac{B_1}{1 + 2\lambda} \max \left\{ \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1}, 1 \right\}.
\]
(b) If \( \tau B_2 \geq B_1^2 \), then
\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1 + 2\lambda)(\tau B_2 + \tau B_1 - B_1^2)}} \quad \text{and} \quad |a_3| \leq \frac{B_1}{1 + 2\lambda} \max \left\{ \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2}, 1 \right\}.
\]
Remark 1.4. Theorem 1.3 is an improvement over the coefficient estimates obtained in [3, 9, 13, 23]. For an analytic function \( \varphi \) of the form (1.2), Kumar et al. [13, Theorem 2.2] obtained a bound on the coefficient \( a_2 \) of the function \( f \in \mathcal{R}_\sigma(\lambda, \varphi) \) for \( \lambda \geq 0 \). In addition, if \( B_2 \in \mathbb{R} \), by the means of the following comparisons, it may be noted that Theorem 1.3 gives an estimate for \( a_2 \) which is smaller than the one given by [13, Theorem 2.2]. We can see that if \( \tau B_2 \leq B_1^2 \) and \( B_2 \leq B_1 \), then

\[
\frac{2B_1 - B_2}{B_1} - \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} = \frac{(B_1 - B_2)(2\tau B_1 + B_1^2 - \tau B_2)}{B_1(B_1^2 - \tau B_2 + \tau B_1)} \geq 0.
\]

Therefore, \( \min \left\{ \frac{2B_1 - B_2}{B_1}, \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} \right\} = \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} \) which implies that the estimate obtained for \( a_2 \) using Theorem 1.3 for this case is less than \( \sqrt{(2B_1 - B_2)/(1 + 2\lambda)} \). Similarly, if \( \tau B_2 \leq B_1^2 \) and \( B_2 \geq B_1 \), then

\[
\frac{B_2}{B_1} - \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} = \frac{(B_1 - \tau B_2)(B_2 - B_1)}{B_1(B_1^2 - \tau B_2 + \tau B_1)} \geq 0.
\]

Next, the case when the conditions \( \tau B_2 \geq B_1^2 \) and \( B_2 \leq B_1 \) hold, it follows that

\[
\frac{2B_1 - B_2}{B_1} - \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2} \geq \frac{B_1(\tau - B_1)^2}{\tau(\tau B_2 + \tau B_1 - B_1^2)} \geq 0
\]

and further, if \( \tau B_2 \geq B_1^2 \) and \( B_2 \geq B_1 \), then the inequality

\[
\frac{B_2}{\tau B_2 + \tau B_1 - B_1^2} - \frac{B_1^2}{B_1^2 - B_1} = \frac{(\tau B_2 - B_1^2)(B_2 + B_1)}{B_1(\tau B_2 + \tau B_1 - B_1^2)} \geq 0
\]

holds.

Now let us consider the class \( \mathcal{R}_{\sigma,\alpha}(\lambda) := \mathcal{R}_\sigma(\lambda,((1 + z)/(1 - z))^\alpha) \) for \( 0 < \alpha \leq 1 \). Clearly, \( B_1 = 2\alpha \) and \( B_2 = 2\alpha^2 \). For a function \( f \) given by (1.1) in the class \( \mathcal{R}_{\sigma,\alpha}(\lambda) \), Theorem 1.3 yields

\[
|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1-\lambda^2 + 2\lambda)}} & \text{if } 1 \leq \lambda \leq 1 + \sqrt{2}, \\ \frac{2\alpha}{\sqrt{(1+\lambda)^2 - \alpha(1-\lambda^2 + 2\lambda)}} & \text{if } \lambda \geq 1 + \sqrt{2}. \end{cases}
\]

It can be verified that if \( 1 \leq \lambda \leq 1 + \sqrt{2}, \) then the bound derived for \( a_2 \) coincides with that obtained by Frasin and Aouf [9, Theorem 2.2], whereas the estimate obtained for \( a_2 \) for the part \( \lambda \geq 1 + \sqrt{2} \) is smaller than that in [9, Theorem 2.2]. Likewise, using Theorem 1.3, we can see that \( |a_3| \leq 2\alpha/(1 + 2\lambda) \) which is less than the bound for \( a_3 \) derived in [9, Theorem 2.2].

Similarly, let \( \varphi(z) = (1 + (1 - 2\beta)z)/(1 - z) \) for \( 0 \leq \beta < 1 \). As a result of Theorem 1.3, the functions in the class \( \mathcal{R}_\sigma(\lambda, \beta) \) satisfy

\[
|a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \beta)}{1 + 2\lambda}} & \text{if } 0 \leq \beta \leq \frac{1 - \lambda^2 + 2\lambda}{2(1 + 2\lambda)}, \\ \frac{2}{(1 - \beta)\sqrt{\frac{2}{\lambda^2 + \beta(1 + 2\lambda)}}} & \text{if } \frac{1 - \lambda^2 + 2\lambda}{2(1 + 2\lambda)} \leq \beta < 1. \end{cases}
\]
and $|a_3| \leq 2(1 - \beta)/(1 + 2\lambda)$. Again, the estimate for $a_2$ so determined for the part when $0 \leq \beta \leq (1 - \lambda^2 + 2\lambda)/(2(1 + 2\lambda))$ is same as that obtained by Frasin and Aouf [9, Theorem 3.2]. For $(1 - \lambda^2 + 2\lambda)/(2(1 + 2\lambda)) \leq \beta < 1$, the estimate for $a_2$, derived using Theorem [13] is refined in comparison with [9, Theorem 3.2]. The estimate for the coefficient $a_3$ obtained using Theorem [13] is smaller than the one in [9, Theorem 3.2]. Moreover, the coefficient estimates derived above for the functions in classes $\mathcal{R}_{\sigma,\lambda}^*(\lambda)$ and $\mathcal{R}_\sigma^*(\lambda, \beta)$ are valid for $\lambda \geq 0$.

Also, Ali et al. [3, Corollary 2.3] gave estimates on the coefficients $a_2$ and $a_3$ of a function $f \in \mathcal{R}_\sigma^*(\varphi)$ of the form (1.1). It may be noted that the estimates for the coefficients $a_2$ and $a_3$ of the function $f \in \mathcal{R}_\sigma^*(\varphi)$ given using Theorem [13] improve the estimates given in [3, Corollary 2.3] provided $\varphi''(0) \in \mathbb{R}$.

Furthermore, the coefficient estimates for the functions in the classes $\mathcal{R}_{\sigma,\lambda}^*$ and $\mathcal{R}_\sigma^*(\beta)$ determined in [23, Theorem 2] and [23, Theorem 1], respectively are particular cases for the above-mentioned estimates.

The next theorem determines the estimates for the initial coefficients for a function in the class $\mathcal{S}_\sigma^*(\varphi)$.

**Theorem 1.5.** Let $f \in \mathcal{S}_\sigma^*(\varphi)$, where $\varphi''(0) \in \mathbb{R}$.

(a) If $B_2 \leq B_1^2$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{B_1^2 + B_1 - B_2}}$$

and

$$|a_3| \leq \max\left\{ \frac{B_1^3}{B_1^2 - B_2 + B_1}, \frac{B_1}{2} \right\}.$$

(b) If $B_2 \leq B_1^2$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{B_2 + B_1 - B_1^2}}$$

and

$$|a_3| \leq \max\left\{ \frac{B_1^3}{B_2 + B_1 - B_1^2}, \frac{B_1}{2} \right\}.$$

**Remark 1.6.** Bohra et al. [5, Corollary 2.3] and Ali et al. [3, Corollary 2.1] gave estimates on the coefficients $a_2$ and $a_3$ of the functions in the class $\mathcal{S}_\sigma^*(\varphi)$. In addition, let us assume that $B_2 \in \mathbb{R}$. By means of inequalities similar to those in Remark [13] we can see that the estimates for the coefficients $a_2$ and $a_3$ of a function $f \in \mathcal{S}_\sigma^*(\varphi)$, obtained using Theorem [13] improve those derived in the above references.

Particularly if $\varphi(z) = ((1 + z)/(1 - z))^\alpha$, $(0 < \alpha \leq 1)$, the Theorem [13] readily yields that for a function $f \in \mathcal{S}_\sigma^*[\alpha]$ of the form (1.1), we have $|a_2| \leq 2\alpha/(\sqrt{\alpha + 1})$, while

$$|a_3| \leq \alpha \quad \text{if} \quad 0 < \alpha \leq 1/3, \quad \text{and} \quad |a_3| \leq 4\alpha^2/(\alpha + 1) \quad \text{if} \quad 1/3 \leq \alpha \leq 1$$

which coincides with the estimates for $a_3$ as mentioned in [18, Theorem 2.1]. For a function $f \in \mathcal{S}_\sigma^*(\beta)$ $(0 \leq \beta < 1)$, a bi–starlike function of order $\beta$, using Theorem [13] we may solve to get

$$|a_2| \leq \sqrt{2(1 - \beta)} \quad \text{if} \quad 0 \leq \beta \leq 1/2, \quad \text{whereas} \quad |a_2| \leq (1 - \beta)\sqrt{2/\beta} \quad \text{if} \quad 1/2 \leq \beta < 1.$$

Further,

$$|a_3| \leq \begin{cases} 
2(1 - \beta) & \text{if} \quad 0 \leq \beta \leq 1/2 \\
2(1 - \beta)^2/\beta & \text{if} \quad 1/2 \leq \beta \leq 2/3 \\
1 - \beta & \text{if} \quad 2/3 \leq \beta < 1.
\end{cases}$$

The bounds for $a_2$ and $a_3$ obtained above are smaller than those given by [17]. Also it can be seen that the bounds obtained as a result of Theorem [13] are an improvement over the ones given by Brannan and Taha [7].
2. Proofs of the main results

We now prove following lemmas which are useful to prove the main result.

**Lemma 2.1.** Let $\xi \in \mathbb{R}$, $\eta > 0$. Let the function $G: \mathbb{R} \to [0, \infty)$ be defined by

$$G(x) := \max\{1, |\eta x - \xi|\}.$$ 

Then

$$\inf_{x,y \in \mathbb{R}} \frac{G(x) + G(y)}{|2 - x - y|} = \begin{cases} \frac{1}{1 - \gamma} & \text{if } \xi \leq \eta \\ \frac{1}{\rho - 1} & \text{if } \xi \geq \eta, \end{cases}$$

where $\gamma := (\xi - 1)/\eta$ and $\rho := (\xi + 1)/\eta$.

**Proof.** The function $G(x)$ can be simplified to

$$G(x) = \begin{cases} \xi - \eta x & \text{if } x \leq \gamma \\ 1 & \text{if } \gamma \leq x \leq \rho \\ \eta x - \xi & \text{if } x \geq \rho. \end{cases}$$

Let $H$ be a function on $\mathbb{R}^2 \setminus \{(x, y) : x + y = 2\}$ defined by

$$H(x, y) := \frac{G(x) + G(y)}{|2 - x - y|}.$$ 

It may be noted that $\lim_{x+y \to 2} H(x, y) = \infty$. Being a non-negative real-valued function, $H$ has a non-negative infimum in $\mathbb{R}$. To find the infimum of the function $H$, we consider the possibilities as $x$ and $y$ vary in the domain of definition. Since $H$ is a symmetric function in the variables $x$ and $y$, the infimum of the function $H$ if $\gamma \leq x \leq \rho$ and $y \leq \gamma$ coincides with that whenever $x \leq \gamma$ and $\gamma \leq y \leq \rho$. Similarly, the conditions $x \geq \rho$, $y \leq \gamma$, and $\gamma \leq x \leq \rho$, $y \geq \rho$ result into the same infimum of the function $H$ as given by $x \leq \gamma$, $y \geq \rho$, and $x \geq \rho$, $\gamma \leq y \leq \rho$, respectively. Therefore, the process of determining the infimum of the function $H$ reduces to finding infimum of $H$ in the following cases:

**Case 1:** $x, y \leq \gamma$.

The function $H$ becomes

$$H(x, y) = \frac{1}{|2 - x - y|} (2\xi - (x + y)\eta).$$

This case may be divided into three subcases *viz.* $\xi \leq \eta$, $\eta < \xi \leq \eta + 1$ and $\xi > \eta + 1$. If $\xi \leq \eta$, it is clear that $\gamma < 1$. Hence, the function $H$ reduces to

$$H(x, y) = \frac{1}{2 - x - y} (2\xi - (x + y)\eta).$$

The function $H$ has no critical points in the set $(-\infty, \gamma) \times (-\infty, \gamma)$ which states that $H$ does not attain its minimum in its set. Thus, the minimum of the function $H$, if it exists, is attained on the boundary; however, $H$ may have its infimum as $x$ or $y$ approach $-\infty$ or both. Along the line $y = \gamma$, it is easy to see that the function $H(x, \gamma)$ is decreasing in $x$ and $\min H(x, \gamma) = H(\gamma, \gamma) = 1/(1 - \gamma)$. Similarly, $\min H(\gamma, y) = 1/(1 - \gamma)$. At the same time

$$\lim_{x \to -\infty} H(x, y) = \lim_{y \to -\infty} H(x, y) = \eta.$$
For $\xi \leq \eta$, since $\min\{\eta, 1/(1 - \gamma)\} = 1/(1 - \gamma)$, we conclude that
\[
\inf H(x, y) = \min H(x, y) = H(\gamma, \gamma) = 1/(1 - \gamma).
\]

Let us now assume that $\eta < \xi \leq \eta + 1$ which implies $\gamma \leq 1$. Being increasing functions in $x$ and $y$, $H(x, \gamma)$ and $H(\gamma, y)$, respectively, do not attain their minimum; however, we have
\[
\inf H(x, y) = \lim_{x \to -\infty} H(x, y) = \lim_{y \to -\infty} H(x, y) = \eta.
\]
Therefore, in this case, $\inf H(x, y) = \eta$.

Suppose $\xi > \eta + 1$ which means $\gamma > 1$. Whenever $x + y < 2$, by virtue of the subcase $\eta < \xi \leq \eta + 1$, we observe that $\inf H(x, y) = \eta$, whereas, if $x + y > 2$, then the function $H$ is given by
\[
H(x, y) = \frac{1}{x + y - 2}(2\xi - (x + y)\eta).
\]
The function $H$ attains its minimum along the edges $x = \gamma$ and $y = \gamma$. It can be verified that the functions $H(x, \gamma)$ and $H(\gamma, y)$ both attain their minimum at the point $(\gamma, \gamma)$ with the minimum value $H(\gamma, \gamma) = 1/(\gamma - 1)$ and on choosing the least amongst the values $\eta$ and $1/(\gamma - 1)$, we infer that
\[
\inf H(x, y) = \begin{cases} 
\eta & \text{if } \eta + 1 < \xi \leq \eta + 2 \\
1/(\gamma - 1) & \text{otherwise}
\end{cases}
\]
provided $\xi > \eta + 1$. In view of the observations made above in each of the subcase, by selecting the least of the corresponding infimum, it follows that
\[
\inf_{x, y \leq \gamma} H(x, y) = \begin{cases} 
\eta & \text{if } \eta \leq \xi \leq \eta + 2 \\
1/|1 - \gamma| & \text{otherwise}.
\end{cases}
\]

**Case 2: $x \leq \gamma$ and $\gamma \leq y \leq \rho$.**

In this case, the function $H$ becomes
\[
H(x, y) = \frac{1}{|2 - x - y|}(\xi + 1 - x\eta) = \frac{\eta}{|2 - x - y|}(\rho - x).
\]

It may be noted that the function $H$ either attains its minimum on the boundary or has its infimum as $x$ approaches $-\infty$.

This case may be partitioned into the subcases $\xi \leq \eta - 1$, $\eta - 1 < \xi \leq \eta$ and $\xi > \eta$. Whenever $\xi \leq \eta - 1$, it can be seen that the functions $H(x, \gamma)$ and $H(x, \rho)$, being decreasing functions of $x$, attain their minimum at the points $(\gamma, \gamma)$ and $(\gamma, \rho)$, respectively with the minimum values $1/(1 - \gamma)$ and $2/(2 - \gamma - \rho)$, respectively. Also, $H(\gamma, y) \geq 1/(1 - \gamma)$ for $\gamma \leq y \leq \rho$. Since $y$ can be a finite real number, the function $H$ may have its infimum as $x$ approaches $-\infty$. Clearly, $\lim_{x \to -\infty} H(x, y) = \eta$. Since the least of the derived values with $\xi \leq \eta - 1$ is $1/(1 - \gamma)$, we conclude that $\inf H(x, y) = \min H(x, y) = H(\gamma, \gamma) = 1/(1 - \gamma)$.

For the case if $\eta - 1 < \xi \leq \eta$, we get that $\min H(x, \gamma) = \min H(\gamma, y) = H(\gamma, \gamma) = 1/(1 - \gamma)$, whereas $H(x, \rho)$ does not attain its minimum. It is easy to verify that
\[
\inf H(x, \rho) = \inf H(x, y) = \lim_{x \to -\infty} H(x, y) = \eta.
\]
Again by simple computations, we can see that \( \inf H(x, y) = 1/(1 - \gamma) \).

Let us assume that \( \xi > \eta \) which is same as \( \gamma + \rho > 2 \). Whenever \( x + y < 2 \), it may be noted that \( H \) does not have a minimum value, while \( \inf H(x, y) = \lim_{x \to -\infty} H(x, y) = \eta \). Further, if \( x + y > 2 \), the function \( H \) attains its minimum on the boundary. The functions \( H(x, \rho) \) and \( H(\gamma, y) \) have the minimum values given by \( H(\gamma, \rho) = 2/(\gamma + \rho - 2) \). In addition, if \( \xi > \eta + 1 \), then we also have \( \min H(x, \gamma) = H(\gamma, \gamma) = 1/(1 - \gamma) \).

Choosing the least of all the values so obtained, we deduce that

\[
\inf_{x \leq \gamma, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } \xi \leq \eta \\
\eta & \text{if } \eta \leq \xi \leq \xi + 1 \\
2/(\gamma + \rho - 2) & \text{if } \xi \geq \xi + 1.
\end{cases}
\]

Working on similar lines, infimum for the function \( H \) for each of the case may be noted as follows:

**Case 3:** \( x \leq \gamma \) and \( y \geq \rho \).
This case can be viewed in four subcases as given by \( \xi \leq \eta - 1, \eta - 1 < \xi \leq \eta, \eta < \xi \leq \eta + 1 \) and \( \xi > \eta + 1 \). For \( \xi \leq \eta - 1 \), we can see that the minimum of the function \( H \) is attained at the point \( (\gamma, \rho) \) with the minimum value \( 2/(2 - \gamma - \rho) \), whereas if \( \eta - 1 < \xi \leq \eta \), then

\[
\inf H(x, y) = \lim_{x \to -\infty} H(x, y) = \lim_{y \to \infty} H(x, y) = \eta.
\]

Solving for the other two parts similarly and selecting the least value, we get

\[
\inf_{x \leq \gamma, y \geq \rho} H(x, y) = \begin{cases} 
\eta & \text{if } \eta - 1 \leq \xi \leq \eta + 1 \\
2/(|2 - \gamma - \rho|) & \text{otherwise}.
\end{cases}
\]

**Case 4:** \( \gamma \leq x, y \leq \rho \).
This case may be partitioned into the subcases \( viz. \ xi \leq \eta - 1, \eta - 1 \leq \xi \leq \eta + 1 \) and \( \xi \geq \eta + 1 \). If \( \xi \leq \eta - 1 \) which indicates \( \rho \leq 1 \), then \( \min H(x, y) = H(\gamma, \gamma) = 1/(1 - \gamma) \). Assuming that \( \eta - 1 \leq \xi \leq \eta + 1 \) which is same as \( \gamma \leq 1 \leq \rho \), we have

\[
\min H(x, y) = \begin{cases} 
H(\gamma, \gamma) = 1/(1 - \gamma) & \text{if } \eta - 1 \leq \xi \leq \eta \\
H(\rho, \rho) = 1/(\rho - 1) & \text{if } \eta \leq \xi \leq \eta + 1.
\end{cases}
\]

In case \( \xi \geq \eta + 1 \) which means \( \gamma \geq 1 \), then \( \min H(x, y) = H(\rho, \rho) = 1/(\rho - 1) \). Hence, we conclude that

\[
\min_{\gamma \leq x, y \leq \rho} H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } \xi \leq \eta \\
1/(\rho - 1) & \text{if } \xi \geq \eta.
\end{cases}
\]

**Case 5:** \( x \geq \rho \) and \( \gamma \leq y \leq \rho \).
Let \( \xi \leq \eta - 1 \) which signifies the condition \( \rho \leq 1 \). In this case \( \min H(x, y) = H(\gamma, \rho) = 2/(2 - \gamma - \rho) \). Suppose that \( \eta - 1 \leq \xi \leq \eta \) that is \( \gamma + \rho \leq 2 \) and \( \rho \geq 1 \) for which we have \( \inf H(x, y) = \lim_{x \to \infty} H(x, y) = \eta \). Given that \( \xi \geq \eta \) that is \( \gamma + \rho \geq 2 \), \( \min H(x, y) = H(\rho, \rho) = 1/(\rho - 1) \). Consequently, we have

\[
\inf_{x \geq \rho, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 
2/(2 - \gamma - \rho) & \text{if } \xi \leq \eta - 1 \\
\eta & \text{if } \eta - 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi.
\end{cases}
\]
Case 6: \( x, y \geq \rho \).

Again, the case may be divided into the subcases given by \( \xi \leq \eta - 1, \eta - 1 < \xi < \eta \) and \( \xi > \eta \). On the similar lines as followed in the case when \( x, y \leq \gamma \), it can be verified that

\[
\inf_{x, y \geq \rho} H(x, y) = \begin{cases} 
\eta & \text{if } \eta - 2 \leq \xi \leq \eta \\
\frac{1}{|\rho - 1|} & \text{otherwise}.
\end{cases}
\]

From the above cases, we use some simple computations to select the least value of all the infimum obtained above. Therefore, it can be concluded that

\[
\inf_{x, y \in \mathbb{R}} H(x, y) = \begin{cases} 
\frac{1}{1 - \gamma} & \text{if } \xi \leq \eta \\
\frac{1}{\rho - 1} & \text{if } \xi \geq \eta
\end{cases}
\]

and the lemma holds. \( \square \)

**Lemma 2.2.** Let \( \xi \in \mathbb{R} \) and \( \eta > 0 \). Let the function \( G: \mathbb{R} \rightarrow [0, \infty) \) be defined as in Lemma 2.1. Then

\[
\inf_{x, y \in \mathbb{R}} \frac{|2 - y|G(x) + |x|G(y)}{|2 - x - y|} = \begin{cases} 
\frac{1}{1 - \gamma} & \text{if } 1 \leq \xi \leq \eta \\
\frac{1}{\rho - 1} & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise,}
\end{cases}
\]

where \( \gamma := (\xi - 1)/\eta \) and \( \rho := (\xi + 1)/\eta \).

**Proof.** Let \( H \) be a function on \( \mathbb{R}^2 \setminus \{(x, y) : x + y = 2\} \) defined by

\[
H(x, y) := \frac{|2 - y|G(x) + |x|G(y)}{|2 - x - y|}.
\]

and \( \lim_{x+y \to 2} H(x, y) = \infty \). Being a non-negative real-valued function, \( H \) has a non-negative infimum. In order to obtain the infimum of the function \( H \), we consider the following cases:

**Case 1:** \( x, y \leq \gamma \).

The function \( H \) becomes

\[
H(x, y) = \frac{1}{|2 - x - y|} (|2 - y|(\xi - \eta x) + |x|(\xi - \eta y)).
\]

Since the function \( H \) has no critical points in the set \((-\infty, \gamma) \times (-\infty, \gamma)\), it does not acquire a minimum value in this region. Thus, the function \( H \) has infimum as \( x \) or \( y \) approach \(-\infty\) or has minimum along the edges \( x = \gamma \) or \( y = \gamma \).

To minimize the function \( H \), this case is divided into the subcases \textit{viz.} \( \xi \leq 1, 1 \leq \xi \leq \eta + 1, \eta + 1 \leq \xi \leq 2\eta + 1 \) and \( 2\eta + 1 \leq \xi \). If \( \xi \leq 1 \) which yields \( \gamma \leq 0 \), the function \( H \) simplifies to

\[
H(x, y) = \xi - \frac{2\eta x(1 - y)}{2 - x - y}.
\]

It can easily be verified that along the edge \( y = \gamma \), the function \( H(x, \gamma) \) is a decreasing function of \( x \), hence attains its minimum at the point \((\gamma, \gamma)\) with the minimum
value \( H(\gamma, \gamma) = 1 \) and so does the function \( H(\gamma, y) \). Also, we may note that the values \( \lim_{x \to -\infty} H(x, y) = \xi - 2\eta(1 - y) \) and \( \lim_{y \to -\infty} H(x, y) = \xi - 2\eta x \) exceed 1. This implies \( \inf H(x, y) = \min H(x, y) = 1 \) whenever \( \xi \leq 1 \).

Now if \( 1 \leq \xi \leq \eta + 1 \) which means \( 0 \leq \gamma \leq 1 \), the case can be split into parts when \( x \leq 0 \) and when \( x \geq 0 \). If \( x \leq 0 \), the function \( H(x, \gamma) \) has its minimum value given by \( H(0, \gamma) = \xi \). Besides, for a fixed value of \( y \), \( \lim_{x \to -\infty} H(x, y) = \xi + 2\eta(1 - y) \), and \( \lim_{y \to -\infty} H(x, y) = \xi - 2\eta x \) for a particular value of \( x \). We see that \( \xi + 2\eta(1 - y) > \xi \) and \( \xi - 2\eta x > \eta \) for \( x \leq 0 \) and \( y \leq \gamma \). Further, if \( x \geq 0 \), then the function \( H \) may be expressed as

\[
H(x, y) = \frac{\xi(2 - y + x) - 2\eta x}{2 - x - y}.
\]

With simple calculations, it can be seen that the function \( H(x, \gamma) \) is an increasing function of \( x \) provided \( \eta \leq 1 \). Therefore, \( \min H(x, \gamma) = H(0, \gamma) = \xi \). Whereas, the condition \( \eta \geq 1 \) implies that \( H(x, \gamma) \) is an increasing function of \( x \) whenever \( \xi \geq \eta \) and is a decreasing function for \( 1 \leq \xi \leq \eta \). Hence, we infer that

\[
\min H(x, \gamma) = \begin{cases} 
H(\gamma, \gamma) = 1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
H(0, \gamma) = \xi & \text{if } \eta \leq \xi \leq \eta + 1.
\end{cases}
\]

Computing in a similar manner along the edge \( x = \gamma \), it may be noted that if \( \eta \leq 1 \), then we have \( \inf H(\gamma, y) = \lim_{y \to -\infty} H(\gamma, y) = \xi \). If \( \eta \geq 1 \), then the infimum of \( H(x, y) \) is given by

\[
\inf H(\gamma, y) = \begin{cases} 
H(\gamma, \gamma) = 1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
\lim_{y \to -\infty} H(\gamma, y) = \xi & \text{if } \eta \leq \xi \leq \eta + 1.
\end{cases}
\]

In addition to this, \( \lim_{y \to -\infty} H(x, y) = \xi \). Consequently, for \( 1 \leq \xi \leq \eta + 1 \), simplifying the values so obtained, it can be seen that if \( \eta \leq 1 \), then \( \inf H(x, y) = \xi \), otherwise we have

\[
\inf H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
\xi & \text{if } \eta \leq \xi \leq \eta + 1.
\end{cases}
\]

The subcase when we have \( \eta + 1 \leq \xi \leq 2\eta + 1 \) which results in \( 1 \leq \gamma \leq 2 \), we further divide it into the parts as \( x \leq 0 \), \( x \geq 0 \) and \( x + y < 2 \), and \( x \geq 0 \) and \( x + y > 2 \). Let \( x \leq 0 \). The function \( H(x, \gamma) \) is increasing, and has the infimum as \( x \) approaches \(-\infty\). Therefore, it is easy to see that \( \inf H(x, \gamma) = \lim_{x \to -\infty} H(x, \gamma) = 2\eta - \xi + 2 \). Since, in this case, the values \( \lim_{x \to -\infty} H(x, y) = \xi + 2\eta(1 - y) \) and \( \lim_{y \to -\infty} H(x, y) = \xi - 2\eta x \) are greater than \( 2\eta - \xi + 2 \), we have \( \inf H(x, y) = 2\eta - \xi + 2 \). Suppose that \( x \geq 0 \) and \( x + y < 2 \), then the function \( H \) becomes

\[
H(x, y) = \frac{\xi(2 - y + x) - 2\eta x}{2 - x - y}.
\]

The function \( H(x, \gamma) \) being an increasing function has minimum given by \( H(0, \gamma) = \xi \) and for \( 0 \leq x \leq \gamma \), we have \( \lim_{y \to -\infty} H(x, y) = \xi \). Thus, infimum of the function \( H(x, y) \), in this situation, is \( \xi \). The part when \( x \geq 0 \) and \( x + y > 2 \), the minimum value of the function \( H(x, y) \) occurs at the point \((\gamma, y)\) with \( H(\gamma, y) = 1/(\gamma - 1) \). Choosing minimum
amongst the values so obtained, for \( \eta + 1 < \xi < 2\eta + 1 \), we observe that if \( \eta \leq 1 \), then 
\[
\min H(x, y) = 2\eta - \xi + 2.
\]
If \( \eta \geq 1 \), then
\[
\inf H(x, y, \gamma) = \begin{cases} 
2\eta - \xi + 2 & \text{if } \eta + 1 \leq \xi \leq \eta + 2 \\
1/(\gamma - 1) & \text{if } \eta + 2 \leq \xi \leq 2\eta + 1.
\end{cases}
\]

The case \( \xi \geq 2\eta + 1 \) which is equivalent to \( \gamma \geq 2 \) may be subdivided into six sections depending upon the signs of \( 2 - y, x \) and \( 2 - x - y \). For the part \( x \leq 0, y \leq 2 \) and \( x + y < 2 \), the function \( H \) may have its infimum as \( x \) or \( y \) approach \(-\infty\). It can be seen that \( \lim_{x \rightarrow -\infty} H(x, y) = \xi + 2\eta(1 - y) \geq \xi - 2\eta \) and \( \lim_{y \rightarrow -\infty} H(x, y) = \xi - 2\eta x \geq \xi \). The case when \( x \geq 0, y \leq 2 \) and \( x + y < 2 \), we note that for \( y < 2 \), \( H(0, y) = \xi \) and at the same time, for \( 0 \leq x \leq \gamma \), we have \( \lim_{y \rightarrow -\infty} H(x, y) = \xi \). Similarly for \( x \geq 0, y \leq 2 \) and \( x + y > 2 \), it can be verified that \( \inf H(\gamma, y) = H(\gamma, 2) = \xi - 2\eta \). Likewise, the minimum of the function \( H(x, y) \) for the case \( x \geq 0 \) and \( y \geq 2 \) is 1. Further, for the part \( x \leq 0, y \geq 2 \) with \( x + y > 2 \), the function \( H(x, \gamma) \) has minimum given by \( H(0, \gamma) = \xi \). If \( x \leq 0, y \geq 2 \) with \( x + y < 2 \), then \( \inf H(x, y) = \xi - 2\eta \). The infimum of the function \( H(x, y) \) is the least of the values for infimum obtained in different cases above. In this way, we see that the \( \min H(x, y) = 1 \) whenever \( \xi > 2\eta + 1 \).

Briefly, we observe that if \( \eta \leq 1 \), then
\[
\inf_{x, y \leq \gamma} H(x, y) = \begin{cases} 
\xi & \text{if } 1 \leq \xi \leq \eta + 1 \\
2\eta - \xi + 2 & \text{if } \eta + 1 \leq \xi \leq 2\eta + 1 \\
1 & \text{otherwise}.
\end{cases}
\]

But if \( \eta \geq 1 \), then
\[
\inf_{x, y \leq \gamma} H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
\xi & \text{if } \eta \leq \xi \leq \eta + 1 \\
2\eta - \xi + 2 & \text{if } \eta + 1 \leq \xi \leq \eta + 2 \\
1/(\gamma - 1) & \text{if } \eta + 2 \leq \xi \leq 2\eta + 1 \\
1 & \text{otherwise}.
\end{cases}
\]

For the rest of the cases, we follow a similar trend by dividing the subcases into parts in accordance with the sign of \( 2 - y, x \) and \( 2 - x - y \). Thus, we derive the minimum of the function \( H \) as follows:

**Case 2:** \( x \leq \gamma \) and \( \gamma \leq y \leq \rho \).

The case if \( \eta \leq 1 \), we can solve to get
\[
\inf_{x \leq \gamma, \gamma \leq y \leq \rho} H(x, y) = 1.
\]
Let \( 1 \leq \eta \leq 2 \). Then
\[
\inf_{x \leq \gamma, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
2\eta - \xi & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise}.
\end{cases}
\]
and the case when $\eta \geq 2$, we have

$$\inf_{x \leq \gamma, y \leq \rho} H(x, y) = \begin{cases} 
1/(1-\gamma) & \text{if } 1 \leq \xi \leq \eta \\
2\eta - \xi & \text{if } \eta \leq \xi \leq \eta + 1 \\
(2 - \rho + \gamma)/(\gamma + \rho - 2) & \text{if } \eta + 1 \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise}.
\end{cases}$$

**Case 3:** $x \leq \gamma$ and $y \geq \rho$.

For $\eta \leq 1$, we have that the function $H(x, y)$ has infimum given by

$$\inf_{x \leq \gamma, y \geq \rho} H(x, y) = \begin{cases} 
2 - \xi & \text{if } 2\eta - 1 \leq \xi \leq \eta \\
2 + \xi - 2\eta & \text{if } \eta \leq \xi \leq 1 \\
1 & \text{otherwise}.
\end{cases}$$

If we have the condition $1 \leq \eta \leq 2$, then

$$\inf_{x \leq \gamma, y \geq \rho} H(x, y) = \begin{cases} 
\xi & \text{if } 1 \leq \xi \leq \eta \\
(2 - \rho + \gamma)/(\gamma + \rho - 2) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise}.
\end{cases}$$

The possibility when $\eta \geq 2$, then it may noted that

$$\inf_{x \leq \gamma, y \geq \rho} H(x, y) = \begin{cases} 
(2 - \rho + \gamma)/(2 - \gamma - \rho) & \text{if } 1 \leq \xi \leq \eta - 1 \\
\xi & \text{if } \eta - 1 \leq \xi \leq \eta \\
2\eta - \xi & \text{if } \eta \leq \xi \leq \eta + 1 \\
(2 - \rho + \gamma)/(\gamma + \rho - 2) & \text{if } \eta + 1 \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise}.
\end{cases}$$

**Case 4:** $\gamma \leq x \leq \rho$ and $y \leq \gamma$.

As in the cases above, we have infimum of the function $H(x, y)$ to be $\xi$ whenever $1 \leq \xi \leq \eta + 1$; $(2 + \rho - \gamma)/(\gamma + \rho - 2)$ whenever $\eta + 1 \leq \xi \leq 2\eta + 1$ and 1 elsewhere provided $\eta \leq 1$. In case $\eta \geq 1$, then

$$\inf_{\gamma \leq x \leq \rho, y \leq \gamma} H(x, y) = \begin{cases} 
1/(1-\gamma) & \text{if } 1 \leq \xi \leq \eta \\
\xi & \text{if } \eta \leq \xi \leq \eta + 1 \\
(2 + \rho - \gamma)/(\gamma + \rho - 2) & \text{if } \eta + 1 \leq \xi \leq 2\eta + 1 \\
1 & \text{otherwise}.
\end{cases}$$

**Case 5:** $\gamma \leq x, y \leq \rho$.

We compute that for the case if $\eta \leq 1$, $\inf H(x, y) = 1$ as $\xi$ ranges over the real line. For the part when $\eta \geq 1$, we have

$$\inf_{\gamma \leq x, y \leq \rho} H(x, y) = \begin{cases} 
1/(1-\gamma) & \text{if } 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise}.
\end{cases}$$
Case 6: \( \gamma \leq x \leq \rho \) and \( y \geq \rho \).

Again with condition \( \eta \leq 1 \), we note that \( \inf H(x, y) = \xi \) for \( \xi \in \mathbb{R} \) and the case when \( 1 \leq \eta \leq 2 \), then \( \inf H(x, y) = \xi \) whenever \( 1 \leq \xi \leq \eta \) and \( \inf H(x, y) = 1/(\rho - 1) \) provided \( \eta \leq \xi \leq 2\eta - 1 \) and the infimum is 1 elsewhere. On the other hand, if \( \eta \geq 2 \), then

\[
\inf_{\gamma \leq x \leq \rho, y \geq \rho} H(x, y) = \begin{cases} 
(2 - \rho + \gamma)(2 - \gamma - \rho) & \text{if } 1 \leq \xi \leq \eta - 1 \\
\xi & \text{if } \eta - 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise.}
\end{cases}
\]

Case 7: \( x \geq \rho \) and \( y \leq \gamma \).

In this case, we may see that

\[
\inf_{x \geq \rho, y \leq \gamma} H(x, y) = \begin{cases} 
(2 + \rho - \gamma)/(2 - \gamma - \rho) & \text{if } -1 \leq \xi \leq \eta - 1 \\
\xi + 2 & \text{if } \eta - 1 \leq \xi \leq \eta \\
2\eta - \xi + 2 & \text{if } \eta \leq \xi \leq \eta + 1 \\
(2 + \rho - \gamma)/(\gamma + \rho - 2) & \text{if } \eta + 1 \leq \xi \leq 2\eta + 1 \\
1 & \text{otherwise.}
\end{cases}
\]

Case 8: \( x \geq \rho \) and \( \gamma \leq y \leq \rho \).

This case is also partitioned as \( \eta \leq 1 \) and \( \eta \geq 1 \). Let \( \eta \leq 1 \). Then the function \( H \) has infimum given by \( (2 + \rho - \gamma)/(2 - \gamma - \rho) \), whenever \(-1 \leq \xi \leq \eta - 1 \) and is given by \( 2\eta - \xi \) if \( \eta - 1 \leq \xi \leq 2\eta - 1 \) and is otherwise 1. If \( \eta \geq 1 \), then it can be seen that

\[
\inf_{x \geq \rho, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 
(2 + \rho - \gamma)/(2 - \gamma - \rho) & \text{if } -1 \leq \xi \leq \eta - 1 \\
\xi & \text{if } \eta - 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise.}
\end{cases}
\]

Case 9: \( x, y \geq \rho \).

For this case, if \( \eta \leq 1 \), then infimum of \( H \) happens to be \( \xi + 2 \) if \(-1 \leq \xi \leq \eta - 1 \); \( 2\eta - \xi \) if \( \eta - 1 \leq \xi \leq 2\eta - 1 \) and 1 elsewhere. For the part \( \eta \geq 1 \), we observe

\[
\inf_{x, y \geq \rho} H(x, y) = \begin{cases} 
1/(\rho - 1) & \text{if } -1 \leq \xi \leq \eta - 2 \\
\xi + 2 & \text{if } \eta - 2 \leq \xi \leq \eta - 1 \\
2\eta - \xi & \text{if } \eta - 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise.}
\end{cases}
\]

Drawing the conclusion by choosing least of all the infimum values obtained above, the infimum of the function \( H \) is determined and is obtained to be

\[
\inf_{x, y \in \mathbb{R}} H(x, y) = \begin{cases} 
1/(1 - \gamma) & \text{if } 1 \leq \xi \leq \eta \\
1/(\rho - 1) & \text{if } \eta \leq \xi \leq 2\eta - 1 \\
1 & \text{otherwise.}
\end{cases}
\]
Using the above-mentioned lemmas we now prove the theorems stated in Section I.

Proof of Theorem 1.3. Let \( f \in \mathcal{R}_\sigma(\lambda, \varphi) \). Using Definition 1.1, we know that there exist two analytic functions \( r, s : \mathbb{D} \to \mathbb{D} \) satisfying \( r(0) = 0 = s(0) \) such that

\[
(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi(r(z)) \quad \text{and} \quad (1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) = \varphi(s(w)).
\]

(2.1)

Define the functions \( p \) and \( q \) by

\[
p(z) := 1 + \frac{r(z)}{1 - r(z)} = 1 + p_1z + p_2z^2 + \cdots \quad \text{and} \quad q(w) := 1 + \frac{s(w)}{1 - s(w)} = 1 + q_1w + q_2w^2 + \cdots.
\]

(2.2)

It may be noted that the functions \( p \) and \( q \) are analytic with positive real part in \( \mathbb{D} \) and \( p(0) = 1 = q(0) \). Using equations (2.1) and (2.2), it is clear that

\[
(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right)
\]

(2.3)

and

\[
(1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) = \varphi\left(\frac{q(w) - 1}{q(w) + 1}\right).
\]

(2.4)

Comparing the coefficients on the both sides of equation (2.3), we have the relations

\[
(1 + \lambda)a_2 = \frac{B_1p_1}{2} \quad \text{and} \quad (1 + 2\lambda)a_3 = \frac{B_1}{2}p_2 + \frac{p_1^2}{4}(B_2 - B_1).
\]

(2.5)

Similarly, using equation (2.4), we get

\[
(1 + \lambda)a_2 = -\frac{B_1q_1}{2} \quad \text{and} \quad (1 + 2\lambda)(2a_2^2 - a_3) = \frac{B_1}{2}q_2 + \frac{q_1^2}{4}(B_2 - B_1).
\]

(2.6)

Some simple calculations in equation (2.5) yield

\[
a_3 - xa_2^2 = \frac{B_1}{2(1 + 2\lambda)}\left(p_2 - \frac{\nu}{2}p_1^2\right),
\]

(2.7)

where \( \nu := x\frac{B_1}{\tau} - \frac{B_2}{B_1} + 1 \) and \( \tau := ((1 + \lambda)^2)/(1 + 2\lambda) \). From \[11\ Lemma 2\], we have

\[
|p_2 - (\nu/2)p_1^2| \leq \max\{2, 2|\nu - 1|\}.
\]

(2.8)

By means of inequality equation (2.7) and (2.8), we get

\[
|a_3 - xa_2^2| \leq \frac{B_1}{1 + 2\lambda} \max\left\{1, \left|x\frac{B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}.
\]

(2.9)

With a similar computation using relation (2.6), it follows that

\[
|a_3 - (2 - y)a_2^2| \leq \frac{B_1}{1 + 2\lambda} \max\left\{1, \left|y\frac{B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}.
\]

(2.10)
Using triangle’s inequality with the inequalities (2.9) and (2.10), we arrive at
\[
|(2 - x - y)a_2^2| \leq |a_3 - xa_2^2| + |a_3 - (2 - y)a_2^2| \leq \frac{B_1}{1 + 2\lambda}(G(x) + G(y)),
\]
where \(G(x) := \max\left\{1, \left|x\frac{B_1}{\gamma} - \frac{B_2}{B_1}\right|\right\}\). Hence, if \(x, y \in \mathbb{R}\), then we have
\[
|a_2|^2 \leq \frac{B_1}{1 + 2\lambda} \inf_{x,y \in \mathbb{R}} \frac{G(x) + G(y)}{|2 - x - y|}.
\]
Since \(B_2 \in \mathbb{R}\), in view of Lemma 2.1 by taking \(\eta := B_1/\tau\) and \(\xi := B_2/B_1\) which leads to \(\gamma = \frac{\tau}{B_1}\left(\frac{B_2}{B_1} - 1\right)\) and \(\rho = \frac{\tau}{B_1}\left(\frac{B_2}{B_1} + 1\right)\), we have that \(|a_2|\) is bounded by
\[
|a_2| \leq \sqrt{\frac{B_1}{1 + 2\lambda}} \begin{cases} \frac{B_1}{\sqrt{B_1^2 - \tau B_2 + \tau B_1}} & \text{if } \frac{B_2}{B_1} \leq \frac{B_1}{\tau} \\ \frac{B_1}{\tau B_2 + \tau B_1 - B_1^2} & \text{otherwise} \end{cases}.
\]
In order to obtain estimate for the coefficient \(a_3\), we again apply triangle’s inequality to the relations (2.9) and (2.10) and infer
\[
|a_3| \leq \frac{B_1}{1 + 2\lambda} \inf_{x,y \in \mathbb{R}} \frac{|2 - y|G(x) + |x|G(y)}{|2 - x - y|}.
\]
By Lemma 2.2 with \(\eta := B_1/\tau\) and \(\xi := B_2/B_1\), we conclude
\[
|a_3| \leq \frac{B_1}{1 + 2\lambda} \begin{cases} \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} & \text{if } 1 \leq \frac{B_2}{B_1} \leq \frac{B_1}{\tau} \\ \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2} & \text{if } \frac{B_2}{B_1} \leq \frac{B_1}{\tau} \leq \frac{2B_1}{\tau} - 1 \\ 1 & \text{otherwise} \end{cases}
\]
which completes the proof of the theorem.

**Illustration 2.3.** Let \(\varphi(z) = (1 + z)/(1 - z)\) and \(\lambda = 1\). For \(\nu \geq \sqrt{2}\), the function \(f_\nu(z) := \nu z/(\nu - z) \in \mathcal{R}_\sigma(1, \varphi)\). The assertion can be justified as follows:
The inverse of the function \(f_\nu\), denoted by \(g_\nu\), is given by \(g_\nu(w) = \nu w/(\nu + w)\). Given that \(\nu \geq \sqrt{2}\), we can see that \(f_\nu\) is univalent and has a univalent inverse in \(\mathbb{D}\). For \(f_\nu \in \mathcal{R}_\sigma(1, (1 + z)/(1 - z))\), it is required that for \(z, w \in \mathbb{D}\), the subordinations
\[
f'_\nu(z) = \frac{\nu^2}{(\nu - z)^2} < \frac{1 + z}{1 - z} \quad \text{and} \quad g'_\nu(w) = \frac{\nu^2}{(\nu + w)^2} < \frac{1 + w}{1 - w}
\]
hold. It may be noted that the function \(f'_\nu(z)\) maps unit disk \(\mathbb{D}\) onto the domain
\[
\left|\sqrt{z} - \frac{\nu^2}{\nu^2 - 1}\right| < \frac{\nu}{\nu^2 - 1}
\]
which lies in the right-half plane if and only if \(\nu \geq \sqrt{2}\). Similarly, \(g'_\nu(w)\) maps \(\mathbb{D}\) onto a domain which is contained in the right-half plane provided \(\nu \geq \sqrt{2}\). Thus, we have
Comparing the coefficients on each side of the above two relations, we get
\[ 1 = \frac{\nu^2}{\nu^2 - 1} \]
which is contained in the right half plane provided \( \nu \geq 1 \). Similar is the case for the mapping \( g_\nu(w)/w \). Therefore, for \( \nu \geq 1 \) and \( z, w \in \mathbb{D} \), the subordinations
\[
\frac{f_\nu(z)}{z} = \frac{\nu}{\nu - 1} \frac{1 + z}{1 - z} \quad \text{and} \quad \frac{g_\nu(w)}{w} = \frac{\nu}{\nu + 1} \frac{1 + w}{1 - w}
\]
hold. Hence, the function \( f_\nu \in \mathcal{R}_\sigma(0, (1 + z)/(1 - z)) \). Theorem 1.3 readily yields \( 1/|\nu| \leq \sqrt{2} \) which clearly is true because of the assumption \( \nu \geq 1 \).

Now for \( \varphi(z) = \sqrt{1 + z} \) and \( \lambda = 0 \), the function \( f_\nu \in \mathcal{R}_\sigma(0, \sqrt{1 + z}) \) if the image of the unit disk under the mappings \( f_\nu(z)/z \) and \( g_\nu(w)/w \) lie in the region bounded by the right of lemniscate of Bernoulli given by \( \{ w : |w^2 - 1| = 1 \} \). By means of [2, Lemma 2.2], it may be noted that if \( \nu \geq \sqrt{2}(\sqrt{2} + 1) \), then the disk (2.11) is contained in the set \( \{ w : |w^2 - 1| < 1 \} \). Thus, we infer that if \( \nu \geq \sqrt{2}(\sqrt{2} + 1) \), then \( f_\nu \in \mathcal{R}_\sigma(0, \sqrt{1 + z}) \). Further, using Theorem 1.3, it is easy to compute that \( 1/|\nu| \leq 1/\sqrt{7} \) which is true as \( \nu \geq \sqrt{2}(\sqrt{2} + 1) \).

Based on the proof of Lemma 2.2, we have the following lemma.

**Lemma 2.4.** Let \( \xi \in \mathbb{R} \) and \( \eta > 0 \). Let the function \( G : \mathbb{R} \to [0, \infty) \) be defined as in Lemma 2.7. Then
\[
\inf_{x,y \in \mathbb{R}} \frac{|3 - y|G(x) + |x + 1|G(y)}{|2 - x - y|} = \begin{cases} \frac{2}{1 - \gamma} & \text{if } 1 - \eta \leq \xi \leq \eta \\ \frac{2}{\rho} & \text{if } \eta \leq \xi \leq 3\eta - 1 \\ 1 & \text{otherwise} \end{cases}
\]

where \( \gamma := (\xi - 1)/\eta \) and \( \rho := (\xi + 1)/\eta \).

**Proof of Theorem 1.3.** Since the function \( f \in \mathcal{S}_\sigma^2(\varphi) \), the Definition 1.2 states that there exist two Schwarz functions \( r \) and \( s \) such that
\[
\frac{zf'(z)}{f(z)} = \varphi(r(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \varphi(s(w)).
\]
Let the functions \( p \) and \( q \) be defined by equation (2.2). Clearly, the functions \( p \) and \( q \) are analytic functions in \( \mathbb{D} \) with positive real part and \( p(0) = 1 = q(0) \). Therefore, equation (2.12) and (2.2) yield
\[
\frac{zf'(z)}{f(z)} = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \varphi \left( \frac{q(w) - 1}{q(w) + 1} \right).
\]
Comparing the coefficients on each side of the above two relations, we get
\[
a_2 = \frac{B_1p_1}{2}, \quad 2a_3 - a_2^2 = \frac{B_2p_1^2}{4} + \frac{B_1}{2} \left( p_2 - \frac{p_1^2}{2} \right).
\]
where

\[ G \]

is easy to see that

\[ 2a_3 - (x + 1)a_2^2 \leq B_1 G(x) \quad \text{and} \quad |2a_3 - (3 - y)a_2^2| \leq B_1 G(y), \quad (2.14) \]

where \( G(x) := \max\{1, |xB_1 - B_2/B_1|\} \). On computing using triangle’s inequality, it is easy to see that

\[ |(2 - x - y)a_2^2| \leq |a_3 - xa_2^2| + |a_3 - (2 - y)a_2^2| \leq B_1(G(x) + G(y)) \]

which implies

\[ |a_2|^2 \leq B_1 \inf_{x,y \in \mathbb{R}} \frac{G(x) + G(y)}{|2 - x - y|}. \]

Since \( B_2 \in \mathbb{R} \), upon taking \( \xi = B_2/B_1 \) and \( \eta = B_1 \), Lemma 2.1 gives

\[ |a_2| \leq \sqrt{\frac{B_1}{1 - \gamma}} \left( \text{if } \frac{B_2}{B_1} \leq B_1 \right) \quad \text{and} \quad \sqrt{\frac{B_1}{\rho - 1}} \left( \text{if } \frac{B_2}{B_1} \geq B_1 \right) \]

where \( \gamma = \frac{1}{B_1} \left( \frac{B_2}{B_1} - 1 \right) \) and \( \rho = \frac{1}{B_1} \left( \frac{B_2}{B_1} + 1 \right) \). Besides, keeping in view the relation \( (2.14) \), we may solve to get

\[ |a_3| \leq \frac{B_1}{2} \inf_{x,y \in \mathbb{R}} \frac{|3 - y|G(x) + |x + 1|G(y)}{|2 - x - y|}. \]

By means of Lemma 2.3 with \( \xi = B_2/B_1 \) and \( \eta = B_1 \) again, on simplifying the above relations, we get the desired estimates for the second and third coefficient of a function in class \( S_\nu^*(\varphi) \).

**Illustration 2.5.** Let \( \varphi(z) = (1 + z)/(1 - z) \). For \( \nu \geq 1 \), the function \( f_\nu(z) := \nu z/(\nu - z) \in S_\nu^*((1 + z)/(1 - z)) \). The function \( f_\nu \) and its inverse, denoted by \( g_\nu \), are univalent in \( \mathbb{D} \) for \( \nu \geq 1 \). For \( f_\nu \in S_\nu^*((1 + z)/(1 - z)) \), the following subordinations must hold:

\[ \frac{zf'_\nu(z)}{f_\nu(z)} = \frac{\nu}{\nu - z} \times \frac{1 + z}{1 - z} \quad \text{and} \quad \frac{wg'_\nu(w)}{g_\nu(w)} = \frac{\nu}{\nu + w} \times \frac{1 + w}{1 - w}. \]

As in Illustration 2.3, the functions \( zf'_\nu(z)/f_\nu(z) \) and \( wg'_\nu(w)/g_\nu(w) \) map the unit disk onto the region contained in the right-half plane if and only if \( \nu \geq 1 \). Hence, \( f_\nu \in S_\nu^*((1 + z)/(1 - z)) \) for \( \nu \geq 1 \). Further, according to Theorem 1.5, it is required that \( 1/|\nu| < \sqrt{2} \) which is true as \( \nu \geq 1 \).

Assuming \( \nu \geq \sqrt{2}(\sqrt{2} + 1) \), using [2] Lemma 2.2, we can see that the mappings \( zf'_\nu(z)/f_\nu(z) \) and \( wg'_\nu(w)/g_\nu(w) \) map the unit disk onto the disks that are contained in the region \( \{w : w^2 - 1 < 1\} \). Hence, the function \( f_\nu \in S_\nu^*(\sqrt{1 + z}) \). In this case, Theorem 1.5 implies that \( 1/|\nu| \leq 1/\sqrt{7} \) which is true as \( \nu \geq \sqrt{2}(\sqrt{2} + 1) \).

**Remark 2.6.** It may be noted that with \( \varphi(z) = (1 + z)/(1 - z) \), whenever \( \nu \geq 1 \), the function \( f_\nu := \nu z/(\nu - z) \in S_\nu^*(\varphi) \) and \( f_\nu \in \mathcal{R}_\sigma(0, \varphi) \) but for \( 1 \leq \nu < \sqrt{2} \), \( f_\nu \not\in \mathcal{R}_\sigma(1, \varphi) \).
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