Dynamical Density Fluctuations around QCD Critical Point Based on Dissipative Relativistic Fluid Dynamics

— possible fate of Mach cone at the critical point *) —

Yuki Minami and Teiji Kunihiro

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

The purpose of this paper is twofold. Firstly, we study the dynamical density fluctuations around the critical point (CP) of Quantum Chromodynamics (QCD) using dissipative relativistic fluid dynamics in which the coupling of the density fluctuations to those of other conserved quantities is taken into account. We show that the sound mode which is directly coupled to the mechanical density fluctuation is attenuated and in turn the thermal mode becomes the genuine soft mode at the QCD CP. We give a speculation on the possible fate of a Mach cone in the vicinity of the QCD CP as a signal of the existence of the CP on the basis of the above findings. Secondly, we clarify that the so called first-order relativistic fluid dynamic equations have generically no problem to describe fluid dynamic phenomena with long wave lengths contrary to a naive suspect whereas even Israel-Stewart equation, a popular second-order equation, may not describe the hydrodynamic mode in general depending on the value of the relaxation time.

Subject Index: 231,519

§1. Introduction

The analysis^2) of the experimental data^3) obtained in Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) has suggested that the matter created by the relativistic heavy ion collisions is well described as almost ideal relativistic fluids with the ratio of shear viscosity \( \eta \) to entropy density \( s \) being tiny. This discovery has prompted great interest in the origin and the true value of the viscosities including the bulk viscosity \( \zeta \) and so on, and people are now enthusiastic in elucidating dissipative effects^4) and thus in constructing a transport theory for strongly coupled systems.^5)–7)

A unique feature of the phase diagram of Quantum Chromodynamics (QCD) is the existence of a critical point (CP), which is yet mainly based on some effective models^8) and a few lattice studies;^9) see Refs.10),11) and 12) for possible variants and alternatives. At the QCD CP, the first order phase transition terminates and turns to a second order phase transition. Around a critical point of a second order transition, we can expect large fluctuations of various quantities, and more importantly there should exist a soft mode associated to the CP of second order.\(^13)\) What is, however, the soft mode of the QCD CP? It has been established now that the QCD CP belongs to the same universality class as the liquid-gas CP and shown that the density fluctuating mode and generically hydrodynamic modes coupled to conserved quantities in the space-like region are a softening mode at the CP;\(^14),15)\) The would-be soft mode, the \( \sigma \) mode, is coupled to the density fluctuation\(^16)\) and becomes a

*) The preliminarily version of this work has been reported in Ref.1)
slaving mode of the density variable;\cite{14,15,17} see Ref. 18) for another argument on the fate of the \(\sigma\) mode around the CP. Furthermore it is also suggested\cite{19,20} that the bulk viscosity should show a divergent behavior around the QCD CP.\footnote{It seems that there exist some refutable arguments in Ref.19), as shown in 20), 21).}

In this paper, we shall show that the ultimate soft mode at the QCD CP may not be a sound mode that is directly related to the dynamical density fluctuation, and that the possible divergent behavior of the viscosities might not be observed through the density fluctuations; the sound modes are attenuated around the CP and would eventually die out at the CP; in turn, the diffusive thermal mode that is coupled to the sound mode would become the soft mode at the QCD CP. We should mention that Fujii and Ohtani\cite{14} emphasized that the thermal mode as well as the density oscillation play the role of the soft modes in the QCD CP on the basis of the time-dependent Ginzburg-Landau formalism. However, the relative importance of these hydrodynamic modes around the QCD CP was not elucidated.

Our investigation is based on an explicit use of the relativistic fluid dynamic equations for a viscous fluid. We shall show that even the so called first-order relativistic fluid dynamic equations have generically no problem to describe fluid dynamical phenomena with long wave lengths contrary to a naive suspect.

In the case of nonrelativistic fluids, the nature of the critical dynamics around the critical point is rather thoroughly elucidated.\cite{22,23,25} The static density fluctuation induces a strong light scattering in the vicinity of the critical point, which is called critical opalescence. The dynamical density fluctuations are investigated on the basis of the (non-relativistic) Navier-Stokes equation.\cite{22,24,25} Note that the fluid dynamics is a dynamical thermodynamics, which tells us that the density fluctuation is in general coupled to the entropy fluctuations creating diffusive thermal mode.

We shall explore how the singularities of the thermodynamic values as well as the transport coefficients affect the dynamical density fluctuations around the QCD CP using relativistic fluid dynamics for a viscous fluid. We examine the spectral function of the density fluctuation in the equilibrium. Our analysis is actually an extension of that made for non-relativistic case\cite{25,26} to the relativistic case. We are not aware of such an analysis so far except for a simple estimate\cite{27} using Euler equation without dissipation in a astronomical context. The use of the relativistic fluid dynamics for describing the dynamics around QCD CP should be interesting in view of the success of the fluid dynamics for RHIC phenomenology and the nature of the QCD CP mentioned above.\cite{28}

Admittedly, the analysis based on fluid dynamics is only valid in the fluid dynamical regime: \(k\xi << 1\), where \(k\) and \(\xi\) denote the typical wave number of the fluid and the correlation length; that is, our discussions on the critical behavior are based on extrapolation from the fluid dynamical to the critical region: \(k\xi >> 1\).\footnote{This extrapolation is known to make good sense for the non-relativistic case.\cite{22}}

A set of the dynamical variables is thus the density \(n\), temperature \(T\) and the fluid velocity \(u^\mu\) \((u_\mu u^\mu = 1; \mu = 0, 1, 2, 3)\), while the sigma mode is not included as a dynamical variable because it is a slaving mode of the density.\cite{14,15} We shall see that the coupling of the density fluctuation with that of the entropy gives rise to an
important effect and is found essential for the description of the dynamics around the CP: It will turn out that the relativistic effects on the spectral function of the density fluctuation appears only in the width of the Lorentzian peaks due to the sound and thermal modes through the modification of the transport coefficients due to relativistic effects. Moreover, our detailed analysis of the critical behavior of the transport coefficients as well as the thermodynamic quantities will show that the possible singular behavior of the transport coefficients with relativistic effects turns out to be masked by the more singular behavior of the specific heats; hence we find that the sound mode will be attenuated, and only the Rayleigh peak due to the thermal fluctuation stays out around the QCD CP, which is found, to our surprise, precisely the same in effect as in the nonrelativistic case.\textsuperscript{22}

The forms of the dissipative relativistic fluid dynamic equations are far from being established, although a great development has been seen in a couple of years.\textsuperscript{5,7} Relativistic fluid dynamic equations are the balance equations for energy-momentum and particle number,

\begin{align*}
\partial_\mu T^{\mu\nu} &= 0, \\
\partial_\mu N^\mu &= 0,
\end{align*}

where $T^{\mu\nu}$ is the energy-momentum tensor and $N^\mu$ the particle current. They are expressed as

\begin{align*}
T^{\mu\nu} &= (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \tau^{\mu\nu}, \\
N^\mu &= nu^\mu + \nu^\mu,
\end{align*}

where $\epsilon$ is the energy density, $P$ the pressure, $u^\mu$ the flow velocity, and $n$ the particle density, the dissipative part of the energy-momentum tensor and the particle current are denoted by $\tau^{\mu\nu}$ and $\nu^\mu$, respectively.

The so-called first-order equations such as Landau\textsuperscript{30} and Eckart\textsuperscript{31} equations are parabolic and formally violates the causality, and hence called acausal. Moreover, the Eckart equation which is defined for the particle frame where the particle current does not have a dissipative part shows a pathological property that the fluctuations around the thermal equilibrium is unstable,\textsuperscript{32} while the Landau equation defined for the energy frame does not show such a pathological behavior. The causality problem is circumvented in the Israel-Stewart equation,\textsuperscript{33} which is a second-order equation with relaxation times incorporated. One should, however, note that the problem of the causality is only encountered when one tries to describe phenomena with fast velocities which is beyond the scope of the fluid dynamics; nobody would actually expect that the fluid dynamics should describe phenomena with a velocity comparable with or greater than the mean velocity $v$ of the constituent particles, though $v$ is still less than the light velocity. The fluid dynamic equations describe the behaviors of the conserved quantities in even larger scale than the mean free path. Thus the phenomena which the fluid dynamics should describe are slowly varying ones with the wavelengths much larger than the mean free path. The problem of causality always takes place in short wavelength region such as shock wave phenomena,\textsuperscript{34} so formally acausal fluid dynamic equations suffice in describing
fluid dynamical phenomena. In fact we will see that the results for fluid dynamical modes with long wave lengths are qualitatively the same irrespective whether the second-order or first-order equations are used.

As for the instability seen in the Eckart equation, a new first-order equation in the particle frame constructed by Tsumura, Kunihiro and Ohnishi (TKO)\(^5\) has no such a pathological behavior.\(^{35}\) We employ Landau,\(^{30}\) Eckart\(^{31}\) and Israel-Stewart (I-S)\(^{33}\) equation as typical equations, and TKO equation.

This paper is organized as follows. In Sec. 2, we calculate the spectral function of the dynamical density fluctuation around thermal equilibrium state. We show that relativistic effects on the spectral function of the density fluctuation appears only in the width of sound modes, irrespective of the choice of the frame. In Sec. 3 we analyze the behavior of the spectral function around the QCD CP with the scaling laws, and show that the sound mode is attenuated while the thermal mode which is coupled to the density fluctuation stands out in the critical region. Section 4 is devoted to discussions in which the fundamental reason is given why the sound mode is attenuated in the critical region in terms of the correlation length \(\xi\) which diverges at the critical point. We furthermore suggest that the attenuation of the sound mode can be used to identify the existence of the QCD CP by the relativistic heavy-ion collisions; a possible suppression or disappearance of a Mach cone can be such an example. The final section is for summary and concluding remarks. We give a detailed derivation of thermodynamic relations in Appendix A, which are used in Sec. 2. In Appendix B, we show explicitly how the pathological behavior in Eckart equation affects the procedure of the calculation of the spectral function of the density fluctuations for an instructive purpose. In Appendix C and D, we present the detailed derivation of the spectral function in TKO equation in the particle frame and Israel-Stewart (I-S) equation, respectively.

§ 2. Analysis of dynamical density fluctuation

In this section, we derive the spectral function of the dynamical density fluctuations using some typical relativistic fluid dynamic equations for a viscous fluid, and discuss relativistic effects on and the frame dependence of the spectral function. A detailed derivation of the spectral function is given for the Landau equation in the first subsection. Leaving the similar detailed derivations to Appendix C and D, we present the results for the Tsumura-Kunihiro-Ohnishi (TKO) equation\(^5\) and the Israel-Stewart equation\(^{33}\) in particle frame in the subsequent subsections.

2.1. In the case of Landau equation (energy frame)

In Landau equation, the dissipative terms are given by

\[
\nu^\mu = \kappa \left( \frac{nT}{w} \right)^2 \partial_{\perp} \left( \frac{\mu}{T} \right),
\]

\[
\tau^{\mu\nu} = \eta [\partial_{\perp} \chi^\nu + \partial_{\perp}^\mu u^\nu - \frac{2}{3} \Delta^{\mu\nu} (\partial_{\perp} \cdot u)] + \zeta \Delta^{\mu\nu} (\partial_{\perp} \cdot u),
\]
where $\eta$ is the shear viscosity, $\zeta$ the bulk viscosity, $\kappa$ the thermal conductivity, $T$ the temperature, $\mu$ the chemical potential, and $w = \epsilon + P$ the enthalpy density. $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$ is the projection operator to the space-like vector and $\partial_\perp^\mu = \Delta^{\mu\nu}\partial_\nu$ the space-like derivative (gradient operator).

We calculate the spectral function of the dynamical density fluctuation around the thermal equilibrium state by the linear approximation for the deviation from the equilibrium. The following calculational procedure is an extension of the non-relativistic case described in the text book Ref.26).

Let us write $n(x) = n_0 + \delta n(x)$, $\epsilon(x) = \epsilon_0 + \delta \epsilon(x)$, $P(x) = P_0 + \delta P(x)$, $\mu(x) = \mu_0 + \delta \mu(x)$, and $u^\mu(x) = u_0^\mu + \delta u^\mu(x)$, where the respective quantities in the equilibrium state are denoted with $0$. For simplicity, let the equilibrium state be the rest frame of the fluid, $u_0^\mu = (1, 0)$; then owing to the relation $u_0^\mu \delta u_\mu = 0$, it is found that $\delta u_\mu$ takes the form $\delta u_\mu = (0, \delta v(x))$ with $\delta v(x)$ to be determined together with other quantities like $\delta n$ etc. Then Landau equation Eqs.(1-1) and (1-2) are reduced to

\[
\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta v = \kappa \frac{n_0}{w_0} \frac{T_0}{w_0} \nabla^2 (\delta P) - \nabla^2 (\delta T),
\]

\[
w_0 \frac{\partial \delta v}{\partial t} - \eta \nabla^2 \delta v - \left( \zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \delta v) + \nabla (\delta P) = 0,
\]

\[
n_0 \frac{\partial \delta s}{\partial t} + \kappa \frac{\nabla \cdot \delta P}{w_0} - \kappa \frac{T_0}{w_0} \nabla^2 (\delta T) = 0,
\]

in the linear approximation. Here we have used the following thermodynamic relations.

\[
d\epsilon = T_0 d(n s) + \mu_0 d n,
\]

\[
dP = n_0 s_0 dT + n_0 d \mu,
\]

\[
w_0 = T_0 n_0 s_0 + n_0 \mu_0,
\]

where $s_0$ is the entropy per the particle number in the thermal equilibrium. Note that the right hand side of Eq. (2-3) and the second term in Eq. (2-5) represent relativistic effects which are absent for the non-relativistic Navier-Stokes equation. In addition, the coefficient of the first term in Eq. (2-4) $w_0$ is replaced by the mass density $\rho_0$ for the Navier-Stokes equation.

Now we have five equations for seven unknown quantities, $\delta n$, $\delta T$, $\delta P$, $\delta s$, and $\delta v$. In order to solve these equations, the thermodynamic quantities $\delta n$, $\delta T$, $\delta P$ and $\delta s$ are interrelated using thermodynamics. Choosing $\delta n$ and $\delta T$ as independent variables, we have

\[
\delta P(x) = \frac{w_0 c_s^2}{n_0 \gamma} \delta n(x) + \frac{w_0 c_s^2 \alpha P}{\gamma} \delta T(x),
\]

\[
\delta s(x) = - \frac{w_0 c_s^2 \alpha P}{n_0^2 \gamma} \delta n(x) + \frac{\epsilon_n}{T_0} \delta T(x),
\]

where the following thermodynamic identities are used,

\[
\left( \frac{\partial P}{\partial n} \right)_T = \frac{w_0 c_s^2}{n_0 \gamma}, \quad \left( \frac{\partial P}{\partial T} \right)_n = \frac{w_0 c_s^2 \alpha P}{\gamma}.
\]
(\frac{\partial s}{\partial n})_T &= -\frac{w_0 c_0^2 \alpha_P}{n_0^2 \gamma}, \\
(\frac{\partial s}{\partial T})_n &= \frac{\tilde{c}_n}{T_0} \tag{2.11}
\end{align*}

where \( \tilde{c}_n = T_0(\partial s/\partial T)_n \) and \( \tilde{c}_P = T_0(\partial s/\partial T)_P \) are the specific heats at constant density and pressure, respectively; \( c_s = (\partial P/\partial s)^{1/2} \) the sound velocity, \( \alpha_P = -(1/n_0)(\partial n/\partial T)_P \) the thermal expansivity at constant pressure, and \( \gamma = \tilde{c}_P/\tilde{c}_n \) the ratio of the specific heats. Then Eqs. (2.3)-(2.5) take the form

\begin{align*}
\{ \frac{\partial}{\partial t} - \kappa \frac{T_0 c_0^2}{w_0 \gamma} \nabla^2 \} \delta n + n_0 \nabla \cdot \delta \mathbf{v} + \kappa \frac{n_0}{w_0} (1 - \frac{c_0^2 \alpha_P T_0}{\gamma}) \nabla^2 \delta T &= 0, \tag{2.12} \\
w_0 \frac{\partial \delta \mathbf{v}}{\partial t} - \eta \nabla^2 \delta \mathbf{v} - (\zeta + \frac{1}{3} \eta) \nabla(\nabla \cdot \delta \mathbf{v}) + \frac{w_0 c_0^2}{n_0 \gamma} \nabla \delta n + \frac{w_0 c_0^2 \alpha_P}{\gamma} \nabla \delta T &= 0, \tag{2.13} \\
\left\{ -\frac{w_0 c_0^2 \alpha_P}{n_0 \gamma} \frac{\partial}{\partial t} + \kappa \frac{c_s^2}{n_0 \gamma} \nabla^2 \right\} \delta n + \left\{ \frac{n_0 \tilde{c}_n}{T_0} \frac{\partial}{\partial t} + \kappa \left( \frac{c_s^2 \alpha_P}{\gamma} - \frac{1}{T_0} \right) \right\} \nabla \delta T &= 0. \tag{2.14}
\end{align*}

To utilize the correlations in the initial time to obtain the correlation function at later time \( t \), we perform Fourier-Laplace transformation

\[ \delta n(k, z) = \int_{-\infty}^{+\infty} dr \int_0^\infty dt \, e^{-zt-ik \cdot r} \delta n(r, t), \ldots \text{etc.} \]

Then we find

\begin{align*}
(z + k^2 \frac{T_0 c_0^2}{w_0 \gamma}) \delta n(k, z) + i n_0 k \cdot \delta \mathbf{v}(k, z) + k^2 \frac{n_0}{w_0} \left( \frac{c_0^2 \alpha_P T_0}{\gamma} - 1 \right) \delta T &= \delta n(k, t = 0), \tag{2.15} \\
z w_0 \delta \mathbf{v}(k, z) + k^2 \eta \delta \mathbf{v}(k, z) + (\zeta + \frac{1}{3} \eta) k(k \cdot \delta \mathbf{v}(k, z)) + i k \frac{w_0 c_0^2}{n_0 \gamma} \delta n(k, z) \\
+ i k \frac{w_0 c_0^2 \alpha_P}{\gamma} \delta T(k, z) &= w_0 \delta \mathbf{v}(k, t = 0), \tag{2.16} \\
-z \frac{w_0 c_0^2 \alpha_P}{n_0 \gamma} - k^2 \frac{c_s^2}{n_0 \gamma} \delta n(k, z) + \left\{ \frac{n_0 \tilde{c}_n}{T_0} - k^2 \frac{c_s^2 \alpha_P}{\gamma} - \frac{1}{T_0} \right\} \delta T(k, z) \\
= -\frac{w_0 c_0^2 \alpha_P}{n_0 \gamma} \delta n(k, t = 0) + \frac{n_0 \tilde{c}_n}{T_0} \delta T(k, t = 0). \tag{2.17}
\end{align*}

It is convenient to divide the velocity into longitudinal and transverse components

\[ \delta \mathbf{v}(k, z) = \delta \mathbf{v}_l(k, z) + \delta \mathbf{v}_t(k, z). \tag{2.18} \]

The transverse component of Eqs. (2.15)-(2.17) reads

\[ zw_0 \delta \mathbf{v}_t(k, z) + k^2 \eta \delta \mathbf{v}_t(k, z) = w_0 \delta \mathbf{v}_t(k, t = 0). \tag{2.19} \]

We shall not treat this equation, which admits a diffusive solution, in the present work since this equation is decoupled to the density fluctuation. We note that a complete treatment based on the mode-mode coupling theory of the critical
Dynamical Density Fluctuations around QCD CP

Dynamics in the close vicinity of the CP involves the diffusive transverse mode as well as the thermal one which is also diffusive as we will see.

The longitudinal component of Eqs. (2.15)-(2.17) can be cast into the following matrix form

\[
\begin{pmatrix}
\delta n(k, z) \\
\delta v_{\|}(k, z) \\
\delta T(k, z)
\end{pmatrix}
= \begin{pmatrix}
\delta n(k, 0) \\
\delta v_{\|}(k, 0) \\
\delta T(k, 0)
\end{pmatrix}
\]

where the matrix \(A\) is defined by

\[
A = \begin{pmatrix}
z + k^2 \frac{T_0 c^2}{w_0 \gamma} & ikn_0 & -k^2 \frac{\kappa n_0}{w_0} \left(1 - \frac{\alpha P c^2 T_0}{\gamma}\right) \\
-ik \frac{\kappa n_0}{w_0} & z + \nu l k^2 & 0 \\
\frac{n_0 c n_0}{w_0 T_0} \left[-z \frac{w_0 T_0 c^2}{\gamma} - k^2 \chi c^2 T_0 n_0\right] & 0 & \frac{n_0 c n_0}{w_0 T_0} \left[z + k^2 \gamma \chi \left(1 - \frac{\alpha P c^2 T_0}{\gamma}\right)\right]
\end{pmatrix}
\]

Here, we have introduced the longitudinal kinetic viscosity \(\nu_l\), and the thermal diffusivity \(\chi\),

\[
\nu_l = \left(\zeta + \frac{4}{3} \eta\right)/w_0, \quad \chi = \frac{\kappa}{n_0 c P}.
\]

Multiplying the inverse \(A^{-1}\) of the matrix \(A\) from the left in Eq. (2.20), we obtain the Fourier-Laplace coefficient of the density fluctuation

\[
\delta n(k, z) = \{(A^{-1})_{11} - \frac{\alpha P c^2}{n_0 \gamma} (A^{-1})_{13}\} \delta n(k, 0) + (A^{-1})_{12} \delta v_{\|}(k, 0)
\]

\[
+ \frac{n_0 c n_0}{T_0 w_0} (A^{-1})_{13} \delta T(k, 0).
\]

Now, we are interested in the spectral function of the dynamical density fluctuation

\[
S_{nn}(k, \omega) \equiv \langle \delta n(k, \omega) \delta n(k, t = 0) \rangle,
\]

where \(\delta n(k, \omega)\) is the Fourier transformation of the density fluctuation defined by

\[
\delta n(k, \omega) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} dt e^{-i \omega t - ik \cdot r} \delta n(r, t),
\]

and \(\langle \rangle\) denotes the thermal average in the equilibrium. We see that \(S_{nn}(k, \omega)\) is obtained by taking the thermal average of Eq. (2.23) multiplied by \(\delta n(k, 0)\). However, note that \(\delta T\) and \(\delta n\) are statistically independent in fluids system which is established in Einstein fluctuation theory,\(^{26}\) and accordingly

\[
\langle \delta T(k, 0) \delta n(k, 0) \rangle = 0,
\]

Similarly,

\[
\langle \delta v_{\|}(k, 0) \delta n(k, 0) \rangle = 0.
\]
we obtain the dynamical density fluctuation at time $t$ second order in $k$

$$S_{nn}(k, \omega) = \{(A^{-1})_{11} - \frac{\alpha \rho c^2}{n_0 \gamma}(A^{-1})_{13}\}\langle \delta n(k, 0)\delta n(k, 0)\rangle. \quad (2.28)$$

The needed matrix elements $(A^{-1})_{11}$ and $(A^{-1})_{13}$ may be calculated using the simple formula $(A^{-1})_{11} = \frac{1}{\det A}(A_{22}A_{33} - A_{23}A_{32})$ and $(A^{-1})_{13} = \frac{1}{\det A}(A_{12}A_{23} - A_{13}A_{22})$. Here, $\det A$ reads

$$\det A = \frac{n_0 \tilde{c}_n}{w_0 T_0} \left[ z^3 + z^2 k^2 \left\{ \gamma \chi + \nu_l + \kappa \frac{T_0 c_s^2}{w_0} - 2 \chi c_s^2 \alpha \rho T_0 \right\} + z k^2 c_s^2 + k^4 c_s^2 \chi + \cdots \right], \quad (2.29)$$

where we have used the thermodynamic identity (A·14), and '...' denotes the higher order terms in $k$ which we assume small because we are interested in the fluid dynamical modes. $\det A$ can be nicely factorized in this approximation,

$$\det A \sim \frac{n_0 \tilde{c}_n}{w_0 T_0} (z + \Gamma_R k^2)(z + \Gamma_B k^2 + ic_s k)(z + \Gamma_B k^2 - ic_s k), \quad (2.30)$$

where

$$\Gamma_R = \chi, \quad (2.31)$$
$$\Gamma_B = \frac{1}{2} \left[ \chi (\gamma - 1) + \nu_l + c_s^2 T_0 (\kappa/w_0 - 2\chi \alpha \rho) \right]. \quad (2.32)$$

Then we can write the Fourier-Laplace coefficient of the density fluctuation to second order in $k$

$$\frac{\delta n(k, z)}{\delta n(k, 0)} \sim \frac{(z + \Gamma_B k^2 + ic_s k)(z + \Gamma_B k^2 - ic_s k) + z k^2 \frac{\kappa c_s^2}{w_0 \gamma} - k^2 c_s^2 \chi}{(z + \Gamma_R k^2)(z + \Gamma_B k^2 + ic_s k)(z + \Gamma_B k^2 - ic_s k)}. \quad (2.33)$$

Performing the inverse Laplace transformation

$$\frac{\delta n(k, t)}{\delta n(k, 0)} = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} dz e^{zt} \frac{\delta n(k, z)}{\delta n(k, 0)},$$

we obtain the dynamical density fluctuation at $t$

$$\delta n(k, t)/\delta n(k, 0) \sim (1 - \frac{1}{\Gamma_B k^2} e^{-\Gamma_R k^2 t} + \frac{1}{\Gamma_B k^2} e^{-\Gamma_R k^2 t} \cos(c_s k t) e^{-\Gamma_B k^2 t}). \quad (2.34)$$

Here, we have retained only the terms in the amplitudes to zeroth order in $k$. Since Eq.(2·34) is the density fluctuation in a stationary process, we can replace the time $t$ by $|t|$. Therefore the Fourier transformation of Eq.(2·34) is given by

$$\frac{\delta n(k, \omega)}{\delta n(k, 0)} = (1 - \frac{1}{\Gamma_B k^2} \frac{2\Gamma_R k^2}{\omega^2 + \Gamma_R^2 k^4} + \frac{1}{\Gamma_B k^2} \frac{\Gamma_B k^2}{(\omega - c_s k)^2 + \Gamma_B^2 k^4} + \frac{\Gamma_B k^2}{(\omega + c_s k)^2 + \Gamma_B^2 k^4}). \quad (2.35)$$
Thus we finally obtain the spectral function of the density fluctuation

\[
S_{nn}(k, \omega) = \langle \delta n(k, \omega) \delta n(k, t = 0) \rangle = \langle (\delta n(k, t = 0))^2 \rangle \left[ (1 - \frac{1}{\gamma}) \frac{2\Gamma_R k^2}{\omega^2 + \Gamma_R^2 k^4} \right.
\]
\[
+ \frac{1}{\gamma} \left\{ \frac{\Gamma_B k^2}{(\omega - c_s k)^2 + \Gamma_B^2 k^4} + \frac{\Gamma_B^2 k^2}{(\omega + c_s k)^2 + \Gamma_B^2 k^4} \right\} \right].
\]

(2.36)

We see that the spectral function have three peaks at frequencies \(\omega = 0\) and \(\omega = \pm c_s k\): The peak at \(\omega = 0\) corresponds to thermally induced density fluctuations. This peak is called the Rayleigh peak, while the two side peaks at \(\omega = \pm c_s k\) correspond to mechanically induced density fluctuation, i.e. sound waves. These two peaks are called Brillouin peaks.

Now let us compare this results with that in the nonrelativistic case: \(^{25}, 26\)

\[
S_{nn}^{NR}(k, \omega) = \langle (\delta n(k, t = 0))^2 \rangle \left[ (1 - \frac{1}{\gamma}) \frac{2\chi k^2}{\omega^2 + \chi^2 k^4} \right.
\]
\[
+ \frac{1}{\gamma} \left\{ \frac{\Gamma_B^{NR} k^2}{(\omega - c_s k)^2 + \Gamma_B^{NR^2} k^4} + \frac{\Gamma_B^{NR^2} k^2}{(\omega + c_s k)^2 + \Gamma_B^{NR^2} k^4} \right\} \right].
\]

(2.37)

where

\[
\Gamma_B^{NR} = \frac{1}{2} [\chi(\gamma - 1) + \nu_l^{NR}],
\]

(2.38)

\[
\nu_l^{NR} = (\zeta + \frac{4}{3}\eta)/\rho_0.
\]

(2.39)

We see that relativistic effects appear only in the width of the Brillouin peaks:

\[
\Gamma_B = \Gamma_B^{MR} + \delta\Gamma_B^{La} \equiv \Gamma_B^{La},
\]

(2.40)

where

\[
\Gamma_B^{MR} = \frac{1}{2} [\chi(\gamma - 1) + \nu_l],
\]

(2.41)

with the longitudinal kinetic viscosity \(\nu_l\) defined in (2.22), and

\[
\delta\Gamma_B^{La} \equiv \frac{1}{2} c_s^2 T_0 (\kappa/w_0 - 2\chi\rho_p).
\]

(2.42)

Firstly, the longitudinal kinetic viscosity is expressed in terms of the enthalpy density \(w_0\) in the relativistic case in place of the mass density \(\rho_0\). We call this modification the minimal relativistic (MR) effect. The other is a genuine relativistic effect \(\delta\Gamma_B^{La}\) which is absent in the non-relativistic case. This part comes from the right hand side of Eq. (2.3) and the second term in Eq. (2.5) which represent relativistic effects and vanishes if we take the light speed \(c \to \infty\). We see that the width of the Rayleigh peak is the same as the non-relativistic case\(^{25}, 26\) and the relativistic effects on the width of the sound modes tend to cancel with each other, so even the relativistic effect on the Brillouin peaks may be moderate.
Fig. 1. The spectral function in Landau and the minimal relativistic equation are shown by the solid line and the dashed line, respectively. The parameters are $k = 0.1[1/fm]$, $\mu_0 = 200[MeV]$, $T_0 = 200[MeV]$, $\eta/(n_0s_0) = \zeta/(n_0s_0) = 0.3$ and $\kappa T_0/(n_0s_0) = 0.6$. Relativistic effects does not show up in the Rayleigh peak but enhance the Brillouin peaks.

In order to see the relativistic effects $\delta \Gamma_{\text{La}}^B$ quantitatively, we calculate thermodynamical values, $c_s, \alpha_P$ and $\gamma$ by the equation of state(EoS) of massless classical ideal gas, $\epsilon = 3P = 3nT$, so we have $\tilde{c}_p = 4, \tilde{c}_n = 3, \alpha_P = 1/T_0$, $c_s = \sqrt{1/3}$ and $s_0 = 4 - \mu_0/T_0$. Then the widths in the minimal relativistic case are given by

$$\Gamma_R = \frac{\kappa}{4n_0}, \quad (2.43)$$

$$\Gamma_{\text{MR}}^B = \frac{1}{24n_0T_0}(4\eta + 3\zeta + \kappa T_0), \quad (2.44)$$

and the relativistic corrections are found to be

$$\delta \Gamma_{\text{La}}^B = -\frac{1}{2}c_s^2T_0(\kappa/w_0 - 2\chi \alpha_P) = -\frac{\kappa}{24n_0}. \quad (2.45)$$

We see that the relativistic effects in the Brillouin peaks is solely due to the thermal conductivity and has a negative value, implying that the Brillouin peaks get enhanced with a smaller width by the relativistic effect. It addition, this relativistic term exactly cancels out with the minimal relativistic part due to the thermal conductivity in Eq.(2.44); thus we see that the thermal conductivity does not affect the Brillouin peaks in net for the massless classical ideal gas.

Figure 1 shows the spectral function Eq.(2.36) and the minimal relativistic case with the parameter set $k = 0.1[1/fm]$, $\mu_0 = 200[MeV]$, $T_0 = 200[MeV]$, $\eta/(n_0s_0) = \zeta/(n_0s_0) = 0.3$ and $\kappa T_0/(n_0s_0) = 0.6$. Note that $n_0s_0$ represents the entropy density in the equilibrium state because $s_0$ is the entropy per particle number.

As is expected, Fig.1 shows that the Brillouin peaks owing to the sound mode is enhanced by the relativistic effects, while the Rayleigh peak owing to the thermal mode is the same as in the non-relativistic case.
2.2. In the case of particle frame

Next let us take fluid dynamic equations in the particle frame. Does any difference arise in the density spectral function depending on the choice of the frame. A typical equation in the particle frame is due to Eckart.\textsuperscript{31)} It is, however, well known that the Eckart equation shows a pathological behavior;\textsuperscript{32)} i.e., when the fluctuations around the thermal equilibrium is described by this equation, the fluctuation tends to diverge as $t$ goes infinity; see Appendix B for a detailed discussion.

Then, does any first-order relativistic fluid dynamic equation for a viscous fluid in the particle frame show such a pathological behavior? It is then noteworthy that a new fluid dynamic equation in the particle frame constructed from the relativistic Boltzmann equation by Tsumura, Kunihiro and Ohnishi (TKO)\textsuperscript{5)} turns out to be a stable one in the sense that any fluctuations around the thermal equilibrium state relax down to recover the equilibrium.\textsuperscript{35)} So let us take TKO equation in the particle frame and examine whether the spectral function of the density fluctuations shows any frame dependence.

In the case of TKO equation in the particle frame, the dissipative terms are given by

$$
\tau^{\mu \nu} = \eta \left[ \partial_{\perp} u^\nu + \partial^\nu u^\mu - \frac{2}{3} \Delta^{\mu \nu} (\partial_{\perp} \cdot u) \right] - \zeta' (3u^\mu u^\nu - \Delta^{\mu \nu})(\partial_{\perp} \cdot u)
$$

$$(2.46)
$$

$$
\nu^\mu = 0,
$$

$$(2.47)$$

where $\zeta' = \zeta/(3\gamma - 4)^2$ with $\gamma$ being the ratio of the specific heats as before.

From the same procedure as taken for the Landau equation, we obtain the spectral function as follows (the detailed derivation is given in Appendix C);

$$
\frac{S_{nn}(k, \omega)}{\langle \delta n(k, t = 0) \rangle^2} = \left( 1 - \frac{1}{\gamma} \right) \frac{2\chi k^2}{\omega^2 + \chi^2 k^4}
$$

$$
+ \frac{1}{\gamma} \left[ \frac{\Gamma_B k^2}{(\omega - c_s k)^2 + \Gamma_B^2 k^4} + \frac{\Gamma_B k^2}{(\omega + c_s k)^2 + \Gamma_B^2 k^4} \right],
$$

$$(2.48)$$

with

$$
\Gamma_B = \frac{1}{2} \left[ \chi (\gamma - 1) + \nu_T^{TKO} - \frac{\alpha P c_s^2}{n_0 \bar{c}_P} (\kappa T_0 + 3\zeta') \right] \equiv \Gamma_T^{TKO},
$$

$$(2.49)$$

where

$$
\nu_T^{TKO} \equiv (\zeta' + \frac{4}{3}\eta)/w_0.
$$

$$(2.50)$$

Note that since the fluid dynamic fluctuations around the equilibrium state is relaxing in TKO equation (see Appendix C), we have obtained the spectral function without difficulty as was in the case of Landau equation. Moreover, the deviation of the width of the Brillouin peaks reads

$$
\delta \Gamma_B^{TKO} \equiv -\frac{\alpha P c_s^2}{2n_0 \bar{c}_P} (\kappa T_0 + 3\zeta'),
$$

$$(2.51)$$
Fig. 2. The spectral function in full relativistic TKO equation (solid line) and its minimal relativistic version (dashed line) with the same parameters as those in Fig. 1. Relativistic effects do not appear in the Rayleigh peak as in Landau case, but more enhance the Brillouin peaks than in the Landau case. Note that the scale of the vertical line is much bigger than that in Fig. 1.

which is definitely negative. So the relativistic effect acts to enhance and sharpen the spectral function of the density fluctuation in the TKO equation in the particle frame in comparison with the minimal relativistic case than in the energy frame; see, however, the next subsection for the Israel-Stewart equation in the particle frame.

If we estimate the relativistic effects using the EoS of massless classical ideal gas, we have

\[ \delta \Gamma^{TKO}_B = -\frac{1}{24n_0}(\kappa + 3\zeta'/T_0). \]  \hspace{1cm} (2.52)

It is noted that the effective bulk viscosity \( \zeta' = \zeta/(3\gamma - 4)^2 \) is finite in the massless case although \( 3\gamma - 4 \to 0 \). The minimal relativistic part of the width is Eq.(2.44) with \( \zeta \) replaced by \( \zeta' \). We see that the relativistic effect due to the thermal conductivity is the same as Landau equation but another relativistic effect due to the bulk viscosity exists. In addition the relativistic effect cancels out with some part of the minimal relativistic contribution due to the bulk viscosity and the thermal conductivity in Eq.(2.44) with \( \zeta \) replaced by \( \zeta' \). That is, the bulk viscosity and the thermal conductivity does not have any contribution in net in TKO equation for the massless classical ideal gas.

The resultant spectral function is shown in Fig.2 together with that of the minimal relativistic case; the parameters are the same as in Fig.1: As in the case of the Landau equation, any relativistic effects does not appear in the Rayleigh peak while the Brillouin peaks are enhanced by them. The enhancement is more prominent than in the Landau case because the bulk viscosity is involved in addition to the thermal conductivity as relativistic effects. However, it is noteworthy that relativistic effects only appears in the Brillouin peaks but not in the Rayleigh peak, irrespective of the
2.3. In the case of Israel-Stewart equation in particle frame

The Israel-Stewart (I-S) equation is a second-order equation where relaxation times are contained and has a form of the telegrapher’s equation. Thus in the I-S equation, the so-called causality problem is formally resolved. However, the fluid dynamic equations are supposed to describe the behaviors of the conserved quantities in even larger scale than the mean free path, and the problem of causality takes place in short wavelength region. Therefore, it is expected that the acausal and causal fluid dynamic equations should give the same description for the phenomena with long wavelengths, or hydrodynamic modes. In fact we shall show that the I-S equation gives completely the same results for the spectral function of the density fluctuations as the Landau equation gives.

In the I-S equation in the particle frame, the dissipative terms are given by

\[ \Pi = -\zeta (\partial_{\mu} u^\mu + \beta_0 u^a \partial_a \Pi - \alpha_0 \partial_{\mu} q^\mu), \]

\[ q^\mu = \kappa T \Delta^{\mu\nu} \left( \frac{1}{T} \partial_\nu T - u^a \partial_a u_\nu - \beta_1 u^b \partial_b q_\nu - \alpha_0 \partial_{\nu} \Pi + \alpha_1 \partial_a \pi^a_\nu \right), \]

\[ \pi^{\mu\nu} = 2\eta \Delta^{\mu\nuab} (\partial_a u_b - \beta_2 u^c \partial_c \pi_{ab} - \alpha_1 \partial_a q_b), \]

with \( u^\mu q_\mu = 0, \pi^{\mu\nu} = \pi^{\nu\mu}, u^\mu \pi^{\mu\nu} = 0 \) and \( \pi^\mu_\mu = 0 \). Here \( \beta_0, \beta_1 \) and \( \beta_2 \) are the relaxation time of the bulk viscosity, the heat flux and the shear viscosity, respectively. \( \alpha_0 (\alpha_1) \) is the coupling of the bulk viscosity and the heat flux (the shear viscosity and the heat flux). \( \Delta^{\mu\nu\rho\sigma} \) is a projector defined by

\[ \Delta^{\mu\nu\rho\sigma} = \frac{1}{2} [\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma}], \]

Applying the similar procedure as in the first-order equations, we find the time-dependent density fluctuation is given by

\[ \delta n(k, t)/\delta n(k, 0) \sim (1 - \frac{1}{\gamma}) e^{-\chi k^2 t} + \frac{1}{\gamma} \cos(c_s k t) e^{-\Gamma_B k^2 t} \]

\[ + O(k^2) \times [e^{-t/\beta_0 \zeta} + e^{-t/2\beta_2 \eta} + e^{-w_0 t/[(\beta_1 w_0 - 1)\kappa T_0]}], \]

with

\[ \Gamma_B = \frac{1}{2} [(\gamma - 1) \chi + \nu_1 + c_s^2 T_0 (\kappa/w_0 - 2 \chi \alpha P)] \equiv \Gamma_B^{IS}, \]

which is exactly the same as \( \Gamma_B^{La} \) given in the Landau equation in the energy frame. See Appendix D for a detailed derivation of these results. We note that the last term in Eq.(2.58)

\[ \exp\left[ -\frac{w_0 t}{(\beta_1 w_0 - 1)\kappa T_0} \right], \]
has an exponent which can be positive depending on the value of the relaxation time $\beta_1$. In fact this term is a remnant existing for the Eckart equation, which as shown in Appendix B, behaves pathologically as

$$\delta n(k, t) \sim \exp\left[\frac{w_0}{\kappa T_0} t\right].$$  

From Eq.(2.60), one can see that if the relaxation time of the heat current is large enough to satisfy the inequality

$$\beta_1 > \frac{1}{w_0},$$  

the density fluctuation would relax to the equilibrium state. On the other hand, if

$$\beta_1 < \frac{1}{w_0},$$  

the density fluctuation will not relax down and the system stays pathological even though the relaxation time is finite; that is, the Israel-Stewart equation in particle frame takes over the pathological behavior of Eckart equation.

Here let us assume that the relaxation time is sufficiently large to satisfy the inequality $\beta_1 > \frac{1}{w_0}$. Then the spectral function can be obtained as

$$S_{nn}(k, \omega) = \langle (\delta n(k, t = 0))^2 \rangle = \left(1 - \frac{1}{\gamma} \right) \frac{2\chi k^2}{\omega^2 + \chi^2 k^4} + \frac{1}{\gamma} \frac{I_B k^2}{(\omega - c_s k)^2 + I_B^2 k^4} + O(k^2) \times \left[ \frac{2/\beta_0 \zeta}{\omega^2 + 1/(\beta_0 \zeta)^2} \right] + \frac{1/\beta_2 \eta}{\omega^2 + 1/(2\beta_2 \eta)^2} + \frac{2w_0/[(\beta_1 w_0 - 1)\kappa T_0]}{\omega^2 + w_0^2/[(\beta_1 w_0 - 1)\kappa T_0]^2}. $$  

(2.64)

Apparently, the spectral function has six peaks including the conventional three peaks, but the new three Lorentzian functions should vanish in the long wavelength limit $k \to 0$, because the strength of these is in the second order of $k$. Therefore Israel-Stewart equation gives completely the same result for the spectral function of the dynamical fluctuations as Landau equation does in the long wavelength limit, as shown in Fig.1. That is, the relaxation times does not affect the result in the fluid dynamical regime.

Now one sees that the two relativistic fluid dynamic equations in the particle frame give different results on the dynamical density fluctuation; I-S equation gives the same result as that given by the equation in the energy frame whereas the TKO equation shows a frame dependence in the spectral peak of the sound mode. What is the origin of the difference? It may be attributed to that of the condition for the dissipative part of the energy-momentum tensor $\delta T^{\mu\nu}$ in these two equations though in the same particle frame: The I-S equation\(^{33}\) is constructed with an ad-hoc postulate on $\delta T^{\mu\nu}$ as in the Eckart equation\(^ {31}\) that $u_\mu \delta T^{\mu\nu} u_\nu = 0$ which the Landau equation in the energy frame does satisfy because $u_\mu \delta T^{\mu\nu} = 0$ by the very definition of the energy frame by Landau and Lifshitz.\(^ {30}\) On the other hand, the
TKO equation\textsuperscript{5}) in the particle frame is derived by a powerful reduction theory called the renormalization-group (RG) method\textsuperscript{36}) from the relativistic Boltzmann equation and it is found\textsuperscript{5), 35}) that their equation satisfies $\delta T^\mu_\mu = 0$ but not $u_\mu \delta T^{\mu\nu} u_\nu = 0$. It is to be noted that the RG method gives the Landau equation for the fluid dynamic equation in the energy frame.\textsuperscript{5}) It is clear that more work is needed to establish the correct fluid dynamic equation in the particle frame.

§3. The behavior around the QCD critical point

We analyze the behavior of the spectral function of the density fluctuations around the QCD CP on the basis of the dynamic as well as static scaling laws.\textsuperscript{22), 24}) We introduce the static critical exponents $\tilde{\gamma}$ and $\tilde{\alpha}$ which are defined as follows

\[
\tilde{c}_n = c_0 t^{-\tilde{\alpha}}, \quad K_T = K_0 t^{-\tilde{\gamma}},
\]

where $t = |(T - T_c)/T_c|$ is a reduced temperature, $c_0$ and $K_0$ are constants and $K_T = (1/n_0) (\partial n/\partial T)_{T=0}$ is the isothermal compressibility. Using the known thermodynamical identities\textsuperscript{22})

\[
\tilde{c}_P = \tilde{c}_n + (T_0/n_0) \left( \frac{\partial P}{\partial T} \right)_n^2 K_T,
\]

\[
\tilde{c}_s^2 = \frac{\tilde{c}_P}{w_0 \tilde{c}_n K_T},
\]

\[
\alpha_P = \left( \frac{\partial P}{\partial T} \right)_n K_T,
\]

we see that the specific heat at constant pressure, the sound velocity and the thermal expansivity at constant pressure has the critical behavior as

\[
\tilde{c}_P \sim \frac{K_0 T_0}{n_0} \left( \frac{\partial P}{\partial T} \right)_n^2 t^{-\tilde{\gamma}},
\]

\[
\tilde{c}_s^2 \sim \frac{T_0}{n_0 w_0 c_0} \left( \frac{\partial P}{\partial T} \right)_n^2 t^{-\tilde{\alpha}},
\]

\[
\alpha_P \sim K_0 \left( \frac{\partial P}{\partial T} \right)_n t^{-\tilde{\gamma}},
\]

respectively. Since the phase transition at the QCD CP belongs to the same universality class $Z_2$ as the liquid-gas CP, the critical exponents are given as $\tilde{\alpha} \sim 0.11$ and $\tilde{\gamma} \sim 1.2$. Recently, it has been argued\textsuperscript{20}) on the basis of the analysis of the liquid-gas CP by Onuki\textsuperscript{37}) that the bulk viscosity may show a singular behavior around the QCD CP. So we introduce the critical exponent $\alpha_\zeta$ as

\[
\zeta = \zeta_0 t^{-\alpha_\zeta},
\]

where $\zeta_0$ is constant. The exponent $\alpha_\zeta$ for the liquid-gas CP is predicted to be $z\nu - \tilde{\alpha}$.\textsuperscript{37}) Here $z$ is the dynamical critical exponent and $\nu$ is the exponent which
represents the singularity of the correlation length. These are given by \( z \sim 3 \) and \( \nu \sim 0.63 \) around the liquid-gas CP.

We denote the exponent of the thermal conductivity by \( a_\kappa \),

\[
\kappa = \kappa_0 t^{-a_\kappa},
\]

where \( \kappa_0 \) is a constant. It is known that \( a_\kappa \sim 0.63 \) around the liquid-gas CP. Let us take it for granted that the exponents \( a_\zeta \) and \( a_\kappa \) at the QCD CP are given by those at the liquid-gas CP. That is, we take

\[
a_\zeta \sim z \nu - \tilde{\alpha} \sim 1.8,
\]

\[
a_\kappa \sim 0.63.
\]

Combining these ingredients including the possible singular behavior of \( \zeta \), we see that the width of the Rayleigh peak behaves as

\[
\Gamma_R \sim \frac{\kappa_0}{K_0 T_0} \left( \frac{\partial T}{\partial P} \right)_n t^{\tilde{\gamma} - a_\kappa}.
\]

Then, we see

\[
\Gamma_R \sim t^{\tilde{\gamma} - a_\kappa},
\]

which shows that the width \( \Gamma_R \) becomes narrow as the QCD CP is approached. We emphasize that this result is independent of the choice of the relativistic fluid dynamic equation or frame.

So far for the width of the Rayleigh peak. How about the width \( \Gamma_B \) of the Brillouin peaks? For the Landau and I-S cases, it has the critical behavior as follows,

\[
\Gamma_B \sim \frac{\zeta_0}{2 w_0} t^{-a_\zeta}.
\]

We note that this singularity comes from that of the bulk viscosity.

In the case of TKO equation, we first note that the critical behavior of the effective bulk viscosity is given by

\[
\zeta' = \frac{\zeta}{(3\tilde{\gamma} - 4)^2} \sim t^{2\tilde{\gamma} - a_\zeta},
\]

which shows that the effective bulk viscosity has a positive exponent and does not show a singular behavior because \( a_\zeta \sim 1.8 \) and \( \tilde{\gamma} \sim 1.2 \). Instead, the singularity of the Brillouin peaks for TKO equation comes from that of the thermal conductivity;

\[
\Gamma_B \sim \Gamma_{\kappa}^{TKO} t^{-(a_\kappa - \tilde{\alpha})},
\]

with

\[
\Gamma_{\kappa}^{TKO} = \frac{\kappa_0}{2c_0 n_0} \left[ 1 - \frac{T_0}{w_0} \left( \frac{\partial P}{\partial T} \right)_n \right].
\]

We note that the strength of the divergence of \( \Gamma_B \) for TKO equation is weaker than that for the Landau and I-S equations.
Fig. 3. The spectral function at $t = 0.5$ and $k = 0.1$ [1/fm]. The solid line represents the Landau/Israel-Stewart case, while the dashed line the TKO case. The strength of the Brillouin peaks becomes small due to the singularity of the ratio of specific heats.

Fig. 4. The spectral function at $t = 0.1$ and $k = 0.1$ [1/fm]. We see that the Brillouin peaks which correspond to sound wave dies out and the difference between the Landau and TKO cases disappears. Note that the scale of the vertical line is much bigger than that of Fig.3.

Anyway, we have confirmed that the width $\Gamma_B$ may diverge at the QCD CP irrespective of the relativistic fluid dynamic equations.

Unfortunately or fortunately, these singular behaviors of the width of the Brillouin peaks around the QCD CP may not be observed: Note that the strengths of the Rayleigh and the Brillouin peaks are given in terms of $\gamma$, the ratio of the specific
heats, which behaves like
\[ \gamma = \tilde{c}_p/\tilde{c}_n \sim t^{-\tilde{\gamma} + \tilde{\alpha}} \to \infty, \]  
(3.18)
in the critical region. Then the strength of the Brillouin peaks is attenuated and only the Rayleigh peak stands out in the critical region, as follows;
\[ S_{nn}(k, \omega) \sim \langle (\delta n(k, t = 0))^2 \rangle \frac{2\Gamma_R k^2}{\omega^2 + \Gamma_R^2 k^4}, \quad (t \sim 0). \]  
(3.19)

Figure 3 and 4 show how the spectral function behaves around the QCD CP for \( k = 0.1 \ [1/fm], \ t = 0.5 \) and \( t = 0.1, \) respectively. We see that the strength of the Brillouin peaks becomes small and dies out as the system approach the QCD CP. In addition, the static correlation function \( \langle (\delta n(k, t = 0))^2 \rangle \) has the critical behavior as
\[ \langle (\delta n(k, t = 0))^2 \rangle \sim \frac{B}{1 + \xi^2 k^2}, \]  
(3.20)
with \( B = TK_T/V, \) where \( V \) is the volume.\(^{38} \) Equation (3.20) would show a singular peak in the forward direction \( k = 0, \) because \( K_T \sim t^{-1.2} \) and \( \xi^2 \sim t^{-1.26}. \) Then the strength of the Rayleigh peak will be most drastically enhanced in the forward angle in the critical region.

The critical behavior on the spectral function of the density fluctuation is summarized as follows.

1. Disappearance of the strength of the sound mode:
The sound mode loses its strength at the QCD CP because of the divergence of the ratio of the specific heats. Thus, we see that the formation of the sound mode itself is suppressed.

2. Divergence of the width of the sound mode \( \Gamma_B: \)
According to Eq.(3.14), the width of the sound mode diverges at the QCD CP, which means that the sound mode is strongly damped and attenuated in the critical region. Therefore we see that the sound mode is very much attenuated even if it is created with a small strength.

3. Narrowing of the width of the thermal mode \( \Gamma_R: \)
The width of the thermal mode vanishes at the QCD CP because of the divergence of the specific heat at constant pressure. This means that the entropy density exhibits critical slowing down and the thermal mode is the critical mode at the QCD CP. That is, the entropy density is the order parameter and mainly governs the critical dynamics at the QCD CP as in the liquid-gas CP.

\section*{4. Discussions}

We have seen that the sound mode would lose its strength and be strongly damped as the system approaches the critical point. But why at all do sound modes die out at the critical point? To answer this question, let \( \xi = \xi_0 t^{-\nu} \) be the correlation
\(^{23), 26}\)
length which diverges as the critical point is approached. If we write the wave length of the sound mode by $\lambda_s$, the fluid dynamic regime is expressed as

$$\xi << \lambda_s,$$  

(4.1)

with which condition the sound mode can develop.\(^{29}\) However, in the vicinity of the critical point, the correlation length $\xi$ becomes very large and eventually goes to infinity, so the above inequality cannot be satisfied, and the sound mode cannot be developed in the vicinity of the critical point.\(^{29}\)

From this argument, we can speculate that phenomena inherently related to the existence of the sound mode may disappear around the critical point. One of such phenomena is the possible Mach cone formation\(^{39}\) by the particle passing through the medium with a speed larger than the sound velocity $c_s$. Such a Mach-cone like particle correlations are observed in the RHIC experiment.\(^{40}\) If such three-particle correlations have been confirmed to be a Mach-cone formation, then the disappearance or suppression of the Mach cone would be a signal that the created matter has passed through the critical region, showing the existence of the QCD critical point. Even if the thermal wake also contributes to the formation of Mach cone,\(^{41}\) a suppression of Mach cone may be expected by the attenuation of the sound mode. So it would be very interesting to see possible variation of the strength of the Mach cone according to the variation of the incident energy of the heavy-ion collisions. In theoretical side, it is an intriguing task to explore the fate of Mach cone with an EoS which admits the existence of the CP.

Precisely speaking, the analysis based on fluid dynamics is, however, only valid in the fluid dynamical regime: $k\xi << 1$, that is, our discussions on the critical behavior are based on extrapolation from the fluid dynamical to the critical region: $k\xi >> 1$.\(^{29}\) This extrapolation is known to make good sense for the non-relativistic case.\(^{22}\) It is well known that the coupling of the thermal mode to the diffusive transverse mode becomes essential for the description of the dynamical critical phenomena in the close vicinity of the CP for the non-relativistic case.\(^{23}\) This coupling can be analyzed by the mode-mode coupling theory.\(^{23}, 24\) Such an analysis is interesting but beyond the scope of the present work.

§5. Summary and concluding remarks

We have explored how the singularities of the thermodynamic values as well as the transport coefficients affect the dynamical density fluctuations around the QCD critical point (CP) using dissipative relativistic fluid dynamics. Our analysis is an extension of those made for non-relativistic case to the relativistic case. We have shown that the sound modes which are directly related to the density fluctuation are enhanced by relativistic effects, but tend to be attenuated around and would eventually die out at the CP and, hence, are not the soft mode. Our analysis based on the relativistic fluid dynamics has shown that the genuine and remaining soft mode at the QCD CP is the diffusive thermal mode. Our analysis also suggests that the possible divergent behavior of the bulk viscosity may not be observed through the density fluctuations. We have also argued that the possible suppression or disappearance of
Mach cone can be used as a signal of the existence of the QCD CP.

We have shown that the relativistic effect on the spectral function of the density fluctuation appears only in the width of the Lorentzian function. More precisely, the relativistic effect enhances only the Brillouin peaks due to the sound mode but not the Rayleigh peak irrespectively of the equations and the frames. Therefore one might tend to think that the choice of the frame is just a matter of convenience and gives no physical difference. However, it is noteworthy that there is a slight difference in the enhancement depending on the choice of the fluid dynamic equation in the particle frame. The spectral function obtained in the Israel-Stewart(I-S) equation gives the same result as that in the Landau equation in the energy frame whereas the equation by Tsumura-Kunihiro-Ohnishi(TKO)\(^5\) gives a more enhanced peaks to the sound mode than in the two equations. We have suggested that the origin of the difference may be attributed to that of the condition for the dissipative part of the energy-momentum tensor \(\delta T^{\mu \nu}\) in these two equations though in the same particle frame. It is certain that more work is needed to establish the correct fluid dynamic equation in the particle frame.\(^{42}\)

We have also elucidated the properties of some of dissipative relativistic fluid dynamic equations: In the case of Eckart equation, the density fluctuation around the equilibrium is unstable and tends to diverge as \(t\) goes infinity. This pathological behavior is due to the heat current induced by the time derivative of the fluid velocity. Since this property is carried over to the Israel-Stewart(I-S) equation in particle frame, the density fluctuation in the I-S equation also behaves pathologically if the relaxation time of the system happens to be small, although it is formally causal; if the relaxation time is large enough, the I-S equation in particle frame gives completely the same spectral function for the density fluctuations as Landau equation gives. That is, formally acausal fluid dynamic equations suffices in description of phenomena in the fluid dynamical regime with long wave lengths if the equation gives a relaxation of the fluctuations around the thermal equilibrium.

In the present work, we have confined ourselves to the study of fluctuations around the thermal equilibrium. To make a direct relevance to the relativistic heavy-ion collisions, study on the fluctuations around expanding flow background like Bjorken flow should be done, which constitute one of the future works. It should be also an intriguing task to explore the fate of Mach cones or shock waves in general in the critical region by explicit calculations.

It is important to study the coupling between the thermal fluctuations and the transverse mode using the mode-mode coupling theory in the close vicinity of the critical region for the relativistic case. The present work constitutes the basis for such a more complete analysis.

Acknowledgments

We would like to thank Kenji Fukushima, Hideo Suganuma, Akira Ohnishi and the members of the Quark-Hadron seminar at Kyoto university for their interest in our work and encouragement. Discussions during the YIPQS international molecule-type workshop on "Non-equilibrium quantum field theories and dynamic critical
phenomena”, March 2009, were useful to complete this work. We acknowledge the participants of the workshop, especially Juergen Berges, Hirotugu Fujii, Berndt Mueller and Misha Stephanov for their interest in this work and comments. We thank Guy Moore for giving our attention to Ref.20). This work was partially supported by a Grant-in-Aid for Scientific Research by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan (Nos. 20540265, 19·07797), by Yukawa International Program for Quark-Hadron Sciences, and by the Grant-in-Aid for the global COE program “The Next Generation of Physics, Spun from Universality and Emergence” from MEXT.

Appendix A

derivation of thermodynamic identities

In this Appendix, we derive thermodynamic identities we have used. Here $s$ is not the entropy density but the entropy per the particle number, $\tilde{c}_n = T_0 (\partial s / \partial T)_n$ and $\tilde{c}_P = T_0 (\partial s / \partial T)_P$ are the specific heats at constant density and pressure, respectively, $c_s = (\partial P / \partial \epsilon)_s^{1/2}$ is the sound velocity, $\alpha_P = - (1/n_0)(\partial n / \partial T)_P$ is the thermal expansivity at constant pressure, and $\gamma = \tilde{c}_P / \tilde{c}_n$ is the ratio of the specific heats. $\frac{\partial (a,b)}{\partial (c,d)}$ represents Jacobian determinant:

$$\frac{\partial (a,b)}{\partial (c,d)} \equiv \frac{\partial a}{\partial c} \frac{\partial b}{\partial d} - \frac{\partial a}{\partial d} \frac{\partial b}{\partial c}. \quad (A.1)$$

A.1. Derivation of $(\partial P/\partial n)_T = (w_0 c_s^2)/(n_0 \gamma)$

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{\partial (P,T)}{\partial (n,T)},$$

$$= \frac{\partial (P,T)}{\partial (P,s)} \frac{\partial (P,s)}{\partial (\epsilon,s)} \frac{\partial (\epsilon,s)}{\partial (n,s)} \frac{\partial (n,s)}{\partial (n,T)},$$

$$= \left( \frac{\partial T}{\partial s} \right)_P \left( \frac{\partial P}{\partial \epsilon} \right)_s \left( \frac{\partial \epsilon}{\partial n} \right)_s \left( \frac{\partial s}{\partial T} \right)_n,$$

$$= \frac{T}{\tilde{c}_P} \cdot c_s^2 \cdot \left( \frac{\partial \epsilon}{\partial n} \right)_s \cdot \frac{\tilde{c}_n}{T},$$

$$= \frac{c_s^2}{\gamma} \left( \frac{\partial \epsilon}{\partial n} \right)_s, \quad (A.2)$$

where

$$\left( \frac{\partial \epsilon}{\partial n} \right)_s = \frac{\partial (\epsilon,s)}{\partial (n,s)},$$

$$= \frac{\partial (\epsilon,s)}{\partial (\epsilon,n)} \frac{\partial (\epsilon,n)}{\partial (n,s)},$$

$$= - \left( \frac{\partial s}{\partial n} \right)_\epsilon \left( \frac{\partial \epsilon}{\partial s} \right)_n. \quad (A.3)$$
Using the first law of thermodynamics
\[ d\varepsilon = Td(ns) + \mu dn, \] (A.4)
we find
\[ \left( \frac{\partial \varepsilon}{\partial (ns)} \right)_n = T \rightarrow \left( \frac{\partial \varepsilon}{\partial s} \right)_n = nT, \] (A.5)
and
\[ s + n \left( \frac{\partial s}{\partial n} \right)_\varepsilon = -\frac{\mu}{T}, \]
\[ \left( \frac{\partial s}{\partial n} \right)_\varepsilon = -\frac{Ans + n\mu}{n^2T} = -\frac{w}{n^2T}, \] (A.6)
where we have used \( \varepsilon = Tns - P + \mu n \) and \( w = \varepsilon + P \).

Then we obtain
\[ \left( \frac{\partial \varepsilon}{\partial n} \right)_s = w, \] (A.7)
and so
\[ \left( \frac{\partial P}{\partial n} \right)_T = \frac{w_0c^2_s}{n_0\gamma}, \] (A.8)

A.2. Derivation of \( \left( \frac{\partial P}{\partial T} \right)_n = \frac{w_0\alpha pc^2_s}{\gamma} \)

\[ \left( \frac{\partial P}{\partial T} \right)_n = \frac{\partial (P, n)}{\partial (T, n)} = \frac{\partial (P, n)}{\partial (T, P)} \frac{\partial (T, P)}{\partial (s, P)} \frac{\partial (s, P)}{\partial (s, \varepsilon)} \frac{\partial (s, \varepsilon)}{\partial (s, n)} \frac{\partial (s, n)}{\partial (T, n)}, \]
\[ = \frac{\partial (P, n)}{\partial (T, P)} \frac{\partial (T, P)}{\partial (s, P)} \frac{\partial (s, P)}{\partial (s, \varepsilon)} \frac{\partial (s, \varepsilon)}{\partial (s, n)} \frac{\partial (s, n)}{\partial (T, n)}, \]
\[ = -\left( \frac{\partial n}{\partial T} \right)_P \left( \frac{\partial T}{\partial s} \right)_P \left( \frac{\partial P}{\partial \varepsilon} \right)_s \left( \frac{\partial \varepsilon}{\partial n} \right)_s \left( \frac{\partial s}{\partial T} \right)_n, \]
\[ = n_0\alpha P \cdot \frac{T}{c_P} \cdot c^2_s \cdot \frac{w_0}{n_0} \cdot \frac{\tilde{c}_n}{T}, \]
\[ = \frac{w_0\alpha pc^2_s}{\gamma}. \] (A.9)

A.3. Derivation of \( \left( \frac{\partial s}{\partial n} \right)_T = -\frac{w_0\alpha pc^2_s}{(n_0^2\gamma)} \)

Assuming that the total particle number \( N \) is constant, we find
\[ \left( \frac{\partial s}{\partial n} \right)_T = \left( \frac{\partial (S/N)}{\partial (N/V)} \right)_T, \]
\[ = \frac{1}{N^2} \frac{\partial V}{\partial (1/V)} \left( \frac{\partial S}{\partial V} \right)_T, \]
\[ = -\frac{1}{n_0^2} \left( \frac{\partial S}{\partial V} \right)_T. \] (A.10)
Dynamical Density Fluctuations around QCD CP

Using Maxwell relations

\[
\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial P}{\partial T} \right)_V. \tag{A.11}
\]

We obtain

\[
\left( \frac{\partial s}{\partial n} \right)_T = -\frac{1}{n_0^2} \left( \frac{\partial P}{\partial T} \right)_V, \tag{A.12}
\]

\[
\left( \frac{\partial s}{\partial n} \right)_T = -\frac{w_0\alpha P c_s^2}{n_0^2\gamma}. \tag{A.12}
\]

where we have used Eq.(A.8).

A.4. Derivation of \((w_0 T_0 \alpha P c_s^2)/(n_0 \tilde{c} P \gamma) = 1 - 1/\gamma\)

\[
\tilde{c}_P = T_0 \frac{\partial(s, P)}{\partial(T, P)}, \tag{B.1}
\]

\[
= T_0 \frac{\partial(s, P)}{\partial(T, n)} \frac{\partial(T, n)}{\partial(T, P)}, \tag{B.1}
\]

\[
= T_0 \left( \frac{\partial s}{\partial T} \right)_n + T_0 \left( \frac{\partial s}{\partial n} \right)_T \left( \frac{\partial n}{\partial T} \right)_P, \tag{B.1}
\]

\[
\tilde{c}_P - \tilde{c}_n = T_0 \left( \frac{\partial s}{\partial n} \right)_T \left( \frac{\partial n}{\partial T} \right)_P, \tag{B.1}
\]

\[
= T_0 \cdot \left( -\frac{w_0 \alpha P c_s^2}{n_0^2 \gamma} \right) \cdot (-n_0 \alpha P). \tag{B.1}
\]

Dividing the both side by \(\tilde{c}_P\), we obtain

\[
1 - \frac{1}{\gamma} = \frac{w_0 T_0 \alpha P c_s^2}{n_0 \tilde{c} P \gamma}. \tag{B.14}
\]

Appendix B

In the case of Eckart equation (particle frame)

In this Appendix, we shall show explicitly how the pathological behavior of the Eckart equation hinders the procedure of the calculation of the spectral function of the density fluctuations for an instructive purpose.

In Eckart equation, the dissipative terms are given by

\[
\tau^{\mu\nu} = \eta \left[ \partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu}(\partial_\perp \cdot u) \right] + \zeta \Delta^{\mu\nu}(\partial_\perp \cdot u), \tag{B.1}
\]

\[
+ \kappa (u^\mu \partial_\perp T + u^\nu \partial_T^\perp T - T u^\mu (u \cdot \partial) u^\nu - Tu^\mu (u \cdot \partial) u^\nu), \tag{B.1}
\]

\[
\nu^\mu = 0. \tag{B.2}
\]
The fourth order term in $z$ in the time derivative of the fluid velocity. We can factorize this to second order in $z$

where $\kappa_{\omega}$ omitted. det $A$ is evaluated to be

$$\det A = \tilde{c}_\omega n_0 \left[ -z^4 \kappa T_0 \right] \left\{ 1 - k^2 (\gamma \kappa T_0) \right\},$$

$$+ z^2 k^2 \left( \gamma \nu_1 + \nu_1 - 2 \kappa c_s^2 T_0 \right) \left( k^2 c_s^2 + k^4 \gamma \nu_1 \right) + k^4 \gamma c_s^2.$$  

The fourth order term in $z$ appears in det $A$ because of the heat current induced by the time derivative of the fluid velocity. We can factorize this to second order in $k$

$$\det A \sim \frac{-\kappa c_s n_0}{w_0} (z - \frac{w_0}{\kappa T_0}) (z + \chi k^2)(z + \Gamma_B k^2 + ic_s k)(z + \Gamma_B k^2 - ic_s k),$$

where

$$\Gamma_B = \frac{1}{2} [(\gamma - 1) \chi + \nu_1 + c_s^2 T_0 (\kappa/\nu_0 - 2 \chi \alpha P)] \equiv \Gamma^P_B.$$
Comparing Eq.(B.11) with the case of Landau equation, we see a new mode is present in the present case, which is found to be an unstable mode so that the density fluctuation increases with $t$ without limit,

$$\delta n(k, t) \sim \exp\left[ \frac{w_0}{\kappa T_0} t \right].$$

Of course, this is an unwanted result because any fluctuation around the equilibrium state should relax down to recover the equilibrium state. This pathological properties of the Eckart equation was first noted by Hiscock and Lindblom.\(^{32}\)

**Appendix C**

---

**Detailed derivation in TKO equation in particle frame**

The linearized TKO equation\(^5\) reads

\[
\begin{align*}
\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta \mathbf{v} &= 0, \quad \text{(C.1)} \\
\frac{\partial \delta \mathbf{v}}{\partial t} - \eta \nabla \delta \mathbf{v} - \frac{1}{3} \eta + \zeta' \nabla \cdot \delta \mathbf{v} + \nabla (\delta P) - \kappa \nabla (\frac{\partial \delta T}{\partial t}) &= 0, \quad \text{(C.2)} \\
\frac{n_0}{\partial \delta s}{\partial t} - 3 \zeta' T_0 \nabla \cdot \delta \mathbf{v} - \frac{\kappa}{T_0} \nabla^2 \delta T &= 0. \quad \text{(C.3)}
\end{align*}
\]

Accordingly, we have

\[
A \begin{pmatrix} \delta n(k, z) \\ \delta v_{||}(k, z) \\ \delta T(k, z) \end{pmatrix} = \begin{pmatrix} \delta n(k, 0) \\ \delta v_{||}(k, 0) - ik \frac{\kappa}{w_0} \delta T(k, 0) \\ -\frac{\alpha \rho c_s^2}{n_0 \gamma} \delta n(k, 0) - ik \frac{3 \zeta'}{w_0} \delta v_{||}(k, 0) + \frac{\alpha \rho c_s^2}{w_0 \gamma} \delta T(k, 0) \end{pmatrix}, \quad \text{(C.4)}
\]

where

\[
A = \begin{pmatrix} ik \frac{c_s^2}{n_0 \gamma} & z + \nu \kappa k^2 & 0 \\ -\frac{\alpha \rho c_s^2}{n_0 \gamma} z + \nu \kappa k^2 & ik \left(-\frac{\kappa}{w_0} z + \frac{\alpha \rho c_s^2}{\gamma} \right) \end{pmatrix}. \quad \text{(C.5)}
\]

Using these result, we have the density fluctuation up to the second order in $k$, as follows,

\[
\frac{\delta n(k, z)}{\delta n(k, 0)} = [(A^{-1})_{11} - \frac{\alpha \rho c_s^2}{n_0 \gamma^2} (A^{-1})_{13}],
\]

\[
\sim \frac{(z + \Gamma_B k^2 + ic_s k)(z + \Gamma_B k^2 - ic_s k) - \frac{1}{\gamma} c_s^2 k^2 + z \kappa k^2}{(z + \chi k^2)(z + \Gamma_B k^2 + Ac_s k)(z + \Gamma_B k^2 - ic_s k)}. \quad \text{(C.6)}
\]

Here

\[
\Gamma_B = \frac{1}{2} [\chi (\gamma - 1) + \nu_t^{TKO} - \frac{\alpha \rho c_s^2}{n_0 \gamma} (\kappa T_0 + 3 \zeta')] \equiv \Gamma_B^{TKO}, \quad \text{(C.7)}
\]

with

\[
\nu_t^{TKO} = \left( \zeta' + \frac{4}{3} \eta \right) / w_0, \quad \text{(C.8)}
\]
Thus the spectral function is now given by
\[
S_{nn}(k, \omega) = \frac{(\delta n(k, t = 0))^2}{(\delta n(k, t = 0))^2} = (1 - \frac{1}{\gamma}) \frac{2\chi k^2}{\omega^2 + \chi^2 k^4} + \frac{1}{\gamma} \frac{\Gamma_B k^2}{(\omega - c_s k)^2 + \Gamma_B^2 k^4} + \frac{\Gamma_B k^2}{(\omega + c_s k)^2 + \Gamma_B^2 k^4}. \tag{C.9}
\]

---

**Appendix D**

**Detailed derivation in the case of I-S equation**

Applying the similar procedure as in the first-order equations, the linearized I-S equation reads:
\[
\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta v = 0, \tag{D.1}
\]
\[
(\epsilon_0 + P_0) \frac{\partial \delta v}{\partial t} + \nabla (\delta P + \delta \Pi) + \frac{\partial \delta q}{\partial t} + \nabla \cdot \delta \pi = 0, \tag{D.2}
\]
\[
n_0 \frac{\partial \delta s}{\partial t} + \frac{1}{T_0} \nabla \cdot \delta q = 0, \tag{D.3}
\]
\[
\delta \Pi + \zeta [\nabla \cdot \delta v + \beta_0 \frac{\partial \delta \Pi}{\partial t} - \alpha_0 \nabla \cdot \delta q] = 0, \tag{D.4}
\]
\[
\delta q - \kappa T_0 \left[-\frac{1}{T_0} \nabla (\delta T) - \frac{\partial \delta v}{\partial t} - \beta_1 \frac{\partial \delta q}{\partial t} - \alpha_0 \nabla (\delta \Pi) - \alpha_1 \nabla \cdot \delta \pi\right] = 0, \tag{D.5}
\]
\[
\delta \pi^{ij} - \eta [\partial^i \delta v^j + \partial^j \delta v^i - 2/3 g^{ij} \nabla \cdot \delta v - 2/3 \delta v^i \nabla \cdot \delta q] = 0. \tag{D.6}
\]

Performing Fourier-Laplace transformation, we find for the longitudinal components,
\[
A = \begin{pmatrix}
\delta n(k, z) \\
\delta v_\parallel(k, z) \\
\delta T(k, z) \\
\delta \Pi(k, z) \\
\delta q_\parallel(k, z) \\
\delta \pi_\parallel(k, z)
\end{pmatrix} = \begin{pmatrix}
\delta n(k, 0) \\
\beta_0 \zeta \delta \Pi(k, 0) \\
\kappa T_0 \delta v_\parallel(k, 0) + \beta_1 \kappa T_0 \delta q_\parallel(k, 0) \\
\delta v_\parallel(k, 0) + (1/w_0) \delta q_\parallel(k, 0) \\
2\eta \beta_2 \delta \pi_\parallel(k, 0)
\end{pmatrix}, \tag{D.7}
\]

where \(\delta q_\parallel = k \cdot \delta q\), \(\delta \pi_\parallel = k^i \cdot k^j \cdot \pi_{ij}\) and
\[
A = \begin{pmatrix}
z & ikn_0 & 0 & 0 & 0 & 0 \\
0 & ik\zeta & 0 & 1 + z\beta_0 \zeta & -ik\alpha_0 \zeta & 0 \\
-k^2 \alpha P/n_0 \gamma & 0 & -z \frac{n_0 \zeta}{w_0 T_0} & 0 & -\frac{ik}{w_0 T_0} & 0 \\
z \kappa T_0 & k \kappa & k \alpha_0 \zeta T_0 & 1 + z \beta_1 \kappa T_0 & k \alpha_1 & 0 \\
k \kappa \chi & z & k^2 \alpha P/n_0 \gamma & \frac{ik}{w_0} & \frac{1}{w_0} & \frac{k}{w_0} \\
0 & \frac{4}{3} \kappa \eta & 0 & 0 & -\frac{4}{3} k \alpha_1 \eta & 1 + 2/3 \beta \eta
\end{pmatrix}. \tag{D.8}
\]
For simplicity, we have neglected the coupling terms $\alpha_0$ and $\alpha_1$, which are expected small and irrelevant for the following argument.

Applying the similar argument done in the case of Landau equation, we have

$$\delta n(k, z) = [(A^{-1})_{11} - \frac{c_*^2 \alpha_P}{n_0 \gamma} (A^{-1})_{13}] \delta n(k, 0).$$  \hspace{1cm}(D.9)

In the present case, $\det A$ is given by

$$\det A = F_1 z^6 + F_2 z^5 + F_3 z^4 + F_4 z^3 + k^2 (F_5 z^4 + F_6 z^3 + F_7 z^2 + F_8 z) + k^4 (F_9 z^2 + F_{10} z + F_{11})],$$  \hspace{1cm}(D.10)

where $F_1 \sim F_{11}$ are defined by

$$F_1 = 2 \beta_0 \beta_2 \zeta \kappa T_0 (\beta_1 w_0 - 1),$$  \hspace{1cm}(D.11)

$$F_2 = 2 \beta_0 \beta_2 \zeta \eta + \kappa T_0 (\beta_1 w_0 - 1) (\beta_0 \zeta + 2 \beta_2 \eta),$$  \hspace{1cm}(D.12)

$$F_3 = \kappa T_0 (\beta_1 w_0 - 1) + w_0 (\beta_0 \zeta + 2 \beta_2 \eta),$$  \hspace{1cm}(D.13)

$$F_4 = w_0,$$  \hspace{1cm}(D.14)

$$F_5 = \beta_1 \zeta \kappa T_0 \left[ \frac{2}{3} \beta_0 + 2 \beta_2 \right] + 2 \beta_0 \beta_2 \zeta \eta w_0 (\gamma \chi - 2 \chi c_*^2 \alpha_P T_0 + c_*^2 \beta_1 \kappa T_0),$$  \hspace{1cm}(D.15)

$$F_6 = \beta_1 \kappa T_0 w_0 \nu_l + w_0 (\beta_0 + 2 \beta_2 \nu_l + 2 \beta_2 + 2 w_0 c_*^2 \beta_0 \beta_2),$$  \hspace{1cm}(D.16)

$$F_7 = w_0 [\gamma \chi - 2 \chi c_*^2 \alpha_P T_0 + c_*^2 \beta_1 \kappa T_0 + \nu_l + c_*^2 (\beta_0 + 2 \beta_2 \eta)],$$  \hspace{1cm}(D.17)

$$F_8 = w_0 c_*^2,$$  \hspace{1cm}(D.18)

$$F_9 = \gamma \chi (\beta_0 \zeta + 2 \beta_2 \eta) + 2 \beta_0 \beta_2 \zeta \eta w_0 c_*^2 \chi,$$  \hspace{1cm}(D.19)

$$F_{10} = w_0 [\gamma \chi \nu_l + c_*^2 \chi (\beta_0 \zeta + 2 \beta_2 \eta)],$$  \hspace{1cm}(D.20)

$$F_{11} = w_0 c_*^2 \chi,$$  \hspace{1cm}(D.21)

which are all independently of $k$ and $z$. As before, $\det A$ can be approximately factorized for small $k$,

$$\det A \sim \frac{n_0 \delta_n}{w_0 T_0} (\beta_0 \zeta z + 1 + O(k^2)) (\beta_1 w_0 - 1) \kappa T_0 + w_0 + O(k^2)]$$

$$\times (2 \beta_2 \eta z + 1 + O(k^2)) (z + \chi k^2)$$

$$\times (z + \Gamma_B k^2 + i c_* k)(z + \Gamma_B k^2 - i c_* k),$$  \hspace{1cm}(D.22)

with

$$\Gamma_B = \frac{1}{2} [\gamma - 1] \chi + \nu_l + c_*^2 T_0 (\kappa/w_0 - 2 \chi \alpha_P) \equiv \Gamma_B^{IS},$$  \hspace{1cm}(D.23)

which will be found to be the width of the Brillouin peaks and is equal to that for the Eckart equation, $\Gamma_B^{Ec}$. One can see that the relativistic effect enter two ways; one is due to thermal conductivity which tends to enhance the width while the thermal expansivity would reduce the width. The fact that the relativistic effects might
possibly enhance the Brillouin width is in contrast to the case of TKO equation where the relativistic effects only tends to reduce the width of the sound modes. As we will see in the text, the net relativistic contribution to \( \Gamma_{IS}^B \) is found to be negative.

References

1) Y. Minami and K. Kunihiro, talk presented at YITP workshop, “Thermal Quantum Field Theory and Their Applications”, September 3-5 (2008), Yukawa Institute of Theoretical Physics, Kyoto, Japan; Soryushiron Kenkyu (mimeographed circular in Japanese) 116 (6) (2009), F39; Y. Minami, Master thesis submitted to Kyoto University, February, 2009; T. Kunihiro, Y. Minami and K. Tsumura, talk presented at Quark Matter 2009, March 30-April 4, Knoxville, Tennessee; arXiv:0907.3388v2 [hep-ph].
2) T. Hirano, U. W. Heinz, D. Elkhart, R. Lacey and Y. Nara, Phys. Lett. B 636 (2006), 299
3) I. Arsene et al. [BRAHMS Collaboration], Nucl. Phys. A 757 (2005), 1; B. B. Back et al., ibid., p.28; J. Adams et al. [STAR Collaboration], ibid. p. 102; K. Adcox et al. [PHENIX Collaboration], ibid. p.184.
4) A. Muronga, Phys. Rev. C 69 (2004), 034903; Phys. Rev. C 76 (2007), 014910; A. K. Chaudhuri, J. Phys. G 35 (2008), 104015.
5) K. Tsumura, T. Kunihiro and K. Ohnishi, Phys. Lett. B 646 (2007), 134.
6) R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP 0804 (2008), 100.
7) As review articles, see, P. Huovinen and P. V. Ruuskanen, Ann. Rev. Nucl. Part. Sci. 56 (2006), 163; J. Y. Ollitrault, Eur. J. Phys. 29 (2008), 275; T. Hirano, N. van der Kolk and A. Bilandzic, “Hydrodynamics and Flow”, to be published in Lecture Notes in Physics, (Springer); arXiv:0808.2684 [nucl-th]. P. Romatschke, arXiv:0902.3663 [hep-ph].
8) M. Asakawa and K. Yazaki, Nucl. Phys. A 504 (1989), 668; A. Barducci, R. Casalbuoni, S. De Curtis, R. Gatto and G. Pettini, Phys. Lett. B 231 (1989), 463; Phys. Rev. D 49 (1994), 426; As review articles, see, M. A. Stephanov, Prog. Theor. Phys. Suppl. 153 (2004), 139 [Int. J. Mod. Phys. A 20 (2005), 4387].
9) S. Ejiri, Phys. Rev. D 77 (2008), 014508; Phys. Rev. D 78 (2008), 074507.
10) M. Kitazawa, T. Koide, T. Kunihiro and Y. Nemoto, Prog. Theor. Phys. 108 (2002), 929; Z. Zhang, K. Fukushima and T. Kunihiro, Phys. Rev. D 79 (2009), 014004; Z. Zhang and T. Kunihiro, Phys. Rev. D 80 (2009), 014015.
11) T. Hatsuda, M. Tachibana, N. Yamamoto and G. Baym, Phys. Rev. Lett. 97 (2006), 122001; N. Yamamoto, M. Tachibana, T. Hatsuda and G. Baym, Phys. Rev. D 76 (2007), 074001.
12) P. de Forcrand and O. Philipsen, JHEP 0701 (2007), 077; P. de Forcrand and O. Philipsen, JHEP 0811 (2008), 012; O. Philipsen, Prog. Theor. Phys. Suppl. 174 (2008), 206; K. Fukushima, Phys. Rev. D 78 (2008), 114019; J. W. Chen, K. Fukushima, H. Kohyama, K. Ohnishi and U. Raha, arXiv:0901.2407 [hep-ph].
13) M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. 81, (1998), 4816; M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D 60 (1999), 114028.
14) H. Fujii, Phys. Rev. D 67 (2003),094018; H. Fujii and M. Ohtani, Phys. Rev. D 70 (2004), 014016; H. Fujii and M. Ohtani, Prog. Theor. Phys. Suppl. 153 (2004), 157; H. Fujii and N. Tanji, J. Phys. G 35 (2008), 104060.
15) D. T. Son and M.A. Stephanov, Phys. Rev. D 70 (2004), 056001.
16) T. Kunihiro, Phys. Lett. B 271 (1991), 395.
17) M. A. Stephanov, Phys. Rev. Lett. 102 (2009), 032301.
18) K. Ohnishi and T. Kunihiro, Phys. Lett. B 632 (2006), 252.
19) F. Karsch, D. Kharzeev and K. Tuchin, Phys. Lett. B 663 (2008), 217.
20) G. D. Moore and O. Saremi, JHEP 0809 (2008), 015.
21) P. Romatschke and D. T. Son, arXiv:0903.3946 [hep-ph].
S. Caron-Huot, Phys. Rev. D 79 (2009), 125009.

22) H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena, (Oxford University Press, Oxford, 1977).

23) K. Kawasaki, Ann. Phys. 61 (1970), 1.

24) A. Onuki, Phase Transition Dynamics, (Cambridge University Press, 2007).

25) L. D. Landau and G. Placzek, Phys. Z. Sowjetunion 5 (1934), 172; L. P. Kadanoff and P. C. Martin, Ann. Phys. 24 (1963), 419; R. D. Mountain, Rev. Mod. Phys. 38(1966), 205.

26) As a comprehensive account of the light scattering by the fluid dynamical density fluctuations see, L. E. Reichl, A Modern Course in Statistical Physics (Wiley-Interscience 1998).

27) A. Sandoval-Villalbazo and R. Maartens, Gen. Rel. Gravit. 37(6) (2005), 1137.

28) See for typical attempts of fluid dynamical simulations of time evolution of the matter created by relativistic heavy-ion collisions, K. Paech, H. Stoecker and A. Dumitru, Phys. Rev. C 68 (2003), 044907; C. Nonaka and M. Asakawa, Phys. Rev. C 71 (2005), 044904.

29) B. I. Halperin and P. C. Hohenberg, Phys. Rev. Lett. 19 (1967), 700; Phys. Rev. 177 (1969), 952; P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49 (1977), 435.

30) L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon, New York, 1959).

31) L. D. Landau and G. Placzek, Phys. Z. Sowjetunion 5 (1934), 172; L. P. Kadanoff and P. C. Martin, Ann. Phys. 24 (1963), 419; R. D. Mountain, Rev. Mod. Phys. 38(1966), 205.

32) As a comprehensive account of the light scattering by the fluid dynamical density fluctuations see, L. E. Reichl, A Modern Course in Statistical Physics (Wiley-Interscience 1998).

33) A. Onuki, Phys. Rev. E 55 (1997), 403.