Cohomological, Poisson structures and integrable hierarchies in tautological subbundles for Birkhoff strata of Sato Grassmannian

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May 7, 2014

Abstract
Cohomological and Poisson structures associated with the special tautological subbundles $TB_{W_1,2,...,n}$ for the Birkhoff strata of Sato Grassmannian are considered. It is shown that the tangent bundles of $TB_{W_1,2,...,n}$ are isomorphic to the linear spaces of 2-coboundaries with vanishing Harrison’s cohomology modules. Special class of 2-coboundaries is provided by the systems of integrable quasilinear PDEs. For the big cell it is the dKP hierarchy. It is demonstrated also that the families of ideals for algebraic varieties in $TB_{W_1,2,...,n}$ can be viewed as the Poisson ideals. This observation establishes a connection between families of algebraic curves in $TB_{W_1,2,...,n}$ and coisotropic deformations of such curves of zero and nonzero genus described by hierarchies of hydrodynamical type systems like dKP hierarchy. Interrelation between cohomological and Poisson structures is noted.

1 Introduction
In this paper we continue the study of algebraic, geometric and other structures arising in the tautological subbundles for the Birkhoff strata of Sato Grassmannian. In the paper [1] it was shown that each Birkhoff stratum $\Sigma_S$ of the Sato Grassmannian $Gr$ contains a subset $W_S$ of points such that for each of these points the corresponding infinite-dimensional linear space (fiber of the tautological subbundle $TB_{W_S}$ associated with $W_S$) is closed with respect to pointwise multiplication. Algebraically all $TB_{W_S}$ are infinite families of infinite dimensional associative commutative algebras. Geometrically each fiber of $TB_{W_S}$ is an algebraic variety and the whole $TB_{W_S}$ is an infinite family of algebraic varieties with each finite-dimensional subvariety being a family of algebraic curves defined by the equations [1]
\[ p_j p_k - \sum_l C^l_{jk} p_l = 0, \quad (1) \]
\[ \sum_l \left( C^l_{jk} C^p_{lm} - C^l_{mk} C^p_{lj} \right) = 0, \quad j, k, m, p \in S \quad (2) \]

where \( C^l_{jk} \) are parameterized by certain quantities \( H^l_{jk} \).

For the big cell \( \Sigma_\emptyset \) of the Sato Grassmannian one has

\[ C^l_{jk} = \delta^l_{j+k} + H^k_{j-l} + H^j_{k-l}, \quad j, k = 0, 1, 2, 3, \ldots \]

and the \( TB_{W_0} \) is the collection of families of normal rational curves (Veronese curves) of the all degrees 2, 3, 4, \ldots. For the stratum \( \Sigma_1 \), each fiber of \( TB_{W_1} \) is the coordinate ring of the elliptic curve and \( TB_{W_1} \) is the infinite family of such rings. For the set \( W_{1,2} \) the \( TB_{W_{1,2}} \) is equivalent to the families of coordinate rings of a special space curve with pretty interesting properties. This family of curves in \( TB_{W_{1,2}} \) contains plane trigonal curve of genus two. For the higher strata \( \Sigma_{1,2,\ldots,n} \) \( (n = 3, 4, 5, \ldots) \) \( TB_{W_{1,2,\ldots,n}} \) contains plane \( (n+1, n+2) \) curve of genus \( n \).

In the present paper local and Poisson structures associated with the subbundles \( TB_{W_\emptyset} \) and the corresponding algebraic curves are discussed. It is shown that the tangent bundles of \( TB_{W_{1,2,\ldots,n}} \) carry the hierarchies of integrable equations. In particular, the tangent subbundle \( TB_{W_\emptyset} \) for the big cell contains the hierarchy of dispersionless Kadomtsev-Petviashvili (dKP) equations. It is demonstrated that the tangent bundles of \( TB_{W_{1,2,\ldots,n}} \) and \( TB_{W_{1,2,\ldots,n}} \) modules \( E_{1,2,\ldots,n} \) are isomorphic to the linear spaces of 2-coboundaries and Harrison’s cohomology modules \( H^2(W, E) \) and \( H^3(W, E) \) vanish. Special classes of 2-cocycles and 2-coboundaries are described by the systems of integrable quasilinear PDEs. For example, a class of 2-coboundaries associated with the subbundle \( TB_{W_\emptyset} \) in the big cell is provided by the dKP hierarchy.

We give also an interpretation of the families of ideals \( I(\Gamma_\infty) \) for families of algebraic curves in \( TB_{W_{1,2,\ldots,n}} \) as the Poisson ideals. It is shown that the family of ideals for the family of normal rational curves in the big cell is the Poisson ideal with respect to a Poisson structure obeying certain constraints. Two sets of canonical variables in such Poisson ideals are used. It is demonstrated that in the Darboux coordinates the above constraints are nothing else than the dKP hierarchy. Similar results remain valid for other strata too. Finally an interrelation between cohomological and Poisson structures of \( TB_{W_{1,2,\ldots,n}} \) is observed.

The results presented here are the extension of those for \( Gr^{(2)} \) \( (2, 3) \) to the general Sato Grassmannian.

The paper is organized as follows. In section 2 we recall the structure of Sato Grassmannian. Tangent subbundles \( TB_{W_{1,2,\ldots,n}} \) and systems of integrable quasilinear PDEs are considered in section 3. In section 4 we study connection between Harrison cohomology and dKP hierarchy. Associated Poisson ideals and coisotropic deformations are discussed in section 4.

## 2 Sato Grassmannian, Birkhoff strata and algebraic varieties

Here we briefly recall, without entering in any technicalities, some basic facts about Sato Grassmannian, its Birkhoff stratification (see e.g., in \cite{4, 5}) and results obtained in \cite{1}. The Sato Grassmannian \( Gr \) can be viewed as the set of all subspaces of the infinite-dimensional set of all formal Laurent series with complex-valued coefficients admitting an algebraic basis \( (w_0(z), w_1(z), \ldots) \) with the basis elements
\[ w_n = \sum_{k = -\infty}^{\infty} H^n_k z^k \]  

of finite order \( n \). The Grassmanian \( G_r \) has a stratified structure \( G_r = \bigcup \Sigma_k \) where the strata \( \Sigma_k \) are subsets in \( G_r \) which are span of basis elements \( \mathfrak{g} \) with particular set of integers \( n \) (see \[4, 5\]). For the big cell the basis is composed by formal Laurent series

\[ p_0 = 1, \quad p_i(z) = z^i + \sum_{k=1}^{\infty} \frac{H^i_k}{z^k}, \quad i = 1, 2, \ldots \]  

This set is identified by the infinite set of symbols \( H^i_j \), \( i, j \geq 1 \). The big cell is a dense set in the Sato Grassmannian. The points outside the big cell form Birkhoff strata. It can be shown that the points in the Birkhoff strata can be constructed changing the basis of the space of Laurent series and, in a suitable way (\[4, 5\]), the queue of the Laurent series. In the simplest case of first Birkhoff stratum \( \Sigma_1 \) the basis of Laurent series is obtained removing the element \( p_1 \) and hence it is of the form

\[ p_0 = 1, \quad p_1(z) = z + \sum_{k=1}^{\infty} \frac{H^1_k}{z^k}, \quad i = 2, 3, \ldots \]  

plus element of the degree \(-1\). As in \( \Sigma_\emptyset \) case, a point of \( \Sigma_1 \) is a span of the Laurent series \( \{p_i\}_{i=0, 1, 2, 3, \ldots} \).

In this paper, following \[1\], will be interested to a particular subsets \( W_S \) of the Birkhoff strata of the Sato Grassmannian. In general, \( W_S \) are composed by the Laurent series closed with respect to point-wise multiplication. In the case of the big cell \( W_\emptyset \) is given by the span of the elements \( \mathfrak{g} \) closed with respect to multiplication, i.e. such that

\[ p_j(z)p_k(z) = \sum_{l=0}^{C_{jk}} C^l_{jk} p_l(z), \quad j, k = 0, 1, 2, \ldots \]  

for some suitable \( C^l_{jk} \). This property is verified iff the coefficients \( H^i_j \) of the formal Laurent series \( p_i \) obey the constraints \[1\]

\[ H^{i+k}_{m+k} - H^{i+k}_{m+k} - H^{k}_{j+m} + \sum_{l=1}^{j-1} H^{i-l}_{m+l} + \sum_{l=1}^{k-1} H^{j-l}_{m-l} + \sum_{l=1}^{m-1} H^{k-l}_{m-l} H^l_i = 0, \quad j, k, m = 1, 2, 3, \ldots \]  

and

\[ C^l_{jk} = \delta_{j+k}^l + H^k_{j-l} + H^l_{j-k}, \quad j, k = 0, 1, 2, 3, \ldots \]  

Under these constraints the Laurent series \( p_i \) can be presented as

\[ p_0 = 1, \]
\[ p_2 = p_1^2 - 2H^1_1, \]
\[ p_3 = p_1^3 - 3H^1_1 p_1 - 3H^1_2, \]
\[ p_4 = p_1^4 - 4H^1_1 p_1^2 - 4H^1_2 p_1 - 4H^2_1 + 2H^2_1, \]
\[ p_5 = p_1^5 - 5H^1_1 p_1^3 - 5H^1_2 p_1^2 - \left( 5H^3_1 - 5H^2_1 \right) p_1 - 5H^1_4 + 5H^1_1 H^1_2, \]

\[ \ldots \]
One has similar results for all the Birkhoff strata. For the first stratum $\Sigma_1$ the subset of Laurent series closed with respect to point-wise product $p_ip_j = \sum_k C_{ij}^k p_k$ can be constructed iteratively. In this case the coefficients $H^i_j$ should satisfy the constraints

$$H^i_{j+l} + H^l_j H^i_{j+1} + H^i_{j-1} H^j_{i+1} + \sum_{n=1}^{l-1} H^i_n H^j_{i-n} =$$

$$H^i_{l+j} + H^l_j H^i_{l+1} + H^i_{l-1} H^j_{l+1} + \sum_{n=2}^{l-1} H^i_{l-n} H^j_l + \sum_{n=2}^{j-1} H^j_{l-n} H^i_l + H^i_{l-1} H^j_{l-1} H^2_l +$$

$$(H^i_j + H^l_j H^i_l + H^i_j H^j_{l-1}) \delta^0_{i,j}, \quad j, k = 2, 3, 4, \ldots, l = -1, 1, 2, 3, \ldots $$

and

$$C_{ij}^l = \delta^i_{l+j} + H^i_{l-1} \delta^i_{l+1} + H^i_{l-1} \delta^i_{j+1} + H^i_{l-1} + H^i_{l-1} H^j_{l+1} + (H^i_j + H^l_j H^i_l + H^i_j H^j_{l-1}) \delta^l_{0,j}.\quad (11)$$

The first $p_i$'s are

$$p_4 = p_2^2 - 2 H^2_{-1} p_3 - H^2_{-2} p_2^2 - 2 H^2_{-2} - 2 H^2_{-1} H^2_1$$

$$p_5 = p_2 p_3 - H^2_{-1} p_2^2 - \left( H^3_{-1} - 2 H^2_{-1} \right) p_3 - \left( H^2_{-1} + H^2_{-1} H^3_{-1} - H^2_{-1} \right) p_2$$

$$- \frac{3}{2} H^2_{-1} H^3_{-1} - \frac{5}{2} H^2_{-2} - \frac{1}{2} H^2_{-1} H^3_{-1} + 2 H^2_{-1} H^2_{-2} + 2 H^2_{-1} H^2_{-2}$$

$$- \frac{1}{2} H^2_{-1} H^3_{-1} + 4 H^3_{-1} H^2_{-1} = 0.\quad (12)$$

The analogue of the subset $W_{\Sigma}$ for $\Sigma_1$ is given by the points of $\Sigma_1$ whose $p_i$'s satisfy (12). The relations (10) naturally defines a set of quadric algebraic varieties in an infinite dimensional space of coordinates $p_i$. Every variety is of genus zero. This property changes in the Birkhoff strata. In the space of coordinates $p_0, p_1, p_2, \ldots$ there are an infinite number of quadric algebraic varieties defined by the relations $p_ip_j = \sum_k C_{ij}^k p_k$. For $W_{\Sigma}$ there is an elliptic (genus 1) curve

$$F_{23}^1 = p_3^2 - p_2^3 + 3 H^2_{-1} p_3 p_2 - 2 H^2_{-1} p_2^2 + \left( H^2_{-1} + 3 H^2_{-1} H^3_{-1} \right) p_3$$

$$- \left( H^2_{-1} + 3 H^2_{-1} H^3_{-1} - 3 H^2_{-1} \right) p_2$$

$$- 2 H^3_{-1} H^3_{-1} + 3 H^3_{-1} + 3 H^2_{-1} H^3_{-1} + 3 H^3_{-1} - \frac{3}{2} H^2_{-1} H^3_{-1}$$

$$- \frac{1}{2} H^2_{-1} H^3_{-1} + 4 H^3_{-1} H^2_{-1} = 0.\quad (13)$$

among them $\Sigma_1$.

3 Tangent subbundles $TB_S$ and systems of integrable quasilinear PDEs

In this paper we will consider only the subsets $W_{1,2,\ldots,n}$ of the Birkhoff strata and the corresponding subbundles $TB_{W_{1,2,\ldots,n}}$. A standard method to analyze local properties of the varieties defined by
equations (12) is to deal with their tangent bundle $T_W$ \([6]-[9]\). Let us denote by $\pi_i$ and $\Delta^i_{jk}$ the corresponding elements of $T_W$ in a point. They are defined, as usual, by the system of linear equations

$$
\pi_j p_k + p_j \pi_k - \sum_l \Delta^l_{jk} p_l - \sum_l C^l_{jk} \pi_l = 0,
$$

$$
\sum_l \left( \Delta^l_{jk} C^p_{lm} + C^l_{jk} \Delta^p_{lm} - \Delta^l_{mk} C^p_{lj} - C^l_{mk} \Delta^p_{lj} \right) = 0, \quad j, k, m, p \in S_{1,2,\ldots,n}.
$$

In more general setting these equations define also a $T^1_B W_{1,2,\ldots,n}$-module $E$.

For the Birkhoff strata the structure constants $C^l_{jk}$ have a special structure being parameterized by $H^l_{jk}$. Consequently $\Delta^l_{jk}$ are also parameterized by $\Delta_{jk}$, i.e. by images of $H^l_{jk}$ in the map $W \to E$, and equations (15) becomes linear equations (15) for $\Delta_{jk}$. Being the elements of $E$ (in particular, the tangent space in a point) $\Delta_{jk}$ admit a natural Ansatz

$$
\Delta_{jk} = \frac{\partial u_k}{\partial x_j}
$$

where $u_k$ is a set of independent coordinates for the variety defined by the associativity condition (2) and $x_j$ is a set of new independent parameters. Under the Ansatz (16) equations (15) take the form of quasilinear PDEs for $u_k$.

These systems of quasilinear PDEs are very special. Let us consider the subbundle $T^1_B W_0$ for the big cell. Since in this case

$$
H^j_{m+k} + H^j_{m+k} - H^k_{j+m} + \sum_{l=1}^{j-1} H^k_{j-l} H^l_{m} + \sum_{l=1}^{k-1} H^l_{k-l} H^l_{m} - \sum_{l=1}^{m-1} H^l_{m-l} H^l_{j} = 0, \quad j, k, m = 1, 2, 3, \ldots.,
$$

one has $\Delta^l_{kj} = \Delta_{kj} - \Delta_{jk}$ and the system (15) is

$$
\Delta_{j+k,m} + \left( - \Delta_{j,m+k} + \sum_{l=1}^{j-1} H^k_{j-l} \Delta_{lm} + \sum_{l=1}^{k-1} H^l_{k-l} \Delta_{lm} - \sum_{l=1}^{m-1} H^l_{m-l} \Delta_{lj} \right) + \left( - \Delta_{k,m+j} + \sum_{l=1}^{j-1} \Delta^k_{j-l} H^l_{lm} + \sum_{l=1}^{k-1} \Delta^l_{k-l} H^l_{lm} - \sum_{l=1}^{m-1} \Delta^l_{m-l} H^l_{jl} \right) = 0.
$$

This system implies

$$
k \Delta_{ik} - i \Delta_{ki} = 0.
$$

Let us rename $H^1_k$ as $u_k$.

**Proposition 3.1** Under the Ansatz

$$
\Delta_{ik} = \frac{\partial u_k}{\partial x_i}, \quad i, k = 1, 2, 3, \ldots.
$$

the system (14) coincides with the dKP hierarchy.
Proof  For $j = 1, k = 2, m = 1$ the system (19) is
\[
\Delta_{31} - \Delta_{13} - \Delta_{22} + 2H_1^1\Delta_{11} = 0
\]  
(22)
while the relations (20) at $i = 1, k = 2$ and $i = 1, k = 3$ are
\[
2\Delta_{12} - \Delta_{21} = 0, \quad 3\Delta_{13} - \Delta_{31} = 0.
\]  
(23)
The ansatz (21) gives
\[
\partial_{x_1}u_1 - \frac{3}{2}\partial_{x_2}u_2 + 3u_1\partial_{x_1}u_1 = 0, \\
2\partial_{x_1}u_2 - \partial_{x_2}u_1 = 0.
\]  
(24)
It is the celebrated dKP (Khoklov-Zaboloskaya) equations (see e.g. [10]-[14], [15]). Similarly for $j = 1, k = 1$ and $m = 3$ the system (19) is
\[
\Delta_{23} - 2\Delta_{14} - 2H_2^1\Delta_{11} - 2H_1^1\Delta_{12} = 0
\]  
(25)
and the relations (20) at $i = 1, k = 4$, and $i = 1, k = 3$ are
\[
\Delta_{41} = 4\Delta_{14}, \quad 3\Delta_{13} = \Delta_{31}.
\]  
(26)
These equations, using also (24), give
\[
\partial_{x_4}u_1 = 2\partial_{x_2}u_3 - 2\partial_{x_1}(u_1u_2) \\
\partial_{x_1}u_3 = \frac{1}{2}\partial_{x_2}u_2 - u_1\partial_{x_1}u_1 \\
2\partial_{x_1}u_2 - \partial_{x_2}u_1 = 0
\]  
(27)
which the second equation in the dKP hierarchy. The higher equations (19), (20) give rise to the higher dKP equations under the ansatz (21). □

The Ansatz (16) is necessary and sufficient condition for the closeness of the differential one-forms $\Omega_k = \sum_{j=0}^{\infty} \Delta_{jk}dx_j$, $k = 1, 2, ...$. Hence, the above observation can be formulated also in the following form.

**Proposition 3.2** Special subbundle $TB_{W_1}$ for which differential one-forms $\Omega_k = \sum_{j=0}^{\infty} \Delta_{jk}dx_j$, $k = 1, 2, ...$ are closed is governed by the dKP hierarchy.

4 Harrison cohomology and dKP Harrison cohomology

Equations (14) and (15) imply certain cohomological properties of the subbundles $TB_{W_1,2,...,n}$. Indeed, if one introduces the bilinear map $\psi(\alpha, \beta)$ with $\alpha, \beta \in TB_{W_1,2,...,n}$ defined by (see e.g. [17])
\[
\psi(p_i, p_k) = \sum_l \Delta_{jk}^l p_l.
\]  
(28)
Then the equations (15) take the form

\[
\]
\begin{align}
\psi(p_j, p_k) - \psi(p_j, p_k) + \psi(p_j, p_k) - \psi(p_j, p_k) = 0,
\end{align}

or equivalently

\begin{align}
\alpha \psi(\beta, \gamma) - \psi(\alpha \beta, \gamma) + \psi(\alpha \beta, \gamma) - \gamma \psi(\alpha, \beta) = 0
\end{align}

where \( \alpha, \beta, \gamma \in TB_{W,1,2,\ldots,n} \) Bilinear maps of such type are called Hochschild 2-cocycles. So, the tangent bundle to the variety of the structure constants \( C_{jk}^l \) is isomorphic to the linear space of the 2-cocycles on \( TB_{W,1,2,\ldots,n} \) (see e.g. [16]). For the commutative algebras this classical results represents a part of the cohomology theory of commutative associative algebras proposed by Harrison in [17]. It is the most appropriate tool to analyze the local properties of the varieties \( W_s \).

Equations (14) gives us an additional information about the 2-cocycle \( \psi(\alpha, \beta) \). Introducing a linear map \( g(\alpha) \) defined by \( g(p_i) = \pi_i \), one rewrites equation (14) as

\begin{align}
p_j g(p_k) + p_k g(p_j) - \psi(p_j, p_k) - g(p_j p_k) = 0
\end{align}

Thus,

\begin{align}
\psi(\alpha, \beta) = \alpha g(\beta) + \beta g(\alpha) - g(\alpha \beta)
\end{align}

with \( \alpha, \beta \in TB_{W,1,2,\ldots,n} \). So

\begin{align}
\psi(\alpha, \beta) = \delta g(\alpha, \beta)
\end{align}

where \( \delta \) is the Hochschild coboundary operation. Hence, \( \psi(\alpha, \beta) \) is a 2-coboundary and one has

**Proposition 4.1** The tangent bundle of the subbundle \( TB_{W,1,2,\ldots,n} \) is isomorphic to the linear space of 2-coboundaries and Harrison's cohomology modules \( H^2(TB_{W,1,2,\ldots,n}, E) \) and \( H^3(TB_{W,1,2,\ldots,n}, E) \) vanish.

This statement is essentially the reformulation for the subbundles \( TB_{W,1,2,\ldots,n} \) of the well-known results concerning the cohomology of commutative associative algebras (see e.g. [17]-[25]). In particular the existence of the 2-cocycle and \( H^2(TB_{W,1,2,\ldots,n}, E) = 0 \) is sufficient condition for the regularity of the point at which it is calculated (see e.g. [17]-[19]).

The above results are valid for all algebraic varieties associated with the Birkhoff strata. The observation made in the previous section shows that among the generic 2-coboundaries considered above there are special ones associated with the integrable equations. For the big cell, as the immediate consequence of the Proposition 3.1, one has

**Proposition 4.2** Solutions of the dKP hierarchy provide us with the class of 2-cocycles and 2-coboundaries defined by

\begin{align}
\psi(p_j, p_k) = \sum_l \left( \frac{\partial u_{j-l}}{\partial x_k} + \frac{\partial u_{k-l}}{\partial x_j} \right) p_l
\end{align}

for the subbundle \( TB_{W,\emptyset} \) in the big cell.

We will refer to such 2-coboundaries as dKP 2-coboundaries. These dKP 2-coboundaries describe local properties of the family of normal rational curves.
In terms of the dKP tau-function $F$ defined by (see e.g. [13, 15])
\[ u_k = H_k^1 = -\frac{1}{k} \frac{\partial F}{\partial x_1 \partial x_k} \] (33)
the whole dKP hierarchy is represented by the celebrated dispersionless Hirota-Miwa equations (see e.g. [13],[15])
\[ -\frac{1}{m} F_{i+k,m} + \frac{1}{m+k} F_{i,k+m} + \frac{1}{i+m} F_{k,i+m} + \sum_{l=1}^{i-1} \frac{1}{m(i-l)} F_{k,i-l} \partial F_{l,m} \]
\[ + \sum_{l=1}^{k-1} \frac{1}{m(k-l)} F_{i,k-l} \partial F_{l,m} - \sum_{l=1}^{m-1} \frac{1}{i(m-l)} F_{k,m-l} \partial F_{l,i} = 0 \] (34)
where $F_{i,k}$ stands for the second-order derivative of $F$ with respect to $x_i$ and $x_k$. So any solution $F$ of the system (34) provides us with the dKP 2-cocycles (and 2-coboundaries) given by
\[ \psi(p_j,p_k) = -\sum_{l=1}^{i-1} \left( \frac{1}{j-l} \frac{\partial^2}{\partial x_k \partial x_j-l} + \frac{1}{k-l} \frac{\partial^2}{\partial x_j \partial x_k-l} \right) \frac{\partial F}{\partial x_1} p_l. \] (35)
This formula shows that the choice (16) corresponds to a simple realization of the map $W_\emptyset \to E$, namely, $F \to \frac{\partial F}{\partial x_1}$ or $H_j^k \to \frac{\partial H_j^k}{\partial x_1}$.

It is evident that all above expressions are well defined only for bounded $\frac{\partial u}{\partial x_k}$. When $\frac{\partial u}{\partial x_k} \to \infty$ the formulas presented above break down and $H^2(W,E) \neq 0$.

For the dKP hierarchy the points where $\frac{\partial u}{\partial x_k} \to \infty$ are the, so-called, breaking points (or points of gradient catastrophe). Such points form the singular sector of the space of solutions of the dKP hierarchy. In this sector the space of variables $x_1, x_2, \ldots$ is stratified and such stratification is closely connected with the Birkhoff stratification. For Burgers-Hopf hierarchy (2-reduction of the dKP hierarchy) and the Grassmannian $Gr^{(2)}$ such situation has been analyzed in [26].

We are confident that similar results hold for other strata too. A complete analysis of the Harrison cohomology of the tautological subbundles for Birkhoff strata of the Grassmannian $Gr^{(2)}$ and corresponding integrable equations has been performed in [2].

5 Families of curves, Poisson ideals and coisotropic deformations

Families of curves, algebraic varieties and families of their ideals considered above can be viewed also as embedded in larger spaces with certain specific properties, for instance, as the coisotropic submanifolds of Poisson manifolds and Poisson ideals, respectively. Recall that a submanifold in the Poisson manifold equipped with the Poisson bracket $\{,\}$ is a coisotropic submanifold if its ideal $I$ is the Poisson ideal (see e.g [27]), i.e.
\[ \{I,I\} \subset I. \] (36)
Relevance of Poisson ideals in the study of (quantum) cohomology of manifolds was observed in the paper [28]. Theory of coisotropic deformations of commutative associative algebras based on the
property [39] has been proposed in [15]. An extension of this theory to general algebraic varieties was given in [25].

Thus let us consider an infinite-dimensional Poisson manifold $P$ with local coordinates $q_1, q_2, q_3, \ldots, y_1, y_2, y_3, \ldots$ endowed with the Poisson bracket defined by the relations

$$\{q_i, q_k\} = 0, \quad \{y_i, y_k\} = 0, \quad \{q_i, q_k\} = J_{ki}, \quad i, k = 1, 2, 3, \ldots$$

(37)

where $J_{ki}$ are certain functions of $p$ and $q$. This choice of the Poisson structure is suggested by the roles that the variables $p_i$ and $q_k$ play in our construction. Jacobi identities for the Poisson structures (37) are given by the system

$$\sum_s J_s \partial_{y_j} J_{kj} - \sum_s J_s \partial_{y_j} J_{ij} = 0,$$

$$\sum_s J_s \partial_{y_j} J_{ik} - \sum_s J_s \partial_{y_k} J_{ij} = 0.$$  

(38)

Then, we consider ideals $\mathcal{I}(\Gamma_\infty)$ of the families of algebraic varieties in $W_{1,2,\ldots,n}$ as ideals in $P$ and require that they are Poisson ideals, i.e. subalgebras

$$\{\mathcal{I}(\Gamma_\infty), \mathcal{I}(\Gamma_\infty)\} \subset \mathcal{I}(\Gamma_\infty).$$

(39)

The property [39] means, in particular, that the Hamiltonian vector fields generated by each member of $\mathcal{I}(\Gamma_\infty)$ are tangent to the coisotropic submanifold with the ideal $\mathcal{I}(\Gamma_\infty)$.

The crucial question now is whether a Poisson structure exists such ideals $\mathcal{I}(\Gamma_\infty)$ obey [39]. Let us begin with the big cell. For the subbundle $\mathcal{TB}_{\Psi}$ the answer is given by

**Proposition 5.1** The family of ideals $I(\Gamma_\infty)$ of the family of normal rational curves in the big cell represents the Poisson ideal in the Poisson manifold endowed with the Poisson brackets [37] with $J_{ik}$ obeying the constraints

$$(J_{i-1} - J_{i-1}^1)_{|\Gamma_\infty} = 0 \quad i, k = 2, 3, 4, \ldots.$$  

(40)

**Proof** To prove [39] it is sufficient to show that for the elements $h_n$ of the basis of $I(\Gamma_\infty)$ one has $\{h_n, h_m\} \subset I(\Gamma_\infty)$. The local coordinates $p_n^* = P_n(-p_1, -\frac{1}{2}p_2, -\frac{1}{3}p_3, \ldots, n = 2, 3, 4, \ldots)$, where $P_n(t_1, t_2, t_3, \ldots)$ are the standard Schur polynomials defined by the formula $\exp(\sum_{n=1}^{\infty} z^n t_n) = \sum_{m=0}^{\infty} z^m P_m(t_1, t_2, t_3, \ldots)$, and canonical basis $h_2^*, h_3^*, h_4^*, \ldots$ given by

$$h_n^* = p_n - H_{n-1}^1, \quad n = 2, 3, 4, \ldots,$$

(41)

i.e. $h_n^* = p_n^* - u_{n-1}, n = 2, 3, 4, \ldots$ are the most convenient for this purpose. In these coordinates one has the identity

$$\{h_n^*, h_m^*\} = J_{n-1,m-1}^* - J_{n,m-1}^*, \quad n, m = 2, 3, 4, \ldots$$

(42)

where $J_{nm}^*$ denotes the Poisson tensor in these coordinates. So the conditions $\{h_n, h_m\} \subset I(\Gamma_\infty)$ is satisfied if and only if the conditions [40] are valid. □

On $\Gamma_\infty$ one has $p_n^* = u_{n-1}, n = 2, 3, 4, \ldots$ and, hence,

$$J_{ik}^*_{|\Gamma_\infty} = \alpha_{ik}(u) + \beta_{ik}(u)p_1^*, \quad i, k = 1, 2, 3, \ldots$$

(43)
where $\alpha_{ik}$ and $\beta_{ik}$ are functions of $u_k$ only. Since $p_i^* \notin I(\Gamma_\infty)$ the conditions (40) are equivalent to

$$\alpha_{i,k-1} = \alpha_{k,i-1}, \quad \beta_{i,k-1} = \beta_{k,i-1}, \quad i, k = 1, 2, 3, \ldots \quad (44)$$

The property (43) indicates that Poisson tensors $J_{ik}^*$ linear in the variables $p_k^*$ could be of particular relevance. Thus let us consider the following class of tensors $J_{ik}^*$

$$J_{ik}^* = - \sum_m \frac{1}{m} J_{mk}(u) p_i^* - m \quad (45)$$

where $J_{mk}(u)$ depend only on the variables $u_1, u_2, u_3, \ldots$. The conditions (40) or (43) are equivalent to the following

$$\frac{1}{m} J_{mn} - \frac{1}{n} J_{nm} = 0,$$

$$\frac{1}{m} J_{m,n-1} - \frac{1}{n} J_{m,n-1} + \sum_{k=1}^{m-2} \frac{1}{k} u_{m,k-1} J_{k,n-1} - \sum_{k=1}^{n-2} \frac{1}{k} u_{n,k-1} J_{k,m-1} = 0, \quad n, m = 1, 2, 3, \ldots \quad (46)$$

Using the well-known property of Schur’s polynomials, i.e. $\partial_{p_k} P_n(p) = P_{n-k}(p)$ which implies that $\partial_{p_i} h = \frac{1}{i} \sum_{q=1}^{i-1} p_{i-q} \partial_{p_q} h$, one easily concludes that the Poisson structure (37) with $J_{ik}^*$ of the form (45) in the coordinates $p_1, p_2, \ldots, u_1, u_2, \ldots$ has the form

$$\{p_i, p_k\} = 0, \quad \{u_i, u_k\} = 0, \quad \{u_i, p_k\} = J_{ki}(u), \quad i, k = 1, 2, 3, \ldots \quad (47)$$

**Observation 5.2** The system (46) is equivalent to the system (19) modulo the associativity conditions (18) and $J_{nm} = \Delta_{nm}$. So there is a strong interrelation between cohomological and Poisson structures associated with the subbundle $TBW_\emptyset$ for the big cell.

This fact has been checked by computer calculations up to $n, m = 11$. We do not have formal proof of this statement.

Note that due to the properties of the Schur polynomials the Poisson tensor (47) is of the form

$$J_{ik}^* = \sum_m J_{mk}(u) \frac{\partial p_i^*}{\partial p_m} \quad (48)$$

A subclass of the Poisson tensors (48) for which $J_{mk}(u) = \frac{\partial u_k}{\partial x_m}$, i.e.

$$J_{ik}^* = \sum_m \frac{\partial p_i^*}{\partial p_m} \frac{\partial u_k}{\partial x_m}, \quad i, k = 1, 2, 3, \ldots \quad (49)$$

where $x_1, x_2, x_3, \ldots$ are new coordinates on $\mathcal{M}$, is of particular interest. First in the coordinate $x_i, p_i$ the Poisson structures (37), (47) take the form

$$\{p_i, p_k\} = 0, \quad \{x_i, x_k\} = 0, \quad \{x_i, p_k\} = \delta_{ki}, \quad i, k = 1, 2, 3, \ldots \quad (50)$$

i.e., the coordinates $p_i, x_i, i = 1, 2, 3, \ldots$ are the Darboux coordinates in $\mathcal{M}$. Second, the Jacobi conditions (38) are identically satisfied for the Ansatz $J_{ik}(u) = \frac{\partial u_k}{\partial x_i}$ while the algebraic constraints...
become the system of quasilinear equations

\[
\frac{1}{m} \frac{\partial u_n}{\partial x_m} - \frac{1}{n} \frac{\partial u_m}{\partial x_n} = 0,
\]

\[
\frac{1}{m} \frac{\partial u_{n-1}}{\partial x_m} - \frac{1}{n} \frac{\partial u_{n-1}}{\partial x_n} + \sum_{k=1}^{n-2} \frac{1}{k} u_{m-k-1} \frac{\partial u_{n-1}}{\partial x_k} - \sum_{k=1}^{n-2} \frac{1}{k} u_{n-k-1} \frac{\partial u_{m-1}}{\partial x_k} = 0, \quad n, m = 1, 2, 3, \ldots.
\] (51)

This system of equations coincides with that derived in [30] in a different manner. It was shown in [30] that the system (51) is equivalent to the dKP hierarchy. This fact provide us with an alternative proof of the Proposition 3.1.

Thus we have

**Observation 5.3** In the Darboux coordinates the system of equations (47), (46) characterizing the Poisson structure for the family of ideals \(I(\Gamma_{\infty})\) is equivalent to the dKP hierarchy with \(x_1, x_2, x_3, \ldots\) and \(u_1, u_2, u_3, \ldots\) playing the role of independent and dependent variables, respectively.

The sets of variables \((p_k^*, u_k)\) and \((p_k^*, x_k)\) play the dual roles in the description of the families of ideals \(I(\Gamma_{\infty})\). The former are canonical from the algebraic viewpoint while the latter are canonical within the interpretation of the family of ideals \(I(\Gamma_{\infty})\) as Poisson ideal. In virtue of the formulas (46), (49) the connection between these two sets of variables is provided by solutions of the dKP hierarchy.

This observation points out the deep interrelation between the theory of Poisson ideals for the families of algebraic curves in Sato Grassmannian and theory of integrable hierarchies and the role of Darboux coordinates in such interconnection. The variables \(x_k, k = 1, 2, 3, \ldots\) are deformation parameters within such an approach. They can be viewed as the local coordinates in the infinite-dimensional base space for coisotropic deformations of the associative algebra (1).

The Darboux coordinates has been used in [15] within the study of coisotropic deformations of the relations (11) viewed as equations defining structure constants of associative algebras. It was shown in [15] that for infinite-dimensional polynomial algebra in the Faà di Bruno basis for which structure constants \(C_{jk}^l\) are given by (17) the coisotropy condition (39) is equivalent to the associativity conditions (18) plus the exactness conditions

\[
\frac{\partial H_{ik}^n}{\partial x_l} = \frac{\partial H_{il}^n}{\partial x_k}, \quad i, l, n = 1, 2, 3, \ldots.
\] (52)

These conditions together with the algebraic relations \(nH_{ik}^n = iH_{il}^n\) imply the existence of a function \(F\) such that [15]

\[
H_{ik}^n = -\frac{1}{m} \frac{\partial^2 F}{\partial x_i \partial x_m}.
\] (53)

With such a form of \(H_{ik}^n\) the associativity conditions (18) are equivalent to the celebrated Hirota-Miwa bilinear equations (34).

This result indicates one more time the importance of the Darboux coordinates in the whole our approach. The detailed analysis of the Poisson structures for ideals of the families of algebraic curves in Birkhoff strata and their connection with the hierarchy of integrable equations will be given in the forthcoming paper.
Here we will present an illustrative example. In the subbundle \( TB_{\Gamma_1} \) one has the ideal

\[
I(\Gamma_\infty^1) = \langle F_{23}^1, h_4^{(1)}, h_5^{(1)}, \ldots \rangle 
\]

where

\[
F_{23}^1 = p_3^2 - p_2^3 - \mu_3 p_2 p_3 - \mu_2 p_3^2 - \mu_2 p_2 - \mu_0, \\
h_4^{(1)} = p_4 - p_2^2 - v_3 p_3 - v_1 p_2 - v_0
\]

and so on (see [29]).

The requirement that the family of ideals (54) is a Poisson ideal gives rise to an infinite hierarchy of systems of PDEs. The simplest of them which is equivalent to the condition \( \{ F_{23}^1, h_4^{(1)} \} \mid_{\Gamma_\infty^1} = 0 \) with the canonical Poisson bracket is given by (see also [29])

\[
\begin{align*}
\frac{\partial \mu_4}{\partial x_4} &= -\frac{2}{3} \frac{\partial}{\partial x_2} (\mu_2 \mu_3) - \frac{5}{9} \mu_4 \frac{\partial \mu_4}{\partial x_2} + \frac{4}{9} \mu_4 \frac{\partial \mu_4}{\partial x_3} + 2 \frac{\partial \mu_4}{\partial x_2} + \frac{4 \partial \mu_3}{\partial x_2} + \frac{4 \partial \mu_3}{\partial x_3} + \frac{\partial \mu_4}{\partial x_2} + \frac{8 \partial}{\partial x_2} \frac{\partial}{\partial x_3} \\
\frac{\partial \mu_3}{\partial x_4} &= -\frac{2}{3} \frac{\mu_4 \mu_3}{\partial x_2} + \frac{v_1}{\partial x_2} + \frac{2 \partial \mu_1}{\partial x_2} - 3 \frac{\partial \mu_3}{\partial x_2} - 2 \frac{\mu_3}{\partial x_2} + \frac{2 \mu_3}{\partial x_3} - \frac{\mu_4}{\partial x_3} + \frac{4 \mu_3}{\partial x_3} \\
\frac{\partial \mu_2}{\partial x_4} &= -\frac{2}{3} \frac{\mu_4 \mu_2}{\partial x_2} + \frac{2 \partial \mu_0}{\partial x_2} + \frac{v_1}{\partial x_2} - \frac{\mu_4}{\partial x_2} - \frac{\mu_4}{\partial x_3} + \frac{2 \mu_4}{\partial x_3} + \frac{2 \mu_4}{\partial x_3} + \frac{2 \mu_2}{\partial x_3} \\
\frac{\partial \mu_1}{\partial x_4} &= -\frac{2}{3} \frac{\mu_4 \mu_1}{\partial x_2} + \frac{2 \partial \mu_0}{\partial x_2} + \frac{v_1}{\partial x_2} - \frac{\mu_4}{\partial x_2} - \frac{\mu_4}{\partial x_3} + \frac{2 \mu_4}{\partial x_3} + \frac{\mu_4}{\partial x_3} + \frac{4 \mu_1}{\partial x_3} \\
\frac{\partial \mu_0}{\partial x_4} &= v_1 - \mu_0 - \frac{2}{3} \frac{\partial \mu_0}{\partial x_3} + \frac{2}{3} \frac{\partial \mu_0}{\partial x_3} - \frac{\mu_2}{\partial x_3} - \frac{\mu_2}{\partial x_3} + \frac{2}{3} \frac{\partial \mu_0}{\partial x_3} + \frac{4 \mu_4}{\partial x_3}
\end{align*}
\]

where \( v_1 = \frac{2}{3} \mu_3 - \frac{4}{9} \mu_4^2 + \frac{4}{9} \frac{\partial \mu_1}{\partial x_3} \) and \( v_0 \) is associated with a gauge freedom of the system.

For the stratum \( \Sigma_{1,2} \) the coisotropy condition (39) is given by pretty large system of equations. For example the condition

\[
\{ C_8, C_9 \} \mid_{\Gamma_\infty^1} = 0
\]

with the Poisson bracket in the Darboux coordinates \((p_3, p_4, p_5, \ldots, x_3, x_4, x_5, \ldots)\) and cyclic variable \( x_3 \), is equivalent to the system

\[
\partial_{x_i} H_5^i = \partial_{x_5} H_4^i, \quad i = 1, 2, 4, 5
\]
where

\[ H_4^4 = 3U_{x_4}U_{x_5}V_{x_4} - 2U_{x_4}V_{x_4}^2 - H_2^4U_{x_4} - 2U_{x_4}^2V_{x_5} - U_{x_4}U_{x_5}^2 + U_{x_4}^4 - H_2^4U_{x_5} + V_{x_5}^2 + V_{x_4}H_4^4, \]

\[ H_5^4 = 2H_2^4V_{x_4} - 6U_{x_5}V_{x_4}^2 + 5V_{x_4}U_{x_5}^2 - \frac{5}{3}V_{x_4}U_{x_4}U_{x_5}^2 - 2U_{x_5}H_2^4 + \frac{4}{3}U_{x_4}U_{x_4}^3 - H_1^4V_{x_5} \]

\[ + 3V_{x_4}U_{x_4}V_{x_5} - 2U_{x_5}U_{x_4}V_{x_5} + \frac{7}{3}V_{x_4}^3 - \frac{4}{3}U_{x_5}^3, \]

\[ H_1^5 = V_{x_4}^2 + 2H_2^4 + U_{x_4}V_{x_5} - 2U_{x_5}V_{x_4} + U_{x_5}^2 - U_{x_4}^3, \]

\[ H_2^5 = 2H_1^4U_{x_4} - V_{x_5}V_{x_4} + V_{x_5}U_{x_5} - V_{x_4}U_{x_4}^2, \]

\[ H_4^5 = 2H_2^4V_{x_4} - 7U_{x_5}V_{x_4}^2 + 6V_{x_4}U_{x_5}^2 - \frac{4}{3}V_{x_4}U_{x_4}U_{x_5}^2 - 2U_{x_5}H_2^4 + \frac{5}{3}U_{x_5}U_{x_4}^3 - H_1^4V_{x_5} \]

\[ + 4V_{x_4}U_{x_4}V_{x_5} - 3U_{x_5}U_{x_4}V_{x_5} + \frac{8}{3}V_{x_4}^3 - \frac{5}{3}U_{x_5}^3 - H_1^2U_{x_4}^2, \]

\[ H_5^5 = 5U_{x_5}V_{x_4}U_{x_4}^2 - 2V_{x_4}U_{x_4}^2 - 3H_2^2U_{x_4}^2 - 4U_{x_4}U_{x_5}^3 - 2U_{x_5}^2U_{x_5}^2 + 2U_{x_5}^5 \]

\[ + 2U_{x_4}U_{x_5}^2 - 2V_{x_4}H_1^4U_{x_4} + H_1^4 + 2V_{x_5}V_{x_4}^2 - 3V_{x_4}V_{x_5}U_{x_5} + V_{x_5}U_{x_5}^2 + H_1^4V_{x_5} \]

(59)

and \( \partial_{x_4}H_2^4 := \partial_{x_5}U \) and \( \partial_{x_4}H_3^1 := \partial_{x_5}U \) and \( \partial_{x_4}H_3^1 := \partial_{x_5}U \).

Finally, the requirement that the family of ideals \( I^2(\Gamma_2^2) \) for hyperelliptic curves is the Poisson ideal with respect to the canonical Poisson bracket gives rise to the infinite hierarchy of hydrodynamical type systems which is equivalent to that found in the paper [26].

Acknowledgments

The authors thank Marco Pedroni and Andrea Previtali for many fruitful discussions. The author also thanks the referees for useful suggestions. This work has been partially supported by PRIN grant no 28002K9KXZ and by FAR 2009 (Sistemi dinamici Integrabili e Interazioni fra campi e particelle) of the University of Milano Bicocca.

References

[1] B. G. Konopelchenko and G. Ortenzi, Birkhoff strata of Sato Grassmannian and algebraic curves, \texttt{arXiv:1005.2053}

[2] B. G. Konopelchenko and G. Ortenzi, “Algebraic varieties in the Birkhoff strata of the Grassmannian Gr(2): Harrison cohomology and integrable systems.” J. Phys. A 44, no. 46, 465201 (2011)

[3] B. G. Konopelchenko and G. Ortenzi, “Birkhoff strata of the Grassmannian Gr(2): Algebraic curves” Theor Math Phys 167 (3), pp. 785-799 (2011)

[4] A. Pressley and G. Segal, Loop groups. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986

[5] G. Segal and G. Wilson, “Loop groups and equations of KdV type.”, Inst. Hautes Études Sci. Publ. Math. No. 61, 5–65 (1985)
[6] W. V. D. Hodge and D. Pedoe, Methods of algebraic geometry, I, New York: Cambridge Univ. Press, (1947)
[7] I.R. Shafarevich, Basic algebraic geometry, I. Berlin: Springer Verlag (1977)
[8] P. Griffiths and J. Harris, Principles of algebraic geometry Pure and Applied Mathematics, New York: John Wiley & Sons (1978)
[9] J. Harris, Algebraic geometry: a first course. Springer-Verlag, Berlin (1992)
[10] V.E. Zakharov, “Benney equations and quasiclassical approximation in the inverse problem method.” (Russian) Funktsional. Anal. i Prilozhen. 14, no. 2, 15–24 (1980)
[11] I. M. Krichever, “The averaging method for two-dimensional “integrable” equations.” (Russian) Funktsional. Anal. i Prilozhen. 22, no. 3, 37–52, 96 (1988) translation in Funct. Anal. Appl. 22, no. 3, 200–213 (1989)
[12] Y. Kodama, “A method for solving the dispersionless KP equation and its exact solutions.” Phys. Lett. A 129, no. 4, 223–226 (1988)
[13] K. Takasaki and T. Takebe, “SDiff(2) KP hierarchy.” Int. J. Mod. Phys. A, 7, 889-922 (1992)
[14] I.M. Krichever, “The τ-function of the universal Whitham hierarchy, matrix models and topological field theories.” Comm. Pure Appl. Math. 47, no. 4, 437–475 (1994)
[15] B. G. Konopelchenko and F. Magri, Coisotropic deformations of associative algebras and dispersionless integrable hierarchies, Commun. Math. Phys., 274, 627-658 (2007)
[16] G. Hochschild, “On the cohomology groups of an associative algebra.” Ann. of Math. 2 46, 58–67 (1945)
[17] D. K. Harrison, “Commutative algebras and cohomology.” Trans. Amer. Math. Soc. 104 191–204 (1962)
[18] M. Gerstenhaber, “On the deformation of rings and algebras.” Ann. of Math. 2 79, 59–103 (1964)
[19] A. Nijenhuis, and Richardson, R.W. Jr., “Commutative algebra cohomology and deformations of Lie and associative algebras.” J. Algebra 9, 42–53 1968
[20] M. Barr, “Harrison homology, Hochschild homology and triples.” J. Algebra 8, 314–323 (1968)
[21] V.P. Palamodov, “Deformations of complex spaces.” (Russian) Uspehi Mat. Nauk 31, no. 3(189), 129–194 (1976)
[22] M. Schlessinger and J. Stasheff, “The Lie algebra structure of tangent cohomology and deformation theory.” J. Pure Appl. Algebra 38, no. 2-3, 313–322 (1985)
[23] S. Gutt, “On some second Hochschild cohomology spaces for algebras of functions on a manifold.” Lett. Math. Phys. 39, no. 2, 157–162 (1997)
[24] M. Kontsevich, “Deformation quantization of algebraic varieties.” Lett. Math. Phys. 56, no. 3, 271–294 (2001)
[25] C. Frønsdal, “Harrison cohomology and abelian deformation quantization on algebraic varieties. 149–161, IRMA Lect. Math. Theor. Phys. 1, de Gruyter, Berlin (2002)
[26] Y. Kodama and B. G. Konopelchenko, “Singular sector of the Burgers-Hopf hierarchy and deformations of hyperelliptic curves” J. Phys.A: Math.Gen., 35, L489-L500 (2002)
[27] A. Weinstein: “Coisotropic calculus and Poisson groupoids.” J. Math. Soc. Japan 40, no. 4, 705–727 (1988)

[28] A. Givental and K. Bumsig, “Quantum cohomology of flag manifolds and Toda lattices.” Comm. Math. Phys. 168, no. 3, 609–641 (1995)

[29] B. G. Konopelchenko and G. Ortenzi, “Coisotropic deformations of algebraic varieties and integrable systems.” J. Phys. A 42, no. 41, 415207, 18 pp. (2009)

[30] B.G. Konopelchenko and F. Magri: “Dispersionless integrable equations as coisotropic deformations: extensions and reductions.” Theor. Math. Physics, 151, 803-819 (2007)