A new dual for quadratic programming and its applications

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Abstract This paper has twofold aims. First, we present a new dual for quadratic programs. In this notion, the dual variables are affine functions. We prove strong duality. As the new dual is intractable, we consider a subset of feasible set. We investigate the properties of this new problem We demonstrate the dual of the new concept is a semi-definite relaxation for the quadratic programs. In the second part, thanks to the new problem, we propose a branch and bound algorithm for concave quadratic programs. We establish that the algorithm enjoys global convergence in finite steps. The effectiveness of the method is shown for numerical problem instances.

Keywords Nonconvex quadratic programming · Duality · Semi-definite relaxation · Branch and bound method · Concave quadratic programming

1 Introduction

We consider the following quadratic program:

\[
\min_{x} \ x^{T}Qx + 2c^{T}x \\
\text{s.t. } Ax \leq b,
\]

(QP)

where \(Q\) is a symmetric \(n \times n\) matrix, \(A\) is an \(m \times n\) matrix, \(c \in \mathbb{R}^{n}\) and \(b \in \mathbb{R}^{m}\). Moreover, throughout the paper, it is assumed that the feasible set, \(X = \{x \in \mathbb{R}^{n} : Ax \leq b\}\), is nonempty and bounded. It is well-known that when \(Q\) is positive semi-definite, (QP) is solvable in polynomial time. Nevertheless, indefinite QPs, in which \(Q\) is an indefinite matrix, are NP-hard even

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for rank-1 problems \cite{22,25}. In this paper, our focus is on non-convex QPs. Duality plays a fundamental role in optimization, from both theoretical and numerical points of view \cite{19}. It serves as a strong tool in stability and sensitivity analysis of optimization problems. Needless to say, for convex problems, duality is employed in the wide class of numerical methods for obtaining or verifying an optimal solution.

It is well-known that the (Lagrangian) dual of a convex QP is also a convex QP. In addition, strong duality holds in this case \cite{3}. However, for non-convex case the dual problem may be trivial i.e. the objective function is equal to minus infinity while the primal has finite optimal solution.

The strong duality holds for convex problems under some constraint qualifications, but this property is still true for some non-convex problems. Optimizing a quadratic function on the level set of one quadratic function is an archetype. S-lemma guarantees strong duality under Slater condition \cite{24}. Due to the widespread applications of S-lemma, many researchers have been extending it to cover larger class of problems; For more information we refer the reader to \cite{24,31}.

In addition to the classical approaches for handling non-convex problems including cutting plane, branch and bound and so on, scholars have introduced two attractive methods which are capable of obtaining global optimal value for some class of problems. One of them that is effective for QPs with quadratic constraints is use of copositive program. It is established that QPs with quadratic constraints can be formulated as a linear optimization problem over the cone of completely positive matrices, see \cite{4} and references therein for more information. Although the new problem is convex, the cone of completely positive matrices is intractable. In fact, the new problem is also NP-hard. Another interesting method is Lasserre hierarchy \cite{17}. This method is able to tackle polynomial optimization problems, that is an optimization of a polynomial function on a given semi-algebraic set. It is well-known that polytopes are Archimedean, so on account of Putinar’s Positivstellensatz theorem, optimal value of \((QP)\) is obtained by the following convex optimization problem:

\[
\max \ell \quad \text{s.t. } x^T Q x + 2 c x - \ell = a_0(x) - \sum_{i=1}^m \sigma_i(x)(A_i x - b_i),
\]

where \(\Sigma[x]\) denotes cone of polynomials which are sums of squares \cite{17}. By virtue of Lasserre hierarchy, the optimal value of \((QP)\) can be obtained by solving the finite number of semi-definite programs. However, the dimension of semi-definite programs may increase dramatically and the provided upper bound respect to the dimension of \((QP)\) is exponential \cite{17}.

As mentioned earlier, a general QP is NP-hard. So, many scholars have proposed approximation methods. One of the interesting approximation methods is semi-definite relaxations. First, semi-definite relaxations were applied for some combinatorial problems \cite{18}. Due to their efficiency, these methods have been extended for QPs with quadratic constraints \cite{21}. For more discussion on the use of the semidefinite relaxations and their comparisons, we refer the
reader to the recent survey [1]. Moreover, semidefinite relaxations have been employed in branch and bound method for solving global QPs [5, 8].

The paper is organized as follows. After reviewing our notations, the new dual for QPs is introduced in Section 2. Some nice properties of the dual are provided. Section 3 is devoted to concave QPs. Thanks to the new dual, we introduce a new branch and bound bound approach. In Section 4, we illustrate the effectiveness of the method by presenting its numerical performance on the some concave QPs.

1.1 Notation

The following notation is used throughout the paper. The n-dimensional Euclidean space is denoted by \( \mathbb{R}^n \). Let \( A_i \) stand for \( i^{th} \) row of matrix \( A \). Vectors are considered to be column vectors. In addition, \( T \) denotes transposition. \( \mathbb{R}_+^n \) denotes non-negative orthant. Notation \( A \succeq B \) implies that matrix \( A - B \) is positive semidefinite. Furthermore, \( A \cdot B \) denotes the inner product of \( A \) and \( B \), i.e., \( A \cdot B = \text{trace}(AB^T) \).

For a set \( X \subseteq \mathbb{R}^n \), we use the notations \( \text{int}(X) \) and \( \text{cone}(X) \) for the interior and the convex conic hull of \( X \), respectively. \( \nabla f(\bar{x}) \) and \( \nabla^2 f(\bar{x}) \) denote gradient and Hessian of \( f \) at \( \bar{x} \).

2 A new dual of quadratic program

In this section, we present a new dual for (QP). Throughout the section, it is assumed that \( X \) is a bounded polyhedral. Inspired by Putinar’s Positivstellensatz theorem, we propose the following convex optimization problem as a dual of (QP),

\[
\begin{align*}
\max \ & \ell \\
\text{s.t.} \ & x^T Q x + 2c^T x - \ell + \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \in P[x], \\
& \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \leq 0 \ \forall x \in X, \quad (2)
\end{align*}
\]

where \( \alpha_i, i = 1, \ldots, m \), are affine functions and \( P[x] \) denotes non-negative polynomials on \( \mathbb{R}^n \). It is readily seen that The above problem is a convex problem with the infinite numbers of constraints. In the rest, we show that problem (2) is feasible and satisfies strong duality. Before we get to the proof, let us bring a lemma.

**Lemma 1** Let \( X = \{ x : Ax \leq b \} \) be a polytope and \( q(x) = x^T Q x + 2c^T x + c_0 \) be a quadratic function. Then there exist affine functions \( \alpha_i \) for \( i = 1, \ldots, m \) such that

\[
x^T Q x + 2c^T x + c_0 = \sum_{i=1}^m \alpha_i(x)(A_i x - b_i).
\]
Proof The existence of affine functions which give the above equality is equivalent to the consistency of the following linear system

$$\frac{1}{2} \sum_{i=1}^{m} \left( \frac{d_i}{f_i} \right) (A_i - b_i) + \frac{1}{2} \sum_{i=1}^{m} \left( \frac{A_i^T}{-b_i} \right) (d_i^T f_i) = \begin{pmatrix} Q & c \\ c & c_0 \end{pmatrix}$$

where variables $\left( \frac{d_i}{f_i} \right), i = 1, ..., m,$ are presentation of affine functions $\alpha_i$ in $\mathbb{R}^{n+1}$. For consistency, it is enough to show that the above system is full rank. On the contrary, let the above linear system is not full rank. So there is a non-zero symmetric matrix $D$ such that

$$\frac{1}{2} \sum_{i=1}^{m} D \cdot \left( \frac{d_i}{f_i} \right) (A_i - b_i) + \frac{1}{2} \sum_{i=1}^{m} D \cdot \left( \frac{A_i^T}{-b_i} \right) (d_i^T f_i) = 0$$

By the trace property of matrices, we have

$$\sum_{i=1}^{m} (A_i - b_i) D \left( \frac{d_i}{f_i} \right) = 0.$$ 

As the vectors $\{ \begin{pmatrix} A_1^T \\ -b_1 \end{pmatrix}, ..., \begin{pmatrix} A_m^T \\ -b_m \end{pmatrix} \}$ generates $\mathbb{R}^{n+1}$, $D$ must be zero. This implies consistency of the linear system and completes the proof.

The following theorem establishes the strong duality.

**Theorem 1** Let $X$ be a polytope. Then optimal value of problems (QP) and (2) are equal.

Proof Theorem follows form Lemma [4].

Although problem (1) is convex, it is not tractable. Indeed, it has infinite number of constraints. So to take advantage of this formulation, we need to adopt a procedure which one can handle the dual problem. One straightforward approach can be restriction of feasible set, that is one optimizes on a subset of the feasible set.

One method is to consider nonnegative affine functions on $X$. In the rest of paper we concentrate on this approach and thanks to it we propose a new algorithm for concave quadratic programs.

If the vertices of $X$ is also available one can consider the following set which is bigger than the aforementioned set. Let $v_1, ..., v_k$ denote vertices of $X$. It is easily seen that the affine functions $\alpha_i(x), i = 1, ..., m,$ which satisfies the following inequalities are feasible for problem (2),

$$\sum_{i=1}^{m} (A_i v_j - b_i) \alpha_i(x) \leq 0, \forall x \in X, j = 1, ..., k.$$ 

Non-homogenous Farkas’ Lemma provides explicit form of affine functions which satisfy the above inequalities. So it can be modeled by a finite number of linear inequalities. It is readily seen if $cone(\{Ax - b : x \in X\}) = -\mathbb{R}^{m+}_+$.
then $\alpha_i(x), i = 1, ..., m$, satisfies the above inequalities are nonnegative affine functions on $X$.

Let $A_+(X) \subseteq \mathbb{R}^{n+1}$ denote the set of non-negative affine function on $X$. It is easily seen that $A_+(X)$ is a polyhedral cone with nonempty interior [20]. To tackle problem (2), we consider the following problem,

$$\max \ell \quad \text{s.t.} \quad x^TQx + 2c^Tx - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_ix - b_i) \in P[x],$$

(3)

Considering affine functions instead of scalers for Lagrange multiplier is not new in the literature. Sturm et al apply an affine function with some properties as a multiplier to extend S-lemma [29]. However, to the best knowledge of author, affine functions have not been applied as dual variables in finite optimization theory.

By virtue of alternative theorem, the problem (3) can be formulated as the following semi-definite program:

$$\max \ell \quad \text{s.t.} \quad \frac{1}{2} \sum_{i=1}^{m} \left( e_i f_i \right) (A_i x - b_i) + \frac{1}{2} \sum_{i=1}^{m} \left( A_i^T f_i \right) (e_i x - b_i) + \left( \begin{array}{c} Q \ c \\ c^T - \ell \end{array} \right) \succeq 0,$$

$$\left( -e_i f_i \right) \in \text{cone}\left\{ \left( \begin{array}{c} A_i^T \\ b_i \end{array} \right), 1 \leq j \leq m, \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}, i = 1, ..., m,$$

It is worth mentioning that under non-emptiness of $X$ problem (3) is equivalent to the above problem. As the second constraint gives $\alpha_i$ explicitly, by replacement, the above problem can be written as follows

$$\max \ell \quad \text{s.t.} \quad \frac{-1}{2}(A^TYA + A^TY^TA) + \frac{1}{2}(Yb + y)^T A b - b^TYb) + \left( \begin{array}{c} Q \ c \\ c^T - \ell \end{array} \right) \succeq 0,$$

$Y \geq 0, y \geq 0$,

where $Y \in \mathbb{R}^{m \times m}$ and $y \in \mathbb{R}^m$. The next lemma shows that the boundedness of $X$ guarantees feasibility of dual problem.

**Proposition 1.** If $X$ is a polytope, then problem (3) is feasible.

**Proof.** As $X$ is bounded, there are scalers $f_i$ such that $A_ix + f_i$ is positive on $X$ for $i = 1, ..., m$, which implies that $A_ix + f_i \in A_+(X)$. Due to the boundedness of $X$, the system $Ad \leq 0$ does not have any non-zero solution. Thus, $H = \sum_{i=1}^{m} A_i^TA_i$ is positive-definite. By choosing $\gamma$ sufficiently large and suitable choice of $l$, $\alpha_i(x) = \gamma (A_ix + f_i), i = 1, ..., m$, and $\ell$ satisfy all constraints of problem (3).
In general, the proposition does not hold for QPs with unbounded feasible set and finite optimal value. The following QP illustrates this point.

\[
\begin{align*}
\min & \quad x_1 x_2 \\
\text{s.t.} & \quad x_1 = 0.
\end{align*}
\]

This dual can be interpreted via S-lemma. For given \( \alpha_i \in A_+(X), i = 1, \ldots, m, \) which are feasible for dual we have \( X \subseteq \{ x : \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \leq 0 \}, \) that is an overestimation of \( X \) via a quadratic function. As \( A_+(X) \) is a cone, the first constraint is merely S-lemma, here it is assumed that \( \text{int}(X) \neq \emptyset. \) In addition, problem (3) realizes the greatest optimal solution \( x^T Q x + 2 c^T x \) on a set of quadratic overestimations of \( X. \)

Moreover, if the interior of \( X \) is nonempty and involves a strictly convex function. It is easily seen that \( \mathcal{N}(X) \) is a closed, convex cone with nonempty interior. Moreover, if the interior of \( X \) is nonempty, then \( \mathcal{N}(X) \) is pointed, i.e. \( \mathcal{N}(X) \cap -\mathcal{N}(X) = \{ 0 \}. \)

\( \mathcal{N}(X) \) can be applied for obtaining a lower approximation of convex envelope of a quadratic function on a given polytope. For (QP), one can consider \( F : X \to \mathbb{R}, \) as a lower estimation of convex envelope, which defined as

\[
F(x) = \{ \max x^T Q x + c^T x + p(x) : p \in \mathcal{N}(X), \nabla^2 p + 2 Q \succeq 0 \}.
\]

Another problem which \( \mathcal{N}(X) \) may be useful is approximation of Löwner-John ellipsoid for a given polytope. For more details on Löwner-John ellipsoid, we refer the interested reader to \[2, 3\]. In the sequel, we will investigate other properties of problem (3).

**Proposition 2** Let \( \bar{x} \) and \( \alpha_i (i = 1, \ldots, m), \bar{l} \) be feasible points of problem (QP) and (3), respectively. Then \( \bar{x}^T Q \bar{x} + 2 c^T \bar{x} \geq \bar{l} \).

**Proof** It follows from Theorem 1.

The aforementioned proposition states that the optimal value (3) is a lower bound for the problem (QP) and vise versa (weak duality). In addition, if one of them is unbounded, the other one is infeasible and it is not the case both problem to be infeasible. However, in general, strong duality does not hold. The next theorem provides sufficient conditions for strong duality. Let \( I(x) \) denote active constraints at \( x, \) i.e. \( I(x) := \{ i : A_i x = b_i \}. \)

**Theorem 2** Let \( X \) be a polytope. Optimal value of (QP) and problem (3) are same if there exist \( \bar{x} \in \text{argmin}_{x \in X} x^T Q x + c^T x \) and \( e_i^T \bar{x} + f_i \in A_+(X) \) for \( i = 1, 2, \ldots, m \) such that

\[
Q + \frac{1}{2} \sum_{i=1}^{m} e_i A_i + \frac{1}{2} \sum_{i=1}^{m} A_i^T e_i^T \succeq 0,
\]

\[
Q \bar{x} + c + \sum_{i \in I(x)} (e_i^T \bar{x} + f_i) A_i^T + \sum_{i \in \{ 1, 2, \ldots, m \} \setminus I(\bar{x})} (A_i \bar{x} - b_i) e_i = 0,
\]

and

\[
e_i^T \bar{x} + f_i = 0, \quad \forall i \in \{ 1, 2, \ldots, m \} \setminus I(\bar{x}).
\]
Proof It is easily seen the above-mentioned conditions imply sufficient optimality conditions for the convex quadratic function $x^T Q x + 2c^T x + \sum_{i=1}^{m}(e_i x + f_i)(A_i x - b_i)$ at $\bar{x}$ with optimal value $\bar{x}^T Q \bar{x} + 2c^T \bar{x}$. In addition, $e_i^T x + f_i, i = 1, 2, ..., m$, and $\ell = \bar{x}^T Q \bar{x} + 2c^T \bar{x}$ are feasible for the dual problem. Thanks to the Proposition 3, the optimal value of both problems are equal and the proof is complete.

Checking the conditions of above theorem for a given point is not difficult. In fact, it can be done by testing feasibility of one semi-definite program, which there are polynomial time algorithms for it. Under convexity, the above theorem holds for any optimal solution and so we have strong duality in convex case. However, in general the theorem may not hold.

Another point concerning this result is that it provides sufficient conditions for global optimality. Some scholars provide sufficient global optimality conditions by virtue of semi-definite relaxation, see [33] and the references therein. As we will see in the sequel that the dual of (3) is a semi-definite relaxation of problem (QP). So most of the given results with a little modification can be applied for problem (3).

In general, it is possible that a semi-definite program does not realize its optimal value [2]. In the following result, we prove that problem (3) achieves its optimal value.

Proposition 3 Let $X$ be a polytope. Then problem (3) achieves its optimal value.

Proof Without loss of generality, we take into account the case that $\text{int}(X) \neq \emptyset$. If $X$ is not full dimensional, it is enough to consider the polytope on the affine space which it generates.

Let $\ell$ be the optimal value of (3). Thus, there exist sequences $\{\alpha_i^k\} \subseteq A_+(X), i = 1, ..., m$, and $\{\ell^k\}$ such that $\ell^k \rightarrow \ell$ and $x^T Q x + 2c^T x - \ell^k + \sum_{i=1}^{m} \alpha_i^k (A_i x - b_i) \in P[x]$. If the sequences $\{\alpha_i^k\}, i = 1, ..., m$, are bounded then, due to the closedness of $P[x]$ and $A_+(X)$, the proof is complete. Otherwise, some sequences are unbounded. Let $\mu_i^k = \frac{\alpha_i^k}{t_k}$ for $i = 1, ..., m$ where $t_k = \max_{1 \leq i \leq m} \|\alpha_i^k\|$. Without loss of generality, we may assume that $\mu_i^k \rightarrow \bar{\mu}_i$ for $i = 1, ..., m$ and at least one them is not equal to zero. Assume that $\bar{\mu}_j \neq 0$. Thus, $q(x) = \sum_{i=1}^{m} \bar{\mu}_i (A_i x - b_i) \in P[x]$. As $\text{int}(X) \neq \emptyset$, there is $\bar{x} \in X$ such that $\bar{\mu}_j (\bar{x}) > 0$ and $A_j \bar{x} < b_j$. Therefore, $q(\bar{x}) < 0$ which contradicts non-negativity of $q$.

It follows from the above proposition that all optimal solutions of problem (3) are bounded whether $\text{int}(X) \neq \emptyset$. One important question may arise about problem (3) is that whether optimal value of (3) for two different representations of polytope $X$ is same. Next theorem gives affirmative answer to the question.

Theorem 3 Let $\{x \in \mathbb{R}^n : \hat{A} x \leq \hat{b}\}$ and $\{x \in \mathbb{R}^n : \hat{A} x \leq \hat{b}\}$ be two different representations of polytope $X$. Then optimal value of problem (3) corresponding to these representations are equal.
Proof Let \( \alpha_i \) (\( i = 1, \ldots, m \)), \( \ell \) be feasible points of the following problem

\[
\begin{align*}
\max \ell \\
\text{s.t. } x^TQx + 2c^Tx - \ell + \sum_{i=1}^{m} \alpha_i(x)(\tilde{A}_i x - \tilde{b}_i) \in P[x], \\
\alpha \in A_+(X), \ i = 1, \ldots, m.
\end{align*}
\]  

(4)

Since the inequality \( \tilde{A}_i x \leq \tilde{b}_i \) is redundant for the system \( \tilde{A} x \leq \tilde{b} \), there are non-negative scalers \( Y_{ij} \) such that

\[
\tilde{A}_i x - \tilde{b}_i = \sum_{j=1}^{m} Y_{ij}(\tilde{A}_j x - \tilde{b}_j) - Y_{i0}.
\]

Similarly, as \( \alpha_i \in A_+(X) \) there are non-negative scalers \( W_{ij} \) such that \( \alpha_i(x) = \sum_{j=1}^{m} W_{ij}(\tilde{A}_j x - \tilde{b}_j) + W_{i0} \). By substituting the variables in the first constraint of (4), we have

\[
x^TQx + 2c^Tx - \ell + \sum_{i=1}^{m} \alpha_i(x)[\sum_{j=1}^{m} Y_{ij}(\tilde{A}_j x - \tilde{b}_j) - Y_{i0}] =
\]

\[
x^TQx + 2c^Tx - \ell + \sum_{j=1}^{m} (\tilde{A}_j x - \tilde{b}_j) \sum_{i=1}^{m} Y_{ij} \alpha_i(x) - \sum_{i=1}^{m} Y_{i0} \alpha_i(x) =
\]

\[
x^TQx + 2c^Tx - \ell + \sum_{j=1}^{m} (\tilde{A}_j x - \tilde{b}_j) \sum_{i=1}^{m} (Y_{ij} + W_{ij}) \alpha_i(x) - \sum_{i=1}^{m} Y_{i0} W_{i0} \in P[x].
\]

Hence, \( \sum_{i=1}^{m}(Y_{ij} + W_{ij}) \alpha_i(x) \in A_+(X) \) (\( i = 1, \ldots, m \)) and \( \ell \) are feasible points of (3) with respect to the representation \( X = \{ x \in \mathbb{R}^n : Ax \leq b \} \). Likewise, we have same result for the representation \( X = \{ x \in \mathbb{R}^n : Ax \leq \tilde{b} \} \). Therefore, the optimal value of (3) will be independent of any representation and the proof is complete.

Similar to the above proof, one can establish that If polytope \( X_1 \) is subset of polytope \( X_2 \), then \( \text{opt}(X_1) \leq \text{opt}(X_2) \). \( \text{opt}(X) \) stands for optimal value (3) with respect to polytopes \( X \). We call this property as inclusion property. In the following proposition, we investigate the value of problem (3) under affine transformation of \( X \).

**Proposition 4** The optimal value of problem (3) is preserved under any affine invertible transformation.

Proof Let \( T \) be an invertible affine transformation on \( \mathbb{R}^n \). We set \( Y = T^{-1}(X) \) and assume that \( T(y) = Hy + d \) for some invertible matrix \( H \) and \( d \in \mathbb{R}^n \). For \( \alpha_i \in A_+(X) \), \( i = 1, \ldots, m \), and \( \ell \) satisfying the following statement

\[
x^TQx + 2c^Tx - \ell + \sum_{i=1}^{m} \alpha_i(x)(A_i x - b_i) \in P[x],
\]
we have
\[ y^T \bar{Q}y + 2c^Ty + c_0 - \ell + \sum_{i=1}^{m} \mu_i(y)(\bar{A}_i y - \bar{b}_i) \in P[y], \]

where \( \mu_i = \alpha_i T \in A_+(Y) \), \( \bar{Q} = H^T QH \), \( \bar{c} = Hc + H^T Qd \), \( c_0 = d^T Qd + c^T d \), \( \bar{A} = AH \) and \( \bar{b} = b - Hd \). The following statement implies that \( \text{opt}(Y) \leq \text{opt}(X) \). Similarly, we have \( \text{opt}(X) \leq \text{opt}(Y) \) which completes the proof.

The following proposition states whether a polytope is singleton then problem (3) provides optimal value of (QP).

**Proposition 5** If \( X = \{\bar{x}\} \), then optimal value of (QP) and (3) are equal.

**Proof** From Proposition 4, we know the optimal value of (3) is independent of translation. So we may assume without loss of generality that \( \bar{x} = 0 \). In addition, thanks to Theorem 3, we may assume that \( X = \{x : x = 0\} \). We define \( \alpha_i, \mu_i \in A_+(X) \) for \( i = 1, ..., n \) corresponding to (QP) as follows:

\[
\alpha_i(x) = \begin{cases} 
0 & c_i \geq 0 \\
(-Qx)_i - 2c_i & c_i < 0
\end{cases} \quad \mu_i(x) = \begin{cases} 
(Qx)_i + 2c_i & c_i \geq 0 \\
0 & c_i < 0
\end{cases}
\]

Thus, \( \bar{x}^T Qx + 2c^T x + \sum_{i=1}^{m} \alpha_i(x)(x_i) + \sum_{i=1}^{m} \mu_i(x)(-x_i) = 0 \). On account of Proposition 3, strong duality holds, which is the desired conclusion.

Since (3) is a convex optimization problem, it is natural to ask which is the dual of it. In the rest of this section, we investigate the dual of problem (3). With a little computation and Shor decomposition, we find the following semi-definite program as a dual of (3),

\[
\begin{align*}
\min & \quad (Q \ c^T) \cdot \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \\
\text{s.t.} & \quad AXA^T - Ax^T - bA^T \geq -bb^T \\
& \quad -Ax \geq -b, \\
& \quad X \succeq \lambda^T.
\end{align*}
\]

Now, we show that above problem is a semi-definite relaxation of (QP). Let \( \bar{x} \) be a feasible point of problem (QP). It is easily seen for point \( \begin{pmatrix} \bar{x} & \bar{x}^T & \bar{x} \end{pmatrix} \) we have \( (Q \ c^T) \cdot \begin{pmatrix} \bar{x} & \bar{x}^T & \bar{x} \end{pmatrix} = \bar{x}^T Q \bar{x} + 2c^T \bar{x} \). In addition,

\[
A \bar{x} \bar{x}^TA^T - A \bar{x}^T - bA^T + bb^T = (A \bar{x} - b)(A \bar{x} - b)^T \geq 0,
\]

and easily seen the other constraints are also satisfied by the given point. It is worth noting that If the interior of polytope X is nonempty, then strong duality holds [2]. In fact, the optimal value of both problems (3) and (5) are equal. As the above problem is dual of (3) it is bounded from below, and there is no need to add some constraints. In general, semi-definite relaxations are
not necessary bounded from below \cite{13}. This relaxation is also called a strong relaxation by some scholars \cite{6}. One interesting point which one can infer in the calculation of the above dual is that the polar cone of \( \mathcal{N}(X) \) is equal to

\[
\text{cone}\{ \begin{pmatrix} X \\ x \\ 1 \end{pmatrix} \geq 0 : AXA^T - Axb^T - bx^T A^T \geq -bb^T, -Ax \geq -b \}. \]

One may wonder about the relationship between this relaxation with other semidefinite relaxations. It is easily seen that problem (5) is Shor relaxation of the following problem

\[
\min x^T Qx + 2c^T x \\
\text{s.t.} -Ax \geq -b,
\]

which is exactly (QP) with \( m^2 \) redundant constraints \cite{28}. Many scholars have applied these redundant constraints or subset of it for tackling QPs. Sherali et al. use the constraints to propose a linear program relaxation \cite{26}. Gorge et al. use convex combination of them as a cut for semi-definite relaxation \cite{13}. We refer the reader to \cite{13} for more information on applications of these redundant constraints in QPs. Moreover, the constraint \( Ax \leq b \) is redundant in the above problem \cite{26} and so it can be removed.

The most important inquiry in duality theory is under which conditions the duality gap can be zero or be as least as possible. In this context, one idea may be the replacement of non-negative affine function with non-negative convex quadratic functions on the given polytope. Similar to the affine case, a new dual can be formulated as a semi-definite program and all results hold for it. However, the number of variables is of the order of \( m^3 \) which makes this formulation less attractive. In addition, numerical implementations showed that the duality gap improvement was not considerable compare to the affine case. Pursuing this method by polynomials with degree of three or more is not practical since checking positivity of a polynomial with degree of greater than two on a given polytope or \( \mathbb{R}^n \) is not easy \cite{17}. However, this procedure can be continued by restriction on the some class of polynomials \cite{16}.

As mentioned earlier, problem (QP) realizes the greatest optimal solution of objective function on \( \mathcal{N}(X) \)'s members and whether \( \text{opt} - x^T Qx - 2c^T x \in \mathcal{N}(X) \) strong duality holds. As a result, enlargement of the aforementioned cone may lead to the improvement of duality gap.

Let \( \bar{x} \in \text{int}(X) \) and \( d \in \mathbb{R}^n \) be an arbitrary non-zero vector. The polytope \( X \) can be partitioned in two polytopes \( X_1 = \{d^T(x - \bar{x}) \leq 0, x \in X \} \) and \( X_2 = \{d^T(x - \bar{x}) \geq 0, x \in X \} \). It is readily seen that

\[
\mathcal{N}(X) \subseteq N(X_1) \cap N(X_2).
\]

The cone \( \mathcal{N}(X) \) is strictly included in \( \mathcal{N}(X_1) \cap \mathcal{N}(X_2) \) when strong duality does not hold. \( \mathcal{N}(X_1) \cap \mathcal{N}(X_2) \) can be characterized by a semi-definite system, but the number variables and constraints will be two times bigger than problem (5). It is easily seen the optimal value depends on the choice of \( \bar{x} \) and
In the same line, one can partition $X$ to $k$ polytopes and fattens $N(X)$ more. In this case, the number of variables will be of $O(km^2)$. For instance, for $\bar{x} \in \text{int}(X)$, one could consider two different hyperplanes which pass through the give point. As a result, the polytope is divided to four polytopes. We will take advantage of partitioning to develop a branch and bound algorithm for concave QPs.

Although problem (3) provides a lower bound for (QP), the number of variables is of $O(m^2)$, which makes this semi-definite program time-consuming in some cases. In the sequel, we propose other problems whose decision variable is less than problem (3). However, this problem does not equip us with a better lower bound.

Consider problem (QP). Let $L$ denote the subspace which is generated by eigenvectors of $Q$ with negative eigenvalues. Corresponding to (QP), we define the following problem

\[
\begin{align*}
\text{max} & \quad \ell \\
\text{s.t.} & \quad x^T Q x + 2c^T x - \ell + \sum_{i=1}^m \alpha_i(x)(A_i x - b_i) \in P[x], \\
& \quad \alpha_i \in \mathcal{A}_+(X), \quad i = 1, \ldots, m, \\
& \quad \nabla \alpha_i \in L, \quad i = 1, \ldots, m.
\end{align*}
\] (6)

The above problem reduces to problem (3) if $Q$ is negative definite. Nevertheless, for the class of problems which $Q$ has only one negative eigenvalue problem (6) has $2m$ variables. As a result, it would be more beneficial from time aspect to tackle this problem instead of (3). In the following proposition, we prove that the optimal value of problem (6) is bounded.

**Proposition 6** Let $X$ be a polytope. Then problem (6) has finite optimal value.

**Proof** Proposition 2 implies the optimal value of problem (6) is either finite or minus infinity. So it suffices to prove the existence of one feasible point. Let vectors $\{v_1, \ldots, v_k\}$ be a basis for $L$. As $X$ is bounded, by virtue of Farkas Lemma, there exist nonnegative constants $Y_{ij}$ such that $v_j = \sum_{i=1}^m Y_{ij} a_i^T$. Moreover, there are $f_j$ such that $v_j^T x + f_j \in \mathcal{A}_+(X)$ for $j = 1, \ldots, k$. For $\gamma$ sufficiently large, the matrix $Q + \gamma \sum_{j=1}^k v_j v_j^T$ is positive semi-definite. As $X$ is bounded, it is readily seen for suitable choice of $\ell$ the affine functions $\alpha_i = \gamma \sum_{j=1}^k Y_{ij}(v_j^T x + f_j), \quad i = 1, \ldots, m,$ satisfy all constraints of (6).

The following example demonstrates that the optimal value of problem (6) may be strictly less than that of problem (3).

**Example 1** Consider the following QP problem:

\[
\begin{align*}
\text{min} & \quad 2x_1 x_2 \\
\text{s.t.} & \quad x_1, x_2 \leq 1 \\
& \quad -x_1, -x_2 \leq 0.
\end{align*}
\]

The optimal value of above problem and (3) is zero. (Take into account $\alpha_1(x) = 0, \alpha_1(x) = 0, \alpha_1(x) = x_2, \alpha_1(x) = x_1, \ell = 0$.) It is readily seen that $\alpha \in \mathcal{A}_+(X)$
and $\nabla \alpha \in L$ if and only if $\alpha \in \text{cone}(\{x_1 - x_2 + 1, -x_1 + x_2 + 1\})$. As a result, for any feasible point of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ell$, we have $\ell \leq \frac{1}{\pi}$.

When we have box constraint, we could also formulate a semi-definite program with fewer decision variable compared to $\mathcal{X}$. In this case, one could consider the affine coefficient of constraint $x_k \leq u_k \ (-x_k \leq -l_k)\alpha_k$, in the following form

$$\alpha_k(x) = e_k x_k + f_i, \quad \alpha_k \in A_+(X),$$

where $e_k \in \mathbb{R}$. Similar to Proposition 1, the lower bound in this case is also finite.

### 3 A new algorithm for concave quadratic optimization

In this section, by virtue of the new dual, we propose a new algorithm for concave QPs. Throughout the section, it is assumed that $X$ is a polytope with nonempty interior and $Q$ is negative semi-definite.

Concave QPs are important both theoretically and practically. Concave QPs appear in many applications, for instance in fixed charge problems, risk management problems [10]. Furthermore, several classes of optimization problem including quadratic assignment problems can be formulated as a concave QP [7]. In addition, it is shown some class of QPs can be reformulated as concave QPs [10].

It is well-known that a concave QP realizes its minimum at some vertices [10]. So the problem is equivalent to the combinatorial problem of optimizing a quadratic function on the vertices of a given polytope. This problem, as mentioned earlier, is NP-hard. Many avenues for tackling concave QPs have been pursued. Common approaches for handling concave QPs are cutting plane methods, successive approximation methods and branch and bound or combination of them [15]. For details of these methods and other approaches, we refer the interested reader to [10, 15, 30].

Here, we propose a branch and bound (B&B) type algorithm for tackling concave QP. Before we pay to the details of algorithm, let us remind a definition.

**Definition 1** Let $\hat{x} \in X$ be a vertex. $\hat{x}$ is called a local vertex optimal if its value is less than or equal to that of adjacent vertices.

As mentioned before, we are developing a B&B algorithm for concave QPs. Branching, bounding and fathoming form main steps of B&B. Common branching approach for QPs is partitioning and for bounding is linear program relaxation; See [15, 27] for more details. Recently, some scholars have employed KKT optimality condition and semi-definite relaxation for branching and bounding, respectively [5, 8].

The outline of the these steps are as follows. Consider (QP). This problem is regarded as the root node of the B&B tree. For branching, we divide the polytope $X$ into two partitions, polytopes $X_1$ and $X_2$ such that
\(X_1 = \{x : x \in X, n^T x \leq n^T \bar{x}\}\) and \(X_2 = \{x : x \in X, n^T x \geq n^T \bar{x}\}\), where \(\bar{x} \in \text{int}(X)\) and \(n \neq 0\) is a random vector in \(\mathbb{R}^n\).

For bounding, we consider the following subproblem of (QP):

\[
\min \ x^T Q x + 2c^T x \\
\text{s.t.} \ Ax \leq b,
\]

where \(A\) and \(b\) are sub-matrix of \(\bar{A}\) and \(\bar{b}\), respectively. Let \(\bar{m}\) denote the number of columns of \(\bar{A}\). To obtain a lower bound for this node, we formulate a problem similar to (3) corresponding to the subproblem and optimal solution \(\bar{l}\) and \(\bar{\alpha}_i, i = 1, \ldots, \bar{m}\) are calculated. Then, the following convex QP is solved

\[
\min \ x^T Q x + 2c^T x + \sum_{i=1}^{\bar{m}} \bar{\alpha}_i (\bar{A}_i x - b_i) \\
\text{s.t.} \ Ax \leq b.
\] (7)

Let \(\hat{x}\) be a solution of problem (7). Next, a local vertex optimal point \(\hat{x} \in X\) is chosen such that \(\hat{x}^T Q \hat{x} + 2c^T \hat{x} \leq \bar{x}^T Q \bar{x} + 2c^T \bar{x}\). This step is not time consuming. In fact, there are some efficient approaches for obtaining \(\hat{x}\) [30]. We use the value \(\hat{x}^T Q \hat{x} + 2c^T \hat{x}\) to update the solution. Moreover, we use \(\bar{l}\) and other lower bounds obtained from fathomed nodes and child nodes to update the lower bound.

Finally, a subproblem is fathomed if the corresponding lower bound is bigger or equal to the optimal value which obtained until this step. Updating optimal solution is straightforward, it is enough to consider the minimum of provided solutions. To update the lower bound, we must consider the lower bound of all fathomed and child nodes. Strictly speaking, it is readily seen for father node \(n_p\) and its two child nodes, \(n_{p_1}\) and \(n_{p_2}\), we have \(l_p \leq \min\{l_{p_1}, l_{p_2}\}\), where \(l_p, l_{p_1}\) and \(l_{p_2}\) stand for the lower bound \(n_p, n_{p_1}\) and \(n_{p_2}\), respectively. Therefore, if \(\{l_{1j}\}_{j=1}^{k}\) and \(\{l_{ij}\}_{i=1}^{o}\) stand for the lower bound fathomed and new nodes, respectively, the new lower bound obtained by the following formula

\[
l = \min\{\min_{i=1}^{k} l_{ij}, \min_{i=1}^{o} l_{ij}\}.
\] (8)

It is easily seen that the lower bound is increasing in the process of algorithm. All steps of the algorithm are presented in Algorithm 1.
Algorithm 1 Branch and Bound Algorithm

1: Initialization:
\[ k = 1, l = -\infty, u = \infty, s = 1, \epsilon > 0, L = \{\}, \Pi_1 = \{(P_1, X)\} \text{ and } \Pi_2 = \{\}\]

2: while \( s = 1 \) do
3: \( \Pi_1 \neq \emptyset \) do
4: \((P_k, X_k) \leftarrow \text{select}(\Pi_1) \text{ and } \Pi_1 \setminus (P_k, X_k)\).
5: Solve semi-definite program \( \mathcal{P} \) corresponding to \((P_k, X_k)\) to get \( l_k \).
6: \( L \leftarrow l_k \) and delete its father lower bound if exists in \( L \).
7: Solve convex QP \( \mathcal{Q} \) and obtain a local vertex optimal \( \hat{x} \).
8: \( u \leftarrow \min(u, x^TQ\hat{x} + 2c^T\hat{x}) \).
9: if \( l_k + \epsilon < u \) then
10: Branch \((P_k, X_k)\) to two subproblems \((P_{k+1}, X_{k+1})\) and \((P_{k+2}, X_{k+2})\).
11: \( \Pi_2 \leftarrow \{(P_{k+1}, X_{k+1}), (P_{k+2}, X_{k+2})\} \) and \( k \leftarrow k + 2 \).
12: end if
13: end while
14: update \( l \) by the formula \( (8) \).
15: if \( \Pi_2 = \emptyset \) or other stoping criteria are satified then
16: \( s = 0 \).
17: else
18: \( \Pi_1 \leftarrow \Pi_2 \) and \( \Pi_2 \leftarrow \{\} \).
19: end if
20: end while

Here, \( \Pi_1 \) and \( \Pi_2 \) stand for the set of subproblems and the set \( L \) contains lower bounds of fathomed and child nodes. Other stoping criteria which mentions in Algorithm 1 can be the maximum number of nodes or the converge rate of optimal value.

In the rest of the section, we are investigating the finite convergent of the algorithm. To this end, we define function \( \Phi : \mathbb{R}^n \times \mathbb{R}^n_+ \rightarrow \mathbb{R} \) where \( \Phi(y, d) \) is defined as the optimal value of the following problem

\[
\max \ell \\
\text{s.t. } x^TQx + 2c^Tx - \ell + \sum_{i=1}^{n} [\alpha_i(x)(x - y_i - d_i) + \mu_i(x)(-x + y_i - d_i)] \in P[x] \\
\alpha_i, \mu_i \in \mathcal{A}_+(X(y, d)), 
\]

where polytope \( X(y, d) = \{x : y - d \leq x \leq y + d\} \). The well-definedness of \( \Phi \) on its domain follows from Proposition 1. The next lemma lists some properties of \( \Phi \).

Lemma 2 The function \( \Phi : \mathbb{R}^n \times \mathbb{R}^n_+ \rightarrow \mathbb{R} \) satisfies the following properties:

(i) For each \( y \in \mathbb{R}^n \), we have \( \Phi(y, 0) = y^TQy + 2c^Ty \);
(ii) \( \Phi \) is continuous on \( X \times \mathbb{R}^n_+ \).

Proof The first part follows immediately from Proposition 3.
For the second part, first, we prove the lower semi-continuity of \( \Phi \). Let \((\bar{y}, d) \in \mathbb{R}^n_+ \) and \( \bar{\alpha}_i, \bar{\mu}_i \in \mathcal{A}_+(X(\bar{y}, d)) \) and \( \ell \) denote an optimal solution of \( \mathcal{P} \). For
every $\epsilon > 0$, there are $\tilde{\alpha}_i, \tilde{\mu}_i \in A_+ (X(\tilde{y}, \tilde{d}))$ such that the quadratic function,

$$x^T Q x + 2 c^T x - (\ell - \epsilon) + \sum_{i=1}^n (\bar{\alpha}_i(x) + \hat{\alpha}_i(x))(x - \bar{y}_i - \bar{d}_i) + (\bar{\mu}_i(x) + \hat{\mu}_i(x))(-x + \bar{y}_i - \bar{d}_i),$$

is strictly convex and positive on $\mathbb{R}^n$. As a result, for small perturbation of $\bar{\alpha}_i, \bar{\mu}_i, \bar{y}$ and $\bar{d}$ the above-mentioned quadratic function belongs to $P(X)$, which implies

$$\liminf_{(y, d) \to (\tilde{y}, \tilde{d})} \Phi(y, d) \geq \ell - \epsilon.$$

As the following limit holds for each $\epsilon > 0$, $\Phi$ is lower semi-continuous at $(\tilde{y}, \tilde{d})$. Now, we prove the upper semi-continuity of $\Phi$. First, we consider the case that $\tilde{d} \in \text{int}(\mathbb{R}_+^n)$. Let the sequence $\{(y_k, d_k)\} \subseteq X \times \mathbb{R}_+^n$ tends to $(\tilde{y}, \tilde{d})$. Assume that $\alpha_i^k, \mu_i^k$ and $\ell^k$ are optimal solution of (3) corresponding to $(y_k, d_k)$. If for each $i = 1, \ldots, n$ the sequences $\{\alpha_i^k\}, \{\mu_i^k\} \subseteq \mathbb{R}^{n+1}$ are bounded, then without loss of generality we may assume that $\alpha_i^k \to \bar{\alpha}_i$, $\mu_i^k \to \bar{\mu}_i$ and $\ell^k \to \ell$. In addition, due to the lower semi-continuity set-valued mapping $X(\cdot, \cdot)$, it is readily seen that $\bar{\alpha}_i, \bar{\mu}_i \in A_+ (X(\tilde{y}, \tilde{d}))$ and

$$x^T Q x + 2 c^T x - \ell + \sum_{i=1}^n [\alpha_i(x)(x - \bar{y}_i - \bar{d}_i) + \mu_i(x)(x - \bar{y}_i - \bar{d}_i)] \in P[x],$$

which implies upper semi-continuity in this case. For the case that some sequences are unbounded, without loss of generality we may assume that $\ell_{k+1}^{-1}(\alpha_i^k) \to \bar{\alpha}_i$ and $\ell_k^{-1}$,$\mu_i^k \to \bar{\mu}_i$, where $\ell_k = \max_{1 \leq i \leq n} \{\|\alpha_i^k\|, \|\mu_i^k\|\}$. Moreover, there exists $\bar{\alpha}_i \neq 0$ and

$$q(x) = \sum_{i=1}^n [\bar{\alpha}_i(x)(x - \bar{y}_i - \bar{d}_i) + \bar{\mu}_i(x)(x - \bar{y}_i - \bar{d}_i)] \in P[x].$$

Similar to the proof of Proposition 3, as int$(X(\tilde{y}, \tilde{d})) \neq \emptyset$, there is $\bar{x}$ such that $q(\bar{x}) < 0$, which contradicts non-negativity of $q$. So, the unboundedness of sequences cannot occur, which establishes the upper semi-continuity of $\Phi$ in this case. Likewise, one can prove that for the case that some components of $\tilde{d}$ are zero, $\Phi$ is upper semi-continuous at $(\tilde{y}, \tilde{d})$ on $X \times \mathbb{R}_+^n$, where $\mathbb{R}_+^n = \{x \in \mathbb{R}_+^n : x_i = 0 \text{ if } d_i = 0\}$. Assume that the sequence $\{(y_k, d_k)\} \subseteq X \times \mathbb{R}_+^n$ tends to $(\tilde{y}, \tilde{d})$. We decompose the sequence $\{d_k\}$ in the form of $d_k = d_k^1 + d_k^2$ such that $d_k^2$ is projection of $d_k$ on $\mathbb{R}_+^n \setminus \{0\}$. It is easily seen $d_k^1 \to d$, so, we have

$$\limsup_{k \to \infty} \Phi(y_k, d_k) \leq \limsup_{k \to \infty} \Phi(y_k, d_k^1) \leq \Phi(\tilde{y}, \tilde{d}).$$

The first inequality is yielded from the inclusion property. Therefore, $\Phi$ is continuous on $X \times \mathbb{R}_+^n$ and the proof is complete.

**Theorem 4** The algorithm is finitely convergent.
Proof Consider the function $\Phi$ on the compact set $X \times (B \cap \mathbb{R}^n_{+})$, where $B$ stands for closed unit ball. Thanks to Lemma\textsuperscript{2} we can infer uniform continuity of $\Phi$ on the given domain. Additionally, we have the following property
\[ \forall \epsilon > 0, \exists \delta > 0, \forall x \in X, \forall d \in \mathbb{R}^n_{+}; \|d\|_{\infty} < 2\delta \Rightarrow |\Phi(x, d) - x^T Q x - 2c^T x| < \epsilon. \]

Since $X$ is compact, the objective function is Lipschitz continuous on it. (Assume Lipschitz modulus to be one.) Let $\Delta \subseteq X$ be a polytope with diameter less than $0.5 \min\{\epsilon, \delta\}$, that is, $\max_{x_1, x_2 \in \Delta} \|x_1 - x_2\| < 0.5 \min\{\epsilon, \delta\}$. As a result, there is $\bar{y} \in X$ and $d \in \mathbb{R}^n_{+}$ such that $\Delta \subseteq X(\bar{y}, d)$ and $\|d\|_{\infty} < 2\delta$. Let $\text{opt}(\Delta)$ denote the optimal value of (3) respect to the $\Delta$. Due to the inclusion property and the provided results, we have
\[
|\text{val}(\Delta) - \min_{x \in \Delta} (x^T Q x + 2c^T x)| \leq |\Phi(\bar{y}, d) - \min_{x \in \Delta} (x^T Q x + 2c^T x)|
\leq |\Phi(\bar{y}, d) - \bar{y}^T Q \bar{y} - 2c^T \bar{y} + \min_{x \in \Delta} (x^T Q x + 2c^T x)|
\leq \epsilon + \epsilon = 2\epsilon.
\]

Since after finite number of branching, every feasible set of subproblem, $\Delta$, is included in $X(y, d)$ for some $y \in X$ and $d \in \mathbb{R}^n_{+}$ ($\|d\|_{\infty} < \min\{\epsilon, \delta\}$), all nodes will be fathomed and algorithm will stop after finite steps.

It is worth noting for having the finite convergent, one should adopt a branching procedure which guarantees the diameters of polytopes tend to zero.

4 Computational results

In this section, we illustrate numerical performance of Algorithm 1 on three groups of test problems. Before we pay to the details of implementation, we should clarify the branching step. To partition full dimensional polytope $X = \{x : Ax \leq b\}$, we chose a hyperplane $L$ such that it passed through Chebyshev center of $X$ and its normal vector selected randomly from unit sphere. It is worth noting that Chebyshev center of $X$ is obtained by solving one linear program \cite{3}. Moreover, the code and the test problems with obtained solutions are publicly available at https://github.com/molsemzamani/quadprog.

We implemented the algorithm using MATLAB 2017b and ran our test problems on an Apple MacBook Air laptop with Intel Core i5 CPU, 1.4 GHz, and 4GB of RAM. To solve semi-definite program \cite{3} we employed CVX, a package for specifying and solving convex programs \cite{14}, and we opted for MOSEK as a selected solver. To solve convex QP \cite{7} we employed MATLAB’s function quadprog. In addition, CPLEX was used for solving linear programs. To evaluate the performance of Algorithm 1, we employed CPLEX 12.8 within MATLAB \cite{9}. We compared them from two aspects, computational time and quality of optimal value. We preferred CPLEX to other solvers as it had the best performance on the test problems \cite{32}. It worth noting that time report of Algorithm 1 involved modeling time which was taken by CVX.
For the first group, we selected twenty concave instances from Globallib folder in [8]. This folder contains all non-convex instances of Globallib test problems [12]. The dimension of problems ranged from five to fifty. We set 1000 for the maximum number of nodes for CPLEX. For algorithm 1, we set 64 for the maximum number of nodes and set $\epsilon = 10^{-5}$. Moreover, we terminated the algorithm if:

(C3) improvement of optimal value in three consecutive layers of B&B tree is less than 5$\epsilon$.

The performance of Algorithm 1 and CPLEX are summarized in Table 1. In the tables, $n$ denotes the dimension of instances and fval and time show the provided optimal value and spent CPU time in second for both methods, respectively. The columns lb and branch list the provided lower bound and the number of used nodes via algorithm 1, respectively. Moreover, STP column signifies which criterion terminated Algorithm 1. Here, criterion c1 denotes that the difference of the obtained optimal value and lower bound is less than $\epsilon$. c2 shows the algorithm used the maximum number of nodes.

The second group of instances involves twenty concave QPs with dense data. The test problems were generated as follows. The feasible set, $X$, was given by the following system

$$Ax \leq 10b, \quad \sum_{i=1}^{n} x_i \leq 100, \quad x \geq 0,$$

where square matrix $A$ was generated by MATLAB’s function randn and vector $b$ was generated by MATLAB’s function rand. randn generates a sample of
Table 2 Dense instances

| Instance | n | CPLEX         | Algorithm 1    |
|----------|---|---------------|---------------|
|          |   | fval | time | fval | time | lower | branch | STP |
| Ex1-40   | 40 | -2286.1 | 72.4 | -2286.1 | 69.0 | -2286.6 | 9 | c3  |
| Ex2-40   | 40 | -3813.5 | 260.2 | -3821.5 | 155.7 | -3961.8 | 25 | c2  |
| Ex3-40   | 40 | -2756.7 | 250.1 | -2756.7 | 35.1 | -2773.3 | 5 | c3  |
| Ex4-40   | 40 | -2341.7 | 69.4 | -2339.5 | 47.4 | -2400.7 | 7 | c3  |
| Ex5-40   | 40 | -3805.9 | 191.8 | -2808.2 | 37.7 | -2815.6 | 5 | c3  |
| Ex6-40   | 40 | -3431.9 | 93.7 | -4341.9 | 36.3 | -4346   | 5 | c3  |
| Ex7-40   | 40 | -2465.5 | 203  | -2465.5 | 35   | -2466.7 | 5 | c3  |
| Ex8-40   | 40 | -2554.7 | 498.4 | -2554.7 | 117  | -2837.5 | 15 | c3  |
| Ex9-40   | 40 | -4599.6 | 46.7 | -4599.6 | 39   | -4600.8 | 5 | c3  |
| Ex10-40  | 40 | -3446.6 | 45   | -3446.6 | 48.6 | -3446.6 | 7 | c1  |
| Ex1-45   | 45 | -4493.2 | 112.1 | -4493.2 | 91.6 | -4614.6 | 9 | c3  |
| Ex2-45   | 45 | -2705.9 | 791.2 | -2705.9 | 192.4 | -2849.2 | 21 | c3  |
| Ex3-45   | 45 | -3057.7 | 840.8 | -3057.8 | 129.1 | -3287.2 | 15 | c3  |
| Ex4-45   | 45 | -2714.2 | 571.0 | -2714.2 | 302.1 | -2941.3 | 31 | c3  |
| Ex5-45   | 45 | -3027.3 | 1140.3 | -3028.2 | 594.1 | -3261.9 | 47 | c2  |
| Ex6-45   | 45 | -2333.1 | 1054.5 | -2354.4 | 62.3 | -2564.9 | 7 | c3  |
| Ex7-45   | 45 | -3391.4 | 543.5 | -3391.4 | 245.4 | -3428.6 | 27 | c3  |
| Ex8-45   | 45 | -1947.9 | 460.5 | -1948.3 | 49.9 | -1955.4 | 5 | c3  |
| Ex9-45   | 45 | -2710.2 | 305.9 | -2710.2 | 105.6 | -2725.6 | 11 | c3  |
| Ex10-45  | 45 | -3087.9 | 491.4 | -3088.0 | 68.4 | -3201.7 | 7 | c3  |

a Gaussian random variable, with mean 0 and standard deviation 1 while rand generates a uniformly distributed random number between 0 and 1. We also generated the vector $c$ via randn function. We generated the square matrix $Q$ with the formula $Q = -2U^TDU$, where $U$ is an orthogonal matrix obtained by singular value decomposition of some random matrix and $D$ is a diagonal matrix whose components chosen by rand function. We generated twenty concave QPs in $\mathbb{R}^{40}$ and $\mathbb{R}^{45}$. The parameters were set as same as the former case. Table 2 reports computational performances of both methods.

For the last group of instances, we regarded concave QPs with sparsity. We selected twenty examples form RandQP folder in [8]. In the most of instances, matrix $Q$ were indefinite. We shifted eigenvalues to guarantee negative definiteness of $Q$. Moreover, we considered the instances without equality constraints. As there were box constraints in all instances, boundedness of feasible set preserved. We set parameters similar to the other cases. Table 3 gives computational performances of both approaches.

Both methods never failed on instances and the provided optimal values were approximately equal in all examples. However, the CPU time of both algorithms differed considerably in many instances. CPLEX outperformed for first and third groups of instance, but Algorithm 1 had better performance on dense problems. As we used CVX for solving problem 3 which tackles optimization problem from scratch, taking information father nodes may decrease computational time of Algorithm 1.
Table 3 Sparse instances

| Instance | n  | CPLEX         | Algorithm 1         |
|----------|----|---------------|---------------------|
|          |    | fval  time   | fval  time  lb  branch  STP |
| qp40-20-2-1 | 40 | -286.28 1.1 | -286.31 13.4 287.24 5 c3 |
| qp40-20-2-2 | 40 | -169.57 3.0 | -169.57 23.2 174.70 7 c3 |
| qp40-20-2-3 | 40 | -151.89 2.6 | -152.31 54.6 161.3 15 c3 |
| qp40-20-3-1 | 40 | -161.32 2.04 | -161.32 12.5 220.46 5 c3 |
| qp40-20-3-2 | 40 | -170.12 4.9 | -169.91 105.9 177.47 31 c3 |
| qp40-20-3-3 | 40 | -101.25 5.5 | -101.25 21.5 104.83 5 c3 |
| qp40-20-3-4 | 40 | -118.07 5.5 | -118.12 78.8 121.44 19 c3 |
| qp40-20-4-1 | 40 | -240.46 7.0 | -240.46 19.0 242.21 7 c3 |
| qp40-20-4-2 | 40 | -168.81 7.9 | -168.81 21.3 170.30 5 c3 |
| qp40-20-4-3 | 40 | -93.51 56.3 | -93.08 35.6 103.52 7 c3 |
| qp50-25-1-1 | 50 | -430.80 2.2 | -430.90 40.69 40.69 7 c3 |
| qp50-25-1-2 | 50 | -131.68 2.6 | -131.88 144.2 137.31 17 c3 |
| qp50-25-1-3 | 50 | -137.46 2.8 | -137.57 398.2 149.54 41 c2 |
| qp50-25-1-4 | 50 | -133.52 3.3 | -133.52 72.5 143.65 7 c3 |
| qp50-25-2-1 | 50 | -269.92 3.3 | -269.92 30.5 278.8 5 c3 |
| qp50-25-2-2 | 50 | -204.39 9.6 | -204.73 475.24 213.48 53 c2 |
| qp50-25-2-3 | 50 | -167.34 7.8 | -167.34 136.4 176.17 15 c3 |
| qp50-25-2-4 | 50 | -129.2 3.0 | -129.2 68.2 137.65 7 c3 |
| qp50-25-3-1 | 50 | -393.76 5.0 | -393.77 181.7 394.08 23 c3 |
| qp50-25-3-2 | 50 | -224.27 17.2 | -224.27 51.7 227.83 7 c3 |

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