Discrimination in Heterogeneous Games

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Abstract

In this paper, we consider coordination and anti-coordination heterogeneous games played by a finite population formed by different types of individuals who fail to recognize their own type but do observe the type of their opponent. We show that there exists symmetric Nash equilibria in which players discriminate by acting differently according to the type of opponent that they face in anti-coordination games, while no such equilibrium exists in coordination games. Moreover, discrimination has a limit: the maximum number of groups where the treatment differs is three. We then discuss the theoretical results in light of the observed behavior of people in some specific psychological contexts.

Keywords: Discrimination, diversity, working memory.

1. Introduction

Heterogeneous games have been studied in Iñarra and Laruelle (2012), Barreira da Silva Rocha and Laruelle (2013) and Iñarra, Laruelle and Zuazo-Garin (2015). These are games where the population is divided into types. A type is some particular feature that makes the individual to be perceived as different by the other individuals in the population. Examples could be a genotype, phenotype or behavior. The division is artificial in the sense that individuals have the same capacities and play the same game. But types can be distinguished and players can adapt their behavior according to the type of the opponent. The second characteristic of such heterogeneous games is that individuals lack self-perception (that is, they do not know their type) but recognize the type of the opponent.

In this paper, we study how individuals discriminate different types of opponents in symmetric Nash equilibria of heterogeneous games. We show that in coordination games players do not behave differently when they face different types of opponents. By contrast in anti-coordination games, equilibria in which players behave differently according to the type of opponent arise. In other words, an artificial division of the population may generate a real discrimination.

In anti-coordination games with three types, there are three kinds of equilibria: non discriminating equilibria, partially discriminating equilibria (where two types are treated equally and one is treated differently) and totally discriminating equilibria (where each of the three types is treated differently). The following question is whether this can be generalized when there are more than three types. The answer is negative: the maximum number of groups where the treatment differs is three. That is, discrimination has a limit: there is no totally discriminating equilibrium when the population is divided into four or more types.

The rest of the paper is organized as follows: in Section 2 we present the model; in Section 3, we obtain all possible symmetric Nash equilibria and Section 4 concludes with a discussion.

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2. Heterogeneous games with m types

Consider a population of \( n \) individuals of \( m \geq 2 \) different types. Let \( T \) be the set of types. The proportion of individuals of type \( t \) in \( T \) is given by \( x_t \left( \sum_{t \in T} x_t = 1 \right) \) with \( 1 < nx_t < n \). That is, we assume that there is strictly more than one individual of each type.

Any pair of individuals (whatever their types) plays the same symmetric game with the normalized matrix\(^1\).

| Cooperate | Defect |
|-----------|--------|
| 0         | \( y \) |
| \( z \)   | 0      |

This corresponds to a matrix of coordination (it is better if players choose the same action) if \( y, z < 0 \) and a matrix of anti-coordination (it is better if players choose different actions) if \( y, z > 0 \). Let

\[
\zeta = \frac{y}{y+z}.
\]

The following assumptions are made concerning the types: (i) Players do not know their own type; and (ii) Players recognize their opponents’ type. In this context, individuals cannot condition their behavior on their type. By contrast they can choose different probabilities of playing “cooperate” when they face different types of opponents. A strategy can thus be represented by \((\alpha_t)_{t \in T}\), where \( \alpha_t \) gives the probability of cooperation when facing an individual of type \( t \).

A strategy \((\alpha_t)_{t \in T}\) is said to be \( l \)-discriminating if there exists an \( l \)-partition\(^2\) of \( T \), \((T_1, \ldots, T_l)\) such that \( \alpha_s = \alpha_l \) for any \( s, t \in T_k \), and \( \alpha_s \neq \alpha_l \) if \( s \in T_j \) and \( t \in T_k \). In this case the probability of cooperation when facing an individual of type \( s \in T_k \) is denoted \( \alpha_k \). The special case \( l = 1 \) corresponds to \( \alpha_t = \alpha \) for any \( t \in T \) and the strategy is referred to as non-discriminating strategy as the probability of cooperation is identical whatever the type of the opponent. When \( l = m \) the strategy is said to be totally discriminating as the probability of cooperation is different for each type of opponent: \( \alpha_i \neq \alpha_j \) for any \( i, j \in T \).

A pair of individuals is selected at random to play the game. This is equivalent to assuming that an individual is randomly picked from a set of size \( n \), and then an opponent is picked from the remaining set of size \( n - 1 \). First consider that the individual is of type \( i \in T \) and an opponent of type \( j \in T \). The probability of the encounter is denoted \( p(i, j) \). We have

\[
p(i, j) = \frac{nx_i x_j}{n-1} \quad \text{and} \quad p(i, i) = \frac{(nx_i - 1)x_i}{n-1}.
\]

If an individual plays \((\alpha_t)_{t \in T}\) while the opponent plays \((\beta_t)_{t \in T}\), the individual sees that the opponent is of type \( j \) (and plays \( \alpha_j \)) while the opponent sees that the individual is of type \( i \) (and plays \( \beta_i \)). The payoff obtained by the individual in this encounter is obtained from matrix \(^1\). It is given by

\[
y, z > 0 \quad \text{if} \quad \gamma y (1 - \beta_i ) + z (1 - \alpha_j) \beta_i = z \beta_i + [y - (y+z) \beta_i] \alpha_j.
\]

This payoff is weighted by the probability of occurrence of the encounter, \( p(i, j) \); and the payoff of all possible encounters are summed to obtain the total expected payoff of the individual.

The expected payoff of an individual who plays \((\alpha_t)_{t \in T}\) while the opponent plays \((\beta_t)_{t \in T}\) is denoted by \( U((\alpha_t, \beta_t)_{t \in T}) \). We have

\[
U((\alpha_t, \beta_t)_{t \in T}) = \sum_{j \in T} \sum_{i \in T} p(i, j) [z \beta_i + y \alpha_j - (y+z) \beta_i \alpha_j].
\]

Substituting \(^2\) in \(^3\), we obtain after some algebra (see the Appendix for details):

\[
U((\alpha_t, \beta_t)_{t \in T}) = \sum_{j \in T} x_j \beta_j + \sum_{j \in T} \frac{x_j}{n-1} \left( (n-1)y + (y+z) \beta_j - n(y+z) \sum_{i \in T} x_i \beta_i \right) \alpha_j.
\]

The heterogeneous game is denoted \( \Gamma(n, T, (x_t)_{t \in T}, y, z) \).

\(^1\)Details of this normalization are given in Eichberger, Haller and Milne (1993).

\(^2\)\( T = T_1 \cup \ldots \cup T_l \) and \( T_j \cap T_k = \emptyset \) for any \( j \neq k \).
Lemma 1. in anti-coordination games. The proof of these results are based on the following lemma.

In coordination games there is only one equilibrium, in mixed strategies.

In anti-coordination games there are three equilibria, two in pure strategies and one in mixed strategies. In anti-

Lemma 2. \[ U((\alpha_t, \beta_t)_{t \in T}) = z \sum_{j \in T} x_j \beta_j + \frac{z}{n - 1} F_j((\beta_t)_{t \in T}) \alpha_j \]

where

\[ F_j((\beta_t)_{t \in T}) = (n - 1)y + (y + z)\beta_j - n(y + z) \sum_{i \in T} x_i \beta_i. \] (5)

The individual chooses \((\alpha_k)_{k \in T}\) in response to the opponent playing \((\beta_t)_{t \in T}\). When the individual sees

Theorem 1. Consider a heterogeneous game \(\Gamma(n, T, (x_t)_{t \in T}, y, z)\). \((\alpha^*, ..., \alpha^*)\) is a symmetric equilibrium

if and only if one of the following condition is satisfied:

1. \(y, z < 0\), and \([\alpha^* = \zeta; \alpha^* = 1; or \alpha^* = 0];\)

2. \(y, z > 0\), and \(\alpha^* = \zeta.\)

That is, if individuals do not take into account the different types, we obtain the classical results. In coordination games there are three equilibria, two in pure strategies and one in mixed strategies. In anti-

coordination games there is only one equilibrium, in mixed strategies.

In coordination games these equilibria are the only ones. Symmetric discriminating equilibria only arise in anti-coordination games. The proof of these results are based on the following lemma.

Lemma 1. Consider a heterogeneous game \(\Gamma(n, T, (x_t)_{t \in T}, y, z)\). If there exists a symmetric equilibrium

\((\alpha^*_t)_{t \in T}\) with \(0 < \alpha^*_t < 1 \) and \(0 < \alpha^*_t < 1 \) for \(i, j \in T\) then \(\alpha^*_t = \alpha^*_j.\)

A direct consequence is that the probability of cooperation at equilibrium can only take three different values: 0, 1 and an intermediate value. Therefore there does not exist any \(l\)-discriminating equilibrium for \(l > 3\).

Theorem 2. Consider a heterogeneous game \(\Gamma(n, T, (x_t)_{t \in T}, y, z)\). There is no totally discriminating equi-

librium for \(m > 3\).

Another less obvious consequence of the lemma is that there is no discriminating equilibrium in coordination games.

\footnote{The other two equilibria in pure strategies are not symmetric equilibria.}
4. Discussion

Setting individuals are rationally able to differentiate at most three partitions of types or three sets of information. As results on the ability to discriminate, items that an individual can store in the memory and remember for a short period of time is around three, in mental tasks, problem solving and planning. In Cowan (2010), it has been discussed why the number of decision and the working memory, i.e., the few temporarily active thoughts. The working memory is used is three. The Nash equilibria found in our results may be related to the mechanism of human individual there are no totally discriminating equilibria: the maximum number of partitions where the treatment differs is three. The main result of this paper is that when we have four types of players or more in a heterogeneous game, such link between our theoretical results and those empirical ones discussed in the latter provide an

Theorem 3. Consider a heterogeneous game $\Gamma(n, T, (x_t)_{t \in T}, y, z)$ with $y, z < 0$. There is no discriminating symmetric equilibrium.

The following theorems permit to find all discriminating equilibria in anti-coordination games. For each partition $(T_1, T_2)$ of $T$, we obtain one 2-discriminating equilibrium with $\alpha_1^* < \alpha_2^*$.

Theorem 4. Consider a heterogeneous game $\Gamma(n, T, (x_t)_{t \in T}, y, z)$ with $y, z > 0$. Let $(T_1, T_2)$ be a partition of $T$. The pair of strategies $((\alpha_1^*)_{t \in T}, (\alpha_2^*)_{t \in T})$ with

$$\alpha_t^* = \begin{cases} \alpha_1^* & \text{if } t \in T_1 \\ \alpha_2^* & \text{if } t \in T_2 \end{cases} \quad \text{and } \alpha_1^* < \alpha_2^*$$

is an equilibrium of game $\Gamma(n, T, (x_t)_{t \in T}, y, z)$ if and only if one of the following condition holds:

1. $\alpha_1^* = \frac{(n-1) \zeta - n \sum_{i \in T_1} x_i}{n \sum_{i \in T_1} x_i} \alpha_2^* = 1$; and $\sum_{i \in T_2} x_i < \left(1 - \frac{1}{n}\right) \zeta$;
2. $\alpha_1^* = 0$; $\alpha_2^* = 1$; and $\left(1 - \frac{1}{n}\right) \zeta < \sum_{i \in T_2} x_i < \left(1 - \frac{1}{n}\right) \zeta + \frac{1}{n}$;
3. $\alpha_1^* = 0$; $\alpha_2^* = \frac{(n-1) \zeta - n \sum_{i \in T_2} x_i}{n \sum_{i \in T_2} x_i} \alpha_2^* = \left(1 - \frac{1}{n}\right) \zeta + \frac{1}{n} < \sum_{i \in T_2} x_i$.

Similarly, for each partition $(T_1, T_2, T_3)$ of $T$, we obtain one 3-discriminating equilibrium with $\alpha_1^* < \alpha_2^* < \alpha_3^*$.

Theorem 5. Consider a heterogeneous game $\Gamma(n, T, (x_t)_{t \in T}, y, z)$ with $y, z > 0$. Let $(T_1, T_2, T_3)$ be a partition of $T$. The pair of strategies $((\alpha_1^*)_{t \in T}, (\alpha_2^*)_{t \in T})$ with

$$\alpha_t^* = \begin{cases} 0 & \text{if } t \in T_1 \\ \frac{(n-1) \zeta - n \sum_{k \in T_2} x_k}{n \sum_{k \in T_2} x_k} & \text{if } t \in T_2 \\ 1 & \text{if } t \in T_3 \end{cases}$$

is an equilibrium of game $\Gamma(n, T, (x_t)_{t \in T}, y, z)$ if and only if

$$\sum_{k \in T_2} x_k < \left(1 - \frac{1}{n}\right) \zeta \quad \text{and} \quad \sum_{i \in T_1} x_i < \left(1 - \frac{1}{n}\right) \left(1 - \zeta\right).$$

Note that the discriminating strategies at the equilibrium always include at least one pure action. Another point worth to mention is that, when $T_2 \neq \emptyset$ in Theorem 5 we obtain the results of Theorem 4 by either setting $T_1 = \emptyset$ or $T_3 = \emptyset$. By contrast when $T_2 = \emptyset$, Theorem 4 is not a special case of Theorem 5.

4. Discussion

The main result of this paper is that when we have four types of players or more in a heterogeneous game, there are no totally discriminating equilibria: the maximum number of partitions where the treatment differs is three. The Nash equilibria found in our results may be related to the mechanism of human individual decision and the working memory, i.e., the few temporarily active thoughts. The working memory is used in mental tasks, problem solving and planning. In Cowan (2010), it has been discussed why the number of items that an individual can store in the memory and remember for a short period of time is around three, despite the reasons for that fact remaining unclear in psychological science. The latter is in line with our results on the ability to discriminate.

Such link between our theoretical results and those empirical ones discussed in the latter provide an additional bridge between the fields of classic and evolutionary game theory. Our results show that individuals are rationally able to differentiate at most three partitions of types or three sets of information. As

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The equilibrium with $\alpha_1^* > \alpha_2^*$ is obtained for partition $(T_2, T_1)$ of $T$. 

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Arthur (1994) points out, under complicated problems, the type of rationality assumed in classic economics demands much of human behavior and breaks down. Beyond a certain level of complexity, human logical capacity ceases to cope and psychologists tend to agree that humans think inductively with bounded rationality, simplifying the problem (Bower and Hilgard, 1981; Holland et al., 1986; Rumelhart, 1980; Schank and Abelson, 1977).

Thus, on the one hand, economic agents do rationally maximize their utility or profit functions, on the other hand, the collection of information on the possible ways that the utility function can be derived and built might be too large for an individual to deal with, making him unable to identify all the possibilities and ending up with a narrower set of strategies available to choose from. As a consequence, when an agent chooses to play some strategy, despite the fact that he selects the one that rationally maximizes his utility, he is not fully aware if he is maximizing or not the utility function that provides him with the largest possible maximized profit. This creates room for the so-called bounded rationality in the literature. The role of natural selection then links evolutionary and classic game theory in dynamic models such as the replicator dynamics by selecting the strategy(ies) which profit maximizing function(s) outperform(s) in the long run, when the static stage-game is repeatedly played over time. Such adaptive process replaces profit maximization at the individual level in classic static games with profit maximization at the overall population level in evolutionary dynamic games.

Certainly, other examples in different fields of science can be found and related to our results, although in a further research paper we suggest the focus might be on understanding how the results and the ways of discriminating change when the players are aware of their own type as well.

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Appendix:

Equation (4): Substituting (2) in (3), we obtain

$$U((\alpha_t, \beta_t)_{t \in T}) = \sum_{j \in T} \sum_{i \neq j} \frac{nx_i x_j}{n-1} [z \beta_i + y \alpha_j - (y + z) \beta_i \alpha_j] + \sum_{j \in T} \frac{(nx_j - 1)x_j}{n-1} [z \beta_j + y \alpha_j - (y + z) \beta_j \alpha_j]$$

$$= \left\{ \begin{array}{ll} \frac{1}{n-1} \sum_{j \in T} \left[ \sum_{i \neq j} n x_i - 1 \right] z \beta_j + \frac{1}{n-1} \sum_{j \in T} \left[ \sum_{i \neq j} n x_i - 1 \right] y \alpha_j & \text{if } y \neq 1 \text{ and } \alpha^* = \alpha^* \neq \alpha^* \text{ for some } i, j \in T; \\
\frac{1}{n-1} \sum_{j \in T} n x_j (y + z) \left[ \sum_{i \neq j} x_j \beta_j \right] \alpha_j + \frac{1}{n-1} \sum_{j \in T} x_j (y + z) \beta_j \alpha_j & \text{if } y = 1 \text{ and } \alpha^* = \alpha^* \neq \alpha^* \text{ for some } i, j \in T. \end{array} \right.$$

$$= \left\{ \begin{array}{ll} z \sum_{j \in T} \beta_j + \sum_{j \in T} \frac{x_j}{n-1} \left[ (n-1)y + (y + z) \beta_j - n(y + z) \sum_{i \in T} x_i \beta_i \right] \alpha_j & \text{if } y > 0 \text{ and } \alpha^* = \alpha^* \neq \alpha^* \text{ for some } i, j \in T; \\
\text{impossible given that } y + z < 0 \text{ and } \alpha^* = \alpha^* \neq \alpha^* \text{ for some } i, j \in T. \end{array} \right.$$
together, \((1 - \frac{1}{n}) \zeta < \sum_{i \in T_2} x_i < (1 - \frac{1}{n}) \zeta + \frac{1}{n}\). (3) We have \(\alpha_1^* = 0\) and \(0 < \alpha_2^* < 1\) if \(F_i((\alpha_i^*)_{i \in T}) = 0\) for \(i \in T_2\) and \(F_j((\alpha_i^*)_{i \in T}) < 0\) for \(j \in T_1\). The equality \(F_i((\alpha_i^*)_{i \in T}) = 0\) leads to \(\alpha_2^* = \frac{(n-1)\zeta}{n \sum_{i \in T_2} x_i - 1}\). Clearly \(\alpha_2^* > 0\) and the inequality \(\alpha_2^* < 1\) requires \((1 - \frac{1}{n}) \zeta + \frac{1}{n} < \sum_{i \in T_2} x_i\). We can easily check that \(F_j((\alpha_i^*)_{i \in T}) < 0\) for \(j \in T_1\). 

**Proof of Theorem 5.** Suppose that there exists a discriminating symmetric equilibrium \(((\alpha_i^*)_{i \in T}, (\alpha_i^*)_{i \in T})\) with \(\alpha_i^* = \alpha_1^*\) for \(t \in T_1\) and \(\alpha_i^* = \alpha_2^*\) for \(t \in T_2\); and \(\alpha_i^* = \alpha_3^*\) for \(t \in T_3\); \(\alpha_1^* < \alpha_2^* < \alpha_3^*\). Then there is only one possibility: \(\alpha_1^* = 0\) and \(\alpha_2^* = \lambda^*\) and \(\alpha_3^* = 1\). The following conditions must hold: (i) \(F_i((\alpha_i^*)_{i \in T}) < 0\) for \(i \in T_1\); (ii) \(F_j((\alpha_i^*)_{i \in T}) = 0\) for \(j \in T_2\); and (iii) \(F_k((\alpha_i^*)_{i \in T}) > 0\) for \(k \in T_3\). Condition \(F_j((\alpha_i^*)_{i \in T}) = 0\) for \(j \in T_2\) gives \((n-1)\zeta - n(y+z)\lambda^* - n(y+z) \sum_{t \in T} x_t \alpha_t^* = 0\). After some algebra, we obtain \(\lambda^* = \frac{(n-1)\zeta - n \sum_{t \in T_2} x_t}{n \sum_{t \in T_2} x_t - 1}\). The condition \(\lambda^* < 1\) can be written as \((n-1)\zeta - n \sum_{t \in T_3} x_t < n \sum_{t \in T_2} x_t - 1\). This gives the condition \(\sum_{t \in T_1} x_i < (1 - \frac{1}{n}) (1 - \zeta)\). Condition \(\lambda^* > 0\) can be written as \((n-1)\zeta - n \sum_{t \in T_3} x_t > 0\) or \(\sum_{t \in T_3} x_t < (1 - \frac{1}{n}) (1 - \zeta)\). Finally two conditions remain to be checked: \(F_i((\alpha_i^*)_{i \in T}) < 0\) for \(i \in T_1\) and \(F_k((\alpha_i^*)_{i \in T}) > 0\) for \(k \in T_3\). Note that for \(i \in T_1, j \in T_2\) and \(k \in T_3\) we have \(F_i((\alpha_i^*)_{i \in T}) - F_j((\alpha_i^*)_{i \in T}) = -(y+z)\lambda^* < 0\) and \(F_k((\alpha_i^*)_{i \in T}) - F_j((\alpha_i^*)_{i \in T}) = (1 - \lambda^*) (y+z) > 0\). Given that \(F_j((\alpha_i^*)_{i \in T}) = 0\), we have \(F_i((\alpha_i^*)_{i \in T}) < 0\) for \(i \in T_1\) and \(F_k((\alpha_i^*)_{i \in T}) > 0\) for \(k \in T_3\).