mth roots of the identity operator and the geometry conjecture

Stephen Simons *

Abstract
In this paper, we give three different new proofs of the validity of the geometry conjecture about cycles of projections onto nonempty closed, convex subsets of a Hilbert space. The first uses a simple minimax theorem, which depends on the finite dimensional Hahn-Banach theorem. The second uses Fan’s inequality, which has found many applications in optimization and mathematical economics. The third uses three results on maximally monotone operators on a Hilbert space.

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1 Introduction

Let $H$ be a Hilbert space and $m \geq 2$. For $j = 1, \ldots, m$, let $C_j$ be a nonempty closed convex subset of $H$ and $P_j$ be the projection of $H$ onto $C_j$. The element $z = (z_1, z_2, \ldots, z_m)$ of $H^m$ is said to be a cycle if

$$z_1 = P_1 z_m, \quad z_2 = P_2 z_1, \quad \ldots, \quad z_m = P_m z_{m-1}.$$ 

The geometry conjecture, which was first formulated in [4, Conjecture 5.1.6] in 1997, stated that there exist $v_1, \ldots, v_m \in H$ such that $v_1 + \cdots + v_m = 0$ and

$$\{z_m : (z_1, \ldots, z_m) \text{ is a cycle}\} = C_m \cap (C_{m-1} + v_m-1) \cap (C_1 + \sum_{i=1}^{m-1} v_i). \quad (1)$$

This conjecture was finally solved in the affirmative in [3, Theorem 9, pp. 7–8]. In this paper, we give a proof of this conjecture which is simpler than that in [3], and extends the result to a more general situation. (See Theorem 7.)

Section 2 contains the algebraic results on which our analysis is based. It discusses certain linear operators on a vector space, $X$, which has no additional structure.

*Department of Mathematics, University of California, Santa Barbara, CA 93106-3080, U.S.A. Email: stesim38@gmail.com.
In Section 3, we assume that $X$ is endowed with a Hilbert space structure, and we assume that the operator $R$ introduced in Section 2 is an isometry. All the subsequent analysis depends on the disarmingly simple Equations (5) and (6), which will be used 6 times in what follows. It is important to realize that the Hilbert space $X$ is not necessarily the same as the Hilbert space, $H$, discussed at the beginning of this introduction — for the material in Section 4, $X$ will be $H^n$.

The first main result in Section 3 is the somewhat technical Lemma 6, which is about the support function of a nonempty closed, convex subset of $X$. Our proof of this goes by way of a simple minimax theorem, which depends on nothing more complicated than the Hahn-Banach theorem in finite dimensional spaces. We state the relevant minimax theorem in Theorem 5. We do not work in the Hilbert space $X$, but the Hilbert subspace $Y$ introduced in Section 2. Theorem 7 is the main result of the paper, and gives a generalization to this more abstract situation of the solution to the geometry conjecture.

In Section 4 we apply Theorem 7 to obtain a solution to the geometry conjecture. (See Theorems 9 and 10.)

In Section 5 we give two other ways of obtaining Theorem 7, the Fan’s inequality approach and the maximally monotone operator approach.

In Theorem 12 we state Fan’s inequality, and show how this can be used instead of Theorem 5 to obtain Lemma 6. This is the Fan’s inequality approach. In Lemma 16, we give a result very similar to Lemma 6 using three (non–trivial) results on maximally monotone operators on a Hilbert space. This corresponds more closely with the method used in [3], and is what we call the maximally monotone operator approach.

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All spaces in this paper will be real.

## 2 The average operator

Let $X$ be a linear space, $R: X \to X$ be linear and $R^m = I_X$. Let

$$A := \frac{1}{m} \sum_{i=1}^{m} R_i.$$ (A stands for average.)

Let $Y$ be the subspace $\{ y \in X : Ay = 0 \}$ of $X$. We now define $S: X \to X$ by $S := R - I_X$ and $Q: Y \to X$ by $Q := \frac{1}{m} \sum_{i=1}^{m-1} iR_i$. (Compare with [3] Equation (28), p. 4.)

**Lemma 1.** We have

$$S(X) \subset Y,$$ (2)

for all $y \in Y$, $Qy \in Y$ and $S(Qy) = y$. (3)

**Proof.** We have

$$mAS = \sum_{i=1}^{m} (R^{i+1} - R^i) = R^{m+1} - R = 0$$
and (2) follows immediately. Note that

\[ mAR = mRA = \sum_{i=1}^{m} R^{i+1} = \sum_{i=2}^{m} R^{i} + R^{m+1} = \sum_{i=2}^{m} R^{i} + R = mA. \]

Thus \( AR = RA = A \). It is easy to deduce from this that, if \( 1 \leq k \leq m \) then

\[ AR^{k} = A \text{ and } R^{k}A = A. \tag{4} \]

Let \( y \in Y \). Then \( A(Qy) = (AQ)y = \frac{1}{m} \sum_{i=1}^{m-1} iAR^{i}y. \) From (1), \( A(Qy) = \frac{1}{m} \sum_{i=1}^{m-1} iAy = 0 \), and so \( Qy \in Y \). Now

\[ mRQy = \sum_{i=1}^{m-1} iR^{i+1}y = \sum_{i=2}^{m} (i-1)R^{i}y = \sum_{i=2}^{m-1} (i-1)R^{i}y + my - y \]

and

\[ mQy = \sum_{i=1}^{m-1} iR^{i}y = Ry + \sum_{i=2}^{m-1} iR^{i}y. \]

Thus

\[ mS(Qy) = mRQy - mQy = \sum_{i=2}^{m-1} (i-1)R^{i}y + my - y - Ry - \sum_{i=2}^{m-1} iR^{i}y = my - y - Ry - \sum_{i=2}^{m-1} R^{i}y = my - mA y = my. \]

The proof of (3) is now completed by dividing by \( m \).

**Remark 2.** Using the same argument as in (3), we can prove that, for all \( y \in Y \), \( QSy = y \). If we define \( S_{0} := S|_{Y} \) then it follows that \( S_{0} \) and \( Q \) are bijections from \( Y \) onto \( Y \) and \( S_{0}^{-1} = Q \).

There are actually stronger versions of some of the relationships established above. (3) can be strengthened to \( SQ = QS = I_{X} - A \). (Compare [2] Proposition 3, p. 4.) It can also be proved that \( AQ = QA = \frac{m-1}{2}A \). (Compare [2] Proposition 2, p. 4.) Finally, \( A \) is a projection, that is to say, \( A^{2} = A \) and, further, \( SA = AS = 0 \).

The material in this section is related to the recent paper [1].

### 3 Results for Hilbert spaces

We now suppose that \( X \) is a Hilbert space. We have the following Theorem with two parts and a one–line proof. Theorem 3(a) is due to Xianfu Wang.

**Theorem 3.** Let \( \mathcal{R} : X \to X \) be a linear map and \( \mathcal{S} := \mathcal{R} - I_{X} \). Then:

(a) \( \mathcal{R} \) is nonexpansive if, and only if, for all \( x \in X \), \( \|Sx\|^{2} \leq -2\langle Sx, x \rangle \).

(b) \( \mathcal{R} \) is an isometry if, and only if, for all \( x \in X \), \( \|Sx\|^{2} = -2\langle Sx, x \rangle \).

**Proof.** Here is the one line: for all \( x \in X \),

\[ \|Sx\|^{2} + 2\langle Sx, x \rangle = \langle Sx, Sx + 2x \rangle = \langle Rx - x, Rx + x \rangle = \|Rx\|^{2} - \|x\|^{2}. \]

That completes the proof. □
We now suppose that $R, Y, S$ and $Q$ are as in Section 2, and that $R$ is a linear isometry. Obviously, $Y$ is closed, so $Y$ is a Hilbert space in its own right. Furthermore, $S$ is continuous on $X$ and $Q$ is continuous on $Y$. From Theorem 3(b),
\[
\|Sx\|^2 = -2\langle Sx, x \rangle.
\] (5)
Now let $y \in Y$. Setting $x = Qy$ in (5) and using (3),
\[
\|y\|^2 = -2\langle y, Qy \rangle.
\] Consequently,
\[
\langle Qy, y \rangle = -\frac{1}{2}\|y\|^2.
\] (6)
In what follows, we suppose that $C$ is a nonempty, closed convex subset of $X$ and $\sigma_C$ is the support function of $C$, defined by $\sigma_C(t) := \sup_{x \in C} \langle x, t \rangle$. Let
\[
D := \{y \in Y : \sigma_C(y) + \frac{1}{2}\|y\|^2 \leq 0\}.
\]
We now give some properties of the set $D$.

Lemma 4. The set $D$ is nonempty, convex and weakly compact.

Proof. Since $D \ni 0$, $D$ is nonempty. Since $\sigma_C + \frac{1}{2}\|\cdot\|^2$ is proper, convex and lower semicontinuous on $Y$, $D$ is convex and closed, hence weakly closed. Fix $c_0 \in C$. Then
\[
d \in D \implies \langle c_0, d \rangle + \frac{1}{2}\|d\|^2 \leq 0 \implies -\|c_0\|\|d\| + \frac{1}{2}\|d\|^2 \leq 0 \implies \|d\| \leq 2\|c_0\|,
\]
so $D$ is bounded and, consequently, weakly compact.

We now state a simple minimax theorem that depends only on the finite dimensional Hahn-Banach theorem. See [13, Theorem 3.2, p. 25] and [12, Theorem 3.1, p. 17]. The proof in [12] is somewhat more direct. There are many more minimax theorems - there is a survey of these in [11].

Theorem 5. Let $B$ be a nonempty convex subset of a vector space, $D$ be a nonempty convex subset of a vector space and $D$ also be a compact Hausdorff topological space. Let $f : B \times D \to \mathbb{R}$ be concave with respect to its first variable, and convex and lower semicontinuous with respect to its second variable. Then
\[
\sup_{b \in B} \min_{d \in D} f(b, d) = \min_{d \in D} \sup_{b \in B} f(b, d).
\] (7)

Lemma 6. There exist $d, e \in Y$ such that
\[
d = Se \in D
\] (8)
and
\[
\text{for all } x \in S^{-1}D, \quad \sigma_C(Se) + \langle Sx - Se, e \rangle - \sigma_C(Sx) \leq 0.
\] (9)

Proof. Define $f : D \times D \to \mathbb{R}$ by
\[
f(b, d) := \sigma_C(d) + \frac{1}{2}\|d\|^2 + \langle b, Qd \rangle - \sigma_C(b).
\]
Then $f$ is concave with respect to its first variable, and convex and lower semi-continuous with respect to its second variable. Let $b$ be an arbitrary element of $D$. Then, from (10),

$$\min_{d \in D} f(b, d) \leq f(b, b) = \sigma_C(b) + \frac{1}{2} \|b\|^2 + \langle b, Qb \rangle - \frac{1}{2} \|b\|^2 = 0.$$  

(11)

Thus $\sup_{b \in D} \min_{d \in D} f(b, d) \leq 0$. From Theorem 5, $\min_{d \in D} \sup_{b \in D} f(b, d) \leq 0$. Consequently, there exists $d \in D$ such that,

$$\sup_{b \in D} \left[ \sigma_C(d) + \frac{1}{2} \|d\|^2 + \langle b, Qd \rangle - \sigma_C(b) \right] \leq 0.$$  

(12)

From (6), $\frac{1}{2} \|d\|^2 = -\langle Qd, d \rangle$, and so we can rephrase the above as

$$\sup_{b \in D} \left[ \sigma_C(d) + \langle b - d, Qd \rangle - \sigma_C(b) \right] \leq 0.$$  

Let $e := Qd \in Y$. From (3), $Se = d \in D$, so (8) is satisfied and,

$$\sup_{b \in D} \left[ \sigma_C(Se) + \langle b - Se, e \rangle - \sigma_C(b) \right] \leq 0.$$  

(9) follows easily from this.

We now come to the main result of this section.

**Theorem 7.** Let $d, e$ be as in Lemma 6, $P_C$ be the projection of $X$ onto $C$ and $z \in C$. Then the following conditions are equivalent:

$$z = P_C Rz$$

(12)

$$z \in S^{-1}D$$

(13)

$$z \in S^{-1}d.$$  

(14)

**Proof.** (12) $\iff$ (13) It is a classical result in Hilbert space theory that (12) is equivalent to the statement “for all $c \in C$, $\langle Rz - z, c - z \rangle \leq 0$”, which is obviously equivalent to

$$\sigma_C(Sz) \leq \langle Sz, z \rangle.$$  

(15)

From (5), $\langle Sz, z \rangle = -\frac{1}{2} \|Sz\|^2$, so (15) is equivalent to: “$\sigma_C(Sz) \leq -\frac{1}{2} \|Sz\|^2$”, i.e., “$\sigma_C(Sz) + \frac{1}{2} \|Sz\|^2 \leq 0$”, which is exactly (13).

(13) $\iff$ (14) If (13) is true then, from the above equivalences, (15) is true. From (14), (13) and (9),

$$\sigma_C(Se) + \langle Sz - Se, e \rangle - \langle Sz, z \rangle \leq \sigma_C(Se) + \langle Sz - Se, e \rangle - \sigma_C(Sz) \leq 0.$$  

Since $z \in C$, $\langle Se, z \rangle \leq \sigma_C(Se)$. Consequently,

$$\langle Se, z \rangle + \langle Sz - Se, e \rangle - \langle Sz, z \rangle \leq 0,$$

from which $\langle S(z - e), z - e \rangle \geq 0$. It now follows from (6) that $\|S(z - e)\| = 0$, and so $Sz = Se = d$, giving (14). Finally, it is obvious that (14)$\implies$ (13).
4 Cycles of projections

Let \( X := H^m, C := C_1 \times \cdots \times C_m \subset H^m, \)
\[
R(x_1, x_2, \ldots, x_m) := (x_m, x_1, \ldots, x_{m-1}),
\]
and \( Y \) and \( d = (d_1, d_2, \ldots, d_m) \in Y \) be as in Lemma 16. \( R \) is clearly a linear isometry and \( R^m = I_X \). Then \( Ad = (\sum_{i=1}^m d_i)(1,1,\ldots,1) \) and so, since \( d \in Y \), \( \sum_{i=1}^m d_i = 0 \). Then, for all \( x = (x_1, x_2, \ldots, x_m) \in X \),
\[
S(x_1, x_2, \ldots, x_m) = (x_m - x_1, x_1 - x_2, \ldots, x_{m-1} - x_m). \tag{16} \]
As we postulated in Theorem 7, \( P_C \) is the projection from \( X \) onto \( C \). For all \( x = (x_1, x_2, \ldots, x_m) \in X \), \( P_C(x) = (P_1x_1, P_2x_2, \ldots, P_mx_m) \), and \( z \) is a cycle exactly when \( z = P_C R z \). (See 3 Equations (12), (14) and (16), p. 3.) We now have a simple characterization of cycles.

**Lemma 8.** \((z_1, z_2, \ldots, z_m)\) is a cycle if, and only if,
for all \( i = 1, \ldots, m \), \( z_i \in C_i \) and, for all \( i = 1, \ldots, m-1 \), \( z_i - z_{i+1} = d_{i+1} \). \tag{17}

**Proof.** Since \( \sum_{i=1}^m d_i = 0 \), it follows from (17) that \( z_m - z_1 = d_1 \) and the result is immediate from Theorem 7 [12] \( \iff \tag{13} \) and (16).

It is now convenient to make a notational change. Let \( v := -R^{m-1} d \), from which \( \sum_{i=1}^m v_i = 0 \) and \( d = -R v \). With this substitution, we get:

**Theorem 9.** \((z_1, z_2, \ldots, z_m)\) is a cycle if, and only if,
for all \( i = 1, \ldots, m, z_i \in C_i \) and, for all \( i = 1, \ldots, m-1 \), \( z_{i+1} = z_i + v_i \). \tag{18}

We now prove the geometry conjecture:

**Theorem 10.** We have
\[
\{ z_m : (z_1, \ldots, z_m) \text{ is a cycle} \} = C_m \cap (C_{m-1} + v_{m-1}) \cap (C_1 + \sum_{i=1}^{m-1} v_i). \tag{19}
\]

**Proof.** \((\subset)\) If \((z_1, z_2, \ldots, z_m)\) is a cycle then, from (18), for all \( k = 1, \ldots, m-1 \),
\[
z_m = z_k + \sum_{i=k}^{m-1} v_i \subset C_k + \sum_{i=k}^{m-1} v_i. \]
Since \( z_m \in C_m \),
\[
z_m \in C_m \cap (C_{m-1} + v_{m-1}) \cap (C_1 + \sum_{i=1}^{m-1} v_i). \tag{20}
\]
Thus we have established the inclusion \((\subset)\) in (19).

\((\supset)\) Suppose, conversely, that \( \zeta \in C_m \cap (C_{m-1} + v_{m-1}) \cap (C_1 + \sum_{i=1}^{m-1} v_i) \).
Then there exists \( w = (w_1, w_2, \ldots, w_m) \in C \) such that
\[
\zeta = w_m = w_{m-1} + v_{m-1} = \cdots = w_2 + \sum_{i=2}^{m-1} v_i \quad \text{and} \quad w_1 + \sum_{i=1}^{m-1} v_i. \tag{21}
\]
From (10), using the facts that \( \sum_{i=1}^m v_i = 0 \) and \( d = -R v \),
\[
Sw = S(w_1, w_2, \ldots, w_m) = (w_m - w_1, w_1 - w_2, \ldots, w_{m-1} - w_m) = (\sum_{i=1}^{m-1} v_i, -v_1, \ldots, -v_{m-1}) = (-v_m, -v_1, \ldots, -v_{m-1}) = -R v = d.
\]
It now follows from Theorem 7 [14] \( \iff \tag{12} \) that \( w = P_C R w \), i.e., \( w \) is a cycle.
Since \( \zeta = w_m \), we have established the inclusion \((\supset)\) in (19).
Remark 11. Even if there are no cycles, Lemma 16 still provides \( d \in Y \) satisfying (8) and (9). If we define \( v := -R^{m-1}d \) as above, \( v \) is known as the generalized gap vector. See [2] for more on this.

5 Other approaches leading to Theorem 7

Theorem 12 below was proved by Fan in [6, Theorem 5, p 525]. It is closely related to fixed point theorems, and has found many applications in optimization and mathematical economics. It first appeared in [7].

Theorem 12. Let \( D \) be a compact, convex subset of a topological vector space, and \( f: D \times D \to \mathbb{R} \). Suppose that \( f \) is quasiconcave with respect to its first variable and lower semicontinuous with respect to its second variable. Then

\[
\min_{d \in D} \sup_{b \in D} f(b, d) \leq \sup_{b \in D} f(b, b).
\]

It was actually proved in (10) and (11) in the proof of Lemma 6 that, for all \( b \in D \), \( f(b, b) = 0 \). Thus, from Theorem 12, \( \min_{d \in D} \sup_{b \in D} f(b, d) \leq 0 \), and we obtain (9) as before. This is the Fan’s inequality approach.

The statement of Lemma 16 below is identical with that of Lemma 6, except that (22) and (23) are in the reverse order to that of (8) and (9), and the quantifier “\( x \in X \)” in (22) is less restrictive than the quantifier “\( x \in S^{-1}D \)” in (9). So Lemma 16 can be used instead of Lemma 6 to obtain Theorem 7. Thus we obtain the maximally monotone operator approach.

The first of the results on maximal monotonicity that we will use is Rockafellar’s maximal monotonicity theorem, [9, Proposition 1, pp. 211–212], (see [15, Theorem 4.6, p. 10] and [16, Theorem 3.2, pp. 634–635] for recent developments), which we state in Theorem 13 below:

Theorem 13. Let \( f \) be a proper, convex lower semicontinuous function on a Banach space. Then the subdifferential of \( f \), \( \partial f \), is maximally monotone.

The second result on maximal monotonicity that we shall use follows from Rockafellar’s sum theorem, [10, Proposition 1, p. 77] (see [14, Theorem 8.4 (a)⇒(d), pp. 1036–1037] for recent developments), which we state in Theorem 14 below:

Theorem 14. Let \( S \) and \( T \) be maximally monotone operators on a Banach space and \( S \) have full domain. Then \( S + T \) is maximally monotone.

The final result on maximal monotonicity that we shall use is Minty’s theorem, [8] or [5, Theorem 21.1, pp. 311], which we state in Theorem 15 below:

Theorem 15. Let \( S \) be a maximally monotone operator on a Hilbert space \( Y \) then there exists \( y \in Y \) such that \( 0 \in y + Sy \).
Lemma 16. There exist \( d, e \in Y \) such that,

\[
\text{for all } x \in X, \quad \sigma_C(Se) + \langle Sx - Se, e \rangle - \sigma_C(Sx) \leq 0, \tag{22}
\]

and

\[
d = Se \in D. \tag{23}
\]

Proof. \( \sigma_C \) is proper, convex and lower semicontinuous on \( Y \). From Theorem 13, \( \partial \sigma_C \) is maximally monotone. From (6), for all \( y \in Y \),

\[
\langle -(Q - \frac{1}{2}I_Y)y, y \rangle = 0
\]

and so \(-Q - \frac{1}{2}I_Y\) is skew with full domain \( Y \) and hence maximally monotone and so, from Theorem 14, \(-Q - \frac{1}{2}I_Y + \partial \sigma_C\) is also maximally monotone, where \( \partial \sigma_C \) is the subdifferential of \( \sigma_C \) in the Hilbert space \( Y \).

Theorem 15, or consideration of the resolvent of \(-2Q - I_Y + 2\partial \sigma_C\) evaluated at 0, provides \( d \in Y \) such that

\[
0 \in \frac{1}{2}d + (-Q - \frac{1}{2}I_Y + \partial \sigma_C)(d), \quad \text{that is to say}, \quad 0 \in -Qd + \partial \sigma_C(d).
\]

Thus

\[
Qd \in \partial \sigma_C(d) \quad \text{and} \quad \sup_{y \in Y} [\sigma_C(d) + \langle y - d, Qd \rangle - \sigma_C(y)] \leq 0.
\]

If \( x \in X \) then, from (2), \( Sx \in Y \) and so

\[
\sup_{x \in X} [\sigma_C(d) + \langle Sx - d, Qd \rangle - \sigma_C(Sx)] \leq 0. \tag{24}
\]

Let \( e := Qd \in Y \). (Compare [3] Equation (40), p. 6.) From [3], \( Se = d \in Y \), so (22) follows from (24). If we substitute \( x = 0 \) in (22), we see that \( \sigma_C(Se) \leq \langle Se, e \rangle \). From [3], \( \langle Se, e \rangle = -\frac{1}{2}\|Se\|^2 \). Thus \( \sigma_C(Se) \leq -\frac{1}{2}\|Se\|^2 \), and (23) follows. \( \square \)

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