A NOTE ON A PROBLEM OF SAFF AND VARGA
CONCERNING THE DEGREE
OF COMPLEX RATIONAL APPROXIMATION
TO REAL VALUED FUNCTIONS
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Saff and Varga [1977, 1978] discovered the surprising fact that given a
real valued function on \([-1, 1]\) one can sometimes obtain a better rational ap­
proximation to \(f\) of a given type by allowing complex coefficients than by re­
stricting the coefficients to be real. In this note we point out a connection
between this result and the “Trefethen effect” (Trefethen [1981a, 1981b]) of near
circular error curves for best approximation on the unit disc, which as Trefethen
has shown is in turn closely related to the Carathéodory-Fejér Theorem and its
generalisation to meromorphic approximation due to Takagi.

We consider a compact simply connected set \(\Omega\) in the complex plane
which is symmetric about the real line (i.e. \(z \in \Omega\) iff \(\bar{z} \in \Omega\)). We assume further
that the complement of \(\Omega\) is simply connected in the extended plane and that
\(\Omega\) is a Faber domain (see e.g. Gaier [1980]). Finally if \(\psi\) denotes the confor­
mal mapping of the complement of the unit disc \(D\) onto the complement of \(\Omega\) such
that \(\psi(\infty) = \infty\), \(\lim_{w \to \infty} \psi(w)/w\) finite real and positive, we assume for sim­pli­
city that \(\psi\) can be extended continuously to the boundary of the disc. All
these properties are, of course, satisfied for the main case of interest here, \(\Omega = I = [-1, 1]\).

\(\mathcal{A}(\Omega)\) will denote the set of functions continuous on \(\Omega\) and analytic at
interior points, and \(\mathcal{A}^R(\Omega)\) the subspace of functions satisfying \(f(\bar{z}) = \overline{f(z)}\).
\(\Psi : \mathcal{A}(D) \to \mathcal{A}(\Omega)\) is the Faber transform (Gaier [1980]); \(E_{mn}^C(f; \Omega)\) the error
of best (Chebyshev) approximation to \(f \in \mathcal{A}^R(\Omega)\) from \(\mathcal{R}_{mn}(\Omega)\), the set of
type \((m, n)\) rational functions with no poles in \(\Omega\), and \(E_{mn}^R(f; \Omega)\) the correspon­
ding error of best approximation from the subset of rational approximations with
real coefficients.

Saff and Varga [1977, 1978] observed that it is possible to find \(f \in \mathcal{A}^R(I)\)
for which \(E_{nn}^C(f; I) < E_{nn}^R(f; I)\) (e.g. \(f(x) = x^2\), \(n = 1\)). Some authors attacked
the case \(n = 1\) by largely geometric arguments and showed among other things