ERGODIC COMPONENTS OF PARTIALLY HYPERBOLIC SYSTEMS

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ABSTRACT. This paper gives a complete classification of the possible ergodic decompositions for certain open families of volume-preserving partially hyperbolic diffeomorphisms. These families include systems with compact center leaves and perturbations of Anosov flows under conditions on the dimensions of the invariant subbundles. The paper further shows that the non-open accessibility classes form a $C^1$ lamination and gives results about the accessibility classes of non-volume-preserving systems.

Invariant measures are important objects in the study of dynamical systems. Often, these measures are ergodic, allowing a single orbit to express the global behaviour of the system. However, this is not always the case. For instance, a Hamiltonian system always possesses a smooth invariant measure, but a generic smooth Hamiltonian yields level sets on which the dynamics are not ergodic [35]. Any invariant measure may be expressed as a linear combination of ergodic measures and while such a decomposition always exists, it is not, in general, tractable to find it. For partially hyperbolic systems, there is a natural candidate for the ergodic decomposition given by the accessibility classes of the system. This paper analyzes certain families of partially hyperbolic systems, characterizing the possible accessibility classes and showing that these coincide with the ergodic components of any smooth invariant measure.

By the classical work of Hopf, the geodesic flow on a surface of negative curvature is ergodic [31]. Further, by the work Anosov and Sinai, the flow is stably ergodic meaning that all nearby flows are also ergodic [1, 2]. Based on these techniques, Grayson, Pugh, and Shub showed that the time-one map of this geodesic flow is also stably ergodic as a diffeomorphism [21]. To prove this, they observed two important properties. The first property is partial hyperbolicity. A diffeomorphism $f$ is partially hyperbolic if there is an invariant splitting of the tangent bundle of the phase space $M$ into three subbundles

$$TM = E^u \oplus E^c \oplus E^s$$

such that vectors in the unstable bundle $E^u$ are expanded by the derivative $Tf$, vectors in the stable bundle $E^s$ are contracted, and these dominate any expansion and contraction of vectors in the center bundle $E^c$. (Appendix A gives a precise definition.) The second property is accessibility. For a point $x \in M$, the accessibility class $AC(x)$ is the set of all points that can be reached from $x$ by
a concatenation of paths, each tangent to either $E^s$ or $E^u$. A system is called accessible if its phase space consists of a single accessibility class. For the geodesic flow, the phase space $M$ is the unit tangent bundle of the surface, $E^c$ is the direction of the flow, and $E^s$ and $E^u$ are given by the horocycles. Grayson, Pugh, and Shub demonstrated that any diffeomorphism near the time-one map of the flow is both partially hyperbolic and accessible and used this to prove its ergodicity. This breakthrough was followed by a number of papers demonstrating stable ergodicity for specific cases of partially hyperbolic systems (see the surveys [25, 46]) and lead Pugh and Shub to formulate the following conjecture [42].

**Conjecture 1.** Ergodicity holds on an open and dense set of volume-preserving partially hyperbolic diffeomorphisms.

They further split this into two subconjectures.

**Conjecture 2.** Accessibility implies ergodicity.

**Conjecture 3.** Accessibility holds on an open and dense set of partially hyperbolic diffeomorphisms (volume-preserving or not).

The Pugh-Shub conjectures have been established in a number of settings. In particular, they are true when the center bundle $E^c$ is one-dimensional [26]. However, there are a number of partially hyperbolic systems which arise naturally and which are not ergodic, leading to the following questions.

**Question.** Is it possible to give an exact description of the set of non-ergodic partially hyperbolic diffeomorphisms?

**Question.** For a non-ergodic partially hyperbolic diffeomorphism, do the ergodic components coincide with the accessibility classes of the system?

This paper answers these questions in the affirmative under certain assumptions on the system. We first give one example as motivation before introducing more general results. Consider on the 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ a diffeomorphism $f$ defined by

$$f(x, y, z) = (2x + y, x + y, z).$$

The eigenvalues are $\lambda < 1 < \lambda^{-1}$ and $f$ is therefore partially hyperbolic. Arguably, this is the simplest partially hyperbolic example one can find. It preserves Lebesgue measure but is not ergodic. Further, there are several ways to construct nearby diffeomorphisms which are also non-ergodic. With a bit of thought, the following methods come to mind.

1. Rotate $f$ slightly along the center direction, yielding a diffeomorphism

   $$(x, y, z) \mapsto (2x + y, x + y, z + \theta)$$

   for some small rational $\theta \in \mathbb{R}/\mathbb{Z}$. 
(2) Compose \( f \) with a map of the form \( (x, y, z) \mapsto (\psi(x, y, z), z) \) for some \( \psi : \mathbb{T}^3 \to \mathbb{T}^2 \).

(3) Perturb \( f \) on a subset of the form \( \mathbb{T}^2 \times X \) where \( X \subseteq \mathbb{S}^1 \).

(4) Conjugate \( f \) with a diffeomorphism close to the identity.

The results of this paper imply that any non-ergodic diffeomorphism in a neighbourhood of \( f \) can be constructed by applying these four steps in this order.

1. Statement of Results

We again refer the reader to the appendix for a list of definitions.

Suppose \( A \) and \( B \) are automorphisms of a compact nilmanifold \( N \) such that \( A \) is hyperbolic and \( AB = BA \). Then, \( A \) and \( B \) define a diffeomorphism

\[
f_{AB} : M_B \to M_B, \quad (v, t) \to (Av, t)
\]

on the manifold

\[
M_B = N \times \mathbb{R}/(v, t) \sim (Bv, t-1).
\]

Call \( f_{AB} \) an \( AB \)-prototype.

Note that every \( AB \)-prototype is an example of a volume-preserving, partially hyperbolic, non-ergodic system. Further, just like the linear example on \( \mathbb{T}^3 \) given above, every \( AB \)-prototype may be perturbed to produce nearby diffeomorphisms which are also non-ergodic.

To consider such perturbations, we use the notion of leaf conjugacy as introduced in [30]. Two partially hyperbolic diffeomorphisms \( f \) and \( g \) are leaf conjugate if there are invariant foliations \( W^c_f \) and \( W^c_g \) tangent to \( E^c_f \) and \( E^c_g \) and a homeomorphism \( h \) such that for every leaf in \( L \) in \( W^c_f \), \( h(L) \) is a leaf of \( W^c_g \) and \( h(f(L)) = g(h(L)) \).

We now define a family of diffeomorphisms which will be the focus of the paper. A partially hyperbolic system \( f : M \to M \) is an \( AB \)-system if it preserves an orientation of the center bundle \( E^c \) and is leaf conjugate to an \( AB \)-prototype.

In order to consider skew-products over infranilmanifolds and systems which do not preserve an orientation of \( E^c \), we also consider the following generalization. A diffeomorphism \( f_0 \) is an infra-\( AB \)-system if an iterate of \( f_0 \) lifts to an \( AB \)-system on a finite cover. To the best of the author’s knowledge, this family of partially hyperbolic diffeomorphisms includes every currently known example of a non-ergodic system with one-dimensional center. Further, there are manifolds on which every conservative partially hyperbolic diffeomorphism is an \( AB \)-system.

**Question 1.1.** Suppose \( f \) is a conservative, non-ergodic, partially hyperbolic \( C^2 \) diffeomorphism with one-dimensional center. Is \( f \) necessarily an infra-\( AB \)-system?

Skew products with trivial bundles correspond to \( AB \)-systems where \( B \) is the identity map. The suspensions of Anosov diffeomorphisms correspond to the case \( A = B \). These are not the only cases, however. For instance, one could take hyperbolic automorphisms \( A, B : \mathbb{T}^3 \to \mathbb{T}^3 \) defined by the commuting matrices
Throughout this paper, the letters $A$ and $B$ will always refer to the maps associated to the AB-system under study, and $N$ and $M_B$ will be the manifolds in the definition. In general, if $f : M \to M$ is an AB-system, $M$ need only be homeomorphic to $M_B$, not diffeomorphic [15, 14].

We show that every conservative AB-system belongs to one of three cases, each with distinct dynamical and ergodic properties.

**Theorem 1.2.** Suppose $f : M \to M$ is a $C^2$ AB-system which preserves a smooth volume form. Then, one of the following occurs.

1. $f$ is accessible and stably ergodic.
2. $E^u$ and $E^s$ are jointly integrable and $f$ is topologically conjugate to $M_B, (v, t) \to (Av, t+\theta)$ for some $\theta$. Further, $f$ is (non-stably) ergodic if and only if $\theta$ defines an irrational rotation.
3. There are $n \geq 1$, a $C^1$ surjection $p : M \to \mathbb{S}^1$, and a non-empty open set $U \subset \mathbb{S}^1$ such that
   - for every connected component $I$ of $U$, $p^{-1}(I)$ is an $f^n$-invariant subset homeomorphic to $N \times I$ and the restriction of $f^n$ to this subset is accessible and ergodic, and
   - for every $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an $f^n$-invariant submanifold tangent to $E^u \oplus E^s$ and homeomorphic to $N$.

Note that the first case can be thought of as a degenerate form of the third case with $U = \mathbb{S}^1$. Similarly, the second case with rational rotation corresponds to $U = \emptyset$.

To give the ergodic decomposition of these systems, we decompose the measure and show that each of the resulting measures is ergodic. Suppose $\mu$ is a smooth measure on a manifold $M$ and $p : M \to \mathbb{S}^1$ is continuous and surjective such that $p_*\mu = m$ where $m$ is Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The Rokhlin disintegration theorem [45] implies that $\mu$ can be written as

$$
\mu = \int_{t \in \mathbb{S}^1} \mu_t \, dm(t)
$$

where the support of each $\mu_t$ is contained in $p^{-1}(t)$. Moreover, this disintegration is essentially unique; if measures $\{\nu_t\}_{t \in \mathbb{S}^1}$ give another disintegration of $\mu$, then $\nu_t = \mu_t$ for $m$-a.e. $t \in \mathbb{S}^1$. For an open interval $I \subset \mathbb{S}^1$ define

$$
\mu_I := \frac{1}{m(I)} \int_I \mu_t \, dm(t).
$$
Note that $\mu_I$ is the normalized restriction of $\mu$ to $p^{-1}(I)$. Then an open subset $U \subset \mathbb{S}^1$ yields a decomposition

$$
\mu = \sum_I m(I) \mu_I + \int_{\mathbb{S}^1 \setminus U} \mu_t \, dm(t)
$$

where $\sum_I$ denotes summation over all of the connected components $I$ of $U$.

**Theorem 1.4.** If $f : M \to M$ is a $C^2$ AB-system and $\mu$ is a smooth, invariant, non-ergodic measure with $\mu(M) = 1$, then there are $n \geq 1$, a $C^1$ surjection $p : M \to \mathbb{S}^1$, and an open set $U \subset \mathbb{S}^1$ such that $p_* \mu = m$ and (1.3) is the ergodic decomposition of $(f^n, \mu)$.

If $f$ is in case (3) of (1.2), then the $n, p,$ and $U$ can be taken to be the same in both theorems. If $f$ is in case (2) and non-ergodic, then $\theta$ is rational, and the map $p$ can be defined by composing the topological conjugacy from $M$ to $M_B$ with a projection from $M_B$ to $\mathbb{S}^1$.

As $f$ preserves $\mu$ and $p_* \mu = m$, it follows that $p(f(x)) = p(x) + q$ for some rational $q \in \mathbb{S}^1$ and all $x$ with $p(x) \notin U$. Because of this, one can derive the ergodic decomposition of $(f, \mu)$ from (1.4). Each component is either of the form $\frac{1}{n} \sum_{j=1}^n \mu_{I_k} + \frac{1}{n} \sum_{j=1}^n \mu_{I_k}$ where if $I_k = (a, b)$ then $I_k = (a + jq, b + jq)$. In (1.4), the ergodic components of $(f^n, \mu)$ are mixing and, in fact, have the Kolmogorov property [12]. The ergodic components of $f$ are mixing if and only if (1.4) holds with $n = 1$.

Using the perturbation techniques of [26], for any AB-prototype $f_{AB}$, rational number $\theta = \frac{k}{n}$, and open subset $U \subset \mathbb{S}^1$ which satisfies $U + \theta = U$, one can construct an example of a volume-preserving AB-system which satisfies (1.4) with the same $n$ and $U$. In this sense, the classification given by (1.2) and (1.4) may be thought of as complete. Versions of these theorems for infra-AB-systems are given in Section 13.

Accessibility also has applications beyond the conservative setting. For instance, Brin showed that accessibility and a non-wandering condition imply that the system is (topologically) transitive [8]. Therefore, we state a version of (1.2) which assumes only this non-wandering condition. For a homeomorphism $f : M \to M$, a wandering domain is a non-empty open subset $U$ such that $U \cap f^n(U)$ is empty for all $n \geq 1$. Let $NW(f)$ be the non-wandering set, the set of all points $x \in M$ which do not lie in a wandering domain.

**Theorem 1.5.** Suppose $f : M \to M$ is an AB-system such that $NW(f) = M$. Then, one of the following occurs.

1. $f$ is accessible and transitive.
2. $E^u$ and $E^s$ are jointly integrable and $f$ is topologically conjugate to $M_B \to M_B$, $(v, t) \to (Av, t + \theta)$ for some $\theta$. Further, $f$ is transitive if and only if $\theta$ defines an irrational rotation.
3. There are $n \geq 1$, a continuous surjection $p : M \to \mathbb{S}^1$, and a non-empty open set $U \subset \mathbb{S}^1$ such that
for every connected component $I$ of $U$, $p^{-1}(I)$ is an $f^n$-invariant subset homeomorphic to $N \times I$, and
for every $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an $f^n$-invariant submanifold tangent to $E^u \oplus E^s$ and homeomorphic to $N$.

The restriction of $f^n$ to a subset $p^{-1}(t)$ or $p^{-1}(I)$ is transitive.

The non-wandering assumption is used in only a few places in the proof and so certain results may be stated without this assumption. For a partially hyperbolic diffeomorphism with one-dimensional center, a us-leaf is a complete $C^1$ submanifold tangent to $E^u \oplus E^s$.

Theorem 1.6. Every non-accessible AB-system has a compact us-leaf.

Theorem 1.7. Suppose $f : M \to M$ is a non-accessible AB-system with at least one compact periodic us-leaf. Then, there are $n \geq 1$, a continuous surjection $p : M \to \mathbb{S}^1$ and an open subset $U \subset \mathbb{S}^1$ with the following properties.

For $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an $f^n$-invariant compact us-leaf. Moreover, every $f$-periodic compact us-leaf is of this form.

For every connected component $I$ of $U$, $p^{-1}(I)$ is $f^n$-invariant, homeomorphic to $N \times I$ and, letting $g$ denote the restriction of $f^n$ to $p^{-1}(I)$, one of three cases occurs:

1. $g$ is accessible,
2. there is an open set $V \subset p^{-1}(I)$ such that
   \[ \overline{g(V)} \subset V, \quad \bigcup_{k \in \mathbb{Z}} g^k(V) = p^{-1}(I), \quad \bigcap_{k \in \mathbb{Z}} g^k(V) = \emptyset, \]
   and the boundary of $V$ is a compact us-leaf, or
3. there are no compact us-leaves in $p^{-1}(I)$, uncountably many non-compact us-leaves in $p^{-1}(I)$, and $\lambda \neq 1$ such that $g$ is semiconjugate to
   \[ N \times \mathbb{R} \to N \times \mathbb{R}, \quad (v, t) \mapsto (Av, \lambda t). \]

It is relatively easy to construct examples in the first two cases above. Section 15 gives an example of the third case. It is based on the discovery by F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures of a non-dynamically coherent system on the 3-torus [28]. Theorem (1.7) corresponds to a rational rotation on an $f$-invariant circle. The following two theorems correspond to irrational rotation.

Theorem 1.8. Suppose $f : M \to M$ is a non-accessible AB-system with no periodic compact us-leaves. Then, there is a continuous surjection $p : M \to \mathbb{S}^1$ and a $C^1$ diffeomorphism $r : \mathbb{S}^1 \to \mathbb{S}^1$ such that

- $NW(f) = p^{-1}(NW(r))$,
- if $t \in NW(r)$ then $p^{-1}(t)$ is a compact us-leaf and $f(p^{-1}(t)) = p^{-1}(r(t))$, and
- if $I$ is a connected component of $\mathbb{S}^1 \setminus NW(r)$, then $f(p^{-1}(I)) = p^{-1}(r(I))$.

In particular, $p^{-1}(I) \subset M$ is a wandering domain.
Theorem 1.9. Suppose $f : M \to M$ is a non-accessible AB-system with no periodic compact $us$-leaves. Then, $f$ is semiconjugate to

$$M_B \to M_B, \quad (v, t) \mapsto (Av, t + \theta)$$

for $\theta$ defining an irrational rotation.

One can construct $C^1$ examples of AB-systems satisfying the conditions of (1.8) and with $\text{NW}(f) \neq M$. For instance, if $r$ is a Denjoy diffeomorphism of the circle, simply consider a direct product $A \times r$ where $A$ is Anosov.

The diffeomorphism $f$ in (1.5)–(1.9) need only be $C^1$ in general. If $f$ is a $C^2$ diffeomorphism, then the surjection $p : M \to S^1$ may be taken as $C^1$. This is a consequence of the following regularity result, proven in Section 11.

Theorem 1.10. For a non-accessible partially hyperbolic $C^2$ diffeomorphism with one-dimensional center, the $us$-leaves form a $C^1$ lamination.

The existence of a $C^0$ lamination was shown in [26].

The next sections discuss how this work relates to other results in partially hyperbolic theory, first for three-dimensional systems in Section 2 and for higher dimensions in Section 3. Section 4 gives an outline of the proof and of the organization of the rest of the paper. The appendix gives precise definitions for many of the terms used in these next few sections.

2. Dimension three

The study of partially hyperbolic systems has had its greatest success in dimension three, where $\dim E^u = \dim E^c = \dim E^s = 1$. Still, in this simplest of cases, a number of important questions remain open. Hertz, Hertz, and Ures posed the following conjecture specifically regarding ergodicity.

Conjecture 2.1. If a conservative partially hyperbolic diffeomorphism in dimension three is not ergodic, then there is a periodic 2-torus tangent to $E^u \oplus E^s$.

They also showed that the existence of such a torus would have strong dynamical consequences. We state this theorem as follows.

Theorem 2.2 ([29]). If a partially hyperbolic diffeomorphism on a three dimensional manifold $M$ has a periodic 2-torus tangent to $E^u \oplus E^s$, then $M$ has solvable fundamental group.

In fact, the theorem may be stated in a much stronger form. See [29] for details.

Work on classifying partially hyperbolic systems has seen some success in recent years, at least for 3-manifolds with "small" fundamental group. This was made possible by the breakthrough results of Brin, Burago, and Ivanov to rule out partially hyperbolic diffeomorphisms on the 3-sphere and prove dynamical coherence on the 3-torus [9, 7]. Building on this work, the author and R. Potrie
Theorem 2.3 ([23]). A conservative partially hyperbolic diffeomorphism on a 3-manifold with solvable fundamental group is (up to finite iterates and finite covers) either

(a) an AB-system,

(b) a skew-product with a non-trivial fiber bundle, or

(c) a system leaf conjugate to an Anosov diffeomorphism.

Further, the ergodic properties of each of these three cases have been examined in detail. Case (a) is the subject of the current paper. Case (b) was studied in [27], where it was first shown that there are manifolds on which all partially hyperbolic systems are accessible and ergodic. Case (c) was studied in [24], which showed that if such a system is not ergodic then it is topologically conjugate to an Anosov diffeomorphism (not just leaf conjugate). It is an open question if such a non-ergodic system can occur. All of these results can be synthesized into the following statement, similar in form to (1.2).

Theorem 2.4. Suppose $M$ is a 3-manifold with solvable fundamental group and $f : M \to M$ is a $C^2$ conservative partially hyperbolic system. Then, (up to finite iterates and finite covers) one of the following occurs.

1. $f$ is accessible and stably ergodic.
2. $E^u$ and $E^s$ are jointly integrable and $f$ is topologically conjugate either to a linear hyperbolic automorphism of $T^3$ or to
   
   $M_B \to (v, t) \to (Av, t + \theta)$

   where $A, B : T^2 \to T^2$ define an AB-prototype and $\theta \in S^1$.

3. There are $n \geq 1$, a $C^1$ surjection $p : M \to S^1$, and a non-empty open set $U \subseteq S^1$ such that
   
   • for every connected component $I$ of $U$, $p^{-1}(I)$ is an $f^n$-invariant subset homeomorphic to $T^2 \times I$ and the restriction of $f^n$ to this subset is accessible and ergodic,
   
   • for every $t \in S^1 \setminus U$, $p^{-1}(t)$ is an $f^n$-invariant 2-torus tangent to $E^u \oplus E^s$.

If (2.1) is true, then this theorem encapsulates every possible ergodic decomposition for a 3-dimensional partially hyperbolic system.

Question 2.5. Is the condition “with solvable fundamental group” necessary in (2.4)?

3. Higher dimensions

We next consider the case of skew products in higher dimension. In related work, K. Burns and A. Wilkinson studied stable ergodicity of rotation extensions...
and of more general group extensions over Anosov diffeomorphisms [11], and M. Field, I. Melbourne, V. Nițică, and A. Török have analyzed group extensions over Axiom A systems, proving results on transitivity, ergodicity, and rates of mixing [16, 17, 36].

In this paper, we use the following definition taken from [20]. Let $\pi : M \to X$ define a fiber bundle on a compact manifold $M$ over a topological manifold $X$. If a partially hyperbolic diffeomorphism $f : M \to M$ is such that the center direction $E^c_f$ is tangent to the fibers of the bundle and there is a homeomorphism $A : X \to X$ satisfying $\pi f = A \pi$, then $f$ is a partially hyperbolic skew product. We call $A$ the base map of the skew product. While $f$ must be $C^1$, $\pi$ in general will only be continuous.

This definition has the benefit that it is open: any $C^1$-small perturbation of a partially hyperbolic skew product is again a partially hyperbolic skew product. This can be proven using the results in [30] and the fact that the base map is expansive. The base map also has the property that it is topologically Anosov [3]. As with smooth Anosov systems, it is an open question if all topologically Anosov systems are algebraic in nature.

**Question 3.1.** If $A$ is a base map of a partially hyperbolic skew product, then is $A$ topologically conjugate to a hyperbolic infranilmanifold automorphism?

We now consider the case where $\dim E^c = 1$ in order to relate skew products to the AB-systems studied in this paper. The following is easily proved.

**Proposition 3.2.** Suppose $f$ is a partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and $E^c$ is one-dimensional and has an orientation preserved by $f$. Then, $f$ is an AB-system if and only if the fiber bundle defining the skew product is trivial.

If we are interested in the ergodic properties of the system, we can further relate accessibility to triviality of the fiber bundle.

**Theorem 3.3.** Suppose $f$ is a partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and $E^c$ is one-dimensional and orientable. If $f$ is not accessible, then the fiber bundle defining the skew product is trivial.

**Corollary 3.4.** Suppose $f$ is a conservative $C^2$ partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and $E^c$ is one-dimensional and has an orientation preserved by $f$. Then, $f$ satisfies one of the three cases of (1.2) and if $f$ is not ergodic, its ergodic decomposition is given by (1.4).

Theorem (3.3) is proved in Section 12. A similar statement, (13.6), still holds when “nilmanifold” is replaced by “infranilmanifold” and the condition on orientability is dropped.
Every partially hyperbolic skew product has compact center leaves and an open question, attributed in [25] to C. C. Pugh, asks if some form of converse statement holds.

**Question 3.5.** Is every partially hyperbolic diffeomorphism with compact center leaves finitely covered by a partially hyperbolic skew product?

This question was studied independently by D. Bohnet, P. Carrasco, and A. Gogolev who gave positive answers under certain assumptions [5, 6, 13, 20]. In relation to the systems studied in the current paper, the following results are relevant.

**Theorem 3.6** ([20]). If \( f \) is a partially hyperbolic diffeomorphism with compact center leaves, and \( \dim E^c = 1 \), \( \dim E^u \leq 2 \), and \( \dim E^s \leq 2 \), then \( f \) is finitely covered by a skew product.

**Corollary 3.7.** Suppose \( f : M \to M \) is a partially hyperbolic diffeomorphism with compact center leaves, \( \dim E^c = 1 \), and \( \dim M = 4 \). If \( f \) is not accessible, then \( f \) is an infra-AB-system.

A compact foliation is uniformly compact if there is a uniform bound on the volume of the leaves.

**Theorem 3.8** ([6]). If \( f \) is a partially hyperbolic diffeomorphism with uniformly compact center leaves and \( \dim E^u = 1 \), then \( f \) is finitely covered by a partially hyperbolic skew product where the base map is a hyperbolic toral automorphism.

**Corollary 3.9.** Suppose \( f \) is a partially hyperbolic diffeomorphism with uniformly compact center leaves and \( \dim E^u = \dim E^c = 1 \). If \( f \) is not accessible, then \( f \) is an infra-AB-system.

In the conservative setting, we may then invoke the results of the current paper to describe the ergodic properties of these systems.

**Question 3.10.** If \( f \) is a non-accessible partially hyperbolic diffeomorphism with compact one-dimensional center leaves, then is \( f \) an infra-AB-system?

Positive answers to both (3.1) and (3.5) would give a positive answer to (3.10).

In his study of hyperbolic flows, Anosov established a dichotomy, now known as the “Anosov alternative” which states that every transitive Anosov flow is either topologically mixing or the suspension of an Anosov diffeomorphism with constant roof function [1, 17]. Ergodic variants of the Anosov alternative have also been studied and the following holds.

**Theorem 3.11** ([38, 10]). For an Anosov flow \( \phi_t : M \to M \), the following are equivalent:

- the time-one map \( \phi_1 \) is not accessible,
- the strong stable and unstable foliations are jointly integrable,
and both imply the flow is topologically conjugate to the suspension of an Anosov
diffeomorphism.

**Corollary 3.12.** Suppose every Anosov diffeomorphism is topologically conjugate
to an infranilmanifold automorphism. Then, every non-accessible time-one map
of an Anosov flow is an infra-AB-system.

Thus, if the conjecture about Anosov diffeomorphisms is true, then the re-
results given in Section 13 will classify the ergodic properties of diffeomorphisms
which are perturbations of time-one maps of Anosov flows. This conjecture is
true when the Anosov diffeomorphism has a one dimensional stable or unstable
bundle [37].

**Corollary 3.13.** Suppose \( f \) is the time-one map of an Anosov flow with
\( \dim E^u_f = 1 \). If \( f \) is not accessible, then it is an AB-system.

### 4. Outline

Most of the remaining sections focus on proving the results listed in Section 1
and we present here an outline of the main ideas.

A partially hyperbolic system has **global product structure** if it is dynamically
coherent and, after lifting the foliations to the universal cover \( \hat{M} \), the following
hold for all \( x, y \in \hat{M} \):

1. \( W^u(x) \) and \( W^{cs}(y) \) intersect exactly once,
2. \( W^s(x) \) and \( W^{cu}(y) \) intersect exactly once,
3. if \( x \in W^{cs}(y) \), then \( W^c(x) \) and \( W^s(y) \) intersect exactly once, and
4. if \( x \in W^{cu}(y) \), then \( W^c(x) \) and \( W^u(y) \) intersect exactly once.

**Theorem 4.1.** Every AB-system has global product structure.

This proof of this theorem is left to Section 14. That section also proves the fol-
lowering.

**Theorem 4.2.** AB-systems form a \( C^1 \)-open subset of the space of diffeomorphisms.

Now assume \( f \) is a non-accessible AB-system. There is a lamination con-
sisting of \( us \)-leaves [26], and this lamination lifts to the universal cover. Global
product structure implies that for a center leaf \( L \) on the cover, every leaf of the
lifted \( us \)-lamination intersects \( L \) exactly once. Each deck transformation maps
the lamination to itself and this leads to an action of the fundamental group on
a closed subset of \( L \) as depicted in Figure 1.

In Section 5, we consider an order-preserving action of a nilpotent group \( G \)
on a closed subset \( \Gamma \subset \mathbb{R} \). We also assume there is \( f \) acting on \( \Gamma \) such that
\( fGf^{-1} = G \). Then, \( f \) and \( G \) generate a solvable group. Solvable groups acting on
the line were studied by Plante [40]. By adapting his results, we prove (5.5) which
(omitting some details for now) states that either \( \text{Fix}(G) \) is non-empty or, up to a
common semiconjugacy from $\Gamma$ to $\mathbb{R}$, each $g \in G$ gives a translation $x \mapsto x + \tau(g)$ and $f$ gives a scaling $x \mapsto \lambda x$.

Instead of applying this result immediately to AB-systems, Section 6 introduces the notion of an "AI-system" which can be thought of as the lift of an AB-system to a covering space homeomorphic to $N \times \mathbb{R}$ where, as always, $N$ is a nilmanifold. Using (5.5), Section 6 gives a classification result, (6.1), for the accessibility classes of AI-systems. Section 7 applies the results for AI-systems to give results about AB-systems and gives a proof of (1.6). The higher dimensional dynamics of the AB-system depend on the one-dimensional dynamics on an invariant circle. Sections 8 and 9 consider the cases of rational and irrational rotation respectively and prove theorems (1.7)–(1.9).

Section 10 gives the proofs of (1.2), (1.4), and (1.5) based on the other results. In order to establish the ergodic decomposition, the lamination of $us$-leaves must be $C^1$. By (1.10), this holds if the diffeomorphism is $C^2$. The proof requires a highly technical application of Whitney’s extension theorem and is given in Section 11. The specific version of this regularity result for AB-systems can be stated as follows.

**Proposition 4.3.** Let $f : M \to M$ be a $C^2$ AB-system. Then, there is a $C^1$ surjection $p : M \to \mathbb{S}^1$ and $U \subset \mathbb{S}^1$ such that the compact $us$-leaves of $f$ are exactly the sets $p^{-1}(t)$ for $t \in \mathbb{S}^1 \setminus U$.

If $S$ is a center leaf which intersects each compact $us$-leaf exactly once, then $p$ may be defined so that its restriction to $S$ is a $C^1$-diffeomorphism.

If $\mu$ is a probability measure given by a $C^1$ volume form on $M$, then $p$ may be chosen so that $p_* \mu$ is Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Section 12 proves (3.3) concerning the triviality of non-accessible skew products. Infra-AB-systems are treated in Section 13.

### 5. Actions on subsets of the line

**Notation.** To avoid excessive parentheses, if $f$ and $g$ are composable functions, we simply write $fg$ for the composition. In this section, $\mu$ is a measure on the real line and $\mu(x, y)$ denotes the measure of the half-open interval $[x, y)$.

Let $\text{Homeo}^+ (\mathbb{R})$ denote the group of orientation-preserving homeomorphisms of the line. If $\Gamma$ is a non-empty closed subset of $\mathbb{R}$, let $\text{Homeo}^+ (\Gamma)$ denote the group of all homeomorphisms of $\Gamma$ which are restrictions of elements of $\text{Homeo}^+ (\mathbb{R})$. That is, $g$ is in $\text{Homeo}^+ (\Gamma)$ if it is a homeomorphism of $\Gamma$ and $g(x) < g(y)$ for $x < y$.

We now adapt results of Plante to this setting.
After lifting to the universal cover, an AB-system has a center leaf $L$ invariant under the lifted dynamics $f$. Each deck transformation $\alpha$ then defines a function $g_{\alpha} : L \rightarrow L$ where $g_{\alpha}(x)$ is the unique point for which $W^s(\alpha(x))$ intersects $W^u(g_{\alpha}(x))$. These functions together with $f$ define a solvable action on a closed subset of $L$ and this action is semiconjugate to an affine action on $\mathbb{R}$.

**Proposition 5.1.** Suppose $\Gamma$ is a non-empty closed subset of $\mathbb{R}$ and $G$ is a subgroup of $\text{Homeo}^+ (\Gamma)$ with non-exponential growth. Then, there is a measure $\mu$ on $\mathbb{R}$ such that

- $\text{supp} \mu \subset \Gamma$,
- $\mu(X) = \mu(g(X))$ for all $g \in G$ and Borel sets $X \subset \mathbb{R}$, and
- if $X \subset \mathbb{R}$ is compact, then $\mu(X) < \infty$.

**Proof.** In the case $\Gamma = \mathbb{R}$, this is a restatement of (1.3) in [40]. One can check that the techniques in [40] and [39] extend immediately to the case $\Gamma \neq \mathbb{R}$. □

**Proposition 5.2.** Let $\Gamma$, $G$, and $\mu$ be as in (5.1) and suppose $\text{Fix}(G)$ is empty. Then there is a non-zero homomorphism $\tau : G \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$\tau(g) = \begin{cases} 
\mu(x, g(x)) & \text{if } x < g(x), \\
0 & \text{if } x = g(x), \\
-\mu(g(x), x) & \text{if } g(x) < x.
\end{cases}$$

**Proof.** Choose any $x \in \mathbb{R}$ and define $\tau$ as above. One can then show that $\tau$ is a non-zero homomorphism and independent of the choice of $x$. See (5.3) of [39] for details. □
Let $f : G \to G$ be an automorphism of $G$ and suppose that $f$ is an automorphism of $G$. Then, $f$ is a bijection and $f$ is measurable.

Proof. The first half of the statement follows as an adaptation of §4 of [40]. Further, if $\lambda \neq 1$, then $f_\mu = \lambda \mu$ by (4.2) of [40]. To prove the final claim, we first show that if $\lambda \neq 1$ then $f$ has a fixed point. Consider $x \in \Gamma$. As $\text{Fix}(G)$ is empty by assumption, there is $g \in G$ such that $x < g(x)$. Then,

$$\mu(x, +\infty) = \mu(x, g^k(x)) = k \tau(g)$$

for all $k \geq 1$. This shows that $\mu(x, +\infty) = \infty$ for any $x \in \mathbb{R}$.

Assume, without loss of generality, that $\lambda < 1$ and $x < f(x)$ for some $x \in \mathbb{R}$. Then,

$$\mu(x, \sup_{k \geq 0} f^k(x)) = \sum_{k=0}^{\infty} \lambda^k \mu(x, f(x)) < \infty$$

and therefore, $x_0 := \sup_{k \geq 0} f^k(x) < \infty$ is a fixed point for $f$. If $h \in \text{Homeo}^+(\mathbb{R})$ commutes with $f$ then for all $k \in \mathbb{Z}$

$$\mu(x_0, h^k(x_0)) = \mu(f(x_0), f h^k(x_0)) = \lambda \mu(x_0, h^k(x_0))$$

which is possible only if $\mu(x_0, h^k(x_0)) = 0$. Then $\mu(x_0, \sup_{k \in \mathbb{Z}} h^k(x_0)) = 0$ and so $\sup h^k(x_0) < \infty$ is a fixed point for $h$. \hfill \Box

We now consider the case where $G$ is a fundamental group of a nilmanifold.

Proposition 5.4. Let $G$ be a torsion-free, finitely-generated, nilpotent group and suppose $\phi \in \text{Aut}(G)$ is such that $\phi(g) \neq g$ for all non-trivial $g \in G$. If $H$ is a $\phi$-invariant subgroup, then $\phi(g H) \neq g H$ for all non-trivial cosets $g H \neq H$.

Proof. First, we show that the function $\psi : G \to G$ defined by $\psi(g) = g^{-1} \phi(g)$ is a bijection. If $G$ is abelian, then $G$ is isomorphic to $\mathbb{Z}^d$ for some $d$ and $\psi$ is an invertible linear map, and hence bijective. Suppose now that $G$ is non-abelian and let $Z$ be its group-theoretic center. Pick some element $g_0 \in G$. As $G/Z$ is of smaller nilpotency class, by induction there is $g \in G$ such that $\psi(g Z) = g_0 Z$ or equivalently $\psi(g) z_0 = g_0$ for some $z_0 \in Z$. As $\psi|_Z$ is an automorphism of $Z$, there is $z \in Z$ such that $\psi(g z) = \psi(g) \psi(z) = \psi(g) z_0 = g_0$. As $h$ was arbitrary, this shows $\psi$ is onto.

To prove injectivity, suppose $\psi(g) = \psi(g')$. By induction, $g' = g z$ for some $z \in Z$. Then,

$$\psi(g) = \psi(g') = \psi(g) \psi(z) \Rightarrow \psi(z) = 1 \Rightarrow z = 1 \Rightarrow g' = g.$$

If $H$ is a $\phi$-invariant subgroup, then $\psi(H) = H$ and the bijectivity of $\psi$ implies that $\psi(g H) \neq H$ for any non-trivial coset. \hfill \Box
The results of J. Franks and A. Manning [19, 18, 34] show that for any Anosov diffeomorphism on a nilmanifold, the resulting automorphism on the fundamental group satisfies the hypotheses of (5.4).

Lemma 5.5. Suppose \( \Gamma \subset \mathbb{R} \), \( G \subset \text{Homeo}^+(\Gamma) \), and \( f \in \text{Homeo}^+(\mathbb{R}) \) are such that

- \( \Gamma \) is closed and non-empty,
- \( G \) is finitely generated and nilpotent,
- \( F : G \to G \) defined by \( F(g)(x) = fgf^{-1}(x) \)
  is a group automorphism with no non-trivial fixed points, and
- \( \text{Fix}(G) \) is empty.

Then, there are

- a closed non-empty subset \( \Gamma_0 \subset \Gamma \),
- a continuous surjection \( P : \mathbb{R} \to \mathbb{R} \),
- a non-zero homeomorphism \( \tau : G \to \mathbb{R} \), and
- \( 0 < \lambda \neq 1 \)

such that for \( x, y \in \mathbb{R} \) and \( g \in G \)

- \( x \leq y \) implies \( P(x) \leq P(y) \),
- \( P g(x) = P(x) + \tau(g) \),
- \( P f(x) = \lambda P(x) \),
- \( \Gamma_0 = \{ x \in \Gamma : g(x) = x \text{ for all } g \in \text{ker } \tau \} \), and
- for each \( t \in \mathbb{R} \), \( P^{-1}(t) \) is either a point \( z \in \Gamma_0 \) or an interval \( [a, b] \) with \( a, b \in \Gamma_0 \).

Moreover, any homeomorphism which commutes with \( f \) has a fixed point in \( P^{-1}(0) \).

Proof. The conditions on \( G \) imply that it has non-exponential growth [22]. Therefore, we are in the setting of the previous propositions. In particular, there are \( \mu \), \( \tau \), and \( \lambda \) as above.

First, suppose that the image \( \tau(G) \) is a cyclic subgroup of \( \mathbb{R} \) in order to derive a contradiction. In this case, the condition \( \tau F = \lambda \tau \) in (5.3) implies that \( \lambda \tau(G) = \tau(G) \) and therefore \( \lambda = 1 \). Then, \( F \) maps a coset of \( \text{ker } \tau \) to itself. As \( \text{Homeo}^+(\Gamma) \) is torsion free, so is \( G \), and by (5.4), \( F \) has a non-trivial fixed point, in contradiction to the hypotheses of the lemma being proved. Therefore, \( \tau(G) \) is non-cyclic.

Consequently, \( \tau(G) \) is a dense subgroup of \( \mathbb{R} \). Further \( \lambda \neq 1 \), as otherwise, one could derive a contradiction exactly as above. By (5.3), \( f \) has at least one fixed point, say \( x_0 \in \mathbb{R} \). Define a function \( P : \mathbb{R} \to \mathbb{R} \) by

\[
P(x) = \begin{cases} 
\mu(x_0, x) & \text{if } x > x_0, \\
0 & \text{if } x = x_0, \\
-\mu(x, x_0) & \text{if } x < x_0.
\end{cases}
\]

By definition, \( P \) is (non-strictly) increasing. The density of \( \tau(G) \) implies that \( P(\mathbb{R}) \) is dense. Then, as a monotonic function without jumps, \( P \) is continuous and therefore surjective. For each \( t \in \mathbb{R} \), the pre-image \( P^{-1}(t) \) is either a point or a closed interval. In either case, one can verify that \( g(P^{-1}(t)) = P^{-1}(t) \) for all
$g \in \ker r$ and therefore the boundary of $P^{-1}(t)$ is in $\Gamma_0$. The other properties of $P$ listed in the lemma are easily verified.

The statement for homeomorphisms commuting with $f$ follows by adapting the proof of (5.3).

6. AI-systems

We now consider partially hyperbolic systems on non-compact manifolds. Suppose $M$ is compact and $f : M \to M$ is partially hyperbolic. Then, any lift of $f$ to a covering space of $M$ is also considered to be partially hyperbolic. Also, any restriction of a partially hyperbolic diffeomorphism to an open invariant subset is still considered to be partially hyperbolic.

Let $A$ be a hyperbolic automorphism of the compact nilmanifold $N$ and $I \subset \mathbb{R}$ an open interval. The $AI$-prototype is defined as $f_{AI} : N \times I \to N \times I$, $(v, t) \mapsto (Av, t)$.

A partially hyperbolic diffeomorphism $f$ on a (non-compact) manifold $\hat{M}$ is an $AI$-system if it has global product structure, preserves the orientation of its center direction, and is leaf conjugate to an $AI$-prototype.

**Theorem 6.1.** Suppose $f : \hat{M} \to \hat{M}$ is an AI-system with no invariant compact us-leaves. Then, either

1. $f$ is accessible,
2. there is an open set $V \subset \hat{M}$ such that
   \[ f(V) \subset V, \quad \bigcup_{k \in \mathbb{Z}} f^K(V) = \hat{M}, \quad \bigcap_{k \in \mathbb{Z}} f^K(V) = \emptyset, \]
   and the boundary of $V$ is a compact us-leaf, or
3. there are no compact us-leaves in $\hat{M}$, uncountably many non-compact us-leaves in $\hat{M}$ and there is $\lambda \neq 1$ such that $f$ is semiconjugate to $N \times \mathbb{R} \to N \times \mathbb{R}$, $(v, t) \mapsto (Av, \lambda t)$.

**Notation.** For a point $x$ on a manifold supporting a partially hyperbolic system, let $W^s(x)$ be the stable manifold through $x$, and $W^u(x)$ the unstable manifold. Then $AC(x)$, the accessibility class of $x$, is the smallest set containing $x$ which satisfies

\[ W^s(y) \cup W^u(y) \subseteq AC(x) \]

for all $y \in AC(x)$. For an arbitrary subset $X$ of the manifold, define

\[ W^s(X) = \bigcup_{x \in X} W^s(x), \quad W^u(X) = \bigcup_{x \in X} W^u(x), \quad \text{and} \quad AC(X) = \bigcup_{x \in X} AC(x). \]

Note that $AC(X)$ may or may not be a single accessibility class.

**Proposition 6.2 ([26]).** Suppose $f$ is a partially hyperbolic system with one-dimensional center on a (not necessarily compact) manifold $M$. For $x \in M$, the following are equivalent:
• $AC(x)$ is not open.
• $AC(x)$ has empty interior.
• $AC(x)$ is a complete $C^1$ codimension one submanifold.

If $L$ is a curve through $x$ tangent to the center direction, then the following are also equivalent to the above:
• $AC(x) \cap L$ is not open in $L$.
• $AC(x) \cap L$ has empty interior in $L$.

If $f$ is non-accessible, the set of non-open accessibility classes form a lamination.

Assumption 6.3. For the remainder of the section, assume $f : \hat{M} \to \hat{M}$ is a non-accessible AI-system.

All of the analysis of this section will be on the universal cover. Let $\tilde{M}$ and $\tilde{N}$ be the universal covers of $M$ and $N$. Then, $f$ and the leaf conjugacy $h$ lift to functions $f : \tilde{M} \to \tilde{M}$ and $h : \tilde{M} \to \tilde{N} \times I$ still denoted by the same letters. Every lifted center leaf of the lifted $f$ is of the form $h^{-1}(v \times I)$ for some $v \in \tilde{N}$. In general, the choice of the lifts of $f$ and $h$ are not unique. They may be chosen, however, so that $hf^{-1}(v \times I) = Av \times I$ where $A : \tilde{N} \to \tilde{N}$ is a hyperbolic Lie group automorphism. As $A$ fixes the identity element of the Lie group, there is a center leaf mapped to itself by $f$. Let $L$ denote this leaf. As $L$ is homeomorphic to $\mathbb{R}$, assume there is an ordering on the points of $L$ and define open intervals $(a, b) \subset L$ for $a, b \in L$ and suprema $\sup X$ for subsets $X \subset L$ exactly as for $\mathbb{R}$.

Define a closed subset
$$\Lambda = \{ t \in L : AC(t) \text{ is not open} \}.$$  

Lemma 6.4. $\Lambda$ is non-empty.

Proof. As $\tilde{M}$ is connected, if all accessibility classes were open, $f$ would be accessible (both on $\hat{M}$ and $\hat{M}$). Therefore, there is at least one non-open accessibility class. By global product structure, this class intersects $L$. \hfill $\Box$

Lemma 6.5. If $t \in \Lambda$, then $AC(t) = W^sW^u(t) = W^uW^s(t)$.

This is an adaptation to the case of global product structure of local arguments used in the proof of (6.2).

Proof. Each center leaf in $\tilde{M}$ is of the form $h^{-1}(v \times I)$ for some $v \in \tilde{N}$. By global product structure, for each $v \in \tilde{N}$, there exist unique points $x_v, y_v, z_v, t_v \in \tilde{M}$ such that
$$x_v \in W^s(t), \quad y_v \in W^u(x_v) \cap h^{-1}(v \times I), \quad z_v \in W^s(y_v), \quad t_v \in W^u(z_v) \cap L.$$  

See Figure 2. These points depend continuously on $v$. As $\tilde{N}$ is connected, the set
$$\{ t_v : v \in \tilde{N} \} \subset L \cap AC(t)$$  

is connected and, by (6.2), has empty interior as a subset of $L$. Therefore, it consists of the single point $t$. This shows that both $W^sW^u(t)$ and $W^uW^s(t)$ intersect each center leaf $h^{-1}(v \times I)$ in the same unique point $y_v$ and so the two
sets are identical. This set is both $s$-saturated and $u$-saturated and so contains $AC(t)$. □

By global product structure, for any $x \in \hat{M}$, there is a unique point $R(x) \in L$ such that $W^u(x)$ intersects $W^s(R(x))$. This defines a retraction, $R: \hat{M} \to L$. By the previous lemma, if $t \in \Lambda$, then $R^{-1}(t) = AC(t)$.

Let $\alpha: \tilde{M} \to \hat{M}$ be a deck transformation of the covering $\tilde{M} \to \hat{M}$. Then, as depicted in Figure 1, $\alpha$ defines a map $g_{\alpha} \in \text{Homeo}^+(\Lambda)$ given by the restriction of $R \circ \alpha$ to $\Lambda$. Define

$$G = \{g_{\alpha} : \alpha \in \pi_1(\hat{M})\}.$$ 

**Lemma 6.6.** $G$ is a finitely generated, nilpotent subgroup of $\text{Homeo}^+(\Lambda)$.

**Proof.** For $\alpha \in \pi_1(\hat{M})$ and $t \in \Lambda$, $g_{\alpha}(t)$ is given by the unique intersection of $\alpha(AC(t))$ and $L$. Then,

$$AC(g_{\alpha}(g_{\beta}(t))) = \alpha(AC(g_{\beta}(t))) = \alpha\beta(AC(t)) = AC(g_{\alpha\beta}(t))$$

shows that $\pi_1(\hat{M}) \to \text{Homeo}^+(\Lambda)$, $\alpha \mapsto g_{\alpha}$ is a group homomorphism. As $\hat{M}$ is homotopy equivalent to the nilmanifold $N$, its fundamental group is finitely generated and nilpotent. □

It is necessary to define $G$ with elements in $\text{Homeo}^+(\Lambda)$ as, in general, the same construction on $L$ will define a subset of $\text{Homeo}^+(L)$ but not a subgroup.

**Lemma 6.7.** For a point $t \in \Lambda$, $AC(t) \subset \tilde{M}$ projects to a compact $us$-leaf in $\hat{M}$ if and only if $t \in \text{Fix}(G)$.

**Proof.** Consider $t \in \Lambda$ and let $\tilde{X} \subset \tilde{M}$ be the image of $AC(t)$ by the covering $\tilde{M} \to \hat{M}$. First, suppose $t \in \text{Fix}(G)$. By global product structure, there is a unique map $\sigma: \tilde{N} \to AC(t)$ such that $h\sigma(v) \in v \times I$ for every $v \in \tilde{N}$. For any deck transformation $\alpha \in \pi_1(\hat{M})$,

$$\alpha(AC(t)) = AC(g_{\alpha}(t)) = AC(t).$$
which implies that \(a\sigma = a\alpha N\) where \(\alpha N\) is the corresponding deck transformation for the covering \(\hat{N} \to N\). It follows that \(\sigma\) quotients to a homeomorphism from the compact nilmanifold \(N\) to \(\hat{X}\) and therefore \(\hat{X}\) is compact.

To prove the converse, suppose \(\hat{X}\) is compact. From the definition of an AI-system, one can see that every center leaf on \(\hat{M}\) is properly embedded. Therefore, \(\hat{X}\) intersects each center leaf in a compact set. If \(\hat{X}\) is the pre-image of \(\hat{X}\) by covering \(\hat{M} \to \hat{M}\), then \(\hat{X}\) intersects each center leaf on \(\hat{M}\) in a compact set. In particular, \(\hat{X} \cap L\) is compact. Note that \(\hat{X} \cap L\) is exactly equal to the orbit \(Gt = \{g(t) : g \in G\}\). Define \(s = \sup Gt\). Then, \(s \in Gt\) by compactness and \(g(Gt) = Gt\) implies \(g(s) = s\) for each \(g \in G\). This shows that \(\{s\} = Gs = Gt\) and therefore \(t = s \in \text{Fix}(G)\).

**Lemma 6.8.** Suppose \(J \subset L\) is an open interval such that \(\partial J \subset \text{Fix}(f) \cap \text{Fix}(G)\). Let \(X\) be the image of \(AC(J)\) by the covering \(\hat{M} \to \hat{M}\). Then, \(f|_X\) is an AI-system.

This lemma is the justification for assuming there are no invariant, compact leaves in (6.1). If such leaves exist, the AI-system can be decomposed into smaller systems.

**Proof.** Assume the subinterval \(J\) in the hypothesis is of the form \(J = (a, b)\) with \(a, b \in L\). Unbounded subintervals of the form \((a, +\infty)\) and \((-\infty, b)\) are handled similarly.

For every center leaf \(h^{-1}(v \times I)\), let \(a_\nu, b_\nu \in I\) be such that \(v \times a_\nu \in h(AC(a))\) and \(v \times b_\nu \in h(AC(b))\). The set \(\hat{X} = \bigcup_{\nu \in \hat{N}} h^{-1}(v \times (a_\nu, b_\nu))\) is \(s\)-saturated, \(u\)-saturated, and contains \(J\). Therefore, \(AC(J) \subset \hat{X}\). By global product structure, one can show that \(\hat{X} \subset AC(J)\), so the two sets are equal. By its construction \(\hat{X}\) is simply connected, and invariant under deck transformations. Therefore, it is the universal cover for \(X\). Global product structure is inherited from \(\hat{M}\). For instance, for \(x, y \in AC(J)\), there is a unique point \(z \in \hat{M}\) such that \(z \in W^s(x) \cap W^c(y)\). Since, \(W^s(x) \subset \hat{X}\), \(z\) is in \(\hat{X}\).

Compose \(h\) with a homeomorphism which maps each \(v \times (a_\nu, b_\nu)\) to \(v \times (0, 1)\) by rescaling the second coordinate. This results in a leaf conjugacy between \(f\) on \(\hat{X}\) and \(A \times \text{id}\) on \(\hat{N} \times (0, 1)\) which quotients down to a leaf conjugacy from \(X\) to \(N \times (0, 1)\).

We now show that if the AI-system has no fixed compact \(us\)-leaves, then it satisfies either case (2) or case (3) of (6.1) depending on whether it has any (non-fixed) compact \(us\)-leaves.

**Lemma 6.9.** If \(\text{Fix}(G)\) is non-empty and \(\text{Fix}(f) \cap \text{Fix}(G)\) is empty, then \(f\) satisfies case (2) of (6.1).

**Proof.** We first show that \(f\) restricted to \(L\) is fixed-point free. Suppose, instead, that \(f(t) = t \in L\). By assumption \(t \notin \text{Fix}(G)\), so let \(J\) be the connected component of \(L \setminus \text{Fix}(G)\) containing \(t\). As \(\text{Fix}(G)\) is \(f\)-invariant, \(f(J) = J\) and each \(s \in \partial J\) is then an element of \(\text{Fix}(f) \cap \text{Fix}(G)\), a contradiction.
Without loss of generality, assume \( t < f(t) \) for all \( t \in L \). Choose some \( t_0 \in \text{Fix}(G) \) and define \( L^+ = \{ t \in L : t > t_0 \} \). Then,

\[
\overline{f(L^+)} \subset L^+, \quad \bigcup_{k \in \mathbb{Z}} f^k(L^+) = L, \quad \text{and} \quad \bigcap_{k \in \mathbb{Z}} f^k(L^+) = \emptyset.
\]

One can then show that the covering \( \tilde{M} \to \hat{M} \) takes \( AC(L^+) \) to an open set \( V \subset \tilde{M} \) which satisfies the second case of (6.1). \( \square \)

**Lemma 6.10.** If \( \text{Fix}(G) \) is empty, then \( f \) satisfies case (3) of (6.1).

**Proof.** In this case, the hypotheses of (5.5) hold with \( \Gamma = \Lambda \). Let \( P : L \to \mathbb{R} \) and \( \tau : G \to \mathbb{R} \) be as in (5.5).

If \( \alpha \in \pi_1(\hat{M}) \) is a deck transformation \( \hat{M} \to \hat{M} \), then \( hah^{-1} \) is equal to \( \alpha_N \times \text{id} \) on \( \hat{N} \times I \) for some deck transformation \( \alpha_N \in \pi_1(\hat{N}) \). As \( \hat{N} \) is a nilmanifold, any homomorphism from \( \pi_1(\hat{N}) \) to \( \mathbb{R} \) defines a unique homomorphism from the nilpotent Lie group \( \hat{N} \) to \( \mathbb{R} \) [33]. This implies that there is a unique Lie group homomorphism \( T : \hat{N} \to \mathbb{R} \) such that \( T\alpha_N(v) = T(v) + \tau(g_{\alpha}) \) for all \( v \in \hat{N} \) and \( \alpha \in \pi_1(\hat{M}) \).

Let \( R : \hat{M} \to L \) be the retraction defined earlier in this section and let \( H : \hat{M} \to \hat{N} \) be the composition of the leaf conjugacy \( h : \hat{M} \to \hat{N} \times I \) with projection onto the first coordinate. Define

\[
Q : \hat{M} \to \mathbb{R}, \quad x \mapsto PR(x) - TH(x).
\]

We will show that \( Q \) quotients to a function \( \hat{M} \to \mathbb{R} \) and use this to construct the semiconjugacy in the last case of (6.1).

First, consider a point \( x \in \hat{M} \) which has a non-open accessibility class. Then, \( R(x) \in \Lambda \) and, for \( \alpha \in \pi_1(\hat{M}) \),

\[
PR(\alpha(x)) = Pg_{\alpha}R(x) = PR(x) + \tau(g_{\alpha})
\]

and

\[
TH(\alpha(x)) = T\alpha_N(H(x)) = TH(x) + \tau(g_{\alpha})
\]

which together show \( Q\alpha(x) = Q(x) \).

Now, consider a point \( x \in \hat{M} \) which has an open accessibility class, and let \( J \subset \hat{M} \) be the connected component of \( W^c(x) \cap AC(x) \) which contains \( x \). The set \( \Gamma_0 \) from (5.5) is a subset of \( \Gamma = \Lambda \) and therefore \( P \) is constant on \( L \setminus \Lambda \). Then, \( PR \) is constant on \( J \) and, by continuity, constant on the closure of \( J \) as well. As \( H \) is constant on center leaves, \( Q = PR - TH \) is also constant on the closure of \( J \). Let \( y \) be a point on the boundary of \( J \). Then, as \( AC(y) \) is non-open, \( Q(x) = Q(y) = Q\alpha(y) = Q\alpha(x) \). This shows that \( Q \) quotients down to a function \( \hat{Q} : \hat{M} \to \mathbb{R} \). A much simpler argument shows that \( H : \hat{M} \to \hat{N} \) quotients down to a function \( \hat{H} : \hat{M} \to N \).

The properties of \( F \) and \( P \) in (5.5) imply that \( TA = \lambda T \) and therefore \( THf = TAH = \lambda TH \). As \( PRf = PfR = APR \), this shows that \( Qf = \lambda Q \). Then, \( \hat{H} \times \hat{Q} \) is the desired semiconjugacy in (6.1). By (5.5), \( P(\Lambda) = \mathbb{R} \) and so \( \Lambda \) is uncountable.
Each \( G \)-orbit of \( \Lambda \) corresponds to a distinct \( us \)-leaf, and so there are uncountably many.

This concludes the proof of (6.1). We note one additional fact which will be used in the next section.

**Corollary 6.11.** If \( \text{Fix}(G) \) is empty, any homeomorphism of \( L \) which commutes with \( f \) has a fixed point.

**Proof.** This follows from the use of (5.5) in the previous proof. \( \square \)

7. \( AB \)-systems

**Assumption 7.1.** In this section, assume \( f : M \to M \) is a non-accessible \( AB \)-system.

The \( AB \)-prototype \( f_{AB} \) has an invariant center leaf which is a circle. By the leaf conjugacy, \( f \) also has an invariant center leaf. Call this leaf \( S \). Note that \( f \) lifts to an \( AI \)-system. This is because the \( AB \)-prototype \( f_{AB} \) lifts to the \( AI \)-prototype \( A \times \text{id} \) on \( N \times \mathbb{R} \). If \( h : M \to M_B \) is the leaf conjugacy, then \( h f h^{-1} \) is homotopic to \( f_{AB} \) and therefore also lifts to \( N \times \mathbb{R} \).

Let \( \pi : \tilde{M} \to M \) be the universal covering, and choose a lift \( \tilde{f} : \tilde{M} \to \tilde{M} \) and \( \tilde{S} \) a connected component of \( \pi^{-1}(S) \) such that \( \tilde{f}(\tilde{S}) = \tilde{S} \). The universal cover \( \tilde{N} \times \mathbb{R} \) of the manifold \( M_B \) has a deck transformation of the form \( (v, t) \mapsto (B v, t - 1) \). Conjugating this by the lifted leaf conjugacy gives a deck transformation \( \beta : \tilde{M} \to \tilde{M} \) and one can assume that \( \beta(\tilde{S}) = \tilde{S} \). Then, \( \tilde{S} \) plays the role of \( L \) in the previous section. Define \( \Lambda = \{ t \in \tilde{S} : AC(\pi(t)) \text{ is not open} \} \) and \( G \) as a subgroup of \( \text{Homeo}^+(\Lambda) \) as in the previous section.

**Lemma 7.2.** \( \text{Fix}(G) \) is non-empty.

**Proof.** This follows from (6.11) since \( \beta \) and \( \tilde{f} \) are commuting diffeomorphisms when restricted to \( \tilde{S} \) and \( \beta \) is fixed-point free. \( \square \)

**Lemma 7.3.** For \( t \in \Lambda \), \( AC(\pi(t)) \subset M \) is compact if and only if \( t \in \text{Fix}(G) \).

**Proof.** If \( t \in \text{Fix}(G) \), then, by (6.7), \( AC(\pi(t)) \) is covered by a compact \( us \)-leaf of the \( AI \)-system and is therefore compact itself.

Conversely, suppose \( t \in \Lambda \) is such that \( AC(\pi(t)) \subset M \) is a compact \( us \)-leaf. Note that as \( \beta(\text{Fix}(G)) = \text{Fix}(G) \) there are \( a, b \in \text{Fix}(G) \) such that \( a < t < b \) in the ordering on \( \tilde{S} \). Then, \( G t \) is contained in \( (a, b) \), a bounded subset of \( \tilde{S} \). Considering the supremum as in (6.7), one shows that \( s := \sup G t \) is in \( \text{Fix}(G) \). Consequently, \( AC(\pi(t)) \) accumulates on \( \pi(s) \) which, as \( AC(\pi(t)) \) is compact, implies \( \pi(s) \in AC(\pi(t)) \) and so there is a deck transformation \( \alpha : \tilde{M} \to \tilde{M} \) such that \( \alpha(s) \in AC(t) \). This implies there is \( k \in \mathbb{Z} \) and \( g \in G \) such that \( t = \beta^k g(s) = \beta^k(s) \in \text{Fix}(G) \). \( \square \)

In this, and the next two sections, define

\[ K = \{ x \in S : AC(x) \subset M \text{ is compact} \}. \]
The last lemma shows that $K = \pi(\text{Fix}(G))$.

**Corollary 7.4.** $K$ is closed and non-empty.

This also completes the proof of (1.6).

**Corollary 7.5.** $K \cap \text{NW}(f|_S)$ is non-empty.

*Proof.* $K$ is non-empty, $f$-invariant, and closed.

**Corollary 7.6.** $f$ has a compact periodic us-leaf if and only if $f|_S$ has rational rotation number.

*Proof.* As a consequence of (7.3), any compact us-leaf $X$ in $M$ intersects $S$ in a unique point $t$. If $f^n(X) = X$ then $f^n(t) = t$ and $f|_S$ has rational rotation number. If, conversely, $f|_S$ has rational rotation number, its non-wandering set consists of periodic points, and a compact periodic leaf exists by (7.5).

The following is also from the last proof.

**Corollary 7.7.** All compact periodic us-leaves have the same period.

**Lemma 7.8.** If $K = S$, then $f$ on $M$ is topologically conjugate to a function $(v, x) \mapsto (Av, \tilde{r}(x))$ defined on the manifold $M_B = N \times \mathbb{R}/(Bv, t) \sim (v, t + 1)$ where $\tilde{r} : \mathbb{R} \to \mathbb{R}$ is a lift of a homeomorphism $r : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ topologically conjugate to $f|_S$.

*Proof.* Let $\phi : \tilde{S} \to \mathbb{R}$ be any homeomorphism such that $\phi \beta(t) = \phi(t) + 1$ for all $t$. Define $\tilde{r}$ as $\phi \tilde{f} \phi^{-1}$. Extend $\phi$ to all of $\tilde{M}$ by making it constant on accessibility classes. As in the proof of (6.10), let $H : \tilde{M} \to \tilde{N}$ be the first coordinate of the lifted leaf conjugacy $h : \tilde{M} \to \tilde{N} \times \mathbb{R}$. Then, the function $H \times \phi : \tilde{M} \to \tilde{N} \times \mathbb{R}$ gives a topological conjugacy between $\tilde{f}$ on $\tilde{M}$ and $A \times \tilde{r}$.

The fundamental group of $M_B$ is generated by deck transformations of the form $(v, t) \mapsto (\alpha_N(v), t)$ or $(v, t) \mapsto (Bv, t - 1)$. Using the fact that $\text{Fix}(G) = \tilde{S}$ and the definition of $\tilde{r}$, one can then show that $H \times \phi$ quotients down to a topological conjugacy defined from $M$ to $M_B$.

**Lemma 7.9.** Suppose $J \subset S$ is an open interval such that $\partial J \subset \text{Fix}(f) \cap K$. Then, $f|_{AC(J)}$ is an AI-system.

*Proof.* Let $\tilde{J}$ be a lift of $J$ to $\tilde{S}$. Then, as $f(J) = J$, $\tilde{f}(\tilde{J}) = \beta^k(\tilde{J})$ for some $k \in \mathbb{Z}$. By replacing the lift $\tilde{f}$ by $\tilde{f} \beta^k$, assume, without loss of generality that $\tilde{f}(\tilde{J}) = \tilde{J}$. As $K = \pi(\text{Fix}(G))$, $\partial \tilde{J} \subset \text{Fix}(\tilde{f}) \cap \text{Fix}(G)$, and so by (6.8), $AC(\tilde{J})$ projects to $X$ on $\tilde{M}$ such that the dynamics on $X$ is an AI-system. As $\tilde{J}$ is contained in a fundamental domain of the covering $\tilde{S} \to S$, one can show that $X$ is contained in a fundamental domain of the covering $\tilde{M} \to M$. Therefore, the dynamics on $\pi(AC(\tilde{J})) = AC(J)$ is an AI-system.
We now give a $C^0$ version of (4.3).

**Lemma 7.10.** There is a continuous surjection $p : M \rightarrow \mathbb{S}^1$ such that $p|_S$ is a homeomorphism, $p|_{W^c(x)}$ is a covering for any center leaf $W^c(x)$ ($x \in M$) and $p$ is constant on each compact accessibility class.

**Proof.** Define $p$ on $S$ so that $p|_S$ maps $S$ to $\mathbb{S}^1$ with constant speed along $S$. Extend $p$ to $AC(K) \cup S$ by making $p$ constant on accessibility classes. Then, for any center leaf $W^c(x)$, let $J$ be a connected component of $W^c(x) \setminus AC(K)$ and define $p$ on $J$ so that $J$ is mapped at constant speed to $\mathbb{S}^1$ and extends continuously to the boundary $\partial J \subset AC(K)$. Transversality of the center foliation and us-lamination implies that $p$ is continuous. The other properties are easily verified. □

Compare this short $C^0$ proof to the $C^1$ proof in Section 11.

We now consider the cases of rational and irrational rotation of $f|_S$ separately in the next two sections.

### 8. Rational Rotation

**Assumption 8.1.** Assume $f$ is a non-accessible AB-system with at least one periodic compact us-leaf.

Let $S$, $K$, and other objects be defined as in Section 7. By (7.7), all compact periodic leaves have the same period. Call this period $n$. Define $K_n = K \cap \text{Fix}(f^n) \subset S$. By (7.4), $K_n$ is closed. Let $p : M \rightarrow \mathbb{S}^1$ be the projection given by (7.10) and define $U \subset \mathbb{S}^1$ as $U = \mathbb{S}^1 \setminus p(K_n)$.

Note that if $t \notin U$, then $p^{-1}(t)$ is an $f^n$-invariant compact us-leaf. Moreover, every such leaf is of this form. This proves the first part of (1.7).

To prove the rest of the theorem, replace $f$ by its iterate $f^n$ and assume $n = 1$. The new $f$ is still an AB-system, albeit with a different “$A$” than before. Now $K_1 = \text{Fix}(f) \cap K \subset S$. If $I$ is a connected component of $U \subset \mathbb{S}^1$, then $p^{-1}(I) \cap S$ is a connected component of $S \setminus K_1$ and (7.9) implies that $f$ restricted to $p^{-1}(I) = AC(\pi(I))$ is an AI-system. Since $I \cap K_n$ is empty, $AC(I)$ contains no invariant compact us-leaves. Therefore, the AI-system falls into one of the cases given in (6.1). As these cases correspond exactly to those given in (1.7), this concludes the proof.

### 9. Irrational Rotation

**Assumption 9.1.** Assume $f$ is a non-accessible AB-system with no periodic compact us-leaves.

Let $S$, $K$ and other objects be defined as in Section 7. By (7.6), $f|_S$ has irrational rotation number.
Lemma 9.2. \( NW(f|_{S}) \subset K \).

Proof. For any \( C^1 \) circle diffeomorphism with irrational rotation, the non-wandering set is minimal. The result then follows from (7.5). \( \square \)

Lemma 9.3. If \( I \) is a connected component of \( S \setminus NW(f|_{S}) \), then \( AC(I) \) is a wandering domain. That is, the sets \( f^k(AC(I)) = AC(f^k(I)) \) are pairwise disjoint for all \( k \in \mathbb{Z} \).

Proof. Let \( J \) be the closure of \( I \). Note that any compact leaf in \( AC(J) \) must be of the form \( AC(t) \) for some \( t \in J \). By the properties of circle diffeomorphisms, the sets \( f^k(J) \) are pairwise disjoint. By the last lemma, \( \partial J \subset K \). If \( AC(J) \) intersects \( AC(f^k(J)) \), then this intersection has a boundary consisting of compact \( u \)-leaves. Such a compact leaf would intersect \( S \) in a point \( t \in J \cap f^k(J) \), a contradiction. \( \square \)

Lemma 9.4. \( NW(f) = AC(NW(f|_{S})) \).

Proof. The last lemma shows \( NW(f) \subset AC(NW(f|_{S})) \).

To prove the other inclusion, suppose \( t \in NW(f|_{S}) \), \( x \in AC(t) \) and \( V \subset M \) is a neighbourhood of \( x \). There is a sequence \( \{n_k\} \) such that \( f^{n_k}(t) \) converges to \( t \). By taking a further subsequence, assume \( f^{n_k}(x) \) converges to some point \( y \in AC(t) \). Let \( D \subset V \) be a small unstable plaque containing \( x \). Then \( f^{n_k}(D) \) is a sequence of ever larger unstable plaques, and

\[
W^u(y) \subset \bigcup_{k} f^{n_k}(D).
\]

Unstable leaves of the Anosov diffeomorphism \( A \) are dense in \( N \) [19]. Therefore, by the leaf conjugacy, \( W^u(y) \) is dense in \( AC(t) \). This shows that some iterate \( f^{n_k}(V) \) intersects \( V \). \( \square \)

Now, let \( p : M \to \mathbb{S}^1 \) be as in (7.10). We may assume \( p|_{S} \) is a \( C^1 \)-diffeomorphism. Define \( r : \mathbb{S}^1 \to \mathbb{S}^1 \) by \( rp(t) = pf(t) \) for all \( t \in S \). Then, (1.8) can be proved from the above lemmas. As \( r \) has irrational rotation number, it is semiconjugate to a rigid rotation \( t \mapsto t + \theta \). Using this and the leaf conjugacy, one can prove (1.9) using an argument similar to the proof of (7.8).

10. PROVING THEOREMS (1.2), (1.4), AND (1.5)

This section gives the proofs of several of the theorems stated in Section 1 based on results proved in other sections.

The proof of (1.5) makes use of a result of Brin regarding transitivity [8]. The following is an extension of this result to the non-compact case, though the proof is in essence the same.

Proposition 10.1 (Brin). Suppose \( f \) is a partially hyperbolic diffeomorphism of a (not necessarily compact) manifold \( M \). If \( V \) is open and \( f(V) = V \subset NW(f) \), then \( V = AC(V) \).

In particular, if \( f \) is accessible and \( NW(f) = M \), then \( f \) is transitive.
Proof. For $\epsilon > 0$ and $y \in M$, let $W^u_\epsilon(y)$ be the set of all points reachable from $y$ by a path tangent to $E^u$ of length less than $\epsilon$.

If $x \in V$, then $x \in NW(f)$ implies there are sequences $\{x_k\}$ and $\{y_k\}$ both converging to $x$ and such that $y_k = f^{j_k}(x_k)$ for some non-zero $j_k \in \mathbb{Z}$. By swapping $x_k$ with $y_k$ if necessary, assume every $j_k$ is positive. If $j_k$ is bounded, then $x$ is periodic, so we may freely assume that $j_k \to +\infty$. As $V$ is open, there is $\epsilon > 0$ such that $W^u_\epsilon(x_k) \subset V$ for all large $k$. The uniform expansion of $E^u$ implies there is $r_k \to \infty$ such that $W^u_i(y_k) \subset f^{j_k}(W^u_i(x_k)) \subset f^{j_k}(V) = V$ and therefore the entire unstable manifold $W^u(x)$ lies in the closure of $V$. This proves $W^u(\overline{V}) = \overline{V}$. Similarly, $W^s(\overline{V}) = \overline{V}$ and so $AC(\overline{V}) = \overline{V}$.

Proof of (1.5). By (10.1), any accessible $f$ satisfies case (1) of (1.5). Therefore, assume that $f$ is non-accessible.

For now, assume $f$ has no periodic compact $us$-leaves, so that (1.8) holds. That theorem, with the assumption $NW(f) = M$, implies that $NW(r) = \mathbb{S}^1$ and that every point in $M$ lies in a compact $us$-leaf. This shows that (7.8) holds and the $r$ in that lemma can be taken as the same $r$ in (1.8). As $NW(r) = \mathbb{S}^1$, $r$ is topologically conjugate to a rigid rotation $t \to t + \theta$ and therefore $f$ satisfies case (2) of (1.5).

For the remainder of the proof, assume $f$ has a periodic compact $us$-leaf, so that (1.7) holds. Let $I$ be a connected component of $U$ and $g : p^{-1}(I) \to p^{-1}(I)$ be as in (1.7). The condition $NW(f) = M$ implies $NW(g) = p^{-1}(I)$. This is only possible in the first of the three cases in (1.7), where $g$ is accessible. Then, $g$ is transitive by (10.1).

If $t \in \mathbb{S}^1 \setminus U$, then $f^n$ restricted to $p^{-1}(I)$ is topologically conjugate to a hyperbolic nilmanifold automorphism and is therefore transitive [19]. Hence, if $U$ is non-empty, the third case of (1.5) is satisfied.

If $U$ is empty, then every $p^{-1}(t)$ is an $f^n$-invariant compact $us$-leaf and (7.8) holds with $r : \mathbb{S}^1 \to \mathbb{S}^1$ topologically conjugate to a rigid rational rotation $t \to t + \theta$. This shows that $f$ is in case (2) of (1.5).

To prove ergodicity of the components of the decomposition given in (1.4), we use results given in [12], [26], and in the classical work of Birkhoff and Hopf. These results were formulated for systems on compact manifolds, but the proofs are local in nature, involving short holonomies along stable and unstable manifolds. The results, therefore, generalize to the non-compact case so long as the measure is still finite.

**Proposition 10.2.** Let $f$ be a homeomorphism of a (not necessarily compact) manifold $M$ and let $C_0(M)$ be the space of continuous functions $M \to \mathbb{R}$ with compact support. Suppose $\mu$ is an invariant measure with $\mu(M) = 1$ and there is an invariant closed submanifold $S$ such that $\mu$ is equivalent to Lebesgue measure on $S$. 

(1) For $\phi \in C_0(M)$ the limits

$$
\phi^s(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi f^k(x) \quad \text{and} \quad \phi^u(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi f^{-k}(x)
$$

exist and are equal $\mu$-almost everywhere.

(2) There is a countable set \( \{\phi_j\}_{j=1}^{\infty} \subset C_0(M) \) (depending only on \( M \)) such that \((f, \mu)\) is ergodic if and only if $\phi_j$ and $\phi_j^u$ are constant $\mu$-almost everywhere for every $j$.

Further, suppose $f$ is a $C^2$ partially hyperbolic diffeomorphism with one dimensional center.

(3) If $\phi \in C_0(M)$, then $\phi^s$ is constant on stable leaves and $\phi^u$ is constant on unstable leaves.

(4) If $S = M$, $X^s, X^u \subset M$ are measurable, and

$$W^s(X^s) = X^s, \quad W^u(X^u) = X^u \quad \text{and} \quad \mu(X^s \cup X^u) = 0,$$

then there is $X \subset M$ measurable such that

$$\mu(X^s \cup X^u) = 0 = \mu(X^u \cup X^s).$$

(5) If $S = M$ and $f$ is accessible, then $(f, \mu)$ is ergodic.

Proof. Item (1) is a re-statement of the classic Birkhoff Ergodic Theorem.

To prove (2), let $\{\phi_j\}$ be a countable set whose linear span is dense in $C_0(M)$ with respect to the supremum norm. As any function in $C_0(S)$ may be extended to a function in $C_0(M)$, the linear span of $\{\phi_j\}$ is dense in $L^1(\mu)$. Suppose the bounded linear operator $\phi \mapsto \phi^s$ on $L^1(\mu)$ takes every element of $\{\phi_j\}$ to the subspace of constant functions. By density, every $\phi \in L^1(\mu)$ is mapped to the same subspace. Therefore $(f, \mu)$ is ergodic. The converse statement in (2) follows directly from the properties of ergodicity.

Proofs of (3)–(5) can be found in both [12] and [26].

Proof of (1.4). As $\mu$ is a finite, $f$-invariant measure which is equivalent to Lebesgue, $NW(f) = M$ by Poincaré recurrence. Let $p$, $n$, and $U$ then be given as in (1.5). By (4.3), assume $p \cdot \mu = m$ where $m$ is Lebesgue measure on $S^1$. Without loss of generality, assume $n = 1$.

For each connected component $I$ of $U$, the set $p^{-1}(I)$ is an accessibility class and therefore $(f, \mu)$ is ergodic by (10.2) where $\mu_I$ is as in (1.3).

Let $\{\phi_j\}_{j=1}^{\infty}$ be as in (10.2) and for $j \in \mathbb{N}$ and $q \in \mathbb{Q}$ define $X^s_{j,q} = \{x \in M : \phi_j^s(x) < q\}$. Define $X^u_{j,q}$ similarly. By items (3) and (4) of (10.2), there is $X_{j,q} = AC(X_{j,q})$ equal mod zero to both $X^s_{j,q}$ and $X^u_{j,q}$. Define a “bad” set $Y$ by

$$Y = \bigcup_{j,q} (X^s_{j,q} \cup X^u_{j,q} \cup X^s_{j,q} \cup X^u_{j,q})$$

and note that $\mu(Y) = 0$. Equation (1.3) implies that there is a “good” set $Z \subset S^1 \setminus U$ such that $U \cup Z$ has full measure in $S^1$ and $\mu((U \cap Z)^1(t)) = 0$ for all $t \in Z$ where
 muslim of the decomposition in (1.3). By (4.3), we may further assume that 
\( \mu_t \) is equivalent to Lebesgue measure on \( p^{-1}(t) \) for all \( t \in Z \).

As \( p^{-1}(t) \) is an accessibility class, every \( X_{j,q} \cap p^{-1}(t) \) is either empty or all of \( p^{-1}(t) \). Therefore for \( t \in Z \), every \( X_{j,q} \) and \( X_{j,q}^u \) either has \( \mu_t \)-measure equal to zero or one, and item (2) of (10.2) implies that \( (f, \mu_t) \) is ergodic. Thus, modulo a set of measures whose combined support has \( \mu \)-measure zero, every measure in \( (1.3) \) is ergodic. This shows that \( (1.3) \) is the ergodic decomposition of \( \mu \).

One might be tempted to prove (1.4) by arguing that for \( t \in U \), \( f \) restricted to \( p^{-1}(t) \) is an Anosov diffeomorphism and therefore the invariant measure \( \mu_t \) is ergodic. The problem is that we have only shown that \( p^{-1}(t) \) is a \( C^1 \) submanifold of \( M \), which is not enough regularity to conclude ergodicity for an Anosov system. Hence, the above proof.

Proof of (1.2). If \( f \) is in case (1) or (3) of (1.5), it is fairly easy to show that \( f \) is also in the corresponding case of (1.2). Therefore, assume \( f \) is in case (2) of (1.5).

If \( \theta \) is rational, then \( (v, t) \to (Av, t + \theta) \) is non-transitive and therefore \( f \) is not ergodic.

Suppose \( \theta \) is irrational and \( f \) is not ergodic. Then there are \( j \in \mathbb{N} \) and \( q \in \mathbb{Q} \) such that the sets \( X_{j,q}^s \), \( X_{j,q}^u \), and \( X_{j,q} \), defined as in the last proof, have neither zero measure nor full measure with respect to the \( f \)-invariant measure \( \mu \). Write \( X = X_{j,q} \). As \( X = AC(X) \), there is \( Y \subset S^1 \) such that \( X = p^{-1}(Y) \) and \( p_* \mu = m \) implies that \( \mu(Y) \) is neither zero nor one. The condition \( p_* \mu = m \) further implies that \( p \) gives a semiconjugacy from \( f \) to a rigid irrational rotation \( R_\theta(x) = x + \theta \) on \( S^1 \). Then, \( f(X) = X \) implies \( R_\theta(Y) = Y \) which contradicts the ergodicity of \( (R_\theta, m) \).

11. Regularity

This section proves (1.10), showing that the \( us \)-lamination of a partially hyperbolic diffeomorphism is \( C^1 \) if the center is one-dimensional and the diffeomorphism is \( C^2 \).

Proposition 11.1. Suppose \( f : M \to M \) is a \( C^2 \) dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center. Then any unstable holonomy \( h \) inside a \( cu \)-leaf is \( C^1 \). Moreover, the derivative of \( h \) tends uniformly to one as the unstable distance between the point \( x \) and its image \( h(x) \) tends to zero.

Proof. That such a holonomy is \( C^1 \) is proved in an erratum [44] to the paper [43]. If \( y \in W^u(x) \) and \( h \) is the holonomy taking \( x \) to \( y \), then adapting the argument in §3 of [41] one can show that the norm of the derivative of \( h \) at \( x \) is given by

\[
J_{xy} = \prod_{n=0}^{\infty} \frac{\| T^c_{f^{-n}(y)}f \|}{\| T^c_{f^{-n}(x)}f \|}
\]

where \( T^c_z f : E^c_z \to E^c_{f(z)} \) is the restriction of the derivative \( T_z f : T_z M \to T_{f(z)} M \).

As \( f \) is \( C^2 \), the derivative \( T_z f \) is Lipschitz in \( z \) and the center bundle \( E^c \) is Hölder
also implies that if \( \rho_\sigma \) sequence are used it is therefore continuous on all of \( \Sigma \). Suppose \( f : M \to M \) is a \( C^2 \) dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center. Suppose \( L_0 \subset M \) is a compact interval inside a center leaf and \( g : L_0 \to \mathbb{R} \) is \( C^1 \). Then \( g \) extends to a \( C^1 \) function defined on a neighbourhood of \( L_0 \) which is constant on us-leaves.

**Proof.** Without loss of generality, assume \( g \) is defined so that \( |g(x) - g(y)| \) is the arc length of the center segment between \( x \) and \( y \). Any other \( C^1 \) function on \( L_0 \) can be constructed by composition with this specific \( g \).

By local product structure and the compactness of \( L_0 \), one may construct a compact set \( C \subset M \) containing \( L_0 \) with the following properties:

- The interior of \( C \) contains the (one-dimensional) interior of \( L_0 \).
- If \( W^c(x) \) is a center leaf, then every connected component of \( W^c(x) \cap C \) is a compact interval, called a “center segment.”
- If \( AC(y) \) is a us-leaf, then every connected component of \( AC(y) \cap C \) is a compact set homeomorphic to a closed ball and called a “us-plaque.”
- Each center segment intersects each us-plaque in exactly one point.
- \( L_0 \) is a center segment.

By a \( C^1 \) change of coordinates, assume that \( C \subset \mathbb{R}^d \).

Let \( \Sigma \subset C \) be the union of all us-plaques, and \( \Sigma' \subset \Sigma \) the union of all us-plaques which are accumulated on by other us-plaques. If \( x \in \Sigma' \), define

\[
D(x) = \lim_{n \to \infty} \frac{\|\sigma_n \cap L_0 - \sigma \cap L_0\|}{\|\sigma_n \cap L - \sigma \cap L\|}
\]

where \( L \) is the center segment through \( x \), \( \sigma \) is the us-plaque through \( x \), and \( \sigma_n \) are us-plaques converging to \( \sigma \). By (11.1), this limit exists, is independent of the sequence \( \sigma_n \) tending to \( \sigma \), and is non-zero. The \( C^1 \) regularity of the holonomies also implies that if \( \rho_n \) is another sequence of us-plaques converging to \( \sigma \), then

\[
D(x) = \lim_{n \to \infty} \frac{\|\sigma_n \cap L_0 - \rho_n \cap L_0\|}{\|\sigma_n \cap L - \rho_n \cap L\|}
\]

so long as \( \sigma_n \neq \rho_n \) for large \( n \). Further, by (11.1), the ratio \( D(L_1 \cap \sigma)/D(L_2 \cap \sigma) \) tends uniformly to one as \( \text{dist}(L_1, L_2) \) tends to zero. As \( D \) is continuous when restricted to each center segment and uniformly continuous on each us-plaque \( \sigma \), it is therefore continuous on all of \( \Sigma' \). Define \( D(x) = 1 \) for all \( x \in L_0 \) and note that this agrees with the above definition on the intersection \( \Sigma' \cap L_0 \). Then, choose a continuous positive extension \( D : \Sigma \cup L_0 \to \mathbb{R} \).

Also extend \( g : L_0 \to \mathbb{R} \) to a function \( g : \Sigma \cup L_0 \to \mathbb{R} \) by making it constant on each us-plaque. To further extend \( g \) to a \( C^1 \) function on all of \( C \), we will...
define for each point \( x \in \Sigma \cup L_0 \) a candidate derivative \( dg_x : \mathbb{R}^d \to \mathbb{R} \) and show that Whitney’s extension theorem applies. Choose an orientation for \( E^c \) and for each \( x \in \Sigma \cup L_0 \), let \( v^c_x \) be the unique oriented unit vector in \( E^c_x \). Define \( dg_x \) as the unique linear map such that \( dg_x(v^c_x) = D(x) \) and \( \ker dg_x = E^u_x \oplus E^s_x \). As both \( D(x) \) and the splitting \( E^u_x \oplus E^s_x \) are continuous in \( x \), the linear map \( dg_x \) is continuous in \( x \).

Define the function \( R : C \times C \to \mathbb{R} \) by

\[
R(x_n, y_n) = \frac{1}{\|y_n - x_n\|} \left( g(y_n) - g(x_n) - dg_{x_n}(y_n - x_n) \right).
\]

To apply Whitney’s extension theorem, one needs to show that for any two sequences \( \{x_n\}_{n=1}^\infty \) and \( \{y_n\}_{n=1}^\infty \) with \( \|x_n - y_n\| \) converging to zero, the sequence \( R(x_n, y_n) \) also converges to zero. If this does not hold, there are sequences \( \{x_n\} \) and \( \{y_n\} \) so that \( R(x_n, y_n) \) is bounded away from zero. Therefore, without loss of generality, one may replace these sequences by subsequences and assume \( x_n \) and \( y_n \) both converge to a point \( q \in C \). We will also restrict to further subsequences as necessary later in the proof.

We prove the convergence in progressively more general cases.

**Case 1.** First, assume \( x_n, y_n, \) and \( q \) are all on the same center segment \( L \neq L_0 \). Let \( \sigma_n, \rho_n \) and \( \sigma \) be such that

\[
\sigma_n \cap L = x_n, \quad \rho_n \cap L = y_n, \quad \text{and} \quad \sigma \cap L = q.
\]

If \( \sigma \notin \Sigma' \), then \( x_n = y_n = q \) for large \( n \). Therefore, assume \( \sigma \in \Sigma' \). Then,

\[
\lim_{n \to \infty} \frac{g(y_n) - g(x_n)}{\|x_n - y_n\|} = \lim_{n \to \infty} \frac{\|\sigma_n \cap L \ominus \rho_n \cap L\|}{\|\sigma_n \cap L - \rho_n \cap L\|} = D(q).
\]

As both the candidate derivative \( dg_x \) and the center direction \( v^c_x \) are continuous in \( x \),

\[
\lim_{n \to \infty} \frac{1}{\|y_n - x_n\|} dg_{x_n}(y_n - x_n) = \left( \lim_{n \to \infty} \|y_n - x_n\| \right) \left( \lim_{n \to \infty} \frac{g(y_n) - g(x_n)}{\|y_n - x_n\|} \right) = d g_q(v^c_q) = D(q).
\]

Therefore, \( \lim_{n \to \infty} R(x_n, y_n) = D(q) - D(q) = 0 \).

**Case 2.** Now, consider the case where \( x_n \) and \( y_n \) are on the same center segment \( L_n \) for each \( n \). Define \( x_n^c \) to be on the same us-plaque as \( x_n \) and the same center segment as \( q \). Define \( y_n^c \) similarly. Then,

\[
g(x_n) - g(y_n) = g(x_n^c) - g(y_n^c).
\]

By (11.1),

\[
\lim_{n \to \infty} \|y_n - x_n\| = 1.
\]

Thus,

\[
\lim_{n \to \infty} \frac{g(y_n) - g(x_n)}{\|y_n - x_n\|} = \lim_{n \to \infty} \frac{g(y_n^c) - g(x_n^c)}{\|y_n^c - x_n^c\|} = D(q)
\]
where the last equality is by the previous case. As before,
\[ \lim_{n \to \infty} \frac{1}{\| y_n - x_n \|} d g_{x_n}(y_n - x_n) = d g_q(v^q) = D(q) \]
and therefore \( \lim_{n \to \infty} R(x_n, y_n) = 0 \).

**Case 3.** Now consider \( x_n \) and \( z_n \) as general sequences in \( \Sigma \) converging to \( q \). Define \( y_n \) as the unique point lying on the same center segment as \( x_n \) and the same \( us \)-plaque as \( z_n \). By taking subsequences, assume
\[ \lim_{n \to \infty} \frac{z_n - y_n}{\| z_n - y_n \|} \]
equals. By continuity of the partially hyperbolic splitting, this limit is in \( E^u_q \oplus E^s_q \).

Therefore,
\[ \lim_{n \to \infty} \frac{1}{\| z_n - y_n \|} d g_{x_n}(z_n - y_n) = (\lim_{n \to \infty} d g_{x_n}) (\lim_{n \to \infty} \frac{z_n - y_n}{\| z_n - y_n \|}) = 0 \]
implying, with \( g(z_n) = g(y_n) \), that
\[ \lim_{n \to \infty} \frac{1}{\| z_n - y_n \|} (g(z_n) - g(y_n) - d g_{x_n}(z_n - y_n)) = 0. \]
By transversality of the foliations, there is a constant \( c_1 > 0 \) such that \( \| z_n - x_n \| \geq c_1 \| z_n - y_n \| \) and therefore
\[ \lim_{n \to \infty} \frac{1}{\| z_n - x_n \|} (g(y_n) - g(x_n) - d g_{x_n}(y_n - x_n)) = 0. \]
Added together, these limits show that \( \lim_{n \to \infty} R(x_n, z_n) = 0 \).

**Case 4.** Now consider the case where \( x_n \in L_0 \) and \( z_n \in \Sigma \) for all \( n \). Define \( y_n \) from \( x_n \) and \( z_n \) exactly as in the last case. Then,
\[ R(x_n, z_n) = \frac{1}{\| z_n - x_n \|} (g(z_n) - g(y_n) - d g_{x_n}(z_n - y_n)) + \]
\[ \frac{1}{\| z_n - x_n \|} (g(y_n) - g(x_n) - d g_{x_n}(y_n - x_n)) \]
and, similar to the previous case, both summands can be shown to converge to zero. The case \( x_n \in \Sigma \) and \( z_n \in L_0 \) is almost identical.

**Case 5.** If both \( \{ x_n \} \) and \( \{ z_n \} \) are in \( L_0 \), then \( \lim_{n \to \infty} R(x_n, z_n) = 0 \) simply by the fact that \( g \) is \( C^1 \) when restricted to \( L_0 \).

**The general case.** The final case to consider is where \( \{ x_n \} \) and \( \{ z_n \} \) are general sequences in \( X = \Sigma \cup L_0 \). By taking subsequences, one can assume each sequence lies either entirely in \( L_0 \) or entirely in \( \Sigma \) and therefore reduce to a previous case. \( \square \)

We now prove the following restatement of (1.10).
Corollary 11.3. If $f : M \to M$ is a non-accessible, partially hyperbolic $C^2$ diffeomorphism with one-dimensional center, the non-open accessibility classes form a $C^1$ lamination. That is, around any point $x \in M$ there is a neighbourhood $V$ and functions $g : V \to \mathbb{R}$ and $\psi : V \to \mathbb{R}^{d-1}$ such that $g \times \psi$ is a $C^1$ embedding and if $AC(y)$ is a $us$-leaf and $\sigma$ a connected component of $AC(y) \cap V$, then $\sigma = g^{-1}(t)$ for some $t \in \mathbb{R}$.

Proof. Define a coordinate chart $\phi \times \psi : V \to \mathbb{R} \times \mathbb{R}^{d-1}$ such that the kernel of the derivative $d\phi : T_x M \to \mathbb{R}$ at $x$ is equal to $E^u_x \oplus E^s_x$. By (11.2), after replacing $V$ by a subset, there is a $C^1$ function $g : V \to \mathbb{R}$ constant on $us$-plaques and such that $g$ and $\phi$ are equal on a center segment through $x$. Then, the derivative of $g \times \psi$ is invertible at $x$ and so, after again replacing $V$ by a subset, $g \times \psi$ is the desired $C^1$ embedding. □

We now proceed to prove (4.3). Recall the definition of an AI-system from Section 6.

Proposition 11.4. Let $f : \hat{M} \to \hat{M}$ be a $C^2$ AI-system and $X \subset \hat{M}$ a compact $us$-leaf. Then, there is a neighbourhood $V$ of $X$, an open subset $U \subset (0, 1)$ and function $p : V \to (0, 1)$ and $\psi : V \to X$ such that $p \times \psi$ is a $C^1$ diffeomorphism and the compact $us$-leaves in $V$ are exactly of the form $p^{-1}(t)$ for $t \notin U$.

Moreover, $p$ restricted to each center segment $L \subset V$ is a $C^1$ diffeomorphism.

Proof. There is a neighbourhood $V$ of $X$ such that inside $V$ each center segment intersects each compact $us$-leaf in a unique point. Therefore, the proofs of the previous results of this section hold as before with compact $us$-leaves now filling the role of $us$-plaques. This gives the existence of $p$ and $\psi$.

As the function $D$ is positive in the proof of (11.2), for $x \in X$ and unit vector $v^c \in E^c_x$ the derivative $dp_x$ of $p$ satisfies $dp_x(v^c) \neq 0$. By continuity, this property holds for all $x$ in a neighbourhood of $X$ and so, by replacing $V$ by a subset, the restriction of $p$ to any center segment $L$ has non-zero derivative along all of $L$. □

As it is a local result, (11.4) also holds for a compact $us$-leaf in an AB-system instead of an AI-system. To go from the local to the global requires a technical lemma which “fills in the gaps” between compact $us$-leaves.

Lemma 11.5. Let $N$ be a $C^1$ manifold, and for $0 < \epsilon < \frac{1}{2}$ define $V_\epsilon = N \times ([0, \epsilon) \cup (1 - \epsilon, 1]) \subset N \times [0, 1]$. If there are $\epsilon > 0$ and a $C^1$ function $g : V_\epsilon \to [0, 1]$ such that

- $\frac{\partial g}{\partial t}|_{(x, t)} > 0$ for all $(x, t) \in V_\epsilon$, and
- $g(x, 0) = 0$ and $g(x, 1) = 1$ for all $x \in N$
then there are \( \delta > 0 \) and a \( C^1 \) function \( h: N \times [0, 1] \to [0, 1] \) such that

- \( h(x, t) = g(x, t) \) for all \((x, t) \in V_\delta, \)
- \((x, t) \mapsto (x, h(x, t))\) is a \( C^1 \) diffeomorphism of \( N \times [0, 1], \)
- if \( x \in N \) satisfies \( g(x, t) = t \) for all \((x, t) \in V_\delta, \) then \( h(x, t) = t \) for all \( t \in [0, 1]. \)

**Proof.** Pick \( \delta > 0 \) small enough that there is a continuous function \( h_0: N \times [0, 1] \to [0, 1] \) which for each \( x \in N \) satisfies the following properties:

- \( t \mapsto h_0(x, t) = \) strictly increasing and linear on each of the intervals \([\delta, 3\delta], \)
  \([3\delta, 1 - 3\delta], \) and \([1 - 3\delta, 1 - \delta]; \)
- \( h_0 \) agrees with \( g \) and \( \frac{\partial h_0}{\partial t} = \) agrees with \( \frac{\partial g}{\partial t} \) at the points of the form \((x, \delta)\)
  and \((x, 1 - \delta).\)

Then, define \( h \) by \( h(x, t) = g(x, t) \) for \((x, t) \in V_\delta, \)
\( h(x, t) = h_0(x, t) \) for \((x, t) \in V_{2\delta} \setminus V_\delta, \)
and \( h(x, t) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} h_0(x, s)\,ds \) otherwise. \( \square \)

**Proposition 11.6.** Let \( f: \hat{M} \to \hat{M} \) be a \( C^2 \) AI-system, and \( J \) a compact interval inside a center leaf such that its endpoints \( x_0 \) and \( x_1 \) lie inside compact us-leaves. Then there are \( r: AC(J) \to AC(x_0) \)
and \( p: AC(J) \to [0, 1] \) such that \( r \times p \) is a \( C^1 \)
diffeomorphism and every compact us-leaf in \( AC(J) \) is of the form \( p^{-1}(t) \) for some \( t \in [0, 1]. \)

**Proof.** By approximating the center bundle \( E^c \) by a \( C^1 \) vector field \( v, \)
one may define a \( C^1 \) flow taking points in \( AC(x_0) \) to points in \( AC(x_1). \)
By rescaling \( v, \)
assume the flow takes each point in \( AC(x_0) \) to a point in \( AC(x_1) \)
in exactly one unit of time. This flow then defines a \( C^1 \) diffeomorphism between \( AC(J) \) and \( AC(x_0) \times [0, 1]. \)
Therefore, we may assume our system is defined on a space of the form \( N \times [0, 1] \)
where \( N \) is a manifold \( C^1 \)-diffeomorphic to \( AC(x_0) \) and that \( r: N \times [0, 1] \to N \)
is given by projection onto the first coordinate. Further assume that the flow \( v \)
and tangent to \( E^c \) on the center leaf containing \( J. \) Then, when viewed as a subset of \( N \times [0, 1], \)
\( J \) is of the form \( J = (x_0) \times [0, 1]. \)

By adapting the arguments in the proofs of (11.2) and (11.4), there is a \( C^1 \)
function \( g: N \times [0, 1] \to [0, 1] \) which is constant on compact us-leaves
and such that \( g(x_0, t) = t \) for all \( t \in [0, 1]. \)

Let \( \Sigma \subset N \times [0, 1] \) be the union of all compact us-leaves. For a point \( z \in N \times [0, 1], \)
let \( v^c_z \) be the oriented unit vector in \( E^c_z. \) Then, due to the construction of \( g \)
as in the proof of (11.2), \( d g_z(v^c_z) \) is positive for all \( z \in \Sigma. \) As \( d g \)
is continuous, there is a \( C^1 \) vector field \( \tilde{v} \) approximating \( v^c \) such that \( d g_z(\tilde{v}(z)) \) is positive for all \( z \in \Sigma. \)
By another \( C^1 \) change of coordinates, assume \( v \) is equal to \( \tilde{v} \)
and therefore \( \frac{\partial g}{\partial t} |_{(x, t)} = d g(x, t)(v(x, t)) \) for all \((x, t) \in N \times [0, 1]. \) By uniform continuity, there is \( \epsilon > 0 \)
such that \( d g_z(v(z)) > 0 \) for all \( z \) at distance at most \( \epsilon \) from \( \Sigma. \) Hence, there
are at most a finite number of regions \( X_i \subset N \times [0, 1] \) such that

- the boundary of \( X_i \) is given by two compact us-leaves,
- there are no compact leaves in the interior of \( X_i, \) and
• \( \frac{\partial g}{\partial t}(x,t) \leq 0 \) for some \((x,t) \in X_i\).

By (11.5), define a \(C^1\) function \(p : N \times [0,1] \to [0,1]\) which is equal to \(g\) everywhere outside of \(\bigcup_i X_i\) and such that \(\frac{\partial p}{\partial t}(x,t) > 0\) for all \((x,t) \in N \times [0,1]\).

Since both \(r\) and \(p\) are submersions, \(r \times p\) has an invertible derivative at every point and is therefore a \(C^1\) diffeomorphism.

\[ \square \]

**Corollary 11.7.** In the setting of (11.6), if \(L \subset \hat{M}\) is a center leaf, then \(p\) and \(r\) may be chosen so that \(p\) restricted to \(L \cap AC(J)\) is a \(C^1\) diffeomorphism onto \([0,1]\).

**Proof.** Take \(J \subset L\) in the previous proof. \[ \square \]

**Corollary 11.8.** In the setting of (11.6), if \(\mu\) is a probability measure given by a continuous volume form on \(AC(J)\), then \(p\) may be chosen so that \(p^* \mu\) is Lebesgue measure on \([0,1]\).

**Proof.** Assume \(\rho : N \times [0,1] \to \mathbb{R}\) is a positive density function such that

\[
\mu(X) = \int_X \rho \, dm_N \times dm
\]

where \(m_N \times m\) is the product of the Lebesgue measures on \(N\) and \([0,1]\).

If \(h : [0,1] \to [0,1]\) is defined by \(h(t) = \mu(p^{-1}([0,t]))\), then

\[
\frac{dh}{dt} \leq \int_{N \times [t]} \rho \, dm_N
\]

is continuous and positive, showing that \(h\) is a \(C^1\) diffeomorphism. Replacing \(p\) with the composition \(hp\), the result is proved. \(\square\)

**Proof of (4.3).** As noted in Section 7, every AB-system \(f : M \to M\) lifts to an AI-system \(\hat{f} : \hat{M} \to \hat{M}\). Moreover, if the AB-system has a compact \(us\)-leaf, the covering \(\hat{M} \to M\) has a fundamental domain which is bounded between two compact leaves \(AC(x)\) and \(\beta(AC(x))\) where \(\beta\) is the deck transformation defined in Section 7. Then, (11.6) applies where the region \(AC(J)\) is exactly this fundamental domain and therefore, there is a \(C^1\) surjection \(p : AC(J) \to [0,1]\). Moreover, the candidate derivative in the application of Whitney’s extension theorem may be chosen so that it agrees on \(AC(x)\) and \(\beta(AC(x))\). Then, \(p\) quotients down to a \(C^1\) function \(M \to S^1\) as desired.

The other statements in (4.3) follow from the above two corollaries. \(\square\)

12. **Skew products**

This sections proves (3.3) showing that non-accessible skew products have trivial fiber bundles.

**Proof of (3.3).** As the base map \(A\) has a fixed point, there is a fiber \(S\) such that \(f(S) = S\). By replacing \(f\) by \(f^2\) if necessary, assume \(f\) preserves the orientation of \(S\). As \(\pi_2(N)\) is trivial (see, for instance, [19]), the long exact sequence of fiber
bundles gives a short exact sequence $0 \to Z \to G \to H \to 0$ where $Z = \pi_1(S)$, $G = \pi_1(M)$, and $H = \pi_1(N)$. By naturality, $f$ induces the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & Z \\
\downarrow \text{id} & & \downarrow f_* \\
0 & \longrightarrow & Z \\
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & G \\
\downarrow f_* & & \downarrow \\
0 & \longrightarrow & H \\
\end{array}
\quad

As can be shown for any circle bundle with oriented fibers, the subgroup $G$ contained in the center of $\tau$ assume

aff(\tilde{g} for all trivial.

Let $\Lambda \subset \tilde{S}$ be the set of all points $t \in \tilde{M}$ such that $AC(t)$ is not open. Then $G$ induces an action on $\Lambda$.

Let $z$ be a non-trivial element of $Z$. Then $z$ may be regarded as a fixed-point free homeomorphism of $\tilde{S}$. By (5.1) and (5.2), there is a homomorphism $\tau : G \to \mathbb{R}$ such that $\tau(z)$ is non-zero. By (5.3), there is $\lambda > 0$ such that $\tau f_*(g) = \lambda \tau(g)$ for all $g \in G$. Since, $f_*(z) = 1$, this implies that $\lambda$ equals one. By rescaling $\tau$, assume $\tau(Z) = \mathbb{Z}$. Then, $\tau : G \to \mathbb{R}$ quotients to a homomorphism $\hat{\tau} : H \to \mathbb{R}/\mathbb{Z}$ and $\hat{\tau} A_* = \hat{\tau}$.

As $A$ is hyperbolic, $A_*$ has no non-trivial fixed points and, by (5.4), no non-trivial fixed cosets. As all of the cosets of $\ker \hat{\tau}$ are fixed by $A_*$, it follows that $\hat{\tau} = 0$. That is, $\tau(G) = \mathbb{Z}$. One can then define a map which takes each $g \in G$ to the unique $z \in Z$ such that $\tau(g) = \tau(z)$. This shows that the exact sequence $0 \to Z \to G \to H \to 0$ splits. Then, $G$ is isomorphic to $H \times Z$ and the bundle is trivial.

In fact, one can find a compact us-leaf directly. Viewing $H$ now as a subgroup of $G$ equal to the kernel of $\tau$, choose a point $x \in \tilde{S}$ and define $y = \sup_{g \in H} g(x)$. Then, with $\mu$ as in (5.1), $\mu(x, y) = 0$ which implies $y < +\infty$. In other words, $y$ is a well-defined point in $\tilde{S}$. Since $y$ is in $\text{Fix}(H)$ it projects to a point in $M$ contained in a compact us-leaf.

\section{13. Infra-AB-systems}

We now consider infra-AB-systems as defined in Section 1.

First, recall the definition of an infranilmanifold. Let $\tilde{N}$ be a simply connected nilpotent Lie group. A diffeomorphism $\phi : \tilde{N} \to \tilde{N}$ is a (right) translation if there is $v \in \tilde{N}$ such that $\phi(u) = u \cdot v$ for all $u \in \tilde{N}$. Let $\text{Trans}(\tilde{N})$ be the group of all translations (which is canonically isomorphic to $\tilde{N}$ itself). Let $\text{Aut}(\tilde{N})$ be the group of all automorphisms of $\tilde{N}$. Then the group of affine diffeomorphisms, $\text{Aff}(\tilde{N})$, is the smallest group containing both $\text{Trans}(\tilde{N})$ and $\text{Aut}(\tilde{N})$. Equivalently,
\( \psi \in \text{Aff}(\bar{N}) \) if and only if there is \( \phi \in \text{Aut}(\bar{N}) \) and \( \nu \in \bar{N} \) such that \( \psi(u) = \phi(u) \cdot \nu \) for all \( u \in \bar{N} \).

If a subgroup \( \Gamma < \text{Aff}(\bar{N}) \) is such that \( \Gamma \cap \text{Trans}(\bar{N}) \) has finite index in \( \Gamma \) and \( N_0 := \bar{N}/\Gamma \) is a compact manifold, then \( N_0 \) is a (compact) infranilmanifold. If \( A \in \text{Aff}(\bar{N}) \) quotients to a function \( A_0 : N_0 \to N_0 \) then \( A_0 \) is also called affine.

**Theorem 13.1.** Suppose \( f_0 \) is a conservative \( C^2 \) infra-AB-system. Then, either

1. \( f_0 \) is accessible and stably ergodic,
2. \( E^u \) and \( E^s \) are jointly integrable and \( f_0 \) is topologically conjugate to an algebraic map, or
3. there are \( n \geq 1 \), a \( C^1 \) surjection \( p_0 \) from \( M_0 \) to either \( S^1 \) or \( S^1/Z_2 \), and a non-empty open subset \( U \subset p_0(M_0) \) with the following properties.

   If \( t \not\in U \) then \( p_0^{-1}(t) \) is an \( f_0^n \)-invariant compact us-leaf homeomorphic to an infranilmanifold. Moreover, every \( f_0 \)-periodic compact us-leaf is of this form.

   If \( I \) is a connected component of \( U \), then \( p_0^{-1}(I) \) is \( f_0^n \)-invariant and homeomorphic to a (possibly twisted) I-bundle over an infranilmanifold.

This theorem is proved at the end of the section and the exact nature of the “algebraic map” in case (2) is given in the proof. Also, as will be evident from the proof, if \( E^c \) is orientable then \( p_0(M_0) = S^1 \). Otherwise, \( p_0(M_0) = S^1/Z_2 \) which is the 1-dimensional orbifold constructed by quotienting \( \mathbb{R} \) by both \( Z \) and the involution \( t \mapsto -t \). This orbifold is homeomorphic to a compact interval. A set \( p^{-1}(I) \) will be twisted (as an I-bundle) if and only if \( I \) is homeomorphic to a half-open interval.

The ergodic decomposition given in (1.4) also generalizes.

**Theorem 13.2.** Let \( f_0 : M_0 \to M_0 \) be a \( C^2 \) infra-AB-system and suppose there is a smooth, \( f_0 \)-invariant, non-ergodic measure \( \zeta \) supported on \( M_0 \). Then, there are \( n \geq 1 \), a \( C^1 \) surjection \( p_0 \) from \( M_0 \) to either \( S^1 \) or \( S^1/Z_2 \), and an open subset \( U \subset p_0(M_0) \) such that

\[
(13.3) \quad \zeta = \sum_I m(I) \zeta_I + \int_{\tau \in U} \zeta_t \, d\tau
\]

is the ergodic decomposition for \( (f_0^n, \zeta) \).

Here, the components \( \zeta_I \) and \( \zeta_t \) of the decomposition are defined analogously to (1.3).

**Proof.** Let \( \pi : M_0 \to M \) be the finite covering and \( f \) an AB-system such that \( \pi f = f_0^m \pi \) for some \( m \geq 1 \). Then, \( \zeta \) lifts to a measure \( \mu \) on \( M \) which (up to rescaling the measure so that \( \mu(M) = 1 \)) satisfies the hypotheses of (1.4). If \( \zeta_t \) is a component of the decomposition (13.3), then its support is a single accessibility class \( X_0 \). If \( X \) is a connected component of \( \pi^{-1}(X_0) \subset M \), then there is an ergodic component \( (f^n, \mu_t) \) of \((f^n, \mu)\) where \( \mu_t \) is supported on \( X \) and such that \( \pi^* \mu_t \) (up to rescaling) is equal to \( \zeta_t \). Ergodicity of \( (f_0^n, \zeta_t) \) then follows from the
ergodicity of \((f^n, \mu_t)\). Ergodicity of components of the form \(\zeta_I\) can be proven similarly. □

The theorems in Section 1 concerning non-conservative AB-systems may also be generalized using techniques similar to those in the proof of (13.1) below. In the interests of brevity, we leave the statements and proofs of these other results to the reader. The following two known results about functions on infranilmanifolds will be useful.

**Lemma 13.4.** If \(N_0\) is an infranilmanifold, there is a nilmanifold \(N\) finitely covering \(N_0\) such that every homeomorphism of \(N_0\) lifts to \(N\).

**Proof.** This follows from the fact that \(\Gamma \cap \text{Trans}(\tilde{N})\) is the unique maximal normal nilpotent subgroup of \(\pi_1(M)\). A proof of this is given in [4], a paper which also contains an infamously incorrect result about maps between infranilmanifolds. (See the discussion in [32].) However, the proof of the above fact about \(\Gamma \cap \text{Trans}(\tilde{N})\) is widely held to be correct. □

**Lemma 13.5.** If a homeomorphism \(B\) on a compact infranilmanifold \(N_0\) commutes with a hyperbolic affine diffeomorphism \(A\), then \(B\) itself is affine.

**Proof.** This follows by a combination of the results of Mal’cev and Franks. First, consider the case where \(N = N_0\) is a nilmanifold. Let \(x\) be a fixed point of \(A\). Then \(y := B(x)\) is also a fixed point of \(A\). Using the standard definition of the fundamental group for based spaces, the diagram

\[
\pi_1(N, x) \xrightarrow{B_*} \pi_1(N, y) \\
\downarrow A_* \quad \downarrow A_* \\
\pi_1(N, x) \xrightarrow{B_*} \pi_1(N, y)
\]

commutes. By [33], there is a unique affine map \(\phi : (N, x) \to (N, y)\) such that \(\phi_* = B_*\). (If \(x \neq y\) one shows this by considering two distinct lattices of the form \(x\Gamma \tilde{x}^{-1}\) and \(y\Gamma \tilde{y}^{-1}\) on the Lie group \(\tilde{N}\) in order to construct a Lie group homomorphism which quotients down to \(\phi\).)

As \(\phi_* A_* = A_* \phi_*\), the uniqueness given in [33] entails that \(\phi A = A\phi\) as functions on \(N\). As \(N\) is aspherical, \(\phi\) is homotopic to \(B\). Then, using that \(A\) is a \(\pi_1\)-diffeomorphism as defined in [19], it follows that \(\phi\) and \(B\) are equal.

Now suppose \(N_0\) is an infranilmanifold. By (13.4), there is a nilmanifold \(N\) and a normal finite covering \(N \to N_0\) such that both \(A\) and \(B\) lift to functions \(N \to N_0\). By abuse of notation, we still call these functions \(A\) and \(B\). As the covering is finite, there is \(j \geq 1\) such that \(A^j \gamma = \gamma A^j\) for every deck transformation \(\gamma\). In particular, there is a deck transformation \(\gamma : N \to N\) such that \(A^j B = B A^j\gamma\). Then, \(A^{jk} B = B(A^j \gamma)^k = B A^{jk} \gamma^k\) for all \(k \in \mathbb{Z}\), and, taking \(k \geq 1\) such that \(\gamma^k\) is the identity, \(A^{jk}\) commutes with \(B\) and the problem reduces to the previous case. □
Proposition 13.6. Suppose \( f_0 \) is a partially hyperbolic skew product where the base map is a hyperbolic infranilmanifold automorphism and \( E^c \) is one-dimensional. If \( f_0 \) is not accessible, it is an infra-AB-system.

Proof. Lift the fiber bundle projection \( \pi : M_0 \to N_0 \) to \( \tilde{\pi} : \tilde{M} \to \tilde{N} \) where \( \tilde{M} \) and \( \tilde{N} \) are the universal covers. Let \( G \) consist of those deck transformations \( \alpha \in \pi_1(M_0) \) which preserve the orientation of the lifted center bundle and for which \( \tilde{\pi} \alpha = \gamma \tilde{\pi} \) for some \( \gamma \in \text{Trans}(\tilde{N}) \). Then, \( G \) is a finite index subgroup of \( \pi_1(M_0) \) defining a finite cover \( M = \tilde{M}/\tilde{\pi}(G) \). If \( f_0 \) is not accessible, then \( f \) is not accessible. The fiber bundle on \( M \) is then trivial by (3.3), implying that \( f^2 \), which preserves the orientation of \( E^c \), is an AB-system. □

We now prove (13.1).

Assumption 13.7. For the remainder of the section, assume \( f : M \to M \) is a non-accessible conservative \( C^2 \) AB-system, \( \pi : M \to M_0 \) is a (not-necessarily normal) finite covering map and that \( f_0 : M_0 \to M_0 \) and \( m \geq 1 \) are such that \( \pi f = f_0^m \pi \).

Note this implies that \( f_0 \) is partially hyperbolic and the splitting on the tangent bundle \( TM_0 \) lifts to the splitting for \( f \) on \( TM \).

For now, make the following additional assumptions, which will be removed later.

Assumption 13.8. Assume until the end of the proof of (13.10) that

- \( E^c \) on \( M_0 \) is orientable;
- \( f_0 \) preserves the orientation of \( E^c \); and
- \( m = 1 \), that is, \( \pi f = f_0 \pi \).

By the assumption \( m = 1 \), both \( f_0 \) and \( f \) can be lifted to the same map \( \tilde{f} \) on the universal cover \( \tilde{M} \).

As \( f \) is an AB-system defined by nilmanifold automorphisms \( A, B : N \to N \), there is a map \( H : \tilde{M} \to \tilde{N} \) whose fibers are the center leaves of \( f \) and where \( \tilde{N} \) is the universal cover of \( N \) and therefore a nilpotent Lie group. Further, \( A \) lifts to a hyperbolic automorphism of \( \tilde{N} \), which we also denote by \( A \), and the leaf conjugacy implies that \( H \tilde{f} = AH \).

Define \( \tilde{S} = H^{-1}(0) \) where 0 is the identity element of the Lie group. Then \( \tilde{S} \) is an \( \tilde{f} \)-invariant center leaf which covers a circle \( S \subset M \) and \( S \) further covers a circle \( S_0 \subset M_0 \). By (1.2), there is a \( C^1 \) surjection \( p : M \to \mathbb{S}^1 \) and a constant \( \theta \in \mathbb{S}^1 \) such that if \( x \in M \) has non-open accessibility class \( AC(x) \) then \( p \) is constant on \( AC(x) \) and \( pf(x) = p(x) + \theta \). By (4.3), assume \( p \) restricted to \( S \) is a \( C^1 \) diffeomorphism. Using \( p \) and the covering \( \pi : M \to M_0 \), define a map

\[
q : M_0 \to \mathbb{S}^1, \quad x_0 \mapsto \sum_{y \in \pi^{-1}(x_0)} p(y).
\]
It follows that if \( x_0 \in M_0 \) has non-open accessibility class \( AC(x_0) \) then \( q \) is constant on \( AC(x_0) \) and \( q_{f_0}(x_0) = q(x_0) + \theta d \) where \( d \) is degree of the covering. Further, \( q \) restricted to \( S_0 \) is a \( C^1 \) covering from \( S_0 \) to \( \mathbb{S}^1 \) (though not necessarily of degree \( d \)). After lifting \( q \) to a map \( \tilde{q} : \tilde{M} \to \mathbb{R} \), there is a homomorphism \( q_* : \pi_1(M_0) \to \mathbb{Z} \) such that \( \tilde{q}\gamma(\tilde{x}) = \tilde{q}(\tilde{x}) + q_*(\gamma) \) for every \( \tilde{x} \in \tilde{M} \) and deck transformation \( \gamma \in \pi_1(M_0) \).

As the deck transformations preserve the lifted center foliation, for each \( \gamma \in \pi_1(M_0) \), there is a unique homeomorphism \( B_\gamma : \tilde{N} \to \tilde{N} \) such that \( H\gamma = B_\gamma H \).

**Lemma 13.9.** \( B_\gamma \in \text{Aff}(\tilde{N}) \) for all \( \gamma \in \pi_1(M_0) \).

**Proof.** We may view \( \pi_1(M) \) as a finite index subgroup of \( \pi_1(M_0) \). The definition of an AB-system implies that \( B_\gamma \in \text{Aff}(\tilde{N}) \) for all \( \gamma \in \pi_1(M) \).

Now consider the subgroups \( K_3 < K_2 < K_1 < \pi_1(M_0) \) defined as follows:

\[
K_1 = \text{the kernel of } q_*,
K_2 = K_1 \cap \pi_1(M), \quad \text{and}
K_3 = \{ \alpha \in K_2 : \alpha \beta K_2 = \beta K_2 \text{ for all } \beta \in K_1 \}.
\]

By its definition, \( K_3 \) is a normal finite index subgroup of \( K_1 \). The lift \( \tilde{f} \) of \( f_0 \) induces a homomorphism \( f_* : \pi_1(M_0) \to \pi_1(M_0) \) given by \( f_*(\gamma) = \tilde{f}_*\gamma\tilde{f}^{-1} \). There is a constant \( c \in \mathbb{R} \) such that

\[
\tilde{q}\tilde{f}(\tilde{x}) = \tilde{q}(\tilde{x}) + c
\]

for all \( \tilde{x} \in \tilde{M} \) with non-open accessibility class. This implies that \( f_*(K_1) = K_1 \).

From this, one can show that \( f_*(K_2) = K_2 \) and therefore \( f_*(K_3) = K_3 \).

Note that \( N_3 := \tilde{N}/\{ B_\gamma : \gamma \in K_3 \} \) is a nilmanifold (which finitely covers the original nilmanifold \( N \)), and the hyperbolic Lie group automorphism \( A : \tilde{N} \to \tilde{N} \) descends to an Anosov diffeomorphism on \( N_3 \).

Suppose \( \gamma \in K_1 \). As \( f_* \) permutes the cosets of \( K_3 \), there is \( j \geq 1 \) such that \( f_*^j(\gamma)K_3 = \gamma K_3 \). This implies that \( A^j \) and \( B_\gamma \) descend to commuting diffeomorphisms on \( N_3 \). Then, by (13.5), \( B_\gamma \) is affine. Thus, we have established the desired result for all \( \gamma \in K_1 \), and further shown that \( N_1 := \tilde{N}/\{ B_\gamma : \gamma \in K_1 \} \) is an infranilmanifold (finitely covered by the original nilmanifold \( N \)).

Now suppose \( \gamma \in \pi_1(M_0) \) is an arbitrary deck transformation. Then

\[
\tilde{q}\tilde{f}_*\gamma\tilde{f}_*^{-1}\gamma^{-1}(\tilde{x}) = \tilde{q}(\tilde{x})
\]

for all \( \tilde{x} \in \tilde{M} \) with non-open accessibility class. This implies that \( f_*(\gamma)K_1 = \gamma K_1 \) and so \( A \) and \( B_\gamma \) descend to commuting diffeomorphisms on \( N_1 \). As \( A \) is hyperbolic, \( B_\gamma \in \text{Aff}(\tilde{N}) \) by (13.5).

If \( f \) is accessible, then clearly \( f_0 \) is accessible. Therefore to prove (13.1), it is enough to consider \( f \) in cases (2) and (3) of (1.2).

**Proposition 13.10.** If \( f \) is in case (3) of (1.2) and \( f_0 \) satisfies assumption (13.8), then \( f_0 \) is in case (3) of (13.1).
Proof. By replacing $f_0$, $f$, and $\tilde{f}$ by iterates, assume $n = 1$ in (1.2) and that the lift $\tilde{f}$ was chosen so that $\tilde{f}(\tilde{x}) = \tilde{x}$ for every accessibility class $\tilde{x} \in \tilde{M}$.

The image of $q_*$ is equal to $\ell \mathbb{Z}$ for some $\ell \geq 1$. Then $p_0 := \frac{1}{\ell} \tilde{q}$ quotients to a function $p_0 : M \to S^1$. As the original $p : M \to S^1$ was $C^1$, the functions $q$, $\tilde{q}$, $p_0$, and $p$ are also $C^1$. Also, $p_0$ is constant on compact $us$-leaves and its restriction to $S_0$ is a $C^1$ covering. If, for some $t \in S^1$, $X_0$ and $Y_0$ are compact $us$-leaves in the pre-image $p_0^{-1}(t)$, then they lift to closed $us$-leaves $\tilde{X}, \tilde{Y} \subset \tilde{M}$ such that $\tilde{p}_0(\tilde{X}) - \tilde{p}_0(\tilde{Y})$ is an integer. By the definition of $\tilde{p}_0$, there is then a deck transformation taking $\tilde{X}$ to $\tilde{Y}$ and so $X_0 = Y_0$. This shows that every compact $us$-leaf in $M_0$ is of the form $p_0^{-1}(t)$ for some $t$.

If $X_0$ is instead an open accessibility class, then its boundary consists of two compact $us$-leaves and from this one can show that $p_0^{-1}(p_0(X_0)) = X_0$.

Note that every accessibility class $X_0$ on $M_0$ is the projection of an accessibility class $\tilde{X}$ on $\tilde{M}$. As $f$ fixes accessibility classes, so does $f_0$. Further, using $K_1$ and $N_1$ as in the proof of the lemma above, $X_0$ is homeomorphic to $\tilde{X}/K_1$. If $\tilde{X}$ is non-open, then $\tilde{X}/K_1$ is homeomorphic to the infranilmanifold $N_1$. If $\tilde{X}$ is open, then $\tilde{X}/K_1$ is an $I$-bundle over $N_1$ where the fibers of the $I$-bundle are segments of center leaves.

This shows that $f_0$ satisfies case (3) of (13.1).

We now remove the additional assumptions above and show that this result still holds.

**Proposition 13.11.** If $f$ is in case (3) of (1.2) and $f_0$ does not satisfy assumption (13.8), then $f_0$ is in case (3) of (13.1).

**Proof.** In case (3) of (13.1), we are free to replace $f_0$ by an iterate. By replacing $f_0$ by $f_0^m$, one can assume $m = 1$. That is, $\pi f = f_0 \pi$. By replacing $f_0$ by $f_0^2$, one can assume $f_0$ preserves the orientation of any orientable bundle. Thus, the only condition to test is when $E^c$ is non-orientable.

Any non-orientable bundle on a manifold lifts to an orientable bundle on a double cover and any bundle-preserving diffeomorphism lifts as well. Therefore, we are free to consider the following situation. As before, $E^c$ is orientable and $f_0$ preserves the orientation, but now there is an involution $\tau : M_0 \to M_0$, such that $\tau$ reverses the orientation of $E^c$ and $\tau$ commutes with $f_0$. As a consequence of this commutativity, $\tau$ preserves the partially hyperbolic splitting of $f_0$. Choose a continuous function $p_1 : M_0 \to S^1$ which satisfies $2p_1(x) = p_0(x) - p_0(\tau(x))$. As $\tau^2$ is the identity, $p_1 \tau(x) = -p_1(x)$ and so $p_1$ descends to a function $p_2 : M_0/\tau \to S^1/\mathbb{Z}_2$.

Since $S^1 \to S^1$, $x \mapsto -x$ has two fixed points, one can show that $\tau$ fixes exactly two accessibility classes on $M_0$. Let $X_0$ be one of these two classes, and lift $\tau$ and $X_0$ to the universal cover to get $\tilde{X}$ and $\tilde{\tau}$ such that $\tilde{\tau}(\tilde{x}) = \tilde{x}$. As $f$ and $\tau$ commute, it follows from an adaptation of (13.9) that $B_\tau \in \text{Aff}(\tilde{N})$. If $X_0$ is compact, then $X_0/\tau$ is homeomorphic to an infranilmanifold. If instead $X_0$ is open,
then $X_0$ is an $I$-bundle over $N_0$ where the fibers are center segments, and $\tau$ reverses the orientation of these fibers. Therefore, $X_0/\tau$ is a twisted $I$-bundle over an infranilmanifold.

This shows that case (3) holds for the quotient of $f_0$ to $M_0/\tau$ where $p_0$ and $U \subset S^1$ are replaced by $p_2$ and $U/\mathbb{Z}_2 \subset S^1/\mathbb{Z}_2$. □

Now consider the situation where $f$ is in case (2) of (1.2). The following proposition shows that $f_0$ is “algebraic” as stated in case (2) and concludes the proof of (13.1).

**Proposition 13.12.** Suppose $f_0$ is an infra-AB-system and $E_u \oplus E^c$ is integrable. Then there is a lift $\tilde{f}_0$ of $f_0$ to the universal cover $\tilde{M}$ and a homeomorphism $h: \tilde{M} \to \tilde{N} \times \mathbb{R}$ such that

$$h_0h^{-1} \in \text{Aff}(\tilde{N}) \times \text{Isom}(\mathbb{R})$$

and

$$h\gamma h^{-1} \in \text{Aff}(\tilde{N}) \times \text{Isom}(\mathbb{R})$$

for every deck transformation $\gamma \in \pi_1(M_0)$.

Here, $\text{Isom}(\mathbb{R})$ is the group of functions of the form $t \mapsto \text{char} \to t + c$ or $t \mapsto -t + c$.

**Proof.** First consider the case where $f_0$ satisfies assumption (13.8) and recall the functions $H: \tilde{M} \to \tilde{N}$ and $\tilde{q}: \tilde{M} \to \mathbb{R}$ defined earlier in this section. By global product structure and the integrability of $E_u \oplus E^c$, $H \times \tilde{q}$ is a homeomorphism. The results already given in this section then show that $h = H \times \tilde{q}$ satisfies the conclusions of the lemma.

If $f_0$ does not satisfy (13.8) and $E^c$ is orientable on $M_0$, then there is $m > 1$ such that $f_0^m$ satisfies (13.8). Let $H$ and $\tilde{q}$ be given for $f_0^m$. Define $a = +1$ if $\tilde{f}_0$ preserves the orientation of $E^c$ and $a = -1$ if $\tilde{f}_0$ reverses the orientation. Define $r: \tilde{M} \to \mathbb{R}$ by $r(x) = \sum_{k=0}^{m-1} a^k \tilde{q} f_0^k(x)$ and take $h = H \times r$.

If $E^c$ is non-orientable on $M_0$, then $f_0$ lifts to a double cover on which $E^c$ is orientable. Then, let $H$ and $r$ be defined as in the previous case. Choose a deck transformation $\tilde{r}: \tilde{M} \to \tilde{M}$ which reverses the orientation of $E^c$ on $\tilde{M}$ and define a function $s: \tilde{M} \to \mathbb{R}$ by $s(x) = r(x) - r(\tilde{r}(x))$ and take $h = H \times s$. □

14. Openness

This section establishes that AB-systems have global product structure and form an open subset of the space of $C^1$ diffeomorphisms.

**Lemma 14.1.** Suppose $G$ is a simply connected nilpotent Lie group. For any distinct $u, v \in G$, there is a unique one-dimensional Lie subgroup $G_{u,v}$ such that $v^{-1}u \in G_{u,v}$. (That is, $u$ lies in the coset $vG_{u,v}$.)

**Proof.** This follows from the fact that for such groups, the exponential map from the Lie algebra to the Lie group is surjective [33]. □
A right-invariant metric on such a group $G$ is a metric $d : G \times G \to [0, \infty)$ such that $d(u, v) = d(u \cdot w, v \cdot w)$ for all $u, v, w \in G$. For such a metric, we define a function $d_1 : G \times G \to [0, \infty)$ where $d_1(u, v)$ is the length of the path from $u$ to $v$ which lies in the coset $vG_u$, given by (14.1). Clearly, $d(u, v) \leq d_1(u, v)$ for all $u, v \in G$. Further, $d_1$ is continuous and the ratio $d_1(u, v) / d(u, v)$ tends uniformly to one as $d(u, v)$ tends to zero. Note that $d_1$ is not a metric on $G$ in general. (If $G = \mathbb{R}^d$ is abelian, however, the coset $uG_1$ is simply the line through $u$ and $v$ and $d = d_1$.)

If $\phi : G \to G$ is an automorphism and $G_1$ is a one dimensional subgroup, then there is $\lambda$ such that $d_1(\lambda u, \lambda v) = \lambda d_1(u, v)$ for all $u, v \in G$ with $u \in vG_1$. This follows because both $G_1$ and $\phi(G_1)$ are Lie groups isomorphic to $\mathbb{R}$ and $d_1$ restricts to a right-invariant metric on either of $G_1$ or $\phi(G_1)$.

**Lemma 14.2.** Suppose $G$ is a simply connected nilpotent Lie group, $d$ is a right-invariant metric, $\{\phi_k\}$ is a sequence of Lie group automorphisms of $G$, $G_1 \subset G$ is a one-dimensional Lie subgroup, $u_0 \in G$, and $v_0 \in u_0G_1$ with $u_0 \neq v_0$.

1. If $\lim_{k \to \infty} d(\phi_k(u_0), \phi_k(v_0)) = 0$, then $\lim_{k \to \infty} d(\phi_k(u), \phi_k(v)) = 0$ for all $u \in G$ and $v \in uG_1$.
2. If $a > 1$ and $\lim_{k \to \infty} a^k d(\phi_k(u_0), \phi_k(v_0)) = 0$, then $\lim_{k \to \infty} a^k d(\phi_k(u), \phi_k(v)) = 0$ for all $u \in G$ and $v \in uG_1$.
3. If $\sup_k d(\phi_k(u_0), \phi_k(v_0)) < \infty$, then $\sup_k d_1(\phi_k(u_0), \phi_k(v)) = 1$ for some $\hat{v} \in u_0G_1$.

**Proof.** Let $\lambda_k$ be such that $d_1(\phi_k(u), \phi_k(v)) = \lambda_k d_1(u, v)$ when $u \in vG_1$. Then in the first item, the two limits hold if and only if $\lambda_k \to 0$ and so one implies the other. For the second item, consider $a^k \lambda_k$. For the final item, if the first supremum is finite, then $\Lambda := \sup_k \lambda_k < \infty$ and one can take $\hat{v} \in v_0G_1$ such that $d_1(\hat{v}, v_0) = 1/\Lambda$.

We now show that every AB-system has global product structure.

**Proof of (4.1).** Let $f : \tilde{M} \to \tilde{M}$ be the lift of the AB-system to the universal cover and $h : \tilde{M} \to \tilde{M}_B$ the lifted leaf conjugacy to the AB-prototype. The functions $f$ and $h$ are written without tildes as all the analysis will be on the universal covers.

Measuring distances on the manifold $\tilde{M}_B$ requires care. The metric $d_{\tilde{M}_B}$ on $\tilde{M}_B$ is defined by lifting a metric from $M_B$. If $p_k = (u_k, s_k)$, and $q_k = (v_k, t_k)$ are sequences in $\tilde{M}_B = \tilde{N} \times \mathbb{R}$, then $d_{\tilde{M}_B}(p_k, q_k)$ may not converge to zero, even if both $d_N(u_k, v_k) \to 0$ on $\tilde{N}$ and $|s_k - t_k| \to 0$ on $\mathbb{R}$. The convergence depends on the exact nature of the automorphism $B$. If $s_k$ and $t_k$ are bounded sequences in $\mathbb{R}$, however, then one can show in this special case that $d_{\tilde{M}_B}(p_k, q_k) \to 0$ if and only if both $d_N(u_k, v_k) \to 0$ on $\tilde{N}$ and $|s_k - t_k| \to 0$ on $\mathbb{R}$.

There is a deck transformation $\beta : \tilde{M}_B \to \tilde{M}_B$ defined by $\beta(v, t) = (Bv, t - 1)$ which is an isometry with respect to $d_{\tilde{M}_B}$. For general $\{p_k\}$ and $\{q_k\}$, let $\{n_k\}$ be the unique sequence of integers such that $0 \leq |s_k - n_k| < 1$ for all $k$. Then,
\( \beta^{n_k}(p_k) \in \tilde{N} \times [0,1) \) for all \( k \) and

\[
    d_{\tilde{M}}(p_k, q_k) = d_{\tilde{M}}(\beta^{n_k}(p_k), \beta^{n_k}(q_k)) \to 0
\]

if and only if both

\[
    d_{\tilde{N}}(B^{n_k}(u_k), B^{n_k}(v_k)) \to 0 \quad \text{and} \quad |s_k - t_k| \to 0.
\]

In what follows, we write \( d \) without a subscript for the metrics on \( \tilde{M}, \tilde{M}_B, \) and \( \tilde{N} \). There is no ambiguity as they are all treated as distinct manifolds. If \( Y \) is a subset of one of these three manifolds, then

\[
    \text{dist}(x, Y) := \inf_{y \in Y} d(x, y).
\]

Also let \( d_s(x, y) \) denote distance measured along the corresponding stable foliation: \( W^s_f \) if \( x, y \in \tilde{M}, W^s_A \) if \( x, y \in \tilde{N} \), and \( W^s_{A \times \text{id}} \) if \( x, y \in \tilde{N} \times \mathbb{R} \). Similarly for \( d_u \) and \( d_c \).

The leaf conjugacy implies that every \( cs \)-leaf of \( f \) intersects a \( cu \)-leaf in a unique center leaf. Therefore, establishing global product structure reduces to showing existence and uniqueness of intersections inside the \( cs \) and \( cu \) leaves.

\textbf{Uniqueness.} Suppose \( x \in \tilde{M} \) and \( x \not\in W^c_f(x) \cap W^s_f(x) \). Then as \( k \to \infty \),

\[
    d_s(f^k(x), f^k(y)) \to 0 \quad \text{and} \quad d_c(f^k(x), f^k(y)) \not\to 0
\]

since if both sequences tended to zero, local product structure would imply that \( x \) and \( y \) were equal. Define \( p_k = h f^k(x) \) and \( q_k = h f^k(y) \). As the leaf conjugacy is uniformly continuous, \( d(p_k, q_k) \to 0 \) and \( d_c(p_k, q_k) \not\to 0 \). If \( p_k = (u_k, s_k) \) and \( q_k = (v_k, t_k) \), then, as noted above,

\[
    d(p_k, q_k) \to 0 \quad \Rightarrow \quad |s_k - t_k| \to 0 \quad \Rightarrow \quad d_c(p_k, q_k) \to 0,
\]

a contradiction.

\textbf{Existence.} Suppose \( x \in \tilde{M} \) lies on a center leaf \( L_0 \) and \( L_1 \subset W^c_f(x) \) is a distinct center leaf. Then \( h(L_0) = \{ v_0 \} \times \mathbb{R} \) and \( h(L_1) = \{ v_1 \} \times \mathbb{R} \) for distinct points \( v_0, v_1 \in \tilde{N} \). As \( L_0 \) and \( L_1 \) are subsets of the same \( cs \)-leaf of \( f \), \( v_0 \) and \( v_1 \) lie on the same stable leaf of \( A \). By (14.1), there is a one-dimensional subgroup \( \tilde{N}_1 \subset \tilde{N} \) such that \( v_0^{-1} \cdot v_1 \in \tilde{N}_1 \). By item (2) of (14.2), the coset \( v_0 \tilde{N}_1 \) is a subset of \( W^s_A(v_0) \).

If \( U_f \) is a small neighbourhood of \( x \) in \( W^f_f(x) \), then \( h(U_f) \subset W^s_A(v_0) \times \mathbb{R} \) and the set \( h(W^s_f(U_f)) = W^s_{A \times \text{id}}(h(U_f)) \) is a neighbourhood of \( h(x) \) in \( W^s_{A \times \text{id}}(v_0) \times \mathbb{R} \). Therefore, if \( v \in W^s_A(v_0) \) is sufficiently close to \( v_0 \), then there is \( y \in W^s_f(x) \) such that \( h(y) \in \{ v \} \times \mathbb{R} \). In particular, let \( v \) be such that \( v \in v_0 \tilde{N}_1 \) and fix such a point \( y \). See Figure 3. Let \( \{|n_k|\} \) be such that \( \beta^{n_k} h f^k(x) \in \tilde{N} \times [0,1) \) for all \( k \). Then,

\[
    d(f^k(x), f^k(y)) \to 0 \quad \Rightarrow \quad d(\beta^{n_k} h f^k(x), \beta^{n_k} h f^k(y)) \to 0 \quad \Rightarrow \quad d(B^{n_k} A^k(v_0), B^{n_k} A^k(v_1)) \to 0
\]

which by (14.2) implies \( d(B^{n_k} A^k(v_0), B^{n_k} A^k(v_1)) \to 0 \).
Then, as $h f^k(L_1) = \{A^k(v_1)\} \times \mathbb{R}$,
\[
\text{dist}(\beta^{nu} h f^k(x), \beta^{nu} h f^k(L_1)) \to 0 \quad \Rightarrow \quad 
\text{dist}(h f^k(x), h f^k(L_1)) \to 0 \quad \Rightarrow \quad 
\text{dist}(f^k(x), f^k(L_1)) \to 0.
\]

Thus, for sufficiently large $k$, $W^s_f(f^k(x))$ intersects $f^k(L_1)$ showing that $W^s_f(x)$ intersects $L_1$.

A sequence $\{x_k\}$ is an $\epsilon$-$c$-pseudoorbit if for each $k \in \mathbb{Z}$ the points $f(x_k)$ and $x_{k+1}$ lie $\epsilon$-close on the same center leaf. A partially hyperbolic system is plaque expansive if there is $\epsilon > 0$ such that if $\{x_k\}$ and $\{y_k\}$ are $c$-$c$-pseudoorbits and $d(x_k, y_k) < \epsilon$ for all $k \in \mathbb{Z}$, then $x_0$ and $y_0$ are on the same local center leaf.

**Theorem 14.3.** Every AB-system is plaque expansive.

Since plaque expansive systems are open in the $C^1$ topology [30], this also proves (4.2).

**Proof.** Let $f : M \to M$ be an AB-system. Let $C > 1$ be a constant to be defined shortly. Since $f$ expands in the unstable direction, there is $\epsilon_0 > 0$ such that if points $x, y, x', y' \in M$ satisfy
\[
\frac{1}{C} \leq d_u(x, y) \leq C, \quad d_c(f(x), x') < \epsilon_0, \quad \text{and} \quad y' \in W^c(f(y)) \cap W^u(x')
\]
then $d_u(x, y) < (1 - \epsilon_0)d_u(x', y')$. This result then also holds for points on the universal cover $\tilde{M}$ where $f$ for the remainder of the proof denotes the lift $f : \tilde{M} \to \tilde{M}$.
Let \( h : \tilde{M} \to \tilde{N} \times \mathbb{R} \) be the lifted leaf conjugacy. Define sets
\[
X = \{(v, w) \in \tilde{N} \times \tilde{N} : \; v \in W^{u}_{\tilde{A}}(w), \; d(v, w) \leq 1\}
\]
and
\[
X_1 = \{(v, w) \in \tilde{N} \times \tilde{N} : \; v \in W^{u}_{\tilde{A}}(w), \; \frac{1}{2} \leq d_1(v, w) \leq 1\}
\]
and a function
\[
D : X \times [-1, 1] \to \mathbb{R}, \quad (v, w, t) \mapsto d_u(h^{-1}(v \times \mathbb{R}), h^{-1}(w \times t)).
\]
That is, \( D(v, w, t) \) is the distance, measured along an unstable leaf of \( f \), between the center leaf \( h^{-1}(v \times \mathbb{R}) \) and the point \( h^{-1}(w \times t) \). Such a function is well-defined and continuous by global product structure.

If \( \alpha : \tilde{N} \to \tilde{N} \) is a deck transformation for the covering \( \tilde{N} \to N \), then \( \alpha \times \text{id} \) is a deck transformation for the covering \( \tilde{M}_B \to M_B \) and one can verify that \( D(\alpha(v), \alpha(w), t) = D(v, w, t) \). Using the compactness of \( N \) and \([-1, 1]\), there is \( C > 1 \) such that
\[
D(X \times [-1, 1]) \subset [0, C] \quad \text{and} \quad D(X_1 \times [-1, 1]) \subset \left[ \frac{1}{C}, C \right].
\]
This defines the constant \( C \) used above.

For some \( \epsilon > 0 \) let \( \{x_k\} \) and \( \{z_k\} \) be \( \epsilon \)-\( c \)-pseudoorbits such that \( d(x_k, z_k) < \epsilon \). By increasing \( \epsilon \) and by sliding the points \( z_k \) along center leaves, assume, without loss of generality, that there is a point \( y_k \) for each \( k \) such that \( x_k \) and \( y_k \) are connected by a short unstable segment and \( y_k \) and \( z_k \) are connected by a short stable segment. By again increasing \( \epsilon \), one can show that \( \{y_k\} \) is a \( c \)-\( c \)-pseudoorbit. We may freely assume that the original \( \epsilon \) was chosen small enough that \( d_\beta(f(x_k), x_{k+1}) < \epsilon_0 \) for all \( k \). We will show that \( x_0 \) and \( y_0 \) lie on the same center leaf. An analogous argument holds for \( y_0 \) and \( z_0 \) which will complete the proof.

Suppose \( x_0 \) and \( y_0 \) lie on distinct center leaves. Then, using \( \beta \) as in the previous proof, there are \( v_x \neq v_y \in \tilde{N} \) and \( \{n_k\} \) such that \( h^n \beta(v_x) \in \{B^n \tilde{A}^k v_x\} \times (-1, 1) \) and \( h^n \beta(v_y) \in \{B^n \tilde{A}^k v_y\} \times (-1, 1) \) for all \( k \in \mathbb{Z} \). This implies that
\[
\sup_k d(B^n \tilde{A}^k v_x, B^n \tilde{A}^k v_y) < \infty.
\]
Let \( \tilde{N}_1 \subset \tilde{N} \) be a one-dimensional subgroup such that \( v_y \in v_x \tilde{N}_1 \). By (14.2), there is \( \hat{v} \in v_x \tilde{N}_1 \) such that
\[
\sup_{k \in \mathbb{Z}} d_1(B^n \tilde{A}^k v_x, B^n \tilde{A}^k \hat{v}) = 1.
\]
By the global product structure of \( f \), there is a unique sequence \( \{\hat{y}_k\} \) in \( \tilde{M} \) such that \( h(\hat{y}_k) \in \{A^n \hat{v}\} \times \mathbb{R} \) and \( \hat{y}_k \in W^{u}_{\tilde{A}}(x_k) \). Then, \( S = \sup_{k \in \mathbb{Z}} d_u(x_k, \hat{y}_k) \) satisfies \( \frac{1}{C} \leq S \leq C \). Let \( k \in \mathbb{Z} \) be such that \( d_u(x_k, \hat{y}_k) > (1 - \epsilon_0)S \). The definition of \( \epsilon_0 \) implies that \( d_u(x_{k+1}, \hat{y}_{k+1}) > S \), a contradiction. \( \square \)
15. The dynamically-incoherent example

This section gives a construction of the example due to Hertz, Hertz, and Ures of a partially hyperbolic system on the 3-torus having an invariant cs-s-torus [28]. For this specific construction, \( E^u \) and \( E^s \) are jointly integrable and the tangent foliation has exactly one compact leaf. The system therefore gives an example of case (3) of (1.7). We use the following to prove the example is partially hyperbolic.

**Proposition 15.1.** Suppose \( f \) is a diffeomorphism of a compact manifold \( M \), \( TM = E^s \oplus E^c \oplus E^u \) is an invariant splitting, and there is \( k > 1 \) such that
\[
\| T f^k v^s_x \| < \| T f^k v^c_x \| < \| T f^k v^u_x \| \quad \text{and} \quad \| T f^k v^s_x \| < 1 < \| T f^k v^u_x \|
\]
for all \( x \in \text{NW}(f) \) and unit vectors \( v^s_x, v^c_x, v^u_x \in E^*_x \). Then, \( f \) is partially hyperbolic.

To prove this, note that if the above inequalities hold on \( \mathbb{T}^3 \), then \( f \) has a linear stable bundle, so we first consider dynamics in dimension two. Define \( \lambda = \frac{1}{2}(1 + \sqrt{5}) \) and functions
\[
\psi : \mathbb{R} \to \mathbb{R}, \ x \mapsto x + \frac{2}{3} \sin x \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (\psi(x), \lambda y + \cos x).
\]
The derivative of \( g \) is
\[
Dg = \begin{pmatrix}
\psi'(x) & 0 \\
-\sin x & \lambda
\end{pmatrix}.
\]
On the vertical line \( x = 0 \), there is an expanding fixed point for \( g \). Through this point is an invariant one-dimensional unstable manifold associated to the larger eigenvalue of \( Dg \). One can show that this unstable manifold may be expressed as the graph of a function \( u : (-\pi, \pi) \to \mathbb{R} \). For now, only consider \( u \) on \([0, \pi]\). By an invariant cone argument, one can show that \( u'(x) < 0 \) for all \( x \in (0, \pi) \). Using that \( \psi'(x) < \lambda \) when \( x \) is close to \( \pi \) and that
\[
\frac{|\lambda t - \sin x|}{|\psi'(x)|} > \frac{\lambda}{|\psi'(x)|} |t| > |t|,
\]
for \( t < 0 \), one can show that \( \lim_{x \to \pi} u'(x) = -\infty \).

Define a foliation \( W^u \) on \([0, \pi] \times \mathbb{R}\) by all graphs of functions of the form \( x \mapsto u(x) + b \) for \( b \in \mathbb{R} \). This foliation is \( g \)-invariant. Reflecting about the \( y \)-axis, extend this to a foliation on \((-\pi, \pi) \times \mathbb{R}\). By including the vertical lines on the boundary, extend this foliation to \([-\pi, \pi] \times \mathbb{R}\) and then, by \( 2\pi \)-periodicity in \( x \), to all of \( \mathbb{R}^2 \). Call this foliation \( W^u \) and let \( E^s \) be the \( C^0 \) line field tangent to it.

Now consider the hyperbolic fixed point of \( g \) on the line \( x = \pi \). Part of the stable manifold of this point is given by the graph of a function \( c : (0, \pi) \to \mathbb{R} \). One can show that \( c'(x) > 0 \) for all \( x \in (0, \pi) \) and, since \( \psi'(0) > \lambda \), that \( \lim_{x \to 0} c'(x) = -\infty \).
+∞. From the definition of g, there is a constant C > 1 such that \( g^{-1} \) maps the region \([-C, C] \times [0, \pi]\) into itself. The stable manifold given by graph(c) must therefore be contained in this region, showing that c is a bounded function and can be continuously extended to all of \([0, \pi]\). By reflection and periodicity, further extend c to a continuous function \( \mathbb{R} \rightarrow \mathbb{R} \) which is differentiable except at \( 2n\pi \) and such that \( g(\text{graph}(c)) = \text{graph}(c) \). By considering translates, \( x \mapsto c(x) + b \), define a foliation \( W^c \) on \( \mathbb{R}^2 \) and let \( E^c \) be the unique continuous line field on \( \mathbb{R}^2 \) which is tangent to \( W^c \) on \( (\mathbb{R} \setminus 2n\pi) \times \mathbb{R} \). As \( u' < 0 < c' \) on \( (0, \pi) \), \( E^u \) and \( E^c \) are transverse.

The matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

has eigenvalues \( \lambda = \frac{1}{2}(1 + \sqrt{5}) \) and \(-\lambda^{-1} \). Therefore, there is a lattice \( \Lambda \subset \mathbb{Z}^2 \) such that \((y, z) \mapsto (\lambda y, -\lambda^{-1} z)\) quotients to an Anosov diffeomorphism on the 2-torus \( \mathbb{R}^2/\Lambda \). Define \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by

\[
f(x, y, z) = (x + \frac{2}{3} \sin x, \lambda y + \cos x, -\lambda^{-1} z)
\]

and a splitting \( E^c \oplus E^u \oplus E^s \) by \( E^s = \frac{\partial}{\partial x} \) and where \( E^c \oplus E^u \) on each \( xy \)-plane is given by the earlier splitting constructed for \( g \). This splitting is \( f \)-invariant and there is a foliation tangent to \( E^u \oplus E^s \). Define \( M = (\mathbb{R} \times \mathbb{R}^2)/(2\pi \mathbb{Z} \times \Lambda) \). Both \( f \) and the splitting descend to \( M \). Here, \( NW(f) < M \) consists of two tori, one tangent to \( E^c \oplus E^s \) and the other tangent to \( E^u \oplus E^s \). Using (15.1), one can verify that \( f \) is partially hyperbolic. It has a foliation tangent to \( E^u \oplus E^s \) with one compact leaf and all other leaves are planes.

This is not an example of an AB-system as there is no invariant foliation tangent to \( E^c \). In the above analysis, the crucial properties needed for the term \( \cos x \) in the formula \( \lambda y + \cos x \) for the second coordinate of \( g \) were that \( \cos' < 0 \) on \((0, \pi)\) and \( \cos'(\pi) \leq 0 = \cos'(0) \). Therefore, replace \( \lambda y + \cos x \) by \( \lambda y + \sin x - x \) in all of the above analysis. As \( \sin x - x \) is an odd function, the resulting function \( c : \mathbb{R} \rightarrow \mathbb{R} \) is odd and its graph is a \( C^1 \) submanifold in \( \mathbb{R}^2 \). Defining \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) now by

\[
f(x, y, z) = (x + \frac{2}{3} \sin x, \lambda y + \sin x - x, -\lambda^{-1} z)
\]

and quotienting by the lattice in \( \mathbb{R}^3 \) generated by \( \{0\} \times \Lambda \) and \((2\pi, \frac{2\pi}{\lambda-1}, 0) \) one constructs a skew product on \( \mathbb{T}^3 \) having a foliation tangent to \( E^u \oplus E^s \) with exactly one compact leaf.

**Appendix A. Definitions**

This appendix defines a number of notions in smooth dynamical theory.

All manifolds considered in this paper are Riemannian manifolds without boundary. Suppose \( f \) is a \( C^1 \) diffeomorphism on a compact manifold and there is a \( T f \)-invariant splitting \( TM = E^u \oplus E^c \oplus E^s \) of the tangent bundle and \( k \geq 1 \) such that \( \|T f^k v^s\| < 1 < \|T f^k v^u\| \) for all unit vectors \( v^s \in E^s \) and \( v^u \in E^u \). If \( E^c \)
is the zero bundle, then \( f \) is an Anosov diffeomorphism. If \( E^u, E^c, \) and \( E^s \) are all non-zero and \( \| Tf^k v^s \| < \| Tf^k v^c \| < \| Tf^k v^u \| \) for all \( p \in M \) and unit vectors \( v^c \in E^c_p, \) \( v^c \in E^c_p, \) and \( v^u \in E^u_p \), then \( f \) is a partially hyperbolic diffeomorphism.

The notion of partially hyperbolicity is also extended to certain non-compact manifolds in Section 6.

A \( C^1 \) flow is an Anosov flow if its time-one map is a partially hyperbolic diffeomorphism with a center bundle given by the direction of the flow.

A partially hyperbolic diffeomorphism \( f \) is dynamically coherent if there are invariant foliations \( W^c_u \) and \( W^c_s \) tangent to \( E^c \oplus E^u \) and \( E^c \oplus E^s \). As a consequence, there is also an invariant center foliation \( W^c \) tangent to \( E^c \). Global product structure is defined in Section 4.

For homeomorphisms \( f : X \to Y \) and \( g : Y \to Z \), a topological semiconjugacy is a continuous surjection \( h : X \to Y \) such that \( h f = g h \). If \( h \) is a homeomorphism, it is a topological conjugacy.

Partially hyperbolic diffeomorphisms \( f \) and \( g \) are leaf conjugate if they are dynamically coherent and there is a homeomorphism \( h \) such that for every center leaf \( L \) of \( f \), \( h(L) \) is a center leaf of \( g \) and \( h f(L) = g h(L) \).

A homeomorphism \( f : M \to M \) is (topologically) transitive if every non-empty open \( f \)-invariant subset of \( M \) is dense in \( M \).

For a homeomorphism \( f : M \to M \), a Borel measure \( \mu \) is invariant if \( \mu(X) = \mu(f(X)) \) for every measurable set \( X \subset M \). The pair \( (f, \mu) \) is ergodic if \( \mu \) is \( f \)-invariant and either \( \mu(X) = 0 \) or \( \mu(X) = 1 \) for every \( f \)-invariant measurable \( X \subset M \). We often write that \( f \) is ergodic or \( \mu \) is ergodic if the context is clear. For brevity, we sometimes say that a system \( f \) with a finite non-probability measure \( \mu \) is ergodic when, to be precise, we should actually say that the pair \( (f, \frac{1}{\mu(M)} \mu) \) is ergodic. A homeomorphism \( f \) is conservative if it has an invariant measure given by a smooth volume form on \( M \). A conservative \( C^2 \) diffeomorphism is stably ergodic if it has a neighbourhood \( \mathcal{U} \) in the \( C^1 \) topology of \( C^1 \) diffeomorphisms such that every conservative \( C^2 \) diffeomorphism in \( \mathcal{U} \) is also ergodic. For a discussion of why the quirky combination of \( C^1 \) and \( C^2 \) regularity is necessary, see [46].

If \( \tilde{N} \) is a simply connected nilpotent Lie group and \( \Gamma \) is a discrete subgroup such that \( N := \tilde{N}/\Gamma \) is a compact manifold, then \( N \) is called a (compact) nilmanifold [33]. If \( \hat{A} : \tilde{N} \to \tilde{N} \) is a Lie group automorphism which descends to \( A : N \to N \), then \( A \) is a nilmanifold automorphism (also called a toral automorphism when \( N = \mathbb{T}^d \)). If \( A \) is Anosov, it is called hyperbolic. Infranilmanifolds and their automorphisms are defined in Section 13.

If \( f : M \to N \) is a continuous function and \( \pi_M : \tilde{M} \to M \) and \( \pi_N : \tilde{N} \to N \) are covering maps, then a lift of \( f \) is a function \( \tilde{f} : \tilde{M} \to \tilde{N} \) such that \( \pi_N \tilde{f} = f \pi_M \). Note that if \( \pi_M \) and \( \pi_N \) are universal covering maps, then at least one such lift exists, but is not unique in general.
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REFERENCES

[1] D. V. Anosov. Geodesic flows on closed Riemannian manifolds with negative curvature. *Proc. Inst. Steklov*, 90:1–235, 1967.
[2] D. V. Anosov and Ja. G. Sinai. Certain smooth ergodic systems. *Uspehi Mat. Nauk*, 22(5 (137)):107–172, 1967.
[3] N. Aoki and K. Hiraide. *Topological theory of dynamical systems*, volume 52 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1994. Recent advances.
[4] L. Auslander. Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups. *Annals of Math.*, 71(3):579–590, 1960.
[5] D. Bohnet. *Partially hyperbolic diffeomorphisms with a compact center foliation with finite holonomy*. PhD thesis, Universität Hamburg, 2011.
[6] D. Bohnet. Codimension one partially hyperbolic diffeomorphisms with a uniformly compact center foliation. *Journal of Modern Dynamics*, 7(4):565–604, 2013.
[7] M. Brin, D. Burago, and S. Ivanov. Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *Journal of Modern Dynamics*, 3(1):1–11, 2009.
[8] M. I. Brin. Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Funktional. Anal. i Priložen.*, 9(1):9–19, 1975.
[9] D. Burago and S. Ivanov. Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *Journal of Modern Dynamics*, 2(4):541–580, 2008.
[10] K. Burns, C. Pugh, and A. Wilkinson. Stable ergodicity and Anosov flows. *Topology*, 39(1):149–159, 2000.
[11] K. Burns and A. Wilkinson. Stable ergodicity of skew products. *Ann. Sci. École Norm. Sup. (4)*, 32(6):859–889, 1999.
[12] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Annals of Math.*, 171(1):451–489, 2010.
[13] P Carrasco. *Compact dynamical foliations*. PhD thesis, University of Toronto, 2010.
[14] F. T. Farrell and A. Gogolev. Anosov diffeomorphisms constructed from \( \pi_1(\text{Diff}(S^n)) \). *J. Topology*, 5(2):276–292, 2012.
[15] F. T. Farrell and L. E. Jones. Anosov diffeomorphisms constructed from \( \pi_1(\text{Diff}(S^n)) \). *Topology*, 17(3):273–282, 1978.
[16] M. Field, I. Melbourne, and A. Török. Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets. *Ergodic Theory Dynam. Systems*, 25(2):517–551, 2005.
[17] M. Field, I. Melbourne, and A. Török. Stability of mixing and rapid mixing for hyperbolic flows. *Ann. of Math. (2)*, 166(1):289–291, 2007.
[18] J. Franks. Anosov diffeomorphisms on tori. *Transactions of the American Mathematical Society*, 145:117–124, 1969.
[19] J. Franks. Anosov diffeomorphisms. *Global Analysis: Proceedings of the Symposia in Pure Mathematics*, 14:61–93, 1970.
[20] A. Gogolev. Partially hyperbolic diffeomorphisms with compact center foliations. *J. Mod. Dyn.*, 5(4):747–769, 2011.
[21] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *Annals of Math.*, 140(2):295–330, 1994.
[22] M. Gromov. Groups of polynomial growth and expanding maps. *Publications Mathématiques de l’IHÉS*, 53(1):53–78, 1981.
[23] A. Hammerlindl and R. Potrie. Classification of partially hyperbolic diffeomorphisms in 3-manifolds with solvable fundamental group. preprint arXiv:1307.4631, 2013.

[24] A. Hammerlindl and R. Ures. Ergodicity and partial hyperbolicity on the 3-torus. Commun. Contemp. Math., 16(4):1350038, 22, 2014.

[25] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. A survey of partially hyperbolic dynamics. "Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow," (eds. G. Forni, M. Lyubich, C. Pugh and M. Shub), pages 103–112, 2007.

[26] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle. Invent. Math., 172(2):353–381, 2008.

[27] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Partial hyperbolicity and ergodicity in dimension three. Journal of Modern Dynamics, 2(2):187–208, 2008.

[28] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. A non-dynamically coherent example on $T^3$. preprint arXiv:1409.0738, 2014.

[29] F.R. Hertz, M.A. Hertz, and R. Ures. Tori with hyperbolic dynamics in 3-manifolds. Journal of Modern Dynamics, 5(1):185–202, 2011.

[30] M. Hirsch, C. Pugh, and M. Shub. Invariant Manifolds, volume 583 of Lecture Notes in Mathematics. Springer-Verlag, 1977.

[31] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. Ber. Verh. Sächs. Akad. Wiss. Leipzig, 91:261–304, 1939.

[32] K. B. Lee and F. Raymond. Rigidity of almost crystallographic groups. In Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), volume 44 of Contemp. Math., pages 73–78. Amer. Math. Soc., Providence, RI, 1985.

[33] A. I. Malcev. On a class of homogeneous spaces. Amer. Math. Soc. Translation, 1951(39):33, 1951.

[34] A. Manning. There are no new Anosov diffeomorphisms on tori. Amer. J. Math., 96(3):422–42, 1974.

[35] L. Markus and K. R. Meyer. Generic Hamiltonian dynamical systems are neither integrable nor ergodic. American Mathematical Society, Providence, R.I., 1974. Memoirs of the American Mathematical Society, No. 144.

[36] I. Melbourne, V. Niţică, and A. Török. Transitivity of Heisenberg group extensions of hyperbolic systems. Ergodic Theory Dynam. Systems, 32(1):223–235, 2012.

[37] S.E. Newhouse. On codimension one Anosov diffeomorphisms. American Journal of Mathematics, 92(3):761–770, 1970.

[38] J.F. Plante. Anosov flows. Amer. J. Math., 94:729–754, 1972.

[39] J.F. Plante. Foliations with measure preserving holonomy. Annals of Math., 102(2):327–361, 1975.

[40] J.F. Plante. Solvable groups acting on the line. Trans. Amer. Math. Soc., 278(4):401–414, 1983.

[41] C. Pugh and M. Shub. Ergodicity of Anosov actions. Invent. Math., 15:1–23, 1972.

[42] C. Pugh and M. Shub. Stable ergodicity and julienne quasi-conformality. Journal of the European Mathematical Society, 2(1):1–52, 2000.

[43] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations. Duke Math. J., 86(3):517–546, 1997.

[44] C. Pugh, M. Shub, and A. Wilkinson. Correction to: “Hölder foliations” [Duke Math. J. 86 (1997), no. 3, 517–546; MR1432307 (97m:58155)]. Duke Math. J., 105(1):105–106, 2000.

[45] V.A. Rokhlin. On the fundamental ideas of measure theory. Translations Amer. Math. Soc., Series 1, pages 1–54, 1952.

[46] A. Wilkinson. Conservative partially hyperbolic dynamics. In Proceedings of the International Congress of Mathematicians, volume 901, pages 1816–1836, 2010.