The Nazarov proof of the non-symmetric Bourgain–Milman inequality

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Abstract

In 2012, Nazarov used Bergman kernels and Hörmander’s $L^2$ estimates for the $\bar{\partial}$-equation to give a new proof of the Bourgain–Milman theorem for symmetric convex bodies and made some suggestions on how his proof should extend to general convex bodies. This article achieves this extension and serves simultaneously as an exposition to Nazarov’s work. A key new ingredient is an affine invariant associated to the Bergman kernel of a tube domain. This gives the first ‘complex’ proof of the Bourgain–Milman theorem for general convex bodies, specifically, without using symmetrization.

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1 Introduction and main result

Let $K \subset \mathbb{R}^n$ be a convex body, that is a compact convex set with non-empty interior. Its polar $K^\circ := \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } x \in K \}$ is convex as it is the intersection of convex sets (half-spaces); it is compact if and only if
\[ 0 \in \text{int } K, \]
(1)
an assumption that we will use very often. Also, $K$ is called symmetric if $K = -K := \{ x \in \mathbb{R}^n : -x \in K \}$.

Let $A \in \text{GL}(n, \mathbb{R})$. While $AK$ and $K$ could have wildly differing volume (with respect to the Lebesgue measure $d\lambda$), the volume of the product body $AK \times (AK)^\circ \subset \mathbb{R}^n \times \mathbb{R}^n$ is equal to that of $K \times K^\circ$. This leads to the following $\text{GL}(n, \mathbb{R})$-invariant functional on convex bodies $\mathbb{R}^n$.

**Definition 1.** Let $K \subset \mathbb{R}^n$ be a convex body satisfying (1). The Mahler volume of $K$ is
\[ \mathcal{M}(K) := n!|K \times K^\circ| = n!|K||K^\circ|. \]

Crude bounds on $\mathcal{M}$ were demonstrated by Mahler already in 1939 [21, (6)]. In 1987, Bourgain–Milman showed that there exists an unspecified but uniform $c > 0$ independent of $K$ and $n$ such that [7, Corollary 6.1]
\[ \mathcal{M}(K) \geq c^n, \]
(2)
that—is aside from determining the best value of $c$—is optimal in an asymptotic sense [30, pp. 149–150]. One of Mahler’s conjectures asserts that $c$ should be 4 if $K$ is also symmetric [21, p. 96], and another that for general convex $K$ [22, (1)] [32, p. 564],
\[ \mathcal{M}(K) \geq \frac{(n+1)^{n+1}}{n!} \]
(3)
(this is, by Stirling’s formula, asymptotic to $e^n$).

In 2012, Nazarov proved (2) with $c = \pi^3/16 \approx 1.937$ for symmetric $K$. His proof was pioneering in that he used complex methods, namely, Bergman kernels and Hörmander’s $\overline{\partial}$-theorem. Moreover, he made some insightful suggestions [26, p. 342] on how his mainly complex arguments should extend to yield (2) for all $K$ with $c = \pi/4 \approx 0.785$ without using symmetrization techniques (but with a sacrifice in the value of the constant, see Remark 3). The purpose of the present article is to give an exposition of his beautiful ideas and carry out the necessary computations mostly following his suggestions. The main result of this article confirms Nazarov’s expected bound, and thus gives the first proof of the Bourgain–Milman theorem for general convex bodies without using symmetrization:
Theorem 2. For all convex bodies $K \subset \mathbb{R}^n$, one has (2) with $c = \pi/4$.

There are several points in the proof of Theorem 2 that differ from Nazarov’s analysis in the symmetric case:

The weight function for Hörmander’s $L^2$-estimates. An interesting issue not present in the symmetric case comes from applying John’s theorem to non-symmetric bodies. This comes up when one constructs the appropriate weight function to be used in Hörmander’s $L^2$-estimates. Yet without the assumption of symmetry, John’s theorem does not guarantee that the maximal ellipsoid contained in the convex body will be centered at the origin. It may even happen that the origin is not at all contained in the maximal ellipsoid. In this latter case, $B^n(a, r) \subset K \subset B^n(a, nr)$, and $0 \notin B^n(a, r)$, i.e., $K^\circ \subset B^n(a, r)^\circ$ which is not a useful bound, since in this case $|B^n(a, r)^\circ| = \infty$. This does not allow for good control on several estimates, e.g., items (iii), (iv) and (v) in Lemma 43. However, using Santaló’s inequality one can overcome this by proving the estimate $K^\circ \subset B^n(0, \frac{2n}{\pi})$ (Lemma 37). It turns out that this argument is not essential for our goal, but we include it since it might be of independent interest; we also give an alternative analytical argument that avoids the geometric arguments involving John’s theorem altogether and relies purely on tensorization (Remark 39).

The bump function. In Nazarov’s approach it is crucial to carefully construct a bump function to be used in setting up a $\partial$-equation whose solution will allow to construct a holomorphic $L^2$ function on a tube domain with good estimates via Hörmander’s theorem. This largely follows Nazarov’s ideas in the symmetric case but the calculations are more involved in the general case. For instance, a difference from the symmetric case is accounting for small perturbations of the center of some bodies, which are no longer centered at the origin. For example, the bump function constructed for the proof of Theorem 2 is no longer supported on $\sigma\delta K_C \setminus \delta K_C$, but rather, in $(\sigma\delta K_C \setminus (\sigma - 1)\delta(a + \sqrt{-1}ta)) \setminus \delta K_C$ (see Lemma 48).

An affine invariant. The proof of Proposition 9 relies on both the weight function and the bump function discussed above. However, once again, because of the possibly awkward position of the non-symmetric body after applying John’s theorem it is necessary to find a quantity that is controlled. Fortunately, we observe the affine invariance of $\mathcal{B}(K) := |K^{\delta K_t} \setminus \mathcal{B}(K)|$ (see Definition 7, Lemma 8, and §4.2) that is new compared to the symmetric case and allows to complete the proof of Proposition 9, and hence of Theorem 2.

Several remarks follow to place Theorem 2 in context.

Remark 3. Applying Nazarov’s complex methods directly to non-symmetric convex bodies—as we do in this article—gives (2) with $c = \pi/4 \approx 0.785$. On the other hand, as pointed out by Nazarov, replacing a given non-symmetric $K$ with an associated symmetrization of $K$ and then applying Nazarov’s bound for symmetric bodies yields a better estimate, namely, $\frac{1}{2}\pi^3/16 \approx 0.968$ [26, p. 342]. This well-known “symmetrization trick” is briefly described in Corollary 56 for the sake of exposition. Thus, the main point of this article, as in Nazarov’s original article, is not to derive the best possible constant but rather to highlight that Nazarov’s complex methods give the first proof of the Bourgain–Milman theorem for general bodies without using symmetrization. For previous proofs that use symmetrization see, e.g., [7, Lemma 3.1], [25, Theorem 1.4], [19, Corollary 1.6], [11, Theorem 1.3].

Remark 4. Blocki recovered one of Nazarov’s estimates on the Bergman kernel (Proposition 9) by providing lower bounds for the Bergman kernel of a convex domain in $\mathbb{C}^n$ [5, Theorem 2]. As we explain in a forthcoming publication [23], even though also Blocki only considered symmetric convex bodies [6, §4], his approach readily applies to the non-symmetric case, yielding another proof of Theorem 2. However, we believe that the approach presented here is more accessible.
Remark 5. The best known constant for \((2)\) in dimensions \(n \geq 4\) with \(K\) symmetric is \(c = \pi\) [19, Corollary 1.6], [4, Theorem 2.1]. The sharp bound \(c = 4\) is due to Mahler in dimension \(n = 2\) [22, (2)], and Iriyeh–Shibata in dimension \(n = 3\) [17, Theorem 1.1] (cf. Fradelizi et al. [9]). For general \(K\), the best known constant is \(c = 2\) for \(n = 3\) and \(c = \pi/2\) for \(n \geq 4\) by the symmetric bound and Corollary 56. In dimension \(n = 2\) the sharp bound \((3)\) is due to Mahler [22, (1)].

Organization. In Section 2 basic facts about Bergman kernels are given, the functional \(B\) is introduced (Definition 7), and its affine invariance is stated (Lemma 8). This is followed by the statement of two key estimates (Propositions 6 and 9) on the Mahler volume \(M\) and on \(B\) involving Bergman kernels of tube domains that lead to the proof of Theorem 2. The key ideas in the proof of Proposition 6 are outlined at the beginning of §3.2 and the detailed proof occupies the remainder of that subsection. It relies on the well-known Paley–Wiener correspondence for tube domains that is carefully derived in §3.1 relying and expanding on several sources [3, §3], [6, §3], [16, §4]. Section 4 derives a lower bound on \(B\) and we refer to the beginning of that section for a detailed step-by-step road-map for the proof. Many of the steps are adaptations (some rather straightforward, some quite technical) of Nazarov’s arguments from the symmetric setting [26, §§5–6], yet several steps are new to the non-symmetric setting and the study of \(B\). First, the affine invariance of \(B(K) = |K|^2 K_{T_K}(\sqrt{-1} b(\int K), \sqrt{-1} b(K))\) is derived in §4.2. Second, a convenient displacement of \(K\) is studied in §4.3. It is at this point in the analysis that the use of Santaló’s theorem occurs (Lemma 37), though we also give an alternative argument that avoids this feature (Remark 39). Third, Nazarov’s plurisubharmonic support function is extended to the non-symmetric setting in §4.4 leading to the definition of the weight function. Next, Lemma 43 in §4.5 contains the properties needed from the weight function for the application of Hörmander’s \(L^2\) estimates. In §4.6 a bump function is constructed to be used in setting up a \(\bar{\partial}\)-equation whose solution will allow to construct a holomorphic \(L^2\) function on the tube domains with good estimates. This follows Nazarov’s ideas in the symmetric case but the calculations are more involved in the general case. The proof of Proposition 9 occupies §4.7 and relies on the ingredients above together with a standard “tensorization trick” for Bergman kernels described in §4.8. Finally, Appendix A serves as an explanation (though not self-contained) of the classical symmetrization trick that has been prevalent in other approaches to the Bourgain–Milman theorem [26, p. 342].

2 Mahler volume and Bergman kernel of tube domains

This section introduces the functional \(B\) (Definition 7) and provides the proof of Theorem 2, modulo two key estimates (Propositions 6 and 9) and the affine invariance of \(B\) (Lemma 8).

2.1 Bergman spaces

Nazarov’s key idea is to reduce the proof of Theorem 2 to the study of the Hilbert space of \(L^2\)-integrable holomorphic functions

\[
A^2(T_K) := \{ f : T_K \to \mathbb{C} : f \text{ holomorphic}, \|f\|_{L^2(T_K)}^2 := \int_{T_K} |f(z)|^2 \, d\lambda(z) < \infty \},
\]
on so-called ‘tube domains over \(K\),

\[
T_K := \mathbb{R}^n + \sqrt{-1}(\text{int } K) \subset \mathbb{C}^n.
\]
Fix a convex body $K \subset \mathbb{R}^n$. For $w \in T_K$, the evaluation map
\[ ev_w \equiv ev_{T_K, w} : f \in A^2(T_K) \mapsto f(w) \in \mathbb{C}, \]
is a bounded linear functional from $A^2(T_K)$ equipped with the $L^2(T_K)$ norm to $(\mathbb{C}, | \cdot |)$: this follows from the holomorphicity of $f$, which implies that $|f|^2$ is subharmonic, so for $\varepsilon > 0$ such that $B_2^{2n}(w, \varepsilon) \subset T_K$, by the mean value inequality,
\[ |ev_{T_K, w}(f)|^2 = |f(w)|^2 \leq \frac{1}{|B_2^{2n}(w, \varepsilon)|} \int_{B_2^{2n}(w, \varepsilon)} |f(z)|^2 \, d\lambda(z) \leq \frac{\|f\|_{L^2(T_K)}^2}{\varepsilon^{2n}|B_2^{2n}(0, 1)|}, \]
so
\[ \|ev_{T_K, w}\|_{(A^2(T_K), L^2(T_K)), (\mathbb{C}, | \cdot |)} := \sup_{0 \neq f \in A^2(T_K)} \frac{|f(w)|}{\|f\|_{L^2(T_K)}} \leq \varepsilon^{-n}|B_2^{2n}(0, 1)|^{-1/2}, \]
that is bounded as claimed.

Thus, the Riesz representation theorem provides
\[ K_{T_K}(\cdot, w) \in A^2(T_K), \tag{5} \]
satisfying
\[ f(w) = \langle f, K_{T_K}(\cdot, w) \rangle_{L^2(T_K)} = \int_{T_K} f(z)\overline{K_{T_K}(z, w)} \, d\lambda(z), \tag{6} \]
for all $f \in A^2(T_K)$. The reproducing kernel $K_{T_K}$, considered as a function on $T_K \times T_K$, is the Bergman kernel of the tube domain. By (5) $K_{T_K}$ is holomorphic in $z$. Since in (6) $f \in A^2(T_K)$, i.e., $f$ is holomorphic, $K_{T_K}$ is also anti-holomorphic in $w$.

### 2.2 Two estimates on the Bergman kernel

In essence, there are two key estimates in the proof of Theorem 2, following a standard preliminary step (described in detail in the proof below) involving a translation by the barycenter
\[ b(K) := \frac{1}{|K|} \int_K x \, dx. \tag{7} \]

The first estimate. The first step is a lower bound on the Mahler volume of the translated body in terms of the Bergman kernel.

**Proposition 6.** For a convex body $K \subset \mathbb{R}^n$ and $a \in \text{int } K$,
\[ M(K - a) \geq \pi^n |K|^2 K_{T_K}(\sqrt{-1}a, \sqrt{-1}a). \]

Proposition 6 leads to the following functional on convex bodies:

**Definition 7.** For a convex body $K \subset \mathbb{R}^n$ let
\[ B(K) := |K|^2 K_{T_K}(\sqrt{-1}b(K), \sqrt{-1}b(K)). \]

An elementary new observation that is crucial for this article is:

**Lemma 8.** $B(K)$ is an affine invariant.
Proposition 6 will be used with $a = b(K)$ (7) since then both sides of
\[ M(K - b(K)) \geq \pi^n B(K) \]  
(8)
are affine invariants and, moreover, the right hand side will be shown to have a uniform lower bound (Proposition 9). The affine invariance of the right-hand side is the content of Lemma 8. To see the affine invariance of the left-hand side is more straightforward: let $S(x) = Ax + b$ for $A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n$, be an affine transformation. Since $b(S(K)) = S(b(K))$ (9),
\[ M(S(K) - b(S(K))) = M(S(K) - S(b(K))) = M(AK + b - Ab(K) - b) = M(A(K - b(K))) = M(K - b(K)), \]
because $M$ is invariant under the action of $GL(n, \mathbb{R})$ and
\[ b(S(K)) = \frac{1}{|AK + b|} \int_{AK + b} x \, dx = \frac{1}{|\det A||K|} \int_{K} (Ay + b) |\det A| \, dy = Ab(K) + b = S(b(K)), \]
since $A$ is linear, and hence commutes with the integral.

Nazarov proves a special case of Proposition 6 for symmetric convex bodies [26, p. 338]:
\[ M(K) \geq |K|^2 K_{Tk}(0, 0) \pi^n. \]  
(10)
We extend his estimate to general convex bodies by observing that his proof does not actually require symmetry (which implies $b(K) = 0$) (see, e.g., Lemma 31). A similar observation was already made by Hultgren who derived a functional version of (10) for convex functions $f$ with \( \int_{\mathbb{R}^n} xe^{-f(x)} \, dx = 0 \) [16, Lemma 11]. The proof of Proposition 6 occupies §3.2. It relies on the well-known Paley–Wiener correspondence for tube domains that is carefully derived in §3.1. Lemma 8 is not part of the proof of Proposition 6 and is proved in §4.2.

The second estimate. So far there is little distinction between symmetric and non-symmetric convex bodies. The essential differentiation between the two cases comes in the next estimate concerning a lower bound on the Bergman kernel on the diagonal:

**Proposition 9.** For $K \subset \mathbb{R}^n$ a convex body,
\[ B(K) \geq 4^{-n}. \]

**Conjecture 10.** For $K \subset \mathbb{R}^n$ a convex body, $B(K) \geq B(\Delta_n)$ where $\Delta_n := \{ x \in [0, \infty)^n : x_1 + \ldots + x_n \leq 1 \}$ is the $n$-dimensional simplex.

**Remark 11.** Nazarov’s analogue of Proposition 9 for symmetric bodies is
\[ K_{Tk}(0, 0) \geq \left( \frac{\pi^2}{16} \right)^n |K_C|, \]  
(11)
where $|K_C| \leq |K|^2$ as in (59) [26, p. 341]. While (11) is sharp [26, p. 342], if one were to replace $|K_C|$ by $|K|^2$ it would no longer be. For us, Proposition 9 is not sharp and would not be sharp even if $|K|^2$ were replaced by $|K_C|$ (which is possible by Lemma 49). Indeed, in dimension $n = 1$, $K = (-1/2, 1/2)$, $K_{Tk}(0, 0) = \pi/4$ and $|K| = 1$ [26, p. 342]. Thus, by affine invariance (Lemma 8)
\[ |K|^2 K_{Tk}(0, 0) = \frac{\pi}{4} \geq \frac{1}{4}, \]
for all symmetric intervals $K$ in $\mathbb{R}$ (i.e., $K \subset \mathbb{R}$ with $b(K) = 0$). Moreover, if $B(K)$ were to be replaced by $|K_C|K_{Tk}(\sqrt{-1}b(K), \sqrt{-1}b(K))$ in Proposition 9, the estimate would still not be sharp since in dimension $n = 1$ the estimate (11) is sharp [26, p. 342].
Remark 12. Blocki conjectured that for symmetric convex bodies $K_T(0,0) \geq (\frac{\pi}{4})^n / |K|^2$, attained for the cube $[-1,1]^n$ (that would imply (2) with $c = \pi^2/4$ for symmetric convex bodies) [5, (7)].

Proof of Theorem 2. For convex bodies $K \subset \mathbb{R}^n$ with $b(K) = 0$, the claim follows from Propositions 6 and 9. In general, for any convex body $K \subset \mathbb{R}^n$ the volume product

$$\inf_{z \in \mathbb{R}^n} \mathcal{M}(K - z) = \mathcal{M}(K - s(K))$$

is minimized at a unique point $s(K) \in \text{int } K$ (called the Santaló point) for which $b((K - s(K))^o) = 0$ [31, (2.3)]. The Mahler volume of the translated body $K - s(K)$ equals that of its polar $(K - s(K))^o$, and the latter is bounded from below by $(\pi/4)^n$ as its barycenter is at the origin.

3 Estimating the Mahler volume

This section culminates in §3.2 in the proof of Proposition 6 that states a lower bound for the Mahler volume in terms of a Bergman kernel. Since the former can be expressed as an integral involving the support function $h_K$ (13) (Claim 30), the gist of the proof is to recognize that the support function has an “$L^1$-cousin” in the form of a functional $\tilde{h}_K$ (Definition 13), that this cousin actually can bound its “$L^\infty$-cousin” $h_K$ (Lemma 31), and that this $\tilde{h}_K$, in turn, is closely related to the Bergman kernel of the tube domain $T_K$ over $K$.

3.1 A Paley-Wiener correspondence

The main result of this subsection is the Paley–Wiener correspondence, Theorem 15, and we mainly follow Berndtsson [3, Proposition 3.1], Blocki [6, Section 3], Hultgren [16, Chapter 4], and Nazarov [26, Section 3], but add more detail as needed.

3.1.1 The Paley–Wiener map

The Mahler volume naturally involves the support function

$$h_K(y) := \sup_{x \in K} \langle x, y \rangle.$$  \hspace{1cm} (13)

The key idea relating $\mathcal{M}(K)$ to $K_T$ is an $L^1$ analogue.

Definition 13. For $K \subset \mathbb{R}^n$ a compact body, let

$$\tilde{h}_K(x) := \log \frac{1}{|K|} \int_K e^{\langle x, y \rangle} dy,$$

and denote by $L^2(\mathbb{R}^n, \tilde{h}_K)$ the class of functions $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|g\|_{L^2(\tilde{h}_K)} := \left( |K| \int_{\mathbb{R}^n} |g(x)|^2 e^{\tilde{h}_K(-2x)} dx \right)^{\frac{1}{2}} < \infty.$$

By compactness of $K$, $h_K \geq \tilde{h}_K$. A key observation is that convexity yields a reverse inequality (Lemma 31). Proposition 6 then readily follows since the Bergman kernel is closely related to $\tilde{h}_K$ by a classical formula (that can be justified by Theorem 15).
Remark 14. Nazarov [26, p. 337] (and Blocki [6, p. 93]) define \( J_K(x) := \int_K e^{-2(x,y)} \, dy \), and work with the class \( L^2(\mathbb{R}^n, J_K) \) of weighted \( L^2 \)-integrable functions \( g : \mathbb{R}^n \to \mathbb{R} \) such that \( \|g\|_{L^2(J_K)} := \left( \int_{\mathbb{R}^n} |g(x)|^2 J_K(x) \, dx \right) < \infty \). Since \( J_K(x) = |K| e^{\bar{h}_K(-2x)} \), \( L^2(\mathbb{R}^n, J_K) = L^2(\mathbb{R}^n, \bar{h}_K) \). Working with \( \bar{h}_K \) makes some of the key estimates more intuitive geometrically (e.g., Lemma 31).

The following Paley–Wiener type theorem establishes an integral representation of the elements of \( A^2(T_K) \) in terms of elements of \( L^2(\mathbb{R}^n, \bar{h}_K) \). Define a map \( \text{PW} \) sending functions in \( L^2(\mathbb{R}^n, \bar{h}_K) \) to functions on \( T_K \),

\[
L^2(\mathbb{R}^n, \bar{h}_K) \ni g \mapsto \text{PW}(g)(w) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(x)e^{\sqrt{-1}(w,x)} \, dx, \quad w \in T_K. \tag{14}
\]

Theorem 15. \( \text{PW} \) is a bijection between \( L^2(\mathbb{R}^n, \bar{h}_K) \) and \( A^2(T_K) \), with \( \|\text{PW}(\cdot)\|_{L^2(T_K)} = \|\cdot\|_{L^2(\bar{h}_K)} \).

Theorem 15 establishes not just an integral representation, but one that is also an isometry between the respective Hilbert space structures. Theorem 15 is well-documented, see, e.g., [3, §3], [6, §3], [16, §4], but the details are scattered and often left to the reader so we provide a streamlined proof for the reader’s convenience. Theorem 15 follows from the following four propositions.

Proposition 16. For \( g \in L^2(\mathbb{R}^n, \bar{h}_K) \) and \( w \in T_K \) the integral \( \int_{\mathbb{R}^n} g(x)e^{\sqrt{-1}(w,x)} \, dx \) converges in \( \mathbb{C} \).

Proposition 17. \( \|\text{PW}(\cdot)\|_{L^2(T_K)} = \|\cdot\|_{L^2(\bar{h}_K)} \).

Proposition 18. \( \text{PW}(L^2(\mathbb{R}^n, \bar{h}_K)) \subset A^2(T_K) \).

Proposition 19. \( A^2(T_K) \subset \text{PW}(L^2(\mathbb{R}^n, \bar{h}_K)) \).

Proof of Theorem 15. By Proposition 16, for any \( g \in L^2(\mathbb{R}^n, \bar{h}_K) \), \( \text{PW}(g) \) is a \( \mathbb{C} \)-valued function on \( T_K \). By Proposition 17, \( \text{PW} \) is injective since it is linear and \( \text{PW}(g) = 0 \) if and only if \( \|\text{PW}(g)\|_{L^2(T_K)} = \|g\|_{L^2(\bar{h}_K)} = 0 \), i.e, \( g = 0 \). Surjectivity follows from Propositions 18 and 19. \( \square \)

3.1.2 Convergence of the integral

For the proof of Propositions 16 we follow Blocki [6, p. 93]. Essentially, the function \( \text{PW}(g) \) can be estimated by putting absolute value inside the integrand. This, naturally, leads to the appearance of \( \bar{h}_K \), which itself can be estimated from below (Lemma 20).

Proof of Proposition 16. Let \( w = \xi + \sqrt{-1}v \in T_K \), that is \( \xi \in \mathbb{R}^n \), \( v \in \text{int} \, K \). By Cauchy–Schwarz,

\[
|\text{PW}(g)(w)| = \left| \int_{\mathbb{R}^n} g(x)e^{\sqrt{-1}(x,w)} \, dx \right| \leq \int_{\mathbb{R}^n} |g(x)|e^{-\langle x,v \rangle} \, dx \\
= \int_{\mathbb{R}^n} |g(x)|\sqrt{|K|e^{\bar{h}_K(-2x)}} \sqrt{|K|e^{\bar{h}_K(-2x)}} \, dx \\
\leq \left( |K| \int_{\mathbb{R}^n} |g(x)|^2 e^{\bar{h}_K(-2x)} \, dx \right)^{\frac{1}{2}} \left( \frac{1}{|K|} \int_{\mathbb{R}^n} e^{-2\langle x,v \rangle-\bar{h}_K(-2x)} \, dx \right)^{\frac{1}{2}}
\]
To estimate the last term, there exists some \( r > 0 \) such that \( v + [-r, r]^n \subset \text{int} \, K \). As a result,

\[
|K| e^{\hat{h}_K(2x)} \geq \int_{v+[r-r]^n} e^{-2(x,y)} \, dy = \prod_{i=1}^n \int_{v_i-r}^{v_i+r} e^{-2x_i y_i} \, dy_i = e^{-2(x,v)} \prod_{i=1}^n \frac{\sinh(2r x_i)}{x_i}. \tag{15}
\]

By Lemma 20,

\[
\frac{1}{|K|} \int_{\mathbb{R}^n} e^{-2(x,v)-\hat{h}_K(-2x)} \, dx \leq \left( \int_{\mathbb{R}} \frac{s}{\sinh(2rs)} \, ds \right)^n = \left( \frac{\pi^2}{8r^2} \right)^n. \tag{16}
\]

Since, \( g \in L^2(\mathbb{R}^n, \hat{h}_K) \), \( \text{PW}(g)(w) \in \mathbb{C} \) for each \( w \in T_K \), proving Proposition 16.

The following was used for the integral of \( s/\sinh(2rs) \).

**Lemma 20.** For \( r > 0 \),

\[
\int_{\mathbb{R}} \frac{t \, dt}{\sinh(2rt)} = \frac{\pi^2}{8r^2}.
\]

**Proof.** Expand the integrand,

\[
\frac{t}{\sinh(2rt)} = \frac{2t}{e^{2rt} - e^{-2rt}} = \frac{2te^{-2rt}}{1 - (e^{-2rt})^2} = \sum_{k=0}^\infty 2te^{-2rt(k+1)}.
\]

Using integration by parts,

\[
\int_0^\infty 2te^{-2rt(k+1)} \, dt = \frac{1}{r(k+1)} \int_0^\infty e^{-2rt(k+1)} \, dt = \frac{1}{2r^2(k+1)^2}.
\]

By (17), (18), and Tonelli’s theorem [10, §2.37] (see Claim 22 below), since the integrand is an even function,

\[
\int_{\mathbb{R}} \frac{t}{\sinh(2rt)} \, dt = 2 \int_0^\infty \frac{t}{\sinh(2rt)} \, dt = 2 \sum_{k=0}^\infty \int_0^\infty 2te^{-2rt(k+1)} \, dt
\]

\[
= 2 \sum_{k=0}^\infty \frac{1}{2r^2(k+1)^2} = \frac{1}{r^2} \sum_{k=1}^{\infty} \frac{3}{4k^2} = \frac{3 \pi^2}{4r^2} 6 = \frac{\pi^2}{8r^2}.
\]

**Remark 21.** Similarly, an \( L^2 \) property for \( e^{-2(\cdot,v)-\hat{h}_K(-2\cdot)} \) can derived (this will be useful in proving the formula for the Bergman kernel of a tube domain (Lemma 32)). By (17),

\[
\frac{t^2}{\sinh(2rt)^2} = \frac{4t^2 e^{-4rt}}{(1 - (e^{-2rt})^2)} = \sum_{k=1}^\infty 4t^2 e^{-4rtk},
\]

thus,

\[
\int_{\mathbb{R}} \frac{t^2}{\sinh(2rt)^2} \, dt = 2 \int_0^\infty \frac{t^2}{\sinh(2rt)^2} \, dt = 2 \sum_{k=1}^\infty \int_0^\infty 4t^2 e^{-4rtk} \, dt = \frac{1}{8r^3} \sum_{k=1}^\infty \frac{1}{k^3} \tag{19}
\]

which is finite. As a result, by (15) and (19), \( \int_{\mathbb{R}^n} \left( e^{-2(x,v)-\hat{h}_K(-2x)} \right)^2 \, dx \) is also finite.
3.1.3 Fourier transform and integration tools

This subsection recalls some elementary real analysis tools that will be used repeatedly throughout.

For \( \xi \in \mathbb{R}^n \),
\[
\hat{g}(\xi) := \int_{\mathbb{R}^n} g(x)e^{-\sqrt{-1}\langle \xi, x \rangle} \, dx,
\]
(20)
is the Fourier transform of \( g \). Strictly speaking (20) requires \( g \in L^1(\mathbb{R}^n) \), but relaxing (20) to hold a.e. one can allow \( g \in L^2(\mathbb{R}^n) \) [15, Theorem 7.1.11]. For \( g \in L^2(\mathbb{R}^n) \), by Fourier inversion [15, (7.1.4)],
\[
g(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{g}(\xi)e^{\sqrt{-1}\langle \xi, x \rangle} \, d\xi.
\]
(21)
Combining (20) and (21), and flipping the sign of \( \xi \),
\[
g(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(s)e^{\sqrt{-1}\langle \xi, x-s \rangle} \, ds \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(s)e^{\sqrt{-1}\langle \xi, s-x \rangle} \, ds \, d\xi.
\]
(22)
Moreover, for \( f \in L^2(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} f(x)g(x) \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi) \, d\xi,
\]
(23)
[15, Theorem 7.1.6] and, in particular,
\[
\|\hat{g}\|_{L^2(\mathbb{R}^n)} = (2\pi)^\frac{n}{2}\|g\|_{L^2(\mathbb{R}^n)},
\]
(24)
the so called Plancherel’s theorem [10, §8.29].

Recall the theorems attributed to Tonelli and Fubini [10, §2.37].

Claim 22. For \( n, m \in \mathbb{N} \), denote by \( v = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \).

(i) For non-negative measurable \( f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty) \),
\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} f(v) \, d\lambda_{n+m}(v) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} f(x, y) \, d\lambda_m(y) \right) \, d\lambda_n(x)
\]
(25)
\[
= \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(x, y) \, d\lambda_n(x) \right) \, d\lambda_m(y).
\]
(ii) For \( f \in L^1(\mathbb{R}^n \times \mathbb{R}^m) \), \( x \mapsto f(x, y) \) is \( L^1 \)-integrable for almost all \( y \in \mathbb{R}^m \), and \( y \mapsto f(x, y) \) is \( L^1 \)-integrable for almost all \( x \in \mathbb{R}^n \) with
\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} f(v) \, d\lambda_{n+m}(v) = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(x, y) \, d\lambda_n(x) \right) \, d\lambda_m(y)
\]
(26)
\[
= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} f(x, y) \, d\lambda_m(y) \right) \, d\lambda_n(x).
\]
(27)

Remark 23. Since part (i) (Tonelli’s theorem) does not assume integrability of \( f \), it is often used to justify part (ii) (Fubini’s theorem): given measurable \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) one may compute any of the two iterated integrals
\[
\int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} |f(x, y)| \, dy \right) \, dx \text{ or } \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} |f(x, y)| \, dx \right) \, dy;
\]
if either is finite, by (25), \( f \in L^1(\mathbb{R}^n \times \mathbb{R}^m) \) justifying the use of Fubini’s theorem (Claim 22 (ii)).
3.1.4 PW is an isometry between $L^2$ spaces

Proposition 16 shows that for any $g \in L^2(\mathbb{R}^n, \mathcal{h}_K)$, PW$(g)$ is a $\mathbb{C}$-valued function on $T_K$. The following shows it is also $L^2$-integrable and has $L^2(T_K)$ norm equal to $\|g\|_{L^2(\mathcal{h}_K)}$ [6, (6)] [3, Proposition 3.1].

**Proof of Proposition 17.** For $w = \xi + \sqrt{1}v \in T_K$,

$$\text{PW}(g)(w) = \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} g(x)e^{\sqrt{-1}(w,x)} \, dx = \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} g(x)e^{-(x,v)}e^{\sqrt{-1}(x,\xi)} \, dx.$$  

Since by Lemma 24 below $x \mapsto (2\pi)^{n/2}g(x)e^{-(x,v)}$ is $L^2$-integrable, by (20) and (21) $x \mapsto (2\pi)^{n/2}g(x)e^{-(x,v)}$ is the Fourier transform of $\xi \mapsto \text{PW}(g)(\xi + \sqrt{-1}v)$. Moreover, by (24)

$$(2\pi)^n \int_{\mathbb{R}^n} |g(x)|^2 e^{-2(x,v)} \, dx = (2\pi)^n \int_{\mathbb{R}^n} |\text{PW}(g)(\xi + \sqrt{-1}v)|^2 \, d\xi. \quad (28)$$

Integrating (28) with respect to $v \in K$ and interchanging the order of integration (by (25)),

$$\|\text{PW}(g)\|^2_{L^2(T_K)} = \int_{\text{int} K} \int_{\mathbb{R}^n} |g(x)|^2 e^{-2(x,v)} \, dx \, dv = |K| \int_{\mathbb{R}^n} |g(x)|^2 e^{\mathcal{h}_K(-2x)} \, dx = \|g\|^2_{L^2(\mathcal{h}_K)},$$

by Definition 13. \qed

**Lemma 24.** For $g \in L^2(\mathbb{R}^n, \mathcal{h}_K)$ and $v \in \text{int} K$, $x \mapsto g(x)e^{-(x,v)}$ is in $L^2(\mathbb{R}^n)$.

**Proof.** As $|s| \leq 2|\sinh s|$ for all $s \in \mathbb{R}$, by (15) there exists $r > 0$ satisfying

$$|K| e^{\mathcal{h}_K(-2x)} \geq e^{2(x,v)} \prod_{i=1}^n \frac{\sinh(2x_i)}{x_i} \geq r^n e^{-2(x,v)}.$$  

Thus,

$$\int_{\mathbb{R}^n} |g(x)|^2 e^{-2(x,v)} \, dx \leq r^{-n}|K| \int_{\mathbb{R}^n} |g(x)|^2 e^{\mathcal{h}_K(-2x)} \, dx = r^{-n}\|g\|_{L^2(\mathcal{h}_K)}^2,$$

proving the lemma. \qed

3.1.5 PW maps to $A^2(T_K)$

By Propositions 16–17, $\text{PW}(L^2(\mathbb{R}^n, \mathcal{h}_K)) \subset L^2(T_K)$. To show that the image of PW is in fact in $A^2(T_K)$ it remains to show that PW$(g)$ is holomorphic. A similar theorem, in a more general setting, was shown by Hultgren [16, Theorem 3]. It is useful to first show PW$(g)$ is continuous.

**Lemma 25.** For $g \in L^2(\mathbb{R}^n, \mathcal{h}_K)$, PW$(g)$ is continuous on $T_K$.

**Proof.** Fix $w = \xi + \sqrt{-1}v \in T_K$ and $\delta > 0$ such that $w + z \in T_K$ for all $z = u + \sqrt{-1}y \in B_{2^n}(0, \delta)$. By Cauchy–Schwarz,

$$|\text{PW}(g)(w + z) - \text{PW}(g)(w)| = \left| \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} g(x)(e^{\sqrt{-1}(x,w+z)} - e^{\sqrt{-1}(x,w)}) \, dx \right|$$

$$\leq \left( \int_{\mathbb{R}^n} |g(x)|^2 e^{\mathcal{h}_K(-2x)} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |e^{\sqrt{-1}(x,w+z)} - e^{\sqrt{-1}(x,w)}|^2 e^{-\mathcal{h}_K(-2x)} \, dx \right)^{\frac{1}{2}}. \quad (29)$$
Moreover, let \( r > 0 \) such that \( v + y + [−r, r]^n \subset \text{int } K \) for all \( z = u + iy \in B_2^{-n}(0, \delta) \). By (16),

\[
\int_{\mathbb{R}^n} |e^{\sqrt{-1}(x,w+z)} - e^{\sqrt{-1}(x,w)}|^2 e^{-\tilde{h}_K(-2x)} \, dx \\
\leq \int_{\mathbb{R}^n} \left( 2|e^{\sqrt{-1}(x,w+z)}| + 2|e^{\sqrt{-1}(x,w)}| \right) e^{-\tilde{h}_K(-2x)} \, dx \\
= 2\int_{\mathbb{R}^n} e^{-2(x,v+y)} e^{-\tilde{h}_K(-2x)} \, dx + 2\int_{\mathbb{R}^n} e^{-2(x,v)} e^{-\tilde{h}_K(-2x)} \, dx \\
\leq 4|K| \left( \frac{|\pi^2|}{8\nu^2} \right)^n,
\]

that is finite and independent of \( z \). So, dominated convergence applies [10, §2.24],

\[
\lim_{z \to 0} \int_{\mathbb{R}^n} |e^{\sqrt{-1}(x,w+z)} - e^{\sqrt{-1}(x,w)}|^2 e^{-\tilde{h}_K(-2x)} \, dx = \\
\int_{\mathbb{R}^n} \lim_{z \to 0} |e^{\sqrt{-1}(x,w+z)} - e^{\sqrt{-1}(x,w)}|^2 e^{-\tilde{h}_K(-2x)} \, dx = 0.
\]

(30)

From (30) and (29) it follows \( \lim_{z \to 0} |\text{PW}(g)(w+z) - \text{PW}(g)(w)| = 0 \), thus \( \text{PW}(g) \) is continuous. \( \square \)

**Proof of Proposition 18.** Let \( g \in L^2(\mathbb{R}^n, \tilde{h}_K) \). By Propositions 16, 17, \( \text{PW}(g) \in L^2(T_K) \). To show \( \text{PW}(g) \) is holomorphic it suffices to show it is holomorphic in each variable separately. As a result, let us take \( n = 1 \). By Lemma 25, \( \text{PW}(g) \) is continuous and hence by Morera’s theorem [1, p. 122] it suffices to show that for any closed smooth curve \( \gamma : [0, 1] \to T_K \) the integral \( \int_{\gamma} \text{PW}(g)(w) \, dw \) vanishes. Let \( \gamma(s) = (x(s), y(s)) \). Since, the image of \( \gamma \) is compact, there exists \( r > 0 \) small enough so that \( \gamma(s) + [−r, r]^2 \subset T_K \), for all \( s \in [0, 1] \) (here we used that (4) involves the *interior* of \( K \)). Thus, (16) holds for all \( y(s) \) and \( s \in [0, 1] \). By Cauchy–Schwarz and (16),

\[
\left| \int_{\gamma} \int_{\mathbb{R}} |g(x)e^{\sqrt{-1}xw}| \, dx \, dw \right| \leq \int_{0}^{1} \int_{\mathbb{R}} |g(x)|e^{-xy(s)}|\gamma'(s)| \, dx \, ds \\
\leq \int_{0}^{1} \left( \int_{\mathbb{R}} |g(x)|^2 e^{\tilde{h}_K(-2x)} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} e^{-2xy(s)} e^{-\tilde{h}_K(-2x)} \, dx \right)^{\frac{1}{2}} |\gamma'(s)| \, ds \\
\leq \text{length}(\gamma) \| g \|_{L^2(T_K)} \frac{\pi}{r\sqrt{8}}.
\]

Since \( g \in L^2(\mathbb{R}^n, \tilde{h}_K) \), it follows that \( \int_{\gamma} \int_{\mathbb{R}} |g(x)e^{\sqrt{-1}xw}| \, dx \, dw \) is finite. Thus, the order of integration in \( \int_{\gamma} \text{PW}(g)(w) \, dw \) can be changed, i.e., by (27),

\[
\int_{\gamma} \text{PW}(g)(w) \, dw = (2\pi)^{-\frac{n}{2}} \int_{\gamma} \int_{\mathbb{R}} g(x)e^{\sqrt{-1}xw} \, dx \, dw = (2\pi)^{-\frac{n}{2}} \int_{\gamma} g(x) \int_{\gamma} e^{\sqrt{-1}xw} \, dw \, dx = 0,
\]

because for each \( x, w \mapsto e^{\sqrt{-1}xw} \) is holomorphic and \( \gamma \) is closed [1, p. 122]. \( \square \)

**3.1.6 PW is surjective**

By Proposition 17, PW is an isometry to its image in \( A^2(T_K) \). To show it surjects onto \( A^2(T_K) \), as \( L^2(\mathbb{R}^n, \tilde{h}_K) \) is complete, it suffices to show that a dense subset of \( A^2(T_K) \) is contained in
PW($L^2(\mathbb{R}^n, \tilde{h}_K))$ (isometry implies that PW maps Cauchy sequences in $L^2(\mathbb{R}^n, \tilde{h}_K)$ to Cauchy sequences in $A^2(T_K)$). The key technical result is Lemma 27 saying that any $f \in A^2(T_K)$ can be approximated by $\{F_j\}_j \in A^2(T_K)$ such that the $\xi$-Fourier transform of $F_j(\xi + \sqrt{-1}y)$ is compactly supported for all $y \in \text{int } K$. The following technical lemma (augmenting the brief discussion by Berndtsson [3, p. 405]) is required to carry out such an approximation.

**Lemma 26.** For $f \in A^2(T_K)$ and $\eta \in L^1(\mathbb{R}^n)$,

$$F(w) := \int_{\mathbb{R}^n} f(w - u)\eta(u) \, du,$$

is holomorphic in $T_K$.

**Proof.**

**Step 1: the integral is bounded.** To show that $F(w) \in \mathbb{C}$ for all $w \in T_K$, set $w_0 := \xi_0 + \sqrt{-1}y_0 \in T_K$, and pick $\varepsilon > 0$ such that $B^{2n}_2(w_0, \varepsilon) \subset T_K$. Since $T_K$ is a tube domain, $B^{2n}_2(\xi + \sqrt{-1}y_0, \varepsilon) \subset T_K$, for all $\xi \in \mathbb{R}^n$. As $f$ is holomorphic, $|f|^2$ is subharmonic,

$$|f(\xi + \sqrt{-1}y_0)|^2 \leq \frac{1}{\varepsilon^{2n}|B^{2n}_2(0, 1)|} \int_{B^{2n}_2(\xi + \sqrt{-1}y_0, \varepsilon)} |f(w)|^2 \, d\lambda(w) \leq \frac{\|f\|_{L^2(T_K)}^2}{\varepsilon^{2n}|B^{2n}_2(0, 1)|},$$

for all $\xi \in \mathbb{R}^n$. Thus,

$$\left| \int_{\mathbb{R}^n} f(w_0 - u)\eta(u) \, du \right| \leq \int_{\mathbb{R}^n} |f(w_0 - u)||\eta(u)| \, du \leq \frac{\|f\|_{L^2(T_K)}\|\eta\|_1}{\varepsilon^n \sqrt{|B^{2n}_2(0, 1)|}},$$

(31)

which shows $F$ is $\mathbb{C}$-valued.

**Step 2: verifying Morera’s criterion.** Holomorphicity follows as in the proof of Proposition 18. Let $\gamma : [0, 1] \to T_K$ be a closed curve in $T_K$. Since its image is compact there exists $\varepsilon > 0$ such that $B^{2n}_2(\gamma(t), \varepsilon) \subset T_K$ for all $t \in [0, 1]$. It follows from (31), that $F$ is bounded on $\gamma([0, 1])$, and hence by the holomorphicity of $f$ and (27),

$$\int_{\gamma} F \, dw = \int_{\mathbb{R}^n} \left( \int_{\gamma} f(w - u) \, dw \right) h(u) \, du = 0. \tag{32}$$

**Step 3: continuity.** It remains to show that $F$ is continuous since then by (32) and Morera’s theorem $F$ is holomorphic [1, p. 122]. For $w \in T_K$, let $\varepsilon > 0$ such that $B^{2n}_2(w, 2\varepsilon) \subset T_K$ and $z \in B^{2n}_2(0, \varepsilon)$. As in (31),

$$|F(w + z) - F(w)| = \left| \int_{\mathbb{R}^n} [f(w + z - u) - f(w - u)]\eta(u) \, du \right| \leq \frac{2\|f\|_{L^2(T_K)}\|\eta\|_1}{\varepsilon^n \sqrt{|B^{2n}_2(0, 1)|}},$$

because $B^{2n}_2(w + z, \varepsilon) \subset T_K$ for all $z \in B^{2n}_2(0, \varepsilon)$. As a result, dominated convergence applies [10, §2.24],

$$\lim_{z \to 0} \left[ F(w + z) - F(w) \right] = \int_{\mathbb{R}^n} \lim_{z \to 0} [f(w + z - u) - f(w - u)]\eta(u) \, du = 0,$$

because $f$ is holomorphic, and hence continuous.

Since $F$ is continuous and (32) holds, by Morera’s theorem $F$ is holomorphic [1, p. 122]. \[\square\]
For \( f \in A^2(T_K) \) and \( y \in \text{int } K \), denote
\[
 f_y(\xi) := f(\xi + \sqrt{-1} y).
\]
Berndtsson claims in a more general setting (replacing \( A^2(T_K) \) by \( A^2(e^{-2\phi}) \), for \( \phi \) a convex function) that the class of functions in \( f \in A^2(T_K) \) with compactly supported Fourier transform \( \hat{f}_y \) for at least one \( y \in \text{int } K \) is dense in \( A^2(T_K) \) [3, p. 405] and gives a brief sketch of a proof. Amplifying his ideas, set
\[
 C := \{ f \in A^2(T_K) : \hat{f}_y \text{ is compactly supported for all } y \in \text{int } K \}. \tag{33}
\]

**Lemma 27.** \( C \) is dense in \( A^2(T_K) \).

**Proof.** Set
\[
 \chi(x) := \begin{cases} 
 e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\
 0, & \text{otherwise.}
\end{cases}
\]
Note \( \chi \in C^\infty(\mathbb{R}^n) \), is supported on \( B_2^2(0, 1) \) with \( 0 \leq \chi \leq 1 \), so \( \chi \in L^1(\mathbb{R}^n) \). Let also
\[
 \psi(x) := (\chi * \chi)(x) = \int_{\mathbb{R}^n} \chi(x - u) \chi(u) \, du,
\]
is smooth, non-negative, supported on \( B_2^2(0, 2) \), with \( \hat{\psi} = (\hat{\chi})^2 \geq 0 \) [10, Theorem 8.22(c)]. Moreover, since \( 0 \leq \chi \leq 1 \), \( \psi \) is bounded by
\[
 \psi(x) = \int_{\mathbb{R}^n} \chi(x - u) \chi(u) \, du \leq \int_{\mathbb{R}^n} \chi(u) \, du = \| \chi \|_{L^1}. \tag{34}
\]
By (24), \( \hat{\chi} \in L^2(\mathbb{R}^n) \) since \( \chi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), with \( \| \hat{\chi} \|_{L^2} = (2\pi)^{n/2} \| \chi \|_{L^2} \). As a result, \( \hat{\psi} \in L^1(\mathbb{R}^n) \) with \( \| \hat{\psi} \|_{L^1} = \| \hat{\chi} \|_{L^2} = (2\pi)^n \| \chi \|_{L^2}^2 \). For \( \varepsilon > 0 \), let
\[
 \eta_{\varepsilon}(x) := \frac{\varepsilon^n}{\int \hat{\psi} (x/\varepsilon)} \hat{\chi}(x/\varepsilon)^2 \in C^\infty(\mathbb{R}^n).
\]
Note \( \eta_{\varepsilon} \) is non-negative and
\[
 \int \eta_{\varepsilon} = 1. \tag{35}
\]
By (21),
\[
 \hat{\eta}_{\varepsilon}(\xi) := \int_{\mathbb{R}^n} \eta_{\varepsilon}(x) e^{-\sqrt{-1} \langle \xi, x \rangle} \, dx = \frac{\varepsilon^n}{\int \hat{\psi} } (2\pi)^n \psi(-\xi/\varepsilon),
\]
is smooth and supported on \( B_2^2(0, 2\varepsilon) \). Let \( f \in A^2(T_K) \). By Lemma 26,
\[
 f_{\varepsilon}(w) := \int_{\mathbb{R}^n} f(w - u) \eta_{\varepsilon}(u) \, du,
\]
is holomorphic in \( T_K \). Moreover, by Cauchy–Schwarz, (25), and (35),
\[
 \| f_{\varepsilon} \|^2_{L^2(T_K)} = \int_{T_K} |f_{\varepsilon}(w)|^2 \, d\lambda(w) = \int_{T_K} \left| \int_{\mathbb{R}^n} f(w - u) \eta_{\varepsilon}(u) \, du \right|^2 \, d\lambda(w)
\]
\[
 \leq \int_{T_K} \left( \int_{\mathbb{R}^n} |f(w - u)|^2 \eta_{\varepsilon}(u) \, du \right) \left( \int_{\mathbb{R}^n} \eta_{\varepsilon}(u) \right) \, d\lambda(w)
\]
\[
 = \int_{T_K} \int_{\mathbb{R}^n} |f(w - u)|^2 \eta_{\varepsilon}(u) \, du \, d\lambda(w)
\]
\[
 = \| f \|^2_{L^2(T_K)} \int \eta_{\varepsilon} = \| f \|^2_{L^2(T_K)}. \tag{36}
\]

Therefore, $f_\varepsilon \in A^2(T_K)$. Furthermore, $f_\varepsilon$ has compactly supported Fourier transform for all $y \in \text{int} \, K$, since $(f_\varepsilon)_y = f_y \ast \eta_\varepsilon$, thus $(\hat{f}_\varepsilon)_y = \hat{f}_y \hat{\eta}_\varepsilon$ [10, Theorem 8.22], which is compactly supported because $\hat{\eta}_\varepsilon$ is.

It remains to show that $f_\varepsilon$ $L^2$-converges to $f$. Observe,

$$
\|f_\varepsilon - f\|_{L^2(T_K)}^2 = \int_{\text{int} \, K} \int_{\mathbb{R}^n} |f_\varepsilon(\xi + \sqrt{-1}y) - f(\xi + \sqrt{-1}y)|^2 \, d\xi \, dy
$$

$$
= \int_{\text{int} \, K} \|f_\varepsilon(y) - f(y)\|_{L^2(\mathbb{R}^n)}^2 \, dy.
$$

But, $\lim_{\varepsilon \to 0} \|f_\varepsilon(y) - f(y)\|_{L^2(\mathbb{R}^n)} = 0$, for almost all $y \in \text{int} \, K$ [10, Theorem 8.14], and by (36) $\|f_\varepsilon(y) - f(y)\|_{L^2(\mathbb{R}^n)}^2 \leq 4\|f_y\|_{L^2(\mathbb{R}^n)}^2$, that is integrable because $\int_{\text{int} \, K} \|f_y\|_{L^2(\mathbb{R}^n)}^2 \, dy = \|f\|_{L^2(T_K)}^2$. Combining this and dominated convergence [10, §2.24], gives $\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^2(T_K)} = 0$. ∎

The next argument is due to Berndtsson [3, pp. 404–405].

**Proof of Proposition 19.** By Propositions 16–18, $\text{PW} : L^2(\mathbb{R}^n, \hat{h}_K) \to A^2(T_K)$ is an isometry. Thus, as remarked at the beginning of §3.1.6, to show it is surjective it suffices to show that its image is dense. By Lemma 27, it is enough to prove the theorem for $f \in A^2(T_K)$ with $\hat{f}_y$ compactly supported for all $y \in \text{int} \, K$. Let $f \in A^2(T_K)$ be such a function, and write $f_y(\xi) := f(\xi + \sqrt{-1}y)$. Since

$$
\|f\|_{L^2(T_K)}^2 = \int_{\text{int} \, K} \int_{\mathbb{R}^n} |f(\xi + \sqrt{-1}y)|^2 \, dx \, dy < \infty,
$$

$f_y$ is $L^2(\mathbb{R}^n)$-integrable for almost all $y \in \text{int} \, K$. In particular, there exists some $y_0 \in \text{int} \, K$ such that $f_{y_0}(\xi) := f(\xi + \sqrt{-1}y_0)$ is $L^2(\mathbb{R}^n)$-integrable. By Fourier inversion (21),

$$
\hat{f}_{y_0}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_{y_0}(x) e^{\sqrt{-1}(\xi,x)} \, dx.
$$

By assumption, $\hat{f}_{y_0} \in L^2(\mathbb{R}^n)$ is compactly supported. Therefore, $\hat{f}_{y_0}(x) e^{(x,y_0)} \in L^2(\mathbb{R}^n)$ is also compactly supported and, in particular, lies in $L^2(\mathbb{R}^n, \hat{h}_K)$. By Propositions 16–18 then,

$$
F(w) := \text{PW}((2\pi)^{-\frac{n}{2}} \hat{f}_{y_0}(x) e^{\sqrt{-1}(x,y_0)})(w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_{y_0}(x) e^{(x,y_0)} e^{\sqrt{-1}(x,w)} \, dx,
$$

is well-defined and holomorphic in $T_K$. By (37) and (38),

$$
F(\xi + \sqrt{-1}y_0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_{y_0}(x) e^{\sqrt{-1}(\xi,x)} \, dx = f(\xi + \sqrt{-1}y_0),
$$

i.e., $F$ agrees with $f$ on $\mathbb{R}^n \times \{y_0\}$. Since they are both holomorphic, by analytic continuation $f \equiv F = \text{PW}((2\pi)^{-\frac{n}{2}} \hat{f}_{y_0}(x) e^{\sqrt{-1}(x,y_0)})$, as desired. ∎

**Remark 28.** In fact, restricting to the larger family (recall (33))

$$
\tilde{\mathcal{C}} := \{f \in A^2(T_K) : \hat{f}_y \text{ is compactly supported for some } y \in \text{int} \, K\} \supset \mathcal{C},
$$

suffices for the proof of Proposition 19 above. This is because for $y_0 \in \text{int} \, K$ such that $\hat{f}_{y_0}$ is compactly supported, by (24), $\xi \mapsto f(\xi + iy_0) \in L^2(\mathbb{R}^n)$, and hence (37) holds.
Claim 30. Perhaps a more intuitive proof for Proposition 19 would be the following. Take an \( f \in A^2(T_K) \). Assume that \( f_y \in L^2(\mathbb{R}^n) \) for all \( y \in \text{int} \, K \). By (21),

\[
f(\xi + \sqrt{-1} y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_y(x) e^{\sqrt{-1} \langle x, \xi \rangle} \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_y(x) e^{\langle x, y \rangle} e^{\sqrt{-1} \langle x, \xi + \sqrt{-1} y \rangle} \, dx.
\]  

(39)

In view of (39), let \( g(x, y) := \hat{f}_y(x) e^{\langle x, y \rangle} \). For \( g \) independent of \( y \), \( f = \text{PW}(g) \) as desired. This is where the holomorphicity of \( f \) comes into play. By (20) and (39),

\[
g(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi + \sqrt{-1} y) e^{\langle x, y \rangle} e^{-\sqrt{-1} \langle \xi, x \rangle} \, d\xi.
\]

Assuming that one can differentiate under the integral sign,

\[
\frac{\partial g}{\partial y} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \frac{\partial f}{\partial y} + x f \right) e^{\langle x, y \rangle} e^{-\sqrt{-1} \langle \xi, x \rangle} \, d\xi.
\]  

(40)

Moreover, \( f \) is holomorphic, thus \( \partial f / \partial \xi = -\sqrt{-1} \partial f / \partial y \), i.e.,

\[
\int_{\mathbb{R}^n} \frac{\partial f}{\partial y} e^{\langle x, y \rangle} e^{-\sqrt{-1} \langle \xi, x \rangle} \, d\xi = \sqrt{-1} \int_{\mathbb{R}^n} \frac{\partial f}{\partial \xi} e^{\langle x, y \rangle} e^{-\sqrt{-1} \langle \xi, x \rangle} \, d\xi
\]

\[
= - \int_{\mathbb{R}^n} x f e^{\langle x, y \rangle} e^{-\sqrt{-1} \langle \xi, x \rangle} \, d\xi,
\]  

(41)

by integration by parts. It follows from (40) and (41) that \( \partial g / \partial y = 0 \), that is, \( g \) is independent of \( y \). Nonetheless, several assumptions were made, that may not hold in general, i.e., \( f_y \) is \( L^2 \)-integrable for all \( y \in \text{int} \, K \) and taking the derivatives under the integral, but this can be made rigorous [6, p. 94].

3.2 Proof of Proposition 6

With the Paley–Wiener correspondence established (Theorem 15) the proof of the lower bound on the Mahler volume in terms of the Bergman kernel conceptually proceeds as follows.

- For \( a \in \text{int} \, K \), \( \mathcal{M}(K - a) \) is the product of \(|K|\) with \( n! |(K - a)^c| \). The latter equals to the integral of \( e^{-\hat{h}_{K-a}(x)} \), where \( h_K \) is the support function of \( K \) (Claim 30).

- Jensen’s inequality provides a lower bound \( e^{\hat{h}_{K-a}(x)} \leq 2^n e^{\hat{h}_{K-a}(2x)} \) (Lemma 31).

- Using the Paley–Wiener correspondence established in the previous section one may verify a formula for \( K_{T_K}(z, w) \) so that \( (2\pi)^n |K| K_{T_K}(\sqrt{-1} a, \sqrt{-1} a) = \int_{\mathbb{R}^n} e^{-\hat{h}_{K-a}(2x)} \, dx \) on the diagonal (Lemma 32).

- By the first step, \( \mathcal{M}(K - a) = |K| \int_{\mathbb{R}^n} e^{-\hat{h}_{K-a}} \) which, by the previous two steps, is bounded below by \( \pi^n |K|^2 K_{T_K}(\sqrt{-1} a, \sqrt{-1} a) \) proving Proposition 6.

The following is a well-known formula for \(|K^c|\) in terms of \( h_K \) [16, (2.3)].

Claim 30. For a convex body \( K \subset \mathbb{R}^n \) satisfying (1), \( \int_{\mathbb{R}^n} e^{-h_K(y)} \, dy = n! |K^c| \).

Jensen’s inequality gives a lower bound on \( e^{-\hat{h}_{K-M}(x)} \) in terms of \( \hat{h}_K \). For a subset \( S \subset \mathbb{R}^n \) denote by

\[
1_S(x) := \begin{cases} 
1 & \text{for } x \in S, \\
0 & \text{otherwise.}
\end{cases}
\]
Lemma 31. For $K \subset \mathbb{R}^n$ a convex body and $a \in K$, $e^{h_{K-a}}(x) \leq 2^n e^{\tilde{h}_{K-a}(2x)}$.

Proof. Assume $b(K) = 0$ for a moment. Note that

$$a = b(K) + a = \frac{1}{|K|} \int_K (u + a) \, du = \frac{1}{|K|} \int_{K+a} v \, dv.$$ 

Fix $y \in K$, $x \in \mathbb{R}^n$, and let $F(u) := e^{(u,x)}$. By Jensen’s inequality [2, Remark A.2.3], for the probability measure $\frac{1}{|K|} \int_{K+a+y} v \, dv$ and the convex function $F$,

$$e^{(y-a,x)} = e^{-2(a,x)} e^{(y+a,x)} = e^{-2(a,x)} e^{\left(\frac{1}{|K|} \int_K (u+y+a) \, du\right)} = e^{-2(a,x)} e^{\left(\frac{1}{|K|} \int_K u + y + a \, du\right)}$$

$$\leq e^{-2(a,x)} \frac{1}{|K|} \int_K F(u + y + a) \, du = e^{-2(a,x)} \frac{1}{|K|} \int_K e^{(u+y+a,x)} \, du$$

$$= e^{-2(a,x)} \frac{1}{|K|} \int_K e^{2\left(\frac{y+a}{2},x\right)} \, du = e^{-2(a,x)} 2^n \frac{1}{|K|} \int_{K+y} e^{2(v,x)} \, dv$$

$$\leq e^{-2(a,x)} 2^n \frac{1}{|K|} \int_K e^{2(v,x)} \, dv = 2^n \frac{1}{|K|} \int_K e^{(v-a,2x)} \, dv = 2^n e^{\tilde{h}_{K-a}(2x)},$$

because $y, a \in K$ thus $(y + a)/2 \in K$ and hence $K/2 + (y + a)/2 \subset K$. Taking supremum over all $y \in K$ yields $e^{h_{K-a}}(x) \leq 2^n e^{\tilde{h}_{K-a}(2x)}$ as desired.

In general, for any $a \in K$ write $a = a - b(K) + b(K)$. By the previous case, since $a - b(K) \in K - b(K)$ and $b(K - b(K)) = 0$

$$e^{h_{K-a}(x)} = e^{h_{K-b(K)-(a-b(K))}(x)} \leq 2^n e^{\tilde{h}_{K-b(K)-(a-b(K))}(2x)} = e^{\tilde{h}_{K-a}(2x)},$$

as desired. \qed

The left-hand side of the inequality in Lemma 31 appears in the integral representation of $\mathcal{K}_{T_K}(\sqrt{-1}a, \sqrt{-1}a)$ since on the diagonal one may explicitly compute

$$\mathcal{K}_{T_K}(\sqrt{-1}a, \sqrt{-1}a) = \frac{1}{(2\pi)^n |K|} \int_{\mathbb{R}^n} e^{-\tilde{h}_{K-a}(2x)} \, dx. \quad (42)$$

This follows from the general formula for the Bergman kernel of a tube domain of a convex body $K_T(z,w)$ (Lemma 32 below) [12, (1.2)] [29, Theorem 2.6], and the following computation:

$$e^{2(a,x)+\tilde{h}_{K-a}(2x)} = \int_K e^{-2(x,y)} \, dy = |K| \int_{K-a} e^{-2(x,y-a)} \, dy$$

$$\leq |K - a| \int_{K-a} e^{-2(x,y)} \, dy = e^{\tilde{h}_{K-a}(2x)}.$$

Lemma 32. For a convex body $K \subset \mathbb{R}^n$,

$$\mathcal{K}_{T_K}(z,w) = \frac{1}{(2\pi)^n |K|} \int_{\mathbb{R}^n} e^{\sqrt{-1}(z-w,x)-\tilde{h}_{K-a}(2x)} \, dx.$$

Proof. For $z = \xi + \sqrt{-1}y \in T_K$ and $w = a + \sqrt{-1}b \in T_K$, since $K$ is convex $(y + b)/2 \in \text{int } K$. Take $r > 0$ such that $(y + b)/2 + [-r,r]^n \subset \text{int } K$. By (16),

$$\int_{\mathbb{R}^n} \left| e^{\sqrt{-1}(z-w,x)-\tilde{h}_{K-a}(2x)} \right| \, dx = \int_{\mathbb{R}^n} e^{-(y+b,x)-\tilde{h}_{K-a}(2x)} \, dx$$

$$\leq \int_{\mathbb{R}^n} e^{-2(y+b,x)-\tilde{h}_{K-a}(2x)} \, dx \leq \left(\frac{\pi^2}{8r^2}\right)^n. \quad (43)$$
As a result, by (43),

$$F(z, w) := \frac{1}{(2\pi)^n |K|} \int_{\mathbb{R}^n} e^{\sqrt{-1}(z \cdot \overline{w}) - \tilde{h}_K(-2x)} \, dx$$

$$= \frac{1}{(2\pi)^n |K|} \int_{\mathbb{R}^n} e^{-(y+b,x) - \sqrt{-1}(a,x) - \tilde{h}_K(-2x)} e^{\sqrt{-1}(\xi,x)} \, dx$$

(44)

converges in $\mathbb{C}$ for all $z, w \in T_K$. In particular, by Remark 21,

$$G(x) := \frac{1}{|K|} e^{-(y+b,x) - \sqrt{-1}(a,x) - \tilde{h}_K(-2x)},$$

is $L^2$-integrable and, by (44), it is the Fourier transform of $\xi \mapsto F(\xi + iy, w)$. Therefore, by (24), $\xi \mapsto F(\xi + iy, w)$ is $L^2$-integrable with

$$\int_{\mathbb{R}^n} |F(\xi + iy, w)|^2 \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} |G(x)|^2 \, dx = \frac{1}{(2\pi)^n |K|^2} \int_{\mathbb{R}^n} e^{-2(y+b,x) - 2\tilde{h}_K(-2x)} \, dx.$$

(45)

Integrating (45) with respect to $y$, by (25) (since the integrand is positive),

$$\int_{T_K} |F(z, w)|^2 \, d\lambda(z) = \frac{1}{(2\pi)^n |K|^2} \int_{\mathbb{R}^n} e^{-2(y+b,x) - 2\tilde{h}_K(-2x)} \, dx \, dy$$

$$= \frac{1}{(2\pi)^n |K|^2} \int_{\mathbb{R}^n} e^{-2(x,b) - 2\tilde{h}_K(-2x)} \left( \int_{\mathbb{R}^n} e^{-2(y,x)} \, dy \right) \, dx$$

$$= \frac{1}{(2\pi)^n |K|^2} \int_{\mathbb{R}^n} e^{-2(x,b) - 2\tilde{h}_K(-2x)} |K| e^{\tilde{h}_K(-2x)} \, dx$$

$$= \frac{1}{(2\pi)^n |K|} \int_{\mathbb{R}^n} e^{-2(x,b) - \tilde{h}_K(-2x)} \, dx = F(w, w)$$

is finite by (44), i.e., $z \mapsto F(z, w) \in L^2(T_K)$.

Moreover, $F$ enjoys a reproducing property. To see why, let $f \in A^2(T_K)$. By Theorem 15, let $g \in L^2(\mathbb{R}^n, \tilde{h}_K)$ be such that $f(z) := \text{PW}(g)(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(x) e^{\sqrt{-1}(z,x)} \, dx$. In particular, $x \mapsto (2\pi)^{\frac{n}{2}} g(x) e^{-(x,y)}$ is the Fourier transform of $\xi \mapsto f(\xi + \sqrt{-1}y)$. By (23),

$$\int_{\mathbb{R}^n} f(z) F(z, w) \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} g(x) e^{-(x,y)} \overline{G(x)} \, dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |K|} \int_{\mathbb{R}^n} g(x) e^{-(b,x) + \sqrt{-1}(a,x) - \tilde{h}_K(-2x)} e^{-2(x,y)} \, dx$$

(46)

Integrating (46) with respect to $y$ over int $K$, by (27),

$$\langle f, F(\cdot, w) \rangle_{L^2(T_K)} = \int_{T_K} f(z) F(z, w) \, d\lambda(z) = \int_{\text{int } K} \int_{\mathbb{R}^n} f(z) F(z, w) \, d\xi \, dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |K|} \int_{\text{int } K} \int_{\mathbb{R}^n} g(x) e^{\sqrt{-1}(w,x) - \tilde{h}_K(-2x)} e^{-2(x,y)} \, dx \, dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |K|} \int_{\mathbb{R}^n} g(x) e^{\sqrt{-1}(w,x) - \tilde{h}_K(-2x)} \left( \int_{\text{int } K} e^{-2(x,y)} \, dy \right) \, dx$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(x) e^{\sqrt{-1}(w,x) - \tilde{h}_K(-2x)} \, dx = f(w).$$
To justify the use of (27), by Cauchy–Schwarz

$$\int_{T_K} |f(z)F(z,w)| \, d\lambda(z) \leq \left( \int_{T_K} |f(z)|^2 \, d\lambda(z) \right)^{\frac{1}{2}} \left( \int_{T_K} |F(z,w)|^2 \, d\lambda(z) \right)^{\frac{1}{2}}$$

is finite.

As a result, by the reproducing properties of $K_{T_K}$ and $F$,

$$F(z,w) = F(w,z) = \langle F(\cdot, z), K_{T_K}(\cdot, w) \rangle = \langle K_{T_K}(\cdot, w), F(\cdot, z) \rangle = K_{T_K}(z,w), \quad (47)$$

because $F$ and $K_{T_K}$ are holomorphic in the first variable. \hfill \square

Combining (42) with Claim 30 and Lemma 31 proves Proposition 6.

**Proof of Proposition 6.** By Claim 30, Lemma 31 and (42),

$$\mathcal{M}(K - a) = |K| |(K - a)^\circ|$$

$$= |K| \int_{\mathbb{R}^n} e^{-h_{K-a}(x)} \, dx$$

$$\geq \frac{|K|}{2^n} \int_{\mathbb{R}^n} e^{-h_{K-a}(-2x)} \, dx$$

$$= \frac{|K|}{2^n} (2\pi)^n |K||K_{T_K}(\sqrt{-1}a, \sqrt{-1}a)$$

$$= \pi^n |K|^2 K_{T_K}(\sqrt{-1}a, \sqrt{-1}a),$$

as desired \hfill \square

### 4 Estimating the Bergman kernel

This section proves Proposition 9. Conceptually, here are the key ideas:

- **Lemma 33** recalls the standard characterization of the Bergman kernel on the diagonal as a supremum involving $L^2$ holomorphic functions. This reduces the proof of Proposition 9 to finding such a function that equals 1 at $\sqrt{-1}b(K)$ and has $L^2$ norm bounded above by $2^n|K|$.

- **Lemma 8**, proved in §4.2, establishes the affine invariance of $B(K)$. This allows to displace $K$ by a convenient affine transformation for the remainder of the proof.

- John’s theorem is used in Lemma 37 to place $K$ in better position via an affine transformation, while Santaló’s inequality ensures good control on $K^\circ$ in the new position.

- A weight function $\phi$ (58) on $T_K$ is constructed (Lemma 43) satisfying the conditions of Proposition 51, with $\phi$ bounded from above, bounded from below away from the origin, and $e^{-\phi}$ is not integrable around the origin.

- A smooth bump function $g : \mathbb{C}^n \rightarrow \mathbb{C}$ is constructed (Lemma 49) with controlled weighted $L^2$-norm over $T_K$ with respect to the weight function of the previous step. While not holomorphic, this bump function has all the other properties one wants in order to estimate the Bergman kernel using Lemma 33.
• Hörmander’s theorem is used to solve for $h$ with $\bar{\partial}h = -\partial g$ and “correct” $g$ to a holomorphic function $f := g + h$ on $T_K$ with the other properties intact, namely, $f(0) = 1$ and bounded weighted $L^2$-norm. This requires several auxiliary estimates, mainly: an earlier upper bound on the weight function (Lemma 43 (ii)) guaranteeing that the non-weighted $L^2$-norm of $h$ is controlled; control on the volume of the support of $g$ (Lemma 43 (v)) guaranteeing that the non-weighted $L^2$-norm of $g$ is controlled; prescribed singularity of $\phi$ at the origin guaranteeing that $h(0) = 0$. Altogether, by Lemma 33, this yields the desired bound on the Bergman kernel up to subexponential terms (Lemma 50).

• Tensorization for Bergman kernels is used to eliminate the subexponential terms in the previous bound (Proposition 54), yielding Proposition 9.

4.1 Bergman kernel on the diagonal

First, recall the standard characterization of the Bergman kernel on the diagonal in terms of the norm of the evaluation functional $A^2(T_K) \ni f \mapsto f(w) \in \mathbb{C}$ [16, (5.2)].

Lemma 33. For $K \subset \mathbb{R}^n$ a convex body and $w \in T_K$,

$$K_{T_K}(w, w) = \|K_{T_K}(\cdot, w)\|^2_{L^2(T_K)} = \sup_{\substack{f \in A^2(T_K) \setminus \{0\}}} \frac{|f(w)|^2}{\|f\|^2_{L^2(T_K)}}.$$  \hspace{1cm} (48)

Proof. The first equality follows from (47). By Cauchy–Schwarz for any $f \in A^2(T_K),$

$$|f(w)| = \left| \int_{T_K} f(z)K_{T_K}(z, w) \, d\lambda(z) \right| \leq \|K_{T_K}(\cdot, w)\|_{L^2(T_K)} \|f\|_{L^2(T_K)},$$

and hence, by (49) and the first equality of (48),

$$\sup_{\substack{f \in A^2(T_K) \setminus \{0\}}} \frac{|f(w)|^2}{\|f\|^2_{L^2(T_K)}} \leq \|K_{T_K}(\cdot, w)\|^2_{L^2(T_K)} = \frac{K_{T_K}(w, w)^2}{\|K_{T_K}(\cdot, w)\|^2_{L^2(T_K)}} \leq \sup_{\substack{f \in A^2(T_K) \setminus \{0\}}} \frac{|f(w)|^2}{\|f\|^2_{L^2(T_K)}},$$

since $z \mapsto K_{T_K}(z, w) \in A^2(T_K)$, proving the second equality in (48).

4.2 Affine invariance of $B(K)$

The discussion of §2.1 for a tube domain $T_K$ carries over to any domain $\Omega \subset \mathbb{C}^n$, to yield a Bergman kernel $K_\Omega(z, w)$, that is the reproducing kernel of the evaluation functional $ev_{\Omega, w} : A^2(\Omega) \to \mathbb{C}$ at $w \in \Omega$.

The following lemma describes how the Bergman kernel behaves under affine transformations.

Lemma 34. Let $\Omega \subset \mathbb{C}^n$ open domain in $\mathbb{C}^n$ and $T(z) = Az + b$, $A \in GL(n, \mathbb{C})$, $b \in \mathbb{C}^n$. For $z, w \in \Omega$, $K_\Omega(z, w) = \det_\mathbb{R} A \cdot K_{T_\Omega}(Tz, Tw)$.

Remark 35. A $\mathbb{C}$-linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ can also be viewed as an $\mathbb{R}$-linear map $\mathbb{R}^{2n} \ni (x, y) \mapsto (\Re(A(x + \sqrt{-1}y)), \Im(A(x + \sqrt{-1}y))) \in \mathbb{R}^{2n}$. Denote by $\det_\mathbb{C} A$ the determinant of the former and $\det_\mathbb{R} A$ the determinant of the latter. Then, $\det_\mathbb{R} A = |\det_\mathbb{C} A|^2$ [8, Lemma 2].
Proof of Lemma 34. Let \( f \in A^2(T\Omega) \). Then,
\[
f(Tw) = \int_{T\Omega} f(\zeta) K_{T\Omega}(\zeta, Tw) \, d\lambda(\zeta) = \int_{\Omega} f(Tz) K_{T\Omega}(Tz, Tw) \det A \, d\lambda(z),
\]
because \( |\det A| = \det A \), since \( \det A = |\det A|^2 \geq 0 \). On the other hand, since \( f \in A^2(T\Omega) \) and \( T \) is a holomorphic map then \( f \circ T \in A^2(\Omega) \). So,
\[
f(Tw) = (f \circ T)(w) = \int_{\Omega} f(Tz) K_{\Omega}(z, w) \, d\lambda(z).
\]
The claim follows by comparing the two equations. \( \square \)

A direct application of Lemma 34 gives the affine invariance of \( B(K) \).

Proof of Lemma 8. Let \( K \subset \mathbb{R}^n \) be a convex body, and \( S(y) = Ay + a, A \in GL(n, \mathbb{R}), a \in \mathbb{R}^n \) be an affine transformation. Consider the embedding of \( K \) in \( T_K \) as \( \{0\} \times \sqrt{-1}K \), and the induced transformation, still denoted by \( S \), \( S : \sqrt{-1}y \mapsto \sqrt{-1}Ay + \sqrt{-1}a \). There is a unique extension of \( S \) to a \( \mathbb{C} \)-linear map on \( \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n = \mathbb{C}^n \), that we still denote by \( S \), \( S(z) := Ax + \sqrt{-1}(Ay + a) \). Note, \( \det S = (\det A)^2 \). By (9),
\[
S(\sqrt{-1}b(K)) = \sqrt{-1}b(AK) + \sqrt{-1}a = \sqrt{-1}b(AK + a) = \sqrt{-1}b(S(K)),
\]
and hence, by Lemma 34,
\[
|K|^2 K_{T_K}(\sqrt{-1}b(K), \sqrt{-1}b(K)) = |K|^2 (\det A)^2 K_{S(T_K)}(S(\sqrt{-1}b(K)), S(\sqrt{-1}b(K)))
\]
\[
= |S(K)|^2 K_{S(T_K)}(\sqrt{-1}b(S(K)), \sqrt{-1}b(S(K))),
\]
because \( S(T_K) = A\mathbb{R}^n + \sqrt{-1}AK + \sqrt{-1}a = \mathbb{R}^n + \sqrt{-1}S(K) = T_{S(K)} \), and \( |S(K)| = |AK + a| = |\det A||K| \). \( \square \)

Remark 36. Motivated by Lemma 8, in a subsequent article [24] we introduce a family of affine invariants involving \( L^p \)-versions of the support function that generalize \( B \). To give a glimpse, setting \( h_{K,p}(x) := \log |[K]|^{-1} \int_K e^{p(x,y)} \, dy/p \), we introduce the \( p \)-Mahler volume \( \mathcal{M}_p(K) := |K| \left( \int_{\mathbb{R}^n} e^{-h_K(y)} \, dy \right) \). Then, \( B(K) = (4\pi)^{-n} \mathcal{M}_1(K) \). This leads to an alternative proof of Lemma 8, that simultaneously generalizes to all \( p > 0 \) (as well as to a generalization of Proposition 6). In [24] we also study extremizers and monotonicity properties of \( \mathcal{M}_p \).

4.3 Repositioning \( K \)

For symmetric bodies, by John’s theorem there exist \( A \in GL(n, \mathbb{R}) \) and \( r > 0 \) such that \( B^2_2(0, r) \subset AK \subset B^2_2(0, r\sqrt{m}) \) and hence \( B^2_2(0, 1/(r\sqrt{m})) \subset (AK)^\circ \subset B^2_2(0, 1/r) \). However, without the assumption of symmetry one cannot be certain that the maximal ellipsoid contained in \( K \) contains \( b(K) \). As a result, John’s theorem alone does not guarantee a good upper bound for elements of \( (K - b(K))^\circ \). Nonetheless, combining John’s theorem [18, Theorem III] with Santaló’s inequality [31, (3.12)] yields the following explicit inclusions.

Lemma 37. For a convex body \( K \subset \mathbb{R}^n \) with \( b(K) = 0 \), there exists \( A \in GL(n, \mathbb{R}) \), \( r > 0 \) and \( a \in AK \subset \mathbb{R}^n \) such that,
\[
B^2_2(a, r) \subset AK \subset B^2_2(0, 2nr),
\]
\[
(AK)^\circ \subset B^2_2(0, 2n/r).
\]
Proof. For the proof of (50) we merely need 0 ∈ int K. By John’s theorem [32, Theorem 10.12.2], there exist A ∈ GL(n, R), a ∈ R^n and r > 0 such that B^n_2 (a, r) ⊂ AK ⊂ B^n_2 (a, nr). Since AK contains the origin, 0 ∈ AK ⊂ B^n_2 (a, nr), |0 − a| = |a| < nr. Thus, for any x ∈ AK, |x| ≤ |x − a| + |a| ≤ 2nr, proving (50).

Since also 0 ∈ (AK)^o, by the same reasoning, there exists ˜r > 0 and b ∈ R^n such that

\[ B^n_2 (b, ˜r) ⊂ (AK)^o ⊂ B^n_2 (b, n ˜r) ⊂ B^n_2 (0, 2n ˜r). \]  

To prove (51), note that b(AK) = 0 by (9). Thus, Remark 38 applies and using (50) and (52),

\[ |B^n_2 (a, r)||B^n_2 (b, ˜r)| ≤ \mathcal{M}(AK) = \inf_{z \in \mathbb{R}^n} \mathcal{M}((AK)^o − z) ≤ |B^n_2 (0, 1)|^2. \]  

Note |B^n_2 (a, r)||B^n_2 (b, ˜r)| = r^n ˜r^n|B^n_2 (0, 1)|^2. Thus, r ˜r ≤ 1 so (51) follows from (52).

The next remark accompanies the proof of Lemma 37.

Remark 38. To justify (53), note that for a convex body K with b(K) = 0 the Santaló point of its polar is at the origin, s(K^o) = 0, since b((K^o) − 0^o) = b(K) = 0 [31, p. 157]. As a result, by Santaló’s inequality [31, (3.12)]

\[ \mathcal{M}(K) = \mathcal{M}(K^o) = \inf_{z \in \mathbb{R}^n} \mathcal{M}(K^o − z) ≤ |B^n_2 (0, 1)|^2. \]  

Remark 39. One may avoid the use of John’s theorem at the cost of slightly less precise estimates down the road. Yet, using tensorization one can still obtain from these Proposition 9. To see this, for a convex body K ⊂ R^n with b(K) = 0 let r, R > 0 such that

\[ B^n_2 (0, r) ⊂ K ⊂ B^n_2 (0, R). \]  

(Of course, r and R depend on K. Moreover, unlike in our previous work that did invoke John’s theorem, R/r depends on K and may not be uniformly controlled.) Now, B^n_2 (0, 1/r) ⊂ K^o ⊂ B^n_2 (0, 1/r), thus for any x ∈ K and t ∈ K^o, |⟨x, t⟩| ≤ |x||t| ≤ R/r, and hence \( r^2 K \times K \subset K_C \subset K \times K \). As a result, since B^n_2 (0, r) ⊂ K,

\[ B^n_2 \left( 0, \frac{r^2}{R^2} \right) = \frac{r}{R\sqrt{2}} B^n_2 (0, r) ⊂ \frac{r}{R\sqrt{2}} B^n_2 (0, r) \times B^n_2 (0, r) \subset \frac{r}{R\sqrt{2}} K \times K \subset K_C. \]  

As in Lemma 46 this shows that dist(\( C^n \setminus (\sigma \delta K_C), \delta K_C \)) ≥ \( \frac{2\sqrt{2}}{\sigma^2} \), which allows for a bump function g : \( C^n \rightarrow \mathbb{R} \) supported on \( \sigma \delta K_C \), equal to 1 in \( \delta K_C \) with |dg| ≤ \( \frac{2\sqrt{2}}{\sigma^2} \). Therefore,

\[ \int_{K_C} |\bar{g}|^2 e^{-\phi} \leq \frac{2\sqrt{2}}{\sigma^2} e^{-2n \log \delta + 2n C \delta (\delta)^2 |K|^2}, \]  

\[ \int_{K_C} |h|^2 \leq \left( \frac{R}{r} \right)^4 \frac{8e^{1+2nC}}{8e^{1+2nC}} (4\sigma^2)^n |K|^2. \]  

As a result, (73) becomes \( |K|^2 K_{T_K} (0, 0) ≥ \left( 1 + \frac{1}{\delta^2} \right)^n \left( \frac{R}{r} \right)^4 \frac{8e^{1+2nC}}{8e^{1+2nC}} (4\sigma^2)^n |K|^2. \) Note that for m ∈ \( \mathbb{N} \),

\[ B^n_{2m} (0, r) ⊂ B^n_2 (0, r) \times \ldots \times B^n_2 (0, r) \subset K^m \subset B^n_2 (0, R) \times \ldots \times B^n_2 (0, R) \subset B^n_{2m} (0, mR). \]  

Therefore,

\[ \left( |K|^2 K_{T_K} (0, 0) \right)^m = |K^m|^2 K_{T_{K^m}} (0, 0) ≥ \left( 1 + \frac{1}{\delta^2} \right)^n \left( \frac{r}{mR} \right) \left( \frac{R}{r} \right)^4 \frac{8e^{1+2nC}}{8e^{1+2nC}}. \]  

(54)

One recovers (73) by first taking m → ∞, then σ → 1 and δ → 0.
4.4 Plurisubharmonic support function

In the symmetric setting, Nazarov invokes a neat trick of using a kind of complexified (plurisubharmonic) support function to construct the weight function. On the tube domain the inner product $\langle z, t \rangle$ (with $z \in T_K$ and $t \in K^\circ$) is, of course, a complex number, lying in the strip $\mathbb{R} + \sqrt{-1}[−1, 1] \subset \mathbb{C}$; trying to consider a naive candidate for a complexified support function of the form $\log \sup_{t \in K^\circ} |\langle z, t \rangle|$ would not quite work as it is unbounded. To overcome this Nazarov composes with the conformal map $\Phi^s(\zeta) = \frac{4\pi e^{\pi/2} \zeta - 1}{e^{\pi/2} \zeta + 1}$ sending the strip $\{ |\text{Im} \zeta| \leq 1 \}$ to the disk of radius $\frac{4\pi}{p}$ [26, p. 339], that leads to the nicely bounded psh function

$$\log \sup_{t \in K^\circ} |\Phi(\langle z, t \rangle)| \quad (55)$$

(that Nazarov modifies by a quadratic term to obtain a desirable weight function). There is a small caveat (not discussed in [26]) that we deal with in Lemma 43 (i): to show that this supremum in (55) is indeed plurisubharmonic, it is necessary to check it is upper semi-continuous.

In the non-symmetric case, the expression $\langle z, t \rangle$ (with $z \in T_K$ and $t \in K^\circ$) now lies in the half-space $\mathbb{R} + \sqrt{-1}[−\infty, 1] \subset \mathbb{C}$; composing with the conformal map from the half-plane $H := \{ \zeta \in \mathbb{C} : \text{Im} \zeta \leq 1 \}$ to the closed disk of radius 2 (as suggested by Nazarov [26, p.342]),

$$\Phi : \{ \text{Im} \zeta \leq 1 \} \rightarrow B^2_2(0, 2), \quad \zeta \mapsto -\frac{2\sqrt{-1}\zeta}{\zeta - 2\sqrt{-1}}, \quad (56)$$

one is again led to the useful complexified support function (55). A simple auxiliary estimate is needed for later calculations:

**Claim 40.** There exists $C > 0$ such that $|\log |\Phi(\zeta)|| - |\log |\zeta|| \leq C|\zeta|$, for all $|\zeta| \leq \frac{1}{2}$.

**Remark 41.** In Nazarov’s symmetric setting he uses (without proof) the estimate $|\log |\Phi^s(\zeta)|| - |\log |\zeta|| \leq C|\zeta|$ when $|\zeta| \leq 1/2$ [26, p. 339].

**Proof.** For

$$\Psi(\zeta) := -\frac{2\sqrt{-1}}{\zeta - 2\sqrt{-1}} = \begin{cases} \frac{\Phi(\zeta)}{\zeta}, & \zeta \neq 0, \\ 1, & \zeta = 0, \end{cases}$$

compute, $\Psi'(\zeta) = \frac{2\sqrt{-1}}{(\zeta - 2\sqrt{-1})^2}$. In particular, $\Psi(0) = 1$ and $\Psi'(0) = -\frac{\sqrt{-1}}{2}$. Since $\Psi(\zeta) \neq 0$ for $|\zeta| \leq \frac{1}{2}$, log $\Psi(\zeta)$ is well-defined and holomorphic. Note,

$$\log \Psi(0) = 0, \quad (\log \Psi')(0) = \frac{\Psi'(0)}{\Psi(0)} = -\frac{\sqrt{-1}}{2},$$

so there exists $g(\zeta)$ holomorphic in $B^2_2(0, \frac{1}{2})$ with $g(0) = -\frac{\sqrt{-1}}{2}$ so that $\log \Psi(\zeta) = \zeta g(\zeta)$. Take $C = \sup_{|\zeta| \leq \frac{1}{2}} |g(\zeta)|$. By the maximum principle, $C \geq |g(0)| = 1/2$. Since $|\log |\Psi|| = \text{Re} \log \Psi$,

$$|\log |\Psi(\zeta)|| \leq |\log |\Psi(0)|| = |\zeta||g(\zeta)| \leq C|\zeta|,$$

for all $|\zeta| \leq \frac{1}{2}$. $\square$
Remark 42. There is flexibility in the choice of the conformal map. For instance, replacing (56) by a map \( \Phi \) from \( \{ \text{Im} \zeta \leq 1 \} \) to the unit ball \( B_2^2(0,1) \) would give \( \Phi'(0) = 1/2 \) and the estimate of Claim 40 would change to \( |\log |\Phi(\zeta)| - \log |\zeta/2|| \leq C|\zeta| \). As the reader may readily verify, the only substantial subsequent changes would be in Lemma 43 (ii) and (iv) which would read \( \phi(z) \leq 1, z \in T_K \), and

\[
\phi(z) \geq 2n \log(\delta/2) - 16Cn^3\sqrt{2}, \quad z \in (\sigma \delta K_C - (\sigma - 1)\delta a) \setminus \delta K_C,
\]

respectively. In any case, the estimate on the \( L^2 \)-norm of \( h \) (70) would remain the same since

\[
\int_{T_K} |h|^2 \, d\lambda \leq e^{\sup_{z \in T_K} \phi(z)} \int_{T_K} |h|^2 e^{-\phi} \, d\lambda,
\]

is bounded above by the product of \( e^{\sup_{T_K} \phi(z)} \) and the weighted \( L^2 \)-norm of \( h \). Changing \( \Phi \) would result in a trade-off between estimates in those two terms. First of all, in this case

\[
e^{\sup_{z \in T_K} \phi(z)} \leq e,
\]

instead of \( e^{1+2n \log^2} = 4^n e \). On the other hand, (66) would become

\[
\int_{T_K} \overline{\partial g} \, e^{-\phi} \, d\lambda \leq \frac{32n^4}{(\sigma - 1)2\delta^2 r^2} (\delta/2)^{-2n} \sigma^{-2n} e^{2n \log \sigma + 16n^3C\delta \sqrt{n}} (\sigma \delta)^{2n} |K|^2 = (4\sigma^2)^n e^o(n) |K|^2,
\]

i.e., \( (\delta)^{-2n} \) is replaced by \( (\delta/2)^{-2n} \), thus (69) becomes

\[
\int_{T_K} |h|^2 e^{-\phi} \, d\lambda \leq 8n^2 r^2 \int_{T_K} \overline{\partial g} \, e^{-\phi} \, d\lambda \leq (4\sigma^2)^n e^o(n) |K|^2,
\]

i.e., \( \sigma^{2n} \) is replaced by \( 4^n \sigma^{2n} \), resulting in the same estimate in (57).

4.5 Weight function

At this point an important reduction is needed. In order to prove Proposition 9, by replacing \( K \) with \( A(K - b(K)) \) and invoking Lemmas 8 and 37, it is enough to restrict to a sub-class of “John convex bodies”:

\[
\mathcal{J} := \{ K \subset \mathbb{R}^n \text{ convex body with } b(K) = 0 \text{ such that the conclusion of Lemma 37 holds with } A = I_n \}.
\]

Thus, for the remainder of the Section, fix \( K \in \mathcal{J} \).

For \( K \in \mathcal{J} \), let \( r > 0 \) and \( a \in \mathbb{R}^n \) be such that (50) and (51) hold. Building on (55), consider the plurisubharmonic weight function on \( T_K \),

\[
\phi(z) := \frac{|\text{Im} z|^2}{4n^2 r^2} + 2n \log \sup_{t \in K^\circ} |\Phi(\langle z, t \rangle)|.
\]

(The advantage of using \( |\text{Im} z|^2 \), and not \( |z|^2 \), is that the resulting \( \phi \) is bounded.) The main properties of \( \phi \) are the content of the next lemma that requires two more pieces of notation. Set

\[
K_C := \{ z \in \mathbb{C}^n : |\langle z, t \rangle| := \sqrt{\langle x, t \rangle^2 + \langle y, t \rangle^2} \leq 1 \text{ for all } t \in K^\circ \},
\]

and

\[
\tilde{a} := a + \sqrt{-1} \alpha \in T_K.
\]
Lemma 43. Let $K \in \mathcal{J}$. Let $\phi$ and $K_C$ be given by (58) and (59).

(i) $\phi$ is plurisubharmonic and satisfies the conditions of Proposition 51 with $\tau = 1/(8n^2r^2)$.

(ii) For all $z \in T_K$, $\phi(z) \leq 2n \log 2 + 1$.

(iii) $e^{-\phi}$ is not locally integrable at $0$:

$$e^{-\phi(z)} \geq e^{-nC} \frac{r^{2n}}{(2n)2^{2n}|z|^{2n}}, \text{ for } z \in T_K \text{ with } |z| \leq (4n)^{-1}r.$$ 

(iv) For $\sigma \in (1, 2)$, $\delta \in (0, \frac{1}{8(1+2\sqrt{2}n^2)})$, $a \in \mathbb{R}^n$ as in Lemma 37, and $\tilde{a} \in \mathbb{C}^n$ as in (60),

$$\phi(z) \geq 2n \log \delta - 16\sqrt{2}C\delta n^3, \quad z \in (\sigma\delta K_C - (\sigma - 1)\tilde{a}) \setminus \delta K_C,$$

and the set $(\sigma\delta K_C - (\sigma - 1)\tilde{a}) \setminus \delta K_C$ is non-empty.

(v) $\frac{1}{4\sqrt{2}n^2}(K \times K) \cap (K_C \subset K \times K)$.

Remark 44. The lemma is a little different from Nazarov’s bounds for symmetric bodies [26, §5–§6]. The most substantial differences are in Lemma 43 (iv), for which $\delta$ needs to move in a smaller range for Claim 40 to hold, and the bound is now for $z \in (\sigma\delta K_C - (\sigma - 1)\tilde{a}) \setminus \delta K_C$ instead of $z \in \sigma\delta K_C \setminus \delta K_C$. Moreover, for Lemma 43 (v) a smaller dilation, dependent on dimension, is necessary for the first inclusion to hold. This is due to Lemma 37, which gives $|\langle x, t \rangle| \leq 4n^2$, for all $x \in K, t \in K^\circ$ (in place of $|\langle x, t \rangle| \leq 1$ for symmetric bodies).

Proof. (i) The plurisubharmonicity of $\log \sup_{t \in K^\circ} |\Phi((z, t))|$ is proved as follows. First, note that $\Phi((z, t))$ is holomorphic in $z$ for all $t \in K^\circ$, thus $\log |\Phi((z, t))|$ is plurisubharmonic on $z$ [15, Example 4.1.10], and so $\sup_{t \in K^\circ} |\Phi((z, t))| = \log \sup_{t \in K^\circ} |\Phi((z, t))|$ is plurisubharmonic if it is upper semi-continuous [20, Theorem 5] (here we used log is increasing). Since log is increasing, it suffices to show that $\sup_{t \in K^\circ} |\Phi((z, t))|$ is upper semi-continuous.

Fix $z_0 \in T_K$ and let $\rho > 0$ such that $B_{2n}^2(z_0, \rho) \subset T_K$. Note that for $z \in B_{2n}^2(z_0, \rho)$ and $t \in K^\circ$, by (51), $|\langle z, t \rangle| \leq (\rho + |z_0|)2nr^{-1}$. On $\{\text{Im} \frac{1}{z^j} \leq 1\} \cap B_{2n}^2(0, (\rho + |z_0|)2nr^{-1})$, since it is compact and $\Phi$ is continuous, $\Phi$ is uniformly continuous. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $\zeta_1, \zeta_2 \in \{\text{Im} \frac{1}{z^j} \leq 1\} \cap B_{2n}^2(0, (\rho + |z_0|)2nr^{-1})$ with $|\zeta_1 - \zeta_2| < \delta$, $|\Phi(\zeta_1) - \Phi(\zeta_2)| < \varepsilon$. For $z \in B_{2n}^2(z_0, \rho)$ with $|z - z_0| < \frac{\delta}{2n}$, and $t \in K^\circ$, by (51),

$$|\langle z, t \rangle - \langle z_0, t \rangle| = |\langle z - z_0, t \rangle| \leq |z - z_0||t| < \frac{r^2}{2n^2} = \delta,$$

thus $|\Phi((z, t)) - \Phi((z_0, t))| < \varepsilon$, i.e., $|\Phi(z, t)| \leq \varepsilon + |\Phi((z_0, t))|$. Taking supremum over $t \in K^\circ$,

$$\sup_{t \in K^\circ} |\Phi((z, t))| \leq \varepsilon + \sup_{t \in K^\circ} |\Phi((z_0, t))|,$$

from which the upper semi-continuity of $\sup_{t \in K^\circ} |\Phi((z, t))|$ follows.

Given that log sup $\sup_{t \in K^\circ} |\Phi((z, t))|$ is plurisubharmonic, compute

$$\sqrt{-1} \Delta \Phi \geq \Delta \left( \frac{1}{4n^2r^2} \bar{\partial \bar{\partial} \text{Im} z} \right)^2 = \frac{1}{4n^2r^2} \bar{\partial \bar{\partial}} \text{Im} z \frac{z \bar{z}}{2} = \frac{1}{8n^2r^2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$ 

(ii) $\Phi$ maps $\{\zeta \in \mathbb{C} : \text{Im} \zeta \leq 1\}$ to $B_{2n}^2(0, 2)$, thus $|\Phi((z, t))| \leq 2$ for all $z \in T_K$ and $t \in K^\circ$. Moreover, for $z = x + \sqrt{-1}y \in T_K$, $y \in K$ so by (50),

$$|\text{Im} z| = |y| \leq 2nr,$$ 

(61)
thus $\phi(z) \leq 1 + 2n \log 2$.

(iii) By Claim 40 there exists $C > 0$ such that $|\log |\Phi(\zeta)| - \log |\zeta|| \leq C|\zeta|$, for all $|\zeta| \leq \frac{1}{2}$. By (51), $|t| \leq 2nr^{-1}$, for all $t \in K^\circ$. By Cauchy–Schwarz, if $z \in T_K$ such that $|z| \leq (4n)^{-1}r$, the inner product $|\langle z,t \rangle| \leq 1/2$ thus,

$$\log |\Phi(\langle z,t \rangle)| \leq C|\langle z,t \rangle| + \log |\langle z,t \rangle| \leq \frac{C}{2} + \log(|z||t|) \leq \frac{C}{2} + \log\left(\frac{2n|z|}{r}\right), \quad z \in T_K \text{ with } |z| \leq r/(4n), t \in K^\circ.$$ 

Thus,

$$2n \log \sup_{t \in K^\circ} |\Phi(\langle z,t \rangle)| \leq nC + 2n \log \frac{2n|z|}{r} = nC + \log \left(\frac{(2n)^2|z|^{2n}}{r^{2n}}\right), \quad z \in T_K \text{ with } |z| \leq r/(4n).$$

Therefore, by (61) and (58),

$$e^{-\phi(z)} \geq e^{-nC} \frac{r^{2n}}{(2n)^2|z|^{2n}}, \quad \text{for all } z \in T_K \text{ with } |z| \leq r/(4n).$$

(iv) Let $z \in ((\sigma\delta K_C - (\sigma - 1)\delta \tilde{a}), i.e., |\langle z + (\sigma - 1)\delta \tilde{a}, t \rangle| \leq \sigma \delta$, for all $t \in K^\circ$. By the triangle inequality,

$$|\langle z,t \rangle| - (\sigma - 1)\delta|\langle \tilde{a}, t \rangle| \leq \sigma \delta, \quad \text{for all } t \in K^\circ. \quad (62)$$

Note that $\langle \tilde{a}, t \rangle = \langle a, t \rangle + \sqrt{-1} \langle \tilde{a}, t \rangle$, thus, by (50), (51), since $a \in K$,

$$|\langle \tilde{a}, t \rangle| = |\langle a, t \rangle| \sqrt{2} \leq |a||t| \sqrt{2} \leq 2nr \frac{2n}{r} \sqrt{2} = 4\sqrt{2}n^2, \quad (63)$$

for all $t \in K^\circ$. Combining (62), (63),

$$|\langle z,t \rangle| \leq \sigma \delta + (\sigma - 1)\delta 4\sqrt{2}n^2 \leq 2\delta + 4\sqrt{2}\delta n^2, \quad \text{for all } t \in K^\circ, \quad (64)$$

since $\sigma < 2$. Note that for $\delta \in (0, \frac{1}{8\sqrt{1+2\sqrt{2}n^2}})$ the right-hand side in (64) is less than or equal to $1/2$, thus Claim 40 applies. Next, assume that in addition also $z \notin (\delta K_C)$, i.e., $z \in ((\sigma\delta K_C - (\sigma - 1)\delta \tilde{a}) \setminus \delta K_C$. In particular, there exists $t_0 \in K^\circ$ such that $|\langle z,t_0 \rangle| > \delta$. As a result, by Claim 40 and (64),

$$\log |\Phi(\langle z,t_0 \rangle)| \geq \log |\langle z,t_0 \rangle| - C|\langle z,t_0 \rangle| \geq \log \delta - 2C\delta - 4\sqrt{2}C\delta n^2 \geq \log \delta - 8\sqrt{2}C\delta n^2,$$

because $2C\delta \leq 4\sqrt{2}C\delta n^2$. From the definition (58) then,

$$\phi(z) \geq 2n \log |\Phi(\langle z,t_0 \rangle)| \geq 2n \log \delta - 16\sqrt{2}C\delta n^3.$$ 

Finally, the set $(\sigma\delta K_C - (\sigma - 1)\delta \tilde{a}) \setminus \delta K_C$ is non-empty by Claim 45 below.

(v) For $z = x + \sqrt{-1}y \in K_C$, $\langle x,t \rangle^2 + \langle y,t \rangle^2 \leq 1$, for all $t \in K^\circ$, thus $\langle x,t \rangle \leq 1$ and $\langle y,t \rangle \leq 1$, for all $t \in K^\circ$, i.e., $x \in K$ and $y \in K$.

For the second inclusion, note that by Cauchy–Schwarz and (50), (51),

$$1 \geq \langle x,t \rangle \geq -|x||t| \geq -2nr \frac{2n}{r} = -4n^2,$$

thus, $|\langle x,t \rangle| \leq 4n^2$, and similarly $|\langle y,t \rangle| \leq 4n^2$, for all $x,y \in K, t \in K^\circ$. As a result, for $x,y \in (4n^2\sqrt{2})^{-1}K$,

$$\langle x,t \rangle^2 + \langle y,t \rangle^2 \leq \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1,$$

that is, $z = x + \sqrt{-1}y \in K_C$. 

\[ \square \]
Claim 45. The set \((\sigma \delta K_C - (\sigma - 1)\delta \bar{a}) \setminus \delta K_C\) in Lemma 43 (iv) is non-empty.

Proof. This follows from Lemma 46 below. Indeed, for any two non-empty sets \(S_1, S_2 \subset \mathbb{R}^k\) with \(S_1 \setminus S_2 = \emptyset\), one has \(S_1 \subset S_2\) and \(0 \leq \text{dist}(\mathbb{R}^k \setminus S_1, S_2) \leq \text{dist}(\mathbb{R}^k \setminus S_2, S_2) = 0\). □

4.6 Bump function

For symmetric bodies, one needs only an easier estimate on the distance between \(\mathbb{C}^n \setminus (\sigma \delta K_C)\) and \(\delta K_C\) since \(\bar{a} = 0 \in \mathbb{C}^n\) (recall (60)), where \(a \in \text{int } K\) as in (50) [16, Lemma 16]. For the general setting the following lower bound on the distance between \(\mathbb{C}^n \setminus (\sigma \delta K_C - (\sigma - 1)\delta \bar{a})\) and \(\delta K_C\) holds. Aside from being useful for the construction of the bump function below, the next lemma is also needed for one of the properties of the weight function above (see Claim 45):

Lemma 46. Let \(K \in \mathcal{J}\). For \(\sigma > 1, \delta > 0, K_C\) as in (59) and \(\bar{a}\) as in (60),

\[
\text{dist}(\mathbb{C}^n \setminus (\sigma \delta K_C - (\sigma - 1)\delta \bar{a}), \delta K_C) \geq \frac{(\sigma - 1)\delta r}{4n^2 \sqrt{2}}.
\]

Proof. By (50), \(B^n_2(a, r) \subset K\), so

\[
B^n_2(\bar{a}, \frac{r}{4n^2 \sqrt{2}}) \subset B^n_2(a, \frac{r}{4n^2 \sqrt{2}}) \times B^n_2(a, \frac{r}{4n^2 \sqrt{2}}) \subset \frac{K \times K}{4n^2 \sqrt{2}} \subset K_C,
\]

by Lemma 43 (v). As a result,

\[
\delta K_C + (\sigma - 1)\delta B^n_2(0, \frac{r}{4n^2 \sqrt{2}}) = \delta K_C + (\sigma - 1)\delta B^n_2(\bar{a}, \frac{r}{4n^2 \sqrt{2}}) - (\sigma - 1)\delta \bar{a}
\]

\[
\subset \delta K_C + (\sigma - 1)\delta K_C - (\sigma - 1)\delta \bar{a}.
\]

To conclude the proof, observe that \(\delta K_C + (\sigma - 1)\delta K_C = \sigma \delta K_C: \sigma \delta K_C \subset \delta K_C + (\sigma - 1)\delta K_C\) since \(\sigma \delta x = \delta x + (\sigma - 1)\delta x\); conversely, for \(\delta x \in \delta K_C\) and \((\sigma - 1)\delta y \in (\sigma - 1)\delta K_C\),

\[
\delta x + (\sigma - 1)\delta y = \sigma \delta \left(\frac{1}{\sigma} x + \frac{\sigma - 1}{\sigma} y\right) \in \sigma \delta K_C,
\]

by convexity. □

Remark 47. Centering at \((\sigma - 1)\delta \bar{a}\) comes from John’s theorem (Lemma 37). Lemma 46 explains why it is necessary to consider \(\sigma \delta K_C - (\sigma - 1)\delta \bar{a}\) instead of \(\sigma \delta K_C\). For the latter, since \(0 \in \text{int } K\), one can still find a ball centered at the origin and contained in \(K\) but with possibly too small of a radius to provide a substantial lower bound on the distance between \(\sigma \delta K_C\) and \(\delta K_C\).

Next, we show that Lemma 46 allows for a bump function \(g\) with good enough control on the derivative (though dependent of \(n\)), supported on \(\sigma \delta K_C - (\sigma - 1)\delta \bar{a}\) equal to 1 in \(\delta K_C\) (in the symmetric setting, the bump function is supported on \(\sigma \delta K_C\)), its gradient is bounded from above independent of the dimension [26, p. 340]). Before stating the result we define some notation concerning 1-forms. For a function \(f\) on \(T_K\), \(df = \partial f + \bar{\partial} f\), with \(\partial f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dz_i\) and \(\bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \overline{x_i}} dz_i\). For a \((0,1)\)-form \(\omega = \sum_j a_j d\overline{z}_j\), set

\[
|\omega|^2 := \sum_j |a_j|^2,
\]

and similarly for \((1,0)\)-forms. Also, \(|df|^2 = |\partial f|^2 + |\bar{\partial} f|^2\). If, in addition, \(f\) only takes real values, \(\partial f/\partial \overline{x_i} = \bar{\partial} f/\partial z_i\) thus \(\bar{\partial} f = \partial f\) and hence

\[
|df|^2 = 2|\partial f|^2, \quad \text{for } f : \mathbb{C}^n \to \mathbb{R}.
\]
Lemma 48. Let $K \in \mathcal{J}$ and let $K_C$ be given by (59). For $\sigma > 1, \delta > 0$, there exists a smooth $g : \mathbb{C}^n \to [0, 1]$, supported on $\sigma \delta K_C - (\sigma - 1) \delta \bar{a}$ such that $g = 1$ on $\delta K_C$ and

$$|dg| \leq \frac{8n^2 \sqrt{2}}{(\sigma - 1) \delta r}.$$ 

Proof. Denote by $R := ((\sigma - 1) \delta r)/(4n^2 \sqrt{2})$. By Lemma 46, $\delta K_C + B_2^n(0, R) \subset \sigma \delta K_C - (\sigma - 1) \delta \bar{a}$. Consider the map

$$G(x) = \begin{cases} 
1, & \text{if } d_{\delta K_C}(x) \leq \frac{R}{4}, \\
2 - \frac{2}{R}(d_{\delta K_C}(x) + \frac{R}{4}), & \text{if } \frac{R}{4} \leq d_{\delta K_C}(x) \leq \frac{3R}{4}, \\
0, & \text{otherwise},
\end{cases}$$

where the distance function of a non-empty set $S \subset \mathbb{R}^k$ is defined by $d_S(x) := \inf_{y \in S} |x - y|$. Thus, $G$ is continuous, equal to 1 in $\delta K_C$, and 0 outside $(\sigma \delta K_C - (\sigma - 1) \delta \bar{a})$. Let $\chi$ a smooth, non-negative function, compactly supported on $B_2^n(0, \frac{R}{4})$, with $\int_{\mathbb{R}^n} \chi \, d\lambda = 1$. The convolution,

$$g(x) := (G \ast \chi)(x),$$

is smooth, $g = 1$ in $\delta K_C$, and $g = 0$ outside $\delta K_C - (\sigma - 1) \delta \bar{a}$. For $S \subset \mathbb{R}^k$ a submanifold of codimension at least 1, $d(d_S)$ is almost everywhere defined with $|d(d_S)| = 1$ [32, Lemma 1.5.9], [27, Example 22]. Hence,

$$|dg| = |dG \ast \chi| \leq \frac{2}{R} = \frac{8n^2 \sqrt{2}}{(\sigma - 1) \delta r},$$

as claimed. \hfill \Box

By Lemmas 43 and 48 the following estimate on the weighted $L^2(T_K)$ norm of derivative of the bump function $\partial g$ holds.

Lemma 49. Let $K \in \mathcal{J}$ and let $K_C$ be given by (59). For $g$ as in Lemma 48 and $\phi$ as in (58), $\sigma \in (1, 2)$ and $\delta \in (0, \frac{1}{8(1+2n^2 \sqrt{2})})$,

$$\int_{T_K} |\partial g|^2 e^{-\phi(z)} \, d\lambda(z) \leq \frac{16n^4}{(\sigma - 1)^2 \delta^2 r^2} e^{2n \log \sigma + 16n^3 C \delta \sqrt{2} |K_C|} \leq \frac{16n^4}{(\sigma - 1)^2 \delta^2 r^2} e^{2n \log \sigma + 16n^3 C \delta \sqrt{2} |K|^2}. \tag{66}$$

Proof. Since $g$ is a real-valued function, by (65), $|\partial g|^2 = |dg|^2/2$. Moreover, $\partial g = 0$ outside $(\sigma \delta K_C - (\sigma - 1) \delta \bar{a})$ and in $\delta K_C$ since $g$ is constant there. As a result, by Lemma 48 and Lemma 43 (iv),

$$\int_{T_K} |\partial g|^2 e^{-\phi} \, d\lambda = \int_{T_K \cap (\sigma \delta K_C - (\sigma - 1) \delta \bar{a}) \setminus (\delta K_C)} |\partial g|^2 e^{-\phi} \, d\lambda$$

$$\leq \int_{(\sigma \delta K_C - (\sigma - 1) \delta \bar{a}) \setminus (\delta K_C)} \frac{16n^4}{(\sigma - 1)^2 \delta^2 r^2} e^{-2n \log \delta + 16C \delta \sqrt{2} |\delta \delta K_C|} \, d\lambda(z)$$

$$\leq \frac{16n^4}{(\sigma - 1)^2 \delta^2 r^2} \delta^{-2n} e^{16n^3 C \delta \sqrt{2} |\delta \delta K_C|} \leq \frac{16n^4}{(\sigma - 1)^2 \delta^2 r^2} \delta^{-2n} e^{2n \log \sigma + 16n^3 C \delta \sqrt{2} (\delta \delta)^2 |K|^2},$$

because $|K_C| \leq |K \times K| = |K|^2$ (Lemma 43 (v)). \hfill \Box
4.7 Constructing the holomorphic function

The weight function $\phi$ (58) and $\omega = \overline{\partial}g$ with $g$ the bump function of Lemma 48 will next be used in conjunction with Hörmander’s theorem (Proposition 51).

For a function $f$ defined on positive integers,

$$f(k) = o(k) \quad \text{if} \quad \lim_{k \to \infty} \frac{f(k)}{k} = 0.$$  \hfill (67)

**Lemma 50.** For $K \in \mathcal{J}$, $B(K) \geq e^{\phi(n)}4^{-n}$.

**Proof.** Let $a \in \mathbb{R}^n$ and $r > 0$ as in Lemma 37. Let also $\tilde{a}$ as in (60), $\sigma \in (1, 2)$ and $\delta := 1/(16n^3)$. The conditions of Lemma 49 are satisfied, so there exists $g$ supported on $(\sigma \delta K_C - (\sigma - 1)\delta \tilde{a})$ with $g = 1$ on $\delta K_C$ and (66) holds. By Proposition 51 and Lemma 43 (i), there exists $h : T_K \to \mathbb{C}$ solving

$$\overline{\partial}h = -\overline{\partial}g,$$  \hfill (68)

so that $\int_{T_K} |h|^2 e^{-\phi} \, d\lambda \leq 8n^2 r^2 \int_{T_K} |\overline{\partial}g|^2 e^{-\phi} \, d\lambda$. Therefore, by (66),

$$\int_{T_K} |h|^2 e^{-\phi} \, d\lambda \leq \frac{2^{15} n^{12}}{(\sigma - 1)^2} e^{2n \log \sigma + C \sqrt{2}} |K|^2,$$  \hfill (69)

because $\delta = 1/(2^4 n^3)$. Moreover, for $z \in T_K$, by Lemma 43 (ii), $\phi(z) \leq 1 + 2n \log 2$, thus by (69),

$$\int_{T_K} |h|^2 \, d\lambda \leq e^{1 + 2n \log 2} \int_{T_K} |h|^2 e^{-\phi} \, d\lambda \leq 4^n \frac{2^{15} n^{12}}{(\sigma - 1)^2} e^{2n \log \sigma + C \sqrt{2}} |K|^2.$$  \hfill (70)

In particular, $\delta < \frac{1}{2}$ so the product $\sigma \delta < 1$. Also, by Lemma 48, $0 \leq g \leq 1$ and is supported on $(\sigma \delta K_C - (\sigma - 1)\delta \tilde{a})$. Thus,

$$\int_{T_K} |g|^2 \, d\lambda \leq |(\sigma \delta K_C - (\sigma - 1)\delta \tilde{a})| = (\sigma \delta)^2 |K_C| \leq |K|^2,$$  \hfill (71)

because $K_C \subset K \times K$ (Lemma 43 (v)), thus $|K_C| \leq |K|^2$.

For $z \in T_K$ consider the holomorphic (by (68)) function

$$f(z) := g(z) + h(z).$$

We claim $f \in A^2(T_K)$. To see that, $|f|^2 = |g + h|^2 \leq 2|g|^2 + 2|h|^2$, hence by (70) and (71), since the right-hand side in (70) is bigger than the right-hand side in (71),

$$\|f\|^2_{L^2(T_K)} := \int_{T_K} |f|^2 \, d\lambda \leq 2 \int_{T_K} |g|^2 \, d\lambda + 2 \int_{T_K} |h|^2 \, d\lambda \leq 4^n \frac{2^{17} n^{12}}{(\sigma - 1)^2} e^{2n \log \sigma + C \sqrt{2}} |K|^2.$$  \hfill (72)

On the other hand, by Lemma 43, $e^{-\phi}$ is comparable to $|z|^{-2n}$ around the origin, which is not integrable. But by (69), $|h|^2 e^{-\phi}$ is integrable in $T_K$, thus $h(0) = 0$. Furthermore, by construction (Lemma 48) $g(0) = 1$, thus $f(0) = g(0) + h(0) = 1$. By (72) and Lemma 33,

$$K_{T_K}(0, 0) \geq \frac{|f(0)|^2}{\|f\|^2_{L^2(T_K)}} \geq \frac{(\sigma - 1)^2 e^{-1 - C \sqrt{2}}}{(4\sigma^2)^n 2^{18} n^{12} |K|^2} \geq e^{\phi(n)} \frac{1}{(4\sigma^2)^n |K|^2}.$$  \hfill (73)

Taking $\sigma \to 1$ proves the lemma. \hfill \(\square\)
Since $K$ is convex, $T_K$ is also convex, and hence a pseudoconvex domain [13, Theorem 4.2.8, Corollary 2.5.6], and Hörmander’s Theorem applies [14, Theorem 2.2.11]:

**Proposition 51.** Suppose $\phi$ is plurisubharmonic and $\sqrt{-1}\partial\bar{\partial}\phi \geq \tau \sum_{i=1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i$ on $T_K$ for some constant $\tau > 0$. Then, for any $(0,1)$-form $\omega$ with $\partial \omega = 0$, there is $h$ in $T_K$ such that $\partial h = \omega$, with
\[
\int_{T_K} |h|^2 e^{-\phi} \, d\lambda(z) \leq \tau^{-1} \int_{T_K} |\omega|^2 e^{-\phi} \, d\lambda(z).
\]

### 4.8 Tensorization and Bergman kernels

The Bergman kernel of the Cartesian product is the product of the Bergman kernels.

**Lemma 52.** For $\Omega_1 \subset \mathbb{C}^n, \Omega_2 \subset \mathbb{C}^m$ domains, and $z_1, w_1 \in \Omega_1, z_2, w_2 \in \Omega_2$,
\[
K_{\Omega_1 \times \Omega_2}(z_1, z_2, w_1, w_2) = K_{\Omega_1}(z_1, w_1)K_{\Omega_2}(z_2, w_2).
\]

**Proof.** First, note that $(z_1, z_2) \mapsto K_{\Omega_1}(z_1, w_1)K_{\Omega_2}(z_2, w_2)$ is holomorphic as the product of holomorphic functions. Moreover, for $w_1 \in \Omega_1, w_2 \in \Omega_2$, by (25),
\[
\int_{\Omega_1 \times \Omega_2} |K_{\Omega_1}(z_1, w_1)K_{\Omega_2}(z_2, w_2)|^2 \, d\lambda(z_1, z_2) = \int_{\Omega_1} |K_{\Omega_1}(z_1, w_1)|^2 \, d\lambda(z_1) \int_{\Omega_2} |K_{\Omega_2}(z_2, w_2)|^2 \, d\lambda(z_2),
\]
i.e., $(z_1, z_2) \mapsto K_{\Omega_1 \times \Omega_2}(z_1, z_2, w_1, w_2) \in L^2(\Omega_1 \times \Omega_2)$. As a result, for $f \in A^2(\Omega_1 \times \Omega_2)$, by Cauchy–Schwarz, the pairing $\langle f, K_{\Omega_1}(\cdot, w_1)K_{\Omega_2}(\cdot, w_2) \rangle_{L^2(\Omega_1 \times \Omega_2)}$ converges in $\mathbb{C}$ and by (27),
\[
\int_{\Omega_1 \times \Omega_2} f(z_1, z_2)\overline{K_{\Omega_1}(z_1, w_1)K_{\Omega_2}(z_2, w_2)} \, d\lambda(z_1, z_2)
= \int_{\Omega_1} \left( \int_{\Omega_2} f(z_1, z_2)\overline{K_{\Omega_2}(z_2, w_2)} \, d\lambda(z_2) \right) \overline{K_{\Omega_1}(z_1, w_1)} \, d\lambda(z_1)
= \int_{\Omega_1} f(z_1, w_2)\overline{K_{\Omega_1}(z_1, w_1)} \, d\lambda(z_1) = f(w_1, w_2).
\]

The same reproducing property holds, by definition, for $K_{\Omega_1 \times \Omega_2}$. By the uniqueness of the Bergman kernel (as in (47)) the claim follows. 

**Remark 53.** For $K \subset \mathbb{R}^n, L \in \mathbb{R}^m$ convex bodies, let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ the coordinates in $\mathbb{R}^{n+m}$. By (27),
\[
b(K \times L) = \frac{1}{|K \times L|} \int_{K \times L} (x, y) \, dx \, dy = \frac{1}{|K||L|} \left( |L| \int_K x \, dx, |K| \int_L y \, dy \right),
\]
that is, $b(K \times L) = (b(K), b(L))$. In particular, $b(K \times L) = 0$ if and only if $b(K) = 0$.

The next proposition shows how to eliminate subexponential terms in (73) [26, §7].

**Proposition 54.** Let $c > 0$, independent of dimension, such that $\mathcal{B}(K) \geq c^{o(n)}c^n$, for all $n$ and all convex bodies $K \subset \mathbb{R}^n$. Then, $\mathcal{B}(K) \geq c^n$, for all convex bodies.
Proof. Fix $n \in \mathbb{N}$ and a convex body $K \subset \mathbb{R}^n$. For each $m \geq 1$, the product $(T_K)^m := \overbrace{T_K \times \ldots \times T_K}^{\text{m-times}} = \mathbb{R}^m + \sqrt{-1} K^m = T_K^m$ is the tube domain of $K^m$. By Remark 53, $b(K^m) = b(K)^m$, thus by Lemma 52 and Definition 7,

$$B(K)^m = \mathcal{K}_{T_K} (\sqrt{-1} b(K), \sqrt{-1} b(K))^m$$

$$= \mathcal{K}_{(T_K)^m} (\sqrt{-1} b(K)^m, \sqrt{-1} b(K)^m)$$

$$= \mathcal{B}(K^m) = \mathcal{K}_{T_{K^m}} (\sqrt{-1} b(K^m), \sqrt{-1} b(K^m)) \geq e^{o(nm)} \frac{c^{nm}}{|K|^{2m}},$$

i.e., $B(K) \geq e^{o(nm)/m} c^n$. Taking $m \to \infty$, by (67), the claim follows. \hfill \Box

4.9 Proof of Proposition 9

Proposition 9 follows from Lemmas 50, 8 and 54.

Proof of Proposition 9. For $K \subset \mathbb{R}^n$ a convex body $b(K - b(K)) = 0$. Therefore, by Lemmas 37 and 50, there exists $A \in GL(n, \mathbb{R})$ such that $|A(K - b(K))|^2 K_{T_A(K - b(K))}(0, 0) \geq e^{o(n)} 4^{-n}$, because $A(K - b(K)) \subset J$. By Lemma 8,

$$|K|^2 \mathcal{K}_{T_K} (\sqrt{-1} b(K), \sqrt{-1} b(K)) = |A(K - b(K))|^2 \mathcal{K}_{T_A(K - b(K))}(0, 0) \geq e^{o(n)} 4^{-n}. \quad (74)$$

Since (74) holds for all convex bodies, by Lemma 54 and Definition 7,

$$\mathcal{B}(K) = |K|^2 \mathcal{K}_{T_K} (\sqrt{-1} b(K), \sqrt{-1} b(K)) \geq 4^{-n},$$

as desired. \hfill \Box

A Symmetrization

In the last line of his paper, Nazarov mentions that while his work should adapt to non-symmetric bodies [26, p. 342],

“Unfortunately, this is well-below the bound you can get by the symmetrization trick”.

This, in conjunction with Remark 3, can be interpreted as follows. Recall the reflection body of a convex body $K$,

$$RK := \text{conv} \{ K \cup (-K) \},$$

is a symmetric convex body of the same dimension satisfying [28, Theorem 3]:

Theorem 55. For $K \subset \mathbb{R}^n$ a convex body satisfying (1), $|RK| \leq 2^n |K|$, with equality if and only if $K$ is a simplex and 0 is a vertex.

Corollary 56. Suppose that $\mathcal{M}(K) \geq c^n/n!$ for all symmetric convex bodies $K \subset \mathbb{R}^n$. Then $\mathcal{M}(K) \geq (c/2)^n/n!$ for all convex bodies $K \subset \mathbb{R}^n$.

Proof. Let $K \subset \mathbb{R}^n$ be a convex body. As in the proof of Theorem 2, there is no loss in assuming $b(K) = 0$, so Theorem 55 applies. Because $K \subset RK$, $|(RK)^\circ| \leq |K^\circ|$, and hence by Theorem 55, $\mathcal{M}(K) \geq 2^{-n} \mathcal{M}(RK)$. Since $RK$ is symmetric then by assumption $\mathcal{M}(RK) \geq c^n/n!$, so $\mathcal{M}(K) \geq (c/2)^n/n!$, as desired. \hfill \Box
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