Bohr’ inequality for large functions

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Abstract

We prove that the Bohr’ radius for large functions is $e^{-\pi}$.

1 Introduction:

In this article, we shall investigate Bohr’s phenomenon for spaces of large functions. By large, we mean analytic function that maps the disk into a hyperbolic domain (it misses at least two finite points). This includes almost all known classes of analytic functions. Here we give a lower bound for Bohr’s radius.

We say a class of analytic functions $\mathcal{F}$ satisfies the Bohr’s phenomenon, if a function $f = \sum_{0}^{\infty} a_n z^n$ is in $\mathcal{F}$, the Bohr’ operator

$$M(f) = \sum_{0}^{\infty} |a_n z^n|$$

is uniformly bounded on some uniform disk $|z| \leq \rho$, with $\rho > 0$. [1], [3].

The largest such radius $\rho$ is called the Bohr’s radius for the class $\mathcal{F}$.

Clearly: 1) $M(f + g) \leq M(f) + M(g)$
2) $M(fg) \leq M(f)M(g)$.
3) $M(1) = 1$.

This makes $F$ into a Banach algebra, with norm $M(f)$. Let us recall two properties for Banach algebra:
1) Bohr’s theorem: If $|f(z)| < 1$, for all $z \in U$, then $M(f) < 1$, when $|z| < 1/3$.
2) Von Neumann inequality: $|p(f(z))| < ||p||_{\infty}$, where $p(z)$ is a polynomial. Von Neumann showed the above inequality is true for the space of bounded operators on a Hilbert space, $L(H)$. [3], [5].
Dixon [5], showed that the space

\[ l^1_\beta = \left\{ x = (x_1, x_2, x_3, \ldots) : \frac{1}{\beta} \sum_{j=1}^{\infty} |x_j| < \infty \right\} , \]

Dixon in [5], showed that the Von Neumann inequality is true, for \( 0 < \beta < 1/3 \) and not satisfied for any \( \beta \geq 1/3 \).

By a large function, we mean an analytic function on the unit disk whose range misses at most two finite points from the complex plane. The following theorem is center to all of this:

**Theorem 1** *(Uniformization Theorem [4])*, If \( D \) is an open set missing at least 2 points then there is a universal cover (conformal) from \( U \) into \( D \). This cover is unique with the normalization \( F(0) = a \) and \( F'(0) > 0 \), for some \( a \in D \).

**Corollary 2** If \( f(z) \) is analytic and maps \( U \) into \( D \) then there is a Schwartz function \( \varphi(z) \) so that

\[ f(z) = F(\varphi(z)). \]

If we denote the universal covering of \( D \) by \( F \), then \( F \) defines a hyperbolic metric on \( D \) defined by

\[ \lambda(F(z)) = \frac{1}{F'(z)} \frac{1}{1 - |z|^2}. \]

(1)

In the following theorem of David Minda, [9], conformal means non-vanishing derivative.

**Theorem 3** Let \( D \) be a hyperbolic domain, with hyperbolic metric \( \lambda(w) \),

and a conformal map \( f : U \to D \) is onto. Then

\[ \lambda(f(z)) = \frac{1}{|f'(z)|} \frac{1}{1 - |z|^2}. \]

Proof: Divide \( D \) into geodiscs. These geodiscs correspond to geodisks in \( U \). Let \( \mu \) be defined as

\[ \mu(f(z)) = \frac{1}{|f'(z)|} \frac{1}{1 - |z|^2}. \]

Then \( \int_{c} \mu(w)|dw| = \int_{c} \mu(f(z))f'(z)|dz| = \int_{c} \frac{1}{1 - |z|^2}|dz| \), For all \( c \) geodiscs in \( U \),

Hence

\[ \int_{\gamma} \mu(w)|dw| = \int_{\gamma} \lambda(w)|dw|, \]

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for all geodisks $\gamma$ in $D$. Hence they are the same.

Let us also recall the modular function: [11], [10],

$$J(z) = 16z \prod_{1}^{\infty} \left[ \frac{1 + z^{2n}}{1 + z^{2n-1}} \right]^{8},$$

$J(z)$ is 0 only at 0 and $J \neq 0, 1, \infty$ on $|z| > 0$. Note that

$$-J(-z) = 16z \sum_{0}^{\infty} A_n z^n, \quad (2)$$

$$A_n > 0.$$

This function is like a Koeba function for large function. We immediately conclude that

$$\max_{|z|=r} |J(z)| = |J(-r)|. \quad (3)$$

It is known that the coefficients $\{A_n\}$ are convex and increasing, [10] and [11]. Hence, by a theorem of Littlewood [8], we have

Lemma 1: if $\sum_{0}^{\infty} a_k$ is subordinate to $-J(-z)$ then $|a_k| \leq 16A_k$, for all $k$.

Also the following will be used:

Lemma II: $J(z)$ has radius of univalence $e^{-\pi/2}$, [10, p85].

In addition, as shown in the proof of Lemma II,

$$|J(-e^{-\pi})| = 1, \quad |J(e^{-\pi})| = 1/2. \quad (4)$$

Consequently, by (1), (2), (3), we conclude that

$$\max_{|z| \leq e^{-\alpha}} |J(z)| = 1. \quad (5)$$

The function

$$Q(z) = J(\exp(-\alpha \frac{1+z}{1-z})), \alpha < 1$$

is a covering map into the domain $C\{0,1\}$ with $Q(0) = J(e^{-\alpha})$.

Other properties can be found in [10].

2 This is our main theorem:

**Theorem 4** Let $F(z) = \sum_{0}^{\infty} a_n z^n$ be analytic on $U$ and suppose that $F(U)$ misses at least
two points then
\[ \sum_{n=1}^{\infty} |a_n z^n| \leq \text{dis}(F(0), \partial F(U)), \]
for \( |z| \leq e^{-\pi} = 4.3214 \times 10^{-2} \).

**Proof.** By the Uniformization Theorem, there is a universal cover (conformal)

\[ G(z) \] onto \( F(U) \), with
\[ G(0) = F(0) = a_0. \]
Then \( F(z) = G(\varphi(z)) \), where \( |\varphi(z)| < 1 \) and \( \varphi(0) = 0 \). Suppose also that \( F(U) \)
misses points \( a, b \), with
\[ |a - a_0| = d(a_0, \partial F(U)). \]

Then the function
\[ g(z) = \frac{F(z) - a}{b - a}, \]
misses 0, 1 and \( g(z) = Q(\psi(z)), \psi(0) = 0 \). Then
\[ g(0) = Q(\psi(0)) = \frac{-a_0 - a}{b - a} = J(e^{-\alpha}). \]

Next, let
\[ h(z) = zg(z), \]

\( h(z) \) is only 0 at 0.
If \( F(z) = \sum_{n=0}^{\infty} a_n z^n. \)
Then
\[ h(z) = \frac{a_0 - a}{b - a} z + \frac{z}{b - a} \sum_{n=1}^{\infty} a_n z^n. \]

If \( \delta = \text{dis}(0, \partial h(U)) \),
then \( \delta = \min |zg(z)| \leq \min |g(z)| = \min \left| \frac{F(z) - a}{b - a} \right| \leq \left| \frac{-a_0 - a}{b - a} \right| = \left| \frac{\text{dis}(F(0), \partial F(U))}{b - a} \right|, \)
hence
\[ \delta \leq \left| \frac{\text{dis}(F(0), \partial F(U))}{b - a} \right|. \]

On the other hand, since \( h(z) \) is 0 only at 0, \( h/\delta = J(\omega) \), \( \omega(0) = 0 \), [10]
and then \( h(z) = \delta J(\omega) \).
Using the hyperbolic identity [4],
\[ \lambda(w) d(w, \partial D) \leq 1, \]
By (2), we conclude that $\lambda(a_0) = \frac{1}{\varphi(a_0)}$ and by (3),

$$\text{dis}(F(0), \partial F(U)) \geq \delta|b-a|.$$  

Then

$$\lambda(a_0) d(a_0, \partial D) \leq 1,$$

$$\frac{1}{|G'(0)|} d(a_0, \partial F(U)) \leq 1,$$

$$\delta|b-a|/|G'(0)| \leq 1$$

and then

$$\delta < \left| \frac{G'(0)}{b-a} \right|.$$  

As the coeff of $J(z)$ are convex increasing, by Littlewood [8],

$$\left| \frac{a_{n-1}}{b-a} \right| < \delta 16A_n < \left| \frac{\text{dis}(F(0), \partial F(U))}{b-a} \right| 16A_n$$

$$|a_{n-1}| < |a| A_n = \text{dis}(F(0), \partial F(U)) A_n$$

$$\sum_{1}^{\infty} |a_{n-1}| r^n \leq \text{dis}(F(0), \partial F(U)) \cdot 16 \sum_{1}^{\infty} |A_n| r^n$$

$$= \text{dis}(F(0), \partial F(U)) \cdot J(-r).$$

(4) and (5) imply that $-J(-r) < 1$ when $r \leq e^{-\pi}$. This proves Theorem 1.

**Corollary 5** If $F(z) = \sum_{0}^{\infty} a_n z^n$ is analytic on $U$ and suppose that $F(U)$ misses at least two points, with $\text{dis}(F(0), \partial F(U)) < 1$ then $F(z)$ satisfies the von Neumann inequality for $|z| \leq e^{-\pi} = 4.3214 \times 10^{-2}$, in other words, for any polynomial $p(z)$

$$p(F(z)) \leq ||p||_\infty.$$  

3 **Harmonic maps:** If $f = \sum_{0}^{\infty} a_n z^n$ is analytic, define the operator

$$M(f) = \sum_{0}^{\infty} |a_n z^n|.$$
Clearly: 1) $M(f + g) \leq M(f) + M(g)$
2) $M(fg) \leq M(f)M(g)$.

**Theorem 6** Let $f(z) = h(z) + \overline{g(z)}$ be harmonic with $h$ missing at least 2 points,

**Theorem 7** $h(0) = a_0$, $g(0) = 0$ and $g'(z) = \mu(z)h'(z)$ then

$$M(f) \leq (1 + |\mu(z)|)d(a_0, \partial(h(U)))$$

for $|z| \leq e^{-\pi} = 4.3214 \times 10^{-2}$.

**Proof.** $M(g) = \int_0^r M(g')dr \leq \int_0^r M(a)M(h')dr$

**Proof.** Let $|z| < 1/3$. Then

$$M(g) \leq \int_0^r M(h')dr = M(h) - |a_0| = M(h - a_0).$$

Let $|z| < e^{-\pi}$, then

$$M(g) < d(a_0, \partial(h(U))),$$

$$M(f) = M(h) + M(g) \leq (1 + |\mu(z)|)d(a_0, \partial(h(U)))$$

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