ON MINKOWSKI MEASURES OF METRIC SPACES

LIANGYI HUANG, HUI RAO*, ZHIYING WEN, AND YANLI XU

ABSTRACT. In this paper, we introduce a new notion called Minkowski measure for a class of metric spaces, which is closely related to the box dimension of the space. If $E$ is a self-similar set satisfying the open set condition, then the Hausdorff measure restricted to $E$ is a Minkowski measure. We show that the Lalley-Gatzouras type self-affine sponges always admit Minkowski measures. Moreover, we show that if a metric space is totally disconnected and possesses Minkowski measures, then the multi-fractal spectrum of the Minkowski measure is a Lipschitz invariant.

1. Introduction

The box dimension may be the most important dimension except the Hausdorff dimension. Unlike the Hausdorff dimension, the box dimension is not defined by a measure; associated to the box dimension, there is a concept called Minkowski content. The Minkowski measurability has been studied by many authors and it is closely related to the so-called fractal drum problem, see Lapidus and Pomerance [21], Falconer [7] and the references therein. The purpose of this paper is, for suitable metric space, to define a class of measures, called Minkowski measures, which describes the Minkowski measurability from another point of view. Moreover, we will explore the applications of Minkowski measures in the analysis of metric spaces.

Let $(X,d_X)$ be a compact metric space and let $A \subseteq X$. A family of balls is called a packing of $A$, if they are disjoint and their centers are located in $A$; is called a $\delta$-packing of $A$ if all the balls are of radius $\delta$. For any $\delta > 0$, we define

$$N_\delta(A) := \max\{\#P; P \text{ is a } \delta\text{-packing of } A\}. \quad (1.1)$$

Then the upper and lower box dimension of $X$ are defined by

$$\dim^B_X = \lim_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}, \quad \dim^B_X = \lim_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}. \quad \text{If the two values coincide, then the common value is called the box dimension and is denoted by } \dim^B_X.$$ 

Two points $x, y \in X$ are said to be $\varepsilon$-equivalent if there exists a sequence $\{x_1 = x, x_2, \ldots, x_{k-1}, x_k = y\} \subseteq X$ such that $d_X(x_i, x_{i+1}) \leq \varepsilon$ for $1 \leq i \leq k - 1$. An

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* The correspondence author.
\( \varepsilon \)-equivalent class is called an \( \varepsilon \)-component of \( X \). Let \( \mathcal{C}(X, \varepsilon) \) be the collection of \( \varepsilon \)-components of \( X \).

We introduce a notion of Minkowski measure as follows.

**Definition 1.1** (Minkowski measure). Let \( X \) be a compact metric space such that \( \beta = \dim_B X \) exists. A Borel measure \( \mu \) on \( X \) is called a Minkowski measure of \( X \) if there is a constant \( M > 0 \), such that for any \( \varepsilon \)-connected component \( R \),

\[
M^{-1}\mu(R) \leq N_\delta(R)\delta^\beta \leq M\mu(R)
\]

holds for \( \delta \) small enough.

Recall that two Borel measures \( \mu \) and \( \nu \) on \( X \) are said to be equivalent, and denoted by \( \mu \sim \nu \), if there exists \( \zeta > 0 \) such that \( \zeta^{-1}\mu(\cdot) \leq \nu(\cdot) \leq \zeta\mu(\cdot) \).

If \( X \) is totally disconnected, then all the Minkowski measures form an equivalent class with respect to the relation \( \sim \). Precisely, we have

**Theorem 1.1.** Let \( X \) be a compact metric space and \( \mu \) be a Minkowski measure of \( X \). If \( X \) is totally disconnected then \( \mu' \) is a Minkowski measure of \( X \) if and only if \( \mu \sim \mu' \).

An iterated function system (IFS) is a family of contractions \( \{\varphi_j\}_{j=1}^N \) on \( \mathbb{R}^d \), and the attractor of the IFS is the unique nonempty compact set \( E \) satisfying \( E = \bigcup_{j=1}^N \varphi_j(E) \) and it is called a self-similar set if all \( \varphi_j \) are similitudes. It is not hard to show that a self-similar set satisfying the OSC always admits Minkowski measures (Theorem 5.1).

**Example 1.2.** Let \( F_\lambda \) be the self-similar set generated by the IFS

\[
\left\{ f_1(x) = \frac{x}{3}, f_2(x) = \frac{x+1}{3}, f_3(x) = \frac{x+\lambda}{3} \right\}.
\]

When \( \lambda \) is an irrational number, it is shown by Kenyon (17) that the Lebesgue measure of \( F_\lambda \) is zero, and it is proved by Hochman (13) that \( \dim_H F_\lambda = \dim_B F_\lambda = 1 \). Therefore the Minkowski content of \( F_\lambda \) is zero, and \( F_\lambda \) does not admit any Minkowski measure by Lemma 2.3.

It is shown (Rao et al. 35) that McMullen-Bedford carpets always admit Minkowski measures. Let \( 2 \leq m < n \) be two integers and denote by \( \text{diag}(n, m) \) the \( 2 \times 2 \) diagonal matrix. Let \( D \subset \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\} \). For \( d \in D \), define

\[
S_d(z) = \text{diag}(n^{-1}, m^{-1})(z + d),
\]

then \( \{S_d\}_{d \in D} \) is an iterated function system (IFS), and its invariant set, which we denote by \( K(n, m, D) \), is called a Bedford-McMullen carpet. See [3, 27]. Let \( s = \dim_H K \). Y. Peres (31) showed that the \( s \)-dimensional Hausdorff measure of \( E \) is infinity apart from the exception case that the Hausdorff dimension and box dimension coincide. Nevertheless, it is shown by Rao et al. 35 that the uniform Bernoulli measure of \( K(n, m, D) \) is a Minkowski measure.

The second part of the paper focus on Lipschitz invariants related to Minkowski measures.
Let $X$ be a metric space and $\mu$ a Borel measure on $X$. Let $Y$ be another metric space and let $f : X \to Y$ be a Borel measurable function. We shall denote by $f^*\mu$ the measure on $Y$ defined by $f^*\mu(B) = \mu(f^{-1}(B))$ for every Borel set $B \subset Y$.

We shall use $B(x, r)$ to denote the open ball with center $x$ and radius $r$.

A metric space $X$ is said to be a doubling space, if there is a constant $C_1 > 0$ such that for any $x \in X$ and $r > 0$, $B(x, 2r)$ contains at most $C_1$ disjoint balls of radius $r$. (See [12]). Clearly doubling property is invariant under bi-Lipschitz maps. We show that

**Theorem 1.2.** Let $X$ and $Y$ be two compact doubling spaces which are totally disconnected. Suppose $\mu$ is a Minkowski measure of $X$. If $f : X \to Y$ is bi-Lipschitz, then $f^*\mu$ is a Minkowski measure of $Y$.

A special case of the above theorem is proved in [35], where both $X$ and $Y$ are Bedford-McMullen carpets. As a corollary of the above theorem, we have

**Corollary 1.3.** Let $X$ and $Y$ be two totally disconnected compact doubling spaces. Suppose $\mu$ and $\nu$ are Minkowski measures of $X$ and $Y$, respectively. If $f : X \to Y$ is bi-Lipschitz, then $f^*\mu \sim \nu$, consequently, $\mu$ and $\nu$ have the same multifractal spectrum, and $\mu$ is doubling if and only if $\nu$ is doubling.

Thirdly, we give a criterion for Minkowski measure, and we show that a large class of self-affine sets admits Minkowski measures.

**Definition 1.4 (Diagonal IFS [5]).** Fix $d \geq 1$, and let $D = 1, \ldots, d$. For each $i \in D$, let $A_i$ be a finite index set, and let $\Phi_i = (\phi_{a,i})_{a \in A_i}$ be a collection of contracting similarities of $[0, 1]$, called the base IFS in coordinate $i$. Let $A = \coprod_{i \in D} A_i$, and for each $a = (a_1, \ldots, a_d) \in A$, consider the contracting affine map $\phi_a : [0, 1]^d \to [0, 1]^d$ defined by the formula

$$\phi_a(x_1, \ldots, x_d) = (\phi_{a,1}(x_1), \ldots, \phi_{a,d}(x_d)),$$

where $\phi_{a,i}$ is shorthand for $\phi_{i,a_i}$ in the formula above. Then we can get

$$\phi_a([0, 1]^d) = \prod_{i \in D} \phi_{a,i}([0, 1]) \subset [0, 1]^d.$$

Given $D \subset A$, we call the collection $\Phi = (\phi_a)_{a \in D}$ a diagonal self-affine sponge.

**Remark 1.5.** (i) If for each $i \in D$, $\{\phi_{a,i}[0, 1); a \in A_i\}$ is a partition of $[0, 1)$, then we call $\Lambda$ a Barański sponge. In particular, if $d = 2$, then we call $\Lambda$ a Barański carpet. Barański ([1]) computed the Hausdorff dimension and box dimension of Barański carpet.

Furthermore, if the contraction ratios of $\phi_{a,i}$’s are all equal, then $\Lambda$ is called a self-affine Sierpiński sponge by Kenyon and Peres. If $d = 2$ in this case, $\Lambda$ is the well known Bedford-McMullen carpet.

(ii) Feng and Wang ([11]) calculated the dimension of Bernoulli measures of two-dimensional diagonal self-affine sponges. They also obtained a formula for the box dimension.
We say that $\Phi$ satisfies the coordinate ordering condition if there exists a permutation $\sigma$ of $\{1, 2, \ldots, d\}$ such that for all $a = (a_1, \ldots, a_d) \in \mathcal{D}$,
\[
|\phi'_{a,\sigma(1)}| > \cdots > |\phi'_{a,\sigma(d)}|.
\]
In this paper, we consider diagonal self-affine sponges satisfying the coordinate ordering condition. For simplicity, we always assume that
\begin{equation}
(1.3) \quad |\phi'_{a,1}| > \cdots > |\phi'_{a,d}|
\end{equation}
for all $a \in \mathcal{D}$ in the sequel. See Figure 1 (c) for an example with $d = 2$.

Let $\pi_j : \mathbb{R}^d \to \mathbb{R}^j$ be the projection map defined by $\pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_j)$. Let
\begin{equation}
(1.4) \quad \Phi_{\{1, \ldots, j\}} = (\phi_{a,1}, \ldots, \phi_{a,j})_{a \in \pi_j(E)},
\end{equation}
which is an IFS on $\mathbb{R}^j$. Here we emphasize that each map occurs at most once in the above IFS.

**Definition 1.6** (Neat projection condition). Let $\Lambda_\Phi$ be a diagonal self-affine sponge satisfies the coordinate ordering condition. We say $\Phi$ satisfies the neat projection condition, if for each $j \in \{1, \ldots, d\}$, the IFS $\Phi_{\{1, \ldots, j\}}$ satisfies the open set condition with the open set $I_j = (0, 1)^j$, that is,
\[
(\phi_{a,\{1, \ldots, j\}}(I_j))_{a \in \pi_j(E)}
\]
is disjoint.

Now let $\Lambda_\Phi$ be a diagonal self-affine sponge satisfies the coordinate ordering condition as well as the neat projection condition, which is a generalization of the Lalley-Gatzouras carpets [20]. We define a sequence $(\beta_j)^d_{j=1}$ inductively as follows.

Let $\beta_1 > 0$ be the unique real number such that
\[
\sum_{f_1 \in \Phi_{\{1\}}} (f_1')^{\beta_1} = 1.
\]
If $\beta_1, \ldots, \beta_{j-1}$ are defined, we define $\beta_j > 0$ to be the unique positive real number such that

$$\sum_{(f_1, \ldots, f_j) \in \Phi_{1, \ldots, j}} \prod_{k=1}^{j} (f'_k)^{\beta_k} = 1.$$ 

For $a \in E$, we define

(1.5) \[ p_a = \prod_{j=1}^{d} (f'_j)^{\beta_j}. \]

Then $(p_a)_{a \in E}$ is a probability weight. We denote by $\mu$ the Bernoulli measure defined by this probability weight. We show that

**Theorem 1.3.** Let $\Lambda_\Phi$ be a diagonal self-affine sponge satisfies the coordinate ordering condition as well as the neat projection condition. Then

$$\dim_B \Lambda_\Phi = \sum_{j=1}^{d} \beta_j.$$ 

Moreover, if $\Lambda_\Phi$ is totally disconnected, then the Bernoulli measure $\mu$ defined by the weight in (1.5) is a Minkowski measure.

The box dimension of Barański measures are computed in [1], but we do not know whether every Barański carpet admits a Minkowski measure. In Section 6, we discuss the Minkowski measures of some symbolic spaces.

The paper is organized as follows.

2. Proof of Theorem 1.1

Let $(X, d)$ be a metric space. Let $\delta > 0$. For a set $A \subset X$, let $U(A, \delta) := \{x \in X; |x - a| \leq \delta \text{ for some } a \in A\}$, which is called the $\delta$-neighbourhood of $A$. Let $\mathcal{H}$ be the collection of all compact subsets of $X$. If $A, B \in \mathcal{H}$, the Hausdorff metric $D$ is defined as follows

$$D(A, B) = \inf \{\delta | A \subset U(B, \delta) \text{ and } B \subset U(A, \delta)\}.$$ 

**Lemma 2.1** ([29](Exercise 7 of §45)). If $X$ is a compact metric space, then the space $\mathcal{H}$ is compact in the Hausdorff metric $D$.

For $\mathcal{F}$ being a collection of subsets of $X$, we use $\sigma(\mathcal{F})$ to denote the $\sigma$-algebra generated by $\mathcal{F}$.

**Lemma 2.2.** Let $X$ be a compact metric space which is totally disconnected. Let $\varepsilon_k \to 0$. Then the family

$$\mathcal{F} = \bigcup_{k \geq 1} \mathcal{C}(X, \varepsilon_k)$$

generates the Borel algebra $\mathcal{B}(X)$. 

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Proof. Denote $\mathcal{C}_k = \mathcal{C}(X, \varepsilon_k)$. Let $U \subset X$ be an open ball. Let

$$U_k = \bigcup \{ A; A \in \mathcal{C}_k \text{ and } A \subset U \}.$$  

Let $x \in U$. Let $A_k$ be the element in $\mathcal{C}_k$ containing $x$. Since $x$ is a trivial point, we claim that $\lim_{k \to \infty} \text{diam}(A_k) = 0$. Suppose on the contrary that $\lim_{k \to \infty} \text{diam}(A_k) > 0$. By Lemma 2.1, there is a subsequence of $\{A_k\}_{k=1}^\infty$ which converges to a compact set $K \subset X$. Recall that a set $E$ is said to be well-chained provided that for any real number $\epsilon > 0$, any two points $a, b \in E$ are $\epsilon$-equivalent. Let $\epsilon > 0$, there exists $k > 0$ such that $\epsilon_k \leq \epsilon$ and $D(A_k, K) \leq \epsilon$. Let $y := y_1, z := y_n$ be any two points of $K$, there exists $\{x_1, \ldots, x_n\} \subset A_k$ such that $d_X(x_i, y_1) \leq \epsilon, d_X(x_n, y_n) \leq \epsilon$ and $d_X(x_i, x_{i+1}) \leq \epsilon(1 \leq i \leq n-1)$. For each $x_i (2 \leq i \leq n-1)$, there exist $y_i (2 \leq i \leq n-1) \in K$ such that $d_X(x_i, y_i) \leq \epsilon(2 \leq i \leq n-1)$. We see that $d_X(y_i, y_{i+1}) \leq 3\epsilon$ for all $1 \leq i \leq n-1$, which implies that $y$ and $z$ are $3\epsilon$-equivalent. So $K$ is well-chained. According to [35] (page 15), every compact well-chained set is connected, thus $K$ is connected, which is a contradiction since $X$ is totally disconnected. This proves that $\lim_{k \to \infty} \text{diam}(A_k) = 0$, and hence $A_k \subset U$ for $k$ large enough. Therefore, $U_k$ increase to $U$. It follows that $U \in \sigma(\mathcal{F})$ and $\mathcal{B}(X) \subset \sigma(\mathcal{F})$. The lemma is proved. \hfill $\Box$

Proof of Theorem 1.1. Let $\mu$ and $\mu'$ be two Minkowski measures of $X$. Then by definition, $\mu(R)$ and $\mu'(R)$ are comparable when $R$ is an $\epsilon$-connected component of $X$. By Lemma 2.2, such $R$ generates $\mathcal{B}(X)$, hence $\mu(R)$ and $\mu'(R)$ are comparable for any Borel subset of $X$. The lemma is proved. \hfill $\Box$

2.1. Remarks on Minkowski content. For $A \subset \mathbb{R}^n$, and $0 \leq \beta \leq n$, the $\beta$-dimensional upper Minkowski content is

$$\mathcal{M}^\beta(A) = \limsup_{\delta \to 0} \frac{\mathcal{L}^n(\{x; d(x, A) < \delta\})}{\delta^{n-\beta}}$$

where $\mathcal{L}^n$ is $n$-dimensional Lebesgue measure.

By taking lower limit instead of upper limit, we define the $m$-dimensional lower Minkowski content $\mathcal{M}^\beta_*(A)$. (See for instance, §3.1 of [9].) The following lemma justifies the name of Minkowski measure.

Lemma 2.3. Let $X$ be a compact subset of $\mathbb{R}^n$ with $\beta = \dim_B X$. If $\mu$ is a Minkowski measure of $X$, then there exists a constant $M_1 > 0$ such that for any $\varepsilon$-connected component $R$, it holds that

$$M_1^{-1} \mu(R) \leq \mathcal{M}^\beta_*(R) \leq \mathcal{M}^\beta(R) \leq M_1 \mu(R).$$

Proof. Define

$$g_\delta(A) = \frac{\mathcal{L}^n(\{x; d(x, A) < \delta\})}{\delta^n}.$$

It is easy to show that

$$g_\delta(A) \asymp N_\delta(A).$$

Hence

$$g_\delta(R) \asymp \mu(R)\delta^{-\beta},$$
which proves the lemma.

3. Proof of Theorem 3.1

Let \( X \) be a compact metric space with geometrically doubling property. A set \( R \subset X \) is called a clopen set if it is closed as well as open. Recall that \( N_\delta(A) \) is the maximum of cardinality of \( \delta \)-packings.

We call \( \{A_1, \ldots, A_k\} \) a partition of \( X \), if \( X = \bigcup_{j=1}^k A_j \) and \( A_j \)'s are disjoint; moreover, we call \( \max_{1 \leq j \leq k} \text{diam}(A_j) \) the size of the partition.

**Theorem 3.1.** Let \( X \) be a compact metric space, let \( \mu \) be a Borel measure of \( X \). Let \((\mathcal{P}_k)_{k \geq 1}\) be a sequence such that

(i) for each \( k \), \( \mathcal{P}_k \) is a partition of \( X \) consisting of compact subsets of \( X \);

(ii) the size of \( \mathcal{P}_k \) tends zero as \( k \to \infty \).

If there is a constant \( M > 0 \) such that for any \( R \in \bigcup_{k \geq 1} \mathcal{P}_k \), \( (1.2) \) holds for \( \delta \) small enough, then \( \mu \) is a Minkowski measure.

**Lemma 3.1.** Let \((\mathcal{P}_k)_{k \geq 1}\) be a sequence satisfying (i) and (ii) in Theorem 3.1. If \( R \subset X \) is a clopen set, then there exists \( k \) such that \( R \) is a union of some elements of \( \mathcal{P}_k \). In particular, there exists \( \varepsilon > 0 \) such that \( R \) is a union of some \( \varepsilon \)-connected components of \( X \).

**Proof.** Clearly, \( X \setminus R \) is clopen since \( R \) is clopen, and both of them are compact. So \( d_X(R, X \setminus R) > 0 \). Notice that the size of \( \mathcal{P}_k \) tends zero as \( k \to \infty \), there exists \( k > 0 \) such that the size of \( \mathcal{P}_k \) is less than \( d_X(R, X \setminus R) \). Thus each element in \( \mathcal{P}_k \) is either in \( R \) or in \( X \setminus R \), which implies that \( R \) is a union of some elements of \( \mathcal{P}_k \). □

**Proof of Theorem 3.1.** Let \( \varepsilon > 0 \). There exists an integer \( N > 0 \) such that for all \( k > N \), the size of \( \mathcal{P}_k \) is less than \( \varepsilon/2 \). Hence each \( R \in \bigcup \mathcal{P}_k \) belongs to one \( \varepsilon \)-connected component of \( X \). Let \( S \in \mathcal{C}(X, \varepsilon) \), then there exist \( k \) and \( R_1, \ldots, R_\ell \in \mathcal{P}_k \) such that \( S = \bigcup_{j=1}^\ell R_j \). Notice that

\[
M^{-1} \delta^{-\beta} \mu(R_j) \leq N_\delta(R_j) \leq M \delta^{-\beta} \mu(R_j), \quad 1 \leq j \leq \ell
\]

for \( \delta \) small enough. Moreover,

\[
\mu(S) = \sum_{j=1}^\ell \mu(R_j) \quad \text{and} \quad \sum_{j=1}^\ell N_\delta(R_j) = N_\delta(S)
\]

for \( \delta \) small enough. Thus we conclude that \( \mu \) is a Minkowski measure. □

**Proof.** Suppose the IFS of \( F \) is \( \{\varphi_i\}_{i=1}^N \) and the corresponding contraction ratios are \( r_1, \ldots, r_N \). Then we have \( \sum_{i=1}^N r_i^s = 1 \) and \( \mu \) is the Bernoulli measure determined by \( (r_1^s, \ldots, r_N^s) \).

Given \( \varepsilon > 0 \), denote \( \rho_\varepsilon := \min_{R_1, R_2 \in \mathcal{C}(F, \varepsilon)} d(R_1, R_2) \). Let \( k \) be the smallest integer such that \( r_{\max}^k \text{diam}(F) < \rho \), where \( r_{\max} = \max\{r_1, \ldots, r_N\} \). Then for \( \ell \geq k \), each \( \ell \)-th cylinder intersects exactly one \( \varepsilon \)-connected component of \( F \).
Now let $\delta < \rho \varepsilon$ small. Define $U = \{ \varphi_\sigma(F) : \sigma = \sigma_1 \ldots \sigma_\ell \text{ with } r_{\sigma_1 \ldots \sigma_{\ell-1}} \leq \delta \}$ but $r_{\sigma_1 \ldots \sigma_{\ell-1}} > \delta \}$. Let $R \in C(F, \varepsilon)$ and write $R := \bigcup_{i=1}^t C_i$, where $C_i \in U$. It is easy to see that $\text{diam}(C_i) \asymp \delta, 1 \leq i \leq t$. Since $F$ satisfying OSC, say, $U$ is the open set of the OSC, we see that $\varphi_\sigma(U)$ contains an open ball. By a volume argument there exists a constant $C > 0$ such that

$$Ct < N_\delta(R) \leq 2^d t.$$ 

By the OSC condition again, we have

$$\frac{t \delta^s}{r_{\max}^s} < \mu(R) = \sum_{i=1}^t \mu(C_i) \leq t \delta^s.$$ 

Therefore, $N_\delta(R) \delta^s \asymp \mu(R)$ and we finish the proof. \hfill \Box

We close this section with an example of compact metric space which is not geometrically doubling.

**Example 3.2.** Let $X_n = \{1, \ldots, n\}$, and $X = \prod_{n=1}^\infty X_n$. For $x = (x_k)_{k \geq 1}, y = (y_k)_{k \geq 1} \in X$, we define $d_X(x, y) = 2^{-|x \wedge y|}$, where $|x \wedge y|$ denotes the length of the maximal common prefix of $x$ and $y$. Then $(X, d_X)$ is compact, but it is not geometrically doubling.

4. **Proof of Theorem 1.2**

Two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are said to be *Lipschitz equivalent*, denoted by $(X, d_X) \sim (Y, d_Y)$, if there exists a bijection $f : X \to Y$ and a constant $C > 0$ such that

$$C^{-1} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C d_X(x, y), \quad \text{for all } x, y \in X;$$

in this case, we call $f$ a bi-Lipschitz map.

We start with two lemmas. Let $\delta > 0$. Recall that $N_\delta(A)$ is defined by (1.1).

**Lemma 4.1.** Let $X$ and $Y$ be two doubling spaces, and let $f : X \to Y$ be a bi-Lipschitz map. Then there exists a constant $C > 0$ such that for any $\delta > 0$,

$$C^{-1} N_\delta(X) \leq N_\delta(Y) \leq C N_\delta(X). \quad (4.1)$$

**Proof.** Let $C_1$ be a Lipschitz constant of $f$. For $r > 0$ and $x \in X$, we have

$$B_{C_1^{-1} r}(f(x)) \subset f(B_r(x)),$$

and it follows that $N_\delta(X) \leq N_{C_1^{-1} \delta}(Y)$.

Since $Y$ is a doubling space, there exists a constant $C_2 > 0$ such that for any $y \in Y$ and $r > 0$, $B(y, r)$ contains at most $C_2$ disjoint balls of radius $C_1^{-1} r/3$.

Let $P_1 = \{ B(x_1, C_1^{-1} \delta), \ldots, B(x_k, C_1^{-1} \delta) \}$ be a $C_1^{-1} \delta$-packing of $Y$ such that $k = N_{C_1^{-1} \delta}(Y)$, let $P_2 = \{ B(y_1, \delta), \ldots, B(y_\ell, \delta) \}$ be a $\delta$-packing of $Y$ such that $\ell = N_\delta(Y)$. Then for any $x_i$, there exists $y_i$ such that $d_Y(x_i, y_i) \leq 2 \delta$. Let $P_3 = \{ B(y_1, 3 \delta), \ldots, B(y_\ell, 3 \delta) \}$; then $\bigcup P_1 \subset \bigcup P_3$. Therefore,

$$N_{C_1^{-1} \delta}(Y) = k \leq C_2 \ell = C_2 N_\delta(Y). \quad (4.2)$$
This proves the first inequality of (4.1). By symmetry, we have the second inequality. □

**Lemma 4.2.** Let $X$ and $Y$ be two compact metric spaces. Let $f : X \to Y$ be a continuous map. Let $S$ be an $\varepsilon$-connected component of $X$. Then there exist $\varepsilon' > 0$ such that $f(S)$ is a union of some $\varepsilon'$-connected components of $Y$.

*Proof.* Let $P$ be a partition of $X$ such that each element of $P$ is an $\varepsilon$-connected component, so $S \in P$ and it is a clopen set in $X$. It is easy to see that $f(P)$ is a partition of $Y$ and $f(S)$ is also clopen. Notice that $d_Y(f(S), Y \setminus f(S)) > 0$, by Lemma 3.1, $f(S)$ is a union of $\varepsilon'$-connected components of $Y$ if we take $\varepsilon' < d_Y(f(S), Y \setminus f(S))/2$. □

**Theorem 4.1.** Let $X$ and $Y$ be two compact doubling spaces. Assume that $X$ and $Y$ have Minkowski measures $\mu_X$ and $\mu_Y$, respectively. Let $f : X \to Y$ be a bi-Lipschitz map. Then there exists $\zeta > 0$, such that, for any $\varepsilon$-connected component $S$ of $X$, it holds that

\[(4.3) \quad \mu_Y(f(S)) \leq \zeta \mu_X(S).\]

*Proof.* Let $S \in C(X, \varepsilon)$. First, by Lemma 4.2, we have

\[(4.4) \quad f(S) = \bigcup_{j=1}^{q} R_j,\]

where $R_j \in C(Y, \varepsilon')$.

Let $\beta$ be the common value of the box dimension of $E$ and $F$. Take $\delta$ small so that (1.2) holds for $S$ as well as $R_1, \ldots, R_q$.

Since $f$ is bi-Lipschitz, by Lemma 4.1, there exists a constant $C_1 > 0$ such that

\[(4.5) \quad N_\delta(S) \geq C_1^{-1} N_\delta \left( \bigcup_{j=1}^{q} R_j \right).\]

Moreover, $\mu_X$ and $\mu_Y$ are Minkowski measures, there exists a constant $C_2 > 0$ such that

\[(4.6) \quad N_\delta(S) \leq C_2 \mu_X(S) \delta^{-\beta}\]

and

\[(4.7) \quad N_\delta \left( \bigcup_{j=1}^{q} R_j \right) \geq C_2^{-1} \delta^{-\beta} \mu_Y \left( \bigcup_{j=1}^{q} R_j \right).\]

Combining (4.5), (4.6) and (4.7), we obtain

\[\mu_Y \left( \bigcup_{j=1}^{q} R_j \right) \leq C_1 C_2 \mu_X(S).\]

The theorem is proved. □
Proof of Theorem 1.2. According to Lemma 2.2, the Borel algebra on X can be generated by
\[ \mathcal{F} = \bigcup_{k \geq 1} \mathcal{C}(X, 1/k), \]
Then by Theorem 4.1 and changing the role of X and Y, we obtain the result. \(\square\)

5. Minkowski measure of self-similar sets with OSC

An IFS \( \{\varphi_j\}_{j=1}^{N} \) is said to satisfy the open set condition (OSC), if there is a bounded nonempty open set \( U \subset \mathbb{R}^d \) such that for any \( 1 \leq i \leq N \), \( \varphi_i(U) \subset U \) and \( \varphi_i(U) \cap \varphi_j(U) = \emptyset \) for \( 1 \leq i \neq j \leq N \). Let \( E \) be the invariant set of an IFS \( \{\varphi_j\}_{j=1}^{N} \). For each \( \sigma \in \{1, \ldots, N\}^k \), \( \varphi_\sigma(E) \) is called a \( k \)-th cylinder of \( E \). Let \( \mathcal{H}^s \) be the s-dimensional Hausdorff measure.

Theorem 5.1. Let \( F \subset \mathbb{R}^d \) be a self-similar set satisfying the OSC. Let \( s = \dim_H F \). Then \( \mu = \mathcal{H}^s|_F \) is a Minkowski measure of \( F \).

Proof. Let \( F \) be a self-similar set generating by the IFS \( \{\varphi_i\}_{i=1}^{N} \). Denote the contraction ratio of \( \varphi_i \) by \( r_i \). Then \( s = \dim_H F \) satisfies \( \sum_{i=1}^{N} r_i^s = 1 \), and \( \mu \) is the Bernoulli measure determined by the probability weight \( (r_1^s, \ldots, r_N^s) \).

Given \( \varepsilon > 0 \), denote \( \rho_\varepsilon := \min_{R_1, R_2 \in \mathcal{C}(F, \varepsilon)} d(R_1, R_2) \). Let \( k \) be the smaller integer such that \( r_{\max}^k \delta r \leq \rho_\varepsilon \), where \( r_{\max} = \max\{r_1, \ldots, r_N\} \). Then for any \( \ell \geq k \), each \( \ell \)-th cylinder intersects exactly one \( \varepsilon \)-connected component of \( F \).

Pick \( \delta < \rho_\varepsilon \). Define
\[ J = \{ \sigma : \text{ where } \sigma = \sigma_1 \ldots \sigma_\ell \text{ with } r_{\sigma_1 \ldots \sigma_\ell} \leq \delta \text{ but } r_{\sigma_1 \ldots \sigma_\ell-1} > \delta \}. \]
Let \( R \in \mathcal{C}(F, \varepsilon) \) and write \( R := \bigcup_{t=1}^{\ell} D_t \), where \( D_t \in \{\varphi_\sigma(F) ; \sigma \in J\} \). It is easy to see that \( r_{\min}^s \delta^s \leq \mu(D_t) \leq \delta^s, 1 \leq i \leq t \).

Since \( F \) satisfies OSC, there exists an open set \( U \) such that \( \phi_i(U) \), \( 1 \leq i \leq N \) are disjoint and are contained in \( U \). Let \( B_0 \) be a ball contained in \( U \). We see that \( \varphi_\sigma(B_0) \), \( \sigma \in J \) are disjoint balls with radius no less than \( \delta r_0 \), where \( r_0 \) is the radius of \( B_0 \). Clearly, there exists a constant \( C > 0 \) such that \( Ct < N_\delta(R) \). On the other hand, by a volume argument there exists a constant \( C' > 0 \) such that \( N_\delta(R) < C't \). So we have
\[ N_\delta(R) \asymp t. \]
By the OSC condition, we have \( \mu(R) = \sum_{t=1}^{T} \mu(D_t) \), which implies that
\[ t\delta^s r_{\min}^s < \mu(R) \leq t\delta^s. \]
Therefore, \( N_\delta(R) \delta^s \asymp \mu(R) \) and we finish the proof. \(\square\)

6. Minkowski measure of diagonal self-affine sponges

Let \( (\Phi_1, \ldots, \Phi_d) \) be a set of basis IFS. Let \( \mathcal{D} \) be a digit set and let \( \Phi = \Phi_{\mathcal{D}} \) be a diagonal IFS. Let \( \Lambda_\Phi \) be the attractor of \( \Phi \). In this section, we always assume that \( \Lambda_\Phi \) satisfies the coordinate ordering condition as well as the neat projection condition. Our goal is to show that the Bernoulli measure \( \mu \) defined in (1.5) is a Minkowski measure of \( \Lambda_\Phi \).
Recall that $\pi_j : \mathbb{R}^d \to \mathbb{R}^j$ is the projection to the first $j$ coordinates, that is, $\pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_j)$. Denote $\Lambda_j = \pi_j(\Lambda)$, which is the limit set of the IFS $\Phi_{\{1, \ldots, j\}}$, see (1.4). For $f = (f_1, \ldots, f_j) \in \Phi_{\{1, \ldots, j\}}$, define

$$p_f = \prod_{k=1}^j (f'_k)^{\beta_k}.$$  

Let $\mu_j$ be the Bernoulli measure on $\Lambda_j$ defined by the weight $(p_f)_{f \in \Phi_{\{1, \ldots, j\}}}$. Especially, $\mu_d$ is the measure $\mu$ in Theorem 1.3. For $\omega = \omega_1 \ldots \omega_k$, we define $\phi_\omega(z) = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_k}(z)$ and call $\phi_\omega([0,1]^d)$ a $k$-th basic pillar of $\Lambda_\Phi$. We will use $S(C)$ to denote the shortest side of a basic pillar $C$.

**Remark 6.1.** Let $C$ and $C'$ be two basic pillar of $\Lambda_\Phi$ of rank $k$, and $\nu$ be a Bernoulli measure of $\Lambda_\Phi$. Then $\nu(C \cap C') = 0$ is always true. See for instance $[35]$.

Two quantities $A$ and $B$ are said to be comparable, and denoted by $A \asymp B$, if there exists a constant $c > 0$ such that $c^{-1} \leq B/A \leq c$. For simplicity, we denote $\mu(R)\delta^{-\beta} \asymp N_\delta(R)$ if (1.2) holds.

According to (1.3), we denote $r_* := \min\{|\phi'_a|; a \in D\}$ and $r^* := \max\{|\phi'_a|; a \in D\}$.

**Proof of Theorem 1.3.** We shall prove by induction on $j$ that

(i) $\dim_B \Lambda_j = \sum_{k=1}^j \beta_k := \alpha_j$;

(ii) for any basic pillar $C$ of $\Lambda_j$, and $\delta < S(C)$,

$$N_\delta(C) \asymp \mu_j(C)\delta^{-\alpha_j}.$$  

(iii) $\mu_j$ is a Minkowski measure of $\Lambda_j$.

If $j = 1$, $\Lambda_1$ is a one-dimensional self-similar set satisfying the open set condition, hence $\beta_1$ is the Hausdorff dimension and box dimension of $\Lambda_1$. Moreover, $\mu_1$ is a multiple of the Hausdorff measure of $\Lambda_1$ and it is a Minkowski measure of $\Lambda_1$. It is an easy matter to show that (ii) holds. So (i)-(iii) holds for $j = 1$.

Now suppose that (i)-(iii) holds for all $j < d$. Especially we have $\dim_B \Lambda_{d-1} = \sum_{k=1}^{d-1} \beta_k := \alpha_{d-1}$ and $\mu_{d-1}$ is a Minkowski measure of $\Lambda_{d-1}$, and

$$N_\delta(C) \asymp \mu_{d-1}(C)\delta^{-\alpha_{d-1}}$$  

for any basic pillar $C$ of $\Lambda_{d-1}$.

First, we show that item (ii) holds. Let $C$ be a basic pillar of $\Lambda_d = \Lambda_\Phi$. We will discuss it in two case.

**Case 1.** $S(C) < \delta/r^*$. In this case, we have $N_\delta(C) \asymp N_\delta(\pi_{d-1}(C'))$. Denote $C' = \pi_{d-1}(C)$, then $C'$ is a basic pillar of $\Lambda_{d-1}$. By induction hypothesis, we have

$$N_\delta(C') \asymp \mu_{d-1}(C')\delta^{-\alpha_{d-1}}.$$
Write \( C = \prod_{k=1}^{d} f_k([0, 1]) \), then \( \delta < S(C) = f'_d < \delta/r_* \). It follows that

\[
N_\delta(C) \asymp \mu_{d-1}(C') \delta^{-\alpha_{d-1}} \\
= \prod_{k=1}^{d-1} (f'_k)^{\beta_k} \delta^{-\alpha_{d-1}} \\
= \prod_{k=1}^{d} (f'_k)^{\beta_k} (f'_d)^{-\beta_d} \delta^{-\alpha_{d-1}} \\
= \prod_{k=1}^{d} (f'_k)^{\beta_k} \delta^{-\beta_d} \delta^{-\alpha_{d-1}} \\
= \mu(C) \delta^{-\alpha_d},
\]

which verifies (ii) in this case.

**Case 2.** The general case.

Let \( \mathcal{V}_\delta \) be the collection of basic pillars \( H \) such that \( S(H) \leq \delta \) but \( S(H') > \delta \) where \( H' \) is the parent of \( H \). Then \( \mathcal{V}_\delta \) is a covering of \( \Lambda_d \), and the elements in \( \mathcal{V}_\delta \) have disjoint interiors. These two properties guarantee that

\[
N_\delta(C) \asymp \sum_{H \in \mathcal{V}_\delta \text{ and } H \subset C} N_\delta(H).
\]

This together with (6.1) imply that

\[
N_\delta(C) \asymp \sum_{H \in \mathcal{V}_\delta \text{ and } H \subset C} \mu(H) \delta^{-\alpha_d} = \mu(C) \delta^{-\alpha_d}.
\]

This proves item (ii).

As a consequence of (ii), we have \( N_\delta(\Lambda_d) \asymp \delta^{-\alpha_d} \), which verifies item (i).

Let \( R \) be an \( \epsilon \)-connected component of \( \Lambda_d \). We choose \( k \) large such that a basic pillar \( C \) of rank \( k \) is either a subset of \( R \), or is disjoint with \( R \). Hence, for \( \delta < \epsilon/4 \), we have

\[
N_\delta(R) \asymp \sum_{C \subset R \text{ and } \text{rank}(C)=k} N_\delta(C) \\
\asymp \sum_{C \subset R \text{ and } \text{rank}(C)=k} \mu(C) \delta^{-\alpha_d} \\
\asymp \mu(R) \delta^{-\alpha_d},
\]

which verifies that \( \mu \) is a Minkowski measure of \( \Lambda_d \). The theorem is proved. \( \square \)

### 7. Minkowski measures of symbolic spaces

Let \( \mathcal{D} \subset \mathbb{Z}^2 \) be a finite set and denote \( N := \# \mathcal{D} \). We will write an element in \( \mathcal{D}^\infty \) as \((x_i, y_i)_{i \geq 1}\), where \((x_i, y_i) \in \mathcal{D}\). For two words \( x \) and \( i \), we denote \( x < i \) if \( x \) is a prefix of \( i \). For \((x, y) \in \mathcal{D}^k\), we call

\[
[x, y] = \{(i, j) \in \mathcal{D}^\infty; \ x < i, y < j\}
\]

a **cylinder** of rank \( k \). Denote \( N := \# \mathcal{D} \). Let \( \mu \) be the measure on \( \mathcal{D}^\infty \) such that for any cylinder \([x, y]\) of rank \( k \),

\[
\mu([x, y]) = 1/N^k,
\]

and we call it the **uniform Bernoulli measure** of \( \mathcal{D}^\infty \).
7.1. Full-symbolic space.

Let $\Sigma$ be a finite set, and let $0 < \xi < 1$. Let $d_\xi$ be the metric on $\Sigma^\infty$ defined by
\[
d_\xi(i,j) = \xi^{i \wedge j}, \quad i, j \in \Sigma^\infty,
\]
where $i \wedge j$ denote the maximal common prefix of $i$ and $j$, and $|W|$ denote the length of a finite word $W$.

Let $2 \leq m \leq n$ be two integers. The metric $\lambda$ on $D^\infty$ is defined as
\[
\lambda((x, y), (x', y')) = \max\{d_{1/n}(x, x'), d_{1/m}(y, y')\}.
\]
We call $(D^\infty, \lambda)$ the full-symbolic space with the parameter $(n, m, D)$. This symbolic space has been used in many works to study the Bedford-McMullen carpets, for example, [27] on dimensions, [16, 19] on multi-fractal analysis, and [41] on Lipschitz classification.

It is seen that a cylinder of rank $k$ of the full-symbolic space is a clopen set, and any two cylinders of rank $k$ have positive distance. Hence, $(D^\infty, \lambda)$ is always totally disconnected.

7.2. Half-symbolic space.

To study the Lipschitz classification of Bedford-McMullen carpets, [41] introduces a quasi-metric $\rho$ on $D^\infty$ as following: Assume that $D^\infty \subset \mathbb{Z} \times \{0, 1, \ldots, m-1\}$.

For $(x, y), (x', y') \in D^\infty$, set
\[
\rho((x, y), (x', y')) = \max\{d_{1/n}(x, x'), r_{1/m}(y, y')\},
\]
where $d_{1/n}$ is defined by (7.2) and
\[
r_{1/m}(y, y') = \left| \sum_{i=1}^{\infty} \frac{y_i}{m^i} - \sum_{i=1}^{\infty} \frac{y'_i}{m^i} \right|.
\]
If for any two distinct points $(x, y), (x', y') \in D^\infty$, it holds that $\rho((x, y), (x', y')) \neq 0$, then we say $D$ satisfies the non-overlapping condition; in this case $(D^\infty, \rho)$ is a metric space, and we call it the half-symbolic space with the parameter $(n, m, D)$.

7.3. Minkowski measure.

It is easy to show that both the full-symbolic space and half-symbolic space are doubling space.

**Theorem 7.1.** The uniform Bernoulli measure is a Minkowski measure of the half-symbolic space $(D^\infty, \rho)$ as well as the full-symbolic space $(D^\infty, \lambda)$.

**Proof.** We will only prove the theorem for the full-symbolic space, the proof for the half-symbolic space is very similar.

Denote $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ be two projections on $\mathbb{Z}^2$. Let $Y = \pi_2(D)$ and denote $s = \# Y$. Let $\beta_1 = \log s / \log m$ and let $\beta_2 > 0$ be the unique positive real number such that
\[
\sum_{d \in D} \left( \frac{1}{m} \right)^{\beta_1} \left( \frac{1}{n} \right)^{\beta_2} = 1.
\]
Then $\beta_2 = \log(N/s)/\log n$. Denote

$$\beta := \beta_1 + \beta_2 = \log_m s + \log_n \frac{N}{s}.$$ 

Let $\mu$ be the uniform Bernoulli measure on $(D^\infty, \lambda)$. Denote $N := \#D$ and let $R$ be a cylinder with rank $k(k \geq 0)$. Then $\mu(R) = \frac{1}{N^k}$. Similar to the proof of Theorem 1.3, one can show that for any $\delta \leq 1/n^k$,

$$(7.5) \quad N_\delta(R \cap D^\infty) \asymp \mu(R) \delta^{-\beta}.$$ 

First, this implies that $\dim_B D^\infty = \beta$.

Notice that if $\delta < 1/n^k$, then a ball with radius $\delta$ can intersect at most $M$ number of cylinders of rank $k$, where $M$ is a constant depends only on $D$, $n$ and $m$. Hence, (7.5) still holds if we replace $R$ by any $\epsilon$-connected component of $(D^\infty, \lambda)$. This proves that $\mu_{\lambda}$ is a Minkowski measure. \hfill \Box

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School of Computer Science and Technology, Beijing Institute of Technology, Beijing, 100081, China
Email address: liangyihuang@bit.edu.cn

Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China
Email address: yzhang@mail.ccnu.edu.cn

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
Email address: wenziz@tsinghua.edu.cn

Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China.
Email address: xu_y1@mails.ccnu.edu.cn