VELOCITY FIELD STATISTICS AND TESSELLATION
TECHNIQUES:
UNBIASED ESTIMATORS OF $\Omega$

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We describe two new – stochastic-geometrical – methods to obtain reliable velocity field statistics from N-body simulations and from any general density and velocity fluctuation field sampled at a discrete set of locations. These methods, the Voronoi tessellation method and Delaunay tessellation method, are based on the use of the Voronoi and Delaunay tessellations of the point distribution defined by the locations at which the velocity field is sampled. Adjusting themselves automatically to the density of sampling points, they represent the optimal estimator for volume-averaged quantities. They are therefore particularly suited for checking the validity of the predictions of quasi-linear analytical density and velocity field perturbation theory through the results of N-body simulations of structure formation. We illustrate the subsequent successful application of the two methods to estimate the bias-independent value of $\Omega$ in the N-body simulations on the basis of the predictions of perturbation theory for the $\Omega$-dependence of the moments and PDF of the velocity divergence in gravitational instability structure formation scenarios with Gaussian initial conditions. We will also shortly discuss practical and complicating issues involved in the obvious extension of the Voronoi and Delaunay method to the analysis of observational samples of galaxy peculiar velocities.

1 Introduction

The study of the large-scale cosmic velocity field is a very promising and crucial area for the understanding of structure formation. The cosmic velocity field is in particularly interesting because of its close relation to the underlying field of mass fluctuations. Indeed, on these large and (quasi)-linear scales the acceleration, and therefore the velocity, of any object is expected to have an exclusively gravitational origin so that it should be independent of its nature, whether it concerns a dark matter particle or a bright galaxy. Moreover, in the linear regime the generic gravitational instability scenario of structure formation predicts that at every location in the Universe the local velocity is related
to the local acceleration, and hence the local mass density fluctuation field, through the same universal function of the cosmic density parameter Ω (Peebles 1980), \( f(Ω) \propto Ω^{0.6} \). Because linear theory provides a good description on scales exceeding a few Megaparsec, the use of this straightforward relation implies the possibility of a simple inversion of the measured velocity field into a field that is directly proportional to the field of local mass density fluctuations. Such a procedure can then be invoked to infer the value of Ω, through a comparison of the resulting field with the field of mass density fluctuations in the same region.

However, such a determination of Ω may be contrived as the estimate of the mass density fluctuation field on the basis of the observed galaxy distribution may offer a biased view of the underlying mass distribution. By lack of a complete and self-consistent physical theory of galaxy formation, the commonly adopted approach is to make the simplifying assumption that the galaxy density \( δ_g \) and the mass density \( δ \) are related via a linear bias factor \( b \), \( δ_g = b δ \). The comparison between the observed galaxy density fluctuation field and the local cosmic velocity field will therefore yield an estimate of the ratio \( β = f(Ω)/b \approx Ω^{0.6}/b \). However, while numerous studies have yielded estimates of \( β \) in the range \( β \approx 0.5 - 1.2 \) (see Dekel 1994, Strauss & Willick 1995, for compilations of results), it has proven very cumbersome to subsequently disentangle the contribution of Ω and \( b \) to the quantity \( β \). In fact, it turns out to be impossible within procedures based on the linearity of the analysed velocity field.

2 Perturbation theory and the Quasi-linear evolution of the velocity field

As long as density and velocity fluctuations are small they grow linear, at a global rate irrespective of the value and location of the perturbation. The dependence on the mass of the fluctuation only expresses itself through the value of the universal cosmic matter density, i.e. through the value of Ω. This specific circumstance will cease to hold as soon as the fluctuations start to acquire values in the order of unity and higher. The larger gravitational acceleration exerted by the more massive structures in the density field then starts to induce the infall of proportionally ever larger amounts of matter and hence to an increase of the growth rate. This leads to a situation in which the growth of the density fluctuation is no longer merely dependent on the global value of Ω, but also on the local value of the density excess or deficit. This in turn implies that the situation during the linear regime of the random density and velocity perturbations retaining the same statistical properties will no longer
Figure 1: The PDF of the velocity divergence $\theta$ for 3 different cosmologies, as a function of $\Omega$ and $\sigma_\theta$.

Persist. Instead, the nonlinear evolution of the fluctuations leads to an increasingly asymmetric evolution of the density and velocity fluctuations, with overdensities collapsing into compact massive objects whose density excess can achieve values exceeding unity by many orders of magnitude, while underdensities expand into empty troughs whose density deficit is naturally limited to be no lower than -1.0.

If the primordial random fluctuation field is characterized by Gaussian statistics, one can apply perturbation theory to analytically compute the development of the stochastic properties of the field in the first quasi-linear stages as mild nonlinear perturbations develop in the random density and velocity field through the action of gravity (see Juszkiewicz & Bouchet 1997 for a review and references). For the purpose of determining the value of $\Omega$, Bernardeau (1994) and Bernardeau et al. (1995) found one of the most straightforward and useful results in the context of perturbation theory. In an extensive body of work he demonstrated the existence of a simple relation between the higher order moments $\langle \theta^p \rangle$ of the divergence $\theta$ of the locally smoothed velocity field $\mathbf{v}$,

$$\theta = \frac{\nabla \cdot \mathbf{v}}{H},$$

(1)
to its second order moment $\langle \theta^2 \rangle$. The coefficient $T_p$ depends on $\Omega$, on the shape of the power spectrum, the geometry of the window function that has been used to filter the velocity and even on the value of the cosmological constant $\Lambda$. For the case of a *top-hat filtered* velocity field, analytical expressions for $T_p$ were derived that showed them to be strongly dependent on the value of $\Omega$, only very weakly dependent on $\Lambda$ and, *very significant* within the present context, completely independent of a linear bias factor $b$. As in practical circumstances it may still be feasible to get reliable estimates of the lower-order moments, this implies that e.g. the relation between the skewness and the second-order moment of $\theta$ may be successfully exploited for the determination of $\Omega$ through the coefficient $T_3$.

$$T_3 \equiv \langle \theta^3 \rangle / \langle \theta^2 \rangle^2 \propto \Omega^{-0.6}.$$  \hspace{1cm} (2)

In fact, perturbation theory allows to infer the strongly $\Omega$-dependent, weakly $\Lambda$-dependent, and $b$-independent expressions for all coefficients $T_p$, and from this complete series of coefficients one can construct the complete Probability Distribution Function $p(\theta)$ (see Bernardeau 1994). An illustration of the changing global shape and behaviour of the complete PDF is evidently more direct into conveying an impression of the way in which the statistical properties of the velocity field depend on the value of $\Omega$ and evolve through gravity. Such an illustration is provided by the linear-log plot of $p(\theta)$ for 3 different cosmologies. Notice that $\Omega$ not only influences the overall shape of the PDF but also the location of its peak – indicated by $\theta_{\text{max}}$ – as well as that of the cutoff at the high positive values of $\theta$. The value of the maximum of $\theta$, at $\theta_{\text{max}}$, is directly related to the expansion of the deepest voids in such a Universe.

3 Volume-averaged quantities

The perturbation formalism discussed above evidently concerns continuous density fields. Practical applications to either N-body simulations or real observational samples, on the other hand, yield velocity fields that are sampled at a finite number of discrete, non-uniformly distributed, locations. This *discreteness* forms a major technical obstacle for the successful application of the theoretical results. The usual approach is to smooth the discrete velocity field by some filter function. Almost without exception the conventional filtering schemes concern a *mass-weighted* velocity field filtered with a filter function $W_M(x, x_0)$

$$v_{\text{mass}}(x_0) \equiv \frac{\int \! dx \, v(x) \rho(x) W_M(x, x_0)}{\int \! dx \, \rho(x) W_M(x, x_0)}.$$  \hspace{1cm} (3)
An example of this is probably one of the most frequently applied class of filtering schemes, involving the interpolation of the velocity field values at the random sampling locations to those at regular grid locations, weighing the contribution by each sampling point by the filter function value. However, the presence of the extra mass-weighting density field factor \( \rho \) would introduce considerable technical repercussions, and therefore analytical results within perturbation theory have been almost exclusively limited to volume-weighted filtered velocity fields \( \tilde{v} \),

\[
\tilde{v}(x_0) \equiv \frac{\int d\mathbf{x} \mathbf{v}(\mathbf{x}) W_V(\mathbf{x}, x_0)}{\int d\mathbf{x} W_V(\mathbf{x}, x_0)},
\]

where \( W_V(\mathbf{x}, x_0) \) is the applied weight function. For a successful practical implementation of the analytical results of perturbation theory it is then of crucial importance to have reliable numerical estimators of volume-averaged quantities. One specific aspect of such estimators is that from the discrete set of sampled velocities they should provide a prescription for the value of the velocity field throughout the whole sampling volume, unlike the conventional mass-weighted schemes that may restrict themselves to estimates at a finite number of positions.

4 Voronoi and Delaunay Tessellations

Ideally, the discrete probing and subsequent filtering of the velocity field should assign to each location in space the value of the velocity at exactly that location. In the infeasible situation of being able to do this all over space this would yield the continuous velocity field that one attempts to reconstruct as good as possible from the finite number of discretely sampled velocities. In other words, one could consider the underlying continuous velocity field as the result of a set of an infinite number of sampled points volume filtered with a filter function \( W_V \) whose filter radius is infinitely small. In the case of a discrete sampling of the field this suggests a velocity field reconstruction procedure that defines the velocity at every point in space by filtering the corresponding density field – which can be considered as the sum of delta functions peaking at each sample location – with a volume weighted filter that also has a filter radius as small as possible. It is then easy to see (see Bernardeau & Van de Weygaert 1996) that this defines a velocity field in which the value of the velocity at every location in space acquires the value of the velocity at the closest point of the discrete velocity sample.
The procedure described above implies nothing else than the concept of the Voronoi tessellation. Such a tessellation consists of a space-filling network of mutually disjoint convex polyhedral cells, the Voronoi polyhedra, each of which delimits the part of space that is closer to the defining point in the discrete point sample set than to any of the other sample points (see Van de Weygaert 1991, 1994, for extensive descriptions and references).

For the application of velocity field analysis, the Voronoi method introduced by Bernardeau & Van de Weygaert (1996) is based on the assumption that the velocity field is uniform within each Voronoi cell of the tessellation, such that the velocity throughout each of the Voronoi polyhedra is equal to that of the sample point defining that cell. This assumption immediately implies that the only non-zero values of the velocity gradients $\partial v_i / \partial x_j$ are localized to the (polygonal) Voronoi walls. For the specific case that the window function for the volume filtering is a top-hat filter, the subsequent computation of the volume averages of the velocity gradients consists of a relatively simple sum of the values of those velocity gradients in each of the walls $k$ intersected or inside the filter sphere weighted by the surface area $A_k$ of the part of the wall located within the sphere. Of special interest to us was for instance the locally volume-filtered velocity divergence $\tilde{\theta}_{\text{Vor}}$, which for a radius $R$ of the filter...
sphere is computed from

$$\tilde{\theta}_{\text{Vor}} = \frac{3}{4\pi R^3} \sum_{\text{walls}_k} A_k \mathbf{n}_k \cdot \frac{(v_{k1} - v_{k2})}{H}. \quad (5)$$

To illustrate the above we refer to the cartoon representation of the two-dimensional Voronoi method in Figure 2.

While the Voronoi method in general leads to good results and in fact was successfully applied to the results of N-body simulations to confirm the predictions of perturbation theory (see Bernardeau & Van de Weygaert 1996), it evidently represents an artificial situation of a discontinuous velocity field. Moreover, its artificial velocity field implies a few limitations to its application. The most significant one of these is that it cannot be applied to filter radii that are smaller than the average Voronoi wall distance. Below those scales the probability that a randomly placed filter sphere does not contain or intersect
any Voronoi wall gets prohibitively high, and therefore it would yield unrealistic zero values for the velocity gradients.

It is therefore best to consider the Voronoi method as a zeroth-order interpolation scheme, and extend the machinery to include a first-order interpolation scheme. It is the Delaunay method that can be regarded as this elaboration and extension towards a multidimensional equivalent of a linear interpolation scheme. It is based on the Delaunay tessellation defined by the points in the sample. This uniquely defined and space-covering network consists of mutually disjoint Delaunay tetrahedra. Each of the Delaunay tetrahedra is defined by four nuclei from the point sample that have a circumscribing sphere that does not contain any of the other nuclei in its interior. In fact, a host of practical, computational and image-processing, applications already employ the interpolation and triangulation qualities of Delaunay tessellations, based on the realization that they represent the near-optimal multidimensional triangulation of a discretely sampled space. For better appreciation, we have also illustrated the Delaunay method through its two-dimensional equivalent in Figure 2, the dashed lines representing the Delaunay tessellation. As they are each others dual, the Delaunay and Voronoi tessellation are closely related. This close relationship is born out by the fact that the centre of the circumscribing sphere of a Delaunay tetrahedron is a vertex in the corresponding Voronoi tessellation.

The first-order velocity field interpolation scheme defined by our Delaunay method (see Bernard et al. & Van de Weygaert 1996) is based on the construction of the velocity field \( \mathbf{v}(M) \) at every location \( M \) in space through linear interpolation between the velocities of the four particles \( A, B, C \) and \( D \) that define the Delaunay cell in which \( M \) is situated. \( \mathbf{v}(M) = \alpha_A \mathbf{v}(A) + \alpha_B \mathbf{v}(B) + \alpha_C \mathbf{v}(C) + \alpha_D \mathbf{v}(D) \), where the weights \( \alpha_j \) are the barycentric weights of the points. The resultant velocity field is one of a field of uniform velocity gradients within each Delaunay cell. The value of each of the nine velocity gradient components \( \partial \mathbf{v}_i / \partial \mathbf{v}_j \) within each Delaunay tetrahedron, and through them the value of the velocity divergence \( \theta \), the shear \( \sigma_{ij} \), and even the vorticity \( \omega_i \), follow immediately through solving the 9 linear equations that one obtains through evaluation of the three independent velocity differences obtained by evaluating the value of the velocity at the four different vertex locations of the tetrahedron. Having defined the values of the velocity gradient, and hence of the velocity itself, over the entire sample volume, the last step in the Delaunay method then consists of determining the volume averaging quantities through top-hat filtering with a filter \( W_{TH} \) that has a characteristic radius \( R \). Essentially this consists of determining the weighted average of the value of \( \theta \), \( \sigma_{ij} \) and/or \( \omega_i \) in a sphere of radius \( R \) centered on a location \( x_0 \), the weights being
the volume of the part of the Delaunay tetrahedron that intersects with the filter sphere.

As the velocity gradients in the Delaunay method will nowhere acquire artificial zero values, it is a much more robust method. Moreover, it is far less memory consuming to store the information on the Delaunay tetrahedra than it is to store the complete geometrical information on Voronoi polyhedra, so that it can be applied to truly big datasets. The one serious disadvantage in its present state is that it is a rather time-consuming method, due to the fact that the calculation of the intersection between randomly shaped and located tetrahedra and spheres turns out to be anything but a trivial problem.

By applying the procedures sketched above to a large number of randomly located filter spheres of a particular filter radius $R$, one obtains a numerically determined probability distribution of the velocity gradients in for example the outcome of N-body simulations. In this way one can check the analytical predictions of perturbation theory, evidently restricted to the early nonlinear stages of such simulations. Moreover, as agreement between the analytical predictions and the numerical inferences by the Voronoi and Delaunay method provides confidence in both, one can apply the tessellation methods to for extending the corresponding statistical study to the highly nonlinear structure formation stages whose velocity field characteristics cannot be addressed by analytical means.

5 Results, Discussion and Prospects

The main incentive for developing the Delaunay and Voronoi method is provided by the wish to be able to infer a bias-independent value of $\Omega$ through comparison of the velocity statistics obtained from the discrete point sample with those of analytical distributions. In figure 3 we show the PDFs of the velocity divergence $\theta$ that were numerically determined by the Delaunay method for a range of N-body simulations, each with a different value of $\Omega$. The solid curve shows the corresponding analytical distribution function $p(\theta)$, for the cosmic epoch with the same dispersion $\sigma_\theta$. For contrast, each of the four frames also contains the dashed curve for the PDF in an Einstein-de Sitter universe with the same value of $\sigma_\theta$. Evidently, the Delaunay method is highly successful in reproducing the correct statistical distribution, and via the relations between the moments of the PDF we indeed obtain very good estimates of $\Omega$.

While figure 3 illustrates the potential power of the tessellation methods, we are obviously motivated to apply them to more practical situations and hence more cumbersome cases where selection and sampling effects and
sampling errors are of crucial influence. In particular we hope to be able to
develop a formalism capable of dealing with observational catalogues of pecu-
liar velocities of galaxies. In previous work (Bernardeau & Van de Weygaert
1996, Bernardeau et al. 1997) we already addressed the issue of diluted samples.
In those cases both methods yielded encouraging results. However, the true
world will present problems ranging from the fact that one can measure galaxy
velocities only along the line of sight to complicated selection effects like differ-
ential Malmquist bias (see e.g. Bertschinger et al. 1990, Dekel, Bertschinger
& Faber 1990). Work on these issues is in progress, but they obviously provide
a considerable complication.

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