Efficient generation of correlated random numbers using Chebyshev-optimal magnitude-only IIR filters

Alejandro Rodriguez\(^1\) and Steven G. Johnson\(^2\)

\(^1\)Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139
\(^2\)Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

We compare several methods for the efficient generation of correlated random sequences (colored noise) by filtering white noise to achieve a desired correlation spectrum. We argue that a class of IIR filter-design techniques developed in the 1970s, which obtain the global Chebyshev-optimum minimum-phase filter with a desired magnitude and arbitrary phase, are uniquely suited for this problem but have seldom been used. The short filters that result from such techniques are crucial for applications of colored noise in physical simulations involving random processes, for which many long random sequences must be generated and computational time and memory are at a premium.

The generation of correlated random sequences, or “colored noise”, is important for many physical simulations involving random processes [1, 2, 3, 4, 5], and often the required computational time and memory is a critical concern. Although many filter-based techniques have been applied to this problem [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], in this brief manuscript we point out that these methods are suboptimal: a global Chebyshev-optimum stable IIR filter for this problem may be designed based on techniques developed in the 1970s [12, 13].

Colored-noise generation is required for many types of numerical simulations, such as thermodynamics [2, 3, 4], laser noise and first-passage time problems [1], and chaotic dynamics [5]. In general, any numerical model involving stochastic differential equations in which there is some background distribution, nonlinearity, or external noise associated with the quantities driving the fluctuations will require the use of colored noise [1, 2, 3, 4]. Such colored noise is most commonly described by a linear process: the output of a time-invariant linear filter applied to white noise [14]. The key implementation question, as we discuss below, is to determine how to apply this filter process. With rare exceptions [15], the underlying random process is almost always Gaussian in nature, if it arises by a physical process governed by the central-limit theorem [16]. It is this nearly ubiquitous case that we focus on here, in which only the correlation function of zero-mean colored noise is of concern, and not any higher-order statistics.

From a computational standpoint, the central problem is that many of these applications require many random sequences to be generated in parallel, and/or very long sequences, imposing performance and memory constraints on the noise generation. This occurs, for example, in the generation of power-law (1/f\(^\alpha\)) “pink noise” for applications ranging from hydrology to music, where very long sequences (whose length may not be known a priori) are often required [9]. Multiple long random sequences also occur in the generation of Rayleigh random variates for simulation of nonisotropic scattering, macromolecular systems, and physical models of mobile multipath radio channels [9, 11, 17]. Perhaps the most stringent requirements for colored-noise generation occur for spatio-temporal noise: where a different random sequence may need to be generated simultaneously for every point in space. This occurs, for example, in modeling of Brownian motion [5], or for simulation of thermal radiation [4, 27].

The fundamental technique for generating correlated random sequences is to start with white noise (uncorrelated random numbers) and to apply a filter whose power spectrum matches the desired correlation spectrum [2, 3, 4, 10, 11]. That is, suppose that we want colored-noise \(y\) with some desired correlation function \(R_m = \langle y_n y_{n+m} \rangle\), the (discrete-time) Fourier transform of which is the correlation spectrum \(R(\omega)\). We then start with white noise \(x\), usually uncorrelated Gaussian random numbers, with zero mean and \(x_n^2 = 1\), and apply...
a filter $H(\omega)$: in frequency domain, $Y(\omega) = H(\omega) X(\omega)$. The desired correlation spectrum is achieved if $|H(\omega)|^2 = R(\omega)$.

Although this filtering operation can be performed entirely in the frequency domain via a fast Fourier transform (FFT) of the data sequences [2, 3, 10], such an approach is often too inefficient. Filtering entirely in the frequency domain requires the entire data sequence $y_n$ to be computed and stored in advance, and if many long sequences are required the storage becomes prohibitive. The alternative is to perform the filtering in the time domain, using either finite-impulse response (FIR) filters or infinite-impulse response (IIR) filters (also called recursive or ARMA filters). Since the generation of colored noise is of interest to many researchers without a background in signal processing, let us briefly review the basics of FIR and IIR filtering, which are covered in more detail by numerous textbooks, e.g. Ref. [18]. FIR filters consist of a finite-length sequence $b_n$ that is convolved with the $x_n$. The Fourier transform of the $b_n$ is the filter $H(\omega)$, but in general a finite-length sequence can only approximate an arbitrary desired spectrum $R(\omega)$. In particular, an FIR filter yields a spectrum $H(\omega)$ which is a polynomial in $z = e^{i\omega}$ with coefficients $b_n$. A better approximation may be obtained by generalizing to a ratio of two polynomials (i.e., rational functions), which leads to IIR filters. An IIR filter is determined by two finite-length sequences, $a_n$, $b_n$ $(n = 0 \ldots N$, $a_0 = 1)$ and $b_n$ $(n = 0 \ldots M)$, that determine $y_n$ via the recurrence (which can be written in several equivalent forms):

$$y_n = \sum_{k=0}^{M} b_k x_{n-k} - \sum_{k=1}^{N} a_k y_{n-k}. \quad (1)$$

That is, $y_n$ is a convolution of $b_n$ with $x_n$ and of $a_n$ with the previous values of $y_n$. The filter design problem is then, given filter orders $N$ and $M$, to find the $a_n$ and $b_n$ that best approximate the desired spectrum.

Therefore, we must choose what type of filter to apply (IIR or FIR) and a filter-design method. Several choices have been previously proposed in the context of colored-noise generation. The simplest method, as we mentioned above, is to just perform a fast Fourier transform (FFT) of the entire desired spectrum multiplied by random phases [2, 3, 10]. This is equivalent to an FIR filter of the same length as the data, designed by the “window” method [18]. One can also employ IIR filters of shorter lengths, designed by a variety of standard methods such as Parks-McClellan [13, 12]. For certain noise problems, an FIR filter may also be designed by analytical methods [2, 3]. In order to shorten the length of the required filter, and thus the memory and time requirements, IIR recursive filters have been proposed [2, 6, 11]. In general, the design of IIR filters is a difficult problem [15, 20], and past approaches to colored noise generation by IIR filtering have used local-optimization [11] or Yule-Walker methods [6] that are not guaranteed to yield the global-optimum filter coefficients. Another difficulty is that not all IIR-filter design techniques are guaranteed to yield a stable filter (one which does not lead to a diverging process). One important exception is exponentially correlated noise, for which an exact first-order IIR filter is known analytically and is commonly used (although typically derived from a stochastic differential equation $y' = -ay + x$ and not recognized as an IIR filter per se) [1, 11, 17, 8]. However, there is a key property of the IIR filter design problem for colored-noise generation that makes optimal filter design practical: the phase of the filter $H(\omega)$ is irrelevant, since it is multiplied in any case by white noise $X(\omega)$ with a random phase.

In particular, we can exploit results by Dudgeon [12] and Rabiner [13], who demonstrated that the global Chebyshev-optimum magnitude-only minimum-phase stable IIR filter-design problem can be efficiently solved by a sequence of linear-programming problems [21]. Alternative methods with similar properties have also been proposed [22, 23], and the Dudgeon and Rabiner technique was recently generalized to multidimensional IIR filters [24]. However, its applicability to the problem of colored-noise generation does not seem to have been appreciated, and in this manuscript we demonstrate that it can yield dramatically more efficient filters than previous approaches.

To demonstrate the efficacy of the various filter-design approaches for colored-noise generation, we consider an example drawn from thermodynamic simulations of gray-body thermal radiation [4]. In this problem, thermal effects are modeled as random fluctuating current sources everywhere in space, with a correlation spectrum $R(\omega) = a\omega/(\exp(a\omega) - 1)$, for a constant $a$ determined by the temperature, based on the Planck distribution. This distribution is shown in Fig. 1, along with the periodogram of a finite-length correlated random sequence generated by the methods in this paper.

An IIR filter is defined by a rational polynomial function in $z = e^{i\omega}$:

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{i\omega k}}{1 + \sum_{k=1}^{N} a_k e^{i\omega k}} \quad (2)$$

where $N$ and $M$ define the filter order and $a_0 = 1$ for convenience. A stable minimum-phase IIR filter has all of its poles and zeros within the unit circle in $z$ (i.e., for $\text{Im} \omega > 0$) [18]. Given a sequence of uncorrelated random numbers $x_n$, the output colored-noise sequence $y_n$ is then given by the recurrence (Eq. 1) above. The required memory, along with the computation time per output, is therefore $O(N + M)$. An FIR filter is the special case
FIG. 2: Plot of Chebychev error, \( L_\infty = \max_\omega |R(\omega) - |H(\omega)|^2| \), vs. \( N + M \) for the Planck distribution \( R(\omega) \) given in Fig. 1. The Chebychev error is plotted for six different filter methods: IIR Yule-Walker (red squares), FIR Parks-McClellan (black crosses), FIR Least-Squares (blue dots), IIR global-optimum filter (gray circles), and two nonlinear conjugate-gradient methods optimizing two different \( L_2 \) norms: (a) \( L_2(|H| - \sqrt{R}) \) (green diamonds) and (b) \( L_2(|H|^2 - R) \) (magenta triangles).

FIG. 3: Plot of \( L_2 \) error, \( L_2(|H| - \sqrt{R}) \), vs. \( N + M \) for the Planck distribution \( R(\omega) \) given in Fig. 1. The \( L_2(|H| - \sqrt{R}) \) error is plotted for six different filter methods: IIR Yule-Walker (red squares), FIR Parks-McClellan (black crosses), FIR Least-Squares (blue dots), IIR global-optimum filter (gray circles), and two nonlinear conjugate-gradient methods optimizing two different \( L_2 \) norms: (a) \( L_2(|H| - \sqrt{R}) \) (green diamonds) and (b) \( L_2(|H|^2 - R) \) (magenta triangles).

\( N = 0 \). A sequence \( y_n \) of length \( K \) from an FIR filter can be generated in \( O(K \log M) \) time instead of \( O(KM) \) by use of fast Fourier transforms (via overlap-add or overlap-save techniques [18]), but the memory requirements are not improved.

In Fig. 2, we plot the \( L_\infty \) (Chebyshev) error, \( \max_\omega |R(\omega) - |H(\omega)|^2| \), as a function of the total filter order \( N + M \), for filters \( H(\omega) \) designed by several techniques. (Here, we employ the Chebyshev error over the entire frequency bandwidth; for other applications, only a subset of the bandwidth may be of interest.) The best method, i.e. smallest \emph{Chebyshev error} for any given-order, is the optimal magnitude-only IIR filter design (implemented using the differential-correction algorithm [12, 21]). The other design methods plotted consist of two FIR and two IIR filter techniques. The two linear-phase FIR filter techniques are (from the Matlab signal-processing toolbox [6, 26]): first, the Parks-McClellan algorithm, which finds the global Chebyshev optimum [18, 19] (black); second, a least-squares FIR optimization (blue), as described in Ref. 20. The two IIR filter designs plotted are: first, a nonlinear conjugate-gradient minimization of a least-squares norm proposed by Ref. 25, and suggested by Ref. 11 for use in generation of Rayleigh-correlated noise (green diamonds); second, another (non-global) optimization technique based on the modified Yule-Walker algorithm in the Matlab.

We suspect that the Chebyshev norm is typically the most appropriate one for physical simulations involving random processes. The reason is that a large error in a narrow bandwidth, which might be allowed by a least-square (\( L_2 \)) norm, could result in a spectral feature that might be mistaken for a spurious physical phenomenon; a large “spike” in error may also interact adversely with nonlinearities in the physics. Nevertheless, in Fig. 3 and Fig. 4, we show two different \( L_2 \) errors, \( L_2(|H| - \sqrt{R}) \) and \( L_2(|H|^2 - R) \), for the same filter designs as in Fig. 2, and the results demonstrate that the Chebyshev-optimal IIR filter is at least comparable, and often superior to, the other methods, even in norms that it does not strictly optimize. In particular, the \( L_2(|H|^2 - R) \) norm of the Chebyshev-optimum IIR filter does better than all other local optimization techniques for most \( N + M \). Further gains could potentially be made, if this is the desired error.
ACKNOWLEDGEMENTS

We are grateful to Alan V. Oppenheim, Stephen Boyd, and Almir Mutapcic for helpful discussions. This work was supported in part by the Materials Research Science and Engineering Center program of the National Science Foundation under award DMR 02-13282, by a Department of Energy (DOE) Computational Science Fellowship under grant DE-FG02-97ER25308, and also by the Paul E. Gray Undergraduate Research Opportunities Program Fund at MIT.

* Electronic address: alexrod7@mit.edu

[1] R. F. Fox, I. R. Gatland, R. Roy, and G. Venuri, “Fast, accurate algorithm for numerical simulation of exponentially correlated colored noise,” Phys. Rev. A, vol. 38, pp. 5938–5940, 1988.
[2] K. Y. Billah and M. Shinozuka, “Numerical method for colored-noise generation and its application to a bistable system,” Phys. Rev. A, vol. 42, pp. 7492–7495, 1990.
[3] J. N. Kasdin, “Discrete simulation of colored noise and stochastic processes and 1/f<sup>a</sup> power law generation,” Proc. IEEE, vol. 83, no. 5, pp. 802–827, 1995.
[4] C. Luo, A. Narayanaswamy, G. Ghen, and J. D. Joannopoulos, “Thermal radiation from photonic crystals: A direct calculation,” Phys. Rev. Lett., vol. 93, no. 21, p. 213905, 2004.
[5] A. Traulsen, K. Lippert, and U. Behn, “Generation of spatiotemporal correlated noise in 1 + 1 dimensions,” Phys. Rev. E, vol. 69, p. 026116–9, 2004.
[6] Y. K. Chan and R. W. Edsinger, “A correlated random numbers generator and its use to estimate false alarm rates of airplane sensor failure detection algorithms,” IEEE Trans. Automatic Control, vol. 26, pp. 676–680, 1981.
[7] R. Manella, “Fast and precise algorithm for computer simulation of stochastic differential equations,” Phys. Rev. A, vol. 40, pp. 3381–3386, 1989.
[8] J. Garcia-Ojalvo, “Generation of spatiotemporal colored noise,” Phys. Rev. A, vol. 46, pp. 4670–4675, 1992.
[9] D. J. Young and N. C. Beaulieu, “The generation of correlated Rayleigh random variates by inverse discrete fourier transform,” IEEE Trans. Comm., vol. 48, pp. 1114–1128, 2000.
[10] K. Lu and J.-D. Bao, “Numerical simulation of generalized langevin equation with arbitrary correlated noise,” Phys. Rev. E, vol. 72, pp. 067701–4, 2005.
[11] C. Komninakis and J. F. Kirshman, “Fast rayleigh fading simulation with an IIR filter and polyphase interpolation,” Satellite Communications, pp. 24–34, 2004.
[12] D. E. Dudgeon, “Recursive filter design using differential correction,” IEEE Trans. Acoust., Speech and Sig. Proc., vol. 22, no. 6, pp. 443–449, 1974.
[13] L. R. Rabiner, N. Y. Graham, and H. D. Helms, “Linear programming design of IIR digital filters with arbitrary magnitude function,” IEEE Trans. Acoust. Speech and Sig. Proc., vol. 22, no. 2, pp. 117–124, 1974.
[14] J. P. Brockwell and A. D. Davis, Introduction to Time Series and Forecasting. New York, NY: Springer, 2002.
[15] S. H. Wio and R. Toral, “Effect of non-Gaussian noise source in a noise induced transition,” Physica D: Nonlinear Phenomena, vol. 193, pp. 161–168, 2003.
[16] F. Reif, Fundamentals of Statistical and Thermal Physics. McGraw-Hill, 1956.
[17] Baddour and Beaulieu, “Autoregressive models for fading channel simulation,” IEEE Trans. Wireless Comm., vol. 4, no. 4, p. 1650, 2005.
[18] A. V. Oppenheim and R. A. Schafer, Discrete-Time Signal Processing: Second Edition. Englewood Cliffs, NJ: Prentice-Hall, 1999.
[19] L. R. Rabiner, J. H. McClellan, and T. W. Parks, “FIR digital filter design techniques using weighted Chebyshev approximation,” Proc. of the IEEE, vol. 63, no. 4, pp. 595–612, 1975.
[20] T. W. Parks and C. S. Burrus, Digital Filter Design. New York: John Wiley and Sons, 1987.
[21] S. Crosara and G. A. Mian, “A note on the design of IIR filters by the differential-correction algorithm,” IEEE Trans. Circ. and Sys., vol. 30, no. 12, pp. 989–996, 1983.
[22] A. Alkhairy, “Design of optimal IIR filters with arbitrary magnitude,” IEEE Trans. Circ. and Sys. II, vol. 42, pp. 618–620, 1995.
[23] X. Zhang and H. Iwakura, “Design of IIR digital filters based on eigenvalue problem,” IEEE Trans. Sig. Proc., vol. 44, no. 6, pp. 1325–1334, 1996.
[24] D. Gorinevsky and S. Boyd, “Optimization-based design and implementation of multi-dimensional zero-phase IIR filters.”
[25] K. Steiglitz, “Computer-aided design of recursive digital filters,” *IEEE Trans. Audio and Electroacoustics*, vol. 18, no. 2, pp. 123–130, 1970.

[26] B. Friedlander and B. Porat, “The modified yule-walker method of ARMA spectral estimation,” *IEEE Trans. Circ. Syst. I*, vol. 53, no. 2, pp. 372–383, 2006.

[27] In the specific problem considered in Ref. [4], the correlation function could be renormalized to an uncorrelated sequence due to the linearity of the system, but this trick would not work in more general nonlinear or nonequilibrium systems.