a-T-menability of Baumslag-Solitar groups

Świątosław R. Gal
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław
http://www.math.uni.wroc.pl/~sgal/

Tadeusz Januszkiewicz
Instytut Matematyczny Polskiej Akademii Nauk
and
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław
http://www.math.uni.wroc.pl/~tjan/

Abstract

The Baumslag-Solitar groups are a-T-menable. This is proved by embedding them into topological groups and studying representation theoretic properties of the latter.

The paper is motivated by the questions of A. Valette.

Topological groups and approximation properties.

We investigate some approximation properties of not necessarily discrete groups. We are mainly interested in Baumslag-Solitar groups which are discrete. However in the process topological groups (Lie groups and the automorphism group of a tree) arise. Therefore in first two sections we clarify some concepts that have been used so far only in context of discrete groups.

Definition 1 (M. Gromov). A locally compact, second countable, compactly generated group $G$ is a-T-menable iff there exists a metrically proper affine isometric action of $G$ on some Hilbert space.

a-T-menability is often referred to as the Haagerup approximation property. It is equivalent to existence of an approximate identity consisting of $C_0$ positive definite functions on $G$. For ample discussion one may consult [CCJJV].

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BS-groups

Let $G \subset \mathfrak{N}$ be a closed subgroup of a locally compact compactly generated topological group $\mathfrak{N}$. Let $i_k: H \to G$ ($k = 1, 2$) be two inclusions onto finite index open subgroups, which are conjugated by an automorphism $\phi$ of $\mathfrak{N}$. The main case of interest is when $G$ is discrete and $\mathfrak{N}$ is Lie group.

**Definition 3.** The $\mathfrak{N}$-BS group $\Gamma$ is the group derived from $(G, H, i_1, i_2)$ by the (topological) HNN construction. In other words, if $G$ is given by the presentation $< S | R >$, $\Gamma$ has the presentation $< S, t | R, ti_1(g)t^{-1} = i_2(g) \forall g \in H >$.

Since we are working with topological presentations, recall that the topology in the HNN extension $\Gamma$ is given by the prebasis $\mathcal{B} = \{ \gamma U \gamma': U \text{ open in } G, \gamma, \gamma' \in \Gamma \}$. It is clear, that $\Gamma$ is a topological group w.r.t. $\mathcal{B}$.

To see that $\mathcal{B}$ is in fact a basis consider $U$ open in $G$. We can decompose $U$ with respect to $i_1(H)$ as follows $U = \bigcup i_1(U_n)g_n$, where $g_n$ are representatives of cosets of $i_1(H)$. Then $tU = \bigcup i_2(U_n)tg_n$. Therefore any set of $\mathcal{B}$ can be written uniquely in form $\bigcup U_{\gamma} \gamma'$ where $U_{\gamma}$ runs over open subsets of $G$ and $\gamma$ runs over the chosen set of representatives of $G \setminus \Gamma$. Thus $\mathcal{B}$ is closed under the intersections, in particular is a basis. The same argument shows that the topology on $G$ induced from $\Gamma$ concides with the original one (ie $G \subset \Gamma$ is an open embedding).

**Examples:**

1. ($\mathfrak{N} = \mathbb{R}$) Baumslag-Solitar group with parameters $p$ and $q$ is given by a following presentation $BS_p^q = \langle x, t | x^p = tx^q t^{-1} \rangle = HNN(\mathbb{Z}, \mathbb{Z}, p, q)$.

2. ($\mathfrak{N} = \mathbb{R}^n$) Torsion free, finitely generated, abelian-by-cyclic groups are exactly ascending $(i_1 = id)$ HNN extensions of $\mathbb{Z}^n$ with $i_2$ given by a $n$ by $n$ matrix with nonzero determinant $[BS]$.

3. $\mathfrak{N}$ a homogeneous nilpotent group (i.e. one admitting a dilating automorphism $\phi$) with a discrete subgroup $G \subset \mathfrak{N}$ such that $\phi(G) \subset G$.

An obvious adaptation of the Bass-Serre theory [S] to the topological context shows that for a topological HNN extension $\Gamma$ of a group $G$ there is a tree $T$ with an edge-transitive $\Gamma$-action such that the vertex stabilizers are conjugated to $G$ and edge stabilizers are conjugated to $H$.

Condition, that $i_k(H)$ are of finite index in $G$ ensures, that $T$ is locally finite. The simplicial automorphism group $\text{Aut}(T)$ carries the natural (compact-open) topology with the basis of neighborhoods of the identity $U_K = \{ g | gv = v \ \forall v \in K \}$ where $K$ runs over the family of the compact subsets $K \subset T$. Denote $j_T: \Gamma \to \text{Aut}(T)$ the homomorphism given by the action.

Define $\tilde{\mathfrak{N}} = \mathbb{Z} \ltimes \mathfrak{N}$, to be the semidirect product given by the $\mathbb{Z}$ action on $\mathfrak{N}$ via $\phi$. There is an obvious homomorphism $j_\mathfrak{N}: \Gamma \to \tilde{\mathfrak{N}}$, which is the identity on $G$ and sends $t$ to the generator of $\mathbb{Z}$.

**Theorem 1.** Let $\Gamma$ be $\mathfrak{N}$-BS group as in Definition 3, and let $j_T$ and $j_\mathfrak{N}$ be the homomorphisms defined above. Then the homomorphism $j = (j_T, j_\mathfrak{N}): \Gamma \to \text{Aut}(T) \times \tilde{\mathfrak{N}}$ is an embedding onto a closed subgroup.
Proof: Observe that since $j_\mathcal{R}$ restricted to $G$ is an embedding onto a closed subgroup (this follows from the fact that $G$ in closed subgroup in $\mathcal{R}$) the same is true for $j$ restricted to $G$. Moreover, if $v$ is the vertex stabilized by $G$, then $j_T(G) = \text{Stab}_v \cap j_T(\Gamma)$.

Let $j(g) = 1$. Since $j_T(g) = 1$, $gv = v$ i.e $g \in G$. Since $j_\mathcal{R}$ restricted to $G$ is embedding, $g = 1$.

Let $\gamma \notin j(\Gamma)$. Since $\Gamma$ acts transitively on $T$, we can multiply $\gamma$ by some element of $j(\Gamma)$ and assume that $\gamma v = v$. Then $(\text{Stab}_v - G) \times \mathcal{R}$ is an open neighbourhood of $\gamma$ omitting $j(\Gamma)$. So $j(\Gamma)$ is closed.

Topologies on $G$ and $j(G)$ coincide. Since $G$ and $j(G)$ are open in $\Gamma$ and $j(\Gamma)$ respectively, the same is true for $\Gamma$ and $j(\Gamma)$.

**Corollary 1.** If $\mathcal{R}$ is a-T-menable then so are $\mathcal{R}$-BS groups.

Proof: The group $\text{Aut}(T)$ is a-T-menable by a result of Haagerup [Ha].

A cyclic extension of a-T-menable group is a-T-menable by [J]. Therefore $\tilde{\mathcal{R}}$ is a-T-menable. It is clear that a product of a-T-menable groups is also a-T-menable and that a closed subgroup of a-T-menable group is also a-T-menable. Thus the corollary follows from Theorem 1. □

**Remark 1.** According to Chapter 3 of [CCJJV] every a-T-menable connected Lie group $\mathcal{R}$ is locally isomorphic to a direct product of an amenable group and copies of $SO(1,n)$ and $SU(1,n)$. A Lie group is amenable if it is compact-by-solvable. Thus the examples 1.-3. are a-T-menable.

**Remark 2.** Another proof of Corollary 1, relying on the study of a-T-menable of amalgams, is given in a forthcoming paper by the first author.

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