Hölder estimates for magnetic Schrödinger semigroups on open subsets of $\mathbb{R}^d$ from mirror coupling

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Abstract

We use the mirror coupling of Brownian motion to show that under a $\beta \in (0, 1)$-dependent Kato type assumption on the possibly nonsmooth electro-magnetic potential, the corresponding magnetic Schrödinger semigroup on an open subset of $\mathbb{R}^d$ has a global $L^p$-to-$C^{0, \beta}$ Hölder smoothing property for all $p \in [1, \infty]$, in particular all eigenfunctions are globally $\beta$-Hölder continuous. This results shows that the eigenfunctions of the Hamilton operator of molecule in a magnetic field are globally $\beta$-Hölder continuous under weak $L^q$-assumptions on the magnetic potential.

1 Introduction

Kato [Kat] has shown that all eigenfunctions of a multi-particle Schrödinger operator $H = -\Delta + W$ in $L^2(\mathbb{R}^{3m})$ with a potential $W : \mathbb{R}^{3m} \to \mathbb{R}$ of the form

$$W(x) = \sum_{1 \leq j \leq m} w_j(x_j) + \sum_{1 \leq j < k \leq m} w_{jk}(x_j - x_k),$$

with $w_j, w_{jk} \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $p \geq 2$

are globally $\beta$-Hölder continuous for all $0 < \beta < 2 - 3/p$. In particular, an application of this result to multi-particle Coulomb type potentials shows that all molecular Hamilton operators (in the infinite mass limit) are globally $\alpha$-Hölder continuous for all $0 < \alpha < 1$. Kato’s proof relies on the Fourier transform and so does not apply directly to Schrödinger operators that are defined on open subsets of $\mathbb{R}^{3m}$ or to magnetic Schrödinger operators (even if one assumes a Coulomb gauge). The aim of this paper is to use probabilistic techniques to find a variant of Kato’s regularity result that applies to the Dirichlet realization

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1 which is satisfied under a suitable $L^q$-assumption on the electro-magnetic potential, where $q$ depends on $\beta$ and the dimension $d$
$H_{\Lambda}(A,V)$ of the magnetic Schrödinger operator with magnetic potential $A : \mathbb{R}^d \to \mathbb{R}^d$ and electric potential $V : \mathbb{R}^d \to \mathbb{R}$, which is defined on an open connected subset $\Lambda \subset \mathbb{R}^d$. To this end we to prove the following result (cf. Theorem 2.5):

Let $\beta \in (0,1)$ and let $C^{0,\beta}(\Lambda)$ denote the semi-normed space of globally $\beta$-Hölder continuous functions on $\Lambda$. Then for all Borel functions $A : \mathbb{R}^d \to \mathbb{R}^d$, $V : \mathbb{R}^d \to \mathbb{R}$ with

$$\max \left( |A|^{1/\beta}, |\text{div}(A)|^{1/\beta} \right) \in \mathcal{K}(\mathbb{R}^d), \quad V \in \mathcal{K}^\beta(\mathbb{R}^d),$$

and all $t > 0$, $p \in [1,\infty]$ one has

$$e^{-tH_{\Lambda}(A,V)} : L^p(\Lambda) \longrightarrow C^{0,\beta}(\Lambda),$$

and the semi-norm of this operator can be estimated explicitly.

Above, $\mathcal{K}^\beta(\mathbb{R}^d)$, $\beta \in [0,1]$, denotes the $\beta$-Kato class (cf. Definition 2.1 below) of Borel functions $\mathbb{R}^d \to \mathbb{R}$ which has been introduced in [Gün2], so that $\mathcal{K}(\mathbb{R}^d) := \mathcal{K}^0(\mathbb{R}^d)$ is the classical Kato class [AizSim] and one has $\mathcal{K}^\beta(\mathbb{R}^d) \subset \mathcal{K}^\alpha(\mathbb{R}^d)$ if $\beta \geq \alpha$. Note also that $H_{\Lambda}(A,V)\Psi = \theta\Psi$ implies $e^{-tH_{\Lambda}(A,V)}\Psi = e^{t\theta}\Psi$, so that one also obtains global $\beta$-Hölder regularity for eigenfunctions.

The proof of (1) uses Brownian mirror coupling techniques (cf. Section 2 for the basic definitions) to deal with the magnetic potential $A$. More precisely, (cf. Theorem 2.3 below), we show:

There exists a universal constant $C < \infty$, such that for every $q \in (1,\infty)$, every Borel function $A : \mathbb{R}^d \to \mathbb{R}^d$ with

$$\max \left( |A|^{2q}, |\text{div}(A)|^q \right) \in \mathcal{K}(\mathbb{R}^d),$$

every $t > 0$, $x \neq y$ in $\mathbb{R}^d$, and every mirror coupling $(X,Y)$ of Brownian motions from $(x,y)$ one has

$$\mathbb{E} \left( |e^{-\mathcal{J}_t(A|X)} - e^{-\mathcal{J}_t(A|Y)}| \right) \leq C(A,t,q)t^{-\frac{1}{2q^*}} |x-y|^\frac{3}{2q^*},$$

where for any Brownian motion $Z$, the process $\mathcal{J}_t(A|Z)$ denotes the magnetic Euclidean action functional (cf. (6) below) which appears in the Feynman-Kac-Itô formula and where the constant $C(A,t,q) < \infty$ can be computed explicitly.

This estimate is then combined with the Feynman-Kac-Itô formula (and perturbation theory to deal with $V$) to finally obtain (1).

Using $L^p$-criteria for the $\beta$-Kato class, we show that this result directly implies the following generalization of Kato’s result for multi-particle Schrödinger operators in $\mathbb{R}^{3n}$ to magnetic multi-particle Schrödinger operators in open subsets of $\mathbb{R}^{3n}$:

Assume there exists $\beta \in (0,1)$, $l \in \mathbb{N}$ and Borel functions $a : \mathbb{R}^3 \to \mathbb{R}^3$, $v_i, v_{ij} : \mathbb{R}^3 \to \mathbb{R}$ with

$$|a|^{2/(1-\beta)}, |\text{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > 3/2, \quad (3)$$

$$v_i, v_{ij} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > \frac{3}{2(1-\beta/2)}. \quad (4)$$
and define a vector potential, resp. a magnetic potential on $\mathbb{R}^{3n}$ through

$$A(x) := \sum_{i=1}^{n} a(x_i), \quad V(x) = \sum_{1 \leq i < j \leq n} v_{ij}(x_i - x_j) + \sum_{i=1}^{n} v_i(x_i).$$

Then, given an open connected subset $\Lambda \subset \mathbb{R}^{3n}$, for all $t > 0$ and $p \in [1, \infty]$ one has

$$e^{-tH_{A,V}} : L^p(\Lambda) \to C^{0,\beta}(\Lambda).$$

To the best of our knowledge, this is the first global Hölder-regularity result for multi-particle magnetic Schrödinger operators that are defined on open subsets. Let us finally explain how this result applies to molecules in a magnetic field: Given $R \in \mathbb{R}^{3n}$, $l \in \mathbb{N}$, $Z \in [0, \infty)$, consider the potential

$$V_{R,Z} : \mathbb{R}^{3n} \to \mathbb{R}, \quad V_{R,Z}(x_1, \ldots, x_n) := -\sum_{i=1}^{n} \sum_{j=1}^{l} \frac{Z_j}{|x_i - R_j|} + \sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j|}.$$ 

Given $a : \mathbb{R}^3 \to \mathbb{R}^3$ (sufficiently well-behaved) set as above $A(x) := \sum_{i=1}^{n} a(x_i)$. Then the operator

$$H_{A,R,Z} := H_{\mathbb{R}^{3n}}(A, V_{R,Z})$$

is the Hamilton operator corresponding to a molecule (in the infinite mass limit) with $l$ protons and with $n$ electrons, where the $j$-th nucleus is located in $R_j$ and has a $\sim Z_j$ protons, and the electrons interact with the magnetic field induced by $A$. Then given an arbitrary $\beta \in (0, 1)$ one has (4) for

$$v_{ij}(x) := 1/|x|, \quad v_i(x) := -\sum_{j=1}^{l} \frac{Z_j}{|x - R_j|},$$

so that the previous result gives that for all $t > 0$ and $p \in [1, \infty]$ one has

$$e^{-tH_{\mathbb{R}^{3n}}(A,V)} : L^p(\mathbb{R}^{3n}) \to C^{0,\beta}(\mathbb{R}^{3n}),$$

as long as

$$|a|^{2/(1-\beta)}, |\text{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^{\infty}(\mathbb{R}^3) \quad \text{for some} \ s > 3/2.$$ 

## 2 Main results

We start by recalling the definition of the mirror coupling of Brownian motions as presented in [HsuStu] and follow their exposition (pages 1-3 therein) closely before presenting our main results. A continuous process $(X,Y)$ with values in $\mathbb{R}^d \times \mathbb{R}^d$ is called a coupling of Brownian motions
from \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\), if \(X\) and \(Y\) are Brownian motions starting in \(x\) and \(y\), respectively. Then, with the coupling time
\[
\tau(X, Y) := \inf\{t > 0 : X_s = Y_s \text{ for all } t > s\},
\]
the coupling \((X, Y)\) is said to be \textit{maximal}, if for all \(t > 0\) one has
\[
P(\tau(X, Y) \geq t) = 1 - \frac{1}{2} \int_{\mathbb{R}^d} |\rho(t, x, z) - \rho(t, y, z)| \, dz,
\]
with
\[(t, b) \mapsto \rho(t, a, b) = (2\pi t)^{-d/2} e^{-\frac{|a-b|^2}{2t}} \]
the transition density of Brownian motion starting in \(a\). The reason for this notion of maximality is that for an arbitrary coupling of Brownian motions one has \(\leq\) in (5).

Let \(x\) and \(y\) be two distinct points of \(\mathbb{R}^d\). Then
\[
N_{x,y} := \{v \in \mathbb{R}^d : \langle v - (x + y)/2, (x - y) |x - y|^{-1}\rangle = 0\},
\]
is the hyperplane orthogonal on and bisecting the segment \(xy\). Furthermore define the affine map
\[
R_{x,y} : \mathbb{R}^d \to \mathbb{R}^d, \quad R_{x,y}v := v - 2 \langle v - (x + y)/2, (x - y) |x - y|^{-1}\rangle (x - y) |x - y|^{-1}.
\]
This is the reflection at the hyperplane \(N_{x,y}\). Let \(L_{x,y}\) be the linear part of \(R_{x,y}\). Note that \(L_{x,y}\) is self-adjoint and idempotent.

A coupling \((X, Y)\) of Brownian motions from \((x, y)\) is called a \textit{mirror coupling}, if
\[
Y_t = \begin{cases} 
R_{x,y}X_t, & t \in [0, \tau_{x,y}(X)], \\
X_t, & t \in (\tau_{x,y}(X), \infty), 
\end{cases}
\]
where
\[
\tau_{x,y}(X) := \inf\{t \geq 0 : X_t \in N_{x,y}\}
\]
is the hitting time of \(X\) with respect to \(N_{x,y}\). In other words, \(Y\) is equal to \(X\) before \(X\) hits \(N_{x,y}\), and is then equal to the reflection of \(X\) at \(N_{x,y}\). It follows that \(\tau(X, Y) = \tau_{x,y}(X)\), which by an explicit calculation of \(P(t \leq \tau_{x,y}(X))\) implies that every mirror coupling is maximal.

Whenever well-defined, we consider the following action functional on the paths of any Brownian motion \(Z\), which depends on a sufficiently regular function \(A : \mathbb{R}^d \to \mathbb{R}^d\):
\[
\mathcal{I}_t(A|Z) := \sqrt{-1} \int_0^t \langle A(Z_s), dZ_s \rangle + \frac{\sqrt{-1}}{2} \int_0^t \text{div}(A)(Z_s) \, ds, \quad t \geq 0.
\]
Above, \(\text{div}(A)\) denotes the divergence of \(A\) (in general, understood in the distributional sense) and the stochastic integral is understood in Itô’s sense.

Let \(P_a\) denote the law of Brownian motion starting in \(a\), which is considered as a probability measure on the space of continuous paths \(\omega : [0, \infty) \to \mathbb{R}^d\). Generalizing the Kato class, the following hierarchy of Kato classes has been introduced in Günl2.
Definition 2.1. Given $\alpha \in [0, 1]$, a Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be in the $\alpha$-Kato class $K^\alpha(\mathbb{R}^d)$, if
\[ \lim_{t \to 0^+} \sup_{z \in \mathbb{R}^d} \int_0^t s^{-\alpha/2} \int \left| f(\omega(s))\right| P_z(d\omega) ds = 0. \]

Remark 2.2. 1) Each $K^\alpha(\mathbb{R}^d)$ is a linear space and $K(\mathbb{R}^d) := K^0(\mathbb{R}^d)$ is the usual Kato class.
2) One has $K^\alpha(\mathbb{R}^d) \subset K^\beta(\mathbb{R}^d)$, if $\alpha \geq \beta$, (7)
3) For all $q \in [1, \infty)$ with $q > d/(2 - \alpha)$ one has $L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subset K^\alpha(\mathbb{R}^d)$.
4) For every linear surjective map $\pi : \mathbb{R}^D \to \mathbb{R}^d$ and every $f \in K^\alpha(\mathbb{R}^d)$ one has $f \circ \pi \in K^\alpha(\mathbb{R}^d)$, cf. \cite{Gun2}.
5) For every $W \in K(\mathbb{R}^d)$, $z \in \mathbb{R}^d$, $t > 0$ one has
\[ \int_0^t |W(\omega(s))| ds < \infty \quad \mathbb{P}_z \text{ a.s. for all } t > 0, \]
and if $W \in K^\beta(\mathbb{R}^d)$, then also \cite{Gun2}
\[ \sup_{z \in \mathbb{R}^d} \int_0^t s^{-\beta/2} \int \left| W(\omega(s))\right| P_z(d\omega) ds < \infty \quad \text{for all } t > 0. \]

The following probabilistic estimate is our main technical result:

Theorem 2.3. There exists a universal constant $C < \infty$, such that for every $q \in (1, \infty)$, every Borel function $A : \mathbb{R}^d \to \mathbb{R}^d$ with
\[ \max \left( |A|^{2q}, |\text{div}(A)|^q \right) \in K(\mathbb{R}^d), \]
every $t > 0$, $x \neq y$ in $\mathbb{R}^d$, and every mirror coupling $(X, Y)$ of Brownian motions from $(x, y)$ one has
\[ \mathbb{E} \left( |e^{-\mathcal{H}_t(A,X)} - e^{-\mathcal{H}_t(A,Y)}| \right) \leq C(A,t,q) t^{-\frac{1}{2q^*}} |x - y|^{\frac{q}{q^*}}, \]
where $1/q^* + 1/q = 1$ and
\[
C(A,t,q) := \left( \sup_{z \in \mathbb{R}^d} \int_0^t \int |A(\omega(s))|^{2q} P_z(d\omega) ds \right)^{\frac{1}{q'}} + \left( \sup_{z \in \mathbb{R}^d} \int_0^t \int \frac{\sqrt{1-1}}{2} |\text{div}(A)(\omega(s))|^q P_z(d\omega) ds \right)^{\frac{1}{q}} < \infty.
\]
Remark 2.4. 1) As every Kato function is locally integrable (cf. Lemma VI.5 c) in [Gün], \( \text{div}(A) \) exists as a distribution in the above situation.

2) Remark 2.2.5) easily shows that \( C(A, t, q) < \infty \) under the assumptions of Theorem 2.3 and that \( \int_0^t A(Z_s) \, dZ_s \) is a continuous \( L^2 \)-martingale for every Brownian motion \( Z \) having a deterministic initial value. In particular, the process \( \mathcal{F}(A|Z) \) is a continuous semimartingale.

3) The function \( t \mapsto C(A, t, q) \) is locally bounded under the assumptions of Theorem 2.3: the easiest way to see this is to refer to Khashminiski’s lemma, which implies that for every \( W \in \mathcal{K}(\mathbb{R}^d) \) one has

\[
\sup_{z \in \mathbb{R}^d} \int \int_0^t W(\omega(s)) \, ds \, \mathbb{P}_z(d\omega) \leq C_W e^{C_W t} \text{ for all } t > 0,
\]

and so trivially

\[
\sup_{z \in \mathbb{R}^d} \int \int_0^t W(\omega(s)) \, ds \, \mathbb{P}_z(d\omega) \leq C_W e^{C_W t} \text{ for all } t > 0.
\]

Proof of Theorem 2.3. Let \( x \neq y \) in \( \mathbb{R}^d \) and \( t > 0 \) be fixed. We set

\[
\tau : = \tau(X, Y) = \tau_{x,y}(X), \quad L := L_{x,y}, \quad R := R_{x,y}.
\]

Given a Brownian motion \( Z \) we split

\[
\mathcal{F}_t(Z) := \mathcal{F}_t(A|Z)
\]

into

\[
\mathcal{F}_t(Z) = \mathcal{F}_t^{\text{Itô}}(Z) + \mathcal{F}_t^{\text{Leb}}(Z),
\]

\[
\mathcal{F}_t^{\text{Itô}}(Z) := \int_0^t \langle A(Z_s), dZ_s \rangle,
\]

\[
\mathcal{F}_t^{\text{Leb}}(Z) := \frac{\sqrt{-1}}{2} \int_0^t \text{div}(A) (Z_s) \, ds.
\]

Clearly we a.s. have

\[
I_t := \int_0^t \mathbb{1}_{s < \tau} \left( \frac{\sqrt{-1}}{2} \text{div}(A) (X_s) - \frac{\sqrt{-1}}{2} \text{div}(A) (RX_s) \right) \, ds
\]

\[
= \mathcal{F}_t^{\text{Leb}}(X) - \mathcal{F}_t^{\text{Leb}}(Y).
\]

Likewise, heuristically, for \( s < \tau \) one has \( dY_s = LdX_s \) and while for \( s \geq \tau \) one has \( dY_s = dX_s \), and we therefore expect that

\[
\mathcal{F}_t^{\text{Itô}}(X) - \mathcal{F}_t^{\text{Itô}}(Y) = M_t. \tag{12}
\]
holds a.s., where
\[
\tilde{A}(x) := A(x) - LA(Rx),
\]
\[
M_t := \int_0^t 1_{\{s < \tau\}} \langle \tilde{A}(X_s), dX_s \rangle.
\]

To show that equation (12) holds, by replacing
\[
A = (A_1, \ldots, A_d)
\]
with the sequence
\[
A_n := (\max(A_1, n), \ldots, \max(A_d, n)), \quad n \in \mathbb{N},
\]
and using the Itô-isometry and dominated convergence, we can assume that \(A\) is bounded. By Theorem 6.5 in [Ste] we have the \(L^2\)-convergence of the dyadic approximations
\[
\mathcal{S}_{t}^{\text{Itô}}(X) - \mathcal{S}_{t}^{\text{Itô}}(Y) = \lim_{n \to \infty} \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \left( \langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du,
\]
\[
M_t = \lim_{n \to \infty} \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} 1_{\{u < \tau\}} \langle \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle du,
\]
where \(t_i := \frac{it}{2^n}\) for \(i = 0, \ldots, 2^n\). We immediately note that in case \(t < \tau\), we have \(Y_s = RX_s\) on \([0, t]\), hence in that case by the above limits we conclude that:
\[
\mathcal{S}_{t}^{\text{Itô}}(X) - \mathcal{S}_{t}^{\text{Itô}}(Y) = M_t, \quad \text{for } t < \tau.
\]

If we now assume that \(\tau \in (t_k, t_{k+1}]\) for some \(k = 0, \ldots, 2^n - 1\), we get the following expressions for the summands in the above limits:

For \(i \leq k - 1\):
\[
\int_{t_{i-1}}^{t_i} \left( \langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du
\]
\[
= \int_{t_{i-1}}^{t_i} \left( \langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(RX_u), RX_{t_{i+1}} - RX_{t_i} \rangle \right) du
\]
\[
= \int_{t_{i-1}}^{t_i} \langle \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle du.
\]

In the last step we used that \(L\) is selfadjoint and \(Rv - Rw = L(v - w)\).

For \(i = k\):
\[
\int_{t_{k-1}}^{t_k} \left( \langle A(X_u), X_{t_{k+1}} - X_{t_k} \rangle - \langle A(Y_u), Y_{t_{k+1}} - Y_{t_k} \rangle \right) du
\]
\[
= \int_{t_{k-1}}^{t_k} \left( \langle A(X_u), X_{t_{k+1}} - X_{t_k} \rangle - \langle A(RX_u), X_{t_{k+1}} - RX_{t_k} \rangle \right) du.
\]
For $i = k + 1$:
\[
\int_{t_k}^{t_{k+1}} \left( \langle A(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle - \langle A(Y_u), Y_{t_{k+2}} - Y_{t_{k+1}} \rangle \right) du
\]
\[
= \int_{t_k}^{t_{k+1}} \langle A(X_u) - A(RX_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du,
\]
\[
= \int_{t_k}^{t_{k+1}} \langle 1_{\{u < \tau\}} \tilde{A}(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du
\]
\[
= \int_{t_k}^{\tau} \langle A(X_u) - LA(RX_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du.
\]

For $i \geq k + 2$ the summands vanish. Compiling these equations allows us to make the following estimates,
\[
\mathbb{E} \left( |\mathcal{S}_t^{\text{Itô}}(X) - \mathcal{S}_t^{\text{Itô}}(Y) - M_t|^2 \right) = \mathbb{E} \left( 1_{\{t \geq \tau\}} |\mathcal{S}_t^{\text{Itô}}(X) - \mathcal{S}_t^{\text{Itô}}(Y) - M_t|^2 \right)
\]
\[
\leq \limsup_{n \to \infty} \mathbb{E} \left( 1_{\{t \geq \tau\}} \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \left( \langle A(X_u) - 1_{\{s < \tau\}} \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du \right)^2
\]
\[
= \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left( 1_{\{\tau \in (t_k, t_{k+1})\}} \left| \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \left( \langle A(X_u) - 1_{\{s < \tau\}} \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du \right|^2 \right)
\]
\[
= \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left( 1_{\{\tau \in (t_k, t_{k+1})\}} \left| \int_{t_k}^{t_{k+1}} \langle A(RX_u), (R - 1) X_{t_{k+1}} \rangle du \right| \right)^2
\]
\[
+ \left( \int_{t_k}^{\tau} \langle \gamma(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du \right)^2,
\]
where $\gamma(z) := LA(Rz) - A(Rz)$. Note that
\[
(R - 1)X_{t_{k+1}} = L(X_{t_{k+1}} - X_\tau) - (X_{t_{k+1}} - X_\tau),
\]
because $RX_\tau = X_\tau$. In particular, since $L$ is self-adjoint and idempotent,
\[
\left| (R - 1)X_{t_{k+1}} \right|^2 = \left| X_{t_{k+1}} - X_\tau \right|^2.
\]
Since $A$ is bounded by some $\kappa > 0$, and so $|\gamma| \leq 2\kappa$, using
\[
(a + b)^2 \leq 2a^2 + 2b^2, \quad t_{j+1} - t_j = \frac{t}{2^n},
\]

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we can thus estimate as follows,

\[
\mathbb{E} \left( | \mathcal{I}_t^\top (X) - \mathcal{I}_t^\top (Y) - M_t |^2 \right) \\
\leq \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \frac{2^{n+2}}{t^2} \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left( \int_{t_{k-1}}^{t_k} |A(RX_u)| \left( (R - \mathbb{1}) X_{t_k} \right) du \right)^2 \right) \\
+ \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \frac{2^{n+2}}{t^2} \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left( \int_{t_k}^{t} |\gamma(X_u)| |X_{t_{k+2}} - X_{t_{k+1}}| du \right)^2 \right) \\
\leq \limsup_{n \to \infty} 4\kappa^2 \sum_{k=0}^{2^n-1} \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left( (R - \mathbb{1}) X_{t_{k+1}} \right)^2 \right) \\
+ 16\kappa^2 \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left| X_{t_{k+2}} - X_{t_{k+1}} \right|^2 \right) \\
\leq 16\kappa^2 \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \left( \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left| X_{t_{k+2}} - X_{t_{k+1}} \right|^2 \right) + \mathbb{E} \left( \mathbb{1}_{\{ \tau \in (t_k, t_{k+1}) \}} \left| X_{t_{k+1}} - X_{t} \right|^2 \right) \right).
\]

Since \( \tau \) is an \( X \)-stopping time, we conclude by the Markov property of \( X \), using

\[
\int |\omega(r) - \omega(s)|^2 \mathbb{P}_x (d\omega) = r - s, \quad r > s > 0
\]

and once more \( t_{j+1} - t_j = \frac{t}{2^n} \), that

\[
\mathbb{E} \left( | \mathcal{I}_t^\top (X) - \mathcal{I}_t^\top (Y) - M_t |^2 \right) \\
\leq 16\kappa^2 \limsup_{n \to \infty} \sum_{k=0}^{2^n-1} \left( \frac{t}{2^n} \mathbb{P} (\tau \in (t_k, t_{k+1})) + \int_{t_k}^{t_{k+1}} \mathbb{E} \left( \left| X_{t_{k+1}} - X_u \right|^2 | \tau = u \right) \tau_* \mathbb{P} (du) \right) \\
\leq 16\kappa^2 \limsup_{n \to \infty} \frac{t}{2^n} \mathbb{P} (t \geq \tau) + 16\kappa^2 \limsup_{n \to \infty} \frac{t}{2^n} \sum_{k=0}^{2^n-1} \int_{t_k}^{t_{k+1}} \tau_* \mathbb{P} (du) \\
= \limsup_{n \to \infty} \frac{32\kappa^2 t}{2^n} \mathbb{P} (t \geq \tau) = 0.
\]

Alltogether, we have found that under the assumptions of the theorem one has

\[
\mathcal{I}_t (X) - \mathcal{I}_t (Y) = M_t + I_t \quad \text{a.s.} \quad (13)
\]

We are now going to estimate the \( L^1 \)-norms of \( M_t \) and \( I_t \). Let us start with \( \mathbb{E} (|I_t|) \): setting

\[
w := \sqrt{-1} \text{div}(A), \quad p := q^*,
\]

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we have

\[
\mathbb{E} (|I_t|) \leq \int_0^t \mathbb{E} \left( |1_{\{s < \tau\}} (w(X_s) - w(RX_s))| \right) \, ds
\]

\[
\leq \left( \int_0^t \mathbb{P} (s < \tau) \, ds \right)^{\frac{1}{p}} \left( \int_0^t \mathbb{E} \left( |w(X_s) - w(RX_s)|^q \right) \, ds \right)^{\frac{1}{q}}
\]

\[
\leq 2 \left( \int_0^t \mathbb{P} (s < \tau) \, ds \right)^{\frac{1}{p}} C_{w,t,q},
\]

where

\[
C_{w,t,q} := \left( \sup_{x \in \mathbb{R}^d} \int_0^t \int |w(\omega(s))|^q \mathbb{P}_z (d\omega) ds \right)^{\frac{1}{q}}.
\]

In view of

\[
\mathbb{P} (\tau > s) \leq \frac{1}{2} \int_{\mathbb{R}^d} \rho(s, x, z) - \rho(s, y, z) \, dz = \frac{2}{\sqrt{2\pi s}} \int_0^{\frac{|x - y|}{2s}} e^{-\frac{u^2}{2}} \, du
\]

\[
\leq (2\pi)^{-1/2} |x - y| s^{-1/2},
\]

we conclude

\[
\mathbb{E} (|I_t|) \leq CC_{w,t,q} t^{-\frac{1}{2p}} |x - y|^{\frac{1}{p}},
\]

where from here on \( C < \infty \) denotes a universal constant whose value may change from line to line. Now let us turn to the estimate for \( \mathbb{E} (|M_t|) \): define

\[
h(r) := \begin{cases} \frac{1}{12} (2 - |r|)^3 + |r|, & |r| \leq 2, \\ |r|, & |r| > 2. \end{cases}
\]

We note that \( h(r) \geq |r| \) and that \( h \) is in \( C^2 (\mathbb{R}) \) with

\[
h''(r) = \begin{cases} 1 - \frac{1}{2} |r|, & |r| \leq 2, \\ 0, & |r| > 2. \end{cases}
\]

In particular we have \( |h''| \leq 1_{[-2,2]} \) and \( |h'| \leq 1 \). We conclude by Itô’s formula

\[
h(M_t) = \frac{1}{2} \int_0^t h''(M_s) \mathbbm{1}_{\{s < \tau\}} \left| \hat{A}(X_s) \right|^2 \, ds + \tilde{M}_t,
\]

where

\[
\tilde{M}_t = \int_0^t h'(M_s) \mathbbm{1}_{\{s < \tau\}} \langle \hat{A}(X_s), dX_s \rangle.
\]
is an \(L^2\)-martingale, as follows from the assumption on \(A\) and the boundedness of \(h'\). Thus
\[
E(\tilde{M}_t) = E(\tilde{M}_0) = 0,
\]
and we have,
\[
E(|M_t|) \leq E(h(M_t))
\]
\[
= \frac{1}{2}E\left(\int_0^t h''(M_s) \mathbb{1}_{\{s<\tau\}} \left|\tilde{A}(X_s)\right|^2 ds\right)
\]
\[
\leq \frac{1}{2}E\left(\int_0^t \mathbb{1}_{\{s<\tau\}} \left|\tilde{A}(X_s)\right|^2 ds\right)
\]
\[
\leq \frac{1}{2}\left(\int_0^t \mathbb{P}(s<\tau) ds\right)^{\frac{1}{q}} \left(\int_0^t E\left(\left|\tilde{A}(X_s)\right|^{2q}\right) ds\right)^{\frac{1}{q}}
\]
\[
\leq \left(\int_0^t \mathbb{P}(s<\tau) ds\right)^{\frac{1}{q}} \left(\sup_{x\in\mathbb{R}^d} \int_0^t |A(\omega(s))|^{2q} \mathbb{P}_x(\omega) ds\right)^{\frac{1}{q}}.
\]
Hence, similarly to the Lebesgue integral \(I_t\), we conclude
\[
E(|M_t|) \leq CC_{A,t,q} t^{-\frac{1}{2p}} |x-y|^{\frac{1}{p}},
\]
where
\[
C_{A,t,q} := \left(\sup_{x\in\mathbb{R}^d} \int_0^t \int |A(\omega(s))|^{2q} \mathbb{P}_x(\omega) ds\right)^{\frac{1}{q}}.
\]
Thus we have shown
\[
E(|\mathcal{I}_t(X) - \mathcal{I}_t(Y)|) \leq C(C_{w,t,q} + C_{A,t,q}) t^{-\frac{1}{2p}} |x-y|^{\frac{1}{p}}.
\]
Finally, noting that for all \(z, z' \in \mathbb{C} \setminus \mathbb{R}\) one has the elementary estimate
\[
|e^z - e^{z'}| \leq C |z - z'|,
\]
and \(\Re(\mathcal{I}_t(Z)) = 0\), the proof is complete.

If \(A : \mathbb{R}^d \to \mathbb{R}^d\) and \(V : \mathbb{R}^d \to \mathbb{R}\) are Borel functions with
\[
\max \left(\{|A|^2, |\text{div}(A)|, |V|\}\right) \in \mathcal{K}(\mathbb{R}^d),
\]
then given any open connected subset \(\Lambda \subset \mathbb{R}^d\), the symmetric sesquilinear form
\[
(\Psi_1, \Psi_2) \mapsto \frac{1}{2} \int_\Lambda \left((\sqrt{-1}\nabla + A)\Psi_1, (\sqrt{-1}\nabla + A)\nabla \Psi_2\right) + \int_\Lambda V \cdot \nabla_1 \cdot \Psi_2 \in \mathbb{C}
\]
in \(L^2(\Lambda)\) with domain of definition \(C^\infty_c(\Lambda)\) is semibounded from below and closable \([\text{BroHunLes}]\).
Thus, the closure of this form induces a self-adjoint semibounded from below operator.
\[ H_\Lambda(A, V) \] in \( L^2(\Lambda) \), which corresponds to Dirichlet boundary conditions. The corresponding magnetic Schrödinger semigroup is given by the Feynman-Kac-Itô formula \[ e^{-tH_\Lambda(A, V)}\Psi(x) = \mathbb{E}\left(1_{\{t < \zeta_\Lambda(Z(x))\}}e^{-\mathcal{H}_t(A|Z(x)) - \int_0^t V(Z(s))ds}\Psi(Z_t(x))\right), \quad \Psi \in L^2(\Lambda), \]

where \( Z(x) \) is an arbitrary Brownian motion in \( \mathbb{R}^d \) starting in \( x \in \Lambda \) and

\[ \zeta_\Lambda(Z(x)) := \inf\{u \geq 0 : Z_u(x) \notin \Lambda\} \]

its first exit time of from \( \Lambda \). Consider the semi-normed space \( C^{0,\beta}_0(\Lambda) \) of globally \( \beta \)-Hölder functions, given by all \( f : \Lambda \rightarrow \mathbb{C} \) with

\[ \|f\|_{0,\beta} := \sup_{a \neq b} |f(a) - f(b)||a - b|^{-\beta} < \infty. \]

Using the Feynman-Kac-Itô formula for \( V = 0 \) with Theorem 2.3 to deal with the magnetic potential \( A \), and perturbation theory to deal with the electric potential \( V \), we can now establish the first main result of this paper:

**Theorem 2.5.** Let \( \beta \in (0, 1) \), let \( A : \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( V : \mathbb{R}^d \rightarrow \mathbb{R} \) be Borel functions which satisfy

\[ \max\left(\frac{|A|^2}{1 + |A|}, |\text{div}(A)|, \frac{1}{1 + |A|}\right) \in \mathcal{K}(\mathbb{R}^d), \quad V \in \mathcal{K}^\beta(\mathbb{R}^d), \quad (15) \]

let \( \Lambda \subset \mathbb{R}^d \) be an open connected subset, and let \( t > 0, p \in [1, \infty] \). Then one has

\[ e^{-tH_\Lambda(A, V)} : L^p(\Lambda) \rightarrow C^{0,\beta}(\Lambda), \]

and there exists a universal constant \( C < \infty \) and a constant \( C_V < \infty \) which only depends on \( V \), such that

\[ \leq \left\| e^{-tH_\Lambda(A, V)} \right\|_{L^p \rightarrow C^{0,\alpha}} \]

\[ \leq CC_V e^{C_V t} C(A, t, (1 - \beta)^{-1}) t^{-\frac{d}{2} - \frac{d}{2p}} + C_V e^{C_V t} t^{-\frac{d}{2} - \frac{d}{2p}} \]

\[ + CC_V e^{C_V t} t^{-\frac{d}{2p}} \int_0^t \left(C(A, s/2, (1 - \beta)^{-1}) s^{-\frac{d}{2}} + s^{-\frac{d}{2}}\right) D_V(s/2)ds, \]

where

\[ C(A, \cdot, (1 - \beta)^{-1}) : (0, \infty) \rightarrow [0, \infty) \]

is the locally bounded function from Theorem 2.3 and

\[ D_V : (0, \infty) \rightarrow [0, \infty], \quad D_V(s) := \sup_{x \in \mathbb{R}^d} \int |V(\omega(s))| \mathbb{P}_x(d\omega). \]
Remark 2.6. 1) Using monotone convergence one finds
\[
\int_0^t s^{-\beta/2} \sup_{z \in \mathbb{R}^d} \int |V(\omega(s))| P_z(d\omega)ds \leq \sup_{z \in \mathbb{R}^d} \int_0^t s^{-\beta/2} \int |V(\omega(s))| P_z(d\omega)ds,
\]
which is finite for all \( t > 0 \) by Remark 2.5, so that a posteriori one also has \( D_V < \infty \) a.e.
2) As our proof shows, the constant \( C_V \) can be chosen to be any constant which satisfies
that for all \( 1 \leq p \leq q \leq \infty \), \( r > 0 \) one has
\[
\| e^{-rH_{A,V}(A)} \|_{L^p \to L^q} \leq C_{V,r}^{-\frac{2}{2}(\frac{1}{p} - \frac{1}{q})} e^{C_V r}.
\]
The existence of such a uniform constant has been shown in [BroHunLes].
3) Using \( \Psi = e^{t\Lambda} e^{-tH_{A,V}} \Phi \) for eigenfunctions \( \Phi \) of \( H_{\Lambda}(A,V) \), one obtains explicit \( L^r \to C^{0,\beta} \)-estimates for eigenfunctions.

Proof of Theorem 2.5. We start by remarking that the assumptions on \( A \) together with Jensen’s inequality, and that \( \mathcal{K}_\beta(\mathbb{R}^d) \subset \mathcal{K}_\beta(\mathbb{R}^d) \) shows that the pair \( (A,V) \) satisfies (14).
Set \( q := 1/(1 - \beta) \in (1, \infty) \) so that \( q^* = 1/\beta \) and pick a mirror coupling \( (X,Y) \) from \( (x,y) \in (\Lambda \times \Lambda) \setminus \text{diag}(\Lambda) \) and set \( \tau := \tau(X,Y) \). Then, given \( r > 0 \), \( \Phi \in L^2 \cap L^\infty(\Lambda) \) we can estimate as follows,
\[
|e^{-rH_{A,0}} \Phi(x) - e^{-rH_{A,0}} \Phi(y)|
\leq E |e^{-\mathcal{F}_r(A|x)} - e^{-\mathcal{F}_r(A|y)}| |\Phi(X_r) - \Phi(Y_r)|
\leq E |e^{-\mathcal{F}_r(A|x)} - e^{-\mathcal{F}_r(A|y)}| |\Phi(X_r) - \Phi(Y_r)|
\leq C(A,r,q) r^{-\frac{1}{q^*}} |x - y|^{\frac{1}{q^*}} \| \Phi \|_{L^\infty} + 2P(r < \tau) \| \Phi \|_{L^\infty}
\leq C(A,r,q) r^{-\frac{1}{q^*}} |x - y|^{\frac{1}{q^*}} \| \Phi \|_{L^\infty} + 2P(r < \tau)^{1/q^*} \| \Phi \|_{L^\infty}
\leq C(A,r,q) r^{-\frac{1}{q^*}} |x - y|^{\frac{1}{q^*}} \| \Phi \|_{L^\infty} + C r^{-\frac{1}{q^*}} |x - y|^{\frac{1}{q^*}} \| \Phi \|_{L^\infty},
\]
where \( C < \infty \) is a universal constant. Thus we have shown
\[
\| e^{-rH_{A,0}} \|_{L^\infty \to C^{0,\beta}} \leq CC(A,r,1/(1 - \beta)) r^{-\frac{2}{q^*}} + C r^{-\frac{2}{q^*}}.
\]
Duhamel’s formula states that
\[
e^{-tH_{A,V}} \Phi = e^{-tH_{A,0}} \Phi + \int_0^t e^{-\frac{s}{2}H_{A,0}} e^{-\frac{s}{2}H_{A,0}} V e^{-(t-s)H_{A,V}} \Phi ds,
\]
In principle one should be more careful here as \( V \) is not bounded; but for the purpose of proving the estimate from Theorem 2.5 one can replace \( V \) with a sequence of potentials \( V_n \) with \( |V_n| \leq |V| \) and \( V_n \to V \) a.e. and take \( n \to \infty \) in the end.
and so
\[
\|e^{-tH_A(A,V)}\|_{C^{0,\beta}} = \|e^{-tH_A(A,0)}\|_{C^{0,\beta}}
\]
\[
+ \int_0^t \|e^{-\frac{t-s}{2}H_A(A,0)}\|_{L^\infty \rightarrow C^{0,\beta}} \|e^{-\frac{t-s}{2}H_A(A,0)V}\|_{L^\infty \rightarrow L^\infty} \|e^{-(t-s)H_A(A,V)}\|_{L^\infty} ds.
\]
There exists {\textbf{BroHunLes}} a constant $C_V$ such that for all $1 \leq p \leq q \leq \infty$, $r > 0$ one has
\[
\|e^{-rH_A(A,V)}\|_{L^p \rightarrow L^q} \leq C Vr^{-\frac{\alpha}{2} - \frac{\beta}{q}} e^{C_Vr},
\]
so that
\[
\|e^{-(t-s)H_A(A,V)}\|_{L^\infty} \leq C_V e^{C_V t} \|\Phi\|_{L^\infty}.
\]
Moreover, by what we have shown above,
\[
\|e^{-\frac{t-s}{2}H_A(A,0)}\|_{L^\infty \rightarrow C^{0,\beta}} \leq CC(A, s/2, 1/(1 - \beta))s^{-\frac{\alpha}{2}} + Cs^{-\frac{\beta}{2}}.
\]
Given $f \in L^\infty(A, x \in \Lambda$, and a Brownian motion $Z(x)$ in $\mathbb{R}^d$ starting in $x$ we have, using $|e^{-\mathcal{L}_{s/2}(A[Z(x)])} = 1$, the estimate
\[
\|e^{-\frac{t-s}{2}H_A(A,0)}Vf(x)\| = \mathbb{E} \left(1_{s/2<\zeta_A(Z(x))}e^{-\mathcal{L}_{s/2}(A[Z(x)])}V(Z_{s/2}(x))f(Z_{s/2}(x))\right) \leq \mathbb{E} \left(V(Z_{s/2}(x))f(Z_{s/2}(x))\right) \leq \int |V(\omega(s/2))| \cdot |f(\omega(s/2))| \mathbb{P}_x(d\omega) \leq \|f\|_{L^\infty} D_V(s/2) \leq \|f\|_{L^\infty},
\]
so that
\[
\|e^{-\frac{t-s}{2}H_A(A,0)V}\|_{L^\infty \rightarrow L^\infty} \leq D_V(s/2).
\]
Combining (16), (18), (19), (20) we have shown that for all $\Phi \in L^\infty(A)$,
\[
\left|e^{-tH_A(A,V)}\Phi(x) - e^{-tH_A(A,V)}\Phi(y)\right| \leq \left(CC(A, t, (1 - \beta)^{-1})t^{-\frac{\alpha}{2}} + Ct^{-\frac{\beta}{2}}
\right.
\]
\[
+ C_V e^{C_V t} \int_0^t \left(C(A, s/2, (1 - \beta)^{-1})s^{-\frac{\alpha}{2}} + s^{-\frac{\beta}{2}}\right) D_V(s/2) ds \|\Phi\|_{L^\infty} |x - y|^\beta.
\]
Using this estimate with $\Phi = e^{-tH_A(A,V)}\Psi$ and (17) shows there exists a constant $C_{V,\beta} < \infty$, which only depends on $V$ and $\beta$, such that
\[
\|e^{-2tH_A(A,V)}\|_{L^p \rightarrow C^{0,\alpha}} \leq C C_V e^{C_V t} C(A, t, (1 - \beta)^{-1})t^{-\frac{\alpha}{2} - \frac{d}{2p}} + C_V e^{C_V t} t^{-\frac{\beta}{2} - \frac{d}{2p}}
\]
\[
+ CC_V e^{C_V t} t^{-\frac{d}{2p}} \int_0^t \left(C(A, s/2, (1 - \beta)^{-1})s^{-\frac{\alpha}{2}} + s^{-\frac{\beta}{2}}\right) D_V(s/2) ds,
\]
which completes the proof.
For the following result consider the linear surjective maps
\[ \pi_j : \mathbb{R}^{3n} \rightarrow \mathbb{R}^3, \quad (x_1, \ldots, x_n) \mapsto x_j, \]
\[ \pi_{ij} := \pi_i - \pi_j : \mathbb{R}^{3n} \rightarrow \mathbb{R}^3, \]
and let \( A : \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}, V : \mathbb{R}^{3n} \rightarrow \mathbb{R} \) be arbitrary functions. Remark 2.2 then shows:

**Corollary 2.7.** Assume there exists \( \beta \in (0, 1), l \in \mathbb{N} \) and Borel functions \( a : \mathbb{R}^3 \rightarrow \mathbb{R}^3, v_i, v_{ij} : \mathbb{R}^3 \rightarrow \mathbb{R} \) with
\[
A = \sum_{i=1}^{n} a \circ \pi_i, \quad V = \sum_{1 \leq i < j \leq n} v_{ij} \circ \pi_{ij} + \sum_{i=1}^{n} v_i \circ \pi_i,
\]
and
\[
|a|^{2/(1-\beta)}, |\text{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > 3/2,
\]
\[
v_i, v_{ij} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > \frac{3}{2(1-\beta/2)}.
\]
Then for all open connected \( \Lambda \subset \mathbb{R}^3 \), \( t > 0 \) and \( p \in [1, \infty) \) one has
\[ e^{-tH_{\mathbb{R}^3n}(A,V)} : L^p(\mathbb{R}^{3n}) \rightarrow C^{0,\beta}(\mathbb{R}^{3n}). \]

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