CAT’S DILEMMA – TRANSITIVITY VS. INTRANSITIVITY

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We study a simple example of a sequential game illustrating problems connected with making rational decisions that are universal for social sciences. The set of chooser’s optimal decisions that manifest his preferences in case of a constant strategy of the adversary (the offering player), is investigated. It turns out that the order imposed by the player’s rational preferences can be intransitive. The presented quantitative results imply a revision of the “common sense” opinions stating that preferences showing intransitivity are paradoxical and undesired.

Keywords: intransitivity; game theory; sequential game.

1. Introduction

The intransitivity can occur in games with three or more strategies if the strategies A, B, C are such that A prevails over B, B prevails over C, and C prevails over A (A > B > C > A). The most known example of intransitivity is the children game “Rock, Scissors, Paper” (R, S, P) where R > S > P > R. The other interesting example of intransitive order is the so-called Condorcet’s paradox, known since XVIIIth century. Considerations regarding this paradox led Arrow in the XXth century to prove the theorem stating that there is no procedure of successful choice that would meet the democratic assumptions [1]. The importance of this result to mathematical political science is comparable to Gödel’s Incompleteness Theorem in logic [2].

It seems logical to choose an order, in a consistent way between things we like. But what we prefer often depends on how the choice is being offered [3, 4]. This paradox was perceived by many researchers and analysts (for instance Stan Ulam described this in his book “Adventures of a Mathematician”, some problems with intransitive options can be found in [5, 6]). On the other hand scientists have a penchant for classifications (rankings) on basis of linear orders and this (we think) follows from...
such intransitive preferences there are so suspicious for many researchers.

In the paper, we present quantitative analysis of a model, which can be illustrated by the Pitts’s experiments with cats, mentioned in the Steinhaus diary [7] (Pitts noticed that a cat facing choice between fish, meat and milk prefers fish to meat, meat to milk, and milk to fish!). This model finds its reflection in the principle of least action that controls our mental and physical processes, formulated by Ernst Mach [8] and referring to Ockham’s razor principle. Pitts’s cat, thanks to the above-mentioned food preferences, provided itself with a balanced diet. In our work, using elementary tools of linear algebra, we obtained the relationship between the optimal cat’s strategy and frequencies of appearance of food pairs. Experiments with rats confirmed Pitts’s observations. Therefore, it is interesting to investigate whether intransitivity of preferences will provide a balanced diet also in a wider sense in more or less abstract situations involving decisions. Maybe in the class of randomized behaviors we will find the more effective ways of nutrition? The following sections constitute an attempt at providing quantitative answer to these questions. The analysis of an elementary class of models of making optimal decision presented below permits only determined behaviors, that is such for which the agent must make the choice.

Through this analysis we wish to contribute to dissemination of theoretical quantitative studies of nondeterministic algorithms of behaviors which are essential for economics and sociology – this type of analysis is not in common use. The geometrical interpretation presented in this article can turn out very helpful in understanding of various stochastic models in use.

2. Nondeterministic cat

Let us assume that a cat is offered three types of food (no. 1, no. 2 and no. 3), every time in pairs of two types, whereas the food portions are equally attractive regarding the calories, and each one has some unique components that are necessary for the cat’s good health. The cat knows (it is accustomed to) the frequency of occurrence of every pair of food and his strategy depends on only this frequency. Let us also assume that the cat cannot consume both offered types of food at the same moment, and that it will never refrain from making the choice. The eight ($2^3$) possible deterministic choice functions $f_k$:

$$f_k : \{(1,0),(2,0),(2,1)\} \to \{0,1,2\}, \quad k = 0, \ldots, 7$$

are defined in Table 1. The functions $f_2$ and $f_5$ determine intransitive orders. The

| $f_k$ | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|------|------|------|------|------|------|------|------|------|
| $f_k(1,0)$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $f_k(2,0)$ | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 |
| $f_k(2,1)$ | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |

parameters $p_k$, $k = 0, \ldots, 7$ give the frequencies of appearance of the choice function
in the nondeterministic algorithm (strategy) of the cat \((\sum_{k=0}^{7} p_k = 1, p_k \geq 0\) for \(k = 0, \ldots, 7\)).

We will show the relationship between the frequency of occurrence of individual type of food in cat’s diet and the frequencies of occurrence of food pairs. Let us denote the frequency of occurrence of the pair \((k, j)\) as \(q_m\), where \(m\) is the number of food that does not occur in the pair \((k, j)\) \((\sum_{m=0}^{2} q_m = 1)\). This denotation causes no uncertainty because there are only three types of food. When the choice methods \(f_k\) are selected nondeterministically, with the respective intensities \(p_k\), the frequency \(\omega_m\), \(m = 0, 1, 2\), of occurrence of individual food in cat’s diet are according to Table 1, given as follows:

- food no. 0: \(\omega_0 = (p_0 + p_1 + p_2 + p_3)q_2 + (p_0 + p_1 + p_4 + p_5)q_1\),
- food no. 1: \(\omega_1 = (p_4 + p_5 + p_6 + p_7)q_2 + (p_0 + p_2 + p_4 + p_6)q_0\),
- food no. 2: \(\omega_2 = (p_2 + p_3 + p_6 + p_7)q_1 + (p_1 + p_3 + p_5 + p_7)q_0\).

Three equalities above can be explained with the help of the conditional probability concept. Let us denote \(B_{3-k} = \{(j, k)\}\), \(P(B_j) = q_j\) and \(C_j = \{j\}\) for \(j, k = 0, 1, 2, j \neq k\). The number \(P(C_k|B_j)\) indicates the probability of choosing the food of number \(k\), when the offered food pair does not contain the food of number \(j\).

Since the events of choosing different pairs of food are disjoint and comprise all the space of elementary events. Hence, for each food chosen, we have the following relation:

\[
\omega_k = P(C_k) = \sum_{j=0}^{2} P(C_k|B_j)P(B_j), \ k = 0, 1, 2. \tag{2}
\]

By inspection of the table of the functions \(f_k, k = 0, \ldots, 7\), we easily get the following relations:

\[
\begin{align*}
P(C_0|B_2) &= P(\sum_{k=0}^{7} f_k(B_2) = 0) = p_0 + p_1 + p_2 + p_3, \\
P(C_0|B_1) &= P(\sum_{k=0}^{7} f_k(B_1) = 0) = p_0 + p_1 + p_4 + p_5, \\
P(C_1|B_0) &= P(\prod_{k=0}^{7} f_k(B_0) = 1) = p_0 + p_2 + p_4 + p_6, \tag{3} \\
P(C_1|B_2) &= P(\prod_{k=0}^{7} f_k(B_2) = 1) = p_4 + p_5 + p_6 + p_7, \\
P(C_2|B_1) &= P(\prod_{k=0}^{7} f_k(B_1) = 2) = p_2 + p_3 + p_6 + p_7, \\
P(C_2|B_0) &= P(\prod_{k=0}^{7} f_k(B_0) = 2) = p_1 + p_3 + p_5 + p_7, \\
\end{align*}
\]

and \(P(C_0|B_0) = P(C_1|B_1) = P(C_2|B_2) = 0\).

Frequency of the least preferred food, that is the function \(\min(\omega_0, \omega_1, \omega_2)\), determines the degree of the diet completeness. Since \(\omega_0 + \omega_1 + \omega_2 = 1\), the most valuable
方式选择食物的猫选择的概率为\(p_0, \ldots, p_7\)，使得函数\(\min(\omega_0, \omega_1, \omega_2)\)取得最大值，即

\[
\omega_0 = \omega_1 = \omega_2 = \frac{1}{3}.
\] 

(4)

任一向量\(\vec{p} = (p_0, \ldots, p_7)\)（或六个条件概率\((P(C_1|B_0), P(C_2|B_0), P(C_0|B_1), P(C_2|B_1), P(C_0|B_2), P(C_1|B_2))\)），当固定三元组\((q_0, q_1, q_2)\)时，满足等式（4）的系统将被称为猫的最优策略。

让我们对这个策略进行更详细的研究，并对其进行几何分析。

对于给定\(q_0, q_1, q_2\)的所有最优策略都计算出来。等式（4）的矩阵形式为：

\[
\begin{pmatrix}
P(C_0|B_2) & P(C_0|B_1) & 0 \\
P(C_1|B_2) & 0 & P(C_1|B_0) \\
0 & P(C_2|B_1) & P(C_2|B_0)
\end{pmatrix}
\begin{pmatrix}
q_2 \\
q_1 \\
q_0
\end{pmatrix} = \frac{1}{3}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\]

(5)

其解为：

\[
q_2 = \frac{1}{3} \left( \frac{P(C_0|B_1) + P(C_1|B_0)}{3} - P(C_0|B_1)P(C_1|B_0) \right),
\]

\[
q_1 = \frac{1}{3} \left( \frac{P(C_0|B_2) + P(C_2|B_0)}{3} - P(C_0|B_2)P(C_2|B_0) \right),
\]

\[
q_0 = \frac{1}{3} \left( \frac{P(C_1|B_2) + P(C_2|B_1)}{3} - P(C_1|B_2)P(C_2|B_1) \right),
\]

(6)

defines a mapping of the three-dimensional cube \([0,1]^3\) in the space of parameters \((P(C_0|B_2), P(C_0|B_1), P(C_1|B_0))\) into a triangle in the space of parameters \((q_0, q_1, q_2)\)，where \(d\) is the determinant of the matrix of parameters \(P(C_j|B_i)\)。The barycentric coordinates \([9]\) of a point of this triangle are interpreted as the probabilities \(q_0, q_1\) and \(q_2\)。These numbers represent the heights \(a, b\) and \(c\) or the areas \(P_{QAB}, P_{QBC}\) and \(P_{QAC}\) of three smaller triangles determined by the point \(Q\) (cf. Fig. 1)，or the lengths of the segments formed by the edges of the triangle by cutting them with the straight lines passing through the point \(Q\) and the opposite vertex of the triangle。Hence e.g. \(q_2 = \frac{q_2}{q_2} = \frac{P_{QBC}}{P_{QAC}} = \frac{|RB|}{|CA|}\)，where the symbol \(|RB|\) represents length of the segment。

下一张图（Fig. 2）展示了三维立方体在这一简单的图像。它确定了频率\(q_m\)的出现情况，其中的个体选择在两种食物在简单中，使得最优策略存在。在为数不多的10,000随机选择的点与相对概率分布的立方体上。概率的等价分割可能可以在拉普拉斯的原则下不充分的原因[10]。在我们的随机化模型中，先验概率的事实是，如果概率之和的\(P(C_j|B_k)\)小于给定数\(\alpha \in [0,1]\)是一个\(\alpha\)。最优解的缺失概率之外的六边形形成阴影部分的图（Fig. 2）是显而易见的，因为明亮的（非点）部分的图代表的区域，对于\(q_0 > \frac{1}{4}\)（或\(q_1 > \frac{1}{4}\)，或\(q_2 > \frac{1}{4}\)），和总频率出现的对\((0,1)\)或\((0,2)\)必须至少\(1\)在为数不多的10,000随机选择的点与相对概率分布的立方体上。
assure the completeness of the diet with respect of the ingredient 0 (but this cannot happen because when $q_0 > \frac{2}{3}$, then $q_1 + q_2 = 1 - q_0 < \frac{1}{3}$).

The system of equations (5) can be transformed into the following form:

$$
\begin{pmatrix}
q_2 & -q_1 & 0 \\
-q_2 & 0 & q_0 \\
0 & q_1 & -q_0
\end{pmatrix}
\begin{pmatrix}
P(C_0|B_2) \\
P(C_2|B_1) \\
P(C_1|B_0)
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{3} - q_1 \\
\frac{1}{3} - q_2 \\
\frac{1}{3} - q_0
\end{pmatrix},
$$

(7)

which allows to write out the inverse transformation to the mapping defined by equations (6). By introducing the parameter $\lambda$ we may write them as follows:

$$
P(C_0|B_2) = \frac{\lambda}{3q_2}, \quad P(C_2|B_1) = \frac{\lambda - 1 + 3q_1}{3q_1}, \quad P(C_1|B_0) = \frac{\lambda + 1 - 3q_2}{3q_0}.
$$

(8)

A whole segment on the unit cube corresponds to one point of the simplex, parameterized by $\lambda$. The range of this representation should be limited to the unit cube, which gives the following conditions for the above subsequent equations:

$$
\lambda \in [0, 3q_2], \quad \lambda \in [1 - 3q_1, 1], \quad \lambda \in [3q_2 - 1, 2 - 3q_1].
$$

(9)
The permitted values of the parameter $\lambda$ form the common part of these segments, hence it is nonempty for:

$$\max(0, 1 - 3q_1, 3q_2 - 1) \leq \min(2 - 3q_1, 3q_2, 1).$$  \hspace{1cm} (10)

Therefore

$$\lambda \in [\max(0, 1 - 3q_1, 3q_2 - 1), \min(2 - 3q_1, 3q_2, 1)].$$  \hspace{1cm} (11)

It may be now noticed that for any triple of probabilities belonging to the hexagon, there exists an optimal solution within a set of parameters $((P(C_0|B_2), P(C_0|B_1), P(C_1|B_0)))$. If we assume the equal measure for each set of frequencies of occurrence of food pairs as the triangle point, then we may state that we deal with optimal strategies in $\frac{2}{3}$ of all the cases (it is the ratio of area of regular hexagon inscribed into an equilateral triangle). The inverse image of the area of frequencies $(q_0, q_1, q_2)$ of food pairs that enable realization of the optimal strategies, which is situated on the cube of all possible strategies, is presented by four consecutive plots in Fig. 3. We present there the same configuration observed from different points of view. The segments on the figures correspond to single points of the frequency triangle of the individual food pairs. The greatest concentration of the segments is observed in two areas of the cube that correspond to intransitive strategies. The bright area in the center of the cube, which may be seen in the last picture, belongs to the effective strategies – effective in the subset of frequencies of a small measure $(q_0, q_1, q_2)$ of the food pairs appearance. Among them, the totally incidental behavior is located, which gives consideration in equal amounts to all the mechanisms of deterministic choice $p_j = p_k = \frac{1}{2}$.

3. Example of an optimal strategy

The formulas that map the triangle into a cube can be used to find an optimal strategy in cases, when the probabilities $(q_0, q_1, q_2)$ of appearance of individual pairs of the products are known. Let us assume that $q_0 = \frac{1}{2}$, $q_1 = \frac{1}{3}$ and $q_2 = \frac{1}{6}$. Then, according to the formulas, we have $P(C_1|B_0) = \frac{1}{3} + \frac{2}{3} \lambda$, $P(C_0|B_2) = 2\lambda$, $P(C_2|B_1) = \lambda$, where $\lambda \in [0, \frac{2}{3}]$. Selecting $\lambda = \frac{1}{3}$ we have: $P(C_0|B_2) = \frac{1}{3}$, $P(C_2|B_1) = \frac{1}{4}$, $P(C_1|B_0) = \frac{1}{6}$. We may now show the solution of equations, e.g.: $p_0 = \frac{1}{2}$, $p_5 = p_7 = \frac{1}{4}$ and $p_j = 0$ for others parameters. We will obtain the following frequencies of occurrence of individual foods in the diet:

$$\omega_0 = (p_0 + p_5)q_1 + p_0q_2 = \frac{1}{4} + \frac{1}{12} = \frac{1}{3},$$

$$\omega_1 = p_0q_0 + (p_5 + p_7)q_2 = \frac{1}{4} + \frac{1}{12} = \frac{1}{3},$$

$$\omega_2 = (p_5 + p_7)q_0 + p_7q_1 = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}. \hspace{1cm} (12)$$

The above calculations of the frequency $\omega_j$ confirm optimality of the indeterministic algorithm determined in this example.

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1See section 4.
Intransitive nondeterministic decisions

In the case of random selections we may talk about order relation *food no. 0 < food no. 1* when from the offered pair (0, 1) we are willing to choose the food no. 1 more often than the food no. 0 \( P(C_0|B_2) < P(C_1|B_2) \). Therefore we have two intransitive orders:

- \( P(C_0|B_2) < \frac{1}{3}, P(C_2|B_1) < \frac{1}{3}, P(C_1|B_0) < \frac{1}{3} \).
- \( P(C_0|B_2) > \frac{1}{3}, P(C_2|B_1) > \frac{1}{3}, P(C_1|B_0) > \frac{1}{3} \).

It is interesting to see in which part of the simplex of parameters \((q_0, q_1, q_2)\) we may take optimal intransitive strategies. They form the six-armed star composed of two triangles, each of them corresponding to one of two possible intransitive orders (Fig. 4). They dominate in the central part of triangle, near point \( q_0 = q_1 = q_2 = \frac{1}{3} \). They form darkened part of area inside the star. Optimal transitive strategies cover the same area of the simplex as all optimal strategies, however they occur less often in the center of the simplex. We illustrated this situation in the next picture (Fig. 5). In areas of high concentration of optimal transitive strategies, one of three frequencies \( q_0, q_1, q_2 \) looses its significance – two from three pairs of the food occur with considerable predominance. We have enough information to be able to compare the applicability range of different types of optimal strategies. Let us assume the same measure of the possibility of occurrence of determined proportion of all three food pairs. This assumption means that the probability of appearance of the situation determined by a point in the triangle-domain of parameters \((q_0, q_1, q_2)\) does not depend on those parameters. Two thirds of strategies are optimal. There
are 33%² of circumstances, which allow for the use of the optimal strategies that belong to the specified intransitive order. There are 44% (4/9) of situations of any order that favor optimal strategies, what follows from the fact that they are measured by the surface of regular star, and its area is equal to double area of the triangle corresponding to one intransitive order reduced by the area of the hexagon inscribed into the star. So we have: \[ \frac{1}{3} + \frac{1}{3} - \frac{2}{9} = \frac{4}{9}. \] Appearance of the number \( \frac{2}{9} \) in the calculation can be easily explained by the observation that the area of the regular six-armed star is two times bigger than the area of the hexagon inscribed into it. This number (22%) is the measure of the events that favor both types of intransitive strategies.

It is worth to stress that in the situation that favors optimal strategies we can always find the strategy that determines the transitive order (see Fig. 5). However, we should remember that this feature concerns only the simple model of the cat’s behavior, and does not have to be true in the cases of more complicated reaction mechanisms.

They are measured by the area of equilateral triangle inscribed into a regular hexagon.
5. Conclusions

In this article, we used a stochastic variant of the principle of least action. Perhaps dissemination of usage of this principle will lead to formulation of many interesting conclusions and observations. We presented a method, which allows successful analysis of intransitive orders that still are surprisingly suspicious for many researchers. More profound analysis of this phenomenon can have importance everywhere where the problem of choice behavior is studied. For instance in economics (description of the customer preference toward products-marketing strategy) or in political science where the problem of voting exists. Analysis of intransitive orders is a serious challenge to those who seek description of our reasoning process.

The quantitative observations from the previous section show that intransitivity, as the way of making the decision, can provide the diet completeness for the cat from our example. Moreover, the intransitive optimal strategies constitute the major part of all optimal strategies. Therefore, it would be wrong to prematurely acknowledge the preferences showing the intransitivity as undesired. Perhaps there are situations, when only the intransitive orders allow obtaining the optimal effects. The most intriguing problem that remains open, is to answer the question whether there exists a useful model of optimal behaviors, which gives the intransitive orders, and for which it would be impossible to specify the transitive optimal strategy of identical action results. Showing the impossibility of building such constructions would cause marginalization of the practical meaning of intransitive orders. On the other hand, indication of this type of models would force us to accept the intransitive ordering.

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Appendix A.

The following mini-program written in the language Mathematica 5.0 generate four plots in Fig. 3.
In[1] := \[c = N\left[\frac{4}{\sqrt{3}}\right];

gencorrect := Module[{a = c \{Random[], Random[]\}, While[a[[2]] > \frac{2}{9},
a = c \{Random[], Random[]\};
setpoint := Module[{q1, a, x, y, v = 1, w = 1}, While[\frac{2}{9} v + \frac{3}{2} w > 1,
a = gencorrect; q1 = a[[2]]; x = a[[1]] - \frac{\sqrt{3}}{2};
y = a[[2]] - \frac{1}{3}; v = Abs[x]; w = Abs[y];
{q1, N[\frac{\sqrt{3}}{2} x - \frac{1}{3} y + \frac{1}{3}]}];
setsegment := Module[{a, q1 = 1, q2 = 1, l1 = 1, l2 = 0},
While[l1 > l2, a = setpoint; q1 = a[[1]]; q2 = a[[2]]; l1 = Max[0, 1 - 3 q1, 3 q2 - 1]; l2 = Min[2 - 3 q1, 3 q2, 1];
{\{\frac{l1}{3 q2}, \frac{1 - l1}{3 q1}, \frac{1 - 3 q2 + l1}{3 (1 - q1 - q2)}\}, \{\frac{l2}{3 q2}, \frac{1 - l2}{3 q1}, \frac{1 - 3 q2 + l2}{3 (1 - q1 - q2)}\}}];

fig[x_, y_, z_] := Show[Graphics3D[{GrayLevel[.0], Thickness[.002],
Table[Line[setsegment], \{2000\}]],
ViewPoint \rightarrow \{x, y, z\}, Axes \rightarrow True,
AxesLabel \rightarrow \{"P(C_0 | B_2)"", "P(C_2 | B_1)"", "P(C_1 | B_0)"\},
Ticks \rightarrow \{\{0, 1\}, \{0, 1\}, \{0, 1\}\},
BoxStyle \rightarrow Dashing[\{.02, .02\}],
AxesStyle \rightarrow Thickness[.005]];
[5] J. Y. Halpern, *Intransitivity and Vagueness*, Principles of Knowledge Representation and Reasoning, Proceedings of the Ninth International Conf., Whistler, Canada (2004); arXiv:cs/0410049.

[6] E. Groes, H. J. Jacobsen, T. Tranas, *Testing the Intransitivity Explanation of the Allais paradox*, Theory and Decision 47 (1999), 229-245.

[7] H. Steinhaus, *Memoirs and Notes* (in Polish), Aneks, London (1992).

[8] E. Mach, *The Science of Mechanics*, Open Court, LaSalle, IL (1960).

[9] *Encyclopaedia of mathematics on cd-rom*, Kluwer Academic Publishers, Dordrecht (1997).

[10] P. Dupont, *Laplace and the Indifference Principle in the 'Essai philosophique des probabilits'*, Rend. Sem. Mat. Univ. Politec. Torino, 36 (1977/78) 125-137.