Fractal Boundaries of Complex Networks

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Abstract

We introduce the concept of boundaries of a complex network as the set of nodes at distance larger than the mean distance from a given node in the network. We study the statistical properties of the boundaries nodes of complex networks. We find that for both Erdös-Rényi and scale-free model networks, as well as for several real networks, the boundaries have fractal properties. In particular, the number of boundaries nodes $B$ follows a power-law probability density function which scales as $B^{-2}$. The clusters formed by the boundary nodes are fractals with a fractal dimension $d_f \approx 2$. We present analytical and numerical evidence supporting these results for a broad class of networks. Our findings imply potential applications for epidemic spreading.
Many complex networks are “small world” due to the very small average distance $d$ between two randomly chosen nodes. Usually $d \sim \ln N$, where $N$ is the number of nodes. Thus, starting from a randomly chosen node following the shortest path, one can reach any other node in a very small number of steps. This phenomenon is called “six degrees of separation” in social networks. That is, for most pairs of randomly chosen people, the shortest “distance” between them is not more than six. Many random network models, such as Erdős-Rényi network (ER), Watts-Strogatz network (WS) and random scale-free network (SF), as well as many real networks, have been shown to possess this small-world property.

Much attention has been devoted to the structural properties of networks within the average distance $d$ from a given node. However, almost no attention has been given to nodes which are at distances greater than $d$ from a given node. We define these nodes as the boundaries of the network and study the ensemble of boundaries. An interesting question is how many “friends of friends of friends etc...” one has at distance greater than the average distance $d$? What is their probability distribution and what is the structure of the boundaries? The boundaries have an important role in several scenarios, such as in the spread of viruses or information in a human social network. If the virus (information) spreads from one node to all its nearest neighbors, and from them to all next nearest neighbors and further on until $d$, how many nodes do not get the virus (information), and what is their distribution with respect to the origin of the infection. Our results may explain why epidemics such as “black death” in medieval Europe stopped before reaching the entire population.

In this Letter, we find theoretically and numerically that the nodes at the boundaries, which are of order $N$, exhibit similar fractal features for many types of networks, including ER and SF models as well as several real networks. Song et al. found that some networks have fractal properties while others do not. Here we show that almost all model and real networks including non-fractal networks have fractal features at their boundaries which are different from Song et al.

Fig. demonstrates our approach and analysis. For each node, we identify the nodes at distance $\ell$ from it as nodes in shell $\ell$. We chose a random origin node and count the number of nodes $B_\ell$ at shell $\ell$. We see that $B_1=10, B_2=11, B_3=13$, etc... We estimate the average distance $d \approx 2.9$ by averaging the distances between all pairs of nodes. After
removing nodes with \( \ell < d = 2.9 \), the network is fragmented into 12 clusters, with sizes
\[ s_3 = \{1, 1, 2, 5, 1, 3, 1, 1, 8, 1, 2, 3 \} . \]

![Illustration of shells and clusters originating from a randomly chosen node](image)

**FIG. 1:** (Color on line) Illustration of shells and clusters originating from a randomly chosen node, which is shown in the center (red). Its neighboring nodes are defined as shell 1, the nodes at distance \( \ell \) are defined as shell \( \ell \). When removing all nodes with \( \ell < 3 \), the remaining network becomes fragmented into 12 clusters.

We begin our study by simulating ER and SF networks, and later present analytical proofs. Fig. 2a shows simulation results for the number of nodes \( B_\ell \) reached from a randomly chosen origin node for an ER network. The results shown are for a single network realization of size \( N = 10^6 \), with average degree \( \langle k \rangle = 6 \) and \( d \approx 7.9 \) [11]. For \( \ell < d \), the cumulative distribution function, \( P(B_\ell) \), which is the probability that shell \( \ell \) has more than \( B_\ell \) nodes, decays exponentially for \( B_\ell > B_\ell^* \), where \( B_\ell^* \) is the maximum typical size of shell \( \ell \) [12].

However, for \( \ell > d \), we observe a clear transition to a power law decay behavior, where \( P(B_\ell) \sim B_\ell^{-\beta} \), with \( \beta \approx 1 \) and the pdf of \( B_\ell \) is \( \tilde{P}(B_\ell) \equiv dP(B_\ell)/dB_\ell \sim B_\ell^{-2} \). Thus, our results suggest a broad “scale-free” distribution for the number of nodes at distances larger than \( d \). This power law behavior demonstrates the fractal nature of the boundaries of network, suggesting that there is no characteristic size and a broad range of sizes can appear in a shell at the boundaries. Further fractal features of the boundaries structure will be shown below.

In SF networks, the degrees of the nodes, \( k \), follow a power law distribution function \( q(k) \sim k^{-\lambda} \), where the minimum degree of the network is chosen to be 2. Fig. 2b shows, for SF networks with \( \lambda = 2.5 \), similar power law results, \( P(B_\ell) \sim B_\ell^{-\beta} \) for \( \ell > d \) as for ER, with
a similar power $\beta \approx 1$. We find similar results also for $\lambda > 3$ (not shown).

![Graphs](image)

FIG. 2: The cumulative distribution function, $P(B_\ell)$, for two random network models: (a) ER network with $N = 10^6$ nodes and $\langle k \rangle = 6$, and (b) SF network with $N = 10^6$ nodes and $\lambda = 2.5$, and two real networks: (c) the High Energy Particle (HEP) physics citations network and (d) the Autonomous System (AS) Internet network. The shells with $\ell > d$ are marked with their shell number. The thin lines from left to right represent shells $\ell = 1, 2...$ respectively, with $\ell < d$. For $\ell > d$, $P(B_\ell)$ follows a power-law distribution $P(B_\ell) \sim B_\ell^{-\beta}$, with $\beta \approx 1$ (corresponding to $\tilde{P}(B_\ell) \sim B_\ell^{-2}$. The appearance of a power law decay only happens for $\ell$ larger than $d \approx 7.9$ for ER and $d \approx 4.7$ for the SF network. The straight lines represent a slope of $-1$.

To test how general is our finding, we also study several real networks (Figs. 2c, 2d), including the High Energy Particle (HEP) physics citations network [14] and the Autonomous System (AS) Internet network [10, 15]. Our results suggest that the fractal properties of the boundaries appear also in both networks, with similar values of $\beta \approx 1$ for $\ell > d$ [16].

Next we ask how many nodes are on average at the boundaries? Are they a finite fraction
FIG. 3: (Color on line) (a) Normalized average number of nodes at shell $\ell$, $\langle B_\ell \rangle/N$, as a function of $\ell - \ln N / \ln \langle k \rangle$ for ER network with $< k >= 6$. For different $N$, the curves collapse. (b) $\bar{k}_\ell + 1$, which is $\langle k^2_\ell \rangle / \langle k_\ell \rangle$, as function of $\ell$ shown for both ER and SF network. (c) The probability distribution function $\tilde{P}(B_\ell)$ in shells $\ell \leq d$ for ER network. For small values of $B_\ell$, $\tilde{P}(B_\ell) \sim B_\ell^\mu$, where $\mu$ depends on the $<k>$ of the network (Eq. (4)). (d) The fraction of nodes outside shell $\ell + m$, $r_{\ell + m}$, as a function of $r_\ell$ for ER network, where $r_\ell$ is calculated for any possible $\ell$. The (red) lines represent the theoretical iteration function (Eq. (6)).

of $N$, or less? In Fig. 3a, we study the mean number $\langle B_\ell \rangle$ in shell $\ell$, and plot $\langle B_\ell \rangle / N$ as function of $\ell - \ln N / \ln \langle k \rangle$ for different values of $N$ for ER network. The term $\ln N / \ln \langle k \rangle$ represents the average distance $d$ of the network [2]. We find that, for different values of $N$, the curves collapse, supporting a relation independent of network size $N$. Since $\langle B_\ell \rangle / N$ is apparently constant and independent of $N$, it follows that $\langle B_\ell \rangle \sim N$, i.e., a finite fraction of $N$ nodes appear at each shell including shells with $\ell > d$. We find similar behavior for SF network with $\lambda = 3.5$ (not shown). The branching factor [13] of the network is
\[ \tilde{k} = \langle k^2 \rangle / \langle k \rangle - 1, \] where the averages are calculated for the entire network. Similarly, we define \[ \tilde{k}_\ell = \langle k^2_\ell \rangle / \langle k_\ell \rangle - 1, \] where the averages are calculated only for nodes in shell \( \ell \). Above the average distance, \( \tilde{k}_\ell + 1 \) decreases with \( \ell \) for both ER and SF networks (Fig. 3b). Thus, at the shells where power law behavior of \( P(B_\ell) \) appears (Fig. 2), the nodes have much lower \( \tilde{k}_\ell + 1 \) compared with the entire network. The approach of \( \tilde{k}_\ell + 1 \) to 1 (ER network) and 2 (SF network) is consistent with a critical behavior at the boundaries of the network [13].

Fig. 3c shows that \( \tilde{P}(B_\ell) \) for \( \ell < d \) and small values of \( B_\ell \) increase as a power law, \( \tilde{P}(B_\ell) \sim B_\ell^\mu \), for ER network, where \( \mu \) depends on \( \tilde{k} \) (supporting the theory developed below). We define the fraction of nodes outside shell \( m \) as \( r_m = 1 - (\sum_{\ell=1}^{m} B_\ell) / N \). There exists a functional relation Eq. (6), which is independent of \( \ell \), between any two \( r_\ell \) and \( r_{\ell+m} \) (\( m = 1, 2, 3, \ldots \)), for ER network in Fig. 3d. Figs. 3c, 3d provide empirical evidences for the theory developed below.

Next, we study the structural properties of the boundaries. Removing all nodes that are within a distance \( \ell > d \) (not including shell \( \ell \)), the network will become fragmented into several clusters (see Fig. 1). We denote the size of those clusters as \( s_\ell \), the number of clusters of size \( s_\ell \) as \( n(s_\ell) \), and the average distance in the clusters as \( d_\ell \) [17]. We find \( n(s) \sim s^{-\theta} \), with \( \theta \approx 3.0 \) (Figs. 4a and 4b). Similar relations are also found for ER and other real networks. The relation between the size of the clusters \( s_\ell \) and their mean distance \( d_\ell \) is shown in Figs. 4c and 4d, for SF (\( \lambda = 2.5 \)) and HEP citations networks respectively. These plots suggest a power law relation, \( s_\ell \sim d_\ell^\varphi \), with \( \varphi \approx 2.0 \). It indicates that the clusters at the boundaries are fractals with fractal dimension \( d_f = 2 \) as percolation clusters at criticality [18]. Note that, for very large clusters their average distances \( d_\ell \) decrease with size, suggesting that the largest clusters are not fractals. We find that the fractal dimension is \( d_f = \varphi \approx 2.0 \) also for ER, SF with \( \lambda = 3.5 \) and several other real networks.

Next we present analytical derivations supporting the above numerical results. We denote the degree distribution of a network as \( q(k) \). In infinitely large network we can neglect loops for \( \ell < d \) and approximate the behavior of forming a network as a branching process [19, 20, 21]. The probability of reaching a node with \( k \) outgoing links through a link is \( \tilde{q}(k) = (k+1)q(k+1)/\langle k \rangle \). The probability of number of branches equals to \( \tilde{q}(k) \). We define the generating function of \( q(k) \) as \( G_0(x) = \sum_{k=0}^{\infty} q(k)x^k \), the generating function of \( \tilde{q}(k) \) as \( G_1(x) = \sum_{k=0}^{\infty} \tilde{q}(k)x^k = G'_0(x)/\langle k \rangle \). For ER networks we have \( G_0(x) = G_1(x) = e^{(k)(x-1)} \).
FIG. 4: The number of clusters of sizes \( s_\ell \), \( n(s_\ell) \), as function of \( s_\ell \) after removing nodes within shell \( \ell \) for: (a) SF network with \( N = 10^6 \) and \( \lambda = 2.5 \), (b) HEP citations network, and \( s_\ell \) as function of average distance \( d_\ell \) of the clusters for (c) SF network with \( N=10^6 \) and \( \lambda = 2.5 \), (d) HEP citations network. The relation between \( n(s_\ell) \) and \( s_\ell \) is characterized by a power law, \( n(s_\ell) \sim s_\ell^{-\theta} \), with \( \theta \approx 3 \). Also, \( s_\ell \) scales with \( d_\ell \) as \( s_\ell \sim d_\ell^\varphi \), with \( \varphi \approx 2 \).

The generating function for the number of nodes, \( B_m \), at the shell \( m \) is \[22\]:

\[ \tilde{G}_m(x) = G_0(G_1(...(G_1(x)))) = G_0(G_1^{m-1}(x)), \]

where \( G_1(G_1(...)) \equiv G_1^{m-1}(x) \) is the result of applying \( G_1(x) \), \( m - 1 \) times. \( \tilde{P}(B_m) \), which is the probability distribution of \( B_m \), is the coefficient of \( x^{B_m} \) in the Taylor expansion of \( \tilde{G}_m(x) \).

For shells with large \( m \) but still much smaller than \( d \), we expect \[22\] that the number of nodes will increase by a factor of \( \tilde{k} \). It is possible to show \[20\] that \( G_1^{m-1}(x) \) converges to a function of the form \( f(((1 - x)\tilde{k}^m) \) for large \( m \) (\( m << d \)), and \( f(x) \) satisfies the Poincaré
The functional relation: 
\[ G_1(f(y)) = f(y\tilde{k}), \]  
where \( y = 1 - x \). The function form of \( f(y) \) can be uniquely determined from Eq. (2).

The solution of \( G_1(f_\infty) = f_\infty \) gives the probability that a link is not connected to the giant component of the network by one of its ends [21]. It is known [20] that \( f(x) \) has an asymptotic functional form, \( f(y) = f_\infty + ay^{-\delta} + 0(y^\delta) \). Expanding both sides of Eq. (2) we obtain:

\[ G_1(f_\infty) + G'_1(f_\infty)ay^{-\delta} = f_\infty + a\tilde{k}^{-\delta}y^{-\delta} + 0(y^\delta). \]  

Since \( G_1(f_\infty) = f_\infty \), we have \( \delta = -\ln G'_1(f_\infty)/\ln \tilde{k} \).

If \( q(1) = 0 \) and \( q(2) \neq 0 \), from \( G_1(f_\infty) = f_\infty \), we have \( f_\infty = 0 \) and \( G'_1(f_\infty) = G''_0(0)/\langle k \rangle = 2q(2)/\langle k \rangle \). If \( q(2) = q(1) = 0 \) (Böttcher case [20], then \( \delta = \infty \), which indicates that \( f(y) \) has an exponential singularity. Therefore, networks with minimum degree \( k_m \geq 3 \) do not exhibit the following properties for \( m << d \), and therefore have no fractal boundaries.

Applying Tauberian like theorems [20, 23] to \( f(y) \), which has a power-law behavior for \( y \to \infty \), Dubuc [24] concluded that the Taylor expansion coefficient of \( \tilde{G}_m(x) \), \( \tilde{P}(B_m) \), behaves as \( B_m^\mu \) with an exponential cutoff at \( B_m^* \sim \tilde{k}^m \). When \( q(1) \neq 0 \) and \( q(2) \neq 0 \), we have \( \mu = \delta - 1 \) and when \( q(1) = 0 \) and \( q(2) \neq 0 \), we have \( \mu = 2\delta - 1 \). Thus the distribution of the number of nodes in the shell \( m \) with \( m << d \) has a power law tail for small values of \( B_m \):

\[ \tilde{P}(B_m) \sim B_m^\mu. \]  

For ER network, Eq. (4) is supported by simulations for \( m \leq d \) in Fig. 3c.

The above considerations are correct only for \( m < d \), for which the depletion of nodes with large degree in the network is insignificant.

In a large network, the shells with \( m >> 1 \) behave almost deterministically and there exists a functional relation between any two shell \( m \) and shell \( n \) with \( n > m \) (a detailed proof will be given elsewhere):

\[ r_n = G_0(G_1^{m}G_0^{-1}(r_m)), \]  

where \( r_n \) is the fraction of nodes outside shell \( n \). It can also be shown that the branching factor in the \( r_n \) fraction of nodes is \( \tilde{k}(r_n) = uG''_0(u)/G'_0(u) \), where \( u = G_0^{-1}(r_n) \).
For ER networks, Eq. (5) yields:

\[ r_{\ell+1} = e^{(k(r_{\ell-1})} = \sum_{\ell=0}^{\infty} q(k) r_{\ell}^k, \] (6)

which is valid for all possible \( \ell \). We test it in Fig. 3d.

When \( m << d \) and \( n >> d \), using the same considerations as before it can be shown that:

\[ r_n = [a\tilde{k}(1 - r_m)]^{-\mu - 1} + r_\infty, \] (7)

where \( r_\infty = G_0(f_\infty) \) is the fraction of nodes not belonging to the giant component of the network, \( a \) is a constant.

Based on Eqs. (4) and (7), expressing \( r_m \) and \( r_n \) in terms of \( B_m \) and \( B_n \), we find that for \( m << d \) and \( n >> d \), \( B_n \sim B_m^{1-\mu/(\mu+1)-1/(\mu+1)} = B_n^{-2} \),

supporting the numerical findings in Fig. 2.

These results are rigorous when \( \tilde{k} \) exists and when the minimum degree \( k_m \leq 2 \). For SF networks with \( \lambda < 3 \), \( \tilde{k} \) diverges for \( N \rightarrow \infty \). But for finite \( N \), \( \tilde{k} \) still exists. Thus the above results can also be applied to the case of \( \lambda < 3 \). For both ER and SF networks with \( k_m \geq 3 \), the power law of \( P(B_n) \) with \( n >> d \) cannot be observed, as we indeed confirm by simulations.

Relating our problem with percolation theory, we can explain the simulation results of probability distribution of cluster size \( s_\ell \). The cluster size distribution in percolation at some concentration \( p \) close to \( p_c \) is determined by the formula [13]:

\[ P_p(s > S) \sim S^{-\tau + 1} \exp(-S|p - p_c|^{1/\sigma}). \] (9)

In the case of random networks the percolation threshold is given by \( p_c = 1/\tilde{k} \). In the exterior of the shell \( n \) (\( n >> d \)), we can estimate \( |p - p_c| \sim (\tilde{k}(r_n) - 1)/\tilde{k} \), where \( \tilde{k}(r_n) \) decreases and reaches the critical percolation value of 1.

The cluster size distribution can be estimated by introducing a sharp exponential cutoff at \( s = S_n^* \sim |\tilde{k}(r_n) - 1|^{1/\tau} \), so that \( P_n(s > S) \sim S^{-\tau + 1} P(S_n^* > S) \), where \( P(S_n^* > S) \) is the probability for a given shell to have \( S_n^* > S \).

Since \( r_n - r_\infty \) has a smooth power law distribution and \( \tilde{k}(r_\infty) < 1 \), the probability that \( |\tilde{k}(r_n) - 1| < S^{-\sigma} = \varepsilon \) is proportional to \( \varepsilon \). Thus \( P(S_n^* > S) \sim S^{-\sigma} \) and \( P_n(s > S) = S^{-\tau + 1-\sigma} \) [25]. Therefore the cluster size distribution follows \( n(s) \sim s^{-(\tau+\sigma)} \).
For ER networks and SF networks with $\lambda > 4$, $\tau = 2.5$ and $\sigma = 0.5$, the above derivations lead to $n(s) \sim s^{-3}$. For SF networks with $2 < \lambda < 4$, $\tau = (2\lambda - 3)/(\lambda - 2)$ and $\sigma = |\lambda - 3|/(\lambda - 2)$ [18]. Thus, for SF network with $\lambda > 3$, there will be $n_s \sim s^{-3}$. We conjecture $n_s \sim s^{-3}$ even for $2 < \lambda < 3$, although in this case $\tilde{k}(r_n)$ does not exist and the above derivations are not valid. Our numerical simulations support these results in Fig. 4a, b.

In summary, we find empirically and analytically that the boundaries of a broad class of complex networks including non-fractal networks [9] have fractal features. Our findings can be applied to the study of epidemics. It implies that a strong decay of the epidemic will happen in the boundaries of human network, due to the low degree of nodes. The fractal clusters at the boundaries, which are connected sparsely to the bulk, may explain why big breakout of epidemical disease (such as the “black death” in medieval Europe) would suddenly stop after affecting a large percentage of population.

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