A UNIQUENESS THEOREM FOR TWISTED GROUPOID C*-ALGEBRAS

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Abstract. We present a uniqueness theorem for the reduced C*-algebra of a twist $\mathcal{E}$ over a Hausdorff étale groupoid $\mathcal{G}$. We show that the interior $\mathcal{I}^G$ of the isotropy of $\mathcal{E}$ is a twist over the interior $\mathcal{I}^G$ of the isotropy of $\mathcal{G}$, and that the reduced twisted groupoid C*-algebra $C^*_r(\mathcal{I}^G; \mathcal{I}^E)$ embeds in $C^*_r(\mathcal{G}; \mathcal{E})$. We also investigate the full and reduced twisted C*-algebras of the isotropy groups of $\mathcal{G}$, and we provide a sufficient condition under which states of (not necessarily unital) C*-algebras have unique state extensions. We use these results to prove our uniqueness theorem, which states that a C*-homomorphism of $C^*_r(\mathcal{G}; \mathcal{E})$ is injective if and only if its restriction to $C^*_r(\mathcal{I}^G; \mathcal{I}^E)$ is injective. We also show that if $\mathcal{G}$ is effective, then $C^*_r(\mathcal{G}; \mathcal{E})$ is simple if and only if $\mathcal{G}$ is minimal.

1. Introduction

1.1. Background. The purpose of this article is to prove a uniqueness theorem for the reduced C*-algebra of a twist $\mathcal{E}$ over a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$. The study of twisted groupoid C*-algebras was initiated by Renault [35], who generalised the construction of twisted group C*-algebras by building full and reduced C*-algebras $C^*(\mathcal{G}, \sigma)$ and $C^*_r(\mathcal{G}, \sigma)$ from a second-countable locally compact Hausdorff groupoid $\mathcal{G}$ admitting a left Haar system and a continuous 2-cocycle $\sigma$ on $\mathcal{G}$ taking values in the complex unit circle $\mathbb{T}$. Renault also realised $C^*(\mathcal{G}, \sigma)$ as a quotient of the C*-algebra associated to the extension of $\mathcal{G}$ by $\mathbb{T}$ defined by the 2-cocycle $\sigma$. This construction was subsequently extended by Kumjian [24] to include C*-algebras over groupoid twists that don’t necessarily arise from continuous 2-cocycles.

More recently, Renault [36] showed that every C*-algebra containing a Cartan subalgebra can be realised as a twisted groupoid C*-algebra, thereby providing a C*-algebraic analogue of Feldman–Moore theory [20, 21, 22]. Renault’s reconstruction theorem is of particular importance to the classification program for C*-algebras, given Li’s recent article [32] showing that every simple classifiable C*-algebra has a Cartan subalgebra (and is therefore a twisted groupoid C*-algebra), and the work of Barlak and Li [7, 8] describing the connections between the UCT problem and Cartan subalgebras in C*-algebras. The increasing interest in twisted groupoid C*-algebras (see, for instance, [3, 6, 10, 13, 16, 17, 18, 25]) has also recently inspired the introduction of twisted Steinberg algebras, which are a purely algebraic analogue of twisted groupoid C*-algebras (see [4, 5]).

Examples of twisted groupoid C*-algebras include the twisted C*-algebras associated to higher-rank graphs introduced by Kumjian, Pask, and Sims [26, 27, 28, 29], and the
more general class of twisted C*-algebras associated to topological higher-rank graphs introduced in the author’s PhD thesis [2].

In this article we prove a uniqueness theorem (Theorem 6.3) for the reduced C*-algebra of a twist \( E \) over a second-countable locally compact Hausdorff étale groupoid \( \mathcal{G} \). In particular, we show that the interior \( \mathcal{I}^E \) of the isotropy of \( \mathcal{E} \) is a twist over the interior \( \mathcal{I}^G \) of the isotropy of \( \mathcal{G} \), and that a C*-homomorphism of the reduced twisted C*-algebra \( C^*_r(\mathcal{G}; \mathcal{E}) \) is injective if and only if its restriction to \( C^*_r(\mathcal{I}^G; \mathcal{I}^E) \) is injective. This is an extension of the analogous result [14, Theorem 3.1(b)] for non-twisted groupoid C*-algebras, and also of the result [2, Theorem 5.3.14] appearing in the author’s PhD thesis for twisted groupoid C*-algebras arising from continuous 2-cocycles on groupoids. Although many of the arguments used in this article are inspired by their non-twisted counterparts, the twisted setting differs significantly enough from the non-twisted setting to warrant independent treatment. In particular, although \( \mathcal{G} \) is an étale groupoid, the twist \( \mathcal{E} \) is not an étale groupoid, and this leads to increased technical complexity in many of our proofs. One interesting corollary of our main theorem is that if \( \mathcal{G} \) is effective, then \( C^*_r(\mathcal{G}; \mathcal{E}) \) is simple if and only if \( \mathcal{G} \) is minimal (see Corollary 6.9).

1.2. Outline. This article is organised as follows. In Section 2 we establish the relevant background and notation, and we recall various well known useful results relating to twists and twisted groupoid C*-algebras. In particular, in Section 2.1 we recall the definition of a twist \( \mathcal{E} \) over a Hausdorff étale groupoid \( \mathcal{G} \), and we show that the interior \( \mathcal{I}^E \) of the isotropy of \( \mathcal{E} \) is a twist over the interior \( \mathcal{I}^G \) of the isotropy of \( \mathcal{G} \). In Section 2.2 we recall Kumjian’s construction of the full and reduced twisted groupoid C*-algebras \( C^*(\mathcal{G}; \mathcal{E}) \) and \( C^*_r(\mathcal{G}; \mathcal{E}) \), and in Proposition 2.15 we describe the relationship between these C*-algebras and Renault’s twisted groupoid C*-algebras arising from continuous 2-cocycles. This result can be used to translate the results of Sections 4 and 6 to the analogous results pertaining to twisted groupoid C*-algebras arising from continuous 2-cocycles that appear in the author’s PhD thesis [2] (see Remark 2.16).

Throughout Section 6 we regularly work with twisted C*-algebras associated to the isotropy groups of \( \mathcal{G} \) (which are discrete, since \( \mathcal{G} \) is étale), and so in Section 3 we restrict our attention to twisted C*-algebras associated to discrete groups in order to establish the necessary preliminaries. In particular, we recall the universal property of the full twisted group C*-algebra \( C^*(\mathcal{G}, \sigma) \) associated to a discrete group \( \mathcal{G} \) and a 2-cocycle \( \sigma \) on \( \mathcal{G} \), and in Theorem 3.2 we translate this universal property to the language of the associated central extension of \( \mathcal{G} \) by \( T \).

In Section 4 we show that the full and reduced twisted C*-algebras of the isotropy groups of \( \mathcal{I}^G \) are quotients of the full and reduced twisted C*-algebras of the groupoid \( \mathcal{I}^G \) itself (see Theorem 4.1). For the full C*-algebra, we quotient \( C^*(\mathcal{I}^G; \mathcal{I}^E) \) by the ideal generated by functions that vanish on the given isotropy group, but surprisingly, it turns out that this is not the correct ideal to quotient by in the reduced setting. Although the discovery of this fact did not cause us any problems when proving our main theorem, it was somewhat unexpected (at least to the author), and so we provide a proof using an example due to Willett [38] of a nonamenable groupoid whose full and reduced C*-algebras coincide (see Theorem 4.8).

A substantial portion of the author’s PhD thesis is dedicated to extending well known results of Anderson [1] about states of unital C*-algebras to the nonunital setting (see [2, Section 5.2]). We reproduce some of this material in Section 5 of the article, as we have been unable to find explicit proofs of these results in the literature, despite them apparently being well known (for instance, they are used in [14]). We apply these results to twisted groupoid C*-algebras in Section 6. The main result of Section 5 is Theorem 5.4,
in which we provide a sufficient compressibility condition under which states of (not necessarily unital) C*-algebras have unique state extensions.

In Section 6 we observe that there is an embedding \(\iota_r\) of \(C_\tau^*(\mathcal{I}^G; \mathcal{E})\) into \(C_\tau^*(\mathcal{G}; \mathcal{E})\), and that if \(\mathcal{I}^G\) is closed, then there is a conditional expectation from \(C_\tau^*(\mathcal{G}; \mathcal{E})\) to \(\iota_r(C_\tau^*(\mathcal{I}^G; \mathcal{E}))\) extending restriction of functions. We also show that these results hold for the full C*-algebras when \(\mathcal{I}^G\) is amenable. We then present our main theorem (Theorem 6.3), which states that a C*-homomorphism \(\Psi\) of \(C_\tau^*(\mathcal{G}; \mathcal{E})\) is injective if and only if the homomorphism \(\Psi \circ \iota_r\) of \(C_\tau^*(\mathcal{I}^G; \mathcal{E})\) is injective. We use this theorem to prove Corollary 6.9, which states that if \(\mathcal{G}\) is effective, then \(C_\tau^*(\mathcal{G}; \mathcal{E})\) is simple if and only if \(\mathcal{G}\) is minimal. The uniqueness theorem also has potential applications to the study of the ideal structure of twisted C*-algebras associated to Hausdorff étale groupoids, and in fact has already been used by the author, Brownlowe, and Sims in \cite{3} to characterise simplicity of twisted C*-algebras associated to Deaconu–Renault groupoids.

2. Preliminaries

In this section we present the necessary background on twists over Hausdorff étale groupoids and the associated (full and reduced) twisted groupoid C*-algebras. Although groupoid C*-algebras were introduced by Renault in \cite{35}, we will frequently reference Sims’ treatise \cite{37} on Hausdorff étale groupoids and their C*-algebras instead, as it aligns more closely with our setting. The results in this section are presumably well known, but we have presented proofs wherever we have been unable to find them in the literature, or whenever we have felt the need to expand on the level of detail given in existing literature. We begin by recalling some preliminaries on groupoids from \cite[Chapter 8]{37}.

Throughout this article, \(\mathcal{G}\) will denote a second-countable locally compact Hausdorff groupoid with unit space \(\mathcal{G}^{(0)}\), which is étale in the sense that the range and source maps \(r, s\) are local homeomorphisms. We refer to such a groupoid as a Hausdorff étale groupoid, and we denote the set of composable pairs in \(\mathcal{G}\) by \(\mathcal{G}^{(2)}\). If \(\mathcal{G}\) is étale, then \(\mathcal{G}\) admits a Haar system, \(\mathcal{G}^{(0)}\) is an open subset of \(\mathcal{G}\), and the range, source, and multiplication maps are all open. We call a subset \(B\) of \(\mathcal{G}\) a bisection if there is an open subset \(U\) of \(\mathcal{G}\) containing \(B\) such that \(r|_U\) and \(s|_U\) are homeomorphisms onto open subsets of \(\mathcal{G}^{(0)}\). Every Hausdorff étale groupoid has a (countable) basis of open bisections. Given subsets \(A, B \subseteq \mathcal{G}\), we write \(AB := \{\alpha\beta : (\alpha, \beta) \in (A \times B) \cap \mathcal{G}^{(2)}\}\) and \(A^{-1} := \{\alpha^{-1} : \alpha \in A\}\), and for \(\gamma \in \mathcal{G}\), we write \(\gamma A := \{\gamma\}A\) and \(A\gamma := A\{\gamma\}\). Given \(u \in \mathcal{G}^{(0)}\), we define \(\mathcal{G}_u := s^{-1}(u), \mathcal{G}_u^* := r^{-1}(u)\), and \(\mathcal{G}_u^u := \mathcal{G}_u \cap \mathcal{G}_u^{(0)}\). For each \(u \in \mathcal{G}^{(0)}\), the relative topology on \(\mathcal{G}_u, \mathcal{G}_u^*, \) and \(\mathcal{G}_u^u\) is discrete, and \(\mathcal{G}_u^u\) is a countable closed subgroup of \(\mathcal{G}\), called an isotropy group. The isotropy subgroupoid of \(\mathcal{G}\) is the groupoid \(\text{Iso}(\mathcal{G}) := \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma)\}\). We write \(\mathcal{I}^\mathcal{G}\) for the topological interior of \(\text{Iso}(\mathcal{G})\), and we note that if \(\mathcal{G}\) is a Hausdorff étale groupoid, then so is \(\mathcal{I}^\mathcal{G}\). Since \(\mathcal{G}^{(0)}\) is open in \(\mathcal{G}\), the unit space of \(\mathcal{I}^\mathcal{G}\) is \(\mathcal{G}^{(0)}\). For each \(u \in \mathcal{G}^{(0)}\), \(\mathcal{I}^\mathcal{G}_u = \mathcal{I}^\mathcal{G} \cap \mathcal{G}_u^u\) is an isotropy group of \(\mathcal{I}^\mathcal{G}\). We say that \(\mathcal{G}\) is effective if \(\mathcal{I}^\mathcal{G} = \mathcal{G}^{(0)}\). We call a subset \(U\) of \(\mathcal{G}^{(0)}\) invariant if \(s(\gamma) \in U \implies r(\gamma) \in U\) for all \(\gamma \in \mathcal{G}\), and we say that \(\mathcal{G}\) is minimal if \(\mathcal{G}^{(0)}\) has no nonempty proper open (or, equivalently, closed) invariant subsets.

\footnote{We acknowledge that this notation is in fact redundant, because using the previously defined notation, we have \(\mathcal{G}_u = \mathcal{G}u, \mathcal{G}_u^* = u\mathcal{G},\) and \(\mathcal{G}_u^u = u\mathcal{G}u,\) for each \(u \in \mathcal{G}^{(0)}\). However, although the notation that omits the subscripts and superscripts is commonly used in the literature and is arguably more intuitive, we choose not to use it here because we feel that expressions for C*-algebras such as \(C_\tau^*(q(u)\mathcal{G}q(u); u\mathcal{E}u)\) look significantly cleaner when written in the form \(C_\tau^*(\mathcal{G}^{(0)}_q(u); E_u)\).}
2.1. Twists over Hausdorff étale groupoids. Groupoid twists and their associated $C^*$-algebras were introduced by Kumjian [24] and subsequently studied by Renault [36]; however, for consistency of terminology and notation, we will continue to reference Sims’ treatise [37]. We begin by recalling the definition of a twist from [37, Definition 11.1.1].

**Definition 2.1.** A twist $(\mathcal{E}, i, q)$ over a Hausdorff étale groupoid $\mathcal{G}$ is a sequence

$$\mathcal{G}^{(0)} \times \mathbb{T} \xleftarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{G},$$

where the groupoid $\mathcal{G}^{(0)} \times \mathbb{T}$ is viewed as a trivial group bundle with fibres $\mathbb{T}$, $\mathcal{E}$ is a locally compact Hausdorff groupoid with unit space $\mathcal{E}^{(0)} = i(\mathcal{G}^{(0)} \times \{1\})$, and the following additional conditions hold.

(a) The maps $i$ and $q$ are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces, and we identify $\mathcal{E}^{(0)}$ with $\mathcal{G}^{(0)}$ via $q|_{\mathcal{E}^{(0)}}$.

(b) The sequence is exact, in the sense that $i(\{x\} \times \mathbb{T}) = q^{-1}(x)$ for each $x \in \mathcal{G}^{(0)}$, $i$ is injective, and $q$ is surjective.

(c) The groupoid $\mathcal{E}$ is a locally trivial $\mathcal{G}$-bundle, in the sense that for each $\alpha \in \mathcal{G}$, there is an open neighbourhood $U_\alpha \subseteq \mathcal{G}$ of $\alpha$, and a continuous map $P_\alpha : U_\alpha \to \mathcal{E}$ such that

- (i) $q \circ P_\alpha = \text{id}_{U_\alpha}$; and
- (ii) the map $\phi_{P_\alpha} : (\beta, z) \mapsto i(r(\beta), z) P_\alpha(\beta)$ is a homeomorphism from $U_\alpha \times \mathbb{T}$ onto $q^{-1}(U_\alpha)$.

(d) The image of $i$ is central in $\mathcal{E}$, in the sense that $i(r(\varepsilon), z) \varepsilon = \varepsilon i(s(\varepsilon), z)$ for all $\varepsilon \in \mathcal{E}$ and $z \in \mathbb{T}$.

We sometimes denote a twist $(\mathcal{E}, i, q)$ over $\mathcal{G}$ simply by $\mathcal{E}$. We call a continuous map $P_\alpha : U_\alpha \to \mathcal{E}$ satisfying condition (c)(i) a (continuous) local section for $q$, and we call a collection $(U_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in \mathcal{G}}$ satisfying condition (c) a local trivialisation of $\mathcal{E}$.

If $\mathcal{G}$ is a discrete group, then a twist over $\mathcal{G}$ as defined above is a central extension of $\mathcal{G}$. It is well known (see, for instance, [15, Theorem IV.3.12]) that there is a one-to-one correspondence between central extensions of a discrete group and 2-cocycles on the group. This result does not hold in general for groupoids (see [33, Section 2] or [10, Section 3]); however, every continuous $\mathbb{T}$-valued 2-cocycle on a groupoid $\mathcal{G}$ does give rise to a twist over $\mathcal{G}$, as we show in Example 2.3. To make sense of this example, we first recall the definition of a groupoid 2-cocycle.

**Definition 2.2.** A continuous $\mathbb{T}$-valued 2-cocycle on a topological groupoid $\mathcal{G}$ is a continuous map $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying

(i) $\sigma(\alpha, \beta) \sigma(\alpha \beta, \gamma) = \sigma(\alpha, \beta \gamma) \sigma(\beta, \gamma)$, for all $\alpha, \beta, \gamma \in \mathcal{G}$ such that $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$; and

(ii) $\sigma(r(\gamma), \gamma) = \sigma(\gamma, s(\gamma)) = 1$, for all $\gamma \in \mathcal{G}$.

**Example 2.3.** Let $\mathcal{G}$ be a Hausdorff étale groupoid and let $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle. Let $\mathcal{E}_\sigma = \mathcal{G} \times_\sigma \mathbb{T}$ be the set $\mathcal{G} \times \mathbb{T}$ endowed with the product topology. The formulae

$$(\alpha, w)(\beta, z) := (\alpha \beta, \sigma(\alpha, \beta) wz) \quad \text{and} \quad (\alpha, w)^{-1} := (\alpha^{-1}, \sigma(\alpha, \alpha^{-1}) w)$$

define multiplication and inversion operations on $\mathcal{E}_\sigma$, under which $\mathcal{E}_\sigma$ is a locally compact Hausdorff groupoid. Let $i_\sigma : \mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E}_\sigma$ be the inclusion map and let $q_\sigma : \mathcal{E}_\sigma \to \mathcal{G}$ be the projection onto the first coordinate. Then $(\mathcal{E}_\sigma, i_\sigma, q_\sigma)$ is a twist over $\mathcal{G}$.

A routine argument shows that if $(\mathcal{E}, i, q)$ is a twist over a Hausdorff étale groupoid, then the formulae

$$z \cdot \varepsilon := i(r(\varepsilon), z) \varepsilon \quad \text{and} \quad \varepsilon \cdot z := \varepsilon i(s(\varepsilon), z)$$

define multiplication and inversion operations on $\mathcal{E}$.
define continuous free left and right actions of $\mathbb{T}$ on $\mathcal{E}$. Centrality of the image of $i$ implies that $z \cdot \varepsilon = \varepsilon \cdot z$ for all $z \in \mathbb{T}$ and $\varepsilon \in \mathcal{E}$. This action has the following additional properties.

**Lemma 2.4.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$ with local trivialisation $(U_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in \mathcal{G}}$.

(a) For each fixed $z \in \mathbb{T}$, the map $\varphi_z : \varepsilon \mapsto z \cdot \varepsilon$ is a homeomorphism of $\mathcal{E}$.

(b) If $\varepsilon, \zeta \in \mathcal{E}$ satisfy $q(\varepsilon) = q(\zeta)$, then there is a unique $z \in \mathbb{T}$ such that $\varepsilon = z \cdot \zeta$.

(c) For each $\alpha \in \mathcal{G}$, there is a unique continuous map $t_\alpha : q^{-1}(U_\alpha) \to \mathbb{T}$ such that $\phi_{P_\alpha}^{-1}(\varepsilon) = (q(\varepsilon), t_\alpha(\varepsilon))$ for all $\varepsilon \in q^{-1}(U_\alpha)$.

**Proof.** For part (a), fix $z \in \mathbb{T}$. Since the action of $\mathbb{T}$ on $\mathcal{E}$ is continuous, $\varphi_z$ is a continuous bijection with inverse $\varphi_{z^{-1}}$, and hence $\varphi_z$ is a homeomorphism. Part (b) is [37, Lemma 11.1.3]. For part (c), fix $\alpha \in \mathcal{G}$ and $\varepsilon \in q^{-1}(U_\alpha)$. Since $\phi_{P_\alpha} : U_\alpha \times \mathbb{T} \to q^{-1}(U_\alpha)$ is a homeomorphism, there is a unique pair $(\beta_\varepsilon, z_\varepsilon) \in U_\alpha \times \mathbb{T}$ such that $\varepsilon = \phi_{P_\alpha}(\beta_\varepsilon, z_\varepsilon) = z_\varepsilon \cdot P_\alpha(\beta_\varepsilon)$. Since $q \circ P_\alpha = \text{id}_{U_\alpha}$, we have $q(\varepsilon) = \beta_\varepsilon$, and since $\phi_{P_\alpha}$ is a homeomorphism, there is a unique continuous map $t_\alpha : q^{-1}(U_\alpha) \to \mathbb{T}$ given by $t_\alpha(\varepsilon) := z_\varepsilon$. \qed

We now show that the continuous local sections of Definition 2.1(c)(i) can always be chosen to be defined on bisections of $\mathcal{G}$, and to map units of $\mathcal{G}$ to units of $\mathcal{E}$.

**Lemma 2.5.** Every twist $(\mathcal{E}, i, q)$ over a Hausdorff étale groupoid $\mathcal{G}$ has a local trivialisation $(B_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in \mathcal{G}}$ such that for all $\alpha \in \mathcal{G}$, $B_\alpha$ is a bisection and $P_\alpha(B_\alpha \cap \mathcal{G}^{(0)}) \subseteq \mathcal{E}^{(0)}$.

**Proof.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$, and let $(U_\alpha, S_\alpha, \psi_{S_\alpha})_{\alpha \in \mathcal{G}}$ be a local trivialisation of $(\mathcal{E}, i, q)$. For each $\alpha \in \mathcal{G}$, let $D_\alpha$ be an open bisection of $\mathcal{G}$ such that $\alpha \in D_\alpha \subseteq U_\alpha$, and define

$$B_\alpha := \begin{cases} D_\alpha \cap \mathcal{G}^{(0)} & \text{if } \alpha \in \mathcal{G}^{(0)} \\ D_\alpha \setminus \mathcal{G}^{(0)} & \text{if } \alpha \notin \mathcal{G}^{(0)}. \end{cases}$$

Since $\mathcal{G}$ is a Hausdorff étale groupoid, $\mathcal{G}^{(0)}$ is clopen, and hence each $B_\alpha$ is an open bisection of $\mathcal{G}$ containing $\alpha$.

There are now two cases to deal with. For the first case, fix $\alpha \in \mathcal{G} \setminus \mathcal{G}^{(0)}$. Define $P_\alpha := S_\alpha|_{B_\alpha}$ and $\phi_{P_\alpha} := \psi_{S_\alpha}|_{B_\alpha \times \mathbb{T}}$. It is clear that $P_\alpha$ is a continuous local section for $q$ satisfying $P_\alpha(B_\alpha \cap \mathcal{G}^{(0)}) \subseteq \mathcal{E}^{(0)}$, and that $\phi_{P_\alpha}$ satisfies Definition 2.1(c)(ii). For the second case, fix $x \in \mathcal{G}^{(0)}$. Then $B_x \subseteq \mathcal{G}^{(0)}$. Define $P_x : B_x \to \mathcal{E}$ by $P_x(y) := (q|_{\mathcal{E}^{(0)}})^{-1}(y)$. Then $P_x$ is continuous because $q|_{\mathcal{E}^{(0)}}$ a homeomorphism and the inclusion map $\mathcal{E}^{(0)} \to \mathcal{E}$ is continuous. It is clear that $q \circ P_x = \text{id}_{B_x}$, and so $P_x$ is a continuous local section for $q$ satisfying $P_x(B_x \cap \mathcal{G}^{(0)}) \subseteq \mathcal{E}^{(0)}$. By Lemma 2.4(c), there is a unique continuous map $t_x : q^{-1}(U_x) \to \mathbb{T}$ such that $\psi_{S_x}^{-1}(\varepsilon) = (q(\varepsilon), t_x(\varepsilon))$ for all $\varepsilon \in q^{-1}(U_x)$. In particular, for all $y \in B_x$, we have

$$\psi_{S_x}^{-1}(P_x(y)) = (q(P_x(y)), t_x(P_x(y))) = (y, t_x(P_x(y))). \quad (2.1)$$

Define $f_x : B_x \times \mathbb{T} \to B_x \times \mathbb{T}$ by $f_x(y, z) := (y, z \cdot t_x(P_x(y)))$. Since $t_x$ and $P_x$ are continuous, $f_x$ is a homeomorphism with inverse $f_x^{-1} : (y, z) \mapsto (y, z \cdot t_x^{-1}(P_x(y)))$. Define $\psi_{S_x} := \psi_{S_x} \circ f_x$. Then $\phi_{P_x}$ is a homeomorphism from $B_x \times \mathbb{T}$ onto $q^{-1}(B_x)$. Fix $(y, z) \in B_x \times \mathbb{T}$. Using the definition of $\psi_{S_x}$ and that $i$ is a homeomorphism for the second equality and using equation (2.1) for the final equality, we see that

$$\phi_{P_x}(y, z) = \psi_{S_x}(y, z \cdot t_x(P_x(y))) = i(y, z) i(y, t_x(P_x(y))) S_x(y) = i(y, z) \psi_{S_x}(y, t_x(P_x(y))) = i(y, z) P_x(y).$$

Thus we have constructed a local trivialisation $(B_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in \mathcal{G}}$ for $(\mathcal{E}, i, q)$ with the desired properties. \qed
Remark 2.6. In some texts (see, for instance, [10, Definition 3.1]), the existence of continuous local sections (and of the induced local trivialisation) is omitted from the definition of a twist \((E, i, q)\), and instead, \(i\) is defined to be a homeomorphism onto the open set \(q^{-1}(G(0))\), and \(q\) is defined to be a continuous open surjection. These conditions imply that \(q\) admits continuous local sections (see [10, Proposition 3.4]), and since \(i\) has a continuous inverse defined on \(q^{-1}(G(0))\), an argument similar to the one used in the proof of [5, Proposition 4.8(c)] shows that these local sections induce a local trivialisation of the twist. Hence the twists of [10, Definition 3.1] are twists in the sense of Definition 2.1. On the other hand, in Lemma 2.7 we show that, given a twist \((E, i, q)\) in the sense of Definition 2.1, the map \(i\) is a homeomorphism onto the open set \(q^{-1}(G(0))\), and the map \(q\) is a continuous open surjection. Hence Definition 2.1 is in fact equivalent to [10, Definition 3.1].

Lemma 2.7. Let \((E, i, q)\) be a twist over a Hausdorff étale groupoid \(G\).

(a) The map \(q\) is a continuous open surjection, and \(G\) has the quotient topology.
(b) The range, source, and multiplication maps on \(E\) are all open.
(c) The map \(i\) is a homeomorphism onto \(q^{-1}(G(0))\), which is an open subset of \(E\).

Proof. For part (a), note that \(q\) is a continuous surjection by Definition 2.1. We first show that \(G\) has the quotient topology. Let \(X\) be a subset of \(G\). If \(X\) is open, then \(q^{-1}(X)\) is open in \(E\), because \(q\) is continuous. Suppose instead that \(q^{-1}(X)\) is open in \(E\). We must show that \(X\) is open in \(G\). Choose a local trivialisation \((U_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in G}\) of \((E, i, q)\). Fix \(\alpha \in X\). The set \(q^{-1}(X) \cap q^{-1}(U_\alpha)\) is an open subset of \(q^{-1}(U_\alpha)\) that is closed under the action of \(T\) on \(E\), and hence its (open) image under \(\phi_{P_\alpha}^{-1}\) is of the form \(V_\alpha \times T\), for some open subset \(V_\alpha\) of \(U_\alpha\). We have

\[V_\alpha = q(\phi_{P_\alpha}(V_\alpha \times T)) = q^{-1}(X) \cap q^{-1}(U_\alpha) = X \cap U_\alpha \subseteq X,\]

and so \(V_\alpha\) is an open neighbourhood of \(\alpha\) contained in \(X\). Hence \(X\) is open in \(G\), and \(q\) is a quotient map. We now show that \(q\) is an open map. Let \(Y\) be an open subset of \(E\).

Since \(G\) has the quotient topology, \(q(Y)\) is open in \(G\) if and only if \(q^{-1}(q(Y))\) is open in \(E\). Recall from Lemma 2.4(a) that for each \(z \in T\), the map \(\varphi_z : \varepsilon \mapsto z \cdot \varepsilon\) is a homeomorphism of \(E\), and so \(\varphi_z(Y)\) is open. Since \(z \in q^{-1}(q(Y))\) if and only if \(z \cdot \varepsilon = z \cdot \zeta\) for some \(z \in T\) and \(\varepsilon, \zeta \in Y\), we have \(q^{-1}(q(Y)) = T \cdot Y = \bigcup_{z \in T} \varphi_z(Y)\), which is open.

For part (b), note that the range and source maps of \(G\) are open because \(G\) is étale. Since \(q\) restricts to a homeomorphism of unit spaces, it follows that the range and source maps of \(E\) are open, and thus the multiplication map on \(E\) is open by [37, Lemma 8.4.11].

For part (c), note that \(i(G(0) \times T) = q^{-1}(G(0))\) is open in \(E\) because \(G(0)\) is open in \(G\). To see that \(i\) is an open map, let \(U \subseteq G(0)\) and \(W \subseteq T\) be open sets, and use Lemma 2.5 to find a local trivialisation \((B_\alpha, P_\alpha, \phi_{P_\alpha})_{\alpha \in G}\) of \((E, i, q)\) such that for all \(\alpha \in G\), \(B_\alpha\) is a bisection of \(G\) and \(P_\alpha(B_\alpha \cap G(0)) \subseteq E(0)\). For each \(x \in U\), define \(D_x := B_x \cap U\), so that \(D_x \subseteq G(0)\) and \(U = \bigcup_{x \in U} D_x\). Fix \(x \in U\). Since \(P_x(D_x) \subseteq E(0)\), we have \(\phi_{P_x}(D_x \times W) = i(D_x \times W)\). By Definition 2.1(c)(ii), \(\phi_{P_x}\) is a homeomorphism onto the open set \(q^{-1}(B_x)\), and so since \(D_x \times W\) is open, it follows that \(i(D_x \times W)\) is open in \(E\). Hence \(i(U \times W) = \bigcup_{x \in U} i(D_x \times W)\) is an open subset of \(E\), and thus \(i\) is an open map. 

\[\Box\]

Definition 2.8. Let \((E, i, q)\) be a twist over a Hausdorff étale groupoid \(G\). Any map \(P : G \to E\) satisfying \(q \circ P = \text{id}_G\) is called a (global) section for \(q\).

The following result shows that there is a one-to-one correspondence between continuous 2-cocycles on a Hausdorff étale groupoid \(G\) and twists over \(G\) admitting continuous global sections. Note that a twist over a Hausdorff étale groupoid need not admit any continuous global sections.
Proposition 2.9. Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(\mathcal{G}\). Suppose that \(P: \mathcal{G} \rightarrow \mathcal{E}\) is a continuous global section for \(q\). Then there is a continuous 2-cocycle \(\sigma: \mathcal{G}(2) \rightarrow \mathbb{T}\) such that \(P(\alpha) P(\beta) = \sigma(\alpha, \beta) \cdot P(\alpha \beta)\) for all \((\alpha, \beta) \in \mathcal{G}(0)\). Let \((\mathcal{E}_\sigma, i_\sigma, q_\sigma)\) be the twist defined in Example 2.3. The map \(\phi_P: \mathcal{E}_\sigma \rightarrow \mathcal{E}\) given by \(\phi_P(\gamma, z) := z \cdot P(\gamma)\) defines an isomorphism of twists, in the sense that \(\phi_P\) is a topological groupoid isomorphism that makes the diagram

\[
\begin{array}{ccc}
\mathcal{G}(0) \times \mathbb{T} & \xrightarrow{\phi_P} & \mathcal{G} \\
\downarrow \gamma & & \downarrow q \\
\mathcal{E} & & \mathcal{E}_\sigma
\end{array}
\]

commute. Moreover, there is a continuous global section \(S: \mathcal{G} \rightarrow \mathcal{E}\) for \(q\) satisfying \(S(\mathcal{G}(0)) \subseteq \mathcal{E}(0)\).

Proof. By [24, Section 4, Fact 1], every continuous global section for \(q\) induces a continuous 2-cocycle satisfying the given formula. It is observed in [24, Section 4, Remark 2] that the map \(\phi_P: (\gamma, z) \mapsto z \cdot P(\gamma)\) defines an isomorphism of the twists \((\mathcal{E}_\sigma, i_\sigma, q_\sigma)\) and \((\mathcal{E}, i, q)\). (Alternatively, see the proof of the analogous result [5, Proposition 4.8] for discrete twists, which holds in our non-discrete setting.) To see that continuous global sections can be chosen to map units to units, observe that \((\mathcal{G}, P, \phi_P)_{a \in \mathcal{G}}\) is a local trivialisation of \(\mathcal{E}\), and so we can apply the argument of Lemma 2.5 (without replacing \(\mathcal{G}\) by bisections in the local trivialisation).

The following result and the subsequent corollary will be frequently used throughout the remainder of the article, without necessarily being explicitly referenced. These results allow us to restrict our attention to twists over the unit space, the interior of the isotropy, or the isotropy groups of a Hausdorff étale groupoid.

Lemma 2.10. Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(\mathcal{G}\). Suppose that \(\mathcal{H}\) is an open or closed étale subgroupoid of \(\mathcal{G}\). Define \(\mathcal{E}_{\mathcal{H}} := q^{-1}(\mathcal{H}), i_{\mathcal{H}} := i|_{\mathcal{H}(0) \times \mathbb{T}}\), and \(q_{\mathcal{H}} := q|_{\mathcal{E}_{\mathcal{H}}}\). Then \(\mathcal{H}\) is a Hausdorff étale groupoid, and \((\mathcal{E}_{\mathcal{H}}, i_{\mathcal{H}}, q_{\mathcal{H}})\) is a twist over \(\mathcal{H}\).

Proof. The argument used in the proof of [12, Lemma 2.7] applies in both the open and closed settings.

Corollary 2.11. Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(\mathcal{G}\).

(a) The groupoid \(q^{-1}(\mathcal{G}(0))\) is a twist over \(\mathcal{G}(0)\).

(b) The isotropy subgroupoid \(\mathcal{I}_\mathcal{E}\) of \(\mathcal{E}\) is a twist over the isotropy subgroupoid \(\mathcal{I}_\mathcal{G}\) of \(\mathcal{G}\).

(c) For each \(u \in \mathcal{E}(0)\), the isotropy group \(\mathcal{I}_\mathcal{E}(u)\) is a twist over the isotropy group \(\mathcal{I}_\mathcal{G}(u)\).

Proof. Part (a) follows immediately from Lemma 2.10 since \(\mathcal{G}(0)\) is open. For part (b), we will show that \(q^{-1}(\mathcal{I}_\mathcal{G}) = \mathcal{I}_\mathcal{E}\), because the result then follows immediately from Lemma 2.10 since \(\mathcal{I}_\mathcal{G}\) is an open étale subgroupoid of \(\mathcal{G}\). We first show that \(q^{-1}(\text{Iso}(\mathcal{G})) = \text{Iso}(\mathcal{E})\). For this, fix \(\varepsilon \in \mathcal{E}\). Since \(q: \mathcal{E} \rightarrow \mathcal{G}\) is a groupoid homomorphism that restricts to a homeomorphism of unit spaces, we have

\[r(q(\varepsilon)) = s(q(\varepsilon)) \iff q(r(\varepsilon)) = q(s(\varepsilon)) \iff r(\varepsilon) = s(\varepsilon),\]

and so \(q(\varepsilon) \in \text{Iso}(\mathcal{G})\) if and only if \(\varepsilon \in \text{Iso}(\mathcal{E})\). Thus \(q^{-1}(\text{Iso}(\mathcal{G})) = \text{Iso}(\mathcal{E})\), as claimed. Since \(\mathcal{I}_\mathcal{G}\) is open and \(q\) is continuous, \(q^{-1}(\mathcal{I}_\mathcal{G})\) is an open subset of \(\mathcal{E}\) contained in \(\text{Iso}(\mathcal{E})\), and so \(q^{-1}(\mathcal{I}_\mathcal{G}) \subseteq \mathcal{I}_\mathcal{E}\). Since \(\mathcal{I}_\mathcal{E}\) is open and \(q\) is an open map by Lemma 2.7(a), \(q(\mathcal{I}_\mathcal{E})\) is an open subset of \(\mathcal{G}\) contained in \(\text{Iso}(\mathcal{G})\), and hence \(\mathcal{I}_\mathcal{E} \subseteq q^{-1}(\mathcal{I}_\mathcal{G})\). Therefore, \(q^{-1}(\mathcal{I}_\mathcal{G}) = \mathcal{I}_\mathcal{E}\). For part (c), fix \(u \in \mathcal{E}(0)\). The proof of part (b) implies that \(q^{-1}(\mathcal{I}_\mathcal{G}(u)) = \mathcal{I}_\mathcal{E}(u)\), and so the result follows from Lemma 2.10 since \(\mathcal{I}_\mathcal{G}(u)\) is a discrete closed subgroupoid of \(\mathcal{I}_\mathcal{G}\).
2.2. Twisted groupoid $C^*$-algebras. In this section we recall Kumjian’s construction (given in [24]) of the full and reduced twisted groupoid $C^*$-algebras associated to a twist over a Hausdorff étale groupoid. We also recall Renault’s construction (given in [35]) of the full and reduced twisted groupoid $C^*$-algebras arising from a continuous 2-cocycle on a Hausdorff étale groupoid, and we describe the relationship between these two constructions in Proposition 2.15.

Given a locally compact Hausdorff space $X$, we write $C(X)$ for the vector space of continuous complex-valued functions on $X$ under pointwise operations. For each $f \in C(X)$, we define $\text{osupp}(f) := f^{-1}(\mathbb{C} \setminus \{0\})$, and we write $\text{supp}(f)$ for the closure of $\text{supp}(f)$ in $X$. We define $C_c(X) := \{f \in C(X) : \text{supp}(f) \text{ is compact}\}$, and we write $C_0(X)$ for the collection of continuous functions on $X$ vanishing at infinity, which is the completion of the subspace $C_c(X)$ with respect to the uniform norm $\|\cdot\|_\infty$.

Suppose that $(\mathcal{E}, i, q)$ is a twist over a Hausdorff étale groupoid $\mathcal{G}$. For each $\gamma \in \mathcal{G}$, the set $q^{-1}(\gamma)$ is homeomorphic to $\mathbb{T}$, since $\mathcal{E}$ is a locally trivial $\mathcal{G}$-bundle. Since the Haar measure on $\mathbb{T}$ is rotation-invariant, pulling it back to $q^{-1}(\gamma)$ gives a measure that is independent of our choice of $\varepsilon \in q^{-1}(\gamma)$. For each $u \in \mathcal{E}^{(0)}$, we endow $\mathcal{E}_u$ with a measure $\lambda_u$ that agrees with these pulled back copies of Haar measure on $q^{-1}(\gamma)$ for each $\gamma \in \mathcal{G}_{\varepsilon_u}$, and so each $q^{-1}(\gamma)$ has measure 1. We define a measure $\lambda^u$ on each $\mathcal{E}_u$ in a similar fashion.

We say that a function $f : \mathcal{E} \to \mathbb{C}$ is $\mathbb{T}$-equivariant if $f(z \cdot \varepsilon) = z f(\varepsilon)$ for all $z \in \mathbb{T}$ and $\varepsilon \in \mathcal{E}$. The collection

$$\Sigma_c(\mathcal{G}; \mathcal{E}) := \{f \in C_c(\mathcal{E}) : f \text{ is } \mathbb{T}\text{-equivariant}\}$$

is a $*$-algebra under pointwise addition and scalar multiplication, multiplication given by the convolution formula

$$(fg)(\varepsilon) := \int_{\mathcal{E}_u(\varepsilon)} f(\varepsilon \xi^{-1}) g(\xi) \, d\lambda_u(\xi) = \int_{\mathcal{E}_v(\varepsilon)} f(\eta) g(\eta^{-1} \varepsilon) \, d\lambda^v(\eta),$$

and involution given by $f^*(\varepsilon) := \overline{f(-\varepsilon)}$. Taking $P : \mathcal{G} \to \mathcal{E}$ to be any (not necessarily continuous) global section for $q$, it follows from the $\mathbb{T}$-equivariance of $f, g \in \Sigma_c(\mathcal{G}; \mathcal{E})$ that for all $\varepsilon \in \mathcal{E},$

$$(fg)(\varepsilon) = \sum_{\alpha \in \mathcal{G}_{P(\varepsilon)^{-1}}} f(\varepsilon P(\alpha)^{-1}) g(P(\alpha)) = \sum_{\beta \in \mathcal{G}_{P^{-1}(\varepsilon)}} f(P(\beta)) g(P(\beta)^{-1} \varepsilon). \quad (2.2)$$

**Remark 2.12.** In [24, Section 2] Kumjian observes that $\Sigma_c(\mathcal{G}; \mathcal{E})$ can alternatively be regarded as a collection of sections of the complex line bundle $(\mathbb{C} \times \mathcal{E})/\mathbb{T}$ over $\mathcal{G}$, but we will only make use of this description when referencing external results that use it.

Although $C_c(\mathcal{G})$ is spanned by functions supported on open bisections of $\mathcal{G}$ (see, for instance, [37, Lemma 9.1.3]), this is not the case for $\Sigma_c(\mathcal{G}; \mathcal{E})$, because $\mathcal{E}$ is not étale. However, we do have the following similar result.

**Lemma 2.13.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Then

$$\Sigma_c(\mathcal{G}; \mathcal{E}) = \text{span}\{f \in \Sigma_c(\mathcal{G}; \mathcal{E}) : q(\text{supp}(f)) \text{ is a bisection of } \mathcal{G}\}.$$

**Proof.** Fix $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$. Since $q(\text{supp}(f))$ is compact, it can be covered by a finite collection $\{B_\gamma : \gamma \in F\}$ of open bisections of $\mathcal{G}$. As in [31, Remark 2.9], let $\{h_\gamma : \gamma \in F\}$ be a partition of unity subordinate to $\{B_\gamma \cap q(\text{supp}(f)) : \gamma \in F\}$. For each $\gamma \in F$, the function $f_\gamma : \varepsilon \mapsto f(\varepsilon) h_\gamma(q(\varepsilon))$ belongs to $\Sigma_c(\mathcal{G}; \mathcal{E})$, and $q(\text{supp}(f_\gamma)) \subseteq \text{supp}(h_\gamma) \subseteq B_\gamma$. Fix $\gamma \in F$. Since $\sum_{\gamma \in F} h_\gamma = 1$, we have $f = \sum_{\gamma \in F} f_\gamma$, which completes the proof. \qed
The full twisted groupoid $C^*$-algebra associated to the pair $(\mathcal{G}, \mathcal{E})$ is defined to be the completion $C^*(\mathcal{G}; \mathcal{E})$ of $\Sigma_c(\mathcal{G}; \mathcal{E})$ with respect to the full norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is a } *\text{-representation of } \Sigma_c(\mathcal{G}; \mathcal{E})\}.$$ 

For each unit $u \in \mathcal{E}^{(0)}$, there is a $*$-representation $\pi_u$ of $\Sigma_c(\mathcal{G}; \mathcal{E})$ on the Hilbert space $L^2(\mathcal{G}_{q(u)}; \mathcal{E}_u)$ consisting of square-integrable $\mathbb{T}$-equivariant functions on $\mathcal{E}_u$, which is given by extension of the convolution formula. We call each $\pi_u$ the regular representation of $\Sigma_c(\mathcal{G}; \mathcal{E})$ associated to $u$, and we write $\pi^L_u$ for the regular representation of $\Sigma_c(\mathcal{T}; \mathcal{E})$ associated to $u$. The reduced twisted groupoid $C^*$-algebra $C^*_r(\mathcal{G}; \mathcal{E})$ is defined to be the completion of $\Sigma_c(\mathcal{G}; \mathcal{E})$ with respect to the reduced norm

$$\|f\|_r := \sup\{\|\pi_u(f)\| : u \in \mathcal{E}^{(0)}\}.$$ 

For all $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$, we have $\|f\|_\infty \leq \|f\|_r \leq \|f\|$, with equality throughout if $q(\text{supp}(f))$ is a bisection of $\mathcal{G}$. If $\mathcal{G}$ is amenable, then the full and reduced norms agree on $\Sigma_c(\mathcal{G}; \mathcal{E})$.

We define

$$D_0 := \{f \in \Sigma_c(\mathcal{G}; \mathcal{E}) : q(\text{supp}(f)) \subseteq \mathcal{G}^{(0)}\} = \{f \in \Sigma_c(\mathcal{G}; \mathcal{E}) : \text{supp}(f) \subseteq i(\mathcal{G}^{(0)} \times \mathbb{T})\},$$

and note that there is a $*$-isomorphism $C_c(\mathcal{G}^{(0)}) \cong D_0$ mapping $h \in C_c(\mathcal{G}^{(0)})$ to the function $f_h : i(x, z) \mapsto z h(x)$ (see [37, Lemma 11.1.9]). This $*$-isomorphism extends to an isomorphism of $C_0(\mathcal{G}^{(0)})$ to the completion $D_0$ of $C^*_r(\mathcal{G}; \mathcal{E})$. There is a faithful conditional expectation $\Phi_r : C^*_r(\mathcal{G}; \mathcal{E}) \to D_0$ that extends restriction of functions in $\Sigma_c(\mathcal{G}; \mathcal{E})$ to the set $q^{-1}(\mathcal{G}^{(0)}) = i(\mathcal{G}^{(0)} \times \mathbb{T})$ (see [36, Proposition 4.3] and [37, Proposition 11.1.13]). We write $\Phi^L_r$ for the corresponding conditional expectation from $C^*_r(\mathcal{G}; \mathcal{E})$ to $D_0$. There is also a conditional expectation from $C^*(\mathcal{G}; \mathcal{E})$ to the completion of $D_0$ in the full norm that extends restriction of functions from $\Sigma_c(\mathcal{G}; \mathcal{E})$ to $q^{-1}(\mathcal{G}^{(0)})$, but this conditional expectation is not necessarily faithful.

We now show that every $*$-homomorphism of $\Sigma_c(\mathcal{G}; \mathcal{E})$ into a $C^*$-algebra $A$ extends uniquely to a $C^*$-homomorphism of $C^*(\mathcal{G}; \mathcal{E})$ into $A$. This is an extension of the well known result [2, Lemma 3.3.22] to the setting of $C^*$-algebras of groupoid twists.

**Lemma 2.14.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Suppose that $A$ is a $C^*$-algebra and that $\phi : \Sigma_c(\mathcal{G}; \mathcal{E}) \to A$ is a $*$-homomorphism. Then $\phi$ extends uniquely to a $C^*$-homomorphism $\overline{\phi} : C^*(\mathcal{G}; \mathcal{E}) \to A$ satisfying $\|\overline{\phi}\| = \|\phi\|$.

**Proof.** The result follows by a similar argument to the proof of [2, Lemma 3.3.22]. \qed

We now recall Renault’s construction of twisted groupoid $C^*$-algebras arising from continuous groupoid 2-cocycles. For our purposes, it suffices to consider a Hausdorff étale groupoid $\mathcal{G}$, although we note that Renault deals with groupoids that are not necessarily étale in [35]. Suppose that $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ is a continuous 2-cocycle. We write $C_c(\mathcal{G}, \sigma)$ for the $*$-algebra consisting of continuous, compactly supported, complex-valued functions on $\mathcal{G}$ equipped with pointwise addition and scalar multiplication, multiplication given by the twisted convolution formula

$$(fg)(\gamma) := \sum_{(\alpha, \beta) \in \mathcal{G}^{(2)}, \delta = \gamma} \sigma(\alpha, \beta)f(\alpha)g(\beta) = \sum_{\eta \in \mathcal{G}_\gamma(\gamma)} \sigma(\gamma^{-1}, \eta)f(\gamma^{-1})g(\eta),$$

and involution given by $f^*(\gamma) := \overline{\sigma(\gamma, \gamma^{-1})f(\gamma^{-1})}$. The full and reduced norms on $C_c(\mathcal{G}, \sigma)$ are defined in a similar fashion to the full and reduced norms on $\Sigma_c(\mathcal{G}; \mathcal{E})$, and by completing $C_c(\mathcal{G}, \sigma)$ with respect to these norms, we obtain the full and reduced twisted groupoid $C^*$-algebras $C^*(\mathcal{G}, \sigma)$ and $C^*_r(\mathcal{G}, \sigma)$, respectively.

In Section 2.1 we showed that every continuous 2-cocycle $\sigma$ on $\mathcal{G}$ gives rise to a twist $\mathcal{E}_\sigma$ over $\mathcal{G}$ by $\mathbb{T}$, and in Proposition 2.9 we showed that every twist $\mathcal{E}$ admitting a continuous
global section gives rise to a continuous 2-cocycle $\sigma$ such that $\mathcal{E}_\sigma \cong \mathcal{E}$. If $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is an isomorphism of twists over $\mathcal{G}$, then a routine argument shows that the map $f \mapsto f \circ \phi$ is an isomorphism from $\Sigma_c(\mathcal{G}; \mathcal{E}_2)$ to $\Sigma_c(\mathcal{G}; \mathcal{E}_1)$. Thus we can use Proposition 2.9 to describe the relationship between twisted groupoid C*-algebras arising from continuous 2-cocycles, and those arising from twists admitting continuous global sections, as follows.

**Proposition 2.15.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Suppose that $P : \mathcal{G} \to \mathcal{E}$ is a continuous global section for $q$. Let $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ be the continuous 2-cocycle induced by $P$, as described in Proposition 2.9. Let $\sigma$ denote the continuous 2-cocycle obtained by composing $\sigma$ with the complex conjugation map on $\mathbb{T}$. There is a *-isomorphism $\psi_P : \Sigma_c(\mathcal{G}; \Sigma) \to C_c(\mathcal{G}, \sigma)$ given by $\psi_P(f) := f \circ P$. This isomorphism extends to C*-isomorphisms $C^*(\mathcal{G}; \mathcal{E}) \cong C^*(\mathcal{G}, \sigma)$ and $C^*_r(\mathcal{G}; \mathcal{E}) \cong C^*_r(\mathcal{G}, \sigma)$.

**Proof.** Let $(\mathcal{E}_\sigma, i_\sigma, q_\sigma)$ be the twist defined in Example 2.3. Since $\mathcal{E} \cong \mathcal{E}_\sigma$ by Proposition 2.9, and since $(\gamma, 1) \mapsto \gamma$ is a continuous global section for $q_\sigma$, the result follows from the “$n = -1$” case of [13, Lemma 3.1(b)]. $lacksquare$

**Remark 2.16.** Using Proposition 2.15, the results in Sections 4 and 6 can be translated into analogous results for twisted groupoid C*-algebras arising from continuous 2-cocycles. Alternatively, we refer the reader to [2, Chapter 5] for the corresponding results written explicitly in this framework.

### 3. A universal property for twisted group C*-algebras

In this section, we describe twisted C*-algebras of countable discrete groups. Since every such group is a Hausdorff étale groupoid, the preliminaries given in Section 2 are all applicable here. In particular, since every twist over a discrete group admits a (trivially continuous) global section, Proposition 2.9 applies to twists over countable discrete groups. Thus, a twisted group C*-algebra can be viewed as arising either from a continuous $\mathbb{T}$-valued 2-cocycle on the group, or from a central extension of the group by $\mathbb{T}$. The main purpose of this section is to translate the universal property for the full twisted C*-algebra $C^*(G, \sigma)$ associated to a countable discrete group $G$ and a $\mathbb{T}$-valued 2-cocycle $\sigma$ on $G$ into the language of the associated central extension $G \times_\sigma \mathbb{T}$ of $G$ by $\mathbb{T}$ (see Theorem 3.2). This result is presumably well known, but we were unable to find an explicit statement of it in the literature.

Suppose that $G$ is a countable discrete group with identity $e$, and that $\sigma : G \times G \to \mathbb{T}$ is a 2-cocycle. Given a C*-algebra $A$ with identity $1_A$, we say that $u : G \to A$ is a $\sigma$-twisted unitary representation of $G$ in $A$ if each $u_g$ is a unitary element of $A$, and $u_g u_h = \sigma(g, h) u_{gh}$ for all $g, h \in G$. This implies that $u_e = 1_A$, and $u_g^* = \overline{\sigma(g, g^{-1})} u_{g^{-1}}$ for each $g \in G$. The map $\delta : G \to C^*(G, \sigma)$ sending each $g \in G$ to the point-mass function $\delta_g$ at $g$ is a $\sigma$-twisted unitary representation of $G$ such that $C^*(G, \sigma) = C^*(\{\delta_g : g \in G\})$, and the following universal property holds: if $B$ is a unital C*-algebra and $u : G \to B$ is a $\sigma$-twisted unitary representation of $G$ in $B$, then there is a homomorphism $\lambda_u : C^*(G, \sigma) \to B$ such that $\lambda_u(\delta_g) = u_g$ for each $g \in G$.

**Proposition 3.1.** Let $(E, i, q)$ be a twist over a countable discrete group $G$. Fix $\varepsilon \in E$, and define $\varepsilon^* : E \to \mathbb{C}$ by

$$
\varepsilon^*(\zeta) := \begin{cases} 
  w & \text{if } \zeta = w \cdot \varepsilon \text{ for some } w \in \mathbb{T} \\
  0 & \text{if } \zeta \neq w \cdot \varepsilon \text{ for all } w \in \mathbb{T}.
\end{cases}
$$
Then $\Sigma_c(G; E) = \text{span}\{\delta^\varepsilon : \varepsilon \in E\}$, and $C^*_c(G; E)$ and $C^*(G; E)$ are both unital with identity $\delta^e$, where $e$ is the identity of $E$. For all $\varepsilon, \zeta \in E$ and $z \in \mathbb{T}$, $\delta^\varepsilon$ is a unitary element of $\Sigma_c(G; E)$ satisfying $(\delta^\varepsilon)^* = \delta^{\varepsilon^{-1}}$, and we have $\delta^\varepsilon_{e, e} = \varepsilon \delta^\zeta_{e, e}$ and $\delta^\zeta_{\varepsilon, \zeta} = \delta^\varepsilon_{\zeta, \zeta}$.

**Proof.** By regarding $\Sigma_c(G; E)$ as sections of a line bundle (as in [12, Section 2]) and by performing some routine calculations, it can be seen that the material following [12, Remark 2.5] implies the result. \qed

In the following theorem, we use the universal property of $C^*(G, \sigma)$ to give a universal property for $C^*(G; E)$.

**Theorem 3.2.** Let $(E, i, q)$ be a twist over a countable discrete group $G$. Suppose that $\{t_\varepsilon : \varepsilon \in E\}$ is a family of unitary elements of a unital $C^*$-algebra $A$ such that $t_{\varepsilon \zeta} = \varepsilon t_\zeta$ and $t_\varepsilon t_\zeta = t_{\varepsilon \zeta}$ for all $\varepsilon, \zeta \in E$ and $z \in \mathbb{T}$. Then there is a homomorphism $\pi_i : C^*(G; E) \to A$ satisfying $\pi_i(\delta^\varepsilon) = t_\varepsilon$ for each $\varepsilon \in E$, where $\delta^\varepsilon$ is defined as in Proposition 3.1.

**Proof.** Recall from Propositions 2.9 and 2.15 that there is a 2-cocycle $\sigma : G \times \sigma \mathbb{T}$ such that $E \cong G \times_\sigma \mathbb{T}$ and $C^*(G; E) \cong C^*(G, \sigma)$. Let $e$ be the identity of $G$, and let $1_A$ be the identity of $A$. By [19, Example 2.8.14], there is a one-to-one correspondence between $\sigma$-twisted unitary representations of $G$ in $A$ and unitary representations $t$ of $G \times_\sigma \mathbb{T}$ in $A$ that satisfy $t_{(e, z)} = \varepsilon 1_A$ for all $z \in \mathbb{T}$. Such representations satisfy $t_{\varepsilon \zeta} = \varepsilon t_\zeta$ and $t_\varepsilon t_\zeta = t_{\varepsilon \zeta}$ for all $\varepsilon, \zeta \in E$. Thus the result follows from the universal property of $C^*(G, \sigma)$. \qed

4. **Twisted $C^*$-algebras associated to the interior of the isotropy of a Hausdorff étale groupoid**

In this section we study the relationships between the full and reduced twisted $C^*$-algebras associated to the interior $\mathcal{I}^\sigma$ of the isotropy of a Hausdorff étale groupoid $\mathcal{G}$ and the full and reduced twisted $C^*$-algebras associated to the isotropy groups of $\mathcal{I}^\sigma$. The results in this section are extensions of the results in [2, Section 5.1] to the setting of $C^*$-algebras of groupoid twists.

We saw in Corollary 2.11 that, given a twist $(\mathcal{E}, i, q)$ over a Hausdorff étale groupoid $\mathcal{G}$, the interior $\mathcal{I}^\varepsilon$ of the isotropy of $\mathcal{E}$ is a twist over $\mathcal{I}^\sigma$, and for each $u \in \mathcal{E}^{(0)}$, the isotropy group $\mathcal{I}_u^\varepsilon := \{\varepsilon \in \mathcal{I}^\varepsilon : r(\varepsilon) = s(\varepsilon) = u\}$ is a twist over the countable discrete group $\mathcal{I}^\sigma_{q(u)}$. Thus, recalling the notation defined in Proposition 3.1, we have

$$\Sigma_c(\mathcal{I}^\sigma_{q(u)}; \mathcal{I}^\varepsilon_u) = \text{span}\{\delta^\varepsilon : \varepsilon \in \mathcal{I}^\varepsilon_u\}.$$ 

**Theorem 4.1.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Fix $u \in \mathcal{E}^{(0)}$, and define $\mathcal{F}_u := \{f \in \Sigma_c(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon) : f|_{\mathcal{I}^\varepsilon_u} \equiv 0\}$.

(a) Let $\mathcal{J}^\sigma_u$ denote the closure of $\mathcal{F}_u$ in the reduced norm. Then $\mathcal{J}^\sigma_u$ is an ideal of $C^*_r(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)$, and there is a surjective $*$-homomorphism

$$\theta^\sigma_u : C^*_r(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)/\mathcal{J}^\sigma_u \to C^*_r(\mathcal{I}^\sigma_{q(u)}; \mathcal{I}^\varepsilon_u)$$

satisfying $\theta^\sigma_u(f + \mathcal{J}^\sigma_u) = f|_{\mathcal{I}^\varepsilon_u}$ for all $f \in \Sigma_c(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)$.

(b) Let $\mathcal{J}_u$ denote the closure of $\mathcal{F}_u$ in the full norm. Then $\mathcal{J}_u$ is an ideal of $C^*(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)$, and there is an isomorphism

$$\theta_u : C^*(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)/\mathcal{J}_u \to C^*(\mathcal{I}^\sigma_{q(u)}; \mathcal{I}^\varepsilon_u)$$

satisfying $\theta_u(f + \mathcal{J}_u) = f|_{\mathcal{I}^\varepsilon_u}$ for all $f \in \Sigma_c(\mathcal{I}^\sigma; \mathcal{I}^\varepsilon)$. 


Remark 4.2. Somewhat surprisingly, the map \( \theta_u^*: C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) / J^u_r \rightarrow C^*_r(\mathcal{I}^\theta_q(u); \mathcal{T}^\xi_u) \) of Theorem 4.1(a) is not in general an isomorphism, unlike the analogue for the full C*-algebras in Theorem 4.1(b). In Theorem 4.8, we prove this by using an example introduced by Willett in [38] of a nonamenable groupoid whose full and reduced C*-algebras coincide.

In order to prove Theorem 4.1, we need several preliminary results. We begin by showing that \( J^u_r \) and \( J_u \) are ideals of \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \) and \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \), respectively.

Lemma 4.3. Let \((E, i, q)\) be a twist over a Hausdorff étale groupoid \( G \). Fix \( u \in E(0) \), and define \( F_u = \{ f \in C_c(\mathcal{I}^\theta; \mathcal{T}^\xi) : f|_{\mathcal{I}^\theta_u} \equiv 0 \} \). Then \( F_u \) is an algebraic ideal of \( C_c(\mathcal{I}^\theta; \mathcal{T}^\xi) \).

Let \( J^u_r \) and \( J_u \) denote the closures of \( F_u \) in the reduced and full norms, respectively. Then \( J^u_r \) is an ideal of \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \), and \( J_u \) is an ideal of \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \).

Proof. It is clear that \( F_u \) is a linear subspace of \( C_c(\mathcal{I}^\theta; \mathcal{T}^\xi) \). To see that \( F_u \) is an algebraic ideal, fix \( f \in F_u \) and \( g \in C_c(\mathcal{I}^\theta; \mathcal{T}^\xi) \). For all \( \epsilon \in \mathcal{T}^\xi_u \), we have \( \epsilon^{-1} \epsilon = \mathcal{T}^\xi_u \), and so \( f^*(\epsilon) = f(\epsilon^{-1}) = 0 \). Thus \( F^* \subseteq F_u \), and thus \( F_u \) is an algebraic ideal of \( C_c(\mathcal{I}^\theta; \mathcal{T}^\xi) \). Since all C*-algebraic operations are continuous, it follows that \( J^u_r \) and \( J_u \) are ideals of \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \) and \( C^*_r(\mathcal{I}^\theta; \mathcal{T}^\xi) \), respectively. \( \square \)

Lemma 4.4. Let \((E, i, q)\) be a twist over a Hausdorff étale groupoid \( G \). Fix \( u \in E(0) \). For each \( \epsilon \in \mathcal{T}^\xi_u \), define \( \delta^\theta_\epsilon \in \Sigma_c(\mathcal{I}^\theta_q(u); \mathcal{T}^\xi_u) \) as in Proposition 3.1. Then for each \( \epsilon \in \mathcal{T}^\xi_u \), there exists \( j_\epsilon \in \Sigma_c(\mathcal{I}^\theta; \mathcal{T}^\xi) \) such that \( j_\epsilon|_{\mathcal{I}^\theta_u} = \delta^\theta_\epsilon \) and \( j_{\omega \epsilon} = \overline{\omega} j_\epsilon \) for each \( \omega \in \mathbb{T} \).

Proof. By Corollary 2.11(b), \( \mathcal{T}^\xi \) is a twist over \( \mathcal{I}^\theta \), and so by Lemma 2.5, we can find a local trivialisation \((B_\alpha, P_\alpha, \phi_\alpha)_{\alpha \in \mathcal{I}^\theta} \) for \( \mathcal{T}^\xi \) such that for each \( \alpha \in \mathcal{I}^\theta \), \( B_\alpha \) is an open bisection of \( \mathcal{I}^\theta \) containing \( \alpha \). Fix \( \epsilon \in \mathcal{T}^\xi_u \). Use Urysohn’s lemma to choose \( h_{q(\epsilon)} \in C_c(\mathcal{I}^\theta) \) such that \( \text{supp}(h_{q(\epsilon)}) \subseteq B_{q(\epsilon)} \) and \( h_{q(\epsilon)}(q(\epsilon)) = 1 \). Recall from Lemma 2.4(c) that for each \( \zeta \in q^{-1}(B_{q(\epsilon)}) \), there is a unique \( z_\zeta \in \mathbb{T} \) such that \( \zeta = \phi_{P_{q(\epsilon)}}(q(\epsilon), z_\zeta) \), and the map \( \zeta \mapsto z_\zeta \) is continuous on \( q^{-1}(B_{q(\epsilon)}) \). Thus \( \zeta \mapsto z_\zeta h_{q(\epsilon)}(q(\zeta)) \) is a continuous map from \( q^{-1}(B_{q(\epsilon)}) \) to \( \mathbb{C} \). Define \( j_\epsilon : \mathcal{T}^\xi \rightarrow \mathbb{C} \) by

\[
    j_\epsilon(\zeta) := \begin{cases} 
      \overline{z_\zeta} h_{q(\epsilon)}(q(\zeta)) & \text{if } \zeta \in q^{-1}(B_{q(\epsilon)}) \\
      0 & \text{if } \zeta \notin q^{-1}(B_{q(\epsilon)}).
    \end{cases}
\]

Then \( \text{supp}(j_\epsilon) \subseteq q^{-1}(\text{supp}(h_{q(\epsilon)})) = \mathbb{T} \cdot P_{q(\epsilon)}(\text{supp}(h_{q(\epsilon)})) \). Since \( \mathbb{T} \) and \( P_{q(\epsilon)}(\text{supp}(h_{q(\epsilon)})) \) are compact sets and the action of \( \mathbb{T} \) on \( \mathcal{T}^\xi_u \) is continuous, \( \text{supp}(j_\epsilon) \) is compact. Since \( j_\epsilon|_{q^{-1}(B_{q(\epsilon)})} \) is continuous and \( q^{-1}(B_{q(\epsilon)}) \) is open, \( j_\epsilon \) is continuous on all of \( \mathcal{T}^\xi_u \). To see that \( j_\epsilon \) is \( \mathbb{T} \)-equivariant, fix \( \zeta \in \mathcal{T}^\xi_u \) and \( w \in \mathbb{T} \). Then \( w \cdot \zeta \in q^{-1}(B_{q(\epsilon)}) \) if and only if \( \zeta \in q^{-1}(B_{q(\epsilon)}) \).

If \( \zeta \in q^{-1}(B_{q(\epsilon)}) \), then \( z_{w \cdot \zeta} = w z_\zeta \), and so \( j_\epsilon(w \cdot \zeta) = \overline{z_\zeta} w z_\zeta h_{q(\epsilon)}(q(\zeta)) = w j_\epsilon(\zeta) \). On the other hand, if \( \zeta \notin q^{-1}(B_{q(\epsilon)}) \), then \( j_\epsilon(w \cdot \zeta) = 0 = w j_\epsilon(\zeta) \). Hence \( j_\epsilon \in \Sigma_c(\mathcal{I}^\theta; \mathcal{T}^\xi_u) \).

We now show that \( j_\epsilon|_{\mathcal{I}^\theta_u} = \delta^\theta_\epsilon \). Fix \( \zeta \in \mathcal{T}^\xi_u \). Suppose first that \( \zeta \notin q^{-1}(B_{q(\epsilon)}) \). Then \( q(\zeta) \neq q(\epsilon) \), and so \( \zeta \neq w \cdot \epsilon \) for all \( w \in \mathbb{T} \). Hence \( j_\epsilon(\zeta) = 0 = \delta^\theta_\epsilon(\zeta) \). Now suppose that \( \zeta \in q^{-1}(B_{q(\epsilon)}) \). Since \( r_{q(\epsilon)} h_{q(\epsilon)}(q(\zeta)) = q(\epsilon) = r_{q(\epsilon)} h_{q(\epsilon)}(q(\zeta)) \) and \( B_{q(\epsilon)} \) is a bisection, we have \( q(\zeta) = q(\epsilon) \). Hence \( h_{q(\epsilon)}(q(\zeta)) = 1 \), and \( \zeta = z_\zeta \cdot P_{q(\epsilon)}(q(\epsilon)) = \overline{z_\zeta} z_\zeta \cdot \epsilon \). Therefore,

\[
    j_\epsilon(\zeta) = \overline{z_\zeta} z_\zeta h_{q(\epsilon)}(q(\zeta)) = \overline{z_\zeta} z_\zeta = \delta^\theta_\epsilon(\zeta),
\]

and so \( j_\epsilon|_{\mathcal{I}^\theta_u} = \delta^\theta_\epsilon \). Finally, for all \( w \in \mathbb{T} \), we have \( q(w \cdot \epsilon) = q(\epsilon) \) and \( \overline{w \cdot \epsilon} = \overline{w} \overline{\epsilon} \), and thus \( j_{w \cdot \epsilon} = \overline{w} j_\epsilon \). \( \square \)
We will use the following lemma to show that the map $\theta_u$ of Theorem 4.1(b) is injective.

**Lemma 4.5.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $G$. Fix $u \in \mathcal{E}^{(0)}$, and let $J_u$ be the ideal of $C^*(\mathcal{I}; \mathcal{I})$ defined in Lemma 4.3. For each $\varepsilon \in \mathcal{I}^c$, define $\varepsilon^\perp \in \Sigma_c(\mathcal{I}_q(u); \mathcal{I}_u)$ as in Proposition 3.1, and use Lemma 4.4 to choose $j_\varepsilon \in \Sigma_c(\mathcal{I}; \mathcal{I}^c)$ such that $j_\varepsilon|_{\mathcal{I}^\perp} = \delta_\varepsilon^\perp$. For any $\varepsilon \in \mathcal{I}_\varepsilon$ and $k_\varepsilon \in \Sigma_c(\mathcal{I}; \mathcal{I}^c)$ satisfying $k_\varepsilon|_{\mathcal{I}^\perp} = \delta_\varepsilon^\perp$, we have $j_\varepsilon - k_\varepsilon \in J_u$. The quotient $C^*$-algebra $C^*(\mathcal{I}; \mathcal{I}^c)/J_u$ is unital with identity $j_u + J_u$, and each $j_\varepsilon + J_u$ is a unitary element of $C^*(\mathcal{I}; \mathcal{I}^c)/J_u$. Moreover, $j_\varepsilon j_\zeta + J_u = j_\varepsilon j_\zeta + J_u$ for all $\varepsilon, \zeta \in \mathcal{I}_\varepsilon$.

**Proof.** For any $\varepsilon \in \mathcal{I}_\varepsilon$ and $k_\varepsilon \in \Sigma_c(\mathcal{I}; \mathcal{I}^c)$ satisfying $k_\varepsilon|_{\mathcal{I}^\perp} = \delta_\varepsilon^\perp$, we have $(j_\varepsilon - k_\varepsilon)|_{\mathcal{I}^\perp} \equiv 0$, and hence $j_\varepsilon - k_\varepsilon \in J_u$.

We now show that $C^*(\mathcal{I}; \mathcal{I}^c)/J_u$ is unital. Fix $f \in \Sigma_c(\mathcal{I}; \mathcal{I}^c)$. For all $\xi \in \mathcal{I}_\xi$, we have

$$(j_\varepsilon f)(\xi) = \int_{\mathcal{I}^\perp} j_\varepsilon(\eta) f(\eta^{-1}\xi) \, d\lambda(\eta) = \int_{\mathcal{I}} \delta_\varepsilon^\perp(w \cdot \varepsilon) f(w \cdot (\varepsilon^{-1}\xi)) \, dw = \int_{\mathcal{T}} \bar{w} \cdot f(\xi) \, dw = f(\xi).$$

Hence $(j_\varepsilon f - f)|_{\mathcal{I}^\perp} \equiv 0$, and so $j_\varepsilon f - f \in J_u$. A similar argument shows that $f j_\varepsilon - f \in J_u$. Thus

$$(f + J_u)(j_\varepsilon + J_u) = f j_\varepsilon + J_u = f + J_u = j_\varepsilon f + J_u = (j_\varepsilon + J_u)(f + J_u),$$

and so $j_\varepsilon + J_u$ is the identity element of $C^*(\mathcal{I}; \mathcal{I}^c)/J_u$.

**Proof of Theorem 4.1.** Fix $f \in \Sigma_c(\mathcal{I}; \mathcal{I})$. Then $f|_{\mathcal{I}^\perp}$ is continuous. We claim that $f|_{\mathcal{I}^\perp} \in \Sigma_c(\mathcal{I}_q(u); \mathcal{I}^\perp)$. We have osupp($f|_{\mathcal{I}^\perp}$) = osupp($f$) $\cap \mathcal{I}^\perp$, and so since $\mathcal{I}^\perp = I|^{(1)}_u$ is closed in $\mathcal{I}$, supp($f$) $\cap \mathcal{I}^\perp$ is a compact subset of $\mathcal{I}^\perp$. Hence supp($f|_{\mathcal{I}^\perp}$) $\subseteq$ supp($f$), and so $f|_{\mathcal{I}^\perp} \in C_c(\mathcal{I}^\perp)$. For all $\varepsilon \in \mathcal{I}^\perp$ and $z \in \mathcal{T}$, we have $z \cdot \varepsilon \in \mathcal{I}^\perp$ if and only if $\varepsilon \in \mathcal{I}^\perp$, and so $f|_{\mathcal{I}^\perp}(z \cdot \varepsilon) = z f|_{\mathcal{I}^\perp}(\varepsilon)$ for all $\varepsilon \in \mathcal{I}^\perp$ and $z \in \mathcal{T}$. Hence $f|_{\mathcal{I}^\perp} \in \Sigma_c(\mathcal{I}_q(u); \mathcal{I}^\perp)$.

For part (a), define $\psi_u^\perp : \Sigma_c(\mathcal{I}; \mathcal{I}^\perp) \to C_c^*(\mathcal{I}_q(u); \mathcal{I}_u^\perp)$ $\times (\mathcal{I}_q(u); \mathcal{I}_u)$ by $\psi_u^\perp(f) := f|_{\mathcal{I}^\perp}$. Routine calculations show that $\psi_u^\perp$ is a $*$-homomorphism that vanishes on $F_u$. We now show that $\psi_u^\perp$ is bounded in the reduced norm. Fix $f \in \Sigma_c(\mathcal{I}; \mathcal{I})$. For each $v \in \mathcal{E}^{(0)}$, let

$$(\pi_u^\perp : \Sigma_c(\mathcal{I}; \mathcal{I}) \to B(L^2(\mathcal{I}_q(u); \mathcal{I}_u)))$$

and

$$(\rho_u^\perp : \Sigma_c(\mathcal{I}_q(u); \mathcal{I}_u) \to B(L^2(\mathcal{I}_q(v); \mathcal{I}_v)))$$
be the regular representations associated to \(v\) of \(\Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\) and \(\Sigma_c(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\), respectively, onto the space of square-integrable \(\mathbb{T}\)-equivariant functions on \(\mathcal{I}^\varepsilon_u\). For each \(v \in \mathcal{E}^{(0)}\), we have \(\pi^F_v(f) = \rho^*_{I^\varepsilon}_u(f|_{I^\varepsilon_u})\), and hence

\[
\left\|\psi^\varepsilon_v(f)\right\| = \left\|f|_{I^\varepsilon_u}\right\| = \left\|\rho^*_{I^\varepsilon}_u(f|_{I^\varepsilon_u})\right\| \leq \sup \left\{\left\|\pi^F_v(f)\right\| : v \in \mathcal{E}^{(0)}\right\} = \|f\|_r.
\]

So \(\psi^\varepsilon_v\) is bounded, and since \(\Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\) is dense in \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon)\), \(\psi^\varepsilon_v\) extends to a \(C^*\)-homomorphism \(\overline{\psi^\varepsilon_v} : C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon) \to C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\). Recall from Lemma 4.3 that \(J^\varepsilon_u\) is an ideal of \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon)\). Since \(\overline{\psi^\varepsilon_v}\) is bounded and vanishes on \(F_u\) by definition, it vanishes on \(J^\varepsilon_u\). Therefore, \(\overline{\psi^\varepsilon_v}\) descends to a \(*\)-homomorphism

\[
\theta^\varepsilon_u : C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon)/J^\varepsilon_u \to C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)
\]
satisfying \(\theta^\varepsilon_u(f + J^\varepsilon_u) = f|_{I^\varepsilon_u}\) for all \(f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\). We claim that \(\theta^\varepsilon_u\) is surjective. Since \(\Sigma_c(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\) is dense in \(C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\), it suffices to show that \(\Sigma_c(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\) is contained in the image of \(\theta^\varepsilon_u\). Recall from Proposition 3.1 that

\[
\Sigma_c(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u) = \text{span}\{\delta^\varepsilon : \varepsilon \in \mathcal{I}^\varepsilon_u\}.
\]

For each \(\varepsilon \in \mathcal{I}^\varepsilon_u\), use Lemma 4.4 to choose \(j_\varepsilon \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\) such that \(j_\varepsilon|_{I^\varepsilon_u} = \delta^\varepsilon\). Then for each \(\varepsilon \in \mathcal{I}^\varepsilon_u\), we have \(\theta^\varepsilon_u(j_\varepsilon + J^\varepsilon_u) = j_\varepsilon|_{I^\varepsilon_u} = \delta^\varepsilon\), and hence \(\theta^\varepsilon_u\) is surjective.

For part (b), define \(\psi_u : \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon) \to C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\) by \(\psi_u(f) := f|_{I^\varepsilon_u}\). The argument used in part (a) shows that \(\psi_u\) is a \(*\)-homomorphism that vanishes on \(F_u\). Thus Lemma 2.14 implies that \(\psi_u\) extends uniquely to a \(C^*\)-homomorphism \(\overline{\psi_u} : C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon) \to C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\). By Lemma 4.3, \(J_u\) is an ideal of \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon)\), and so a similar argument to the one used in part (a) shows that there is a surjective \(*\)-homomorphism

\[
\theta_u : C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon)/J_u \to C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)
\]
satisfying \(\theta_u(f + J_u) = f|_{I^\varepsilon_u}\) for all \(f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\). To see that \(\theta_u\) is injective, recall from Lemma 4.4 that for each \(\varepsilon \in \mathcal{I}^\varepsilon_u\), there exists \(j_\varepsilon \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon)\) such that \(j_\varepsilon|_{I^\varepsilon_u} = \delta^\varepsilon\) and \(j_{\varepsilon z} = \overline{z} j_\varepsilon\) for each \(z \in \mathbb{T}\). By Lemma 4.5, \(\{j_\varepsilon + J_u : \varepsilon \in \mathcal{I}^\varepsilon_u\}\) is a family of unitary elements of \(C^*_{r}(\mathcal{I}^\varepsilon_u)/J_u\) such that \((j_\varepsilon + J_u)(j_\zeta + J_u) = j_{\varepsilon \zeta} + J_u\) and \(j_{\varepsilon z} + J_u = \overline{z} j_\varepsilon + J_u\) for all \(\varepsilon, \zeta \in \mathcal{I}^\varepsilon_u\) and \(z \in \mathbb{T}\). Hence Theorem 3.2 implies that there is a homomorphism

\[
\eta_u : C^*_{r}(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u) \to C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)/J_u
\]
satisfying \(\eta_u(\delta^\varepsilon) = j_\varepsilon + J_u\) for each \(\varepsilon \in \mathcal{I}^\varepsilon_u\), where \(\delta^\varepsilon \in \Sigma_c(\mathcal{I}^0_{q(u)}; \mathcal{I}^\varepsilon_u)\) is defined as in Proposition 3.1. We claim that \(\eta_u \circ \theta_u\) is the identity map on \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)/J_u\). To see this, observe that since \(\{f + J_u : f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)\}\) is a dense subspace of \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)/J_u\) and since \(\eta_u \circ \theta_u\) is continuous, it suffices to show that \(\eta_u(\theta_u(f + J_u)) = f + J_u\) for all \(f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)\). Fix \(f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)\). Since \(\eta_u\) is linear, we have

\[
\eta_u(\theta_u(f + J_u)) = \eta_u(f|_{I^\varepsilon_u}) = \eta_u\left(\sum_{\varepsilon \in F} c_\varepsilon \delta^\varepsilon\right) = \sum_{\varepsilon \in F} c_\varepsilon \eta_u(\delta^\varepsilon) = \left(\sum_{\varepsilon \in F} c_\varepsilon j_\varepsilon\right) + J_u. \tag{4.1}
\]

Since \((f - \sum_{\varepsilon \in F} c_\varepsilon j_\varepsilon)|_{I^\varepsilon_u} = f|_{I^\varepsilon_u} - \sum_{\varepsilon \in F} c_\varepsilon \delta^\varepsilon \equiv 0\), we have \(f - \sum_{\varepsilon \in F} c_\varepsilon j_\varepsilon \in J_u\). Thus we deduce from equation (4.1) that \(\eta_u(\theta_u(f + J_u)) = f + J_u\). Therefore, \(\eta_u \circ \theta_u\) is the identity map on \(C^*_{r}(\mathcal{I}^0; \mathcal{I}^\varepsilon_u)/J_u\), and so \(\theta_u\) is injective. \(\square\)
Corollary 4.6. Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(\mathcal{G}\). Fix \(u \in \mathcal{L}^{(0)}\).

Recall from Proposition 3.1 the definition of the identity element \(\delta_u^\mathcal{G}\) of both \(C^*_r(\mathcal{G}; \mathcal{E})\) and \(C^*(\mathcal{G}; \mathcal{E})\). Recall from Theorem 4.1 the definitions of the ideals \(J^i_u\) of \(C^*_r(\mathcal{G}; \mathcal{E})\) and \(J_u\) of \(C^*(\mathcal{G}; \mathcal{E})\), and the maps

\[
\theta^r_u : C^*_r(\mathcal{G}; \mathcal{E})/J^r_u \to C^*_r(\mathcal{G}_q(u); \mathcal{E}_u) \quad \text{and} \quad \theta_u : C^*(\mathcal{G}; \mathcal{E})/J_u \to C^*(\mathcal{G}_q(u); \mathcal{E}_u).
\]

(a) The map \(Q^r_u : C^*_r(\mathcal{G}; \mathcal{E}) \to C^*_r(\mathcal{G}_q(u); \mathcal{E}_u)\) given by \(Q^r_u(a) := \theta^r_u(a + J^r_u)\) is a surjective \(*\)-homomorphism. For every \(g \in \Sigma_c(\mathcal{G}; \mathcal{E})\) satisfying \(q(\text{supp}(g)) \subseteq \mathcal{G}^{(0)}\) and \(g(u) = 1\), we have \(Q^r_u(g) = \delta_u^\mathcal{G}\).

(b) The map \(Q_u : C^*(\mathcal{G}; \mathcal{E}) \to C^*(\mathcal{G}_q(u); \mathcal{E}_u)\) given by \(Q_u(a) := \theta_u(a + J_u)\) is a surjective \(*\)-homomorphism. For every \(g \in \Sigma_c(\mathcal{G}; \mathcal{E})\) satisfying \(q(\text{supp}(g)) \subseteq \mathcal{G}^{(0)}\) and \(g(u) = 1\), we have \(Q_u(g) = \delta_u^\mathcal{G}\).

Proof. We will only prove part (a), as the proof of part (b) is identical. The map \(Q^r_u\) is a surjective \(*\)-homomorphism because it is the composition of \(\theta^r_u\) and the quotient map from \(C^*_r(\mathcal{G}; \mathcal{E})\) to \(C^*_r(\mathcal{G}_q(u); \mathcal{E}_u)\). Suppose that \(g \in \Sigma_c(\mathcal{G}; \mathcal{E})\) satisfies \(q(\text{supp}(g)) \subseteq \mathcal{G}^{(0)}\) and \(g(u) = 1\). By the definition of \(Q^r_u\) and \(\theta^r_u\), we have \(Q^r_u(g) = \theta^r_u(g + J^r_u) = g|_{\mathcal{E}_u}\). So we must show that \(g|_{\mathcal{E}_u} = \delta_u^\mathcal{G}\). Fix \(e \in \mathcal{E}_u\). If \(e \in q^{-1}(\mathcal{G}^{(0)})\), then \(q(e) \in \mathcal{G}^{(0)} \cap \mathcal{I}_e^\mathcal{G}\), and so \(q(e) = g(u)\). Thus \(e = z_e \cdot u = i(q(u), z_e)\) for some unique \(z_e \in \mathcal{T}\), and so since \(g\) is \(\mathcal{T}\)-equivariant, we have \(g(e) = g(z_e \cdot u) = z_e g(u) = z_e = \delta^\mathcal{G}_u(e)\). If \(e \notin q^{-1}(\mathcal{G}^{(0)})\), then \(e \notin \mathcal{G}^{(0)}\), then \(\mathcal{G}^{(0)}\) is not amenable. However, by \([38, \text{Lemmas 2.7 and 2.8}]\), \(C^*(\mathcal{G})\) is isomorphic to \(C^*_r(\mathcal{G})\). □

In our proof of Theorem 4.1(b), we used the universal property of \(C^*(\mathcal{G}_q(u); \mathcal{E}_u)\) given in Theorem 3.2 to show that \(\theta_u : C^*(\mathcal{G}; \mathcal{E})/J_u \to C^*(\mathcal{G}_q(u); \mathcal{E}_u)\) is injective, but this argument doesn’t work in the reduced setting because the universal property doesn’t hold. In fact, somewhat surprisingly, even though \(\theta_u\) is always an isomorphism, there exist examples of groupoids for which the map \(\theta^r_u : C^*_r(\mathcal{G}; \mathcal{E})/J^r_u \to C^*_r(\mathcal{G}_q(u); \mathcal{E}_u)\) of Theorem 4.1(a) is not an isomorphism. One such example, due to Willett \([38]\), comes from the class of HLS groupoids constructed in \([23, \text{Section 2}]\) (see also \([38, \text{Definition 2.2}]\)).

Before presenting Willett’s example, we first recall the construction of an HLS groupoid.

Suppose that \((K_n)_{n \in \mathbb{N}}\) is an approximating sequence for a discrete group \(\Gamma\), as defined in \([38, \text{Definition 2.1}]\). For each \(n \in \mathbb{N}\), define \(\Gamma_n := \Gamma/K_n\), and \(\Gamma_{\infty} := \Gamma\). For each \(n \in \mathbb{N} \cup \{\infty\}\), denote the identity of the group \(\Gamma_n\) by \(e_{\Gamma_n}\). Define

\[
\mathcal{G} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} \{n\} \times \Gamma_n,
\]

and equip \(\mathcal{G}\) with the groupoid operations coming from the group structure on the fibres over each \(n \in \mathbb{N} \cup \{\infty\}\). Then \(\mathcal{G}^{(0)} = \{(n, g) \in \mathcal{G} : g = e_{\Gamma_n}\}\), and for all \((n, g) \in \mathcal{G}\), we have \(r(n, g) = (n, e_{\Gamma_n}) = s(n, g)\). Thus \(\text{Iso}(\mathcal{G}) = \mathcal{G}\). In \([38, \text{Definition 2.2}]\), Willett endows this groupoid \(\mathcal{G}\) with a topology, under which it is a second-countable locally compact Hausdorff étale groupoid, called the \(\text{HLS groupoid}\) associated to the approximated group \((\Gamma, (K_n)_{n \in \mathbb{N}})\).

Example 4.7. Let \(F_2\) denote the free group on two generators. For each \(n \in \mathbb{N}\), define

\[
K_n := \bigcap \{\ker(\phi) : \phi : F_2 \to \Gamma\text{ is a group homomorphism, }|\Gamma| \leq n\}. \quad (4.2)
\]

By \([38, \text{Lemma 2.8}]\), \((K_n)_{n \in \mathbb{N}}\) is an approximating sequence for \(F_2\). Since \(F_2\) is not amenable, \([38, \text{Lemma 2.4}]\) shows that the HLS groupoid \(\mathcal{G}\) associated to the approximated group \((F_2, (K_n)_{n \in \mathbb{N}})\) is not amenable. However, by \([38, \text{Lemmas 2.7 and 2.8}]\), \(C^*(\mathcal{G})\) is isomorphic to \(C^*_r(\mathcal{G})\).
Taking $\mathcal{E}$ to be the trivial twist $\mathcal{G} \times \mathbb{T}$ over the HLS groupoid $\mathcal{G}$ described in Example 4.7, and $u := ((\infty, e_{F_2}), 1) \in \mathcal{E}(0)$, we now prove that the map $\theta_u^* : C_\ast^r(\mathcal{I}^g ; \mathcal{I}^e)/J_u \to C_\ast^r(\mathcal{I}^g_u ; \mathcal{I}^e_u)$ of Theorem 4.1(a) is not injective. Note that since $\mathcal{E}$ is trivial, we omit it from our notation in Theorem 4.8, and we identify $\Sigma_c(\mathcal{I}^g ; \mathcal{I}^e)$ with $C_c(\mathcal{I}^g)$ and $C_\ast^r(\mathcal{I}^g ; \mathcal{I}^e)$ with $C_\ast^r(\mathcal{I}^g_u)$ via the $*$-isomorphism defined in Proposition 2.15.

**Theorem 4.8.** For each $n \in \mathbb{N}$, let $K_n$ be defined as in equation (4.2), and let $\mathcal{G}$ be the HLS groupoid associated to $(F_2, (K_n)_{n \in \mathbb{N}})$, as described in Example 4.7. Consider the unit $u := ((\infty, e_{F_2}) \in \mathcal{G}(0)$, where $e_{F_2}$ is the identity element of $F_2$. The map $\theta_u^* : C_\ast^r(\mathcal{I}^g) / J_u \to C_\ast^r(\mathcal{I}^g_u)$ of Theorem 4.1(a) is not injective.

**Proof.** Define $\Gamma := F_2$. Since $\text{Iso}(\mathcal{G}) = \mathcal{G}$, we have $\mathcal{I}^g_u = \mathcal{I}^g_{e} = \{\infty\} \times \mathbb{R} \cong \Gamma$. Let $\Upsilon : C_\ast^r(\mathcal{I}^g) \to C_\ast^r(\mathcal{I}^g)$ be the unique homomorphism that restricts to the identity map on $C_c(\mathcal{I}^g)$. By [38, Lemmas 2.7 and 2.8] and [2, Corollary 3.3.23], $\Upsilon$ is an isomorphism, and so the full and reduced $C^*$-norms coincide on $C_c(\mathcal{I}^g)$. Thus $\Upsilon|_{J_u} : J_u \to J_u$ is an isomorphism. Observe that for all $a, b \in C_\ast^r(\mathcal{I}^g)$ with $a - b \in J_u$, we have $\Upsilon(a) - \Upsilon(b) = \Upsilon(a - b) \in J_u$. Hence there is a map $\Upsilon_u : C_\ast^r(\mathcal{I}^g)/J_u \to C_\ast^r(\mathcal{I}^g)/J_u$ satisfying $\Upsilon_u(a + J_u) = \Upsilon(a) + J_u$, for all $a \in C_\ast^r(\mathcal{I}^g)$. Since $\Upsilon : C_\ast^r(\mathcal{I}^g) \to C_\ast^r(\mathcal{I}^g_u)$ and $\Upsilon|_{J_u} : J_u \to J_u$ are isomorphisms, it is clear that $\Upsilon_u$ is also an isomorphism.

Let $\Xi_u : C_\ast^r(\mathcal{I}^g_u) \to C_\ast^r(\mathcal{I}^g_u)$ be the unique homomorphism that restricts to the identity map on $C_c(\mathcal{I}^g_u)$. Since $\Gamma = F_2$ is a nonamenable group, we know that $\Xi_u$ is not injective. For all $f \in C_c(\mathcal{I}^g_u)$, we have

$$\theta_u^*(\Upsilon_u(f + J_u)) = \theta_u^*(\Upsilon(f) + J_u) = \theta_u^*(f + J_u) = f|_{I^g_u} = \Xi_u(f|_{I^g_u}) = \Xi_u(\theta_u(f + J_u)).$$

Therefore, the maps $\theta_u^* \circ \Upsilon_u$ and $\Xi_u \circ \theta_u$ agree on the set $\{f + J_u : f \in C_c(\mathcal{I}^g_u)\}$, which is a dense subspace of $C_\ast^r(\mathcal{I}^g_u)/J_u$. Since these maps are homomorphisms of $C^*$-algebras, they are continuous, and hence they agree on all of $C_\ast^r(\mathcal{I}^g_u)/J_u$. Therefore, $\theta_u^* = \Xi_u \circ \theta_u \circ \Upsilon_u^{-1}$. Since $\theta_u \circ \Upsilon_u^{-1}$ is surjective and $\Xi_u$ is not injective, it follows that $\theta_u^*$ is not injective. $\square$

5. **Unique state extensions**

In this short section we provide a sufficient compressibility condition under which a (pure) state of a $C^*$-subalgebra $B$ of a (not necessarily unital) $C^*$-algebra $A$ has a unique (pure) state extension to $A$. This result is an extension of a well known result of Anderson about unital $C^*$-algebras (proved in the paragraph preceding [1, Theorem 3.2]), but we were unable to find a proof of it in the literature, and so we present one here.

We begin by introducing some notation.

**Notation 5.1.** Given a state $\phi$ of a $C^*$-algebra $A$, we define

$$\mathcal{M}_\phi := \{a \in A : \phi(ax) = \phi(xa) = \phi(a)\phi(x) \text{ for all } x \in A\}$$

and

$$\mathcal{U}_\phi := \{a \in A : \rho(\phi(a)) = \|a\| = 1\}.$$

**Remark 5.2.** At the bottom of page 304 of [1], Anderson observes that if $\phi$ is a state of a unital $C^*$-algebra $A$, then $\mathcal{U}_\phi \subseteq \mathcal{M}_\phi$. In fact, the same is true when $A$ is nonunital. To see this, let $\phi^+$ be the unique state extension of $\phi$ to the minimal unitalisation $A^+$ of $A$. Then for all $a \in \mathcal{U}_\phi$, we have $(a, 0) \in \mathcal{M}_{\phi^+}$ and $\mathcal{M}_{\phi^+} \subseteq \mathcal{M}_\phi$, and it follows that $a \in \mathcal{M}_\phi$.

We now recall Anderson’s compressibility condition given in [1, Section 3].

**Definition 5.3.** Let $A$ be a $C^*$-algebra and let $B$ be a $C^*$-subalgebra of $A$. Suppose that $\phi$ is a state of $B$. We say that $A$ is $B$-compressible modulo $\phi$ if, for each $a \in A$ and $\epsilon > 0$, there exists $b \in \mathcal{U}_\phi \subseteq B$ and $c \in B$ such that $b \geq 0$ and $\rho(C\phi) < \epsilon$. 

We thank the anonymous referee for providing a more elegant proof of the following theorem than the one that appeared in the initial preprint of this article.

**Theorem 5.4.** Let $A$ be a $C^*$-algebra and let $B$ be a $C^*$-subalgebra of $A$. If $\phi$ is a state of $B$ such that $A$ is $B$-compressible modulo $\phi$, then $\phi$ has a unique state extension to $A$. If $\phi$ is a pure state, then so is its unique extension.

*Proof.* Let $\phi$ be a state of $B$ such that $A$ is $B$-compressible modulo $\phi$. Let $B^+$ denote the minimal unitisation of $B$, and let $\phi^+$ be the unique state extension of $\phi$ to $B^+$, which is given by $\phi^+(b, \lambda) := \phi(b) + \lambda$ for all $(b, \lambda) \in B \times \mathbb{C}$. We claim that $A^+$ is $B^+$-compressible modulo $\phi^+$. To see this, fix $(a, \lambda) \in A^+$ and $\epsilon > 0$. Then $a \in A$, and since $A$ is $B$-compressible modulo $\phi$, there exists $b \in U_\phi \subseteq B$ and $c \in B$ such that $b \geq 0$ and $\|bab - c\| < \epsilon$. Let $b' := (b, 0) \in B^+$ and $c' := (\lambda b^2 + c, 0) \in B^+$. Then $b' \in U_{\phi^+}$ and $b' \geq 0$, and we have

$$\|b'(a, \lambda)b' - c'\| = \|(bab + \lambda b^2, 0) - (\lambda b^2 + c, 0)\| = \|(bab - c, 0)\| = \|bab - c\| < \epsilon,$$

as claimed. Thus [1, Theorem 3.2] implies that $\phi^+$ extends uniquely to a state $\overline{\phi}$ of $A^+$. By identifying $A$ and $B$ with their images under the isometric inclusions into $A^+$ and $B^+$, respectively, we obtain a state $\overline{\phi} := \overline{\phi}^+|_A$ of $A$ that satisfies $\overline{\phi}|_B = \phi^+|_B = \phi$.

To see that $\overline{\phi}$ is unique, suppose that $\psi$ is a state of $A$ that satisfies $\psi|_B = \phi$. Since $\overline{\phi}^+$ is the unique state extension of $\phi$ to $A^+$, it must also be the unique state extension of $\psi$ to $A^+$, and hence $\psi = \phi$.

Finally, if $\phi$ is a pure state of $B$, then [9, II.6.3.2] implies that the unique state extension of $\phi$ to $A$ is also a pure state. \hfill $\square$

6. A Uniqueness Theorem for Reduced Twisted Groupoid $C^*$-Algebras

In this section we prove that there is an embedding $\iota_r$ of the twisted $C^*$-algebra $C^*_r(G^0; \mathcal{I}^\xi)$ associated to the interior of the isotropy of a Hausdorff étale groupoid $G$ into $C^*_r(G; \mathcal{E})$ (see Proposition 6.1). We use this result to prove our uniqueness theorem (Theorem 6.3), which states that a $C^*$-homomorphism $\Psi$ of $C^*_r(G; \mathcal{E})$ is injective if and only if $\Psi \circ \iota_r$ is injective. We then use our uniqueness theorem to prove Corollary 6.9, which states that if $G$ is effective, then $C^*_r(G; \mathcal{E})$ is simple if and only if $G$ is minimal. With the exception of Corollary 6.9, the results in this section are extensions of the results in [2, Section 5.3] to the setting of $C^*$-algebras of groupoid twists.

**Proposition 6.1.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $G$. There is a homomorphism $\iota_\mathcal{E} : C^*(\mathcal{I}^0; \mathcal{I}^\xi) \to C^*(G; \mathcal{E})$ such that

$$\iota(f)(\epsilon) = \begin{cases} f(\epsilon) & \text{if } \epsilon \in \mathcal{I}^\xi \\ 0 & \text{if } \epsilon \notin \mathcal{I}^\xi \end{cases}$$

for all $f \in \Sigma_c(\mathcal{I}^0; \mathcal{I}^\xi)$ and $\epsilon \in \mathcal{E}$. We have $\iota(\Sigma_c(\mathcal{I}^0; \mathcal{I}^\xi)) \subseteq \Sigma_c(G; \mathcal{E})$, and $\iota$ descends to an injective homomorphism $\iota_r : C^*_r(\mathcal{I}^0; \mathcal{I}^\xi) \to C^*_r(G; \mathcal{E})$. If $\mathcal{I}^0$ is amenable, then $\iota$ is also injective.

*Proof.* A standard argument shows that $\iota|_{\Sigma_c(\mathcal{I}^0; \mathcal{I}^\xi)}$ is a *-homomorphism with image contained in $\Sigma_c(G; \mathcal{E})$, and hence Lemma 2.14 shows that it extends uniquely to a $C^*$-homomorphism $\iota : C^*(\mathcal{I}^0; \mathcal{I}^\xi) \to C^*(G; \mathcal{E})$. Since $\mathcal{I}^\xi$ is an open subgroupoid of $G$ and $\mathcal{I}^\xi = q^{-1}(\mathcal{I}^0)$ by the proof of Corollary 2.11(b), [12, Lemma 2.7] implies that $\iota$ descends to an injective homomorphism $\iota_r : C^*_r(\mathcal{I}^0; \mathcal{I}^\xi) \to C^*_r(G; \mathcal{E})$. For the final claim, suppose
that $\mathcal{G}$ is amenable, and fix $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$. Using [37, Theorem 11.1.11] for the first equality and that $\iota_\tau$ is isometric for the second equality, we see that
\[
\|f\| = \|f\|_* = \|\iota_\tau(f)\|_* = \|\iota(f)\|_* \leq \|\iota(f)\|_* \leq \|f\|_,
\]
and hence $\iota$ is injective.

We now prove that if $\mathcal{G}$ is closed, then the map that restricts functions in $\Sigma_c(\mathcal{G}; \mathcal{E})$ to $\mathcal{E}$ extends to a conditional expectation from $C^*_r(\mathcal{G}; \mathcal{E})$ to $\iota_\tau(C^*_r(\mathcal{G}; \mathcal{E}))$, and that it also extends to a conditional expectation from $C^*_r(\mathcal{G}; \mathcal{E})$ to $\iota(C^*(\mathcal{G}; \mathcal{E}))$ if $\mathcal{G}$ is amenable.

Although we do not actually use this result in this article, we include it here as we believe it could have various future applications; for example, for constructing KK-classes that may be valuable in K-theory calculations or noncommutative geometry. Moreover, the result has already been used in [3, Section 6] in order to prove a simplicity characterisation for twisted C*-algebras of Deaconu–Renault groupoids.

Note that the hypothesis of $\mathcal{G}$ being closed in $\mathcal{G}$ is indeed necessary, because there do exist Hausdorff étale groupoids for which this is not the case (see, for instance, [14, Example 4.7]), and because [11, Lemma 3.4] shows that if $\mathcal{G}$ is not closed, then there is no conditional expectation $C^*_r(\mathcal{G}; \mathcal{E}) \to \iota_\tau(C^*_r(\mathcal{G}; \mathcal{E}))$.

**Lemma 6.2.** Let $\mathcal{G}$ be a Hausdorff étale groupoid such that $\mathcal{G}$ is closed in $\mathcal{G}$, and let $(\mathcal{E}, i, q)$ be a twist over $\mathcal{G}$. For all $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$, we have $f|_{\mathcal{E}} \in \Sigma_c(\mathcal{G}; \mathcal{E})$. Let $\iota_\tau : C^*_r(\mathcal{G}; \mathcal{E}) \to C^*_r(\mathcal{G}; \mathcal{E})$ be the homomorphism of Proposition 6.1. There is a conditional expectation $\Psi^\tau : C^*_r(\mathcal{G}; \mathcal{E}) \to \iota_\tau(C^*_r(\mathcal{G}; \mathcal{E}))$ satisfying $\Psi^\tau(f) = \iota_\tau(f)$ for all $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$, and $\Psi^\tau \circ \iota_\tau = \iota_\tau$.

**Lemma 6.3.** (C*-uniqueness theorem). Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Let $\iota_\tau : C^*_r(\mathcal{G}; \mathcal{E}) \to C^*_r(\mathcal{G}; \mathcal{E})$ be the injective homomorphism of Proposition 6.1, and define $M_\tau := \iota_\tau(C^*_r(\mathcal{G}; \mathcal{E}))$. Suppose that $A$ is a C*-algebra and that $\Psi : C^*_r(\mathcal{G}; \mathcal{E}) \to A$ is a C*-homomorphism. Then $\Psi$ is injective if and only if $\Psi \circ \iota_\tau$ is an injective C*-homomorphism of $C^*_r(\mathcal{G}; \mathcal{E})$.
A and B and a surjective \(*\)-homomorphism \(Q: A \rightarrow B\), we say that a state \(\psi\) of \(A\) factors through \(B\) if there exists a state \(\phi\) of \(B\) such that \(\psi = \phi \circ Q\). It follows from [34, Theorems 3.3.1 and 3.3.3] that in this setting, a state \(\psi\) of \(A\) factors through \(B\) if and only if \(\ker(Q) \subseteq \ker(\psi)\).

**Proposition 6.4.** Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(G\). Let

\[
\iota_r: C^*_r(\mathcal{I}^G; \mathcal{I}^\mathcal{E}) \rightarrow C^*_r(\mathcal{G}; \mathcal{E}) \quad \text{and} \quad \iota: C^*(\mathcal{I}^G; \mathcal{I}^\mathcal{E}) \rightarrow C^*(\mathcal{G}; \mathcal{E})
\]

be the homomorphisms of **Proposition 6.1**, and define

\[
M_r := \iota_r(C^*_r(\mathcal{I}^G; \mathcal{I}^\mathcal{E})) \quad \text{and} \quad M := \iota(C^*(\mathcal{I}^G; \mathcal{I}^\mathcal{E})).
\]

Suppose that \(u \in \mathcal{E}^{(0)}\) satisfies \(\mathcal{E}^u = \mathcal{I}^\mathcal{E}_u\).

(a) If \(\varphi_r\) is a state of \(M_r\) such that \(\varphi_r \circ \iota_r\) factors through \(C^*_r(\mathcal{G}_q^{(u)}; \mathcal{E}^u)\), then \(\varphi_r\) has a unique state extension to \(C^*_r(\mathcal{G}; \mathcal{E})\).

(b) If \(\varphi\) is a state of \(M\) such that \(\varphi \circ \iota\) factors through \(C^*(\mathcal{G}(q^{(u)}); \mathcal{E}^u)\), then \(\varphi\) has a unique state extension to \(C^*(\mathcal{G}; \mathcal{E})\).

In order to prove **Proposition 6.4**, we need the following two preliminary results. The first of these results is a generalisation of [14, Lemma 3.3(b)] to the twisted setting.

**Lemma 6.5.** Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(G\). Suppose that \(u \in \mathcal{E}^{(0)}\) satisfies \(\mathcal{E}^u = \mathcal{I}^\mathcal{E}_u\). For each \(f \in \Sigma_c(\mathcal{G}; \mathcal{E})\), there exists \(g \in \Sigma_c(\mathcal{I}^G; \mathcal{I}^\mathcal{E})\) satisfying \(g \geq 0\), \(q(\supp(g)) \subseteq \mathcal{G}^{(0)}\), \(\|g\| = \|g\|_r = g(u) = 1\), and \(\supp(gfg) \subseteq \mathcal{I}^\mathcal{E}\).

**Proof.** First observe that since \(\mathcal{E}^u = \mathcal{I}^\mathcal{E}_u\), we have \(\mathcal{G}^{(u)}_q = \mathcal{I}^\mathcal{E}_{(u)_q}\). Fix \(f \in \Sigma_c(\mathcal{G}; \mathcal{E})\). By **Lemma 2.13**, we can write \(f = \sum_{D \in F} f_D\), where \(F\) is a finite collection of open bisections of \(G\) such that for each \(D \in F\), \(f_D \in \Sigma_c(\mathcal{G}; \mathcal{E})\) and \(q(\supp(f_D)) \subseteq D\). Choose open neighbourhoods \(\{V_D \subseteq \mathcal{G}^{(0)} : D \in F\}\) of \(q(u)\) as in the proof of [14, Lemma 3.3(b)], so that \((V_D \cap D) \cap q(\supp(f_D)) \subseteq \mathcal{I}^\mathcal{E}\) for each \(D \in F\). Let \(V := \bigcap_{D \in F} V_D\). Then \(V\) is an open neighbourhood of \(q(u)\) contained in \(\mathcal{G}^{(0)}\). Now use Urysohn’s lemma to choose \(b \in C_c(V)\) such that \(b \geq 0\) and \(b(q(u)) = \|b\|_\infty = 1\). Since \(\mathcal{G}^{(0)}\) is a bisection of \(G\), it follows that \(\|b\| = \|b\|_r = \|b\|_\infty = 1\) (see, for instance, [37, Corollary 9.3.4]). For each \(\varepsilon \in q^{-1}(\mathcal{G}^{(0)})\), there are unique elements \(v_\varepsilon \in \mathcal{G}^{(0)}\) and \(z_\varepsilon \in \mathbb{T}\) such that \(\varepsilon = i(v_\varepsilon, z_\varepsilon)\). Define \(g: \mathcal{I}^\mathcal{E} \rightarrow \mathbb{C}\) by

\[
g(\varepsilon) := \begin{cases} 
z_\varepsilon b(v_\varepsilon) & \text{if } \varepsilon \in q^{-1}(\mathcal{G}^{(0)}) \\
0 & \text{if } \varepsilon \notin q^{-1}(\mathcal{G}^{(0)}). \end{cases}
\]

It follows immediately from our choice of \(b\) and construction of \(g\) that \(g(u) = b(q(u)) = 1\) and that \(q(\supp(g)) \subseteq V \subseteq \mathcal{G}^{(0)}\). By [37, Lemma 11.1.9], there is an isomorphism from \(C_0(\mathcal{G}^{(0)})\) to \(D_0^G := \{h \in \Sigma_c(\mathcal{I}^G; \mathcal{I}^\mathcal{E}) : q(\supp(h)) \subseteq \mathcal{G}^{(0)}\}\) that maps \(b\) to \(g\), and thus \(g \in \Sigma_c(\mathcal{I}^G; \mathcal{I}^\mathcal{E})\) and \(g \geq 0\). By [37, Theorem 11.1.11], this isomorphism extends to (isometric) isomorphisms of \(C_0(\mathcal{G}^{(0)})\) onto the full and reduced \(*\)-completions of \(D_0^G\), and therefore, \(\|g\| = \|g\|_r = \|b\|_\infty = 1\). Finally, for each \(D \in F\), we have

\[
q(\supp(gfg)) \subseteq q(\supp(g_1)q(\supp(f_D))q(\supp(g)) \subseteq (VDV) \cap q(\supp(f_D)) \subseteq \mathcal{I}^\mathcal{E}
\]

by construction, and since \(f = \sum_{D \in F} f_D\), it follows that \(\supp(gfg) \subseteq q^{-1}(\mathcal{I}^\mathcal{E}) = \mathcal{I}^\mathcal{E} \). \(\square\)

The next result is an extension of [14, Lemma 3.5] to the twisted setting.

**Lemma 6.6.** Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \(G\). Let

\[
\iota_r: C^*_r(\mathcal{I}^G; \mathcal{I}^\mathcal{E}) \rightarrow C^*_r(\mathcal{G}; \mathcal{E}) \quad \text{and} \quad \iota: C^*(\mathcal{I}^G; \mathcal{I}^\mathcal{E}) \rightarrow C^*(\mathcal{G}; \mathcal{E})
\]
be the homomorphisms of Proposition 6.1, and define
\[ M_r := \iota_r \left( C_r^{*} \left( \mathcal{I}^0; \mathcal{I}^\xi \right) \right) \quad \text{and} \quad M := \iota \left( C^{*} \left( \mathcal{I}^0; \mathcal{I}^\xi \right) \right). \]

Suppose that \( u \in \mathcal{E}^{(0)} \) satisfies \( \mathcal{E}^u = \mathcal{I}^\xi \).

(a) Fix \( \varepsilon > 0 \) and \( a \in C_r^{*} \left( \mathcal{G}; \mathcal{E} \right) \). There exist \( b, c \in M_r \) satisfying \( b \geq 0 \), \( \|b\|_r = 1 \), \( \|b-a-f\|_r < \varepsilon \), and \( \varphi_r(b) = 1 \) for every state \( \varphi_r \) of \( M_r \) such that \( \varphi_r \circ \iota_r \) factors through \( C^{*} \left( G_q^{(u)}; E^\xi \right) \). If \( a \) is positive, then \( c \) can be taken to be positive.

(b) Fix \( \varepsilon > 0 \) and \( a \in C^{*} \left( \mathcal{G}; \mathcal{E} \right) \). There exist \( b, c \in M \) satisfying \( b \geq 0 \), \( \|b\| = 1 \), \( \|b-a-f\|_r < \varepsilon \), and \( \varphi(b) = 1 \) for every state \( \varphi \) of \( M \) such that \( \varphi \circ \iota \) factors through \( C^{*} \left( G_q^{(u)}; E^\xi \right) \). If \( a \) is positive, then \( c \) can be taken to be positive.

**Proof.** Both parts follow from Corollary 4.6 in the same way, and so we will only prove part (a). First observe that since \( \mathcal{E}^u = \mathcal{I}^\xi \), we have \( G_q^{(u)} = \mathcal{I}^\xi \). Let
\[ Q_r^U : C_r^{*} \left( \mathcal{I}^\xi; \mathcal{I}^\xi \right) \to C_r^{*} \left( \mathcal{I}^\xi; \mathcal{I}^\xi \right) = C_r^{*} \left( G_q^{(u)}, E^\xi \right). \]
be the surjective \(*\)-homomorphism of Corollary 4.6(a). Since \( \Sigma_c(\mathcal{G}; \mathcal{E}) \) is dense in \( C^{*} \left( \mathcal{G}; \mathcal{E} \right) \), we can choose \( f \in \Sigma_c(\mathcal{G}; \mathcal{E}) \) such that \( \|a-f\|_r \leq \varepsilon \). If \( a \) is a positive, a standard C*-algebraic argument (see, for instance, [2, Lemma 5.3.10]) shows that \( f \) can also be taken to be positive. Use Lemma 6.5 to choose \( g \in \Sigma_c(\mathcal{I}^\xi; \mathcal{I}^\xi) \) satisfying \( g \geq 0 \), \( \supp(gf) \subseteq G_q^{(u)} \), \( \|g\| = \|g\|_r = g(u) = 1 \), and \( \supp(gf \mathcal{G}) \subseteq \mathcal{I}^\xi \). Define \( b := \iota_r(g) \) and \( c := bf \). Then \( \supp(c) \subseteq \mathcal{I}^\xi \), \( b, c \in M_r \), \( b \geq 0 \), and \( b(u) = g(u) = 1 \). If \( f \geq 0 \), then it follows that \( c \geq 0 \). Since \( q(supp(b)) \subseteq G_q^{(u)} \), [37, Theorem 11.1.11] implies that \( \|b\| = \|b\|_r = \|\iota_r(g)\|_r = \|g\|_r = 1 \), and hence
\[ \|b-a-f\|_r < \varepsilon. \]
Suppose that \( \varphi_r \) is a state of \( M_r \) such that \( \varphi_r \circ \iota_r \) factors through \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \). Then there is a state \( \psi_r \) of \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \) such that \( \varphi_r \circ \iota_r = \psi_r \circ Q_r^U \). By Corollary 4.6(a), we have \( Q_r^U(g) = \delta^\xi_u \), which is the identity element of \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \) defined in Proposition 3.1. Thus, since \( \psi_r \) is unital, we have \( \varphi_r(b) = \psi_r(\iota_r(g)) = \psi_r(Q_r^U(g)) = \psi_r(\delta^\xi_u) = 1 \). \qed

**Proof of Proposition 6.4.** Both parts follow from Lemma 6.6 in the same way, and so we will only prove part (a). Suppose that \( \varphi_r \) is a state of \( M_r \) such that \( \varphi_r \circ \iota_r \) factors through \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \). Recalling the terminology defined in Definition 5.3, \( C_r^{*} \left( \mathcal{G}; \mathcal{E} \right) \) is \( M_r \)-compressible modulo \( \varphi_r \), by Lemma 6.6(a), and so by Theorem 5.4, \( \varphi_r \) has a unique state extension to \( C_r^{*} \left( \mathcal{G}; \mathcal{E} \right) \). \qed

We need the following two additional results in order to prove Theorem 6.3.

**Lemma 6.7.** Let \((\mathcal{E}, i, q)\) be a twist over a Hausdorff étale groupoid \( \mathcal{G} \). Suppose that \( u \in \mathcal{E}^{(0)} \) satisfies \( \mathcal{E}^u = \mathcal{I}^\xi \), and let \( Q_r^E : C_r^{*} \left( \mathcal{I}^0; \mathcal{I}^\xi \right) \to C_r^{*} \left( \mathcal{I}^\xi_q; \mathcal{I}^\xi \right) = C_r^{*} \left( G_q^{(u)}, E^\xi \right) \) be the surjective \(*\)-homomorphism of Corollary 4.6(a). Let \( \iota_r : C_r^{*} \left( \mathcal{I}^\xi; \mathcal{I}^\xi \right) \to C_r^{*} \left( \mathcal{G}; \mathcal{E} \right) \) be the injective homomorphism of Proposition 6.1, and define \( M_r := \iota_r \left( C_r^{*} \left( \mathcal{I}^\xi; \mathcal{I}^\xi \right) \right) \). Let \( \phi \) be a state of \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \), and define \( \psi : M_r \to \mathbb{C} \) by \( \psi(\iota_r(a)) := \phi(Q_r^E(a)) \). Then \( \psi \) is a state of \( M_r \), and \( \psi \circ \iota_r \) is a state of \( C_r^{*} \left( \mathcal{I}^\xi; \mathcal{I}^\xi \right) \) that factors through \( C_r^{*} \left( G_q^{(u)}, E^\xi \right) \). If \( \phi \) is a pure state, then \( \psi \) and \( \psi \circ \iota_r \) are also pure states.

**Proof.** Since \( Q_r^E \) is a \( C^* \)-homomorphism and \( \phi \) is a state, it is clear that \( \psi \) and \( \psi \circ \iota_r \) are positive bounded linear functionals. To see that \( \psi \) and \( \psi \circ \iota_r \) are states, we must show that \( \|\psi\| = \|\psi \circ \iota_r\| = 1 \). Using Lemma 6.5, we can find \( g \in \Sigma_c(\mathcal{I}^\xi; \mathcal{I}^\xi) \) satisfying
q(supp(g)) \subseteq \mathcal{G}^{(0)}$ and $g(u) = 1$, and hence Corollary 4.6(a) implies that $Q^r_u(g)$ is the identity element of $\mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u))$. Therefore, since $\phi$ is a state of $\mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u))$, we have

$$||\psi|| \geq ||\psi \circ \iota_r|| \geq |\psi(\iota_r(g))| = |\phi(Q^r_u(g))| = 1,$$

and so $||\psi|| = ||\psi \circ \iota_r|| = 1$. Thus $\psi$ is a state of $M_r$, and $\psi \circ \iota_r = \phi \circ Q^r_u$ is a state of $\mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E)$ that factors through $\mathcal{E}_u$. Let $u$ be the injective homomorphism $f \mapsto f(u)$ on $\mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u))$.

Proof. It is clear that $\psi_1 = \phi \circ Q^r_u = \psi \circ \iota_r$ and $\psi_2 = \phi \circ Q^r_u = \psi \circ \iota_r$, and so $\psi \circ \iota_r$ is a pure state. A similar argument shows that $\psi$ is also a pure state. \hfill \Box

Before we present the next result, we recall from [36, Proposition 4.3] the existence of the faithful conditional expectations $\Phi_r : \mathcal{C}^*_r(\mathcal{G}; \mathcal{E}) \to \mathcal{C}^*_r(\mathcal{G}^{(0)}; q^{-1}(\mathcal{G}^{(0)})) \cong \mathcal{C}_0(\mathcal{G}^{(0)})$ and $\Phi_r^E : \mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E) \to \mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u); q^{-1}(\mathcal{G}^{(0)})) \cong \mathcal{C}_0(\mathcal{G}^{(0)})$ extending restriction of functions.

**Lemma 6.8.** Let $(\mathcal{E}, i, q)$ be a twist over a Hausdorff étale groupoid $\mathcal{G}$. Suppose that $u \in \mathcal{E}^{(0)}$ satisfies $\mathcal{E}_u = \mathcal{I}^G$. Let $\iota_r : \mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E) \to \mathcal{C}^*_r(\mathcal{G}; \mathcal{E})$ be the injective homomorphism of Proposition 6.1, and define $M_r := \iota_r(\mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E))$. Let $ev_u$ be the evaluation map $f \mapsto f(u)$ on $\mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u); q^{-1}(\mathcal{G}^{(0)})) \cong \mathcal{C}_0(\mathcal{G}^{(0)})$. Then

(a) $ev_u \circ \Phi_r^E$ is a state of $\mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E)$ that factors through $\mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u));$

(b) $ev_u \circ (\Phi_r|_{M_r})$ is a state of $M_r$ that satisfies $ev_u \circ (\Phi_r|_{M_r}) \circ \iota_r = ev_u \circ \Phi_r^E$; and

(c) $ev_u \circ \Phi_r^E$ is the unique state extension of $ev_u \circ (\Phi_r|_{M_r})$ to $\mathcal{C}^*_r(\mathcal{G}; \mathcal{E})$.

**Proof.** It is clear that $ev_u \circ \Phi_r^E, ev_u \circ (\Phi_r|_{M_r})$, and $ev_u \circ \Phi_r^E$ are positive bounded linear functionals since they are composed of positive bounded linear maps. To see that they are states, use Lemma 6.5 to find $g \in \Sigma_c(\mathcal{I}^G; \mathcal{I}^E)$ such that $g(u) = 1$. Then

$$|\Phi_r^E(g)(u)| = |\Phi_r^E(g)(u)| = |g(u)| = 1,$$

and it follows that

$$||ev_u \circ \Phi_r^E|| = ||ev_u \circ (\Phi_r|_{M_r})|| = ||ev_u \circ \Phi_r^E|| = 1.$$

Thus $ev_u \circ \Phi_r^E, ev_u \circ (\Phi_r|_{M_r})$, and $ev_u \circ \Phi_r^E$ are states.

Since $ev_u \circ (\Phi_r|_{M_r}) \circ \iota_r$ and $ev_u \circ \Phi_r^E$ agree on $\Sigma_c(\mathcal{I}^G; \mathcal{I}^E)$, which is dense in $\mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E)$, it follows that $ev_u \circ (\Phi_r|_{M_r}) \circ \iota_r = ev_u \circ \Phi_r^E$. Thus part (b) holds.

For part (a), define $\mathcal{H} := \mathcal{I}^G$ and $\mathcal{E}_H := q^{-1}(\mathcal{H}) = \mathcal{I}^G_H$. Let $Q^r_H : \mathcal{C}^*_r(\mathcal{I}^G; \mathcal{I}^E) \to \mathcal{C}^*_r(\mathcal{H}; \mathcal{E}_H) = \mathcal{C}^*_r(\mathcal{G}^{(u)}_q(\mathcal{E}_u); \mathcal{E}_u)$ be the surjective *-homomorphism of Corollary 4.6(a), and
let $\Phi^H_r : C^*_r(\mathcal{H} ; \mathcal{E}_H) \to C^*_r(\mathcal{H}^{(0)} ; q^{-1}(\mathcal{H}^{(0)})) \cong C_0(\mathcal{H}^{(0)})$ be the conditional expectation extending restriction of functions. To see that $ev_u \circ \Phi^T_r$ factors through $C^*_r(\mathcal{H} ; \mathcal{E}_H)$, we will find a state $\phi_u$ of $C^*_r(\mathcal{H} ; \mathcal{E}_H)$ such that $\phi_u \circ Q^u_r = ev_u \circ \Phi^T_r$. We have $\mathcal{H}^{(0)} = \{ q(u) \}$, and by a similar argument to the one above, $\phi_u := ev_u \circ \Phi^H_r$ is a state of $C^*_r(\mathcal{H} ; \mathcal{E}_H)$.

For all $f \in \Sigma_c(\mathcal{I}^G ; \mathcal{I}^E)$, we have $(\phi_u \circ Q^u_r)(f) = \phi_u\left( f |_{\mathcal{I}^G} \right) = f(u) = (ev_u \circ \Phi^T_r)(f)$. Since $\Sigma_u(\mathcal{I}^G ; \mathcal{I}^E)$ is dense in $C^*_r(\mathcal{I}^G ; \mathcal{I}^E)$, it follows that $\phi_u \circ Q^u_r = ev_u \circ \Phi^T_r$, as required.

We conclude by proving part (c). By parts (a) and (b), $ev_u \circ (\Phi_r |_{M_r}) \circ \iota_r = ev_u \circ \Phi^T_r$ is a state of $C^*_r(\mathcal{I}^G ; \mathcal{I}^E)$ that factors through $C^*_r(\mathcal{G}_{q(u)}^G ; \mathcal{E}_u^G)$. Therefore, Proposition 6.4(a) implies that $ev_u \circ (\Phi_r |_{M_r})$ extends uniquely to a state of $C^*_r(\mathcal{G} ; \mathcal{E})$. Thus, since $ev_u \circ \Phi_r$ is an extension of $ev_u \circ (\Phi_r |_{M_r})$ to $C^*_r(\mathcal{G} ; \mathcal{E})$, it must be the unique state extension. □

Proof of Theorem 6.3. Since $\iota_r$ is injective, it is clear that if $\Psi$ is injective, then so is the homomorphism $\Psi \circ \iota_r$. We prove the converse. For this, suppose that $\Psi \circ \iota_r$ is injective. Then $\Psi$ is injective on the subalgebra $M_r$ of $C^*_r(\mathcal{G} ; \mathcal{E})$. Let

$$X^G := \{ u \in \mathcal{E}^{(0)} : \mathcal{E}^u = \mathcal{I}^E_u \} \quad \text{and} \quad X^G := q(X^G) = \{ x \in \mathcal{G}^{(0)} : \mathcal{G}^x = \mathcal{I}^G_u \}.$$

For each $u \in X^G$, let $S_u$ be the collection of pure states $\varphi$ of $M_r$ such that $\varphi \circ \iota_r$ is a pure state of $C^*_r(\mathcal{I}^G ; \mathcal{I}^E)$ that factors through $C^*_r(\mathcal{G}_{q(u)}^G ; \mathcal{E}_{q(u)}^G)$. Define $S := \bigcup_{u \in X^G} S_u$.

By Proposition 6.4(a), each $\varphi \in S$ extends uniquely to a state $\varphi$ of $C^*_r(\mathcal{G} ; \mathcal{E})$. For each $\varphi \in S$, let $\pi_{\varphi}$ be the GNS representation of $C^*_r(\mathcal{G} ; \mathcal{E})$ associated to $\varphi$. To see that $\Psi$ is injective on $C^*_r(\mathcal{G} ; \mathcal{E})$, it suffices by [14, Theorem 3.2], to show that $\pi_S := \bigoplus_{\varphi \in S} \pi_{\varphi}$ is faithful on $C^*_r(\mathcal{G} ; \mathcal{E})$. For this, fix $a \in C^*_r(\mathcal{G} ; \mathcal{E})$ such that $\pi_S(a) = 0$. Then $\pi_{\varphi}(a) = 0$ for every $\varphi \in S$. Let $\Phi_r : C^*_r(\mathcal{G} ; \mathcal{E}) \to C^*_r(\mathcal{G}^{(0)} ; q^{-1}(\mathcal{G}^{(0)}))$ be the faithful conditional expectation extending restriction of functions. To see that $a = 0$, it suffices to show that $\Phi_r(a^*a) = 0$, because $\Phi_r$ is faithful. Suppose, for contradiction, that $\Phi_r(a^*a) \neq 0$. Then $\Phi_r(a^*a) > 0$, because $\Phi_r$ is positive. Let $\Xi_u \in C_0(\mathcal{G}^{(0)})$ be the image of $\Phi_r(a^*a)$ under the isomorphism from $C^*_r(\mathcal{G}^{(0)} ; q^{-1}(\mathcal{G}^{(0)}))$ to $C_0(\mathcal{G}^{(0)})$ given in [37, Theorem 11.1.11]. Then $\Xi_u > 0$, since $\Phi_r(a^*a) > 0$. Let $Y_u := \Xi_u^{-1}((0, \infty))$. Since $\Xi_u$ is continuous, $Y_u$ is an open subset of $\mathcal{G}^{(0)}$. Thus, since $X^G$ is dense in $\mathcal{G}^{(0)}$ by [14, Lemma 3.3(a)], we have $Y_u \cap X^G \neq \emptyset$. Choose $x \in Y_u \cap X^G$, and let $u := i(x, 1)$. Then $u \in (q_{\mathcal{E}^{(0)}})^{-1}(X^G) = X^G$, and $\Phi_r(a^*a)(u) = \Xi_u(x) > 0$. Fix $\epsilon > 0$ such that $\Phi_r(a^*a)(u) > \epsilon$.

Since $\mathcal{E}^u = \mathcal{I}^E_u$ and $a^*a \geq 0$, we know by Lemma 6.6(a) that there exist $b, c \in M_r$ such that $b, c \geq 0$,

$$\|ba^*ab - c\|_r < \frac{\epsilon}{2}, \quad (6.3)$$

and $\varphi(b) = \|b\|_r = 1$ for every (not necessarily pure) state $\varphi$ of $M_r$ such that $\varphi \circ \iota_r$ factors through $C^*_r(\mathcal{G}_{q(u)}^G ; \mathcal{E}_{q(u)}^G)$. Let $(e_{\lambda})_{\lambda \in \Lambda}$ be an approximate identity for $C^*_r(\mathcal{G} ; \mathcal{E})$, and for each $\varphi \in S_u$, let $N_{\varphi} := \{ f \in C^*_r(\mathcal{G} ; \mathcal{E}) : \varphi(f^*f) = 0 \}$ be the null space for $\varphi$. Since the GNS representation $\pi_{\varphi}$ satisfies $\pi_{\varphi}(a) = 0$, we have $\pi_{\varphi}(ba^*ab) = 0$, and so by the GNS construction, we have

$$\pi_{\varphi}(ba^*ab) = \lim_{\lambda \in \Lambda} \left( \pi_{\varphi}(ba^*ab)(e_{\lambda} + N_{\varphi}) \right) = \pi_{\varphi}(ba^*ab) = 0, \quad \text{for each } \varphi \in S_u. \quad (6.4)$$

Together, equations (6.3) and (6.4) imply that for all $\varphi \in S_u$,

$$|\varphi(c)| = |\pi_{\varphi}(c)| \leq |\pi_{\varphi}(ba^*ab)| + |\pi_{\varphi}(ba^*ab)| \leq \|c - ba^*ab\|_r < \frac{\epsilon}{2}. \quad (6.5)$$

Let $Q^G_u : C^*_r(\mathcal{I}^G ; \mathcal{I}^E) \to C^*_r(\mathcal{I}^G_{q(u)} ; \mathcal{I}^E_u) = C^*_r(\mathcal{G}_{q(u)}^G ; \mathcal{E}_{q(u)}^G)$ be the surjective $*$-homomorphism of Corollary 4.6(a). Since $c$ is a positive element of $M_r$, $Q^G_u(i_{-1}(c))$ is a positive element of
By Lemma 6.8(c),

\[ |\phi(Q_u^*(t_r^{-1}(c)))| = \|Q_u^*(t_r^{-1}(c))\|_r. \]  

(6.6)

Define \( \psi : M_r \to \mathbb{C} \) by \( \psi(t_r(h)) := \phi(Q_u^*(h)) \). By Lemma 6.7, \( \psi \) is a pure state of \( M_r \) and \( \psi \circ \iota_r \) is a pure state of \( C^*_r(G^{(q)}; E_u) \) that factors through \( C^*_r(G^{(q)}; E_u) \). Hence \( \psi \in S_u \), and so equations (6.6) and (6.5) imply that

\[ \|Q_u^*(t_r^{-1}(c))\|_r = |\phi(Q_u^*(t_r^{-1}(c)))| = |\psi(c)| < \frac{\epsilon}{2}. \]  

(6.7)

Let \( \text{ev}_u \) be the evaluation map \( f \mapsto f(u) \) on \( C^*_r(G^{(0)}; q^{-1}(G^{(0)})) \), and let \( \rho_u := \text{ev}_u \circ (\Phi_r|_{M_r}) \). By parts (a) and (b) of Lemma 6.8, \( \rho_u \) is a state of \( M_r \) and \( \rho_u \circ \iota_r \) is a state of \( C^*_r(T^c; \mathcal{T}^c) \) that factors through \( C^*_r(G^{(q)}; E_u) \). Hence there is a state \( \kappa_u \) of \( C^*_r(G^{(q)}; E_u) \) such that \( \rho_u \circ \iota_r = \kappa_u \circ Q_u^* \). Thus, using equation (6.7) for the final inequality, we obtain

\[ |\rho_u(c)| = |\kappa_u(Q_u^*(t_r^{-1}(c)))| \leq \|Q_u^*(t_r^{-1}(c))\|_r < \frac{\epsilon}{2}. \]  

(6.8)

By Lemma 6.8(c), \( \overline{\rho}_u := \text{ev}_u \circ \Phi_r \) is the unique state extension of \( \rho_u \) to \( C^*_r(G; E) \). By our choice of \( b \), we have \( \overline{\rho}_u(b) = \rho_u(b) = \|b\|_r = 1 \). Thus, recalling the notation defined in Notation 5.1, we have \( b \in \mathcal{M}_{\overline{\rho}_u} \), and so Remark 5.2 implies that \( b \in \mathcal{M}_{\overline{\rho}_u} \). Therefore,

\[ \overline{\rho}_u(ba^*ab) = \overline{\rho}_u(b) \overline{\rho}_u(a^*a) \overline{\rho}_u(b) = \overline{\rho}_u(a^*a). \]  

(6.9)

Using equation (6.9) for the second equality, we obtain

\[ \Phi_r(a^*a)(u) = |\overline{\rho}_u(a^*a)| = |\overline{\rho}_u(ba^*ab)| \leq |\overline{\rho}_u(ba^*ab) - \overline{\rho}_u(c)| + |\overline{\rho}_u(c)| \leq \|ba^*ab - c\|_r + |\rho_u(c)|. \]  

(6.10)

Together, equations (6.10), (6.3), and (6.8) imply that

\[ \Phi_r(a^*a)(u) \leq \|ba^*ab - c\|_r + |\rho_u(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \]

which contradicts the inequality (6.2). Thus, we deduce that \( \Phi_r(a^*a) = 0 \), which completes the proof. \( \square \)

We conclude with a characterisation of simplicity of reduced twisted C*-algebras of effective Hausdorff étale groupoids. This result is presumably well known (for instance, if \( G \) is minimal and effective, then [30, Theorem 7.26] implies that \( C^*_v(G; E) \) is simple), but we present a proof as an example of an application of Theorem 6.3.

Note that if the groupoid \( G \) is not effective, then characterising simplicity of \( C^*_v(G; E) \) in terms of \( G \) and \( E \) is a much harder problem; see [28, Remark 8.3].

**Corollary 6.9.** Let \((E, i, q)\) be a twist over an effective Hausdorff étale groupoid \( G \). Then \( C^*_v(G; E) \) is simple if and only if \( G \) is minimal.

**Proof.** Suppose that \( G \) is not minimal. Then there exists a nonempty proper closed invariant subset \( K \) of \( G^{(0)} \). Define \( G_K := s^{-1}(K) \). Since \( K \) is closed and invariant, \( G_K \) is a closed étale subgroupoid of \( G \) with unit space \( G_K^{(0)} = K \). Hence Lemma 2.10 implies that \( q^{-1}(G_K) \) is a twist over \( G_K \). The restriction map \( f \mapsto f|_{q^{-1}(G_K)} \) is a *-homomorphism from \( \Sigma_c(G; E) \) to \( \Sigma_c(G_K; q^{-1}(G_K)) \), and so it extends to a C*-homomorphism \( \text{res}_K : C^*_v(G; E) \to C^*_v(G_K; q^{-1}(G_K)) \). Since \( G_K \) is nonempty, \( \text{res}_K \) is not the zero map, and since \( G_K \neq G \), \( \text{res}_K \) is not injective. Hence \( \ker(\text{res}_K) \) is a nonzero proper ideal of \( C^*_v(G; E) \), and so \( C^*_v(G; E) \) is not simple.

For the converse, suppose that \( G \) is minimal. Let \( D_r \) denote the completion of the set \( D_0 := \{ f \in \Sigma_c(G; E) : q(\text{supp}(f)) \subseteq G^{(0)} \} \) with respect to the reduced norm. Since \( G \) is
effective, we have $I^G = G^{(0)}$ and $I^E = q^{-1}(G^{(0)})$. Let $\iota_r : C^*_r(G^{(0)}; q^{-1}(G^{(0)})) \to C^*_r(G; E)$ be the injective homomorphism of Proposition 6.1. Then $\iota_r(C^*_r(G^{(0)}; q^{-1}(G^{(0)}))) = D_r$.

Let $I$ be a nonzero ideal of $C^*_r(G; E)$. Then there is a C*-homomorphism $\Psi$ of $C^*_r(G; E)$ such that $I = \ker(\Psi)$. Since $I$ is nonzero, $\Psi$ is not injective, and hence Theorem 6.3 implies that $\Psi \circ \iota_r$ is not injective either. Thus $J := \ker(\Psi \circ \iota_r)$ is a nonzero ideal of $C^*_r(G^{(0)}; q^{-1}(G^{(0)}))$, and we have $\iota_r(J) = I \cap D_r$. To see that $C^*_r(G; E)$ is simple, we must show that $I = C^*_r(G; E)$. We know by [36, Theorem 5.2] that $D_r$ contains an approximate identity for $C^*_r(G; E)$ (see also, [37, Proposition 11.1.14]), and so it suffices to show that $\iota_r(J) = D_r$, because then $D_r \subseteq I$, and it follows that $I = C^*_r(G; E)$.

Recall from [37, Theorem 11.1.11] that there is an isomorphism $\Upsilon : C_0(G^{(0)}) \to D_r$ such that $\Upsilon(f)(i(x, z)) = z f(x)$ for all $f \in C_0(G^{(0)})$ and $(x, z) \in G^{(0)} \times \mathbb{T}$. Define

$$F := \{ x \in G^{(0)} : f(x) = 0 \text{ for all } f \in \Upsilon^{-1}(\iota_r(J)) \}.$$

Since $\Upsilon^{-1}(\iota_r(J))$ is a nonzero ideal of $C_0(G^{(0)})$, $F$ is a proper closed subset of $G^{(0)}$, and

$$\Upsilon^{-1}(\iota_r(J)) = \{ f \in C_0(G^{(0)}) : f(x) = 0 \text{ for all } x \in F \}.$$

To see that $\iota_r(J) = D_r$, we will prove the equivalent statement that $\Upsilon^{-1}(\iota_r(J)) = C_0(G^{(0)})$.

Suppose that $F \neq \emptyset$. We will derive a contradiction by showing that $F = G^{(0)}$. Since $G$ is minimal, the only closed invariant subsets of $G^{(0)}$ are $\emptyset$ and $G^{(0)}$ itself, and so it suffices to show that $F$ is invariant. For this, fix $x \in F$, and suppose that $\gamma \in G$ satisfies $s(\gamma) = x$. We must show that $r(\gamma) \in F$. For this, fix $f \in \Upsilon^{-1}(\iota_r(J))$. We must show that $f(r(\gamma)) = 0$. Use Lemma 2.5 to choose a local trivialisation $(B_\alpha, P_\alpha, \phi_\alpha)_\alpha \in \mathcal{G}$ of $E$ such that each $B_\alpha$ is a bisection of $G$. Use Urysohn’s lemma to choose $h \in C_c(G)$ such that $\text{supp}(h) \subseteq B_\gamma$ and $h(\gamma) = 1$. Recall from Lemma 2.4(c) that for each $\varepsilon \in q^{-1}(B_\gamma)$, there is a unique $z_\varepsilon \in \mathbb{T}$ such that $\varepsilon = \phi_\alpha(q(\varepsilon), z_\varepsilon)$, and the map $\varepsilon \mapsto z_\varepsilon$ is continuous on $q^{-1}(B_\gamma)$. Thus $\varepsilon \mapsto z_\varepsilon h(q(\varepsilon))$ is a continuous map from $q^{-1}(B_\gamma)$ to $\mathbb{C}$. Define $g : E \to \mathbb{C}$ by

$$g(\varepsilon) := \begin{cases} z_\varepsilon h(q(\varepsilon)) & \text{if } \varepsilon \in q^{-1}(B_\gamma) \\ 0 & \text{if } \varepsilon \notin q^{-1}(B_\gamma). \end{cases}$$

Since $P_\gamma(\gamma) = \phi_\alpha(\gamma, 1)$, we have $g(P_\gamma(\gamma)) = h(\gamma) = 1$. By a similar argument to the one used in the proof of Lemma 4.4 to show that $x_\alpha \in \Sigma_\alpha(I^G; I^E)$, we see that $g \in \Sigma_\alpha(G; E)$. Since $\text{supp}(g) \subseteq q^{-1}(B_\gamma)$ and $\text{supp}(\Upsilon(f)) \subseteq q^{-1}(G^{(0)})$, it follows from equation (2.2) and the fact that $B_\gamma$ is a bisection that

$$q(\text{supp}(g^* \Upsilon(f) g)) \subseteq B_\gamma^{-1} G^{(0)} B_\gamma = s(B_\gamma) \subseteq G^{(0)},$$

and thus $g^* \Upsilon(f) g \in D_r$. Since $\Upsilon(f)$ is an element of the ideal $I$, it follows that $g^* \Upsilon(f) g \in I \cap D_r = \iota_r(J)$. Therefore, since $x \in F$, we have

$$\left( g^* \Upsilon(f) g \right)(i(x, 1)) = \Upsilon^{-1} \left( g^* \Upsilon(f) g \right)(x) = 0. \quad (6.11)$$

Observe that $i(x, 1) = s(P_\gamma(\gamma)) = P_\gamma(\gamma)^{-1} i(r(\gamma), 1) P_\gamma(\gamma)$. Thus, since $q(\text{supp}(g))$ is contained in the bisection $B_\gamma$, equation (2.2) implies that

$$\left( g^* \Upsilon(f) g \right)(i(x, 1)) = g^* \left( P_\gamma(\gamma)^{-1} \right) \Upsilon(f)(i(r(\gamma), 1)) g(P_\gamma(\gamma)) = f(r(\gamma)). \quad (6.12)$$

Together, equations (6.11) and (6.12) imply that $f(r(\gamma)) = 0$, and so $F$ is invariant. Thus $F = G^{(0)}$, which is a contradiction, because $\Upsilon^{-1}(\iota_r(J))$ is a nonzero ideal of $C_0(G^{(0)})$. Therefore, we must have $F = \emptyset$, and so $\Upsilon^{-1}(\iota_r(J)) = C_0(G^{(0)})$, as required. \hfill \Box
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