ON THE AZUMAYA LOCUS OF ALMOST COMMUTATIVE
ALGEBRAS

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Abstract. We prove a general statement which implies the coincidence
of the Azumaya and smooth loci of the center of an algebra in positive
characteristic, provided that the spectrum of its associated graded al-
gebra has a large symplectic leaf. In particular, we show that for a
symplectic reflection algebra smooth and the Azumaya loci coincide.

1. Introduction

Throughout, we will fix a ground field $k$, which will be assumed to be
algebraically closed with positive characteristic $p$. In studying the represen-
tation theory of an associative $k$-algebra $A$ with a large center (i.e. $A$
is a finitely generated module over its center), it is important to understand
the Azumaya locus of $A$. Recall that the Azumaya locus of $A$ is defined to
be the subset of Spec $\mathcal{Z}(A)$ (where $\mathcal{Z}(A)$ denotes the center of $A$) consisting
of all prime ideals $I \in \text{Spec } \mathcal{Z}(A)$ such that the localized algebra $A_I$ is an
Azumaya algebra. If $A$ is a prime Noetherian ring with its center $\mathcal{Z}(A)$
being finitely generated over $k$, then a character $\chi : \mathcal{Z}(A) \to k$ belongs to
the Azumaya locus if and only if $A_\chi = A \otimes_{\mathcal{Z}(A)} k$ affords an irreducible
representation whose dimension is the largest possible dimension for an irre-
ducible $A$-module, and this largest possible dimension is the PI-degree of $A$
[\cite{BG} Proposition 3.1]. In particular, the square of PI-degree of $A$ is
equal to the rank of $A$ over $\mathcal{Z}(A)$. It is well-known that if the algebra $A$ is smooth,
then the Azumaya locus is contained in the smooth locus of Spec $\mathcal{Z}(A)$ [\cite{BG}].
Thus, the natural question is when are these two open subsets of Spec $\mathcal{Z}(A)$
equal. In this direction, there is a general result (which will be crucial for us) due to Brown and Goodearl [\cite{BG} Theorem 3.8.], which states that if $A$
is a prime Noetherian ring which is Auslander-regular and Cohen-Macaulay,
such that the complement of the Azumaya locus in Spec $\mathcal{Z}(A)$ has codim-
ension at least 2, then the Azumaya and smooth loci coincide. Using this
result, Brown and Goodearl [\cite{BG}] showed that the Azumaya locus coincides
with the smooth locus when $A$ is either the universal enveloping algebra of
a reductive Lie algebra in a very good characteristic, quantized algebra of
functions, quantized enveloping algebras at roots of unity. Also, Brown and

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Changtong [BC] have proved the similar result for rational Cherednik algebras [BC], and we have shown an analogous result for infinitesimal Hecke algebras of $\mathfrak{sl}_2$ [T].

Given examples above, the natural question is what do the above examples have in common, and whether there is a general condition which will imply that the Azumaya and smooth loci coincide. In this paper we give such a statement. Namely, we will show that if the algebra $A$ can be equipped with a positive filtration such that the associated graded algebra has an open symplectic leaf whose complement has codimension greater or equal to 2, then the Azumaya locus of $A$ has the complement of codimension at least 2.

The combination of this with the above mentioned result of Brown-Goodearl enables us to state a general result about the coincidence of the Azumaya and smooth loci, in particular we will show that for a symplectic reflection algebra the Azumaya locus coincides with the smooth locus.

2. General results

Let $A$ be an associative algebra over $k$ equipped with a nonnegative filtration $A_n \subset A_{n+1} \subset \cdots$, $n \in \mathbb{Z}_+$ such that the associated graded algebra $\text{gr} A$ is commutative. Let $d$ be the largest number such that $[A_n, A_m] \subset A_{n+m-d}$ for all $n, m$. Then $\text{gr} A$ becomes equipped with the following natural Poisson bracket: for any homogeneous elements $a \in \text{gr} A_n, b \in \text{gr} A_m$, one defines their Poisson bracket $\{a, b\}$ to be the symbol of $[a', b'] \in \text{gr} A_{n+m-d}$, where $a' \in A_n, b' \in A_m$ are arbitrary lifts of $a, b$.

Similarly, if $A$ is a flat associative algebra over $k[t]$, such that $A/tA$ is commutative, then commutator bracket of $A$ induces a Poisson bracket on $A/tA$ in the standard way: If $d$ is the largest integer such that $A/tA$ is commutative, then for any $a, b \in A/tA$, one puts $\{a, b\} = \frac{1}{t^d}[a', b']$, where $a', b'$ are arbitrary lifts of $a, b$ in $A$.

We start by recalling some terminology and a result by Bezrukavnikov and Kaledin [BK], which will play a crucial role in our proof.

Definition 2.1. [BK] A central quantization of a (commutative) Poisson algebra $B$ is a flat associative $k[[t]]$-algebra $B'$, such that $B = B'/tB'$ and Poisson bracket of $B$ is induced from the commutator bracket of $B$, and the quotient map from the center of $B'$ to the Poisson center of $B$ is surjective.

Here is the result that we are going to use.

Theorem 2.1 ([BK] Proposition 1.24). If under the assumptions of the above definition, $\text{Spec} B$ is a symplectic variety, then $B'[t^{-1}]$ is a simple algebra over its center.

The following result will be crucial.

Theorem 2.2. Let $A$ be an associative algebra over $k$ equipped with an increasing algebra filtration $k = A_0 \subset A_1 \subset \cdots$, such that $\text{gr} A$ is a commutative finitely generated domain, and the smooth locus of $X = \text{Spec} \text{gr} A, U \subset X$ is a symplectic variety with respect to the Poisson bracket of $\text{gr} A$ and the
codimension of the singular locus $Z = X - U$ in $X$ is $\geq 2$. If $\text{gr} Z(A) = (\text{gr} A)^p$, then the complement of the Azumaya locus of $Z(A)$ inside the smooth locus of $Z(A)$ has codimension $\geq 2$.

**Proof.** Let $f \in (\text{gr} A)^p$ be a nonzero homogeneous element which vanishes on the singular locus of $\text{Spec} \text{gr} A$, so $f(Z) = 0$. Thus, by the assumption $\text{Spec}(\text{gr} A_f)$ is a symplectic variety. Let us consider an element $g \in Z(A)$ such that $\sigma(g) = f$ (from now on $\sigma$ will denote the principal part of an element with respect to the filtration). Let us put $S = A_g, \deg(g) = d, d$ is a positive integer. Let us consider an induced filtration on $A_g$ coming from the filtration on $A$, namely $\deg g^{-1} = -d$. Then, $\text{gr} S = (\text{gr} A)_f$. Let us consider the Rees algebra of $S : R(S) = \sum S_m t^m \subset S[t, t^{-1}]$, where $S_m$ denotes the set of elements of $S$ of the filtration degree $\leq m$. Clearly $R(A)$ is a finitely generated module over its center $\text{Z}(R(A)) = R(A)$ (since $R(A)$ is positively graded), and since $R(S)$ is the localization $R(A)$ by a central element $gt^d$, we see that $R(S)$ is finite over its center $\text{Z}(R(S))$.

Let us complete $R(S)$ with respect to $t \in R(S)$. Denote this completion by $\overline{R(S)}$, so $\overline{R(S)} = \varprojlim R(S)/t^n R(S)$ . We have that $\overline{R(S)}$ is a flat module over $k[[t]], \overline{R(S)}/t \overline{R(S)} = (\text{gr} A)_f$. Thus by Theorem 2.11 $\overline{R(S)[t^{-1}]}$ is simple over $\overline{R(\text{Z}(S))}[t^{-1}].$ Notice that $\overline{R(S)} = R(S) \otimes_{\text{Z}(R(S))} \overline{R(\text{Z}(S))}$, so

$$\overline{R(S)[t^{-1}]} = R(S)[t^{-1}] \otimes R(\text{Z}(S))[t^{-1}] = \text{Spec} R(S)[t^{-1}] = \text{Spec} R(\text{Z}(S))[t^{-1}],$$

where we use the embedding $i : \text{Z}(S) \to \overline{R(\text{Z}(S))}$. Thus, we see that if $I \in \text{Spec} Z(A)$ is a prime ideal of height 1 in the smooth locus of $Z(A)$ such that $g \notin I$ and $I$ belongs to the image of $i^* : \text{Spec} R(\text{Z}(S))[t^{-1}] \to \text{Spec} Z(A)$, then there is a faithfully flat base change $Z(A)_I \to B,$ where $B = \overline{R(\text{Z}(S))}[t^{-1}]$.

Since $Z(A)_I$ is a regular local ring of dimension 1 and $A_I$ is torsion-free over $Z(A)_I,$ we get that $A_I$ is projective over $Z(A)_I$. Therefore, $A_I$ is Azumaya, so $I$ belongs to the Azumaya locus of $Z(A)$.

Since we want to show that all primes of height 1 from the smooth locus of $Z(A)$ belong to the Azumaya locus, it is enough to show that for any such prime $I \in \text{Spec} Z(A)$, there exists $f \in (\text{gr} A)^p$ such that $f(Z) = 0$, and $I \in i^* \text{Spec} R(\text{Z}(S))[t^{-1}]$, for some $g \in A$ with $\sigma(g) = f$. Notice that $\overline{R(\text{Z}(S))}[t^{-1}]$ is a subring of $\overline{R(\text{Z}(S))}((t))$, where $\overline{R(S)}$ is a completion of the filtered ring $\text{Z}(S) = Z(A)_g$ with respect to negative degree subspaces, meaning that $\overline{R(S)}$ is the inverse limit $\text{Z}(S)/\text{Z}(S)_n$ as $n \to -\infty$, where $\text{Z}(S)_n$ denoted the $n$-th degree filtration subspace of $\text{Z}(S).$ More precisely, elements of $\overline{R(\text{Z}(S))}[t^{-1}]$ are of the form $\sum_s s_i t^i$, where $s_i \in \overline{R(S)}$ such that limit of $i - \deg(s_i)$ is $\infty$ as $i \to \infty$, and $s_i = 0$ for sufficiently small $i << 0$. Thus, if $j : \text{Z}(S) \to \overline{R(\text{Z}(S))}$ denotes the embedding, then if $I \in \text{Im}(j^*), j^* : \text{Spec} \overline{R(\text{Z}(S))} \to \text{Spec} \text{Z}(S))$, then $I$ belongs to the Azumaya
locus. Invertible elements of $\overline{Z}(S)$ are precisely those elements whose principal symbol is a power of $f$ times an element of $k^*$. It follows from the following trivial lemma that a prime ideal $I$ of height 1 is in the image of $j^*$ if and only if $I$ contains no elements whose principal symbol is a power of $f$ up to an element of $k^*$.

**Lemma 2.1.** Let $j : R \to S$ be an embedding of integral domains, and let $I$ be a prime ideal of height 1 of $R$ which belongs to the smooth locus of $\text{Spec } R$. If $I \cap S^* = \emptyset$, then $I \in \text{im}(j^*) : \text{Spec } S \to \text{Spec } R$.

**Proof.** Without loss of generality we may assume that $R$ is a local ring with $I$ being the maximal ideal. Therefore, $I = (g)$ for some $g \in R$. Since $g \notin S^*$, there is a prime ideal $J \in \text{Spec } S$ such that $g \in J$. Therefore, $I = j^{-1}(J)$. \hfill $\square$

Now what we want follows from the following well-known fact.

**Lemma 2.2.** Let $B$ be a nonnegatively filtered finitely generated commutative algebra over $k$, and let $I \subset B$ be an ideal. Then $\text{ht}(I) = \text{ht} \text{gr } I$.

**Proof.** We have that the Gelfand-Kirillov dimension of $B/I$ equals to that of $\text{gr } B/\text{gr } I = \text{gr}(B/I)$, which implies that $\text{ht}(I) = \text{ht}(\text{gr } I)$, since $\text{dim } B = \text{dim } \text{gr } B$. \hfill $\square$

Note that if $\text{Spec } \text{gr } A$ consists of finitely many symplectic leaves, and $\text{gr } Z(A) = (\text{gr } A)^p$, then assumptions of Theorem 2.2 are satisfied.

We will also use the following

**Lemma 2.3.** Let $M$ be a finitely generated positively filtered module over a positively filtered commutative algebra $H$, such that $\text{gr } H$ is a domain and $\text{gr } M$ is finite over $\text{gr } H$. Then the rank of $M$ over $H$ is equal to the rank of $\text{gr } M$ over $\text{gr } H$.

**Proof.** First of all, it is easy to see that the rank of $\text{gr } M$ over $\text{gr } H$ is $\leq \text{rank}_H M$. Indeed, if $x_1, \ldots, x_n \in H$ are elements such that their principal parts $\sigma(x_1), \ldots, \sigma(x_n) \in \text{gr } M$ are $H$-linearly independent, then $x_1, \ldots, x_n$ are $H$-linearly independent in $M$. Now let us consider $R(M)$, the Rees module of $M$ over the Rees algebra $R(H)$. $R(M)$ is a finitely generated $R(H)$-module. From the fact that $R(M)/(t - \lambda) = M, \lambda \neq 0, R(M)/tR(M) = \text{gr } M$ we see that the generic dimension of fibers of $R(M)$ over $R(H)$ have dimension equal to $\text{rank}_H M$. Therefore, by the semi-continuity, we conclude that $\text{rank}_{\text{gr } H} \text{gr } M \geq \text{rank}_H M$, so we are done. \hfill $\square$

**Lemma 2.4.** Suppose that an algebra $A$ satisfies all the assumptions of Theorem 2.2 except that $\text{gr } Z(A) = (\text{gr } A)^p$. If $\text{gr } A$ is normal, then $\text{gr } Z(A) \subset (\text{gr } A)^p$. 

Proof. Let $f \in Z(A)$, then its top symbol $\bar{f}$ belongs to the Poisson center of $gr A$. Let $U \subset X$ be the smooth locus of $X$. By the assumption $U$ is a symplectic variety and $\bar{f}|U$ belongs to the Poisson center of $O(U)$. Therefore, $df|_U = 0$, hence, there exists $g \in O(U)$, such that $\bar{f}|_U = g^\frac{q}{p}$. Since $X - U$ has codimension $\geq 2$ in $X$ and $X$ is normal, $g$ extends to the whole $X$. Thus, that $f \in (gr A)^p$.

\[ \square \]

**Theorem 2.3.** Let $A$ be an algebra over $k$ equipped with an increasing algebra filtration $k = A_0 \subset A_1 \subset \cdots \subset A_n \cdots$, such that $gr A$ is a finitely generated smooth commutative domain over $k$. Suppose that there exist a central subalgebra of $A$, $Z_0 \subset Z(A)$, such that $gr A/(gr A)(gr Z_0)_+$ is a domain whose smooth locus is a symplectic variety under the natural Poisson bracket and whose complement has codimension $\geq 2$. Assume moreove that $gr A/(gr A)(gr Z_0)_+$ is normal. If $(gr A)^p \subset gr Z(A)$, then the smooth and the Azumaya locus of $A$ coincide, and the PI-degree of $A$ is $p^{d}$, where $2d = \dim (gr A)/(gr A)(gr Z_0)_+$.

**Proof.** Let $\chi_0 : Z_0 \rightarrow k$ be a character, then we can consider the quotient algebra $A_{\chi_0} = A \otimes k$. Then, $A_{\chi_0}$ comes equipped with the natural filtration induced from $A$ and $gr A_{\chi_0} = gr A/(gr Z_0)_+$, also lemma [2.3] with the assumption $Z(A) \subset (gr A)^p$ implies that $gr Z(A_{\chi_0}) = (gr A/(gr Z_0)_+)^p$. Thus, we may apply our proposition, which implies that the Azumaya locus of $Z(A_{\chi_0})$ has complement of codimension at least 2. Now we claim that a character $\chi : Z(A) \rightarrow k$ belongs to the Azumaya locus of $Z(A)$ if and only if the corresponding character $\chi_0 : Z(A_{\chi_0}) \rightarrow k$ belongs to the Azumaya locus of $A_{\chi_0}$, where $\chi_0$ is the restriction of $\chi$ on $Z_0$. It is enough to check that the PI-degree of $A$ is equal to the PI-degree of $A_{\chi_0}$ for any character $\chi_0 : Z_0 \rightarrow k$. But this is clear because the PI-degree of $A_{\chi_0}$, which is is equal to the square root of the dimension of the generic fiber of $A_{\chi_0}$ ([BG]), is greater or equal to the PI-degree of $A$ (the square root of the dimension of the generic fiber of $A$) by the semi-continuity of the rank of fibers of $A$. On the other hand, the PI-degree of $A$ is the largest dimension of an irreducible module of $A$, which is greater or equal to the largest possible dimension of an irreducible module of $A_{\chi_0}$, which is precisely the PI-degree of $A_{\chi}$.

So, let $U \subset Spec Z(A)$ be the Azumaya locus of $A$. We have a map $f : Spec Z(A) \rightarrow Spec Z_0$ corresponding to the inclusion $Z_0 \subset Z(A)$. Let us denote by $Y$ the complement of the Azumaya locus $Y = Spec Z(A) - U$. Then for any closed point $\chi \in Spec Z_0$, the intersection $f^{-1}(\chi) \cap Y$ has codimension at least 2 in $f^{-1}(\chi)$. So, the codimension of $Y$ in Spec $Z(A)$ is at least 2. Now, since $A$ is Auslander-regular and Cohen-Macaulay, the above mentioned result of Brown-Goodearl ([BG]) implies the coincidence of the Azumaya and smooth loci.

By lemma [2.3] the PI degree of $A$ is equal to the rank of $gr A/(gr A)(gr Z_0)_+$ over $(gr A/(gr A)(gr Z_0)_+)^p$, which is $p^d$, where $2d$ is the Krull dimension of $gr A/(gr A)(gr Z_0)_+$. 

3. Applications to symplectic reflection algebras and enveloping algebras

Let us recall the definition of a symplectic reflection algebra. Let $V$ be a symplectic $k$-vector space with the symplectic form $\omega : V \times V \to k$. An element $g \in Sp(V)$ is called a symplectic reflection if $\text{rank}(\text{Id} - g) = 2$. To a symplectic reflection $s \in Sp(V)$ one may associate a skew-symmetric form $\omega_s : V \times V \to k$ which coincides with $\omega$ on $\text{Im}(\text{Id} - s)$ and is 0 on $\text{Ker}(\text{Id} - s)$. Let $G \subset Sp(V)$ be a finite group generated by symplectic reflections. To a $G$-invariant function $c : S \to k$ and $t \in k$, where $S \subset G$ is the subset of symplectic reflection of $G$, Etingof and Ginzburg [EG] associated an algebra (called a symplectic reflection algebra) $H_{t,c}$ which is defined as a quotient of $k[G] \ltimes T(V)$ by the relations

$$[x, y] = t\omega(x, y) + \sum_{s \in S} \omega_s(x, y)c(s)s.$$

There is a filtration on $H_{t,c}$ deg $v = 1$, $v \in V$, deg $g = 0$, $g \in G$. The crucial property is that $\text{gr} H_{t,c} = k[G] \ltimes \text{Sym} V$ [EG].

The following theorem answers positively two questions raised by Brown-Changtong [BC] Questions 6.1, 6.4, and proved by them in the case of a rational Cherednik algebra.

**Theorem 3.1.** Let $H_{t,c}$ be a symplectic reflection algebra associated to $G \subset Sp(V), \dim V = 2n, t \neq 0$ and $p$ does not divide the order of $G$, then the smooth and the Azumaya loci of the center of $H_{t,c}$ coincide, and the PI-degree of $H_{t,c}$ (the maximal dimension of an irreducible module) is equal to $p^n |G|$. 

**Proof.** Let us consider $U_{t,c} = eH_{t,c}e$, the spherical subalgebra of the symplectic reflection algebra $H_{t,c}$, where $e = \frac{1}{|G|} \sum_{g \in G} g$ is the symmetrizing idempotent of $G$. By a theorem of Etingof [BFG], $\text{gr} Z(U_{t,c}) = ((\text{Sym} V)^G)^p = (\text{gr} U_{t,c})^p$. Theorem 2.2 can be applied to $U_{t,c}$, since $\text{gr} U_{t,c} = S(V)^G \subset \text{Sym} V$ has finitely many symplectic leaves by a result of Brown-Gordon [BG] Proposition 7.4. So, we get that for all prime ideals $I$ of the smooth locus of $Z(U_{t,c})$ of height 1, the algebra $U_{t,c}I$ is Azumaya. But by a result of Brown-Changtong [BC] Corollary 3.6., $(H_{t,c})_I$ is Morita equivalent to $(U_{t,c})_I$, so the complement of the Azumaya locus in the smooth locus of $H_{t,c}$ has codimension $\geq 2$. Since $H_{t,c}$ is Auslander-regular and Cohen-Macaulay ([BC]), by Brown-Goodearl [BG], smooth and the Azumaya loci coincide for $Z(H_{t,c})$. Lemma 2.3 applied to $M = H_{t,c}, H = Z(H_{t,c})$ implies that the PI-degree is independent of $c$, so we may take $c = 0$, in which case the desired statement is clear.

$\square$
Applying the considerations from the previous section, we may take $A$ to be the enveloping algebra of a semi-simple Lie algebra $\mathfrak{g}$, and take $Z_0 \subset Z(A)$ to be the subalgebra obtained by the symmetrization map applied to the generators of $\text{(Sym} \mathfrak{g})^G$, in other words $Z_0$ is the reduction modulo $p$ of the usual characteristic 0 central elements of $\mathfrak{Ug}$. Then $\text{gr } A/(\text{gr } Z_0)_+ = \text{Sym} \mathfrak{g}/(\text{Sym} \mathfrak{g})^G$ is the ring of coinvariants, which is the ring of functions on the nilpotent cone of $\mathfrak{g}^*$ [Ko], which as a Poisson variety consists of finitely many symplectic leaves. Thus, assumptions of Theorem 2.3 are satisfied, as a result we obtain that the Azumaya locus of $\mathfrak{Ug}$ coincides with the smooth locus of its center, a theorem of Brown-Goodearl [BC]. Note that we did not use any modular representation theory of $\mathfrak{g}$.

A standard example of an almost commutative algebra for which the Azumaya locus does not coincide with the smooth locus is the enveloping algebra of the Heisenberg Lie algebra: A Lie algebra $\mathfrak{g}$ with a basis $z, x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations $[x_i, y_j] = \delta_{ij} z, [z, x_i] = [z, y_j] = 0$, where $\delta_{ij}$ is the Kronecker symbol. Then, the center of $\mathfrak{Ug}$ is the polynomial algebra $k[z, x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p]$, but the Azumaya locus is the set of characters which do not vanish on $z$.

The above theorem can also applied to infinitesimal Hecke algebras of $\mathfrak{sl}_2$ [T]. We expect many applications of the above result for other infinitesimal Hecke algebras.

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