Conformal dynamical equivalence and applications

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Abstract. The “Conformal Dynamical Equivalence” (CDE) approach is briefly reviewed, and some of its applications, at various astrophysical levels (Sun, Solar System, Stars, Galaxies, Clusters of Galaxies, Universe as a whole), are presented. According to the CDE approach, in both the Newtonian and general-relativistic theories of gravity, the isentropic hydrodynamic flows in the interior of a bounded gravitating perfect-fluid source are dynamically equivalent to geodesic motions in a virtual, fully defined fluid source. Equivalently, the equations of hydrodynamic motion in the former source are functionally similar to those of the geodesic motions in the latter, physically, fully defined source. The CDE approach is followed for the dynamical description of the motions in the fluid source. After an observational introduction, taking into account all the internal physical characteristics of the corresponding perfect-fluid source, and based on the property of the isentropic hydrodynamic flows (quite reasonable for an isolated physical system), we examine a number of issues, namely, (i) the classical Newtonian explanation of the celebrated Pioneer-Anomaly effect in the Solar System, (ii) the possibility of both the attractive gravity and the repulsive gravity in a non-quantum Newtonian framework, (iii) the evaluation of the masses - theoretical, dynamical, and missing - and of the linear dimensions of non-magnetized and magnetized large-scale cosmological structures, (iv) the explanation of the flat-rotation curves of disc galaxies, (v) possible formation mechanisms of winds and jets, and (vi) a brief presentation of a conventional approach - toy model to the dynamics of the Universe, characterized by the dominant collisional dark matter (with its subdominant luminous baryonic “contamination”), correctly interpreting the cosmological observational data without the need of the notions dark energy, cosmological constant, and universal accelerating expansion.

1. Introduction and Motivation
According to many current observational data, the realistic picture and morphology of an astrophysical-cosmological structure differs greatly from its corresponding optical picture. The Solar...
System, for example, viewed as a star accompanied by a number of nearby small objects and planets, differs from the one observed embedded in the solar wind, the outer boundaries of which extend up to the corresponding outer boundaries of the winds from nearby stars. Also, far beyond the Kuiper Belt, extending out of the orbits of the outer planets Neptune and Pluto, there is an enormous spherical array of icy worlds orbiting the Sun, called the Oort Comet Cloud, whose linear dimensions are of the order of $10^6$ AU. This means that the Solar System extends up to half the distance of the alpha-Proxima Centauri, the star nearest to the Sun. Furthermore, according to far-ultraviolet spectroscopic observations of the properties of the highly-ionized oxygen, of the hot gas, and of the high-velocity clouds in the halo of the Milky Way galaxy and the Local Group \cite{1} (see also \cite{2} and references therein), there exists an extended, hot ($T > 10^6$ K), and sufficiently diffuse Galactic Corona, previously undetected by other means (e.g., X-rays), far beyond the nearby halo detected previously. This implies that the Galactic Corona, and hence the Milky Way galaxy itself, possibly extends out to the Magellanic Clouds. So, the expected linear dimensions of the Milky Way galaxy are at least 200 kpc, almost ten times larger than its optical linear dimensions ($\sim 30$ kpc), and even larger on the basis of at least some of the high-velocity clouds at estimated distances as large as 850 kpc \cite{3}. If such a result could be considered as typical, then the mutual (minimal) distance of the outer boundaries of the Milky Way and the nearby Andromeda Galaxy, at a distance of approximately 600 kpc from it, is of the order of their linear dimensions, approximately 200 kpc. Such a result for neighboring galaxies is generally believed to be valid for typical clusters of galaxies, in which the mutual distances of neighboring members are of the order of a few hundred kpc. Under such conditions, the linear dimensions of galaxies with haloes and coronas (in the range 0.1 kpc – 1 Mpc, depending on the type of the galaxies, e.g., elliptical, spiral, irregular) is not necessarily much smaller than their mutual distances. As a consequence, the dynamical description of the observed motions in a typical cosmological structure is not that of a system of gravitating point masses (namely, no tidal interactions), but rather of the form of hydrodynamic flows in a more or less continuous gravitating source.

Moreover, the structure of the interstellar medium itself in elliptical galaxies and of the intracluster medium in cluster of galaxies present special interest. Thus, on the one hand, the interstellar medium consists primarily of hot ($T > 10^6$ K) plasma; several molecular clouds with dimensions 20-50 pc and masses $\sim 10^8 m_\odot$ in it, are partly associated with HII regions and embedded in a lower-density interstellar medium; the masses ($\sim 10^8 - 10^9 m_\odot$) of the central dark objects in elliptical galaxies are only a few percent of the molecular-gas mass in the sub-kilo parsec region, as implied by the observational study of water-masering sources \cite{4} - \cite{7}. On the other hand, a cluster of galaxies can be treated \cite{8} as an approximately isothermal sphere of hot ionized hydrogen (of dimensions a few Mpc, temperature, which can be as high as $\sim 10^6$ K, number density of electrons $\sim 10^4$ cm$^{-3}$, and number-density of molecular clouds $\sim 10^4$ to $10^3$ cm$^{-3}$); the intracluster gas contains X-rays, fills the space between the galaxies, occupies much of the cluster’s volume, and the X-ray luminosities fall in the range $10^{43}$ to $10^{45}$ erg sec$^{-1}$; the mass of the intracluster medium exceeds the total mass of the luminous parts of the cluster’s galaxies by a factor of several. (However, one must not forget that the surveys for the largest cosmological features so far cover only 0.001% of the observable universe!!!). In this spirit, it is interesting that, as a consequence of the improvement of the observing techniques, currently we witness in various ways (see, e.g., \cite{9} – \cite{12}; or even continuous citations-reports, prior the official publication) continuous observational detections of many new galaxies (e.g., small ones each orbiting another larger and brighter galaxy, or numerous, tidally-disrupted and densely-populated galaxies (ultra-compact dwarf; UCD galaxies) inhabiting at the central regions of huge clusters of galaxies, like the Fornax and the Virgo galaxy clusters), of intergalactic globular clusters (beyond the $\sim 150$ ones surrounding our Milky Way galaxy) and even stars \cite{13}, and also of hot filamentary networks connecting and surrounding the galaxies of a cluster of galaxies, seen in the ultraviolet but unseen in the optical, infrared and radio wavelengths \cite{2}, \cite{3}.

According to all the above, the morphology of galaxies is very different from the simple picture of spiral, elliptical, or, even irregular galaxies, in the sense that the galaxies (as well as the
clustering of galaxies, and, possibly, the second-order clusters - super clusters - of galaxies are almost spherically symmetric, very complex, practically continuous, and of much larger linear dimensions cosmological structures than previously assumed. This situation becomes even more interesting, considering, in particular, the shapes of galactic dark-matter haloes derived from gravitational-lensing studies, N-body simulations, stellar tracers (e.g., globular clusters), HI studies and polar rings, studies of satellites (e.g., various tails), as well as from X-ray, and absorption lines studies (also, in the same volume, see, particularly, the contributions by Fillipi and Sepulveda, Ryden and Tinker, Zepf, Sparke, Johnston, Buote, Bowen). The observable Universe differs very much from the simple picture of a collection of galaxies (or higher-order cosmological structures), in which the mutual distances of the neighboring members are much larger than their linear dimensions. Consequently, the constituents elements of the Universe and the Universe as a whole, can quite satisfactorily be treated dynamically as continuous gravitational systems and, more specifically, bounded, gravitating perfect-fluid sources, the physical-dynamical description of which is very well established in both the Newtonian and the general-relativistic levels. So, we arrive at the very crucial result, that the motions of and in these constituents should be considered as hydrodynamic flows rather than geodesic motions. It is exactly this result, that enables us, through the dynamical-equivalence approach, to recast the geodesic motions, now taking into account the contribution to the observationally determined mass of the fluid source of all of its internal physical characteristics, not only its mass density, as sources of the observed motions.

Also, we recall [15], [16] that, according to the observations of the cosmic microwave background by the Wilkinson Microwave Anisotropy Probe (WMAP), the Cosmos is believed to be composed of heavy elements (0.03 %), ghostly neutrinos (0.3 %), stars (0.5 %), free hydrogen and helium (4 %) (and so baryonic matter ~5%), dark matter (22 %), and dark energy (73 %). Finally, we point out that the cosmological fluid is mostly considered as collisionless, and that, in the context of a cosmology dominated by a cosmological constant, it is not currently known what definitely this unknown is, and whether it is strictly constant, increasing, or decreasing with time.

In view of the above, we have applied the model of a bounded, continuous, gravitating perfect-fluid source of classical hydrodynamics or/and magneto hydrodynamics, in order to describe the isentropic motions in the large-scale cosmological structures, all their sources, and their consequences. This is in accordance with the approach of the Conformal Dynamical Equivalence (CDE) between the hydrodynamic flows and the geodesic motions, proposed and further developed and elaborated by the author and collaborators (for details and applications, see [17] – [38]). Exactly, the results so far of the study group on this subject will be briefly presented here.

The structure of this article is as follows: After an observational introduction in the beginning of the present Section 1, we recall in the next Section 2 that, in general, the conservation of rest mass (baryon-number conservation) and the conservation of entropy are not mutually independent requirements of general relativity, a very important result frequently overlooked. Then, in Section 3, we first outline the dynamical equivalence in the case of the Newtonian theory of gravity. We prove that the isentropic flows in a bounded, gravitating perfect-fluid source are dynamically equivalent to geodesic motions in a generalized scalar potential (V). This generalized potential is explicitly expressed in terms of the standard gravitational potential (U, satisfying the standard gravitational Poisson equation), mass density, and, additionally, the internal physical characteristics of the gravitating source, namely, mass density, isotropic pressure and internal thermodynamic energy density. This enables us to define, through a Poisson-type equation for the generalized potential, the generalized mass-energy density, \( \rho_V \) (and the corresponding generalized mass, \( m_V \)) producing the generalized potential. As an astrophysical byproduct-application, we prove in Section 3, that, in the Newtonian theory of gravity for a static configuration, the mass-density and pressure functional laws cannot be chosen arbitrarily from each other, but their choice depends on each other’s one. This can considerably limit the form of the (widely used in the literature, Plummer-type) mass-density law of an isentropic and isothermal equation of state.
In Section 4, it is proved that the generalized mass-energy density, ρV, can be positive/vanishing/negative, and that the generalized mass-energy, mV, can, similarly, be positive/vanishing/negative. As a consequence, based on the form of the generalized Euler-Newton equations of motion, we prove that, in the Newtonian theory of gravity, a positive/vanishing/negative generalized mass-energy density, ρV, implies a spatially decreasing/non-changing/increasing acceleration, and, hence, dominance of the inwards acceleration (attractive gravity)/non changing acceleration/outwards acceleration (repulsive gravity), all of them stemming from the relative importance of the mass density and of the other internal physical characteristics of the fluid source.

In Section 5, based on the results of Section 4, we explore the discrimination of the regions of prevalence of the attractive gravity and repulsive gravity, leading to the determination of the so-called inversion distance, the latter’s importance in determining-explaining the conventional linear dimensions of the source considered, as well as its temperature dependence.

In Section 6, we apply the results of the dynamical equivalence of Section 3 to a typical inhomogeneous (with a Plummer-type mass-density configuration), and isothermal and isentropic perfect-fluid-cosmological structure. We explore the relative importance of the source’s mass, m (theoretically evaluated through the Plummer-type density law) and its observationally determined mass, mV (dynamical mass) and the latter’s dependence on the mass density, ρ, (and mass, m) and on the fluid’s internal physical characteristics. This relation enables us to prove that, under conditions generally met in the physical Universe, the mass m of the cosmological structure generally differs from, and actually it is much larger than the observationally determined mass mV (dynamical mass). In this way, we propose a possible, classical (namely, not quantum-mechanically oriented) partial explanation of the missing-mass problem.

In Section 7, the distinction between the theoretical mass, m, and the observationally determined (dynamical) mass, mV, (under the assumption and use of planar, equatorial, circular geodesics in disc galaxies), allow us to propose a classical explanation of the flat-rotation curves problem. Additionally, the dependence of the rotational velocity on the structure’s (spatially constant) temperature (beyond its gravitational field) is critically examined.

In Section 8, we outline the use of the CDE approach in presenting a classical explanation of the celebrated Pioneer Anomaly Effect in the Solar System and, also, we provide the first analytical determination of the true linear dimensions of the Solar System (it extends up to almost half the distance to our closest star α-Proxima Centauri), and the determination of the internal thermodynamic energy at the near and far regions of the Solar System.

In Section 9, based on the notions of the repulsive gravity and inversion distance, we outline the use of the CDE approach towards providing a possible theoretical physical explanation of the mechanism of formation of winds and jets, with possible applications to the phenomena of the expansion of a red-giant star and a supernova.

In Section 10, we outline the use of the CDE approach in the determination of the (cosmologically-dynamically important) masses of the globular clusters of galaxies, composed of both dark matter and baryonic matter, by relating theory and observational results.

In Section 11, we outline the generalization of the CDE approach in the case of a magnetized gravitating perfect-fluid source and the importance of the new magnetic contribution.

In Section 12, we outline the general-relativistic cosmological applications of the CDE, in the case of a collisional-dark-matter model. We prove that the extra energy, needed to compromise for the currently available cosmological data, can be compensated by the energy of the internal motions of the collisional cosmological fluid, that the post-recombination Universe remains ever-decelerating, and also that the explicit form of the equation of state for the cosmological fluid can be determined. In other words, in the collisional-dark-matter model, no dark energy and no acceleration of the universal expansion are necessary. Also, we independently verify the particularly interesting facts, that, in the context of the standard, although, physically, less probable, collisionless-dark-matter model, the extra dark-energy component (to compromise the cosmic-microwave-radiation-background data) is, in fact, needed, the distant light sources appear to be dimmer than expected, and the expansion of the Universe
keeps balancing between acceleration and deceleration, depending on the (fully and explicitly
determined appropriate) value of the cosmological red-shift parameter. Therefore, these results of the
collisionless-dark-matter model seem to be a misinterpretation of the cosmological observational data
by an observer, who (although living in a Universe filled, mainly, with, collisional dark matter) insists
in adopting the traditional (collisionless-dark matter) and less physical approach.
Finally, we conclude in Section 13.

2. Adiabaticity and Isentropicty versus Baryon Conservation in General Relativity
It is known that, in general relativity, the field equations for a gravitating perfect-fluid source
\[ G_{ik} = -\frac{8\pi G}{c^4} T_{ik} \]  
are supplied by the equations of motion
\[ T_{ik}^\text{\textquoteleft\textprime} = 0 \]  
In Eqs. (2.1) and (2.2), G is the gravitational constant and c the velocity of light in vacuum and, in the
standard notation, \( G_{ik} \) is the Einstein tensor,
\[ T_{ik} = (\varepsilon + p)u_i u_k - pg_{ik} \] 
is the fluid’s energy-momentum tensor, where \( g_{ik} \) are the covariant components of the metric tensor, \( u_i \)
the covariant components of the four-velocity, a semi-colon denotes covariant derivative, and the
mass-energy density \( \varepsilon \) is assumed (see, e.g., [39], pp. 81–84, 90-94) to split as
\[ \varepsilon = \rho c^2 + \rho \Pi \] 
with the proper quantities \( \rho \), \( p \) and \( \rho \Pi \) being, respectively, the rest-mass density, proper isotropic
pressure, and internal specific-energy density of the fluid source, the latter being the thermodynamic
energy density that changes during the expansions and/or contractions of the fluid.
A direct consequence of Eqs. (2.2) and (2.4) is [40]
\[ \varepsilon \, u^i + (\varepsilon + p)u_i = 0 \]  
or, equivalently,
\[ \left[ 1 + \frac{1}{c^2} \left( \Pi + \frac{p}{\rho} \right) \right] \left( \rho u^i \right)_\text{\textquoteleft\textprime} + \frac{1}{c^2} \left( \Pi_j + p \left( \frac{1}{\rho} \right)_j \right) \left( \rho u^i \right)_\text{\textquoteleft\textprime} = 0 \]  
where a comma denotes usual partial derivative with respect to the space-time coordinates.
Therefore, in general relativity, the conservation of rest-mass (baryon-number conservation)
required by the equation
\[ (\rho u^i)_\text{\textquoteleft\textprime} = 0 \]  
is compatible with the equations of motion if, and only if,
\[ \left[ \Pi_j + p \left( \frac{1}{\rho} \right)_j \right] u^j = 0 \] 
This last equation is obviously satisfied under the assumption that the fluid’s hydrodynamic
flows are adiabatic, in other words, the net change, dQ, of the thermal content of a fluid’s volume
element, defined through the first classical axiom of thermodynamics
\[ dQ = d\Pi + pd \left( \frac{1}{\rho} \right) \] is always zero, namely,
\[ dQ = Q'u' = d\Pi + p\frac{d}{\rho} \left( \frac{1}{\rho} \right) = 0 \] 
whence Eq. (2.7) is identically satisfied.

Proceeding further to isentropic flows, it might be helpful to recall here that in reaching the general result (2.9) we can also use (as, e.g., in [17]) the equilibrium hydrodynamics hypothesis, in the form of the constancy of the entropy, \( S \), along the flow lines [41], [42], namely, in view of the second classical thermodynamic axiom,

\[ dQ = TdS \] 

\[ 0 = dS = S'u' \]

\[ = \frac{1}{T} dQ = \frac{1}{T} \left[ d\Pi + p\frac{d}{\rho} \left( \frac{1}{\rho} \right) \right] \]

\[ = \frac{1}{T} \left[ \Pi_j + p\left( \frac{1}{\rho} \right)_j \right] u^j \]

In fact, in relativistic astrophysics and cosmology, we usually assume \( S \) to be a group invariant [43], [44], whence

\[ S'u' = 0, \text{ for every } u' \text{ and } T \]

or, equivalently,

\[ \Pi_j + p\left( \frac{1}{\rho} \right)_j = 0 \]

direct consequence of which is the adiabaticity condition (2.9) (or (2.7)).

According to all the above, the conservation of rest-mass (Eq. 2.6) and the conservation of entropy (Eq. 2.13) are not independent requirements in the framework of general relativity. The reason of their independence in the Newtonian limit \( \frac{1}{c^2} \to 0 \), is that Eq. (2.5b) reduces simply to the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

We emphasize that isentropicity (and, hence, also adiabaticity) of the hydrodynamic flows could be physically necessary and useful. Actually, then, beyond the constancy of the matter content (the number of baryons) of a finite fluid volume element (Eqs. (2.6) or (2.14)), also its thermodynamic content remains unchanged along the flow line, in accordance with such a usual assumption for an isolated physical system.

As emphasized by Chandrasekhar [40], the conservation of rest-mass (2.6), under the assumption of the absence of any dissipative mechanisms in the fluid source, should be considered as a fundamental physical law supplementing the field equations (2.1). If, according to [40], we furthermore assume that the only baryons present are protons and neutrons of practically equally rest masses, the conservation law (2.6) is equivalent to the equally fundamental law of the conservation of the baryon-number (practically Eq. (2.6), with \( \rho \) replaced by the baryon-number density). Consequently, if we assume that the rest-mass of a baryon remains constant, we conclude that the conservation of rest-mass (2.6) is actually expressing the conservation of the baryonic mass.

All these physical results concerning the adiabaticity or/and isentropicity of the hydrodynamic flows and its dependence on the conservation of rest-mass has long ago been emphasized by Chandrasekhar [40], but, in our opinion, since then, they, unfortunately, have been
highly overlooked, especially in astrophysical applications. Here, we shall rely heavily on the adiabaticity of the hydrodynamic flows and, hence, on their dynamical equivalence to geodesic motions (see, e.g., [17]).

3. Newtonian Dynamical Equivalence of Hydrodynamic Flows and Geodesics

It has been suggested [17] that, in both the Newtonian and the general-relativistic theories of gravity, it is possible to give to the equations of the hydrodynamic flow motions in the interior of a bounded gravitating perfect-fluid source the form of the equations of geodesic motions.

In this Section 3, we limit ourselves to the Newtonian case (for the relativistic hydrodynamic case, see [17], [33], [37], [38] and for the magneto-hydrodynamic case, see [34], [35]; see also Sections 2 and 12 herewith). More precisely, in the case of the interior of a non-magnetized fluid, the Newtonian equations of the motion of a test particle (geodesic motion in the interior of the fluid) are

$$\frac{d\tilde{U}}{dt} = \nabla U$$

(3.1)

where the Newtonian gravitational potential $U$ obeys the Newtonian field equation (Poisson’s equation)

$$\nabla^2 U = -4\pi G\rho$$

(3.2)

with $\rho$ being the gravitational source’s mass-density function.

On the other hand, the Euler’s equations for the hydrodynamic flow motion of a fluid volume element (in the interior of the non-magnetized fluid), are

$$\frac{d\tilde{U}}{dt} = \nabla U - \frac{1}{\rho} \nabla p$$

(3.3)

supplied by the continuity equation (2.14).

These two kinds of motion in the interior of the fluid source differ from each other due to existence of the pressure-gradient term in Eq. (3.3). We remark that the continuity equation (2.14), as an expression of the conservation of the fluid’s total mass, corresponds, physically, to the assumed constancy of the test-particle’s mass.

Now we assume that the fluid-volume element’s hydrodynamic flow is adiabatic and isentropic, whence, there is no exchange of heat quantities (Q) between the fluid-volume element and the rest of the fluid source. Then, according to Eq. (2.9),

$$\frac{dQ}{dt} = 0 = \frac{d\Pi}{dt} + p \frac{d}{dt} \left( \frac{1}{\rho} \right)$$

or, equivalently,

$$\frac{\partial \Pi}{\partial t} + p \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) + \tilde{\nu} \cdot \left[ \nabla \Pi + p \nabla \left( \frac{1}{\rho} \right) \right] = 0$$

(3.4)

In analogy to Eq. (2.12), for $dQ$ to be identically zero, Eq. (3.4) should be viewed as an identity for every value of the three velocity, $\tilde{\nu}$, of the volume element, so that the following two equations

$$\nabla \Pi + p \nabla \left( \frac{1}{\rho} \right) = 0,$$

$$\frac{\partial \Pi}{\partial t} + p \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) = 0$$

(3.5a,b)

suffice to hold simultaneously. Notice that, for a time-independent source ($\frac{\partial}{\partial t} = 0$), Eqs. (3.4) and (3.5a) are equivalent to each other.

In view of Eq. (3.5a), the Euler’s equations of motion (3.3) are written in the form

$$\frac{d\tilde{U}}{dt} = \nabla V$$

(3.6)
where the generalized (scalar) potential \( V \) is defined as
\[
V = U - \left( \Pi + \frac{p}{\rho} \right).
\]
Eqs. (3.6) are of the same functional form as the geodesic equations of motion (3.1), namely, the acceleration three-vector is equal to the gradient of a scalar potential. Therefore, the adiabatic perfect-fluid hydrodynamic flows are dynamically equivalent to the geodesic motions in the generalized potential \( V \). Interestingly enough, \( p \) and \( \Pi \), beyond \( \rho \), appear now as sources of geodesic motion, in contrast to the non-adiabatic-fluid case, in which only \( \rho \) (through \( U \), Eq. (3.2)) produces geodesic motions.

Next we might ask: What is the mass-energy density \( \rho_v \), producing the generalized potential \( V \)? Obviously, in view of the corresponding Newtonian “geodesic” equations (3.1) and (3.2), the density \( \rho_v \) can be defined through the Poisson-type field equation
\[
\nabla^2 V = -4\pi G \rho_v.
\]
The generalized field equation (3.8), and, hence, the introduction of \( \rho_v \) is a natural consequence of the generalized equations of motion (3.6) (a property shared by the general-relativistic equations (2.1) and (2.2), but not of classical Newtonian gravity). In contrast to this, the standard Poisson equation (3.2), is not a consequence of the Euler’s equations (3.3), but, independently of the latter, it, simply complements them.

It is straightforward to prove, with the aid of Eq. (3.7) along with Eqs. (3.2) and (3.5a), that \( \rho_v = \rho + \rho_i \) (3.9)

where \( \rho_i = \frac{1}{4\pi G} \nabla^2 \left( \Pi + \frac{p}{\rho} \right) = \frac{1}{4\pi G} \left[ \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) \right] \) (3.10)

will be called the internal-mass density, so that Eq. (3.8) is written in the form
\[
\nabla^2 V = -4\pi G (\rho + \rho_i).
\]

It is obvious that the mass \( m_v \), which corresponds to the density \( \rho_v \) producing the generalized potential \( V \), is
\[
m_v = \int \rho_v d^3x
\]
where \( v \) is the three-dimensional volume of the source considered. In view of Eqs. (3.9) and (3.12),
\[
m_v = m + m_i
\]
namely, the mass \( m_v \) differs from the mass
\[
m = \int \rho d^3x
\]
by the internal mass
\[
m_i = \int \rho_i d^3x
\]

Hence, the extra ingredient \( \rho_i \) to the generalized mass density, \( \rho_v \), stemming from the source’s internal physical characteristics, results in an extra mass, independent of the mass \( m \) of the source (namely, the volume integral of \( \rho \)), the so-called internal mass (namely, the volume integral of \( \rho_i \)), being also a part of the observationally determined mass (the dynamical mass; see Section 6, below).
We note that, in general, the mass $m_v$ is not constant. If we would wish to postulate the conservation of $m_v$

$$\frac{dm_v}{dt} = 0 \iff m + m_i = \text{const.} \quad (3.16)$$

this would be equivalent to the generalized continuity equation

$$\frac{\partial \rho_v}{\partial t} + \nabla \cdot (\rho_v \vec{v}) = 0 \quad (3.17)$$

which, in turn, generalizes the classical continuity equation (2.14). The constancy of $m + m_i$ permits $m$ and $m_i$ to change in time. For both to be constant in time, the corresponding two continuity equations must hold simultaneously and independently of each other at every moment. In any case, the possibly assumed necessity of Eqs. (3.16) and (3.17) would limit the character and properties of the physical source considered, probably, in a quite decisive way.

The astrophysical and cosmological significance of the above results (3.12 – 3.15) lies in the fact that (see Sections 6 and 7, below) the mass determined at any moment with the aid of the geodesic motions in the generalized potential $V$ is not simply the mass $m$, determined with the aid of the geodesics in the field of the gravitational potential $U$, as it is generally believed, but the mass $m_v$, which is different from $m$. Moreover, it is obvious that $V$ is not the standard gravitational potential $U$ “modified by hand”; instead it is a natural generalization of $U$ resulting simply from the assumption of the isentropicity of the hydrodynamic flows, and, so, we are not dealing here with a modified-gravity approach. Also, in the generalized potential $V$, the volume element moves as a point mass, but, now, carrying along all the internal physical characteristics of the fluid source (not only its mass density). We emphasize that both the above two remarks are of special importance in the context of both the Newtonian and the general-relativistic theories of gravity (see also Section 12, below).

In the special case of a static distribution $\vec{v} = 0$, $\frac{d\vec{v}}{dt} = 0$, from Eqs. (3.6) and (3.8) we obtain

$$\nabla V = 0, \quad \nabla^2 V = 0, \quad \rho_v = 0 \quad (3.18)$$

a situation similar to the motion in «vacuum», and also

$$\nabla \cdot \left( \frac{1}{\rho} \nabla \rho \right) = 4\pi G \rho \quad (3.19)$$

The physical significance of Eq. (3.19) is best seen in the case of a spherically-symmetric fluid distribution with on equation of state of the form

$$p = f(\rho) \quad (3.20)$$

Then, Eq. (3.19) is written as

$$\left( \frac{1}{\rho} \frac{df}{d\rho} \right) \rho' + \frac{2}{r} \left( \frac{1}{\rho} \frac{df}{d\rho} \right) \rho' + \frac{d}{d\rho} \left( \frac{1}{\rho} \frac{df}{d\rho} \right) (\rho')^2 = -4\pi G \rho \quad (3.21)$$

where a prime denotes total derivative with respect to the radial distance $r$.

Eq. (3.21) can be seen as either a consistency condition between Eqs. (3.22) and (3.20), for a given density distribution $\rho = \rho(r)$, or, as a differential equation to be solved for $\rho(r)$. Therefore, for static and spherically-symmetric gravitating perfect-fluid sources, under adiabatic conditions, the functional form of an arbitrarily chosen equation of state $f(\rho)$ depends on the functional form of the mass-density distribution law $\rho(r)$. In other words, the laws of the two parameters, mass density and pressure, cannot be chosen arbitrarily from each other, but their choice depends on each other’s one.
As an application we consider the *adiabatic and isothermal* (adiabatic index \( \gamma = 1 \)) equation of state of the form (6.1) below and a *Plummer-type rest-mass density* of the form (6.2) below. In this case, Eq. (3.21) reduces to

\[
\frac{d^2 \rho}{d \rho} + \frac{2 \rho'}{\rho} \left( \frac{\rho'}{\rho} \right)^2 = -\frac{4\pi G}{\kappa} \rho
\]  

(3.22)
or, furthermore,

\[
2n \left( 1 + \frac{r^2}{r_0^2} \right)^{-2} \frac{r^2}{r_0^2} - \frac{3n}{r_0^2} \left( 1 + \frac{r^2}{r_0^2} \right)^{-1} = -\frac{4\pi G}{\kappa} \rho_0 \left( 1 + \frac{r^2}{r_0^2} \right)^{\frac{n}{2}}
\]  

(3.23)

Applying Eq. (3.23) for \( r \sim 0 \), \( r = r_0 \) and \( r = R \) (the radius of the distribution), and comparing the values of \( \kappa \) in these three cases, we conclude that *necessarily*

\[
n = 2 \frac{\ln 3}{\ln 2} = 3.169
\]  

(3.24a)

and, since \( n \) is, by assumption, an integer, finally

\[
n = 3 \quad \text{(only!!!)}
\]  

(3.25b)

Such a relation between the adiabatic index \( \gamma \) (here 1) and \( n \) has already been derived independently and more generally by Kleidis and Spyrou [17], namely,

\[
\gamma \leq 1 + \frac{1}{n}
\]  

(3.26)

and so \( n \) in Eq. (6.2) below is, practically, the adiabatic index \( \gamma \) (see also Section 5 of [17]).

4. Attractive Gravity and Repulsive Gravity in Newtonian Gravity

Now we remark that, contrary to the case of the equations of motion (3.1), the equations of motion (3.6) do not necessarily imply an inwards acceleration solely (namely, only attractive gravity), because

\[
\nabla \cdot \left( \frac{d \mathbf{v}}{dt} \right) = \nabla \cdot (\nabla V) = \nabla^2 V = -4\pi G \rho_v
\]  

(4.1)

Therefore

\[
\nabla \cdot \left( \frac{d \mathbf{v}}{dt} \right) \gtrless 0 \Leftrightarrow \rho_v \gtrless 0
\]  

(4.2)
or, equivalently,

Spatially decreasing \( \frac{d \mathbf{v}}{dt} \) implies \( \rho_v > 0 \) \((\nabla^2 V = -4\pi G \rho_v < 0)\)

(4.3a)

Spatially non-changing \( \frac{d \mathbf{v}}{dt} \) implies \( \rho_v = 0 \) \((\nabla^2 V = 0)\)

(4.3b)

Spatially increasing \( \frac{d \mathbf{v}}{dt} \) implies \( \rho_v < 0 \) \((\nabla^2 V = -4\pi G \rho_v > 0)\)

(4.3c)

and vice versa. According to the above,

\[ \rho_v > 0 \] implies dominance of inwards acceleration (attracting gravity)

(4.4a)

\[ \rho_v = 0 \] implies non changing (spatially) acceleration

(4.4b)

\[ \rho_v < 0 \] implies dominance of outwards acceleration (antigravity, repulsive gravity)

(4.4c)
The significance of the above results (4.1) – (4.4) is that, in contrast to the general belief, 
*they reveal the possibility of repulsive gravity in the framework of the Newtonian gravity.* Obviously, 
this possibility of repulsive gravity is a consequence of the CDE approach, according to which the 
source’s internal physical characteristics appear to act sources of gravitational geodesic motions.

Furthermore, it is rather straightforward from the definition (3.10) for \( \rho \), that, in realistic 
cases, \( \rho \) cannot be a constant or vanishing (see Section 6, below) and, hence, \( \rho \), can, in principle, have 
any sign, \( \rho \geq 0 \), as stated above.

### 5. The Inversion Distance and the CDE Approach

We recall that the generalized mass-density (3.9) producing the geodesic motions (3.6) and (3.7) can 
be either positive, or negative, or even vanish. This implies the possibility of a spatially-increasing, or 
spatially-decreasing, or even spatially-unchanging acceleration, depending on the distance from the 
centre of the source as compared to the so-called *inversion distance*.

Next, using Eqs. (3.2, for \( n=3 \)), (3.9) and (3.10), we find

\[
\frac{-\rho_i}{\rho} = \frac{2 + \alpha^2}{\alpha^2} \tag{5.1}
\]

where

\[
z = \left(1 + \frac{r^2}{r_0^2}\right)^{\frac{1}{2}} \geq \sqrt{2}, \quad (r \geq r_0) \tag{5.2}
\]

and

\[
\alpha = \frac{3kT}{\mu m_u} \frac{1}{4\pi G \rho_0 r_0^2} \tag{5.3}
\]

Then, first we notice that, by standard theoretically-derived assumption, the radius of the 
innermost stable circular orbit (identified with \( r_0 \)) around a central Schwarzschild black hole of mass \( M_c \) is three times the black hole’s gravitational radius, \( R_s \),

\[
r_0 = 3R_s = \frac{6GM_c}{c^2} \tag{5.4a}
\]

If we approximate [28] \( M_c \) as

\[
M_c = \frac{4}{3} \pi r_0^3 \rho (r = r_0) = \frac{4}{3} \pi r_0^3 \frac{\rho_0}{2^2} \tag{5.4a}
\]

we find

\[
4\pi G \rho_0 r_0^2 = \sqrt{2}c^2, \tag{5.4b}
\]

whence

\[
\rho_0 = 6.439 \times 10^4 (M_c / M_{\odot})^{-2} gr.cm^{-3} \tag{5.4c}
\]

and

\[
\alpha = \frac{3kT}{\sqrt{2} \mu m_u c^2} \tag{5.5}
\]

Although the choice (5.4a) seems arbitrary, the conditions (5.4a,b,c) can be compatible with the 
assumption of a black hole as a fictitious homogeneous dark object of mass \( M_c \), radius \( r_0 \) and *mean* 
mass density equal to \( \rho (r=r_0) \). Actually, in the case of a super-massive black hole of mass \( M_c \sim 10^9 \)
Mo, we find $r_0 \sim 10^{15}$ cm, whence the condition (5.4b) implies $\rho_0 \sim 10^{-14}$ gr.cm$^{-3}$, namely, about ten orders of magnitude larger than the mass density of the Milky Way in the solar neighborhood ($10^{-24}$ gr.cm$^{-3}$).

Then, under the conditions assumed up to now, we notice that the condition
\[
\frac{-\rho_i}{\rho} \gtrsim 1 \quad \text{(equivalently, } 0 \gtrsim \rho + \rho_i = \rho) \quad (5.6)
\]
is equivalent to
\[
\varphi(z) = z^2 - \frac{1}{\alpha} z + 2 \gtrsim 0 \quad (5.7)
\]
The discriminant
\[
\Delta = \frac{1}{\alpha^2} - 8 \quad (5.8)
\]
of the binomial $\varphi(z)$ satisfies
\[
\Delta \gtrsim 0 \quad \text{when } \bar{T} \lesssim T_{\text{lim}} \quad (5.9)
\]
with (see, however, also Eqs. (5.3) - (5.5))
\[
T_{\text{lim}} = \frac{m_\mu c^2}{6k} \sim 1.817 \times 10^{12} K \quad (5.10)
\]
where the threshold temperature $T_{tt}$ is defined via Eq. (7.4c). Therefore (see also Eqs. (4.21)),

For $T \geq T_{\text{lim}}$ ($\Delta \leq 0$), $\varphi(z) > 0$, $\rho + \rho_i < 0$ \quad (5.11)

For $T < T_{\text{lim}}$ ($\Delta > 0$), from the two solutions of the Eq. (5.7) only one, the largest ($z \gtrsim \sqrt{2}$), is acceptable, namely,
\[
r = r_{\text{inv}}; \frac{r_{\text{inv}}}{r_0} = \left( \frac{1}{4} \frac{1}{\alpha} + \left( \frac{1}{\alpha^2} - 8 \right)^{1/2} \right)^2 - 1 \quad (5.12)
\]
such that
a) If $r < r_{\text{inv}}$, then $\rho + \rho_i > 0$ \quad (inwards acceleration or attractive gravity dominates) \quad (5.13a)
b) If $r = r_{\text{inv}}$, then $\rho + \rho_i = 0$ \quad (spatially unchanging acceleration) \quad (5.13b)
c) If $r > r_{\text{inv}}$, then $\rho + \rho_i < 0$ \quad (outwards acceleration or repulsive gravity dominates) \quad (5.13c)

In the special case $\Delta \gg 0$, namely, $T \ll T_{\text{lim}}$, which is believed to apply usually in almost all of the cosmological large-scale structures, Eqs. (5.12) and (5.2) reduce to, respectively,
\[
\frac{r_{\text{inv}}}{r_0} = \frac{1}{\alpha} = \frac{\sqrt{2}}{3} \frac{\mu m_\mu c^2}{kT} \quad (5.14)
\]
with
\[
\frac{z_{\text{inv}}}{r_0} = \frac{r_{\text{inv}}}{r_0} = x_{\text{inv}}, \quad \text{for } r \gg r_0 \quad (5.15)
\]
### Astronomical System

| System                  | $r_0$          | Inversion Distance $r_{inv}$ | Numerical Values                      |
|-------------------------|----------------|-----------------------------|----------------------------------------|
| **Solar System**        | $R_\odot$     | $\frac{2.39 \times 10^8}{T_{(2)}/\mu}$ AU $\Rightarrow$ $2.39 \times 10^7$ AU, for $\frac{T}{\mu} \sim 10^3$ K |
|                         | (6.96X10^{10} cm) |                             |                                        |
| **Galaxy**              | $3R_\odot$   | $\frac{1.48}{T_{(6)}/\mu}$ pc $\Rightarrow$ 14.76 Kpc, for $\frac{T}{\mu} \sim 10^2$ K (Conventional Outskirts of the Milky Way) |
| (Central Black Hole)    | $M_c \sim 10^6 \, M_\odot$ |                             |                                        |
|                         | $\sim 8.86 \times 10^{11}$ cm) |                             |                                        |
| **Cluster of Galaxies** | $3R_\odot$   | $\frac{14.76}{T_{(8)}/\mu}$ Kpc $\Rightarrow$ 14.76 Mpc, for $\frac{T}{\mu} \sim 10^5$ K (Conventional Dimensions of a Cluster of Galaxies) |
| (Central Galaxy)        | $M_c \sim 10^{12} \, M_\odot$ |                             |                                        |
|                         | $\sim 8.86 \times 10^{17}$ cm) |                             |                                        |
| **Supercluster of**    | $3R_\odot$   | $\frac{1.48}{T_{(9)}/\mu}$ Kpc $\Rightarrow$ 147.56 Mpc, for $\frac{T}{\mu} \sim 10^4$ K (Conventional Dimensions of a Super-cluster of Galaxies) |
| **Galaxies**            | (Central Galaxy) |                             |                                        |
|                         | Black Hole    |                             |                                        |
|                         | $M_c \sim 10^{12} \, M_\odot$ |                             |                                        |
|                         | $\sim 8.86 \times 10^{17}$ cm) |                             |                                        |

**Table 1:** The inversion distance for typical cosmological structures

Representative values of the inversion distance, $r_{inv}$, based on Eqs. (5.14) and (5.15), are shown in Table 1, and are seen to be comparable to the currently accepted linear dimensions of the corresponding structure. This means that the real dimensions of the structure are much larger, than thought up to now.

We conclude that, for increasing r, smaller than the inversion distance $r_{inv}$, the gravitational repulsion of the negative internal mass is smaller than the gravitational attraction of the positive rest-mass. The matter accelerates inwards and eventually reaches an equilibrium situation due to the action of some kind of pressure (here, thermal pressure). At the inversion distance, $r_{inv}$, the gravitational repulsion and gravitational attraction cancel each other. Beyond the inversion distance an inversion occurs, namely, the gravitational repulsion dominates over the gravitational attraction, and matter accelerates outwards. Therefore, we conclude that, in general, mass should exist also beyond the conventional limits of the structure.
6. A Simplified Classical Treatment of the Problem of the Missing Mass

Today, the general cosmological belief is that the (dominant) dark matter coexists with its (subdominant, small) baryonic "contamination". The latter is tightly bounded to the former, and the two constitute the cosmological gravitating (perfect) fluid, in the sense that on top of a volume element of dark matter "sits" the baryonic (luminous) matter. So, in the spirit of CDE approach, the cosmological structures, e.g., the galaxies, should be treated as finite volume elements of the cosmological gravitating fluid in the context of a continuous Universe (in contrast to a collection of gravitating point masses (galaxies) at mutual distances large compared to their linear dimensions). In using the results of Section 3 (as in Sections 10 and 12 below), we note that, generally, \( \rho \) is the total mass-energy density, including both the dark and the baryonic matter, and the equation of state \( p(\rho) = \kappa \rho \), with \( \kappa \) being a positive constant, refers to the total pressure, namely, due to both the dark and baryonic masses. However, in this Section and for reasons of clarifying the physical importance of the results of Sections 3-5, we shall treat the source's matter density and pressure as simply of baryonic origin, and examine the relative importance of the (simply baryonic) mass-density, \( \rho \), and the internal-mass density, \( \rho_i \), in astronomical and cosmological structures, and its consequences.

Generally, all the physical parameters describing an astrophysical or cosmological source can be functions of both space and time. Here we shall limit ourselves to the case of physical parameters being explicitly functions of only the spatial coordinates. So, we assume that a gravitating fluid source (of absolute temperature \( T \) and mean molecular weight \( \mu \)) is spherically-symmetric of radius \( R \), and is characterized by an adiabatic and isothermal (adiabatic index \( \gamma = 1 \)) equation of state, such that

\[
p = \frac{kT}{\mu m_H} \rho 
\]

where \( k \) and \( m_H \) are, respectively, the Boltzmann constant and rest-mass of the atomic hydrogen, obviously compatible with the isentropicity conditions (3.5a,b). Furthermore, in choosing the baryonic-mass-density distribution law, we shall rely on the well-known observation-based "universal profile" of Navarro, Frenk and White [45], [46] and the Hernquist [47] profile (see also [48] – [50]) for the density distribution law of clusters of galaxies. So we shall adopt a special case of the above profiles, namely, the Plummer-type density described by

\[
\rho(r) = \rho_0 \left(1 + \frac{r^2}{r_0^2}\right)^{-\frac{n}{2}} \quad (r_0 > 0, \rho_0 > 0, n: \text{a positive integer})
\]

Then using, additionally, the Gauss-Stokes theorem, from Eqs. (3.13) - (3.15) we find

\[
m_I = -\frac{nkT}{G\mu m_H} \frac{R^3}{r_0^2} \left(1 + \frac{R^2}{r_0^2}\right) < 0
\]

whence

\[
m_I (R >> r_0) = -\frac{nkT}{G\mu m_H} R < 0
\]

Then, for \( n=3 \) and for every upper limit \( x_{\text{max}} \), we find

\[
\frac{-m_I}{m} = 1.946 \times 10^{-6} \frac{x_{\text{max}}}{\ln x_{\text{max}}} \frac{T_{(7)}}{\mu}
\]

where

\[
x = \frac{r}{r_0}, \quad x_{\text{max}} = \frac{R}{r_0}, \quad r_0 = 3R_s = 2.872 \times 10^{-5} M_{(8)} \text{pc}
\]

\( R_s, T_{(7)} \) and \( M_{(8)} \) being, respectively, the Schwarzschild radius of the central dark object, the region’s (constant) temperature in units \( 10^7 \) K and the mass-energy, \( M_{(8)} \), of the central dark object in units of
Therefore, in this case, the internal mass is definitely negative and, absolutely, it diverges with the source’s linear dimensions.

Under such conditions, Eq. (3.13) takes on the form

\[ m = m_i + (-m_i) > m_i \]  \hspace{1cm} (6.7)

Therefore, the baryonic mass \( m \), determined with the aid of the geodesic motions in the gravitational potential \( U \), is larger than the mass \( m_i \), determined with the aid of the geodesic motions in the generalized gravitational potential \( V \) (for a proof of the above theorem in the relativistic case, see [33]). Obviously, it is the geodesic motions in the potential \( V \) that must be used rather than those in the gravitational potential \( U \), because in the former (\( V \)) all the internal physical characteristics of the gravitating source are taken into account as sources of the observed motions. In other words, for simply physical reasons, as mass determined on the basis of geodesic motions, the mass \( m_i \) must be used, not the mass \( m \). But then the baryonic mass, \( m \), of the (thermal) source is larger than the mass \( m_i \), determined observationally.

We shall apply the above results to galactic and cosmological levels (for details see [17], [22] – [31], [37], [38]). Thus in the case of masering galaxies (e.g., NGC 4258 and NGC 1068) and non-masering galaxies (e.g., NGC 4261) we find that, depending on the linear dimensions of the circumnuclear region considered (ranging from the sub-parsec up to the kiloparsec), \(-m_i\) is not always negligible compared to the mass \( M_c \) of the central dark object \((10^5 < -m_i / M_c < 10^5)\). On the other hand, \(-m_i\) can be comparable to the total rest-mass, \( m \), of the circumnuclear region \((10^3 < -m_i / m < 0.6)\; \text{the upper bound 0.6 corresponding to the mass of the black hole believed to lurk in the Milky Way’s nuclear region).}

Specifically, in the case of the Milky Way, for the linear dimensions \( R \) of the region considered being in the range 0.1 pc and 100 kpc, the ratio \(-m_i/m\) falls in the range \((10^2 \text{ to } 10^5)\; T/\mu\). This means that, e.g., in the innermost region of dimensions 0.1 pc, this ratio can be larger than unity, provided that the temperature there is at least \(10^5\; \text{K}\), and also that considering the broader region of dimensions 100 kpc, the same ratio can be at least \(10^3\) for temperature of the order \(10^5\; \text{K}\).

It is very interesting that the above conditions can generally be met in the physical Universe. Thus, as mentioned in the Section 1, the interstellar medium in elliptical galaxies consists primarily of hot \((T > 10^5\; \text{K})\) plasma; several molecular clouds with dimensions 20-50 pc and masses \(-10^6\; \text{solar masses}\) \((m_\odot)\) are partly associated with HII regions and embedded in a lower-density interstellar medium; the masses \((-10^8\; m_\odot)\) of the central dark objects in galaxies are only a few percent of the molecular-gas mass in the sub kiloparsec region. All the above data are in accordance with the modern aspect of a galaxy, different from that deduced on the basis of simply the galaxy’s optical view.

Analogously, in the case of typical clusters of galaxies with dimensions [11] of a few Mpc, the ratio \(-m_i/m\) falls in the range \((10^2 \text{ to } 10^5)\; T/\mu\) and so it can be larger than unity, provided that \(T/\mu > 10^4\; \text{to} 10^5\; \text{K}\). Again this is generally true, because, as mentioned in Section 1, a cluster of galaxies can be treated as an approximately isothermal sphere of hot ionized hydrogen (of dimensions a few Mpc, temperature \(-10^5\; \text{K}\), number density of electrons \(-10^{-3}\; \text{cm}^{-3}\), and number density of molecular clouds \(-10^{-4}\; \text{to} 10^{-3}\; \text{cm}^{-3}\); the intracluster gas contains X-rays, fills the space between the galaxies, occupies much of the cluster’s volume, and the X-ray luminosities fall in the range \(10^{43}\) to \(10^{45}\) erg. sec\(^{-1}\); the mass of the intracluster medium exceeds the total mass of the luminous parts of the cluster’s galaxies by a factor of several.

Finally, in the case of a second-order cluster (cluster of clusters or super-cluster) of galaxies, the ratio \(-m_i/m\) is of the order of \(10^2\; T/\mu\), and so it can exceed unity, for \(T/\mu > 10^5\; \text{K}\), a condition that again is met (notice, however, that current observations do not seem to support the idea of a spherically-symmetric super-cluster of galaxies; on this, see also, especially, [23], [28]).
| $r_{\text{max}}$  | $-m/m$ | $-m/m \geq 1$ |
|-----------------|--------|----------------|
| 0.1 pc          | $T(\gamma)/\mu \geq 18.831$ , $T(\mu)/\mu \geq 1.883 \times 10^8 K$ | 0.053 $T(\gamma)/\mu$ |
| 1 pc            | $T(\gamma)/\mu \geq 2.223$ , $T(\mu)/\mu \geq 2.223 \times 10^7 K$ | 0.449 $T(\gamma)/\mu$ |
| 100 pc          | $T(\gamma)/\mu \geq 2.903 \times 10^{-2}$ , $T(\mu)/\mu \geq 2.903 \times 10^5 K$ | 34.453 $T(\gamma)/\mu$ |
| 2 kpc           | $T(\gamma)/\mu \geq 1.672 \times 10^{-3}$ , $T(\mu)/\mu \geq 1.672 \times 10^3 K$ | 597.979 $T(\gamma)/\mu$ |
| 15 kpc          | $T(\gamma)/\mu \geq 2.428 \times 10^{-4}$ , $T(\mu)/\mu \geq 2.428 \times 10^4 K$ | 4.119 x $10^3 T(\gamma)/\mu$ |
| 30 kpc          | $T(\gamma)/\mu \geq 1.248 \times 10^{-4}$ , $T(\mu)/\mu \geq 1.248 \times 10^4 K$ | 8.012 x $10^3 T(\gamma)/\mu$ |
| 100 kpc         | $T(\gamma)/\mu \geq 3.922 \times 10^{-5}$ , $T(\mu)/\mu \geq 3.921 \times 10^5 K$ | 2.549 x $10^4 T(\gamma)/\mu$ |
| 200 kpc         | $T(\gamma)/\mu \geq 2.012 \times 10^{-5}$ , $T(\mu)/\mu \geq 2.012 \times 10^5 K$ | 4.969 x $10^4 T(\gamma)/\mu$ |

**Table 2:** Relative importance $-m_i/m$, of the negative internal mass, $m_i$, and the positive rest mass $m$, for the Milky Way: $\displaystyle \frac{-m_i}{m} = 1.946 \times 10^{-6} \frac{x_{\text{max}}}{\ln x_{\text{max}}} \frac{T(\gamma)}{\mu}$, $M(8) = 10^2$, $r_0 = 2.872 \times 10^{-7}$ pc

From all the above we conclude that, using the geodesic motions (inside and outside of) the cosmological structures for the observational determination of masses, in conjunction with the CDE approach, the extra (negative) mass $m_i$ (which is meaningless in the context of the standard geodesic motion (3.1) of a test particle) appears in the observationally determined dynamical mass and, in many cases, it can largely exceed the mass of the gas of the (corresponding region of the) large-scale cosmological structure under consideration. *We consider this as an interesting contribution towards clarifying the concept of the missing mass.* The above results, for different values of $M$ (and $r_0$), are shown analytically in Tables 2 and 3.
A typical cluster of galaxies: 
\[ \frac{m_i}{m} = 1.946 \times 10^{-6} \frac{x_{\text{max}}}{\ln x_{\text{max}}} \frac{T_{(7)}}{T_{(\mu)}} , \quad r_0 = 2.872 \times 10^{-5} M_{(8)} \text{pc} \]

| $M_{(8)}$ | $r_{\text{max}}$ | $-m/m$ | $-m_i/m \geq 1$ |
|---------|-----------------|--------|-----------------|
| 1 Mpc   | $2.792 \times 10^3 T_{(7)}/\mu$ |        |                 |
| 10      | $T_{(7)}/\mu \geq 3.582 \times 10^{-4}$, $T/\mu \geq 3.5821 \times 10^3$ K |        |                 |
| $10^2$  | $34.453 T_{(7)}/\mu$ |        |                 |
| $T_{(7)}/\mu \geq 2.903 \times 10^{-2}$, $T/\mu \geq 2.903 \times 10^5$ K |        |                 |

A typical 2nd order cluster of galaxies: 
\[ \frac{m_i}{m} = 1.946 \times 10^{-6} \frac{x_{\text{max}}}{\ln x_{\text{max}}} \frac{T_{(7)}}{T_{(\mu)}} , \quad r_0 = 2.872 \times 10^{-5} M_{(8)} \text{pc} \]

| $M_{(8)}$ | $r_{\text{max}}$ | $-m/m$ | $-m_i/m \geq 1$ |
|---------|-----------------|--------|-----------------|
| 100 Mpc | $2.7917 \times 10^3 T_{(7)}/\mu$ |        |                 |
| $10^2$  | $T_{(7)}/\mu \geq 3.582 \times 10^{-4}$, $T/\mu \geq 3.5821 \times 10^3$ K |        |                 |

Table 3: Relative importance $-m_i/m$, of the negative internal mass, $m_i$, and the positive rest mass $m$, for: (i) A typical cluster of galaxies and (ii) a typical 2nd order cluster of galaxies.

Finally, we stress that our method has been based on the assumption that only the baryonic density is considered, while the density of dark matter is ignored. However, our method can be generalized, in the spirit of Sections 10 and 12 below, so as to take into account both the above densities. This subject is currently under scrutiny.

7. Dynamical Masses and Flat Rotation Curves of Disc Galaxies
In this Section, we shall demonstrate that the flat rotational curves of the disk galaxies can naturally be explained, based on the generalized Euler’s equations of motion (3.6), in which $\rho$ is again simply the baryonic density. Thus, the case of a spherically-symmetric perfect-fluid source of generalized density $\rho(r)$, the velocity, $u(r)$, of an equatorial, circular geodesic orbital motion at a distance $r$ from the center is

\[ u_f^2(r) = \frac{Gm_i(r)}{r} = \frac{G}{r} \left[ m(r) + m_i(r) \right] \]  \hspace{1cm} (7.1a)

replacing the standard one (on the basis of the equations of motion (3.1))

\[ u^2(r) = \frac{Gm(r)}{r}, \]  \hspace{1cm} (7.1b)
where, we emphasize, that, by definition, the masses \( m(r) \) and \( m_v(r) \) are the volume integrals of the corresponding densities \( \rho(r) \) and \( \rho_v(r) \), manifesting themselves in the corresponding Poisson equations, (3.2) and (3.8).

We recall that, for a homogeneous source of density \( \rho \), Eq.(7.1b) reduces to

\[
\nu(r) = \left(\frac{4\pi G \rho}{3}\right)^\frac{1}{2} r
\]  
(7.2)

Eq. (7.2), with the exception of the circumnuclear region, is in contradiction with observations. On the other hand, for \( \rho_v \) to be different than \( \rho \), an isothermal gravitating perfect-fluid cannot be homogeneous. Moreover, as deduced from the 21 cm radiation, in disk galaxies the HI clouds, at different distances from the center of the galaxy, all orbit at more or less the same speed. This can be explained, in the present context, however, if we assume a non-homogeneous, spherically-symmetric source and use Eq. (6.2), whence

\[
m(r) = 4\pi\rho_0 r^3 \left[ \frac{1}{\sqrt{2}} - \ln \left(1 + \sqrt{2}\right) - \frac{r}{r_0} \ln \left(\frac{r^2}{r_0^2} + 1\right) + \ln \left(\frac{r}{r_0} + \sqrt{\frac{r^2}{r_0^2} + 1}\right) \right]
\]  
(7.3a)

\[
\equiv 4\pi\rho_0 r_0^3 \ln \frac{r}{r_0} \quad \text{for} \quad r \gg r_0
\]  
(7.3b)

If, additionally, we assume the equation of state, Eq. (6.1), we find, in analogy to Eqs. (7.3) (for \( n=3 \)),

\[
m_v(r) = -\frac{3kT}{G\mu m_\mu} r^3 \left(1 + \frac{r^2}{r_0^2}\right)
\]  
(7.4)

Inserting Eqs. (7.3) and (7.1a) into Eq. (7.8a) and using also the condition (5.4a,b) we find

\[
\nu^2_j(x) = \sqrt{2} \frac{1}{x} \ln \left(1 + \sqrt{x^2 + 1}\right) - \frac{\sqrt{2}}{\sqrt{x^2 + 1}} - \frac{1}{x} \ln \left(1 + \sqrt{2}\right) - \frac{1}{\sqrt{2}} - \gamma \frac{x^2}{x^2 + 1}
\]  
(7.5)

where we have put

\[
x = \frac{r}{r_0}, \quad \gamma = \frac{3kT}{\mu m_\mu c^2} = 2.75 \times 10^{-13} \frac{T}{\mu}
\]  
(7.6)

The interest in Eq. (7.5) is, among others, in the explicit dependence of \( \nu_j(x) \) on the distribution’s temperature.

A more precise expression, however, for the velocity \( \nu_j(x) \), to be used for comparison with observational data, should take into account the mass, \( M_c \), of the central dark object, beyond of the density \( \rho_c \). The velocity \( \nu_{BH}(r) \), due to the mass \( M_c \), of an equatorial, circular geodesic orbit at the distance, from the center is
\[ u_{\text{rot}}^2 = \frac{GM_c}{r} = \frac{GM_c}{3R_c \gamma} = \frac{GM_c}{3 \times \frac{2GM}{c^2} x} = \frac{c^2}{6x} \]  

(7.7)

So, the total circular velocity, \( V_c \), is

\[
V_c = \left( u_j^2 + u_{\text{rot}}^2 \right)^{1/2}
\]

\[
= c \left\{ \sqrt{\frac{1}{x} \ln \left( 1 + \sqrt{x^2 + 1} \right)} - \frac{\sqrt{2}}{\sqrt{x^2 + 1}} \left[ \ln \left( 1 + \sqrt{2} \right) - \frac{1}{\sqrt{2}} \gamma - \frac{x^2}{x^2 + 1} + \frac{1}{6x} \right] \right\}^{1/2}
\]

(7.8a)

or

\[
V_c \left( \frac{100 \text{ km}}{\text{s}} \right) = 2.989 \times 10^5 \left\{ \sqrt{\frac{1}{x} \ln \left( 1 + \sqrt{x^2 + 1} \right)} - \frac{\sqrt{2}}{\sqrt{x^2 + 1}} \left[ \ln \left( 1 + \sqrt{2} \right) - \frac{1}{\sqrt{2}} \gamma - \frac{x^2}{x^2 + 1} + \frac{1}{6x} \right] \right\}^{1/2}
\]

(7.8b)

or, equivalently,

\[
V_c = 2.989 \times 10^5 \left\{ \sqrt{\frac{1}{x} \ln \left( 1 + \sqrt{x^2 + 1} \right)} - \frac{\sqrt{2}}{\sqrt{x^2 + 1}} \left[ \ln \left( 1 + \sqrt{2} \right) - \frac{1}{\sqrt{2}} \gamma - \frac{x^2}{x^2 + 1} + \frac{1}{6x} \right] - \frac{3k_B T}{\mu m_{\text{H}} c^2} \frac{x^2}{x^2 - 1} + \frac{1}{6x} \right\}^{1/2} \text{ km/s}
\]

(7.8c)

In the form (7.8a) or (7.8b) or (7.8c), the velocity \( V_c \) is, generally, preferable than in the form (7.5), especially close to the central dark object. In Figure 1, the form of the velocity \( V_c \) is plotted as a function of \( x \), for \( T/\mu \approx 10^2 \). The expected flat character of the rotation curve is prominent, resembling the case of the Milky Way (\( V_c \approx 200 \text{ km/s} \)). Similar results are presented in Figure 2 (the group of curves in the middle), for several values of the temperature, namely, \( T/\mu \approx 1.82 \times 10^6 \) (\( \gamma \approx 5.01 \times 10^{-7} \)), \( T/\mu \approx 1.45 \times 10^6 \) (\( \gamma \approx 3.99 \times 10^{-7} \)), \( T/\mu \approx 2.75 \times 10^5 \), \( T/\mu \approx 2.75 \times 10^5 \) (\( \gamma \approx 2.75 \times 10^{-8} \)), and finally, \( T/\mu \approx 10^3 \) (\( \gamma \approx 2.75 \times 10^{-10} \)). The fourth case, in particular, is in good agreement with, e.g., the typical velocity-curve for a Sc spiral galaxy [9]. The main difference, beyond the different scale, between the theoretically-derived velocity curve and the observed one, refers to the points close to the center, which are not considered in the theoretical treatment.
It is remarkable that the “flat” part of the rotation curve correspond to the same range of values of \( x \left( >10^{10} \gg 1 \right) \) and, generally speaking, for such large values of \( x \left( >10^{10} \gg 1 \right) \), only a slight (but not vanishing) dependence of \( V_c \) on the temperature is noticed. Also, in our case, the theoretically derived flat rotation curve continues up to the distance of at least 200 kpc \( (x \geq 10^{12}) \). In the case of the Milky Way galaxy, such a distance (beyond the Magellanic Clouds!!!) defines the galaxy’s real dimensions, as contrasted to its conventional radius \( \sim 15 \) kpc [1]. (For an independent method of determination of the true linear dimensions, applied in the case of the Solar System and based on the vanishing of the generalized acceleration \( \nabla V \) (Eq. (3.6)), see Section 8 below; and also Table 1 for the inversion distance). We consider the results of Sections 6 and 7 as a partial classical solution to the dark-matter problem. Obviously, as with Section 6, our method can be generalized, in the spirit of Sections 10 and 12 below, so as to include both the dark-matter and the baryonic densities. This subject is currently under scrutiny.
Figure 2: Flat rotation curves for several values of the temperature, in large-scale structures.

8. A Classical Treatment of the Pioneer-Anomaly Effect in the Solar System (with K. Zagkouris)

The Pioneer 10 and 11 space probes have been launched in 1970s and are currently travelling outwards our Solar system. There has been a measured deviation in their (and other space probes in the Solar System) speeds, which shows a small extra inward acceleration towards the Sun, amounting in, approximately, the above space probe (and not only them), during their lifetime up to now, “loosing” a distance approximately equal to the distance of the Moon from the Earth. This extra acceleration has been given the name the Pioneer Anomaly Effect. Currently, no universally accepted explanation exists for this anomaly. In order to solve this mysterious acceleration, many theories have been proposed, most of which concentrate in observational errors, recording errors, new gravity sources in the Solar system and, finally, even new physics.

We studied the Pioneer Anomaly Effect in the framework of the CDE approach. In our study we use a model of the Solar System with the spherically-symmetric (or, equivalently, point-mass) Sun of mass $M_s$ at its centre and an extra mass density (generally, both baryonic and dark matter)
dispersed around it. The mass-density profile is described by a Plummer-type equation (see Eq. (6.2)). In the numerical applications, just for simplicity, we treat the extra, dispersed mass as a baryonic isothermal perfect fluid source (nevertheless, this may not be necessary, because the same steps can be followed for any mixture of dark matter and baryonic matter). For this system, the generalized Euler’s hydrodynamic equations for the spherically-symmetric total acceleration of the isentopic motion are written in the form

\[
\gamma(r) = -\frac{G}{r^2} \left[ M_{c} + m_{e}(r) \right] = \gamma_{c} + \gamma_{e} 
\]

(8.1)

the two parts on the r.h.s. being due to, respectively, the central spherically-symmetric Sun and the extra mass (fluid) dispersed around it.

Furthermore, recalling that in the CDE approach the fluid volume element moves as a test particle, in accordance with the generalized laws (3.6) and (3.7), namely, but, now, carrying along all the physical characteristics of the fluid source, we identify the (Pioneer or other) space probe with the above volume element-test particle, moving as above.

From our analysis, applied to a baryonic and isothermal mass distribution, dispersed around the central Sun, we end up with some interesting results two of which are worth mentioning here. The first result is that we can actually evaluate an extra acceleration at the region of the Pioneer space probe and the numerical results are in agreement with the observed data, namely, the extra acceleration evaluated in our analysis is equal to \( \gamma = -8.74311 \times 10^{-8} \text{ cm s}^{-2} \), while the measured extra acceleration is \( \gamma = -8.74 \pm 1.33 \times 10^{-8} \text{ cm s}^{-2} \). We consider the above result as a quite acceptable classical explanation of the celebrated Pioneer Anomaly Effect in the Solar System.

Figure 3: The extra acceleration \( \gamma_{e} \) as a function of the distance from the Sun.
The behaviour, with the distance from the Sun, of the evaluated extra acceleration is shown in Figures 3 and 4, and the corresponding result for the total acceleration is shown in Figure 5. The extra acceleration is an inward acceleration up to certain distance, after which it becomes an outward acceleration. In contrast, the total acceleration is always an inward acceleration vanishing at some far distance, \( R \). This last result permits the first theoretical analytical determination of the true linear dimensions of the Solar System, \( R \), as the solution of the algebraic equation (for \( r \)) \( \gamma = 0 \), for \( r = R \).

![The acceleration \( \gamma_i \) in the region of 40 - 1000 AU](image_url)

**Figure 4:** The extra acceleration \( \gamma_i \) as a function of the distance from the Sun.

In fact, for \( T/\mu = 1 \), the Solar System is found to extend up to, approximately, the distance 100 kAU, which, remarkably enough, is the radius of the *Oort Cloud*, and amounts to almost half the distance to our closest star \( \alpha \)-Proxima Centauri. This result shows that the Solar system is not as small as we thought up to now, but it may actually be a continuous entity that almost touches the nearest neighbouring stellar system.

Finally, we note that our model permits also the evaluation of thermodynamic parameters, like the internal thermodynamic energy (per unit mass), \( \Pi \), whose values at the surface of the Sun is \( \Pi(r_0) = 4.01 \times 10^6 \text{cm}^2 \text{s}^{-2} \) and at the Solar System’s boundary is \( \Pi(R) = 1.43 \times 10^9 \text{cm}^2 \text{s}^{-2} \). Of course, our model permits the analogous treatment of other space probes in the Solar System (e.g., *New Horizons*). For further details and results on the applications of the CDE approach to the Solar System, see [36].
9. A possible theoretical explanation of the formation mechanism of winds, jets, supergiant stars, and supernovae
As described in Sections 4 and 5, the generalized gravity can be repulsive, provided that the generalized density $\rho_v$ is negative. In this way, using the notion of the inversion distance, we can describe and justify the outwards acceleration necessary for the formation of winds and jets. Also, possible applications to the phenomena of the expansion of a red-giant star and a supernova are considered. This subject is, currently, under scrutiny.

10. Determination of the masses of clusters of galaxies (with M. Plionis and S. Basilakos)
Of obvious special cosmological importance and interest is the determination of the masses of the globular clusters of galaxies. Usually, the total gravitating mass (dark matter and baryons) of a typical cluster of galaxies is classically being treated using the hydrodynamic properties of the inter-cluster gas. More specifically (Plionis, private communication), for a cluster in hydrostatic equilibrium, from the combined use of the Poisson equation for the total gravitational potential (due to the total mass density, namely, dark matter and baryons), an equation of state for the gas (namely, the total pressure due only to the density of the gas – i.e., its baryonic mass), and an assumed Plummer-type gas-density profile (with, e.g., $n = 3$), all plugged into the Euler equation of hydrostatic motions, the hydrodynamic mass and the total mass-density are evaluated. Similar results are obtained for the same perfect-fluid cluster, using the idea of the CDE approach.

The latter method can be generalized for a non-static cluster of galaxies, so as to include the total mass-density and the total pressure of the non-static source, namely, considering the contributions to them of both the dark matter and the baryonic matter, and, additionally, using for the non-static source the notions of the spectral shift and the generalized dynamical mass. More precisely, assuming an equation of state in terms of the total density and total pressure,

$$p = w \rho c^2, \quad w > 0$$

we evaluate the generalized density $\rho$ in terms of the mass density of the baryonic mass

$$\rho_v = \rho + \frac{w c^2}{4 \pi G} \left[ \nabla \cdot \left( \frac{\nabla \rho}{\rho} \right) \right], \quad m_v = \int \rho_v d^3 x$$
Then, according to an assumed total-mass-density profile, by its volume integration, we evaluate the generalized mass $m_{v} = \int \rho_{v} d^{3}x$. Independently, this same generalized (namely, the dynamical) mass, under the assumption of circular equatorial motions is directly related to the spectral shift, $z$,

$$z^2 = \frac{G m_{v}(r)}{c^2 r}$$

immediately obtained from observations.

Therefore the whole game here, finally, is an interplay of the (assumed) constants appearing in the equation of state and the assumed total-mass-density profile. Next, using an observationally derived baryonic-gas-profile law (see also [46]), we can, by subtraction, evaluate the dark-mass-density profile and, then, by the latter’s volume integration, evaluate the dark mass itself, and compare it with observational data, thus limiting the values of the constants appearing in the equation of state and the assumed total-mass-density profile. This subject is under scrutiny currently.

11. CDE in classical magneto hydrodynamics (with C. G. Tsagas)

The validity of the CDE approach in classical hydrodynamics (Section 3) has been extended into the realm of classical magneto hydrodynamics (MHD). This can be accomplished as follows. First, by a suitable redefinition of the pressure of the magnetized perfect-fluid source, so as to include also the magnetic pressure, the MHD equations of motion are put in the form of the Euler’s hydrodynamic equations of motion. The above result is possible not for an arbitrary magnetic field $B$, but only for those characterized by a “zero-curl Lorentz force”. Then, by a suitable redefinition of the internal thermodynamic energy of the magnetized fluid and of the condition of the adiabatic-isentropic motions, the above derived Euler’s equations are put in the form of Newton’s gravitational equations of motion. Specifically, the MHD equations of motion are written as

$$\frac{d\mathbf{V}}{dt} = \mathbf{\nabla} \cdot \rho, \quad \mathbf{\nabla} \times \mathbf{V} = -4\pi G \rho_{\nu}, \quad \rho_{\nu} = \rho + \rho_{i(p)} + \rho_{i(B)}$$

with the magnetically generalized mass-density (third equation above) containing also a magnetic contribution $\rho_{i(B)}$, which becomes important in the central region of a magnetized large-scale cosmological structure (for further details, see [34], [35]; see also [18], [20]).

12. Cosmology, collisional dark matter and CDE (with K. Kleidis)

This Section 12 is a short presentation of the talk, at the same Conference, given by the first of the authors (K. Kleidis; A Conventional Form of Dark Energy; see the detailed program of the talk(s) at the official site of the Conference (http://neb14.physics.uoi.gr/index/files/program.htm).

The current cosmological picture, known as the concordance model [51], includes two major unresolved issues: (i) According to the observational data on the temperature-variations of the cosmic microwave background, the Universe can be described, adequately, by a spatially-flat Robertson-Walker (RW) model (see, e.g., [52]) and, therefore, it must contain a considerably larger amount of energy, than the equivalent to the total rest-mass density of its matter content does. (ii) The cosmologically-distant indicators (standard candles) appear to be dimmer than expected [53], [54], something which has been accommodated by the assumption that, recently, the Universe entered into a phase of accelerated expansion [55]–[57].

In order to compromise the above mentioned observational results, as well as to deal with them into a unified theoretical framework, an extra, dark energy (DE) component, of negative pressure, has been introduced [58], occasionally referred to as quintessence [59]. However, there may be also another, more conventional explanation, which can be revealed, most appropriately, with the aid of the conformal dynamical equivalence technique [17], [27], [33], [35].
Nowadays, a lot of accumulated evidence suggests that, more than 85% (by mass) of the matter in the Universe consists of non-luminous and non-baryonic material [60]. Its name, dark matter (DM), reflects our ignorance on the exact nature of this constituent. Although we do not know for certain how the DM came to be formed, a sizeable relic abundance of weakly interacting massive particles (WIMPs) is generally expected to have been produced as a by-product of the Universe's hot youth [61]. Apart from their exact nature, the scientific community used to argue that, the WIMPs should be collisionless [62], [63]. However, many recent results from high-energy-particle tracers, such as the ATIC [64] and PAMELA [65], combined with those of the Wilkinson microwave anisotropy probe (WMAP) survey [66], have revealed an unusually-high electron - positron production in the Universe, much more than what is anticipated by supernovae explosions or cosmic-ray collisions. These results have led many scientists to argue that, among the best candidate-sources of these high-energy events are the annihilations of WIMPs (see, e.g., [67]–[76], for an extensive, though incomplete list), i.e., that the DM constituents can be slightly collisional [77]–[80]. If this is true, it could affect our perception on the nature of the DE (in connection, see, e.g., [81]–[85]).

Based on the perception that the DM constituents can be slightly collisional, we have assumed that, the matter-energy content which drives the evolution of the Universe (being modeled by a spatially-flat RW space-time) at every post-recombination epoch, is in the form of a perfect fluid with positive pressure,

\[ p = w \rho c^2, \]

where \( \rho \) is the rest-mass density of the Universe matter content and \( w \) is a dimensionless constant. Now, together with all the other physical characteristics, the energy of this fluid's internal motions, \( \Pi \), is (also) taken into account as a source of the universal gravitational field and the total energy-density of the Universe matter-energy content is written in the form (2.4). The dynamical evolution of this model is governed by the Friedmann equation with vanishing cosmological constant, which is written in the form

\[
\left( \frac{H}{H_0} \right)^2 = \Omega_M \left( \frac{S}{S_0} \right)^3 \left[ 1 + \frac{\Pi_0}{c^2} + 3w \ln \left( \frac{S_0}{S} \right) \right].
\]

In Eq. (12.1), \( H = \frac{S'}{S^2} \) is the Hubble parameter, \( S(\eta) \) is the scale factor in terms of the conformal time, \( \eta \), and a prime denotes differentiation with respect to \( \eta \). Moreover, the constants \( H_0, S_0 \) and \( \Pi_0 \) are considered as denoting the corresponding present-time values, while \( \Omega_M = \frac{\rho}{\rho_c} \approx 0.3 \) is the rest-mass density parameter, which measures \( \rho \) in terms of the corresponding critical quantity, \( \rho_c = \frac{3H_0^2}{8\pi G} \), with \( G \) being Newton’s universal constant of gravitation. At the present epoch, we obtain

\[
\Pi_0 = \left( \frac{1}{\Omega_M} - 1 \right) c^2.
\]

This result is quite interesting: It suggests that, in principle, the energy of the internal motions of the collisional-DM fluid can account for the (extra) DE, so that, at the present epoch, the total-density parameter is

\[
\Omega = \frac{\epsilon_0}{\epsilon_c} = \frac{\rho_c c^2}{\rho_c c^2} + \frac{\rho_c \Pi_0}{\rho_c c^2} = 1,
\]

where \( \epsilon_c = \rho_c c^2 \) is the energy density equivalent to \( \rho_c \). Accordingly, we have been able to determine the "correct" form of the scale factor, which (under the assumption that the DM is collisional) governs the evolution of the Universe, as follows
\[ S(\eta) = S_0 \left( \frac{\eta}{\eta_0} \right)^{\frac{2}{1+3w\Omega_m}}, \quad (12.4) \]

which, for \( w \neq 0 \), is the natural generalization of the Einstein-de Sitter (EdS) model. Next, we have attempted to determine what is realized by someone who, although living in a collisional-DM model, insists in adopting the (traditional) collisionless-DM approach.

To do so, we have applied the \textit{conformal dynamical equivalence} technique [17]. With the aid of this technique, we have found the (conformal) transformation, which relates the collisional-DM description of a cosmological model (the scale factor \( S(\eta) \)) to the corresponding collisionless-DM approach (i.e., to the scale factor of a pressureless Universe, \( R(\eta) \)). In terms of the cosmological redshift, the corresponding \textit{conformal factor} is written in the form

\[ f(z) = 1 + w\Omega_m[1 + 3ln(1+z)]. \quad (12.5) \]

With this "tool" at hand, we have explored the way that, a supporter of the collisionless-DM scenario interprets the observations carried out in a collisional-DM Universe. The result of the "debate" between collisional- and collisionless-DM approach is, definitely, in favor of the former. In particular, in the collisional-DM Universe there is a \textit{characteristic value}, \( z_c \), of the cosmological redshift, above which, the luminosity distance of the various light-emitting sources becomes larger than what is realized by an observer who treats the DM as a pressureless fluid. In other words, from the point of view of someone who (although living in a collisional-DM model) insists in adopting the (traditional) collisionless-DM approach, the cosmologically-distant indicators, located at \( z > z_c \), seem to \textit{lie farther} (i.e., they appear to be \textit{dimmer}) than expected. Clearly, the similarity between the \textit{characteristic value} \( z_c \), and the (observationally-determined) \textit{transition redshift}, \( z_t \), that signals the onset of dimming of the SNe Ia standard candles [56], is more than obvious. On the other hand, after the thermodynamical content of a collisional-DM fluid is taken into account, the theoretical curve representing the \textit{distance modulus}, \( \mu(z) \) (green solid line), fits the \textit{Hubble diagram} of an extended sample of SN Ia standard candles [86] to high accuracy (cf. Fig. 6), in contrast to the corresponding collisionless-DM quantity, \( \tilde{\mu}(\tilde{z}) \), being either appropriately (orange solid line) or falsely (dashed line) expressed in terms of the \textit{truly measured quantity}, \( z \). At the same time, as far as a supporter of the collisionless-DM scenario is concerned, the Universe appears to be \textit{either accelerating or decelerating}, depending on the value of the cosmological redshift.
At the same time, as far as a supporter of the collisionless-DM scenario is concerned, the Universe appears to be either accelerating or decelerating, depending on the value of the cosmological redshift. In this case, the quantity $w$, which, in the collisional-DM approach, parameterizes the various isothermal flows, plays also another (more interesting) role: As we have found, for $w \geq 0.24$, there exists a (theoretically-determined) transition value, $z_t$, of the (collisionless-DM-oriented) cosmological redshift, $z$, such that, for $z < z_t$, the (correspondingly-oriented) deceleration parameter, $q$, becomes negative, i.e., from the point of view of someone who adopts the (traditional) collisionless-DM approach, the Universe is accelerating. Accordingly, taking into account the observational result that, the transition redshift between accelerated and decelerated expansion is set at the value $z_t \approx 0.46$ of the truly measured quantity $z$, we have determined the exact value of $w$, for which the collisional-DM approach to the post-recombination Universe is compatible with observations, namely $w \approx 0.31$. This result implies that, compatibility of the collisional-DM approach with the observational data currently available, suggests that, the DM itself consists of relativistic particles, i.e., $w \approx \frac{1}{3} = 0.33$.

In conclusion, the assumption that the DM constituents can be both collisional and relativistic, could provide a reasonable and conventional explanation for some open aspects of modern Cosmology, including:

(i) The extra (dark) energy needed to flatten the Universe: It can be compensated by the energy of the internal motions of the collisional-DM fluid.
(ii) The observed dimming of the SNe Ia standard candles and the apparent accelerated expansion of the Universe: Both of them can be due to the misinterpretation of several cosmologically-relevant parameters by those observers who, although living in a collisional-DM Universe, insist in adopting the collisionless-DM approach. Therefore, before inventing any new theory, it is useful to allow for a suitable use of the (so far) neglected degrees of freedom (hydrodynamic flows, pressure, energy of the internal motions, etc.). As we have shown, these internal physical characteristics can reveal their influence on several parameters of cosmological significance (scale factor, cosmological redshift, luminosity distance, Hubble and deceleration parameters) and yield a consistent alternative to the currently-accepted DE concept.

We consider our proposed cosmological model, with all its advantages and disadvantages, as a quite reasonable classical interpretation-explanation of the observational cosmological data and of the non-necessity of notions like dark energy, cosmological constant and accelerating universal expansion. This, in short, is expressed as: “Dark Matter: Yes! Dark Energy: No!” (for further details, see [37], while, for a slightly different approach, see [23]).

13. Discussion, Concluding Remarks, and Outlook

We have presented the theoretical concept of the “Conformal Dynamical Equivalence” approach, and used it for treating, in a unified and classically (namely, in a non-quantum) way, some interesting problems at various astrophysical levels (Sun, Solar System, Stars, Galaxies, Clusters of Galaxies, Universe as a whole).

The dynamical equivalence refers to the formal functional similarity between the adiabatic-isentropic hydrodynamic flows in a physical bounded gravitating perfect-fluid source and the geodesic motions in a virtual, fully defined one; and such problems are related to the celebrated Pioneer-Anomaly Effect in the Solar System, missing mass, dynamical mass, flat-rotation curves of disc galaxies, masses of magnetized and/or non-magnetized large-scale cosmological structures, possible formation mechanisms of winds and jets, dark matter, repulsive/attractive gravity, non-necessity of the cosmological constant, the dark energy, and the accelerating expansion of the Universe, as a whole.

The essential background idea at the heart of the classically (i.e., not quantum-mechanically oriented) unifying CDE approach is that, all the internal physical characteristics of the source-cosmological structure (beyond its rest-mass density) are taken into account as sources of geodesic motions. Treating the internal physical characteristics in this way was a known feature of the general-relativistic theory of gravity for geodesic motions, but not of the Newtonian one. Here, based on the CDE approach, we proved that, in the Newtonian theory of gravity, the internal physical characteristics can act as sources of geodesic motion, and examined some physical consequences of this fact. The interest in this fact stems from the methods and assumptions made in estimating the masses of astrophysical, galactic, and cosmological objects. In such mass determinations, observational techniques are used, which are based on the assumption of geodesic motions.

According to recent observational data, large amounts of hot gas exist in hydrodynamic (more generally, hydro-magnetic) flows, in, e.g., AGNs, or the extended haloes of galaxies, or in clusters and super clusters of galaxies and their haloes. In spite of this, however, the observational determination of e.g. the central masses and of the total masses of galaxies, and of clusters and super clusters of galaxies is based on the assumption of purely geodesic motions, namely, motions under the effect of gravity alone. However, obviously, in the presence of non-gravitational sources, due to, say, pressure gradients, viscosity or an inhomogeneous magnetic field, the standard geodesic motion in a purely gravitation field is no longer sustainable and adequate, and the hydrodynamic (or, more generally, hydro-magnetic) description is more appropriate. Therefore the emerging question is how appropriate the standard mass measurements are. In exactly this spirit we used the idea of the dynamical equivalence, and, for various cosmological structures, we took into account all the source’s internal physical characteristics as sources of geodesic motions, we evaluated various forms of mass (rest mass, baryonic mass, internal mass, total mass, dynamical mass), and explored their relative
importance. Thus, for the isothermal baryonic source, we proved that the baryonic mass is larger than the observationally determined (dynamical) mass, and that the (negative) internal mass can be (absolutely) larger than the baryonic mass, and, as such, it cannot be ignored. As a consequence, under these assumptions, there should be a plenitude of baryonic matter in the cosmological structures. These results enabled us to apply the same ideas in explaining the problems of the missing mass and of the flat-rotation curves, to explore the notion of repulsive gravity, but also, to question the notions of cosmological constant, dark energy, and accelerating expansion of the Universe.

As a result of the present treatment and in accordance with especially high-frequency electromagnetic observational data, for every cosmological structure, a new picture of a hot and quite extended structure emerges. This is determined by its temperature and chemical composition, and by the distinction between the actual linear dimensions of the structure and the (temperature-and chemical composition-dependent, much smaller) inversion distance (expressing the structure’s conventional linear dimensions). Interestingly, the region between the inversion distance and the actual (and larger) linear dimensions of the structure is the region of dominance of repulsive gravity over attractive gravity. Especially in the case of a super cluster of galaxies, the actual dimensions are much larger than its conventional dimensions. So, the interesting result is reached that, now, the super cluster appears as a hot and extended structure, resembling the typical picture of the whole cosmological Universe. Such a picture of a cosmological structure conforms with its more luminous, as denser, central regions, and its fainter, as more rarefied, outer regions (both assumed as having the same temperature and mean molecular weight). It is quite probable that this new picture excludes the existence of third (and higher)-order clusters of galaxies.

As for the general-relativistic cosmological applications of the CDE approach, we point out that, in the collisional-dark-matter model, the extra energy, needed to compromise for the currently available cosmological data, can be compensated by the energy of the internal motions of the collisional cosmological fluid, that the post-recombination Universe remains ever-decelerating, and, also that the explicit form of the equation of state for the cosmological fluid can be determined. In other words, in the collisional-dark-matter model, no dark energy and no acceleration of the universal expansion are necessary. The interest in these results is enhanced by proving that, in the context of the less probable collisionless-dark-matter model, the extra (dark)-energy component (to compromise the cosmic-background-microwave-radiation data) is, in fact, needed, the distant light sources appear to be dimmer than expected, and the expansion of the Universe keeps balancing between acceleration and deceleration, depending on the (fully and explicitly determined appropriate) value of the cosmological red-shift parameter. Therefore, these results of the collisionless-dark-matter model seem to be a misinterpretation of the cosmological observational data by an observer, who (although living in a Universe filled, mainly, with, collisional dark matter) insists in adopting the traditional (collisionless-dark matter) approach.

To the extent of our knowledge, the CDE approach is classical and novel, and it is characterized by simplicity of thought, inner consistency, conformity with observational data and generally acceptable theoretical predictions. Also it does not constitute any kind of modifications of gravity (Newtonian and relativistic), and it extends the realm of validity of the classical geodesic motions.

In spite of all its advantages above, in our treatment there exist many deficiencies. Thus, one can argue against the fundamental assumptions of the time-independence of the Newtonian theoretical framework used here and of the isentropicity of the hydrodynamic flow motions, although such assumptions are in line with the most usual assumption of an isolated astronomical system. Also, beyond the assumption of the (greatly simplifying) spherical symmetry, one can argue against the use of the Plummer-type mass-density profile, although such a density profile is a particular case of more general “universal” density profiles, which are supported by observations, and so they have been widely applied in the literature. We believe, that the choice of physically and mathematically reasonable density profiles will not affect our present conclusions considerably, and, in any case, it will not affect our fundamental and conceptually important conclusion, that the theoretically evaluated mass is different from the observationally determined (dynamical) mass. The same is true.
for the consequences of the isotropic, isothermal, and adiabatic equation of state assumed here, which enables one to evaluate the internal specific energy density in terms of the mass density, temperature and mean molecular weight of the distribution, and its temporal and spatial variations. Finally, although less probable and least justifiable, one can question even the necessity of the collisional dark matter itself. Such objections and disadvantages imply the need of more intense research work, both theoretical and observational, on these subjects. Therefore, for the future, it remains to be seen whether these novel ideas on the role and importance of the internal physical characteristics of the large-scale cosmological structures and on the new interpretation of basic cosmological observations will attract the interest of the academic society and, so, perhaps, it will be acceptable as new scientific knowledge concerning the past and future history and evolution of the Universe and its constituents, and their interpretation.

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References

[1] Sembach K 2002 Space Telescope Science Institute Newsletter 19 19
[2] Nicastro F et al. 2003 Nature 421 719
[3] Braun R and Burton W B 2001 A&A 375 219
[4] Hagiwara Y, Diamond P J, Nakai N and Kawabe R 2000 A&A 360 49
[5] Hagiwara Y, Henkel C, Menten K M and Nakai N 2001 A&A 560 L37
[6] Hagiwara Y, Diamond P J and Miyoshi M 2002 A&A 383 65
[7] Tarchi A, Henkel C, Peck A B and Menten K M 2002 A&A 385 1049
[8] Felten J E, Gould R J, Stein W A and Woof N J 1996 ApJ 146 955
[9] Begeman K, Broeils A and Sanders R H 1991 MNRAS 249 523
[10] Navarro J F 2002 in The Shapes of Galaxies and their dark Halos, Yale University Workshop, Ed. P Natarajan, World Scientific Publishing, pp. 114-118.
[11] Jones C, David L, Forman B, Markevich M, Murray S, van Speybroeck L and Vikhilin in Chandra News 10 1
[12] Simon J D, Robishaw T and Blitz L 2003 “Dark Matter in Dwarf Galaxies: The First Dark Galaxy?” in Satellites and Tidal Streams, ASP Conference Series, Eds. F Prada, D Martinez-Delgado and T Mahoney
[13] Alves J and Homeir N 2003 ApJ 589 L45
[14] Ellis R S 2002, in The Shapes of Galaxies and their dark Halos, Yale University Workshop, Ed. P Natarajan, World Scientific Publishing, pp. 254-262
[15] Livio M 2003 Space Telescope Science Newsletter 20 16
[16] Page L et al. 2003 ApJ Suppl. Series 148 39
[17] Kleidis K and Spyrou N K 2000 Class. Quantum Grav. 17 2965
[18] Spyrou N K 1997a in The Physics of Ionized Gases, Proceedings of the 18th Summer School and International Symposium on the Physics of Ionized Gases (Kotor, Yugoslavia, 2-6 September 1996), Eds. B Vujicic, S Djurovic and I Puric, Institute of Physics, Novi Sad Univ. Yugoslavia, pp. 417-446
[19] Spyrou N K 1997b in The Earth and the Universe, Volume in Honorem L Mavrides, Eds. G Asteriadis, A Bandellas, M Contadakis, K Katsambalos, A Papademetriou and I Tziavos, Thessaloniki, Greece, pp. 277-291
[20] Spyrou N K 1997c Facta Universitatis 4 7
[21] Spyrou N K 1999 in Current Issues of Astronomical and Planetary Environmental Concern, Proceedings of International Seminar (Thessaloniki, 6-7 April 1999), Ed. N K Spyrou, Astronomy Department, Aristoteleion University of Thessaloniki, Thessaloniki, Greece, pp.23-35

[22] Spyrou N K 2001 “Conformal Invariance and the Nature of Cosmological Structures”, Invited Talk, Proceedings of the Conference on Applied Differential Geometry-General Relativity and the Workshop on Global Analysis, Differential Geometry and Lie Groups, Thessaloniki, 27 June-1July 2001, eds. G Tsagas, C Udriste and D B Papadopoulos, pp. 101-107.

[23] Spyrou N K 2002 “On the determination of the masses of cosmological structures”, in Modern Theoretical and Observational Cosmology, Proceedings of the 2nd Hellenic Cosmology Meeting, Athens, 19-20 April 2001, Eds. M Plionis and S Cotsakis, Kluwer Academic Publishers, Dordrecht, 276, pp.35-43

[24] Spyrou N K 2003 “Conformal Dynamical Equivalence and the Cosmological Expansion of a Realistic Universe”, in Proceedings of the 10th Conference Recent Developments in Gravity, Eds. K Kokkotas and N Stergioulas, Kluwer Academic Publishers, Dordrecht, pp.90-96

[25] Spyrou N K 2004a, “A classical treatment of the dark-matter and flat-rotation-curves problems”, in Proceedings of the 6th Hellenic Astronomical Conference (Penteli, Athens), September 2003, Ed. P G Laskarides (and the Edition Office of the University of Athens), Athens, pp. 229-234.

[26] Spyrou N K 2004b “A classical treatment of the problems of dark energy, dark matter and accelerating expansion”, in Proceedings of the Conference Recent Developments in Gravity (NEB) XI (Mytilini, Lesvos, Hellas) 2-6 June 2004, Eds. S Cotsakis and J Myritzis; published as Spyrou 2005.

[27] Spyrou N K 2005 JPhCS 8 122, http://www.iop.org/EJ/toc/1742-55.1.6596/8/1

[28] Spyrou N K 2006 “Large-Scale Cosmological Structures: A New Look”, in Proceedings of the International “Workshop Cosmology and Gravitational Physics”, Thessaloniki, December 15-16, 2005, Eds. N K Spyrou, N Stergioulas, and C G Tsagas, pp. 57-66

[29] Spyrou N K 2008 “Negative mass and repulsive gravity in Newtonian theory and consequences”, in Proceedings of the Conference Recent Developments in Gravity (NEB) XIII (Thessaloniki, Hellas) 4-6 June 2008 (in press)

[30] Spyrou N K 2009 “Negative mass and repulsive gravity in Newtonian theory and consequences”, in Proceedings of the 2008 International Conference in honour of John Hadjidemetriou, Litohoron, Greece, Eds. H Varvoglis and Z Knezevic, Aristoteleion University of Thessaloniki, Thessaloniki, Greece, and Astronomical Observatory, Belgrade, Serbia, pp.181-184

[31] Spyrou N K 2010 “Conformal Dynamical Equivalence and Applications”, in Proceedings of the Conference Recent Developments in Gravity (NEB) XIV (Ioannina, Hellas) 8-12 June 2010; see also Kleidis and Spyrou 2010a (in press)

[32] Kleidis K and Spyrou N K 1999 “On the Nature of Nuclear Galactic Masses” in Proceedings of JENAM 1999, Samos, Greece.

[33] Spyrou N K and Tsagas C G 2004, Class Quantum Grav. 21 2435

[34] Spyrou N K and Tsagas C G 2008 MNRAS 388 187

[35] Spyrou N K and Tsagas C G 2010 Cent. Eur. J. Phys. 8 989

[36] Spyrou N K and Zagkouris K 2010 “The Solar System: A New Look”, Aristoteleion University of Thessaloniki, Thessaloniki, Greece (Preprint)

[37] Kleidis K and Spyrou N K 2010a “A Conventional Form of Dark Energy”, in Proceedings of the Conference Recent Developments in Gravity (NEB) XIV (Ioannina, Hellas) 8-12 June 2010

[38] Kleidis K and Spyrou N K 2010b “Collisional Dark Matter: A conventional approach to the dark-energy concept (Preprint)

[39] Fock V 1959 ‘The theory of space, time and gravitation” (London: Pergamon Press)
[40] Chandrasekhar S 1969 *ApJ* **158** 45
[41] Taub A 1959 *Arch. Ration. Mech. Anal.* **3** 312
[42] Misner C, Thorne K S and Wheeler J A 1973 “*Gravitation*” (San Francisco: Freeman)
[43] Anderson J 1967 “*Principles of Relativistic Physics*” (New York: Academic Press)
[44] Ryan M and Shepley L 1975 “*Homogeneous Relativistic Cosmologies*” (Cambridge: Cambridge University Press)
[45] Navarro J F, Frenk C S and White S D M 1995 *MNRAS* **275** 720
[46] Navarro J F, Frenk C S and White S D M 1997 *ApJ* **490** 493
[47] Hernquist L 1990 *ApJ* **356** 359
[48] Dehnen W 1995 *MNRAS* **274** 919
[49] Syer D and White S D M 1998 *MNRAS* **293** 337
[50] Zhao H S 1996 *MNRAS* **278** 488
[51] Spergel D N et al. [WMAP Collaboration] 2003 *ApJ Suppl. Series* **148** 175
[52] de Bernardis P et al. [BOOMERanG Collaboration] 2000 *Nature* **404** 955
[53] Riess A G et al. [High-z Supernova Search Team] 1998 *Astron. J.* **116** 1009
[54] Perlmutter S et al. [Supernova Cosmology Project Group] 1999 *ApJ* **517** 565
[55] Riess A G et al. [High-z Supernova Search Team] 2001 *ApJ* **560** 49
[56] Riess A G et al. [High-z Supernova Search Team] 2004 *ApJ* **607** 665
[57] Tonry J L et al. [High-z Supernova Search Team] 2003 *ApJ* **594** 1
[58] Caldwell R, Dave R and Steinhardt P J 1998 *Phys. Rev. Lett.* **80** 1528
[59] Carroll S M 1998 *Phys. Rev. Lett.* **81** 3067
[60] Komatsu E et al. [WMAP Collaboration] 2009 *ApJ Suppl. Series* **180** 330
[61] Kolb E W and Turner M S 1990 “*The Early Universe*” (Menlo Park: Addison-Wesley)
[62] Olive K A 2003 *TASI Lectures on dark matter*, arXiv: 0301505 [astro-ph]
[63] Hooper D 2009 *TASI 2008 Lectures on dark matter*, arXiv: 0901.4090 [hep-ph]
[64] Chang J et al. [ATIC Collaboration] 2008 *Nature* **456** 362
[65] Adriani O et al. [PAMELA Collaboration] 2009 *Nature* **458** 607
[66] Hooper D, Finkbeiner D P and Dobler G 2007 *Phys. Rev.* **D 76** 083012
[67] Barger V, Keung W Y, Marfatia D and Shaughnessy G 2008, *Phys. Lett.* **B 672** 141
[68] Baushev A 2008, “*Dark matter annihilation at cosmological redshifts: Possible relic signal from weakly interacting massive particles annihilation*”, arXiv: 0806.3108 [astro-ph]
[69] Bergstrom L, Bringmann T and Edsjo J 2008 *Phys. Rev.* **D 78** 103520
[70] Cholis I, Goodenough L, Hooper D, Simet M and Weiner N 2008a “*High-energy positrons from annihilating dark matter*”, arXiv: 0809.1683 [hep-ph]
[71] Cholis I, Dobler G, Finkbeiner D P, Goodenough L and Weiner N 2008b, “*The case for a 700+ GeV WIMP: Cosmic-ray spectra from ATIC and PAMELA*”, arXiv: 0811.3641 [astro-ph]
[72] Cirelli M and Strumia A 2008 “*Minimal dark-matter predictions and the PAMELA positron excess*”, arXiv: 0808.3867 [astro-ph]
[73] Regis M and Ullio P 2008 *Phys. Rev.* **D 78** 3505
[74] Fornasa M, Pieri L, Bertone G and Branchini E 2009 “*Anisotropy probe of galactic and extra-galactic dark matter annihilations*”, arXiv: 0901.2921 [astro-ph]
[75] Fox P J and Poppitz E 2009 *Phys. Rev.* **D 79** 083528
[76] Zurek K M 2009 *Phys. Rev.* **D 79** 115002
[77] Spergel D N and Steinhardt P J 2000 *Phys. Rev. Lett.* **84** 3760
[78] Arkani-Hamed N, Finkbeiner D P, Slatyer T R and Weiner N 2009 *Phys. Rev.* **D 79** 015014
[79] Cirelli M, Kadastik M, Raidal M and Strumia A 2009 *Nucl. Phys.* **B 813** 1
[80] Cohen T and Zurek K 2010 *Phys. Rev. Lett.* **104** 101301
[81] Zimdahl W, Schwarz D J, Balakin A B and Pavon D 2001 *Phys. Rev.* **D 64** 3501
[82] Balakin A B, Pavon D, Schwarz D J and Zimdahl W 2003 *New J. Phys.* **5** 85
[83] Lima J A S, Silva F E and Santos R C 2008 *Class. Quantum Grav.* **25** 205006
[84] Basilakos S and Plionis M 2009 *A&A* **507** 47
[85] Basilakos S and Plionis M 2010 “Interactive dark matter as an alternative to dark energy”, in Invisible Universe: Conference Proceedings, AIP Conference Proceedings 1241, 721

[86] Davis T M et al. [ESSENCE Supernova Survey Team] 2007 ApJ 666 71