Quantum Galois theory for finite groups

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Dong and Mason [DM1] initiated a systematic research for a vertex operator algebra with a finite automorphism group, which is referred to as the “operator content of orbifold models” by physicists [DVVV]. The purpose of this paper is to extend one of their main results. We will assume that the reader is familiar with the vertex operator algebras (VOA), see [B], [FLM].

Throughout this paper, $V$ denotes a simple vertex operator algebra, $G$ is a finite automorphism group of $V$, $C$ denotes the complex number field, and $Z$ denotes rational integers. Let $H$ be a subgroup of $G$ and $\text{Irr}(G)$ denote the set of all irreducible $CG$-characters. In their paper [DM1], they studied the sub VOA $V^H = \{v \in V : h(v) = v \text{ for all } h \in H\}$ of $H$-invariants and the subspace $V^\chi$ on which $G$ acts according to $\chi \in \text{Irr}(G)$. Especially, they conjectured the following Galois correspondence between sub VOAs of $V$ and subgroups of $G$ and proved it for an Abelian or dihedral group $G$ [DM1, Theorem 1] and later for nilpotent groups [DM2], which is an origin of their title of [DM].

**Conjecture (Quantum Galois Theory)** Let $V$ be a simple VOA and $G$ a finite and faithful group of automorphisms of $V$. Then there is a bijection between the subgroups of $G$ and the sub VOAs of $V$ which contains $V^G$ defined by the map $H \to V^H$.

Our purpose in this paper is to prove the above conjecture. Namely, we will prove:

**Theorem 1** Let $V$ be a simple VOA and $G$ a finite and faithful group of automorphisms of $V$. Then there is a bijection between the subgroups of $G$ and the sub VOAs of $V$ which contains $V^G$ defined by the map $H (\leq G) \to V^H (\geq V^G)$. 

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We adopt the notation and results in [DM1] and [DLM]. Especially, the following result in [DLM] is the main tool for our study.

**Theorem 2 (DLM, Corollary 2.5)** Suppose that $V$ is a simple VOA and that $G$ is a finite and faithful group of automorphisms of $V$. Then the following hold:

(i) For $\chi \in \text{Irr}(G)$, each $V^\chi$ is a simple module for the $G$-graded VOA $C_G \otimes V^G$ of the form

$$V^\chi = M_{\chi} \otimes V_{\chi}$$

where $M_{\chi}$ is the simple $C_G$-module affording $\chi$ and where $V_{\chi}$ is a simple $V^G$-module.

(ii) The map $M_{\chi} \rightarrow V_{\chi}$ is a bijection from the set of simple $C_G$-modules to the set of inequivalent simple $V^G$-modules which are contained in $V$.

It was proved in [DM1, Lemma 3.2] that the map $H(\leq G) \rightarrow V^H (\geq V^G)$ is injective. Therefore, it is sufficient to show that for any sub VOA $W$ containing $V^G$, there is a subgroup $H$ of $G$ such that $W = V^H$. Our first purpose is to transform the assumption of quantum Galois conjecture to the following purely group theoretic condition:

**Hypotheses (A)** Let $G$ be a finite group and $\{M_{\chi} : \chi \in \text{Irr}(G)\}$ be the set of all nonisomorphic simple modules of $G$. Assume $M_{1_G}$ is a trivial module. Let $R$ be a subspace of $M = \bigoplus_{\chi \in \text{Irr}(G)} M_{\chi}$ containing $M_{1_G}$. Assume that for any $G$-homomorphism $\pi : M \otimes M \rightarrow M$, $\pi(R \otimes R) \subseteq R$.

Let’s show how to transform the assumption of quantum Galois conjecture to Hypotheses (A). We first explain the way to give a relation between $V$ and $M$. Let introduce a relation $u \sim v$ for two nonzero elements $u, v \in V$ if there is an element $w \in V^G$ such that $v = w_n u$ or $u = w_n v$ for some $n \in \mathbb{Z}$. Extend this relation into an equivalent relation $\equiv$ as follows:

$u \equiv v$ if there are $u^1, \ldots, u^m \in V$ such that $u \sim u^1 \sim \ldots \sim u^m \sim v$.

This equivalent relation implies that for any $G$-homomorphism $\phi : V \rightarrow M_{\chi}$, the image of each equivalent class is uniquely determined up to scalar times. Also since $V_{\chi}$ is spanned by $\{v_n s_{\chi} : v \in V^G, n \in \mathbb{Z}\}$ for some $s_{\chi} \in V_{\chi}$ by [DM1 Proposition 4.1], $\phi(V_{\chi}) = C\phi(s_{\chi})$ is a subspace of dimension at most one.
Let’s start the transformation. We recall
\[ V = \oplus_{\lambda}(M_{\lambda} \otimes V_{\lambda}) \]
from Theorem 2 ([DLM, Corollary 2.5]). We may view \( M_{\lambda} \) as a subspace of \( V \). Set \( M = \oplus M_{\lambda} \). Let \( W \) be a sub VOA containing \( V^G \). Let \( U_{\lambda} \) be the \( V^G \)-subspace of \( W \) whose composition factors are all isomorphic to \( V_{\lambda} \). Then \( U_{\lambda} = W \cap (M_{\lambda} \otimes V_{\lambda}) \) and so \( W = \oplus_{\lambda} U_{\lambda} \). Set
\[ R_{\lambda} = \{ m \in M_{\lambda} | m \otimes V_{\lambda} \subseteq W \} \]
and \( R = \oplus R_{\lambda} \). In particular, we have
\[ W = \oplus_{\lambda}(R_{\lambda} \otimes V_{\lambda}) \]
Replacing \( M_{\lambda} \) by \( L(k_1)...L(k_r)M_{\lambda} \cong M_{\lambda} \) if necessary, we may think that \( R = \oplus R_{\lambda} \) and \( M = \oplus M_{\lambda} \) are subspaces of a homogeneous part \( V_p \) of \( V \) for some \( p \). Let \( \{ v^1, ..., v^n \} \) be a basis of \( R \) and \( \{ v^1, ..., v^s, ..., v^n \} \) be a basis of \( M \).
The proof of Lemma 3.1 in [DM1] shows that \( \{ Y(v^i, z)v^j : i, j = 1, ..., n \} \) is a linearly independent set. Define \( \pi_m : M \times M \to V_m \) by \( \pi_m(v^i \times v^j) = (v^i)_m v^j \).
Since \( \{ Y(v^i, z)v^j \} \) is a linearly independent set, \( \cap_{m \in \mathbb{Z}} \text{Ker}(\pi_m) = 0 \). Since \( M \times M \) has only a finite dimension, there is a finite set \( \{ a, a + 1, ..., b \} \) of integers such that \( \cap_{m=a}^{b} \text{Ker}(\pi_m) = 0 \). Since the grade of \( (v^i)_m v^j \) depends only on \( m \) and \( (v^i)_m v^j \) and \( (v^h)_m' v^k \) belong to different homogeneous spaces for \( m \neq m' \), the map \( \pi : M \times M \to V \) given by \( \pi(v^i \times v^j) = \sum_{m=a}^{b} (v^i)_m(v^j) \) is injective. Set \( E = \text{Im}(\pi) =< \sum_{m=a}^{b} (v^i)_m(v^j) : i, j = 1, ..., n > \). Clearly, \( E \) is a \( G \)-invariant subspace of \( V \). Decompose \( V \) into a direct sum \( V = E \oplus E' \) of \( E \) and some \( G \)-submodule \( E' \) of \( V \). Define \( \mu : V = E \oplus E' \to M \times M \) by \( \mu(e + e') = \pi^{-1}(e) \) for \( e \in E, e' \in E' \). This is a \( G \)-epimorphism. Since \( W \) is a sub VOA, the any products \( u, v \) of two elements \( u, v \) of \( W \) are in \( W \). Hence, \( \pi(R \times R) \subseteq W \) and so the image \( \mu(W) \) contains \( R \times R \). Therefore, for any \( G \)-homomorphism \( \phi : M \otimes M \to M \), we have \( \phi(R \times R) \subseteq \phi(\mu(W)) \). On the other hand, for any \( G \)-homomorphism \( \psi : V \to M \), we have \( \psi(W) \subseteq R \) since \( W = \oplus_{\lambda}(R_{\lambda} \otimes V_{\lambda}) \). Hence, we have
\[ \phi(R \times R) \subseteq \phi(\mu(W)) = (\phi \mu)(W) \subseteq R \]
for any \( G \)-homomorphism \( \phi : M \otimes M \to M \). Namely, \( R \) satisfies the Hypotheses (A).

We will next prove the following group theoretic problem:
Theorem 3 Let $G$ be a finite group and $\{M_{\chi} : \chi \in \text{Irr}(G)\}$ be the set of all simple modules of $G$. Assume $M_1 = C$ is a trivial module. Let $R$ be a subspace of $M = \bigoplus_{\chi \in \text{Irr}(G)} M_{\chi}$ containing $M_1$. Assume that $R$ satisfies the following condition: for any $G$-homomorphism $\pi : M \otimes M \to M$, $\pi(R \otimes R) \subseteq R$. Then there is a subgroup $G_1$ of $G$ such that $R = M^{G_1}$.

Proof. Consider the group algebra $CG$, and write $CG = \bigoplus_{\chi \in \text{Irr}(G)} M_{\chi}\chi(1)$ as a left $G$-module. We define a subspace $S$ of $CG$ by

$$S = \bigoplus_{\chi \in \text{Irr}(G)} (R \cap M_{\chi})\chi(1).$$

Then $S$ is a right ideal of $CG$. Note that $R = \bigoplus_{\chi} (R \cap M_{\chi})$, since $M_1 = C \subseteq R$ and $\pi_{\chi}(C \otimes R) \subseteq R$ for a projection $\pi_{\chi} : C \otimes M \to M_{\chi} \subseteq M$. Thus $S$ satisfies that $\pi'(S \otimes S) \subseteq S$ for any $G$-homomorphism $\pi' : CG \otimes CG \to CG$.

We define a new product $\circ$ in $CG$ by

$$\left(\sum_{g \in G} a_g g\right) \circ \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} a_g b_g g,$$

where $a_g, b_g \in C$. Then $(CG, \circ)$ is a semisimple commutative algebra with the identity $\sum_{g \in G} g$. We write the identity by $1^\circ$.

Define $\pi' : CG \otimes CG \to CG$ by

$$\pi'\left(\left(\sum_{g \in G} a_g g\right) \otimes \left(\sum_{g \in G} b_g g\right)\right) = \sum_{g \in G} a_g b_g g.$$ 

Then $\pi'$ is a $G$-homomorphism, and so $\pi'(S \otimes S) \subseteq S$. This means that $S$ is a subalgebra of $(CG, \circ)$.

Since $R$ contains the trivial module, $1^\circ$ belongs to $S$. Consider the primitive idempotent decomposition of $1^\circ$ in $S$. Then there exists a partition $G = \cup_i G_i$ such that $\sum_{g \in G_i} g$ is a primitive idempotent in $S$ for any $i$. Put $e_i = \sum_{g \in G_i} g$, then $S = \bigoplus_i C e_i$ since $S$ is semisimple and commutative. Assume $1_G \in G_1$. For $h \in G_1$, $e_1 h$ is in $S$ since $S$ is a right ideal in $CG$. By the form of $S$, $e_1 h$ is a sum of some $e_i$'s. But $h \in G_1 h$, so $e_1 h = e_1$. This means $G_1$ is a subgroup of $G$. Similarly $G_i$ is a left coset of $G_1$ in $G$ for any $i$.

Now $S = CG^{G_1}$, and so $R = V^{G_1}$. The proof is completed. \qed
Let’s go back to the proof of Theorem 1 (the quantum Galois theory). By the above theorem 3, there is a subgroup $H$ of $G$ such that $R_\chi = M_\chi^H$ and so

$$U_\chi = M_\chi^H \otimes V_\chi = (M_\chi \otimes V_\chi)^H.$$ 

Hence, $W = V^H$. This completes the proof of Theorem 1.

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