THE $\mathcal{LS}$ METHOD FOR THE CLASSICAL GROUPS 
IN POSITIVE CHARACTERISTIC AND 
THE RIEMANN HYPOTHESIS

LUIS ALBERTO LOMELÍ

Abstract. We provide a definition for an extended system of $\gamma$-factors for products of generic representations $\tau$ and $\pi$ of split classical groups or general linear groups over a non-archimedean local field of characteristic $p$. We prove that our $\gamma$-factors satisfy a list of axioms (under the assumption $p \neq 2$ when both groups are classical groups) and show their uniqueness (in general). This allows us to define extended local $L$-functions and root numbers. We then obtain automorphic $L$-functions $L(s, \tau \times \pi)$, where $\tau$ and $\pi$ are globally generic cuspidal automorphic representations of split classical groups or general linear groups over a global function field. In addition to rationality and the functional equation, we prove that our automorphic $L$-functions satisfy the Riemann Hypothesis.

INTRODUCTION

Let $G_m$ and $G_n$ denote either split classical groups or general linear groups of ranks $m$ and $n$, respectively. Let $k$ be a global function field with finite field of constants $F_q$ and ring of ad`eles $\mathbb{A}_k$. We present a theory of automorphic $L$-functions $L(s, \tau \times \pi)$, where $\tau$ and $\pi$ are globally generic cuspidal automorphic representations of $G_m(\mathbb{A}_k)$ and $G_n(\mathbb{A}_k)$.

The case of $G_m = GL_m$ and $G_n$ a classical group is made possible by our work on the Langlands-Shahidi method in positive characteristic for the classical groups [10, 11]. Already in this case, the $\mathcal{LS}$ method has particularly interesting applications. In addition to the Ramanujan conjecture for the classical groups over function fields established in [10], the zeros of $L(s, \tau \times \pi)$ satisfy $\Re(s) = 1/2$.

We note that the case of $G_m = GL_m$ and $G_n = GL_n$ gives rise to Rankin-Selberg factors. Indeed, we include a treatise of this case in a self contained manner within the Langlands-Shahidi method in [11]. And, we provide a short proof of the equality of local factors when defined via different methods in [4]. Thanks to the work of Lafforgue on the Langlands correspondence for $GL_N$ over function fields, the Riemann Hypothesis is available for $L$-functions of products of cuspidal automorphic representations of $GL_m(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_k)$ [8]. In this very important case, all of our results are available with no restriction on the characteristic of $k$.

Notice that the case of two classical groups $G_m$ and $G_n$, is treated for the first time here in positive characteristic. We provide axioms for an extended system of $\gamma$-factors, $L$-functions and root numbers which cover all of the above mentioned cases for $G_m$ and $G_n$. We first establish existence and uniqueness of $\gamma$-factors, Theorem 1.5, and then include existence and uniqueness of local $L$-functions and $\varepsilon$-factors in Theorem 4.3. The theory is complete under the assumption char($k$) $\neq 2$. 

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We begin by introducing notation that is useful when dealing with systems of $\gamma$-factors, $L$-functions and root numbers. Local factors $\gamma(s, \tau \times \pi, \psi)$, $L(s, \tau \times \pi)$ and $\varepsilon(s, \tau \times \pi, \psi)$ are defined on the local class $\mathfrak{ls}(p, G_m, G_n)$, while global $L$-functions and root numbers

$$L(s, \tau \times \pi) = \prod_v L(s, \tau_v \times \pi_v), \quad \varepsilon(s, \tau \times \pi) = \prod_v \varepsilon(s, \tau_v \times \pi_v, \psi_v)$$

are defined on the global class $\mathcal{LS}(p, G_m, G_n)$ (see §1.1 and §1.2). We often write $\mathfrak{ls}(p)$ and $\mathcal{LS}(p)$ when there is no need to specify the groups $G_m$ and $G_n$.

**Theorem.** Automorphic $L$-functions on $\mathcal{LS}(p)$, $p \neq 2$, satisfy the following properties:

(i) (Rationality). $L(s, \tau \times \pi)$ converges on a right half plane and has a meromorphic continuation to a rational function on $q^{-s}$.

(ii) (Functional equation). $L(s, \tau \times \pi) = \varepsilon(s, \tau \times \pi)L(1 - s, \tilde{\tau} \times \tilde{\pi})$.

(iii) (Riemann Hypothesis) The zeros of $L(s, \tau \times \pi)$ are contained in the line $\Re(s) = 1/2$.

Let us now give a more detailed description of the contents of the article. Theorem 1.5 concerns the existence an uniqueness of a system of $\gamma$-factors on $\mathfrak{ls}(p)$. The theorem is true with no assumption on $p$ for $\mathfrak{ls}(p, GL_m, G_n)$. However, we assume $p \neq 2$ in order to produce $\gamma$-factors on $\mathfrak{ls}(p, G_m, G_n)$ when both $G_m$ and $G_n$ are classical groups. Extended $\gamma$-factors satisfy several local properties including a twin multiplicativity property when the representations are obtained via parabolic induction, as well as a stability property on $\mathfrak{ls}(p, GL_1, G_n)$ with respect to highly ramified characters of $GL_1$. Globally, $\gamma$-factors make an important appearance in the functional equation for partial $L$-functions on $\mathcal{LS}(p)$.

All of §2 is devoted to a proof of the existence part of Theorem 1.5. With the Langlands-Shahidi method now complete for the split classical groups in positive characteristic, we give a straightforward and linearly ordered presentation of $\gamma$-factors on $\mathfrak{ls}(p, GL_m, G_n)$. We recall the basic definitions in §2.1. Then, our results on the Siegel Levi case for the split classical groups [11] are summarized in §2.2. The Siegel Levi case allows us to define exterior and symmetric square $\gamma$-factors, which are uniquely characterized and are proved to be in accordance with the local Langlands conjecture for $GL_m$ in [4]. Filling any gaps that were left in [10], the case of $\mathfrak{ls}(p, GL_m, G_n)$ is presented in §2.3 with no assumption on the characteristic. The new case of $\mathfrak{ls}(p, G_m, G_n)$, with both $G_m$ and $G_n$ classical groups, is developed in §2.4 under the assumption $p \neq 2$.

There are fundamental differences in local-to-global arguments between global fields of characteristic zero and characteristic $p$. In §3, we use a variation of a result of Vignéras that allows us to lift a local pair of irreducible supercuspidal generic representations $\tau_0$ and $\pi_0$ to a global pair of globally generic cuspidal automorphic representations $\tau$ and $\pi$ (see Proposition 3.1). We remark that, over number fields, Shahidi makes a crucial improvement upon the local-to-global argument of Henniart and Vignéras by incorporating the archimedean theory that is available at infinite places (Proposition 5.1 of [13]). Over function fields, the main difference is due to the fact that all places are non-archimedean; a place at infinity plays the role of the archimedean places. As a further remark, in the case of two general linear groups
we have a much more precise local-to-global result [4], which allows us to remove stability from the list of properties required in the characterization [6].

The uniqueness part of Theorem 1.5 is proved in § 3 over a global function field with no restriction on \( p \). We first treat the case of \( \mathfrak{L}(p, G_m, G_n) \) in § 3.2, where we can use stability of \( \gamma \)-factors combined with the Grundwald-Wang theorem of class field theory. We then proceed to the general case in § 3.3. Our method of proof resembles the approach taken in [10], and we refer to the introduction for further remarks on the local-to-global argument over a global function field (see also § 3).

Building upon extended \( \gamma \)-factors, we axiomatize local \( L \)-functions and root numbers on \( \mathfrak{L}(p) \) in § 4.1. Additional properties of \( \gamma \)-factors are recorded in § 4.2, these include a local functional equation for which we provide a proof. We extend Theorem 1.5 to a theorem on extended \( \gamma \)-factors, local \( L \)-functions and root numbers in § 4.3. Finally, we establish our main global results in § 4.4, under the assumption \( p \neq 2 \). Theorem 4.4 includes rationality, the functional equation and the Riemann Hypothesis for automorphic \( L \)-functions on \( \mathcal{L}S(p) \).

Our general results are possible since we have established a Langlands functorial lift from globally generic cuspidal automorphic representations \( \pi \) of a classical group \( G_n \) to automorphic representations \( \Pi \) of \( GL_N \) [10]. The integer \( N \) is obtained from the table of § 1.3, by minimally embedding the connected component of the Langlands dual group \( ^L G_n \) of \( G_n \) into \( GL_N(\mathbb{C}) \). The image of functoriality can be further expressed as an isobaric sum

\[
\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,
\]

where each \( \Pi_i \) is a self-dual cuspidal automorphic representation of \( GL_{n_i} \). They satisfy the additional condition that, for each \( i \), a partial \( L \)-function \( L^\mathfrak{L}(s, \Pi_i, r \circ \rho_{n_i}) \) has a pole at \( s = 1 \), where \( r = \wedge^2 \) or \( \text{Sym}^2 \) depending on the classical group [17]. Exterior and symmetric square \( L \)-functions, and related \( \gamma \)-factors, are thoroughly studied in [4] [11]. Furthermore, [11] also develops the necessary theory for the Siegel Levi case of quasi-split Unitary groups. This leads us to Asai \( \gamma \)-factors and \( L \)-functions, which are uniquely characterized in [5]. They play a similar role for the quasi-split unitary groups, as the exterior and symmetric square \( L \)-functions do for the split classical groups, when describing the image of the functorial lift of [12] as an isobaric sum.

I would like to thank Freydoon Shahidi for many insightful mathematical conversations. Thanks are also due to Guy Henniart; our collaborative work greatly influenced the axiomatization of the system of \( \gamma \)-factors, \( L \)-functions and \( \varepsilon \)-factors presented in this paper. I would also like to thank the Département de Mathématiques d’Orsay, Université Paris-Sud 11, for their hospitality during a short visit to conduct research in May-June of 2012. Part (iii) of Theorem 4.4 for the case of \( \mathcal{L}S(p, G_m, G_n) \), \( G_n \) a classical group, was an unpublished result of the author since the summer of 2009. The opportunity presented itself to write its proof together with the extended case of \( \mathcal{L}S(p, G_m, G_n) \) during this visit.

1. Extended \( \gamma \)-factors

Let \( G_n \) be either the general linear group \( GL_n \) or a split classical group of rank \( n \). Given a ring \( R \) and an algebraic group \( G \) defined over \( R \), we often let \( G \) denote its group of rational points. Given a non-archimedean local field \( F \), let \( O_F \) denote its ring of integers, \( p_F \) its maximal ideal, \( \varpi_F \) a uniformizer, and \( q_F \) the cardinality of
its residue field. Given a global function field field $k$, we let $q$ denote the cardinality of its field of constants; for a place $v$ of $k$, we let $q_v$ be the cardinality of the residue field of $k_v$. Given a representation $\sigma$, we let $\tilde{\sigma}$ denote its contragredient.

1.1. **Local notation.** Let $\mathfrak{ls}(p, G_m, G_n)$ denote the class of quadruples $(F, \tau, \pi, \psi)$ consisting of: a non-archimedean local field $F$ of characteristic $p$; irreducible generic representations $\tau$ of $G_m = G_m(F)$ and $\pi$ of $G_n = G_n(F)$; and, a non-trivial character $\psi$ of $F$.

Given $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, G_m, G_n)$, we call it tempered (resp. discrete series, supercuspidal) if $\tau$ and $\pi$ are tempered representations (resp. discrete series, supercuspidal).

1.2. **Global notation.** Let $\mathcal{LS}(p, G_m, G_n)$ denote the class of quadruples $(k, \tau, \pi, \psi)$ consisting of: a global function field $k$ of characteristic $p$; globally generic cuspidal automorphic representations $\tau = \otimes_v \tau_v$ of $G_m = G_m(k_v)$ and $\pi = \otimes_v \pi_v$ of $G_n = G_n(k_v)$; and, a non-trivial character $\psi = \otimes_v \psi_v$ of $k \setminus k_v$.

**Remark.** We often write $\mathfrak{ls}(p)$ and $\mathcal{LS}(p)$ when there is no need to specify the groups $G_m$ and $G_n$.

1.3. **$L$-groups, principal series and partial $L$-functions.** The connected components of the $L$-groups for the split classical groups are embedded into an appropriate dual group of $GL_N$ according to the following table:

| $G_n$     | $L G_0^0 \hookrightarrow L GL_N^0$ | $GL_N$ |
|----------|------------------------------------|--------|
| $SO_{2n+1}$ | $\text{Sp}_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $SO_{2n}$   | $\text{SO}_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $Sp_{2n}$   | $\text{SO}_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$ | $GL_{2n+1}$ |

We include the possibility of $G_n = GL_n$, where we take $N = n$. Also, we let $\rho_n$ denote the standard representation of $GL_n(\mathbb{C})$.

Let $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, G_m, G_n)$ and assume that $\tau$ and $\pi$ are unramified principal series. Then, the Satake parametrization gives semisimple conjugacy classes $\{A_\tau\}$ in $L G_0^0 \hookrightarrow GL_M(\mathbb{C})$ and $\{B_\pi\}$ in $L G_0^0 \hookrightarrow GL_N(\mathbb{C})$. Then, $L$-functions for unramified principal series representations are defined by

$$L(s, \tau \times \pi) = \frac{1}{\det(I - \rho_M(A_\tau) \otimes \rho_N(B_\pi)q^{-s})}.$$ 

Given $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, G_m, G_n)$, we take $S$ to be a finite set of places of $k$ such that $\tau$, $\pi$ and $\psi$ are unramified for $v \notin S$. The corresponding partial $L$-functions are defined by

$$L^S(s, \tau \times \pi) = \prod_{v \notin S} L(s, \tau_v \times \pi_v).$$

We can show that partial $L$-functions converge on a right half plane; in fact, they have a meromorphic continuation to a rational function on $q^{-s}$.
1.4. **Axioms for a system of γ-factors.** The Langlands-Shahidi method in positive characteristic allows us to produce a system of rational functions \( \gamma(s, \tau \times \pi, \psi) \in \mathbb{C}(q_F^{-s}) \) on \( \mathfrak{I}(p, \text{GL}_m, \text{G}_n) \). In this article, we concoct a system of extended γ-factors on \( \mathfrak{I}(p, \text{G}_m, \text{G}_n) \). Extended γ-factors can be characterized by a list of local properties together with their role in the global functional equation.

(i) (Naturality). Let \((F, \tau, \pi, \psi) \in \mathfrak{I}(p)\), and let \( \eta : F' \to F \) be an isomorphism of local fields. Let \((F', \tau', \pi', \psi') \in \mathfrak{I}(p)\) be the quadruple obtained from \((F, \tau, \pi, \psi)\) via \( \eta \). Then

\[
\gamma(s, \tau \times \pi, \psi) = \gamma(s, \tau' \times \pi', \psi').
\]

(ii) (Isomorphism). Let \((F, \tau, \pi, \psi) \in \mathfrak{I}(p)\). If \((F, \tau', \pi', \psi') \in \mathfrak{I}(p)\) is such that \( \tau \simeq \tau' \) and \( \pi \simeq \pi' \), then

\[
\gamma(s, \tau \times \pi, \psi) = \gamma(s, \tau' \times \pi', \psi).
\]

(iii) (Compatibility with class field theory). Let \((F, \chi_1, \chi_2, \psi) \in \mathfrak{I}(p, \text{GL}_1, \text{GL}_1)\). Then

\[
\gamma(s, \chi_1 \times \chi_2, \psi) = \gamma(s, \chi_1 \chi_2, \psi).
\]

(iv) (Multiplicativity). Let \((F, \tau_0, \pi_j, \psi) \in \mathfrak{I}(p, \text{G}_{m_0}, \text{G}_{n_0})\), \(0 \leq i \leq d, 0 \leq j \leq e; \text{G}_{m_0} \text{ and } \text{G}_{n_0} \text{ can be classical groups or general linear groups; } \text{G}_{m_i} = \text{GL}_{m_i} \text{ and } \text{G}_{n_j} = \text{GL}_{n_j} \text{ for } 1 \leq i \leq d, 1 \leq j \leq e. \) Set \( m = \sum m_i \) and \( n = \sum n_j \). If \( \text{G}_{m_0} \) (resp. \( \text{G}_{n_0} \)) is a classical group, take \( \text{G}_m \) (resp. \( \text{G}_n \)) to be a classical group of the same type. Let \( \text{P}_m \) (resp. \( \text{P}_n \)) be the standard parabolic subgroup of \( \text{G}_m \) (resp. \( \text{G}_n \)) with Levi \( \text{M}_m = \prod_{i=1}^d \text{GL}_{m_i} \times \text{G}_{m_0} \) (resp. \( \text{M}_n = \prod_{j=1}^e \text{GL}_{n_j} \times \text{G}_{n_0} \)).

First, assume \( m_0 \geq 1 \) and \( n_0 \geq 1 \). Let \( \pi \) be the generic constituent of

\[
\text{ind}_{\text{P}_m}(\tau_1 \otimes \cdots \otimes \tau_d \otimes \tau_0),
\]

and let \( \pi \) be the generic constituent of

\[
\text{ind}_{\text{P}_n}(\pi_1 \otimes \cdots \otimes \pi_e \otimes \pi_0).
\]

When \( m_0 = 0 \) (resp. \( n_0 = 0 \)) we make the following conventions: take \( \tau_0 \) (resp. \( \pi_0 \)) to be the trivial character of \( \text{GL}_1(F) \) if \( \text{G}_m \) (resp. \( \text{G}_n \)) is a symplectic group; in all other cases, we interpret \( \gamma \)-factors involving \( \tau_0 \) (resp. \( \pi_0 \)) to be trivial.

(iv.a) If both \( \text{G}_m \) and \( \text{G}_n \) are classical groups, then

\[
\gamma(s, \tau \times \pi, \psi) = \gamma(s, \tau_0 \times \pi_0)
\]

\[
\times \prod_{i=1}^d \gamma(s, \tau_i \times \pi_0, \psi) \gamma(s, \pi_0 \times \tau_0, \psi) \prod_{j=1}^e \gamma(s, \tau_0 \times \pi_j, \psi) \gamma(s, \tau_0 \times \pi_j, \psi)
\]

\[
\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \gamma(s, \tau_h \times \pi_l, \psi) \gamma(s, \tau_h \times \pi_l, \psi) \gamma(s, \pi_l \times \tau_l, \psi) \gamma(s, \pi_l \times \tau_l, \psi).
\]

(iv.b) If \( \text{G}_m = \text{GL}_m \) and \( \text{G}_n \) is a classical group, then

\[
\gamma(s, \tau \times \pi, \psi) = \prod_{i=0}^d \gamma(s, \tau_i \times \pi_0, \psi) \times \prod_{j=0}^d \prod_{j=1}^e \gamma(s, \tau_i \times \pi_j, \psi) \gamma(s, \tau_i \times \pi_j, \psi).
\]
(iv.c) If \( G_m = \text{GL}_m \) and \( G_n = \text{GL}_n \), then
\[
\gamma(s, \tau \times \pi, \psi) = \prod_{i,j} \gamma(s, \tau_i \times \pi_j, \psi).
\]

(v) (Dependence on \( \psi \)). Let \((F, \tau, \pi, \psi) \in \mathfrak{Is}(p, G_m, G_n)\). Given \( a \in F^\times \), let \( \psi^a \) denote the character of \( F \) given by \( \psi^a(x) = \psi(ax) \). Then
\[
\gamma(s, \tau \times \pi, \psi^a) = \omega_\tau(a)^h \omega_\pi(a)^l |a|^{\frac{h(l(s-\frac{1}{2}))}{2}} \gamma(s, \tau \times \pi, \psi),
\]
where \( h = 2m \) if \( G_m = \text{SO}_{2m} \), \( \text{SO}_{2m+1} \); \( h = 2m+1 \) if \( G_m = \text{Sp}_{2m} \); \( h = m \) if \( G_m = \text{GL}_m \); and, similarly for \( l \), depending on \( G_n \).

(vi) (Stability). Let \((F, \eta, \pi_i, \psi) \in \mathfrak{Is}(p, \text{GL}_1, G_n)\), \( i = 1, 2 \), where \( \pi_1 \) and \( \pi_2 \) have the same central character and \( \eta \) is highly ramified. Then
\[
\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi).
\]

(vii) (Functional equation). Let \((k, \tau, \pi, \psi) \in \mathcal{LS}(p)\), then
\[
L^S(s, \tau \times \pi) = \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v)L^S(1 - s, \tilde{\tau} \times \tilde{\pi}).
\]

1.5. Theorem. There exists a system of \( \gamma \)-factors on \( \mathfrak{Is}(p, \text{GL}_m, G_n) \) satisfying properties (i) – (vii). If \( p \neq 2 \), there exists a system of \( \gamma \)-factors on \( \mathfrak{Is}(p, G_m, G_n) \) satisfying properties (i) – (vii). Any system of \( \gamma \)-factors satisfying properties (i) – (vii) is uniquely determined.

2. Existence

A treatise of \( \gamma \)-factors, \( L \)-functions and root numbers for general linear groups is presented in a self contained manner within the Langlands-Shahidi method in [11] and the appendix [6]. We now complete the study begun in [10] for the cases involving split classical groups.

2.1. The Langlands-Shahidi local coefficient for the split classical groups in positive characteristic. Let \( G \) be a split classical group of rank \( l \) and let \( B = TU \) be the Borel subgroup of upper triangular matrices with maximal torus \( T \) and unipotent radical \( U \). Let \( P \) be the standard maximal parabolic subgroup of \( G \) with maximal Levi \( M = \text{GL}_m \times G_n \), \( l = m + n \), and unipotent radical \( N \). Given \((F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \text{GL}_m, G_n)\), we can form a smooth irreducible generic representation \( \sigma = \tau \otimes \pi \) of \( M \).

Let \( \Sigma \) denote the roots of \( G \) with respect to the maximal torus \( T \), \( \Sigma^+ \) the positive roots, and \( \Delta \) a basis fixed by our choice of Borel subgroup. Let \( N_\alpha \) denote the one parameter subgroup associated to \( \alpha \in \Sigma \). The surjection \( N \rightarrow N/\prod_{\alpha \in \Sigma^+ - \Delta} N_\alpha \simeq \prod_{\alpha \in \Delta} N_\alpha \) allows us to extend the character \( \psi \) of \( F \) to a non-degenerate character \( \psi \) of \( N \) by setting \( \psi(\sum_{\alpha \in \Delta} x_\alpha) = \prod_{\alpha \in \Delta} \psi(x_\alpha) \). Let \( \alpha_m \in \Delta \) be such that \( P \) is the maximal parabolic subgroup corresponding to \( \Delta - \{\alpha_m\} \). Then, we let \( \psi_M \) be the character of \( U_M = U \cap M \) obtained from \( \psi \) via the surjection \( U_M \rightarrow \prod_{\alpha \in \Delta - \{\alpha_m\}} N_\alpha \).

For every \( \alpha \in \Delta \), we fix an embedding for the corresponding semisimple rank one group \( G_\alpha \) into \( G \) and fix a representative \( w_\alpha \) for the corresponding Weyl group element as in [11]. We abuse notation and identify Weyl group elements with their fixed representatives. Let \( w_0 = w_l w_{l,M} \), where \( w_l \) is the longest Weyl group element of \( G \) and \( w_{l,M} \) is the longest Weyl group element with respect to
Then, the non-degenerate characters \( \psi \) of \( N \) and \( \psi_M \) of \( U_M \) are \( w_0 \)-compatible, i.e., \( \psi(u) = \psi_M(w_0^{-1}uw_0) \) for \( u \in U_M \).

Let \( \sigma = \tau \otimes \tilde{\pi} \) be a \( \psi_M \)-generic representation of \( M = \text{GL}_m(F) \times G_n \). We then let \( I(s, \sigma) \) be the unitarily induced representation

\[
\text{ind}^G_F((\det(\cdot))^s, \tau \otimes \tilde{\pi}).
\]

Let \( V(s, \sigma) \) denote the space of \( I(s, \sigma) \). If \( \lambda_M \) is a Whittaker functional for \( \sigma \), then \( I(s, \sigma) \) is \( \psi \)-generic for the Whittaker functional \( \lambda_\psi \) given by

\[
\lambda_\psi(s, \sigma)f = \int_N \lambda_M(w_0^{-1}n)\overline{\psi}(n)\,dn,
\]

where \( f \in V(s, \sigma) \). The integral on the right hand side converges as a principal value integral over compact open subgroups of \( N \).

We also have an intertwining operator \( \Lambda(s, \sigma, w_0) : V(s, \sigma) \to V(-s, w_0^{-1}(\sigma)) \), given by

\[
\Lambda(s, \sigma, w_0)f(g) = \int_N f(w_0^{-1}ng)\,dn,
\]

where we write \( w_0(\sigma) \) for the representation given by \( w_0(\sigma)(x) = \sigma(w_0^{-1}xw_0) \). It converges for \( \Re(s) \gg 0 \) and extends to a rational operator on \( q_F^{-s} \).

The local coefficient is then defined using the uniqueness property of Whittaker models and the relationship

\[
\lambda_\psi(s, \sigma)f = C_\psi(s, \sigma, w_0)\lambda_\psi(-s, w_0(\sigma))\Lambda(s, \sigma, w_0)f.
\]

The local coefficient \( C_\psi(s, \sigma, w_0) \) is a rational function on \( q_F^{-s} \).

2.2. The case of a Siegel Levi subgroup. The case of a Siegel Levi subgroup \( M \simeq \text{GL}_n \) of \( G_n \) is studied in [4] when \( G_n = \text{SO}_{2n} \) or \( \text{SO}_{2n+1} \). In these cases the Langlands-Shahidi method allows us to study exterior square and symmetric square \( L \)-functions; the case of a Siegel Levi subgroup of \( \text{Sp}_{2n} \) is included in [11]. These cases provide an important step in the Langlands-Shahidi method for the split classical groups. Given an irreducible generic representation \( \tau \) of \( \text{GL}_n(F) \), we define

\[
C_\psi(s, \tau, w_0) = \begin{cases} 
\gamma(s, \tau, \text{Sym}^2\rho_n, \psi) & \text{if } G = \text{SO}_{2n+1} \\
\gamma(s, \tau, \wedge^2\rho_n, \psi) & \text{if } G = \text{SO}_{2n}.
\end{cases}
\]

Here, \( \rho_n \) is the standard representation of \( \text{GL}_n(C) \), the dual group of \( \text{GL}_n \), and \( \gamma(s, \tau, \psi) \) is a Godement-Jacquet \( \gamma \)-factor. For unramified principal series representations, the above definition agrees with the Satake parametrization and provides a definition of exterior square and symmetric square local factors for general smooth representations. We now state the main result of [4], which shows that Langlands-Shahidi \( \gamma \)-factors are in accordance with the local Langlands correspondence in positive characteristic [9].

**Theorem.** Let \( \tau \) be an irreducible smooth representation of \( \text{GL}_n(F) \) and let \( \sigma \) be an \( n \)-dimensional Frobenius-semisimple \( \ell \)-adic representation of the Weil-Deligne group in the isomorphism class \( \sigma(\tau) \) corresponding to \( \tau \) via the local Langlands correspondence. Then

\[
\gamma(s, \tau, r \circ \rho_n, \psi) = \gamma(s, r \circ \sigma, \psi),
\]

where \( r = \text{Sym}^2 \) or \( \wedge^2 \).
For the remaining case of a Siegel Levi subgroup of a split classical group, let $\gamma(s, \tau, \psi)$ denote a Godement-Jacquet $\gamma$-factor. Then

$$C_{\psi}(s, \tau, w_0) = \gamma(s, \tau, \psi)\gamma(2s, \tau, \lambda_2^2 \rho_n, \psi)$$ if $G = Sp_{2n}$.

2.3. The case of $\mathfrak{sl}(p, GL_m, G_n)$. The study of $\gamma$-factors, $L$-functions and root numbers on $\mathfrak{sl}(p, GL_m, G_n)$ and $LS(p, GL_m, G_n)$ was begun in [10]. We now gather the necessary results that establish the existence part of Theorem [13] in these cases; we use the conventions of [11] regarding Weyl group element representatives and normalization of Haar measures. Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p, GL_m, G_n)$ and let $\sigma = \tau \otimes \tilde{\pi}$.

We first assume that $\sigma$ is a $\psi_M$-generic representation of $M$.

Having defined exterior and symmetric square $\gamma$-factors, which are in accordance with the local Langlands correspondence for $GL_n$, the $\gamma$-factors $\gamma(s, \tau \times \pi, \psi)$ are defined via the local coefficient:

$$C_{\psi}(s, \sigma, w_0) = \begin{cases} 
\gamma(s, \tau \times \pi, \psi)\gamma(2s, \tau, \text{Sym}^2 \rho_n, \psi) & \text{if } G = SO_{2n+1} \\
\gamma(s, \tau \times \pi, \psi)\gamma(2s, \tau, \lambda_2^2 \rho_n, \psi) & \text{if } G = Sp_{2n} \text{ or } SO_{2n}.
\end{cases}$$

Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p, GL_m, G_n)$, with $\sigma = \tau \otimes \tilde{\pi}$ $\psi_M$-generic. An isomorphism of local fields $\eta : F' \to F$, takes normalized Haar measures on $N(F)$ to normalized Haar measures on $N(F')$. Hence, the local coefficients $C_{\psi}(s, \sigma, w_0)$ and $C_{\psi'}(s, \sigma', w_0)$ are equal. Also, we have an isomorphism property for the local coefficient given two isomorphic $\psi_M$-generic representations $\sigma$ and $\sigma'$ of $M$. Thus, properties (i) and (ii) follow for $\psi_M$-generic $\sigma$ and $\sigma'$. Property (iii) is included in the list of semisimple rank one cases of [11]. Property (iv.b) for the classical groups is Theorem 6.7 of [10], where it is explicitly stated.

We now discuss the relationship for $\gamma$-factors as the character varies. Given a triple $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p, GL_m, G_n)$, the representation $\sigma = \tau \otimes \tilde{\pi}$ is generic with respect to a non-degenerate character $\chi_M$ of $U_M$. We will take $\chi$ to be a non-degenerate character of $U$, which is $w_0$-compatible with $\chi_M$. Now, given the group $G$, we can embed it into a group $\tilde{G}$ with Borel subgroup $B = TU$. The group $\tilde{G}$ has the same derived group as $G$ and has $H^1(Z_G) = \{1\}$ (see § 5 of [10]). Thus, $\tilde{T}$ acts transitively on the set of non-degenerate characters of $U$. Let $t \in T$ be such that the non-degenerate character $\tilde{\psi}_M$ of $U_M$ is equal to $\chi_{t, M}$, where $\chi_M(u) = \chi_M(t^{-1}ut)$. The character $\chi$ is taken so that $\psi = \chi_t$ on $U$, and $w_0$-compatibility is preserved for the action of $t \in \tilde{T}$ on the non-degenerate characters. Let $\sigma_t$ be defined by $\sigma_t(m) = \sigma(t^{-1}mt)$. It is a $\psi_M$-generic representation. Then, we have $\gamma$-factors defined on all of $\mathfrak{sl}(p, GL_m, G_n)$ via the local coefficient as follows

$$\gamma(s, \pi, \psi) = C_{\psi}(s, \pi_t, w_0).$$

We have a formula for the local coefficient when the character $\psi$ varies. Written explicitly for $\gamma$-factors gives property (v) on $\mathfrak{sl}(p, GL_m, G_n)$.

Property (vi), stability of $\gamma$-factors in positive characteristic, is the content of Theorem 6.12 of [10] for the split classical groups. This includes the case of characteristic 2.

Finally, let $(F, \tau, \pi, \psi) \in LS(p, GL_m, G_n)$. The crude functional equation for the local coefficient of split classical groups over function fields, Theorem 5.14 of [loc. cit.], reads

$$LS(s, \tau \times \pi)L^S(2s, \tau, r \circ \rho_m) = \prod_{v \in S} C_{\psi_v}(s, \tau_v \otimes \tilde{\pi}_v, w_0)L^S(1-s, \tilde{\tau} \times \tilde{\pi})L^S(1-2s, \tilde{\tau}, r \circ \rho_m),$$
where \( r = \text{Sym}^2 \) or \( \wedge^2 \) depending on the classical group and we use the conventions of \([11]\).

For every \( v \in S \) we have that

\[
C_{\psi_v} = \gamma(s, \tau_v \times \pi_v, \psi_v)\gamma(2s, \tau_v, r \circ \rho_m, \psi_v).
\]

The study of exterior and symmetric square \( \gamma \)-factors begun in \([10]\) is completed in \([11]\). We have the functional equation:

\[
L^S(s, \tau, r \circ \rho_m) = \prod_{v \in S} \gamma(s, \tau_v, r \circ \rho_m, \psi_v)L^S(1 - s, \tilde{\tau}, r \circ \rho_m).
\]

Combining this equation with the crude functional equation of the local coefficient gives property (vii) for the corresponding \( \gamma \)-factors:

\[
L^S(s, \tau \times \pi) = \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v)L^S(1 - s, \tilde{\tau} \times \tilde{\pi}).
\]

Therefore, we can conclude that the existence part of Theorem 1.5 holds in the case of \( \mathfrak{sl}(p, \text{GL}_m, G_n) \). Notice that no assumption on the characteristic is made for this part of the theorem.

2.4. The general case. A system of \( \gamma \)-factors on \( \mathfrak{sl}(p, \text{GL}_m, G_n) \) gives a system of \( \gamma \)-factors on \( \mathfrak{sl}(p, G_m, \text{GL}_n) \) via the relationship

\[
\gamma(s, \pi \times \tau, \psi) := \gamma(s, \tau \times \pi, \psi),
\]

for \((F, \pi, \tau, \psi) \in \mathfrak{sl}(p, G_m, \text{GL}_n)\). We now build upon § 9 of \([10]\), which is written under the assumption \( p \neq 2 \). We make this assumption for the rest of this section.

Also, we focus on the case of two classical groups \( G_m \) and \( G_n \); we let \( M \) and \( N \) denote the positive integers obtained from \( m \) and \( n \) via the table on § 1.3.

First, let \((F, \pi, \tau, \psi) \in \mathfrak{sl}(p, G_m, G_n)\) be such that \( \tau \) is supercuspidal. By Proposition 9.4 of \([10]\), there exists a generic representation \( T \) of \( \text{GL}_M \) such that

\[
\gamma(s, \tau \times \rho, \psi) = \gamma(s, T \times \rho, \psi),
\]

for every supercuspidal representation \( \rho \) of \( \text{GL}_n(F) \). The representation \( T \) is unique due to Théorème 1.1 of \([3]\); it is called the local functorial lift of \( \tau \).

An irreducible generic discrete series representation \( \tau \) of a classical group can be described in terms of its inducing data

\[
\tau \hookrightarrow \text{ind}^{G_m}_{F_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta'_1 \otimes \cdots \otimes \delta'_c \otimes \tau_0),
\]

where the \( \delta_i \)'s and the \( \delta'_i \)'s are essentially square integrable representations of \( \text{GL}_m(F) \) and \( \tau_0 \) is an irreducible generic supercuspidal representation of \( G_{m_0} \), with \( G_{m_0} \) is a classical group of the same type as \( G_m \). Following the results of Moeglin-Tadić \([13]\), this is made precise in equation (9.1) of \([10]\). The case of \( m_0 = 0 \) is allowed, with appropriate interpretations for the corresponding formulas. If \( T_0 \) is the local functorial lift of \( \tau_0 \), then the local functorial lift \( T \) of \( \tau \) is the generic constituent of

\[
\text{ind}^{G_m}_{F_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta'_1 \otimes \cdots \otimes \delta'_c \otimes T_0 \otimes \delta''_1 \otimes \cdots \otimes \delta''_c \otimes \delta_1 \cdots \otimes \delta_1).
\]

The representation \( T \) is a self-dual tempered representation of \( \text{GL}_M(F) \).

Now, an irreducible generic tempered representation of \( G_m \) is a constituent of

\[
\text{ind}^{G_m}_{F_m}(\delta_1 \otimes \cdots \delta_d \otimes \tau_0),
\]
where the $\delta_i$’s are discrete series representations of $GL_m(F)$ and $\tau_0$ is a generic discrete series of $G_m$, where $G_m$ is a classical group of the same kind as $G_m$.

Then, the local functorial lift $T$ of $\tau$ is given by

$$\text{ind}_{P_m}^{G_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes T_0 \otimes \tilde{\delta}_d \otimes \tilde{\delta}_1),$$

where $T_0$ is the local functorial lift of the generic discrete series representation $\tau_0$.

An arbitrary irreducible generic representation $\tau$ of $G_m$ can be described via the
work of Muić on the standard module conjecture $[14]$. Then,

$$\tau = \text{ind}_{P_m}^{G_m}(\tau_1 \nu^{\tau_1} \otimes \cdots \otimes \tau_d \nu^{\tau_d} \otimes \tau_0).$$

Here, each $\tau_i$ is a tempered representation of $GL_m(F)$; $\tau_0$ is a generic tempered representation of $G_{m_0}$, where $G_{m_0}$ is a classical group of the same kind as $G_m$; and, $\nu = |\text{det}(\cdot)|_F$. If $G_m = SO_{2n}$, and $\tau_0$ is the trivial representation of $G_{m_0}(F)$ and $m_d = 1$, the above formula needs the following modification

$$\tau = \text{ind}_{P_m}^{G_m}(\tau_1 \nu^{\tau_1} \otimes \cdots \otimes \tau_d \nu^{\tau_d}),$$

where we have $0 < |r_d| < r_{d-1} < \cdots < r_1$. In all other cases, it is given by equation (2.5), where the exponents can be taken so that $0 < r_d < \cdots < r_1$.

The local functorial lift $T$ of $\tau$ is then given as the unique irreducible quotient of

$$\text{ind}_{P_m}^{GL_m(F)}(\tau_1 \nu^{\tau_1} \otimes \cdots \otimes \tau_d \nu^{\tau_d} \otimes T_0 \otimes \tilde{\tau}_d \nu^{-\tau_d} \otimes \cdots \otimes \tilde{\tau}_1 \nu^{\tau_1}),$$

where $T_0$ is the local functorial lift of $\tau_0$, with appropriate modifications if the induced representation is given by (2.6). The local lift has the property that

$$\gamma(s, \tau \times \rho, \psi) = \gamma(s, T \times \rho, \psi),$$

for any irreducible generic representation $\rho$ of $GL_n(F)$.

With a description of the local image of functoriality, we can now obtain a system of extended $\gamma$-factors. Given $(F, \tau, \pi, \psi) \in \mathcal{I}(p, G_m, G_n)$ let $(F, T, \pi, \psi) \in \mathcal{I}(p, GL_M, G_n)$ be such that $T$ is the local functorial lifts of $\tau$. Then, we define

$$\gamma(s, \tau \times \pi, \psi) := \gamma(s, T \times \pi, \psi).$$

We have that $\gamma(s, \tau \times \pi, \psi) = \gamma(s, \pi \times \tau, \psi)$ for every $(F, \tau, \pi, \psi) \in \mathcal{I}(p, G_m, G_n)$. Furthermore, if $(F, T, \Pi, \psi) \in \mathcal{I}(p, GL_M, GL_N)$, where $T$ and $\Pi$ are the local functorial images of $\tau$ and $\pi$, then

$$\gamma(s, \tau \times \pi, \psi) = \gamma(s, T \times \Pi, \psi).$$

It is now an exercise to show that properties (i) and (ii) are verified; property (iii) remains as before; and, our definition is indeed compatible with multiplicativity, property (iv). The dependence on $\psi$ can now be obtained from the corresponding property for $\gamma(s, T \times \Pi, \psi)$ (see property (iv) in the main Theorem of $[6]$). And, stability remains as before.

Given $(k, \tau, \pi, \psi) \in \mathcal{L}(p, G_m, G_n)$ let $T$ and $\Pi$ be the functorial lifts of $\tau$ and $\pi$. This is possible via Theorem 9.1 of $[10]$. In fact, the work of Ginzburg, Rallis and Soudry allows us to give a precise description of the image of functoriality $[17]$. The global functorial lift $T$ of $\tau$ can be expressed as an isobaric sum

$$T = T_1 \boxplus \cdots \boxplus T_d,$$

where each $T_i, 1 \leq i \leq d$, is a unitary self-dual cuspidal automorphic representation of $GL_{M_i}(A_k)$. Also, $\Pi_i \nmid \Pi_j$ whenever $i \neq j$. Furthermore, if $S$ is a sufficiently large finite set of places of $k$, then
(i) $L^S(s, \Pi_k, \wedge^2 \rho_m)$ has a pole at $s = 1$, if $G_m = \text{SO}_{2m+1}$;
(ii) $L^S(s, \Pi_k, \text{Sym}^2 \rho_m)$ has a pole at $s = 1$, if $G_m = \text{SO}_{2m}$ or $\text{Sp}_{2m}$.

We can similarly express the global functorial lift $\Pi$ of $\pi$ as an isobaric sum.

The functional equation for extended $\gamma$-factors can then be obtained from the above description of the global image and the functional equation for Rankin-Selberg $\gamma$-factors, i.e.,

$$L^S(s, \tau \times \pi) = L^S(s, T \times \Pi)$$

$$= \prod_{v \in S} \gamma(s, T_v \times \Pi_v, \psi_v)L^S(1 - s, T \times \Pi)$$

$$= \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v)L^S(1 - s, \tau \times \pi),$$

for every $(k, \tau, \pi, \psi) \in \mathcal{L}(p, G_m, G_n)$.

3. Uniqueness

In the cases involving classical groups, we use a variation of a local-to-global result of Vignéras, which follows from the proof of Theorem 2.2 of [18]. We note that, over a global function field, a place at infinity plays the role that archimedean places play over number fields; the notion of an automorphic representation over function fields is independent of the choice of place at infinity. To prove the following proposition, we start with a local field $F$ and take a global field $k$ such that $k_{v_0} \simeq F$ at a place $v_0$ of $k$. We fix two different places $v_1$ and $v_2$ over the same function field $k$. Then one can apply the observation made on p. 469 of [loc. cit.] to the construction of globally generic cuspidal automorphic representations $\tau$ and $\pi$ from the local representations $\tau_0$ and $\pi_0$. Throughout this section, we impose no restriction on $p$.

3.1. Proposition. Let $(F, \tau_0, \pi_0, \psi_0) \in \mathfrak{L}(p, G_m, G_n)$ be supercuspidal. Then, there exists a $(k, \tau, \pi, \psi) \in \mathcal{L}(p, G_m, G_n)$ and a set of places $S = \{v_0, v_1, v_2\}$ of $k$ such that

(i) $k_{v_0} \simeq F$;
(ii) $\tau_{v_0} \simeq \tau_0$ and $\pi_{v_0} \simeq \pi_0$;
(ii) $\tau_v$ is an unramified principal series for $v \notin \{v_0, v_1\}$;
(iv) $\pi_v$ is an unramified principal series for $v \notin \{v_0, v_2\}$.

3.2. Uniqueness for $\mathfrak{L}(p, GL_1, G_n)$. Let $\gamma$ be a rule on $\mathfrak{L}(p, GL_1, G_n)$ which assigns to every $(F, \chi, \pi, \psi) \in \mathfrak{L}(p, GL_1, G_n)$ a rational function on $q_F^{-s}$ satisfying properties (i)-(vii). Using property (iv), we can reduce to the case when $\pi$ is supercuspidal.

Given $(F, \chi_0, \pi_0, \psi_0) \in \mathfrak{L}(p, GL_1, G_n)$ supercuspidal we can lift it to a global $(k, \chi, \pi, \psi) \in \mathcal{L}(p, GL_1, G_n)$ where $\pi_v$ is an unramified principal series for $v \notin \{v_0, v_1\}$ as in Proposition 3.1. However, in this situation we can take a character $\chi_{v_0}$ of $k_{v_0}^{\times}$ which is isomorphic to $\chi_0$ and a highly ramified character $\chi_{v_2}$ of $k_{v_2}^{\times}$. We then apply the Grundwald-Wang theorem of class field theory [1], in order to lift $\chi_{v_0}$ and $\chi_{v_2}$ to a grössencharakter $\chi : k^{\times} \backslash k_k^{\times} \rightarrow \mathbb{C}^{\times}$. From properties (i) and (ii) we have that

$$\gamma(s, \chi_{v_0} \times \pi_{v_0}, \psi_{v_0}) = \gamma(s, \chi_{v_0} \times \pi_{v_0}, \psi_{v_0}),$$

and we can assume $\psi_{v_0}$ is obtained from $\psi_0$ using property (v) if necessary.
For every \( v \notin \{ v_0, v_2 \} \), we have that
\[
\text{ind}_{B_n}^G(\mu_1, v \otimes \cdots \otimes \mu_n, v),
\]
where \( \mu_1, v, \ldots, \mu_n, v \) are unramified characters. If \( G_n = \text{SO}_{2n} \) or \( \text{SO}_{2n+1} \), then
\[
\gamma_i(s, \chi_v \times \pi, \psi) = \prod_{i=1}^{n} \gamma_i(s, \chi_v \mu_i, v, \psi) \gamma_i(s, \chi_v \mu_i^{-1}, v).
\]
And, if \( G_n = \text{Sp}_{2n} \), then
\[
\gamma_i(s, \chi_v \times \pi, \psi) = \gamma(s, \chi_v, \psi) \prod_{i=1}^{n} \gamma(s, \chi_v \mu_i, v, \psi) \gamma(s, \chi_v \mu_i^{-1}, v).
\]
The \( \gamma \)-factors on the right hand side of the previous two equations are abelian \( \gamma \)-factors of class field theory. Hence, the rule \( \gamma \) is uniquely determined at these places.

At the place \( v_2 \), let \( \xi_1, \ldots, \xi_n \), be characters of \( \text{GL}_1(F) \) such that the restriction of \( \xi_1 \otimes \cdots \otimes \xi_n \) to the center of \( G_n \) agrees with the central character \( \omega_{\pi} \) of \( \pi \). Let \( \tau_{v_2} \) be the generic constituent of
\[
\text{ind}_{B_n}^G(\xi_1 \otimes \cdots \xi_n).
\]
Since we have \( \chi_{v_2} \) sufficiently ramified we can use property (vi) to obtain
\[
\gamma(s, \chi_{v_2} \times \pi_{v_2}, \psi_{v_2}) = \gamma(s, \chi_{v_2} \times \tau_{v_2}, \psi_{v_2}).
\]
Then, using multiplicativity, \( \gamma(s, \chi_{v_2} \times \tau_{v_2}, \psi_{v_2}) \) can be written as a product of abelian \( \gamma \)-factors similar to equations (3.1) and (3.2) above. Any system of \( \gamma \)-factors satisfying properties (i)-(vi) of the Theorem gives the same result at \( v_2 \).

Now, let \( S \) be a finite set of places of \( k \) including \( v_0 \) and such that \( \chi_v, \pi_v, \) and \( \psi_v \) are unramified for \( v \notin S \). Then, property (vii) gives
\[
L^S(s, \tau \times \pi) = \gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma(s, \tau_v \times \pi_v, \psi_v) L^S(1-s, \tilde{\tau}_v \times \tilde{\pi}_v).
\]
Since \( \gamma \)-factors are determined for every \( v \notin S - \{v_0\} \) by the above discussion, we can conclude that \( \gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0}) \) is completely determined by equation (3.3).

3.3. Uniqueness in general. Let \( \gamma \) be a rule on \( \mathfrak{g}(p) \) which assigns to every quadruple \( (F, \tau, \pi, \psi) \in \mathfrak{g}(p) \) a rational function on \( q_F^{1/2} \) satisfying properties (i)-(vii). Using property (iv), we can reduce to the supercuspidal case.

Take a fixed supercuspidal \( (F, \tau_0, \pi_0, \psi_0) \in \mathfrak{g}(p, G_m, G_n) \) and lift it to a global \( (k, \tau, \pi, \psi) \in \mathcal{L}S(p, G_m, G_n) \) via Proposition 3.3 properties (i), (ii), and (v). Let \( B_m \) (resp. \( B_m \)) be the Borel subgroup of \( G_m \) (resp. \( G_n \)) of upper triangular matrices. For every \( v \notin \{ v_0, v_1 \} \), let \( \chi_{1, v}, \ldots, \chi_{m, v} \) be unramified characters of \( \text{GL}_1(k_v) \) such that \( \tau_v \) occurs as a subrepresentation of the unitarily induced representation
\[
\text{ind}_{B_m}^{G_m} (\chi_{1, v} \otimes \cdots \otimes \chi_{m, v}).
\]
For every \( v \notin \{ v_0, v_2 \} \), let \( \mu_{1, v}, \ldots, \mu_{n, v} \) be unramified characters of \( \text{GL}_1(k_v) \) such that \( \pi_v \) occurs as a subrepresentation of the unitarily induced representation
\[
\text{ind}_{B_m}^{G_m} (\mu_{1, v} \otimes \cdots \otimes \mu_{n, v}).
\]
Take \( v \notin S \), then both \( \tau_v \) and \( \pi_v \) are unramified. We can in then use properties (iii) and (iv) to show that:
\[(a) \text{ If both } G_m \text{ and } G_n \text{ are classical groups, then}
\]
\[
\gamma(s, \tau_v \times \pi_v, \psi_v) = \prod_{i=1}^{d} \gamma(s, \chi_i, v \mu_{0,v}, \psi_v) \prod_{j=1}^{e} \gamma(s, \chi_0, v \mu_{j,v}, \psi_v)
\]
\[
\times \prod_{1 \leq h \leq d \leq i \leq e} \gamma(s, \chi_h, v \mu_{j,v}, \psi_v) \gamma(s, \chi_h, v \mu_{i,v}^{-1}, \psi_v) \gamma(s, \chi_{h,v} \mu_{j,v}, \psi_v) \gamma(s, \chi_{h,v} \mu_{i,v}^{-1}, \psi_v),
\]
\[
\text{where we take } \gamma(s, \chi_0, v \mu_{j,v}, \psi_v) \text{ and } \gamma(s, \chi_0, v \mu_{j,v}^{-1}, \psi_v) \text{ (resp. } \gamma(s, \chi_i, v \mu_{0,v}, \psi_v) \text{ and } \gamma(s, \chi_i, v \mu_{0,v}^{-1}, \psi_v)),
\]
\[
\text{to be trivial if } G_m \text{ (resp. } G_n) \text{ is a special orthogonal group; and, we take } \chi_0 = 1 \text{ (resp. } \mu_0 = 1) \text{ if } G_m \text{ (resp. } G_n) \text{ is a symplectic group.}
\]

\[(b) \text{ If } G_m = GL_m \text{ and } G_n \text{ is a classical group, then}
\]
\[
\gamma(s, \tau_v \times \pi_v, \psi_v) = \prod_{i=0}^{d} \gamma(s, \chi_i, v \mu_{0,v}, \psi_v) \times \prod_{i=1}^{d} \gamma(s, \chi_i, v \mu_{j,v}, \psi_v) \gamma(s, \chi_i, v \mu_{j,v}^{-1}, \psi_v),
\]
\[
\text{where we take } \gamma(s, \chi_j, v \mu_{0,v}, \psi_v) \text{ to be trivial if } G_n \text{ is a special orthogonal}
\]
\[
\text{group; and, we take } \mu_0 = 1 \text{ if } G_n \text{ is a symplectic group.}
\]

\[(c) \text{ If } G_m = GL_m \text{ and } G_n = GL_n, \text{ then}
\]
\[
\gamma(s, \tau_v \times \pi_v, \psi_v) = \prod_{i,j} \gamma(s, \chi_i, v \mu_{j,v}, \psi_v).
\]

Any system of \(\gamma\)-factors satisfying properties (i)-(iv) gives \(\gamma(s, \tau_v \times \pi_v, \psi_v), \nu \notin S\), as a product of abelian \(\gamma\)-factors of Tate’s thesis as above. Hence, it is uniquely determined at these places.

The remaining possibility, at either \(v_1\) or \(v_2\), is that one representation is unramified while the other remains arbitrary. For concreteness, assume \(\tau_{v_1}\) and \(\pi_{v_2}\) are unramified while \(\tau_{v_2}\) and \(\pi_{v_1}\) remain arbitrary. Then \(\tau_{v_1}\) and \(\pi_{v_2}\) are constituents of representations via unitary parabolic induction from a product of unramified characters as before. The multiplicativity property of \(\gamma\)-factors gives \(\gamma(s, \tau_{v_1} \times \pi_{v_2}, \psi_{v_2})\) as a product of \(\gamma\)-factors of the form \(\gamma(s, \chi_{v_1} \times \pi_{v_1}, \psi_{v_1})\), where \(\chi_{v_1}\) is a character of \(GL_1(k_{v_1})\). Similarly, multiplicativity gives an expression for \(\gamma(s, \tau_{v_2} \times \pi_{v_1}, \psi_{v_1})\) as a product of \(\gamma\)-factors of the form \(\gamma(s, \tau_{v_2} \times \mu_{v_2}, \psi_{v_2})\), where \(\mu_{v_2}\) is a character of \(GL_1(k_{v_2})\). In these cases, \(\gamma\)-factors are uniquely determined as shown in § 3.2 where property (vi) is used.

At places where \(\psi_v\) may be ramified, we can use property (v) to compute \(\gamma\)-factors with respect to an unramified character. The functional equation for \(\gamma\)-factors gives,
\[
L^S(s, \tau \times \pi) = \gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0}) \prod_{\nu \in S \setminus \{v_0\}} \gamma(s, \tau_{\nu} \times \pi_{\nu}, \psi_{\nu}) L^S(1 - s, \tilde{\tau}_v \times \tilde{\pi}_v).
\]

Since partial \(L\)-functions are uniquely determined, and we have shown that \(\gamma\)-factors are uniquely determined at places other than \(v_0\), we can conclude that \(\gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0})\) is uniquely determined.

4. Extended \(L\)-functions and \(\varepsilon\)-factors

We now turn towards the defining properties of \(L\)-functions and \(\varepsilon\)-factors. We assume that \(p \neq 2\), which is necessary to study the case \(\mathfrak{s}\langle p, G_m, G_n \rangle\), when \(G_m\) and \(G_n\) are both classical groups.
4.1. Axioms for a local system of $L$-functions and root numbers. With a system of $\gamma$-factors on $\mathfrak{sl}(p)$ satisfying properties (i)-(vii), we can proceed to define rational functions $L(s, \tau \times \pi)$ and monomials $\varepsilon(s, \tau \times \pi, \psi)$ on the variable $q_F^{-s}$ for every $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$.

(viii) (Tempered $L$-functions). For $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$ tempered, let $P_{\tau \times \pi}(t)$ be the unique polynomial with $P_{\tau \times \pi}(0) = 1$ and such that $P_{\tau \times \pi}(q_F^{-s})$ is the numerator of $\gamma(s, \tau \times \pi, \psi)$. Then

$$L(s, \tau \times \pi) = P_{\tau \times \pi}(q_F^{-s})^{-1}.$$ 

(ix) (Holomorphy of tempered $L$-functions). Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$ be tempered. Then, $L(s, \tau \times \pi)$ is holomorphic and non-zero for $\Re(s) > 0$.

(x) (Tempered $\varepsilon$-factors). Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$ be tempered, then

$$\varepsilon(s, \tau \times \pi, \psi) = \gamma(s, \tau \times \pi, \psi) \frac{L(s, \tau \times \pi)}{L(1 - s, \tau \times \pi)}.$$ 

(xi) (Twists by unramified characters). Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p, \text{GL}_m, \text{G}_n)$, then

$$L(s + s_0, \tau \times \pi) = L(s, \varepsilon(\text{det}(\cdot))^{s_0}_F \tau \times \pi)$$

$$\varepsilon(s + s_0, \tau \times \pi, \psi) = \varepsilon(s, \varepsilon(\text{det}(\cdot))^{s_0}_F \tau \times \pi, \psi).$$

(xii) (Multiplicativity). Let $(F, \tau_i, \pi_j, \psi) \in \mathfrak{sl}(p, \text{GL}_{m_i}, \text{GL}_{n_j})$, $1 \leq i \leq d, 1 \leq j \leq e$, be quasi-tempered, and let $(F, \tau_0, \pi_0, \psi) \in \mathfrak{sl}(p, \text{G}_{m_0}, \text{G}_{n_0})$ be tempered. Let $m = \sum m_i$ and $n = \sum n_j$. Let $\text{G}_m$ and $\text{G}_n$ be of the same type as $\text{G}_{m_0}$ and $\text{G}_{n_0}$. Let $\text{P}_m$ and $\text{P}_n$ be parabolic subgroups of $\text{G}_m$ and $\text{G}_n$ with Levi subgroups $M_m \simeq \prod_{i=1}^d \text{GL}_{m_i} \times \text{G}_{m_0}$ and $M_n \simeq \prod_{j=1}^e \text{GL}_{n_j} \times \text{G}_{n_0}$. Suppose that $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$ is such that $\tau$ is the generic constituent of

$$\text{ind}_{\text{P}_m}^{\text{G}_m} (\tau_1 \otimes \cdots \otimes \tau_d \otimes \tau_0),$$

and $\pi$ is the generic constituent of

$$\text{ind}_{\text{P}_n}^{\text{G}_n} (\pi_1 \otimes \cdots \otimes \pi_e \otimes \pi_0).$$

When $m_0 = 0$ (resp. $n_0 = 0$) we make the following conventions: take $\tau_0$ (resp. $\pi_0$) to be the trivial character of $\text{GL}_1(F)$ if $\text{G}_m$ (resp. $\text{G}_n$) is a symplectic group; in all other cases, we interpret local factors involving $\tau_0$ (resp. $\pi_0$) to be trivial.

(xii.a) If both $\text{G}_m$ and $\text{G}_n$ are classical groups, then

$$L(s, \tau \times \pi) = L(s, \tau_0 \times \pi_0)$$

$$\times \prod_{i=1}^d L(s, \tau_i \times \pi_0) L(s, \tilde{\tau}_i \times \pi_0) \prod_{j=1}^e L(s, \tau_0 \times \pi_j) L(s, \tilde{\tau}_0 \times \tilde{\pi}_j)$$

$$\times \prod_{i=1}^d \prod_{j=1}^e L(s, \tau_i \times \pi_j) L(s, \tilde{\tau}_i \times \pi_j) L(s, \tau_i \times \tilde{\pi}_j) L(s, \tilde{\tau}_i \times \tilde{\pi}_j),$$

where $\tilde{\tau}_i$ and $\tilde{\pi}_j$ are the generic constituents of $\tau_i$ and $\pi_j$, respectively.
4.2. Additional properties of $\gamma$-factors. The following properties are satisfied by any system of $\gamma$-factors on $\mathfrak{I}(p)$ satisfying properties (i)-(vii).

(xiii) (Local functional equation). Let $(F, \tau, \pi, \psi) \in \mathfrak{I}(p)$, then
\[ \gamma(s, \tau \times \pi, \psi) \gamma(1 - s, \pi \times \pi, \psi) = 1. \]

(xiv) (Twists by unramified characters). Let $(F, \tau, \pi, \psi) \in \mathfrak{I}(p, \text{GL}_m, \mathbf{G}_n)$, then
\[ \gamma(s + s_0, \tau \times \pi, \psi) = \gamma(s, |\text{det}(\cdot)|_{p, F}^{s_0} \tau \times \pi, \psi). \]

(xv) (Commutativity). Let $(F, \tau, \pi, \psi) \in \mathfrak{I}(p)$, then
\[ \gamma(s, \tau \times \pi, \psi) = \gamma(s, \pi \times \tau, \psi). \]

We now give a proof of property (xiii) following the proof of uniqueness given in § 3. Notice that it is a property of abelian $\gamma$-factors: if $(F, \chi, \mu, \psi) \in \mathfrak{I}(p, \text{GL}_1, \mathbf{G}_n)$, then
\[ \gamma(s, \chi \mu, \psi) \gamma(1 - s, \chi^{-1} \mu^{-1}, \psi) = 1, \]
see for example equation (1.3) of [11]. First, we prove the local functional equation for the case of $\mathfrak{I}(p, \text{GL}_1, \mathbf{G}_n)$. We can reduce to the supercuspidal case via multiplicativity. Lift a supercuspidal $(F, \chi_0, \pi_0, \psi_0) \in \mathfrak{I}(p, \text{GL}_1, \mathbf{G}_n)$ to $(k, \chi, \pi, \psi) \in \mathcal{LS}(p, \text{GL}_1, \mathbf{G}_n)$ as in § 3.2. Notice that the method of proof gives an expression for every $\gamma(s, \pi \times \chi, \psi_0)$, $v \notin \{v_0\}$, as a product of abelian $\gamma$-factors. Then, the local functional equation at $v_0$ follows from the global functional equation applied twice.
Now, in the proof of uniqueness for the general case of § 3.3, let $(k, \tau, \pi, \psi) \in LS(p, G_m, G_n)$ be the global quadruple obtained from the supercuspidal quadruple $(F, \tau_0, \pi_0, \psi_0) \in \mathfrak{ls}(p, G_m, G_n)$. The method of proof gives an expression for every $\gamma(s, \tau_0 \times \pi_0, \psi_0, v) \neq \{v_0\}$, in terms of $\gamma$-factors for $\mathfrak{ls}(p, G_{l_1}, G_n)$, $\mathfrak{ls}(p, G_m, GL_1)$ or $\mathfrak{ls}(p, GL_1, GL_1)$. In all of these cases, the local functional equation holds. Hence, it follows at $v_0$ by applying the global functional equation twice.

We leave the proofs of properties (xiv) and (xv) as exercises.

4.3. **Theorem.** It $p \neq 2$, there exists a system of local factors on $\mathfrak{ls}(p)$ satisfying properties (i)-(xii). Any system of local factors on $\mathfrak{ls}(p)$ satisfying properties (i)-(xii) is uniquely determined.

**Proof.** We have already established the existence and uniqueness part of the theorem concerning properties (i)-(vii). We now construct local $L$-functions and $\varepsilon$-factors.

We first treat the tempered case, where property (viii) is taken as the definition of a local $L$-function. Notice that the multiplicativity property of $\gamma$-factors gives the multiplicativity property of local $L$-functions in the tempered case.

We now prove property (ix) for the new case of two classical groups $G_m$ and $G_n$. Let $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, G_m, G_n)$ be tempered. The representation $\tau$ is a constituent of

$$\text{ind}_{P_m}^G(\delta_1 \otimes \cdots \otimes \delta_d \otimes \tau_0),$$

as in equation (2.3) with $\delta_i, i = 1, \ldots, d$, generic discrete series representations of $GL_m(F)$, and $\tau_0$ a generic discrete series representation of $G_{m_0}$. Similarly, $\pi$ is a constituent of

$$\text{ind}_{P_n}^G(\rho_1 \otimes \cdots \otimes \rho_e \otimes \tau_0),$$

with $\rho_j, j = 1, \ldots, e$, generic discrete series representations of $GL_{m_j}(F)$, and $\tau_0$ a generic discrete series representation of $G_{n_0}$.

Let $T_0$ and $\Pi_0$ be the local functorial lifts of $\tau_0$ and $\tau_0$ given by equation (2.2). Notice that they are self-dual tempered representations of $GL_{M_0}(F)$ and $GL_{N_0}(F)$.

The local functorial lift $T$ of $\tau$ is given by

$$\text{ind}_{P_m}^G(\delta_1 \otimes \cdots \otimes \delta_d \otimes T_0 \otimes \tilde{\delta}_1 \otimes \cdots \otimes \tilde{\delta}_1)$$

and the lift $\Pi$ of $\pi$ is given by

$$\text{ind}_{P_n}^G(\rho_1 \otimes \cdots \otimes \rho_e \otimes \Pi_0 \otimes \tilde{\rho}_e \otimes \cdots \otimes \tilde{\rho}_1).$$

Then

$$L(s, \tau \times \pi) = L(s, T_0 \times \Pi_0) \prod_{i=1}^d L(s, \delta_i \times \Pi_0) L(s, \tilde{\delta}_i \times \Pi_0) \prod_{j=1}^e L(s, T_0 \times \rho_j) L(s, T_0 \times \tilde{\rho}_j)$$

$$\times \prod_{i=1}^d \prod_{j=1}^e L(s, \delta_i \times \rho_j) L(s, \tilde{\delta}_i \times \rho_j) L(s, \delta_i \times \tilde{\rho}_j) L(s, \tilde{\delta}_i \times \tilde{\rho}_j).$$

Each $L$-function on the right hand side is a Rankin-Selberg $L$-function for products of representations of general linear groups, known to be holomorphic for $\Re(s) > 0$. Hence, the extended $L$-function $L(s, \tau \times \pi)$ is holomorphic for $\Re(s) > 0$. Thus, our local $L$-functions satisfy property (ix) in the tempered case.

Next, for tempered $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p)$, property (x) is taken as the definition of root numbers. Then $\varepsilon(s, \tau \times \pi, \psi)$ is a monomial in $q_F^{-s}$. For this, we use the
local functional equation of $\gamma$-factors, property (xiii), together with property (ix) to ensure that no cancellations occur on the strip $0 < \Re(s) < 1$.

Having defined local $L$-functions and root numbers for tempered representations, they are then defined on $\mathfrak{sl}(p)$ with the aid of Langlands classification. In the case of $G_m = GL_m$ and $G_n = GL_n$, we include a treatment of $L$-functions and $\varepsilon$-factors in [11] in a self contained manner within the $\mathcal{LS}$-method. The definition given by equations (7.6) of [loc. cit.] is in accordance with the definition of Rankin-Selberg local factors [7]. We will now define extended $L$-functions and root numbers on $\mathfrak{sl}(p, GL_m, G_n)$, when both $G_m$ and $G_n$ are classical groups. Obtaining explicit defining relations for local $L$-functions and $\varepsilon$-factors on $\mathfrak{sl}(p, GL_m, G_n)$, with $G_n$ a classical group, is left as an exercise.

Assume that both $G_m$ and $G_n$ are classical groups. Let $\tau$ be given by equation (2.5)

$$\text{ind}_{\pi_m}^{G_m}(\tau_1 \nu^{r_1} \otimes \cdots \otimes \tau_d \nu^{r_d} \otimes \tau_0),$$

where each $\tau_i$ is a tempered representation of $GL_m(F)$ and $\tau_0$ is a generic tempered representation of $G_{m_0}$. With the appropriate modifications if $\tau$ is given by equation (2.6). Similarly, $\pi$ is given by

$$\text{ind}_{\pi_n}^{G_n}(\pi_1 \nu^{r_1} \otimes \cdots \otimes \pi_d \nu^{r_d} \otimes \pi_0),$$

where each $\pi_i$ is a tempered representation of $GL_n(F)$ and $\pi_0$ is a generic tempered representation of $G_{n_0}$. With appropriate modifications if $\pi$ is given by equation (2.7).

Let $(F, \tau, \pi, \psi) \in \mathfrak{sl}(p)$, then we define local $L$-functions by

$$L(s, \tau \times \pi) = L(s, \tau_0 \times \pi_0) \times \prod_{i=1}^{d} L(s + r_i, \tau_i \times \pi_0)L(s - r_i, \tilde{\tau}_i \times \pi_0) \prod_{j=1}^{e} L(s + t_j, \tau_0 \times \pi_j)L(s - t_j, \tau_0 \times \tilde{\tau}_j)$$

$$\times \prod_{i=1}^{d} \prod_{j=1}^{e} L(s + r_i + t_j, \tau_i \times \pi_j)L(s - r_i + t_j, \tilde{\tau}_i \times \pi_j)$$

$$\times L(s + r_i - t_j, \tau_i \times \tilde{\tau}_j)L(s - r_i - t_j, \tilde{\tau}_i \times \tilde{\tau}_j),$$

and local root numbers by

$$\varepsilon(s, \tau \times \pi, \psi) = \varepsilon(s, \tau_0 \times \pi_0, \psi) \times \prod_{i=1}^{d} \varepsilon(s + r_i, \tau_i \times \pi_0, \psi)\varepsilon(s - r_i, \tilde{\tau}_i \times \pi_0, \psi) \prod_{j=1}^{e} \varepsilon(s + t_j, \tau_0 \times \pi_j, \psi)\varepsilon(s - t_j, \tau_0 \times \tilde{\tau}_j, \psi)$$

$$\times \prod_{i=1}^{d} \prod_{j=1}^{e} \varepsilon(s + r_i + t_j, \tau_i \times \pi_j, \psi)\varepsilon(s - r_i + t_j, \tilde{\tau}_i \times \pi_j, \psi)$$

$$\times \varepsilon(s + r_i - t_j, \tau_i \times \tilde{\tau}_j, \psi)\varepsilon(s - r_i - t_j, \tilde{\tau}_i \times \tilde{\tau}_j, \psi).$$

It is now possible to show that our construction is compatible with properties (xi) and (xii). Notice that the definition of local $L$-functions and root numbers is based on an extended system of $\gamma$-factors, and only uses special cases of properties (viii)-(xii). Hence, any system of extended $\gamma$-factors, local $L$-functions and root numbers satisfying properties (i)-(xii) is uniquely determined.
4.4. **Theorem.** Assume that \( p \neq 2 \). For every \((k, \tau, \pi, \psi) \in \mathcal{LS}(p)\), let

\[
L(s, \tau \times \pi) = \prod_v L(s, \tau_v \times \pi_v), \quad \varepsilon(s, \tau \times \pi) = \prod_v \varepsilon(s, \tau_v \times \pi_v, \psi_v).
\]

Automorphic \( L \)-functions satisfy the following properties:

(i) (Rationality). \( L(s, \tau \times \pi) \) converges on a right half plane and has a meromorphic continuation to a rational function on \( q^{-s} \).

(ii) (Functional equation). \( L(s, \tau \times \pi) = \varepsilon(s, \tau \times \pi) L(1 - s, \bar{\tau} \times \bar{\pi}) \).

(iii) (Riemann Hypothesis) The zeros of \( L(s, \tau \times \pi) \) are contained in the line \( \Re(s) = 1/2 \).

**Proof.** First, given \((F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \text{GL}_m, \text{G}_n)\), we know that partial \( L \)-functions \( L^S(s, \tau \times \pi) \) converge on a right half plane. The rationality argument of Harder for Eisenstein series [2], allows us to give an automorphic forms proof that \( L^S(s, \tau \times \pi) \) is a rational function on \( q^{-s} \), Corollary 6.6 of [10]. Now, notice that each local \( L \)-function in the product

\[
\prod_{v \in S} L(s, \tau_v \times \pi_v)
\]

is a rational function on \( q_v^{-s} = q^{-\deg(v)s} \). Hence, property (i) follows for completed automorphic \( L \)-functions in this case. Also, the definition of local \( L \)-functions and \( \varepsilon \)-factors at the places \( v \in S \) can be incorporated into the functional equation satisfied by \( \gamma \)-factors in order to obtain property (ii) for global \( L \)-functions and \( \varepsilon \)-factors on \( \mathcal{LS}(p, \text{GL}_m, \text{G}_n) \).

Next, we treat the case of \( \mathcal{LS}(p, \text{G}_m, \text{G}_n) \), with both \( \text{G}_m \) and \( \text{G}_n \) classical groups. Let \((k, \tau, \pi, \psi) \in \mathcal{LS}(p, \text{G}_m, \text{G}_n)\). The functorial lift of Theorem 9.1 of [10] is compatible with the local functorial lift at every place \( v \) of \( k \). The construction of the local lift is reviewed in § 2.4. Equation (2.9) enables us to write the global lifts \( T \) of \( \tau \) and \( \Pi \) of \( \pi \) as isobaric sums

\[
(4.2) \quad T = T_1 \boxplus \cdots \boxplus T_d \quad \text{and} \quad \Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_e.
\]

We have that

\[
L^S(s, \tau \times \pi) = L^S(s, T \times \Pi),
\]

which converge on a right half plane and have a meromorphic continuation to a rational function on \( q^{-s} \). At the remaining places, extended \( L \)-functions \( L(s, \tau_v \times \pi_v) \) are rational on \( q_v^{-s} \). Hence, the completed automorphic \( L \)-function \( L(s, \tau \times \pi) \) satisfies property (i). The way local factors are defined can be coupled with the functional equation satisfied by extended \( \gamma \)-factors in order to establish property (ii) for automorphic \( L \)-functions on \( \mathcal{LS}(p, \text{G}_m, \text{G}_n) \).

Finally, let \((k, \tau, \pi, \psi) \in \mathcal{LS}(p, \text{G}_m, \text{G}_n)\). If both \( \text{G}_m \) and \( \text{G}_n \) are classical groups, take \( T \) and \( \Pi \) to be the global functorial lifts of \( \tau \) and \( \pi \) of equation (4.2). If \( \text{G}_m = \text{GL}_m \), we just take \( T = \tau \). Then

\[
L(s, \tau \times \pi) = L(s, T \times \Pi) = \prod_{i,j} L(s, T_i \times \Pi_j),
\]

where \((k, T_i, \Pi_j, \psi) \in \mathcal{LS}(p, \text{GL}_m, \text{GL}_n)\), for \( 1 \leq i \leq d, 1 \leq j \leq e \). Thanks to the work of Lafforgue on the global Langlands conjecture for \( \text{GL}_N \) over function fields, each Rankin-Selberg \( L \)-function \( L(s, T_i \times \Pi_j) \) satisfies the Riemann Hypothesis (see Théorème VI.10(ii) of [3]). Hence, so does \( L(s, \tau \times \pi) \). We conclude that automorphic \( L \)-functions on \( \mathcal{LS}(p) \) satisfy the Riemann Hypothesis.
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