Efficient quantum error-correction protocol which can be realized easily

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In this work, we construct one efficient quantum error-correction protocol against the general quantum noise, just with the two three-qubit quantum error-correction codes. The physical resources costed by this protocol is multiple of the physical resources costed to realize the three-qubit quantum error-correction, with no increase in complexity. We have given the error-correction threshold for different cases, and discussed how to realize the optimal error-correction. We believe it will be helpful for realizing quantum error-correction in physical system.

In quantum computation and communication, quantum error-correction (QEC) developed from classic schemes to preserving coherent states from noise and other unexpected interactions. It was independently discovered by Shor and Steane [1, 2]. The QEC codes were proved independently by Bennett, DiVincenzo, Smolin and Wootters [3], and by Knill and Laflamme [4]. QEC codes are introduced as active error-correction. The nine-qubit code was discovered by Shor, called the Shor code. The seven-qubit code was discovered by Steane, called the Steane code. The five-qubit code was discovered by Bennett, DiVincenzo, Smolin and Wootters [3], and independently by Laflamme, Miquel, Paz and Zurek [5].

There are many constructions for specific classes of quantum codes. Rains, Hardin, Shor and Sloane [6] have constructed interesting examples of quantum codes lying outside the stabilizer codes. Gottesman [7] and Rains [8] constructed non-binary codes and consider fault-tolerant computation with such codes. Aharonov and Ben-Or [9] construct non-binary codes using interesting techniques based on polynomials over finite fields, and also investigate fault-tolerant computation with such codes. Approximate QEC can lead to improved codes was shown by Leung, Nielsen, Chuang and Yamamoto [10].

Calderbank and Shor [11], and Steane [12] used ideas from classical error-correction to develop the CSS (Calderbank-Shor-Steane) codes. Calderbank and Shor also stated and proved the Gilbert-Varshamov bound for CSS codes. Gottesman [13] invented the stabilizer formalism, and used it to define stabilizer codes, and investigated some of the properties of some specific codes. Independently, Calderbank, Rains, Shor and Sloane [14] invented an essentially equivalent approach to QEC based on ideas from classical coding theory.

QEC codes are introduced as active error-correction. Another way, passive error-avoiding techniques contain decoherence-free subspaces [15–17] and noiseless subsystem [18–20]. Recently, it has been proven that both the active and passive QEC methods can be unified [21–23]. So, more QEC codes means more options for suppressing noise, and more options for optimizing the performance of QEC. Meanwhile, for efficient quantum communication and computation, one efficient and easy to implement QEC protocol should be found.

In this work, we construct one efficient quantum error-correction protocol against the general quantum noise, just with the two three-qubit quantum error-correction codes. One of them is the bit-flip code,

$$|0_\mathcal{L}\rangle = |000\rangle,$$

$$|1_\mathcal{L}\rangle = |111\rangle. \quad (1)$$

the state $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded to $|\tilde{\psi}\rangle = a|000\rangle + b|111\rangle$.

When bit-flip error occurred on one qubit, measuring with the stabilizer $\langle Z_1Z_2, Z_2Z_3 \rangle$. $X_1|\tilde{\psi}\rangle = a|100\rangle + b|011\rangle, S_1 = -1, 1; X_2|\tilde{\psi}\rangle = a|010\rangle + b|101\rangle, S_2 = -1, -1; X_3|\tilde{\psi}\rangle = a|001\rangle + b|110\rangle, S_3 = 1, -1.$

Here, $S_a$ is the measurement result for the $n$-qubit. We should notice that if bit-phase-flip error occurred on one qubit, the measurement results with the stabilizer $\langle Z_1Z_2, Z_2Z_3 \rangle$ are

$$Y_1|\tilde{\psi}\rangle = i[a|100\rangle - b|011\rangle], S_1 = -1, 1;$$

$$Y_2|\tilde{\psi}\rangle = i[a|010\rangle - b|101\rangle], S_2 = -1, -1;$$

$$Y_3|\tilde{\psi}\rangle = i[a|001\rangle - b|110\rangle], S_3 = 1, -1.$$

The measurement results for each qubit occurred one-qubit bit-flip or bit-phase-flip errors are the same, which means the bit-flip code can be used to correct one-qubit bit-flip or bit-phase-flip error in single-error environment, and not in mixed noise environment because the bit-flip and bit-phase-flip errors can not be identified simultaneously.

With another code, the state $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded to $|\tilde{\psi}\rangle = a|0_\mathcal{L}\rangle + b|1_\mathcal{L}\rangle$.

$$|0_\mathcal{L}\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle),$$

$$|1_\mathcal{L}\rangle = \frac{1}{2}(|111\rangle + |100\rangle + |010\rangle + |001\rangle). \quad (2)$$

The three-qubit code in Eq. (2) is one three-qubit phase-flip code, and the errors $\{I, Z_1, Z_2, Z_3\}$ with the stabilizer $\langle X_1X_2, X_2X_3 \rangle$. The logical $Z$ operator is $Z_1Z_2Z_3$, and the logical $X$ operator is one of $X_1X_2X_3$, $X_1$, $X_2$, and $X_3$.

For the three-qubit code in Eq. (2), the state $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded to $|\tilde{\psi}\rangle = a|000\rangle + b|011\rangle + |101\rangle + |110\rangle + b|111\rangle + |100\rangle + |010\rangle + |001\rangle$. When phase-flip error occurred on one qubit, measuring with the stabilizer...
For these cases, only one of the codes in Eq. (1) and Eq. (2) is applied, the error-correction threshold is $f_0 = 0.91518$, which is obtained when the three Pauli errors have the same weight. Here, we should abandon the Pauli-Y error if it had the lowest weight. On the other hand, the inner recovery must be $U_m = y$ when the weight of the Pauli-Y is not the lowest, and the outer code should against the rest higher weight Pauli error.

For depolarizing channel, we list the change of process matrix to show the exact performance of the QEC protocol in Table I. In level-1, $i = 1, j = 2, U_m = y, V_m = z$. In level-2, $i = 2, j = 1, U_m = z, V_m = x$. In level-3, $i = 1, j = 2, U_m = x, V_m = z$. As comparison, the performance of the Shor code in Table II.

As shown in tables, both protocols are efficient. The biggest difference between the protocol from Eq. (4) and the Shor code (five-qubit code and Steane code) is that the physical resources costed by this protocol is multiple of the physical resources costed to realize the three-qubit QEC. It just need quantum gate operations as simple as realizing three-qubit quantum gate operations as simple as realizing three-qubit resources costed to realize the three-qubit QEC. It just need
TABLE I. Results of 5-level concatenated QEC from Eq. (4), where the initial process matrix $\lambda_{L,L=0}$ is $\lambda_{II} = 0.92, \lambda_{XX} = \lambda_{ZZ} = \lambda_{YY} = \frac{0.08}{9},$ and $L$ is the concatenation level.

| $\lambda - L$ | Shor code $(\lambda_{II}, \lambda_{XX}, \lambda_{ZZ}, \lambda_{YY})$ |
|---------------|-------------------------------------------------------------|
| $0$           | $(0.92, \frac{0.08}{9}, \frac{0.08}{9}, \frac{0.08}{9})$       |
| $1 - in$      | $(0.852345, 0.00411496, 0.139425, 0.00411496)$              |
| $1 - out$     | $(0.923232, 0.0208713, 0.0524821, 0.00341433)$             |
| $2 - in$      | $(0.922795, 0.0681813, 0.00782988, 0.00119405)$            |
| $2 - out$     | $(0.960219, 0.0131944, 0.0260905, 0.00057664)$             |
| $3 - in$      | $(0.957846, 0.0400713, 0.00196852, 0.00011471)$            |
| $3 - out$     | $(0.989099, 0.00467851, 0.00618629, 0.0000363724)$         |
| $4 - in$      | $(0.985875, 0.0140098, 0.000113806, 1.87668 \times 10^{-6})$ |
| $4 - out$     | $(0.99907, 0.000583257, 0.000346744, 2.2364 \times 10^{-7})$|
| $5 - in$      | $(0.989859, 1.01981 \times 10^{-6}, 0.00104018, 0)$       |
| $5 - out$     | $(0.999994, 3.06284 \times 10^{-6}, 3.24367 \times 10^{-6}, 0)$ |

TABLE II. Results of 5-level concatenated QEC with the Shor code, where the initial process matrix $\lambda_{L,L=0}$ is $\lambda_{II} = 0.92, \lambda_{XX} = \lambda_{ZZ} = \lambda_{YY} = \frac{0.12}{27},$ and $L$ is the concatenation level.

| $\lambda - L$ | Shor code $(\lambda_{II}, \lambda_{XX}, \lambda_{ZZ}, \lambda_{YY})$ |
|---------------|-------------------------------------------------------------|
| $0$           | $(0.92, \frac{0.08}{3}, \frac{0.08}{3}, \frac{0.08}{3})$       |
| $1$           | $(0.934261, 0.01414538, 0.02211, 0.00217564)$               |
| $2$           | $(0.968208, 0.0153426, 0.0159628, 0.00048603)$             |
| $3$           | $(0.990936, 0.00683566, 0.00220061, 0.0000274494)$         |
| $4$           | $(0.999438, 0.000140051, 0.000421678, 1.84203 \times 10^{-7})$|
| $5$           | $(0.999995, 4.79366 \times 10^{-6}, 1.76974 \times 10^{-7}, 0)$|

QEC, which is

$$C_l = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 \ldots$$  

(5)

Here, $l$ is the level of the QEC protocol, $2l$ is number of terms of the cost, and 1 represents the cost of realizing three-qubit QEC. With no complicated quantum gate operations like others.

In realization of QEC, some works have been done [24, 25]. Why the QEC protocol from Eq. (4) can be efficient, the reason may raise from the ability to change the logical error when applying inner QEC, and the outer QEC is good at against it. Meanwhile, it is easy for realizing just because it is with repeated implementation of three-qubit QEC. Similar phenomena for other codes have been noted in [26–28], but the realization need more complicated quantum gate operations. Once the QEC with the three-qubit code had been realized, the realization with this protocol is nearer. Based on the realization, any one-qubit error can be corrected, and fault-tolerant quantum computation may be realized in practical quantum computer.