ALGEBRO-GEOMETRIC INVARIANTS OF FG GROUPS
(THE PROFILE OF A REPRESENTATION VARIETY)

S. Liriano, S. Majewicz

ABSTRACT. If \( G \) is a finitely generated group (fg), and \( A \) is a complex affine algebraic group, then the space \( \text{Hom}(G, A) \) admits the structure of an algebraic variety, not necessarily irreducible, denoted briefly by \( R_A(G) \). The invariants of \( R_A(G) \) are independent of the finite set \( X \) of generators for \( G \) chosen. In this communication we define the profile function, \( P_d(R_A(G)) \), of the representation variety over an algebraic group \( A \) of a fg group \( G \) to be

\[
P_d(R_A(G)) = (N_d(R_A(G)), \ldots, N_0(R_A(G))),
\]

where \( N_i(R_A(G)) \) stands for the number of irreducible components of \( R_A(G) \) of dimension \( i \), \( 0 \leq i \leq d \), and \( d = \text{Dim}(R_A(G)) \). We then unleash this invariant in the study of fg groups and prove various results. In particular, we show that if \( G \) is the fundamental group of an orientable surface group of genus \( g \geq 1 \), then

\[
(**) \quad P_d(R_{\text{SL}(2,\mathbb{C})}(G)) \neq P_d(R_{\text{PSL}(2,\mathbb{C})}(G)).
\]

We also show that \( ** \) holds for \( G \) a torus knot group with presentation \( < x, y; x^p = y^t > \) where \( p, t \) are both \( \geq 3 \), and that \( ** \) also holds when \( G \) is a the fundamental group of a compact non-orientable surface of genus \( g \geq 3 \). Additionally, we show that if a group \( G \) can be \( n + 1 \) generated and admits a presentation \( < x_1, \ldots, x_n, y; W = y^p > \), where \( W \) is a non-trivial word in \( F_n = < x_1, \ldots, x_n > \), and \( A = \text{PSL}(2, \mathbb{C}) \), then \( \text{Dim}(R_A(G)) = \text{Max}\{3n, \text{Dim}(R_A(G')) + 2\} \leq 3n + 1 \), where \( G' = < x_1, \ldots, x_n; W = 1 > \). We also give a condition guaranteeing that the algebraic variety is reducible.

Historical Remark.

We mention briefly only a few developments bearing some resemblance to this work, keeping in mind that this account is anything but thorough; in passing, let us remark that the study of spaces of representations of fg groups can be traced back to the work of H. Poincare [LM]. We begin with the work of W. Goldman, who in the eighties produced results giving the number of connected components of \( \text{Hom}(G, L) \), where \( L \) is a Lie group and \( G \) an orientable surface group; for example, in [GW] a result was laid out yielding the number of connected components in the space of representations of the fundamental group of an oriented surface in the \( n \)-fold covering group of \( \text{PSL}(2, \mathbb{R}) \) in terms of \( n \) and the genus of the surface. We recall, however, that counting connected components of a complex variety is fundamentally different from counting its irreducible components. For example, consider the vanishing set \( V \) of \( xy = 0 \) in \( \mathbb{C}^2 \). It is easy to see that \( V \) has two irreducible components and only one connected component. More recently, in the nineties,
Rapinchuk, et al, [RBC] produced a paper where amongst numerous interesting results, they established the absolute irreducibility of $R_A(\Gamma_g)$, the representation variety of the fundamental group of a compact orientable surface of genus $g$ for $A$ either $GL(n, K)$, or $SL(n, K)$, where $K$ is a field of characteristic 0; they also gave formulas for the dimension of their corresponding representation varieties involving the genus, and the dimension of the relevant algebraic group. Subsequently, Benyash-Krivetz and Chernousov, [BC], obtained formulas yielding the number of irreducible components of $R_A(\Gamma_g)$, where $\Gamma_g$ is a compact non-orientable surface group of genus $g$, and $A$ is either $GL(n, K)$, or $SL(n, K)$, where $K$ is a field of characteristic 0. Consequently, Liriano in [L3] produced a formula giving the number of four dimensional irreducible components of $R_{SL(2, \mathbb{C})}(G)$ for groups $G$ having presentation $< x, y; x^p = y^q >$ where $p, q$ are positive integers, and pointed out that when such a group is the fundamental group of a torus knot complement in $\mathbb{R}^3$, that its number of four dimensional irreducible components equals the genus of the corresponding torus knot.

**Introduction**

The study of invariants of representation varieties over $SL(2, \mathbb{C})$ of parafree groups\(^1\) has provided ample evidence to justify the claim that invariants of groups associated with spaces of representation over algebraic groups can be highly sensitive, even when launched in classes of groups exhibiting pronounced structural affinities [L4], [L2]. In this paper an invariant, $P_n(R_A(G))$, is introduced and we call it “the profile function of the representation variety of $G$ over the algebraic group $A$”, or more swiftly, “the profile of $G$ over $A$”. This invariant that counts all irreducible components of all dimensions in $R_A(G)$ is then deployed, in what may be described as ‘doubly close settings’; by that it is meant that not only is the invariant studied over groups that are known to be structurally very close to each other, for example, free groups and surface groups, (or more generally, the class of fully residually free groups), but also over algebraic groups that are also structurally similar; for example, $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$. Even when we observe this doubly close condition, if you will, rather remarkable divergence in the respective values of the invariant is obtained.

More formally, define the profile function $P_d(R_A(G))$ of the representation variety over an algebraic group $A$ for a fg group $G$ to be

$$P_d(R_A(G)) = (N_d(R_A(G)), \ldots, N_0(R_A(G))),$$

where $N_i(R_A(G))$ stands for the number of irreducible components of $R_A(G)$ of dimension $i$, $0 \leq i \leq d$, and $d = \text{Dim}(R_A(G))$. From the start, let’s make it clear that it is possible to take the profile function $P_n(R_A(G))$ at any integral value $n \geq \text{Dim}(R_A(G))$ with the understanding that all values to the left of the leftmost non-zero entry are to be discarded. In this paper the following results are obtained:

**Theorem A.** Let $w$ be a non-trivial freely reduced word in the commutator subgroup of $F_n = < x_1, \ldots, x_n >$, and let $G = < x_1, \ldots, x_n; \ w = 1 >$. If $R_{SL(2, \mathbb{C})}(G)$

---

\(^1\)A group is termed parafree of rank $r$ and deviation $d$, if it has the same lower central sequence as a free group of rank $r$ and if the difference between the minimum number of generators of $G$ and $n$ is the number $d$. For a more detailed account on parafree groups consult [B1], [B2].
is an irreducible variety and $V_{-1} = \{ \rho \mid \rho \in R_{SL(2, \mathbb{C})}(F_n), \rho(w) = -I \} \neq \emptyset$,  then $P_d(R_{SL(2, \mathbb{C})}(G)) \neq P_d(R_{PSL(2, \mathbb{C})}(G))$.

**Theorem B.** Let $w$ be a non-trivial freely reduced word in the free group on \{x_1, \ldots, x_n\} with even exponent sum on each generator, and with exponent sum not equal to zero on at least one generator. Suppose $G = \langle x_1, \ldots, x_n ; w = 1 \rangle$. If $R_{SL(2, \mathbb{C})}(G)$ is an irreducible variety, then $P_d(R_{SL(2, \mathbb{C})}(G)) \neq P_d(R_{PSL(2, \mathbb{C})}(G))$.

An immediate consequence of the above results, together with results found in [RBC] and [BC], is that when $G$ is the fundamental group of a compact orientable, or a compact non-orientable surface of genus $g \geq 1, g \geq 3$, respectively, then $R_{SL(2, \mathbb{C})}(G)$ is reducible and $P_d(R_{SL(2, \mathbb{C})}(G)) \neq P_d(R_{PSL(2, \mathbb{C})}(G))$. In [BC] it was shown that $R_{SL(2, \mathbb{C})}(G)$ is irreducible when $G$ is a closed non-orientable surface group of genus $g \geq 3$, and in [RBC] it was shown that $R_{SL(2, \mathbb{C})}(G)$ is irreducible when $G$ is a closed orientable surface group of genus $g \geq 1$.

Furthermore, we show that if $G$ is a torus knot group with presentation $\langle a, b ; a^p = b^t \rangle$, where both $p, t \geq 3$, then $P_d(R_{SL(2, \mathbb{C})}(G)) \neq P_d(R_{PSL(2, \mathbb{C})}(G))$. In particular, the genus of these torus knots does not equal $N_d(R_{PSL(2, \mathbb{C})}(G))$, a result established in [L3] for the representation varieties over $SL(2, \mathbb{C})$ for all torus knot groups.

To illustrate the sudden changes of invariants that can arise when looking at the representation variety of a group $G$ over one algebraic group as opposed to another, consider the following: let $G = \langle x, y ; [x, y^p]y^2 = 1 \rangle$, and define $[a, b] = aba^{-1}b^{-1}$. Then $\dim(R_{SL(2, \mathbb{C})}(G)) = 3$, and $P_4(R_{SL(2, \mathbb{C})}(G)) = (2, 0, 0, 0)$, while $\dim(R_{PSL(2, \mathbb{C})}(G)) = 5$, and $P_5(R_{PSL(2, \mathbb{C})}(G)) = (1, 0, 1, 0, 0)$. Had we defined $[a, b] = a^{-1}b^{-1}ab$ the resulting profile functions would have been different as well. In fact, given a fixed arbitrary integer $t$ there exists infinitely many finitely generated groups $G$ with $\dim(R_{SL(2, \mathbb{C})}(G)) = 0$ and $\dim(R_{PSL(2, \mathbb{C})}(G)) \geq t$. In this paper we prove the following theorem quite useful in handling many cyclically pinched one relator presentations like those of compact non-orientable surface groups, torus knot groups, or many of the parafree groups introduced in [B1].

**Theorem C.**

1. Let $G = \langle X, y ; W(X) = y^p \rangle$, where $Card(X) = n, p > 1$, and $W(X)$ is a non-trivial freely reduced word in the free group on $X$. If $A = PSL(2, \mathbb{C})$, then $\dim(R_A(G)) = \max\{3n, \dim(R_A(G')) + 2\} \leq 3n + 1$, where $G' = \langle X ; W(X) = 1 \rangle$.

2. If $\dim(R_A(G')) + 2 \geq 3n$ in (1), then $R_A(G)$ is reducible.

In this paper it is also shown that if a group $G$ embeds into an algebraic group $A$, then the profile function of $R_A(G)$, and the profile function of $R_A(G/N(w))$, where $N(w)$ is the normal closure of a non-trivial word $w$ in $G$, are always different. This is equivalent to saying that the vanishing set of the ideal associated with the subgroup $N(w)$ in the coordinate algebra of $R_A(G)$ is always non-trivial; think of it as variation of Hilbert’s Weak Nullstellensatz. This result particularly applies to fg non-abelian fully residually free groups, a class that is known to contain all but a finite number of fundamental groups of compact orientable, and non-orientable surfaces. In passing, observe that if a group $G$ does not embed into an algebraic group $A$, then the profile function need not change when going to a non-trivial
quotient of \( G \); a good example is the Baumslag-Solitar group \( G = \langle x, y; x^{-1}y^2x = y^3 \rangle \). This group does not embed into any affine group since it is non-hopfian. Thus, there is at least one non-trivial word \( w \) in \( G \) such that \( R_A(G) \) has the same profile function as \( R_A(G/N(w)) \).

Given the positive tone of the foregoing, it is but prudent one mentions that the next section begins with the rather sobering result: “Given any algebraic group \( A \), there are continuously many non-isomorphic finitely generated groups with isomorphic representation varieties over \( A \)”.

Indeed, no invariant of representation varieties over \( A \) can discriminate these groups. To assist the reader in placing the above in a more nuanced perspective, we recall that there are infinitely many rank two parafree groups of deviation one with equi-dimensional non-isomorphic \( SL(2, \mathbb{C}) \) representation varieties; see \([L4]\). Indeed, this seems at odds with the commonly received understanding that parafree groups are very much like free groups, especially when one is told that any \( n \)-generated group \( G \) with \( \dim(R_{SL(2, \mathbb{C})}(G)) = 3n \) is free. In fact, if the criteria that the deviation be one is discarded, then given a parafree group of rank 2, there exists infinitely many parafree groups also of rank 2 having representation varieties of dimension greater than any apriori chosen integer \([L2]\).

**One: The Profile Function**

Unless specified, the term “algebraic group” is assumed to mean a complex affine algebraic group, and all algebraic varieties will always be complex, even if some of the results in the sequel hold in far more general settings, as many do. Having made that clear, given any algebraic group \( A \) with identity \( I \), and a group \( G \) generated by a finite set of generators \( X = \{x_1, \cdots, x_n\} \), the set \( \text{Hom}(G, A) \) brings with it the structure of an algebraic variety\(^3\). Indeed, this is simple to see since \( \text{Hom}(G, A) \) is nothing but \( \bigcap W_i^{-1}(I) \), where each \( W_i^{-1}(I) \) is the fiber over \( I \) pertaining to the regular map given by \( P_{W_i(X)} : A^n \to A \); this map arises when one evaluates in the algebraic group \( A^n \) the \( i \)-th relation of a possibly infinite set of relators \( \{W_1 = 1, \cdots, W_i = 1, \cdots\} \) obtained from a presentation of \( G \) on \( X \). This algebraic variety goes by the name “the representation variety” of \( G \) over \( A \), and we denote it by \( R_A(G) \).

Fortunately, the Hilbert Basis Theorem guarantees that only a finite number of the relators of \( G \) on the generating set \( X \) are necessary in defining the algebraic variety \( R_A(G) \). An immediate consequence of this is the following:

**Lemma 1.1.** Given any algebraic group \( A \) there exists an uncountable set \( S \) of pairwise non-isomorphic 2-generated groups with the property that any two groups in \( S \) have isomorphic representation varieties over the algebraic group \( A \).

**Proof.** Denote the set of two generated groups by \( S_2 \). It is well known that there are continuously many non-isomorphic 2-generated groups. However, the set of finitely presentable groups is countable since the set of all finite subsets of a countable set injects into the set of all finite subsets of the natural numbers \( \mathbb{N} \); in particular, this holds for the set of all finite words made from the elements of a finite set of

\(^3\)The term “algebraic variety” applies to any algebraic set, whether reducible or not.
symbols. Thus it follows that the set, $N_2$, of finitely presentable groups having two
generators is countable. So there exists a bijection between $N_2$ and $\mathbb{N}$; this bijection
can be used to list all possible finitely presented 2-generated groups thus:

\[ \zeta = P_1, P_2, P_3, \ldots \]

Many of the groups in this infinite sequence may be isomorphic, but that poses no
difficulty. Now, with the sequence of presentations in $\zeta$ associate a sequence $R_A(\zeta)$
of representation varieties over $A$ thus:

\[ R_A(\zeta) = R_A(P_1), R_A(P_2), R_A(P_3), \ldots \]

By the Hilbert Basis Theorem, $R_A(\zeta)$ contains all of the possible representation
varieties of 2-generated groups over the algebraic group $A$. Introduce an equivalency
relation $\sim$ on the sequence of algebraic varieties in $R_A(\zeta)$ given by $R_A(P_i) \sim R_A(P_j)$
iff $R_A(P_i) \cong R_A(P_j)$, and in the process give rise to a sequence of equivalency classes
$R_A(\zeta)/\sim$. Further, define a mapping $\Phi$ from the set $S_2$ of all 2-generated groups
to $R_A(\zeta)/\sim$ given by $\Phi(G) = [R_A(G)]$, where $[R_A(G)]$ is the equivalency class
of $R_A(G)$ in $R_A(\zeta)/\sim$. Clearly, $S_2$ is nothing but the union of all the fibers over
points in $R_A(\zeta)/\sim$, and $R_A(\zeta)/\sim$ is countable. It follows then that the fiber over
at least one point of $R_A(\zeta)/\sim$ contains an uncountable number of non-isomorphic
groups since, by the Axiom of Choice, the countable union of countable sets is
countable. Let $S$ be the uncountable fibre over that point. Then all of the groups
in $S$ have isomorphic representation varieties over $A$.

Given the negativity of the above, it is fortunate that one can easily show that
an $n$-generated group $G$ is free iff $\dim(R_A(G)) = 3n$, where $A$ is an irreducible
algebraic group containing a free group of rank 2. This was proven in [L4] for
$SL(2, \mathbb{C})$, but the proof can be readily generalized to an arbitrary irreducible affine
group containing a free group of rank 2.

In fact, one can do better; we next show that even when the algebraic group $A$
is reducible, the representation variety of a fg free group $F_n$ of rank $n$ over $A$ has
remarkable properties provided that $A$ contains a group isomorphic to $F_2$, the free
group of rank 2. To this end it will be necessary to introduce what we shall call the
"Profile Function $P_m$" from the space $\mathcal{S}_m$ of all algebraic varieties of dimension at
most $m$ over a field $K$ to $\mathbb{Z}_+^{m+1}$ given by:

\[ P_m(V) = (N_m(V), N_{m-1}(V), \ldots, N_0(V)), \]

where $V$ is in $\mathcal{S}_m$, and $N_i(V)$ is the function that counts the number of irreducible
components of dimension precisely $i$ in $V$. As will become amply clear in the sequel,
this invariant will prove to be quite useful in the study of finitely generated groups,
but first it is important that we introduce some additional notions surrounding it.
Before doing so, let’s make it clear that the profile function for an algebraic variety
of dimension $m$ can be taken at values $n \geq m$, provided that all entries to the left
of the leftmost non-zero entry are discarded.

**Definition 1.1 (The Profile of a FG Group Over $A$).** If $G$ is a finitely gen-
erated group, then the *profile of $G$* over an affine algebraic group $A$ is defined to be
$P_d(R_A(G))$, where $d = \dim(R_A(G))$. 
Definition 1.2 (Same Profile). Two algebraic varieties $V$ and $W$ have the same profile if they have the same dimension $d$, and $P_d(V) = P_d(W)$.

A trivial consequence of Lemma 1.1 is that given any algebraic group $A$, there exists an uncountable set $S_2$ of pairwise non-isomorphic 2-generated groups having the property that for each pair of different groups $G_i, G_j \in S_2$ the profile of the representation varieties of $G_i$ and $G_j$ over $A$, respectively, are the same.

Definition 1.3 (Proper Relative Descent, Relative Descent). An algebraic variety $W$ descends properly relative to an algebraic variety $V$ if there exists a regular map $i: W \to V$ such that $i(W) \subseteq V$, and what will be called Property No. 1 holds: $P_d(i(W)) = P_d(W)$.

One says that $W$ descends relative to an algebraic variety $V$ if $i(W) \subseteq V$ and Property No. 1 holds. Obviously, given a finitely generated group $G$, the representation variety over an algebraic group $A$ of any quotient group $G/N$ descends relative to $R_A(G)$, but as the Baumslag-Solitar group $< x, y; xyx^{-1} = y^2 >$, shows, the descent needs not be proper.

Definition 1.4 (Proper Relative Falling, Relative Falling). An algebraic variety $T$ is said to have properly falling profile relative to an algebraic variety $V$ if there exists an algebraic variety $W$ descending properly relative to $V$ with $c = Dim(T) = Dim(W)$ and such that $P_c(T) = P_c(W)$.

$T$ is said to be falling relative to $V$ if the algebraic variety $W$ above only descends relative to $V$, $c = Dim(T) = Dim(W)$, and $P_c(T) = P_c(W)$.

Theorem 1.1. If the free group of rank 2 embeds into the not necessarily irreducible algebraic group $A$, then an $n$-generated group $G$ is free of rank $n$ provided that $Dim(R_A(G)) = c$ and $P_c(R_A(G)) = P_c(R_A(F_n))$, where $c = nDim(A)$.

Proof. The fact that $Dim(R_A(G)) = Dim(R_A(F_n))$ implies that $G$ is not free of rank less than $n$. Let’s assume that $G$ is not free of rank $n$. Then there is at least one non-trivial freely reduced word $w$ in $G$ that is a relator of $G$ and involves some of the $n$-generators of $G$. Since, by assumption, $A$ contains a free group of rank 2, it contains a free group on rank $m$ for any $m \geq 2$. Let $\{g_1, g_2, \ldots, g_t\}$ generate a free subgroup $F_t$ in $A$ of rank $t$ which is greater than or equal to the number of generators of $G$ appearing in the word $w$. Then $w(g_1, g_2, \ldots, g_t) \neq 1$ for any replacement of the generators of $G$ appearing in $w$ with different generators in $\{g_1, g_2, \ldots, g_t\}$. It must be so then that $R_A(G)$ descends properly relative to $R_A(F_n)$, a contradiction. So $G$ has to be a free group of rank $n$.

Theorem 1.2. Let the fg generated group $G$ have a faithful representation in the algebraic group $A$, and let $N$ be a proper normal subgroup of $G$. Then $R_A(G/N)$ descends properly relative to $R_A(G)$.

Proof. There is a point in $R_A(G)$ corresponding to the embedding of $G$ into $A$. Since $N$ is proper there is at least one word $w$ that is not the identity in $G$. Let $\rho$ be an embedding of $G$ in $A$, then $\rho(w) \neq I$ in $A$. Consequently, the irreducible component containing $\rho$ in $R_A(G)$ cannot be an irreducible component of the same dimension in $R_A(G/N)$. It follows then that its dimension is smaller, and thus at least one of the entries in the profile function $P_c(R_A(G))$ changes from that of
\( P_c(R_A(G/N)) \). Since \( R_A(G/N) \) injects properly into \( R_A(G) \) it must be so that the profile function of \( R_A(G/N) \) descends properly relative to the profile function of \( R_A(G/N) \), which is what we set out to prove.

**Corollary 1.1.** Let \( G_0, G_1, G_2, G_3, \ldots \) be a sequence of fg groups having the property that each \( G_i \) embeds into the algebraic group \( A \), and that \( G_{i+1} \) is a proper quotient group of \( G_i \) for all values of \( i \) in the sequence. There exists an integer \( k \) such that for all integers \( t \geq k \), \( G_k \cong G_i \).

**Proof.** Suppose this is not the case. Then \( R_A(G_0) \) contains an infinite sequence of representation varieties corresponding to the sequence of fg groups \( G_1, G_2, G_3, \ldots \) with the property that \( R_A(G_{i+1}) \) descends properly relative to \( R_A(G_i) \) for each value of \( i \). But since the Zariski topology is Noetherian this leads to a contradiction. Thus the associated sequence of representation varieties must stabilize, and since each of the \( G_i \) embeds into \( A \), it must be the case that the sequence of groups also stabilizes.

Next a lemma is introduced which is quite useful in giving a lower bound for the dimension of representation varieties of finitely generated groups.

**Lemma 1.2.** Let \( R_A(G) \) be the representation variety of a group \( G \) with a presentation of deficiency \( d \geq 0 \) in an algebraic group \( A \) of dimension \( t \). Then all irreducible components of \( R_A(G) \) have dimension greater than or equal to \( dt \). In particular, \( \text{Dim}(R_A(G)) \geq dt \).

**Proof.** We will apply the following theorem of Chevalley found in [RS]: Let \( \phi : X \to Y \) be a dominant morphism of irreducible varieties, and set \( r = \text{Dim}X - \text{Dim}Y \). Let \( W \) be a closed irreducible subset of \( Y \). If \( Z \) is an irreducible component of \( \phi^{-1}(W) \) which dominates \( W \), then \( \text{Dim}Z \geq \text{Dim}W + r \). In particular, if \( y \in \phi(X) \), each component of \( \phi^{-1}(y) \) has dimension at least \( r \).

One may employ the stated result when dealing with a \( t \) dimensional reducible algebraic group \( A \) since all irreducible components are equi-dimensional. Set \( W = I^m \), where \( I^m \) is the identity in the \( tm \) dimensional algebraic group \( A^m \). We will assume that \( G \) has a presentation on \( n \) generators and \( m \) non-trivial relators with \( n - m = d \geq 0 \). The \( m \) relators in the presentation of \( G \) can be used, via evaluation of arbitrary \( n \) tuples from \( A^n \), to give rise to a regular map \( \phi : A^n \to A^m \) whose fiber over \( I^m \) is nothing but \( R_A(G) \). Let \( A_0^m \) be the irreducible component in \( A^m \) containing the identity. Then \( \phi(A^n) \) dominates at least \( I^m \) since the identity of \( A^n \) maps onto it. In particular, \( \phi \) dominates an irreducible \( q \) dimensional subvariety \( M_0 \) containing the identity in \( \phi(A^n) \cap A_0^m \). Obviously, \( I^m \) is a closed irreducible subset of \( M_0 \). Now, if \( r = tn - q \), it follows by the result stated that every irreducible component \( Z \) that dominates \( W \) has \( \text{Dim}Z \geq \text{Dim}W + r \). In particular, \( \text{Dim}Z \geq tn - tm \) since \( r \geq tn - tm \). In other words, \( \text{Dim}Z \geq t(\text{def}(G)) \), where \( \text{def}(G) \) is the deficiency of the presentation of \( G \) chosen.

**Lemma 1.3.** Let \( A \) be an irreducible algebraic group of dimension \( d \) and let \( G \) have a presentation of deficiency \( n - m \). If \( \text{Dim}(R_A(G)) = d(n - m) \), then for some positive integer \( i \),

\[
P_{d(n-m)}(R_A(G)) = (i, 0, \ldots, 0).
\]
Proof. A group with irreducible components of dimension \( \geq d(n-m) \) by Lemma 1.2. But, \( \text{Dim}(R_A(G)) = d(n-m) \). It follows then that there are only zeros in the rest of the entries of the profile function of its representation variety.

Example. Let \( G = \langle x, y, z \mid x^p y^q z^t = 1 \rangle \), where \( p, q, \) and \( t \) are integers greater than one with at least one of them greater than two. If \( A = SL(2, \mathbb{C}) \), we have that \( R_A(G) = (i, 0, 0, 0, 0, 0, 0) \), where \( i \) is an integer greater than one. This follows directly from Theorem 1.2 of [L4]. It turns out, by a result in [BC], that if all of the integers are exactly 2 then \( i = 1 \). As we shall see in the sequel, the situation changes if \( A = PSL(2, \mathbb{C}) \).

**Theorem 1.3.** Let \( G \) be the fundamental group of either a compact orientable surface of genus \( g \geq 1 \) or a non-orientable surface of genus \( g \geq 4 \), and let \( A = SL(2, \mathbb{C}) \). Suppose that \( N \) is a non-trivial normal subgroup in \( G \). Let \( \text{Dim}(R_A(G)) = d \), then

1. \( P_d(R_A(G)) \) has a one in its left most entry and zeros in all other entries,
2. \( N_d(R_A(G/N)) = 0 \), and
3. \( P_d(R_A(G/N)) \neq P_d(R_A(G)) \).

Proof. By [RBC], we know that compact orientable surface groups of genus \( g \geq 1 \) have irreducible representation varieties in \( SL(2, \mathbb{C}) \), and that the same holds for non-orientable surface groups of genus \( g \geq 3 \). This proves (1). Now the orientable surface groups of genus \( g = 1 \) embed into \( SL(2, \mathbb{C}) \), and by [BGSP] all of the orientable surface groups of genus \( g \geq 2 \) as well as all the non-orientable surface groups of genus \( g \geq 4 \) embed into \( SL(2, \mathbb{C}) \) since they are fully residually free and \( SL(2, \mathbb{C}) \) is a connected complex semisimple Lie group; so Theorem 1.2 yields (2) together with (1). Part (3) is a result of (2).

**Theorem 1.4.** Let \( A \) be a connected complex semisimple algebraic group, \( G \) a non-abelian fg fully residually free group, and \( N \) a non-trivial normal subgroup of \( G \). Then \( P_d(R_A(G)) \neq P_d(R_A(G/N)) \).

Proof. By [BGSP], any connected semisimple Lie group contains a dense copy of any non-abelian fully residually free group. Thus \( G \) embeds into \( A \). The result follows from Theorem 1.2.

**Proposition 1.1.** Let \( G \) be a finitely generated free abelian group, \( A = SL(2, \mathbb{C}) \), and suppose that \( \text{Dim}(R_A(G)) = d \), and that \( N \) is non-trivial normal subgroup of \( G \). Then \( P_d(R_A(G)) \neq P_d(R_A(G/N)) \).

Proof. Without loss of generality assume that \( G \) is generated by \( \{x_1, \ldots, x_n\} \). Let \( L \) be the list of the first \( n \) odd primes denoted thus: \( p_1, p_2, \ldots, p_n \). With each prime \( p_k \) in the list \( L \) associate a two by two matrix \( m_k \) with the property that \( m_k \) has entries \( a_{i,j} = 0 \) if \( i \neq j \), \( a_{1,1} = p_k \), and \( a_{2,2} = \frac{1}{p_k} \). The matrix \( m_k \) lies in \( A \), and it is straightforward to see that the homomorphism \( \phi : G \rightarrow A \) given by \( \phi(x_i) = m_i \) is an embedding of \( G \) into \( A \). The conclusion of the proposition follows from Theorem 1.2.

**Two: Projection of Solutions and Word Equations**

In this section we will assume that all algebraic groups are irreducible. If \( A \) is an algebraic group over an algebraically closed field of characteristic zero, \( N \) is a
closed normal subgroup of $A$, and $\pi : A \rightarrow A/N$ is the projection map, then it is well known that $A/N$ is an algebraic group. If $N$ is finite, then it lies in the center of $A$ whenever $A$ is irreducible. Note that every finite normal subgroup of an algebraic group is necessarily closed.

**Definition 2.1.** Let $A$ be an algebraic group, $N$ a finite normal subgroup of $A$, and $\pi : A \rightarrow A/N$ the projection map. If $S \subseteq A$, then define $\pi(S) = \{\pi(s) \mid s \in S\}$ to be the projection of $S$ in $A/N$.

**Proposition 2.1.** Let $S \subseteq A$ be an algebraic set, then $\pi(S)$ is also an algebraic set.

**Proof.** The projection map $\pi : A \rightarrow A/N$ is a surjective open and closed regular map and thus, since $N$ is finite, maps closed algebraic sets of $A$ to closed algebraic sets of $A/N$.

The next result is a standard fact in the theory of algebraic groups. We omit the proof.

**Proposition 2.2.** Let $G$ and $G'$ be algebraic groups with finite normal subgroups $H$ and $H'$, respectively. Then $G/H \times G'/H'$ and $(G \times G')/(H \times H')$ are isomorphic as algebraic groups.

**Induction yields:**

**Corollary 2.1.** Let $\{G_i\}$ be a finite collection of algebraic groups and $\{H_i\}$ be corresponding finite normal subgroups, respectively, where $i \in \{1, \ldots, n\}$. Then the Cartesian product of the $\{G_i\}$’s modulo the Cartesian product of the $\{H_i\}$’s is isomorphic, as an algebraic group, to the Cartesian product of the $\{G_i/H_i\}$’s.

**Notation.** Denote the canonical projection map from an algebraic group $A$ to the algebraic group $A/H$ by $\pi$. Please note that under the proper context $\pi$ shall also denote the canonical projection map from $A^n$ to $(A/H)^n$.

**Proposition 2.3.** Let $H$ be a finite normal subgroup of the algebraic group $A$ and let $\pi : A^n \rightarrow (A/H)^n$ be the projection map. Then:

1. $\text{Dim}(A/H) = \text{Dim}(A)$
2. $\text{Dim}(A^n) = \text{Dim}((A/H)^n) = n\text{Dim}(A)$
3. If $V$ is an algebraic set in $A^n$, then $\text{Dim}(V) = \text{Dim}(\pi(V))$.
4. If $V$ is an irreducible algebraic set, then $\pi(V)$ is also irreducible.

**Proof.**

1. Under the conditions stipulated, $\text{Dim}(A/H) = \text{Dim}(A) - \text{Dim}(H)$; but $H$ is finite, and thus has dimension zero.
2. The dimension of the product of a finite number of algebraic varieties is the sum of the dimension of its factors. By (1), it is the case that $\text{Dim}(A/H) = \text{Dim}(A)$. It follows that $\text{Dim}(A^n) = \text{Dim}((A/H)^n) = n\text{Dim}(A)$.
3. The map $\pi$ is regular and has finite fiber over each point of $\pi(V)$. It follows that $\text{Dim}(V) = \text{Dim}(\pi(V))$.
4. If $V$ is irreducible and $\pi(V)$ is not, then the irreducibility of $V$ is contradicted since the projection map $\pi$ is a closed regular map.
Given an algebraic group $A$, a free group $F_n$ on $\{x_1, \ldots, x_n\}$, a non-trivial word $w = w(x_1, \ldots, x_n)$ in $F_n$, and the expression $w = h$, where $h \in A$, one can think of $n$-tuples $(m_1, \ldots, m_n)$ of elements of $A$ as solutions to the equation $w = h$ provided that an orderly replacement in $w$ of each $x_i$ with $m_i$ makes valid the statement $w(m_1, \ldots, m_n) = h$. This is the same as saying that there exists a representation $\rho \in R_A(F_n)$ having the property that $\rho(w) = h$. Both ways of looking at the situation will be used interchangeably in the sequel. Note that it is not necessary that all generators of $F_n$ appear in the word $w$.

**Theorem 2.1.** Let $A$ be an algebraic group and $H$ a finite normal subgroup of $A$ of order $m$. Let $w$ be a non-trivial freely reduced word in the free group $F_n$ with basis $\{a_1, \ldots, a_n\}$. Then the solutions of the equation $w = 1$ in $A/H$ are given by the projections to $A/H$ of the solutions in $A$ of the equations $\bigcup_{i=1}^{m} \{ w = h_i \mid h_i \in H \}$.

**Proof.** If $(a_1, \ldots, a_n)$ is a solution in $A$ to the equation $w = h_i$, with $h_i \in H$, then it is clear that $\pi(w(a_1, \ldots, a_n)) = \pi(h_i)$, where $\pi : A \to A/H$ is the projection map. Thus, $\pi(w(a_1, \ldots, a_n)) = 1$. And so solutions in $A$ of $w = h_i$ ($h_i \in H$) correspond to solutions over $A/H$ of $w = 1$. Now suppose that $w(b_1, \ldots, b_n) = 1$, where $b_i \in A/H$; then each $b_i$ corresponds to a coset $c_iH$, for some $c_i \in A$. In particular, $w(c_1, \ldots, c_n) = t$, where $t \in \pi^{-1}(1)$; but $\pi^{-1}(1) = H$.

**Proposition 2.4.** $V$ has falling profile, or properly falling profile relative to $\pi^{-1}(V)$, where $V \subseteq (A/H)^n$ is a closed algebraic set.

**Proof.** $\pi^{-1}(V)$ is Zariski closed, and since $H$ is finite, the coordinate algebra of $\pi^{-1}(V)$ is an integral extension of the coordinate algebra of $V$. Consequently, over any prime ideal of the coordinate algebra of $V$ there is a prime ideal of the coordinate algebra of $\pi^{-1}(V)$. In particular, for every prime ideal chain of length $l$ in the coordinate algebra of $V$ there corresponds a prime ideal chain in the coordinate algebra of $\pi^{-1}(V)$ at least as long as $l$. Because the Krull dimension of a coordinate algebra corresponding to an algebraic variety is nothing but the dimension of the variety, the result follows.

**Example.** Let $A = SL(2, \mathbb{C})$ and $A/H = PSL(2, \mathbb{C})$. Let $V = R_{PSL(2,\mathbb{C})}(\mathbb{Z}_2)$. Then it is easy to see that $Dim(V) = 2$ and $P_2(V) = (1, 0, 1)$.

Notice, however, that $P_2(\pi^{-1}(V)) = P_2(\{m \mid m^2 = \pm 1, m \in SL(2, \mathbb{C})\})$, the right side of which is nothing but $P_2(R_{SL(2,\mathbb{C})}(\mathbb{Z}_2) \cup \{m \mid m \in SL(2, \mathbb{C}), tr(m) = 0\}) = (1, 0, 2)$. Thus, $V$ has properly falling profile relative to $\pi^{-1}(V)$.

**Three: Representation Varieties in $PSL(2, \mathbb{C})$ for a Class of One-Relator Groups**

The next theorem gives a method for computing $Dim(R_A(G))$, where $A = PSL(2, \mathbb{C})$, for certain classes of one relator cyclically pinched groups, and also gives conditions guaranteeing the reducibility of the corresponding representation variety.

**Theorem 3.1.**

1. Let $G = \langle X, y \rangle; W(X) = y^n$, where $Card(X) = n, p > 1$, and let $W(X)$ be a non-trivial freely reduced word in the free group on $X$. Then $Dim(R_A(G)) = \max\{3n, Dim(R_A(G')) + 2\} \leq 3n + 1$, where $G' = \langle X; W(X) = 1 \rangle$. 
2. If $Dim(R_A(G')) + 2 \geq 3n$ in (1), then $R_A(G)$ is reducible.
Before proceeding we shall adopt the following notation.

Notation. Given an algebraic group \( \mathfrak{A} \), an element \( a \in \mathfrak{A} \), and an integer \( p \), denote by \( \Omega_\mathfrak{A}(p,a) \) the closed set \( \{ m \mid m^p = a, m \in \mathfrak{A} \} \). When the context is clear \( \Omega_\mathfrak{A}(p,a) \) will be simply denoted as \( \Omega(p,a) \).

Example. Here’s an illustration of how Theorem 3.1 may be used. Consider the group \( G = \langle x, y, z ; [x, y] = z^p \rangle \), where \( p \geq 2 \). We will compute \( \text{Dim}(R_A(G)) \).

By Theorem 3.1, \( \text{Dim}(R_A(G)) = \text{Max}(6, \text{Dim}(R_A(G')) + 2) \leq 3n + 1 \), where \( G' = \langle x, y ; [x, y] = 1 \rangle \). Using \([RBC]\) we know that \( R_{SL(2,C)}(G') \) is irreducible and four dimensional. By Proposition 2.3 (3), we know that \( \pi(R_{SL(2,C)}(G')) \) maps to a four dimensional component in \( PSL(2, C)^2 \). By Theorem 2.1, it is the case that \( \pi(V) \) is also in \( R_A(G') \), where \( V = \{ (m_1, m_2) \mid [m_1, m_2] = -I, m_1, m_2 \in SL(2, C) \} \).

Thus, we must compute \( \text{Dim}(\pi(V)) \). It suffices to compute \( \text{Dim}(V) \). It is easy to deduce that \( tr(m_1) = tr(m_2) = 0 \). Thus, \( V \) is a subvariety of the four dimensional irreducible variety \( \Omega(2, -I) \times \Omega(2, -I) \subset SL(2, C) \times SL(2, C) \). Because there is at least one pair of matrices \( (m_1, m_2) \in \Omega(2, -I) \times \Omega(2, -I) \) that is not\(^4\) in \( V \), we have that \( \text{Dim}(V) < 4 \). And so \( \text{Dim}(\pi(V)) < 4 \). We immediately know then that \( \text{Dim}(R_A(G)) = 6 \). Incidently, we know by Lemma 1.2 that the precise dimension of \( \pi(V) \), if \( \pi(V) \) is an irreducible component of \( R_A(G') \), has to be three since the deficiency of the presentation \( G' \) is one, and \( PSL(2, C) \) is of dimension three. Theorem 3.1 also guarantees for us the reducibility of \( R_A(G) \).

Incidently, when \( p = 2 \), the above group is isomorphic to \( \langle x, y, z ; x^2y^2z^2 = 1 \rangle \), the closed non-orientable surface group of genus three since it is known that a closed non-orientable surface groups of genus \( k \geq 3 \) can be given the presentation \( \langle x_1, x_2, \ldots, x_k ; [x_1, x_2]x_3^2 \cdots x_k^2 \rangle \). By a result of \([BC]\), we have that \( R_{SL(2,C)}(\langle x, y, z ; [x, y] = z^2 \rangle) \) is irreducible, something that is not the case for \( R_A(\langle x, y, z ; [x, y] = z^2 \rangle) \).

Proof of (1) of Theorem 3.1.
The proof will employ the following lemmas stated independently for transparency.

Lemma 3.1. \( \text{Dim}(R_A(\mathbb{Z}_t)) = 2 \), for \( t \geq 2 \).

Proof. This is a special case of Theorem 4.4 below.

The next lemma will prove quite useful in the sequel.

Lemma 3.2. A dominating and regular map from an affine variety to an irreducible variety contains an open set in its image.

Proof. Standard fact; see \([MD]\).

At this juncture, the stage is set for the proof of the theorem. We begin by introducing the following projection map from \( R_A(G) \) to \( R_A(F_n) \):

\[ \Phi : R_A(G) \to R_A(F_n) \]

given by \( \Phi(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_n) \).

\(^4\)Example: Take \( m_1 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \) and \( m_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \) such that \( a^2 \neq -b^2 \).
Let $V = \{ \rho \in R_A(G) \mid \rho(W(X)) = I \}$; then it is easy to see that $\Phi(V) = R_A(G')$, and consequently, since $G'$ is of deficiency $(n-1)$ by Lemma 1.2, that $\text{Dim}(R_A(G')) \geq \text{Dim}(\text{PSL}(2, \mathbb{C}))(n-1)$.

Now, since $G'$ is not free on $n$ generators, given that $W$ is a freely reduced non-trivial word, it follows that $\text{Dim}(R_A(G')) < \text{Dim}(\text{PSL}(2, \mathbb{C}))(n-1)$. Thus,

\begin{equation}
3(n-1) \leq \text{Dim}(\Phi(V)) < 3n.
\end{equation}

Furthermore, for every point $m$ in $\Phi(V)$ one has that $\text{Dim}(\Phi^{-1}(m)) = 2$, a consequence of Lemma 3.1. It follows that

\begin{equation}
\text{Dim}(V) = \text{Dim}(\Phi(V)) + 2 = \text{Dim}(R_A(G')) + 2.
\end{equation}

By (3-1), we have

\begin{equation}
\text{Dim}(R_A(G')) + 2 \leq 3n + 1.
\end{equation}

Let $O_2 = R_A(F_n) - \Phi(V)$. Clearly, $O_2$ is an open set in $R_A(F_n)$. Furthermore, if $m \in O_2$ lies in $\Phi(R_A(G)) \cap O_2$ then it is the case that $\text{Dim}(\Phi^{-1}(m)) = 0$. By Lemma 3.2, $\Phi(R_A(G))$ contains an open set in $O$ in $R_A(F_n)$. Thus, the irreducibility of $R_A(F_n)$ insures that $O_2 \cap \Phi^{-1}(m)$ is an open set, and $\Phi^{-1}(m) \subseteq \{ R_A(G) - V \}$ is open and dense in at least one component of largest dimension in $\{ R_A(G) - V \}$. Moreover, $\text{Dim}(\Phi^{-1}(m)) = 0$ for all $m \in O_1$, and thus $\text{Dim}(\Phi^{-1}(O_1)) = 3n$. This immediately implies that $\text{Dim}(R_A(G) - V) = 3n$.

Since

$$R_A(G) = \{ R_A(G) - V \} \cup \{ V \},$$

we have $\text{Dim}(R_A(G)) = \text{Max}\{ \text{Dim}(R_A(G) - V), \text{Dim}(V) \} = \text{Max}\{3n, \text{Dim}(R_A(G')) + 2\}$. Therefore, $\text{Max}\{3n, \text{Dim}(R_A(G')) + 2\} \leq 3n + 1$ by (3-3) or (3-2). The proof of (1) is complete.

To complete the proof, it is necessary to introduce the following lemma:

**Lemma 3.3.** Let $W$ be an algebraic variety of dimension $n$ and $V''$ a proper subvariety of $W$. If $\text{Dim}(W - V'') = n$ and $\text{Dim}(V'') \geq n$, then $W$ is reducible.

The upshot of Lemma 3.3 is that if $\text{Dim}(R_A(G')) + 2 \geq 3n$, then $R_A(G)$ is reducible. The proof of Theorem 3.1 is now complete if the lemmas assumed can be established.

**Proof of Lemma 3.1.** By Proposition 2.3, $\pi$ leaves the dimensions fixed. By Theorem 2.1, $R_A(\mathbb{Z}_p) = \pi(\{ \Omega(p, I) \cup \Omega(p, -I) \})$. Thus, $\text{Dim}(R_A(\mathbb{Z}_p)) = \text{Dim}(\{ \Omega(p, I) \cup \Omega(p, -I) \})$. But by Proposition 4.2, $\text{Dim}(\{ \Omega(p, I) \cup \Omega(p, -I) \}) = 2$ whenever an integer $p \geq 2$ is chosen. This completes the proof.

**Proof of Lemma 3.3.** Assume that $W$ is irreducible of dimension $n$. Then $\text{Dim}(W - V'') = n$ implies $\text{Dim}(V'') < n$, else a contradiction would arise.
FOUR: ALGEBRAIC VARIETIES WITH CONDITIONS

If $A$ is an algebraic group and $t \in A$, then the left translation $\rho_t : A \to A$ given by $\rho_t(v) = tv$ is an isomorphism of the algebraic variety $A$ onto itself. Next a condition is defined insuring that certain closed sets in $A$ are mapped to themselves by translations $\rho_t$ associated with any element $t$ in the center of the algebraic group $A$. We could define this for more general settings, but the case $A = R_{SL(2,\mathbb{C})}(F_n)$ is enough for our treatment.

**Definition 4.1. (The $\pm$ Condition).** An algebraic variety $V \subseteq SL(2,\mathbb{C})^n$ satisfies the $\pm$ Condition if whenever $v = (m_1, \ldots, m_n)$ is in $V$, then so is $(\pm m_1, \ldots, \pm m_n)$.

So an algebraic variety $V$ satisfies the $\pm$ Condition provided it is mapped to itself by all isomorphisms of $SL(2,\mathbb{C})^n$ induced by central translations.

**Proposition 4.1.** Let $w$ be a non-trivial freely reduced word in $F_n = \langle x_1, \ldots, x_n \rangle$, the free group of rank $n$. Let $V_{-1} = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = -I \}$, and $V_1 = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = I \}$. Suppose that each of these two algebraic varieties satisfies the $\pm$ Condition. If $\pi : R_{SL(2,\mathbb{C})}(F_n) \to R_{PSL(2,\mathbb{C})}(F_n)$ is the canonical projection map, the following holds: $\pi(V_{-1}) \cap \pi(V_1) = \emptyset$ and disconnected, provided that $\pi(V_{-1})$ and $\pi(V_1)$ are non-empty. In particular, $R_{PSL(2,\mathbb{C})}(\langle x_1, \ldots, x_n : w = 1 \rangle)$ is disconnected, and so are the sets $V_{-1}$ and $V_1$ in $R_{SL(2,\mathbb{C})}(F_n)$.

**Proof.** The space $R_A(F_n)$ is nothing but the algebraic group $A^n$, where $A$ is an algebraic group. Since $\pi$ is a closed regular map, $\pi(V)$ is closed when $V$ is a closed algebraic set in $SL(2,\mathbb{C})^n$, as are the algebraic sets $V_{-1}$ and $V_1$. Let $v \in V_{-1}$. We must show that $\pi(v)$ is not in $\pi(V_1)$. Suppose it lies in $\pi(V_1)$; so $\pi^{-1}(v)$ is a solution to the $SL(2,\mathbb{C})$ equation $w(x_1, \ldots, x_n) = I$ obtained from the word $w$. This is a contradiction since $\pi^{-1}(v)$ must satisfy the $\pm$ Condition and it is a solution to the $SL(2,\mathbb{C})$ equation $w(x_1, \ldots, x_n) = -I$; so it can’t be a solution to $w(x_1, \ldots, x_n) = I$. Using the same argument a contradiction arises when we assume that some $v \in V_1$ lies in $\pi(V_{-1})$. Because the projection map is also an open map $\pi(V_{-1}) \cup \pi(V_1)$ is the disjoint union of two open sets in $R_{PSL(2,\mathbb{C})}(F_n)$ when both sets are non-empty; under such an assumption, it follows that their union is disconnected, and in particular $R_{PSL(2,\mathbb{C})}(\langle x_1, \ldots, x_n : w = 1 \rangle)$ is a disconnected variety since $R_{PSL(2,\mathbb{C})}(\langle x_1, \ldots, x_n : w = 1 \rangle) = \pi(V_{-1}) \cup \pi(V_1)$. The sets $V_{-1}$ and $V_1$ are disconnected since a regular map is continuous in the Zariski topology, and their image under $\pi$ is disconnected as established above. It follows that they could not have been connected in the first place.

The $\pm$ Condition for an algebraic variety $V$ is $SL(2,\mathbb{C})^n$ is equivalent to the following mapping criteria stipulated in the next lemma.

**Lemma 4.1.** An algebraic variety $W$ in $R_{SL(2,\mathbb{C})}(F_n)$ satisfies the $\pm$ Condition iff $\pi^{-1}(\pi(W)) = W$, where $\pi : SL(2,\mathbb{C})^n \to PSL(2,\mathbb{C})^n$ is the canonical projection.

**Proof.** Suppose an arbitrary point $(m_1, \ldots, m_n) \in W$ is chosen, and that $W$ satisfies the $\pm$ Condition. Then $\pi^{-1}(\pi(m_1, \ldots, m_n)) = \pi^{-1}([m_1] \cdots [m_n]) = (\pm m_1, \ldots, \pm m_n) \in W$, where $[m_i]$ denotes the equivalence class of $m_i$ in $PSL(2,\mathbb{C})$. Since $W$ is the union of all the fibers $\pi^{-1}(v)$ of points $v$ in $\pi(W)$, it follows that $\pi^{-1}(\pi(W)) = W$. Now, suppose that $\pi^{-1}(\pi(W)) = W$. If $(m_1, \ldots, m_n) \in W$, then $\pi^{-1}(\pi(m_1, \ldots, m_n)) \in W$. However, $\pi^{-1}(\pi(m_1, \ldots, m_n)) = (\pm m_1, \ldots, \pm m_n)$. Therefore, $(\pm m_1, \ldots, \pm m_n) \in W$ and the $\pm$ Condition is met.
Example. Consider the word $x^2$ in the free group of rank one, and the resulting equations $x^2 = I$, and $x^2 = -I$ in $SL(2, \mathbb{C})$. It is not difficult to see that the variety $V_1$ obtained as solutions to the first equation is $\{ \pm I \}$, and that $V_1$ satisfy the $\pm$ Condition; $V_1$ is a reducible variety with two points as irreducible components; the individual components are permuted by the isomorphism of $SL(2, \mathbb{C})$ induced by left translation with the element $-I$. Now, let $V_{-1}$ be the solutions to the second equation $x^2 = -I$; then $V_{-1}$ consists of all matrices of trace zero, and since $V_{-1}$ is the orbit under conjugation of a matrix of trace other than $\pm 2$, it is an irreducible variety of $SL(2, \mathbb{C})$. Notice also that each point of the variety satisfies the $\pm$ Condition as well, and that left translation by the element $-I$ maps this variety to itself. Observe that $\pi(V_1)$ is the identity element in $PSL(2, \mathbb{C})$, and $\pi(V_{-1})$ are the equivalence classes of all matrices of trace zero in $PSL(2, \mathbb{C})$. So no component of $V_1$ mapped into a component of $V_{-1}$, and no component of $V_{-1}$ mapped into a component of $V_1$. Incidentally, notice also that one of the components of $V_1$ collapsed, making $\pi(V_1)$ an irreducible variety.

Theorem 4.1. Let $w$ be a non-trivial freely reduced word in the commutator subgroup of $F_n = \langle x_1, \ldots, x_n \rangle$, and $G = \langle x_1, \ldots, x_n ; w = 1 \rangle$. If $R_{SL(2,\mathbb{C})}(G)$ is an irreducible variety and $V_{-1} = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = -I \} \neq \emptyset$, then $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$.

Proof. If $w$ is a non-trivial freely reduced word in the commutator subgroup of $F_n$ then $V_{-1} = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = -I \}$, and similarly $V_1 = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = I \}$ are properly contained in $R_A(F_n)$, for any algebraic group where $F_n$ embeds. In particular, this is the case for $SL(2, \mathbb{C})$ since by Sanov, [SN], free groups embed in it. Now, because the word $w$ is in the commutator subgroup, and $\{ I, -I \}$ is the center of $SL(2, \mathbb{C})$, it can be checked that the algebraic varieties $V_{-1}$, and $V_1$ satisfy the $\pm$ Condition; for, by Wicks [WJ] when an arbitrary generator $x$ of $F_n$ appears in $w$, it is also the case that the inverse of the generator appears, and the number of occurrences of $x$ in $w$ is exactly equal to the number of occurrences of $x^{-1}$. Now, by assumption the algebraic variety $R_{SL(2,\mathbb{C})}(G)$ is irreducible, and non-empty since it contains at least the identity of the algebraic group $R_{SL(2,\mathbb{C})}(F_n)$. One can easily see that the algebraic variety $V_1 = R_{SL(2,\mathbb{C})}(G)$. The set $\pi(V_1)$ is non-empty, and by Proposition 2.3, the algebraic set $\pi(V_1)$ is also irreducible and of the same dimension as its pre-image; now by Proposition 4.1, $\pi(V_{-1}) \cap \pi(V_1) = \emptyset$ and by assumption $V_{-1}$ is non-empty and thus neither is $\pi(V_{-1})$. Thus $\pi(V_{-1})$ and $\pi(V_1)$ share no irreducible components of any dimension. So $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(\pi(V_{-1}) \cup \pi(V_1))$. But, by Theorem 2.1, it is the case that $R_{PSL(2,\mathbb{C})}(G) = \pi(V_{-1}) \cup \pi(V_1)$. It follows then that $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$, which is what we set out to show.

Corollary 4.1. Let $G$ be the fundamental group of a compact orientable surface of genus $g \geq 1$, then $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$, where $d = \text{Dim}(R_A(G))$, and $A$ is either $SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C})$.

Proof. A presentation of such a group can be found with its relator $w$ in the commutator subgroup of the free group on its $2g$ generators; thus we can obtain two algebraic varieties $V_1$, and $V_{-1}$ in $SL(2, \mathbb{C})^{2g}$ satisfying the $\pm$ Condition by Theorem 4.2. Now, by [RBC] we know that $R_{SL(2,\mathbb{C})}(G)$ is irreducible. The result follows once we show that $V_{-1}$ is not empty. Well, there are matrices $(m_1, m_2)$ in $SL(2, \mathbb{C})^2$ with $[m_1, m_2] = -I$, where $[m_1, m_2]$ stands for the commutator of
the matrices $m_1$ and $m_2$. It follows that a word in the free group $F_{2g}$ of the type 
$x_1, x_2, [x_3, x_4], \ldots, [x_{2g-1}, x_{2g}]$ can be evaluated over $SL(2, \mathbb{C})^{2g}$ to equal $-I$ simply
by mapping each of the generators $x_i$ for $i < 2g - 1$ to $I$ and setting $x_{2g-1} = m_1$ and $x_{2g} = m_2$. So the algebraic variety $V_{-1}$ is not empty; the result follows.

**Theorem 4.2.** Let $w$ be a non-trivial and freely reduced word in the free group $F_n$ with even exponent sum on each generator. Let $G = \langle x_1, \ldots, x_n; w = 1 \rangle$. If $R_{SL(2,\mathbb{C})}(G)$ is an irreducible variety and $V_{-1} = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = -I \} \neq \emptyset$, then $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$.

**Proof.** Essentially the same argument as was given in the proof of Theorem 4.1 applies in this setting. If the exponent sum of each generator in the word $w$ is even. Then since $\{ \pm I \}$ is the center of $SL(2, \mathbb{C})$, substituting $-I$ for any generator $x_i$ appearing in $w$ it is the same as raising $-I$ to an even power, and thus results in $I$. Thus a replacement as stipulated by the $\pm$ Condition does not affect the algebraic varieties $V_{-1}$ or $V_1$. The irreducibility of $R_{SL(2,\mathbb{C})}(\langle x_1, \ldots, x_n; w = 1 \rangle)$, and the equi-dimensionality between $R_{SL(2,\mathbb{C})}(\langle x_1, \ldots, x_n; w = 1 \rangle)$ and its image under $\pi$ guarantees that if the set $V_{-1}$ is non-empty $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$, an immediate consequence of Proposition 4.1.

The next lemma guarantees that $V_{-1}$ is always non-empty provided that a minor assumption on the word $w$ is made.

**Lemma 4.2.** Let a freely reduced word $w$ in $F_n = \langle x_1, \ldots, x_n \rangle$ be such that it has even exponent sum not equal to zero on one of the generators appearing in it, then the algebraic variety $V_{-1} = \{ \rho | \rho \in R_{SL(2,\mathbb{C})}(F_n), \rho(w) = -I \} \neq \emptyset$. If, in addition, $V_{-1}$ and $V_1$ each satisfy the $\pm$ Condition, then $\dim(V_{-1}) \geq 3(n-1)$.

**Proof.** Assume that $x_i$ is a generator with even exponent sum other than zero in $w$. It will be shown that there is at least one point in $V_{-1}$. Since $-I$ is its own inverse we can assume that $x_i$ has exponent sum $s \geq 2$. Then, in $w$ substitute all occurrences of $x_i$ with a value $m \in \mathbb{C}(s, -I)$, and all other variables replace with the matrix $I$. Then the point $(I, I, \ldots, m, I, \ldots)$, where $m$ corresponds to the $x_i$ entry, is a point in $V_{-1}$. Now, if in addition $V_{-1}$ and $V_1$ each satisfy the $\pm$ Condition, we have the consequences of Proposition 4.1 guaranteeing that for $G = \langle x_1, \ldots, x_n; w = 1 \rangle$, the irreducible components of $R_{PSL(2,\mathbb{C})}(G)$ contained in $V_{-1}$ are disjoint from those of $V_1$. By Lemma 1.2, all irreducible components of $R_{PSL(2,\mathbb{C})}(G)$ have dimension $\geq 3(n-1)$. We are done.

**Corollary 4.2.** Let $G$ be the fundamental group of an compact non-orientable surface of genus $g \geq 3$, then $P_d(R_{SL(2,\mathbb{C})}(G)) \neq P_d(R_{PSL(2,\mathbb{C})}(G))$, where $d = \dim(R_A(G))$, and $A$ is either $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C})$.

**Proof.** Obviously a presentation of such group can be found with its relator $W$ giving rise to two algebraic varieties $V_{-1}$ and $V_1$ in $SL(2,\mathbb{C})$ satisfying the $\pm$ Condition. By [BC] we know that $R_A(G)$ is irreducible when $A = SL(2,\mathbb{C})$. By Proposition 4.1, such a group has a reducible representation variety over $PSL(2,\mathbb{C})$. The result follows.

**Definition 4.2 (Lift of a Representation).** Let $\rho \in R_{PSL(2,\mathbb{C})}(G)$ be a representation of a finitely generated group $G$. Then $\rho$ is said to have a lift if there exists $\phi \in R_{SL(2,\mathbb{C})}(G)$ such that $\pi \circ \phi = \rho$.

**Observation.** If a group $G$ is generated by the set $\{x_1, \ldots, x_n\}$, then $R_A(G)$ is in one-to-one correspondence with the set $\{ (\rho(x_1), \ldots, \rho(x_n)) \in A^n | \rho \in R_A(G) \}$. 

(REPRESENTATION VARIETY) 15
Theorem 4.3. Let $G$ be a one relator $n$-generated group whose relator is a non-trivial word $w$ in $F_n = <x_1, \ldots, x_n>$ which gives rise to algebraic varieties $V_1$ and $V_{-1}$, as in Proposition 4.1, satisfying the $\pm$ Condition. Then no representation $\rho \in R_{PSL(2,\mathbb{C})}(G)$ with $(\rho(x_1), \ldots, \rho(x_n)) \in \pi(V_{-1})$ has a lift.

Proof. Suppose that a representation $\rho \in R_{PSL(2,\mathbb{C})}(G)$ with $(\rho(x_1), \ldots, \rho(x_n)) \in \pi(V_{-1})$ has a lift. Then, there exists a homomorphism $\phi : G \rightarrow SL(2,\mathbb{C})$ such that $\pi \circ \phi = \rho$. In particular, using the observation above, we have after extending $\phi$ to the $n$-tuple $(x_1, \ldots, x_n)$ that $\pi(\phi(x_1), \ldots, x_n)$ equals $(\rho(x_1), \ldots, \rho(x_n)) = m = (m_1, \ldots, m_n) \in PSL(2,\mathbb{C})^n$, and it is the case that $\pi^{-1}(m) = (\pm m_1, \ldots, \pm m_n)$. But since $V_{-1}$ satisfies the $\pm$ Condition, the $n$-tuples of matrices $\pi^{-1}(m)$ all lie in $V_{-1}$. Now $(\phi(x_1), \ldots, \phi(x_n)) \in \pi^{-1}(\rho(x_1), \ldots, \rho(x_n))$. But since $V_{1}$ satisfies the $\pm$ Condition, this implies that $\phi(w) = -1$, a contradiction since $w$ is the relator of $G$. So no representation $\rho$ in $\pi(V_{-1})$ can be lifted.

Lemma 4.3. Let $V_1$ and $V_2$ be non-trivial algebraic varieties in $SL(2,\mathbb{C})^n$ which satisfy the “Minus Condition” stipulated below.

The Minus Condition:

1. If $m = (m_1, m_2, \ldots, m_n) \in V_1$ then $-m \not\in V_2$.
2. If $m = (m_1, m_2, \ldots, m_n) \in V_2$ then $-m \in V_1$.

Then $V_1$ and $V_2$ are isomorphic, $\pi(V_1) = \pi(V_2)$, and if either $V_1$ or $V_2$ is irreducible, then $\pi(V_1 \cup V_2)$ is irreducible in $PSL(2,\mathbb{C})^n$.

Proof. Clearly, two algebraic varieties meeting the Minus Condition are isomorphic since the condition gives rise to a bijective correspondence originating from a regular map whose inverse is also a regular map. That $\pi(V_1) = \pi(V_2)$ follows from the fact that for each point $m \in V_1$, the point $-m$ is in $V_2$, and conversely, and the pair of points $m, -m$ map to a single point in $PSL(2,\mathbb{C})^n$. The irreducibility follows since the image of an irreducible variety under a regular map should also be irreducible.

Example. Let $G = <x, y, z; x^2 = 1>$. It is easy to see that $R_{SL(2,\mathbb{C})}(G) = \{I \times SL(2,\mathbb{C})^2\} \cup \{-I \times SL(2,\mathbb{C})^2\}$. Let $V_1 = I \times SL(2,\mathbb{C})^n$, and let $V_2 = -I \times SL(2,\mathbb{C})^n$. Then $V_1$ and $V_2$ meet the Minus Condition, and at least one of the $V_i$ ($i = 1, 2$) is irreducible. Consequently, $\pi(R_{SL(2,\mathbb{C})}(G))$ is irreducible. In fact, since $R_{PSL(2,\mathbb{C})}(G)$ is six dimensional, with irreducible components $V_1$, and $V_2$ respectively, $P_6(\pi(R_{SL(2,\mathbb{C})}(G)) = (2,0,0,0,0,0)$. Further, as a consequence of the earlier statements in this example, $P_6(\pi(R_{PSL(2,\mathbb{C})}(G)) = (1,0,0,0,0,0,0)$. It is readily seen, however, that $R_{PSL(2,\mathbb{C})}(G)$ consists of more than $\pi(R_{SL(2,\mathbb{C})}(G))$.

In fact it contains an eight dimensional irreducible variety, namely $\pi(V_3)$, where $V_3 = \{(m_1, m_2, m_3) | m_i \in SL(2,\mathbb{C}), tr(m_1) = 0\}$. Now, by Proposition 4.1, it is the case that since the word $x^2$ is a non-trivial word in the free group of rank 3 on the generators $\{x, y, z\}$, and that since the $SL(2,\mathbb{C})$ solutions $S_1$, and $S_{-1}$ to the equations $x^2 = I$, and $x^2 = -I$, respectively, satisfy the $\pm$ Condition of Proposition 4.1, that $\pi(S_i \cup S_{-1})$ is a reducible variety. So $\pi(S_1)$ does not collapse onto $\pi(S_j)$, where $i \neq j$ and $i, j \in \{1, -1\}$. In fact, $\pi(S_{-1})$ is irreducible since $S_{-1}$ is irreducible and the projection map is regular. Putting the foregoing together yields for example that $P_6(\pi(R_{PSL(2,\mathbb{C})}(G)) = (1,0,1,0,0,0,0,0)$. Next we compute the profile function of $R_{PSL(2,\mathbb{C})}(\mathbb{Z}_n)$, when $n \geq 2$. Clearly, by Lemma 3.1, $\dim(R_{PSL(2,\mathbb{C})}(\mathbb{Z}_n)) = 2$. Before starting it is prudent to introduce some notational convention.
Notation.

(1) If \( x \in SL(2, \mathbb{C}) \), denote by \([x]\) the equivalency class corresponding to \( x \) in \( PSL(2, \mathbb{C}) \).

(2) For positive integers \( p, q, \) and \( n \), let \( \text{diag}(p, q) \) stand for the \( 2 \times 2 \) matrix
\[
\begin{pmatrix}
e^{\frac{2\pi i}{n}} & 0 \\
0 & e^{\frac{2\pi i}{n}}
\end{pmatrix},
\]
where \( 0 < p < 2n, \ p \neq n, \) and \( 0 < q < 2n, \ q \neq n \).

(3) \( x \sim y \) shall mean that \( x \) is conjugate to \( y \) (i.e. \( x \) and \( y \) are in the same orbit under the action of conjugation).

Theorem 4.4.

(1) \( P_2(R_{PSL(2, \mathbb{C})}(\mathbb{Z}_n)) = (\frac{\varphi}{2}, 0, 1) \), if \( n \) is even.

(2) \( P_2(R_{PSL(2, \mathbb{C})}(\mathbb{Z}_n)) = (\frac{\varphi - 1}{2}, 0, 1) \), if \( n \) is odd.

Proof. It is clear that there are no components of dimensions two or one and only one component of dimension zero for the case of \( n = 1 \). We assume throughout the proof that \( n > 1 \).

We shall first count the number of two dimensional irreducible components of \( R_{PSL(2, \mathbb{C})}(\mathbb{Z}_n) \). Let \( \pi : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C}) \) be the projection map defined by \( \pi(a) = [a] \). Then \( R_{PSL(2, \mathbb{C})}(\mathbb{Z}_n) \) is the image under \( \pi \) of the \( SL(2, \mathbb{C}) \) algebraic set \( \Omega(n, I) \cup \Omega(n, -I) \) by Theorem 2.1. Clearly, by way of the structural invariance of the canonical forms between the non-singular matrices \( m \) and \( m^{\frac{1}{n}} \), solutions to the equations \( x^n = \pm I \) are all diagonalizable with spectrum \( n^{th} \) roots of 1, or \(-1\), depending respectively on whether \( x^n = I \) or \( x^n = -I \) is being considered. Additionally, because with the exception of the points \( I \) and \(-I\), the orbit of any diagonalizable matrix in \( SL(2, \mathbb{C}) \) is a two dimensional irreducible variety [L2], computing the two dimensional irreducible components in \( PSL(2, \mathbb{C}) \) of \( R_{PSL(2, \mathbb{C})}(\mathbb{Z}_n) \) will prove to be merely an application of Lemma 4.3 to the two dimensional irreducible components in \( SL(2, \mathbb{C}) \) of \( \Omega(n, I) \cup \Omega(n, -I) \) obeying the Minus Condition.

For any \( n = 2, 3, 4, \ldots \), an element of \( \Omega(n, I) \) or \( \Omega(n, -I) \) is similar to a matrix of the form \( \text{diag}(p, -p) \) since \( 0 < p < 2n, \ p \neq n \). Notice that there are \( 2n - 2 \) such matrices. We would like to determine the number of conjugacy classes of \( \text{diag}(p, -p) \) in \( PSL(2, \mathbb{C}) \). To begin with, observe that \( \text{diag}(p, -p) \sim \text{diag}(-p, p) \) since there is only one conjugacy class of matrices of any fixed given trace other than \( \pm 2 \) in \( SL(2, \mathbb{C}) \). Consequently, the following identity is established:

\[
\text{(4.1)} \quad \text{diag}(p, -p) \sim \text{diag}(2n - p, p - 2n)
\]
Observe that there is a total of \( n - 1 \) such conjugacy equivalences. Hence, in \( PSL(2, \mathbb{C}) \), we have that

\[
\text{(4.2)} \quad [\text{diag}(p, -p)] \sim [\text{diag}(2n - p, p - 2n)]
\]
for \( p = 1, 2, \ldots, n - 1 \). This implies that there are at most \( n - 1 \) conjugacy classes. Next we make use of the identity

\[
\text{(4.3)} \quad \text{diag}(p, -p) = -\text{diag}(p - n, n - p)
\]
for $1 \leq p \leq n - 1$. In particular. Note that the equality (4.3) projects via $\pi$ to $[\text{diag}(p, -p)] = [-\text{diag}(p - n, n - p)] = [\text{diag}(p - n, n - p)]$ in $\text{PSL}(2, \mathbb{C})$. Hence, we have the obvious conjugacy relation in $\text{PSL}(2, \mathbb{C})$:

$$\text{(4.4)} \quad [\text{diag}(p, -p)] \sim [\text{diag}(p - n, n - p)]$$

Thus, if $n \geq 2$ is even, there are precisely $\frac{n}{2} - 1$ unique relations of the form (4.3), up to conjugacy, for $1 \leq p \leq \frac{n}{2} - 1$. We deduce, by (4.4), that there are $n - 1 - \left(\frac{n}{2} - 1\right) = \frac{n}{2}$ conjugacy classes.

If $n > 2$ is odd, then there are $\frac{n - 1}{2}$ unique relations of the form (4.3), up to conjugacy, for $1 \leq p \leq \frac{n - 1}{2}$. Again, (4.4), we find that there are $n - 1 - \left(\frac{n - 1}{2}\right) = \frac{n - 1}{2}$ conjugacy classes.

All two dimensional irreducible components of $R_{\text{PSL}(2, \mathbb{C})}(\mathbb{Z}_n)$ are now accounted for. Obviously, there are no conjugacy classes of dimension one in $\text{SL}(2, \mathbb{C})$, nor in $\text{PSL}(2, \mathbb{C})$; so the number of one dimensional components of $R_{\text{PSL}(2, \mathbb{C})}(\mathbb{Z}_n)$ is zero.

The number of zero dimensional components is another matter. The algebraic set $\Omega(n, I) \cup \Omega(n, -I)$ in $\text{SL}(2, \mathbb{C})$, when $n$ is even, has exactly two zero dimensional irreducible components, namely $\pm I$, and these two components satisfy the Minus Condition. So, in $\text{PSL}(2, \mathbb{C})$, they correspond to exactly one zero dimensional component by Lemma 4.3. When $n$ is odd, $\Omega(n, I) \cup \Omega(n, -I)$ again has two zero dimensional irreducible components satisfying the Minus Condition. This completes the proof.

Compare Theorem 4.4 with the following:

**Proposition 4.2.**

1. $P_2(R_{\text{SL}(2, \mathbb{C})}(\mathbb{Z}_n)) = \left(\frac{n}{2}, 0, 2\right)$, if $n$ is even.
2. $P_2(R_{\text{SL}(2, \mathbb{C})}(\mathbb{Z}_n)) = \left(\frac{n - 1}{2}, 0, 1\right)$, if $n$ is odd.

**Proof.** Clearly there are no one dimensional components. When $n$ is even, $\pm I$ are the only zero dimensional components. When $n$ is odd, $I$ is the only zero dimensional component. The number of two dimensional components when $n$ is even and when $n$ is odd were computed in [L3]. The result thus follows.

We will employ Theorem 4.4 in computing the profile function for $R_{\text{PSL}(2, \mathbb{C})}(G)$, where $G$ is the free product of two finite cyclic groups; this will prove useful in our study of the profile functions of $R_{\text{PSL}(2, \mathbb{C})}(G)$, when $G$ is a torus knot group. The next lemma will prove indispensable.

**Lemma 4.4.** Let $V$ and $W$ be algebraic varieties with $\dim(V) = d_1$ and $\dim(W) = d_2$. Then

1. $\dim(V \times W) = d_1 + d_2$ and
2. $N_{d_1 + d_2}(V \times W) = N_{d_1}(V)N_{d_2}(W)$.

**Proof.** This is an elementary result, the proof of which may be deduced from the basic properties of irreducible varieties and and their Krull dimension. See, for example, [MD].

The next lemma is stated merely for completion, and its proof will also be omitted, as it is an elementary exercise. As usual, $G_1 \ast G_2$ denotes the free product of the groups $G_1$ and $G_2$. 

Lemma 4.5. If $G_1$ and $G_2$ are fg groups and $A$ is an algebraic group, then $R_A(G_1 \ast G_2) = R_A(G_1) \times R_A(G_2)$.

The next theorem gives the profile function for the $PSL(2, \mathbb{C})$ representation variety of the free product of two cyclic groups of finite order. Besides being an interesting result in its own right, it will also be indispensable in our study of the profile function over $PSL(2, \mathbb{C})$ for the representation varieties associated with a class of one relator groups containing the class of torus knot groups.

Theorem 4.5. Let $\mathbb{Z}_k$ be the cyclic group of order $k \geq 2$, and let $A = PSL(2, \mathbb{C})$. Then $P_4(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$ equals:

1. $\left(\frac{m+n}{4}, 0, \frac{m+n}{2}, 0, 1\right)$ if both $m$ and $n$ are even.
2. $\left(\frac{m(n-1)}{4}, 0, \frac{m+n-1}{2}, 0, 1\right)$ if $m$ is odd and $n$ is even.
3. $\left(\frac{m(n-1)}{4}, 0, \frac{m+n-1}{2}, 0, 1\right)$ if $m$ is even and $n$ is odd.
4. $\left(\frac{(m-1)(n-1)}{4}, 0, \frac{m+n-2}{2}, 0, 1\right)$ if both $m$ and $n$ are odd.

Proof. First, recall that for $k \geq 2$, $\dim(R_A(\mathbb{Z}_k)) = 2$, and that $\dim(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) = 4$ by Lemma 4.5. We calculate $N_i(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$, for $0 \leq i \leq 4$, using Theorem 4.4, Lemma 4.4, and Lemma 4.5.

1. $N_0(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) = 1$ since $N_0(R_A(\mathbb{Z}_m)) = N_0(R_A(\mathbb{Z}_n)) = 1$.
2. $N_1(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) = 0$ since $N_0(R_A(\mathbb{Z}_m)) = 1$ and $N_1(R_A(\mathbb{Z}_n)) = 0$ (similarly, $N_1(R_A(\mathbb{Z}_m)) = 0$ and $N_0(R_A(\mathbb{Z}_n)) = 1$).
3. $N_2(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$ depends on parities of $m$ and $n$. If $m$ and $n$ are even, then
   
   \begin{align*}
   N_2(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) &= N_2(R_A(\mathbb{Z}_m))N_0(R_A(\mathbb{Z}_n)) + N_0(R_A(\mathbb{Z}_m))N_2(R_A(\mathbb{Z}_n)) + \\
   &+ N_1(R_A(\mathbb{Z}_m))N_1(R_A(\mathbb{Z}_n)) = \frac{m+n}{2}.
   \end{align*}

   Similarly, $N_2(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$ equals either $\frac{m+n}{2}$ (if both $m$ and $n$ are even).

4. $N_3(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$ equals either
   
   \begin{align*}
   N_3(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) &= N_3(R_A(\mathbb{Z}_m))N_1(R_A(\mathbb{Z}_n)) + \\
   &+ N_1(R_A(\mathbb{Z}_m))N_3(R_A(\mathbb{Z}_n)) + N_1(R_A(\mathbb{Z}_m))N_2(R_A(\mathbb{Z}_n)) + N_2(R_A(\mathbb{Z}_m))N_1(R_A(\mathbb{Z}_n)) = 0.
   \end{align*}

5. Similarly, $N_4(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n))$ equals
   
   \begin{align*}
   N_4(R_A(\mathbb{Z}_m \ast \mathbb{Z}_n)) &= N_4(R_A(\mathbb{Z}_m))N_0(R_A(\mathbb{Z}_n)) + \\
   &+ N_0(R_A(\mathbb{Z}_m))N_4(R_A(\mathbb{Z}_n)) + N_1(R_A(\mathbb{Z}_m))N_3(R_A(\mathbb{Z}_n)) + \\
   &+ N_3(R_A(\mathbb{Z}_m))N_1(R_A(\mathbb{Z}_n)) = \frac{m+n}{4}.
   \end{align*}

Next a theorem is proven, a special case of which gives the number of four dimensional irreducible components of a torus knot group. The theorem is a consequence of Theorem 3.1, Theorem 4.4, Lemma 4.4, and some of the ideas developed in [L3] once the necessary adjustments for the projective linear group are made. An immediate consequence of the theorem is that if $G$ is a torus knot group, then $N_4(R_{PSL(2, \mathbb{C})}(G))$ does not equal to genus of the torus knot, as is the case when $PSL(2, \mathbb{C})$ is replaced by $SL(2, \mathbb{C})$.

Proposition 4.3. Let $p, t$ be positive integers, $G_{pt} = \langle x, y \mid x^p = y^t \rangle$, and $A = PSL(2, \mathbb{C})$. Then $N_4(R_A(G_{pt}))$ equals:

1. $\left(\frac{(p-1)(t-1)}{4}, 0, \frac{p+t}{2}, 0, 1\right)$ if both $p$ and $t$ are odd.
2. $\left(\frac{pt}{4}, 0, \frac{p+t}{2}, 0, 1\right)$ if $p$ is even and $t$ is odd.
3. $\left(\frac{tp}{4}, 0, \frac{p+t}{2}, 0, 1\right)$ if $p$ is odd and $t$ is even.
4. $\left(\frac{pt}{4}, 0, \frac{p+t}{2}, 0, 1\right)$ if both $p$ and $t$ are even.
Proof. By Theorem 3.1 and Theorem 4.4, $\dim(R_A(G_{pt})) = 4$ and $R_A(G_{pt})$ is reducible. It can be easily seen that

$$R_A(G_{pt}) = \{(m_1, m_2) \mid m_1 \in PSL(2, \mathbb{C}), m_2 \in \Omega(t, m_1^p)\}.$$

Now consider the set $S = \{(m_1, m_2) \mid m_1^p = 1\} \subseteq R_A(G_{pt})$. Suppose $\phi : R_A(G_{pt}) \rightarrow R_A(F_1)$ is the projection onto the first coordinate. It is straightforward to see that $R_A(G_{pt}) - S$ maps under the map $\phi$ onto a quasi-affine variety $Q$ of $R_A(F_1)$ having the property that the fiber over every point in $Q$ is zero dimensional and thus $\dim(R_A(G_{pt}) - S) = 3$. By Theorem 4.4, every point $m \in \phi(S)$ has a two dimensional fiber. It can readily be seen that $\phi(S)$ is just the set $\Omega(p, I)$ and as a consequence, by Theorem 4.4, it is also a two dimensional closed subset of $R_A(F_1)$. It follows then that the set $S$ is a four dimensional set and that it contains all the irreducible four dimensional components of $R_A(G_{pt})$. Closer inspection reveals that the set $S$ is nothing but $\Omega(p, I) \times \Omega(t, I)$, and consequently, we have that

$$N_4(S) = N_2(R_A(Z_p)) \times N_2(R_A(Z_t)).$$

And this leads to the desired result.

For the sake of comparison, we state the result established in [L3] for the groups $G_{pt}$ as in the above theorem.

**Theorem 4.6 ([L3]).** Let $G_{pt} = < x, y ; x^p = y^t >$, where $p, t$ are integers greater than one, and $A = SL(2, \mathbb{C})$. Then

1. $N_4(R_A(G_{pt})) = \left(\frac{(p-2)(t-2)+pt}{4}\right)$ if both $p, t$ are even and
2. $N_4(R_A(G_{pt})) = \left(\frac{(p-1)(t-1)}{2}\right)$ if either $p$ or $t$ is odd.

The next theorem is an immediate consequence of Proposition 4.3.

**Theorem 4.7.** If $p, t \geq 3$ are integers, then $P_4(R_{SL(2,\mathbb{C})}(G_{pt})) \neq P_4(R_{PSL(2,\mathbb{C})}(G_{pt}))$.

**Proof.** Each of the equations has integral solutions only when

1. $\left(\frac{(p-2)(t-2)+pt}{4}\right)$, when $p = 2$ or $t = 2$,
2. $\left(\frac{p-1}{2}\right)$, when $p = 2$, and
3. $\left(\frac{t-1}{2}\right)$, when $t = 2$.

Every torus knot group has a presentation of the type $< x, y ; x^p = y^t >$, where $p$ and $t$ are relatively prime integers. Immediate consequences of Theorem 4.7 are the following:

**Corollary 4.3.** If a torus knot group has presentation $< x, y ; x^p = y^t >$, where $p, t \geq 3$ are relatively prime integers, then $N_4(R_{PSL(2,\mathbb{C})}(G_{pt}))$ is not equal to the genus of the torus knot.

**Proof.** $N_4(R_{SL(2,\mathbb{C})}(G_{pt}))$, in such a case of the $p$ and $t$, is equal to the genus of the torus knot, [L3]. By Theorem 4.7, $N_4(R_{SL(2,\mathbb{C})}(G_{pt})) \neq N_4(R_{PSL(2,\mathbb{C})}(G_{pt}))$.

**Corollary 4.4.** Let $G$ be a torus knot group with presentation $< x, y ; x^p = y^t >$ with $p, t \geq 3$, then $P_4(R_{SL(2,\mathbb{C})}(G)) \neq P_4(R_{PSL(2,\mathbb{C})}(G))$.

**Proof.** This is a direct consequence of Theorem 4.7 since, for such values of $p$ and $t$, the number of four dimensional irreducible components are not the same. So the profile function does not agree at dimension four.
Lemma 4.6. Let $G$ be a one relator $n$-generated group with non-trivial relator $w = 1$. Let $V_{-1}$ and $V_1$ be all representations of the free group $F_n$ in $SL(2, \mathbb{C})$ mapping $w$ to $-1$, and $w$ to $1$, respectively. If $V_1$ and $V_{-1}$ satisfy the Minus Condition, then every representation $\rho \in R_{PSL(2, \mathbb{C})}(G)$ has a lift.

Proof. Assume that $V_1 = \{ \beta | \beta \in R_{SL(2, \mathbb{C})}(F_n), \beta(w) = I \}$ and $V_{-1} = \{ \beta | \beta \in R_{SL(2, \mathbb{C})}(F_n), \beta(w) = -I \}$ satisfy the Minus Condition. We will show that every representation $\rho$ of $G$ into $PSL(2, \mathbb{C})$ has a lift. Using the observation following Definition 4.2, $R_{PSL(2, \mathbb{C})}(G)$ is in one-to-one correspondence with the set \( \{(m_1, \ldots, m_n) \in PSL(2, \mathbb{C})^n \mid w(m_1, \ldots, m_n) = I \} \). We know that $R_{PSL(2, \mathbb{C})}(G) = \pi(V_{-1}) \cup \pi(V_1)$ and that, as algebraic varieties, $V_{-1} \cong V_1$ since $V_{-1}$ and $V_1$ satisfy the Minus Condition. Suppose that some arbitrary representation $\rho \in R_{PSL(2, \mathbb{C})}(G)$ is given. Then we can think of $\rho$ as a point $(m_1, \ldots, m_n)$ in $PSL(2, \mathbb{C})^n$. Now assume that $\pi^{-1}(\rho) = (m'_1, \ldots, m'_n)$ lies in $V_{-1}$; then, since $V_{-1}$ and $V_1$ satisfy the Minus Condition, we have that $(-m'_1, \ldots, -m'_n)$ lies in $V_1$, and thus $w(-m'_1, \ldots, -m'_n) = I$. Thus, by the observation following Definition 4.2, we obtain a representation $\phi : G \rightarrow SL(2, \mathbb{C})$ if we let $\phi(x_1) = -m'_1, \ldots, \phi(x_n) = -m'_n$. Now, $\pi \circ \phi = \rho = (m_1, \ldots, m_n)$. So $\phi$ is a lift of $\rho$, which is what we were after. If, on the other hand, $\pi^{-1}(\rho) = (m'_1, \ldots, m'_n)$ lies in $V_1$ we have nothing to prove since it clearly has a lift.

We finish with the following result quite useful in comparing profile functions for certain groups.

Theorem 4.8. Suppose that $w$ in $F_n = < x_1, \ldots, x_n >$ is a freely reduced and non-trivial word, and $G = < x_1, \ldots, x_n ; w = 1 >$. Suppose that $V_{-1} = \{ \rho | \rho \in R_{SL(2, \mathbb{C})}(F_n), \rho(w) = -I \}$, and $V_1 = \{ \rho | \rho \in R_{SL(2, \mathbb{C})}(F_n), \rho(w) = I \}$ are algebraic varieties that satisfy the Minus Condition. Then $R_{PSL(2, \mathbb{C})}(G)$ falls, or falls properly relative to $R_{SL(2, \mathbb{C})}(G)$.

Proof. This is a consequence of the fact that $V_{-1}$ and $V_1$ are isomorphic, and that each of these varieties project via $\pi$ onto $R_{PSL(2, \mathbb{C})}(G)$. We know that $V_1 = R_{SL(2, \mathbb{C})}(G)$, and that over every point of $R_{PSL(2, \mathbb{C})}(G)$ the fiber is finite. So every irreducible component corresponds to at least one irreducible component above having the same dimension. The result thus follows.

As an example, consider the cyclic group $\mathbb{Z}_n$, when $n$ is odd. The presentation of such a group arises from a word $x^n$ in the free group $F_1$ that leads to varieties $V_{-1}$ and $V_1$ satisfying the Minus Condition, and as a consequence we have that $P_2(R_A(\mathbb{Z}_n)) = (\frac{n-1}{2}, 0, 1)$ for $A \in \{ SL(2, \mathbb{C}), PSL(2, \mathbb{C}) \}$. So the representation variety over $PSL(2, \mathbb{C})$ of $\mathbb{Z}_n$ falls relative to the representation variety of $\mathbb{Z}_n$ over $SL(2, \mathbb{C})$. To find examples of where the representation variety falls properly, torus groups like $< x, y; x^p y = 1 >$, where $p$ is even and $t$ is odd, may be used. In contrast, notice that the representation variety of $\mathbb{Z}_2$ over $PSL(2, \mathbb{C})$ does not fall properly or otherwise relative the representation variety of $\mathbb{Z}_2$ over $SL(2, \mathbb{C})$.

References

[B1] Baumslag, G., Groups with the same lower central sequence as a relatively free group I, the groups., Trans. Amer. Math. Soc. 129 (1967), 308 – 321.

[B2] Baumslag, G., Groups with the same lower central sequence as a relatively free group II, properties., Trans. Amer. Math. Soc. 142 (1969), 507 – 538.
[BGSP] Breuillard E., Gelander T., Souto J., Storm T., Dense embeddings of surface groups, Geometry And Topology 10 (2006), 1373-1389.

[BC] Benyash-Krivets V. V., Chernousov V. I., Representation varieties of the fundamental groups of non-orientable surfaces, SB MATH 188 (7) (1997), 997-1039.

[GW] Goldman, William, Topological components of spaces of representations, Inventiones Mathematicae 93 (1988), 557-607.

[LM] Lubotzky, A. and Magid, A., Varieties of Representations of Finitely Generated Groups, Memoirs of the AMS 58 (1985), no. 336.

[L1] Liriano, S., Algebraic geometric invariants for a class of one-relator groups, J. Pure and Appl. Algebra 132 (1998), 105-118.

[L2] Liriano, S., Krull dimension and deviation in parafree groups, (to appear) Communications in Algebra (arXiv:math/0612102).

[L3] Liriano, S., Irreducible components in an algebraic variety of representations of a class of one-relator groups, Internat. J. Algebra Comput. 9 (1999), 129-133.

[L4] Liriano, S., Algebraic geometric invariants of parafree groups, Internat. J. Algebra Comput. 17, No 1 (2007), 155-169.

[MD] Mumford, D., The Red Book of Varieties and Schemes, vol. 1358, Lecture Notes in Mathematics, Springer-Verlag, 1980.

[RBC] Rapinchuk A.S., Benyash-Krivets V.V., Chernousov V.I., Representation varieties of the fundamental groups of compact orientable surfaces, Israel J. Math. 93 (1996), 29-71.

[RS] Rittatore A., Santos W.F., Actions and Invariants of Algebraic Groups, vol. 269, Taylor & Francis Group, LLC, 2005.

[SN] Sanov, I.N., A property of a representation of a free group, Dokl. Akad. Nauk SSSR 57 (1947), 657-659.

[WJ] Wicks, M. J., Commutators in free products, J. London Math. Soc. 37 (1962), 433-444.

Author’s Emails:
S. Liriano: SAL21458@yahoo.com
S. Majewicz: smajewicz@kbcc.cuny.edu