1. **Introduction.** Fix a field $k$ and a commutative ring $\Lambda$ (with 1). Assume either that $k$ is perfect and admits resolution of singularities, or that $k$ is arbitrary and the exponential characteristic\(^{1}\) of $k$ is invertible in $\Lambda$. Write $DM(k; \Lambda)$ for the triangulated category of motives with $\Lambda$-coefficients (see §2).

We write ‘scheme’ in lieu of ‘separated scheme of finite type over $k$’. For a group scheme $G$, a $G$-torsor will mean a faithfully flat $G$-invariant morphism $X \to Y$, such that the canonical morphism $G \times X \to X \times_Y X$ is an isomorphism. In this situation, we say that the quotient of the $G$-action on $X$ exists, and we set $X/G = Y$. The trivial torsor over a scheme $X$ is the projection $G \times X \to X$, with $G$ acting on $G \times X$ via multiplication on $G$. A reductive group will mean a smooth affine group scheme $G$ such that every smooth connected unipotent subgroup of $G \times \bar{k}$ is trivial, where $\bar{k}$ is the algebraic closure of $k$.

**Theorem 1.1.** Let $G$ be a connected split reductive group, and let $B \subset G$ be a Borel subgroup. Let $X$ be a scheme with $G$-action. Let $t(G)$ denote the torsion index of $G$. If $t(G)$ is invertible in $\Lambda$, then there is an isomorphism

$$M^c(X_{hB}) \simeq M^c(X_{hG}) \otimes M^c(G/B)$$

in $DM(k; \Lambda)$ that commutes with smooth pullbacks and localization triangles.

Here $M^c(X_{hG})$, $M^c(X_{hB})$ denote the $G$-equivariant and $B$-equivariant (Borel-Moore) motives, respectively, of $X$ (in the sense of [T4]; see §3).

The proof of Theorem 1.1 is essentially the same as that of the analogous statement for the cohomology of topological spaces (for instance, compare with [M] and [T3, Theorem 16.1]). No claim to originality is being made.

**Example 1.2.** If $X = \text{Spec}(k)$, then Theorem 1.1 states

$$M^c(BB) \simeq M^c(BG) \otimes M^c(G/B),$$

where $M^c(BB)$ and $M^c(BG)$ are the motives of the classifying spaces of $B$ and $G$, respectively (see [T4] and §3). This is a motivic analogue of the usual splitting principle (working with groups over the complex numbers say):

$$H^\ast(BB) \simeq H^\ast(BG) \otimes H^\ast(G/B),$$

where cohomology is with coefficients in a ring $\Lambda$ in which $p$ is invertible for all primes $p$ such that $H^\ast(G; \mathbb{Z})$ has $p$-torsion.

\(^{1}\) If $\text{char}(k) = 0$, then the exponential characteristic of $k$ is 1. If $\text{char}(k) > 0$, then the exponential characteristic of $k$ is $\text{char}(k)$. 

2. **Motives.** The category $DM(k; \Lambda)$ is the monoidal triangulated category of Nisnevich motivic spectra over $k$, obtained by applying $\mathbb{A}^1$-localization and $\mathbb{P}^1$-stabilization to the derived category of (unbounded) complexes of Nisnevich sheaves with transfer [CD1, Definition 11.1.1]. The primary references for the properties of $DM(k; \Lambda)$ that we use are [V], [CD1] and [CD2]. The assumption that $k$ is perfect and admits resolution of singularities stems from the treatment in [V].

The alternate assumption that $k$ is arbitrary, but the exponential characteristic is invertible in $\Lambda$, stems from [V, §9], amongst other things, extend the constructions of [V] under this alternate hypothesis.

There is a covariant functor $X \mapsto M^e(X)$ from the category of schemes and proper morphisms to $DM(k; \Lambda)$ (see [V, §2.2]; alternatively, in the notation of [CD1], we have $M^e(X) = a_* a! \Lambda$, where $a : X \to \text{Spec}(k)$ is the structure morphism). The functor $M^e(X)$ behaves like a Borel-Moore homology theory in the following sense. If $f : X \to Y$ is a smooth morphism with fibres of dimension $r$ (the morphism $f$ is allowed to have some fibres empty), then there is a map (compatible with composition of morphisms):

$$f^* : M^e(Y)(r)[2r] \to M^e(X),$$

where $(j)$ denotes tensoring with the $j$-th Tate twist (see [CD1, (11.1.2.2)]) and $[i]$ denotes the $i$-th shift functor (available in any triangulated category). If $f$ is a vector bundle, then $f^*$ is an isomorphism.

Let $i : Z \hookrightarrow X$ be a closed immersion, and let $j : U \rightarrow X$ be the open immersion of the complement $U = X \smallsetminus Z$. Then $i_*$ and $j^*$ fit into a canonical distinguished triangle [V, §2.2], [CD2, Corollary 5.9, Theorem 5.11], the localization triangle,

$$M^e(Z) \xrightarrow{i_*} M^e(X) \xrightarrow{j^*} M^e(U) \xrightarrow{\partial}$$

The category $DM(k; \Lambda)$ is a symmetric monoidal triangulated category, and

$$M^e(X \times Y) = M^e(X) \otimes M^e(Y).$$

The motive $M^e(\text{Spec}(k))$ is the unit object. Although notationally abusive, it is convenient to set

$$\Lambda = M^e(\text{Spec}(k)).$$

Let $H_i^M(X; \Lambda(j))$ denote the motivic cohomology groups of $X$, as defined in [CD1, §11.2]. These are contravariant functors from the category of schemes to $\Lambda$-modules. By [CD1, Example 11.2.3], if $X$ is smooth and equidimensional, then motivic cohomology determines the Chow ring $CH^i(X)$ of $X$:

$$H^i_M(X; \Lambda(j)) = CH^i(X) \otimes_{\mathbb{Z}} \Lambda.$$

For an arbitrary scheme $X$, each $e \in H^i(X; \Lambda(j))$ determines a canonical map

$$e \cap : M^e(X)(-j)[-i] \to M^e(X).$$

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2 The category $DM^{eff}(k; \Lambda)$ of [V] embeds into $DM(k; \Lambda)$ as a full and faithful subcategory [CD1, Example 11.1.3]. Utilizing $DM(k; \Lambda)$ (instead of $DM^{eff}(k; \Lambda)$) is dictated by the need for arbitrary direct sums and compact generation [V, Corollary 3.5.5]. Both of these properties are required to make sense of equivariant motives (see [14] and §3).
Example 2.1. Let $f: L \to X$ be a line bundle. Write $i: X \hookrightarrow L$ for the zero section. Let $c_1(L) \in H^2_{et}(X; \Lambda(1))$ be the first Chern class of $L$ [CD1, Definition 11.3.2]. Then $c_1(L) \cap$ is the composition

$$M^c(X)(-1)[-2] \xrightarrow{i^*} M^c(L)(-1)[-2] \xrightarrow{f^{* - 1}} M^c(X).$$

3. Equivariant motives, following B. Totaro. Let $G$ be an affine group scheme. Let $X$ be a scheme with $G$-action. Let $\cdots \to V_2 \to V_1$ be a sequence of surjections of $G$-equivariant vector bundles over $X$. Write $n_i$ for the rank of $V_i$. Let $U_i \subset V_i$ be a $G$-stable open subscheme such that $V_{i+1} \setminus U_{i+1}$ is contained in the inverse image of $V_i \setminus U_i$, and such that the quotient $U_i/G$ exists. Assume that the codimension of $V_i \setminus U_i$ goes to infinity with $i$. The equivariant motive $M^c(X_{hG})$ is defined to be the homotopy limit of the sequence

$$\cdots \to M^c(U_2/G)(-n_2)[-2n_2] \to M^c(U_1/G)(-n_1)[-2n_1].$$

The definition of $M^c(X_{hG})$ is independent (up to a not necessarily unique isomorphism) of all the choices involved [T4, Theorem 8.5]. If $X$ satisfies any of the conditions of [EG, Proposition 23], then such vector bundles exist (also see [T1, Remark 1.4] and [T4, §8]). Whenever we speak of $M^c(X_{hG})$, we implicitly, and without further comment, assume that such vector bundles exist (including in the statement of Theorem 1.1).

Since $M^c(X_{hG})$ is defined as a (homotopy) limit of ordinary motives $M^c(U_i/G)$, the functorial properties of ordinary motives (pullback, pushforward, localization triangles, etc.) extend to the equivariant setup. By construction, $M^c(X_{hG})$ satisfies equivariant descent: if the quotient of the $G$-action on $X$ exists, then

$$M^c(X_{hG}) \simeq M^c(X/G).$$

We set

$$M^c(BG) = M^c(\text{Spec}(k)_{hG}).$$

Example 3.1. Let $V_i$ be the direct sum of $i$-copies of the natural 1-dimensional representation of $G_m$. Let $U_i = V_i - \{0\}$. Then $U_i/G_m \simeq \mathbb{P}^{i-1}$. We have [T4, Lemma 8.7]:

$$M^c(BG_m) \simeq \prod_{i \leq -1} \Lambda(i)[2i].$$

4. Restriction to a subgroup. Let $G$ be an affine group scheme. Let $H \subset G$ be a closed subgroup. If the quotient $X/G$ exists, then the quotient $X/H$ exists. If $G$ is smooth, then we have a pullback

$$M^c(X/G)(\dim(G/H))[2\dim(G/H)] \to M^c(X/H).$$

This family of pullbacks, one for each such $X$, yields a map, restriction,

$$\text{res}_G^H: M^c(Y_{hG})(\dim(G/H))[2\dim(G/H)] \to M^c(Y_{hH}),$$

for any scheme $Y$ with $G$-action. Restriction commutes with smooth pullbacks and localization triangles.
5. **Chow ring of a classifying space.** Let $G$ be an affine group scheme. Following [T1], define the Chow ring $CH^*_G$ of the classifying space of $G$ as follows. Let $V$ be a representation of $G$ over $k$. Let $U \subset V$ be an open subscheme such that the quotient $U/G$ exists, and such that $V - U$ has codimension greater than 1. Then $CH^*_G = CH^*(U/G)$. This definition is independent of all the choices involved [T1, Theorem 1.1] and gives a well-defined ring $CH^*_G$. It follows from the definition that each $e \in CH^*_G$ determines a canonical map

$$e\cap: M^c(X_{hG})(-i)[-2i] \to M^c(X_{hG}),$$

for a scheme $X$ with an action of $G$.

By faithfully flat descent, each representation of $G$ over $k$ determines a vector bundle over the schemes $U/G$ used to define $CH^*_G$. Consequently, each such representation has Chern classes in $CH^*_G$.

**Example 5.1.** Let $T$ be a split torus. Let $\chi$ be a character of $T$, with first Chern class $c^1(\chi) \in CH^1_T$. Let $X$ be a scheme with $T$-action. Then $\chi$ determines an equivariant line bundle $L_\chi \to X$. If the quotient $X/T$ exists, then $L_\chi$ descends to a line bundle $\tilde{L}_\chi \to X/T$. In this situation, the map

$$c_1(\chi)\cap: M^c(X_{hT})(-1)[-2] \to M^c(X_{hT})$$

is the composition

$$M^c(X_{hT})(-1)[-2] \xrightarrow{\sim} M^c(X/T)(-1)[-2] \xrightarrow{c_1(L_\chi)\cap} M^c(X/T) \xrightarrow{\sim} M^c(X_{hT}),$$

where the first and last isomorphisms are taken to be inverse to each other.

6. **The torsion index.** Let $G$ be a connected split reductive group over $k$. Let $B \subset G$ be a Borel subgroup. The *torsion index* of $G$ is the smallest integer $t(G) \in \mathbb{Z}_{>0}$ such that the image of the map $CH^*_B \to CH^*(G/B)$ contains $t(G) \cdot CH^{\dim(G/B)}(G/B)$. The natural map,

$$CH^*_B \otimes_\mathbb{Z} \mathbb{Z}[t(G)^{-1}] \to CH^*(G/B) \otimes_\mathbb{Z} \mathbb{Z}[t(G)^{-1}],$$

is surjective. According to [G, Théorème 2], for any $G$-torsor $X \to Y$, there is a non-empty open subscheme $U \subset Y$ along with a finite étale morphism $V \to U$ of degree invertible in $\mathbb{Z}[t(G)^{-1}]$, such that $X$ is trivial over $V$.

**Example 6.1.** The group $GL_n$ has torsion index 1.

7. **Proof of Theorem 1.1.** (Compare with [M] and the proof of [T3, Theorem 16.1]). Pick elements $e_1, \ldots, e_n \in CH^*(BB) \otimes_\mathbb{Z} \Lambda$, of homogeneous degree, that restrict to a basis of $CH^*(G/B) \otimes_\mathbb{Z} \Lambda$. Write $d_i$ for the degree of $e_i$. Set $d = \dim(G/B)$. For each $e_i$, consider the composition

$$M^c(X_{hG})(d - d_i)[2(d - d_i)] \xrightarrow{\text{res}^G_{BB}} M^c(X_{hB})(-d_i)[-2d_i] \xrightarrow{e_i\cap} M^c(X_{hB}).$$

Summing these, we obtain a map

$$\bigoplus_i M^c(X_{hG})(d - d_i)[2(d - d_i)] \to M^c(X_{hB}).$$

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3 The point is that, if one stays away from primes that divide $t(G)$, then all the challenges of ‘étale descent’ for equivariant Chow groups disappear. For further information on the torsion index, [T2] is highly recommended.
By the Bruhat decomposition, this may be rewritten as a map
\[ \theta : M^c(X_{hB}) \otimes M^c(G/B) \to M^c(X_{hB}). \]
The map \( \theta \) commutes with smooth pullbacks and localization triangles. We will show \( \theta \) is an isomorphism. It suffices to demonstrate this under the assumption that the quotient \( X/G \) exists. Via the isomorphisms \( M^c(X_{hG}) \simeq M^c(X/G) \) and \( M^c(X_{hB}) \simeq M^c(X/B) \), the map \( \theta \) yields a map
\[ \theta_X : M^c(X/G) \otimes M^c(G/B) \to M^c(X/B). \]
If \( X \to X/G \) is the trivial \( G \)-torsor, then \( \theta_X \) is manifestly an isomorphism. In general, there exists a non-empty open subscheme \( U \subset X/G \), along with a finite étale morphism \( f : V \to U \) of degree invertible in \( \mathbb{Z}[t(G)^{-1}] \), such that \( X \) pulled back to \( V \) is the trivial \( G \)-torsor (see §6). The map \( f_*f^* : M^c(U) \to M^c(U) \) is the degree of \( f \) times the identity (as follows from \([\text{CD}_1, A.5(6)]\) and \([\text{CD}_1, \text{Proposition } 11.2.5]\)). Consequently, \( \theta_U \) is an isomorphism. Now let \( Z = X - U \) be the closed complement (with reduced scheme structure say). Then, by virtue of the localization triangle, it suffices to show \( \theta_Z \) is an isomorphism. This follows from an induction on dimension (the base case has been dealt with by the above considerations).

8. A complement. The Chow ring \( CH^*_G \) acts on \( M^c(X_{hB}) \). Under the isomorphism
\[ M^c(X_{hB}) \simeq M^c(X_{hG}) \otimes M^c(G/B), \]
this action on the right hand side is the action of \( CH^*_G\) on \( M^c(G/B) \).

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