Capacity of Gaussian Many-Access Channels

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Abstract

Classical multiuser information theory studies the fundamental limits of models with a fixed (often small) number of users as the coding blocklength goes to infinity. This work proposes a new paradigm, referred to as many-user information theory, where the number of users is allowed to grow with the blocklength. This paradigm is motivated by emerging systems with a massive number of users in an area, such as machine-to-machine communication systems and sensor networks. The focus of the current paper is the many-access channel model, which consists of a single receiver and many transmitters, whose number increases unboundedly with the blocklength. Moreover, an unknown subset of transmitters may transmit in a given block and need to be identified. A new notion of capacity is introduced and characterized for the Gaussian many-access channel with random user activities. The capacity can be achieved by first detecting the set of active users and then decoding their messages.

I. INTRODUCTION

Classical information theory characterizes the fundamental limits of communication systems by studying the asymptotic regime of infinite coding blocklength. The prevailing models in multiuser information theory assume a fixed (usually small) number of users, where fundamental limits as the coding blocklength goes to infinity are studied. Even in the large-system analysis of multiuser systems [1]–[3], the blocklength is sent to infinity before the number of users is sent to infinity. In some sensor networks and emerging machine-to-machine communication systems, a massive and ever-increasing number of wireless devices with bursty traffic may need to share...
the spectrum in a given area. This motivates us to rethink the assumption of fixed population of fully buffered users. Here we propose a new many-user paradigm, where the number of users is allowed to increase without bound with the blocklength.

In this paper, we introduce the many-access channel (MnAC) to model systems consisting of a single receiver and many transmitters, the number of which is comparable to or even larger than the blocklength. We study the asymptotic regime where the number of transmitting devices \((k)\) increases with the blocklength \((n)\). The model also accommodates random access, namely, it allows each transmitter to be active with certain probability in each block. We assume synchronous transmission in the model, while the capacity of strong asynchronous MnAC was studied in [9].

In general, the classical theory does not apply to systems where the number of users is comparable or larger than the blocklength, such as in a machine-to-machine communication system with many thousands of devices in a given cell. One key reason is that, for many functions of two variables \(f\), \(\lim_{k \to \infty} \lim_{n \to \infty} f(k, n) \neq \lim_{n \to \infty} f(n, n)\), i.e., letting \(k \to \infty\) after \(n \to \infty\) may yield a different result than letting \(n\) and \(k = k_n\) (as a function of \(n\)) simultaneously tend to infinity. Moreover, the traditional notion of rate in bits per channel use is ill-suited for the task in the many-user regime as noted (for the Gaussian multiaccess channel) in [10, pp. 546-547] by Cover and Thomas, “when the total number of senders is very large, so that there is a lot of interference, we can still send a total amount of information that is arbitrary large even though the rate per individual sender goes to 0.”

Capacity of the conventional multiaccess channel is well understood [11]–[13]. The capacity can be established using the fact that joint typicality holds with high probability as the blocklength grows to infinity. This argument, however, does not directly apply to models where the number of users also goes to infinity. Specifically, joint typicality requires the simultaneous convergence of the empirical joint entropy of every subset of the input and output random variables to the corresponding joint entropy. Even though convergence holds for every subset due to the law of large numbers, the asymptotic equipartition property is not guaranteed because the number of those subsets increases exponentially with the number of users [14]. Resorting to strong typicality does not resolve this because the empirical distribution over an increasing alphabet

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2The only existing model of this nature is found in [6], in which the authors sought for uniquely-decodable codes for a noiseless binary adder channel with the number of users increasing with the blocklength.
(due to increasing number of users) does not converge.

In general, the received signal of the Gaussian MnAC is a noisy superposition of the codewords chosen by the active users from their respective codebooks. The detection problem boils down to identifying codewords based on their superposition. It is closely related to sparse recovery, also known as compressed sensing, which has been studied in a large body of works [15]–[24]. Information-theoretic limits of exact support recovery was considered in [18], and stronger necessary and sufficient conditions have been derived subsequently [20], [21], [24]. Using existing results in the sparse recovery literature, it can be shown that the message length (in bits) that can be transmitted reliably by each user should be in the order of $\Theta(n (\log k_n)/k_n)$.

In this paper, we provide a sharp characterization of the capacity of Gaussian many-access channels as well as the user identification cost. As an achievable scheme, each user’s transmission consists of a signature that identifies the user, followed by a message-bearing codeword. The decoder first identifies the set of active users based on the superposition of their unique signatures. (This is in fact a compressed sensing problem [25], [26].) It then decodes the messages from the identified active users. The length of the signature matches the capacity penalty due to user activity uncertainty. The proof techniques find their roots in Gallager’s error exponent analysis [27]. Also studied is a more general setup where groups of users have heterogeneous channel gains and activity patterns. Again, separate identification and decoding is shown to achieve the capacity region.

Unless otherwise noted, we use the following notational conventions: $x$ denotes a scalar, $\mathbf{x}$ denotes a column vector, and $\mathbf{X}$ denotes a matrix. The corresponding uppercase letters $X$, $\mathbf{X}$, and $\mathbf{X}$ denote the corresponding random scalar, random vector and random matrix, respectively. Given a set $A$, let $\mathbf{x}_A = (x_i)_{i \in A}$ denote the subset of variables of $\mathbf{x}$ whose indices are in $A$ and let $\mathbf{X}_A = (x_i)_{i \in A}$ be the matrix formed by columns of $\mathbf{X}$ whose indices are in $A$. Let $x_n \leq y_n$ denote $\lim \sup_{n \to \infty} (x_n - y_n) \leq 0$. That is, $x_n$ is essentially asymptotically dominated by $y_n$. All logarithms are natural. The binary entropy function is denoted as $H_2(p) = -p \log p - (1 - p) \log (1 - p)$.

The rest of the paper is organized as follows. Section II presents the system model and main capacity results. Section III gives the proof of converse for the MnAC capacity. Section IV proves the random user identification cost. Section V shows that the MnAC capacity is achievable using separate identification and decoding. Section VI discusses the challenges of applying successive
decoding in MnAC. Section VII analyzes the capacity of MnAC with heterogeneous channel gains and activity patterns. Concluding remarks are given in Section VIII.

II. SYSTEM MODEL AND MAIN RESULTS

Let \( n \) denote the number of channel uses, i.e., the blocklength. Let the number of users be a function of \( n \) and be explicitly denoted as \( \ell_n \), so that it is tied to the blocklength. The received symbols in a block form a column vector of length \( n \):

\[
Y = \sum_{k=1}^{\ell_n} S_k(w_k) + Z
\]

(1)

where \( w_k \) is the message of user \( k \), \( S_k(w_k) \in \mathbb{R}^n \) is the corresponding \( n \)-symbol codeword, and \( Z \) is a Gaussian noise vector with independent standard Gaussian entries. Suppose each user accesses the channel independently with identical probability \( \alpha_n \) during any given block. If user \( k \) is inactive, it is thought of as transmitting the all-zero codeword \( S_k(0) = 0 \).

Definition 1. Let \( S_k \) and \( \mathcal{Y} \) denote the input alphabet of user \( k \) and the output alphabet, respectively. An \((M, n)\) symmetric code with power constraint \( P \) for the MnAC channel \((S_1 \times S_2 \times \ldots \times S_{\ell_n}; P_Y|S_1, \ldots, S_{\ell_n}, \mathcal{Y})\) consists of the following mappings:

1) The encoding functions \( E_k : \{0, 1, \ldots, M\} \rightarrow S_k^n \) for every user \( k \in \{1, \ldots, \ell_n\} \), which maps any message \( w \) to the codeword \( s_k(w) = [s_{k1}(w), \ldots, s_{kn}(w)]^T \). In particular, \( s_k(0) = 0 \), for every \( k \). Every codeword \( s_k(w) \) satisfies the power constraint:

\[
\frac{1}{n} \sum_{i=1}^{n} s_{ki}^2(w) \leq P.
\]

(2)

2) Decoding function \( D : \mathcal{Y}^n \rightarrow \{0, 1, \ldots, M\}^{\ell_n} \), which is a deterministic rule assigning a decision on the messages to each possible received vector.

The average error probability of the \((M, n)\) code is:

\[
P_e^{(n)} = P \{ D(Y) \neq (W_1, \ldots, W_{\ell_n}) \},
\]

(3)

where \( W_1, \ldots, W_{\ell_n} \) are independent, and for every \( k \in \{1, \ldots, \ell_n\} \),

\[
P \{ W_k = w \} = \begin{cases} 
1 - \alpha_n, & w = 0, \\
\alpha_n \frac{M}{M}, & w \in \{1, \ldots, M\}.
\end{cases}
\]

(4)
The preceding model reduces to the conventional \( \ell \)-user multiaccess channel in the special case where \( \ell_n = \ell \) is fixed and \( \alpha_n = 1 \) as the blocklength \( n \) varies.

A. The Message-Length Capacity

**Definition 2** (Asymptotically achievable message length). We say a positive nondecreasing sequence of message lengths \( \{v(n)\}_{n=1}^{\infty} \), or simply, \( v(\cdot) \), is asymptotically achievable for the MnAC if there exists a sequence of \( (\lceil \exp(v(n)) \rceil, n) \) codes according to Definition 1 such that the average error probability \( P_v^{(n)} \) given by (3) vanishes as \( n \to \infty \).

It should be clear that by asymptotically achievable message length we really mean a function of the blocklength. The base of \( \exp(\cdot) \) should be consistent with the unit of the message length. If the base of \( \exp(\cdot) \) is 2 (resp. \( e \)), then the message length is measured in bits (resp. nats).

**Definition 3** (Symmetric message-length capacity). For the MnAC channel described by (1), a positive nondecreasing function \( B(n) \) of the blocklength \( n \) is said to be the symmetric message-length capacity of the MnAC channel if, for any \( 0 < \epsilon < 1 \), \( (1 - \epsilon)B(n) \) is an asymptotically symmetric achievable message length, whereas \( (1 + \epsilon)B(n) \) is not.

For the special case of a (conventional) multiaccess channel, the symmetric capacity \( B(n) \) in Definition 3 is asymptotically linear in \( n \), so that \( \lim_{n \to \infty} B(n)/n \) is equal to the symmetric capacity of the multiaccess channel (in, e.g., bits per channel use). From this point on, by “capacity” we mean the message-length capacity in contrast to the conventional capacity. In many-user information theory, \( B(n) \) need not grow linearly with the blocklength.

Let \( \mathbf{S}_k = [\mathbf{S}_k(1), \ldots, \mathbf{S}_k(M)] \) denote the matrix consisting of all but the first all-zero codeword of user \( k \). Let \( \mathbf{S} = [\mathbf{S}_1, \ldots, \mathbf{S}_\ell_n] \in \mathbb{R}^{n \times (\ell_n M)} \) denote the concatenation of the codebooks of all users. For ease of analysis, we often use the following equivalent model for the Gaussian MnAC (1):

\[
\mathbf{Y} = \mathbf{S}\mathbf{X} + \mathbf{Z},
\]

where \( \mathbf{Z} \) is defined as in (1) and \( \mathbf{X} \in \mathbb{R}^{\ell_n M} \) is a vector indicating the codewords transmitted by the users. Specifically, \( \mathbf{X} = [\mathbf{X}_1^T, \mathbf{X}_2^T, \ldots, \mathbf{X}_{\ell_n}^T]^T \), where \( \mathbf{X}_k \in \mathbb{R}^M \) indicates the codeword
transmitted by user $k$, $k = 1, \ldots, \ell_n$, i.e.,

$$X_k = \begin{cases} 
0 & \text{with probability } 1 - \alpha_n \\
e_m & \text{with probability } \frac{\alpha_n}{M}, \quad m = 1, \ldots, M
\end{cases}$$

(6)

where $e_m$ is the binary column $M$-vector with a single 1 at the $m$-th entry. Let

$$\mathcal{X}_M = \left\{ x = [x_1^T, \ldots, x_{\ell_n}^T]^T : x_i \in \{0, e_1, \ldots, e_m\}, \text{ for every } i \in \{1, \ldots, \ell\} \right\}.$$  

(7)

The signal $X$ must take its values in $\mathcal{X}_{M}^\ell$.

The following theorem is a main result of the paper.

**Theorem 1** (Symmetric capacity of the Gaussian many-access channel). Let $n$ denote the coding blocklength, $\ell_n$ denote the total number of users, and $\alpha_n$ denote the probability a user is active, independent of other users. Suppose $\ell_n$ is nondecreasing with $n$ and

$$\lim_{n \to \infty} \alpha_n = \alpha \in [0, 1].$$

(8)

Denote the average number of active users as

$$k_n = \alpha_n \ell_n.$$  

(9)

Then the symmetric message-length capacity $B(n)$ of the Gaussian many-access channel described by (1), with each user’s SNR being no greater than $P$, is characterized as follows:

1) Suppose $\ell_n$ and $k_n$ are both unbounded, $k_n = O(n)$, and

$$\ell_n e^{-\delta k_n} \to 0$$

(10)

for every $\delta > 0$. Let $\theta$ denote the limit of

$$\theta_n = \frac{2 \ell_n H_2(\alpha_n)}{n \log(1 + k_n P)},$$

(11)

which may be $\infty$.

If $\theta < 1$, then

$$B(n) = \frac{n}{2k_n} \log(1 + k_n P) - \frac{H_2(\alpha_n)}{\alpha_n}.$$  

(12)
If $\theta > 1$, then a user cannot send even 1 bit reliably.

If $\theta = 1$, then message length $\frac{en}{2k_n} \log(1 + k_n P)$ is not achievable for any $\epsilon > 0$.

2) If $\ell_n$ is unbounded and $k_n$ is bounded, then message length $\epsilon n$ is not achievable for any $\epsilon > 0$.

3) If $\ell_n$ is bounded, i.e., $\ell_n = \ell < \infty$ for sufficiently large $n$, then

$$B(n) = \begin{cases} \frac{n}{2} \log(1 + P) & \text{if } \alpha = 0, \\ \frac{n}{2\ell} \log(1 + \ell P) & \text{if } \alpha > 0. \end{cases}$$

(13)

A heuristic understanding of the expression of $B(n)$ in (12) is as follows: If a genie-aided receiver revealed the set of active users to the receiver, the total number of bits that can be communicated through the MnAC with $k_n$ users would be approximately $(n/2) \log(1 + k_n P)$, so that the symmetric capacity is

$$B_1(n) = \frac{n}{2k_n} \log(1 + k_n P).$$

(14)

The total uncertainty in the activity of all $\ell_n$ users is $\ell_n H_2(\alpha_n) = k_n H_2(\alpha_n)/\alpha_n$, so the capacity penalty on each of the $k_n$ active users is $H_2(\alpha_n)/\alpha_n$. If every user is always active, i.e., $\alpha_n = 1$, the penalty term is zero and the capacity resembles that of a multiaccess channel.

By the current definition, the symmetric capacity (12) can be reduced to

$$B'(n) = \frac{n}{2k_n} \log k_n - \frac{H_2(\alpha_n)}{\alpha_n},$$

(15)

because $\log(1 + k_n P) = \log k_n + o(\log k_n)$. We prefer the form of (12) for its connection to the original capacity formula for the Gaussian multiaccess channel.

Fig. 1 illustrates the capacity $B(n)$ given by (12) in the special case where $P = 10$ (i.e., the SNR is 10 dB), $k_n = n/4$, with different scalings of user number $\ell_n$. The purpose is to show the trend of the capacity as the blocklength increases rather than the capacity at finite length. The message-length capacity $B(n)$ scales sub-linearly in $n$. Moreover, $B(n)$ depends on the scaling of $k_n$ and $\ell_n$, whose effects cannot be captured by the conventional multiaccess channels. In particular, if $\ell_n$ grows too quickly (e.g., $\ell_n = n^3$), an average user cannot transmit a single bit reliably.

The assumptions in Case 1) of Theorem 1 prohibit two uninteresting cases: i) The average number of active users $k_n$ grows faster than linear in the blocklength $n$; and ii) the total number...
of users $\ell_n$ grows exponentially in $n$. For example, if $k_n = n(\log n)^2$, an average user will not be able to transmit a single bit reliably as $n$ increases to infinity.

Time sharing with power allocation, which can achieve the capacity of the conventional multiaccess channel [10], is inadequate for the MnAC in general. For example, if $k_n = 2n$, not a single channel use can be guaranteed for every active user. Moreover, if $k_n = n$ and each user applies all energy in a single exclusive channel use, the resulting data rate is generally poor.

### B. The User Identification Cost

As a by-product in the proof of Theorem 1, we can derive the fundamental limits of random user identification (without data transmission), where every user is active with certain probability and the receiver aims to detect the set of active users. To quantify the cost of user identification, we denote the total number of users as $\ell$ and let other parameters depend on $\ell$. (This is in contrast to the setting in Section II-A.) The probability of a user being active is denoted as $\alpha_\ell$, and the average number of active users is denoted as $k_\ell = \alpha_\ell \ell$. Suppose $n_0$ symbols are used for user identification purpose. Let $X^a \in \mathbb{R}^\ell$ be a random vector, which consists of independent and identically distributed (i.i.d.) Bernoulli entries with mean $\alpha_\ell$. Then the received signal is

$$Y^a = S^a X^a + Z^a,$$

(16)

where $Z^a$ consists of $n_0$ i.i.d. standard Gaussian entries, and $S^a = [S_1^a, \cdots, S_\ell^a]$ with $S_k^a \in \mathbb{R}^{n_0}$ being the signature of user $k$. Moreover, the realization of the signature must satisfy the following
power constraint:
\[ \frac{1}{n_0} \sum_{i=1}^{n_0} (s_{k_i}^a)^2 \leq P. \] (17)

**Definition 4 (Minimum user identification cost).** We say the identification is erroneous in case of any miss or false alarm. For the channel described by (16), the minimum user identification cost is said to be \( n(\ell) \) if \( n(\ell) > 0 \) and for every \( 0 < \epsilon < 1 \), the probability of erroneous identification vanishes as \( \ell \to \infty \) if the signature length \( n_0 = (1 + \epsilon)n(\ell) \), whereas the error probability is strictly bounded away from zero if \( n_0 = (1 - \epsilon)n(\ell) \).

As in the case of capacity, the definition focuses on the asymptotics of \( \ell \to \infty \), so the minimum cost function \( n(\cdot) \) is not unique. The random user identification problem has been studied in the context of compressed sensing problem [18], [28]. The following theorem gives a sharp characterization of how many channel uses \( n_0 \) are needed for reliable identification.

**Theorem 2 (Minimum identification cost through the Gaussian many-access channel).** Let the total number of users be \( \ell \), where each user is active with the same probability. Suppose the average number of active users \( k_\ell \) satisfies
\[ \lim_{\ell \to \infty} \ell e^{-\delta k_\ell} = 0 \] (18)
for every \( \delta > 0 \). Let
\[ n(\ell) = \frac{\ell H_2(k_\ell/\ell)}{\frac{1}{2} \log(1 + k_\ell P)}. \] (19)

Suppose \( n(\ell)/k_\ell \) has finite limit or diverges to infinity. The asymptotic identification cost is characterized as follows:

1) If \( \lim_{k_\ell \to \infty} n(\ell)/k_\ell > 0 \), then the minimum user identification cost is \( n(\ell) \).

2) If \( \lim_{k_\ell \to \infty} n(\ell)/k_\ell = 0 \), then a signature length of \( n_0 = \epsilon k_\ell \) yields vanishing error probability for any \( \epsilon > 0 \); on the other hand, if \( n_0 \leq (1 - \epsilon)n(\ell) \), then the identification error cannot vanish as \( \ell \to \infty \).

Note that (18) implies \( k_\ell \to \infty \) as \( \ell \to \infty \). In the special case where \( k_\ell = \lceil \ell^{1/d} \rceil \) for some \( d > 1 \), the minimum user identification cost is \( n(\ell) = 2(d - 1)k_\ell + o(k_\ell) \), which is linear in the...
number of active users. The minimum cost function $n(\ell)$ is illustrated in Fig. 2.

In the following, we first prove the converse of Theorem 1 which can be particularized to prove the converse of Theorem 2. Then we prove the achievability of Theorem 2 which is an essential step leading to the achievability of Theorem 1 eventually.

III. PROOF OF THE CONVERSE OF THEOREM 1

We prove the converse for the three cases in Theorem 1 respectively.

A. Converse for Case 1): unbounded $\ell_n$ and unbounded $k_n$

This proof requires more work than a straightforward use of Fano’s inequality, because the size of the joint input alphabet may increase rapidly with the blocklength. To overcome this difficulty, define for every given $\delta \in (0, 1)$,

$$B^\ell_m(\delta, k) = \{x \in X^\ell_m : 1 \leq ||x||_0 \leq (1 + \delta)k\},$$

which can be thought of as an $\ell_0$ ball but the origin. Since $X$ in (5) is a binary vector, whose expected support size is $k_n$, it is found in $B^\ell_M(\delta, k_n)$ with high probability for large $n$.

Based on the input distribution described in Section II

$$H(X) = \ell_n H(X_1) = \ell_n(H_2(\alpha_n) + \alpha_n \log M).$$
Let $E = 1 \{ \hat{X} \neq X \}$ indicate whether the receiver makes an error, where $\hat{X}$ is the estimation of $X$. Consider an $(M, n)$ code satisfying the power constraint (2) with $P_e^{(n)} = P\{ E = 1 \}$. The input entropy $H(X)$ can be calculated as

\[
H(X) = H(X|Y) + I(X;Y) \tag{22}
\]

\[
= H(X, 1 \{ X \in B^e_M(\delta, k_n) \} | Y) + I(X;Y) \tag{23}
\]

\[
= H(1 \{ X \in B^e_M(\delta, k_n) \} | Y) + H(X|1 \{ X \in B^e_M(\delta, k_n) \}, Y) + I(X;Y), \tag{24}
\]

where we used the chain rule of the entropy to obtain (24). Because the error indicator $E$ is determined by $X$ and $Y$, we can further obtain

\[
H(X) = H(1 \{ X \in B^e_M(\delta, k_n) \} | Y) + H(X, E|Y, 1 \{ X \in B^e_M(\delta, k_n) \}) + I(X;Y) \tag{25}
\]

\[
= H(1 \{ X \in B^e_M(\delta, k_n) \} | Y) + H(E|Y, 1 \{ X \in B^e_M(\delta, k_n) \})
\]

\[
+ H(X|E, Y, 1 \{ X \in B^e_M(\delta, k_n) \}) + I(X;Y) \tag{26}
\]

\[
\leq H_2(P \{ X \in B^e_M(\delta, k_n) \}) + H_2(P_e^{(n)})
\]

\[
+ H(X|E, Y, 1 \{ X \in B^e_M(\delta, k_n) \}) + I(X;Y) \tag{27}
\]

\[
\leq 2 \log 2 + H(X|E, Y, 1 \{ X \in B^e_M(\delta, k_n) \}) + I(X;Y). \tag{28}
\]

In the following, we will upper bound $I(X;Y)$ and $H(X|E, Y, 1 \{ X \in B^e_M(\delta, k_n) \})$.

**Lemma 1.** Suppose $X$ and $Y$ follow the distribution described by (5), then

\[
I(X;Y) \leq \frac{n}{2} \log (1 + k_n P). \tag{29}
\]

**Proof.** See Appendix A. \( \square \)

**Lemma 2.** Suppose $X$ and $Y$ follow the distribution described by (5). If $k_n$ is an unbounded sequence satisfying (10), then for large enough $n$,

\[
H(X|E, Y, 1 \{ X \in B^e_M(\delta, k_n) \}) \leq 4P_e^{(n)}(k_n \log M + k_n + \ell_n H_2(\alpha_n)) + \log M. \tag{30}
\]

**Proof.** See Appendix B. \( \square \)
Combining (21), (28), and Lemmas 1 and 2 we can obtain
\[
\ell_n H_2(\alpha_n) + k_n \log M \leq 2 \log 2 + 4P_e^{(n)}(\alpha_n) k_n \log M + k_n \alpha_n + \ell_n H_2(\alpha_n) + n \log(1 + k_n P).
\] (31)

Dividing both sides of (31) by \(k_n\) and rearranging the terms, we have
\[
(1 - 4P_e^{(n)}) \log M - \frac{1}{k_n} \log M + \left(1 - 4P_e^{(n)}\right) \frac{H_2(\alpha_n)}{\alpha_n} \leq B_1(n) + \frac{2 \log 2}{k_n} + 4P_e^{(n)},
\] (32)
where \(B_1(n)\) is defined as (14). Since \(k_n \to \infty\), we have for large enough \(n\),
\[
\left(1 - 4P_e^{(n)} - \frac{1}{k_n}\right) \left(\log M + \frac{H_2(\alpha_n)}{\alpha_n}\right) \leq B_1(n) + \delta + 4P_e^{(n)}.
\] (33)

Since \(P_e^{(n)}\) vanishes and \(k_n \to \infty\) as \(n\) increases and \(\delta\) can be chosen arbitrarily small, according to (33), given any \(\epsilon > 0\), there exists some \(\delta\) and for large enough \(n\) such that the following holds:
\[
\log M \leq (1 + \epsilon)B_1(n) - \frac{H_2(\alpha_n)}{\alpha_n}
\] (34)
\[
= (1 + \epsilon - \theta_n)B_1(n),
\] (35)
where \(\theta_n\) is defined as (11), whose limit is denoted as \(\theta\). Since (35) holds for arbitrary \(\epsilon\), if \(\theta > 1\), there exists a small enough \(\epsilon\) such that \(\log M < 0\) for large enough \(n\). It implies \(B(n) = 0\), meaning that an average user cannot send a single bit of information reliably. If \(\theta = 1\), then (35) implies that for large enough \(n\), \(\log M < \epsilon B_1(n)\) for any \(\epsilon > 0\).

If \(\theta < 1\), \(B(n)\) given by (12) can be written as
\[
B(n) = (1 - \theta_n)B_1(n).
\] (36)

The message length can be further upper bounded as
\[
\log M \leq \left(1 + \frac{\epsilon}{1 - \theta_n}\right) B(n),
\] (37)
which implies \(\log M \leq (1 + \epsilon)B(n)\) for any arbitrarily small \(\epsilon\). Thus, the converse for Case 1)
is established.

We have the following result on the “overhead factor” $\theta_n$.

**Proposition 1.** Let $\theta_n$ be defined as in (11). Consider the regime $k_n = \Theta(n)$. The following holds as $n \to \infty$:

1) If $\ell_n = \lceil an \rceil$ for some constant $a > 0$, then $\theta_n \to 0$ as $n \to \infty$.

2) If $\ell_n = \lceil an^d \rceil$ for some constant $a > 0$, $d > 1$ and $c = \lim_{n \to \infty} \frac{k_n}{n}$, then $\theta_n \to 2c(d - 1)$.

**Proof.** The proof is straightforward from (11) as $n \to \infty$. □

Proposition 1 demonstrates the overhead of active user identification as a function of the growth rate of $\ell_n$. When $\ell_n$ grows linearly in $n$, the cost of detecting the set of active users is negligible when amortized over $n$ channel uses. On the other hand, when $\ell_n$ grows too quickly in $n$, $\theta_n$ could be larger than 1, meaning that an average user cannot even transmit a single bit reliably over a block. For user identification not to use up all channel uses, we need

$$d < 1 + \frac{1}{2} \limsup_{n \to \infty} \frac{n}{k_n}.$$  \hspace{1cm} (38)

This explains the capacity trends in Fig. 1.

**B. Converse for Case 2): unbounded $\ell_n$ and bounded $k_n$**

The converse claim is basically that no linear growth in message length is achievable. Suppose that, to the contrary, $\limsup_{n \to \infty} B(n)/n = C$ for some $C > 0$. There must exist some $k_0 \geq 1$ such that $\frac{1}{2k_0} \log(1 + k_0 P) < C$. Then $C$ is at least the symmetric capacity of the conventional multiaccess channel with $k_0$ users. However, as $n \to \infty$, there is a non-vanishing probability that the number of active users is greater than $2k_0$. Letting each user transmit a message length of $B(n)$ would yield a strictly positive error probability. Hence the converse is proved.

**C. Converse for Case 3): bounded $\ell_n$**

If $\alpha_n \to 0$, a transmitting user sees no interference with probability $(1 - \alpha_n)^{\ell_n - 1} \to 1$. The converse is obvious because $\frac{1}{2} \log(1 + P)$ is the conventional capacity for the point-to-point channel. The achievable message length cannot exceed $\frac{1}{2} \log(1 + P)$ asymptotically.
If $\alpha_n \to \alpha > 0$, the number of active users is a binomial random variable. (The channel is nonergodic.) The probability that all $\ell$ users are active is $\alpha^\ell > 0$. Hence the converse follows from the symmetric rate $\frac{1}{2\ell} \log(1 + \ell P)$ for the conventional multiaccess channel with $\ell$ users.

IV. PROOF OF THEOREM 2

In this section, we prove the converse and achievability of the minimum user identification cost (Theorem 2). It is a crucial step in the proof of the achievability part of Theorem 1.

A. Converse

In either of the two cases in Theorem 2 it suffices to show that the probability of error cannot vanish if $n_0 = (1 - \epsilon)n(\ell)$ for any $0 < \epsilon < 1$. The converse of Theorem 2 follows exactly from that of Theorem 1 by replacing $M = 1$ and letting $n = n_0$. According to (34), in order to achieve vanishing error probability for random user identification, for any $0 < \epsilon < 1$,

$$\frac{n_0}{2k_\ell} \log\left(1 + k_\ell P\right) \geq \frac{H_2(\alpha_\ell)}{\alpha_\ell}. \quad (39)$$

Therefore, the length of the signature must satisfy

$$n_0 > \frac{\ell H_2(\alpha_\ell)}{2} \frac{1}{\log\left(1 + k_\ell P\right)} \quad (40)$$

for sufficiently large $\ell$.

B. Achievability

Let $n(\ell)$ be given by (19). Pick an arbitrary fixed $\epsilon \in (0, P)$. In the following, we will show that we can achieve vanishing error probability in identification using signature length

$$n_0 = \begin{cases} 
(1 + \epsilon) n(\ell), & \text{if } \lim_{k_\ell \to \infty} n(\ell)/k_\ell > 0 \\
\epsilon k_\ell, & \text{if } \lim_{k_\ell \to \infty} n(\ell)/k_\ell = 0.
\end{cases} \quad (41)$$

We provide a user identification scheme whose error probability is upper bounded by $e^{-c_k\ell}$ for some positive constant $c$ dependent on $\epsilon$. Let the signatures of each user $S^a_k$ be generated according to i.i.d. Gaussian distribution with zero mean and variance

$$P' = P - \epsilon. \quad (42)$$
The receiver searches the binary activity vector that best explains the received signal. We restrict
the search to be among all binary $\ell$-vectors whose weight does not exceed the average $k_\ell$ by a
small fraction, and formulate it as an optimization problem:

$$\begin{align*}
\text{minimize} & \quad \| Y^a - S^a x \|_2^2 \\
\text{subject to} & \quad x \in \{0, 1\}^\ell \\
& \quad \sum_{i=1}^\ell x_i \leq (1 + \delta_\ell) k_\ell,
\end{align*}$$

where $\delta_\ell$ controls the search region of $x$. We choose $\delta_\ell$ to be some monotone decreasing sequence
such that $\delta_\ell^2 k_\ell$ is unboundedly increasing and $\delta_\ell \log k_\ell \to 0$. Specifically, we let

$$\delta_\ell = k_\ell^{-\frac{1}{3}}.$$  

(44)

Denote $E_d$ as the event of detection error and $F_j$ as the event that the signature of the $j$-th
user violates the power constraint (2), $j = 1, \ldots, \ell$. The probability of error in the stage of
activity identification $P_e(\ell)$ is thus calculated as

$$P_e(\ell) \leq P \left( E_d \cup \left( \bigcup_{j \in \{1, \ldots, \ell\}} F_j \right) \right)$$

$$\leq P \{ E_d \} + \ell P \{ F_1 \}$$

(45)

(46)

using the union bound and the fact that all codewords are identically distributed.

Furthermore,

$$\ell P \{ F_1 \} = \ell P \left\{ \sum_{i=1}^{n_0} (S_{1i}^a)^2 > n_0 P \right\}$$

$$\leq \ell e^{-cn_0},$$

(47)

(48)

where $c$ is some positive number (which depends on $\epsilon$) due to large deviation theory for the
sum of i.i.d. Gaussian random variables [29]. In either case of (41), $n_0 \geq_\ell ak_\ell$ for some $a > 0$, so
we imply

$$\ell P \{ F_1 \} \leq_\ell \ell e^{-\delta k_\ell}$$

(49)
for some $\delta > 0$, which vanishes as $\ell \to \infty$ by assumption (18).

We next derive an upper bound of the probability of detection error $P\{E_d\}$. Clearly,

$$P\{E_d\} = E\{P\{E_d|X^a\}\}$$

$$\leq P\{X^a \notin B_1^\ell(\delta_\ell, k_\ell)\} + \sum_{x \in B_1^\ell(\delta_\ell, k_\ell)} P\{E_d|X^a = x\}P\{X^a = x\}. \quad (51)$$

The support size of the transmitted signal $X^a$ given by (16) follows the binomial distribution $\text{Bin}(\ell, k_\ell/\ell)$. By the Chernoff bound for binomial distribution [30],

$$P\{X^a \notin B_1^\ell(\delta_\ell, k_\ell)\} = P\left\{\sum_{i=1}^{\ell} X^a_i > (1 + \delta_\ell)k_\ell\right\} + P\left\{\sum_{i=1}^{\ell} X^a_i = 0\right\}$$

$$\leq \exp\left(-k_\ell\delta_\ell^2/3\right) + (1 - k_\ell/\ell)^\ell, \quad (52)$$

which vanishes due to (44) and the fact that $(1 - k_\ell/\ell)^\ell$ vanishes for unbounded $k_\ell$. In other words, the number of active user is smaller than $(1 + \delta_\ell)k_\ell$ with high probability. In order to prove Theorem 2, it suffices to show that the second term on the right-hand side (RHS) of (51) vanishes.

Pick arbitrary $x^* \in B_1^\ell(\delta_\ell, k_\ell)$. Let its support be $A^*$, which must satisfy $1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell$. We write $P\{E_d|X^a = x^*\}$ interchangeably with $P\{E_d|A^*\}$, because there is a one-to-one mapping between $x^*$ and $A^*$. In the remainder of this subsection, we analyze the decoding error probability conditioned on a fixed $A^*$ and drop the conditioning on $A^*$ for notational convenience, i.e., $P\{E_d\}$ implicitly means $P\{E_d|A^*\}$. The randomness lies in the signatures $S^a$ and the received signal $Y^a$ from $x^*$. Define

$$T_A = \left\|Y^a - \sum_{i \in A^*} S^a_i\right\|^2_2 - \left\|Y^a - \sum_{i \in A} S^a_i\right\|^2_2. \quad (54)$$

According to the decoding rule (43), a detection error may occur only if there is some $A \subseteq \{1, \cdots, \ell\}$ such that $A \neq A^*$, such that $|A| \leq (1 + \delta_\ell)k_\ell$, and $T_A \leq 0$. Hence,

$$E_d \subseteq \bigcup_{A \subseteq \{1, \cdots, \ell\}: \quad |A| \leq (1 + \delta_\ell)k_\ell, A \neq A^*} \{T_A \leq 0\}. \quad (55)$$

In the following, we divide the exponential number of error events in (55) into a relatively small number of classes. We will show that the probability of error of each class vanishes and
so does the overall error probability. Specifically, we write the union over $A$ according to the cardinality of the sets $A^* \cap A$ and $A\backslash A^*$. Let $w_1 = |A_1|$ and $w_2 = |A_2|$, where $A_1 = A^* \backslash A$ represents the set of misses and $A_2 = A \backslash A^*$ represents the set of false alarms. Then $(w_1, w_2)$ must satisfy $w_1 \leq |A^*|$, $w_2 \leq |A|$, and $|A^*| + w_2 = |A| + w_1$. According to the decoding rule (43), $(w_1, w_2)$ must be found in the following set:

$W^{(\ell)} = \left\{ (w_1, w_2) : w_1 \in \{0, 1, \ldots, |A^*|\}, w_2 \in \{0, 1, \ldots, (1 + \delta \ell) k \ell \}, w_1 + w_2 > 0, |A^*| + w_2 \leq (1 + \delta \ell) k \ell + w_1 \right\}$.  

Further define the event $E_{w_1,w_2}$ as

$E_{w_1,w_2} = \bigcup_{(w_1, w_2) \in W^{(\ell)}} \{ T_A \leq 0 \}$.  

By (55), $E_d \subseteq \bigcup_{(w_1, w_2) \in W^{(\ell)}} E_{w_1,w_2}$. Hence

$P\{E_d\} \leq \sum_{(w_1, w_2) \in W^{(\ell)}} P\{E_{w_1,w_2}\}$.  

We will show that when $\ell$ is large enough, there exists some constant $c_0 > 0$ such that $P\{E_{w_1,w_2}\} \leq e^{-k \ell c_0}$ for all $(w_1, w_2) \in W^{(\ell)}$.

Define

$A_1(w_1) = \{ A_1 : A_1 \subseteq A^*, |A_1| = w_1 \}$

and

$A_2(w_2) = \{ A_2 : A_2 \subseteq \{1, \cdots, \ell\} \backslash A^*, |A_2| = w_2 \}$.  

Fig. 3: The set relationship.
Then for any $A$ leading to an error event in $\mathcal{E}_{w_1,w_2}$ specified by (57), it can be written as $A = A_2 \cup (A^* \setminus A_1)$, for some $A_1 \in A_1(w_1)$ and $A_2 \in A_2(w_2)$. Therefore, (57) gives

$$
\mathcal{E}_{w_1,w_2} = \bigcup_{A_1 \in A_1(w_1)} \bigcup_{A_2 \in A_2(w_2)} \{T_A \leq 0\},
$$

(61)

which implies

$$
1 \{\mathcal{E}_{w_1,w_2}\} \leq \sum_{A_1 \in A_1(w_1)} \left( \sum_{A_2 \in A_2(w_2)} 1 \{T_A \leq 0\} \right)^\rho \sum_{A_1 \in A_1(w_1)} \left( \sum_{A_2 \in A_2(w_2)} 1 \{T_A \leq 0\} \right)^\rho
$$

(62)

for every $\rho \in [0, 1]$. As a result,

$$
P \{\mathcal{E}_{w_1,w_2}\} = E \{1 \{\mathcal{E}_{w_1,w_2}\}\}
$$

(63)

$$
\leq \sum_{A_1 \in A_1(w_1)} E \left\{ \left( \sum_{A_2 \in A_2(w_2)} 1 \{T_A \leq 0\} \right)^\rho \right\}
$$

(64)

where the expectation is taken over $(\mathcal{S}^a, Y^a)$. We further calculate the expectation by first conditioning on $(\mathcal{S}^a_{A^*}, Y^a)$ as follows:

$$
P \{\mathcal{E}_{w_1,w_2}\} \leq \sum_{A_1 \in A_1(w_1)} E \left\{ E \left\{ \left( \sum_{A_2 \in A_2(w_2)} 1 \{T_A \leq 0\} \right)^\rho \mathcal{S}^a_{A^*}, Y^a \right\} \right\}
$$

(65)

$$
\leq \sum_{A_1 \in A_1(w_1)} E \left\{ \left[ E \left\{ \left( \sum_{A_2 \in A_2(w_2)} 1 \{T_A \leq 0\} \mathcal{S}^a_{A^*}, Y^a \right\} \right\} \right]^\rho \right\},
$$

(66)

where the expectation is taken first with respect to the probability measure $p_{\mathcal{S}^a_{1,\ldots,t}\setminus A^*}\mathcal{S}^a_{A^*}, Y^a$ and then with respect to the probability measure $p_{\mathcal{S}^a_{A^*}, Y^a}$; and Jensen’s inequality is applied in (66) to the concave function $x^\rho$, $0 < \rho \leq 1$. Note that $\mathcal{S}^a_{1,\ldots,t}\setminus A^*$ and $\mathcal{S}^a_{A^*}$ are independent and $Y^a$ only depends on $\mathcal{S}^a_{A^*}$, we have $p_{\mathcal{S}^a_{1,\ldots,t}\setminus A^*}\mathcal{S}^a_{A^*}, Y^a(\mathcal{S}_{a}, y) = p_{\mathcal{S}^a_{1,\ldots,t}\setminus A^*}(\mathcal{S}_{a})$. The inner expectation in (66) is taken with respect to the probability measure $p_{\mathcal{S}^a_{A^*}}$ for each $A_2 \in A_2(w_2)$. Since the entries of $\mathcal{S}^a$ are i.i.d., the inner expectation yields identical results for all $A_2 \in A_2(w_2)$ and the outer expectation yields identical results for all $A_1 \in A_1(w_1)$.

The number of choices for $A_1$ is $\binom{\lvert A^* \rvert}{w_1}$, whereas the number of choices for $A_2$ is no greater
than \((\ell_{w_2})\). Therefore, we apply the union bound to obtain

\[
P \{ \mathcal{E}_{w_1,w_2} \} \leq \binom{|A^*|}{w_1} \binom{\ell}{w_2} \mathbb{E} \left\{ \left[ \mathbb{E} \{ 1 \{ T_A \leq 0 \} \left| \mathcal{S}_{A^*}^a, Y^a \right. \} \right]^{\rho} \right\},
\]

(67)

where \(A\) is now a fixed representative choice with \(|A^*\setminus A| = w_1\) and \(|A\setminus A^*| = w_2\).

We will obtain an upper bound of the detection error probability by further upper bounding \(\mathbb{E} \{ 1 \{ T_A \leq 0 \} \left| \mathcal{S}_{A^*}^a, Y^a \right. \}\). Let

\[
p_{Y|S_A}(y_i|s_{A,i}) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( y_i - \sum_{k \in A} s_{ki} \right)^2 \right).
\]

(68)

The conditional distribution of \(y\) given that the codewords \(s_A\) are transmitted is given by

\[
p_{Y|S_A}(y|s_A) = \prod_{i=1}^n p_{Y|S_A}(y_i|s_{A,i}), \quad \text{where } n \text{ is the dimension of } y.
\]

Then for any \(\lambda \geq 0\), the following holds due to (54):

\[
\mathbb{E} \left\{ 1 \{ T_A \leq 0 \} \left| \mathcal{S}_{A^*}^a, Y^a \right. \right\} = \mathbb{E} \left\{ 1 \left\{ \frac{p_{Y|S_A}(Y^a|s_{A^*}^a)}{p_{Y|S_A}(Y^a|s_A^a)} \geq 1 \right\} \left| \mathcal{S}_{A^*}^a, Y^a \right. \right\}
\]

(69)

\[
\leq \mathbb{E} \left\{ \left( \frac{p_{Y|S_A}(Y^a|s_{A^*}^a)}{p_{Y|S_A}(Y^a|s_A^a)} \right)^\lambda \left| \mathcal{S}_{A^*}^a, Y^a \right. \right\}
\]

(70)

\[
= p_{Y|S_A}(Y^a|s_{A^*}^a) \mathbb{E} \left\{ p_{Y|S_A}(Y^a|s_A^a) \left| \mathcal{S}_{A^*}^a, Y^a \right. \right\},
\]

(71)

where (71) follows because \((\mathcal{S}_{A^*}^a, Y)\) is independent of \(\mathcal{S}_{A^a}^a\). For every function \(g(S_{A^*}^a, Y^a)\),

\[
\mathbb{E} \{ g(S_{A^*}^a, Y^a) \} = \int_{\mathbb{R}^n_0} \mathbb{E} \{ g(S_{A^*}^a, Y^a) p_{Y|S_A}(Y^a|s_{A^*}^a) \} \, dy.
\]

Combining (67) and (71) yields

\[
P \{ \mathcal{E}_{w_1,w_2} \} \leq \binom{|A^*|}{w_1} \binom{\ell}{w_2} \int_{\mathbb{R}^n_0} \mathbb{E} \left\{ p_{Y|S_A}(y|s_{A^*}^a) \left( \mathbb{E} \left\{ p_{Y|S_A}(y|s_A^a) \left| \mathcal{S}_{A^*}^a \right. \right\} \right)^\rho \right\} \, dy.
\]

(72)

Due to the memoryless channel property, i.e., \(p_{Y|S_A}(y|s_A^a) = \prod_{i=1}^{n_0} p_{Y|S_A}(y_i|s_{A,i}^a)\), we obtain

\[
P \{ \mathcal{E}_{w_1,w_2} \} \leq \binom{|A^*|}{w_1} \binom{\ell}{w_2} (m_{\lambda,\rho}(w_1, w_2))^{n_0}
\]

(73)

where

\[
m_{\lambda,\rho}(w_1, w_2) = \int_{\mathbb{R}} \mathbb{E} \left\{ p_{Y|S_A}(y|s_{A^*}^a) \left( \mathbb{E} \left\{ p_{Y|S_A}(y|s_A^a) \left| \mathcal{S}_{A^*}^a \right. \right\} \right)^\rho \right\} \, dy.
\]

(74)
The first two terms of the RHS of (73) can be upper bounded as [10, Page 353]

\[
\left(\frac{|A^*|}{w_1}\right)\left(\frac{\ell}{w_2}\right)^\rho \leq \exp\left(|A^*|H_2\left(\frac{w_1}{|A^*|}\right) + \rho \ell H_2\left(\frac{w_2}{\ell}\right)\right).
\] (75)

Moreover, by the Gaussian distribution of the codewords, the last term of the RHS of (73) can be explicitly calculated (see Appendix C) to obtain

\[
m_{\lambda, \rho}(w_1, w_2) = \exp\left(\frac{1 - \rho}{2} \log(1 + \lambda w_2 P') - \frac{1}{2} \log\left(1 + \lambda(1 - \lambda \rho) w_2 P' + \lambda \rho(1 - \lambda \rho) w_1 P'\right)\right).
\] (76)

Therefore, by (73)-(76),

\[
\mathbb{P}\{\mathcal{E}_{w_1, w_2}\} \leq \exp\left(-k_\ell h_{\lambda, \rho}(w_1, w_2)\right),
\] (77)

where

\[
h_{\lambda, \rho}(w_1, w_2) = \frac{n_0}{2k_\ell} \log\left(1 + \lambda(1 - \lambda \rho) w_2 P' + \lambda \rho(1 - \lambda \rho) w_1 P'\right) - \frac{(1 - \rho)n_0}{2k_\ell} \log(1 + \lambda w_2 P') - \frac{|A^*|}{k_\ell} H_2\left(\frac{w_1}{|A^*|}\right) - \frac{\rho \ell}{k_\ell} H_2\left(\frac{w_2}{\ell}\right).
\] (78)

To show the capacity achievability, we next show that by choosing \(\lambda\) and \(\rho\) properly, for large enough \(\ell\), \(h_{\lambda, \rho}(w_1, w_2)\) is strictly greater than some positive constant for all \((w_1, w_2) \in \mathcal{W}(\ell)\).

**Lemma 3.** Fix \(\epsilon \in (0, P)\). Let \(P' = P - \epsilon\). Let \(n(\ell)\) be given by (19) and \(n_0\) be given by (41). Suppose \(n(\ell)/k_\ell\) has finite limit or diverges to infinity. There exists \(\ell^* > 0\) and \(c_0 > 0\) such that for every \(\ell \geq \ell^*\) the following holds: If the true signal \(x^a \in \mathcal{B}_\ell'(\delta_\ell, k_\ell), i.e., 1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell, then for every \((w_1, w_2) \in \mathcal{W}(\ell)\) with \(\mathcal{W}(\ell)\) defined as in (56), there exist \(\lambda \in [0, \infty)\) and \(\rho \in [0, 1]\) such that

\[
h_{\lambda, \rho}(w_1, w_2) \geq c_0.
\] (79)

**Proof.** See Appendix D

Lemma 3 and (77) imply

\[
\mathbb{P}\{\mathcal{E}_{w_1, w_2} \mid A^*\} \leq e^{-c_0k_\ell},
\] (80)
for all $\ell \geq \ell^*$, $(w_1, w_2) \in W^{(\ell)}$, and $1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell$. Then as long as $\ell \geq \ell^*$, for any $x \in B_1^{(\delta_\ell, k_\ell)}$,

$$P\{E_d | X^n = x\} \leq \sum_{(w_1, w_2) \in W^{(\ell)}} P\{E_{w_1, w_2} | X^n = x\} \leq \sum_{(w_1, w_2) \in W^{(\ell)}} e^{-c_0 k_\ell} \leq 4k_\ell^2 e^{-c_0 k_\ell},$$

(83)

where (83) is due to $w_1 \leq 2k_\ell$ and $w_2 \leq 2k_\ell$. Therefore, the first term on the RHS of (51) vanishes as $\ell$ increases. So does $P\{E_d\}$. Thus we can achieve arbitrarily reliable identification with SNR $P' = P - \epsilon$ and signature length $n_0$ given by (41). Since $\epsilon$ can be arbitrarily small, the achievability of Theorem 2 is established.

V. PROOF OF THE ACHIEVABILITY OF THEOREM 1

A. Achievability for Case 3) with bounded $\ell_n$

As $\ell_n$ is nondecreasing, $\ell_n \to \ell$ for some constant $\ell$. If $\alpha_n \to \alpha > 0$, with some positive probability all $\ell$ users are active. Hence the achievability capacity follows from the result for the conventional multiaccess channel with $\ell$ users.

If $\alpha_n \to 0$, a transmitting user experiences a single-user channel with probability $(1 - \alpha_n)^{\ell_n - 1} \to 1$. Therefore, it can achieve a vanishing error probability with the conventional capacity for the point-to-point channel.

B. Achievability for Case 1) and Case 2) with unbounded $\ell_n$

We first assume unbounded $k_n$ and establish the achievability result. The case of bounded $k_n$ is then straightforward.

We consider a two-stage approach: In the first stage, the set of active users are identified based on their unique signatures. In the second stage, the messages from the active users are decoded. Let $\theta_n$ and its limit $\theta$ be defined as in Theorem 1. We consider the cases of $\theta = 0$ and $\theta > 0$ at the same time. Fix $\epsilon \in (0, \min(1, P))$. Specifically, the following scheme is used:
Fig. 4: Codebook structure. Each user maintains $M$ codewords with each consisting of a message-bearing codeword prepended by a signature.

- **Codebook construction:** The codebooks of the $\ell_n$ users are generated independently. Let

$$n_0 = \begin{cases} \epsilon n, & \text{if } \theta = 0 \\ (1 + \epsilon) \theta n, & \text{otherwise} \end{cases}$$

(84)

For user $k$, codeword $s_k(0) = 0$ represents silence. User $k$ also generates $M$ codewords as follows. First, generate $M$ random sequences of length $n - n_0$, each according to i.i.d. Gaussian distribution with zero mean and variance $P' = P - \epsilon$. Then generate one signature of length $n_0$ with i.i.d. $\mathcal{N}(0, P')$, denoted by $S_k^a$, and prepend this signature to every codeword to form $M$ codewords of length $n$. In other words, the $w$-th codeword of user $k$ takes the shape of $S_k(w) = \left( S_k^a, s_k^b(w) \right)$. The matrix of the concatenated codebooks of all users is illustrated in Fig. 4.

- **Transmission:** For user $k$ to be silent, it is equivalent to transmitting $s_k(0)$. Otherwise, to send message $w_k \neq 0$, user $k$ transmits $S_k(w_k)$.

- **Channel:** Each user is active independently with probability $\alpha_n$. The active users transmit simultaneously. The received signal is $Y$ given by (5).

- **Two-stage detection and decoding:** Upon receiving $Y$, the decoder performs the following:

  1. Active user identification: Let $Y^a$ denote the first $n_0$ entries of $Y$, corresponding to the superimposed signatures of all active users subject to noise. $Y^a$ is mathematically described by (16). The receiver estimates $X^a$ according to (43). The output of this stage is
a set $A \subseteq \{1, \cdots, \ell_n\}$ that contains the detected active users.

(2) Message decoding: Let $Y^b$ denote the last $n - n_0$ entries of $Y$, corresponding to the superimposed message-bearing codewords. The receiver solves the following optimization problem:

$$\begin{align*}
\text{minimize} & \quad ||Y^b - S^b \left[ x_1^T, \cdots, x_{\ell_n}^T \right]^T ||^2 \\
\text{subject to} & \quad x_k \in X_{M}, k = 1, \cdots, \ell_n \\
& \quad x_k = 0, \forall k \notin A \\
& \quad x_k \neq 0, \forall k \in A
\end{align*}$$

(86)

(87)

(88)

(89)

Basically the receiver performs the maximum likelihood decoding for the set of users in the purported active user set $A$. The position of 1 in each recovered nonzero $x_k$ indicates the message from user $k$.

**Theorem 3** (Achievability of the Gaussian many-access channel). Let $\theta_n$ be defined as (11) and $B(n)$ be defined as (12). Suppose $\lim_{n \to \infty} \theta_n < 1$. For the MnAC given by (1), for any given constant $\epsilon \in (0, 1)$, the message length of $(1 - \epsilon)B(n)$ is asymptotically achievable using the preceding scheme.

The remainder of this section is devoted to the proof of Theorem 3. In Section V-C, we show that the set of active users can be accurately identified in stage 1. In Section V-D, we show that the users’ messages can be accurately decoded in stage 2 assuming knowledge of the active users. The results are combined in Section V-E to establish the achievability part of Theorem 3.

C. Optimal User Identification

We shall invoke Theorem 2 (proved in Section IV) to quantify the cost of reliable user identification. To adapt to the notation in this section, we apply Theorem 2 with $\ell$ and $k_\ell$ being replaced by $\ell_n$ and $k_n$, respectively. With the change of notations, $n(\ell)$ as defined in Theorem 2 can be written as

$$n(\ell) = \frac{\ell_n H_2(k_n/\ell_n)}{\frac{1}{2} \log(1 + k_n P)}$$

$$= \theta_n n,$$

(90)

(91)
where $\theta_n$ is given by (11).

According to Theorem 2 choosing the signature length $n_0 = (1 + \epsilon)\theta_n n$ and $n_0 = \epsilon k_n$ yields vanishing error probability in user identification for the case of $\lim_{n \to \infty} \theta_n n/k_n > 0$ and $\lim_{n \to \infty} \theta_n n/k_n = 0$, respectively, where $\epsilon \in (0, 1)$ is an arbitrary constant. In the following, we make use of this result to prove that choosing $n_0$ according to (84) guarantees reliable user identification.

First, consider $\theta = 0$. By (84), the signature length is $n_0 = \epsilon n$ for some $\epsilon$. In the case of $\lim_{n \to \infty} \theta_n n/k_n > 0$, since $\theta_n$ vanishes, it must have $n_0 \geq_n (1 + \epsilon)\theta_n n$. In the case of $\lim_{n \to \infty} \theta_n n/k_n = 0$, since $k_n = O(n)$, $n_0 = \epsilon n$ implies $n_0 \geq_n \epsilon' k_n$ for some $\epsilon' > 0$. By Theorem 2 the choice of $n_0$ is sufficient for reliable user identification.

Second, consider $\theta > 0$. By (84), the signature length is $n_0 = (1 + \epsilon)\theta_n n$. Since $k_n = O(n)$, it must have $\lim_{n \to \infty} \theta_n n/k_n > 0$. Thus, the signature length $n_0$ obviously achieves reliable user identification by Theorem 2.

D. Achieving the Capacity of MnAC with Known User Activities

In previous work [7], we studied the capacity of the Gaussian MnAC where all users are always active and the number of users is sublinear in the blocklength, i.e., $k_n = o(n)$. In that case, random coding with Feinstein’s suboptimal decoding, which suffices to achieve the capacity of conventional multiaccess channel capacity, can achieve the capacity of the Gaussian MnAC. Proving the capacity achievability for faster scaling of the number of active users is much more challenging, mainly because the exponential number of possible error events prevents one from using the simple union bound. Here, we derive the capacity of the MnAC for the case where the number of users may grow as quickly as linearly with the blocklength by lower bounding the error exponent of the error probability due to maximum-likelihood decoding. The results also complement a related study of many-broadcast models in [14].

**Theorem 4** (Capacity of the Gaussian many-access channel without random access). For the MnAC with $k_n$ always-active users, suppose the number of channel uses is $n$ and the number of users $k_n$ grows as $O(n)$, the symmetric capacity is

$$B_1(n) = \frac{n}{2k_n} \log(1 + k_n P).$$

(92)
In particular, for any \( \epsilon \in (0,1) \), there exists a sequence of codebooks with message lengths (in nats) \( B_1(n)(1-\epsilon) \) such that the average error probability is arbitrarily small for sufficiently large \( n \).

In the following, we will prove Theorem 4. We can model the MnAC with known user activities using (5) with \( \alpha_n = 1 \), i.e., \( k_n = \ell_n \). Upon receiving the length-\( n \) vector \( y \), we estimate \( x = [x_1^T, \ldots, x_{k_n}^T]^T \) using the maximum likelihood decoding:

\[
\begin{align*}
&\text{minimize} \quad ||y - sx||^2 \\
&\text{subject to} \quad x_k = e_m, \quad \text{for some } m = 1, \ldots, M.
\end{align*}
\]

(93)

(94)

Define \( F_j \) as the event that user \( j \)'s codeword violates the power constraint (2), \( j = 1, \ldots, k_n \). Define \( E_k \) as the error event that \( k \) users are received in error. Suppose \( P\{E_k|A^*\} \) is the probability of \( E_k \) given that the true signal is \( x^* \) with support \( A^* \). By symmetry of the codebook construction, the average error probability can be calculated as

\[
P_e^{(n)} \leq P\left\{ \bigcup_{k=1}^{k_n} E_k \cup \bigcup_{j=1}^{k_n} F_j \right\} \quad \text{subject to} \quad x_k = e_m, \quad \text{for some } m = 1, \ldots, M.
\]

(95)

(96)

Let \( A \) be the support of the estimated \( x \) according to the maximum likelihood decoding. Define \( A_1 \) and \( A_2 \) in the same manner as that in Section [V-C] i.e., \( A_1 = A^* \setminus A \) and \( A_2 = A \setminus A^* \). In this case, \( |A| = |A^*| = k_n \) and \( |A_2| = |A_1| = k \). Further denote \( \gamma = k/k_n \) as the fraction of users subjected to errors. Then we write \( P\{E_k|A^*\} \) and \( P\{E_{\gamma}|A^*\} \) interchangeably. In the following analysis, we consider a fixed \( A^* \) and drop the conditioning on \( A^* \) for notational convenience.

Following similar arguments leading to (73), letting \( \lambda = \frac{1}{1+\rho} \) and considering \( \binom{k_n}{\gamma k_n} \) possible sets of \( A_1 \) and \( M^{\gamma k_n} \) possible sets of \( A_2 \), we have

\[
P\{E_{\gamma}\} \leq \binom{k_n}{\gamma k_n} M^{\gamma k_n} \rho \left( \int _{\mathbb{R}} E \left\{ \frac{1}{P_{Y|S_A}(y|S_{A^*})} \left( E \left\{ \frac{1}{P_{Y|S_A}(y|S_A)} \right| S_{A^*} \right) \right)^{\rho} dy \right)^n
\]

(97)

(98)
By symmetry, \( E \left\{ \mathbb{E}_{Y | S_A}(y | S_A) \mathbb{E}_{S_A} \right\} = E \left\{ \mathbb{E}_{Y | S_A}(y | S_A^* \triangleleft A) \mathbb{E}_{S_A^* \triangleleft A} \right\}, \) which results in

\[
P \{ \mathcal{E}_\gamma \} \leq \left( \frac{k_n}{\gamma k_n} \right) M^{\gamma k_n \rho} \exp(-n E_0(\gamma, \rho)),
\]

(99)

where \( E_0(\gamma, \rho) \) is defined by

\[
E_0(\gamma, \rho) = -\log \left( \int_{\mathbb{R}} \mathbb{E} \left\{ \left( \mathbb{E}_{Y | S_A}(y | S_A) \right)^{\frac{1}{\rho+1}} \mathbb{E}_{S_A^*} \right\} dy \right).
\]

(100)

By the inequality \( \left( \frac{k_n}{\gamma k_n} \right) \leq \exp(k_n H_2(\gamma)) \), we can further upper bound \( P \{ \mathcal{E}_\gamma \} \) as

\[
P \{ \mathcal{E}_\gamma \} \leq \exp \left[ -n f(\gamma, \rho) \right],
\]

(101)

where

\[
f(\gamma, \rho) = E_0(\gamma, \rho) - \gamma \rho k_n v(n) - \frac{k_n}{n} H_2(\gamma),
\]

(102)

and \( v(n) = \log M \). Intuitively, \( E_0(\gamma, \rho) \) in (101) is an achievable error exponent for the error probability caused by a particular \( A \) being detected in favor of \( A^* \) and the terms \( k_n H_2(\gamma) + \gamma \rho k_n v(n) \) correspond to the cardinality of all possible \( A \) leading to the error event \( \mathcal{E}_\gamma \).

By particularizing (76) with \( w_1 = w_2 = \gamma k_n \) and \( \lambda = \frac{1}{1+\rho} \), we can derive \( E_0(\gamma, \rho) \) explicitly as

\[
E_0(\gamma, \rho) = -\log m_{\lambda, \rho}(w_1, w_2) |_{w_1 = w_2 = \gamma k_n, \lambda = \frac{1}{1+\rho}}
\]

(103)

\[
= \frac{\rho}{2} \log \left( 1 + \frac{\gamma k_n P'}{\rho + 1} \right).
\]

(104)

The achievable error exponent for \( P(\mathcal{E}_\gamma) \) is determined by the minimum error exponent over the range of \( \gamma \), i.e.,

\[
E_r = \min_{\frac{1}{k_n} \leq \gamma \leq 1} \max_{0 \leq \rho \leq 1} f(\gamma, \rho).
\]

(105)

The following Lemma is key to establishing Theorem 4.

**Lemma 4.** Let \( M \) be such that the message length \( v(n) = \log M \) is given by

\[
v(n) = \frac{n}{2k_n} \log(1 + k_n P').
\]

(106)
Suppose \( k_n = O(n) \), there exists \( n^* \) and \( c_0 > 0 \) such that for every \( n \geq n^* \),

\[
P\{E_k | A^*\} \leq e^{-c_0 n}
\]  

(107)

holds uniformly for all \( 1 \leq k \leq k_n \) and for all \( |A^*| \).

**Proof.** See Appendix E. □

Due to Lemma 4, for large enough \( n \),

\[
\sum_{k=1}^{k_n} P\{E_k | A^*\} \leq k_n e^{-c_0 n}
\]  

(108)

which vanishes as \( n \) increases. Moreover, following the same argument as (48), the second term of the RHS of (96) vanishes and hence \( P_{e^{(n)}} \) given by (96) can be proved to vanish. As a result, Theorem 4 is established.

### E. Achieving the Capacity of MnAC with On-off Random Access

In this subsection, we combine the results of Section V-C and Section V-D to prove the achievability result for Case 1) and Case 2) in Theorem 3. We first prove the case of unbounded \( k_n \), and the case of bounded \( k_n \) follows naturally. Let \( \theta \) denote the limit of \( \theta_n \).

**Case 1): unbounded \( \ell_n \) and unbounded \( k_n \).**

We further divide this case into two sub-cases.

**Sub-case a: \( 0 < \theta < 1 \).**

We need to show that the message length \( (1 - \epsilon)B(n) \) is asymptotically achievable for any fixed \( \epsilon \in (0, 1) \).

The detection errors are caused by activity identification error or message decoding error. It has been shown by (53) that with high probability the number of active users is no more than \( (1 + \delta_n)k_n \). As a result, Theorem 2 and Theorem 4 conclude that the message length

\[
\frac{(1 - \epsilon')(n - n_0)}{2(1 + \delta_n)k_n} \log (1 + (1 + \delta_n)k_n P),
\]  

(109)

where \( n_0 = (1 + \epsilon')\theta_n n \), is asymptotically achievable for any \( \epsilon' > 0 \).

In order to prove the achievability, it suffices to show that there exists \( \epsilon' \) such that the message
length given by (109) is asymptotically greater than
\[(1 - \epsilon)B(n) = \frac{(1 - \epsilon)(1 - \theta_n)n}{2k_n} \log (1 + k_nP). \quad (110)\]

The intuition of proof is that for sufficiently large \(n\), \((1 + \delta_n)k_n\) is approximately \(k_n\), and we can always find a small enough \(\epsilon'\) such that \((1 - \epsilon')(n - n_0)\) is greater than \((1 - \epsilon)(1 - \theta_n)n\).

We choose some small enough \(\epsilon' > 0\) such that
\[ (1 - \epsilon')^2 - \epsilon'(1 - \epsilon')^2 \frac{1 + \theta}{1 - \theta} > 1 - \epsilon. \quad (111)\]

This is feasible because the left-hand side of (111) is equal to 1 if \(\epsilon' = 0\).

Since \(\log (1 + (1 + \delta_n)k_nP) / \log(1 + k_nP) \to 1\) and \(\delta_n \to 0\) as \(n\) increases, we have
\[ \frac{\log (1 + (1 + \delta_n)k_nP)}{(1 + \delta_n)} \geq n (1 - \epsilon') \log(1 + k_nP). \quad (112)\]

The difference between (109) and \((1 - \epsilon)B(n)\) is calculated as
\[
\begin{align*}
\frac{(1 - \epsilon')(n - n_0)}{2(1 + \delta_n)k_n} & \log (1 + (1 + \delta_n)k_nP) - (1 - \epsilon)B(n) \\
\geq & \frac{1}{1 - \theta_n} \left[ (1 - \epsilon')^2 (1 - n_0/n) - (1 - \epsilon) \right] B(n) \\
= & \left[ (1 - \epsilon')^2 - \epsilon'(1 - \epsilon')^2 \frac{\theta_n}{1 - \theta_n} - (1 - \epsilon) \right] B(n) \\
\geq & \frac{1}{1 - \theta} \left[ (1 - \epsilon')^2 - \epsilon'(1 - \epsilon')^2 \frac{1 + \theta}{1 - \theta} - (1 - \epsilon) \right] B(n)
\end{align*}
\quad (113)\]

where (115) is due to \(\theta_n \leq n (1 + \theta)/2\). By (111), the RHS of (115) is greater than zero. It means that for large enough \(n\), the achievable message length (109) is greater than \((1 - \epsilon)B(n)\), which establishes the achievability.

\textbf{Sub-case b:} \(\theta = 0\).

The proof for the case of vanishing \(\theta_n\) is analogous. We need to show that message length \((1 - \epsilon)B_1(n)\) is asymptotically achievable for any fixed \(\epsilon \in (0, 1)\).

The number of active users is no more than \((1 + \delta_n)k_n\) with high probability. As a result, Theorem 2 and Theorem 4 conclude that the message length
\[
\frac{(1 - \epsilon')(n - n_0)}{2(1 + \delta_n)k_n} \log (1 + (1 + \delta_n)k_nP), \quad (116)
\]
where $n_0 = \epsilon' n$, is asymptotically achievable for any $\epsilon' > 0$.

In order to prove Theorem 3, it suffices to show that there exists $\epsilon'$ such that the message length given by (116) is asymptotically greater than

\[(1 - \epsilon)B_1(n) = (1 - \epsilon) \frac{n}{2k_n} \log (1 + k_n P) \quad (117)\]

Choose some small enough $\epsilon' > 0$ such that

\[(1 - \epsilon')^3 > (1 - \epsilon). \quad (118)\]

The difference between (116) and $(1 - \epsilon)B_1(n)$ is calculated as

\[
\begin{align*}
\frac{(1 - \epsilon')(n - n_0)}{2(1 + \delta_n)k_n} \log (1 + (1 + \delta_n)k_n P) - (1 - \epsilon)B_1(n) \\
\geq_n \left[ (1 - \epsilon')^2 (1 - n_0 / n) - (1 - \epsilon) \right] B_1(n) \\
= \left[ (1 - \epsilon')^3 - (1 - \epsilon) \right] B(n),
\end{align*}
\]

where (119) is due to (112). By the choice of $\epsilon'$ given by (118), (120) is greater than zero. It concludes that for large enough $n$, the achievable message length (116) is greater than $(1 - \epsilon)B_1(n)$, which establishes the achievability.

**Case 2): unbounded $\ell_n$ and bounded $k_n$**

In the case of unbounded $\ell_n$ and bounded $k_n$, there is nonvanishing probability that the number of active users is equal to any finite number. The number of active users is no longer fewer than $(1 + \delta_n)k_n$ with high probability. Let $s_n$ be any increasing sequence. There is high probability that the number of users is fewer than $(1 + \delta_n)s_n$. As a result, by treating $s_n$ as the unbounded $k_n$ as in Case 1), we can apply the established achievable results for Case 1). The achievability result for Case 2) is summarized in the following theorem.

**Theorem 5.** Let $s_n$ be any increasing sequence satisfying $s_n = O(n)$, $\ell_n e^{-\delta s_n} \to 0$ for every $\delta > 0$ and

\[
\lim_{n \to \infty} \frac{2\ell_n H_2(s_n / \ell_n)}{n \log(1 + s_n P)} < 1.
\]

(121)
Then any message length given by
\[
(1 - \epsilon) \left( \frac{n}{2s_n} \log(1 + s_n P) - H_2(s_n/\ell_n) \right)
\]
(122)
is asymptotically achievable.

Proof. See Appendix F.

VI. ON SUCCESSIVE DECODING FOR MANY-ACCESS CHANNELS

In conventional multiaccess channels, the sum capacity can be achieved by successive decoding. A natural question is: Can the sum capacity of the MnAC be achieved using successive decoding? We consider the system model where all users have the same power constraints, assuming no random activity and the number of users being \( k_n = an \) for some \( a > 0 \). We provide a negative answer for the case where Gaussian random codes are used and successive decoding is applied. Throughout the discussion in this section, we do not seek to achieve the symmetric capacity, but the sum capacity achieved by successive decoding.

Suppose Gaussian random codes are used, i.e., each user generates its codewords as i.i.d. Gaussian random variables with zero mean and variance \( P \). Thus the codewords of other users look like Gaussian noise to any given user. The first user to be decoded has the largest interference from all the other \( k_n - 1 \) users and its signal-to-interference-plus-noise ratio (SINR) is \( Q = \frac{P}{1 + (k_n - 1)P} \). Suppose the first user transmits with message length
\[
v(n) = (1 - \epsilon)nC,
\]
(123)
where \( C = \frac{1}{2} \log(1 + Q) \). We will show that the error probability is strictly bounded from zero. The intuition is that the error probability usually decays at the rate of \( \exp(-\delta nC) \), where \( \delta \) is some positive constant dependent on \( \epsilon \). In the MnAC setting, if the interference due to many users is so large that \( nC \) converges to a finite constant, the error exponent is not large enough to drive the error probability to zero as the blocklength increases.

Lemma 5. Suppose Gaussian random codes are used and successive decoding is applied. There exist universal constants \( A_1 > 0 \) and \( A_2 > 0 \), such that the error probability of the first user is
lower bounded as
\[ P_e^{(n)} \geq Q(x) e^{-\frac{A_1 T x^3}{S^{3/2}}} \left( 1 - \frac{A_2 T x}{S^{3/2}} \right) - e^{-(\lambda-1)(n-1)\epsilon C}, \]  
(124)

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du, \) \( S = 2nQ(2+Q), \)
\[ x = \frac{2(\lambda\epsilon n + 1 - \lambda\epsilon)C(1+Q)}{\sqrt{S}}, \]  
(125)

and
\[ T = nE \left\{ (-Q(1-Z^2) - 2\sqrt{QZ})^3 \right\} \]  
(126)

with \( Z \) being a standard Gaussian random variable.

**Proof.** See Appendix G.

Let \( k_n = an \) for some constant \( a > 0 \). Then, as \( n \to \infty \),
\[ nQ \to \frac{1}{a}, \]  
(127)
\[ S \to \frac{4}{a}, \]  
(128)
\[ T \to 0, \]  
(129)
\[ nC \to \frac{1}{2a}, \]  
(130)
\[ x \to \frac{\epsilon\lambda}{2\sqrt{a}}. \]  
(131)

Therefore,
\[ \lim_{n\to\infty} P_e^{(n)} \geq Q \left( \frac{\epsilon\lambda}{2\sqrt{a}} \right) - e^{-(\lambda-1)\epsilon C}. \]  
(132)

Using the lower bound of \( Q(x) \geq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{x} - \frac{1}{x^3} \right) e^{-x^2/2} \), it can be seen that when the exponential term is dominating, there exist some small enough \( \lambda\epsilon \) such that the first term in (132) is greater than the second term. In this case, the error probability is strictly bounded away from zero. Fig. 5 plots the numerical results of the RHS of (132) for different values of \( a \) and \( \lambda \). It can be seen that for the different values of \( a \), there exists some \( \lambda \) that makes the lower bound of error probability (132) strictly greater than zero.
VII. MANY-ACCESS CHANNEL WITH HETEROGENEOUS USER GROUPS

In this section, we will generalize the characterization of capacity region to the case where groups of users have heterogeneous channel gains and activity patterns. Suppose \( \ell_n \) users can be divided into a finite number of \( J \) groups, where group \( j \) consists of \( \beta^{(j)} \ell_n \) users with \( \sum_{j=1}^{J} \beta^{(j)} = 1 \). Further assume every user in group \( j \) has the same power constraint \( P^{(j)} \). Each user in group \( j \) transmits with probability \( \alpha^{(j)} n \). We refer to such MnAC with heterogeneous channel gains and activity patterns as the configuration \( \left( \left\{ \alpha^{(j)} n \right\}, \left\{ \beta^{(j)} \right\}, \left\{ P^{(j)} \right\}, \ell_n \right) \). The error probability is defined as the probability that the receiver incorrectly detects the message of any user in the system. The problem is what is the maximum achievable message length for users in each group such that the average error probability vanishes.

**Definition 5** (Asymptotically achievable message length tuple). Consider a MnAC of configuration \( \left( \left\{ \alpha^{(j)} n \right\}, \left\{ \beta^{(j)} \right\}, \left\{ P^{(j)} \right\}, \ell_n \right) \). A sequence of \( \left( [\exp(v^{(1)}(n))] , \cdots , [\exp(v^{(J)}(n))] , n \right) \) code for this configuration consists of a \( \left( [\exp(v^{(j)}(n))] , n \right) \) symmetry code for every user in group \( j \) according to Definition 4, \( j = 1, \cdots , J \). We say a message length tuple \( \left( v^{(1)}(n) , \cdots , v^{(J)}(n) \right) \) is asymptotically achievable if there exists a sequence of \( \left( [\exp(v^{(1)}(n))] , \cdots , [\exp(v^{(J)}(n))] , n \right) \) codes such that the average error probability vanishes as \( n \to \infty \).

**Definition 6** (Capacity region of the many-access channel). Consider a MnAC of configuration \( \left( \left\{ \alpha^{(j)} n \right\}, \left\{ \beta^{(j)} \right\}, \left\{ P^{(j)} \right\}, \ell_n \right) \). The capacity region is the set of asymptotically achievable message
length tuples. In particular, for every \((B^{(1)}(n), \ldots, B^{(J)}(n))\) in the capacity region, if the users transmit with message length tuple \(((1 - \epsilon)B^{(1)}(n), \ldots, (1 - \epsilon)B^{(J)}(n))\), the average error probability vanishes as \(n \to \infty\). If any user transmits with message length outside the capacity region, reliable communication cannot be achieved.

**Theorem 6.** Consider a MnAC of configuration \(\left\{\alpha^{(j)}(n)\right\}, \left\{\beta^{(j)}(n)\right\}, \left\{P^{(j)}(n)\right\}, \ell_n\). Suppose \(\ell_n \to \infty\) and \(\alpha^{(j)}(n) \to \alpha^{(j)} \in [0, 1]\). Let the average number of active users in group \(j\) be \(k^{(j)}(n) = \alpha^{(j)}(n)\beta^{(j)}(n)\ell_n = O(n)\), such that \(\ell_n e^{-\delta k^{(j)}(n)} \to 0\) for every \(\delta > 0\) and every \(j = 1, \ldots, J\). Let \(\theta^{(j)}(n)\) be defined as

\[
\theta^{(j)}(n) = \frac{2\beta^{(j)}(n)\ell_n H_2\left(\frac{\alpha^{(j)}(n)}{n}\right)}{n \log k^{(j)}(n)}. \tag{133}
\]

and let \(\theta^{(j)}\) denote its limit. Suppose \(\log k^{(j_1)}(n)/\log k^{(j_2)}(n) \to 1\) for any \(j_1, j_2 \in \{1, \ldots, J\}\). If \(\sum_{j=1}^J \theta^{(j)} < 1\), then the message length capacity region is characterized as

\[
\sum_{j=1}^J k^{(j)}(n)B^{(j)}(n) \leq \frac{n}{2} \log \left(\sum_{j=1}^J k^{(j)}(n)\right) - \sum_{j=1}^J \beta^{(j)}(n)\ell_n H_2\left(\frac{\alpha^{(j)}(n)}{n}\right). \tag{134}
\]

If \(\sum_{j=1}^J \theta^{(j)} > 1\), then some user cannot transmit a single bit reliably.

It is interesting to note that as far as the asymptotic message lengths are concerned, the impact of the transmit power is inconsequential. Also, the only limitation on the message is their weighted average. This is in contrast to the classical multiaccess channel, where the sum rate of each subset of users is subject to a separate upper bound in general.

**A. Converse**

The proof of converse follows similarly as in Section III. We only sketch the proof here. Consider the system model described by (5). Suppose the message length transmitted by each user in group \(j\) is \(v^{(j)}(n), j = 1, \ldots, J\). Let \(\tilde{X}_j\) denote a vector, which stacks the vectors \(X_k\), for all \(k\) belonging to group \(j\). Since there are a total of \(\beta^{(j)}\ell_n\) users in group \(j\) and the distributions of \(X_k\) are the same for all \(k\) in the same group \(j\), we have

\[
H\left(\tilde{X}_j\right) = \beta^{(j)}\ell_n H(X_k) = \beta^{(j)}\ell_n \left(H_2\left(\frac{\alpha^{(j)}(n)}{n}\right) + \alpha^{(j)}(n)v^{(j)}(n)\right). \tag{135}
\]

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Define $\mathcal{J} \subseteq \{1, \cdots, J\}$. Further denote $\bar{X}_{\mathcal{J}}$ as the vector consisting of $\{\bar{X}_j : j \in \mathcal{J}\}$. Thus,

$$H(\bar{X}_{\mathcal{J}}) = \sum_{j \in \mathcal{J}} H(\bar{X}_j). \quad (137)$$

Applying the chain rule, we have

$$H(\bar{X}_{\mathcal{J}}) = I(\bar{X}_{\mathcal{J}}; Y) + H(\bar{X}_{\mathcal{J}}|Y) = H(\bar{X}_{\mathcal{J}}|\bar{X}_{\{1,\ldots,J\}\setminus \mathcal{J}}) - H(\bar{X}_{\mathcal{J}}|Y) + H(\bar{X}_{\mathcal{J}}|Y) \quad (138)$$

$$\leq I(\bar{X}_{\mathcal{J}}; Y|\bar{X}_{\{1,\ldots,J\}\setminus \mathcal{J}}) + H(\bar{X}_{\mathcal{J}}|Y). \quad (139)$$

Following the argument in Lemma II, we have

$$I(\bar{X}_{\mathcal{J}}; Y|\bar{X}_{\{1,\ldots,J\}\setminus \mathcal{J}}) \leq \frac{n}{2} \log \left(1 + \sum_{j \in \mathcal{J}} k_n^{(j)} P^{(j)}\right). \quad (140)$$

In order to achieve vanishing error probability, following the argument in Lemma II, we have

$$H(\bar{X}_{\mathcal{J}}|Y) = o \left(\sum_{j \in \mathcal{J}} k_n^{(j)} v^{(j)}(n) + \beta^{(j)} \ell_n H_2(\alpha_n^{(j)})\right). \quad (141)$$

Combining (136), (137), (140), (141), and (142), we have for large enough $n$,

$$(1 - \epsilon) \sum_{j \in \mathcal{J}} [k_n^{(j)} v^{(j)}(n) + \beta^{(j)} \ell_n H_2(\alpha_n^{(j)})] \leq \frac{n}{2} \log \left(1 + \sum_{j \in \mathcal{J}} k_n^{(j)} P^{(j)}\right), \quad (142)$$

for every $\epsilon > 0$.

Since the power in each group is bounded, we have

$$\log \left(1 + \sum_{j \in \mathcal{J}} k_n^{(j)} P^{(j)}\right) / \log \left(\sum_{j \in \mathcal{J}} k_n^{(j)}\right) \to 1 \quad \text{as } n \text{ increases.}$$

Thus, (143) implies that for every $\epsilon > 0$ and every $\mathcal{J} \subseteq \{1, \cdots, J\}$,

$$\sum_{j \in \mathcal{J}} k_n^{(j)} v^{(j)}(n) \leq (1 + \epsilon) \frac{n}{2} \log \left(\sum_{j \in \mathcal{J}} k_n^{(j)}\right) - \sum_{j \in \mathcal{J}} \beta^{(j)} \ell_n H_2(\alpha_n^{(j)}). \quad (144)$$

As in (15), we have dropped the power terms in the capacity expression to ease the rest of the proof. By (144), we have

$$\sum_{j \in \mathcal{J}} k_n^{(j)} v^{(j)}(n) \leq \left[1 + \epsilon - \sum_{j \in \mathcal{J}} \theta_n^{(j)} e^{(\mathcal{J},j)}\right] \frac{n}{2} \log \left(\sum_{j \in \mathcal{J}} k_n^{(j)}\right), \quad (145)$$
where
\[ \xi_n^{(\mathcal{J}, j)} = \frac{\log \left( k_n^{(j)} \right)}{\log \left( \sum_{j \in \mathcal{J}} k_n^{(j)} \right)} . \]  

(146)

For any \( \mathcal{J}_1, \mathcal{J}_2 \subseteq \{1, \cdots, J\} \), we have
\[ \frac{\log \left( \min_{j \in \mathcal{J}_1} k_n^{(j)} \right)}{\log \left( \max_{j \in \mathcal{J}_2} k_n^{(j)} \right) + \log J} \leq \frac{\log \left( \sum_{j \in \mathcal{J}_1} k_n^{(j)} \right)}{\log \left( \sum_{j \in \mathcal{J}_2} k_n^{(j)} \right)} \leq \frac{\log \left( \max_{j \in \mathcal{J}_1} k_n^{(j)} \right) + \log J}{\log \left( \min_{j \in \mathcal{J}_2} k_n^{(j)} \right)} . \]  

(147)

Taking the limit of \( n \to \infty \) on both sides of (147), by the assumption that \( \log \frac{k_n^{(j_1)}}{k_n^{(j_2)}} \to 1 \) for any \( j_1, j_2 \), we have
\[ \frac{\log \left( \sum_{j \in \mathcal{J}_1} k_n^{(j)} \right)}{\log \left( \sum_{j \in \mathcal{J}_2} k_n^{(j)} \right)} \to 1 . \]  

(148)

It implies that \( \xi_n^{(\mathcal{J}, j)} \to 1 \) for all \( j \in \mathcal{J} \). If \( \sum_{j=1}^J \theta^{(j)} > 1 \), particularizing (145) with \( \mathcal{J} = \{1, \cdots, J\} \) implies that for large enough \( n \), \( v^{(j)}(n) = 0 \) for all \( j = 1, \cdots, J \).

If \( \sum_{j=1}^J \theta^{(j)} < 1 \), the achievable message length can be further upper bounded as
\[ \sum_{j \in \mathcal{J}} k_n^{(j)} v^{(j)}(n) \leq \left( 1 + \frac{\epsilon}{1 - \sum_{j \in \mathcal{J}} \theta_n^{(j)} \xi_n^{(\mathcal{J}, j)}} \right) B_{\mathcal{J}}(n) , \]  

(149)

where
\[ B_{\mathcal{J}}(n) = \frac{n}{2} \log \left( \sum_{j \in \mathcal{J}} k_n^{(j)} \right) - \sum_{j \in \mathcal{J}} \beta^{(j)}_n H_2 \left( \alpha_n^{(j)} \right) . \]  

(150)

Applying (149) with \( \mathcal{J} = \{1, \cdots, J\} \) and \( \xi_n^{(\mathcal{J}, j)} \to 1 \), the achievable message length tuple must satisfy
\[ \sum_{j \in \{1, \cdots, J\}} k_n^{(j)} v^{(j)}(n) \leq (1 + \epsilon) B_{\{1, \cdots, J\}}(n) \]  

(151)
for every $\epsilon > 0$. Thus, the converse part of Theorem 6 is established.

Note that by (149), any achievable message length tuple must satisfy

$$\sum_{j \in \mathcal{J}} k^{(j)}(n) v^{(j)}(n) \leq (1 + \epsilon) B_{\mathcal{J}}(n)$$

for all $\mathcal{J} \subseteq \{1, \cdots, J\}$. However, in the regime of unbounded $k_n$, (149) implies that these constraints are dominated by the one for $\mathcal{J} = \{1, \cdots, J\}$, because $B_{\mathcal{J}}(n) \geq n B_{\{1,\cdots,J\}}(n)$ for all $\mathcal{J} \subseteq \{1, \cdots, J\}$.

**B. Achievability**

We need to prove that the region of the achievable message length tuple covers the region specified by (134). In particular, we will show that the message length tuple satisfying

$$\sum_{j=1}^{J} k^{(j)}(n) v^{(j)}(n) \leq (1 - \epsilon) \left[ \frac{n}{2} \log \left( \sum_{j=1}^{J} k^{(j)}(n) \right) - \sum_{j=1}^{J} \beta^{(j)} \ell_n H_2(\alpha^{(j)}_n) \right]$$

is asymptotically achievable for every $\epsilon > 0$.

One achievable scheme is to detect active users in each group and their transmitted messages in a time-division manner. In particular, in the first stage, we let users in group 1 transmit the signatures before group 2, and so on. The signature length transmitted by users in group $j$ is $n^{(j)}_0$, $j = 1, \cdots, J$. In the second stage, we let each group share the remaining time resource $n - \sum_{j=1}^{J} n^{(j)}_0$. Users in group 1 transmit their message-bearing codewords before group 2, and so on. The time resource allocated to group $j$ in the second stage is $\phi_j \left( n - \sum_{j=1}^{J} n^{(j)}_0 \right)$, where $\phi_j \geq 0$ and $\sum_{j=1}^{J} \phi_j = 1$. At the receiver side, the receiver performs user identification according
to the group order, and then decode the transmitted messages according to the group order. The overall scheme is illustrated in Fig. 6.

Let \( \theta_n^{(j)} \) be given by (133), which can be regarded as the fraction of resource to detect the active users in group \( j \). According to Theorem 2 and Theorem 4, the message length tuple satisfying

\[
v^{(j)}(n) = (1 - \epsilon') \phi^{(j)} n - \sum_{j' = 1}^{J} n^{(j')} n_0 \log \left( \frac{k_n^{(j)}}{2k_n^{(j')}} \right),
\]

where

\[
n^{(j)}_0 = \begin{cases} 
(1 + \epsilon'/2) \theta^{(j)} n, & \text{if } \theta^{(j)} > 0 \\
\epsilon' n, & \text{if } \theta^{(j)} = 0
\end{cases}
\]

is achievable for every \( \epsilon' \in (0, 1) \).

If \( \theta^{(j')} > 0 \), by (148),

\[
\frac{n^{(j')}_0}{2} \log(k_n^{(j)}) = (1 + \epsilon'/2) \beta^{(j')} \ell_n H_2 \left( \alpha^{(j')} n \right) \log \left( \frac{k_n^{(j)}}{2k_n^{(j')}} \right)
\]

\[
\leq n \left( 1 + \epsilon' \right) \beta^{(j')} \ell_n H_2 \left( \alpha^{(j')} n \right).
\]

If \( \theta^{(j')} = 0 \),

\[
\frac{n^{(j')}_0}{2} \log(k_n^{(j)}) = \frac{\epsilon' n}{2J} \log(k_n^{(j)}).
\]

Therefore,

\[
\sum_{j' = 1}^{J} \frac{n^{(j')}_0}{2} \log(k_n^{(j)}) \leq n \frac{\epsilon' n}{2J} \log(k_n^{(j)}) + \sum_{j' = 1}^{J} (1 + \epsilon') \beta^{(j')} \ell_n H_2 \left( \alpha^{(j')} n \right).
\]

By (154), the achievable message length is calculated as

\[
k_n^{(j)} v^{(j)}(n) \geq n \left( 1 - \epsilon' \right) \phi^{(j)} \left[ (1 - \epsilon'/2) \frac{n}{2} \log(k_n^{(j)}) - (1 + \epsilon') \sum_{j = 1}^{J} \beta^{(j)} \ell_n H_2 \left( \alpha^{(j)} n \right) \right] \]

\[
\geq n \left( 1 - \epsilon' \right) \phi^{(j)} \left[ (1 - \epsilon') \frac{n}{2} \log \left( \sum_{j = 1}^{J} k_n^{(j)} \right) - (1 + \epsilon') \sum_{j = 1}^{J} \beta^{(j)} \ell_n H_2 \left( \alpha^{(j)} n \right) \right].
\]
According to \((161)\), there must exist some small enough \(\epsilon'\) such that for large enough \(n\),
\[
\kappa_n^{(j)} v^{(j)}(n) \geq \phi^{(j)}(1 - \epsilon) \left[ \frac{n}{2} \log \left( \sum_{j=1}^{J} k^{(j)}_n \right) - \sum_{j=1}^{J} \beta^{(j)} H_2(\alpha^{(j)}_n) \right] \tag{162}
\]
for all \(j = 1, \ldots, J\).

Since \((162)\) holds for any \(\phi_j > 0\), by varying the convex combination due to \(\phi^{(j)}\), \(j = 1, \ldots, J\), the region spanned by the achievable message tuple \((154)\) covers the region specified by \((153)\). The achievability result is thus established.

VIII. Conclusion

In this paper, we have proposed a model of many-access channel, where the number of users scales with the coding blocklength as a first step towards the study of many-user information theory. New notions of message length and symmetric capacity have been defined. The symmetric capacity of a many-access channel is shown to be a function in the channel uses, consisting of two terms. The first term is the symmetric capacity of many-access channel with knowledge of the set of active users and the second term can be regarded as the cost of user identification in random access channels. Separate identification and decoding has been shown to be capacity achieving. The detection scheme can be extended to achieve the capacity region of a many-access channel with a finite number of groups experiencing different channel gains.

The results presented in this paper reveal the capacity growth in the asymptotic regime. The holy grail is a many-user information theory for finite but large number of users and finite but large block length that applies accurately in practice. The challenge of developing such a theory is difficult to overestimate (see, e.g., \([31]\), \([32]\)).

The many-access channel model together with the capacity result and the compressed sensing based identification technique will provide insights for the optimal design in emerging applications with massive sporadic access \([33]–[35]\), such as in the Internet of Things and machine-to-machine communication, where the number of devices in a cell may far exceed the blocklength.

APPENDIX A
PROOF OF LEMMA \([1]\)

To upper bound the input-output mutual information of the white Gaussian noise channel, it suffices to identify the power constraint on the input signal \(\mathbf{X}\) based on the power constraint \((2)\).
on \( s \) and the structure of the binary vector \( X \).

According to the distribution of \( X \), we can obtain the marginal distribution of \( X_i, i = 1, \cdots, M\ell_n \), as \( P\{X_i = 0\} = 1 - \frac{\alpha_n}{M} \) and \( P\{X_i = 1\} = \frac{\alpha_n}{M} \). Therefore, \( E\{X_i\} = \frac{\alpha_n}{M} \) and

\[
E\{X_iX_j\} = \begin{cases} \frac{\alpha_n}{M} & \text{if } i = j \\ 0 & \text{if } i \neq j, i, j \in I(\ell) \text{ for some } \ell \end{cases},
\]

where we let the indices corresponding to transmitter \( \ell \) be \( I(\ell) = \{ (\ell - 1)M + 1, \cdots, \ell M \} \), \( \ell = 1, \cdots, \ell_n \). Thus, the covariance matrix \( K = E\{(X - EX)(X - EX)^T\} \) can be calculated as

\[
K_{ij} = \begin{cases} \frac{\alpha_n}{M} \left( 1 - \frac{\alpha_n}{M} \right) & i = j, \\ - \left( \frac{\alpha_n}{M} \right)^2 & i \neq j, i, j \in I(\ell) \text{ for some } \ell, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \text{tr}(\cdot) \) find the trace of a matrix. The power constraint on the codewords induces the power constraint on \( sX \) as

\[
\text{tr} \left( sKs^T \right) = \text{tr} \left( Ks^Ts \right) = \sum_{i=1}^{M\ell_n} \sum_{j=1}^{M\ell_n} \sum_{k=1}^{n} K_{ij}s_{ki}s_{kj}
\]

\[
= \sum_{k=1}^{n} \left[ \frac{\alpha_n}{M} \left( 1 - \frac{\alpha_n}{M} \right) \sum_{i=1}^{M\ell_n} s_{ki}^2 - \left( \frac{\alpha_n}{M} \right)^2 \sum_{\ell=1}^{\ell_n} \sum_{i \neq j, i, j \in I(\ell)} s_{ki}s_{kj} \right]
\]

\[
= \sum_{k=1}^{n} \left[ \frac{\alpha_n}{M} \sum_{i=1}^{M\ell_n} s_{ki}^2 - \left( \frac{\alpha_n}{M} \right)^2 \sum_{\ell=1}^{\ell_n} \sum_{i \in I(\ell)} \sum_{j \in I(\ell)} s_{ki}s_{kj} \right]
\]

\[
\leq \frac{n\alpha_n}{M} \sum_{i=1}^{M\ell_n} \frac{1}{n} \sum_{k=1}^{n} s_{ki}^2
\]

\[
\leq k_n n P,
\]
where (169) is due to
\[\sum_{i \in I(\ell)} \sum_{j \in I(\ell)} s_{ki} s_{kj} = \left( \sum_{i \in I(\ell)} s_{ki} \right)^2 \geq 0, \tag{171}\]
and the last inequality is due to the power constraint \(\frac{1}{n} \sum_{k=1}^{n} s_{ki}^2 \leq P\).

Since \(X \rightarrow sX \rightarrow Y\) forms a Markov chain, we can obtain an upper bound of \(I(X; Y)\) as
\[I(X; Y) \leq I(sX; Y) \tag{172}\]
\[\leq \max_{\text{tr}(sKs^T) \leq k_n P} I(sX; Y) \tag{173}\]
\[\leq \frac{n}{2} \log(1 + k_n P), \tag{174}\]
where (174) follows by the results on parallel Gaussian channels [10, Chapter 10].

**APPENDIX B**

**PROOF OF LEMMA 2**

Conditioned on \(E = 0\), \(H(X|E = 0, Y, 1 \{X \in B^e_M(\delta, k_n)\}) = 0\). Therefore, we can obtain
\[H(X|E, Y, 1 \{X \in B^e_M(\delta, k_n)\}) = H(X|E = 1, Y, X \notin B^e_M(\delta, k_n)) P\{E = 1, X \notin B^e_M(\delta, k_n)\}
+ H(X|E = 1, Y, X \in B^e_M(\delta, k_n)) P\{E = 1, X \in B^e_M(\delta, k_n)\}.\tag{175}\]

We upper bound the first term on the right hand side of (175) as follows: \(X\) can take at most \((M+1)^{\ell_n}\) values and \(\|X\|_0\) follows the binomial distribution \(\text{Bin}(\ell_n, \alpha_n)\) with mean \(\ell_n \alpha_n = k_n\), then \(P\{X \notin B^e_M(\delta, k_n)\}\) can be upper bounded by \(e^{-c(\delta)k_n}\) [30], where \(c(\delta)\) is some constant depending on \(\delta\) by the large deviations for binomial distribution. Then
\[H(X|E = 1, Y, X \notin B^e_M(\delta, k_n)) P\{E = 1, X \notin B^e_M(\delta, k_n)\} \leq e^{-c(\delta)k_n \ell_n \log(M + 1)} \tag{176}\]
\[\leq n \log M. \tag{177}\]

For the second term on the RHS of (175), \(P\{E = 1, X \in B^e_M(\delta, k_n)\} \leq P^{(n)}\) and
\[H(X|E = 1, Y, X \in B^e_M(\delta, k_n)) \leq \log |B^e_M(\delta, k_n)|. \tag{178}\]
The cardinality of \( B_{M}^{\ell_n}(\delta, k_n) \) is

\[
|B_{M}^{\ell_n}(\delta, k_n)| = \sum_{j=1}^{(1+\delta)k_n} \binom{\ell_n}{j} M^j
\]

(179)

\[
\leq (1 + \delta)k_n M^{(1+\delta)k_n} \max_{1 \leq j \leq (1+\delta)k_n} \binom{\ell_n}{j}.
\]

(180)

If \((1 + \delta)k_n \geq \frac{\ell_n}{2}\), then

\[
\max_{1 \leq j \leq (1+\delta)k_n} \binom{\ell_n}{j} \leq 2^{\ell_n}
\]

(181)

\[
\leq \exp(2(1 + \delta)k_n \log 2).
\]

(182)

If \((1 + \delta)k_n < \frac{\ell_n}{2}\), then

\[
\max_{1 \leq j \leq (1+\delta)k_n} \binom{\ell_n}{j} \leq \binom{\ell_n}{(1 + \delta)k_n}
\]

(183)

\[
\leq \exp(\ell_n H_2((1 + \delta)\alpha_n)).
\]

(184)

We further upper bound \( H_2((1 + \delta)\alpha_n) \) in terms of \( H_2(\alpha_n) \). By the mean value theorem, there exists some \( \gamma'_n \) in between \( \alpha_n \) and \((1 + \delta)\alpha_n\) such that

\[
H_2((1 + \delta)\alpha_n) - H_2(\alpha_n) = \delta \alpha_n \log \frac{1 - \gamma'_n}{\gamma'_n},
\]

(185)

where \( \log \frac{1-x}{x} \) is the first order derivative of \( H_2(x) \). Since \( \log \frac{1-x}{x} \) is decreasing in \( x \), we have

\[
H_2((1 + \delta)\alpha_n) - H_2(\alpha_n) \leq \delta \alpha_n \log \frac{1 - \alpha_n}{\alpha_n} \leq \delta H_2(\alpha_n).
\]

(186)

As a result,

\[
\log |B_{M}^{\ell_n}(\delta, k_n)| \leq \log ((1 + \delta)k_n) + (1 + \delta)k_n \log M + 2(1 + \delta)k_n \log 2 + (1 + \delta)\ell_n H_2(\alpha_n).
\]

(187)

For large enough \( n \), we have \( \log ((1 + \delta)k_n) \leq (1 + \delta)k_n \). Then

\[
H(X | E = 1, X \in B_{M}^{\ell_n}(\delta, k_n), Y) \leq 4(k_n \log M + k_n + \ell_n H_2(\alpha_n)).
\]

(188)

Combining (175), (177) and (188) yields the lemma.
APPENDIX C
DERIVATION OF (78)

We will derive the closed-form expression of (74), which is calculated as

\[
m_{\lambda, \rho}(w_1, w_2) = \int_{\mathbb{R}} E \left\{ p_{Y|S_A}^{1-\lambda \rho}(y|S_A^a) \left( E \left\{ p_{Y|S_A}^{\lambda}(y|S_A) \left| S_A^a \right. \right\} \right)^\rho \right\} dy \tag{189}
\]

\[
= \int_{\mathbb{R}} E \left\{ p_{Y|S_A}^{1-\lambda \rho}(y|S_A^a) \left( E \left\{ p_{Y|S_A}^{\lambda}(y|S_A) \left| S_A^a \right. \right\} \right)^\rho \right\} dy \tag{190}
\]

\[
= \int_{\mathbb{R}} E \left\{ p_{Y|S_A}^{1-\lambda \rho}(y|S_A^a) \left( E \left\{ p_{Y|S_A}^{\lambda}(y|S_A) \left| S_A^a \right. \right\} \right)^\rho \right\} dy \tag{191}
\]

where (190) follows because \( A \cap A^* = A^* \setminus A_1 \).

Let \( Z_1 = \sum_{k \in A_1} S_k^a \), \( Z_2 = \sum_{k \in A_2} S_k^a \) and \( Z_3 = \sum_{k \in A^* \setminus A_1} S_k^a \). Since \( |A_1| = w_1 \) and \( |A_2| = w_2 \), we have \( Z_1 \sim \mathcal{N}(0, \lambda v_1) \), \( Z_2 \sim \mathcal{N}(0, \lambda v_2) \), \( Z_3 \sim \mathcal{N}(0, \lambda v_3) \), where \( v_1 = w_1P' \), \( v_2 = w_2P' \) and \( v_3 = (|A^*| - w_1)P' \).

We can write

\[
E \{ p_Y^{\lambda|S_A}(y|S_A^a) \left| S_A^a \right. \} = E \left\{ \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-Z_3)^2}{2}} \right)^\lambda \left| Z_3 \right. \right\} \tag{192}
\]

\[
= \int_{\mathbb{R}} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-Z_3)^2}{2}} \right)^\lambda \frac{1}{\sqrt{2\pi} v_2} e^{-\frac{z_2^2}{2v_2}} dz_2 \tag{193}
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^\lambda \sqrt{\frac{t_3}{v_2}} e^{\frac{\mu_3^2}{2t_3}} e^{-\frac{\lambda(y-Z_3)^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi} t_3} e^{-\frac{(z_2-\mu_3)^2}{2t_3}} dz_2 \tag{194}
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^\lambda \sqrt{\frac{t_3}{v_2}} e^{\frac{\mu_3^2}{2t_3}} e^{-\frac{\lambda(y-Z_3)^2}{2}}, \tag{195}
\]

where \( \frac{1}{t_3} = \lambda + \frac{1}{v_2} \) and \( \mu_3 = \lambda(y - Z_3)t_3 \).
Similarly,

\[
E \left\{ p_{Y|S_A}(y|S_{A^\perp}) | S_{A^\perp}^a \right\} = E \left\{ \left( \frac{1}{\sqrt{2\pi}} \right)^{1-\lambda\rho} e^{-(1-\lambda\rho)(y-Z_3)^2} \right\} \\
= \left( \frac{1}{\sqrt{2\pi}} \right)^{1-\lambda\rho} \int_\mathbb{R} e^{-(1-\lambda\rho)(y-Z_3)^2} \frac{1}{\sqrt{2\pi}v_1} e^{-\frac{z_1^2}{2v_1}} dz_1
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^{1-\lambda\rho} \sqrt{\frac{t_4}{v_1}} e^{\frac{\mu_4^2}{2v_1}} e^{-\frac{(1-\lambda\rho)(y-Z_3)^2}{2}} \int_\mathbb{R} e^{-\frac{z_1^2}{2t_4}} dz_1
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^{1-\lambda\rho} \sqrt{\frac{t_4}{v_1}} e^{\frac{\mu_4^2}{2v_1}} e^{-\frac{(1-\lambda\rho)(y-Z_3)^2}{2}},
\]

where \( \frac{1}{t_4} = 1 - \lambda\rho + \frac{1}{v_1} \) and \( \mu_4 = (1 - \lambda\rho)(y - Z_3)t_4 \).

Then

\[
(E \left\{ p_{Y|S_A}(y|S_{A^\perp}) | S_{A^\perp}^a \right\})^\rho E \left\{ p_{Y|S_A}(y|S_{A^\perp}) | S_{A^\perp}^a \right\} = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\frac{t_3}{v_1}} e^{\frac{\mu_3^2}{2v_1}} \right)^\rho \sqrt{\frac{t_4}{v_1}} e^{\frac{\mu_4^2}{2v_1}} \frac{(y-Z_3)^2}{2}.
\]

Plugging \( \mu_3, t_3, \mu_4 \) and \( t_4 \) yields \( \frac{\mu_3^2}{t_3} = \frac{\lambda^2v_2(y-Z_3)^2}{1+\lambda\rho v_2} \) and \( \frac{\mu_4^2}{t_4} = \frac{(1-\lambda\rho)^2(y-Z_3)^2v_1}{1+(1-\lambda\rho)v_1} \). Let \( t_0 = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\frac{t_4}{v_1}} \right)^\rho \sqrt{\frac{t_3}{v_1}} \) and \( \frac{1}{t_5} = 1 - \frac{\rho\lambda^2v_2}{1+\lambda\rho v_2} - \frac{(1-\lambda\rho)^2v_1}{1+(1-\lambda\rho)v_1} \). We have

\[
\int_\mathbb{R} E \left\{ E \left\{ p_{Y|S_A}(y|S_{A^\perp}) | S_{A^\perp}^a \right\} \right\} dy = t_0 \int_\mathbb{R} \int_\mathbb{R} \left( \frac{1}{\sqrt{2\pi}v_3} \right)^{1-\lambda\rho} e^{-\frac{z_3^2}{2v_3}} \frac{\rho\lambda^2v_2(y-Z_3)^2}{2(1+\lambda\rho v_2)} + \frac{(1-\lambda\rho)^2(y-Z_3)^2v_1}{2(1+(1-\lambda\rho)v_1)} - \frac{(y-Z_3)^2}{2} \right) dz_3 dy
\]

\[
= t_0 \int_\mathbb{R} \int_\mathbb{R} \frac{1}{\sqrt{2\pi}v_3} e^{-\frac{z_3^2}{2v_3}} e^{-\frac{(y-Z_3)^2}{2t_5}} dy dz_3 \frac{1}{\sqrt{2\pi}t_5} e^{-\frac{(y-Z_3)^2}{2t_5}} dy dz_3
\]

\[
= t_0 \int_\mathbb{R} \int_\mathbb{R} \frac{1}{v_3} e^{-\frac{z_3^2}{2v_3}} \frac{1}{\sqrt{2\pi}t_5} e^{-\frac{(y-Z_3)^2}{2t_5}} dy dz_3
\]

\[
= \left( \frac{t_3}{v_1} \right)^\rho \frac{t_4}{v_1} \int_\mathbb{R} \int_\mathbb{R} e^{\frac{1}{t_A} - \frac{1}{t_5}} \int_\mathbb{R} \int_\mathbb{R} e^{\frac{1}{t_5} - \frac{1}{t_A}} dy dz_3
\]

\[
= (1 + \lambda\rho v_2)^{-\rho/2} \left( \frac{1 + \lambda v_2}{1 + \lambda(1 - \lambda\rho)v_2 + \lambda\rho(1 - \lambda\rho)v_1} \right)^{1/2}.
\]

Therefore, \( m_{\lambda,\rho}(w_1, w_2) \) is given by (76).
We first establish the following two lemmas that will be useful in the proof.

**Lemma 6.** Suppose (18) holds, i.e., \( \lim_{\ell \to \infty} \ell e^{-\delta k_\ell} = 0 \) for every \( \delta > 0 \), then for every constant \( \bar{w} \geq 0 \),

\[
\lim_{\ell \to \infty} \frac{\ell}{k_\ell} H_2 \left( \frac{\bar{w}}{\ell} \right) = 0. \tag{206}
\]

**Proof.** The case of \( \bar{w} = 0 \) is trivial. Suppose \( \bar{w} > 0 \). Since \( \bar{w}/\ell \to 0 \),

\[
\frac{\ell}{k_\ell} H_2 \left( \frac{\bar{w}}{\ell} \right) = \frac{\ell}{k_\ell} \left( \bar{w} \log \frac{\ell}{\bar{w}} - \left(1 - \frac{\bar{w}}{\ell} \right) \log \left(1 - \frac{\bar{w}}{\ell} \right) \right) \tag{207}
\]

\[
\leq \frac{\ell}{k_\ell} \left( \bar{w} \log \frac{\ell}{\bar{w}} + \left(1 - \frac{\bar{w}}{\ell} \right) \frac{2\bar{w}}{\ell} \right) \tag{208}
\]

\[
\leq \frac{\bar{w}}{k_\ell} \left( \log \ell - \log \bar{w} + 2 \right). \tag{209}
\]

Since \( \ell e^{-\delta k_\ell} \to 0 \) for every \( \delta > 0 \), we have \( \ell \leq \ell e^{\delta k_\ell} \), so that \( \log \ell \leq \delta k_\ell \). This implies \( (\log \ell)/k_\ell \to 0 \), so that the right hand side of (209) vanishes. \( \square \)

**Lemma 7.** Suppose (18) holds for every \( \delta > 0 \). Let \( A > 0 \), \( B > 0 \) and \( \bar{w} \geq 1 \) be constants. Let \( \{a_\ell\} \) and \( \{b_\ell\} \) be two sequences that satisfy \( b_\ell \leq a_\ell \), \( \lim_{\ell \to \infty} \frac{k_\ell}{a_\ell} = a \in [0, \infty) \), and \( \lim_{\ell \to \infty} \frac{k_\ell}{b_\ell} = b \in (0, \infty) \). Let \( A_\ell \) be a sequence that satisfies \( \lim \inf_{\ell \to \infty} A_\ell = A \). Define \( h_\ell(\cdot) \) on \( [0, a_\ell] \) as

\[
h_\ell(w) = A_\ell \log(1 + Bw) - \frac{a_\ell}{k_\ell} H_2 \left( \frac{w}{a_\ell} \right). \tag{210}
\]

Let \( w_\ell^* \) achieve the global minimum of \( h_\ell(\cdot) \) restricted to \( [\bar{w}, b_\ell] \). For large enough \( \ell \), either \( w_\ell^* = \bar{w} \) or \( w_\ell^* \in [c b_\ell, b_\ell] \), where

\[
c = \min \left\{ \frac{bA}{64(1 + Aa)}, 1 \right\}. \tag{211}
\]

**Proof.** The function \( h_\ell(w) \) is equal to the difference of two concave functions. Its first two derivatives on \( (0, a_\ell) \) are:

\[
h'_\ell(w) = \frac{A_\ell B}{1 + Bw} + \frac{1}{k_\ell} \log \frac{w}{a_\ell - w}. \tag{212}
\]
and

\[ h''_\ell(w) = \frac{a_\ell}{k_\ell w(a_\ell - w)} - \frac{A_\ell B^2}{(1 + Bw)^2} \]  

\[ = \frac{a_\ell g_\ell(w)}{k_\ell w(a_\ell - w)(1 + Bw)^2}, \]  

(214)

where

\[ g_\ell(w) = (B^2 + k_\ell A_\ell B^2/a_\ell)w^2 + (2B - k_\ell A_\ell B^2)w + 1. \]  

(215)

Due to (18), \( k_\ell \to \infty \) as \( \ell \to \infty \). For large enough \( \ell \), \( g_\ell(0) = 1 \), \( g_\ell(1) = -A_\ell B^2 k_\ell + A_\ell B^2 k_\ell/a_\ell + (B + 1)^2 < 0 \), and \( g_\ell(a_\ell) = (B a_\ell + 1)^2 > 0 \). Moreover, the minimum of the quadratic function \( g_\ell(w) \) is achieved at:

\[ v_\ell = \frac{k_\ell A_\ell B - 2}{2B(1 + k_\ell A_\ell /a_\ell)}. \]  

(216)

Since \( \frac{1}{2} k_\ell A_\ell B \geq 2 \), we have \( k_\ell A_\ell B - 2 \geq \frac{1}{2} k_\ell A_\ell B \). Also, \( A_\ell k_\ell /a_\ell \leq 1 + 2Aa \). We have

\[ \frac{v_\ell}{b_\ell} \geq \ell \frac{\frac{1}{2} k_\ell A_\ell B }{2B(1 + A_\ell /a_\ell)} \]  

\[ \geq \ell \frac{1}{2} \left( \frac{1}{2} b \right) \frac{\frac{1}{2} A}{2(2 + 2Aa)} \]  

\[ = \frac{bA}{32(1 + Aa)}. \]  

(219)

Note that \( b_\ell \to \infty \) and (219) implies \( v_\ell \to \infty \). For large enough \( \ell \), since \( h''_\ell(w) < 0 \) for every \( w \in [\bar{w}, v_\ell] \), \( h_\ell(w) \) is concave over \([\bar{w}, v_\ell]\). Since \( v_\ell/b_\ell \geq 2c \), we have either \( w^*_\ell = \bar{w} \) or \( w^*_\ell \in [cb_\ell, b_\ell] \) for large enough \( \ell \).

The general idea for proving Lemma 3 is to divide \( \mathcal{W}(\ell) \) into two regions based on whether the error probability is dominated by false alarms or miss detections, and to lower bound \( h_{\lambda, \rho}(w_1, w_2) \) given by (78) for \((w_1, w_2)\) in those two regions separately. It is crucial to note that Lemma 3 claims the existence of a uniform lower bound of \( h_{\lambda, \rho}(w_1, w_2) \), i.e., \( \ell^* \) is such that for all \( \ell \geq \ell^* \), \( h_{\lambda, \rho}(w_1, w_2) \geq c_0 \) regardless of \((w_1, w_2)\), which in general depend on \( \ell \).
Define
\[ \phi_\ell = \frac{n(\ell)}{k_\ell} = \frac{2\ell H_2(\alpha_\ell)}{k_\ell \log(1 + k_\ell P')}, \] (220)
which can be regarded as the identification cost per active user. Let
\[ \phi = \lim_{\ell \to \infty} \phi_\ell, \] (221)
which may be \(\infty\). As \(\phi \geq 0\), we prove the cases of \(\phi > 0\) and \(\phi = 0\) separately.

A. The case of \(\phi > 0\)

In this case, by (41), the signature length is \(n_0 = (1 + \epsilon) \phi_\ell k_\ell\). As we shall see, if the number of false alarms \(w_2 = |A\setminus A^*|\) is small, the error probability is dominated by miss detections; whereas for relatively large \(w_2\), the error probability is dominated by false alarms.

Define the following positive constant:
\[ \bar{w} = \max \left\{ \frac{4}{P'} e^{(8+4\epsilon)/\phi}, 1 \right\}. \] (222)
We will derive lower bounds of \(h_{\lambda,\rho}(w_1, w_2)\) for the cases of \(0 \leq w_2 \leq \bar{w}\) and \(\bar{w} < w_2 \leq (1+\delta_\ell)k_\ell\) separately.

1) The case of \(0 \leq w_2 \leq \bar{w}\): Recall that \(\rho \in [0, 1]\) and \(\lambda \in [0, \infty)\) can be chosen arbitrarily to yield a lower bound. We shall always choose them to satisfy \(0 \leq \lambda\rho \leq 1\). This implies that
\[ \log (1 + \lambda(1 - \lambda\rho)w_2 P' + \lambda\rho(1 - \lambda\rho)w_1 P') \geq \frac{1}{2} \log (1 + \lambda(1 - \lambda\rho)w_2 P') + \frac{1}{2} \log (1 + \lambda\rho(1 - \lambda\rho)w_1 P'). \] (223)
In this case, a lower bound of \(h_{\lambda,\rho}(w_1, w_2)\) can be splitted into two parts as
\[ h_{\lambda,\rho}(w_1, w_2) \geq g^1_{\lambda,\rho}(w_1) + g^2_{\lambda,\rho}(w_2), \] (224)
where
\[ g^1_{\lambda,\rho}(w_1) = \frac{n_0}{4k_\ell} \log (1 + \lambda\rho(1 - \lambda\rho)w_1 P') - \frac{|A^*|}{k_\ell} H_2 \left( \frac{w_1}{|A^*|} \right) \] (225)
and
\[ g_{\lambda,\rho}(w_2) = \frac{n_0}{4k_{\ell}} \log \left( 1 + \lambda (1 - \lambda \rho) w_2 P' \right) - \frac{(1 - \rho)n_0}{2k_{\ell}} \log \left( 1 + \lambda w_2 P' \right) - \frac{\rho \ell}{k_{\ell}} H_2 \left( \frac{w_2}{\ell} \right). \] (226)

Note that \( g_{\lambda,\rho}(0) = g_{\lambda,\rho}(0) = 0 \). However, since \((w_1, w_2) \in W(\ell)\), they cannot be 0 simultaneously. In the following, we lower bound \( g_{\lambda,\rho}(w_1) \) for \( w_1 \geq 1 \) and \( g_{\lambda,\rho}(w_2) \) for \( w_2 \geq 1 \). Then \( h_{\lambda,\rho}(w_1, w_2) \) can be lower bounded by the minimum of the two lower bounds of \( g_{\lambda,\rho}(w_1) \) and \( g_{\lambda,\rho}(w_2) \).

Choose \( \lambda = 2/3 \) and \( \rho = 3/4 \). We have
\[ g_{\lambda,\rho}(w_2) = \frac{n_0}{4k_{\ell}} \log \left( 1 + \frac{w_2 P'}{3} \right) - \frac{n_0}{8k_{\ell}} \log \left( 1 + \frac{2w_2 P'}{3} \right) - \frac{3\ell}{4k_{\ell}} H_2 \left( \frac{w_2}{\ell} \right). \] (227)

Since \((1 + x)^r \leq 1 + rx \) for \( r \in [0, 1] \), we have
\[ \log(1 + rx) \geq r \log(1 + x) \] (228)
for \( x \geq 0 \) and the equality is achieved only if \( x = 0 \). Letting \( r = 1/2, x = 2w_2 P'/3 \), we can see that for \( w_2 > 0 \),
\[ \log \left( 1 + \frac{w_2 P'}{3} \right) > \frac{1}{2} \log \left( 1 + \frac{2w_2 P'}{3} \right). \] (229)

Define a positive constant
\[ \epsilon' = \min_{1 \leq w_2 \leq \bar{w}} \frac{\phi}{8} \left[ \log \left( 1 + \frac{w_2 P'}{3} \right) - \frac{1}{2} \log \left( 1 + \frac{2w_2 P'}{3} \right) \right]. \] (230)

By Lemma 6, \( \frac{\ell}{k_{\ell}} H_2(\bar{w}/\ell) \) vanishes as \( \ell \) increases. We can find some \( \ell_0 > 2\bar{w} \) such that for all \( \ell \geq \ell_0, \phi_{\ell} > \phi/2 \) and \( \frac{3\ell}{4k_{\ell}} H_2(\bar{w}/\ell) \leq \epsilon'/2 \).

For every \( \ell \geq \ell_0 \), we have \( H_2(\bar{w}/\ell) \leq H_2(\bar{w}/\ell) \) for \( 1 \leq w_2 \leq \bar{w} \) and thus \( g_{\lambda,\rho}(w_2) \) is lower bounded as
\[ g_{\lambda,\rho}(w_2) \geq \frac{\phi_{\ell}}{4} \left[ \log \left( 1 + \frac{w_2 P'}{3} \right) - \frac{1}{2} \log \left( 1 + \frac{2w_2 P'}{3} \right) \right] - \frac{3\ell}{4k_{\ell}} H_2(\bar{w}/\ell) \] (231)
\[ \geq \epsilon' - \frac{\epsilon'}{2} \] (232)
\[ = \frac{\epsilon'}{2}. \] (233)
Meanwhile,
\[
g_{1/3,3/4}^1(w_1) = \frac{(1 + \epsilon)\phi_\ell}{4} \log \left( 1 + \frac{w_1 P'}{4} \right) - \frac{|A^*|}{k_\ell} H_2 \left( \frac{w_1}{|A^*|} \right). \tag{234}
\]

When \( w_1 \geq 1 \), we shall invoke Lemma 7 to show that the minimum of the RHS of (234) is achieved at either \( w_1 = 1 \) or some value close to \( k_\ell \). Define
\[
a = \min \left\{ \frac{\phi_\ell}{16} \log \left( 1 + \frac{P'}{4} \right), 1 \right\} \tag{235}
\]

We consider the following three cases separately:

**case a):** \( 1 \leq |A^*| \leq ak_\ell, 1 \leq w_1 \leq |A^*| \) \tag{236}

**case b):** \( ak_\ell \leq |A^*| \leq (1 + \delta_\ell)k_\ell, ak_\ell/2 \leq w_1 \leq |A^*| \) \tag{237}

**case c):** \( ak_\ell \leq |A^*| \leq (1 + \delta_\ell)k_\ell, 1 \leq w_1 \leq ak_\ell/2 \) \tag{238}

For every \( \ell \geq \ell_0 \), \( g_{1/3,3/4}^1(w_1) \) in case a) is lower bounded as
\[
g_{1/3,3/4}^1(w_1) \geq \frac{(1 + \epsilon)\phi_\ell}{4} \log \left( 1 + \frac{P'}{4} \right) - a \tag{239}
\]

\[
\geq \phi \frac{1}{8} \log \left( 1 + \frac{P'}{4} \right) - a \tag{240}
\]

\[
\geq \phi \frac{1}{16} \log \left( 1 + \frac{P'}{4} \right). \tag{241}
\]

In case b), \( g_{1/3,3/4}^1(w_1) \) is lower bounded as
\[
g_{1/3,3/4}^1(w_1) \geq \frac{(1 + \epsilon)\phi_\ell}{4} \log \left( 1 + \frac{ak_\ell P'}{8} \right) - (1 + \delta_\ell), \tag{242}
\]

which grows without bound as \( \ell \) increases.

In case c), \( w_1/|A^*| \leq 1/2 \). Since \( H_2(\cdot) \) is increasing on \([0, 1/2]\), by (234),
\[
\begin{align*}
g_{1/3,3/4}^1(w_1) & \geq \frac{(1 + \epsilon)\phi_\ell}{4} \log \left( 1 + \frac{w_1 P'}{4} \right) - \frac{(1 + \delta_\ell)k_\ell}{k_\ell} H_2 \left( \frac{w_1}{ak_\ell} \right) \tag{243} \\
& \geq 2 \left[ \frac{(1 + \epsilon)\phi_\ell}{8} \log \left( 1 + \frac{w_1 P'}{4} \right) - \frac{ak_\ell}{k_\ell} H_2 \left( \frac{w_1}{ak_\ell} \right) \right]. \tag{244}
\end{align*}
\]

Applying Lemma 7 with \( A_\ell = (1 + \epsilon)\phi_\ell/8, B = P'/4, a_\ell = ak_\ell, \bar{w} = 1 \) and \( b_\ell = ak_\ell/2 \), we conclude that there exists \( \ell_1 \) such that for all \( \ell \geq \ell_1 \), the RHS of (244) restricted to \( w_1 \in [1, ak_\ell/2] \)
achieves the minimum either at 1 or on \([cak_\ell/2, a\ell/2]\) for some \(c \in (0, 1]\). Moreover, \(H_2\left(\frac{1}{a\ell}\right)\) vanishes as \(\ell\) increases. There exists some \(\ell_2\) such that for all \(\ell \geq \ell_2\), \(H_2\left(\frac{1}{a\ell}\right) \leq \frac{1}{2} \log \left(1 + \frac{P'}{4}\right)\) and \(\phi_\ell \geq \phi/2\).

For every \(\ell \geq \max\{\ell_1, \ell_2\}\), if the minimum of the RHS of (244) is achieved at 1, then \(g^{1/4}_{2/3, 3/4}(w_1)\) in case c) is lower bounded as

\[
g^{1/4}_{2/3, 3/4}(w_1) \geq \frac{\phi_\ell}{4} \log \left(1 + \frac{P'}{4}\right) - 2H_2\left(\frac{1}{a\ell}\right)
\geq \frac{\phi}{8} \log \left(1 + \frac{P'}{4}\right) - 2H_2\left(\frac{1}{a\ell}\right)
\geq \frac{\phi}{16} \log \left(1 + \frac{P'}{4}\right). \tag{247}
\]

For every \(\ell \geq \max\{\ell_1, \ell_2\}\), if the minimum of the RHS of (244) is achieved on \([cak_\ell/2, a\ell/2]\), then \(g^{1/4}_{2/3, 3/4}(w_1)\) in case c) is lower bounded as

\[
g^{1/4}_{2/3, 3/4}(w_1) \geq \frac{\phi_\ell}{4} \log \left(1 + \frac{cak_\ell P'}{8}\right) - 2, \tag{248}
\]

which grows without bound as \(\ell\) increases.

By (241), (242), (247) and (248), it concludes that for all \(\ell \geq \max\{\ell_0, \ell_1, \ell_2\}\), \(g^{1/4}_{2/3, 3/4}(w_1) \geq \frac{\phi}{16} \log \left(1 + \frac{P'}{4}\right)\) for all \(1 \leq w_1 \leq |A^*|\) and for all \(1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell\). Combining the lower bound of \(g^{1/4}_{2/3, 3/4}(w_2)\) given by (233), we conclude that for all \(\ell \geq \max\{\ell_0, \ell_1, \ell_2\}\) and for all \((w_1, w_2) \in \mathcal{W}(\ell)\) with \(0 \leq w_2 \leq \bar{w}\), \(h_{2/3, 3/4}(w_1, w_2)\) can be uniformly lower bounded as

\[
h_{2/3, 3/4}(w_1, w_2) \geq \min \left\{\frac{e'}{2}, \frac{\phi}{16} \log \left(1 + \frac{P'}{4}\right)\right\}. \tag{249}
\]

2) The case of \(\bar{w} < w_2 \leq (1 + \delta_\ell)k_\ell\): Letting \(\lambda = 1/2\) and \(\rho = 1\) in (78), and using the fact that \(w_1 \geq 0\) and \(|A^*|/k_\ell \leq 2\), we have

\[
h_{1/2, 1}(w_1, w_2) \geq \frac{n_0}{2k_\ell} \log \left(1 + \frac{w_2P'}{4}\right) - \frac{\ell}{k_\ell} H_2\left(\frac{w_2}{\ell}\right) - \frac{|A^*|}{k_\ell} H_2\left(\frac{w_1}{|A^*|}\right)
\geq \frac{(1 + e)\phi_\ell}{2} \log \left(1 + \frac{w_2 P'}{4}\right) - \frac{\ell}{k_\ell} H_2\left(\frac{w_2}{\ell}\right) - 2. \tag{251}
\]

Applying Lemma 7 with \(A_\ell = (1 + e)\phi_\ell/2, B = P'/4, a_\ell = \ell\) and \(b_\ell = (1 + \delta_\ell)k_\ell\), we can conclude that there exists some \(\ell_3\) such that for all \(\ell \geq \ell_3\), the minimum of the RHS of (251) restricted to \([\bar{w}, (1 + \delta_\ell)k_\ell]\) is achieved either at \(\bar{w}\) or on \([c\ell k_\ell, (1 + \delta_\ell)k_\ell]\), for some \(c \in (0, 1]\).
Moreover, by Lemma \ref{lemma:phi} there exists some $\ell_4$ such that for all $\ell \geq \ell_4$, $\frac{\ell}{k_\ell} H_2(\bar{w}/\ell) \leq 1$ and $\phi_\ell > \phi/2$.

For every $\ell \geq \max\{\ell_3, \ell_4\}$, if the minimum of the RHS of \eqref{eq:original} is achieved at $\bar{w}$, then $h_{1/2,1}(w_1, w_2)$ is uniformly lower bounded as

$$h_{1/2,1}(w_1, w_2) \geq \frac{\phi}{4} \log \left( 1 + \frac{\bar{w}P'}{4} \right) - 2 \geq \epsilon.$$ \hfill (252)

For every $\ell \geq \max\{\ell_3, \ell_4\}$, if the minimum of the RHS of \eqref{eq:original} is achieved on $[ck_\ell, (1+\delta_\ell)k_\ell]$, we consider two cases:

\begin{align*}
\text{case a): } &\ell > 2(1 + \delta_\ell)k_\ell \quad \text{(254)} \\
\text{case b): } &\ell \leq 2(1 + \delta_\ell)k_\ell \quad \text{(255)}
\end{align*}

In case a), $w_2/\ell < 1/2$. Since $H_2(\cdot)$ is increasing on $[0, 1/2]$, by \eqref{eq:original}, we have

$$h_{1/2,1}(w_1, w_2) \geq \frac{(1 + \epsilon)\phi_\ell}{2} \log \left( 1 + \frac{ck_\ell P'}{4} \right) - \frac{\ell}{k_\ell} H_2 \left( \frac{1 + \delta_\ell k_\ell}{\ell} \right) - 2 \geq \frac{(1 + \epsilon)\phi_\ell}{2} \log \left( 1 + \frac{ck_\ell P'}{4} \right) - (1 + \delta_\ell) \frac{\ell}{k_\ell} H_2 \left( \frac{k_\ell}{\ell} \right) - 2 \geq \frac{\phi_\ell}{2} \left[ (1 + \epsilon) \log \left( 1 + \frac{ck_\ell P'}{4} \right) - (1 + \delta_\ell) \log(1 + k_\ell P') \right] - 2,$$ \hfill (256)

where \eqref{eq:257} follows from \eqref{eq:186}, and \eqref{eq:258} is due to \eqref{eq:220}. By \eqref{eq:44}, $\delta_\ell \log(1 + k_\ell P')$ vanishes as $k_\ell$ increases. Moreover,

$$\lim_{k_\ell \to \infty} \log \left( 1 + \frac{ck_\ell P'}{4} \right) - \log(1 + k_\ell P') = \log(c/4).$$ \hfill (259)

Thus, the RHS of \eqref{eq:258} grows without bound (uniformly for $(w_1, w_2)$) as $\ell$ increases.

In case b), by \eqref{eq:original}, we have

$$h_{1/2,1}(w_1, w_2) \geq \frac{(1 + \epsilon)\phi_\ell}{2} \log \left( 1 + \frac{ck_\ell P'}{4} \right) - \frac{\ell}{k_\ell} - 2 \geq \frac{(1 + \epsilon)\phi_\ell}{2} \log \left( 1 + \frac{ck_\ell P'}{4} \right) - 5,$$ \hfill (260)

which grows without bound (uniformly for $(w_1, w_2)$) as $\ell$ increases.
By (253), (258) and (261), we conclude that for all $\ell \geq \max\{\ell_3, \ell_4\}$,

$$h_{1/2,1}(w_1, w_2) \geq \epsilon$$  \hspace{1cm} (262)

uniformly for all $0 \leq w_1 \leq |A^*|$, $\bar{w} \leq w_2 \leq (1 + \delta_\ell)k_\ell$, and $1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell$.

Combining (249) and (262), we conclude that Lemma 3 holds for the case of $\phi > 0$ with $\ell^* = \max\{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4\}$.

B. The case of $\phi = 0$

In this case, $n_0 = \epsilon k_\ell$ by (41). We let $\lambda = 3/5$, $\rho = 5/6$. Note that (224) - (226) remain true in this case.

Consider first $g_{3/5,5/6}^2(w_2)$. By (228), we have

$$\log \left(1 + \frac{3w_2P'}{10}\right) \geq \frac{1}{2} \log \left(1 + \frac{3w_2P'}{5}\right).$$  \hspace{1cm} (263)

Thus,

$$g_{3/5,5/6}^2(w_2) = \frac{\epsilon}{4} \log \left(1 + \frac{3w_2P'}{10}\right) - \frac{\epsilon}{12} \log \left(1 + \frac{3w_2P'}{5}\right) - \frac{5\ell}{6k_\ell}H_2 \left(\frac{w_2}{\ell}\right)$$  \hspace{1cm} (264)

$$\geq \frac{\epsilon}{24} \log \left(1 + \frac{3w_2P'}{5}\right) - \frac{5\ell}{6k_\ell}H_2 \left(\frac{w_2}{\ell}\right).$$  \hspace{1cm} (265)

Applying Lemma 7 with $A_\ell = \epsilon/20$, $B = 3P'/5$, $a_\ell = \ell$, $b_\ell = (1 + \delta_\ell)k_\ell$, we conclude that there exists some $\ell_5$ such that for all $\ell \geq \ell_5$, the minimum of the RHS of (265) restricted to $w_2 \in [1, (1 + \delta_\ell)k_\ell]$ is achieved at either 1 or on $[ck_\ell, (1 + \delta_\ell)k_\ell]$ for some $c \in (0, 1]$. Moreover, by Lemma 6, there exists some $\ell_6$ such that for all $\ell \geq \ell_6$, $\frac{5\ell}{6k_\ell}H_2 \left(\frac{1}{\ell}\right) \leq \frac{\epsilon}{48} \log \left(1 + \frac{3P'}{5}\right)$.

For every $\ell \geq \max\{\ell_5, \ell_6\}$, if the minimum of the RHS of (265) is achieved at 1, then $g_{3/5,5/6}^2(w_2)$ is lower bounded as

$$g_{3/5,5/6}^2(w_2) \geq \frac{\epsilon}{24} \log \left(1 + \frac{3P'}{5}\right) - \frac{5\ell}{6k_\ell}H_2 \left(\frac{1}{\ell}\right)$$  \hspace{1cm} (266)

$$\geq \frac{\epsilon}{48} \log \left(1 + \frac{3P'}{5}\right).$$  \hspace{1cm} (267)

For every $\ell \geq \max\{\ell_5, \ell_6\}$, if the minimum of the RHS of (265) is achieved on $[ck_\ell, (1 + \delta_\ell)k_\ell]$,
we consider two cases:

\text{case a): } \ell > 2(1 + \delta \ell)k \ell \tag{268}

\text{case b): } \ell \leq 2(1 + \delta \ell)k \ell. \tag{269}

In case a), $w_2/\ell < 1/2$. Since $H_2(\cdot)$ is increasing on $[0, 1/2]$, we have

\begin{align*}
g_{5/5,6}^2(w_2) &\geq \frac{\epsilon}{24} \log \left( 1 + \frac{3ck \ell P'}{5} \right) - \frac{5\ell}{6k \ell} H_2 \left( \frac{(1 + \delta \ell)k \ell}{\ell} \right) \\
&\geq \frac{\epsilon}{24} \log \left( 1 + \frac{3ck \ell P'}{5} \right) - (1 + \delta \ell) \frac{5\ell}{6k \ell} H_2 \left( \frac{k \ell}{\ell} \right) \\
&= \frac{\epsilon}{24} \log \left( 1 + \frac{3ck \ell P'}{5} \right) - (1 + \delta \ell) \frac{5\phi \ell}{12} \log (1 + k \ell P) \\
&= \left[ \frac{\epsilon}{24} - (1 + \delta \ell) \frac{5\phi \ell}{12} \log (1 + k \ell P) \right] \log \left( 1 + \frac{3ck \ell P'}{5} \right). \tag{273}
\end{align*}

where (271) is due to (186). Since $\phi \ell \to 0$, we have

\[(1 + \delta \ell) \frac{5\phi \ell}{12} \log (1 + k \ell P) \to 0. \tag{274}\]

The right hand side of (273) thus grows without bound (uniformly for all $w_2$) as $\ell$ increases.

In the case b), we have

\begin{align*}
g_{3/5,6}^2(w_2) &\geq \frac{\epsilon}{24} \log \left( 1 + \frac{3ck \ell P'}{5} \right) - \frac{5\ell}{6k \ell} \\
&\geq \frac{\epsilon}{24} \log \left( 1 + \frac{3ck \ell P'}{5} \right) - \frac{10}{3}. \tag{276}
\end{align*}

which grows without bound (uniformly for all $w_2$) as $k$ increases.

By (267), (273) and (276), we conclude that for all $\ell \geq \max\{\ell_5, \ell_6\}$,

\[g_{3/5,6}^2(w_2) \geq \frac{\epsilon}{48} \log \left( 1 + \frac{3P'}{5} \right) \tag{277}\]

holds uniformly for all $1 \leq w_2 \leq (1 + \delta \ell)k \ell$.

Consider next $g_{3/5,6}^1(w_1)$.

\[g_{3/5,6}^1(w_1) = \frac{\epsilon}{4} \log \left( 1 + \frac{w_1 P'}{4} \right) - \frac{|A^*|}{k \ell} H_2 \left( \frac{w_1}{|A^*|} \right). \tag{278}\]
Define
\[ a = \min \left\{ \frac{\epsilon}{8} \log \left( 1 + \frac{P'}{4} \right), 1 \right\}. \] (279)

We consider the following three cases:

- **case a):** \(1 \leq |A^*| \leq ak, 1 \leq w_1 \leq |A^*|\) (280)
- **case b):** \(ak \leq |A^*| \leq (1 + \delta_k)k, ak/2 \leq w_1 \leq |A^*|\) (281)
- **case c):** \(ak \leq |A^*| \leq (1 + \delta_k)k, 1 \leq w_1 \leq ak/2.\) (282)

In case a), \(g_{3/5,5/6}^1(w_1)\) is uniformly lower bounded as
\[ g_{3/5,5/6}^1(w_1) \geq \frac{\epsilon}{4} \log \left( 1 + \frac{P'}{4} \right) - a \geq \frac{\epsilon}{8} \log \left( 1 + \frac{P'}{4} \right). \] (283)

In case b), \(g_{3/5,5/6}^1(w_1)\) is uniformly lower bounded as
\[ g_{3/5,5/6}^1(w_1) \geq \frac{\epsilon}{4} \log \left( 1 + \frac{akP'}{8} \right) - (1 + \delta_k), \] (285)

which grows without bound as \(k\) increases.

In case c), \(w_1/|A^*| \leq 1/2.\) Since \(H_2(\cdot)\) is increasing on \([0, 1/2],\) we have
\[ g_{3/5,5/6}^1(w_1) \geq \frac{\epsilon}{4} \log \left( 1 + \frac{w_1P'}{4} \right) - (1 + \delta_k)H_2 \left( \frac{w_1}{ak} \right). \] (286)

Applying Lemma 7 with \(A = a\epsilon/8, B = P'/4, a = ak, \bar{w} = 1\) and \(b = ak/2,\) we conclude that there exists some \(\ell_7\) such that for all \(\ell \geq \ell_7,\) the RHS of (287) restricted to \(w_1 \in [1, ak/2]\) achieves minimum either at 1 or on \([cak/2, ak/2]\) for some \(c \in (0, 1].\) Moreover, there exists some \(\ell_8\) such that for all \(\ell \geq \ell_8, H_2 \left( \frac{1}{ak} \right) \leq \frac{\epsilon}{16} \log \left( 1 + \frac{P'}{4} \right).\)

For every \(\ell \geq \max\{\ell_7, \ell_8\},\) if the minimum of the RHS of (287) is achieved at \(w_1 = 1,\) then
\( g_{3/5,5/6}(w_1) \) in case c) is lower bounded as
\[
g_{3/5,5/6}(w_1) \geq \frac{\epsilon}{4} \log \left( 1 + \frac{P'}{4} \right) - 2H_2 \left( \frac{1}{ak_\ell} \right) - \frac{\epsilon}{8} \log \left( 1 + \frac{P'}{4} \right).
\]
(288)

For every \( \ell \geq \max\{\ell_7, \ell_8\} \), if the minimum is achieved on \([cak_\ell/2, ak_\ell/2]\), then \( g_{3/5,5/6}(w_1) \) in case c) is uniformly lower bounded as
\[
g_{3/5,5/6}(w_1) \geq \frac{\epsilon}{4} \log \left( 1 + \frac{ack_\ell P'}{8} \right) - 2,
\]
(290)
which grows without bound as \( k_\ell \) increases.

By (284), (285), (289) and (290), it concludes that for all \( \ell \geq \max\{\ell_7, \ell_8\} \),
\[
g_{3/5,5/6}(w_1) \geq \frac{\epsilon}{8} \log \left( 1 + \frac{P'}{4} \right)
\]
(291)
holds uniformly for all \( 1 \leq w_1 \leq |A^*| \). Combining the lower bound of \( g_{3/5,5/6}(w_2) \) given by (277), we conclude that for all \( \ell \geq \max\{\ell_5, \ell_6, \ell_7, \ell_8\} \), and all \( 1 \leq |A^*| \leq (1 + \delta_\ell)k_\ell \),
\[
h_{2/3,3/4}(w_1, w_2) \geq \min \left\{ \frac{\epsilon}{48} \log \left( 1 + \frac{3P'}{5} \right), \frac{\epsilon}{8} \log \left( 1 + \frac{P'}{4} \right) \right\}
\]
(292)
holds uniformly for all \( (w_1, w_2) \in W^{(\ell)} \). Consequently, Lemma 3 is established for the case of \( \phi = 0 \). Combining the results of Appendix D-A and Appendix D-B proves Lemma 3.

APPENDIX E

PROOF OF LEMMA 4

The lemma was proved for \( k_n = o(n) \) in [7]. In this paper, we prove the achievability result for \( k_n = O(n) \). Throughout the proof, we focus on the case where \( k_n \) grows without bound as \( n \) increases, because the case of bounded \( k_n \) was included in [7].

Let \( f(\gamma, \rho) \) be defined as (102). Choosing \( \rho = 1 \), we have
\[
f(\gamma, 1) = \frac{1}{2} \log \left( 1 + \frac{\gamma k_n P'}{2} \right) - \frac{(1 - \epsilon)\gamma}{2} \log(1 + k_n P') - \frac{k_n}{n} H_2(\gamma).
\]
(293)
Denote \( c_n = k_n/n \) and \( c = \lim \sup_{n \to \infty} c_n \). By differentiating \( f(\gamma, 1) \) with respect to \( \gamma \), we have

\[
\frac{df(\gamma, 1)}{d\gamma} = \frac{k_n P'}{4 + 2\gamma k_n P'} - \frac{1 - \epsilon}{2} \log(1 + k_n P') + \frac{k_n}{n} \log \frac{\gamma}{1 - \gamma},
\]

and

\[
\frac{d^2 f(\gamma, 1)}{d\gamma^2} = \frac{c_n}{\gamma(1 - \gamma)} - \frac{(k_n P')^2}{2(2 + \gamma k_n P')^2}.
\]

Note that \( k_n = O(n) \), \( k_n \) is increasing without bound and \( \gamma \geq 1/k_n \). Evidently,

\[
8c_n \leq k_n P'^2 / 4 \leq \frac{1}{4} (k_n P')^2 \gamma.
\]

Therefore, for sufficiently large \( n \),

\[
8c_n k_n P' \gamma + 8c_n \leq \frac{1}{2} (k_n P')^2 \gamma
\]

holds uniformly for all \( \gamma \in [1/k_n, 1] \). Thus, for sufficiently large \( n \),

\[
\frac{d^2 f(\gamma, 1)}{d\gamma^2} = \frac{(1 + 2c_n)\gamma^2 (k_n P')^2 - (k_n P')^2 \gamma + 8c_n k_n P' \gamma + 8c_n}{2(2 + \gamma k_n P')^2 \gamma(1 - \gamma)}
\]

\[
\leq \frac{(1 + 2c_n)\gamma^2 (k_n P')^2 - (k_n P')^2 \gamma + \frac{1}{2} (k_n P')^2 \gamma}{2(2 + \gamma k_n P')^2 \gamma(1 - \gamma)}
\]

\[
= \frac{[(1 + 2c_n)\gamma - 1/2] (k_n P')^2}{2(2 + \gamma k_n P')^2 \gamma(1 - \gamma)}
\]

\[
\leq \frac{[(1 + 4c)\gamma - 1/2] (k_n P')^2}{2(2 + \gamma k_n P')^2 \gamma(1 - \gamma)}
\]

holds uniformly for all \( \gamma \).

We pick the constant \( \gamma' = \frac{1/2}{1 + 4c} \). Since \( 0 \leq c < \infty \), we have \( 0 < \gamma' \leq 1/2 \). By (302), for sufficiently large \( n \), \( \frac{d^2 f(\gamma, 1)}{d\gamma^2} < 0 \) holds uniformly for all \( 1/k_n \leq \gamma \leq \gamma' \). It means \( f(\gamma, 1) \) is concave over \( \gamma \in [1/k_n, \gamma'] \). Therefore, there exists some \( N_0 \) such that for all \( n \geq N_0 \),

\[
\min_{1/k_n \leq \gamma \leq 1} f(\gamma, 1) = \min \left\{ f(1/k_n, 1), \min_{\gamma' \leq \gamma \leq 1} f(\gamma, 1) \right\}.
\]
If the minimum is achieved at $\gamma = 1/k_n$, we have

$$f(1/k_n, 1) = \frac{1}{2} \log \left( 1 + \frac{P'}{2} \right) - \frac{(1 - \epsilon)}{2k_n} \log(1 + k_n P') - \frac{k_n}{n} H_2 \left( \frac{1}{k_n} \right).$$  \hfill (304)

Since $(1/k_n) \log(1 + k_n P')$ and $k_n H_2(1/k_n)$ vanishes as $k_n$ increases, there exists $N_1$ such that for all $n \geq N_1$,

$$f(1/k_n, 1) \geq \frac{1}{4} \log \left( 1 + \frac{P'}{2} \right).$$  \hfill (305)

If the minimum is achieved on $[\gamma', 1]$, we can lower bound $f(\gamma, 1)$ as

$$f(\gamma, 1) \geq \frac{1}{2} \log \left( 1 + \frac{\gamma'k_n P'}{2} \right) - \frac{(1 - \epsilon)}{2} \log(1 + k_n P') - \frac{k_n}{n}.$$  \hfill (306)

Since $\log \left( 1 + \frac{\gamma'k_n P'/2}{2} \right) - \log(1 + k_n P')$ and $k_n/n$ converge to some constants, the lower bound given by (306) grows without bound as $n$ increases.

In summary, combining (303), (305) and (306), it concludes that for all $n \geq \max\{N_0, N_1\}$ and all $|A^*|$, the error exponent is lower bounded

$$E_r \geq \min_{1/k_n \leq \gamma \leq 1} f(\gamma, 1)$$

$$\geq \frac{1}{4} \log \left( 1 + \frac{P'}{2} \right).$$  \hfill (308)

The lemma is thus established.

**APPENDIX F**

**PROOF OF THEOREM 5**

Unlike the case of unbounded $k_n$, there is a nonvanishing probability that the number of active users is zero. Let $A^*$ denote the set of active users and $E_d$ denote the event of detection error. Given an increasing sequence $s_n$ satisfying the conditions specified in Theorem 5, the overall error probability can be calculated as

$$P\{E_d\} \leq P\{|A^*| > s_n\} + P\{E_d \mid 1 \leq |A^*| \leq s_n\} + P\{E_d \mid |A^*| = 0\}. \hfill (309)$$

By the Chernoff bound for binomial distribution [30], the probability that the number of active
users is greater than \( s_n \) is calculated as

\[
P \{|A^*| > s_n\} \leq \exp \left( -k_n(s_n/k_n - 1)^2/3 \right),
\]

which vanishes as \( s_n \) grows without bound.

Note that the sequence \( s_n \) satisfies \( \ell_n e^{-\delta s_n} \to 0 \) for every \( \delta > 0 \) and

\[
\lim_{n \to \infty} \frac{2s_n H_2(s_n/\ell_n)}{n \log(1 + s_n P)} < 1,
\]

which are the regularity conditions for unbounded \( k_n \) as specified in Case 1) of Theorem [1]. The error probability \( P \{E_d \mid |A^*| \leq s_n\} \) vanishes by following exactly the same as the analysis for the case of unbounded \( k_n \) (i.e., Case 1) by treating \( s_n \) as an unbounded \( k_n \).

We consider the identification error when \( |A^*| = 0 \). If no user is active, the received signal in the first \( n_0 \) channel uses is purely noise, i.e., \( Y^a = Z^a \). By the user identification rule (43) with \( k_n \) replaced by \( s_n \), a detection error occurs if at least one user is claimed to be active. The detection error probability can be calculated as

\[
P \{E_d \mid |A^*| = 0\} \leq \sum_{w=1}^{(1+\delta_n)s_n} \binom{\ell_n}{w} P \left\{ \left\| Z^a - \sum_{i=1}^{w} S^a_{i} \right\|^2 \leq \|Z^a\|^2 \right\}.
\]

Let \( \bar{S} = \sum_{i=1}^{w} S^a_{i} \). The entries of \( \bar{S} \) are i.i.d. according to \( \mathcal{N}(0, wP') \). We have

\[
P \left\{ \left\| Z^a - \sum_{i=1}^{w} S^a_{i} \right\|^2 \leq \|Z^a\|^2 \right\} = P \left\{ \sum_{i=1}^{n_0} Z^a_i \bar{S}_i \geq \frac{1}{2} \|\bar{S}\|^2 \right\}
\]

\[
= E \left\{ P \left\{ \sum_{i=1}^{n_0} Z^a_i \bar{S}_i \geq \frac{1}{2} \|\bar{S}\|^2 \right\} \mid \bar{S} \right\}.
\]

Conditioned on \( \bar{S} \), \( \sum_{i=1}^{n_0} Z^a_i \bar{S}_i \sim \mathcal{N}(0, \|\bar{S}\|^2) \). Therefore,

\[
E \left\{ P \left\{ \sum_{i=1}^{n_0} Z^a_i \bar{S}_i \geq \frac{1}{2} \|\bar{S}\|^2 \right\} \mid \bar{S} \right\} \leq E \left\{ Q \left( \frac{\|\bar{S}\|}{2} \right) \right\} \leq \frac{e^{\frac{-\|\bar{S}\|^2}{8}}}{(1 + wP'/4)^{-n_0/2}}
\]

where (316) is due to \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-u^2/2)du \leq e^{-x^2/2} \), and (317) follows because \( \|\bar{S}\|^2/wP \) is chi-squared distributed with \( n_0 \) degrees of freedom and \( E \{e^{tX}\} = (1 - 2t)^{-n/2} \)
for a chi-squared distributed variable $X$ with $n$ degrees of freedom.

Combining (312), (314) and (317), the detection error probability for $|A^*| = 0$ can be upper bounded as

$$
P \{ \mathcal{E}_d ||A^*|| = 0 \} \leq \sum_{w=1}^{(1+\delta_n)s_n} \exp \left( \ell_n H_2(\ell_n) - \frac{n_0}{2} \log(1 + wP'/4) \right). \tag{318}$$

Let $\theta_n$ be given by (11) with $k_n$ replaced by $s_n$ and define $\theta = \lim_{n \to \infty} \theta_n$. By the choice of the signature length given by (84), $n_0 \geq \delta n$, where $\delta = \min(\epsilon, \theta(1+\epsilon)/2)$. For a large enough $n$, the error probability can be further upper bounded as

$$
P \{ \mathcal{E}_d ||A^*|| = 0 \} \leq \sum_{w=1}^{(1+\delta_n)s_n} \exp (-s_n h(w)), \tag{319}$$

where

$$h(w) = \frac{\delta n}{2s_n} \log(1 + wP'/4) - \frac{\ell_n}{s_n} H_2 \left( \frac{w}{\ell_n} \right). \tag{320}$$

Note that $s_n = O(n)$. Applying Lemma 7 with $\ell = n$, $\bar{w} = 1$, $A_n = \delta n/(2s_n)$, $k_n = s_n$, $a_n = \ell_n$ and $b_n = (1+\delta_n)s_n$, we conclude that for large enough $n$, the minimum of $h(w)$ restricted to $[1, (1+\delta_n)s_n]$ is achieved either at 1 or $[cs_n, (1+\delta_n)s_n]$ for some $0 < c \leq 1$.

As long as $s_n$ satisfies the conditions as specified in Theorem 5, $\frac{\ell_n}{s_n} H_2 (1/\ell_n)$ vanishes as $n$ increases by Lemma 6. For large enough $n$, if the minimum of $h(w)$ is achieved at $w = 1$, $h(w)$ is uniformly lower bounded by some constant $c_0 > 0$. If the minimum of $h(w)$ is achieved on $[cs_n, (1+\delta_n)s_n]$, it implies that $h(w)$ grows without bound. It concludes that there exists some $N_0$, such that for all $n \geq N_0$, $h(w)$ is uniformly lower bounded by $c_0$ for all $1 \leq w \leq (1+\delta_n)s_n$.

By (319), there exists some $N_0$ and $c_0 > 0$ such that for all $n \geq N_0$,

$$
P \{ \mathcal{E}_d ||A^*|| = 0 \} \leq (1+\delta_n)s_n e^{-c_0s_n}. \tag{321}$$

Therefore, $P \{ \mathcal{E}_d ||A^*|| = 0 \}$ vanishes as the blocklength $n$ increases. Since the three terms on the RHS of (309) all vanish, the overall detection error probability also vanishes.
Since the users adopt Gaussian random codes, by treating the other users as interference, the first user to be decoded effectively sees Gaussian noise with variance $1 + (k_n - 1)P$. In order to prove the lemma, we show that the error probability of any $\left\lfloor \exp(v(n)) \right\rfloor, n$ code for the first user, where the message length $v(n)$ is given by (123), is lower bounded by some positive constant.

Let $P_m(v(n), n)$ denote the average error probability for the first user achieved by the best channel code of blocklength $n$ with message length $v(n)$, where each codeword satisfies the maximal power constraint (2). Let $P_e(v(n), n)$ denote the average error probability for the first user achieved by the best channel code of blocklength $n$ with message length $v(n)$, where each codeword satisfies the equal power constraint, i.e., each codeword lies on a power-sphere $\sum_{i=1}^{n} s_{ki} = nP$. According to [36, eq. (83)], we have

$$P_m(v(n-1), n-1) \geq P_e(v(n-1), n). \quad (322)$$

We will lower bound $P_e(v(n-1), n)$ in order to show that $P_m(v(n), n)$ is strictly bounded away from zero for $v(n)$ given by (123).

Let $\lambda > 1$ be an arbitrary constant. Following the notations in [37, eq. (13)], let the decoding threshold be $\gamma = (n-1)(1 - \lambda \epsilon)C$, $P'_Y$ be the distribution of $n$ i.i.d. Gaussian random variables with zero mean and variance $1 + k_n P$, $P_Y|X = [\sqrt{P}, \ldots, \sqrt{P}]$ be the distribution of $n$ i.i.d. Gaussian random variables with mean $\sqrt{P}$ and variance $1 + (k_n - 1)P$, and $\beta_1 - \epsilon_n \left( P_Y|X = [\sqrt{P}, \ldots, \sqrt{P}], P'_Y \right)$, where $\beta_\alpha(P, P')$ is the minimum error probability of the binary hypothesis test under hypothesis $P'$ if the error probability under hypothesis $P$ is not larger than $1 - \alpha$. The error probability $P_e(v(n-1), n)$ is lower bounded as (see also [37, eq. (88)])

$$P_e(v(n-1), n) \geq P \left\{ \frac{1}{2(1 + Q)} \sum_{i=1}^{n} Q(1 - Z_i^2) + 2\sqrt{Q}Z_i \leq -\lambda \epsilon n C - (1 - \lambda \epsilon)C \right\} - e^{-(\lambda - 1)(n-1)\epsilon C}. \quad (323)$$

We will follow a similar step as in [37] to further calculate the RHS of (323). Let $X_i = -Q(1 - Z_i^2) - 2\sqrt{Q}Z_i$, where $Z_i$ are i.i.d. standard Gaussian random variables. Then $EX_i = 0$. 

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By recalling Rozovsky’s large deviation result [37, Theorem 5], we have

$$P\left\{ \sum_{i=1}^{n} X_i > x\sqrt{S} \right\} \geq Q(x) e^{-\frac{A_1 T x^3}{S^{3/2}}} \left( 1 - \frac{A_2 T x}{S^{3/2}} \right), \quad (324)$$

where $A_1, A_2$ are some universal constants, $S = \sum_{i=1}^{n} E|X_i|^2$, and $T = \sum_{i=1}^{n} E|X_i|^3$ which is equivalent to (126).

Then the first term in (323) can be calculated as

$$P\left\{ \frac{1}{2(1+Q)} \sum_{i=1}^{n} Q(1-Z_i^2) + 2\sqrt{Q}Z_i \leq -(\lambda\epsilon n C - (1-\lambda\epsilon)C) \right\} = P\left\{ \sum_{i=1}^{n} X_i \geq x\sqrt{S} \right\}, \quad (325)$$

where $x = \frac{2(\lambda\epsilon n + 1 - \lambda\epsilon)C(1+Q)}{\sqrt{S}}$.

We can derive that $S = 2nQ(2+Q)$. Since $Q = \frac{P}{1+(k_n-1)P} \to 0$ as $n$ increases, we have

$$E|X_i|^3 = O\left( Q^{3/2} \right). \quad (326)$$

Moreover, since $k = an$, we have $T = O\left( nQ^{3/2} \right)$ and therefore $T$ tends to zero as $n$ increases.

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