Two-parameter Quantum Affine Algebra of Type $C_n^{(1)}$,
Drinfeld Realization and Vertex Representation

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Abstract. The two-parameter quantum vertex operator representation of level-one is explicitly constructed for $U_{r,s}(C_n^{(1)})$ based on its two-parameter Drinfeld realization we give. This construction will degenerate to the one-parameter case due to Jing-Koyama-Misra ($\text{J} \text{K} \text{M}^2$) when $rs = 1$.

1. Introduction

In 2000, the study of two-parameter quantum groups was revitalized by a series of work for type $A$ of Benkart and Witherspoon [BW1, BW2, BW3] originally obtained by Takeuchi [T]. A systematic study afterwards both on the structures and finite-dimensional representation theory of two-parameter quantum groups for the semisimple Lie algebras of any other types can be seen [BGH1, BGH2, BH, HP1, HP2, HS, HS1, HW1, HW2], etc. In 2004, Hu, Rosso and Zhang [HRZ] began to investigate the two-parameter quantum affine algebra of type $A_n^{(1)}$, and obtained the two-parameter version of the celebrated Drinfeld realization in the case of $U_{r,s}(\hat{sl}_n)$, as well as proposed for the first time the quantum affine Lyndon basis as a monomial basis in the affine case. A general insight ([HZ]) for handling the two-parameter quantum affine algebras of untwisted types in a unified manner had been found when the first author visited ICTP early in 2006, that is, the $\tau$-invariant generating functions for the two-parameter version (where $\tau$ is the involution as a $Q$-antiautomorphism) successfully served as a defining tool of the Drinfeld realization formalism in a compact form avoiding the case-by-case manner. As a valid verification of such defining relations for Drinfeld realizations, the quantum two-parameter vertex representations of level one for the simply-laced cases $X_n^{(1)}$, where $X = A, D, E$, had been established there. Also for type $G_2^{(1)}$, the validness of our definition for the two-parameter Drinfeld realization can be well-checked in the level of its two-parameter quantum vertex representation, see [GHZ].

As two generalizations of [HZ] to the two-parameter quantum affine algebras of twisted types $X^{(r)}$ for $r = 2, 3$, the readers can consult [JZ1], to the two-parameter

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quantum toroidal algebras, please refer to [JZ2], where the authors gave a McKay correspondence formalism of the vertex representations obtained in [HZ].

It was known that the theory of two-parameter quantum affine algebras has been developed with some analogous stories as in the one-parameter counterpart such as Drinfeld realization theorem with a different argument approach [HRZ, HZ, JZ1], fermionic realization [JZ3] and a finite-dimensional representation theory [JZ4]. The quantum vertex representations of one-parameter quantum affine algebras for the untwisted types were first constructed by Frenkel-Jing [FJ] that confirmed the Drinfeld’s celebrated conjectural “new realization” [Dr2] in the level of vertex representations, even though it was not proved until Beck gave his rigorous proof for the untwisted types by generalizing the Lusztig’s braid automorphisms suitable for the quantum affine algebras based on the work of Damiani [D] and Levendorskii-Soibelman-Stukopin [LSS]. Afterwards, the quantum vertex representation theory had been established in many works, for instance, see [Be0, J1, JM, J2, JKM1, JKM2] and references therein, etc.

The goal of the current paper is to construct the level-one vertex representation of two-parameter quantum affine algebra of \( U_{r,s}(C_n^{(1)}) \), which also verifies our unified defining formalism for the two-parameter version of the Drinfeld realization for the multiply-laced cases in the level of two-parameter quantum vertex representations.

The paper is organized as follows. In section 2, We first give Drinfeld-Jimbo presentation of two-parameter quantum affine algebra \( U_{r,s}(C_n^{(1)}) \) in the sense of Hopf algebra. The Drinfeld realization of two-parameter quantum affine algebra \( U_{r,s}(C_n^{(1)}) \) is given in section 3. Furthermore we present and prove the Drinfeld theorem between the above two realizations. In section 4, we start from the two-parameter enlarged quantum Heisenberg algebra and introduce a quasi-cocycle. Then we construct the level-one quantum vertex representation of two-parameter quantum affine algebras \( U_{r,s}(C_n^{(1)}) \). This construction will degenerate to the one-parameter case due to Jing-Koyama-Misra [JKM2] when \( rs = 1 \).

2. Quantum Affine Algebra \( U_{r,s}(C_n^{(1)}) \) and Drinfeld Double

2.1. Structure of \( U_{r,s}(C_n^{(1)}) \). Let \( K = \mathbb{Q}(r^\frac{1}{2}, s^\frac{1}{2}) \) denote a field of rational functions with two-parameters \( r^{\frac{1}{2}}, s^{\frac{1}{2}} \) (\( r \neq \pm s \)). Let \( g \) be the finite-dimensional complex simple Lie algebra of type \( C_n \), with Cartan subalgebra \( h \) and Cartan matrix \( A = (a_{ij})_{i,j \in I} \) (\( I = \{1, 2, \cdots, n\} \)). Fix coprime integers \( (d_i)_{i \in I} \) such that \( (d_i)_{i \in I} \) is symmetric. Assume \( \Phi \) is a finite root system of type \( C_n \) with \( \Pi \) a base of simple roots. Regard \( \Phi \) as a subset of a Euclidean space \( E = \mathbb{R}^n \) with an inner product \( (\ , \ ) \). Let \( \epsilon_1, \epsilon_2, \cdots, \epsilon_n \) denote an orthonormal basis of \( E \). Let \( \alpha_1 = \epsilon_i - \epsilon_{i+1}, \alpha_n = 2\epsilon_n \) be the simple roots of the simple Lie algebra \( sp_{2n} \), \( \{\alpha_i^\vee\} \) and \( \{\lambda_i\} \), the sets of, respectively, simple coroots and fundamental weights. \( \mathcal{Q} = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i \) is the root lattice. Let \( \theta \) be the highest root and \( \delta \) denote the primitive imaginary root of \( C_n^{(1)} \). Take \( \alpha_0 = \delta - \theta \), then \( \mathcal{I} = \{\alpha_i | i \in I_0 = \{0, 1, \cdots, n\}\} \) is a base of simple roots of the affine Lie algebra \( C_n^{(1)} \). Let \( \hat{\mathcal{Q}} = \bigoplus_{i=0}^n \mathbb{Z} \alpha_i \) denote the root lattice of \( C_n^{(1)} \).

Let \( c \) be the canonical central element of the affine Lie algebra of type \( C_n^{(1)} \). Define a nondegenerate symmetric bilinear form \( (\ | \ ) \) on \( h^* \) satisfying

\[
(\alpha_i | \alpha_j) = d_ia_{ij}, \quad (\delta | \alpha_i) = (\delta | \delta) = 0, \quad \text{for all } i, j \in I_0,
\]
where \((d_0, d_1, \ldots, d_n) = (1, 1/2, \ldots, 1/2, 1)\). Denote \(r_i = r^{d_i}, s_i = s^{d_i}\) for \(i = 0, 1, \ldots, n\).

Given two sets of symbols \(W = \{\omega_0, \omega_1, \ldots, \omega_n\}, W' = \{\omega'_0, \omega'_1, \ldots, \omega'_n\}\). Define the structural constants matrix \((\omega_i', \omega_j)_{(n+1) \times (n+1)}\) of type \(C^{(1)}_n\) by

\[
\begin{pmatrix}
rs^{-1} & r^{-1} & 1 & \ldots & 1 & rs \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 1 & \ldots & r^\frac{1}{2} s^{-\frac{1}{2}} & r^{-1} \\
0 & 0 & \ldots & rs^{-\frac{1}{2}} & s \\
\end{pmatrix}
\]

**Definition 2.1.** The two-parameter quantum affine algebra \(U_{r,s}(C^{(1)}_n)\) is a unital associative algebra over \(K\) generated by the elements \(e_j, f_j, \omega^\pm_1, \omega'^\pm_1 \ (j \in I_0)\), \(\gamma^\pm_{\frac{1}{2}}, \gamma'^\pm_{\frac{1}{2}}, D^\pm_1, D'^\pm_1\), satisfying the following relations:

\[\gamma^\pm_{\frac{1}{2}}, \gamma'^\pm_{\frac{1}{2}}\] are central with \(\gamma = \omega_j, \gamma' = \omega'_j\), such that \(\gamma \gamma' = (rs)^c\). The \(\omega^\pm_1, \omega'^\pm_1\) all commute with one another and \(\omega_i \omega^{-1}_i = \omega'^i \omega'^{-1}_i = 1, [\omega^\pm_1, D^\pm_1] = [\omega'^\pm_1, D'^\pm_1] = [D^\pm_1, D'^\pm_1] = 0\).

\(\hat{C}(1)\) For \(0 \leq i \leq n\) and \(1 \leq j < n\),

\[
D e_i D^{-1} = r^{\delta_{ij}} e_i, \\
D f_i D^{-1} = r^{-\delta_{ij}} f_i,
\]

\(\hat{C}(2)\) For \(0 \leq i \leq n\) and \(1 \leq j < n\),

\[
D e_i D^{-1} = s^{\delta_{ij}} e_i, \\
D f_i D^{-1} = s^{-\delta_{ij}} f_i,
\]

\(\hat{C}(3)\) For \(i, j \in I_0\), then we have:

\[
[e_i, f_j] = \frac{\delta_{ij}}{r_i - s_i} (\omega_i - \omega'_i).
\]

\(\hat{C}(5)\) For all \(1 \leq i \neq j \leq n\) but \((i, j) \not\in \{(0, n), (n, 0)\}\) such that \(a_{ij} = 0\), then we have:

\[
[e_i, e_j] = [f_i, f_j] = 0, \\
e_n e_0 = rs e_0 e_n, \\
f_0 f_n = rs f_n f_0.
\]
(Č6) For $1 \leq i \leq n - 2$, there are the following $(r, s)$-Serre relations:

\[
e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + (rs) e_{i+1} e_i^2 = 0,
\]

\[
e_0^2 e_1 - (r+s) e_0 e_0 + rs e_0 e_0^2 = 0,
\]

\[
e_{i+1} e_i - (r^{-1}_i + s^{-1}_i) e_{i+1} e_i + (r^{-1}_i s^{-1}_i) e_i e_{i+1}^2 = 0,
\]

\[
e_{n-1}^2 e_{n-1} - (r^{-1} + s^{-1}) e_{n-1} e_{n-1} + (r^{-1} s^{-1}) e_{n-1} e_{n-1}^2 = 0,
\]

\[
e_{n-1} e_n - (r+rs)^{\frac{1}{2}} + s e_{n-1} e_{n-1} + (rs)^{\frac{1}{2}} (r + (rs)^{\frac{1}{2}} + s) e_{n-1} e_{n-1}^2 - (rs) e_n e_{n-1}^2 = 0,
\]

\[
e_i^3 e_0 - (r^{-1} + (rs)^{\frac{1}{2}} + s^{-1}) e_i e_0 e_1 + (rs)^{\frac{1}{2}} (r^{-1} + (rs)^{\frac{1}{2}} + s^{-1}) e_i e_0 e_1^2 - (rs)^{\frac{1}{2}} e_0 e_i^3 = 0.
\]

(Č7) For $1 \leq i \leq n - 2$, there are the following $(r, s)$-Serre relations:

\[
f_{i+1} f_i^2 - (r+s) f_i f_{i+1} f_i + (rs) f_i^2 f_{i+1} = 0,
\]

\[
f_0 f_0^2 - (r+s) f_0 f_1 f_0 + rs f_0^2 f_1 = 0,
\]

\[
f_i f_{i+1}^2 - (r^{-1}_i + s^{-1}_i) f_i f_{i+1} f_{i+1} + (r^{-1}_i s^{-1}_i) f_{i+1} f_{i+1} f_i = 0,
\]

\[
f_{n-1} f_n^2 - (r^{-1} + s^{-1}) f_{n-1} f_{n-1} f_n + (r^{-1} s^{-1}) f_n^2 f_{n-1} = 0,
\]

\[
f_n f_{n-1}^3 - (r+rs)^{\frac{1}{2}} f_{n-1} f_n f_{n-1} + (rs)^{\frac{1}{2}} (r + (rs)^{\frac{1}{2}} + s) f_{n-1} f_n f_{n-1} - (rs)^{\frac{1}{2}} f_n f_{n-1} = 0,
\]

\[
f_0 f_1^3 - (r^{-1} + (rs)^{\frac{1}{2}} + s^{-1}) f_1 f_0 f_1^2 + (rs)^{\frac{1}{2}} (r^{-1} + (rs)^{\frac{1}{2}} + s^{-1}) f_1 f_0 f_1 - (rs)^{\frac{1}{2}} f_1 f_0^2 = 0.
\]

$U_{r,s}(C_n^{(1)})$ is a Hopf algebra with the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ defined below: for $i \in I_0$, we have

\[
\Delta(\gamma_i^{\pm \frac{1}{2}}) = \gamma_i^{\pm \frac{1}{2}} \otimes \gamma_i^{\pm \frac{1}{2}}, \quad \Delta(\gamma_i^{\pm 1}) = \gamma_i^{\pm 1} \otimes \gamma_i^{\pm 1},
\]

\[
\Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}, \quad (D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1},
\]

\[
\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes e_i + 1 \otimes f_i,
\]

\[
\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(\gamma_i^{\pm \frac{1}{2}}) = \varepsilon(\gamma_i^{\pm 1}) = 0, \quad \varepsilon(D^{\pm 1}) = 0, \quad \varepsilon(\gamma_i) = 1,
\]

\[
S(\gamma_i^{\pm \frac{1}{2}}) = \gamma_i^{\mp \frac{1}{2}}, \quad S(\gamma_i^{\pm 1}) = \gamma_i^{\mp 1}, \quad S(D^{\pm 1}) = D^{\mp 1}, \quad S(D^{\mp 1}) = D^{\pm 1},
\]

\[
S(e_i) = -\omega_i^{\pm 1} e_i, \quad S(f_i) = -f_i \omega_i^{\pm 1}, \quad S(\omega_i) = \omega_i^{\pm 1}, \quad S(\omega_i) = \omega_i^{\pm 1}.
\]

2.2. Triangular decomposition of $U_{r,s}(C_n^{(1)})$.

Corollary 2.2. $U_{r,s}(C_n^{(1)}) \cong U_{r,s}(\widehat{\mathfrak{g}}) \otimes U^0 \otimes U_{r,s}(\widehat{\mathfrak{g}})$, as vector spaces, where $U^0 = K[\omega_0^{\pm 1}, \omega_1^{\pm 1}, \ldots, \omega_n^{\pm 1}, \omega_0^{\pm 1}, \omega_1^{\pm 1}, \omega_n^{\pm 1}, \ldots, \omega_n^{\pm 1}]$, and $U_{r,s}(\widehat{\mathfrak{g}})$ (resp. $U_{r,s}(\widehat{\mathfrak{g}})$) is the subalgebra generated by $e_i$ (resp. $f_i$) for all $i \in I_0$. $\hat{\mathfrak{g}}$ (resp. $\hat{\mathfrak{g}}$) is the Borel Hopf subalgebra of $U_{r,s}(C_n^{(1)})$ generated by $e_j, \omega_j^{\pm 1}, \gamma_j^{\pm \frac{1}{2}}, D^{\pm 1}$ (resp. $f_j, \omega_j^{\pm 1}, \gamma_j^{\pm \frac{1}{2}}, D^{\pm 1}$) with $j \in I_0$.

Definition 2.3. Let $\tau$ be the $Q$-algebra anti-automorphism of $U_{r,s}(C_n^{(1)})$ such that $\tau(r) = s$, $\tau(s) = r$, $\tau(\omega_i, \omega_j) = 1$, and

\[
\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(\omega_i) = \omega_i, \quad \tau(\omega_i) = \omega_i,
\]

\[
\tau(\gamma) = \gamma', \quad \tau(\gamma') = \gamma, \quad \tau(D) = D', \quad \tau(D') = D.
\]

Then $\hat{\mathfrak{g}} = \tau(\hat{\mathfrak{g}})$ with those induced defining relations from $\hat{\mathfrak{g}}$, those cross relations in (C2)–(C4), (C5) and (C6) are antisymmetric with respect to $\tau$. 

3. Drinfeld Realization of \( U_{r,s}(C_n^{(1)}) \)

3.1. Drinfeld Realization. In this subsection, we describe the two-parameter Drinfeld quantum affinization of \( U_{r,s}(C_n) \), which will play an important role in the vertex representation of Section 4. Briefly denote \( \langle i, j \rangle := \langle \omega_i, \omega_j \rangle \).

**Definition 3.1.** Let \( U_{r,s}(C_n^{(1)}) \) be the unital associative algebra over \( \mathbb{K} \) generated by the elements \( x_i^\pm(k), a_i(\ell), \omega_i^\pm, \omega_i'^\pm, \gamma_i^\pm, \gamma_i'^\pm, D_i^\pm, D_i'^\pm \) \( (i = 1, \ldots, n, \quad k, k' \in \mathbb{Z}, \quad l, l' \in \mathbb{Z}\setminus\{0\}) \) with the following defining relations:

\[
(D1) \quad \gamma_i^\pm, \gamma_i'^\pm \text{ are central with } \gamma_i' = (rs)^e \text{ and } \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1 \quad (i \in I),
\]

\[
[\omega_i^\pm, \omega_j^\pm] = [\omega_i'^\pm, \omega_j'^\pm] = [\omega_i^\pm, \omega_i'^\pm] = [\omega_i'^\pm, D_i'^\pm] = [\omega_i'^\pm, D_i'^\pm] = [D_i'^\pm, D_i'^\pm] = 0, \quad \text{for } i, j \in I.
\]

\[
(D2) \quad [a_i(\ell), a_j(\ell')] = \delta_{\ell, \ell'} \frac{(\gamma_i^\prime)^{\ell} (rs)^{\ell + (\alpha_i(\alpha_j))}}{|\ell|} \frac{\ell (\alpha_i | \alpha_j)}{r - s}, \quad \frac{\ell (\alpha_i | \alpha_j)}{r - s}.
\]

\[
(D3) \quad [a_i(\ell), \omega_i'^\pm] = [a_i(\ell), \omega_i'^\pm] = 0.
\]

\[
(D4) \quad D x_i^\pm(k) D^{-1} = s_i^k x_i^\pm(k), \quad D' x_i^\pm(k) D'^{-1} = s_i^k x_i^\pm(k),
\]

\[
D a_i(\ell) D^{-1} = r_i^\ell a_i(\ell), \quad D' a_i(\ell) D'^{-1} = r_i^\ell a_i(\ell).
\]

\[
(D5) \quad \omega_i x_i^\pm(k) \omega_i^{-1} = (i, j)^\pm x_j^\pm(k), \quad \omega_i' x_i^\pm(k) \omega_i'^{-1} = (i, j)^\pm x_j^\pm(k).
\]

\[
(D6_1) \quad [a_i(\ell), x_j^\pm(k)] = \frac{(rs)^{\ell (\alpha_i | \alpha_j)}}{|\ell|} [\ell (\alpha_i | \alpha_j)] x_j^\pm(\ell + k), \quad \text{for } \ell > 0,
\]

\[
(D6_2) \quad [a_i(\ell), x_j^\pm(k)] = \frac{(rs)^{-\ell (\alpha_i | \alpha_j)}}{|\ell|} [\ell (\alpha_i | \alpha_j)] x_j^\pm(\ell + k), \quad \text{for } \ell < 0.
\]

\[
(D7) \quad x_i^\pm(k+1) x_j^\pm(k') - (j, i)^\pm x_j^\pm(k) x_i^\pm(k+1) = -(j, i)^\pm x_j^\pm(k) x_i^\pm(k+1) = -\left((j, i)^\pm x_j^\pm(k+1) x_i^\pm(k) - (i, j)^\pm x_j^\pm(k) x_i^\pm(k+1)\right),
\]

\[
(D8) \quad [x_i^\pm(k), x_j^\pm(k')] = \frac{\delta_{ij}}{r_i - s_i} \left(\gamma_i^{-k} \gamma_i'^{-k} \psi_i(k + k') - \gamma_i'^{-k} \gamma_i'^{-k} \varphi_i(k + k')\right),
\]

where \( \psi_i(m), \varphi_i(m) \) \( (m \in \mathbb{Z}_{\geq 0}) \) with \( \psi_i(0) = \omega_i \) and \( \varphi_i(0) = \omega_i' \) are defined by:

\[
\sum_{m=0}^{\infty} \psi_i(m) z^m = \omega_i \exp \left((r-s) \sum_{\ell=1}^{\infty} a_i(\ell) z^{-\ell}\right), \quad (\psi_i(-m) = 0, \forall m > 0);
\]

\[
\sum_{m=0}^{\infty} \varphi_i(-m) z^m = \omega_i' \exp \left(-(r-s) \sum_{\ell=1}^{\infty} a_i(-\ell) z^\ell\right), \quad (\varphi_i(m) = 0, \forall m > 0).
\]

\[
(D9_1) \quad \text{Sym}_{n_1, n_2} \left(x_i^\pm(n_1)x_i'^\pm(n_2)x_j^\pm(k) - (r_i^\pm + s_i^\pm) x_i^\pm(n_1)x_j^\pm(k)x_i'^\pm(n_2)\right) = 0,
\]

for \( a_{ij} = -1 \) and \( 1 \leq i < j \leq n; \)

\[
(D9_2) \quad \text{Sym}_{n_1, n_2} \left(x_i^\pm(n_1)x_i'^\pm(n_2)x_j^\pm(k) - (r_i^\pm + s_i^\pm) x_i^\pm(n_1)x_j^\pm(k)x_i'^\pm(n_2)\right) = 0,
\]

for \( a_{ij} = -1 \) and \( 1 \leq j < i \leq n; \)
(D9) \( Sym_{n_1,n_2,n_3} \left( x_i^\pm(n_1)x_i^\pm(n_2)x_i^\pm(n_3)x_j^\pm(k) \right. \)
\( - (r_i^{\pm 2}+(r_is_i)^{\pm 1}(+)s_i^{\pm 2})x_i^\pm(n_1)x_i^\pm(n_2)x_j^\pm(k)x_i^\pm(n_3) \)
\( + (r_is_i)^{\pm 1}(r_i^{\pm 2}+(r_is_i)^{\pm 1}(+)s_i^{\pm 2})x_i^\pm(n_1)x_j^\pm(k)x_i^\pm(n_2)x_i^\pm(n_3) \)
\( - (r_is_i)^{\pm 3}x_j^\pm(k)x_i^\pm(n_1)x_i^\pm(n_2)x_i^\pm(n_3) \right) = 0, \quad \text{for } a_{ij} = -2. \)

Remark 3.2. Notice that the defining relations (D7), (D8) can be written equivalently by virtue of generating functions (see [HZ]) as follows:

(D7') \( (z-(\langle i,j \rangle^{\pm \frac{1}{2}}w)x_i^\pm(z)x_j^\pm(w) = (\langle j,i \rangle^{\pm 1}z-(\langle j,i \rangle^{-1}^{\mp \frac{1}{2}}w)x_j^\pm(w)x_i^\pm(z), \)
(D8') \[ x_i^\pm(z), x_j^\pm(w) = \frac{\delta_{ij}}{(r_i-s_i)zw}(\delta(\frac{w\gamma_{i,j}^{-1}}{z})\psi_i(w\gamma_{i,j}^{-\frac{1}{2}}) - \delta(\frac{w\gamma_{i,j}^{-1}}{z})\phi_i(w\gamma_{i,j}^{-\frac{1}{2}})), \]
where \( \delta(z) = \sum_{n \in \mathbb{Z}} x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_i^\pm(n)z^{-n}, \psi_i(z) = \sum_{m \in \mathbb{Z}_+} \psi_i(m)z^{-m}, \) and \( \phi_i(z) = \sum_{n \in -\mathbb{Z}_+} \phi_i(n)z^{-n}. \)

As one of crucial observations of considering the compatibilities of the defining system above, we have

Proposition 3.3. There exists the \( \mathbb{Q} \)-algebra antiautomorphism \( \tau \) of \( U_{r,s}(C_n^{(1)}) \) such that \( \tau(r) = s, \tau(s) = r, \tau(\langle \omega_i',\omega_j \rangle^{\pm 1}) = \langle \omega_j',\omega_i \rangle^{\mp 1} \) and

\[ \tau(\omega_i) = \omega_i', \quad \tau(\omega_j') = \omega_i, \quad \tau(\gamma) = \gamma', \quad \tau(\gamma') = \gamma, \quad \tau(D) = D', \quad \tau(D') = D, \]
\[ \tau(x_i^\pm(m)) = x_j^\mp(-m), \quad \tau(a_i(\ell)) = a_i(-\ell), \]
\[ \tau(\psi_i(m)) = \phi_i(-m), \quad \tau(\phi_i(-m)) = \psi_i(m), \]
and \( \tau \) preserves each defining relation (Dn) in Definition 3.1 for \( n = 1, \ldots, 9. \)

Remark 3.4. The defining relations (D1)—(D9) ensure that \( U_{r,s}(C_n^{(1)}) \) has a triangular decomposition:

\[ U_{r,s}(C_n^{(1)}) = U_{r,s}(\mathbb{N}^+) \bigotimes U_{r,s}^{0}(C_n^{(1)}) \bigotimes U_{r,s}(\overline{n}), \]
where \( U_{r,s}(\mathbb{N}^+) = \bigoplus_{\alpha \in Q^+} U_{r,s}(\mathbb{N}^+) \) is generated respectively by \( x_i^\pm(k) \) \((i \in I)\), and \( U_{r,s}^{0}(C_n^{(1)}) \) is the subalgebra generated by \( \omega_i^{\pm 1}, \omega_i^{\pm \frac{1}{2}}, \gamma^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1} \) and \( a_i(\pm \ell) \) for \( i \in I, \ell \in \mathbb{N}. \) Namely, \( U_{r,s}^{0}(C_n^{(1)}) \) is generated by the toral subalgebra \( U_{r,s}(C_n^{(1)})^0 \) and the quantum Heisenberg subalgebra \( H_{r,s}(C_n^{(1)}) \) generated by those quantum imaginary root vectors \( a_i(\pm \ell) \) \((i \in I, \ell \in \mathbb{N}). \)

3.2. Drinfeld Isomorphism. To obtain the isomorphism between the above two realizations for the two-parameter quantum affine algebra \( U_{r,s}(C_n^{(1)}) \), we need the following notations and definitions.
**Definition 3.5.** ([HRZ]) The quantum Lie brackets \([a_1, \ldots, a_s]_{(q_1, \ldots, q_{s-1})}\) are defined recursively by

\[
[a_1, a_2]_{v_1} = a_1a_2 - v_1 a_2a_1,
\]

\[
[a_1, a_2, \ldots, a_s]_{(v_1, v_2, \ldots, v_{s-1})} = [a_1, \ldots, a_{s-1}, a_s]_{v_1} (v_2, \ldots, v_{s-1}),
\]

\[
[a_1, a_2, \ldots, a_s]_{(v_1, v_2, \ldots, v_{s-1})} = ([a_1, a_2]_{v_1} \cdots, a_{s-1}) (v_2, \ldots, v_{s-2}),
\]

for \(q_i \in \mathbb{K}^* = \mathbb{K}\{0\} \).

For \(q \in \mathbb{K}^*\), the following identities follow from the definition

\[
[a, bc]_v = [a, b]_q c + q b [a, c]_{\frac{q}{q}}, \tag{3.1}
\]

\[
[ab, c]_v = a [b, c]_q + q [a, c]_{\frac{q}{q}} b, \tag{3.2}
\]

\[
[a, [b, c]_u]_v = [[a, b]_q, c]_{\frac{q}{q}} + q [b, [a, c]_{\frac{q}{q}}]_{\frac{q}{q}}, \tag{3.3}
\]

\[
[[a, b], c]_u]_v = [a, [b, c]_q]_{\frac{q}{q}} + q [a, [c, b]_{\frac{q}{q}}]_{\frac{q}{q}}. \tag{3.4}
\]

In particular, we have (see [HRZ, HZ])

\[
[a, [b_1, \ldots, b_s]_{(v_1, \ldots, v_{s-1})}] = \sum_i [b_1, \ldots, [a, b_i], \ldots, b_s]_{(v_1, \ldots, v_{s-1})}, \tag{3.5}
\]

\[
[a, a, b]_{(u, v)} = a^2 b - (u+v) aba + (uv) ba^2 = (uv)[b, a, a]_{(u-1, v-1)}, \tag{3.6}
\]

where \([n]_{u,v} = \frac{q^{u+v} - q^{-1}}{q - q^{-1}}, [n]_{u,v}! := [n]_{u,v} \cdots [2]_{u,v}[1]_{u,v}, [\frac{n}{m}]_{u,v} := \frac{[n]_{u,v}}{[m]_{u,v} [n-m]_{u,v}}. \tag{3.6}
\]

**Definition 3.6.** ([HRZ]) For \(\alpha, \beta \in \dot{Q}^+\) (the positive root lattice of \(C_n\)), and \(x_\alpha^\pm (k), x_\beta^\pm (k') \in U_s \mathfrak{h}(\hat{\mathfrak{g}}^\pm)\), define the affine quantum Lie bracket as follows:

\[
[x_\alpha^\pm (k), x_\beta^\pm (k')]_{(\omega^\alpha, \omega^\beta)^{\pm 1}} := x_\alpha^\pm (k) x_\beta^\pm (k') - (\omega^\alpha, \omega^\beta)^{\pm 1} x_\beta^\pm (k') x_\alpha^\pm (k).
\]

By the definition above, formula (D7) will take the convenient form as

\[
[x_i^\pm (k), x_j^\pm (k'+1)]_{(i, i)^{\pm 1}} = -((i, i)^{-1})^{\pm 1} \frac{q}{r} [x_i^\pm (k'), x_i^\pm (k+1)]_{(i, i)^{\pm 1}}. \tag{3.7}
\]

Especially, we have

\[
[x_2^0 (0), x_1^{-1} (1)]_{\frac{r}{s}} = - (rs)^{\frac{1}{2}} [x_1^{-1} (0), x_2^0 (1)]_{\frac{r}{s}}, \tag{3.8}
\]

By (3.6) & (3.8), the \((r, s)\)-Serre relations (D9\_1), (D9\_2) & (D9\_3) for \(n_i = n_j = \ell \) in the case of \(a_{ij} = -1\) and \(a_{n-1,n} = -2\) can be reformulated respectively as:

\[
[x_i^\pm (\ell), x_j^\pm (\ell), x_i^\pm (k+1)]_{(s_{i,i}^1, r_{i,i}^1, s_{i,i}^2)} = 0, \tag{3.9}
\]

\[
[x_i^\pm (\ell), x_j^\pm (\ell), x_i^\pm (k)]_{(s_{i,i}^1, r_{i,i}^1)} = 0, \tag{3.10}
\]

\[
[x_n-i^{-1}(\ell), x_n-i^{-1}(\ell), x_n^{-1}(\ell), x_n^0(k)]_{(s_{n-i-1}^2, s_{n-i-1}^1, s_{n-i}^1, s_{n-i}^2)} = 0. \tag{3.11}
\]

For \(2 \leq i \leq n - 1\), let us set some notations for later use,

\[
x_{\alpha_i+1}^{-}(1) = [x_i^{-} (0), \ldots, x_2^{-} (0), x_1^{-} (1)]_{(s_{i,i}, s_{i,i}^1, s_{i,i}^2)} \tag{3.12}
\]

\[
x_{\alpha_i-1}^{-}(1) = [x_n^{-} (0), \ldots, x_2^{-} (0), x_1^{-} (1)]_{(s_{i,i}^1, s_{i,i}^2)} \tag{3.13}
\]
\[ x^{-}_{\beta_1,1}(1) = \left\{ x^{-}_{1}(0), x^{-}_{2}(0), \ldots, x^{-}_{n}(0), x^{-}_{1}(1) \right\}_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

\[ x^{-}_{\beta_1,1}(1) = \left\{ x^{-}_{1}(0), x^{-}_{2}(0), \ldots, x^{-}_{n}(0), x^{-}_{1}(0), x^{-}_{1}(1) \right\}_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

**Remark 3.7.** In particular, we denote that,

\[ x_{\beta}^{-}(-1) := x_{\beta_1,1}^{-}(-1) \]

\[ x_{\beta}^{-}(1) := x_{\beta_1,1}^{-}(1) \]

The following two lemmas will be used in the proof of the main theorem of this section.

**Lemma 3.8.** Using the above notations, one has,

\[ [x^{-}_{i}(0), x^{-}_{\alpha_{i,1}}(1)]_{r_{\alpha_{i,1}}} = 0, \text{ for } 2 \leq i \leq n - 1; \]

\[ [x^{-}_{i}(0), x^{-}_{\alpha_{i,i+1}}(1)] = 0, \text{ for } 2 \leq i \leq n. \]

**Proof:** Using the above notations and (3.3), one has

\[ [x^{-}_{i}(0), x^{-}_{\alpha_{i,1}}(1)]_{r_{\alpha_{i,1}}} \]

(by definition)

\[ = [x^{-}_{i}(0), [x^{-}_{i}(0), x^{-}_{i-1}(0), x^{-}_{\alpha_{i,i-1}}(1)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{r_{\alpha_{i,1}}} \]

(using (3.3))

\[ = [x^{-}_{i}(0), [x^{-}_{i}(0), x^{-}_{i-1}(0)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

\[ + [x^{-}_{i}(0), x^{-}_{i-1}(0), [x^{-}_{i}(0), x^{-}_{\alpha_{i,i-2}}(1)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

\[ = 0 \text{ by (3.3)} \]

\[ [x^{-}_{i}(0), x^{-}_{\alpha_{i,i+1}}(1)]_{r^{-}, s^{-}, \ldots, r^{-}, s^{-}} \]

(by definition)

\[ = [x^{-}_{i}(0), [x^{-}_{i+1}(0), x^{-}_{i}(0), x^{-}_{\alpha_{i,i-1}}(1)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{r^{-}, s^{-}, \ldots, r^{-}, s^{-}} \]

(using (3.3))

\[ = [x^{-}_{i}(0), [x^{-}_{i+1}(0), x^{-}_{i}(0)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

\[ + [x^{-}_{i}(0), [x^{-}_{i+1}(0), x^{-}_{\alpha_{i,i-1}}(1)]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}}]_{s^{-}, \ldots, s^{-}, r^{-}, \ldots, r^{-}, (rs)^{-}} \]

\[ = 0 \text{ by (3.3)} \]
\[ [x_i^{-}(0), x_{\beta_{i+1}}^{-}(1)]_{s_{-}\frac{1}{2}} = 0; \]
\[ [x_i^{-}(0), x_{\beta_{1,i+2}}^{-}(1)] = 0; \quad \text{for} \quad 2 \leq i \leq n - 2; \]
\[ [x_i^{-}(0), x_{\beta_{i}}^{-}(1)]_{s_{-}\frac{1}{2}} = 0; \quad \text{for} \quad 2 \leq i \leq n - 1; \]
\[ [x_n^{-}(0), x_{\alpha_{n}}^{-}(1)]_{r} = 0. \]

**Proof.** (3.18): It is easy to obtain that

\[
\begin{align*}
[x_{i}^{-}(0), x_{\beta_{1,i+1}}^{-}(1)]_{s_{-}\frac{1}{2}} &= [x_{i}^{-}(0), x_{\beta_{1,i+2}}^{-}(1)] \quad \text{(by definition)} \\
&= [x_{i}^{-}(0), x_{\beta_{1,i+2}}^{-}(1)] \quad \text{for} \quad 2 \leq i \leq n - 2; \\
&= [x_{n}^{-}(0), x_{\alpha_{n}}^{-}(1)]_{r} \quad \text{for} \quad 2 \leq i \leq n - 1; \\
&= 0. \quad \text{by (3.10)}
\end{align*}
\]

(3.19): One has

\[
\begin{align*}
[x_{i}^{-}(0), x_{\beta_{i+1}}^{-}(1)] &= [x_{i}^{-}(0), x_{\beta_{i+2}}^{-}(1)] \quad \text{(by definition)} \\
&= [x_{i}^{-}(0), x_{\beta_{i+2}}^{-}(1)] \quad \text{(using (3.3))} \\
&= 0. \quad \text{by (3.17)}
\end{align*}
\]
We can get without difficulty:

\[ [x_i^-(0), x_{\alpha_1}^-(1)]_{s+1} \quad \text{(by definition)} \]

\[
= [x_{\alpha_1}^-(0), [x_{\alpha_1}^-(0), x_{\alpha_1}^-(1)]_{s+1}]_{r-s+1} \quad \text{(using (3.3))}
\]

\[
= [x_{\alpha_1}^-(0), [x_{\alpha_1}^- i, x_{\alpha_1}^-(1)]_{r-s} x_{\alpha_1}^-(1)]_{r-s} \quad \text{(using (3.3))}
\]

\[
+ r^{-\frac{1}{2}} [x_{\alpha_1}^-(0), x_{\alpha_1}^- i, x_{\alpha_1}^-(1)]_{r-s} x_{\alpha_1}^-(1)]_{r-s} \quad \text{(=0 by (3.9))}
\]

\[
+ s^{-\frac{1}{2}} [x_{\alpha_1}^-(0), x_{\alpha_1}^- i, x_{\alpha_1}^-(1)]_{r-s} x_{\alpha_1}^-(1)]_{r-s} \quad \text{(=0 by (3.10))}
\]

\[
= 0.
\]

By calculating, we obtain

\[ [x_{\alpha_1}^-(0), x_{\alpha_1}^-(1)]_{r} \quad \text{(by definition)} \]

\[
= [x_{\alpha_1}^-(0), [x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x_{\alpha_1}^- i, x\]

\[
= 0.
\]

Now we turn to give one of our main theorems as follows.

**Theorem 3.10.** For non-simply-laced Lie algebra $C_{n}^{(1)}$, there exists an algebra isomorphism $\Psi : U_{r,s}(C_{n}^{(1)}) \rightarrow U_{r,s}(C_{n}^{(1)})$ defined by: for $i \in I$,

\[
\begin{align*}
\omega_i & \mapsto \omega_i \\
\omega_i' & \mapsto \omega_i' \\
\omega_0 & \mapsto \gamma^{-1} \omega_0^{-1} \\
\omega_0' & \mapsto \gamma^{-1} \omega_0'^{-1} \\
\gamma^{\pm 1} & \mapsto \gamma^{\pm 1} \\
\gamma' & \mapsto \gamma' \\
D^{\pm 1} & \mapsto D^{\pm 1} \\
D' & \mapsto D' \\
e_i & \mapsto x_i^+(0) \\
f_i & \mapsto x_i^+(0) \\
e_0 & \mapsto a x_{\theta}^-(1) \cdot \gamma^{-1} \omega_0^{-1} \\
f_0 & \mapsto \tau(a x_{\theta}^+(1) \cdot \gamma^{-1} \omega_0^{-1}) = a (\gamma^{-1} \omega_0^{-1}) \cdot x_{\theta}^-(1)
\end{align*}
\]
We shall check that elements

Proof of Theorem 3.12. for the simply-laced cases, please see (3.18). Theorems are the same as Theorems B and C in (3.19). We only prove the Theorem 3.13, and the last two relations involving $i$ are left to the reader.

**Remark 3.11.** We note that $\tau(a) = a$.

**3.3. The proof of Theorem 3.10.** In this subsection, we prove Theorem 3.10. Let $E_i, F_i (i \in I_0)$ and $\omega, \omega'$ denote the images of $e_i, f_i (i \in I_0)$ and $\omega, \omega'$ in the algebra $U_{r,s}(C_n^{(1)})$ respectively.

Denote by $U'_{r,s}(C_n^{(1)})$ the subalgebra of $U_{r,s}(C_n^{(1)})$ generated by $E_i, F_i, \omega_i^\pm 1, \omega_i^\pm 1 (i \in I_0), \gamma^\pm \frac{1}{2}, \gamma^\pm \frac{1}{2}, D^{\pm 1}$ and $D'^{\pm 1}$, that is,

$$U'_{r,s}(C_n^{(1)}) := \left\langle E_i, F_i, \omega_i^\pm 1, \omega_i^\pm 1, \gamma^\pm \frac{1}{2}, \gamma^\pm \frac{1}{2}, D^{\pm 1}, D'^{\pm 1} \mid i \in I_0 \right\rangle.$$

**Theorem 3.12.** $\Psi : U_{r,s}(C_n^{(1)}) \rightarrow U'_{r,s}(C_n^{(1)})$ is an epimorphism.

**Theorem 3.13.** $U'_{r,s}(\mathfrak{g}) = U_{r,s}(\mathfrak{g})$.

**Theorem 3.14.** There exists a surjective $\Phi : U'_{r,s}(\mathfrak{g}) \rightarrow U_{r,s}(\mathfrak{g})$ such that $\Psi \Phi = \Phi \Psi = 1$.

Therefore, to prove the Drinfeld Isomorphism Theorem is equivalent to prove the above three Theorems. We only prove the Theorem 3.13, and the last two theorems are the same as Theorems B and C in [HZ], which are left to the reader.

**Proof of Theorem 3.12.** We shall check that elements $E_i, F_i, \omega_i, \omega'_i (i \in I_0), \gamma^\pm \frac{1}{2}, \gamma^\pm \frac{1}{2}, D, D'$ satisfy the defining relations of (3.1) – (3.7) of $U_{r,s}(C_n^{(1)})$. At first, the defining relations of $U_{r,s}(C_n^{(1)})$ imply that $E_i, F_i, \omega_i, \omega'_i (i \in I)$ generate a subalgebra of $U_{r,s}(C_n^{(1)})$ that is isomorphic to $U_{r,s}(C_n)$. So we are left to check the relations involving $i = 0$.

For the proof of the relations of (3.1) – (3.3), it is almost the same as those for the simply-laced cases, please see ([HZ]). Here we only give the proof of the relations of (3.4) – (3.7).

For (3.4): at first, when $i \neq 0$, we see that

$$[ E_0, F_i ] = [ a x^{-1}_0(1) (\tau^{-1} \omega_0^{-1}), x^{-1}_i(0) ] = -a [ x^{-1}_0(0), x^{-1}_i(1) ](\omega_i, \omega_0)^{-1} (\gamma^{-1} \omega_0^{-1}).$$

By the $(r, s)$-Serre relations, we claim

**Lemma 3.15.** $[ x^{-1}_i(0), x^{-1}_0(1) ](\omega_i, \omega_0)^{-1} = 0$, for $i \in I$.

**Proof.** (I) When $i = 1$, $(\omega_1, \omega_0) = s$ and $(\omega'_1, \omega_0) = s^{-1}$: we first notice that

$$x^{-1}_0(1) = x^{-1}_{\tilde{\beta}_{1,3}}(1) = [ x^{-1}_1(0), x^{-1}_2(0) x^{-1}_{\tilde{\beta}_{1,3}}(1) ]_{(r^{-\frac{1}{2}}, (rs)^{-\frac{1}{2}})}$$

$$= [ [ x^{-1}_1(0), x^{-1}_2(0) ]_{r^{-\frac{1}{2}}, (rs)^{-\frac{1}{2}}}, x^{-1}_{\tilde{\beta}_{1,3}}(1) ]_{(rs)^{-\frac{1}{2}}}$$

$$+ r^{-\frac{1}{2}} [ x^{-1}_2(0), [ x^{-1}_1(0), x^{-1}_{\tilde{\beta}_{1,3}}(1) ]_{rs}^{-\frac{1}{2}} ] = 0$$

By (3.18).

$$= [ [ x^{-1}_1(0), x^{-1}_2(0) ]_{r^{-\frac{1}{2}}, (rs)^{-\frac{1}{2}}}, x^{-1}_{\tilde{\beta}_{1,3}}(1) ]_{(rs)^{-\frac{1}{2}}}.$$
As a result, it is no difficult to see that
\[
[x_\bar{1}^- (0), x_\theta^- (1)] = [x_\bar{1}^- (0), [x_\bar{1}^- (0), x_2^- (0)] \cdot x_\bar{1}^- (1)]_{r - \frac{s}{2}, s - 1} \quad \text{(using (3.3))}
\]
\[
= [[x_\bar{1}^- (0), x_\bar{1}^- (0), x_2^- (0)]_{r - \frac{s}{2}, s - 1}, x_\bar{1}^- (1)]_{r - \frac{s}{2}, s - 1} \quad \text{(= 0 by (3.10))}
\]
\[
+ [[x_\bar{1}^- (0), x_2^- (0)]_{r - \frac{s}{2}, s - 1}, [x_\bar{1}^- (0), x_\bar{1}^- (1)]_{r - \frac{s}{2}, s - 1}] = 0 \quad \text{(= 0 by (3.18))}
\]
\[
= 0.
\]

(II) When \(2 \leq i \leq n - 1\), \(\langle \omega'_i, \omega_0 \rangle = 1\), that is, \(\langle \omega'_i, \omega_0 \rangle = 1\). Using (3.3) and the \((r, s)\)-Serre relations, one has
\[
[x_\bar{i}^- (0), x_\theta^- (1)] \quad \text{(by definition)}
\]
\[
= [x_\bar{i}^- (0), [x_\bar{i}^- (0), \cdots, x_{i-2}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, \cdots, r - \frac{s}{2}, (r, s) - \frac{s}{2}}]
\]
\[
= [x_\bar{i}^- (0), x_{i-2}^- (0), [x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, \cdots, r - \frac{s}{2}, (r, s) - \frac{s}{2}}]
\]

Thus, it remains to check that \([x_\bar{i}^- (0), x_{\bar{1}^- (1)}] = 0\).

Actually, it is easy to get
\[
[x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, s - 1} \quad \text{(by definition)}
\]
\[
= [x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, s - 1} \quad \text{(using (3.3))}
\]
\[
= [x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, s - 1} \quad \text{(using (3.3))}
\]
\[
+ r^{\frac{1}{2}} [x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{1 + 1 - \frac{s}{2}, s - 1} \quad \text{(= 0 by (3.19))}
\]
\[
= [[x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, s - 1}, x_\bar{1}^- (1)]_{r - \frac{s}{2}, s - 1} \quad \text{(= 0 by (3.19))}
\]
\[
= s^{\frac{1}{2}}[[x_{\bar{1}^- (1)} - 1, x_\bar{i}^- (0), x_{\bar{1}^- (1)}]_{s - \frac{s}{2}, r - \frac{s}{2}} \quad \text{(using (3.20))}
\]
\[
+ [[x_{\bar{1}^- (1)} - 1, x_\bar{1}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, s - 1} \quad \text{(by definition)}
\]
which implies that \((1 + r^{\frac{1}{2}}) x_{\bar{1}^- (0), x_{\bar{1}^- (1)}] = 0\). Then if \(r \neq -s\), we get the required relation
\[
(3.22) \quad [x_\bar{i}^- (0), x_{\bar{1}^- (1)}] = 0.
\]

(III) When \(i = n\), \(\langle \omega'_n, \omega_0 \rangle = (r, s)^{-1}\), that is, \(\langle \omega'_n, \omega_0 \rangle = rs\). Applying (3.3) and the \((r, s)\)-Serre relations, we deduce from direct calculations
\[
[x_n^- (0), x_\theta^- (1)]_{rs} \quad \text{(by definition)}
\]
\[
= [x_n^- (0), [x_n^- (0), \cdots, x_{n-2}^- (0), x_{\bar{1}^- (1)}]_{r - \frac{s}{2}, \cdots, r - \frac{s}{2}, (r, s) - \frac{s}{2}}]_{rs}
\]
\[
= [x_n^- (0), x_{n-2}^- (0), [x_n^- (0), x_{\bar{1}^- (1)}]_{rs} \quad \text{(= 0 by (3.20))}
\]
It remains to show the relation \([x_n^- (0), x_{\bar{1}^- (1)}]_{rs} = 0\).
In fact,

\[
[x^n_\alpha(0), x^{-}_{\alpha - 1 n - 1}(1)]_{rs} \quad \text{(by definition)}
\]

\[
= [x^n_\alpha(0), [x^{-}_{n - 1}(0), x^n_\alpha(0), x^{-}_{\alpha - 1 n - 1}(1)]_{(s, r^{-1})}]_{rs} \quad \text{(using (3.7))}
\]

\[
= [x^n_\alpha(0), [x^{-}_{n - 1}(0), x^n_\alpha(0)]_{r^{-1}}, x^{-}_{\alpha - 1 n - 1}(1)]_{(r^{-1}, s, r^{-2})} \quad \text{(using (3.7))}
\]

\[
+ r^{-1}[x^n_\alpha(0), x^{-}_{n - 1}(0), [x^{-}_{n - 1}(0), x^{-}_{\alpha - 1 n - 1}(1)]_{r^{-2}}] \quad (= 0 \text{ by (3.14))}
\]

\[
= \left[[x^n_\alpha(0), [x^{-}_{n - 1}(0), x^n_\alpha(0)]_{r^{-1}}, x^{-}_{\alpha - 1 n - 1}(1)]_{r^{-1}}\right]_{rs} \quad (= 0 \text{ by (D92))}
\]

\[
+ s[x^{-}_{n - 1}(0), [x^{-}_{n - 1}(0), x^{-}_{\alpha - 1 n - 1}(1)]_{s}, x^{-}_{\alpha - 1 n - 1}(1)]_{r^{-2}} \quad \text{(by definition & (3.21))}
\]

\[
= s[x^{-}_{n - 1}(0), [x^{-}_{n - 1}(0), x^{-}_{\alpha - 1 n - 1}(1)]_r]_{r^{-2}} \quad (= 0 \text{ by (3.21))}
\]

\[
+ rs[x^{-}_{n - 1}(0), x^{-}_{\alpha - 1 n - 1}(1)]_{r^{-2}}, x^{-}_{n - 1}(0)]_{r^{-2}} \quad \text{(by definition)}
\]

\[
= rs[x^{-}_{\alpha - 1 n - 1}(1), x^{-}_{n - 1}(0)]_{r^{-2}}.
\]

Expanding the two sides of the above result, one has

\[
(1 + r^{-2}s^2)[x^n_\alpha(0), x^{-}_{\alpha - 1 n - 1}(1)]_{rs} = 0.
\]

Therefore, if \( r \neq s \), \( [x^n_\alpha(0), x^{-}_{\alpha - 1 n - 1}(1)]_{rs} = 0 \).

This complete the proof of Lemma 3.15.

Next, we turn to check the relation

**Lemma 3.16.** \( [E_0, F_0] = \frac{\omega - \omega^0}{r-s} \).

**Proof.** Using (D1) and (D5), one has

\[
\begin{align*}
[E_0, F_0] &= (rs)^{-\frac{n-2}{2}}[2]^{-1}_1[x_{\theta}^{-}](1) \gamma^{-1} \omega^{-1} \gamma^{-1} \omega^0^{-1} x_{\theta}^{+}(-1) \\
&= (rs)^{-\frac{n-2}{2}}[2]^{-1}_1[x_{\theta}^{-}](1) x_{\theta}^{+}(-1) \cdot (\gamma^{-1} \gamma^{-1} \omega^{-1} \omega^0^{-1} \omega^{-1}).
\end{align*}
\]

First recall the notations,

\[
\begin{align*}
x_{\theta}^{-}(1) &= [x_1^{-}(0), \cdots, x_n^{-}(0), x^{-}_{\alpha - 1 n - 1}(1)]_{(s, r^{-1}, \ldots, r^{-\frac{1}{2}}, \ldots, (rs)^{-\frac{1}{2}})}, \\
x_{\theta}^{+}(-1) &= [[[x_{\alpha - 1 n - 1}^{+}(-1), x_{\alpha - 1 n - 1}^{-}(0)]_{r^{-\frac{1}{2}}, \cdots, s^{-\frac{1}{2}}, \cdots, x_{2}^{+}(0)]_{r^{-\frac{1}{2}}, \cdots, x_{1}^{+}(0)}]_{(rs)^{-\frac{1}{2}}}.
\end{align*}
\]

Owing to the result of the case of \( A_{n-1}^{(1)} \), one has

\[
\begin{align*}
[x^{-}_{\alpha - 1 n - 1}(1), x^{+}_{\alpha - 1 n - 1}(-1)] &= \frac{\gamma \omega^0_{\alpha - 1 n - 1} - \gamma \omega_{\alpha - 1 n - 1}}{r^{\frac{1}{2}} - s^{\frac{1}{2}}}.
\end{align*}
\]

\]
Next, it is easy to get
\[
[x_{\alpha_1 n}^{-1}(1), x_{\alpha_1 n}^+(-1)] \quad \text{(by definition)}
\]
\[
= \left[ x_{\alpha_1 n}^{-1}(0), x_{\alpha_1 n}^{-1}(-1), x_{\alpha_1 n}^+(0) \right] \quad \text{(using (3.3))}
\]
\[
= \left[ \left[ x_{\alpha_1 n}^{-1}(0), x_{\alpha_1 n}^{-1}(-1), x_{\alpha_1 n}^{-1}(1) \right], x_{\alpha_1 n}^+(0) \right] \quad \text{(using (3.3) & (D8))}
\]
\[
+ \left[ [x_{\alpha_1 n}^{-1}(0), x_{\alpha_1 n}^{-1}(-1), x_{\alpha_1 n}^{-1}(1)] s, x_{\alpha_1 n}^+ (0) \right] \quad \text{(using (3.24), (D5), (D8))}
\]
\[
+ \left[ x_{\alpha_1 n}^+(1), [x_{\alpha_1 n}^+ (0), x_{\alpha_1 n}^{-1}(1)], x_{\alpha_1 n}^{-1}(1) \right] s, \quad \text{(using (D8), (D5), (3.24))}
\]
\[
= \frac{\gamma' \omega_{\alpha_1 n} - \gamma_{\alpha_1 n} \omega_n}{r^2 - s^2} + \frac{\gamma' \omega_{\alpha_1 n} - \gamma_{\alpha_1 n} \omega_n}{r^2 - s^2}.
\]

Furthermore, we obtain
\[
[x_{\beta_1 n}^{-1}(1), x_{\beta_1 n}^+(-1)] \quad \text{(by definition)}
\]
\[
= \left[ x_{\beta_1 n}^{-1}(0), x_{\beta_1 n}^{-1}(-1), x_{\beta_1 n}^+(0) \right] \quad \text{(using (3.3))}
\]
\[
= \left[ \left[ x_{\beta_1 n}^{-1}(0), x_{\beta_1 n}^{-1}(1) \right], x_{\beta_1 n}^+(0) \right] \quad \text{(using (3.3) & (D5))}
\]
\[
+ \left[ [x_{\beta_1 n}^{-1}(0), x_{\beta_1 n}^{-1}(1)] s, x_{\beta_1 n}^+ (0) \right] \quad \text{(using (D5) & (D8))}
\]
\[
+ \left[ x_{\beta_1 n}^+(1), [x_{\beta_1 n}^+ (0), x_{\beta_1 n}^{-1}(1)], x_{\beta_1 n}^{-1}(1) \right] s, \quad \text{(using (D8) & (D5))}
\]
\[
= \frac{(rs)^{\frac{1}{2}} \gamma' \omega_{\beta_1 n}, \omega_{\beta_1 n} - \omega_{\beta_1 n} (-1)}{r^2 - s^2} + \frac{(rs)^{\frac{1}{2}} \gamma' \omega_{\beta_1 n} - \gamma_{\beta_1 n} \omega_n}{r^2 - s^2}.
\]

Repeating the above step, we get by directly calculating
\[
[3.25] \quad x_{\beta_1 n}^{-1}(1), x_{\beta_1 n}^+(-1) = (rs)^{\frac{2\omega_n}{r^2 - s^2}} \frac{\gamma' \omega_{\beta_1 n} - \gamma_{\beta_1 n}}{r^2 - s^2}.
\]

As a result, we get the required conclusion:
\[
[x_{\beta_1 n}^{-1}(1), x_{\beta_1 n}^+(-1)] \quad \text{(by definition)}
\]
\[
= \left[ x_{\beta_1 n}^{-1}(0), x_{\beta_1 n}^{-1}(1) \right], x_{\beta_1 n}^+(0) \right] \quad \text{(using (3.3))}
\]
\[
+ \left[ [x_{\beta_1 n}^{-1}(0), x_{\beta_1 n}^{-1}(1)] s, x_{\beta_1 n}^+ (0) \right], \quad \text{(using (3.3), (D8) & (D5))}
\]
\[
+ \left[ x_{\beta_1 n}^+(1), [x_{\beta_1 n}^+ (0), x_{\beta_1 n}^{-1}(1)], x_{\beta_1 n}^{-1}(1) \right] s, \quad \text{(using (D8) & (D5))}
\]
\[
= \frac{(rs)^{\frac{1}{2}} \gamma' \omega_{\beta_1 n}, \omega_{1} - \omega_{1} (-1)}{r^2 - s^2} + \frac{(rs)^{\frac{1}{2}} \gamma' \omega_{\beta_1 n} - \gamma_{\beta_1 n}}{r^2 - s^2}.
\]
Thus, we arrive at the last step

\[ [E_0, F_0] = \frac{\gamma^{-1} \omega^{-1}_\theta - \gamma^{-1} \omega^{-1}_\theta}{r - s}. \]

The proof is complete. \(\square\)

For (C6): when \(i \cdot j \neq 0\), (D9) implies that the corresponding generators satisfy exactly those \((r, s)\)-Serre relations in \(U_{r,s}(\mathbb{C}_n^{(1)})\), so it is enough to check the \((r, s)\)-Serre relations involving \(i \cdot j = 0\).

**Lemma 3.17.**

1. \(E_n E_0 = rs E_0 E_n\),
2. \(E_0^2 E_1 - (r + s) E_0 E_1 + rs E_1 E_0^2 = 0\),
3. \(E_0 E_0 (r + (r + s) \frac{1}{2} + s) E_1 E_0 E_1^2 + (r + (r + s) \frac{1}{2} + s) E_0^2 E_0 E_1 - (r + s) \frac{3}{2} E_1 E_0 = 0\),
4. \(F_0 F_n = rs F_n F_0\),
5. \(F_1 F_1^2 - (r + s) F_1 F_0 + rs F_0^2 = 0\),
6. \(F_1^2 F_0 (r + (r + s) \frac{1}{2} + s) F_1^2 F_0 F_1 + (r + (r + s) \frac{1}{2} + s) F_1 F_0 F_2 - (r + s) \frac{3}{2} F_0 F_1 = 0\).

**Proof.** Here we only check the first and third \((r, s)\)-Serre relations, and the rest are left to the readers.

1. (by definition)

\[ E_n E_0 - rs E_0 E_n = [x_n(0), x_0(1)\gamma^{-1} \omega^{-1}_\theta]_{rs} \quad \text{(using (D5))} \]

= \[ [x_n(0), x_0(1)] \cdot \gamma^{-1} \omega^{-1}_\theta \quad \text{(by definition)} \]

= \[ [x_n(0), \cdots, x_{n-1}(0), x_n(0), x_n(0), x_n(1)]_{(s, r - \frac{1}{2}, \cdots, r - \frac{1}{2}, (rs) - \frac{1}{2})} \]

\[ = [x_n(0), \cdots, [x_{n-1}(0), x_{n-1}(1)]_{(r - \frac{1}{2}, \cdots, r - \frac{1}{2}, (rs) - \frac{1}{2})} \omega_n = 0 \text{ by (3.16)} \]

= 0.

2. At the same time, we consider that

\[ [E_1, x_0(1)] \quad \text{(using (D5))} \]

\[ = [x_1(0), x_0(1), [x_{1}(0), [x_0(1)]_{(r - \frac{1}{2}, \cdots, r - \frac{1}{2}, (rs) - \frac{1}{2})} = - (r + s) \frac{3}{2} \gamma^{-1} \omega^{-1} [x_0(0), x_{32}(1)]_{(r - 1) \omega}. \]

In terms of the above result, it is easy to see that

\[ E_0 E_0^2 (r + (r + s) \frac{1}{2} + s) E_1 E_0 E_1^2 + (r + (r + s) \frac{1}{2} + s) E_0 E_0 (r - 1) \frac{1}{2} E_1 E_0 = (r + s) \frac{3}{2} E_0^2 E_1 - (r + s) \frac{3}{2} E_0 E_1 \]

\[ = (r + s) \frac{3}{2} \left( E_0^2 x_0(1) - (r - 1) \frac{1}{2} + s) E_0^2 x_0(1) \right) + (r - 1) \frac{1}{2} + s) E_1 x_0(1) E_0^2 - (r - 1) \frac{1}{2} x_0(1) E_0^3 \left(\gamma^{-1} \omega^{-1}_\theta \right) \]

\[ = (r + s) \frac{3}{2} \left[ \left[ E_1, E_0^2 \right] (E_0^2, x_0(1)) \right]_{(r - 1) \frac{1}{2} + s) \omega \left(\gamma^{-1} \omega^{-1}_\theta \right) \]

\[ = - (r + s) \frac{3}{2} [2] (r - 1) \omega \left(\gamma^{-1} \omega^{-1}_\theta \right) \quad \text{by (3.13) \& (D8)} \]

\[ = 0. \]
The proof is complete. □

For (C7): the verification is analogous to that of (C6). Hence, we establish the Drinfeld isomorphism for the two-parameter quantum affine algebra of type $C^{(1)}$.

4. Vertex representation

4.1. Enlarged quantum Heisenberg algebra. In order to obtain the vertex representation of $U_{r,s}(C_n^{(1)})$, first we need the following construction.

The associative algebra $U_{r,s}(\hat{h})$, generated by $\{ a_i(m), \gamma^\pm \}, \gamma^\pm, \gamma_{ij}^\pm \mid m \in \mathbb{Z}\setminus\{0\}$, $i = 1, 2 \cdots, n$, is called the two-parameter quantum Heisenberg algebra, where $a_i(m)$ ($i = 1, 2, \cdots, n$) and $\gamma_{ij}^\pm, \gamma_{ij}^{\pm 1}$ satisfy (D3), as well as the following

\[
\begin{align*}
[a_i(m), a_j(l)] & = \delta_{m+l,0} \frac{(\gamma \gamma')^{(rs-m|\alpha_i|\alpha_j)} - m|\alpha_i|\alpha_j}{m} \cdot \frac{\gamma^{|m|} - \gamma'^{|m|}}{r-s}, \\
(b_i(m), b_j(l)) & = [a_i(m), a_j(l)], \\
(a_i(m), b_j(l)) & = 0, \\
[a_i(0), a_j(0)] & = [b_i(0), b_j(0)] = 0.
\end{align*}
\]

Define an enlarged quantum Heisenberg algebra $U_{r,s}(\hat{b})$, which is an associative algebra generated by the subalgebra $U_{r,s}(\hat{h})$, together with a coupled bosonic operators $b(m)$ ($m \in \mathbb{Z}^*$) satisfying the following relations:

\[
[\tilde{b}_i(m), \tilde{b}_j(l)] = [a_i(m), a_j(l)],
\]

(4.2)

4.2. Fock space. In order to construct the Fock space, one can take a copy of the root lattice of $A_{n-1}$ as the sublattice $\tilde{Q} = Q[A_{n-1}]$ of short roots of the root lattice $Q$ for type $C_n$. The basis of $\tilde{Q}$ will be denoted by $\tilde{a}_i, i = 1, \cdots, n-1$. Thus

\[
(\tilde{a}_i|\tilde{a}_j) = (a_i|a_j) = \delta_{ij} - \frac{1}{2}\delta_{|i|-|j|,1}.
\]

We also consider the associated weight lattice $\tilde{P} = P[A_{n-1}]$ defined by the inner product.

The Fock space $\mathcal{F}$ is defined to be the tensor product of the symmetric algebra generated by $a_i(-m)$’s, $b_i(-m)$’s ($m \neq 0$) and the group algebra generated by $e^\lambda \otimes e^\lambda$ such that $(a_i|\lambda) \pm (\tilde{a}_i|\tilde{\lambda}) \in \mathbb{Z}$ for each $i \in \{1, \cdots, n\}$, where $\lambda \in P$ and $\tilde{\lambda} \in \tilde{P}$. Note that we treat $\tilde{a}_n = 0$.

The actions of $a_i(m)$’s and $b_i(m)$’s with $m \neq 0$ on $\mathcal{F}$ are obtained by viewing the Fock space $\mathcal{F}$ as some quotient space of the Heisenberg algebra tensored with the group algebras of $P$ and $\tilde{P}$. The operators $a_i(0), b_i(0), e^\alpha, e^{\tilde{\alpha}}$ act on $\mathcal{F}$ by

\[
\begin{align*}
a_i(0)e^\lambda e^{\tilde{\lambda}} & = (a_i|\lambda)e^\lambda e^{\tilde{\lambda}}, & b_i(0)e^\lambda e^{\tilde{\lambda}} & = (\tilde{a}_i|\tilde{\lambda})e^\lambda e^{\tilde{\lambda}}, \\
e^\alpha e^\lambda e^{\tilde{\lambda}} & = e^{\alpha + \lambda}e^{\tilde{\lambda}}, & e^{\tilde{\alpha}}e^\lambda e^{\tilde{\lambda}} & = e^\lambda e^{\tilde{\alpha} + \lambda}, \\
\omega_i \cdot (v \otimes e^\beta) & = (\beta, i)v \otimes e^\beta, & \omega_i' \cdot (v \otimes e^\beta) & = (i, \beta)^{-1}v \otimes e^\beta
\end{align*}
\]

and define

\[
D(r)(v \otimes e^\beta) = r^\beta v \otimes e^\beta, \quad D(s)(v \otimes e^\beta) = s^\beta v \otimes e^\beta.
\]

It is easy to see that $a_i(m), b_j(l), e^\alpha, e^{\tilde{\alpha}}$ commute with each other except that

\[
[a_i(0), e^{\alpha_j}] = (a_i|\alpha_j)e^{\alpha_j}, \quad [b_i(0), e^{\tilde{\alpha}_j}] = (\tilde{a}_i|\tilde{\alpha}_j)e^{\tilde{\alpha}_j}.
\]
4.3. **Normal ordering.** The normal product : : is defined as usual:

\[ a_i(m)a_j(l) := a_i(m)a_j(l) \quad \text{(if } m \leq l) \text{, or } a_j(l)a_i(m) \quad \text{(if } m > l) \text{,} \]

\[ e^\alpha a_i(0) := a_i(0)e^\alpha := e^\alpha a_i(0) \text{,} \]

\[ e^{\delta \beta} b_i(0) := b_i(0)e^{\delta \beta} := e^{\delta \beta} b_i(0) \text{,} \]

and similarly for product involving the \( b_i(m) \).

4.4. **Quasi-cocycle.** Let \( \varepsilon(\ ,\ ) : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{K} \) be the quasi-cocycle such that

\[ \varepsilon(\alpha, \beta + \theta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \theta), \]

\[ \varepsilon(\alpha + \beta, \theta) = \varepsilon(\alpha, \theta)\varepsilon(\beta, \theta)(-1)^{(\alpha + \beta - \beta - \theta)}. \]

where the \( - \) is the projection from \( \mathbb{P} \) to \( \tilde{\mathbb{P}} \) defined by

\[ \alpha = \sum_{i=1}^{n} m_i\alpha_i \in \mathbb{P} \quad \text{and} \quad m_i = m_i \mod 2. \]

We construct such a quasi-cocycle directly by

\[ \varepsilon(\alpha_i, \alpha_j) = \begin{cases} ( -1 )^{\alpha_i \alpha_j} ( r_i s_j )^{\frac{\alpha_j}{r_i}} & \text{if } \alpha_i + \alpha_j \in \Phi, i > j; \\ - ( r_i s_j )^{\frac{1}{2}} & \text{if } i = j; \\ 1 & \text{if } \text{other pairs (}i, j)). \end{cases} \]

It is easy to verify that the quasi-cocycle satisfies all the defining relations. In particular, we have

\[ \varepsilon(\alpha_i, \alpha_j) \varepsilon(\alpha_j, \alpha_i) = \begin{cases} ( -1 )^{2(\alpha_i \alpha_j)} ( r_i s_j )^{\frac{\alpha_i \alpha_j}{r_i s_j}} & \text{if } 1 \leq i, j \leq n - 1; \\ -(r_s)^{\frac{1}{2}} & \text{otherwise.} \end{cases} \]

For \( \alpha \in \mathbb{P} \), we define the operators \( \varepsilon_{\alpha} \) on \( \mathcal{V} \) such that

\[ \varepsilon_{\alpha} e^{\lambda} e^{\lambda} = \varepsilon(\alpha, \lambda) e^{\lambda} e^{\lambda}. \]

For simplicity, we denote \( \varepsilon_i = \varepsilon_{\alpha_i} \), for \( i = 1, \cdots, n \).

4.5. **Vertex operators.** We can now introduce the main vertex operators:

\[ Y_i^\pm(z) = \exp(\pm \sum_{k=1}^{\infty} \frac{a_i(-k)}{k} s^{\pm \frac{1}{2}} z^k) \times \exp(\pm \sum_{k=1}^{\infty} \frac{b_i(k)}{k} r^{\pm \frac{1}{2}} z^{-k} e^{\pm \alpha_i \alpha_i(0) \varepsilon_i}, \]

\[ U_j^+(z) = \exp(\sum_{k=1}^{\infty} \frac{b_j(-k)}{k} z^k) \exp(-\sum_{k=1}^{\infty} \frac{b_j(k)}{k} z^{-k} e^{\delta_j \beta_j(0) (r s)^{-\frac{1}{2}}}, \]

\[ U_j^-(z) = \exp(-\sum_{k=1}^{\infty} \frac{b_j(-k)}{k} z^k) \exp(\sum_{k=1}^{\infty} \frac{b_j(k)}{k} z^{-k} e^{-\delta_j \beta_j(0) (r s)^{-\frac{1}{2}}}, \]

\[ Z_j^+(z) = U_j^+(s^{1/2} z) + (-1)^{2\alpha_j(0)} U_j^-(r^{1/2} z), \]

\[ Z_j^-(z) = U_j^+(r^{1/2} z) + (-1)^{2\alpha_j(0)} U_j^-(s^{-1/2} z), \]

where \( i \in \{1, \cdots, n\}, j \in \{1, \cdots, n - 1\} \). For simplicity, we define \( Z_n^\pm(z) = 1 \).
4.6. Vertex representation of $U_{r,s}(C_n^{(1)})$.

**Theorem 4.1.** The Fock space $\mathcal{V}$ is a $U_{r,s}(C_n^{(1)})$-module of level 1 under the action defined by

\[
\begin{align*}
\gamma^\pm \Psi & \mapsto r^\pm \Psi, & \gamma'^\pm \Psi & \mapsto s^\pm \Psi, \\
D & \mapsto D(r), & D' & \mapsto D(s), \\
\omega_i & \mapsto \omega_i, & \omega'_i & \mapsto \omega'_i, \\
a_i(m) & \mapsto a_i(m), \\
x_i^{\pm}(z) & \mapsto Y_i^{\pm}(z), & i = 1, \cdots, n.
\end{align*}
\]

**Proof of Theorem 4.1.** In what follows, we check Theorem 4.1 using the similar but more detailed techniques than in the one-parameter setting (cf. [JKM2]). Let $X_i^{\pm}(z)$ denote the images of $x_i^{\pm}(z)$ ($i \in \{1, 2, \cdots, n\}$) in the algebra $U_{r,s}(C_n^{(1)})$ under the mapping $\pi$, respectively. Therefore we have to check $X_i^{\pm}(z)$, $\Psi_j(z)$ and $\Phi_j(z)$ satisfy relations (D1)—(D9). Firstly, it is clear that relations (D1)—(D6) are true by the construction, which can be verified similarly to [HZ]. We are left to show relations (D7)—(D9).

Firstly, we give the notation formally, for $a \in \mathbb{K}$,

(4.5) \[(1 - z)^{\frac{a}{r,s}} := \exp(- \sum_{n \geq 1} \frac{[an]}{n[n]} z^n).\]

In particular,

(4.6) \[(1 - z)^{\frac{1}{r,s}} = (1 - z),\]

(4.7) \[(1 - z)^{\frac{1}{r,s}} = \left( (1 - (rs)^{-1} z) \right)^{-1}.\]

**Lemma 4.2.** With the above notations, one has

(4.8) \[(1 - (rs)^{\frac{1}{r,s}} z)^{\frac{1}{r,s}} (1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}}) \frac{1}{r,s} = (1 - (r^{-1}s)^{\frac{1}{r,s}} z)^{-1};\]

(4.9) \[(1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z) \frac{1}{r,s} (1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}}) \frac{1}{r,s} = \frac{1 - (rs)^{\frac{1}{r,s}} z}{1 - (r^{-1}s)^{\frac{1}{r,s}} z};\]

(4.10) \[(1 - (rs)^{\frac{1}{r,s}} a z) \frac{1}{r,s} (1 - (rs)^{\frac{1}{r,s}} a z) \frac{1}{r,s} = 1;\]

(4.11) \[(1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z) \frac{1}{r,s} (1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z) \frac{1}{r,s} = (1 - (rs)^{-1} s^{-1} z);\]

(4.12) \[(1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z) \frac{1}{r,s} (1 - (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z) \frac{1}{r,s} = (1 - (rs)^{-1} r^{-1} z).\]

**Proof.** (4.8): using the notation (4.5), one directly gets

\[
\begin{align*}
LHS &= \exp\left( - \sum_{n \geq 1} \frac{[-\frac{1}{n}]}{n[n]} (rs)^{\frac{1}{r,s}} z^n \right) \cdot \exp\left( - \sum_{n \geq 1} \frac{[-\frac{1}{n}]}{n[n]} (rs)^{\frac{1}{r,s}} (r^{-1}s)^{\frac{1}{r,s}} z^n \right) \\
&= \exp\left( - \sum_{n \geq 1} \frac{[-\frac{1}{n}]}{n[n]} (rs)^{\frac{1}{r,s}} (1 + (r^{-1}s)^{\frac{1}{r,s}} z^n) \right) \\
&= \exp\left( \sum_{n \geq 1} \frac{r^{-\frac{1}{r,s}} s^{\frac{1}{r,s}}}{n} z^n \right) = RHS.
\end{align*}
\]

The others are similar, which are left to the readers. \[\Box\]
For the later use, we will write

\[(z - w)_{r,s}^a = (1 - \frac{w}{z})_{r,s}^a \cdot z^a.\]

Before checking relations (D7)–(D9), we list the operator product expansions.

**Lemma 4.3.** With the above notation, the following formulas hold

\[
Y_i^\pm (z) Y_j^\mp (w) := Y_i^\pm (z) Y_j^\pm (w) : \begin{cases} 
1, & (\alpha_i | \alpha_j) = 0; \\
(z - (rs)_{r,s}^i (r^{-1}s)_{r,s}^j w)^{-\frac{1}{2}} \cdot \epsilon(\alpha_i, \alpha_j)^{\pm 1}, & (\alpha_i | \alpha_j) = -\frac{1}{2}; \\
(z - (r^{-1}s)_{r,s}^i \pm \frac{1}{2} w)^{-1} \cdot \epsilon(\alpha_i, \alpha_j)^{\pm 1}, & (\alpha_i | \alpha_j) = -1; \\
(z - w) (z - (r^{-1}s)_{r,s}^\pm w)^{-1} \cdot \epsilon(\alpha_i, \alpha_j)^{\pm 1}, & (\alpha_i | \alpha_j) = 1; \\
\end{cases}
\]

\[(4.13)\]

\[
Y_i^\pm (z) Y_j^\mp (w) := Y_i^\pm (z) Y_j^\mp (w) : \begin{cases} 
1, & (\alpha_i | \alpha_j) = 0; \\
(z - (rs)_{r,s}^i (r^{-1}s)_{r,s}^j w)^{\frac{1}{2}} \cdot \epsilon(\alpha_i, \alpha_j)^{\mp 1}, & (\alpha_i | \alpha_j) = -\frac{1}{2}; \\
(z - (r^{-1}s)_{r,s}^i \pm \frac{1}{2} w)^{1} \cdot \epsilon(\alpha_i, \alpha_j)^{\mp 1}, & (\alpha_i | \alpha_j) = -1; \\
(z - w) (z - (r^{-1}s)_{r,s}^\pm w)^{1} \cdot \epsilon(\alpha_i, \alpha_j)^{\mp 1}, & (\alpha_i | \alpha_j) = 1; \\
\end{cases}
\]

\[(4.14)\]

\[
U_i^\pm (z) U_j^\mp (w) := U_i^\pm (z) U_j^\mp (w) : \begin{cases} 
1, & (\alpha_i | \alpha_j) = 0; \\
(z - (rs)_{r,s}^i w)^{-\frac{1}{2}}, & (\alpha_i | \alpha_j) = -\frac{1}{2}; \\
(z - w)^{-1}, & (\alpha_i | \alpha_j) = -1; \\
(z - w), & (\alpha_i | \alpha_j) = 1; \\
(z - (rs^{-1})_{r,s}^i \mp \frac{1}{2} w) (z - (r^{-1}s)_{r,s}^\pm w), & (\alpha_i | \alpha_j) = 2. \\
\end{cases}
\]

\[(4.15)\]

\[
U_i^\pm (z) U_j^\mp (w) := U_i^\pm (z) U_j^\mp (w) : \begin{cases} 
1, & (\alpha_i | \alpha_j) = 0; \\
(z - (rs)_{r,s}^i w)^{\frac{1}{2}}, & (\alpha_i | \alpha_j) = -\frac{1}{2}; \\
(z - w), & (\alpha_i | \alpha_j) = -1; \\
(z - w)^{-1}, & (\alpha_i | \alpha_j) = 1; \\
(z - (rs^{-1})_{r,s}^i \mp \frac{1}{2} w)^{-1} (z - (r^{-1}s)_{r,s}^\pm w)^{-1}, & (\alpha_i | \alpha_j) = 2. \\
\end{cases}
\]

\[(4.16)\]
PROOF. (4.13) follows from the following

\[ LHS = Y_i^\pm(z)Y_j^\pm(w) \]

\[ = \exp(\pm \sum_{k=1}^{\infty} \frac{a_i(-k)}{|k|} z^k) \times \exp(\mp \sum_{k=1}^{\infty} \frac{a_j(k)}{|k|} \tilde{z}^{-k}) e^{\pm \alpha z^{\pm \alpha_i(0) \varepsilon_i}} \times \exp(\mp \sum_{k=1}^{\infty} \frac{a_j(-k)}{|k|} \tilde{w}^k) e^{\pm \alpha_j z^{\pm \alpha_j(0) \varepsilon_j}} \]

\[ = \epsilon(\alpha_i, \alpha_j) \pm 1 : Y_i^\pm(z)Y_j^\pm(w) : \exp(-\sum_{k=1}^{\infty} \frac{(r-1) \pm \frac{\mp}{k^2}}{a_i(k), a_j(-k)} |(\frac{w}{z})^k| z^{(\alpha_i|\alpha_j)}) \]

\[ = RHS. \]

(4.14) follows from the following

\[ LHS = Y_i^\pm(z)Y_j^\mp(w) \]

\[ = \exp(\pm \sum_{k=1}^{\infty} \frac{a_i(-k)}{|k|} z^k) \times \exp(\mp \sum_{k=1}^{\infty} \frac{a_j(k)}{|k|} \tilde{z}^{-k}) e^{\pm \alpha z^{\pm \alpha_i(0) \varepsilon_i}} \times \exp(\mp \sum_{k=1}^{\infty} \frac{a_j(-k)}{|k|} \tilde{w}^k) e^{\mp \alpha_j z^{\mp \alpha_j(0) \varepsilon_j}} \]

\[ = \epsilon(\alpha_i, \alpha_j) \pm 1 : Y_i^\pm(z)Y_j^\mp(w) : \exp(-\sum_{k=1}^{\infty} \frac{(r) \mp \frac{\pm}{k^2}}{a_i(k), a_j(-k)} |(\frac{w}{z})^k| z^{-(\alpha_i|\alpha_j)}) \]

\[ = RHS. \]

(4.15) follows from the following

\[ LHS = U_i^\pm(z)U_j^\pm(w) \]

\[ = \exp(\pm \sum_{k=1}^{\infty} \frac{b_i(-k)}{|k|} z^k) \times \exp(\mp \sum_{k=1}^{\infty} \frac{b_j(k)}{|k|} \tilde{z}^{-k}) e^{\pm \alpha z^{\pm \alpha_i(0) \varepsilon_i}} \times \exp(\mp \sum_{k=1}^{\infty} \frac{b_j(-k)}{|k|} \tilde{w}^k) e^{\mp \alpha_j z^{\mp \alpha_j(0) \varepsilon_j}} \]

\[ = U_i^\pm(z)U_j^\pm(w) : \exp(-\sum_{k=1}^{\infty} \frac{1}{|k|^2} [b_i(k), b_j(-k)] |(\frac{w}{z})^k| z^{(\alpha_i|\alpha_j)}) = RHS. \]

(4.16) follows from the following

\[ LHS = U_i^\pm(z)U_j^\mp(w) \]

\[ = \exp(\pm \sum_{k=1}^{\infty} \frac{b_i(-k)}{|k|} z^k) \times \exp(\mp \sum_{k=1}^{\infty} \frac{b_j(k)}{|k|} \tilde{z}^{-k}) e^{\pm \alpha z^{\pm \alpha_i(0) \varepsilon_i}} \times \exp(\mp \sum_{k=1}^{\infty} \frac{b_j(-k)}{|k|} \tilde{w}^k) e^{\mp \alpha_j z^{\mp \alpha_j(0) \varepsilon_j}} \]

\[ = U_i^\pm(z)U_j^\mp(w) : \exp(-\sum_{k=1}^{\infty} \frac{1}{|k|^2} [b_i(k), b_j(-k)] |(\frac{w}{z})^k| z^{-(\alpha_i|\alpha_j)}) = RHS. \]
Now we proceed to check relation (D7), which holds from the following Proposition.

**Proposition 4.4.**

\[
(z - ((i, j) | (j, i))^{\pm \frac{1}{2}} w) X_{i}^{+}(z) X_{j}^{+}(w) = ((j, i)^{\pm 1} z - ((j, i) | (j, i))^{-1}^{\pm \frac{1}{2}} w) X_{j}^{+}(w) X_{i}^{+}(z).
\]

**Proof.** The proof will be carried out in the following two cases, the other cases can be checked similarly.

(i) For the case of \((\alpha_{i} | \alpha_{j}) = -1\), let us first consider

\[
X_{n-1}^{-}(z) X_{n}^{-}(w) =: X_{n-1}^{-}(z) X_{n}^{-}(w) : (z - (r^{-1} s)^{\frac{1}{2}} w)^{-1}.
\]

On the other hand, one gets

\[
X_{n}^{-}(w) X_{n-1}^{-}(z) = -(r s)^{\frac{1}{2}} : X_{n}^{-}(w) X_{n-1}^{-}(z) : (w - (r^{-1} s)^{-\frac{1}{2}} z)^{-1}.
\]

It holds that

\[
(z - (r s^{-1})^{\frac{1}{2}} w) X_{n-1}^{-}(z) X_{n}^{-}(w) = (s^{-1} z - (r s)^{-\frac{1}{2}} w) X_{n}^{-}(w) X_{n-1}^{-}(z).
\]

(ii) For the case of \((\alpha_{i} | \alpha_{j}) = -\frac{1}{2}\), that is, \((\alpha_{i} | \alpha_{i+1}) = -\frac{1}{2}\). For \(i = 1, \cdots, n - 2\), by (4.14)–(4.17), one has

\[
X_{i}^{+}(z) X_{i+1}^{+}(w)
= \varepsilon(\alpha_{i}, \alpha_{i+1}) : Y_{i}^{+}(z) Y_{i+1}^{+}(w) : (z - (r s)^{\frac{1}{2}} (r^{-1} s)^{\frac{1}{2}} w)^{-\frac{1}{2}}.
\]

Using (4.9)–(4.11), one easily gets

\[
X_{i}^{+}(z) X_{i+1}^{+}(w)
= : U_{i}^{+}(z s^{\frac{1}{2}} z U_{i+1}^{+} w - r s^{\frac{1}{2}} z u_{i}^{+} (\alpha_{i} | \alpha_{i+1})^{\frac{1}{2}}.
\]

Using (4.9)–(4.11), one easily gets

\[
X_{i}^{+}(z) X_{i+1}^{+}(w)
= : U_{i}^{+}(z s^{\frac{1}{2}} z U_{i+1}^{+} w - r s^{\frac{1}{2}} z u_{i}^{+} (\alpha_{i} | \alpha_{i+1})^{\frac{1}{2}}.
\]

Using (4.14)–(4.17), one has

\[
X_{i}^{+}(z) X_{i+1}^{+}(w)
= : U_{i}^{+}(z s^{\frac{1}{2}} z U_{i+1}^{+} w - r s^{\frac{1}{2}} z u_{i}^{+} (\alpha_{i} | \alpha_{i+1})^{\frac{1}{2}}.
\]

Using (4.9)–(4.11), one easily gets

\[
X_{i}^{+}(z) X_{i+1}^{+}(w)
= : U_{i}^{+}(z s^{\frac{1}{2}} z U_{i+1}^{+} w - r s^{\frac{1}{2}} z u_{i}^{+} (\alpha_{i} | \alpha_{i+1})^{\frac{1}{2}}.
\]
On the other hand, it holds similarly

\[
X_{i+1}^+(w)X_i^+(z) = -(rs)^{-\frac{1}{2}} : Y_{i+1}^+Y_i^+ : (U_{i+1}^+(s^\frac{1}{2}w)U_i^+(s^\frac{1}{2}z) : (w-(r^{-1}s)^\frac{1}{2})^{-1} \cdot s^{-\frac{1}{2}}
+ U_{i+1}^-(r^\frac{1}{2}w)U_i^+(s^\frac{1}{2}z) : (-1)^{2\alpha_{i+1}(0)+1} \cdot r^\frac{1}{2}
+ U_{i+1}^+(s^\frac{1}{2}w)U_i^-(r^\frac{1}{2}z) : (-1)^{2\alpha_i(0)}w-(rs^{-1})^\frac{1}{2}z
+ U_{i+1}^-(r^\frac{1}{2}w)U_i^-(r^\frac{1}{2}z) : (-1)^{2\alpha_i(0)+2\alpha_{i+1}(0)+1}(w-(r^{-1}s)^\frac{1}{2})^{-1} \cdot r^{-\frac{1}{2}}.
\]

As a consequence of the above relations, we get our required result

\[
(z-(r^{-1}s)^\frac{1}{2}w)X_{i+1}^+(z)X_i^+(w) = (s^\frac{1}{2}z-(rs)^\frac{1}{2}w)X_{i+1}^+(w)X_i^+(z).
\]

The “-” part of relation (D7) can be verified similarly, which is left to the readers. \(\square\)

Now we focus on checking relation (D8).

**Lemma 4.5.**

\[
[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{(r_i-s_i)zw} \left( \delta(zw^{-1}s)\psi_i(wz^\frac{1}{2}) - \delta(zw^{-1}r)\phi_i(ws^\frac{1}{2}) \right).
\]

**Proof.** It suffices to consider the four cases: \((\alpha_i|\alpha_j) = -1/2\), \((\alpha_i|\alpha_j) = 1\), \((\alpha_i|\alpha_j) = 1\), and \((\alpha_i|\alpha_j) = 2\). Here we only give the proof for the first two cases, since the other cases are either immediate or similar to our previous considerations.

(i) For \((\alpha_i|\alpha_j) = -1/2\), that is, \(j = i + 1\) or \(i - 1\). Firstly, consider \(i = 1, \ldots, n-1\), using (4.14)–(4.17), together with (4.11)–(4.13), one immediately gets

\[
X_{i+1}^+(z)X_i^+(w) = Y_{i+1}^+(z)Y_i^+(w) :
\]

\[
U_{i+1}^+(s^\frac{1}{2}z)U_i^+(r^\frac{1}{2}w) : s^{-\frac{1}{2}}
+ U_{i+1}^-(s^\frac{1}{2}z)U_i^+(r^\frac{1}{2}w) : (-1)^{2\alpha_{i+1}(0)}(s^\frac{1}{2}z - r^{-\frac{1}{2}}s^{-\frac{1}{2}}w)
+ U_{i+1}^-(r^\frac{1}{2}z)U_i^+(r^\frac{1}{2}w) : (-1)^{2\alpha_i(0)+1}(r^\frac{1}{2}z - r^{-\frac{1}{2}}s^{-\frac{1}{2}}w)
+ U_{i+1}^-(r^\frac{1}{2}z)U_i^+(s^{-\frac{1}{2}}w) : (-1)^{2\alpha_i(0)+2\alpha_{i+1}(0)+1}r^{-\frac{1}{2}}.
\]

Using the same method, it is easy to see that

\[
X_{i+1}^+(w)X_i^+(z) = (rs)^{-\frac{1}{2}} : Y_{i+1}^-Y_i^+(z) : \epsilon_0(\alpha_{i+1}, \alpha_i)
\]

\[
U_{i+1}^-(r^\frac{1}{2}w)U_i^+(s^\frac{1}{2}z) : r^\frac{1}{2}
+ U_{i+1}^-(s^{-\frac{1}{2}}w)U_i^+(s^\frac{1}{2}z) : (-1)^{2\alpha_{i+1}(0)+1}(s^{-\frac{1}{2}}z - r^\frac{1}{2}s^\frac{1}{2}w)
+ U_{i+1}^-(r^\frac{1}{2}w)U_i^-(r^\frac{1}{2}z) : (-1)^{2\alpha_i(0)}(r^{-\frac{1}{2}}z - r^\frac{1}{2}s^\frac{1}{2}w)
+ U_{i+1}^-(s^{-\frac{1}{2}}w)U_i^-(r^\frac{1}{2}z) : (-1)^{4\alpha_i(0)}s^\frac{1}{2}.
\]

Thus it is easy to get \([X_i^+(z), X_{i+1}^+(w)] = 0\).
(ii) For the case of \(1 \leq i = j \leq n - 1\), using (4.13)-(4.16), it follows from (4.11)-(4.13) that
\[
X_i^+(z)X_i^-(w) = (rs)^{\frac{i}{2}} : Y_i^+(z)Y_i^-(w) : \\
\left(: U_i^+(s^{\frac{i}{2}}z)U_i^+(r^{-\frac{i}{2}}w) : s^{\frac{i}{2}} \right) \\
+ : U_i^+(s^{\frac{i}{2}}z)U_i^-(s^{-\frac{i}{2}}w) : (-1)^{2a_i(0)} \frac{1}{(s^{\frac{i}{2}}z - s^{-\frac{i}{2}}w)(z - (rs)^{-\frac{i}{2}}w)} \\
+ : U_i^-(r^{\frac{i}{2}}z)U_i^+(r^{-\frac{i}{2}}w) : (-1)^{2a_i(0)} \frac{1}{(r^{\frac{i}{2}}z - r^{-\frac{i}{2}}w)(z - (rs)^{-\frac{i}{2}}w)} \\
+ : U_i^-(r^{\frac{i}{2}}z)U_i^-(s^{-\frac{i}{2}}w) : (-1)^{4a_i(0)}s^{-\frac{i}{2}} \right).
\]
At the same time, we actually have
\[
X_i^-(w)X_i^+(z) = (rs)^{-\frac{i}{2}} : Y_i^-(w)Y_i^+(z) : \\
\left(: U_i^-(r^{-\frac{i}{2}}w)U_i^+(s^{\frac{i}{2}}z) : r^{-\frac{i}{2}} \right) \\
+ : U_i^-(s^{-\frac{i}{2}}w)U_i^+(s^{\frac{i}{2}}z) : (-1)^{2a_i(0)} \frac{1}{(s^{-\frac{i}{2}}w - s^{\frac{i}{2}}z)(w - (rs)^{\frac{i}{2}}z)} \\
+ : U_i^+(r^{\frac{i}{2}}z)U_i^-(r^{\frac{i}{2}}z) : (-1)^{2a_i(0)} \frac{1}{(r^{\frac{i}{2}}z - r^{\frac{i}{2}}z)(w - (rs)^{\frac{i}{2}}z)} \\
+ : U_i^+(s^{\frac{i}{2}}z)U_i^-(r^{\frac{i}{2}}w) : (-1)^{4a_i(0)}s^{-\frac{i}{2}} \right).
\]
Therefore, it is easy to see that
\[
[X_i^+(z), X_i^-(w)] \\
= (rs)^{-\frac{i}{2}} : U_i^+(s^{\frac{i}{2}}z)U_i^-(s^{-\frac{i}{2}}w)Y_i^+(z)Y_i^-(w) : \left(\frac{1}{(s^{\frac{i}{2}}z - s^{-\frac{i}{2}}w)(z - (rs)^{-\frac{i}{2}}w)} \right) \\
- \frac{1}{(s^{-\frac{i}{2}}w - s^{\frac{i}{2}}z)((rs)^{-\frac{i}{2}}w - z)} \\
+ (rs)^{-\frac{i}{2}} : U_i^-(r^{\frac{i}{2}}z)U_i^+(r^{-\frac{i}{2}}w)Y_i^+(z)Y_i^-(w) : \left(\frac{1}{(r^{\frac{i}{2}}z - r^{-\frac{i}{2}}w)(z - (rs)^{-\frac{i}{2}}w)} \right) \\
- \frac{1}{(r^{-\frac{i}{2}}w - r^{\frac{i}{2}}z)((rs)^{-\frac{i}{2}}w - z)} \\
=: U_i^+(s^{\frac{i}{2}}z)U_i^-(s^{-\frac{i}{2}}w)Y_i^+(z)Y_i^-(w) : \left(\frac{(rs)^{\frac{i}{2}}}{(r^{\frac{i}{2}}z - s^{\frac{i}{2}}z)zw} \right) \left(\delta(s^{-1}w^{-1}z) - \delta((rs)^{-\frac{i}{2}}w) \right) \\
+ : U_i^-(r^{\frac{i}{2}}z)U_i^+(r^{-\frac{i}{2}}w)Y_i^+(z)Y_i^-(w) : \left(\frac{(rs)^{\frac{i}{2}}}{(r^{-\frac{i}{2}}w - s^{-\frac{i}{2}}w)zw} \right) \left(\delta((rs)^{-\frac{i}{2}}w) - \delta(r^{-1}w^{-1}) \right) \\
= \frac{1}{(r_i - s_i)zw} \left(\psi_i(ws)^{\frac{i}{2}} \delta(ws^{-1}z) - \phi_i(ws^{-\frac{i}{2}}) \delta(ws^{-1}z) \right),
\]
where we have used the following results
\[
: U_i^+(s^{-\frac{i}{2}}w)U_i^-(s^{\frac{i}{2}}w)Y_i^+(s^{-1}w)Y_i^-(w) := (rs)^{-\frac{i}{2}}\psi_i(ws^{\frac{i}{2}}); \\
: U_i^-(r^{-\frac{i}{2}}w)U_i^+(r^{\frac{i}{2}}w)Y_i^+(r^{-1}w)Y_i^-(w) := (rs)^{-\frac{i}{2}}\phi_i(ws^{-\frac{i}{2}}).
\]
This completes the proof. \(\square\)
Proof of Serre relations (D9). (i) For \( i = 1, \cdots, n - 2 \), let us write the operators \( X_i^\pm(z) \) as a sum of two terms:

\[
X_i^+(z) = \sum_{\epsilon = \pm} Y_i^+(z) U^\epsilon_U(z r^{(i-1)/8} s^{(i+1)/4}) (-1)^{(1-\epsilon) a_i(0)}
\]

\[= X_{i+1}^+(z) + X_{i-1}^+(z),\]

\[
X_i^-(z) = \sum_{\epsilon = \pm} Y_i^-(z) U^\epsilon_U(z r^{-(i+1)/8} s^{(i-1)/4}) (-1)^{(1-\epsilon) a_i(0)}
\]

\[= X_{i+1}^-(z) + X_{i-1}^-(z).\]

From the relations \([1,13],[1.10]\), it follows that

\[
X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) =: X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) : (-1)^{(\epsilon_1-\epsilon_2)/2}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2)(r^{1/4} s^{1/4} z_1 - r^{-1/4} s^{1/4} z_2)^{\epsilon_1} r^{(s z_1 - s z_2)^2/4} s^{(2s z_1 - 2s z_2)^2/4} s^{(2s z_1 - 2s z_2)}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4},\]

where we include the sign factor \((-1)^{(1-\epsilon) a_i(0)}\) and \( \epsilon_i \) in the normal ordered product.

Similar normal product computation gives that

\[
X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) =: X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) : (-1)^{(\epsilon_1-\epsilon_2)/2}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2)(r^{1/4} s^{1/4} z_1 - r^{-1/4} s^{1/4} z_2)^{\epsilon_1} r^{(s z_1 - s z_2)^2/4} s^{(2s z_1 - 2s z_2)^2/4} s^{(2s z_1 - 2s z_2)}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4},\]

One may now use above results to get that

\[
X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) : (-r^{1/2} + s^{1/2}) X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) X_{i+2}^+(z_2)
\]

\[\times X_{i+1}^+(z_2) =: X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) : (z_1 - (r^{-1}s)^{1/4} z_2)(r^{1/4} s^{1/4} z_1 - r^{-1/4} s^{1/4} z_2)^{\epsilon_1} r^{(s z_1 - s z_2)^2/4} s^{(2s z_1 - 2s z_2)^2/4} s^{(2s z_1 - 2s z_2)}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4}
\]

\[\times (z_1 - (r^{-1}s)^{1/4} z_2) w^{1/4} (z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4},\]

(4.17) \[
\frac{(z_2 - (r^{-1}s)^{1/4} z_2) w^{1/4}}{w} + [2] \frac{(r^{-1}s)^{1/4} X_{i+1}^+(z_1) X_{i+2}^+(z_2) X_{i+1}^+(w) : z_1 - (r^{-1}s)^{1/4} z_2 w^{1/4} \times}
\]

\[\frac{(w - (r^{-1}s)^{1/4} z_2) w^{1/4}}{w - (r^{-1}s)^{1/4} z_2} + [((r^{-1}s)^{1/4} w - z_1) w^{1/4} \times z_1 - (r^{-1}s)^{1/4} z_2 w^{1/4} \times}
\]

\[\frac{(w - (r^{-1}s)^{1/4} z_2) w^{1/4}}{w - (r^{-1}s)^{1/4} z_2}.\]
where each term in the parentheses corresponds to the above relations for the three normal products, and we have also used the relation:

\[ :X^+_{it_1}(z_1)X^+_{it_2}(z_2)X^+_{i+1,e}(w) := -(rs)^{\frac{1}{2}} :X^+_{it_1}(z_1)X^+_{i+1,e}(w)X^+_{it_2}(z_2) :. \]

We check the claim for two cases \( \epsilon_1 \neq \epsilon_2 \) and \( \epsilon_1 = \epsilon_2 \):

(a) For \( \epsilon_1 \neq \epsilon_2 \), one has

\[ X^+_{it_1}(z_1)X^+_{it_2}(z_2)X^+_{i+1,e}(w) - (s^{\frac{1}{2}} + s^{\frac{1}{2}})X^+_{it_1}(z_1)X^+_{i+1,e}(w)X^+_{it_2}(z_2) \]

(4.18)

\[ + (rs)^{\frac{1}{2}}X^+_{i+1,e}(w)X^+_{it_1}(z_1)X^+_{it_2}(z_2) + (z_1 \leftrightarrow z_2, \epsilon_1 \leftrightarrow \epsilon_2) = 0. \]

The claim of this case is verified by checking four cases for \( \epsilon, \epsilon_i \), which are all similar and relied upon the following important identity [JKM1]:

(4.19) \( (z_1 - tw)(z_2 - tw) + (t + t^{-1})(z_1 - tw)(w - tz_2) + (w - tz_1)(w - t^2 z_2), \)

for any \( t \in \mathbb{C} \).

Take \( \epsilon = 1, \epsilon_1 = -\epsilon_2 = 1 \) for example, the parentheses in (4.17) is simplified to the following expression times \( \prod (z_1 - (r^{-1}s)^{\frac{1}{2}} w)^{-1}. (w - (r^{-1}s)^{\frac{1}{2}} z_2)^{-1}. \)

\[ (z_1 - (r^{-1}s)^{\frac{1}{2}} w)(z_2 - (r^{-1}s)^{\frac{1}{2}} w) + (rs)^{-\frac{1}{2}} [2_i (z_1 - (r^{-1}s)^{\frac{1}{2}} w)(w - (r^{-1}s)^{\frac{1}{2}} z_2) \]

\[ + (w - (r^{-1}s)^{\frac{1}{2}} z_1)(w - (r^{-1}s)^{\frac{1}{2}} w)] \]

\[ = ((r^{-1}s)^{\frac{1}{2}} - (r^{-1}s)^{\frac{1}{2}})w(z_1 - (r^{-1}s)^{\frac{1}{2}} z_2). \]

Under the symmetry \( (z_1, \epsilon_1) \leftrightarrow (z_2, \epsilon_2) \) it follows that the claim holds due to

\[ w(z_1 - (r^{-1}s)^{\frac{1}{2}} z_2) + w((r^{-1}s)^{\frac{1}{2}} z_2 - z_1) = 0. \]

(b) We now turn to the other four cases with \( \epsilon_1 = \epsilon_2 \). For the case \( \epsilon = \epsilon_1 = \epsilon_2 = -1 \), using the identity (4.19) again to simplify the parentheses in (4.17), the contraction function in the Serre relation becomes

\[ s(z_1 - (r^{-1}s)^{\frac{1}{2}} z_2)(z_1 - z_2) \left( \frac{(z_1 - (r^{-1}s)^{\frac{1}{2}} w)(z_2 - (r^{-1}s)^{\frac{1}{2}} w)}{(z_1 - (r^{-1}s)^{\frac{1}{2}} w)(z_2 - (r^{-1}s)^{\frac{1}{2}} w)} \right) \]

\[ - [2_i (rs)^{-\frac{1}{2}} z_1 - (r^{-1}s)^{\frac{1}{2}} w + (r^{-1}s)^{-\frac{1}{2}} ] \]

\[ = \frac{s(r^{-1}s)^{\frac{1}{2}} - (r^{-1}s)^{\frac{1}{2}} w(z_1 - z_2)(z_1 - (r^{-1}s)^{\frac{1}{2}} w)(z_1 - (r^{-1}s)^{\frac{1}{2}} z_2)}{(z_1 - (r^{-1}s)^{\frac{1}{2}} w)(z_2 - (r^{-1}s)^{\frac{1}{2}} w)}, \]

which is anti-symmetric under \( (z_1 \leftrightarrow z_2) \), hence the sub-Serre relation is proved in this case. That is,

\[ X^+_{it_1}(z_1)X^+_{it_1}(z_2)X^+_{i+1,e}(w) - (s^{\frac{1}{2}} + s^{\frac{1}{2}})X^+_{it_1}(z_1)X^+_{i+1,e}(w)X^+_{it_1}(z_2) \]

\[ + (rs)^{\frac{1}{2}}X^+_{i+1,e}(w)X^+_{it_1}(z_1)X^+_{it_1}(z_2) + (z_1 \leftrightarrow z_2) = 0. \]

Combining this sub-Serre relation with (4.13), we prove the Serre relation for \( a_{i,i+1} = a_{i+1,i} = -1 \).

We remark that the case \( a_{n-1,n} = -1 \) is easily proved by using the identity (4.19) with \( t = (r^{-1}s)^{\frac{1}{2}} \).

Finally, let us show the fourth order Serre relation with \( a_{n,n-1} = -2 \).
\[ \text{Sym}_{z_1, z_2, z_3} \left( (X_{n-1}^+)(z_1)X_{n-1}^+ (z_2)X_{n-1}^+ (z_3)X_n^+ (w) \right) \]

(4.20)

\[-(r+(s))^\frac{1}{2} X_{n-1}^+ (z_1)X_{n-1}^+ (z_2)X_n^+ (w)X_{n-1}^+ (z_3) \]
\[+(rs)^\frac{1}{2} (r+(s))^\frac{1}{2} + s)X_{n-1}^+ (z_1)X_n^+ (w)X_{n-1}^+ (z_2)X_{n-1}^+ (z_3) \]
\[-(rs)^\frac{1}{2} X_n^+ (w)X_{n-1}^+ (z_1)X_{n-1}^+ (z_2)X_{n-1}^+ (z_3) \] = 0.

Firstly, using the relations (4.13)-(4.16), it is not difficult to get

\[ X_{n-1, \epsilon_1} (z_1)X_{n-1, \epsilon_2} (z_2)X_n^+ (w)X_{n-1, \epsilon_3} (z_3) \]
\[=: X_{n-1, \epsilon_1} (z_1)X_{n-1, \epsilon_2} (z_2)X_n^+ (w) \]
\[\prod_{1 \leq i < j \leq 3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]
\[\prod_{i=1}^{3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]

Similarly, it follows from the relations (4.13)-(4.16),

\[ X_{n-1, \epsilon_1} (z_1)X_{n-1, \epsilon_2} (z_2)X_n^+ (w)X_{n-1, \epsilon_3} (z_3) \]
\[=: X_{n-1, \epsilon_1} (z_1)X_{n-1, \epsilon_2} (z_2)X_n^+ (w) \]
\[\prod_{1 \leq i < j \leq 3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]
\[\prod_{i=1}^{3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]

\[ X_{n-1, \epsilon_2} (z_1)X_n^+ (w)X_{n-1, \epsilon_3} (z_3) \]
\[=: X_{n-1, \epsilon_1} (z_1)X_{n-1, \epsilon_2} (z_2)X_n^+ (w)X_{n-1, \epsilon_3} (z_3) \]
\[\prod_{1 \leq i < j \leq 3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]
\[\prod_{i=1}^{3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]

As a consequence, pulling out the common normal product, (4.19) becomes

\[ \text{Sym}_{z_1, z_2, z_3} \left( (z_1^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \right) \]
\[\prod_{i=1}^{3} (z_i^2 + (r+1)^{\frac{1}{2}} s^{\frac{1}{2}} z_i - r^{\frac{1}{2}} s^{\frac{1}{2}} z_i j)^{\epsilon_{ij}} \]

which is almost the same as that of the one-parameter case with \( q = (rs)^{-\frac{1}{2}} \) ([JKM2]). Thus we complete the proof of Theorem 4.1.
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