The geometry and DSZ quantization four-dimensional supergravity

C. Lazaroiu 1 · C. S. Shahbazi 2,3

Received: 9 February 2021 / Revised: 9 December 2022 / Accepted: 15 December 2022 /
Published online: 30 December 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
We implement the Dirac–Schwinger–Zwanziger integrality condition on four-dimensional classical ungauged supergravity and use it to obtain its duality-covariant, gauge-theoretic, differential-geometric model on an oriented four-manifold $M$ of arbitrary topology. Classical bosonic supergravity is completely determined by a submersion $\pi$ over $M$ equipped with a complete Ehresmann connection, a vertical Euclidean metric and a vertically polarized flat symplectic vector bundle $\Xi$. Building on these structures, we implement the Dirac–Schwinger–Zwanziger integrality condition through the choice of an element in the degree-two sheaf cohomology group with coefficients in a locally constant sheaf $L \subset \Xi$ valued in the groupoid of integral symplectic spaces. We show that these data determine a Siegel principal bundle $P_t$ of fixed type $t \in \mathbb{Z}^{n^v}$ whose connections provide the global geometric description of the local electromagnetic gauge potentials of the theory. Furthermore, we prove that the Maxwell gauge equations of the theory reduce to the polarized self-duality condition determined by $\Xi$ on the connections of $P_t$. In addition, we investigate the continuous and discrete U-duality groups of the theory, characterizing them through short exact sequences and realizing the latter through the gauge group of $P_t$ acting on its adjoint bundle. This elucidates the geometric origin of U-duality, which we explore in several examples, illustrating its dependence on the topology of the fiber bundles $\pi$ and $P_t$ as well as on the isomorphism type of $L$. 

C. S. Shahbazi
cshahbazi@mat.uned.es; carlos.shahbazi@uni-hamburg.de
C. Lazaroiu
lcalin@theory.nipne.ro

1 Department of Theoretical Physics, Horia Hulubei National Institute for Physics and Nuclear Engineering, Bucharest, Magurele, Romania
2 Departamento de Matemáticas, UNED Madrid, Reino de España, Madrid, Spain
3 Fakultät Mathematik, Hamburg Universität, Bundesrepublik, Deutschland
Keywords  Mathematical supergravity · Abelian gauge theory · Electromagnetic duality · Symplectic vector bundles

Mathematics Subject Classification  Primary 53C80; Secondary 83E50

1 Introduction

The main goal of this article is to construct the gauge-theoretic and duality-covariant global differential geometric model of the universal bosonic sector of four-dimensional ungauged supergravity and study its U-duality group. In order to do so, we will implement the Dirac–Schwinger–Zwanziger (DSZ) integrality condition [15, 33, 36] on the gauge sector of the classical theory and we will interpret the result geometrically. This procedure is conceptually analogous to the implementation of the Dirac integrality condition on the field strengths of classical Maxwell theory in order to identify the notion of gauge field with the notion of connection on a principal U(1) bundle, only that implemented in a remarkably more complex theory that requires more sophisticated sheaf cohomology groups. Although this type of schemes is usually referred to as DSZ quantization conditions in the literature, we generally prefer the term DSZ integrality condition since quantization has a very different meaning in of quantum field theory. In the following we will use both terms interchangeably.

The local formulation of classical four-dimensional supergravity theories has been studied intensively in the physics literature, see, for instance, the seminal references [2, 3, 7, 8, 10, 11, 13, 14] and the reviews and books [4, 18, 20, 30, 32]. Such theories share a universal bosonic sector, which is subject to increasingly stringent constraints according to the number of supersymmetry generators of the theory. The global classical geometric formulation of the universal bosonic sector of supergravity was obtained in [25, 26], see also [29] for a mathematically rigorous approach to supergravity based on supergeometry. In [24–26], it was found that the local structure of supergravity does not suffice to determine the theory on spacetimes which are not simply connected. As show in loc. cit., the global formulation of the classical universal bosonic sector of ungauged supergravity on an oriented four-manifold \( M \) is a generalized Einstein-Section-Maxwell theory, which is determined by the following data:

- A flat scalar bundle \((\pi, \mathcal{H}, \mathcal{G})\), where \(\pi: X \to M\) is a submersion equipped with a complete flat Ehresmann connection \(\mathcal{H}\) and a Euclidean vertical metric \(\mathcal{G}\), i.e., a Euclidean metric on the vertical bundle of \(\pi\) which is invariant under the parallel transport of \(\mathcal{H}\).
- A duality bundle \(\Delta = (S, \omega, \mathcal{D})\), i.e., a flat symplectic vector bundle over the total space \(X\) of \(\pi\).
- A vertical polarization \(\mathcal{J}\) on \(\Delta\), i.e., a taming of \((S, \omega)\) which is invariant under the extended parallel transport induced by \(\mathcal{H}\) and \(\mathcal{D}\).

The configuration space of the bosonic supergravity theory determined by a tuple \((\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})\) as introduced above is the set of triples \((g, s, \mathcal{F})\), where \(g\) is a Lorentzian metric on \(M\), \(s\) is a global section of \(\pi\) and \(\mathcal{F} \in \Omega^2(M, S^0)\) is a two-form on \(M\) taking values in the pull-back \(S^0\) of \(S\) by \(s\) which is covariantly closed with
respect to the pullback connection $D^s$. The classical equations of motion of the theory were given naturally in terms of aforementioned geometric structures as explained in [26]. In particular, the Maxwell equations (i.e., the equations of motion for $\mathcal{F}$) correspond to the polarized self-duality condition determined by the taming $\mathcal{J}$.

The description of the gauge sector in terms of field strengths $\mathcal{F}$ is unsatisfactory when coupling the theory to quantized charged particles. Indeed, the Aharonov–Bohm effect [1] implies that this sector should admit a global description in terms of gauge potentials, which are expected to be modeled by connections $A$ on an appropriate principal bundle $P$ defined on $M$. To determine this bundle, we implement the Dirac–Schwinger–Zwanziger (DSZ) integrality condition on the gauge sector. We then show that this condition implies that $P$ is a Siegel bundle in the sense of [27], i.e., a principal bundle whose structure group is the automorphism group of an integral symplectic affine torus. The latter is isomorphic to a certain semidirect product of an even-dimensional torus group with a modified Siegel modular group. This process parallels the DSZ quantization of Abelian gauge theories with manifest electromagnetic duality, developed in [27], which depends on a Siegel system $Z$ on $X$. The latter was defined in [27] as a local system of finitely generated free Abelian groups whose structure group reduces to a modified Siegel modular group and which is isomorphic to $\Delta$ upon tensorization with $\mathbb{R}$ over $\mathbb{Z}$. We define a classical configuration $(g, s, \mathcal{F})$ to be integral if the cohomology class of $\mathcal{F}$ with respect to the de Rham differential twisted by $D^s$ belongs to the integral lattice defined by the second cohomology group of $M$ with coefficients in $\mathbb{Z}^s$, the pull-back of $Z$ by $s$. By the results of [27], any element of $H^2(M, \mathbb{Z}^s)$ is the twisted Chern class of a Siegel bundle defined on the total space of $\pi$. Using this fact, we show that the DSZ quantization of classical bosonic supergravity is determined by the following data:

- A flat scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$.
- A Siegel bundle $P_t$ of type $t$ defined on the total space $X$ of $\pi$.
- A vertical polarization $\mathcal{J}$ on the adjoint bundle $\text{ad}(P_t)$ of $P_t$.

The configuration space of the DSZ quantization of bosonic supergravity determined by $(\pi, \mathcal{H}, \mathcal{G})$ and $(P_t, \mathcal{J})$ is given by triples $(g, s, A)$, where $g$ and $s$ are as defined above and $A$ is a connection on $P_t^\mathcal{H}$, which describes both the electric and magnetic potentials of the theory. The Maxwell equations become a first-order condition on the connections of $P_t$ which depends on the Hodge operator of $g$ and the polarization $\mathcal{J}$. In particular, this gives the global and duality-covariant equations of motion of the universal bosonic sector of four-dimensional ungauged supergravity on $(P_t, \mathcal{J})$ in terms of the variables $(g, s, A)$.

Using this geometric and gauge-theoretic formulation of bosonic supergravity, we study its group of continuous and discrete U-duality transformations, which we characterize through short exact sequences involving the group of automorphisms of $P_t$ and its adjoint bundle. In general, these groups can differ markedly from their local counterparts considered in the physics literature [22]. In this regard, we emphasize the dependence of U-duality groups on the type of the corresponding Siegel modular group and of the isomorphism class of the Siegel system $Z$, a point which does not

---

1 This is a classical notion in the theory of symplectic lattices see for instance [12] or [27, Appendix B] for more details.
appear to have been noticed in the supergravity literature. In particular, the explicit computation of the discrete U-duality groups becomes a hard arithmetic problem in the theory of automorphisms of local systems. As an application of the framework that we develop, we show that the group of discrete electromagnetic duality transformations of a four-dimensional supergravity is the discrete remnant of the unbased group of automorphisms of $P_t$. This clarifies the geometric origin of U-duality in supergravity in terms of a particular class of gauge transformations of $P_t$.

The geometric formulation described in this paper provides the basis of the mathematical framework necessary to investigate the differential-geometric problems arising in four-dimensional supergravity. In particular, it allows for a mathematically rigorous formulation of the geometric constraints imposed by supersymmetry through the corresponding Killing spinor equations, in the spirit of [9], thus opening the way for developing the mathematical theory of supergravity supersymmetric solutions and moduli spaces of such.

2 Classical bosonic supergravity

In this section, we recall the construction of generalized Einstein-Section-Maxwell theories on an oriented four-manifold $M$ given in [25, 26]. These give the global differential-geometric model of the universal bosonic sector of four-dimensional supergravity (also called classical geometric bosonic supergravity or classical bosonic supergravity for short, see [9]).

2.1 Preparations

Let $M$ be an oriented and connected four-manifold. We start by introducing the geometric data needed to formulate classical bosonic supergravity on $M$, a detailed account of which was given in [26].

**Definition 2.1** A scalar bundle of rank $n_s$ on $M$ is a triple $(\pi, \mathcal{H}, \mathcal{G})$ consisting of:

- A smooth submersion $\pi : X \to M$, where $X$ is a connected and oriented differentiable manifold of dimension $n_s + 4$.
- A complete Ehresmann connection $\mathcal{H} \subset TX$ on $\pi$.
- A vertical Euclidean metric, i.e., a Euclidean metric $\mathcal{G}$ defined on the vertical bundle $V \subset TX$ of $\pi$ which is preserved by the parallel transport of $\mathcal{H}$. Recall that the vertical bundle $V \subset TX$ is defined as the kernel of the differential map $d\pi : TX \to TM$.

We say that a scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$ is flat if $\mathcal{H}$ is Frobenius integrable.

In the following, we denote by $O_m$ the restriction of any geometric structure $O$ defined on $X$ to the fiber $X_m$ of $\pi$ at $m \in M$.

**Remark 2.2** Since the Ehresmann connection $\mathcal{H}$ of a scalar bundle is complete and its parallel transport preserves the vertical metric $\mathcal{G}$, the fibers $(X_m, \mathcal{G}_m)$ of $\pi$ are isomorphic to each other as Riemannian manifolds. By the results of [17], it follows
that $\pi$ is a fiber bundle associated with a principal bundle with structure group given by the isometry group of $(X_m, G_m)$. Notice that $X_m$ is typically non-compact in physics applications.

**Definition 2.3** A scalar bundle $(\pi, \mathcal{H}, G)$ is:

- **Topologically trivial** if $\pi$ is topologically trivial as fiber bundle, i.e., $X$ is diffeomorphic with $M \times M$ for some manifold $M$ and $\pi$ identifies with the projection $\text{pr} : M \times M \to M$ on the first factor.
- **Holonomy-trivial** if the holonomy of $\mathcal{H}$ is trivial.

**Remark 2.4** Every holonomy-trivial scalar bundle is topologically trivial. Moreover, its Ehresmann connection identifies with the pull-back of $TM$ through the projection $\text{pr} : M \times M \to M$.

Let $P(M)$ and $P(X)$, respectively, be the sets of piece-wise smooth paths in $M$ and $X$ defined on the unit interval. Given a scalar bundle $(\pi, \mathcal{H}, G)$, let $T$ be the parallel transport defined by $\mathcal{H}$, which associates with a path $\gamma \in P(M)$ the diffeomorphism:

$$T_\gamma : X_{\gamma(0)} \xrightarrow{\sim} X_{\gamma(1)},$$

obtained by parallel transport along $\mathcal{H}$.

**Definition 2.5** A duality bundle $\Delta = (S, \omega, D)$ over $\pi$ is a triple $(S, \omega, D)$ where $S$ is a vector bundle on $X$, $\omega$ is a symplectic structure on $S$ and $D$ is a flat connection on $S$ preserving $\omega$. We denote the rank of $S$ by $2n_v$.

**Remark 2.6** As shown in [26], a duality bundle of rank $2n_v$ corresponds locally to a supergravity theory coupled to $n_v$ vector multiplets.

Let $(\pi, \mathcal{H}, G)$ be a scalar bundle over $M$ and $\Delta = (S, \omega, D)$ be a duality bundle on $\pi$, which we shall also call a duality bundle over $(\pi, \mathcal{H}, G)$. For any $m \in M$, let $(S_m, \omega_m, D_m)$ be the restriction of $(S, \omega, D)$ to the fiber $X_m = \pi^{-1}(m)$. This is a flat symplectic vector bundle on $X_m$ and hence a duality structure on the latter as defined in [25]. For any path $\Gamma \in P(X)$ in the total space $X$ of $\pi$, we denote by $\Upsilon_\Gamma : S_{\Gamma(0)} \to S_{\Gamma(1)}$ the parallel transport of $D$ along $\Gamma$. Since $D$ is a symplectic connection, $\Upsilon_\Gamma$ is a symplectomorphism between the symplectic vector spaces $(S_{\Gamma(0)}, \omega_{\Gamma(0)})$ and $(S_{\Gamma(1)}, \omega_{\Gamma(1)})$. For any $\gamma \in P(M)$, let $\tilde{\gamma}_x \in P(X)$ be the horizontal lift of $\gamma$ starting at the point $x \in X_{\gamma(0)}$. By the definition of $T$, we have $\tilde{\gamma}_x(1) = T_\gamma(x)$.

**Definition 2.7** The extended horizontal transport along a path $\gamma \in P(M)$ is the unbased isomorphism of flat symplectic vector bundles $T_\gamma : S_{\gamma(0)} \to S_{\gamma(1)}$ defined by:

$$T_\gamma(x) \overset{\text{def.}}{=} \Upsilon_{\tilde{\gamma}_x} : S_x \to S_{T_\gamma(x)}, \quad \forall x \in X_{\gamma(0)},$$

which linearizes the Ehresmann transport $T_\gamma : X_{\gamma(0)} \to X_{\gamma(1)}$ along $\gamma$. 

Springer
Given a duality bundle $\Delta = (S, \omega, D)$, a (compatible) taming $J \in \text{Aut}_b(S)$ on $\Delta$ is a complex structure on $S$ which tames the symplectic pairing $\omega$, i.e., it satisfies the compatibility condition:

$$\omega(J\xi_1, J\xi_2) = \omega(\xi_1, \xi_2), \quad \forall (\xi_1, \xi_2) \in S \times_X S,$$

and the positivity condition:

$$\omega(\xi, J\xi) > 0, \quad \forall \xi \in \hat{S},$$

where $\hat{S}$ is the complement of the image of the zero section in $S$.

**Definition 2.8** A taming on the duality bundle $\Delta$ over $(\pi, H, G)$ is called \textit{vertical} if it is preserved by the extended horizontal transport $T$, i.e., if $T_\gamma : (S_\gamma(0), \omega_\gamma(0), J_\gamma(0)) \to (S_\gamma(1), \omega_\gamma(1), J_\gamma(1))$ an isomorphism of tamed symplectic vector bundles for all $\gamma \in P(M)$.

**Remark 2.9** As explained in [25], a vertical taming on $\Delta$ is equivalent to a positive Lagrangian sub-bundle of the complexification of $(S, \omega)$ which is preserved by the complexified extended horizontal transport.

As in [25, 26], we refer to the pair $\Xi = (\Delta, J)$ consisting of a duality bundle $\Delta$ defined on $(\pi, H, G)$ and a vertical taming $J$ as an \textit{electromagnetic bundle} on $(\pi, H, G)$. We will refer to a choice scalar bundle $(\pi, H, G)$ together with a choice of electromagnetic bundle $\Xi$ as a \textit{scalar-electromagnetic bundle} $\Phi$, that is:

$$\Phi \overset{\text{def.}}{=} (\pi, H, G, \Xi).$$

As shown in [25, 26], the universal bosonic sector of supergravity defined on $M$ is determined by the choice of a scalar-electromagnetic bundle. Morphisms of duality and electromagnetic bundles are defined in the natural way (see [26]). Note that standard bundle theory implies that isomorphism classes of duality bundles over a fixed submersion $\pi : X \to M$ are in one to one correspondence with the character variety:

$$\mathcal{M}_d(X) \overset{\text{def.}}{=} \text{Hom}(\pi_1(X), \text{Sp}(2n_v, \mathbb{R}))/\text{Sp}(2n_v, \mathbb{R}).$$

**Remark 2.10** In general, the character variety above has positive dimension, giving a moduli space of inequivalent duality bundles. This implies [26] that one can construct an uncountable infinity of globally inequivalent bosonic geometric supergravities which are however all locally equivalent.

**Remark 2.11** A duality bundle $\Delta = (S, \omega, D)$ is called \textit{topologically trivial} if the vector bundle $S$ is trivial, i.e., if it admits a global frame. It is called \textit{symplectically trivial} if the $(S, \omega) \in \Delta$ is symplectically trivial, i.e., if $S$ admits a global symplectic frame. Finally, we say that $\Delta$ is \textit{holonomy trivial} if the holonomy of $D$ is the trivial group. Holonomy-triviality implies symplectic triviality, which in turn implies topological triviality. If $X$ is simply connected, then every duality bundle is holonomy trivial.
Smooth sections of the submersion $\pi : X \to M$ are called scalar sections. For every scalar section $s : M \to X$ we use a superscript $s$ to denote the bundle pull-back by $s$ and the subscript $s$ to denote push-forward by $s$ in the appropriate category. For instance, $\Delta^s = (S^s, \omega^s, D^s)$ denotes the bundle pull-back of $\Delta = (S, \omega, S)$ by $s$, which is a flat symplectic vector bundle over $M$. Similarly, $\Xi^s = (\Delta^s, \mathcal{J}^s)$ denotes the pull-back of $\Xi = (\Delta, \mathcal{J})$ by $s$, which is an electromagnetic structure on $M$ in the sense of [25]. Let $\Phi$ be a scalar-electromagnetic bundle on $M$. For every Lorentzian metric $g$ on $M$ and every scalar section $s \in \Gamma_1(\pi)$, consider the isomorphism of vector bundles:

$$\star_{g, \mathcal{J}} : \bigwedge^* TM \otimes S^s \to \bigwedge^* TM \otimes S^s,$$

defined through $\star_{g, \mathcal{J}} = \star_g \otimes \mathcal{J}^s$. Since both $\star_g$ and $\mathcal{J}^s$ square to minus the identity, this restricts to an involutive automorphism:

$$\star_{g, \mathcal{J}} : \bigwedge^2 TM \otimes S^s \to \bigwedge^2 TM \otimes S^s$$

which gives a direct sum decomposition into eigenbundles corresponding to the eigenvalues $+1$ and $-1$:

$$\bigwedge^2 TM \otimes S^s = \left( \bigwedge^2 TM \otimes S^s \right)_+ \oplus \left( \bigwedge^2 TM \otimes S^s \right)_-.$$

Here the subscript denotes the sign of the corresponding eigenvalue of $\star_{g, \mathcal{J}}$. The spaces of smooth global sections of these sub-bundles are denoted by $\Omega^2_\pm(M, S^s)$, and their elements are called polarized (anti)-self-dual $S^s$-valued 2-forms with respect to $\mathcal{J}^s$. We have:

$$\Omega^2(M, S^s) = \Omega^2_+(M, S^s) \oplus \Omega^2_-(M, S^s),$$

The flat symplectic connection $D^s$ of $\Delta^s$ defines an exterior covariant derivative acting on $S^s$-valued forms defined on $M$, which we denote by:

$${d}_{D^s} : \Omega(M, S^s) \to \Omega(M, S^s).$$

This operator squares to zero since $D^s$ is flat. We denote its cohomology groups by $H^k(M, \Delta^s)$ and the corresponding total cohomology by $H(M, \Delta^s)$. For every scalar section $s \in \Gamma(\pi)$, we denote by $\mathcal{G}^s_\Delta$ the sheaf of flat sections of $\Delta^s$, defined as follows:

$$\mathcal{G}^s_\Delta(U) \overset{\text{def}}{=} \{ \xi \in \Gamma(U, S^s) \mid D^s \xi = 0 \},$$

for any open set $U \subset M$. This is a locally constant sheaf of symplectic vector spaces of rank $2n_v$, whose stalk is isomorphic to the typical fiber of $\Delta$. Since the sheaf of smooth $S^s$-valued forms is acyclic, there exists a natural isomorphism of graded vector spaces:

$$H(M, \Delta^s) \simeq H(M, \mathcal{G}^s_\Delta),$$

Springer
where $H(M, \mathcal{G}_\Delta)$ is the sheaf cohomology of $\mathcal{G}_\Delta$. Note that the definition of an electromagnetic bundle $\Xi = (\Delta, J)$ does not require $D \in \Delta$ to be compatible with $J$, a fact which is crucial for recovering the correct local description of bosonic geometric supergravity. The failure of $D$ to be compatible with $J$ is measured by the fundamental form of an electromagnetic bundle.

**Definition 2.12** Let $\Phi$ be an scalar-electromagnetic bundle. The fundamental form $\Psi$ of $\Xi$ is the following $\text{End}(S)$-valued one-form defined on $X$:

$$
\Psi \overset{\text{def.}}{=} D J \in \Omega^1(X, \text{End}(S)).
$$

**Remark 2.13** For every $v \in \Gamma(TX)$, the endomorphism $\Psi(v) \in \text{End}(S) = \Gamma(\text{End}(S))$ is $J$-antilinear $Q$-symmetric, where the scalar product $Q$ is the Euclidean metric induced by $\omega$ and $J$ on $S$ as follows:

$$
Q(\xi_1, \xi_2) \overset{\text{def.}}{=} \omega(\xi_1, J \xi_2) \quad \forall (\xi_1, \xi_2) \in S \times X S,
$$

See [25, 26] for more details.

**Definition 2.14** An electromagnetic bundle $\Xi$ is called unitary if $\Psi = 0$.

To describe the universal bosonic sector of 4d supergravity, we introduce three natural operations which are determined by a choice of a scalar-electromagnetic bundle $\Phi$, a Lorentzian metric $g$ on $M$ and a scalar section $s \in \Gamma(\pi)$.

**Definition 2.15** The twisted exterior pairing $(\cdot, \cdot)_g, Qs$ is the unique pseudo-Euclidean scalar product on $\wedge T^*M \otimes S^s$ which satisfies:

$$
(\rho_1 \otimes \xi_1^s, \rho_2 \otimes \xi_2^s)_g, Qs = (\rho_1, \rho_2)_g Q^s(\xi_1^s, \xi_2^s) = (\rho_1, \rho_2)_g Q^s(\xi_1, \xi_2)
$$

for all $\rho_1, \rho_2 \in \Omega(M)$ and all $\xi_1, \xi_2 \in \Gamma(S^s)$.

Given any vector bundle $W$ on $M$, we extend this trivially to a $W$-valued pairing (which for simplicity we denote by the same symbol) between the bundles $W \otimes \wedge T^*M \otimes S^s$ and $\wedge T^*M \otimes S^s$. Thus:

$$
(w \otimes \eta_1, \eta_2)_g, Qs = w \otimes (\eta_1, \eta_2)_g, Qs, \quad \forall w \in \Gamma(W), \ \forall \eta_1, \eta_2 \in \Omega(M, S^s).
$$

**Definition 2.16** The inner $g$-contraction of (2,0)-tensors is the bundle morphism $\otimes_g : (\otimes^2 T^*M)^{\otimes 2} \to \otimes^2 T^*M$ uniquely determined by the condition:

$$
(\alpha_1 \otimes \alpha_2) \otimes_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_4)_g \alpha_1 \otimes \alpha_3, \quad \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in T^*M.
$$

We define the inner $g$-contraction of two-forms to be the restriction of $\otimes_g$ to $\wedge^2 T^*M \otimes \wedge^2 T^*M \subset (\otimes^2 T^*M)^{\otimes 2}$.
Definition 2.17 The twisted inner contraction of $S^s$-valued two-forms is the unique morphism of vector bundles:

$$\bigotimes Q^s : \wedge^2 T^*M \otimes S^s \times_M \wedge^2 T^*M \otimes S^s \to \bigotimes^2(T^*M)$$

which satisfies:

$$(\rho_1 \otimes s_1) \bigotimes Q^s (\rho_2 \otimes s_2) = Q^s(s_1, s_2)\rho_1 \bigotimes g \rho_2,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $s_1, s_2 \in \Gamma(S^s)$.

2.2 The configuration space and equations of motion

We are ready to give the geometric formulation of the universal bosonic sector of 4d classical supergravity, whose global solutions can be interpreted locally as geometric classical supergravity U-folds [24, 26].

Definition 2.18 Let $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Xi)$ be a scalar-electromagnetic bundle on an oriented four-manifold $M$. The configuration space of the universal bosonic sector determined by $(M, \Phi)$ is the set:

$$\text{Conf}(\Phi) \overset{\text{def}}{=} \left\{ (g, s, \mathcal{F}) \mid g \in \text{Lor}(M), \ s \in \Gamma(\pi), \ \mathcal{F} \in \Omega^2_{d\Gamma^1 \text{cl}}(M, S^s) \right\},$$

where $\text{Lor}(M)$ denotes the set of Lorentzian metrics on $M$ and $\Omega^2_{d\Gamma^1 \text{cl}}(M, S^s)$ denotes the set of $D^s$-closed 2-forms on $M$ valued in $S^s$.

Remark 2.19 In general, the isomorphism class of $S^s$ depends on the scalar section $s \in \Gamma(\pi)$.

Given a scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$, the complete Ehresmann connection $\mathcal{H}$ can be described through a one-form $C \in \Omega^1(X, \mathcal{V}) = \text{Hom}(TX, \mathcal{V})$ which restricts to the identity on $\mathcal{V} \subset TX$ and satisfies the condition $C \circ C = C$. Thus $C : TX \to \mathcal{V}$ is a projection of the tangent bundle of $X$ onto $\mathcal{V}$. The horizontal distribution $\mathcal{H}$ is recovered as the kernel of $C$. Given $(g, s, \mathcal{F}) \in \text{Conf}(\Phi)$, we define the vertical first fundamental form $(s_C^*\mathcal{G}) \in \Gamma(T^*M \otimes T^*M)$ through [34, 35]:

$$(s_C^*\mathcal{G})(v_1, v_2) \overset{\text{def}}{=} \mathcal{G}(C \circ ds(v_1), C \circ ds(v_2)), \quad (v_1, v_2) \in TM \times_M TM;$$

which depends explicitly on $C$ or, equivalently, $\mathcal{H}$. The trace of this tensor with respect to $g$ is called the vertical tension of the section $s \in \Gamma(\pi)$. For ease of notation, we define $d^C s(v) = C \circ ds(v) \in \mathcal{V}$, where $v \in TM$. Notice that $d^C s \in \Omega^1(M, \mathcal{V})$. Denote by $\text{Lor}(X)$ the set of Lorentzian metrics on $X$. Every Lorentzian metric $g$ on $M$ can be lifted to $\mathcal{H}$ using the isomorphism of vector bundles $(d\pi|_\mathcal{H}) : \mathcal{H} \simeq TM$ given by the restriction of $d\pi$ to $\mathcal{H}$. Thus, there exists a natural map (see [26]):

$$h : \text{Conf}(\Phi) \to \text{Lor}(X), \quad (g, s, \mathcal{F}) \mapsto h(g) \overset{\text{def}}{=} \pi^*g + \mathcal{G},$$
where \( h(g) \) is written using the direct sum decomposition \( TX = \mathcal{H} \oplus \mathcal{V} \). The Lorentzian metric \( h(g) \) enters the equations of motion of bosonic supergravity, as explained below. Note that, equipped with the lifted metric \( h(g) \), \( \pi : (X, h(g)) \to (M, g) \) becomes a Lorentzian submersion. Given \((g, s, \mathcal{F}) \in \text{Conf}(\Phi)\), we denote by \( \nabla^{h(g)} \) the Levi-Civita connection defined by \( h(g) \) on \( X \). For every scalar section \( s : M \to X \), we denote by \( \nabla^{\Phi(g,s)} \) the connection on \( TM \otimes \mathcal{V} \) given by the tensor product of the Levi-Civita connection \( \nabla^g \) of \( g \) with the vertical projection of the pull-back by \( s \) of the connection \( \nabla^{h(g)} \). We then have:

\[
\nabla^{\Phi(g,s)} d_C s \in \Gamma(T^*M \otimes T^*M \otimes \mathcal{V}),
\]

as explained in [34, 35]. In particular:

\[
\text{Tr}_g \left( \nabla^{\Phi(g,s)} d_C s \right) \in \Gamma(\mathcal{V}).
\]

Given \((g, s, \mathcal{F}) \in \text{Conf}(\Phi)\), we recall that \( \Psi^s \in \Gamma((\mathcal{V})^* \otimes \text{End}(S^s)) \) denotes the pull-back of \( \Psi \in \Omega^1(X, \text{End}(S)) \) by \( s : M \to X \). Hence, \( (\Psi^s)^\sharp g \in \Gamma(\mathcal{V} \otimes \text{End}(S^s)) \). For further reference we introduce symbol \( (\Psi^s)^\sharp g \mathcal{F}_A \in \Omega^2(M, \mathcal{V} \otimes S^s) \), which by definition denotes the action of \( \Psi^s \) on \( \mathcal{F}_A \) as an endomorphism of \( S^s \) while tensoring with \( \mathcal{V}^s \).

**Definition 2.20** Let \( \Phi \) be a scalar-electromagnetic bundle on \( M \). The universal bosonic sector defined by \( \Phi \) on \( M \) is described by following system of partial differential equations for triples \((g, s, \mathcal{F}) \in \text{Conf}(\Phi)\):

- The Einstein equations:
  \[
  \text{Ric}^g - \frac{g}{2} \text{R}^g = \frac{1}{2} \text{Tr}_g (s_c^s G) g - s_c^s G + 2 \mathcal{F} \ominus Qs \mathcal{F},
  \]
  where \( \text{Ric}^g \) and \( \text{R}^g \) are, respectively, the Ricci tensor and Ricci scalar of \( g \), while \( \text{Tr}_g \) denotes trace with respect to \( g \).

- The scalar equations:
  \[
  \text{Tr}_g \left( \nabla^{\Phi(g,s)} d_C s \right) = \frac{1}{2} \left( \ast g, (\Psi^s)^\sharp g \mathcal{F} \right)_{g, Qs}.
  \]

- The Maxwell equations:
  \[
  \ast g \left. \mathcal{F} \right. = \mathcal{F}.
  \]

We denote by \( \text{Sol}(\Phi) \subset \text{Conf}(\Phi) \) the set of solutions to these equations.

**Remark 2.21** The configuration space \( \text{Conf}(\Phi) \) is formulated using the *field strength* two-forms instead of the appropriate notion of gauge potential, as required by the Aharonov–Bohm effect [1]. The latter suggests that the gauge potentials of the theory should be described by connections on an appropriate principal bundle. To identify
this bundle, we must impose an appropriate DSZ integrality condition on the field strength $\mathcal{F}$. We consider this condition and its geometric interpretation in Sect. 3.

The fact that the formulation given above reduces \textit{locally} to the usual formulas of local bosonic supergravity found in the physics literature was proved in detail in references [25, 26], to which we refer the reader for further details. It is not known if this theory can be supersymmetrized when $\mathcal{H}$ is not flat, although the Killing spinor equations can be formulated exactly as in the case when $\mathcal{H}$ is flat.

2.3 The classical U-duality group

In this section, we characterize the \textit{global} U-duality group of the bosonic supergravity associated with a fixed scalar electromagnetic bundle $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \mathcal{J})$. Given a duality bundle $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ let $\text{Aut}(\mathcal{S})$ denote the group of all \textit{unbased} automorphisms of the vector bundle $\mathcal{S}$. Let $f_u \in \text{Diff}(\mathcal{X})$ be the diffeomorphism covered by $u \in \text{Aut}(\mathcal{S})$. Moreover, let $\text{Aut}(\Delta)$ be the group of those unbased automorphisms of $\mathcal{S}$ which preserve both $\omega$ and $\mathcal{D}$:

$$\text{Aut}(\Delta) \overset{\text{def}}{=} \{ u \in \text{Aut}(\mathcal{S}) \mid \omega^u = \omega, \mathcal{D}^u = \mathcal{D} \}.$$ 

Let $\text{Aut}_\pi(\Delta)$ be the subgroup consisting of all elements of $\text{Aut}(\Delta)$ which cover based automorphisms of the fiber bundle $\pi$:

$$\text{Aut}_\pi(\Delta) \overset{\text{def}}{=} \{ u \in \text{Aut}(\Delta) \mid f_u \in \text{Aut}_b(\pi) \} = \{ u \in \text{Aut}(\Delta) \mid \pi \circ f_u = \pi \}.$$ 

We have a short exact sequence of groups:

$$1 \rightarrow \text{Aut}_b(\Delta) \rightarrow \text{Aut}_\pi(\Delta) \rightarrow \text{Aut}_0^b(\pi) \rightarrow 1,$$

where $\text{Aut}_0^b(\pi) \subset \text{Aut}_b(\pi)$ is the subgroup of those automorphisms of $\pi$ which are covered by elements of $\text{Aut}(\Delta)$.

Given a scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$ and an element $u \in \text{Aut}_\pi(\Delta)$, the fiber bundle automorphism $f_u \in \text{Aut}_b(\pi)$ covered by $u$ acts as a gauge transformation on $\mathcal{H}$ through push-forward $\mathcal{H}_u \overset{\text{def}}{=} (f_u)_* \mathcal{H}$. Similarly, since $f_u$ is an automorphism of $\pi$ covering the identity, the push-forward of $\mathcal{G}$ by $f_u$ defines a new vertical Riemannian metric $\mathcal{G}_u \overset{\text{def}}{=} (f_u)_* \mathcal{G}$ on $\pi$ such that $(X_m, \mathcal{G}_u)_m$ is isometric to $(X_m, (\mathcal{G}_u)_m)$ for all $m \in M$.

Given an electromagnetic bundle $\mathcal{J} = (\Delta, \mathcal{J})$, push-forward by $f_u$ produces another electromagnetic bundle which we denote by $(\Delta_u, \mathcal{J}_u)$. Given a scalar-electromagnetic bundle $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})$, the system:

$$\Phi_u \overset{\text{def}}{=} (\pi, \mathcal{H}_u, \mathcal{G}_u, \Delta_u, \mathcal{J}_u)$$

is a scalar-electromagnetic bundle with the same underlying submersion $\pi : X \rightarrow M$. If $\mathcal{C} \in \Omega^1(\mathcal{X}, \mathcal{V})$ is the connection one-form associated with $\mathcal{H}$, then the natural push-forward $f_{us}\mathcal{C} \in \Omega^1(\mathcal{X}, \mathcal{V})$ is the connection one-form associated with $\mathcal{H}_u$. 

\[ Springer \]
Remark 2.22 Since elements of $\text{Aut}(\mathcal{S})$ may cover non-trivial diffeomorphisms of $X$, the pull-back or push-forward operations must be dealt with care (see [25]). Explicitly, define the following action of $\text{Aut}(\mathcal{S})$ on sections of $\mathcal{S}$:

$$u \cdot \xi = u \circ \xi \circ f_u^{-1} : M \to \mathcal{S}, \quad u \in \text{Aut}(\mathcal{S}), \quad \xi \in \Gamma(\mathcal{S}).$$

This gives an isomorphism of real vector spaces $u : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ for every element $u \in \text{Aut}(\mathcal{S})$. We have $\omega^u = \omega$ if and only if:

$$(\omega^u)(\xi_1, \xi_2) \overset{\text{def}}{=} \omega(u \cdot \xi_1, u \cdot \xi_2) \circ f_u = \omega(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in \Gamma(\mathcal{S}).$$

Likewise, we have $\mathcal{D}^u = \mathcal{D}$ if and only if:

$$\mathcal{D}^u_v(\xi) \overset{\text{def}}{=} u^{-1} \cdot \mathcal{D}_{f_u \circ v}(u \cdot \xi) = \mathcal{D}_v(\xi), \quad \forall \xi \in \Gamma(\mathcal{S}), \quad \forall v \in \Gamma(TX),$$

where $f_u \circ v = df_u(v) \circ f_u^{-1}$ and $df_u : TX \to TX$ is the ordinary differential of $f_u \in \text{Diff}(X)$. Recall that if $v \in \Gamma(TX)$ then $df_u(v)$ is not a vector field on $X$ but a section of $TX$ along $f_u$, whereas $f_u \circ v \in \Gamma(TX)$ is again a vector field on $X$. To illustrate the inner workings of the pull-backed connection $D^u$, we verify that it satisfies the Leibniz identity:

$$\mathcal{D}^u_v(\kappa \xi) = u^{-1} \cdot \mathcal{D}_{f_u \circ v}(u \cdot (\kappa \xi)) = u^{-1} \cdot \mathcal{D}_{f_u \circ v}(u \cdot \xi)$$

$$= u^{-1} \cdot \left( d \left( \kappa \circ f_u^{-1} \right) (f_u \circ v) \cdot u \cdot \xi \right)$$

$$+ u^{-1} \cdot \left( \kappa \circ f_u^{-1} \mathcal{D}_{f_u \circ v}(u \cdot \xi) \right) = u^{-1} \cdot \left( d\kappa (v \circ f_u^{-1}) \cdot u \cdot \xi \right)$$

$$+ u^{-1} \cdot \left( \kappa \circ f_u^{-1} \mathcal{D}_v(u \cdot \xi) \right) = d\kappa (v) \xi + \kappa \mathcal{D}^u_v(\xi),$$

where $\kappa \in C^\infty(X)$ is a function on $X$. On the other hand, the push-forward of $\mathcal{J}$ by $u \in \text{Aut}(\mathcal{S})$ is given by:

$$\mathcal{J}_u(\xi) \overset{\text{def}}{=} \left( u \cdot \mathcal{J}(\xi) \right) = u \circ \mathcal{J}(\xi),$$

for every $\xi \in \Gamma(\mathcal{S})$.

Given a duality bundle $\Delta$ over the scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$, every element $u \in \text{Aut}(\Delta)$ maps a triplet of the form:

$$(g, s, \mathcal{F}) \in \text{Lor}(M) \times \Gamma(\pi) \times \Omega^2(M, \mathcal{S}^e),$$

to a triplet of the form:

$$\mathcal{A}_u(g, s, \mathcal{F}) \overset{\text{def}}{=} (g, f_u \circ s, u \cdot \mathcal{F}) \in \text{Lor}(M) \times \Gamma(\pi) \times \Omega^2(M, \mathcal{S}^{f_u(s)}),$$

where $dot$ denotes the natural action of $\text{Aut}(\mathcal{S})$ on $\mathcal{S}^e$-valued forms.
Remark 2.23 Recall that \( F_m \in \wedge^2 T^*_m M \otimes S^*_m \) or, equivalently:

\[
F_m \in \wedge^2 T^*_m M \otimes S_{s(m)}.
\]

The push-forward \( u \cdot \mathcal{F} \in \Omega^2(M, S^u(s)) \) of \( \mathcal{F} \in \Omega^2(M, S^s) \) by \( u \) produces a \( S^u(s) \)-valued two-form on \( M \) whose value at \( m \in M \) is given by:

\[
(u \cdot \mathcal{F})_m = u_{s(m)}(\mathcal{F}_m) \in \wedge^2 T^*_m M \otimes S^u(s(m))
\]

where \( u_{s(m)} : S_{s(m)} \rightarrow S^u(s(m)) \) acts trivially on the two-form components of \( \mathcal{F} \).

Given \( u \in \text{Aut}(\Delta) \), the map \( \mathcal{A}_u \) defined above need not preserve the configuration space \( \text{Conf}(\Phi) \) defined by a fixed scalar-electromagnetic bundle \( \Phi = (\pi, \mathcal{H}, \mathcal{G}, \Xi) \). Instead we have the following result.

Theorem 2.24 Let \( \pi : X \rightarrow M \) be a smooth submersion. For every connection \( \mathcal{H} \), vertical metric \( \mathcal{G} \) and electromagnetic bundle \( \Xi \) on \( \pi \), an element \( u \in \text{Aut}_\pi(\Delta) \) defines a bijection:

\[
\mathcal{A}_u : \text{Conf}(\Phi) \sim \text{Conf}(\Phi_u), \quad (g, s, \mathcal{F}) \mapsto (g, f_u \circ s, u \cdot \mathcal{F}),
\]

which restricts to a bijection:

\[
\mathcal{A}_u : \text{Sol}(\Phi) \sim \text{Sol}(\Phi_u),
\]

between the solution spaces of the bosonic supergravities associated with \( \Phi \) and \( \Phi_u \) on \( (\pi, \mathcal{H}, \mathcal{G}) \).

Proof Assume that \( (g, s, \mathcal{F}) \in \text{Conf}(\Phi) \), where \( \Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J}) \) and \( u \in \text{Aut}_\pi(\Delta) \) covers \( f_u \in \text{Aut}_\mathcal{J}(X) \). Clearly, \( f_u \circ s : M \rightarrow X \) is again a section of \( \pi \) since \( f_u : X \rightarrow X \) is covers the identity over \( M \). On the other hand, \( u \cdot \mathcal{F} \) is by construction a two-form on \( M \) taking values in \( S^u(s) \) whence \( (g, s, \mathcal{F}) \in \text{Conf}(\Phi_u) \). The fact that this map takes solutions to solutions follows by a computation that involves several different pull-backs through unbased automorphisms of fiber bundles. The reader is referred to [25, Appendix D] for a detailed account of the operations involved. Assume that \( (g, s, \mathcal{F}) \in \text{Sol}(\Phi) \). For the Einstein Eq. (1), we compute:

\[
s^*_\mathcal{G} = (f_u^{-1} \circ f_u \circ s)^*_\mathcal{G} = (f_u \circ s)^* \left( \left( f_u^{-1} \right)^*_\mathcal{G} \right) = (f_u \circ s)^* f_u \circ s \mathcal{G}.
\]

On the other hand, we have:

\[
\mathcal{F} \otimes Q^u \mathcal{F} = \left( u^{-1} \cdot u \cdot \mathcal{F} \right) \otimes Q^u \left( u^{-1} \cdot u \cdot \mathcal{F} \right) = \left( u \cdot \mathcal{F} \right) \otimes Q^u f_u(s) (u \cdot \mathcal{F}),
\]

where \( Q^u f_u(s) \) denotes the bilinear form on \( S^s \) defined as follows:

\[
Q^u f_u(s) (\xi f_u(s), \xi f_u(s)) = \omega (\xi (f_u(s)), \mathcal{J} (f_u(s)) \xi (f_u(s))).
\]
and where $\xi f_u(s) \in S f_u(s)$ for every $\xi \in S$. Combining Eqs. (5) and (6) together with the fact that the left hand side of Eq. (1) is invariant under $u$ we obtain that $(g, f_u \circ s, u \cdot \mathcal{F})$ satisfies the Einstein equations with respect to the scalar-electromagnetic structure $(\pi, G, \mathcal{H}_u, \Delta_u, \mathcal{J}_u)$. For the scalar Eq. (2), we compute:

$$\nabla \Phi(g, s) d^C_s = \nabla \Phi(g, s) d^C_s \left( f_u^{-1} \circ f_u \circ s \right) = \nabla \Phi_u(g_u \cdot f_u(s)) d^C_{f_u(s)}(f_u \circ s).$$

Similarly, using the fact that $u \in \text{Aut}(\Delta)$ preserves the flat connection $\mathcal{D}$ determined by $\Delta$ together with Eq. (6), we obtain:

$$(* \mathcal{F}, \Psi^s \mathcal{F})_{g, Q^s} = \left( *(u \cdot \mathcal{F}), \Psi^s_{f_u(s)}(u \cdot \mathcal{F}) \right)_{g, Q^s_{f_u(s)}},$$

where we have defined:

$$\Psi^s_{f_u(s)} = (\mathcal{D} \mathcal{J}_u) f_u(s).$$

Hence, $(g, f_u \circ s, u \cdot \mathcal{F})$ satisfies the scalar equations associated with the scalar-electromagnetic structure $\Phi_u = (\pi, G, \mathcal{H}_u, \Delta_u, \mathcal{J}_u)$. Since the flatness condition for $\mathcal{F}$ is linear in $\mathcal{F}$ it is enough to verify it on an homogenous element of the form $\mathcal{F} = \alpha \otimes \xi^s$, where $\alpha \in \Omega^s(M)$ and $\xi^s$ is the pull-back by $s$ of a section $\xi \in \Gamma(S)$. We compute:

$$d_{\mathcal{D} f_u(s)}(u \cdot \mathcal{F}) = d_{\mathcal{D} f_u(s)}(\alpha \otimes u \cdot \xi^s) = d_{\mathcal{D} f_u(s)}(\alpha \otimes (u \circ \xi \circ f_u^{-1} \circ f_u(s))) = d\alpha \otimes (u \cdot \xi^s) + \alpha \otimes \mathcal{D} f_u(s)(u \circ \xi \circ f_u^{-1}) f_u(s) = d\alpha \otimes (u \cdot \xi^s) + \alpha \otimes (\mathcal{D}(u \circ \xi \circ f_u^{-1})) f_u(s) = d\alpha \otimes (u \cdot \xi^s) + \alpha \otimes u \cdot \mathcal{D}^s \xi^s = u \cdot d\mathcal{D}_\mathcal{F} = 0,$$

where we have used that $u \in \text{Aut}_\pi(\Delta)$ preserves the symplectic connection $\mathcal{D}$. Whence $u \cdot \mathcal{F}$ is flat with respect to $\mathcal{D} f_u(s)$. On the other hand, the Maxwell equation is also linear in $\mathcal{F}$; hence, it is enough to verify it on an homogenous element $\mathcal{F} = \alpha \otimes \xi^s$. We obtain:

$$* g_{\mathcal{J}_u(s)}(u \cdot \mathcal{F}) = * g_{\mathcal{J}_u(s)}(u \cdot \xi^s) = * g_{\mathcal{J}_u(s)}(u \circ \xi \circ f_u^{-1} \circ f_u(s)) = * g_{\mathcal{J}_u(s)}(u \circ \xi \circ f_u^{-1} \circ f_u(s)) = * g_{\mathcal{J}_u(s)}(u \circ \mathcal{J}(\xi) \circ f_u^{-1} \circ f_u(s)) = * g_{\mathcal{J}_u(s)}(u \cdot \mathcal{J}(\xi^s) = u \cdot * g_{\mathcal{J}_u(s)} \mathcal{F} = u \cdot \mathcal{F},$$

whence $(g, f_u \circ s, u \cdot \mathcal{F})$ also satisfies the Maxwell equations associated with $(\pi, G, \mathcal{H}_u, \Delta_u, \mathcal{J}_u)$. Thus $(g, f_u \circ s, u \cdot \mathcal{F}) \in \text{Sol}(\pi, \mathcal{H}_u, G_u, \Delta_u, \mathcal{J}_u)$ and reversing the previous relations it is easy to see that the map $(g, s, \mathcal{F}) \mapsto (g, f_u \circ s, u \cdot \mathcal{F})$ is a bijection. □
Remark 2.25 The group Aut\(\pi(\Delta)\) is the global counterpart of the so-called pseudo-duality group introduced in [23] as the direct product of the symplectic group and the diffeomorphism group of the simply connected open set on which the theory is considered. When both \(\Delta\) and \(\pi\) are non-trivial, the group Aut\(\pi(\Delta)\) can differ markedly from the local pseudo-duality group of loc. cit.

The following statement follows from [16, Lemma 4.2.8].

Lemma 2.26 Let \(\Delta\) be a duality bundle over the submersion \(\pi : X \to M\) and consider a point \(x \in X\). Then, there exists a canonical isomorphism:

\[
\text{Aut}_b(\Delta) = C(\text{Hol}_x(\mathcal{D}), \text{Aut}(S_x, \omega_x)),
\]

where \(\text{Hol}_x(\mathcal{D})\) is the holonomy group of \(\mathcal{D}\) at \(x\), \(\text{Aut}(S_x, \omega_x) \simeq \text{Sp}(2n_v, \mathbb{R})\) is the automorphism group of the fiber \((S_x, \omega_x) = (S, \omega)|_x\) and \(C(\text{Hol}_x(\mathcal{D}), \text{Aut}(S_x, \omega_x))\) denotes the centralizer of \(\text{Hol}_x(\mathcal{D})\) in \(\text{Aut}(S_x, \omega_x)\).

Fixing \(x \in X\), this shows that (4) is isomorphic with the exact sequence:

\[
1 \to C(\text{Hol}_x(\mathcal{D}), \text{Aut}(S_x, \omega_x))) \to \text{Aut}_{\pi(\Delta)} \to \text{Aut}_0^b(\pi) \to 1.
\]

Given a scalar bundle \((\pi, \mathcal{H}, \mathcal{G})\) and an electromagnetic bundle \(\Xi_1 = (\Delta, \mathcal{J})\) we next introduce a subgroup of \(\text{Aut}_{\pi(\Delta)}\) which preserves the Ehresmann connection \(\mathcal{H}\), the vertical metric \(\mathcal{G}\) and the vertical taming \(\mathcal{J}\). This subgroup gives the global counterpart of the group of continuous U-dualities studied traditionally in the supergravity literature.

Definition 2.27 Let \(\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})\) be a scalar-electromagnetic bundle on \(M\). The classical U-duality group of \(\Phi\) is the subgroup \(U(\Phi)\) of \(\text{Aut}_{\pi(\Delta)}\) consisting of those elements which preserve the Ehresmann connection \(\mathcal{H}\), the vertical metric \(\mathcal{G}\) and the vertical taming \(\mathcal{J}\):

\[
U(\Phi) \overset{\text{def.}}{=} \{ u \in \text{Aut}_{\pi(\Delta)} \mid \mathcal{H}_u = \mathcal{H}, \ \mathcal{G}_u = \mathcal{G}, \ \mathcal{J}_u = \mathcal{J} \}.
\]

Similarly, we denote by \(U_0(\Phi) \subset U(\Phi)\) the subgroup of \(U(\Phi)\) consisting of those elements that cover diffeomorphisms of \(X\) isotopic to the identity. Let \(\text{Aut}_b(\Xi) \subset \text{Aut}_b(\Delta)\) be the group based automorphisms of \(\Xi\) which consists of those vector bundle automorphisms of \(\mathcal{S}\) which cover the identity and preserve \(\omega, \mathcal{D}\) and \(\mathcal{J}\). The U-duality group fits into a short exact sequence:

\[
1 \to \text{Aut}_b(\Xi) \to U(\Phi) \to \text{Aut}^0_b(\pi, \mathcal{H}, \mathcal{G}) \to 1,
\]

where \(\text{Aut}^0_b(\pi, \mathcal{H}, \mathcal{G}) \subset \text{Aut}_b(\pi)\) denotes the subgroup of those based automorphisms of \(\pi\) that can be covered by elements of \(U(\Phi)\) and preserve both the Ehresmann connection \(\mathcal{H}\) and the vertical metric \(\mathcal{G}\). If the scalar bundle \((\pi, \mathcal{H}, \mathcal{G})\) is flat, then the group \(\text{Aut}_b(\pi(\mathcal{H}, \mathcal{G}))\) is finite-dimensional by Lemma 2.26, which in turn implies that \(U(\Phi)\) is a finite-dimensional Lie group. In general, this group is markedly different.
from the U-duality group traditionally considered in the local formulation of the theory. The main feature of the latter is that maps solutions to solutions and hence it can be used as a solution generating mechanism. This key property also holds for \( U(\Phi) \) as a consequence of Theorem 2.24.

**Corollary 2.28** The action \( \Lambda \) of the U-duality group \( U(\Phi) \) preserves both \( \text{Conf}(\Phi) \) and \( \text{Sol}(\Phi) \), i.e., it maps configurations to configurations and solutions to solutions. Moreover, if the scalar bundle \((\pi, \mathcal{H}, \mathcal{G}) \in \Phi\) is flat, then \( U(\Phi) \) is a finite-dimensional Lie group.

For further reference, we introduce the following definition.

**Definition 2.29** The classical U-duality transformation defined by an element \( u \in U(\Phi) \) is the bijection \( \Lambda_u : \text{Sol}(\Phi) \to \text{Sol}(\Phi) \).

### 3 The Dirac–Schwinger–Zwanziger integrality condition

This section discusses the geometric model obtained by imposing the DSZ integrality condition on the universal bosonic sector of four-dimensional supergravity defined by a fixed scalar-electromagnetic bundle. This condition depends on the choice of a Dirac system for the underlying duality bundle \( \Delta \) and of the choice of an integral cohomology class in \( H^2(M, \Delta) \), where integrality is defined relative to that Dirac system.

#### 3.1 The vector space of integral field strengths

The DSZ quantization condition of local supergravity is implemented using a full symplectic lattice. Similarly, we implement the DSZ quantization of the universal bosonic sector defined by a scalar-electromagnetic bundle \( \Phi \) in terms of a smoothly varying fiber-wise choice of full symplectic lattices for the underlying duality bundle \( \Delta \), as proposed in [25]. Recall that a full lattice \( \Lambda \) in a \( 2n \)-dimensional symplectic vector space \( (V, \omega) \) is called symplectic if the restriction of the symplectic pairing \( \omega \) to \( \Lambda \) takes integer values. Such lattices are characterized up to symplectomorphism by their type \( t \in \text{Div}^n \) (see [12, Proposition 1.1]), where:

\[
\text{Div}^n \overset{\text{def.}}{=} \{ t = (t_1, \ldots, t_n) \in \mathbb{Z}_{>0}^n \mid t_1 \mid t_2 \mid \ldots \mid t_n \}\,.
\]

Any full symplectic lattice of type \( t \in \text{Div}^n \) in \((V, \omega)\) admits a basis \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \) such that:

\[
\omega(\lambda_i, \mu_j) = t_j \delta_{ij}, \quad \omega(\lambda_i, \lambda_j) = \omega(\mu_i, \mu_j) = 0 \quad \forall i, j = 1, \ldots, n
\]

and any element \( t \in \text{Div}^n \) is realized as the type of some full symplectic lattice. The symplectic lattice \( \Lambda \) is called principal if \( t = \delta_n \overset{\text{def.}}{=} (1, \ldots, 1) \). The modified Siegel modular group of type \( t \in \text{Div}^n \) is the subgroup \( \text{Sp}(2n, \mathbb{Z}) \subset \text{Sp}(2n, \mathbb{R}) \simeq \text{Aut}(V, \omega) \)
consisting of those symplectic transformations which preserve the standard lattice of type \( t \) in \( \mathbb{R}^{2n} \). We have \( \text{Sp}_{2n}(2n, \mathbb{Z}) = \text{Sp}(2n, \mathbb{Z}) \) and \( \text{Sp}(2n, \mathbb{Z}) \subset \text{Sp}(2n, \mathbb{Z}) \) for all \( t \in \text{Div}^n \).

**Definition 3.1** Let \( \Delta = (S, \omega, D) \) be a duality bundle on the scalar bundle \((\pi, \mathcal{H}, \mathcal{G})\) with submersion \( \pi : X \to M \). A Dirac system on \( \Delta \) is a smooth fiber sub-bundle \( j : \mathcal{L} \hookrightarrow S \) of full symplectic lattices in \((S, \omega)\) which is preserved by the parallel transport \( T \) of the flat connection \( D \) in the sense that we have:

\[
T_\gamma(\mathcal{L}|_{\gamma(0)}) = \mathcal{L}|_{\gamma(1)}
\]

for any piece-wise smooth path \( \gamma \in \mathcal{P}(X) \). The common type of these fiberwise symplectic lattices is called the type of \( \mathcal{L} \). A pair:

\[
\Delta \overset{\text{def.}}{=} (\Delta, \mathcal{L}),
\]

consisting of a duality bundle \( \Delta \) and a choice of Dirac system \( \mathcal{L} \) for \( \Delta \) is called an integral duality bundle.

For every \( x \in X \), the fiber \((S_x, \omega_x, \mathcal{L}_x)\) of an integral duality bundle \( \Delta = (\Delta, \mathcal{L}) \) with \( \Delta = (S, \omega, D) \) is an integral symplectic space as defined in [27, Appendix B]. All fibers of \( \Delta \) are isomorphic as integral symplectic spaces, hence their type does not depend on \( x \in X \) since we assume that \( X \) is connected.

**Remark 3.2** The existence of a Dirac system is obstructed. A duality bundle \( \Delta = (S, \omega, D) \) of rank \( 2n_v \) admits a Dirac system of type \( t \in \text{Div}^n \) if and only if the structure group of \( S \) can be reduced from \( \text{Sp}(2n, \mathbb{R}) \) to \( \text{Sp}_t(2n_v, \mathbb{Z}) \). We say that \( \Delta \) is semiclassical if it admits a Dirac system.

**Definition 3.3** Let \( \Delta_1 = (\Delta_1, \mathcal{L}_1) \) and \( \Delta_2 = (\Delta_2, \mathcal{L}_2) \) be two integral duality bundles on \( M \). A morphism of integral duality bundles from \( \Delta_1 \) to \( \Delta_2 \) is a morphism of duality bundles \( f : \Delta_1 \to \Delta_2 \) such that \( f(\mathcal{L}_1) = \mathcal{L}_2 \).

**Remark 3.4** Given a Dirac system \( \mathcal{L} \) for a duality bundle \( \Delta = (S, \omega, D) \), let \( \mathcal{G}_\Delta \overset{\text{def.}}{=} \mathcal{L}(\mathcal{L}) \) be the locally constant sheaf of continuous sections of the discrete fiber bundle \( \mathcal{L} \). This is a subsheaf of the sheaf \( \mathcal{G}_\Delta \) of flat sections of \((S, D)\) whose stalk at \( x \in X \) identifies with the symplectic lattice \( \mathcal{L}_x \subset S_x \).

For every scalar section \( s \in \Gamma(\pi) \), let \( \mathcal{G}_\Delta^s \overset{\text{def.}}{=} s^*(\mathcal{G}_\Delta) \) be the locally constant sheaf on \( M \) obtained as the pullback of \( \mathcal{G}_\Delta \) through \( s \). The sheaf cohomology groups \( H^k(M, \mathcal{G}_\Delta^s) \) are naturally isomorphic with the cohomology groups \( H^k(M, L^s) \) of \( M \) with coefficients in the local system \( L^s = s^*(\mathcal{L}) \) and play a crucial role in what follows.

**Definition 3.5** An integral electromagnetic bundle is a pair:

\[
\Xi \overset{\text{def.}}{=} (\mathcal{E}, \mathcal{L}),
\]
where $Ξ$ is an electromagnetic bundle on $M$ and $L$ is Dirac system for the duality bundle of $Ξ$. An integral scalar-electromagnetic bundle on $M$ is a pair:

$$\Phi \overset{\text{def}}{=} (\Phi, L),$$

where $\Phi$ is a scalar-electromagnetic bundle on $M$ and $L$ is a Dirac system for the duality bundle of $\Phi$.

Given an integral electromagnetic bundle $Ξ = (Ξ, L)$ with integral duality structure $Δ = (S, ω, D, L)$ over a submersion $π : X \to M$, the quotient:

$$\mathcal{X}_Δ \overset{\text{def}}{=} S/L$$

is a flat fibration over $X$ by symplectic torus groups. The taming $J$ of $Δ$ makes this into a fibration by polarized Abelian varieties which, however, need not be flat since $J$ is not flat unless the underlying electromagnetic bundle $Ξ$ is unitary. The sheaf $\mathcal{G}_{\mathcal{X}_Δ}$ of smooth flat sections of $\mathcal{X}_Δ$ fits into a short exact sequence of sheaves of Abelian groups defined on $X$:

$$0 \to \mathcal{G}_Δ \overset{j}{\to} \mathcal{G}_Δ \to \mathcal{G}_{\mathcal{X}_Δ} \to 0.$$

which pulls-back to a short exact sequence of sheaves of Abelian groups defined on $M$:

$$0 \to \mathcal{G}_Δ^s \overset{j^s}{\to} \mathcal{G}_Δ^s \to \mathcal{G}_{\mathcal{X}_Δ}^s \to 0.$$

The latter induces a long exact sequence in sheaf cohomology, of which we are interested in the following portion:

$$\ldots \to H^1(M, \mathcal{G}_{\mathcal{X}_Δ}^s) \to H^2(M, \mathcal{G}_Δ^s) \overset{j^s}{\to} H^2(M, \mathcal{G}_Δ^s) \to H^2(M, \mathcal{G}_{\mathcal{X}_Δ}^s) \to \ldots .$$

**Definition 3.6** The charge lattice of the integral scalar-electromagnetic structure $Ξ = (Ξ, L)$ relative to the scalar section $s ∈ Γ(π)$ is the lattice:

$$L^s_Ξ \overset{\text{def}}{=} j^s_*(H^2(M, \mathcal{G}_Δ^s)) \subset H^2(M, \mathcal{G}_Δ^s).$$

Elements of this lattice are called integral cohomology classes.

It can be shown that $L^s_Ξ$ is a full lattice in $H^2(M, \mathcal{G}_Δ^s)$ (see [27, Proposition 2.24]). Given an integral scalar-electromagnetic bundle $Φ$, we implement DSZ quantization by restricting the configuration space $\text{Conf}(Φ)$ to a subset $\text{Conf}(Φ) \subset \text{Conf}(Φ)$ obtained by imposing an integrality condition on the elements of $\text{Conf}(Φ)$. This is the appropriate implementation of the DSZ quantization condition in our situation.
Definition 3.7 Let $\Phi$ be an integral scalar-electromagnetic bundle. The integral configuration space $\text{Conf}(\Phi)$ of defined by $\Phi$ is the set:

$$\text{Conf}(\Phi) \overset{\text{def.}}{=} \{(g, s, F) \in \text{Conf}(M, \Phi) \mid [F] \in 2\pi L_s^I\}.$$

The integral solution space $\text{Sol}(\Phi) \subset \text{Conf}(\Phi)$ defined by $\Phi$ is the set:

$$\text{Sol}(\Phi) \overset{\text{def.}}{=} \text{Sol}(\Phi) \cap \text{Conf}(\Phi).$$

For further reference, we introduce a refinement of the previous definition.

Definition 3.8 Let $\Phi$ be an integral scalar-electromagnetic bundle and let $\mathcal{V} \in H^2(X, \mathcal{G}_A)$. The framed integral configuration space $\text{Conf}(\mathcal{V}, \Phi)$ with framing $\mathcal{V}$ of the classical geometric supergravity theory associated with $\Phi$ is defined as the following subset of $\text{Conf}(\Phi)$:

$$\text{Conf}(\mathcal{V}, \Phi) \overset{\text{def.}}{=} \{(g, s, F) \in \text{Conf}(\Phi) \mid [F] = 2\pi j_s^i(\mathcal{V}^s)\},$$

The framed integral solution space $\text{Sol}(\mathcal{V}, \Phi) \subset \text{Conf}(\mathcal{V}, \Phi)$ is the set:

$$\text{Sol}(\mathcal{V}, \Phi) \overset{\text{def.}}{=} \text{Sol}(\Phi) \cap \text{Conf}(\mathcal{V}, \Phi).$$

Definition 3.9 The arithmetic U-duality group of an integral scalar-electromagnetic structure $\Phi = (\Phi, \mathcal{L})$ is the subgroup of $U(\Phi)$ defined through:

$$U(\Phi) \overset{\text{def.}}{=} \{u \in U(\Phi) \mid u(\mathcal{L}) = \mathcal{L}\}.$$

Similarly $U_o(\Phi) \overset{\text{def.}}{=} \{u \in U_o(\Phi) \mid u(\mathcal{L}) = \mathcal{L}\}$.

The arithmetic U-duality group $U(\Phi)$ is the global counterpart of the arithmetic U-duality group of local supergravity normally considered in the physics literature [22, 31]. We remark that the supergravity literature seems to have considered thus far only holonomy trivial Dirac systems $\mathcal{L}$ of principal type, though there is a priori no physical or mathematical reason to make that assumption. We will consider some simple examples of arithmetic U-duality groups in Sect. 5. More elaborated examples will be considered in a separate publication.

4 The DSZ quantization of 4d bosonic supergravity

In this section, we describe the geometric and gauge-theoretic formulation of the universal bosonic sector of 4d supergravity implied by the DSZ quantization condition. This formulation can be constructed through a step-by-step process as done in [27] for Abelian gauge theory. Instead of going through the details of that process, which are similar to those in [27], we give the description of the theory in its final form,
verifying then that it satisfies the appropriate DSZ quantization (see Theorem 4.5). The key ingredient occurring in the construction is a Siegel bundle, a special kind of principal bundle which was introduced and discussed in detail in [27, Section 3] and forms a particular case of the more general notion of principal bundle with weakly Abelian structure group studied in [28], to which we refer the reader for background and further details. Given \( t \in \text{Div}^{n_v} \), we define the following disconnected Lie group:

\[
\text{Aff}_t \overset{\text{def.}}{=} U(1)^{2n_v} \times \text{Sp}_t(2n, \mathbb{Z}),
\]

where \( U(1)^{2n_v} \simeq \mathbb{R}^{2n_v} / \mathbb{Z}^{2n_v} \) is an affine torus group of dimension \( 2n_v \). The group \( \text{Aff}_t \) identifies with the set \( U(1)^{2n_v} \times \text{Sp}_t(2n, \mathbb{Z}) \) equipped with the multiplication rule:

\[
(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 a_2, \gamma_1 \gamma_2), \quad \forall \ a_1, a_2 \in U(1)^{2n_v}, \ \forall \ \gamma_1, \gamma_2 \in \text{Sp}_t(2n, \mathbb{Z}).
\]

The modified Siegel modular group \( \text{Sp}_t(2n, \mathbb{Z}) \) coincides with the automorphism group of the standard integral symplectic space \( (\mathbb{R}^{2n_v}, \omega_{n_v}, \wedge_t) \) of type \( t \), where \( \omega_{n_v} \) is the standard symplectic form on \( \mathbb{R}^{2n_v} \) and:

\[
\Lambda_t \overset{\text{def.}}{=} \mathbb{Z}^{n_v} \oplus \bigoplus_{i=1}^{n_v} t_i \mathbb{Z} \subset \mathbb{R}^{2n_v}
\]

is the standard symplectic lattice of type \( t \) (see [27, Appendix B]). Moreover, \( \text{Aff}_t \) coincides with the group of affine symplectomorphisms of the \( 2n_v \)-dimensional symplectic torus \( (\mathbb{R}^{2n_v} / \Lambda_t, \Omega_t) \), whose symplectic form \( \Omega_t \) is induced by \( \omega_{n_v} \). The connected component of the identity in \( \text{Aff}_t \) is the torus group \( U(1)^{2n_v} \), while the group of components of \( \text{Aff}_t \) is the discrete group \( \text{Sp}_t(2n, \mathbb{Z}) \), which is infinite and non-Abelian when \( n_v > 0 \).

**Definition 4.1** A Siegel bundle \( P_t \) of rank \( n_v \) and type \( t \in \text{Div}^{n_v} \) on \( X \) is a principal bundle defined on \( X \) with structure group \( \text{Aff}_t \). A based isomorphism of Siegel bundles is a based isomorphism of principal bundles.

Let \( (\pi, \mathcal{H}, \mathcal{G}) \) be a scalar bundle with submersion \( \pi : X \rightarrow M \) and consider a Siegel bundle \( P_t \) of rank \( n_v \) and type \( t \in \text{Div}^{n_v} \) over \( X \). As shown in [27], the adjoint bundle of \( P_t \) admits a natural structure of integral duality bundle of type \( t \) which we denote by \( \Delta(P_t) \). By definition, a vertical taming \( J \) of \( P_t \) is a vertical taming of \( \Delta(P_t) \). Given such a taming, the pair \( (P_t, J) \) is called a (positively) polarized Siegel bundle (cf. [27, 28]). The integral electromagnetic bundle \( \Xi(P_t, J) \) determined by \( (P_t, J) \) is defined through:

\[
\Xi(P_t, J) \overset{\text{def.}}{=} (\Delta(P_t), J).
\]

Given a scalar section \( s \in \Gamma(\pi) \), we denote by \( P_t^s \) the pullback of \( P_t \) by \( s \), which becomes a Siegel bundle over \( M \). Similarly, we denote by \( \Delta(P_t^s) \) and \( \Xi(P_t^s, J^s) \) the integral duality and integral electromagnetic bundles defined by \( P_t^s \) and \( J^s \), which
coincide with the $s$-pullbacks of the corresponding bundles defined by $P$ and $J$ on $X$. When necessary, we will write:

$$\Delta(P^s_t) = (S^s, \omega^s, D^s).$$

Let $\text{Conn}(P^s_t)$ be the affine space of connections on $P^s_t$. Elements of this space are invariant one-forms on $P_t$ mapping the fundamental vector fields of $P_t$ to their generators in $\text{aff}_t$, where $\text{aff}_t \simeq \mathbb{R}^{2n_v}$ is the Lie algebra of $\text{Aff}_t$, which has trivial Lie bracket. The adjoint curvature of a connection $A \in \text{Conn}(P^s_t)$ will be denoted by $\mathcal{A} \in \Omega^2(M, S^s)$. This bundle-valued 2-form is $d_D^s$-closed by the Bianchi identity since (by the results of [27, 28]) all connections on $P^s$ induce the same connection on $S^s$, which coincides with the connection induced by $D^s$ on the adjoint bundle of $P^s_t$.

Thus,

$$d_D^s\mathcal{A} = 0.$$

**Definition 4.2** A scalar-Siegel bundle of rank $n_v$ and type $t \in \text{Div}^{n_v}$ over $M$ is a system $\zeta \overset{\text{def.}}{=} (\pi, \mathcal{H}, \mathcal{G}, P_t)$, where $(\pi, \mathcal{H}, \mathcal{G})$ is a scalar bundle over $M$ with submersion $\pi : X \to M$ and $P_t$ is a Siegel bundle of rank $n_v$ and type $t \in \text{Div}^{n_v}$ defined on $X$.

Given a vertical taming $J$ of $\Delta(P_t)$, the system $\zeta \overset{\text{def.}}{=} (\Psi, J)$ is called a polarized scalar-Siegel bundle of rank $n_v$ and type $t$ over $M$.

**Definition 4.3** Let $\zeta \overset{\text{def.}}{=} (\pi, \mathcal{H}, \mathcal{G}, P_t, J)$ be a polarized scalar-Siegel bundle over $M$.

The configuration space of the bosonic supergravity defined by $\zeta$ is the set:

$$\text{Conf}(\zeta) = \{ (g, s, A) \mid g \in \text{Lor}(M), \ s \in \Gamma(\pi), \ A \in \text{Conn}(P^s_t) \}.$$ 

The universal bosonic sector of four-dimensional supergravity determined on $M$ by $\zeta$ is defined through the following system of partial differential equations for triples $(g, s, A) \in \text{Conf}(\zeta)$:

- The Einstein equations:

$$\text{Ric}^g - \frac{g}{2} \text{R}^g = \frac{1}{2} \text{Tr}_g(s^s_c G) g - s^s_c G + 2 D_A \otimes Q^s D_A.$$ \hspace{1cm} (7)

- The scalar equations:

$$\text{Tr}_g(\nabla^{\Phi(g,s)} d^c s) = \frac{1}{2} (\ast D_A, \ast D_A) g, Q^s.$$ \hspace{1cm} (8)

- The Maxwell equations:

$$\ast_g D_A = D_A,$$ \hspace{1cm} (9)

whose set of solutions we denote by $\text{Sol}(\zeta) \subset \text{Conf}(\zeta)$. 

\begin{flushright}
\copyright Springer
\end{flushright}
Connections $\mathcal{A}$ satisfying Eq. (9) will be called polarized self-dual, following the terminology introduced in [27] in the context of Abelian gauge theory.

**Remark 4.4** The Maxwell equations of the bosonic gauge sector of local supergravity are given by a system of second-order partial differential equations for a number $n_v$ of electromagnetic local gauge potentials whose curvatures satisfy a generalization of the Maxwell equations. This is locally equivalent with the description given by the first order global Eq. (9), which reduces locally to a system of first-order partial differential equations for $2n_v$ local gauge fields, both electric and magnetic (considered up to gauge transformations of the principal bundle $P_t$).

The Bianchi identity and polarized self-duality condition imply that the gauge potential of any solution $(g, s, A) \in \text{Sol}(\mathcal{A})$ automatically satisfies the following second order differential equation of Yang–Mills type:

$$d_{D^s} g, \star_{g} \mathcal{F}_A = 0.$$ 

These differ from the usual Yang–Mills equations since $\mathcal{F}_A$ involves both electric and magnetic degrees of freedom while the equations themselves involve the pulled-back taming $\mathcal{J}$.

**Theorem 4.5** Let $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Delta, \mathcal{J})$ be an integral scalar-electromagnetic bundle of type $t$. For every framed integral configuration space $\text{Conf}(\mathcal{W}, \Phi)$ there exists a vertically polarized Siegel bundle $(P_t, \mathcal{J})$ on $(\pi, \mathcal{H}, \mathcal{G})$ such that $\Delta = \Delta(P_t)$ and the twisted Chern class $c(P_t)$ of $P_t$ satisfies $c(P_t) = \mathcal{W}$. Moreover, the map:

$$\text{Sol}(\pi, \mathcal{H}, \mathcal{G}, P_t, \mathcal{J}) \rightarrow \text{Sol}(\mathcal{W}, \Phi), \quad (g, s, A) \mapsto (g, s, \mathcal{F}_A),$$

is surjective.

**Proof** Given $\Delta$ and $\mathcal{W}$, it follows from the results of [5, 6] (see also [28]) that there exists a Siegel bundle $P_t$ of type $t$ (unique up to isomorphism) whose twisted Chern class $c(P_t)$ equals $\mathcal{W}$ and whose adjoint bundle is isomorphic to $\Delta$ as an integral duality bundle. The vertical taming $\mathcal{J}$ in $\Phi$ makes $P_t$ into a polarized Siegel bundle. On the other hand, the curvature of any connection $A \in \text{Conn}(P_t^s)$ defines a $d_{D^s}$-cohomology class $[\mathcal{F}_A]_{D^s}$ in $H^2(M, \mathcal{G}_\Delta^s)$ since, as remarked earlier:

$$d_{D^s} \mathcal{F}_A = d_A \mathcal{F}_A = 0.$$

Given any other connection $A'$ on $P_t$ we have:

$$A' = A + \bar{\tau},$$

for a unique horizontal and invariant one-form $\bar{\tau} \in \Omega^1(P_t, \mathfrak{aff}_t)$. Therefore, the curvatures of $\mathcal{F}_A$ and $\mathcal{F}_{A'}$ are related as follows:

$$\mathcal{F}_{A'} = \mathcal{F}_A + d_{D^s} \tau \in \Omega^2(M, S^c).$$

---

See [27, 28] for its precise definition.
where $\tau \in \Omega^1(M, \mathcal{S}^\ast)$ is uniquely determined by $\bar{\tau} \in \Omega^1(P_t, \text{aff}_t)$. This implies that the cohomology class $[\mathcal{F}_A]_{\mathcal{D}^\tau} \in H^2(M, \mathcal{S}_A^\ast)$ does not depend on the connection $\mathcal{A}$. A similar argument shows that the cohomology class $[\mathcal{F}_A]_{\mathcal{D}^\tau}$ is invariant under automorphisms of $P_t$ and therefore only depends on the isomorphism class of the latter. This is further elaborated in [28] to show that $[\mathcal{F}_A]_{\mathcal{D}^\tau}$ is equal to the real twisted Chern class of $P_t$ as follows:

$$[\mathcal{F}_A]_{\mathcal{D}^\tau} = 2\pi j_x^s(c(P_t^s)) \in L^{\ast}_A.$$ Since $c(P_t) = \mathfrak{H}$ by construction, we immediately conclude that:

$$[\mathcal{F}_A]_{\mathcal{D}^\tau} = 2\pi j_x^s(\mathfrak{H}^\tau) \in L^{\ast}_A.$$

Hence, $(g, s, \mathcal{F}_A)$ belongs to $\text{Sol}(\mathfrak{H}, \Phi)$ for all $(g, s, \mathcal{A}) \in \text{Conf}(\pi, \mathcal{H}, \mathcal{G}, P_t, \mathcal{F})$. Furthermore, every element in $\text{Sol}(\mathfrak{H}, \Phi)$ is of the form $(g, s, \mathcal{F}_A)$ for some $(g, s, \mathcal{A}) \in \mathfrak{Sol}(\pi, \mathcal{H}, \mathcal{G}, P_t, \mathcal{F})$. An explicit way to prove this is to use a good open cover $M \subset \{U_a\}_{a \in I}$ of $M$. Then, given $(g, s, \mathcal{F}) \in \text{Sol}(\mathfrak{H}, \Phi)$, the restriction:

$$\mathcal{F}_a \overset{\text{def.}}{=} \mathcal{F}|_{U_a} = dA_a, \quad A_a \in \Omega^1(U_a, \mathbb{R}^{2n_v}) \quad a \in I,$$

is $d_{\mathcal{D}^\tau}$-exact and hence $d$-exact, since we can trivialize $\Delta$ over $U_a$ as the latter is simply connected. The family of one-forms $\{A_a\}$ taking values in $\mathbb{R}^{2n_v}$ can be shown to define a connection on $P_t^s$ whose curvature is precisely $\mathcal{F}$ and hence we conclude. \hfill \Box

The previous theorem shows that Definition 4.3 realizes geometrically the DSZ quantization of the universal bosonic supergravity sector defined by $\Phi$ since it shows that, given a Dirac system for the duality structure of $\Phi$, every element in the solution space $\text{Sol}(\Phi, \mathcal{L})$ can be realized through a Lorentz metric on $M$, a section of $\pi$ and a gauge potential $\mathcal{A} \in \text{Conn}(P_t^s)$ for some Siegel bundle $P_t^s$ on $X$. The latter is the novel geometric object attached to the DSZ quantization condition.

### 5 The electromagnetic U-duality group

In this section, we investigate the gauge U-duality group of the DSZ quantization of bosonic supergravity, which yields a natural extension of its arithmetic U-duality group and provides the geometric interpretation of U-duality transformations as gauge transformations.

Fix a vertically polarized Siegel bundle $(P_t, \mathcal{F})$ on the total space $X$ of a scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$ and let $\text{Aut}(P_t)$ be the automorphism group of $P_t$. For every $u \in \text{Aut}(P_t)$, denote by $\text{ad}_u : \Delta(P_t) \to \Delta(P_t)$ the automorphism of the integral duality structure $\Delta(P_t)$ defined by $u$. Let $\text{Aut}_\pi(P_t) \subset \text{Aut}(P_t)$ be the subgroup formed by all elements of $\text{Aut}(P_t)$ which cover based automorphisms of the fiber bundle $\pi$, that is:

$$\text{Aut}_\pi(P_t) \overset{\text{def.}}{=} \{u \in \text{Aut}(P_t) \mid f_u \in \text{Aut}_\pi(\pi)\} = \{u \in \text{Aut}(P_t) \mid \pi \circ f_u = \pi\}.$$
We have the a short exact sequence of groups:

$$1 \rightarrow \text{Aut}_b(P_t) \rightarrow \text{Aut}_\pi(P_t) \rightarrow \text{Aut}_b^0(\pi) \rightarrow 1,$$

where $\text{Aut}_b^0(\pi)$ is the subgroup of $\text{Aut}_b(\pi)$ formed by those based automorphisms of $\pi$ that can be covered by elements of $\text{Aut}(P_t)$.

**Definition 5.1** Let $(P_t, \mathcal{J})$ be a vertically polarized Siegel bundle over the scalar-bundle $(\pi, \mathcal{H}, \mathcal{G})$. The **gauge U-duality group** $U(\zeta)$ of the polarized scalar-Siegel bundle $\zeta = (\pi, \mathcal{H}, \mathcal{G}, P_t, \mathcal{J})$ is the subgroup of $\text{Aut}_\pi(P_t)$ consisting of those elements which preserve the Ehresmann connection $\mathcal{H}$, the metric $\mathcal{G}$ and the vertical taming $\mathcal{J}$:

$$U(\zeta) \overset{\text{def}}{=} \{ u \in \text{Aut}_\pi(P_t) \mid \mathcal{H}_u = \mathcal{H}, \ G_u = \mathcal{G}, \ \mathcal{J}_u = \mathcal{J} \}.$$  

Similarly, we denote by $U_o(\zeta) \subset U(\zeta)$ the subgroup of elements that cover automorphisms of $\pi$ isotopic to the identity. We will also refer to $U_o(\zeta)$ as the gauge U-duality group. Let $\text{Aut}_b(P_t, \mathcal{J})$ be the subgroup of $\text{Aut}_b(P_t)$ consisting of those based automorphisms of $P_t$ which preserve $\mathcal{J}$. The gauge U-duality group fits into a short exact sequence:

$$1 \rightarrow \text{Aut}_b(P_t, \mathcal{J}) \rightarrow U(\zeta) \rightarrow \text{Aut}_b^0(\pi, \mathcal{H}, \mathcal{G}) \rightarrow 1,$$

where $\text{Aut}_b^0(\pi, \mathcal{H}, \mathcal{G}) \subset \text{Aut}_b^0(\pi)$ is the subgroup consisting of those based automorphisms of $\pi$ that can be covered by elements of $U(\zeta)$ and preserve the Ehresmann connection $\mathcal{H}$ and the metric $\mathcal{G}$.

The main feature of the local U-duality group of a local supergravity theory is that maps solutions to solutions and thus can be used as a solution generating mechanism. This key property also holds for $U(\zeta)$ as we show below. Let:

$$(g, s, \mathcal{A}) \in \mathcal{S}ol(\zeta).$$

Recall that $\mathcal{A} \in \text{Conn}(P_t^s)$ is a connection on the pull-back of $P_t$ through $s \in \Gamma(\pi)$. An element $u \in \text{Aut}_\pi(P_t)$ which covers $f_u \in \text{Aut}_b(\pi)$ acts on $(g, s, \mathcal{A})$ through:

$$u \cdot (g, s, \mathcal{A}) = (g, f_u(s), \mathcal{A}_u),$$

where $f_u(s) = f_u \circ s \in \Gamma(\pi)$ and $\mathcal{A}_u$ is the push-forward of $\mathcal{A}$ by the based isomorphism of $P_t^s$ naturally associated with $u$ as follows:

$$u_m : (P_t^s)_m = (P_t)_{s(m)} \rightarrow \left( P_t^{f_u(s)} \right)_m = (P_t)_{f_u(s(m))}, \quad p \mapsto u_{s(m)}(p)$$

for all $m \in M$. Notice that $\mathcal{A}_u$ is a connection on the bundle $P_t^{f_u(s)}$, where the latter denotes the pull-back of $P_t$ by the section $f_u(s) \in \Gamma(s)$. Denote by:

$$\zeta_u = (\pi, \mathcal{H}_u, \mathcal{G}_u, P_t, \mathcal{J}_u)$$

$\square$ Springer
the push-forward of $\zeta_u$ of $\zeta$ by $u \in \text{Aut}_\pi(P_t)$.

**Corollary 5.2** Let $\zeta = (\pi, \mathcal{H}, \mathcal{G}, P_t, \mathcal{F})$ be a polarized scalar-Siegel bundle with submersion $\pi : X \to M$. Every element $u \in \text{Aut}(P)$ defines a bijection of sets:

$$A_u : \text{Conf}(\zeta) \to \text{Conf}(\zeta_u), \quad (g, s, A) \mapsto (g, f_u(s), A_u),$$

which restricts to a bijection:

$$A_u : \text{Sol}(\zeta) \to \text{Sol}(\zeta_u).$$

In particular, if $u \in U(\zeta)$ then $A_u : \text{Sol}(\zeta) \to \text{Sol}(\zeta)$ preserves the solution space of the given supergravity theory.

**Proof** The result follows directly from Theorem 2.24 upon noticing that:

$$\mathcal{F}_{A_u} = u \cdot \mathcal{F}_A,$$

which shows that $\mathcal{F}_A$ transforms as the field strength $\mathcal{F}$ considered in Sect. 2 in the classical formulation of the theory. \qed

For further reference, we introduce the following definition.

**Definition 5.3** The gauge *U-duality transformation* induced by $u \in U(\zeta)$ is the bijection $A_u : \text{Sol}(\zeta) \to \text{Sol}(\zeta)$.

We have a canonical morphism of groups:

$$\text{ad} : U(\zeta) \to U(\Phi(\zeta)), \quad u \mapsto \text{ad}_u,$$

where $\Phi(\zeta)$ is the integral scalar-electromagnetic bundle determined by the polarized scalar-Siegel bundle $\zeta$. This morphism associates with $u$ the automorphism of the adjoint bundle of $P_t$ defined canonically by the latter.

**Definition 5.4** The *continuous subgroup* of the gauge U-duality group $U(\zeta)$ is:

$$C(\zeta) \overset{\text{def.}}{=} \ker(\text{ad}) \subset U(\zeta).$$

Similarly, $C_o(\zeta) \overset{\text{def.}}{=} \ker(\text{ad}|_{U_o(\zeta)}) \subset U_o(\zeta)$.

The classical U-duality group was shown to be a finite-dimensional Lie group in Sect. 2.3 when the scalar bundle is flat. This is no longer true for the gauge U-duality group. Instead, if the rank of $P_t \in U(\zeta)$ is positive $U(\zeta)$ is an extension of the arithmetic duality group $U(\Phi(\zeta))$ by an *infinite-dimensional* abelian group, a fact that allows us to pinpoint the geometric origin of U-duality.
Proposition 5.5 The gauge U-duality group \( U_o(\zeta) \) fits into a short exact sequence:

\[
1 \rightarrow C_o(\zeta) \hookrightarrow U_o(\zeta) \xrightarrow{\alpha \otimes} U_o(\Phi(\zeta)) \rightarrow 1,
\]

where \( \Phi(\zeta) \) is the polarized integral scalar-electromagnetic bundle determined by \( \zeta \).

Proof Since it is clear that the natural inclusion \( C(\zeta) \hookrightarrow U(\zeta) \) is injective and the map \( \alpha \otimes \) is a homomorphism, it suffices to prove that \( \alpha \otimes \) is surjective. Write the Dirac system \( L \) in \( \zeta \) as an associated bundle \( L = P_t \times_{\ell} \mathbb{Z}^{2n} \) to \( P_t \) through the natural representation \( \ell \) of \( \text{Aff}_t \) on \( \mathbb{Z}^{2n} \).

Let \( \phi \in U(\Phi(\zeta)) \) be an automorphism of the scalar-electromagnetic bundle \( \Phi(\zeta) \) associated with \( \zeta \) and covering the diffeomorphism \( f_\phi \in \text{Diff}(M) \). Since the latter is a diffeomorphism isotopic to the identity, the principal bundles \( P_t \) and \( P_{t_0} \) covering \( f_\phi \), where the latter denotes the pull-back of \( P_t \) by \( f_\phi \), are isomorphic. Fix such an isomorphism, which is equivalent to fixing an automorphism \( u_\phi' : P_t \rightarrow P_t \) covering \( f_\phi \). Then, for every \( [p, v] \in L \) there exists a unique map \( \mathcal{B}_p \in \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^{2n} \) such that:

\[
\phi([p, v]) = [u_\phi'(p), \mathcal{B}_p(v)].
\]

Since \( \phi \) is a linear automorphism of the integral duality structure determined by \( \zeta \), it follows that the map \( \mathcal{B}_p \in \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^{2n} \) is a linear automorphism of the standard symplectic lattice of type \( t \in \mathbb{Z} \) and therefore belongs to \( \text{Aff}_t \). Furthermore, independence of the representative in \( [p, v] \in L \) in the definition of \( \mathcal{B}_p \) implies:

\[
\mathcal{B}_{px} = x^{-1} \circ \mathcal{B}_p \circ x,
\]

for every \( x \in \text{Aff}_t \). Hence, the assignment \( p \mapsto \mathcal{B}_p \) defines a smooth map \( \mathcal{B} : P_t \rightarrow \text{Aff}_t \) which is equivariant with respect to the adjoint action. Therefore, we have:

\[
\phi([p, v]) = [u_\phi'(p) \mathcal{B}_p, (v)],
\]

and the automorphism \( u_\phi : P_t \rightarrow P_t \) defined as follows:

\[
u_\phi(p) \overset{\text{def}}{=} u_\phi'(p) \mathcal{B}_p, \quad p \in P_t,
\]

covers \( f_\phi \in \text{Diff}(M) \) and satisfies \( \alpha \otimes (u_\phi) = \phi \) by construction. Hence, \( \alpha \otimes \) is surjective and thus we conclude. \( \square \)

It is clear that \( \alpha \otimes u \) is trivial when \( u \in C(\zeta) \). Intuitively speaking, elements in \( C(\zeta) \) behave as gauge transformations on a principal torus bundle and therefore act trivially on the curvature of any connection. In fact, the arithmetic U-duality group \( U(\Phi(\zeta)) \) identifies with the discrete remnant (in the sense of [28]) of the gauge group \( \text{Aut}(P_t) \), which shows that U-dualities in supergravity are but gauge transformations of the underlying Siegel bundle, a fact that elucidates their geometric origin. We discuss next a few examples. An in-depth study of the gauge U-duality group will be presented in a separate publication.
5.1 Rank-zero Siegel bundle

Let \((\pi, H, G)\) be a scalar bundle over \(M\) with submersion \(\pi : X \to M\) and consider the rank zero Siegel bundle \(P_0 = (\text{id}_X : X \to X)\) on \(X\) (which is necessarily trivial). In this case \(\text{Aff}_t\) is the trivial group and we have:

\[
\text{Aut}(P_0) = \text{Diff}(X), \quad \text{Aut}_\pi(P_0) = \text{Aut}_b(\pi), \quad \text{Aut}_b(P_0) = \{\text{id}_X\},
\]

as well as:

\[
\Delta(P_0) = X \times \{0\}.
\]

Let \(\zeta_0 = (\pi, H, G, P_0, J_0)\), where \(J_0 \overset{\text{def.}}{=} \text{id}_{\Delta(P_0)}\). Then,

\[
U(\zeta_0) = \text{Aut}_b^0(\pi, H, G) = \{u \in \text{Aut}_b(\pi) \mid H_u = H, \ G_u = G\}.
\]

Lemma 2.26 shows that \(U(\zeta_0)\) is isomorphic to the commutant of the holonomy group of \(H\) inside the isometry group of the typical fiber of \((M, G)\) of \((\pi, G)\). When the holonomy of \(H\) is trivial, \(U(\zeta_0)\) reduces to the orientation-preserving isometry group \(\text{Iso}(M, G)\) of the scalar manifold but is in general different. In particular, when the holonomy of \(H\) is full, that is, equal to \(\text{Iso}(M, G)\), then \(U(\zeta_0)\) is isomorphic to the center of \(\text{Iso}(M, G)\) and hence possibly trivial. This gives a simple and explicit example illustrating how the supergravity duality group may differ from its local counterpart considered in the literature, which in this case would correspond always with \(\text{Iso}(M, G)\).

5.2 Rank zero scalar bundle

Let \((\pi, H, G)\) be the rank zero scalar bundle, that is, \(X = M\), \(\pi : M \to M\) is the identity map, \(H = TM\) is canonically identified with the tangent bundle of \(M\) and \(G\) is the trivial metric on the rank zero vector bundle over \(M\). Then, the isometry group of the typical fiber of \((\pi, H, G)\) is the trivial group whence the short exact sequence:

\[
1 \to \text{Aut}_b(P_t, J) \to U(\zeta) \to \text{Aut}_b^0(\pi, H, G) \to 1,
\]

reduces to an isomorphism of groups \(U(\zeta) = \text{Aut}_b(P_t, J)\) where \((P_t, J)\) is a polarized Siegel bundle over \(M\). Therefore, the gauge U-duality group reduces to the gauge group of the Siegel bundle underlying the given bosonic supergravity. This corresponds, in fact, with the electromagnetic gauge duality group of the abelian gauge theory determined by \((P_t, J)\) as explained in detail in [27].
5.3 Holonomy-trivial scalar bundle

Consider a holonomy-trivial scalar bundle \((\pi, \mathcal{H}, \mathcal{G})\) in the presentation:

\[ X = M \times M, \quad \mathcal{H} = TM^{pr_1}, \]

where \(M\) is an oriented \(n_s\)-dimensional manifold and \(pr_1: M \times M \to M\) is the canonical projection onto the first factor. In this situation \(Y = TM^{pr_2}\), where \(pr_2: M \times M \to M\) is the canonical projection onto the second factor, and the vertical metric \(\mathcal{G}\) descends to a Riemannian metric on \(M\) which we denote by the same symbol for ease of notation. Furthermore, consider the vertically polarized Siegel bundle \((P_t, J)\) obtained by pull-back through \(pr_2: M \times M \to M\) of a vertically polarized Siegel bundle on \((M, \mathcal{G})\), which we denote again by \((P_t, J)\) for ease of notation. Then:

\[ \text{Aut}_b(\pi) = \text{Diff}(M), \]

where \(\text{Diff}(M)\) the group of oriented diffeomorphisms of \(M\). In particular, we obtain the following short exact sequence:

\[ 1 \to \text{Aut}_b(P_t) \to \text{Aut}(P_t) \to \text{Diff}_0(M) \to 1, \]

where \(\text{Diff}_0(M)\) denotes the subgroup of \(\text{Diff}(M)\) that can be covered by elements in \(\text{Aut}(P_t)\). Here \(P_t\) is considered as a Siegel bundle over \(M\). In this case, the gauge U-duality group is:

\[ U(\zeta) = \{ u \in \text{Aut}(P_t) \mid G_u = G, \ J_u = J \}, \]

and fits into the short exact sequence:

\[ 1 \to \text{Aut}_b(P_t, J) \to U(\zeta) \to \text{Iso}_0(M, \mathcal{G}) \to 1, \]

where \(\text{Iso}_0(M, \mathcal{G})\) is group of those isometries of \((M, \mathcal{G})\) which can be covered by elements in \(U(\zeta)\). Assume in addition that \(P_t\) is topologically trivial and write:

\[ P_t = M \times \text{Aff}_t = M \times \left[ U(1)^{2n_v} \rtimes \text{Sp}_t(2n_v, \mathbb{Z}) \right]. \]

in a fixed trivialization. We have:

\[ \text{Aut}_b(P_t) = C^\infty(M, U(1)^{2n_v} \rtimes \text{Sp}_t(2n_v, \mathbb{Z})), \]

\[ \text{Aut}(P_t) = \text{Diff}(M) \times C^\infty(M, U(1)^{2n_v} \rtimes \text{Sp}_t(2n_v, \mathbb{Z})). \]

Since \(\text{Sp}_t(2n, \mathbb{Z})\) is discrete, we find:

\[ C^\infty(M, U(1)^{2n_v} \rtimes \text{Sp}_t(2n_v, \mathbb{Z})) = C^\infty(M, U(1)^{2n_v}) \rtimes \text{Sp}_t(2n_v, \mathbb{Z}). \]
as well as:

\[
U(\zeta) \overset{\text{def.}}{=} \left\{ (f, u_T, \Omega) \in \text{Iso}(M, G) \ltimes (C^\infty(M, U(1)^{2n_v}) \times \text{Sp}_t(2n_v, \mathbb{Z})) \mid \Omega M^{-1} = J \circ f \right\}.
\]

In particular, the morphism \( \text{ad} : U(\zeta) \to U(\Phi(\zeta)) \) is given explicitly by:

\[
\text{ad}(f, u_T, \Omega) = (f, \Omega) \in \text{Iso}(M, G) \ltimes \text{Sp}_t(2n_v, \mathbb{Z}),
\]

The short exact sequence:

\[
1 \to C^\infty(M, U(1)^{2n_v}) \hookrightarrow U(\zeta) \xrightarrow{\text{ad}} U(\Phi(\zeta)) \to 1,
\]

shows how \( C(\zeta) = C^\infty(M, U(1)^{2n_v}) \) captures the non-discrete gauge transformations in \( U(\zeta) \), which act trivially on the adjoint bundle of \( P \). Consequently, we have:

\[
U(\Phi(\zeta)) \overset{\text{def.}}{=} \left\{ (f, \Omega) \in \text{Iso}(M, G) \times \text{Sp}_t(2n_v, \mathbb{Z}) \mid \Omega J M^{-1} = J \circ f \right\},
\]

which illustrates the explicit dependence of the arithmetic U-duality group on the type \( t \in \text{Div}^n \) of its underlying Siegel bundle \( P_t \).

Acknowledgements We thank Vicente Cortés and Tomás Ortín for useful comments and discussions. The work of C. I. L. was supported by grant IBS-R003-S1. The work of C.S.S. is supported by the Germany Excellence Strategy Quantum Universe—390833306 and the 2022 Leonardo Grant for Researchers and Cultural Creators, BBVA Foundation.

References

1. Aharonov, Y., Bohm, D.: Significance of electromagnetic potentials in the quantum theory. Phys. Rev. 115, 485 (1959)
2. Andrianopoli, L., Bertolini, M., Ceresole, A., D’Auria, R., Ferrara, S., Fre, P., Magri, T.: N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map. J. Geom. Phys. 23, 111 (1997)
3. Andrianopoli, L., D’Auria, R., Ferrara, S.: U duality and central charges in various dimensions revisited. Int. J. Mod. Phys. A 13, 431 (1998)
4. Aschieri, P., Ferrara, S., Zumino, B.: Duality rotations in nonlinear electrodynamics and in extended supergravity. Riv. Nuovo Cim. 31, 625 (2008)
5. Baraglia, D.: Topological T-duality for general circle bundles. Pure Appl. Math. Q. 10(3), 367–438 (2014)
6. Baraglia, D.: Topological T-duality for torus bundles with monodromy. Rev. Math. Phys. 27(3), 1550008 (2015)
7. Ceresole, A., D’Auria, R., Ferrara, S.: The Symplectic structure of N=2 supergravity and its central extension. Nucl. Phys. Proc. Suppl. 46, 67 (1996)
8. Ceresole, A., D’Auria, R., Ferrara, S., Van Proeyen, A.: Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity. Nucl. Phys. B 444, 92 (1995)
9. Cortés, V., Lazaroiu, C.I., Shahbazi, C.S.: \( N = 1 \) Geometric Supergravity and chiral triples on Riemann surfaces. Commun. Math. Phys. (2019)
10. Cremmer, E., Ferrara, S., Girardello, L., Van Proeyen, A.: Yang-Mills theories with local supersymmetry: Lagrangian, transformation laws and superhiggs effect. Nucl. Phys. B 212, 413 (1983)
11. Cremmer, E., Ferrara, S., Girardello, L., Van Proeyen, A.: Coupling supersymmetric Yang-Mills theories to supergravity. Phys. Lett. 116B, 231 (1982)
12. Debarre, O.: Tores et variétés abéliennes complexes, EDP Sciences (2000)
13. de Wit, B., Lauwers, P.G., Van Proeyen, A.: Lagrangians of N=2 supergravity - matter systems. Nucl. Phys. B 255, 569 (1985)
14. de Wit, B., Van Proeyen, A.: Potentials and symmetries of general gauged N=2 supergravity: Yang-Mills models. Nucl. Phys. B 245, 89 (1984)
15. Dirac, P.A.M.: Quantised singularities in the electromagnetic field. Proc. R. Soc. Lond. A 133(821), 60–72 (1931)
16. Donaldson, S.K., Kronheimer, P.B.: The Geometry of Four-Manifolds. Oxford Mathematical Monographs, Oxford University Press, Oxford (1997)
17. Ehresmann, C.: Les connexions infinitesimales dans un espace fibre differentiable, Colloque de topologie (espaces fibres), Bruxelles. Georges Thone. Liege 1951, 29–55 (1950)
18. Freedman, D.Z., Van Proeyen, A.: Supergravity. Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge (2012)
19. Gaillard, M.K., Zumino, B.: Duality rotations for interacting fields. Nucl. Phys. B 193, 221 (1981)
20. Gallerati, A., Trigiante, M.: Introductory lectures on extended supergravities and gaugings. Springer Proc. Phys. 176, 41–109 (2016)
21. Galli, P., Ortin, T., Perz, J., Shahbazi, C.S.: Non-extremal black holes of N=2, d=4 supergravity. JHEP 1107, 041 (2011)
22. Hull, C., Townsend, P.: Unity of superstring dualities. Nucl. Phys. B 438, 109–137 (1995)
23. Hull, C., Van Proeyen, A.: Pseudoduality. Phys. Lett. B 351, 188–193 (1995)
24. Lazaroiu, C.I., Shahbazi, C.S.: Geometric U-folds in four dimensions. J. Phys. A 51(1), 015207 (2018)
25. Lazaroiu, C.I., Shahbazi, C.S.: Generalized einstein-scalar-maxwell theories and locally geometric U-folds. Rev. Math. Phys. 30(05), 1850012 (2018)
26. Lazaroiu, C.I., Shahbazi, C.S.: Section sigma models coupled to symplectic duality bundles on Lorentzian four-manifolds. J. Geom. Phys. 128, 58 (2018)
27. Lazaroiu, C.I., Shahbazi, C.S.: The Duality Covariant Geometry and DSZ Quantization of Abelian Gauge Theory, to appear in Advances in Theoretical and Mathematical Physics
28. Lazaroiu, C.I., Shahbazi, C.S.: The classification of weakly abelian principal bundles, preprint
29. Liu, C.H., Yau, S.T.: Grothendieck Meeting [Wess & Bagger]: [Supersymmetry and Supergravity: IV, V, VI, VII, XXII] (2nd ed.) Reconstructed in Complexified \( \mathbb{Z}/2 \)-graded \( C^\infty \)-Algebraic Geometry, I. Construction Under Trivialization of Spinor Bundle, preprint arXiv:2002.11987
30. Lopes Cardoso, G., Mohaupt, T.: Special geometry. Hessian structures and applications. Phys. Rep. 855, 1–141 (2020)
31. Mizoguchi, S., Schroder, G.: On discrete U duality in M theory. Class. Quant. Grav. 17, 835–870 (2000)
32. Ortín, T.: Gravity and Strings, Cambridge Monographs on Mathematical Physics, 2nd edn. Cambridge University Press, Cambridge (2015)
33. Schwinger, J.S.: Magnetic charge and quantum field theory. Phys. Rev. 144, 1087–1093 (1966)
34. Wood, C.M.: The Gauss section of a Riemannian immersion. J. Lond. Math. Soc. (2) 33(1), 157–168 (1986)
35. Wood, C.M.: Harmonic sections and Yang - Mills fields. Proc. Lond. Math. Soc. (3) 54(3), 544–558 (1987)
36. Zwanziger, D.: Quantum field theory of particles with both electric and magnetic charges. Phys. Rev. 176, 1489–1495 (1968)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.