A REPRESENTATION THEORETIC APPROACH TO KOHNEN'S PLUS SPACE OF MODULAR FORMS OF HALF INTEGRAL WEIGHT

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Abstract. In this paper, we define a notion of pseudo-spherical type for the two fold central extension of $\text{SL}_2(\mathbb{Q}_2)$. We relate this definition to some results in classical modular forms of half integral weights.

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers. For every place $v$ of $\mathbb{Q}$ let $\mathbb{Q}_v$ denote the corresponding local field. Then $\mathbb{Q}_v = \mathbb{R}$ or $\mathbb{Q}_p$ for a prime $p$. The group $\text{SL}_2(\mathbb{Q}_v)$ has a non-trivial two-fold central extension

$$1 \to \mu_2 \to G(\mathbb{Q}_v) \to \text{SL}_2(\mathbb{Q}_v) \to 1$$

where $\mu_2 = \{\pm 1\}$. Recall that an irreducible representation of $G(\mathbb{Q}_v)$ is called genuine if the central subgroup $\mu_2$ acts faithfully on it. Gelbart’s book [G2] contains a basic theory of genuine representations of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ for $p \neq 2$. Our intent is to develop a theory in the case of $G(\mathbb{Q}_2)$. The main difference between $G(\mathbb{Q}_2)$ and $G(\mathbb{Q}_p)$ for $p \neq 2$ lies in the fact that the central extension splits over $\text{SL}_2(\mathbb{Z}_p)$ for $p \neq 2$. In particular, we have a subgroup $K_p \subseteq G(\mathbb{Q}_p)$ isomorphic to $\text{SL}_2(\mathbb{Z}_p)$ under the natural projection from $G(\mathbb{Q}_p)$ to $\text{SL}_2(\mathbb{Q}_p)$ for every $p \neq 2$. A genuine representation $\pi$ of $G(\mathbb{Q}_p)$ is called unramified if it contains a non-zero $K_p$-fixed vector.

Assume now that $p = 2$. Let $K$ denote the full inverse image of $\text{SL}_2(\mathbb{Z}_2)$ in $G(\mathbb{Q}_2)$. In this case the central extension splits over a smaller subgroup. More precisely, we have a subgroup $K_1(4) \subseteq K$ isomorphic to the subgroup of $\text{SL}_2(\mathbb{Z}_2)$ given by the following congruence:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{4}$$

In this paper we completely describe genuine irreducible representations of $G(\mathbb{Q}_2)$ containing non-zero $K_1(4)$-fixed vectors. More precisely, in Section 3 we describe a Hecke algebra $H(\gamma)$ which captures the structure of all representations generated by $K_1(4)$-fixed vectors and with a fixed central character $\gamma$. In Section 4 we show that $H(\gamma)$ is isomorphic to the Iwahori-Matsumoto Hecke algebra for $\text{PGL}_2(\mathbb{Q}_2)$. In this way we get a correspondence between (some) representations of $G(\mathbb{Q}_2)$ and representations of $\text{PGL}_2(\mathbb{Q}_2)$. We call this correspondence a local Shimura correspondence.

1991 Mathematics Subject Classification. 11F70, 22E50, 22E55.
In Section 5 we show that the compact group $K$ has exactly two irreducible genuine representations, with the fixed central character $\gamma$, containing non-zero $K_1(4)$-fixed vectors. These representations are denoted by $V(2)$ and $V(-1)$ and have dimensions 2 and 4, respectively. We show that a representation $\pi$ of $G(\mathbb{Q}_2)$ has $V(2)$ as a $K$-type if and only if it corresponds to an unramified representation of $\text{PGL}_2(\mathbb{Q}_2)$, by the local Shimura correspondence. Thus, it is natural to define unramified representations of $G(\mathbb{Q}_2)$ to be those that contain $V(2)$ as a $K$-type, and we call $V(2)$ a pseudo-spherical type.

We should point out that the center of $G(\mathbb{Q}_2)$ is a cyclic group of order 4. Thus, we have two different genuine central characters $\gamma$ and two classes of unramified representations. This is analogous to the case of the real group $G(\mathbb{R})$, where the weights $-1/2$ and $1/2$ are called pseudo-spherical types.

We then apply our local results in a global setting in Section 8. Let $\mathbb{A}$ be the ring of adeles and let $G(\mathbb{A})$ be the two-fold cover of $\text{SL}_2(\mathbb{A})$. Let $r > 1$ be an odd integer. Let $\pi = \bigotimes \pi_v$ be a genuine cuspidal automorphic representation such that

- $\pi_\infty$ is a holomorphic discrete series representation with the lowest weight $r/2$.
- $\pi_p$ is unramified for all $p \neq 2$.
- $\pi_2$ contains a non-zero $K_1(4)$-fixed vector.

Every such $\pi$ corresponds to a Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$, the space of cuspidal modular forms of weight $r/2$. Roughly speaking, a function $f = \bigotimes f_v$ in $\pi$ such that $f_\infty$ is a lowest weight vector in $\pi_\infty$, $f_p$ is $K_p$-fixed and $f_2$ is $K_1(4)$-fixed, gives naturally a modular form in $S_{r/2}(\Gamma_0(4))$. Since the space of $K_1(4)$-fixed vectors in $\pi_2$ is 2-dimensional, unless $\pi_2$ is a Steinberg representation, the cuspidal automorphic representation $\pi$ gives rise to a two-dimensional Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$. We can pick a line in this subspace by taking $f_2$ to be in the $K$-type isomorphic to $V(2)$. In this way we get a representation-theoretic description of Kohnen’s plus space $S_{r/2}(\Gamma_0(4))$ [Ko]. Finally, we show that the global Shimura correspondence is compatible with our local Shimura correspondence at the place $p = 2$.

2. Double cover of $\text{SL}_2(\mathbb{Q}_v)$

We will describe the double cover $G(\mathbb{Q}_v)$ in (1). If we fix a section $s : \text{SL}_2(\mathbb{Q}_v) \to G(\mathbb{Q}_v)$ then $G(\mathbb{Q}_v)$ can be identified with the set $\text{SL}_2(\mathbb{Q}_v) \times \mu_2$, and the group law on $G(\mathbb{Q}_v)$ is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = \left((g_1 g_2, \epsilon_1 \epsilon_2 \sigma_v(g_1, g_2))\right)$$

where $\sigma_v(g_1, g_2)$ is a cocycle which depends on $s$. Following [G2] we make the following choice of the cocycle $\sigma_v$. Let $(\ , \ )_v$ be the Hilbert symbol over $\mathbb{Q}_v$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Q}_v)$ we define

$$x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0; \end{cases} \quad \text{and} \quad s(g) = \begin{cases} (c, d)_v & \text{if } v \text{ is a finite prime, } cd \neq 0 \text{ and } \text{ord}(c) \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases}$$
Then
\[ \sigma_v(g_1, g_2) = (x(g_1 g_2) x(g_1), x(g_1 g_2) x(g_2)) y s(g_1) s(g_1) s(g_1, g_2). \]

An advantage of this particular section is that \( K_p = s(\text{SL}_2(\mathbb{Z}_p)) \) is a subgroup in \( G(\mathbb{Q}_p) \) if \( p \neq 2 \). If \( p = 2 \), we define
\[
K_1(4) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), 1 \right\} \in \text{SL}_2(\mathbb{Z}_2) \times \{ \pm 1 \} : a \in 1 + 4\mathbb{Z}_2, c \in 4\mathbb{Z}_2 \right\}.
\]

By Proposition 2.14 in [2], \( K_1(4) \) is a compact subgroup of \( G(\mathbb{Q}_2) \).

A smooth representation of \( G(\mathbb{Q}_v) \) is called genuine if \( \mu_2 \) acts non-trivially. If \( p \) is an odd prime number, a smooth genuine representation of \( G(\mathbb{Q}_p) \) is called unramified if it contains a vector fixed by \( K_p \). A vector fixed by \( K_p \) is called a spherical vector.

If \( p = 2 \), a smooth genuine representation is called tamely ramified if it contains a vector fixed by \( K_1(4) \). Unfortunately \( \text{SL}_2(\mathbb{Z}_2) \) does not split in \( G(\mathbb{Q}_2) \) so we could not define spherical vectors in the same manner as those for odd primes. The objective of this paper is to motivate and define spherical vectors of genuine representations of \( G(\mathbb{Q}_2) \).

We set up some notations for the later sections. For \( u \in \mathbb{Q}_v \) and \( t \in \mathbb{Q}_v^\times \), we define the following elements in \( \text{SL}_2(\mathbb{Q}_v) \):

\[
\mathcal{Z}(u) = \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right), \quad \mathcal{Y}(u) = \left( \begin{array}{cc} 1 & 0 \\ u & 1 \end{array} \right), \quad \mathcal{W}(t) = \left( \begin{array}{cc} 0 & t \\ -t & 0 \end{array} \right) \quad \text{and} \quad \mathcal{H}(t) = \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right).
\]

Let \( x(u) = s(\mathcal{Z}(u)), y(u) = s(\mathcal{Y}(u)), w(t) = s(\mathcal{W}(t)) \) and \( h(t) = s(\mathcal{H}(t)) \) in \( G(\mathbb{Q}_v) \). Note that

\[ h(t) h(s) = h(ts)(t, s)v. \]

Let \( N = \{ x(u) : u \in \mathbb{Q}_v \} \), \( \bar{N} = \{ y(u) : u \in \mathbb{Q}_v \} \) and \( T \) be the subgroup of \( G \) generated by elements \( h(t) \).

3. Hecke Algebra at \( p = 2 \)

We fix \( p = 2 \) throughout Sections 3 to 4. We will denote \( G(\mathbb{Q}_2) \) by \( G \) and \( K_1(4) \) by \( K_1 \). The objective of these sections is to classify genuine representations of \( G \) containing a non-zero vector fixed by \( K_1 \).

Let \( M \) be the center of \( G \). It is a cyclic group of order 4 generated by \( h(-1) \). (Note that \( h(-1) h(-1) = (-1, -1)_2 = -1 \in \mu_2 \).) Thus, a genuine central character \( \gamma \) is determined by its value on \( h(-1) \), which is a fourth root of 1. Let \( K \) and \( K_0 \) be the open compact subgroups in \( G \) equal to the inverse images of \( \text{SL}_2(\mathbb{Z}_2) \) and

\[
\left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}_2) : c \in 4\mathbb{Z}_2 \right\}
\]

respectively. Let \( K(4) \subset K_1 \) denote the principal congruence subgroup. It is the image under the section \( s \) of the subgroup of \( \text{SL}_2(\mathbb{Z}_2) \) consisting of matrices congruent to 1 modulo 4. We have \( K \supset K_0 \supset K_1 \supset K(4) \) and \( K_0 = M \times K_1 \). We extend the central character \( \gamma \) to \( K_0 \), so that it is trivial on \( K_1 \). Given a smooth representation \( (\pi, V) \) of \( G \), we denote

\[ V^\gamma := \{ v \in V : \pi(k_0)v = \gamma(k_0)v \ \text{for all} \ k_0 \in K_0 \}. \]
Let $\mathcal{R}(G, \gamma)$ denote the category of admissible smooth (necessarily genuine) representations $V$ of $G$ such that $V^\gamma$ generates $V$ as a $G$-module.

Next we define the corresponding Hecke algebra. Let $C_c(G)$ denote the set of locally constant, compactly supported functions on $G$. Let

$$H(\gamma) = \{ f : C_c(G) : f(k_0gk'_0) = \tau(k_0)f(g)\tau(k'_0) \text{ for all } k_0, k'_0 \in K_0 \}.$$ 

For $f_1, f_2 \in H(\gamma)$, we define

$$f_1 \cdot f_2(g_0) = \int_G f_1(g)f_2(g^{-1}g_0)dg = \int_G f_1(g_0g)f_2(g^{-1})dg$$

where $dg$ is the Haar measure on $G$ such that the measure of $K_0$ is 1. Then $H(\gamma)$ is a $\mathbb{C}$-algebra. For $f \in H(\gamma)$ and $v \in V$, we have

$$\pi(f)v = \int_G f(g)v dg \in V^\gamma.$$ 

In this way $V^\gamma$ is a left $H(\gamma)$-module. Let $\mathcal{R}(H(\gamma))$ denote the category of finite dimensional left $H(\gamma)$-modules. We have a functor $A : \mathcal{R}(G, \gamma) \rightarrow \mathcal{R}(H(\gamma))$ given by $V \mapsto V^\gamma$. Since the group $K_0$ has a triangular decomposition

$$K_0 = (K_0 \cap N)(K_0 \cap T)(K_0 \cap N)$$

the functor $A$ is an equivalence of categories. This follows, in essence, from [Ca], Corollary 3.3.6 (see also [Bo] and Theorem 4.2 in [BZ]).

Our immediate goal is to understand the structure of $H(\gamma)$. The character $\gamma$ of the center $M$ extends to a character $\gamma$ of $T$ which is trivial on $K_1 \cap T$ and $\gamma(h(2^n)) = 1$ for all $n \in \mathbb{Z}$. Let us abbreviate

$$\gamma(t) = \gamma(h(t)).$$

We define

$$\zeta = \frac{1 + \gamma(-1)}{\sqrt{2}}.$$ 

Note that $\zeta$ is a primitive 8-th root of 1. The character $\gamma$ of $T$ is invariant under conjugation by $w = w(1)$. We can now extend the character $\gamma$ from $T$ to the normalizer $N_G(T)$ by defining $\gamma(w) = \zeta$.

We define some functions in $H(\gamma)$. For $g$ in $N_G(T)$ we set $X_g$ to be the function supported on $K_0gK_0$ such that

$$X_g(k_0gk'_0) = \tau(k_0)\tau(g)\tau(k'_0)$$

for all $k_0, k'_0 \in K_0$. Note that this definition depends only on the image of $g$ in the affine Weyl group $W_a := N_G(T)/(T \cap K_0)$.

**Proposition 1.** Functions $X_g$ for $g$ in $W_a$ form a basis of $H(\gamma)$.

**Proof.** We need first to determine the $K_0$-double cosets in $G$. This can be easily determined in $SL_2(\mathbb{Q}_2)$ using the row-column reduction. In addition to $h(2^n)$ and $w(2^{-n})$ the double coset representatives are:

$$y(2), h(2^n)y(2), y(2)h(2^{-n}), y(2)w(2^{-n}), w(2^{-n})y(2) \text{ and } y(2)w(2^{-n})y(2).$$
where \( n \geq 1 \) in all cases. We claim that the Hecke algebra is not supported on these cosets.

**Lemma 2.** The commutator of \( x(2) \) and \( y(2) \) modulo the principal congruence subgroup \( K(4) \) is equal to \(-1 \in \mu_2\).

**Proof.** This can be easily checked using the multiplication rule. It also follows from applying Corollary 2.9 in [St] to the ring \( A = \mathbb{Z}/4\mathbb{Z} \).

Now we can easily finish the proof of proposition. Indeed if \( f \) is in \( H(\gamma) \) then

\[
f(y(2)) = f(y(2)x(2)) = -f(x(2)y(2)) = -f(y(2))
\]

by the above lemma. This implies that \( f \) must vanish on \( y(2) \). Other cases are dealt with in the same manner.

Let \( \ell : N_G(T) \to \mathbb{Z} \) be defined by \( \ell(g) = \log_2(n) \) where \( n \) is the number of left (or right) \( K_0 \)-cosets in the double coset \( K_0gK_0 \). In other words, the volume of \( K_0gK_0 \) is \( 2^{\ell(g)} \). For example, \( w(2^{-1}) \) normalizes \( K_0 \), so \( \ell(w(2^{-1})) = 1 \).

**Proposition 3.** For every integer \( n \) we have \( \ell(h(2^n)) = 2|n| \) and \( \ell(w(2^{-n})) = 2|1-n| \). More precisely, we have the following decompositions of double co-sets:

(i) If \( n \geq 0 \),

\[
K_0h(2^n)K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} x(u)h(2^n)K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0h(2^n)y(4u).
\]

(ii) If \( n \geq 1 \),

\[
K_0h(2^{-n})K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} y(4u)h(2^{-n})K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0h(2^{-n})x(u).
\]

(iii) If \( n \geq 0 \),

\[
K_0w(2^n)K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} x(u)w(2^n)K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} K_0w(2^n)x(u).
\]

(iv) If \( n \geq 1 \),

\[
K_0w(2^{-n})K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} y(4u)w(2^{-n})K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} K_0w(2^{-n})y(4u).
\]

**Proof.** This is an easy consequence of the decomposition \( K_0 = (K_0 \cap N)(K_0 \cap T)(K_0 \cap N) \). Details are left to the reader.

We record the following tautological lemma:

**Lemma 4.** Let \( g_1 \) and \( g_2 \) be two elements in \( N_G(T) \). If \( \ell(g_1g_2) = \ell(g_1) + \ell(g_2) \) then \( X_{g_1} \cdot X_{g_2} = X_{g_1g_2} \).

Let

\[
\begin{cases}
T_n = X_{h(2^n)} \\
U_n = X_{w(2^{-n})}.
\end{cases}
\]
Proposition 5. \( T_w = \sqrt{2}^{-1} U_0 \). We have the following identities where \( m, n \) are any integers unless specified otherwise.

(i) \( (T_w + 1)(T_w - 2) = 0 \).
(ii) \( U_1 \cdot U_1 = 1 \).
(iii) If \( m, n \geq 0 \), or \( m, n \leq 0 \) then \( T_m \cdot T_n = T_{m+n} \).
(iv) \( U_1 \cdot T_n = U_{n+1} \) and \( T_n \cdot U_1 = U_{1-n} \).
(v) \( U_1 \cdot U_n = T_{n-1} \) and \( U_n \cdot U_1 = T_{1-n} \).

Proof. All statements except the first follow from Lemma 4. For (i) we need to show \( T_w^2 = T_w \cdot T_w = T_w + 2 \). Since \( T_w^2 \) is supported in \( K \) this is equivalent to \( T_w^2(1) = 2 \) and \( T_w^2(w(1)) = T_w(w(1)) \). Suppose \( f_1, f_2 \in H(\gamma) \) where \( f_1 \) is supported on \( K_0 \cdot K_0 = \bigcup_{i=1}^n r_i K_0 \) (disjointed union). Then

\[
f_1 \cdot f_2(g) = \sum_{i=1}^s f_1(r_i) f_2(r_i^{-1} g).
\]

We can apply this observation to \( f_1 = f_2 = T_w \). Proposition 3 (the case \( n = 0 \) in (iii)) gives a decomposition of \( K(0) \cdot K_0 \) into single cosets. Hence

\[
T_w^2(g) = \sum_{u \mod 4} T_w(x(u)w(1)) \cdot T_w(w(-1)x(-u)g).
\]

If \( g = 1 \), this gives \( T_w^2(1) = 4T_w(w(1)) \cdot T_w(w(-1)) \). Since \( T_w(w(1)) = 2^{-1/2} \zeta \) and \( T_w(w(-1)) = 2^{-1/2} \zeta \), we obtain that \( T_w^2(1) = 2 \). If \( g = w(1) \), then

\[
T_w^2(w(1)) = T_w(w(1)) \sum_{u \mod 4} T_w(y(u)).
\]

If \( u = 0 \) or \( 2 \) then \( y(u) \) is not in \( K_0 w(1) K_0 \) and \( T_w(y(u)) = 0 \). If \( u = \pm 1 \), then \( y(u) = x(u)w(-u)x(u) \) and we can rewrite

\[
T_w^2(w(1)) = T_w(w(1))[T_w(w(1)) + T_w(w(-1))] = T_w(w(1)).
\]

This proves (i).

Here is the main result of this section.

Theorem 6. The Hecke algebra \( H(\gamma) \) is generated by \( T_w \) and \( U_1 \) as an abstract \( \mathbb{C} \)-algebra modulo the relations

(a) \( (T_w - 2)(T_w + 1) = 0 \) and
(b) \( U_1^2 = 1 \).

Proof. Suppose \( H \) is the abstract algebra generated by \( U_0 = \sqrt{2} T_w \) and \( U_1 \) modulo the relations (a) and (b). We have a natural homomorphism of \( \mathbb{C} \)-algebras \( B : H \to H(\gamma) \). By Proposition 4 \( H(\gamma) \) is spanned by \( T_n \) and \( U_n \) and by Proposition 5 these elements are generated by \( U_0 \) and \( U_1 \). This shows that \( B \) is surjective. It remains to show that \( B \) is injective. Suppose \( h \in H \) is in the kernel of \( B \). Since \( U_0 \) and \( U_1 \) satisfy quadratic relations, \( h = \sum_i c_i u_i \) where \( c_i \in \mathbb{C} \) and \( u_i \in H \) is of the form \( U_1 U_0 U_1 U_0 \ldots \) or \( U_0 U_1 U_0 U_1 \ldots \). Since \( U_0 U_1 = T_1 \), \( B(u_i) \) is either \( T_n, T_n U_1 = U_{1-n}, U_1 T_n = U_{n+1} \), or \( U_1 T_n U_1 = T_{-n} \). These
elements have disjointed supports as functions in $H(\gamma)$. Therefore $B(h) = \sum_i c_i B(u_i) = 0$ implies that $c_i = 0$ and $h = 0$. This proves that $B$ is an injection and Theorem 6 (a) and (b) (see [Ma]). This gives the next corollary.

**Proposition 7.** The element $Z := \frac{P_1}{2} + (\frac{P_2}{2})^{-1}$ belongs to the center of $H(\gamma)$.

**Proof.** By Proposition 5, $T_1$ and $U_1$ generate $H(\gamma)$. Clearly $Z$ commutes with $T_1$. It suffices to show that $Z$ commutes with $U_1$. Since $T_1 = U_0 U_1$ we can use quadratic relations satisfied by $U_0$ and $U_1$ to write

$$2Z = U_0 U_1 + U_1 U_0 - 2^{1/2} U_1.$$

Hence $Z$ commutes with $U_1$. This proves the proposition.

**Proposition 8.** For $n \geq 0$, $T_n$ is an invertible element in the algebra $H(\gamma)$.

**Proof.** Note that the quadratic relations satisfied by $U_0$ and $U_1$ imply that $U_0$ and $U_1$ are invertible. Since $T_1 = U_0 U_1$, $T_1$ is also invertible. Hence $T_n = T_1^n$ is invertible.

Suppose $(\pi, V)$ is a representation in $\mathcal{R}(G, \gamma)$. Let $(V_N)^\gamma = \{v \in V_N : \pi_{V_N}(t)v = \gamma(t)v \text{ for all } t \in K_0 \cap T\}$. The invertibility of $T_n$ implies (see Lemma 4.7 in [Bo]):

**Corollary 9.** Suppose $(\pi, V)$ is a representation in $\mathcal{R}(G, \gamma)$. Then the canonical map $V^\gamma \to (V_N)^\gamma$ is a bijection. In particular $V_N$ is nonzero, and $V$ cannot be a supercuspidal representation.

4. **Local Shimura correspondence**

Let $G' = \text{PGL}_2(\mathbb{Q}_2)$. Let $I$ be its Iwahori subgroup and let $H'$ be its Iwahori-Hecke algebra. Let $T'_w$ and $U'_1$ denote the characteristic functions of

$$I \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I \text{ and } I \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} I$$

respectively. Then $H'$ is the abstract $\mathbb{C}$-algebra generated by $T'_w$ and $U'_1$ satisfying the same relations as Theorem 6 (a) and (b) (see [Ma]). This gives the next corollary.

**Corollary 10.** The Hecke algebras $H(\gamma)$ and $H'$ are isomorphic $\mathbb{C}$-algebras.

Let $\mathcal{R}(H')$ denote the category of finite dimensional representations of $H'$. Let $\mathcal{R}(G', I)$ denote the category of admissible smooth representations $V$ of $G'$ such that $V^I$ generates $V$ as a $G'$-module. By [Bo] and [BZ], the functor $V \mapsto V^I$ is an equivalence of categories from $\mathcal{R}(G', I)$ to $\mathcal{R}(H)$. The isomorphism in Corollary 10 establishes an equivalence of categories between $\mathcal{R}(H(\gamma))$ and $\mathcal{R}(H')$. Hence the following four categories are equivalent:

$$\mathcal{R}(G, \gamma) \simeq \mathcal{R}(H(\gamma)) \simeq \mathcal{R}(H') \simeq \mathcal{R}(G', I).$$

If $V$ is a representation in $\mathcal{R}(G, \gamma)$, then we call the corresponding representation in $\mathcal{R}(G', I)$ the local Shimura lift of $V$. We denote it by $\text{Sh}(V)$.

**Proposition 11.** Let $V$ be a representation in $\mathcal{R}(G, \gamma)$. Then the following statements are equivalent.
(i) The local Shimura lift $\text{Sh}(V)$ is a spherical representation of $G'$.
(ii) The action of $T'_w$ on $\text{Sh}(V)'$ has an eigenvalue 2.
(iii) The action of $T'_w$ on $V^\gamma$ has an eigenvalue 2.

Proof. The projection map to $G'(\mathbb{Z}_2)$-fixed vectors in $\text{Sh}(V)$ is given by $\frac{1}{3}(T'_w + 1)$, since $T'_w + 1$ is the characteristic function of $G'(\mathbb{Z}_2)$ and the volume of $G'(\mathbb{Z}_2)$ is 3. It follows that a $G'(\mathbb{Z}_2)$-fixed vector is an eigenvector of $T'_w$ with eigenvalue 2. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Corollary 10.

The above proposition motivates the following definition.

Definition. Let $V$ be a smooth representation of $G$. An eigenvector of $T_w$ in $V^\gamma$ with an eigenvalue 2 is called a $\gamma$-spherical vector. The representation is called a $\gamma$-unramified or $\gamma$-spherical representation if it contains a $\gamma$-spherical vector.

5. Pseudo-spherical representation of $K$ at $p = 2$

We retain the notations in Sections 3 and 4 where $p = 2$. In the previous section we defined a representation $V$ of $G$ to be unramified if $V^\gamma \neq 0$ and $T_w$ has an eigenvalue 2 on $V^\gamma$. In this section we shall reinterpret this condition in terms of representations of $K$. We shall see that $K$ has only two irreducible representations $E$ such that $E^\gamma \neq 0$. For both representations $E^\gamma$ is one dimensional and they are distinguished by the action of $T_w$ on $E^\gamma$. That eigenvalue can be either 2 or $-1$, so we shall use the eigenvalue to denote the representations by $V(2)$ and $V(-1)$. Their dimensions are 2 and 4, respectively. Thus, a representation of $G$ is unramified if and only if it contains the two-dimensional $K$-type $V(2)$, which we may call a pseudo-spherical type.

If $E^\gamma \neq 0$ then, by Frobenius reciprocity, the $K$-type $E$ is a summand of a six dimensional induced representation

$$I_K(\gamma) := \text{Ind}_{K_0}^K \gamma = \{ \phi : K \to \mathbb{C} : \phi(k_0k) = \gamma(k_0)\phi(k) \text{ for all } k \in K, k_0 \in K_0 \}.$$ 

Here the group $K$ acts on it by right translation. We denote this action by $\pi_R$. Let $H_K(\gamma)$ denote the subalgebra of $H(\gamma)$ consisting of functions supported on $K$. We have the action of $H(\gamma)$ on $I_K(\gamma)^\gamma$, also denoted by $\pi_R$. By Proposition III, $H_K(\gamma) = \mathbb{C}1 \oplus \mathbb{C}T_w$ and it is a commutative subalgebra. The algebra $H_K(\gamma)$ is anti-isomorphic to the algebra $H_K(\bar{\gamma})$ via the map $f \mapsto \hat{f}$ where

$$\hat{f}(g) = f(g^{-1}).$$

For $f \in H_K(\gamma)$ and $\phi \in I_K(\gamma)$, we set

$$(\pi_L(f)\phi)(g) = \int_K f(k)\phi(k^{-1}g)dk \text{ for all } g \in K.$$ 

This action commutes with the right action $\pi_R$ of $K$ on $I_K(\gamma)$ and

$$H_K(\gamma) = \text{End}_K(I_K(\gamma)).$$

Note that $I_K(\gamma)^\gamma = H(\bar{\gamma})$. The actions $\pi_L$ and $\pi_R$ of $H(\bar{\gamma})$ and $H(\gamma)$ on $I_K(\gamma)^\gamma = H(\bar{\gamma})$ are related by $\pi_L(\hat{f}) = \pi_R(f)$. 


We define the functions $F_{-1} := \frac{1}{3}(2-T_w)$ and $F_2 := \frac{1}{3}(T_w+1)$ in $H_K(\gamma)$. Then $\{F_{-1}, F_2\}$ is a basis of idempotents of $H_K(\gamma)$.

For $j = -1, 2$, let $V(j) = \pi_L(\hat{F}_j)I_K(\gamma)$. In other words $V(j)$ is the eigenspace of $\pi_L(T_w)$ on $I_K(\gamma)$ corresponding to the eigenvalue $j$. Note that $\hat{F}_j \in V(j)$ and $\pi_R(T_w)\hat{F}_j = j\hat{F}_j$. In particular $\hat{F}_2$ is a $\gamma$-spherical vector.

**Proposition 12.** 
(i) We have $I_K(\gamma) = V(-1) \oplus V(2)$ where each summand is an irreducible representation of $K$.
(ii) We have $\dim V(-1) = 4$ and $\dim V(2) = 2$.
(iii) The $K$-submodule $V(2)$ contains a $\gamma$-spherical vector $\hat{F}_2$. The space of $\gamma$-spherical vectors is one dimensional.
(iv) The $K$-submodule $V(-1)$ does not have any $\gamma$-spherical vector.

**Proof.** Since $\dim \text{End}(I_K(\gamma)) = 2$, both $V(-1)$ and $V(2)$ are irreducible $K$-modules. This proves (i).

In order to compute the dimensions of $V(-1)$ and $V(2)$ we need the following lemma.

**Lemma 13.** The operator $\pi_L(T_w)$ as an element in $\text{End}_K(I_K(\gamma))$ has trace 0.

**Proof.** For $g \in K$, let $\phi_g \in I_K(\gamma)$ such that $\phi_g$ is supported on $K_0g$ and $\phi_g(k_0g) = \gamma(k_0)$. Let $S$ be a set of representatives of $K_0\backslash K$, then $\{\phi_g : g \in S\}$ is a basis of $I_K(\gamma)$. In order to prove the lemma, it suffices to show that $(\pi_L(T_w)\phi_g)(g) = 0$. Indeed, this shows that the matrix of $\pi_L(T_w)$ in the basis $\phi_g$ has all diagonal entries equal 0. Note that $\pi_L(T_w)\phi_g$ is supported on $K_0w(1)K_0g$. If $(\pi_L(T_w)\phi_g)(g) \neq 0$, then $g \in K_0w(1)K_0g$ and $1 \in K_0w(1)K_0$. Since $K_0 \neq K_0w(1)K_0$, this is a contradiction. The lemma is proved.

We have $\dim V(2) + \dim V(-1) = \dim I_K(\gamma) = |K : K_0| = 6$. By the above lemma, $2 \dim V(2) - \dim V(-1) = 0$. This implies $\dim V(-1) = 4$ and $\dim V(2) = 2$ and proves Proposition 12(ii). We have $I_K(\gamma) = H_K(\gamma)$ and $\pi_R(F_j)I_K(\gamma) = \mathbb{C}\hat{F}_j$ for $j = -1, 2$. The vector $\hat{F}_2$ is $\gamma$-spherical while $\hat{F}_{-1}$ is not. This proves Parts (iii) and (iv).

**Theorem 14.** A smooth representation $V$ of $G$ with central character $\gamma$ is $\gamma$-unramified if and only if there is a nontrivial $K$-module homomorphism $l : V(2) \to V$. A vector in $V$ which is a scalar multiple of $l(\hat{F}_2)$ is a $\gamma$-spherical vector of $V$.

**Proof.** A $\gamma$-spherical vector in $V$ would generated a representation of $K$ where every irreducible $K$-submodule is isomorphic to an irreducible submodule of $I_K(\gamma)$. Now the theorem follows from Proposition 12.

6. UNRAMIFIED PRINCIPAL SERIES REPRESENTATIONS AT $p = 2$

In this section, we continue to assume $p = 2$ and notations as in Sections 3 to 5. We will show that $\gamma$-unramified representations appear as submodules of principal series representations.

We recall the character $\gamma$ of $T$ in Section 3. Let $(\pi_s, I(\gamma, s))$ be the normalized induced principal series representation where $I(\gamma, s)$ is the set of smooth functions $\phi : G \to \mathbb{C}$
It remains to determine $c \in \mathbb{Q}$ satisfying
\[
\phi(\varepsilon x(u)h(t)g) = \varepsilon \gamma(t)|t|^{s+1}\phi(g)
\]
for all $\varepsilon \in \mu_2$, $u \in \mathbb{Q}_2$ and $t \in \mathbb{Q}_2^\times$. The group $G$ acts by left translation $(\pi_s(g)\phi)(g') = \phi(g'g)$.

**Proposition 15.** An irreducible $\gamma$-unramified representation $V$ is isomorphic to a submodule of some $I(\gamma, s)$.

**Proof.** By Corollary 9, $(V_N)^\gamma$ is nonzero. Hence there is a nontrivial $T$-homomorphism $V_N \to \gamma\nu^{s+1}$ for some $s \in \mathbb{C}$. Here $\nu$ is the character $\nu(h(t)) = |t|$. By Frobenius reciprocity, there is a nontrivial map $V \to I(\gamma, s)$ which is an injection because $V$ is irreducible. \hfill $\square$

We recall that $K(4)$ is the principal congruence subgroup in $K_1$. Restricting functions $\phi$ in $I(\gamma, s)$ to $K$ gives a natural isomorphism of $K$-modules
\[
l : I_K(\gamma) \to I(\gamma, s)^{K(4)}.
\]

**Theorem 16.** The $K$-types $V(2)$ and $V(-1)$ appear with multiplicity one in $I(\gamma, s)$. The space $I(\gamma, s)^\gamma$ is 2-dimensional. It is spanned by $l(\hat{F}_2)$ and $l(\hat{F}_{-1})$. \hfill $\square$

We will describe a scalar multiple $\phi_j$ of $l(\hat{F}_j) \in V_j$ which is more convenient for later calculations. Let $d_2 = 1$ and $d_{-1} = -2$, and define $\phi_j$ be the unique vector in $I(\gamma, s)$ whose restriction to $K$ is given by
\[
\phi_j(k) = \begin{cases} \frac{d_j \gamma(k)}{2^{-1/2} \zeta \gamma(k_0 k')} & \text{if } k \in K_0, \\ 0 & \text{otherwise.} \end{cases}
\]

We define an intertwining map $M(s) : I(\gamma, s) \to I(\gamma, -s)$ by
\[
(M(s)\phi)(g) = \int_{\mathbb{Q}_2} \phi(w(1)x(u)g)du
\]
where $g$ is in $G$ and $du$ is the Haar measure on $\mathbb{Q}_2$ such that the measure of $\mathbb{Z}_2$ is 1.

**Proposition 17.** We have
\[
M(s)\phi_2 = \frac{\zeta}{\sqrt{2}} \left( \frac{1 - \frac{1}{2} (2^{-2s})}{1 - 2^{-2s}} \right) \phi_2 \quad \text{and} \quad M(s)\phi_{-1} = -\frac{\zeta}{2\sqrt{2}} \left( \frac{1 - 2(2^{-2s})}{1 - 2^{-2s}} \right) \phi_{-1}.
\]

**Proof.** Since the vector $\phi_j$ is unique up to a scalar in $I_K(\gamma)$, $M(s)\phi_j = c\phi_j$ for some $c \in \mathbb{C}$. It remains to determine $c = d_j^{-1}M(s)\phi_j(1)$.

If $u \notin \mathbb{Z}_2$, then $w(1)x(u) = (-1, u)_2 \cdot x(-u^{-1})h(u^{-1})y(u^{-1})$. We write $u^{-1} = 2^m v$ where $v \in \mathbb{Z}_2^\times$ and $m \geq 1$. Recall that $\gamma(t) = \gamma(h(t))$. Then
\[
M(s)\phi_j(1) = \int_{\mathbb{Z}_2} \phi_j(w(1)x(u))du + \sum_{m=1}^\infty 2^{m-1} \int_{\mathbb{Z}_2^\times} (-1, 2^m v)_2 \phi_j(h(2^m v)y(2^m v))d^x v
\]
\[
= 2^{-1/2} \zeta + \sum_{m=1}^\infty 2^{-m-1} \int_{\mathbb{Z}_2^\times} (-1, v)_2 \gamma(2^m v)\phi_j(y(2^m v))d^x v
\]
where \( d^x v \) is the Haar measure of \( \mathbb{Z}_2^x \) with total measure 1. Now \( \phi_j(y(2^m v)) = 0 \) if \( m = 1 \) and it is equal to 1 if \( m \geq 2 \). Since \( \gamma(2^m v) = \gamma(2^m) \gamma(v)(2^m, v) \) and \( \gamma(2^m) = 1 \), we can rewrite

\[
M(s) \phi_j(1) = 2^{-1/2} \zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \int_{\mathbb{Z}_2} (2, v)^m_2 (-1, v) \gamma(v) d^x v
\]

\[
= 2^{-1/2} \zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \frac{1}{4} \sum_{v \in (\mathbb{Z}/8\mathbb{Z})^x} (2, v)^m_2 (-1, v) \gamma(v).
\]

The sum \( \sum_{v \in (\mathbb{Z}/8\mathbb{Z})^x} \) on the right is zero if \( m \) is odd, and equals \( \sqrt{2} \zeta \) if \( m \) is even. Finally adding up all the terms gives the constant \( c \) and the lemma.

Let \( s_0 = \frac{1}{2} \) or \( \frac{1}{2} + \frac{\pi i}{\log 2} \). From the above proposition, \( \phi_{-1} \) lies in the kernel of \( M(s_0) \) so \( I(\gamma, s_0) \) is reducible. Indeed \( I(\gamma, s_0) \) has a unique irreducible quotient which is an even Weil representation.

**Definition.** Let \( s_0 = \frac{1}{2} \) or \( \frac{1}{2} + \frac{\pi i}{\log 2} \). The kernel of \( M(s_0) \) is called the **Steinberg representation** of \( G(\mathbb{Q}_2) \). We shall denote this representation by \( St(\epsilon) \) where \( \epsilon = \pm 1 \) such that \( 2^{s_0} = \epsilon \sqrt{2} \).

We claim that \( St(\epsilon) \) is an irreducible representation of \( G(\mathbb{Q}_2) \). Indeed by Section 6 in [LS], for every \( s \in \mathbb{C} \), we have

\[
I(\gamma, s)^{s_0} \cong \gamma| \cdot |^{s+1} \oplus \gamma| \cdot |^{-s+1}
\]

where \( I(\gamma, s)^{s_0} \) is the semi-simplification of \( I(\gamma, s) \) as a \( T \)-module. Hence \( I(\gamma, s) \) has at most length 2. The claim now follows because \( St(\epsilon) \) is a proper submodule of \( I(\gamma, s_0) \). Also see Section 7 of [Sa].

**Corollary 18.** The even Weil representation contains the irreducible \( K \)-module \( V(2) \). It is a \( \gamma \)-unramified representation. The Steinberg representation contains the irreducible \( K \)-module \( V(-1) \). □

**Proposition 19.** Let \( Z = \frac{T}{2} + (\frac{T}{2})^{-1} \) be the central element in the Hecke algebra \( H(\gamma) \) as in Proposition 7. Then \( \pi_s(Z) \) acts on \( I(\gamma, s)^{\gamma} \) as the scalar \( 2^s + 2^{-s} \).

**Proof.** By Corollary 9 the natural projection of \( I(\gamma, s) \) on \( I(\gamma, s)_N \) gives an isomorphism of \( I(\gamma, s)^{\gamma} \) and \( I(\gamma, s)_N \). From the decomposition of \( K_0 h(2)K_0 \) into single \( K_0 \)-cosets (Proposition 8(i)) it follows that the action of \( T_1 \) on \( I(\gamma, s)^{\gamma} \) corresponds to the action of \( 4 \pi_s N(h(2)) \) on \( I(\gamma, s)_N \). By 23 the eigenvalues of \( \frac{T}{2} \) are \( 2^s \) and \( 2^{-s} \). This proves the proposition. □

**Corollary 20.** An irreducible \( \gamma \)-unramified representation is uniquely determined by the eigenvalue of the action of \( Z \) on its \( \gamma \)-spherical vector.

**Proof.** Suppose the irreducible \( \gamma \)-unramified representation is a subquotient of both \( I(\gamma, s) \) and \( I(\gamma, s') \). Then by Proposition 19 \( 2^s + 2^{-s} = 2^{s'} + 2^{-s'} \) which implies \( 2^s = 2^{s'} \) or \( 2^s = 2^{-s'} \). By Proposition 27 both \( I(\gamma, s) \) and \( I(\gamma, -s) \) have the same irreducible \( \gamma \)-unramified subquotient. This proves the corollary. □
Corollary 21. The Steinberg representation $St(\epsilon)$ corresponds to the one dimensional representation of $H(\gamma)$ given by $T_w = -1$ and $U_1 = -\epsilon$.

Proof. We know that $T_w = -1$ on $St(\epsilon)^\gamma$. It remains to compute the action of $U_1$. Since $St(\epsilon)$ is a subquotient of $I(\gamma, s_0)$ where $2^{s_0} = \epsilon \sqrt{2}$, the central element $Z$ acts on $St(\epsilon)$ by the scalar $\epsilon(2^{1/2} + 2^{-1/2})$. By [19] we have $2^{1/2}Z = T_wU_1 + U_1T_w - U_1$. Hence $U_1 = -\epsilon$ as claimed.

Let $V$ be an irreducible $\gamma$-unramified representation. By Proposition 15, we may assume that $V$ is the unique $\gamma$-unramified subquotient of $I(\gamma, s)$ for some $s \in \mathbb{C}$. By Proposition 11 its local Shimura lift $V' = Sh(V)$ is an unramified irreducible representation of $G' = PGL_2(\mathbb{Q}_2)$. Let $B'$ be the Borel subgroup of $G'$. We may realize $V'$ as the unramified irreducible subquotient of the normalized induced principal series representation $(\pi'_s, I'(t))$ with trivial central character. Here $I'(t) = \text{Ind}_{B'}^{G'} \omega^t$ (normalized induction) where $\omega$ is the character $\omega \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right) = |a_1/a_2|$.

Theorem 22. If $V$ is the unique $\gamma$-unramified irreducible subquotient of $I(\gamma, s)$, then its local Shimura lift $Sh(V)$ is the unique unramified irreducible subquotient of $I'(s)$.

Proof. Assume that $Sh(V)$ is a subquotient of $I'(t)$. By Proposition 19 the central operator $Z$ in $H(\gamma)$ acts on $I(\gamma, s)^\gamma$ by the scalar $2^s + 2^{-s}$. The corresponding operator $Z'$ in the algebra $H'$ acts on $I'(t)$ by $2^t + 2^{-t}$. Thus, $2^s + 2^{-s} = 2^t + 2^{-t}$. Solving the equation gives $2^s = 2^t$ or $2^s = 2^{-t}$. Both $I'(t)$ and $I'(-t)$ have the same irreducible subquotients so we may set $s = t$. This proves the theorem.

Corollary 23. The principal series representation $I(\gamma, s)$ is reducible if and only if $s = \frac{1}{2}$ or $\frac{1}{2} + \frac{i\pi}{\log 2}$.

Proof. Let $V$ be the $\gamma$-unramified irreducible subquotient of $I(\gamma, s)$. Let $W$ be the unramified irreducible subquotient of $I'(s)$. Then $V = I(\gamma, s)$ if and only if $\dim V^\gamma = 2$. By Theorem 22, $\dim V^\gamma = \dim W'$. Now $\dim W' = 2$ if and only if $I'(s)$ is irreducible. Finally $I'(s)$ is irreducible if and only if $s \neq \frac{1}{2}$ and $\frac{1}{2} + \frac{i\pi}{\log 2}$.

7. Automorphic forms

In this section we first review a connection between automorphic forms and classical modular forms of half integral weight. This is mostly a well known material that can be found in Chapters 2 and 3 of [22], and in [3]. We then transfer the action of the Hecke algebra $H(\gamma)$ to the setting of classical modular forms.

Let $\mathbb{A} = \prod_v \mathbb{Q}_v$ be the ring of adeles over $\mathbb{Q}$. We recall $K_p$, $s(g)$ and the cocycle $\sigma_v$ defined in Section 2. Let $G(\mathbb{A}) = \text{SL}_2(\mathbb{A}) \times \{\pm 1\}$ as a set. For $g_1 = (g_{1,v}), g_2 = (g_{2,v}) \in \text{SL}_2(\mathbb{A})$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, the group law on $G(\mathbb{A})$ is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\sigma(g_1, g_2))$$
where \( \sigma(g_1, g_2) = \prod_v \sigma_v(g_1, g_2) \). Then \( pr : G(\mathbb{A}) \to SL_2(\mathbb{A}) \) given by \( pr(g, \epsilon) = g \) is a two-fold cover which splits over the subgroup \( SL_2(\mathbb{Q}) \). Since \( SL_2(\mathbb{Q}) \) is perfect this splitting is unique and it is given by \( s_Q : SL_2(\mathbb{Q}) \to G(\mathbb{A}) \)

\[
s_Q(g) = (g, s_\mathbb{A}(g))
\]

where \( s_\mathbb{A}(g) = \prod_v s(g_v) \).

We also need a description of a maximal compact subgroup in \( G(\mathbb{R}) \). Let

\[
k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SL_2(\mathbb{R})
\]

for \(-\pi < \theta \leq \pi\). Then \( K_\infty := \{ k(\theta) : -\pi < \theta \leq \pi \} \) is a maximal compact subgroup in \( SL_2(\mathbb{R}) \). Let \( K_\infty = \{ k(\theta) : -2\pi < \theta \leq 2\pi \} \) where

\[
k(\theta) = \begin{cases} (k(\theta), 1) & \text{if } -\pi < \theta \leq \pi, \\ (k(\theta), -1) & \text{if } -2\pi < \theta \leq -\pi \text{ or } \pi < \theta \leq 2\pi. \end{cases}
\]

Then \( K_\infty \) is a maximal compact subgroup of \( G(\mathbb{R}) \) and \( pr(K_\infty) = K_\infty \). If \( r \) is an odd integer, then \( k(\theta) \mapsto e^{i\frac{\pi}{r} \theta} \) defines a genuine character of \( K_\infty \).

Let \( A_{r/2}(4) \) denote the set of functions \( \varphi \) in \( L^2(SL_2(\mathbb{Q}) \setminus G(\mathbb{A})) \) satisfying the following properties:

1. \( \varphi(gk_1) = \varphi(g) \) for all \( k_1 \in K_1(4) \prod_{p \neq 2, \infty} K_p \),
2. \( \varphi(gk_0) = \gamma(k_0)\varphi(g) \) for all \( k_0 \in K_0 \) in \( G(\mathbb{Q}_2) \) where \( \gamma(-1) = -i^r \),
3. \( \varphi(gr(\theta)) = e^{i\frac{\pi}{r} \theta} \varphi(g) \),
4. as a function on \( G(\mathbb{R}) \), \( \varphi \) is smooth and satisfies \( \Delta \varphi = -\frac{r}{4}(\frac{r}{4} - 1) \varphi \) where \( \Delta \) is the Casimir operator and
5. \( \varphi \) is cuspidal, ie. \( \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \varphi(x(u))du = 0 \) for all \( g \in G(\mathbb{A}) \).

A basis of \( A_{r/2}(4) \) arises from cuspidal automorphic representations \( \pi = \otimes_v \pi_v \) of \( G(\mathbb{A}) \) such that \( \pi_\infty \) is a holomorphic discrete series representation with the lowest weight \( r/2 \), \( \pi_p \) is unramified for all \( p \neq 2 \), and \( \pi_2 \) contains a \( K_1(4) \)-fixed vectors. In particular, \( \pi_2 \neq 0 \) for some central character \( \gamma \). Note that \( \gamma \) is determined by \( r \). Indeed, since the local components of \( s_Q(h(-1)) \) for \( v \neq \infty, 2 \) are contained in \( K_p \), \( \varphi(1) = \varphi(s_Q(h(-1))) = \gamma(-1)e^{i\pi r/2} \varphi(1) \) and we get \( \gamma(-1) = -i^r \).

Let \( \mathcal{H} \) be the complex upper half plane. For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \), \( g = (g, \epsilon) \in G(\mathbb{R}) \) and \( z \in \mathcal{H} \), we define

\[
 gz = g z = \frac{az + b}{cz + d}.
\]

We define a holomorphic function on \( \mathcal{H} \) by

\[
 J(g, z) = J((g, \epsilon), z) := \epsilon (cz + d)^{1/2}.
\]

Here we choose \( w^{1/2} \) such that \(-\frac{\pi}{2} < \arg(w^{1/2}) \leq \frac{\pi}{2}\). We call \( J(g, z) \) a factor of automorphy. By Lemma 3.3 in \([G2]\), it satisfies \( J(gg', z) = J(g, g'z)J(g', z) \) for any two \( g \) and \( g' \) in \( G(\mathbb{R}) \).
Define a congruence subgroup $\Gamma_0(4)$ by
\[ \Gamma_0(4) := G(\mathbb{R}) \cap (s_\mathbb{Q}(SL_2(\mathbb{Q}))) \cdot K_0(4) \cdot \prod_{p \neq 2} K_p. \]
Similarly, define $\Gamma_1(4) \subseteq \Gamma_0(4)$ by replacing $K_0(4)$ with $K_1(4)$. Let $S_{r/2}(\Gamma_0(4))$ and $S_{r/2}(\Gamma_1(4))$ be the spaces of classical modular forms of weight $r/2$. By page 183 in [KloSavin], $S_{r/2}(\Gamma_0(4)) = S_{r/2}(\Gamma_1(4))$. We will denote this space by $S_{r/2}(4)$.

By Proposition 3.1 in [G2], there is a bijection $Q : A_{r/2}(4) \to S_{r/2}(4)$ which we will recall below: Given $\varphi \in A_{r/2}(4)$, then
\[ (Q\varphi)(z) = \varphi(g_\infty)J(g_\infty, i)^r \]
where $z = g_\infty i \in \mathcal{H}$. Conversely, given $f \in S_{r/2}(4)$. Let $g \in G(\mathbb{A})$. By Lemma 3.2 in [G2], $g = g_\mathbb{Q}g_\infty k$ for some $g_\mathbb{Q} \in s_\mathbb{Q}(SL_2(\mathbb{Q})), g_\infty \in G(\mathbb{R})$ and $k \in K_1(4) \prod_{p \neq 2, \infty} K_p$. Then
\[ (Q^{-1} f)(g) = f(g_\infty(i))J(g_\infty, i)^{-r}. \]
Using the bijection $Q$, we define another bijection between the spaces of operators
\[ q : \text{End}_C(A_{r/2}(4)) \to \text{End}_C(S_{r/2}(4)) \]
by $q(L) = QLQ^{-1}$. Since the Hecke algebra $H(\gamma)$ defined in Section 3 acts on $A_{r/2}(4)$ it is of interest to reinterpret this action in terms of classical modular forms.

**Proposition 24.** Let $U_1$ and $T_1$ be the operators in the local Hecke algebra $H(\gamma)$ where $\gamma(-1) = -i^r$. Recall that $\zeta = \frac{1 - i^r}{\sqrt{2}}$. For $f(z) \in S_{r/2}(4)$, we have
\begin{align*}
(i) \quad (q(U_1)f)(z) &= \zeta(2z)^{-r/2} f \left( -\frac{1}{4z} \right) \quad \text{and} \\
(ii) \quad (q(T_1)f)(z) &= 2^{-r/2} \sum_{u=0}^{3} f \left( \frac{z + u}{4} \right). 
\end{align*}

**Proof.** (i) Suppose $\varphi = Q^{-1}(f) \in A_{r/2}(4)$. For every place $v$, let $w_v = w(2^{-1})$ be the element in $G(\mathbb{Q}_v)$ defined in Section 2. By Proposition 3(iv),
\[ (U_1 \varphi)(g_\infty) = \int_{K_0w_2K_0} U_1(k) \varphi(g_\infty k) dk = U_1(w_2) \varphi(g_\infty w_2) = \zeta \varphi(g_\infty w_2). \]
Next, consider $w(2^{-1})$ in $SL_2(\mathbb{Q})$. By (2.30) in [G2], $s_\mathbb{Q}(w(2^{-1})) = \prod w_v$. Since $\varphi$ is left $SL_2(\mathbb{Q})$-invariant, and right $K_p$-invariant for $p \neq 2$,
\[ \zeta \varphi(g_\infty w_2) = \zeta \varphi(s_\mathbb{Q}(w(2^{-1}))^{-1} g_\infty w_2) = \zeta \varphi(\prod_{v \neq 2} w_v^{-1} g_\infty) = \zeta \varphi(w_\infty^{-1} g_\infty). \]
Applying $Q$ to the above equation gives (i). Part (ii) is proved analogously. \qed
8. Kohnen’s plus space

Hecke eigenforms in $S_{r/2}(4)$ correspond to cuspidal automorphic representations $\pi$ such that $\pi_\infty$ is a discrete series representation of lowest weight $\frac{r}{2}$. $\pi_p$ is unramified for all $p \neq 2$, and $\pi_2$ has $K_1(4)$-fixed vectors. In particular, $\pi_2^* \neq 0$ for the central character $\gamma(-1) = -i^r$. If $\pi_2$ is a principal series representation then $\pi_2^*$ is 2-dimensional and therefore the corresponding Hecke eigenspace in $S_{r/2}(4)$ is also 2-dimensional. Kohnen’s plus space is introduced to resolve this ambiguity. In terms of the space of automorphic functions $A_{r/2}(4)$, it is clear what to do. Decompose

$$A_{r/2}(4) = A^+_{r/2}(4) \oplus A^-_{r/2}(4)$$

where $A^+_{r/2}(4)$ is the eigenspace of the local Hecke operator $T_w$ with the eigenvalue 2, while $A^-_{r/2}(4)$ is the eigenspace with the eigenvalue $-1$. Since a presence of the eigenvalue 2 for $T_w$ acting on $\pi_2$ eliminates a possibility that $\pi_2$ is a Steinberg representation, we see that there is a one to one correspondence between Hecke eigenforms in $A^+_{r/2}(4)$ and cuspidal automorphic representations $\pi$ (as above) such that $\pi_2$ is a $\gamma$-unramified representation.

The classical Kohnen plus space is (essentially) $Q(A^+_{r/2}(4))$ as it will be explained in a moment. Niwa defines two operators $T_4$ and $W_4$ on $S_{r/2}(4)$ [Ni]:

$$(T_4 f)(z) = \frac{1}{4} \sum_{u=0}^{3} f \left( \frac{z + u}{4} \right)$$

and

$$(W_4 f)(z) = (-2iz)^{-r/2} f \left( -\frac{1}{4z} \right).$$

Note that $W_4^2 = 1$. Let $\kappa = \frac{r-1}{2}$. Niwa shows that the operator $W = (-1)^{\frac{r^2-1}{8}}2^{1-\kappa}W_4T_4$ on $S_{r/2}(4)$ satisfies the quadratic relation

$$(W + 1)(W - 2) = 0.$$ 

Kohnen defines $S^+_{r/2}(4)$ and $S^-_{r/2}(4)$ to be the eigenspaces of $W$ on $S_{r/2}(4)$ of eigenvalues 2 and $-1$ respectively [Ko]. Proposition 24 says that

$$\begin{cases}
q(U_1) = (-1)^{\frac{r^2-1}{8}}W_4 \\
q(T_1) = 2^{\frac{3}{2}-\kappa}T_4
\end{cases}$$

where the sign $(-1)^{\frac{r^2-1}{8}}$ is the quotient of $\zeta = \frac{1+i^r}{\sqrt{2}}$ and $i^\frac{r}{2} = \left( \frac{1+i^r}{\sqrt{2}} \right)^r$. Since $T_w = \sqrt{2}^{-1}T_1U_1$ it follows that $q(T_w)$ and $W$ are conjugates of each other by $W_4$. Thus the Kohnen’s plus space is simply a conjugate of our space:

$$Q(A^+_{r/2}(4)) = W_4(S^+_{r/2}(4)).$$

Since $W_4$ commutes with the classical Hecke operators $T_p$ for $p \neq 2$, $Q(A^+_{r/2}(4))$ and $S^+_{r/2}(4)$ are isomorphic as $\mathbb{C}[T_3, T_5, \ldots]$-modules.

There is another description of $S^+_{r/2}(4)$ in terms of Fourier coefficients. It consists of the cusp forms whose $n$-th Fourier coefficient vanishes whenever $(-1)^\kappa n \equiv 2, 3 \pmod{4}$.

\[1\] In Kohnen’s paper [Ko], the operator is $T_4W_4$ acting on the right, ie $T_4$ acts first and $W_4$ follows.
Kohnen also defines a Hecke operator $T_4^+$ which preserves $S_{r/2}^+(4)$ in the following way: For $f(z) = \sum_n a_n q^n \in S_{r/2}^+(4)$, we set $(T_4^+ f)(z) = \sum_n b_n q^n$ where the sum is taken over integers $n > 0$ and $(-1)^{r/2} n \equiv 0, 1 \pmod 4$, and
\[
b_n = a_{4n} + \left(\frac{(-1)^{r/2}}{2}\right) 2^{r-1} a_n + 2^{r-2} a_{n/4}.
\]
Here $a_{n/4} = 0$ if $n$ is not a multiple of 4. The large parenthesis denotes the Legendre symbol.

We can now formulate and prove our main global results.

**Theorem 25.** There is a one to one correspondence between Hecke eigenforms $f$ in $S_{r/2}^+(4)$ and irreducible cuspidal automorphic representations $\pi = \otimes_v \pi_v$ in $L^2(\SL_2(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}))$ such that

1. $\pi_\infty$ is the discrete series representation of $G(\mathbb{R})$ with the lowest weight $\frac{r}{2}$.
2. $\pi_p$ is unramified for all odd primes $p$.
3. $\pi_2$ is $\gamma$-unramified where $\gamma(-1) = -i^r$.
4. If $T_4^+ f = \lambda_2 f$, then a $\gamma$-spherical vector in $\pi_2$ is an eigenvector for $Z = T_\frac{r}{2} + (T_\frac{r}{2})^{-1}$ with the eigenvalue $2^{1-r} \lambda_2$.

Note that $\lambda_2$ determines the eigenvalue of $Z$ on a $\gamma$-spherical vector which in turn determines $\pi_2$ uniquely by Corollary 24.

**Proof.** The first three statements are clear, since $Q^{-1}(W_4 f)$ is a Hecke eigenform in $A_{r/2}^+(4)$ which is contained in a cuspidal automorphic representation $\pi$ with these properties. It remains to show (iv). We need the following lemma.

**Lemma 26.** Let $f$ be in $S_{r/2}^+(4)$. Then $T_4^+ f = 2^{l-1} q(Z) f$.

**Proof.** Recall that $T_1$ is invertible by Proposition 8. Hence, it suffices to show that
\[
2^{2-r/2} q(T_1) T_4^+ = q(T_1^2 + 4).
\]
If $f(z) = \sum_{n=1}^\infty a_n q^n \in S_{r/2}(4)$, then, by Proposition 24, $(q(T_1) f)(z) = 2^{2-r/2} \sum_{n=0}^\infty a_{4n} q^n$.

Thus, if $f(z) \in S_{r/2}^+(4)$, then one computes
\[
2^{2-r/2} (q(T_1) T_4^+ f)(z) = (q(T_1^2 + 4) f)(z) = \sum_n (2^{1-r} a_{16n} + 4a_n) q^n.
\]
This proves the lemma.

Now we can finish the proof of Theorem 25. If $T_4^+ f = \lambda f$ then Lemma 26 implies that $Q^{-1}(f)$ is an eigenform for $Z$ with the eigenvalue $2^{1-r/2} \lambda_2$. Since $W_4 = (-1)^{r/2+1} q(U_1)$ and $Z$ commutes with $U_1$, $Q^{-1}(W_4 f)$ is also an eigenform for $Z$ with the same eigenvalue. This completes the proof of Theorem 25.

If $f$ is a Hecke eigenform in $S_{r/2}^+(4)$ then by Theorem 1(iii) in [Ko] the corresponding Shimura lift $f' = \text{Sh}(f)$ is a Hecke eigenform in $S_{r-1}(\SL_2(\mathbb{Z}))$. Recall that $G' = \PGL_2$. There is a one to one correspondence between Hecke eigenforms $f'$ in $S_{r-1}(\SL_2(\mathbb{Z}))$ and
irreducible cuspidal automorphic representations $\pi' = \otimes_v \pi'_v$ in $L^2(G'(\mathbb{Q}) \backslash G'(\mathbb{A}))$ such that $\pi'_\infty$ is a discrete series representation with the lowest weight $r - 1$ and $\pi'_p$ is unramified for all primes $p$. See Proposition 3.1 in [G1]. We recall the local Shimura lift $Sh(\pi_2)$ in Proposition 11 of a $\gamma$-unramified representation $\pi_2$ of $G(\mathbb{Q}_2)$. The following corollary gives a precise representation-theoretic description of the Shimura correspondence at the place $p = 2$.

**Corollary 27.** Let $f$ be a Hecke eigenform in $S^+_{r/2}(4)$.

(i) Let $\pi = \otimes_v \pi_v$ be the cuspidal automorphic representation corresponding to $f$ in Theorem 25.

(ii) Let $\pi' = \otimes_v \pi'_v$ be the cuspidal automorphic representations of $L^2(G'(\mathbb{Q}) \backslash G'(\mathbb{A}))$ corresponding to the Hecke eigenform $f' = Sh(f)$ in $S_{r-1}(SL_2(\mathbb{Z}))$.

Then $Sh(\pi_2) = \pi'_2$.

**Proof.** If $T^+_4 f = \lambda_2 f$ then by Theorem 1(ii) in [Ko], $T_2 f' = \lambda_2 f'$ where $T_2$ is the classical Hecke operator action on $S_{r-1}(SL_2(\mathbb{Z}))$. By Proposition 5.2.1 in [G1] one checks that $\pi'_2$ is indeed isomorphic to $Sh(\pi_2)$. □

Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A})$ as in Theorem 25 and $\pi'$ be the corresponding cuspidal automorphic representation of $G'(\mathbb{A})$ as in Corollary 27. By the Ramanujan conjecture, proved by Deligne, $\pi'_2 = Sh(\pi_2)$ is a tempered irreducible unramified representation so $\pi'_2 = I'(s)$ for some $s \in i\mathbb{R}$. This implies that $\pi_2 = I(\gamma, s)$ by Theorem 22 and Corollary 23. In particular $\pi'_2$ is an irreducible $H(\gamma)$-module of dimension 2. It corresponds under $Q$ to a two dimensional subspace of $S^+_{r/2}(4)$ spanned by a line in $S^+_{r/2}(4)$ and a line in $S^-_{r/2}(4)$.

On the other hand, if $\pi_2 = St(\epsilon)$ is a Steinberg representation of $G(\mathbb{Q}_2)$ (see the definition before Corollary 18), then $\pi$ corresponds under $Q$ to an Hecke eigenform in $S^-_{r/2}(4)$. More precisely, we have the following theorem:

**Theorem 28.** There is a one to one correspondence between Hecke eigenforms $f$ in $S^-_{r/2}(4)$ such that $W_4 f = -\epsilon (-1)^{\frac{r-1}{2}} f$, for some $\epsilon = \pm 1$, and irreducible cuspidal automorphic representations $\pi = \otimes_v \pi_v$ in $L^2(SL_2(\mathbb{Q}) \backslash G(\mathbb{A}))$ such that

1. $\pi_\infty$ is the discrete series representation of $G(\mathbb{R})$ with the lowest weight $\frac{r}{2}$.
2. $\pi_p$ is unramified for all odd primes $p$.
3. $\pi_2$ is the Steinberg representation $St(\epsilon)$.

**Proof.** Recall, by Corollary 21 that $T_w$ and $U_1$ act on one-dimensional space $St(\epsilon)\gamma$ by $-1$ and $-\epsilon$. The theorem now follows from Proposition 24 and the definition of $S^-_{r/2}(4)$. □

**Acknowledgment.** The first author would like to thank the hospitality of the University of Utah while part of this paper was written. The second author is supported by an NSF grant DMS-0551846.
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