Renormalization and cyclotron resonance
in bilayer graphene with weak electron-hole asymmetry

K. Shizuya
Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

Cyclotron resonance in bilayer graphene is studied with weak electron-hole asymmetry, suggested by experiment, taken into account and with the focus on many-body corrections that evade Kohn’s theorem. It is shown by direct calculation that the theory remains renormalizable to $O(e^2)$ in the presence of electron-hole asymmetry parameters, and a general program to carry out renormalization for graphene under a magnetic field is presented. Inclusion of electron-hole asymmetry in part improves the theoretical fit to the existing data and the data appear to indicate the running of the renormalized velocity factor with the magnetic field, which is a key consequence of renormalization.

PACS numbers: 73.22.Pr,73.43.Lp,76.40.+b

I. INTRODUCTION

Graphene supports charge carriers that behave as Dirac fermions, which, in a magnetic field, lead to a particle-hole symmetric and unequally-spaced pattern of Landau levels. Accordingly, graphene gives rise to a variety of cyclotron resonance, both intraband and interband, with resonance energies varying from one resonance to another. This is in sharp contrast to standard quantum Hall systems with a parabolic energy dispersion, where cyclotron resonance takes place between adjacent Landau levels, hence at a single frequency $\omega_c = eB/m^*$, which, according to Kohn’s theorem, is unaffected by electron-electron interactions. The nonparabolic spectra in graphene offer the challenge of detecting many-body corrections to cyclotron resonance.

Theoretical studies over the past few years have revealed some notable features of quantum corrections to cyclotron resonance in graphene and bilayer graphene. The genuine many-body corrections arise from vacuum polarization, specific to graphene, which diverges logarithmically at short wavelengths. This means that one has to carry out renormalization properly, as in quantum electrodynamics, to extract observable results. In particular, for bilayer graphene it turns out that both the leading intralayer and interlayer coupling strengths undergo renormalization and that their renormalized strengths run with the magnetic field. Bilayer graphene is marked with the unique property that its band gap is externally controllable.

Experiment has so far verified, via infrared spectroscopy, some basic features of cyclotron resonance in monolayer and bilayer graphene. The data for the monolayer show a good symmetry between the electron and hole bands but generally show no clear sign of the many-body effect, except for a datum. Indeed, a comparison between some leading intraband and interband cyclotron resonances revealed a small deviation in excitation energy, consistent with the presence of many-body corrections roughly in magnitude and sign.

The situation is quite different for bilayer graphene, for which only a limited number of data are available so far. The data on intraband resonances show a weak electron-hole asymmetry, and generally defy a good fit by theory. Actually one has to employ different values of the velocity factor $v$ to fit the electron data and hole data separately.

Earlier Raman spectroscopy and subsequent infrared spectroscopy of bilayer graphene under zero magnetic field also revealed a significant asymmetry between the conduction and valence bands, mainly due to subleading intraand inter-layer couplings $\Delta$ and $\gamma_4$.

The purpose of this paper is to reexamine cyclotron resonance in bilayer graphene, with possible electron-hole and valley asymmetries taken into account. It is not clear a priori whether the renormalizability of the low-energy effective theory is maintained in the presence of electron-hole asymmetry, since the asymmetry parameters (especially $\gamma_4$) critically modify the ultraviolet structure of the theory. We show that the theory indeed remains renormalizable to $O(e^2)$ (at least), and that the renormalization counterterms depend on $\gamma_4$ in a nontrivial way. We present a general algorithm to carry out renormalization for graphene under a magnetic field, executable even numerically. Inclusion of electron-hole asymmetry parameters partially improves the theoretical fit to the existing data, and the fit in turn suggests some nontrivial modification of the spectra of the zero-mode and pseudo-zero-mode Landau levels specific to bilayer graphene.

In Sec. II we briefly review the effective theory of bilayer graphene and examine the effect of electron-hole and valley asymmetries. In Sec. III we study the Coulombic many-body corrections to cyclotron resonance, with a focus on renormalization and its consequences. Section IV is devoted to a summary and discussion.

II. BILAYER GRAPHENE

Bilayer graphene consists of two coupled honeycomb lattices of carbon atoms, arranged in Bernal $A’B$ stacking. The electrons in it are described by four-component
spinor fields on the four inequivalent sites \((A, B)\) and \((A', B')\) in the bottom and top layers, and their low-energy features are governed by the two inequivalent Fermi points \(K\) and \(K'\) in the Brillouin zone. The intralayer coupling \(\gamma_0 \equiv \gamma_{AB} \sim 3\) eV is related to the Fermi velocity \(v = (\sqrt{3}/2) a_L \gamma_0 / h \sim 10^6\) m/s (with \(a_L = 0.246\) nm) in monolayer graphene. The interlayer couplings\(^{25,26}\) \(\gamma_1 \equiv \gamma_{AB} \sim 0.4\) eV and \(\gamma_3 \equiv \gamma_{AB'} \sim 0.1\) eV are one-order of magnitude weaker than \(\gamma_0\). Actually, interlayer hopping via the \((A', B)\) dimer bonds modifies the intralayer linear spectra to yield quasi-parabolic spectra\(^1\) in the low-energy branches \(|\epsilon| < \gamma_1\).

The bilayer Hamiltonian with the leading intra- and inter-layer couplings \(v \propto \gamma_0\) and \(\gamma_1\) lead to electron-hole symmetric spectra. Infrared spectroscopy\(^{25}\) of bilayer graphene, however, has detected some weak asymmetry between the electron and hole bands, such as (i) the energy difference \(\Delta \approx 18\) meV between the \(A\) and \(B\) sublattices within the same layer and (ii) the next-nearest-neighbor interlayer coupling \(\gamma_4 \equiv \gamma_{AA'} = \gamma_{BB'} \approx 0.04\gamma_0\).

The effective Hamiltonian with such intra- and inter-layer couplings is written as\(^{14,29}\)

\[
H^{bi} = \int d^2 x \left[ \frac{\Psi^\dagger \mathcal{H}_+ \Psi + \Psi^\dagger A H_- \Psi}{\sqrt{\epsilon}} \right],
\]

\[
\mathcal{H}_+ = \begin{pmatrix}
\frac{1}{2} v & v p & \frac{v^2}{2}
\frac{v}{2} & v p & \frac{v^2}{2}
\frac{v^2}{2} & \frac{v}{2} & \frac{v^2}{2}
\end{pmatrix},
\]

where \(p = p_x + i p_y\), \(p^\dagger = p_x - i p_y\). Here \(\Psi = (\psi_A, \psi_{B'}, \psi_{A'}, \psi_B)^t\) stands for the electron field at the \(K\) valley, with \(A\) and \(B\) referring to the associated sublattices; \(u\) stands for the interlayer bias, which opens a tunable gap\(^{15}\) between the \(K\) and \(K'\) valleys. We ignore the effect of trigonal warping \(\propto \gamma_3\) which, in a strong magnetic field, causes only a negligibly small level shift.\(^{23}\) We also ignore weak Zeeman coupling and, for conciseness, suppress the electron spin. Our definition of \(v_4 \equiv -(\gamma_4 / \gamma_0) v\) differs in sign and by factor \(v\) from the one \((v_4 \rightarrow \gamma_4 / \gamma_0)\) in the literature\(^{23,27,28}\), this choice is made simply for notational convenience.

The Hamiltonian \(H_-\) at another \((K')\) valley is given by \(H_+\) with \((v, \psi, u) \rightarrow (v, -\psi, -u)\), and acts on a spinor of the form \(\Psi = (\psi_{B'}, \psi_A, \psi_{B'}, \psi_{A'})^t\). Note that \(H_+\) is unitarily equivalent to \(H_-\) with the sign of \(u\) reversed,

\[
U^\dagger H_+ U = H_- | -u\rangle
\]

with \(U = \text{diag}(1, 1, -1, -1)\). This implies that the electronic spectrum at the \(K'\) valley is obtained from the spectrum at the \(K\) valley by reversing the sign of \(u\); in particular, the spectra at the two valleys are the same for \(u = 0\). Nonzero interlayer voltage \(u \neq 0\) thus acts as a valley-symmetry breaking.

We adopt the set of experimental values\(^{26}\)

\[
v \approx 1.1 \times 10^6\) m/s, \(\gamma_1 \approx 0.4\) meV,
\]

\[
v_4 / v \equiv -\gamma_4 / \gamma_0 \approx -0.04, \quad \Delta \approx 18\) meV,
\]

in what follows. Full account is also taken of the effect of interlayer bias \(u\). For notational simplicity, however, we often present analytical expressions only for \(u = 0\).

The Hamiltonian \(H^{bi}\) gives rise to four bands with electron and hole spectra, which, for \(u = 0\), read

\[
e_4(p) = \sqrt{v^2 + (\gamma_4 / \gamma_0)^2} + \gamma_4 / \gamma_0,
\]

\[
e_3(p) = \sqrt{v^2 + (\gamma_4 / \gamma_0)^2} - \gamma_4 / \gamma_0,
\]

\[
e_2(p) = -\sqrt{v^2 + (\gamma_4 / \gamma_0)^2} + \gamma_4 / \gamma_0,
\]

\[
e_1(p) = -\sqrt{v^2 + (\gamma_4 / \gamma_0)^2} - \gamma_4 / \gamma_0,
\]

where \(v_{\pm} \equiv v \pm v_4\) and \(\gamma_{\pm} \equiv \gamma_1 \pm \Delta\). Note that \(v_4\) and \(\Delta\) effectively modify \(v\) and \(\gamma_1\), respectively, in a manner different for electrons and holes; the spectra are electron-hole asymmetric unless \(v_4 = \Delta = 0\). These band spectra acquire nonzero valley gaps for \(u \neq 0\).

Let us place bilayer graphene in a strong uniform magnetic field \(B_z = -B < 0\) normal to the sample plane; we set, in \(\mathcal{H}_\pm, p \rightarrow \Pi = p + e A\) and \(p^\dagger \rightarrow \Pi^\dagger\) with \(A = A_x + i A_y = B y\), and denote the magnetic length as \(\ell = 1 / \sqrt{e B}\). It is easily seen that the eigenmodes of \(H_+\) have the structure

\[
\Psi_n = \left( |n\rangle b_{n,1}^{(1)}, |n - 2\rangle b_{n,1}^{(2)}, |n - 1\rangle b_{n,1}^{(3)}, |n - 1\rangle b_{n,1}^{(4)} \right)^t
\]

with \(n = 0, 1, 2, \ldots\), where only the orbital eigenmodes are shown using the standard harmonic-oscillator basis \(|n\rangle\) (with the understanding that \(|n\rangle = 0\) for \(n < 0\)). The coefficients \(b_n = (b_{n,1}^{(1)}, b_{n,1}^{(2)}, b_{n,1}^{(3)}, b_{n,1}^{(4)})^t\) for \(n = 2, 3, \ldots\) are given by the eigenvectors of the reduced Hamiltonian

\[
\mathcal{H}_\text{red} = \omega_c \left( \begin{pmatrix}
\frac{1}{2} u' & -\frac{1}{2} u' & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}}
\frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}}
\frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}}
\frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}} & \frac{r}{\sqrt{n+1}}
\end{pmatrix} \right),
\]

where

\[
\omega_c \equiv \sqrt{2} v / \ell \approx 36.3 \times v [10^6\) m/s] \sqrt{B[T]} \) meV,
\]

with \(v\) measured in units of \(10^6\) m/s and \(B\) in tesla, is the characteristic cyclotron energy for monolayer graphene; \(r \equiv v_4 / v = -\gamma_4 / \gamma_0 \approx -0.04\), \(\gamma' \equiv \gamma_1 / \omega_c\), \(\Delta / \omega_c\), \(u' \equiv u / \omega_c\).

The energy eigenvalues \(\epsilon_n\) of \(\mathcal{H}_\text{red}\) are determined from the secular equation, which, for \(u = 0\), reads

\[
n(n-1)(1 - r^2)^2 - \left[ \gamma'^2 - d^2 + c_n (1 + r^2) \right] \epsilon^2 + (\epsilon'^2)^2
\]

\[-c_n (2r \gamma' - d (1 + r^2)) \epsilon' - 2d (\epsilon'^2) = 0,
\]

with \(c_n = 2n - 1\) and \(\epsilon' \equiv \epsilon_n / \omega_c\). We first consider the \(u = 0\) case. Let us denote the four solutions of the secular equation as \(\epsilon_{\pm n} < \epsilon_{n} < \epsilon_{n} < \epsilon_{n}^\prime\) for each integer \(n \geq 2\), so that the index \(\pm n\) reflects the sign of the energy eigenvalues. For \(n = 0\) \(H^{bi}\) has an obvious zero
eigenvalue $\epsilon_0 = 0$, with the eigenvector $b_0 = (1, 0, 0, 0)^t$. For $n = 1$ the secular equation \( (2.8) \) is reduced to a cubic equation in $\epsilon'$, excluding $\epsilon' = 0$, and leads to three solutions, which, for the present choice \( (2.8) \) of parameters, are given, e.g., by $\epsilon' = (-3.96, 0.029, 4.29)$ at $B = 10$ T; we thus denote the corresponding eigenvalues as $(\epsilon_{-1}, \epsilon_1, \epsilon_1')$. This eigenvalue $\epsilon_1$ changes sign if one sets $d \to -d$ and $r \to -r$ simultaneously. Thus the assignment of $\epsilon_{\pm 1}$ in general depends on the choice of asymmetry parameters $(\gamma_4, \Delta)$ and also on the interlayer bias $u$. Actually, for zero bias $u = 0$, $\epsilon_1$ deviates from zero as $u_4$ and $\Delta$ develop. In this sense, the $n = 1$ Landau level is a pseudo-zero-mode level while the $n = 0$ level is a genuine zero-mode level.

As $u$ is turned on, these $n = (0, \pm 1)$ levels go up or down oppositely at the two valleys; e.g., $\epsilon_0|K = -\epsilon_0|K' = u/2$. Their spectra vary linearly with $u$ while other levels $(|n| \geq 2)$ get shifted only slightly. Interestingly, for $0 < |u| \ll \omega_\gamma \epsilon_1 \geq \epsilon_0$ at one valley while $\epsilon_{|n|=1}$ at another valley crosses $\epsilon_0$, from below with increasing magnetic field $B$. A nonzero $u$ thus critically spoils the valley symmetry of the $(n = (0, \pm 1))$ sector.

The spectra $\epsilon_n$ and $\epsilon_{-n}$ with $n \geq 1$ form the high-energy branches of the electron and hole Landau levels, respectively; $|\epsilon^\pm| \gg \gamma_1$. Let us combine $\pm n$ into the low-energy branch of Landau levels $\{\epsilon_n\} = \{\ldots, \epsilon_{-3}, \epsilon_{-2}, \epsilon_0, \epsilon_1, \epsilon_2, \ldots\}$, and denote the three branches $(\epsilon_{-n}, \epsilon_n, \epsilon_n')$ as

$$\epsilon_n = \omega_\gamma \eta_n (\gamma', d, r, u'), \quad \epsilon_n' = \omega_\gamma \eta_n' (\gamma', d, r, u'). \quad (2.9)$$

These $(\eta_n, \eta_n') \sim \epsilon'$ are uniquely determined from $\mathcal{H}_{col}$ of Eq. \( (2.0) \) or from Eq. \( (2.8) \) as functions of $(\gamma', d, r, u')$.

One can thereby construct the associated eigenvectors, which, e.g., for $u = 0$ and $|n| \geq 1$, read $b_n = (b_n^{(1)}, b_n^{(2)}, b_n^{(3)}, b_n^{(4)})^t = b_n^{(1)} (1, \beta_n^{(2)}, \beta_n^{(3)}, \beta_n^{(4)})^t$ with

$$\beta_n^{(3)} = -[n] - \eta_n (\eta_n - d) + ([n] - 1) r^2, \quad \beta_n^{(4)} = [\gamma' \eta_n + (2|n| - 1) r] / \sqrt{|n| G_n}, \quad \beta_n^{(2)} = \sqrt{|n| - 1} (\beta_n^{(3)} + r \beta_n^{(4)}) / \eta_n, \quad (2.10)$$

where $G_n = \gamma' + r \{\eta_n - d + ([n] - 1) (1 - r^2) / \eta_n\}$ and $b_n^{(1)} = 1 / \sqrt{1 + \sum_{i=2,3,4} (\beta_n^{(i)})^2}$. These expressions are equally valid for eigenvectors belonging to $\epsilon_n'$, with $\eta_n \to \eta_n'$.

The Landau-level spectrum $\epsilon_n = \omega_\gamma \eta_n (\gamma', d, r, u')$ depends on the magnetic field $B$ in a nontrivial manner through the dimensionless quantities $\gamma'$, $d$ and $u'$. Actually, for the choice of $\gamma_1 = 404$ meV, $\Delta = 18$ meV, $r = -0.04$ and $u = 0$, the electron and hole spectra differ considerably, as shown in Fig. 1. In particular, the zero-mode level $(n = 0)$ remains intact while the pseudo-zero-mode level $(n = 1)$ gets shifted, e.g., by $\sim 8$ meV as $B$ is increased from $0$ to $20$ T. The Landau gaps are generally larger for electrons than holes; this is readily understood from the behavior of $\epsilon_3(p) \approx (v_0^2 / \gamma') p^2$ for $p \to 0$,

which implies that the effective mass $m^e \sim \gamma' / (2 v_0^2)$ is smaller for electrons. The asymmetry between the electron Landau levels $L_n$ and hole levels $L_{-n}$ becomes more prominent for higher levels $|n| \geq 2$.

Cyclotron resonance in bilayer graphene is governed by the selection rule $\Delta |n| = \pm 1$, and there are two classes of transitions, (i) $L_{\pm(n-1)} \leftrightarrow L_n$ and (ii) $L_n \leftrightarrow L_{\pm(n-1)}$.
(with \( n \geq 2 \)), which are distinguished by the use of circularly-polarized light. See Fig. 1 (c).

As for experiment, Henriksen et al measured, via infrared spectroscopy, cyclotron resonance in bilayer graphene in magnetic fields up to 18T. They observed intraband resonances, which are identified with the \( L_2 \leftarrow L_1, L_3 \leftarrow L_2, L_4 \leftarrow L_3 \) and \( L_5 \leftarrow L_4 \) transitions at filling factor \( \nu = 4, 8, 12 \) and 16, respectively, and the corresponding hole resonances at \( \nu = -4, -8, -12 \) and -16, together with a significant asymmetry between the electron and hole data.

Such data from Ref. 23 are reproduced in Fig. 2. Also included are the zeroth-order Landau gaps of Fig. 1 (b), which apparently fit the experimental data reasonably well, except for the \( \nu = 4 \) electron data on the \( L_2 \leftarrow L_1 \) resonance, which deviates considerably.

It is worth discussing the effect of interlayer bias \( u \) here. The \( n = (0, 1) \) levels are very sensitive to \( u \) and may easily acquire valley gaps for \( u \neq 0 \) while other levels \( |n| \geq 2 \) are relatively inert as long as \( |u| \ll \omega_c \). Cyclotron resonances involving the \( n = 1 \) level, i.e., (1 \( \leftarrow \) -2) and (2 \( \leftarrow \) 1) resonances, therefore tend to be affected by \( u \). Actually, with \( u \approx 20 \text{meV} \) one can apparently fit the data for those resonances at one valley. The asymmetry, however, is reversed at another valley. Nonzero \( u \) may thus broaden the observed widths of the (1 \( \leftarrow \) -2) and (2 \( \leftarrow \) 1) resonances but would not account for their asymmetry.

It is clear now that one should treat those resonances separately from the rest of the resonances, which are barely sensitive to \(|u| \ll \omega_c \). The former and latter are also different in their sensitivities to electron-hole asymmetry \( \propto \Delta \) and \( \nu_4 \). See Fig. 1 (b) again. It shows that for \( u = 0 \) the 2 \( \leftarrow \) 1 and 1 \( \leftarrow \) -2 gaps barely differ while other Landau gaps exhibit significant asymmetry between the electron and hole bands. In this sense, Fig. 2 shows us that \( \Delta \approx 18 \text{meV} \) and \( \nu_4 \approx -0.04 \nu \), obtained from independent experiments, account for the electron-hole asymmetry between the \( \nu = (8, 12, 16) \) data and the \( \nu = (-8, -12, -16) \) data reasonably well. It is a nontrivial fact that this single set of parameters can fit the electron and hole data simultaneously.

Poor fitting to the \( \nu = 4 \) data, on the other hand, would suggest that the spectrum of the pseudo-zero-mode level \((|n|=1)\) is further modified\(^{24-25}\) by some other sources. Actually it is expected theoretically\(^{22,23}\) that the \( n = (0, \pm 1) \) sector of bilayer graphene has nontrivial dynamics due to orbital mixing and supports characteristic collective excitations, orbital pseudospin waves. It would be interesting to study how the electron-hole asymmetry affects the detailed structure of this special sector.

III. CYCLOTRON RESONANCE AND MANY-BODY CORRECTIONS

In this section we study the many-body corrections to cyclotron resonance, with emphasis on how to carry out renormalization. The Coulomb interaction is written as

\[
H_{\text{Coul}} = \frac{1}{2} \sum_{\mathbf{p}} \text{v}_{\mathbf{p}} : \rho_{-\mathbf{p}} \rho_{\mathbf{p}} : ,
\]

where \( \rho_{\mathbf{p}} \) is the Fourier transform of the electron density \( \rho = \Psi^\dagger \Psi \), \( v_{\mathbf{p}} = 2 \pi \alpha / (\epsilon_0 |\mathbf{p}|) \) is the Coulomb potential with \( \alpha = e^2 / (4 \pi \epsilon_0) \approx 1/137 \) and the substrate dielectric constant \( \epsilon_0 \); \( \sum_\mathbf{p} = \int d^2 \mathbf{p} / (2\pi)^2 \).

The Landau-level structure is made explicit by passing to the \( |n, y_0 \rangle \) basis (with \( y_0 = \ell^2 p_x \)) via the expansion \( \langle \Psi(x, t) | \Psi(x, t) \rangle = \sum_{n,y_0} (x|n, y_0 \rangle \psi_n(y_0, t) \rangle \); remember that fields \( \psi_n \) carry (suppressed) spin and valley indices. The Hamiltonian \( H^{\text{bi}} \) is thereby rewritten as

\[
H^{\text{bi}} = \int dy_0 \sum_{n=-\infty}^{\infty} \psi_n^\dagger (y_0, t) \epsilon_n \psi_n (y_0, t),
\]

and the charge density \( \rho_{-\mathbf{p}} (t) = \int d^2 \mathbf{x} e^{i \mathbf{p} \cdot \mathbf{x}} \rho \) as\(^{31}\)

\[
\rho_{-\mathbf{p}} = \sum_{k,n=-\infty}^{\infty} \rho_{k,n} \rho_{\mathbf{p}}^{k,n} = \sum_{k,n=-\infty}^{\infty} g_{\mathbf{p}}^{k,n} R_{\mathbf{p}}^{k,n} ,
\]

where \( g_{\mathbf{p}}^{k,n} = e^{-i \mathbf{p}^2/4} ; \mathbf{r} = (i \ell^2 \partial / \partial y_0, y_0) \) stands for the center coordinate with uncertainty \( \delta \mathbf{r}/\mathbf{p} = i \ell^2 \). The charge operators \( R_{\mathbf{p}}^{k,n} \) obey two \( W_\infty \) algebras\(^{22}\) associated with intralevel center-motion and interlevel mixing of electrons.

The coefficient matrix \( g_{\mathbf{p}}^{k,n} \) is constructed from the knowledge of the eigenvectors \( \mathbf{b}_n \),

\[
g_{\mathbf{p}}^{k,n} = b_{\mathbf{p}}^{(1)} b_n^{(1)} f|k,n| + b_k^{(2)} b_n^{(2)} f|k,-2|n| \]

\[+ (b_k^{(3)} b_n^{(3)} + b_k^{(4)} b_n^{(4)}) f|k-1|n|-1| ,\]

where

\[
f_{\mathbf{p}}^{k,n} = \sqrt{n!} \sqrt{2} \ell^2 \pi^{(k-n)} L_n^{(k-n)} \left( \frac{2}{\ell^2 \mathbf{p}^2} \right) \]

for \( k \geq n \geq 0 \), and \( f_{\mathbf{p}}^{k,n} \) are the same at the two valleys, i.e., \( g_{\mathbf{p}}^{k,n} |K = g_{\mathbf{p}}^{k,n} |K \) with \( (b_n^{(1)}, b_n^{(2)}, b_n^{(3)}, b_n^{(4)}) |K = (b_n^{(1)}, b_n^{(2)}, -b_n^{(3)}, -b_n^{(4)}) |K \). This follows from the unitary equivalence\(^{22}\) of the Hamiltonians \( H_K |\pm \rangle \) and the invariance of the charge density \( \rho_{\mathbf{p}} \) under \( U \) there.

The Coulombic correction to cyclotron resonance in graphene to \( O(\alpha/\ell) \) was calculated earlier\(^{22}\) using the
single-mode approximation. Here we consider cyclotron resonance (at integer filling \( \nu \)) from the filled \( n \)th Landau level \((L_n)\) to the empty \( n \)th level \((L_0)\) at zero momentum transfer \( k = 0 \), where no mixing takes place in spin and valley. The cyclotron-resonance energy for a general \( L_b \leftarrow L_a \) transition with the Landau levels filled up to \( n = n_f \) \((a \leq n_f < b)\) is written as

\[
\epsilon_{\text{exc}}^{b-a} = \epsilon_b - \epsilon_a + \Delta \epsilon^{b-a},
\]

with the correction

\[
\Delta \epsilon^{b-a} = \sum_p \epsilon_p^{2} \left( \sum_{n \leq n_i} \left( |g_p^{2n}|^2 - |g_p^{2n}|^2 \right) - g_p^{2b} g_p^{-a} \right),
\]

(3.7)
diagonal in spin and valley. As shown by Fig. 1 (c), \( n_f = -4, -3, -2, 1, 2, \ldots \) correspond to the filling factor \( \nu = -12, -8, -4, 4, 8, 12, \ldots \), respectively.

One can now substitute Eq. (3.7) into this formula and calculate the Coulomb corrections with the effect of electron-hole asymmetry taken into account. There is, however, one technical problem to solve. The term \( \sum_{n \leq n_i} \left( |g_p^{2n}|^2 - |g_p^{2n}|^2 \right) \) in Eq. (3.7) refers to quantum fluctuations of the filled states and actually diverges logarithmically with the number \( N_L \to \infty \) of filled Landau levels in the valence band (or the Dirac sea).

One has to handle such ultraviolet (UV) divergences by renormalization of the basic parameters \((\nu, \nu_4, \gamma_1, \gamma)\) and, if necessary, the tunable parameter \( u \). In the electron-hole symmetric case, \( \nu \) and \( \gamma_1 \) turn out to be renormalized in the same way, i.e., \( \nu = Z_{\nu} \nu^{\text{ren}} \) and \( \gamma_1 = Z_{\gamma} \gamma_1^{\text{ren}} \). Actually, it is not clear a priori if the theory remains renormalizable in the absence of symmetry parameters \( \nu_4 \) and \( \Delta \). If inclusion of \( \nu_4 \) and \( \Delta \) were to yield a new type of divergence unremovable by rescaling of the existing parameters, the theory would lose renormalizability (or one would have to introduce a new parameter to remove the divergence and, if necessary, repeat this process). We prove by direct calculations below that the theory is renormalizable to \( O(\alpha/\ell) \).

The key to this problem of renormalization is to note that the magnetic field supplies only a long-wavelength cutoff through the magnetic length \( \ell = 1/\sqrt{\epsilon B} \), leaving the UV structure of the theory intact. One can therefore first look into the theory in free space \((B = 0)\) and determine the UV structure of the Coulomb exchange corrections. Such corrections are written as \( \sum_k v_k i S(p) + k \), a convolution of the photon propagator \( v_k = 2\pi\alpha/\epsilon_b |k| \) and the instantaneous electron propagator \( \langle \Psi \Psi \rangle_{\text{inst}} = iS(p) \). Their UV structure is thus read from the asymptotic behavior of \( S(p) \).

The resulting divergences are then absorbed into the counterterms \( \delta \nu, \delta \gamma_1, \delta \nu_4, \delta \Delta \) and \( \delta u \), generated by rescaling

\[
v = Z_{\nu} \nu^{\text{ren}} = \nu^{\text{ren}} + \delta \nu,
\]

\[
\gamma_1 = \gamma_1^{\text{ren}} + \delta \gamma_1, \ldots
\]

(3.8)

where “ren” refers to renormalized parameters. See Appendix A for such an analysis of divergences. Here we quote only the result: (i) The counterterm for velocity factor \( \nu \) turns out to be the same as in the electron-hole symmetric case (and in the case of monolayer graphene as well),

\[
\delta \nu = (Z_{\nu} - 1) \nu^{\text{ren}} \sim - (\alpha/8\epsilon_b) \log \Lambda^2,
\]

(3.9)

where \( \Lambda \) stands for the momentum cutoff which is related to the Dirac-sea cutoff so that \( \Lambda \approx 2N_L/\ell^2 \). (ii) Remarkably, \( \nu_4 \) and \( u \) remain finite,

\[
\delta \nu_4 = \delta u = 0,
\]

(3.10)

and require no renormalization, \( \nu_4 = \nu_4^{\text{ren}} \) and \( u = u^{\text{ren}} \).

(iii) The dimensional parameters \( \gamma_1 \) and \( \Delta \) are mixed under renormalization,

\[
\delta \gamma_1 = (\gamma_1^{\text{ren}} - r^{\text{ren}} \Delta^{\text{ren}}) \delta \nu/\nu^{\text{ren}},
\]

\[
\delta \Delta = (\Delta^{\text{ren}} - r^{\text{ren}} \gamma_1^{\text{ren}}) \delta \nu/\nu^{\text{ren}},
\]

(3.11)

where \( h[r] = 1/(1 - r^2) \) and \( r^{\text{ren}} = \nu_4^{\text{ren}}/\nu^{\text{ren}} \). Note that the counterterms are highly nonlinear in \( r^{\text{ren}} \approx \nu_4 \).

One can now pass to the \( B \neq 0 \) case with these counterterms. Let us denote by \( H_+^{\text{ren}} \) the Hamiltonian \( H_+ \) of Eq. (2.1) in magnetic field \( B \) with \((\nu, \nu_4, \gamma_1, \Delta, u)\) replaced by \((\nu_4^{\text{ren}}, \nu_4^{\text{ren}}, \gamma_1^{\text{ren}}, \Delta^{\text{ren}}, u^{\text{ren}})\), and write its spectrum as \( \epsilon_n^{\text{ren}} = u^{\text{ren}} \eta_n (\gamma_1^{\text{ren}}, \Delta^{\text{ren}}, \nu_4^{\text{ren}}, \nu_4^{\text{ren}}) \) with \( u^{\text{ren}} = \sqrt{\nu_4^{\text{ren}}/\ell} \), etc., in obvious notation; see Eq. (2.21). Suppose now that we start with \( H_+^{\text{ren}} \) and calculate Coulomb corrections to \( O(\alpha/\ell) \). The divergences we encounter are removed by the counterterms formally written as \( \delta_{\text{ct}} H_+^{\text{ren}} \), where the differential operator

\[
\delta_{\text{ct}} = \delta \nu \frac{\partial}{\partial \nu^{\text{ren}}} + \delta \gamma_1 \frac{\partial}{\partial \gamma_1^{\text{ren}}} + \delta \Delta \frac{\partial}{\partial \Delta^{\text{ren}}}
\]

(3.12)

acts on \( H_+^{\text{ren}} \). For the related reduced Hamiltonian \( H_+^{\text{ren}} \), defined as in Eq. (2.20), the counterterm is also written as \( \delta_{\text{ct}} H_+^{\text{ren}} \). If, for example, one is to subtract divergences from the \( O(\alpha/\ell) \) correction to the spectrum \( \epsilon_n^{\text{ren}} \), the required counterterm is obtained from the expectation value of \( \delta_{\text{ct}} H_+^{\text{ren}} \), which equals \( \delta_{\text{ct}} \epsilon_n^{\text{ren}} \), the variation of the eigenvalue itself. One can easily handle it numerically by writing

\[
\delta_{\text{ct}} \epsilon_n^{\text{ren}} = (b_n)_{\text{T}} \cdot \delta_{\text{ct}} \hat{H}_+^{\text{ren}} \cdot b_n^\dagger.
\]

(3.13)

Rewriting \( \delta_{\text{ct}} \) in favor of renormalized parameters, \( \delta_{\text{ct}} = (\delta \nu/\nu^{\text{ren}}) (\delta \nu/\nu^{\text{ren}} + \Delta) \), yields

\[
\delta_{\text{ct}} \epsilon_n^{\text{ren}} = \omega_n^{\text{ren}} (\delta \nu/\nu^{\text{ren}}) (\eta_n + \varphi \eta),
\]

\[
\varphi = - \frac{\partial}{\partial \varphi} \frac{r(\varphi - \gamma) - d}{1 - r^2} \frac{\partial}{\partial \gamma}
\]

(3.14)

for conciseness, we have suppressed “ren” in \( \varphi \).
of Eq. (3.7), which, when integrated over

FIG. 3: (a) Momentum profiles of the many-body corrections

c−2−→3|B| − c−2−→3|B=10T

(b) c−2−→3|B| − c−2−→3|B=10T reveals the running of v|B.

With this in mind, let us rewrite Eq. (3.14) as

\[ \epsilon_{\text{exc}}^{b-a} = \epsilon_b^{\text{ren}} - \epsilon_a^{\text{ren}} + (\Delta \epsilon_b^{a \text{ren}}), \]

where the renormalized correction \( \langle \Delta \epsilon_b^{a \text{ren}} \rangle \equiv \delta_{\text{ct}}^{\text{ren}} - \delta_{\text{ct}}^{\text{a ren}} + \Delta \epsilon_a^{b \text{ ren}} \) is now made finite. Writing the counterterm as \( \delta_{\text{ct}}^{\text{ren}} - \delta_{\text{ct}}^{\text{a ren}} = \eta^{b-a} \omega_{\text{ren}} (\delta v/\epsilon_{\text{ren}}^{\text{a ren}}) \), with

\[ \eta^{b-a} \equiv \eta_b - \eta_a + D(\eta_b - \eta_a), \]

and setting \( \Delta \epsilon^{b-a}/\eta^{b-a} \equiv V_c \epsilon^{b-a} \), in units of the characteristic Coulomb energy

\[ V_c \equiv \alpha/(\epsilon_b \ell) \approx (56.1/\epsilon_b) \sqrt{B[T] \text{ meV}}, \]

yields the expression

\[ (\Delta \epsilon_b^{a \text{ ren}}) = \eta^{b-a} \{ V_c \epsilon^{b-a} + (\sqrt{2/\ell}) \delta v \}. \]

This reveals that the UV divergence is common to all ratios \( \Delta \epsilon^{b-a}/\eta^{b-a} \equiv V_c \epsilon^{b-a} \), independent of \((b, a)\) and \((v, v_4, \gamma_1, \Delta)\); the dimensionless quantities \( \epsilon^{b-a} \) have the structure \( \epsilon^{b-a} = (\sqrt{2}/8) \log(\Lambda^2/B) + F^{b-a}(\gamma^{\text{ren}}, D^{\text{ren}}, \ldots) \), where \( F^{b-a} \) denote finite corrections.

Figure 3 (a) shows for some typical channels, with \( v_4/v = -0.04 \) at \( B = 10T \) and decreases gradually with increasing \( B \). The leading correction \( \epsilon^{\text{ren}} \approx B - \epsilon^{\text{a ren}} |B_0| \sim -(\sqrt{2}/8) \log(B/B_0) \) is logarithmic but corrections coming from finite terms \( F^{b-a} \) are equally important for relatively low magnetic fields. For definiteness let us take \( L_2 \leftarrow L_3 \) as the reference channel, as chosen experimentally. 

For this channel the contribution from the low-momentum region decreases with \( B \), as seen from the \( (−2 \leftarrow −3) \) profiles for \( B = (10T, 20T) \) in Fig. 3 (a), and numerically the correction is roughly doubled over the range \( 10T \lesssim B \lesssim 30T \), as shown in Fig. 3 (b).

One can multiply it by factor \( \alpha/(\sqrt{2} \epsilon_b v) \sim 1.5/\epsilon_b \sim 0.3 \) (with \( \epsilon_b \sim 5 \)) to estimate the rate of decrease in \( v|B \) with \( B \), which is about 10% for \( B = 10T \rightarrow 20T \).

The renormalized Coulombic corrections in all other channels are thereby fixed uniquely,

\[ \langle \Delta \epsilon_b^{a \text{ ren}} \rangle = V_c \eta^{b-a} \{ (\epsilon^{b-a} - \epsilon^{b-a}) |B \}. \]

These observable corrections are essentially calculated from the profiles in the low-momentum region \( \ell |\mathbf{p}| \lesssim 15 \).

It is enlightening to write the resonance energies as

\[ \epsilon_{\text{exc}}^{b-a} = (\eta_b - \eta_a) \left[ \omega_{\text{ren}}^{c} + V_c \Delta \epsilon_b^{a} \right], \]

\[ \Delta \epsilon_b^{a} = \frac{\eta^{b-a} \epsilon^{b-a}}{\eta_b - \eta_a} \{ \epsilon^{b-a} - \epsilon^{b-a} |B \}, \]

so that the Coulombic corrections \( V_c \Delta \epsilon_b^{a} \) seemingly arise relative to \( \omega_{\text{ren}} \). Using the set of parameters in Eq. (2.23), one finds, for some typical intraband channels,

\[ \Delta \epsilon^{5.4} \quad \nu = 16 \quad -0.317, \quad (-0.259), \]

\[ \Delta \epsilon^{4.3} \quad \nu = 12 \quad -0.223, \quad (-0.175), \]

\[ \Delta \epsilon^{3.2} \quad \nu = 8 \quad -0.073, \quad (-0.040), \]

\[ \Delta \epsilon^{2.1} \quad \nu = 4 \quad 0.650, \quad (0.533), \]

\[ \Delta \epsilon^{1.2} \quad \nu = 4 \quad 0.897, \quad (0.675), \]

\[ \Delta \epsilon^{1.3} \quad \nu = 8 \quad 0, \quad (0), \]

\[ \Delta \epsilon^{4.5} \quad \nu = 12 \quad -0.165, \quad (-0.150), \]

\[ \Delta \epsilon^{4.5} \quad \nu = 16 \quad -0.275, \quad (-0.247), \]

at \( B = 10T \) \((B = 20T) \). For comparison, setting \( v_4 = \Delta = 0 \) yields the electron-hole symmetric values \( \Delta \epsilon^{1.2} = 0.815, \Delta \epsilon^{2.3} = 0, \Delta \epsilon^{3.4} = -0.159 \) and \( \Delta \epsilon^{5.4} = -0.303 \) at \( B = 10T \). Similarly, some interband channels yield

\[ \Delta \epsilon^{3.2} \quad \nu = 8 \quad 0.353, \quad (0.375), \]

\[ \Delta \epsilon^{2.3} \quad \nu = 8 \quad 0.370, \quad (0.363), \]

\[ \Delta \epsilon^{3.4} \quad \nu = 12 \quad 0.261, \quad (0.278). \]
Note that those corrections are ordered regularly in magnitude for a sequence of resonances; see also the theoretical curves for $\epsilon_{\text{exc}}$ in Figs. 4(b) and 4(d).

Our formula (3.22) summarizes the effect of renormalization in a concise form and is also useful in analyzing the experimental results. One may rescale the observed excitation energies $\epsilon_{\text{exc}}^{\text{tot}}$ in the form $\epsilon_{\text{exc}}^{\text{tot}}/(\eta_b - \eta_a)$ and plot them in units of $1/\ell \propto \sqrt{B}$ for each given value of $B$. The Coulombic many-body effect will then be seen as a variation in characteristic velocity $v_{\text{ren}}|_B [1 + O(V_c)]$ from one resonance to another, and a deviation of $\omega_{\text{ren}}$ from $\sqrt{B}$ behavior would indicate the running of $v_{\text{ren}}$ with $B$.

Figures 4(a) and 4(c) show such plots for the series of cyclotron resonance reported in Ref. 24. Both electron and hole data are plotted in units of $\omega_c = \sqrt{2}v_0/\ell \approx 40\sqrt{B/|T|}$ meV (with reference velocity $v_0 = 1.1 \times 10^6$ m/s kept fixed). These plots offer a closer look into the plots in Fig. 2.

They are to be compared with Figs. 4(b) and 4(d), which illustrate how each resonance would behave with $B$, according to Eq. (3.22), for $V_c \approx 11\sqrt{B/|T|}$ meV (or $\epsilon_b \approx 5$). The $\nu = -8$ curve for $\epsilon_{\text{exc}}^{\text{ren}} |_{B \approx 10}$, in particular, represents the running of $v_{\text{ren}}|_B$ according to Eq. (3.19) (normalized to 1 at $B_0 = 14T$). These theoretical curves and experimental data look similar but differ in details. They are not quite consistent, but there are some notable features: (i) The 2 $\leftrightarrow$ 1 and 1 $\leftrightarrow$ −2 resonances (the $\nu = \pm 4$ data) appear distinct from the rest, especially in their variation with $B$. In addition, the $\nu = \pm 4$ theoretical curves are separated from the rest by appreciable Coulombic gaps, but such a gap is not seen in the hole data. This would indicate, as noted in Sec. II, that the $n = (0,1)$ sector in bilayer graphene is significantly modified from the naive one we have supposed.

(ii) The $\nu = (8,12,16)$ electron data and the $\nu = (-8,-12,-16)$ hole data show a general trend to decrease with $B$, consistent with possible running of $v_{\text{ren}}$ with $B$. Such (∼ logarithmic) running of $v_{\text{ren}}|_B$ is a direct consequence of renormalization and is thus the key signature of the Coulomb interaction. In both electron and hole data $v_{\text{ren}}$ appears to run in the same way at a rate somewhat faster than naively expected. Such enhanced running could in part be attributed to possible quantum screening of the Coulomb interaction in graphene such that $\epsilon_b$ is effectively larger for lower $B$.

Interband cyclotron resonance was recently observed by Orlita et al. in bilayer inclusions in multilayer epitaxial graphene on the C-face of SiC. They identify some $n - 1 \leftrightarrow -n$ [or $n \leftrightarrow -(n - 1)$] resonances with $n \geq 3$ and obtain, via fitting, $v \approx 1.02 \times 10^6$ m/s and $\gamma_1 \approx 385$ meV, which are somewhat smaller than those for bilayer graphene.

Some of their data are analyzed according to our for-
mula (3.22) in Fig. 5; there we have set \( \gamma_4 = \Delta = 0 \) since this experiment searched for no intraband resonances which would clarify a possible electron-hole asymmetry. The data appear to indicate slight running of \( v_{\text{ren}} \) with \( B \), far slower than in the data in Fig. 4 on bilayer graphene. This suggests that the Coulomb interaction could be significantly weaker (or more efficiently screened) in multilayered epitaxial graphene than in exfoliated bilayer graphene.

IV. SUMMARY AND DISCUSSION

Experiment suggests that bilayer graphene has intrinsic electron-hole asymmetry due to subleading intralayer and interlayer couplings. In this paper we have studied cyclotron resonance in bilayer graphene with such asymmetry taken into account. The set of asymmetry parameters, \( \Delta \approx 18 \) meV and \( \gamma_4/\gamma_0 \approx 0.04 \) derived from independent measurements, entails a considerable modification of Landau levels in bilayer graphene and improves the theoretical fit to the data on cyclotron resonance between higher levels \( |n| \geq 2 \) in both electron and hole bands. In contrast, the fit to the data on \( (2 \leftarrow 1) \) and \( (1 \leftarrow 2) \) resonances appears somewhat puzzling, and this suggests that the zero- and pseudo-zero-mode Landau levels \( n = (0, \pm 1) \) are further affected by some sources other than \( \Delta \) and \( \gamma_4 \). It would be important to clarify, both theoretically and experimentally, the detailed structure of this special sector in bilayer graphene.

The Coulombic many-body corrections to cyclotron resonance in graphene, unlike in standard quantum-Hall systems, are afflicted with UV divergences, and one has to carry out renormalization to extract genuine observable corrections. We have shown how to perform renormalization for bilayer graphene under a magnetic field by first constructing necessary counterterms in free space. This renormalization program, formulated analytically, can equally be handled numerically in practical calculations by use of the reduced matrix Hamiltonian \( \hat{H}_{\text{red}}^{\text{ren}} \) in Eq. (A2) and counterterm \( \delta_{\text{ct}} \hat{H}_{\text{red}}^{\text{ren}} \) in Eq. (3.13). As a further illustration, we present the renormalization program for monolayer graphene with a possible valley gap in Appendix B.

Equation (3.22) summarizes the effect of renormalization on cyclotron-resonance energies in a neat and concise form. This formula is also useful in analyzing the experimental data; it magnifies possible effects of the Coulombic corrections per channel and running of the renormalized velocity \( v_{\text{ren}} \) with \( B \), as we have seen in Sec. III. In particular, the nearly logarithmic running of \( v_{\text{ren}} \) is a direct consequence of renormalization, specific to graphene.\(^{30,41}\) It is remarkable that such a renormalization effect is apparently seen in the data.

More detailed measurements of cyclotron resonance, both intraband and interband ones, are highly desired to pin down the many-body effects as well as the structure of the zero-mode and pseudo-zero-mode sector in bilayer graphene.

Acknowledgments

This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan (Grant No. 21540265).

Appendix A: Analysis of divergence

In this appendix we examine the UV structure of the Coulomb exchange correction. Let us first look at \( \mathcal{H}_+ \) in Eq. (2.1) and construct the electron propagator in free space,

\[
\langle \Psi(x)\Psi^\dagger(x') \rangle = \langle x|i/(i\partial_t - \mathcal{H}_+)|x' \rangle \quad (A1)
\]

with \(|x\rangle \equiv |\pi\rangle\). We divide \( \mathcal{H}_+ \) into \( 2 \times 2 \) blocks,

\[
\mathcal{H}_+ = \begin{pmatrix} m \sigma_3 & v P + v_4 Q \\ v P + v_4 Q^\dagger & -m \sigma_3 + \Delta + \gamma_1 \sigma_1 \end{pmatrix} \quad (A2)
\]

where we have set \( m \equiv u/2; P = p^1 \sigma_+ + p \sigma_- \) with \( p = p_x + i p_y \) and \( \sigma_{\pm} = (\sigma_1 \pm i \sigma_2)/2 \); \( Q = \text{diag}(p^1, p) \); the unit matrix \( 1 \), which, e.g., multiplies \( \omega \). We go to the Fourier \((p, \omega)\) space and invert \((\omega - \mathcal{H}_+)\) in this \( 2 \times 2 \) block form. A direct calculation yields \( \langle \Psi^\dagger \rangle_{jk} = i N_{jk}/D \) with

\[
N_{11} = \Gamma + \{(v^2 + v_4^2) \gamma_1 + 2v_4 \langle \omega - \Delta \rangle\} P \sigma_1 P + m \{\Xi - 2 \Delta \omega + \Delta^2 - \gamma_1^2\} \sigma_3,
\]

\[
N_{22} = \Gamma - 2 \langle (v^2 - v_4 \gamma_1) \rangle P^2 + \omega \{\gamma_1^2 + \Delta^2\} \Xi + \langle v^2 \rangle \sigma_3 + \{\gamma_1 (\omega^2 - m^2) + 2v_4 \langle \omega_4 \rangle \sigma_1),
\]

\[
N_{12} = \langle v^2 \rangle - v_4 v_4 p^2 + m^2 P + 2m v_4 \sigma_3 P + v_4 \Xi Q + (v_4 \gamma_1 - v \Delta) \langle \omega + m \sigma_3 \rangle P + (v_4 \gamma_1 - v \Delta) \langle \omega + m \sigma_3 \rangle Q, \quad (A3)
\]

and \( N_{21} = (N_{12})^\dagger \), where

\[
\Gamma = \omega \{(\omega - \Delta^2) - (v^2 + v_4^2) \Delta^2 - m^2 - \gamma_1^2\} + \langle (v^2 + v_4^2) \Delta - 2v_4 \gamma_1 \rangle P^2, \quad (A4)
\]

\[
\Xi = \omega^2 + v^2 \gamma_1 \langle p^2 \rangle - m^2; \quad (A4)
\]

\[
v_4 \equiv v^2 + v_4 \quad \text{and} \quad P \sigma_1 P = (p^1)^2 \sigma_+ + p^2 \sigma_- \quad \text{. The denominator} \quad D = \det(\omega - \mathcal{H}_+) \quad \text{is cast in the form}
\]

\[
D = \{\omega^2 - \Delta^2 - (v^2 + v_4^2) \Delta^2 - m^2\}^2 - (\gamma_1^2 + 2v_4 \langle \omega_4 \rangle \sigma_1)^2 - m^2 (4v^2 \langle p^2 \rangle^2 - \gamma_1^2 - \Delta^2), \quad (A5)
\]
which, for $u = 2m \to 0$, leads to the band spectra $\{\epsilon_i\}$ in Eq. (2.4).

The Coulomb exchange correction to $O(\alpha/\ell)$ is written as the convolution $\sum_k v_k iS(p+k)$ of $v_k = 2\pi \alpha/|k|$ and the instantaneous limit of the electron propagator $\langle \Psi \Psi^\dagger \rangle_{\omega=p} = \int (d\omega/2\pi) \langle \Psi \Psi^\dagger \rangle_{\omega=p} = iS(p)$. In particular, divergences arise from the portion of $S(p)$, that decreases like $1/|p|$ or slower for $p \to \infty$, and we shall focus on that portion.

Integration over $\omega$, with the standard boundary condition, is readily carried out, yielding, e.g.,

$$
\int d\omega \frac{1}{2\pi D(\omega)} = \frac{i(a+b+c+d)}{(a+c)(b+c)(a+d)(b+d)}, 
$$

(A6)

where $D(\omega) = (\omega - a)(\omega - b)(\omega - c)(\omega + d)$ (a, b, c, d > 0) is short for $D$ in Eq. (A5). This implies that both $\gamma_1$ and $\Delta$ undergo infinite renormalization. Actually, this leading form $\gamma_1$ and $\Delta$ lead to no di-

Portion has the asymptotic structure

$$
\Gamma = \frac{v\gamma_1 - v\Delta}{4v_+v_-|p|} (1/\sqrt{p^2 + M^2}) \sigma_1 + \frac{v\Delta - v\gamma_1}{2v_+v_-|p|} + \cdots 
$$

(A10)

This implies that both $\gamma_1$ and $\Delta$ undergo infinite renormalization. Evaluating the convolution integral $\int d^2k v_k S(p+k)$ with momentum cutoff $\Lambda$ eventually leads to the counterterms in Eqs. (B14) and (B15).

**Appendix B: monolayer graphene with a valley gap**

In this appendix we outline the renormalization prescription for monolayer graphene with a possible valley gap $M$. The effective Hamiltonian is written as

$$
\mathcal{H}_+ = v p \cdot \mathbf{\sigma} + M \sigma_3 
$$

(B1)

at one valley and acts on a two-component spinor of the form $\Psi = (\psi_A, \psi_B)$. One can pass to another valley by setting $M \to -M$ and $\Psi \to \Psi' = (-\psi_B, \psi_A)$. Using the instantaneous propagator

$$
\langle \Psi \Psi^\dagger \rangle_{\omega=p} = \frac{1}{2} \frac{v p \cdot \mathbf{\sigma} + M \sigma_3}{\sqrt{p^2 + M^2}} 
$$

(B2)

one can calculate the Coulombic selfenergy correction to $O(\alpha)$ and find divergences of the form

$$
\delta v \sim -(\alpha/8\epsilon_b) \log \Lambda^2 \quad \delta M = 2(M^{\text{ren}}/v_0^{\text{ren}}) \delta v, 
$$

(B3)

which implies that the mass gap $M$, as well as $v$, undergoes renormalization.

Let us now pass to the $B \neq 0$ case and denote the zeroth order Landau-level spectrum as $\epsilon_n = \omega_c^{\text{ren}} \eta_n$ with

$$
\eta_n = \text{sign}[n] \sqrt{|n| + (M^{\text{ren}}/\omega_c^{\text{ren}})^2}. 
$$

(B4)

Letting $\delta \epsilon_t = \delta v \partial \epsilon_t^{\text{ren}} + \delta M \partial \epsilon_t^{\text{ren}}$ act on $\epsilon_n$, then yields the counterterm $\delta\epsilon_t^{\text{ren}}$. In particular, one finds that $\mathcal{D}\eta_n$ in Eq. (B14) is now replaced by

$$
\mathcal{D}\eta_n \to (M^{\text{ren}}/\omega_c^{\text{ren}})^2/\eta_n. 
$$

(B5)

With Eqs. (B14) and (B15), velocity and mass renormalization for monolayer graphene in a magnetic field is carried out according to formula (2.22); we have checked numerically that this renormalization program works correctly.
1 K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature (London) 438, 197 (2005).
2 Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, Nature (London) 438, 201 (2005).
3 Y. Zhang, Z. Jiang, J.P. Small, M.S. Purewal, Y.-W. Tan, M. Fazlollahi, J.D. Chudow, J.A. Jaszcak, H. L. Stormer, and P. Kim, Phys. Rev. Lett. 96, 136806 (2006).
4 N. H. Shon and T. Ando, J. Phys. Soc. Jpn. 67, 2421 (1998);
5 Y. Zheng and T. Ando, Phys. Rev. B 65, 245420 (2002).
6 N. M. R. Peres, F. Guinea, and A. H. Castro Neto, Phys. Rev. B 73, 125411 (2006).
7 W. Kohn, Phys. Rev. 123, 1242 (1961).
8 For many-body corrections in quantum Hall systems, see K. Asano and T. Ando, Phys. Rev. B 58, 1485 (1998). Kohn’s theorem is also discussed for graphene by R. Roldán, J.-N. Fuchs, and M. O. Goerbig, Phys. Rev. B 82, 205418 (2010).
9 D. S. L. Abergel and V. I. Fal’ko, Phys. Rev. B 75, 155430 (2007).
10 A. Iyengar, J. Wang, H. A. Fertig, and L. Brey, Phys. Rev. B 75, 125430 (2007).
11 Yu. A. Bychkov and G. Martinez, Phys. Rev. B 77, 125417 (2008).
12 S. Viola Kusminskiy, D. K. Campbell, and A. H. Castro Neto, Euro. Phys. Lett. 85, 58005 (2009).
13 K. Shizuya, Phys. Rev. B 81, 075407 (2010).
14 K. S. Novoselov, E. McCann, S. V. Morozov, V. I. Fal’ko, M. I. Katsnelson, U. Zeitler, D. Jiang, F. Schedin, and A. K. Geim, Nat. Phys. 2, 177 (2006).
15 E. McCann and V. I. Fal’ko, Phys. Rev. Lett. 96, 086805 (2006).
16 T. Ohta, A. Bostwick, T. Seyller, K. Horn, and E. Rotenberg, Science 313, 951 (2006).
17 E. McCann, Phys. Rev. B 74, 161403(R) (2006).
18 E. V. Castro, K. S. Novoselov, S. V. Morozov, N. M. R. Peres, J. M. B. Lopes dos Santos, J. Nilsson, F. Guinea, A. K. Geim, and A. H. Castro Neto, Phys. Rev. Lett. 99, 216802 (2007).
19 J. B. Oostinga, H. B. Heersche, X. Liu, A. F. Morpurgo, and L. M. K. Vandersypen, Nature Mater. 7, 151 (2008).
20 M. Mucha-Kruczyński, E. McCann, and V. I. Fal’ko, Solid State Commun. 149, 1111 (2009).
21 Z. Jiang, E. A. Henriksen, L. C. Tung, Y.-J. Wang, M. E. Schwartz, M. Y. Han, P. Kim, and H. L. Stormer, Phys. Rev. Lett. 98, 197403 (2007).
22 R. S. Deacon, K.-C. Chuang, R. J. Nicholas, K. S. Novoselov, and A. K. Geim, Phys. Rev. B 76, 081406(R) (2007).
23 E. A. Henriksen, P. Cadden-Zimansky, Z. Jiang, Z. Q. Li, L.-C. Tung, M. E. Schwartz, M. Takita, Y.-J. Wang, P. Kim, and H. L. Stormer, Phys. Rev. Lett. 104, 067404 (2010).
24 E. A. Henriksen, Z. Jiang, L.-C. Tung, M. E. Schwartz, M. Takita, Y.-J. Wang, P. Kim, and H. L. Stormer, Phys. Rev. Lett. 100, 087403 (2008).
25 M. Orlita, C. Faugeras, J. Borysiuk, J. M. Baranowski, W. Strupinski, M. Sprinkle, C. Berger, W. A. de Heer, D. M. Basko, G. Martinez, and M. Potemski, Phys. Rev. B 83, 125302 (2011).
26 L. M. Malard, J. Nilsson, D. C. Elias, J. C. Brant, F. Plentz, E. S. Alves, A. H. Castro Neto, and M. A. Pimenta, Phys. Rev. B 76, 201401(R) (2007).
27 L. M. Zhang, Z. Q. Li, D. N. Basov, and M. F. Fogler, Z. Hao, and M. C. Martin, Phys. Rev. B 78, 235408 (2008).
28 Z. Q. Li, E. A. Henriksen, Z. Jiang, Z. Hao, M. C. Martin, P. Kim, H. L. Stormer, and D. N. Basov, Phys. Rev. Lett. 102, 037403 (2009).
29 A. B. Kuzmenko, E. van Heumen, D. van der Marel, P. Lerch, P. Blake, K. S. Novoselov, and A. K. Geim, Phys. Rev. B 79, 115441 (2009).
30 J. Nilsson, A. H. Castro Neto, F. Guinea, and N. M. R. Peres, Phys. Rev. B 78, 045405 (2008).
31 D. S. L. Abergel and T. Chakraborty, Phys. Rev. Lett. 102, 056807 (2009).
32 T. Misumi and K. Shizuya, Phys. Rev. B 77, 195423 (2008); K. Shizuya, Phys. Rev. B 75, 245417 (2007).
33 Numerically, for $u = 5$meV crossing $\epsilon_{\alpha} \approx \epsilon_{\alpha}^{\text{ren}}$ takes place at $B \approx 10T$.
34 Y. Barlas, R. Côté, K. Nomura, and A. H. MacDonald, Phys. Rev. Lett. 101, 097601 (2008).
35 K. Shizuya, Phys. Rev. B 79, 165402 (2009).
36 R. Côté, J. Lambert, Y. Barlas, and A. H. MacDonald, Phys. Rev. B 82, 035445 (2010).
37 S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B 33, 2481 (1986).
38 J. González, F. Guinea, and M. A. H. Vozmediano, Nucl. Phys. B 424, 595 (1994).
39 For an eigenmode $|\psi_{\alpha}\rangle$ of $H$ with eigenvalue $\epsilon_{\alpha}$ one generally finds that $\langle \psi_{\alpha} | \delta H | \psi_{\alpha} \rangle = \delta \epsilon_{\alpha}$ for an arbitrary variation $\delta$.
40 Recently the effect of renormalization on the electronic spectrum has been measured for suspended graphene, in which $\nu^{\text{ren}}$ runs with the electron density $\rho$ as $v^{\text{ren}}(\rho) \approx v^{\text{ren}}(0) + (\alpha/8\pi^2) \log(\rho/\rho_0)$. See, D. C. Elias, R. V. Gorbachev, A. S. Mayorov, S. V. Morozov, A. A. Zhukov, P. Blake, K. S. Novoselov, A. K. Geim, and F. Guinea, preprint [arXiv:1104.1396].
41 Renormalization of electronic interactions in bilayer graphene under zero magnetic field was discussed by Y. Barlas and K. Yang, Phys. Rev. B 80, 161408(R) (2009); O. Vafek and K. Yang, Phys. Rev. B 81, 041401(R) (2010); R. Nandkishore and L. Levitov, Phys. Rev. B 82, 115431 (2010).