Generalization of GANs under Lipschitz continuity and data augmentation

Khoat Than* Nghia Vu†

Abstract

Generative adversarial networks (GANs) have been being widely used in various applications. Arguably, GANs are really complex, and little has been known about their generalization. In this paper, we make a comprehensive analysis about generalization of GANs. We decompose the generalization error into an explicit composition: generator error + discriminator error + optimization error. The first two errors show the capacity of the player's families, are irreducible and optimizer-independent. We then provide both uniform and non-uniform generalization bounds in different scenarios, thanks to our new bridge between Lipschitz continuity and generalization. Our bounds overcome some major limitations of existing ones. In particular, our bounds show that penalizing the zero- and first-order informations of the GAN loss will improve generalization, answering the long mystery of why imposing a Lipschitz constraint can help GANs perform better in practice. Finally, we show why data augmentation penalizes the zero- and first-order informations of the loss, helping the players generalize better, and hence explaining the highly successful use of data augmentation for GANs.

1 Introduction

In Generative Adversarial Networks (GAN) (Goodfellow et al., 2014), we want to train a discriminator \(D \in \mathcal{D}\) and a generator \(G \in \mathcal{G}\) by solving the following problem:

\[
\min_{G \in \mathcal{G}} \max_{D \in \mathcal{D}} \mathbb{E}_{x \sim P_d} \log(D(x)) + \mathbb{E}_{z \sim P_z} \log(1 - D(G(z)))
\]  

(1)

where \(P_d\) is a data distribution that generates real data, and \(P_z\) is some noise distribution. \(G\) is a mapping that maps a noise \(z\) to a point in the data space. After training, \(G\) can be used to generate novel but realistic data.

Since its introduction (Goodfellow et al., 2014), a significant progress has been made for developing GANs and for interesting applications (Hong et al., 2019). Some recent works (Brock et al., 2019; Zhang et al., 2019; Karras et al., 2020b) can train a generator that produces synthetic images of extremely high quality. Nevertheless, little has been known about the generalization of the trained players. Two major limitations of existing works are: (i) The distance-based bounds (Arora et al., 2017; Zhang et al., 2018; Jiang et al., 2019; Husain et al., 2019) for generalization error depend on the capacity of the family, which defines the distance between two distributions, sometimes leading to trivial bounds; (ii) Existing uniform bounds (Zhang et al., 2018; Jiang et al., 2019) mix the capacity of the generator family into optimization error. Those limitations prevent us from fully understanding and identifying the key factors that contribute to the generalization of GANs.

This work has the following contributions:

\(\triangleright\) We make a bridge between Lipschitz continuity of a loss and generalization of a hypothesis. Basically, we show that the learnt hypothesis can generalize well if the training loss is Lipschitz continuous w.r.t input.

*<khoattq@soict.hust.edu.vn>; Hanoi University of Science and Technology; work performed while visiting VinAI Research, Vietnam.
†Hanoi University of Science and Technology
We decompose the generalization error of GANs into an explicit composition: generator error + discriminator error + optimization error. The first two errors show the capacity of families \((\mathcal{D}, \mathcal{G})\), are irreducible and optimizer-independent. Thanks to the above bridge, upper bounds on those irreducible errors can be derived explicitly.  We then provide both distance-based and loss-based bounds for GAN generalization. Our bounds overcome the two major limitations of existing works. In particular, our bounds show that penalizing the zero- and first-order informations of the GAN loss will improve generalization, answering the long mystery of why imposing a Lipschitz constraint can help GANs to generalize well in practice.

Finally, we show why data augmentation (DA) penalizes the zero- and first-order informations of the loss, helping the players generalize better. Furthermore, DA requires both the zero- and first-order informations of the fake distribution to match those of the real distribution, leading to faster training for \(G\). Those findings are intriguing and will support the highly successful uses of data augmentation for GANs (Zhao et al., 2020a,b).

Organization: We will review related work in the next section. Section 3 presents the bridge between Lipschitz continuity and generalization. In Section 4, we analyze the generalization error in various aspects. We also point out some issues of existing works. Section 5 discusses the role of data augmentation for GANs.

2 Related work

Generalization: There are few efforts to analyze the generalization for GANs using the notion of neural distance, \(d_{\mathcal{D}}(P_d, P_g)\) which is the distance of two distributions \((P_d, P_g)\). Arora et al. (2017) show that we can bound the quantity \(|d_{\mathcal{D}}(P_d, \hat{P}_g) − d_{\mathcal{D}}(P_d, P_g)|\), where \((\hat{P}_d, \hat{P}_g)\) are empirical versions of \((P_d, P_g)\). Zhang et al. (2018) and Jiang et al. (2019) provide uniform bounds on generalization error by analyzing \(|d_{\mathcal{D}}(P_d, P_g) − \min_{G} d_{\mathcal{D}}(P_d, P_g)|\). Their bounds base on the assumption of \(|d_{\mathcal{D}}(P_d, \hat{P}_g) − \min_{G} d_{\mathcal{D}}(P_d, P_g)| \leq \epsilon_{og}\), for a fixed and small \(\epsilon_{og}\). However, as shown in Section 4.2, \(\epsilon_{og}\) depends not only on optimization error of an optimizer given a training set of finite size, but also on the capacity of generator family. In other words, (Zhang et al., 2018; Jiang et al., 2019) mix the capacity of the generator family into optimization error. A small \(\epsilon_{og}\) implies that the generator family is implicitly assumed to be strong and each \(G \in \mathcal{G}\) has a high complexity. A major limitation of (Zhang et al., 2018; Jiang et al., 2019) is the ignorance of such complexity and hence does not provide a clear understanding of GANs’ generalization.

A common limitation of neural distance-based bounds (Arora et al., 2017; Zhang et al., 2018; Jiang et al., 2019) is the dependence on the notion of distance \(d_{\mathcal{D}}(\cdot, \cdot)\) which relies on the best \(D \in \mathcal{D}\) for measuring proximity between two distributions. The distance between two distributions may be small even when the two are far away (Arora et al., 2017). This is because there exists a perfect discriminator \(D\), whenever \(\mu\) and \(\nu\) do not have overlapping supports (Arjovsky and Bottou, 2017). In those cases, a distance-based bound may be trivial. As a result, existing distance-based bounds are insufficient to see the generalization of GANs. Our work decomposes the generalization error into an explicit composition which includes the capacity of both families \((\mathcal{D}, \mathcal{G})\). We then analyze the capacity of \((\mathcal{D}, \mathcal{G})\) and take it into account to provide new (non-uniform and uniform) bounds. Surprisingly, the generalization of GANs can be revealed directly from their loss.

Qi (2020) shows a generalization bound for their proposed Loss-Sensitive GAN. Nonetheless, it is nontrivial to make their bound to work with other GAN losses. Wu et al. (2019) show that the discriminator will generalize if the learning algorithm is differentially private. Their concept of differential privacy basically requires that the learnt hypothesis will change negligibly if the training set slightly changes. Such a requirement is known as algorithmic stability (Xu et al., 2010) and is nontrivial to assure in practice. In contrast, we show that under the assumption of Lipschitz continuity of the loss w.r.t input, both \(D\) and \(G\) generalize well in the classical sense. Our bounds apply to a large class of GANs.

Lipschitz continuity, stability, and generalization: Lipschitz continuity naturally appears in the formulation of Wasserstein GAN (WGAN) (Arjovsky et al., 2017). It was then quickly recognized as a key to improve various GANs (Fedus et al., 2018; Lucic et al., 2018; Mescheder et al., 2018; Kurach et al., 2019; Jenni and Favaro, 2019; Wu et al., 2019; Zhou et al., 2019; Qi, 2020; Chu et al., 2020). Gradient penalty
are two popular techniques to constraint the Lipschitz continuity of $D$ or $G$ w.r.t their inputs. Some other works (Mescheder et al., 2017; Nagarajan and Kolter 2017; Sanjabi et al., 2018; Nie and Patel, 2019) suggest to control the Lipschitz continuity of $D$ or $G$ w.r.t their parameters. Those works empirically found that Lipschitz continuity can help improving stability and generalization of GANs. However, it has long been a mystery of why imposing a Lipschitz constraint can help GANs to generalize well. This work provides an answer for it.

Data augmentation for GANs: Data augmentation (Shorten and Khoshgoftaar 2019) has been playing a crucial role in various areas. Some recent works (Zhao et al., 2020a,b; Zhang et al., 2020) found that it is really beneficial to exploit data augmentation for training GANs. However, there is a lack of theory to explain those observations. This work will fill this gap.

3 Lipschitz continuity and Generalization

In this section, we will review an important result by Xu and Mannor (2012) about the close relation between the robustness and generalization of a learning algorithm. We then show how robustness connects to Lipschitz continuity. Finally we point out why imposing a Lipschitz constraint on the loss will lead to generalization.

Consider a learning problem specified by a hypothesis class $H$, an instance set $Z$, and a loss function $f : H \times Z \to \mathbb{R}$ which is bounded by a constant $C$. Given a distribution $P_z$ defined on $Z$, the quality of a hypothesis is measured by its expected loss $F(h) = \mathbb{E}_{z \sim P_z}[f(h;z)]$. Since $P_z$ is unknown, we need to rely on a finite training sample $S = \{z_1, ..., z_m\} \subset Z$ and often work with the empirical loss $F_S(h) = \frac{1}{m} \sum_{z \in S} f(h;z)$. A learning algorithm $A$ will pick a hypothesis based on input $S$. One can interpret a learning algorithm as a mapping from a subset of $Z$ to a hypothesis.

Let $Z = \bigcup_{i=1}^K Z_i$ be a partition of $Z$ into $K$ disjoint subsets. We use the following definition about robustness.

**Definition 1 (Robustness).** An algorithm $A$ is $(K, \epsilon)$-robust, for $\epsilon(\cdot) : Z^m \to \mathbb{R}$, if the following holds for all $S \in Z^m$: $\forall s \in S, \forall z \in Z, \forall i \in \{1, ..., K\}$, if $s, z \in Z_i$ then $|f(A(S), s) - f(A(S), z)| \leq \epsilon(S)$.

Basically, a robust algorithm will learn a hypothesis which ensures that the losses of two similar data instances should be the same. A small change in the input leads to a small change in the loss of the learnt hypothesis. In other words, the robustness ensures that each testing sample which is close to the training dataset will have a similar loss with that of the closest training samples. Therefore, the hypothesis $A(S)$ will generalize well over the areas around $S$.

**Theorem 1 (Xu and Mannor, 2012).** If a learning algorithm $A$ is $(K, \epsilon)$-robust, and the training data $S$ is an i.i.d. sample from distribution $P_z$, then for any $\delta \in (0,1]$ we have the following with probability at least $1 - \delta$: $|F(A(S)) - F_S(A(S))| \leq \epsilon(S) + C\sqrt{(K \log 4 - 2 \log \delta)/m}$.

This theorem formally makes the important connection between robustness and generalization of an algorithm. Essentially, an algorithm will generalize if it is robust. One important implication of this result is that we should ensure the robustness of a learning algorithm in practice. However, it is nontrivial to do so.

Let us have a closer look at robustness. $\epsilon(S)$ in fact bounds the amount of change in the loss with respect to a change in the input given a fixed hypothesis. This observation suggests that robustness closely resembles the concept of Lipschitz continuity. Remember that a function $y : Z \to Y$ is said to be $L$-Lipschitz continuous if $d_y(y(z), y(z')) \leq Ld_z(z, z')$ for any $z, z' \in Z$, where $d_z$ is a metric on $Z$, $d_y$ is a metric on $Y$, and $L \geq 0$ is the Lipschitz constant. Therefore, we establish the following connection between robustness and Lipschitz continuity. (The proof appears in Appendix A.)

**Lemma.** Given any constant $\lambda > 0$, consider a loss $f : H \times Z \to \mathbb{R}$, where $Z \subset \mathbb{R}^n$ is compact, $B = \text{diam}(Z) = \max_{z, z' \in Z} ||z - z'||_\infty, K = [B^n \lambda^{-n}]$. If $f(A;z)$ is $L$-Lipschitz continuous w.r.t input $z$, then algorithm $A$ is $(K, L\lambda)$-robust.
Theorem 3 (Lipschitz continuity ⇒ Generalization). If a loss $f(A; z)$ is $L$-Lipschitz continuous w.r.t. input $z$ in a compact set $Z \subseteq \mathbb{R}^n$, and the training data $S$ is an i.i.d. sample from distribution $P_z$, then $|F(A(S)) - F_S(A(S))|$ is upper-bounded by

1. $L\lambda + C \sqrt{[\ln n \log 4 - 2\log \delta]/m}$ with probability at least $1 - \delta$, for any constants $\delta \in (0, 1]$ and $\lambda \in (0, B]$.

2. $(L + 2C) m^{-\alpha/n}$ with probability at least $1 - 2\exp(-0.5m^\alpha)$, for any $\alpha \leq n/(2 + n)$.

This theorem tells that Lipschitz continuity is the key to ensure an algorithm $A$ (and the learnt hypothesis) to generalize. $A$ may generalize better as the Lipschitz constant of the loss decreases. Note that there is a tradeoff between the Lipschitz constant and the expected loss $F(A(S))$ of the learnt hypothesis. A smaller $L$ means that both $f$ and hypothesis $h = A(S)$ are getting simpler and flatter, due to $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial h} = 0$, and hence may increase $F(h)$. In contrast, a decrease of $F(h)$ may require $h$ to be more complex and hence may increase the Lipschitz constants of both $f$ and $h$. It is worth noting that the Lipschitz constant of the loss also depends on how fast the loss changes w.r.t. $h$.

The learning theory community has been trying various ways, e.g., Rademacher complexity, algorithmic robustness, uniform stability [Mohri et al., 2018], where the characteristics of the hypothesis space or learning algorithms are used to provide bounds on generalization. The bridge between Lipschitz continuity and generalization in Theorem 3 provides a novel perspective: the structural properties of the input and output domains of the loss function can be exploited to analyze generalization. Note that those properties directly relate to the zero (C) and first-order (L) information of the loss.

4 Generalization of GANs

This section presents a comprehensive analysis on generalization of GANs. We first present non-uniform bounds which are consequences of Theorem 3. Next we show how generalization error of GANs can be explicitly decomposed into different factors. Such a decomposition help us to further disentangle the capacity of generator and discriminator families, and their contributions to GANs’ generalization. We point out why existing works remain unsolved issues, and this work overcomes those drawbacks. Uniform bounds in different scenarios will be derived.

Let $S = \{x_1, ..., x_k, z_1, ..., z_m\}$ consist of $k$ i.i.d. samples from real distribution $P_d$ defined on a compact set $Z_x \subseteq \mathbb{R}^{n_x}$ and $m$ i.i.d. samples from noise distribution $P_z$ defined on a compact set $Z_z \subseteq \mathbb{R}^{n_z}$. $P_d$ and $P_z$ be the empirical distributions defined from $S$ respectively. Let $v(D, G, x, z) = \psi_1(D(x)) + \psi_2(1 - D(G(z)))$ be the loss defined from a real example $x \sim P_d$, a noise $z \sim P_z$, a discriminator $D \in D$, and a generator $G \in \mathcal{G}$. Different choices of the measuring functions $(\psi_1, \psi_2)$ will lead to different GANs. For example, the vanilla GAN (Goodfellow et al., 2014) uses $\psi_1(x) = \psi_2(x) = \log(x)$; WGAN (Arjovsky et al., 2017) uses $\psi_1(x) = \psi_2(x) = x$; LSGAN (Mao et al., 2017, 2019) uses $\psi_1(x) = -(x + a)^2$, $\psi_2(x) = -(x + b)^r$ for some constants $a, b$; EBGAN (Zhao et al., 2017) uses $\psi_1(x) = x$, $\psi_2(x) = \max(0, r - x)$ for some constant $r$. We will often work with the following losses:

$$V(P_d, P_z, D, G) = \mathbb{E}_{x \sim P_d, z \sim P_z} v(D, G, x, z)$$
$$V(P_d, \hat{P}_z, D, G) = \mathbb{E}_{x \sim P_d} \psi_1(D(x)) + \frac{1}{m} \sum_{z \in S} \psi_2(1 - D(G(z)))$$
$$V(\hat{P}_d, P_z, D, G) = \frac{1}{k} \sum_{x \in S} \psi_1(D(x)) + \mathbb{E}_{z \sim P_z} \psi_2(1 - D(G(z)))$$
$$V(\hat{P}_d, \hat{P}_z, D, G) = \frac{1}{k} \sum_{x \in S} \psi_1(D(x)) + \frac{1}{m} \sum_{z \in S} \psi_2(1 - D(G(z)))$$
In the latter discussions, we will often use the following assumptions and notation $L = L_\psi L_d L_g$ which upper bounds the Lipschitz constant of the loss $v(D, G, x, z)$.

**Assumption 1.** $\psi_1$ and $\psi_2$ are $L_\psi$-Lipschitz continuous w.r.t. their inputs on a compact domain and upper-bounded by constant $C \geq 0$.

**Assumption 2.** Each generator $G \in \mathcal{G}$ is $L_G$-Lipschitz continuous w.r.t its input $z$ over a compact set $Z_z \subset \mathbb{R}^n$ with diameter $B_z = \max_{z, z' \in Z_z} ||z - z'||_\infty$.

**Assumption 3.** Each discriminator $D \in \mathcal{D}$ is $L_D$-Lipschitz continuous w.r.t its input $x$ over a compact set $Z_x \subset \mathbb{R}^n$ with diameter $B_x = \max_{x, x' \in Z_x} ||x - x'||_\infty$.

### 4.1 Non-uniform bounds on generalization errors

We first concern on upper bounds of the generalization error of a particular generator when it is learned from a finite training noise data. To see generalization of both players, observe that $|V(P_d, P_z, D, G) - V(P_\hat{d}, P_\hat{z}, D, G)| = \frac{1}{K} \sum_{x \in S} \psi_1(D(x)) + \frac{1}{m} \sum_{z \in S} \psi_2(1 - D(G(z))) - \mathbb{E}_{x \sim P_d, z \sim P_z} v(D, G, x, z) \leq \mathbb{E}_{x \sim P_d, \psi_2(1 - D(G(z))) - \mathbb{E}_{x \sim P_d, \psi_1(D(x))} + \mathbb{E}_{x \sim P_d, \psi_1(D(x))}$. Therefore Theorem 3 provides the following bounds.

**Corollary 2.** Given the assumptions \[\mathcal{A} \mathcal{B} \mathcal{C}, \mathcal{A} \mathcal{B} \mathcal{C},\] $|V(P_d, P_z, D, G) - V(P_\hat{d}, P_\hat{z}, D, G)|$ is upper-bounded by

1. $L_\lambda \frac{C}{L_\lambda} \sqrt{\left(\frac{L_\lambda}{L_\lambda} \log 4 - 2 \log \delta \right) / m}$ with probability at least $1 - 2\delta$, for any constants $\delta \in (0, 1)$ and $\lambda \in (0, B_z]$;
2. $(L_B + 2C)m^{-\alpha/n}$ with probability at least $1 - 2 \exp(-0.5m^\alpha)$, for any $\alpha \leq n/(2 + n)$.
may be more practical than existing ones, since \( D \) is not trained to optimality before training \( G \) in most implementations of GANs.

Given the optimal \( D^* = \arg \max_{D \in \mathcal{D}} V(P_d, P_\hat{z}, D, G) \) from the family \( \mathcal{D} \) of discriminators, training \( G \) is to minimize \( V(P_d, P_\hat{z}, D^*, G) \). Goodfellow et al. (2014) showed that it is equivalent to finding a \( G^* \) that minimizes the Jensen-Shannon divergence \( d_{JS}(P_d, P_\hat{g}) \), due to \( V(P_d, P_\hat{z}, D^*, G) = -\log 4 + 2d_{JS}(P_d, P_\hat{g}) \), where \( P_\hat{z} \) is the result of putting samples from \( P_\hat{z} \) through generator \( G \). Different choices of the loss \( v(D, G, x, z) \) will lead to different divergences, e.g., \( d_{JS}(P_d, P_\hat{g}) \) in saturating GAN, total variation distance in EBGAN, Wasserstein distance in WGAN, \( \chi^2 \)-divergence in LSGAN.

**Theorem 4.** Assume the assumptions \([1, 2, 3]\) and let \( \mu, \nu \) be the divergence between two distributions \( \mu, \nu \), for some universal constants \( c_1 > 0 \) and \( c_2 \). Then the errors \( |d_{D}(P_d, P_\hat{g}) - d_{D}(\hat{P}_d, \hat{P}_g)| \) and \( |d_{D}(P_d, P_\hat{g}) - d_{D}(P_\hat{d}, P_\hat{g})| \) are upper-bounded by

1. \( c_1 L \lambda + c_1 C \sqrt{\left( \eta_{P_d}^2 \lambda^{-\alpha} \log 4 - 2 \log \delta \right) / m} \) with probability at least \( 1 - \delta \), for any constants \( \delta \in (0, 1] \) and \( \lambda \in (0, 1] \);

2. \( c_1 (L B_2 + 2C)m^{-\alpha} / n \) with probability at least \( 1 - 2 \exp(-0.5m^\alpha) \), for any \( \alpha \leq n/(2 + n) \).

**Corollary 3.** Assume the assumptions \([1, 2, 3]\) and let \( \mu, \nu \) be the divergence between two distributions \( \mu, \nu \), for some universal constants \( c_1 > 0 \) and \( c_2 \). Then the errors \( |d_{D}(P_d, P_\hat{g}) - d_{D}(\hat{P}_d, \hat{P}_g)| \) and \( |d_{D}(P_d, P_\hat{g}) - d_{D}(P_\hat{d}, P_\hat{g})| \) are upper-bounded by

1. \( c_1 L \lambda + c_1 C \sqrt{\left( \eta_{P_d}^2 \lambda^{-\alpha} \log 4 - 2 \log \delta \right) / m} \) with probability at least \( 1 - \delta \), for any constants \( \delta \in (0, 1] \) and \( \lambda \in (0, 1] \);

2. \( c_1 (L B_2 + 2C)m^{-\alpha} / n \) with probability at least \( 1 - 2 \exp(-0.5m^\alpha) \), for any \( \alpha \leq n/(2 + n) \).

It is worth observing that, in general, \( d_{D}(\cdot, P_\hat{g}) \) and \( d_{D}(\cdot, \hat{P}_\hat{g}) \) correspond to different optimal discriminators. Therefore, the generalization bounds in Corollary 3 are nontrivial, and their proofs are given in Appendix B.

**Comparison with existing non-uniform bounds:** Arora et al. (2017) analyze the generalization for GANs in terms of \( F \)-divergence \( d_F(\cdot, \cdot) \), which coincides with \( d_D(\cdot, \cdot) \) under some suitable choices of \( (V, c_1, c_2) \). They analyze the generalization by bounding \( d_F(P_d, P_\hat{g}) - d_F(P_d, \hat{P}_d) \). Note that this quantity may not be meaningful in cases that both \( d_F(P_d, P_\hat{g}) \) and \( d_F(P_\hat{d}, P_\hat{g}) \) are small, but \( \hat{P}_d \) is still far from \( P_d \). The distance between two distributions may be small even when the two are far away (Arora et al., 2017). This is because there exists a perfect discriminator \( D \), whenever \( \mu \) and \( \nu \) do not have overlapping supports (Arjovsky and Bottou, 2017). In those cases, \( G \) does not generalize well. Another recent work by Husain et al. (2019) shows \( d_F(P_d, P_\hat{g}) - d_F(P_\hat{d}, \hat{P}_\hat{g}) \leq O(m^{-1/2}) \) (details in Appendix B). Note that their bound faces the same limitation with that by Arora et al. (2017). Qi (2020) shows a generalization bound of the generator for their Loss-Sensitive GAN, which contains a Lipschitz regularizer for the generator and the margin between the real and fake distributions. Nonetheless, it is nontrivial to make their bound to work with other GAN losses.

### 4.2 Uniform bounds and Capacity of the players

Next we consider uniform bounds in order to see the goodness of a particular \( G \) compared to the optimal generator. A recent analysis in (Zhang et al., 2018) provides a closer look at generalization of \( G \) by analyzing \( |d_F(P_d, P_\hat{g}) - \min_G d_F(P_d, P_\hat{g})| \). They show \( |d_F(P_d, P_\hat{g}) - \min_G d_F(P_d, P_\hat{g})| \leq \mathcal{R}_m(F, P_d) + \mathcal{O}(m^{-1/2}) + \epsilon_{og} \), where \( \mathcal{R}_m(F, P_d) \) is the Rademacher complexity of family \( F, \epsilon_{og} \) is the (assumed) error of minimizing \( d_F(P_d, P_\hat{g}) \) w.r.t \( G \). Jiang et al. (2019) further improve the bound to \( \mathcal{O}(m^{-1/2}) + \epsilon_{og} \) for a special family of generators with bounded spectral norms. Using the same assumption, we show the followings in Appendix B.

**Theorem 4.** Assume the assumptions \([1, 2, 3]\) and let \( (P_d, P_\hat{g}) \) be defined on a sample of size \( m \). Assume further that \( |d_D(P_d, P_\hat{g}) - \min_G d_D(P_d, P_\hat{g})| \leq \epsilon_{og} \). Then \( |d_D(P_d, P_\hat{g}) - \min_G d_D(P_d, P_\hat{g})| \) is upper-bounded by

1. \( 2c_1 L \nu_0 L_0 \lambda + 2c_1 C \sqrt{\left( \eta_{P_d}^2 \lambda^{-\alpha} \log 4 - 2 \log \delta \right) / m} + \epsilon_{og} \), with probability at least \( 1 - \delta \), for any constants \( \delta \in (0, 1] \) and \( \lambda \in (0, 1] \);

2. \( \mathcal{O}(m^{-\alpha \epsilon / m^\alpha}) + \epsilon_{og} \), with probability at least \( 1 - 2 \exp(-0.5m^\alpha) \), for any \( \alpha \leq n/(2 + n) \).

---

1. With some abuse of notation, we will use \( P_\hat{z} \) and \( P_\hat{g} \) interchangeably, where \( P_\hat{g} \) is the result of pushing forward \( P_\hat{z} \) through generator \( G \).

2. That is: \( d_F(P_d, P_\hat{g}) \leq \min_G d_F(P_d, P_\hat{g}) + \epsilon_{og} \).
The generalization bounds in Theorem 4 apply for any particular generator $G$ and are similar with those in [Zhang et al., 2018; Jiang et al., 2019]. This is the reason for why the bounds are of order $O(m^{-\alpha_x/n_x})$, where $n_x$ is the dimensionality of the real data. Such uniform bounds are significantly worse than those in Corollaries 1 and 3, but do not need the assumption on Lipschitzness of $G$.

Unsolved issue: The bounds in [Zhang et al., 2018; Jiang et al., 2019] and Theorem 4 base on the assumption of a fixed and small $\epsilon_{og}$ [Zhang et al., 2018; Jiang et al., 2019] called $\epsilon_{og}$ optimization error. However, we will see that $\epsilon_{og}$ depends on not only optimization quality but also the capacity of generator family. By definition, $\epsilon_{og}$ bounds the error when solving for $G^* = \arg \min_G d_D(\hat{P}_d, P_g)$ and therefore depends on $(m, n, n_x)$ of unknown degree. In practice, an optimizer has to solve problem $\min_{\mathcal{G}} d_D(\hat{P}_d, P_g)$ given a finite number of (real, noise) samples, instead of problem $\min_{\mathcal{G}} d_D(\hat{P}_d, P_g)$ which is defined from the population distribution $P_g$. It is worth observing the decomposition $d_D(\hat{P}_d, P_g) = d_D(\hat{P}_d, \hat{P}_g) + \min_G d_D(\hat{P}_d, P_g)$. This suggests that $\epsilon_{og}$ hides two intriguing errors:

**Optimization error:** $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| = \epsilon_o$

**Generator error:** $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| = \epsilon_g$

While $\epsilon_o$ tells the goodness of approximate solution $G$ returned by an optimizer after solving $\min_{\mathcal{G}} d_D(\hat{P}_d, \hat{P}_g)$, $\epsilon_g$ reveals the generalization of the optimal generator solely learnt from an empirical dataset. Note that $\epsilon_o$ is optimizer-independent and shows the capacity of the chosen generator family. A small $\epsilon_{og}$ implies a small $\epsilon_g$, meaning that the generator family is implicitly assumed to be strong and each $G \in \mathcal{G}$ has a high complexity. A major limitation of [Zhang et al., 2018; Jiang et al., 2019] is the ignorance of such complexity and hence does not provide a clear understanding of GANs’ generalization.

Fortunately, our bridge between Lipschitz continuity and generalization in Theorem 3 allows us to remove the above limitation, to provide explicit bounds for $\epsilon_g$, and to take them into our analysis. Appendix B shows the followings.

**Theorem 5 (Generator error).** Given the assumptions 1, 2, 3, the errors $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)|$ and $|\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, P_g)|$ are upper-bounded by

1. $c_1 L \lambda + c_1 C \sqrt{(\|B^2\lambda^{-n}\| \log 4 - 2 \log \delta)/m}$ with probability at least $1 - \delta$, for any constants $\delta \in (0, 1]$ and $\lambda \in (0, B_x]$;
2. $c_1 (LB_z + 2C) m^{-\alpha/n}$ with probability at least $1 - 2 \exp(-0.5 m^\alpha)$, for any $\alpha \leq n/(2 + n)$.

Surprisingly, the generalization error of both optimal generators $G_{d^*_o} = \arg \min_{\mathcal{G}} d_D(\hat{P}_d, \hat{P}_g)$ and $G_g^* = \arg \min_{\mathcal{G}} d_D(P_d, \hat{P}_g)$ is bounded above by $O(m^{-\alpha/n})$. In practice the dimensionality $n$ of the noise is often significantly small compared to $n_x$ of real data. Meanwhile, the bound of $O(m^{-\alpha/n_x})$ for any particular $G$ is substantially worse than that of $G_{d^*_o}$ and $G_g^*$. To the best of our knowledge, this is the first work that shows the existence of generators whose generalization errors do not face with the curse of dimensionality $n_x$ of real data.

**Theorem 6 (Discriminator error).** Given the assumptions 1, 2, 3, the errors $|\min_G d_D(\hat{P}_d, P_g)|$ and $|\min_G d_D(P_d, P_g)|$ are upper-bounded by

1. $c_1 L \psi_{d^*} \lambda + c_1 C \sqrt{(\|B^2\lambda^{-n_x}\| \log 4 - 2 \log \delta)/m}$ with probability at least $1 - \delta$, for any constants $\delta \in (0, 1]$ and $\lambda \in (0, B_x]$;
2. $c_1 (L \psi_{d^*} B_z + 2C) m^{-\alpha_x/n_x}$ with probability at least $1 - 2 \exp(-0.5 m^{\alpha_x})$, for any $\alpha_x \leq n_x/(2 + n_x)$.

[Zhu et al., 2020] show that $\epsilon_{og} \geq c_3 m^{-3/n}$ even for isotropic gaussian $P_d = N(0, I)$, for some constant $c_3$. 

7
In the proof of this theorem (Appendix B) we observe that both $|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)|$ and $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)|$ are bounded above by $c_1 \max_{D \in \mathcal{D}} |\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))|$ which is independent with any optimizer. Therefore we can see them as Discriminator error, which represents the capacity of the discriminator family.

Now consider the joint error of both families. Observe that $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq |\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| + |\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)|$. Theorems 5 and 6 result in the following.

**Theorem 7 (Joint error).** Given the assumptions 1, 2, 3, and $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq |\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| + |\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)|$. Theorems 5 and 6 imply the following.

One can observe that $|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)|$ measures the generalization error of the best players when trained from a finite number of samples only, and such error does not depend on any optimizer. Hence it represents the Joint capacity of both generator and discriminator families.

### 4.3 From optimization error to generalization

Next we make a bridge between optimization error to generalization. In practice, we always have to train a generator from a finite number of samples (both real data and noises). As a result, it would be more meaningful if we can bound the generalization error of $G$ when trained from $S$ of size $m$. This requires us to take into account the optimization error when solving for $G'_{dy} = \arg\min_G d_D(P_d, \hat{P}_g)$.

We observe that $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| \leq |d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)| + |\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)| \leq \epsilon_o + \epsilon_g$. Interestingly, we can explicitly decompose the generalization error of $G$ into the sum of optimization error and generator error which is irreducible for the chosen generator family. Theorem 5 provides the bound $\epsilon_g = O(m^{-\alpha/n})$.

**Theorem 8 (G generalization).** Assume the assumptions 1, 2, 3 and $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| \leq \epsilon_o$. For any $\alpha \leq n/(2 + n)$, with probability at least $1 - 2\exp(-0.5m^\alpha)$, we have

$$|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| \leq O(m^{-\alpha/n}) + \epsilon_o.$$

Next we consider the generalization of a particular $D$. Observe that $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq |d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| + |\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq \epsilon_o + \epsilon_d$, where $\epsilon_d = |\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, P_g)|$. Theorem 6 implies the following.

**Theorem 9 (D generalization).** Assume the assumptions 1, 2, 3 and $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, \hat{P}_g)| \leq \epsilon_o$. For any $\alpha_x \leq n_x/(2 + n_x)$, with probability at least $1 - 2\exp(-0.5m^{\alpha_x})$, we have

$$|d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| \leq O(m^{-\alpha_x/n_x}) + \epsilon_o.$$  

In the extreme case, we would like to bound the error $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)|$. Note that $|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq |d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| + |\min_G d_D(P_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| + |\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)|$. Therefore

$$|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq \epsilon_o + \epsilon_g + \epsilon_d. \quad (2)$$

Zhang et al. (2018) used the bound $\max_{D \in \mathcal{D}} |\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))| \leq R_m(F, P_d) + O(\sqrt{\log(1/\delta)/m})$. This bound can be used in Theorem 6. Nonetheless, we did not use it to avoid dependence on $R_m(F, P_d)$.  

8
Interestingly, we can explicitly decompose the generalization of \((D, G)\) into different kinds of error. Such a decomposition helps us to clearly see different factors that contribute to the generalization of GANs: a small \(\epsilon_g\) means the strong family of generators, a small \(\epsilon_d\) means the strong family of discriminators, while \(\epsilon_o\) represents the strength of the chosen optimizer. Such a decomposition suggests different ways to improve generalization bounds for GANs.

From the decomposition (2), Theorems 5 and 6, we arrive at the following bounds for both players in the extreme cases where the training set for both players is finite.

**Theorem 10.** Assume the assumptions in Theorem 3. For any \(\alpha \leq n/(2 + n), \alpha_x \leq n_x/(2 + n_x)\), with probability at least \(1 - 2 \exp(-0.5m^\alpha) - 2 \exp(-0.5m^{\alpha_x})\),

\[
|d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, P_g)| \leq O(m^{-\alpha/n} + m^{-\alpha_x/n_x} + \epsilon_o)
\]

We have analyzed the generalization in GANs for various scenarios, which goes beyond existing works to understand GANs, and connects generalization with optimization aspects. Nonetheless, Theorems 8, 9, and 10 still depend on the notion of distance \(d_D(\cdot, \cdot)\) which relies on the best discriminator when measuring proximity between two distributions. The distance between two distributions may be small even when the two are far away (Arora et al., 2017), leading to trivial bounds. As a result, the use of discriminator-based distances may not help us to fully understand generalization of various GANs. This is a major limitation of existing works (Arora et al., 2017; Zhang et al., 2018; Jiang et al., 2019; Husain et al., 2019; Zhu et al., 2020) and those in Theorems 5 and 10.

**Loss-based bounds:** Next, we remove the limitation of distance-based bounds by directly analyzing the GAN loss. Denote \((D_o, G_o)\) be the approximate solution returned by an optimizer after solving \(\min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G)\). It is easy to see that

\[
|V(\hat{P}_d, \hat{P}_z, D_o, G_o) - \min_G \max_D V(P_d, P_z, D, G)|
\leq |V(\hat{P}_d, \hat{P}_z, D_o, G_o) - \min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G)|
+ |\min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G) - \min_G \max_D V(P_d, P_z, D, G)|
+ |\min_G \max_D V(\hat{P}_d, P_z, D, G) - \min_G \max_D V(P_d, P_z, D, G)|
\]

(3)

The optimization error of an optimizer is quantified by \(|V(\hat{P}_d, \hat{P}_z, D_o, G_o) - \min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G)|\). Such a quantity is more realistic than the one used in Theorems 8, 9, and 10. By definition from Corollary 7, \(|\min_G d_D(\hat{P}_d, P_d) - \min_G d_D(P_d, P_g)| = c_1|\min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G) - \min_G \max_D V(P_d, P_z, D, G)|\) reveals the generator error, while \(|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| = c_1|\min_G \max_D V(\hat{P}_d, P_z, D, G) - \min_G \max_D V(P_d, P_z, D, G)|\) reveals the discriminator error. Theorems 5 and 6 imply the followings.

**Theorem 11.** Given the assumptions 4, 7, and 8, for any \(\alpha \leq n/(2 + n), \alpha_x \leq n_x/(2 + n_x)\),

(Generator error) with probability at least \(1 - 2 \exp(-0.5m^\alpha)\), we have

\[
|\min_G \max_D V(\hat{P}_d, P_z, D, G) - \min_G \max_D V(\hat{P}_d, P_z, D, G)| \leq (LB_z + 2C)m^{-\alpha/n}
\]

(Discriminator error) with probability at least \(1 - 2 \exp(-0.5m^{\alpha_x})\), we have

\[
|\min_G \max_D V(\hat{P}_d, P_z, D, G) - \min_G \max_D V(P_d, P_z, D, G)| \leq (L_y L_d b_x + 2C)m^{-\alpha_x/n_x}
\]

Combining the decomposition (3) with Theorem 11 will lead to the following result.

**Theorem 12.** Assume the assumptions 1, 2, 3 and \(|V(\hat{P}_d, \hat{P}_z, D_o, G_o) - \min_G \max_D V(\hat{P}_d, \hat{P}_z, D, G)| \leq \epsilon_o\). For any \(\alpha \leq n/(2 + n), \alpha_x \leq n_x/(2 + n_x)\), with probability at least \(1 - 2 \exp(-0.5m^\alpha) - 2 \exp(-0.5m^{\alpha_x})\),

\[
|V(\hat{P}_d, \hat{P}_z, D_o, G_o) - \min_G \max_D V(P_d, P_z, D, G)| \leq (LB_z + 2C)m^{-\alpha/n} + (L_y L_d b_x + 2C)m^{-\alpha_x/n_x} + \epsilon_o
\]
We have provided a comprehensive picture about the generalization of GANs, relying on very mild conditions of Lipschitz continuity. One can further tighten our bounds by exploiting more properties of the generator (discriminator) family to reduce the generator (discriminator) error. Optimization error is another way to improve our bounds.

4.4 Discussions and Sample-efficient bounds for Autoencoders

Various works (Guo et al., 2019; Jenni and Favaro, 2019; Qi, 2020; Arjovsky et al., 2017; Gulrajani et al., 2017; Roth et al., 2017; Miyato et al., 2018; Zhou et al., 2019; Thanh-Tung et al., 2019; Jiang et al., 2019; Tanielian et al., 2020; Xu et al., 2020) try to ensure Lipschitz continuity of the discriminator or generator or both. The most popular techniques are gradient penalty (Gulrajani et al., 2017) and spectral normalization (Miyato et al., 2018). Those two techniques are really useful for different losses (Fedus et al., 2018) and high-capacity architectures (Kurach et al., 2019). From a large-scale evaluation, Kurach et al. (2019) found that gradient penalty can help the performance of GANs but does not stabilize the training, whereas using spectral normalization on $G$ only is insufficient to ensure stability (Brock et al., 2019). Some recent large-scale generators (Brock et al., 2019; Zhang et al., 2019; Karras et al., 2020b) use gradient penalty or spectral normalization to ensure their successes. Those empirical observations without a theory poses a long mystery of why imposing a Lipschitz constraint can help GANs to perform better. This work provides an answer for it. Indeed, Theorem 12 and Corollary 2 show that a Lipschitz constraint on one player ($D$ or $G$) only can help, but may be not enough. A penalty on the zero-order ($C$) and first-order ($L_ψ, L_d, L_g$) informations of the loss can lead to better generalization. Our experiments with spectral normalization in Appendix F also confirm this.

4.4.1 Lipschitz penalty for reducing sample complexity

The bounds in Theorem 12 and Corollary 2 are quite loose in terms of the dependence on the dimensionality. The main reason is that we did not take any special properties of the hypothesis family into account. We can tighten those bounds and make them more efficient in terms of sample complexity.

Indeed, consider the upper bound of $L_ψ L_h + C \sqrt{A/m}$ in Theorem 3 and Corollaries 1, 2 for any hypothesis $h \in \mathcal{H}$, where $A$ is a constant, $L_ψ$ is the Lipschitz constant of the loss w.r.t. $h$, and $L_h$ is the Lipschitz constant of $h$. The first term depends on Lipschitzness while the later does not. Take a fully-connected feedforward neural network $h$ with $T$ layers and weight matrices $W_1, ..., W_T$, Miyato et al. (2018) show $L_h \leq \prod_{i=1}^{T} || W_i ||_σ$ when using 1-Lipschitz activations (e.g. ReLU), $|| \cdot ||_σ$ denotes spectral norm. If $|| W_i ||_σ < 1$, then $L_h \leq r^T$ which is exponentially small. So the upper bound will be $L_ψ A r^T + C \sqrt{A/m}$, which is significantly better than those in Theorem 3 and Corollaries 1, 2. Note that the condition of $|| W_i ||_σ < 1$ is often satisfied as observed by Miyato et al. (2018) when spectral normalization is used. Our experiments with data augmentation (and noises) in Appendix D and spectral normalization in Appendix F also show that the Lipschitz constant of the loss is often small for models with good performance.

4.4.2 Tightness of the bounds

Note that our bounds in Theorem 3 are not tight in terms of sample complexity and dimensionality. In general, Zhu et al. (2020) show a lower bound of $m^{-3/n}$, which is much lower than our bound of $m^{−1/(n+2)}$. This suggests a large room for future improvement. The previous paragraph discusses one way to tighten our bounds for spectrally normalized networks. It is worth noting that our upper bound $O(m^{−1/(n+2)} + m^{−1/(n_g+2)})$ for GANs surpasses the best bound $O(m^{−1/(1.5n}) + m^{−1/(1.5n_g)})$ in the GAN literature (Husain et al., 2019).

4.4.3 Sample-efficient bounds for Autoencoders

Husain et al. (2019) did a great job at connecting GANs and Autoencoder models. They showed that the generator objective in $f$-GAN (Nowozin et al., 2016) is upper bounded by the objective of Wasserstein
Autoencoders (WAE) (Tolstikhin et al., 2018). Under some suitable conditions, the two objectives equal. They further showed the generalization bound of $O(m^{-1/s_d} + m^{-1/s_g})$, where $s_d > d^*(P_d)$ (the 1-upper Wasserstein dimension of $P_d$) and $s_g > d^*(P_g)$. We show in Appendix C.1 that $s_d > 1.5n_x$, $s_g > 1.5n$ even for a simple distribution, where $n_x$ is the dimensionality of real data, and $n$ is the dimensionality of latent codes. Therefore their bound becomes $O(m^{-1/(n+2)} + m^{-1/(n+2)})$. As a consequence, our results provide tighter generalization bounds for both GANs and Autoencoder models.

5 Why does data augmentation impose a Lipschitz constraint?

In this section, we study a perturbed version of GANs and point out why data augmentation (DA) penalizes the zero-order and first-order informations of the loss, and hence improves generalization for GANs.

Consider the following formulation:

$$\min_G \max_D \mathbb{E}_{x \sim p_d}[\log D(x + \epsilon)] + \mathbb{E}_{z \sim p_z}[\log(1 - D(G(z) + \epsilon))]$$

where $\epsilon = \sigma u$ and $u$ follows a distribution with mean 0 and covariance matrix $I$, $\sigma$ is a non-negative constant. Note that when $u$ is the Gaussian noise, the formulation (4) turns out to be the noisy version of GAN (Arjovsky and Bottou [2017]).

Noise penalizes the zero-order information: One difference between (1) and (4) is the input for the discriminator. The loss $v(x, G(z)) = \log D(x) + \log(1 - D(G(z)))$ for each sample $(x, z)$ in (1) is replaced by $\mathbb{E}_u [v(x + \epsilon, G(z) + \epsilon)]$ in (4). Observe that $|\mathbb{E}_{\epsilon \in \mathcal{B}} [v(x + \epsilon, G(z) + \epsilon)]| \leq \max_{\epsilon \in \mathcal{B}} |v(x + \epsilon, G(z) + \epsilon)|$ for any compact set $\mathcal{B} \subseteq \mathbb{R}^x$, meaning the expectation likely reduces the zero-order information. As a result, adding noises is a way to penalize the zero-order information of the loss.

Noise penalizes the Jacobian norms: Adding noises to the discriminator inputs corresponds to making a convolution to real and fake distributions (Roth et al., 2017; Arjovsky and Bottou, 2017). Let $p_{d*}(x) = \mathbb{E}_u[p_d(x + \epsilon)], p_{g*}(x) = \mathbb{E}_z[p_g(x + \epsilon)]$ be the density functions of the convoluted distributions $P_{d*}, P_{g*}$, respectively. We rewrite $V(D, G) = \mathbb{E}_{x \sim P_d}[\mathbb{E}_u \log D(x + \epsilon)] + \mathbb{E}_{z \sim P_z}[\mathbb{E}_u \log(1 - D(G(z) + \epsilon))] = \mathbb{E}_{x \sim P_{d*}} \log D(x) + \mathbb{E}_{z \sim P_{g*}} \log(1 - D(x))$. Given a fixed $G$, the optimal discriminator is $D^*(x) = \frac{p_{d*}(x)}{p_{d*}(x) + p_{g*}(x)}$ according to Arjovsky and Bottou (2017). Training $G$ is to minimize $V_c(D^*, G)$. By using the same argument as Goodfellow et al. (2014), one can see that training $G$ is equivalent to minimizing the Jensen-Shannon divergence $d_{JS}(P_{d*}, P_{g*})$. Appendix D shows

$$\sqrt{d_{JS}(P_{d*}, P_g)} \leq \sqrt{d_{JS}(P_{d*}, P_{g*})} + \sqrt{o(\sigma)}$$

where $o(\sigma)$ satisfies $\lim_{\sigma \to 0} \frac{o(\sigma)}{\sigma} = 0$. They suggest that for a fixed $\sigma$, minimizing $d_{JS}(P_{d*}, P_{g*})$ implies minimizing both $d_{JS}(P_d, P_{g*})$ and $d_{JS}(P_{d*}, P_g)$.

Lemma 13. Let $J_x(f)$ be the Jacobian of $f(x)$ w.r.t its input $x$. Assume the density functions $p_d$ and $p_g$ are differentiable everywhere in $\mathbb{R}^x$. For any $x \in \mathbb{R}^x$, we have:

1. $[p_{d*}(x) - p_g(x)]^2 + o(\sigma^2) = [p_d(x) - p_g(x) + o(\sigma)]^2 + \sigma^2 \mathbb{E}_u [u^T J_x^T(p_d)J_x(p_d)u]$.

2. $[p_d(x) - p_{g*}(x)]^2 + o(\sigma^2) = [p_d(x) - p_g(x) - o(\sigma)]^2 + \sigma^2 \mathbb{E}_u [u^T J_x^T(p_g)J_x(p_g)u]$.

3. $[p_{d*}(x) - p_{g*}(x)]^2 + o(\sigma^2) = [p_d(x) - p_g(x) + o(\sigma)]^2 + \sigma^2 \mathbb{E}_u [u^T J_x^T(p_d - p_g)J_x(p_d - p_g)u]$.

Lemma 14. If $u \sim \mathcal{N}(0, I)$ then $\mathbb{E}_u [u^T A^T Au] = \text{trace}(A^T A) = \|A\|_F^2$ for any given matrix $A$, where $\|\cdot\|_F$ denotes the Frobenious norm.
The proof of Lemma 13 appears in Appendix D. While Lemma 14 comes from a well-known result.

When training $G$, we are trying to minimize the expected norms of the Jacobians of the densities induced by $D$ and $G$. Indeed, training $G$ will minimize $d_{JS}(P_{dres}, P_{gres})$, and thus also minimize $d_{JS}(P_{d}, P_{g})$ and $d_{JS}(P_{dres}, P_{g})$ according to (5) and (6). Because $\sqrt{d_{JS}}$ is a proper distance, minimizing $d_{JS}(P_{d}, P_{g})$ leads to minimizing $\mathbb{E}_{x \sim P_{d}}[(p_{d}(x) - p_{g}(x))^2]$. Combining this observation with Lemma 13, we find that training $G$ requires both $\mathbb{E}_{x \sim P_{d}}(p_{d}(x) - p_{g}(x) + o(\sigma))^2$ and $\mathbb{E}_{x \sim P_{d}}u^TJ_{x}(p_{g})J_{x}(p_{g})u$ to be small. As a result, $\mathbb{E}_{x \sim P_{d}}[|J_{x}(p_{g})|^2]_{\mathbb{F}}$ should be small due to Lemma 14. A larger $\sigma$ encourages a smaller Jacobian norm, meaning the flatter learnt distribution. A small $\sigma$ enables us to learn complex distributions. The optimal $D^*(x) = \frac{p_{dres}(x)}{p_{dres}(x) + p_{gres}(x)}$ suggests that a penalty on $J_{x}(p_{g})$ will lead to a penalty on the Jacobian of $D$.

It is also useful to observe that adding noises to real data $(x)$ only will require $d_{JS}(P_{dres}, P_{g})$ to be small, whereas adding noises to fake data $(G(z))$ only will require $d_{JS}(P_{d}, P_{gres})$ to be small. Lemma 13 suggests that adding noises to real data only does not make any penalty on $p_{g}$. Further, if noises are used for both real and fake data, we are making penalties on both $J_{x}(p_{g})$ and $J_{x}(p_{d} - p_{g})$. Note that a small $J_{x}(p_{d} - p_{g})$ implies $J_{x}(p_{d}) \approx J_{x}(p_{g})$. As a consequence, training GAN by the loss (4) will require both the zero-order $(p_{g})$ and first-order $(J_{x}(p_{g}))$ informations of the fake distribution to match those of the real distribution. This is surprising. The (implicit) appearance of the first-order information of $p_{d}$ can help the GAN training to converge faster, due to the ability to use more information from $p_{d}$. On the other hand, the use of noise in (4) penalizes the zero- and first-order informations of the loss, and hence can improve the generalization of $D$ and $G$, following Theorem 12.

**Connection to data augmentation:** Note that each input for $D$ in (4) is perturbed by an $\epsilon$. When $\epsilon$ has a small norm, each $x' = x + \epsilon$ is a local neighbor of $x$. Noise is a common way to make perturbation and can lead to stability for GANs (Arjovsky and Bottou, 2017). Another way to make perturbation is data augmentation, including translation, cutout, rotation. The main idea is to make another version $x'$ of an original image $x$ such that $x'$ should preserve the semantic of $x$. By this way, $x'$ belongs to the neighborhood of $x$ in some senses, and can be represented as $x' = x + \epsilon$ for some $\epsilon$. Those observations suggest that when training $D$ and $G$ from a set of original and augmented images (Zhao et al., 2020a,b), we are working with an empirical version of (4). Note that our proofs for inequalities (4) and Lemma 13 apply to a larger contexts than Gaussian noise, meaning that they can apply to different kinds of data augmentation.

Some recent works (Karras et al., 2020a; Tran et al., 2021) show that data augmentation (DA) for real data only will be problematic, meanwhile using DA for both real and fake data can significantly improve GANs (Zhao et al., 2020a,b). Our Lemma 13 agrees with those observations: DA for fake data only poses a penalty on Jacobian of $p_{g}$, while DA for real data only does no penalty on $p_{g}$. Different from prior works, Lemma 13 shows that DA for both real and fake data poses a penalty on $J_{x}(p_{g})$, and requires $J_{x}(p_{g}) \approx J_{x}(p_{d})$. In other words, DA requires the zero- and first-order informations of $p_{g}$ to match those of $p_{d}$, while also penalizes the zero- and first-order informations of the loss for better generalization of $D$ and $G$. This is a novel finding.

6 Conclusion

We have discussed the generalization of GANs under more general settings than existing studies, and pointed out a simple way to improve generalization for the players in GANs. Our work provides a theoretically grounded explanation for the highly successful applications of Lipschitz constraints e.g., spectral normalization or gradient penalty. We further suggest data augmentation to be an effective alternative which is extremely cheap in practice, and pointing out some intriguing properties of data augmentation.
References

Martin Arjovsky and Leon Bottou. Towards principled methods for training generative adversarial networks. In International Conference on Learning Representations, 2017.

Martin Arjovsky, Soumith Chintala, and Leon Bottou. Wasserstein generative adversarial networks. In Proceedings of the 34th International Conference on Machine Learning, 2017.

Sanjeev Arora, Rong Ge, Yingyu Liang, Tengyu Ma, and Yi Zhang. Generalization and equilibrium in generative adversarial nets (gans). In International Conference on Machine Learning, pages 224–232, 2017.

Haim Avron and Sivan Toledo. Randomized algorithms for estimating the trace of an implicit symmetric positive semi-definite matrix. Journal of the ACM (JACM), 58(2):1–34, 2011.

Andrew Brock, Jeff Donahue, and Karen Simonyan. Large scale gan training for high fidelity natural image synthesis. In International Conference on Learning Representations, 2019.

Casey Chu, Kentaro Minami, and Kenji Fukumizu. Smoothness and stability in gans. In International Conference on Learning Representations, 2020.

William Fedus, Mihaela Rosca, Balaji Lakshminarayanan, Andrew M Dai, Shakir Mohamed, and Ian Goodfellow. Many paths to equilibrium: Gans do not need to decrease a divergence at every step. In International Conference on Learning Representations, 2018.

Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In Advances in Neural Information Processing Systems, pages 2672–2680, 2014.

Ishaan Gulrajani, Faruk Ahmed, Martin Arjovsky, Vincent Dumoulin, and Aaron C Courville. Improved training of wasserstein gans. In Advances in Neural Information Processing Systems, pages 5767–5777, 2017.

Tianyu Guo, Chang Xu, Boxin Shi, Chao Xu, and Dacheng Tao. Smooth deep image generator from noises. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 3731–3738, 2019.

Yongjun Hong, Uiwon Hwang, Jaeyoon Yoo, and Sungroh Yoon. How generative adversarial networks and their variants work: An overview. ACM Computing Surveys (CSUR), 52(1):1–43, 2019.

Hisham Husain, Richard Nock, and Robert C Williamson. A primal-dual link between gans and autoencoders. In Advances in Neural Information Processing Systems, volume 32, pages 415–424, 2019.

Michael F Hutchinson. A stochastic estimator of the trace of the influence matrix for laplacian smoothing splines. Communications in Statistics - Simulation and Computation, 18(3):1059–1076, 1989.

Simon Jenni and Paolo Favaro. On stabilizing generative adversarial training with noise. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 12145–12153, 2019.

Haoming Jiang, Zhehui Chen, Minshuo Chen, Feng Liu, Dingding Wang, and Tuo Zhao. On computation and generalization of generative adversarial networks under spectrum control. In International Conference on Learning Representations, 2019.

Tero Karras, Miika Aittala, Janne Hellsten, Samuli Laine, Jaakko Lehtinen, and Timo Aila. Training generative adversarial networks with limited data. In Advances in Neural Information Processing Systems, 2020a.
Tero Karras, Samuli Laine, Miika Aittala, Janne Hellsten, Jaakko Lehtinen, and Timo Aila. Analyzing and improving the image quality of stylegan. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 8110–8119, 2020b.

Karol Kurach, Mario Lučić, Xiaohua Zhai, Marcin Michalski, and Sylvain Gelly. A large-scale study on regularization and normalization in gans. In International Conference on Machine Learning, pages 3581–3590, 2019.

Mario Lucic, Karol Kurach, Marcin Michalski, Sylvain Gelly, and Olivier Bousquet. Are gans created equal? a large-scale study. In Advances in Neural Information Processing Systems, pages 700–709, 2018.

Xudong Mao, Qing Li, Haoran Xie, Raymond YK Lau, Zhen Wang, and Stephen Paul Smolley. Least squares generative adversarial networks. In Proceedings of the IEEE International Conference on Computer Vision, pages 2794–2802, 2017.

Xudong Mao, Qing Li, Haoran Xie, Raymond YK Lau, Zhen Wang, and Stephen Paul Smolley. On the effectiveness of least squares generative adversarial networks. IEEE Transactions on Pattern Analysis and Machine Intelligence, 41(12):2947–2960, 2019.

Lars Mescheder, Sebastian Nowozin, and Andreas Geiger. The numerics of gans. In Advances in Neural Information Processing Systems, pages 1825–1835, 2017.

Lars Mescheder, Andreas Geiger, and Sebastian Nowozin. Which training methods for gans do actually converge? In International Conference on Machine Learning, pages 3481–3490, 2018.

Takeru Miyato, Toshiki Kataoka, Masanori Koyama, and Yuichi Yoshida. Spectral normalization for generative adversarial networks. In International Conference on Learning Representations, 2018.

Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. Foundations of Machine Learning. MIT Press, 2018.

Vaishnavh Nagarajan and J Zico Kolter. Gradient descent gan optimization is locally stable. In Advances in Neural Information Processing Systems, pages 5585–5595, 2017.

Weili Nie and Ankit Patel. Towards a better understanding and regularization of gan training dynamics. In Conference on Uncertainty in Artificial Intelligence (UAI), 2019.

Sebastian Nowozin, Botond Cseke, and Ryota Tomioka. f-gan: Training generative neural samplers using variational divergence minimization. In Advances in Neural Information Processing Systems, pages 271–279, 2016.

Guo-Jun Qi. Loss-sensitive generative adversarial networks on lipschitz densities. International Journal of Computer Vision, 128(5):1118–1140, 2020.

Kevin Roth, Aurelien Lucchi, Sebastian Nowozin, and Thomas Hofmann. Stabilizing training of generative adversarial networks through regularization. In Advances in Neural Information Processing Systems, pages 2018–2028, 2017.

Maziar Sanjabi, Jimmy Ba, Meisam Razaviyayn, and Jason D Lee. On the convergence and robustness of training gans with regularized optimal transport. In Advances in Neural Information Processing Systems, pages 7091–7101, 2018.

Connor Shorten and Taghi M Khoshgoftaar. A survey on image data augmentation for deep learning. Journal of Big Data, 6(1):60, 2019.

Ugo Tanielian, Thibaut Issenhuth, Elvis Dohmatob, and Jeremie Mary. Learning disconnected manifolds: a no gan’s land. In International Conference on Machine Learning, 2020.
Hoang Thanh-Tung, Truyen Tran, and Svetha Venkatesh. Improving generalization and stability of generative adversarial networks. In *International Conference on Learning Representations*, 2019.

Ilya Tolstikhin, Olivier Bousquet, Sylvain Gelly, and Bernhard Schoelkopf. Wasserstein auto-encoders. In *International Conference on Learning Representations*, 2018.

Ngoc-Trung Tran, Viet-Hung Tran, Ngoc-Bao Nguyen, Trung-Kien Nguyen, and Ngai-Man Cheung. On data augmentation for gan training. *IEEE Transactions on Image Processing*, 30:1882–1897, 2021.

Bingzhe Wu, Shiwan Zhao, Chaochao Chen, Haoyang Xu, Li Wang, Xiaolu Zhang, Guangyu Sun, and Jun Zhou. Generalization in generative adversarial networks: A novel perspective from privacy protection. In *Advances in Neural Information Processing Systems*, pages 307–317, 2019.

Huan Xu and Shie Mannor. Robustness and generalization. *Machine learning*, 86(3):391–423, 2012.

Huan Xu, Constantine Caramanis, and Shie Mannor. Robust regression and lasso. *IEEE Transactions on Information Theory*, 56(7):3561–3574, 2010.

Kun Xu, Chongxuan Li, Huanshu Wei, Jun Zhu, and Bo Zhang. Understanding and stabilizing gans’ training dynamics with control theory. In *Proceedings of the 37th International Conference on Machine Learning*, 2020.

Han Zhang, Ian Goodfellow, Dimitris Metaxas, and Augustus Odena. Self-attention generative adversarial networks. In *International Conference on Machine Learning*, pages 7354–7363, 2019.

Han Zhang, Zizhao Zhang, Augustus Odena, and Honglak Lee. Consistency regularization for generative adversarial networks. In *International Conference on Learning Representations*, 2020.

Pengchuan Zhang, Qiang Liu, Dengyong Zhou, Tao Xu, and Xiaodong He. On the discrimination-generalization tradeoff in gans. In *International Conference on Learning Representations*, 2018.

Junbo Zhao, Michael Mathieu, and Yann LeCun. Energy-based generative adversarial networks. In *International Conference on Learning Representations*, 2017.

Shengyu Zhao, Zhijian Liu, Ji Lin, Jun-Yan Zhu, and Song Han. Differentiable augmentation for data-efficient gan training. In *Advances in Neural Information Processing Systems*, 2020a.

Zhengli Zhao, Zizhao Zhang, Ting Chen, Sameer Singh, and Han Zhang. Image augmentations for gan training. *arXiv preprint arXiv:2006.02595*, 2020b.

Zhiming Zhou, Jiadong Liang, Yuxuan Song, Lantao Yu, Hongwei Wang, Weinan Zhang, Yong Yu, and Zhihua Zhang. Lipschitz generative adversarial nets. In *International Conference on Machine Learning*, pages 7584–7593, 2019.

Banghua Zhu, Jiantao Jiao, and David Tse. Deconstructing generative adversarial networks. *IEEE Transactions on Information Theory*, 66(11):7155–7179, 2020. doi: 10.1109/TIT.2020.2983698.
A  Lipschitz continuity ⇒ Generalization

Proof of Lemma 2: It is easy to see that there exist \( K = [(B/\lambda)^n] \) disjoint \( n \)-dimensional cubes, each with edge length of \( \lambda \), satisfying that their union covers \( Z \) completely since \( Z \) is compact. Let \( C_k \) be one of these cubes, indexed by \( k \), and \( Z_k = Z \cap C_k \). We can write \( Z = \bigcup_{k=1}^{K} Z_k \).

Consider any \( s, z \in Z \). If both \( s \) and \( z \) belong to the same \( Z_k \) for some \( k \), we have \( |f(A; s) - f(A; z)| \leq L||s - z||_\infty \leq L\lambda \) due to the Lipschitz continuity of \( f \), completing the proof. □

Proof of Theorem 3: The first bound comes from combining Theorem 1 and Lemma 2. The second statement is an application of the first one by taking \( \lambda = Bm^{-\alpha/n} \) and \( \delta = 2 \exp(-0.5m^{-\alpha}) \). Indeed,

\[
|F(A(S)) - F_S(A(S))| \leq L\lambda + \frac{C}{\sqrt{m}} \sqrt{|B^n\lambda^{-n}| \log 4 - 2 \log \delta} \\
\leq LBm^{-\frac{\alpha}{n}} + \frac{C}{\sqrt{m}} |m^n| \log 4 - \log 4 + m^\alpha \\
\leq LBm^{-\frac{\alpha}{n}} + \frac{C}{\sqrt{m}} m^\alpha \log 4 + m^\alpha \\
\leq LBm^{-\frac{\alpha}{n}} + Cm^{-\frac{\alpha}{n}} \sqrt{(1 + \log 4)m^{-1+\frac{n+2}{n}}} \\
\leq (LB + 2C)m^{-\frac{\alpha}{n}}
\]

The last inequality holds because \(-1 + \frac{n+2}{m} \leq 0\) and hence \( m^{-1+\frac{n+2}{n}} \leq 1 \), completing the proof. □

B  Proofs of the main theorems for GANs

Proof of Corollary 3: Observe that

\[
|d_D(P_d, P_g) - d_D(P_d, \hat{P}_g)| = c_1 |\max_{D \in \mathcal{D}} V(P_d, P_z, D, G) - \max_{D \in \mathcal{D}} V(P_d, \hat{P}_z, D, G)| \\
\leq c_1 |\max_{D \in \mathcal{D}} V(P_d, P_z, D, G) - V(P_d, \hat{P}_z, D, G)| \\
\leq c_1 |\max_{D \in \mathcal{D}} |E_{z \sim P_z} \psi_2(1 - D(G(z))) - E_{z \sim \hat{P}_z} \psi_2(1 - D(G(z)))| |
\]

We have used Lemma 15 to derive (8) from (7). Since \( \hat{P}_z \) is an empirical version of \( P_z \), for any particular \( D \), applying Theorem 3 will provide the generalization bounds for \( |E_{z \sim P_z} \psi_2(1 - D(G(z))) - E_{z \sim \hat{P}_z} \psi_2(1 - D(G(z)))| \) and hence \( |d_D(P_d, P_g) - d_D(P_d, \hat{P}_g)| \). The same arguments can be done for \( |d_D(P_d, P_g) - \min_G d_D(P_d, P_g)| \), completing the proof. □

Proof of Theorem 4: We observe that

\[
|d_D(P_d, P_g) - \min_G d_D(P_d, P_g)| \\
\leq |d_D(P_d, \hat{P}_g) - d_D(P_d, \hat{P}_g)| + |d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(\hat{P}_d, P_g)| + |\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)|
\]

Note that

\[
|d_D(P_d, \hat{P}_g) - d_D(\hat{P}_d, \hat{P}_g)| = c_1 |\max_{D \in \mathcal{D}} V(P_d, \hat{P}_z, D, G) - \max_{D \in \mathcal{D}} V(\hat{P}_d, \hat{P}_z, D, G)| \\
\leq c_1 |\max_{D \in \mathcal{D}} V(P_d, \hat{P}_z, D, G) - V(\hat{P}_d, \hat{P}_z, D, G)| \\
\leq c_1 |\max_{D \in \mathcal{D}} |E_{x \sim P_d} \psi_1(D(x)) - E_{x \sim \hat{P}_d} \psi_1(D(x))| |
\]

We can write

\[
\leq c_1 L\psi L_d \lambda + \frac{c_1 C}{\sqrt{m}} \sqrt{|B^n\lambda^{-n}r| \log 4 - 2 \log \delta}
\]
where we have used Lemma \ref{lem:15} to derive (10) from (9), and Theorem \ref{thm:3} to obtain (12) with probability at least $1 - \delta$, for any constants $\delta \in (0, 1]$ and $\lambda \in (0, B_x]$.

Next we consider
\[
|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| = c_1 |\max_{D \in \mathcal{D}} \min_G V(\hat{P}_d, P_z, D, G) - \max_{D \in \mathcal{D}} \min_G V(P_d, P_z, D, G)|\ (13)
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - \min_G V(P_d, P_z, D, G)|
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\] (14)\(\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\] (15)
where we have used Lemma \ref{lem:15} to derive (14) from (13) and (15) from (14). Note that
\[
|V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)| = |\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))|
\leq L \psi(L_d + \frac{C}{\sqrt{m}} \sqrt{|B_x^{n^w} \lambda^{-n^w}|} \log 4 - 2 \log \delta).
\]
As a result, we obtain
\[
|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| \leq c_1 L \psi(L_d + \frac{c_1 C}{\sqrt{m}} \sqrt{|B_x^{n^w} \lambda^{-n^w}|} \log 4 - 2 \log \delta).
\]
Combining (12), (18), and the assumption of $|d_D(\hat{P}_d, P_g) - \min_G d_D(\hat{P}_d, P_g)| \leq \epsilon_{og}$, we conclude that
\[
|d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| \leq 2c_1 L \psi(L_d + 2c_1 C \sqrt{|B_x^{n^w} \lambda^{-n^w}|} \log 4 - 2 \log \delta)/m + \epsilon_{og}.
\]

The second statement of this theorem is an application of the first one by taking $\lambda = B_x m^{-a_x/n}$ and $\delta = 2 \exp(-0.5m^a)$, similar with the proof of Theorem \ref{thm:3} completing the proof. \(\square\)

**Proof of Theorem \ref{thm:5}**  We observe that
\[
|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| = c_1 |\max_{D \in \mathcal{D}} \min_G V(\hat{P}_d, P_z, D, G) - \max_{D \in \mathcal{D}} \min_G V(P_d, P_z, D, G)|
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - \min_G V(P_d, P_z, D, G)|
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\] (19)\(\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\) (20)\(\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\) (21)
where we have used the results of Lemma \ref{lem:14} Note that
\[
|V(\hat{P}_d, P_z, D, G) - V(\hat{P}_d, P_z, D, G)| = |\mathbb{E}_{z \sim P_z} \psi_2(1 - D(G(z))) - \mathbb{E}_{z \sim P_z} \psi_2(1 - D(G(z)))|
\leq L \lambda + C \sqrt{|B_x^{n^w} \lambda^{-n^w}|} \log 4 - 2 \log \delta)/m,
\]
by applying Theorem \ref{thm:3} As a result, we obtain
\[
|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| \leq c_1 L \lambda + c_1 C \sqrt{|B_x^{n^w} \lambda^{-n^w}|} \log 4 - 2 \log \delta)/m
\]
with probability at least $1 - \delta$, for any constants $\delta \in (0, 1]$ and $\lambda \in (0, B_x]$. The same bound can be similarly derived for $|\min_G d_D(P_d, P_g) - \min_G d_D(P_d, P_g)|$.

The second statement is an application of the first one by taking $\lambda = B_x m^{-a/n}$ and $\delta = 2 \exp(-0.5m^a)$, similar with the proof of Theorem \ref{thm:3} completing the proof. \(\square\)

**Proof of Theorem \ref{thm:6}**  We observe that
\[
|\min_G d_D(\hat{P}_d, P_g) - \min_G d_D(P_d, P_g)| = c_1 |\max_{D \in \mathcal{D}} \min_G V(\hat{P}_d, P_z, D, G) - \max_{D \in \mathcal{D}} \min_G V(P_d, P_z, D, G)|
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - \min_G V(P_d, P_z, D, G)|
\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\] (24)\(\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\) (25)\(\leq c_1 \max_{D \in \mathcal{D}} |\min_G V(\hat{P}_d, P_z, D, G) - V(P_d, P_z, D, G)|
\) (26)\(\leq c_1 \max_{D \in \mathcal{D}} |\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))|
\]
(27)\[
17
where we have used Lemma 15 to derive (29) from (28), and (30) from (29).

Similarly we have

\[
|\min_G d_D(\hat{P}_d, \hat{P}_g) - \min_G d_D(P_d, \hat{P}_g)| = c_1 \max \min V(\hat{P}_d, \hat{P}_z, D, G) - \max \min V(P_d, \hat{P}_z, D, G)\]

\[
\leq c_1 \max \min V(\hat{P}_d, \hat{P}_z, D, G) - \min V(P_d, \hat{P}_d, D, G)\]

\[
\leq c_1 \max \max V(\hat{P}_d, \hat{P}_z, D, G) - V(P_d, \hat{P}_z, D, G)\]

\[
\leq c_1 \max \max \left[\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))\right]\]

where we have used Lemma 15 to derive (29) from (28), and (30) from (29).

According to Theorem 3, \(\mathbb{E}_{x \sim P_d} \psi_1(D(x)) - \mathbb{E}_{x \sim P_d} \psi_1(D(x))\) is bounded by \(L_{\phi} L_d \lambda + C \sqrt{\left|B_{\phi}^n \lambda^{-n} \right| \log 4 - 2 \log \delta / m}\) with probability at least \(1 - \delta\), for any constants \(\delta \in (0, 1]\) and \(\lambda \in (0, B_{\phi}]\). Combining this fact with (27) and (31) will complete the proof. □

**Lemma 15.** Assume that \(h_1\) and \(h_2\) are continuous functions defined on a compact set \(Z_x\). Then

\[
\left|\min_{x \in Z_x} h_1(x) - \max_{x \in Z_x} h_2(x)\right| \leq \max_{x \in Z_x} |h_1(x) - h_2(x)|
\]

**Proof:** Denote \(x_1^* = \arg\max_{x \in Z_x} h_1(x), x_2^* = \arg\max_{x \in Z_x} h_2(x)\). It is easy to see that

\[
|h_1(x_2^*) - h_2(x_2^*)| = |h_1(x_1^*) - h_2(x_2^*)| \leq |h_1(x_1^*) - h_2(x_1^*)|
\]

Therefore

\[
\left|\min_{x \in Z_x} h_1(x) - \max_{x \in Z_x} h_2(x)\right| \leq \max_{x \in Z_x} |h_1(x) - h_2(x)|
\]

We can rewrite \(\min_{x \in Z_x} h_1(x) - \max_{x \in Z_x} h_2(x)\) as \(-\max_{x \in Z_x} (-h_1(x)) + \max_{x \in Z_x} (-h_2(x))\), which is bounded by \(\max_{x \in Z_x} |h_1(x) - h_2(x)|\), completing the proof. □

**C Sample-efficient bounds for Autoencoders**

Husain et al. (2019) did a great job at connecting GANs and Autoencoder models. They showed that the generator objective in f-GAN Nowozin et al. (2016) is upper bounded by the objective of Wasserstein Autoencoders (WAE) Tolstikhin et al. (2018). Under some suitable conditions, the two objectives equal. They further showed the generalization bound of

\[
O(m^{-1/s_d} + m^{-1/s_g}),
\]

where \(s_d > d^*(P_d)\) (the 1-upper Wasserstein dimension of \(P_d\)) and \(s_g > d^*(P_g)\). We show in Appendix C.1 that \(s_d > 1.5n_x, s_g > 1.5m\) even for a simple distribution, where \(n_x\) is the dimensionality of real data, and \(n\) is the dimensionality of latent codes. Therefore their bound becomes \(O(m^{-1/(5n_x)} + m^{-1/5m})\), which is significantly worse than our bounds in Corollary 3. As a consequence, our results provide tighter generalization bounds for both GANs and Autoencoder models.

**C.1 How large is 1-upper Wasserstein dimension?**

This part provides an example of why 1-upper Wasserstein dimension is not small. Before that we need to take the following definitions from Husain et al. (2019).

**Definition 2 (Covering number).** For a set \(S \subseteq \mathbb{R}^n\), we denote \(N_\eta(S)\) be the \(\eta\)-covering number of \(S\), which is the smallest non-negative integer \(m\) such that there exists closed balls \(B_1, B_2, ..., B_m\) of radius \(\eta\) with \(S \subseteq \bigcup_{i=1}^m B_i\).

For any distribution \(P\), the \((\eta, \tau)\)-dimension is \(d_\eta(P, \tau) := \frac{\log N_\eta(P, \tau)}{-\log \eta}\), where \(N_\eta(P, \tau) := \inf \{N_\eta(S) : P(S) \geq 1 - \tau\}\).
We will need the following discussion about local linearity and approximation before presenting the proofs where we have used Taylor’s theorem allows us to write
\[ \sigma \geq \epsilon \text{ linear everywhere. Let } \]

Consider a function \( f \) relating to some claims on data augmentation.

\[ \text{Taylor’s theorem allows us to write} \]

\[ \sigma \geq \epsilon \text{ linear everywhere. Let } \]

First of all, we need to see the region \( S \) such that \( P(S) \geq 1 - \eta^{\frac{s}{4}} \). Since \( P \) is a Gaussian, the Birnbaum-Raymond-Zuckerman inequality tells that \( \Pr(||x||_2^2 \geq n\eta^{-\frac{s}{4}}) \leq \eta^{\frac{s}{4}} \). It implies that \( \Pr(||x||_2^2 \leq n\eta^{-\frac{s}{4}}) \geq 1 - \eta^{\frac{s}{4}} \). In other words, \( S \) is the following ball:

\[ S = \{ x \in \mathbb{R}^n : ||x||_2^2 \leq n\eta^{-\frac{s}{4}} \} \]

As a consequence, we can lower bound the covering number of \( S \) as

\[ N_\eta(S) \geq \left( \frac{\sqrt{n\eta^{-\frac{s}{4}}}}{\eta} \right)^n = \left( n\eta^{-\frac{3s+4}{4}} \right)^{\frac{n}{2}} \]

By definition we have

\[ N_\eta(P,\eta^{\frac{s}{4}}) = \inf \{ N_\eta(S) : P(S) \geq 1 - \eta^{\frac{s}{4}} \} \geq \left( n\eta^{-\frac{3s+4}{4}} \right)^{\frac{n}{2}} \]

Next we observe that

\[ d_\eta(P,\eta^{\frac{s}{4}}) = \frac{1}{-\log \eta} \log N_\eta(P,\eta^{\frac{s}{4}}) \]

\[ \geq \frac{1}{-\log \eta} \left[ \frac{n}{2} \log n + \frac{n}{2} \left( \frac{-3s+4}{s-2} \right) \log \eta \right] \]

\[ \geq \frac{n}{2} \left( \frac{3s-4}{s-2} \right) - \frac{n}{2} \log \eta \log n \]

Therefore

\[ \limsup_{\eta \to 0} d_\eta(P,\eta^{\frac{s}{4}}) \geq \frac{n}{2} \left( \frac{3s-4}{s-2} \right) \]

The definition of \( d^*(P) \) requires \( \limsup_{\eta \to 0} d_\eta(P,\eta^{\frac{s}{4}}) \leq s \) and \( s \in (2, \infty) \). Those requirements imply \( \frac{n}{2} \left( \frac{3s-4}{s-2} \right) \leq s \), and thus \( s \geq \frac{1}{4} \left( 4 + 3n + \sqrt{4 + 3n^2 - 32n} \right) > \frac{3n}{4} \). As a result, \( d^*(P) > 1.5n \).

D Data augmentation poses a Lipschitz constraint

We will need the following discussion about local linearity and approximation before presenting the proofs relating to some claims on data augmentation.

D.1 Local linearity

Consider a function \( f : \mathbb{R}^n \to \mathbb{R} \) which is differentiable everywhere in its domain. \( f \) is also called locally linear everywhere. Let \( \epsilon = \sigma u \), where \( u \) follows a distribution with mean 0 and covariance matrix \( I \), \( \sigma \geq 0 \), \( J_x(f) \) be the Jacobian of \( f \) w.r.t its input \( x \). Considering \( f(x + \epsilon) = f(x + \sigma u) \) as a function of \( \sigma \), Taylor’s theorem allows us to write \( f(x + \sigma u) = f(x) + \sigma J_x(f) u + o(\sigma) \). Therefore,

\[ \mathbb{E}_\epsilon[f(x + \epsilon)] = \mathbb{E}_\epsilon[f(x) + \sigma J_x(f) u + o(\sigma)] \]

\[ = f(x) + o(\sigma) + \sigma \mathbb{E}_u[J_x(f) u] \]

\[ = f(x) + o(\sigma), \]

where we have used \( \mathbb{E}_u[J_x(f) u] = 0 \) due to \( \mathbb{E}_u[u] = 0 \) and the independence of the elements of \( u \). As \( \sigma \to 0 \), we have \( \mathbb{E}_\epsilon[f(x + \epsilon)] \to f(x) \).
D.2 Proofs for inequalities (5, 6)

Consider the Jensen-Shannon divergence $d_{JS}(P_{dsc}, P_{gsc})$. Since $\sqrt{d_{JS}}$ is a proper distance, we have the following triangle inequalities:

$$\sqrt{d_{JS}(P_{dsc}, P_g)} \leq \sqrt{d_{JS}(P_{dsc}, P_{gsc})} + \sqrt{d_{JS}(P_{gsc}, P_g)}$$ \hfill (40)

$$\sqrt{d_{JS}(P_d, P_{gsc})} \leq \sqrt{d_{JS}(P_{dsc}, P_{gsc})} + \sqrt{d_{JS}(P_{dsc}, P_d)}$$ \hfill (41)

Next we will show that $d_{JS}(P_{gsc}, P_g) = o(\sigma)$. The following expression comes from a basic property of Jensen-Shannon divergence:

$$d_{JS}(P_{gsc}, P_g) = H\left(\frac{P_{gsc} + P_g}{2}\right) - \frac{1}{2}H(P_{gsc}) - \frac{1}{2}H(P_g),$$ \hfill (42)

where $H(P)$ denotes the entropy of distribution $P$.

Denote $o(\cdot), o_1(\cdot), o_2(\cdot), o_3(\cdot)$ be some functions of $\sigma$ satisfying $\lim_{\sigma \to 0} \frac{o(\sigma)}{\sigma} = 0$. Appendix D.1 suggests that $p_{gsc}(x) = E_\sigma[p_g(x + \epsilon)] = p_g(x) + o_1(\sigma)$ and $\log(p_g(x) + o_1(\sigma)) = \log(p_g(x)) + o_2(\sigma)$ by using Taylor’s theorem for $\sigma$. Therefore:

$$-\frac{1}{2}H(P_g) = \frac{1}{2} \int p_g(x) \log p_g(x) dx. \hfill \text{(43)}$$

$$-\frac{1}{2}H(P_{gsc}) = \frac{1}{2} \int p_{gsc}(x) \log p_{gsc}(x) dx = \frac{1}{2} \int p_{gsc}(x) \log[p_g(x) + o_1(\sigma)] dx$$

$$= \frac{1}{2} \int p_{gsc}(x)[\log p_g(x) + o_2(\sigma)] dx$$

$$= \frac{1}{2} \int p_{gsc}(x) \log p_g(x) dx + \frac{1}{2} o_2(\sigma). \hfill \text{(44)}$$

$$H\left(\frac{P_{gsc} + P_g}{2}\right) = -\int \frac{p_{gsc}(x) + p_g(x)}{2} \log \left(\frac{p_{gsc}(x) + p_g(x)}{2}\right) dx$$

$$= -\int \frac{p_{gsc}(x) + p_g(x)}{2} \log \left(\frac{2p_g(x) + o_1(\sigma)}{2}\right) dx$$

$$= -\int \frac{p_{gsc}(x) + p_g(x)}{2} \log\left(p_g(x) + \frac{1}{2} o_1(\sigma)\right) dx$$

$$= -\int \frac{p_{gsc}(x) + p_g(x)}{2} \left[\log p_g(x) + \frac{1}{2} o_2(\sigma)\right] dx$$

$$= -\frac{1}{2} \int [p_{gsc}(x) + p_g(x)] \log p_g(x) dx - \frac{o_3(\sigma)}{4} \int [p_{gsc}(x) + p_g(x)] dx$$

$$= -\frac{1}{2} \int [p_{gsc}(x) + p_g(x)] \log p_g(x) dx - \frac{1}{2} o_3(\sigma)$$

$$= -\frac{1}{2} \int p_{gsc}(x) \log p_g(x) dx - \frac{1}{2} \int p_g(x) \log p_g(x) dx - \frac{1}{2} o_3(\sigma). \hfill \text{(45)}$$

From equations (42, 43, 44, 45) we can conclude $d_{JS}(P_{gsc}, P_g) = o(\sigma)$. Similar arguments can be done to prove $d_{JS}(P_{dsc}, P_d) = o(\sigma)$. Combining those with (40) and (41), we arrive at

$$\sqrt{d_{JS}(P_{dsc}, P_g)} \leq \sqrt{d_{JS}(P_{dsc}, P_{gsc})} + \sqrt{o(\sigma)}$$ \hfill (46)

$$\sqrt{d_{JS}(P_d, P_{gsc})} \leq \sqrt{d_{JS}(P_{dsc}, P_{gsc})} + \sqrt{o(\sigma)}. \hfill (47)$$

20
D.3 Proof of Lemma 13

For any $x \in \mathbb{Z}_d$, it is worth remembering that $p_{d+e}(x) = E_e[p_d(x + e)]$. Consider

$$Y = p_d(x + e) - p_g(x) = p_d(x + \sigma u) - p_g(x),$$

which is a function of $u$. Since $u$ follows a distribution with mean 0 and covariance which is the identity matrix, $Y$ is a random variable. Due to $p_d(x + e) = p_d(x) + \sigma J_x(p_d)u + o(\sigma)$ from Appendix D.1, we can express the variance of $Y$ as

$$\text{Var}(Y) = E_u(Y^2) - (E_u(Y))^2 = E_u\left[(p_d(x + e) - p_g(x))^2\right] - (E_u(p_d(x + e) - p_g(x)))^2 \quad (48)$$

$$= E_u\left[p_d(x) - p_g(x) + o(\sigma) + \sigma J_x(p_d)u\right]^2 - [p_{d+e}(x) - p_g(x)]^2 \quad (49)$$

$$= E_u\left[p_d(x) - p_g(x) + o(\sigma)\right]^2 + 2\sigma[p_d(x) - p_g(x) + o(\sigma)]E_u[J_x(p_d)u] + \sigma^2E_u\left[u^T J_x^T(p_d)J_x(p_d)u\right] - [p_{d+e}(x) - p_g(x)]^2 \quad (50)$$

$$= [p_d(x) - p_g(x) + o(\sigma)]^2 + \sigma^2E_u[u^T J_x^T(p_d)J_x(p_d)u] - [p_{d+e}(x) - p_g(x)]^2 \quad (51)$$

Since $p_g(x)$ does not depend on $e = \sigma u$, we have $\text{Var}(Y) = \text{Var}(p_d(x + e)) = \text{Var}(p_d(x + \sigma u))$ which is bounded above by $C\text{Var}(\sigma u) = C\sigma^2$, for some $C \geq 0$. Combining this with (52) will result in

$$[p_{d+e}(x) - p_g(x)]^2 + C\sigma^2 \geq [p_d(x) - p_g(x) + o(\sigma)]^2 + \sigma^2E_u[u^T J_x^T(p_d)J_x(p_d)u] \quad (53)$$

completing the first statement. The second and third statements can be proven similarly.

E Evaluation of data augmentation for GANs

There is a tradeoff in data augmentation. Making augmentation from a larger region around a given image implies a larger $\sigma$. Lemmas 13 and 14 suggest that the Jacobian norms should be smaller, meaning the flatter learnt distributions. Hence, too large region for augmentation may result in underfitting. On the other hand, augmentation in a too small region (a small $\sigma$) allows the Jacobian norms to be large, meaning the learnt distributions can be complex. As $\sigma \to 0$, no regularization is used at all.

This section will provide some empirical evidences about those analyses. We first evaluate the role of $\sigma$ when doing augmentation by simple techniques such as translation. We then evaluate the case of augmentation by adding noises. Two models are used in our evaluations: Saturating GAN (Goodfellow et al., 2014) and LSGAN (Mao et al., 2017).

E.1 Experimental setups

The architectures of $G$ and $D$ are specified in Figure 1 which follow PyTorch-GAN/blob/master/implementations/gan/gan.py. We use this architecture with Spectral normalization (Miyato et al., 2018) for $D$ in all experiments of GAN and LSGAN. Note that, for LSGAN, we remove the last Sigmoid layer in $D$.

We use MNIST dataset which has 60000 images for training and 5000 images for testing. During the testing phase, 5000 new noises are sampled randomly at every epoch minibatch to compute some metrics. For the derivative of $D$ with respect to its input, the input includes 25000 fake images and 25000 real images. Before fetching into $D$, both real and fake images are converted to tensor size (1, 28, 28), rescaled to (0, 1) and normalized with $mean = 0.5$ and $std = 0.5$. The noise input of $G$ has 100 dimensions and is sampled from normal distribution $\mathcal{N}(0, I)$. We use Adam optimizer with $\beta_1 = 0.5, \beta_2 = 0.999, lr = 0.0002, \text{batchsize} = 64$. 
Figure 1: The architectures of $G$ and $D$ with the negative slope of LeakyReLU is 0.2

Figure 2: Some behaviors of GAN (first two subfigures) and LSGAN (last two subfigures) with different $\sigma$ for augmentation. Both $\frac{\partial D}{\partial x}$ and $\frac{\partial V}{\partial z}$ are measured along the training process. $\|\cdot\|_F$ denotes the Frobenious norm.

E.2 The role of $\sigma$ for data augmentation

In this experiment, the input of $D$ which includes real and fake images are augmented using translation. The shifts in horizontal and vertical axis are sampled from discrete uniform distribution within interval $[-s, s]$, where $s = 2$ corresponds to $\sigma = \sqrt{2}$, $s = 4$ corresponds to $\sigma = \sqrt{6.67}$, and $s = 8$ corresponds to $\sigma = \sqrt{24}$.

**Jacobian norms of $D$ and loss $V$:** Figure 2 shows the results. It can be seen from the figure that the higher $\sigma$ provides smaller Frobenius norms of Jacobian of both $D$ and $V$. Such behaviors appear in both GAN and LSGAN, which is consistent with our theory.

**Jacobian norms of $G$:** To see the effect of data augmentation on $G$, we need to fix $D$ when training $G$. Therefore we did the following steps: (i) train both $G$ and $D$ for 100 epochs, (ii) then keeping $D$ fixed, we further train $G$ to measure its Jacobian norm along the training progress. We chose $\sigma \in \{\sqrt{6.67}, \sqrt{14}, \sqrt{24}\}$ and augmented 64, 96, 128 times for each image respectively.

Figure 3 shows the results. We observe that a higher $\sigma$ provides smaller Jacobian norm of $G$. Interestingly, as the norm decreases as training $G$ more, suggesting that $G$ gets simpler.
Figure 3: Some behaviors of GAN and LSGAN with different $\sigma$ for augmentation. $\frac{\partial G}{\partial z}$ is measured along the training process.

Figure 4: Some behaviors of GAN (first two subfigures) and LSGAN (last two subfigures) when augmenting images by adding noises. Both $\frac{\partial D}{\partial x}$ and $\frac{\partial V}{\partial z}$ are measured along the training process.

E.3 Augmentation by adding noises

In this experiment, the input of $D$ which includes real and fake images are augmented by adding Gaussian noise $N(0, \sigma)$. We choose $\sigma \in \{0.01, 0.1, 0.5\}$ and augment $\{16, 64, 128\}$ times for each image respectively.

Jacobian norms of $D$ and loss $V$: Figure 4 shows the results after 500 epochs. It can be seen from the figure that the higher $\sigma$ provides smaller Jacobian norms. This is consistent with our theoretical analysis. In comparison with using translation, adding Gaussian noise makes the Jacobian norms in both GAN and LSGAN more stable.

Jacobian norms of $G$: We did the same procedure as for the case of image translation to see how large the norm of $\frac{\partial G}{\partial z}$ is. We choose $\sigma \in \{0.5, 2, 4\}$. Figure 5 show the results. The same behaviour can be observed. Larger $\sigma$ often leads to smaller norms. It is worth noting that the Jacobian norm will be zero as $\sigma$ is too large. In this case both $D$ and $G$ may be over-penalized. Those empirical results support well our theory.

F Evaluation of spectral normalization

This section presents an evaluation on the effect of Lipschitz constraint by using spectral normalization (SN) (Miyato et al., 2018). We use Saturating GAN and LSGAN with four scenerios: no penalty; SN for $G$ only; SN for $D$ only; SN for both $D$ and $G$. The setting for our experiments appears in subsection E.1.

The results appear in Figure 6. When no penalty is used, we observe that the gradients of the loss
Figure 5: Some behaviors of GAN and LSGAN when augmenting images by adding noises. $\frac{\partial G}{\partial z}$ is measured along the training process.

tend to increase in magnitude while both $D$ and $G$ are hard to reach optimality. When SN is used for $G$, it seems that $G$ has been over-penalized since the gradient norms are almost zero, meaning that $G$ may be underfitting. This behavior appears in both GAN and LSGAN, and was also observed before (Brock et al., 2019). The most successful case is the use of SN for $D$ only. We observe that both players seem to reach the Nash equilibrium. The gradient norms of the loss are relatively stable and small in the course of training, while the quality of fake image (measured by FID) can be better than the other cases. Furthermore both the loss $V_g$ and $\|\frac{\partial V_g}{\partial z}\|_F$ of the generator are stable and belong to small domains, suggesting that the use of SN for $D$ can help us to penalize the zero- and first-order informations of the loss.

Our experiments suggest three messages which agree well with our theory in Theorem 12. Firstly, when no penalty is used, the Lipschitz constant of a hypothesis may be large in order to well fit the training data. In this case the generalization may not be good. Secondly, we can get stuck at underfitting if a penalty on Lipschitzness is overused. The reason is that a heavy penalty can result in a small Lipschitz constant (thus simpler hypothesis), meanwhile a too simple hypothesis may cause a large optimization error. Hence, the generalization is not good in this case. Thirdly, when an appropriate penalty is used, we can obtain both a small Lipschitz constant and small optimization error which lead to better generalization.
Figure 6: Some behaviors of Saturating GAN and LSGAN in different situations. $V_g$ is the loss for training the generator, and FID measures the quality of generated images, the lower the better.