The Cauchy problem of a periodic 2-component $\mu$-Hunter-Saxton system in Besov spaces

Jingjing Liu

Department of Mathematics and Information Science, Zhengzhou University of Light Industry, 450002 Zhengzhou, China

Abstract: This paper is concerned with the local well-posedness and the precise blow-up scenario for a periodic 2-component $\mu$-Hunter-Saxton system in Besov spaces. Moreover, we state a new global existence result to the system. Our obtained results for the system improve considerably earlier results.

Keywords: periodic Besov spaces, periodic 2-component $\mu$-Hunter-Saxton system, local well-posedness, blow-up scenario, global existence.

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1 Introduction

Recently, a new 2-component system was introduced by Zuo in [21] as follows:

$$\begin{cases}
\mu(u)_t - u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - u_{xxx} + \rho \rho_x - \gamma_1 u_{xxx}, \\
\rho_t = (\rho u)_x + 2\gamma_2 \rho_x, \\
u(0, x) = u_0(x), \\
\rho(0, x) = \rho_0(x), \\
u(t, x + 1) = u(t, x), \\
\rho(t, x + 1) = \rho(t, x),
\end{cases} \quad (1.1)$$

where $\mu(u) = \int_{\mathbb{S}} u dx$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2$. By integrating both sides of the first equation in the system (1.1) over the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and using the periodicity of $u$, one obtain

$$\mu(u_t) = \mu(u)_t = 0.$$
This yields the following periodic 2-component $\mu$-Hunter-Saxton system:

\[
\begin{cases}
-u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \rho \rho_x - \gamma_1 u_{xxx}, \\
\rho_t = (\rho u)_x + 2\gamma_2 \rho_x, \\
u(0, x) = u_0(x), \\
\rho(0, x) = \rho_0(x), \\
u(t, x + 1) = u(t, x), \\
\rho(t, x + 1) = \rho(t, x),
\end{cases}
\]

with $\gamma_i \in \mathbb{R}$, $i = 1, 2$. This system is a 2-component generalization of the generalized Hunter-Saxton equation obtained in [14]. The author [21] shows that this system is both a bihamiltonian Euler equation and a bivariational equation. Moreover, the geometric background of the system (1.1) has been comprehensively studied by Escher in [9] recently.

Obviously, (1.1) is equivalent to (1.2) under the condition $\mu(u_t) = \mu(u)x = 0$. In this paper, we will study the system (1.2) under the assumption $\mu(u_t) = \mu(u)x = 0$.

For $\rho \equiv 0$ and $\gamma_i = 0, i = 1, 2$ and replacing $t$ by $-t$, the system (1.2) reduces to the generalized Hunter-Saxton equation (named $\mu$-Hunter-Saxton equation or $\mu$-Camassa-Holm equation) as follows:

\[
-u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx},
\]

which is obtained and studied in [14]. Moreover, the periodic $\mu$-Hunter-Saxton equation and the periodic $\mu$-Degasperis-Procesi equation have also been studied in [10, 15] recently. It is worthy to note that the $\mu$-Hunter-Saxton equation has a very close relation with the periodic Hunter-Saxton and Camassa-Holm equations [13].

For $\rho \equiv 0$, $\gamma_i = 0, i = 1, 2$, $\mu(u) = 0$ and replacing $t$ by $-t$, the system (1.2) becomes to a 2-component periodic Hunter-Saxton system. Its peakon solutions and the Cauchy problem have been analysed and discussed in [6] and [18], respectively.

The system (1.2) has been studied in [16] in Sobolev spaces $H^s(S) \times H^{s-1}(S)$, $s \geq 2$ recently. The authors established the local well-posedness in $H^s(S) \times H^{s-1}(S)$, $s \geq 2$, by Kato’s semigroup theory, derived the precise blow-up scenario, presented some blow-up results for strong solutions and gave a global existence result to the system. Inspired by the study of the local well-posedness and blow-up criterion to the Camassa-Holm equation in [7, 8] and the local well-posedness and blow-up criterion to the two-component Camassa-Holm equation in [12, 13], we will discuss the system (1.2) in Besov spaces. Our obtained local well-posedness, blow-up criterion and global existence results for the system improve considerably earlier results in [16]. Moreover, a very interesting result in this paper is Lemma 3.3. Using this lemma, we will give a explicit proof of the continuity of solution with respect to the initial data when
establish the local well-posedness of the system (1.2) in Besov spaces. In my opinion, this proof is new and necessary.

Our paper is organized as follows. In Section 2, we recall some basic facts of periodic Besov spaces and the transport equation theory. In Section 3, we establish the local well-posedness of the initial value problem associated with the system (1.2). In Section 4, we derive the precise blow-up scenario of strong solution to the system (1.2) and present a new global existence result for strong solutions to the system (1.2) with certain initial profiles.

Notation Given a Banach space $Z$, we denote its norm by $\| \cdot \|_Z$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations if there is no ambiguity. Let $u^{(k)}$ stand for $k$th derivative of $u$ and let $\ast$ denote the convolution.

2 Preliminaries

In this section, we will recall some basic facts on periodic Besov spaces and the transport equation theory. We refer to [1, 7, 8, 17] for the elementary properties of them. Here, we only display some facts which will be used later.

Proposition 2.1 ([1, 7, 8, 17, 11]). (Littlewood-Paley decomposition) Let $B = \{ \xi \in \mathbb{R}, |\xi| \leq \frac{4}{3} \}$ and $C = \{ \xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{5}{3} \}$. There exist two radial functions $\chi \in C_\infty^c (B)$ and $\varphi \in C_\infty^c (C)$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \ \xi \in \mathbb{R}.$$  

For $u \in D'(\mathbb{S})$, let

$$\Delta_q u = 0 \text{ for } q \leq -2, \quad \Delta_{-1} u = \sum_{\beta \in \mathbb{Z}} \chi(\beta) \hat{u}_\beta e^{2\pi i \beta x},$$

$$\Delta_q u = \sum_{\beta \in \mathbb{Z}} \varphi(2^{-q}\beta) \hat{u}_\beta e^{2\pi i \beta x} \quad \text{for } q \geq 0$$

and

$$S_q u = \sum_{-1 \leq p \leq q-1} \Delta_p u.$$  

A direct computation implies, for any $u \in D'(\mathbb{S})$ and $v \in D'(\mathbb{S})$, the following properties hold:

$$\Delta_p \Delta_q u \equiv 0 \quad \text{if } |p - q| \geq 2,$$

$$\Delta_q (S_{p-1} u \Delta_p v) \equiv 0 \quad \text{if } |p - q| \geq 5.$$  

Moreover,

$$\| \Delta_q u \|_{L^p} \leq C \| u \|_{L^p}$$

3
for some constant $C$ independent of $q$.

**Definition 2.1** ([11]). (Besov spaces) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The periodic Besov spaces $B^s_{p,r}(\mathbb{S})$ is defined by

$$B^s_{p,r}(\mathbb{S}) = \{ u \in \mathcal{D}'(\mathbb{S}); \| u \|_{B^s_{p,r}(\mathbb{S})} < \infty \},$$

where

$$\| u \|_{B^s_{p,r}(\mathbb{S})} = \begin{cases} \left( \sum_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q u \|_{L^p} \right)^{\frac{1}{r}}, & \text{for } r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q u \|_{L^p}, & \text{for } r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r} = \cap_{s \in \mathbb{R}} B^s_{p,r}$. The Sobolev spaces correspond to $H^s = B^s_{2,2}$.

**Proposition 2.2** ([11]). The following properties hold:

1. Density: for $1 \leq p, r \leq \infty$, we have $\mathcal{D}(\mathbb{S}) \subset B^s_{p,r} \subset \mathcal{D}'(\mathbb{S})$. Moreover, if $p, r < \infty$, then the set of all trigonometric polynomial is dense in $B^s_{p,r}(\mathbb{S})$.

2. Sobolev embedding: if $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$. If $s_1 < s_2$, $1 \leq p \leq +\infty$ and $1 \leq r_1, r_2 \leq +\infty$, then $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$. Moreover, if $r_1 = r_2$, then the embedding is compact.

3. Algebraic properties: ($B^s_{p,r}$ is an algebra) $B^s_{p,r} \hookrightarrow L^\infty$ if ($s > \frac{1}{p}$ or ($s \geq \frac{1}{p}$, $r=1$)).

4. Fatou property: if $(u^n)_{n \in \mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to $u$ in $\mathcal{D}'(\mathbb{S})$, then $u \in B^s_{p,r}$ and

$$\| u \|_{B^s_{p,r}} \leq \liminf_{n \to \infty} \| u^n \|_{B^s_{p,r}}.$$

5. Complex interpolation: if $u \in B^s_{p,r} \cap B^s_{p',r'}$ and $\theta \in [0, 1]$, $1 \leq p, r \leq \infty$, then $u \in B^{\theta s+(1-\theta)s}_{p,r}$ and

$$\| u \|_{B^{\theta s+(1-\theta)s}_{p,r}} \leq \| u \|_{B^s_{p,r}}^{\theta} \| u \|_{B^s_{p',r'}}^{1-\theta}.$$  

6. The lifting property: let $u \in \mathcal{D}'(\mathbb{S})$ and $\alpha \in \mathbb{R}$. Then $u \in B^s_{p,r}$ if and only if

$$\sum_{\beta \neq 0} e^{2\pi i \beta x} (i\beta)^\alpha \hat{u}_\beta \in B^{s-\alpha}_{p,r}.$$

7. Let $s > 0$. Then $u \in B^{s+1}_{p,r}$ if and only if $u$ is differentiable a.e. and $u' \in B^s_{p,r}$.

**Lemma 2.1** ([11]). Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\min\{\frac{1}{p}, 1-\frac{1}{p}\}$. Let $v$ be a vectorfield such that $\partial_x v$ belongs to $L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{1}{p}$ or to $L^1([0, T]; B_{p,r}^\infty)$ otherwise. Suppose also that $f_0 \in B^{s+1}_{p,r}$, $g \in L^1([0, T]; B^{s}_{p,r})$ and that $f \in L^\infty([0, T]; B^{s}_{p,r}) \cap C([0, T]; \mathcal{D}'(\mathbb{S}))$ solves the following linear transport equation

$$\begin{cases} \partial_t f + v \partial_x f = g, \\ f|_{t=0} = f_0, \\ f(t, x+1) = f(t, x). \end{cases}$$

(T)
Then there exists a constant $C$ depending only on $s, p$, such that the following statements hold for all $t \in [0, T]$:

(i). 

\[
\|f(t)\|_{B^{s,r}_p} \leq \|f_0\|_{B^{s,r}_p} + \int_0^t \|g(\tau)\|_{B^{s,r}_p} d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{B^{s,r}_p} d\tau,
\]

or hence

\[
\|f\|_{B^{s,r}_p} \leq e^{CV(t)} \left( \|f_0\|_{B^{s,r}_p} + \int_0^t e^{-CV(\tau)} \|g(\tau)\|_{B^{s,r}_p} d\tau \right),
\]

where

\[
V(t) = \begin{cases} 
\int_0^t \|\partial_x v(\tau, \cdot)\|_{B^{s-1}_p} d\tau, & \text{if } s < 1 + \frac{1}{p}, \\
\int_0^t \|\partial_x v(\tau, \cdot)\|_{B^{s-1}_p} d\tau, & \text{if } s > 1 + \frac{1}{p} \text{ or } \{s = 1 + \frac{1}{p} \text{ and } r = 1\}.
\end{cases}
\]

(ii). If $f = c_1(v + c_2)$ with $c_1, c_2 \in \mathbb{R}$, then for all $s > 0$, the estimates in (i) hold with $V(t) = \int_0^t \|\partial_\tau v(\tau)\|_{L^\infty} d\tau$.

(iii). If $r < +\infty$, then $f \in C([0, T]; B^{s,r}_p)$ for all $s' < s$.

**Lemma 2.2** ([17] [13]). Let $0 < \sigma < 1$. Suppose that $f_0 \in H^\sigma$, $g \in L^1([0, T]; H^\sigma)$, $v, \partial_x v \in L^1([0, T]; L^\infty)$ and that $f \in L^\infty([0, T]; H^\sigma) \cap C([0, T]; D'(S))$ solves the 1-dimensional linear transport equation

\[
\begin{aligned}
\partial_t f + v \partial_x f &= g, \\
f\big|_{t=0} &= f_0, \\
f(t, x + 1) &= f(t, x).
\end{aligned}
\]  

(\text{T})

Then $f \in C([0, T]; H^\sigma)$. More precisely, there exists a constant $C$ depending only on $\sigma$ and such that the following statement holds:

\[
\|f(t)\|_{H^\sigma} \leq \|f_0\|_{H^\sigma} + C \int_0^t \|g(\tau)\|_{H^\sigma} d\tau + C \int_0^t \|f(\tau)\|_{H^\sigma} V'(\tau) d\tau
\]

or hence

\[
\|f\|_{H^\sigma} \leq e^{CV(t)} \left( \|f_0\|_{H^\sigma} + \int_0^t \|g(\tau)\|_{H^\sigma} d\tau \right)
\]

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_\tau v(\tau)\|_{L^\infty}) d\tau$.

**Lemma 2.3** ([17] [8]). Let $(p, p_1, r) \in [1, +\infty]^3$. Assume that $s > -\min\{\frac{1}{p_1}, \frac{1}{p}\}$ or $s > -1 - \min\{\frac{1}{p_1}, \frac{1}{p}\}$ if $\partial_x v = 0$ with $p' = (1 - \frac{1}{p})^{-1}$. Let $f_0 \in B^{s,r}_{p_1}$ and $g \in L^1([0, T]; B^{s,r}_{p_1})$. Let $v$ be a time dependent vector field such that $v \in L^p([0, T]; B^{-M\infty}_{\infty,\infty})$ for some $p > 1$, $M > 0$ and $\partial_x v \in L^1([0, T]; B^{-p_1}_{p_1,\infty} \cap L^\infty)$ if $s < 1 + \frac{1}{p_1}$, and $\partial_x v \in L^1([0, T]; B^{-p_1}_{p_1,\infty})$ if $s > 1 + \frac{1}{p_1}$ or $s = 1 + \frac{1}{p_1}$ and $r = 1$. Then the transport equations (T) has a unique solution $f \in L^\infty([0, T]; B^{s,r}_p) \cap (\cap_{s' < s} C([0, T]; B^{s',r}_p))$ and the inequalities of Lemma 2.1 hold. If,
moreover, \( r < \infty \), then we have \( f \in C([0,T];B^s_{p,r}) \).

**Lemma 2.4** ([3, 7, 11, 13]). (1-D Moser-type estimates) Assume that \( 1 \leq p, r \leq +\infty \), the following estimates hold:

1. for \( s > 0 \),
   \[
   \|fg\|_{B^{s}_{p,r}} \leq C(\|f\|_{B^{s}_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^{s}_{p,r}}\|f\|_{L^\infty});
   \]
2. for \( s > 0 \),
   \[
   \|f \partial_x g\|_{H^s} \leq C(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\partial_x g\|_{H^s});
   \]
3. for \( s_1 \leq \frac{1}{p}, s_2 > \frac{1}{p} (s_2 \geq \frac{1}{p} \text{ if } r = 1) \text{ and } s_1 + s_2 > 0, \)
   \[
   \|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}},
   \]
where \( C \) is constant independent of \( f \) and \( g \).

## 3 Local well-posedness

In this section, we will establish the local well-posedness for the Cauchy problem of the system (1.2) in Besov spaces and then get our main result in \( H^s \times H^{s-1}, s > \frac{3}{2} \).

Note that \( \mu(u)_t = \mu(u)_x = 0 \). Then we let
\[
\mu_0 = \mu(u_0) = \mu(u) = \int_\mathbb{S} u(t, x)dx.
\]
We now provide the framework in which we shall reformulate the system (1.2). We rewrite the system (1.2) as follows:

\[
\begin{aligned}
&u_t - (u + \gamma_1)u_x = \partial_x(\mu - \partial^2_x)^{-1} (2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2),
&\quad t > 0, \quad x \in \mathbb{R},

&\rho_t - (u + 2\gamma_2)\rho_x = u_x \rho,
&\quad t > 0, \quad x \in \mathbb{R},

&u(0, x) = u_0(x),
&\quad x \in \mathbb{R},

&\rho(0, x) = \rho_0(x),
&\quad x \in \mathbb{R},

&u(t, x+1) = u(t, x),
&\quad t \geq 0, \quad x \in \mathbb{R},

&\rho(t, x+1) = \rho(t, x),
&\quad t \geq 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

If we denote \( P(D) \) as the Fourier integral operator with the Fourier multiplier \(-\frac{2\pi i \beta}{\delta(\beta) + 4\pi^2 \beta^2}\) with
\[
\delta(\beta) = \begin{cases}
1, & \beta = 0, \\
0, & \beta \neq 0,
\end{cases}
\]
then the system (3.1) equivalent to
\[
\begin{aligned}
&u_t - (u + \gamma_1)u_x = P(D)(2\mu_0u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), \quad t > 0, \ x \in \mathbb{R}, \\
&\rho_t - (u + 2\gamma_2)\rho_x = u_x\rho, \quad t > 0, \ x \in \mathbb{R}, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
&\rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \\
&u(t, x + 1) = u(t, x), \quad t \geq 0, \ x \in \mathbb{R}, \\
&\rho(t, x + 1) = \rho(t, x), \quad t \geq 0, \ x \in \mathbb{R}.
\end{aligned}
\] (3.2)

Moreover, combining Proposition 2.2 (6) and
\[
P(D)u = \sum_{\beta \in \mathbb{Z}} e^{2\pi i \beta x} \hat{P}(D)u_\beta = \sum_{\beta \in \mathbb{Z}} e^{2\pi i \beta x} \left( -\frac{2\pi i \beta}{\delta(\beta) + 4\pi^2 \beta^2} \right) \hat{u}_\beta
\]
we have if \( u \in B_{p,r}^s \), then \( \|P(D)u\|_{p^{s+1}} \leq C\|u\|_{B_{p,r}^s} \).

On the other hand, integrating both sides of the first equation in (1.2) with respect to \( x \), we obtain
\[
\begin{align*}
u_{tx} &= -2\mu_0u + \frac{1}{2}u_x^2 + uu_{xx} - \frac{1}{2}\rho^2 + \gamma_1u_{xx} + a(t), \\
\end{align*}
\]
where
\[
a(t) = 2\mu(u)^2 + \frac{1}{2}\int_S (u_x^2 + \rho^2)dx.
\]
Using the system (1.2), we have
\[
\frac{d}{dt} \int_S (u_x^2 + \rho^2)dx = 0.
\]

By \( \mu(u)_t = \mu(u_t) = 0 \), we have
\[
\frac{d}{dt} a(t) = 0.
\]

For convenience, we let
\[
\mu_1 := \left( \int_S (u_x^2 + \rho^2)dx \right)^{\frac{1}{2}} = \left( \int_S (u_{0,x}^2 + \rho_0^2)dx \right)^{\frac{1}{2}}
\]
and write \( a := a(0) \) henceforth. Thus,
\[
\begin{align*}
u_{tx} &= -2\mu_0u + \frac{1}{2}u_x^2 + uu_{xx} - \frac{1}{2}\rho^2 + \gamma_1u_{xx} + a
\end{align*}
\] (3.3)
is a valid reformulation of the first equation in (1.2).

**Definition 3.1.** For \( T > 0, \ s \in \mathbb{R} \) and \( 1 \leq p, r \leq +\infty \), we set
Theorem 3.1. Suppose that \( 1 \leq p, r \leq +\infty \) and \( s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2}\} \) with \( s \neq 2 + \frac{1}{p} \). Given \( z_0 = (u_0, \rho_0) \in B_{p,r}^s \times B_{p,r}^{s-1}, \) there exists a time \( T > 0 \) and a unique solution \( z = (u, \rho) \) to the system (3.2) such that \( z \in E_{s,r}^p(T) \times E_{s-1,r}^p(T) \). Moreover, the mapping \( z_0 \rightarrow z \) is continuous from \( B_{p,r}^s \times B_{p,r}^{s-1} \) into \( C([0, T]; B_{p,r}^s \times B_{p,r}^{s-1}) \) for every \( s' < s \) if \( r = +\infty \) and \( s' = s \) otherwise.

Uniqueness with respect to the initial data is an immediate consequence of the following result.

Lemma 3.1. Let \( 1 \leq p, r \leq +\infty \) and \( s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2}\} \) with \( s \neq 2 + \frac{1}{p}, 3 + \frac{1}{p} \), then

\[
E_{p,r}^s(T) = C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1}) \text{ if } r < +\infty,
\]

\[
E_{p,\infty}^s(T) = L^\infty([0, T]; B_{p,\infty}^s) \cap \text{Lip}([0, T]; B_{p,\infty}^{s-1}) \text{ and } E_{p,r}^s = \cap_{T>0} E_{p,r}^s(T).
\]

The local well-posedness result of the system (1.2) in \( B_{p,r}^s \) and \( E_{p,r}^s(T) \) can be stated as follows:

\[
E_{p,r}^s(T) = C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1}) \text{ if } r < +\infty,
\]

\[
E_{p,\infty}^s(T) = L^\infty([0, T]; B_{p,\infty}^s) \cap \text{Lip}([0, T]; B_{p,\infty}^{s-1}) \text{ and } E_{p,r}^s = \cap_{T>0} E_{p,r}^s(T).
\]
(3) if \( r \neq 1 \) and \( s = 3 + \frac{1}{p} \), then
\[
\|u^1 - u^2\|_{B^{s,r}_p} + \|\rho^1 - \rho^2\|_{B^{s,r}_p} \\
\leq e^{C \int_0^t \|u\|_{B^s_{p,r}} + \|\rho\|_{B^s_{p,r}} + \|\partial_x u\|_{B^{s-1}_{p,r}} + \|\partial_x \rho\|_{B^{s-1}_{p,r}} + \|\mu\|_{B^{s-1}_{p,r}} + \|\nu\|_{B^{s-1}_{p,r}}) d\tau
\]
\[
(\|u^1_0 - u^2_0\|_{B^{s,r}_p} + \|\rho^1_0 - \rho^2_0\|_{B^{s,r}_p} + C\|\mu^0\|_{B^{s-1}_{p,r}} + \|\nu^0\|_{B^{s-1}_{p,r}} + \|\mu_0\|_{B^{s-1}_{p,r}}) d\tau
\]
\[
+Ce^{C \int_0^t \|u\|_{B^s_{p,r}} + \|\rho\|_{B^s_{p,r}} + \|\partial_x u\|_{B^{s-1}_{p,r}} + \|\partial_x \rho\|_{B^{s-1}_{p,r}} + \|\mu\|_{B^{s-1}_{p,r}} + \|\nu\|_{B^{s-1}_{p,r}}) d\tau
\]
\[
\left(\|u^{12}_t\|_{B^{s,r}_p} + \|\rho^{12}_t\|_{B^{s,r}_p} + C\|\mu^1 - \mu^2\|_{L^\infty} \right) \leq \left(\|\rho^1(t)\|_{B^{s,r}_p} + \|\rho^2(t)\|_{B^{s,r}_p} \right)^{1-\theta}
\]

**Proof** Denote \( u^{12} = u^1 - u^2 \), \( \rho^{12} = \rho^1 - \rho^2 \). It is obvious that
\[
u^{12} \in L^\infty([0,T];B^s_{p,r}) \cap C([0,T];B^{s-1}_{p,r}), \quad \rho^{12} \in L^\infty([0,T];B^{s-1}_{p,r}) \cap C([0,T];B^{s-2}_{p,r}),
\]
and \((u^{12}, \rho^{12})\) solves the transport equation:

\[
\begin{cases}
\partial_t u^{12} - (u + \gamma_1)\partial_x u^{12} = u^{12}\partial_x u^2 + F(t, x) & t > 0, x \in \mathbb{R}, \\
\partial_t \rho^{12} - (u + 2\gamma_2)\partial_x \rho^{12} = u^{12}\partial_x \rho^2 + \rho^{12}\partial_x u^1 + \rho^2\partial_x u^{12} & t > 0, x \in \mathbb{R}, \\
u^{12}(0, x) = u^1(x) - u^2(x) = u^{12}(x), & x \in \mathbb{R}, \\
\rho^{12}(0, x) = \rho^1(x) - \rho^2(x) = \rho^{12}(x), & x \in \mathbb{R}, \\
u^{12}(t, x + 1) = u^{12}(t, x), & t \geq 0, x \in \mathbb{R}, \\
\rho^{12}(t, x + 1) = \rho^{12}(t, x), & t \geq 0, x \in \mathbb{R},
\end{cases}
\]

where

\[
F(t, x) = P(D)(2\mu^0 u^{12} + 2(\mu^0 - \mu^2)u^2 + \frac{1}{2}\partial_x u^{12}\partial_x u^1 + u^{12}u^2 + \frac{1}{2}\rho^{12}(\rho^1 + \rho^2)).
\]

(1) If \( r = 1 \), \( s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2}\} \) or \( r \neq 1 \), \( s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2}\} \) but \( s \neq 2 + \frac{1}{p}, 3 + \frac{1}{p} \), noting that for \( w \in B^s_{p,r} \) with \( s > 1 + \frac{1}{p} \), then \( \|\partial_x w\|_{B^{s-1}_{p,r}} \leq C\|w\|_{B^{s}_{p,r}} \). Applying Lemma 2.1 and the fact that \( \|\partial_x w\|_{B^{s-3}_{p,r}} \leq C\|\partial_x w\|_{B^{s-2}_{p,r}} \leq C\|w\|_{B^{s}_{p,r}} \), we have

\[
e^{-C \int_0^t \|u(x)\|_{L^p} d\tau} \|u^{12}(t)\|_{B^{s-1}_{p,r}} \leq \|u^0\|_{B^{s-1}_{p,r}}
\]
\[
+ \int_0^t e^{-C \int_0^\tau \|u(x)\|_{L^p} d\tau} \cdot \left(\|u^{12}\|_{L^p} u^{12} + \|F(\tau)\|_{L^{p^{-1}}_{p,r}}\right) d\tau
\]
\[
(3.4)
\]
and

\[
e^{-C \int_0^t \|u(x)\|_{L^p} d\tau} \|\rho^{12}(t)\|_{B^{s-2}_{p,r}}
\]
\[
\leq \|\rho^0\|_{B^{s-2}_{p,r}} + \int_0^t e^{-C \int_0^\tau \|u(x)\|_{L^p} d\tau} \cdot \left(\|u^{12}\|_{L^p} \rho^{12} + \|\rho^{12}\|_{L^p} u^{12} + \|\rho^2\|_{L^p} u^{12}\right) d\tau.
(3.5)
\]
For $s > 1 + \frac{1}{p}$, $B^{s-1}_{p,r}$ is an algebra according to Proposition 2.3 (3), so we have
\[
\|u^{12}\partial_x u^2\|_{B^{s-1}_{p,r}} \leq \|u^{12}\|_{B^{s-1}_{p,r}} \|\partial_x u^2\|_{B^{s-1}_{p,r}} \leq \|u^{12}\|_{B^{s-1}_{p,r}} \|u^2\|_{B^{s}_{p,r}}.
\]

By the property of $P(D)$, we have
\[
\|P(D)(2\mu_0 u^{12} + 2(\mu_0^2 - \mu_0^2)u^2)\|_{B^{s-1}_{p,r}} \leq C\|2\mu_0 u^{12} + 2(\mu_0^2 - \mu_0^2)u^2\|_{B^{s-2}_{p,r}} \\
\leq C(\|u^{12}\|_{B^{s-1}_{p,r}} + \|u^2\|_{B^{s}_{p,r}}).
\]

Moreover, note that $B^{s-2}_{p,r}$ is an algebra with $s - 2 > \frac{1}{p}$. If $s - 2 \leq \frac{1}{p}$, then combining $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and Lemma 2.4, we get
\[
\|P(D)(\mu_0 u^{12})(u^1 + u^2)\|_{B^{s-2}_{p,r}} \leq C\|\partial_x u^{12}\|_{B^{s-2}_{p,r}} \\
\leq C\|\partial_x u^{12}\|_{B^{s-2}_{p,r}} \left(\|\partial_x u^1\|_{B^{s-1}_{p,r}} + \|\partial_x u^2\|_{B^{s-1}_{p,r}}\right) \\
\leq C\|u^{12}\|_{B^{s-1}_{p,r}} \left(\|u^1\|_{B^{s}_{p,r}} + \|u^2\|_{B^{s}_{p,r}}\right).
\]

The inequalities above imply:
\[
\|u^{12}\partial_x u^2\|_{B^{s-1}_{p,r}} + \|F(\tau)\|_{B^{s-1}_{p,r}} \leq C(\|u^{12}\|_{B^{s-1}_{p,r}} + \|\rho^{12}\|_{B^{s-2}_{p,r}}) \\
(\|u^1\|_{B^{s}_{p,r}} + \|u^2\|_{B^{s}_{p,r}} + \|\rho^1\|_{B^{s-1}_{p,r}} + \|\rho^2\|_{B^{s-1}_{p,r}} + |\mu_0|) \\
+ C|\mu_0 - \mu_0^2|\|u^2\|_{B^{s}_{p,r}}.
\]

While thanks to Lemma 2.4, we have
\[
\|u^{12}\partial_x \rho^2\|_{B^{s-2}_{p,r}} \leq C\|u^{12}\|_{B^{s-1}_{p,r}} \|\partial_x \rho^2\|_{B^{s-2}_{p,r}} \leq C\|u^{12}\|_{B^{s-1}_{p,r}} \|\rho^2\|_{B^{s-1}_{p,r}},
\]
\[
\|\rho^{12}\partial_x u^1\|_{B^{s-2}_{p,r}} \leq C\|\rho^{12}\|_{B^{s-2}_{p,r}} \|\partial_x u^1\|_{B^{s-1}_{p,r}} \leq C\|\rho^{12}\|_{B^{s-2}_{p,r}} \|u^1\|_{B^{s}_{p,r}},
\]

and
\[
\|\rho^2\partial_x u^{12}\|_{B^{s-2}_{p,r}} \leq C\|\rho^2\|_{B^{s-1}_{p,r}} \|\partial_x u^{12}\|_{B^{s-2}_{p,r}} \leq C\|\rho^2\|_{B^{s-1}_{p,r}} \|u^{12}\|_{B^{s-1}_{p,r}}.
\]

It then follows that
\[
\|u^{12}\partial_x \rho^2\|_{B^{s-1}_{p,r}} + \|\rho^{12}\partial_x u^1\|_{B^{s-1}_{p,r}} + \|\rho^2\partial_x u^{12}\|_{B^{s-1}_{p,r}} \\
\leq C(\|u^{12}\|_{B^{s-1}_{p,r}} + \|\rho^{12}\|_{B^{s-1}_{p,r}})(\|u^1\|_{B^{s}_{p,r}} + \|\rho^2\|_{B^{s-1}_{p,r}}).
\]

(3.7)
Thus, combining (3.4)-(3.7), we have
\[
e^{-C \int_0^t \| u^1(\tau) \|_{B^s_{p,r}}^2 d\tau} \left( \| u^{12}(t) \|_{B^s_{p,r}} + \| \rho^{12}(t) \|_{B^s_{p,r}}^2 \right) \\
\leq \| u_0^{12} \|_{B^s_{p,r}} + \| \rho_0^{12} \|_{B^s_{p,r}}^2 + C \int_0^t \| \mu_0^1 - \mu_0^2 \| u^2(\tau) \|_{B^s_{p,r}}^2 d\tau \\
+ C \int_0^t e^{-C \int_0^\tau \| u^1(\sigma) \|_{B^s_{p,r}}^2 d\sigma'} \left( \| u^{12}(\tau) \|_{B^s_{p,r}} + \| \rho^{12}(\tau) \|_{B^s_{p,r}}^2 \right) \\
\cdot \left( \| u^1 \|_{B^s_{p,r}} + \| u^2 \|_{B^s_{p,r}} + \| \rho^1 \|_{B^s_{p,r}} + \| \rho^2 \|_{B^s_{p,r}} + |\mu_0^1| \right) d\tau.
\]
That is
\[
w(t) \leq v(t) + C \int_0^t w(\tau) u(\tau) d\tau
\]
with
\[
w(t) = e^{-C \int_0^t \| u^1(\tau) \|_{B^s_{p,r}}^2 d\tau} \left( \| u^{12}(t) \|_{B^s_{p,r}} + \| \rho^{12}(t) \|_{B^s_{p,r}}^2 \right),
\]
\[
v(t) = \| u_0^{12} \|_{B^s_{p,r}} + \| \rho_0^{12} \|_{B^s_{p,r}}^2 + C|\mu_0^1 - \mu_0^2| \int_0^t \| u^2(\tau) \|_{B^s_{p,r}}^2 d\tau,
\]
and
\[
u(t) = \| u^1 \|_{B^s_{p,r}} + \| u^2 \|_{B^s_{p,r}} + \| \rho^1 \|_{B^s_{p,r}} + \| \rho^2 \|_{B^s_{p,r}} + |\mu_0^1|.
\]
This completes the proof of (1) by applying Gronwall’s inequality.

(2) If $r \neq 1$ and $s = 2 + \frac{1}{p}$, we will use the interpolation method to deal with it. Indeed, if we choose $s_1 \in (\max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2} \} - 1, s - 1)$, $s_2 \in (s - 1, s)$ and $\theta = \frac{s_2 -(s-1)}{s_2 - s_1} \in (0, 1)$, then $s - 1 = \theta s_1 + (1 - \theta)s_2$. According to Proposition 2.3 (5), we have
\[
\| u^{12}(t) \|_{B^s_{p,r}} \leq \| u^{12}(t) \|_{B^{s_1}_{p,r}}^{\theta} \| u^{12}(t) \|_{B^{s_2}_{p,r}}^{1-\theta} \leq \left( \| u^1 \|_{B^{s_1}_{p,r}} + \| u^2 \|_{B^{s_2}_{p,r}} \right)^{1-\theta} \| u^{12}(t) \|_{B^{s_1}_{p,r}}^{\theta}.\]
Since $s_1 + 1 > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2} \}$ and $s_1 + 1 < s = 2 + \frac{1}{p}$, the estimate in case (1) for $\| u^{12}(t) \|_{B^{s}_{p,r}}$ holds. On the other hand, thanks to $s - 2 = \frac{1}{p} < 1 + \frac{1}{p}$, we have (3.5) holds. Consequently, the estimate for $\| \rho^{12}(t) \|_{B^{s}_{p,r}}$ in case (1) can also hold true. Hence, we can get the desired result.

(3) For the critical case $s = 3 + \frac{1}{p}$, noting that $s - 1 = 2 + \frac{1}{p} > 1 + \frac{1}{p}$, we have (3.4) holds. So the estimate for $\| u^{12}(t) \|_{B^{s}_{p,r}}$ in case (1) holds here. The left proof is very similar to that of case (2). Therefore, we complete our proof of Lemma 3.1.

Next, we will construct the approximate solutions to (3.2).

**Lemma 3.2.** Let $u_0, \rho_0, p, r$ and $s$ be as in the statement of Theorem 3.1. Assume that $u^0 = \ldots$
\(\rho^0 = 0\). Then there exists a unique sequence of smooth functions \((u^n, \rho^n)_{n \in N} \in C(\mathbb{R}^+; B^\infty_{p,r} \times B^\infty_{p,r})\) solving the following linear transport equation by induction:

\[
\begin{aligned}
&\partial_t u^{n+1} - (u^n + \gamma_1) \partial_x u^{n+1} = F^n(t, x), \\
&\partial_t \rho^{n+1} - (u^n + 2\gamma_2) \partial_x \rho^{n+1} = \rho^n \partial_x u^n, \\
&u^{n+1}(0, x) = u_0^{n+1}(x) = S_{n+1}u_0, \\
&\rho^{n+1}(0, x) = \rho_0^{n+1}(x) = S_{n+1}\rho_0, \\
&u^{n+1}(t, x + 1) = u^{n+1}(t, x), \\
&\rho^{n+1}(t, x + 1) = \rho^{n+1}(t, x),
\end{aligned}
\]

(Tn)

where \(F^n(t, x) = P(D)(2\rho_0^{n+1}u^n + \frac{1}{2}(\partial_x u^n)^2 + \frac{1}{2}(\rho^n)^2)\) with \(\mu_0^{n+1} = \int_S u_0^{n+1} dx\). Moreover, there exists a positive \(T\) such that the solutions satisfy:

(i) \((u^n, \rho^n)_{n \in N}\) is uniformly bounded in \(E^s_{p,r}(T) \times E^{s-1}_{p,r}(T)\).

(ii) \((u^n, \rho^n)_{n \in N}\) is a Cauchy sequence in \(C([0, T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r})\).

**Proof**

For convenience, we assume that \(r \neq 1\) here. In fact, Theorem 3.2 corresponds to \(p = r = 2\). Since all the data \(S_{n+1}u_0\) and \(S_{n+1}\rho_0\) belong to \(B^\infty_{p,r}\), Lemma 2.3 enables us to show by induction that for all \(n \in N\), the equation \((T_n)\) has a unique global solution which belongs to \(C(\mathbb{R}^+, B^\infty_{p,r} \times B^\infty_{p,r})\). Note that

\[|\mu_0^{n+1}| \leq \int_S |u_0^{n+1}| dx \leq \|S_{n+1}u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}.
\]

Similar to the proof of Lemma 3.1, by \(s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}, \frac{3}{2}\}\) with \(s \neq 2 + \frac{1}{p}\), we have the following inequalities for all \(n \in N\),

\[
e^{-C \int_0^t \|u^n(\tau)\|_{B^p_{p,r}} d\tau} |\|u^{n+1}(t)\|_{B^p_{p,r}} |
\]

\[
\leq \|u_0\|_{B^p_{p,r}} + \frac{C}{2} \int_0^t e^{-C \int_0^\tau \|u^n(\tau')\|_{B^p_{p,r}} d\tau'} \left( \|u^n\|_{B^p_{p,r}} + \|u^n\|_{B^p_{p,r}}^2 + \|\rho^n\|_{B^p_{p,r}}^2 \right) d\tau
\]

(3.8)

and

\[
e^{-C \int_0^t \|u^n(\tau)\|_{B^p_{p,r}} d\tau} |\|\rho^{n+1}(t)\|_{B^{p-1}_{p,r}} |
\]

\[
\leq \|\rho_0\|_{B^{p-1}_{p,r}} + \frac{C}{2} \int_0^t e^{-C \int_0^\tau \|u^n(\tau')\|_{B^p_{p,r}} d\tau'} \|u^n\|_{B^p_{p,r}} \|\rho^n\|_{B^{p-1}_{p,r}} d\tau.
\]

(3.9)

Hence we have

\[
e^{-C \int_0^t \|u^n(\tau)\|_{B^p_{p,r}} d\tau} \left( \|u^{n+1}(t)\|_{B^p_{p,r}} + \|\rho^{n+1}(t)\|_{B^{p-1}_{p,r}} \right)
\]

\[
\leq \left( \|u_0\|_{B^p_{p,r}} + \|\rho_0\|_{B^{p-1}_{p,r}} \right) + \frac{C}{2} \int_0^t e^{-C \int_0^\tau \|u^n(\tau')\|_{B^p_{p,r}} d\tau'} \left( \|u^n\|_{B^p_{p,r}} + \|\rho^n\|_{B^{p-1}_{p,r}} + 1 \right) d\tau.
\]

(3.10)
Denoting $l^n(t) = ||u^n(t)||_{B^s_{p,r}} + ||\rho^n(t)||_{B^{s-1}_{p,r}}$, $L = ||u_0||_{B^s_{p,r}} + ||\rho_0||_{B^{s-1}_{p,r}}$, we have

$$l^{n+1}(t) \leq e^{C_0} \int_0^t ||u^n(\tau)||_{B^s_{p,r}} d\tau \left( L + \frac{C}{2} \int_0^t e^{-C_0} ||u^n(\tau)||_{B^s_{p,r}} d\tau \right) + e^{C_0} \int_0^t \rho^n(\tau) d\tau.$$ 

Let us choose a $T > 0$ such that $T < \min\{\frac{1}{4CL}, \frac{1}{2C}\}$ and prove by induction that for all $t \in [0, T]$

$$l^n(t) \leq \frac{2L}{1 - 4CLt}.$$ 

Note that

$$\int_\tau^t ||u^n(\tau)||_{B^s_{p,r}} d\tau' \leq -\frac{1}{2C} \ln \frac{1 - 4CLt}{1 - 4CL\tau}$$

with $l^n(t) \leq \frac{2L}{1 - 4CLt}$. A direct computation implies

$$l^{n+1}(t) \leq \frac{2L}{1 - 4CLt},$$

Therefore, $(u^n, \rho^n)_{n \in N}$ is uniformly bounded in $C([0, T]; B^s_{p,r} \times B^{s-1}_{p,r})$. Using the fact that $B^{s-1}_{p,r}$ with $s > 1 + \frac{1}{p}$ is an algebra, together with Lemma 2.4, one can see that

$$(u^n + \gamma_1)\partial_x u^{n+1}, \ P(D)(2\mu_0^{n+1} u^n + \frac{1}{2}(\partial_x u^n)^2 + \frac{1}{2}(\rho^n)^2)$$

are uniformly bounded in $C([0, T]; B^{s-1}_{p,r})$, and

$$(u^n + 2\gamma_2)\partial_x \partial_x u^{n+1}, \ \rho^n \partial_x u^n$$

are uniformly bounded in $C([0, T]; B^{s-2}_{p,r})$. Hence using the equations $(T_n)$, we have

$$(\partial_t u^{n+1}, \partial_t \rho^{n+1}) \in C([0, T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r})$$

are uniformly bounded, which yields that the sequence $(u^n, \rho^n)_{n \in N}$ is uniformly bounded in $E^s_{p,r}(T) \times E^{s-1}_{p,r}(T)$.

Next, we show that $(u^n, \rho^n)_{n \in N}$ is a Cauchy sequence in $C([0, T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r})$. In fact, according to the equations $(T_n)$, we obtain that, for all $m, n \in N$

$$(\partial_t - (u^{n+m} + \gamma_1)\partial_x)(u^{n+m+1} - u^{n+1}) = (u^{n+m} - u^n)\partial_x u^{n+1} + P(D)(2\mu_0^{n+m+1}(u^{n+m} - u^n)) + P(D)(2(\mu_0^{n+m+1} - \mu_0^{n+1})u^n) + \frac{1}{2}P(D)((\partial_x u^{n+m} - \partial_x u^n)(\partial_x u^{n+m} + \partial_x u^n)) + \frac{1}{2}P(D)((\rho^{n+m} - \rho^n)(\rho^{n+m} + \rho^n))$$

(3.12)

and

$$(\partial_t - (u^{n+m} + 2\gamma_2)\partial_x)(\rho^{n+m+1} - \rho^{n+1}) = (u^{n+m} - u^n)\partial_x \rho^{n+1} + \rho^{n+m} \partial_x (u^{n+m} - u^n) + (\rho^{n+m} - \rho^n)\partial_x u^n.$$ 

(3.13)
Applying Lemma 2.1, together with the fact that $B^{s-1}_{p,r}$ is an algebra and the property of the operator $P(D)$, we get for $t \in [0,T]$

\[
e^{-C \int_0^t \|u^{n+m}(\tau)\|_{B^{s}_{p,r}}^2 \, d\tau} \|(u^{n+m+1} - u^{n+1})(t)\|_{B^{s-1}_{p,r}}
\]
\[\leq \|u_0^{n+m+1} - u_0^{n+1}\|_{B^{s-1}_{p,r}} + C \int_0^t e^{-C \int_0^\tau \|u^{n+m}(\tau')\|_{B^{s}_{p,r}}^2 \, d\tau'} \||u^{n+m} - u^n\|_{B^{s-1}_{p,r}}(\|u^{n+1}\|_{B^{s}_{p,r}} + 1 + \|u^n(\tau)\|_{B^{s}_{p,r}} + \|u^{n+m}(\tau)\|_{B^{s}_{p,r}})
+ \|\rho^{n+m} - \rho^n\|_{B^{s-1}_{p,r}}(\|\rho^{n+1}\|_{B^{s}_{p,r}} + \|\rho^{n+m}\|_{B^{s}_{p,r}}) + \|\mu_0^{n+m+1} - \mu_0^{n+1}\|\|u^n\|_{B^{s}_{p,r}} \, d\tau \tag{3.14}
\]

and

\[
e^{-C \int_0^t \|u^{n+m}(\tau)\|_{B^{s}_{p,r}}^2 \, d\tau} \|(\rho^{n+m+1} - \rho^{n+1})(t)\|_{B^{s-2}_{p,r}}
\]
\[\leq \|\rho_0^{n+m+1} - \rho_0^{n+1}\|_{B^{s-2}_{p,r}} + C \int_0^t e^{-C \int_0^\tau \|u^{n+m}(\tau')\|_{B^{s}_{p,r}}^2 \, d\tau'} \||u^{n+m} - u^n\|_{B^{s-1}_{p,r}}(\|\rho^{n+1}\|_{B^{s}_{p,r}} + \|\rho^{n+m}\|_{B^{s}_{p,r}})
+ \|\rho^{n+m} - \rho^n\|_{B^{s-2}_{p,r}}(\|\rho^{n+1}\|_{B^{s}_{p,r}} + \|\rho^{n+m}\|_{B^{s}_{p,r}}) \tag{3.15}
\]

By Proposition 2.1 and Definition 2.3, we have

\[u_0^{n+m+1} - u_0^{n+1} = S_{n+m+1}u_0 - S_{n+1}u_0 = \sum_{q=n+1}^{n+m} \Delta_q u_0,
\]
\[\rho_0^{n+m+1} - \rho_0^{n+1} = S_{n+m+1}\rho_0 - S_{n+1}\rho_0 = \sum_{q=n+1}^{n+m} \Delta_q \rho_0,
\]

Moreover,

\[\|\sum_{q=n+1}^{n+m} \Delta_q u_0\|_{B^{s-1}_{p,r}} \leq C2^{-n}\|u_0\|_{B^{s}_{p,r}},
\]

and

\[\|\sum_{q=n+1}^{n+m} \Delta_q \rho_0\|_{B^{s-2}_{p,r}} \leq C2^{-n}\|\rho_0\|_{B^{s-1}_{p,r}},
\]

see [19] for detailed computations. Since $(u^n, \rho^n)_{n \in \mathbb{N}}$ is uniformly bounded in $E^{s}_{p,r}(T) \times E^{s-1}_{p,r}(T)$, combining (3.14)-(3.15), we get a constant $C_T$ independent of $n, m$ such that for all $t \in [0,T]$

\[h_{n+1}^m(t) \leq C_T \left(2^{-n} + \int_0^t h^n_m(\tau) \, d\tau + \|\mu_0^{n+m+1} - \mu_0^{n+1}\|\|u^n\|_{B^{s-1}_{p,r}} + \|\rho^{n+m} - \rho^n\|_{B^{s-2}_{p,r}} \right) \tag{3.16}
\]

with

\[h^n_m(t) = \|(u^{n+m} - u^n)(t)\|_{B^{s-1}_{p,r}} + \|(\rho^{n+m} - \rho^n)(t)\|_{B^{s-2}_{p,r}}.
\]
Arguing by induction with respect to the index $n$, one can easily prove that
\[
\begin{align*}
&
\frac{h_{n+1}^m(t)}{h_n^m(t)} \\
&
\leq C_T \left( 2^{-n} \sum_{k=0}^{n} \frac{(2TC_T)^k}{k!} + \frac{(TC_T)^{n+1}}{(n+1)!} + \sum_{k=0}^{n} \mu_0^{n+n-k+1} - \mu_0^{n-k+1}\frac{(C_T)^k}{k!} \right) \\
&
\leq \left( C_T \sum_{k=0}^{n} \frac{(2TC_T)^k}{k!} \right) 2^{-n} + C_T \frac{(TC_T)^{n+1}}{(n+1)!} \\
&
\quad + C_T \sum_{k=0}^{n} \mu_0^{n+n-k+1} - \mu_0^{n-k+1}\frac{(C_T)^k}{k!},
\end{align*}
\]
which implies that $(u^n, \rho^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r})$. This completes the proof of Lemma 3.2.

Following the proof of Theorem 4.24 in [2], we obtain the following result, which is crucial in the proof of the continuity of solution with respect to the initial data.

**Lemma 3.3.** Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Suppose that $(p, r) \in [1, \infty]^2$, $r < +\infty$, $s > 1 + {1 \over p}$ or $s \geq 1 + {1 \over p}$, $r = 1$. Given a sequence $\{a^n\}_{n \in \overline{\mathbb{N}}}$ of periodic continuous bounded functions on $[0, T] \times \mathbb{S}$ with $\partial_x a^n \in C([0, T]; B^{s-1}_{p,r})$ and for some $\alpha(t) \in L^1([0, T])$
\[
||\partial_x a^n||_{B^{s-1}_{p,r}} \leq \alpha(t), \quad \text{for all } t \in [0, T], \quad n \in \mathbb{N}.
\]
Assume that $\{v^n\}_{n \in \mathbb{N}} \in L^\infty([0, T]; B^{s-1}_{p,r})$ is the solution of
\[
\begin{align*}
\partial_t v^n + a^n \partial_x v^n &= f, \\
v^n|_{t=0} &= v_0, \\
v^n(t, x + 1) &= v^n(t, x),
\end{align*}
\]
with $v_0 \in B^{s-1}_{p,r}$, $f \in C([0, T], B^{s-1}_{p,r})$. If $a^n \to a^\infty$ in $L^1([0, T]; B^{s-1}_{p,r})$ as $n \to \infty$, then the sequence $\{v^n\}_{n \in \mathbb{N}}$ tends to $v^\infty$ in $C([0, T], B^{s-1}_{p,r})$ as $n \to \infty$.

**Proof** We first consider $v_0 \in B^{s}_{p,r}$ and $f \in C([0, T], B^{s}_{p,r})$. Note that $r < \infty$. By Lemma 2.1 (iii), we have $\{v^n\}_{n \in \mathbb{N}} \in C([0, T]; B^{s}_{p,r})$ in this particular case. From (3.17), we get
\[
\partial_t (v^n - v^\infty) + a^n \partial_x (v^n - v^\infty) = (a^\infty - a^n) \cdot \partial_x v^\infty.
\]
Applying Lemma 2.1, we have
\[
\begin{align*}
||v^n - v^\infty||_{B^{s-1}_{p,r}} &\leq \int_0^t e^{C f'_r ||\partial_x a^n(\tau)||_{B^{s-1}_{p,r}} d\tau} ||(a^\infty - a^n) \cdot \partial_x v^\infty||_{B^{s-1}_{p,r}} d\tau \\
&\leq \int_0^t e^{C f'_r ||\partial_x a(\tau)||_{B^{s-1}_{p,r}} d\tau} ||a^\infty - a^n||_{B^{s-1}_{p,r}} ||\partial_x v^\infty||_{B^{s-1}_{p,r}} d\tau \\
&\leq \int_0^t e^{C f'_r ||\partial_x a(\tau)||_{B^{s-1}_{p,r}} d\tau} ||a^\infty - a^n||_{B^{s-1}_{p,r}} ||v^\infty||_{B^{s}_{p,r}} d\tau.
\end{align*}
\]
\]

15
Since $a^n \to a^\infty$ in $L^1([0,T]; B^{s-1}_{p,r})$ as $n \to \infty$, we have $v^n \to v^\infty$ in $C([0,T]; B^{s-1}_{p,r})$ as $n \to \infty$.

Now, we will discuss the general case $v_0 \in B^{s-1}_{p,r}$, $f \in C([0,T], B^{s-1}_{p,r})$. For all $n \in \mathbb{N}$ and $q \in \mathbb{N}$, we consider the following equation:

\[
\begin{aligned}
    \partial_t v^n_q + a^n \partial_x v^n_q &= S_q f, \\
    v^n_q|_{t=0} &= S_q v_0, \\
    v^n_q(t, x + 1) &= v^n_q(t, x).
\end{aligned}
\]

(3.18)

On one hand, since all the date $S_q v_0$ and $S_q f$ belongs to $B^{\infty}_{p,r}$, the step above implies

\[
v^n_q \to v^\infty_q \quad \text{in} \quad C([0,T]; B^{s-1}_{p,r}) \quad \text{as} \quad n \to \infty.
\]

(3.19)

On the other hand, for $n \in \mathbb{N}$ and $q \in \mathbb{N}$, subtracting (3.18) from (3.17) gives

\[
\begin{aligned}
    \partial_t (v^n - v^n_q) + a^n \partial_x (v^n - v^n_q) &= f - S_q f, \\
    (v^n - v^n_q)|_{t=0} &= v_0 - S_q v_0, \\
    (v^n - v^n_q)(t, x + 1) &= (v^n - v^n_q)(t, x).
\end{aligned}
\]

It follows that

\[
\begin{aligned}
    \|v^n - v^n_q\|_{B^{s-1}_{p,r}} &\leq e^{C \int_0^t \alpha(\tau) d\tau} \left( \|v_0 - S_q v_0\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C \int_0^{\tau'} \alpha(\tau'') d\tau''} \|f - S_q f\|_{B^{s-1}_{p,r}} d\tau \right).
\end{aligned}
\]

By the definition of $S_q$ in Proposition 2.1 and the Lebesgue dominated convergence theorem, we have for all $n \in \mathbb{N}$,

\[
v^n_q \to v^n \quad \text{in} \quad C([0,T]; B^{s-1}_{p,r}) \quad \text{as} \quad q \to \infty.
\]

(3.20)

Note that

\[
\|v^n - v^\infty\|_{B^{s-1}_{p,r}} \leq \|v^n - v^n_q\|_{B^{s-1}_{p,r}} + \|v^n_q - v^\infty\|_{B^{s-1}_{p,r}} + \|v^\infty - v^\infty\|_{B^{s-1}_{p,r}}.
\]

For fixed $q$ large enough, letting $n$ tend to infinity, then combining (3.19) and (3.20), we have the desired result.

Next, we give the proof of Theorem 3.1.

**Proof** According to Lemma 3.2 (ii), we have that $z^n = (u^n, \rho^n)_{n \in \mathbb{N}}$ converges to some function $z = (u, \rho) \in C([0,T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r})$. Next, we will prove that $z = (u, \rho)$ satisfies Theorem 3.1.

Firstly, we will claim that $z = (u, \rho)$ is indeed a solution of the system (3.2). Obviously, $z = (u, \rho)$ satisfies (3.2) in the sense of $\mathcal{D}'([0,T] \times \mathbb{R})$. Combining (i) and (ii) in Lemma 3.2 and using the interpolation estimate (5) in Proposition 2.3, we have that $z^n = (u^n, \rho^n)_{n \in \mathbb{N}}$ is a Cauchy consequence in $C([0,T]; B^{s'}_{p,r} \times B^{s'-1}_{p,r})$, for any $s' < s$. Moreover,

\[
z^n \to z, \quad \text{as} \quad n \to \infty, \quad \text{in} \quad C([0,T]; B^{s'}_{p,r} \times B^{s'-1}_{p,r}).
\]
Therefore,
\[(u + \gamma_1)u_x + P(D)(2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) \quad \text{and} \quad (u + 2\gamma_2)\rho_x + u_x\rho\]
is continuous to \(z = (u, \rho)\) in \(C([0, T]; B_{p,r}^{s_0-1} \times B_{p,r}^{s_0-2})\). Taking limit in \((T_n)\), we can see that \(z\) solves the system (3.2) in the sense of \(C([0, T]; B_{p,r}^{s'} \times C([0, T]; B_{p,r}^{s'-2})\) for all \(s' < s\). Furthermore, combining
\[
\int_S u^n(x)\,dx = \int_S u_0^n(x)\,dx \to \int_S u_0(x)\,dx = \mu_0
\]
and
\[
\int_S u^n(x)\,dx \to \int_S u(x)\,dx
\]
as \(n \to \infty\), we know that \(u\) satisfies \(\mu(u)_t = 0\).

Secondly, we will prove that \(z \in E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)\). Lemma 3.2 and Proposition 2.2 (4) guarantee that \(z = (u, \rho)\) belongs to \(L^\infty([0, T]; B_{p,r}^s \times B_{p,r}^{s-1})\). It follows that the right-hand side of the equation
\[
u_t - (u + \gamma_1)u_x = P(D)(2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)
\]
belongs to \(L^\infty([0, T]; B_{p,r}^{s_0})\) and the right-hand side of the equation
\[
\rho_t - (u + 2\gamma_2)\rho_x = u_x\rho
\]
belongs to \(L^\infty([0, T]; B_{p,r}^{s_0-1})\). By Lemma 2.1 (iii), we have that \(z = (u, \rho) \in C([0, T]; B_{p,r}^{s'} \times B_{p,r}^{s'-1})\) for any \(s' \leq s\). Using the system (3.2) again, we have \(z \in E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)\).

Thirdly, we will prove the continuity of solution with respect to the initial data. At first, the continuity with respect to the initial data in
\[
C([0, T]; B_{p,r}^{s'} \times B_{p,r}^{s'-1}) \cap C^1([0, T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2}), \quad \forall s' < s
\]
can be obtained by Lemma 3.1 and a simple interpolation argument. Then we will prove that the continuity holds true up to index \(s\). By the argument before, we know there is a \(B_{p,r}^s \times B_{p,r}^{s-1}\)-neighborhood \(B_{z_0}\) of \(z_0 = (u_0, \rho_0)\) and some \(T > 0\) such that for any \(v_0 \in B_{z_0}\), the system (3.2) with initial data \(v_0\) has a solution \(v \in C([0, T]; B_{p,r}^{s'} \times B_{p,r}^{s'-1}) \cap C^1([0, T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2})\).

For \(n \in \mathbb{N}\), consider a sequence of data \(z_0^n = (u_0^n, \rho_0^n) \in B_{z_0}\) satisfying \(z_0^n \to z_0^\infty := z_0\) in \(B_{p,r}^s \times B_{p,r}^{s-1}\). Then we have the corresponding solutions \(z^n = (u^n, \rho^n) \in C([0, T]; B_{p,r}^s \times B_{p,r}^{s-1}) \cap C^1([0, T]; B_{p,r}^{s-1} \times B_{p,r}^{s-2})\) satisfy
\[
(E_n)
\]
\[
\begin{align*}
  u_t^n - (u^n + \gamma_1)u_x^n &= P(D)(2\mu_0^n u^n + \frac{1}{2}(u^n_x)^2 + \frac{1}{2}(\rho^n)^2), \\
  \rho_t^n - (u^n + 2\gamma_2)\rho_x^n &= u_x^n\rho^n, \\
  u^n(0, x) &= u_0^n(x), \\
  \rho^n(0, x) &= \rho_0^n(x), \\
  u^n(t, x + 1) &= u^n(t, x), \\
  \rho^n(t, x + 1) &= \rho^n(t, x).
\end{align*}
\]
Next, we will prove
\[ z^n = (u^n, \rho^n) \rightarrow z = (u, \rho) \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r} \times B^{s, -1}_{p, r}) \quad \text{as} \quad n \rightarrow \infty. \]

Since \( u^n \rightarrow u \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r}) \), it suffices to prove that
\[ u^n_x \rightarrow u_x \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r}) \quad \text{and} \quad \rho^n \rightarrow \rho \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r}). \]

For this purpose, differentiating the first equation in \((E_n)\) with respect to \(x\), we have
\[
\begin{align*}
\begin{cases}
(u^n_t)_x - (u^n + \gamma_1)(u^n)_x = F^n, \\
\rho^n_t - (u^n + 2\gamma_2)\rho^n_x = u^n_x \rho^n,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
u^n(0, x) = u_0^n(x), \\
\rho^n(0, x) = \rho_0^n(x), \\
u^n(t, x + 1) = u^n(t, x), \\
\rho^n(t, x + 1) = \rho^n(t, x),
\end{cases}
\end{align*}
\]
where \( F^n = (u^n_t)^2 + \partial_x P(D)(2\mu_0^n u^n + \frac{1}{2}(u^n_x)^2 + \frac{1}{2}(\rho^n)^2) \) with \( \mu_0^n = \int_\Omega u^n_0(x)dx \). Taking \( u^n_x = w^n + v^n \) and \( \rho^n = f^n + g^n \), we have
\[
\begin{align*}
\begin{cases}
w^n_t - (u^n + \gamma_1)w^n_x = F, \\
f^n_t - (u^n + 2\gamma_2)f^n_x = u_x, \\
w^n(0, x) = u_0, \\
f^n(0, x) = \rho_0, \\
w^n(t, x + 1) = w^n(t, x), \\
f^n(t, x + 1) = f^n(t, x),
\end{cases}
\end{align*}
\]
where \( F = u^n_x^2 + \partial_x P(D)(2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) \). By the first system above and Lemma 3.3, we have
\[ w^n \rightarrow w = u_x \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r}) \quad \text{and} \quad f^n \rightarrow \rho \quad \text{in} \quad C([0, T]; B^{s, -1}_{p, r}). \]

By Lemma 3.2 and \( z \in E^{s}_{p, r}(T) \times E^{s, -1}_{p, r}(T) \), we obtain that there is a positive constant \( M \) such that \( \|z^n\|_{B^{s, r}_{p, r} \times B^{s, r}_{p, r}} \leq M \) and \( \|z\|_{B^{s, r}_{p, r} \times B^{s, r}_{p, r}} \leq M \). By the second system above and Lemma 2.1, we have
\[
\|v^n\|_{B^{s, -1}_{p, r}} \leq e^{\text{CMT}} \left( \|u^n_{0x} - u_{0x}\|_{B^{s, -1}_{p, r}} + \int_0^t \|F^n - F\|_{B^{s, -1}_{p, r}} d\tau \right), \tag{3.21}
\]
and
\[
\|g^n\|_{B^{s, -1}_{p, r}} \leq e^{\text{CMT}} \left( \|\rho^n_{0x} - \rho_{0x}\|_{B^{s, -1}_{p, r}} + \int_0^t \|u^n_x \rho^n - u_{x} \rho\|_{B^{s, -1}_{p, r}} d\tau \right). \tag{3.22}
\]
Noticing that for \( s > 1 + \frac{1}{p} \), \( B^{s, -1}_{p, r} \) is an algebra, we have
\[
\|(u^n_x)^2 - u_x^2\|_{B^{s, -1}_{p, r}} \leq \|u^n_x + u_x\|_{B^{s, -1}_{p, r}} \|u^n_x - u_x\|_{B^{s, -1}_{p, r}} \leq 2M \|u^n_x - u_x\|_{B^{s, -1}_{p, r}}.
\]
By the property of $P(D)$, we have
\[
\|\partial_x P(D)(2\mu_0^n u^n - 2\mu_0 u)\|_{B^{s-1}_{p,r}} \\
\leq \|2\mu_0^n u^n - 2\mu_0 u\|_{B^{s-1}_{p,r}} \\
\leq 2|\mu_0^n - \mu_0|\|u^n\|_{B^{s}_{p,r}} + |\mu_0|\|u^n - u\|_{B^{s-1}_{p,r}} \\
\leq 2M|\mu_0^n - \mu_0| + |\mu_0|\|u^n - u\|_{B^{s-1}_{p,r}} ,
\]
\[
\|\partial_x P(D)(\frac{1}{2}(u_x^n)^2 - \frac{1}{2}u_x^2)\|_{B^{s-1}_{p,r}} \leq \frac{1}{2}\|u_x^n)^2 - u_x^2\|_{B^{s-1}_{p,r}} \leq M\|u_x^n - u_x\|_{B^{s-1}_{p,r}} ,
\]
\[
\|\partial_x P(D)(\frac{1}{2}(\rho^n)^2 - \frac{1}{2}\rho^2)\|_{B^{s-1}_{p,r}} \leq \frac{1}{2}\|\rho^n)^2 - \rho^2\|_{B^{s-1}_{p,r}} \\
\leq \frac{1}{2}|\rho^n + \rho|_{B^{s-1}_{p,r}}\|\rho^n - \rho\|_{B^{s-1}_{p,r}} \\
\leq M\|\rho^n - \rho\|_{B^{s-1}_{p,r}} .
\]

It then follows that
\[
\|F^n - F\|_{B^{s-1}_{p,r}} \\
\leq 3M\|u_x^n - u_x\|_{B^{s-1}_{p,r}} + M\|\rho^n - \rho\|_{B^{s-1}_{p,r}} \\
+ 2M|\mu_0^n - \mu_0| + |\mu_0|\|u^n - u\|_{B^{s-1}_{p,r}} \\
\leq 3M\|v^n\|_{B^{s-1}_{p,r}} + 3M\|u^n - u_x\|_{B^{s-1}_{p,r}} + M\|g^n\|_{B^{s-1}_{p,r}} \\
+ M\|f^n - \rho\|_{B^{s-1}_{p,r}} + 2M|\mu_0^n - \mu_0| + |\mu_0|\|u^n - u\|_{B^{s-1}_{p,r}} .
\]

(3.23)

Moreover,
\[
\|u_x^n\rho^n - u_x\|_{B^{s-1}_{p,r}} \\
\leq \|(u_x^n - u_x)\rho^n\|_{B^{s-1}_{p,r}} + \|u_x(\rho^n - \rho)\|_{B^{s-1}_{p,r}} \\
\leq M\|u_x^n - u_x\|_{B^{s-1}_{p,r}} + M\|\rho^n - \rho\|_{B^{s-1}_{p,r}} \\
\leq M\|v^n\|_{B^{s-1}_{p,r}} + M\|u^n - u_x\|_{B^{s-1}_{p,r}} + M\|g^n\|_{B^{s-1}_{p,r}} + M\|f^n - \rho\|_{B^{s-1}_{p,r}} .
\]

(3.24)

Combining (3.21)-(3.24), we get
\[
\|v^n\|_{B^{s-1}_{p,r}} + \|g^n\|_{B^{s-1}_{p,r}} \leq e^{CMT}\{\|u_{0x} - u_{0x}\|_{B^{s}_{p,r}} + \|\rho_0^n - \rho_0\|_{B^{s-1}_{p,r}} \\
+ \int_0^t [4M\|v^n\|_{B^{s-1}_{p,r}} + \|g^n\|_{B^{s-1}_{p,r}}] + 4M\|u^n - u_x\|_{B^{s-1}_{p,r}} \\
+ 2M\|f^n - \rho\|_{B^{s-1}_{p,r}} + 2M|\mu_0^n - \mu_0| + |\mu_0|\|u^n - u\|_{B^{s-1}_{p,r}}]d\tau}.
\]

Note that
\[
z_0^n \to z_0^\infty := z_0 \text{ in } B^{s}_{p,r} \times B^{s-1}_{p,r} ,
\]

19
\[ w^n \rightarrow w = u_x \text{ in } C([0,T]; B^{s-1}_{p,r}) \ , \quad f^n \rightarrow \rho \text{ in } C([0,T]; B^{s-1}_{p,r}), \]

and

\[ u^n \rightarrow u \text{ in } C([0,T]; B^{s-1}_{p,r}). \]

By the Lebesgue dominated convergence theorem, we have

\[
\lim_{n \to \infty} \left( \|v^n\|_{B^{s-1}_{p,r}} + \|g^n\|_{B^{s-1}_{p,r}} \right) \leq 4Me^{CMT} \int_0^t \lim_{n \to \infty} \left( \|v^n\|_{B^{s-1}_{p,r}} + \|g^n\|_{B^{s-1}_{p,r}} \right) d\tau.
\]

Applying Gronwall’s inequality, we have

\[ v^n \rightarrow 0 \text{ in } C([0,T]; B^{s-1}_{p,r}) \quad \text{and} \quad g^n \rightarrow 0 \text{ in } C([0,T]; B^{s-1}_{p,r}), \]

which implies

\[ u^n_x \rightarrow u_x \text{ in } C([0,T]; B^{s-1}_{p,r}) \quad \text{and} \quad \rho^n \rightarrow \rho \text{ in } C([0,T]; B^{s-1}_{p,r}). \]

This completes the proof of Theorem 3.1.

Since the Sobolev space \( H^s = B^s_{2,2} \), Theorem 3.1 implies that if \((u_0, \rho_0) \in H^s \times H^{s-1}\) with \( s > \frac{3}{2} \) and \( s \neq \frac{5}{2} \), we can obtain the local well-posedness to the system (3.2) in \( H^s \times H^{s-1} \) with \( \frac{3}{2} < s \neq \frac{5}{2} \). Combining the corresponding local well-posedness result in [16] (where \( s \geq 2 \) is obtained) and letting \( p = r = 2 \) in Theorem 3.1, we get the following main result of this section:

**Theorem 3.2.** Given \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s > \frac{3}{2}, \) there exists a maximal \( T = T(\|z_0\|_{H^s \times H^{s-1}}) > 0, \) and a unique solution \( z = (u, \rho) \) to the system (3.2)(or (1.2)) such that

\[ z = z(\cdot, z_0) \in C([0,T]; H^s \times H^{s-1}) \cap C^1([0,T]; H^{s-1} \times H^{s-2}). \]

Moreover, the solution depends continuously on the initial data, i.e. the mapping

\[ z_0 \rightarrow z(\cdot, z_0) : H^s \times H^{s-1} \rightarrow C([0,T]; H^s \times H^{s-1}) \cap C^1([0,T]; H^{s-1} \times H^{s-2}) \]

is continuous.

## 4 The precise blow-up scenario

In this section, we present the precise blow-up scenario and global existence for solutions to the system (3.2) in Sobolev spaces.

**Lemma 4.1** ([10], [4]). If \( f \in H^1(S) \) is such that \( \int_S f(x)dx = 0, \) then we have

\[
\max_{x \in S} f^2(x) \leq \frac{1}{12} \int_S f_x^2(x) dx.
\]
Note that \( \int_S (u(t, x) - \mu_0)dx = \mu_0 - \mu_0 = 0 \). By Lemma 4.1, we find that
\[
\max_{x\in S}[u(t, x) - \mu_0]^2 \leq \frac{1}{12} \int_S u_x^2(t, x)dx \leq \frac{1}{12} \mu_1^2.
\]
So we have
\[
\|u(t, \cdot)\|_{L^\infty(S)} \leq |\mu_0| + \frac{\sqrt{3}}{6} \mu_1.
\] (4.1)

Consider now the following initial value problem
\[
\begin{cases}
q_t = u(t, -q) + 2\gamma_2, & t \in [0, T), \\
q(0, x) = x, & x \in \mathbb{R},
\end{cases}
\] (4.2)
where \( u \) denotes the first component of the solution \( z \) to the system (3.2). Then we have the following two useful lemmas.

Similar to the proof of Lemma 4.1 in [20], applying classical results in the theory of ordinary differential equations, one can obtain the following result on \( q \) which is crucial in the proof of blow-up scenarios.

**Lemma 4.2** ([16]). Let \( u \in C([0, T); H^s) \cap C^1((0, T); H^{s-1}), s > \frac{3}{2} \). Then Eq.(4.2) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with
\[
q_s(t, x) = \exp \left(- \int_0^t u_x(s, -q(s, x))ds \right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

**Lemma 4.3** ([16]). Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{3}{2} \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (3.2). Then we have
\[
\rho(t, -q(t, x))q_x(t, x) = \rho_0(-x), \quad \forall (t, x) \in [0, T) \times S.
\] (4.3)
Moreover, if there exists \( M > 0 \) such that \( u_x \leq M \) for all \( (t, x) \in [0, T) \times S \), then
\[
\|\rho(t, \cdot)\|_{L^\infty} \leq e^{MT} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0, T).
\]

Our next result implies that the wave breaking to the system (3.2) is determined only by the slope of \( u \) but not the slope of \( \rho \).

**Theorem 4.1.** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1} \) with \( s > \frac{3}{2} \) and let \( T \) be the maximal
existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to the system (3.2), which is guaranteed by Theorem 3.2. If $T < \infty$, then
\[ \int_0^T \| \partial_x u(\tau) \|_{L^\infty} d\tau = \infty. \]

The proof of the theorem is similar to that of Theorem 4.1 in [13, 19], so we omit it here. Next, we first recall a useful lemma before giving the next result.

**Lemma 4.4** ([5]). Let $t_0 > 0$ and $v \in C^1([0, t_0); H^2(\mathbb{R}))$. Then for every $t \in [0, t_0)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with
\[ m(t) := \inf_{x \in \mathbb{R}} \{ v_x(t, x) \} = v_x(t, \xi(t)), \]
and the function $m$ is almost everywhere differentiable on $(0, t_0)$ with
\[ \frac{d}{dt} m(t) = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, t_0). \]

**Remark 4.1.** If $v \in C^1([0, t_0); H^s(\mathbb{R}))$, $s > \frac{3}{2}$, then Lemma 4.4 also holds true. Meanwhile, Lemma 4.4 works analogously for
\[ M(t) := \sup_{x \in \mathbb{R}} \{ v_x(t, x) \}. \]

Our next result describes the precise blow-up scenario for sufficiently regular solutions to the system (3.2). This result for the system improve considerably the earlier result in [16].

**Theorem 4.2.** Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > \frac{5}{2}$, and let $T$ be the maximal existence time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to the system (3.2) with the initial $z_0$. Then the the maximal existence time $T$ is finite if and only if
\[ \limsup_{t \to T} \sup_{x \in \mathbb{S}} \{ u_x(t, x) \} = +\infty. \]

**Proof** On one hand, by Sobolev’s imbedding theorem it is clear that if
\[ \limsup_{t \to T} \sup_{x \in \mathbb{S}} \{ u_x(t, x) \} = +\infty, \]
then the the maximal existence time $T < \infty$. 

22
On the other hand, assume that the maximal existence time $T$ is finite and there exists a $M > 0$ such that
\[ u_x(t, x) \leq M, \quad \forall (t, x) \in [0, T) \times \mathbb{S}. \] (4.4)

Then, from Lemma 4.3, we have
\[ \|\rho(t, \cdot)\|_{L^\infty} \leq e^{MT} \|\rho_0\|_{L^\infty}, \quad \forall t \in [0, T). \] (4.5)

Let \( m(t) = \min_{x \in \mathbb{S}} \{u_x(t, x)\} \). It follows from Remark 4.1 that there is a point \((t, \xi(t)) \in [0, T) \times \mathbb{S}\) such that \( m(t) = u_x(t, \xi(t)) \). Moreover, \( u_{xx}(t, \xi(t)) = 0 \). Evaluating (3.3) on \((t, \xi(t))\) we get
\[
\frac{d}{dt}m(t) = -2\mu_0 u(t, \xi(t)) + \frac{1}{2} m^2(t) - \frac{1}{2} \rho^2(t, \xi(t)) + a \geq -2\mu_0 u(t, \xi(t)) - \frac{1}{2} \rho^2(t, \xi(t)).
\]

By (4.1) and (4.5), we get
\[
\frac{d}{dt}m(t) \geq -2|\mu_0|(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1) - \frac{1}{2} (e^{MT} \|\rho_0\|_{L^\infty})^2.
\]

Integrating this inequality on \((0, t)\), we have
\[
m(t) \geq m(0) - \left(2|\mu_0|(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1) + \frac{1}{2} (e^{MT} \|\rho_0\|_{L^\infty})^2\right)T.
\]

That is,
\[
\min_{x \in \mathbb{S}} u_x(t, x) \geq \min_{x \in \mathbb{S}} u'_x(0) - \left(2|\mu_0|(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1) + \frac{1}{2} (e^{MT} \|\rho_0\|_{L^\infty})^2\right)T,
\]
which together with (4.4) and \( T < \infty \) implies that
\[
\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty.
\]
This contradicts Theorem 4.1.

Furthermore, if \( \gamma_1 = 2\gamma_2 \), then we get the following sharper conclusion for \( s \).

**Theorem 4.3.** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{3}{2}, \) and let \( T \) be the maximal existence time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to the system (3.2) with the initial \( z_0 \). Assume \( \gamma_1 = 2\gamma_2 \), then the maximal existence time \( T \) is finite if and only if
\[
\lim_{t \to T} \sup_{x \in \mathbb{S}} \{u_x(t, x)\} = +\infty.
\]

23
Proof} On one hand, by Sobolev’s imbedding theorem it is clear that if
\[
\limsup_{t \to T} \sup_{x \in S} \{u_x(t, x)\} = +\infty,
\]
then the maximal existence time \( T < \infty \).

On the other hand, assume that the maximal existence time \( T \) is finite and there exists a \( M > 0 \) such that
\[
u_x(t, x) \leq M, \quad \forall (t, x) \in [0, T) \times S. \tag{4.6}
\]
Then, from Lemma 4.3, we have
\[
\|\rho(t, \cdot)\|_{L^\infty} \leq e^{MT}\|\rho_0\|_{L^\infty}, \quad \forall t \in [0, T). \tag{4.7}
\]
By (4.2) and the condition \( \gamma_1 = 2\gamma_2 \), we have
\[
\frac{du_x(t, -q(t, x))}{dt} = u_{xt}(t, -q(t, x)) - u_{xx}(t, q(t, x))q(t, x)
= (u_x - (u + \gamma_1)u_{xx})(t, -q(t, x)). \tag{4.8}
\]
Evaluating (3.3) on \((t, -q(t, x))\) we get
\[
\frac{du_x(t, -q(t, x))}{dt}
= -2\mu_0 u(t, -q(t, x)) + \frac{1}{2}u_x^2(t, -q(t, x)) - \frac{1}{2}\rho^2(t, -q(t, x)) + a
\geq -2\mu_0 u(t, -q(t, x)) - \frac{1}{2}\rho^2(t, -q(t, x)).
\]
Similar to the proof of Theorem 4.2, we obtain
\[
\min_{x \in S} u_x(t, x) \geq \min_{x \in S} u_x'(0) - \left(2\mu_0(|\mu_0| + \frac{\sqrt{3}}{2}\mu_1) + \frac{1}{2}\left(e^{MT}\|\rho_0\|_{L^\infty}\right)^2\right)T,
\]
which together with (4.6) and \( T < \infty \) implies that
\[
\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty.
\]
This contradicts Theorem 4.1.

Next, we state an improved global existence theorem, which improves the result of the global solutions in [16], where the special case \( s = 2 \) is required. However, the proof of this improved result is the same as the proof of corresponding result in [16], so we omit it here.

**Theorem 4.4.** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{3}{2}, \) and \( T \) be the maximal time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to the system (3.2) with the initial data \( z_0 \). If \( \gamma_1 = 2\gamma_2 \) and \( \rho_0(x) \neq 0 \) for all \( x \in S \), then the corresponding solution \( z \) exists globally in time.
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