A REMARK ON THE MEAN-FIELD DYNAMICS OF MANY-BODY BOSONIC SYSTEMS WITH RANDOM INTERACTIONS

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ABSTRACT. The mean-field limit for the dynamics of bosons with random interactions is rigorously studied. It is shown that, for interactions that are almost surely bounded, the many-body quantum evolution can be replaced in the mean-field limit by a single particle nonlinear evolution that is described by the Hartree equation. This is an Egorov-type theorem for many-body quantum systems with random interactions.

1. Introduction

This work is a modest contribution to the mathematical theory of the mean-field limit for bosons with random interactions. There has been substantial developments in the study of the mean-field dynamics of bosons with deterministic interactions. Early results were proven by Hepp in [1], see also [2]. A different approach based on the reduced density matrix was developed in [3] and was substantially extended to more general potentials and to the derivation of the Gross-Pitaevskii equation in [4], [5], [6], [7], [8]. Recently, a new approach was developed in [9], which gives convergence estimates in the mean-field limit that are uniform in Planck’s constant $\hbar$, see also [10].

While the mean-field dynamics for bosons with deterministic interactions has attracted considerable interest, the question of the mean-field dynamics of bosons with random interactions has not been addressed, yet. Many-body bosonic systems with random interactions are relevant to concrete physical systems, such as inhomogeneous nonlinear optical media, or Bose-Einstein experiments where irregular fluctuations in currents inside conductors close to the condensate induce via Feshbach resonances inhomogeneous interactions between the bosons, see [11] for a description of the latter; also [12] and references therein. Here, we give a simple recipe for extending the deterministic mean-field analysis to the case of random interactions (and in the presence of a random potential).

1.1. The model. Consider the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, such that the probability space $\Omega$ has a generic point $\omega$ and is endowed with measure $\mu$. Define on this space the random field

$$v(x, \omega) : \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R},$$
such that \( v \) is measurable in \( x \in \mathbb{R}^3 \) and \( \omega \in \Omega \), and is almost surely in \( L^\infty(\mathbb{R}^3) \), i.e. there exists \( \Omega_0 \subset \Omega \) such that \( \mu(\Omega_0) = 1 \) and, for all \( \omega \in \Omega_0 \), \( v(\cdot, \omega) \in L^\infty(\mathbb{R}^3) \).

A concrete example of \( v \) that satisfies the above conditions is \( v(x, \omega) = v_1(x) + v_2(x, \omega) \), such that \( v_1 \in L^\infty \) and \( v_2 \) is Gaussian with finite mean and variance. For a measurable and integrable function \( f \) on \( \Omega \), we define the expectation value of \( f \) as

\[
\mathbb{E}(f) := \int f(\omega) \mu(d\omega).
\]

We consider the \( N \)-body random Schrödinger operator

\[
H^N \equiv H^N_\omega := - \sum_{i=1}^N \Delta_i + \frac{1}{N} \sum_{1 \leq i < j \leq N} v(x_i - x_j, \omega),
\]

where \( \Delta = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} \) is the 3-dimensional Laplacian and \( \omega \in \Omega \). Here, we work in units where Planck’s constant \( \hbar = 1 \) and the mass of each particle is \( m = \frac{1}{2} \).

We note that the analysis below is uniform in \( \hbar \). The Hamiltonian \( H^N \) acts on the Hilbert space \( \mathcal{H}^N := L^2_S(\mathbb{R}^{3N}) \), the symmetrization of \( L^2(\mathbb{R}^{3N}) \), which is the space of pure states for a system of \( N \) nonrelativistic bosons.

The quantum dynamics of the \( N \)-body system is described by the Schrödinger equation

\[
i \partial_t \Psi^N(t) = H^N \Psi^N(t),
\]

with an initial condition \( \Psi^N(t = 0) = \Psi^{N,0} \in L^2_S(\mathbb{R}^{3N}) \).

Together with the dynamics defined above, the \( N \)-body system is described by a kinematical algebra of “observables”. For \( p \leq N \), a \( p \)-particle observable is described by an operator \( a^{(p)} \in \mathcal{B}(\mathcal{H}^{(p)}) \), where \( \mathcal{B}(\mathcal{H}^{(p)}) \) is the algebra of bounded operators on \( \mathcal{H}^{(p)} = L^2_S(\mathbb{R}^{3p}) \). By the nuclear theorem, one can associate with \( a^{(p)} \) a tempered distribution kernel in \( \mathcal{S}'(\mathbb{R}^{3p} \times \mathbb{R}^{3p}) \), \( \alpha^{(p)}(x_1, \ldots, x_p; y_1, \ldots, y_p) := \alpha^{(p)}(x_p; y_p) \), such that

\[
(a^{(p)} \varphi^{(p)})(x_p) = \int_{\mathbb{R}^{3p}} \alpha^{(p)}(x_p; y_p) \varphi^{(p)}(y_p) \, dy_p,
\]

where \( \varphi^{(p)}(y_p) \in L^2_S(\mathbb{R}^{3p}) \). We associate to \( a^{(p)} \) an operator \( A^N(a^{(p)}) \) acting on \( \mathcal{H}^{(N)} \) that is given by

\[
(A^N(a^{(p)})) \Psi(x_1, \ldots, x_N) = \frac{N!}{N^p(N - p)!} (P_S a^{(p)} \otimes I^{(N-p)}) P_S \Psi(x_1, \ldots, x_N),
\]

where \( \Psi(x_1, \ldots, x_N) \in L^2_S(\mathbb{R}^{3N}) \) and \( P_S \) is the projection onto the symmetric subspace \( L^2_S(\mathbb{R}^{3N}) \) of \( L^2(\mathbb{R}^{3N}) \). It follows from (3) and (4) that the map

\[
A^N : \mathcal{B}(\mathcal{H}^{(p)}) \to \mathcal{B}(\mathcal{H}^{(N)}), \quad 1 \leq p \leq N,
\]
is linear, such that
\begin{equation}
\|A^N(a^{(p)})\|_{B(H^N)} \leq \|a^{(p)}\|_{B(H^p)},
\end{equation}
\begin{equation}
A^N(a^{(p)})^* = A^N(a^{(p)*}).
\end{equation}

In the Heisenberg picture, the evolution of $A^N \in B(H^N)$ is given by
\begin{equation}
\alpha^N_t(A^N) := e^{iH^N t} A^N e^{-iH^N t}, \quad t \in \mathbb{R}.
\end{equation}
Since $v$ is almost surely bounded, $H^N$ is almost surely self-adjoint on the symmetrized Sobolev space $H^N_\alpha(\mathbb{R}^{3N})$, and hence the propagator $e^{-iH^N t}$, $t \in \mathbb{R}$, is almost surely unitary. Moreover, it follows from the fact that the pointwise limit of measurable functions is itself measurable, [13], and the Trotter product formula, [14], that
\begin{equation}
\langle \otimes_{j=1}^N \psi_j(x_j), \alpha^N_t(A^N) \otimes_{j=1}^N \psi_j(x_j) \rangle, \quad A^N \in B(H^N), \quad \psi_j \in L^2(\mathbb{R}^3)
\end{equation}
is $\omega$-measurable.

We now introduce the classical evolution. The Hartree equation is given by
\begin{equation}
i\partial_t \psi_t = -\Delta \psi_t + (v \ast |\psi_t|^2) \psi_t,
\end{equation}
with the initial condition $\psi_{t=0} = \phi \in L^2(\mathbb{R}^3)$. It follows from Duhamel’s formula for $\psi_t$ and the fact that $v \in L^\infty$ almost surely, that global solutions of [7] in $L^2$ exist almost surely, such that $\|\psi_t\|_{L^2} = \|\phi\|_{L^2}$ with probability 1, for all $t \in \mathbb{R}$, (see for example [13] for the case when $v \in L^\infty$). It also follows from Duhamel’s formula that the random variable
\begin{equation}
\langle \otimes_{i=1}^p \psi_t, A^{(p)} \otimes_{i=1}^p \psi_t \rangle, \quad A^{(p)} \in B(H^{(p)})
\end{equation}
is $\omega$-measurable.

1.2. **Statement of the main result.** We are in a position to state the main result.

**Theorem 1.** Given $a^{(p)}$, $A^N(a^{(p)})$ and $\alpha^N$ as above, suppose that the initial state of the $N$-body system is a normalized coherent (product) state $\Psi^{N,0}(x_1, \cdots, x_N) = \otimes_{i=1}^N \phi(x_i)$, $\phi \in L^2(\mathbb{R}^3)$. Then, for fixed $t \geq 0$,
\begin{equation}
\lim_{N \to \infty} \mathbb{E}(\langle \Psi^{N,0}, \alpha^N_t(A^N(a^{(p)})) \Psi^{N,0} \rangle) = \mathbb{E}(\langle \otimes_{i=1}^p \psi_t, a^{(p)} \otimes_{i=1}^p \psi_t \rangle),
\end{equation}
where $\psi_t$ satisfies the Hartree equation [7] with initial condition $\psi_{t=0} = \phi$.

We note that the analysis below can be easily extended to study the mean-field dynamics of bosons in a random external potential that is almost surely smooth, polynomially bounded and positive, and to investigate the semi-classical limit of the dynamics under additional assumptions on the decay of the interaction, as in [9]. Furthermore, the analysis below can be applied “in toto” to extend the results of [10] and [16] to the case of random interactions.
2. Proof of Theorem 1

The proof of Theorem 1 follows effectively from an application of the dominated convergence theorem, see [13], and Theorem 1.1 in [9]. In what follows, we drop the explicit dependence on the time \( t \) in the notation, since we fix it.

**Proof.** We introduce the random variables

\[
X_N^{(p)} := \langle \Psi_N^N, \alpha_t^N (A_N^N (a^{(p)})) \Psi_N^N \rangle
\]

and

\[
X^{(p)} := \langle \otimes_{i=1}^p \psi_t, a^{(p)} \otimes_{i=1}^p \psi_t \rangle.
\]

The claim of the theorem is equivalent to the statement

\[
\lim_{N \to \infty} \mathbb{E}(X_N^{(p)}) = \mathbb{E}(X^{(p)}).
\]

We divide the proof of (8) into several steps.

**Step 1. Uniform integrability.** We want to show that

\[
\lim_{\beta \to \infty} \mathbb{E}(|X_N^{(p)}|_{|X_N^{(p)}| \geq \beta}) = 0,
\]

uniformly in \( N \in \mathbb{N} \).

We have from (5) and the fact that the quantum time-evolution is almost surely unitary, that

\[
|X_N^{(p)}| \leq \|a^{(p)}\|_{B(\mathcal{H}^{(p)})} < 2\|a^{(p)}\|_{B(\mathcal{H}^{(p)})} < \infty, \text{ almost surely,}
\]

uniformly in \( N \in \mathbb{N} \). For \( \beta > 0 \), it follows from (10) that

\[
|X_N^{(p)}|_{|X_N^{(p)}| \geq \beta} \leq |X_N^{(p)}| < 2\|a^{(p)}\|_{B(\mathcal{H}^{(p)})} < \infty, \text{ almost surely,}
\]

uniformly in \( N \in \mathbb{N} \). The dominated convergence theorem together with (11) give (9).

**Step 2. Mean-field limit with probability 1.** It follows from the fact that the particle interaction \( v \in L^\infty \) almost surely and Theorem 1.1 in [9] that, for fixed \( t > 0 \),

\[
X_N^{(p)} \xrightarrow{N \to \infty} X^{(p)} \text{ almost surely.}
\]

**Step 3.** It follows from Fatou’s lemma, [13], and (10), that

\[
\mathbb{E}(|X^{(p)}|) \leq \liminf_N \mathbb{E}(|X_N^{(p)}|) \leq \limsup_N \mathbb{E}(|X_N^{(p)}|) < 2\|a^{(p)}\|_{B(\mathcal{H}^{(p)})} < \infty,
\]

uniformly in \( N \in \mathbb{N} \). We also have that

\[
|X^{(p)}|_{|X^{(p)}| \geq \beta} \leq |X^{(p)}|,
\]

which together with (13) and the dominated convergence theorem, imply that

\[
\lim_{\beta \to \infty} \mathbb{E}(|X^{(p)}|_{|X^{(p)}| \geq \beta}) = 0.
\]
Step 4. Convergence as $N \to \infty$. We introduce the random variable
\[ Y_N^{(p)} := |X(p) - X_N^{(p)}|. \]
Note that it suffices to show that $\mathbb{E}(Y_N^{(p)}) \to 0$ as $N \to \infty$, from which (8) follows by the triangular inequality.

It follows from (12), Step 2, that
\[ Y_N^{(p)} \xrightarrow{N \to \infty} 0 \quad \text{almost surely}. \]

We decompose $Y_N^{(p)}$ into two parts,
\[ Y_N^{(p)} = Y_N^{(p),<\beta} + Y_N^{(p),\geq \beta}, \]
where $Y_N^{(p),<\beta} := Y_N^{(p)} 1_{|Y_N^{(p)}|<\beta}$ and $Y_N^{(p),\geq \beta} := Y_N^{(p)} 1_{|Y_N^{(p)}|\geq \beta}$, for $\beta > 0$.

Since $Y_N^{(p),<\beta} < \beta$, (15) together with the dominated convergence theorem imply that
\[ \lim_{N \to \infty} \mathbb{E}(Y_N^{(p),<\beta}) = 0. \]
Furthermore, since
\[ Y_N^{(p),\geq \beta} \leq 2|X(p)|1_{|X(p)|\geq \beta/2} + 2|X_N^{(p)}|1_{|X_N^{(p)}|\geq \beta/2}, \]
it follows from (9) and (14) that
\[ \lim_{\beta \to \infty} \mathbb{E}(Y_N^{(p),\geq \beta}) = 0, \]
uniformly in $N \in \mathbb{N}$.

Given $\epsilon > 0$, (17) implies that there exists a finite $\beta_0 > 0$ such that
\[ \sup_N \mathbb{E}(Y_N^{(p),\geq \beta_0}) < \epsilon/2. \]
Moreover, (16) implies that there exists a positive integer $N_0$ such that, for all $N \geq N_0$,
\[ \mathbb{E}(Y_N^{(p),<\beta_0}) < \epsilon/2. \]
It follows that
\[ \mathbb{E}(Y_N^{(p)}) = \mathbb{E}(Y_N^{(p),<\beta_0}) + \mathbb{E}(Y_N^{(p),\geq \beta_0}) < \epsilon \]
for $N \geq N_0$. Therefore, $\mathbb{E}(Y_N^{(p)}) \xrightarrow{N \to \infty} 0$.

By the triangular inequality,
\[ |\mathbb{E}(X(p)) - \mathbb{E}(X_N^{(p)})| \leq \mathbb{E}(Y_N^{(p)}) \xrightarrow{N \to \infty} 0, \]
which gives the claim of the theorem. \qed
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