DIMENSION OF GRAPHOIDS OF RATIONAL VECTOR-FUNCTIONS

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Abstract. Let \( F \subset \mathbb{R}(x,y) \) be a countable family of rational functions of two variables with real coefficients. Each rational function \( f \in F \) can be thought as a continuous function \( f : \text{dom}(f) \to \mathbb{R} \) taking values in the projective line \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \) and defined on a cofinite subset \( \text{dom}(f) \) of the torus \( \mathbb{R}^2 \). Then the family \( F \) determines a continuous vector-function \( \mathcal{F} : \text{dom}(\mathcal{F}) \to \mathbb{R}^F \) defined on the dense \( G_\delta \)-set \( \text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f) \) of \( \mathbb{R}^2 \). The closure \( \bar{\Gamma}(\mathcal{F}) \) of its graph \( \Gamma(\mathcal{F}) = \{(x,f(x)) : x \in \text{dom}(\mathcal{F})\} \) in \( \mathbb{R}^2 \times \mathbb{R}^F \) is called the graphoid of the family \( \mathcal{F} \). We prove the graphoid \( \bar{\Gamma}(\mathcal{F}) \) has topological dimension \( \dim(\bar{\Gamma}(\mathcal{F})) = 2 \). If the family \( F \) contains all linear fractional transformations \( f(x,y) = \frac{a x + b}{c x + d} \) for \( (a,b) \in \mathbb{Q}^2 \), then the graphoid \( \bar{\Gamma}(\mathcal{F}) \) has cohomological dimension \( \dim_G(\bar{\Gamma}(\mathcal{F})) = 1 \) for any non-trivial 2-divisible abelian group \( G \). Hence the space \( \bar{\Gamma}(\mathcal{F}) \) is a natural example of a compact space that is not dimensionally full-valued and by this property resembles the famous Pontryagin surface.

1. Introduction

Let \( X,Y \) be topological spaces and \( f : \text{dom}(f) \to Y \) be a function defined on a subset \( \text{dom}(f) \subset X \). Such a function \( f \) will be called a partial function on \( X \). The closure \( \bar{\Gamma}(f) \) of the graph

\[ \Gamma(f) = \{(x, f(x)) : x \in \text{dom}(f)\} \]

of \( f \) in the Cartesian product \( X \times Y \) will be called the graphoid of \( f \). The graphoid \( \bar{\Gamma}(f) \) determines a multi-valued function \( \bar{f} : X \to Y \) assigning to each point \( x \in X \) the (possibly empty) subset \( \bar{f}(x) = \{ y \in Y : (x,y) \in \bar{\Gamma}(f) \} \). It is clear that \( \bar{\Gamma}(f) \) coincides with the graph \( \Gamma(f) = \{(x,y) \in X \times Y : y \in \bar{f}(x)\} \) of the multi-valued function \( \bar{f} : X \to Y \). Also it is clear that \( f(x) \in \bar{f}(x) \) for each \( x \in \text{dom}(f) \). The multi-valued function \( \bar{f} \) is called the graphoid extension of the partial function \( f \). The set \( \text{dom}(\bar{f}) = \{ x \in X : \bar{f}(x) \neq \emptyset \} \) will be called the domain of \( \bar{f} \). If the space \( Y \) is compact, then the projection \( \text{pr}_X : \bar{\Gamma}(f) \to X \) is a perfect map \([1] \), which implies that the multi-valued map \( \bar{f} \) is upper semi-continuous in the sense that for any open subset \( U \subset Y \) the preimage \( \bar{f}^{-1}(U) \) is open in \( X \).

In this paper we shall study topological properties of the graphoids of rational vector-functions. By a rational function of \( k \) variables we understand a partial function \( f : \text{dom}(f) \to \mathbb{R}^k \) of the form

\[ f(x_1, \ldots, x_k) = \frac{p(x_1, \ldots, x_k)}{q(x_1, \ldots, x_k)} \]

where \( p \) and \( q \) are two relatively prime polynomials of \( k \) variables. The rational function \( f = \frac{p}{q} \) is defined on the open dense subset

\[ \text{dom}(f) = \mathbb{R}^k \setminus (p^{-1}(0) \cap q^{-1}(0)) \]

of \( \mathbb{R}^k \) and takes its values in the projective real line \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \) (carrying the topology of one-point compactification of the real line \( \mathbb{R} \)).

By \( \mathbb{R}(x_1, \ldots, x_k) \) we denote the field of rational functions of \( k \) variables with coefficients in the field \( \mathbb{R} \) of real numbers. Each rational function \( f \in \mathbb{R}(x_1, \ldots, x_k) \) will be thought as a partial function defined on the open dense subset \( \text{dom}(f) \) of the \( k \)-dimensional torus \( \mathbb{R}^k \) with values in the projective line \( \mathbb{R} \).

By a rational vector-function we understand any family \( \mathcal{F} \subset \mathbb{R}(x_1, \ldots, x_k) \) of rational functions.

If \( \mathcal{F} \) is countable, then the intersection \( \text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f) \) is a dense \( G_\delta \)-set in \( \mathbb{R}^k \). So, \( \mathcal{F} \) can be thought as a partial function

\[ \mathcal{F} : \text{dom}(\mathcal{F}) \to \mathbb{R}^F, \quad F : x \mapsto (f(x))_{f \in \mathcal{F}}. \]

Its graphoid \( \bar{\Gamma}(\mathcal{F}) \) is a closed subset of the compact Hausdorff space \( \mathbb{R}^k \times \mathbb{R}^F \) and its graphoid extension \( \bar{\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^F \) is an upper semi-continuous multi-valued function with \( \text{dom}(\bar{\mathcal{F}}) = \mathbb{R}^k \). For every \( f \in \mathcal{F} \) the composition \( \text{pr}_f \circ \bar{\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^f \) of \( \bar{\mathcal{F}} \) with the projection \( \text{pr}_f : \mathbb{R}^F \to \mathbb{R}, \text{pr}_f : (x_g)_{g \in \mathcal{F}} \mapsto x_f \), coincides with the graphoid extension \( \bar{f} \) of the rational function \( f \).

For uncountable families \( \mathcal{F} \subset \mathbb{R}(x_1, \ldots, x_k) \) this approach to defining \( \bar{\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^F \) does not work properly as \( \text{dom}(\bar{\mathcal{F}}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f) \) can be empty. This problem can be fixed as follows.

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Let $\mathcal{F}$ be a family of partial functions $f : \text{dom}(f) \to Y$ defined on subsets $\text{dom}(f)$ of a topological space $X$. By the graphoid extension of $\mathcal{F}$ we understand the multi-valued function $\bar{\mathcal{F}} : X \to Y$ assigning to each point $x \in X$ the set $\bar{\mathcal{F}}(x)$ of all points $y = (y_f)_{f \in \mathcal{F}} \in Y^X$ such that for any neighborhood $O(x) \subset X$ of the point $x$, any finite subfamily $\mathcal{E} \subset \mathcal{F}$, and neighborhoods $O(y_f) \subset Y$ of the points $y_f$, $f \in \mathcal{E}$, there is a point $x' \in O(x) \cap \bigcap_{f \in \mathcal{E}} \text{dom}(f)$ such that $f(x') \in O(y_f)$ for all $f \in \mathcal{E}$. The graph

$$
\Gamma(\mathcal{F}) = \{(x, y) \in X \times Y^X : y \in \mathcal{F}(x)\}
$$

of the multi-valued function $\bar{\mathcal{F}}$ is called the graphoid of the family $\mathcal{F}$. The set $\text{dom}(\bar{\mathcal{F}}) = \{x \in X : \mathcal{F}(x) \neq \emptyset\}$ is called the domain of $\bar{\mathcal{F}}$.

If the family $\mathcal{F}$ is empty, then $\mathcal{F}^X = Y^\emptyset$ is a singleton and the graphoid $\Gamma(\mathcal{F})$ coincides with $X \times Y^\emptyset$.

It can be shown that for any family of rational functions $\mathcal{F} \subset \mathbb{R}(x_1, \ldots, x_k)$ its graphoid extension $\bar{\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^\mathcal{F}$ has the following properties:

1. $\bar{\mathcal{F}}$ is upper semi-continuous;
2. $\text{dom}(\bar{\mathcal{F}}) = \mathbb{R}^k$;
3. for any subfamily $\mathcal{E} \subset \mathcal{F}$ and the coordinate projection $\text{pr}_E : \mathbb{R}^\mathcal{F} \to \mathbb{R}^E$ the composition $\text{pr}_E \circ \bar{\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^E$ coincides with the graphoid extension $\mathcal{E}$ of $E$;
4. If $\text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{E}} \text{dom}(f)$ is dense in $\mathbb{R}^k$, then $\bar{\mathcal{F}}$ coincides with the graphoid extension of the partial function $\mathcal{F} : \text{dom}(\mathcal{F}) \to \mathbb{R}^\mathcal{F}$.

In this paper we shall consider the following problem.

**Problem 1.1.** Given a family of rational functions $\mathcal{F} \subset \mathbb{R}(x_1, \ldots, x_k)$, study topological (and dimension) properties of the graphoid $\Gamma(\mathcal{F}) \subset \mathbb{R}^k \times \mathbb{R}^\mathcal{F}$ of $\mathcal{F}$.

A precise question: Has $\Gamma(\mathcal{F})$ the topological dimension $\dim(\Gamma(\mathcal{F})) = k$?

This problem was motivated by the problem of studying the topological structure of the space of real places of a field of rational functions, posed in [3] and partly solved in [14], [8]. In this paper we shall answer Problem 1.1 for $k \leq 2$.

In fact, the case $k = 1$ is trivial: each rational function $f \in \mathbb{R}(x)$ admits a continuous extension to $\mathbb{R}$ and can be thought as a continuous function $f : \mathbb{R} \to \mathbb{R}$. Then any family $\mathcal{F} \subset \mathbb{R}(x)$ can be thought as a continuous function $\bar{\mathcal{F}} : \mathbb{R} \to \mathbb{R}^\mathcal{F}$. Its graphoid extension $\bar{\mathcal{F}}$ coincides with $\mathcal{F}$. Consequently, the graphs $\Gamma(\mathcal{F}) = \Gamma(f)$ are homeomorphic to the projective real line $\mathbb{R}$ and hence, $\dim(\Gamma(\mathcal{F})) = \dim(\mathbb{R}) = 1$.

The case of two variables is much more difficult. The following theorem is the main result of this paper and has a long and technical proof that exploits tools of Real and Complex Analysis, Algebraic Geometry, Algebraic Topology, Dimension Theory, General Topology, and Combinatorics. This theorem has been applied in [2] for evaluating the dimension of the space of real places of some function fields.

**Theorem 1.2.** For any family of rational functions $\mathcal{F} \subset \mathbb{R}(x, y)$ its graphoid $\Gamma(\mathcal{F}) \subset \mathbb{R}^2 \times \mathbb{R}^\mathcal{F}$ has covering topological dimension $\dim(\Gamma(\mathcal{F})) = 2$.

This theorem reveals only a part of the truth about the dimension of $\Gamma(\mathcal{F})$. The other part says that for sufficiently rich families $\mathcal{F}$ the graphoid $\Gamma(\mathcal{F})$ has cohomological dimension $\dim_G(\Gamma(\mathcal{F})) = 1$ for any 2-divisible abelian group $G$! So, $\Gamma(\mathcal{F})$ is a natural example of a compact space which is not dimensionally full-valued. A classical example of this sort is the Pontryagin surface: a surface with glued Möbius bands at each point of a countable dense set, see [11] §4.7.

The covering and cohomological dimensions are partial cases of the extension dimension $\dim \Gamma(\mathcal{F})$ defined as follows. We say that the extension dimension of a topological space $X$ does not exceed a topological space $Y$ and write $e$-$\dim(X) \leq Y$ if each continuous map $f : A \to Y$ defined on a closed subspace $A$ of $X$ can be extended to a continuous map $\bar{f} : X \to Y$. By Theorem 3.2.10 of [10], a compact Hausdorff space $X$ has covering dimension $\dim(X) \leq n$ for some $n \in \omega$ if and only if $e$-$\dim(X) \leq S^n$ where $S^n$ stands for the $n$-dimensional sphere.

On the other hand, for a non-trivial abelian group $G$, a compact topological space $X$ has cohomological dimension $\dim_G(X) \leq n$ if and only if $\dim(X) \leq K(G, n)$ where $K(G, n)$ is the Eilenberg-MacLane complex of $G$ (this is a CW-complex having all homotopy groups trivial except for the $n$-th homotopy group $\pi_n(K(G, n))$ which is isomorphic to $G$, see [11] §4.2). It is known [9] that $\dim_G(X) \leq \dim(X)$ for each abelian group $G$ and $\dim(X) = \dim(X)$ for any finite-dimensional compact space $X$. A group $G$ is called 2-divisible if for each $x \in G$ there is $y \in G$ with $y^2 = x$.

Theorem 1.2 is completed by the following

**Theorem 1.3.** If a family of rational functions $\mathcal{F} \subset \mathbb{R}(x, y)$ contains a family of linear fractional transformations

$$
\left\{ \frac{x-a}{y-b} : (a, b) \in D \right\},
$$

for some dense subset $D$ of $\mathbb{R}^2$, then the graphoid $\bar{\mathcal{F}}$ of $\mathcal{F}$ has cohomological dimensions $\dim_G(\Gamma(\mathcal{F})) = \dim(\Gamma(\mathcal{F})) = 2$ and $\dim_G(\Gamma(\mathcal{F})) = 1$ for any non-trivial 2-divisible abelian group $G$.
Theorems 2.2 and 2.3 will be proved in Sections 6 and 7. The main instrument in the proof of these theorems is Theorem 3.1 describing the local structure of the graphoid extension \( \bar{F} \) of a finite family of rational functions \( F \subset \mathbb{R}(x,y) \). Section 2 contains some notation and preliminary information, necessary for the proof of Theorem 3.1.

2. Preliminaries

This section has preliminary character and contains notations and facts necessary for understanding the proof of Theorem 3.1.

2.1. Notation and Terminology. For two points \( a, b \in \mathbb{R}^2 \) by \([a,b] = \{(1-t)a + tb : t \in [0,1] \}\) we shall denote the affine segment connecting \( a \) and \( b \) and by

\[
\alpha_{a,b} : [0,1] \to [a,b], \quad \alpha_{a,b} : t \mapsto (1-t)a + tb,
\]

the corresponding affine map. Let also \([a,b] = [a,b] \setminus \{a,b\}\) be the open segment with the end-points \( a, b \). For a subset \( A \subset \mathbb{R}^2 \) and a real number \( t \) let \( tA = \{ta : a \in A\} \) be a homothetic copy of \( A \). By \( \mathbf{0} = (0,0) \) we denote the origin of the plane \( \mathbb{R}^2 \).

Two points \( a, b \) of a subset \( B \subset \mathbb{R}^2 \setminus \{\mathbf{0}\} \) are called neighbour points of \( B \) if \( a \neq b \) and \( a,b \) are unique points of the set \( B \) that lie in the convex cone \( \{ua + vb : u,v \geq 0\} \). By \( \mathcal{N}(B) \) (resp. \( \mathcal{N}\{B\} \)) we denote the family of ordered pairs \( (a,b) \in B^2 \) (resp. unordered pairs \( (a,b) \subset B \)) of neighbour points of \( B \). This family will often occur in the proof of Theorem 3.1 below, so this is an important notion.

A subset \( A \) of a metric space \((X,d)\) is called an \( \varepsilon \)-net in \( X \) if for each point \( x \in X \) there is a point \( a \in A \) with \( d(x,a) < \varepsilon \). For a point \( z \) of a metric space \((X,d)\) and \( \varepsilon > 0 \) let \( B(z,\varepsilon) = \{x \in X : d(x,z) < \varepsilon\} \), \( \bar{B}(z,\varepsilon) = \{x \in X : d(x,z) \leq \varepsilon\} \), and \( S(z,\varepsilon) = \{x \in X : d(x,z) = \varepsilon\} \) denote respectively the open \( \varepsilon \)-ball, closed \( \varepsilon \)-ball and \( \varepsilon \)-sphere centered at the point \( z \).

A map \( f : X \to Y \) between topological spaces \( X, Y \) is monotone if \( f^{-1}(y) \) is connected for each \( y \in Y \). It is easy to see that for a connected subspace \( X \subset \mathbb{R} \) a function \( f : X \to \mathbb{R} \) is monotone if and only if \( f \) is either non-increasing or non-decreasing.

On the extended real line \( \mathbb{R} = \mathbb{R} \cup \{\infty\} \) we shall consider the metric \( d \) inherited from the complex plane \( \mathbb{C} \) after the identification of \( \mathbb{R} \) with the unit circle \( T = \{z \in \mathbb{C} : |z| = 1\} \) with help of stereographic projection that maps \( T \setminus \{i\} \) onto the real line \( \mathbb{R} \). In the metric \( d \) the extended real line \( \mathbb{R} \) has diameter 2. Observe that each (open or closed) ball in the metric space \((\mathbb{R},d)\) is connected.

By an arc we understand a topological copy of the closed interval \([0,1]\). An arc \( A \in \mathbb{R}^n \) is called a monotone arc if for each \( i \in n \) the coordinate projection \( pr_i : A \to \mathbb{R} \) is a monotone map.

2.2. Pusieux-analytic functions. A function \( \varphi : A \to \mathbb{R} \) defined on a subset \( A \subset \mathbb{R} \) is called analytic if for every \( a \in A \) there are \( \varepsilon > 0 \) and real coefficients \( (c_n)_{n \in \mathbb{Z}} \) such that \( \sum_{n=0}^{\infty} |c_n| \varepsilon^n < \infty \) and for every \( x \in A \) with \( |x-a| < \varepsilon \) we get \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \).

Let \( \varepsilon \) be a positive real number. A function \( \varphi : [0,\varepsilon] \to \mathbb{R} \) is called Pusieux-analytic if \( \varphi([0,\varepsilon]) \) is analytic and there are \( m \in \mathbb{N}, \delta \in (0,\varepsilon) \) and an analytic function \( \psi : [0,\sqrt{\delta}] \to \mathbb{R} \) such that \( \varphi(x) = \psi(\sqrt{x}) \) for all \( x \in [0,\delta] \). The smallest number \( m \) with this property is called the Pusieux denominator of \( \varphi \). In a neighborhood of zero a Pusieux-analytic function \( \varphi(x) \) develops into a series \( \sum_{k=0}^{\infty} c_k x^{\frac{k}{m}} \) called the Newton-Pusieux series of \( \varphi \), see [4, 8.3]. The interval \([0,\varepsilon]\) will be called the domain of the Pusieux analytic function \( \varphi \) and will be denoted by \( \text{dom}(\varphi) \).

The Uniqueness Theorem for analytic functions (see e.g., [13]) implies the following Uniqueness Theorem for Pusieux-analytic functions.

**Theorem 2.1.** Two Pusieux-analytic functions \( f, g : [0,\varepsilon] \to \mathbb{R} \) are equal if and only if the set \( \{x \in [0,\varepsilon] : f(x) = g(x)\} \) is infinite.

The Pusieux analyticity can be also introduced for functions defined on an interval \([-\varepsilon,0]\). Namely, we say that a function \( \varphi : [-\varepsilon,0] \to \mathbb{R} \) is Pusieux-analytic if the function \( \psi : [0,\varepsilon] \to \mathbb{R}, \psi : x \mapsto \varphi(-x) \), is Pusieux analytic.

Two Pusieux analytic function \( \varphi, \varphi^* \) are called conjugate if they have the same Pusieux denominator \( m \) and for some analytic function \( \psi : (\varepsilon,\delta) \to \mathbb{R} \) we get

\[
\{(t^m, \psi(t)) : |t| < \delta\} = \{(x, \varphi(x)) : x \in \text{dom}(\varphi) \cap (-\delta^m, \delta^m)\} \cup \{(x, \varphi^*(x)) : x \in \text{dom}(\varphi^*) \cap (-\delta^m, \delta^m)\}.
\]

It can be shown that the Pusieux denominator \( m \) of two conjugate Pusieux analytic functions \( \varphi, \varphi^* \) is odd if and only if \( \text{dom}(\varphi) \cap \text{dom}(\varphi^*) = \{0\} \).

For example, the Pusieux analytic functions \( \varphi_1 : [-\varepsilon,0] \to \mathbb{R}, \varphi_1 : x \mapsto x^\frac{3}{2}, \) and \( \varphi_1^* : [0,\varepsilon] \to \mathbb{R}, \varphi_1^* : x \mapsto x^\frac{3}{2}, \) are conjugate and have Pusieux denominator 3.

The Pusieux analytic functions \( \varphi_2 : [0,\varepsilon] \to \mathbb{R}, \varphi_2 : x \mapsto x^\frac{3}{2}, \) and \( \varphi_2^* : [0,\varepsilon] \to \mathbb{R}, \varphi_2^* : x \mapsto -x^\frac{3}{2}, \) are conjugate and have Pusieux denominator 2.
Lemma 2.2. If \( \varphi, \varphi^* \) are two conjugate Pusieux analytic functions, then for any rational function \( f \in \mathbb{R}(x, y) \) the limits \( \lim_{x \to 0} f(x, \varphi(x)) \) and \( \lim_{x \to 0} f(x, \varphi^*(x)) \) exist and are equal.

Proof. The lemma is trivial if the rational function \( f \) is constant. If \( f \) is not constant we can write it as the fraction \( f = \frac{p}{q} \) of two relatively prime polynomials \( p \) and \( q \). Observe that for each analytic function \( \psi : [-\delta, \delta] \to \mathbb{R} \) and any \( m \in \mathbb{N} \) the functions \( p(t^m, \psi(t)) \) and \( q(t^m, \psi(t)) \) are analytic and hence develop into Maclaurin series at a neighborhood of zero. This fact can be used to show that a (finite or infinite) limit

\[
\lim_{t \to 0} f(t^m, \psi(t)) = \lim_{t \to 0} \frac{p(t^m, \psi(t))}{q(t^m, \psi(t))}
\]

exists.

Now let \( m \) be the Pusieux denominator of the conjugated Pusieux-analytic functions \( \varphi \) and \( \varphi^* \) and \( \psi : [-\delta, \delta] \to \mathbb{R} \) be an analytic function such that

\[
\{(t^m, \psi(t)) : |t| < \delta \} = \{(x, \varphi(x)) : x \in \text{dom}(\varphi) \cap (-\delta^m, \delta^m)\} \cup \{(x, \varphi^*(x)) : x \in \text{dom}(\varphi^*) \cap (-\delta^m, \delta^m)\}.
\]

It follows that

\[
\lim_{x \to 0} f(x, \varphi(x)) = \lim_{t \to 0} f(t^m, \psi(t)) = \lim_{x \to 0} f(x, \varphi^*(x)).
\]

\( \square \)

2.3. A local structure of a plane algebraic curve. In this section we recall the known description of the local structure of an algebraic curve.

By an algebraic curve we understand a set of the form

\[ p^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}, \]

where \( p \in \mathbb{R}(x, y) \) is a non-zero polynomial of two variables with real coefficient. The polynomial \( p \) in this definition can be also replaced by a non-zero rational function \( r = \frac{p}{q} \) where \( p \) and \( q \) are two relatively prime polynomials. In this case the symmetric difference of the algebraic curves \( r^{-1}(0) = \{(x, y) \in \text{dom}(r) : r(x, y) = 0\} \) and \( p^{-1}(0) \) lies in the intersection \( q^{-1}(0) \cap p^{-1}(0) \), which is finite according to the classical Bézout theorem or [41 6.1] or [13 5.7].

We are going to describe the structure of an algebraic curve \( A \subset \mathbb{R}^2 \) at a neighborhood of zero \( 0 = (0, 0) \).

By \( K = (-1, 1)^2 \) we shall denote the open square with side 2 centered at the origin \( 0 \) of the plane and by \( \tilde{K} \) and \( K_0 \) its closure and its boundary in the plane \( \mathbb{R}^2 \). Let \( K_0 = \tilde{K} \setminus \{0\} \) be the square \( K \) with removed centrum and \( \{\pm1\}^2 = \{-1, 1\}^2 \) be the set of the vertices of the square.

Next, decompose the square \( \tilde{K} \) into four triangles:

\[
\begin{align*}
K_N &= \{(x, y) \in \tilde{K} : |x| \leq y\}, \\
K_W &= \{(x, y) \in \tilde{K} : |y| \leq -x\}, \\
K_S &= \{(x, y) \in \tilde{K} : |x| \leq -y\}, \\
K_E &= \{(x, y) \in \tilde{K} : |y| \leq x\},
\end{align*}
\]

whose indices \( N, W, S, E \) correspond to the directions: North, West, South and East.

A subset \( C \subset \mathbb{R}^2 \) is called an east \( \varepsilon \)-elementary curve if \( C \subset \varepsilon K_E \) and \( C = \{(x, \varphi(x)) : x \in (0, \varepsilon)\} \) for a (unique) Pusieux-analytic function \( \varphi : [0, \varepsilon] \to \mathbb{R} \). The Pusieux denominator \( m \) of \( \varphi \) will be called the Pusieux denominator of \( C \).

An east \( \varepsilon \)-elementary curve \( C \) is drawn on the following picture:
The definitions of north, west, and south $\varepsilon$-elementary curves can be obtained by “rotating” the definition of an east $\varepsilon$-elementary curve.

Namely, let $R_{\frac{\pi}{2}} : (x, y) \mapsto (y, -x)$ be the clockwise rotation of the plane on the angle $\frac{\pi}{2}$. Then $R_{\pi} = R_{\frac{\pi}{2}} \circ R_{\frac{\pi}{2}}$ and $R_{\frac{\pi}{2}} = R_{\pi} \circ R_{\frac{\pi}{2}}$ are the clockwise rotations of the plane by the angles $\pi$ and $\frac{3\pi}{2}$, respectively.

A subset $C \subset \mathbb{R}^2$ is called north (resp. west, south) $\varepsilon$-elementary curve if $R_{\frac{\pi}{2}}(C)$ (resp. $R_{\pi}(C)$, $R_{\frac{3\pi}{2}}(C)$) is an east $\varepsilon$-elementary curve. A subset $C \subset \mathbb{R}^2$ will be called an $\varepsilon$-elementary curve if $C$ is an east, north, west or south $\varepsilon$-elementary curve.

We shall exploit the following fundamental fact describing the local structure of a plane algebraic curve, see [4 §8.3] or [13 §16].

**Theorem 2.3.** For any algebraic curve $A \subset \mathbb{R}^2$ there is $\varepsilon > 0$ such that the intersection $A \cap \varepsilon K_\partial$ has finitely many connected components and each of them is an $\varepsilon$-elementary curve.

For an algebraic curve $A \subset \mathbb{R}^2$ the number $\varepsilon > 0$ satisfying the condition of Theorem 2.3 will be called $A$-small.

For an $A$-small number $\varepsilon$ each connected components of $A \cap \varepsilon K$ is an $\varepsilon$-elementary curve called an $\varepsilon$-branch of $A$. Each $\varepsilon$-branch $C$ of $A$ has a conjugated $\varepsilon$-branch $C^*$ of $A$ defined as follows.

Assume first that the $\varepsilon$-branch $C$ is an east $\varepsilon$-elementary curve. Then $C = \{(x, \varphi(x)) : x \in (0, \varepsilon]\}$ for some Pusieux-analytic function $\varphi : [0, \varepsilon] \to [-\varepsilon, \varepsilon]$ with Pusieux denominator $m$. For the function $\varphi$ there exist a positive $\delta \leq \sqrt{\varepsilon}$ and an analytic function $\psi : [-\delta, \delta] \to [-\varepsilon, \varepsilon]$ such that $\varphi(x) = \psi(\sqrt{\varepsilon})$ for all $x \in [0, \delta]$.

If $m$ is odd, then the formula $\varphi^*(x) = \psi(-\sqrt{\varepsilon})$ determines a Pusieux-analytic function $\varphi^* : [-\delta^m, 0] \to \mathbb{R}$, which is conjugate to $\varphi$. If $m$ is even, then the conjugate function $\varphi^* : [0, \delta^m] \to \mathbb{R}$ is defined by the formula $\varphi^*(x) = \psi(-\sqrt{\varepsilon})$.

We claim that the graph $\{(x, \varphi^*(x)) : x \in \text{dom}(\varphi^*)\}$ lies in some $\varepsilon$-branch $C^*$ of the algebraic curve $A$. Find a polynomial $p \in \mathbb{R}(x, y)$ such that $A = p^{-1}(0)$. Taking into account that $C \subset A$, we conclude that $p(x, \varphi(x)) = 0$ for all $x \in [0, \varepsilon]$ and hence $p(t^m, \psi(t)) = 0$ for all $t \in [0, \delta]$. Taking into account that the formula $f(t) = p(t^m, \psi(t))$ determines an analytic function $f : [-\delta, \delta] \to \mathbb{R}$, which is zero on $[0, \delta]$, we conclude that $f \equiv 0$.

If $m$ is odd, then for every $x \in [-\delta^m, 0] = \text{dom}(\varphi^*)$ and $t = -\sqrt{x}$, we get $p(x, \varphi^*(x)) = p(t^m, \psi(t)) = 0$. If $m$ is even, then for every $x \in [0, \delta^m] = \text{dom}(\varphi^*)$ and $t = -\sqrt{x}$, we get $p(x, \varphi^*(x)) = p(t^m, \psi(t)) = 0$. Therefore the graph $\{(x, \varphi^*(x)) : x \in \text{dom}(\varphi^*) \setminus \{0\}\}$ lies in the algebraic curve $A$ and being a connected subset of $A \cap \varepsilon K_\partial$ lies in a unique branch $C^*$, which is called the conjugate $\varepsilon$-branch of the $\varepsilon$-branch $C$. Observe that the conjugate branch $C^*$ is an east $\varepsilon$-elementary curve if $m$ is even and west if $m$ is odd.

By analogy we can define conjugate branches of north, west and south $\varepsilon$-branches of the algebraic curve $A$. Since the conjugate Pusieux analytic curves are not equal, the conjugated $\varepsilon$-branches of $A$ are disjoint.

So, the intersection $A \cap \varepsilon K_\partial$ decomposes into the union of conjugated branches and hence contains an even number of connected components. This is a crucial observation which will be used in the proof of the inequality $\text{dim} \Gamma(F) \geq 2$ in Theorem 1.2.

## 2.4. Degree of maps between circles.

In this section we recall some basic information about the degree of maps between circles. Since the degree has topological nature, instead of the circle we can consider the boundary $K_\partial$ of the square $K = (-1, 1)^2$ in the plane $\mathbb{R}^2$.

We assume that the reader knows Elements of Singular Homology Theory with coefficients in an abelian group $G$ at the level of Chapter 2 of Hatcher’s monograph [11]. In particular, we assume that the reader knows the definition of the first homology group $H_1(X; G)$ of a topological space $X$ and also that each continuous map $f : X \to Y$ induces a homomorphism $f_* : H_1(X; G) \to H_1(Y; G)$ of the corresponding homology groups. It is well-known that the first homology group $H_1(K_\partial; G)$ of the (topological) circle $K_\partial$ is isomorphic to the group $G$, see [11 p.153]. In particular, for the infinite cyclic group $G = \mathbb{Z}$ the first homology group $H_1(K_\partial; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

Observe that each homomorphism $h : \mathbb{Z} \to \mathbb{Z}$ is of the form $h(n) = d \cdot n$ for some integer number $d$ called the degree of the homomorphism $h$. By the degree of a continuous map $f : K_\partial \to K_\partial$ we understand the degree of the induced homomorphism $f_* : H_1(K_\partial; \mathbb{Z}) \to H_1(K_\partial; \mathbb{Z})$. 


Theorem 3.1. Let $f$ be a map, then the restriction $f|\overline{\partial}K_0 \to K_0$ of $f$ is $\mathbb{Z}_2$-trivial if and only if it has even degree.

Proof. Let $\sigma: [0,1] \to K_0$ be a continuous map such that $\sigma(0) = \sigma(1)$ and $\sigma([0,1] : [0,1] \to K_0)$ is bijective. By [11, 2.23], its homology class $[\sigma]$ is a generator of the homology group $H_1(K_0; \mathbb{Z}_2)$, which is isomorphic to $\mathbb{Z}_2$.

Let $X = \overline{K} \setminus B(\varepsilon, z)$ and $f_\sigma: H_1(X; \mathbb{Z}_2) \to H_1(K_0; \mathbb{Z}_2)$ denote the homomorphism between the first homology groups, induced by the map $f: X \to K_0$. Let $i: K_0 \to X$ denote the identity embedding.

For every $z \in Z$ consider the singular simplex $\sigma_z: [0,1] \to S(z, \varepsilon)$, $\sigma_z: t \mapsto z + \varepsilon \sigma(t)$, whose homology class is a generator of the homology group $H_1(S(z, \varepsilon); \mathbb{Z}_2)$ which is isomorphic to $\mathbb{Z}_2$. Since the composition $f|S(z, \varepsilon): S(z, \varepsilon) \to K_0$ is $\mathbb{Z}_2$-trivial, $f(\sigma_z) = 0$.

It is easy to show that the 1-cycle $\sigma - \sum_{z \in Z} \sigma_z$ is equal to the boundary of some singular 2-chain in $X$. Consequently, $f_\sigma(\sigma_z) = 0$, which means that the map $f|K_0 = f \circ i$ is $\mathbb{Z}_2$-trivial.

3. Resolving the singularity of a rational vector-function

In this section given a finite non-empty family $F \subset \mathbb{R}(x,y)$ thought as a rational vector-function, we study the local structure of its canonical multi-valued extension $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ at a neighborhood of an arbitrary point $(a,b) \in \mathbb{R}^2$.

The principal result of this section is the following structure theorem.

Theorem 3.1. Let $F \subset \mathbb{R}(x,y)$ be a non-empty finite family of rational functions and $\overline{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be its graphoid extension. There is $\varepsilon > 0$ such that for every $\varepsilon \in (0, \varepsilon]$ there is a homeomorphism $h: \overline{\varepsilon K_0} \setminus \frac{\varepsilon}{2}K_0 \rightarrow K_0$ such that

1. $\varepsilon K_0 \subset \text{dom}(F)$,
2. $h|\varepsilon K_0 = \text{id}$,
3. For every $f \in F$ the composition $f \circ h: \varepsilon K_0 \setminus \frac{\varepsilon}{2}K_0 \rightarrow \varepsilon K_0$ has a continuous extension $\tilde{f}_h: \varepsilon K_0 \setminus \frac{\varepsilon}{2}K_0 \rightarrow \text{R}$,
4. The functions $\tilde{f}_h, f \in F$, compose a continuous extension $\overline{F}_h = (\tilde{f}_h)_{f \in F}: \varepsilon K_0 \setminus \frac{\varepsilon}{2}K_0 \rightarrow \mathbb{R}^2$ of $F \circ h$ such that $F_h(\frac{\varepsilon}{2}K_0) = \overline{F}(0)$.
5. There is a finite subset $B_0$ of $\varepsilon K_0$ containing the set $\{-\varepsilon, \varepsilon\}^2$ of vertices of $\varepsilon K$ such that for any neighbor points $a, b$ of $\frac{\varepsilon}{2}B_0$ and every $f \in F$ the restriction $\tilde{f}_h|\{a, b\} \to \mathbb{R}$ is monotone and the image $\tilde{f}_h|\{a, b\}$ lies in one of the segments $[0,1], [-1,0], [1, \infty], [\infty, -1]$ composing the circle $\mathbb{R}$.
6. The set $\overline{F}(0)$ is either a singleton or a finite union of monotone arcs in $\mathbb{R}^2$.
7. For any continuous map $g: \overline{F}(0) \rightarrow K_0$ the composition $g \circ \tilde{F}_h: \varepsilon K_0 \setminus \frac{\varepsilon}{2}K_0 \rightarrow K_0$ is $\mathbb{Z}_2$-trivial.

Proof. We lose no generality assuming that all functions $f \in F$ are not constant. Observe that for each rational function $f = \frac{p}{q} \in F \subset \mathbb{R}(x,y)$ the set $\mathbb{R}^2 \setminus \text{dom}(f) \subset p^{-1}(0) \cap q^{-1}(0)$ is finite according to the classical theorem of Bézout [4, 6.1] (which says that for two relatively prime polynomials $p, q \in \mathbb{R}(x,y)$ the algebraic curves $p^{-1}(0)$ and $q^{-1}(0)$ have finite intersection). This implies that the set $\text{dom}(F) = \bigcap_{f \in F} \text{dom}(f)$ is cofinite in $\mathbb{R}^2$ (i.e., has finite complement in $\mathbb{R}^2$).

In the family $F$ consider the subfamilies:

- $F_x$ of rational functions $f \in F$ with non-zero partial derivative $f_x = \frac{\partial f}{\partial x}$,
- $F_y$ of rational functions $f \in F$ with non-zero partial derivative $f_y = \frac{\partial f}{\partial y}$.

Let $C_0 = \{0, 1, -1, \infty\}$, $X = \{(x,y) \in \mathbb{R}^2 : x^2 = y^2\}$, and consider the algebraic curve $A_0 = X \cup \bigcup_{f \in F_x} f_x^{-1}(0) \cup \bigcup_{f \in F_y} f_y^{-1}(0) \cup \bigcup_{f \in F} f^{-1}(C_0)$. 
Using Theorem 2.3 choose an $A_0$-small number $\varepsilon \in (0, 1)$ such that $\varepsilon K_0 \subset \text{dom}(F)$. For this number $\varepsilon$ the intersection $A_0 \cap \varepsilon K_0$ decomposes into even number of pairwise disjoint $\varepsilon$-elementary curves. Since the set $\mathbb{R}^2 \setminus \text{dom}(F)$ is finite, we can assume that $\varepsilon$ is so small that $\varepsilon K_0 \subset \text{dom}(F)$. Now, given any real number $\varepsilon \in (0, \varepsilon]$ we shall construct a homeomorphism $h : \varepsilon K \setminus \frac{1}{2} K \to \varepsilon K_0$ that satisfies the conditions (1)–(7) of Theorem 5.1.

Let us recall that $d$ stands for the metric on the extended real line $\bar{\mathbb{R}}$ identified with the unit circle in the complex plane via the stereographic projection. This metric induced the max-metric

$$d^F((x_f, y_f)) = \max_{f \in F} d(x_f, y_f)$$

on the $F$-torus $\bar{\mathbb{R}}^F$.

Using Theorem 2.3 by induction we can construct a sequence of algebraic curves $(A_n)_{n=1}^{\infty}$, a sequence of real numbers $(\varepsilon_n)_{n=1}^{\infty}$ and a sequence of finite subsets $(C_n)_{n\in \omega}$ of $\bar{\mathbb{R}}$ such that for every $n \in \mathbb{N}$ the following conditions hold:

1. $0 < \varepsilon_n < \min\{\varepsilon_{n-1}, 2^{-n}\}$;
2. the number $\varepsilon_n$ is $A_n$-small;
3. the set $C_{n+1}$ contains $C_n$ and is a finite $2^{-n}$-net in $(\bar{\mathbb{R}}, d)$;
4. $A_{n+1} = A_n \cup \bigcup_{f \in F} f^{-1}(C_{n+1})$.

Let $\varepsilon_0 = \varepsilon$. Now we are ready to construct a homeomorphism $h : \varepsilon K \setminus \frac{1}{2} K \to \varepsilon K_0$ required in Theorem 5.1. This homeomorphism will be recursively defined with help of the algebraic curves $A_n$, $n \in \omega$. For every $n \in \omega$ consider the finite set $B_n = A_n \cap \varepsilon_n K_0$ in the boundary of the square $\varepsilon_n K$. It follows from $X \subset A_0$ that the set $B_0$ contains the set $\{ -\varepsilon_n, \varepsilon_n \}^2$ of vertices of the square $\varepsilon_n K$. For each point $b \in B_n$ there is a unique $\varepsilon_n$-elementary branch $C_b$ of the algebraic curve $A_n$ such that $\{ b \} = C_b \cap A_n$.

For every $n \in \omega$ let $\delta_n = \varepsilon + \varepsilon_n^2$ and observe that $\lim_{n \to \infty} \delta_n = \varepsilon$. Let $h_{-1}$ be the identity map of $\varepsilon K_0$. By induction, for every $n \in \omega$ we shall define a subset $B'_n \subset \delta_n K_0$ and a homeomorphism $h_n : \delta_n K \setminus \delta_{n+1} K \to \varepsilon K \setminus \varepsilon_{n+1} K$ such that:

5. $h_n((\varepsilon + \varepsilon_n^2, \varepsilon_n) K_0) = t K_0$ for each $t \in [\varepsilon_{n+1}, \varepsilon_n]$;
6. $B' = h_{n-1}^{-1}(B_n) \subset \delta_n K_0$;
7. for any $b \in B'_n$ we get $h_n((\varepsilon + \varepsilon_n^2, \varepsilon_n) K_0) = C_b \setminus \varepsilon_{n+1} K$ where $b = h_{n-1}(b) \in B_n$;
8. for any neighbor points $a$, $b$ of $B'_n$ and any $t \in [\varepsilon_n, \varepsilon]$ the map $h_n|_{\{ t \} \times b}$ is affine, which means that $h_n((1-u)t + u b) = (1-u)h_n(ta) + uh_n(t b)$ for all $u \in [0,1]$.

The conditions (6)–(8) imply that for every $n \in \omega$ we get $h_{n-1}|_{\delta_n K_0} = h_n|_{\delta_n K_0}$. So, we can define a homeomorphism $h : \varepsilon K \setminus \frac{1}{2} K \to \varepsilon K_0$ letting $h|_{\varepsilon K \setminus \delta_{n+1} K} = h_n$ for all $n \in \omega$. The properties (5)–(8) of the homeomorphisms $h_n$ imply that the homeomorphism $h$ has the following properties for every $n \in \omega$:

9. $h_n((\varepsilon + \varepsilon_n^2, \varepsilon_n) K_0) = t K_0$ for each $t \in (0, \epsilon]$;
10. $B'_n = h^{-1}(B_n)$;
11. for any $b \in B'_n \subset \delta_n K_0$ we get $h((\varepsilon + \varepsilon_n^2, \varepsilon_n) K_0) = C_b$ where $b = h(b) \in B_n$;
12. for any neighbor points $a$, $b$ of $B'_n$ and any $t \in [\varepsilon_n, \varepsilon]$ the map $h|_{\{ t \} \times b}$ is affine.

Moreover the choice of the algebraic curve $A_0$ guarantees that for any neighbor point $a$, $b \in B_0 = B'_0$, any $t \in (1/2, 1]$, and any function $f \in F$

13. the restriction $f \circ h|_{\{ t \} \times b}$ is either constant or injective (this follows from $f_x^{-1}(0) \cup f_y^{-1}(0) \subset A_0$) and
14. the image $f \circ h|_{\{ t \} \times b}$ lies in one of segments $[0, 1]$, $[-1, 0]$, $[1, \infty]$, $[\infty, -1]$ composing the projective line $\mathbb{R}$ (this follows from $f^{-1}(\{ 0, 1, -1, -\infty \}) \subset A_0$).

Now we shall prove the statements (1)–(7) of Theorem 5.1. In fact, the statements (1) and (2) follow from the choice of $\varepsilon = \varepsilon_0$ and the definition of $h|_{\delta_n K_0} = h_{-1}$. The other statements will be proved in a series of claims and lemmas.

In the following claim (that proves the statement (3) of Theorem 5.1) on the plane $\mathbb{R}^2$ we consider the metric

$$\rho((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$$

generated by the norm $||(x, y)|| = \max\{|x|, |y|\}$. In this metric the square $K$ is just the open unit ball centered at $0$.

Claim 3.2. For every $f \in F$ the map $f \circ h : \varepsilon K \setminus \frac{1}{2} K \to \bar{\mathbb{R}}$ is uniformly continuous and hence admits a continuous extension $\tilde{f}_h : \varepsilon K \setminus \delta_{m+1} K \to \bar{\mathbb{R}}$.

Proof. Given any $\eta > 0$, we should find $\tau > 0$ such that for any two points $x, x' \in \varepsilon K \setminus \frac{1}{2} K$ with $\rho(x, x') < \tau$ we get $d(f \circ h(x), f \circ h(x')) < \eta$. Choose a natural number $m \in \mathbb{N}$ such that $2^{-m+2} < \eta$. By the uniform continuity of the function $f$ on the compact set $\varepsilon K \setminus \delta_{m+1} K$, there exists a real number $\tau_1 > 0$ such that for any points
Lemma 3.7. Let \( (x, x') \in \varepsilon K \setminus \delta_{m+1}K \) with \( \rho(x, x') < \tau_1 \) we have \( d(f \circ h(x), f \circ h(x')) < \eta \). Let \( \tau_2 = \delta_m - \delta_{m+1} \) be equal to the smallest distance between the squares \( \delta_m K_\partial \) and \( \delta_{m+1} K_\partial \).

Now let us consider the finite set \( \frac{1}{2m}B'_m \subset \frac{\varepsilon}{2}K_\partial \) and put

\[
\tau_3 = \min \{ \rho(a', b') : a', b' \in \frac{1}{2m}B'_m, a' \neq b' \}.
\]

We claim that the number \( \tau = \min \{ \tau_1, \tau_2, \tau_3 \} \) has the required property.

The choice of \( \tau \) implies that any two points \( x, x' \in \varepsilon K \setminus \frac{\varepsilon}{2}K \) with \( \rho(x, x') < \tau \) either both lie in \( \varepsilon K \setminus \delta_{m+1}K \) and by the definition of \( \tau_1 \) this implies that \( d(f \circ h(x), f \circ h(x')) < \eta \), or they both lie in the same trapezoid \( T_{ab} \), bounded by the lines \( \delta_m K_\partial, \frac{\varepsilon}{2}K_\partial, \frac{1}{2m+1}a, \frac{1}{2m+1}b \), where \( a, b \) are neighbor points in \( B'_m \), or, at least, in such two adjacent trapezoids. The interior \( T_{ab} \setminus \partial T_{ab} \) of the trapezoid \( T_{ab} \) is a connected set whose image \( h(T_{ab} \setminus \partial T_{ab}) \) does not intersect the algebraic curve \( A_m \) while the image \( f \circ h(T_{ab} \setminus \partial T_{ab}) \) does not intersect the \( 2^{-m+1} \)-net \( C_m \) in \( \mathbb{R} \). Consequently, \( \text{diam} f \circ h(T_{ab}) = \text{diam} f \circ h(T_{ab} \setminus \partial T_{ab}) < 2^{-m+1} \) and \( d(f \circ h(x), f \circ h(x')) \leq 2 \cdot 2^{-m+1} < \eta \). □

The functions \( \tilde{f}_h, f \in \mathcal{F} \), compose a continuous function \( \bar{F}_h = (\tilde{f}_h)_{f \in \mathcal{F}} : \varepsilon K \setminus \frac{\varepsilon}{2}K \to \bar{K}^F \) that extends the composition \( \mathcal{F} \circ h : \varepsilon K \setminus \frac{\varepsilon}{2}K \to \bar{K}^F \). Let \( \tilde{F}_\partial = \bar{F}_h(\frac{\varepsilon}{2}K_\partial) \) be the restriction of \( \bar{F}_h \) onto the boundary square \( \frac{\varepsilon}{2}K_\partial \). Also let \( h : \varepsilon K \setminus \frac{\varepsilon}{2}K \to \mathbb{R} \) be the continuous extension of the homeomorphism \( h \) and observe that \( h^{-1}(0) = \frac{\varepsilon}{2}K_\partial \).

The following claim completes the proof of the statement (4) of Theorem 3.1.

Claim 3.3. \( \bar{F}(0) = \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \) is a Peano continuum.

Proof. First, we are going to show that \( \bar{F}(0) = \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \).

Let \( y \in \bar{F}(0) \). This means that there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset \varepsilon K_\partial \) such that \( \lim_{n \to \infty} (x_n, \mathcal{F}(x_n)) = (0, y) \). By the compactness of \( \varepsilon K \setminus \frac{\varepsilon}{2}K \), the sequence \( \{h^{-1}(x_n)\}_{n \in \mathbb{N}} \subset \varepsilon K \setminus \frac{\varepsilon}{2}K \) contains a subsequence \( \{h^{-1}(x_{n_k})\}_{k \in \mathbb{N}} \) that converges to some point \( z \in \frac{\varepsilon}{2}K_\partial = h^{-1}(0) \). The continuity of the map \( \bar{F}_h \) guarantees that

\[
y = \lim_{k \to \infty} \mathcal{F}(x_{n_k}) = \lim_{k \to \infty} \bar{F}_h(h^{-1}(x_{n_k})) = \bar{F}_h(z) \in \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial).
\]

The converse inclusion \( \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \subset \bar{F}(0) \) is obvious. The equality \( \bar{F}(0) = \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \) and the continuity of \( \bar{F}_\partial \) implies that \( \bar{F}(0) \) is a Peano continuum. □

Now we prove the statement (5) of Theorem 3.1. We recall that \( B'_0 = B_0 = A_0 \cap \varepsilon K_\partial \). The conditions (13), (14) imply the following:

Claim 3.4. For every \( f \in \mathcal{F} \) and neighbor points \( a, b \) of the set \( \frac{1}{2}B_0 \) the map \( \tilde{f}_h|[a, b] \) is monotone and its image lies in one of the segments: \([0, 1]\), \([-1, 0]\), \([1, \infty]\), \([-\infty, -1]\) composing the projective line \( \bar{R} \).

Claim 3.4 implies:

Claim 3.5. For any neighbor points \( a, b \) of the set \( \frac{1}{2}B_0 \), the map \( \bar{F}_\partial|[a, b] \) is monotone and \( \bar{F}_\partial|[a, b] \) is either a singleton or a monotone arc in \( \bar{K}^F \).

Claim 3.5 implies that for any neighbor points \( a, b \) of the set \( \frac{1}{2}B_0 \) and any point \( y \in \bar{F}_\partial|[a, b] \) the preimage \( (\bar{F}_\partial|[a, b])^{-1}(y) \) is either a singleton or an arc. Let

\[
Y_{a,b} = \{ y \in \bar{F}_h|[a, b] : |(\bar{F}_\partial|[a, b])^{-1}(y)| > 1 \}.
\]

Since \([a, b]\) does not contain uncountably many disjoint arcs, the set \( Y_{a,b} \) is at most countable and so is the set

\[
Y = \bigcup \{Y_{a,b} : (a, b) \in \mathcal{N}(\frac{1}{2}B_0)\}.
\]

The definition of the set \( Y \) implies:

Claim 3.6. For each \( y \in \bar{K}^F \setminus Y \) the preimage \( \bar{F}^{-1}_\partial(y) \) is finite.

The last statement of Theorem 3.1 is proved in the following lemma, which is the most difficult part of the proof of Theorem 3.1.

Lemma 3.7. For any map \( g : \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \to \frac{\varepsilon}{2}K_\partial \) the composition \( g \circ \bar{F}_\partial : \frac{\varepsilon}{2}K_\partial \to \frac{\varepsilon}{2}K_\partial \) is \( \mathbb{Z}_2 \)-trivial.

Proof. Since homotopic maps have the same degrees, it suffices to show that the map \( g \circ \bar{F}_\partial \) is homotopic to some \( \mathbb{Z}_2 \)-trivial map \( \bar{g} : \frac{\varepsilon}{2}K_\partial \to \frac{\varepsilon}{2}K_\partial \). The construction of such a map \( \bar{g} \) is rather long and require some preliminary work, in particular, introducing some notation.

We recall that \( \rho \) stands for the max-metric on the plane \( \mathbb{R}^2 \), \( d \) denotes the metric of the projective line \( \bar{K} \). The latter metric induces the max-metric \( d^F \) on the \( \mathcal{F} \)-torus \( \mathbb{R}^F \).

By the uniform continuity of the map \( g \) there is \( \delta > 0 \) such that for any points \( x, y \in \bar{F}(0) = \bar{F}_\partial(\frac{\varepsilon}{2}K_\partial) \) with \( d^F(x, y) \leq \delta \) we get \( \rho(g(x), g(y)) < \varepsilon \).
Find $m \in \mathbb{N}$ such that $2^{-m+1} < \delta$ and consider the $\delta$-net $C_m$ in $\tilde{\mathbb{R}}$ (which appeared in the construction of the homeomorphism $h$). The definition of the set $C_m$ guarantees that $C_0 = \{0, 1, -1, \infty\} \subset C_m$. The set $C_m$ induces the disjoint cover
\[
\mathcal{C} = \{\{c\} : c \in C_m\} \cup \{(a, b) : (a, b) \in \mathcal{N}(C_m)\}
\]
of the projective line $\tilde{\mathbb{R}}$.

It follows that the closure $\bar{C}$ of each set $C \in \mathcal{C}$ is a connected subset that lies in one of the intervals: $[0, 1], [0, -1], [1, \infty], [\infty, -1]$. So, we can endow each segment $\bar{C}$ with the linear order inherited from the extended real line $[-\infty, \infty]$.

Now for each set $C \in \mathcal{C}$ consider the rational homeomorphism $\mu_C : \tilde{\mathbb{R}} \to \tilde{\mathbb{R}}$ defined by formula:
\[
\mu_C(x) = \begin{cases} 
  x & \text{if } C \subset [0, 1]; \\
  x + 1 & \text{if } C \subset [-1, 0); \\
  1 - x^{-1} & \text{if } C \subset (1, \infty); \\
  -x^{-1} & \text{if } C \subset (\infty, -1).
\end{cases}
\]

Observe that $\mu_C(\bar{C}) \subset [0, 1]$ and the restriction $\mu_C|C : \bar{C} \to [0, 1]$ is strictly increasing (with respect to the linear order on $\bar{C}$ inherited from $[-\infty, +\infty]$).

The cover $\mathcal{C}$ induces the disjoint cover
\[
\Pi\mathcal{C}^F = \left\{ \prod_{f \in F} C_f : (C_f)_{f \in F} \in \mathcal{C}^F \right\}
\]
of the $\mathcal{F}$-torus $\tilde{\mathbb{R}}^F$ by cubes of various dimensions. Since $C_m$ is a $\delta$-net, for each cube $C \in \Pi\mathcal{C}^F$ its closure $\bar{C}$ has diameter $< \delta$ (with respect to the metric $d^F$). Consequently, the image $g(\bar{C} \cap \mathcal{F}(0))$ has $\rho$-diameter $\text{diam} g(\bar{C} \cap \mathcal{F}(0)) < \varepsilon$ and hence lies in some topological arc
\[
I_C \subset \varepsilon \frac{1}{2} K_{\bar{\partial}}.
\]

For each cube $C = \prod_{f \in F} C_f \in \Pi\mathcal{C}^F$ consider the embedding
\[
\mu_C : \prod_{f \in F} \bar{C}_f \to [0, 1]^F; \quad \mu_C : (x_f)_{f \in F} \mapsto (\mu_{C_f}(x_f))_{f \in F}.
\]

And now the final portion of definitions and notations which should be digested before the start of the proof of Lemma 3.7

A pair $(a, b)$ of distinct points of $\tilde{\mathbb{R}} K_{\bar{\partial}}$ is called $\mathcal{F}$-admissible if $[a, b] \subset \tilde{\mathbb{R}} K_{\bar{\partial}}$ and for every $f \in F$ there is a (unique) set $C_{a,b}^f \in \mathcal{C}$ such that $\bar{f}_h([a, b]) \subset C_{a,b}^f$ and the restriction $\bar{f}_{h|[a, b]} : [a, b] \to C_{a,b}^f$ is monotone. It is clear that the product
\[
C_{a,b} = \prod_{f \in F} C_{a,b}^f \subset \tilde{\mathbb{R}}^F
\]
is an element of the cover $\Pi\mathcal{C}^F$ of $\tilde{\mathbb{R}}^F$.

For each pair $(a, b)$ of $\mathcal{F}$-admissible points of $\tilde{\mathbb{R}} K_{\bar{\partial}}$ consider the sets
\[
\mathcal{F}_{a,b}^< = \{ f \in \mathcal{F} : \bar{f}_h(a) < \bar{f}_h(b) \}, \\
\mathcal{F}_{a,b}^> = \{ f \in \mathcal{F} : \bar{f}_h(a) > \bar{f}_h(b) \}, \\
\mathcal{F}_{a,b}^\geq = \{ f \in \mathcal{F} : \bar{f}_h(a) = \bar{f}_h(b) \}, \\
\mathcal{F}_{a,b} = \mathcal{F} \setminus \mathcal{F}_{a,b} = \mathcal{F}_{a,b}^< \cup \mathcal{F}_{a,b}^>.
\]

Two $\mathcal{F}$-admissible ordered pairs $(a, b), (a’, b’)$ of neighbor points of the set $\tilde{\mathbb{R}} K_{\bar{\partial}}$ are called $\mathcal{F}$-coherent if
\[
\mathcal{F}_{a,b}^< = \mathcal{F}_{a’,b’}^<, \quad \mathcal{F}_{a,b}^> = \mathcal{F}_{a’,b’}^>, \quad \mathcal{F}_{a,b}^\geq = \mathcal{F}_{a’,b’}^\geq, \quad \text{and } C_{a,b} = C_{a’,b’}.
\]

Two unordered pairs $\{a, b\}, \{a’, b’\}$ of neighbor points of the set $B$ are called $\mathcal{F}$-coherent if the ordered pair $(a, b)$ is $\mathcal{F}$-coherent either to $(a’, b’)$ or to $(b’, a’)$.

It is easy to check that the $\mathcal{F}$-coherence relation is an equivalence relation on the family of $\mathcal{F}$-admissible (un)ordered pairs of points of $\tilde{\mathbb{R}} K_{\bar{\partial}}$.

Claim 3.8. There is a finite subset $D \subset \tilde{\mathbb{R}} K_{\bar{\partial}}$ such that:

1. $\frac{1}{2} B_0 \subset D$.
2. Any pair $(a, b)$ of neighbor points of $D$ is $\mathcal{F}$-admissible.
3. Two unordered pairs $\{a, b\}, \{a’, b’\}$ of neighbor points of $D$ are $\mathcal{F}$-coherent provided $\bar{\mathcal{F}}_{\partial}([a, b]) \cap \bar{\mathcal{F}}_{\partial}([a’, b’]) \neq \emptyset$. 
Proof. For every neighbor points \( a, b \) of the set \( \frac{1}{2}B_0 \), consider the disjoint cover \( D_{a,b} = \{ [a, b] \cap F^{-1}(C) : C \in \Pi F \} \) of the affine interval \([a, b]\) by convex subsets of \([a, b]\). The convexity of the sets of the cover \( D_{a,b} \) follows from the monotonicity of the maps \( f_k \) of the set \( F_{\tilde{\partial}}([a, b]) \) in the affine set \([a, b]\) and its boundary \( \partial D_{a,b} \) in \( \tilde{\mathcal{K}} \), which consists of at most two points.

Claim 3.9. \( \tilde{\partial} (\partial D_{a,b}^{a'}) = \tilde{\partial} (\partial D_{b,b'}^{a'}) \). If the intersection \( \tilde{\partial} (\partial D_{a,b}^{a'}) \cap \tilde{\partial} (\partial D_{b,b'}^{a'}) \) contains more than one point, then doubletons \( \partial D_{a,b}^{a'} \) and \( \partial D_{b,b'}^{a'} \) are \( \mathcal{F} \)-coherent.

Proof. The claim is trivial if the intersection \( Z = \tilde{\partial} (\partial D_{a,b}^{a'}) \cap \tilde{\partial} (\partial D_{b,b'}^{a'}) \) contains at most one point. So, assume that this intersection contains more than one point. It follows that \( Z \subset C_{a,a'} = C_{b,b'} \) and hence for each \( f \in \mathcal{F} \) the sets \( C_{a,a'}^f \) and \( C_{b,b'}^f \) coincide and carry the same linear order.

Choose two points \( y = (y_f)_{f \in \mathcal{F}} \) and \( y' = (y'_f)_{f \in \mathcal{F}} \) in \( Z \subset \mathbb{R}^\mathcal{F} \) for which the set \( \mathcal{F}_{y,y'}^f = \{ f \in \mathcal{F} : y_f \neq y'_f \} \) has maximal possible cardinality. It is clear that \( \mathcal{F}_{y,y'}^f = \mathcal{F}_{y',y}^f \cup \mathcal{F}_{y,y'}^f \), where \( \mathcal{F}_{y,y'}^f = \{ f \in \mathcal{F} : y_f < y'_f \} \) and \( \mathcal{F}_{y,y'}^f = \{ f \in \mathcal{F} : y_f > y'_f \} \).

Choose two points \( x_a, x'_a \in [a, a'] \) such that \( y_a = \tilde{\partial} (x_a) \) and \( y'_a = \tilde{\partial} (x'_a) \). Exchanging the points \( a, a' \) by their places, if necessary, we can assume that the intervals \([a, x_a] \) and \([x'_a, a'] \) have empty intersection. Choose unique points \( z_a \in [a, x_a] \) and \( z'_a \in [x'_a, a'] \) such that \( z_a, z'_a = \partial D_{a,a'}^{b,b'} \). Since \([x_a, x'_a] \subset [a, z'_a] \), the monotonicity of the functions \( f_k \) implies that \( \mathcal{F}_{y,y'} \subset \mathcal{F}_{z_a,z'_a} \) and hence \( \mathcal{F}_{y,y'} = \mathcal{F}_{z_a,z'_a} \) by the maximality of \( \mathcal{F}_{y,y'} \). This fact, combined with the monotonicity of the functions \( f_k \) and the choice of the order of the points \( a, a' \) implies that \( \mathcal{F}_{z_a,z'_a} = \mathcal{F}_{y,y'} \) and \( \mathcal{F}_{z_a,z'_a} = \mathcal{F}_{y,y'} \). Let \( y_a = \tilde{\partial} (z_a) \) and \( y'_a = \tilde{\partial} (z'_a) \).

Now do the same for the pair \([b, b']\); choose two points \( x_b, x'_b \in [b, b'] \) such that \( y_b = \tilde{\partial} (x_b) \) and \( y'_b = \tilde{\partial} (x'_b) \). Replacing the points \( b, b' \) by their places, if necessary, we can assume that the intervals \([b, x_b] \) and \([x'_b, b'] \) have no common points. Choose unique points \( z_b \in [b, x_b] \) and \( z'_b \in [x'_b, b'] \) such that \( z_b, z'_b = \partial D_{b,b'}^{a,a'} \) and let \( y_b = \tilde{\partial} (z_b) \) and \( y'_b = \tilde{\partial} (z'_b) \). It follows that \( \mathcal{F}_{z_b,z'_b} = \mathcal{F}_{z_b,z'_b} \) and \( \mathcal{F}_{z_b,z'_b} = \mathcal{F}_{z_b,z'_b} \).

We claim that \( y_a = y_b \) and \( y'_a = y'_b \). Assume first that \( y_a \neq y_b \). Find a point \( u_a \in [z_a, z'_a] \) such that \( \tilde{\partial} (u_a) = y_a \) and a point \( u_b \in [z_b, z'_b] \) such that \( \tilde{\partial} (u_b) = y_b \). Since \( y_a \neq y_b \), there is a function \( f \in \mathcal{F} \) such that \( \text{pr}_f (y_a) \neq \text{pr}_f (y_b) \) where \( \text{pr}_f : \mathbb{R}^\mathcal{F} \to \mathbb{R} \) denotes the projection onto the \( f \)-th coordinate. On the set \( \text{pr}_f (Z) \) consider a linear order inherited from the set \( C_{a,a'}^f = C_{b,b'}^f \). We lose no generality assuming that \( \text{pr}_f (y_a) < \text{pr}_f (y_b) \). Then by the monotonicity of the function \( f_k || \{ a, a' \} \), we get

\[
\tilde{f}_k (z_a) = \text{pr}_f (y_a) < \text{pr}_f (y_b) = \tilde{f}_k (u_a) \leq \tilde{f}_k (z'_a)
\]

and hence \( f \in \mathcal{F}_{z_a,z'_a} = \mathcal{F}_{y,y'} \). On the other hand, the monotonicity of the function \( f_k || \{ b, b' \} \) implies

\[
\tilde{f}_k (z_b) = \text{pr}_f (y_b) > \text{pr}_f (y_a) = \tilde{f}_k (u_b) \geq \tilde{f}_k (z'_b),
\]

and \( f \in \mathcal{F}_{z_b,z'_b} = \mathcal{F}_{y,y'} \). This is a desired contradiction that proves the equality \( y_a = y_b \). By analogy we can prove the equality \( y'_a = y'_b \).

Now we see that the equality \( \tilde{\partial} (\partial D_{a,a'}^{b,b'}) = \{ y_a, y'_a \} = \{ y_b, y'_b \} = \tilde{\partial} (\partial D_{b,b'}^{a,a'}) \) implies that the doubletons \( \partial D_{a,a'}^{b,b'} \) and \( \partial D_{b,b'}^{a,a'} \) are \( \mathcal{F} \)-coherent.

Define the set \( D_{n+1} \) as the union

\[
D_{n+1} = D_n \cup \{ \partial D_{a,a'}^{b,b'} : (a, a'), (b, b') \in \mathcal{N}(D_n) \}.
\]

For every \( n \in \mathcal{N} \) consider the function \( p_n : \mathcal{N}(D_{n+1}) \to \mathcal{N}(D_n) \) assigning to each ordered pair \((a, a')\) of neighbor points of the set \( D_{n+1} \) a unique ordered pair \((b, b')\) of neighbor points of \( D_n \) such that \([a, a'] \subset [b, b']\) and \([a, b] \cap [a', b'] = \emptyset\). For \( n \leq m \) consider the composition

\[
p^m_n = p_n \circ \cdots \circ p_{m-1} : \mathcal{N}(D_m) \to \mathcal{N}(D_n).
\]
Claim 3.10. There is \( n \in \omega \) such that for any \( m \geq n \) any pair \((a', b') \in \mathcal{N}(D_m)\) is \( \mathcal{F}\)-coherent to the pair \((a, b) = p_{m-1}(a', b')\).

Proof. The proof of this claim relies on the König Lemma [12, 14.2], which says that a tree \( T \) is finite provided each element of \( T \) has finite degree and each branch of \( T \) is finite. Let us recall that a tree is a partially ordered set (poset) \((T, \leq)\) with the smallest element such that for each \( t \in T \), the set \( \{s \in T : s \leq t\} \) is well-ordered by the relation \( \leq \). For each \( t \in T \), the order type of \( \{s \in T : s \leq t\} \) is called the height of \( t \). The height of \( T \) itself is the least ordinal greater than the height of each element of \( T \). The degree of an element \( t \in T \) is the number of immediate successors of \( t \) in \( T \). The root of a tree \( T \) is the unique element of height 0. A branch of a tree \( T \) is a maximal linearly ordered subset of \( T \).

Now consider the tree \( T = \{\emptyset\} \cup \bigcup_{n \in \omega} \mathcal{N}(D_n) \). The partial order on \( T \) is defined as follows. Given two vertices \((a, a') \in \mathcal{N}(D_n)\) and \((b, b') \in \mathcal{N}(D_m)\) of \( T \), we write \((a, a') \leq (b, b')\) if \( n \leq m \) and \((a, a') = p_n(b, b')\). The set \( \emptyset \) is the root of \( T \) and is smaller that any other non-empty element of \( T \). It is clear that each vertex of the tree \( T \) has finite degree.

The monotonicity of the maps \( \tilde{f}_n[|a, a'|] \) for \((a, a') \in T\) implies the following fact:

Claim 3.11. For any two vertices \((a, a') \leq (b, b')\) of the tree \( T \) we get \( \mathcal{F}_{a,a'} = \mathcal{F}_{b,b'} \). Moreover, the pairs \((a, a')\) and \((b, b')\) are \( \mathcal{F}\)-coherent if and only if \( \mathcal{F}_{a,a'} = \mathcal{F}_{b,b'} \).

Now consider the subtree \( T' \subset T \) consisting of the root of \( T \) and all pairs \((a, a') \in \mathcal{N}(D_n) \subset T \) that are not \( \mathcal{F}\)-coherent to some pair \((b, b') \in \mathcal{N}(D_{n+1}) \subset T \) with \( p_n(b, b') = (a, a')\). Claim 3.11 implies that each branch of the tree \( T' \) has finite length \( \leq |\mathcal{F}| + 1 \). By König Lemma, the subtree \( T' \) is finite. Consequently, there is \( n \in \mathbb{N} \) such that \( T' \cap \mathcal{N}(D_m) = \emptyset \) for all \( m \geq n - 1 \). This implies that for every \( m \geq n \), each pair \((a, a') \in \mathcal{N}(D_m)\) is \( \mathcal{F}\)-coherent to the pair \((b, b') = p_{m-1}(a, a')\). This completes the proof of Claim 3.10.

Let \( D = D_{n+1} \) where the number \( n \) is taken from Claim 3.10. It is clear that the set \( D \) satisfies the conditions (1) and (2) of Claim 3.8. The condition (3) is verified in the following claim.

Claim 3.12. Two unordered pairs \( \{a, a'\}, \{b, b'\} \in \mathcal{N}(D) \) of neighbor points of the set \( D = D_{n+1} \) are \( \mathcal{F}\)-coherent if \( \mathcal{F}_h[|a, a'|] \cap \mathcal{F}_h[|b, b'|] \neq \emptyset \).

Proof. We shall consider two cases (and several subcases).

1. The intersection \( \mathcal{F}_h[|a, a'|] \cap \mathcal{F}_h[|b, b'|] \) contains more than one point. By Claim 3.9 the doubletons \( \partial D^b_{a,a'} \) and \( \partial D^a_{b,b'} \) are \( \mathcal{F}\)-coherent. Take a pair of neighbor points \((a_{n+2}, a'_{n+2}) \in \mathcal{N}(D_{n+2})\) such that \( |a_{n+2}, a'_{n+2}| \subset \mathcal{D}^b_{a,a'} \), and \( p_{n+1}(a_{n+2}, a'_{n+2}) = (a, a')\). The choice of the number \( n \) guarantees that the pairs \((a_{n+2}, a'_{n+2})\) and \((a, a')\) are \( \mathcal{F}\)-coherent. Taking into account that \( |a_{n+2}, a'_{n+2}| \subset \operatorname{conv} \mathcal{D}^b_{a,a'} \subset |a, a'| \), we conclude that the pair \((a, a')\) is \( \mathcal{F}\)-coherent to the doubleton \( \partial D^b_{a,a'} \). By analogy we can prove that the pair \((b, b')\) is \( \mathcal{F}\)-coherent to the doubleton \( \partial D^a_{b,b'} \). Now we see that the \( \mathcal{F}\)-coherence of the doubletons \( \partial D^b_{a,a'} \) and \( \partial D^a_{b,b'} \) implies the \( \mathcal{F}\)-coherence of the pairs \( \{a, a'\}\) and \( \{b, b'\}\).

2. The intersection \( \mathcal{F}_h[|a, a'|] \cap \mathcal{F}_h[|b, b'|] \) is a singleton containing a unique point \( y \). If both sets \( \partial D^b_{a,a'} \) and \( \partial D^a_{b,b'} \) are doubletons, then we can use the equality \( \mathcal{F}_h(\partial D^b_{a,a'}) = \{y\} = \mathcal{F}_h(\partial D^a_{b,b'}) \), which implies that the doubletons \( \partial D^b_{a,a'} \) and \( \partial D^a_{b,b'} \) are \( \mathcal{F}\)-coherent and proceed as in the preceding case.

2a. Now assume that \( \partial D^b_{a,a'} \) is a singleton. Let \((a_n, a'_n) = p_{n+1}(a, a')\) and \((b_n, b'_n) = p_{n+1}(b, b')\). The choice of the number \( n \) guarantees that the pair \((a_n, a'_n)\) is \( \mathcal{F}\)-coherent to \((a, a')\) and \((b_n, b'_n)\) is \( \mathcal{F}\)-coherent to \((b, b')\). It follows that the intersection \( \mathcal{F}_h[|a_n, a'_n|] \cap \mathcal{F}_h[|b_n, b'_n|] \supset \mathcal{F}_h[|a, a'|] \cap \mathcal{F}_h[|b, b'|] = \{y\} \) is not empty. If this intersection is a singleton, then the convex set \( D^b_{a,a'} \) is also a singleton (in the opposite case, the set \( D^b_{a,a'} = D^b_{a,a'} \cap |a, a'| \) cannot be a singleton). In this case the singleton \( \partial D^b_{a,a'} = \partial D^b_{a,a'} \) belongs to the set \( D = D_{n+1} \) and is disjoint with the open interval \(|a, a'|\), which contradicts \( y \in \mathcal{F}_h(\mathcal{F}_h[|a, a'|]) \). This proves that the intersection \( \mathcal{F}_h[|a_n, a'_n|] \cap \mathcal{F}_h[|b_n, b'_n|] \) is not a singleton. Proceeding as in the case 1, we can show that the pairs \( \{a_n, a'_n\}\) and \( \{b_n, b'_n\}\) are \( \mathcal{F}\)-coherent and so are the pairs \( \{a, a'\}\) and \( \{b, b'\}\) (which are \( \mathcal{F}\)-coherent to the pairs \( \{a_n, a'_n\}\) and \( \{b_n, b'_n\}\), respectively).

2b. In case \( \partial D^a_{b,b'} \) is a singleton, we can proceed by analogy with the case 2a.

Now we are ready to prove that the composition \( g \circ \mathcal{F}_h \) is homotopic to some a map \( \tilde{g} : \frac{\pi}{2}K_\partial \to \frac{\pi}{2}K_\partial \) of even degree. It suffices to define \( \tilde{g} \) on each segment \(|a, b|\) connecting two neighbor points of the set \( D \).
We recall that by $\mathcal{N}\{D\}$ we denote the family of unordered pairs of neighbor points of the set $D$. The family $\mathcal{N}\{D\}$ decomposes into pairwise disjoint equivalence classes consisting of $\mathcal{F}$-coherent pairs. Denote by $\hat{\mathcal{N}}\{D\}$ the family of these equivalence classes. For each equivalence class $E \in \hat{\mathcal{N}}\{D\}$ let

$$
\hat{E} = \bigcup \{ \{a, b\} : \{a, b\} \in E \} \text{ and } \partial E = \bigcup \{ \{a, b\} : \{a, b\} \in E \}.
$$

It is clear that $\hat{\mathcal{N}}K_\partial = \bigcup \{ \hat{E} : E \in \hat{\mathcal{N}}\{D\} \}$.

For each equivalence class $E \in \hat{\mathcal{N}}\{D\}$ we are going to construct a specific map $\hat{g}_E : \hat{E} \to \hat{\mathcal{N}}K_\partial$ such that $\hat{g}_E|\partial E = g \circ \mathcal{F}_\partial|\partial E$ and $\hat{g}_E$ is homotopic to $g \circ \mathcal{F}_\partial|\hat{E}$. This map $\hat{g}_E$ will have a specific algebraic structure which will help us to evaluate the degree of the unified map $\hat{g} = \bigcup \{ \hat{g}_E : E \in \hat{\mathcal{N}}\{D\} \}$.

So, fix an equivalence class $E \in \hat{\mathcal{N}}\{D\}$. Since any two unordered pairs from $E$ are $\mathcal{F}$-coherent, we can choose a function $\gamma : E \to D^2$ assigning to each unordered pair $\{a, b\} \in E$ one of ordered pairs $(a, b)$ or $(b, a)$ so that for any unordered pairs $\{a, b\}, \{a', b'\} \in \hat{\mathcal{N}}\{D\}$ the ordered pairs $\gamma(\{a, b\})$ and $\gamma(\{a', b'\})$ are $\mathcal{F}$-coherent. Let $\hat{E} = \gamma(E) \subset \mathcal{N}\{D\}$ and $\hat{\mathcal{N}}\{D\} = \{ \hat{E} : E \in \hat{\mathcal{N}}\{D\} \}$. The $\mathcal{F}$-coherence of any two pairs $(a, b), (a', b') \in \hat{E}$ implies that $\mathcal{F}_a^\gamma = \mathcal{F}_a', \mathcal{F}_b^\gamma = \mathcal{F}_b'$, and $\mathcal{C}_a = \mathcal{C}_b$. So, we can put

$$
\mathcal{F}_a^\gamma = \mathcal{F}_a^\gamma, \mathcal{F}_a^\gamma = \mathcal{F}_a^\gamma, \mathcal{F}_b^\gamma = \mathcal{F}_b^\gamma,
$$

$$
\mathcal{C}_a = \mathcal{C}_a, \mathcal{C}_b = \mathcal{C}_b, \mathcal{C}_a = \mathcal{C}_b
$$

where $(a, b) \in \hat{E}$ is any pair. We recall that $I_{(a,b)}$ is an arc in $\mathcal{F}_{(a,b)}$ that contains the set $g(\mathcal{E} \cap \mathcal{F}(0))$.

For every $f \in \mathcal{F}$ consider the number $\varepsilon_f \in \{0, 1, -1\}$ defined by the formula

$$
\varepsilon_f = \begin{cases} 
1 & \text{if } f \in \mathcal{F}_E^\gamma \setminus \mathcal{F}_E^\gamma, \\
0 & \text{if } f \in \mathcal{F}_E^\gamma, \\
-1 & \text{if } f \in \mathcal{F}_E^\gamma.
\end{cases}
$$

Taking into account that the subset $\mu_E \circ \mathcal{F}_\partial(\partial E) \subset \mu_E(C_E) \subset [0, 1]^{\mathcal{F}}$ is finite, it is easy to find a sequence of positive real numbers $(\alpha_f)_{f \in \mathcal{F}}$ such that the linear map

$$
\lambda_E : [0, 1]^{\mathcal{F}} \to \mathbb{R}, \quad \lambda_E : (x_f)_{f \in \mathcal{F}} \mapsto \sum_{f \in \mathcal{F}} \varepsilon_f \alpha_f x_f
$$

is injective on the set $\mu_E \circ \mathcal{F}_\partial(\partial E)$.

Claim 3.13. For each pair $\{a, b\} \in E$ the map $\lambda_E \circ \mu_E$ is injective on the set $\mathcal{F}_\partial[\{a, b\}]$.

Proof. Assume that $\lambda_E \circ \mu_E(y) = \lambda_E \circ \mu_E(y')$ for some points $y = (y_f)_{f \in \mathcal{F}}$ and $y' = (y'_f)_{f \in \mathcal{F}}$ in $\mathcal{F}_\partial[\{a, b\}]$. Choose two points $x, x' \in [a, b]$ such that $y = \mathcal{F}_\partial(x)$ and $y' = \mathcal{F}_\partial(x')$. On the interval $[a, b]$ we consider the linear order such that $a < b$. We lose no generality assuming that $x < x'$ with respect to this order. The monotonicity of the maps $\mathcal{F}_f$, $f \in \mathcal{F}$, imply that

- $\mathcal{F}_f(x) \leq \mathcal{F}_f(x')$ for each $f \in \mathcal{F}_E^\gamma$;
- $\mathcal{F}_f(x) \geq \mathcal{F}_f(x')$ for each $f \in \mathcal{F}_E^\gamma$;
- $\mathcal{F}_f(x) = \mathcal{F}_f(x')$ for each $f \in \mathcal{F}_E^\gamma$.

Taking into account these inequalities, the increasing property of the maps $\mu_C$ and the choice of the numbers $\varepsilon_f$, $f \in \mathcal{F}$, we conclude that

$$
\varepsilon_f \cdot \mu_C(\mathcal{F}_f(x)) \leq \varepsilon_f \cdot \mu_C(\mathcal{F}_f(x')) \quad \text{for all } f \in \mathcal{F}.
$$

Consequently,

$$
\lambda_E \circ \mu_E(y) = \lambda_E \circ \mu_E \circ \mathcal{F}_\partial(x) = \sum_{f \in \mathcal{F}} \alpha_f \varepsilon_f \cdot \mu_C(\mathcal{F}_f(x)) \leq \sum_{f \in \mathcal{F}} \alpha_f \varepsilon_f \cdot \mu_C(\mathcal{F}_f(x')) = \lambda_E \circ \mu_E \circ \mathcal{F}_\partial(x') = \lambda_E \circ \mu_E(y').
$$

Taking into account that $\lambda_E \circ \mu_E(y) = \lambda_E \circ \mu_E(y')$, we conclude that $y_f = \mathcal{F}_f(x) = \mathcal{F}_f(x') = y'_f$ for all $f \in \mathcal{F}_E^\gamma$. For each $f \in \mathcal{F}_E^\gamma$ the function $\mathcal{F}_f|[a, b]$ is constant and hence $y_f = \mathcal{F}_f(x) = \mathcal{F}_f(x') = y'_f$. Consequently, $y = y'$, which means that the map $\lambda_E \circ \mu_E$ is injective on $\mathcal{F}_\partial[\{a, b\}]$. \qed
Let us recall that by $Y$ we denote the countable set of points $y \in \mathbb{R}^F$ with infinite preimage $\bar{F}_y^{-1}(y)$. The following claim plays a crucial role in the proof of Lemma 3.7.

Claim 3.14. For any $y \in \mathbb{R} \setminus \lambda_E \circ \mu_E(\bar{F}_y(\partial E) \cup Y)$ the preimage $D_y = (\lambda_E \circ \mu_E \circ \bar{F}_y|\bar{E})^{-1}(y)$ is finite and contains even number of points.

**Proof.** If $D_y$ is empty, then there is nothing to prove. So, we assume that the set $D_y$ is not empty. Claims 3.13 and 3.4 imply that for any pair $(a, b) \in E$ the intersection $D_y \cap [a, b]$ contains at most one point. This point belongs to the interior $[a, b]$ as $y \notin \lambda_E \circ \mu_E(\bar{F}_y(\partial E))$.

Since $\mu_E \circ \bar{F}_y(\partial E) \subset \mu_E(\bar{C}_E) \subset \{0,1\}^F \subset \mathbb{R}^F$, the preimage $W = (\mu_E \circ \bar{F}_y)^{-1}(\mathbb{R}^F)$ is an open neighborhood of the set $\bar{E} \in \varepsilon K \setminus \frac{1}{2} K$ and

$$\eta_E = \lambda_E \circ \mu_E \circ \bar{F}_y|W : W \to \mathbb{R}$$

is a well-defined continuous map.

Observe that the formula

$$\mathcal{R}(u, v) = \lambda_E \circ \mu_E \circ \bar{F}(u, v) = \sum_{f \in F} \alpha_f \varepsilon_f \cdot \mu_{C_E}(f(u, v))$$

determines a rational function on $\mathbb{R}^2$. We claim that this rational function is not constant. Indeed, the set $D_y$, being not empty, contains some point $x$ which lies in the interval $[a, b]$ for some pair $(a, b) \in E$. Since $\eta_E(x) = y \neq \eta_E(a)$, we can find $t > 1$ such that $ta, tx \in W$ and $\eta_E(ta) \neq \eta_E(tx)$. Now we see that

$$\mathcal{R}(h(ta)) = \eta_E(ta) \neq \eta_E(tx) = \mathcal{R}(h(tx)),$$

which means that the rational function $\mathcal{R}$ is not constant. So, it is legal to consider the plane algebraic curve $A_y = \mathcal{R}^{-1}(y)$.

The choice of the point $y$ guarantees that $y \notin \eta_E(\partial E)$. Since $W$ is an open neighborhood of $\bar{E}$ in $\varepsilon K \setminus \frac{1}{2} K$ and $\lim_{m \to \infty} \delta_m = \frac{1}{2}$, there is a number $m \in \mathbb{N}$ so large that:

- $[1, \frac{2d_E}{\varepsilon}] : \bar{E} \subset W,$
- $y \notin \eta_E([1, \frac{2d_E}{\varepsilon} \cdot \partial E]),$ and
- the number $\varepsilon_m$ is $A_y$-small.

Let $A_y$ denote the family of connected components of the set $A_y \cap \varepsilon_m K$. Since $\varepsilon_m$ is $A_y$-small each set $A \in A_y$ is an $\varepsilon_m$-elementary branch of the algebraic curve $A_y$. By $A^* \in A_y$ we shall denote its conjugate $\varepsilon_m$-branch.

For each $\varepsilon_m$-elementary branch $A \in A_y$ the preimage $B = h^{-1}(A)$ is a curve in the “square annulus” $\delta_m K \setminus \frac{1}{2} K$. Let $B^* = h^{-1}(A^*)$ be the “conjugate” curve to $B = h^{-1}(A)$. Now consider the family $B_y = \{h^{-1}(A) : A \in A_y\}$ that decomposes into pairs of conjugate curves.

Claim 3.15. For any curve $B \in B_y$ and its closure $\bar{B}$ in $\mathbb{R}^2$ the intersection $\bar{B} \cap \frac{1}{2} K$ is a non-empty convex subset of $\frac{1}{2} K$ such that $\bar{B} \cap \bar{E} \subset D_y$. If the intersection $\bar{B} \cap \bar{E}$ is not empty, then it is a singleton.

**Proof.** We lose no generality assuming that the $\varepsilon_n$-elementary curve $A = h(B) \in A_y$ is an east $\varepsilon_m$-elementary curve. The construction of the homeomorphism $h$ guarantees that the curve $B$ coincides with the graph of some continuous function defined on the interval $[\frac{1}{2} \delta_m, \delta_m]$. This implies that the intersection $\bar{B} \cap \frac{1}{2} K$ is a non-empty closed convex subset that lies in the east side $\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]$ of the square $\frac{1}{2} K$. Taking into account that $\eta_E(W \cap B) = \{y\}$, we conclude that $\eta_E(W \cap B) = \{y\}$ and hence $\bar{B} \cap \bar{E} \subset D_y$.

If $\bar{B} \cap \bar{E}$ is not empty, then it is a singleton because $\bar{B} \cap \frac{1}{2} K$ is convex, does not meet the set $\partial E$, and the intersection $\bar{B} \cap \bar{E} \subset D_y$ is finite.

Let

$$B_y^E = \{B \in B_y : \bar{B} \cap \bar{E} \neq \emptyset\} \text{ and } A_y^E = \{h(B) : B \in B_y^E\}.$$

For every $B \in B_y^E$ let $\pi(B)$ be the unique point of the intersection $\bar{B} \cap \bar{E} \subset D_y$.

The following claim completes the proof of Claim 3.14 showing that the set $|D_y| = |B_y^E|$ has even cardinality.

Claim 3.16. 

1. For any pair $(a, b) \in E$ and $t \in (1, \frac{2d_E}{\varepsilon}]$ the segment $[ta, tb]$ meets at most one set $B \in B_y^E$.
- The function $\pi|B_y^E : B_y^E \to D_y$ is bijective.
- For each curve $B \in B_y^E$ its conjugate curve $B^*$ belongs to $B_y^E$, so the cardinality $|B_y^E|$ is even.

**Proof.** Assume that for some pair $(a, b) \in E$ and some $t \in (1, \frac{2d_E}{\varepsilon}]$ the segment $[ta, tb]$ meets two distinct curves $B, B^* \in B_y^E$ at some points $u, u'$, respectively. We lose no generality assuming the points $u, u'$ are ordered so that $[ta, u] \cap [u', tb] = \emptyset$. Since $D \supset \frac{1}{2} B_0$ there are two neighbor points $a_0, b_0 \in \frac{1}{2} B_0$ such that $[a, b] \subset [a_0, b_0]$ and $[a_0, a] \cap [b, b_0] = \emptyset$. 

Now consider the points \( h(u), h(u') \in A_y \cap (t-1)\varepsilon K_\omega \) and observe that \([h(u), h(u')] \subset [h(ta), h(tb)] \subset [h(ta_0), h(tb_0)].\) The property (13) of the set \( B_0 \) guarantees that for every function \( f \in \mathcal{F} \) the restriction \( f[h(ta), h(tb)] \) is constant or is injective. In particular, for each function \( f \in \mathcal{F}_{a,b} \) the restriction \( f[h(u), h(u')] \) is injective. Then the choice of the sign \( \varepsilon_f \), guarantees that \( \varepsilon_f f(h(u)) < \varepsilon_f f(h(u')) \) and then
\[
y = \lambda_E \circ \mu_E \circ \mathcal{F}(h(u)) < \lambda_E \circ \mu_E \circ \mathcal{F}(h(u')) = y,
\]
which is the desired contradiction.

2. First we check that the function \( \pi|B_y^E \) is injective. Assume that \( \pi(B) = \pi(B') \) for two distinct curves \( B, B' \in B_y^E.\) Let \( x \in \pi(B) = \pi(B') \in \bar{E} \cap D_y \) and find an ordered pair \((a, b) \in \bar{E} \) such that \( x \in [a, b] \). The connectedness of the curves \( B \) and \( B' \) implies that for some \( t \in (1, 2\delta_m) \) the segment \([ta, tb] \) intersects both curves \( B \) and \( B' \) which is forbidden by Claim 3.13 1).

Now we prove that the function \( \pi|B_y^E \) is surjective. Fix any point \( x \in D_y \) and find an ordered pair \((a, b) \in \bar{E} \) such that \( x \in [a, b] \). Since \( y \notin \eta_E\{(a, b)\} \), we conclude that \( \mathcal{F}_h(a) \neq \mathcal{F}_h(x) \neq \mathcal{F}_h(b) \). Then the choice of the signs \( \varepsilon_f \), \( f \in \mathcal{F} \), guarantees that \( \eta_E(a) < \eta_E(x) = y < \eta_E(b) \). Choose a number \( t \in (1, 2\delta_m) \) such that
\[
\eta_E(x) - \eta_E(a) = \eta_E(b) - \eta_E(x). 
\]
It follows that the point \( y \) belongs to \( \eta_E\{(ta, tb)\} \) and the segment \([ta, tb] \) meets the preimage \( B = h^{-1}(A) \) of some \( \varepsilon_m \)-branch \( A \) of the algebraic curve \( A_y.\) Taking into account that \( A \) is an \( \varepsilon_m \)-elementary curve and the intervals \([a, ta]\) and \([b, tb]\) do not intersect \( B \), we conclude that the curve \( B \) has a limit point \( \pi(B) \) in the singleton \([a, b] \cap D_y = \{x\}.\)

3. Take any curve \( B \in B_y^E \) and consider its conjugate curve \( B^* \). Choose any point \( x^* \in \bar{B} \cap \frac{1}{2}K_\vartheta.\) For the points \( x = \pi(B) \) and \( x^* \) find pairs \( \{a, b\}, \{a^*, b^*\} \in N(D) \) such that \( x \in [a, b] \) and \( x^* \in [a^*, b^*] \). It follows from \( B \in B_y^E \) that the pair \( \{a, b\} \in E.\) We need to show that the pair \( \{a^*, b^*\} \) also belongs to \( E, \) which means that \( \{a^*, b^*\} \) is \( \mathcal{F} \)-coherent to \( \{a, b\} \). This will follow from Claim 3.4.3 as soon as we check that \( \mathcal{F}_h(x) = \mathcal{F}_h(x^*).\)

Since the curves \( B \) and \( B^* \) are conjugated, their images \( A = h(B) \) and \( A^* = h(B^*) \) are conjugated \( \varepsilon_m \)-branches of the algebraic curve \( A_y.\) Lemma 2.2 implies that
\[
\lim_{A^*_2 \to 0} \mathcal{F}(z) = \lim_{A^*_2 \to 0} \mathcal{F}(z).
\]
Using the continuity of the function \( \mathcal{F}_h \) at the points \( x \) and \( x^* \), we see that
\[
\mathcal{F}_h(x) = \lim_{B \to x} \mathcal{F}_h(u) = \lim_{B \to x} \mathcal{F}(h(u)) = \lim_{B \to x} \mathcal{F}(v) = \lim_{B \to x} \mathcal{F}(v) = \lim_{B \to x} \mathcal{F}(h(u)) = \lim_{B \to x} \mathcal{F}_h(u) = \mathcal{F}_h(x^*). \]
\]

Now we can continue the proof of Lemma 3.7.

Choose a finite subset \( N_E \subset \mathbb{R} \) such that
\begin{itemize}
  \item the convex hull \( \text{conv}(N_E) \) of \( N_E \) contains the compact subset \( \eta_E(\bar{E}) \) of \( \mathbb{R}; \)
  \item \( \eta_E(\partial E) \subset N_E; \)
  \item for any neighbor points \( a, b \) of \( \eta_E(\partial E) \) the interval \([a, b]\) has non-empty intersection with the set \( N_E.\)
\end{itemize}

Fix a continuous map \( \varphi_E : \mathbb{R} \to I_E \subset \frac{1}{2}K_\vartheta \) such that
\begin{itemize}
  \item \( \varphi_E \circ \eta_E(\partial E) = g \circ \mathcal{F}_h(\partial E); \)
  \item for any neighbor points \( a, b \) of the set \( N_E \) the restriction \( \varphi|[a, b] : [a, b] \to I_E \) is injective.
\end{itemize}
Finally, put
\[
\tilde{g}_E = \varphi_E \circ \eta_E : \bar{E} \to I_E \subset \frac{1}{2}K_\vartheta.
\]

Taking into account that \( \tilde{g}_E(\bar{E}) \cup g \circ \mathcal{F}(\bar{E}) \subset I_E \) and \( \tilde{g}_E(\partial E) = g \circ \mathcal{F}(\partial E), \) we see that the maps \( \tilde{g}_E, g \circ \mathcal{F}(E) \) are homotopic by a homotopy \( h_E : \bar{E} \times [0, 1] \to I_E \) such that
\begin{itemize}
  \item \( h_E(x, 0) = \tilde{g}_E(x), \ h_E(x, 1) = g \circ \mathcal{F}(x) \) for all \( x \in \bar{E} \) and
  \item \( h_E(x, t) = \tilde{g}_E(x) = g \circ \mathcal{F}(x) \) for all \( x \in \partial E \) and \( t \in [0, 1]. \)
\end{itemize}

The maps \( \tilde{g}_E, E \in N(D), \) compose a map \( \tilde{g} : \frac{1}{2}K_\vartheta \to \frac{1}{2}K_\vartheta \) defined by \( \tilde{g}|\bar{E} = \tilde{g}_E \) for \( E \in \bar{N}(D). \)

Also, the homotopies \( h_E : \bar{E} \times [0, 1] \to I_E \subset \frac{1}{2}K_\vartheta, \ E \in \bar{N}(D), \) compose a homotopy \( h : \frac{1}{2}K_\vartheta \times [0, 1] \to \frac{1}{2}K_\vartheta \) between the maps \( \tilde{g} \) and \( g \circ \mathcal{F}.\)

The proof of Lemma 3.7 is finished by the following claim.

**Claim 3.17.** The map \( \tilde{g} \) is \( \mathbb{Z}_2 \)-trivial.
Proof. To show that the map \( \hat{g} \) is \( \mathbb{Z}_2 \)-trivial, we shall apply Lemma 2.4. Pick any point \( y_0 \in \frac{\mathbb{Z}}{2} K_0 \) which does not belong to the countable set

\[
\bigcup \{ \tilde{g}_E(\partial E) \cup \varphi(N_E) \cup \varphi_E \circ \lambda_E \circ \mu_E(Y \cap \tilde{C}_E) : E \in \mathcal{N} \{ D \} \}.
\]

For every equivalence class \( E \in \mathcal{N} \{ D \} \) consider the set \( Y_E = \varphi_E^{-1}(y_0) \) which is finite by the choice of the function \( \varphi_E \). By Claim 3.14 for every point \( y \in Y_E \) the preimage \( D_y = (\lambda_E \circ \mu_E \circ F_h)(y) \) is finite and contains even number of points. Since \( y_0 \notin \partial(\tilde{g}(\tilde{E})) \), we get \( D_y \subset \tilde{E} \setminus \partial E \). Then the preimage \( \tilde{g}_E^{-1}(y_0) = \bigcup_{y \in Y_E} D_y \) lies in \( E \setminus \partial E \) and contains even number of points. Unifying these preimages, we conclude that the preimage

\[
\tilde{g}^{-1}(y_0) = \bigcup \{ \tilde{g}_E^{-1}(y_0) : E \in \mathcal{N} \{ D \} \}
\]

has even cardinality and lies in the set \( \frac{\mathbb{Z}}{2} K_0 \setminus D \).

It remains to check that each point \( x \in \tilde{g}^{-1}(y_0) \) has a neighborhood \( U_x \subset \frac{\mathbb{Z}}{2} K_0 \) such that the map \( \hat{g}|U_x \) is monotone. Find two neighbor points \( a, b \) of the set \( D \) such that \( x \in [a, b] \). Let \( E \) be the \( \mathcal{F} \)-coherence class of the pair \( \{a, b\} \). By Claims 3.5 and 3.13 the map \( \varphi_E \circ \lambda_E \circ \mu_E \circ F_h \) is monotone. Now consider the point \( y = \lambda_E \circ \mu_E \circ F_h(x) \) and observe that \( y \notin N_E \) (as \( \varphi_E(y) = y_0 \notin \varphi(E(N_E)) \)). The choice of the function \( \varphi_E \) guarantees that the point \( y \) has a neighborhood \( V_y \subset \mathbb{R} \setminus N_E \) such that the restriction \( \varphi(E)|V_y \) is injective (and hence monotone). Then the neighborhood \( U_x = (\lambda_E \circ \mu_E \circ F_h)(a, b) \) lies in \( V_y \) has the desired property: the restriction \( \hat{g}|U_x = \hat{g}_E|U_x = \varphi_E \circ \lambda_E \circ \mu_E \circ F_h(U_x) \) is monotone.

\[\square\]

4. INVERSE SPECTRA

In the proof of Theorems 1.2 and 1.3 we shall widely use the technique of inverse spectra, see [9, 5]. Formally speaking an inverse spectrum in a category \( \mathcal{C} \) is a contravariant functor \( S : \Sigma \to \mathcal{C} \) from a directed partially ordered set \( \Sigma \) to the category \( \mathcal{C} \). A partially ordered set (briefly a poset) \( \Sigma \) is called directed if for any elements \( \alpha, \beta \in \Sigma \) there is an element \( \gamma \in \Sigma \) such that \( \gamma \geq \alpha \) and \( \gamma \geq \beta \). Each poset \( \Sigma \) can be identified with a category whose objects are elements of \( \Sigma \) and two objects \( \alpha, \beta \in \Sigma \) are linked by a single morphism \( \alpha \rightarrow \beta \) if and only if \( \alpha \leq \beta \).

An inverse spectrum \( S : \Sigma \to \mathcal{C} \) can be written directly as the family \( \{X_{\alpha}, \rho_{\alpha}, \Sigma\} \) consisting of objects \( X_{\alpha} \) of the category \( \mathcal{C} \), indexed by elements \( \alpha \) of the poset \( \Sigma \), and bonding morphisms \( \rho_{\alpha} : X_{\beta} \to X_{\alpha} \) defined for any indices \( \alpha \leq \beta \) in \( \Sigma \), so that for any indices \( \alpha \leq \beta \leq \gamma \) in \( \Sigma \) the following two conditions are satisfied:

- \( \rho_{\alpha} = \rho_{\beta} \circ \rho_{\alpha} \)
- \( \rho_{\alpha} \) is the identity morphism of \( X_{\alpha} \).

Inverse spectra over a poset \( \Sigma \) in a category \( \mathcal{C} \) form a category \( \mathcal{C}_\Sigma \) whose morphisms are natural transformations of functors. In other words, for two inverse spectra \( S = \{X_{\alpha}, \rho_{\alpha}, \Sigma\} \) and \( S' = \{X'_{\alpha}, \pi_{\alpha}, \Sigma\} \) a morphism \( f : S \rightarrow S' \) in \( \mathcal{C}_\Sigma \) is a family of morphisms \( f = \{f_{\alpha} : X_{\alpha} \rightarrow X'_{\alpha} \}_{\alpha \in \Sigma} \) of the category \( \mathcal{C} \) such that for any indices \( \alpha \leq \beta \) in \( \Sigma \) the following square is commutative:

\[
\begin{array}{ccc}
X_{\beta} & \xrightarrow{f_{\beta}} & X'_{\beta} \\
\downarrow{\rho_{\alpha}} & & \downarrow{\pi_{\alpha}} \\
X_{\alpha} & \xrightarrow{f_{\alpha}} & X'_{\alpha}
\end{array}
\]

There is a functor \( (\cdot)_{\Sigma} : \mathcal{C} \rightarrow \mathcal{C}_\Sigma \) assigning to each object \( X \) of \( \mathcal{C} \) the inverse spectrum \( X_{\Sigma} = \{X_{\alpha}, \rho_{\alpha}, \Sigma\} \) where \( X_{\alpha} = X \) and \( \rho_{\alpha} \) is the identity map of \( X \) for all \( \alpha \leq \beta \) in \( \Sigma \). To each morphism \( f : X \rightarrow Y \) of the category \( \mathcal{C} \) the functor \( (\cdot)_{\Sigma} \) assigns the morphism \( f_{\Sigma} = \{f_{\alpha} \}_{\alpha \in \Sigma} \) where \( f_{\alpha} = f \) for all \( \alpha \in \Sigma \).

For an inverse spectrum \( S : \Sigma \rightarrow \mathcal{C} \) its limit is a pair \( (X, p) \) consisting of an object \( X \) of \( \mathcal{C} \) and a morphism \( p = \{p_{\alpha} \}_{\alpha \in \Sigma} : X_{\Sigma} \rightarrow S \) in the category \( \mathcal{C}_\Sigma \) such that for any other pair \( (Z, \pi) \) consisting of an object \( Z \) of \( \mathcal{C} \) and a morphism \( \pi = \{\pi_{\alpha} \}_{\alpha \in \Sigma} : Z_{\Sigma} \rightarrow S \) there is a unique morphism \( f : Z \rightarrow X \) such that \( \pi = p \circ f_{\Sigma} \). This definition implies that a limit \( (X, p) \) of \( S \) if exists, is unique up to the isomorphism. Because of that the space \( X \) is denoted by \( \lim S \) and called the limit of the inverse spectrum \( S \).

In this paper we shall be mainly interested in inverse spectra in the category \( \text{CompEpi} \) of compact Hausdorff spaces and their continuous surjective maps. In this case, each inverse spectrum \( S = \{X_{\alpha}, \rho_{\alpha}, \Sigma\} \) has a limit \((X, p)\) consisting of the closed subspace

\[
X = \{ (x_{\alpha})_{\alpha \in \Sigma} \in \prod_{\alpha \in \Sigma} X_{\alpha} : \rho_{\alpha}(x_{\beta}) = x_{\alpha} \text{ for all } \alpha \leq \beta \text{ in } \Sigma \}
\]
of the Tychonoff product $\prod_{\alpha \in \Sigma} X_\alpha$ and the morphism $p = (p_\alpha)_{\alpha \in \Sigma} : X_\Sigma \to S$ where $p_\alpha : X \to X_\alpha$, $p_\alpha : (x_\alpha)_{\alpha \in \Sigma} \mapsto x_\alpha$, is the $\alpha$-th coordinate projection.

Using the technique of inverse spectra, we shall reduce the problem of investigation of the graphoid $\Gamma(F)$ of an arbitrary family $F \subset \mathbb{R}(x_1, \ldots, x_k)$ to studying the graphoids $\Gamma(\alpha)$ of finite subfamilies $\alpha$ of $F$. Namely, given any family $F \subset \mathbb{R}(x_1, \ldots, x_k)$ of rational functions of $k$-variables, consider the set $\Sigma = [F]^{<\omega}$ of finite subfamilies of $F$, partially ordered by the inclusion relation $\subset$. Endowed with this relation, $\Sigma = [F]^{<\omega}$ becomes a directed poset. For any elements $\alpha \leq \beta$ of $\Sigma$ (which are finite subsets $\alpha \subset \beta$ of $F$) we can consider the coordinate projection $p_\alpha : \Gamma(\beta) \to \Gamma(\alpha)$. In such a way we obtain the inverse spectrum $S_F = \{\Gamma(\alpha), p_\alpha^\beta, \Sigma\}$ consisting of graphoids of finite subfamilies of $F$. For each finite subset $\alpha \in \Sigma$ of $F$ the limit projection $p_\alpha : \Gamma(F) \to \Gamma(\alpha)$ coincides with the corresponding coordinate projection (we recall that $\Gamma(F) = \mathbb{R}^2 \times \mathbb{R}^n$ while $\Gamma(\alpha) \subset \mathbb{R}^2 \times \mathbb{R}^n$).

The crucial fact that follows from the definition of $\Gamma(F)$ is the following lemma:

**Lemma 4.1.** The graphoid $\Gamma(F)$ together with the limit projections $p_\alpha : \Gamma(F) \to \Gamma(\alpha)$, $\alpha \in \Sigma$, is the limit of the inverse spectrum $S_F = \{\Gamma(\alpha), p_\alpha^\beta, \Sigma\}$ consisting of graphoids $\Gamma(\alpha)$ of finite subfamilies $\alpha \subset F$.

5. Extension dimension of limit spaces of inverse spectra

In this section we shall evaluate the extension dimension of limit spaces of inverse spectra in the category $\text{CompEpi}$. This information will be then used in the proofs of Theorems 1.2 and 1.3.

We shall say that a topological space $Y$ is an absolute neighborhood extensor for compact Hausdorff spaces and write $Y \in \text{ANE(Comp)}$ if each map $f : A \to Y$ defined on a closed subspace $A$ of a compact Hausdorff space $X$ has a continuous extension $\tilde{f} : N(A) \to Y$ defined on a neighborhood $N(A)$ of $A$ in $X$.

Let us recall that a space $X$ has extension dimension $e\dim(X) \leq Y$ if each map $f : A \to Y$ defined on a closed subspace $A$ of $X$ has a continuous extension $\tilde{f} : X \to Y$.

The following lemma should be known in Extension Dimension Theory but we could not find a precise reference. So, we have decided to give a proof for convenience of the reader.

**Lemma 5.1.** Let $(X, (p_\alpha))$ be a limit of an inverse spectrum $S = \{X_\alpha, p_\beta^\alpha, \Sigma\}$ in the category $\text{CompEpi}$. The limit space $X$ has extension dimension $e\dim(X) \leq Y$ for some space $Y \in \text{ANE(Comp)}$ if and only if for any $\alpha \in \Sigma$ and a map $f_\alpha : A_\alpha \to Y$ defined on a closed subspace $A_\alpha$ of the space $X_\alpha$ there are an index $\beta \geq \alpha$ in $\Sigma$ and a map $f_\beta : A_\beta \to Y$ extending the map $f_\alpha : A_\alpha \to Y$ defined on a closed subset $A_\alpha$ of $X_\beta$.

**Proof.** First we prove the “if” part of the lemma. To prove that $X$ has extension dimension $e\dim(X) \leq Y$, fix a map $f : A \to Y$ defined on a closed subset $A$ of the space $X$. Embed the space $X$ into a Tychonoff cube $[0,1]^\kappa$. Since $Y \in \text{ANE(Comp)}$, the map $f$ admits a continuous extension $\tilde{f} : O(A) \to Y$ defined on an open neighborhood $O(A)$ of $A$ in $[0,1]^\kappa$. Next, find a closed neighborhood $\tilde{A} \subset O(A)$ of $A$ in $[0,1]^\kappa$.

Let $U$ be a cover of $[0,1]^\kappa$ by open convex subsets such that

$$\text{St} (\tilde{A}, U) := \bigcup \{ U \in U : \tilde{A} \cap U \neq \emptyset \} \subset O(A).$$

**Claim 5.2.** There is an index $\alpha \in \Sigma$ and a continuous map $r_\alpha : \tilde{A} \to O(A)$ defined on the closed subset $\tilde{A} = p_\alpha (\tilde{A})$ of $X_\alpha$ such that the map $r_\alpha \circ p_\alpha : \tilde{A} \to O(A)$ nears to the identity embedding $\tilde{A} \to O(A)$ in the sense that for each $x \in \tilde{A}$ the doubleton $\{ x, r_\alpha (p_\alpha (x)) \}$ lies in some set $U \in U$.

**Proof.** Let $V$ be an open cover of $[0,1]^\kappa$ that star-refines the cover $U$ (the latter means that for every $V \in V$ its $V$-star $\text{St}(V, V)$ lies in some set $U \in U$).

It is well-known that the topology of the limit space $X$ of the spectrum $S$ is generated by the base consisting of the sets $p_\alpha^{-1}(U_{\alpha})$ where $\alpha \in \Sigma$ and $U_{\alpha}$ is an open set in $X_\alpha$. Here $p_\alpha : X \to X_\alpha$ stands for the limit projection.

This fact allows us to find for every $z \in \tilde{A}$ an index $\alpha_z \in \Sigma$ and an open neighborhood $W_z \subset X_\alpha$ of $p_\alpha (z)$ such that the neighborhood $p_\alpha^{-1}(W_z)$ of $z$ lies in some set $V_z \in V$. The open cover $\{ p_\alpha^{-1}(W_z) : z \in \tilde{A} \}$ of the compact subset $\tilde{A}$ admits a finite subcover $\{ p_\alpha^{-1}(W_z) : z \in F \}$. Here $F$ is a suitable finite subset of $\tilde{A}$. Since the index set $\Sigma$ is directed, there is an index $\alpha \in \Sigma$ such that $\alpha \geq \alpha_z$ for all $z \in F$. Changing the sets $W_z$ by $(p_\alpha^{-1})^{-1}(W_z)$, we can assume that $\alpha_z = \alpha$ for all $z \in F$. Then $W = \{ W_z : z \in F \}$ is an open cover of the closed subset $\tilde{A} = p_\alpha (\tilde{A})$ of the compact space $X_\alpha$. Let $\{ \lambda_z : \tilde{A} \to [0,1] \}_{z \in F}$ be a partition of unity, subordinated to the cover $W$ in the sense that $\lambda_z^{-1}([0,1]) \subset W_z$ for all $z \in F$.

Consider the map $r_\alpha : \tilde{A} \to [0,1]^\kappa$ defined by

$$r_\alpha(x) = \sum_{z \in F} \lambda_z(x) \cdot z.$$

We claim that this map has the required property: $r_\alpha \circ p_\alpha : \tilde{A} \to O(A)$.
Given any point \( x \in \hat{A} \), consider the finite set \( E = \{ z \in F : \lambda_z(p_\alpha(x)) > 0 \} \). It follows that
\[
r_\alpha(p_\alpha(x)) = \sum_{z \in E} \lambda_z(p_\alpha(x)) \cdot z.
\]

Observe that for every \( z \in E \) we get
\[
x \in p_\alpha^{-1}(p_\alpha(x)) \subset p_\alpha^{-1}\left( \lambda_z^{-1}\left( [0,1] \right) \right) \subset p_\alpha^{-1}(W_z) \subset V_z
\]
and hence
\[
E \cup \{ x \} \subset \bigcup_{z \in E} V_z \subset St(x,V) \subset U
\]
for some open convex set \( U \in \mathcal{U} \). The convexity of the set \( U \) guarantees that this set contains the following convex combination:
\[
r_\alpha(p_\alpha(x)) = \sum_{z \in E} \lambda_z(p_\alpha(x)) \cdot z.
\]

Claim 5.2 implies that
\[
r_\alpha(\hat{A}_\alpha) = r_\alpha \circ p_\alpha(\hat{A}) \subset St(\hat{A},U) \subset O(A),
\]
so the composition \( f_\alpha = \hat{f} \circ r_\alpha : \hat{A}_\alpha \to Y \) is a well-defined continuous map. By our assumption, there is an index \( \beta \geq \alpha \) and a continuous map \( \hat{f}_\beta : \hat{X}_\beta \to Y \) that extends the map \( f_\beta = f_\alpha \circ p_\beta^{\delta}(\hat{A}) = p_\beta^{-1}(\hat{A}_\alpha) \supset p_\beta(\hat{A}) \).

Observe that
\[
\hat{f}_\beta \circ p_\beta(\hat{A}) = \hat{f}_\beta \circ p_\beta(\hat{A}) = f_\beta \circ p_\beta^{\delta} \circ p_\beta|\hat{A} = f_\alpha \circ p_\beta(\hat{A}) = \hat{f} \circ r_\alpha \circ p_\alpha|\hat{A}.
\]

Using the Urysohn Lemma, choose a continuous function \( \xi : X \to [0,1] \) such that \( \xi(A) \subset \{1\} \) and \( X \setminus \hat{A} \subset \xi^{-1}(0) \).

Claim 5.2 implies that for every \( x \in \hat{A} \) the convex combination \( \xi(x)x + (1-\xi(x))r_\alpha(p_\alpha(x)) \) lies in \( St(\hat{A},U) \subset O(A) \) so, the function \( \hat{f} : X \to Y \),
\[
\hat{f} : x \mapsto \left\{ \begin{array}{ll}
\tilde{f}(\xi(x)x + (1-\xi(x))r_\alpha(p_\alpha(x))) & \text{if } x \in \hat{A} \\
\hat{f}_\beta \circ p_\beta(x) & \text{if } x \in \xi^{-1}(0), \\
\hat{f}_\beta \circ p_\beta(x) = \tilde{f} \circ r_\alpha \circ p_\alpha(x) & \text{if } x \in \xi^{-1}(0) \cap \hat{A},
\end{array} \right.
\]
is a well-defined continuous extension of the map \( f = \hat{f}|A \), witnessing that \( e\text{-dim}(X) \leq Y \).

Now we prove the “only if” part of the lemma. Assume that \( e\text{-dim}(X) \leq Y \). Fix an index \( \alpha \in \Sigma \) and a continuous map \( f_\alpha : A_\alpha \to Y \) defined on a closed subset \( A_\alpha \subset X_\alpha \). We need to find an index \( \beta \geq \alpha \) in \( \Sigma \) and a continuous map \( f_\beta : X_\beta \to Y \) that extends the map \( f_\beta = f_\alpha \circ p_\beta^{\delta}(\hat{A}_\alpha) \subset p_\beta(\hat{A}) \).

Since \( Y \in \text{ANE(Comp)} \), the map \( f_\alpha \) admits a continuous extension \( \tilde{f}_\alpha : \hat{A}_\alpha \to Y \) defined on a closed neighborhood \( \hat{A}_\alpha \) of \( A_\alpha \) in \( X_\alpha \). Then \( \hat{A} = p_\alpha^{-1}(\hat{A}_\alpha) \) is a closed neighborhood of the closed set \( A = p_\alpha^{-1}(A_\alpha) \) in \( X \). Since \( e\text{-dim}(X) \leq Y \), the map \( \tilde{f}_\alpha \circ p_\alpha | \hat{A} \) has a continuous extension \( \hat{f} : X \to Y \).

Embed the compact Hausdorff space \( K = \tilde{f}(X) \subset Y \) in a Tychonoff cube \( [0,1]^\kappa \) of a suitable weight \( \kappa \). Since \( Y \in \text{ANE(Comp)} \), the identity embedding \( K \to Y \) admits a continuous extension \( \psi : O(K) \to Y \) defined on an open neighborhood \( O(K) \) of \( K \) in \( [0,1]^\kappa \). Let \( \mathcal{U} \) be a cover of \( [0,1]^\kappa \) by open convex subsets such that \( St(K,\mathcal{U}) \subset O(K) \).

Repeating the argument of Claim 5.2 we can find an index \( \beta \geq \alpha \) in \( \Sigma \) and a continuous map \( f_\beta : X_\beta \to [0,1]^{\kappa} \) such that the composition \( f_\beta \circ p_\beta \) is \( \mathcal{U}\)-near to the map \( \hat{f} : X \to K \subset [0,1]^{\kappa} \).

Consider the closed neighborhood \( \hat{A}_\beta = (p_\beta^{\delta})^{-1}(\hat{A}_\alpha) \supset p_\beta(\hat{A}) \) of the set \( A_\beta = (p_\beta^{\delta})^{-1}(A_\alpha) \) in the space \( X_\beta \). Using the Urysohn Lemma, choose a continuous function \( \xi : X_\beta \to [0,1] \) such that \( A_\beta \subset \xi^{-1}(1) \) and \( X_\beta \setminus \hat{A} \subset \xi^{-1}(0) \).

Given any point \( y \in \hat{A}_\beta \), choose a point \( x \in p_\beta^{-1}(y) \subset \hat{A} \) (which exists by the surjectivity of the limit projection \( p_\beta \)), and observe that
\[
\{ \tilde{f}_\alpha \circ p_\beta(y), f_\beta(y) \} = \{ \tilde{f}_\alpha \circ p_\alpha(x), f_\beta \circ p_\beta(x) \} = \{ \tilde{f}(x), f_\beta \circ p_\beta(x) \} \subset U \subset St(K,\mathcal{U}) \subset O(K)
\]
for some set \( U \in \mathcal{U} \) according to the choice of the map \( f_\beta \). Then the convex combination \( \xi(x)\tilde{f}_\alpha(p_\alpha(x)) + (1-\xi(x))f_\beta(x) \) also belongs to \( U \subset O(K) \), which implies that the map \( \hat{f}_\beta : X_\beta \to Y \),
\[
\hat{f}_\beta(x) = \left\{ \begin{array}{ll}
\psi(\xi(x)\tilde{f}_\alpha(p_\beta(x)) + (1-\xi(x))f_\beta(x)) & \text{if } x \in \hat{A}_\beta \\
\psi(f_\beta(x)) & \text{if } x \in \xi^{-1}(0) \\
\psi(f_\beta(x)) = \psi \circ \tilde{f}_\alpha \circ p_\beta(x) & \text{if } x \in \hat{A}_\beta \cap \xi^{-1}(0)
\end{array} \right.
\]
is a well-defined continuous extension of the map \( \psi \circ \tilde{f}_\alpha \circ p_\beta | A_\beta = f_\alpha \circ p_\beta^{\delta} | A_\beta \).

Lemma 5.4 implies the following known fact on preservation of extension dimension by inverse limits.
Lemma 6.2. Let \((X, (p_\alpha))\) be a limit of an inverse spectrum \(S = \{X_\alpha, p_\alpha^0, \Sigma\}\) in the category \(\text{CompEpi}\). The limit space \(X\) has extension dimension \(e\text{-}\dim(X) \leq Y\) for some space \(Y \in \text{ANE(Comp)}\) provided that \(e\text{-}\dim(X_\alpha) \leq Y\) for all \(\alpha \in \Sigma\).

By [10] 3.2.9, a compact Hausdorff space \(X\) has covering dimension \(\dim X \leq n\) if and only if \(e\text{-}\dim(X) \leq S^n\) where \(S^n\) denotes the \(n\)-dimensional sphere. This fact combined with Corollary 5.3 yields the following well-known fact [10] 3.4.11:

Corollary 5.3. Let \((X, (p_\alpha))\) be a limit of an inverse spectrum \(S = \{X_\alpha, p_\alpha^0, \Sigma\}\) in the category \(\text{CompEpi}\). The limit space \(X\) has dimension \(\dim(X) \leq n\) for some \(n \in \omega\) provided that \(\dim(X_\alpha) \leq n\) for all \(\alpha \in \Sigma\).

6. Proof of Theorem 1.2

In this section we present a proof of Theorem 1.2. Given any non-empty family of rational functions \(F \subset \mathbb{R}(x, y)\) we need to prove that the graphoid \(\Gamma(F)\) has dimension \(\dim(\Gamma(F)) = 2\).

Lemma 6.1. The graphoid \(\Gamma(F)\) has dimension \(\dim(\Gamma(F)) \leq 2\).

Proof. By Lemma 6.1 the graphoid \(\Gamma(F)\) is homeomorphic to the limit space of the inverse spectrum \(\mathcal{S}_F = \{\Gamma(\alpha), p_\alpha^0, [\mathcal{F}]_\alpha^\omega\}\) that consists of graphoids \(\Gamma(\alpha)\) of finite subfamilies \(\alpha \subset \mathcal{F}\). Now Corollary 5.4 will imply that \(\dim(\Gamma(F)) \leq 2\) as soon as we check that \(\dim(\Gamma(\emptyset)) \leq 2\) for any finite subfamily \(\emptyset \subset \mathcal{F}\).

Since \(\emptyset\) is finite, the set \(\text{dom}(\emptyset) = \bigcap_{f \in \emptyset} \text{dom}(f)\) is cofinite in \(\mathbb{R}^2\). Identify the family \(\emptyset\) with the partial continuous function \(\emptyset : \text{dom}(\emptyset) \to \mathbb{R}^2\), \(\emptyset : x \mapsto (f(x))_{f \in \emptyset}\) and let \(\emptyset\) be the graphoid extension of \(\emptyset\). Then \(\Gamma(\emptyset) = \Gamma(\emptyset)\) and hence \(\Gamma(\emptyset) = \Gamma(\emptyset) = \Gamma(\emptyset) \cup \bigcup \{\{z\} \times \emptyset(z) : z \in \mathbb{R}^2 \setminus \text{dom}(\emptyset)\}\).

Theorem 5.4 implies that for every point \(z \in \text{dom}(\emptyset)\) the set \(\emptyset(z)\) has dimension \(\dim(\emptyset(z)) \leq 1\). Since the graph \(\Gamma(\emptyset)\) is homeomorphic to the cofinite set \(\mathbb{R}^2 \setminus \text{dom}(\emptyset)\), it has dimension \(\dim(\Gamma(\emptyset)) \leq \dim(\mathbb{R}^2) = 2\). Now Theorem of Sum [10] 1.5.3 implies that \(\dim(\emptyset(z)) \leq \dim(\Gamma(\emptyset)) \setminus \text{dom}(\emptyset)\) \(\leq 2\).

\(\square\)

Lemma 6.2. \(\dim(\Gamma(F)) \geq 2\).

Proof. Since \(\dim(\Gamma(F)) \leq 1\) if and only if \(e\text{-}\dim(\Gamma(F)) \leq K_\emptyset\), it suffices to check that \(e\text{-}\dim(\Gamma(F)) \not\leq K_\emptyset\). To prove this fact, we shall apply Lemma 5.1. By Lemma 4.1 the graphoid \(\Gamma(F)\) is the limit of the spectrum \(\mathcal{S}_F = \{\Gamma(\alpha), p_\alpha^0, [\mathcal{F}]_\alpha^\omega\}\). The smallest element of the poset \([\mathcal{F}]_\alpha^\omega\) is the empty set. Its graphoid \(\Gamma(\emptyset)\) can be identified with the torus \(\mathbb{R}^2\).

Let \(A_\emptyset = 2K \setminus K\) where \(K = (-1, 1)^2\) is the open square in the plane \(\mathbb{R}^2\) endowed with the max-norm \(\|(x, y)\| = \max\{|x|, |y|\}\).

Consider the map \(f : A_\emptyset \to K_\emptyset, f : (x, y) \mapsto \frac{(x, y)}{\|(x, y)\|}\). Assuming that \(e\text{-}\dim(\Gamma(F)) \leq K_\emptyset\), and applying Lemma 5.1 we can find a finite subset \(\beta \subset \mathcal{F}\) and a continuous map \(f_\beta : \Gamma(\emptyset) \to K_\emptyset\) that extends the map \(f_\emptyset \circ p_\emptyset^0[A_\emptyset : A_\emptyset \to K_\emptyset]\) where \(A_\emptyset = (p_\emptyset^0)^{-1}(A_\emptyset)\). The finite family \(\beta \subset \mathcal{F}\) thought as a partial function \(\beta : \text{dom}(\beta) \to \mathbb{R}^2\) is defined on a cofinite subset \(\text{dom}(\beta)\) of \(\mathbb{R}^2\). So, we can find a real number \(t \in [1, 2]\) such that \(tK_\emptyset \subset \text{dom}(\beta)\). Consider the finite set \(Z = tK \setminus \text{dom}(\beta)\) in \(tK\).

Using Theorem 4.1 we can find \(\varepsilon > 0\) so small that

1. \(tK_\emptyset \cap B(z, \varepsilon) = \emptyset\),
2. \(B(z, \varepsilon) \cap B(z', \varepsilon) = \emptyset\) for any distinct points \(z, z' \in Z\);
3. there is a homeomorphism \(h : tK \setminus B(Z, \varepsilon) \to tK \setminus Z\) such that
   a. \(h\) is the identity on the set \(tK \setminus B(Z, \varepsilon)\),
   b. \(h\) has continuous extension \(\tilde{h} : tK \setminus B(Z, \varepsilon) \to tK \setminus Z\) such that \(\tilde{h}^{-1}(z) = S(z, \varepsilon)\) for every \(z \in Z\);
   c. the composition \(\beta \circ h : tK \setminus B(Z, \varepsilon) \to \mathbb{R}^2\) has a continuous extension \(\tilde{\beta}_h : tK \setminus B(Z, \varepsilon) \to \mathbb{R}^2\);
   d. for every \(z \in Z\) and any map \(\varphi_z : \beta(z) \to K_\emptyset\) the composition \(\varphi_z \circ \tilde{\beta}_h[S(z, \varepsilon) : S(z, \varepsilon)] \to K_\emptyset\) is \(Z_2\)-trivial.

It follows that the map \(\psi = (\tilde{h}, \tilde{\beta}_h) : tK \setminus B(Z, \varepsilon) \to \Gamma(\emptyset), \psi : z \mapsto (\tilde{h}(z), \tilde{\beta}_h(z))\), is continuous and for every \(z \in Z\) the map \(f_\emptyset \circ \psi[S(z, \varepsilon) : S(z, \varepsilon)] \to K_\emptyset\) is \(Z_2\)-trivial. Then by Lemma 2.3, the map \(\psi[tK_\emptyset : tK_\emptyset \to K_\emptyset] \to K_\emptyset\) is also \(Z_2\)-trivial, which is impossible as this map is a homeomorphism, which induces an isomorphism of the homology groups \(H_1(tK_\emptyset; \mathbb{Z}_2)\) and \(H_1(K_\emptyset; \mathbb{Z}_2)\). This contradiction completes the proof of Lemma 6.2.

\(\square\)
7. Proof of Theorem 1.3

Assume that \( F \subset \mathbb{R}(x, y) \) is a family of rational functions, containing a family of linear fractional transformations

\[
\left\{ \frac{x-a}{y-b} : (a, b) \in D \right\}
\]

for some dense subset \( D \) of \( \mathbb{R}^2 \).

By Theorem 1.2 \( \dim(\Gamma(F)) = 2 \). By Alexandroff Theorem [6, 1.4], \( \dim_2(X) = \dim(X) \) for each finite-dimensional compact Hausdorff space \( X \). Consequently, \( \dim_2(\Gamma(F)) = \dim(\Gamma(F)) = 2 \).

Now let \( G \) be a non-trivial 2-divisible abelian group. We need to show that \( \dim_G(\Gamma(F)) = 1 \). To see that \( \dim_G(\Gamma(F)) > 0 \), take any Eilenberg-MacLane complex \( K(G, 0) \), for example, take the group \( G \) endowed with the discrete topology. Since the space \( \Gamma(F) \) is connected, any injective map \( f : A \rightarrow G \) defined on a doubleton \( A = \{a, b\} \subset \Gamma(F) \) has no continuous extension \( \bar{f} : \Gamma(F) \rightarrow G \), which means that \( e\dim(\Gamma(F)) \leq G \) and \( \dim_G(\Gamma(F)) \leq 0 \).

The inequality \( \dim_G(\Gamma(F)) \leq 1 \), which is equivalent to \( e\dim(\Gamma(F)) \leq K(G, 1) \), follows from the subsequent a bit more general result:

**Lemma 7.1.** \( e\dim(\Gamma(F)) \leq Y \) for any path-connected space \( Y \in ANE(\text{Comp}) \) with 2-divisible fundamental group \( \pi_1(Y) \).

**Proof.** To show that \( e\dim(\Gamma(F)) \leq Y \) we shall apply Lemma 5.1. By Lemma 4.1 the graphoid \( \Gamma(F) \) is homeomorphic to the limit space of the inverse spectrum \( S_F = \{\Gamma(\alpha), p_{\alpha}^a[,\alpha] < \omega\} \). Given a finite subset \( \alpha \in [\alpha] < \omega \) and a map \( f_\alpha : A_\alpha \rightarrow Y \) defined on a closed subset \( A_\alpha \) of the graphoid \( \Gamma(\alpha) \), we need to find a finite subset \( \beta \supset \alpha \) of \( F \) and a continuous function \( \beta : \bar{\Gamma}(\beta) \rightarrow Y \) that extends the map \( f_\alpha \circ p_{\alpha}^a[A_\beta] \) defined on the set \( A_\beta = (p_{\alpha}^a)^{-1}(A_\alpha) \).

We can think of the family \( \alpha \subset F \) as a partial function \( \alpha \circ \text{dom}(\alpha) \rightarrow \mathbb{R}^\circ \) defined on the cofinite set \( \text{dom}(\alpha) \) in \( \mathbb{R}^\circ \). Let \( \bar{\alpha} : \mathbb{R}^2 \rightarrow \mathbb{R}^\circ \) be the graphoid extension of \( \alpha \). Its graph \( \Gamma(\bar{\alpha}) \) coincides with the graphoid \( \Gamma(\alpha) \) of \( \alpha \).

By Theorem 3.16, for every point \( z \) of the finite set \( Z = \{p, \infty\} \cup \{\mathbb{R}^2 \setminus \text{dom}(\alpha)\} \) the image \( \bar{\alpha}(z) \) is a singleton or a finite union of arcs. Consequently, the set \( \bar{\alpha}(\mathcal{Z}) = \bigcup_{z \in \mathcal{Z}} \{z\} \times \bar{\alpha}(z) \) is a finite union of singletons or arcs. Using the path-connectedness of the space \( Y \in ANE(\text{Comp}) \), we can extend the map \( f_\alpha \) to a continuous map \( f_\alpha' : A_\alpha \cup \bar{\alpha}(\mathcal{Z}) \rightarrow Y \).

Since \( Y \in ANE(\text{Comp}) \), the map \( f_\alpha' : A_\alpha \cup \bar{\alpha}(\mathcal{Z}) \rightarrow Y \) has a continuous extension \( \bar{f}_\alpha : \bar{A}_\alpha \rightarrow Y \) defined on the closed neighborhood \( \bar{A}_\alpha \) of the set \( A_\alpha \cup \bar{\alpha}(\mathcal{Z}) \) in \( \Gamma(\alpha) \). The boundary \( \partial \bar{A}_\alpha \) of \( \bar{A}_\alpha \) in \( \Gamma(\alpha) \) is a compact subset of \( \Gamma(\bar{\alpha}) \setminus \bar{\alpha}(\mathcal{Z}) \subset \Gamma(\alpha) \). The projection \( p_{\alpha}^a : \Gamma(\bar{\alpha}) \rightarrow \mathbb{R}^2 \) maps homeomorphically the graph \( \Gamma(\alpha) \) onto the cofinite subset \( \text{dom}(\alpha) \) of the torus \( \mathbb{R}^2 \).

Replacing \( A_\alpha \) by a smaller (more regular) neighborhood, if necessary, we can assume that the boundary \( \partial \bar{A}_\alpha \) is a topological graph, that is, a finite union of arcs that are disjoint or meet by their end-points. Adding to \( A_\alpha \) a finite union of arcs, we can enlarge the set \( A_\alpha \) to a closed set \( A_\alpha \subset \Gamma(\alpha) \) whose boundary \( \partial \bar{A}_\alpha \) is a topological graph such that

- the family \( C \) of connected components of \( \bar{\Gamma}(\alpha) \setminus \bar{A}_\alpha \) is finite and
- for each connected component \( C \in \mathcal{C} \) the closure \( \bar{C} \) is homeomorphic to the closed square \( \bar{K} = [-1, 1]^2 \).

Using the path-connectedness of the space \( Y \in ANE(\text{Comp}) \), we can extend the map \( \bar{f}_\alpha \) to a continuous map \( \bar{f}_\alpha : \bar{A}_\alpha \rightarrow Y \).

For every connected component \( C \in \mathcal{C} \) use the density of the set \( D \) in \( \mathbb{R}^2 \) and find a point \((a_C, b_C) \in D \cap C \). Now consider the finite subfamily

\[
\beta = \alpha \cup \left\{ \frac{x-a_C}{y-b_C} : C \in \mathcal{C} \right\} \subset \mathcal{F},
\]

which determines a partial continuous function \( \beta : \text{dom}(\beta) \rightarrow \mathbb{R}^\circ \) defined on the cofinite set \( \text{dom}(\alpha) \setminus (\{p, \infty\} \cup \{(a_C, b_C) : C \in \mathcal{C}\}) \) of \( \mathbb{R}^2 \).

We claim that there is a continuous function \( \bar{f}_\beta : \bar{\Gamma}(\beta) \rightarrow Y \) that extends the map \( \bar{f}_\alpha \circ p_{\alpha}^a[A_\beta] \) defined on the set \( A_\beta = (p_{\alpha}^a)^{-1}(A_\alpha) \).

Put \( A_\beta = (p_{\alpha}^a)^{-1}(A_\alpha) \) and observe that the complement \( \bar{\Gamma}(\beta) \setminus A_\beta \) is the union of connected components \( C_\beta = (p_{\alpha}^a)^{-1}(C), C \in \mathcal{C} \), which are graphoids of the rational functions \( \frac{x-a_C}{y-b_C} \) restricted to the open 2-disks \( p_{\alpha}^a(C) \). Such graphoids are homeomorphic to the open Möbius band.

For every \( C \in \mathcal{C} \) the closure \( \bar{C} \) of \( C_\beta \) in \( \bar{\Gamma}(\beta) \), being homeomorphic to the closed Möbius band, is homeomorphic to the quotient space of the “square annulus” \( K \setminus \frac{1}{2}K \) by the equivalence relation that identifies the pairs of opposite points on the inner boundary square \( \frac{1}{2}K \). Let \( g_C : \bar{K} \setminus \frac{1}{2}K \rightarrow \bar{C}_\beta \) be the corresponding quotient map.

Fix a continuous map \( \sigma : [0, 1] \rightarrow \bar{K} \) such that

- \( \sigma(0) = \sigma(1) \),
- \( \sigma \mid [0, 1) : [0, 1) \rightarrow K \) is bijective,
- for any \( t \in [0, \frac{1}{2}] \) the points \( \sigma(t + \frac{1}{2}) = -\sigma(t) \).
The map $\gamma_C = \bar{f}_\alpha \circ p^\beta \circ q_C \circ \sigma : [0, 1] \to Y$ determines a loop in $Y$, whose equivalence class is an element of the fundamental group $\pi_1(Y)$ of $Y$. Since the group $\pi_1(Y)$ is 2-divisible, there is a loop $\delta_C : [0, 1] \to Y$ such whose square $\delta^2_C : [0, 1] \to Y$,

$$\delta^2_C : t \mapsto \begin{cases} \delta_C(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \delta_C(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

is homotopic to the loop $\gamma_C$ by a homotopy that does not move the points 0 and 1.

Now consider the loop $\bar{\gamma}_C = \bar{f}_\alpha \circ p^\beta \circ q_C \mid K_\varnothing : K_\varnothing \to Y$ and observe that $\gamma_C = \bar{\gamma}_C \circ \sigma$. Let $\delta^2_C : \frac{1}{2}K_\varnothing \to Y$ be a unique map such that $\delta^2_C \circ \frac{1}{2} = \delta^2_C$. Here $\frac{1}{2} = [0, 1] \to \frac{1}{2}K_\varnothing$ is the point $\frac{1}{2}(t)$ of the square $\frac{1}{2}K_\varnothing$. The homotopy between the loops $\gamma_C$ and $\delta^2_C$ allows us to find a continuous map $h_C : \bar{K} \setminus \frac{1}{2}K_\varnothing \to Y$ such that $h_C|K_\varnothing = \bar{\gamma}_C$ and $h_C|\frac{1}{2}K_\varnothing = \delta^2_C$.

The definition of $\sigma$ and $\delta^2_C$ guarantees that $\delta^2_C(x) = \delta^2_C(-x)$ for any point $x \in \frac{1}{2}K_\varnothing$. Hence there is a unique continuous map $h_C : C^\beta \to Y$ such that $h_C = h_C \circ q_C$. It follows from $h_C|K_\varnothing = \bar{\gamma}_C$ that $h_C|\partial C^\beta = \bar{f}_\alpha \circ p^\beta|\partial C^\beta$. This implies that the map $f_\beta : \Gamma(\beta) \to Y$,

$$f_\beta(x) = \begin{cases} \bar{f}_\alpha \circ p^\beta(x) & \text{if } x \in \bar{A}_\beta, \\ h_C(x) & \text{if } x \in C^\beta \text{ for some } C \in \mathcal{C}, \end{cases}$$

is a well-defined continuous extension of the map $f_\alpha \circ p^\beta|A_\beta$.

8. Some Open Problems

In light of Theorem 4.3, the following problem arises naturally:

**Problem 8.1.** Has the graphoid $\Gamma(F)$ of any family $F \subset \mathbb{R}(x,y)$ the cohomological dimension $\dim_C(\Gamma(F)) = 2$ for any abelian group $G$ that is not 2-divisible?

The answer to this problem is affirmative if the following problem has an affirmative answer.

**Problem 8.2.** Let $F \subset \mathbb{R}(x,y)$ be a finite family, $\bar{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be its graphoid extension, and $z \in \mathbb{R}^2$ be an arbitrary point. Is $F(z)$ a singleton or a finite union of analytic arcs in $\mathbb{R}^2$.

An arc $A$ in $\mathbb{R}^n$ is called analytic if $A = \bar{\alpha}([0,1])$ for some vector function $\bar{\alpha} : [0,1] \to \mathbb{R}^n$ that has analytic coordinate functions $\alpha_1, \ldots, \alpha_n : [0,1] \to \mathbb{R}$. Here we identify the projective line $\mathbb{R}$ with the unit circle on plane via the stereographic projection.

In case of positive answer to Problem 5.2, the proof of the inequality $\dim(\Gamma(F)) \geq 2$ can be much simplified (Lemma 3 with its extremely long proof will not be required).

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