Some classes of projectively and dually flat Finsler spaces with Randers change

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Abstract

In this paper, we consider Randers change of some special $(\alpha, \beta)$--metrics. First we find the fundamental metric tensor and Cartan tensor of these Randers changed $(\alpha, \beta)$--metrics. Next, we establish a general formula for inverse of fundamental metric tensors of these metrics. Finally, we find the necessary and sufficient conditions under which the Randers change of these $(\alpha, \beta)$--metrics are projectively and locally dually flat.

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1 Introduction

According to S. S. Chern [5], Finsler geometry is just Riemannian geometry without quadratic restriction. Now a days, Finsler geometry is an interesting and active area of research for both pure and applied reasons [2]. Though there has been a lot of development in this area, still there is a huge scope of research work in Finsler geometry. The concept of $(\alpha, \beta)$--metric was introduced by M. Matsumoto [9] in 1972. For a general Finsler metric $F$, C. Shibata [17] introduced the notion of $\beta$--change in $F$, i.e., $\bar{F} = f(F, \beta)$ in 1984. Recall [6] that a Finsler metric $F$ on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ is called projectively flat if and only if all the geodesics are straight in $\mathcal{U}$. The concept
of dually flatness in Riemannian geometry was given by Amari and Nagaoka in [1] while studying information geometry. Information geometry provides mathematical science with a new framework for analysis. Information geometry is an investigation of differential geometric structure in probability distribution. It is also applicable in statistical physics, statistical inferences etc. Z. Shen [16] extended the notion of dually flatness in Finsler spaces. Since then, many authors ([3], [12], [15]) have worked on this topic.

The current paper is organized as follows:
In second section, we give basic definitions and examples of some special Finsler spaces with \((\alpha, \beta)\)–metrics obtained by Randers change. In section 3, we find fundamental metric tensors \(\bar{g}_{ij}\) and Cartan tensors \(\bar{C}_{ijk}\) for these metrics. In section 4, we find a generalized formula for the inverse \(\bar{g}^{ij}\) of fundamental metric tensor \(\bar{g}_{ij}\). In sections 5 and 6 we find the necessary and sufficient conditions for Randers change of some special Finsler spaces with \((\alpha, \beta)\)–metrics to be projectively and locally dually flat respectively.

2 Preliminaries

Though there is vast literature available for Riemann-Finsler geometry, here we give some basic definitions, examples and results required for subsequent sections.

**Definition 2.1.** Let \(M\) be an \(n\)-dimensional smooth manifold, \(T_x M\) the tangent space at \(x \in M\), and
\[
T M := \bigsqcup_{x \in M} T_x M
\]
be the tangent bundle of \(M\) whose elements are denoted by \((x, y)\), where \(x \in M\) and \(y \in T_x M\).
A Finsler structure on \(M\) is a function
\[
F : TM \rightarrow [0, \infty),
\]
with the properties:

- **Regularity:** \(F\) is \(C^{\infty}\) on the slit tangent bundle \(TM \setminus \{0\}\).
- **Positive homogeneity:** \(F(x, \lambda y) = \lambda F(x, y) \forall \lambda > 0\).
- **Strong convexity:** The \(n \times n\) matrix \((g_{ij}) = \left[\frac{1}{2} F^2 \right]_{ij}^p g^p\) is positive-definite at every point of \(TM \setminus \{0\}\).
A smooth manifold $M$ together with the Finsler structure $F$, i.e., $(M, F)$ is called Finsler space and the corresponding geometry is called Finsler geometry.

Next, we recall following definition:

**Definition 2.2.** An $(\alpha, \beta)$ metric $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$ with $\phi = \phi(s)$ a smooth positive function on some symmetric interval $(-b_0, b_0)$, is a Finsler metric $F$, which is positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1–form.

**Lemma 2.1.** [6] An $(\alpha, \beta)$–norm $F = \alpha \phi \left( \frac{\beta}{\alpha} \right)$ is said to be a Minkowski norm for any Riemannian metric $\alpha$ and 1–form $\beta$ with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ satisfies the following conditions:

$$
\phi(s) > 0, \; \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \; \phi(s) - s\phi'(s) > 0,
$$

where $b$ is an arbitrary numbers satisfying $|s| \leq b < b_0$.

There are so many classical examples of $(\alpha, \beta)$–metrics, below we mention few of them: Randers metric, Kropina metric, generalized Kropina metric, Z. Shen’s square metric, Matsumoto metric, exponential metric, infinite series metric.

Recall [17] that

**Definition 2.3.** Let $(M, F)$ be an $n$–dimensional Finsler space. Then a metric $\bar{F} = f(F, \beta) + \beta$ constructed via a $\beta$–change is called Randers change of $(\alpha, \beta)$–metric.

Next, we construct some special Finsler metrics via Randers change of $(\alpha, \beta)$ metrics. Our further studies will be based on these metrics.

1. Kropina-Randers change of $(\alpha, \beta)$–metric:

   We know that $F = \frac{\alpha^2}{\beta}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1–form, is Kropina metric. Applying $\beta$–change and Randers change simultaneously to this metric, we obtain a new metric $\bar{F} = \frac{F^2}{\beta} + \beta$, which we call Kropina-Randers changed metric.
2. Generalized Kropina-Randers change of \((\alpha, \beta)\)-metric:

We know that \( F = \frac{\alpha^{m+1}}{\beta m} \) (\( m \neq 0, -1 \)), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1–form, is generalized Kropina metric. Applying \( \beta \)–change and Randers change simultaneously to this metric, we obtain a new metric \( \bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \), \( m \neq 0, -1 \), which we call generalized Kropina-Randers changed metric.

3. Square-Randers change of \((\alpha, \beta)\)-metric:

We know that \( F = (\alpha + \beta)^2 \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1–form, is Z. Shen’s square metric. Applying \( \beta \)–change and Randers change simultaneously to this metric, we obtain a new metric \( \bar{F} = \frac{(F + \beta)^2}{F} + \beta \), which we call Square-Randers changed metric.

4. Matsumoto-Randers change of \((\alpha, \beta)\)-metric:

Again \( F = \frac{\alpha^2}{\alpha - \beta} \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1–form, is a well known Matsumoto metric. Applying \( \beta \)–change and Randers change simultaneously to this metric, we obtain a new metric \( \bar{F} = \frac{F^2}{F - \beta} + \beta \), which we call Matsumoto-Randers changed metric.

5. Exponential-Randers change of \((\alpha, \beta)\)-metric:

The metric \( F = \alpha e^{\beta/\alpha} \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1–form, is called exponential metric. Applying \( \beta \)–change and Randers change simultaneously to this metric, we obtain a new metric \( \bar{F} = Fe^{\beta/F} + \beta \), which we call exponential-Randers changed metric.

6. Randers change of infinite series \((\alpha, \beta)\)-metric:

The metric \( F = \frac{\beta^2}{\beta - \alpha} \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1–form, is called infinite series metric. Applying \( \beta \)–change and Randers change simultaneously to this metric, we obtain a new metric \( \bar{F} = \frac{\beta^2}{\beta - F} + \beta \), which we call infinite series-Randers changed metric.
Definition 2.4. Let \((M, F)\) be an \(n\)-dimensional Finsler space. If \(F = \sqrt{A}\), where \(A = a_{i_1i_2}(x)y^{i_1}y^{i_2}\) with \(a_{i_1i_2}(x)\) symmetric in both the indices, then \(F\) is called square-root Finsler metric.

We use following notations in the subsequent sections:

\[
\begin{align*}
\frac{\partial \bar{F}}{\partial x^i} &= \bar{F}_{x^i}, \quad \frac{\partial \bar{F}}{\partial y^i} = \bar{F}_{y^i}, \quad \frac{\partial A}{\partial x^i} = A_{x^i}, \quad \frac{\partial A}{\partial y^i} = A_i, \quad A_{x^i y^j} y^i = A_{0j}, \\
\frac{\partial \beta}{\partial x^i} &= \beta_{x^i}, \quad \frac{\partial \beta}{\partial y^i} = b_i \text{ or } \beta_i, \quad \beta_{x^i y^j} y^i = \beta_{0j} \text{ etc.}
\end{align*}
\]

3 Fundamental Metric Tensors and Cartan Tensors

Definition 3.1. Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. The function

\[
g_{ij} := \left(\frac{1}{2} F^2\right) y^i y^j = FF_{y^i y^j} + F_{y^i} F_{y^j} = h_{ij} + \ell_i \ell_j
\]

is called fundamental tensor of the metric \(F\).

Definition 3.2. Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. Then its Cartan tensor is defined as

\[
C_{ijk}(y) = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \left(F^2\right) y^i y^j y^k,
\]

which is symmetric in all the three indices \(i, j, k\).

Next, we find fundamental metric tensor and Cartan tensor for all the Randers changed \((\alpha, \beta)\)-metrics constructed in the previous section. Kropina-Randers changed metric is

\[
\bar{F} = \frac{F^2}{\beta} + \beta.
\] (3.1)

Differentiating (3.1) w.r.t. \(y^i\), we get

\[
\bar{F}_{y^i} = \frac{2F}{\beta} F_{y^i} - \frac{F^2 - \beta^2}{\beta^2} b_i.
\] (3.2)

Differentiation of (3.2) further w.r.t. \(y^j\) gives

\[
\bar{F}_{y^i y^j} = \frac{2}{\beta} g_{ij} \frac{2F}{\beta^2} \left(b_i F_{y^j} + b_j F_{y^i}\right) + \frac{2F^2}{\beta^3} b_i b_j.
\] (3.3)
Now,
\[
\bar{g}_{ij} = \bar{F} F_{y^{i}y^{j}} + F_{y^{i}} F_{y^{j}} = \left\{ \frac{F^2}{\beta + \beta} \right\} \left\{ \frac{1}{\beta} g_{ij} \frac{2F}{\beta^2} (b_i F_{y^{i}} + b_j F_{y^{j}}) + \frac{2F^2}{\beta^3} b_i b_j \right\} + \left\{ \frac{2F}{\beta} F_{y^{i}} - \frac{F^2 - \beta^2}{\beta^2} b_i \right\} \left\{ \frac{2F}{\beta} F_{y^{j}} - \frac{F^2 - \beta^2}{\beta^2} b_j \right\}.
\]

Simplifying, we get
\[
\bar{g}_{ij} = \frac{2(F^2 + \beta^2)}{\beta^2} g_{ij} \frac{4F^3}{\beta^3} (b_i F_{y^{i}} + b_j F_{y^{j}}) + \frac{4F^2}{\beta^2} F_{y^{i}} F_{y^{j}} + \left( \frac{3F^4}{\beta^4} + 1 \right) b_i b_j.
\]

Hence, we have following:

**Proposition 3.1.** Let \((M, \bar{F})\) be an \(n\)–dimensional Finsler space with \(\bar{F} = \frac{F^2}{\beta} + \beta\) as a Kropina-Randers changed metric. Then its fundamental metric tensor is given by equation (3.4).

Next, we find Cartan tensor for Kropina-Randers changed metric.

By definition, we have
\[
2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^{k}}.
\]

From the equation (3.4), we get
\[
2\bar{C}_{ijk} = \frac{\partial}{\partial y^{k}} (\bar{g}_{ij}) = \frac{2(F^2 + \beta^2)}{\beta^2} g_{ij} \frac{4F^3}{\beta^3} (b_i F_{y^{i}} + b_j F_{y^{j}}) + \frac{4F^2}{\beta^2} F_{y^{i}} F_{y^{j}} + \left( \frac{3F^4}{\beta^4} + 1 \right) b_i b_j
\]

\[
= \frac{4(F^2 + \beta^2)}{\beta^2} C_{ijk} + h_{ij} \left( \frac{-4F^2}{\beta^3} b_k + \frac{4}{\beta^2} y_k \right) + h_{jk} \left( \frac{-4F^2}{\beta^3} b_i + \frac{4}{\beta^2} y_i \right) + h_{ki} \left( \frac{-4F^2}{\beta^3} b_j + \frac{4}{\beta^2} y_j \right)
\]

\[
- \frac{12F^4}{\beta^5} \left\{ b_i b_j b_k - \frac{\beta}{F^2} (b_i b_k y_j + b_j b_k y_i + b_k b_i y_j) - \frac{\beta^3}{F^4} y_i y_j y_k + \frac{\beta^2}{F^4} (y_i y_k b_j + y_i y_j b_k + y_j y_i b_k) \right\}
\]

6
\[
\begin{aligned}
&= \frac{4(F^2 + \beta^2)}{\beta^2} C_{ijk} - \frac{4F^2}{\beta^3} \sum_{\text{cyclic sum}} \left( b_k - \frac{\beta}{F^2} y_k \right) - \frac{12F^4}{\beta^5} \prod_{\text{cyclic product}} \left( b_i - \frac{\beta}{F^2} y_i \right). \\
\end{aligned}
\]

After simplification, we get
\[
\bar{C}_{ijk} = \frac{2(F^2 + \beta^2)}{\beta^2} C_{ijk} - \frac{2F^2}{\beta^3} \left( h_{ij} m_k + h_{jk} m_i + h_{ki} m_j \right) - \frac{6F^4}{\beta^5} \left( m_i m_j m_k. \right) \\
(3.5)
\]

The above discussion leads to the following proposition.

**Proposition 3.2.** Let \((M, \bar{F})\) be an \(n\)–dimensional Finsler space, with \(\bar{F} = \frac{F^2}{\beta} + \beta\) as a Kropina-Randers changed Finsler metric. Then its Cartan tensor is given by equation (3.5).

Next, we find fundamental metric tensor for generalized Kropina-Randers changed metric
\[
\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta, \quad m \neq 0, -1. \\
(3.6)
\]

Differentiating (3.6) w.r.t. \(y^i\), we get
\[
\bar{F}_{y^i} = (m+1) \frac{F^m}{\beta^m} F_{y^i} + \left( 1 - m \frac{F^{m+1}}{\beta^{m+1}} \right) b_i. \\
(3.7)
\]

Differentiation of (3.7) further w.r.t. \(y^j\) gives
\[
\bar{F}_{y^i y^j} = (m+1) \frac{F^m}{\beta^m} F_{y^i y^j} + m(m+1) \frac{F^{m-1}}{\beta^m} F_{y^i} F_{y^j} - m(m+1) \frac{F^m}{\beta^{m+1}} \left( b_i F_{y^j} + b_j F_{y^i} \right) + m(m+1) \frac{F^{m+1}}{\beta^{m+2}} b_i b_j. \\
(3.8)
\]

Now,
\[
\bar{g}_{ij} = \bar{F} \bar{F}_{y^i y^j} + \bar{F}_{y^i} \bar{F}_{y^j} \\
= \left\{ \frac{F^{m+1}}{\beta^m} + \beta \right\} \left\{ (m+1) \frac{F^m}{\beta^m} F_{y^i y^j} + m(m+1) \frac{F^{m-1}}{\beta^m} F_{y^i} F_{y^j} - m(m+1) \frac{F^m}{\beta^{m+1}} \left( b_i F_{y^j} + b_j F_{y^i} \right) + m(m+1) \frac{F^{m+1}}{\beta^{m+2}} b_i b_j \right\} \\
+ \left\{ (m+1) \frac{F^m}{\beta^m} F_{y^i} + \left( 1 - m \frac{F^{m+1}}{\beta^{m+1}} \right) b_i \right\} \left\{ (m+1) \frac{F^m}{\beta^m} F_{y^j} + \left( 1 - m \frac{F^{m+1}}{\beta^{m+1}} \right) b_j \right\}
\]
After simplification, we get

\[ \bar{g}_{ij} = (m + 1) \left( \frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}} \right) g_{ij} - (m + 1) \left( 2m \frac{F^{2m+1}}{\beta^{2m+1}} + (m - 1) \frac{F^{m}}{\beta^{m}} \right) \left( b_i F y^j + b_j F y_i \right) + \left( m + 1 \right) \left( 2m \frac{F^{2m}}{\beta^{2m}} + (m - 1) \frac{F^{m}}{\beta^{m-1}} \right) F y^i F y^j + \left( 1 + m(2m + 1) \frac{F^{2m+2}}{\beta^{2m+2}} + m(m - 1) \frac{F^{m+1}}{\beta^{m+1}} \right) b_i b_j. \]

(3.9)

Hence, we have the following proposition.

**Proposition 3.3.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta\) \((m \neq 0, -1)\) as generalized Kropina-Randers changed metric. Then its fundamental metric tensor is given by equation (3.9).

Next, we find Cartan tensor for generalized Kropina-Randers changed metric.

By definition, we have

\[ 2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}. \]

From the equation (3.9), we get

\[ 2\bar{C}_{ijk} = \frac{\partial}{\partial y^k} (\bar{g}_{ij}) = 2(m + 1) \left( \frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}} \right) C_{ijk} + (m + 1) \left\{ 2m \frac{\beta F^{2m-2} y_k - F^{2m} b_k}{\beta^{2m+1}} + (m - 1) \frac{\beta F^{m-3} y_k - F^{m-1} b_k}{\beta^{m}} \right\} \left( h_{ij} \frac{y k}{F^2} \right) - (m + 1) \left( 2m \frac{F^{2m+1}}{\beta^{2m+1}} + (m - 1) \frac{F^{m}}{\beta^{m}} \right) \left( \frac{b_i h_{jk} + b_j h_{ik}}{F} \right) - m(m + 1) \left\{ 2(2m + 1) \frac{\beta F^{2m-1} y_k - F^{2m+1} b_k}{\beta^{2m+2}} + (m - 1) \frac{\beta F^{m-2} y_k - F^{m} b_k}{\beta^{m+1}} \right\} \left( \frac{b_i y j + b_j y_i}{F^2} \right) + (m + 1) \left( 2m \frac{F^{2m}}{\beta^{2m}} + (m - 1) \frac{F^{m-1}}{\beta^{m-1}} \right) \left( \frac{y k h_{ij} + y j h_{ik}}{F^2} \right) + (m + 1) \left\{ 4m^2 \frac{\beta F^{2m-2} y_k - F^{2m} b_k}{\beta^{2m+1}} + (m - 1)^2 \frac{\beta F^{m-3} y_k - F^{m-1} b_k}{\beta^{m}} \right\} \frac{y i y j}{F^2} + m(m + 1) \left\{ 2(2m + 1) \frac{\beta F^{2m} y_k - F^{2m+2} b_k}{\beta^{2m+3}} + (m - 1) \frac{\beta F^{m-1} y_k - F^{m+1} b_k}{\beta^{m+2}} \right\} b_i b_j. \]
\[ \begin{align*}
&= 2(m + 1) \left( \frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}} \right) C_{ijk} - (m + 1) \frac{F^{m-1}}{\beta^m} \left( 2m \frac{F^{m+1}}{\beta^{m+1}} + (m - 1) \right) \sum_{\text{cyclic sum}} h_{ij} \left( b_k - \frac{\beta}{F^2} y_k \right) \\
&\quad - m \frac{F^{m+1}}{\beta^{m+2}} \left( 2m + 1 \right) \left( 2m + 2 \right) \frac{F^{m+1}}{\beta^{m+1}} + (m^2 - 1) \prod_{\text{cyclic product}} \left( b_i - \frac{\beta}{F^2} y_i \right) .
\end{align*} \]

After simplification, we get

\[ \hat{C}_{ijk} = (m + 1) \left( \frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}} \right) C_{ijk} \]

\[ - m \frac{F^{m+1}}{\beta^{m+2}} \left( 2m + 1 \right) \left( 2m + 2 \right) \frac{F^{m+1}}{\beta^{m+1}} + (m - 1) \left( h_{ij} m_k + h_{jk} m_i + h_{ki} m_j \right) \]

\[ - m \frac{F^{m+1}}{\beta^{m+2}} \left( 2m + 1 \right) \left( 2m + 2 \right) \frac{F^{m+1}}{\beta^{m+1}} + (m^2 - 1) \right) m_i m_j m_k. \]

(3.10)

The above discussion leads to the following proposition.

**Proposition 3.4.** Let \((M, \bar{F})\) be an \(n\)–dimensional Finsler space with \(\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \quad (m \neq 0, -1)\) as generalized Kropina-Randers changed metric. Then its Cartan tensor is given by equation (3.10).

Next, we find fundamental metric tensor for square-Randers changed metric

\[ \bar{F} = \frac{(F + \beta)^2}{F} + \beta. \quad (3.11) \]

Differentiating (3.11) w.r.t. \(y^i\), we get

\[ \bar{F}_{y^i} = \left( 1 - \frac{\beta^2}{F^2} \right) F_{y^i} + \left( \frac{2\beta}{F} + 3 \right) b_i. \quad (3.12) \]

Differentiation of (3.12) further w.r.t. \(y^j\) gives

\[ \bar{F}_{y^i y^j} = \left( 1 - \frac{\beta^2}{F^2} \right) F_{y^i y^j} - \frac{2\beta}{F^2} \left( b_i F_{y^j} + b_j F_{y^i} \right) + \frac{2\beta^2}{F^3} F_{y^i} F_{y^j} + \frac{2}{F} b_i b_j. \quad (3.13) \]

Now,

\[ \bar{g}_{ij} = \bar{F} \bar{F}_{y^i y^j} + \bar{F}_{y^i} \bar{F}_{y^j} \]

\[ = \left\{ F + \frac{3\beta}{F} \right\} \left\{ \left( 1 - \frac{\beta^2}{F^2} \right) F_{y^i y^j} - \frac{2\beta}{F^2} \left( b_i F_{y^j} + b_j F_{y^i} \right) + \frac{2\beta^2}{F^3} F_{y^i} F_{y^j} + \frac{2}{F} b_i b_j \right\} \]

\[ \quad + \left\{ \left( 1 - \frac{\beta^2}{F^2} \right) F_{y^i} + \frac{2\beta}{F} + 3 \right\} b_i \left\{ \left( 1 - \frac{\beta^2}{F^2} \right) F_{y^j} + \frac{2\beta}{F} + 3 \right\} b_j . \]
Simplifying, we get

\[
\bar{g}_{ij} = \left(1 + \frac{3\beta}{F} - \frac{3\beta^3}{F^3} - \frac{\beta^4}{F^4}\right)g_{ij} + \left(3 - \frac{4\beta^3}{F^3} - \frac{9\beta^2}{F^2}\right)(b_i F_{y^j} + b_j F_{y^i}) \\
+ \left(-\frac{3\beta}{F} + \frac{9\beta^3}{F^3} + \frac{4\beta^4}{F^4}\right)F_{y^i} F_{y^j} + \left(11 + \frac{18\beta}{F} + \frac{6\beta^2}{F^2}\right)b_i b_j.
\]  

(3.14)

Hence, we have following:

**Proposition 3.5.** Let \((M, F)\) be an \(n\)-dimensional Finsler space with \(F = \frac{(F + \beta)^2}{F} + \beta\) as a Square-Randers changed metric. Then its fundamental metric tensor is given by equation (3.14).

Next, we find Cartan tensor for square-Randers changed metric.

By definition, we have

\[2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.\]

From the equation (3.14), we get

\[
2\bar{C}_{ijk} = \frac{\partial}{\partial y^k}(\bar{g}_{ij})
\]

\[
= 2\left(1 + \frac{3\beta}{F} - \frac{3\beta^3}{F^3} - \frac{\beta^4}{F^4}\right)C_{ijk} \\
+ \left\{\frac{3(F^2 b_k - \beta y_k)}{F^3} - \frac{9(\beta^2 F^2 b_k - \beta^3 y_k)}{F^6} - \frac{4(\beta^3 F^2 b_k - \beta^4 y_k)}{F^9}\right\} \left(\frac{h_{ij} + y_i y_j}{F^2}\right) \\
- \left(3 - \frac{4\beta^3}{F^3} - \frac{9\beta^2}{F^2}\right)\left(\frac{b_i h_{jk} + b_j h_{ik}}{F}\right) \\
- \left\{\frac{-12(\beta^2 F^2 b_k - \beta^3 y_k)}{F^5} - \frac{18(\beta F^2 b_k - \beta^2 y_k)}{F^4}\right\} \left(\frac{b_i y_j + b_j y_i}{F}\right) \\
+ \left(-\frac{3\beta}{F} + \frac{9\beta^3}{F^3} + \frac{4\beta^4}{F^4}\right)\left(\frac{y_i h_{jk} + y_j h_{ik}}{F^2}\right) \\
+ \left\{\frac{16(\beta^3 F^2 b_k - \beta^4 y_k)}{F^6} + \frac{27(\beta^2 F^2 b_k - \beta^3 y_k)}{F^5} - \frac{3(F^2 b_k - \beta y_k)}{F^3}\right\} \frac{y_i y_j}{F^2} \\
+ \left\{\frac{18(F^2 b_k - \beta y_k)}{F^3} + \frac{12(\beta F^2 b_k - \beta^2 y_k)}{F^4}\right\} b_i b_j
\]
\[
\begin{align*}
&= 2 \left( 1 + \frac{3\beta}{F} - \frac{3\beta^3}{F^3} - \frac{\beta^4}{F^4} \right) C_{ijk} + \left( \frac{3}{F} - \frac{9\beta^2}{F^3} - \frac{4\beta^3}{F^4} \right) \sum_{\text{cyclic sum}} h_{ij} \left( b_k - \frac{\beta}{F^2} y_k \right) \\
&\quad + \left( \frac{18}{F} + \frac{12\beta}{F^2} \right) \prod_{\text{cyclic product}} \left( b_i - \frac{\beta}{F^2} y_i \right).
\end{align*}
\]

After simplification, we get
\[
\begin{align*}
\bar{C}_{ijk} &= \left( 1 + \frac{3\beta}{F} - \frac{3\beta^3}{F^3} - \frac{\beta^4}{F^4} \right) C_{ijk} + \frac{1}{2} \left( \frac{3}{F} - \frac{9\beta^2}{F^3} - \frac{4\beta^3}{F^4} \right) \left( h_{ij} m_k + h_{jk} m_i + h_{ki} m_j \right) \\
&\quad + \left( \frac{9}{F} + \frac{6\beta}{F^2} \right) m_i m_j m_k.
\end{align*}
\]

(3.15)

The above discussion leads to the following proposition.

**Proposition 3.6.** Let \( (M, \bar{F}) \) be an \( n \)–dimensional Finsler space with \( \bar{F} = \frac{(F + \beta)^2}{F} + \beta \) as a Square-Randers changed metric. Then its Cartan tensor is given by equation (3.15).

Next, we find fundamental metric tensor for Matsumoto-Randers changed metric
\[
\bar{F} = \frac{F^2}{F - \beta} + \beta.
\]

(3.16)

Differentiating (3.16) w.r.t. \( y^i \), we get
\[
\bar{F}_{y^i} = \frac{F^2 - 2\beta F}{(F - \beta)^2} F_{y^i} + \left( \frac{F^2}{(F - \beta)^2} + 1 \right) b_i.
\]

(3.17)

Differentiation of (3.17) further w.r.t. \( y^i \) gives
\[
\bar{g}_{y^i y^j} = \frac{1}{(F - \beta)^3} \left\{ (F^3 - 3\beta F^2 + 2\beta^2 F) F_{y^i y^j} - 2\beta F (b_i F_{y^j} + b_j F_{y^i}) + 2\beta^2 F_{y^i} F_{y^j} + 2F^2 b_i b_j \right\}.
\]

(3.18)

Now,
\[
\bar{g}_{ij} = \bar{F}_{y^i y^j} + \bar{F}_{y^i} \bar{F}_{y^j}
\]
\[
= \frac{F^2 + \beta F - \beta^2}{(F - \beta)^4} \left\{ (F^3 - 3\beta F^2 + 2\beta^2 F) F_{y^i y^j} - 2\beta F (b_i F_{y^j} + b_j F_{y^i}) + 2\beta^2 F_{y^i} F_{y^j} + 2F^2 b_i b_j \right\}
\]
\[
+ \left\{ \frac{F^2 - 2\beta F}{(F - \beta)^2} F_{y^i} + \left( \frac{F^2}{(F - \beta)^2} + 1 \right) b_i \right\} \left\{ \frac{F^2 - 2\beta F}{(F - \beta)^2} F_{y^j} + \left( \frac{F^2}{(F - \beta)^2} + 1 \right) b_j \right\}.
\]
After simplification, we get
\[
\tilde{g}_{ij} = \frac{1}{(F - \beta)^4} \left\{ (F^4 - 2\beta F^3 - 2\beta^2 F^2 + 5\beta^3 F - 2\beta^4)g_{ij} + (2F^4 - 8\beta F^3 + 3\beta^2 F^2) (b_i F_{y^j} + b_j F_{y^i}) \\
+ (-2\beta F^3 + 8\beta^2 F^2 - 3\beta^3 F) F_{y^i y^j} + (6F^4 - 6\beta F^3 + 6\beta^2 F^2 - 4\beta^3 F + \beta^4) b_i b_j \right\}.
\]

Hence, we have following

**Proposition 3.7.** Let \((M, \tilde{F})\) be an \(n\)-dimensional Finsler space with \(\tilde{F} = \frac{F^2}{F - \beta} + \beta\) as a Matsumoto-Randers changed metric. Then its fundamental metric tensor is given by equation (3.19).

Next, we find Cartan tensor for Matsumoto-Randers changed metric.

By definition, we have
\[
2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.
\]

From the equation (3.19), we get
\[
2\tilde{C}_{ijk} = \frac{\partial}{\partial y^k} (g_{ij})
= 2\frac{(F^4 - 2\beta F^3 - 2\beta^2 F^2 + 5\beta^3 F - 2\beta^4)}{(F - \beta)^4} C_{ijk}
+ \left\{ \frac{-\beta (2F^2 - 8\beta F + 3\beta^2)}{F(F - \beta)^4} y_k + \frac{2F^3 - 8\beta F^2 + 3\beta^2 F}{(F - \beta)^4} b_k \right\} \left( h_{ij} + \frac{y_i y_j}{F^2} \right)
+ \left\{ \frac{2F^4 - 8\beta F^3 + 3\beta^2 F^2}{(F - \beta)^4} \right\} \left( \frac{b_i h_{jk} + b_j h_{ik}}{F} \right)
+ \left\{ \frac{18\beta^2 F - 6\beta^3}{(F - \beta)^5} y_k + \frac{-18\beta F^3 + 6\beta^2 F^2}{(F - \beta)^5} b_k \right\} \left( \frac{b_i y_j + b_j y_i}{F^2} \right)
+ \left\{ \frac{18\beta^2 F^2 - 2\beta F^3 - 3\beta^3 F}{(F - \beta)^4} \right\} \left( \frac{y_i h_{jk} + y_j h_{ik}}{F^2} \right)
+ \left\{ \frac{-6\beta F^2 + 16\beta^2 F - 3\beta^3}{(F - \beta)^4} \right\} \left( \frac{y_j y_i}{F^2} \right)
+ \left\{ \frac{-2F^3 + 16\beta F^2 - 9\beta^2 F}{(F - \beta)^4} + \frac{4 (-2\beta F^3 + 8\beta^2 F^2 - 3\beta^3 F)}{(F - \beta)^5} \right\} b_k \right\} \frac{y_i y_j}{F^2}
+ \left\{ \frac{-18\beta F^3 + 6\beta^2 F^2}{F(F - \beta)^5} y_k + \frac{18F^4 - 6\beta F^3}{(F - \beta)^5} b_k \right\} b_i b_j.
\]
\[
\begin{align*}
&= 2(F^4 - 2\beta F^3 - 2\beta^2 F^2 + 5\beta^3 F - 2\beta^4) C_{ijk} \\
&+ \frac{2F^3 - 8\beta F^2 + 3\beta^2 F}{(F - \beta)^4} \sum_{\text{cyclic sum}} h_{ij} \left( b_k \frac{\beta}{F^2 y_k} \right) + \frac{18F^4 - 6\beta F^3}{(F - \beta)^5} \prod_{\text{cyclic product}} \left( b_i \frac{\beta}{F^2 y_i} \right)
\end{align*}
\]

Simplifying, we get
\[
\tilde{C}_{ijk} = \frac{F^4 - 2\beta F^3 - 2\beta^2 F^2 + 5\beta^3 F - 2\beta^4}{(F - \beta)^4} C_{ijk}
+ \frac{2F^3 - 8\beta F^2 + 3\beta^2 F}{2(F - \beta)^4} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{9F^4 - 3\beta F^3}{(F - \beta)^5} m_i m_j m_k.
\]

The above discussion leads to the following proposition.

**Proposition 3.8.** Let \((M, \tilde{F})\) be an \(n\)-dimensional Finsler space with \(\tilde{F} = \frac{F^2}{F - \beta} + \beta\) as a Matsumoto-Randers changed metric. Then its Cartan tensor is given by equation (3.20).

Next, we find fundamental metric tensor for exponential-Randers changed metric
\[
\tilde{F} = F e^{\beta/F} + \beta.
\]

Differentiating (3.21) w.r.t. \(y^i\), we get
\[
\tilde{F}_{y^i} = \left( 1 - \frac{\beta}{F} \right) e^{\beta/F} F_{y^i} + \left( 1 + e^{\beta/F} \right) b_i.
\]

Differentiation of (3.22) further w.r.t. \(y^i\) gives
\[
\tilde{F}_{y^i y^j} = e^{\beta/F} \left\{ \left( 1 - \frac{\beta}{F} \right) F_{y^i y^j} - \frac{\beta}{F^2} (b_i F_{y^j} + b_j F_{y^i}) + \frac{\beta^2}{F^3} F_{y^i} F_{y^j} + \frac{1}{F} b_i b_j \right\}.
\]

Now,
\[
\tilde{g}_{ij} = \tilde{F}_{y^i y^j} + \tilde{F}_{y^i} \tilde{F}_{y^j}
= \left( F e^{\beta/F} + \beta \right) e^{\beta/F} \left\{ \left( 1 - \frac{\beta}{F} \right) F_{y^i y^j} - \frac{\beta}{F^2} (b_i F_{y^j} + b_j F_{y^i}) + \frac{\beta^2}{F^3} F_{y^i} F_{y^j} + \frac{1}{F} b_i b_j \right\}
+ \left\{ \left( 1 - \frac{\beta}{F} \right) e^{\beta/F} F_{y^j} + \left( 1 + e^{\beta/F} \right) b_j \right\} \left\{ \left( 1 - \frac{\beta}{F} \right) e^{\beta/F} F_{y^i} + \left( 1 + e^{\beta/F} \right) b_i \right\}.
\]
Simplifying, we get

$$
\bar{g}_{ij} = \left( e^{\beta/F} + \frac{\beta}{F} \right) e^{\beta/F} \left( 1 - \frac{\beta}{F} \right) g_{ij} - e^{\beta/F} \left\{ -1 + \frac{\beta}{F} + \frac{\beta^2}{F^2} + \left( -1 + \frac{2\beta}{F} \right) e^{\beta/F} \right\} (b_i F_{y^j} + b_j F_{y^i}) \\
+ \frac{\beta}{F} e^{\beta/F} \left\{ -1 + \frac{\beta}{F} + \frac{\beta^2}{F^2} + \left( -1 + \frac{2\beta}{F} \right) e^{\beta/F} \right\} F_{y^i} F_{y^j} + \left\{ 1 + 2e^{2\beta/F} + \left( 2 + \frac{\beta}{F} \right) e^{\beta/F} \right\} b_i b_j.
$$

(3.24)

Hence, we have following:

**Proposition 3.9.** Let $\bar{M}, F$ be an $n-$dimensional Finsler space with $F = Fe^{\beta/F} + \beta$ as an exponential-Randers changed metric. Then its fundamental metric tensor is given by equation (3.24).

Next, we find Cartan tensor of exponential-Randers changed metric. By definition, we have

$$
2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.
$$

From the equation (3.24), we get

$$
2\bar{C}_{ijk} = \frac{\partial \bar{g}_{ij}}{\partial y^k} \\
= 2 \left( e^{\beta/F} + \frac{\beta}{F} \right) e^{\beta/F} \left( 1 - \frac{\beta}{F} \right) C_{ijk} \\
+ \frac{e^{\beta/F}}{F} \left\{ \left( 1 + e^{\beta/F} \right) - \frac{\beta}{F} \left( 1 + \frac{\beta}{F} + 2e^{\beta/F} \right) \right\} \left( b_k - \frac{\beta}{F^2} y_k \right) (h_{ij} + \frac{y_i y_j}{F^2}) \\
- \frac{\beta e^{\beta/F}}{F} \left\{ 1 - \frac{\beta}{F} - \frac{\beta^2}{F^2} + \left( -1 + \frac{2\beta}{F} \right) e^{\beta/F} \right\} \left( h_{ik} y_j + h_{jk} y_i \right) \frac{1}{F^2} y^i y^j \left( b_k - \frac{\beta}{F^2} y_k \right) \\
+ \left\{ -1 + \frac{\beta}{F} + \frac{4\beta^2}{F^2} + \frac{\beta^3}{F^3} \left( 1 + \frac{2\beta}{F} + \frac{4\beta^2}{F^2} \right) e^{\beta/F} \right\} \frac{y_i y_j e^{\beta/F}}{F^3} \left( b_k - \frac{\beta}{F^2} y_k \right) \\
+ \frac{e^{\beta/F}}{F} \left\{ 1 - \frac{\beta}{F} + \frac{\beta^2}{F^2} + \left( -1 + \frac{2\beta}{F} \right) e^{\beta/F} \right\} (b_i h_{jk} + b_j h_{ik}) \\
- \frac{\beta e^{\beta/F}}{F^3} \left\{ 3 + \frac{\beta}{F} + 4e^{\beta/F} \right\} (b_i y_j + b_j y_i) \left( b_k - \frac{\beta}{F^2} y_k \right) \\
+ \frac{e^{\beta/F}}{F} \left\{ 3 + \frac{\beta}{F} + 4e^{\beta/F} \right\} \left( b_k - \frac{\beta}{F^2} y_k \right) b_i b_j.
$$
\[ C_{ijk} = 2 \left( e^{\beta/F} + \frac{\beta}{F} \right) e^{\beta/F} \left( 1 - \frac{\beta}{F} \right) C_{ijk} \]

\[ + \frac{e^{\beta/F}}{F} \left\{ 1 - \frac{\beta}{F} - \frac{\beta^2}{F^2} + \left( 1 - \frac{2\beta}{F} \right) e^{\beta/F} \right\} \sum_{\text{cyclic sum}} h_{ij} \left( b_k - \frac{\beta}{F^2} y_k \right) \]

\[ + \frac{e^{\beta/F}}{F} \left\{ 3 + \frac{\beta}{F} + 4 e^{\beta/F} \right\} \prod_{\text{cyclic product}} \left( b_i - \frac{\beta}{F^2} y_i \right). \]

After simplifying, we get

\[ C_{ijk} = \left( e^{\beta/F} + \frac{\beta}{F} \right) e^{\beta/F} \left( 1 - \frac{\beta}{F} \right) C_{ijk} \]

\[ + \frac{e^{\beta/F}}{2F} \left\{ 1 - \frac{\beta}{F} - \frac{\beta^2}{F^2} + \left( 1 - \frac{2\beta}{F} \right) e^{\beta/F} \right\} \left( h_{ij} m_k + h_{jk} m_i + h_{ki} m_j \right) \]

\[ + \frac{e^{\beta/F}}{2F} \left\{ 3 + \frac{\beta}{F} + 4 e^{\beta/F} \right\} m_i m_j m_k. \]

(3.25)

The above discussion leads to the following proposition.

**Proposition 3.10.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = F e^{\beta/F} + \beta\) as an exponential-Randers changed metric. Then its Cartan tensor is given by equation (3.25).

Next, we find fundamental metric tensor for Randers change of infinite series metric

\[ \bar{F} = \frac{\beta^2}{\beta - F} + \beta. \]

(3.26)

Differentiating (3.26) w.r.t. \( y^i \), we get

\[ \bar{F}_{y^i} = \frac{\left( \beta^2 - 2\beta F \right) b_i + \beta^2 F_{y^i}}{(\beta - F)^2} + b_i. \]

(3.27)

Differentiation of (3.27) further w.r.t. \( y^j \) gives

\[ \bar{F}_{y^i y^j} = \frac{\beta^2 \left( \beta - F \right) F_{y^i y^j} - 2 \beta F (b_i F_{y^j} + b_j F_{y^i}) + 2 \beta^2 F_{y^i} F_{y^j} + 2 F^2 b_i b_j}{(\beta - F)^3}. \]

(3.28)
Now,
\[
\bar{g}_{ij} = \bar{F}_{y_{i}y_{j}} + \bar{F}_{y_{i}}\bar{F}_{y_{j}} = \frac{\beta^2}{\beta - F} + \beta \left\{ \frac{\beta^2 (\beta - F) F y_{i} y_{j} - 2 \beta F (b_{i} F_{y_{i}} + b_{j} F_{y_{j}}) + 2 \beta^2 F_{y_{i}}^2 y_{j} + 2 F^2 b_i b_j}{(\beta - F)^3} \right\} + \frac{(2 \beta^2 - 4 \beta F + F^2) b_i + \beta^2 F y_{j}}{(\beta - F)^2} \left\{ \frac{(2 \beta^2 - 4 \beta F + F^2) b_j + \beta^2 F y_{i}}{(\beta - F)^2} \right\}.
\]

Simplifying, we get
\[
\bar{g}_{ij} = \frac{1}{(\beta - F)^4} \left\{ \frac{\beta^3 (2 \beta - F) (\beta - F)}{F} y_{ij} + (2 \beta^4 - 8 \beta^3 F + 3 \beta^2 F^2) (b_{i} F_{y_{i}} + b_{j} F_{y_{j}}) \right. \\
+ (2 \beta^5 / F + 8 \beta^4 - 3 \beta^3 F) F y_{i} F y_{j} + (4 \beta^4 - 16 \beta^3 F + 24 \beta^2 F^2 - 10 \beta F^3 + F^4) b_i b_j \left. \right\}.
\]

Hence, we have following:

**Proposition 3.11.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{\beta^2}{\beta - F} + \beta\) as an infinite series-Randers changed metric. Then its fundamental metric tensor is given by equation (3.29).

Next, we find Cartan tensor of Randers change of infinite series metric.

By definition, we have
\[
2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.
\]

From the equation (3.29), we get
\[
2\tilde{C}_{ijk} = \frac{\partial \bar{g}_{ij}}{\partial y^k} = \frac{2 \beta^3 (2 \beta - F)}{F (\beta - F)^3} C_{ijk} + \frac{2 \beta^4 - 8 \beta^3 F + 3 \beta^2 F^2}{F (\beta - F)^4} \left( b_{k - \beta} y_{j} \right) \left( h_{ij} + \frac{y_{i} y_{j}}{F^2} \right) \\
+ \frac{\beta^3 (-2 \beta^2 + 8 \beta F - 3 F^2)}{F^3 (\beta - F)^4} \left( h_{i} y_{j} + h_{j} y_{i} \right) \\
+ \frac{\beta^2 (-2 \beta^3 + 10 \beta^2 F - 29 \beta F^2 + 9 F^3)}{F^3 (\beta - F)^5} y_{i} y_{j} \left( b_{k - \beta} y_{k} \right) \\
+ \frac{2 \beta^4 - 8 \beta^3 F + 3 \beta^2 F^2}{F (\beta - F)^4} \left( b_{i} h_{j k} + b_{j} h_{i k} \right) - \frac{6 \beta F (F - 3 \beta)}{F (\beta - F)^5} \left( b_{i} y_{j} + b_{j} y_{i} \right) \left( b_{k - \beta} y_{k} \right) \\

16
After simplification, we get

\[
\bar{C}_{ijk} = \frac{\beta^3 (2\beta - F)}{F (\beta - F)^3} C_{ijk} + \frac{\beta^4 - 8\beta^3 F + 3\beta^2 F^2}{2F (\beta - F)^4} \sum_{\text{cyclic sum}} h_{ij} \left( b_k - \frac{\beta}{F^2} y_k \right)
\]

\[
+ \frac{6F^3 (F - 3\beta)}{(\beta - F)^5} \prod_{\text{cyclic product}} \left( b_i - \frac{\beta}{F^2} y_i \right).
\]

Hence, we have the following proposition.

**Proposition 3.12.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = Fe^{\beta/F} + \beta\) as an infinite series-Randers changed metric. Then its Cartan tensor is given by equation (3.30).

### 4 General formula for inverse \(\bar{g}^{ij}\) of fundamental metric tensor \(\bar{g}_{ij}\):

Let us put

\[
\frac{2(F^2 + \beta^2)}{\beta^2} = \rho_0, \quad \frac{4F^3}{\beta^3} = \rho_1, \quad \frac{4F^2}{\beta^2} = \rho_2, \quad \frac{3F^4}{\beta^4} + 1 = \rho_3
\]

in the equation (3.34).

Then the fundamental metric tensor \(\bar{g}_{ij}\) for Kropina-Randers changed Finsler metric \(\bar{F} = \frac{F^2}{\beta} + \beta\) takes the following form

\[
\bar{g}_{ij} = \rho_0 g_{ij} + \rho_1 \left( b_i \frac{y_j}{F} + b_j \frac{y_i}{F} \right) + \rho_2 \frac{y_i y_j}{F} + \rho_3 b_i b_j
\]

\[
= \rho_0 \left\{ g_{ij} + \frac{\rho_4}{\rho_0} \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right) + \left( \frac{\rho_4}{\rho_0} \right) \frac{y_i y_j}{F} \right\} + \left( \frac{\rho_3 - \rho_1}{\rho_0} \right) b_i b_j,
\]

i.e.,

\[
\bar{g}_{ij} = \rho_0 \left\{ g_{ij} + \lambda \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right) + \mu \frac{y_i y_j}{F} + \nu b_i b_j \right\},
\]

\[(4.1)\]
where \( \lambda = \frac{\rho_1}{\rho_0}, \mu = \frac{\rho_2 - \rho_1}{\rho_0} \) and \( \nu = \frac{\rho_3 - \rho_1}{\rho_0}. \) \( (4.2) \)

Similarly, for all other metrics constructed in section two, the fundamental metric tensors obtained in equations (3.9), (3.14), (3.19), (3.24) and (3.29) respectively, can be written in the form of equation (4.1).

Next, we find the inverse metric tensor \( \bar{g}^{ij} \) of fundamental metric tensor \( \bar{g}_{ij} \) for the metric \( F. \)

Let

\[
m_{ij} = g_{ij} + \lambda \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right)
= g_{ij} + \lambda c_i c_j,
\]

where \( c_i = b_i + \frac{y_i}{F}. \)

Define

\[
c^2 = g^{ij} c_i c_j \\
= g^{ij} \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right)
= b^2 + \frac{2\beta}{F} + 1.
\]

Next,

\[
\det m_{ij} = (1 + \lambda c^2) \det g_{ij} \\
= X \det g_{ij},
\]

where \( X = 1 + \lambda c^2 = 1 + \lambda \left( b^2 + \frac{2\beta}{F} + 1 \right). \) \( (4.3) \)

Also,

\[
m^{ij} = g^{ij} - \frac{\lambda}{1 + \lambda c^2} c^i c^j = g^{ij} - \frac{\lambda}{X} A^{ij},
\]

where

\[
A^{ij} = c^i c^j \\
= g^{ik} c_k g^{jk} c_k \\
= g^{ik} \left( b_k + \frac{y_k}{F} \right) g^{jk} \left( b_k + \frac{y_k}{F} \right) \\
= \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right).
\]

18
Then equation (4.1) becomes
\[ \bar{g}_{ij} = \rho_0 \left\{ m_{ij} + \mu \frac{y_i y_j}{F} + \nu b_i b_j \right\}. \] (4.4)

Further, assume that
\[ n_{ij} = m_{ij} + \mu d_i d_j, \text{ where } d_i = \frac{y_i}{F}. \] (4.5)

Define
\[ d^2 = m_{ij} d_i d_j \]
\[ = \left\{ g^{ij} - \frac{\lambda}{X} \left( b^i b^j + b^i y^j + y^i y^j \right) \right\} \frac{y^i y^j}{F} \]
\[ = 1 - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right)^2. \]

Next,
\[ \det n_{ij} = (1 + \mu d^2) \det m_{ij} \]
\[ = Y X \det g_{ij}. \]

where \( Y = 1 + \mu d^2 = 1 + \mu \left\{ 1 - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right)^2 \right\}. \) (4.6)

Also,
\[ n_{ij} = m_{ij} - \frac{\mu}{1 + \mu d^2} d^i d^j \]
\[ = g^{ij} - \frac{\lambda}{X} A^{ij} - \frac{\mu}{Y} B^{ij} \]
\[ = g^{ij} - \frac{\lambda}{X} \left( b_i + \frac{y_i}{F} \right) \left( b_j + \frac{y_j}{F} \right) - \frac{\mu}{Y} B^{ij} \] (4.7)

where
\[ B^{ij} = d^i d^j \]
\[ = m^{th} d_h m^{jk} d_k \]
\[ = \frac{1}{F^2} \left\{ 1 - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) \right\} y^i y^j + \frac{1}{F} \left\{ -\frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) + \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right)^2 \right\} (y^i b^j + y^j b^i) \]
\[ + \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right)^2 b^i b^j. \] (4.8)
From the equations (4.4) and (4.5), we get
\[ \tilde{g}_{ij} = \rho_0 \{ n_{ij} + \nu b_i b_j \} , \]  
(4.9)

Define
\[ \tilde{b}^2 = n_{ij} b_i b_j \]
\[ = b^2 - \frac{\lambda}{X} \left( b^2 + \frac{\beta}{F} \right)^2 - \frac{\mu}{Y} \left( 1 - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) \right)^2 \]
\[ - 2b^2 \frac{\beta}{F} \frac{\mu}{Y} \left( - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) + \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right) \right)^2 \]
\[ - b^4 \frac{\mu}{Y} \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right)^2 \].
(4.10)

From equation (4.9), we have
\[ \det \tilde{g}_{ij} = \rho_0^n \left( 1 + \nu \tilde{b}^2 \right) \det n_{ij} , \]
and the inverse \( \tilde{g}^{ij} \) of \( \tilde{g}_{ij} \) is given by
\[ \tilde{g}^{ij} = \frac{1}{\rho_0} \left( n^{ij} - \frac{\nu}{1 + \nu b^2} b^i b^j \right) . \]
(4.11)

From the equations (4.7), (4.8) and (4.11), we get
\[ \tilde{g}^{ij} = \frac{1}{\rho_0} \left\{ \tilde{g}^{ij} + \frac{1}{F} \left[ - \frac{\lambda}{X} - \frac{\mu}{Y} \left( - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) + \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right) \right)^2 \right] \right\} \left( y^i b^j + y^j b^i \right) \]
\[ + \frac{1}{F^2} \left[ - \frac{\lambda}{X} - \frac{\mu}{Y} \left( 1 - \frac{\lambda}{X} \left( 1 + \frac{\beta}{F} \right) \right)^2 \right] y^i y^j + \left[ - \frac{\lambda}{X} - \frac{\mu}{Y} \frac{\lambda^2}{X^2} \left( 1 + \frac{\beta}{F} \right)^2 - \frac{\nu}{1 + \nu b^2} \right] b^i b^j \}, \]
(4.12)
where the values of \( \lambda, \mu \) and \( \nu \) are obtained by equation (4.2), \( X \) by equation (4.3), \( Y \) by equation (4.6) and \( \tilde{b} \) by equation (4.10).

The above discussion leads to the following theorem.

**Theorem 4.1.** Let \( (M, F) \) be an \( n \)-dimensional Finsler space. Let \( \tilde{F} = f(F, \beta) \) be a Finsler metric obtained by Randers \( \beta \)-change of a Finsler metric \( F = f(\alpha, \beta) \), where \( \alpha = \sqrt{a_{ij}(x)y^i y^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a \( 1 \)-form. If \( \tilde{g}_{ij} \) is the fundamental metric tensor of \( \tilde{F} \) given by (4.7), then its inverse fundamental metric tensor, \( \tilde{g}^{ij} \) is given by (4.12).
5 Projective flatness of Finsler metrics

The notion of projective flatness is one of the important topics in differential geometry. The real starting point of the investigations of projectively flat metrics is Hilbert’s fourth problem \[8\]. During the International Congress of Mathematicians, held in Paris(1900), Hilbert asked about the spaces in which the shortest curves between any pair of points are straight lines. The first answer was given by Hilbert’s student G. Hamel in 1903. In \[7\], Hamel found the necessary and sufficient conditions in order that a space satisfying a system of axioms, which is a modification of Hilbert’s system of axioms for Euclidean geometry, be projectively flat. After Hamel, many authors (see \[3, 10,...,15, 18\]) have worked on this topic. Here we find necessary and sufficient conditions for Randers change of some special \((\alpha, \beta)\)-metrics to be projectively flat. For this, first we discuss some definitions and results related to projective flatness.

**Definition 5.1.** Two Finsler metrics \(F\) and \(\bar{F}\) on a manifold \(M\) are called projectively equivalent if they have same geodesics as point sets, i.e., for any geodesic \(\bar{\sigma}(\bar{t})\) of \(\bar{F}\), there is a geodesic \(\sigma(t) := \bar{\sigma}(\bar{t}(t))\) of \(F\), where \(\bar{t} = \bar{t}(t)\) is oriented re-parametrization, and vice-versa.

Next, recall (\[6, 7\]) the following theorem:

**Theorem 5.1.** Let \(F\) and \(\bar{F}\) be two Finsler metrics on a manifold \(M\). Then \(F\) is projectively equivalent to \(\bar{F}\) if and only if
\[
F_{x^k y^j} y^k - \bar{F}_{x^k} = 0.
\]
Here the spray coefficients are related by \(G^i = \bar{G}^i + P y^i\), where \(P = \frac{F_{x^k y^j}}{2F}\).

**Definition 5.2.** Let \(\bar{F}\) be a standard Euclidean norm on \(\mathbb{R}^n\). Then spray coefficients vanish, i.e.,
\[
\bar{G}^i = 0.
\]
Since in \(\mathbb{R}^n\), geodesics are straight lines, we have

**Definition 5.3.** For a Finsler metric \(F\) on an open subset \(U\) of \(\mathbb{R}^n\), the geodesics of \(F\) are straight lines if and only if the spray coefficients satisfy
\[
G^i = P y^i.
\]
Definition 5.4. A Finsler metric $F$ on an open subset $U$ of $\mathbb{R}^n$ is called projectively flat if and only if all the geodesics are straight in $U$, and a Finsler metric $F$ on a manifold $M$ is called locally projectively flat, if at any point, there is a local co-ordinate system $(x^i)$ in which $F$ is projectively flat.

Therefore, by theorem (5.1) and definition (5.4), we have

**Theorem 5.2.** A Finsler metric $F$ on an open subset $U$ of $\mathbb{R}^n$ is projectively flat if and only if it satisfies the following system of differential equations

$$F_{x^k y^\ell} y^k - F_{x^\ell} = 0.$$ 

Here $G^i = Py^i$, where $G^i$ is spray coefficient and the scalar function $P = \frac{F_{x^k y^k}}{2F}$ is called projective factor of $F$.

In further calculations, we assume that $F^2 = A$, i.e., $F$ is a square root Finsler metric.

First we find necessary and sufficient conditions for Kropina-Randers changed Finsler metric $\bar{F} = F^2 \beta^i + \beta_i$ to be projectively flat.

Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{A}{\beta} + \beta.$$

Differentiating (5.1) w.r.t. $x^k$, we get

$$\bar{F}_{x^k} = \frac{A_{x^k}}{\beta} - \frac{A}{\beta^2} \beta_{x^k} + \beta_{x^k}.$$ (5.2)

Differentiation of (5.2) further w.r.t. $y^\ell$ gives

$$\bar{F}_{x^k y^\ell} = \frac{A_{x^k y^\ell}}{\beta} - \frac{A_{x^k}}{\beta^2} \beta_{y^\ell} - \frac{A}{\beta^2} \beta_{x^k y^\ell} - \frac{A_{\beta_{x^k y^\ell}}}{\beta^2} \beta_{x^k} + 2\frac{A}{\beta^3} \beta_{x^k} \beta_{\ell} + \beta_{x^k y^\ell}. \quad (5.3)$$

Contracting (5.3) with $y^k$, we get

$$\bar{F}_{x^k y^\ell} y^k = \frac{A_{0\ell}}{\beta} - \frac{A_{0\ell}}{\beta^2} - \frac{A_{\beta_{0\ell}}}{\beta^2} + \frac{2A}{\beta^3} \beta_{0\ell} \beta_{\ell} + \beta_{0\ell}$$

$$= \frac{1}{\beta^3} \left\{ \beta^2 A_{0\ell} - \beta A_{0\ell} - \beta A_{0\ell} - \beta A_{0\ell} + 2A \beta_{0\ell} + \beta^3 \beta_{0\ell} \right\}.$$
From the equation (5.2), we have
\[ \bar{F}_{\ell} = \frac{A_{\ell}}{\beta} - \frac{A}{\beta^2} \beta_{\ell} + \beta_{\ell} = \frac{1}{\beta^3} \{ \beta^2 A_{\ell} - A \beta \beta_{\ell} + \beta^3 \beta_{x,\ell} \}. \]

We know that \( \bar{F} \) is projectively flat if and only if
\[ \bar{F}_{x, k} y^k - \bar{F}_{x,\ell} = 0, \]
i.e.,
\[ \frac{1}{\beta^3} \{ \beta^2 A_{0\ell} - \beta A_0 \beta_{\ell} - \beta A \beta_{0\ell} - \beta A_{\ell} \beta_0 + 2 A \beta_0 \beta_{\ell} + \beta^3 \beta_{0\ell} \} - \frac{1}{\beta^3} \{ \beta^2 A_{x,\ell} - A \beta \beta_{x,\ell} + \beta^3 \beta_{x,\ell} \} = 0, \]
i.e.,
\[ A \{ 2 \beta_0 \beta_{\ell} + \beta (\beta_{x,\ell} - \beta_0 \ell) \} + \{ \beta^3 (\beta_{0\ell} - \beta_{x,\ell}) + \beta^2 (A_{0\ell} - A_{x,\ell}) - \beta (A_0 \beta_{\ell} + A_{\ell} \beta_0) \} = 0. \]

From the above equation, we conclude that \( \bar{F} \) is projectively flat if and only if following two equations are satisfied.
\[ 2 \beta_0 \beta_{\ell} + \beta (\beta_{x,\ell} - \beta_0 \ell) = 0 \quad (5.4) \]
\[ \beta^3 (\beta_{0\ell} - \beta_{x,\ell}) + \beta^2 (A_{0\ell} - A_{x,\ell}) - \beta (A_0 \beta_{\ell} + A_{\ell} \beta_0) = 0 \quad (5.5) \]

Above discussion leads to the following theorem.

**Theorem 5.3.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \( \bar{F} = \frac{F^2}{\beta} + \beta \) as a Kropina-Randers changed metric. Then \( \bar{F} \) is projectively flat if and only if equations (5.4) and (5.5) are satisfied.

Next, we find necessary and sufficient conditions for generalized Kropina-Randers changed Finsler metric
\[ \bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \quad (m \neq 0, -1) \]
to be projectively flat.

Let us put \( F^2 = A \) in \( \bar{F} \), then
\[ \bar{F} = \frac{A^{(m+1)/2}}{\beta^m} + \beta. \quad (5.6) \]

Differentiating (5.6) w.r.t. \( x^k \), we get
\[ \bar{F}_{x^k} = \frac{m+1}{2 \beta^m} A^{(m-1)/2} A_{x^k} - \frac{m}{\beta^{m+1}} A^{(m+1)/2} \beta_{x^k} + \beta_{x^k}. \quad (5.7) \]
Differentiation of (5.7) further w.r.t. $y^\ell$ gives

$$
\bar{F}_{x^k y^\ell} = \frac{m+1}{2b^m} A^{(m-1)/2} A_{x^k y^\ell} + \frac{m^2-1}{4b^m} A^{(m-3)/2} A_{x^k} A_{\ell} - \frac{m(m+1)}{2b^{m+1}} A^{(m-1)/2} A_{x^k} \beta_{\ell}
\ 
- \frac{m}{b^{m+1}} A^{(m+1)/2} \beta_{x^k y^\ell} - \frac{m(m+1)}{2b^{m+1}} A^{(m-1)/2} \beta_{x^k} A_{\ell} + \frac{m(m+1)}{b^{m+2}} \beta_{\ell} A^{(m+1)/2} \beta_{x^k} + \beta_{x^k y^\ell}.
$$

(5.8)

Contracting (5.8) with $y^k$, we get

$$
\bar{F}_{x^k y^k}^{y^k} = \frac{m+1}{2b^m} A^{(m-1)/2} A_{0\ell} + \frac{m^2-1}{4b^m} A^{(m-3)/2} A_{0} A_{\ell} - \frac{m(m+1)}{2b^{m+1}} A^{(m-1)/2} A_{0} \beta_{\ell}
\ 
- \frac{m}{b^{m+1}} A^{(m+1)/2} \beta_{0\ell} - \frac{m(m+1)}{2b^{m+1}} A^{(m-1)/2} \beta_{0} \beta_{\ell} + \beta_{0\ell}
\ 
= \frac{m+1}{2b^{m+2}} A^{(m-3)/2}
\left\{ \beta^2 A A_{0\ell} + \frac{m-1}{2} \beta^2 A A_0 A_\ell - m \beta A A_0 \beta_\ell - \frac{2m}{m+1} \beta A^2 \beta_\ell
\ 
- m \beta A \beta_\ell A + 2m \beta_\ell A^2 \beta_0 + \frac{2}{m+1} b^{m+2} A^{(-m+3)/2} \beta_{0\ell} \right\}.
$$

From the equation (5.7), we get

$$
\bar{F}_{x^\ell} = \frac{m+1}{2b^m} A^{(m-1)/2} A_{x^\ell} - \frac{m}{b^{m+1}} A^{(m+1)/2} \beta_{x^\ell} + \beta_{x^\ell}
\ 
= \frac{m+1}{2b^{m+2}} A^{(m-3)/2}
\left\{ \beta^2 A A_{x^\ell} - \frac{2m}{m+1} \beta A^2 \beta_{x^\ell} + \frac{2}{m+1} b^{m+2} A^{(-m+3)/2} \beta_{x^\ell} \right\}.
$$

We know that $\bar{F}$ is projectively flat if and only if $\bar{F}_{x^k y^k}^{y^k} - \bar{F}_{x^\ell} = 0$

i.e.,

$$
\frac{m+1}{2b^{m+2}} A^{(m-3)/2}
\left\{ \beta^2 A A_{0\ell} + \frac{m-1}{2} \beta^2 A A_0 A_\ell - m \beta A A_0 \beta_\ell - \frac{2m}{m+1} \beta A^2 \beta_\ell - m \beta A \beta_\ell A + 2m \beta_\ell A^2 \beta_0
\ 
+ \frac{2}{m+1} b^{m+2} A^{(-m+3)/2} \beta_{0\ell} \right\} - \left\{ \beta^2 A A_{x^\ell} - \frac{2m}{m+1} \beta A^2 \beta_{x^\ell} + \frac{2}{m+1} b^{m+2} A^{(-m+3)/2} \beta_{x^\ell} \right\} = 0,
$$

which implies

$$
\frac{m-1}{2} \beta^2 A A_{0\ell} + A \{ \beta^2 (A_{0\ell} - A_{x^\ell}) - m \beta (A_{0\ell} + \beta_0 A_\ell) \} + 2m A^2 \left\{ \frac{\beta}{m+1} (\beta_{x^\ell} - \beta_0) + \beta_0 \beta_\ell \right\}
\ 
+ \frac{2}{m+1} b^{m+2} A^{(-m+3)/2} (\beta_{0\ell} - \beta_{x^\ell}) = 0.
$$
From the above equation, we conclude that $\bar{F}$ is projectively flat if and only if following four equations are satisfied.

\[ \frac{m-1}{2} \beta^2 A_0 A_\ell = 0 \] (5.9)

\[ \beta^2 (A_0 \ell - A_x \ell) - m \beta (A_0 \beta_\ell + \beta_0 A_\ell) = 0 \] (5.10)

\[ \frac{\beta}{m+1} (\beta_x \ell - \beta_0 \ell) + \beta_\ell \beta_0 = 0 \] (5.11)

\[ \beta_0 \ell = \beta_x \ell. \] (5.12)

Further, from the equation (5.9), we see that either $m = 1$ or $A_0 A_\ell = 0$.

Now, if $m = 1$, then (5.11) reduces to (5.4) and (5.10), (5.12) reduce to (5.5).

But the equations (5.4) and (5.5) are necessary and sufficient conditions for Kropina-Randers changed Finsler metric to be projectively flat. Therefore, we exclude the case $m = 1$ for general case.

Then

\[ A_0 A_\ell = 0 \] (5.13)

Also from the equations (5.11) and (5.12), we get

\[ \beta_0 \beta_\ell = 0 \] (5.14)

Above discussion leads to the following theorem.

**Theorem 5.4.** Let $(M, \bar{F})$ be an $n$-dimensional Finsler space with $F = F^{m+1} + \beta (m \neq -1, 0, 1)$ as generalized Kropina-Randers changed metric. Then $\bar{F}$ is projectively flat if and only if equations (5.10), (5.12), (5.13) and (5.14) are satisfied.

Next, we find necessary and sufficient conditions for square-Randers changed Finsler metric

\[ \bar{F} = \frac{(F + \beta)^2}{F} + \beta \]

to be projectively flat.

Let us put $F^2 = A$ in $\bar{F}$, then

\[ \bar{F} = A^{1/2} + \frac{\beta^2}{A^{1/2}} + 3\beta. \] (5.15)
Differentiating (5.15) w.r.t. \( x^k \), we get

\[
\bar{F}_{x^k} = \frac{1}{2} A^{-1/2} A_{x^k} + 2 \beta A^{-1/2} \beta_{x^k} - \frac{1}{2} \beta^2 A^{-3/2} A_{x^k} + 3 \beta_{x^k}. \tag{5.16}
\]

Differentiation of (5.16) w.r.t. \( y^f \) gives

\[
\bar{F}_{x^k y^f} = \frac{1}{2} A^{-1/2} A_{x^k y^f} - \frac{1}{4} A^{-3/2} A_{f} A_{x^k} + 2 \beta A^{-1/2} \beta_{x^k y^f} + 2 \beta_{f} A^{-1/2} \beta_{x^k} - \beta A^{-3/2} A_{f} \beta_{x^k} - \frac{1}{2} \beta^2 A^{-3/2} A_{x^k} + 3 \beta_{x^k y^f}.
\]

Contracting (5.17) with \( y^k \), we get

\[
\bar{F}_{x^k y^f y^k} = \frac{1}{2} A^{-1/2} A_{x^k A^0_{f}} - \frac{1}{4} A^{-3/2} A_{f} A_0 + 2 \beta A^{-1/2} \beta_{f} A_0 + 2 \beta_{f} A^{-1/2} \beta_0 - \beta A^{-3/2} A_{f} \beta_0 - \frac{1}{2} \beta^2 A^{-3/2} A_{f} - \beta \beta_0 - \beta A^{-5/2} A_{f} A_0 + 3 \beta_{f}
\]

From the equation (5.10), we get

\[
\bar{F}_{x^f} = \frac{1}{2} A^{-1/2} A_{x^f} + 2 \beta A^{-1/2} \beta_{x^f} - \frac{1}{2} \beta^2 A^{-3/2} A_{x^f} + 3 \beta_{x^f}
\]

We know that \( \bar{F} \) is projectively flat if and only if \( \bar{F}_{x^k y^k} - \bar{F}_{x^f} = 0 \), i.e.,

\[
\frac{1}{4} A^{-5/2} \left[ 2 A^2 A_{x^f} + 8 \beta A^2 \beta_{x^f} - 2 \beta^2 A A_{x^f} + 12 A^{5/2} \beta_{x^f} \right] = 0,
\]

i.e.,

\[
3 \beta^2 A_{x^f} + 2 A \{ \beta^2 (A_{x^f} - A_{0^f}) - 2 \beta (A_{x^f} + A_{0^f}) - A_{x^f} \} + 2 A^2 \{ (A_{0^f} - A_{x^f}) + 4 \beta A_{0^f} + 4 \beta (A_{0^f} - A_{x^f}) \} + 12 A^{5/2} \{ \beta_{0^f} - \beta_{x^f} \} = 0.
\]
From the above equation, we conclude that \( \bar{F} \) is projectively flat if and only if following four equations are satisfied.

\[
A_\ell A_0 = 0 \tag{5.18}
\]

\[
\beta^2 (A_x \ell - A_0 \ell) - 2\beta (A_\ell \beta_0 + \beta_\ell A_0) - A_\ell A_0 = 0 \tag{5.19}
\]

\[
4\beta (\beta_0 \ell - \beta_x \ell) + A_0 \ell - A_x \ell + 4\beta_\ell \beta_0 = 0 \tag{5.20}
\]

\[
\beta_0 \ell = \beta_x \ell. \tag{5.21}
\]

Above discussion leads to the following theorem.

**Theorem 5.5.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{(F + \beta)^2}{F} + \beta\) as a square-Randers changed metric. Then \(\bar{F}\) is projectively flat if and only if the equations (5.18), (5.19), (5.20), and (5.21) are satisfied.

Next, we find necessary and sufficient conditions for Matsumoto-Randers changed Finsler metric \(\bar{F} = F^2 F - \beta\) to be projectively flat.

Let us put \(F^2 = A\) in \(\bar{F}\), then

\[
\bar{F} = \frac{A}{\sqrt{A - \beta}} + \beta. \tag{5.22}
\]

Differentiating (5.22) w.r.t. \(x^k\), we get

\[
\bar{F}_x^k = \frac{1}{\sqrt{A - \beta}} A_x^k - \frac{\sqrt{A}}{2(\sqrt{A - \beta})^2} A_x^k + \frac{A}{(\sqrt{A - \beta})^2} \beta_x^k + \beta_x^k. \tag{5.23}
\]

Differentiation of (5.23) further w.r.t. \(y^\ell\) gives

\[
\bar{F}_{x^k y^\ell} = \frac{1}{\sqrt{A - \beta}} A_{x^k y^\ell} - \frac{1}{(\sqrt{A - \beta})^2} A_{x^k} \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) - \frac{\sqrt{A}}{2(\sqrt{A - \beta})^2} A_{x^k y^\ell}

- \frac{1}{4\sqrt{A}(\sqrt{A - \beta})^2} A_\ell A_x^k + \frac{\sqrt{A}}{(\sqrt{A - \beta})^3} \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) A_x^k + \frac{A}{(\sqrt{A - \beta})^2} \beta_{x^k y^\ell}

+ \frac{1}{(\sqrt{A - \beta})^2} A_\ell \beta_x^k - \frac{2A}{(\sqrt{A - \beta})^3} \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) \beta_x^k + \beta_{x^k y^\ell}. \tag{5.24}
\]
From the equation (5.23), we get

\[ F_{x^k y^l} y^k = \frac{1}{\sqrt{A - \beta}} A_{0\ell} - \frac{1}{(\sqrt{A - \beta})^2} A_0 \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) - \frac{\sqrt{A}}{2(\sqrt{A - \beta})^2} A_{0\ell} \]

\[- - \frac{1}{4\sqrt{A}(\sqrt{A - \beta})^2} A_\ell A_0 + \frac{\sqrt{A}}{(\sqrt{A - \beta})^3} \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) A_0 + \frac{A}{(\sqrt{A - \beta})^2} \beta_{0\ell} \]

\[ + \frac{1}{(\sqrt{A - \beta})^2} A_\ell \beta_0 - \frac{2A}{(\sqrt{A - \beta})^3} \left( \frac{1}{2\sqrt{A}} A_\ell - \beta_\ell \right) \beta_0 + \beta_{0\ell}. \]

Simplifying, we get

\[ F_{x^k y^l} y^k = \frac{1}{4\sqrt{A}(\sqrt{A - \beta})^3} \left\{ 8A^2 \beta_{0\ell} + 2A^{3/2} (A_{0\ell} + 4\beta_0 \beta_\ell - 8\beta_0 \beta_\ell) + 6A \left( 2\beta^2 \beta_{0\ell} - \beta A_{0\ell} \right) \]

\[ + \sqrt{A} \left( -4\beta^3 \beta_{0\ell} + 4\beta^2 A_{0\ell} - 4\beta (A_0 \beta_\ell + A_\ell \beta_0) - A_0 A_\ell \right) + 3\beta A_0 A_\ell \right\}. \]

From the equation (5.23), we get

\[ F_{x^\ell} = \frac{1}{\sqrt{A - \beta}} A_{x^\ell} - \frac{\sqrt{A}}{2(\sqrt{A - \beta})^2} A_{x^\ell} + \frac{A}{(\sqrt{A - \beta})^2} \beta_{x^\ell} + \beta_{x^\ell} \]

\[ = \frac{8A^2 \beta_{x^\ell} + 2A^{3/2} (A_{x^\ell} - 8\beta_{x^\ell}) - 6A (\beta A_{x^\ell} - 2\beta^2 \beta_{x^\ell}) + \sqrt{A} (4\beta^2 A_{x^\ell} - 4\beta^3 \beta_{x^\ell})}{4\sqrt{A}(\sqrt{A - \beta})^3}. \]

We know that \( F \) is projectively flat if and only if \( F_{x^k y^l} y^k - F_{x^\ell} = 0 \), i.e.,

\[ \frac{1}{4\sqrt{A}(\sqrt{A - \beta})^3} \left\{ 8A^2 \beta_{0\ell} + 2A^{3/2} (A_{0\ell} + 4\beta_0 \beta_\ell - 8\beta_0 \beta_\ell) + 6A \left( 2\beta^2 \beta_{0\ell} - \beta A_{0\ell} \right) \]

\[ + \sqrt{A} \left( -4\beta^3 \beta_{0\ell} + 4\beta^2 A_{0\ell} - 4\beta (A_0 \beta_\ell + A_\ell \beta_0) - A_0 A_\ell \right) + 3\beta A_0 A_\ell \right\} = 0, \]

i.e.,

\[ 8A^2 (\beta_{0\ell} - \beta_{x^\ell}) + 2A^{3/2} \left\{ A_{0\ell} - A_{x^\ell} + 4\beta_0 \beta_\ell + 8\beta (\beta_{x^\ell} - \beta_{0\ell}) \right\} + 6A \left\{ 2\beta^2 (\beta_{0\ell} - \beta_{x^\ell}) + \beta (A_{x^\ell} - A_{0\ell}) \right\} \]

\[ + \sqrt{A} \left\{ 4\beta^3 (\beta_{x^\ell} - \beta_{0\ell}) + 4\beta^2 (A_{0\ell} - A_{x^\ell}) - 4\beta (A_0 \beta_\ell + A_\ell \beta_0) - A_0 A_\ell \right\} + 3\beta A_0 A_\ell = 0. \]
From the above equation, we conclude that $\bar{F}$ is projectively flat if and only if following five equations are satisfied.

\[
\beta_{0\ell} = \beta_{x\ell} 
\]  
(5.25)

\[
A_{0\ell} - A_{x\ell} + 4\beta_{0\beta} + 8\beta (\beta_{x\ell} - \beta_{0\ell}) = 0 
\]  
(5.26)

\[
2\beta^2 (\beta_{0\ell} - \beta_{x\ell}) + \beta (A_{x\ell} - A_{0\ell}) = 0 
\]  
(5.27)

\[
4\beta^3 (\beta_{x\ell} - \beta_{0\ell}) + 4\beta^2 (A_{0\ell} - A_{x\ell}) - 4\beta (A_{0\beta} + A_{\beta}) - A_{0}A_{\ell} = 0 
\]  
(5.28)

\[
A_{0}A_{\ell} = 0 
\]  
(5.29)

Further, from the equations (5.25) and (5.27), we get

\[
A_{x\ell} = A_{0\ell}. 
\]  
(5.30)

Again from the equations (5.25), (5.30) and (5.26), we get

\[
\beta_{0}\beta_{\ell} = 0, 
\]  
(5.31)

and from the equations (5.25), (5.30), (5.29) and (5.28), we get

\[
A_{0}\beta_{\ell} + A_{\ell}\beta_{0} = 0. 
\]  
(5.32)

Above discussion leads to the following theorem.

**Theorem 5.6.** Let $(M, \bar{F})$ be an $n$-dimensional Finsler space with $\bar{F} = F^2 + \beta$ as a Matsumoto-Randers changed metric. Then $\bar{F}$ is projectively flat if and only if the following equations are satisfied:

\[
A_{0}A_{\ell} = 0, A_{0\ell} = A_{x\ell}, \beta_{0}\beta_{\ell} = 0, \beta_{0\ell} = \beta_{x\ell}, A_{\ell}\beta_{0} + A_{0}\beta_{\ell} = 0. 
\]

Next, we find necessary and sufficient conditions for exponential-Randers changed Finsler metric

\[
\bar{F} = Fe^{\beta/F} + \beta 
\]

to be projectively flat.

Let us put $F^2 = A$ in $\bar{F}$, then

\[
\bar{F} = \sqrt{A}e^{\beta/\sqrt{A}} + \beta. 
\]  
(5.33)
Differentiation of (5.34) w.r.t. \(x^k\), we get
\[
\bar{F}_{x^k} = e^{\beta/\sqrt{\lambda}} \beta_{x^k} - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_{x^k} + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_{x^k} + \beta_{x^k}.
\] (5.34)

Differentiation of (5.34) further w.r.t. \(y^\ell\) gives
\[
\bar{F}_{x^k y^\ell} = e^{\beta/\sqrt{\lambda}} \beta_{x^k y^\ell} + e^{\beta/\sqrt{\lambda}} \beta_{x^k} \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_{x^k y^\ell} - \frac{1}{2A} e^{\beta/\sqrt{\lambda}} A_{x^k} \beta_\ell
\]
\[
+ \frac{\beta}{2A^2} e^{\beta/\sqrt{\lambda}} A_{x^k} A_\ell - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_{x^k} \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_{x^k} \beta_\ell
\]
\[
- \frac{1}{4A^{3/2}} e^{\beta/\sqrt{\lambda}} A_{x^k} A_\ell + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_{x^k} \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) + \beta_{x^k y^\ell}.
\] (5.35)

Contracting (5.35) with \(y^k\), we get
\[
L_{x^k y^k} = e^{\beta/\sqrt{\lambda}} \beta_{0\ell} + e^{\beta/\sqrt{\lambda}} \beta_0 \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_{0\ell} - \frac{1}{2A} e^{\beta/\sqrt{\lambda}} A_0 \beta_\ell
\]
\[
+ \frac{\beta}{2A^2} e^{\beta/\sqrt{\lambda}} A_{0\ell} - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_0 \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_0 \beta_\ell
\]
\[
- \frac{1}{4A^{3/2}} e^{\beta/\sqrt{\lambda}} A_0 \beta_\ell + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_0 \left( \frac{1}{\sqrt{A}} \beta_\ell - \frac{\beta}{2A^{3/2}} A_\ell \right) + \beta_{0\ell}
\]
\[
= \frac{1}{4A^{5/2}} \left[ 4A^{5/2} \left( e^{\beta/\sqrt{\lambda}} + 1 \right) \beta_{0\ell} + 2A^2 e^{\beta/\sqrt{\lambda}} (2\beta_{0\ell} + A_{0\ell}) - 2\beta A^{3/2} e^{\beta/\sqrt{\lambda}} A_{0\ell}
\]
\[
- A e^{\beta/\sqrt{\lambda}} \left( A_{0\ell} + 2\beta (A_{0\ell} + A_{0\ell}) \right) + A^{1/2} \beta e^{\beta/\sqrt{\lambda}} A_{0\ell} \right].
\]

From the equation (5.34), we get
\[
\bar{F}_{x^\ell} = e^{\beta/\sqrt{\lambda}} \beta_{x^\ell} - \frac{\beta}{2A} e^{\beta/\sqrt{\lambda}} A_{x^\ell} + \frac{1}{2\sqrt{A}} e^{\beta/\sqrt{\lambda}} A_{x^\ell} + \beta_{x^\ell}
\]
\[
= \frac{1}{4A^{5/2}} \left[ 4A^{5/2} \left( e^{\beta/\sqrt{\lambda}} + 1 \right) \beta_{x^\ell} + 2A^2 e^{\beta/\sqrt{\lambda}} (2\beta_{x^\ell} + A_{x^\ell}) - 2\beta A^{3/2} e^{\beta/\sqrt{\lambda}} A_{x^\ell}
\]
\[
- A e^{\beta/\sqrt{\lambda}} \left( A_{0\ell} + 2\beta (A_{0\ell} + A_{0\ell}) \right) + A^{1/2} \beta e^{\beta/\sqrt{\lambda}} A_{0\ell} \right]
\]
We know that \(\bar{F}\) is projectively flat if and only if \(\bar{F}_{x^k y^k} y^k - \bar{F}_{x^\ell} = 0\),

i.e.,
\[
\frac{1}{4A^{5/2}} \left[ 4A^{5/2} \left( e^{\beta/\sqrt{\lambda}} + 1 \right) \beta_{0\ell} + 2A^2 e^{\beta/\sqrt{\lambda}} (2\beta_{0\ell} + A_{0\ell}) - 2\beta A^{3/2} e^{\beta/\sqrt{\lambda}} A_{0\ell}
\]
\[
- A e^{\beta/\sqrt{\lambda}} \left( A_{0\ell} + 2\beta (A_{0\ell} + A_{0\ell}) \right) + A^{1/2} \beta e^{\beta/\sqrt{\lambda}} A_{0\ell} \right]
\]
\[
- \frac{1}{4A^{5/2}} \left[ 4A^{5/2} \left( e^{\beta/\sqrt{\lambda}} + 1 \right) \beta_{x^\ell} + 2A^2 e^{\beta/\sqrt{\lambda}} A_{x^\ell} - 2\beta A^{3/2} e^{\beta/\sqrt{\lambda}} A_{x^\ell} \right] = 0,
\]

30
i.e.,
\[
4A^{5/2} \left( e^{\beta/\sqrt{A}} + 1 \right) (\beta_0 - \beta_x) + 2A^2 e^{\beta/\sqrt{A}} (2\beta_0 \beta_x + A_0 - A_x) - 2\beta A^{3/2} e^{\beta/\sqrt{A}} (A_0 - A_x)
\]
\[
- A e^{\beta/\sqrt{A}} \left\{ A_0 A_x + 2\beta (A_0 \beta_x + A_0 \beta_0) \right\} + A^{1/2} e^{\beta/\sqrt{A}} A_0 A_x + \beta^2 e^{\beta/\sqrt{A}} A_0 A_x = 0.
\]

From the above equation, we conclude that \( \bar{F} \) is projectively flat if and only if following five equations are satisfied.

\[
\beta_0 = \beta_x \quad (5.36)
\]
\[
2\beta_0 \beta_x + A_0 - A_x = 0 \quad (5.37)
\]
\[
A_0 - A_x = 0 \quad \Rightarrow \quad A_0 = A_x \quad (5.38)
\]
\[
A_0 A_x + 2\beta (A_0 \beta_x + A_0 \beta_0) = 0 \quad (5.39)
\]
\[
A_0 A_x = 0 \quad (5.40)
\]

Further, from the equations (5.37) and (5.38), we get

\[
\beta_0 \beta_x = 0,
\]

and from the equations (5.39) and (5.40), we get

\[
A_0 \beta_x + A_x \beta_0 = 0.
\]

The above discussion leads to the following theorem.

**Theorem 5.7.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \( \bar{F} = F e^{\beta/F} + \beta \) as an exponential-Randers changed metric. Then \( \bar{F} \) is projectively flat if and only if the following equations are satisfied:

\[
A_0 A_x = 0, \quad A_0 = A_x, \quad \beta_0 \beta_x = 0, \quad \beta_0 = \beta_x, \quad A_0 \beta_0 + A_0 \beta_x = 0.
\]

Next, we find necessary and sufficient conditions for infinite series-Randers changed Finsler metric

\[
\bar{F} = \frac{\beta^2}{\beta - F} + \beta
\]

31
to be projectively flat.
Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{\beta^2}{\beta - \sqrt{A}} + \beta.$$  (5.41)

Differentiating (5.41) w.r.t. $x^k$, we get

$$\bar{F}_x = \frac{2\beta}{\beta - \sqrt{A}} \beta_x - \frac{\beta^2}{(\beta - \sqrt{A})^2} \beta_x^2 + \frac{\beta^2}{2\sqrt{A}(\beta - \sqrt{A})^2} A_{x^k} + \beta_{x^k}.  \tag{5.42}$$

Differentiation of (5.42) further w.r.t. $y^l$ gives

$$\bar{F}_{x^{k}y^l} = \frac{2\beta}{\beta - \sqrt{A}} \beta_{x^k y^l} + \frac{2}{\beta - \sqrt{A}} \beta_{x^k} \beta_{y^l} - \frac{2\beta}{(\beta - \sqrt{A})^2} \beta_{x^k} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) - \frac{\beta^2}{(\beta - \sqrt{A})^2} \beta_{x^k y^l}
- \frac{2\beta}{(\beta - \sqrt{A})^2} \beta_{x^k} \beta_{y^l} + \frac{2\beta^2}{(\beta - \sqrt{A})^3} \beta_{x^k} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) + \frac{\beta^2}{2\sqrt{A}(\beta - \sqrt{A})^2} A_{x^k y^l}
+ \frac{\beta}{\sqrt{A}(\beta - \sqrt{A})^2} A_{x^k} \beta_{y^l} - \frac{\beta^2}{4A^{3/2}(\beta - \sqrt{A})^2} A_{x^k} A_{\ell} - \frac{\beta^2}{\sqrt{A}(\beta - \sqrt{A})^3} A_{x^k} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) + \beta_{x^k y^l}.  \tag{5.43}$$

Contracting (5.43) with $y^k$, we get

$$L_{x^k y^l y^k} = \frac{2\beta}{\beta - \sqrt{A}} \beta_{0\ell} + \frac{2}{\beta - \sqrt{A}} \beta_{0} \beta_{\ell} - \frac{2\beta}{(\beta - \sqrt{A})^2} \beta_{0} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) - \frac{\beta^2}{(\beta - \sqrt{A})^2} \beta_{0 \ell}
- \frac{2\beta}{(\beta - \sqrt{A})^2} \beta_{0} \beta_{\ell} + \frac{2\beta^2}{(\beta - \sqrt{A})^3} \beta_{0} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) + \frac{\beta^2}{2\sqrt{A}(\beta - \sqrt{A})^2} A_{0 \ell}
+ \frac{\beta}{\sqrt{A}(\beta - \sqrt{A})^2} A_{0} \beta_{\ell} - \frac{\beta^2}{4A^{3/2}(\beta - \sqrt{A})^2} A_{0} A_{\ell} - \frac{\beta^2}{\sqrt{A}(\beta - \sqrt{A})^3} A_{0} \left( \beta_{\ell} - \frac{1}{2\sqrt{A}} A_{\ell} \right) + \beta_{0 \ell}.$$

Simplifying, we get

$$L_{x^k y^l y^k} = \frac{1}{4A^{3/2}(\beta - \sqrt{A})^3} \left[ -4A^3 \beta_{0\ell} + A^{5/2} \left( 20 \beta_{0\ell} + 8 \beta_0 A_{\ell} \right) - 24A^2 \beta^2 \beta_{0\ell}
+ A^{3/2} \left( 8 \beta^3 \beta_{0\ell} - 2 \beta^2 A_{0\ell} - 4 \beta A_{0} \beta_{\ell} + A_{\ell} \beta_{0} \right) \right] + 2 \beta^3 A A_{0\ell} + 3A^{1/2} \beta^2 A_{0} A_{\ell} - \beta^3 A_{0} A_{\ell}.  \tag{5.44}$$

32
From the above equation (5.42), we get

\[
\tilde{F}_{x^t} = \frac{2\beta}{\beta - \sqrt{A}} \beta_{x^t} - \frac{\beta^2}{(\beta - \sqrt{A})^2} \beta_{x^t} + \frac{\beta^2}{2\sqrt{A}(\beta - \sqrt{A})^2} A_{x^t} + \beta_{x^t}
\]

\[
= \frac{2\beta^2 - 4\sqrt{A}\beta + A}{(\beta - \sqrt{A})^2} \beta_{x^t} + \frac{\beta^2}{2\sqrt{A}(\beta - \sqrt{A})^2} A_{x^t}
\]

\[
= \frac{1}{4A^{3/2}(\beta - \sqrt{A})^3} \left[ -4A^3 \beta_{0\ell} + 20A^{5/2} \beta_{x^t} - 24A^2 \beta^2 \beta_{x^t} + A^{3/2} \left\{ 8\beta^3 \beta_{x^t} - 2\beta^2 A_{x^t} \right\} + 2A\beta^3 A_{x^t} \right].
\]

We know that \(\tilde{F}\) is projectively flat if and only if \(\tilde{F}_{x^t y^k} y^k - \tilde{F}_{x^t} = 0\), i.e.,

\[
\frac{1}{4A^{3/2}(\beta - \sqrt{A})^3} \left[ -4A^3 \beta_{0\ell} + 20A^{5/2} \left\{ 20\beta \beta_{0\ell} + 8\beta_{0\beta} \right\} - 24A^2 \beta^2 \beta_{0\ell} + A^{3/2} \left\{ 8\beta^3 \beta_{0\ell} - 2\beta^2 A_{0\ell} \right\}
\]

\[
- 4\beta \left( A_0 \beta_{0\ell} + 4A_{0\ell} \right) \right\} + 2a^3 A\beta_{0\ell} + 3A^{1/2} A_0 A_{0\ell} - \beta^3 A_0 A_{0\ell} \right\} - \frac{1}{4A^{3/2}(\beta - \sqrt{A})^3} \left[ -4A^3 \beta_{x^t}\right.
\]

\[
+ 20A^{5/2} \beta_{x^t} - 24A^2 \beta^2 \beta_{x^t} + A^{3/2} \left\{ 8\beta^3 \beta_{x^t} - 2\beta^2 A_{x^t} \right\} + 2A\beta^3 A_{x^t} \right] = 0.
\]

Simplifying, we get

\[
4A^3 \left\{ \beta_{x^t} - \beta_{0\ell} \right\} + 4A^{5/2} \left\{ 5\beta \left( \beta_{0\ell} - \beta_{x^t} \right) + 2\beta_{0\beta} \right\} + 24A^2 \beta^2 \left\{ \beta_{x^t} - \beta_{0\ell} \right\}
\]

\[
+ 2A^{3/2} \left\{ 4\beta^3 \left( \beta_{0\ell} - \beta_{x^t} \right) + \beta^2 \left( A_{x^t} - A_{0\ell} \right) - 2\beta \left( A_0 \beta_{0\ell} + A_{0\ell} \beta_{0} \right) \right\} + 2A\beta^3 \left\{ A_{0\ell} - A_{x^t} \right\}
\]

\[
+ 3A^{1/2} \beta^2 A_0 A_{0\ell} - \beta^3 A_0 A_{0\ell} = 0.
\]

From the above equation, we conclude that \(\tilde{F}\) is projectively flat if and only if following five equations are satisfied.

\[
\beta_{x^t} - \beta_{0\ell} = 0 \implies \beta_{x^t} = \beta_{0\ell} \quad (5.44)
\]

\[
5\beta \left( \beta_{0\ell} - \beta_{x^t} \right) + 2\beta_{0\beta} = 0 \quad (5.45)
\]

\[
4\beta^3 \left( \beta_{0\ell} - \beta_{x^t} \right) + \beta^2 \left( A_{x^t} - A_{0\ell} \right) - 2\beta \left( A_0 \beta_{0\ell} + A_{0\ell} \beta_{0} \right) = 0 \quad (5.46)
\]

\[
A_{0\ell} - A_{x^t} = 0 \implies A_{0\ell} = A_{x^t} \quad (5.47)
\]
Further, from the equations (5.44) and (5.45), we get
\[ \beta_0 \beta_\ell = 0 \]  
(5.49)
and from the equations (5.44), (5.47) and (5.46), we get
\[ A_0 \beta_\ell + A_\ell \beta_0 = 0. \]  
(5.50)

Above discussion leads to the following theorem.

**Theorem 5.8.** Let \((M, \tilde{F})\) be an \(n\)-dimensional Finsler space with \(\tilde{F} = \beta^2 - \beta + \beta\) as an infinite series-Randers changed metric. Then \(\tilde{F}\) is projectively flat if and only if the following equations are satisfied:

\[ A_0 A_\ell = 0, \quad A_0 \ell = A_\ell \ell, \quad \beta_0 \beta_\ell = 0, \quad \beta_0 \ell = \beta_\ell \ell, \quad A_\ell \beta_0 + A_0 \beta_\ell = 0. \]

6 Dually flatness of Finsler metrics

Firstly, we recall [16] the following definition:

**Definition 6.1.** A Finsler metric \(F\) on a smooth \(n\)-dimensional manifold \(M\) is called locally dually flat if, at any point, there is a standard co-ordinate system \((x^i, y^\ell)\) in \(TM\), \((x^i)\) called adapted local co-ordinate system, such that
\[ L_{\ell k} y^k y^\ell - 2L_{\ell \ell} = 0, \quad \text{where} \quad L = F^2. \]

Next, we find the necessary and sufficient conditions for locally dually flatness of all the metrics constructed via Randers change in section two.

First, we find necessary and sufficient conditions for Kropina-Randers changed Finsler metric
\[ \tilde{F} = \frac{F^2}{\beta} + \beta \]
to be locally dually flat.

Let us put \(F^2 = A\) in \(\tilde{F}\), then
\[ \tilde{F} = \frac{A}{\beta} + \beta. \]

\[ L = \tilde{F}^2 = \frac{A^2}{\beta^2} + 2A + \beta^2. \]  
(6.1)
Differentiating (6.1) w.r.t. $x^k$, we get
\[ L_{x^k} = \frac{2A}{\beta^2} A_{x^k} - \frac{2A^2}{\beta^3} \beta_{x^k} + 2A_{x^k} + 2\beta \beta_{x^k}. \] (6.2)

Differentiation of (6.2) further w.r.t. $y^\ell$ gives
\[ L_{x^k y^\ell} = \frac{2A}{\beta^2} A_{x^k y^\ell} + \frac{2A}{\beta^2} A_{x^k} A_\ell - \frac{4A}{\beta^3} \beta_\ell A_{x^k} - \frac{2A^2}{\beta^3} \beta_{x^k y^\ell} - \frac{4A}{\beta^3} A_\ell \beta_{x^k} + \frac{6A^2}{\beta^4} \beta_\ell \beta_{x^k} + 2A_{x^k y^\ell} \\
+ 2\beta \beta_{x^k y^\ell} + 2\beta \beta_{x^k}. \] (6.3)

Contracting (6.3) with $y^k$, we get
\[ L_{x^k y^\ell} y^k = \frac{2A}{\beta^2} A_{0\ell} + \frac{2A}{\beta^2} A_0 A_\ell - \frac{4A}{\beta^3} A_{0\ell} - \frac{2A^2}{\beta^3} \beta_{0\ell} - \frac{4A}{\beta^3} A_{0\ell} + \frac{6A^2}{\beta^4} \beta_\ell \beta_{0\ell} + 2A_{0\ell} + 2\beta \beta_{0\ell} + 2\beta \beta_0 \\
= \frac{2}{\beta^4} \left[ A^2 (3\beta_\ell \beta_0 - \beta_0 \beta_\ell) + A (\beta^2 A_{0\ell} - 2\beta (\beta_\ell A_0 - A_\ell \beta_0)) + \beta^5 \beta_0 \ell + \beta^4 (A_{0\ell} + \beta_\ell \beta_0) + \beta^2 A_0 A_\ell \right]. \]

Further, equation (6.2) can be rewritten as
\[ 2L_{x^\ell} = \frac{4A}{\beta^2} A_{x^\ell} - \frac{4A^2}{\beta^3} \beta_{x^\ell} + 4A_{x^\ell} + 4\beta \beta_{x^\ell} \\
= \frac{2}{\beta^4} \left[ -2A^2 \beta_{x^\ell} + 2A \beta^2 A_{x^\ell} + 2\beta^4 A_{x^\ell} + 2\beta^5 \beta_{x^\ell} \right]. \]

We know that $\bar{F}$ is locally dually flat if and only if $L_{x^k y^\ell} y^k - 2L_{x^\ell} = 0$, i.e.,
\[ \frac{2}{\beta^4} \left[ A^2 (3\beta_\ell \beta_0 - \beta_0 \beta_\ell) + A (\beta^2 A_{0\ell} - 2\beta (\beta_\ell A_0 - A_\ell \beta_0)) + \beta^5 \beta_0 \ell + \beta^4 (A_{0\ell} + \beta_\ell \beta_0) + \beta^2 A_0 A_\ell \right] \\
- \frac{2}{\beta^4} \left[ 2A^2 \beta_{x^\ell} - 2A^2 \beta_{x^\ell} + 2\beta^4 A_{x^\ell} + 2\beta^5 \beta_{x^\ell} \right] = 0, \]
i.e.,
\[ A^2 (3\beta_\ell \beta_0 - \beta_0 \beta_\ell + 2\beta \beta_{x^\ell}) + A \left\{ \beta^2 (A_{0\ell} - 2A_{x^\ell}) - 2\beta (\beta_\ell A_0 - A_\ell \beta_0) \right\} + \beta^4 (A_{0\ell} + \beta_\ell \beta_0 - 2A_{x^\ell}) + \beta^2 A_0 A_\ell = 0. \]

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following three equations are satisfied.
\[ 3\beta_\ell \beta_0 - \beta_0 \beta_\ell + 2\beta \beta_{x^\ell} = 0 \] (6.4)
\[ \beta^2 (A_0 \ell - 2A_x) - 2 \beta (\beta \ell A_0 - A_\ell) = 0 \] (6.5)

\[ \beta^5 (\beta \ell - 2 \beta_x) + \beta^4 (A_0 + \beta \ell \beta_0 - 2A_x) + \beta^2 A_0 A_\ell = 0 \] (6.6)

Hence, we have the following theorem.

**Theorem 6.1.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{F^2}{\beta} + \beta\) as a Kropina-Randers changed metric. Then \(\bar{F}\) is locally dually flat if and only if equations (6.4), (6.5) and (6.6) are satisfied.

Next, we find necessary and sufficient conditions for generalized Kropina-Randers changed Finsler metric

\[ \bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \] (m \neq 0, -1)

to be locally dually flat.

Let us put \(F^2 = A\) in \(\bar{F}\), then

\[ \bar{F} = \frac{A^{(m+1)/2}}{\beta^m} + \beta \]

\[ L = \frac{A^{m+1}}{\beta^{2m}} + 2 \frac{A^{(m+1)/2}}{\beta^{m-1}} + \beta^2. \] (6.7)

Differentiating (6.7) w.r.t. \(x^k\), we get

\[ L_{x^k} = (m+1)\frac{A^m}{\beta^{2m}} A_{x^k} - 2m \frac{A^{m+1}}{\beta^{2m+1}} \beta_{x^k} + (m+1) \frac{A^{(m-1)/2}}{\beta^{m-1}} A_{x^k} - 2(m-1) \frac{A^{(m+1)/2}}{\beta^m} \beta_{x^k} + 2 \beta \beta_{x^k}. \] (6.8)

Differentiation of (6.8) further w.r.t. \(y^\ell\) gives

\[ L_{x^k y^\ell} = (m+1) \frac{A^m}{\beta^{2m}} A_{x^k y^\ell} + m(m+1) \frac{A^{m-1}}{\beta^{2m}} A_{x^k} A_{y^\ell} - 2m(m+1) \frac{A^m}{\beta^{2m+1}} A_{x^k} \beta_{y^\ell}
- 2m \frac{A^{m+1}}{\beta^{2m+1}} \beta_{x^k y^\ell} - 2(m+1) \frac{A^m}{\beta^{2m+1}} A_{x^k} \beta_{y^\ell} + 2m(2m+1) \frac{A^{m+1}}{\beta^{2m+2}} \beta_{x^k} \beta_{y^\ell}
+ (m+1) \frac{A^{(m-1)/2}}{\beta^{m-1}} A_{x^k y^\ell} + \frac{m^2 - 1}{\beta^{m-1}} A_{x^k} A_{y^\ell} - (m^2 - 1) \frac{A^{(m-1)/2}}{\beta^m} A_{x^k} \beta_{y^\ell}
- 2(m-1) \frac{A^{(m+1)/2}}{\beta^m} \beta_{x^k y^\ell} - (m^2 - 1) \frac{A^{(m-1)/2}}{\beta^m} \beta_{x^k} \beta_{y^\ell}
+ 2 \beta \beta_{x^k y^\ell} + 2 \beta \beta_{x^k} \beta_{y^\ell}. \] (6.9)
Contracting \((6.9)\) with \(y^k\), we get
\[
L_{x^k y^l} y^k = (m+1) \frac{A^m}{\beta_{2m+1}} A_{0\ell} + m(m+1) \frac{A^{m-1}}{\beta_{2m}} A_{\ell} A_0 - 2m(m+1) \frac{A^m}{\beta_{2m+1}} A_0 \beta_\ell
\]
\[
- 2m \frac{A^{m+1}}{\beta_{2m+1}} \beta_\ell - 2m(m+1) \frac{A^m}{\beta_{2m+1}} A_\ell \beta_0 + 2m(2m+1) \frac{A^{m+1}}{\beta_{2m+2}} \beta_0 \beta_\ell
\]
\[
+ (m+1) \frac{A^{m-1/2}}{\beta_{m-1}} A_{0\ell} + \frac{m^2 - 1}{\beta_{m-1}} A^{(m-3/2)} A_0 A_\ell - (m^2 - 1) \frac{A^{(m-1)/2}}{\beta_{m+1}} A_0 \beta_\ell
\]
\[
- 2(m-1) \frac{A^{m+1/2}}{\beta_{m}} \beta_0 \beta_\ell - (m^2 - 1) \frac{A^{(m-1/2)}}{\beta_{m+1}} A_{0\ell} + 2m(m-1) \frac{A^{(m+1/2)}}{\beta_{m+1}} \beta_0 \beta_\ell
\]
\[+ 2\beta_0 \beta_\ell + 2\beta_\ell \beta_0 .\]

After simplification, we get
\[
L_{x^k y^l} y^k = \frac{A^{m-1}}{\beta_{2m+2}} \left[ (m+1) A^2 A_{0\ell} + m(m+1) \beta^2 A_\ell A_0 - 2m(m+1) A_\beta A_0 \beta_\ell - 2mA^2 \beta \beta_0 \ell \right]
\]
\[
- 2m(m+1) A_\beta A_\ell \beta_0 + 2m(2m+1) A^2 \beta_0 \beta_\ell + \frac{A^{(m-3/2)}}{\beta_{m+1}} \left[ (m+1) A^2 A_{0\ell} \right]
\]
\[
+ \frac{m^2 - 1}{2} \beta^2 A_0 A_\ell - (m^2 - 1) A_\beta A_0 \beta_\ell - 2(m-1) A^2 \beta_0 \beta_\ell - (m^2 - 1) A_\beta A_\ell \beta_0
\]
\[
+ 2m(m-1) A^2 \beta_0 \beta_\ell \right] + 2\beta_0 \beta_\ell + 2\beta_\ell \beta_0 .
\]

Further, equation \((6.8)\) can be rewritten as
\[
2L_{x^\ell} = \frac{A^{m-1}}{\beta_{2m+2}} \left[ (m+1) A^2 A_{0\ell} + 4mA^2 \beta \beta_\ell \right]
\]
\[
+ \frac{A^{(m-3/2)}}{\beta_{m+1}} \left[ 2(m+1) A^2 A_{0\ell} - 4(m-1) A^2 \beta \beta_\ell \right]
\]
\[+ 4\beta \beta_\ell .
\]

We know that \( \tilde{F} \) is locally dually flat if and only if \( L_{x^k y^l} y^k - 2L_{x^\ell} = 0 \),

i.e.,
\[
\frac{A^{m-1}}{\beta_{2m+2}} \left[ (m+1) A^2 A_{0\ell} + m(m+1) \beta^2 A_\ell A_0 - 2m(m+1) A_\beta A_0 \beta_\ell - 2mA^2 \beta \beta_0 \ell - 2m(m+1) A_\beta A_\ell \beta_0 + 2m(2m+1) A^2 \beta_0 \beta_\ell \right]
\]
\[
+ \frac{A^{(m-3/2)}}{\beta_{m+1}} \left[ (m+1) A^2 A_{0\ell} + m^2 - 1 \right] \beta^2 A_0 A_\ell - (m^2 - 1) A_\beta A_0 \beta_\ell
\]
\[
- 2(m-1) A^2 \beta_0 \beta_\ell - (m^2 - 1) A_\beta A_\ell \beta_0 + 2m(m-1) A^2 \beta_0 \beta_\ell + 2\beta_0 \beta_\ell + 2\beta_\ell \beta_0
\]
\[= \frac{A^{m-1}}{\beta_{2m+2}} \left[ 2(m+1) A^2 A_{0\ell} - 4m A^2 \beta \beta_\ell \right]
\]
\[= \frac{A^{m-1}}{\beta_{2m+2}} \left[ 2m(1) A^2 A_{0\ell} - 4(m-1) A^2 \beta \beta_\ell \right]
\]
\[= 4\beta \beta_\ell = 0 ,
\]
i.e.,

\[ \frac{A^{m-1}}{\beta^{2m+2}} \left[ 2mA^2 \left\{ \beta (2\beta_{x}\ell - \beta_{0}\ell) + (2m + 1)\beta_{0}\beta_{t} \right\} + (m + 1)A \left\{ \beta^2 (A_{0}\ell - 2A_{x}\ell) - 2m\beta (A_{t}\beta_{0} + A_{0}\beta_{t}) \right\} 
+ m(m + 1)\beta^2 A_{t}A_{0} \right] \left[ 2(m - 1)A^2 \left\{ \beta (2\beta_{x}\ell - \beta_{0}\ell) + m\beta_{0}\beta_{t} \right\} + (m + 1)A \left\{ \beta^2 (A_{0}\ell - 2A_{x}\ell) - (m - 1)\beta (A_{0}\beta_{t} + \beta_{0}A_{t}) \right\} + \frac{m^2 - 1}{2} \beta^2 A_{t}\ell A_{0} \right] + 2\beta (A_{0}\ell - 2\beta_{x}\ell) + 2\beta_{t}\beta_{0} = 0. \]

From the above equation, we conclude that \( \bar{F} \) is locally dually flat if and only if following seven equations are satisfied.

1. \( \beta (2\beta_{x}\ell - \beta_{0}\ell) + (2m + 1)\beta_{0}\beta_{t} = 0 \) (6.10)
2. \( \beta^2 (A_{0}\ell - 2A_{x}\ell) - 2m\beta (A_{t}\beta_{0} + A_{0}\beta_{t}) = 0 \) (6.11)
3. \( m(m + 1)\beta^2 A_{t}A_{0} = 0 \implies A_{t}A_{0} = 0 \) (6.12)
4. \( (m - 1) \left\{ \beta (2\beta_{x}\ell - \beta_{0}\ell) + m\beta_{0}\beta_{t} \right\} = 0 \) (6.13)
5. \( \beta^2 (A_{0}\ell - 2A_{x}\ell) - (m - 1)\beta (A_{0}\beta_{t} + \beta_{0}A_{t}) = 0 \) (6.14)
6. \( (m^2 - 1)\beta^2 A_{t}A_{0} = 0 \implies (m - 1)A_{0}A_{t} = 0 \) (6.15)
7. \( \beta (\beta_{0}\ell - 2\beta_{x}\ell) + \beta_{t}\beta_{0} = 0. \) (6.16)

Further, from the equation (6.13), we see that either \( m = 1 \) or \( \beta (2\beta_{x}\ell - \beta_{0}\ell) + m\beta_{0}\beta_{t} = 0 \). Now, if \( m = 1 \), then (6.10) reduces to (6.4), (6.11) reduces to (6.5), and (6.12), (6.14), (6.16) reduce to (6.6). But the equations (6.4), (6.5) and (6.6) are necessary and sufficient conditions for Kropina-Randers changed Finsler metric to be locally dually flat. Therefore, we assume that \( m \neq 1 \) for the general case. Then

\[ \beta (2\beta_{x}\ell - \beta_{0}\ell) + m\beta_{0}\beta_{t} = 0. \] (6.17)

From the equations (6.10) and (6.17), we get

\[ \beta_{0}\beta_{t} = 0. \]
Again from the equation (6.10), we get
\[ \beta_{0\ell} = 2\beta_{x\ell}. \]

Also from the equations (6.11) and (6.14), we get
\[ A_\ell \beta_0 + A_0 \beta_\ell = 0. \]

Again from the equation (6.11), we get
\[ A_0 \ell = 2A_{x\ell}. \]

Above discussion leads to the following theorem.

**Theorem 6.2.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \) \((m \neq -1, 0, 1)\) as generalized Kropina-Randers changed metric. Then \(\bar{F}\) is locally dually flat if and only if the following equations are satisfied:

\[ A_0 A_\ell = 0, \ A_0 \ell = 2A_{x\ell}, \ \beta_0 \beta_\ell = 0, \ \beta_{0\ell} = 2\beta_{x\ell}, \ A_\ell \beta_0 + A_0 \beta_\ell = 0. \]

Next, we find necessary and sufficient conditions for square-Randers changed Finsler metric
\[ \bar{F} = \frac{(F + \beta)^2}{F} + \beta \]

To be locally dually flat.

Let us put \(F^2 = A\) in \(\bar{F}\), then
\[ \bar{F} = A^{1/2} + \frac{\beta^2}{A^{1/2}} + 3\beta. \]

\[ L = \bar{F}^2 = \frac{\beta^4}{A} + \frac{6\beta^3}{A^{1/2}} + 6\beta A^{1/2} + A + 11\beta^2. \]  \hspace{1cm} (6.18)

Differentiating (6.18) w.r.t. \(x^k\), we get
\[ L_{,x^k} = \frac{4\beta^3}{A} \beta_{,x^k} - \frac{\beta^4}{A^2} A_{,x^k} + \frac{18\beta^2}{A^{1/2}} \beta_{x^k} - \frac{3\beta^3}{A^{3/2}} A_{x^k} + 6A^{1/2} \beta_{x^k} + \frac{3\beta}{A^{1/2}} A_{x^k} + A_{x^k} + 22\beta \beta_{x^k}. \]  \hspace{1cm} (6.19)
Differentiation of (6.19) further w.r.t. $y^f$ gives
\[
L_{x^k y^f} = \frac{4\beta^3}{A} \beta_{x^k y^f} + \frac{12\beta^2}{A} \beta_{x^k} \beta_{\ell} - \frac{4\beta^3}{A^2} \beta_{x^k} A_{\ell} - \frac{\beta^4}{A^2} A_{x^k y^f} - \frac{4\beta^3}{A^2} A_{x^k} \beta_{\ell} + \frac{2\beta^4}{A^3} A_{x^k} A_{\ell} + \frac{18\beta^2}{A^{1/2}} \beta_{x^k y^f} \\
+ \frac{36\beta}{A^{1/2}} \beta_{x^k} \beta_{\ell} - \frac{9\beta^2}{A^{3/2}} \beta_{x^k} A_{\ell} - \frac{3\beta^3}{A^{3/2}} A_{x^k y^f} - \frac{9\beta^2}{A^{3/2}} A_{x^k} \beta_{\ell} + \frac{9\beta^3}{2A^{5/2}} A_{x^k} A_{\ell} + 6A^{1/2} \beta_{x^k y^f} \\
+ \frac{3}{A^{1/2}} \beta_{x^k} A_{\ell} + \frac{3}{A^{1/2}} A_{x^k y^f} + \frac{3}{A^{1/2}} A_{x^k} \beta_{\ell} - \frac{3\beta}{2A^{3/2}} A_{x^k} A_{\ell} + A_{x^k y^f} + 22\beta_{x^k y^f} + 22\beta_{x^k} \beta_{\ell}.
\]
Simplifying, we get
\[
L_{x^k y^f} = \frac{1}{A^3} \left[ 4\beta^3 A^2 \beta_{x^k y^f} + 12\beta^2 A^2 \beta_{x^k} \beta_{\ell} - 4\beta^3 A \beta_{x^k} A_{\ell} - \beta^4 A A_{x^k y^f} - 4\beta^3 A A_{x^k} \beta_{\ell} + 2\beta^4 A A_{x^k} A_{\ell} \right] \\
+ \frac{3}{2A^{3/2}} \left[ 12\beta^3 A^2 \beta_{x^k y^f} + 24\beta A^2 \beta_{x^k} \beta_{\ell} - 6\beta^2 A \beta_{x^k} A_{\ell} - 2\beta^3 A A_{x^k y^f} - 6\beta^2 A A_{x^k} \beta_{\ell} + 3\beta^3 A A_{x^k} A_{\ell} \right] \\
+ 4\beta A^3 \beta_{x^k y^f} + 2A^2 \beta_{x^k} A_{\ell} + 2A^2 A \beta_{x^k} \beta_{\ell} - \beta A A_{x^k} A_{\ell} + A_{x^k y^f} + 22\beta_{x^k y^f} + 22\beta_{x^k} \beta_{\ell}.
\]
Contracting (6.20) with $y^k$, we get
\[
L_{x^k y^k y^f} = \frac{1}{A^3} \left[ 4\beta^3 A^2 \beta_{0\ell} + 12\beta^2 A^2 \beta_{0} \beta_{\ell} - 4\beta^3 A \beta_{0} A_{\ell} - \beta^4 A A_{0} y^f - 4\beta^3 A A_{0} \beta_{\ell} + 2\beta^4 A A_{0} A_{\ell} \right] \\
+ \frac{3}{2A^{3/2}} \left[ 12\beta^2 A^2 \beta_{0} y^f + 24\beta A^2 \beta_{0} \beta_{\ell} - 6\beta^2 A \beta_{0} A_{\ell} - 2\beta^3 A A_{0} y^f - 6\beta^2 A A_{0} \beta_{\ell} + 3\beta^3 A A_{0} A_{\ell} \right] \\
+ 4\beta A^3 \beta_{0} y^f + 2A^2 \beta_{0} A_{\ell} + 2A^2 \beta_{0} \beta_{\ell} - \beta A A_{0} A_{\ell} + A_{0} y^f + 22\beta_{0} y^f + 22\beta_{0} \beta_{\ell}.
\]
Further, equation (6.19) can be rewritten as
\[
2L_{x^f} = \frac{1}{A^3} \left[ 8\beta^3 A^2 \beta_{x^f} - 2\beta^4 A A_{x^f} \right] + \frac{3}{2A^{3/2}} \left[ 24\beta^2 A^2 \beta_{x^f} - 4\beta^3 A A_{x^f} + 8\beta^3 \beta_{x^f} + 4\beta A^2 A_{x^f} \right] \\
+ 2A_{x^f} + 44\beta \beta_{x^f}.
\]
We know that $\bar{F}$ is locally dually flat if and only if $L_{x^k y^f y^k} - 2L_{x^f} = 0$, 

40
From the above equation, we conclude that
\[
\beta^3 (\beta_0 \ell - 2 \beta_{x\ell}) + 3 \beta^2 \beta_0 \beta_{\ell} = 0
\]
\[(6.21)\]
\[
\beta^4 (2A_{x\ell} - A_0 \ell) - 4 \beta^3 (\beta_0 A_{\ell} + A_0 \beta_{\ell}) = 0
\]
\[(6.22)\]
\[
A_0 A_{\ell} = 0
\]
\[(6.23)\]
\[
\beta_0 \ell - 2 \beta_{x\ell} = 0 \quad \Rightarrow \quad \beta_0 \ell = 2 \beta_{x\ell}
\]
\[(6.24)\]
\[
6 \beta^2 (\beta_0 \ell - 2 \beta_{x\ell}) + \beta (A_0 \ell - 2 A_{x\ell} + 12 \beta_0 \beta_{\ell}) + (\beta_0 A_{\ell} + A_0 \beta_{\ell}) = 0
\]
\[(6.25)\]
\[
2 \beta^3 (2A_{x\ell} - A_0 \ell) - 6 \beta^2 (\beta_0 A_{\ell} + A_0 \beta_{\ell}) - \beta A_0 A_{\ell} = 0
\]
\[(6.26)\]

After simplification, we get
\[
\frac{1}{A^3} \left[ 4 \beta^3 A^2 \beta_0 \ell + 12 \beta^2 A^2 \beta_0 \beta_{\ell} - 4 \beta^3 A_0 \beta_{\ell} - \beta^4 A A_0 \ell - 4 \beta^3 A_0 A_{\ell} + 2 \beta^4 A_0 A_{\ell} \right]
\]
\[
+ \frac{3}{2 A^{5/2}} \left[ 12 \beta^2 A^2 \beta_0 \ell + 24 \beta A^2 \beta_0 \beta_{\ell} - 6 \beta^2 A_0 \beta_{\ell} - 2 \beta^3 A A_0 \ell - 6 \beta^2 A_0 A_{\ell} + 3 \beta^3 A_0 A_{\ell} \right]
\]
\[
+ 4 A^3 \beta_0 \ell + 2 A^2 \beta_0 A_{\ell} + 2 \beta A^2 A_0 \ell + 2 A^2 A_0 A_{\ell} - \beta A A_0 A_{\ell} \right] + A_0 \ell + 22 \beta_0 A_{\ell} + 22 \beta_0 \beta_{\ell}
\]
\[
- \frac{1}{A^3} \left[ 8 \beta^3 A^2 \beta_{x\ell} - 2 \beta^4 AA_{x\ell} \right] - \frac{3}{2 A^{5/2}} \left[ 24 \beta^2 A^2 \beta_{x\ell} - 4 \beta^3 A A_{x\ell} + 8 \beta^3 A_{x\ell} + 4 \beta A^2 A_{x\ell} \right]
\]
\[
- 2 A_{x\ell} - 44 \beta \beta_{x\ell} = 0.
\]

From the above equation, we conclude that \( F \) is locally dually flat if and only if following eight equations are satisfied.
\[ A_0 A_\ell = 0 \] (6.27)

\[ A_{0\ell} - 2 A_{x\ell} + 22 \beta_0 \beta_\ell + 22 \beta (\beta_{0\ell} - 2 \beta_{x\ell}) = 0. \] (6.28)

Further, from the equations (6.21) and (6.24), we get

\[ \beta_0 \beta_\ell = 0. \] (6.29)

Again from the equations (6.29), (6.24) and (6.28), we get

\[ A_{0\ell} = 2 A_{x\ell}, \] (6.30)

and from the equations (6.30), (6.23) and (6.26), we get

\[ \beta_0 A_\ell + A_0 \beta_\ell = 0. \] (6.31)

Above discussion leads to the following theorem.

**Theorem 6.3.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = \frac{(F + \beta)^2}{F} + \beta\) as a square-Randers changed metric. Then \(\bar{F}\) is locally dually flat if and only if the following equations are satisfied:

\[ A_0 A_\ell = 0, \ A_{0\ell} = 2 A_{x\ell}, \ \beta_0 \beta_\ell = 0, \ \beta_0 = 2 \beta_{x\ell}, \ A_\ell \beta_0 + A_0 \beta_\ell = 0. \]

Next, we find necessary and sufficient conditions for Matsumoto-Randers changed Finsler metric

\[ \bar{F} = \frac{F^2}{F - \beta} + \beta \]

to be locally dually flat.

Let us put \(F^2 = A\) in \(\bar{F}\), then

\[ \bar{F} = \frac{A}{\sqrt{A} - \beta} + \beta. \]

\[ L = \bar{F}^2 = \frac{A^2}{(\sqrt{A} - \beta)^2} + \frac{2A\beta}{\sqrt{A} - \beta} + \beta^2. \] (6.32)

Differentiating (6.32) w.r.t. \(x^k\), we get

\[ L_{x^k} = \frac{2A}{(\sqrt{A} - \beta)^2} A_{x^k} - \frac{A^{3/2}}{(\sqrt{A} - \beta)^3} A_{x^k} + \frac{2A^2}{(\sqrt{A} - \beta)^3} \beta_{x^k} + \frac{2\beta}{\sqrt{A} - \beta} A_{x^k} + \frac{2A}{\sqrt{A} - \beta} \beta_{x^k} - \frac{\sqrt{A} \beta}{(\sqrt{A} - \beta)^2} A_{x^k} + \frac{2A\beta}{(\sqrt{A} - \beta)^2} \beta_{x^k} + 2 \beta \beta_{x^k}. \] (6.33)
Differentiation of (6.33) further w.r.t. \(y^\ell\) gives

\[
L_{x^k y^\ell} = \frac{2A}{(\sqrt{A} - \beta)^2} A_{x^k y^\ell} + \frac{2}{(\sqrt{A} - \beta)^2} A_{x^k A_\ell} - \frac{4A}{(\sqrt{A} - \beta)^3} A_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right)
- \frac{A^{3/2}}{(\sqrt{A} - \beta)^3} A_{x^k y^\ell} - \frac{3A^{1/2}}{2(\sqrt{A} - \beta)^3} A_{x^k A_\ell} + \frac{3A^{3/2}}{(\sqrt{A} - \beta)^4} A_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right)
+ \frac{2A^2}{(\sqrt{A} - \beta)^3} \beta_{x^k y^\ell} + \frac{4A}{(\sqrt{A} - \beta)^3} \beta_{x^k A_\ell} - \frac{6A^2}{(\sqrt{A} - \beta)^4} \beta_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right)
+ \frac{2\beta}{\sqrt{A} - \beta} A_{x^k y^\ell} + \frac{2}{\sqrt{A} - \beta} A_{x^k} \beta_{\ell} - \frac{2\beta}{(\sqrt{A} - \beta)^2} A_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right)
+ \frac{2\beta}{\sqrt{A} - \beta} \beta_{x^k y^\ell} + \frac{2}{\sqrt{A} - \beta} \beta_{x^k A_\ell} - \frac{2\beta}{(\sqrt{A} - \beta)^2} \beta_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right)
- \frac{\sqrt{A} \beta}{(\sqrt{A} - \beta)^2} A_{x^k y^\ell} - \frac{\sqrt{A}}{(\sqrt{A} - \beta)^2} A_{x^k \beta_{\ell}} - \frac{\beta}{2\sqrt{A}(\sqrt{A} - \beta)^2} A_{x^k A_\ell}
+ \frac{2\sqrt{A} \beta}{(\sqrt{A} - \beta)^3} A_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right) + \frac{2\beta A_\ell}{(\sqrt{A} - \beta)^2} \beta_{x^k y^\ell} + \frac{2A}{(\sqrt{A} - \beta)^2} \beta_{x^k A_\ell} + \frac{2\beta}{(\sqrt{A} - \beta)^2} \beta_{x^k A_\ell}
- \frac{4\beta A_\ell}{(\sqrt{A} - \beta)^3} \beta_{x^k} \left( \frac{1}{2\sqrt{A}} A_{\ell} - \beta \right) + 2\beta \beta_{x^k y^\ell} + 2\beta_{x^k} \beta_{\ell}.
\]

(6.34)

Contracting (6.34) with \(y^k\), we get

\[
L_{x^k y^k} = \frac{2A}{(\sqrt{A} - \beta)^2} A_{0^k} + \frac{2}{(\sqrt{A} - \beta)^2} A_{0^k A_\ell} - \frac{2\sqrt{A}}{(\sqrt{A} - \beta)^3} A_{0^k} A_\ell + \frac{4A}{(\sqrt{A} - \beta)^3} A_{0^k} \beta_{\ell}
- \frac{A^{3/2}}{(\sqrt{A} - \beta)^3} A_{0^k} - \frac{3A^{1/2}}{2(\sqrt{A} - \beta)^3} A_{0^k A_\ell} + \frac{3A}{2(\sqrt{A} - \beta)^4} A_{0^k} A_\ell - \frac{3A^{3/2}}{(\sqrt{A} - \beta)^4} A_{0^k} \beta_{\ell}
+ \frac{2A^2}{(\sqrt{A} - \beta)^3} \beta_{0^k} + \frac{4A}{(\sqrt{A} - \beta)^3} \beta_{0^k A_\ell} - \frac{6A^2}{(\sqrt{A} - \beta)^4} \beta_{0^k} A_\ell + \frac{2\beta A_\ell}{(\sqrt{A} - \beta)^2} \beta_{0^k y^\ell} + \frac{2\beta}{(\sqrt{A} - \beta)^2} \beta_{0^k y^\ell} + \frac{2\beta}{(\sqrt{A} - \beta)^2} \beta_{0^k} y^\ell.
\]
After simplification, we get

\[ L_{x^ky^k} = \frac{1}{2\sqrt{A}(\sqrt{A} - \beta)^4} \left[ 8A^3 \beta_0 \epsilon + 2A^{5/2} \left\{ (A_0 \epsilon + 10\beta_0 \epsilon + 2\beta_0 \epsilon) - 4\beta \beta_0 \epsilon \right\} 
+ 4A^2 \left\{ (A_0 \beta_0 \epsilon + \beta_0 A_0 \epsilon) - \beta (A_0 \epsilon + 4\beta_0 \beta_0 \epsilon + 2\beta_0 \epsilon) - 3\beta^2 \beta_0 \epsilon \right\} 
- 4A^{3/2} \left\{ 4\beta (A_0 \beta_0 \epsilon + \beta_0 A_0 \epsilon) + \beta_0^2 (A_0 \epsilon - 5\beta_0 \beta_0 \epsilon - \beta_0 \epsilon) - 6\beta^2 \beta_0 \epsilon \right\} 
- 2A \left\{ \beta A_0 \epsilon - 3\beta^2 (A_0 \beta_0 \epsilon + \beta_0 A_0 \epsilon) - \beta^3 (5A_0 \epsilon - 8\beta_0 \beta_0 \epsilon) + 8\beta^4 \beta_0 \epsilon \right\} 
+ 4A^{1/2} \left\{ 2\beta^2 A_0 \epsilon + \beta^4 (\beta_0 \beta_0 \epsilon - A_0 \epsilon) + \beta^5 \beta_0 \epsilon \right\} - 3\beta^3 A_0 \epsilon \right] \].

Further, equation (6.33) can be rewritten as

\[ 2L_{x^\ell} = \frac{4A}{(\sqrt{A} - \beta)^2} A_{x^\ell} - \frac{2A^{3/2}}{(\sqrt{A} - \beta)^3} A_{x^\ell} + \frac{4A^2}{(\sqrt{A} - \beta)^3} B_{x^\ell} + \frac{4\beta}{\sqrt{A} - \beta} A_{x^\ell} + \frac{4A}{\sqrt{A} - \beta} B_{x^\ell} 
- \frac{2\sqrt{A} \beta}{(\sqrt{A} - \beta)^2} A_{x^\ell} + \frac{4A \beta}{(\sqrt{A} - \beta)^2} B_{x^\ell} + 4\beta B_{x^\ell} 
\]

\[ = \frac{1}{2\sqrt{A}(\sqrt{A} - \beta)^4} \left[ 16A^3 \beta_{x^\ell} + 2A^{5/2} \left\{ 2A_{x^\ell} - 8\beta \beta_{x^\ell} \right\} - 4A^2 \left\{ 2\beta A_{x^\ell} + 6\beta^2 \beta_{x^\ell} \right\} 
- 4A^{3/2} \left\{ 2\beta^2 A_{x^\ell} - 12\beta^3 \beta_{x^\ell} \right\} + 2A \left\{ 10\beta^3 A_{x^\ell} - 16\beta^4 \beta_{x^\ell} \right\} - 4A^{1/2} \left\{ 2\beta^4 A_{x^\ell} - 2\beta^5 \beta_{x^\ell} \right\} \right] \].

We know that \( \bar{F} \) is locally dually flat if and only if \( L_{x^k y^k} y^k - 2L_{x^\ell} = 0 \),
From the above equation, we conclude that \( F \) is locally dually flat if and only if following seven equations are satisfied.

\[
\beta_{0\ell} = 2\beta_{x\ell} \quad (6.35)
\]

\[
(A_{0\ell} - 2A_{x\ell} + 10\beta_0\beta_\ell + 2\beta_{0\ell}) - 4\beta(\beta_{0\ell} - 2\beta_{x\ell}) = 0 \quad (6.36)
\]

\[
(A_0\beta_\ell + \beta_0A_\ell) - \beta(A_{0\ell} - 2A_{x\ell} + 4\beta_0\beta_\ell + 2\beta_{0\ell}) - 3\beta^2(\beta_{0\ell} - 2\beta_{x\ell}) = 0 \quad (6.37)
\]
\[4\beta (A_0\beta + \beta_0 A_\ell) + \beta^2 (A_{0\ell} - 2A_{x\ell} - 5\beta_0\beta_\ell - \beta_0\ell) - 6\beta^3 (\beta_0\ell - 2\beta_x\ell) = 0\]  \hspace{1cm} (6.38)

\[\beta A_0 A_\ell - 3\beta^2 (A_0\beta + \beta_0 A_\ell) - \beta^3 (5A_{0\ell} - 10A_{x\ell} - 8\beta_0\beta_\ell) + 8\beta^4 (\beta_0\ell - 2\beta_x\ell) = 0\]  \hspace{1cm} (6.39)

\[2\beta^2 A_0 A_\ell + \beta^4 (\beta_0\beta_\ell - A_{0\ell} + 2A_{x\ell}) + \beta^5 (\beta_0\ell - 2\beta_x\ell) = 0\]  \hspace{1cm} (6.40)

\[A_0 A_\ell = 0.\]  \hspace{1cm} (6.41)

Solving above seven equations, we get

\[A_0 A_\ell = 0, \ A_{0\ell} = 2A_{x\ell}, \ \beta_0\beta_\ell = 0, \ \beta_0\ell = 0 = 2\beta_x\ell, \ A_\ell\beta_0 + A_0\beta_\ell = 0.\]

Above discussion leads to the following theorem.

**Theorem 6.4.** Let \( (M, \bar{F}) \) be an \( n \)-dimensional Finsler space with \( \bar{F} = F^2 / \sqrt{F - \beta} + \beta \) as a Matsumoto-Randers changed metric. Then \( \bar{F} \) is locally dually flat if and only if the following equations are satisfied:

\[A_0 A_\ell = 0, \ A_{0\ell} = 2A_{x\ell}, \ \beta_0\beta_\ell = 0, \ \beta_0\ell = 0 = 2\beta_x\ell, \ A_\ell\beta_0 + A_0\beta_\ell = 0.\]

Next, we find necessary and sufficient conditions for exponential-Randers changed Finsler metric

\[\tilde{F} = Fe^{\beta/F} + \beta\]

to be locally dually flat.

Let us put \( F^2 = A \) in \( \bar{F} \), then

\[\tilde{F} = \sqrt{A}e^{\beta/\sqrt{A}} + \beta.\]

\[L = \tilde{F}^2 = A e^{2\beta/\sqrt{A}} + 2\sqrt{A}e^{\beta/\sqrt{A}} + \beta^2.\]  \hspace{1cm} (6.42)
Differentiating (6.42) w.r.t. $x^k$, we get

$$L_{x^k} = Ae^{2\beta/\sqrt{A}} \left( \frac{2}{\sqrt{A}} \beta_x - \frac{\beta}{A^{3/2}} A_{x^k} \right) + e^{2\beta/\sqrt{A}} A_{x^k} + \frac{\beta}{\sqrt{A}} e^{\beta/\sqrt{A}} A_{x^k} + 2\sqrt{A} e^{\beta/\sqrt{A}} \beta_{x^k}$$

$$+ 2\sqrt{A} \beta e^{\beta/\sqrt{A}} \left( \frac{1}{\sqrt{A}} \beta_x - \frac{\beta}{2A^{3/2}} A_{x^k} \right) + 2\beta \beta_{x^k}$$

$$= 2\sqrt{A} e^{2\beta/\sqrt{A}} \beta_{x^k} - \frac{\beta}{\sqrt{A}} 2e^{\beta/\sqrt{A}} A_{x^k} + e^{2\beta/\sqrt{A}} A_{x^k} + \frac{\beta}{\sqrt{A}} e^{\beta/\sqrt{A}} A_{x^k} + 2\sqrt{A} e^{\beta/\sqrt{A}} \beta_{x^k}$$

$$+ 2\beta e^{\beta/\sqrt{A}} \beta_{x^k} - \frac{\beta^2}{A} e^{\beta/\sqrt{A}} A_{x^k} + 2\beta \beta_{x^k}.$$  

(6.43)

Differentiation of (6.43) further w.r.t. $y^\ell$ gives

$$L_{x^k y^\ell} = 2\sqrt{A} e^{2\beta/\sqrt{A}} \beta_{x^k y^\ell} + \frac{1}{\sqrt{A}} e^{2\beta/\sqrt{A}} \beta_y A_{\ell} + 2\sqrt{A} e^{2\beta/\sqrt{A}} \beta_{x^k} \left( \frac{2}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{A^{3/2}} A_{\ell} \right) - \frac{\beta}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_{x^k y^\ell}$$

$$- \frac{1}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_{x^k} \beta_{\ell} + \frac{\beta}{2A^{3/2}} e^{2\beta/\sqrt{A}} A_{x^k} A_{\ell} - \frac{\beta}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_{x^k} \left( \frac{2}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{A^{3/2}} A_{\ell} \right)$$

$$+ e^{2\beta/\sqrt{A}} A_{x^k} \beta_{\ell} + 2\sqrt{A} e^{2\beta/\sqrt{A}} \beta_{x^k} \left( \frac{1}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{2A^{3/2}} A_{\ell} \right) + 2\sqrt{A} e^{\beta/\sqrt{A}} \beta_{x^k y^\ell} + \frac{1}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_{x^k} \beta_{\ell}$$

$$- \frac{\beta}{2A^{3/2}} e^{\beta/\sqrt{A}} A_{x^k} A_{\ell} + \frac{\beta}{\sqrt{A}} e^{\beta/\sqrt{A}} A_{x^k} \left( \frac{1}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{2A^{3/2}} A_{\ell} \right) + 2\beta e^{\beta/\sqrt{A}} \beta_{x^k y^\ell} + 2\beta e^{\beta/\sqrt{A}} \beta_{x^k \beta_{\ell}}$$

$$+ 2\beta e^{\beta/\sqrt{A}} \beta_{x^k} \left( \frac{1}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{2A^{3/2}} A_{\ell} \right) - \frac{\beta^2}{A} e^{\beta/\sqrt{A}} A_{x^k y^\ell} - \frac{2\beta}{A} e^{\beta/\sqrt{A}} A_{x^k} \beta_{\ell} + \frac{\beta^2}{A^2} e^{\beta/\sqrt{A}} A_{x^k} A_{\ell}$$

$$- \frac{\beta^2}{A} e^{\beta/\sqrt{A}} A_{x^k} \left( \frac{1}{\sqrt{A}} \beta_{\ell} - \frac{\beta}{2A^{3/2}} A_{\ell} \right) + 2\beta \beta_{x^k y^\ell} + 2\beta \beta_{x^k \beta_{\ell}}.$$  

(6.44)

Contracting (6.44) with $y^k$, we get

$$L_{x^k y^k} = 2\sqrt{A} e^{2\beta/\sqrt{A}} \beta_{0\ell} + \frac{1}{\sqrt{A}} e^{2\beta/\sqrt{A}} \beta_0 A_{\ell} + 4e^{2\beta/\sqrt{A}} \beta_0 \beta_{\ell} - \frac{2\beta}{A} e^{\beta/\sqrt{A}} \beta_0 A_{\ell} - \frac{\beta}{A} e^{\beta/\sqrt{A}} A_0 A_{\ell}$$

$$- \frac{1}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_0 \beta_{\ell} + \frac{\beta}{2A^{3/2}} e^{2\beta/\sqrt{A}} A_0 A_{\ell} - \frac{2\beta}{A} e^{\beta/\sqrt{A}} A_0 A_{\ell} + \frac{\beta^2}{A^2} e^{\beta/\sqrt{A}} A_0 A_{\ell}$$

$$+ e^{2\beta/\sqrt{A}} A_0 \ell + \frac{2}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_0 \beta_{\ell} - \frac{\beta}{A^{3/2}} e^{2\beta/\sqrt{A}} A_0 A_{\ell} + \frac{\beta}{\sqrt{A}} e^{\beta/\sqrt{A}} A_0 A_{\ell} + \frac{1}{\sqrt{A}} e^{\beta/\sqrt{A}} A_0 \beta_{\ell}$$

47
Further, equation (6.43) can be rewritten as

\[
- \frac{\beta}{2 A^{3/2}} e^{\beta/\sqrt{A}} A_0 A_\ell + \frac{\beta}{A} e^{\beta/\sqrt{A}} A_0 \beta_0 + \frac{\beta^2}{2 A^2} A e^{\beta/\sqrt{A}} A_0 A_\ell + 2 \sqrt{A} e^{\beta/\sqrt{A}} A_0 A_\ell \\
+ \frac{1}{\sqrt{A}} e^{\beta/\sqrt{A}} \beta_0 A_\ell + 2 e^{\beta/\sqrt{A}} \beta_0 \beta_\ell - \frac{\beta}{A} e^{\beta/\sqrt{A}} A_0 A_\ell + 2 \beta e^{\beta/\sqrt{A}} A_0 A_\ell + 2 e^{\beta/\sqrt{A}} A_0 A_\ell \\
+ \frac{\beta}{\sqrt{A}} e^{\beta/\sqrt{A}} \beta_0 A_\ell - \frac{\beta^2}{A^{3/2}} e^{\beta/\sqrt{A}} A_0 A_\ell - \frac{\beta^2}{A} e^{\beta/\sqrt{A}} A_0 A_\ell + \frac{\beta^2}{A^2} e^{\beta/\sqrt{A}} A_0 A_\ell \\
- \frac{\beta^2}{A^{3/2}} e^{\beta/\sqrt{A}} A_0 \beta_\ell + \frac{\beta^3}{2 A^{5/2}} e^{\beta/\sqrt{A}} A_0 A_\ell + 2 \beta \beta_0 \beta_\ell.
\]

Simplifying, we get

\[
L_{x^k y^k} = \frac{1}{2 A^{5/2}} \left[ 4 A^3 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_0 + 2 A^{5/2} \left\{ 2 \beta \left( e^{\beta/\sqrt{A}} \beta_0 + \beta_0 \right) + e^{23/\sqrt{A}} A_0 \right\} \\
+ 4 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_0 + 2 e^{\beta/\sqrt{A}} A_0 \right\} + 2 e^{\beta/\sqrt{A}} A_0 \left\{ \beta \left( 1 - e^{\beta/\sqrt{A}} \right) A_0 + 2 \beta \beta_0 \beta_\ell \right\} \\
+ \left( e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_0 + \beta_0 A_\ell) \right\} - 2 e^{\beta/\sqrt{A}} A^{3/2} \left\{ \beta^2 A_0 + \beta \left( e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_0 + \beta_0 A_\ell) \right\} \\
- e^{\beta/\sqrt{A}} A \left\{ 2 (A_0 \beta_0 + \beta_0 A_\ell) + \beta \left( e^{\beta/\sqrt{A}} + 1 \right) A_0 A_\ell \right\} + \beta^2 e^{\beta/\sqrt{A}} \left( 2 e^{\beta/\sqrt{A}} + 1 \right) \sqrt{A} A_0 A_\ell \\
+ \beta^3 e^{\beta/\sqrt{A}} A_0 A_\ell \right].
\]

Further, equation (6.33) can be rewritten as

\[
2 L_{x^k x^\ell} = 4 \sqrt{A} e^{2\beta/\sqrt{A}} A_{x^\ell} + \frac{2 \beta}{\sqrt{A}} e^{2\beta/\sqrt{A}} A_{x^\ell} + 2 e^{2\beta/\sqrt{A}} A_{x^\ell} + 2 \sqrt{A} e^{\beta/\sqrt{A}} A_{x^\ell} + 4 \sqrt{A} e^{\beta/\sqrt{A}} A_{x^\ell} \\
+ 4 \beta e^{\beta/\sqrt{A}} A_{x^\ell} + 2 e^{\beta/\sqrt{A}} A_{x^\ell} + 4 \beta \beta_{x^\ell} \\
= 4 \sqrt{A} e^{2\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{x^\ell} + 2 e^{2\beta/\sqrt{A}} A_{x^\ell} + 4 \beta \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{x^\ell} \\
- \frac{2 \beta}{\sqrt{A}} e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} - 1 \right) A_{x^\ell} - \frac{2 \beta^2}{A} e^{\beta/\sqrt{A}} A_{x^\ell} \\
= \frac{1}{2 A^{5/2}} \left[ 8 A^3 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{x^\ell} + 2 \beta \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{x^\ell} \right] \\
- 4 \beta A^2 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} - 1 \right) A_{x^\ell} - 4 \beta^2 A^{3/2} e^{\beta/\sqrt{A}} A_{x^\ell} \right].
\]

We know that $\tilde{F}$ is locally dually flat if and only if $L_{x^k y^k} - 2 L_{x^\ell} = 0$. 

48
i.e.,

\[
\frac{1}{2A^{5/2}} \left[ 4A^3 e^{3/\sqrt{A}} \left( e^{3/\sqrt{A}} + 1 \right) \beta_{0\ell} + 2A^{5/2} \left\{ 2\beta \left( e^{3/\sqrt{A}} \beta_{0\ell} + \beta_{0\ell} \right) + e^{2\beta/\sqrt{A}} A_{0\ell} \right\} 
+ 4e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{0\ell} + 2\beta_0 \beta_\ell \right\} + 2e^{\beta/\sqrt{A}} A^2 \left\{ \beta \left( 1 - e^{\beta/\sqrt{A}} \right) A_{0\ell} + 2\beta_0 \beta_\ell \right\} 
+ \left( e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_\ell + \beta_0 A_\ell) \right] 
- 2e^{\beta/\sqrt{A}} A^{3/2} \left\{ \beta^2 A_{0\ell} + \beta \left( 2e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_\ell + \beta_0 A_\ell) \right\} 
- 2\beta_0 A^2 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} - 1 \right) A_{x\ell} - 4\beta_0 A^3 e^{\beta/\sqrt{A}} A_{x\ell} \right] = 0.
\]

Simplifying, we get

\[
4A^3 e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \left\{ \beta_{0\ell} - 2\beta_{x\ell} \right\} 
+ 2A^{5/2} \left\{ 2\beta \left( e^{\beta/\sqrt{A}} + 1 \right) (\beta_{0\ell} - 2\beta_{x\ell}) + e^{2\beta/\sqrt{A}} (A_{0\ell} - 2A_{x\ell}) + 4e^{\beta/\sqrt{A}} \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{0\ell} + 2\beta_0 \beta_\ell \right\} 
+ 2e^{\beta/\sqrt{A}} A^2 \left\{ \beta \left( 1 - e^{\beta/\sqrt{A}} \right) (A_{0\ell} - 2A_{x\ell}) + 2\beta_0 \beta_\ell \right\} + \left( e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_\ell + \beta_0 A_\ell) \}
- 2e^{\beta/\sqrt{A}} A^{3/2} \left\{ \beta^2 (A_{0\ell} - 2A_{x\ell}) + \beta \left( 2e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta_\ell + \beta_0 A_\ell) \right\} 
- e^{\beta/\sqrt{A}} A \left\{ 2(A_0 \beta_\ell + \beta_0 A_\ell) + \beta \left( e^{\beta/\sqrt{A}} + 1 \right) A_0 A_\ell \right\} 
+ \beta^2 e^{\beta/\sqrt{A}} \left( 2e^{\beta/\sqrt{A}} + 1 \right) \sqrt{A} A_0 A_\ell + \beta^3 e^{\beta/\sqrt{A}} A_0 A_\ell = 0.
\]

From the above equation, we conclude that $\tilde{F}$ is locally dually flat if and only if following seven equations are satisfied.

\[
\left( e^{\beta/\sqrt{A}} + 1 \right) \left\{ \beta_{0\ell} - 2\beta_{x\ell} \right\} = 0 \implies \beta_{0\ell} = 2\beta_{x\ell} \quad \text{(6.45)}
\]

\[
2\beta \left( e^{\beta/\sqrt{A}} + 1 \right) (\beta_{0\ell} - 2\beta_{x\ell}) + e^{2\beta/\sqrt{A}} (A_{0\ell} - 2A_{x\ell}) + 4\beta \left( e^{\beta/\sqrt{A}} + 1 \right) \beta_{0\ell} + 2\beta_0 \beta_\ell = 0 \quad \text{(6.46)}
\]
\( \beta \left( \left( 1 - e^{\beta/\sqrt{A}} \right) (A_0 \ell - 2A_x \ell) + 2\beta_0 \beta \ell \right) + \left( e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta + \beta_0 A_\ell) = 0 \)  
(6.47)

\[ \beta^2 (A_0 \ell - 2A_x \ell) + \beta \left( 2e^{\beta/\sqrt{A}} + 1 \right) (A_0 \beta + \beta_0 A_\ell) = 0 \]  
(6.48)

\[ 2 (A_0 \beta + \beta_0 A_\ell) + \beta \left( e^{\beta/\sqrt{A}} + 1 \right) A_0 A_\ell = 0 \]  
(6.49)

\( \left( 2e^{\beta/\sqrt{A}} + 1 \right) A_0 A_\ell = 0 \implies A_0 A_\ell = 0. \)  
(6.50)

Further, from the equations (6.49) and (6.50), we get

\[ A_0 \beta + \beta_0 A_\ell = 0 \]  
(6.51)

Again, from the equations (6.48) and (6.51), we get

\[ A_0 \ell = 2A_x \ell, \]  
(6.52)

and from the equations (6.47), (6.51) and (6.52), we get

\[ \beta_0 \beta_\ell = 0. \]  
(6.53)

Above discussion leads to the following theorem.

**Theorem 6.5.** Let \((M, \bar{F})\) be an \(n\)-dimensional Finsler space with \(\bar{F} = Fe^{\beta/F} + \beta\) as an exponential-Randers changed metric. Then \(\bar{F}\) is locally dually flat if and only if the following equations are satisfied:

\[ A_0 A_\ell = 0, \ A_0 \ell = 2A_x \ell, \ \beta_0 \beta_\ell = 0, \ \beta_0 \ell = 2\beta_\ell \ell, \ A_\ell \beta_0 + A_0 \beta_\ell = 0. \]

Next, we find necessary and sufficient conditions for infinite series-Randers changed Finsler metric

\[ \bar{F} = \frac{\beta^2}{\beta - \sqrt{A}} + \beta \]

to be locally dually flat.

Let us put \(F^2 = A\) in \(\bar{F}\), then

\[ \bar{F} = \frac{\beta^2}{\beta - \sqrt{A}} + \beta. \]

\[ L = \bar{F}^2 = \frac{\beta^4}{(\beta - \sqrt{A})^2} + \frac{2\beta^3}{\beta - \sqrt{A}} + \beta^2. \]  
(6.54)
Differentiating (6.54) w.r.t. $x^k$, we get

$$L_{x^k} = \frac{4\beta^3}{(\beta - \sqrt{A})^2} \beta x^k - \frac{2\beta^4}{(\beta - \sqrt{A})^3} \left( \beta x^k - \frac{1}{2\sqrt{A}} A x^k \right) + \frac{6\beta^2}{\beta - \sqrt{A}} \beta x^k - \frac{2\beta^3}{(\beta - \sqrt{A})^2} \left( \beta x^k - \frac{1}{2\sqrt{A}} A x^k \right) + 2\beta \beta x^k$$

$$= -\frac{2\beta^3\sqrt{A}}{(\beta - \sqrt{A})^3} \beta x^k + \frac{6\beta^2}{\beta - \sqrt{A}} \beta x^k + 2\beta \beta x^k + \frac{\beta^4}{\sqrt{A}(\beta - \sqrt{A})^3} A x^k + \frac{\beta^3}{\sqrt{A}(\beta - \sqrt{A})^2} A x^k. \tag{6.55}$$

Differentiation of (6.55) further w.r.t. $y^\ell$ gives

$$L_{x^k y^\ell} = -\frac{2\beta^3\sqrt{A}}{(\beta - \sqrt{A})^3} \beta x^k y^\ell - \frac{6\beta^2\sqrt{A}}{(\beta - \sqrt{A})^3} \beta x^k \beta y^\ell - \frac{\beta^3}{\sqrt{A}(\beta - \sqrt{A})^3} \beta x^k A_y^\ell + \frac{6\beta^3\sqrt{A}}{(\beta - \sqrt{A})^4} \beta x^k \left( \beta y^\ell - \frac{A_y^\ell}{2\sqrt{A}} \right)$$

$$+ \frac{6\beta^2}{\beta - \sqrt{A}} \beta x^k y^\ell + \frac{12\beta}{\sqrt{A}} \beta x^k \beta y^\ell - \frac{6\beta^2}{(\beta - \sqrt{A})^2} \beta x^k \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right) + 2\beta \beta x^k y^\ell + 2\beta y^\ell \beta x^k$$

$$+ \frac{\beta^4}{\sqrt{A}(\beta - \sqrt{A})^3} A x^k y^\ell + \frac{4\beta^4}{\sqrt{A}(\beta - \sqrt{A})^3} A x^k \beta y^\ell - \frac{6\beta^3}{2A^{3/2}(\beta - \sqrt{A})^3} A x^k A_y^\ell$$

$$- \frac{3\beta^4}{\sqrt{A}(\beta - \sqrt{A})^4} A x^k \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right) + \frac{\beta^3}{\sqrt{A}(\beta - \sqrt{A})^2} A x^k y^\ell + \frac{3\beta^4}{\sqrt{A}(\beta - \sqrt{A})^4} A x^k \beta y^\ell$$

$$- \frac{\beta^3}{2A^{3/2}(\beta - \sqrt{A})^2} A x^k A_y^\ell - \frac{2\beta^3}{\sqrt{A}(\beta - \sqrt{A})^3} A x^k \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right). \tag{6.56}$$

Contracting (6.56) with $y^k$, we get

$$L_{x^k y^\ell} y^k = -\frac{2\beta^3\sqrt{A}}{(\beta - \sqrt{A})^3} \beta_0^\ell - \frac{6\beta^2\sqrt{A}}{(\beta - \sqrt{A})^3} \beta_0 \beta_0^\ell - \frac{\beta^3}{\sqrt{A}(\beta - \sqrt{A})^3} \beta_0 A_y^\ell + \frac{6\beta^3\sqrt{A}}{(\beta - \sqrt{A})^4} \beta_0 \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right)$$

$$+ \frac{6\beta^2}{\beta - \sqrt{A}} \beta_0 y^\ell + \frac{12\beta}{\sqrt{A}} \beta_0 \beta y^\ell - \frac{6\beta^2}{(\beta - \sqrt{A})^2} \beta_0 \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right) + 2\beta \beta_0 y^\ell + 2\beta_0 y^\ell \beta$$

$$+ \frac{\beta^4}{\sqrt{A}(\beta - \sqrt{A})^3} A_0 y^\ell + \frac{4\beta^4}{\sqrt{A}(\beta - \sqrt{A})^3} A_0 \beta y^\ell - \frac{6\beta^3}{2A^{3/2}(\beta - \sqrt{A})^3} A_0 A_y^\ell$$

$$- \frac{3\beta^4}{\sqrt{A}(\beta - \sqrt{A})^4} A_0 \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right) + \frac{\beta^3}{\sqrt{A}(\beta - \sqrt{A})^2} A_0 y^\ell + \frac{3\beta^4}{\sqrt{A}(\beta - \sqrt{A})^4} A_0 \beta y^\ell$$

$$- \frac{\beta^3}{2A^{3/2}(\beta - \sqrt{A})^2} A_0 A_y^\ell - \frac{2\beta^3}{\sqrt{A}(\beta - \sqrt{A})^3} A_0 \left( \beta y^\ell - \frac{1}{2\sqrt{A}} A_y^\ell \right).$$
Further, equation (6.55) can be rewritten as

$$L_{x^k y^k} = \frac{1}{2A^{3/2}(\beta - \sqrt{A})^4} \left[ 4A^{7/2} \left( \beta^2 \partial_0 - 2\beta_0 \partial_\ell \right) - 28A^3 \left\{ \beta^2 \partial_0 + 2\beta_0 \beta_\ell \right\} + 8A^{5/2} \left\{ 8\beta^3 \partial_0 - 10\beta^2 \partial_0 \beta_\ell + 2A^2 \left\{ -28\beta^4 \partial_0 \beta_\ell + 8\beta^3 \partial_0 + 3\beta_0 \partial_0 A_\ell + 3\beta_0 \beta_0 A_\ell + 2A^3 \left\{ 8\beta^5 \partial_0 + 10\beta^4 \partial_0 \beta_\ell - 3\beta_0 \partial_0 A_\ell - 8\beta^3 (A_0 \beta_\ell + \beta_0 A_\ell) \right\} + A \left\{ 4\beta^5 A_0 \partial_0 + 4\beta^4 (A_0 \beta_\ell + \beta_0 A_\ell) - 3\beta^3 A_0 A_\ell \right\} + 8\beta^4 \sqrt{A} A_0 A_\ell - 2\beta^3 A_0 A_\ell \right] \right].$$

Simplifying, we get

$$L_{x^\ell} = -\frac{4\beta^3 \sqrt{A}}{(\beta - \sqrt{A})^3} \beta x^\ell + \frac{12\beta^2}{\beta - \sqrt{A}} \beta x^\ell + 4\beta^3 \beta x^\ell + \frac{2\beta^3}{\sqrt{A}(\beta - \sqrt{A})^3} A x^\ell + \frac{2\beta^3}{\sqrt{A}(\beta - \sqrt{A})^2} A x^\ell$$

$$= \frac{1}{2A^{3/2}(\beta - \sqrt{A})^4} \right[ 8\beta A^{7/2} \beta x^\ell - 56A^3 \beta^2 \beta x^\ell + 128A^{5/2} \beta^3 \beta x^\ell - 2A^2 \left\{ 56\beta^3 \beta x^\ell - 2\beta^3 A x^\ell \right\} + 2A^3 \left\{ 16\beta^5 \beta x^\ell - 6\beta^4 A x^\ell \right\} + 8\beta^5 A A x^\ell \right].$$

We know that $F$ is locally dually flat if and only if $L_{x^k y^k} - 2L_{x^\ell} = 0$,

i.e.,

$$\frac{1}{2A^{3/2}(\beta - \sqrt{A})^4} \right[ 4A^{7/2} \left\{ \beta^2 \partial_0 - 2\beta_0 \partial_\ell \right\} - 28A^3 \left\{ \beta^2 \partial_0 + 2\beta_0 \beta_\ell \right\} + 8A^{5/2} \left\{ 8\beta^3 \partial_0 - 10\beta^2 \partial_0 \beta_\ell + 2A^2 \left\{ -28\beta^4 \partial_0 \beta_\ell + 8\beta^3 \partial_0 + 3\beta_0 \partial_0 A_\ell + 3\beta_0 \beta_0 A_\ell \right\} + 2A^3 \left\{ 8\beta^5 \partial_0 + 10\beta^4 \partial_0 \beta_\ell - 3\beta_0 \partial_0 A_\ell - 8\beta^3 (A_0 \beta_\ell + \beta_0 A_\ell) \right\} + A \left\{ 4\beta^5 A_0 \partial_0 + 4\beta^4 (A_0 \beta_\ell + \beta_0 A_\ell) - 3\beta^3 A_0 A_\ell \right\} + 8\beta^4 \sqrt{A} A_0 A_\ell - 2\beta^3 A_0 A_\ell \right]$$

$$+ 2A^3 \left\{ 16\beta^5 \beta x^\ell - 6\beta^4 A x^\ell \right\} + 8\beta^5 A A x^\ell \right] = 0.$$
After simplification, we get

\[
4A^{7/2}\left\{ \beta (\beta_0 \ell - 2\beta_{x\ell}) + 2\beta_0 \beta_\ell \right\} - 28A^3 \beta \left\{ \beta (\beta_0 \ell - 2\beta_{x\ell}) + 2\beta_0 \beta_\ell \right\}
\]

\[
+ 2A^2 \beta^2 \left\{ -28\beta^2 (\beta_0 \ell - 2\beta_{x\ell}) - 40\beta_0 \beta \ell + \beta (A_0 \ell - 2A_{x\ell}) + 3 (A_0 \beta_\ell + \beta_0 A_\ell) \right\}
\]

\[
+ 8A^{5/2} \beta^2 \left\{ 8\beta (\beta_0 \ell - 2\beta_{x\ell}) + 15\beta_0 \beta_\ell \right\}
\]

\[
+ 2A^{3/2} \beta^3 \left\{ 8\beta^2 (\beta_0 \ell - 2\beta_{x\ell}) + 10\beta_0 \beta_\ell - 3\beta (A_0 \ell - 2A_{x\ell}) - 8 (A_0 \beta_\ell + \beta_0 A_\ell) \right\}
\]

\[
+ A \beta^3 \left\{ 4\beta^2 (A_0 \ell - 2A_{x\ell}) + 4\beta (A_0 \beta_\ell + \beta_0 A_\ell) - 3A_0 A_\ell \right\} + 8\beta^4 \sqrt{AA_0 A_\ell} - 2\beta^5 A_0 A_\ell = 0.
\]

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following six equations are satisfied.

\[
\beta (\beta_0 \ell - 2\beta_{x\ell}) + 2\beta_0 \beta_\ell = 0 \quad (6.57)
\]

\[
-28\beta^2 (\beta_0 \ell - 2\beta_{x\ell}) - 40\beta_0 \beta \ell + \beta (A_0 \ell - 2A_{x\ell}) + 3 (A_0 \beta_\ell + \beta_0 A_\ell) = 0 \quad (6.58)
\]

\[
8\beta (\beta_0 \ell - 2\beta_{x\ell}) + 15\beta_0 \beta_\ell = 0 \quad (6.59)
\]

\[
8\beta^2 (\beta_0 \ell - 2\beta_{x\ell}) + 10\beta_0 \beta_\ell - 3\beta (A_0 \ell - 2A_{x\ell}) - 8 (A_0 \beta_\ell + \beta_0 A_\ell) = 0 \quad (6.60)
\]

\[
4\beta^2 (A_0 \ell - 2A_{x\ell}) + 4\beta (A_0 \beta_\ell + \beta_0 A_\ell) - 3A_0 A_\ell = 0 \quad (6.61)
\]

\[
A_0 A_\ell = 0 \quad (6.62)
\]

Further, from the equations (6.57) and (6.59), we get

\[
\beta_0 \ell = 2\beta_{x\ell} \quad (6.63)
\]

and

\[
\beta_0 \beta_\ell = 0. \quad (6.64)
\]

Again, from the equations (6.63), (6.64), (6.58) and (6.60), we get

\[
A_0 \ell = 2A_{x\ell} \quad (6.65)
\]

and

\[
A_0 \beta_\ell + \beta_0 A_\ell = 0. \quad (6.66)
\]

Above discussion leads to the following theorem.
Theorem 6.6. Let \((M, F)\) be an \(n\)-dimensional Finsler space with \(F = \frac{\beta^2}{\beta - F} + \beta\), as an infinite series-Randers changed metric. Then \(F\) is locally dually flat if and only if the following equations are satisfied:

\[
A_0 A_\ell = 0, \quad A_0 \beta = 2 A_\beta \beta \ell, \quad \beta_0 \beta = 0, \quad \beta_0 \ell = 2 \beta_\beta \ell, \quad A_\ell \beta_0 + A_0 \beta_\ell = 0.
\]

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