Anomalous relativistic tunneling and exotic point interactions

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Abstract – We examine the one-dimensional quantum scattering of a Dirac particle off relativistic potential barriers. With proper considerations of the Dirac sea, the existence of anomalous tunneling at zero incident energy is revealed for a particular type of relativistic potential having the same magnitudes and opposite signs for scalar and vector components. It is also shown that this leads to an exotic short-range limit of the potentials.

Since its conception, Dirac equation has been plagued by difficulties related to the Dirac sea, and it is only after the proper taming of this monster of negative continuous spectra that useful results are extracted from it. This fact is demonstrated again recently by two examples, one on the resolution of the Klein paradox [1,2], and the other on the resolution of Plesset’s no-bound-state problem [3–5]. In light of the resurgent interest in the Dirac equation in condensed-matter physics [6,7], as well as of the traditional interest in nuclear physics [8], we believe it is timely to report, in this letter, a new result concerning a subtle effect of negative spectra on the simple one-dimensional potential scattering of a Dirac particle off relativistic potential barriers.

Here, we show that a proper consideration of the Dirac sea has intriguing ramifications for low-energy scattering matrices. In particular, it is shown to cause anomalous tunneling at zero incident energy for “$S = -V$ potentials”, a particular type of relativistic potential having the same magnitudes and opposite signs for scalar and vector components of the potential.

We further show, by considering the short-range limit of this potential scattering, that it leads to an exotic low-pass Gaussian wave filter whose non-relativistic kinematics limit is the “delta-prime” point interaction, that causes discontinuity in the quantum wave function itself but not in its derivative [9], which is distinct from the conventional delta-function point interaction.

We start by considering the Dirac equation in one dimension that takes the following two-component form:

$$
\begin{pmatrix}
\varphi' \\
\chi'
\end{pmatrix} = \begin{pmatrix}
0 & m + \varepsilon + S - V \\
m - \varepsilon + S + V & 0
\end{pmatrix} \begin{pmatrix}
\varphi \\
\chi
\end{pmatrix},
$$

where $m$, $\varepsilon$ stand for the mass and relativistic energy of a Dirac particle, and $S$ and $V$ are scalar and (time component of) vector potentials. We only treat time-symmetric systems, so the spatial component of the vector potential is absent. The prime signifies the spatial derivative $\frac{d}{dx}$.

We first consider the one-dimensional potential barrier of constant height located in the positive $x$ region, formally given by

$$
V(x) = v \Theta(x), \quad S(x) = s \Theta(x),
$$

where $\Theta(x)$ is a Heaviside step function which is unity for $x > 0$ and zero for $x < 0$. We define the “mass excluded” energy $w$ by

$$
w = \varepsilon - m,
$$

which we assume to be positive. The spectra of a Dirac particle inside the potential barrier is composed of two disjoint continuous spectra separated by a gap. Some examples of the spectra are shown in fig. 1. The scattering wave functions at $x < 0$ and $x > 0$ are given, respectively, as

$$
\begin{align*}
\left(\begin{array}{c}
\varphi \\
\chi
\end{array}\right) &= \left(\begin{array}{c}
\frac{1}{ik} \frac{1}{m + \varepsilon} \\
-\frac{1}{ik} \frac{1}{m + \varepsilon}
\end{array}\right) e^{ikx} - R \left(\begin{array}{c}
\frac{1}{ip} \frac{1}{m + \varepsilon + s - \varepsilon} \\
\frac{1}{ip} \frac{1}{m + \varepsilon + s - \varepsilon}
\end{array}\right) e^{-ikx}, \\
\left(\begin{array}{c}
\varphi \\
\chi
\end{array}\right) &= T \left(\begin{array}{c}
\frac{1}{ip} \frac{1}{m + \varepsilon + s - \varepsilon}
\end{array}\right) e^{ipx},
\end{align*}
$$

with the free momentum $k = \sqrt{\varepsilon^2 - m^2}$ and the momentum $p = \sqrt{(\varepsilon - v)^2 - (m + s)^2}$ inside the potential barrier. Expression (4) is valid for the case of continuous spectra for the barrier region $\varepsilon > |m + s| + v$. For the case of
\[ |m + s| > \varepsilon > -|m + s| + v, \] we need to make a replacement, \( p = \kappa x \) with \( \kappa = \sqrt{-\varepsilon - v + (m + s)^2} \). The case of \( \varepsilon < -|m + s| + v \) corresponds to a particle under the Dirac sea, for which a \( p < -p \) is needed in (4), but this case is soon to be shown irrelevant. Reflection and transmission rate are given by the squared absolute values of coefficients \( R \) and \( T \), respectively, when \( p \) is real. When \( \kappa \) is real, on the other hand, \( T \) is the amplitude of the wave function at the classically forbidden region.

Expressing the momenta in terms of the energy \( w \), we have
\[
\begin{align*}
k &= \sqrt{w(w + 2m)}, \\
p &= \sqrt{(w - s - v)(w + 2m + s - v)}. \\
\end{align*}
\]

The matching of the wave functions (4) at \( x = 0 \) gives
\[ 1 - R = T, \quad g(1 + R) = T \]
with
\[
\begin{align*}
g &= \sqrt{\frac{w(w + 2m + s - v)}{(w + 2m)(w - s - v)}} \quad Q = \frac{1}{\sqrt{i}} \sqrt{\frac{w(w + 2m + s - v)}{(w + 2m)(s + v - w)}} Q, \\
\end{align*}
\]
Here, \( Q \) is the Giachetti-Sorace factor given by
\[
\begin{align*}
Q &= 1 - \Theta(v - s - 2m - w)\Theta(s + v - w),
\end{align*}
\]
that represents the exclusion of the wave function to the barrier region \( x > 0 \) when the energy \( w \) hits the negative energy spectra of the Dirac equation with potentials \( s \) and \( v \). It is technically obtained from the proper connection condition \( \varphi(0-) = 0, \chi(0-) = \text{const} \), which is obtainable as the \( n \to \infty \) limit of the \((x - 1)^n\) potential, for which \( \varphi(0-) = 0, \chi(0-) = \text{const} \) is found to be the correct condition [5]. A fuller picture of this peculiar boundary condition may require the treatment of the problem with a proper field theoretical setup as in [2], where the exclusion factor \( Q \) could be understood as a result of many-body Pauli blocking.

The solution of the problem (6) is elementary, and we have
\[
T = \frac{2g}{1 + g}, \quad R = \frac{1 - g}{1 + g}.
\]

For large enough \( w \), that satisfies the condition \( w > |m + s| - m + v \), the spectrum inside the potential region is continuous, and we have partial transmission and reflection specified by (9) with (7). We naturally have a unitarity relation \(|R|^2 + |T|^2 = 1\). As we decrease \( w \) down to the threshold energy \( w = v - m + |m + s| \), \( p \) approaches zero, and \( g \) becomes either zero (if \( s < -m \) and therefore \( w = v - s - 2m \)) or infinity (if \( s > -m \) and therefore \( w = v + s \)). They, respectively, correspond to the perfect reflection \( R = 1 \) with Dirichlet boundary \( \varphi(0-) = 0 \) or \( R = -1 \) with Neumann boundary \( \chi(0-) = 0 \).

Below this threshold, \( |m + s| - m + v > w > -|m + s| - m + v \) (or \( |\varepsilon - v| < |m + s| \) if it occurs with positive \( \varepsilon \)), we have an exponential wave function with decay constant \( \kappa \).

The full reflection \(|R| = 1\) with quantum penetration \( 0 < T < 2 \) to the classically forbidden area is observed. Note that there is no problem in having \(|T| > 1\) in this case, since the unitarity is guaranteed by the decaying wave function \( e^{-\kappa x} \). At the “Dirac sea” threshold, \( w = v - m - |m + s| \), \( \kappa \) approaches zero, and \( g \) becomes either infinity (if \( s < -m \) and therefore \( w = v + s \)) or zero (if \( s > -m \) and therefore \( w = v - s - 2m \)). They, respectively, correspond to the perfect reflection \( R = -1 \) with Neumann boundary \( \chi(0-) = 0 \) or \( R = 1 \) with Dirichlet boundary \( \varphi(0-) = 0 \).

Below the Dirac sea threshold, \( w < -|m + w| - m + v \) we have the perfect reflection with the Dirichlet condition \( R = 1 \) as a result of the Giachetti-Sorace factor.

When the energy \( w \) approaches 0, we have \( \kappa \to 0 \), that signifies the quantum penetration length to the classically forbidden region \( x > 0 \) becoming infinite. However, (7) tells us that we have \( g = 0 \), and thus no penetration amplitude \( T = 0 \) and, as a result, the perfect “classical” reflection \( R = 1 \).

The above statements are true in the generic case, depicted in the left-hand graph in fig. 2, for example, but there is an exception to the case when potentials \( v \) and \( s \) are related by \( s + v = 0 \), depicted in the right-hand graph of fig. 2. In this case, there is a cancellation in the expression for \( g \), (7), and we have
\[
\begin{align*}
g &= \sqrt{\frac{w + 2m - 2v}{w + 2m}} \quad \frac{1}{\sqrt{i}} \sqrt{\frac{2v - w - 2m}{w + 2m}} \quad (10)
\end{align*}
\]
which is finite as a result of the “merging” of the \( w = v + s \) line, on which \( g = \infty \) holds, and of the \( w = 0 \) line, on which \( g = 0 \) holds.

This means that for this special case of opposite-sign but equal-magnitude scalar and vector potential, \( s + v = 0 \), we have a singular infinite-range penetration limit \( \kappa \to 0 \) with finite amplitude \( 0 < |T| < 2 \) for zero-energy barrier reflection \( w \to 0 \). We rush to note that this poses no paradox of any sort, since we still have full reflection \(|R| = 1\) albeit with some non-trivial phase for \( R \). This is a subtle but an exotic exception, nonetheless, whose significance soon becomes obvious in the following.

We now consider the one-dimensional scattering by the square well of the constant height potential with spatial...
The scattering wave functions at the gap. Mass is set to be $m = 1$. Among the lines separating the spectra, the solid line represents $w = v + s$, on which we have $g = \infty$ and the other line $w = v - s - 2m$, on which we have $g = 0$. At the crossing point of these two lines, $g$ takes the value $g = \sqrt{\frac{2m}{v - s}}$.

![Fig. 3: Schematic representations of the relativistic potential barrier scattering.](image)

An elementary calculation yields the following expressions for transmission and reflection amplitudes:

$$
T = \frac{e^{-ikL}}{\cos pL - \frac{1}{2}(1/g + g) \sin pL},
$$

$$
R = \frac{-\frac{1}{2}(1/g - g) \sin pL}{\cos pL - \frac{1}{2}(1/g + g) \sin pL}.
$$

This expression is literally valid for the energy $w > |m + s| + v$. For the energy $|m + s| + v > w > -|m + s| + v$, we have to make the replacement $p = ix$ as before, which will result in the replacements $\cos pL \to \cosh \kappa L$ and $\sin pL \to \sinh \kappa L$ in (14). This expression is reduced to $T = 0, R = 1$ for the energy $w < -|m + s| + v$ with which we hit the Dirac sea spectra inside the potential barrier, where we have $Q = 0$, thus $g = 0$.

We look at the low-energy limit of the scattering matrix $T$. Generically, for the case of $s \neq -v$, we have the quantity $g$ that approaches to zero as we take the $w \to 0$ limit, causing the divergence of $1/g$, which guarantees the perfect reflection

$$
T \to 0, \quad R \to 1 \quad \text{as} \quad w \to 0 \quad (s + v \neq 0).
$$

This is simply an exact expression of the intuitive statement that a generic obstacle works as a reflecting block for a low-energy projectile, or in other words, if we hit any barrier too slowly, we are bound to get reflected all the time.

However, for the special case of $s + v = 0$, $g$ takes the form (10) after cancellation of $w$ in both denominator and numerator of (7), and we have $g = \text{finite}$ and $\kappa \to 0$ (or $p \to 0$) as we take the $w \to 0$ limit. We therefore obtain, from (14), a peculiar limit

$$
T \to 1, \quad R \to 0 \quad \text{as} \quad w \to 0 \quad (s + v = 0),
$$

which signifies an anomalous full transmission at zero energy. This is particularly intriguing for the case of the decaying wave in the gap region, in which $\kappa$ is real, where the decaying length $1/\kappa$ becomes infinity at $w \to 0$ limit.

The situation is immediately understood by inspecting the illustrations in fig. 4. Here, the graph on the left depicts a generic case that has a normal perfect reflection at the $w \to 0$ limit, while the graph on the right shows the anomalous zero-energy transparency. The reason behind this transparency lies in the enhanced long-range tunneling inside the barrier, which occurs because, at $w \to 0$, the energy approaches to the threshold of negative continuous spectra that exists right below $w = 0$ for $s + v = 0$ potentials. The presence of the Dirac sea not only induces the perfect reflection for $w > 0$ with $v > |s|$, for example, it also affects the decaying length and induces the anomalous tunneling, and transmission at the $w \to 0$ limit for the case of $s + v = 0$ potentials.

Let us now consider the $L \to 0$ limit of relativistic scattering. A straightforward limit will, of course, lead to...
the disappearance of the barrier, $T \to 1$. The limit $L \to 0$
with constant volume integrals,

$$v = \frac{\bar{v}}{L}, \quad s = \frac{\bar{s}}{L} \quad (L \to 0),$$

on the other hand, leads us to

$$T = \frac{1}{\cos \beta + \frac{1}{\sqrt{3}} \sin \beta \left[(\bar{v} + \bar{s})/K + (\bar{v} - \bar{s})K\right]} \Theta(\bar{v} + |\bar{s}|),$$

$$R = \frac{i}{\sqrt{3}} \sin \beta \left[(\bar{v} + \bar{s})/K - (\bar{v} - \bar{s})K\right] \Theta(-\bar{v} - |\bar{s}|)$$

with $\beta = \sqrt{\bar{v}^2 - \bar{s}^2} = i \sqrt{\bar{s}^2 - \bar{v}^2}$ and $K = \sqrt{\frac{w}{w + 2m}}$.

We can interpret this result in terms of relativistic point interactions specified by the boundary condition which is a most general time-reversal symmetric one [10]

$$\begin{pmatrix} \varphi(0) \\ \chi(0) \end{pmatrix} = \begin{pmatrix} \alpha & u_- \\ u_+ & \alpha \end{pmatrix} \begin{pmatrix} \varphi(-0) \\ \chi(-0) \end{pmatrix}$$

with $\alpha^2 - u_+ u_- = 1$. The scattering off the point interaction (19) is given by

$$T = \frac{1}{\alpha + \frac{i}{2}[u_+/K - u_- K]},$$

$$R = -\frac{i}{2}[u_+/K + u_- K]/\alpha + \frac{1}{2}[u_+/K - u_- K],$$

which allows the identifications

$$u_+ = (\bar{s} + \bar{v}) \frac{\sin \beta}{\beta}, \quad u_- = (\bar{s} - \bar{v}) \frac{\sin \beta}{\beta}.$$  

The special cases $\bar{s} = \bar{v}$ and $\bar{s} = -\bar{v}$ can be considered as the limiting cases of (18), and we have, for $\bar{s} = \bar{v}$

$$T = \frac{1}{1 + i\bar{v}/K}, \quad R = \frac{i\bar{v}/K}{1 + i\bar{v}/K}.$$  

while, for $\bar{s} = -\bar{v}$, we have

$$T = \frac{1}{1 + i\bar{v}/K}, \quad R = \frac{-i\bar{v}K}{1 + i\bar{v}/K}. \quad (23)$$

If we take the non-relativistic limit in kinematics, $K \to k/(2m)$, these two cases are exactly identical to the scattering form delta and delta-prime point interactions [9], which represent high-pass and low-pass wave filters, respectively. Note that constructing non-standard point interactions, which results in (23), within the non-relativistic framework involves highly singular procedures [11, 12].

Finally, we ask the question whether we can construct an analogue of the phenomena we have found in the framework of the non-relativistic Schrödinger equation. We rewrite the Dirac equation (1) by eliminating the small component $\chi$ in the form

$$-\frac{d}{dx} \frac{1}{2m^*} \frac{d}{dx} \varphi + U \varphi = w \varphi,$$  

with effective mass $m^*$ and potential $U$ defined by

$$m^* = m + \frac{w}{2} + \frac{S - V}{2}, \quad U = S + V. \quad (25)$$

Assuming the conditions $w \ll m$ and $|S - V| \ll m$, we obtain the Schrödinger equation with effective potential which is given by the sum of vector and scalar potentials. This is nothing but the true non-relativistic limit. However, we obtain a non-standard low-energy limit by assuming the non-relativistic kinematics $w \ll m$ in conjunction with strong relativistic potentials $|S - V| \sim m$. Specifically, we can reproduce an anomalous transmission from the Schrödinger equation (24) by setting $S = -V$, which results in $m^* \approx m - V$ and $U = 0$. This means that we can construct a purely non-relativistic model of anomalous scattering and delta-prime point interaction with just effective mass and no potential. Readers are warned, however, that this non-relativistic analogue scheme works only to the extent that when $S - V$ is negative in sign and so large, we obtain a negative value for the effective mass $m^*$. For this bona fide relativistic dynamics, the non-relativistic analogue (24) does not make sense, and therefore does not exist.

We have shown that a set of relativistic potentials having the property of the same strength but opposite signs for scalar and vector components displays anomalous full tunneling and transparency at zero energy, while the barrier starts functioning at higher energy. The short-range limit of this phenomenon leads to a smooth relativistic realization of an exotic point interaction, the delta-prime one, that conventionally requires singular and esoteric constructions within non-relativistic dynamics. It has been pointed out [13] that in three dimensions, the “$S = -V$” relativistic potentials have an esoteric property called pseudospin symmetry [14] that has been found to play an important role in the degeneracy structure of nuclear levels. Current work shows that
there is yet another aspect to this pseudospin symmetric limit of relativistic potentials, which is revealed only in one-dimensional systems.

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