Right regular triples of semigroups

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Abstract

Let \( M(S; \Lambda; P) \) denote a Rees \( I \times \Lambda \) matrix semigroup without zero over a semigroup \( S \), where \( I \) is a singleton. If \( \theta_S \) denotes the kernel of the right regular representation of a semigroup \( S \), then a triple \( A, B, C \) of semigroups is said to be right regular, if there are mappings \( A \xleftarrow{P} B \xrightarrow{P'} C \) such that \( M(A; B; P)/\theta_{M(A; B; P)} \cong M(C; B; P') \). In this paper we examine right regular triples of semigroups.

Keywords: semigroup, congruence, Rees matrix semigroup

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1 Introduction and motivation

The notion of right regular triples of semigroups is defined in [18], where a special type of Rees matrix semigroups without zero over semigroups are examined. A triple \( A, B, C \) of semigroups is said to be right regular, if there are mappings

\[
A \xleftarrow{P} B \xrightarrow{P'} C
\]

such that the factor semigroup \( M(A; B; P)/\theta_{M(A; B; P)} \) is isomorphic to the semigroup \( M(C; B; P') \), where \( \theta_{M(A; B; P)} \) is the kernel of the right regular representation of the semigroup \( M(A; B; P) \). In [18] it is proved that if \( A, B, C \) are semigroups such that \( A/\theta_A \cong B \) and \( B/\theta_B \cong C \), then the triple \( A, B, C \) is right regular. There is also an example given for a right regular triple \( A, B, C \) of semigroups such that none of the conditions \( A/\theta_A \cong B \) and \( B/\theta_B \cong C \) are fulfilled. These results motivate us to investigate right regular triples of semigroups. In this paper we examine the connection between the structure of semigroups belonging to a right regular triples of semigroups, and present quite a few examples of right regular triples of semigroups.
2 Preliminaries

By a semigroup we mean a multiplicative semigroup, that is, a nonempty set endowed with an associative multiplication.

A nonempty subset $L$ of a semigroup $S$ is called a left ideal of $S$ if $SL \subseteq L$. The concept of a right ideal of a semigroup is defined analogously. A semigroup $S$ is said to be left (resp., right) simple if $S$ itself is the only left (resp., right) ideal of $S$. A semigroup $S$ is left (resp., right) simple if and only if $Sa = S$ (resp., $aS = S$) for every $a \in S$.

A semigroup $S$ is called left cancellative if $xa = xb$ implies $a = b$ for every $x, a, b \in S$. A left cancellative and right simple semigroup is called a right group. A semigroup satisfying the identity $ab = b$ is called a right zero semigroup. By [2, Theorem 1.27.], a semigroup is a right group if and only if it is a direct product of a group and a right zero semigroup.

In [6, Theorem 1], it is shown that a semigroup $S$ is embedded in an idempotent-free left simple semigroup if and only if $S$ is idempotent-free and satisfies the condition: for all $a, b, x, y \in S$, $xa = xb$ implies $ya = yb$.

Using the terminology of [15], a semigroup $S$ satisfying this last condition is called a left equalizer simple semigroup. In other words, a semigroup $S$ is left equalizer simple if, for arbitrary elements $a, b \in S$, the assumption that $xa = xb$ is satisfied for some $x \in S$ implies that $ya = yb$ is satisfied for all $y \in S$. By [15, Theorem 2.1], a semigroup $S$ is left equalizer simple if and only if the factor semigroup $S/\theta_S$ is left cancellative.

A nonempty subset $I$ of a semigroup $S$ is called an ideal of $S$ if $I$ is a left ideal and a right ideal of $S$. A semigroup $S$ is called simple if $S$ itself is the only ideal of $S$. By [2, Lemma 2.2.], a semigroup $S$ is simple if and only if $SaS = S$ for every $a \in S$.

Let $S$ be a semigroup and $I$ be an ideal of $S$. We say that the homomorphism $\varphi : S \to I$ is a retract homomorphism [12, Definition 1.44], if it leaves the elements of $I$ fixed. In this case, $I$ is called a retract ideal of $S$, and $S$ is a retract extension of $I$ by the Rees factor semigroup $S/I$.

A transformation $\varrho$ of a semigroup $S$ is called a right translation of $S$ if $(xy)\varrho = x(y\varrho)$ is satisfied for every $x, y \in S$. For an arbitrary element $a$ of a semigroup $S$, $\varrho_a : x \mapsto xa$ ($x \in S$) is a right translation of $S$ which is called an inner right translation of $S$ corresponding to the element $a$. For an arbitrary semigroup $S$, the mapping $\Phi_S : a \mapsto \varrho_a$ is a homomorphism of $S$ into the semigroup of all right translations of $S$. The homomorphism $\Phi_S$ is called the right regular representation of $S$. For an arbitrary semigroup $S$,
let $\theta_S$ denote the kernel of $\Phi_S$. It is clear that $(a, b) \in \theta_S$ for elements $a, b \in S$ if and only if $xa = xb$ for all $x \in S$. A semigroup $S$ is called left reductive if $\theta_S$ is the identity relation on $S$. Thus $\theta_S$ is faithful if and only if $S$ is left reductive. The congruence $\theta_S$ plays an important role in the investigation of the structure of the semigroup $S$. In [4], the author characterizes semigroups $S$ for which the factor semigroup $S/\theta_S$ is a group. In [5], semigroups $S$ are characterized for which the factor semigroup $S/\theta_S$ is a right group. In [14], Theorem 2, a construction is given which shows that every semigroup $S$ can be obtained from the factor semigroup $S/\theta_S$ by using this construction. In [17], the authors study the probability that two elements which are selected at random with replacement from a finite semigroup have the same right matrix.

If $S$ is a semigroup, $I$ and $\Lambda$ are nonempty sets, and $P$ is a $\Lambda \times I$ matrix with entries $P(\lambda, i)$, then the set $\mathcal{M}(S; I, \Lambda; P)$ of all triples $(i, s, \lambda) \in I \times S \times \Lambda$ is a semigroup under the multiplication $(i, s, \lambda)(j, t, \mu) = (i, sP(\lambda, j)t, \mu)$. According to the terminology in [2, §3.1], this semigroup is called a Rees $I \times \Lambda$ matrix semigroup without zero over the semigroup $S$ with $\Lambda \times I$ sandwich matrix $P$. In [18], Rees matrix semigroups $\mathcal{M}(S; I, \Lambda; P)$ without zero over semigroups $S$ satisfying $|I| = 1$ are in the focus. In our present paper we also use such type of Rees matrix semigroups, which will be denoted by $\mathcal{M}(S; \Lambda; P)$. In this case the matrix $P$ can be considered as a mapping of $\Lambda$ into $S$, and so the entries of $P$ will be denoted by $P(\lambda)$. If the element of $I$ is denoted by 1, then the element $(1, s, \lambda)$ of $\mathcal{M}(S; \Lambda; P)$ can be considered in the form $(s, \lambda)$; the operation on $\mathcal{M}(S; \Lambda; P)$ is $(s, \lambda)(t, \mu) = (sP(\lambda)t, \mu)$.

For notations and notions not defined but used in this paper, we refer the reader to books [2], [8], and [12].

3 Results

**Theorem 1** If $A, B, C$ is a right regular triple of semigroups such that $A$ is right simple, then $C$ is also right simple.

**Proof.** Assume that $A, B, C$ is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A; B; P)} \cong \mathcal{M}(C; B; P')$.

Assume that $A$ is right simple. Let $(a_1, b_1), (a_2, b_2) \in \mathcal{M}(A; B; P)$ be arbitrary elements. Since $A$ is right simple, we have $aP(b_1)A = A$, and so there
is an element $\xi \in A$ such that $a_1P(b_1)\xi = a_2$ and

$$(a_1, b_1)(\xi, b_2) = (a_2, b_2).$$

Hence the Rees matrix semigroup $\mathcal{M}(A; B; P)$ is right simple. As every homomorphic image of a right simple semigroup is right simple, the Rees matrix semigroup $\mathcal{M}(C; B; P')$ is right simple. Let $c, \eta \in C$ be an arbitrary elements. Then, for any $b \in B$, $(c, b)\mathcal{M}(C; B; P') = \mathcal{M}(C; B; P')$, and so

$$(c, b)(u, v) = (\eta, b)$$

for some $(u, v) \in \mathcal{M}(C; B; P')$. Hence $cP'(b)u = \eta$. Thus $cC = C$ for every $c \in C$. Then $C$ is right simple.

**Theorem 2** If $A, B, C$ is a right regular triple of semigroups such that $A$ is a right group, then $C$ is also a right group.

**Proof.** Assume that $A, B, C$ is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A; B; P)} \cong \mathcal{M}(C; B; P').$$

Assume that $A$ is a right group, that is, right simple and left cancellative. By the proof of Theorem 1, the semigroups $\mathcal{M}(A; B; P)$ and $C$ are right simple. Let $(a, b), (a_1, b_1), (a_2, b_2) \in \mathcal{M}(A; B; P)$ be arbitrary elements with

$$(a, b)(a_1, b_1) = (a, b)(a_2, b_2).$$

Then

$$(aP(b)a_1, b_1) = (aP(b)a_2, b_2),$$

that is,

$$aP(b)a_1 = aP(b)a_2 \quad \text{and} \quad b_1 = b_2.$$  

As $A$ is left cancellative, we get $a_1 = a_2$, and so

$$(a_1, b_1) = (a_2, b_2).$$

Hence the semigroup $\mathcal{M}(A; B; P)$ is left cancellative. As $\mathcal{M}(A; B; P)$ is also right simple, it is a right group. From the left cancellativity of $\mathcal{M}(A; B; P)$ it follows that $\theta_{\mathcal{M}(A; B; P)} = \iota_{\mathcal{M}(A; B; P)}$. Thus the semigroup $\mathcal{M}(C; B; P')$ is left cancellative. Assume $xc_1 = xc_2$ for elements $x, c_1, c_2 \in C$. Let $b \in B$
be arbitrary. As $C$ is right simple, there are elements $u, v \in C$ such that $P(b)u = c_1$ and $P(b)v = c_2$. Thus

$$xP(b)u = xP(b)v.$$ 

Then, for an arbitrary $b' \in B$,

$$(x, b)(u, b') = (x, b)(v, b')$$

is satisfied in $M(C; B; P)$. As $M(C; B; P)$ is left cancellative, we get $u = v$, from which it follows that $c_1 = c_2$. Hence $C$ is left cancellative. By the above, $C$ is right simple. Consequently $C$ is a right group. \qed

**Theorem 3** If $A, B, C$ is a right regular triple of semigroups such that $A$ is simple, then $C$ is also simple.

**Proof.** Assume that $A, B, C$ is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$M(A; B; P) / \theta_{M(A; B; P)} \cong M(C; B; P').$$

Assume that $A$ is simple. Let $(a, b), (u, v) \in M(A; B; P)$ and $z \in B$ be an arbitrary elements. Then $AP(z)aP(b)A = A$ implies that there are elements $\xi, \eta \in A$ such that $\xi P(z)aP(b)\eta = u$ and so

$$(\xi, z)(a, b)(\eta, v) = (u, v).$$

Hence the Rees matrix semigroup $M(A; B; P)$ is simple. As every homomorphic image of a simple semigroup is simple, the Rees matrix semigroup $M(C; B; P')$ is simple.

Let $c_1, c_2 \in C$ and $b_1, b_2 \in B$ be arbitrary elements. Then

$$M(C; B; P')(c_1, b_1)M(C; B; P') = M(C; B; P'),$$

and so there are elements $(x, \xi), (y, \eta) \in M(C; B; P')$ such that

$$(xP(\xi)c_1P(b_1)y, \eta) = (x, \xi)(c_1, b_1)(y, \eta) = (c_2, b_2).$$

Hence

$$xP(\xi)c_1P(b_1)y = c_2.$$ 

Thus

$$Cc_1C = C$$

for every $c_1 \in C$. Then $C$ is simple. \qed

The next proposition is used in the proof of Theorem 4.
Proposition 1 Let $A$ be a semigroup, $\Lambda$ be an arbitrary nonempty set and $P : \Lambda \mapsto A$ is an arbitrary mapping. If $A$ is left equalizer simple, then the Rees matrix semigroup $\mathcal{M}(A; \Lambda; P)$ is also left equalizer simple.

Proof. Suppose that $A$ is a left equalizer simple semigroup, $\Lambda$ is a nonempty set and $P : \Lambda \mapsto A$ is a mapping. Take $(a_1, b_1), (a_2, b_2), (a, b) \in \mathcal{M}(A; \Lambda; P)$. Suppose that

$$(a, b)(a_1, b_1) = (a, b)(a_2, b_2).$$

This means that

$$(aP(b)a_1, b_1) = (aP(b)a_2, b_2) \iff aP(b)a_1 = aP(b)a_2 \text{ and } b_1 = b_2.$$ 

Since $A$ is left equalizer simple we have that, for all $x \in A$ and $y \in \Lambda :$

$$xP(y)a_1 = xP(y)a_2,$$

hence,

$$(x, y)(a_1, b_1) = (x, y)(a_2, b_2).$$

Thus, $\mathcal{M}(A; \Lambda; P)$ is a left equalizer simple semigroup.

Theorem 4 Let $A, B, C$ be a right regular triple of semigroups such that $P' : B \mapsto C$ is surjective. If $A$ is left equalizer simple, then $C$ is left cancellative.

Proof. Assume that $A, B, C$ is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; \Lambda; P) / \theta_{\mathcal{M}(A; B; P)} \cong \mathcal{M}(C; B; P').$$

From Proposition 1, we have that $\mathcal{M}(A; B; P)$ is a left equalizer simple semigroup, and hence $\mathcal{M}(C; B; P')$ is left cancellative by [15, Theorem 2.1].

Now, take $x, c_1, c_2 \in C$ such that $xc_1 = xc_2$. Since $P'$ is surjective, there exists $b \in B$ such that $P'(b) = x$. Then $P'(b)c_1 = P'(b)c_2$. Let $c \in C$ be arbitrary, then

$$(c, b)(c_1, b) = (cP'(b)c_1, b) = (cP'(b)c_2, b) = (c, b)(c_2, b).$$

Since $\mathcal{M}(C; B; P')$ is left cancellative, $(c_1, b) = (c_2, b)$, hence $c_1 = c_2$. Thus $C$ is left cancellative.
**Theorem 5** Let $A, B, C$ be a right regular triple of semigroups such that $C$ is left commutative. If $A$ is left equalizer simple, then $C$ is left cancellative.

**Proof.** From the proof of Theorem 4, we know that $\mathcal{M}(C; B; P')$ is left cancellative. Again, take $x, c_1, c_2 \in C$ such that $xc_1 = xc_2$. Then for arbitrary $b \in B$,

$$P'(b)x_1 = P'(b)x_2.$$  

Since $C$ is left commutative,

$$xP'(b)c_1 = xP'(b)c_2,$$

and then

$$(x, b)(c_1, b) = (x, b)(c_2, b).$$

$\mathcal{M}(C; B; P')$ is left cancellative, thus we get $c_1 = c_2$, and that $C$ is left cancellative.

\end{proof}

**Theorem 6** Let $A, B, C$ be a right regular triple of semigroups such that $P : B \mapsto A$ is surjective. If $A$ is left reductive, then $C$ is also left reductive.

**Proof.** Assume that $A, B, C$ is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A; B; P)} \cong \mathcal{M}(C; B; P').$$

Assume, that $A$ is a left reductive semigroup, and $(a_1, b_1), (a_2, b_2) \in \mathcal{M}(A; B; P)$ are elements such that

$$\forall(x, y) \in \mathcal{M}(A; B; P) : (x, y)(a_1, b_1) = (x, y)(a_2, b_2).$$

This means that

$$xP(y)a_1 = xP(y)a_2 \quad \text{and} \quad b_1 = b_2.$$  

Since $A$ is left reductive, we get that

$$\forall y \in B : P(y)a_1 = P(y)a_2.$$  

In this case, $P$ is a surjective mapping, hence using again that $A$ is left reductive, we have $a_1 = a_2$. We conclude that $(a_1, b_1) = (a_2, b_2)$, and thus $\mathcal{M}(A; B; P)$ is left reductive.
We know, that if $S$ is a left reductive semigroup, then $\theta_S = \iota_S$. This means, that $\mathcal{M}(A; B; P) \cong \mathcal{M}(C; B; P')$, hence $\mathcal{M}(C; B; P')$ is also left reductive.

Now suppose that $c_1, c_2 \in C$ are such elements, that

$$\forall c \in C : \ c c_1 = c c_2.$$ 

Take two elements, $(c_1, b), (c_2, b)$ from $\mathcal{M}(C; B; P')$. For arbitrary $(x, y) \in \mathcal{M}(C; B; P')$ we have:

$$(x, y)(c_1, b) = (x P'(y)c_1, b) = (x P'(y)c_2, b) = (x, y)(c_2, b).$$

In the second equality, we used the assumption that $\forall c \in C : \ c c_1 = c c_2$.

Since $\mathcal{M}(C; B; P')$ is left reductive, we have $(c_1, b) = (c_2, b)$, and thus $c_1 = c_2$.

We conclude that $C$ is left reductive.

Let $A$ be a semigroup and $B$ be a nonempty set. For a mapping $P$ of $B$ into $A$, let $\alpha_P$ denote the following relation on $A$:

$$\alpha_P = \{(a_1, a_2) \in A \times A : (\forall a \in A)(\forall b \in B) \ a P(b)a_1 = a P(b)a_2\}.$$ 

It is clear that $\alpha_P$ is a right congruence on $A$.

**Remark 1** It is clear that if $P$ is a mapping of a semigroup $B$ into a semigroup $A$ such that $\alpha_P$ is the identity relation on $A$, then $\theta_{\mathcal{M}(A, B, P)}$ is the identity relation on $\mathcal{M}(A; B; P)$, and hence the triple $A, B, A$ is right regular.

Let $A, B, C$ be semigroups and $P : B \to A$, $P' : B \to C$ be arbitrary mappings. We shall say that the triple $A, B, C$ is right regular with respect to the couple $(P, P')$ if $\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A, B, P)} \cong \mathcal{M}(C; B; P')$.

**Theorem 7** Let $A$ and $B$ be arbitrary semigroups, and $P$ be a mapping of $B$ into $A$ such that $\alpha_P$ is a congruence on $A$. Then the triple $A, B, A/\alpha_P$ is right regular with respect to $(P, P')$, where $P'$ is defined by $P' : b \mapsto [P(b)]_{\alpha_P}$ for every $b \in B$.

**Proof.** Let $\Phi$ be the mapping of the Rees matrix semigroup $M = \mathcal{M}(A; B; P)$ onto the Rees matrix semigroup $\mathcal{M}(A/\alpha_P; B; P')$ defined by

$$\Phi : (a, b) \mapsto ([a]_{\alpha_P}, b).$$
For arbitrary elements \((a_1, b_1), (a_2, b_2)\) of \(M\), we have
\[
\Phi((a_1, b_1)(a_2, b_2)) = \Phi((a_1 P(b_1)a_2, b_2)) = ([a_1 P(b_1)]_{\alpha_P}, b_2) =
\]
\[
= ([a_1]_{\alpha_P} P(b_1)[a_2]_{\alpha_P}, b_2) = ([a_1]_{\alpha_P} P'(b_1)[a_2]_{\alpha_P}, b_2) =
\]
\[
= ([a_1]_{\alpha_P}, b_1) ([a_2]_{\alpha_P}, b_2) = \Phi((a_1, b_1)) \Phi((a_2, b_2)).
\]
Hence, \(\Phi\) is a homomorphism. It is clear that \(\Phi\) is surjective. We show that the kernel \(\ker \Phi\) of \(\Phi\) is the kernel of the right regular representation of \(M\). For elements \((a_1, b_1)\) and \((a_2, b_2)\) of \(M\), the equation
\[
(a, b)(a_1, b_1) = (a, b)(a_2, b_2)
\]
is satisfied for every \(a \in A\) and every \(b \in B\) if and only if
\[
(aP(b)a_1, b_1) = (aP(b)a_2, b_2),
\]
that is
\[
\Phi((a_1, b_1)) = \Phi((a_2, b_2)).
\]
Thus, \(\ker \Phi = \theta_M\) which proves our theorem.

A semigroup satisfying the identity \(axyb = ayxb\) is called a medial semigroup. It is easy to see that if \(A\) is a medial semigroup, then, for an arbitrary semigroup \(B\) and an arbitrary mapping of \(B\) into \(A\), the right congruence \(\alpha_P\) is a congruence on \(A\). Thus we have the following corollary.

**Corollary 1** Let \(A\) be a medial semigroup. Then, for an arbitrary semigroup \(B\) and an arbitrary mapping \(P\) of \(B\) into \(A\), the triple \(A, B, A/\alpha_P\) is right regular, where \(P'\) is defined in Theorem 7.

If \(\rho\) is an arbitrary congruence on a semigroup \(S\), then \(\rho^* = \{(a, b) \in S \times S : (\forall s \in S)(sa, sb) \in \rho\}\) (defined in [15]) is also a congruence on \(S\) which is called the right colon congruence of \(\rho\).

**Remark 2** If \(P\) is a mapping of a nonempty set \(B\) onto a semigroup \(A\), then \(\alpha_P \supseteq \theta_A^*\). If \(P\) is surjective, then \(\alpha_P = \theta_A^*\).

Remark 2 and Theorem 7 imply the following corollary.
Corollary 2 Let \( A \) be an ideal of a semigroup \( B \) such that there is a surjective homomorphism \( P \) of \( B \) onto \( A \). Let \( P' \) denote the mapping of \( B \) onto \( A/\theta_A^* \) defined in the following way: \( P': b \mapsto [P(b)]_{\theta_A^*} \) for every \( b \in B \). Then the triple \( A, B, A/\theta_A^* \) is right regular with respect to \( (P, P') \).

Since the projective homomorphism \( P_A: (a, b) \mapsto a \) of the direct product \( A \times B \) of semigroups \( A \) and \( B \) is surjective, Remark 2 and Theorem 7 imply the following corollary.

Corollary 3 For arbitrary semigroups \( A \) and \( B \), the triple \( A, A \times B, A/\theta_A^* \) is right regular with respect to the couple \( (P_A, P') \), where \( P_A \) denotes the projection homomorphism \( P_A: (a, b) \mapsto a \) and \( P': A \times B \to A/\theta_A^* \) is defined by \( P': (a, b) \mapsto [a]_{\theta_A^*} \).

Theorem 8 Let \( A \) and \( B \) be arbitrary semigroups, and \( \varphi \) be a mapping of \( A \) into \( B \) such that \( \alpha_\varphi \) is a congruence on \( B \). Then the triple \( A \times B, A/\theta_A^* \times B/\alpha_\varphi \) is right regular with respect to the couple \( (P_A, P') \), where \( P_A \) is defined by \( P_A: a \mapsto (a, \varphi(a)) \) and \( P' \) is defined by \( P': a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\alpha_\varphi}) \).

Proof. Suppose that \((((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4)) \in \theta_M, \) where \( M = \mathcal{M}(A \times B; A; P_A) \). This means that, for every \( x, x' \in A \) and \( y \in B \),

\[
((x, y), x')((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4)) \iff ((x, y), x')((a_1, b_1), (a_3, b_3), (a_4)) \iff ((x' a_1, y \varphi(x) b_1), (a_2)) = ((x' a_3, y \varphi(x') b_3), (a_4)).
\]

The equality holds if and only if

\[
x x' a_1 = x x' a_3, \quad y \varphi(x) b_1 = y \varphi(x') b_3, \quad a_2 = a_4,
\]

that is

\[
(a_1, a_3) \in \theta_A^*, \quad (b_1, b_3) \in \alpha_\varphi, \quad a_2 = a_4 \quad (1)
\]

Let \( \Phi \) be the mapping of \( \mathcal{M}(A \times B; A; P_A) \) into \( \mathcal{M}(A/\theta_A^* \times B/\alpha_\varphi; P') \) defined by \( \Phi: ((a, b), a') \mapsto ([a]_{\theta_A^*}, [b]_{\alpha_\varphi}, a') \) for every \( a, a' \in A \) and every \( b \in B \). Since

\[
\Phi(((a_1, b_1), (a_2, b_3), (a_4))) = \Phi((a_1 a_2 a_3, b_1 \varphi(a_2) b_3), (a_4)) =
\]

\[
= ([a_1 a_2 a_3]_{\theta_A^*}, [b_1 \varphi(a_2) b_3]_{\alpha_\varphi}, a_4) = ([a_1]_{\theta_A^*}, [b_1]_{\alpha_\varphi}, a_2) ([a_3]_{\theta_A^*}, [b_3]_{\alpha_\varphi}, a_4) =
\]
\[
\Phi(((a_1, b_1), (a_2)), ((a_3, b_3), (a_4)))
\]

for every \(a_1, a_2, a_3, a_4 \in A\) and \(b_1, b_3 \in B\), \(\Phi\) is a homomorphism. It is clear that \(\Phi\) is a surjective.

Since \(((a_1, b_1), (a_2)), ((a_3, b_3), (a_4)) \in \ker \Phi\) if and only if all three conditions in (1) are satisfied, we have \(\ker \Phi = \theta_M\) and this proves our theorem. \(\square\)

If \(\varphi : A \mapsto B\) defined in Theorem 8 is surjective, then \(\alpha_{\varphi} = \theta_B^*\) by Remark 2, and thus we have the following corollaries:

**Corollary 4** Let \(A\) and \(B\) be semigroups, and \(\varphi\) be a surjective mapping of \(A\) onto \(B\). Then the triple \(A \times B, A/\theta_A^* \times B/\theta_B^*\) is right regular with respect to the couple \((P_A, P')\), where \(P_A\) is defined by \(P_A : a \mapsto (a, \varphi(a))\) and \(P'\) is defined by \(P' : a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\theta_B^*})\).

**Corollary 5** Let \(A\) be a semigroup, and \(B\) be a retract ideal of \(A\). Let \(\varphi\) be a retract homomorphism of \(A\) onto \(B\). Then the triple \(A \times B, A/\theta_A^* \times B/\theta_B^*\) is right regular with respect to the couple \((P_A, P')\), where \(P_A\) is defined by \(P_A : a \mapsto (a, \varphi(a))\) and \(P'\) is defined by \(P' : a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\theta_B^*})\).

If \(B\) is an ideal of a semigroup \(A\) such that \(B\) is a group, then \(\varphi_B : A \mapsto B\) defined by \(\varphi_B(a) = ae\) \((a \in A)\) is a retract homomorphism of \(A\) onto \(B\), where \(e\) denotes the identity element of the group \(B\).

**Corollary 6** Let \(A\) be a semigroup and \(B\) be an ideal of \(A\) such that \(B\) is a group. Then the triple \(A \times B, A/\theta_A^* \times B\) is right regular with respect to the couple \((P_A, P')\), where \(P_A\) is defined by \(P_A : a \mapsto (a, \varphi_B(a))\) and \(P'\) is defined by \(P' : a \mapsto ([a]_{\theta_A^*}, \varphi_B(a))\); here \(\varphi_B\) denotes the above surjective homomorphism of \(A\) onto \(B\).

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