On the Inequivalence of Weak Localization 
and Coherent Backscattering

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Abstract

We define a current-conserving approximation for the local conductivity tensor of a disordered system which includes the effects of weak localization. Using this approximation we show that the weak localization effect in conductance is not obtained simply from the diagram corresponding to the coherent back-scattering peak observed in optical experiments. Other diagrams contribute to the effect at the same order and decrease its value. These diagrams appear to have no semiclassical analogues, a fact which may have implications for the semiclassical theory of chaotic systems. The effects of discrete symmetries on weak localization in disordered conductors is evaluated and compared to results from chaotic scatterers.
I. INTRODUCTION

The weak localization (WL) effect has played a seminal role in both theoretical and experimental work on quantum transport in disordered conductors. On the theoretical side it enters the scaling theory of localization in a fundamental manner [1], and on the experimental side it provides the best measurement of the phase-coherence length for electrons at low temperature [2]. Finally, through the normal-metal Aharonov-Bohm effect, it provides a very direct measurement of quantum interference of normal electrons on the scale of this phase-coherence length [3]. The quantitative theory of weak-localization is based on impurity-averaged perturbation theory in the small parameter \((k_f l)^{-1}\) \((k_f\) is the fermi wavevector and \(l\) is the elastic mean free path) [4]. Many experiments over the past decade have tested this theory in detail and found excellent agreement with the measured low-temperature variation of the conductance with temperature and magnetic field.

The weak localization effect arises due to time-reversal (TR) symmetry. It is often described [2] by noting that a diffusing electron wave has an enhanced probability of returning to its starting point (compared to a classical particle) due to the constructive interference of time-reversed pairs of trajectories forming closed loops. This enhancement of the return probability then leads to reduced conductance with respect to the classical Drude value. Breaking time-reversal symmetry eliminates this interference causing a positive magnetoconductance (in the simplest case in which spin-orbit scattering is negligible [2]). Although WL is an interference effect and hence clearly non-classical, it is often described as a semi-classical effect [2,5,6] which can be understood simply by adding the amplitudes for motion along classical trajectories with appropriate phases, and then squaring the amplitude. Since recent theoretical advances in semi-classical quantum theory have shown quantitative success in other fields, e.g. atomic physics [7]; it is of some interest to see if weak localization can be quantitatively obtained from a summation over classical paths. Recently two of the authors addressed this problem in a treatment of WL in ballistic conductors and numerically found contributions to the WL effect which they could not obtain from a semiclassical calculation [8,9]. It was noted [9] that a similar contribution apparently entered in the WL theory of disordered conductors, but had not been explicitly discussed or calculated in the literature. We present such a calculation below and discuss its implications. Reference [8] also discovered a sensitivity of WL in ballistic conductors to discrete spatial symmetries; we evaluate this sensitivity for disordered conductors below.

Well before the discovery of weak localization in disordered conductors the increase in backscattering probability due to time-reversal symmetry had been noted both in optics [10,11] and in nuclear scattering [12]. In the former this effect is referred to as coherent backscattering (CB) and in the latter case as the elastic enhancement factor. In optics one clearly observes a peak approaching a factor of two in height in the backwards direction [13–15] (if the polarization is preserved). Recently the CB effect in optics has been hypothesized [16] to explain the surge in brightness of astronomical bodies when in opposition, and this hypothesis is believed to have significant implications for planetary science. Although known for some time, the detailed study of CB in optics was stimulated by WL theory and the two terms (weak localization and coherent backscattering) are often used interchangeably. In this work we emphasize that the two effects are not equivalent: diagrams which make a negligible contribution to the differential scattering cross-section make a contribution
to the weak localization conductance of the same order and of opposite sign as the diagram which gives rise to the CB peak.

II. CURRENT CONSERVATION FOR EXACT CONDUCTIVITY AND CONDUCTANCE

In linear response theory the quantum expectation value of the local DC current density induced by an electric field \( \mathbf{E}(\mathbf{r}') \) is

\[
< \mathbf{J}(\mathbf{r}) > = \int d\mathbf{r}' \sigma(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')
\]

where \( \sigma \) is the Kubo local conductivity tensor, which may be expressed in terms of the exact eigenfunctions and eigenenergies for a given disorder configuration \([17–19]\). The general expression for \( \sigma \) in the presence of a magnetic field, \( B \), may be divided into a term which only depends on states at the fermi surface (at \( T=0 \)) and is symmetric in \( B \), and a term which depends on all states below the fermi surface and is anti-symmetric in \( B \) \([19]\). The weak localization effect relates to average properties of conductivity and conductance which are symmetric in \( B \), hence the anti-symmetric term may be neglected. With this simplification we may write

\[
\sigma(\mathbf{r}, \mathbf{r}') = -\frac{e^2 \hbar^3}{16\pi m^2} \Delta G(\varepsilon_f, \mathbf{r}, \mathbf{r}') \hat{D} \ast \hat{D} \Delta G(\varepsilon_f, \mathbf{r}', \mathbf{r}),
\]

where \( \Delta G(E, \mathbf{r}, \mathbf{r}') \equiv G^+(\mathbf{r}, \mathbf{r}') - G^-(\mathbf{r}, \mathbf{r}') \), \( G^\pm \) is the advanced (retarded) Green functions for the single particle Schrödinger equation, and \( \hat{D} \) represents the antisymmetric gauge-invariant derivative:

\[
f(x) \hat{D} g(x) = f(x) [D g(x)] - g(x) [D f(x)],
\]

\( D = \nabla - (e i / \hbar c) A(\mathbf{r}). \) Since WL occurs at relatively weak magnetic fields it turns out to be sufficient to make the further approximation \( \hat{D} \approx \nabla \) which we do henceforth. With this set of approximations DC current conservation for the local current density, \( \nabla \cdot < \mathbf{J}(\mathbf{r}) > = 0 \), is equivalent \([19]\) to the condition \( \nabla \cdot \sigma = \nabla' \cdot \sigma = 0 \).

Transport experiments do not directly measure the local conductivity tensor; instead they measure conductance or resistance. In the simplest case of a two-probe measurement, in which the voltage drop is measured between the current source and sink, the conductance, \( g \), is just the inverse of the resistance and is related to the local conductivity by integration over the cross-sections at the interface between the sample and the leads \([19]\):

\[
g = \int dS_1 \int dS_2 \hat{n}_1 \cdot \sigma(\mathbf{r}, \mathbf{r}') \cdot \hat{n}_2
\]

where \( \hat{n}_1, \hat{n}_2 \) are unit normal vectors to the cross-sections of the two leads.

There are two different forms of Eq. (3) often used in the literature; the first reexpresses conductance in terms of scattering coefficients, and the second reexpresses conductance as an average over the sample volume. In the first case, the cross-sections are taken just outside the sample in order to express \( \sigma \) in terms of the exact asymptotic wavefunctions which involve the transmission and reflection amplitudes. Straightforward manipulations worked
out by Economou and Soukolis [17] and by Fisher and Lee [18] then show that Eq. (3) is equivalent to the two-probe Landauer formula:

\[ g = \frac{e^2 N_c}{h} \sum_{i,j} T_{ij} \]  

(4)

where \( T_{ij} \) is the transmission coefficient between the propagating states \( i,j \) in the leads (assumed identical) and there are \( N_c \) such states at \( \varepsilon_f \). Below we will discuss extensively the contribution to the WL effect of the transmission coefficients which are diagonal or off-diagonal in the mode indices \( i,j \).

In the second case one uses the relation \( \nabla \cdot \sigma = \nabla' \cdot \sigma = 0 \) to show that \( g \) is independent of the location of the cross-section in Eq. (3) and then integrates over both \( x, x' \) to obtain

\[ g = -\frac{e^2 \hbar^3}{16\pi m^2 L_x^2} \int d\mathbf{r} d\mathbf{r}' \Delta G(\varepsilon_f, \mathbf{r}, \mathbf{r}') \nabla_x \nabla' x' \Delta G(\varepsilon_f, \mathbf{r}', \mathbf{r}), \]  

(5)

where the integrations are over the entire sample. This volume-averaged form for the conductance has been used in most previous studies of WL; as we will see it leads to a much simpler set of diagrams for the average conductance than does Eq. (4). We note that the volume-averaged form is only valid for two-probe conductance, and treatments of four-probe resistance measurements must rely on Eqs. (3) and (4) [19–21]. This is particularly important for mesoscopic samples, where the lead geometry may have significant influence on the measurements.

III. CURRENT-CONSERVING APPROXIMATIONS FOR \( < \sigma > \)

A. Impurity-Averaged Perturbation Theory

Disordered conductors are typically characterized by their statistical properties when averaged over an ensemble of impurity configurations. As noted above, the WL effect appears in the average conductance; the primary analytic tool for calculating this effect is impurity-averaged perturbation theory in the small parameter \( (k_f l)^{-1} \). This perturbation theory is equivalent to that for electrons with a static interaction, except for the absence of closed fermion loops (which is explained naturally in the replica formulation of the problem). As in any diagrammatic expansion, it is crucial to evaluate a set of diagrams at each order which maintains the conservation laws present in the exact theory. In this case the important conserved quantity is current, and we seek a set of diagrams for \( < \sigma(\mathbf{r}, \mathbf{r}') > \) which both describes the WL effect and satisfies

\[ \nabla \cdot < \sigma >= \nabla' \cdot < \sigma >= 0, \]  

(6)

where henceforth both the angle brackets and overbars will denote the average over impurity configurations.

We shall adopt the standard white-noise model for the statistics of the impurity potential: \( < V(\mathbf{r}) > = 0, < V(\mathbf{r}) V(\mathbf{r}') > = c_i u^2 \delta(\mathbf{r} - \mathbf{r}') \), where \( c_i \) is the density of impurities and all
higher cumulants are assumed zero. The usual approximation (see e.g. Ref. [28]) for the average Green function is the self-consistent Born approximation (SCBA) which is shown diagrammatically in Fig. 1; it may be expressed for our case by the self-consistent equation:

$$[E - c_i u^2 \mathcal{G}(\mathbf{r}, \mathbf{r}) + \frac{\nabla^2}{2m} \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

(7)

Although we will consider finite disordered samples connected to ordered leads (and this will be important for certain aspects of the calculation) the violation of translational symmetry at the boundaries is negligible for the average Green function. This is first because \( \mathcal{G} \) decays exponentially for \( |\mathbf{r} - \mathbf{r}'| > l \) and is insensitive to the boundaries over most of the sample, and secondly because even at the boundaries corrections to the average Green function due to the interface only appear at next order in \((k_f l)^{-1}\). Since the boundary effects are found to be negligible in Eq. (7) we may treat the system as translationally invariant on average, hence \( \mathcal{G}(\mathbf{r}, \mathbf{r}, E) \) is just a complex function of energy, independent of \( \mathbf{r} \). For weak magnetic field even the energy-dependence is negligible and one finds \( c_i u^2 \mathcal{G}^{\pm}(\mathbf{r}, \mathbf{r}) = \mp i/2\tau \), where \( \tau \) is the elastic scattering time. It then follows from Eq. (7) that

$$\mathcal{G}^\pm(\mathbf{r}, \mathbf{r}', E) = G_0^\pm(\mathbf{r}, \mathbf{r}', E \pm i/2\tau),$$

(8)

where \( G_0 \) is the Green function of the problem in the absence of disorder. Substituting this relation into the known form of \( G_0 \) leads to the exponential decay with \( l \) mentioned above.

B. Current Conservation in the Ladder Approximation

At leading order in \((k_f l)^{-1}\) it is well-known [22] that the SCBA for the average Green function will yield a current-conserving approximation for the conductivity if the velocity vertices are dressed by the infinite sum of ladder diagrams (see Fig. 2). This ladder sum \( L(\mathbf{r}, \mathbf{r}') \) satisfies the integral equation:

$$L(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') + \int d\mathbf{r}_1 L_0(\mathbf{r}, \mathbf{r}_1) L(\mathbf{r}_1, \mathbf{r}')$$

(9)

where the kernel \( L_0(\mathbf{r}, \mathbf{r}') = c_i u^2 |\mathcal{G}^\pm(\mathbf{r}, \mathbf{r}')|^2 \).

The proofs of current conservation in various approximations to \( < \mathcal{G}(\mathbf{r}, \mathbf{r}') > \) rely on properties of the velocity vertex under the divergence operation. The most general (bare) velocity vertex which can appear has the form

$$V_0(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) = G^+(\mathbf{r}_1, \mathbf{r}) \frac{\hat{\nabla}}{2m|} G^-(\mathbf{r}, \mathbf{r}_2)$$

(10)

where \( \mathbf{r}_1, \mathbf{r}_2 \) are normally integrated over in evaluating the diagram and \( \hat{\nabla} \) is the anti-symmetric gradient operator. To prove current conservation for any set of diagrams we must take the divergence of each diagram in the chosen set and show that the resulting sum vanishes. Since the \( \mathbf{r}, \mathbf{r}' \) dependence in each diagram resides solely in the vertices we need only evaluate the divergence of the relevant vertex and then convolve this with the rest of
the diagram. Following Hershfield [22], we can regard the operation of taking the divergence as generating a new set of diagrams with the velocity vertex replaced by a “divergence vertex” which simply designates the point at which the divergence was taken (we denote this divergence vertex by a black dot in the figures); the resulting set of diagrams must sum to zero if the approximation conserves current.

The divergence of the vertex $V_0$ may be evaluated using Eq. (7) to substitute for the factors $\nabla^2 G/2m$ which arises after differentiation.

$$\nabla \cdot V_0(r, r_1, r_2) = \frac{1}{\tau} G^+(r_1, r)G^-(r, r_2) - iG^-(r, r_2)\delta(r - r_1) + iG^+(r_1, r)\delta(r - r_2). \quad (11)$$

The diagrams generated by this procedure are shown in Fig. 3. We see that in general taking the divergence of a bare vertex in a conductivity diagram generates three diagrams: the first and second diagrams have one of the Green functions deleted and the divergence vertex moved to $r_1, r_2$ respectively, and the third is identical to the original conductivity diagram except that the divergence vertex replaces the velocity vertex and is multiplied by a factor $1/\tau$.

Now consider dressing the bare velocity vertex with an impurity line as shown in Fig. 4a. In this case the bare vertex connects to the same point, $r_0$; if we set $r_1 = r_2 = r_0$ in Eq. (11) the second and third terms combine using the identity $-iC\mu^2 [G^-(r, r) - \overline{G}^+(r, r)] = -1/\tau$ and yield a single diagram with the impurity line removed as shown in Fig. 4a. A similar result occurs when the divergence is taken of a bare vertex dressed by two impurity lines. It is evident from the figure that the first diagram generated from the singly dressed vertex is the negative of the second diagram generated from the doubly dressed vertex and will cancel if these two diagrams are summed. It is very useful at this point to define the dressed vertex $\tilde{V}$ as the vertex $V_0$ dressed with a full ladder sum (note that the bare vertex appears as the first term of $\tilde{V}$). Because of the cancellation of successive diagrams generated by taking the divergence of a velocity vertex connected to a ladder we obtain the simple result

$$\nabla \cdot \tilde{V}(r, r_1, r_2) = -iG^-(r, r_2)\delta(r - r_1) + iG^+(r_1, r)\delta(r - r_2). \quad (12)$$

This result is shown diagrammatically in Fig. 4b. The divergence operation on the dressed vertex generates only two diagrams, one with the point $r$ shifted to the point $r_1$, the other with it shifted to the point $r_2$, with the analyticity of the Green function reversed, and with opposite sign. These two diagrams will have the same number of impurity lines.

With this identity it is easy to confirm current conservation in the ladder approximation for $<\sigma>$ (this approximation is represented diagrammatically in Fig. 5). Taking the divergence at the dressed vertex generates two diagrams with a single Green function multiplied by $\delta(r - r')$ attached to a bare velocity vertex. The antisymmetric derivative of these diagrams vanishes due to the symmetry of the Green functions under interchange of $r, r'$. Thus, as claimed, the divergence of $<\sigma>$ in this approximation is zero. It is worth noting that this argument establishing current conservation remains valid even at the interface of between the sample and the leads.
C. Current Conservation Including Weak Localization

The ladder approximation to \(<\sigma>\) discussed in the last section does not, however, describe the WL effect: the WL effect only appears at next order in \((k_f l)^{-1}\). We have not found in the literature an explicitly current-conserving set of diagrams for \(<\sigma>\) which upon integration yields the WL contribution to the conductance. However, it is known \[2\] that the volume-averaged expression for \(g\) yields the full effect when one evaluates only the diagram with one crossed ladder (Fig. 2b) without vertex corrections; this implies that all other volume-averaged diagrams at this order either vanish or sum to zero. Thus the WL effect in the total conductance may be calculated without determining an explicit current-conserving set for \(<\sigma>\). However in order to evaluate individual transmission or reflection coefficients \(<T_{ij}>\), \(<R_{ij}>\) and determine their separate contributions to \(<g>\), the identification of such a set is necessary. A current-conserving approximation which includes the WL effect is represented by the diagrams shown in Fig. 6. This approximation will be shown to conserve current \emph{exactly} (if the SCBA is used for the average Green function) by a generalization of the type of argument \[22\] used for the simple ladder approximation to \(<\sigma>\) and for conductance fluctuations in multi-lead structures. The set given includes diagrams of higher order in \((k_f l)^{-1}\) than needed for the WL effect; however, inclusion of the higher-order diagrams actually expedites the proof. (We will not evaluate all diagrams explicitly anyway). In these diagrams the crossed insertion represents the time-reversed or crossed ladder shown in Fig. 2b, while the number on each insertion represents the number of impurity scatterings. We will show that the divergence of this set for a fixed total number of impurity lines is zero; summing over these sets for all numbers of impurity lines in the ladder yields a complete set of current-conserving diagrams containing the crossed ladder sum (and hence the WL effect).

The proof that the set of diagrams shown in Fig. 6a,b satisfy \(\nabla \cdot <\sigma> = 0\) for each \(n\) is as follows. The set shown consists of a crossed ladder with \(n\) scatterings and all diagrams containing \(n - j\) scatterings in the crossed ladder and \(j\) crossed lines dressing the crossed ladder on one of the two Green’s functions, for \(j = 1, 2 \ldots n\). (For \(j > 1\) we have included diagrams of lower order in \((k_f l)^{-1}\).) It is helpful to redraw these diagrams as shown in the figures; in this way it is clear that this set is generated simply by moving one of the vertices \(\tilde{V}\) around the figure, passing through the impurity lines one by one. Without loss of generality, we may take the divergence at the “moving” vertex. As just shown, taking the divergence of each diagram generates \emph{two} diagrams, one in which the vertex is moved to the impurity line immediately to the left, the other in which the sign is reversed and the vertex is moved to the impurity line immediately to the right (and in both cases the ladder dressing is removed). This alternation in sign leads to a cancellation between the “right” diagram generated from the conductivity diagram with \(j = m\) and the the “left” diagram generated by the one with \(j = m + 1\). This cancellation proceeds around the figure until the only remaining divergence diagram is the one where the divergence vertex has moved all the way around the crossed ladder and lies adjacent to the remaining dressed velocity vertex attached at \(r'\). This diagram may be exactly cancelled by adding to the conductivity the set of diagrams shown in Fig. 6b consisting of a crossed ladder self-energy insertion to one of the Green functions dressed by an ordinary ladder. The divergence of this new set cancels internally except for the diagram shown in Fig. 6b. This diagram is exactly
the negative of the remaining divergence diagram from the first set, except that it doesn’t
contain the bare vertex. Thus the sum of the divergence of both sets cancels exactly, except
for the diagram with the bare velocity vertex evaluated at \( r = r' \). However we noted above
that such a diagram vanishes when the antisymmetric derivative with respect to \( r' \) is taken.
Thus we have shown that the sum of the two sets does indeed yield a current-conserving
approximation to \( \langle \sigma \rangle \).

We note that this operation of “moving a vertex around the figure” is a quite general
way of constructing a current-conserving set of diagrams from a given diagram. In fact,
the additional ladder dressings on the crossed ladder that were needed to complete the
cancellation can be regarded as the result of moving the vertex through the ladder dressing
on the other vertex \( \tilde{V} \) (the one attached to the point \( r' \) in Fig. 6a.

D. Limitations of Diffuson and Cooperon Approximations

The exact ladder sum would be obtained by solving the integral equation (9), however
in most applications to disordered conductors one is interested in the solution in the limit
of long wavelengths, so one assumes that \( L(r, r') \) is slowly-varying compared to \( L_0 \) and one
expands \( L(r_1, r') \) for \( r_1 \approx r \) to obtain a differential equation for \( L(r, r') \) which has the form
of a diffusion equation \[28\]. For the simplest case of zero frequency and temperature this
takes the form

\[
- D \tau \nabla^2 d(r, r') = \delta(r - r'),
\]

where \( D \) is the elastic diffusion constant, \( l^2/d\tau \), and we denote the long wavelength approxi-
mation to the ladder sum by \( d(r, r') \). We will henceforth refer to the exact solution of
Eq. (9) as the ladder approximation or ladder sum while the solution to Eq. (13) will be
referred to as the diffuson approximation (the terms are often used interchangeably in the
literature). In the absence of a magnetic field the crossed ladder sum of Fig. 2b satisfies
the same integral equation as the ladder sum, and thus the long-wavelength approximation,
which we denote as the cooperon, satisfies the same diffusion equation. However, in contrast
to the diffuson, the cooperon is sensitive to weak magnetic fields, which enter the diffusion
equation by the minimal substitution for particles of charge \( 2e \), \( i\nabla \to i\nabla - 2eA/\hbar c \) \[3,28\].
The appearance of the magnetic field in this way leads to the suppression of weak localization
by a magnetic field through the elimination of the diffusion pole.

Although the diffuson/cooperon approximation to the ladder/crossed-ladder sums is ade-
quate for evaluating the conductivity tensor for widely separated points \( r, r' \), it causes a
breakdown of local current conservation. With appropriate care this breakdown can be
shown not to affect the value of the average transmission coefficients and hence of the con-
ductance, essentially because one is only evaluating the diffuson at widely separated points.
However the diffuson/cooperon approximation is not adequate to calculate the average re-
fection coefficients (a fact which has been known in the optics literature for some time \[23\]).
The reason for this is that a major contribution to reflection involves short trajectories of
only a few mean free paths in length, for which one needs an accurate expression for \( L(r, r') \)
on the scale of \( l \). But the exact short wavelength behavior of the ladder \( L(r, r') \) is known (for
an infinite medium) \[24\] and differs substantially from that of the diffuson approximation.
As a result, although the diffusion approximation allows a correct evaluation of $< T >$ it fails for $< R >$ (the average total reflection) and hence $< T > + < R > \neq N_c$, and the local violation of current conservation shows up as a global violation of unitarity. We now describe this problem in some detail.

First we discuss the average total transmission, $< T >$, which is just $< g > / (e^2/h)$ by Eq. (4). If one starts from the volume-averaged expression for $g$ it is well-known that the Drude-Sommerfeld average conductance is correctly obtained from the leading order diagram in $(k_f l)^{-1}$ within the white-noise model, i.e. one finds $< g > = \sigma_0 A / L$ where $\sigma_0 = ne^2 \tau / m$ and $A, L$ are the cross-sectional area and length of the conductor. It is easy to show that this conductance corresponds to an average transmission $< T > = (\pi/2) N_c l / L$ (in two dimensions). In the volume-averaged approach, however, all the ladder diagrams for the conductance vanish from isotropy so the result is insensitive to the failure of the diffusion approximation at short distances.

In contrast, when evaluating conductance from the transmission coefficients the ladder diagrams do not vanish since the conductivity tensor is only integrated over a surface. We have not found in the literature confirmation that calculating $< T > = \sum_{i,j} N_c < T_{ij} >$ in the diffusion approximation yields the correct result as well. We now confirm that it does, although only after improving the standard approximation for the diffusion boundary conditions. In volume-averaged calculations the diffusion is taken to vanish on the surface of the conductor connected to the leads (this is supposed to correspond to the boundary condition of a very large metallic contact). However if this approximation is made and if the point $r$ is chosen at the surface, then $\langle \sigma(r, r') \rangle$ will not conserve current even for $r'$ far in the bulk. It can be shown however that if the diffusion is set to zero a distance of order the mean free path outside the interface this property is restored for the total flux crossing the interface. In two dimensions the correct distance is $\pi l / 4$ outside the sample. This modified boundary condition is required to get the correct average transmission coefficients; to get correct reflection coefficients even this approximation is inadequate and a new approximation is needed which is discussed below.

The average transmission and reflection coefficients to leading order are given by the diagram consisting of a single diffusion between two current vertices. To examine individual $T_{ij}, R_{ij}$ we must evaluate these diagrams for a given modes $i, j$ which requires integrating over the transverse directions $(y, y')$ with the mode wavefunctions while staying in real-space for the current direction $(x, x')$. This is the method we use henceforth in the paper when evaluating diagrams explicitly; all calculations are done in two dimensions for convenience. Upon transforming to mode space, the differential operator at the current vertices $\nabla / 2mi$ simply is replaced by the longitudinal velocity,

$$< T_{ij} > = v_i v_j < G^+(x = 0, i; x = L_x, j) G^-(x = L_x, j; x = 0, i) >$$

where $v_i, v_j$ are velocities for the relevant modes. We need the 1PGF which becomes

$$G^+(x, i; x', j) = \delta_{i,j} \int \frac{dp_x}{2\pi} e^{ip_x(x-x')} \frac{e^{ip_x(x-x')}}{\varepsilon_f - p^2/2m + i/2\tau}$$

$$= -i \delta_{i,j} e^{-ip_j |x-x'| - |x-x'|/(2v_j \tau)} / v_j$$

(15a)
where \( p^2 \) is the sum of squares of transverse and longitudinal momenta and \( p_j = \sqrt{2m\varepsilon_f - q_j^2}, q_j = (j\pi/L_y)^2 \). Finally, the propagation from a given mode at \( x = 0 \) to a point \( r' \) is given using this Green’s function and projecting from a mode to a y-position. We obtain

\[
G^+(x, \ j; r') = -\frac{1}{v_j} e^{ip_j|x-x'|-|x-x'|/(2v_j\tau)} \sin(q_jy')/(2/L_y)^{1/2}
\]

(16)

where the factor of \((2/L_y)^{1/2}\) is added for normalization. In evaluating the diagram for \( < T_{ij} > \) we note that the phase factors involving \( x, x' \) cancel since the advanced and retarded Green’s functions have opposite phases. The phases coming from \( y, y' \) vary on the length scale \( k_f^{-1} \), while the diffuson varies on at most the length scale \( l \). So, we can replace the y-phase factor arising from the square of \( \sin(q_jy') \) by its average. One finds:

\[
<T_{ij}>= c_i u^2 \frac{1}{L_y^2} \int_0^{L_x} \int_0^{L_x} dx dx' \left[ v_i v_j \frac{1}{v_i^2} \frac{1}{v_j^2} e^{-x/(v_i\tau)} e^{-x'/(v_j\tau)} \right] dy \int dy' d(x, y; x', y')
\]

(17)

where \( x \) is measured from the left-hand side of the sample and \( x' \) from the right-hand side. The integral of the 2d diffuson over \( y, y' \) satisfies a 1d diffusion equation which has the simple solution: \( d(x, x') = (x + \frac{\pi}{4} l)(x' + \frac{\pi}{4} l)(2L_y/l^2 L_x) \), where the final factor of \( L_y \) arises from the transverse integration and we have dropped terms lower order in \((l/L_x)\). The integral above can now be integrated by parts; we obtain after some algebra and dropping exponentially small terms:

\[
<T_{ij}>= \frac{2}{\pi N L_x} \frac{l}{v_i} (v_i + \pi/4)(v_i + \pi/4).
\]

(18)

Summing over the modes \( i, j \) using the relation \( m v_j = \sqrt{p^2 - (j\pi/L_y)^2} \) yields the correct result

\[
\sum_{i,j}^{N_c} < T_{ij} > = N_c \frac{\pi}{2} \frac{l}{L_x}.
\]

(19)

If global current conservation were satisfied in the diffuson approximation we should find \( \sum_{i,j}^{N_c} < R_{ij} >= N_c (1 - \pi l/2L_x) \approx N_c \). However it is shown in the Appendix that exactly the same calculation for \( < R_{ij} > \) yields the result \( \sum_{i,j}^{N_c} < R_{ij} > \approx 0.73 N_c \). As noted above, this failure of global current-conservation arises from the fact that reflection coefficients require consideration of the conductivity tensor on scales of order the mean free path, and the diffuson approximation to the ladder sum is not accurate on this scale.

E. Compensating for the Diffuson Approximation

The most elegant resolution to this problem would be to work with the full ladder sum or some improved approximation to it, instead of the diffuson. A partial improvement to the diffuson approximation is given in the Appendix, however it is not sufficient to restore current conservation to this order in \((k_f l)^{-1}\). Although an exact evaluation of the crossed-ladder
sum (which differs from the ladder sum only in the absence of the single scattering diagram), does exist \[24\] in momentum space, it is not clear how to obtain a real space solution that can be used in a sample \textit{with boundaries}. The corresponding real space differential equation would be of infinite degree, complicating the specification of boundary values. The analogous problem in classical diffusion, essentially the Milne problem, has been solved \[26\], but the solution appears too complicated to be of any practical use in this context.

An alternative approach to restoring current conservation is to find an approximation for the current vertex when connected to the ladder sum which restores current conservation when the ladder is approximated by the diffuson. This was essentially the approach taken in the work of Kane et al. \[20,21\], where they treat the bare conductivity bubble as proportional to \(\delta(r - r')\). However calculation of the total reflection (for which the points \(r, r'\) may coincide) requires a slightly more careful treatment to avoid evaluating this delta function at zero. First we note that current conservation in the ladder approximation to \(<\sigma>\) followed from the cancellation between the term \((1/\tau)|G^+(r, r')|^2\) arising from \(\nabla \cdot V_0(r, r')\) and that arising from the divergence of the bare vertex dressed by the ladder sum. We can find a current-conserving approximation when the ladder sum is replaced by the diffuson if we can modify \(V_0\) in the dressed vertex so as to maintain the result that the divergence of the dressed vertex is \(- (1/\tau)|G^+(r, r')|^2\). The required modification can be determined by the following argument. Note that the bare vertex depicted in Fig. 3 with \(r_1 = r_2 = r'\) is a vector function of \(r - r'\) with zero curl. Hence it may be written as the gradient of a scalar function \(F(r - r')\). Therefore the divergence of the dressed vertex becomes

\[
\int \text{d}r'\text{d}r'' \nabla^2 F(r - r')d(r', r'')|G^+(r, r')G^{-}(r', r_1)|G^-(r'', r_2).
\] (20)

Because \(F\) is a function of \(r - r'\), integrating by parts twice shifts the Laplacian onto the diffuson and allows use of the basic equation \(\nabla^2 d(r'' - r') = -(1/D\tau)\delta(r'' - r')\) to give the result

\[
-\frac{1}{D\tau} \int \text{d}r'' F(r - r'')G^+(r'', r_1)G^-(r'', r_2).
\] (21)

It follows that current conservation requires \(F(r - r') = D\delta(r - r')\) or equivalently that \(\nabla F = V_0(r, r') = D\nabla\delta(r - r')\). Of course \(V_0(r - r')\) is known explicitly; it is an antisymmetric function of range \(l\), not of zero range. Hence as claimed above the diffuson approximation does not exactly conserve current. However it is possible to let the mean-free path tend to zero in the vertex Green functions such that current conservation is restored; i.e. such that the vertex behaves as the gradient of a delta function. We note that for reflection it is not possible to let all bare vertices approach the gradient of a delta function because of the necessity of evaluating the vertex at \(r = r'\); only the vertex connected to the diffuson is taken to be of zero range.

The desired limit is achieved formally by changing the mean free path in these Green functions from \(l\) to \(l' \to 0\), rescaling \(k_f\) to \((k_f)l/l'\), and rescaling the Green functions by the overall factor \(l/l'\). The antisymmetric derivative operator makes the vertex in direction \(i\) an odd function of \((r_i - r'_i)\) and an even function of other coordinates, where \(r_i\) is the \(i\)th position coordinate of the derivative operator and \(r'_i\) is the \(i\)th position coordinate of the other point to which the Green’s functions in the vertex are connected. It is straightforward
to check that the new vertex vanishes when the function against which it is convoluted vanishes to zeroth and first order in any coordinate, and that the vertex in direction $i$ does not vanish when convoluted against a linear function of $(r_i - r'_i)$. This means that the vertex acts on a polynomial in the coordinates to select out only terms linear in one coordinate and constant in the other two. Therefore the vertex does act as the gradient of a delta function, and it is again easy to check that the weight of the delta function is exactly the diffusion constant, $D$, as required for current conservation. A quick way of confirming that the weight is correct is to note that for points on opposite sides of the sample the difference between using a vertex of range $l$ and the new zero range vertex introduces only exponentially small corrections in $l/L_x$. Since we confirmed above that the diffusion approximation with the old vertex gives exactly the correct value for $<T>$ to leading order, it follows that the leading behavior of the new vertex must be quantitatively correct.

Using this limiting procedure on the bare vertex allows us to calculate the correct average total reflection probability using the diffusion approximation (and employing a real-space formulation analogous to Eq. (3)), i.e. we find $<R> = N_c$ to leading order in $l/L_x$. However this procedure has significant limitations. It only works for evaluation of real-space diagrams or diagrams with one vertex in real-space and one in mode space (the one not taken to be zero range). We have not found a sensible current-conserving approximation incorporating the diffusion approximation which will allow us to evaluate the individual $<R_{ij}>$, so explicit calculations of reflection coefficient diagrams typically contain errors of order unity in prefactors, although the dependence on parameters such as $k_f$, $l$, and $N_c$ is correct.

IV. EFFECTS OF TIMEREVERSAL-SYMMETRY BREAKING ON REFLECTION

A. Change in $<R>$ due to TR Symmetry

Although individual diagrams for $<R_{ij}>$ cannot be evaluated precisely due to the inaccuracy of the diffusion approximation, we can evaluate the sum of all the diagrams over $i, j$ from current conservation. Since we have proven the diagrams of Fig. 6 form a current-conserving set we may replace each diagram by its volume average after summing over $i, j$. In this case one is essentially using the relationship $<R> = N_c - <T>$ and calculating conductance diagrams. All diagrams except the one with the single crossed ladder vanish; this is the familiar diagram which upon evaluation in the cooperon approximation to the crossed ladder gives $\delta <T>_W = -1/3$ \[27\] in the limit of $L_x \gg L_y$ (quasi-1d sample). The only question raised by our derivation is whether one expects the cooperon approximation to be adequate here as again the points $r, r'$ may coincide, and in fact standard treatments \[28\] express the WL correction in terms of $c(r, r)$. However the volume-averaging again appears to rescue this approximation. Any diagram with just a few scatterings will be of order $l/L_x$, and may be neglected. It is only the diagrams with a number of impurity scatterings $n \sim (L_x/l)^2$ which are long-range enough to give a result of order unity, and for such diagrams the vertex may indeed be treated as short range and the cooperon approximation is adequate. Since a weak magnetic field kills the cooperon contribution, the change in total
reflection due to breaking TR symmetry is \( \delta < R > = < R(B) > - < R(0) > = -1/3 \). This leads to a positive magnetoconductance which saturates at a value \( < g(B) > = < g(B = 0) > \equiv \delta g_B = (+1/3)(\epsilon^2/h) \).

**B. Evaluation of \( \delta R_{ij} \)**

Although we cannot evaluate all reflection diagrams precisely, we can accurately estimate the diagonal terms which correspond to the coherent backscattering effect discussed in the introduction. The simplest diagram is just the crossed ladder with no dressings and a fixed mode \( i, j \) at each vertex. For \( i \neq j \) this vanishes using Eq. (17) due to the orthogonality of different mode wavefunctions; for \( i = j \) the diagram is equal to the ladder sum for \( < R_{ii} > \) except for the absence of the single-scattering diagram. Hence we have \( \sum_i^{N_c} < R_{ii} >_{coop} \equiv \delta R_D \approx 1 \). Thus we see that the diagonal reflection is enhanced in the presence of TR symmetry by roughly a factor of two, confirming our statement that this diagram is the mode-space manifestation of the coherent backscattering peak. This diagram makes no contribution to off-diagonal reflection. Therefore if coherent backscattering and WL were equivalent one would expect the WL effect in conductance to be \( \epsilon^2/h \) and not \( (1/3)\epsilon^2/h \) as we just found. The other dressed diagrams for \( < R_{ij} > \) are not diagonal in mode-space and must add up to the difference \( (-2/3)\epsilon^2/h \).

Unfortunately, due to the inaccuracy of the cooperon approximation there is no reason that explicit evaluation of these diagrams will yield precisely \( (-2/3)\epsilon^2/h \). Thus we simply show that there exist diagrams with the appropriate order of magnitude and sign to give the expected effect. We note that all the other diagrams we have introduced which include dressings on the cooperon are of order \( 1/N_c^2 \). Thus after summing over all modes they will indeed give a contribution of order unity. We evaluate the simplest of the diagrams (see Fig. 8) which yields a negative contribution to reflection coefficients. This will illustrate the general properties of these diagrams and also shows that time-reversal symmetry can indeed reduce the off-diagonal reflection (as well as enhancing the diagonal reflection).

The diagram has an overall negative sign arising from the presence of two extra advance Green functions with no compensating retarded functions (each 1PGF gives a factor \( i \)). There are two such diagrams (the second is obtained by dressing the top of the diagram) with the same value. The cooperon is non-zero only if the transverse momentum on one side is the same magnitude as on the other side. This does not hold exactly for the crossed ladder, but it is still sharply peaked about reflection into the same mode. To lowest order in \((k_f l)^{-1}\) we have that the \( x \)-position of the impurity dressing must be less than that of both ends of the cooperon. The integration over the \( x \)-position of this point then will not introduce any additional phase. After integrating by parts and some algebra we find that this diagram and its conjugate contribute to reflection from mode \( i \) to mode \( j \)

\[
-4 \frac{(c_i u_2)^2}{l^2 L_y^2 v_i^2 v_j^2} \left( \frac{\pi}{4} \frac{\tau^3 v_i^2 v_j^2}{v_i + v_j} + 2 \frac{v_i^3 v_j^3 \tau^4}{(v_i + v_j)^2} \right). \tag{22}
\]

After some simplifications this becomes

\[
-\frac{1}{N_c^2 \pi^2} \left( \frac{\pi}{4} \frac{v_f}{v_i + v_j} + 2 \frac{v_i v_j}{(v_i + v_j)^2} \right). \tag{23}
\]
As noted above, the contribution is negative and of order $1/N^2$. This diagram is typical of the off-diagonal contributions to reflection coefficients in the presence of time-reversal symmetry, i.e. it vanishes on volume averaging and is of order $1/N^2$, but gives a total contribution to reflection of the same order as the bare crossed ladder diagram because there is no restriction $i = j$.

Such contributions are hard to understand from a (naive) semiclassical point of view. Since the diffuson and cooperon parts of such diagrams are typically represented in real-space, it has become common in the literature \cite{27,28} to interpret each diagram as representing a type of interference between paths. With this interpretation the cooperon represents a time-reversed pair of counter-propagating paths which form a closed loop, consistent with the well-known interpretation of the CB peak mentioned in the introduction. However it must be kept in mind that the “paths” represented by such diagrams are not classical paths as they would be in a true semiclassical theory. Impurity-averaged perturbation theory in real-space allows any piece-wise linear path to contribute, and this complicates their semiclassical interpretation. For example, diagrams of the type shown in Fig. 6 in which the vertices are dressed with ladder insertions would correspond to two paths which diffuse together to the same point in the sample, but then diverge and traverse a loop in opposite directions, before possibly rejoining. Such a pair of paths is impossible classically, so the flux-sensitivity they impart to the reflection coefficients has no straightforward semiclassical interpretation. Although the white-noise model we have used has no simple classical limit, and one might speculate that it is in some respects untypical, it is noteworthy that such off-diagonal correlations are found numerically in simple chaotic systems \cite{8,9}. Our work makes it clear that such contributions are essentially required by current conservation, and to our knowledge current conservation has not been proven for the semiclassical approximation to the scattering coefficients. Thus we may hope that a deeper understanding of these results will arise from progress in the study of current conservation within the semiclassical approximation.

V. SYMMETRY EFFECTS

Our above analysis of the off-diagonal contribution to the WL effect in disordered conductors was motivated by numerical results obtained for the WL effect in ballistic conductors with chaotic classical dynamics \cite{8,9}. These numerical studies also revealed another aspect of the WL effect which apparently has not been considered previously, the sensitivity of the effect to spatial symmetry. A particularly striking result of this work was a reduction of the WL positive magnetococonductance for ballistic junctions with double reflection symmetry; in this case the magnitude of the WL obtained numerically was slightly positive, but consistent with zero to the statistical accuracy of the calculations. Unlike the off-diagonal contribution to WL, which exists in both ordered and disordered media, the symmetry-induced contributions obviously do not exist in typical disordered media unless some very special process enforces a symmetry in the random potential. However it is still helpful to consider an ensemble of potentials which are random except for the presence of discrete spatial symmetries since in this case analytic calculations may be performed which provide insight into the origin of the sensitivity to symmetry and also may resolve the intriguing
question of whether certain symmetries can actually eliminate or reverse the sign of the WL effect. Thus we study the WL effect in symmetric “random” potentials in this section.

It is possible to consider general spatial symmetry groups using the approach developed below. However, we find that only symmetry operations which are their own inverse give significant contributions; thus, we treat only single or double reflection symmetries. For definiteness we define the x-direction to be the direction of current flow and consider systems with reflection symmetry across the x-axis, the y-axis, or both. The symmetry operations on the random potential are represented by operators $R_x, R_y$ and in the double reflection case $R_{xy} = R_x R_y$. As before, we restrict ourselves to the two-dimensional case but extension to higher dimensions is trivial. The notation $r_x, r_y, r_{xy}$ will be used to represent the point related to $r$ by the relevant symmetry operator (the subscripts will be omitted when it is not necessary to distinguish the different symmetry operations).

With this modification the effective interaction mediated by the impurity potential no longer conserves momentum. If momentum $p$ is lost by scattering off an impurity, $p$ may be gained at the reflected impurity. As a result, the 1PGF is no longer diagonal in momentum space. However the modifications of the 1PGF will be negligible for our purposes. If we compare the contribution to the self-energy from scattering off the same impurity twice to that due to scattering from the impurity and its reflected impurity (see Fig. 9), the latter is only important if the impurity is within a few mean free paths of the reflection plane. This means that changes in the 1PGF produced by the new diagram will produce changes in the conductance that, after volume averaging, are lower order in $l/L$, where $L$ is a typical sample dimension. Furthermore, even for two impurities lying near the symmetry plane one can verify that the phase of the Green function will not vary to first order in the transverse displacement of the impurity if the scattering is off the same impurity but will vary to first order when the scattering is from the reflected impurity (due to the length of the segment propagating from one to the other). As a result, the latter diagram will be reduced by the factor $(k_f l)^{-1/2}$. Thus we may use the usual impurity-averaged 1PGF in evaluating the conductance even in the presence of symmetry.

Even though the 1PGF is unchanged the conductance is modified because new generalized ladder and crossed-ladder vertex corrections occur due to the presence of symmetry. An effect of this general type is fairly obvious from the semiclassical point of view in which the WL arises for constructive interference of paths with the same phase; in the presence of symmetry more such paths arise. In particular, if we interpret the real-space diagrams as representing paths, a crossed-ladder insertion in which the lower line is the precise reflection of the upper line (see Fig. 10) represents paths related by TR and reflection symmetry (or symmetries). We refer to this as a reflected crossed-ladder diagram and evaluate its contribution to the conductance in the cooperon approximation. These new diagrams arising due to symmetry in general have different numerical values than the standard WL diagram due both to changes in the value of the short-ranged current vertex and to changes in the value of the relevant cooperon insertion. We first evaluate the changes in the current vertex. We recall the standard procedure [28] for evaluating the usual crossed-ladder diagram.
(in the cooperon approximation) in real-space. The vertices of the diagram are treated as short-ranged and replaced with $V_0 \delta(r - r')$, where $V_0$ is the volume-average of the vertex. The diagram then becomes proportional to the integral of $c(r, r)$ over the whole sample. To evaluate the coefficient $V_0$ for the standard diagram, note that the vertices give a factor of

$$G^+(r, r_1)v_x(r_1)G^-(r_1, r')G^-(r, r_2)v_x(r_2)G^+(r_2, r')$$  \hspace{1cm} (24)$$

After reversing the order of arguments to $G^-(r, r_2)$ and $G^+(r_2, r')$ (which is permitted in the presence of time-reversal symmetry) and integrating over all the coordinates $r, r', r_1, r_2$ this becomes

$$- Tr(G^+ v_x G^+ G^+ v_x G^+).$$  \hspace{1cm} (25)$$

In the reflected crossed-ladder diagram, one or more of the Green’s functions in the vertex connect to a point $\bar{r}$ instead of $r$ and similarly for $r'$. The new trace may then be written

$$- Tr(G^+ v_x G^+ R_\mu G^+ v_x G^+ R_\mu),$$  \hspace{1cm} (26)$$

where $\mu = x, y, xy$. All the operators $R_\mu$ commute with $G$ since $G(\bar{r}, r') = G(r, \bar{r}')$; however while $R_x$ will also commute with $v_x$, $R_y$ will anti-commute. In addition all the $R_\mu$ satisfy $R_\mu^2 = 1$. Hence we may move one of the symmetry operators through the trace by commutation or anticommutation until it is adjacent to the other at which point the product becomes unity. Therefore we find that if $\mu = y, xy$ the coefficient of the new vertex is $-V_0$ whereas if $\mu = x$ it is just $V_0$.

The new conductance diagrams involve the reflected cooperon $c(\bar{r}, r)$. However it is easy to see that $c = c$ because the kernel of the integral equation defining the reflected crossed-ladder $G^{(x)}(r, r')G^{(-)}(\bar{r}, \bar{r}')$ is identical to that defining the usual crossed-ladder given the symmetry of the Green function under reflection. Hence the only difference from the standard diagram (other than the possible sign change of the vertex) is that the new diagrams are proportional to the volume average of $c(r, \bar{r})$ rather than $c(r, r)$. For a rectangular sample of length $L_x$ and width $L_y$ with the standard boundary conditions and no magnetic field we may write the cooperon in a spectral representation where the eigenfunctions of the diffusion equations are the normalized product of sine and cosine functions chosen to vanish at $x = \pm L_x/2$ and to have zero derivative at $y = \pm L_y/2$. Thus each term in the spectral sum will have the same value as in sum for $c(r, r)$ or same magnitude but opposite sign, depending on whether the mode or modes reflected are even or odd under that reflection operation. For the most relevant case, that of a quasi-one-dimensional sample ($L_x \gg L_y$), the cooperon is dominated by the terms with zero wavevector in the y-direction. In the standard cooperon sum these terms are proportional to $\sum_{n=1}^{\infty} 1/n^2$. If the symmetry present is either $R_y$ or $R_{xy}$ then this sum must be replaced by $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ due to the alternating symmetry of the eigenfunctions in the x-direction. The first sum is equal to $\zeta(2) = \pi^2/6$, while the second is reduced by a factor of two.

In addition to the fully-reflected crossed-ladder diagrams there are in principle other vertex corrections introduced by the imposition of symmetry. For example there are partially-reflected crossed-ladder diagrams in which some of the impurities on the lower Green function.
line are reflected and some are not. It may be shown that these diagrams are lower order in \((k_f l)\) and \(l/L\). Reflected ladder diagrams will give no contribution to the conductivity as can be seen by realizing that each vertex gives a contribution proportional to \(Tr(G^+v_xG^-R)\) which clearly vanishes: if \(R\) anti-commutes with \(x\), it trivially vanishes, while if \(R\) commutes with \(x\), it vanishes after integrating over momenta.

In summary, if \(R\) reflects across the y-axis or both the x-axis and the y-axis, for a quasi-1D sample, the diagram gives a contribution of opposite sign and one half the magnitude as the normal WL diagram. If \(R\) reflects across the x-axis only, the diagram contributes with the same magnitude and sign as the normal WL diagram. If there is 4-fold symmetry in the sample, the total WL effect is the sum of that for the non-symmetric case, plus diagrams in which \(R\) reflects across the x-axis alone, the y- axis alone, and both axes. As we reviewed above, the normal WL diagram is equal to \(-1/3\) (in units of \(e^2/h\), neglecting spin); this gives the difference between the average conductance and the Drude average conductance (obtained from the diffuson diagrams). If we define \(\delta g_\mu\), \(\mu = x, y, xy\) as the difference between the Drude conductance and the average conductance in the presence of both TR and the relevant spatial symmetry our results imply:

\[
\delta g_x = -2/3, \quad \delta g_y = -1/6, \quad \delta g_{xy} = -1/3.
\]

(27)

Thus we arrive at several non-trivial conclusions. First, spatial symmetries do not reverse the sign of the WL effect; in all cases the conductance is reduced from its Drude value. Second, relative to the case of no symmetry, symmetry across the direction of current flow enhances WL, whereas symmetry perpendicular to the direction of current flow decreases WL. Finally, the combination of both symmetries leads to no change in the WL. All of these predictions can (and will) be tested numerically by averaging the conductance over realizations of the random potential which have the requisite symmetries and comparing them to the average without any symmetry. Thus we have in principle a means to “measure” the magnitude of the WL effect without imposition of a magnetic field.

On the other hand in experiments on disordered conductors the WL effect is measured by the difference between the conductance at \(B=0\) and at a magnetic field sufficient to destroy TR symmetry. If additional spatial symmetries are present the situation becomes more complex since in a uniform magnetic field a combination of spatial reflection plus TR may still yield an anti-unitary symmetry. This fact is familiar in the context of quantum chaos where it has been given the name “false TR symmetry breaking” \cite{29}. In particular, a single spatial reflection (either \(R_x\) or \(R_y\)) will reverse the sense of circulation (see Fig. 11) of any closed loop; since the TR operator will again reverse the sense of circulation (in a uniform field) a closed loop and its time-reversed, reflected partner will enclose the same flux and not be “dephased” by the field. Hence the single reflection diagrams will not be eliminated by a uniform magnetic field and will not contribute to the quantity \(\delta_{\alpha} = <g(B)> - <g(B = 0)>\), which is the experimentally measured WL effect under normal conditions. The standard unreflected diagram is of course always eliminated by the field, so we conclude that the WL effect \textit{as usually measured} will be unchanged by the imposition of single reflection symmetry and in quasi-1D will be \(\delta_{\alpha} = 1/3\). However, double reflection symmetry will cause a closed loop to return to its original sense of circulation, hence TR plus double reflection will reverse the sense and such a diagram \textit{will} be eliminated by a magnetic field. As shown
above, the double reflection diagram has one half the magnitude and opposite sign to the standard diagram, thus the measured WL effect $\delta g_B = 1/6$ and is reduced from its value in the absence of symmetry. Therefore the effect of the imposition of symmetry on the average conductance at zero field and on the field-dependent part of the conductance $\delta g_B$ differ significantly.

These results may be tested numerically in two contexts. First for disordered quasi-1D conductors with random potentials chosen to have no symmetry and then the symmetries $R_x, R_y, R_{xy}$. Since the average conductance with no symmetry already includes the standard WL effect it is more convenient to compare to the predictions for a quantity $\delta g_s \equiv <g>_\text{symm} - <g>_\text{non-symm}$. In addition we can compare the predictions of the theory to the numerical results obtained for ordered cavities with chaotic classical dynamics (the results which motivated this study). In this case we have no microscopic analytic theory at present. However, several studies \cite{30,31,32} have used random matrix theory in either its Hamiltonian \cite{30} or S-matrix \cite{31,32} form to show that the standard (non-symmetric) WL effect is 0.25, and this is consistent with recent numerical results \cite{8,9,31}. Hence a plausible extension of our results for quasi-1D disordered systems would rescale $\delta g_s, \delta g_B$ by the factor $3/4$; so for example we would expect a chaotic structure with double reflection symmetry to have $\delta g_B = 1/8$ instead of $1/6$. In the chaotic case the averages are obtained by averaging over many values of the fermi energy as described in Ref. \cite{8}. Because of non-universal effects associated with short trajectories in the chaotic case, it is more difficult to compare two structures which are the same except for the imposition of spatial symmetry; however we have devised a method of analyzing the numerical results which we believe substantially overcomes this problem \cite{33}. The numerical results were all obtained by the recursive Green function method \cite{34,35} which has been widely used in previous studies of disordered conductors and more recently in the study of ordered but chaotic microstructures \cite{9}.

Our numerical results are presented in Tables I and II and compared to the analytic theory. It is worth noting several features of these results. First, the sign of the effect is always as expected from the theory; in particular the positive value of $\delta g_s$ for symmetry perpendicular to the current as contrasted with the negative value for symmetry parallel is found both in the disordered and chaotic cases. Second, the magnitude of the effects found numerically tend to be smaller than that predicted by the theory, but are internally consistent. For example the theory predicts that $\delta g_B$ should be the same for the three cases of no symmetry, symmetry parallel to current and symmetry perpendicular to the current. In both the disordered and chaotic systems these three cases closely agree even though the magnitudes found are $30 - 40\%$ low \cite{36}. Finally, the case of four-fold symmetry tends to have the greatest discrepancy from the theory, particularly in the chaotic structures. Overall, in our view the numerical results are in good qualitative agreement with the analytic theory and in reasonable quantitative agreement.

VI. SUMMARY AND CONCLUSIONS

We have presented a current-conserving approximation for the the local conductivity tensor which includes weak localization corrections and used it to show that the weak localization correction to the conductance has important contributions not attributable to
coherent backscattering. The existence of such contributions, which do not have simple semiclassical analogues, complicates the interpretation of weak localization as compared to coherent backscattering. Numerical results indicate that such contributions occur both for disordered and chaotic conductors, raising questions about the possibility of a complete semiclassical theory of weak localization which are at present unanswered. A related effect, relevant to weak localization in chaotic conductors, is the sensitivity of the effect to discrete symmetries. We showed that for disordered conductors reflection symmetries can either increase or decrease the weak localization effect, but cannot reverse its sign. Again these analytic results for disordered conductors are consistent with numerical results from chaotic conductors. The results relating to symmetry suggest the possibility of using the weak localization effect as a diagnostic of the symmetry of a ballistic microstructure.

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APPENDIX A: EVALUATION OF REFLECTION COEFFICIENTS WITHIN THE DIFFUSON APPROXIMATION

We evaluate the reflection coefficients within the diffuson approximation, using the boundary conditions of no net flux entering the sample, as discussed above. Since the purpose of this exercise is to demonstrate that the diffusion approximation is insufficient to handle reflection, we calculate only the total reflection coefficients to lowest order in \((k_f l)^{-1}\), or, equivalently, to lowest order in \(1/N_c\) where \(N_c\) is the number of transverse modes, equal to \(L_yk_f/\pi\). The diagram which we must evaluate is shown in Fig. 7. The simplest method of evaluating this diagram is to use real-space in the x-direction and momentum-space in the y-direction as described in the text in section (3.4). We obtain in this case the same integral as in equation (17) except that now we measure point \(x'\) from the same side of the sample as point \(x\). In this case the diffuson, integrated over transverse positions, becomes:

\[
d(x, x') = \left\{ \begin{align*}
\frac{2}{l^2}L_y(x + \pi l/4)(L_x + \pi l/4 - x') \quad &\text{for } x < x' \\
\frac{2}{l^2}L_y(x' + \pi l/4)(L_x + \pi l/4 - x) \quad &\text{for } x' < x.
\end{align*} \right.
\]

To lowest order in \((l/L_x)\) we may replace this by \((2/l^2)(\pi l/4 + \min(x, x'))\). For transmission coefficients we were guaranteed that one end of the diffuson would be to the right of the other due to the exponential decay of the Green’s functions on the vertices; here we are not. The integration domain may be broken into \(x < x'\) and \(x > x'\) and the integral may be performed by parts. We obtain:

\[
<R_{ij}> = \frac{2}{\pi N} \left( \frac{\pi}{4} + \frac{v_i v_j}{(v_i + v_j)v_f} \right).
\]

19
Performing the sum over $i, j$ numerically we obtain $\approx (0.73)N_c$, instead of the value $N_c$ (up to corrections of order $l/L$) as required by current conservation and the known value of the average conductance.

We know that calculating these diagrams using the correct ladder sum (instead of the diffuson approximation to it) will give the correct answer as we have proved that the ladder approximation conserves current. Since the exact ladder sum is not analytically tractable, we instead improve on the diffuson approximation by explicitly evaluating the single scattering term in the ladder sum, and then evaluating a diagram with a single scattering dressing a diffuson. Thus the single-scattering contribution is evaluated exactly, and all double and higher scattering contributions are evaluated in the diffuson approximation. The single scattering can be done simply, giving a total contribution of $\approx 0.21N_c$. The diffuson approximation for the second and higher scatterings is more tedious to evaluate but is found to sum to $\approx 0.63N_c$. Hence the total average reflection in this improved approximation has increased to $\approx 0.84N_c$. In principle this method could be extended to even higher scatterings and an even better approximation for the average reflection will be obtained.
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[33] In the chaotic ballistic case, the numerical analysis was complicated by the fact that
changing the symmetry of the structure introduced a small but significant change in
the classical transmission. In order to obtain results not influenced by the change in
the classical transmission, we adopted the following procedure. (1) Fit a straight line to
$T(k)$. The slope, denoted $\mathcal{T}$, has been previously shown to be the classical transmission
probability [3]. (2) Subtract the classical transmission, $\delta T(k) \equiv T(k) - \mathcal{T}kW/\pi$. (3)
Average $\delta T(k)$ over $k$ for the different symmetry classes and compare them.

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[36] One can think of many reasons that the agreement between numerics and analytic
theory is imperfect by this amount for the chaotic conductors. For example, in this case
our argument for neglecting the change in the 1PGF due to symmetry could fail as the
size of the structure effectively sets the mean-free path. However it is puzzling that the
results for the disordered case are consistently low by such an amount. There are certain
parametric inequalities such as $l \ll L_x, L_y$ which are assumed in the analytic theory
but are hard to satisfy strongly in the numerical simulations. Perhaps these account for
the discrepancy; however in the case of the conductance fluctuations, which is similar
in many respects, one finds agreement between analytic theory and numerics to 5%
accuracy with little effort.
FIGURES

FIG. 1. Self-consistent Born approximation (SCBA) to the average one particle Green’s function. Dashed lines represent impurity scatterings, thin lines represent bare Green’s functions, thick lines represent Green’s functions after averaging.

FIG. 2. Expansion for ladder sum. b) Expansion for crossed ladder sum. The + and - signs denote advanced and retarded Green’s functions, L and X the ladder and crossed ladder sums.

FIG. 3. Divergence of a general vertex in a conductivity diagram. The arrow represents the operation of taking the divergence, the wavy line at the vertex represents a current operator and the solid dot indicates a current vertex deleted by the divergence operation as described in the text.

FIG. 4. a) Divergence of a vertex dressed with impurity scatterings. b) Divergence of bare vertex plus vertex dressed with ladder sum, represented by $\tilde{V}$.

FIG. 5. The ladder approximation to conductivity.

FIG. 6. Current conserving set of diagrams containing a crossed ladder with n scatterings. The triangular vertex denotes the dressed vertex $\tilde{V}$. a) The first set of diagrams as described in the text. b) The second set as described in the text.

FIG. 7. Diffuson diagram used to calculate contributions to the transmission coefficients $T_{ij}$ to lowest order in $1/N_c$.

FIG. 8. The simplest negative contribution to $R_{ij}$ due to time reversal symmetry, evaluated in the cooperon approximation.

FIG. 9. The SCBA in the presence of symmetry. The corrections to the SCBA self-energy due to symmetry (last diagram) are found to be negligible.

FIG. 10. Crossed ladder diagram with one side spatially reflected, denoted by the symbol $\times$.

FIG. 11. Effect of reflection on sense of a path. Single reflection reverses sense, double reflection preserves it.

FIG. 12. Improved form of the diffuson approximation as discussed in the Appendix.
TABLE I. The effect of reflection symmetry on weak localization in quasi-1D disordered conductors. The results of numerical calculation are in good qualitative agreement with the analytic theory. The numerical results were obtained for the standard model (34,35) of a tight-binding Hamiltonian with on-site disorder. A rectangular strip with the following parameters was used: length/width = 8, width=120, disorder strength =0.9, $k_f a = 1.61$, mean-free-path=25, $BA/\phi_0 = 0$ or 10, and number of configurations=300.

| Symmetry Type          | $\delta g_s$ (theory) | $\delta g_s$ (num.) | $\delta g_B$ (theory) | $\delta g_B$ (num.) |
|------------------------|------------------------|----------------------|------------------------|----------------------|
| No Symmetry            | 0                      | —                    | 0.333                  | 0.237 ± 0.026        |
| Symmetry $\parallel$ to Current ($R_x$) | $-0.333$              | $-0.272 ± 0.037$     | 0.333                  | 0.211 ± 0.036        |
| Symmetry $\perp$ to Current ($R_y$)  | $+0.167$              | $0.129 ± 0.037$      | 0.333                  | 0.273 ± 0.036        |
| 4-fold Symmetry ($R_x R_y$)  | 0                     | $0.031 ± 0.051$      | 0.167                  | 0.059 ± 0.059        |

TABLE II. The effect of reflection symmetry on weak localization in chaotic ballistic conductors. The theoretical results are obtained by scaling the results for the disordered conductors by a factor of $3/4$ while the numerical results are obtained for stadium-like billiards, the billiard in Fig. 1 of Ref. [8] for the asymmetric case and its symmetry generated analogs. For the numerical results, the average [33] was taken over the energy range $k_f W / \pi \in [4, 25]$ ($[4, 15]$) for the structures with (without) $R_x$ symmetry, the magnetic field used was $BA/\phi_0 = 0$ or 2, and $k_f a \approx 1.0$. Note the good qualitative agreement except in the four-fold symmetric case.

| Symmetry Type          | $\delta g_s$ (theory) | $\delta g_s$ (num.) | $\delta g_B$ (theory) | $\delta g_B$ (num.) |
|------------------------|------------------------|----------------------|------------------------|----------------------|
| No Symmetry            | 0                      | —                    | 0.25                   | 0.147 ± 0.018        |
| Symmetry $\parallel$ to Current ($R_x$) | $-0.25$               | $-0.245 ± 0.038$     | 0.25                   | 0.148 ± 0.020        |
| Symmetry $\perp$ to Current ($R_y$)  | $+0.125$              | $0.170 ± 0.030$      | 0.25                   | 0.161 ± 0.019        |
| 4-fold Symmetry ($R_x R_y$)  | 0                     | $0.229 ± 0.043$      | 0.125                  | 0.043 ± 0.031        |