Can high-mode magnetohydrodynamic waves propagating in a spinning macrospicule be unstable due to the Kelvin–Helmholtz instability?

I. Zhelyazkov · R. Chandra

© Springer

Abstract We investigate the conditions at which high-mode magnetohydrodynamic (MHD) waves propagating in a spinning solar macrospicule can become unstable with respect to the Kelvin–Helmholtz instability (KHI). We consider the macrospicule as a weakly twisted cylindrical magnetic flux tube moving along and rotating around its axis. Our study is based on the dispersion relation (in complex variable) of MHD waves obtained from the linearized MHD equations of incompressible plasma for the macrospicule and cool (zero beta) plasma for its environment. That dispersion equation is solved numerically at appropriate input parameters to find out an instability region/window that accommodates suitable unstable wavelengths of the order of macrospicule’s width. It is established that a $m = 52$ MHD mode propagating in a macrospicule with width of 6 Mm, axial velocity of 75 km s$^{-1}$ and rotating one of 40 km s$^{-1}$ can become unstable against KHI with instability growth times of 2.2 and 0.57 min at 3 and 5 Mm unstable wavelengths, respectively. These growth times are much shorter than the macrospicule lifetime of around 15 min. An increase/decrease in the width of the jet would change the KHI growth times remaining more or less of the same order when are evaluated at wavelengths equal to the width/radius of the macrospicule. It is worth noticing that the excited MHD modes are supper-Alfvénic waves. A change in the background magnetic field can lead to another MHD mode number $m$ that ensures the required instability window.

Keywords: Magnetohydrodynamics; MHD waves and instability; Solar macrospicules

1. Introduction

Macrospicules were first detected four decades ago by Bohlin et al. (1975) from He ii 304 Å spectroheliograms, obtained with the NRL extreme-ultraviolet slitless spectro-
graph during a Skylab mission. Macrospicules are usually seen as jets 5–15′′ in diameter, 5–50′′ in length which move outward into the corona at speeds of 10–150 km s\(^{-1}\) and having lifetime of 5–45 min. An essential step in observing macrospicules was done on using the Coronal Diagnostic Spectrometer (CDS) onboard the Solar and Heliospheric Observatory (SOHO; Domingo et al., 1995). First scientific results on using CDS were reported during 1997 and an extensive review of obtained physical parameters of observed macrospicules can be found in Harrison et al. (1997). Pike and Mason (1998) analyzing macrospicules’ observations from 9 April 1996 to 4 April 1997 first reported that macrospicules can be considered as spinning columns with rotating velocity of 20–50 km s\(^{-1}\). Banerjee et al. (2000) on using SOHO’s Extreme ultraviolet Imaging Telescope (EIT) images in the O\(\text{v}\) 629 Å line detected a giant macrospicule at the limb on 15 July 1999 and were able to follow its dynamical structure. Parenti et al. (2002) from the analysis of a sequence of SOHO/CDS observations obtained off-limb in the south polar coronal hole on 6 March 1998 detected a jet-like feature visible in the chromospheric and low transition region lines. According to these authors, the macrospicule was found to have a density of the order of \(10^{10}\) cm\(^{-3}\) and a temperature of about \(2–3 \times 10^5\) K. The initial outflow velocity near the limb was over 80 km s\(^{-1}\). Kamio et al. (2010) on using Hinode (Kosugi et al., 2007) Extreme-ultraviolet Imaging Spectrometer (EIS; Culhane et al., 2007) and the Solar Ultraviolet Measurements of Emitted Radiation instrument (SUMER; Wilhelm et al., 1995) on the SOHO were able to measure the line of sight (LOS) motions of both macrospicule and coronal jets. At the same time, with the help of the X-Ray Telescope (XRT; Golub et al., 2007) on Hinode and Sun–Earth Connection Coronal and Heliospheric Investigation (SECCHI) instrument suite (Howard et al., 2008) on the Solar Terrestrial Relations Observatory (STEREO; Kaiser et al., 2008) the authors traced the evolution of the coronal jet and the macrospicule. The upward propagating and rotating velocities of the macrospicule, averaged over 10 min, between 02:36 UTC and 02:46 UTC, proved to be \(130 \pm 30\) and \(25 \pm 5\) km s\(^{-1}\), respectively. Scullion et al. (2010) explored the nature of macrospicule structures, both off-limb and on-disk, on using the high resolution spectroscopy obtained with the SOHO/SUMER instrument. The authors reported on finding high velocity features observed simultaneously in spectral lines formed in the mid-transition region, N\(\text{iv}\) 765 Å spectral line \((1.4 \times 10^5\) K), and in the low corona, Ne\(\text{viii}\) 770 Å spectral line \((6.3 \times 10^5\) K), where in the hot Ne\(\text{viii}\) 770 Å line the flow speed reached \(\approx 145\) km s\(^{-1}\). Majarska et al. (2011) performing multi-instrument co-observations with the SUMER/SOHO and with the EIS/SOT/XRT/Hinode at the north pole on 2009 April 28 and 29 detected three rotating macrospicules. Their main result is that although very large and dynamic, these spicules do not appear in spectral lines formed at temperatures above 300 000 K. In the same year, 2011, Murawski et al. (2011) had presented the first numerical simulation of macrospicule formation by implementing the VAL-C model of solar temperature. On using the FLASH code, these authors solved the two-dimensional ideal magnetohydrodynamic equations to model a macrospicule, whose physical parameters match those of a solar spicule observed at the north polar region in the 304 Å of the Atmospheric Imaging Assembly (AIA; Lemen et al., 2012) onboard the Solar Dynamics Observatory (SDO; Pesnell et al., 2012) on 3 August 2010. The essence of this numerical simulation is that the solar macrospicules can be triggered by velocity pulses launched from the chromosphere. Another mechanism for the origin of macrospicules was proposed by Kayshap et al.
Kelvin–Helmholtz instability in a rotating macrospicule (2013) who numerically modeled the triggering of a macrospicule and a jet observed by AIA/SDO on 2010 November 11 in the north polar corona. The spicule, considering as a magnetic flux tube that undergoes kinking, reached up to ~40 Mm in the solar atmosphere with a projected speed of ~95 km s\(^{-1}\). The simulation results, obtained in the same manner as in Murawski et al., 2011 show that reconnection-generated velocity pulse in the lower solar atmosphere steepens into slow shock and the cool plasma is driven behind it in the form of macrospicule. Loboda and Bogachev (2017) on using high-cadence EUV observations obtained by the Lebedev Institute of Physics TESIS solar observatory performed a detailed investigation of the axial plasma motions in solar macrospicules. They used a one-dimensional hydrodynamic method to reconstruct the evolution of the internal velocity field of 18 macrospicules and have found that 15 of them followed parabolic trajectories with high precision which correspond closely to the obtained velocity fields. Two articles, namely those of Bennett and Erdélyi (2015) and Kiss et al. (2017) summarize the origin, evolution, and physical parameters of a large number of observed jets: 101 macrospicules in Bennett and Erdélyi, 2015, and 301 in Kiss et al., 2017. The first data set covers observations over a 2.5 year long time interval, while the second database is built on the observations of macrospicules that occurred between 2010 June and 2015 December, that is, over a 5.5 year long time interval. Table 2 in Kiss et al., 2017 contains the main macrospicule characteristics obtained from ‘old’ and ‘new’ observations and here we will cite some averaged values, notably lifetime of 16.75 ± 4.5 min, maximum width of 6.1 ± 4 Mm, and average upflow velocity of 73.14 ± 25.92 km s\(^{-1}\). According to the authors, the maximum length is overestimated, having the magnitude of 28.05 ± 7.67 Mm.

Solar spicules, like every magnetically structured entity in the solar atmosphere, support the propagation of different kinds of MHD waves—for a review of a large number of observational and theoretical explorations of oscillations and waves in spicules see, for instance, Zaqarashvili and Erdélyi (2009). The axial mass flow in spicules can be the reason for making the propagating MHD modes unstable due to a velocity jump at spicule’s surface and the instability that occurs is of the Kelvin–Helmholtz kind. Recall that the Kelvin–Helmholtz instability (KHI) is a purely hydrodynamic phenomenon and it arises at the interface of two fluid layers that move with different speeds (see, e.g., Chandrasekhar, [1961])—then a strong velocity shear comes into being near the thin interface region of these two fluids forming a vortex sheet that becomes unstable to the spiral-like perturbations at small spatial scales. It is worth noticing that the magnetic field plays an important role in the instability developing, notably a strong enough magnetic filed can suppress the KHI. Solar spicules are usually modeled as moving untwisted cylindrical magnetic flux tubes and the vortex sheet emerging near tube’s boundary may become unstable against the KHI provided that its axial velocity exceeds some critical/threshold value (Ryu et al., 2000). This vortex sheet, during the nonlinear stage of KHI, causes the conversion of the directed flow energy into a turbulent energy making an energy cascade at smaller spatial scales. First theoretical modelings of KHI in the relatively faster Type-II spicules (De Pontieu et al., 2007) (see, e.g., Zhelyazkov, 2012; Zhelyazkov, 2013; Ajabshirizadeh et al., 2015; Ebadi, 2016) show that the magnitudes of the threshold flow velocities at which KHI rises critically depend on the density contrast defined as \(\rho_c/\rho_i = \eta\), where \(\rho_i\) is the spicule’s plasma density, and \(\rho_c\) is that of surrounding magnetized plasma. The study of KHI in a spicule at \(\eta = 0.01\) (Zhelyazkov, 2012; Zhelyazkov, 2013) shows that the critical velocity for the instability
onset of the kink \((m = 1)\) MHD mode must be higher than \(876 \text{ km s}^{-1}\), which speed is generally not accessible for Type-\(ii\) spicules. A decrease in the density contrast, say, taking \(\eta = 0.02\), leads to a decrease of the threshold speed to \(711 \text{ km s}^{-1}\), which is still too high. Ajabshirizadeh et al. (2015) exploring the KHI of the kink \((m = 1)\) mode in Type-\(ii\) spicules at much lower value of the density contrast \((\approx 0.1)\) also claim that the required threshold flow velocity is higher than assumed by them spicule’s speed of \(150 \text{ km s}^{-1}\) (see also a comment to their results in Zhelyazkov et al., 2016). A 2D MHD numerical modeling (on using the Athena3d code) of KHI in solar spicules performed by Ebadi (2016) shows that at some selected densities and flow speeds one can ‘observe’ a KHI-type onset and transition to turbulent flow in spicules. 

Recent studies (De Pontieu et al., 2014; Iijima and Yokoyama, 2017) show that all small-scale jets in solar chromosphere and transition region should possess some twist of their magnetic fields. In particular, Iijima and Yokoyama (2017) performing a 3D MHD simulation of the formation of solar chromospheric jets with twisted magnetic field lines were able to produce a tall chromospheric jet with a maximum height of \(10–11 \text{ Mm}\) and lifetime of \(8–10 \text{ min}\). These authors also found that the produced chromospheric jet forms a cluster with a diameter of several Mm with finer strands and claim that the obtained results imply a close relationship between the simulated jet and solar spicules. The magnetic field twist might weakly change the value of the critical flow velocity required for instability onset of the kink \((m = 1)\) MHD mode propagating along Type-\(i\) or Type-\(ii\) spicules, but hardly will dramatically diminish it—anyway this has to be checked. The case of spinning magnetically twisted macrospicules is more specific—as Zaqarashvili et al. (2015) had established, in axially moving and rotating around their axes solar jets one can be excited high \((m \geq 2)\) MHD modes only. The aim of our study is to see at what wave mode number \(m\) one can expect the occurrence of KHI in a rotating macrospicule at wavelengths of the unstable mode comparable to its width/\(\text{radius}\). As a model we use the macrospicule observed on 8 March 1997 at 00:02 \(\text{UT}\) (Pike and Mason, 1998). The axial velocity of that jet is \(75 \text{ km s}^{-1}\), while its rotating speed, we evaluate to be \(40 \text{ km s}^{-1}\). In addition, we will investigate how the magnitude of jet’s width will influence KH instability characteristics of the excited high MHD mode.

Our article is organized as follows: in the Observations section we present the observational data. The magnetic field topology of a moving and rotating cylindrical flux tube modeling the solar macrospicule along with its physical parameters and the normal mode dispersion relation of the excited wave are given in Section 3. Section 4 is devoted to the numerical solutions to the wave dispersion relation and to the discussion of obtained results. The new findings and comments on future improvements of KHI studies in solar spinning macrospicules can be found in the last section.

2. Observations

Pike and Mason (1998) did a statistical study of the dynamics of solar transition region features, i.e. like macrospicules. These features were observed on the solar disk as well as on the solar limb. For their investigation, as we have mentioned in the Introduction section, they used the data from the Coronal Diagnostic Spectrometer on board SOHO. In their article, these authors discuss the unique CDS observations of a macrospicule
first reported by Pike and Harrison [1997] along with their own (Pike and Mason) observations from the normal incidence spectrometer (NIS), which covers the wavelength range from 307 to 379 Å and that from 513 to 633 Å on using a microchannel plate/CCD combination detector. The details of macrospicule events observed near the limb are given in TABLE I in Pike and Mason, 1998, while those of macrospicule events observed on the disk are presented in TABLE II. The main finding of their study was the rotation in these features. Their conclusion was based on the red/blue shifted emission on either side of the macrospicules’ axes and the detected rotation assuredly plays an important role in the dynamics of the transition region. Our choice for modeling the event observed on 8 March 1997 at 00:02 UT (see TABLE II) is the circumstance that that macrospicule possesses, more or less, the basic characteristics of the observed over the years tornado-like jets.

3. Geometry, physical parameters, and wave dispersion relation

Our model of the macrospicule is similar to our previous articles Zhelyazkov et al. (2018a) and Zhelyazkov and Chandra (2018b), notably it is a considered as a cylindrical weakly twisted magnetic flux tube with radius \( a \) and homogeneous density \( \rho_i \) moving with velocity \( U \). The tube is surrounded by a plasma with homogeneous density \( \rho_e \) being embedded in a homogeneous magnetic field \( B_e \), which, in cylindrical coordinates \((r, \phi, z)\), has only a \( z \) component, that is, \( B_e = (0, 0, B_e) \). (Here the label ‘i’ is abbreviation for interior, while the label ‘e’ means exterior.) The internal magnetic field and the flow velocity, by contrast, are both twisted and can be generally presented by the vectors \( B_i = (0, B_i \phi(r), B_i z) \) and \( U = (0, U_\phi(r), U_z) \), respectively. We note, the \( B_i z \) and \( U_z \), are constant. Concerning the azimuthal magnetic and flow velocity components, evaluated at the tube interface, we assume that they are constants, equal to \( B_\phi = Aa \) and \( U_\phi = \Omega a \), respectively. In particular, \( \Omega \) is the macrospicule angular speed. Thus, in equilibrium the rigidly rotating, constant-magnetic-pitch column, which model the macrospicule, should satisfy the force-balance equation (see, e.g., Chandrasekhar, 1961; Goossens et al., 1992)

\[
\frac{d}{dr} \left( p_t + \frac{B_i^2}{2\mu} \right) = \frac{\rho_i U_i^2}{r} - \frac{B_{i\phi}}{r},
\]

where \( \mu \) is the plasma permeability and \( p_t = p_i + B_i^2/2\mu \) is the total (thermal + magnetic) pressure, in which \( B_i^2 = B_i \phi(r)^2 + B_i z \). Above equation says that the total pressure radial gradient should balance the centripetal/centrifugal force and the force due to curved magnetic field lines. Upon integrating Equation (1) from 0 to the tube radius \( a \), bearing in mind the linear dependence of both \( U_\phi \) and \( B_{i\phi} \) on \( r \), one obtains that

\[
p_t(a) = p_t(0) + \frac{1}{2} \rho_i U_i^2(a) \left( \frac{B^2_{i\phi}(a)}{2\mu} \right),
\]

where \( p_t(0) = p_i(0) + B_i^2(0)/2\mu \). (Integration can be in principle performed from 0 to any \( r \) to get the radial profile of the total pressure inside the tube—for an equivalent expression of \( p_t(r) \), derived from integration of momentum equilibrium equation for the equilibrium variables, see Eq. (2) in Zhelyazkov et al., 2018a.) Above internal total...
pressure (evaluated at the tube boundary) must be balanced by the total pressure of the surrounding plasma which implies that

\[ p_i(0) + B_i^2 \frac{e}{2\mu} - B_i^2 \frac{\varepsilon}{2\mu} + \frac{1}{2} p_i U_\phi^2 = p_e + B_i^2 \frac{e}{2\mu}. \]

Recall that \( B_\phi = B_{i0}(a) \). Obtained total pressure balance equation can be presented in the form

\[ p_i + \frac{1}{2} p_i U_\phi^2 + B_i^2 \frac{e}{2\mu} (1 - \varepsilon_1^2) = p_e + B_i^2 \frac{e}{2\mu}, \tag{2} \]

where \( p_i \) is the thermal/plasma pressure at the tube axis and \( \varepsilon_1 \equiv B_\phi/B_e = \Lambda \alpha / B_e \) is the magnetic field twist parameter. In a similar way we introduce \( \varepsilon_2 \equiv U_\phi / U_e \), which characterizes the jet velocity twist. In above equation, \( p_e \) denotes the thermal pressure in the environment. The choice of plasma and environment parameters must be such that the total pressure balance equation (2) is satisfied. It is important to note that in our case \( \varepsilon_2 \) is defined by observationally measured rotational and axial velocities while \( \varepsilon_1 \) is a parameter, which has to be specified when using Equation (2).

Since Pike and Mason (1998) do not provide any data concerning the macrospicule and its environment electron number densities, \( n_i \) and \( n_e \), respectively, based on measurements of such chromospheric jets, we assume that \( n_i = 1.0 \times 10^{10} \text{ cm}^{-3} \) and \( n_e = 1.0 \times 10^9 \text{ cm}^{-3} \) to have at least one order denser jet with respect to the surrounding plasma. We take macrospicule temperature to be \( T_i = 5.0 \times 10^5 \text{ K} \), while that of its environment is typically equal to 1 MK, that is, \( T_e = 1.0 \times 10^6 \text{ K} \). This choice of electron number densities and electron temperatures defines the density contrast \( \eta \equiv n_i / n_e = 0.1 \), and the sound speeds in both media: \( c_{si} = 83.0 \) and \( c_{se} = 117.3 \text{ km s}^{-1} \), respectively.

Assuming a relatively weak internal magnetic field twist \( \varepsilon_1 = 0.005 \) and a background magnetic field \( B_\phi = 5 \text{ G} \), from the total pressure balance equation (2) one obtains the Alfvén speeds \( v_{Ai} = 60.6 \) and \( v_{Ae} = 344.7 \text{ km s}^{-1} \), respectively, the ratio of axial magnetic fields, \( b \equiv B_e / B_i = 1.798 \), as well as the two plasma betas, \( \beta_i = 2.248 \) and \( \beta_e = 0.139 \). We note that the Alfvén speed inside the macrospicule is defined (and computed) as \( v_{Ai} = B_i / \sqrt{\mu_0 p_i} \), while both plasma betas are evaluated from the ratio \( c_{AI}^2 / c_e^2 \) (multiplied by 6/5), where the Alfvén speeds are calculated with the full magnetic fields. The basic physical parameters of the macrospicule and its environment are summarized in Table 1.

Recall that the macrospicule axial speed is \( U_i = 75 \text{ km s}^{-1} \), while its rotational one is \( U_\phi = 40 \text{ km s}^{-1} \). We assume that the macrospicule width is \( \Delta l = 6 \text{ Mm} \), its height \( H = 28 \text{ Mm} \), and lifetime of the order of 15 min. All these data

| Medium      | Temperature (MK) | Electron density \((\times 10^{10} \text{ cm}^{-3})\) | Plasma beta |
|-------------|------------------|-----------------------------------------------|-------------|
| Macrospicule| 0.5              | 1.0                                           | 2.248       |
| Environment | 1.0              | 0.1                                           | 0.139       |

Table 1. Macrospicule’s and its environment physical parameters derived at background magnetic field \( B_i = 5 \text{ G} \).
Kelvin–Helmholtz instability in a rotating macrospicule

are more or less similar to the averaged macrospicules’ parameters discussed in Kiss et al., [2017].

Dispersion relation of high-mode \((m \geq 2)\) MHD waves traveling in magnetized axially moving and rotating jet in Zaqarashvili et al., [2015] was obtained under the assumption that both media (the jet and its environment) are incompressible plasmas. As seen from Table 1, macrospicule’s plasma beta is larger than 1 and jet’s medium can be treated as a nearly incompressible fluid (Zank and Matthaeus, [1993]). On the other hand, the plasma beta of the surrounding magnetized plasma is less than 1 and it is more adequate to consider it as a cool medium. This implies that the wave dispersion relation, derived in Zaqarashvili et al., [2015], has to be slightly modified. We will skip the derivation of that modified equation from the basic MHD equations, that has been done in Zhelyazkov et al., (2018a). Here we provide its final form—the full derivation of the wave dispersion relation is presented in the Appendix. As is logical to expect that equation has to be expressed in terms of modified Bessel functions of first and second kind, \(I_m\) and \(K_m\), and their derivatives with respect to functions arguments, \(I'_m\) and \(K'_m\), that is,

\[
\frac{(\sigma^2 - \omega_{Ai}^2) F_m(\kappa_i a) - 2m (\sigma \Omega + A \omega_{Ai} / \sqrt{\mu \rho_i})}{\rho_i (\sigma^2 - \omega_{Ai}^2) - 4 \rho_i (\sigma \Omega + A \omega_{Ai} / \sqrt{\mu \rho_i})^2} = \frac{\rho_e (\sigma^2 - \omega_{Ae}^2) - (\rho_i \Omega^2 - A^2 / \mu) P_m(\kappa_e a)}{P_m(\kappa_e a)},
\]

where

\[
F_m(\kappa_i a) = \frac{\kappa_i a I_m(\kappa_i a)}{I_m(\kappa_i a)} \quad \text{and} \quad P_m(\kappa_e a) = \frac{\kappa_e a K'_m(\kappa_e a)}{K_m(\kappa_e a)}.
\]

Here,

\[
\kappa_i^2 = k_i^2 \left[ 1 - 4 \left( \frac{\sigma \Omega + A \omega_{Ai} / \sqrt{\mu \rho_i}}{\sigma^2 - \omega_{Ai}^2} \right)^2 \right] \quad \text{and} \quad \kappa_e^2 = k_e^2 \left( 1 - \frac{\omega^2}{\omega_{Ae}^2} \right)
\]

are the squared wave amplitude attenuation coefficients in both media, in which

\[
\omega_{Ai} = \left( \frac{m r B_{d\parallel} + k_z B_{d\perp}}{\sqrt{\mu \rho_i}} \right) / \sqrt{\mu \rho_i} \quad \text{and} \quad \omega_{Ae} = k_z B_e / \sqrt{\mu \rho_e}
\]

are the corresponding local Alfvén frequencies, and

\[
\sigma = \omega - \frac{m}{r} U_{d\parallel} - k_z U_{d\perp}
\]

is the Doppler-shifted wave frequency in the macrospicule. The solutions to dispersion relation (3) are given and used in the next section.

4. Numerical results and discussion

Since we are looking for instability of MHD high \((m \geq 2)\) modes in the macrospicule–coronal plasma system, we assume that the angular wave frequency, \(\omega\), is a complex
quantity, while the propagating wave number, \( k_z \), is a real quantity. To perform the numerical task, we normalize the velocities with respect to the Alfvén speed inside the macrospicule, \( v_{Ai} \), and the lengths with respect to \( a \) (the tube radius). The normalization of Alfvén local and Doppler-shifted frequencies along with the Alfvén speed in the environment requires the usage of both twist parameters, \( \varepsilon_1, \varepsilon_2 \), as well as the magnetic fields ratio \( b = B_e/B_i \). The normalized axial flow velocity is presented by the Alfvén Mach number \( M_A = U_z/v_{Ai} \). Thus, the input parameters in the numerical solving the transcendental equation (3) (in complex variables) are:

\[
\begin{align*}
  m, \quad \eta, \quad \varepsilon_1, \quad \varepsilon_2, \quad b, \quad & \quad M_A.
\end{align*}
\]

It has been derived in Zaqarashvili et al., 2015 that the instability in an untwisted rotating flux tube at sub-Alfvénic jet velocities can occur if

\[
\frac{a^2 \Omega^2}{v_{Ai}^2} > \frac{1 + \eta}{1 + |m\eta| |m| - 1} \left(\frac{1 + b^2}{1 + b^2}\right).
\]

As seen, each rotating jet can be unstable for any mode number \( m \geq 2 \). This inequality is applicable to slightly twisted spinning jets provided that the magnetic field twist \( \varepsilon_1 \) is relatively small, say between 0.001 and 0.005. Here, we make an important assumption, notably that the deduced from observations axial velocity of the macrospicule, \( U_z \), is the threshold speed for the KHI onset. Then, for fixed values of \( m, \eta, U_\phi = \Omega a, v_{Ai}, \) and \( b = B_e/B_i \), the above inequality defines the right-hand-side limit of the instability range on the \( k_z a \)-axis

\[
(k_z a)_{rhs} < \left(\frac{U_\phi}{v_{Ai}} \frac{1 + |m\eta| |m| - 1}{1 + \eta} \frac{1 + b^2}{1 + b^2}\right)^{1/2}.
\]

This inequality says that the instability can occur for all dimensionless wavenumbers \( k_z a \) less than \( (k_z a)_{rhs} \). However, one can talk for instability if the unstable wavelength is shorter than the height of the jet. This requirement allows us to define the left-hand-side limit of the instability region:

\[
(k_z a)_{lhs} > \frac{\pi \Delta f}{H}.
\]

In our case this limit is equal to 0.673. Numerical computations show that for small MHD mode numbers, \( m \), the instability range/window is relatively narrow and the shortest unstable wavelengths, \( A_{KH} = \pi \Delta f/k_z a \), that can be computed are much larger than the macrospicule width. Such long wavelengths are not comfortable for instability detection/observation. The rapidly developed vortex-like structures at jet’s boundary have the size of the width/radius of the flux tube (see, for instance, Fig. 1 in Zhelyazkov et al., 2013a). Thus, we should look for such instability regions which would contain the expected unstable wavelengths. A noticeable extension of the instability range can be achieved via increasing the wave mode number. If we wish, for example, to have an unstable wavelength \( \lambda_{KH} = 3 \text{ Mm} \) (the half width of our macrospicule), we have to find out that mode number \( m \) whose instability window will accommodate \( k_z a = 2\pi \) (the dimensionless wavenumber that corresponds to \( \lambda_{KH} = 3 \text{ Mm} \)). An estimation of the required mode number for a \( \varepsilon_1 = 0.005 \)-rotating flux tube can be obtained by presenting
the instability criterion in the form
\[ \eta|m|^2 + (1 - \eta)|m| - 1 - \frac{(k_a a^2 (1 + \eta) (1 + b^2)}{(U_B/v_Ai)^2} > 0. \]  

(7)

With \( \eta = 0.1, k_a a = 2\pi, U_B = 40 \text{ km s}^{-1}, v_Ai = 60.6 \text{ km s}^{-1}, \) and \( b = 1.798, \) the above equation yields \( m = 64. \) This magnitude is, however, overestimated—our numerical calculations show that the appropriate MHD wave mode number that accommodates the unstable wavelength of 3 Mm (\( k_a a = 2\pi \)) is \( m = 52. \)

Hence, the input parameters in the numerical task of solving Equation (3) are: \( m = 52, \eta = 0.1, b = 1.798, \epsilon_1 = 0.005, \epsilon_2 = 0.53, \) and \( M_A = 1.24. \) The results are pictured in Figure 1. From this plot we can calculate the instability characteristics of the

![Figure 1](image1.png)

**Figure 1.** (Left panel) Dispersion curve of the \( m = 52 \) MHD mode propagating along a twisted incompressible macrospicule at \( \eta = 0.1, b = 1.798, M_A = 1.24, \epsilon_1 = 0.005, \) and \( \epsilon_2 = 0.53. \) (Right panel) Normalized growth rate curve of the \( m = 52 \) MHD mode computed at the same input parameters as in the left panel.

![Figure 2](image2.png)

**Figure 2.** (Left panel) Dispersion curves of the unstable \( m = 52 \) MHD mode propagating along a twisted incompressible macrospicule at \( \eta = 0.1, \epsilon_2 = 0.53, \) and the following values of \( \epsilon_1 \) (from right to left): 0.005, 0.1, 0.15, 0.175, 0.195, and 0.202085 (red curve in the right plot). Alfvén Mach numbers for these curves are respectively 1.24, 1.23, 1.22, 1.22, 1.21, and 1.2119. (Right panel) Growth rates of the unstable \( m = 52 \) mode for the same input parameters. The azimuthal magnetic field that corresponds to \( \epsilon_1 = 0.202085 \) (the instability window with zero width) and stops the KHI onset is equal to 0.57 G.

For \( m = 52 \) MHD mode for two unstable wavelengths, equal to 3 and 5 Mm (\( k_a a = 6\pi/5 \)), respectively. The KHI wave growth rate, \( \gamma_{KH} \), growth time, \( \tau_{KH} = 2\pi/\gamma_{KH} \), as well as

---

SOLA: SOLA-D-18-00174R4.tex; 31 December 2018; 3:41; p. 9
Table 2. Kelvin–Helmholtz instability characteristics of the $m = 52$ MHD mode at $\lambda_{KH} = 4$ Mm for three different widths of the macrospicule correspondingly equal to 8, 6, and 4 Mm.

| $\Delta \ell$ (Mm) | $\gamma_{KH}$ ($\times 10^{-3}$ s$^{-1}$) | $\tau_{KH}$ (min) | $v_{ph}$ (km s$^{-1}$) | $\varepsilon_{1}^{cr}$ | $B_{\phi}^{cr}$ (G) |
|-------------------|--------------------------------------|------------------|-----------------------|-------------------|------------------|
| 8                 | 36.28                                 | 2.9              | 361                   | 0.19843           | 0.55             |
| 6                 | 156.48                                | 0.67             | 459                   | 0.202085          | 0.57             |
| 4                 | 246.70                                 | 0.35             | 654                   | 0.205465          | 0.58             |

wave velocity, $v_{ph}$, calculated from the graphics in Figure 1 for the aforementioned wavelengths are:

For $\lambda_{KH} = 3$ Mm we obtain

$$\gamma_{KH} \approx 48.38 \times 10^{-3} \text{ s}^{-1}, \quad \tau_{KH} \approx 2.2 \text{ min}, \quad v_{ph} \approx 361 \text{ km s}^{-1},$$

while at $\lambda_{KH} = 5$ Mm we get

$$\gamma_{KH} \approx 184.8 \times 10^{-3} \text{ s}^{-1}, \quad \tau_{KH} \approx 0.57 \text{ min}, \quad v_{ph} \approx 556 \text{ km s}^{-1}.$$  

As seen, at both wavelengths phase velocities are supersonic. The two growth times of 2.2 and ~0.6 min are reasonable taking into account that the macrospicule lifetime is about 15 minutes, that is, the KHI at the selected wavelengths is rather fast.

A specific property of instability $k_{z}a$ ranges is that for a fixed $m$ their widths depend upon the magnetic field twist parameter $\varepsilon_{1}$. A discussion of that dependence is provided in Zhelyazkov and Chandra, 2018b and here we quote it, namely “with increasing the value of $\varepsilon_{1}$, the instability window becomes narrower and at some critical magnetic field twist its width equals zero. This circumstance implies that for $\varepsilon_{1} \geq \varepsilon_{1}^{cr}$ there is no instability, or, in other words, there exists a critical azimuthal magnetic field $B_{\phi}^{cr} = \varepsilon_{1}^{cr} B_{i}$ that suppresses the instability onset”. In the next Figure 2 a series of dispersion and dimensionless wave phase velocity growth rates for various increasing magnetic field twist parameter’s values has been plotted. Note that each larger $\varepsilon_{1}$ implies an increase in $B_{\phi}$.

The red dispersion curve in the right panel of Figure 2 has been obtained for $\varepsilon_{1}^{cr} = 0.202085$ with $M_{A} = 1.2119$ and it visually defines the left-hand-side limit of all other instability ranges. The azimuthal magnetic field $B_{\phi}^{cr}$ that stops the KHI is equal to 0.57 G.

The instability $k_{z}a$-range of the $m = 52$ MHD mode pictured in Figure 1 allows us to investigate how the width of the macrospicule will affect the KHI characteristics for a fixed instability wavelength. Such an appropriate wavelength is $\lambda_{KH} = 4$ Mm.

We will calculate (and plot) the instability growth rate, $\gamma_{KH}$, the instability development/growth time, $\tau_{KH}$, and the wave phase velocity of the $m = 52$ mode. Our choice for macrospicule’s widths is: 8, 6, and 4 Mm, respectively. Note that the unstable 4 Mm wavelength has three different positions on the $k_{z}a$-axis, notably $k_{z}a = 2\pi$ for $\Delta \ell = 8$ Mm, $k_{z}a = 1.5\pi$ for $\Delta \ell = 6$ Mm, and $k_{z}a = \pi$ for $\Delta \ell = 4$ Mm (see Figure 3). The basic KHI characteristics of the $m = 52$ MHD mode at the wavelength $\gamma_{KH} = 4$ Mm at the aforementioned three different macrospicule’s widths are presented
in Table 2. The most striking result concerns the instability growth/developing time: it is only approximately half a minute at $\Delta \ell = 4 \, \text{Mm}$ and approximately 7 times longer (2.9 min) when the jet’s width is 8 Mm. It is not surprising, bearing in mind the shape of the dispersion curve of the $m = 52$ MHD mode, that the wave phase velocities of the unstable mode quickly increase from 361 km s$^{-1}$ at the widest macrospicule to 654 km s$^{-1}$ at the narrowest one. In that Table, in addition, we also give the critical magnetic field twist parameter, $\varepsilon_1^{cr}$, at which the size of the instability range of given jet becomes equal to zero. Those three limiting wave growth rate curves are plotted in three

![Figure 3](image1.png)

**Figure 3.** (Left panel) Dispersion curve of the $m = 52$ MHD mode propagating along a twisted incompressible macrospicule at $\eta = 0.1$, $b = 1.798$, $M_A = 1.24$, $\varepsilon_1 = 0.005$, and $\varepsilon_2 = 0.53$. The marked with purple, cyan, and green vertical lines $k_z\alpha$-positions correspond to $\lambda_{KH} = 4 \, \text{Mm}$ for three different macrospicule’s widths, equal to 8, 6, and 4 Mm, respectively. (Right panel) Normalized growth rate curve of the $m = 52$ MHD mode propagating along a twisted incompressible macrospicule at the same input parameters as in the left panel.

![Figure 4](image2.png)

**Figure 4.** Growth rate curves of the $m = 52$ MHD mode propagating along a twisted incompressible macrospicule at $\eta = 0.1, \varepsilon_2 = 0.53$ and 8 different values of $\varepsilon_1$ equal to 0.005 (orange curve), 0.1 (green curve), 0.15 (purple curve), 0.175 (cyan curve) 0.195 (blue curve), 0.19843 (red curve), 0.202085 (black curve), and 0.205465 (cyan curve), respectively. The growth rate curves computed with the last three values of the magnetic field twist parameter $\varepsilon_1$ correspond to the three different macrospicule’s widths, equal to 8, 6, and 4 Mm, respectively. The Alfvén Mach numbers used at the computation of those three curves are accordingly equal to 1.2128, 1.2119, and 1.211.
Table 3. Kelvin–Helmholtz instability growth times of the $m = 52$ MHD mode at wavelengths equal to the half width of the jet and at $(\Delta \ell/2 + 2)$ Mm for three different widths of the macrospicule correspondingly equal to 8, 6, and 4 Mm.

| $\Delta \ell$ (Mm) | $\tau_{KH}(\Delta \ell/2)$ (min) | $\tau_{KH}((\Delta \ell/2 + 2)$ Mm) (min) |
|-------------------|---------------------------------|---------------------------------|
| 8                 | 2.9                             | 0.80                            |
| 6                 | 2.2                             | 0.57                            |
| 4                 | 1.4                             | 0.35                            |

different colors in Figure[3] from left to right the cyan curve corresponds to $\Delta \ell = 4$ Mm, the black one to 6 Mm, and the red curve to the macrospicule’s width of 8 Mm. It is interesting to observe that the critical azimuthal magnetic field components, $B_\phi^c$, that would stop the KHI appearance have very close magnitudes, roughly 0.6 G. The plots of the dimensionless dispersion curves for the last three values of $\varepsilon_1$ in Figure[4] similar to those seen in the left panel of Figure[2] are practically not usable—they possess very large values (in the range of 80–140) corresponding of extremely high wave phase velocities.

Another interesting observation is that the KHI growth/developing times of the $m = 52$ MHD mode, evaluated at wavelengths equal to the half width of the macrospicule (and, say, at $(\Delta \ell/2 + 2)$ Mm), for the three aforementioned widths are practically of the same order. This observation is illustrated in Table[3]. One sees that at the thinnest jet both growth times are the shortest ones while at the thickest jet they are relatively longer. An observational evaluation of the macrospicule’s width will naturally help in finding the appropriate conditions for KHI onset of the corresponding high MHD mode.

It is interesting to see how the change of the background magnetic field, $B_e$, will modify the picture. Let set $B_e = 4.8$ G (a magnetic field at which the total balance equation[3] is satisfied)—with the same input parameters for electron number densities and electron temperatures, we have new values for Alfven speeds, plasma betas, and magnetic fields ratio, notably $v_{Ae} = 52.36 \text{ km s}^{-1}$, $v_{Ai} = 330.9 \text{ km s}^{-1}$, $\beta_i = 3.01$, $\beta_e = 0.15$, and $b = 1.998$. With this new internal Alfven speed, $v_{Ai}$, the Alfven Mach number has a little bit higher magnitude, $M_A = 1.43$. It turns out that the appropriate MHD mode number that provides the required width of the instability window at $\Delta \ell = 6$ Mm is $m = 48$. Performing the calculations with $m = 48$, $\eta = 0.1$, $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.53$, $b = 1.998$, and $M_A = 1.43$ we get dispersion and growth rate curves very similar to those pictured in Figure[1]. We are not going to plot those curves but will compare the instability characteristics for both mode numbers, 48 and 52, respectively, at the instability wavelength of 3 and 5 Mm. The results of this comparison are shown in Table[4].

As seen, the KHI characteristics for the two mode numbers of 48 and 52 are very close. One can claim that at each MHD mode number which ensures an instability
Table 4. Kelvin–Helmholtz instability characteristics of the $m = 48$ and 52 MHD modes at $\lambda_{KH} = 3$ and 5 Mm for the macrospicule’s width equal to 6 Mm.

| Mode number $m$ | $\gamma_{KH}$ ($\times 10^{-3}$ s$^{-1}$) | $\tau_{KH}$ (min) | $v_{ph}$ (km s$^{-1}$) | $\varepsilon_1$ | $B_0^2$ (G) |
|----------------|--------------------------------------|------------------|-----------------------|--------------|--------------|
| $\lambda_{KH} = 3$ Mm | | | | | |
| 48 | 49.23 | 2.1 | 338.0 | 0.23494 | 0.58 |
| 52 | 48.38 | 2.2 | 361.0 | 0.202085 | 0.57 |
| $\lambda_{KH} = 5$ Mm | | | | | |
| 48 | 172.96 | 0.61 | 518.0 | 0.23494 | 0.48 |
| 52 | 184.85 | 0.57 | 556.5 | 0.202085 | 0.58 |

window width like that shown in Figure 1 will yield similar KHI characteristics as those in Table 4.

5. Summary and conclusion

In this article, we have established the conditions under which high MHD modes excited in a solar macrospicule can become unstable against Kelvin–Helmholtz instability. We model the jet as an axially moving and rotating around its axis weakly twisted cylindrical magnetic flux tube of radius $a$ and homogeneous density $\rho_i$ surrounded by coronal plasma with homogeneous magnetic field $B_e$ and density $\rho_e$. The twist of the internal magnetic field $B_i$ is characterized by the ratio $B_i/\rho_i = \varepsilon_1$, where the components of the internal magnetic field, $B_i$, are evaluated at the tube radius, $a$. In a similar way, the macrospicule’s velocity twist is specified by the ratio $U_\phi/U_z$, where $U_\phi$ and $U_z$ are the rotational and axial speeds of the jet. Along with these two parameters, the density contrast $\rho_e/\rho_i = \eta$ plays an important role in the modeling. Our choice of that parameter is 0.1 because we assume that the macrospicule should be at least one order denser than its environment. Other important physical parameters are the plasma betas of both media—their values tell us how to treat each medium, as incompressible or cool plasma—the general case of compressible media is still intractable from a theoretical point of view. Electron temperatures and background/jet magnetic field, at given density contrast, actually control the values of plasma betas. With $T_e = 1.0$ MK, $T_i = 500,000$ K, and $B_e = 5$ G, as well as $U_\phi = 40$ km s$^{-1}$ and $\varepsilon_1 = 0.005$, from the total pressure balance equation we obtain $\beta_i \approx 2.25$ and $\beta_e \approx 0.14$. These plasma beta values imply that one can treat the macrospicule’s medium as incompressible plasma while the surrounding magnetized plasma may be considered as a cool medium (Zank and Matthaeus, 1993). It is worth noticing, however, that a decrease in the macrospicule temperature to 300,000 K will diminish $\beta_i$ to 0.71 thus making jet’s plasma treatment as incompressible medium problematic. When the two media are treated as cool magnetized plasmas an acceptable modeling of KHI in our macrospicule would require the excitation of a higher than the $m = 52$ MHD mode.
Dispersion equation (3) yields unstable solutions for each mode number \( m \geq 2 \). For relatively small MHD mode numbers, however, the shortest unstable wavelengths that can be ‘extracted’ at \( k_z a \)-positions near the right-hand-side limit (5) of the instability window are too long to be comparable with the sizes of KH vortex-like blobs appearing at macrospicule’s interface. Reliable unstable wavelengths are achieved at the excitation of very high MHD modes. This is not surprising for chromospheric–TR jets—for instance, Kuridze et al. (2016) investigating the dynamics and stability of small-scale rapid redshifted and blueshifted excursions, appearing as high-speed jets in the wings of the \( \alpha \) line, explain their short lifetimes (few seconds) as a result of arising KHI in excited high-mode MHD waves. To achieve growth times of a few seconds on using dispersion equation similar to (3) (but with \( \kappa_c = k_c \)) it was necessary to assume azimuthal mode numbers up to 100. In our case \( m = 52 \) makes the instability region wide enough to accommodate unstable wavelengths of at least 3 Mm and KHI growth time of 2.2 min. This growth time becomes shorter with increasing the wavelength; for example at \( \gamma_{KH} = 4 \) Mm the growth time is around 0.7 min, while at \( \gamma_{KH} = 5 \) Mm it is equal to \( \approx 0.6 \) min. We also have studied how small variations of the macrospicule’s width affect the instability development time of the excited \( m = 52 \) mode—a decrease of \( \Delta \ell \) to 4 Mm yields to generally shorter growth times, while an increase of macrospicule’s width to 8 Mm gives longer instability growth times. Except through the change of the azimuthal mode number \( m \), the width of the instability window can be regulate by increasing/decreasing the magnetic field twist parameter \( \varepsilon_1 \). A progressive increase of the magnetic field twist parameter, \( \varepsilon_1 \), yields to a further decrease of the instability region width and at some critical \( \varepsilon_1 \) the width becomes equal to zero, that is, there is no instability at all. This \( \varepsilon_1^{cr} \) defines an azimuthal internal magnetic field component, \( B_{\phi}^{cr} \), that stops the KHI appearance. For our macrospicule this critical magnetic field is relatively small—it is equal to 0.6 G. Such a small azimuthal field component of the twisted magnetic field might be a reason for the inability to easy observe/detect KH features in spinning macrospicules. The shape of the dimensionless wave dispersion curve, shown in the left panels of Figures 1 and 3 tells us that the excited \( m = 52 \) MHD mode is a super-Alfvénic wave whose phase velocity grows very fast with the wavelength’s increasing. That is why, a realistic modeling of unstable MHD modes requires the finding of such an azimuthal mode number, \( m \), which will ensure an instability region with a width that should contain the expected unstable wavelengths near to its right-hand-side limit \((k_z a)_{rhs}\). A change in the background magnetic field, \( B_e \), can influence the MHD mode number \( m \), which would yield an instability region similar/identical to that seen in Figures 1 and 3. For instance, at a weaker environment magnetic field \( B_e = 4.8 \) G an almost identical instability window can be achieved with the excitation of the \( m = 48 \) MHD wave. The KHI characteristics at this mode number are very close to those obtained at the excitation of the 52 mode number—such a comparison can be seen in Table 4.

The results obtained in this article can be influenced by assuming more complicated velocity and magnetic field profiles, as well as, radially inhomogeneous plasma densities. The latter will involve the appearance of continuous spectra and resonant wave absorption which shall modify in some extent the KHI characteristics. The nonlinearity leads to the saturation of the KHI growth, and formation of nonlinear waves (Miura, 1984). Our approach, nonetheless is flexible enough and can yield reasonable growth times of observationally detected Kelvin–Helmholtz instabilities in solar atmospheric
Kelvin–Helmholtz instability in a rotating macrospicule jets. The next step of its improvement is to include compressibility in governing MHD equations. Arising KHI in the small-scale chromospheric jets, like macrospicules, and the triggered by it wave turbulence can contribute to the coronal heating and to the energy balance in the solar transition region.

Appendix

A. Derivation of the wave dispersion relation

Recall that we model the spinning macrospicule as a rotating and axially moving twisted magnetic flux tube of radius \( r = a \). In cylindrical coordinate system the magnetic and velocity fields inside the jet are supposed to be

\[
(0, B_\phi(r), B_z) \quad \text{and} \quad (0, U_\phi(r), U_z),
\]

respectively. Linearized ideal MHD equations, which govern the incompressible dynamics of perturbations in the rotating jet are

\[
\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{U} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{U} = -\nabla p_{\text{tot}} + \frac{(B_\phi \cdot \nabla) b}{\rho_i} + \frac{(b \cdot \nabla) B_\phi}{\rho_i \mu},
\]

(8)

\[
\frac{\partial}{\partial t} b - \nabla \times (\mathbf{v} \times B_\phi) - \nabla \times (\mathbf{U} \times b) = 0,
\]

(9)

\[
\nabla \cdot \mathbf{v} = 0,
\]

(10)

\[
\nabla \cdot \mathbf{b} = 0,
\]

(11)

where \( \mathbf{v} = (v_r, v_\phi, v_z) \) and \( \mathbf{b} = (b_r, b_\phi, b_z) \) are the perturbations of fluid velocity and magnetic field, respectively, and \( p_{\text{tot}} \) is the perturbation of the total pressure \( p_i \).

Assuming that all perturbations are proportional to \( g(r) \exp \left[ i(\omega t + m \phi + k z) \right] \) with \( g(r) \) being just a function of \( r \), from the above set of equations one obtains the following equations:

\[
-i \sigma v_r - 2 \frac{U_\phi}{r} v_\phi - i \frac{f_\phi}{\mu_\phi} b_r + \frac{2 B_\phi}{\mu_\phi r} b_\phi = -i \frac{1}{\rho_i} \frac{dp_{\text{tot}}}{dr},
\]

(12)

\[
-i \sigma v_\phi + \frac{1}{r} \frac{d}{dr} \left( r U_\phi \right) v_r - i \frac{f_\phi}{\mu_\phi} b_\phi - \frac{1}{\mu_\phi r} \frac{d}{dr} \left( r B_\phi \right) b_r = -i \frac{1}{\rho_i} \frac{m}{r} p_{\text{tot}},
\]

(13)

\[
-i \sigma v_z - i \frac{f_\phi}{\mu_\phi} b_z = -i \frac{1}{\rho_i} k z p_{\text{tot}},
\]

(14)

\[
-i \sigma b_r - i f_\phi v_\phi = 0,
\]

(15)

\[
-i \sigma b_\phi - r \frac{d}{dr} \left( \frac{U_\phi}{r} \right) b_r - i f_\phi v_\phi + r \frac{d}{dr} \left( \frac{B_\phi}{r} \right) v_r = 0,
\]

(16)
\[
-\sigma i b_z - if_B v_z = 0, \quad (17)
\]
\[
\left( \frac{d}{dr} + \frac{1}{r} \right) v_r + \frac{i m}{r} v_\phi + ik_z v_z = 0, \quad (18)
\]
where
\[
\sigma = \omega - \frac{m}{r} U_\phi - k_z U_z \quad (19)
\]
is the Doppler-shifted frequency and
\[
f_B = \frac{m}{r} B_\phi + k_z B_z. \quad (20)
\]
It is now convenient to introduce the Lagrangian displacement, \( \xi \), and express it via the fluid velocity perturbation through the relation (Chandrasekhar, 1961)
\[
v = \frac{\partial \xi}{\partial t} + (U \cdot \nabla) \xi - (\xi \cdot \nabla) U, \quad \text{which yields}
\]
\[
v_r = -\sigma i \xi_r, \quad v_\phi = -\sigma i \xi_\phi - \frac{d}{dr} \left( \frac{U_\phi}{r} \right) \xi_r, \quad v_z = -i \sigma \xi_z. \quad (21)
\]
In terms of \( \xi \), Equations (12)–(18) can be rewritten as
\[
\left[ \sigma^2 - \omega_{Al}^2 - r \frac{d}{dr} \left( \frac{U_\phi^2}{r^2} \right) + \frac{1}{\mu \rho_i} \frac{d}{dr} \left( \frac{B_\phi^2}{r^2} \right) \right] \xi_r - 2i \left( \sigma - \omega_{Al} \right) \frac{U_\phi}{r} \xi_r = \frac{1}{\rho_i} \frac{d p_{tot}}{dr}, \quad (22)
\]
\[
\left( \sigma^2 - \omega_{Al}^2 \right) \xi_\phi + 2i \left( \frac{U_\phi}{r} + \frac{B_\phi f_B}{\mu \rho_i} \right) \xi_r = i \frac{m}{\rho_i} r \xi_\phi \xi_r, \quad (23)
\]
\[
\left( \sigma^2 - \omega_{Al}^2 \right) \xi_z + i \frac{m}{r} \xi_\phi + ik_z \xi_z = 0, \quad (24)
\]
where
\[
\omega_{Al} = \sqrt{\frac{f_B}{\mu \rho_i}}. \quad (26)
\]
is the local Alfvén frequency inside the jet.
Excluding \( \xi_\phi \) and \( \xi_z \) from these equations, we obtain
\[
\rho_i \left( \sigma^2 - \omega_{Al}^2 \right) \frac{d \xi_\phi}{dr} + \xi_\phi = 2 \rho_i d_z \frac{m}{r} \xi_r = \left( \frac{m^2}{r^2} + k_z^2 \right) p_{tot}, \quad (27)
\]
\[
\rho_i d_z \xi_r = \left( \sigma^2 - \omega_{Al}^2 \right) \frac{d p_{tot}}{dr} - 2 \frac{m}{r} d_z p_{tot}. \quad (28)
\]
where

\[ d_1 = \left( \sigma^2 - \omega_{\Lambda i}^2 \right)^3 - r \left( \sigma^2 - \omega_{\Lambda i}^2 \right) \left[ \frac{d}{dr} \left( \frac{U_\phi^2}{r^2} \right) - \frac{1}{\mu \rho_i} \frac{d}{dr} \left( \frac{B_{\phi i}^2}{r^2} \right) \right] - 4d_2^2, \quad d_2 = \sigma \frac{U_\phi}{r} + \frac{B_{\phi i} f_i}{\mu \rho_i} \frac{r}{r} \]

By presenting \( \xi_i \) from Equation (28) in terms of \( p_{tot} \) and inserting it into Equation (27) we obtain the following equation for the total pressure perturbation:

\[
\left[ \left( \sigma^2 - \omega_{\Lambda i}^2 \right) \frac{d}{dr} + \frac{\sigma^2 - \omega_{\Lambda i}^2 + 2md_2}{r} \right] \times \left[ \frac{\sigma^2 - \omega_{\Lambda i}^2}{d_1} \frac{dp_{tot}}{dr} - \frac{2md_2 p_{tot}}{d_1 r} \right] - \left( \frac{m^2}{r^2} + \kappa_i^2 \right) p_{tot} = 0. \tag{29}
\]

This equation can be significantly simplified by considering that the rotation and the magnetic twist of the jet are uniform, that is,

\[ U_\phi(r) = \Omega r \quad \text{and} \quad B_{\phi i}(r) = A r, \tag{30} \]

where \( \Omega \) and \( A \) are constants. In such a case Equation (29) takes the form of the Bessel equation

\[
\frac{d^2 p_{tot}}{d r^2} + \frac{1}{r} \frac{dp_{tot}}{dr} - \left( \frac{m^2}{r^2} + \kappa_i^2 \right) p_{tot} = 0, \tag{31}
\]

where

\[ \kappa_i^2 = k_i^2 \left[ 1 - 4 \left( \frac{\sigma \Omega + A \omega_{\Lambda i}/\sqrt{\mu \rho_i}}{\sigma^2 - \omega_{\Lambda i}^2} \right)^2 \right]. \tag{32} \]

The equation governing plasma dynamics outside the jet (without twist and velocity field, that is, \( A = 0, \Omega = 0, \) and \( U_\phi = 0 \)) is the same Bessel equation, but \( \kappa_i \) is replaced by \( k_i \).

Inside the jet, the solution to Equation (31) bounded on the jet axis is the modified Bessel function of the first kind,

\[ p_{tot}(r \leq a) = \alpha_i I_{\kappa_i}(\kappa_i r), \tag{33} \]

where \( \alpha_i \) is a constant.

Outside the jet, the solution bounded at infinity is the modified Bessel function of the second kind,

\[ p_{tot}(r > a) = \alpha_e K_{\kappa_i}(\kappa_i r), \tag{34} \]

where \( \alpha_e \) is a constant.

To get the dispersion equation governing the propagation of MHD modes along the jet, one needs to merge the solutions at the jet surface through boundary conditions.

It is well-known that for non-rotating and untwisted magnetic flux tubes the boundary conditions are the continuity of Lagrangian radial displacement and total pressure perturbation at the tube surface (Chandrasekhar, 1961), that is,

\[ \xi_i |_{r=a} = \xi_e |_{r=a} \quad \text{and} \quad p_{tot} |_{r=a} = p_{tot e} |_{r=a}, \tag{35} \]
where total pressure perturbations \( p_{\text{tot}i} \) and \( p_{\text{tot}e} \) are given by Equations (33) and (34), respectively.

When the magnetic flux tube is twisted and still non-rotating and the twist has discontinuity at the tube surface, then the Lagrangian total pressure perturbation is continuous and the boundary conditions are (see, e.g., Bennet et al., 1999; Zaqarashvili et al., 2010; Zaqarashvili et al., 2014)

\[
\frac{\partial \xi_r}{\partial r} |_{r=a} = \frac{\partial \xi_\theta}{\partial r} |_{r=a} = 0, \quad \text{and} \quad p_{\text{tot}i} - \frac{B_{\text{tot}i}^2}{\mu a} \xi_r |_{r=a} = p_{\text{tot}e} \xi_r |_{r=a}. \tag{36}
\]

The second term in the second boundary condition stands for the pressure from the magnetic tension force.

If a non-twisted \( (B_\phi = 0) \) tube rotates and the rotation has discontinuity at the tube surface, the boundary condition for the Lagrangian total pressure perturbation has similar form as the second boundary condition in Equation (36)

\[
\frac{\partial \xi_r}{\partial r} |_{r=a} = \frac{\partial \xi_\theta}{\partial r} |_{r=a} = 0, \quad \text{and} \quad p_{\text{tot}i} + \frac{\rho_i U_\phi^2}{a} \xi_r |_{r=a} = p_{\text{tot}e} \xi_r |_{r=a}. \tag{37}
\]

Here, the second term that describes the contribution of centrifugal force in the pressure balance can be derived from Equation (28) for \( B_\phi = 0 \) by multiplying that equation by \( dr \), and considering the limit of \( dr \to 0 \) through the boundary \( r = a \), one obtains the relation \( d \left[ p_{\text{tot}} + \left( \rho_i U_\phi^2 / a \right) \xi_r \right] = 0 \), or, equivalently, the second boundary condition in the above equation. Hence, the boundary condition for the Lagrangian total pressure perturbation in rotating and magnetically twisted flux tubes has the form

\[
p_{\text{tot}i} + \left( \frac{\rho_i U_\phi^2}{a} - \frac{B_{\text{tot}i}^2}{\mu a} \right) \xi_r |_{r=a} = p_{\text{tot}e} \xi_r |_{r=a}. \tag{38}
\]

In the case of uniform rotation and magnetic field twist, Equation (30), the boundary conditions for Lagrangian radial displacement \( \xi_r \) and total pressure perturbation \( p_{\text{tot}} \) are

\[
\frac{\partial \xi_r}{\partial r} |_{r=a} = \frac{\partial \xi_\theta}{\partial r} |_{r=a} = 0, \quad \text{and} \quad p_{\text{tot}i} + a \left( \rho_i \Omega^2 - \frac{A^2}{\mu} \right) \xi_r |_{r=a} = p_{\text{tot}e} \xi_r |_{r=a}. \tag{39}
\]

Using these boundary conditions we obtain the dispersion equation of normal MHD modes propagating in rotating and axially moving twisted magnetic flux tubes (Zaqarashvili et al., 2015)

\[
\frac{\left( \sigma^2 - \omega_{\lambda i}^2 \right) F_m(k_a) - 2m \left( \sigma \Omega + A \lambda_i / \sqrt{\mu \rho_i} \right) \rho_i \left( \sigma^2 - \omega_{\lambda i}^2 \right)^2 - 4 \mu_i \left( \sigma \Omega + A \lambda_i / \sqrt{\mu \rho_i} \right) \right)^2}{\rho_e \left( \sigma^2 - \omega_{\lambda e}^2 \right) - \left( \rho_i \Omega^2 - A^2 / \mu \right) P_m(k_a)} = \frac{P_m(k_a)}{P_m(k_a)} \tag{40},
\]

where

\[
F_m(k_a) = \frac{\kappa_A k_{\lambda i} \xi_r(k_a)}{l_m(k_a)}, \quad P_m(k_a) = \frac{k_{\lambda i} K_m(k_a)}{K_{\lambda i} - K_{\lambda e}} \xi_r(k_a) \quad \text{and} \quad \omega_{\lambda e} = \frac{k_{\lambda e} B_e}{\sqrt{\mu \rho_e}}.
\]
We note that in the case of non-rotating twisted tube ($\Omega = 0$) above Equation (40) recovers the well-known dispersion relation of normal MHD modes propagating in cylindrical twisted jets (see, for instance, Zhelyazkov and Zaqarashvili, 2012a). If the environment medium is cool plasma as is the case of our macrospicule, where the thermal pressure $p_\text{e} = 0$, the $k_a$ in $P_m(k_a)$ must be replaced by $k_a[1 - (\omega/\omega_{Ae})^2]^{1/2}$, which yields the used wave dispersion relation (3).

Acknowledgments Our work was supported by the Bulgarian Science Fund under project DNTS/INDIA 01/7. The authors are indebted to the anonymous reviewer for pointing out them a mathematical error and for his/her helpful and constructive comments and suggestions that contributed to improving the final version of the manuscript.

Disclosure of Potential Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

Ajabshirizadeh, A., Ebadi, H., Vekalati, R.E., Molaverdikhani, K.: 2015, The possibility of Kelvin-Helmholtz instability in solar spicules. Astrophys. Space Sci. 367, 33.

Banerjee, D., O’Shea, E., Doyle, J.G.: 2000, Giant macro-spicule as observed by CDS on SOHO. Astron. Astrophys. 355, 1152.

Bennett, K., Roberts, B., Narain, U.: 1999, Waves in Twisted Magnetic Flux Tubes. Sol. Phys. 185, 41.

Bennett, S.M., Erdelyi, R.: 2015, On the Statistics of Macrospicules. Astrophys. J. 808, 135.

Bohlin, J.D., Vogel, S.N., Purcell, J.D., Sheeley, Jr., N.R., Tousey, R., VanHoosier, M.E.: 1975, A Newly Observed Solar Feature: Macrospicules in He II 304 A. Astrophys. J. 197, L133.

Chandrasekhar, S.: 1961, Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, Chap. 11.

Culhane, J.L., Harra, L.K., James, A.M., et al.: 2007, The EUV Imaging Spectrometer for Hinode. Sol. Phys. 243, 19.

De Pontieu, B., McIntosh, S., Hansteen, V.H., et al.: 2007, A Tale of Two Spicules: The Impact of Spicules on the Magnetic Chromosphere. Proc. Astron. Soc. Japan 59, S655.

De Pontieu, B., Rouppe van der Voort, L., McIntosh, S.W., et al.: 2014, On the prevalence of small-scale twist in the solar chromosphere and transition region. Science 346, 1255372.

Domingo, V., Fleck, B., Poland, A.I.: 1995, SOHO: The Solar and Heliospheric Observatory. Space Sci. Rev. 72, 81.

Ebadi, H.: 2016, Kelvin-Helmholtz instability in solar spicules. Iranian J. Phys. Res. 355, 1152.

Goossens, M., Hollweg, J., Sakurai, T.: 1992, Resonant behaviour of MHD waves on magnetic flux tubes. III. Effect of equilibrium twist. Sol. Phys. 138, 233.

Harrison, R.A., Fluidra, A., Pike, C.D., Payne, J., Thompson, W.T., Poland, A.I., Breeveld, E.R., Breeveld, A.A., Culhane, J.L., Kjeldseth-Moe, O., Huber, M.C.E., Aschenbach, B.: 1997, High-Resolution Observations of the extreme ultraviolet Sun. Sol. Phys. 170, 123.

Howard, T.A., Moses, J.D., Vourlidas, A., et al.: 2008, Sun Earth Connection Coronal and Heliospheric Investigation (SECCHI). Space Sci. Rev. 136, 67.

Iijima, H., Yokoyama, T.: 2017, Three-dimensional Magnetohydrodynamic Simulation of the Formation of Solar Chromospheric Jets with Twisted Magnetic Field Lines. Astrophys. J. 848, 38.

Kaiser, M.L., Kucera, T.A., Davila, J.M., et al.: 2008, The STEREO Mission: An Introduction. Space Sci. Rev. 136, 5.

Kayshap, P., Srivastava, A.K., Munawar, K., Tripathi, D.: 2013, Origin of Macrospicule and Jet in Polar Corona by a Small-Scale Kinked Flux Tube. Astrophys. J. 770, L3.

Kiss, T.S., Gyenge, N., Erdelyi, R.: 2017, Systematic Variations of Macrospicule Properties Observed by SDO/AIA over Half a Decade. Astrophys. J. 835, 47.

Kosugi T., Matsuzaki, K., Sakao, T., et al.: 2007, The Hinode (Solar-B) Mission: An Overview. Sol. Phys. 243, 3.

Kuridze, D., Zaqarashvili, T.V., Henriquez, V., Mathioudakis, M., Keenan, F.P., Hanslmeier, A.: 2016, Kelvin–Helmholtz Instability in Solar Chromospheric Jets: Theory and Observation. Astrophys. J. 830, 133.
Lemen, J.R., Title, A.M., Akin, D.J., et al.: 2012, The Atmospheric Imaging Assembly (AIA) on the Solar Dynamics Observatory (SDO). Sol. Phys. 275, 17.
Lehoda, I.P., Bogachev, S.A.: 2017, Plasma dynamics in solar macrospicules from high-cadence extreme-UV observations. Sol. Phys. 597, A78.
Madjarska, M.S., Vannanathan, K., Doyle, J.G.: 2011, Can coronal hole spicules reach coronal temperatures? Astron. Astrophys. 532, L1.
Miura, A.: 1984, Anomalous transport by magnetohydrodynamic Kelvin-Helmholtz instabilities in the solar wind-magnetosphere interaction. J. Geophys. Res. 89, 801.
Murawski, K., Srivastava, A.K., Zaqarashvili, T.V.: 2011, Numerical simulations of solar macrospicules. Astron. Astrophys. 535, A58.
Parenti, S., Bromage, B.J.I., Bromage, G.E.: 2002, An erupting macrospicule: Characteristics derived from SOHO-CDS spectroscopic observations. Astron. Astrophys. 384, 303.
Pessnell, W.D., Thompson, B.J., Chamberlin, P.C.: 2012, The Solar Dynamics Observatory (SDO). Sol. Phys. 275, 3.
Pike, C.D., Harrison, R.A.: 1997, Euv Observations of a Macrospicule: Evidence for Solar Wind Acceleration? Sol. Phys. 175, 457.
Pike, C.D., Mason, H.E.: 1998, Rotating Transition Region Features Observed with the SOHO Coronal Diagnostic Spectrometer. Sol. Phys. 182, 333.
Ryu, D., Jones, T.W., Frank, A.: 2000, The Magnetohydrodynamic Kelvin-Helmholtz Instability: A Three-Dimensional Study of Nonlinear Evolution. Astrophys. J. 545, 475.
Scullion, E., Doyle, J.G., Erdélyi, R.: 2010, A spectroscopic analysis of macrospicules. Mem. S.A.It. 81, 737.
Wilhelm, K., Curdt, W., Marsch, E., et al.: 1995, SUMER - Solar Ultraviolet Measurements of Emitted Radiation. Solar Phys. 162, 189.
Zank, G.P., Matthaeus, W.H.: 1993, Nearly incompressible fluids. II: Magnetohydrodynamics, turbulence, and waves. Phys. Fluids 5, 257.
Zaqarashvili, T.V., Erdélyi, R. 2009, Oscillations and Waves in Solar Spicules. Space Sci. Rev. 149, 355.
Zaqarashvili, T.V., Díaz, A.J., Oliver, R., Ballester, J.L.: 2010, Instability of twisted magnetic tubes with axial mass flows. Astron. Astrophys. 516, A84.
Zaqarashvili, T.V., Vörös, Z., Zhelyazkov, I.: 2014, Kelvin-Helmholtz instability of twisted magnetic flux tubes in the solar wind. Astron. Astrophys. 561, A62.
Zaqarashvili, T.V., Zhelyazkov, I., Ofman, L.: 2015, Stability of Rotating Magnetized Jets in the Solar Atmosphere. I. Kelvin–Helmholtz Instability. Astrophys. J. 813, 123.
Zhelyazkov, I.: 2012, Magnetohydrodynamic waves and their stability status in solar spicules. Astron. Astrophys. 537, A124.
Zhelyazkov, I., Zaqarashvili, T.V.: 2012a, Kelvin-Helmholtz instability of kink waves in photospheric twisted flux tubes. Astron. Astrophys. 547, A14.
Zhelyazkov, I.: 2013, Kelvin–Helmholtz Instability of Kink Waves in Photospheric, Chromospheric, and X-Ray Solar Jets. In: Zhelyazkov, I., Mishonov, T. (eds) Space Plasma Physics: Proceedings of the 4th School and Workshop on Space Plasma Physics. AIP Conf. Proc. 1551, 150.
Zhelyazkov, I., Chandra, R., Srivastava, A.K.: 2016, Kelvin–Helmholtz instability in an active region jet observed with Hinode. Astrophys. Space Sci. 361, 51.
Zhelyazkov, I., Zaqarashvili, T.V., Ofman, L., Chandra, R.: 2018a, Kelvin–Helmholtz instability in a twisting solar polar coronal hole jet observed by SDO/AIA. Adv. Space Res. 61, 628.
Zhelyazkov, I., Chandra, R.: 2018b, High mode magnetohydrodynamic waves propagation in a twisted rotating jet emerging from a filament eruption. Mon. Not. R. Astron. Soc. 478, 5505.