ON THE CLOSURE OF TRANSLATION-DILATION INVARIANT LINEAR SPACES OF POLYNOMIALS

J. M. ALMIRA AND L. SZÉKELYHIDI

Abstract. Assume that a linear space of real polynomials in \( d \) variables is given which is translation and dilation invariant. We show that if a sequence in this space converges pointwise to a polynomial, then the limit polynomial belongs to the space, too.

1. Introduction

At the 49th International Symposium on Functional Equations in Graz, Maria-terost, Austria, 2011 and later at the 14th International Conference on Functional Equations and Inequalities in Będlewo, Poland, 2011 the second author proposed the following problem: Assume that \( V \) is a linear space of real polynomials in \( n \) variables which is translation invariant. Suppose moreover that the sequence \( (p_n) \) in \( V \) converges pointwise to a polynomial \( p \). Is it true that \( p \) is in \( V \)? Despite several efforts of different researchers this question has still remained open. In this note we solve the problem in the positive for the special case when \( V \) is a translation-dilation invariant linear space of polynomials.

2. Translation and dilation invariant subspaces

In this paper \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}[x] \) denotes the polynomial ring in the variables \( x = (x_1, x_2, \ldots, x_d) \), where \( d \) is a positive integer. We shall use the following standard notation. For every \( a = (a_1, a_2, \ldots, a_d) \) and \( x = (x_1, x_2, \ldots, x_d) \) in \( \mathbb{R}^d \) we let \( a \cdot x = (a_1 x_1, a_2 x_2, \ldots, a_d x_d) \). For every multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \) in \( \mathbb{N}^d \) we write \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \) and \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \), where we use the convention \( 0^0 = 1 \). For convenience the monomial \( x \mapsto x^\alpha \) will be denoted by \( q_\alpha \) for each multi-index \( \alpha \). Also we introduce the notation \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_d! \) for every multi-index \( \alpha \) and we shall use the partial order \( \alpha \preceq \beta \) componentwise. We shall use the notation \( \mathbb{N} \) for the set \( \mathbb{N} \cup \{0\} \) and we extend the order relation \( \leq \) from \( \mathbb{N} \) to \( \mathbb{N}^d \), as well as the partial order relation \( \preceq \) from \( \mathbb{N}^d \) to \( \mathbb{N}^d \) in the obvious manner. The initial section corresponding to \( \alpha \) in \( \mathbb{N}^d \) with respect to the partial order \( \leq \) will be denoted by \( [\alpha] \). In other words

\[
[\alpha] = \{ \beta \in \mathbb{N}^d : \beta \preceq \alpha \}.
\]

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We also introduce the spaces \( B_{[\alpha]} = \text{span}\{x^\beta : \beta \in [\alpha]\} \). In addition we shall use the standard notation for partial differential operators

\[
P(\partial) = \sum_\alpha c_\alpha \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d}
\]

whenever \( P : \mathbb{R}^d \to \mathbb{R} \) is the polynomial \( P(x) = \sum_\alpha c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d} \). In particular, we have \( q_\alpha(\partial) = \partial^\alpha \).

Also we shall use difference operators with multi-index notation. For every \( k = 1, 2, \ldots, d \) the symbol \( \Delta_k \) denotes the partial difference operator on polynomials acting on the \( k \)-th variable with increment 1, that is

\[
\Delta_k p(x) = p(x + e_k) - p(x)
\]

for each \( x \), where \( e_k \) is the element in \( \mathbb{R}^d \) whose \( k \)-th component is 1, all the others are 0. In addition, \( e \) is the element whose all components are 1. Then \( \Delta \) is the vector difference operator defined by \( \Delta = (\Delta_1, \Delta_2, \ldots, \Delta_d) \). Using this notation we have

\[
\Delta^\alpha p = \Delta_1^{\alpha_1} \Delta_2^{\alpha_2} \ldots \Delta_d^{\alpha_d} p
\]

for every multi-index \( \alpha \). Then the meaning of \( P(\Delta) \) is obvious for every polynomial \( P \) in \( d \) variables. In particular, we can write \( q_\alpha(\Delta) = \Delta^\alpha \).

Given a function \( f : \mathbb{R}^d \to \mathbb{R} \) and \( y \) in \( \mathbb{R}^d \) we denote by \( \tau_y \), resp. \( \sigma_y \) the functions defined by

\[
\tau_y f(x) = f(x + y), \quad \sigma_y f(x) = f(x \cdot y)
\]

whenever \( y \) is in \( \mathbb{R}^d \). We call \( \tau_y \), resp. \( \sigma_y \) translation, resp. dilation by \( y \), further \( \tau_y f \), resp. \( \sigma_y f \) the translate, resp. the dilate of \( f \) by \( y \). A set \( H \) of real functions on \( \mathbb{R}^d \) is called translation invariant, resp. dilation invariant, if \( \tau_y f \) is in \( H \), resp. \( \sigma_y f \) is in \( H \) for each \( f \) in \( H \) and for every \( y \) in \( \mathbb{R}^d \). If \( H \) is both translation and dilation invariant, then we call it translation-dilation invariant, or shortly a TDI-set. Given a function \( f \) on \( \mathbb{R}^d \) the intersection of all translation invariant, resp. dilation invariant, resp. translation-dilation invariant linear spaces including \( f \) is denoted by \( \tau(f) \), resp. \( \sigma(f) \), resp. \( \tau\sigma(f) \). Clearly, these are linear spaces, and \( \tau(f) \) is translation invariant, \( \sigma(f) \) is dilation invariant, further \( \tau\sigma(f) \) is translation-dilation invariant. TDI-subspaces have been studied extensively and the following classification of closed TDI-subspaces has been proved in [4].

**Theorem 1.** Every closed subspace of \( C(\mathbb{R}^d) \) which is invariant under all mappings \( S_{a,b} = \sigma_a \tau_b \) with arbitrary \( a,b \) in \( \mathbb{R}^d \) is the closure of the linear span of the union of the sets \( B_{[n_k]} \) for a finite set of points \( n_k \) in \( \mathbb{N}^d \).

We note that here "closed" refers to the topology of the uniform convergence on compact sets.

In this paper we study translation-dilation linear spaces of polynomials. We begin with some preliminary lemmas. The first one is a standard result in Algebra [3 Chapter 5], but we include the proof for the sake of completeness.

**Lemma 2.** Let \( S \) be a set of different multi-indices in \( \mathbb{N}^d \). Then the set of monomials \( \{q_\alpha : \alpha \in S\} \) is linearly independent.

**Proof.** Indeed, if \( c_\alpha \) is a real number for each \( \alpha \) in \( S \) such that we have

\[
\sum_\alpha c_\alpha q_\alpha(x) = 0
\]
for each $x$ in $\mathbb{R}^d$, and $\alpha_0$ is any element in $S$, then we apply the differential operator $\partial^\alpha_0$ to the above equation. We obtain

$$
\sum_{\alpha\in S} c_\alpha \partial_\alpha q_\alpha(x) = \sum_{\alpha\geq\alpha_0, \alpha\in S} c_\alpha \frac{\alpha!}{\alpha_0!} x^{\alpha-\alpha_0} = 0
$$

for each $x$ in $\mathbb{R}^d$. Now we substitute $x = 0$. If for some $\alpha$ in $S$, $\alpha > \alpha_0$, then the corresponding term is zero, hence the above sum is equal to $c_{\alpha_0} \frac{\alpha_0!}{\alpha_0!}$ which implies $c_{\alpha_0} = 0$.

**Lemma 3.** Let $p : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. Then $\tau(p)$ is generated by all partial derivatives of $p$.

**Proof.** The statement follows immediately from the Taylor Formula

$$
\tau_y p = \sum_{|\alpha| \leq \deg p} \frac{1}{\alpha!} \partial^\alpha p q_\alpha(y)
$$

which holds for each polynomial in $\mathbb{R}[x]$ and for every $x$ in $\mathbb{R}^d$. Indeed, this formula shows that every translate of $p$ is a linear combination of partial derivatives of $p$. On the other hand, let $s$ denote the number of different multi-indices $\alpha$ with $\alpha \leq \deg p$.

As the monomials $q_\alpha$ are linearly independent for different multi-indices $\alpha$ there exist elements $y_j$ for $j = 1, 2, \ldots, s$ such that the quadratic matrix $(q_\alpha(y_j))$ is regular (see, for example, [1, Chapter 14]). Substituting $y_j$ for $y$ in the above equation we obtain a system of linear equations with regular matrix from which we can express $\partial^\alpha p$ as a linear combination of the translates $\tau_{y_j} p$ of $p$, hence all these partial derivatives belong to $\tau(p)$.

**Lemma 4.** Let $p : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. Then $\sigma(p)$ is generated by all monomials of the form $\partial^\alpha p(0)x^\alpha$.

**Proof.** We use Taylor Formula [1] again. We have for $y = 0$

$$
p(x) = \sum_{|\alpha| \leq \deg p} \frac{1}{\alpha!} \partial^\alpha p(0)x^\alpha
$$

which means that it is enough to show that every monomial term on the right side is in $\sigma(p)$. In other words, we have to show that $\partial^\alpha p(0)q_\alpha$ is in $\sigma(p)$ for each $\alpha$ with $|\alpha| \leq \deg p$. By [2], we have

$$
\sigma_\lambda p = \sum_{|\alpha| \leq \deg p} \frac{1}{\alpha!} q_\alpha(\lambda) \partial^\alpha p(0)q_\alpha
$$

for each $\lambda$ in $\mathbb{R}^d$. As the functions $\frac{1}{\alpha!} q_\alpha$ are linearly independent for different multi-indices $\alpha$ there exist elements $\lambda_j$ for $j = 1, 2, \ldots, s$ such that the quadratic matrix $(\frac{1}{\alpha!} q_\alpha(\lambda_j))$ is regular. Substituting $\lambda_j$ for $\lambda$ in the above equation we obtain a system of linear equations with regular matrix from which we can express $\partial^\alpha p(0)q_\alpha$ as a linear combination of the dilates $\sigma_{\lambda_j} p$ of $p$, hence all these monomials belong to $\sigma(p)$.

**Corollary 5.** A linear space of real polynomials in several variables is dilation invariant if and only if it admits a basis formed by monomials.

**Proof.** Obvious.
Corollary 6. Let \( p : \mathbb{R}^d \to \mathbb{R} \) be a polynomial. Then \( \tau \sigma(p) \) is generated by all monomials of the form \( x^\beta \) such that, for a certain multi-index \( \alpha, \beta \preceq \alpha \) and \( \partial^\alpha p(0) \neq 0 \). In particular, if \( \partial^\alpha p(0) \neq 0 \), then \( \tau \sigma(p) \) includes \( x^\beta \) for each \( \beta \) with \( \beta \preceq \alpha \).

Proof. Obvious, by Lemma 3 and Lemma 4.

We introduce the following notation: for a subset \( H \) of \( \mathbb{R}[x] \) we let
\[
\Omega_H = \{ \alpha \in \mathbb{N}^d : x^\alpha \text{ is in } H \}.
\]

Our main theorem follows.

Theorem 7. If a sequence in a TDI-subspace in the polynomial ring \( \mathbb{R}[x] \) converges pointwise to a polynomial, then this polynomial belongs to the subspace, too.

Proof. Let us first assume that \( V \) is a TDI-subspace of \( \mathbb{R}[x] \) such that there exist natural numbers \( N_1, \ldots, N_d \) satisfying
\[
\Omega_V = \bigcup_{k=1}^d Z_k,
\]
where \( Z_k = \{ \alpha \in \mathbb{N}^d : \alpha_i \leq N_i \} \) for \( i = 1, 2, \ldots, d \).

Let the sequence \( (p_n)_{n \in \mathbb{N}} \) of the TDI-subspace \( V \) in \( \mathbb{R}[x] \) converge pointwise to the polynomial \( p \). Our assumption on \( \Omega_V \) implies that, for \( n = 0, 1, \ldots \) we can write \( p_n \) in the following form
\[
p_n(x_1, x_2, \ldots, x_d) = \sum_{k=1}^d \sum_{i=0}^{N_k} f_{n,k,i}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d) x_k^i
\]
with some polynomials \( f_{n,k,i} \) arbitrary polynomials in \( d - 1 \) variables.

Proving by contradiction we assume that \( p \) is not in \( V \). We shall use the following notation: for \( k = 1, 2, \ldots, d \) let \( e_k \) the element of \( \mathbb{R}^d \) whose \( k \)-th component is 1, all the others are 0.

For each \( x \in \mathbb{R}^d \) we have
\[
\lim_{n \to \infty} \sum_{k=1}^d \sum_{i=0}^{N_k} f_{n,k,i}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d) x_k^i = p(x_1, x_2, \ldots, x_d)
\]
and \( \partial^\alpha p(0) \neq 0 \) for some \( \alpha > N \). We apply the difference operator \( \Delta^N \) on both sides of this equation. It is easy to see that
\[
\Delta^N p(x_1, x_2, \ldots, x_d) = x_1 x_2 \cdots x_d \cdot h(x_1, x_2, \ldots, x_d)
\]
with some nonzero \( h \) in \( \mathbb{R}[x] \). On the other hand, on the left hand side we have
\[
\lim_{n \to \infty} \sum_{k=1}^d \sum_{i=0}^{N_k} \Delta^N f_{n,k,i}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d) x_k^i = \lim_{n \to \infty} \sum_{k=1}^d F_{n,k}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)
\]
with some polynomials \( F_{n,k} \) for \( n = 0, 1, \ldots \) and \( k = 1, 2, \ldots, d \). Now we substitute successively \( x_j = 1 \) for \( j = 1, 2, \ldots, d \) and we let \( n \to \infty \) to obtain
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\[ x_2 \cdot x_3 \cdots x_d h(1, x_2, \cdots, x_d) = \]
\[ \lim_{n \to \infty} F_{n,1}(x_2, x_3, \ldots, x_d) + F_{n,2}(1, x_3, \ldots, x_d) + \cdots + F_{n,d}(1, x_2, \ldots, x_d-1) \]
\[ x_1 \cdot x_3 \cdots x_d \ h(x_1, 1, \ldots, x_d) = \]
\[ \lim_{n \to \infty} F_{n,1}(1, x_3, \ldots, x_d) + F_{n,2}(1, x_3, \ldots, x_d) + \cdots + F_{n,d}(x_1, \ldots, x_d-1) \]
\[ \vdots \]
\[ x_1 \cdot x_2 \cdots x_{d-1} \ h(x_1, \ldots, x_{d-1}, 1) = \]
\[ \lim_{n \to \infty} F_{n,1}(x_2, \ldots, x_{d-1}, 1) + F_{n,2}(x_1, x_3, \ldots, x_{d-1}, 1) + \cdots + F_{n,d}(x_1, \ldots, x_{d-1}) \]

We can write this system of equations in the more compact form

\[ \lim_{n \to \infty} \sum_{k=1}^{d} F_{n,k}(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{k-1}, \bar{x}_k, x_{k+1}, \ldots, x_d) = \]
\[ x_1 \cdot x_2 \cdots \hat{x}_j \cdots x_d \ h(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_d) \]
for \( j = 1, 2, \ldots, d \) where \( \hat{x}_j \) means that \( x_j = 1 \) and \( \bar{x}_k \) means that the variable \( x_k \)
is missing. Now we sum up these equations for \( j = 1, 2, \ldots, d \). On the left hand side we recover the sum

\[ \sum_{k=1}^{d} F_{n,k}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d), \]

which has the limit \( x_1 x_2 \cdots x_d \cdot h(x_1, x_2, \ldots, x_d) \) as \( n \) tends to infinity, and a sum \( w_n(x_1, x_2, \ldots, x_d) \) of polynomials each of them depending on \( d-2 \) out of the \( d \) variables only. Obviously, we have

\[ g(x_1, \ldots, x_d) = \lim_{n \to \infty} w_n(x_1, x_2, \ldots, x_d) = \]
\[ \sum_{j=1}^{d} \left( h(x_1, \ldots, x_{j-1}, 1, x_j, \ldots, x_d) \prod_{k \neq j} x_k \right) - h(x_1, \ldots, x_d) \prod_{k=1}^{d} x_k. \]

As every term of the sum defining \( w_n \) depends on \( d-2 \) variables only, it follows that with \( \beta = (1, 1, \ldots, 1, 0) \) we have \( \Delta^\beta g = \lim_{n \to \infty} \Delta^\beta w_n = 0 \). As \( g \) is a polynomial, this implies \( \partial^\beta g = 0 \). On the other hand, the assumption \( h \neq 0 \) implies that

\[ h(x_1, \ldots, x_d) = \sum_{k=0}^{s} c_k(x_1, x_2, \cdots, x_{d-1}) x_d^k \]

for some natural number \( s \) and polynomials \( c_k \) in \( d-1 \) variables for \( k = 1, 2, \ldots, s \), not all of them being identically zero. A simple computation shows that

\[ 0 = \partial^\beta (x_1 x_2 \cdots x_{d-1} h(x_1, \cdots, x_{d-1}, 1) - x_1 x_2 \cdots x_d h(x_1, \cdots, x_d)), \]

hence, substituting equation (4) into this formula, we obtain

\[ 0 = \partial^\beta \left( \sum_{k=0}^{s} x_1 x_2 \cdots x_{d-1} c_k(x_1, x_2, \cdots, x_{d-1}) \right) \]
\[ - \sum_{k=0}^{s} \partial^\beta (x_1 x_2 \cdots x_{d-1} c_k(x_1, x_2, \cdots, x_{d-1})) x_d^{k+1}. \]
It follows that
\[ \partial^\beta (x_1 x_2 \cdots x_{d-1} c_k(x_1, x_2, \cdots, x_{d-1})) = 0, \]
for \( k = 0, 1, \ldots, s \). Now we assume that
\[ c_k(x_1, \cdots, x_{d-1}) = \sum_{|\alpha| \leq t} a_{\alpha, k} x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} \]
is not identically zero. Then
\[ x_1 x_2 \cdots x_{d-1} c_k(x_1, x_2, \cdots, x_{d-1}) = \sum_{|\alpha| \leq t} a_{\alpha, k} x_1^{\alpha_1+1} \cdots x_{d-1}^{\alpha_{d-1}+1}, \]
and
\[ \partial^\beta (x_1 x_2 \cdots x_{d-1} c_k(x_1, x_2, \cdots, x_{d-1})) = \sum (\alpha_1 + 1) \cdots (\alpha_{d-1} + 1) a_{\alpha, k} x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}}, \]
which vanishes identically if and only if all coefficients \( a_{\alpha, k} \) are zero. Hence the polynomial \( h \) vanishes identically, which contradicts our assumptions. This proves the result for the very special case when \( \Omega \) admits a decomposition of the form
\[ \Omega = \bigcup_{k=1}^d Z_i, \]
where \( Z_i = \{ \alpha \in \mathbb{N}^d : \alpha_i \leq N_i \} \) for \( i = 1, 2, \ldots, d \).

Now let \( V \) be an arbitrary TDI-subspace of \( \mathbb{R}[x] \) and let the sequence \( (p_n)_{n \in \mathbb{N}} \) of elements of \( V \) converge pointwise to the polynomial \( p \). Assume that
\[ p(x) = \sum_{|\gamma| \leq \deg(p)} a_{\gamma} x^\gamma \]
with \( a_{\alpha} \neq 0 \) for some \( \alpha = (\alpha_1, \cdots, \alpha_d) \) which is not in \( \Omega \). We define
\[ \hat{\Omega} = \{ \beta : \beta_k < \alpha_k \text{ for at least one } k \}, \]
and \( \hat{V} = \text{span}\{x^\alpha : \alpha \in \hat{\Omega}\} \).

Then \( \hat{V} \) is a TDI-subspace of \( \mathbb{R}[x] \) and \( \Omega \subseteq \hat{\Omega} \). Indeed, assuming that \( \beta \) is in \( \Omega \) with \( \beta \notin \hat{\Omega} \) gives \( \beta_k \geq \alpha_k \) for all \( k \), so \( \alpha \leq \beta \), hence \( \alpha \) is in \( \Omega \), a contradiction.

Obviously, \( \hat{\Omega} = \bigcup_{k=1}^d Z_k \), where \( Z_k = \{ \beta : \beta_k < \alpha_k \} \). Hence, by computations we made above, we conclude that \( p \) is in \( \hat{V} \) which, by our construction, is impossible. This proves the theorem.

Note that without translation invariance Theorem 7 fails to hold. Indeed, if \( \mathcal{P} \) denotes the set of prime numbers, M"untz theorem guarantees that the linear span \( V \) of the monomials \( x \mapsto x^p \) with \( p \) in \( \mathcal{P} \cup 2\mathcal{P} \cup \{0\} \) is dense in \( C[a,b] \) over any interval \([a,b] \). In particular, we can find a sequence \( (p_n)_{n \in \mathbb{N}} \) in \( V \) such that
\[ \|p_n - x^{100}\|_{C[\cdot, \cdot, \cdot]} < \frac{1}{n} \]
holds for \( n = 1, 2, \ldots \). Obviously, \( p_n \) converges pointwise to \( x \mapsto x^{100} \), which is not in \( V \). Furthermore, by Lemma 5 we have that \( V \) is dilation invariant.

Having proved our main result for TDI-spaces of polynomials a comparison between these spaces and the class of translation invariant spaces of polynomials should be of interest. Here we present an extremal example.
Example 8. Let $V$ be the set of all polynomials in two variables of the form $(x, y) \mapsto p(x + y)$, where $p$ is any polynomial in a single variable. Then, by the Binomial Theorem, $V$ is translation invariant, and all monomials $x^a y^b$ with positive integers $a, b$ belong to $W$, the smallest TDI-space which contains $V$. It follows that $W$ is the set of all polynomials in two variables, and the co-dimension of $V$ in $W$ is infinite.

Finally, let us comment that in this example a result analogous to Theorem 7 can easily be proved for the space $V$.

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Departamento de Matemáticas, Universidad de Jaén, E.P.S. Linares, Campus Científico Tecnológico de Linares, Cinturón Sur s/n, 23700 Linares, Spain
E-mail address: jmalmira@ujaen.es

Institute of Mathematics, University of Debrecen, Egyetem tér 1, 4032 Debrecen, Hungary — Department of Mathematics, University of Botswana, 4775 Notwane Rd, Gaborone, Botswana
E-mail address: lszekelyhidi@gmail.com