Normal density and moment of inertia of a moving superfluid

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Abstract

In this work, the normal density $\rho_n$ and moment of inertia of a moving superfluid are investigated. We find that, even at zero temperature, there exists a finite normal density for the moving superfluid. When the velocity of superfluid reaches sound velocity, the normal density becomes total mass density $\rho$, which indicates that the system losses superfluidity. At the same time, the Landau’s critical velocity also becomes zero. The existence of the non-zero normal density is attributed to the coupling between the motion of superflow and density fluctuation in transverse directions. With Josephson relation, the superfluid density $\rho_s$ is also calculated and the identity $\rho_s + \rho_n = \rho$ holds. Further more, we find that the finite normal density also results in a quantized moment of inertia in a moving superfluid trapped by a ring. The normal density and moment of inertia at zero temperature could be verified experimentally by measuring the angular momentum of a moving superfluid in a ring trap.

Keywords: superfluidity, Bose–Einstein condensate, normal density, moment of inertia

1. Introduction

Superfluidity is one of the most striking characteristics of liquid helium-4 at low temperature, which can flow a narrow tube without any energy dissipation [1, 2]. The superfluidity usually has close connections with Bose–Einstein condensation [3–5]. Tisza [6] and Landau [7] proposed a two-fluid theory to explain the superfluidity of liquid Helium-4. The basic idea is that the whole liquid with density $\rho$ is divided into two kinds of liquids, i.e. the superfluid part with density $\rho_s$ and the normal part with density $\rho_n$, and the identity $\rho = \rho_s + \rho_n$ holds. The normal one behaves as usual liquid, which has viscosity and transports entropy; while the superfluid part has no viscosity and can display the superfluidity.

For usual bosonic quantum liquid, for example, liquid Helium-4 or dilute atomic Bose–Einstein condensate, it is believed that, at zero temperature, the superfluid density is the whole liquid density, i.e. $\rho_s = \rho$, and the normal density vanishes, i.e. $\rho_n = 0$ [8, 9]. If an impurity moves with a velocity, which is smaller than the Landau’s critical velocity, it would not feel any drag force and the motion of the impurity is completely dissipationless. The superfluidity is usually characterized by a non-zero Landau’s critical velocity and a finite superfluid density. However, the Landau’s critical velocity would decrease if superfluid moves with respect to the laboratory frame. It is natural to ask the following question: under this circumstance, does the superfluid density decrease or is there any normal component in the moving superfluid?
In this paper, we try to answer this question by considering a superfluid that moves with a uniform velocity \( u \) in a long open-ended tube. The normal density is calculated with transverse current–current correlation function. We find that, for a moving superfluid, even at zero temperature, there exists a finite normal density, which is proportional to the square of velocity, i.e. \( \rho_n \propto u^2 \). The finite normal density also brings about a quantized moment of inertia in a ring trap. The superfluid density is also calculated with an independent method, i.e. Josephson relation. It is found that the superfluid density is proportional to the product of sound velocities of two opposite directions and the identity \( \rho_s + \rho_n = \rho \) still holds. The non-zero normal density could be verified by measuring the angular momentum in a ring trap.

The paper is organized as follows. In section 2, we give the hydrodynamic equations for a moving superfluid (Bose–Einstein condensate) and apply standard quantized rules to get phase and density fluctuation operators in terms of phonon’s creation and annihilation operators. In section 3, the superfluid density is calculated with Josephson’s relation, and compare it with that from phase-twist method. In section 4, we calculate the normal density with transverse current–current correlation function. In section 5, a quantized moment of inertia is obtained in a ring trap. A summary is given in section 6.

### 2. Hydrodynamic equations

The Hamiltonian for a dilute Bose atomic gas is \([10, 11]\)

\[
\hat{H} = \hat{H}_0 + \hat{V}_{\text{int}},
\]

\[
\hat{H}_0 = \int d^3r \hat{\psi}^\dagger(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + \hat{g}(r) \right) \hat{\psi}(r),
\]

\[
\hat{V}_{\text{int}} = \frac{g}{2} \int d^3r \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r),
\]

where \( \hat{H}_0 \) and \( \hat{V}_{\text{int}} \) are single-particle Hamiltonian and interaction between atoms, respectively. \( \hat{\psi}(r) \) is bosonic field operator, \( m \) is atomic mass, \( g = 4\pi\hbar^2a_s/m \) is interaction strength, and \( a_s \) is s-wavelong scattering length. At zero temperature, superfluidity and Bose–Einstein condensation would occur and can be characterized by a nonzero order parameter \( \psi \equiv \langle \hat{\psi}(r) \rangle = \sqrt{n_0}e^{i\theta} \), with condensate density \( n_0 \) and phase \( \theta \). For weakly interacting Bose gas, the quantum depletion is very small, so the condensate density \( n_0 \) is approximately equal to the total particle number density \( n \), i.e. \( n_0 \approx n \) \([11]\).

The order parameter satisfies the time-dependent Gross–Pitaevskii equation

\[
i\hbar \partial_t \psi(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + \hat{g}(r) \right) \psi(r, t) + \hat{g}(r)\psi(r, t)^2 \psi(r, t).
\]

Substituting \( \psi(r) = \sqrt{n(r)}e^{i\theta(r)} \) into equation (2), we obtain the hydrodynamic equations, which include a continuity equation for mass and a Euler’s dynamic equation \([16]\), i.e.

\[
\partial_t \rho + \nabla \cdot \mathbf{j} = 0,
\]

\[
\partial_t \psi + (\mathbf{v} \cdot \nabla) \psi = -\frac{\nabla p}{\rho},
\]

with mass density \( \rho(r) = nm(r) \), mass current density \( \mathbf{j} = \rho \mathbf{v} \), velocity of superflow \( \mathbf{v} \equiv \hbar \nabla \theta/m \), and pressure \( p = gn^2/2 \equiv g\rho^2/(2m) \) at zero temperature. In the derivation of equations (3) and (4), we have neglected the quantum pressure term \( (\hbar^2 \nabla^2/2m) \) \([10–12]\). In addition, the velocity is the spatial gradient of phase, so the superfluid velocity must satisfy the condition of non-rotation, i.e. \( \nabla \times \mathbf{v}(r) = 0 \).

In the following, we would set the Plank’s constant \( \hbar = 1 \), mass \( m = 1 \), and the system volume \( V = 1 \) unless stated otherwise.

When a superfluid moves along positive x-axis direction with velocity \( u \) with respect to a stationary tube, there exists a large energy barrier between the superflow state and the stationary state (true thermodynamic equilibrium state). In such a case, the moving superfluid becomes a metastable state \([9, 13, 14]\) (or quasi-equilibrium state \([15]\)). Near the metastable state, we linearize the hydrodynamic equations (3) and (4) as

\[
\partial_t \delta \rho + u \partial_x \delta \rho + \rho \nabla \cdot \delta \mathbf{v} = 0,
\]

\[
\partial_t \delta \mathbf{v} + u \partial_x \delta \mathbf{v} = -\frac{\nabla p}{\rho},
\]

with density fluctuation \( \delta \rho \), velocity fluctuation \( \delta \mathbf{v} \), and average density \( \rho \). In the following parts of the paper, we will label the average density \( \rho \) with \( \rho \) to simplify the notations. Based on the relationship between the velocity and the phase of condensate, i.e. \( \delta \mathbf{v} = \nabla \delta \theta \), the linearized hydrodynamic equations become

\[
\partial_t \delta \rho + u \partial_x \delta \rho + \rho \nabla^2 \delta \theta = 0,
\]

\[
\partial_t \delta \mathbf{v} + u \partial_x \delta \mathbf{v} = -\frac{\partial p}{\rho \partial \theta} \nabla \delta \rho = -\frac{c^2}{\rho} \nabla \delta \rho,
\]

where \( c = \sqrt{\partial p/\partial \theta} = \sqrt{\rho} \) is the sound velocity when superfluid is at rest.

From equations (7) and (8), we get the phonon state energy:

\[
\omega_q = c_\mathbf{q} \mathbf{q},
\]

where the sound velocity \( c_\mathbf{q} = c \pm u \cos \alpha \) and \( \alpha \) is the angle between the direction of momentum \( \mathbf{q} \) and positive x-axis direction. For two opposite directions (\( \pm \mathbf{q} \)), we get two different sound velocities, \( c_\pm = c \pm u \cos \alpha \), and two different energies, \( \omega_\pm = |c \pm u \cos \alpha|q \). This is because when the superfluid moves (\( u \neq 0 \)), due to the Doppler effects, the sound velocities would be different for two opposite spatial directions \([16]\). We see that both the sound velocities and phonon state energies are direction-dependent. In comparison with the usual stationary case, the Landau’s critical velocity of the moving superfluid decreases as \( v_c = (\omega_q/q)_{\text{heim}} = c - u \).

In addition, we can write an effective Hamiltonian for the linearized equations (7) and (8).
\[ H = \frac{1}{2} \int d^3r \left\{ \rho \left[ (\partial_\delta \delta)^2 + (\partial_\delta \delta')^2 + (\partial_\delta \delta')^2 \right] + 2u_\delta \rho \partial_\delta \delta + \frac{c^2(\partial \rho)^2}{\rho} \right\}. \]  

(10)

In comparison with that of the usual stationary superfluid [17], the effective Hamiltonian (10) has an extra cross term of the density and phase fluctuations, \( 2u_\delta \rho \partial_\delta \delta \), which originates from the finite superfluid velocity \( u_\delta \). Similarly as the usual stationary superfluid [17], by using Poisson brackets \{ \delta \theta(r), \delta \rho(r') \} = -i \delta^3(r-r') \) [18], we can get the linearized hydrodynamic equations (7) and (8) from the Hamilton’s equation.

After replacing the above classic physical quantities with their corresponding operators, the results of quantization can be obtained by considering the canonical commutator relation \[ [\delta \theta(r), \delta \rho(r')] = -i \delta^3(r-r') \). For example, we can expand the phase operator \( \delta \theta \) and the density operator \( \delta \rho \) in terms of single phonon’s annihilation and creation operators, i.e.

\[
\delta \theta(r, t) = \sum_{q \neq 0} [A_q \hat{c}_q e^{i(q \cdot r - \omega_q t)} + A_q^\dagger \hat{c}_q^\dagger e^{-i(q \cdot r - \omega_q t)}],
\]

\[
\delta \rho(r, t) = \sum_{q \neq 0} [B_q \hat{c}_q e^{i(q \cdot r - \omega_q t)} + B_q^\dagger \hat{c}_q^\dagger e^{-i(q \cdot r - \omega_q t)}],
\]

(11)

where \( C_q (C_q^\dagger) \) are annihilation (creation) operator for single phonon states, and coefficients \( A_q \) and \( B_q \) need to be determined. From the continuity equation (7) (upgrading it as an operator equation), i.e.

\[
\partial_\delta \delta - i \delta \theta + u_\delta \partial_\delta \rho + \rho \nabla^2 \delta \theta = 0,
\]

(12)

we get \( -icB_q = \mu \rho A_q \). From the commutation relation \[ [\delta \theta(r), \delta \rho(r')] = -i \delta^3(r-r') \), we get \( A_q B_q^\dagger = -i/2 \) and then \( A_q = -i \sqrt{c/(2 \rho \omega)} \) and \( B_q = \sqrt{\rho c/(2 \omega)} \). Based on equation (11), the density and phase fluctuations in momentum space are written as \( \rho \neq 0 \)

\[
\hat{\rho}_q = \sqrt{\frac{\rho \omega}{2c}} \left( C_q + C_q^\dagger \right), \quad \hat{\theta}_q = -i \sqrt{\frac{c}{2 \rho \omega}} \left( C_q - C_q^\dagger \right).
\]

(13)

On the other hand, at low energy, the bosonic field operator can be written as [18]

\[
\hat{\psi}(r) = \langle \psi \rangle e^{i \delta \theta(r)} \simeq \langle \psi \rangle \left[ 1 + i \delta \theta(r) + \cdots \right].
\]

(14)

Consequently, in terms of phonon’s operators, the field operator in momentum space takes the following form of

\[
\hat{\psi}_q = i \langle \psi \rangle \hat{\rho}_q = \langle \psi \rangle \sqrt{\frac{c}{2 \rho \omega}} \left( C_q - C_q^\dagger \right).
\]

(15)

The above formulas would be useful in the following discussions.

### 3. Superfluid density

The superfluid density can be given by the Josephson relation [19–22], i.e.

\[
\rho_s(q) = \lim_{q \to 0} \frac{n_0}{q^2} G(q, 0),
\]

(16)

where \( G(q, \omega) = \sum_n \frac{\langle 0 | q | n \rangle^2}{\omega_n - \omega - i \epsilon} \) is normal Green’s function with excitation state \( | n \rangle \) and excitation energy \( \omega_n = E_n - E_0 \) [18], and \( n_0 = \langle | \psi \rangle^2 \) is condensate density.

With equation (15), we calculate the matrix element of \( \hat{\psi}^\dagger \langle q \rangle \) between the ground state \( | 0 \rangle \) and the single phonon state \( | n \rangle = | q \rangle \), i.e.

\[
\langle 0 | \hat{\psi}^\dagger \langle q \rangle | 0 \rangle = \langle \psi \rangle \sqrt{\frac{c}{2 \rho \omega}}, \quad \langle 0 | \hat{\psi}^\dagger \langle q \rangle | q \rangle = -\langle \psi \rangle^* \sqrt{\frac{c}{2 \rho \omega}},
\]

and thus the Green’s function \( G(q, 0) = -\frac{\langle \psi \rangle^* \sqrt{\frac{c}{2 \rho \omega}} + \langle \psi \rangle \sqrt{\frac{c}{2 \rho \omega}}}{2 \rho c^2 q^2} \). Therefore, the superfluid density

\[
\rho_s(q) = \frac{\rho c^2 q^2}{c^2} \left[ 1 - \frac{\rho_0^2 \cos^2(\alpha)}{c^2} \right].
\]

(17)

In the above equation, we have used the fact that the single phonon states have dominant contributions to the Green’s function \( G(q, 0) \), while the contributions of the multiple phonon states can be neglected as \( q \to 0 \). For a usual stationary superfluid, i.e. \( u = 0 \) \( (c_\perp = c = c) \), the superfluid density \( \rho_s(q) \) is equal to the total density \( \rho \). However, for a moving superfluid \( (u \neq 0) \), the superfluid density is smaller than the total density. When the velocity of superfluid is equal to the sound velocity, i.e. \( u = c \), the superfluid density of x-axis direction would vanish, i.e. \( \rho_s(\hat{x}) = 0 \) \( (\alpha = 0 \) in equation (17) \), where \( \hat{x} \) is a unit vector in positive x-axis direction. In the next section, with current–current correlation function, we will show that when superfluid density becomes zero, the normal density reaches its maximum value, i.e. \( \rho_n = \rho \).

Due to the anisotropy of the sound velocity, the superfluid density is usually a second-order tensor in three dimensional space [22], namely, \( \rho_s = \text{diag}(\rho_s(\hat{x}), \rho_s(\hat{\perp}) = \rho, \rho_s(\hat{\perp}) = \rho) \). The superfluid density in \( q \)'s direction can be given by a tensor contraction, i.e.

\[
\rho_s(q) = \hat{q} \cdot \rho_s \cdot \hat{q} = \rho_s(\hat{x}) \cos^2(\alpha) + \rho_s(\hat{\perp}) \sin^2(\alpha),
\]

(18)

where \( \hat{q} = q/q \) is the unit vector in \( q \)-direction.

In many cases, the superfluid density is usually defined with a phase twist method [23], i.e.

\[
\tilde{\rho}_s(\hat{x}) = \frac{2 \Delta E}{(\delta \theta/L)^2},
\]

where \( \delta \theta \) is phase difference between two ends of tube filled with liquid, \( L \) is tube length, and \( \Delta E \) is energy cost due to the phase gradient between two ends. We further assume that the density fluctuation is negligible \( (\delta \rho \equiv 0) \), and the phase gradient is constant \( (\partial \delta \theta = \delta \theta/L) \) for a superflow state. Thus, the energy in equation (10) is

\[
\Delta E = \frac{\rho(\delta \theta/L)^2}{2}.
\]

(20)

So, we obtain the superfluid density \( \tilde{\rho}_s(\hat{x}) = \rho \). The above result shows the superfluid density defined by using the phase twist method would diverge as \( \rho \rightarrow 0 \).
twist method is the total density $\rho$, which is not consistent with the result from the Josephson relation. Only when $a = 0$ ($c_{-q} = c_{-q}$), these two methods yield the same result.

4. Normal density

In this section, we will calculate the normal density $\rho_n$ with current–current correlation function [8]. Let us assume there exists a long straight open ended tube filled with a moving superfluid with a uniform speed $u$ along positive $x$-axis direction. In addition, we also assume the tube moves at a small velocity $v$ along $x$-axis direction. If the superfluid reaches equilibrium (or quasi-equilibrium) with the tube wall, the normal part would be dragged by the tube wall and moves with tube at the same velocity $v$. Then, the resulting mass current density due to the motion of the tube would be

$$\delta j_x = \rho_n v. \quad (21)$$

The above equation can be viewed as a definition of the normal density $\rho_n$. Here the normal density $\rho_n$ can be calculated with the transverse current–current correlation function [24],

$$\rho_n(x) = \frac{1}{\pi \hbar} \sum_{q \neq 0} \left\{ \frac{|(0)\hat{j}_{x=\vec{q}=0}|^2}{\omega_{\vec{q}0}} + \frac{|(0)\hat{j}_{x=-\vec{q}=0}|^2}{\omega_{\vec{q}0}} \right\}, \quad (22)$$

where $\hat{j}_{x=\vec{q}}$ is current fluctuation operator in momentum space and vector $\vec{q}_\perp$ is ‘transverse’ with respect to $x$-direction, i.e. $\vec{x} \cdot \vec{q}_\perp = 0$ ($\vec{q}_\perp$ and $\vec{x}$ are perpendicular to each other).

The calculation of the normal density with the transverse current–current correlation function amounts to ask whether the superfluid system could have response to a transverse probe or not [8].

The current fluctuation operator $\hat{j}_{x=\vec{q}}$ can be obtained conveniently from hydrodynamic equations. For example, from equation (12), we read off the effective current fluctuation operators for low energy phonon states:

$$\delta j_x = u\phi \delta \rho, \quad \delta j_y = \rho \phi_x \delta \theta, \quad \delta j_z = \rho \phi_y \delta \theta. \quad (23)$$

In momentum space, they take the form of

$$\hat{j}_{x=\vec{q}} = u\phi \delta \rho + i \rho \phi_x \delta \theta, \quad 
\hat{j}_{y=\vec{q}} = \rho \phi_y \delta \theta, \quad \hat{j}_{z=\vec{q}} = i \rho \phi_y \delta \theta. \quad (24)$$

In contrast to the usual stationary superfluid, here we see the current fluctuation operator $\hat{j}_{x=\vec{q}}$ includes an extra density fluctuation term $\rho \phi_x \delta \theta$, which would result in a finite normal density $\rho_n$. Without loss of generality, taking the transverse direction $\vec{q} = \vec{q}_\perp = q\hat{y}$ ($\hat{y}$ is a unit vector of $y$-axis direction) for instance, we get the normal density of $x$-axis direction,

$$\rho_n(x) = \frac{u^2}{c^2} = \rho^2 \kappa, \quad (25)$$

with the compressibility $\kappa$, which satisfies a sum rule of density–density correlation function [26]

$$\rho^2 \kappa = \lim_{q \rightarrow 0} \sum_{\vec{q}_\perp} \left\{ \frac{|(0)\rho_{\vec{q}=\vec{q}_\perp}|^2}{\omega_{\vec{q}0}} + \frac{|(0)\rho_{-\vec{q}=\vec{q}_\perp}|^2}{\omega_{\vec{q}0}} \right\}$$

$$= \lim_{q \rightarrow 0} \frac{2|(0)\rho_{\vec{q}=\vec{q}_\perp}|^2}{\omega_{\vec{q}0}^2} = \frac{\rho}{c^2}. \quad (26)$$

In the compressibility sum rule, we have used equation (13) and the fact that the single phonon state dominates the contribution as $q \rightarrow 0$ and the contributions of phonon states of two transverse directions ($-\vec{q}_\perp$ and $+\vec{q}_\perp$) are the same. Taking equations (17) and (26) into account, the superfluid density can be written, in terms of two sound velocities and compressibility, as

$$\rho_s(q) = \rho \frac{c_{\perp=\vec{q}} c_{\perp=0}}{c^2} = \rho \kappa c_{\perp=\vec{q}} c_{\perp=0}, \quad (27)$$

which is consistent with equation (15) in reference [25].

The above results show that, for a moving superfluid ($u \neq 0$), there would exist a finite normal density even at zero temperature. The existence of the finite normal density has a close connection with density fluctuation. More precisely speaking, the finite normal density results from the coexistence of motion of superflow and density fluctuation in transverse directions. It is worthwhile to emphasis that the above conclusion is quite universal for a generic superfluid system. This is because equation (25) indicates that, as long as the moving superfluid has a finite positive compressibility, i.e. $\kappa > 0$ (the system is thermodynamically stable against density fluctuation), there would exist a finite normal density. In addition, from equations (17) and (25), we see the identity $\rho_s(x) = \rho_n(x) = \rho$ still holds (in $x$-direction).

The existence of the finite normal density can be understood as follows. Since the density fluctuation $\delta \rho$ appears in the current fluctuation $\delta j_x$, the occurrence of the density fluctuation always causes a current change. For example, when a mass density fluctuation occurs in a superfluid, such a density fluctuation would drift away along main stream, which results in the change of current. When $u = c$, the normal density is equal to the total density, i.e. $\rho_n = \rho$, which shows that the total density becomes normal, the superfluid density vanishes (for the case of $c = 0$ in equation (17)), and the system losses its superfluidity. At the same time, the vanishing of the superfluid density is also consistent with the vanishing of the Landau’s critical velocity, i.e. $v_c = c - u = 0$.

In the above discussions, we have assumed the Landau’s critical velocity is determined solely by the sound velocity (e.g. for Bose–Einstein condensate of dilute atomic gas). In addition, a moving impurity with a velocity $u > c$ in a stationary condensate would begin to dissipate energy and experience a drag force [27]. However, if the Landau’s critical velocity is not determined by the sound velocity (e.g. by roton minimum in liquid helium), when the superfluid moves faster than the Landau’s critical velocity, there may exist condensate of roton excitations [28] or occur a second order phase transition between a uniform phase to a spatially periodic state [29, 30].

5. Quantized momentum of inertia

In the following part, we recover the notations: mass $m$, Plank’s constant $\hbar$ and system volume $V$. Here let us assume

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7 We note that there should be a factor $\rho_s$ rather than $\rho$ in right-hand side of equation (15) in reference [25].
the superfluid moves in a ring trap with width $d$, and the radius of ring is $R$ [9]. At the same time, the trap geometrical parameters satisfy $d \ll R$. Due to the constraint of ring geometry, the allowed circulation of the superfluid velocity must take quantized value $n2\pi \hbar /m$ [11], and the velocity should be given by

$$u = \frac{nh}{mR},$$

(28)

where integer $n = 0, 1, 2, 3, \ldots$.

Assuming the angular velocity of trap perpendicular to the plane of ring is $\delta \omega$ and the rotating axis is through the center of ring, therefore the linear velocity $v = \delta \omega R$. Due to existence of the finite normal density, the normal part would be dragged by the rotating trap. The angular momentum arising from the slow rotation is

$$\delta L = \delta \omega I,$$

(29)

with moment of inertia

$$I = \rho_0 VR^2.$$

(30)

Combining it with equations (25) and (28), the moment of inertia turns out to be

$$I = \frac{n^2 M \hbar^2}{m^2 c^2} = 2n^2 M \xi^2,$$

(31)

where $M = \rho V$ is total mass of superfluid and $\xi = \hbar/(\sqrt{2mc})$ is healing length [11]. The above equation shows the moment of inertia is quantized in unit of $M \xi^2$, which is the direct consequence of quantized circulation of velocity field in a non-simply connected region.

For a stationary superfluid ($u = 0$ and $n = 0$), the normal density vanishes, the moment of inertia and angular momentum would be zero, which corresponds to the Hess–Fairbank effect [9] that the superfluid part always keeps stationary in liquid helium as long as the angular velocity of rotating bucket is small enough [31]. However, when $u \neq 0$, a moving superfluid can respond to the slow rotation and has a non-zero moment of inertia even at zero temperature.

To avoid the formation of vortices in the ring, the velocity of superfluid $u$ should be smaller than the Feynman’s critical velocity, $v_F = \ln((d/\xi)/\hbar)/(md)$ [32]. On the other hand, the superfluid velocity should be also smaller than the sound velocity, i.e. $u < c$. So the integer $n$ in equation (28) should satisfy $n < \ln((d/\xi)/\hbar)$ and $n < mcR/\hbar$. The long lifetime persistent current states (a moving superfluid) in ring geometry have been experimentally created [33–36]. In principle, the finite normal density and non-zero moment of inertia could be detected by imparting non-zero angular momentum into atom gas [37].

We would like emphasize that, in order to observe the above effects, the moving superfluid should be in thermal equilibrium with the ring trap. So the trap wall should not be perfectly rotationally symmetrical (e.g. with some roughness), otherwise there is no angular momentum which could be imparted into the superflow when the ring trap moves. Taking sodium atom $^{23}$Na as an example [34], one can choose the typical experimental parameters: ring trap radius $R = 10 \mu$m, ring width $d = 3 \mu$m, chemical potential $\mu = \hbar \times 2\pi \times 1200$ Hz, then healing length $\xi \approx 0.42 \mu$m. Under the above conditions, the maximum $n \approx 5$, the quantized momentum of inertia is $I = (2n^2 \xi^2 /R) \mu \approx 0.1\mu$. where $I_c = MR^2$ is classical value of momentum of inertia of atomic gas in the ring. If one chooses a smaller $d$, then maximum $n$ would be larger, then the resultant momentum of inertia $I$ would also become larger.

6. Summary

In conclusion, we find that there exists non-zero normal density for a moving superfluid even at zero temperature, which is proportional to the square of velocity of the superfluid. The finite normal density can be attributed to the coupling between the motion of superflow and density fluctuation in transverse direction. The finite normal density also results in a quantized moment of inertia in a moving superfluid trapped by a ring. In addition, we find that the superfluid density is proportional to product of sound velocities of two opposite directions and the identity $\rho_n + \rho_h = \rho$ still holds. It is expected that the finite normal density and non-zero moment of inertia of a moving superfluid in ring trap may be measured experimentally by using optical method in atomic gases in near future.

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