Provably Efficient Reinforcement Learning with Aggregated States

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Abstract

We establish that an optimistic variant of Q-learning applied to a finite-horizon episodic Markov decision process with an aggregated state representation incurs regret $\tilde{O}(\sqrt{HM^5K} + \epsilon HK)$, where $H$ is the horizon, $M$ is the number of aggregate states, $K$ is the number of episodes, and $\epsilon$ is the largest difference between any pair of optimal state-action values associated with a common aggregate state. Notably, this regret bound does not depend on the number of states or actions. To the best of our knowledge, this is the first such result pertaining to a reinforcement learning algorithm applied with nontrivial value function approximation without any restrictions on the Markov decision process.

1 Introduction

Value function learning with aggregated state representations has long served a foundational subject in reinforcement learning (RL). With such a representation, the set of state-action pairs is partitioned and an agent learns an approximation to the state-action value function for which value is constant across each partition. In this technical note, we design and study a variant of Q-learning that applies with such a representation. To simplify analysis, we restrict attention to finite-horizon episodic Markov decision processes (MDP).

Our analysis – which builds on recent work [3] pertaining to tabular representations – leads to an $\tilde{O}(\sqrt{HM^5K} + \epsilon HK)$ regret bound, where $H$ is the horizon, $M$ is the number aggregate states, $K$ is the number of episodes, and $\epsilon$ is the largest difference between any pair of optimal state-action values associated with a common aggregate state. Notably, this regret bound does not depend on the number of states or actions. To the best of our knowledge, this is the first such result pertaining to a reinforcement learning algorithm applied with nontrivial value function approximation without any restrictions on the Markov decision process.

Several existing results bear related implications on reinforcement learning with an aggregated state representation. Wen and Van Roy established that, if the transition and rewards of the underlying MDP are deterministic, and the optimal state-action value function lies in a prescribed hypothesis class, then an optimistic algorithm selects optimal actions in all but a number of episodes polynomial in the eluder dimension of the function class [6]. Though this result applies to hypothesis classes much more general than those arising from state aggregation, the requirement that transitions be deterministic is very restrictive relative to our setting. Jiang \textit{et al.} proposed a novel statistic, the Bellman rank, which captures the interrelation between the function class and the MDP dynamics [2]. They also showed the existence of an algorithm whose regret scales quadratically with the Bellman rank. Jin \textit{et al.} [4], Yang and Wang [7], and Zanette \textit{et al.} [8] demonstrated that, under linear function approximations, there exist algorithms that enjoy regret bounds polynomial in the rank of the MDP transition kernel. These results are all applicable to our state aggregation setting, which can be seen as a special case of linear function approximation. However, if no assumption is made with respect to the MDP dynamics, both the Bellman rank and the rank of the MDP transition kernel can be as large as the total number of states.
2 Preliminaries

We consider an agent sequentially interacting with an environment with state space $S$ and action space $A$. Each episode of interaction consists of $H$ stages, and produces a sequence

$$s_1, a_1, \ldots, s_H, a_H,$$

(1)

where for $h = 1, \ldots, H$, $s_h \in S$ is the system state in which the agent resides at the beginning of stage $h$, and $a_h \in A$ is the action taken by the agent after she observes $s_h$. For simplicity, we assume that at the beginning of each episode, the system is reset to a deterministic state $s_1$. The dynamics of the system is governed by the transition kernels $P_h$, which specifies the distribution of the next state, given the current system state and the action that the agent takes, i.e.

$$P^s_a(s') = \mathbb{P}(s_{h+1} = s' \mid s_h = s, a_h = a).$$

(2)

A deterministic reward $R_h(s, a)$ is associated with the state-action pair $(s, a)$ at stage $h$. We assume that the rewards are bounded in $[0, 1]$. At the final stage $H$, the episode terminates after the agent takes $a_H$ in state $s_H$, and realizes the reward $R_H(s_H, a_H)$. The goal of the agent is to maximize the total reward accrued in an episode, namely the episodic return, defined as $\sum_{h=1}^H R_h(s_h, a_h)$.

At each stage, a learning algorithm prescribes a specific distribution over $A$, from which the agent draws the next action. Such a sequential prescription is called a policy. The policy is called deterministic if each distribution is concentrated on one single action. In this work, we only consider deterministic policies, which can be concretized as mappings from $S$ to $A$. We say that the agent follows policy $\pi$, if for all $h = 1, \ldots, H$,

$$a_h = \pi_h(s_h).$$

(3)

The value function of policy $\pi$ is defined as the expected cumulative reward realized by the agent when she follows $\pi$, namely

$$V^\pi_h(s) = \mathbb{E}\left[\sum_{i=h}^H R_i(s_i, \pi_i(s_i)) \mid s_h = s\right],$$

(4)

where the expectation is taken over all possible transitions. We can also define the state-action value function or $Q$-function of policy $\pi$ as

$$Q^\pi_h(s, a) = R_h(s, a) + \mathbb{E}\left[\sum_{i=h+1}^H R_i(s_i, \pi_i(s_i)) \mid s_h = s, a_h = a\right].$$

(5)

According to the dynamic programming theory [1], there exists an optimal policy $\pi^*$, such that $V^\pi_h(s)$ is maximized for every $s$, and at the same time $\pi^*$ is deterministic. We denote the value function corresponding to the optimal policy as $V^*$, and define

$$Q^*_h(s, a) = R_h(s, a) + (P_h V^*_{h+1})(s, a),$$

(6)

where we use the notation

$$(P_h V)(s, a) = \mathbb{E}_{s' \sim P^s_a} [V(s')].$$

(7)

Since $Q^*_h(s, a)$ is the maximum realizable return when the agent starts from $s$ and takes action $a$ at stage $h$, sometimes we also call it the “ground-truth” value of the state-action pair $(s, a)$. From the optimality of $V^*$, we have that

$$V^*_h(s) = \max_{a \in A} Q^*_h(s, a), \quad \forall h = 1, \ldots, H, \ s \in S.$$

(8)

1. We assume that the rewards are deterministic only to streamline the analysis. All our results apply without change to environments with stochastic but bounded rewards.
It is worth noting that under our bounded rewards assumption, we have that
\[ 0 \leq V_\pi^h, V_\pi^* \leq H, \] (9)
for all \( h = 1, \ldots, H \) and policy \( \pi \).

The goal of an RL algorithm is to identify a good policy through consecutive interaction with the environment, with no prior knowledge of the environment dynamics \( P \) and \( R \). One commonly used metric to evaluate the performance of an algorithm is cumulative regret, which is the sum of the suboptimalities of the output policy across all episodes. More formally, let \( \{\pi_1, \ldots, \pi_K\} \) be the sequence of policies output by the algorithm in each episode, the cumulative regret is defined as
\[
\text{Regret}(K) = \sum_{k=1}^{K} V_1^*(s_1) - V_1^{\pi_k}(s_1).
\] (10)

State aggregation reduces complexity and accelerates learning by aggregating state-action pairs. This involves partitioning the set of state-action pairs into \( M \) subsets. Each subset can be thought of as an aggregate state, and we will use a value function representation that maintains one value estimate per aggregate state. Let \( \Phi \) be the set of aggregate states, and \( \phi_h : S \times A \mapsto \Phi \) be the mapping from state-action pair to an aggregate state at stage \( h \). Formally, we define state aggregation as follows.

**Definition 1.** We say that \( \Phi \) is an \( \epsilon \)-error aggregation of an MDP, if for any \( s, s' \in S, \ a, a' \in A \) and \( h \in [H] \), if \( \phi_h(s, a) = \phi_h(s', a') \), then
\[
|Q_h^*(s, a) - Q_h^*(s', a')| \leq \epsilon.
\] (11)

**Remark 1.** If \( \Phi \) is a 0-error aggregation of an MDP, then we say that the aggregate states are sufficient, in the sense that the value of an aggregate state exactly represents the values of all state-action pairs mapped to it. This corresponds to the case where the actual value function lies in the hypothesis function class. As we will show later, only under this case can we guarantee that the algorithm \( AQ-UCB \), introduced in Section 3, finds the optimal policy as \( K \to \infty \).

### 3 Algorithm and Main Results

We first present the algorithm \( AQ-UCB \), which is a Q-learning algorithm under state aggregation, equipped with UCB-type exploration. The algorithm maintains a sequence of Q-function estimates \( \{\hat{Q}^k\}_{h \in [H], k \geq 1} \). Since the state-action pairs mapped to the same aggregate state are not distinguished in the algorithm, each estimate \( \hat{Q} \) is a mapping from \( \Phi \) to real values. Without loss of generality, we let \( \Phi = [M] \), where we henceforth use \( [n] \) to denote the set \( \{1, \ldots, n\} \). We also introduce the notation
\[
\hat{V}_h^k(s) = \max_{a \in A} \hat{Q}^k_h(\phi_h(s, a)).
\] (12)
Algorithm 1: AQ-UCB

input: $S, A, H, \{\phi_h\}_{h=1}^H, s_1$, positive constants $\beta_1, \beta_2, \ldots$

1. Define constants $\alpha_i \leftarrow (H + 1)/(H + t)$, $t = 1, 2, \ldots$
2. Initialize $N_h^0(m) = 0, \bar{Q}_h^0(m) = H$ for all $h \in [H]$ and $m \in [M]$.
3. Randomly draw the first trajectory $s_{01}^0, a_{01}^0, \ldots, s_{0H}^0, a_{0H}^0$, where $s_{01}^0 = s_1$.

for episode $k = 1, \ldots, K$ do
    for stage $h = 1, \ldots, H$ do
        if $m = \phi_h(s_{k-1}^h, a_{k-1}^h)$ then
            $N_h^k(m) \leftarrow N_h^{k-1}(m) + 1$
            $\hat{Q}_h^k(m) \leftarrow r_h^k + V_{h+1}(s_{h+1}^{k-1}) + \beta N_h^k(m) \cdot \frac{1}{\sqrt{N_h^k(m)}}$
            $\bar{Q}_h^k(m) \leftarrow (1 - \alpha N_h^k(m)) \cdot \hat{Q}_h^{k-1}(m) + \alpha N_h^k(m) \cdot \bar{Q}_h^k(m)$
            $\hat{Q}_h^k(m) \leftarrow \min \{\hat{Q}_h^k(m), H\}$
        else
            $N_h^k(m) \leftarrow N_h^{k-1}(m)$
            $\hat{Q}_h^k(m) = \hat{Q}_h^{k-1}(m)$
            $\bar{Q}_h^k(m) = \bar{Q}_h^{k-1}(m)$
        end
        $s_{k+1}^h \leftarrow s_1$
    end

for stage $h = 1, \ldots, H$ do
    Take action $a_h^k \leftarrow \arg\max_{a \in A} \bar{Q}_h^k(\phi_h(s_h^k, a))$
    receive reward $r_h^k$ and next state $s_{h+1}^k$
end

The variables that will be encountered frequently in the analysis are:

- The number of visits to the aggregate state $m$ in stage $h$, from episode 1 to episode $k$: $N_h^k(m)$;
- The incrementally updated state-action value estimate: $\hat{Q}_h^k(m)$;
- The state-action value estimate, truncated by $H$: $\bar{Q}_h^k(m)$.

In each episode, the algorithm computes the above three variables, and samples a new trajectory greedily with respect to $\bar{Q}_h^k$. Assuming that the computation of the aggregation mapping $\phi$ takes $O(1)$ time and the storage of $\phi$ consumes $O(1)$ memory, the time complexity of AQ-UCB is then $O(MAKH)$, where the factor $A$ denotes the cardinality of $A$ and comes from taking argmax over the action space. The space complexity of AQ-UCB is $O(MKH)$. The specific choice of stepsizes $\alpha_i = (H + 1)/(H + t)$, which we borrow from $[3]$, assigns more weight on the recent updates and is crucial in preventing the on-policy error from exploding. Our main result is Theorem 1, the proof of which is deferred to the appendix.

**Theorem 1.** Suppose $\Phi$ is an $\epsilon$-error aggregation of the underlying MDP. We have that, for any $\delta > 0$, if we run $K$ episodes of algorithm AQ-UCB with

$$\beta_i = H^2 \sqrt{\log \frac{HK^2}{\delta}} + \frac{\epsilon}{2} \cdot \sqrt{i}, \quad i = 1, 2, \ldots,$$

then with probability at least $1 - \delta$,

$$\text{Regret}(K) \leq 24H^6M K \log \frac{3HK}{\delta} + 12 \sqrt{2H^3K \log \frac{3}{\delta}} + 3H^2M + 6\epsilon \cdot HK.$$  \hspace{1cm} (14)

**Remark 2.** When the aggregate states are sufficient, i.e. $\epsilon = 0$, Theorem 1 shows that the cumulative regret of AQ-UCB is $\bar{O}(\sqrt{HK^3})$, which translates into $\bar{O}(\sqrt{H^4MT})$ if we let $T = HK$ be the total running.
stages of the agent. When applied to the tabular representation case where \( M = SA \), the bound matches the Hoeffding-type regret bound in \([3]\). Compared with the \( \tilde{O}(\sqrt{dH^3T}) \) regret of linear MDPs in \([4]\), our result trades off \( d \) with \( H^2 \). However, our result does not require any specific transition properties with respect to the environment.

**Remark 3.** In the case with model misspecification where \( \epsilon > 0 \), the regret bound in Theorem \( 1 \) has an extra term \( O(\epsilon HK) \), which shows that the performance of the policy that the algorithm ultimately finds depends on the misspecification error \( \epsilon \). The term matches the linear term in \([4]\) with respect to the orders of \( H \) and \( K \), and, as is shown in \([3]\), is the best that we can hope for with a TD-based algorithm. It is also shown in \([5]\) that there could be an extra \( H \) factor if we use a replay buffer to store the trajectories from the past episodes, and uniformly sample trajectories from the replay buffer to update the Q-function estimates.

Finally, under our episodic setting with state aggregation, we have the following regret lower bound. This is a direct implication of Theorem 3 in \([3]\).

**Theorem 2.** There exists a problem instance and a 0-error aggregation scheme with \( M \) aggregate states, such that the expected cumulative regret of any algorithm is \( \Omega(\sqrt{H^3MK}) \).

**Proof.** From Theorem 3 in \([3]\), there exists an episodic MDP instance with \( S \) states, \( A \) actions and horizon length \( H \), such that the expected regret of any learning algorithm is at least \( \Omega(\sqrt{H^2SAT}) \). Consider the aggregation scheme that assigns each state-action pair to an individual aggregate state at each stage, with \( M = SA \) aggregate states per stage. Apparently such an aggregation is 0-error, and any learning algorithm has to incur \( \Omega(\sqrt{H^3MK}) \) regret in \( K \) episodes.

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**References**

[1] Bellman, Richard. 1966. Dynamic Programming. *Science, 153*(3731), 34–37.

[2] Jiang, Nan, Krishnamurthy, Akshay, Agarwal, Alekh, Langford, John, & Schapire, Robert E. 2017. Contextual Decision Processes with Low Bellman Rank Are PAC-Learnable. Pages 1704–1713 of: *Proceedings of the 34th International Conference on Machine Learning-Volume 70*. JMLR.org.

[3] Jin, Chi, Allen-Zhu, Zeyuan, Bubeck, Sebastien, & Jordan, Michael I. 2018. Is Q-Learning Provably Efficient? Pages 4863–4873 of: *Advances in Neural Information Processing Systems*.

[4] Jin, Chi, Yang, Zhuoran, Wang, Zhaoran, & Jordan, Michael I. 2019. Provably Efficient Reinforcement Learning with Linear Function Approximation. *arXiv preprint arXiv:1907.05388*.

[5] Van Roy, Benjamin. 2019. *On Value Function Learning*. Stanford University MS&E 338 Course Note.

[6] Wen, Zheng, & Van Roy, Benjamin. 2017. Efficient Reinforcement Learning in Deterministic Systems with Value Function Generalization. *Mathematics of Operations Research, 42*(3), 762–782.

[7] Yang, Lin F, & Wang, Mengdi. 2019. Reinforcement Learning in Feature Space: Matrix Bandit, Kernels, and Regret Bound. *arXiv preprint arXiv:1905.10389*.

[8] Zanette, Andrea, Brandfonbrener, David, Pirotta, Matteo, & Lazaric, Alessandro. 2019. Frequentist Regret Bounds for Randomized Least-Squares Value Iteration. *arXiv preprint arXiv:1911.00567*.
A Decomposition of On-Policy Errors

In this section we build a general recursive relationship for the on-policy estimation errors. Throughout
we will assume that the state aggregation is $\epsilon$-error with $\epsilon \geq 0$. The case where $\epsilon = 0$ will be handled in
Appendix B and the case where $\epsilon > 0$ will be handled in Appendix C. To simplify notations, we will use
$N(s,a)$, $\hat{Q}(s,a)$ and $\tilde{Q}(s,a)$ to refer to $N(\phi(s,a))$, $\hat{Q}(\phi(s,a))$ and $\tilde{Q}(\phi(s,a))$, respectively.

We first define a set of notations that facilitate our analysis. Recall our choice of stepsize is that
$$\alpha_t = \frac{H + 1}{H + t}, \quad t = 1, 2, \ldots$$

Let
$$\alpha^0_t = \prod_{j=1}^{t} (1 - \alpha_j),$$
and
$$\alpha^i_t = \alpha_i \prod_{j=i+1}^{t} (1 - \alpha_j), \quad i = 1, \ldots, t - 1$$
with $\alpha^t_t = \alpha_t$. Apparently, for any $t \geq 1$,
$$\sum_{i=0}^{t} \alpha^i_t = 1.$$ 

The following lemma from [3] offers a few useful properties of $\alpha^i_t$.

Lemma 1. (Lemma 4.1 in [3]) We have that
(a) For every $t \geq 1$, $1 - \frac{1}{\sqrt{t}} \leq \sum_{i=1}^{t} \alpha^i_t \leq \frac{2}{\sqrt{t}}$;
(b) For every $t \geq 1$, $\max_{i \in [t]} \alpha^i_t \leq \frac{2H}{t}$ and $\sum_{i=1}^{t} (\alpha^i_t)^2 \leq \frac{2H}{t}$;
(c) For every $i \geq 1$, $\sum_{t=i}^{\infty} \alpha^i_t = 1 + \frac{1}{\sqrt{t}}$.

For each $(s,a) \in S \times A$, let
$$1 \leq \tau_{1, s, a} < \tau_{2, s, a} < \ldots$$
be the episodes in which the aggregate state $\phi_h(s,a)$ is visited in stage $h$. Under these notations, we have that
$$\hat{Q}^k_h(s,a) = \alpha^0_{N^k_h(s,a)} \cdot H + \sum_{j=1}^{N^k_h(s,a)} \alpha^j_{N^k_h(s,a)} \left[ \tau_{t, s, a} + \tilde{V}_{h+1} \left( s_{h+1} \right) \right].$$

For the sake of simplicity, we also define a set of notations for on-policy quantities as follows
$$\hat{n}^k_h = N^k_h(s_h, a_h), \quad \hat{\tau}^k_{h, s, a} = \tau_{h, s, a, a_h}.$$
Then there should be
\[ \hat{V}_k^k(s_k^k) - V_h^*(s_k^k) \leq \hat{Q}_h^k(s_k^k, a_k^k) - Q_h^k(s_k^k, a_k^k) \]
\[ \leq \hat{Q}_h^k(s_k^k, a_k^k) - Q_h^k(s_k^k, a_h^k) \]
\[ \leq \alpha_{n_h^k} \cdot (H - Q_h^k(s_k^k, a_h^k)) + \]
\[ \sum_{j=1}^{\hat{n}_k^j} \alpha_{n_h^j}^2 \left[ r_h^{s_k^j} + V_{h+1}^h(s_{h+1}^{s_k^j}) + \frac{\beta}{\sqrt{j}} - Q_h^k(s_k^k, a_h^k) \right] \]
\[ = \alpha_{n_h^k} \cdot (H - Q_h^k(s_k^k, a_h^k)) + \]
\[ \sum_{j=1}^{\hat{n}_k^j} \alpha_{n_h^j}^2 \left[ r_h^{s_k^j} + V_{h+1}^h(s_{h+1}^{s_k^j}) + \frac{\beta}{\sqrt{j}} - Q_h^k(s_k^k, a_h^k) \right] \]
\[ \leq \alpha_{n_h^k} \cdot (H - Q_h^k(s_k^k, a_h^k)) + \epsilon + \]
\[ \sum_{j=1}^{\hat{n}_k^j} \alpha_{n_h^j}^2 \left[ V_{h+1}^h(s_{h+1}^{s_k^j}) - V_{h+1}^h(s_{h+1}^{s_k^j}) \right] + \]
\[ \sum_{j=1}^{\hat{n}_k^j} \alpha_{n_h^j}^2 \left[ \frac{\beta}{\sqrt{j}} + V_{h+1}^h(s_{h+1}^{s_k^j}) - P_h V_{h+1}^h(s_{h+1}^{s_k^j}, a_{h}^{s_k^j}) \right], \]
where in (22) we use the fact that
\[ a_h^k = \arg\max_a \hat{Q}_h^k(s_k^k, a) \quad \text{and} \quad V_h^*(s_k^k) \geq Q_h^k(s_k^k, a_h^k). \] (25)

Equation (23) results from Definition 1 and that \((s_k^k, a_h^k)\) and \((s_{h}^{s_k^j}, a_{h}^{s_k^j})\) are mapped to the same aggregate state, and in (24) we apply
\[ Q_h^k(s_{h}^{s_k^j}, a_{h}^{s_k^j}) = r_h^{s_k^j} + P_h V_{h+1}^h(s_{h+1}^{s_k^j}, a_{h}^{s_k^j}). \] (26)

We first handle the term \(q_2\) in (24). From Lemma 1 we have that
\[ \sum_{j=1}^{\hat{n}_h^j} \alpha_{n_h^j} \cdot \frac{\beta}{\sqrt{j}} \leq \frac{2\beta}{\sqrt{\hat{n}_h^k}}. \] (27)

And also notice the fact that
\[ P_h V_{h+1}^h(s_{h+1}^{s_k^j}, a_{h}^{s_k^j}) = \mathbb{E} \left[ V_{h+1}^h(s_{h+1}^{s_k^j}) \mid s_{h+1}^{s_k^j}, a_{h}^{s_k^j} \right]. \] (28)

From Azuma-Hoeffding inequality, there should be, with probability at least \(1 - \delta\),
\[ \sum_{j=1}^{\hat{n}_h^j} \alpha_{n_h^j}^2 \left[ V_{h+1}^h(s_{h+1}^{s_k^j}) - P_h V_{h+1}^h(s_{h+1}^{s_k^j}, a_{h}^{s_k^j}) \right] \leq H \cdot \sqrt{2 \sum_{j=1}^{\hat{n}_h^j} (\alpha_{n_h^j}^2)^2 \cdot \log \frac{1}{\delta}}. \] (29)
Recall that Lemma 1 gives us

\[ \sum_{j=1}^{n^*_h} (\alpha^j_{n^*_h})^2 \leq \frac{2H}{n^*_h}. \]  

(30)

Hence with probability at least \( 1 - \delta \),

\[ \sum_{j=1}^{n^*_h} \alpha^j_{n^*_h} \left[ V^*_h(s_{h+1}^{x^{k,j}_{h+1}}) - P_h V^*_h(s_{h+1}^{x^{k,j}_{h+1}}, a_{h}^{x^{k,j}_{h+1}}) \right] \leq \frac{2H^2}{\sqrt{n^*_h}} \cdot \sqrt{\log \frac{1}{\delta}}. \]  

(31)

By scaling \( \delta \) to \( \delta/HK \) and taking a union bound over \( h \in [H] \) and \( k \in [K] \), we have that with probability at least \( 1 - \delta \), for all \( (h, k) \in [H] \times [K] \),

\[ q_2 \leq \frac{1}{\sqrt{n^*_h}} \cdot \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right), \]  

(32)

We denote this event as \( E_1 \).

Notice that the term \( q_1 \) is recursive. By letting

\[ \chi^k_h = \hat{V}^k_h(s^*_h) - V^*_h(s^*_h) \]  

be the on-policy estimation error at stage \( h \) in episode \( k \), we can write \( q_1 \) as

\[ q_1 = \sum_{j=1}^{n^*_h} \alpha^j_{n^*_h} \cdot \chi^j_{h+1}. \]  

(34)

Combining (24), (32) and (34), we have that, under event \( E_1 \), for all \( h \in [H] \) and \( k \in [K] \),

\[ \chi_h^k \leq \alpha^0_{n^*_h} \cdot H + \sum_{j=1}^{n^*_h} \alpha^j_{n^*_h} \cdot \chi^{x^{k,j}_{h+1}}_{h+1} + \frac{1}{\sqrt{n^*_h}} \cdot \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \epsilon. \]  

(35)

Thus we have a recursive inequality for the on-policy estimation error of Algorithm 1.

However, we also need to show that the value function estimations are with high probability optimistic, in order to guarantee that the algorithm does not get stuck in a bad policy forever. In fact, from (20), for
any \((s, a) \in S \times A\) and \(h \in [H]\)

\[
\tilde{Q}_h^*(s, a) - Q_h^*(s, a) = \alpha_{N_h^k(s, a)}^0 \cdot (H - Q_h^*(s, a)) + \sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ r_h^s + \beta \sqrt{\frac{j}{N_h^k(s, a)}} - Q_h^*(s, a) \right]
\]

\[
= \alpha_{N_h^k(s, a)}^0 \cdot (H - Q_h^*(s, a)) + \sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ r_h^s + \beta \sqrt{\frac{j}{N_h^k(s, a)}} - Q_h^*(s, a) \right]
\]

\[
\geq \alpha_{N_h^k(s, a)}^0 \cdot (H - Q_h^*(s, a)) - \epsilon + \sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ V_{h+1}(s_{h+1}, a_{h+1}) - V_h^*(s_{h+1}) \right]
\]

\[
\geq \alpha_{N_h^k(s, a)}^0 \cdot (H - Q_h^*(s, a)) - \epsilon + \sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ V_{h+1}(s_{h+1}, a_{h+1}) - V_h^*(s_{h+1}) \right]
\]

\[
Q_h^*(s_{h+1}, a_{h+1}) = r_h^s + \beta \sqrt{\frac{j}{N_h^k(s, a)}} + \hat{P}_h V_{h+1}^*(s_{h+1}) + \hat{P}_h V_{h+1}(s_{h+1}, a_{h+1})
\]

where (36) results from Definition 1 and Lemma 1 and (37) uses the fact that

\[
Q_h^*(s_{h+1}, a_{h+1}) = r_h^s + \beta \sqrt{\frac{j}{N_h^k(s, a)}} + \hat{P}_h V_{h+1}^*(s_{h+1}) + \hat{P}_h V_{h+1}(s_{h+1}, a_{h+1})
\]

Similar with (31), from Azuma-Hoeffding inequality, we have that with probability at least \(1 - \delta\),

\[
\sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ V_{h+1}^*(s_{h+1}) - P_h V_{h+1}^*(s_{h+1}, a_{h+1}) \right] \geq -\frac{2H^2}{\sqrt{N_h^k(s, a)}} \cdot \sqrt{\log \frac{1}{\delta}}.
\]

Noticing that \(Q_h^*(s, a) \leq H\), by taking a union bound over \((h, k) \in [H] \times [K]\), we arrive at, with probability at least \(1 - \delta\),

\[
\tilde{Q}_h^*(s, a) - Q_h^*(s, a) \geq \sum_{j=1}^{N_h^k(s, a)} \alpha_{N_h^k(s, a)}^j \left[ V_{h+1}^*(s_{h+1}) - V_{h+1}^*(s_{h+1}) \right] + \frac{1}{\sqrt{N_h^k(s, a)}} \cdot \left( 2\beta - 2H^2 \sqrt{\frac{HK}{\delta}} \right) - \epsilon.
\]

In the following, we will split up the cases in which \(\epsilon = 0\) and \(\epsilon > 0\), and finish the proofs respectively.

9
B Proof of Theorem 1 without Misspecification Error

Following the analysis in Appendix A, when the aggregation is 0-error, we have that, with probability at least $1 - \delta$, for all $\hat{h} \in [H]$ and $k \in [K]$,

$$\chi^k_{\hat{h}} \leq \tilde{V}^k_{\hat{h}} - Q^k_{\hat{h}}(s^k_{\hat{h}}, a^k_{\hat{h}}) \leq \alpha^0_{\hat{n}^k_{\hat{h}}} \cdot H + \sum_{j=1}^{\hat{n}^k_{\hat{h}}} \alpha^j_{N^k_{\hat{h}}(s, a)} \cdot \chi^k_{\hat{h}+1} + \frac{1}{\sqrt{\hat{n}^k_{\hat{h}}}} \cdot \left(2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}}\right), \quad (41)$$

and also with probability at least $1 - \delta$, for all $(s, a) \in \mathcal{S} \times \mathcal{A}$, $h \in [H]$ and $k \in [K]$,

$$\tilde{Q}^k_{\hat{h}}(s, a) - Q^k_{\hat{h}}(s, a) \geq \sum_{j=1}^{N^k_{\hat{h}}(s, a)} \alpha^j_{N^k_{\hat{h}}(s, a)} \left[\tilde{V}^k_{\hat{h}+1}(s^k_{\hat{h}+1}, a^k_{\hat{h}+1}) - V^k_{h+1}(s^k_{h+1})\right] + \frac{1}{\sqrt{N^k_{\hat{h}}(s, a)}} \cdot \left(2\beta - 2H^2 \sqrt{\log \frac{HK}{\delta}}\right). \quad (42)$$

Let $\mathcal{E}_2$ be the joint occurrence of both events, which happens with probability at least $1 - 2\delta$. Further, let

$$\beta = H^2 \sqrt{\log \frac{HK}{\delta}} \quad (43)$$

Then, under $\mathcal{E}_2$, we have that at the final stage $H$, for all $(s, a) \in \mathcal{S} \times \mathcal{A}$ and $k \in [K]$,

$$\tilde{Q}^k_{H}(s, a) - Q^k_{H}(s, a) \geq \frac{1}{\sqrt{N^k_{H}(s, a)}} \cdot \left(2\beta - 2H^2 \sqrt{\log \frac{HK}{\delta}}\right) \geq 0. \quad (44)$$

Considering that $\tilde{Q}^k_{H}(s, a) = \tilde{Q}^k_{H}(s, a)$ whenever $\tilde{Q}^k_{H}(s, a) \leq H$ and $Q^k_{H}(s, a) \leq H$, we have that

$$\tilde{Q}^k_{H}(s, a) \geq Q^k_{H}(s, a), \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A}, k \in [K]. \quad (45)$$

Therefore,

$$\tilde{V}^k_{H}(s) \geq V^k_{H}(s), \quad \forall s \in \mathcal{S}, k \in [K]. \quad (46)$$

Suppose that at stage $h$,

$$\tilde{V}^k_{h}(s) \geq V^k_{h}(s), \quad \forall s \in \mathcal{S}, k \in [K]. \quad (47)$$

Then from $\Delta 2$,

$$\tilde{Q}^k_{h-1}(s, a) - Q^k_{h-1}(s, a) \geq \frac{1}{\sqrt{N^k_{h-1}(s, a)}} \cdot \left(2\beta - 2H^2 \sqrt{\log \frac{HK}{\delta}}\right) \geq 0. \quad (48)$$

This leads to

$$\tilde{Q}^k_{h-1}(s, a) \geq Q^k_{h-1}(s, a), \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A}, k \in [K], \quad (49)$$

and further, by maximizing over $a \in \mathcal{A}$ on both sides,

$$\tilde{V}^k_{h-1}(s) \geq V^k_{h-1}(s), \quad \forall s \in \mathcal{S}, k \in [K]. \quad (50)$$

Therefore, by the induction principle, we have that, under $\mathcal{E}_2$ with $\beta$ chosen by $\Delta 3$,

$$\tilde{V}^k_{h}(s) \geq V^k_{h}(s), \quad \forall s \in \mathcal{S}, h \in [H], k \in [K]. \quad (51)$$
Hence we have

\[ \eta^k_h = \hat{V}^k_h(s^k_h) - V^\pi_h(s^k_h) \]  

(52)

and

\[ \xi^k_{h+1} = [P_h V^*_{h+1}(s^k_h, a^k_h) - P_h V^\pi_h(s^k_h, a^k_h)] - [V^*_{h+1}(s^k_{h+1}) - V^\pi_h(s^k_{h+1})]. \]  

(53)

Then, under \( \mathcal{E}_2 \), \( (51) \) translates into

\[
\eta^k_h = \hat{V}^k_h(s^k_h) - Q^*_h(s^k_h, a^k_h) + Q^*_h(s^k_h, a^k_h) - V^\pi_h(s^k_h) \\
\leq \alpha^k_{h+1} \cdot H + \sum_{j=1}^{\hat{n}^k_{h+1}} \alpha^j_{n^k_h} \cdot \chi^k_{h+1} + \frac{1}{\sqrt{n^k_h}} \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \\
P_h V^*_h(s^k_h, a^k_h) - P_h V^\pi_h(s^k_h, a^k_h) \\
= \alpha^k_{h+1} \cdot H + \sum_{j=1}^{\hat{n}^k_{h+1}} \alpha^j_{n^k_h} \cdot \chi^k_{h+1} + \frac{1}{\sqrt{n^k_h}} \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \xi^k_{h+1} + (n^k_{h+1} - 1),
\]  

(54)

where \( (54) \) comes from \( (51) \) and the fact that

\[ Q^*_h(s^k_h, a^k_h) - V^\pi_h(s^k_h) = Q^*_h(s^k_h, a^k_h) - Q^*_h(s^k_h, a^k_h) = P_h V^*_h(s^k_h, a^k_h) - P_h V^\pi_h(s^k_h, a^k_h). \]

Summing over \( k = 1, \ldots, K \), there is

\[
\sum_{k=1}^{K} \eta^k_h \leq \sum_{k=1}^{K} \alpha^0_{n^k_h} \cdot H + \sum_{k=1}^{K} \sum_{j=1}^{\hat{n}^k_{h+1}} \alpha^j_{n^k_h} \cdot \chi^k_{h+1} + \frac{1}{\sqrt{n^k_h}} \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \\
\sum_{k=1}^{K} \xi^k_{h+1} + \sum_{k=1}^{K} n^k_{h+1} - \sum_{k=1}^{K} \chi^k_{h+1}.
\]  

(56)

Notice that

\[
\sum_{k=1}^{K} \sum_{j=1}^{\hat{n}^k_{h+1}} \alpha^j_{n^k_h} \cdot \chi^k_{h+1} \leq \sum_{\tau=1}^{K} \chi^k_{\tau+1} \cdot \sum_{t=1}^{\infty} \alpha^t_{1} \leq \left( 1 + \frac{1}{H} \right) \sum_{\tau=1}^{K} \chi^k_{\tau+1}.
\]  

(57)

Hence we have

\[
\sum_{k=1}^{K} \eta^k_h \leq \sum_{k=1}^{K} \alpha^0_{n^k_h} \cdot H + \frac{1}{H} \sum_{k=1}^{K} \chi^k_{h+1} + \sum_{k=1}^{K} \frac{1}{\sqrt{n^k_h}} \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \\
\sum_{k=1}^{K} \xi^k_{h+1} + \sum_{k=1}^{K} n^k_{h+1} \\
\leq \sum_{k=1}^{K} \alpha^0_{n^k_h} \cdot H + \left( 1 + \frac{1}{H} \right) \sum_{k=1}^{K} \eta^k_{h+1} + \sum_{k=1}^{K} \frac{1}{\sqrt{n^k_h}} \left( 2\beta + 2H^2 \sqrt{\log \frac{HK}{\delta}} \right) + \sum_{k=1}^{K} \xi^k_{h+1},
\]  

(58)

where the second step is because \( V^*_h(s^k_h) \geq V^\pi_h(s^k_h) \) leads to that

\[ \chi^k_h \leq \eta^k_h, \quad \forall h \in [H], k \in [K]. \]  

(59)
Expanding the recursion, and plugging in our choice of $\beta$ in (43), we have that

$$\sum_{k=1}^{K} \eta_1^k \leq \left(1 + \frac{1}{H}\right)^H \cdot \left(\sum_{k=1}^{K} \sum_{h=1}^{H} \alpha_0^{\hat{n}_k} \cdot H + \sum_{k=1}^{K} \sum_{h=1}^{H} \xi_h^k + 4H^2 \sqrt{\log \frac{HK}{\delta}} \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{\hat{n}_h^k}\right). \quad (60)$$

As with $z_1$, since there is

$$\alpha_0^{\hat{n}_h^k} = \begin{cases} 0 & \text{if } \hat{n}_h^k > 1 \\ 1 & \text{if } \hat{n}_h^k = 1 \end{cases}, \quad (61)$$

we have that

$$z_1 \leq H^2 M. \quad (62)$$

Considering that each $\xi_h^k$ is a zero-mean random variable in $[-2H, 2H]$, by Azuma-Hoeffding inequality, with probability at least $1 - \delta$,

$$z_2 \leq 4 \sqrt{2H^3 K \log \frac{1}{\delta}}. \quad (63)$$

For $h \in [H]$ and $m \in [M]$, let $g_{h,m}$ be the number of times aggregate state $m$ is visited at stage $h$ during the entire process of the algorithm. There should be

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{\hat{n}_h^k} = \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{j=1}^{g_{h,m}} \frac{1}{\sqrt{j}} \leq \sum_{h=1}^{H} \sum_{m=1}^{M} 2\sqrt{g_{h,m}} \leq 2 \sqrt{HM \cdot \sum_{h=1}^{H} \sum_{m=1}^{M} g_{h,m}} = 2\sqrt{H^2 MK}. \quad (64)$$

Therefore

$$z_3 \leq 8 \sqrt{H^5 MK \log \frac{HK}{\delta}}. \quad (65)$$

Finally, the cumulative regret can be bounded as follows

$$\text{Regret}(K) = \sum_{k=1}^{K} V_1^*(s_1) - V_1^{\pi_k}(s_1) \leq \sum_{k=1}^{K} \eta_1^k \leq 3(z_1 + z_2 + z_3), \quad (66)$$

where we used the fact that $(1 + 1/H)^H < 3$. This is the result that we desire.
C Proof of Theorem 1 with Misspecification Error

When the aggregation is $\epsilon$-error with $\epsilon > 0$, the optimism factor $\beta$ has to be large enough to offset the maximum misspecification $\epsilon$. By letting

$$
\beta_{N_k(s,a)} = H^2 \sqrt{\log \frac{HK}{\delta}} + \frac{\epsilon}{2} \cdot \sqrt{N_k(s,a)},
$$

(67)

from (40), we have that with probability at least $1 - \delta$, for all $h \in [H]$ and $k \in [K]$,

$$
\hat{Q}_k^h(s,a) - Q_k^h(s,a) \geq \sum_{j=1}^{N_k(s,a)} \alpha_{N_k(s,a)}^j \left[ \hat{r}_{h+1}^j(s_h+1,a) - V_{h+1}^*(s_{h+1}) \right].
$$

(68)

Following the same induction analysis as in (44)–(50), we can conclude that with probability at least $1 - \delta$, for all $h \in [H], k \in [K]$ and $s \in S$,

$$
\hat{V}_k^h(s) \geq V_k^*(s).
$$

(69)

Plugging in our choice of $\beta$ in (67), the recursive upper bound for the on-policy error (35) becomes

$$
\chi_k^h \leq \alpha_0^k \hat{n}_k^h \cdot H + \sum_{j=1}^{\hat{n}_k^h} \alpha_{\hat{n}_k^h}^j \cdot \hat{x}_{h+1}^j + \frac{1}{\sqrt{\hat{n}_k^h}} \cdot 4H^2 \sqrt{\log \frac{HK}{\delta}} + 2\epsilon.
$$

(70)

Let $\eta_k^h$ and $\xi_k^h$ be defined as in (52) and (53), respectively. The recursion (70) thus translates into

$$
\eta_k^h \leq \alpha_0^k \hat{n}_k^h \cdot H + \sum_{j=1}^{\hat{n}_k^h} \alpha_{\hat{n}_k^h}^j \cdot \hat{x}_{h+1}^j + \frac{1}{\sqrt{\hat{n}_k^h}} \cdot 4H^2 \sqrt{\log \frac{HK}{\delta}} + \xi_{h+1}^k + (\eta_{h+1}^k - \chi_{h+1}^k) + 2\epsilon.
$$

(71)

Summing over $k = 1, \ldots, K$ and considering (57) and (59), we have

$$
\sum_{k=1}^{K} \eta_k^h \leq \sum_{k=1}^{K} \alpha_0^k \hat{n}_k^h \cdot H + \left( 1 + \frac{1}{H} \right) \sum_{k=1}^{K} \eta_{h+1}^k + \sum_{k=1}^{K} \frac{1}{\sqrt{\hat{n}_k^h}} \cdot 4H^2 \sqrt{\log \frac{HK}{\delta}} + \sum_{k=1}^{K} \xi_{h+1}^k + 2\epsilon K.
$$

(72)

Unfolding the recursion over $\sum_{k=1}^{K} \eta_k^h$ with respect to $h$, we arrive at

$$
\sum_{k=1}^{K} \eta_1^k \leq \left( 1 + \frac{1}{H} \right) \sum_{k=1}^{K} \sum_{h=1}^{H} \alpha_0^k \cdot H + \sum_{k=1}^{K} \sum_{h=1}^{H} \eta_{h+1}^k + 4H^2 \sqrt{\log \frac{HK}{\delta}} \cdot \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{\hat{n}_k^h} + 2\epsilon KH.
$$

(73)

The rest of the proof follows from the same steps as in (60)–(65).