On the Comparison of Context-Free Grammars

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Abstract

In this paper we consider the problem of context-free grammars comparison from the analysis point of view. We show that the problem can be reduced to numerical solution of systems of nonlinear matrix equations. The approach presented here forms a basis for probabilistic comparison algorithms oriented to automatic assessment of students’ answers in computer science.

1 Introduction

In this paper we consider a language on an alphabet $V_T$, a set of valid words $W(V_T) = V_T^*$. Being this set usually infinite, it is necessary to use grammars as a mechanism for definition of the languages. We are going to consider only context-free grammars, covering context-free languages. The capability of writing correct grammars is an essential task in computer science (used, for example, in the creation of programming languages, compilers, etc.).

Assessment of student’s answers in computer science is a very time-consuming activity. Computer-assisted assessment is a natural way to reduce the time spent by the teachers in their assessment task (see, e.g., [3, 2]). This paper deals with assessment in the theory of context-free grammars. Its main objective is to create a theoretical basis for probabilistic algorithms allowing one to decide if two context-free grammars are equivalent or not.

It is well-known that the equivalence of two context-free grammars is an undecidable problem [9]. This is true if we consider the problem as an algebraic one. In this paper we show that the problem admits a solution if considered as a problem of analysis. Such a situation is not new. For example, the result of work [13] can be interpreted as an impossibility to prove the Fermat’s Last Theorem using arithmetic means. On the other hand, the famous paper by Wiles [12] contains the proof, but it is based on ideas of continuous mathematics.

The problem of context-free grammars equivalence was an object of intensive studies [4, 5]. For example, it was solved when the equivalence is understood in structural sense [8], and some practical algorithms for grammars equivalence checking were developed (see [6, 7], and the references therein).
In order to present the methodology adapt in this paper, let us consider the following simple example. Let the language \( L \) be
\[
L = c, ab, acb, accb, acccb, \ldots
\]
According to [10] we can write a formal power series
\[
S = c + ab + acb + accb + \ldots \tag{1}
\]
corresponding to this language. The language \( L \) can be generated by the grammar
\[
S \to aAb \mid c \\
A \to cA \mid \varepsilon
\]
The following system of formal equations corresponds to this grammar
\[
S = aAb + c \tag{2} \\
A = cA + \varepsilon \tag{3}
\]
Formally applying the iteration method to this system we obtain series (1). Below we define a transform that attributes a matrix meaning to formal power series (1). Namely,
- any terminal letter \( a, b, c \), is substituted by an \((N \times N)\)-matrix \( \mu_a, \mu_b, \mu_c \);
- the nonterminal symbols \( S \) and \( A \) are substituted by \((N \times N)\)-matrix variables \( S(\mu) \) and \( A(\mu) \);
- the formal sum and product are substituted by the matrix ones;
- the empty word \( \varepsilon \) is substituted by the \((N \times N)\) identity matrix.

Then to \( S \) corresponds a matrix \( S(\mu) = S(\mu_a, \mu_b, \mu_c) \) calculated as the sum of the matrix series
\[
S(\mu) = \mu_c + \mu_a \mu_b + \mu_a \mu_c \mu_b + \mu_a \mu_c \mu_b + \mu_a \mu_c \mu_b + \ldots
\]
In order to effectively compute this sum we numerically solve the system of matrix equations
\[
S(\mu) = \mu_a A(\mu) \mu_b + \mu_c \\
A(\mu) = \mu_c A(\mu) + I
\]
obtained applying the transform to formal system (2) and (3). In the same way, in general case of a grammar with the terminal alphabet \( V_T = \{a_1, \ldots, a_n\} \), one can calculate the matrix \( S(\mu) = S(\mu_{a_1}, \ldots, \mu_{a_n}) \).

The main result proved in this paper (Distinguishability Theorem I) shows that if two languages \( L_1 \) and \( L_2 \) generated by context-free grammars are different, then there exists a matrix substitution \( \mu_{a_1}, \ldots, \mu_{a_n} \) such that
\[
S_1(\mu_{a_1}, \ldots, \mu_{a_n}) \neq S_2(\mu_{a_1}, \ldots, \mu_{a_n})
\]
(Note that the languages with different ambiguities we consider as different languages.)
This property allows one to construct probabilistic tools for comparison of context-free grammars. Namely, we calculate $S_1(\mu_{a_1}, \ldots, \mu_{a_n})$ and $S_2(\mu_{a_1}, \ldots, \mu_{a_n})$ for a sufficiently large number of matrix substitutions and if for all substitutions the equality $S_1(\mu_{a_1}, \ldots, \mu_{a_n}) = S_2(\mu_{a_1}, \ldots, \mu_{a_n})$ is satisfied, then we conclude that the grammars are equivalent. In this paper we do not discuss the details of such algorithms, the number of substitutions needed to conclude that the grammars are equivalent with some probability, etc. This will be the subject of further research. We would like to note that according to our experience one $(2 \times 2)$ or $(3 \times 3)$ random matrix substitution is enough to distinguish between two different context-free grammars.

This paper continues the research started in [1] where we considered distinguishability based on $2 \times 2$-matrices. Due to negligence, the distinguishability theorem was not clearly formulated. We remedy this situation below (Theorem 5).

Note also that the idea to use matrices to study formal power series is not new (cf. [11]), but the approach presented in [11] is rather different from the one discussed here.

The paper is organized as follows. In Section 2 we prove the distinguishability theorems. Section 3 contains convergence analysis of the iteration method for nonlinear matrix equations. The limitations of the method are discussed in Section 4. Section 5 contains a brief conclusion.

## 2 Distinguishability theorems

Any context-free language can be defined in terms of a formal power series with associative but not commutative variables [9, 10]. Let $V_T$ be the terminal alphabet, $W(V_T)$ the set of words over $V_T$, and $Z_+$ the set of nonnegative integers. A map $\phi : W(V_T) \to Z_+$ defines a formal power series

$$S = \sum_{P \in W(V_T)} \phi(P) P. \quad (4)$$

Let $\mu$ be a map from $V_T$ to the set $R^{N \times N}$ of $N \times N$-matrices. By $P(\mu)$ we will denote the matrix obtained substituting the letters $a_i \in P$ by the matrices $\mu_i$ and calculating the respective matrix product, i.e., if $P = a_{i_1}, \ldots, a_{i_n}$, then $P(\mu) = \mu_{i_1} \cdots \mu_{i_n}$. If the series

$$s(\mu) = \sum_{P \in W(V_T)} \phi(P) P(\mu) \quad (5)$$

converges, its sum is an $N \times N$-matrix. The following distinguishability theorems form a theoretical basis for probabilistic assessment algorithms.

### 2.1 General distinguishability theorems

**Theorem 1** (distinguishability I). Let $S_1$ and $S_2$ be two different formal series corresponding to context free grammars. Then there exist a positive integer $N$ and a matrix substitution $\mu : V_T \to R^{N \times N}$ such that $S_1(\mu) \neq S_2(\mu)$.

To prove the theorem we need two auxiliary lemmas.

**Lemma 2.** Let $U$ and $V$ be two finite sets of words of length $N$. Then there exist a positive integer $N$ and a matrix substitution $\mu : V_T \to R^{N \times N}$ such that $U(\mu) \neq V(\mu)$. 

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Proof. Let \(a_{i_1} \ldots a_{i_l} a_{i_l} \ldots a_{i_t} \in U \cup V\), where \(l \leq N\). We say that \(a_{i_1} \ldots a_{i_l}\) is a sub-word. Denote the set of sub-words by \(S\). Let us consider the set of unit orthogonal vectors \(\{e_0\} \cup \{e_{i_1 \ldots i_l} \mid a_{i_1} \ldots a_{i_t} \in S\}\) in the space of adequate dimension \(N\). We define linear operators \(\mu_i, i = 1, T\) by:

\[
\mu_i e_0 = \begin{cases} e_i, & a_i \in S \\ 0, & \text{otherwise} \end{cases}
\]

\[
\mu_i e_{i_1 \ldots i_l} = \begin{cases} e_{i_1 \ldots i_l}, & a_i a_{i_1} \ldots a_{i_t} \in S \\ 0, & \text{otherwise} \end{cases}
\]

Let \(a_{i_1} \ldots a_{i_N} \in U \cup V\) be a word. The corresponding linear operator has the form \(\mu_{i_1} \ldots \mu_{i_N} \in R^{N \times N}\). Obviously we have \(\mu_{i_1} \ldots \mu_{i_N} e_0 = e_{i_1 \ldots i_N}\). Hence we get

\[
U(\mu)e_0 = \sum_{(a_{i_1} \ldots a_{i_N}) \in U} e_{i_1 \ldots i_N} \neq \sum_{(a_{i_1} \ldots a_{i_N}) \in V} e_{i_1 \ldots i_N} = V(\mu)e_0,
\]

Indeed, the sets of orthogonal vectors in the two sums are different. \(\Box\)

**Lemma 3.** The number of words of length \(N\) generated by a context-free grammar does not exceed \(Cq^N\), where the constants \(C\) and \(q\) depend on the grammar.

**Proof.** Without loss of generality the grammar has normal Greibach form. Then the grammar has \(M\) productions of the form \(A \rightarrow aA_{i_1} \ldots A_{i_l}\), where \(A, A_{i_1}, \ldots, A_{i_l} \in V_N, a \in V_T\) and \(l \leq L\). Without loss of generality \(ML > 1\). The application of all possible sequences of \(N\) productions generates no more than

\[
ML + (ML)^2 + \ldots + (ML)^N = ML \left(\frac{(ML)^N - 1}{ML - 1}\right) < \frac{ML}{ML - 1}(ML)^N
\]

words. This set of words contains all words of length \(N\). Indeed, any sequence of \(N + 1\) productions contains at least \(N + 1\) terminal symbols. \(\Box\)

**Proof of Theorem.** Since the series are different, they admit the following representation:

\[
S_1 = S_0 + U + R_1 \quad \text{and} \quad S_2 = S_0 + V + R_2,
\]

where \(S_0\) is the part of coinciding words of length less than or equal to \(N\), \(U\) and \(V\) are different parts composed of words with length equal to \(N\), and \(R_1\) and \(R_2\) contain terms with words of length greater than \(N\). By Lemma\(^2\) there exist a positive integer \(N\) and a matrix substitution \(\mu : V_T \rightarrow R^{N \times N}\) such that \(U(\mu) \neq V(\mu)\). Let \(t > 0\). Then we have

\[
\Delta(t) = S_1(t\mu) - S_2(t\mu) = t^N(U(\mu) - V(\mu)) + (R_1(t\mu) - R_2(t\mu)).
\]

The norms of the matrices \(\mu_i, i = 1, T\), constructed in Lemma\(^2\) do not exceed \(\sigma > 0\). By Lemma\(^3\) we obtain

\[
\|R_1(t\mu)\| \leq C_1(q_1\sigma t)^{N+1} \quad \text{and} \quad \|R_2(t\mu)\| \leq C_2(q_2\sigma t)^{N+1}.
\]

Let \(C = \max\{C_1, C_2\}\), \(q = \max\{q_1, q_2\}\) and \(t < 1/(q\sigma)\). Then we get

\[
\|R_1(t\mu) - R_2(t\mu)\| \leq 2C ((q\sigma t)^{N+1} + (q\sigma t)^{N+2} + \ldots) = 2c(q\sigma)^{N+1}\frac{t^{N+1}}{1 - q\sigma t},
\]
From this and (6) we see that \( \Delta(t) \neq 0 \) whenever \( t > 0 \) is sufficiently small. \( \square \)

In many situations it suffices to consider matrix substitutions \( \mu : V_T \to R^{2 \times 2} \). We associate with the symbols \( a_i \in V_T, i = 1, I \), pairs of variables \( u_i \) and \( v_i \), \( i = 1, I \). Let

\[
U = \sum_{\{(k_N^{m},...,k_1^{m})|m=1,M\}} a_{k_N^m} \ldots a_{k_1^m} \quad \text{and} \quad V = \sum_{\{(k_N^{m'},...,k_1^{m'})|m=1,M\}} a_{k_N^{m'}} \ldots a_{k_1^{m'}}
\]

be two sets of words. Consider two sets of associated polynomials

\[
\mathcal{P}_U = \left\{ \prod_{j=l+1}^{N} u_{k_{j}^{m}} v_{k_{j}^{m}} \mid l = 1, N \right\}, \quad \text{and} \quad \mathcal{P}_V = \left\{ \prod_{j=l+1}^{N} u_{k_{j}^{m'}} v_{k_{j}^{m'}} \mid l = 1, N \right\}.
\]

(Here \( u_{k_{N+1}^{m}} = u_{k_{N+1}^{m'}} = 1 \).)

We say that \( U \) and \( V \) satisfy condition (\( \mathcal{P} \)) if \( \mathcal{P}_U \neq \mathcal{P}_V \).

**Lemma 4.** Assume that \( U \) and \( V \) satisfy condition (\( \mathcal{P} \)), then there exist matrix substitution \( \mu : V_T \to R^{2 \times 2} \) such that \( U(\mu) \neq V(\mu) \).

**Proof.** Consider the matrices

\[
\mu_i = \begin{pmatrix} u_i & v_i \\ 0 & 1 \end{pmatrix}.
\]

By induction we easily obtain

\[
\prod_{i=1}^{N} \mu_{k_i} = \begin{pmatrix} \prod_{i=1}^{N} u_{k_i} & \sum_{i=1}^{N} \prod_{j=i+1}^{N} u_{k_j} v_{k_i} \\ 0 & 1 \end{pmatrix}.
\]

From this representation we see that \( \mathcal{P}_U \neq \mathcal{P}_V \) implies \( U(\mu) \neq V(\mu) \). \( \square \)

**Theorem 5** (distinguishability II). Let \( S_1 \) and \( S_2 \) be formal series corresponding to context-free grammars. Assume that the series admit the following representation:

\[
S_1 = S_0 + U + R_1 \quad \text{and} \quad S_2 = S_0 + V + R_2,
\]

where \( S_0 \) is the part of coinciding words of length less than or equal to \( N \), \( U \) and \( V \) are different parts composed of words with length equal to \( N \), and \( R_1 \) and \( R_2 \) contain terms with words of length greater than \( N \). If \( U \) and \( V \) satisfy condition \( \mathcal{P} \), then there exists a matrix substitution \( \mu : V_T \to R^{2 \times 2} \) such that \( S_1(\mu) \neq S_2(\mu) \).

**Proof.** Using Lemma 4 and following the proof of Theorem 3 we obtain the result. \( \square \)
2.2 Examples

Let \( U = \{aab, bab\} \) and \( V = \{aba, bba\} \). In this case condition \( \mathcal{P} \) is satisfied since \( u_1 u_1 v_2 \in \mathcal{P}_U \) and \( u_1 u_1 v_2 \notin \mathcal{P}_V \). On the other hand, there exist languages/grammars that cannot be distinguished with the help of \( 2 \times 2 \)-matrices. For example, using Maxima computer algebra system it is easy to show that for any choice of \( 2 \times 2 \)-matrices \( \mu_1 \) and \( \mu_2 \) we have

\[
\mu_1 \mu_1 \mu_2 \mu_1 + \mu_1 \mu_2 \mu_1 \mu_2 + \mu_2 \mu_1 \mu_2 \mu_1 = \mu_1 \mu_1 \mu_2 \mu_2 + \mu_1 \mu_2 \mu_2 \mu_1 + \mu_2 \mu_1 \mu_1 \mu_2 .
\]

Therefore the languages

\[
\{aabba, abaab, babaa\} \quad \text{and} \quad \{aabab, abbaa, baaba\}
\]

(7) cannot be distinguished using \( 2 \times 2 \)-matrices. However substituting \( 3 \times 3 \)-matrices it is easy show that the languages (7) are different.

2.3 Comparison of short words

In many situations the difference between two grammars can be detected comparing short words through substitution of \( 2 \times 2 \)-matrices. We fulfilled about \( 47 \cdot 10^6 \) tests to analyze finite languages over terminal alphabet \( \{a, b, c\} \) containing no more than three words of length less than or equal to five. We found that only the following languages cannot be distinguished using \( 2 \times 2 \)-matrix substitutions. Namely, the pair of the languages

\[
S_1 : aabca|abaac|bacaa \quad \text{and} \quad S_2 : aabac|abaca|baaca
\]

(8)

and other pairs obtained as the result of permutation of the letters \( \{a, b, c\} \) and/or substitution of \( c \) by \( a \) or \( b \). We shall denote this set of pairs of languages by \( L \).

This proves the following theorem.

**Theorem 6** (distinguishability III). Let \( S_1 \) and \( S_2 \) be formal series corresponding to context-free grammars over the terminal alphabet \( \{a, b, c\} \) and let \( N \leq 5 \). Assume that the series admit the following representation:

\[
S_1 = S_0 + U + R_1 \quad \text{and} \quad S_2 = S_0 + V + R_2,
\]

where \( S_0 \) is the part of coinciding words of length less than or equal to \( N \), \( U \) and \( V \) are different parts composed of no more than three words with length equal to \( N \), and \( R_1 \) and \( R_2 \) contain terms with words of length greater than \( N \). If the pair \( U \) and \( V \) does not coincide with one of the pairs from \( L \), then there exists a matrix substitution \( \mu : V_T \rightarrow R^{2 \times 2} \) such that \( S_1(\mu) \neq S_2(\mu) \).

Proof. Following the proof of Theorem 1 we obtain the result. \( \Box \)

Note that the pair of languages (8) can be used to construct examples of infinite languages that cannot be distinguished using \( 2 \times 2 \)-matrices. One of such examples is the pair

\[
S \rightarrow aabAa \mid abaaA \mid baAaa
\]

\[
A \rightarrow aA \mid b
\]
and

\[ S \rightarrow aabaA \mid abAaa \mid baaAa \]
\[ A \rightarrow aA \mid b \]

Note also that in all the grammars of several programming languages and of educational exercises that we analyzed, the distinguishability was always possible using \((2 \times 2)\)-matrices.

### 3 Systems of nonlinear matrix equations

It is well-known \([9, 10]\) that to any context-free grammar there corresponds a system of nonlinear equation that allows one to obtain the respective formal power series via successive iterations. The terms of the series are the words of the respective language. This correspondence between series and systems of nonlinear equations makes it possible to effectively compute the sums of the series for any \(N \times N\)-matrix substitution. Let \(X_i, i = 0, m, \) be the nonterminals of a context-free grammar and let \(P_{ij}, i = 0, m, j = 1, I_i, \) be the words that appear in the right-hand sides of productions with the left-hand sides \(X_i.\)

The system of equations corresponding to the grammar has the form

\[ X_1 = P_{11}^1 + \ldots + P_{1i}^1, \]
\[ \vdots \]
\[ X_m = P_{m1}^m + \ldots + P_{mi}^m. \]

Substituting the symbols of the terminal alphabet \(a_k \in P_{ij}\) by matrices \(\mu_k,\) we obtain a system of nonlinear matrix equations with unknowns \(X_i.\) This system, \(X = F(X),\) can be solved using the iterative process \(X_{k+1} = F(X_k), X_0 = 0,\) or using the Newton method. As we shall see in the sequel the convergence of the method of successive iterations can be guaranteed for a large class of grammars, for example for the grammars in Chomsky and Greibach normal forms. Note that for regular grammars system \([9]\) is linear.

#### 3.1 Convergence of successive iterations

Let us consider a context-free grammar with productions

\[ X_i \rightarrow P^{0,i}_j, \quad i = 1, n, \quad j = 1, J_i^P \]
\[ X_i \rightarrow p^i_j, \quad i = 1, n, \quad j = 1, J_i^p, \]

where \(P^{0,i}_j \in W(V_N \cup V_T) \setminus W(V_T)\) and \(p^i_j \in W(V_T), p^i_j \neq \varepsilon.\) We assume that the words \(P_j^i\) contain more than one symbol. (For example, a grammar in Chomsky or Greibach normal form is of this type.) The structure of these words can be described in the following manner:

\[ P^{l,i}_j = q^{l+1,i}_j X_{k^{l+1,i}_j} P^{l+1,i}_j, \quad l = 1, L_j \]

where \(P^{l,i}_j \in W(V_N \cup V_T) \cup \{\emptyset\}\) and \(q^{l,i}_j \in W(V_T) \cup \{\emptyset\}.\) The corresponding system of equations has the following structure:

\[ X_i = F_i(X_1, X_2, \ldots, X_n) = \sum_j P^{0,i}_j + \sum_j p^i_j \quad (10) \]
To simplify the notations we denote by $P$ the matrix $P(\mu)$ obtained substituting the symbols $a_i$ and $X_i \in P$ by the matrices $\mu_i$ and $\xi_i$, respectively. We use the notation $\hat{P}$ when the symbols $X_i \in P$ are substituted by the matrices $\hat{\xi}_i$. Let $X = (X_1, \ldots, X_n)$ and $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_n)$ be two collections of $n$ matrices $\mathcal{N} \times \mathcal{N}$. Then, using our notations, we have

\[
\begin{align*}
\mathcal{P}_{j}^{0,i} - \hat{\mathcal{P}}_{j}^{0,i} &= q_j^{1,i}(X_{k_j^{1,i}}\mathcal{P}_j^{1,i} - \hat{X}_k^{1,i}\hat{\mathcal{P}}_j^{1,i}) = q_j^{1,i}(X_{k_j^{1,i}} - \hat{X}_k^{1,i})(\mathcal{P}_j^{1,i} - \hat{\mathcal{P}}_j^{1,i}) \\
&= q_j^{1,i}(X_{k_j^{1,i}} - \hat{X}_k^{1,i})\mathcal{P}_j^{1,i} + \hat{X}_k^{1,i}(q_j^{2,i}(X_{k_j^{2,i}} - \hat{X}_k^{2,i})\mathcal{P}_j^{2,i} + \hat{X}_k^{2,i}(\mathcal{P}_j^{2,i} - \hat{\mathcal{P}}_j^{2,i})) \\
&= \ldots = Y_j^{1,i}(X_{k_j^{1,i}} - \hat{X}_k^{1,i})Z_j^{1,i} + \ldots + Y_j^{n,i}(X_{k_j^{n,i}} - \hat{X}_k^{n,i})Z_j^{n,i},
\end{align*}
\]

where $Y_j^{l,i}, Z_j^{l,i} \in W(\mathcal{V}_N \cup \mathcal{V}_T)$ and $n_j$ is the number of nonterminals in the word $\mathcal{P}_j^{0,i}$.

Assume that the norms of all matrices $a_i$, $X_i$, and $\hat{X}_i$ do not exceed $\delta > 0$ and that $\hat{n}\delta < 1$, where $\hat{n} = \sum_{i,j} n_j$. Then from the representation (11) we obtain

\[
\max_{i=1,n} ||F_i(X) - F_i(\hat{X})|| \leq \hat{n}\delta \max_{j=1,n} ||X_j - \hat{X}_j||
\]  

(12)

Set $\hat{X} = 0$. Then if $||X_i|| < \delta$, we get

\[
\max_{i=1,n} ||F_i(X)|| \leq \hat{n}\delta \max_{j=1,n} ||X_j|| < \hat{n}\delta^2 < \delta.
\]

Let us consider a closed ball $\mathcal{B} = \{X \mid \max_{j=1,n} ||X_j|| \leq \delta\}$ in the space of matrices $X = (X_1, \ldots, X_n)$. We have proved that $F(\mathcal{B}) \subset \mathcal{B}$ and that $F$ is a contracting map. Hence there exists a unique fixed point $\hat{X} = F(\hat{X}) \in \mathcal{B}$. This fixed point is the limit of the sequence of iterations

\[
X^{k+1} = F(X^k), \quad k = 0, 1, \ldots, \quad X^0 = 0.
\]  

(13)

Thus we have the following result.

**Theorem 7 (Convergence).** Assume that the system of nonlinear equations corresponding to a context-free grammar has form (10) and that the words with nonterminal symbols, $P_j$, contain more than one symbol. Then substituting the terminal symbols by matrices with a sufficiently small norm (less than $\delta$), we can guarantee the convergence of the sequence (13) to a unique solution of the matrix system (10).

**Example** Let us consider the grammar

\[
\begin{align*}
S &\rightarrow SaA \mid a \\
A &\rightarrow cSd \mid b
\end{align*}
\]

(14)

The respective system of equations is

\[
\begin{align*}
S &= F_S = SaA + a, \\
A &= F_A = cSd + b.
\end{align*}
\]

(15)

The conditions of Theorem 7 are satisfied (the words $SaA$ and $cSd$ contain more than one symbol). Therefore the system can be solved using the iteration method whenever the symbols $a$, $b$, $c$, and $d$ are replaced by matrices with a sufficiently small norm.
3.2 Other cases where the iteration method can be used

In many cases the grammar may have productions of the form $A \rightarrow B$. The iteration method can be applied also to the corresponding system of equations after some transformation. Namely, assume that the system has the form

$$X = F(X) + \Lambda X,$$  \hspace{1cm} (16)

where $F$ has form (10) considered above and $\Lambda : R^{(N \times N)^n} \rightarrow R^{(N \times N)^n}$ is a linear operator such that there exists the inverse $(I - \Lambda)^{-1}$. Then system (16) is equivalent with the system

$$X = (I - \Lambda)^{-1}F(X).$$

Obviously the map $X \rightarrow (I - \Lambda)^{-1}F(X)$ is contracting and transforms $B = \{X \mid \max_{j=1,N} \|X_j\| \leq \delta\}$ into $B$, whenever the terminal symbols are replaced by matrices with a sufficiently small norms.

Example

Let us consider the grammar

$$S \rightarrow SaA \mid A$$
$$A \rightarrow cSd \mid b$$  \hspace{1cm} (17)

which is a correct solution of an exercise. (This example is taken from [1].) The corresponding system of equations reads:

$$S = SaA + A,$$
$$A = cSd + b.$$  \hspace{1cm} (18)

Below we present three other possible answers.

Alternative correct solution  The following grammar is different but generates the same language:

$$S \rightarrow AaS \mid A$$
$$A \rightarrow cSd \mid b$$  \hspace{1cm} (19)

The corresponding system of equations is

$$S = AaS + A,$$
$$A = cSd + b.$$  \hspace{1cm} (20)

Wrong answer  The following grammar does not generate the same language (does not generate the word $cbabd$):

$$S \rightarrow SaA \mid A$$
$$A \rightarrow cAd \mid b$$  \hspace{1cm} (21)

The corresponding system is

$$S = SaA + A,$$
$$A = cAd + b.$$  \hspace{1cm} (22)
**Ambiguous grammar** The following grammar generates the same language but is ambiguous (the word *baba* can be generated by different ways):

\[
S \rightarrow SaS \mid A \\
A \rightarrow cSd \mid b
\]  

(23)

The corresponding system has the form

\[
S = SaS + A,
\]

\[
A = cSd + b.
\]  

(24)

System (18) corresponding to grammar (17) is equivalent to the system

\[
\begin{pmatrix}
S \\
A
\end{pmatrix} = \begin{pmatrix}
I & I \\
0 & I
\end{pmatrix}^{-1} \begin{pmatrix}
SaA \\
cSd + b
\end{pmatrix}.
\]

(Here \(I\) is the identity matrix.) Replacing the symbols \(a, b, c,\) and \(d\) by matrices with sufficiently small norm we get a system that can be solved using the iteration method. In the same way we can transform and solve other systems. Starting the iterative process with \(S = A = 0\) and solving systems (18), (20), (22), and (24) we see that the difference between \(S\) components of solution of systems (18) and (20) is zero, while for the pair (18) and (22) or (18) and (24) the difference is not zero. This allows one to clearly distinguish between right and wrong answers.

Note that the interval where the components of the matrices are generated must be (a) sufficiently small in order to guarantee the convergence of the iterative process, (b) big enough to distinguish between two different languages.

Another important case deals with the grammars having productions of the form \(A \rightarrow \varepsilon\). In this case some equations have the form

\[X_i = F_i(X_1, X_2, \ldots, X_n) + I\]

Introducing new variables \(Y_i = X_i - I\) in many situations it is possible to transform the system to a system satisfying conditions of Theorem 7.

**Example**

Let us consider the grammar

\[
S \rightarrow SaA \mid b \\
A \rightarrow cSd \mid \varepsilon
\]

The corresponding system of nonlinear equations is

\[
S = SaA + b \\
A = cSd + I
\]

Introducing new variable \(B = A - I\) we obtain the system

\[
S = SaB + Sa + b \\
B = cSd
\]

which can be solved by iteration method.
4 Limitations of iteration method

In some cases the iteration method cannot distinguish between two different grammars. This happens when we consider grammars with very long words from $W(V_T)$. The point is that the computer precision is not sufficient to correctly compute products of many small numbers. Consider the following grammar

$$S \to AS \mid BS \mid B \nonumber$$
$$A \to a_1a_2\ldots a_n;$$
$$B \to a_1|a_2|\ldots|a_n; \nonumber$$

The corresponding system of matrix equations reads

$$S = AS + BS + B, \nonumber$$
$$A = a_1a_2\ldots a_n, \nonumber$$
$$B = a_1 + a_2\ldots + a_n. \nonumber$$

Let the second grammar be

$$S \to AS \mid BS \mid B \nonumber$$
$$A \to a_2a_1\ldots a_n;$$
$$B \to a_1|a_2|\ldots|a_n; \nonumber$$

with the corresponding system of equations

$$S = AS + BS + B, \nonumber$$
$$A = a_2a_1\ldots a_n, \nonumber$$
$$B = a_1 + a_2\ldots + a_n. \nonumber$$

Systems (26) and (28) are equivalent to the equations

$$S = a_1a_2\ldots a_nS + (a_1 + a_2 + \ldots + a_n)S + (a_1 + a_2 + \ldots + a_n) \nonumber$$

and

$$S = a_2a_1\ldots a_nS + (a_1 + a_2 + \ldots + a_n)S + (a_1 + a_2 + \ldots + a_n), \nonumber$$

respectively. To guarantee the convergence of iterations we have to impose the restriction

$$\|a_1\| + \|a_2\| + \ldots + \|a_n\| = \alpha < 1. \nonumber$$

From the inequality of arithmetic and geometric means we get

$$\sqrt[n]{\|a_1\| \ldots \|a_n\|} \leq \frac{\|a_1\| + \ldots + \|a_n\|}{n}. \nonumber$$

Hence for sufficiently large $n$ we have

$$\|a_1\| \ldots \|a_n\| \leq \frac{\alpha^n}{n^n} < \delta, \nonumber$$

where $\delta > 0$ is the computer precision. Thus the computer interprets the iterative processes

$$S_{k+1} = a_1a_2\ldots a_nS_k + (a_1 + a_2 + \ldots + a_n)S_k + (a_1 + a_2 + \ldots + a_n) \nonumber$$
and
\[ S_{k+1} = a_2a_1 \cdots a_nS_k + (a_1 + a_2 + \cdots + a_n)S_k + (a_1 + a_2 + \cdots + a_n) \]
as the same process
\[ S_{k+1} = (a_1 + a_2 + \cdots + a_n)S_k + (a_1 + a_2 + \cdots + a_n) \]
and, therefore, the method does not allow to distinguish between two grammars. However, the method can be applied to real size grammars with a satisfactory result (see [1]).

5 Conclusion

In this paper we addressed the problem of context-free grammars comparison from the analysis point of view. A substitution of terminal letters by matrices allows one to reduce the comparison problem to numerical solution of systems of nonlinear matrix equations. Besides the elegance of the process, this method constitutes a solid base for construction of algorithms and tools for probabilistic comparison of context-free grammars. Our experiments with a built prototype show that the use of this probabilistic comparison method with (2 × 2) and (3 × 3)-matrices in problems appearing in e-learning framework and even in cases of large grammars is very efficient.

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