DYNAMICS OF THE QCD STRING WITH LIGHT AND HEAVY QUARKS

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Abstract

The generalization of the effective action [1] of the quark–antiquark system in the
confining vacuum is performed for the case of arbitrary quark masses. The interaction of
quarks is described by the averaged Wilson loop for which we use the minimal area law
asymptotics.

The system is quantized by the path integral method and the quantum Hamiltonian is
obtained. It contains not only quark degrees of freedom but also the string energy density.

As well as in the equal masses case [1] two dynamical regimes are found [2]: for
large orbital excitations ($l \gg 1$) the system is represented as rotating string, which leads
to asymptotically linear Regge trajectories, while at small $l$ one obtains a potential-like
relativistic or nonrelativistic regime.

In the limiting cases of light-light and heavy-light mesons a unified description is
developed [2]. For the Regge trajectories one obtains nearly straight-line patterns with
the slope very close to $1/2\pi\sigma$ and $1/\pi\sigma$ correspondingly. The upper bound on the light
quark(s) masses which doesn’t change considerably this property of the trajectories is also
found.

1 Introduction

Recently the new approach to study nonperturbative large distance dynamics of quark-
antiquark system in the confining vacuum has been developed and the Hamiltonian for the
case of equal quark masses has been obtained [1]. In the present paper we consider the general
case of arbitrary quark masses. The effective Hamiltonian of the system is derived (for a short
report see [2]) and properties of its spectrum are analysed.

With the help of vacuum correlator formalism [3] we represent gauge invariant Green function
of the $q\bar{q}$ system in a form where all dynamics of the interaction is described by the averaged
Wilson loop operator. Starting from the QCD Lagrangian and making use of the minimal area
law
law asymptotics for the Wilson loop we arrive to the string picture of $q - \bar{q}$ interaction. The appearance of the string at large distances has been discussed for a long time \cite{4,5} (for recent calculations of the colour-tube or string dynamics see ref. \cite{5} ) and here the world sheet of the string coincides with the surface appearing in the area law of Wilson loop.

According to the vacuum correlators method \cite{6,3} it is the minimal surface which enters the area law asymptotics. This approximation provides a good calculation scheme at least for the consideration of the leading Regge trajectories.

As in \cite{3,1} the minimal surface is approximated as the world sheet of the straight line connecting positions of quark and antiquark with the same time in the meson rest frame. We discuss the dynamical motivations of this ansatz for the area, which leads to a local in time string interaction usually postulated in the flux tube models picture \cite{5}. In this approximation the string (minimal string) may rotate and oscillate longitudinally (stretching and expanding).

We simplify our problem by disregarding effects due to the quark spins and additional quark pairs creation having in mind to come back to them in subsequent papers and perform path integral quantization of the system of quark-antiquark with arbitrary masses connected by the minimal string.

It turns out that our effective action can be represented in a form postulated in accordance with the flux tube picture in ref.\cite{5} without direct connection to the QCD Lagrangian.

We generalize the procedure for determination of the center of masses coordinate as compared with that of the equal masses case \cite{1} where the symmetry between quarks simplifies the consideration. Due to the string contribution to the kinetic part of the action, we are to exploit the condition, that the center of masses coordinate is decoupled from the relative one.

As well as in the equal masses case we find two dynamical regimes of the ”minimal” QCD string with quarks of arbitrary masses. They are distinguished by the energy- momentum distribution of nonperturbative gluonic fields along the string: for large orbital momenta $l \gg n_r$ the main part of the energy (orbital momentum) is contained in the rotating string and the resulting spectrum in the leading approximation coincides with that of the pure string. For low values of $l$ ($l \ll n_r$) the dynamics is described by a relativistic or nonrelativistic (depending on quark masses) potential-like approach with almost inert contribution from the string. The transition between these two regimes is very smooth and the relativization of heavy quark(s) due to increase of $l, n_r$ is briefly discussed.

For two limiting cases of heavy-light and light-light mesons a unified description is developed. The resulting spectra in both cases are very simple: practically linear leading Regge-trajectories and nearly parallel to them daughter trajectories, corresponding to radial excitations with the slopes $1/2\pi\sigma$ and $1/\pi\sigma$ correspondingly. We estimate also the upper bound on the light quark(s) masses, which don’t change considerably the slope of the trajectories.

We note that this pattern for the trajectories is very close to one obtained by numerical quantization \cite{5} of the same system which was performed without qualitative consideration of the underlying regimes of quark-string interaction.

The modification of heavy quarkonia Hamiltonian due to the string contribution to the orbital momentum is considered. The corresponding correction to the energy is estimated and we find the conditions under which it is comparable with the first relativistic correction.

The paper is organized in the following way. In Section 2 we derive the effective action of the $qq$ system at large distances and discuss the basic approximations, which enable one to reduce initial field theory problem to the local three dimensional quantum mechanics of quarks connected by the straight line string. In Section 3 we represent this action in a gaussian with
respect to the quark coordinates form. The Hamiltonian for the "minimal" QCD string with quarks is obtained in Section 4 and in Section 5 we analyse the properties of its spectrum.

In Appendix A intermediate steps of the transformation to the gaussian representation are considered. Technical details for calculation of the first corrections to zero order pure potential and pure string approximations for the dynamics are collected in Appendixes B and C correspondingly.

2 Effective QCD action of the $q\bar{q}$ system at large distances

We start with the Green function for spinless $q\bar{q}$-system in quenched approximation, which in the Feynman-Schwinger representation has the following form [3]

$$G(x,\bar{x} \mid y,\bar{y}) = \int_0^\infty ds \int_0^\infty d\bar{s}e^{-s-K-\bar{K}} DzD\bar{z} < W(C) >_A ,$$

where $W(C)$ is the usual Wilson loop operator

$$W(C) = tr P \exp [ig \int_C A_\mu dz_\mu]$$

with

$$K = m_1^2 s + \frac{1}{4} \int_0^s \dot{z}_\mu^2(t) dt , \quad \dot{z}_\mu = \frac{d\dot{z}_\mu(t)}{dt}$$

and the analogous for $\bar{K}$.

The closed contour $C$ consists of initial and final pieces $[x,\bar{x}] , \ [y,\bar{y}]$ entering the boundary conditions and paths $z(t) , \ \bar{z}(\bar{t})$ of the quark and antiquark.

At large distances $R > T_g \sim 0.2-0.3 \ fm$ (where $T_g$ is the vacuum gluon correlation length) one gets from the cumulant expansion for Wilson loop asymptotics (omitting perimeter-type terms always coming from the quark-mass renormalization ) [6,3]

$$< W(C) > \approx \exp(-\sigma S)$$

where $S$ is the area of the minimal surface inside the contour $C$.

This asymptotics is well confirmed by lattice simulations and allows one to take advantage of the approximate reduction of gluodynamics to the formation of buozeen string between quarks. As a consequence we are left at this step with the action depending only on quark coordinates $z_\mu, \bar{z}_{\bar{\mu}}$ (since the minimal surface is determined by the form of these trajectories).

The reparametrization from proper times $t,\bar{t}$ to the Euclidean times $z_0 \equiv \tau , \ \bar{z} \equiv \bar{\tau}$ performed as in [1] yields

$$dt = \frac{dz_0}{\dot{z}_0} , \quad d\bar{t} = \frac{d\bar{z}_0}{\dot{\bar{z}}_0}$$

which amounts [1] to the following substitution in eq.(3) for kinetic terms

$$K \rightarrow K' = \int_0^\tau d\tau \left[ \frac{m_1^2}{2 \mu_1(\tau)} + \mu_1(\tau) \{ 1 + \bar{z}^2(\tau) \} \right],$$
\[ ds Dz_0 \to D\mu_1 \]

and the analogous for $\bar{K}$. Here we have introduced path integration over the new functions, playing, as we will see, the role of dynamical quark masses

\[ \mu_1(\tau) = \frac{T}{2s} \dot{z}_0(\tau) , \quad \mu_2(\tau) = \frac{T}{2s} \dot{\bar{z}}_0(\tau) \] (7)

and $T = \frac{1}{2}(x_0 + \bar{x}_0 - y_0 - \bar{y}_0)$.

Let us discuss the approximate reduction, which we apply to the dynamics of zero components $z_0$, $\bar{z}_0$. Initial path integral representation takes into account all trajectories, including those with the backward motion in time $d\tau = \dot{z}_0(\gamma) d\gamma < 0$, where the signs of $\dot{z}_0$, $\dot{\bar{z}}_0$ are negative. In the Fock space the backward in time pieces of trajectories are responsible for the creation of additional $q\bar{q}$-pair. Here we neglect this backward motion of $q, \bar{q}$ trajectories since the backtracking in time quark is dragging with itself also the string and this enlarges the action due to the formation of foldings on the string world sheet.

With respect to the space–time picture of the evolution it means that we don’t take into account breaking of our string into several ones due to the quark pairs creation from the vacuum (in addition to the neglect of such breaking due to $q\bar{q}$ pairs from quark determinant which corresponds to the quenched approximation (1) we started from).

In what follows this no-backtracking time approximation leads to reduction of quark dynamics to that of the valence quark (connected by the string) Fock sector, where the conditions

\[ \mu_1(\tau) > 0 , \quad \mu_2(\tau) > 0 \] (8)

are valid and the transformations (5) are unique.

Now one is to find reasonable ansatz for the minimal surface of Wilson loop (4) in terms of $z$ and $\bar{z}$. As in [1,2] we introduce standard compact notations

\[ \xi \equiv \{\tau, \beta\} , \quad g_{ab}(\xi) \equiv \partial_a w_\mu \partial_b w^\mu , \quad a, b = \tau, \beta \] (9)

so that the area $S$ is

\[ S = \int d^2 \xi \sqrt{\text{det} g} , \] (10)

where $w_\mu(\tau, \beta)$ are the coordinates of the string world surface, and

\[ \dot{w}_\mu = \frac{\partial w_\mu}{\partial \tau} , \quad w'_\mu = \frac{\partial w_\mu}{\partial \beta} . \]

As well as in [7,1] we can use in the valence quark sector (8) the approximation that the minimal surface for given paths $z_\mu(\tau)$, $\bar{z}_\mu(\tau)$ is determined by eq.(10) with $w_\mu$ given by straight lines, connecting points $z_\mu(\tau)$ and $\bar{z}_\mu(\tau)$ with the same $\tau$, i.e. we exploit the instantaneous approximation to the interaction (10).

\[ z_\mu = (\tau , \bar{z}) , \quad \bar{z}_\mu(\tau) = (\tau , \bar{z}) , \] (11)

and

\[ w_\mu(\tau, \beta) = z_\mu(\tau) \cdot \beta + \bar{z}_\mu(\tau)(1 - \beta) , \quad 0 \leq \beta \leq 1 . \] (12)
We note that this approximation to the surface geometry is inspired by two limiting cases which are of special interest below: in the case \( l = 0 \) one can make use of the flat dynamics of quarks and in the limit \( l \to \infty \) quarks and the string are moving along typical trajectories of the double helicoid type \([7]\), for which the minimal area indeed is formed by the straight-lines. It amounts effectively to the elimination of the second time from the problem and corresponds to the instantaneous formation of the string in accordance with a position of quarks.

Combining the results we obtain the total effective action in the form

\[
A \equiv K' + \bar{K}' + \sigma \int_0^T d\tau \int_0^1 d\beta \sqrt{\text{det}g} \equiv K' + \bar{K}' + A_{\text{str}}.
\]

where the kinetic terms of quarks are

\[
K' + \bar{K}' = \int_0^T d\tau \frac{1}{2} \left[ \left( \frac{m_1^2}{\mu_1(\tau)} + \mu_1(\tau) \left\{ 1 + \dot{z}_1^2(\tau) \right\} \right) + \left( \frac{m_2^2}{\mu_2(\tau)} + \mu_2(\tau) \left\{ 1 + \dot{z}_2^2(\tau) \right\} \right) \right] (14)
\]

We emphasize, that the presence of the terms \( K' + \bar{K}' \) violates the condition that the ends of the string move with the velocity of light.

We stress \([1]\) also, that it is the valence-quark approximation \((8)\) together with the equal time straight-line ansatz \((12)\), which enables one to reduce at large interquark disances nonlocal four dimensional dynamics \((1)\) to the local in time three dimensional dynamics \((13)\).

After integration over \( \mu_1, \mu_2 \) our action \((13)\) obtained from QCD Lagrangian coincides with that of ref.\([5]\) postulated in accordance with the flux tube picture of the interaction.

But as we will see our way of formulation and solution of the Hamiltonian problem is different from numerical one used in \([5]\). The main advantage of our analitical approach is the possibility to reveal underlieing dynamical regimes of quark-string interaction discussed in Section 5.

### 3 Gaussian representation for the effective action

A direct procedure of quantization of eq.\((13)\) is difficult due to the square root term and as in \([1, 2]\) we use the auxiliary fields approach \([8]\) to get rid of it.

The string part of action \( A_{\text{str}} \) in \((13)\) is equivalent \([1]\) (for the straight line ansatz) to the following one quadratic in quark coordinates\([8]\)

\[
A_{\text{str}} = \int_0^T d\tau \int_0^1 d\beta \frac{\nu}{2} \left[ \dot{w}^2 + (\sigma/\nu)^2 w'^2 - 2\eta(\dot{w}w') + \eta^2 w'^2 \right] (15)
\]

where the field \( \nu \) will play the role of the string energy density and the conditions \((11)\) for \( z_{\mu}(\tau), \bar{z}_{\mu}(\tau) \) are implied.

The auxiliary fields \( \nu(\tau, \beta) \geq 0 \) and \( \eta(\tau, \beta) \) are integrated out together with \( \mu_i(\tau) \) in the full path integral representation for \( G \)

\[
G = \int D\bar{z}DzD\nu D\eta D\mu_1 D\mu_2 e^{-A}. \]


We emphasize that (up to the preexponential factor, which is immaterial in our case for the derivation of effective action) the integration over $\mu_1, \mu_2, \nu$ and $\eta$ effectively amounts to the replacement of them by their extremum values [1].

In the valence quark Fock sector (8), (12) one can introduce relative $r_\mu$ and center of masses $R_\mu$ coordinates in a selfconsistent way and fix the rest frame of the meson. Contrary to the equal masses case [1] here one is to use explicitly the condition that $\dot{R}_\mu$ is decoupled from $\dot{r}_\mu$.

First we note that the action (13), (15) must not contain an explicit dependence on $R_\mu$ (for the total momentum $P_\mu$ to be conserved). Therefore the relative coordinate is to be defined as

$$ r_\mu = w'_\mu = z_\mu(\tau) - \bar{z}_\mu(\tau). $$  (17)

Second, to diagonalize the quadratic in $\dot{z}$, $\dot{\bar{z}}$ kinetic part of $A$ in such a way that the Jacobian

$$ \frac{\partial (R, r)}{\partial (z, \bar{z})} = 1 $$  (18)

we are to introduce

$$ R_\mu = \zeta(\tau)z_\mu(\tau) + (1 - \zeta(\tau))\bar{z}_\mu(\tau) $$  (19)

where the parameter $\zeta(\tau)$ is determined from the condition, that $\dot{R}$ is decoupled from $\dot{r}$.

Actually the action can be rewritten in terms of $\dot{R}_\mu, \dot{r}_\mu, r_\mu$ as

$$ A = \int_0^T d\tau \left[ \frac{m_1^2}{2\mu_1} + \frac{m_2^2}{2\mu_2} + \frac{1}{2} \left\{ a_1\dot{R}^2 + 2a_2(\dot{R}\dot{r}) - 2c_1(\dot{R}r) - 2c_2(\dot{r}r) + a_3r^2 + a_4r^2 \right\} \right] $$  (20)

where we have used $w_\mu = R_\mu + (\beta - \zeta)r_\mu$ to express $\dot{w}$ in terms of $\dot{R}$, $\dot{r}$ and the following notations have been introduced

$$ a_1 = \int_0^1 d\beta (\mu_1 + \mu_2 + \nu), \quad a_3 = \int_0^1 d\beta \left( \mu_1(1 - \zeta)^2 + \mu_2\zeta^2 + (\beta - \zeta)^2\nu \right), $$

$$ a_2 = \int_0^1 d\beta \left( \mu_1 - \zeta(\mu_1 + \mu_2) + (\beta - \zeta)\nu \right), \quad a_4 = \int_0^1 d\beta \left( \frac{a_2^2}{\nu} + \eta^2\nu \right), $$

$$ c_1 = \int_0^1 d\beta \eta\nu, \quad c_2 = \int_0^1 d\beta \eta(\beta - \zeta)\eta\nu. $$  (21)

For the diagonalization of the kinetic part of $A$ one is to put $a_2 = 0$, which leads to

$$ \zeta(\tau) = \frac{\mu_1(\tau) + <\beta>}{\mu_1(\tau) + \mu_2(\tau) + \int \nu(\tau, \beta)d\beta} $$  (22)

where

$$ <\beta> = \frac{\int \beta\nu d\beta}{\int \nu d\beta}. $$  (23)

It should be stressed, that eq.(22) for $\zeta(\tau)$ which determines the “center of masses” coordinate (19) involves not only quark dynamical masses $\mu_i$ but also the dynamical string energy density $\nu(\tau, \beta)$. This corresponds to the string contribution to the kinetic part of action (20).
Eqs.(19), (22) generalize the equal masses case \( m_1 = m_2 \) [1] where one gets for the extremal values of \( \mu_i \)

\[
\mu_1(\tau) = \mu_2(\tau).
\]  

(24)

Since the extremal values of the function \( \nu(\beta) \) (that only contribute to the action) are even under the exchange \( \beta - \frac{1}{2} \to -(\beta - \frac{1}{2}) \) (due to the symmetry under permutation of the ends of the string) one obtains from (22), (24) for \( m_1 = m_2 \).

\[
\int (\beta - 1/2) \nu d\beta = 0,
\]  

(25)

\[
\zeta(\tau) = \frac{1}{2},
\]  

(26)

\[
\tilde{R}(\tau) = \frac{1}{2} \left( \tilde{z}(\tau) + \tilde{\bar{z}}(\tau) \right).
\]  

(27)

In the end of this section let us fix the frame of reference. For this purpose we perform usual canonical transformation from \( \dot{\tilde{R}} \) to the total momentum \( \tilde{P} \)

\[
\int D\tilde{R} \exp \left[ i \int L(\tilde{R},...)d\tau \right] = \int D\tilde{R} D\tilde{P} \exp \left[ i \int \left\{ \tilde{P} \dot{\tilde{R}} - H(\tilde{P},...) \right\} d\tau \right]
\]  

(28)

where \( H(\tilde{P},...) = \tilde{P} \dot{\tilde{R}} - L(\tilde{R},...) \) and conserved total momentum is defined as

\[
\tilde{P} = \frac{\partial L(\tilde{R},...)}{\partial \dot{\tilde{R}}}. \tag{29}
\]

Meson rest frame corresponnds to \( \tilde{P} = 0 \). Taking into account conditions (11) leading to

\[
\dot{R}_0(\tau) = 1, \quad r_0(\tau) = 0
\]  

(30)

we obtain for the effective action \( \tilde{A} \), equal to \( H(\tilde{P} = 0,...) \)

\[
\tilde{A} = \frac{1}{2} \int_0^T d\tau \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \frac{1}{a_1} \left\{ a_3 a_1 r^2 - 2c_2 a_1 (r \dot{r}) + (a_4 a_1 - c_1^2) r^2 + a_1^2 \right\} \right]. \tag{31}
\]

4 Derivation of the Hamiltonian for the ”minimal” QCD string with quarks

Action (31) contains (through the dependence of \( a_i, c_i \)) the auxiliary functions \( \mu_1, \mu_2, \nu, \eta \). In contrast to the first three ones (playing the roles of quark dynamical masses and string energy density respectively as one can see for example from eq. (22) the function \( \eta(\tau,\beta) \) is an intermediate one. To derive the effective Hamiltonian for the minimal QCD-string with quarks
we perform gaussian integration ($\tilde{A}$ is quadratic form in $\eta$) over $\eta$ to obtain the effective action $A'$

\[ \int D\eta(\tau, \beta) \exp[-\tilde{A}] \sim e^{-A'}. \] (32)

It amounts to the substitution into $\tilde{A}$ the extremal value of $\eta$ (see Appendix A for details)

\[ \eta_{\text{ext}}(\tau, \beta) = \left( \frac{\dot{\vec{r}} \times \vec{r}}{r^2} \right) \frac{\vec{r}^2}{2} \left( \beta - \frac{\mu_1}{\mu_1 + \mu_2} \right) \] (33)

which leads to the following definition of $A'$

\[ A' = \frac{1}{2} \int_0^T d\tau \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \mu_1 + \mu_2 + \int \nu d\beta + \left\{ \mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int (\beta - \zeta)^2 \nu d\beta \right\} \times \right. \]

\[ \times \left. \frac{[\dot{\vec{r}} \times \vec{r}]^2}{r^2} + \tilde{\mu} \frac{(\dot{\vec{r}} \times \vec{r})^2}{r^2} + r^2 \int \frac{\sigma^2 d\beta}{\nu} \right] \] (34)

where

\[ \tilde{\mu} = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}. \]

Expression (34) forms the basis of our further calculations.

To perform canonical transformation from $\dot{\vec{r}}$ to $\vec{p}$ we separate longitudinal and transverse with respect to $\vec{r}$ components of $\dot{\vec{r}}$

\[ \dot{\vec{r}}^2 = \frac{1}{r^2} \left\{ (\dot{\vec{r}} \times \vec{r})^2 + [\dot{\vec{r}} \cdot \vec{r}]^2 \right\} \] (35)

and obtain for the longitudinal and transverse components of the momentum respectively

\[ \vec{p}_r^2 \equiv \left( \frac{\vec{p} \times \vec{r}}{r^2} \right)^2 = \tilde{\mu}^2 \frac{[\dot{\vec{r}} \times \vec{r}]^2}{r^2} \] (36)

\[ \vec{p}_T^2 \equiv \left( \frac{\vec{p} \times \vec{r}}{r^2} \right)^2 = \left( \mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int_0^1 d\beta (\beta - \zeta)^2 \nu \right) \frac{[\dot{\vec{r}} \times \vec{r}]^2}{r^2}. \] (37)

We note that the string energy density $\nu$ doesn’t contribute into the longitudinal component of the momentum, because the momentum density of the string

\[ P_{\mu}^{\text{str}} = \frac{\partial \left( \sigma \sqrt{\dot{w}^2 w'^2 - (\dot{w}w')^2} \right)}{\partial \dot{w}_{\mu}(\tau, \beta)} \] (38)

is orthogonal to $w'_{\mu} = r_{\mu}$

\[ \left( P^{\text{str} w'} \right) = \left( P^{\text{str} r} \right) = 0. \] (39)

The standard derivation of $H$ from the action (13) yields (in the Minkowski space-time) for the Hamiltonian in functional integral

\[ H(p, r, \nu, \mu_1, \mu_2) = \frac{1}{2} \left[ \frac{(\vec{p}_r^2 + m_1^2)}{\mu_1} + \frac{(\vec{p}_r^2 + m_2^2)}{\mu_2} + \mu_1 + \mu_2 + r^2 \int_0^1 \frac{\sigma^2 d\beta}{\nu} + \right. \]

\[ \left. + \right. \]
\[
\int_0^1 \nu d\beta + \left( \frac{\hat{L}^2}{\vec{r}^2} / \left( \mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \frac{1}{2} \int d\beta (\beta - \zeta)^2 \nu \right) \right) \right] \]

where \( \hat{L}^2 = \vec{p}^2 \vec{r}^2 \).

This is the resulting Hamiltonian for the "minimal" QCD string with quarks. For the case of equal masses \( m_1 = m_2 \), which amounts to the possibility of substitutions (24) - (27), it has been derived in ref.[1]. Approximate expressions for eq.(40) with disregard of string contribution to the orbital momentum were obtained in [7].

Let us stress again, that eq.(34) contains auxiliary fields \( \mu_1, \mu_2, \nu \). After integration over them in the full path integral representation (16) only saddle points values \( \mu_i^{ext}(\tau), \nu^{ext}(\tau, \beta) \)

\[
\frac{\partial H}{\partial \mu_i(\tau)} \bigg|_{\mu_i=\mu_i^{ext}, \nu=\nu^{ext}} = 0, \quad i = 1, 2 ; \quad \frac{\delta H}{\delta \nu(\tau, \beta)} \bigg|_{\nu=\nu^{ext}, \mu_i=\mu_i^{ext}} = 0 \]

(41)

contribute [1] in the effective Hamiltonian. The conditions (41) lead to the following equations for the extremal values of auxiliary fields

\[
\frac{\vec{p}^2 + m_1^2}{\mu_1^2(\tau)} = 1 - l(l + 1)/\vec{r}^2 \left( \frac{(1 - \zeta)^2}{a_3^2} - \frac{1}{\mu_1^2} \right), \]

(42)

\[
\frac{\vec{p}^2 + m_2^2}{\mu_2^2(\tau)} = 1 - l(l + 1)/\vec{r}^2 \left( \frac{\zeta^2}{a_3^2} - \frac{1}{\mu_2^2} \right), \]

(43)

\[
\frac{\sigma^2}{\nu^2(\tau, \beta)} \vec{r}^2 = 1 - l(l + 1)/\vec{r}^2 \left( \frac{\beta - \zeta)^2}{a_3^2} \right), \]

(44)

where \( a_3 = \mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int d\beta (\beta - \zeta)^2 \nu d\beta \) and \( \zeta \) is defined by eq.(22).

Only after substitution of these extremal values into the path integral Hamiltonian (40) one is to construct (performing proper Weil ordering [9]) the operator Hamiltonian acting on the wave functions.

5 Dynamical regimes of the "minimal" QCD string with quarks

In this section we perform analysis of dynamics of the "minimal" QCD string with quarks. First we consider the case when both (heavy) quarks are nonrelativistic. We calculate the string contribution to the orbital momentum of heavy quarkonia and find the domain of quantum numbers \( l, n_r \), where the corresponding correction to the energy is comparable with the leading relativistic correction. Second we analyse another limit of one or both light quarks \( (m_i \lesssim \sqrt{\sigma}) \) and find two relativistic regimes for large and small orbital momentums. Finally we discuss finite mass effects and the transition from nonrelativistic dynamics of heavy quark(s) to the relativistic one.

A. The limit of nonrelativistic potential dynamics
Let us consider the heavy quarks limit of the QCD string with quarks
\[ m_1, m_2 \gg \sqrt{\sigma}. \]

In order to deal with a realistic case we impose also the condition that the characteristic distance \( < r > \) between heavy quarks is more than the vacuum gluon correlation length \( T_g \sim 0.2 \text{fm} \) and the dynamics is governed by the confining potential. It amounts to the following constraint on the masses
\[ < r > \sim (2\tilde{m}\sigma)^{-1/3} \gtrsim T_g, \]
where \( \tilde{m} \) is the reduced quark mass.

In this case one easily gets in the leading order from eqs.(42)-(44)
\[ \mu^\text{ext}_i = m_i, \quad i = 1, 2; \]
\[ \nu^\text{ext} = \sigma |\vec{r}|, \]
\[ \zeta = \frac{m_1}{m_1 + m_2}, \]
with the second term in right hand sides of eqs.(42)-(44) being much less than all others. Therefore we obtain neglecting all relativistic and string corrections the nonrelativistic Hamiltonian in the form (for large distances between quarks)
\[ H = m_1 + m_2 + \frac{1}{2\tilde{m}} \vec{p}^2 + \sigma |\vec{r}|, \]  
\[ \text{(47)} \]
where
\[ \tilde{m} = \frac{m_1 m_2}{m_1 + m_2}. \]

The first corrections to eq.(47) can be easily obtained from the eqs.(40)-(44) with the result (for equal masses \( m_1 = m_2 = m \) case)
\[ \Delta H^\text{rel} = -\frac{1}{4 m^3} \frac{p^4}{m^2} - \frac{l(l+1)\sigma}{6m^2 r}. \]  
\[ \text{(48)} \]
The first term comes from the expansion of the ordinary square root \( 2\sqrt{\vec{p}^2 + m^2} \) and the second one corresponds to the string contribution to the orbital momentum (see the next subsection for a more detailed discussion).

Simple estimates (see Appendix B for details) give the following results for \( \Delta E^\text{rel} \)
\[ \Delta E^\text{rel}_{n_r,l} = -\frac{E_{n_r,l}^{(0)}}{36m} - \frac{l(l+1)\sigma^2}{4m^2 E_{n_r,l}^{(0)}}, \]  
\[ \text{(49)} \]
where \( E_{n_r,l}^{(0)} \) is the eigenvalue of the nonrelativistic Hamiltonian (47) after subtraction of heavy masses. One can represent \( E_{n_r,l}^{(0)} \) in the form [7]
\[ E_{n_r,l}^{(0)} = (2m)^{-1/3} \sigma^{2/3} a_{n_r,l}, \]  
\[ \text{(50)} \]
where \( a_{n_r,l} \) is the eigenvalue of the corresponding dimensionless Hamiltonian. Taking for example \( n_r = 0, l = 1, a_{0,1} = 3.36 [7] \) we obtain that for not very large \( l \neq 0, n_r \) the string contribution (the second term in eqs. (48), (49) ) is of order of the first relativistic correction.
Making use of the asymptotical expression \[7\]
\[a_{n,r,l} \approx \left( \frac{n}{2} \right)^{2/3} \left( 3 + O \left( \frac{2n_r}{n} \right) \right), \quad l \gg n_r,\]
where \(n = n_r + l + 1\) we conclude that for large \(l \gtrsim n_r\) these corrections remain to be comparable while for large \(n_r \gg l\) the relativistic one becomes obviously dominant.

In conclusion we note that the string correction to the nonrelativistic Hamiltonian (47) was also considered in ref.[10].

**B. Relativistic potential regime of the system for small orbital momenta**

Let us consider relativistic dynamics of our system and prove the existence of two relativistic dynamical regimes of the ”minimal” QCD string with quarks: potential one for small orbital momenta (or \(n_r \gg l\)) and the string one for large \(l \gg n_r\), which join each other very smothly.

For the equal masses case \(m_1 = m_2\) this analysis has been performed in ref.[1] and here we generalize it for the system with arbitrary masses of quarks.

We begin our consideration of the general case with derivation of the relativistic Hamiltonian for zero orbital momentum \(l = 0\). As we shall prove it provides a good zero approximation for not very large \(l\) and for excitations with radial quantum numbers \(n_r \gg l\).

For the case \(l = 0\) eqs.(42)-(44) give

\[
\mu_i(\tau) = \sqrt{\vec{p}^2 + m_i^2}, \quad i = 1, 2, \quad \nu(\tau, \beta) = \sigma | \vec{r} | \]

so that the Hamiltonian (40) finally has the form

\[
H(\vec{p}, \vec{r}) = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + \sigma | \vec{r} | ,
\]

where \(\vec{p}^2 = \vec{p}_r^2 = (\vec{p}_r^2 + \vec{L}^2 / \vec{r}^2)^2\).

We note again, that in the case of \(l = 0\) the string doesn’t contribute into the kinetic terms and is responsible for the inert potential term \(\sigma | \vec{r} |\).

Expression (54) (valid strictly speaking only for \(l = 0\)) is widely used in the context of the so-called ”relativistic quark models” [11] for arbitrary \(l\). The approximate version of this Hamiltonian was derived in ref.[7].

Now we consider the case of small values of \(l\) or \(l \ll n_r\). To develop a perturbation expansion for the spectrum it appears convenient to recover in expression (40) the dependence on the total momentum \(\vec{p}^2 = (\vec{p}_r^2 + \vec{L}^2 / \vec{r}^2)\) with the result for \(H\)

\[
H(p, r, \nu, \mu_i) = \frac{1}{2} \left[ \mu_1 + \mu_2 + \frac{\vec{p}^2 + m_1^2}{\mu_1} + \frac{\vec{p}^2 + m_2^2}{\mu_2} + \int_0^1 \frac{\sigma^2 d\beta}{\nu} \vec{r}^2 + \right.
\]

\[
+ \left. \int_0^1 d\beta \nu + \left\{ \vec{L}^2 / \vec{r}^2 \left( \frac{1}{\mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int_0^1 d\beta (\beta - \zeta)^2 \nu} \right) - \frac{1}{\mu} \right\} \right]
\]

where \(\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\). 11
As well as in [1] we will demonstrate now that for not very large values of \( l \) (or for \( l \ll n_r \)) the term in curly brackets corresponding to the string contribution to the (orbital) kinetic part of the expression (40)

\[
H^{(1)} = \mathcal{L}^2 / \vec{r}^2 \left( \frac{1}{\mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int_0 1 d\beta (\beta - \zeta)^2 \nu} - \frac{1}{\mu} \right)
\]

(56)
can be treated as a perturbation to \( H^{(0)} = H - H^{(1)} \).

In zero approximation one again obtains

\[
\mu_1^{(0)} = \sqrt{\vec{p}^2 + m_1^2}, \quad \mu_2^{(0)} = \sqrt{\vec{p}^2 + m_2^2},
\]

\[
\nu^{(0)} = \sigma | \vec{r} |.
\]

(57)

and

\[
H^{(0)} = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + \sigma | \vec{r} |
\]

(58)

where \( \vec{p}^2 = (\vec{p}_r^2 + \vec{L}^2 / r^2) \).

This potential like regime (small \( l \)) corresponds (for \( m_i^2 \lesssim \sigma \)) to

\[
< \mu_i > \sim \frac{1}{2} < \nu >
\]

(59)

with nearly pure inert contribution from the string. It is important that even for \( m_i = 0 \) the quark-string interaction generates dynamical quark masses \( \mu_i \approx \sqrt{\sigma} \).

The first order correction \( \varepsilon^{(1)} \) is determined by the average of \( H^{(1)} \) over wave functions of Hamiltonian (58) and can be represented in the form

\[
\varepsilon^{(1)} = < \frac{1}{2} \frac{\sigma^2 l(l+1)}{\nu^{(0)}^2} \left( \frac{1}{\tilde{\mu}^{(0)}} - \frac{1}{\mu} \right) >
\]

(60)

where we have introduced

\[
\tilde{\mu}^{(0)} = \frac{\mu_1^{(0)} \mu_2^{(0)}}{\mu_1^{(0)} + \mu_2^{(0)}}, \quad \tilde{\mu}^{(0)} = \frac{\mu_1^{(0)} \mu_2^{(0)}}{\mu_1^{(0)} + \mu_2^{(0)}}
\]

(61)

with \( \mu_i^{(0)} = \mu_i^{(0)} + \frac{1}{2} \nu^{(0)} \).

To separate problems we first consider two limiting cases: heavy-light meson \((h.l)\) with

\[
\mu_2 = m_2 \to \infty, \quad m_1 = 0
\]

(62)

and light-light meson \((l.l)\) with

\[
m_1 = m_2 = 0.
\]

(63)

where a unified description can be developed. Effects connected to the presence of finite masses will be discussed in subsection D.

In these two limiting cases zero order Hamiltonian (58) takes a simple form [11]

\[
H^{(0)} - m_2 = K \sqrt{\vec{p}^2 + \sigma | \vec{r} |}
\]

(64)
where $K = 2$ for $l.l$ mesons and $K = 1$ for $h.l$ ones (with the subtraction of the heavy mass for $h.l$ systems). The corresponding spectra can be expressed in the form [11]

$$\varepsilon_{nl}^{(0)} = \sqrt{K E_{nl}^{(0)}} , \quad (65)$$

where

$$\left( E_{nl}^{(0)} \right)^2 = \pi \sigma \left( 2n_r + \frac{\lambda(n_r)}{\pi} l + \frac{3}{2} + \delta(n_r, l) \right) , \quad (66)$$

with $\lambda(n_r) \approx 4$ and $\delta(n_r, l)$ is a small correction. Neglecting deflection (56) one would get for the slope

$$\frac{dl}{dM^2} = 1/4K\sigma \quad (67)$$

and the results of this approximation (for the leading trajectory) are shown as crosses in Fig.1. As we shall see deflection (56) from the “relativistic” Hamiltonian (58) gives rise to a change $\sim 25\%$ in slopes of Regge trajectories for (heavy-)light mesons, so that they are very close to the asymptotical string slopes $1/K\pi\sigma$ from the very beginning of the trajectories.

To calculate matrix elements of $H^{(1)} = H - H^{(0)}$ over the wave functions of Hamiltonian (64) we will neglect as well as in [1,2] the dispersion of $|\vec{p}|$ and $|\vec{r}|$, i.e. will perform the substitution

$$<|\vec{p}|^m \cdot |\vec{r}|^n> \rightarrow <|\vec{p}|^m> \cdot <|\vec{r}|^n> \quad (68)$$

In Appendix B it is shown that within the accuracy $\sim 10\%$ the value of $\varepsilon^{(1)}$ can be calculated in the form

$$\varepsilon_{nl}^{(1)} = \sqrt{K E_{nl}^{(1)}} , \quad (69)$$

where

$$E_{nl}^{(1)} = -\frac{\sigma^2 l(l+1)}{E_{nl}^{(0)^2}} . \quad (70)$$

Therefore in this approximation one gets for not very large $l$

$$(\varepsilon_{nl} - m_2)^2 = K \left( E_{nl}^{(0)} + E_{nl}^{(1)} \right)^2 \quad (71)$$

(with disregard of the term $\sim (\varepsilon^{(1)})^2$, since it exceeds the accuracy of the first order calculations) and the corresponding predictions for $(\varepsilon_{nl} - m_2)^2/K\pi\sigma$ are shown by open circles in Fig.1. For the leading trajectory and not very large $l$ ($l \lesssim 5$)

$$\frac{\varepsilon^{(1)}}{\varepsilon^{(0)}} \lesssim 0.05 , \quad (72)$$

with the decreasing effect for the daughter trajectories. But we stress again that as for the change in the slope this correction leads to a deflection of the order $\sim 20 - 25\%$ as compared with pure potential result (67).

To observe the transition into another regime we consider the dependence of solution (44) for $\nu(\tau, \beta)$ on $\beta$. Taking for simplicity the case $m_1 = m_2 \leq \sqrt{\sigma}$ (and therefore $\mu_1 = \mu_2, \zeta = 1/2$) one can estimate the term $\sim \vec{L}^2$ in eq. (44) as one giving a correction to $\nu^{(0)}$ of eqs.(57) (in the same way as we have done for eq. (60) ) with the result

$$<l(l+1)\over a_3^2> (\beta - 1/2)^2 \approx \frac{36l(l+1)}{\pi(2n_r + \frac{4}{3}l + \frac{3}{2})^2} (\beta - 1/2)^2 . \quad (73)$$
In the limit $l \gg n_r$ comparing the unity in right hand side of eq.(44) with the asymptotical value of expression (73)

$$2(\beta - 1/2)^2$$

we encounter a large deflection (due to increase of the string contribution to the orbital momentum) from zero order equation leading to expr.(57). This transition is governed obviously by the parameter

$$\frac{l}{2n_r + 3/2} \quad (74)$$

To take it into account we are to start for large orbital momenta $l \gg n_r$ with another zero approximation for Hamiltonian (40).

C. Relativistic string regime for large orbital momenta

Let us consider the case of

$$l \gg n_r \quad (75)$$

where as we will prove light quarks in the meson carry only a small part of the total energy (orbital momentum)

$$< \nu > \gg < \mu_i > \quad (76)$$

where $i = 1, 2$ for l.l. meson (with $m_1, m_2 \lesssim \sqrt{\sigma}$) and $i = 1$ ($m_1 \sim \sqrt{\sigma}, \mu_2 = m_2 \to \infty$) for h.l. meson. For equal masses case $m_1 = m_2$ it was shown in ref.[1].

For $l \gg n_r$ one can exploit the quasiclassical condition

$$< (r - < r >)^2 > \ll < r >^2 \quad (77)$$

and expand [1] the $r^2$-depending part of Hamiltonian (40) around the extremum

$$r_i^2 = \left[ \frac{l(l+1)(\sigma^2 \int d\beta/\nu)^{-1}}{((\mu_1(1-\zeta)^2 + \mu_2\zeta^2 + \int_0^1 d\beta(\beta - \zeta)^2\nu)} \right]^{1/2} \quad (78)$$

so that in gaussian approximation

$$H = \frac{1}{2} \left[ \frac{p_r^2 + m_i^2}{\mu_1} + \frac{p_r^2 + m_i^2}{\mu_2} + \mu_1 + \mu_2 + \int \nu d\beta + \right. \left. + 2 \left( \frac{l(l+1)\int_0^\sigma d\beta}{\nu} \right)^{1/2} + 4 \int \frac{\sigma^2 d\beta}{\nu} (r - r_i)^2 \right] \cdot (79)$$

As in the case of small $l$ we first concentrate on the two limiting cases (62), (63). For $m_1 = m_2$ the symmetry between quarks allows one [1] to restrict the class of auxiliary fields by

$$\mu_1 = \mu_2 \quad \int (\beta - 1/2) \nu d\beta = 0 \quad (80)$$

and consequently

$$\zeta = 1/2 \quad (81)$$
For \( \mu_2 \to m_2 \to \infty \) one obviously gets in the leading order
\[
\zeta = 0 , \quad \mu_2 = m_2 . \tag{82}
\]

Let perform in the limit (75) the expansion of Hamiltonian (79) in powers of \( \mu/\nu \). In zero approximation one gets (neglecting radial dynamics, \( m_i \) and \( \mu_i \) of light quarks)
\[
(H^{(0)} - m_2) = \frac{1}{2} \int \nu d\beta + \left( \frac{\sigma^2 l(l + 1) \int \frac{d\beta}{f(\beta - (K - 1)/2)^2 \nu d\beta}}{f(\beta - (K - 1)/2)} \right)^{1/2} , \tag{83}
\]
where \( K = 2 \) for l.l. case and \( K = 1 \) for h.l. one. We emphasize, that in this approximation one recovers the pure straight-line string Hamiltonian [12, 1] without radial excitations, which will appear only as a correction to expression (83).

The extremal value of \( \nu^{(0)}(\beta) \) one can find in the same way as it has been done for l.l. case in [12, 1].
\[
\nu^{(0)} = \left( \frac{4K\sigma\sqrt{l(l+1)}}{\pi} \right)^{1/2} \frac{1}{\sqrt{1 - \left( K \left( \beta - \frac{K-1}{2} \right) \right)^2}} . \tag{84}
\]
This energy distribution along the string leads to the following total energy in zero approximation
\[
(\varepsilon^{(0)})^2 = K\pi\sigma\sqrt{l(l+1)} . \tag{85}
\]
Expression (84) for the energy density can be easily interpreted as a rotation of homogeneous distribution (57) of potential case if one recognises that factor
\[
\frac{1}{\sqrt{1 - \left( K \left( \beta - \frac{K-1}{2} \right) \right)^2}} = \frac{1}{\sqrt{1 - v^2(\beta)}}
\]
represents the standard Lorentz-factor, with \( v(\beta) = K \left( \beta - \frac{K-1}{2} \right) \) playing the role of the velocity of the corresponding elementary piece of the string.

For l.l. system
\[
- v(0) = v(1) = 1 \tag{86}
\]
and for h.l. system one has one half of the string
\[
v(0) = 0 , \quad v(1) = 1 . \]

Let us find the leading correction \( H^{(1)} \) to zero approximation (83) and recover our starting condition \( < \mu_i > \ll \nu > \) (for \( m^2_i \leq \sqrt{\sigma} \)). Again we first discuss two limiting cases (62), (63) for
\[
m_1 = m_2 = 0 ,
\]
or
\[
m_1 = 0 , \quad m_2 \to \infty .
\]

In Appendix C it is shown, that expanding eq.(79) in \( \mu/\nu \) up to the first nonvanishing terms one obtains (after integration over \( \mu \)) the following expression for \( H^{(1)} \)
\[
H^{(1)}_r = \sqrt{K} \left( \frac{3^{4/5}}{4} \left( \pi \sigma \right)^{1/2} l^{-3/10} \right) \left( \left( p^2_x + c\Sigma_i m_i^2 \right)^{2/3} + a^2 \right) , \tag{87}
\]
with
\[ \mu^{(0)} = \left( \frac{\varepsilon^{(0)}}{3K} \left( \frac{p_r^2}{K} + \sum_i m_i^2 \right) \right)^{1/3}, \]
where the sum is over \( i = 1, 2 \) for l.l. and \( i = 1 \) for h.l. mesons, \( c = \frac{3^{12/5}}{\pi \sigma} \) and we have introduced a dimensionless variables
\[ \frac{r_r - r_l}{x} = \frac{p_x}{p_r} = \sqrt{K} 3^{2/5} (\pi \sigma)^{-1/2} l^{1/10}, \quad p_x^2 = -\partial_x^2 \]
and as before \( K = 2, 1 \) for l.l. and h.l. systems correspondingly. For the case of \( K = 2 \) this expression was obtained in ref. [1].
Within the accuracy \( \sim 5 - 10\% \) the spectrum of \( H^{(1)}_r \) can be calculated (see Appendix C for details) in the form
\[ \varepsilon^{(1)}_{nr,l} = \sqrt{K} \pi \sigma \left( \frac{3}{4} \right)^{1/5} \frac{5}{4} (n_r + 1/2)^{4/5} l^{-3/10} \]
and in accordance with our initial assumption (76) one indeed obtains
\[ \frac{\mu}{\nu} \sim \left( \frac{n_r + 1/2}{l} \right)^{2/5} \ll 1 \]
in the limit (75) we are dealing with.
The resulting spectrum for \( M^2/K\pi\sigma \), where
\[ M = \varepsilon^{(0)} + \varepsilon^{(1)} \]
is shown by open squares in Fig.1 and doesn’t depend on \( K \) (again we are to disregard the term \( \sim (\varepsilon^{(1)})^2 \) since it exceeds the accuracy of the calculations). As we argue in Appendix C the exact value of \( (\varepsilon^{(1)})_{\text{exact}} \approx 0.9 \varepsilon^{(1)} \) and we take it into account in this figure. It follows from Figure 1 that low \( l \) and high \( l \) approximations join very smoothly forming (within the accuracy \( \sim 5\% \)) Regge trajectories with the slope
\[ \frac{dM^2}{dl} = K\pi\sigma \]
and intercepts, determined from the Hamiltonian (64)
\[ M^2(l = 0) = K\pi\sigma \left( 2n_r + \frac{3}{2} + \delta \right). \]
The values for \( M^2 \) are very close to results found numerically in ref. [5] (black dots in Fig.1). A small deflection between them is out of the accuracy of the first order calculations we have done. A short discussion of similarities and differences between our approach and that of ref. [5] is postponed till the Conclusion.
It is important to stress, that ”minimal” QCD string combines together properties of both the string models (string like slope (92) ) and ones of the potential models (existence of radial excitations, which lead to the intercept (93) ). We have found both in the l.l. and in h.l. systems
two dynamical regimes distinguished by the contribution of the string into the (orbital) kinetic part of the action: in the potential regime for small $l$ or

$$\frac{l}{2n_r + 1} \ll 1$$  

the string constitutes inert linear potential with $\nu(\tau, \beta)$ close to the homogeneous distribution (57); in the string regime for

$$\frac{l}{2n_r + 1} \gg 1$$  

the main part of the energy (orbital momentum) is carried by the string with energy density $\nu(\tau, \beta)$ given in the leading order by expression (84).

D. Mass effects in the minimal QCD string with quarks

To analyse the dynamics for arbitrary quark masses case we first consider l.l. mesons with

$$m_1 \lesssim \sqrt{\sigma}, \quad m_2 \lesssim \sqrt{\sigma}$$  

and h.l. systems with

$$m_1 \lesssim \sqrt{\sigma}, \quad m_2 \gg \sqrt{\sigma}.$$  

Our main aim here will be to estimate upper bound on light masses, which don’t change considerably $K\pi\sigma$ slope (92) of Regge trajectories for massless light quark(s), discussed in the previous section. We also calculate in this case leading mass corrections to the spectrum.

As in the previous section we concentrate on two limiting regimes (94), (95).

In the potential regime (94) zero approximation as well as in the previous section is described by Hamiltonian (the subtraction of the heavy mass in h.l. system is implied)

$$H^{(0)} = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + \sigma | \vec{r} |$$  

with the spectrum [11] in the case of $m \ll \langle | \vec{p} | \rangle \sim E^{(0)}$

$$\varepsilon^{(0)} \approx \varepsilon^{(0)} \left(1 + \sum_i \frac{m_i^2}{E^{(0)2}} \right), \quad \varepsilon^{(0)} = \sqrt{KE^{(0)}},$$  

where $E^{(0)}$ is given by (66) and the sum is over $i = 1, 2$ for l.l. or $i = 1$ for h.l. mesons.

We calculate the correction $\varepsilon^{(1)} = < H - H^{(0)}> \rangle$ in the same way as for the massless case (70) of the previous section (see Appendix B for details).

Making use of the virial theorem and neglecting the dispersion of $\mu, \nu$ one gets in the leading order of $\left( \frac{\sum m_i^2}{E^{(0)2}} \right)$

$$\langle \mu^{(0)} \rangle \approx \varepsilon^{(0)} \left(1 + 3 \frac{\sum_i m_i^2}{E^{(0)2}} \right), \quad \langle \nu^{(0)} \rangle \approx \frac{\varepsilon^{(0)}}{2} \left(1 - \frac{\sum_i m_i^2}{E^{(0)2}} \right).$$  

(100)
The substitution of (100) in (60) gives for the correction \( \varepsilon^{(1)} \) in massive case

\[
\frac{\varepsilon^{(1)}}{\varepsilon^{(0)}} = -\frac{l(l + 1)\sigma^2}{E^{(0)}4} \left( 1 - 4 \frac{\sum_i m_i^2}{E^{(0)}2} \right).
\]  

(101)

Therefore we have (up to the term \( \sim m^2 \varepsilon^{(1)} \varepsilon^{(0)} \)) for the mass squared \( M^2 = (\bar{\varepsilon}^{(0)} + \varepsilon^{(1)})^2 \)

\[
M^2 = K \left[ E^{(0)}2 \left( 1 - \frac{l(l + 1)\sigma^2}{E^{(0)}4} \right)^2 + 2 \sum_i m_i^2 (1 + 3 \frac{l(l + 1)\sigma^2}{E^{(0)}4}) \right].
\]  

(102)

Let us estimate quark mass corrections to the slope of the trajectory. The first term of eq. (102) gives in accordance with expression (92)

\[
\frac{dM^2}{dl} \approx K \pi \sigma.
\]

For the leading Regge trajectory the deflection of the slope from the massless case can be expressed in the form

\[
\Delta \frac{dM^2}{dl} \approx \frac{8}{3\pi^3} f(l) \frac{\sum_i m_i^2}{\sigma}
\]

(103)

where \( f(l) = \frac{1}{(1 + \frac{3\pi^3}{8\sigma})} \) with \( f(1) \approx 4 \).

Hence the slope of leading Regge trajectory is not essentially changed when

\[
\sum_{i=1}^{K} m_i^2 \ll \frac{3\pi^3}{2} \sigma \approx 40\sigma
\]

(104)

where \( K = 2 \) for l.l. and \( K = 1 \) for h.l. system. For the daughter trajectories the restriction for the masses is even weaker.

We emphasize that restriction (104) is much more moderate as compared with that for the deflection of intercept (93) (the second term in the r.h.s. of eq.(99)) to be small, which amounts to

\[
\sum_{i=1}^{K} m_i^2 \ll \frac{3\pi^3}{2} \sigma
\]

(105)

To complete analysis let us consider the case \( l \gg n_r \) when radial quark dynamics can be considered as a perturbation [1].

We start with expression (87) for the Hamiltonian \( H^{(1)}_r \) which determines the corrections to zero (pure string) approximation (83)-(85). As it is shown in Appendix C the string asymptotics (where \( <\nu> \gg <\mu_{tight}> \) ) is valid for

\[
\frac{l}{n_r + \sum_{i=1}^{K} m_i^2 \pi\sigma + \frac{1}{2}} \gg 1.
\]

(106)

Keeping this condition satisfied we first consider the case of nonrelativistic radial dynamics when \( p_r^2 \ll \sum_i m_i^2 \), which is achieved as we shall see if

\[
\frac{l}{(n_r + \frac{1}{2})^6} \gg \left( \frac{\pi\sigma}{\sum_i m_i^2} \right)^5.
\]

(107)
In this limit one obtains (see Appendix C for details) the spectrum of $H^{(1)}_r$ in the form:

$$
\varepsilon^{(1)}_{n_r,l} = \sqrt{K} \left( \frac{3^{4/3}}{4} \left( \sum_i m_i^2 \right)^{2/3} (\pi \sigma l)^{-1/6} + \left( \frac{3}{8} \right)^{1/6} \frac{(\pi \sigma)^{2/3}}{(\sum_i m_i^2)^{1/6}} l^{-1/3} \left( n_r + \frac{1}{2} \right) \right). \quad (108)
$$

It is easy to make sure, that condition (106) is equivalent to $\varepsilon^{(1)}_{n_r,l}/\varepsilon^{(0)}_l \ll 1$. Making comparison between $p^2_r$ and $\sum_i m_i^2$ one recovers that in the regime (106), (107) radial dynamics is indeed "nonrelativistic":

$$
<p^2_r> \sim (\sum_i m_i^2 (\pi \sigma)^5)^{1/6} l^{-1/6} (n_r + \frac{1}{2}) \ll \sum_i m_i^2. \quad (109)
$$

We note that condition (109) follows also from the requirement, that the first term of eq.(108) is much more than the second one.

To calculate the spectrum in the opposite asymptotics of string regime (106)

$$
<p^2_r> \gg m^2 \quad (110)
$$

and merge smoothly the massless case (89) one is to consider (see Appendix C) the following domain of $l, n_r$

$$
\frac{l}{(n_r + \frac{1}{2})^{10/3}} \ll \left( \frac{\pi \sigma}{\sum_i m_i^2} \right)^5, \quad (111)
$$

where the spectrum of $H^{(1)}_r$ has the form

$$
\varepsilon^{(1)}_{n_r,l} = \sqrt{K} \left( \left( \frac{3}{4} \right)^{1/5} \frac{5}{4} (\pi \sigma)^{1/6} l^{-3/10} (n_r + 1/2)^{4/5} + \right) \quad (112)
$$

$$
+ \left( \frac{3}{4} \right)^{2/5} \sum_i m_i^2 (\pi \sigma)^{-1/2} l^{-1/10} (n_r + 1/2)^{-2/5} \right). 
$$

Restriction (110), (111) comes from the condition, that the first term of eq. (112) (which is the massless correction (89) of the previous section) is much more, than the second one. Due to condition (106) we again have the restriction $\varepsilon^{(1)}_{n_r,l}/\varepsilon^{(0)}_l \ll 1$ to be satisfied.

To conclude this section let us discuss briefly the transition from nonrelativistic dynamics of heavy quark(s) (45) in the heavy-heavy or heavy-light mesons at small $n_r, l$ to the large $n_r, l$ regime, when both quarks become relativistic. We concentrate on the connection between this transition and the transition from the potential regime of subsection A to the string one. As we easily conclude from eq. (66) the relativization, $\tilde{p}^2 \sim m^2$, appears for

$$
\left( n_r + \frac{2}{\pi l} \right) \sim \frac{2}{\pi} \frac{(m_1^2 + m_2^2)}{\sigma}. \quad (113)
$$

If excitation of the system corresponds to $n_r \gg l$, then one achieves the potential relativistic dynamics, described by the Hamiltonian (58) with the small string corrections (60). In the opposite case of orbital relativization $l \gtrsim n_r$ as it is clear from the discussion of eq. (73) there is a considerable deflection from pure potential regime (58). Therefore in this case both quarks start to become relativistic in the regime intermediate between potential one (58) and string
one (83) (valid under the condition (106) ) where the dependence of $\nu$ on $\beta$ is considerable but still differs from the asymptotical one (84).

6 CONCLUSIONS

Let us summarize the results. We obtain the generalization of Hamiltonian [1] for the spinless quark and antiquark in the confining QCD vacuum for the case of arbitrary quark masses. Starting from the QCD Lagrangian we make use the minimal area law asymptotics for the averaged Wilson loop which leads to the appearence of the minimal QCD string connecting quarks. The string contributes (for $l > 0$) in the kinetic part of the effective action and therefore in order to introduce the center of masses and relative coordinates we are to use explicitly the condition that they are decoupled from each other. Introducing the auxiliary fields we represent the action in the quadratic form with respect to the quark coordinates and diagonalization of the kinetic part of the action leads to the proper definition of the total momentum $\vec{P}$. Choosing the rest frame of the meson $\vec{P} = 0$ we arrive (neglecting pathes of quarks with backtracking in time) at the effective Hamiltonian. Additionally to quark coordinates it contains the auxiliary fields playing the roles of the quark dynamical masses $\mu_i(\tau)$ and the string energy density $\nu(\tau, \beta)$. Integration over them amounts to the substitution of their extremal values after which one is to construct the operator of Hamiltonian acting on the wave functions.

The interaction between quarks and the string gives rise to appearence of two different dynamical regimes (as well as for the equal masses case [1]), which are distinguished by the string contribution into the kinetic part of the Hamiltonian. For the low orbital momenta ($l \ll n_r$) dynamics is described in the leading order by the relativistic linear potential Hamiltonian with almost inert contribution from the string constituting potential $\sigma |\vec{r}|$. In the opposite limit of $l \gg n_r$ the system behaves as the rotating string which carries the main part of the orbital momentum and energy. The transition between these regimes is smooth and relativization of heavy quark(s) due to the increase of $l$ is also considered.

We develop the unified description of the heavy-light and light-light mesons and prove that in these limiting cases Regge trajectories are nearly straight line with the slope close to $1/K\pi\sigma$ ($K = 1, 2$ for h.l. and l.l mesons respectively). The upper bound on the light mass(es) which don’t change considerably the slope is obtained and the leading mass corrections are calculated for both regimes. It appears that the slope is much less sensitive to increase of quark masses as compared with the intersepts of the trajectories.

We note that our results for the spectrum are very close to that obtained by the numerical quantization of the same action [5] postulated without derivation from QCD Lagrangian. We stress that as compared with [5] our approach gives the qualitative picture of underlieing regimes of quark-string interaction. Also it enables one to avoid in the leading order the complications of Weil ordering [9] in Hamiltonian (40). In zero approximation Hamiltonian both for small $l$ ($l \ll n_r$) (58) and for $l \gg n_r$ (83) is the sum of the terms depending either only on the momentums or space coordinates which doesn’t require any Weil ordering. The need for this ordering appears only for the correction terms (56), (87) and calculation of them can be performed with the neglect of the dispersion (Weil ordering). Such approximate procedure gives the value for the leading corrections within more than 10% of accuracy. In [5] the numerical procedure makes it difficult to keep track of the ordering and relies on the smallness of the dispersion (see forthcoming paper for a more detailed discussion).
We also consider the modification of the heavy quarkonia Hamiltonian arising from the string contribution (at distances $\gtrsim T_g$) to the orbital momentum. We estimate the energy correction due to this effect and find conditions under which it is comparable with the first relativistic correction.

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Appendix A

In this Appendix we perform the gaussian integration (32) over $\eta(\tau, \beta)$ to go over from effective action (31)

\[
\tilde{A} = \frac{1}{2} \int_0^T d\tau \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \frac{1}{a_1} \{a_3a_1\vec{r}^2 - 2c_2a_1(\vec{r}\vec{r}) + (a_4a_1 - c_1^2)\vec{r}^2 + a_1^2 \} \right] \tag{A.1}
\]

to eq.(34) expressed in terms of physical quantities: dynamical masses $\mu_1, \mu_2$ energy density along the string $\nu(\tau, \beta)$ and relative coordinates.

In what follows we make use of the fact that gaussian integration effectively amounts to substitution of the extremum value of $\eta_{ext}$ into eq.(A.1). On the basis of the definitions (21) of $a_i$ and $c_i$ we obtain from (A.1) the following extremum condition for $\eta$.

\[
2 \int_0^1 d\beta \nu \left[ \nu^2(a_1\eta - c_1) - (\nu \dot{\nu})a_1(\beta - \zeta) \right] \delta \eta = 0 \tag{A.2}
\]

that results in

\[
\eta_{ext} = \frac{c_1}{a_1} + \frac{(\nu \dot{\nu})}{\nu^2} (\beta - \zeta) \tag{A.3}
\]

with $c_1 = \int_0^1 d\beta \eta_{ext} \nu$.

Resolution of (A.3) finally gives

\[
\eta_{ext} = \frac{(\nu \dot{\nu})}{\nu^2} \left( \beta - \frac{\mu_1}{\mu_1 + \mu_2} \right) \tag{A.4}
\]

Expression (A.4) leads to the following $c_i^{ext}$

\[
c_1^{ext} = \frac{(\nu \dot{\nu})}{\nu^2} \left( \zeta - \frac{\mu_1}{\mu_1 + \mu_2} \right) a_1 \tag{A.5}
\]

\[
c_2^{ext} = \frac{(\nu \dot{\nu})}{\nu^2} \left( a_3 - \mu_1 + \frac{\mu_1^2}{\mu_1 + \mu_2} \right),
\]

where we make use of

\[
\int_0^1 d\beta \nu = a_1 - (\mu_1 + \mu_2),
\]

\[
\int_0^1 d\beta \nu \beta = \zeta a_1 - \mu_1, \tag{A.6}
\]

\[
\int_0^1 d\beta \nu \beta^2 = a_3 - \mu_1 + \zeta^2 a_1.
\]

To calculate (A.1) with $\eta = \eta_{ext}$ we note that for quadratic in $\eta$ form one has the following relation at the extremum (A.2)

\[
(a_4^{ext} a_1 - (c_1^{ext})^2) \nu^2 = (\nu \dot{\nu}) c_2^{ext} a_1, \tag{A.7}
\]
where we have separated $\eta$-dependent part of $a_4$ introducing $\bar{a}_4 = \int_0^1 \eta^2 \nu d\beta$.

With the help of (A.7) one concludes that $\eta$-dependent part of the action (A.1) amounts to the following contribution into $\bar{A}$

$$- (\bar{r} \dot{\bar{r}}) \epsilon_2^{ext} a_1 ,$$

that results finally in the effective action (34) for the rest frame

$$A'(\mu_1, \mu_2, \nu) = \frac{1}{2} \int_0^T d\tau \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + \mu_1 + \mu_2 + (\mu_1 (1 - \zeta)^2 + \mu_2 \zeta^2 + \int_0^1 d\beta (\beta - \zeta)^2 \nu) \times \\
\times \frac{[\dot{\bar{r}} \times \bar{r}]^2}{\bar{r}^2} + \frac{(\dot{\bar{r}} \bar{r})^2}{\bar{r}^2} \bar{\mu} + \bar{r}^2 \int_0^1 d\beta \frac{\sigma^2}{\nu} + \int_0^1 d\beta \nu \right]$$

(A.9)

where $\bar{\mu} = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$ is the reduced dynamical mass.
Appendix B

In this Appendix we calculate first order correction (56) to the relativistic Hamiltonian (58). Making use of the approximation (68), proposed in [1], we neglect in (56) the dispersion of $|\vec{p}|$ and $|\vec{r}|$ (and disregard also Weil ordering [9]) and express $\varepsilon^{(1)} = \langle H^{(1)} \rangle$ in terms of $\langle |\vec{p}| \rangle$, $\langle |\vec{r}| \rangle$ averaged over wave functions of zero order Hamiltonian (58) and connected to $E^{(0)}$ of eq.(66). For this purpose we first evaluate $\langle \mu^{(0)}_1 \rangle$, $\langle \mu^{(0)}_2 \rangle$ (determined by eqs.(57)) for $m_i^2 \lesssim \sqrt{\sigma}$ and then substitute them to eq.(60) for $\varepsilon^{(1)}$. Also we will develop the procedure to estimate the accuracy of this approximation and find that account of nonzero dispersions (Weil ordering) would change the result for $\varepsilon^{(1)}$ less than for 10%.

First we use the virial theorem for zero order Hamiltonian (58)

$$\langle \vec{p}^2 \rangle = \sigma \langle |\vec{r}| \rangle$$

(B.1)

where the average is performed over the wave functions of (58).

Neglecting the dispersion one easily obtains the system of equations for $\langle \mu^{(0)}_1 \rangle$, $\langle \mu^{(0)}_2 \rangle$ and $\langle \nu^{(0)} \rangle$

$$\begin{cases}
\langle \mu^{(0)}_1 \rangle - m_1^2 \langle \mu^{(0)}_1 \rangle = \langle \nu^{(0)} \rangle \\
\varepsilon^{(0)}_m = \langle \mu^{(0)}_1 \rangle + \langle \mu^{(0)}_2 \rangle = \langle \nu^{(0)} \rangle \\
\langle \mu^{(0)}_2 \rangle - m_2^2 = \langle \mu^{(0)}_2 \rangle - m_2^2
\end{cases}$$

(B.2)

where the first one follows directly from (B.1), the second one is just the definition of the eigenvalues of eq.(58) and the last equation ensures the condition that the total momentum of the meson is equal to zero.

To obtain from eqs.(B.2) the values of $\langle \mu^{(0)}_i \rangle$ and $\langle \nu^{(0)} \rangle$ with the account of leading mass corrections $\sim \sum_{i=1}^{K} m_i^2 / \sigma$ it is sufficient to consider the reduced system of equations in two limiting cases of h.l. (62) and l.l. (63) mesons

$$\langle \mu^{(0)} \rangle = -\frac{\sum_{i=1}^{K} m_i^2}{K} = \frac{\langle \nu^{(0)} \rangle}{K}$$

(B.3)

$$\sqrt{KE^{(0)}} \cdot \left(1 + \frac{\sum_{i=1}^{K} m_i^2}{E^{(0)}^2} \right) = K \langle \mu^{(0)} \rangle = \langle \nu^{(0)} \rangle$$

where $K = 2$ for l.l. system and $K = 1$ for h.l. one. The solution of the system (B.3) gives

$$\begin{align*}
\langle \mu^{(0)} \rangle &\approx \frac{\varepsilon^{(0)}_m}{2} \left(1 + 3 \left(\frac{\sum_i m_i^2}{E^{(0)}}\right)\right) \\
\langle \nu^{(0)} \rangle &\approx \frac{\varepsilon^{(0)}_m}{2} \left(1 - \left(\frac{\sum_i m_i^2}{E^{(0)}}\right)\right)
\end{align*}$$

(B.4)

where $\varepsilon^{(0)} = \sqrt{KE^{(0)}}$, defined in (66).

Substitution of these values into the eqs.(60), (61) results in the following value for $\langle H^{(1)} \rangle = \varepsilon^{(1)}$

$$\langle H^{(1)} \rangle = \varepsilon^{(1)} = \frac{\varepsilon^{(1)}}{\varepsilon^{(0)}} = -\frac{l(l+1)\sigma^2}{E^{(0)}^4} \left(1 - 4 \sum_i m_i^2 \frac{E^{(0)}^2}{E^{(0)}^2}\right)$$

(B.5)
where the summation over \( i \) runs from 1 to \( K \).

In the rest of this appendix we estimate the effect of dispersion for the calculation of \( \varepsilon^{(1)} \) and find it to be less than 10\% even for the ground state of the Hamiltonian (58). For this purpose we first represent Hamiltonian (58) (taking for the sake of definitness \( K = 1 \)) in the form containing auxiliary fields \( \mu(\tau) \) and \( \nu(\tau) \)

\[
H = \frac{1}{2} \left( \frac{1}{\mu}(\vec{p}^2 + m^2) + \mu \right) + \frac{1}{2} \left( \frac{\sigma^2}{\nu} \vec{r}^2 + \nu \right).
\]

Integration over \( \mu(\tau) \) and \( \nu(\tau) \) amounts effectively [1] (as in the main text) to the insertion of their extremal values

\[
\mu_{\text{ext}} = \sqrt{\vec{p}^2 + m^2}, \quad \nu_{\text{ext}} = \sigma \mid \vec{r} \mid
\]

which enables to recover eq.(58) for h.l. system.

To evaluate the effect of dispersion it is sufficient to restrict the integrations \( D\mu D\nu \) by that over \( \mu \) and \( \nu \) independent of time. Actually such approximation gives (for not very large quantum numbers \( l, n_r \)) deflection in the spectrum (and therefore in dispersion) less then 5 – 10 \% (see [3] for the discussion). As a result within this accuracy we reduce initial Hamiltonian (58) to one of the oscillator type. Our strategy is to compare matrix elements \( < \mid \vec{p}^n \mid \vec{r}^m > \) with their approximate counterparts \( < \mid \vec{p} \mid n < \mid \vec{r} \mid m > \) where averaging will be performed over our oscillator version of eq.(58). For this purpose one is to obtain [3] first the energy \( E(\mu, \nu) \) as the function of \( \mu \) and \( \nu \). After this we find the extremal values of \( \mu \) and \( \nu \) from the conditions

\[
\frac{\partial E}{\partial \mu} (\mu_{\text{ext}}, \nu_{\text{ext}}) = 0, \quad \frac{\partial E}{\partial \nu} (\mu_{\text{ext}}, \nu_{\text{ext}}) = 0,
\]

and substitute them into the wave functions in order to calculate the matrix elements under consideration.

Along these lines we have for the spectrum of the Hamiltonian (B.6)

\[
E(\mu, \nu) = \frac{m^2}{2\mu} + \frac{\mu}{2} + \frac{\nu}{2} + \varepsilon_n
\]

\[
\varepsilon_n = \frac{\sigma}{\sqrt{\mu\nu}} \left( n + \frac{3}{2} \right)
\]

so that

\[
\mu_{\text{ext}} = \sqrt{\left( n + \frac{3}{2} \right)} \sigma, \quad \nu_{\text{ext}} = \sqrt{\left( n + \frac{3}{2} \right)} \sigma, \quad \omega_{\text{ext}} = \sqrt{\frac{\sigma}{\left( n + \frac{3}{2} \right)}}
\]

and finally in zero order of \( m^2 \) one obtains the following spectrum

\[
E^{(0)2} = 4 \left( n + \frac{3}{2} \right) \sigma
\]

with \( n = 2n_r + l \). We note that expression (B.11) is reasonably close to the exact one (66) \((K = 1)\) and even gives the correct Regge slope (67).

The wave functions of the approximate version of Hamiltonian (B.6) in the momentum space have the standard form

\[
a_{nlm}(p) = C_p \exp \left( -\frac{\alpha^2 p^2}{2} \right) Y_{lm}(\theta, \varphi) F \left( \frac{n - l}{2}, l + \frac{3}{2}, \frac{\alpha^2 p^2}{2} \right)
\]

(B.12)
where \( \alpha = \left( \frac{\sigma^2 \rho_{\text{ext}}}{\rho_{\text{ext}}} \right)^{1/4} \) and \( C \) is the normalization constant.

To take the simplest example let us estimate the dispersion contribution to the evaluation of the first order correction \( \Delta \varepsilon \sim \left( \frac{m^2}{p} \right) \) in the relativistic expansion of the energy \( \varepsilon_{nl}^{(0)} \) for a heavy-light meson (neglecting the string contribution (56) to the orbital momentum). Keeping only the first term in the expansion one starts with

\[
\varepsilon_{nl}^{(0)} = \langle \sqrt{\vec{p}^2 + m^2 + \sigma} \mid \vec{r} \mid \rangle \approx \langle \sqrt{\vec{p}^2 + \sigma} \mid \vec{r} \mid \rangle + \frac{1}{2} \langle \frac{m^2}{\sqrt{\vec{p}^2}} \rangle \equiv \varepsilon_{nl}^{(0)} + \Delta \varepsilon \quad (B.13)
\]

where \( \varepsilon_{nl}^{(0)} \) is given by eq.(64) with \( K = 1 \). One is to compare now two quantities \( 1/\langle \sqrt{\vec{p}^2} \rangle \) and \( \langle 1/\sqrt{\vec{p}^2} \rangle \).

For this purpose we consider \( n_r = 0 \), since in this case the discrepancy (for a given \( l \)) maximal. For such quantum numbers wave function (B.12) is reduced with the help of Cummer formula to the simple form

\[
a_{nlm}(p) = C \cdot p \exp \left( -\alpha^2 p^2 \right) Y_{lm}(\theta, \varphi) \quad (B.14)
\]

with \( C^2 = \frac{2\alpha^{2l+3/2}}{\Gamma(l+\frac{3}{2})} \). The dispersion contribution \( \Delta \varepsilon^D \) is defined in the following way

\[
\Delta \varepsilon = \frac{1}{2} m^2 < \frac{1}{\mu(0)} > - \frac{1}{2} m^2 \left( < \frac{1}{\mu(0)} > - \frac{1}{\mu(0)} > \right) \equiv \Delta \varepsilon^{D=0} + \Delta \varepsilon^D \quad (B.15)
\]

with \( E^{(0)} = 2 < \mu(0) >= 2 < \sqrt{\vec{p}^2} \) given by (B.11). The difference \( \Delta \varepsilon^D \) between corresponding matrix elements can be approximately evaluated with the help of wave functions (B.14) with the result

\[
< \frac{1}{\sqrt{\vec{p}^2}} > \approx \frac{1}{\sqrt{< \mu(0) > < \omega(0) >}} \cdot \frac{\Gamma(l+\frac{3}{2})}{\Gamma(l+2)} \quad (B.16)
\]

where \( \mu^{(0)}, \omega^{(0)} \) are defined in (B.10).

Making use of Vallis formula for gamma functions (which is valid strictly speaking for \( l \gg 1 \), but still gives a required accuracy for \( l \geq 1 \) ) we obtain for the dispersion contribution

\[
\Delta \varepsilon^D = \frac{m^2}{\sqrt{\sigma}} \cdot \frac{\sqrt{l}}{8(l+1) \left( l+\frac{1}{2} \right)} \quad (B.17)
\]

Expression (B.17) as compared with the \( \Delta \varepsilon^{D=0} \) gives the correction to \( \Delta \varepsilon \) of order of \( \sim 10\% \). The dispersion correction in the slope occurs even smaller (to the analogy with the ratio of quark mass contribution (105) in the energy and in the Regge slope (104), discussed in the main text). In conclusion of this Appendix we note that relative contribution of the dispersion from higher powers of \( \mu \) and \( \nu \) to the energy is also either of the same order or even smaller.
Appendix C

In this Appendix we derive expression (87) for Hamiltonian $H_r^{(1)}$ which describes radial dynamics in the limit of large $l$ and calculate the corrections to the spectrum of pure string zero approximation (85). We exploit the method, developed in [1] and consider in a unified way two limiting cases of the full straight-line string with $m_1 = m_2 = m$, $\mu_1 = \mu_2 = \mu$ ($K = 2$) and the half of the string with $m_1 = m$, $m_2 \to \infty$ ($K = 1$). We start here with Hamiltonian (79) of the main text. Our strategy will be to expand (in two limiting cases under consideration) the following part of (79)

$$2 \left( \frac{l(l+1) \int \frac{a^2 a \rho}{\mu_2^2 + \mu_2^2 + \int d\beta \mu_2 (\beta - \zeta)^2}}{\mu_2^2} \right)^{1/2} = 2 \left( \frac{l(l+1) \int \frac{a^2 a \rho}{K + \int d\beta (\beta - \frac{K}{2} \bar{r})^2}}{\mu_2^2} \right)^{1/2} \approx$$

$$\approx 2 \left( \frac{a(l(l+1) \int \frac{a \rho}{\beta - \frac{K}{2} \bar{r}})}{\int d\beta (\beta - \frac{K}{2} \bar{r})^2} \right)^{1/2} + (-1) \frac{\hbar}{\mu} \frac{(a(l(l+1) \int \frac{a \rho}{\beta - \frac{K}{2} \bar{r}}))^{1/2}}{\int d\beta (\beta - \frac{K}{2} \bar{r})^2} + \frac{3}{4} \left( \frac{\hbar}{\mu} \right)^2 \frac{(a(l(l+1) \int \frac{a \rho}{\beta - \frac{K}{2} \bar{r}}))^{1/2}}{\int d\beta (\beta - \frac{K}{2} \bar{r})^2}$$

(C.1)

and then integrate this approximate expression over $\mu$ to end up with Hamiltonian (87).

After expansion (C.1) the terms in eq.(79) independent of $\mu_i$ are collected into the pure string Hamiltonian (83) which gives zero approximation to the problem. To obtain the leading radial corrections one is to find the extremal value (84) for $\nu^{(0)}$ from eq.(83) (in the way similar to that of [12, 1]) and substitute it into $H$ with the result

$$H = (K \pi \sigma \sqrt{l(l+1)})^{1/2} + \frac{1}{2} \left[ \frac{\tilde{\mu}^2 + m^2}{\mu/2} + \frac{3K^2 \mu^2}{2l(K \pi \sigma l)^{1/2}} + \frac{(\pi \sigma)^2 (r - r_0)^2}{2l(K \pi \sigma l)^{1/2}} \right] \equiv \epsilon_i^{(0)} + H_r^{(1)}$$

(C.2)

where the first term corresponds to the pure string result (85). Introducing new variables

$$\tilde{\mu} = \sqrt{K} \mu, \quad \tilde{p} = \sqrt{K} p, \quad \tilde{r} - \tilde{r}_0 = \frac{r - r_0}{\sqrt{K}}$$

(C.3)

one obtains the unified radial Hamiltonian $H_r^{(1)} = H - \epsilon_i^{(0)}$ for h.l. and l.l. mesons

$$H_r = \frac{1}{2} \sqrt{K} \left[ \frac{\tilde{p}^2 + \sum_{i=1}^K m_i^2}{\tilde{\mu}} + \frac{3 \tilde{\mu}^2}{2(\pi \sigma l)^{1/2}} + \frac{(\pi \sigma)^2 (\tilde{r} - \tilde{r}_0)^2}{2(\pi \sigma l)^{1/2}} \right] \equiv \sqrt{K} \tilde{H}_r$$

(C.4)

so that $H = \epsilon_i^{(0)} + H_r^{(1)} \equiv \sqrt{K} (E_i^{(0)} + \tilde{H}_r^{(1)})$. After the integration over $\tilde{\mu}$, which amounts to the substitution of its extremal value

$$\tilde{\mu} = \left[ \frac{\tilde{p}^2 + \sum_{i=1}^K m_i^2}{3(\pi \sigma l)^{1/2}} \right]^{1/3}$$

(C.5)

we arrive at the effective radial Hamiltonian

$$H_r^{(1)} = \frac{1}{2} \sqrt{K} \left[ \frac{3^{1/3}}{2} \cdot \frac{\tilde{p}^2 + \sum_{i=1}^K m_i^2}{(\pi \sigma l)^{1/6}} + \frac{(\pi \sigma)^2 (\tilde{r} - \tilde{r}_0)^2}{2(\pi \sigma l)^{1/2}} \right]$$

(C.6)
Introducing instead of \( \tilde{r} - \tilde{r}_0 \) a new dimensionless variable \( x \)

\[
\frac{(\tilde{r} - \tilde{r}_0)}{x} = \frac{p_{x}}{\tilde{p}_{r}} = \frac{3^{2/5} \cdot l^{1/10}}{(\pi \sigma)^{1/2}}
\]

one can represent \( H_{r}^{(1)} \) in the following form

\[
H_{r}^{(1)} = \sqrt{K} \left( \frac{3^{4/5} (\pi \sigma)^{1/2}}{4} l^{-3/10} \right) \left[ \left( \frac{p_{x}^2 + 3^{4/5} l^{1/5} \sum_{i=1}^{K} m_{i}^2}{\pi \sigma} \right)^{2/3} + x^2 \right]
\]

which is convenient for numerical calculations.

In order to obtain an approximation to the spectrum \( \varepsilon_{n_{r}, l}^{(1)} \) of \( H_{r}^{(1)} \) we consider (C.4) for the restricted class of functions \( \tilde{\mu} \) independent on \( \tau \). This procedure usually gives the accuracy about 5% - 10% for the states with not very large \( n \) (see also the discussion in Appendix B) [1], [3]. To this end [7] we first evaluate \( \varepsilon_{n_{r}, l}^{(1)} \) for a given \( \tilde{\mu} \)

\[
\varepsilon_{n_{r}, l}^{(1)}(\tilde{\mu}) = \sqrt{K} \left[ \frac{(\pi \sigma)}{(2(\pi \sigma l)^{1/2})^{1/2}} \cdot \frac{\tilde{\mu}^{3/2}}{\mu^{3/2}} \left( n_{r} + \frac{1}{2} \right) + \frac{\sum_{i=1}^{K} m_{i}^2}{\tilde{\mu}^{2}} \right] \equiv \sqrt{K} E_{n_{r}, l}^{(1)}(\tilde{\mu})
\]

The integration over \( \tilde{\mu} \), as well as in the general case, amounts to the substitution of its extremal value \( \frac{d\varepsilon_{n_{r}, l}^{(1)}}{d\tilde{\mu}} |_{\tilde{\mu}_{ext}} = 0 \) in (C.9). The extremal condition has the form

\[
\frac{(\pi \sigma)}{(2(\pi \sigma l)^{1/2})^{1/2}} \cdot \frac{\tilde{\mu}^{3/2}}{\mu^{3/2}} \left( n_{r} + \frac{1}{2} \right) + \frac{\sum_{i=1}^{K} m_{i}^2}{\tilde{\mu}^{2}} = 3 \frac{\tilde{\mu}^{2}}{(\pi \sigma l)^{1/2}}
\]

and is difficult to solve analytically. Therefore we find the solution in two different asymptotics. The first one corresponds to the relativistic radial dynamics

\[
< p_{r}^2 > \quad \Rightarrow \quad \sum_{i=1}^{K} m_{i}^2
\]

that in the leading order gives

\[
\tilde{\mu} = \left[ \frac{(\pi \sigma)^{5/4} l^{1/4}}{3 \sqrt{2}} \left( n_{r} + \frac{1}{2} \right) \right]^{2/5}
\]

and therefore the spectrum (C.9) can be represented in the form

\[
E_{n_{r}, l}^{(1)} = E_{n_{r}, l}^{(1,0)} + E_{n_{r}, l}^{(1,1)}
\]

with

\[
E_{n_{r}, l}^{(1,0)} = \frac{5}{4} \cdot \left( \frac{3}{4} \right)^{1/5} \left( \frac{\pi \sigma}{l^{1/2}} \right)^{1/2} l^{-3/10} \left( n_{r} + \frac{1}{2} \right)^{4/5}
\]

\[
E_{n_{r}, l}^{(1,1)} = \left( \frac{3}{4} \right)^{2/5} \sum_{i=1}^{K} m_{i}^2 \frac{l^{-1/10}}{\pi \sigma^{1/2}} \left( n_{r} + \frac{1}{2} \right)^{-2/5}
\]

We note that \( E_{n_{r}, l}^{(0)} \) is the radial correction corresponding to the massless quark case (84) which was evaluated in [1].
To improve the accuracy we take into account that usually the value of $\varepsilon$ obtained according to the time-independent $\mu$ ansatz (C.9), (C.10) is 1.1 times larger than the exact one (see [3] for a comparison). Therefore one is to multiply eqs.(C.13) additionally by a factor $\approx 0.9$.

The condition (C.11) can be reformulated as $E_{nr,l}^{(1,1)}/E_{nr,l}^{(1,0)} \ll 1$ with the result

$$
\left(\frac{\sum K_{i=1} m_i^2}{\pi \sigma}\right)^5 \ll \frac{(n_r + \frac{1}{2})^6}{l} \quad (C.14)
$$

Together with the condition, that zero order contribution (80), $E_{str}^{(0)} = \sqrt{\pi \sigma (l(l+1))^{1/2}}$, is much more than $E_{nr,l}^{(1,0)}$ of eq.(C.13)

$$
l/(n_r + 1/2) \gg 1 \quad (C.15)
$$

one obtains the domain of quantum numbers where expression for the spectrum (C.13) is valid.

In the opposite nonrelativistic asymptotics for radial dynamics

$$
< p_r^2 > \ll \sum_{i=1}^{K} m_i^2 \quad (C.16)
$$

one obtains in the leading order

$$
\bar{\mu} = \left[ \frac{\sum K_{i=1} m_i^2 (\pi \sigma l)^{1/2}}{3} \right]^{1/3} \quad (C.17)
$$

which gives rise to

$$
E_{nr,l}^{(1,0)} = \frac{3^{4/3}}{4} \cdot \frac{(\sum m_i^2)^{2/3}}{\pi \sigma l^{1/3}} \quad (C.18)
$$

As well as in the previous case the requirements of selfconsistency, $E_{nr,l}^{(1,0)} \gg E_{nr,l}^{(1,1)}$, $E_{str}^{(0)} \gg E_{nr,l}^{(0)}$, determine the conditions

$$
\left(\frac{\sum K_{i=1} m_i^2}{\pi \sigma}\right)^5 \gg \frac{(n_r + 1/2)^6}{l} \quad (C.19)
$$

and

$$
l \gg \left(\sum_{i=1}^{K} m_i^2 / \pi \sigma\right) \quad (C.20)
$$

respectively.

To summarize we calculated radial corrections in different asymptotics of the string regime which is valid under conditions (C.15), (C.20) that can be expressed in the unified way

$$
\frac{l}{n_r + \sum m_i^2 / \pi \sigma + \frac{1}{2}} \gg 1. \quad (C.21)
$$
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Figure Captions

Fig. 1. Regge trajectories of heavy-light and light-light mesons (with $m_q = 0$ for light quarks). Crosses are the results of the approximation of eqs. (64) - (66) and open circles are calculated with an account of the leading correction, eqs. (69) - (71) (they are not shown for $l = 5$ since they practically coincide with open squares). Open squares are the predictions of the large $l$ approximation (91) with an account of the correction (89). Black dots are the results of the numerical calculation of ref. [5].
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