SOME OBSERVATIONS ON THE NUMERICAL INDEX AND
THE POLYNOMIAL NUMERICAL INDEX

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Abstract. In this paper, we study both the numerical index and the
polynomial numerical index. First, we give a sufficient condition for a
Banach space $X$ to have lushness. Second, we study the relation between
the renormings of a Banach space and the $k$-order polynomial numerical
index. This shows that every real Banach spaces of dimension greater
than 1 can be renormed to have $2$-order polynomial numerical index $\alpha$
for any $\alpha \in [0, 1/18)$.

1. Introduction

The numerical index of a Banach space is a constant relating the norm and
the numerical radius of operators on the space. Let us recall the definitions.
Given a real or complex Banach space $X$, $B_X$ is the closed unit ball and $S_X$
is the unit sphere of $X$. The dual space will be denoted by $X^*$, and $L(X)$ is the
Banach algebra of all bounded linear operators on $X$.

The numerical radius of an operator $T \in L(X)$ is given by
\[ v(T) = \sup \{|x^*(Tx)| : x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}. \]
We see that $v$ is a seminorm on $L(X)$, and $v(T) \leq \|T\|$ for every $T \in L(X)$. The numerical index of the Banach space $X$ is given by
\[ n(X) = \inf \{v(T) : T \in S_{L(X)}\}. \]
It is easy to see that $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$
for every $T \in L(X)$, and so $v$ is a equivalent norm to the operator norm if and
only if $n(X) > 0$. We can find some related survey and papers in [7, 12, 21].

Later, the numerical index was extended to $k$-homogeneous polynomials, $k$-
order polynomial numerical index, in 2006 by Choi et al. [5] as follows
\[ n^{(k)}(X) = \inf \{v(P) : P \in S_{P^k(X; X)}\}. \]

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$P(\mathbb{K}^X; X)$ denotes the space of $k$-homogeneous polynomials, and numerical radius is defined by

$$v(P) = \sup\{|x^*(Px)| : x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}.$$ 

Recent results on polynomial numerical indices can be found in [6, 13, 14, 15, 16, 17].

Our aim in this paper is to give some observations on these numerical indices. In Section 2, we study the Banach space with numerical index 1. There were many studies to find a geometric characterization of a Banach space $X$ with numerical index 1 without using operators. Lushness was recently introduced in [4] as a geometrical property of a Banach space which ensures that the space has numerical index 1. Before lush Banach spaces were studied, the basic examples of Banach spaces with numerical index 1 had been known to be almost-CL-spaces [18, 19]. In fact, lushness is weakest among many isometric properties in the literature which are sufficient conditions for a Banach space to have numerical index 1.

For $x^* \in S_{X^*}$ and $0 < \epsilon < 1$, the set of the form

$$S(B_X, x^*, \epsilon) = \{x \in B_X : \text{Re} \ x^*(x) > 1 - \epsilon\}$$

is called a slice of the unit ball.

**Definition 1.1.** A Banach space $X$ is said to be lush if for every $x, y \in S_X$ and for every $\epsilon > 0$ there is a slice $S = S(B_X, x^*, \epsilon) \subset B_X$, $x^* \in S_{X^*}$, such that $x \in S$ and $\text{dist}(y, \text{aconv}(S)) < \epsilon$ where $\text{aconv}(S) = \{\sum_{i=1}^n \alpha_i s_i : \sum |\alpha_i| \leq 1, \ s_i \in S\}$.

On the other hand, a Banach space $X$ is said to have the alternative Daugavet property (ADP for short) if the norm identity

$$\max_{|\omega|=1} \|\text{Id} + \omega T\| = 1 + \|T\|$$

holds for every rank-one operators $T \in L(X)$. Since every Banach space with numerical index 1 has the ADP, every lush space has the ADP.

We recall that a convex combination of slices of a convex bounded subset $A$ of a Banach space $X$ is a subset of $A$ of the form $\sum_{i=1}^n \lambda_i S_i$, where $\lambda_i > 0$, \( \sum \lambda_i = 1 \) and the $S_i$’s are slices of $A$. For a closed convex bounded subset $A$ of a Banach space $X$, we say that $A$ has small combinations of slices [10] (SCS for short) if every slice of $A$ contains convex combinations of slices of $A$ with arbitrarily small diameter. In this section, we observe that for a Banach space $X$ if $B_X$ has SCS, then the lushness is equivalent to the ADP.

In Section 3, we study the relation between the renormings of a Banach space and the $k$-order polynomial numerical index. C. Finet et al. [8] investigated the values of the numerical index which can be obtained by renorming the space. Their main theorem is that for any Banach space $X$ the set of numerical indices of Banach spaces which are isomorphic to $X$ is an interval. We get the same result for the case of $k$-homogeneous polynomials. More over we conclude that
for any Banach space $X$ and $k \in \mathbb{N}$ the set of $k$-order polynomial numerical indices of Banach spaces which are isomorphic to $X^{**}$ is a subset of the set of $k$-order polynomial numerical indices of Banach spaces which are isomorphic to $X$.

2. SCS and numerical index 1

It is well known that for a Banach space $X$ with the Radon-Nikodým property the lushness and the ADP are equivalent (see [3, Remark 2.2] and [20, Remark 6]). A. Avilés et al. [1] generalized this fact for a strongly regular Banach space $X$. We recall that a closed convex bounded subset $A$ of a Banach space is said to be strongly regular if every non-empty convex subset $L$ of $A$ contains a convex combination of slices of $L$ of arbitrarily small diameter, and we say that $X$ is strongly regular if $B_X$ is strongly regular. Hence, the unit ball of every strongly regular Banach space has SCS.

A. Avilés et al. showed the followings. (1) For a Banach space $X$, a separable closed convex bounded subset $A$ of $X$ having SCS is a slicely countably determined (SCD for short) set. (2) Every Banach space $X$ with the ADP whose unit ball is a SCD set is lush. Hence, we see that separable Banach space with ADP whose unit Ball has SCS is lush. This implies that every strongly regular Banach space with ADP is lush, because lushness and the ADP are separably determined.

The aim of this section is to generalize the last result. Precisely, every Banach space $X$ with the ADP whose unit ball $B_X$ has SCS is lush.

**Proposition 2.1** ([1, Proposition 4.2]). Let $K(X^*)$ be the weak$^*$-closure in $X^*$ of $\text{ext}(B_X)$, and for every slice $S$ of $B_X$ and every $\epsilon > 0$, we write

$$D(S, \epsilon) = \{ y^* \in K(X^*) : S \cap T(S(B_X, y^*, \epsilon)) \neq \phi \}.$$  

Then, $X$ has the ADP if and only if for every $\epsilon > 0$ and every sequence $\{S_n : n \in \mathbb{N}\}$ of slices of $B_X$, the set $\bigcup_{n \in \mathbb{N}} D(S_n, \epsilon)$ is weak$^*$-dense in $K(X^*)$.

**Proposition 2.2** ([10, Corollary III.7]). Let $D$ be a closed convex bounded subset of a Banach space $X$. Suppose $D$ has SCS, then for any $x \in D$ and any $\epsilon > 0$ there exist positive scalars $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ and slices $S_1, \ldots, S_n$ of $D$ such that $\sum_{i=1}^n \alpha_i S_i \subset B(x, \epsilon)$.

**Theorem 2.3.** Every Banach space $X$ with the ADP whose unit ball $B_X$ has SCS is lush.

**Proof.** Fix any $x, y \in S_X$, and $\epsilon > 0$. From Proposition 2.2, there exist some positive numbers $\alpha_i > 0$ and slices $S_i$ such that $\sum_{i=1}^n \alpha_i = 1$ and $\sum_{i=1}^n \alpha_i S_i \subset B(y, \epsilon)$. From Proposition 2.1,

$$A = \{ y^* \in S_{X^*} : S_i \cap T(S(B_X, y^*, \epsilon)) \neq \phi, 1 \leq i \leq n \}$$

is a weak$^*$-dense set of $K(X^*)$ and hence $A$ is norming set. Choose $y^* \in A$ such that $\text{Re} y^*(x) > 1 - \epsilon$, then for every $1 \leq i \leq n$ there exist $x_i$ such that
Theorem 3.1 (Theorem 18.3) The mapping of bounded subset of $X$, the set of numerical indices of Banach spaces which are isomorphic to $x_{122}$ SUN KWANG KIM.

Since it is known that a unit ball which is dentable or huskable has SCS, we get the following.

Corollary 2.4. Every Banach space $X$ with the ADP whose unit ball $B_X$ is dentable or huskable is lush.

3. Renormings and the polynomial numerical index

Before we start, let us fix some notations. Given Banach space $X$, $\mathcal{E}(X)$ denote the set of all equivalent norms on the Banach space $X$, and for $p \in \mathcal{E}(X)$ it will be convenient to denote also by $p$ the dual norm of $p$. $\mathcal{E}(X)$ is an arcwise connected metric space when endowed with the distance given by

$$d(p, q) = \log(\min\{\alpha \geq 1 : p \leq \alpha q, \ q \leq \alpha p\})$$

We give a norm on $X \times X^*$ by $\|(x, f)\| = \|x\| + \|f\|$, and let $\Pi_p(X) = \{(x, x^*) \in X \times X^* : x^*(x) = p(x) = p(f) = 1\}$. $M(E)$ denote the set of all non-void bounded closed subsets of a metric space $E$ with the distance $d(A, B) = \max\{\sup_{x \in A} d(x, B), \ \sup_{x \in B} d(x, A)\}$.

Theorem 3.1 ([2, Theorem 18.3]). The mapping $p \to \Pi_p(X)$ is a continuous mapping of $\mathcal{E}(X)$ into $M(X \times X^*)$, and is uniformly continuous on each bounded subset of $M(X \times X^*)$.

From this theorem, C. Finet et al. [8] showed that for any Banach space $X$ the set of numerical indices of Banach spaces which are isomorphic to $X$, $\mathcal{N}(X)$, is an interval. Moreover, they conclude that $\mathcal{N}(X^*) \subset \mathcal{N}(X)$ and so $\mathcal{N}(X^*) \subset \mathcal{N}(X)$. In this section, we get the same result for polynomials.

First, we can easily check that the proof of the above theorem is also valid for any norm $\|(x, f)\| = a\|x\| + b\|f\|$ on $X \times X^*$, $a, b > 0$, and so we get the following corollary.

Hereafter, $v_p(P)$ is the numerical radius of $P$ in the space $(X, p)$, and $n^{(k)}(X, p)$ is the $k$-order polynomial numerical index of $(X, p)$. That is,

$$v_p(P) = \sup\{|x^*(Px)| : (x, x^*) \in \Pi_p\},$$

$$n^{(k)}(X, p) = \inf\{v_p(P) : P \in P^k(X; X), \|P\| = 1\}.$$

And let us define

$$\mathcal{N}^{(k)}(X) = \{n^{(k)}(X, p) : p \in \mathcal{E}(X)\}.$$

Corollary 3.2. Let $P \in P^k(X; X)$. The mapping $p \in \mathcal{E}(X) \to v_p(P)$ is continuous and uniformly continuous on bounded subsets.

Proof. Let $\epsilon > 0$ be given. with $k > 1$, define $G_k = \{p \in \mathcal{E}(X) : \mu(p, \|\|) < \alpha\}$. Using Theorem 3.1, there exists $\delta > 0$ such that

$$p, q \in G_k, \ d(p, q) < \delta \text{ implies } d(\Pi_p(X), \Pi_q(X)) < \epsilon.$$
For any \( \eta > 0 \), we can find \( (x, x^*) \in \Pi_p(X) \) so that \( |x^* P x| > v_p(P) - \eta \). Then, 
\[
d((x, x^*), \Pi_p) < \epsilon \text{ and so there exists } (y, y^*) \in \Pi_p \text{ with } \frac{k}{(k-1)!} \|x - y\| + \|x^* - y^*\| < \epsilon.
\]
Then,
\[
|y^*(Py) - x^*(Px)| \\
\leq |y^*(Py) - x^*(Py)| + |x^*(Py) - x^*(Px)| \\
\leq \|y - x\|\|P\|\|y\|^k + |x^*\sum_{j=1}^{k} P(y^{k-j+1}, x^{j-1}) - P(y^{k-j}, x^j)| \\
\leq \|y - x\|\|P\|e^{\alpha k} + k \frac{k!}{k!} \|x - y\|\|P\|\|x\|^{k-j}\|y\|^{j-1} \\
\leq e^{\alpha k}\|P\|\epsilon.
\]
Hence, \( v_p(P) - \eta < v_q(P) + e^{\alpha k}\|P\|\epsilon \). This equation is symmetric and \( \eta \) is arbitrary, so \( |v_p(P) - v_q(P)| < e^{\alpha k}\|P\|\epsilon \). \( \square \)

With the same proof of [8, proposition 2] we can get the following.

**Corollary 3.3.** Given a Banach space \( X \), the mapping \( p \to n^{(k)}(X, p) \) from \( \mathcal{E}(X) \) to \( \mathbb{R} \) is continuous. Hence, \( \mathcal{N}^{(k)}(X) \) is an interval.

**Proposition 3.4.** Let \( X \) be a Banach space of dimension greater than 1. Then \( 0 \in \mathcal{N}^{(k)}(X) \) for the real case and \( k^{(1-k)} \in \mathcal{N}^{(k)}(X) \) for the complex case.

*Proof.* Take a two-dimensional subspace \( Y \) of \( X \), and rite \( X = Y \bigoplus Z \) for suitable \( Z \). For the real Banach spaces, we know that \( n^{(k)}(\ell_2) = 0 \) for every \( k \in \mathbb{N} \) (see [9, Example 2.1]). We can see that \( X \simeq \ell_2 \bigoplus \mathbb{C} \) and that \( n^{(k)}(\ell_2 \bigoplus \mathbb{C}) = 0 \) from [5, Proposition 2.8]. For the complex Banach spaces, for each \( k \) there exists a 2-dimensional Banach space \( Y_k \) with \( n^{(k)}(Y_k) = k^{(1-k)} \) [11]. By the same argument in the real case, we can finish our proof. \( \square \)

From the facts that \( n^{(k)}(X) \geq k^{(1-k)} \) for every complex Banach space and \( n^{(k)}(X^{**}) \leq n^{(k)}(X) \) [5, Theorem 2.3, Corollary 2.15] we get our main result in this section.

**Corollary 3.5.** For every Banach space \( X \), \( \mathcal{N}^{(k)}(X^{**}) \subset \mathcal{N}^{(k)}(X) \).

The followings are the spaces whose polynomial numerical indices have been known. In the complex case, \( n^{(k)}(C_0(L)) = 1 \) for every \( k \in \mathbb{N} \) and every locally compact space \( L \), and \( n^{(2)}(\ell_1) \leq 1/2 \). In the real case, \( n^{(2)}(c_0) = n^{(2)}(\ell_1) = n^{(2)}(\ell_\infty) = 1/2 \). Hence, we can see that \( [k^{(1-k)}, 1] = \mathcal{N}^{(k)}(C_0(L)) \) for complex case, 
\[
[0, 1/2] \subset \mathcal{N}^{(2)}(c_0), \mathcal{N}^{(2)}(\ell_1), \mathcal{N}^{(2)}(\ell_\infty) \text{ for real case.}
\]

Now we recall an isometric condition, called *property \( \beta \)* that was introduced by Lindenstrauss [15].
Proposition 3.7. Let \( \rho \) constant

Definition 3.6. A Banach space \( X \) is said to have the property \( \beta \) with constant \( \rho \) if there are two sets \( \{ x_i : i \in I \} \subset S_X, \{ x^*_i : i \in I \} \subset S_{X^*} \) and \( 0 \leq \rho < 1 \) such that the following conditions hold:

1. \( x^*_i(x_i) = 1, \forall i \in I \).
2. \( |x^*_i(x_j)| \leq \rho \) if \( i, j \in I, i \neq j \).
3. \( \|x_i\| = \sup_{x \in I}(|x^*_i(x)|) \) for all \( x \in X \).

The spaces \( c_0 \) and \( \ell_{\infty} \) satisfy the above property. In both cases \( \rho = 0 \).

Finet et al. [8] calculated the numerical index of a Banach space which has the property \( \beta \) with constant \( \rho \) to get the fact that \([0,1/3] \subset N'(X)\) in the real case and \([e^{-1},1/2] \subset N'(X)\) in the complex case. Similarly, we calculate the 2-order polynomial numerical index of a Banach space which has the property \( \beta \) with constant \( \rho \), and find some intervals which contained in \( N^2(X) \).

Proposition 3.7. Let \( X \) be a real Banach space which has the property \( \beta \) with constant \( \rho \) and write \( 0 < \lambda = (1 - \rho)/(1 + \rho) \leq 1 \). Then

\[
n^{(2)}(X) \geq \frac{\lambda^2}{2}.
\]

Proof. The proof follows from a modification of the argument in the one of Proposition 5 in [8], but for the sake of completeness we give here its details.

Let \( \{ x_i : i \in I \} \subset S_X, \{ x^*_i : i \in I \} \subset S_{X^*} \) be the sets which appear in the definition of the property \( \beta \) with constant \( \rho \). Given \( \epsilon > 0 \) and \( P \in P^2(X; X) \), choose \( x \in S_X \) so that \( \|Px\| > \|P\| - \epsilon \). There exist some \( j \in I \) such that \( |x^*_j(Px)| > \|P\| - \epsilon \). Define \( t = (1 + \lambda x^*_j(x))/2, z = \lambda x + (1 - \lambda x^*_j(x))x_j, \) and \( w = \lambda x - (1 + \lambda x^*_j(x))x_j \).

We can see easily that \( x^*_j(z) = 1, x^*_j(w) = -1, z - w = 2x_j, \) and \( \|z\| = \|w\| = 1 \). Since \( \lambda x = tz + (1 - t)w \), we get

\[
|x^*_j(Px)| = |x^*_j(P(tz + (1 - t)w))| > \lambda^2(\|P\| - \epsilon).
\]

Hence, from the fact that \( 2P(z,w) = Pz + Pw - P(z-w) \) we get

\[
\lambda^2(\|P\| - \epsilon) < t^2|x^*_j(Pz)| + (1 - t)^2|x^*_j(Pw)| + 2t(1 - t)|x^*_jP(z,w)|
\]

\[
\leq t|x^*_j(Pz)| + (1 - t)|x^*_j(Pw)| + 4t(1 - t)|x^*_jPz|,
\]

where \( \tilde{P} \) is the associated symmetric bilinear map of \( P \). From the facts \( t + (1 - t) + 4t(1 - t) \leq 2 \) and \( t, 1 - t \geq 0 \), we can see that one of \( |x^*_j(Pz)|, |x^*_j(Pw)|, |x^*_jPz| \) must bigger than \( \frac{\lambda^2}{2}(\|P\| - \epsilon) \). This implies that \( vi(P) \geq \frac{\lambda^2}{2} \|P\| \). \( \square \)

For the complex case, we can get improved result.

Proposition 3.8. Let \( X \) be a complex Banach space which has the property \( \beta \) with constant \( \rho \) and write \( 0 < \eta = (1 - \rho) \leq 1 \). Then

\[
n^{(2)}(X) \geq \frac{\eta^2}{2}.
\]
Proof. Similarly to the case of [8, Proposition 6], we follow the proof of Proposition 3.7. By multiplying by some scalar, we can assume that $x^*_j(x) \in \mathbb{R}$ and so we can find $a \in [0,1]$ such that $|\eta x^*_j(x) + ai| = |\eta x^*_j(x) - ai| = 1$. We use vectors $z = \eta x + aix_j$ and $w = \eta x - aix_j$, then $z - w = 2aix_j$. The rest of the proof is analogous to the one of Proposition 3.7.

From the fact that every real Banach space can be renormed to have property $\beta$ with constant less than $\frac{1}{2} + \epsilon$ for every $\epsilon > 0$, we get the following corollary by using Proposition 3.7.

**Corollary 3.9.** Let $X$ be a real Banach space of dimension greater than 1. Then $[0, 1/18) \subset N^{(2)}(X)$.

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