Sharp Bounds in Stochastic Network Calculus

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Abstract

The practicality of the stochastic network calculus (SNC) is often questioned on grounds of potential looseness of its performance bounds. In this paper it is uncovered that for bursty arrival processes (specifically Markov-Modulated On-Off (MMOO)), whose amenability to per-flow analysis is typically proclaimed as a highlight of SNC, the bounds can unfortunately indeed be very loose (e.g., by several orders of magnitude off). In response to this uncovered weakness of SNC, the (Standard) per-flow bounds are herein improved by deriving a general sample-path bound, using martingale based techniques, which accommodates FIFO, SP, EDF, and GPS scheduling. The obtained (Martingale) bounds gain an exponential decay factor of $O\left(e^{-\alpha n}\right)$ in the number of flows $n$. Moreover, numerical comparisons against simulations show that the Martingale bounds are remarkably accurate for FIFO, SP, and EDF scheduling; for GPS scheduling, although the Martingale bounds substantially improve the Standard bounds, they are numerically loose, demanding for improvements in the core SNC analysis of GPS.

I. INTRODUCTION

Several approaches to the classical queueing theory have emerged over the past decades. For instance, matrix analytic methods (MAM) not only provide a unified treatment for a large class of queueing systems, but they also lend themselves to practical numerical solutions; two key ideas are the proper accounting of the repetitive structure of underlying Markov processes, and the use of linear algebra rather than classic methods based on real analysis (see Neuts [42] and Lipsky [37]). Another unified approach targeting broad classes of queueing problems is the stochastic network calculus (SNC) (see Chang [10] and Jiang and Liu [28]), which can be regarded as a mixture between the deterministic network calculus conceived by Cruz [19] (see Le Boudec and Thiran [6]) and the effective bandwidth theory (see Kelly [30]). Because SNC solves queueing problems in terms of bounds, it is often regarded as an unconventional approach, especially by the queueing theory community.

MAM and SNC could be (slightly) compared by the way they apply to queues with fluid arrivals. In their simplest form, fluid arrival models were defined as Markov-Modulated On-Off (MMOO) processes by Anick, Mitra, and Sondhi [1], and were significantly extended thereafter, especially for the purpose of modelling the increasingly prevalent voice and video traffic in the Internet. By relating fluid models and Quasi-Birth-Death (QBD) processes, Ramaswami has argued that MAM can lend themselves to numerically more accurate solutions than spectral analysis methods [46]. In turn, SNC can produce alternative solutions with negligible numerical complexity, but these are arguably less relevant than exact solutions (simply because they are expressed as bounds). What does, therefore, justify more than two decades of research in SNC?

The answer lies in two key features of SNC: scheduling abstraction and convolution-form networks (see Ciucu and Schmitt [16]). The former expresses the ability of SNC to compute per-flow (or per-class) queueing metrics for a large class of scheduling algorithms, and in a unified manner. Concretely, given a flow $A$ sharing a queueing system with other flows, the characteristics of the scheduling algorithm are first abstracted away in the so-called service process; thereafter, the derivation of queueing metrics for the flow $A$ is scheduling independent. Furthermore, the per-flow results can be extended in a straightforward manner from a single queue to a large class of queueing networks (typically feed-forward), which are amenable to a convolution-form representation in an appropriate algebra.

By relying on these two features, SNC could tackle several open queueing network problems. The typical scenario involves the computation of end-to-end (e2e) non-asymptotic performance bounds (e.g., on the delay distribution) of a single flow crossing a tandem network and sharing the single queues with some other flows. Such scenarios were solved for a large class of arrival processes (see, e.g., Ciucu et al. [14], 8 and Fidler [25] for MMOV processes, and Liebeherr et al. [55] for heavy-tailed and self-similar processes). Another important solution was given for the e2e delay distribution in a tandem (packet) network with Poisson arrival and exponential packet sizes, by circumventing Kleinrock’s independence assumption on the regeneration of packet sizes at each node (see Burchard et al. [7]). Other fundamentally difficult problems include the performance analysis of stochastic networks implementing network coding (see Yuan et al. [54]), the delay analysis of wireless channels under Markovian assumptions (see Zheng et al. [56]), the delay analysis of multi-hop fading channels (see Al-Zubaidy et al. [57]), or bridging information theory and queueing theory by accounting for the stochastic nature and delay-sensitivity of real sources (see Lübben and Fidler [59]).

Based on its ability to partially solve fundamentally hard queueing problems (i.e., in terms of bounds), SNC is justifiably proclaimed as a valuable alternative to the classical queueing theory (see Ciucu and Schmitt [16]). At the same time, SNC is also justifiably questioned on the tightness of its bounds. While the asymptotic tightness generally holds (see Chang [10], p. 291, and Ciucu et al. [14]), doubts on the bounds’ numerical tightness shed skepticism on the practical relevance of SNC.
This skepticism is supported by the fact that SNC largely employs the same probability methods as the effective bandwidth theory, which was argued to produce largely inaccurate results for non-Poisson arrival processes (see Choudhury et al. 12). Moreover, although the importance of accompanying bounds by simulations has already been recognized in some early works (see Zhang et al. 55 for the analysis of GPS), the SNC literature is scarce in that respect.

In this paper we reveal what is perhaps ‘feared’ by SNC proponents and expected by others: the bounds are very loose for the class of MMOO processes, which is very relevant as these can be tuned for various degrees of burstiness. In addition to providing numerical evidence for this fact (the bounds can be off by arbitrary orders of magnitude, e.g., by factors as large as 100 or even 1000), we also prove that the bounds are asymptotically loose in multiplexing regimes. Concretely, we (analytically) prove that the bounds are ‘missing’ an exponential decay factor of $O(e^{-\alpha n})$ in the number of flows $n$, where $\alpha > 0$; this missing factor was conjectured through numerical experiments in Choudhury et al. 12 in the context of effective bandwidth results (which scale identically as the SNC bounds).

While this paper convincingly uncovers a major weakness in the SNC literature, it also shows that the looseness of the bounds is generally not inherent in SNC but it is due to the ‘temptatious’ but ‘poisonous’ elementary tools from probability theory leveraged in its application. We point out that such methods have also been employed in the effective bandwidth literature dealing with scheduling; see Courcoubetis and Weber 17 for FIFO, Berger and Whitt 2 and Wischik 53 for SP, Sivaraman and Chiussi 50 for EDF, and Zhang et al. 55 and Bertsimas et al. 3 for WFQ. Unlike the SNC results, which are given in terms of non-asymptotic bounds, the corresponding effective bandwidth results are typically given in larger buffer asymptotics; while exactly capturing the asymptotic decay rate, they fail to capture the extra $O(e^{-\alpha n})$ decay factor pointed out by Choudhury et al. 12 or by Botvich and Duffield 5.

To fix the weakness of existing SNC bounds, and also of existing effective bandwidth asymptotic results in scheduling scenarios, this paper leverages more advanced tools (i.e., martingale based techniques) and derives new Martingale bounds improving dramatically to the point of almost matching simulation results. We show the improvements for per-flow delay bounds in FIFO, SP, EDF, and WFQ scheduling scenarios with MMOO flows, and in addition we prove the existence of the conjectured $O(e^{-\alpha n})$ decay factor. We point out that extensions to more general Markovian arrivals are immediate (see Appendix A); due to their increased complexity, however, the generalized results do not easily lend themselves to visualizing the $O(e^{-\alpha n})$ decay factor uncovered herein for MMOO flows.

The sharp bounds obtained in this paper are the first in the conventional stochastic network calculus literature, i.e., involving service processes which decouple scheduling from the analysis. Their significance, relative to existing sharp bounds in the effective bandwidth literature (e.g., Duffield 23 and Chang 10, pp. 339-343, using martingale inequalities, or Liu et al. 58 by extending an approach of Kingman involving integral inequalities 55), is that they apply at the per-flow level for various scheduling; in turn, existing sharp bounds only apply at the aggregate level. In other words, our sharp bounds generalize existing ones by accounting for FIFO, SP, EDF, and WFQ scheduling.

The rest of the paper is structured as follows. In Section II we identify, at an intuitive level, the elementary tool from probability theory which is ‘responsible’ for the very loose (Standard) bounds in SNC. In Section III we describe the queuing model and some necessary SNC formalisms. The core of the paper is Section IV which computes the improved (Martingale) and reviews the existing (Standard) SNC per-flow delay bounds in multiplexing scenarios with MMOO flows; both analytical and numerical comparisons of the bounds are further explored. Concluding remarks are presented in Section V.

II. Three Bounding Steps in SNC and One Pitfall

This section overview the SNC bounding approach to compute per-flow queueing metrics for broad classes of arrivals and scheduling. In addition to identifying three major steps in this approach, it is conveyed by means of a simple example that careless bounding can lend itself to impractical results.

Towards this end, we consider a simplified queueing systems in which a (cumulative) arrival process $A(t)$ shares with some other flows a server with capacity $C$ and infinite queue length. We are particularly interested in the complementary distribution of $A(t)$’s backlog process $B(t)$, which is bounded in SNC for some $t, \sigma \geq 0$ according to

$$\mathbb{P}(B(t) > \sigma) \leq \mathbb{P} \left( \sup_{0 \leq s \leq t} \{A(s) - S(s, t)\} > \sigma \right). \tag{1}$$

Here, $A(s, t) := A(t) - A(s)$ is the bivariate extension of $A(t)$, whereas $S(s, t)$ is another bivariate process, called a service process, encoding the information about the server, the scheduling, and the other arrival processes that $A(t)$ shares the server with. In the simplest setting with no other arrivals, $S(s, t) = C(t-s)$ and Eq. (1) (with equality) recovers Reich’s equation.

In another setting in which $A(t)$ receives the lowest priority, should the server implement a static priority (SP) scheduler, then $S(s, t) = C(t-s) - A_c(s, t)$, where $A_c(s, t)$ denotes the other arrivals at the server.

Eq. (1) typically continues in SNC by invoking the Union Bound, i.e.,

$$\mathbb{E} \left( \sum_{s=0}^{t} \mathbb{P}(A(s) - S(s, t) > \sigma) \right). \tag{2}$$
The probability events can be further computed either by 1) convolving the distribution functions of $A(s, t)$ and $S(s, t)$, when available, and under appropriate independence assumptions, or by following a more elegant procedure using the Chernoff bound, i.e.,

$$E_{t,s=0} \leq \sum_{s=0}^{t} E \left[ e^{\theta(A(s,t) - S(s,t))} \right] e^{-\theta \sigma},$$

for some $\theta > 0$. The expectation can be split into a product of expectations, according to the statistical independence properties of $A(s, t)$ and $S(s, t)$, and the sum can be further reduced to some canonical form.

Eqs. (1)-(3) outline three major bounding steps. The first is ‘proprietary’ to SNC, in the sense that it involves the unique construction of a ‘proprietary’ service process $S(s,t)$ which decouples scheduling from analysis. The next two follow general purpose methods in probability theory, which are applied in the same form in the effective bandwidth theory, except that $S(s,t)$ is now a random process rather than a constant-rate function.

The second step in particular reveals a convenient mathematical continuation of Eq. (1). The reason for this ‘temptatious’ step to be consistently invoked in SNC stems from the ‘freedom’ of seeking for bounds rather than exact results. As we will show over the rest of this section, and of the paper, this ‘temptatious’ step is also ‘poisonous’ in the sense that it lends itself to very loose bounds for a class of queuing scenarios which is being proclaimed in SNC as a highlight of its scope.

To convey insight into this direction, let us consider the stationary but non-ergodic process

$$A(s,t) = (t-s)X \forall 0 \leq s \leq t,$$

where $X$ is a Bernoulli random variable taking values in $\{0, 2\}$, each with probabilities $1 - \varepsilon > .5$ and $\varepsilon > 0$. Assume also that $S(s,t) = t-s$. Clearly, for $\sigma > 0$ and for sufficiently large $t$, the backlog process satisfies

$$P(B(t) > \sigma) = \varepsilon.$$

In turn, the application of the bound from Eq. (2) lends itself to a bogus bound, i.e.,

$$P(B(t) > \sigma) \leq \varepsilon t,$$

for $\sigma < 1$ (for $\sigma \geq 1$, the bound diverges as well). The underlying reason behind this bogus result is that the Union Bound from Eq. (2) is agnostic to the statistical properties of the increments of the arrival process $A(s,t)$.

The construction of $A(s,t)$ from Eq. (4) is meant to convey insight into the poor performance of the Union Bound for arrivals with correlated increments, such as MMOO processes. Within the same class, another relevant arrival process is the fractional Brownian motion which has long-range correlations; the analysis of such processes was done either by approximations (e.g., Norros [43]) or by using the Union Bound (e.g., Rizk and Fidler [37]). The rest of the paper will unequivocally reveal that the Union Bound leads to very loose per-flow bounds for MMOO processes.

The Union Bound can however lend itself to reasonably tight bounds when $X_s := A(s,t)$’s are rather uncorrelated (see Talagrand [52]). Shroff and Schwartz [49] argued that the effective bandwidth theory yields reasonable bounds only for Poisson processes. Moreover, Ciucu [13] provided numerical evidence that SNC lends itself to reasonably tight bounds for Poisson arrivals as well.

III. Queueing Model

This section introduces the queueing model and necessary SNC formalisms. The time model is continuous. Consider a stationary (bivariate) arrival process $A(s,t)$ defined as

$$A(s,t) := \int_u^t a(u)du \forall 0 \leq s \leq t, \quad A(t) := A(0,t),$$

where $a(s) \forall s \geq 0$ is the increment process.

According to Kolmogorov’s extension theorem, the one-side (stationary) process $\{a(s) : 0 \leq s < \infty\}$ can be extended to a two-side process $\{a(s) : -\infty < s < \infty\}$ with the same distribution. For convenience, we often work with the reversed cumulative arrival process $A'(s,t)$ defined as

$$A'(s,t) := \int_u^t a(-u)du \forall 0 \leq s \leq t, \quad A'(t) := A'(0,t).$$

This definition is identical with that of $A(s,t)$, except that the time direction is reversed.

Working with time reversed processes is particularly convenient in that the steady-state queueing process (say in a queueing system with constant-rate capacity $C$ fed by the one-side increment process $a(s)$) can be represented by Reich’s equation

$$Q = \sup_{t \geq 0} \{A'(t) - Ct\}.$$
Then we overview the corresponding Standard bounds in SNC. Lastly, we compare these bounds both asymptotically, as well as against simulations.

The evaluation of $Q$ needs an additional stability condition, e.g., $\limsup_{t \to \infty} \frac{A(t)}{t} < C$ a.s. (see Chang [10], pp. 293-294); this condition is fulfilled by the (stronger) Loynes’ condition, i.e., $a(s)$ is also ergodic and $\lim_{t \to \infty} \frac{A(t)}{t} = E[a(1)] < C$ a.s.

In this paper we mostly consider the queueing system depicted in Figure 1. Two cumulative arrival processes $A_1(t)$ and $A_2(t)$, each containing $n_1$ and $n_2$ sub-flows, are served by a server with constant-rate $C = nc$, where $n = n_1 + n_2$. The parameter $c$ is referred to as the per-(sub)flow capacity, and will be needed in the context of asymptotic analysis. For clarity, $A_1(t)$ and $A_2(t)$ will also be suggestive referred to as the through and cross (aggregate) flows, respectively. The data units are infinitesimally small and are referred to as bits. The queue has an infinite size capacity, and is assumed to be stable.

The performance measure of interest is the virtual delay process for the (through) flow $A_1(t)$, defined as

$$W_1(t) := \inf \{d \geq 0 : A_1(t-d) \leq D_1(t)\}; \quad \forall t \geq 0,$$

where $D_1(t)$ is the corresponding departure process of $A_1(t)$ (see Figure 1). The attribute virtual expresses the fact that $W_1(t)$ models the delay experienced by a virtual data unit departing at time $t$. Note that $W_1(t)$ is the horizontal distance between the curves $A_1(t)$ and $D_1(t)$, starting backwards from the point $(t, D_1(t))$ in the Euclidean space.

In stochastic network calculus, queueing performance metrics (e.g., bounds on the distribution of the delay process $W_1(t)$) are derived by constructing service curve processes, which relate the departure and arrival processes by a (min, +) convolution. For instance, in the case of $A_1(t)$ and $D_1(t)$, the corresponding service process is a stochastic process $S_1(s, t)$ such that

$$D_1(t) \geq A_1 * S_1(t) \quad \forall t \geq 0,$$

where ‘*’ is the (min, +) convolution operator defined for all sample-paths as $A_1 * S_1(t) := \inf_{0 \leq s \leq t} \{A_1(s) + S_1(s, t)\}$.

The service process $S_1(s, t)$ typically encodes the information about the cross aggregate $A_2(t)$ and the scheduling algorithm; other information such as the packet size distribution is omitted herein in accordance to the infinitesimal data units assumption. Conceptually, the service process representation from Eq. (5) encodes $A_1(t)$’s own service view, as if it was alone at the network node (i.e., not competing for the service capacity $C$ with other flows). Although the representation is not exact due to the inequality from Eq. (5), it suffices for the purpose of deriving upper bounds on the distribution of $W_1(t)$. The driving key property is that Eq. (5) holds for all arrival processes $A_1(t)$. Due to this property, the service representation in SNC is somewhat analogous with the impulse-response representation of signals in linear and time invariant (LTI) systems (see Ciucu and Schmitt [16] for a recent discussion on this analogy).

In this paper we will compute the distribution of the through-aggregate’s delay process $W_1(t)$ for four distinct scheduling algorithms at the server, i.e., First-In-First-Out (FIFO), Static Priority (SP), Earliest-Deadline-First (EDF), and Generalized Processor Sharing (GPS). The enabling service process $S_1(s, t)$ for the delay computations, for each of scheduling algorithms, will be presented in Section 4.4.4-A.

### IV. SNC Bounds for MMOO Processes

In this section we consider the queueing scenario from Figure 1 in which the sub-flows comprising $A_1(t)$ and $A_2(t)$ are Markov-Modulated On-Off (MMOO) processes. Because such processes can be tuned for various degrees of burstiness, they are particularly relevant for both modelling purposes and testing the tightness of related performance bounds.

After defining the MMOO processes, we derive Martingale bounds for the distribution of $W_1(t)$ for FIFO, SP, EDF, and GPS scheduling. Then we overview the corresponding Standard bounds in SNC. Lastly, we compare these bounds both asymptotically, as well as against simulations.

**Fig. 2. A Markov-modulated On-Off (MMOO) process**
Each MMOO sub-flow is modulated by a continuous time Markov process $Z(t)$ with two states denoted by 0 and 1, and transition rates $\mu$ and $\lambda$ as depicted in Figure 2. The cumulative arrival process for each sub-flow is defined as

$$A'(s,t) := \int_{u=s}^{t} Z(u)P \, du \quad \forall 0 \leq s \leq t \,, \quad A'(t) := A'(0,t) \,,$$

where $P > 0$ is the peak rate. In other words, $A'(t)$ models a data source transmitting with rates 0 and $P$ while $Z(t)$ delves in the 0 and 1 states, respectively. The steady-state ‘On’ probability is $p := \frac{\mu}{\lambda + \mu}$ and the average rate is $pP$.

![Fig. 3. A Markov-modulated process for the aggregation of $n$ homogeneous MMOOs](image)

When $n$ such statistically independent sources are multiplexed together then the corresponding modulating Markov process, denoted with abuse of notation as $Z(t)$ as well, has the states $\{0, 1, \ldots, n\}$ and the transition rates as depicted in Figure 3. The cumulative arrival process for the aggregate flow is defined identically as for each sub-flow, i.e.,

$$A(s,t) := \int_{u=s}^{t} Z(u)P \, du \quad \forall 0 \leq s \leq t \,, \quad A(t) := A(0,t) \,.$$

Note that, by definition, $A(s,t)$ is continuous.

### 4. Martingale Bounds

Recall our main goal of deriving bounds on the distribution of the through aggregate’s delay process $W_1(t)$ for the FIFO, SP, EDF, and GPS scheduling scenarios in the network model from Figure 1. We start this section with a general result enabling the analysis of all four scheduling scenarios, and then analyze each separately.

**Theorem 1:** (Martingale Sample-Path Bound) Consider the single-node queueing scenario from Figure 1 in which $n$ sub-flows are statistically independent MMOO processes with transition rates $\mu$ and $\lambda$, and peak rate $P$, and starting in the steady-state. The aggregate arrival processes are $A_1(t)$ and $A_2(t)$, each being modulated by the (stationary) Markov processes $Z_1(t)$ and $Z_2(t)$ with $n_1$ and $n_2$ states, respectively, with $n_1 + n_2 = n$. Assume that the utilization factor $\rho := \frac{\lambda \mu}{\lambda + \mu}$ satisfies $\rho < 1$ for stability, where $p$ is the steady-state ‘On’ probability; assume also that $P > c$ to avoid a trivial scenario with zero delay. Then the following sample-path bound holds for all $0 \leq u \leq t$ and $\sigma$

$$\mathbb{P}\left( \sup_{0 \leq s < t-u} \left\{ A_1(s,t-u) + A_2(s,t) - C(t-s) \right\} > \sigma \right) \leq K^n e^{-\gamma(C_1 u + \sigma)}$$

where $C_1 = n_1 c$, $K = \rho \left( \frac{\rho \rho + 1 - \rho}{(1-\rho)} \right)^{\frac{2}{3} - 1}$, and $\gamma = \frac{(\lambda + \mu)(1-\rho)}{P^2 - c}$.

We point out that the crucial element in the sample-path bound from Eq. (8) is the parameter $u$, which can be explicitly tuned depending on the scheduling algorithm for the bits of $A_1(t)$ and $A_2(t)$. From a conceptual point of view, the parameter $u$ encodes the information about the underlying scheduling, whereas the theorem further enables the per-flow delay analysis for several common scheduling algorithms: FIFO, SP, and EDF (see Subsections IV-A1–IV-A3).

The obtained delay bounds generalize the delay bounds previously obtained by Palmowski and Rolski [44], by further accounting for several scheduling algorithms. The bounds from [44] can be recovered by applying Theorem 3 with $A_2(t) = 0$ (i.e., no cross traffic and thus no scheduling being considered). The key to the proof of Theorem 3 is the construction of a single martingale $M_t$ from two others suitably shifted in time; this subtle construction, together with the scheduling abstraction feature of SNC, are instrumental to the per-flow analysis for the different scheduling algorithms. Furthermore, the sample-path bound from Eq. (8) follows from a standard technique based on the Optional Sampling theorem, applied to the martingale $M_t$; for relevant definitions and results related to martingales we refer to the Appendix.B. Also, for the generalization of Theorem 3 to general Markov fluid processes we refer to Appendix.A: as mentioned in the Introduction, however, the generalized result does not lend itself to visualizing the conjectured $O(e^{-c n})$ decay factor, for which reason we mainly focus on MMOO processes.

**Proof:** Fix $u \geq 0$ and $\sigma$. For convenience, let us bound the probability from Eq. (8) by shifting the time origin and using the time-reversed representation of arrival processes described in Section III i.e.,

$$\mathbb{P}\left( \sup_{t>u} \left\{ A_1'(u,t) + A_2'(u,t) - C(t-u) \right\} + A_2'(u) - C_2 u > C_1 u + \sigma \right)$$

1More exactly, [44] gives backlog bounds at the aggregate level which can be immediately translated into delay bounds, given the fixed server capacity for the whole aggregate.
where $C_2 = n_2c$. This representation is possible because the underlying Markov modulating processes of $A_1(t)$ and $A_2(t)$, i.e., $Z_1(t)$ and $Z_2(t)$, respectively, are time-reversible processes (see, e.g., Mandjes [40], p. 57); the reversibility is a consequence of the fact that $Z_1(t)$ and $Z_2(t)$ are stationary birth-death processes (see Kelly [29], pp. 10-11). Denote by $Z_1^i(t)$ and $Z_2^i(t)$ the time-reversed versions of $Z_1(t)$ and $Z_2(t)$, respectively.

Given the previous probability event we define the stopping time

$$T := \inf \{ t > u : A_1^i(u, t) + A_2^i(u, t) - C(t - u) + A_2^i(u) - C_2^i u > C_1 u + \sigma \} .$$

This construction is motivated by the fact that $\mathbb{P}(T < \infty)$ is exactly the probability from Eq. (37). The goal of the rest of the proof is to bound $\mathbb{P}(T < \infty)$.

Let $\mathbb{P}_{i,j}$ denote the underlying probability measure conditioned on $Z_1^i(u) = i$ and $Z_2^j(0) = j$, for $0 \leq i \leq n_1$ and $0 \leq j \leq n_2$. Denote also the stationary probability vectors of $Z_1^i(u)$ and $Z_2^j(u)$ by $(\pi_{1,0}, \ldots, \pi_{1,n_1})$ and $(\pi_{2,0}, \ldots, \pi_{2,n_2})$, respectively.

Next we define the following two processes

$$M_1(t) := e^{-\theta(Z_1^i(t)+\gamma)} \int_u^t (e^{C_1 s} - C_1) ds \quad \forall t \geq u$$

$$M_2(t) := e^{-\theta(Z_2^j(t)+\gamma)} \int_u^t (e^{C_2 s} - C_2) ds \quad \forall t \geq 0 ,$$

where $\theta := \log \frac{P_{0,0}}{e}$. Note that $\theta < 0$ due to the stability condition $\rho < 1$.

According to Palmowski and Rolski [44], both $M_1(t)$ and $M_2(t)$ are martingales with respect to (wrt) $\mathbb{P}_{i,j}$ and the natural filtration (for the original result see Ethier and Kurtz [24], p. 175). Moreover, according to Lemmas 3 and 2 from the Appendix, the following process

$$M_t := \begin{cases} 
M_2(t) & t \leq u \\
M_1(t)M_2(t) & t > u
\end{cases}$$

is also a martingale (note that $M_1(u) = 1$, by construction).

Because $T$ may be unbounded, we need to construct the bounded stopping times $T \wedge k$ for all $k \in \mathbb{N}$. For these times, the Optional Sampling theorem (see Theorem 4 in the Appendix) yields

$$E_{i,j} [M_0] = E_{i,j} [M_{T \wedge k}] ,$$

for all $k \in \mathbb{N}$, where the expectations are taken wrt $\mathbb{P}_{i,j}$. Using $E_{i,j} [M_0] = 1$ and according to the construction of $M_2(t)$ we further obtain for $k > u$

$$1 \geq E_{i,j} [M_{T \wedge k} I_{\{T \leq k\}}] \geq e^{-\theta(Z_1^i(T)+\gamma)} \mathbb{E}_{i,j} (C_1 u + \sigma) \mathbb{P}_{i,j} (T \leq k) ,$$

where $I_{\{\cdot\}}$ denotes the indicator function. The first term in the product follows from $\theta < 0$ and

$$(Z_1^i(T) + Z_2^j(T)) P \geq C_1 + C_2 ,$$

according to the construction of $T$ from Eq. (38) and the continuity property of the arrival processes. The second term follows from $\gamma > 0$ and the construction of $T$.

By deconditioning on $i$ and $j$ (note that $Z_1^i(T)$ and $Z_2^j(0)$ are in steady-state by construction) we obtain

$$\mathbb{P}(T \leq k) \leq \sum_{i,j} \pi_{1,i} \pi_{2,j} e^{\theta(C_1 + C_2 - (i+j))} e^{-\gamma (C_1 u + \sigma)} .$$

Using the identities

$$\sum_{i=0}^{n_1} \pi_{1,i} e^{\theta(\gamma - i)} = K^{n_1}$$

and

$$\sum_{j=0}^{n_2} \pi_{2,j} e^{\theta(C_2 - j)} = K^{n_2}$$

(see [44]) and taking $k \to \infty$ we finally obtain that

$$\mathbb{P}(T < \infty) \leq K^n e^{-\gamma (C_1 u + \sigma)} ,$$

which completes the proof. \hfill \Box

In the following we fix $0 \leq d \leq t$ and derive bounds on $\mathbb{P}(W_1(t) > d)$ for FIFO, SP, EDF, and GPS scheduling; the derivations follow more or less directly by instantiating the parameters of Theorem 3 for each scheduling case.
1) FIFO: The FIFO server schedules the data units of \( A_1(t) \) and \( A_2(t) \) in the order of their arrival times.

To derive a bound on the distribution of the through aggregate’s (virtual) delay process \( W_1(t) \), we rely on a service process construction for FIFO scheduling, as mentioned in Section III. We use the service process from Cruz [21] extended to bivariate stochastic processes, i.e.,

\[
S_1(s, t) = [C(t - s) - A_2(s, t - x)]_+ I_{\{t - s > x\}} ,
\]

(10)

for some fixed \( x \geq 0 \) and independent of \( s \) and \( t \) (for a proof, in the slightly simpler case of univariate processes, see Le Boudec and Thiran [6], pp. 177-178; for a more recent and general proof see Liebeherr et al. [35]). By notation, \( [y]_+ := \max\{y, 0\} \) for some real number \( y \).

Using the equivalence of events

\[ W_1(t) > d \iff A_1(t - d) > D_1(t) , \]

and also the service process representation from Eq. (5), we can bound the distribution of \( W_1(t) \) as follows

\[
\mathbb{P}(W_1(t) > d) \\
\mathbb{P}(A_1(t - d) > A_1 \ast S_1(t)) \\
= \mathbb{P}\left( \sup_{0 \leq s < t - d} \{A_1(s, t - d) - [C(t - s) - A_2(s, t - x)]_+ I_{\{t - s > x\}}\} > 0 \right) .
\]

(11)

Here we restricted the range of \( s \) from \([0, t]\) to \([0, t - d]\), using the positivity of the ‘\([\cdot]_+\)’ operator and the monotonicity of \( A_1(s, t) \).

Because \( x \) is a free parameter in the FIFO service process construction from Eq. (10), let us choose \( x = d \). With this choice it follows from above that

\[
\mathbb{P}(W_1(t) > d) \\
\leq \mathbb{P}\left( \sup_{0 \leq s < t - d} \{A_1(s, t - d) + A_2(s, t - d) - C(t - s)\} > 0 \right) .
\]

By applying Theorem [8] with \( u = 0 \) and \( \sigma = Cd \), we get the following

**Martingale Delay Bound (FIFO):**

\[
\mathbb{P}(W_1(t) > d) \leq K^n e^{-\gamma Cd} ,
\]

(12)

where \( K \) and \( \gamma \) are given in Theorem [5]. Note that the bound is invariant to the number of sub-flows \( n_1 \), which is a property characteristic to a virtual delay process (for FIFO); such a dependence will be established by changing the measure from a virtual delay process to a packet delay process (see Section IV-E).

2) SP: Here we consider an SP server giving higher priority to the data units of the cross flow \( A_2(t) \). We are further interested in the delay distribution of the lower priority flow; the case of the higher priority flow is a consequence of the previous FIFO result.

We follow the same procedure of first encoding \( A_1(t) \)’s service view in a service process, e.g., (see Fidler [25]),

\[
S_1(s, t) = C(t - s) - A_2(s, t) ,
\]

(13)

now in the case of SP scheduling.

To bound the distribution of \( W_1(t) \) we continue the first two lines of Eq. (11) as follows

\[
\mathbb{P}(W_1(t) > d) \\
\mathbb{P}\left( \sup_{0 \leq s < t - d} \{A_1(s, t - d) + A_2(s, t - d) - C(t - s)\} > 0 \right) .
\]

By applying Theorem [8] with \( u = d \) and \( \sigma = 0 \), we get the following

**Martingale Delay Bound (SP):**

\[
\mathbb{P}(W_1(t) > d) \leq K^n e^{-\gamma C_1 d} ,
\]

(14)

where \( K \) and \( \gamma \) are given in Theorem [5]. Note that, as expected, the SP delay bound recovers the FIFO delay bound from Eq. (12) when there is no cross aggregate, i.e., in the case when \( C_1 = C \).
3) EDF: An EDF server associates the relative deadlines $d_1^t$ and $d_2^t$ with the data units of $A_1(t)$ and $A_2(t)$, respectively. Furthermore, all data units are served in the order of their remaining deadlines, even when they are negative (we do not consider data unit losses).

A service process for $A_1(t)$ is for some $x > 0$
\[
S_1(s, t) = [C(t - s) - A_2(s, t - x + \min\{x, y\})]_+ I_{\{t - s > x\}},
\]  
where $y = d_1^t - d_2^t$ (see Liebeherr et al. [36]). This service process generalizes the FIFO one from Eq. (10) (which holds for $y = 0$, i.e., the associated deadlines to the flows are equal), and it also generalizes a previous EDF service process by Li et al. [34] (which is restricted to $x = 0$).

To derive a bound on $P (W_1(t) > d)$, for some $d \geq 0$, let us first choose $x := d$, as we did for FIFO. Next we distinguish two cases depending on the sign of $y$.

If $y \geq 0$ then the continuation of Eq. (11) is
\[
P (W_1(t) > d)
\leq P \left( \sup_{0 \leq s < t - d} \left\{ A_1(s, t - d) + A_2(s, t - d + \min\{d, y\}) - C(t - s) \right\} > 0 \right).\]

By changing the variable $t \leftarrow t + d - \min\{d, y\}$ we get
\[
P (W_1(t) > d)
\leq P \left( \sup_{0 \leq s < t - \min\{d, y\}} \left\{ A_1(s, t - \min\{d, y\}) + A_2(s, t) - C(t - s + d - \min\{d, y\}) \right\} > 0 \right).\]

(we point out that as we are looking for the steady-state distribution of $W_1(t)$, we can omit the technical details of writing $W_1(t + d - \min\{d, y\})$ above.) We can now apply Theorem 3 with $u = \min\{d, y\}$ (note that both $d$ and $y$ are positive) and $\sigma = C(d - \min\{d, y\})$, and get the following

Martingale Delay Bound (EDF) ($d_1^t \geq d_2^t$ Case):
\[
P \left( W_1(t) > d \right) \leq K^n e^{\gamma C_2 \min\{d_1^t - d_2^t, d\}} e^{-\gamma C d},
\]
where $K$ and $\gamma$ are given in Theorem 3.

The second case, i.e., $y < 0$, is slightly more complicated. The reason is that $\min\{d, y\} = y$ (see Eq. (15)) such that the continuation of Eq. (11) becomes
\[
P \left( \sup_{0 \leq s < t - d} \left\{ A_1(s, t - d) - [C(t - s) - A_2(s, t - d + y)]_+ I_{\{t - s > d\}} \right\} > 0 \right).\]

Note that when $s \in [t - d + y, t - d)$, then one must consider $A_2(s, t - d + y) := 0$ according to the conventions from [36]. Therefore, one must perform the splitting $[0, t - d) = [0, t - d + y) \cup [t - d + y, t - d)$; thereafter, by changing the variable $t \leftarrow t + d$, the continuation of Eq. (17) is
\[
P \left( \sup_{0 \leq s < t + y} \left\{ A_2(s, t + y) + A_1(s, t) - C(t - s) \right\}, \sup_{t + y \leq s < t} \left\{ A_1(s, t) - C(t - s) \right\} \right) > C d
\]
\[
\leq P \left( \sup_{0 \leq s < t + y} \left\{ A_2(s, t + y) + A_1(s, t) - C(t - s) \right\} > C d \right) + P \left( \sup_{0 \leq s < t} \left\{ A_1(s, t) - C(t - s) \right\} > C d \right)
\]

In the third line we applied the Union Bound [sic], which is conceivably tight because the two elements in the ‘max’ are rather uncorrelated. Moreover, we extended the left margin in the last supremum (in the fourth line), as we are looking for upper bounds, whereas the martingale argument from Theorem 3 is insensitive to where the left margin starts.

The last two probabilities can be directly evaluated with Theorem 4. For the first one we set $u = -y$ (note that $y$ is now negative) and $\sigma = C d$. For the second one we set $u = 0$, $n_2 = 0$, $\sigma = C d$, and we properly rescale the per-flow capacity $c$ and utilization factor $\rho$ (see below). In this way we get the following

Martingale Delay Bound (EDF) ($d_1^t < d_2^t$ Case):
\[
P \left( W_1(t) > d \right) \leq K^n e^{\gamma C_2 (d_1^t - d_2^t)} e^{-\gamma C d} + K^n e^{-\gamma C d},
\]
with the same $K$ and $\gamma$ from Theorem 3 whereas $K'$ and $\gamma'$ are obtained alike $K$ and $\gamma$, but after rescaling $c' \leftarrow \frac{n_1 + n_2}{n_1 n_2} c$ and $\rho' = \frac{n_1 + n_2}{n_1} \rho$.

Note that the first EDF bound from Eq. (16) recovers the FIFO bound when the associated deadlines are equal, i.e., when $d_1^t = d_2^t$. In turn, the second EDF bound from Eq. (16) would also recover the FIFO bound, but only by dispensing with the unnecessary splitting of the interval $[0, t - d)$ since $y = 0$. 


where $K$.

Next, by applying Theorem 3 with Martingale Delay Bound (GPS) and Standard bounds on the distribution of turn, by relying on the SP service process, finite delays can be obtained.

which corresponds to the minimum service guarantee by the GPS property from Eq. (19). Unfortunately, this service process does not capture the full server capacity allocated to the flow $A_1(t)$ when the other flow is not backlogged, and it thus conceivably leads to loose bounds.

Nevertheless, despite the pessimistic outlook of relying on the service process from Eq. (20), we will compute the Martingale bounds.

An improved service process was constructed by Li et al. [34], but it requires an additional concavity assumption on the flow $A_2(t)$; therefore, it does not apply in our setting. An improvement in the general case (i.e., for any types of arrivals) can be obtained when the SP service process from Eq. (13) is larger than the one from Eq. (20); note that the SP service process holds by default. The improvement can be substantial for small values of $\phi$, and it indeed, in the extreme case when $\phi = 0$, the GPS service process from Eq. (20) would predict infinite delays as the system would be (wrongly) perceived in overload; in turn, by relying on the SP service process, finite delays can be obtained.

To derive the Martingale delay bound, we continue the first two lines of Eq. (11) as follows

Next, by applying Theorem 3 with $C := \phi_1 C$, $u := d$, $\sigma := \phi_1 Cd$, and $n_2 := 0$, in this order, we obtain the following Martingale Delay Bound (GPS):

where $K$ and $\gamma$ are given in Theorem 3, note that the new utilization factor is $\rho := \frac{n_1 p^*}{\phi_1 C}$.

B. Standard Bounds

Here we briefly review the standard (per-flow) delay bounds obtained with SNC for FIFO, SP, EDF, and GPS scheduling algorithms. These bounds will be compared, both analytically and numerically, against the Martingale bounds computed so far.

We assume that for each MMOO sub-flow $A_0(t)$ the corresponding Markov process $Z(t)$ (with two states, from Figure 2) starts in the steady-state, i.e.,

where $p$ was defined earlier, i.e., $p = \frac{1}{1 + \mu}$. The computation of the Standard delay bounds relies on the moment generating function (MGF) of $A_0(t)$, which can be written for all $t \geq 0$ and some $\theta > 0$ as (see Courcoubetis and Weber [13])

where $w' = \frac{\lambda r + \mu (r_0 - P)}{(r_\theta - r_\phi)(\lambda + \mu)}$, $w = -\frac{\lambda r + \mu (P - r_0)}{(r_\theta - r_\phi)(\lambda + \mu)}$, $r_\phi' = \frac{b - \sqrt{b^2 + 2\Delta}}{2\lambda + 2\mu}$, $r_\phi = \frac{b - \sqrt{b^2 + 2\Delta}}{2\lambda + 2\mu}$, $b = \lambda + \mu - \theta P$, and $\Delta = b^2 + 4\mu \theta P$. Since $w' + w = 1$ and $r_\theta \leq r_\phi$, it follows that

which is the typical approximation of the MGF in the SNC literature by a single (dominant) exponential. The rate $r_\theta$ corresponds to the effective bandwidth; it is non-decreasing in $\theta$ and satisfies

i.e., $r_\theta$ is between the average and the peak rate of a single MMOO process.

The next result gives a common sample-path bound which will be used to compute the Standard delay bounds for all three scheduling algorithms; the result parallels the one from Theorem 3 which was used for the Martingale bounds.
where the Martingale ones from Eqs. (12), (14), and (21), except that the sample-path bound from Theorem 3 is replaced by the one

\[ P \left( \sup_{0 \leq s < t - u} \{ A_1(s, t - u) + A_2(s, t) - C(t - s) \} > \sigma \right) \]

where \( L = \frac{e^c}{e^{\theta C}} \) and \( r_0 \) was defined prior to Eq. (22).

The proof proceeds by first discretizing the sample-path event and then by using standard arguments in SNC and effective bandwidth theory based on the Union and Chernoff bounds (i.e., the second and third bounding steps discussed in Section II). Similar proofs have been given for various sample-path events (see, e.g., [14]).

Proof: Fix \( 0 \leq u \leq t, \sigma, \) and \( \theta > 0 \) such that \( c > r_0 \). Consider the free parameter \( r_0 > 0 \) for discretizing the event from Eq. (22) at the points \( j = \lfloor \frac{t - u}{r_0} \rfloor \) for all \( 0 \leq s \leq t \). Using the monotonicity of the arrival processes (e.g., \( A_1(s, t - u) \leq A_1(t - u - (j + 1) r_0, t - u) \)), we can bound the probability event from Eq. (23) by

\[ P \left( \bigcup_{j \geq 0} \{ A_1(t - u - (j + 1) r_0, t - u) + A_2(t - u - (j + 1) r_0, t) - C(t - (t - u - j r_0)) > \sigma \} \right) \]

The derivations relied first on the Union Bound, then on the Chernoff bound applied to the MGF bound from Eq. (22), and finally on the inequality \( \sum_{j \geq 1} e^{-j a} \leq \frac{1}{a} \) for some \( a > 0 \). Since \( r_0 \) is a free parameter, the last bound can be optimized with \( r_0 = \frac{1}{\theta} \). Finally, taking the minimum over \( \theta \) completes the proof.

Next we list the standard bounds on \( A_1(t) \)'s virtual delay for FIFO, SP, EDF, and GPS scheduling. They are obtained alike the Martingale ones from Eqs. (12), (14), and (21), except that the sample-path bound from Theorem 3 is replaced by the one from Theorem 2

\[ \text{FIFO} : \ldots \leq \inf_{\{ \theta ; c > r_0 \}} L e^{-\theta C d} \quad (24) \]

\[ \text{SP} : \ldots \leq \inf_{\{ \theta ; c > r_0 \}} L e^{-\theta (C - n r_0) d} \quad (25) \]

\[ \text{EDF}^1 : \ldots \leq \inf_{\{ \theta ; c > r_0 \}} L e^{\theta n r_0 \min(d_1^x - d_2^x, d)} e^{-\theta C d} \quad (26) \]

\[ \text{EDF}^2 : \ldots \leq \inf_{\{ \theta ; c > r_0 \}} L e^{\theta (C - n r_0)(d_1^x - d_2^x)} e^{-\theta C d} + \inf_{\{ \theta ; c > r_0 \}} L' e^{-\theta C d} \quad (27) \]

\[ \text{GPS} : \ldots \leq \inf_{\{ \theta ; c > r_0 \}} L e^{-\theta c r_1 C d} \quad (28) \]

where \( L \) is given in Theorem 2 for FIFO and SP, and \( L = \frac{e^c}{e^{\theta C}} \) for GPS. We mention that \( \text{EDF}^1 \) corresponds to the case \( d_1^x > d_2^x \) (see the Martingale bound from Eq. (16)), and \( \text{EDF}^2 \) to the complementary case (see Eq. (18)). Moreover, for \( \text{EDF}^2 \), \( c' \) is the rescaled value of \( c \) (as in the Martingale bound from Eq. (18)), whereas \( L' \) is defined like \( L \) but with \( c \) replaced by \( c' \). We also point out that the GPS bound is roughly the same as the one computed by Zhang et al. [55], for more general arrival classes and without using the service process concept (which was still to be properly formalized by Cruz [20] a year after).

C. Asymptotic Decay Rates Comparison

Here we compare the asymptotic decay rates (denoted by \( \eta \)) of the Martingale and Standard bounds on \( P(W_1(t) > d) \), which can be isolated by taking the limit

\[ \eta := - \lim_{d \to \infty} \frac{\log P(W_1(t) > d)}{d} \quad . \]

Table I lists the asymptotic decay rates for FIFO, SP, EDF, and GPS scheduling. For the Standard bounds, the \( \theta^* \)'s are the solutions to the optimization problems from Eqs. (23), (24), respectively. Next we discuss the bounds by grouping them into two groups.
Delay Bounds / Scheduling | Martingale | Standard
---|---|---
FIFO, EDF | $\gamma C$ | $\theta^* C$
SP | $\gamma C_1$ | $\theta^* (C - n_2 r_{\theta^*})$
GPS | $\gamma \phi_1 C$ | $\theta^* \phi_1 C$

**TABLE I**

Asymptotic decay rates for $P(W_1(t) > d)$; for FIFO, EDF, and GPS: $\gamma = \theta^*$

1) **FIFO, EDF, and GPS:** We only focus on FIFO; the other two are analogous.

For the Standard FIFO bound, $\theta^*$ is the unique solution of the equation

$$r_{\theta} = c,$$

(29)

where $r_{\theta}$ is the effective bandwidth from Eq. (22). This is a standard result in the effective bandwidth literature (see, e.g., Glynn and Whitt [26] or Chang [10], p. 291).

Next we show that $\theta^* = \gamma$, i.e., the asymptotic decay rates are equal. To this end, we first recall from [18] that $r_{\theta}$ is derived by solving the eigenvalue problem

$$Q_{\theta} x = \zeta_{\theta} I x$$

(30)

where $x$ stands for the eigenvector and

$$Q_{\theta} := \left( \begin{array}{cc} -\mu & \mu \\ \lambda & -\lambda + P\theta \end{array} \right), \quad I := \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),$$

for some $\theta > 0$. If $\zeta_{\theta}$ denotes the spectral radius of $Q_{\theta}$ (corresponding to $\omega_2$ in [18], Section 3), then the effective bandwidth is defined as $r_{\theta} = \frac{\mu}{\zeta_{\theta}}$ (see Eq. (22)).

In turn, $\gamma$ from the Martingale bound satisfies the generalized eigenvalue problem (see [44])

$$Q_0 y = \gamma D y,$$

(31)

where $y$ stands for the eigenvector and

$$D := \left( \begin{array}{cc} c & 0 \\ 0 & c - p \end{array} \right).$$

Using the positivity of $\gamma$ (see Theorem 3 and applying the Separation Lemma from Sonneveld [51], it follows that $\gamma$ is the spectral radius. Furthermore, by relating the eigenvalue problems from Eqs. (30) and (31) through the equality

$$Q_{\gamma} - c\gamma I = Q_0 - \gamma D,$$

it follows that $c\gamma = \zeta_{\gamma}$. Finally, since $\theta^*$ is the unique solution of Eq. (29), whereas $r_{\theta^*} = \frac{\mu}{\zeta_{\theta^*}}$ by definition, it follows that $\theta^* = \gamma$. This result is particularly important as it illustrates an explicit solution for the effective bandwidth equation from Eq. (29).

2) **SP:** Unlike in the FIFO case, there is no immediate explicit solution for the optimal value $\theta^*$. Because $c = r_{\gamma}$, as shown above, we can observe that

$$\gamma C_1 = \gamma \left( C - n_2 r_{\theta^*} \right),$$

which indicates a similarity in the two asymptotic decay rates. Unfortunately, due the form of the optimization problem from Eq. (25), we will solely resort to numerical comparisons between the Martingale and Standard bounds.

In conclusion, in large buffer (or, here, delay) asymptotic regimes, FIFO, EDF, and GPS exhibit the same tail behavior for both the Standard and Martingale bounds; according to numerical results, the same holds for SP (see Section IV-E). Next we consider many-sources asymptotics, which are also commonly discussed in the literature (e.g., Mazumdar [41]).

D. Many-Sources Asymptotics Comparison

We consider the following scaling scheme: the total number of flows $n$ is scaled up, whereas the rest of the parameters, i.e., the utilization factor $\rho$, the per-flow rate $r_0 = pP$, and the per-flow capacity $c$ remain unchanged.

Let us first observe that the factors $K$ (defined in Theorem 3) and $L$ (defined in Theorem 2) from the two sets of bounds satisfy

$$K < 1 \text{ and } L > 1.$$
The second property is immediate. In turn, for the factor $K$, note that $0 < \frac{0 - \rho}{\rho} < 1$ and the functions $f(x) := \frac{0 - x}{\rho}$ and $g(x) := \frac{x - 0}{\rho}$ are non-increasing on $x \in (0, \rho)$. Thus, by the composition of the power function (defined on the interval $(0,1)$) and the exponential function, the function $h(x) := \left(\frac{0 - x}{\rho}\right)^{\frac{x}{\rho}}$ is non-increasing. Moreover, $\lim_{x \to 0} h(x) = \rho^{-1}$, and thus $K < 1$.

Table II illustrates the scaling laws of the Martingale and Standard delay bounds for the four scheduling algorithms. The factors $\alpha > 0$ and $\eta > 0$ are invariant to $n$ and can be fitted for each individual case; e.g., in the case of FIFO, $\alpha = -\log K$ and $\eta = \gamma c$. We remark that all pairs of bounds have the same asymptotic decay rate $\eta$. The critical observation is that, unlike the Standards bounds, the Martingale bounds have an additional factor $e^{-\alpha n}$ decaying exponentially with $n$. This scaling behavior was indicated by Choudhury et al. [12] by numerical evaluations. We point out that [12] further indicated an additional factor $\beta > 0$, invariant to $n$, which is however not captured by the Martingale bounds.

### E. Numerical Comparisons

In this section we compare the Martingale and Standard bounds against simulations and also in asymptotic regimes. The parameters of a single MMOO source are $\lambda = 0.5$, $\mu = 0.1$, and $P = 1$ (the average ‘Off’ period is five fold the average ‘On’ period).

1) Bounds vs. Simulations: We consider two utilization levels ($\rho = 0.75$ and $\rho = 0.9$), and two degrees of multiplexing ($n_1 = n_2 = 5$ and $n_1 = n_2 = 10$). The packet sizes in a packet-level simulator are set to 1; fractional packet sizes are additionally set when the dwell times in the states of the Markov process from Figure 3 are not integers. The simulator measures the delays of the through flow’s first $10^7$ packets, and it discards the first $10^6$. For numerical confidence, 100 independent simulations are being run and the results are presented as box-plots.

For the soundness of the comparisons against simulations, it is important to remark that the delay analysis so far concerned the virtual delay process $\hat{W}_1(t)$, which corresponds to the delay of a through flow’s infinitesimal unit, should it depart, or equivalently arrive, at time $t$; more concretely, we note that the bounds computed with SNC on virtual delays are identical, should they concern a virtual arrival or departure unit. In the packet level simulator, however, it is the packet delay process which is being measured, and which is denoted here by $\hat{W}_1(n)$ (the index ‘$n$’ corresponds to the packet number for the through flow). Therefore, one has to properly perform a suitable change of probability measures in order to provide meaningful numerical comparisons.

We follow a Palm calculus argument and relate the measure of the virtual delay process to that of the packet delay process (see Shakkottai and Srikant [48]). For convenience, we work in reversed time and focus on time $0$ where steady-state is assumed to be reached. Denoting $\hat{W}_1 := \hat{W}_1(0)$, we can write by conditioning

\[
\begin{align*}
P (W_1 > d) = & \ P (W_1 > d \mid a_1(0) > 0) \ P (a_1(0) > 0) + P (W_1 > d \mid a_1(0) = 0) \ P (a_1(0) = 0) \\
\geq & \ P (W_1 > d \mid a_1(0) > 0) \ P (a_1(0) > 0) \\
= & \ P (\hat{W}_1 > d) \ P (a_1(0) > 0) ,
\end{align*}
\]

(32)

where $a_1(0)$ denotes the instantaneous arrivals of the through flow at time $0$, and $\hat{W}_1$ denotes the steady-state packet delay process of the through flow. Note that for the inequality we eliminated the second term in the sum above.

Therefore,

\[
P (\hat{W}_1 > d) \leq \frac{1}{1 - (1 - p)^n} P (W_1 > d) .
\]

(33)

(Recall that $p$ is the steady-state ‘On’ probability of the MMOO process from Figure 2.)

Below we compare the distribution of $\hat{W}_1$ against the one of the measured (simulated) delay process. All the Martingale and Standard bounds which we compute for FIFO, SP, EDF, and GPS scheduling are scaled up by the additional prefactor from Eq. (33) needed for the change of measure. We note that this scaling is conservative because of the inequality from Eq. (32).

Figure 4 illustrates the comparisons for FIFO scheduling (recall Eqs. (12) and (24) for the Martingale and Standard delay bounds, which are scaled as in Eq. (33)); the $y$-axis uses a log scale. The irregular tail behavior (including the presence
of many outliers (at $\rho = 90\%$) of the box-plots is due to the restriction of the simulation runs to $10^7$ packets. All four scenarios, corresponding to various utilizations and multiplexing, clearly indicate that the Standard bounds are very loose, as they overestimate the simulation results by a factor of roughly $10^3$ at $75\%$ utilization (see (a) and (c)), and even $10^3$ at $90\%$ utilization (see (b) and (d)). In turn, the Martingale bounds are reasonably accurate. We suspect that the slight loss of accuracy for $n_1 = n_2 = 5$ (in (a) and (b)) stems from the conservative change of measure from Eq. (32); indeed, at $n_1 = n_2 = 10$ (in (c) and (d)) one can notice an increase in accuracy due to the lesser role played by the change of measure prefactor from Eq. (33).

The same observations hold for SP scheduling, as indicated by Figure 5; recall the Martingale and Standard delay bounds from Eq. (14) and (25), respectively, which are again scaled as in Eq. (33). Moreover, note that the SP delays increase roughly by a factor of 2 relative to the FIFO delays, due to the setting $n_1 = n_2$; the same factor predominates at higher multiplexing regimes as well. While this is an indication that scheduling can matter, we refer to Section IV-E3 for a specific asymptotic scenario (i.e., with many flows) in which scheduling does not matter.

The tightness of the Martingale bounds, in contrast to the looseness of Standard bounds, further holds in the case of EDF scheduling, for both cases (i.e., $d_1^* > d_2^*$ and $d_1^* < d_2^*$), as illustrated in Figure 6; recall the Martingale and Standard delay bounds from Eqs. (16)-(18) and Eqs. (26)-(27), respectively. Note that the bending of the curves, e.g., in (a), is due to the choice of $d_1^*$ and $d_2^*$; the bounds behave like the SP bounds for $d \leq d_1^* - d_2^*$, and asymptotically like the FIFO bounds thereafter.

The previous numerical illustrations conceivably indicate the tightness of the ‘proprietary’ SNC bounding step from Eq. (1), in the case of FIFO, SP, and EDF scheduling. More concretely, the state-of-the-art service processes for these scheduling algorithms (see Eqs. (10), (13), and (15)) appear to be tight, as long as they are used in conjunction with the ‘right’ tools from

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2Outliers are depicted in the box-plots with the ‘+’ symbol; on each box, the central mark is the median, and the edges of the box are the 25th and 75th percentiles.

3The long stretch of the box-plots and the presence of many outliers is caused by the choice of $10^7$ arrivals, in order to illustrate the need for very long simulation runs (e.g., $10^8$ arrivals, in which case the box-plots would significantly shrink and most outliers would disappear).
probability theory. The same illustrations also confirm the pitfall identified in Section II concerning the potential looseness of the Union Bound in the case of bursty processes.

The positive side of our exposition so far (i.e., the Martingale bounds fix the very loose Standard bounds) is disturbed, however, in the case of GPS scheduling (with $\phi_1 = \phi_2 = 0.5$). Indeed, let us refer to Figure 7 showing the corresponding numerical comparisons (recall the Martingale and Standard bounds from Eqs. (21) and (28)); we mention that the packet-level simulator implements WFQ scheduling, i.e., the packetized version of GPS (see Keshav [31], pp. 238-242). Unlike in the FIFO, SP, and EDF cases, the Martingale bounds are now very loose. This indicates that, in addition to the Union Bound bounding step, the ‘proprietary’ SNC bounding step from Eq. (1) can also be very loose. In other words, the service process for GPS scheduling from Eq. (20) needs to be significantly improved.

2) Asymptotic comparisons: Here we illustrate the bounds in a many sources asymptotic regime: we fix the target delay bound $d = 5$ and plot the per-flow violation probabilities $P(\tilde{W}_1 > d)$ by increasing the total number of flows $n$. For visualization purposes we restrict to FIFO, SP, and one case of EDF. The important observation from Figure 8 is that, for all settings (e.g., FIFO, $\rho = 75\%$), the Standard and Martingale bounds diverge. In other words the discrepancy between the Standard and Martingale bounds grows arbitrarily large (e.g., by a factor of $10^9$ at $n = 10^5$). This observation supports the scaling laws from Table II. In the light of this rather pessimistic evidence concerning the Standard bounds, the immediate question is whether they can still have any practical relevance; next we clarify this concern.

3) The Standard bounds can (sometimes) be useful: Here we show that, despite the poor scaling in multiplexing regimes, the Standard bounds can be practically relevant. To this end, we consider the following connection admission control problem: given the server capacity $C$, the target (per-flow) delay bound $d$, and a violation probability $\varepsilon$, determine the maximum number of flows $n$ to meet all the three constraints.

Figure 9 shows the achievable utilization (i.e., $\rho = \frac{npP}{C}$) as a function of the capacity $C$ for FIFO and SP, two delay targets...
\( P(\tilde{W}_1 > d) \)

\( P(\tilde{W}_1 > d) \)

\( P(\tilde{W}_1 > d) \)

\( P(\tilde{W}_1 > d) \)

Fig. 6. EDF delay bounds \((n_1 = n_2 = 10, \rho = 75\% \text{ in (a) and (c), and } \rho = 90\% \text{ in (b) and (d)})\)

\( d = 1 \) and \( d = 10 \), and two violation probabilities \( \varepsilon = 10^{-3} \) and \( 10^{-9} \). The important observation is that once \( d \) and \( \varepsilon \) are both fixed, then by increasing the capacity \( C \) (i.e., 'making room for sufficient statistical multiplexing to kick in') the Martingale and Standard bounds converge to one, albeit at different rates.

The convergence for both FIFO and SP is a manifestation of the assertion from the literature that "scheduling has only a limited impact" (e.g., on admission control) (see, e.g., Li et al. [34]); such an assertion, however, does not generally hold (see, e.g., Figures 4, 5, and 8).

We conclude that the Standard bounds can (sometimes) be nearly optimal and thus be practically relevant. This remark is typically presented in the literature in the form: the admissible region based on the Standard bounds converges to the admissible region based on averages (see, e.g., Boorstyn et al. [4] or Li et al. [34]). Variations of the underlying scaling regime, enabling such conclusions, are often adopted in the literature in order to expose the Standard bounds in a favorable light.

V. Conclusions

In this paper, we have put our finger in a wound of the stochastic network calculus: the lingering issue of the tightness of the SNC bounds. To some degree, this issue had been evaded by the SNC literature for some time although it is a, if not the, crucial one. In fact, we demonstrated that the typical (Standard SNC) way of calculating performance bounds results in loose delay bounds for several scheduling disciplines (FIFO, SP, EDF, and GPS) as well as for various multiplexing regimes. This becomes particularly obvious when comparing the (Standard SNC) analytical results to simulation results, where discrepancies up to many orders of magnitude can be observed. So, we strongly confirm the often rumored conjecture about SNC’s looseness.

Yet, the paper does not stop at these bad news, but in an attempt to understand the problems of Standard SNC, which mainly lie in not properly accounting for the correlation structure of the arrival processes (by coarse usage of the Union bound), we find a new way to calculate performance bounds using the SNC framework based on martingale techniques. Here, SNC still serves as the "master method", yet the Union bound is substituted by the usage of martingale inequalities, to make a long
story short. Comparing the new Martingale SNC bounds to the simulation results shows that they are remarkably close in most cases, which rehabilitates the SNC as a general framework for performance analysis. So, the SNC can arguably still be regarded as a valuable methodology with the caveat that it has to be used with the right probabilistic techniques in order not to arrive at practically irrelevant results. As usual, there is a “but” to such a general statement: this is the issue about GPS for which even the Martingale bounds stay loose (though again improving by orders of magnitude over the Standard bounds). This may be indeed a case where SNC as master method fails, because it might be wrong to separate the service process derivation from the arrivals of the flow under investigation. On the other hand, this is very speculative and it may still be possible to find a service process that tightly captures the GPS characteristics. We leave this interesting open issue for future work.

Other related challenges include extensions to general Markov arrival processes (e.g., by using the general result from [44]), self-similar arrival processes, and the multi-node case (e.g., by accounting for the martingale representation from [9]).

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Fig. 9. Achievable utilization as a function of the capacity $C$ ($n_1 = n_2$)

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A. Extension to General Markov Fluid Processes

Here we present the generalization of Theorem 3 to the case when the arrival processes \( A_1(t) \) and \( A_2(t) \) are general Markov fluid processes.

Consider the queueing model from Figure 1 in which \( A_1(t) \) and \( A_2(t) \) are two cumulative Markov fluid processes served at constant-rate \( C \). Each process \( A_k(t) \) is modulated by a reversible Markov process \( Z_k(t) \) with \( n_k + 1 \) states, generator \( Q_k = (q_{k,i,j})_{i,j=0,\ldots,n_k} \), equilibrium distribution \( \pi_k = (\pi_{k,0}, \ldots, \pi_{k,n_k}) \), arrival rates \( r_k = (r_{k,0}, \ldots, r_{k,n_k}) \), and increment process \( a_k(s) = r_k Z_k(s) \) \( \forall s \geq 0 \). We remark that if \( Z_k(t) \) were not reversible, then one could consider as input the corresponding reversed processes for the sake of expressing Reich’s equation as

\[
Q = \sup_{t \geq 0} \{ A_1(t) + A_2(t) - Ct \} .
\]

For each arrival process \( A_k(t) \) we consider the generalized eigenvalue problem

\[
Q_k \mathbf{h}_k = -\gamma_k \mathbf{u}_k \mathbf{h}_k \quad k = 1, 2 ,
\]

(34)

where \( Q_k \) are the generators, \( \mathbf{u}_k \) are diagonal matrices with \( (u_{k,0}, u_{k,1}, \ldots, u_{k,n_k}) \) on the diagonal, and where

\[
u_{k,j} = r_{k,j} - C_k
\]

are the instantaneous queueing drifts for \( j = 0, 1, \ldots, n_k \). Here, \( C_1 \) and \( C_2 \) are positive values such that \( C_1 + C_2 = C \), and can be regarded as the per-class allocated capacity.

Assuming the per-class stability conditions

\[
\sum_{j=0}^{n_k} \pi_{k,j} u_{k,j} < 0, \quad k = 1, 2 ,
\]

(35)

Lemma 5.1 from [44] guarantees the existence of real generalized eigenvalues \(-\gamma_k\) (as the ones with the biggest negative real parts) and also of the generalized eigenvectors \( \mathbf{h}_k = (h_{k,0}, h_{k,1}, \ldots, h_{k,n_k})^\top \) with positive coordinates. Thus, \( \gamma_k > 0 \) for \( k = 1, 2 \).

Theorem 3: (A General Sample-Path Bound) Consider the single-node queueing scenario from Figure 1 and the solutions for the generalized eigenvalue problems from Eq. (34). Then the following sample-path bound holds for all \( 0 \leq u \leq t \) and \( \sigma \)

\[
P \left( \sup_{0 \leq s \leq t - u} \{ A_1(s, t - u) + A_2(s, t) - C(t - s) \} > \sigma \right) \leq \inf_{0 \leq \gamma \leq \min \gamma_k} \inf_{C_1 + C_2 = C} K e^{-\gamma(C_1 u + \sigma)} ,
\]

(36)

where \( K = \sum_{i,j} \pi_{i,j} \pi_{j,k} h_{i,j} h_{j,k} \) whereas the condition \( C_1 + C_2 = C \) is subject to the stability conditions from Eq. (35).

Let us make several observations the two infimum operators. The parameter \( \gamma \) in the former infimum reconciles the different burstiness of the two non-necessarily homogenous flows \( A_1(t) \) and \( A_2(t) \), loosely expressed through the exponential decay factor. The extreme optimal value \( \gamma = \min \{ \gamma_1, \gamma_2 \} \) is attained for \( \sigma \to \infty \); in turn, an optimization after \( \gamma \) is necessary in finite regimes of \( \sigma \). In turn, due to the implicit expression of \( K \) in terms of the (generalized) eigenvectors from Eq. (34), which depend on \( C_1 \) and \( C_2 \), the optimal values for the former infimum are not apparent and hence numerical optimizations must be invoked.

The theorem generalizes Theorem 3 to the case of general and not-necessarily homogenous Markov fluid processes (recall that Theorem 3 is restricted to multiplexed homogenous Markov-modulated On-Off processes). The theorem also generalizes the seminal result of Palmowski and Rolski [44] (see Proposition 5.1 therein), restricted to \( A_2(t) = 0 \); more details on the extent of our generalization will be provided after the proof.

Proof: Fix \( u \geq 0 \) and \( \sigma \). Since the two arrival processes are reversible, we can rewrite the probability from Eq. (36), by shifting the time origin, as

\[
P \left( \sup_{t > u} \{ A_1(u, t) + A_2(t) - Ct \} > \sigma \right)
\]

\[
= P \left( \sup_{t > u} \{ A_1(u, t) + A_2(u, t) - C(t - u) \} + A_2(u) - C_2 u > C_1 u + \sigma \right) ,
\]

(37)

Given the last probability event we construct the stopping time

\[
T := \inf \left\{ t > u : A_1(u, t) + A_2(u, t) - C(t - u) + A_2(u) - C_2 u > C_1 u + \sigma \right\} .
\]

(38)
For the rest of the proof we bound \( P(T < \infty) \), which exactly characterizes the probability from Eq. (37).

Let \( P_{i,j} \) denote the underlying probability measure conditioned on \( Z_1(u) = i \) and \( Z_2(0) = j \), for \( 0 \leq i \leq n_1 \) and \( 0 \leq j \leq n_2 \). Next we define the following two processes

\[
\begin{align*}
\tilde{M}_{1,t} &:= \frac{h_{1,Z_1(t)}}{h_{1,i}} e^{\int_0^t \psi(x,Z_1(s)) ds} \quad \forall t \geq u \quad \text{and} \\
\tilde{M}_{2,t} &:= \frac{h_{2,Z_2(t)}}{h_{2,j}} e^{\int_0^t \psi(x,Z_2(s)) ds} \quad \forall t \geq 0 .
\end{align*}
\]

\( M_1(t) \) and \( M_2(t) \) are supermartingales with respect to \( P_{i,j} \) and the natural filtration (see Ethier and Kurtz [24], p. 175). Considering the solution of the generalized eigenvalue problem from Eq. (34), we can rewrite

\[
\begin{align*}
\tilde{M}_{1,t} &= \frac{h_{1,Z_1(t)}}{h_{1,i}} e^{\gamma_1 \int_0^t \psi(u,Z_1(s)) ds} \quad \forall t \geq u \quad \text{and} \\
\tilde{M}_{2,t} &= \frac{h_{2,Z_2(t)}}{h_{2,j}} e^{\gamma_2 \int_0^t \psi(u,Z_2(s)) ds} \quad \forall t \geq 0 .
\end{align*}
\]

For \( 0 \leq \gamma \leq \min \{ \gamma_1, \gamma_2 \} \) we consider the transformations

\[
M_{k,t} = \tilde{M}_{k,t}^{\gamma_k} \quad k = 1, 2 .
\]

Denoting by \( F_{k,s} \) the natural filtrations of \( M_{k,t} \) we can write for \( 0 \leq s \leq t \)

\[
E[M_{k,t} \mid F_{k,s}] = E\left[\tilde{M}_{k,t}^{\gamma_k} \mid F_{k,s}\right] \leq E\left[\tilde{M}_{k,t} \mid F_{k,s}\right]^{\gamma_k} = M_{k,s} ,
\]

where the first inequality is due to Jensen’s inequality (applied to the concave function \( \psi(x) = x^{\gamma_k} \) for \( x \geq 0 \)) and the second inequality is due to the supermartingale property of \( \tilde{M}_{k,t} \). Therefore, the new processes \( M_{k,t} \) are also supermartingales; we point out that there construction is motivated by the need of having the same decay, i.e., \( \gamma \), in the corresponding exponentials.

Next we invoke Lemmas 3 and 2 from the Appendix and obtain that the process

\[
M_t := \begin{cases} M_{2,t} & , \quad t \leq u \\ M_{1,t} M_{2,t} & , \quad t > u \end{cases}
\]

is also a supermartingale (note that \( M_{1,u} = 1 \) by definition). It can be explicitly written as

\[
M_t = \begin{cases} \left( \frac{h_{2,Z_2(t)}}{h_{2,j}} \right)^{\gamma_2} e^{\gamma(A_2(t) - C_2 t)} , & t \leq u \\ \left( \frac{h_{1,Z_1(t)}}{h_{1,i}} \right)^{\gamma_1} \left( \frac{h_{2,Z_2(t)}}{h_{2,j}} \right)^{\gamma_2} e^{\gamma(A_1(u,t) + A_2(t) - C t)} , & t > u \end{cases}
\]

Referring now to the stopping time \( T \), which may be unbounded, we construct the bounded stopping times \( T \wedge v \) for all \( v \in \mathbb{N} \). For these times, the Optional Sampling theorem (see Theorem 4 in the Appendix) yields

\[
E_{i,j} [M_0] = E_{i,j} [M_{T \wedge v}] ,
\]

for all \( v \in \mathbb{N} \), where the expectations are taken wrt the underlying probability measure \( P_{i,j} \). Moreover, from the definition of \( T \) as an infimum over a set, it holds for \( v \geq 0 \) that

\[
(u_{1,Z_1(T)} + u_{2,Z_2(T)}) I_{T \leq v} \geq 0 ,
\]

where \( I_{\{ \cdot \}} \) denotes the indicator function. Using now that \( E_{i,j} [M_0] = 1 \) we obtain for \( v > u \)

\[
1 \geq E_{i,j} [M_{T \wedge v} I_{T \leq v}] \geq \min \left( \frac{h_{1,t_1}}{h_{1,i}} \right)^{\gamma_1} \left( \frac{h_{2,t_2}}{h_{2,j}} \right)^{\gamma_2} e^{(C_1 u + \sigma) \gamma} P_{i,j} (T \leq v) ,
\]

where the \( \text{min} \) operator is taken over the set \( \{ (l_1, l_2) : u_{1,l_1} + u_{2,l_2} \geq 0 \} \) according to Eq. (39).

Finally, by deconditioning on \( i \) and \( j \) (recall that \( Z_1(u) \) and \( Z_2(0) \) are in steady-state by construction) we obtain

\[
P(T \leq v) \leq K e^{-\gamma(C_1 u + \sigma)} .
\]

Letting \( v \to \infty \) and optimizing after \( \gamma \) completes the proof. \( \square \)
1) Relationship to State-of-the-Art Bound: Here we show that the bound from Theorem 5 improves the state-of-the-art result from [44]. In the particular case of an arrival flow consisting of \( n \) multiplexed Markov-modulated On-Off processes, the improvement is of the order \( O(K^n) \) for some constant \( 0 < K < 1 \).

The state-of-the-art bound from [44] concerns the distribution of the (stationary) queue size occupancy for a single Markov fluid process served at rate \( C \). To fit this scenario in Theorem 5 we let \( A_2(t) = 0 \) and \( u = 0 \), and for convenience we drop the index in the parameters of the remaining process \( A_1(t) \). Our bound states that

\[
P(Q > \sigma) \leq \sum_{i=1}^{\min_{h_i \geq 0} h_i} \pi_i e^{-\gamma \sigma}
\]

where \( \gamma \) and \( h = (h_0, h_1, \ldots, h_n)^T \) are the solution of the generalized eigenvalue problem from Eq. (44). In turn, the bound from [44] states that

\[
P(Q > \sigma) \leq \sum_{i=1}^{\min_{h_i \geq 0} h_i} \pi_i h_i e^{-\gamma \sigma}.
\]

The bound from Eq. (40) is clearly tighter than the bound from Eq. (41); see the additional constraint on the ‘min’ operator in Eq. (40). Next we give the order of the improvement when the process \( A(t) \) is a superposition of \( n \) Markov-modulated On-Off processes. Each sub-process is modulated by a Markov process \( Z(t) \) with two states, denoted by ‘On’ and ‘Off’, and which communicate at rates \( \lambda \) and \( \mu \). While in the ‘On’ state, each sub-process generates data units at a constant rate \( P \). In this case, according to Eq. (40), the queue size distribution is bounded by

\[
P(Q > \sigma) \leq K^n e^{-\gamma \sigma},
\]

for some \( 0 < K < 1 \); see Theorem 1 in [15], which recovers Theorem 2.1 from [44]. In turn, the pre-factor from Eq. (41) satisfies \( \sum_{i=1}^{\min_{h_i \geq 0} h_i} \pi_i h_i \geq 1 \), whence the \( O(K^n) \) improvement of the bound from Eq. (40) over the one from Eq. (41).

2) Relationship to Effective Bandwidth: Here we establish a fundamental relationship between the decay rate \( \gamma \) and the effective bandwidth.

We consider a single arrival process \( A_1(t) \) for which we drop the index, i.e., \( A(t) \). The effective bandwidth of \( A(t) \) is defined for \( \theta > 0 \) as

\[
\alpha(\theta, t) := \frac{1}{t \theta} \log E\left[ e^{\theta A(t)}\right],
\]

and let \( \alpha_\theta := \lim_{t \to \infty} \alpha(\theta, t) \) (see Kelly [30]); with abuse of notation \( \alpha_\theta \) will be called the effective bandwidth of \( A(t) \).

Lemma 1: (\( \gamma \) vs. \( \alpha_\theta \)) Let the scenario from Theorem 4 with a single arrival process \( A(t) \) having effective bandwidth \( \alpha(\theta, t) \). If \( \alpha_\theta \) is differentiable then

\[
\alpha_\gamma = C.
\]

Proof: From the construction of \( \gamma \) from the generalized eigenvalue problem from Eq. (34), for which we drop the indexes, we have that

\[
(Q + \gamma u) h = 0.
\]

Let the diagonal matrix \( V \) with \( (r_0 \gamma, r_1 \gamma, \ldots, r_n \gamma) \) on the diagonal, and construct the matrix

\[
Q_\gamma := Q + V.
\]

Then it holds that

\[
Q_\gamma x = \alpha_\gamma I x
\]

where \( \alpha, \gamma \) is the spectral radius of \( Q_\gamma \) and \( x \) is the corresponding (positive) eigenvector (see Kesidis et al., Sec. 3, [32]).

Let us now observe that

\[
Q + \gamma u = Q_\delta - C \gamma I.
\]

Combining with Eqs. (33) and (44) we obtain that

\[
0 = (Q_\gamma - \alpha_\gamma I) x = (Q_\gamma - C \gamma I) h.
\]

Therefore, \( C \gamma \) is an eigenvalue for the eigenvalue problem from Eq. (44) and thus

\[
\alpha_\gamma \geq C,
\]

since by construction \( \alpha_\gamma \) is the corresponding spectral radius.

To show the converse, i.e., \( \alpha_\gamma \leq C \), consider the exact asymptotic decay of the distribution of the queue occupancy (of \( A(t) \) when fed at a queue with capacity \( C \)), i.e.,

\[
\lim_{\sigma \to \infty} \frac{1}{\sigma} P(Q > \sigma) = -\theta^*,
\]
where \( \alpha \theta^* = C \) (see Kelly, Eq. (3.21), [30]). In other words, \( \theta^* \) is the exact asymptotic decay rate. As Theorem 3 predicts \( \gamma \) as a decay rate, in terms of an upper bound, it follows that \( \gamma \leq \theta^* \). Finally, since \( \alpha \theta \) is increasing in \( \theta \) (see Chang, p. 241, [10]), it follows that \( \alpha \gamma \leq \alpha \theta^* = C \), completing thus the proof that \( \alpha \gamma = C \).

\( \square \)

B. Review of Martingale Results

Here we summarize some definitions and results related to martingales, which are needed in the paper.

Consider a (continuous-time) stochastic process \( (X_t)_{t \geq 0} \) defined on some joint probability space \((\Omega, \mathcal{F}, P)\). A filtration is a family \( \{\mathcal{F}_t : t \geq 0\} \) of sub-\( \sigma \)-fields of \( \mathcal{F} \) such that \( \mathcal{F}_s \subseteq \mathcal{F}_t \) for all \( 0 \leq s \leq t \). We are particularly interested in the natural filtration generated by \( X_t \), i.e., \( \mathcal{F}_X^t := \sigma(\{X_s : s \leq t\}) \).

**Definition 1:** (Stopping Time) A stopping time, wrt a filtration \( \mathcal{F}_t \), is a non-negative random variable \( T \) such that \( \{T \leq t\} \in \mathcal{F}_t \) for all \( t \geq 0 \).

In this paper we are particularly interested in the first passage time \( T = \inf_{t \geq 0} \{X_t \geq x\} \) for some non-negative \( x \). To avoid technical considerations related to conditions under which \( T \) is a stopping time, we assume throughout that the sample-paths \( X_t(\omega) \) are right-continuous \( \forall \omega \in \Omega \); this assumption is implicitly fulfilled by the definition of the arrival process from Eq. (6).

**Definition 2:** (Martingale) A continuous time process \( X_t \) is a martingale wrt the natural filtration \( \mathcal{F}_X^t \) if
1) \( E[|X_t|] < \infty \forall t \geq 0 \) and
2) \( E[X_t | \mathcal{F}_s] = X_s \forall 0 \leq s \leq t \).

The next three results are needed in the proof of the main theorem in the paper.

**Lemma 2:** (Optional Switching) Consider that \( X_t \) and \( Y_t \) are martingales wrt \( \mathcal{F} := \mathcal{F}_X^Y \), and assume that \( X_u = Y_u \) for some \( u \geq 0 \). Then the process
\[
Z_t = \begin{cases} 
X_t & \text{if } t < u \\
Y_t & \text{if } t \geq u
\end{cases}
\]

is a martingale wrt \( \mathcal{F} \).

For the corresponding result in discrete time see Grimmett and Stirzaker [27], p. 488.

**Proof:** According to the martingale definition, the non-trivial property to prove is
\[
E[Z_t | \mathcal{F}_s] = Z_s,
\]
for \( t \geq u \) and \( s < u \). Indeed, we have according to the tower property of conditional expectation
\[
E[Z_t | \mathcal{F}_s] = E[E[Z_t | \mathcal{F}_u] | \mathcal{F}_s] = E[Z_u | \mathcal{F}_s] = E[X_u | \mathcal{F}_s] = Z_s,
\]
which completes the proof. \( \square \)

**Lemma 3:** (Product of Independent Martingales) Consider that \( X_t \) and \( Y_t \) are independent martingales wrt \( \mathcal{F} := \mathcal{F}_X^Y \). Then \( X_t Y_t \) is a martingale wrt \( \mathcal{F} \).

**Theorem 4:** (Optional Sampling Theorem)(see [24], p. 61) If \( X_t \) is a right-continuous martingale and \( T \) is a finite stopping time wrt \( \mathcal{F}_X^T \), then
\[
E[X_T] = E[X_0].
\]