Group independent color decomposition of next-to-leading order matrix elements for $e^+e^- \rightarrow$ four partons

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Abstract

We present the next-to-leading order partonic cross sections involving an electroweak vector boson and four massless partons (quarks, gluons or long living gluinos) in a general gauge theory with a simple Lie Group. The vector boson couples to a massless lepton pair and a quark-antiquark pair. The cross sections are given in terms of group independent kinematical functions multiplying the eigenvalues of the Casimir operators of the Lie group. This kind of color decomposition is required for the calculation of $O(\alpha_s^3)$ corrections to the group independent kinematical functions in the four-jet production cross sections in electron-positron annihilation. The knowledge of these corrections facilitates the simultaneous precision measurement of the strong coupling and the color charge factors using the four-jet LEP or SLC data as well as the test whether these data favour or exclude the existence of a light gluino.

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Electron-positron annihilation into four jets is a very clean way of measuring the basic properties of Quantum Chromodynamics (QCD), the theory of strong interactions. This process allows the simultaneous measurement of the strong coupling $\alpha_s$ and the eigenvalues of the Casimir operators of the underlying symmetry group, the color factors. Such a measurement provides a very stringent test of QCD because $\alpha_s$ is the only free parameter in the theory, while the color factors show whether the dynamics is indeed described by an SU(3) symmetry. However, in order to carry out such a full measurement the theoretical prediction of perturbative QCD has to be known at next-to-leading order accuracy. Lacking this sort of precision of the theoretical description, so far the experiments used four-jet data for measuring the ratios of the color...
factors $C_A/C_F$ and $T_R/C_F$ that are relatively insensitive to higher order corrections [1]. The $\alpha_s$ measurements require the fixing the normalization of four-jet observables, which however, suffers large radiative corrections. For example, the experimental four-jet rate measured as a function of the jet resolution parameter was about a factor of two higher than the leading order prediction.

Recent theoretical developments make possible the next-to-leading order calculation of four-jet quantities. There are now several general methods available for the cancellation of infrared divergences that can be used for setting up a Monte Carlo evaluation of next-to-leading order partonic cross sections [2]. The other vital piece of new information is the one-loop matrix elements for the relevant QCD subprocesses — i.e. for the production of four-quarks and for the production of two quarks and two gluons in electron-positron annihilation — that are now available [3–6]. In refs. [3,5] Campbell, Glover and Miller make FORTRAN programs for the next-to-leading order squared matrix elements of the $e^+e^- \rightarrow \gamma^* \rightarrow \bar{q}qQQ$ and $\bar{q}qqg$ processes publicly available. In refs. [4,6] Bern, Dixon, Kosower and Wienzierl give analytic formulæ for the helicity amplitudes of the same processes with the $e^+e^- \rightarrow Z^0 \rightarrow$ four partons channel included as well. The helicity amplitudes for the processes $e^+e^- \rightarrow \bar{q}qggg$ and $e^+e^- \rightarrow \bar{q}qQQg$ have been known for a long time [7]. Using the helicity amplitudes in refs. [4,6,7], Dixon and Signer calculated the next-to-leading order corrections to four-jet fractions for various clustering algorithms [8], as well as to angular distributions [9] using a general purpose partonic Monte Carlo program [10]. As a result, the agreement between the data and the theoretical prediction for the four-jet rates improved significantly. In the calculation of these four-jet fractions the underlying gauge group is assumed to be SU($N_C$) and the results are presented in the form of an expansion in the number of colors. This way of presenting the results is sufficient if one is interested in the four-jet rate and will certainly find important application in LEP2 physics, where the $e^+e^- \rightarrow (\gamma, Z) \rightarrow 4$ jets events are the main background to $e^+e^- \rightarrow W^+W^- \rightarrow 4$ jets events, but not for increasing the precision in measuring the color charge factors.

One can measure the color charges directly using any kind of four-jet observable, provided the partonic cross section is given in a factorized form of eigenvalues of the Casimir operators and group independent kinematical factors. Such a decomposition has been known for a long time at tree level [11]. In this letter we give the necessary decomposition of the virtual corrections. Combining these with similarly decomposed real corrections, we can obtain the full $O(\alpha_s^3)$ corrections to the group independent kinematical functions in the four-jet production cross sections in electron-positron annihilation [12].

An additional piece of motivation for using a group independent color decomposition comes from the suggestion that the measurement of the color charges may be influenced by the production of light, colored, but electrically neutral
fermionic particles, such as gluinos [13]. The effect of the presence of light gluinos on the strong coupling is to increase $\alpha_s(M_Z)$ by 2% [14]. It would be interesting to see if the presence of light gluinos influence the value of the color charges more noticeably. In order to make possible a next-to-leading order analysis of this effect, we also present the one-loop helicity amplitudes for the $e^+e^- \rightarrow (\gamma, Z) \rightarrow \bar{q}q\tilde{g}\tilde{g}$ process. We have used these amplitudes to show that in order to obtain the most significant light gluino exclusion limit from four-jet data, it is advantageous to define jets in the range of small resolution parameter [15]. For example, in an analysis based upon ALEPH data for four jet rates [16] using Durham clustering algorithm [17] at $y_{cut} = 0.002$ we were able to exclude the existence of light gluinos at the 95% confidence level [15].

In presenting the new color decomposition, we rely on the work of Bern, Dixon, Kosower and Wienzierl who published the explicit form of the one-loop helicity amplitudes for the QCD subprocesses [4,6]. We use the same notation as these articles and introduce new ones but to the extent that is necessary.

Our aim is to give the next-to-leading order squared matrix elements for the $e^+e^- \rightarrow \bar{q}q\bar{p}p$ processes ($p = Q, g$, or $\tilde{g}$) in terms of color factors multiplied by group independent kinematic functions. In order to find a group independent decomposition of the squared matrix element, we have to give different color decompositions of the one-loop helicity amplitudes for the various processes than those presented in refs. [4,6], where the color charge information has been lost by the use of the SU($N_C$) Fierz identity and SU($N_C$) relations $C_F = (N_C^2 - 1)/N_C$, $C_A = 2N_C$ (for $T_R = 1$). In the new color decomposition we can only use the defining relation of the Lie algebra

$$[T^a,T^b]_{ij} = i \sum_{c=1}^{N_A} f^{abc}T^c_{ij} , (1)$$

and the definition of the quadratic Casimir invariants $C_F$, $C_A$ and $T_R$:

$$\sum_{a=1}^{N_A} (T^a T^a)_{ij} = C_F \delta_{ij} , \quad \sum_{a=1}^{N_A} (F^a F^a)_{cd} = C_A \delta_{cd} , \quad \text{Tr}(T^a T^b) = T_R \delta^{ab} , (2)$$

where $N_A$ is the dimension of the adjoint representation of the gauge group, $T^a$ are the generators in the fundamental and $F^a$ are those in the adjoint representation. The latter are related to the structure constans via $F^a_{bc} = -i f^{abc}$. Careful analysis of the color structure of the individual Feynman diagrams contributing to a given process shows that the color charge information can completely be recovered from the primitive amplitudes of refs. [4,6] for the QCD subprocesses, while in the case of two-quark two-gluino production minor modification of the partial amplitudes is necessary. In order we can use those primitive amplitudes we use $T_R = 1$ normalization.
Let us first consider the $e^+e^- \to \bar{q}qQQ$ process. At tree level the new color decomposition of the $A^\text{tree}_6(1, 2, 3, 4; 5, 6)$ helicity amplitude is

$$A^\text{tree}_6(1, 2, 3, 4; 5, 6) = 2e^2g^2 \sum_{c=1}^{N_A} T_{qq}^c T_{QQ}^c$$

$$\times \left[ C_q^{h_qh_e} \tilde{A}_{6;0}(1^{h_q}, 2^{h_q}, 3^{h_q}, 4^{h_q}, 5^{h_e}, 6^{h_e}) + C_Q^{h_Qh_e} \tilde{A}_{6;0}(3^{h_q}, 4^{h_q}, 5^{h_Q}, 6^{h_e}) \right],$$

where $e$ is the QED coupling, $g$ the QCD coupling $h_x$ is the helicity of particle $x$ and the coupling factors $C^{h_Qh_e}_Q$ depend on the charge $Q^0$ of quark $Q$ in units of $e$ and on the left- and right-handed couplings of the electron ($v^e_{L,R}$) and of the quark ($v^Q_{L,R}$) as follows:

$$C_{Q}^{++} = -Q^0 + v^Q_{L}v^e_{R}P_{Z}(s_{56}), \quad C_{Q}^{+-} = -Q^0 + v^Q_{R}v^e_{L}P_{Z}(s_{56}), \quad (4)$$

$$C_{Q}^{-+} = -Q^0 + v^Q_{L}v^e_{R}P_{Z}(s_{56}), \quad C_{Q}^{--} = -Q^0 + v^Q_{R}v^e_{L}P_{Z}(s_{56}). \quad (5)$$

The left- and right-handed couplings of the electron and quark are the standard ones as given in ref. [4]. $\tilde{A}_{6;0}$ denotes the tree-level partial amplitude $A^\text{tree}_6$ of ref. [4], where it was defined to include a photon propagator. The ratio $P_{Z}(s_{56}) = s_{56}/(s_{56} - M_Z^2 + i\Gamma_Z M_Z)$ appearing in the coupling factors replaces this photon propagator with a $Z$ propagator ($M_Z$ and $\Gamma_Z$ are the mass and width of the $Z^0$ boson).

At one loop the new color decomposition of the helicity amplitudes is

$$A^\text{1-loop}_6(1, 2, 3, 4; 5, 6) = 2e^2g^4$$

$$\times \left\{ C_q^{h_qh_e} \tilde{A}_{6;1}(1^{h_q}, 2^{h_q}, 3^{h_q}, 4^{h_q}, 5^{h_e}, 6^{h_e}) \sum_{c=1}^{N_A} T_{qq}^c T_{QQ}^c$$

$$+ \tilde{A}_{6;2}(1^{h_q}, 2^{h_q}, 3^{h_q}, 4^{h_q}, 5^{h_e}, 6^{h_e}) \sum_{c,d=1}^{N_A} (T^cT^d)_{qq}(T^dT^c)_{QQ} \right\}$$

$$+ C_Q^{h_Qh_e} \tilde{A}_{6;1}(3^{h_Q}, 4^{h_Q}, 1^{h_q}, 2^{h_q}, 5^{h_e}, 6^{h_e}) \sum_{c=1}^{N_A} T_{qq}^c T_{QQ}^c$$

$$+ \tilde{A}_{6;2}(3^{h_Q}, 4^{h_Q}, 1^{h_q}, 2^{h_q}, 5^{h_e}, 6^{h_e}) \sum_{c,d=1}^{N_A} (T^cT^d)_{qq}(T^dT^c)_{QQ} \right\}$$

$$+ C_{ax}^{h_e} \tilde{A}_{6;3}(1^{h_q}, 2^{h_q}, 3^{h_q}, 4^{h_q}, 5^{h_e}, 6^{h_e}) \sum_{c=1}^{N_A} T_{qq}^c T_{QQ}^c \right\},$$

where $C_{ax}^{h_e}$ vanishes for the photon and $W^\pm$ boson, while for the $e^+e^- \to Z^0 \to$
The gluino is a majorana fermion in the adjoint representation of the gauge group and does not couple to the vector bosons directly. Therefore, the $e^+e^- \rightarrow \bar{q}qQQ$ process it is

$$C_{ax}^+ = \frac{v^e_R}{\sin 2\theta_W} P_Z(s_{56}) \ , \quad C_{ax}^- = \frac{v^e_L}{\sin 2\theta_W} P_Z(s_{56}) \ , \quad (7)$$

with $\theta_W$ being the Weinberg angle. In eq. (6) we used the notation $A_{6;i}$ for the partial amplitudes in order to distinguish them from the $A_{6;i}$ functions introduced in ref. [4], where the basic gauge invariant classes of colorless amplitudes, the 'primitive amplitudes' are also given explicitly. Our new partial amplitudes can be expressed in terms of the same primitive amplitudes multiplied by color factors. The expressions depend on the helicities of the partons. There are only two independent helicity configurations, which can be taken to be $(1^+_q, 2^+_Q, 3^3_Q, 4^+_\bar{q}, 5^-_q, 6^+_\bar{q})$, and from which the amplitudes of the other helicity configurations can be obtained [4].

Formulas (3) and (6) apply to the case of unequal quark flavors, $q \neq Q$. The equal flavor amplitude may be obtained from the unequal-flavor formula by subtracting the same formula with $q$ and $Q$ exchanged and then setting $Q = q$ in all the coupling constant prefactors [4].

The explicit expressions for the $A_{6;i}$ partial amplitudes in terms of primitive amplitudes are (we suppress the 5, 6 labels of the lepton pair)

$$\tilde{A}_{6:1}(1^+_q, 2^+_Q, 3^-_Q, 4^-_\bar{q}) = \quad -C_F A_{6\ell}^s(2, 3, 1, 4) + \frac{C_A}{2} \left( A_{6\ell}^s(2, 3, 1, 4) - A_{6\ell}^s(1, 3, 2, 4) \right)$$

$$+ A_{6\ell}^{++}(1, 2, 3, 4) - N_f \left( A_{6\ell}^{++}(1, 2, 3, 4) + A_{6\ell}^{++}(1, 2, 3, 4) \right) \ , \quad (8)$$

$$\tilde{A}_{6:2}(1^+_q, 2^+_Q, 3^-_Q, 4^-_\bar{q}) = A_{6\ell}^{++}(1, 2, 3, 4) + A_{6\ell}^{--}(1, 3, 2, 4) \ , \quad (9)$$

$$\tilde{A}_{6:3}(1^+_q, 2^+_Q, 3^-_Q, 4^-_\bar{q}) = A_{6\ell}^{ax}(1, 4, 2, 3) \ , \quad (10)$$

while for the other helicity configuration

$$\tilde{A}_{6:1}(1^+_q, 2^-_Q, 3^+_Q, 4^-_\bar{q}) = \quad C_F A_{6\ell}^s(3, 2, 1, 4) - \frac{C_A}{2} \left( A_{6\ell}^s(3, 2, 1, 4) + A_{6\ell}^{++}(1, 3, 2, 4) \right)$$

$$+ A_{6\ell}^{+-}(1, 2, 3, 4) - N_f \left( A_{6\ell}^{+-}(1, 2, 3, 4) + A_{6\ell}^{+-}(1, 2, 3, 4) \right) \ , \quad (11)$$

$$\tilde{A}_{6:2}(1^+_q, 2^-_Q, 3^+_Q, 4^-_\bar{q}) = A_{6\ell}^{++}(1, 3, 2, 4) + A_{6\ell}^{--}(1, 2, 3, 4) \ , \quad (12)$$

$$\tilde{A}_{6:3}(1^+_q, 2^-_Q, 3^+_Q, 4^-_\bar{q}) = -A_{6\ell}^{ax}(1, 4, 3, 2) \ . \quad (13)$$

The gluino is a majorana fermion in the adjoint representation of the gauge group and does not couple to the vector bosons directly. Therefore, the $e^+e^- \rightarrow$
\( \bar{q}q\bar{g}\bar{g} \) subprocesses has similar color decomposition to that of the \( e^+e^- \rightarrow \bar{q}q\bar{q}\bar{Q} \) subprocess. At tree level this decomposition reads

\[
A^{\text{tree}}_6(1_q, 2_g, 3_{\bar{g}}, 4_q; 5_e, 6_e) = 2e^2g^2C_q^{h_e}h_e \tilde{A}_{6:0}(1^h_q, 2^{h_2}_h, 3^{h_3}_h, 4^{h_4}_h, 5^{h_5}_h, 6^{h_6}_h) \sum_{c=1}^{N_A} T^c_{qq} F^c_{\bar{g}2\bar{g}3} ,
\]

while at one-loop it is

\[
A^{1\text{-loop}}_6(1_q, 2_g, 3_{\bar{g}}, 4_q; 5_e, 6_e) = 2e^2g^4 \times \left\{ C_q^{h_e} \left[ \tilde{A}_{6:1}(1^h_q, 2^{h_2}_h, 3^{h_3}_h, 4^{h_4}_h, 5^{h_5}_h, 6^{h_6}_h) \sum_{c=1}^{N_A} T^c_{qq} F^c_{\bar{g}3\bar{g}2} \right. \\
+ \tilde{A}_{6:2}(1^h_q, 2^{h_2}_h, 3^{h_3}_h, 4^{h_4}_h, 5^{h_5}_h, 6^{h_6}_h) \sum_{c,d=1}^{N_A} (T^cT^d)_{q\bar{q}}(F^d F^c)_{\bar{g}3\bar{g}2} \\
+ C_{A_3}^{h_e} \left[ \tilde{A}_{6:3}(1^h_q, 2^{h_2}_h, 3^{h_3}_h, 4^{h_4}_h, 5^{h_5}_h, 6^{h_6}_h) \sum_{c=1}^{N_A} T^c_{\bar{q}q} F^c_{\bar{g}3\bar{g}2} \right] \right\} .
\]

The \( \tilde{A}_{6:1} \) partial amplitudes for the \( e^+e^- \rightarrow \bar{q}q\bar{g}\bar{g} \) process are closely related to those of the \( e^+e^- \rightarrow \bar{q}q\bar{Q}\bar{Q} \) process. In fact, the \( \tilde{A}_{6:2} \) and \( \tilde{A}_{6:3} \) amplitudes are exactly the same,

\[
\tilde{A}_{6:2}(1^+_q, 2^{\pm}_g, 3^{\mp}_{\bar{g}}, 4^-_{\bar{q}}) = \tilde{A}_{6:2}(1^+_q, 2^{\pm}_Q, 3^{\mp}_Q, 4^-_{\bar{q}}) ,
\]

\[
\tilde{A}_{6:3}(1^+_q, 2^{\pm}_g, 3^{\mp}_{\bar{g}}, 4^-_{\bar{q}}) = \tilde{A}_{6:3}(1^+_q, 2^{\pm}_Q, 3^{\mp}_Q, 4^-_{\bar{q}}) ,
\]

while the \( \tilde{A}_{6:1} \) amplitudes differ in terms arising from fermion bubble and parent triangle diagrams [4],

\[
\tilde{A}_{6:1}(1^+_q, 2^{\pm}_g, 3^{\mp}_{\bar{g}}, 4^-_{\bar{q}}) = \tilde{A}_{6:1}(1^+_q, 2^{\pm}_Q, 3^{\mp}_Q, 4^-_{\bar{q}}) - (C_F - C_A) A^\Delta_{6}^{\pm\pm}(1, 2, 3, 4) \\
- N_\tilde{g} \frac{C_A}{2} \left( A^\Delta_{6}^{\pm\pm}(1, 2, 3, 4) + A^\Delta_{6}^{\pm\pm}(1, 2, 3, 4) \right) ,
\]

where \( N_\tilde{g} \) is the number of light gluino flavors and

\[
A^{\Delta_{6}^{\pm\pm}}_{6}(1, 2, 3, 4) = c_T A^{\text{tree}, \pm\pm}_{6}(1, 2, 3, 4) \left[ -\frac{1}{\varepsilon^2} \left( \frac{\mu^2}{s_{23}} \right)^\varepsilon - \frac{3}{2\varepsilon} \left( \frac{\mu^2}{s_{23}} \right)^\varepsilon - \frac{7}{2} \right] ,
\]

\[ p \]
with $c_f$ and $A_6^{\text{tree,++}}$ given in ref. [4]. We remark here that gluinos are Majorana fermions therefore, the cross section requires an identical-particle factor of $\frac{1}{2}$ in the phase-space measure.

Finally, the new color decomposition of the helicity amplitudes for the $e^+e^- \rightarrow \bar{q}qgg$ process at tree level is given by

$$A_6^{\text{tree}}(1_q, 2_g, 3_g, 4_q; 5_e, 6_e) = 2e^2g^2$$

$$\times C_q^{hc} \left[ A_{6:0}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he})(T^{g2}T^{g3})_{qq} + A_{6:6}(1_{hq}, 3_{h3}, 2_{hq}, 4_{hq}, 5_{he}, 6_{he})(T^{g3}T^{g2})_{qq} \right],$$

and at one-loop it is

$$A_6^{\text{1-loop}}(1_q, 2_g, 3_g, 4_q; 5_e, 6_e) = 2e^2g^4$$

$$\times \left\{ C_q^{hc} \left[ A_{6:1}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he})(T^{g2}T^{g3})_{qq} + A_{6:2}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he})(T^{g3}T^{g2})_{qq} + A_{6:3}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he}) \sum_{c,d=1}^{N_A} (T^c T^d)_{qq} (F^d F^c)_{g3g2} \right] + \sum_{f=1}^{N_f} \frac{1}{2} \left( C_{q_f}^{hc} + C_{q_f}^{-hc} \right) \tilde{A}_{6:4}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he}) \right\}$$

$$\times \sum_{c=1}^{N_A} \left[ \text{Tr}(T^{g2}T^{g3}T^c)_{qq} + \text{Tr}(T^{g3}T^{g2}T^c)_{qq} \right]$$

$$+ C_{ax}^{hc} \left[ A_{6:5}^{A}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he})(T^{g2}T^{g3})_{qq} + A_{6:5}^{B}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he})(T^{g3}T^{g2})_{qq} + A_{6:5}^{C}(1_{hq}, 2_{hq}, 3_{h3}, 4_{hq}, 5_{he}, 6_{he}) \right]$$

$$\times \sum_{c=1}^{N_A} \left[ \text{Tr}(T^{g2}T^{g3}T^c)_{qq} + \text{Tr}(T^{g3}T^{g2}T^c)_{qq} \right] \right\}.$$ 

The partial amplitudes $\tilde{A}_{6;i}$ can easily be constructed from the primitive amplitudes of ref. [6]:

$$\tilde{A}_{6:1}(1_q^+, 2_g^{h2}, 3_g^{h3}, 4_q^-) = C_F A_6(1_q, 4_q; 3, 2)$$

$$+ \frac{C_A}{2} \left[ A_6(1_q, 2, 3, 4_q) - A_6(1_q, 4_q; 3, 2) - A_6(1_q, 4_q; 3, 2) \right]$$

$$+ \left[ A_6^{f++}(1_q, 2, 3, 4_q) - N_f \left( A_6^{f++}(1_q, 2, 3, 4_q) + A_6^{f++}(1_q, 2, 3, 4_q) \right) \right].$$
\[
\tilde{A}_{6;2}(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = C_F A_6(1_q, 4_q; 2, 3)
\]
\[
+ \frac{C_A}{2} [A_6(1_q, 3, 2, 4_q) - A_6(1_q, 4_q; 2, 3)]
\]
\[
+ A_6^{l++}(1_q, 3, 2, 4_q) - N_f \left( A_6^{f,+-}(1_q, 3, 2, 4_q) + A_6^{i,++}(1_q, 3, 2, 4_q) \right),
\]
\[
\tilde{A}_{6;3}(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = A_{6;3}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}),
\]
\[
\tilde{A}_{6;4}(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = A_{6;4}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}),
\]
\[
\tilde{A}_{6;5}^A(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = \frac{1}{2} \left[ A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) - A_{6;5}^{ax}(1_q^+, 4_q^-, 3_g^{h_3}, 2_g^{h_2}) 
\right.
\]
\[
+ A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) \right],
\]
\[
A_{6;5}^B(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = \frac{1}{2} \left[ A_{6;5}^{ax}(1_q^+, 4_q^-, 3_g^{h_3}, 2_g^{h_2}) - A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) 
\right.
\]
\[
+ A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) \right],
\]
\[
A_{6;5}^C(1_q^+, 2_g^{h_2}, 3_g^{h_3}, 4_q^-) = \frac{1}{2} \left[ A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) + A_{6;5}^{ax}(1_q^+, 4_q^-, 3_g^{h_3}, 2_g^{h_2}) 
\right.
\]
\[
- A_{6;5}^{ax}(1_q^+, 4_q^-, 2_g^{h_2}, 3_g^{h_3}) \right].
\]

Before closing the discussion of the helicity amplitudes, we should remark that in order to take into account the effect of light gluinos in a next-to-leading order calculation fully, one has to make the change

\[
N_f \rightarrow N_f + \frac{C_A}{2} N_{\tilde{g}}
\]

in eqs. (8), (11) and (22), (23), where in SU($N_C$) theory $C_A = 2N_C$ for $T_R = 1$.

As a next step we use eqs. (3), (6), (14), (15), and (20), (21) to give a gauge independent decomposition of the next-to-leading order squared matrix elements. The general expression for these matrix elements is

\[
d\sigma_6^{\text{NLO, virtual}} = 2 \sum \sum \text{Re} \left[ A_6^{\text{tree}} \ast A_6^{1-\text{loop}} \right].
\]

We want to evaluate the color sum in such a way that the group independent information is maintained. For this purpose, we use the commutation relation (1) together with the definition of the quadratic Casimirs, eq. (2) to derive the necessary Lie-algebra relations. In the case of production of two unequal flavor quark pairs these are:

\[
\sum_{a, b=1}^{N_A} \text{Tr}(T^a T_{1b}) \text{Tr}(T^a T_{1b}) = N_C C_F,
\]

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\[
\sum_{a,b,c=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger c}) \text{Tr}(T^{\dagger c} T^b T^a) \equiv C_3 ,
\]

(32)

where in eq. (32) we introduced \(C_3\), the square of a cubic Casimir, that cannot be reduced to an expression of the quadratic Casimirs \(C_F\) and \(C_A\). In SU(\(N_C\)) \(C_3\) takes the value \((N_C^2 - 1)(N_C^2 - 2)/N_C\). In the case of production of two equal flavor quarks, in addition to relations (31) and (32), we need two more color sums:

\[
\sum_{a,b=1}^{N_A} \text{Tr}(T^a T^b T^b T^{\dagger b}) = N_C C_F \left( C_F - \frac{C_A}{2} \right) ,
\]

(33)

\[
\sum_{a,b,c=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger c} T^b T^a T^{\dagger c}) = N_C C_F \left( C_F - \frac{C_A}{2} \right)^2 .
\]

(34)

When a quark and a gluino pair appears in the final state, we use relations

\[
\sum_{a,b=1}^{N_A} \text{Tr}(T^a T^b) \text{Tr}(F^a F^{\dagger b}) = N_C C_F C_A ,
\]

(35)

\[
\sum_{a,b,c=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger c}) \text{Tr}(F^{\dagger c} F^b F^a) = N_C C_F \left( \frac{C_A}{2} \right)^2 .
\]

(36)

Finally, for the production of a quark pair and two gluons the additional necessary color sums are

\[
\sum_{a,b=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger b} T^{\dagger a}) = N_C C_F^2 ,
\]

(37)

\[
\sum_{a,b,g_2,g_3=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger g_3} T^{\dagger g_2}) (F^b F^a)_{g_3 g_2} = N_C C_F \left( \frac{C_A}{2} \right)^2 ,
\]

(38)

\[
\sum_{a,b,g_2,g_3=1}^{N_A} \text{Tr}(T^a T^b T^{\dagger g_2} T^{\dagger g_3}) (F^b F^a)_{g_3 g_2} = 0 ,
\]

(39)

and

\[
\sum_{g_2,g_3,c=1}^{N_A} \text{Tr}(T^{g_2} T^{g_3} T^c) \text{Tr}(T^{\dagger g_2} T^{\dagger g_3} T^{\dagger c}) = C_3 - C_F \frac{C_A}{2} N_C .
\]

(40)

Using eqs. (31-40), one can evaluate the color sum in eq. (30) easily. The resulting differential cross sections can be written as a quadratic form:
\[
\frac{1}{\sigma_0} \text{d}\sigma_{\text{NLO, virtual}}^0 (\vec{p}) = \left( \frac{\alpha_s C_F}{2\pi} \right)^3 \times \left[ C_0(\vec{p}) + C_x(\vec{p}) x + C_y(\vec{p}) y + C_z(\vec{p}) z + C_{xx}(\vec{p}) x^2 + C_{xy}(\vec{p}) x y + C_{yy}(\vec{p}) y^2 \right].
\]

In this equation \( \sigma_0 \) denotes the Born cross section for the process \( e^+ e^- \rightarrow \bar{q}q \), \( \vec{p} \) is the collection of the final state momenta, and \( x, y \) and \( z \) are ratios of eigenvalues of the Casimir operators defined as

\[
x = \frac{C_A}{C_F}, \quad y = \frac{T_R}{C_F} = \frac{N_C}{N_A}, \quad z = \frac{C_3}{N_C C_F^2}.
\]

These ratios are the sole quantities together with the overall normalization that carry group information. The coefficients \( C_i(\vec{p}) \) are the group independent kinematical functions that depend on the subprocess. Their explicit expressions are quite complicated, but can be obtained straightforwardly using the formulas of this letter. These expressions will be published in the form of C++ code in the partonic Monte Carlo program DEBRECEN.

In this letter we have shown explicitly how to rewrite the one-loop helicity amplitudes of the \( e^+ e^- \rightarrow (\gamma, Z) \rightarrow 4 \) partons processes in a form from which the group independent color decomposition of the next-to-leading order squared matrix elements for these processes can be obtained. These kinematical functions for the QCD subprocesses can completely be written in terms of the primitive amplitudes given in refs. [4] and [6]. In the case of the \( e^+ e^- \rightarrow \bar{q}q\bar{g}\bar{g} \) subprocess one has to modify the primitive amplitudes of the four-quark subprocess slightly. We also presented the general structure of the next-to-leading order partonic cross sections in terms of group independent kinematical functions multiplying ratios of eigenvalues of the Casimir operators of the Lie group. This kind of color decomposition together with a similar decomposition of the \( e^+ e^- \rightarrow (\gamma, Z) \rightarrow 5 \) partons squared matrix elements is required for the calculation of \( O(\alpha_s^3) \) corrections to the group independent kinematical functions in the four-jet production cross sections in electron-positron annihilation. We anticipate that these corrections will improve our knowledge of the basic parameters of QCD.

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