A SHORT NOTE ON SELF-DUALITY OF GOPPA CODES ON ELLIPTIC AND HYPERELLIPITIC FUNCTION FIELDS

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Abstract. In this note, we investigate Goppa codes which are constructed by means of Elliptic function field and Hyperelliptic function field. We also give a simple criterion for self-duality of these codes.

1. Introduction

A linear code is a subspace of the \( n \)-dimensional standard vector space \( \mathbb{F}_q^n \) over a finite field \( \mathbb{F}_q \). Such codes are used for transmission of information. A linear code \( C \) is called self-dual if \( C = C^\perp \), where \( C^\perp \) is the dual of \( C \) with respect to Euclidean scalar product on \( \mathbb{F}_q^n \). Self-dual codes are an important class of linear codes.

It was observed by Goppa in 1975 that we can use algebraic function fields over \( \mathbb{F}_q \) to construct a class of linear codes by choosing a divisor and some rational places of algebraic function field over \( \mathbb{F}_q \). In this note, we investigate codes which are constructed by means of elliptic and hyperelliptic function fields. This class of codes provide non-trivial examples of geometric Goppa codes.

For self-dual geometric Goppa codes, Driencourt [2] and Stichtenoth [6] showed a criterion, which is too complex to apply. In [4], Xing gave a simple criterion for self-duality of Goppa codes over elliptic function field with base field of characteristic 2. There are some difficulties in generalising the Xing’s criterion to the field of characteristic not equal to 2. Using Xing’s idea and results from [3], we give a simple criterion for self-duality of Goppa codes over elliptic function field and hyperelliptic function field of characteristic not equal to 2.

2. Preliminaries

2.1. Goppa code. Goppa’s construction is described as follows:

Let \( F/\mathbb{F}_q \) be an algebraic function field of genus \( g \). Let \( P_1, \cdots, P_n \) be pairwise distinct places of \( F/\mathbb{F}_q \) of degree 1. Let \( D = P_1 + \cdots + P_n \) and \( G \) be a divisor of \( F/\mathbb{F}_q \) such that \( \text{supp}(G) \cap \text{supp}(D) = \emptyset \). The geometric Goppa code \( C_L(D, G) \) associated with \( D \) and \( G \)

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is defined by
\[ C_{\mathcal{L}}(D, G) := \{(x(P_1), \cdots, x(P_n))| \ x \in \mathcal{L}(G)\} \subseteq \mathbb{F}_q^n. \]

Then, \( C_{\mathcal{L}}(D, G) \) is an \([n, k, d]\) code with parameters \( k = \text{dim}(G) - \text{dim}(G - D) \) and \( d \geq n - \text{deg}(G) \).

Another code can be associated with the divisors \( G \) and \( D \) by using local components of Weil differentials. We define the code \( C_\Omega(D, G) \subseteq \mathbb{F}_q^n \) by
\[ C_\Omega(D, G) := \{(\omega_P(1), \cdots, \omega_P(1))| \ \omega \in \Omega_F(G - D)\}. \]

Then, \( C_\Omega(D, G) \) is an \([n, k', d']\) code with parameters \( k' = i(G - D) - i(G) \) and \( d' \geq \text{deg}(G) - (2g - 2) \).

The dual code of \( C_{\mathcal{L}}(D, G) \) is \( C_\Omega(D, G) \) i.e. \( C_\Omega(D, G) = C_{\mathcal{L}}(D, G)^\perp \). Let \( \eta \) be a Weil differential such that \( \nu_P(\eta) = -1 \) and \( \eta_P(1) = 1 \) for \( i = 1, \cdots, n \), then \( C_{\mathcal{L}}(D, G)^\perp = C_\Omega(D, G) = C_{\mathcal{L}}(D, D - G + (\eta)). \)

2.2. Elliptic Function Field.

Definition 2.1. An algebraic function field \( F/K \) (where \( K \) is the full constant field of \( F \)) is said to be an elliptic function field if the following conditions hold:

- the genus of \( F/K \) is \( g = 1 \), and
- there exists a divisor \( A \in D_F \) with \( \text{deg} \ A = 1 \).

The following theorems characterize elliptic function field over \( K \) (where \( \text{char} \ K \neq 2 \)).

Theorem 2.2 (\cite{5}, Chapter VI). Let \( F/K \) be an elliptic function field. If \( \text{char} \ K \neq 2 \), there exist \( x, y \in F \) such that \( F = K(x, y) \) and
\[ y^2 = f(x) \in K[x] \]
with a square-free polynomial \( f(x) \in K[x] \) of degree 3.

Theorem 2.3 (\cite{5}, Chapter VI). Suppose that \( F = K(x, y) \) with
\[ y^2 = f(x) \in K[x] \]
where \( f(x) \) is a square-free polynomial of degree 3. Consider the decomposition \( f(x) = c \prod_{i=1}^r p_i(x) \) of \( f(x) \) into monic irreducible polynomials \( p_i(x) \in K[x] \) with \( 0 \neq c \in K \). Denote by \( P_i \in \mathbb{P}_{K(x)} \) the place of \( K(x) \) corresponding to \( p_i(x) \), and by \( P_\infty \in \mathbb{P}_{K(x)} \) the pole of \( x \). Then the following holds:

1. \( K \) is the full constant field of \( F \), and \( F/K \) is an elliptic function field.
2. The extension \( F/K(x) \) is cyclic of degree 2. The places \( P_1, \cdots, P_r \) and \( P_\infty \) are ramified in \( F/K(x) \); each of them has exactly one extension in \( F \), say \( Q_1, \cdots, Q_r \) and \( Q_\infty \), and we have \( e(Q_j|P_j) = e(Q_\infty|P_\infty) = 2 \), \( \deg Q_j = \deg P_j \) and \( \deg Q_\infty = 1 \).
Let \( F/K \) be an algebraic function field, if \( \text{deg} f \equiv 0 \mod 2 \), then for \( G = -P_\infty \) and \( H = -P_\infty - P_\alpha \) we have \( C_L(D, G) = C_L(D, H - G) \) but \( H \neq 2G \). In this case, we observed that we can use following theorem to determine the self-duality of Goppa codes over elliptic and hyperelliptic function fields.
Definition 3.2. We call two divisors $G$ and $H$ equivalent with respect to $D$ if there exists $u \in F$ such that $H = G + (u)$ and $u(P_i) = 1$, for all $i = 1, \ldots, n$.

Theorem 3.3 (\cite{3}, Corollary 4.15). Suppose $n > 2g + 2$. Let $G$ and $H$ be divisors of same degree $m$ on a curve of genus $g$. If $C_L(D, G)$ is not equal to 0 nor to $\mathbb{F}_q^n$ and $2g - 1 < m < n - 1$, then $C_L(D, G) = C_L(D, H)$ if and only if $G$ and $H$ are equivalent with respect to $D$.

We start with $K = \mathbb{F}_q$, $q$ large enough and characteristic of $K \neq 2$. Let $F/K$ be an elliptic function field. Then, $F = K(x, y)$ with $x, y \in F$ such that

$$y^2 = f(x) \in K[x]$$

with a square-free polynomial $f(x) \in K[x]$ of degree 3.

Consider the decomposition $f(x) = c \prod_{i=1}^r p_i(x)$ of $f(x)$ into monic irreducible polynomials $p_i(x) \in K[x]$ with $0 \neq c \in K$. Denote by $P_i \in \mathbb{P}_K(x)$ the place of $K(x)$ corresponding to $p_i(x)$, and by $P_\infty \in \mathbb{P}_K(x)$ the pole of $x$.

Let $n > 4$ an even positive integer. Let $R_1, \ldots, R_{\frac{n}{2}}$ be places of degree 1 of $K(x)$ such that

- $R_i \notin \{P_1, \ldots, P_r\}$ (1 $\leq i \leq \frac{n}{2}$)
- For each $i$, $R_i$ has exactly two extensions in $F$ say, $S_{i,1}$ and $S_{i,2}$.

Let $D = \sum_{i=1}^{\frac{n}{2}} S_{i,1} + S_{i,2}$. Let $g(x) \in K[x]$ such that $(g(x))_0 = D$ in $F$. Let $G$ be a divisor of $F$ of degree $\frac{n}{2}$. Let $\eta$ be a differential in $F$ defined by

$$\eta = \frac{g'(x)dx}{g(x)}.$$ 

Then, we get $\nu_P(\eta) = -1$ and $res_P(\eta) = 1$ for all $P \in supp(D)$. Therefore, $C_L(D, G)^\perp = C_L(D, D + (\eta) - G)$.

Now, $D + (\eta) = D + (g'(x)) + (dx) - (g(x)) = D + (g'(x)) - 4Q_\infty + Q_1 + \cdots + Q_r + Q_\infty + nQ_\infty - D = (g'(x)) + (n-3)Q_\infty + Q_1 + \cdots + Q_r$. Then the condition for self-duality of $C_L(D, G)$ is given by the following theorem.

Theorem 3.4. With all conditions as above, $C_L(D, G)$ is self-dual if and only if $(g'(x)) = 2G - (u) - (n-3)Q_\infty - Q_1 - \cdots - Q_r$ for some $u \in F$ such that $u(P) = 1$ for $P \in supp(D)$.

Proof. $C_L(D, G)$ is self-dual iff $C_L(D, G) = C_L(D, G)^\perp = C_L(D, D + (\eta) - G)$.

By \cite{3} corollary 4.15,

$$C_L(D, G) = C_L(D, D + (\eta) - G)$$

$\Leftrightarrow G = D + (\eta) - G + (u)$, for some $u \in F$ such that $u(P) = 1$ for each $P \in supp(D)$

$\Leftrightarrow (g'(x)) = 2G - (u) - (n-3)Q_\infty - Q_1 - \cdots - Q_r$
4. Self-duality of geometric Goppa codes over Hyperelliptic function field $F/K$ of genus 2 with char $K \neq 2$

Let $K = \mathbb{F}_q$, $q$ large enough and characteristic of $K \neq 2$. Let $F/K$ be a hyperelliptic function field of genus 2. Then there exist $x, y \in F$ such that $F = K(x, y)$ and

$$y^2 = f(x) \in K[x]$$

with a square-free polynomial $f(x)$ of degree 5. Then the places $P \in \mathbb{P}_{K(x)}$ corresponding to all zeroes of $f(x)$ and the pole of $x$ ramify in $F/K(x)$.

Let $n > 6$ be an even positive integer. Let $P_1, \ldots, P_r$ be zeros of $f(x)$ and $P_\infty$ pole of $x$. Therefore, $P_1, \ldots, P_r, P_\infty$ ramify in $F/K(x)$. Let $R_1, \ldots, R_{\frac{n}{2}}$ be places of degree 1 of $K(x)$ such that

- $R_i \notin \{P_1, \ldots, P_r\}, 1 \leq i \leq \frac{n}{2}$
- For each $i$, $R_i$ has exactly two extensions in $F$ say, $S_{i,1}$ and $S_{i,2}$.

Let $D = \sum_{i=1}^{\frac{n}{2}} S_{i,1} + S_{i,2}$. Let $g(x) \in K[x]$ such that $(g(x))_0 = D$ in $F$. Let $G$ be a divisor of $F$ of degree $\frac{n}{2} + 1$. Let $\eta$ be a differential in $F$ defined by

$$\eta = \frac{g'(x)dx}{g(x)}.$$ 

Then, we get $\nu_P(\eta) = -1$ and $res_P(\eta) = 1$ for all $P \in supp(D)$. Therefore, $C_\mathcal{L}(D, G)^\perp = C_\mathcal{L}(\overline{D}, \overline{D} + (\eta) - G)$.

Now, $D + (\eta) = D + (g'(x)) + (dx) - (g(x)) = D + (g'(x)) - 4Q_\infty + Q_1 + \cdots Q_r + Q_\infty + nQ_\infty - D = (g'(x)) + (n - 3)Q_\infty + Q_1 + \cdots Q_r$. Then the condition for self-duality of $C_\mathcal{L}(D, G)$ is given by the following theorem.

**Theorem 4.1.** With all conditions as above, $C_\mathcal{L}(D, G)$ is self-dual if and only if $(g'(x)) = 2G - (u) - (n - 3)Q_\infty - Q_1 - \cdots - Q_r$ for some $u \in F$ such that $u(P) = 1$ for $P \in supp(D)$.

**Proof.** $C_\mathcal{L}(D, G)$ is self-dual iff $C_\mathcal{L}(D, G) = C_\mathcal{L}(D, G)^\perp = C_\mathcal{L}(\overline{D}, \overline{D} + (\eta) - G)$.

By [3] corollary 4.15,

$$C_\mathcal{L}(D, G) = C_\mathcal{L}(D, D + (\eta) - G)$$

$\iff G = D + (\eta) - G + (u)$ for some $u \in F$ such that $u(P) = 1$ for each $P \in supp(D)$

$\iff (g'(x)) = 2G - (u) - (n - 3)Q_\infty - Q_1 - \cdots - Q_r$. 

$\square$

5. Concluding Remarks

In this note, we have investigated Goppa codes over Elliptic and Hyperelliptic function fields with base field of characteristic not equal to 2. We gave a simple criterion for self-duality of these codes.
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