$h^1 \neq h_1$ FOR ANDERSON T-MOTIVES

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Abstract. Let $M$ be an Anderson t-motive of dimension $n$ and rank $r$. It has two invariants $h^1(M), h_1(M)$, both are $\leq r$. There is a theorem (Anderson): $h^1(M) = r \iff h_1(M) = r$; in this case $M$ is called uniformizable. It is natural to expect that always $h^1(M) = h_1(M)$. Nevertheless, we explicitly construct a counterexample. Further, we answer a question of D. Goss: is it possible that two Anderson t-motives that differ only by a nilpotent operator $N$ are of different uniformizability type, i.e. one of them is uniformizable and other not? We give an explicit example that this is possible.

0. Statement of the problem.

Let $q$ be a power of a prime number $p$ and $\mathbb{F}_q$ the finite field of order $q$. The field $\mathbb{F}_q(\theta)$ is its field of rational functions, it is the functional field analog of $\mathbb{Q}$. There is a valuation $\text{ord}$ on $\mathbb{F}_q(\theta)$ defined by the condition $\text{ord} \theta = -1$.

The field of the Laurent series $\mathbb{F}_q((1/\theta))$ is the completion of $\mathbb{F}_q(\theta)$ in the topology defined by $\text{ord}$. It is the functional field analog of $\mathbb{R}$. Let $\mathbb{C}_\infty$ be the completion of the algebraic closure of $\mathbb{F}_q((1/\theta))$, it is the functional field analog of $\mathbb{C}$. By definition, $\mathbb{C}_\infty$ is complete. It is also algebraically closed ([G], Proposition 2.1).

Let $M$ be an Anderson t-motive of dimension $n$ and rank $r$ over $\mathbb{C}_\infty$ and $E$ the dual object - a T-module ([G]). We use notations $H^1(M)$ and $H_1(M)$ instead of $H_1(E)$ of [G], 5.9.11.3 (see below for the definitions). Both $H^1(M)$ and $H_1(M)$ are free $\mathbb{F}_q[T]$-modules. Their dimensions are denoted by $h^1(M), h_1(M)$ respectively. We have $h^1(M), h_1(M) \leq r$. There exists a pairing $\pi : H_1(M) \otimes_{\mathbb{F}_q[T]} H^1(M) \to \mathbb{F}_q[T]$. We have

Theorem 0.1. (Anderson, [A]; [G], 5.9.14). $h^1(M) = r \iff h_1(M) = r$. In this case $\pi$ is perfect over $\mathbb{F}_q[T]$.

The t-motives satisfying this condition are called uniformizable.

1991 Mathematics Subject Classification. 11G09.

Key words and phrases. Anderson t-motives; Uniformizability degree.

Thanks: The authors are grateful to FAPESP, São Paulo, Brazil for a financial support (process No. 2017/19777-6). The first author is grateful to SNPq, Brazil, to RFBR, Russia, grant 16-01-00577a (Secs. 1-4), and to Russian Science Foundation, project 16-11-10002 (Secs. 5-8) for a financial support.

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Unlike the standard notations, $H^1$ is a covariant functor in $M$ and $H_1$ is contravariant, because initially they were defined for a dual category of T-modules.
We have a natural question. Let $M$ be any t-motive.

**Question 0.2.** (a) Is always $h_1(M) = h_1(M)$?

(b) If $h_1(M) = h_1(M)$, what is the type of $\pi$? It can be either perfect over $\mathbb{F}_q[T]$, or perfect only over $\mathbb{F}_q(T)$, or non-perfect.

(c) If $\pi$ is not perfect, what is its possible rank? Can it take all possible values from 0 to $\min(h_1(M), h_1(M))$, or not? Maybe it is always equal to $\min(h_1(M), h_1(M))$? Maybe it is never 0 (if $h_1(M) \neq 0$)?

It is natural to expect that the answer to (a) is yes. Nevertheless, we construct a counterexample (Theorems 4.1, 4.7). Also, we give an answer to a question of D. Goss (Theorem 5.1).

**Structure of the paper.** In Section 1 we give definitions and elementary properties of Anderson t-motives and their $H^1, H_1$. In Section 2 we define affine equations — the main computational tool to calculate $h^1$ of t-motives. In Section 3 we show how to reduce a problem of finding of $h^1(M)$ to a solution of an affine equation. In Section 4 we apply this method to an explicitly defined t-motive to get a counterexample to (0.2a). In Section 5 we consider an example of a t-motive giving answer to a question of D. Goss. In Appendix we consider calculations of Section 3 for some other types of t-motives, in order to confirm Conjecture 2.13.

**Further research.** We want to answer Question 0.2c. Namely, we can consider more examples of Anderson t-motives $M$ given by the formula (1.11), $n = 2$, $N = 0$. We can find $h^1, h_1$ for (most of) them. For $M$ such that $0 < h^1(M) \neq h_1(M) < 4$ we can find explicitly the rank of the pairing $\pi$ and to look what is its value: $\min(h_1(M), h_1(M))$; $\neq 0$; or not.

Further, we give in Section 5 an example of $M_0, M_1$ given by the formula (1.11) such that the matrices $A$ for $M_0$, $M_1$ coincide, $N$ for $M_0$ is 0 while $N$ for $M_1$ is not 0, and such that $M_0$ is non-uniformizable, while $M_1$ is uniformizable. Are there examples $M$ such that the situation is inverse, i.e. $M_0$ is uniformizable and $M_1$ is non-uniformizable?

Finally, we can try to find $h^1, h_1$ for all $M$ described by (1.11), first for $n = 2$. This problem is also related to a problem whether the lattice map of pure uniformizable t-motives is an isomorphism (or near-isomorphism), or not. The first step to a solution of this problem was made in [GL].

1. **Definitions.** Let $\mathbb{C}_\infty[T, \tau]$ be the Anderson ring, i.e. the ring of non-commutative polynomials in two variables $T, \tau$ over $\mathbb{C}_\infty$ satisfying the following relations (here $a \in \mathbb{C}_\infty$):

$$Ta = aT, \quad T\tau = \tau T, \quad \tau a = a^q \tau$$

(1.1)

Subrings of $\mathbb{C}_\infty[T, \tau]$ generated by $\tau$, resp. $T$ are denoted by $\mathbb{C}_\infty\{\tau\}$ (the ring of non-commutative polynomials in one variable), resp. $\mathbb{C}_\infty[T]$ (the ordinary ring of (commutative) polynomials in one variable).

**Definition 1.2.** ([G], 5.4.2, 5.4.18, 5.4.16). A t-motive$^3$ $M$ is a left $\mathbb{C}_\infty[T, \tau]$-module which is free and finitely generated as both $\mathbb{C}_\infty[T]$-, $\mathbb{C}_\infty\{\tau\}$-module and such that

$$\exists m = m(M) \text{ such that } (T - \theta)^m M/\tau M = 0$$

(1.2.1)

$^3$Terminology of Anderson; Goss calls these objects abelian t-motives.
The dimension of \( M \) over \( \mathbb{C}_\infty\{\tau\} \) (resp. \( \mathbb{C}_\infty[T] \)) is denoted by \( n \) (resp. \( r \)), these numbers are called the dimension and rank of \( M \).

We can consider \( M \) as a \( \mathbb{C}_\infty[T] \)-module with the action of \( \tau \). Hence, sometimes for \( x \in M \) we shall write \( \tau(x) \) (action) instead of \( \tau x \) (multiplication).

We shall need the explicit matrix description of \( t \)-motives. First, let \( e_* = (e_1, \ldots, e_n)^t \) be the vector column of elements of a basis of \( M \) over \( \mathbb{C}_\infty\{\tau\} \). There exists a matrix \( \mathfrak{A} \in M_n(\mathbb{C}_\infty\{\tau\}) \) such that

\[
Te_* = \mathfrak{A}e_*; \quad \mathfrak{A} = \sum_{i=0}^l \mathfrak{A}_i \tau^i \text{ where } \mathfrak{A}_i \in M_n(\mathbb{C}_\infty) \tag{1.3}
\]

Condition (1.2.1) is equivalent to the condition

\[
\mathfrak{A}_0 = \theta I_n + N \tag{1.3.1}
\]

where \( N \) is a nilpotent matrix, and the condition \( \{\mathfrak{m}(M)\} \) can be taken to \( 1 \) is equivalent to the condition \( N = 0 \).

Second, let \( f_* = (f_1, \ldots, f_r)^t \) be the vector column of elements of a basis of \( M \) over \( \mathbb{C}_\infty[T] \). There exists a matrix \( Q \in M_r(\mathbb{C}_\infty[T]) \) such that

\[
Qf_* = \tau f_* \tag{1.4}
\]

We use some definitions of [G], 5.9.10. First, we denote by \( \mathbb{C}_\infty\{T\} \) a subring of \( \mathbb{C}_\infty[[T]] \) formed by series \( \sum_{i=0}^\infty a_i T^i \) such that \( \lim_{i \to 0} a_i = 0 \) (\( \iff \) \( \text{ord } a_i \to +\infty \)). \( \tau \) acts on \( \mathbb{C}_\infty\{T\} \) by the formula \( \tau(\sum_{i=0}^\infty a_i T^i) = \sum_{i=0}^\infty a_i \tau^i \).

**Notation.** For \( z = \sum_{i=0}^\infty \lambda_i T^i \in \mathbb{C}_\infty[[T]] \) we denote \( z^{(k)} := \sum_{i=0}^\infty \lambda_i^k T^i = \tau^k(z) \).

Now, we define

\[
M[[T]] := M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty[[T]], \quad M\{T\} := M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty\{T\} \tag{[G], 5.9.11.1}
\]

We have: \( \tau \) acts on both \( M\{T\}, M[[T]] \) by the standard formula of the action of an operator on tensor product (see [G], 5.9.11.1).

**Definition 1.5.** \( H^1(M) = M\{T\}^\tau = M[[T]]^\tau \cap M\{T\} \) (G), 5.9.11.2.

It is a free \( \mathbb{F}_q[T] \)-module.

Using the above basis \( f_* \), we can identify matrices row \( Y \in M_{1 \times r}(\mathbb{C}_\infty\{T\}) = \mathbb{C}_\infty\{T\}^r \) with elements of \( M\{T\} \): \( Y \mapsto Y \cdot f_* \in M\{T\} \) where \( Y \cdot f_* \) is the product of \( 1 \times r \) and \( r \times 1 \) matrices. Under this identification, (1.4) gives us immediately

\[
Y \in H^1(M) \iff Y^{(1)} Q = Y \tag{1.6}
\]

(1.5) can be considered as \( H^1(M) = (M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty\{T\})^\tau \). Analogously, we have (follows immediately from [G], 5.9.25)

\[
H_1(M) = (\text{Hom}_{\mathbb{C}_\infty[T]}(M, \mathbb{C}_\infty\{T\}))^\tau
\]
(again \(\tau\) acts on Hom by the standard manner). We can identify matrices column \(X \in \mathbb{C}_\infty \{T\}^r\) with elements of \(\text{Hom}_{\mathbb{C}_\infty \{T\}}(M, \mathbb{C}_\infty \{T\})\): if \(\varphi : M \to \mathbb{C}_\infty \{T\}\) is a map then the corresponding \(X := \begin{pmatrix} \varphi(f_1) \\ \vdots \\ \varphi(f_r) \end{pmatrix}\). Under this identification, we have immediately

\[
X \in H_1(M) \iff QX = X^{(1)} \tag{1.7}
\]

Now we shall assume that \(M\) is such that \(Q \in \text{GL}_r(\mathbb{C}_\infty \{T\})\) (this assumption holds for all pure \(M\), all \(M\) described in \([L07]\), Definitions 11.2, 11.3, and probably for all \(M\)). In this case the pairing \(\pi\) is defined as follows:

\[\pi(X, Y) = YX\]

Really, we have \((YX)^{(1)} = YQ^{-1}QX = YX\), i.e. \(YX \in \mathbb{F}_q[T]\).

By definition, the dual t-motive \(M'\) (see \([L07]\), 1.8 for a definition, 1.10.1 for an explicit formula) is defined in terms of its \(Q\)-matrix by the equation

\[Q(M') = (T - \theta)Q^{-1}\]

(the dual t-motive has nothing common with \(E(M)\) - the \(T\)-module associated to \(M\); not all \(M\) have dual).

**Proposition 1.8.** If the dual t-motive \(M'\) exists then there exists an isomorphism

\[H_1(M) \to H_1(M') \tag{1.9}\]

and hence \(H_1(M) \to H_1(M')\), because \((M')' = M\).

**Proof.** We identify \(H_1\), resp. \(H_1\) with \(Y, X\) as above. Let \(\Xi \in \mathbb{C}_\infty \{T\}\) be from \([G]\), Example 5.9.36, p. 172; it is a simplest solution to the equation

\[\Xi = (T - \theta)\Xi^{(1)}\]

it is defined uniquely up to a factor from \(\mathbb{F}_q^*\). We have \(\Xi^{-1} \in \mathbb{C}_\infty \{T\}\). Hence, we get: \(Y\) is a root to (1.6) for \(M \iff \Xi^{-1}Y^t\) is a root to (1.7) for \(M'\). This gives a formula for the map (1.9) in coordinates: \(Y \mapsto \Xi^{-1}Y^t\). \(\square\)

1.10. We shall find a counterexample to (0.2a) among t-motives defined by the equation (1.3) such that \(l = 2, \mathfrak{A}_2 = I_n\). We denote \(\mathfrak{A}_1\) by \(A\), hence (1.3) has the form

\[Te_* = (\theta I_n + N)e_* + A\tau e_* + \tau^2 e_* \tag{1.11}\]

This t-motive is denoted by \(M(A) = M(A, N)\). The rank of \(M(A)\) is \(2n\), a basis \(f_*\) can be chosen as

\[
\begin{pmatrix}
  e_1 \\
  \vdots \\
  \tau(e_1) \\
  \vdots \\
  \tau(e_n)
\end{pmatrix}
\]

The matrix \(Q\) of \(M(A)\) in this basis is

\[
\begin{pmatrix}
  0 & I_n \\
  (T - \theta)I_n - N & -A
\end{pmatrix}
\]

(entries are \(n \times n\)-blocks).
If \( N = 0 \) then \( (M(A)) = M(-A) \). Since t-motives \( M(A), M(-A) \) are isomorphic, we get \( (M(A))^t = M(A^t) \). Hence, in order to prove that not always \( h^1(M) = h_1(M) \), it is sufficient to find a matrix \( A \) such that \( h^1(M(A)) \neq h^1(M(A^t)) \).

2. Affine equations.

After elimination of some unknowns the equations (1.6), (1.7) can be transformed to the below equation (2.1). Let us give some general definitions and elementary results concerning such equations.

Let \( r, n \geq 1, x_1, \ldots, x_n \geq 0, a_\gamma \in \mathbb{C}_\infty \) for \( \gamma = 0, \ldots, r, b_\beta \gamma \in \mathbb{C}_\infty \) for \( \beta = 1, \ldots, n, \gamma = 0, \ldots, x_\beta \) are coefficients, \( x_0, x_1, x_2, \ldots \in \mathbb{C}_\infty \) are unknowns. An \( i \)-th affine equation \( (i = 0, 1, 2, \ldots) \) has the form (here \( x_j = 0 \) for \( j < 0 \)):

\[
\sum_{\gamma=0}^r a_\gamma x_i^{\gamma} + \sum_{\beta=1}^n b_\beta \gamma x_i^{\gamma} = 0
\]  

(2.1)

We claim \( a_0 \neq 0 \) (separability), \( a_r \neq 0 \), \( b_\beta, x_\beta \neq 0 \). The set of terms \( a_\gamma x_i^{\gamma} \) is called the head of the equation, the set of other terms is called the tail of the equation. We can assume \( a_r = 1 \), and a substitution \( x_i \rightarrow \lambda x_i \) gives a change of coefficients \( a_\gamma \rightarrow \lambda^{-\gamma} a_\gamma, b_\beta \rightarrow \gamma^{-\gamma} b_\beta \). We can also assume \( \forall \beta < x_\beta < r \).

The system (2.1) is solved consecutively: for \( i = 0 \) the tail is 0, the set of \( x_0 \) is a \( \mathbb{F}_q \)-vector subspace of \( \mathbb{C}_\infty \) of dimension \( r \), denoted by \( S_0 \). Let \( x_0, x_1, \ldots, x_\alpha \) be a solution to (2.1) for \( i = 0, 1, \ldots, \alpha \). For \( i = \alpha + 1 \) the equation (2.1) has the form

\[
\sum_{\gamma=0}^r a_\gamma x^{\alpha+1} + \mathbb{W} = 0
\]  

(2.2)

where \( W = W(\alpha; x_{\alpha+1-n}, \ldots, x_\alpha) \in \mathbb{C}_\infty \) is obtained by substitution of \( x_{\alpha+1-n}, \ldots, x_\alpha \) to the tail members. The set of solutions to (2.2) (for \( x_0, x_1, \ldots, x_\alpha \) fixed) is an affine space over \( \mathbb{F}_q \), with the base vector space \( S_0 \) (this explains the terminology).

Let \( x_0, x_1, \ldots \) be a solution to (2.1). We associate it an element \( \{x\} := \sum_{i=0}^\infty x_i T^i \in \mathbb{C}_\infty[[T]] \) which (by abuse of language) will be also called a solution to (2.1).

\( \mathbb{C}_\infty[[T]] \) is a \( \mathbb{C}_\infty[T, \tau] \)-module. We can consider the multiplication by elements of \( \mathbb{C}_\infty[T, \tau] \) as an action of an operator on \( \mathbb{C}_\infty[[T]] \). From this point of view, we can consider (2.1) as an equation (here \( P \in \mathbb{C}_\infty[T, \tau] \)):

\[
P(\{x\}) = 0
\]  

(2.2a)

where \( \{x\} \) is as above and \( P := \sum_{\gamma=0}^r a_\gamma \tau^{\gamma} + \sum_{\beta=1}^n b_{\beta} \tau^{\gamma} \).  

**Proposition 2.3.** The set of solutions to (2.1) in \( \mathbb{C}_\infty[[T]] \) is a \( \mathbb{F}_q[[T]] \)-vector space ( = free module) of dimension \( r \). Moreover, let \( \{x_1\}, \ldots, \{x_r\} \) be solutions to (2.1). We denote \( \{x_i\} = \{x_{i0}, x_{i1}, \ldots\} \). Then \( \{x_1\}, \ldots, \{x_r\} \) is a basis of the set of solutions to (2.1) over \( \mathbb{F}_q[[T]] \) if \( x_{10}, \ldots, x_{r0} \) is a basis of \( S_0 \) over \( \mathbb{F}_q \). □

The solutions to (2.1) belonging to \( \mathbb{C}_\infty\{T\} \) are called small solutions. They form a vector space over \( \mathbb{F}_q[T] \). Its dimension is called the dimension of (2.1).
The below propositions are not necessary for the proof of Theorems 4.1, 4.7, 5.1. They are given for completeness and for possible future applications.

**Definition 2.4.** Let \( \{x\} = (x_0, x_1, \ldots) \) be a solution to (2.1). It is called simple (or of simple type) if for all \( i_0 \) we have: ord’s of all tail members of the equation (2.1), \( i = i_0 \) for this \( \{x\} \) (i.e. obtained while we substitute \( x_0, x_1, \ldots, x_{i_0-1} \)) are different. An equation (2.1) is called simple if all its solutions are simple.

Particularly, if the tail contains one term then the equation is trivially simple.

For simple equations we can easily find ord \( x_i \). Really, we find all possible ord \( x_0 \) treating the Newton polygon of the head of (2.1) for \( i = 0 \). To pass from \( i \) to \( i + 1 \), we get that ord \( W \) (where \( W \) is from (2.2)) is the minimum of the ord’s of the tail terms. Again using the Newton polygon of the head of (2.1) and ord \( W \), we get ord’s of all possible \( x_{i+1} \).

We see that the simplicity of (2.1) depends only on ord’s of \( a_{\gamma}, b_{\beta+} \). They belong to \( \mathbb{Q} \cup \mathbb{C} \); for any \( i \) the condition of non-simplicity imposes linear relations on ord’s of \( a_{\gamma}, b_{\beta+} \). We conjecture that for given \( r, n \) there are only finitely many such relations, i.e. "almost all" equations are simple.

Let \( x_0 \) be a fixed solution to (2.1) for \( i = 0 \).

**Definition 2.5.** A solution \( \{x\} = (x_0, x_1, x_2, \ldots) \) to (2.1) is called a minimal chain generated by \( x_0 \) if it satisfies the following condition: \( \forall i_0 > 0 \) we have: \( x_{i_0} \) is a solution to ((2.1), \( i = i_0 \)) corresponding to the leftmost segment of the Newton polygon of ((2.1), \( i = i_0 \)) for \( x_0, x_1, x_2, \ldots, x_{i_0-1} \) considered as parameters of ((2.1), \( i = i_0 \)), i.e. ord \( x_{i_0} \) has the maximal possible value amongst ord’s of solutions to ((2.1), \( i = i_0 \)) for fixed \( x_0, x_1, x_2, \ldots, x_{i_0-1} \).

A minimal chain generated by \( x_0 \) can be either simple or not. If \( x_0 \) is fixed then a minimal chain generated by \( x_0 \) is not unique even if (2.1) is simple. But if there exists a simple minimal chain generated by \( x_0 \) (where (2.1) can be simple or not) then all minimal chains generated by \( x_0 \) are simple, and the sequence ord \( x_1, \text{ord} x_2, \ldots \) is uniquely defined by ord \( x_0 \).

**Proposition 2.6.** Let \( x_0 \in S_0 \) be such that its minimal chain \( \{x\} = x_0, x_1, x_2, \ldots \) is simple. Let \( \{y\} = y_0, y_1, y_2, \ldots \) be another simple solution to (2.1). Then ord \( y_0 \leq \text{ord} x_0 \) (resp. ord \( y_0 < \text{ord} x_0 \)) implies: \( \forall i \) we have: ord \( y_i \leq \text{ord} x_i \) (resp. ord \( y_i < \text{ord} x_i \)).

**Proof.** Immediate, by induction. We consider the case ord \( y_0 \leq \text{ord} x_0 \) (for the case ord \( y_0 < \text{ord} x_0 \) the proof is the same). Let this proposition be true for some \( i \). We consider the equation (2.2) for \( i + 1 \). We denote by \( W_x \), resp. \( W_y \) the term \( W \) in (2.2) for the sets \( x_0, x_1, x_2, \ldots, x_i \), resp. \( y_0, y_1, y_2, \ldots, y_i \). Simplicity of \( \{x\}, \{y\} \) implies that ord \( W_x \), ord \( W_y \) = minimum of the ord’s of the corresponding terms of the tail. Hence, because ord \( y_j \leq \text{ord} x_j \) for \( j = 1, \ldots, i \), we have ord \( W_y \leq \text{ord} W_x \), i.e. the leftmost vertex of the Newton polygon for (2.2) for \( y_0, y_1, y_2, \ldots, y_i \) is below or equal to the leftmost point of the Newton polygon for (2.2) for \( x_0, x_1, x_2, \ldots, x_i \). These two Newton polygons are the convex hulls of these points having \( x \)-coordinate 0, and other points corresponding to the head of (2.1) which are the same for \( \{x\}, \{y\} \). This means that the inclination of the leftmost segment of the Newton polygon ( = \(-\text{ord of the root}\) for (2.2) for \( y_0, y_1, y_2, \ldots, y_i \) (denoted by inc\(_{i+1}(y)\)
is $\geq$ of the inclination of the leftmost segment of the Newton polygon for (2.2) for $x_0, x_1, x_2, \ldots, x_i$ (denoted by $\text{inc}_{i+1}(x)$).

Inclinations of other sides of the Newton polygon for (2.2) for $y_0, y_1, y_2, \ldots, y_i$ are $\geq \text{inc}_{i+1}(y)$, hence $\geq \text{inc}_{i+1}(x)$. This means that $\text{ord } y_{i+1} \leq \text{ord } x_{i+1}$. □

**Proposition 2.6a.** Let $\{x\}$ be as above, and let $\{y\} = y_0, y_1, y_2, \ldots$ be a minimal chain of $y_0$. Then $\text{ord } y_0 \geq \text{ord } x_0$ (resp. $\text{ord } y_0 > \text{ord } x_0$) implies: $\forall i$ we have: $\text{ord } y_i \geq \text{ord } x_i$ (resp. $\text{ord } y_i > \text{ord } x_i$).

**Proof.** By induction, similar to the proof of Proposition 2.6. □

**Corollary 2.7.** Let (2.1) be simple. If for minimal chains for all $x_0$ we have $\lim_{i \to \infty} \text{ord } x_i \neq +\infty$ then the dimension of (2.1) is 0.

**Conjecture 2.9.** Let (2.1) be simple, and let $x_{10}, x_{20}, \ldots, x_{r0}$ be a $\mathbb{F}_q$-basis of $S_0$. The dimension of (2.1) is equal to the quantity of $\alpha$ such that the minimal chain of $x_{\alpha 0}$ is a small solution.

**Idea of the proof.** We can assume that $\text{ord } x_{10} \geq \text{ord } x_{20} \geq \cdots \geq \text{ord } x_{r0}$. Let $k$ be maximal number such that a minimal chain of $x_{k0}$ (denoted by $\{x_k\}$) is a small solution. According Proposition 2.6, we have that a minimal chain of $x_{\alpha 0}$ is a small solution iff $\alpha \leq k$. We must prove that minimal chains of $x_{10}, x_{20}, \ldots, x_{k0}$ form a $\mathbb{F}_q[T]$-basis of the set of small solutions. They are $\mathbb{F}_q[[T]]$- and hence $\mathbb{F}_q[T]$-linearly independent. Let $\{y\} = (y_0, y_1, \ldots)$ be a small solution. We apply Proposition 2.6 for $\{x\} = \{x_{k+1}\}$; it gives us that $y_0$ is a $\mathbb{F}_q$-linear combination of $x_{10}, x_{20}, \ldots, x_{k0}$, i.e. $y_0 = \sum_{i=1}^{k} c_i x_i$ where $c_i \in \mathbb{F}_q$ are coefficients. We can consider $\{y\}(1) := (\{y\} - \sum_{i=1}^{k} c_i \{x_i\})/T$ which is also a small solution. Applying the same operation to $\{y\}(1)$ we get $\{y\}(2)$ etc. As a result, we get that $\{y\}$ is a $\mathbb{F}_q[[T]]$-linear combination of $x_{10}, x_{20}, \ldots, x_{k0}$. We need to show that $\{y\}$ is a $\mathbb{F}_q[T]$-linear combination of $x_{10}, x_{20}, \ldots, x_{k0}$. This is an exercise for a student; we need this fact only for a case $k = 1$, and $\forall i$ ord $x_{1,i+1} > \text{ord } x_{1i}$ where it is obvious (see proof of Lemma 4.6).

There is a result for equations whose tail consists of one term:

**2.11.** Let the only tail term be $b_{1k} x_i^{q^k}$ for some fixed $k$, and let $a_r = 1$.

We denote $\alpha_i := \text{ord } a_i$, $\beta := \text{ord } b_{1k}$.

**Proposition 2.12.** Let 2.11 hold, and let $i$ be $x$-coordinate of the right end of the leftmost segment of the Newton polygon of the head of (2.1). Then

$$\frac{\alpha_0 - \alpha_i}{q^i - 1} \leq \frac{\alpha_0 - \beta}{q^k - 1} \quad (2.12.1)$$

$$\iff$$ the dimension of (2.1) is 0.

**Proof.** We denote $y_0 := \text{ord } x_{10}$ where $x_{10}$ be a root to (2.1), $i = 0$ corresponding to the leftmost segment of the Newton polygon of the head of (2.1). We have $y_0 = \frac{\alpha_0 - \alpha}{q^i - 1}$. Let us consider the equation (2.1), $i = 1$ for this value of $x_{10}$. Ord of its free term is $\beta + q^k y_0$. The negation of condition (2.12.1) is equivalent to $\beta + q^k y_0 - \alpha_0 > y_0$. Hence, if (2.12.1) does not hold then the leftmost segment of the Newton polygon of (2.1), $i = 1$ is the segment $(0, \beta + q^k y_0); (1, y_0)$. 7
Let $x_{11}$ be the root to to (2.1), $i=1$ corresponding to this segment. We denote $y_1 := \text{ord} x_{11} = \beta + q^k y_0 - \alpha_0$. We have $y_1 > y_0$. Hence, for $i=2$ the leftmost segment of the Newton polygon of (2.1), $i=2$ is the segment $(0, \beta + q^k y_1); (1, y_0)$.

Continuing the process of finding the minimal chain corresponding to $x_{10}$ we get a solution $\sum_{j=0}^{\infty} x_{1j} T^j$. We denote $y_\gamma := \text{ord} x_{1\gamma}$, they satisfy a recurrent relation $y_{\gamma+1} = \beta + q^k y_\gamma - \alpha_0$. A formula for $y_\gamma$ is

$$y_\gamma = \frac{\alpha_0 - \beta}{q^k - 1} + \left(\frac{\alpha_0 - \alpha_i}{q^i - 1} - \frac{\alpha_0 - \beta}{q^k - 1}\right) q^{k \gamma}$$

it is proved immediately by induction. We get that if (2.12.1) does not hold then the dimension of (2.1) is $>0$.

Let us assume that (2.12.1) holds. In this case $\beta + q^k y_0 - \alpha_0 \leq y_0$, hence $y_1 \leq y_0$. We get by induction that $\forall \gamma$ we have $y_\gamma \leq y_0$, hence the minimal chain generated by $x_{10}$ is not small. Proposition 2.6 implies that the dimension of (2.1) is 0. $
$

Now we can formulate

**Conjecture 2.13.** Let $M$ be a t-motive of rank $r$ and dimension $n$. Equations (1.6), (1.7) with vector unknowns $Y$, $X$ respectively, after eliminations of some entries of $Y$, $X$, can be transformed to an equation of type (2.2a) having the same $r$, $n$.

Before proving this conjecture, it is necessary to formalize its statement (what is an elimination?) and to prove that $n$ is an invariant of (2.1), i.e. the set of solutions to (2.1) defines $n$ uniquely. Further, we have: $r$ of (2.1) is the dimension over $\mathbb{F}_q[[T]]$ of the set of all solutions to (2.1), which is equal to the dimension over $\mathbb{F}_q[[T]]$ of $M[[T]]$ which is $r$ of $M$. Justification for coincidence of two $n$’s for $r=4$, resp. 5, and $n=2$ is given by equations (3.8) - (3.10), resp. (A2.1). See also Remark A3.

### 3. Affine equation corresponding to a t-motive.

Let $n=2$. We consider t-motives given by (1.11), where either $N = 0$ or $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e. $N = \varepsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ where $\varepsilon = 0$ or 1. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

To find $h^1(M(A))$ we use formula (1.6). More exactly, we eliminate 3 unknowns in (1.6), we get an affine equation and find its dimension. Namely, let $Y = (y_{11}, y_{12}, y_{21}, y_{22})$ be from (1.6) for $M(A)$. We denote $Y = (y_1, y_2)$ as a block matrix where $y_1 = (y_{11}, y_{12}), y_2 = (y_{21}, y_{22})$. (1.6) written in a 2-block form is\footnote{We give here all steps of these elementary calculations in order to simplify verification.}

$$\begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \end{pmatrix} \begin{pmatrix} 0 \\ (T - \theta) I_2 - \varepsilon N - A \end{pmatrix} = (y_1, y_2), \text{ i.e.}$$

$$y_2^{(1)} ((T - \theta) I_2 - \varepsilon N) = y_1, \quad y_1^{(1)} - y_2^{(1)} A = y_2$$

Hence,

$$y_2 = y_2^{(2)} ((T - \theta^q) I_2 - \varepsilon N) - y_2^{(1)} A$$

Substituting $y_2 = (y_{21}, y_{22})$ to (3.1) we get

$$y_{21} = y_{21}^{(2)} (T - \theta^q) - y_{21}^{(1)} a_{11} - y_{22}^{(1)} a_{21}$$
\[ y_{22} = -\varepsilon y_{21}^{(2)} + y_{22}^{(2)}(T - \theta^q) - y_{21}^{(1)}a_{12} - y_{22}^{(1)}a_{22} \] (3.3)

Now we eliminate \( y_{22} \) from (3.2), (3.3). Assuming \( a_{21} \neq 0 \) we get from (3.2):

\[ y_{22}^{(1)} = -\frac{1}{a_{21}}y_{21} + \frac{T - \theta^q}{a_{21}}y_{22}^{(2)} - \frac{a_{11}}{a_{21}}y_{21}^{(1)} \] (3.4)

and hence

\[ y_{22}^{(2)} = -\frac{1}{a_{21}}y_{21}^{(1)} + \frac{T - \theta^q}{a_{21}}y_{22}^{(3)} - \frac{a_{11}}{a_{21}}y_{21}^{(2)} \] (3.5)

\[ y_{22}^{(3)} = -\frac{1}{a_{21}}y_{21}^{(2)} + \frac{T - \theta^q}{a_{21}}y_{22}^{(4)} - \frac{a_{11}}{a_{21}}y_{21}^{(3)} \] (3.6)

From (3.3) we get

\[ y_{22}^{(1)} = -\varepsilon y_{21}^{(3)} + y_{22}^{(3)}(T - \theta^q) - y_{21}^{(2)}a_{12} - y_{22}^{(2)}a_{22} \] (3.7)

Substituting (3.4) - (3.6) to (3.7) we get

\[ \frac{(T - \theta^q)(T - \theta^q)}{a_{21}^2}y_{21}^{(4)} + [(-\frac{a_{11}^q}{a_{21}^2} - \frac{a_{12}^q}{a_{21}^2})(T - \theta^q) - \varepsilon]y_{21}^{(3)} + \]

\[ \frac{T - \theta^q}{a_{21}} - \frac{T - \theta^q}{a_{21}^2} + \frac{a_{11}a_{22}}{a_{21}^2} - \frac{a_{12}}{a_{21}}]y_{21}^{(2)} + (\frac{a_{11}}{a_{21}} + \frac{a_{12}}{a_{21}})y_{21}^{(1)} + \frac{1}{a_{21}}y_{21} = 0 \] (3.8)

We denote \( y_{21} \) by \( x = \sum_{i=0}^{\infty} x_i T^i \) where \( x_i \in \mathbb{C}_\infty \). Substituting this formula to (3.8) we get an affine equation of type (2.1) having \( r = 4, n = 2, \kappa_1 = \kappa_2 = 4 \), and \( a_\gamma, b_\beta \) are the following:

\[
\begin{align*}
    a_4 &= \frac{\theta q^3 + q^2}{a_{21}^2}; & a_3 &= \frac{a_{11} q^2 \theta q^2}{a_{21}^2} + \frac{a_{22} \theta q^2}{a_{21}^2} - \varepsilon; & a_2 &= \frac{\theta q^2}{a_{21}} + \frac{\theta q^2}{a_{21}^2} + \frac{a_{11} a_{22}}{a_{21}^2} - \frac{a_{12}}{a_{21}}; \\
    a_1 &= \frac{a_{11}}{a_{21}} + \frac{a_{22}}{a_{21}}; & a_0 &= \frac{1}{a_{21}}; \\
    b_{14} &= -\frac{\theta q^3 + q^2}{a_{21}^2}; & b_{13} &= \frac{a_{11}}{a_{21}^2} + \frac{a_{22}}{a_{21}^2}; & b_{12} &= \frac{1}{a_{21}} + \frac{1}{a_{21}^2}; & b_{24} &= \frac{1}{a_{21}^2}.
\end{align*}
\] (3.9)

Hence, the equation (2.1) has the form (we do not want to get \( \forall \beta \kappa_\beta < r \))

\[
\frac{\theta q^3 + q^2}{a_{21}^2} x_i^4 + [\frac{a_{11}^q}{a_{21}^2} + \frac{a_{22}^q}{a_{21}^2}] \theta q^2 - \varepsilon] x_i^3 + \left( \frac{\theta q}{a_{21}} + \frac{\theta q}{a_{21}^2} + \frac{a_{11} a_{22}}{a_{21}^2} - \frac{a_{12}}{a_{21}} \right) x_i^2 + \left( \frac{a_{11}}{a_{21}} + \frac{a_{22}}{a_{21}^2} \right) x_i + \frac{1}{a_{21}} x_i
\]

\[\frac{\theta q^3 + q^2}{a_{21}^2} x_{i-1}^4 - \frac{a_{11}}{a_{21}^2} x_{i-1}^3 + \frac{1}{a_{21}} x_{i-1}^2 + \frac{1}{a_{21}^2} x_{i-2}^4 = 0 \] (3.10)

**Remark.** There exists another form to write (3.10):

\[
[\theta q^2 \tau^2 + a_{22}^q \tau + 1 - \tau^2 T] \left( \frac{1}{a_{21}} (\theta q^2 \tau^2 + a_{11} \tau + 1 - \tau^2 T) \right) (x) = a_{12}^q \tau^2 (x)
\]
which is much more "agreeable" than the form (3.10). We do not know how to apply this form and what is its generalization to the cases $n > 2$.

4. Not always $h^1(M) = h_1(M)$.

Let us consider the case $q = 2$, $n = 2$, $\varepsilon = 0$. We fix the following matrix $A = \left(\begin{array}{cc} \theta & \theta^6 \\ \theta^2 & 0 \end{array}\right)$. The below calculations show that $h_1(M(A)) = 1$, $h^1(M(A)) = 0$.

**Theorem 4.1.** For the above $A$ we have $h^1(M(A)) = 0$.

**Proof.** For this $A$ numbers $a_i$ of (3.9) are:

$$a_4 = \theta^{20}, \quad a_3 = \theta^{16}, \quad a_2 = \theta^4, \quad a_1 = \theta^3, \quad a_0 = \theta^2$$

$$\text{ord } a_4 = -20, \quad \text{ord } a_3 = -16, \quad \text{ord } a_2 = -4, \quad \text{ord } a_1 = -3, \quad \text{ord } a_0 = -2$$

Later "equation (3.10)" will mean the equation (3.10) with these values of $a_i$ (and also values of $b_{ij}$ coming from $A$, see (4.4.1), (4.4.2) below). The Newton polygon for (3.10), $i = 0$ has vertices $(1, -2); (8, -16); (16, -20)$. We denote elements of a $\mathbb{F}_2$-basis of $S_0$ by $x_j, j = 1, 2, 3, 4$, and solutions to (3.10) over them by $\{x_j\} = \sum_{j=0}^{\infty} x_j T^j$. We have $\text{ord } x_j = 2$, for $j = 1, 2, 3$ and $\text{ord } x_4 = \frac{1}{2}$.

The equation (3.10) is not simple (see below), hence we need one more term for $x_4$.

**Lemma 4.2.** $x_4 = \theta^{-\frac{1}{2}} + \theta^{-\frac{29}{16}} + \Delta_{41}$ where $\text{ord } \Delta_{41} > \frac{29}{16}$ (the value of $\Delta_{41}$ depends on the value of $x_4$; all of them have $\text{ord } > \frac{29}{16}$).

**Proof.** First, we let

$$x_4 = \theta^{-\frac{1}{2}} + \Delta_{40} \quad (4.2.1)$$

where $\Delta_{40}$ is a new unknown. Substituting (4.2.1) to (3.10), $i = 0$, we get

$$\sum_{j=0}^{4} a_j \Delta_{40}^{2j} + \sum_{j=0}^{4} a_j (\theta^{-\frac{1}{2}})^{2j} = 0 \quad (4.2.2)$$

We have

$$\sum_{j=0}^{4} a_j (\theta^{-\frac{1}{2}})^{2j} = \theta^{20}\theta^{-8} + \theta^{16}\theta^{-4} + \theta^4\theta^{-2} + \theta^3\theta^{-1} + \theta^2\theta^{-\frac{1}{2}} = \theta^\frac{3}{2}$$

Hence, the Newton polygon of (4.2.2) has vertices $(0, -\frac{3}{2}); (8, -16); (16, -20)$ and $\text{ord } \Delta_{40} = \frac{29}{16}$ (8 values), $\frac{1}{2}$ (8 values). There exist values of $\Delta_{40}$ such that $\theta^{-\frac{1}{2}} + \Delta_{40} = \sum_{j=1}^{3} c_j x_j$ where $c_j \in \mathbb{F}_2$. For these $\Delta_{40}$ we have that their ord's are $\frac{1}{2}$, exactly 8 values. Values of $\Delta_{40}$ such that $\text{ord } \Delta_{40} = \frac{29}{16}$ give us 8 new solutions to (3.10), $i = 0$.

We can let

$$\Delta_{40} = \theta^{-\frac{29}{16}} + \Delta_{41} \quad (4.2.3)$$

Substituting (4.2.3) to (4.2.2) we get

$$\sum_{j=0}^{4} a_j \Delta_{41}^{2j} + \sum_{j=0}^{4} a_j (\theta^{-\frac{29}{16}})^{2j} + \theta^{\frac{3}{2}} = 0 \quad (4.2.4)$$
We have
\[ \sum_{j=0}^{4} a_j (\theta - \frac{d}{2})^{2j} + \theta^2 = \theta^{20} \theta^{-29} + \theta^{16} \theta^{-29} + \theta^4 \theta^{-29} + \theta^3 \theta^{-29} + \theta^2 \theta^{-29} + \theta \theta^{-29} = \theta \frac{d}{2} + \delta \]
where \( \text{ord} \delta > 0 \). Hence, the Newton polygon of (4.2.2) has vertices \((0, \frac{3}{16})\); \((8, -16)\); \((16, -20)\) and \(\Delta_{41} = \frac{16 - \frac{d}{8}}{8}\) (8 values), \(\frac{1}{2}\) (8 values). Case \(\text{ord} \Delta_{41} = \frac{1}{2}\) gives us again values \(\sum_{j=1}^{3} c_j x_{j0}\) as earlier, case \(\Delta_{41} = \frac{16 - \frac{d}{8}}{8}\) gives us the desired, because \(\frac{16 - \frac{d}{8}}{8} > \frac{29}{16}\). □

**Remark 4.3.** Let us denote by \(C_{\infty,s}\) a subfield of \(C_{\infty}\) formed by series generated by rational powers of \(\theta^{-1}\), with coefficients in \(F_2\). More exactly, let \(\alpha_1 < \alpha_2 < \alpha_3\ldots\) be a sequence of rational numbers such that \(\lim \alpha_i = +\infty\), and \(c_i \in F_2\) coefficients. By definition, \(C_{\infty,s}\) is a proper subfield of \(C_{\infty}\) formed by all sums \(\sum_{i=1}^{\infty} c_i \theta^{-\alpha_i}\). A well-known example of \(\forall \in C_{\infty} - C_{\infty,s}\) is a root to the equation
\[ x^2 + x + \theta^2 = 0 \]  
(4.3.0)
(here \(q = 2\)). Really, in a formal ring we have
\[ \forall = \theta + \theta^\frac{1}{4} + \theta^\frac{1}{8} + \theta^\frac{1}{16} + \ldots \]  
(4.3.1)
but this series \(\not\in C_{\infty,s}\). We let \(\forall_i = \theta + \theta^\frac{1}{4} + \theta^\frac{1}{8} + \theta^\frac{1}{16} + \ldots + \theta^\frac{1}{2^{2i}} + \delta_{in}, i = 1, 2\) (there are two roots: (4.3.0) is separable). We have \(\delta_{in}\) is a root to
\[ y^2 + y + \theta^\frac{1}{2^{2i}} = 0 \]
and hence both \(\delta_{1n}, \delta_{2n}\) have \(\text{ord} = -\frac{1}{2^{2n+1}}\). This shows once again that the series (4.3.1) does not converge to \(\forall\).

This is a model example. For \(x_{40}\) we have the same phenomenon, but first two terms of the equality \(x_{40} = \theta^{-\frac{1}{2}} + \theta^{-\frac{20}{8}} + \Delta_{41}\) is sufficient for our purpose.

4.4. The same arguments as in Lemma 4.2 applied to \(x_{j0}, j = 1, 2, 3\), show that \(x_{j0} = c_j \theta^{-2} + \Delta_j \in C_{\infty,s}\) where \(c_1, c_2, c_3\) form a basis of \(\mathbb{F}_2/\mathbb{F}_2\) and \(\text{ord} \Delta_j > 2\).

Let us fix some \(x_{40}\) and let us consider its minimal chain. We have
\[ b_{14} = \theta^{16} + \theta^{12}, \quad b_{13} = \theta^{12}, \quad b_{12} = \theta^{8} + \theta^{2} \]  
(4.4.1)
Hence,
\[ \text{ord} b_{14} = -16, \quad \text{ord} b_{13} = -12, \quad \text{ord} b_{12} = -8 \]

Hence,
\[ \sum_{k=2}^{4} b_{1k} x_{04}^{2k} = (\theta^{16} + \theta^{12})(\theta^{-\frac{1}{2}} + \theta^{-\frac{20}{8}} + \Delta_{41})^{16} + \theta^{12}(\theta^{-\frac{1}{2}} + \theta^{-\frac{20}{8}} + \Delta_{41})^{8} + \theta^{8}(\theta^{-\frac{1}{2}} + \theta^{-\frac{20}{8}} + \Delta_{41})^{4} = \theta^{6} + \delta \]
where \(\text{ord} \delta = -4\). This means that the Newton polygon of (3.10), \(i = 1\) has vertices \((0, -6)\); \((8, -16)\); \((16, -20)\) and \(\text{ord} x_{41} = \frac{5}{4}\) (the minimal value of \(x_{41}\)).
Finally,

\[ b_{24} = \theta^8, \quad \text{ord} \ b_{24} = -8 \quad (4.4.2) \]

Now we can use induction:

**Lemma 4.4a.** There exists a solution \( \sum_{i=0}^{\infty} x_{4i} T^i \) to (3.10) having \( \text{ord} \ x_{4i} = 2 - \frac{3}{2^{1+r}} \).

**Proof.** We showed that this is true for \( i = 0, 1 \). Let us show that if this is true for \( i = \alpha - 2, \alpha - 1 \) then this is true for \( i = \alpha \). We have

\[
\text{ord} \ (b_{14} x_{4,i-1}^{16}) = 16 - \frac{48}{2^\alpha}; \quad \text{ord} \ (b_{13} x_{4,i-1}^8) = 4 - \frac{24}{2^\alpha};
\]

\[
\text{ord} \ (b_{12} x_{4,i-1}^4) = -\frac{12}{2^\alpha}; \quad \text{ord} \ (b_{24} x_{4,i-1}^{16}) = 24 - \frac{48}{2^{\alpha-1}}
\]

For \( \alpha \geq 2 \) the minimal of these four numbers is the third one (really, for \( \alpha = 2 \) this is true; for \( \alpha \geq 3 \) the third number is negative while the first, second and forth are positive), hence the Newton polygon of \((3.10), \ i = \alpha)\) has vertices \((0, -\frac{12}{2^\alpha}); \ (8, -16); \ (16, -20)\) and \( \text{ord} \ x_{4\alpha} = (16 - \frac{12}{2^\alpha})/8 = 2 - \frac{3}{2^{1+r}}. \ □ \)

**Lemma 4.5.** For \( j = 1, 2, 3 \) there exist solutions \( \{x_j\} \) over \( x_{j0} \) having \( \forall \ i \ \text{ord} \ x_{ji} = 2 \).

**Proof.** For \( j = 1, 2, 3 \) we have \( \text{ord} \ x_{j0} = 2 \), hence for these \( j 
\]

\[
\text{ord} \ (b_{14} x_{j0}^{16}) = 16; \quad \text{ord} \ (b_{13} x_{j0}^8) = 4; \quad \text{ord} \ (b_{12} x_{j0}^4) = 0
\]

and the Newton polygon of (3.10), \( i = 1, j = 1, 2, 3 \) has vertices \((0, 0); \ (8, -16); \ (16, -20)\) and \( \text{ord} \ x_{j1} \) can be chosen 2.

The same situation holds for \( i \geq 2 \). We have

\[
\text{ord} \ (b_{24} x_{j0}^{16}) = 24
\]

hence by induction we get that there are solutions \( \sum_{i=0}^{\infty} x_{ji} T^i \) of (3.10), \( j = 1, 2, 3 \), having \( \text{ord} \ x_{ji} = 2 \). □

So, we got 4 basis solutions \( \{x_j\} = \sum_{i=0}^{\infty} x_{ji} T^i, \ j = 1, ..., 4 \). Any solution to (3.10) is \( \sum_{j=1}^{4} C_j \{x_j\} \) where \( C_j \in \mathbb{F}_2[[T]] \) (Proposition 2.3).

**Lemma 4.6.** The set \( \sum_{j=1}^{4} C_j \{x_j\} \) does not contain small solutions (here and below — except the zero solution).

**Proof.** Let us assume that \( \exists C_1, ..., C_4 \) such that \( \sum_{j=1}^{4} C_j \{x_j\} \) is a small solution. We consider \( S_{123} := \sum_{j=1}^{3} C_j \{x_j\} \), we denote \( S_{123} = \sum_{i=0}^{\infty} \bar{x}_{1,2,3; i} T^i \). We have: \( \text{ord} \ \bar{x}_{1,2,3; i} \geq 2 \), because \( \forall \ i \) elements \( \bar{x}_{1,2,3; i} \) are linear combinations of \( x_{jk} \) for \( j = 1, 2, 3, k \leq i \) with coefficients in \( \mathbb{F}_2 \).

Further, we denote \( S_4 := C_4 \{x_4\} = \sum_{i=0}^{\infty} \bar{x}_{4; i} T^i \). Lemma 4.4a shows that \( \forall \ i \ \text{ord} \ x_{4i} \) are different and \( \frac{1}{2} \leq \text{ord} \ x_{4i} < 2 \), hence \( \forall \ i \) we have \( \frac{1}{2} \leq \text{ord} \ \bar{x}_{4; i} < 2 \). This means that \( \sum_{j=1}^{4} C_j \{x_j\} = S_{123} + S_4 \) cannot be a small solution. □

This gives us a proof of Theorem 4.1. Really, if \( H^1(M(A)) \neq 0 \) then equation (3.10) has a small solution. □
Remark. It is easy to show that if \( \bar{x}_{1,2;0} \neq 0 \) then \( \forall i > 0 \) we have ord \( \bar{x}_{1,2;3;i} = 2 \). Really, let us assume that \( \exists i \) such that ord \( \bar{x}_{1,2;3;i} > 2 \). We choose minimal such \( i \), and denote it by \( i_0 \). Condition \( \bar{x}_{1,2;3;0} \neq 0 \) implies \( i_0 \geq 1 \). The calculation of Lemma 4.5 shows that — because for \( i = 0, \ldots, i_0 - 1 \) we have ord \( \bar{x}_{1,2;3;i} = 2 \), we have either ord \( \bar{x}_{1,2;3;i_0} = 2 \) or ord \( \bar{x}_{1,2;3;i_0} = \frac{1}{2} \). The condition ord \( \bar{x}_{1,2;3;i_0} = \frac{1}{2} \) contradicts to ord \( \bar{x}_{1,2;3;i} \geq 2 \), and the condition ord \( \bar{x}_{1,2;3;i_0} = 2 \) contradicts to the choice of \( i_0 \).

**Theorem 4.7.** For the above \( A \) we have \( h^1(M(A^t)) = 1 \).

**Proof.** We have \( A^t = \begin{pmatrix} \theta^6 & \theta^{-2} \\ 0 & 0 \end{pmatrix} \) The numbers \( a_i \) of (3.10) for \( A^t \) are:

\[
\begin{align*}
a_4 &= \theta^{-12}, & a_3 &= \theta^{-16}, & a_2 &= \theta^{-20}, & a_1 &= \theta^{-5}, & a_0 &= \theta^{-6} \\
\text{ord } a_4 &= 12, & \text{ord } a_3 &= 16, & \text{ord } a_2 &= 20, & \text{ord } a_1 &= 5, & \text{ord } a_0 &= 6
\end{align*}
\]

Now "equation (3.10)" will mean the equation (3.10) with these values of \( a_i \) and the below values of \( b_{ij} \). The Newton polygon for (3.10), \( i = 0 \) has vertices (1,6); (2,5); (16,12), hence \( \text{ord } x_{10} = 1 \), and for \( j = 2,3,4 \) we have ord \( x_{j0} = -\frac{1}{2} \). We have:

\[
\begin{align*}
b_{14} &= \theta^{-16} + \theta^{-20}, & b_{13} &= \theta^{-20}, & b_{12} &= \theta^{-6} + \theta^{-24}, & b_{24} &= \theta^{-24} \\
\text{ord } b_{14} &= 16, & \text{ord } b_{13} &= 20, & \text{ord } b_{12} &= 6, & \text{ord } b_{24} &= 24
\end{align*}
\]

hence the Newton polygon for (3.10), \( i = 1 \), \( x_0 = x_{10} \) has vertices (0,10); (1,6); (2,5); (16,12), and we can choose \( x_{11} \) having ord \( = 4 \).

**Lemma 4.8.** For \( \{x_1\} \) we have: ord \( x_{1n} = 4^n \).

**Proof.** Induction. Let the lemma hold for \( n = i \) and \( n = i + 1 \). Then it holds for \( n = i + 2 \). Really,

\[
\begin{align*}
\text{ord } b_{14}x_{1,i+1}^{16} &= 16 + 16 \cdot 4^{i+1}, & \text{ord } b_{13}x_{1,i+1}^8 &= 20 + 8 \cdot 4^{i+1} \\
\text{ord } b_{12}x_{1,i+1}^4 &= 6 + 4 \cdot 4^{i+1}; & \text{ord } b_{24}x_{1,i}^{16} &= 24 + 16 \cdot 4^i
\end{align*}
\]

The minimal of these four numbers is the third one, it is \( 6 + 4^{i+2} \), this is the \( y \)-coordinate of the vertex of the Newton polygon having \( x = 0 \). Hence, ord \( x_{i+2} = 6 + 4^{i+2} - 6 = 4^{i+2} \). \( \square \)

**Lemma 4.9.** Let \( \{x\} = \sum_{i=0}^{\infty} x_iT^i \) be any solution to (3.10) over \( x_0 \) such that ord \( x_0 = -\frac{1}{2} \). Then \( \forall i > 0 \) we have: ord \( x_i = -\frac{1}{2} \).

**Proof.** Immediate, by induction. In notations of Lemma 4.8, we have

\[
\begin{align*}
\text{ord } b_{14}x_{i+1}^{16} &= 8; & \text{ord } b_{13}x_{i+1}^8 &= 16; \\
\text{ord } b_{12}x_{i+1}^4 &= 4; & \text{ord } b_{24}x_{i}^{16} &= 16
\end{align*}
\]

hence the Newton polygon of (3.10) for any \( i \) is one segment \( (0, 4) - (16, 12) \), hence the lemma. \( \square \)

**Lemma 4.10.** Any small solution to (3.10) belongs to \( \mathbb{F}_q[T]\{x_1\} \).
Proof. Let $y = (y_0, y_1, \ldots)$ be a small solution. Lemma 4.9 implies that $y_0 \in \mathbb{F}_2x_{10}$, i.e. $\exists k_0 \in \mathbb{F}_2$ such that $y_0 = k_0 x_{10}$. Let us consider $y - k_0 \{x_1\}$. It is a small solution, its first term is 0, hence we can divide it by $T$. Now we continue the process: there exists $k_1 \in \mathbb{F}_2$ such that the first term of $(y - k_0 \{x_1\})/T$ is $k_1 x_{10}$. We consider $((y - k_0 \{x_1\})/T - k_1 \{x_1\})/T$ etc. As a result, we get that $\exists K := \sum_{n=0}^{\infty} k_n T^n \in \mathbb{F}_2[[T]]$ such that $y = \{x_1\} K$. It is easy to see that $K \in \mathbb{F}_2[T]$. Really, let $k_n \neq 0$. In this case the ord of the $n$-th term of $\{x_1\} K$ is 1 (= ord $x_{10}$), because $\forall i > 0$ we have ord $x_{10} >$ ord $x_{1i}$. Condition that $y$ is small implies that there exists only finitely many $k_n$ such that $k_n \neq 0$. □

Remark. There is another proof of this lemma. In notations of Lemma 4.6, let $\sum_{j=1}^{4} C_j \{x_j\}$ be a small solution. We denote $C_j = \sum_{i=0}^{\infty} c_{ji} T^i$, where $c_{ji} \in \mathbb{F}_2$. Like in Lemma 4.6, we denote $S_{234} := \sum_{i=2}^{4} C_j \{x_j\} = \sum_{i=0}^{\infty} x_{2,3,4;i} T^i$ and $S_1 := C_1 \{x_1\} = \sum_{i=0}^{\infty} x_{1i} T^i$. If $S_{234} \neq 0$ then there exists the minimal $i_0$ such that $x_{2,3,4;i_0} \neq 0$. This implies $\forall j = 2, 3, 4, \forall k < i_0 c_{jk} = 0$ and $\exists j = 2, 3, 4$ such that $c_{j,i_0} \neq 0$. This means that ord $x_{2,3,4;i_0} = -\frac{1}{2}$, and hence, according Lemma 4.9, $\forall i \geq i_0$ we have ord $x_{2,3,4;i} = -\frac{1}{2}$. Like in the proof of Lemma 4.6, we get that $\forall i$ ord $x_{1i} \geq 1$, hence the sum $S_1 + S_{234}$ cannot be a small solution. The only exception is $S_{234} = 0$. If $C_1 \not\subseteq \mathbb{F}_2[T]$ then there exists infinitely many $i$ such that ord $x_{1i} = 1$ (again because all ord $x_{1i}$ are different), hence for a small solution $C_1$ must belong to $\mathbb{F}_2[T]$.

4.11. End of the proof. Lemma 4.10 implies that $h^1(M(A^4)) = 1$. Really, (3.10) for the present case means that the set of small $y_{21}$ of Section 3 has dimension 1. Further, (3.4) shows that if $y_{21}$ is small then $y_{22}$ is also small, and (3.0) shows that $y_1 = (y_{11}, y_{12})$ are also small. □

5. A question of D. Goss.

Prof. David Goss wrote ([G1])

One last question: Let $\phi$ be a t-motive where $\phi_T$ has infinitesimal part $T + N$ where $N$ is unipotent. Define $\phi$ to be generated by $\tilde{\phi}_T$ where $\tilde{\phi}_T$ has exactly the same coefficients as $\phi_T$ but where $N$ is now set to 0. What is the relationship between these two objects? if one is uniformizable what about the other? And then what would be the relationship between the lattices? etc.

David

We give an example that two t-motives mentioned above can be of different uniformizability type. We use notations of (1.11), $n = 2$, $N$ of Section 3, i.e. $N = \varepsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ where $\varepsilon = 0$ or 1. We consider the case $q > 2$, $A := \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$ where $a_{21}$ is any number having ord $a_{21} = -\frac{a_{21}^2}{q-1}$, and $a_{12}$ satisfies

$$\frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^2} - a_{12}^q = 0$$

(this expression is $a_2$ of (3.9)).

Theorem 5.1. For $\varepsilon = 0$ the t-motive $M(A, N)$ is non-uniformizable, while for $\varepsilon = 1$ it is uniformizable.
Proof. For \( \varepsilon = 0 \), resp. \( 1 \) we denote the corresponding \( M(A, N) \) by \( M_0 \), resp. \( M_1 \). The coefficients (3.9) become

\[
\begin{align*}
  a_4 &= \frac{\theta q^3 + q^2}{a_{21}^2}; \quad a_3 = -\varepsilon; \quad a_2 = a_1 = 0; \quad a_0 = \frac{1}{a_{21}} \\
  \text{ord } a_4 &= \frac{q^2}{q-1}; \quad \text{ord } a_3 = 0 (\varepsilon = 1); \quad \text{ord } a_3 = +\infty (\varepsilon = 0); \quad \text{ord } a_0 = \frac{q^2}{q-1} \\
  b_{14} &= -\frac{\theta q^3 + q^2}{a_{21}^2}; \quad b_{13} = 0; \quad b_{12} = \frac{1}{a_{21}} + \frac{1}{a_{21}^2}; \quad b_{24} = \frac{1}{a_{21}^2} \\
  \text{ord } b_{14} &= \frac{q^2}{q-1} + q^2; \quad \text{ord } b_{12} = \frac{q^2}{q-1}; \quad \text{ord } b_{24} = \frac{q^4}{q-1}
\end{align*}
\]

Let us show \( M_0 \) is not uniformizable. The Newton polygon of ((3.10), \( i = 0 \)) is a segment whose ends have coordinates

\[
(1, \frac{q^2}{q-1}), (q^4, \frac{q^2}{q-1}),
\]

hence all \( x_0 \neq 0 \) have \( \text{ord } = 0 \). We get by induction by \( i \) that for any \( i_0 > 0 \) the Newton polygon for the equation ((3.10), \( i = i_0 \)) is a segment whose ends have coordinates

\[
(0, \frac{q^2}{q-1}), (q^4, \frac{q^2}{q-1}),
\]

hence (if \( x_0 \neq 0 \)) for all \( i \) we have \( \text{ord } x_i = 0 \). This means that \( H^1(M_0) = 0 \).

Now let us show \( M_1 \) is uniformizable. The vertices of the Newton polygon for the equation ((3.10), \( i = 0 \)) have coordinates

\[
(1, \frac{q^2}{q-1}), (q^3, 0), (q^4, \frac{q^2}{q-1}),
\]

hence the set of solutions to ((3.10), \( i = 0 \)) is a \( \mathbb{F}_q \)-vector space of dimension 4 having a basis \( x_{10}, ..., x_{40} \) such \( \text{ord } x_{j0} = \frac{q^2}{(q-1)(q^3-1)} \) for \( j \leq 3 \), \( \text{ord } x_{40} = \frac{1}{(q-1)(q^2-1)} \).

Let us consider the equation ((3.10), \( i = 1 \)) for \( x_0 = x_{40} \) (the ”worst” case). We have:

\[
\begin{align*}
  \text{ord } b_{14} x_{40}^4 &= \frac{q^4 - 2q^3}{(q-1)^2}; \quad \text{ord } b_{12} x_{40}^2 &= \frac{q^3 - q^2 - q}{(q-1)^2} \\
  \text{Hence, for } q > 2 \text{ the vertices of the Newton polygon for the equation ((3.10), } i = 1 \text{, } x_0 = x_{40} \text{ have coordinates}
\end{align*}
\]

\[
(0, \frac{q^3 - q^2 - q}{(q-1)^2}), (q^3, 0), (q^4, \frac{q^2}{q-1}),
\]

hence ((3.10), \( i = 1 \)) has a solution \( x_{41} \) having \( \text{ord } = \frac{q^2 - q^{-1}}{(q-1)^2} \).

Hence, we have

\[
\begin{align*}
  \text{ord } b_{14} x_{41}^4 &= \frac{2q^4 - 2q^3 - q^2}{(q-1)^2}; \quad \text{ord } b_{12} x_{41}^2 &= \frac{q^3 - q - 1}{(q-1)^2}; \quad \text{ord } b_{24} x_{40}^4 &= \frac{q^5 - q^4 - q^3}{(q-1)^2}
\end{align*}
\]

For \( q > 2 \) the minimal of these 3 numbers is \( \frac{q^3 - q - 1}{(q-1)^2} \), hence the vertices of the Newton polygon for ((3.10), \( i = 2 \)) have coordinates.
(0, q^3 - q - 1, (1, q^2, (q^3, 0), (q^4, q^2 - 1))

hence ((3.10), i = 2) has a solution x_{42} having ord = \frac{q^3 - q - 1}{(q-1)^2}. This is sufficient to use induction. Namely, let us prove that

(*) \forall n \geq 1 we have ord x_{4n} \geq \frac{1}{2} q^{2(n-2)}.

(*) holds for n = 1, 2, hence to prove (*) for all n it is sufficient to prove:

**Lemma 5.2.** Let (*) hold for n = i and n = i + 1. Then it holds for n = i + 2.

**Proof** is straightforward. We have

\begin{align*}
\text{ord } b_{14} x_{4, i+1}^2 & \geq \frac{q^3}{q-1} \cdot \frac{1}{2} \cdot q^{2(i-1)} \cdot q^4 > \frac{1}{2} q^{2i+4}, \\
\text{ord } b_{12} x_{4, i+1}^2 & \geq \frac{q^3}{q-1} \cdot \frac{1}{2} \cdot q^{2(i-1)} \cdot q^2 > \frac{1}{2} q^{2i+1} \\
\text{ord } b_{24} x_{4, i}^2 & \geq \frac{q^3}{q-1} \cdot \frac{1}{2} \cdot q^{2(i-2)} \cdot q^4 > \frac{1}{2} q^{2i+3}
\end{align*}

We have \(\frac{1}{2} q^{2i+1} - \frac{q^3}{q-1} > \frac{1}{2} q^{2i}\) hence the lemma. \(\square\)

Calculations for minimal chains of \(x_j, j \leq 3\), are similar: the \(y\)-coordinate of the leftmost vertex of the Newton polygon for (3.10), any \(i\) is greater than the same coordinate for the above minimal chain of \(x_{40}\). Alternatively, we can prove that the above minimal chain of \(x_{40}\) is simple, and to apply Proposition 2.6a. The details are left to the reader.

Finally, the same arguments as in (4.11) show that if we have 4 linearly independent small solutions \(y_{21}\) to (3.10), then they give 4 linearly independent small solutions to (3.0a). This means that \(M_1\) is uniformizable. \(\square\)

**Appendix.** Here we give an analog of calculations of Section 3 for a non-pure \(t\)-motive of dimension 2 and rank 5. In terminology of [L07], it is a standard-1 \(t\)-motive from ([L07], 11.1) having \(\lambda_1 = 3\), \(\lambda_2 = 2\). (1.3) for it is the following:

\[
T \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \tau \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \left( \begin{array}{cc} b_1 & 0 \\ b_2 & 1 \end{array} \right) \tau^2 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \tau^3 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}
\]

Its \(f^*_i\)-basis can be chosen as

\[
\begin{pmatrix}
e_1 \\ e_2 \\ \tau e_1 \\ \tau^2 e_1
\end{pmatrix}. \quad \text{The matrix } Q \text{ in this basis is}
\begin{pmatrix}
0_{32} & I_3 \\
0 & T - \theta & -a_{21} & -a_{22} & -b_2 \\
T - \theta & 0 & -a_{11} & -a_{12} & -b_1
\end{pmatrix}
\]

where \(0_{32}, I_3\) are respectively \(3 \times 2, 3 \times 3\)-blocks. We have

\[
y^{(1)}_5 = -\frac{1}{a_{12}} y_4 - \frac{a_{22}}{a_{12}} y^{(1)}_4 + \frac{T - \theta^q}{a_{12}} y^{(2)}_4 \quad (A1.1)
\]

\[
y_5 + b_1 y^{(1)}_5 + a_{11}^q y^{(2)}_5 - (T - \theta^q) y^{(3)}_5 + b_2 y^{(1)}_4 + a_{21}^q y^{(2)}_4 = 0 \quad (A2)
\]

Further, we transform:

\[
y^{(1)}_5 + b_1 y^{(2)}_5 + a_{11}^q y^{(3)}_5 - (T - \theta^q) y^{(4)}_5 + b_2 y^{(2)}_4 + a_{21}^q y^{(3)}_4 = 0 \quad (A2.1)
\]
\[ y_5^{(2)} = \frac{1}{a_{12}^q} y_4^{(1)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(2)} + T - \theta^q y_4^{(3)} \] (A1.2)

\[ y_5^{(3)} = \frac{1}{a_{12}^q} y_4^{(1)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(3)} + T - \theta^q y_4^{(4)} \] (A1.3)

\[ y_5^{(4)} = \frac{1}{a_{12}^q} y_4^{(1)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(4)} + T - \theta^q y_4^{(5)} \] (A1.4)

Now we substitute (A1.1) – (A1.4) to (A2.1) in order to eliminate \( y_5 \):

\[-\frac{1}{a_{12}} y_4^{(1)} + \frac{T - \theta^q}{a_{12}} y_4^{(2)} + b_1^q (-\frac{1}{a_{12}} y_4^{(1)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(2)} + \frac{T - \theta^q}{a_{12}} y_4^{(3)}) \]

\[+ a_{12}^q (-\frac{1}{a_{12}} y_4^{(2)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(3)} + \frac{T - \theta^q}{a_{12}} y_4^{(4)}) - (T - \theta^q)^3 (-\frac{1}{a_{12}} y_4^{(3)} - \frac{a_{12}^q}{a_{12}^q} y_4^{(4)} + \frac{T - \theta^q}{a_{12}} y_4^{(5)}) \]

\[+ b_2^q y_4^{(2)} + a_{21}^q y_4^{(3)} = 0 \] (A2.1)

We get an equation of type (2.2a) having \( r = 5, n = 2 \), that supports Conjecture 2.13.

**Remark A3.** For t-motives with higher \( r \), \( n \) the similar calculation gives us systems of type

\[ \sum_{\gamma=0}^{r_{11}} a_{1\gamma} x^{(\gamma)} + \sum_{\gamma=0}^{r_{12}} b_{1\gamma} y^{(\gamma)} = 0 \] (A3.1.1)

\[ \sum_{\gamma=0}^{r_{21}} a_{2\gamma} x^{(\gamma)} + \sum_{\gamma=0}^{r_{22}} b_{2\gamma} y^{(\gamma)} = 0 \] (A3.1.2)

where \( a_{ij}, b_{ij} \in \mathbb{C}_\infty(T), \ x, \ y \in \mathbb{C}_\infty[[T]] \) are unknowns. For elimination of an unknown from this system we can use the theory of the \( p \)-resultant, see, for example, [G], Section 1.5. We expect that for any explicitly given t-motive (for example for standard-2 t-motives of [L07], 11.2) the similar calculations will give us a proof of Conjecture 2.13 for them.

**Example A3.2.** Let us consider \( M \) from (1.11) for \( n = 3, N = 0 \). Notations and calculations are similar to the ones of Section 3. We write \( Y = (y_1, y_2) \) as a block matrix where \( y_1 = (y_{11}, y_{12}, y_{13}), y_2 = (y_{21}, y_{22}, y_{23}) \). Analog of (3.1) is the same, (3.2) and (3.3) become

\[ y_{21} = y_{21}^{(2)} (T - \theta^q) - y_{21}^{(1)} a_{11} - y_{22}^{(1)} a_{21} - y_{23}^{(1)} a_{31} \] (A3.2.1)

\[ y_{22} = y_{22}^{(2)} (T - \theta^q) - y_{21}^{(1)} a_{12} - y_{22}^{(1)} a_{22} - y_{23}^{(1)} a_{32} \] (A3.2.2)

\[ y_{23} = y_{23}^{(2)} (T - \theta^q) - y_{21}^{(1)} a_{13} - y_{22}^{(1)} a_{23} - y_{23}^{(1)} a_{33} \] (A3.2.3)

Analog of (3.4) - (3.6) become

\[ y_{23}^{(1)} = -\frac{1}{a_{31}} y_{21} + \frac{T - \theta^q}{a_{31}} y_{21}^{(2)} - \frac{a_{11}}{a_{31}} y_{21}^{(1)} - \frac{a_{21}}{a_{31}} y_{22}^{(1)} \] (A3.2.4)
\[ y_{23}^{(2)} = -\frac{1}{a_{31}}y_{21}^{(1)} + \frac{T - \theta q^2}{a_{31}^2}y_{21}^{(3)} - \frac{a_{11}q}{a_{31}^3}y_{21}^{(2)} - \frac{a_{21}q}{a_{31}^2}y_{22}^{(2)} \]  
\[ y_{23}^{(3)} = -\frac{1}{a_{31}}y_{21}^{(2)} + \frac{T - \theta q^3}{a_{31}^2}y_{21}^{(4)} - \frac{a_{11}q^2}{a_{31}^3}y_{21}^{(3)} - \frac{a_{21}q^2}{a_{31}^2}y_{22}^{(3)} \]  

Applying \( \tau \) to (A3.2.3) we get an analog of (3.7):

\[ y_{23}^{(1)} = y_{23}^{(3)}(T - \theta q^2) - y_{21}^{(2)}a_{13}q - y_{22}^{(2)}a_{23}q - y_{23}^{(2)}a_{33}q \]  

Substituting (A3.2.4) - (A3.2.6) to ((A3.2.2) and (A3.2.7) we get an equation of type (A3.1.1) - (A3.1.2) with unknowns \( y_{21}, y_{22}, r_{11}, r_{12}, r_{21}, r_{22} \) respectively 2, 2, 4, 3 (and \( b_{20} = 0 \)). We can eliminate one of these two unknowns either using the theory of the \( p \)-resultant, or we can enlarge this system (A3.1.1) - (A3.1.2) applying \( \tau, \tau^2, \tau^3 \) to (A3.1.1) and \( \tau \) to (A3.1.2). We shall get a matrix equation

\[
\mathcal{A} \begin{pmatrix} y_{21} \\ y_{21}^{(1)} \\ \vdots \\ y_{21}^{(5)} \end{pmatrix} + \mathcal{B} \begin{pmatrix} y_{22} \\ y_{22}^{(1)} \\ \vdots \\ y_{22}^{(5)} \end{pmatrix} = 0 \tag{A3.3}
\]

where \( \mathcal{A}, \mathcal{B} \in M_{6 \times 6}(\mathbb{C}_\infty[T]) \). We have \( \mathcal{A} \in GL_6(\mathbb{C}_\infty(T)) \); multiplying (A3.3) from the left by the first line of \( \mathcal{A}^{-1} \) we get an expression of \( y_{21} \) as a linear combination of \( y_{22}, y_{22}^{(1)}, \ldots, y_{22}^{(5)} \). Substituting this expression to (A3.1.1) we get an equation

\[ \sum_{\gamma=0}^{7} c_\gamma y_{21}^{(\gamma)} = 0 \]

for the unknown \( y_{21} \), where \( c_\gamma \in \mathbb{C}_\infty[T] \).

Why the maximal value of \( \gamma \) is 7 and not 6, and finding of the value of \( n \) (see (2.1)) is a subject of further calculations.

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