Objective Bayesian analysis for the differential entropy of the Gamma distribution

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Summary

The use of entropy related concepts goes from physics, such as in statistical mechanics, to evolutionary biology. The Shannon entropy is a measure used to quantify the amount of information in a system, and its estimation is usually made under the frequentist approach. In the present paper, we introduce a fully objective Bayesian analysis to obtain this measure’s posterior distribution. Notably, we consider the gamma distribution, which describes many natural phenomena in physics, engineering, and biology. We reparametrize the model in terms of entropy, and different objective priors are derived, such as Jeffreys prior, reference prior, and matching priors. Since the obtained priors are improper, we prove that the obtained posterior distributions are proper and their respective posterior means are finite. An intensive simulation study is conducted to select the prior that returns better results in terms of bias, mean square error, and coverage probabilities. The proposed approach is illustrated in two datasets, where the first one is related to the Achaemenid dynasty reign period, and the second data describes the time to failure of an electronic component in the sugarcane harvest machine.

1 Introduction

In recent years, there has been a growing interest in estimating different metrics of information theory related to parametric distributions. The Shannon entropy, also known as differential entropy, introduced by Claude Shannon [30], is an essential quantity that measures the amount of available information or uncertainty outcome of a random process. Given a density function \( f(x|\alpha, \beta) \), the differential entropy is given by

\[
H(\alpha, \beta) = \mathbb{E}(-\log f(x|\alpha, \beta)).
\]

(1)

The differential entropy depends on the distribution parameters, and, given a sample, it is necessary to be estimated. The commonly used method to estimate the parameters is the maximum likelihood approach due to its one-to-one invariance property. Hence, we need only to estimate the parameters of the original model and plug-in the entropy function. Under this approach many authors have derived the
estimators for different distributions such as, Weibull \[9\], Inverse Weibull \[33\], Log-logistic \[13\] and for the exponential distribution with different shift origin \[20\], to list a few.

A major drawback of the maximum likelihood inference is that the obtained estimates are usually biased for small samples \[11\]. Another concern happens under small samples when constructing the confidence intervals for the parameters since such intervals are not precise and may not return good coverage probabilities. In this case, the maximum likelihood estimation (MLE) skewness study is essential to assess the quality of the interval \[12\]. To overcome these limitations, we can use objective Bayesian methods. In this context, the inference for the parameters of the gamma distribution have been discussed earlier under this approach by Miller \[23\], Sun and Ye \[31\], Berger et al. \[6\], and Louzada and Ramos \[21\]. Moreover, Ramos et al. \[27\] revised the most common objective priors and provided sufficient and necessary conditions for the obtained posteriors and their higher moments to be proper.

Although the authors have obtained different joint posterior distributions for the parameters of interest, the obtained posterior means can not be directly plunged in the Shannon entropy. Under the Bayesian approach, it is necessary to obtain the posterior distribution of the entropy measure. In this context, Shakhatreh \[29\] recently derived different posterior distributions using objective priors for the entropy assuming a Weibull distribution. On the other hand, the cited distribution’s entropy expression is not as complicated as the gamma distribution’s entropy expression. With this in mind, in this paper, focusing on the gamma distribution, we derive the posterior distributions using objective priors, such as Jeffreys prior \[18\], reference priors \[5–7\], and matching priors \[32\], and prove that the obtained posteriors are proper and can be used to construct the posterior distributions of the Shannon entropy. Moreover, even if the posterior distribution is proper, the posterior mean can be infinite, which is undesirable, and thus we shall also prove that the obtained posterior means for the entropy measure are finite. Finally, the credibility intervals are obtained to construct accurate interval estimates.

The gamma distribution considered here is a two-parameter family of distribution among the most well-known distribution used to model different stochastic processes and to make statistical inferences, and has received attention from different fields. It surfaces in many areas of applications, including financial analysis \[10\], climate analysis \[17\], reliability analysis \[19\], machine learning \[19\], and physics \[14\]. Particularly, the gamma distribution includes the exponential distribution, Erlang distribution, and chi-square distribution as special cases.

A random variable \(X\) follows a gamma distribution, if its probability density function, parametrized by a shape parameter \(\alpha > 0\) and scale parameter \(\beta > 0\), is given by,

\[
f(x | \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,
\]

where \(\Gamma(\phi) = \int_0^\infty e^{-x} x^{\phi-1} dx\) is the gamma function.

The paper is organized as follows. Section 2 presents the maximum likelihood estimators for the gamma distribution parameters and the Shannon Entropy computation. Section 3 presents the objective Bayesian analysis using objective priors for the Shannon entropy parameter’s reparametrized posterior distribution. Section 4 provides a simulation study to select the best objective prior. In Section 5, the methodology is illustrated on a real dataset. Some final comments are given in Section 6.
2 Frequentist approach

The classical inference (frequentist) is a commonly used approach to conduct parameter estimation of a particular distribution. In this case, the parameter is treated as fixed, and the MLE is commonly used to obtain the estimates. The MLE has good asymptotic properties, such as invariance, consistency, and efficiency. This procedure search the parameter space of $\theta$ where the maximum likelihood $\hat{\theta} = \sup_\theta L(\theta|x)$ is obtained. Here our main aim is to obtain the estimate of a function of the parameters. Hence, firstly we need to obtain the entropy measure, mathematically defined as 

$$H(\theta) = \mathbb{E}[-\log f(x|\theta)],$$

which quantifies the amount of uncertainty in the data $x$. Besides, it should be noted that a higher realization of $H$ indicates more uncertainty.

The entropy $H$ of the gamma density is given by

$$H(\alpha, \beta) = -\int_0^\infty \log \left( \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-\beta x\} \right) f(x|\alpha, \beta) dx = \alpha - \log(\beta) + \log(\Gamma(\alpha)) + (1 - \alpha) \psi(\alpha),$$

where $\psi(k) = \frac{\partial}{\partial k} \log \Gamma(x)$ is the digamma function.

Now, consider a change of variable by setting $W = \alpha$, which implies $H = W - \log(\beta) + \log(\Gamma(W)) + (1 - W)\psi(W)$. The aim of the transformation is to obtain a likelihood of $H$ and $W$ instead of $\alpha$ and $\beta$. Therefore, if $X_1, \ldots, X_n$ are a complete sample from (2) then the likelihood function of $H$ and $W$ is given as

$$L(W, H | x) = \frac{\delta(W, H)^n W}{\Gamma(W)^n} \left( \prod_{i=1}^n x_i^W \right) \exp\left\{-\delta(W, H) \sum_{i=1}^n x_i \right\},$$

where $\delta(W, H) = \exp(W + \log \Gamma(W) + (1 - W)\psi(W) - H)$.

The log-likelihood function is given by

$$l(W, H | x) = W \log(\delta(W, H)) - \log(\Gamma(W)) + W \sum_{i=1}^n \log(x_i) - \delta(W, H) \sum_{i=1}^n x_i. \quad (5)$$

The MLEs for the parameters are obtained by directly maximizing the log-likelihood function $\ell(\lambda, \phi; t)$. Hence, after some algebraic manipulations the MLEs $\hat{W}$ and $\hat{H}$ are obtained from the solution of

$$\frac{\partial l(W, H | x)}{\partial W} = \log(\delta(W, H)) - \psi(W) - \sum_{i=1}^n \log(x_i) + \sigma \left( W - \delta(W, H) \sum_{i=1}^n x_i \right)$$

$$\frac{\partial l(W, H | x)}{\partial H} = -W + \delta(W, H) \sum_{i=1}^n x_i$$

where $\sigma = 1 + (1 - W)\psi'(W)$. The solutions for these equations provide the maximum likelihood estimators for the entropy of the gamma distributions, $\hat{H}$ and $\hat{W}$. Since equation (3) cannot be solved easily using a closed-form solution, numerical techniques must estimate the true parameters.

Following [22], the MLEs are asymptotically normally distributed with a joint bivariate normal distribution given by

$$\left( \hat{W}_{MLE}, \hat{H}_{MLE} \right) \sim N_2 [(W, H), I^{-1}(W, H)] \quad \text{as} \quad n \to \infty,$$
where \( I(W,H) \) is the Fisher information matrix for the reparametrized model given by

\[
I(W,H) = \begin{bmatrix}
\psi'(W) - 2\sigma + W\sigma^2 & 1 - \sigma W \\
1 - \sigma W & W
\end{bmatrix},
\]

and \( \psi'(W) \) is the derivative of \( \psi(W) \), called the trigamma function.

In the present paper, we are only interested in \( H \), and thus, given \( 0 < a < 1 \) and using the element \((I(W,H)^{-1})_{22}\), we can conclude that the confidence interval for the estimate of the entropy measure with a confidence level of \( 100(1-a)\% \) for \( H \) is given by

\[
\hat{H} - Z_a \sqrt{(1-W)^2\psi'(W) + 2 - W} < H < \hat{H} + Z_a \sqrt{(1-W)^2\psi'(W) + 2 - W},
\]

where \( a \) is the significance level and \( Z_a \) is the \( \frac{a}{2} \)-th percentile of the standard normal distribution.

### 3 Bayesian Inference

Here, the parameter \( \theta \) is considered as a random variable and the distribution that represents knowledge about \( \theta \) is referred to as a prior distribution and defined by \( \pi(\theta) \). The distribution \( \pi(\theta) \) provides the knowledge or uncertainty about \( \theta \) before obtaining the sample data \( x \). After the data \( x \) is observed, a natural way of combining the resulting information from the a priori the distribution and the likelihood function is done by the Bayes’ theorem, resulting in the posterior distribution of \( \theta \) given \( x \). In a Bayesian framework, Ramos et al. [27] analyzed the properties of the posterior distribution of the gamma distribution parameters and stated the conditions for this distribution to have proper posterior and finite moments.

To obtain the posterior distributions for the \( H \) parameter, we can consider the one-to-one invariance property of the Jeffreys prior, reference prior, and matching prior, and thus we only need to obtain the Jacobian matrix related to the reparametrization from \( \alpha \) and \( \beta \) to \( H \) and \( W \). After some algebraic manipulations, we can conclude that the parameters \( \beta \) and \( \alpha \) can be written as

\[
\beta = \exp(W + \log(\Gamma(W)) + (1 - W)\psi(W) - H) \quad \text{and} \quad \alpha = W,
\]

and thus, from the relations

\[
\frac{\partial \alpha}{\partial H} = 0, \quad \frac{\partial \alpha}{\partial W} = 1, \quad \frac{\partial \beta}{\partial H} = -\beta \quad \text{and} \quad \frac{\partial \beta}{\partial W} = \left(1 + (1 - W)\psi^{(1)}(W)\right)\beta,
\]

it follows that the Jacobian matrix (\( J \)) relative to the change of variable will be given by

\[
J = \begin{bmatrix}
\frac{\partial \alpha}{\partial \pi} & \frac{\partial \alpha}{\partial \pi} & 0 \\
\frac{\partial \beta}{\partial \pi} & \frac{\partial \beta}{\partial \pi} & 1 \\
-\beta & \sigma \beta
\end{bmatrix},
\]

where \( \sigma = 1 + (1 - W)\psi'(W) \).

The use of objective priors plays an essential role in Bayesian analysis where the data provide the dominant information, and the posterior distribution is not overshadowed by prior information. Such priors allow us to conduct objective Bayesian inference. On the other hand, in most situations, they
are not proper prior distributions and may lead to improper posterior, invalidating the analysis since we cannot compute the normalizing constant. Therefore, we need to check if the obtained posterior (and posterior mean) is proper (or finite). The priors for the entropy and its related posterior distributions will be discussed in the next subsections.

Before we derive the priors and posterior distributions, hereafter, we shall always assume that there are at least two distinct data \( t_i \), that is, there exists \( 1 \leq i < j \leq n \) such that \( t_i \neq t_j \). Additionally, before we proceed, we present below a definition and proposition that will be used to prove that the obtained posteriors are proper. In the following let \( \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \) denote the extended real number line and let \( \mathbb{R}^+ \) denote the strictly positive real numbers. The following definition is a special case from the one presented in [25] and will play an important role in proving that the analyzed posterior distributions and posterior means are proper.

**Definition 3.1.** Let \( a \in \mathbb{R} \), \( g : \mathcal{U} \to \mathbb{R}^+ \) and \( h : \mathcal{U} \to \mathbb{R}^+ \), where \( \mathcal{U} \subset \mathbb{R} \) and suppose that \( \lim_{x \to a} \frac{g(x)}{h(x)} = c \in \mathbb{R} \). Then, if \( c > 0 \), we say that \( g(x) \propto x \to a h(x) \).

Regarding the above definition, we have the following proposition from [25].

**Proposition 3.2.** Let \( g : (a,b) \to \mathbb{R}^+ \) and \( h : (a,b) \to \mathbb{R}^+ \) be continuous functions in \( (a,b) \subset \mathbb{R} \), where \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \), and let \( c \in (a,b) \). Then \( g(x) \propto x \to a h(x) \) implies in \( \int_c^a g(t) \, dt \propto \int_c^a h(t) \, dt \) and \( g(x) \propto x \to b h(x) \) implies in \( \int_b^c g(t) \, dt \propto \int_b^c h(t) \, dt \).

### 3.1 Jeffreys prior

Jeffreys [18] described a procedure to achieve an objective prior, which is invariant under one-to-one monotone transformations. The invariant property of the Jeffreys prior has been widely exploited to make statistical inferences from its posterior distribution numerical analysis. The prior construction is based on the square root of the determinant of the Fisher information matrix \( I(\alpha, \beta) \). Thus, the Jeffreys prior to the gamma distribution is given by

\[
\pi_1(\alpha,\beta) \propto \sqrt{\alpha \psi'(\alpha) - 1}. \tag{9}
\]

Additionally, from the determinant of the Fisher information, or using the change of variables over the Jeffreys prior we have

\[
\pi_1(H,W) \propto \sqrt{W \psi'(W) - 1}. \tag{10}
\]

Finally, the joint posterior distribution for \( H \) and \( W \) produced by the Jeffreys prior is

\[
\pi_1(H,W|x) \propto \frac{\delta(W,H)^n W^{nW} \sqrt{W \psi'(W) - 1}}{\Gamma(W)^n} \left\{ \prod_{i=1}^n x_i^W \right\} \exp \left\{ -\delta(W,H) \sum_{i=1}^n x_i \right\}. \tag{11}
\]

**Theorem 3.3.** The posterior density \( \{11\} \) is proper for all \( n \geq 2 \).
Proof. Using the change of variables \( \exp(-H) = u \Leftrightarrow du = -\exp(-H)dH \) and denoting \( \delta_1(W) = \exp(W + \log(\Gamma(W)) + (1 - W)\psi(W)) \) it follows that

\[
d_1(x) \propto \int_0^\infty \int_{-\infty}^\infty \pi_1(H, W|x) \, dHdW
\]

\[
\propto \int_0^\infty \int_0^\infty \frac{\delta_1(W)^nW u^{nW-1} \sqrt{W \psi'(W)} - 1}{\Gamma(W)^n} \left\{ \prod_{i=1}^n x_i^W \right\} \exp\left\{ -\delta_1(W)u \sum_{i=1}^n x_i \right\} \, dudW
\]

\[
= \int_0^\infty \frac{\delta_1(W)^nW \sqrt{W \psi'(W)} - 1}{\Gamma(W)^n} \left\{ \prod_{i=1}^n x_i^W \right\} \int_0^\infty u^{nW-1} \exp\left\{ -\delta_1(W)\left( \sum_{i=1}^n x_i \right) u \right\} \, dudW
\]

\[
= \int_0^\infty \sqrt{W \psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \frac{\Gamma(nW)}{\Gamma(W)^n} dW = \int_0^1 g_1(W)dw + \int_1^\infty g_1(W)dw,
\]

where \( g_1(W) = \sqrt{W \psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \frac{\Gamma(nW)}{\Gamma(W)^n} > 0 \) for all \( W \in (0, \infty) \). Now, according to \( \ref{25} \ref{28} \), we have \( \frac{\Gamma(nW)}{\Gamma(W)^n} \underset{W \to 0^+}{\to} W^{n-1} \) and \( \sqrt{W \psi'(W)} - 1 \underset{W \to 0^+}{\to} W^{-1/2} \) and since

\[
\lim_{W \to 0^+} \frac{\left\{ \prod_{i=1}^n x_i^W \right\}}{\left( \sum_{i=1}^n x_i \right)^n W} = 1 \Rightarrow \frac{\left\{ \prod_{i=1}^n x_i^W \right\}}{\left( \sum_{i=1}^n x_i \right)^n W} \to 1
\]

it follows by Proposition \( \ref{3.2} \) that

\[
\int_0^1 g_1(W)dw \propto \int_0^1 W^{-1/2} \times 1 \times W^{n-1} \, dW < \infty.
\]

Moreover, due to \( \ref{25} \ref{28} \) we have \( \frac{\Gamma(nW)}{\Gamma(W)^n} \underset{W \to \infty}{\to} n^n W^{(n-1)/2} \) and \( \sqrt{W \psi'(W)} - 1 \underset{W \to \infty}{\to} W^{-1/2} \), and since \( x_i \) are not all equal, due to the inequality of the arithmetic and geometric means we have \( q = \log \left( \frac{\sum_{i=1}^n x_i}{\sqrt[n]{\prod_{i=1}^n x_i}} \right) > 0 \) and thus it follows that

\[
\frac{\left\{ \prod_{i=1}^n x_i^W \right\}}{\left( \sum_{i=1}^n x_i \right)^n W} = \left( \frac{\sum_{i=1}^n x_i}{\sqrt[n]{\prod_{i=1}^n x_i}} \right)^{-nW} W^{-nW} = \exp(-nqW)n^{-W}.
\]

Therefore, from Proposition \( \ref{3.2} \) it follows that

\[
\int_1^\infty g_1(W)dw \propto \int_1^\infty W^{-1/2} \times \exp(-nqW)n^{-W} \times n^n W^{(n-1)/2} \, dW
\]

\[
= \int_1^\infty W^{n/2-1} \exp(-nqW) \, dW = \frac{\Gamma(n/2)}{(nq)^{n/2}} < \infty,
\]

which concludes the proof.

\( \square \)

**Theorem 3.4.** The posterior mean of \( H \) relative to \( \{14\} \) is finite for any \( n \geq 2 \).

**Proof.** Doing the change of variables \( \exp(-H) = u \Leftrightarrow du = -\exp(-H)dH \) and denoting \( \delta_1(W) = \]
\[
\exp(W + \log(\Gamma(W)) + (1 - W)\psi(W)), \text{ it follows that}
\]
\[
E_1[H|x] \propto \int_0^\infty \int_{-\infty}^\infty H\pi_1(\alpha, \beta|x) \, dHdW
\]
\[
= \int_0^\infty \int_{-\infty}^\infty \log(u) \frac{\delta_1(W)^n u^{n(W-1)} \sqrt{W\psi'(W)} - 1}{\Gamma(W)^n} \left\{ \prod_{i=1}^n x_i^W \right\} \exp \left\{ -\delta_1(W)u \sum_{i=1}^n x_i \right\} dudW
\]
\[
= \int_0^\infty \frac{\delta_1(W)^n \sqrt{W\psi'(W)} - 1}{\Gamma(W)^n} \left\{ \prod_{i=1}^n x_i^W \right\} \int_0^\infty \left( -\log(u) \right) u^{nW-1} \exp \left\{ -\delta_1(W) \left( \sum_{i=1}^n x_i \right) u \right\} dudW.
\]
Moreover, from the identity \( \psi(z)\Gamma(z) = \Gamma'(z) = \int_0^\infty \log(t) t^{z-1} e^{-t} \, dt = 1/a^z \log(a)(\psi(z)\Gamma(z) - \log(a)\Gamma(z)) \) and thus, letting \( |\cdot| \) denote the absolute value operator and letting \( \delta_2(W) = |\psi(nW)| + |\log(\Gamma(W))| + (1 + W)|\psi(W)| + W + \log(\sum_{i=1}^n x_i) \) for all \( W > 0 \), and using the triangle inequality we have
\[
|E_1[H|x]| \propto \left| \int_0^\infty \left( \psi(nW) - \log \left( \delta_1(W) \sum_{i=1}^n x_i \right) \right) \sqrt{W\psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \Gamma(nW) \, dW \right|
\]
\[
\leq \int_0^\infty \left| \psi(nW) - \log \left( \delta_1(W) \sum_{i=1}^n x_i \right) \right| \sqrt{W\psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \Gamma(nW) \, dW
\]
\[
\leq \int_0^\infty \delta_2(W) \sqrt{W\psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \Gamma(nW) \, dW = \int_0^1 h_1(W)dW + \int_1^\infty h_1(W)dW,
\]
where \( h_1(W) = \delta_2(W) \sqrt{W\psi'(W)} - 1 \left\{ \prod_{i=1}^n x_i^W \right\} \Gamma(nW) \) for all \( W > 0 \).

We shall now prove that \( \delta_2(W) \propto W^{-1} \) and \( \delta_2(W) \propto W\log(W) \). Indeed, notice that \( \delta_2(W) \geq W > 0 \) for \( W > 0 \). Moreover, since due to Abramowitz [1] we have \( \lim_{W \to 0^+} W\Gamma(W) = 1 \) and \( \lim_{W \to 0^+} -W\psi(W) = 1 \) it follows that
\[
\lim_{W \to 0^+} \frac{\psi(nW)}{W^{1-n}} = \lim_{W \to 0^+} \frac{1}{n} [nW] \psi(nW) = \frac{1}{n}
\]
\[
\lim_{W \to 0^+} \frac{\log(\Gamma(W))}{W^{-1}} = \lim_{W \to 0^+} |W \log(\Gamma(W)) - W \log(W)| = 0 \cdot \log(1) - 0 = 0
\]
\[
\lim_{W \to 0^+} \frac{(1 + W)\psi(W)}{W^{-1}} = \lim_{W \to 0^+} (1 + W) |W\psi(W)| = 1 \text{ and}
\]
\[
\lim_{W \to 0^+} \frac{W + \log(\sum_{i=1}^n x_i)}{W^{-1}} = \lim_{W \to 0^+} \left( W^2 + W \log \left( \sum_{i=1}^n x_i \right) \right) = 0
\]
and thus
\[
\lim_{W \to 0^+} \frac{\delta_2(W)}{W^{-1}} = 1/n + 1 \Rightarrow \delta_2(W) \propto W^{1/n} \frac{1}{W}.
\]
On the other hand, since due to Abramowitz [1] we have \( \lim_{W \to \infty} \frac{\psi(W)}{\log(W)} = 1 \), it follows from the L’Hopital rule that
\[
\lim_{W \to \infty} \frac{\log(\Gamma(W))}{W(\log(W) + 1)} = \lim_{W \to \infty} \frac{(\log(\Gamma(W)))'}{W(\log(W) + 1)'} = \lim_{W \to \infty} \frac{\psi(W)}{\log(W)} = 1,
\]
and therefore, considering $W \geq 1$ we have
\[
\lim_{W \to \infty} \frac{|\psi(W)|}{W (\log(W) + 1)} = \lim_{W \to \infty} \frac{1}{W (1 + \log(W) - 1)} \left| \psi(W) \right| \log(W) = 0,
\]
\[
\lim_{W \to \infty} \frac{|\log(\Gamma(W))|}{W (\log(W) + 1)} = \lim_{W \to \infty} \left| \frac{\log(\Gamma(W))}{W (\log(W) + 1)} \right| = 1,
\]
\[
\lim_{W \to \infty} \frac{(1 + W) |\psi(W)|}{W (\log(W) + 1)} = \lim_{W \to \infty} (1 + W^{-1}) \left| \frac{1}{(1 + \log(W) - 1)} \psi(W) \right| = 1, \text{ and}
\]
\[
\lim_{W \to \infty} \frac{W + |\log \left( \sum_{i=1}^{n} x_i \right) |}{W (\log(W) + 1)} = \lim_{W \to \infty} \left( \frac{1}{\log(W) + 1} + \left| \frac{\log \left( \sum_{i=1}^{n} x_i \right) }{W (\log(W) + 1)} \right| \right) = 0,
\]
and thus
\[
\lim_{W \to \infty} \frac{\delta_2(W)}{W (\log(W) + 1)} = 2 \Rightarrow \delta_2(W) \propto W^{-1} \text{ as } W \to \infty.
\]

Therefore, combining the obtained proportionality $\delta_2(W) \propto W^{-1}$ with the proportionality proved in Theorem 3.3 and using Proposition 3.2 we have
\[
\int_{0}^{1} h_1(W) dW \propto \int_{0}^{1} W^{-1} \times W^{-1/2} \times 1 \times W^{n-1} dW < \infty.
\]

Finally, using the proportionality $\delta_2(W) \propto W^{-1} \log(W)$, letting $q > 0$ be as in the proof of Theorem 3.3 and using that $\log(W) + 1 \leq \exp(\log(W)) = W$ for $W \geq 1$, it follows from the proportionality proved during Theorem 3.3 and from Proposition 3.2 that
\[
\int_{1}^{\infty} h_1(W) dW \propto \int_{1}^{\infty} W (\log(W) + 1) \times W^{-1/2} \times \exp (-nqW) n^{-nW} \times n^{nW} W^{(n-1)/2} dW
\]
\[
\leq \int_{1}^{\infty} W^{(n/2+2) - 1} \exp (-nqW) \ dW = \frac{\Gamma(n/2 + 2)}{(nq)^{n/2+2}} < \infty,
\]
which concludes the proof. \hfill \square

In order to sample for the posterior distribution we obtain that the marginal posterior distributions of $W$ is given by
\[
\pi_1(W | \mathbf{x}) \propto \sqrt{W} \psi(W) - \frac{\Gamma(nW)}{\Gamma(W)^n} \left( \frac{\sqrt{\prod_{i=1}^{n} x_i}}{\sum_{i=1}^{n} x_i} \right)^{nW},
\]
and the conditional posterior distribution of $H$ is given by
\[
\pi_1(H | W, \mathbf{x}) \propto \exp \left\{ -nWH - \delta(W, H) \sum_{i=1}^{n} x_i \right\}.
\]

### 3.2 Reference prior

Bernardo [7] discussed a different approach to obtain a new class of objective priors, named as reference priors. Further, many studies were presented to develop formal and rigorous definitions to derive such class of prior distributions under different contexts [2-6]. The reference prior is obtained by maximizing the Kullback-Leibler (KL) divergence assuming some regularity conditions. The idea of the expected posterior information to the prior allows the data to have the maximum influence on the posterior distributions. The reference priors have essential properties such as consistent sampling, consistent marginalization, and one-to-one transformation invariance [8]. The reference priors may depend on the order of the parameters of interest. Hence, for the gamma distribution, we have two distinct priors that are presented below.
3.2.1 Reference prior when $\beta$ is the parameter of interest

The reference prior when $\beta$ is the parameter of interest and $\alpha$ is the nuisance parameter is given by

$$\pi_2(\alpha, \beta) \propto \frac{\sqrt{\psi'(\alpha)}}{\beta}.$$  \hspace{1cm} (12)

Thus, using the Jacobian transformation it follows that the related reference prior is given by

$$\pi_2(W, H) \propto \sqrt{\psi(W)}.$$  \hspace{1cm} (13)

Finally, the joint posterior distribution for $H$ and $W$, produced by the reference prior $[20]$, is given by

$$\pi_2(W, H|x) \propto \delta(W, H)^n H \sqrt{\psi(W)} \frac{\Gamma(nW)}{\Gamma(W)^n} \prod_{i=1}^{n} x_i^H \exp \left\{ -\delta(W, H) \sum_{i=1}^{n} x_i \right\}. \hspace{1cm} (14)$$

**Theorem 3.5.** The posterior density $[14]$ is proper for all $n \geq 2$.

**Proof.** Doing the change of variables $\exp(-H) = w \Leftrightarrow dw = -\exp(-H) dH$, denoting $\delta_1(W) = \exp(W + \log(\Gamma(W))) + (1 - W)\psi(W)$ and proceeding analogously as in the proof of Theorem 3.3 we have

$$d_2(w) \propto \int_{-\infty}^{\infty} \int_{0}^{\infty} \pi_2(W, H|x) dHdW \propto \int_{0}^{1} g_2(W) dw + \int_{1}^{\infty} g_2(W) dw,$$

where $g_2(W) = \sqrt{\psi(W)} \frac{\{\prod_{i=1}^{n} x_i^W\} \Gamma(nW)}{(\sum_{i=1}^{n} x_i)^n \Gamma(W)^n} > 0$ for all $W \in (0, \infty)$. Now, according to $[25, 28]$, we have $\frac{\Gamma(nW)}{\Gamma(W)^n} \propto W^{n-1}$ and $\sqrt{\psi(W)} \propto W^{-1}$, and since we proved in Theorem 3.3 that

$$\frac{\{\prod_{i=1}^{n} x_i^W\}}{(\sum_{i=1}^{n} x_i)^n W} \propto 1,$$

it follows from Proposition 3.2 that

$$\int_{0}^{1} g_2(W) dw \propto \int_{0}^{1} W^{-1} \times 1 \times W^{n-1} dW < \infty.$$

Moreover, from Abramowitz $[11]$ we have $\sqrt{\psi(W)} \propto W^{1/2}$, which combined with $\sqrt{W\psi(W)} - 1 \propto W^{-1/2}$ implies in $\sqrt{\psi(W)} \propto W^{1/2} \sqrt{W\psi(W)} - 1$. Therefore it follows that $g_2(W) \propto g_1(W)$, and by Proposition 3.2 it follows that

$$\int_{1}^{\infty} g_2(W) dw \propto \int_{1}^{\infty} g_1(W) dW < \infty,$$

which concludes the proof. $\square$

**Theorem 3.6.** The posterior mean of $H$ relative to $[14]$ is finite for all $n \geq 2$.

**Proof.** Proceeding analogously as in the proof of Theorem 3.3 it follows that

$$|E_2[H|x]| \propto \int_{0}^{\infty} \int_{-\infty}^{\infty} H \pi_2(H, W|x) dHdW \leq \int_{0}^{\infty} \delta_2(W) \sqrt{\psi(W)} \frac{\{\prod_{i=1}^{n} x_i^W\} \Gamma(nW)}{(\sum_{i=1}^{n} x_i)^n \Gamma(W)^n} dW = \int_{0}^{1} h_2(W) dw + \int_{1}^{\infty} h_2(W) dw,$$

where $h_2(W)$ is the related reference prior.
where $\delta_2(W)$ is the same as defined in the proof of Theorem 3.4 and
\[
h_2(W) = \delta_2(W) \sqrt{\psi'(W)} \left\{ \prod_{i=1}^{n} x_i^{W} \right\} \frac{\Gamma(nW)}{\left( \sum_{i=1}^{n} x_i \right)^{nW}} \Gamma(W)^n.
\]
Since in the proof of Theorem 3.4 we showed that $\delta_2(W) \propto W \to 0^+ W^{-1}$, together with the proportionalities proved in Theorem 3.3 and Proposition 3.2 we have
\[
\int_{0}^{1} h_2(W) dW \propto \int_{0}^{1} W^{-1} \times W^{-1} \times 1 \times W^{n-1} dW < \infty.
\]
Finally, from the proof of Theorem 3.5 we know that $\sqrt{\psi'(W)} \propto W \to \infty \sqrt{W \psi(W)} - 1$, which implies directly that $h_2(W) \propto h_1(W)$, and thus from Proposition 3.2 it follows that
\[
\int_{1}^{\infty} h_2(W) dW \propto \int_{1}^{\infty} h_1(W) dW < \infty,
\]
which concludes the proof.

The marginal posterior distributions of $W$ is given by
\[
\pi_2(W|x) \propto \sqrt{\psi'(W)} \frac{\Gamma(nW)}{\Gamma(W)^n} \left( \frac{\sqrt{\prod_{i=1}^{n} x_i}}{\sum_{i=1}^{n} x_i} \right)^{nW}.
\]
Moreover, the conditional posterior distribution of $H$ is given by
\[
\pi_2(H|W,x) \propto \exp \left\{ -nWH - \delta(W;H) \sum_{i=1}^{n} x_i \right\}.
\]

### 3.2.2 Reference prior when $\alpha$ is the parameter of interest

The reference prior when $\alpha$ is the parameter of interest and $\beta$ is the nuisance parameter is given by
\[
\pi_3(\alpha, \beta) \propto \frac{1}{\beta} \sqrt{n} \frac{\psi'(\alpha) - 1}{\alpha}.
\]
Therefore, in terms of the reparametrized model, the reference prior when $W$ is the parameter of interest and $H$ is the nuisance parameter is given by
\[
\pi_3(W,H) \propto \sqrt{\frac{W \psi'(W) - 1}{W}}.
\]
Finally, the joint posterior distribution for $\alpha$ and $\beta$, produced by the reference prior (16) is given by
\[
\pi_3(W,H|x) \propto \frac{W \psi'(W) - 1}{W} \frac{\delta(W,H)^{nW}}{\Gamma(W)^n} \left\{ \prod_{i=1}^{n} x_i^{W} \right\} \exp \left\{ -\delta(W,H) \sum_{i=1}^{n} x_i \right\}.
\]

**Theorem 3.7.** The posterior density (17) is proper for all $n \geq 2$. 

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Finally, the joint posterior distribution for \( W \) and \( H \) is given by

\[
W \propto \frac{W^{\psi(W)} - 1}{\sqrt{W}} \prod_{i=1}^{n} x_i^W \frac{\Gamma(nW)}{\Gamma(W)^n}.
\]

Moreover, the conditional posterior distribution of \( H \) is given by

\[
\pi_3(H|W,x) \propto \exp\left\{ -nW - \frac{1}{H} \sum_{i=1}^{n} x_i \right\}.
\]

### 3.3 Matching priors

Tibshirani \cite{32} considered a different method to obtain a class of one parameter non-informative prior distribution with nuisance parameters. Letting \( \pi(\theta_1, \theta_2) \) be a prior distribution with the parameter of interest \( \theta_1 \) and a nuisance \( \theta_2 \), the proposed approach requires that the resulting credible interval of the posterior distribution for \( \theta_1 \) have a frequentist coverage accurate to \( O(n^{-1}) \), that is, it requires that

\[
P \left[ \theta_1 \leq \theta_1^{1-\alpha}(\pi;X)||\theta_1, \theta_2 \right] = 1 - \alpha - O(n^{-1}),
\]

where \( \theta_1^{1-\alpha}(\pi;X)||\theta_1, \theta_2 \) denotes the \((1-\alpha)\)th quantile of the posterior distribution of \( \theta_1 \). The priors that satisfy \( (18) \) up to \( O(n^{-1}) \) are known as matching priors. Under parametric orthogonality, Mukerjee & Dey \cite{24} discussed sufficient and necessary conditions for a class of Tibshirani priors to be a matching prior up to \( o(n^{-1}) \). Sun and Ye \cite{31} derived a Berger and Bernardo’s \cite{2} forward and backward reference prior for a two-parameter exponential family, and further showed that the reference prior are special cases of the matching priors. For a gamma distribution, they showed that the reference prior \( (16) \) is a matching prior when \( \beta \) is set as a nuisance parameter and \( \alpha \) is the interest parameter, and proved that there exist no matching prior up to order \( O(n^{-1}) \). Again, they showed that the reference prior \( (16) \) is a matching prior when \( \beta \) is the interest parameter and \( \alpha \) is the nuisance parameter with order \( O(n^{-1}) \) and proved that there exists a matching prior up to order \( O(n^{-1}) \). The cited matching prior is defined as

\[
\pi_4(\alpha, \beta) \propto \frac{\alpha^{\psi(\alpha)} - 1}{\beta \sqrt{\alpha}}.
\]

Thus, the reparametrized version of the proposed matching prior is given by

\[
\pi_4(H, \delta(W, H)) \propto \frac{W^{\psi(W)} - 1}{\sqrt{W}} \prod_{i=1}^{n} x_i^W \frac{\Gamma(nW)}{\Gamma(W)^n} \left\{ \prod_{i=1}^{n} x_i^H \right\} \exp\left\{ -\delta(W, H) \sum_{i=1}^{n} x_i \right\}.
\]

Finally, the joint posterior distribution for \( H \) and \( W \), produced by the matching prior \( (20) \) is given by

\[
\pi_4(W, H|x) \propto \delta(W, H)^n \frac{W^{\psi(W)} - 1}{\sqrt{W}} \frac{\Gamma(nW)}{\Gamma(W)^n} \left\{ \prod_{i=1}^{n} x_i^H \right\} \exp\left\{ -\delta(W, H) \sum_{i=1}^{n} x_i \right\}.
\]
Theorem 3.9. The posterior density \( \pi(W) \) is proper for all \( n \geq 2 \).

Proof. Doing the change of variables \( \exp(-H) = u \leftrightarrow du = - \exp(-H) dH \), denoting \( \delta_1(W) = \exp(W + \log(\Gamma(W))) + (1 - W)\psi(W) \) and proceeding analogously as in the proof of Theorem 3.3 we have

\[
d_4(w) \propto \int_0^\infty \int_0^\infty \pi_4(W,H|x) dHdW \propto \int_0^1 g_4(W)dw + \int_1^\infty g_4(W)dw,
\]

where \( g_4(W) = \frac{(W\psi'(W)-1)}{\sqrt{W}} \frac{\{\prod_{i=1}^n x_i^w\}}{\Gamma(W)^n} \Gamma(nW) > 0 \) for all \( W \in (0,\infty) \). Now, according to [25, 28], we have

\[
\frac{\Gamma(nW)}{\Gamma(W)} \xrightarrow{W \to 0^+} W^{n-1} \quad \text{and} \quad \frac{\sqrt{W}\psi'(W) - 1}{W} \xrightarrow{W \to 0^+} W^{-1/2},
\]

which implies in particular that \( \frac{(W\psi'(W)-1)}{\sqrt{W}} \xrightarrow{W \to 0^+} W^{-3/2} \), and since we already proved in Theorem 3.3 that \( \frac{\{\prod_{i=1}^n x_i^n\}}{(\sum_{i=1}^n x_i)^nW} \xrightarrow{W \to 0^+} 1 \), it follows by Proposition 3.2 that

\[
\int_1^\infty g_4(W)dw \propto \int_1^\infty W^{-3/2} \cdot 1 \cdot W^{-n-1} dW < \infty.
\]

Moreover, due to [25, 28] we have \( \frac{\Gamma(nW)}{\Gamma(W)} \xrightarrow{W \to \infty} n^W W^{(n-1)/2} \) and \( \frac{\sqrt{W}\psi'(W) - 1}{W} \xrightarrow{W \to \infty} W^{-3/2} \), which implies in particular that \( \frac{W\psi'(W)-1}{\sqrt{W}} \xrightarrow{W \to \infty} W^{-3/2} \), and since we already proved in Theorem 3.3 that

\[
\frac{\{\prod_{i=1}^n x_i^n\}}{(\sum_{i=1}^n x_i)^nW} = \exp(-nqW) n^{-nW}, \quad \text{where} \quad q = \log \left( \frac{\sqrt[n]{\sum_{i=1}^n x_i}}{\Gamma(nW)} \right) > 0,
\]

by Proposition 3.2 it follows that

\[
\int_1^\infty g_4(W)dw \propto \int_1^\infty W^{-3/2} \cdot \exp(-nqW) n^{-nW} \cdot n^W W^{(n-1)/2} dW
\]

\[
= \int_1^\infty W^{(n/2-1)} \cdot \exp(-nqW) dW = \frac{\Gamma(n/2 - 1)}{(nq)^{n/2-1}} < \infty,
\]

which concludes the proof. \( \square \)

Theorem 3.10. The posterior mean of \( H \) relative to \( \pi(W) \) is finite for all \( n \geq 2 \).

Proof. Proceeding analogously as in the proof of Theorem 3.3 it follows that

\[
|E_4|H|W| \leq \int_0^\infty \int_0^\infty H \pi_4(H,W|x) dHdW \leq \int_0^\infty \delta_2(W)(W\psi'(W) - 1) \frac{\{\prod_{i=1}^n x_i^n\}}{(\sum_{i=1}^n x_i)^nW} \frac{\Gamma(nW)}{\Gamma(W)^n} dW = \int_0^1 g_4(W)dw + \int_1^\infty g_4(W)dw,
\]

where \( \delta_2(W) \) is given as in the proof of Theorem 3.4 and

\[
h_4(W) = \delta_2(W)(W\psi'(W) - 1) \frac{\{\prod_{i=1}^n x_i^n\}}{(\sum_{i=1}^n x_i)^nW} \frac{\Gamma(nW)}{\Gamma(W)^n}.
\]

Since in the proof of Theorem 3.4 we showed that \( \delta_2(W) \xrightarrow{W \to 0^+} W^{-1} \), together with the proportionalities proved in Theorem 3.3 and Proposition 3.2 we have

\[
\int_0^1 h_4(W)dw \propto \int_0^1 W^{-1} \cdot W^{-3/2} \cdot 1 \cdot W^{-n-1} dW < \infty.
\]
Moreover, letting \( q > 0 \) as in the proof of Theorem 3.3, since we proved during the proof of Theorem 3.4 that \( \delta_2(W) \propto W(\log(W) + 1) \) and since \( \log(W) + 1 \leq \exp(\log(W)) = W \) for \( W \geq 1 \) it follows from Proposition 3.2 that

\[
\int_1^\infty h_4(W)dW \propto \int_1^\infty W(\log(W) + 1) \times W^{-3/2} \times \exp(-nqW) n^{-nW} \times n^n W^{(n-1)/2} dW
\leq \int_1^\infty W^{(n/2+1)-1} \exp(-nqW) dW = \frac{\Gamma(n/2 + 1)}{(nq)^{n/2+1}} < \infty,
\]

which concludes the proof. \( \square \)

The marginal posterior distributions of \( W \) is given by

\[
\pi_4(W|x) \propto W \psi'(W) - 1 \Gamma(nW) \left( \frac{\sqrt{n} \prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} \right)^{nW}.
\]

Moreover, the conditional posterior distribution of \( H \) is given by

\[
\pi_4(H|W,x) \propto \exp \left\{ -nWH - \delta(W,H) \sum_{i=1}^n x_i \right\}.
\]

4 Simulation Study

A Monte Carlo simulation study is conducted to quantify and compare the different non-informative priors’ impact on the entropy measure’s posterior distribution. The Bias and Mean Square Error (MSE) were used to identify the prior that provides the posterior distribution with posterior estimates closer to the true value. These metrics are given by

\[
\text{Bias}_H = \frac{1}{N} \sum_{i=1}^N (\hat{H}_i - H) \quad \text{and} \quad \text{MSE}_H = \frac{1}{N} \sum_{i=1}^N (\hat{H}_i - H)^2,
\]

where \( N = 10,000 \) is the number of samples used to estimate the MLE and posterior quantities of interest. Here, we used the posterior mean as the Bayes estimate due to its good properties. The estimates of \( W \) are not presented since we only considered \( W \) as an auxiliary parameter to conduct the Jacobian transformation, and therefore we are not interested in its respective estimates.

In addition to the Bias and MSE, the coverage probabilities \( CP \) were also presented. Such metrics were obtained from the Bayesian credibility intervals (CI) and the asymptotic confidence intervals of \( H \). The nominal level assumed was 0.95, i.e., we expect an adequate procedure to compute the confidence/credibility intervals should return coverage probabilities closer to 0.95. Regarding the Bias and MSE, the best approach among the selected ones should return the Bias and MSE closest to zero.

The Newton-Raphson iterative method was used to maximize the likelihood in order to obtain the MLE. For a fair comparison, the initial values used to start the iterative procedures were the same values as the used to generate the samples. In real applications, there is a need to set initial values. To this end,
we can use the closed-form maximum a posteriori estimator derived by Louzada and Ramos \[21\] given by

\[
\hat{\alpha} = \left( \frac{n - 2.9}{n} \right) \frac{n \sum_{i=1}^{n} t_i}{(n \sum_{i=1}^{n} t_i \log(t_i) - \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \log(t_i))} \tag{23}
\]

and

\[
\hat{\beta} = \frac{1}{n^2} \left( \frac{n \sum_{i=1}^{n} t_i \log(t_i) - \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \log(t_i)}{n \sum_{i=1}^{n} t_i \log(t_i) - \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \log(t_i)} \right). \tag{24}
\]

Therefore, the initial values for \( H \) and \( W \) are computed from \( \tilde{H} = \hat{\alpha} - \log(\hat{\beta}) + \log \Gamma(\hat{\alpha}) + (1 - \hat{\alpha})\psi(\hat{\alpha}) \) and \( \tilde{W} = \hat{\alpha} \).

In the Bayesian framework, the posterior distribution’s marginal densities involve double integrals to obtain the normalizing constants. Therefore, the MCMC approach was adopted to obtain the posterior estimates. Moreover, the Metropolis-Hastings algorithm was adopted to simulate quantities of interest from the posterior densities. The first 500 samples were discarded in the burn-in stage for each data simulated, and 5000 iterations were further conducted. It was considered a thin parameter of 5 to avoid significant autocorrelation among the sample, returning at the end - 1000 simulated values for each marginal distribution. The Geweke diagnostics \[15\] was considered to confirm the convergence of chains under a confidence level of 95%. The generated samples were used to estimate the posterior mean and the credibility intervals, resulting in 10,000 estimates for \( H \) and \( W \).

The R software (R Core Development Team) was used for the simulation, where the codes can be obtained on request from the corresponding author. For \( n = (20, \ldots, 120) \), only the results sets at \((\alpha, \beta) = (4, 2)\) and \((\alpha, \beta) = (2, 0.5)\) were presented, which leded respectively to \( H = 1.33 \) and \( H = 2.27 \). However, the results were similar for different \( \alpha \) and \( \beta \) and therefore were not presented here. For each sample from the posterior distribution, the posterior mode and the credible intervals were evaluated for \( \alpha, \beta, \) and \( H \).

Tables 1 and 2 present the Bias, MSEs and \( CP_{95\%} \) for the MLE and Bayesian estimators of the entropy measure \( H \). In particular, the results revealed that:

1. For all the parameter estimators, the Bias and MSE approach zero for large \( n \), which implies asymptotic unbiasedness, i.e., the Bias approaches zero, and the MSE decreases as the number of samples increases.

2. The posterior means using the reference prior 1 and 2 were superior to the posterior means using Jeffreys prior and MLE. However, the posterior mean using reference prior 2 was consistently superior to the posterior mean using reference prior 1. The said performance is validated through the coverage probability informed by the CI. Additionally, the coverage probability was high for all the estimators, and the credibility of the interval increases with sample size.

3. For all estimators, the highest drop in Bias and MSE was observed when the sample size increased from 20 to 30.

4. Overall, the results show that the MLE performed worst, given its high bias and MSE. On the other hand, the posterior estimates using the matching prior provided Bayes estimates with smaller Bias and MSE; it was considered the most adequate prior to estimating \( H \).
Table 1: The Bias(MSE) from the estimates of \( \mu \) considering different values of \( n \) with \( N = 100,000 \) simulated samples, using the estimation methods: 1 - MLE, 2 - Jeffreys’s rule, 3 - Reference 1 prior, 4 - Reference 2 prior, and 5 - Tibshirani Prior.

| \( \theta \) | n  | MLE            | Jeffreys’s       | Reference 1 | Reference 2 | Tibshirani |
|------|----|----------------|------------------|-------------|-------------|------------|
|      | 20 | 0.0509(0.0374) | 0.0374(0.0355)  | 0.0359(0.0354) | 0.0251(0.0339) | 0.0114(0.0326) |
|      | 30 | 0.0329(0.0234) | 0.0233(0.0225)  | 0.0221(0.0224) | 0.0148(0.0218) | 0.0056(0.0213) |
|      | 40 | 0.0260(0.0170) | 0.0185(0.0165)  | 0.0176(0.0164) | 0.0121(0.0161) | 0.0052(0.0158) |
|      | 50 | 0.0215(0.0139) | 0.0153(0.0136)  | 0.0147(0.0135) | 0.0102(0.0133) | 0.0047(0.0131) |
|      | 60 | 0.0154(0.0113) | 0.0101(0.0111)  | 0.0097(0.0111) | 0.0059(0.0109) | 0.0014(0.0108) |
|      | 70 | 0.0171(0.0099) | 0.0125(0.0097)  | 0.0121(0.0097) | 0.0090(0.0095) | 0.0049(0.0094) |
|      | 80 | 0.0118(0.0083) | 0.0078(0.0081)  | 0.0074(0.0081) | 0.0047(0.0081) | 0.0012(0.0080) |
|      | 90 | 0.0107(0.0073) | 0.0073(0.0072)  | 0.0068(0.0072) | 0.0044(0.0071) | 0.0012(0.0071) |
|      | 100| 0.0090(0.0067) | 0.0058(0.0067)  | 0.0054(0.0066) | 0.0032(0.0066) | 0.0005(0.0066) |
|      | 110| 0.0085(0.0061) | 0.0056(0.0061)  | 0.0053(0.0061) | 0.0033(0.0060) | 0.0007(0.0060) |
|      | 120| 0.0084(0.0055) | 0.0056(0.0055)  | 0.0054(0.0055) | 0.0036(0.0054) | 0.0012(0.0054) |

According to the simulation results, the posterior distribution with associated matching prior leads to the most precise results with the least bias and MSE. The cited prior outperforms other objective priors and ML estimates considered in this study, and therefore should be chosen as the most appropriate prior for inference. Besides, the posterior estimates obtained from the matching have superior theoretical properties, such as invariance under one-to-one parameter transformation, consistent sampling, and consistency under marginalization. Therefore, we conclude that the posterior estimates derived from the matching prior distribution are more appropriate and superior in making inferences of the gamma distribution’s population parameter. To conduct the Bayesian analysis with the proposed Bayes estimator, we have presented a function in R that can be used for this purpose, where the details can be seen in Appendix A.
Table 2: The CP_{95\%} from the estimates of $\mu$ and $\Omega$ considering different values of $n$ with $N = 10,000,000$ simulated samples, using the estimation methods: 1 - MLE, 2 - Jeffreys’s rule, 3 - Reference 1 prior, 4 - Reference 2 prior, and 5 - Tibshirani Prior.

| $\theta$ | n   | MLE | Jeffreys | Ref W | Ref H | Tibshirani |
|----------|-----|-----|----------|-------|-------|------------|
|          | 20  | 0.923 | 0.932   | 0.936 | 0.943 | 0.951      |
|          | 30  | 0.937 | 0.940   | 0.942 | 0.947 | 0.953      |
|          | 40  | 0.941 | 0.946   | 0.946 | 0.952 | 0.955      |
|          | 50  | 0.936 | 0.942   | 0.943 | 0.945 | 0.950      |
|          | 60  | 0.941 | 0.946   | 0.946 | 0.948 | 0.951      |
| $H = 1.33$ | 70  | 0.937 | 0.945   | 0.942 | 0.946 | 0.946      |
|          | 80  | 0.943 | 0.948   | 0.948 | 0.949 | 0.950      |
|          | 90  | 0.946 | 0.952   | 0.952 | 0.954 | 0.955      |
|          | 100 | 0.945 | 0.950   | 0.948 | 0.950 | 0.951      |
|          | 110 | 0.940 | 0.945   | 0.945 | 0.946 | 0.948      |
|          | 120 | 0.949 | 0.953   | 0.954 | 0.954 | 0.956      |

$H = 2.27$

|          | 20  | 0.946 | 0.951   | 0.952 | 0.956 | 0.957      |
|          | 30  | 0.938 | 0.945   | 0.946 | 0.951 | 0.953      |
|          | 40  | 0.937 | 0.941   | 0.941 | 0.946 | 0.947      |
|          | 50  | 0.939 | 0.945   | 0.946 | 0.948 | 0.949      |
|          | 60  | 0.941 | 0.951   | 0.951 | 0.951 | 0.954      |
|          | 70  | 0.943 | 0.954   | 0.956 | 0.955 | 0.957      |
|          | 80  | 0.944 | 0.956   | 0.957 | 0.959 | 0.959      |
|          | 90  | 0.945 | 0.954   | 0.956 | 0.956 | 0.956      |
|          | 100 | 0.946 | 0.960   | 0.960 | 0.962 | 0.963      |
|          | 110 | 0.943 | 0.956   | 0.956 | 0.956 | 0.957      |
|          | 120 | 0.948 | 0.960   | 0.960 | 0.960 | 0.961      |

5 Application

5.1 Achaemenid dynasty

In this section, the proposed model is applied in the Achaemenid dynasty’s rule time to quantify the variability in the Persian Empire’s political institutions. The Achaemenid dynasty of the Achaemenid empire was the royal house of the ancient Persians who ruled over Persia kingdom. The authority is passed to the descendant of the same bloodline. The Persian Empire was build and expanded through military conquest to extend political control to a broader territory and people. They set out for wars to enlarge territories, resulting in an imbalance in the political institutions. Despite the governmental strategic techniques introduced in Cyrus the Great, the Persian dynasty suffered several reoccurring internal political conflicts, assassinations, and wars from internal and external entities, which shaped the political institutions over the years.
Consequentially, the conflicts influenced each emperor’s tenor duration and the pattern of a new emperor’s ascendance. A stable government often has a long time interval between successive emperors, whereas an unstable government has a short tenor period. The induced patterns can be accounted for using statistical tools such as entropy. The more frequent new emperors ascend the throne, the higher the entropy and the more unstable the government. For the sake of comparison, the proposed model was applied to the Roman Empire timeline data, which was also analyzed by Ramos et al. [26].

Figure 1: Timeline (BC) containing the data, time series plot of the posterior distribution of the entropy and autocorrelation plot for the same distribution.

Figure 2 presents the timeline of the Achaemenid dynasty (top panel), the time series of the Bayesian estimate of the entropy $H$ (down left panel), and the autocorrelation plot (downright panel). The time series and the autocorrelation plot indicate the chain’s convergence, which was also confirmed by the Geweek test [15]. It is worth mentioning that the Kolmogorov-Smirnov (KS) test was used (statistic $D = 0.21$) to confirm if the data can be assumed to follow a gamma distribution.

Using the posterior distribution obtained from the matching prior, the Bayes estimate of the Achaemenid dynasty’s entropy is 4.13 with a 95% credible interval of (3.55; 4.73). Moreover, with the same prior, the posterior estimate for the Roman Empire is 3.08 with a 95% credible interval of (2.80; 3.36). Although the estimated entropies obtained for the Roman Empire were high, the Achaemenid dynasty had a higher entropy, which implies that the Achaemenid dynasty political institution is more volatile compared to the Roman Empire. That is, the time between the successive emperor is significantly different, shorter, and irregular for the Achaemenid dynasty, which signifies instability in their political institutions. These results support the historian’s claim that the Achaemenid Empire set out for wars and consequently were exposed to internal and external insurgency, resulting in instability and high entropy.
5.2 Harvest Sugarcane machine

Sugarcane farming is pertinent to Brazil’s economic growth and has heavily contributed to its Gross Domestic Product (GDP). The production process involves an automated harvesting mechanism, and the interest of the sugarcane farmers is to sustain its harvesting mechanism for an extended period. Moreover, the production chain must be kept in stable conditions to avoid fluctuation in production and prevent wastage due to the deterioration of sugarcane. This application modeled the entropy of the harvest machine failure times in a Bayesian framework, using the Tibshirani prior, to quantify the production process’s irregularity. A high entropy indicates a severe irregularity in the production process. Otherwise, the production process is steady.

The considered data was collected from January 2015 till August 2017, which corresponds to 2.5 harvests and describe twenty-one failure times in days of the suspension of a harvester sugarcane machine: 11, 19, 36, 4, 8, 11, 39, 74, 168, 27, 116, 3, 34, 1, 46, 12, 2, 56, 14, 52, 14.

Figure 2 presents the time series of the Bayesian estimation of the average entropy (left panel) and the correlation lags (right panel). The convergence of the estimate was tested using the Geweek test, and the KS test was used (statistic D = 0.12) to confirm if the data can be assumed to follow a gamma distribution.

The Bayes estimate of the entropy for the failure time of harvest sugarcane machine is 4.55 with a 95% credible interval of (4.04; 5.09). The estimated entropy is significantly high, which implies instability in the production process. Moreover, the process is disrupted in an irregular time duration, which causes the harvest output to be unpredictable and unreliable. The harvesting machine must regularly pass a thorough maintenance check within the harvester’s life circle to keep a steady production flow.

6 Final Remarks

The entropy concept was considered first in statistical thermodynamics, and it was further modified to be applied in other fields. In information theory, the Shannon entropy can measure the uncertainty of a random process. In statistical inference, the Shannon entropy parameters are determined by the maximum
likelihood approach (MLE). However, using this approach, the results are biased for small samples, and the confidence intervals may not achieve the desired coverage probabilities as the asymptotic assumptions are not fulfilled. In the present paper, we introduced a fully objective Bayesian analysis to obtain the Shannon entropy’s posterior distribution to overcome this limitation.

We considered objective priors where the obtained posterior distributions are not overshadowed by prior information. The posterior distributions were derived assuming the Jeffreys prior, reference priors, and matching priors, all invariant under one-to-one transformations. Since the obtained priors are improper, they could lead to improper posteriors, which is undesirable. We proved that the obtained posteriors are proper distributions to address this issue and thus can be used to conduct the Bayesian analysis. The posterior mean was considered a Bayes estimator, and since they may not exist or not be finite, we also proved that the posterior means are finite for any sample sizes. Hence, four posterior distributions were proposed to conduct inference. An intensive simulation study was conducted in order to select one Bayesian estimator or the MLE. The posterior distribution using the matching prior returned better results in terms of bias, mean square error, and coverage probabilities compared with other methods, while the MLE returned the worst results.

We analyzed the gamma distribution’s particular case, a more flexible and general model than the exponential distribution, and has been applied to describe many real phenomena. Although we considered the gamma distribution’s particular case, our approach is general and can be further extended for any probability distribution function.

The proposed Bayes estimator was implemented in an R language, available in the appendix, to estimate the Shannon entropy measure. We apply the implementation to estimate the entropy related to the Achaemenid dynasty’s rule time, which returned a higher value compared with the Roman Empire. It shows that the change in the throne in the Achaemenid dynasty was less probable than the ones in the Roman Empire and indicates a significant instability in their political institutions, which probably contributed to its fall. Further, we analyzed the time until the failure of suspension of the harvest sugarcane machine, where its entropy was estimated using the Bayesian approach.

There are a large number of possible extensions of the current work. Other distributions can be considered assuming the same context, and the Bayes estimator of the Shannon entropy can be derived. Different entropy measure types, such as Hartley, Rényi, and Tsallis entropy, could also be estimated under a Bayesian approach. We plan to explore this line of research in the future.

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Appendix A - Code of the Metropolis-Hasting algorithm within Gibbs

The code in R necessary to obtain the posterior mean of $H$ using the matching prior is given by:

```r
library(coda)
###########################################################################
### Gibbs with Metropolis-Hasting algorithm ###
### R: Iteration Number; burn: Burn in; jump: Jump size; b= Control generation values ###
### posteriorW and posteriorH: logarithm of posterior densities ###
###########################################################################
MCMC<-function(t,R,burn,jump,cW=1,seH=0.2) {
    posteriorW <- function (W) {
        p<-lgamma(n*W)-n*lgamma(W) +0.5*log(W*trigamma(W)-1)-0.5*log(W)+(W)*sum(log(t))-(n*W)*log(sum(t))
        return(p) }
    posteriorH <- function (H,W) {
        delta<-exp(W+lgamma(W)+(1-W)*digamma(W)-H)
        p<-n*W*log(delta)-delta*sum(t)
        return(p) }
    n<-length(t)
    vH<-length(R+1); vW<-length(R+1)
    beta<-(1/(n*(n-1)))*(n*sum((t)*log(t))-sum(t)*sum(log(t)))
    vW[1]<-(n-2.9)/(((n*sum((t)*log(t))/sum(t))-sum(log(t))))
    vH[1]<-vW[1]-log(beta)+lgamma(vW[1])+(1-vW[1])*digamma(vW[1])
    c1<-rep(0,times=R) ; i<-1
    while (i<=R) {
        prop1<-rgamma(1,shape=cW*vW[i],rate=cW)
        d1<-dgamma(vW[i],shape=cW*prop1,rate=cW,log=TRUE)-dgamma(prop1,shape=cW*vW[i],rate=cW,log=TRUE)
        ratio1<-posteriorW(prop1)-posteriorW(vW[i])+d1
        has<-min(1,exp(ratio1)); u1<-runif(1)
        if (is.finite(has)) {
            if (u1<has) {vW[i+1]<-prop1} else {vW[i+1]<-vW[i]}}
        else {vW[i+1]<-vW[i]} ; c1[i]<-0
        prop2<- rnorm(1,vH[i],seH)
        d2<-dnorm(vH[i], mean=prop2, sd=seH, log = TRUE)-dnorm(prop2, mean=vH[i], sd=seH, log = TRUE)
        ratio2<-posteriorH(prop2,vW[i+1])-posteriorH(vH[i],vW[i+1]) +d2
        has<-min(1,exp(ratio2)); u2<-runif(1)
        if (u2<has & is.double(has)) {
            vH[i+1]<-prop2 ; c1[i]<-1} else {
            vH[i+1]<-vH[i] ; c1[i]<-0
            i<-i+1 }

    vH<- vH[seq(burn,R,jump)]; H<-mean(vvH)
    ge1<-abs(geweke.diag(vvH)$z[1])
    pr1i<-quantile(vvH, probs = 0.025, na.rm = FALSE, names = FALSE,type = 7)
    prls<-quantile(vvH, probs = 0.975, na.rm = FALSE, names = FALSE,type = 7)
    return(list(acep=(1-sum(c1)/length(c1)),H=H, LCI_H=pr1i, UCI_H=prls, Geweke.statistics=ge1))
}
########################################################################
## Example ###
palpha<-2; pbeta=0.5; n=50 #Parameters
t<-rgamma(n, palpha, rate=pbeta) # data vector
```

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#Calling the function
MCMC(t,R=2000,burn=500,jump=5)

#Output in R
$acep      ##Acceptance rate
[1] 0.4795

$H         ##Posterior mean of H
[1] 4.880441

$LCI_H     ##Lower credibility interval of H
[1] 1.880321

$UCI_H     ##Upper credibility interval of H
[1] 2.32128

$Geweke.statistics ## Geweke Statistics
1.054945