Quantum Amplitudes in Black-Hole Evaporation
II. Spin-0 Amplitude

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Abstract

This second paper, on spin-0 amplitudes, is based on the underlying results and methods outlined in the preceding Paper I, which describes the complex approach to quantum amplitudes in black-hole evaporation. The main result of the present paper is a computation of the quantum amplitude for a given slightly anisotropic configuration of a scalar field $\phi$ on a space-like hypersurface $\Sigma_F$ at a very late time $T$, given also (for simplicity) that the initial data for gravity and the scalar field at an initial surface $\Sigma_I$ are taken to be spherically symmetric. In particular, this applies to perturbations of spherically-symmetric collapse to a black hole, starting from a diffuse, nearly-stationary configuration, where the bosonic part of the Lagrangian consists of Einstein gravity and a massless, minimally-coupled real scalar field $\phi$. As described in Paper I, Feynman’s $+i\epsilon$ approach is taken; here, this involves a rotation into the complex: $T \to |T| \exp(-i\theta)$, with $0 < \theta \leq \pi/2$. A complex solution of the classical boundary-value problem is expected to exist, provided $\theta > 0$; although for $\theta = 0$ (Lorentzian time-separation), the classical boundary-value problem is badly posed. Once the amplitude is found for $\theta > 0$, one can take the limit $\theta \to 0^+$ to find the Lorentzian amplitude. The paper also includes a discussion of adiabatic solutions of the scalar wave equation, and of the dimensionality of certain quantities which occur frequently in the project as a whole, for which these two initial papers establish the underlying results.

1 Introduction

In this treatment of spin-0 quantum amplitudes in black-hole evaporation, a concrete calculation is given, based on the general procedure outlined in the previous Paper I on the complex approach to such amplitudes [1], and in [2]; see also [3]. Consider asymptotically-flat boundary data $(h_{ij}, \phi)_I$ and $(h_{ij}, \phi)_F$, given on two space-like boundary hypersurfaces $\Sigma_I$ and $\Sigma_F$, which are separated

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by a large Lorentzian proper-time interval $T$, as measured near spatial infinity. Here, $(h_{ij})_{I,F} = (g_{ij})_{I,F}$ $(i,j = 1,2,3)$ gives the intrinsic Riemannian 3-metric on the surfaces $\Sigma_I$ and $\Sigma_F$, where $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ gives the space-time 4-metric. As in Paper I, we assume, for the sake of definiteness, that the bosonic part of the Lagrangian contains Einstein gravity with a minimally-coupled real massless scalar field $\phi$, as posed above in the boundary data. One would like to calculate the quantum amplitude corresponding to these data. It is certainly impossible to compute this amplitude directly via a semi-classical expansion, that is, as approximately $\exp(iS_{\text{class}})$, where $S_{\text{class}}$ is the Lorentzian action of a classical solution of the coupled Einstein/scalar field equations, subject to the boundary data above; typically there is no such classical solution, since the boundary-value problem for hyperbolic equations is badly posed [1,4,5]. Instead, following Feynman’s $+i\epsilon$ prescription [6], we rotate the asymptotic time-interval $T$ into the complex: $T \rightarrow |T|\exp(-i\theta)$, for $0 < \theta \leq \pi/2$. If the resulting complex boundary-value problem, up to gauge, is strongly elliptic [7], one would expect existence and uniqueness properties as for real elliptic equations. In particular, this should give semi-classically a quantum amplitude proportional to $\exp(iS_{\text{class}}) = \exp(-I_{\text{class}})$, where $I$ denotes the Euclidean action, provided that the Lagrangian is invariant under local supersymmetry – it appears that meaningful quantum amplitudes can only be defined in that case [5,8,9]. Following Sec.I.2, one would expect such a semi-classical amplitude to have finite loop corrections in a locally-supersymmetric field theory of supergravity coupled to supermatter, where the loop corrections will only contribute significantly for boundary data involving Planckian energies. For a pure supergravity theory, the semi-classical amplitude is expected to be exact [5,8,9]. Finally, to obtain the amplitude for a Lorentzian time-separation $T$, one takes the limit of the amplitude as $\theta \rightarrow 0_+$.

The concrete calculation in this paper concerns the quantum amplitude corresponding to emission of scalar radiation in the black-hole collapse problem, as measured by non-trivial non-spherical perturbations of $\phi$ on the final surface $\Sigma_F$, which is chosen to be at an extremely late time $T$, so as to intersect all the outgoing radiation. To simplify matters, for the purpose of the present calculation, we assume that there are no gravitons on $\Sigma_F$, that is, that there are no non-spherical perturbations present in the final gravitational data $h_{ijF}$. Equivalently, we assume here that $h_{ijF}$ describes an exactly spherically-symmetric spatial gravitational field. Later on, we shall compute the corresponding amplitudes for the opposite case, for which the only non-trivial perturbations in the final data are gravitational, residing in the 3-metric $h_{ijF}$, whereas $\phi_F$ is taken to be exactly spherically symmetric; this gives the amplitudes for generic spin-2 perturbations. Further to simplify matters, we shall assume, in this paper for spin-0 and in subsequent work for spin-2, that the initial data $(h_{ij},\phi)_I$, representing the gravitational 3-metric and scalar field at an early time before collapse, are exactly spherically symmetric. Of course, once one has the above amplitudes for weak spin-0 and spin-2 fluctuations separately, one can combine them.

In Sec.2 of the present paper, we consider the $(t,r)$-dependent decoupled lin-
ear equations for the functions $R_{\ell m}$ appearing in the decomposition of the scalar wave equation into angular harmonics. Here, $\ell = 0, 1, 2, \ldots$, with $m$ subject to $-\ell \leq m \leq \ell$, are the usual angular-momentum quantum numbers. These equations for $R_{\ell m}(t, r)$ simplify considerably in the adiabatic limit, in which $R_{\ell m}(t, r)$ oscillates much more rapidly than does the background spherically-symmetric geometry $\gamma_{\mu \nu}$. In this limit, the mode equation resembles the corresponding equation for a massless scalar field in an exact Schwarzschild geometry [10,11], except that the gravitational field and potential vary slowly with time. This allows one to approximate much of the classical space-time by means of the Vaidya metric [12]. Sec.3 treats the boundary conditions for suitable radial functions on the final surface $\Sigma_F$. It also begins the process of evaluating the second-linearised classical action $S_{\text{class}}^{(2)}$ corresponding to spin-0 perturbations on $\Sigma_F$, as needed in finding the quantum amplitude. In Sec.4, the analytic behaviour of $S_{\text{class}}^{(2)}$ is studied in detail, as a function of complex $T$, leading to an expression (2.1,4.16) for the amplitude for purely spin-0 perturbations on $\Sigma_F$, found by taking the limit $\theta \to 0^+$, with $T = |T| \exp(-i\theta)$. Sec.5 is concerned, as a practical or efficiency measure, with the question of extracting standard dimensionful factors from all quantities appearing in the calculations, so that all equations can be understood more clearly, reduced to relations between dimensionless quantities. Sec.6 contains a short Conclusion.

2 Adiabatic radial functions

In this paper, we are mainly concerned with evaluating the scalar-field contribution to quantum amplitudes for diffuse weak-field gravitational/scalar data on the final surface, which is separated from the initial surface by the 'Euclidean time-interval' $\tau$, or equivalently the 'Lorentzian time-interval' $T$, related by $\tau = iT$. Here, as in Eq.(I.2.3), we shall generally take $T$ itself to be complex, of the form $T = |T| \exp(-i\theta)$, where $0 < \theta \leq \pi/2$, in order that the classical boundary-value problem should be well-posed. For such amplitudes, following Section I.5, we need to compute

$$\text{Amplitude} = (\text{const.}) \times \exp \left\{ i S_{\text{class}} \left[ (h_{ij}, \phi)_{I} ; (h_{ij}, \phi)_{F} ; T \right] \right\}, \quad (2.1)$$

where $h_{ij} = g_{ij}$ denotes the intrinsic 3-metric on the bounding surface, and

$$S_{\text{class}} \left[ \ldots \right] = \frac{1}{32\pi} \left( \int_{\Sigma_F} - \int_{\Sigma_I} \right) d^3x \pi^{ij} h_{ij} + \frac{1}{2} \left( \int_{\Sigma_F} - \int_{\Sigma_I} \right) d^3x \pi_{\phi} \phi - M T \quad (2.2)$$

gives the Lorentzian classical action, appropriate to specifying $h_{ij}$ and $\phi$ on both bounding surfaces, each surface having the same ADM mass $M$, where the Lorentzian time-separation between them is $T$. Equivalently, one may write $\text{Amplitude} = (\text{const.}) \times \exp(-I_{\text{class}})$, where $I_{\text{class}} = -i S_{\text{class}}$ is the 'Euclidean classical action'.
In the present context of nearly-spherical collapse to a black hole, where
the matter source is a massless scalar field \( \phi \), we expand the fields about the
spherically-symmetric background solution \((\gamma_{\mu\nu}, \Phi)\), as:

\[
g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \ldots ,
\]

\[
\phi = \Phi + \phi^{(1)} + \phi^{(2)} + \ldots .
\]

The classical action then takes the perturbative form

\[
S_{\text{class}} = S_{\text{class}}^{(0)} + S_{\text{class}}^{(2)} + S_{\text{class}}^{(3)} + \ldots ,
\]

where \( S_{\text{class}}^{(0)} \) is the background action, and the next-order correction \( S_{\text{class}}^{(2)} \) is of
second order in small perturbations. Explicitly [Eq.(I.5.10)],

\[
S_{\text{class}}^{(2)} = \frac{1}{32\pi} \left( \int_{\Sigma_F} - \int_{\Sigma_I} \right) d^3x \pi^{(1)ij} h_{ij}^{(1)} + \frac{1}{2} \left( \int_{\Sigma_F} - \int_{\Sigma_I} \right) d^3x \pi^{(1)}_\phi \phi^{(1)}. \tag{2.6}
\]

The quantity of physical interest is the amplitude corresponding to the weak-field non-spherical data \((h_{ij}^{(1)}), (\phi^{(1)})_F\) on the final surface, given here (approximately) by \(\exp(iS_{\text{class}}^{(2)})\). For simplicity of exposition, we shall from now on assume that the initial data are chosen to be exactly spherically-symmetric, given only by \((\gamma_{ij}, \Phi)_I\). Equivalently, we now take

\[
(h_{ij}^{(1)})_I = 0, \quad (\phi^{(1)})_I = 0. \tag{2.7}
\]

Then, the amplitude \(\exp(iS_{\text{class}})\) will only depend on the contributions at the
final surface \(\Sigma_F\) in Eq.(2.6), [which themselves depend on \((h_{ij}^{(1)}, \phi^{(1)}); T\)]. As a
practical matter, we could easily put \((h_{ij}^{(1)}, \phi^{(1)})_I\) back into the calculations that follow. Physically, the analogous step of ‘turning back on the early-time perturbations’ corresponds, in ‘particle language’ rather than in the ‘field language’ being used in this paper, to the inclusion of extra particles in the in-states, together with the original spherical collapsing matter, and asking for the late-time consequences. This was first carried out by Wald [13].

In this paper, we concentrate on the scalar-field contribution to the quantum amplitude \(\exp(iS_{\text{class}})\). That is, we need to compute

\[
S_{\text{class, scalar}}^{(2)} = \frac{1}{2} \int_{\Sigma_F} d^3x \pi^{(1)}_\phi \phi^{(1)} , \tag{2.8}
\]

where the linearised perturbations \((h_{\mu\nu}^{(1)}, \phi^{(1)})\) obey the linearised field equations (I.3.20-22.25) about the spherically-symmetric background \((\gamma_{\mu\nu}, \Phi)\). Here, \((h_{\mu\nu}^{(1)}, \phi^{(1)})\) must agree with the prescribed final data \((h_{ij}^{(1)}, \phi^{(1)})_F\) at the fi-
nal surface \(\Sigma_F\), and be zero at the initial surface \(\Sigma_I\). In the Riemannian case, with a real Euclidean time-interval \(\tau\) between \(\Sigma_I\) and \(\Sigma_F\), or in the case (I.2.3) of a complex time-interval \(T = \tau \exp(-i\theta)\) between \(\Sigma_I\) and \(\Sigma_F\), where
\[0 < \theta < \pi/2\], one expects that this linear boundary-value problem will be well-posed. The other, gravitational, contribution

\[S^{(2)}_{\text{class, grav}} = \frac{1}{32\pi} \int_{\Sigma_F} d^3x \, \pi^{(1)ij} h^{(1)}_{ij}\]  

(2.9)

will be studied subsequently.

As described in Section I.4, at late times the perturbed scalar field equation reduces to

\[\nabla^\mu \nabla_\mu \phi^{(1)} = 0\]  

(2.10)

with respect to the spherically-symmetric background \(\gamma_{\mu\nu}\). Here, in contrast to the case of Secs. I.3.4, it is most natural to describe the gravitational field in the Lorentzian case:

\[ds^2 = -e^{b(t,r)} dt^2 + e^{a(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\]  

(2.11)

Making the mode decomposition with respect to 'Lorentzian coordinates' \((t, r, \theta, \phi)\):

\[\phi^{(1)}(t, r, \theta, \phi) = \frac{1}{r} \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell Y_{\ell m}(\Omega) R_{\ell m}(t, r)\]  

(2.12)

one arrives at the \((\ell, m)\) mode equation:

\[\left(e^{(b-a)/2} \partial_r \right)^2 R_{\ell m} - \left(\partial_t \right)^2 R_{\ell m} - \frac{1}{2} \left(\partial_t (a - b) \right) \left(\partial_t R_{\ell m} \right) - V_\ell(t, r) R_{\ell m} = 0\]  

(2.13)

Here,

\[V_\ell(t, r) = e^{b(t,r)} \left(\ell(\ell+1) + \frac{2m(t,r)}{r}\right)\]  

(2.14)

where \(m(t, r)\) is defined by

\[\exp(-a(t,r)) = 1 - \frac{2m(t,r)}{r}\]  

(2.15)

For high frequencies of oscillation, as measured in the nearly-Lorentzian case, it becomes simpler to understand the solutions of the mode equation (2.13). Noting that the present \((t, r)\)-coordinate description is obtained by replacing \(\tau\) by \(it\), consider a solution \(R_{\ell m}(t, r)\) of the form

\[R_{\ell m}(t, r) \sim \exp(ikt) \xi_{\ell m}(t, r)\]  

(2.16)

where \(\xi_{\ell m}(t, r)\) varies 'slowly' with respect to \(t\). In particular, at spatial infinity, where \(r \to \infty\), \(R_{\ell m}(t, r)\) is required to reduce to a flat space-time separated solution in which \(\xi_{\ell m}(t, r)\) loses its \(t\)-dependence [see Eqs.(2.18,21) below].

We are studying the boundary-value problem for real scalar perturbations \(\phi^{(1)}\), as functions of \((t, r, \theta, \phi)\), or equivalently for real functions \(R_{\ell m}(t, r)\) as in Eqs.(2.13-15), subject to the initial condition \(\phi^{(1)} \mid_{t=0} = 0\) and a final
condition in which \( \phi^{(1)} |_{t=T} \) is prescribed, where again the final time \( T \) is of the form \( T = |T| \exp(-i\theta) \), for \( 0 < \theta \leq \pi/2 \). For ease of visualisation, one may regard \( \theta \) as being small and positive. Were the propagation simply in flat space-time, the solution would be of the form

\[
\phi^{(1)} = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} dk \ a_{k\ell m}(r) \frac{\sin(kt)}{\sin(kT)} Y_{\ell m}(\Omega) ,
\]

(2.17)

where the \( \{a_{k\ell m}\} \) are real coefficients and each function \( \xi_{k\ell m}(r) \) is proportional to a spherical Bessel function \( r j_{\ell}(kr) \) \[14\]. In our gravitational-collapse case, \( \xi_{k\ell m} \) becomes a function of \( t \) as well as of \( r \), but otherwise the pattern remains:

\[
\phi^{(1)} = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} dk \ a_{k\ell m}(t, r) \frac{\sin(kt)}{\sin(kT)} Y_{\ell m}(\Omega) .
\]

(2.18)

Here, the \( \{a_{k\ell m}\} \) characterise the final data: they can be constructed from the given \( \phi^{(1)} |_{t=T} \) by inverting Eq.(2.18). The functions \( \xi_{k\ell m}(t, r) \) are defined in the adiabatic or large–\( |k| \) limit, as in the previous paragraph, via Eq.(2.16), where \( R_{\ell m}(t, r) \) obeys the mode equation (2.13).

More precisely, provided that \( k \) is large, in the sense that the adiabatic approximation

\[
|k| >> \frac{1}{2} |\dot{a} - \dot{b}| ,
\]

(2.19)

\[
|k| >> \frac{\xi_{k\ell m}}{\xi_{k\ell m}} , \quad k^2 >> \frac{\xi_{k\ell m}}{\xi_{k\ell m}} ,
\]

(2.20)

holds, the mode equation reduces approximately to

\[
e^{(b-a)/2} \frac{\partial}{\partial r} \left( e^{(b-a)/2} \frac{\partial \xi_{k\ell m}}{\partial r} \right) + (k^2 - V_\ell) \xi_{k\ell m} = 0 .
\]

(2.21)

Here, of course, the functions \( e^{(b-a)/2} \) and \( V_\ell \) do still vary with the time-coordinate \( t \), but only adiabatically or ‘slowly’.

As described further in [2,12,17], the geometry in the space-time region is expected to be approximated very accurately by a Vaidya metric [12], corresponding to a luminosity in the radiated particles which varies slowly with time. Such a metric can be put in the diagonal form

\[
e^{-a} = 1 - \frac{2m(t,r)}{r} ; \quad e^b = \left( \frac{\dot{m}}{f(m)} \right)^2 e^{-a} ,
\]

(2.22)

where \( m(t,r) \) is a slowly-varying function, with \( \dot{m} = (\partial m/\partial t) \), and where the function \( f(m) \) depends on the details of the radiation. Then Eq.(2.19) implies that

\[
|k| >> \frac{\dot{m}}{m} ,
\]

(2.23)
provided that $2m(t, r) < r < 4m(t, r)$. In this case, the rate of change of the
metric with time is slow compared to the typical frequencies of the radiation;
further, the time-variation scale of the background space-time $\gamma_{\mu\nu}(x)$ is much
greater than the period of the waves. With frequencies of magnitudes $|k| \sim m^{-1}$
dominating the radiation, and with $|m|$ of order $m^{-2}$, the adiabatic
approximation is equivalent to $m^2 \gg 1$, which corresponds to the semi-classical
approximation. If, as expected, $m^3$ is a measure of the time taken by the
hole to evaporate, then $r < 4m \ll m^3$, provided that $m^2 \gg 1$. Thus, in the
large-$k$ approximation used in deriving Eq.(2.21), it is valid at lowest order to
neglect time-derivatives of the background metric, out to radii small compared
with the evaporation time of the hole and with the time since the hole was
formed.

It is natural to define a generalisation $r^*$ of the standard Regge-Wheeler
coordinate $r_s^*$ for the Schwarzschild geometry [10,16], according to
\[
\frac{\partial}{\partial r^*} = e^{(b-a)/2} \frac{\partial}{\partial r} .
\] (2.24)
Under the above conditions, the time-dependence of $r^*(t, r)$ is negligibly small,
and one has $r^* \sim r_s^*$ for large $r$, where
\[
r_s^* = r + 2M \ln \left( \frac{r}{2M} - 1 \right)
\] (2.25)
is the Regge-Wheeler coordinate, expressed in terms of the Schwarzschild radial
coordinate $r$. In terms of the variable $r^*$, the approximate (adiabatic) mode
equation (2.21) reads
\[
\frac{\partial^2 \xi_{k\ell m}}{\partial r^{*2}} + \left( k^2 - V_\ell \right) \xi_{k\ell m} = 0 .
\] (2.26)

3 Boundary conditions

We consider here, for definiteness, a set of suitable radial functions $\{\xi_{k\ell m}(r)\}$
on the final surface $\Sigma_F$. As above, since the mode equation (2.13) does not depend
on the quantum number $m$, we may choose $\xi_{k\ell m}(r) = \xi_{k\ell}(r)$, independently
of $m$.

We seek a complete set, such that any smooth perturbation field $\phi^{(1)}(T, r, \theta, \phi)$
as in Eqs.(2.12,13), restricted to the final surface $\{t = T\}$, of rapid decay near
spatial infinity, can be expanded in terms of the $\xi_{k\ell m}(r)$. The 'left' boundary
condition on the radial functions $\{\xi_{k\ell}(r)\}$ is that of regularity at the origin
$\{r = 0\}$:
\[
\xi_{k\ell}(0) = 0 .
\] (3.1)
The solution to the radial equation which is regular near the origin is:
\[
\xi_{k\ell}(r) = r \phi_{k\ell}(r) \propto r j_\ell(kr) \propto \left\{ \left( (\text{const.}) \times (kr)^{\ell+1} \right) + O((kr)^{\ell+3}) \right\} ,
\] (3.2)
where, again, the \( j_\ell \) are spherical Bessel functions [14], and we have assumed that, for small \( r \), \( m(r) \sim r^3 \), and have neglected \( O(r^2) \) terms. These radial functions are purely real, for real \( k \) and \( r \). For \( k \) purely real and positive, the radial functions describe standing waves, which, for mode time-dependence \( e^{\pm i k t} \), have equal amounts of 'ingoing' and 'outgoing' radiation.

For the 'right' boundary condition, note that the potential \( V_\ell(r) \) vanishes sufficiently rapidly as \( r \to \infty \) that a real solution to Eq.(2.26) behaves according to
\[
\xi_{k\ell}(r) \sim \left( z_{k\ell} \exp(i kr^*_s) + z_{k\ell}^* \exp(-i kr^*_s) \right) \quad (3.3)
\]
as \( r \to \infty \). Here the \( z_{k\ell} \) are certain dimensionless complex coefficients, which can be determined via the differential equation by using the regularity at \( \{ r = 0 \} \). The (approximately) conserved Wronskian for Eq.(2.26), together with Eq.(3.3), and the property
\[
\lim_{r^*_s \to \infty} e^{i(k-k')r^*_s} \left( \xi_{k\ell}^* \xi_{k'\ell} + \xi_{k\ell} \xi_{k'\ell}^* \right) = i\pi \delta(k-k') \quad , (3.4)
\]
give the normalisation condition, for \( -\infty < k, k' < \infty \) and \( R_\infty \to \infty \):
\[
\int_0^{R_\infty} dr \ e^{(a-b)/2} \xi_{k\ell}(r) \xi_{k'\ell}(r) \bigg|_{\Sigma_F} = 2\pi |z_{k\ell}|^2 \left( \delta(k-k') + \delta(k+k') \right) \quad . (3.5)
\]
This normalisation is only possible in an adiabatic approximation. Note that the radial functions \( \{ \xi_{k\ell} \} \) form a complete set only for \( k > 0 \), as a result of our boundary conditions.

The above result makes it possible to evaluate the perturbative massless-scalar contribution to the total classical Lorentzian action \( S_{\text{class}} = S_{\text{class}}^{(0)} + S_{\text{class}}^{(2)} + \ldots \) of Eq.(2.5), with \( S_{\text{class}}^{(2)} \) given by Eq.(2.6). This contribution, namely \( S_{\text{class, scalar}}^{(2)} \) of Eq.(2.8), is then given in the notation of Eq.(2.12) by
\[
S_{\text{class}}^{(2)}[\phi^{(1)}; T] \ = \ \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{R_\infty} dr \ e^{(a-b)/2} \left. R_{\ell m} \left( \partial_t R^*_{\ell m} \right) \right|_T , \quad (3.6)
\]
since
\[
\int d\Omega \ Y_{\ell m} Y^*_{\ell' m'} = \delta_{\ell \ell'} \delta_{m m'} . \quad (3.7)
\]
Within the adiabatic approximation above, and using Eq.(3.5), this gives the frequency-space form of the classical action:
\[
S_{\text{class}}^{(2)}[\{a_{k\ell m}\}; T] = \pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} dk \ k |z_{k\ell}|^2 |a_{k\ell m} + a_{-k\ell m}|^2 \cot(kT) , \quad (3.8)
\]
in terms of the final data \( \{ a_{k\ell m} \} \).
From a mathematical point of view, one would expect to work only with the set of square-integrable scalar wave-functions on the final boundary \( \Sigma \), that is, the set \( L^2(\mathbb{R}^3, dr e^{(a-b)/2}) \). To express this, define

\[
\psi_{\ell m}(r) = r \int d\Omega \ Y_{\ell m}(\Omega) \phi^{(1)}(t, r, \Omega) \bigg|_{t=T}.
\]

(3.9)

Then the square-integrability condition reads

\[
\frac{1}{2\pi} \sum_{\ell m} \int_0^{R\infty} dr \ e^{(a-b)/2} |\psi_{\ell m}(r)|^2 < \infty ,
\]

(3.10)

or, equivalently,

\[
\sum_{\ell m} \int_{-\infty}^{\infty} dk \ \left| z_{k\ell} \right|^2 |a_{k\ell m} + a_{-k\ell m}|^2 < \infty .
\]

(3.11)

The left-hand sides of Eqs.(3.10,11) are in fact equal. This arises from the completeness property

\[
e^{(a-b)/2} \int_{-\infty}^{\infty} dk \ \frac{\xi_{k\ell}(r) \xi_{k\ell}(r')}{|z_{k\ell}|^2} = 4\pi \delta(r-r')
\]

(3.12)

and the inverse of Eq.(2.18):

\[
a_{k\ell m} + a_{-k\ell m} = \frac{1}{2\pi |z_{k\ell}|} \int_0^{R\infty} dr \ e^{(a-b)/2} \xi_{k\ell}(r) \psi_{\ell m}(r) .
\]

(3.13)

From a physical point of view, one expects also that taking scalar boundary data which are not square-integrable will lead to various undesirable properties, such as infinite total energy of the system, or an infinite or ill-defined action.

4 Analytic continuation

The perturbative classical scalar action \( S_{\text{class}}^{(2)} \) of Eq.(3.8) was derived subject to the adiabatic approximation, and also to the requirement that the time-interval \( T \) between the initial and final surfaces, measured at spatial infinity, is complex, of the form \( T = |T| \exp(-i\theta) \), provided that \( 0 < \theta \leq \pi/2 \). In this case, the term \( k \cot(kT) \) in the integrand of Eq.(3.8) remains bounded for \( 0 < k < \infty \), and one expects to obtain a finite complex-valued action \( S_{\text{class}}^{(2)} \left[ \{a_{k\ell m}\}; T \right] \), given square-integrable data \( \phi^{(1)} \) on the final surface \( \Sigma \). Further, the dependence of the complex function \( S_{\text{class}}^{(2)} \left[ \{a_{k\ell m}\}; T \right] \) on the complex variable \( T \) is expected to be complex-analytic in this domain \( (0 < \theta \leq \pi/2) \), and, following Feynman [6], ordinary Lorentzian-signature quantum amplitudes should be given by the limiting behaviour of \( \exp(iS_{\text{class}}^{(2)}) \) as \( \theta \to 0_+ \).
If, on the other hand, one restricts attention to the exactly Lorentzian-signature case ($\theta = 0$), then the integral in Eq. (3.8) will typically diverge, due to the simple poles on the real-frequency axis at

$$k = k_n = \frac{n\pi}{T} , \quad (4.1)$$

($n = 1, 2, \ldots$).

Clearly, in order to be able to treat this classical Einstein/massless-scalar boundary-value problem in a way which is analytically sensible, one has to allow $T$ to be displaced into the complex as above, whether by a small or a large angle $\theta$ of rotation. The spherically-symmetric ‘background’ 4-geometry $\gamma_{\mu\nu}$ and scalar field $\Phi$ will typically be complex. One might hope that the general (asymmetric) boundary-value problem of this type would admit a unified mathematical description, based on the (conjectured) property that the boundary-value problem is strongly elliptic [7], up to gauge.

Given that $T = |T| \exp(-i\delta)$ is slightly complex, with $0 < \delta \ll 1$, consider an integral such as Eq. (3.8) for $S_{\text{class}}^{(2)} \{a_{k\ell m}\}; T$. Write this as

$$J = \sum_{\ell m} \int_0^\infty dk \ f_{\ell m}(k) \ \cot(kT) , \quad (4.2)$$

where

$$f_{\ell m}(k) = \pi k \ |z_{k\ell}|^2 \ |a_{k\ell m} + a_{-k\ell m}|^2 . \quad (4.3)$$

There are infinitely many simple poles of the integrand at $k = k_n$ ($n = 1, 2, \ldots$), just above the positive real $k$-axis. We then deform the original contour $C$ along the positive real $k$-axis into three parts, $C_\epsilon$, $C_R$ and $C_\alpha$, where $0 < \alpha \ll 1$. The contour $C_\epsilon$ lies in the lower half-plane, half-encircling each of the simple poles near the positive real $k$-axis, with radius $\epsilon$. The curve $C_R$, also in the lower half-plane, is an arc of a circle $|k| = R$ of large radius. The curve $C_\alpha$ is part of the radial line $\arg(k) = -\alpha$. We write

$$J = \sum_{\ell m} \int_{C_\alpha + C_R - C_\epsilon} dk \ f_{\ell m}(k) \ \cot(kT)$$

$$= J_\alpha + J_R + J_\epsilon . \quad (4.4)$$

Starting with the integral $J_R$, one finds

$$|J_R| \leq \sum_{\ell m} \int_0^\alpha d\theta \ R \ |f_{\ell m}(R, \theta)| \ \coth(|T|R \sin \theta) , \quad (4.5)$$

where $k = Re^{-i\theta}$ on $C_R$, and we have used $|\cot(kT)| \leq \coth(|T|R \sin \theta)$. When the limit $R \to \infty$ is eventually taken, one expects that the contribution from $C_R$ to the total action should vanish: this requires that $|f_{\ell m}(k)|$ should decay at least as rapidly as $|k|^{-2}$, as $|k| \to \infty$. In fact, on dimensional grounds, one expects that

$$|f_{\ell m}(k)| \sim |k|^{-3} \quad (4.6)$$
as $|k| \to \infty$. To see this, rewrite the radial equation (2.26) in terms of the operator

$$
\mathcal{L}_\ell = e^{(b-a)/2} \frac{d}{dr} \left( e^{(b-a)/2} \frac{d}{dr} \right) - V_\ell(r),
$$

which is self-adjoint with respect to the inner product (3.5). Then note that Eq.(3.13) can be rewritten as

$$
a_{k\ell m} + a_{-k\ell m} = -\frac{1}{2\pi k^2 |z_{k\ell}|^2} \int_0^{R_{\infty}} dr \ e^{(a-b)/2} \xi_{k\ell}(r) \mathcal{L}_\ell \psi_{\ell m}(r). \quad (4.8)
$$

We have used the boundary condition (3.1) and assumed that $\psi_{\ell m}(r)$ dies out at large $r$. The form (4.8) is just an expression of the self-adjointness of the radial equation. Now consider the dimensions of the quantities involved – treated explicitly in Sec.5 below. In particular, $\psi_{\ell m}(r)$ has dimensions of length and $|z_{k\ell}|^2$ is dimensionless. In the limit $R_{\infty} \to \infty$, and for large $k$ (so taking a WKB approximation for the radial functions), the integral in Eq.(4.8) can only involve the dimensionless frequency $2Mk$, where $M$ is the total mass (true ADM mass) of the space-time. This gives the desired behaviour (4.6) at large $|k|$.

The contour $C_\epsilon$ gives a purely imaginary contribution to the total Lorentzian action; also (below) the curve $C_\alpha$ gives a complex contribution. We shall interpret the quantity $\exp[-2 \text{ Im}(S)]$, up to normalisation, as describing the conditional probability density over the final boundary data. To compute $J_\epsilon$, we assume that $f_{\ell m}(k)$ is analytic in a neighbourhood of $k = \sigma_n$, where

$$
\sigma_n = \frac{n\pi}{|T|}, \quad (4.9)
$$

for $n = 1, 2, \ldots$. Then

$$
J_\epsilon = -\lim_{\epsilon \to 0} \sum_{\ell m} \int_{C_\epsilon} dk \ f_{\ell m}(k) \cot(k|T|)
= \frac{i\pi}{|T|} \sum_{\ell m} \sum_{n=1}^{\infty} f_{\ell m}(\sigma_n). \quad (4.10)
$$

For the curve $C_\alpha$, one has

$$
J_\alpha = -\sum_{\ell m} \int_R^0 dk \ |k| \ e^{-i\alpha} f_{\ell m}(|k|, \alpha) \cot(|k| e^{-i\alpha} |T|). \quad (4.11)
$$

We shall need the properties [14]

$$
\cot(x) = \sum_{n=-\infty}^{\infty} \frac{1}{x - n\pi}, \quad (4.12)
$$

and

$$
\frac{1}{(x-a \pm i\epsilon)} = \text{P.P.} \frac{1}{x-a} \mp i\pi \delta(x-a), \quad (4.13)
$$

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where \( P.P. \) denotes the principal part. Assuming that \( f_{\ell m}(k) \) is regular along \( C_\alpha \), one has, for small \( \alpha \):

\[
J_{\alpha}^{(1)} = \lim_{\alpha \to 0^+} \sum_{\ell m} \sum_{n=1}^{\infty} \int_0^R d|k| \frac{f_{\ell m}(|k|, \alpha)}{|kT| - n\pi - i\alpha} \]

\[ (4.14) \]

In the further limit \( R \to \infty \), this gives

\[
J_{\alpha} = P.V. + \frac{i\pi}{|T|} \sum_{\ell m} \sum_{n=1}^{\infty} f_{\ell m}(\sigma_n), \]

\[ (4.15) \]

where P.V. denotes the principal-value part of the integral.

Using Eqs.(4.4,5,10,15), the classical action for massless scalar-field perturbations, in the case that \( T = |T|\exp(-i\delta) \) is very slightly complex, is

\[
S_{\text{class}}^{(2)} \left[ \{a_{k\ell m}\}; |T| \right] = \text{real part} + \frac{2i\pi}{|T|} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=1}^{\infty} f_{\ell m}(\sigma_n)
\]

\[ = \text{real part} + \frac{2i\pi^2}{|T|} \sum_{\ell mn} \sigma_n |z_{n\ell}|^2 |a_{\ell m} + a_{-\ell m}|^2. \]

\[ (4.16) \]

The real part of \( S_{\text{class}}^{(2)} \) is, of course, also calculable from the equations above. It contains the principal-value term and the real part of Eq.(4.10). The main, semi-classical contribution to the quantum amplitude is then \( \exp\left(iS_{\text{class}}^{(2)} \left[ \{a_{k\ell m}\}; |T| \right] \right) \).

The probability distribution for final configurations involves only \( \text{Im}(S_{\text{class}}^{(2)}) \); the more probable configurations will have \( S_{\text{class}}^{(2)} \) lying only infinitesimally in the upper half-plane. Whether probable or not, those final configurations \( \{a_{k\ell m}\} \) which contribute to the probability distribution must yield finite expressions in the infinite sums over \( n\ell \) in Eq.(4.16). There will be a corresponding restriction when the data are instead described in terms of the spatial configurations \( \{\psi_{\ell m}(r)\} \). Also, as can be seen in [17], the complex quantities \( z_{n\ell}(a_{n\ell m} + a_{-n\ell m}) \) appearing in Eq.(4.16) are related to Bogoliubov transformations between initial and final states, thus providing a further characterisation of the finiteness of \( \text{Im}(S_{\text{class}}^{(2)}) \) in Eq.(4.16).

With regard to the sum over \( \ell \) in Eq.(4.16), one imagines that a cut-off \( \ell_{\max} \) can be provided by the radial equation (2.26). In the region where \( (V_\ell(r) - k^2) > 0 \), one has exponentially growing radial functions, whereas for \( (V_\ell(r) - k^2) < 0 \), one has oscillatory radial functions. One defines \( \ell_{\max} \) by \( (V_{\ell_{\max}}(r) - k^2) = 0 \), and restricts attention mainly to oscillatory solutions.

When one has both initial and final non-zero Dirichlet data labelled by 'coordinates' \( \{a_{k\ell m}^{(I)}\} \) and \( \{a_{k\ell m}^{(F)}\} \), the perturbative classical scalar action \( S_{\text{class}}^{(2)} \) includes separate terms of the form (4.16) for the initial and final data. But \( S_{\text{class}}^{(2)} \) also includes a cross-term between \( a_{k\ell m}^{(I)} \) and \( a_{k\ell m}^{(F)} \), which represents the correlation or mixing between the initial and final data. The total action will
naturally be symmetric in $a^{(I)}_{k\ell m}$ and $a^{(F)}_{k\ell m}$, and the coefficients $z_{n\ell}$ will be the same (they are time-independent) up to a phase. For large $|T|$, the cross-term becomes negligible, and one has two independent contributions to the classical action, one being a functional of $\{a^{(I)}_{k\ell m}\}$, the other of $\{a^{(F)}_{k\ell m}\}$.

5 Dimensional analysis

In Sec.4 above, we made use of a dimensional argument, in order to obtain the estimate (4.6) of the rate of fall-off of $|f_{\ell m}(k)|$ for large $|k|$. In the present Section, we give a more thorough treatment of the dimensionality of the main quantities appearing in the preceding paper I, this Paper II and subsequent work.

The classical perturbative Einstein equations are schematically of the form $L^{-2} \sim (\partial \phi^{(1)})^2$, where $L$ is a length-scale characteristic of the gravitational field. Hence, the massless-scalar perturbations $\phi^{(1)}$ are dimensionless. Similarly, one finds that the metric perturbations $h^{(1)}_{\mu\nu}$ are dimensionless. From this, one deduces that the functions $\xi_{k\ell m}(t, r)$ of Eq.(2.16), appearing in the adiabatic approximation, are also dimensionless, as are the complex coefficients $z_{k\ell}$ of Eq.(3.3).

In describing dimensionful quantities, let us denote by $M$ the ADM mass of the 'space-time'. We recall again the distinction which it was necessary to draw in Sec.I.5 and in [3] between the naive mass, computed from the fall-off of the spatial metric $g_{ij}$ of a hypersurface badly embedded near spatial infinity, and the true ADM mass. Here, $M$ denotes the true ADM mass, as given by the standard definition for a hypersurface which approaches spatial infinity in the familiar way [10]. Suppose that the time-separation $T$ at spatial infinity obeys $|T| \gg 2M$, as will normally be the case; further, let the radius $R_\infty$ tend to infinity. Then, in our treatment of massless perturbations, $M$ can be regarded as setting a reference length-scale.

We now write

$$x = 2Mk$$

(5.1)

for the dimensionless frequency. Further, recall that the coordinates $\{a_{k\ell m}\}$ of Eq.(2.18) for the perturbative scalar field on the final surface $\Sigma_F$ are defined irrespective of $T$. From Eq.(2.18), it follows that each $a_{k\ell m}$ scales as $M^2$. That is, one can write

$$a_{k\ell m} = M^2 y_{\ell m}(x) \ ,$$

(5.2)

for some (smooth) dimensionless complex functions $y_{\ell m}(x)$. This property can also be derived from Eq.(3.13). By following these arguments, one can write the imaginary part of the total classical action as

$$\text{Im}\left(S^{(2)}_{\text{class}}[\{y_{\ell m}\}; T]\right) = \left(\frac{\pi}{2}\right) M^2 \sum_{n\ell m} x_n (\Delta x_n) |z_{n\ell}|^2 |y_{n\ell m}|^2 \ ,$$

(5.3)

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\[ x_n = 2M\sigma_n, \quad \sigma_n = \frac{n\pi}{|T|}, \quad (5.4) \]
\[ \Delta x_n = \frac{2\pi M}{|T|}. \quad (5.5) \]

Clearly, some understanding of the numerical magnitude of the sum in Eq.(4.16) is necessary, if \( \exp[-2 \text{Im}(S_{\text{class}}^{(2)})] \) is to be interpreted as a conditional probability density. (Indeed, the requirement that these probabilities sum to 1 is part of the requirement of unitarity!) In this lowest-order perturbative approximation, the probability density above for scalar fluctuations is Gaussian; the same holds in the present Einstein/massless-scalar case for the distribution of gravitational-wave fluctuations on the final surface \( \Sigma_F \); this is treated in more detail in [18]. The Gaussian property is a perturbative aspect of a more general property – arising in the Positive Action conjecture [19,20]. For simplicity of exposition, let us consider this for the case of Riemannian 4-metrics, although we expect some version to hold in the more general complex case. Suppose that, as in the ‘Euclidean’ path integral, we take all smooth 4-metrics \( g_{\mu\nu} \), scalar fields \( \phi, \ldots \) which agree with Dirichlet boundary data \( h_{ij}, \phi, \ldots \) prescribed on a compact boundary \( \partial V \), bounding a compact manifold-with-boundary \( V \). The Riemannian action functional \( I[g_{\mu\nu}; \phi; \ldots] \) may be seen to take arbitrarily negative values, when one applies suitable high-frequency conformal transformations \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \) to the metric [19]. But, at least for Einstein gravity with the simplest topology, the classical action \( I_{\text{class}} \) is a non-negative functional of the boundary data [20]. Similar positivity properties can be investigated in the case of Einstein gravity coupled to matter.

This dimensional analysis sets a natural normalisation for the (equal) quantities on the left-hand sides of Eqs.(3.10,11). Using Eq.(5.2), one has
\[
2 \sum_{\ell m} \int_0^\infty dk \ |z_{k\ell}|^2 |a_{k\ell m} + a_{-k\ell m}|^2 = M^3 \sum_{\ell m} \int_0^\infty dx \ |z_{\ell}(x)|^2 |y_{\ell m}(x)|^2 \propto t_0.
\]
(5.6)
The last line applies to semi-classical collapse and subsequent evaporation of a non-rotating black hole, where \( t_0 \propto M^3 \) is the time taken for complete evaporation [15].

6 Conclusion

In this paper, we have derived the quantum amplitude [through Eq.(4.16)] for a spherically-symmetric configuration \((h_{ij}, \phi)\) on the initial surface \( \Sigma_I \) to become a given configuration \((h_{ij}, \phi)\) on the final surface \( \Sigma_F \), with Lorentzian time-interval \( T \) at spatial infinity, provided that the final 3-dimensional metric \( h_{ijF} \) is also spherically symmetric. In the amplitude, which is of the form \( (\text{const.}) \times \exp(iS_{\text{class}}^{(2)}) \), the action \( S_{\text{class}}^{(2)} \) depends approximately quadratically on
the (non-spherical) perturbative part of the final data \( \phi_F \). Further, \( S^{(2)}_{\text{class}} \) has both a real part and an imaginary part. The imaginary part leads to a Gaussian probability density \(|\Psi|^2 \propto \exp(-2\text{Im}(S^{(2)}_{\text{class}}))\), while the real part gives rapid oscillations through the phase of the quantum amplitude or wave function \( \Psi \). Corresponding quantum amplitudes for spin-1 and spin-2 fields have been calculated [18], while the fermionic massless spin-\( \frac{1}{2} \) case is summarised in [21].

The familiar description of black-hole evaporation in terms of Bogoliubov coefficients occurs in many works [22-24]. Following the 'Complex Approach' and the 'Spin-0 Amplitude' for black-hole evaporation, we next make a connection between the description and calculation of quantum amplitudes in the present paper and the alternative description in terms of Bogoliubov coefficients [17]. Further, we treat the approximation of the radiative part of the space-time by the Vaidya metric [12], related to the description of adiabatic scalar-field solutions in Sec.2 of the present paper. Yet another description of quantum amplitudes will later be provided, relating Eq.(4.16) to coherent and squeezed states; this gives a more general conceptual framework with which we have already examined the amplitudes of the present paper and the higher-spin amplitudes [18,21].

References

[1] A.N.St.J.Farley and P.D.D’Eath, ‘Quantum Amplitudes in Black-Hole Evaporation: I. Complex Approach’, submitted for publication, 2005.
[2] A.N.St.J.Farley, ‘Quantum Amplitudes in Black-Hole Evaporation’, Cambridge Ph.D. dissertation, approved 2002 (unpublished); A.N.St.J.Farley and P.D.D’Eath, Phys Lett. B, 601, 184 (2004).
[3] M.K.Parikh and F.Wilczek, Phys. Rev. Lett. 85, 5042 (2000); M.Parikh, Gen. Relativ. Gravit. 36, 2419 (2004).
[4] P.R.Garabedian, Partial Differential Equations, (Wiley, New York) (1964).
[5] P.D.D’Eath, Supersymmetric Quantum Cosmology, (Cambridge University Press, Cambridge) (1996).
[6] R.P.Feynman and A.R.Hibbs, Quantum Mechanics and Path Integrals, (McGraw-Hill, New York) (1965).
[7] W.McLean, Strongly Elliptic Systems and Boundary Integral Equations, (Cambridge University Press, Cambridge) (2000); O.Reula, ‘A configuration space for quantum gravity and solutions to the Euclidean Einstein equations in a slab region’, Max-Planck-Institut für Astrophysik, MPA, 275 (1987).
[8] P.D.D’Eath, ‘Loop amplitudes in supergravity by canonical quantization’, in Fundamental Problems in Classical, Quantum and String Gravity, ed. N.Sánchez (Observatoire de Paris) 166 (1999), [hep-th/9807028].
[9] P.D.D’Eath, ‘What local supersymmetry can do for quantum cosmology’, in The Future of Theoretical Physics and Cosmology, eds. G.W.Gibbons, E.P.S.Shellard and S.J.Rankin (Cambridge University Press, Cambridge) 693 (2003).
[10] C.W.Misner, K.S.Thorne and J.A.Wheeler, *Gravitation*, (Freeman, San Francisco) (1973).

[11] J.A.H.Futterman, F.A.Handler and R.A.Matzner, *Scattering from Black Holes* (Cambridge University Press, Cambridge) (1988).

[12] P.C.Vaidya, Proc. Indian Acad. Sci. A33, 264 (1951); R.W.Lindquist, R.A.Schwartz and C.W.Misner, Phys. Rev. 137, 1364 (1965); A.N.St.J.Farley and P.D.D’Eath, ‘Vaidya Space-Time and Black-Hole Evaporation’, submitted for publication (2005).

[13] R.M.Wald, Phys. Rev. D 13, 3176 (1976).

[14] M.Abramowitz and I.A.Stegun, *Handbook of Mathematical Functions*, (Dover, New York) (1964).

[15] D.N.Page and S.W.Hawking, Astrophys. J. 206, 1 (1976).

[16] T.Regge and J.A.Wheeler, Phys. Rev. 108, 1063 (1957).

[17] A.N.St.J.Farley and P.D.D’Eath, Phys. Lett B 613, 181 (2005).

[18] A.N.St.J.Farley and P.D.D’Eath, Class. Quantum. Grav. 22, 2765 (2005).

[19] G.W.Gibbons, S.W.Hawking and M.J.Perry, Nucl. Phys. B138, 141 (1978).

[20] R.Schoen and S.-T.Yau, Phys. Rev. Lett. 42, 547 (1979).

[21] A.N.St.J.Farley and P.D.D’Eath, Class. Quantum Grav. bf 22, 3001 (2005).

[22] S.W.Hawking, Commun. Math. Phys. 43, 199 (1975).

[23] N.D.Birrell and P.C.W.Davies, *Quantum fields in curved space*, (Cambridge University Press, Cambridge) (1982).

[24] V.P.Frolov and I.D.Novikov, *Black Hole Physics*, (Kluwer Academic, Dordrecht) (1998).