On the dimension formula for the hyperfunction solutions of some holonomic D-modules

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Abstract

In this short note we improve a recent dimension formula of Takeuchi for the dimension of the hyperfunction solutions of some holonomic D-modules. Besides the constructibility result and the local index formula of Kashiwara for the holomorphic solution complex, we only use a vanishing theorem of Lebeau together with a simple calculation in terms of constructible functions.

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1 Introduction

One of the basic results about holonomic D-modules is the constructibility result of Kashiwara \[6\], that the holomorphic solution complex

\[ \text{Sol}(\mathcal{M}) := \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \]

of a holonomic D-module \(\mathcal{M}\) on the complex manifold \(X\) is a bounded complex analytically constructible complex of sheaves of complex vector spaces with finite dimensional stalks (compare also with \[7\] chapter 5, \[10\] thm.4.5.8, p.458 and \[14\] chap.III). In particular, the function

\[ \chi(\mathcal{M}) : X \to \mathbb{Z} ; x \mapsto \chi(\text{Sol}(\mathcal{M})_x) \] (1)

is well defined and complex analytically constructible. Here \(\chi\) is the usual Euler characteristic. Moreover, one has by Kashiwara \[5\] the following beautiful description of this local index in terms of the characteristic cycle of the holonomic D-module \(\mathcal{M}\) (see \[7\] thm.6.3.1, p.127, cor.6.3.4, p.128 and \[2\] thm.2, p.574):

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Theorem 1.1 ((local index formula)). Let $Y_j$ be finitely many distinct irreducible closed complex analytic subsets of $X$ such that the characteristic variety $\text{char}(\mathcal{M})$ of $\mathcal{M}$ can be estimated by

$$\text{char}(\mathcal{M}) \subset \bigcup_j T^*_Y X, \quad \text{with} \quad T^*_Y X := \text{cl}(T^*_{Y_{reg}} X)$$

the closure of the conormal bundle to the regular part $Y_{reg}$ of the irreducible complex analytic subset $Y \subset X$. Then

$$\chi(\mathcal{M})(x) = \sum_j (-1)^{d_j} \cdot m_j \cdot Eu_{Y_j}(x),$$

with $d_j$ the complex codimension of $Y_j$, $Eu_{Y_j}$ the famous Euler obstruction of $Y_j$ as defined by MacPherson (cf. [13]) and $m_j$ the (generic) multiplicity of $\mathcal{M}$ along $T^*_Y X$.

Note that $Eu_Y = 1_Y$, if $Y \subset X$ is a closed complex analytic submanifold. So a very special case of the local index formula is given as in [7, ex. on p.129] by the

**Example 1.1.** Suppose all $Y_j$ in the estimate (2) are closed connected complex analytic submanifolds of $X$. Then

$$\chi(\mathcal{M})(x) = \sum_j (-1)^{d_j} \cdot m_j \cdot 1_{Y_j}(x),$$

with $d_j$ the complex codimension of $Y_j$ and $m_j$ the (generic) multiplicity of $\mathcal{M}$ along $T^*_Y X$.

Let us now consider the case that $X$ is the complexification of the real analytic manifold $M$, with $i : M \hookrightarrow X$ the closed inclusion. Assume $M$ is purely n-dimensional. Then the sheaf complex

$$R\Gamma_M(\mathcal{O}_X)[n] \simeq Ri_*i^!(\mathcal{O}_X)[n]$$

is concentrated in degree zero, with

$$\mathcal{B}_M := h^0( i^!(\mathcal{O}_X)[n] ) \otimes or_M$$

the sheaf of Sato’s hyperfunctions on $M$, and $or_M$ the orientation sheaf of $M$.

Then the hyperfunction solution complex

$$\text{Rhom}_{i^!D_X}(i^*\mathcal{M}, \mathcal{B}_M) \simeq i^!\text{Sol}(\mathcal{M}) \otimes or_M[n]$$

of a holonomic D-module $\mathcal{M}$ on $X$ is subanalytically constructible on $M$ with finite dimensional stalks (compare [7, thm.5.1.7, p.115]). So it is natural to ask for a corresponding index formula like (3) or (4).
Remark 1.1. The same constructibility result is true for the solutions

\[ \text{Rhom}_{i^*D^X}(i^*M, A_M) \simeq i^*\text{Sol}(M) \]

in the sheaf \( A_M = i^*O_X \) of real analytic functions on \( M \). If \( M \) is a regular holonomic D-module on \( X \), then one also has isomorphisms (see [8, cor.8.3, cor.8.5, p.360] or [1, p.326]):

\[ \text{Rhom}_{i^*D^X}(i^*M, B_M) \simeq \text{Rhom}_{i^*D^X}(i^*M, Db_M) \]

and

\[ \text{Rhom}_{i^*D^X}(i^*M, A_M) \simeq \text{Rhom}_{i^*D^X}(i^*M, C^\infty_M), \]

with \( Db_M \) (or \( C^\infty_M \)) the sheaf of distributions (or smooth functions) on \( M \).

The following counterpart of (4) is the main result of this note:

**Theorem 1.2 ((local dimension formula)).** Let \( M_j \) be finitely many distinct closed real analytic submanifolds of \( M \) such that the characteristic variety \( \text{char}(M) \) of the holonomic D-module \( M \) on \( X \) can be estimated by

\[ \text{char}(M) \subset \bigcup_j T^*_Y X, \]

with \( Y_j \subset X \) the complexification of \( M_j \). Assume the \( Y_j \) are irreducible (i.e. connected), with \( Y_j \cap M = M_j \). Then one has for \( x \in M \):

\[ \dim_C \left( \text{hom}_{i^*D^X}(i^*M, B_M)_x \right) = \sum_j m_j \cdot 1_{M_j}(x), \]

with \( m_j \) the (generic) multiplicity of \( M \) along \( T^*_Y X \).

This is indeed a counterpart of (4). The estimate (6) implies by a theorem of Lebeau [12] (compare also with [3] thm.2.1, rem., p.531 and [3] ex.(1), p.533) the vanishing result

\[ \text{Ext}^k_{i^*D^X}(i^*M, B_M)_x = 0 \quad \text{for all } k \geq 1 \]

so that

\[ \dim_C \left( \text{hom}_{i^*D^X}(i^*M, B_M)_x \right) = \chi \left( \text{Rhom}_{i^*D^X}(i^*M, B_M)_x \right). \]

Theorem 1.2 answers affirmatively a question asked (or better, discussed) in [17] rem.3.5 at the end of a recent paper of Takeuchi [17], where he proves the dimension formula (7) under the special assumption, that in suitable local coordinates \((M, x) \simeq (\mathbb{R}^n, 0)\) the \( M_j \) are linear subspaces (passing through \( x = 0 \)).

Note that this special case already covers (locally) the one-dimensional case \((X, M, x) \simeq (\mathbb{C}, \mathbb{R}, 0)\), with \( M \) a holonomic D-module such that

\[ \text{char}(M) \subset T^*_y X \cup T^*_X X. \]
In this case one gets back a classical result of Kashiwara \[4, \text{thm.4.2.7, p.69}\] (cf. \[7, \text{cor.3.2.36(b), p.88-89}\]) and Komatsu \[11\]:

\[
\dim_C \left( \hom_{i^* D_X} \left( i^* \mathcal{M}, \mathcal{B}_M \right)_x \right) = d + d',
\]

with \(d\) or \(d'\) the multiplicity of \(\mathcal{M}\) along \(T_{\{x\}} X\) or \(T_X X\).

Let \(j : X \setminus M \rightarrow M\) be the open inclusion of the complement of \(M\) in \(X\). Then the proof given in \[17, \text{sec.3}\] is based on the distinguished triangle

\[
Ri^* i^! \text{Sol}(\mathcal{M})[n] \longrightarrow \text{Sol}(\mathcal{M})[n] \longrightarrow Rj_* j^* \text{Sol}(\mathcal{M})[n] \longrightarrow \text{Sol}(\mathcal{M})[n] \rightarrow [1]. \tag{10}
\]

Moreover, he uses the micro-local theory of the characteristic cycles \(CC(\cdot)\) for subanalytically constructible complexes of sheaves (as in \[9, 10, 15, 16\]), in particular a deep result of Schmid-Vilonen \[15\] about a description of \(CC(Rj_* j^* \text{Sol}(\mathcal{M}))\) in terms of \(CC(\text{Sol}(\mathcal{M}))\).

In the next section we explain our simple proof of theorem \[16, 2\] which doesn’t make use of this sophisticated micro-local theory of characteristic cycles. Instead of this, we use the observation that the calculation of

\[
\chi( (i^! F)_x ) \quad \text{for} \quad F = \text{Sol}(\mathcal{M})
\]
can be done in terms of subanalytically constructible functions, i.e. the functor \(i^!\) induces a corresponding (unique) \(\mathbb{Z}\)-linear transformation for the abelian groups \(CF(\cdot)\) of subanalytically constructible functions such that the following diagram commutes (compare \[10, \text{sec.9.7}\] and \[16, \text{sec.2.3}\]):

\[
\begin{array}{ccc}
K_0(X) & \xrightarrow{i^!} & K_0(M) \\
\chi_X \downarrow & & \downarrow \chi_M \\
CF(X) & \xrightarrow{i^!} & CF(M).
\end{array}
\tag{11}
\]

Here \(K_0(\cdot)\) is the Grothendieck group of subanalytically constructible (complexes of) sheaves with finite dimensional stalks, with \(\chi\) induced by taking stalkwise the Euler characteristic.

Then the calculation of

\[
\chi( (i^! \text{Sol}(\mathcal{M}) \otimes or_M[n])_x ) = (-1)^n \cdot i^! \left( \chi_X (\text{Sol}(\mathcal{M})) \right)(x)
\]

becomes an easy exercise by the local index theorem and example \[13\] since the \(M_j\) and therefore also the \(Y_j\) are closed submanifolds!

If we allow in the estimate \[6\] also singular subspaces, then we get at least the following weak parity version of the local index theorem:
Theorem 1.3 ((local index formula for hyperfunctions)). Let \( M_j \) be finitely many distinct real analytic subspaces of \( M \) such that the characteristic variety \( \text{char}(M) \) of the holonomic \( D \)-module \( M \) on \( X \) can be estimated as in (6), with \( Y_j \subset X \) the complexification of \( M_j \). Assume the \( Y_j \) are irreducible, with \( Y_j \cap M = M_j \). Then one has for \( x \in M \):

\[
\chi\left( \text{Rhom}_{\mathcal{D}X}(i^*M, B_M)_x \right) = \sum_j m_j \cdot \text{Eu}_{Y_j}(x) \mod 2, \tag{13}
\]

with \( \text{Eu}_{Y_j} \) the Euler obstruction of \( Y_j \) and \( m_j \) the (generic) multiplicity of \( M \) along \( T^*_{Y_j}X \).

2 Constructible functions

In this final section we give the proof of theorem 1.2 and 1.3 in terms of constructible functions. Let us start with the proof of theorem 1.2.

By the estimate (6) and example 1.1 we get

\[
\chi_X(\text{Sol}(\mathcal{M})) = \sum_j (-1)^{d_j} \cdot m_j \cdot 1_{Y_j},
\]

with \( d_j \) the complex codimension of \( Y_j \) and \( m_j \) the (generic) multiplicity of \( M \) along \( T^*_{Y_j}X \). By linearity of \( i^! \) on the level of constructible functions one also has

\[
i^!(\chi_X(\text{Sol}(\mathcal{M}))) = \sum_j (-1)^{d_j} \cdot m_j \cdot i^!(1_{Y_j}).
\]

Then the dimension formula (7) follows from (5), (9), (12) and the simple formula

\[
i^!(1_{Y_j}) = (-1)^{(n-d_j)} \cdot 1_{M_j}. \tag{14}
\]

The formula (14) corresponds by the commutative diagram (11) to the base change formula

\[
i^!(R\kappa_* C_{Y_j}) \simeq R\kappa'_* i^! C_{Y_j}
\]

for the cartesian diagram of inclusions

\[
\begin{array}{ccc}
M & \xrightarrow{i} & X \\
\kappa' & & \kappa \\
\downarrow & & \downarrow \\
M_j & \xrightarrow{i'} & Y_j
\end{array}
\]

Note that \( i^! C_{Y_j} \simeq C_{M_j}[-(n-d_j)] \) locally on \( M_j \), since \( M_j \) is a closed submanifold of \( Y_j \) of real codimension equal to the complex dimension \( n - d_j \) of \( Y_j \).
For the proof of the parity formula \(13\) in theorem \(1.3\) it is enough to show
\[
\chi\left( i^*R_j, j^*\text{Sol}(M)_{x} \right) \equiv 0 \mod 2 \text{ for all } x \in M.
\] (15)

Use the local index formula \(3\) and the distinguished triangle \(10\). But this follows from the fact that the constructible function
\[
\chi_X(\text{Sol}(M)) \mod 2
\]
is invariant under the complex conjugation acting on the complexification \(X\) of \(M\) (with fixed point set \(M\)).

More precisely, by [16, lem.1.1.1, p.27] one gets the description:
\[
\chi\left( i^*R_j, j^*\text{Sol}(M)_{x} \right) = \chi\left( R\Gamma(M_{f,x}, \text{Sol}(M)) \right),
\]
with
\[
M_{f,x} := \{||x|| \leq \delta, f = w\} \text{ for } 0 < w << \delta << 1
\]
(i.e. for \(w, \delta\) small, with \(w\) also small compared to \(\delta\)) a local right Milnor fiber of the function \(f\) at \(x\), defined in local coordinates
\[
(X, M, x) \simeq (\mathbb{C}^n, \mathbb{R}^n, 0) \text{ by } z = (z_1, \ldots, z_n) \mapsto f(z) := \sum_{k=1}^{n} \text{im}(z_k)^2.
\]
Here \(\text{im}(\cdot)\) is the imaginary part, with the complex conjugation acting on \((\mathbb{C}^n, \mathbb{R}^n, 0)\) in the usual way. This conjugation leaves the compact semi-analytic set \(M_{f,x}\) invariant without any fixed point! But the Euler characteristic
\[
\chi\left( R\Gamma(M_{f,x}, \text{Sol}(M)) \right) \mod 2
\]
can be calculated in terms of \(\mathbb{Z}_2\)-valued constructible functions:
\[
\chi\left( R\Gamma(M_{f,x}, \text{Sol}(M)) \right) \equiv (c \circ \pi)_* \alpha \mod 2,
\]
with
\[
\alpha := \chi_{M_{f,x}}(\text{Sol}(M)|M_{f,x}) \mod 2 \in CF(M_{f,x}, \mathbb{Z}_2),
\]
\[
\pi : M_{f,x} \to M_{f,x/\text{conj.}} \text{ the quotient and } c : M_{f,x/\text{conj.}} \to \{\text{pt}\}
\]
a constant map. Here \((c \circ \pi)_*\) is induced by \(R(c \circ \pi)_*\), similarly as in [11] by the commutative diagram (compare [10, sec.9.7] and [10, sec.2.3]):
\[
\begin{align*}
K_0(M_{f,x}) & \xrightarrow{R(c \circ \pi)_*} K_0(\{\text{pt}\}) \simeq \mathbb{Z} \\
\chi_{M_{f,x}} \mod 2 & \xrightarrow{(c \circ \pi)_*} \chi(\{\text{pt}\}) \mod 2 \\
CF(M_{f,x}) & \xrightarrow{(c \circ \pi)_*} CF(\{\text{pt}\}) \simeq \mathbb{Z} \\
\mod 2 & \xrightarrow{\mod 2} \mod 2
\end{align*}
\] (16)
with $CF(\cdot, \mathbb{Z}_2)$ the corresponding abelian group of $\mathbb{Z}_2$-valued subanalytically constructible functions.

Then $(c \circ \pi)_* = c_* \circ \pi_*$ by functoriality. But $\pi_*(\alpha) \equiv 0$, since $\alpha$ is invariant under the conjugation $\text{conj.}$, with $\pi : M_{f,x} \to M_{f,x}/\text{conj.}$ an unramified covering of degree two. Of course, here it is important to work with $\mathbb{Z}_2$-valued constructible functions.

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