The Deformation Space of Calabi-Yau $n$-folds with Canonical Singularities Can Be Obstructed.

Mark Gross

February, 1994

Department of Mathematics
Cornell University
Ithaca, NY 14850
mgross@math.cornell.edu

§0. Introduction.

The Bogomolov-Tian-Todorov theorem ([10] and [12]) states that a non-singular $n$-fold $X$ with $c_1(X) = 0$ has unobstructed deformation theory, i.e. the moduli space of $X$ is non-singular. This theorem was reproven using algebraic methods by Ran in [7]. Since then, it has been proven for Calabi-Yau $n$-folds with various mild forms of isolated singularities: ordinary double points by Kawamata [5] and Tian [11], Kleinian singularities by Ran [8], and finally, in the case of threefolds, arbitrary terminal singularities by Namikawa in [6]. Now the most natural class of singularities in the context of Calabi-Yau $n$-folds are canonical singularities. Indeed, if $X$ is a Calabi-Yau $n$-fold with terminal singularities, and $f : X \to Y$ is a birational contraction, $Y$ normal, then $Y$ has canonical singularities. Thus the natural question to ask is: is the deformation space of Calabi-Yau $n$-folds with canonical singularities unobstructed?

Given the history of this problem presented above, it appears worthwhile to give a counterexample to this most general question. We give an example of a Calabi-Yau $n$-fold $X$ with the simplest sort of dimension 1 canonical singularities, and show that $X$ lies in the intersection of two distinct families of Calabi-Yau $n$-folds. One is a family of generically non-singular Calabi-Yaus, and the other is a family of Calabi-Yaus which generically have terminal singularities. (In the case $n = 3$, these are also non-singular.) In particular, the point of the moduli space corresponding to $X$ is in the intersection of two components of moduli space, and hence has obstructed deformation theory.

We do not address the issue of isolated singularities here. That issue is more of a local one, and the obstructedness of Calabi-Yaus with isolated singularities is related to the obstructedness of the singularities themselves. We will explore this in a future paper, and give applications to smoothing Calabi-Yaus with canonical singularities.
Notation: If $\mathcal{F}$ is a vector bundle on a variety, we use Grothendieck’s convention for $\mathbb{P}(\mathcal{F})$, so $\mathbb{P}(\mathcal{F})$ denotes $\text{Proj}(S(\mathcal{F}))$. By a Calabi-Yau $n$-fold, we mean a normal $n$-dimensional projective variety $X$ over the complex numbers with at worst canonical singularities, $\omega_X \cong \mathcal{O}_X$, and $h^1(\mathcal{O}_X) = 0$.

§1. Two Families of Calabi-Yau $n$-folds.

Let’s start by defining two distinct families of Calabi-Yau $n$-folds. Let $n$ be an integer which is at least 3. Let $P_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}^{(n+1)}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and $P_2 = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}^{(n-1)}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. For each $P_i$, the Picard group is generated by $t$, the class of $\mathcal{O}_{\mathbb{P}^1}(1)$, and $f$, the class of a fibre of the projection $\pi: P_i \to \mathbb{P}^1$. The canonical class of $P_i$ is then $K_{P_i} = -(n+1)t - 2f$. For an element $s \in H^0(\omega_{P_i}^{-1})$, we denote the zero locus of $s$ by $X_i(s) \subseteq P_i$. For a general $s$ we have $X_1(s)$ non-singular. However, for $n > 3$, this is not true for $X_2(s)$.

We need to examine the structure of the singularities of $X_2(s)$. First, let’s look at $P_2$ in more detail. There is a section $C \subseteq P_2$ of the bundle $\pi: P_2 \to \mathbb{P}^1$ given by the inclusion $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)) \subseteq \mathbb{P}(\mathcal{E})$ induced by the quotient map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0$. We denote by $\mathcal{I}_C^p$ the $p$th power of the ideal sheaf of $C$ in $P_2$.

Let $F^p \subseteq S^{n+1}\mathcal{E}$ be the subbundle given by

$$F^p = \bigoplus_{i=0}^{n+1} S^i(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes S^{n+1-i}(\mathcal{O}_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

This yields a filtration of $S^{n+1}\mathcal{E}$:

$$S^{n+1}\mathcal{E} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n+1} \supseteq F^{n+2} = 0$$

such that $F^p/F^{p+1} \cong S^p(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes S^{n+1-p}(\mathcal{O}_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. This is the natural filtration on $S^{n+1}\mathcal{E}$ induced by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

via [3], II, Ex. 5.16 c).

Let

$$G^p = \bigoplus_{i=0}^{n+1-p} S^i(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes S^{n+1-i}(\mathcal{O}_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

We have $S^{n+1}\mathcal{E} \cong G^{n+2-p} \oplus F^p$. 

2
Lemma 1.1.

(i) There is a natural isomorphism $\pi^*(\mathcal{I}_C^p \otimes \omega_{P_2}^{-1}) \cong G^p(2)$ inducing a commuting diagram

$$
\begin{array}{ccc}
H^0(\mathcal{I}_C^p \otimes \omega_{P_2}^{-1}) & \cong & H^0(\omega_{P_2}^{-1}) \\
\downarrow & & \downarrow \\
H^0(G^p(2)) & \cong & H^0((S^{n+1}E)(2))
\end{array}
$$

(ii) Let $V = H^0(\mathcal{I}_C^{-1} \otimes \omega_{P_2}^{-1}) \subseteq H^0(\omega_{P_2}^{-1})$. If $s \in V$ is a general element, then $X_2(s)$ is a Calabi-Yau $n$-fold with canonical singularities along $C$ and is non-singular elsewhere. In addition, the natural map

$$
\psi : S^{n-1}H^0(\mathcal{O}_{P_2}(t)) \otimes S^2H^0(\mathcal{O}_{P_2}(t+f)) \to H^0(\omega_{P_2}^{-1})
$$

has image $\text{im}\ \psi = V \subseteq H^0(\omega_{P_2}^{-1})$.

(iii) If $s \in H^0(\omega_{P_2}^{-1})$ is a general element, then $X_2(s)$ is non-singular outside of $C$ and has singularities generically of multiplicity $\left\lfloor \frac{n}{2} \right\rfloor$ along $C$. If $n = 3$, $X_2(s)$ will be non-singular.

Proof: (i) Let $V_m$ be an $m$-dimensional vector space, and let $P \in \mathbf{P}(V)$ be a point. Giving $P$ is the same thing as giving a one-dimensional quotient space $V_1$ of $V_m$, or an exact sequence

$$
(*) \quad 0 \to V_{m-1} \to V_m \to V_1 \to 0.
$$

Now $V_m = H^0(\mathcal{O}_{\mathbf{P}(V_m)}(1))$, and $V_{m-1} \subseteq V_m$ is the subset of linear forms which vanish at the point $P$. We have a filtration of $S^dV_m = H^0(\mathcal{O}_{\mathbf{P}(V_m)}(d))$,

$$
S^dV_m = W^0 \supseteq \cdots \supseteq W^{d+1} = 0,
$$

with $W^p/W^{p+1} \cong S^pV_{m-1} \otimes S^{d-p}V_1$. $W^p$ in fact then consists of $d$-forms vanishing to order at least $p$ at $P$, so we see that $W^p = H^0(\mathcal{I}_P^p(1))$.

This can all be relativized. In the situation of part (i), the curve $C$ comes from the split exact sequence

$$
(**) \quad 0 \to \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}^{-1} \to \mathcal{E} \to \mathcal{O}_{\mathbf{P}^1}(-1) \to 0,
$$

which corresponds to the sequence $(*)$. The corresponding filtration of $S^{n+1}E$ is then

$$
S^{n+1}E = G^0 \supseteq \cdots \supseteq G^{n+2} = 0.
$$
The relativized statement corresponding to $H^0(T^p_2(d)) = W^p$ is $\pi_*T^p_2((n + 1)t) \cong G^p$, and so $\pi_*T^p_C \otimes \omega^{-1} \cong G^p(2)$. Now $H^0(\omega^{-1}_{t}) \cong H^0(\pi_*(\omega^{-1}_{t})) = H^0((S^{n+1}\mathcal{E})(2))$, and $H^0(T^p_C \otimes \omega^{-1}_{t}) \cong H^0(\pi_*(T^p_C \otimes \omega^{-1}_{t})) \cong H^0(G^p(2)) \subseteq H^0((S^{n+1}\mathcal{E})(2))$. This gives the desired diagram.

(ii) Let $s$ be a general element of $V$. For any $s$, we have $H^1(O_{X_2(s)}) \cong H^2(\omega_{P_2}) = 0$. Thus, to show that $X_2(s)$ is a Calabi-Yau $n$-fold with canonical singularities, since $K_{X_2(s)} = 0$ by adjunction, it is enough to show that there is a resolution of singularities $\tilde{X}_2(s) \to X_2(s)$ such that $K_{\tilde{X}_2(s)} = 0$.

Let $b : \tilde{P}_2 \to P_2$ be the blow-up of $P_2$ along $C$ with exceptional divisor $E$. The proper transform $\tilde{X}_2(s)$ of $X_2(s)$ for $s \in V$ will be an element of the linear system $|(n + 1)b^*t + 2b^*f - (n - 1)E|$. If this linear system is base-point-free, then for general $s \in V$, $\tilde{X}_2(s) \to X_2(s)$ will be a resolution of singularities. Furthermore, $K_{\tilde{P}_2} = -(n + 1)b^*t - 2b^*f + (n - 1)E$, so $K_{\tilde{X}_2(s)} = 0$.

To show that $|(n + 1)b^*t + 2b^*f - (n - 1)E|$ is base-point-free, it is enough to show that $|b^*t - E|$ is base-point-free and that $|t + f|$ is base-point-free, for then so is $|(n - 1)(b^*t - E) + 2b^*(t + f)|$.

It is easy to see, in general, that if

$$\mathcal{E} = \bigoplus_{i=1}^{n} O_{P^1}(a_i)$$

is a vector bundle over $P^1$, and $t = c_1(O_{P(E)}(1))$, then the base locus of $|t|$ is $P(\mathcal{F}) \subseteq P(\mathcal{E})$, where

$$\mathcal{F} = \bigoplus_{i \text{ with } a_i < 0} O_{P^1}(a_i),$$

with the inclusion induced by the natural surjection $\mathcal{E} \to \mathcal{F}$. Thus, in particular, for $P_2$, we see that the base locus of $|t + f|$ is empty, and the base locus of $|t|$ is the curve $C$.

Now to see that $|b^*t - E|$ is base-point-free, note that we have an exact sequence

$$0 \to O_{P_2}(t - f) \to O_{P_2}(t) \to O_f(t) \to 0$$

with $f \cong P^n$. Now $h^1(O_{P_2}(t - f)) = h^1(\pi_*O_{P_2}(t - f)) = h^1(\mathcal{E}(-1)) = 1$, and similarly $h^1(O_{P_2}(t)) = 0$, so the image of $H^0(O_{P_2}(t))$ in $H^0(O_f(t)) = H^0(O_{P^n}(1))$ is codimension one. This image must yield the linear system of hyperplanes in $f$ which contain the point $C \cap f$. Now after blowing up $C$, the linear system $|b^*t - E|$ is isomorphic, via proper
transform, to $|t|$, and thus its restriction to the proper transform of $f$ is now base-point-free. Since there is a divisor of type $f$ through any point in $P_2$, this shows that $|b^*t - E|$ is base-point-free. This proves the first statement.

For the second, first note that the linear system $|\text{im } \psi| \subseteq |-K_{P_2}|$ is spanned by reducible divisors consisting of a union of $n - 1$ divisors in $|t|$ and $2$ divisors in $|t + f|$. We have seen that $C$ is contained in any divisor in $|t|$, so that this reducible divisor in $|-K_{P_2}|$ contains $C$ to order at least $n - 1$. Thus $\text{im } \psi \subseteq V$.

Let $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1}^{n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Then

$$G^{n-1}(2) \cong (S^{n+1}\mathcal{E}')'(2) \oplus (S^n\mathcal{E}')'(1) \oplus S^n\mathcal{E}'',$$

and since $\mathcal{E}'$ is generated by global sections over $\mathbb{P}^1$, it is easy to see that the maps

$$V_1 = S^{n+1}(H^0(\mathcal{E}')) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \to H^0((S^{n+1}\mathcal{E}')'(2)) \subseteq V,$$

$$V_2 = S^n(H^0(\mathcal{E}')) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0((S^n\mathcal{E}')'(1)) \subseteq V,$$

and

$$V_3 = S^n(H^0(\mathcal{E}')) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}) \to H^0(S^n\mathcal{E}') \subseteq V$$

are all surjective, and so the map $\psi' : V_1 \oplus V_2 \oplus V_3 \to V$ is surjective.

Now the linear system $|\psi'(V_1)| \subseteq |V|$ is spanned by divisors consisting of a sum of $n + 1$ divisors in $|t|$ and two in $|f|$; the linear system $|\psi'(V_2)| \subseteq |V|$ is spanned by divisors consisting of a sum of $n$ divisors in $|t|$, one in $|f|$, and the divisor $D \in |t + f|$ given by the inclusion of $\mathcal{O}_{\mathbb{P}^1}(-1)$ in $\mathcal{E}$ coming from the splitting of the sequence (**). The linear system $|\psi'(V_3)| \subseteq |V|$ is spanned by divisors consisting of a sum of $n - 1$ divisors in $|t|$ and $2D$. All these divisors are contained in $|\text{im } \psi|$, so $V \subseteq \text{im } \psi$. Thus $V = \text{im } \psi$.

(iii) By (i), we have a filtration

$$H^0(\omega_{P_2}^{-1}) = H^0(G^0(2)) \supseteq H^0(G^1(2)) \supseteq \cdots \supseteq H^0(G^{n+2}(2)) = 0,$$

with $H^0(G^0(2))$ consisting of those sections of $H^0(\omega_{P_2}^{-1})$ which vanish to order at least $p$ along $C$. A simple calculation shows that

$$H^0(G^{\lfloor n/2 \rfloor}(2)) = H^0(G^{\lfloor n/2 \rfloor - 1}(2)) = \cdots = H^0(G^0(2)),$$

but that

$$H^0(G^{\lfloor n/2 \rfloor + 1}(2)) \neq H^0(G^{\lfloor n/2 \rfloor}(2)).$$
Thus the general element of $| - K_{P_2}|$ has singularities generically of multiplicity $\lfloor n/2 \rfloor$ along $C$. Part (ii) shows that the general element of $| - K_{P_2}|$ has no singularities outside of $C$.

We now restrict to the case $n = 3$. Let $S \subseteq P_2$, $S \equiv P(\mathcal{O}_{P_1}(-1) \oplus \mathcal{O}_{P_1}(1))$, be the surface determined by a surjection $\mathcal{E} \to \mathcal{O}_{P_1}(-1) \oplus \mathcal{O}_{P_1}(1) = \mathcal{E}''$. As the exact sequence

$$0 \to \mathcal{O}_{P_1}^2 \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is split, $S^4 \mathcal{E}''$ is a direct summand of $S^4 \mathcal{E}$. The map $H^0(S^4 \mathcal{E} \otimes \mathcal{O}_{P_1}(2)) \to H^0(S^4 \mathcal{E}'' \otimes \mathcal{O}_{P_1}(2))$ is then surjective, and this coincides with the restriction map $H^0(\omega_{P_2}^{-1}) \to H^0(\omega_{P_2}^{-1}|_S)$ via the diagram

$$
\begin{array}{ccc}
H^0(\omega_{P_2}^{-1}) & \cong & H^0(\pi_*\omega_{P_2}^{-1}) \\
\downarrow & & \downarrow \\
H^0(\omega_{P_2}^{-1}|_S) & \cong & H^0(\pi_*\omega_{P_2}^{-1}|_S) \\
\end{array}
\cong H^0(S^4 \mathcal{E}'' \otimes \mathcal{O}_{P_1}(2))
$$

Now $-K_{P_2}|_S \sim 4C + 6f$ where Pic $S$ is spanned by $C$, which has self-intersection $-2$ on $S$, and $f$. By [3], V, 2.18, a general member of $|4C + 6f|$ consists of a sum of $C$ and an irreducible non-singular curve of type $3C + 6f$, disjoint from $C$. Thus the general element of $|4C + 6f|$ is non-singular, and $X_2(s) \cap S$ is non-singular for general $s \in H^0(\omega_{P_2}^{-1})$. Thus $X_2(s)$ is non-singular along $C$.

§2. The Example.

We are now ready to give our example of a Calabi-Yau $n$-fold with canonical singularities with singular Kuranishi space. A versal Kuranishi space exists for any compact complex space by [1] or [2]. By our definition of a Calabi-Yau $n$-fold and [4], (8.6), $\text{Hom}(\Omega^1_X, \mathcal{O}_X) \cong H^1(\mathcal{O}_X) = 0$, and so by [9] this versal Kuranishi space is universal for a Calabi-Yau $n$-fold.

Returning to the setup of §1, as $\text{Ext}^1(\mathcal{O}_{P_1}(1) \oplus \mathcal{O}_{P_1}^{-1}, \mathcal{O}_{P_1}(-1))$ is one dimensional, there is a universal extension bundle $\mathcal{F}$ on $\mathbb{A}^1 \times \mathbb{P}^1$, for which $\mathcal{F}|_{0 \times \mathbb{P}^1} \cong \mathcal{O}_{P_1}(-1) \oplus \mathcal{O}_{P_1}^{-1} \oplus \mathcal{O}_{P_1}(1)$, and $\mathcal{F}|_{t \times \mathbb{P}^1} = \mathcal{O}_{P_1}^{n+1}$ for $t \in \mathbb{A}^1 - \{0\}$. This yields a family of $\mathbb{P}^n$-bundles over $\mathbb{P}^1$ via $\mathbb{P}(\mathcal{F}) \to \mathbb{A}^1 \times \mathbb{P}^1$. If we take $\mathcal{X} \subseteq \mathbb{P}(\mathcal{F})$ to be the zero locus of a section of $\omega_{\mathbb{P}(\mathcal{F})/\mathbb{A}^1}$, we would presumably obtain a family of Calabi-Yau $n$-folds $\mathcal{X} \to \mathbb{A}^1$, the general fibre being contained in $P_1$, but the fibre over $0 \in \mathbb{A}^1$ being contained in $P_2$. Now we can apply the following Lemma.

**Lemma 2.1.** If there exists a flat family $\mathcal{X} \to \mathcal{S}$ with $0 \in \mathcal{S}$ a point with $\mathcal{X}_0 \cong X_2(s)$ for some $s \in H^0(\omega_{P_2}^{-1})$ with $\mathcal{X}_0$ a Calabi-Yau $n$-fold, and $\mathcal{X}_t$ isomorphic to a non-singular member of $| - K_{P_1}|$ for $t \in \mathcal{S} - \{0\}$, then the Kuranishi space at $\mathcal{X}_0$ is singular.
Proof: The Kuranishi space of $\mathcal{X}_0$ must contain a subspace $\mathcal{M}_1$ corresponding to deformations of $\mathcal{X}_0$ to non-singular elements of $| - K_{P_1}|$, and a subspace $\mathcal{M}_2$ corresponding to deformations of $\mathcal{X}_0$ to elements of $| - K_{P_2}|$.

$X_1(s')$ is non-singular for general $s' \in H^0(\omega_{P_1}^{-1})$, so the dimension of the Kuranishi space for $X_1(s')$ can be calculated by calculating the dimension of its tangent space, which is $H^1(T_{X_1(s')})$. A simple calculation shows that this coincides with the value for the dimension of the Kuranishi space one would expect via a naive dimension counting of the number of moduli in $P_1$:

$$\dim \mathcal{M}_1 = h^1(T_{X_1(s)}) = h^0(\omega_{P_1}^{-1}) - 1 - \dim \text{Aut}(P_1).$$

Furthermore $\mathcal{M}_1$ must be an irreducible component of the Kuranishi space of $\mathcal{X}_0$.

Now the dimension of $\mathcal{M}_2$ is at least

$$h^0(\omega_{P_2}^{-1}) - 1 - \dim \text{Aut}(P_2).$$

An automorphism of $P_2$ is induced by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ and an automorphism of the bundle $\mathcal{E}$. An automorphism of $\mathcal{E}$ is induced by an $(n + 1) \times (n + 1)$ matrix of forms over $\mathbb{P}^1$, of which $(n - 1)^2 + 2$ of these entries are constant forms, $2(n - 1)$ are linear forms, and one is a quadratic form. The other entries must be zero. This gives a dimension $(n + 1)^2 + 1$ set of matrices, or $(n + 1)^2$ dimensional modulo scalars. Thus $\dim \text{Aut}(P_2) = (n + 1)^2 + 3 = \dim \text{Aut}(P_1) + 1$. Meanwhile $h^0(\omega_{P_2}^{-1}) \geq h^0(\omega_{P_1}^{-1}) + 1$, with equality holding if and only if $n = 3$. So, for $n > 3$, $\dim \mathcal{M}_2 > \dim \mathcal{M}_1$, and the Kuranishi space must have at least two irreducible components meeting at $\mathcal{X}_0$, $\mathcal{M}_1$ being one of them, and the other containing $\mathcal{M}_2$.

If $n = 3$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are the same dimension, so this argument does not suffice. However, if $\mathcal{M}_1$ and $\mathcal{M}_2$ coincide then there would be a non-singular Calabi-Yau $X_2(s_2)$ isomorphic to $X_1(s_1)$ for some $s_1$, $s_2$ via an isomorphism $\alpha : X_1(s_1) \to X_2(s_2)$. But such an isomorphism would have to preserve the cubic intersection form, with $(\alpha^*D)^3 = D^3$ for $D \in \text{Pic} X_2(s_2)$. It is then easy to see that $\alpha^*t = t$ and $\alpha^*f = f$ is the only possibility for $\alpha^* : \text{Pic} X_2(s_2) \to \text{Pic} X_1(s_1)$. But $t$ is a nef divisor on $X_1(s_1)$ but not on $X_2(s_2)$, so there is no such isomorphism. Thus $\mathcal{M}_1$ and $\mathcal{M}_2$ are two distinct components of the Kuranishi space at $\mathcal{X}_0$.

Thus, in any event, the Kuranishi space is reducible, hence singular, at $\mathcal{X}_0$. •

Thus, to construct our desired counterexample, we just need to get control of the singularities of $\mathcal{X}_0$ to ensure that they are no worse than canonical singularities. We do this by showing that we can construct a family $\mathcal{X} \to \mathbb{A}^1$ as above with $\mathcal{X}_0 \cong X_2(s)$ for any $s \in V$. 

7
Theorem 2.2. Let \( s \in V \subseteq H^0(\omega_{P_2}^{-1}) \). Then the Kuranishi space of \( X_2(s) \) is singular at \( X_2(s) \).

Proof. By Lemma 2.1, it is enough to show that \( X_2(s) \) is deformation equivalent to \( X_1(s') \) for general \( s' \). We use the following construction:

Let \( \mathcal{F} \) be the universal extension bundle over \( T = \mathbb{A}^1 \times \mathbb{P}^1 \), as at the beginning of this section. Let \( p_1 \) and \( p_2 \) be the projections of \( T \) onto the first and second factors respectively. We set \( \mathcal{O}_T(1) = p_2^* \mathcal{O}_{\mathbb{P}^1}(1) \). Let \( \mathcal{P} = \mathcal{P}(\mathcal{F}), \pi : \mathcal{P} \to T \) the projection, and denote by \( t \) the class of \( \mathcal{O}_T(1) \), and \( f \) the class of \( \pi^* \mathcal{O}_T(1) \).

Lemma 2.3. Let \( x \in \mathbb{A}^1 \), and let

\[
\phi_x : p_1*\pi_* \mathcal{O}_\mathcal{P}((n + 1)t + 2f) \otimes k(x) \to H^0(\pi_* \mathcal{O}_\mathcal{P}((n + 1)t + 2f)|_{x \times \mathbb{P}^1})
\]

be the restriction map, where \( k(x) \) is the residue field of \( \mathbb{A}^1 \) at \( x \). Then \( \phi_x \) is surjective if \( x \neq 0 \), and \( \text{im } \phi_0 = V \subseteq H^0(\omega_{P_2}^{-1}) = H^0(\pi_* \mathcal{O}_\mathcal{P}((n + 1)t + 2f)|_{0 \times \mathbb{P}^1}) \), where \( V \) is the subspace of Lemma 1.1, (ii).

Proof: First we have \( \pi_* \mathcal{O}_\mathcal{P}((n + 1)t + 2f) = (S^{n+1} \mathcal{F})(2) \). For \( x \neq 0 \), \( (S^{n+1} \mathcal{F})(2)|_{x \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)^N \) for suitable \( N \), so by Grauert’s Theorem, \( [3], \text{III, 12.9} \), \( \phi_x \) is surjective.

For \( x = 0 \), first note that the maps

\[
p_1*\pi_* \mathcal{O}_\mathcal{P}(t) \otimes k(x) \to H^0(\pi_* \mathcal{O}_\mathcal{P}(t)|_{0 \times \mathbb{P}^1})
\]

and

\[
p_1*\pi_* \mathcal{O}_\mathcal{P}(t + f) \otimes k(x) \to H^0(\pi_* \mathcal{O}_\mathcal{P}(t + f)|_{0 \times \mathbb{P}^1})
\]

are surjective, again by Grauert’s Theorem. By Lemma 1.1, (ii), this shows that \( \text{im } \phi_0 \) contains \( V \).

By construction, we have

\[
0 \to \mathcal{O}_T(-1) \to \mathcal{F} \to \mathcal{O}_T^{n-1} \oplus \mathcal{O}_T(1) \to 0.
\]

Let

\[
S^{n+1} \mathcal{F} \supseteq \mathcal{F}^0 \supseteq \cdots \supseteq \mathcal{F}^{n+2} = 0
\]

be the filtration of \( S^{n+1} \mathcal{F} \) induced by the above extension, and let

\[
G^p = S^{n+1} \mathcal{F}/\mathcal{F}^{n+2-p}.
\]
For $p = n - 1$, this yields the sequence

$$0 \to \mathcal{F}^3(2) \to (S^{n+1} \mathcal{F})(2) \to \mathcal{G}^{n-1}(2) \to 0.$$ 

Restricting this to $0 \times \mathbb{P}^1$, this sequence splits to obtain

$$S^{n+1} \mathcal{E} \cong \mathcal{F}^3(2) \oplus \mathcal{G}^{n-1}(2).$$

Now $p_1^* \mathcal{F}^3(2) = 0$ since $\mathcal{F}^3(2)|_{x \times \mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^N$ for some suitable $N$, $x \neq 0$, and the map $H^0(\mathcal{G}^{n-1}(2)) \otimes k(0) \to H^0(\mathcal{G}^{n-1}(2))$ is an isomorphism by Grauert’s Theorem. This yields the following diagram:

\[
\begin{array}{ccc}
0 & \to & p_1^*(S^{n+1} \mathcal{F})(2) \otimes k(0) \\
\downarrow_{\phi_0} & & \downarrow_{\beta \cong} \\
0 & \to & H^0(F^3(2)) & \to & H^0((S^{n+1} \mathcal{E})(2)) & \to & H^0(\mathcal{G}^{n-1}(2))
\end{array}
\]

Since $\text{im} \phi_0 \subseteq V = H^0(\mathcal{G}^{n-1}(2))$, yet at the same time this diagram shows $\phi_0$ injects into $H^0(\mathcal{G}^{n-1}(2))$, we see $\phi_0$ has as its image exactly $V = H^0(\mathcal{G}^{n-1}(2))$.

To prove Theorem 2.2, let $s_0 \in V \subseteq H^0((S^{n+1} \mathcal{F})(2)|_{0 \times \mathbb{P}^1})$ be any element. We can then lift $\phi_0^{-1}(s_0)$ to a section $s$ of $S^{n+1} \mathcal{F}(2)$ such that $s|_{1 \times \mathbb{P}^1} = s_1 \in H^0((S^{n+1} \mathcal{F})(2)|_{1 \times \mathbb{P}^1})$ is a general section, and $s|_{0 \times \mathbb{P}^1} = s_0$, by Lemma 2.3. Let $\mathcal{X} \subseteq \mathcal{P}$ be the zero-locus of the corresponding section of $\mathcal{O}_\mathcal{P}((n+1)t+2f)$. The projection $\mathcal{X} \to \mathbb{A}^1$ gives a family of Calabi-Yau’s over $\mathbb{A}^1$, with $\mathcal{X}_1 = X_1(s_1)$, and $\mathcal{X}_0 = X_2(s_0)$.

Bibliography

[1] Douady, A., “Le Problème des Modules Locaux pour les Espaces C-Analytiques Compacts,” Ann. scient. Éc. Norm. Sup., 4e série, 7, (1974) 569–602.

[2] Grauert, H., “Der Satz von Kuranishi für Kompakte Komplexe Räume,” Inv. Math., 25, (1974) 107–142.

[3] Hartshorne, R., Algebraic Geometry, Springer-Verlag 1977.

[4] Kawamata, Y., “Minimal Models and the Kodaira Dimension of Algebraic Fiber Spaces,” J. Reine Angew. Math, 363, (1985) 1–46.

[5] Kawamata, Y., “Unobstructed Deformations— A Remark on a Paper of Z. Ran,” J. Algebraic Geometry, 1, (1992) 183–190.

[6] Namikawa, Y., “On Deformations of Calabi-Yau Threefolds with Terminal Singularities,” preprint, 1993.

[7] Ran, Z., “Deformations of Manifolds with Torsion or Negative Canonical Bundle,” J. of Algebraic Geometry, 1, (1992) 279–291.
[8] Ran, Z., “Deformations of Calabi-Yau Kleinfolds,” in *Essays in Mirror Symmetry*, (ed. S.-T. Yau) Int. Press, Hong Kong, (1992) 451–457.

[9] Schlessinger, M., “Functors on Artin Rings,” *Trans. Amer. Math. Soc.*, **130**, (1968), 208-222.

[10] Tian, G., “Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Petersson-Weil Metric,” in *Mathematical Aspects of String Theory*, 629-646, ed. S.-T. Yau, World Scientific, Singapore, 1987.

[11] Tian, G., “Smoothing 3-folds with trivial canonical bundle and ordinary double points,” in *Essays in Mirror Symmetry*, (ed. S.-T. Yau) Int. Press, Hong Kong, (1992) 458–479.

[12] Todorov, A., “The Weil-Petersson Geometry of the Moduli Space of $SU(n \geq 3)$ (Calabi-Yau) Manifolds I,” *Commun. Math. Phys* **126**, (1989) 325-346.