The Universal Equation of State
near
the Critical Point of QCD

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Abstract

We study the universal properties of the phase diagram of QCD near the critical point using the exact renormalization group. For two-flavour QCD and zero quark masses we derive the universal equation of state in the vicinity of the tricritical point. For non-zero quark masses we explain how the universal equation of state of the Ising universality class can be used in order to describe the physical behaviour near the line of critical points. The effective exponents that parametrize the growth of physical quantities, such as the correlation length, are given by combinations of the critical exponents of the Ising class that depend on the path along which the critical point is approached. In general the critical region, in which such quantities become large, is smaller than naively expected.
1 Introduction

The most prominent feature of the phase diagram of QCD at non-zero temperature and baryonic density is the critical point that marks the end of the line of first-order phase transitions. (For a review see ref. [1].) Its exact location and the size of the critical region around it determine its relevance for the heavy-ion collision experiments at RHIC and LHC.

There exist several analytical studies of the phase diagram near the critical point [2]–[12]. Lattice simulations are a source of precise information for the details of the phase diagram. Recent studies [13, 14, 15] have overcome the difficulties associated with simulating systems with non-zero chemical potential, even though the continuum limit or realistic quark masses have not been reached yet.

The critical equation of state encodes all the physical information for a system near a critical point in the presence of an external source. In the case of QCD, the parameters that can be adjusted in order to approach the critical point are the temperature \( T \) and baryonic chemical potential \( \mu \). For two-flavour QCD, the role of the external source \( j \) is played by the current quark mass that breaks explicitly the chiral symmetry of the QCD Lagrangian. If isospin-breaking effects are neglected, the source is a (linear) function of the common mass of the light quarks.

An analytical description of the critical region requires the use of the renormalization group. The study of first-order phase transitions is easier within the Wilsonian (or exact) formulation [16]. We shall use the formalism of the effective average action \( \Gamma_k \) [17, 18], that results by integrating out (quantum or thermal) fluctuations above a certain cutoff \( k \). The dependence of \( \Gamma_k \) on \( k \) is given by an exact flow equation [19, 20]. When \( k \) is lowered to zero \( \Gamma_k \) becomes the effective action. The effective average action can by used in order to describe very efficiently the universal and non-universal aspects of second-order phase transitions, as well as strong or weak first-order phase transitions [21].

In order to discuss the phase diagram we have to employ an appropriate order parameter. The usual choice is the quark-antiquark condensate that is associated with the spontaneous breaking of the chiral symmetry of the QCD Lagrangian. For two flavours an equivalent description uses as effective degrees of freedom four mesonic scalar fields (the \( \sigma \)-field and the three pions \( \pi_i \)), arranged in a \( 2 \times 2 \) matrix

\[
\Phi = \frac{1}{2} (\sigma + i \vec{\pi} \cdot \vec{\tau}),
\]

where \( \tau_i (i = 1, 2, 3) \) denote the Pauli matrices. The various interactions must be invariant under a global \( SU(2)_L \times SU(2)_R \) symmetry acting on \( \Phi \). If the \( \sigma \)-field (that corresponds to a condensate \( \bar{u}u + \bar{d}d \)) develops an expectation value the symmetry is broken down to \( SU(2)_{L+R} \). The explicit breaking of the chiral symmetry through the current quark masses can be incorporated as well. It corresponds to the interaction with an external source through a term \( -j \sigma \) in the Lagrangian.

The full QCD dynamics cannot be described by an effective Lagrangian involving only mesonic fields. More complicated effective descriptions, such as the linear quark-meson
model, have been used for the discussion of the phase diagram for low chemical potential [22, 23, 24]. In ref. [25] it was shown that the expected structure of the phase diagram for two-flavour QCD can be reproduced within this model. In particular, in the absence of an external source (zero current quark mass), the phase diagram contains a line of first-order phase transitions and a line of second-order ones, that meet at a tricritical point. The second-order phase transitions belong to the $O(4)$ universality class, while the tricritical point is associated with the Gaussian fixed point and is described by mean-field theory. In the presence of a source the first-order line ends at a critical point. The second order phase-transition at this point belongs to the Ising universality class. The reason is that all degrees of freedom other than the $\sigma$-field remain massive and decouple at low energy scales. For varying external source we obtain a line of second-order phase transitions, that starts at the tricritical point and marks the boundary of a surface of first-order transitions.

The universal properties associated with the lines of second-order phase transitions in the phase diagram do not depend on the effective model employed for the description of the physical system (the mesonic model described above, a discretized version of the QCD Langrangian employed in lattice simulations, etc). The crucial ingredient is the inclusion of the degrees of freedom that become critical (massless) at a transition, and the symmetries that characterize the effective interactions [26]. For the same reason the inclusion of the more massive quarks is not expected to modify the universal behaviour. It may only shift the location of the various lines on the phase diagram.

The purpose of this paper is to derive the universal equation of state of QCD in the region of the phase diagram near the line of critical points. We concentrate on the dependence on the quantities that can be varied. In the following section we describe a simplified model that has a phase diagram similar to that of QCD at relatively small values of the chemical potential. In section 3 we discuss the phase diagram in the context of mean-field theory. In section 4 we describe the renormalization-group approach. We also determine the initial conditions for the renormalization-group flow in the context of the quark-meson model. In section 5 we derive the equation of state in the absence of an external source. In section 6 we derive the equation of state for a non-zero external source in the vicinity of the line of critical points. In the final section 7 we discuss the implications of our results for the possibility of observing the universal critical behaviour of QCD in heavy-ion experiments.

2 The model

As we are interested only in the universal characteristics near the second-order phase transitions, we need only discuss a simplified model that preserves the basic structure of the phase diagram of two-flavour QCD. We consider a Lagrangian in three dimensions that contains standard kinetic terms for the fields $\sigma$, $\pi_i (i = 1, 2, 3)$, and the potential

$$V(\rho) = m^2 \rho + \frac{1}{2} \lambda \rho^2 + \frac{1}{3} g \rho^3,$$

(2)
with $\rho = \text{Tr} \left( \Phi^\dagger \Phi \right) = (\sigma^2 + \pi^2)/2$. The couplings $m^2$, $\lambda$, $g$ are considered functions of the temperature $T$ and the chemical potential $\mu$. We assume that $g$ is always positive, so that the potential is bounded from below.

In ref. [25] it was shown that such a potential emerges in the quark-meson model after the integration of the temperature fluctuations with non-zero Matsubara frequencies. The fermions are completely integrated out as their lowest Matsubara frequency is non-zero. The only mode that survives at energy scales below the temperature is the zero-mode of the mesonic field. Thus the theory becomes effectively three-dimensional. The effective potential around the origin has a structure similar to that of eq. (2). In section 4 we calculate the effective potential in the context of an effective model of QCD, the linear quark-meson model.

The various fixed points that determine the structure of the phase diagram appear during the integration of the low-energy fluctuations of the zero mode. Theories for which the renormalization-group flow is dominated by an infrared fixed point become insensitive to the details of the ultraviolet formulation. For this reason the simplified model we are considering is sufficient for the discussion of the universal properties of the phase diagram of QCD near the lines of second-order phase transitions.

The effect of a current quark mass can be taken into account by adding a term $-j\sigma$, with $j$ a (linear) function of the mass.

## 3 Mean-field theory

It is instructive to discuss the phase diagram of our simplified model neglecting the field fluctuations. We allow for variations of $m^2$ and $\lambda$, while we assume that the coupling $g$ remains constant. This is a good approximation of the behaviour of the potential studied in ref. [25].

We begin by studying the phase diagram in the absence of an external source ($j = 0$). Let us consider first the case $\lambda > 0$. For $m^2 > 0$ the minimum of the potential is located at $\sigma = \pi_i = 0$ and the system is invariant under an $O(4)$ symmetry. For $m^2 < 0$ the minimum moves away from the origin. Without loss of generality we take it along the $\sigma$ axis. The symmetry is broken down to $O(3)$. The $\pi$'s, which play the role of the Goldstone fields, become massless at the minimum. If $m^2(T, \mu)$ has a zero at a certain value $T = T_{cr}$, while $\lambda(T_{cr}, \mu) > 0$, the system undergoes a second-order phase transition at this point. The minimum of the potential behaves as $\sigma_0 \sim |T - T_{cr}|^{1/2}$ slightly below the critical temperature. The critical exponent $\beta$ takes the mean-field value $\beta = 1/2$.

Let us consider now the case $\lambda < 0$. It is easy to check that, as $m^2$ increases from negative to positive values (through an increase of $T$ for example), the system undergoes a first-order phase transition. For $|\lambda|$ approaching zero the phase transition becomes progressively weaker: The discontinuity in the order parameter (the value of $\sigma$ at the minimum) approaches zero. For $\lambda = 0$ the phase transition becomes second order. The minimum of the potential behaves as $\sigma_0 \sim |T - T_{cr}|^{1/4}$. The critical exponent $\beta$ takes the value $\beta = 1/4$ for this particular point.
Let us assume now for simplicity that $\lambda$ is a decreasing function of $\mu$ only, and has a zero at $\mu = \mu_*$. If we consider the phase transitions for increasing $T$ and fixed $\mu$ we find a line of second-order phase transitions for $\mu < \mu_*$, and a line of first-order transitions for $\mu > \mu_*$. The two lines meet at the special point $(T_*, \mu_*)$, where $T_* = T_{cr}$ for $\mu = \mu_*$. This point is characterized as a tricritical point. (For a review of the theory of tricritical points see ref. [31].) In the general case $\lambda$ will be a decreasing function of a linear combination of $\mu$ and $T$. The structure of the phase diagram remains the same. Simply there is a linear combination of $T$ and $\mu$ that generates displacements along the lines of first- and second-order phase transitions near the tricritical point, and a different one that moves the system through the phase transition.

If a source term $-j\sigma$ is added to the potential of eq. (2) the $O(4)$ symmetry is explicitly broken. The phase diagram is modified significantly even for small $j$. The second-order phase transitions, observed for $\lambda > 0$, disappear. The reason is that the minimum of the potential is always at a value $\sigma \neq 0$ that moves close to zero for increasing $T$. For small $j$ the mass term at the minimum approaches zero at a value of $T$ near what we defined as $T_{cr}$ for $j = 0$. However, no genuine phase transition appears. Instead we observe an analytical crossover.

The line of first-order transitions persists for $j \neq 0$. However, at the critical temper-
nature the minimum jumps discontinuously between two non-zero values of \( \sigma \) and never becomes zero. The line ends at a new special point, whose nature can be examined by considering the \( \sigma \)-derivative of the potential \( V(\rho) \) of eq. (2). For a first-order phase transition to occur, \( \partial V / \partial \sigma \) must become equal to \( j \) for three non-zero values of \( \sigma \). This requires \( \partial^2 V / \partial \sigma^2 = 0 \) at two values of \( \sigma \). The critical point corresponds to the situation that all these values merge to one point. At the minimum \( \sigma^* \) of the potential \( \tilde{V}(\sigma; j) = V(\sigma, \pi_i = 0) - j\sigma \) at the critical point, we have:

\[
d \tilde{V} / d\sigma = \frac{d^2 \tilde{V}}{d\sigma^2} = \frac{d^3 \tilde{V}}{d\sigma^3} = 0.
\]

It can be checked that this requires \( \lambda < 0 \) and can be achieved by fine-tuning \( m^2 \) and \( \lambda \) for given \( j \). The minimum \( \sigma^* \) is then completely determined. At the critical point we expect a second-order phase transition for the deviation of \( \sigma \) from \( \sigma^* \). As \( d^4 \tilde{V} / d\sigma^4 \neq 0 \) at the critical point, the critical exponent \( \beta \) takes the value \( \beta = 1/2 \). The \( \pi \)'s are massive, because \( m^2_\pi = \partial^2 V(\rho) / \partial \pi^2 \). The potential \( \tilde{V} \) of eq. (2) is obtained from the effective action if only two terms are retained: the potential and a kinetic term that includes a \( \Phi \)-independent wavefunction renormalization \( Z_k \) [21]. The anomalous dimension \( \eta = -d(\ln Z_k) / dt \) describes the scale dependence of \( Z_k \), and can be determined starting from the exact flow equation. It becomes constant when the flow approaches a fixed point [21].

4 The renormalization-group flow

The most efficient way to study a physical system near a phase transition is through the effective potential. In the formulation of the renormalization group which we are employing, the dependence of the potential of a three-dimensional theory on a “coarse-graining” scale \( k \) is described by the equation [19, 21]

\[
\frac{\partial}{\partial t} u_k(\tilde{\sigma}) = -3u_k + \frac{1}{2}(1 + \eta)\tilde{\sigma} u'_k + \frac{1}{4\pi^2} \left[ \int_0^\infty (u_k' + 3\int_0^\infty (u_k'/\tilde{\sigma}) \right],
\]

where \( t = \ln(k/\Lambda) \) and we have defined the dimensionless quantities

\[
u_k = k^{-3}U_k, \quad \tilde{\sigma} = k^{-4}Z_k^{3/2}\sigma.
\]

Primes denote derivatives with respect to \( \tilde{\sigma} \). The scale-dependent potential \( U_k \) results from the integration of the field fluctuations with characteristic momenta larger than \( k \). In the limit \( k \to 0 \) it becomes equal to the effective potential. It is possible that the three-dimensional theory described by the potential \( U_k \) has its origin in the dimensional reduction of a more complicated, higher-dimensional theory. This situation arises during the study of the phase diagram of QCD at non-zero temperature \( T \) and baryonic chemical potential \( \mu \) (see below).

The above equation can be derived from an exact flow equation if only two terms are retained in the effective action: the potential and a kinetic term that includes a \( \Phi \)-independent wavefunction renormalization \( Z_k \) [21]. The data dependence of \( Z_k \) can be determined starting from the exact flow equation. It becomes constant when the flow approaches a fixed point [21].
The “threshold” function $l_3^d(w)$ is a particular case of

$$l_3^d(w) = \frac{n}{2} k^{2n-d+1} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \int d^d\tilde{q} \frac{\partial P_k}{\partial k}(P_k + w k^2)^{-(n+1)}.$$

The inverse propagator for massless fields $P_k$ has been modified so as to forbid the propagation of modes with momenta below the “coarse-graining” scale $k$. Several forms of the modified propagator are possible, which correspond to sharper or smoother cutoffs of the low-momentum modes [21].

The continuous reduction of $k$, from the ultraviolet scale $\Lambda$ to zero, permits the gradual integration of fluctuations with different characteristic momenta. The function $l_3^d(w)$ falls off for large values of $w$ following a power law. As a result it introduces a threshold behaviour for the contributions of massive modes to the evolution equation. When the running squared mass of a massive mode becomes much larger than the scale $k^2$, the mode decouples and the respective contribution vanishes. In eq. (3) we distinguish contributions from two different types of fields: the radial mode (the $\sigma$-field) and the three Goldstone modes (the pions). Their masses are expressed through the derivatives of the effective potential.

The initial condition that is needed for the solution of the evolution equation (3) is given by eq. (2). The potential at the ultraviolet scale $\Lambda$ is identified with the classical (or bare) potential: $U_\Lambda = V$. The integration of field fluctuation with momenta between $\Lambda$ and zero leads the determination of the effective potential: $U_{eff} = U_0 \equiv U$.

The QCD phase transition is associated with the breaking of the chiral symmetry through a non-zero expectation value for the mesonic $\sigma$-field. The connection with the model we are studying is made through the phenomenon of dimensional reduction at non-zero temperature. At energy scales below the temperature a four-dimensional theory can be described in terms of effective three-dimensional degrees of freedom. In a Fourier decomposition of a scalar field only the term with zero Matsubara frequency (the zero mode) survives. The other modes decouple, as they get a mass proportional to the temperature. The fermions do not have a zero mode and they decouple completely.

In the context of the quark-meson model with two flavours an analytical expression has been derived for the form of the potential at a scale $k = \Lambda = T$ [25]. It is

$$U'_\Lambda(\rho; T, \mu) \simeq \lambda_{0R} \left[ \rho - \rho_{0R} - \frac{3}{4\pi^2} T^2 \left( \frac{\sqrt{\pi}}{\theta_2} - \frac{\pi^2}{3} \right) \right] + I_F(\rho; T, \mu),$$

where

$$I_F(\rho; T, \mu) = -N_c \frac{h^4}{16\pi^2} \rho \ln \left( \frac{k^2 \rho}{2\Lambda^2} \right) + V'_F(\rho; T, \mu) + V'_F(\rho; T, -\mu),$$

which for the pion mass is

$$I_F(\rho; T, \mu) = \frac{h^2}{2\pi^2} \rho \ln \left( \frac{k^2 \rho}{2\Lambda^2} \right) + V'_F(\rho; T, \mu) + V'_F(\rho; T, -\mu).$$
\[ V_F(\rho; T, \mu) = -4N_c T \int_{-\infty}^{\infty} \frac{d^3q}{(2\pi)^3} \ln \left( 1 + \exp \left[ -\frac{1}{T} \left( \sqrt{\vec{q}^2 + h^2\rho/2 - \mu} \right) \right] \right). \] (8)

Here \( \rho_{0R} \) is a renormalized expectation value for the mesonic field at zero temperature and chemical potential, while \( \lambda_{0R} \) is the renormalized mesonic quartic coupling. The term \( \sim T^2 \) in eq. (6) results from the integration of the scalar thermal fluctuations with non-zero Matsubara frequencies. The effect of the fermionic quantum and thermal fluctuations is incorporated in the quantity \( I_F(\rho; T, \mu) \). Even at zero temperature, the fermionic quantum fluctuations give a contribution (the first term in the r.h.s. of eq. (7)) that has to be taken into account when defining the vacuum of the zero-temperature theory. The quantity \( V_F(\rho; T, \mu) \) is the free energy of a fermionic gas at non-zero temperature and chemical potential. We have taken into account only two quark flavours (corresponding to the light \( u \)- and \( d \)-quarks), as well as their antiparticles through \( V_F(\rho; T, -\mu) \). The parameter \( N_c = 3 \) stands for the number of colours. The scale \( \tilde{\Lambda} \) corresponds to the energy scale at which the quark-meson model becomes an effective description of QCD. It is close to 650 GeV [22], while \( T \simeq 200 \) MeV. The constituent quark mass is given by \( M_q^2 = h^2\rho/2 \).

We emphasize that the above expressions have been derived for small \( \lambda_{0R} \) and \( h \). If we would like to make contact with realistic mesonic physics we must reproduce the correct pion decay constant \( f_\pi \), the \( \sigma \)-field mass and the constituent quark mass at zero temperature and chemical potential. If we ignore the logarithmic contribution in \( I_F(\rho; T, \mu) \), we have to set \( \sqrt{2\rho_{0R}} \simeq f_\pi \approx 87 \) MeV, \( m_\sigma \simeq \sqrt{2\lambda_{0R}\rho_{0R}} \simeq 600 \) MeV, \( M_q = \sqrt{h^2\rho_{0R}/2} \simeq 300 \) MeV. This leads to \( \lambda, h = \mathcal{O}(10) \). For this reason the essentially perturbative expressions (6)–(8) must be used only for qualitative discussions of the behaviour of realistic QCD. Whenever we make use of these expressions in the following we assume that \( \lambda_{0R} \) and \( h \) are small, consistently with their derivation.

The quantity \( I_F(\rho; T, \mu) \) can be expanded in powers of \( \rho \) as

\[ I_F(\rho; T, \mu) = \frac{h^2N_c}{12} \left( T^2 + \frac{3}{\pi^2}\mu^2 \right) + \frac{h^4N_c}{16\pi^2} \left[ -1 + 2\gamma_e + \ln \left( \frac{\tilde{\Lambda}^2}{4T^2} \right) \right] + 2 \text{Li}^{(1,0)} \left( 0, -\exp \left( \frac{\mu}{T} \right) \right) + 2 \text{Li}^{(1,0)} \left( 0, -\exp \left( -\frac{\mu}{T} \right) \right) \rho + \mathcal{O}(\rho^2), \] (9)

where \( \gamma_e \simeq 0.5772 \) is the Euler-Mascheroni constant and \( \text{Li}^{(l_1, l_2)}(n, x) \) denotes the \( l_1 \)-th and \( l_2 \)-th partial derivative of the polylogarithmic function \( \text{Li}(n, x) \) with respect to \( n \) and \( x \) respectively. For \( \mu = 0 \) we have \( \text{Li}^{(1,0)}(0, -1) = -\ln(\pi/2)/2 \) and we recover the known expressions of ref. [27].

Making use of eqs. (6), (9) we can write the potential \( U_\Lambda \) in the form of eq. (2) with\(^1\)

\[ m^2 = -\lambda_{0R}\rho_{0R} + \left[ \frac{\lambda_{0R}}{4} \left( 1 - \frac{3}{\pi^2} \right) + \frac{h^2N_c}{12} \right] T^2 + \frac{h^2N_c}{4\pi^2}\mu^2. \] (10)

\(^1\)The quartic coupling of the effective three-dimensional theory is given by \( \lambda T \). We do not refer to this coupling, as its additional temperature dependence is eliminated trivially when we discuss the phase diagram in terms of the observables of the four-dimensional theory.
Figure 2: The evolution of the rescaled potential for $m^2 = 0.0213$, $\lambda = -0.335$, $g = 1$.

\[
\lambda = \lambda_0 k + \frac{h^4 N_c}{16\pi^2} \left[-1 + 2\gamma_e + \ln \left(\frac{\tilde{\Lambda}^2}{4T^2}\right) + 2 \text{Li}^{(1,0)} \left(0, -\exp \left(\frac{\mu}{T}\right)\right) + 2 \text{Li}^{(1,0)} \left(0, -\exp \left(-\frac{\mu}{T}\right)\right)\right]. \tag{11}
\]

The function $\text{Li}^{(1,0)}(0, -\exp(x)) + \text{Li}^{(1,0)}(0, -\exp(-x))$ is monotonically decreasing, and it takes the value $-\ln(\pi/2)$ for $x = 0$. These expressions demonstrate how the phase diagram that we discussed in the previous section can emerge in the context of QCD. Even if the mass term is negative at $T = \mu = 0$, it can become positive for sufficiently large $T$ or $\mu$. For constant $T$, the quartic coupling is a decreasing function of $\mu$. For small $h$ the tricritical point is expected to appear for values of $T$ and $\mu$ such that $\lambda \simeq 0$.

5 The equation of state near the tricritical point

5.1 The crossover behaviour

The phase structure in the absence of an external source ($j = 0$) can be determined in complete analogy to ref. [25]. There is a line of second-order phase transitions which can be approached for fixed positive values of $\lambda$, $g$ in eq. (2) by fine-tuning the mass term. During the renormalization-group flow the potential approaches a form characteristic of the fixed point of the $O(4)$ universality class. Subsequently the system deviates towards the broken or symmetric phase. The influence of the fixed point results in a behaviour that can be characterized by universal quantities, such as critical exponents. For sufficiently
negative values of \( \lambda \) the phase transitions become first-order. Again, an appropriate fine-tuning of the mass term leads to an effective potential with two coexisting phases. No fixed point is approached during the flow.

The lines of first- and second-order phase transitions meet at a point characterized by specific values of \( m^2 \) and \( \lambda \). The evolution of the potential in the proximity of this point is depicted in fig. 2. The initial condition is given by eq. (2). For the plot we integrated eq. (3) with \( \eta = 0 \) (and, therefore, \( Z_k = 1 \)). We have used a smooth infrared cutoff in the propagator \( P_k \) of eq. (5), similarly to ref. [25]. The anomalous dimension is very small (\( \eta \simeq 0.03 \)) in this model and setting it equal to zero gives a good approximation to the exact solution. In fig. 2 we plot the quantity \( u_k'/\tilde{\sigma} \) as a function of \( \tilde{\sigma}^2/2 \) for \( \pi_i = 0 \) and decreasing values of the scale \( k \). In an alternative notation, that makes the \( O(4) \) symmetry apparent, the figure depicts \( (dU_k/d\rho)/k^2 \) as a function of \( \tilde{\rho} = \rho/k \), with \( \rho \) defined below eq. (2).

Two quantities must be fine-tuned for the type of evolution shown in fig. 2. We fixed \( g = 1 \) and varied \( \lambda \). (All quantities are given in units of the ultraviolet scale \( \Lambda \).) For every value of \( \lambda \) we fine-tuned \( m^2 \) so as to be very close to the phase transition. For \( \lambda = -0.335 \), \( m_{cr}^2 \simeq 0.0213 \) (the latter quantity is determined with a precision of 15 significant figures) we are in the immediate vicinity of the tricritical point and very close to a second-order phase transition.

Initially the potential is very flat in the range of \( \tilde{\sigma} \) we are considering, so that its derivative is almost zero. The system is very close to the Gaussian fixed point. As \( k \)
Figure 4: The evolution of the rescaled potential for $m^2 = 0.0217$, $\lambda = -0.338$, $g = 1$.

becomes smaller another fixed point is approached, characteristic of the $O(4)$ universality class. Eventually the system moves towards the symmetric phase ($u_k' / \tilde{\sigma}$ becomes positive at the origin). In the limit $k \to 0$ we obtain the effective potential, from which we can derive renormalized quantities such as the location of the vacuum and the field mass.

For systems close to a second-order phase transition, we can parametrize physical quantities in terms of critical exponents. In our example, we can keep $\lambda$ and $g$ fixed with the values $\lambda = -0.335$, $g = 1$ and vary the mass term according to $m^2_R = m^2_{cr} + \delta m^2$. For $\delta m^2 > 0$ the renormalized mass term at the origin can be parametrized as $m^2_R \sim (\delta m^2)^{2\nu}$. For $\delta m^2 < 0$ the minimum of the potential is located at a non-zero field value that can be parametrized as $\sigma_{0R} \sim |\delta m^2|^{-\beta}$. (The quantity $\delta m^2$ replaces the quantity $T - T_{cr}$ that appears when the phase transition is discussed in the context of the fundamental four-dimensional theory at non-zero temperature.)

The effective critical exponents $\nu$, $\beta$ are depicted in fig. 3 as a function of $\delta m^2$. We observe that for relatively large $\delta m^2$ they take the values $\beta = 0.25$, $\nu = 0.5$. These are the mean-field predictions that we derived in section 2. They are the expected values of these exponents, as predicted by the general theory of tricritical points [31]. For large $\delta m^2$ the potential never approaches the $O(4)$ fixed point. It stays close to the Gaussian one, before moving directly towards the broken or the symmetric phase. For smaller $\delta m^2$ system feels the attraction of the $O(4)$ fixed point. For $\delta m^2 \to 0$ the potential spends a large part of its evolution near this fixed point. As a result it loses memory of its initial form near the Gaussian fixed point. The critical exponents take the values characteristic of the $O(4)$ universality class $\beta = 0.41$, $\nu = 0.82$. The curves of fig. 3 are typical

\textsuperscript{2}The most accurate known values for these exponents are close to $\beta = 0.39$, $\nu = 0.75$ [37]. The values
examples of crossover curves. They describe the variation of universal quantities, such as the critical exponents, as the relative influence of two fixed points changes.

In fig. 4 we display the evolution of \( u'_k/\tilde{\sigma} \) for a theory with \( m^2 \approx 0.0217, \lambda = -0.338, \nu = 1 \). The slight decrease of the value of \( \lambda \) relative to fig. 3 modifies drastically the evolution. Initially the potential stays very close to the Gaussian fixed point, but eventually moves away from it. Its final form is characteristic of a first-order phase transition as was discussed in detail in ref. [25]. This transition is very weak, as the discontinuity in the order parameter is several orders of magnitude smaller than \( \Lambda \).

The tricritical point is located on the plane \((m^2, \lambda)\) in the region between the points \((0.0213, -0.335)\) and \((0.0217, -0.338)\) for \( g = 1 \). For the critical values \( m_t^2 \) and \( \lambda_t \) the solution of the evolution equation stays close to the Gaussian fixed point for an infinitely long “time” \( t \). As a result the critical behaviour is determined completely by mean-field theory. For example, the critical exponent \( \beta \) stays equal to 0.25 arbitrarily close to the phase transition. The relative influence of the two fixed points depends on the magnitude of two parameters that can be taken as \( \delta m^2 \) and \( \delta \lambda = \lambda - \lambda_t \). For fixed \( \delta \lambda \) and \( \delta m^2 \to 0 \) the \( O(4) \) fixed point dominates, while for fixed \( \delta m^2 \) and \( \delta \lambda \to 0 \) only the Gaussian fixed point is approached. All this is consistent with the general theory of tricritical points [31].

\section*{5.2 The universal equation of state}

All the physical information near a second-order phase transition can be encoded in the equation of state. This describes the relation between the order parameter (in our case the field expectation value), the parameters that control the distance from the phase transition (such as \( \delta m^2 \)) and the external source. In the case of one relevant parameter the equation of state can be cast in the form [28, 3]

\[
\delta j = \frac{dU}{d\sigma} = \sigma |\sigma|^{\delta - 1} f(x), \quad x = \frac{\delta m^2}{|\sigma|^{1/\beta}},
\]

(12)

with \( U(\sigma) \) the effective potential. For sufficiently small \( \delta m^2 \), so that the \( O(4) \) fixed point is approached, the function \( f(x) \) takes a universal form \( f_{O(4)}(x) \) that has been computed in ref. [24]. The critical exponents \( \delta, \beta \), as well as the other critical exponents and amplitudes encoded in \( f(x) \), take values characteristic of this universality class: \( \delta \approx 4.8, \beta \approx 0.39 \). (Our approximate calculation above gave \( \beta \approx 0.41 \)).

For a theory that only approaches the Gaussian fixed point the equation of state can be inferred more easily. As we saw earlier, because of the smallness of the various couplings near the Gaussian fixed point, the mean-field description is sufficient. For a more detailed understanding of the renormalization-group flow, we assume that the potential \( U_k \) can be approximated by a polynomial, similarly to eq. (2)

\[
U_k(\bar{\rho}) = m^2(k)\rho + \frac{1}{2} \lambda(k)\rho^2 + \frac{1}{3} g(k)\rho^3 + \frac{1}{4} q(k)\rho^4 + ...
\]

(13)

we are quoting agree with the results of an analysis based on the exact renormalization group within the limited truncation we are employing [21].
Starting from the flow equation (3) we can derive evolution equations for the various parameters [20]

\[
\begin{align*}
\frac{dm^2(k)}{dk} &= -\frac{3}{2\pi^2} \lambda^3 \left( \frac{m^2}{k^2} \right) \\
\frac{d\lambda(k)}{dk} &= \frac{3}{\pi^2} \lambda^2 l_3^2 \left( \frac{m^2}{k^2} \right) - \frac{4}{\pi^2} g l_3^3 \left( \frac{m^2}{k^2} \right) \\
\frac{dg(k)}{dk} &= -\frac{15}{2\pi^2} \lambda^3 l_3^3 \left( \frac{m^2}{k^2} \right) + \frac{27}{2\pi^2} \lambda g l_2^3 \left( \frac{m^2}{k^2} \right) - \frac{15}{\pi^2} q l_1^3 \left( \frac{m^2}{k^2} \right).
\end{align*}
\] (14) (15) (16)

(The anomalous dimension is zero near the Gaussian fixed point.) The most important terms in the above equations are the term in the r.h.s. of the first equation and the last term in the r.h.s. of the second equation. These generate contributions to the renormalization of \(m^2\) and \(\lambda\) that are proportional to the ultraviolet scale \(\Lambda\). The last term in the r.h.s. of the third equation does not generate significant contributions because \(h = 0\) in the bare potential of eq. (2).

For a potential that is given by eq. (2) for \(k = \Lambda\), and a flow that stays in the vicinity of the Gaussian fixed point, the integration of the evolution equations is expected to renormalize significantly only \(m^2\) and \(\lambda\). The coupling \(g\) is expected to receive only logarithmic corrections (it corresponds to a marginal operator) cut off by the mass term. Moreover, these corrections are expected to reduce \(g(k)\) and shift the potential closer to the Gaussian fixed point [29]. These conclusions are verified by the numerical integration of the full evolution equation (3).

If we neglect the logarithmic corrections a good approximation of the effective potential is

\[
U(\rho) = m_R^2 \rho + \frac{1}{2} \lambda_R \rho^2 + \frac{1}{3} g \rho^3.
\] (17)

The tricritical point corresponds to the theory with \(\lambda_R = 0\), for which the equation of state can be written as

\[
\frac{4j}{g} = \sigma^5 \left( 1 + \frac{4m_R^2}{\sigma^4} \right).
\] (18)

Comparison with eq. (12) indicates that for the tricritical point \(\delta m^2 = 4m_R^2/g\), \(\delta j = 4j/g\), \(\delta = 5\), \(\beta = 1/4\), \(f_{tr}(x) = 1 + x\). This is in agreement with the values of the exponents in fig. 3 for relatively large \(\delta m^2\).

For \(\lambda_R \neq 0\) the equation of state can be written in terms of the potential of eq. (17) with the identification \(\delta m^2 = m_R^2\), \(\delta \lambda = \lambda_R\). We expect a crossover away from the tricritical point for \(\delta \lambda > \delta m^2\). This is apparent in fig. (3) in the transition of the effective exponent \(\beta\) from the tricritical-point value \(\beta = 0.25\) to a value close to 0.5. Of course, the

\[\text{It must be pointed out that no significant infrared effects are expected near the Gaussian fixed point because of the decoupling of the various modes at low enough scales. This happens in the symmetric phase, around which we expanded the potential in eq. (13). However, if the vacuum is at a non-zero value of the field the contributions of the three massless pions (the Goldstone modes) never decouple. This leads to interesting effects that have been discussed in ref. [30], for example.}\]
asymptotic value is determined by the \( O(4) \) fixed point that attracts the flows after they leave the Gaussian fixed point. In analogy with the crossover observed in fig. 3 for the critical exponents, we expect the critical equation of state to change continuously between the form characteristic of the tricritical point and the one of the \( O(4) \) universality class, as \( \lambda \) is varied.

6 The equation of state near the critical point

6.1 The vicinity of the tricritical point

As we have seen in section 3, for non-zero external source \( j \) and negative \( \lambda \) it is possible to have a second-order phase transition at the end of a line of first-order phase transitions. The order parameter is the deviation of the field from a certain non-zero expectation value. This type of transition is not associated with the breaking of a symmetry, as the original symmetry is explicitly broken by the source.

For the part of the line of critical points in the vicinity of the tricritical point the potential is approximately given by eq. (17) (neglecting logarithmic corrections) and the analysis of section 3 based on mean-field theory is sufficient. As we saw there, we must define a new effective potential \( \tilde{U}(\sigma;j) = U(\sigma, \pi_i = 0) - j_* \sigma \) whose minimum is located away from the origin. The value of the source \( j_* \) must be defined appropriately.

In order to derive the equation of state we shift the field: \( \sigma \to \sigma + \epsilon \). The value of \( \sigma_* \) can be determined by requiring that at the critical point the minimum of the potential is located at \( \sigma = 0 \). As we saw in section 3, at the critical point we have \( d\tilde{U}/d\sigma = d^2\tilde{U}/d\sigma^2 = d^3\tilde{U}/d\sigma^3 = 0 \) at the minimum. The last equality leads to \( \sigma_*^2 = -3\lambda_R/(5g) \), indicating that the existence of a critical point requires \( \lambda_R < 0 \).

On a surface of constant \( \lambda_R \) and \( \sigma_* \) (we assume a fixed renormalized \( g \) throughout the paper) the effective potential for the shifted field can be written as

\[
\delta j = F_1(\epsilon, \delta m^2) + (\sigma + \epsilon)^3 \left( \frac{\delta m^2}{(\sigma + \epsilon)^2} + 1 \right) + O((\sigma + \epsilon)^4),
\]

where we have redefined the mass term according to \( m^2 = m_*^2 + \delta \tilde{m}^2 \), with \( m_*^2 = 5g\sigma_*^4/4 \).

If we repeat the calculation for a different value of \( \lambda_R \), the form of the above equation remains the same. The only parameter that changes is \( \sigma_* \) through its dependence on \( \lambda_R \) given above. The field \( \sigma \) corresponds now to deviations from the new value of \( \sigma_* \).

It is convenient to define \( \sigma_* = \sigma_{*0} - \epsilon \), with \( \sigma_{*0} \) being kept fixed and \( \epsilon \) accounting for variations of \( \lambda_R \) around a constant value. Moreover, the field \( \sigma \) can be redefined as the deviation from \( \sigma_{*0} \). Then, eq. (19) retains its form with the replacements \( \sigma_* \to \sigma_{*0} - \epsilon \), \( \sigma \to \sigma + \epsilon \), and can be written as

\[
\delta j = F_1(\epsilon, \delta m^2) + (\sigma + \epsilon)^3 \left( \frac{\delta m^2}{(\sigma + \epsilon)^2} + 1 \right) + O((\sigma + \epsilon)^4),
\]

where \( j = j_* + \tilde{\delta}j \), \( j_* = 2g\sigma_{*0}^5/3 \), and \( \tilde{\delta}j = 3\tilde{\delta}j/(5g\sigma_{*0}^2) \), \( \delta m^2 = 3\delta \tilde{m}^2/(5g\sigma_{*0}^2) \). We assume that \( |\sigma| \) takes values in a range of a few \( |\epsilon| \ll \sigma_{*0} \). The function \( F_1(\epsilon, \delta m^2) \)
is zero for \( \epsilon = \delta m^2 = 0 \). Near the tricritical point it takes the form 
\[ F_1(\epsilon, \delta m^2) \approx -2\sigma_0^2 \epsilon + 4\sigma_0 \epsilon^2 - 4\epsilon^3 + \sigma_0 \delta m^2. \]

It must be emphasized that the above expression is a rather complicated way to rewrite the potential of eq. (17). As a result, the phase diagram as a function of \( \delta m^2, \delta j \) and \( \epsilon \) is expected to have the same structure as the one discussed in section 3. This can be verified easily. There is a line of critical points parametrized by \( \epsilon \) for \( \delta m^2 = \delta j = 0 \). A particular point can be approached on a surface of constant \( \epsilon \). For example, on the surface \( \epsilon = 0 \) the critical point can be approached along the line \( \delta j = F_1(0, \delta m^2) = \sigma_0 \delta m^2 \). The resulting second-order phase transition is parametrized by the critical exponents of mean-field theory: \( \beta = 1/2, \gamma = 1 \). For \( \epsilon = 0, \delta m^2 < 0 \) the potential has two minima and there are two possible vacuum states. A first-order transition from the vicinity of one vacuum to the other takes place when \( \delta j \) is varied through \( \sigma_0 \delta m^2 \). For \( \delta m^2 > 0 \) the transition from positive to negative field expectation values is continuous, but the mass term never becomes zero. This situation corresponds to an analytical crossover. The qualitative picture remains the same for other values of \( \epsilon \).

The term \( F_1(\epsilon, \delta m^2) \) in eq. (20) can be absorbed in the source term \( \delta j \) through a redefinition of the potential. However, our choice of \( \delta m^2, \delta j \) gives the most transparent connection with the parameters used for the phase diagram of QCD. The term \( \delta j \) corresponds to deviations of the explicit symmetry-breaking term in the bare Lagrangian from some constant value. In this sense it corresponds to variations of the current quark mass. The terms \( \epsilon, \delta m^2 \) are related to parameters of the theory that do not break explicitly the original symmetry. In the QCD case, they are functions of the temperature and the baryonic chemical potential.

Within the quark-meson model, that constitutes an effective description of low-energy QCD, the parameters \( \epsilon, \delta m^2 \) can be expressed as functions of \( \delta T \) and \( \delta \mu \), where \( \delta T = T - T_* \), \( \delta \mu = \mu - \mu_* \) are the deviations from the values that correspond to the second-order phase transition for a given current quark mass. We can integrate eqs. (14)–(16) from \( k = \Lambda = T \) down to \( k = 0 \) using (10), (11) as initial conditions for \( m^2 \) and \( \lambda \), and taking \( g = \mathcal{O}(h^6) \approx 0 \). The only significant contribution resulting from this integration is a term \( 3\lambda_0 R T^2/(4\pi^{3/2}) \) that must be added to the mass term \( m^2 \) [25]. We find

\[ \delta m^2 \sim \left( \lambda_0 R + \frac{h^2 N_c}{3} \right) T_* \delta T + \frac{h^2 N_c}{\pi^2} \mu_* \delta \mu \]

\[ \epsilon \sim \frac{1}{T_*} \delta T + \text{Li}^{(1,1)} \left( 0, -\exp \left( \frac{\mu_*}{T_*} \right) \right) \exp \left( \frac{\mu_*}{T_*} \right) \left( \frac{1}{\mu_*} \delta \mu - \frac{\mu_*}{T_*^2} \delta T \right) \]

\[ + \text{Li}^{(1,1)} \left( 0, -\exp \left( -\frac{\mu_*}{T_*} \right) \right) \exp \left( -\frac{\mu_*}{T_*} \right) \left( -\frac{1}{\mu_*} \delta \mu + \frac{\mu_*}{T_*^2} \delta T \right). \]

In the following subsection we shall see that it is possible to cast the equation of state in a form similar to eq. (20) even away from the tricritical point. An important observation is that there are many ways to approach a critical point. The universal properties of

\[ ^4 \text{Here } h \text{ is the Yukawa coupling that determines the constituent quark mass. We assume that it is small, consistently with the derivation of eqs. (10), (11).} \]
the physical system (such as the values of the critical exponents) are not the same along every path. Contrary to the case discussed above, the critical exponents can take effective values different from the expected ones. For example, if the critical point is approached on the surface $\delta j = 0$, along a line $\epsilon = \epsilon(\delta m^2)$ such that $F_1(\epsilon(\delta m^2), \delta m^2) = 0$, we obtain $\sigma + \epsilon(\delta m^2) \sim |\delta m^2|^{1/2}$ for the location of the minimum of the potential when $\delta m^2 < 0$. The mass term at the vacuum scales $\sim |\delta m^2|$, and the exponent $\gamma$ takes the value $\gamma = 1$. If the critical point is approached along a line $\epsilon = \epsilon(\delta m^2)$ such that $F_1(\epsilon(\delta m^2), \delta m^2) \neq 0$, we obtain $\sigma + \epsilon(\delta m^2) \sim |\delta m^2|^{1/3}$ for the location of the minimum of the potential when $\delta m^2 < 0$. The mass term at the vacuum scales $\sim |\delta m^2|^{2/3}$, and the exponent $\gamma$ takes an effective value $\gamma_{\text{eff}} = 2/3$. In the case of the quark-meson model, it is obvious from the form of eqs. (21), (22) that the generic trajectory that passes through the critical point has $F_1(\epsilon(\delta m^2), \delta m^2) \neq 0$ with high probability.

These findings are consistent with the general analysis of critical behaviour in the vicinity of tricritical points [31]. According to this analysis one must introduce more than one scaling fields [32, 33, 34], and the critical behaviour depends on their relative magnitude. We emphasize, however, that the values that we derived above do not correspond to a tricritical theory. The reason is that we are studying the theory near the line of critical points. The part of this line that lies in the vicinity of the tricritical point is described by mean field theory. As a result we find the mean-field exponents of the critical and not the tricritical theory.

6.2 Away from the tricritical point

Far from the Gaussian fixed point the theory is renormalized significantly and the potential cannot be approximated by a simple polynomial any more. The flow equation (3) describes the evolution of the potential as the scale $k$ is lowered from $\Lambda$ to zero. The last term in the r.h.s. of this equation includes contributions from the radial mode (the $\sigma$ field) and the three Goldstone modes (the pions). In the presence of an external source the ground state of the system is located at a point $\sigma_*$ where the potential has a non-zero derivative and satisfies $dU/d\sigma = j_*$. For zero anomalous dimension, the argument of the second “threshold” function is expected to become asymptotically $u_k(\tilde{\sigma}) \to (j_*/\sigma_*)/k^2$ for $k \to 0$. As a result the pions are expected to decouple and their contribution to the evolution to switch off. (A small anomalous dimension does not affect this conclusion.) This decoupling behaviour has been demonstrated explicitly in ref. [25], and we shall not repeat the analysis here.

Deep in the infrared, after the pion decoupling, the flow equation takes the form

$$
\frac{\partial}{\partial t} u_k(\tilde{\sigma}) = -3u_k + \frac{1}{2}(1 + \eta) \tilde{\sigma} u_k' + \frac{1}{4\pi^2} \ell_0^3 (u_k^\sigma) \,.
$$

Apart from the Gaussian fixed point, the only known $t$-independent solution of this equation is the Ising fixed point, for which $\partial u_I/\partial t = 0$ [36]. This solution has a $Z_2$ symmetry $\sigma \leftrightarrow -\sigma$. Whether the fixed point will be approached depends on the initial condition for the potential. If the potential at some scale $k = \tilde{\Lambda}$ after pion decoupling is $Z_2$-symmetric,
the subsequent flow preserves the symmetry. One needs to fine-tune only one parameter, the mass term, in order to approach the Ising fixed point. The theory has only one relevant parameter.

However, in the presence of a source the potential \( \tilde{U}_\Lambda(\sigma; j) = U_\Lambda(\sigma, \pi_i = 0) - j_* \sigma \) does not possess a \( Z_2 \) symmetry and the flow becomes more complicated. If we expand \( \tilde{U}_k \) after shifting the field by a \( k \)-independent value \( \sigma \rightarrow \sigma_* + \sigma \), we obtain a polynomial with several even and odd powers of \( \sigma \). The crucial question is how many of them need to be fine-tuned in order to approach the Ising fixed point.

In order to answer this question we must study the small perturbations around the Ising fixed-point solution \( u_I \) and examine how many have negative eigenvalues so that they grow in the infrared. In linear perturbation theory around the fixed point \( (u_k = u_I + \delta u_k) \) the evolution equation becomes

\[
\frac{\partial}{\partial \tilde{t}} \delta u_k(\tilde{\sigma}) = -3 \delta u_k + \frac{1}{2} (1 + \eta) \tilde{\sigma} \delta u'_k - \frac{1}{4 \pi^2} \delta u''_k l^3_1 (u'_I), \tag{24}
\]

where we have used the property \( l^d_1(w) = -d[l^d_0(w)]/dw \) of the “threshold” functions defined in eq. (5).

There is one even solution \( \delta u_{1k} \) of the above equation, whose eigenvalue \( \lambda_1 \) is related to the exponent \( \nu \) in the Ising universality class through \( \lambda_1 = 1/\nu \). The simplest odd perturbation is \( \delta u_{2k} = c_2 \tilde{\sigma} \), with an eigenvalue \( \lambda_2 = -(5 - \eta)/2 \). There is one more odd perturbation with negative eigenvalue. It is given by \( \delta u_{3k} = c_3 \partial u_I / \partial \tilde{\sigma} \) and its eigenvalue is \( \lambda_3 = -(1 + \eta)/2 \) [37, 38]. This perturbation corresponds to the symmetry under field shifts of the evolution equation before the rescalings by \( k \) in eqs. (4) [39]. It is not expected to generate genuinely new physical behaviour, as it corresponds to a redundant operator [40]. The dominant non-trivial behaviour near the Ising fixed point is associated with the even perturbation. The two odd perturbations we discussed are related either to a shift of the source term (that is linear in \( \sigma \)) or a field shift. All other perturbations have positive eigenvalues and are expected to become negligible in the infrared.

In spite of the apparent absence of strong effects associated with the odd perturbations, the number of parameters that need to be fine-tuned in order to approach the fixed point is three in the absence of a \( Z_2 \) symmetry. In the model we are considering, with a bare potential given by eq. (2), we have three parameters at our disposal: \( m^2, \lambda, j \). Moreover, the field value \( \sigma_* \) around which we expand the potential is not determined \( a \) priori. If we fix it arbitrarily, we need to fine-tune all three of \( m^2, \lambda, j \) in order to approach the Ising fixed point. Equivalently, for an arbitrary value of \( j \) the fixed point can be approached by fine-tuning \( m^2, \lambda, \sigma_* \). We have verified this conclusion through the numerical integration of the evolution equation. If the three-dimensional theory results from a more fundamental theory related to QCD, such as the quark-meson model, the parameters in the potential are a function of the temperature \( T \) and the chemical potential \( \mu \), while the source term \( j \) is proportional to the current quark mass. Then we conclude that for given \( j \) the fine-tuning of \( T \) and \( \mu \) can lead to a second-order phase transition. By varying \( j \) a line of second-order phase transitions appears in the phase diagram.

We are interested in the universal equation of state near the phase transition. For
this reason we need to discuss the evolution of the potential after the fixed point has been approached. The fine-tuning of the bare parameters guarantees that the system will remain near the fixed point for a long “time” $t$. However, if the fine-tuning is not perfect, the perturbations with negative eigenvalues will start growing eventually. We can solve the evolution equation (23) if we split $u_k$ into even and odd parts as $u_k = u_k^e + u_k^o$ and assume $u_k^e \ll u_k^o$. This assumption is a good approximation, especially in the vicinity of the fixed point, because the symmetric part includes the fixed-point contribution $u_f$. The even part satisfies eq. (23), while for the odd part we use the ansatz $u_k^o(\tilde{\sigma}) = f_2(t) \dot{\tilde{\sigma}} + f_3(t) u_k^e(\tilde{\sigma})$. This leads to the equations

$$\frac{df_2}{dt} = -\frac{1}{2}(5 - \eta)f_2,$$

$$\frac{df_3}{dt} = -\frac{1}{2}(1 + \eta)f_3$$

whose solution is $f_2 = c_2k^{-5/2}Z_k^{-1/2}$, $f_3 = c_3k^{-1/2}Z_k^{1/2}$, with $Z_k$ the wavefunction renormalization and $c_2$, $c_3$ arbitrary constants. This solution is consistent with our previous discussion of the perturbations near the fixed point (where $\eta$ is constant), but also describes the final evolution of the potential away from it.

For $k \to 0$ we obtain for the effective potential

$$\tilde{U}(\sigma) = U(\sigma) - j \sigma = U^s(\sigma) + c_2 \sigma + c_3 U^st(\sigma) \simeq U^s(\sigma + c_3) + c_2 \sigma.$$  

We know that $U^s(\sigma)$ is related to the universal equation of state of the Ising model through eq. (12). This implies that we can parametrize the equation of state similarly to eq. (20)

$$\delta j = \frac{d\tilde{U}}{d\sigma} = F_2(\epsilon, \delta m^2) + (\sigma + \epsilon)|\sigma + \epsilon|^{\delta - 1}f_{Z_2}(x), \quad x = \frac{\delta m^2}{|\sigma + \epsilon|^{1/\delta}}.$$  

where $\delta m^2, \epsilon \to 0$ and $|\sigma|$ takes values in a range of a few $|\epsilon|$. The universal function $f_{Z_2}(x)$ is specified by the Ising universality class [21, 37]. The critical exponents are $\beta = 0.33, \delta = 4.8$. For an effective theory resulting from QCD, the parameters $\delta m^2, \epsilon$ are functions of the temperature and chemical potential, while $\delta j$ corresponds to deviations of the quark mass from a constant value.

The appearance of the function $F_2(\epsilon, \delta m^2)$ can be understood as follows: For a potential expanded around a fixed value $\sigma_{*0}$, the parameters $c_3$ (related to $\epsilon$), $\delta m^2$ (used in the parametrization of the symmetric part) are functions of the parameters $m^2, \lambda$ of the bare theory (or $T, \mu$ for QCD). The parameter $c_2$ depends on $j_*$ as well, but in a trivial way: $c_2 = -j_* + G(m^2, \lambda)$. The choice of $j_*$ does not affect the renormalization flow because the evolution equation involves second functional derivatives with respect to the field. As a result, a term linear in $\sigma$ does not change through the evolution and can be added directly to the effective potential. On the other hand, if the bare potential is not $Z_2$-symmetric its first derivative at the origin changes during the evolution and can become non-zero, even if it was zero for the bare theory. Expressing $m^2, \lambda$ in terms of $\delta m^2, \epsilon$ leads to the expression $c_2 = -j_* + F_2(\epsilon, \delta m^2)$. The fine-tuning of $j_*$ amounts to
choosing it so that it cancels \( F_2(0,0) \). This leads to eq. (28), where \( F_2(0,0) = 0 \) and \( \delta j \) measures deviations of the source from \( j_* \). The function \( F_2(\epsilon, \delta m^2) \) can be determined for a given bare theory similarly to the case of eq. (20). A simple analytical calculation is not feasible, as perturbation theory is not applicable away from the tricritical point. The evolution equation has to be integrated numerically while performing a triple fine-tuning of the initial conditions so that the Ising fixed point is approached.

As expected the odd perturbations do not lead to the introduction of a new universal function. They simply generate a field shift. Our result is in agreement with ref. [41]. In that study the bare potential was assumed to have a very simple form (a polynomial of fourth degree). Our result provides the generalization to the case of an arbitrary bare potential.

Our conclusions are not expected to be altered when higher terms in the derivative expansion are taken into account. The form of the odd perturbations will remain the same. The first perturbation corresponds to a term linear in the field. The introduction of such a term does not modify the exact flow equation for the effective action [19, 21] before the rescaling by \( k \) in eqs. (4), as this equation involves second functional derivatives with respect to the field. As a result the implications of such a term are independent of the truncation. Similarly, the second odd perturbation is related to a redundant operator that leads to field shifts. Again, its effects are expected to be independent of the truncation.

In analogy with the crossover for the equation of state between the form characteristic of the tricritical point (eq. (18)) and that of the \( O(4) \) universality class (eq. (12)) in the case with \( j = 0 \), we expect a different crossover as a function of \( j_* \). For \( j_* \to 0 \) the universal equation of state near the critical point is given by eq. (20), while for large \( j_* \) it takes the form of eq. (28). A crossover is expected between the two forms as \( j_* \) is varied continuously.

### 7 Implications for QCD

Because of its universal form, eq. (28) is expected to describe the critical behaviour of all physical systems whose phase diagram includes a critical point similar to that of our toy model. For two-flavour QCD the connection has been made more explicit in ref. [25]. It was shown that a potential whose leading terms are given by eq. (2) results, at energy scales below the temperature, from the integration of the fermionic contributions in a model of quarks coupled to mesons. The subsequent decoupling of the pions at lower scales leads to a critical theory with only one massless field, the \( \sigma \) field. The universal equation of state is given by eq. (28), with \( \delta m^2, \epsilon \) proportional to deviations of the temperature \( \delta T \) and the baryonic chemical potential \( \delta \mu \) from certain values. The source term \( \delta j \) is proportional to the deviation of the quark mass from a constant value.

It is obvious that in experimental situations the critical point of QCD can be approached only along the surface \( \delta j = 0 \). The details of the experiment (center of mass energy, type of colliding nuclei) determine the effective temperature and chemical potential. Information from various experiments can be used in order to approach the critical
point along a curve $\epsilon = \epsilon(\delta m^2)$. An important question is whether the universal properties of the critical system (such as scaling parametrized by critical exponents) are observable. The solution of eq. (28) with $\delta j = 0$, $\epsilon = \epsilon(\delta m^2)$ determines the location of the vacuum. The function $F_2(\epsilon(\delta m^2), \delta m^2)$ is a regular function around the point $(0,0)$ and can be Taylor expanded. (An explicit example is given for $F_1(\epsilon, \delta m^2)$ below eq. (20).) As $F_2(0,0) = 0$ the leading term is $c \delta m^2$, with $c$ some constant. (We have assumed $\epsilon = \epsilon(\delta m^2)$.) The solution of eq. (28) is $|\sigma + \epsilon(\delta m^2)| = |c \delta m^2 / D|^{1/\delta}$, where $D = f(0)$. This solution emerges because the critical exponents $\delta = 4.8$, $\beta = 0.33$ satisfy $\beta \delta > 1$.

The “unrenormalized” mass term $d^2 \tilde{U} / d\sigma^2$ (equal to the inverse susceptibility) scales as $|\sigma + \epsilon(\delta m^2)|^{\delta - 1} \sim |\delta m^2|^{(\delta - 1)/\delta}$. This implies that the effective exponent $\gamma_{\text{eff}}$ takes the value $\gamma_{\text{eff}} = (\delta - 1)/\delta = 0.79$. This should be compared with the standard value $\gamma = 1.24$ in the Ising universality class [21, 37]. For the exponent $\nu$ parametrizing the divergence of the correlation length we find $\nu_{\text{eff}} = 0.40$. This is a consequence of the scaling law $\nu = \gamma / (2 - \eta)$ that relates $\nu$, $\gamma$ and the small anomalous dimension $\eta = 0.036$. The standard value of $\nu$ in the Ising universality class is $\nu = 0.63$.

It must be pointed out that the mass term may scale with the standard value for the exponent along certain paths that approach the critical point. For example, for $\epsilon = 0$, $\delta j = F_2(0, \delta m^2)$ the standard scaling is obtained. However, this approach to the critical point is unphysical as the quark mass cannot be altered. Another possible path has $\delta j = 0$ and $\epsilon(\delta m^2)$ given by the solution of the equation $F(\epsilon, \delta m^2) = 0$. Again, such a fine-tuned path is unlikely to be realized experimentally. These conclusions are in agreement with previous studies of the tricritical and critical points, in which the arguments were based on scaling relations [35].

The effective values for the exponents that we derived above are smaller than the standard ones in the Ising universality class by approximately 40%. This implies that the divergence of quantities such as the correlation length or the susceptibility along the experimentally accessible paths is much slower than the naive expectation. This conclusion is a consequence of the fact that the external source (the explicit symmetry breaking term) cannot be altered, as it is proportional to the quark mass. The smallness of the effective exponents implies that the universal behaviour is not easily accessible. The critical point must be approached very closely before the divergences of the correlation length or the susceptibility become apparent.

As a final comment we mention that the inclusion of the strange quark is not expected to modify our conclusions. It affects primarily the location of the line of critical points, while the general structure of the phase diagram remains the same. The general conclusion is that there is a direction in the phase diagram that is not under experimental control. It is related to the explicit breaking effects of the chiral symmetry ($SU(3)_L \times SU(3)_R$ in the case of three flavours) that are associated with the masses of the quarks. As a result, the experimental approach to a critical point is expected to follow a path that displays the effective exponents we derived above.

**Acknowledgements:** We would like to thank T. Morris, M. Stephanov, E. Vicari and C. Wetterich for many useful discussions.
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