How to determine the sign of a valuation on $\mathbb{C}[x, y]$?

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Abstract

Given a divisorial discrete valuation centered at infinity on $\mathbb{C}[x, y]$, we show that its sign on $\mathbb{C}[x, y]$ (i.e. whether it is negative or non-positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$) is completely determined by the sign of its value on the last key form (key forms being the avatar of key polynomials of valuations [Mac36] in ‘global coordinates’). The proof involves computations related to the cone of curves on certain compactifications of $\mathbb{C}^2$ and gives a characterization of the divisorial valuations centered at infinity whose skewness can be interpreted in terms of the slope of an extremal ray of these cones, yielding a generalization of a result of [FJ07]. A by-product of these arguments is a characterization of valuations which ‘determine’ normal compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity in terms of an associated ‘semigroup of values’.

1 Introduction

Notation 1.1. Throughout this section $k$ is a field and $R$ is a finitely generated $k$-algebra.

In algebraic (or analytic) geometry and commutative algebra, valuations are usually treated in the local setting, and the values are always positive or non-negative. Even if it is a priori not known if a given discrete valuation $\nu$ is positive or non-negative on $R \setminus k$, it is evident how to verify this, at least if $\nu(k \setminus \{0\}) = 0$: one has only to check the values of $\nu$ on the $k$-algebra generators of $R$. For valuations centered at infinity however, in general it is non-trivial to determine if it is negative or non-positive on $R \setminus k$:

Example 1.2. Let $R := \mathbb{C}[x, y]$ and for every $\epsilon \in \mathbb{R}$ with $0 < \epsilon < 1$, let $\nu_\epsilon$ be the valuation (with values in $\mathbb{R}$) on $\mathbb{C}(x, y)$ defined as follows:

$$\nu_\epsilon(f(x, y)) := -\deg_x \left( f(x, y)|_{y=x^{5/2}+x^{-1}+\xi x^{-5/2}-}\right) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},$$

where $\xi$ is a new indeterminate and $\deg_x$ is the degree in $x$. Direct computation shows that

$$\nu_\epsilon(x) = -1, \quad \nu(y) = -5/2, \quad \nu_\epsilon(y^2 - x^5) = -3/2, \quad \nu_\epsilon(y^2 - x^5 - 2x^{-1}y) = \epsilon.$$

Is $\nu_\epsilon$ negative on $\mathbb{C}[x, y]$? Let $g := y^2 - x^5 - 2x^{-1}y$. The fact that $\nu_\epsilon(g) > 0$ does not seem to be of much help for the answer (especially if $\epsilon$ is very small), since $g \notin \mathbb{C}[x, y]$ and $\nu_\epsilon(xg) < 0$. However, $g$ is precisely the last key form (Definition 2.4) of $\nu_\epsilon$ (see Example 2.8), and therefore Theorem 1.3 implies that $\nu_\epsilon$ is non-positive on $\mathbb{C}[x, y]$, i.e. no matter how small $\epsilon$ is, there exists $f_\epsilon \in \mathbb{C}[x, y]$ such that $\nu_\epsilon(f_\epsilon) > 0$. 

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In this article we settle the question of how to determine if a valuation centered at infinity is negative or non-positive on $R$ for the case that $R = \mathbb{C}[x, y]$. At first we describe how this question arises naturally in the study of algebraic completions of affine varieties:

Recall that a divisorial discrete valuation (Definition 2.2) $\nu$ on $R$ is centered at infinity iff $\nu(f) < 0$ for some $f \in R$, or equivalently iff there is an algebraic completion $\bar{X}$ of $X := \text{Spec} R$ (i.e. $\bar{X}$ is a complete algebraic varieties containing $X$ as a dense open subset) and an irreducible component $C$ of $\bar{X} \setminus X$ such that $\nu$ is the order of vanishing along $C$. On the other hand, one way to construct algebraic completions of the affine variety $X$ is to start with a degree-like function on $R$ (the terminology is from [Mon10b] and [Mon10a]), i.e. a function $\delta : R \to \mathbb{Z} \cup \{-\infty\}$ which satisfy the following ‘degree-like’ properties:

P1. $\delta(f + g) \leq \max\{\delta(f), \delta(g)\}$, and
P2. $\delta(fg) \leq \delta(f) + \delta(g)$,

and construct the graded ring

$$R^\delta := \bigoplus_{d \geq 0} \{ f \in R : \delta(f) \leq d \} \cong \sum_{d \geq 0} \{ f \in R : \delta(f) \leq d \} t^d$$  (2)

where $t$ is an indeterminate. It is straightforward to see that $\bar{X}^\delta := \text{Proj} R^\delta$ is a projective completion of $X$ provided the following conditions are satisfied:

Proj-1. $R^\delta$ is finitely generated as a $k$-algebra, and
Proj-2. $\delta(f) > 0$ for all $f \in R \setminus k$.

A fundamental class of degree-like functions are divisorial semidegrees which are precisely the negative of divisorial discrete valuations centered at infinity - they serve as ‘building blocks’ of an important class of degree-like functions (see [Mon10b], [Mon10a]). Therefore, a natural question in this context is:

**Question 1.3.** Given a divisorial semidegree $\delta$ on $R$, how to determine if $\delta(f) > 0$ for all $f \in R \setminus k$? Or equivalently, given a divisorial discrete valuation $\nu$ on $R$ centered at infinity, how to determine if $\nu(f) < 0$ for all $f \in R \setminus k$?

In this article we give a complete answer to Question 1.3 for the case $k = \mathbb{C}$ and $R = \mathbb{C}[x, y]$ (note that the answer for the case $R = \mathbb{C}[x]$ is obvious, since the only discrete valuations centered at infinity on $\mathbb{C}[x]$ are those which map $x - \alpha \mapsto -1$ for some $\alpha \in \mathbb{C}$). More precisely, we consider the sequence of key forms (Definition 2.4) corresponding to semidegrees, and show that

**Theorem 1.4.** Let $\delta$ be a divisorial semidegree on $\mathbb{C}[x, y]$ (i.e. $-\delta$ is a divisorial discrete valuation on $\mathbb{C}[x, y]$ centered at infinity) and let $g_0, \ldots, g_{n+1}$ be the key forms of $\delta$ in $(x, y)$-coordinates. Then

1. $\delta$ is non-negative on $\mathbb{C}[x, y]$ iff $\delta(g_{n+1})$ is non-negative.
2. $\delta$ is positive on $\mathbb{C}[x, y]$ iff one of the following holds:
   - (a) $\delta(g_{n+1})$ is positive,
(b) \( \delta(g_{n+1}) = 0 \) and \( g_k \not\in \mathbb{C}[x,y] \) for some \( k, 0 \leq k \leq n+1 \), or
(b') \( \delta(g_{n+1}) = 0 \) and \( g_{n+1} \not\in \mathbb{C}[x,y] \).

Moreover, conditions (b) and (b') are equivalent.

Remark 1.5. The key forms of a semidegree \( \delta \) on \( \mathbb{C}[x,y] \) are counterparts in \((x,y)\)-coordinates of the key polynomials of \( \nu := -\delta \) introduced in [Mac36] (and computed in local coordinates near the center of \( \nu \)). The basic ingredient of the proof of Theorem 1.4 is the algebraic contratability criterion of [Mon13] which uses key forms. We note that key forms were already used in [FJ07] (without calling them by any special name).

Remark 1.6. The key forms of a semidegree can be computed explicitly from any of the alternative presentations of the semidegree (see e.g. [Mon13, Algorithm 3.24] for an algorithm to compute key forms from the generic Puiseux series (Definition 2.13) associated to the semidegree). Therefore Theorem 1.4 gives an effective way to determine if a given semidegree is positive or non-negative on \( \mathbb{C}[x,y] \).

Trees of valuations centered at infinity on \( \mathbb{C}[x,y] \) were considered in [FJ07] along with a parametrization of the tree called skewness \( \alpha \). The notion of skewness has an ‘obvious’ extension to the case of semidegrees, and using this definition one of the assertions of [FJ07, Theorem A.7] can be reformulated as the statement that the following identity holds for a certain subtree of semidegrees \( \delta \) on \( \mathbb{C}[x,y] \):

\[
\alpha(\delta) = \inf \left\{ \frac{\delta(f)}{d_\delta \deg(f)} : f \text{ is a non-constant polynomial in } \mathbb{C}[x,y] \right\}, \tag{3}
\]

where

\[
d_\delta := \max\{\delta(x), \delta(y)\}. \tag{4}
\]

It is observed in [Jon12, Page 121] that in general the relation in (3) is satisfied with \( \leq \), and “it is doubtful that equality holds in general.” Example 3.1 shows that the equality indeed does not hold in general. It is not hard to see that \( \alpha(\delta) \) can be expressed in terms of \( \delta(g_{n+1}) \) (see (13)), and using that expression we give a characterization of the semidegrees for which (3) holds true:

Theorem 1.7. Let \( \delta \) be a semidegree on \( \mathbb{C}[x,y] \) and \( g_0, \ldots, g_{n+1} \) be the corresponding key forms. Then (3) holds iff one of the following assertions is true:

1. \( \delta(g_{n+1}) \geq 0 \), or
2. \( \delta(g_{n+1}) < 0 \) and \( g_k \in \mathbb{C}[x,y] \) for all \( k, 0 \leq k \leq n+1 \), or
2'. \( \delta(g_{n+1}) < 0 \) and \( g_{n+1} \in \mathbb{C}[x,y] \).

Moreover, the ‘inf’ in right hand side of (3) can be replaced by ‘min’ iff \( g_{n+1} \in \mathbb{C}[x,y] \) iff \( g_k \in \mathbb{C}[x,y] \) for all \( k, 0 \leq k \leq n+1 \); in this case the minimum is achieved with \( f = g_{n+1} \).

1 Under the assumptions of Lemma A.12 of [FJ07], the polynomials \( U_j \) constructed in Section A.5.3 of [FJ07] are precisely the key forms of \( -\nu \).

2 In [FJ07] the skewness \( \alpha \) was defined only for valuations \( \nu \) centered at infinity which satisfied \( \min\{\nu(x), \nu(y)\} = -1 \). Here for a semidegree \( \delta \), we define \( \alpha(\delta) \) to be the skewness of \( -\delta/d_\delta \) (where \( d_\delta \) is as in (4)) in the sense of [FJ07].
**Remark 1.8.** The right hand side of (3) can be interpreted as the *slope* of one of the extremal rays of the *cone of affine curves* in a certain compactification (namely the compactification of Proposition 2.10) of \( \mathbb{C}^2 \) associated to \( \delta \).

Our final result is the following corollary of the arguments in the proof of Theorem 1.4 which answers a question of Professor Peter Russell.

**Corollary 1.9.** Let \( \delta \) be a semidegree on \( \mathbb{C}[x, y] \). Define

\[
S_\delta := \{(\deg(f), \delta(f)) : f \in \mathbb{C}[x, y] \setminus \{0\}\} \subseteq \mathbb{Z}^2, \tag{5}
\]

and \( C_\delta \) be the cone over \( S_\delta \) in \( \mathbb{R}^2 \). Then

1. \( \delta \) determines an analytic compactification of \( \mathbb{C}^2 \) iff the positive \( x \)-axis is not contained in the closure \( \bar{C}_\delta \) of \( C_\delta \) in \( \mathbb{R}^2 \).

2. \( \delta \) determines an algebraic compactification of \( \mathbb{C}^2 \) iff \( C_\delta \) is closed in \( \mathbb{R}^2 \) and the positive \( x \)-axis is not contained in \( C_\delta \).

**Remark 1.10.** The phrase “\( \delta \) determines an algebraic (resp. analytic) compactification of \( \mathbb{C}^2 \)” means “there exists a (necessarily unique) normal algebraic (resp. analytic) compactification \( \bar{X} \) of \( X := \mathbb{C}^2 \) such that \( C_\infty := \bar{X} \setminus X \) is an irreducible curve and \( \delta \) is the order of pole along \( C_\infty \).” In particular, \( \delta \) determines an algebraic compactification of \( \mathbb{C}^2 \) iff \( \delta \) satisfies conditions [Proj-1] and [Proj-2].

**Remark 1.11.** \( S_\delta \) is isomorphic to the *global Enriques semigroup* (in the terminology of [CPRL02]) of the compactification of \( \mathbb{C}^2 \) from Proposition 2.10. Also, the assertions of Corollary 1.9 remain true if in (5) \( \deg \) is replaced by any other semidegree which determines an algebraic completion of \( \mathbb{C}^2 \) (e.g. a weighted degree with positive weights).

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**2 Preliminaries**

**Notation 2.1.** Throughout the rest of the article we write \( X := \mathbb{C}^2 \) with polynomial coordinates \((x, y)\) and let \( \bar{X}^{(0)} \cong \mathbb{P}^2 \) be the compactification of \( X \) induced by the embedding \((x, y) \mapsto [1 : x : y] \), so that the semidegree on \( \mathbb{C}[x, y] \) corresponding to the line at infinity is precisely on \( \bar{X}^0 \) is \( \deg \), where \( \deg \) is the usual degree in \((x, y)\)-coordinates.

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\[^3\text{Prof. Russell's question was motivated by the correspondence established in [Mon13] between normal algebraic compactifications of } \mathbb{C}^2 \text{ with one irreducible curve at infinity and algebraic curves in } \mathbb{C}^2 \text{ with one place at infinity. Since the semigroup of poles of planar curves with one place at infinity are very special (see e.g. [Abh78], [SS94]), he asked if similarly the semigroups of values of semidegrees which determine normal algebraic compactifications of } \mathbb{C}^2 \text{ can be similarly distinguished from the semigroup of values of general semidegrees. While Example 3.2 shows that they can not be distinguished only by the values of the semidegree itself, Corollary 1.10 shows that it can be done if paired with degree of polynomials.} \]
2.1 Divisorial discrete valuations, semidegrees, key forms, and associated compactifications

Definition 2.2 (Divisorial discrete valuations). A discrete valuation on \( \mathbb{C}(x, y) \) is a map \( \nu : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z} \) such that for all \( f, g \in \mathbb{C}(x, y) \setminus \{0\} \),

1. \( \nu(f + g) \geq \min\{\nu(f), \nu(g)\} \),
2. \( \nu(fg) = \nu(f) + \nu(g) \).

A discrete valuation \( \nu \) on \( \mathbb{C}(x, y) \) is called divisorial iff there exists a normal algebraic surface \( Y_\nu \) equipped with a birational map \( \sigma : Y_\nu \to X^0 \) and a curve \( C_\nu \) on \( Y_\nu \) such that for all non-zero \( f \in \mathbb{C}[x, y] \), \( \nu(f) \) is the order of vanishing of \( \sigma^*(f) \) along \( C_\nu \). The center of \( \nu \) on \( X^0 \) is \( \sigma(C_\nu) \). \( \nu \) is said to be centered at infinity (with respect to \( (x, y) \)-coordinates) iff the center of \( \nu \) on \( X^0 \) is contained in \( X^0 \setminus X \); equivalently, \( \nu \) is centered at infinity iff there is a non-zero polynomial \( f \in \mathbb{C}[x, y] \) such that \( \nu(f) < 0 \).

Definition 2.3 (Divisorial semidegrees). A divisorial semidegree on \( \mathbb{C}(x, y) \) is a map \( \delta : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z} \) such that \(-\delta\) is a divisorial discrete valuation.

Definition 2.4 (cf. definition of key polynomials in [FJ04, Definition 2.1], also see Remark 2.6 below). Let \( \delta \) be a divisorial semidegree on \( \mathbb{C}(x, y) \) such that \( \delta(x) > 0 \). A sequence of elements \( g_0, g_1, \ldots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y] \) is called the sequence of key forms for \( \delta \) if the following properties are satisfied:

P0. \( g_0 = x, g_1 = y \).

P1. Let \( \omega_j := \delta(g_j), 0 \leq j \leq n + 1 \). Then

\[
\omega_{j+1} < \alpha_j \omega_j = \sum_{i=0}^{j-1} \beta_{j,i} \omega_i \quad \text{for } 1 \leq j \leq n, \]

where

(a) \( \alpha_j = \min\{\alpha \in \mathbb{Z}_{>0} : \alpha \omega_j \in \mathbb{Z}\omega_0 + \cdots + \mathbb{Z}\omega_{j-1}\} \) for \( 1 \leq j \leq n \),

(b) \( \beta_{j,i} \)’s are integers such that \( 0 \leq \beta_{j,i} < \alpha_i \) for \( 1 \leq i < j \leq n \) (in particular, \( \beta_{j,0} \)’s are allowed to be negative).

P2. For \( 1 \leq j \leq n \), there exists \( \theta_j \in \mathbb{C}^* \) such that

\[
g_{j+1} = g_j^{\alpha_j} - \theta_j g_0^{\beta_{j,0}} \cdots g_{j-1}^{\beta_{j,j-1}}. \]

P3. Let \( y_1, \ldots, y_{n+1} \) be indeterminates and \( \omega \) be the weighted degree on \( B := \mathbb{C}[x, x^{-1}, y_1, \ldots, y_{n+1}] \) corresponding to weights \( \omega_0 \) for \( x \) and \( \omega_j \) for \( y_j \), \( 0 \leq j \leq n + 1 \) (i.e. the value of \( \omega \) on a polynomial is the maximum ‘weight’ of its monomials). Then for every polynomial \( g \in \mathbb{C}[x, x^{-1}, y] \),

\[
\delta(g) = \min\{\omega(G) : G(x, y_1, \ldots, y_{n+1}) \in B, \ G(x, g_1, \ldots, g_{n+1}) = g\}. \quad (6)
\]

Theorem 2.5. There is a unique and finite sequence of key forms for \( \delta \).

Remark 2.6. Let \( \delta \) be as in Definition 2.4. Set \( u := 1/x \) and \( v := y/x^k \) for some \( k \) such that \( \delta(y) < k\delta(x) \), and let \( \tilde{g}_0 = u, \tilde{g}_1 = v, \tilde{g}_2, \ldots, \tilde{g}_{n+1} \in \mathbb{C}[u, v] \) be the key polynomials of \( \nu := -\delta \) in \( (u, v) \)-coordinates. Then the key forms of \( \delta \) can be computed from \( \tilde{g}_j \)’s as follows:

\[
g_j(x, y) := \begin{cases} x & \text{for } j = 0, \\ x^{k \deg_v(\tilde{g}_j)} \tilde{g}_j(1/x, y/x^k) & \text{for } 1 \leq j \leq n + 1. \end{cases} \quad (7)
\]
Theorem 2.5 is an immediate consequence of the existence of key polynomials (see e.g. [FJ04 Theorem 2.29]).

**Example 2.7.** Let \((p, q)\) be integers such that \(p > 0\) and \(\delta\) be the weighted degree on \(\mathbb{C}(x, y)\) corresponding to weights \(p\) for \(x\) and \(q\) for \(y\). Then the key forms of \(\delta\) are \(x, y\).

**Example 2.8.** Let \(\epsilon := \frac{q}{2p}\) for positive integers \(p, q\) such that \(q < 2p\) and \(\delta_\epsilon\) be the semidegree on \(\mathbb{C}(x, y)\) defined as follows:

\[
\delta_\epsilon(f(x, y)) := 2p \deg_x \left( f(x, y) \Big|_{y=x^{5/2}+x^{-1}+\xi x^{-5/2}} \right) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},
\]

where \(\xi\) is a new indeterminate and \(\deg_x\) is the degree in \(x\). Note that \(\delta_\epsilon = -2p\nu_\epsilon\), where \(\nu_\epsilon\) is from Example 1.2 (we multiplied by \(2p\) to simply make the semidegree integer valued).

Then the sequence of key forms of \(\delta_\epsilon\) is \(x, y, y^2 - x^5, y^2 - x^5 - 2x^{-1}y\).

The following property of key forms can be proved in a straightforward way from their defining properties.

**Proposition 2.9.** Let \(\delta\) and \(g_0, \ldots, g_{n+1}\) be as in Definition 2.4 and \(d_\delta\) be as in (4). Define

\[
m_\delta := \gcd \left( \delta(g_0), \ldots, \delta(g_n) \right).
\]

Then

\[
m_\delta \delta(g_{n+1}) \leq d_\delta^2.
\]

Moreover, (10) is satisfied with an equality iff \(\delta = \deg\).

**Proposition 2.10** ([Mon11 Propositions 4.2 and 4.7]). Given a divisorial semidegree \(\delta\) on \(\mathbb{C}[x, y]\) such that \(\delta \neq \deg\) and \(\delta(x) > 0\), there exists a unique compactification \(\bar{X}\) of \(\mathbb{C}^2\) such that

1. \(\bar{X}\) is projective and normal.
2. \(\bar{X}_\infty := \bar{X} \setminus X\) has two irreducible components \(C_1, C_2\).
3. The semidegree on \(\mathbb{C}[x, y]\) corresponding to \(C_1\) and \(C_2\) are respectively \(\deg\) and \(\delta\).

Moreover, all singularities \(\bar{X}\) are rational. Let \(g_0, \ldots, g_{n+1}\) be the key forms of \(\delta\). Then the inverse of the matrix of intersection numbers \((C_i, C_j)\) of \(C_i\) and \(C_j\), \(1 \leq i, j \leq 2\), is

\[
\mathcal{M} = \left( \begin{array}{cc} 1 & d_\delta \\ d_\delta & m_\delta \delta(g_{n+1}) \end{array} \right),
\]

where \(d_\delta\) and \(m_\delta\) are as in respectively (11) and (9).

We will use the following result which is an immediate corollary of [Mon13 Proposition 4.2].

**Proposition 2.11.** Let \(\delta\), \(\bar{X}\), and \(C_1, C_2\) be as in Proposition 2.10. Let \(g_0, \ldots, g_{n+1}\) be the key forms of \(\delta\). Then the following are equivalent:

1. there is a (compact algebraic) curve \(C\) on \(\bar{X}\) such that \(C \cap C_1 = \emptyset\).
2. $g_k$ is a polynomial for all $k$, $0 \leq k \leq n + 1$.

3. $g_{n+1}$ is a polynomial.

The following is the main result of [Mon13]:

**Theorem 2.12.** Let $\delta$ be a divisorial semidegree on $\mathbb{C}[x,y]$ such that $\delta(x) > 0$ and $g_0, \ldots, g_{n+1}$ be the key forms of $\delta$. Then $\delta$ determines a normal algebraic compactification of $\mathbb{C}^2$ (in the sense of Remark 1.10) iff $\delta(g_{n+1}) > 0$ and $g_{n+1}$ is a polynomial.

### 2.2 Degree-wise Puiseux series

Note that the proof of Theorem 1.4 does not use the material of this subsection. Proposition 2.20 and Corollary 2.22 are used in the proof of $\delta(g_{n+1}) < 0$ case of Theorem 1.7.

**Definition 2.13** (Degree-wise Puiseux series). The field of degree-wise Puiseux series in $x$ is

$$\mathbb{C}(\langle x \rangle) := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j=0}^{\infty} a_j x^{j/p} : k, p \in \mathbb{Z}, p \geq 1 \right\},$$

where for each integer $p \geq 1$, $\mathbb{C}((x^{-1/p}))$ denotes the field of Laurent series in $x^{-1/p}$. Let $\phi = \sum_{q \leq 0} a_q x^{q/p}$ be degree-wise Puiseux series where $p$ is the polydromy order of $\phi$, i.e. $p$ is the smallest positive integer such that $\phi \in \mathbb{C}((x^{-1/p}))$. Then the conjugates of $\phi$ are $\phi_j := \sum_{q \leq 0} a_q \zeta^j x^{q/p}, 1 \leq j \leq p$, where $\zeta$ is a primitive $p$-th root of unity. The usual factorization of polynomials in terms of Puiseux series implies the following

**Theorem 2.14.** Let $f \in \mathbb{C}[x,y]$. Then there are unique (up to conjugacy) degree-wise Puiseux series $\phi_1, \ldots, \phi_k$, a unique non-negative integer $m$ and $c \in \mathbb{C}^*$ such that

$$f = cx^m \prod_{i=1}^{k} \prod_{\phi_i \text{ conjugate of } \phi_i} (y - \phi_i(x)).$$

**Proposition 2.15** ([Mon11 Theorem 1.2]). Let $\delta$ be a divisorial semidegree on $\mathbb{C}(x,y)$ such that $\delta(x) > 0$. Then there exists a degree-wise Puiseux polynomial (i.e. a degree-wise Puiseux series with finitely many terms) $\phi_{\delta} \in \mathbb{C}(\langle x \rangle)$ and a rational number $r_{\delta} < \text{ord}_x(\phi_{\delta})$ such that for every polynomial $f \in \mathbb{C}[x,y]$,

$$\delta(f) = \delta(x) \deg_x \left( f(x,y) \right)_{y = \phi_{\delta}(x) + \xi^{r_{\delta}}} ,$$

where $\xi$ is an indeterminate.

**Definition 2.16.** If $\phi_{\delta}$ and $r_{\delta}$ are as in Proposition 2.15, we say that $\tilde{\phi}_{\delta}(x,\xi) := \phi_{\delta}(x) + \xi^{r_{\delta}}$ is the generic degree-wise Puiseux series associated to $\delta$.

**Example 2.17.** Let $(p,q)$ are integers such that $p > 0$ and $\delta$ be the weighted degree on $\mathbb{C}(x,y)$ corresponding to weights $p$ for $x$ and $q$ for $y$. Then $\tilde{\phi}_{\delta} = \xi^{q/p}$ (i.e. $\phi_{\delta} = 0$).

**Example 2.18.** Let $\delta_e$ be the semidegree from Example 2.8. Then $\tilde{\phi}_{\delta} = x^{5/2} + x^{-1} + \xi x^{-5/2}.$
The following result, which is an immediate consequence of [Mon11, Proposition 4.2, Assertion 2], connects degree-wise Puiseux series of a semidegree with the geometry of associated compactifications.

**Proposition 2.19.** Let $\delta$, $X$, $C_1$, $C_2$ be as in Proposition 2.10 and let $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be the generic degree-wise Puiseux series associated to $\delta$. Assume in addition that $\delta$ is not a weighted degree, i.e. $\phi_\delta(x) \neq 0$. Pick $f \in \mathbb{C}[x, y] \setminus \{0\}$ and let $C_f$ be the curve on $X$ which is the closure of the curve defined by $f$ on $\mathbb{C}^2$. Then $C_f \cap C_1 = \emptyset$ iff the degree-wise Puiseux factorization of $f$ is of the form

$$f = \prod_{i=1}^k \prod_{\phi_{ij} \text{ is a conjugate of } \phi_i} (y - \phi_{ij}(x)), \quad \text{where each } \phi_i \text{ satisfies}$$

$$\phi_i(x) - \phi_\delta(x) = c_i x^{r_\delta} + \text{l.o.t.}$$

for some $c_i \in \mathbb{C}$ (where l.o.t. denotes lower order terms in $x$).

The following result gives some relations between degree-wise Puiseux series and key forms of semidegrees, and follows from standard properties of key polynomials (in particular, the first 3 assertions follow from [Mon13, Proposition 3.28] and the last assertion follows from the first; a special case of the last assertion (namely the case that $\delta(y) \leq \delta(x)$) was proved in [Mon11, Identity (4.6)]).

**Proposition 2.20.** Let $\delta$ be a divisorial semidegree on $\mathbb{C}(x, y)$ such that $\delta(x) > 0$. Let $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be the generic degree-wise Puiseux series associated to $\delta$ and $g_0, \ldots, g_{n+1}$ be the key forms of $\delta$. Then

1. There is a degree-wise Puiseux series $\phi$ with

$$\phi(x) - \tilde{\phi}_\delta(x) = cx^{r_\delta} + \text{l.o.t.}$$

for some $c \in \mathbb{C}$ (where l.o.t. denotes lower order terms in $x$) such that the degree-wise Puiseux factorization of $g_{n+1}$ is of the form

$$g_{n+1} = \prod_{\phi^* \text{ is a conjugate of } \phi} (y - \phi^*(x)).$$

2. Let the Puiseux pairs [Mon13, Definition 3.11] of $\phi_\delta$ be $(q_1, p_1), \ldots, (q_l, p_l)$ (if $\delta_0 \in \mathbb{C}((1/x))$, then simply set $l = 0$). Set $p_0 := 1$. Then

$$\deg(g_{n+1}) = \begin{cases} 1 & \text{if } \phi_\delta = 0, \\
\max\{1, \deg_x(\phi_\delta)\} p_0 p_1 \cdots p_l & \text{otherwise.} \end{cases}$$

3. Write $r_\delta$ as $r_\delta = q_{l+1}/(p_0 \cdots p_l p_{l+1})$, where $p_{l+1}$ is the smallest integer $\geq 1$ such that $p_0 \cdots p_l p_{l+1} r_\delta$ is an integer. Let $d_\delta$ and $m_\delta$ be as in respectively [1] and [9]. Then

$$m_\delta = p_{l+1},$$

$$d_\delta = \begin{cases} \max\{p_1, q_1\} & \text{if } \phi_\delta = 0, \\
\max\{1, \deg_x(\phi_\delta)\} p_0 p_1 \cdots p_{l+1} & \text{otherwise.} \end{cases}$$
4. Let the skewness $\alpha(\delta)$ of $\delta$ be defined as in footnote 2. Then
\[
\alpha(\delta) = m_\delta \delta(g_{n+1})/d_\delta^2 = \begin{cases} \min\{p_i/q_i\} & \min\{\delta(x), \delta(y)\}/d_\delta \quad \text{if } \delta_\delta = 0, \\ \max\{p_i/q_i\} & \delta(g_{n+1})/d_\delta \deg(g_{n+1}) \quad \text{otherwise}. \end{cases} \tag{15}
\]

The following lemma is a consequence of Assertion 1 of Proposition 2.20 and the definition of generic degree-wise Puiseux series of a semidegree. It follows via a straightforward, but cumbersome induction on the number of Puiseux pairs of the degree-wise Puiseux roots of $f$, and we omit the proof.

**Lemma 2.21.** Let $\delta$ be a divisorial semidegree on $\mathbb{C}(x, y)$ such that $\delta(x) > 0$. Let $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^\delta$ be the generic degree-wise Puiseux series associated to $\delta$ and $g_0, \ldots, g_{n+1}$ be the key forms of $\delta$. Then for all $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$,
\[
\frac{\delta(f)}{\deg(f)} \geq \frac{\delta(g_{n+1})}{\deg(g_{n+1})}. \tag{16}
\]

Now assume in addition that $\delta$ is not a weighted degree, i.e. $\phi_\delta(x) \neq 0$. Then the (16) holds with equality iff $f$ has a degree-wise Puiseux factorization as in (13).

Combining Propositions 2.11 and 2.19 and Lemma 2.21 yields the following

**Corollary 2.22.** Consider the set-up of Proposition 2.11 assume in addition that $\delta$ is not a weighted degree. Then the Assertions 1 to 3 of Proposition 2.11 are equivalent to the following statement

4. There exists $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$ which satisfies (16) with equality.

3 Proofs

**Proof of Theorem 1.4.** W.l.o.g. we may (and will) assume that $\delta \neq \deg$. Let $\bar{X}$ be the projective compactification of $X$ from Proposition 2.10. In the notations of Proposition 2.10 the matrix of intersection numbers $(C_i, C_j)$ of $C_i$ and $C_j$, $1 \leq i, j \leq 2$, is:
\[
I = \frac{1}{d_\delta^2 - m_\delta \delta(g_{n+1})} \begin{pmatrix} -m_\delta \delta(g_{n+1}) & d_\delta \\ d_\delta & -1 \end{pmatrix} \tag{17}
\]

We consider the 3 possibilities of the sign of $\delta(g_{n+1})$ separately:

**Case 1:** $\delta(g_{n+1}) > 0$. In this case (10) and (17) imply that $(C_1, C_1) < 0$, so $C_0$ is contractible by a criterion of Grauert [Bad01, Theorem 14.20], i.e. there is a map $\pi: \bar{X} \to \bar{X}'$ of normal analytic surfaces such that $\pi(C_1)$ is a point and $\pi|X \setminus C_1$ is an isomorphism. In particular $\delta$ is the pole along the irreducible curve at infinity on the compactification $\bar{X}'$ of $X := \mathbb{C}^2$. Consequently $\delta$ is positive on all non-constant polynomials in $\mathbb{C}[x, y]$.

**Case 2:** $\delta(g_{n+1}) = 0$. In this case $(C_1, C_1) = 0$. Note that $C_1$ is a $\mathbb{Q}$-Cartier divisor, since all singularities of $\bar{X}$ are rational. It follows that $C_1$ is a nef $\mathbb{Q}$-Cartier divisor. Consequently there can not be any effective curve $C$ on $\bar{X}$ linearly equivalent to $a_1C_1 + a_2C_2$ with $a_2 < 0$, for in that case $(C_1, C) = a_2(C_1, C_2) < 0$, which is impossible. Since the curve determined by a polynomial $f \in \mathbb{C}[x, y]$ is linearly equivalent to $\deg(f)C_1 + \delta(f)C_2$, it follows that $\delta(f) \geq 0$ for all $f \in \mathbb{C}[x, y]$.
Case 3: $\delta(g_{n+1}) < 0$. In this case $(C_1, C_1) > 0$. It follows that $C_1$ is in the interior of the cone of curves on $\bar{X}$ [Kol96, Lemma II.4.12], and consequently there are effective curves of the form $C := a_1 C_1 - a_2 C_2$ with $a_1, a_2 \in \mathbb{Z}_{\geq 0}$ such that $a_2/a_1$ is sufficiently small. But such a curve is the closure in $\bar{X}$ of the curve on $\mathbb{C}^2$ defined by some $f \in \mathbb{C}[x, y]$ such that $\deg(f) = a_1$ and $\delta(f) = -a_2$. In particular, $\delta(f) < 0$, as required.

Assertion 1 of Theorem 1.4 follows from Proposition 2.11 and the conclusions of the above 3 cases. Assertion 2 follows from Assertion 1, Proposition 2.11 and the observation from (17) that if $\delta(g_{n+1}) = 0$, then for every $f \in \mathbb{C}[x, y]$, $\delta(f) = 0$ iff the closure in $\bar{X}$ of the curve on $\mathbb{C}^2$ defined by $f$ does not intersect $C_1$.

Proof of Theorem 1.7. W.l.o.g. we may (and will) assume that $\delta \neq \deg$. Let $\bar{X}$ be the projective compactification of $X$ from Proposition 2.10. We continue to use the notation of Proposition 2.10 and divide the proof into separate cases depending on $\delta(g_{n+1})$.

Case 1: $\delta(g_{n+1}) \geq 0$. In this case Assertion 1 of Theorem 1.4 implies that the cone of curves on $\bar{X}$ is generated (over $\mathbb{R}_{\geq 0}$) by $C_1$ and $C_2$. It follows that the nef cone of $\bar{X}$ is

$$\text{Nef}(\bar{X}) = \{ a_1 C_1 + a_2 C_2 : a_1, a_2 \in \mathbb{R}_{\geq 0}, (a_1 C_1 + a_2 C_2, C_i) \geq 0, 1 \leq i \leq 2 \}$$

$$= \{ a_1 C_1 + a_2 C_2 : a_1, a_2 \in \mathbb{R}_{\geq 0}, a_1 d_1 \geq a_2 \geq a_1 m_\delta \delta(g_{n+1})/d_\delta \} \quad \text{(using (17))}$$

In particular, the ‘lower edge’ of $\text{Nef}(\bar{X})$ is the half line $\{(a_1, a_2) \in \mathbb{R}^2_{\geq 0} : a_2 = a_1 m_\delta \delta(g_{n+1})/d_\delta \}$. Since any nef divisor is a limit of ample divisors and large multiples of ample divisors have global sections, it follows that

$$\frac{m_\delta}{d_\delta} \delta(g_{n+1}) = \inf \left\{ \frac{\delta(f)}{\deg(f)} : f \text{ is a non-constant polynomial in } \mathbb{C}[x, y] \right\}. \quad \text{(18)}$$

It follows from (15) and (18) that (3) holds with equality in this case, as required.

Case 2: $\delta(g_{n+1}) < 0$. In this case it follows as in Case 3 of the proof of Theorem 1.4 that $C_1$ is in the interior of the cone $\text{NE}(\bar{X})$ of curves on $\bar{X}$. [Kol96, Lemma II.4.12] implies that $\text{NE}(\bar{X})$ has an edge of the form $\{r(C_1 - aC_2) : r \geq 0\}$ for some $a > 0$, and moreover, there exists $r > 0$ such that $rC_1 - arC_2$ is linearly equivalent to some irreducible curve $C$ on $\bar{X}$. Pick $g \in \mathbb{C}[x, y]$ such that $C \cap \mathbb{C}^2$ is the zero set of $g$. Then $\deg(g) = r$ and $\delta(g) = -ar$. Since the ‘other’ edge of $\text{NE}(\bar{X})$ is spanned by $C_2$, it follows that for all $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$,

$$\frac{\delta(f)}{\deg(f)} \geq -\frac{\delta(g)}{\deg(g)}. \quad \text{(19)}$$

The assertions of Theorem 1.7 now follow from the conclusions of the above 2 cases together with (13), Lemma 2.21 and Corollary 2.22.

Proof of Corollary 1.8. We continue to assume that $\delta \neq \deg$ and use the notations of the proof of Theorem 1.7. Note that $\delta$ determines an analytic compactification of $\mathbb{C}^2$ iff $C_1$ is contractible iff $(C_1, C_1) < 0$ (by Grauert’s criterion [B˘ ad01, Theorem 14.20]) iff $\delta(g_{n+1}) > 0$.

\footnote{Even though [Kol96, Lemma II.4.12] is proved for only non-singular surfaces, its proof goes through for arbitrary normal surfaces using the intersection theory due to [Mum61].}
(due to (17)). Assertion 1 of Corollary 1.9 then follows from identity (18) and the observation from Case 2 of the proof of Theorem 1.7 that if \( \delta(g_{n+1}) < 0 \), then the positive \( x \)-axis is in the interior of \( C_3 \). For Assertion 2 note that Lemma 2.22 and identity (18) together imply that the ‘lower edge’ of \( \text{Nef}(\tilde{X}) \) is spanned by an effective curve iff \( g_{n+1} \) is a polynomial. Assertion 2 now follows from the preceding sentence and Theorem 2.12. \( \square \)

Example 3.1 (An example where (3) does not hold). Let \( \delta \) be the semidegree on \( \mathbb{C}(x, y) \) defined as follows:

\[
\delta(f(x, y)) := \deg_x \left( f(x, y)|_{y=x^{-1}+\xi x^{-2}} \right) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},
\]

where \( \xi \) is an indeterminate. Then the key forms of \( \delta \) are \( x, y, y - x^{-1} \), and therefore (13) implies that

\[
\alpha(\delta) = \frac{\delta(y - x^{-1})}{\deg(y - x^{-1})} = -2.
\]

(20)

Now consider the surface \( \tilde{X} \) from Proposition 2.10 and let \( C \) be the closure in \( \tilde{X} \) of the curve \( y = 0 \) on \( \mathbb{C}^2 \). Then in the notation of Proposition 2.10 \( C \) is linearly equivalent to \( \deg(y)C_1 + \deg(y)C_2 = C_1 - C_2 \). It follows from (17) that \( (C, C) = -1/3 < 0 \), so that [Kol96, Lemma II.4.12] implies that \( C \) spans an edge of the cone of curves on \( \tilde{X} \), i.e. the polynomial \( g \) from Case 2 of the proof of Theorem 1.7 is \( y \). It then follows from identities (19) and (20) that

\[
\inf \left\{ \frac{\delta(f)}{d_{\delta} \deg(f)} : f \in \mathbb{C}[x, y] \setminus \mathbb{C} \right\} = \frac{\delta(g)}{d_{\delta} \deg(g)} = -1 > \alpha(\delta).
\]

Example 3.2 (The semigroup of values does not distinguish semidegrees that determine algebraic compactifications of \( \mathbb{C}^2 \)). Let \( \delta \) be the semidegree on \( \mathbb{C}(x, y) \) defined as follows:

\[
\delta(f(x, y)) := 2 \deg_x \left( f(x, y)|_{y=x^{-5/2}+x^{-1}+\xi x^{-3/2}} \right) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},
\]

where \( \xi \) is an indeterminate. Then the key forms of \( \delta \) are \( x, y, y^2 - x^5, y^2 - x^5 - 2x^{-1}y \), with corresponding \( \delta \)-values 2, 5, 3, 1. Since \( \delta \)-value of the last key polynomial is positive, it follows from the arguments of the proof of Corollary 1.9 that \( \delta \) determines an analytic compactification of \( \mathbb{C}^2 \). But the last key form of \( \delta \) is not a polynomial, so that the compactification determined by \( \delta \) is not algebraic (Theorem 2.12). On the other hand, it follows from our computation of the values of \( \delta \) and Corollary 2.22 that the semigroup of values of \( \delta \) on polynomials is

\[
N_\delta := \{ \delta(f) : f \in \mathbb{C}[x, y] \} = \{2, 3, 4, \ldots \}.
\]

Now let \( \delta' \) be the weighted degree on \( (x, y) \)-coordinates corresponding to weights 2 for \( x \) and 3 for \( y \). Then \( \delta' \) determines an algebraic compactification of \( \mathbb{C}^2 \), namely the weighted projective surface \( \mathbb{P}^2(1, 2, 3) \). But \( N_\delta = N_{\delta'} \).

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