LOGARITHMIC COEFFICIENTS OF SOME CLOSE-TO-CONVEX FUNCTIONS

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Abstract

The logarithmic coefficients $\gamma_n$ of an analytic and univalent function $f$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalisation $f(0) = 0 = f'(0) - 1$ are defined by $\log(f(z)/z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n$. In the present paper, we consider close-to-convex functions (with argument 0) with respect to odd starlike functions and determine the sharp upper bound of $|\gamma_n|, n = 1, 2, 3,$ for such functions $f$.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0 = f'(0) - 1$. Any function $f$ in $\mathcal{A}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

The class of univalent (that is, one-to-one) functions in $\mathcal{A}$ is denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is called starlike (respectively, convex) if $f(\mathbb{D})$ is starlike (respectively, convex) with respect to the origin. Let $\mathcal{S}^*$ and $\mathcal{C}$ denote the classes of starlike and convex functions in $\mathcal{S}$, respectively. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if $\text{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if $\text{Re}(1 + zf''(z)/f'(z)) > 0$ for $z \in \mathbb{D}$. From the above it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is said to be close-to-convex with argument $\alpha$ and with respect to $g$ if

$$\text{Re}\left(e^{i\alpha} \frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$
Let $K_{\alpha}(g)$ denote the class of all such functions. Let

$$K(g) := \bigcup_{\alpha \in (-\pi/2, \pi/2)} K_{\alpha}(g) \quad \text{and} \quad K_{\alpha} := \bigcup_{g \in S^*} K_{\alpha}(g)$$

be the classes of close-to-convex functions with respect to $g$ and close-to-convex functions with argument $\alpha$, respectively. The class

$$K := \bigcup_{\alpha \in (-\pi/2, \pi/2)} K_{\alpha} = \bigcup_{g \in S^*} K(g)$$

is the class of all close-to-convex functions. It is well known that every close-to-convex function is univalent in $D$ (see [5]). Geometrically, $f \in K$ means that the complement of the image domain $f(D)$ is the union of nonintersecting half-lines.

For a function $f \in S$, the logarithmic coefficients $\gamma_n (n = 1, 2, \ldots)$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in D. \quad (1.2)$$

Bazilevich first noticed that the logarithmic coefficients are essential in the coefficient problem of univalent functions. In [2, 3], he gave estimates in terms of the positive Hayman constant (see [10]) for how close the coefficients $\gamma_n (n = 1, 2, \ldots)$ of the functions of class $S$ are to the relative logarithmic coefficients of the Koebe function $k(z) = z/(1 - z)^2$. He also estimated $\sum_{n=1}^{\infty} n|\gamma_n|^2 \gamma^{2n}$, which after multiplication by $\pi$ is equal to the area of the image of the disc $|z| < r < 1$ under the function $\frac{1}{2} \log(f(z)/z)$ for $f \in S$. The celebrated de Branges’ inequalities (the former Milin conjecture) for univalent functions $f$ state that

$$\sum_{k=1}^{n} (n-k+1)|\gamma_k|^2 \leq \sum_{k=1}^{n} \frac{n+1-k}{k}, \quad n = 1, 2, \ldots,$$

with equality if and only if $f(z) = e^{-i\theta}k(e^{i\theta}z)$, $\theta \in \mathbb{R}$ (see [4]). De Branges [4] used this inequality to prove the celebrated Bieberbach conjecture. Moreover, the de Branges’ inequalities have also been the source of many other interesting inequalities involving logarithmic coefficients of $f \in S$ such as (see [6])

$$\sum_{k=1}^{\infty} |\gamma_k|^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

More attention has been given to the results in an average sense (see [5, 6, 14]) than the exact upper bounds for $|\gamma_n|$ for functions in the class $S$ and few exact upper bounds for $|\gamma_n|$ have been established. For the Koebe function $k(z) = z/(1 - z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function $k(z)$ plays the role of an extremal function for most of the extremal problems in the class $S$, it is expected that $|\gamma_n| \leq 1/n$ for functions in $S$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function $f$ in the class $S$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [5, Theorem 8.4]).
By differentiating (1.2) and equating coefficients,
\[
\gamma_1 = \frac{1}{2} a_2, \quad (1.3) \\
\gamma_2 = \frac{1}{2} (a_3 - \frac{1}{2} a_2^2), \quad (1.4) \\
\gamma_3 = \frac{1}{2} (a_4 - a_2 a_3 + \frac{1}{2} a_2^3). \quad (1.5)
\]

If \( f \in S \), then \(|\gamma_1| \leq 1\) follows from (1.3). Using the Fekete–Szegő inequality [5, Theorem 3.8] in (1.4), it is easy to obtain the sharp estimate
\[
|\gamma_2| \leq \frac{1}{2} (1 + 2e^{-2}) = 0.635 \ldots
\]

For \( n \geq 3 \), the problem seems much harder and no significant upper bounds for \(|\gamma_n|\) when \( f \in S \) appear to be known.

For functions in the class \( S^* \), by the analytic characterisation \( \text{Re}(zf'(z)/f(z)) > 0 \) for \( z \in \mathbb{D} \), it is easy to prove that \(|\gamma_n| \leq 1/n \) for \( n \geq 1 \) and equality holds for the Koebe function \( k(z) = z/(1 - z)^2 \). The inequality \(|\gamma_n| \leq 1/n \) for \( n \geq 2 \) for functions in the class \( \mathcal{K} \) was claimed in a paper of Elhosh [7]. However, Girela [8] pointed out an error in the proof of Elhosh [7] and, hence, the result is not substantiated. Indeed, Girela proved that for each \( n \geq 2 \), there exists a function \( f \in \mathcal{K} \) such that \(|\gamma_n| > 1/n \). Recently, it has been proved [15] that \(|\gamma_3| \leq \frac{7}{12} \) for functions in \( \mathcal{K}_0 \) (close-to-convex functions with argument 0) with the additional assumption that the second coefficient of the corresponding starlike function \( g \) is real. But this estimate is not sharp, as pointed out in [1], where the authors proved that \(|\gamma_3| \leq \frac{1}{18} (3 + 4 \sqrt{2}) = 0.4809 \) for functions in \( \mathcal{K}_0 \) without the additional assumption that the second coefficient of the corresponding starlike function \( g \) is real. In the same paper, the authors also determined the sharp upper bound \(|\gamma_3| \leq \frac{1}{233} (28 + 19 \sqrt{19}) = 0.4560 \) for close-to-convex functions with argument 0 and with respect to the Koebe function and conjectured that this upper bound is also true for the whole class \( \mathcal{K}_0 \) (see also [13]).

Let \( S_2^* \) denote the class of odd starlike functions and \( F \) the class of close-to-convex functions with argument 0 and with respect to odd starlike functions. That is,
\[
F = \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{D}, \text{for some } g \in S_2^* \right\}.
\]

It is important to note that the class \( F \) is rotationally invariant. In the present article, we determine the sharp upper bound of \(|\gamma_n|, n = 1, 2, 3\), for functions in \( F \).

2. Main results

Let \( P \) denote the class of analytic functions \( P \) of the form
\[
P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (2.1)
\]
such that \( \text{Re} \, P(z) > 0 \) in \( \mathbb{D} \). Functions in \( P \) are sometimes called Carathéodory functions. To prove our main results, we need some preliminary lemmas.
Lemma 2.1 [5, page 41]. For a function \( P \in \mathcal{P} \) of the form (2.1), the sharp inequality
\[ |c_n| \leq 2 \text{ holds for each } n \geq 1. \]
Equality holds for the function \( P(z) = (1 + z)/(1 - z) \).

Lemma 2.2 [12]. Let \( P \in \mathcal{P} \) be of the form (2.1) and \( \mu \) be a complex number. Then
\[ |c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}. \]
The result is sharp for the functions \( P(z) = (1 + z^2)/(1 - z^2) \) and \( P(z) = (1 + z)/(1 - z) \).

Lemma 2.3 [11]. Let \( P \in \mathcal{P} \) be of the form (2.1). Then there exist \( x, t \in \mathbb{C} \) with \( |x| \leq 1 \) and \( |t| \leq 1 \) such that
\[ 2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{and} \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t. \]

Theorem 2.4. Let \( f \in \mathcal{F} \) be of the form (1.1). Then there exist \( x, t \in \mathbb{C} \) with \( |x| \leq 1 \) and \( |t| \leq 1 \) such that
\[ 2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{and} \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t. \]

The inequalities are sharp.

Proof. Let \( f \in \mathcal{F} \) be of the form (1.1). Then there exist an odd starlike function \( g(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1} \) and a Carathéodory function \( P \in \mathcal{P} \) of the form (2.1) with
\[ zf'(z) = g(z)P(z). \]
Comparing the coefficients on both sides of (2.2),
\[ a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}(b_3 + c_2) \quad \text{and} \quad a_4 = \frac{1}{4}(b_3 c_1 + c_3). \]
Substituting \( a_2, a_3 \) and \( a_4 \) given by (2.3) in (1.3), (1.4) and (1.5) and simplifying,
\[ \gamma_1 = \frac{3}{2}a_2 = \frac{3}{4}c_1, \]
\[ \gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2) = \frac{1}{6}b_3 + \frac{1}{6}(c_2 - \frac{3}{8}c_1^2), \]
\[ 2\gamma_3 = a_4 - a_2a_3 + \frac{1}{3}a_2^3 = \frac{1}{24}(2b_3 c_1 - 3c_3^2 - 4c_1 c_2 + 6c_3). \]

By Lemma 2.1, it follows from (2.4) that \( |\gamma_1| \leq \frac{1}{2} \) and equality holds for a function \( f \) defined by \( zf'(z) = g(z)P(z) \), where \( g(z) = z/(1 - z^2) \) and \( P(z) = (1 + z)/(1 - z) \). Since \( g \) is an odd starlike function, \( |b_3| \leq 1 \) (see [9, Ch. 4, Theorem 3, page 35]). Using Lemma 2.2, it follows from (2.5) that
\[ |\gamma_2| \leq \frac{1}{6}|b_3| + \frac{1}{6}|c_2 - \frac{3}{8}c_1^2| \leq \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = \frac{1}{2} \]
equality holds for a function \( f \) defined by \( zf'(z) = g(z)P(z) \), where \( g(z) = z/(1 - z^2) \) and \( P(z) = (1 + z^2)/(1 - z^2) \).

From (2.6), after writing \( c_2 \) and \( c_3 \) in terms of \( c_1 \) with the help of Lemma 2.3,
\[ 48\gamma_3 = 2c_1 b_3 + \frac{1}{2}c_1^3 + c_1 x(4 - c_1^2) - \frac{3}{2} c_1 x^2(4 - c_1^2) + 3(4 - c_1^2)(1 - |x|^2)t, \]
where \( |x| \leq 1 \) and \( |t| \leq 1 \). Since the class \( \mathcal{F} \) is invariant under rotation, without loss of generality we can assume that \( c_1 = c \), where \( 0 \leq c \leq 2 \). Taking the modulus on both the sides of (2.7) and then applying the triangle inequality and \( |b_3| \leq 1 \),
\[ 48|\gamma_3| \leq 2c + \frac{1}{2}c^3 + cx(4 - c^2) - \frac{3}{2}cx^2(4 - c^2) + 3(4 - c^2)(1 - |x|^2), \]
where we have also used the fact that \(|\eta| \leq 1\). Let \(x = re^{i\theta}\), where \(0 \leq r \leq 1\) and \(0 \leq \theta \leq 2\pi\). For simplicity, write \(\cos \theta = p\). Then

\[
48|\gamma_3| \leq |\psi(c, r)| + |\phi(c, r, p)| =: F(c, r, p), \tag{2.8}
\]

where \(\psi(c, r) = 2c + 3(4 - c^2)(1 - r^2)\) and

\[
\phi(c, r, p) = \left(\frac{1}{4}c^6 + c^2r^2(4 - c^2)^2 + \frac{9}{4}c^2r^4(4 - c^2)^2 + c^4(4 - c^2)r^2 \right. \\
\left. - \frac{3}{2}c^4r^2(4 - c^2)(2p^2 - 1) - 3c^2(4 - c^2)p^3\right)^{1/2}.
\]

Thus, we need to find the maximum value of \(F(c, r, p)\) over the rectangular cube \(R := [0, 2] \times [0, 1] \times [-1, 1]\). By elementary calculus,

\[
\max_{0 \leq r \leq 1} \psi(0, r) = \psi(0, 0) = 12, \quad \max_{0 \leq r \leq 1} \psi(2, r) = 4, \quad \max_{0 \leq r \leq 2} \psi(c, 0) = \psi(\frac{1}{2}, 0) = \frac{37}{3},
\]

\[
\max_{0 \leq c \leq 2} \psi(c, 1) = \psi(2, 1) = 4 \quad \text{and} \quad \max_{(c, r) \in [0, 2] \times [0, 1]} \psi(c, r) = \psi(\frac{1}{2}, 0) = \frac{37}{3}.
\]

We first find the maximum value of \(F(c, r, p)\) on the boundary of \(R\), that is, on the six faces of the rectangular cube \(R\). On the face \(c = 0\), we have \(F(0, r, p) = \psi(0, r)\) for \((r, p) \in R_1 := [0, 1] \times [-1, 1]\). Thus,

\[
\max_{(r, p) \in R_1} F(0, r, p) = \max_{0 \leq r \leq 1} \psi(0, r) = \psi(0, 0) = 12.
\]

On the face \(c = 2\), we have \(F(2, r, p) = 8\) for \((r, p) \in R_1\). On the face \(r = 0\), we have \(F(c, 0, p) = 2c + 3(4 - c^2)\) for \((c, p) \in R_2 := [0, 2] \times [-1, 1]\). Note that \(F(c, 0, p)\) is independent of \(p\). Thus, by using elementary calculus it is easy to see that

\[
\max_{(c, p) \in R_2} F(c, 0, p) = F\left(\frac{1}{2}(3 - \sqrt{6}), 0, p\right) = \frac{8}{3}(9 + \sqrt{6}) = 12.3546.
\]

On the face \(r = 1\), we have \(F(c, 1, p) = \psi(c, 1) + |\phi(c, 1, p)|\) for \((c, p) \in R_2\). We first prove that \(\phi(c, 1, p) \neq 0\) in the interior of \(R_2\). On the contrary, if \(\phi(c, 1, p) = 0\) in the interior of \(R_2\), then

\[
|\phi(c, 1, p)|^2 = \left|\frac{1}{2}c^3 + ce^{i\theta}(4 - c^2) - \frac{3}{2}c e^{2i\theta}(4 - c^2)\right|^2 = 0,
\]

giving the simultaneous equations

\[
\frac{1}{2}c^3 + cp(4 - c^2) - \frac{3}{2}c(4 - c^2)(2p^2 - 1) = 0 \quad \text{and} \quad c(4 - c^2) \sin \theta - \frac{3}{2}c(4 - c^2) \sin 2\theta = 0.
\]

On further simplification, this reduces to

\[
\frac{1}{2}c^2 + p(4 - c^2) - \frac{3}{2}(4 - c^2)(2p^2 - 1) = 0 \quad \text{and} \quad 1 - 3p = 0,
\]

which is equivalent to \(p = 1/3\) and \(c^2 = 6\). This contradicts the range of \(c \in (0, 2)\). Thus, \(\phi(c, 1, p) \neq 0\) in the interior of \(R_2\).
Next, we find the maximum value $F(c, 1, p)$ in the interior of $R_2$. Suppose that $F(c, 1, p)$ has a maximum at an interior point of $R_2$. At such a point $\partial F(c, 1, p)/\partial c = 0$ and $\partial F(c, 1, p)/\partial p = 0$. From $\partial F(c, 1, p)/\partial p = 0$ (for points in the interior of $R_2$), a straightforward calculation gives

$$p = \frac{2(c^2 - 3)}{3c^2}.$$  \hspace{1cm} (2.9)

Substituting this value of $p$ in $\partial F(c, 1, p)/\partial c = 0$ and further simplification gives

$$2c - 3c^3 + \sqrt{6(c^2 + 2)} = 0.$$  

Taking the last term to the right-hand side and squaring on both the sides yields

$$9c^6 - 12c^4 - 2c^2 - 12 = 0.$$ \hspace{1cm} (2.10)

This equation has exactly one root in $(0, 2)$, which can be shown using the well-known Sturm theorem for isolating real roots and hence for the sake of brevity we omit the details. By solving the equation (2.10) numerically, we obtain the approximate root $1.3584$ in $(0, 2)$ and the corresponding value of $p$ obtained from (2.9) is $-0.4172$. Thus, the extremum points of $F(c, 1, p)$ in the interior of $R_2$ lie in a small neighbourhood of the points $A_1 = (1.3584, 1, -0.4172)$ (on the plane $r = 1$). Clearly, $F(A_1) = 9.3689$.

Since the function $F(c, 1, p)$ is uniformly continuous on $R_2$, the value of $F(c, 1, p)$ would not vary too much in the neighbourhood of the point $A_1$.

Next, we find the maximum value of $F(c, 1, p)$ on the boundary of $R_2$. Clearly, $F(0, 1, p) = 0$, $F(2, 1, p) = 8$,

$$F(c, 1, -1) = \begin{cases} 
2c + c(10 - 3c^2) & \text{for } 0 \leq c \leq \sqrt{\frac{10}{3}}, \\
2c - c(10 - 3c^2) & \text{for } \sqrt{\frac{10}{3}} < c \leq 2
\end{cases}$$

and

$$F(c, 1, 1) = \begin{cases} 
2c + c(2 - c^2) & \text{for } 0 \leq c \leq \sqrt{2}, \\
2c - c(2 - c^2) & \text{for } \sqrt{2} < c \leq 2
\end{cases}$$

By using elementary calculus,

$$\max_{0 \leq c \leq 2} F(c, 1, -1) = F\left(\frac{2 \sqrt{3}}{3}, 1, -1\right) = \frac{16 \sqrt{3}}{3} = 9.2376$$  \hspace{1cm} and

$$\max_{0 \leq c \leq 2} F(c, 1, 1) = F\left(\frac{2 \sqrt{3}}{3}, 1, 1\right) = \frac{16 \sqrt{3}}{9} = 3.0792.$$  

Therefore,

$$\max_{(c, p) \in R_2} F(c, 1, p) \approx 9.3689.$$  

On the face $p = -1$,

$$F(c, r, -1) = \begin{cases} 
\psi(c, r) + \eta_1(c, r) & \text{for } \eta_1(c, r) \geq 0, \\
\psi(c, r) - \eta_1(c, r) & \text{for } \eta_1(c, r) < 0,
\end{cases}$$
where $\eta_1(c, r) = c^3(3r^2 + 2r + 1) - 4cr(3r + 2)$ and $(c, r) \in R_3 := [0, 2] \times [0, 1]$. To find the maximum value of $F(c, r, -1)$ in the interior of $R_3$, we need to solve the pair of equations $\partial F(c, r, -1) / \partial c = 0$ and $\partial F(c, r, -1) / \partial r = 0$ in the interior of $R_3$, but it is important to note that $\partial F(c, r, -1) / \partial c$ and $\partial F(c, r, -1) / \partial r$ may not exist at points in $S_1 = \{(c, r) \in R_3 : \eta_1(c, r) = 0\}$. Solving this pair of equations,

$$
\max_{(c, r) \in \text{int} R_3 \setminus S_1} F(c, r, -1) = F\left(\frac{1}{3}(\sqrt{82} - 8), \frac{1}{57}(\sqrt{82} - 5), -1\right)
= \frac{4}{81}(41 \sqrt{82} - 121) = 12.359.
$$

Now we find the maximum value of $F(c, r, -1)$ on the boundary of $R_3$ and on the set $S_1$. Note that

$$
\max_{(c, r) \in S_1} F(c, r, -1) \leq \max_{(c, r) \in R_3} \psi(c, r) = \frac{37}{3} = 12.33.
$$

On the other hand, by using elementary calculus as before,

$$
\max_{0 \leq r \leq 1} F(0, r, -1) = \max_{0 \leq r \leq 1} 12(1 - r^2) = F(0, 0, -1) = 12, \quad \max_{0 \leq r \leq 1} F(2, r, -1) = 8,
$$

$$
\max_{0 \leq c \leq 2} F(c, 0, -1) = \max_{(c, p) \in R_2} F\left(\frac{2}{3}(3 - \sqrt{6}), 0, -1\right) = 8 \left(\frac{9}{2} + \sqrt{6}\right) = 12.3546,
$$

$$
\max_{0 \leq c \leq 2} F(c, 1, -1) = F\left(\frac{2 \sqrt{3}}{3}, 1, -1\right) = \frac{16 \sqrt{3}}{3} = 9.2376.
$$

Hence, by combining the above cases,

$$
\max_{(c, r) \in R_3} F(c, r, -1) = F\left(\frac{1}{3}(\sqrt{82} - 8), \frac{1}{57}(\sqrt{82} - 5), -1\right)
= \frac{4}{81}(41 \sqrt{82} - 121) = 12.359.
$$

On the face $p = 1$,

$$
F(c, r, 1) = \begin{cases}
\psi(c, r) + \eta_2(c, r) & \text{for } \eta_2(c, r) \geq 0, \\
\psi(c, r) - \eta_2(c, r) & \text{for } \eta_2(c, r) < 0,
\end{cases}
$$

where $\eta_2(c, r) = c^3(3r^2 - 2r + 1) - 4cr(3r - 2)$ for $(c, r) \in R_3$. To find the maximum value of $F(c, r, 1)$ in the interior of $R_3$, we need to solve the pair of equations $\partial F(c, r, 1) / \partial c = 0$ and $\partial F(c, r, 1) / \partial r = 0$ in the interior of $R_3$, but again it is important to note that $\partial F(c, r, 1) / \partial c$ and $\partial F(c, r, 1) / \partial r$ may not exist at points in the set $S_2 = \{(c, r) \in R_3 : \eta_2(c, r) = 0\}$. Solving this pair of equations,

$$
\max_{(c, r) \in \text{int} R_3 \setminus S_2} F(c, r, 1) = F\left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right)
= \frac{4}{81}(95 + 23 \sqrt{46}) = 12.3947.
$$
Now we find the maximum value of \( F(c, r, 1) \) on the boundary of \( R_3 \) and on the set \( S_2 \). By noting that (see earlier cases)
\[
\begin{align*}
\max_{(c,r)\in S_2} F(c, r, 1) &\leq \max_{(c,r)\in R_3} \psi(c, r) = \frac{37}{3} = 12.33, \\
\max_{0\leq r\leq 1} F(0, r, 1) & = 12, \quad \max_{0\leq r\leq 1} F(2, r, 1) = 8, \\
\max_{0\leq c\leq 2} F(c, 0, 1) & = \frac{5}{3}(9 + \sqrt{6}) = 12.3546, \\
\max_{0\leq c\leq 2} F(c, 1, 1) & = \frac{16\sqrt{3}}{9} = 3.0792 \\
\end{align*}
\]
and combining all the cases,
\[
\max_{(c,r)\in R_3} F(c, r, 1) = F\left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right) = \frac{4}{81}(95 + 23\sqrt{46}) = 12.3947.
\]
Let \( S' = \{(c, r, p) \in R : \phi(c, r, p) = 0\} \). Then
\[
\max_{(c,r,p)\in S'} F(c, r, p) \leq \max_{(c,r)\in R_3} \psi(c, r) = \psi(0, \frac{1}{3}) = \frac{37}{3} = 12.33.
\]
We prove that \( F(c, r, p) \) has no maximum value at any interior point of \( R \setminus S' \). Suppose that \( F(c, r, p) \) has a maximum value at an interior point of \( R \setminus S' \). At such a point \( \partial F(c, r, p)/\partial c = \partial F(c, r, p)/\partial r = \partial F(c, r, p)/\partial p = 0 \). Note that the partial derivatives may not exist at points in \( S' \). From \( \partial F(c, r, p)/\partial p = 0 \) (for points in the interior of \( R \setminus S' \)), a straightforward but laborious calculation gives
\[
p = \frac{3c^2r^2 + c^2 - 12r^2}{6c^2r}.
\]
Substituting this value of \( p \) in \( \partial F(c, r, p)/\partial r = 0 \) and simplifying,
\[
(4 - c^2)r(\sqrt{6(c^2 + 2)} - 6) = 0.
\]
This equation has no solution in the interior of \( R \setminus S' \) and hence \( F(c, r, p) \) has no maximum in the interior of \( R \setminus S' \).

On combining all the above cases,
\[
\max_{(c,r,p)\in R} F(c, r, p) = F\left(\frac{1}{3}(8 - \sqrt{46}), \frac{1}{75}(11 - \sqrt{46}), 1\right) = \frac{4}{81}(95 + 23\sqrt{46}) = 12.3947
\]
and hence, from (2.8),
\[
|\gamma_3| \leq \frac{1}{972}(95 + 23\sqrt{46}) = 0.2582. \tag{2.11}
\]
We now show that the inequality (2.11) is sharp. An examination of the proof shows that equality holds in (2.11) if we choose \( b_3 = 1, \ c_1 = c = \frac{1}{3}(8 - \sqrt{46}), \ x = \frac{1}{75}(11 - \sqrt{46}) \) and \( t = 1 \) in (2.7). For such values of \( c_1, x \) and \( t \), Lemma 2.3 gives
\[ c_2 = \frac{1}{27}(134 - 19 \sqrt{46}) \quad \text{and} \quad c_3 = \frac{2}{243}(721 - 71 \sqrt{46}). \] A function \( P \in \mathcal{P} \) having the first three coefficients \( c_1, \, c_2 \) and \( c_3 \) as above is given by

\[ P(z) = (1 - 2\lambda) \left( \frac{1 + z}{1 - z} + \lambda \frac{1 + uz}{1 - uz} + \lambda \frac{1 + \bar{u}z}{1 - \bar{u}z} \right) \]

\[ = 1 + \frac{1}{3}(8 - \sqrt{46})z + \frac{1}{27}(134 - 19 \sqrt{46})z^2 + \frac{2}{243}(721 - 71 \sqrt{46})z^3 + \cdots , \]

(2.12)

where \( \lambda = \frac{1}{10}(-4 + \sqrt{46}) \) and \( u = \alpha + i \sqrt{1 - \alpha^2} \) with \( \alpha = \frac{1}{18}(-1 - \sqrt{46}) \). Hence, equality holds in (2.11) for a function \( f \) which is defined by \( zf'(z) = g(z)P(z) \), where \( g(z) = z/(1 - z^2) \) and \( P(z) \) is given by (2.12). This completes the proof. \( \square \)

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