Team Games Optimality Conditions of Distributed Stochastic Differential Decision Systems with Decentralized Noisy Information Structures

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Abstract

We consider a team game reward, and we derive a stochastic Pontryagin's maximum principle for distributed stochastic differential systems with decentralized noisy information structures. Our methodology utilizes the semi martingale representation theorem, variational methods, and backward stochastic differential equations. Furthermore, we derive necessary and sufficient optimality conditions that characterize team and person-by-person optimality of decentralized strategies.

Finally, we apply the stochastic maximum principle to several examples from the application areas of communications, filtering and control.

Index Terms. Team Games, Distributed Systems, Decentralized, Optimality Conditions, Maximum Principle.

I. INTRODUCTION

We derive necessary and sufficient team game optimality conditions for distributed stochastic differential systems with decentralized noisy information structures. For noiseless information
structures, analogous optimality conditions are derived recently in [1] utilizing the representation of Hilbert space semi martingales and the stochastic Pontryagin’s maximum principle of partially observed stochastic differential systems developed in [2]. However, the results obtained in [1] for decentralized noiseless information structures are not necessarily applicable to decentralized noisy information structures. In fact, there are certain technicalities that must be addressed when dealing with noisy information structures, which are inherited from the centralized fully observable versus partially observable stochastic optimal control [3]–[12]. The main underlying assumption for centralized information structures, is that the acquisition of the information is centralized or the information acquired at different locations is communicated to each decision maker or control.

When the system model consist of multiple decision makers, and the acquisition of information and its processing is decentralized or shared among several locations, then the different decision makers actions are based on different information [13]. We call the information available for such decisions, ”decentralized information structures or patterns” [14], [15]. When the system model is dynamic, consisting of an interconnection of at least two subsystems, and the decisions are based on decentralized information structures, we call the overall system a ”distributed system with decentralized information structures”.

Over the years several specific forms of decentralized information structures are analyzed mostly in discrete-time [13]–[25], and more recently [26]–[34]. However, at this stage the only systematic framework addressing optimality conditions for distributed systems with decentralized information structures is the one reported in [1] for decentralized noiseless information structures.

In this paper, we consider a team game reward [22], [25], [35]–[37], and we derive necessary and sufficient optimality conditions for distributed stochastic differential systems with decentralized noisy information structures. Our methodology utilizes the semi martingale representation theorem, variational methods, and generalizes the concepts utilized in [1], [2] to derive optimality conditions for nonlinear stochastic distributed systems with decentralized noiseless information structures. From the practical point of view, the results of this part give optimality conditions in terms of forward and backward stochastic differential equations, and a Hamiltonian, called
"Hamiltonian System of Equations", which we use to compute the optimal decentralized decision strategies of several examples from the application areas of communications and control.

The specific objectives of this paper are the following.

(a) Derive team games Pontryagin’s stochastic minimum principle (necessary conditions of optimality) for distributed stochastic systems with decentralized noisy information structures;

(b) Introduce assumptions so that the necessary conditions of optimality in (a) are also sufficient, and relate the optimality conditions to person-by-person optimality conditions;

(c) Apply the stochastic minimum principle to several examples from the application areas of communication and control.

The rest of the paper is organized as follows. In Section II we formulate the distributed stochastic system with decentralized information structures. In Section III we introduce the variational equation and discuss its application in decentralized filtering and control. Section IV is devoted to the development of stochastic optimality conditions for team games with decentralized information structures, consisting of necessary and sufficient conditions of optimality. In Section V we apply the minimum principle to various examples. The paper is concluded with some comments on possible extensions of our results.

II. DISTRIBUTED STOCHASTIC DIFFERENTIAL TEAM GAMES

In this section we introduce the mathematical formulation of distributed stochastic differential systems, the noisy information structures available to the decision makers, and the definitions of collaborative decisions via team game optimality and person-by-person optimality. Although, the stochastic differential systems are driven by the Decision Makers (DMs) actions, our analysis includes unforced stochastic differential systems modeling distributed estimation. Therefore, the term ”decision maker” is used for distributed control as well as distributed estimation.

The formulation presupposes a fixed probability space with filtration, \( \left( \Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in [0,T]\}, \mathbb{P} \right) \) satisfying the usual conditions, that is, \( (\Omega, \mathcal{F}, \mathbb{P}) \) is complete, \( \mathcal{F}_{0,0} \) contains all \( \mathbb{P} \)-null sets in \( \mathcal{F} \). Throughout we assume that all filtrations are right continuous and complete \([38]\). Define \( \mathcal{F}_T \triangleq \{\mathcal{F}_{0,t} : t \in [0,T]\} \).
In our derivations we make extensive use of the following spaces. Let \( L^2_{FT}([0, T], \mathbb{R}^n) \subset L^2(\Omega \times [0, T], d\mathbb{P} \times dt, \mathbb{R}^n) \equiv L^2([0, T], L^2(\Omega, \mathbb{R}^n)) \) denote the space of \( \mathbb{F}_T \)-adapted random processes \( \{ z(t) : t \in [0, T] \} \) such that
\[
\mathbb{E} \int_{[0,T]} |z(t)|^2 d\mathbb{P} < \infty,
\]
which is a sub-Hilbert space of \( L^2([0, T], L^2(\Omega, \mathbb{R}^n)) \). Similarly, let \( L^2_{FT}([0, T], L^2(\mathbb{R}^m, \mathbb{R}^n)) \subset L^2([0, T], L^2(\Omega, L^2(\mathbb{R}^m, \mathbb{R}^n))) \) denote the space of \( \mathbb{F}_T \)-adapted \( n \times m \) matrix valued random processes \( \{ \Sigma(t) : t \in [0, T] \} \) such that
\[
\mathbb{E} \int_{[0,T]} |\Sigma(t)|^2 L^2(\mathbb{R}^m, \mathbb{R}^n) dt \triangleq \mathbb{E} \int_{[0,T]} tr(\Sigma^*(t)\Sigma(t)) dt < \infty.
\]

\textbf{A. Distributed Stochastic System}

Next, we introduce the mathematical formulation of the stochastic system. On the fixed probability space \( (\Omega, \mathbb{F}, \{ \mathbb{F}_0, \mathbb{F}_t : t \in [0, T] \}, \mathbb{P}) \) we are given a distributed stochastic dynamical decision system. It consists of an interconnection of \( N \) subsystems, and each subsystem \( i \) has, state space \( \mathbb{R}^{n_i} \), DM action space \( A^i \subset \mathbb{R}^{d_i} \), an exogenous state noise space \( \mathbb{W}^i \triangleq \mathbb{R}^{m_i} \), an exogenous measurement noise space \( \mathbb{B}^i \triangleq \mathbb{R}^{k_i} \), and initial state \( x^i(0) = x^i_0 \), defined by

(S1) \( x^i(0) = x^i_0 \): an \( \mathbb{R}^{n_i} \)-valued Random Variable;
(S2) \( \{ W^i(t) : t \in [0, T] \} \): an \( \mathbb{R}^{m_i} \)-valued standard Brownian motion which models the exogenous state noise, adapted to \( \mathbb{F}_T \), independent of \( x^i(0) \);
(S3) \( \{ B^i(t) : t \in [0, T] \} \): an \( \mathbb{R}^{k_i} \)-valued standard Brownian motion which models the exogenous measurement noise, adapted to \( \mathbb{F}_T \), independent of \( \{ W^i(t) : t \in [0, T] \} \).

The DM \( \{ u^i : i \in \mathbb{Z}_N \} \) take values in closed convex subsets of metric spaces \( \{ (\mathbb{M}^i, d) : i \in \mathbb{Z}_N \} \). The decentralized partial information structure available to DM \( u^i \) is generated by noisy observation
\[
y^i(t) = \int_0^t h^i(s, x^1(s), \ldots x^N(s), y^1(s), \ldots y^N(s)) ds + \int_0^t D^i(s) dB^i(s), \quad t \in [0, T], \quad \forall i \in \mathbb{Z}_N,
\]
(1)
where \( x^i \in \mathbb{R}^{n_i} \) is the state of subsystem \( i \) for \( i = 1, \ldots, N \). Notice that (1) models a channel with memory and feedback. Each subsystem is described by finite dimensional coupled stochastic
differential equations as follows.

\[ dx^i(t) = f^i(t, x^i(t), u^i_t)dt + \sigma^i(t, x^i(t), u^i_t)dW^i(t) + \sum_{j=1, j\neq i}^{N} f^{ij}(t, x^j(t), u^j_t)dt + \sum_{j=1, j\neq i}^{N} \sigma^{ij}(t, x^j(t), u^j_t)dW^j(t), \quad x^i(0) = x^i_0, \quad t \in (0, T], \quad \forall i \in \mathbb{Z}_N. \]

(2)

For decentralized communication and filtering applications the right side of (2) is independent of the DMs \( u^i, i = 1, \ldots, N \).

Since we considered a strong strong formulation, we define the filtration \( \mathbb{F}_T \triangleq \{ \mathbb{F}_{0,t} : t \in [0, T] \} \) as follows. Introduce the \( \sigma \)-algebras

\[ \mathbb{F}^i_{0,t} \triangleq \sigma \left\{ (x^i(0), W^i(s), B^i(s)) : 0 \leq s \leq t \right\}, \quad \mathcal{G}^{y^i,u}_{0,t} \triangleq \sigma \left\{ y^i(s) : 0 \leq s \leq t \right\}, \quad t \in [0, T], \quad i = 1, \ldots, N, \]

and the minimum \( \sigma \)-algebras generated by these as follows

\[ \mathbb{F}_{0,t} \triangleq \bigvee_{i=1}^{N} \mathbb{F}^i_{0,t}, \quad \mathcal{G}^{y,u}_{0,t} \triangleq \bigvee_{i=1}^{N} \mathcal{G}^{y^i,u}_{0,t}, \quad t \in [0, T]. \]

Next, we introduce the admissible sets of decentralized decision strategies considered in this paper.

(FIS): Feedback Information Structures. Let \( \mathcal{G}^{y^i,u}_{T} \triangleq \{ \mathcal{G}^{y^i,u}_{0,t} : t \in [0, T] \} \subset \{ \mathbb{F}_{0,t} : t \in [0, T] \} \) denote the information available to DM \( i, \forall i \in \mathbb{Z}_N \). The admissible set of decentralized feedback strategies for DM \( i \) is defined by

\[ 
\mathbb{U}^{y^i,u}[0, T] \triangleq \left\{ u^i \in L^2_{\mathcal{G}^{y^i,u}_T}([0, T], \mathbb{R}^{d_i}) : u^i_t \in \mathcal{A}^i \subset \mathbb{R}^{d_i}, \ a.e. t \in [0, T], \ \mathbb{P} - a.s. \right\}, \quad \forall i \in \mathbb{Z}_N,
\]

(3)

where \( \mathbb{U}^{y^i,u}[0, T] \) is a closed convex subset of \( L^2_{\mathbb{F}_T}([0, T], \mathbb{R}^n) \), for \( i = 1, 2, \ldots, N \).

Thus, an \( N \) tuple of DM strategies is by definition

\[ (u^1, u^2, \ldots, u^N) \in \mathbb{U}^{y,u}(0, T] \triangleq \times_{i=1}^{N} \mathbb{U}^{y^i,u}[0, T], \]

and hence it is a family of \( N \) functions, say, \( \left( \mu^1_t(\cdot), \mu^2_t(\cdot), \ldots, \mu^N_t(\cdot) \right), t \in [0, T], \) which are nonanticipative with respect to the information structures \( \{ \mathcal{G}^{y^i,u}_{0,t} : t \in [0, T] \}, i = 1, 2, \ldots, N \).

The information structure of each DM is decentralized, and may be generated by local or global subsystem observables.
**(IIS): Innovations Information Structures.** Let \( G_{I^u} \overset{\Delta}{=} \sigma\{I^i(t) : 0 \leq t \leq T\} \) denote the information available to DM \( i, \forall i \in \mathbb{Z}_N \), where \( \{I^i(t) : t \in [0, T]\} \) is the innovations of the process \( \{y^i(t) : t \in [0, T]\} \) defined by

\[
I^i(t) \overset{\Delta}{=} y^i(t) - \int_0^t \mathbb{E}\left\{ h^i(s, x^1(s), \ldots, x^N(s), y^1(s), \ldots, y^N(s)) | G_{0,s}^{y^i,u} \right\} ds, \ t \in (0, T], \ \forall i \in \mathbb{Z}_N,
\]

(4)

The admissible set of decentralized innovations strategies for DM \( i \) is defined by

\[
\mathbb{U}^{I^u}[0, T] \overset{\Delta}{=} \left\{ u^i \in \mathbb{U}^{y^i}[0, T] : u^i \text{ is } G_{0,t}^{I^u} \text{ adapted a.e.} t \in [0, T], \mathbb{P} - a.s. \right\}.
\]

(5)

An \( N \) tuple of DM strategies is by definition \( (u^1, \ldots, u^N) \in \mathbb{U}^{(N), I^u}[0, T] \)

Define the augmented vectors by

\[
W \overset{\Delta}{=} (W^1, \ldots, W^N) \in \mathbb{R}^m, \ B \overset{\Delta}{=} (B^1, \ldots, B^N) \in \mathbb{R}^k, \ u \overset{\Delta}{=} (u^1, \ldots, u^N) \in \mathbb{R}^d, \ x \overset{\Delta}{=} (x^1, \ldots, x^N) \in \mathbb{R}^n.
\]

The distributed stochastic system dynamics are described in compact form by

\[
dx(t) = f(t, x(t), u_t) dt + \sigma(t, x(t), u_t) dW(t), \quad x(0) = x_0, \quad t \in (0, T],
\]

(6)

where \( f : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n \) denotes the drift and \( \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \)

the diffusion coefficients.

The distributed observation equations are described by the observation equation

\[
y(t) = \int_0^t h(s, x(s), y(s)) ds + \int_0^t D^s y(s) dB(s), \quad t \in [0, T],
\]

(7)

where \( h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k \) is a function of the observations \( \{y(t) : 0 \leq t \leq T\} \).

**B. Pay-off Functional and Team Games**

Consider the distributed system (6), (7) with decentralized partial information structures. Given a \( u \in \mathbb{U}^{(N), y^u}[0, T] \), define the reward or performance criterion by

\[
J(u) \equiv J(u^1, u^2, \ldots, u^N) \overset{\Delta}{=} \mathbb{E}\left\{ \int_0^T \ell(t, x(t), u_t) dt + \varphi(x(T)) \right\},
\]

(8)

where \( \ell : [0, T] \times \mathbb{R}^n \times \mathbb{U}^{(N)} \longrightarrow (\mathbb{R}^\infty, \mathbb{R}^\infty) \) denotes the running cost function and \( \varphi : \mathbb{R}^n \longrightarrow (\mathbb{R}^\infty, \mathbb{R}^\infty) \), the terminal cost function. Notice that the performance of the strategies is graded by a single pay-off functional.

The distributed stochastic team optimization problem with \( N \) DM is defined below.
Problem 1. (Team Optimality) Given the pay-off functional (8), constraints (6), (7) the $N$ tuple of strategies $u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in U^{(N)}y^u[0,T]$ is called team optimal if it satisfies
\begin{equation}
J(u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \leq J(u^1, u^2, \ldots, u^N), \quad \forall u \triangleq (u^1, u^2, \ldots, u^N) \in U^{(N)}y^u[0,T]
\end{equation}
(9)
Any $u^o \in U^{(N)}y^u[0,T]$ satisfying (9) is called an optimal decision strategy (or control) and the corresponding $x^o(\cdot) \equiv x(\cdot; u^o(\cdot))$, $y^o(\cdot) \equiv y(\cdot; u^o(\cdot))$ (satisfying (6), (7)) are called an optimal state process and observation process, respectively.
Similarly, for $u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in U^{(N)}y^u[0,T]$.

By definition, Problem 1 is a dynamic team problem with each DM having a different information structure (decentralized). An alternative approach to handle such problems with decentralized information structures is to restrict the definition of optimality to the so-called person-by-person equilibrium.

Define
\begin{equation}
\bar{J}(v, u^{-i}) \triangleq J(u^1, u^2, \ldots, u^{i-1}, v, u^{i+1}, \ldots, u^N), \quad \forall \in \in \mathbb{Z}_N.
\end{equation}

Problem 2. (Person-by-Person Optimality) Given the pay-off functional (8), constraints (6), (7) the $N$ tuple of strategies $u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in U^{(N)}y^u[0,T]$ is called person-by-person optimal if it satisfies
\begin{equation}
\bar{J}(u^{i,o}, u^{-i,o}) \leq \bar{J}(u^i, u^{-i,o}), \quad \forall u^i \in U^{y^u_i}[0,T], \quad \forall i \in \mathbb{Z}_N.
\end{equation}
(10)
Similarly for $u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in U^{(N)}y^u[0,T]$.

The interpretation of (10) is that the variation of the $i$-th player is done while the rest of the players assume their optimal strategies.

In the next remark, the previous team games formulation is discussed in the context of distributed estimation.

Remark 1. In distributed estimation each subsystem is described by unforced coupled stochastic
differential equations

\[ dx^i(t) = f^i(t, x^i(t))dt + \sigma^i(t, x^i(t))dW^i(t) + \sum_{j=1, j \neq i}^{N} f^{ij}(t, x^j(t))dt + \sum_{j=1, j \neq i}^{N} \sigma^{ij}(t, x^j(t))dW^j(t), \quad x^i(0) = x^i_0, \quad t \in (0, T], \quad \forall i \in \mathbb{Z}_N, \quad (11) \]

while the observations for each subsystem are described by (1). The distributed estimation objective is to determine an \( N \) tuple of decision strategies \( u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in \cup^{N}_{y} [0, T] \) which is team optimal or person-by-person optimal (according to Problems 1, 2) subject to constraints (11), (1). This distributed estimation problem formulated via team theory, is a generalization of the static team theory discussed in [20], [22], [23], [36]. However, we point out that for distributed filtering there is no reason to consider innovations information structures.

### III. Strong Solutions and Variational Equation

In this section we introduce assumptions which will allow us to show existence of strong \( \mathbb{F}_T \)-adapted continuous solutions to (6) and (7). We also introduce the variational equation which is utilized to derive the stochastic minimum principle using the methodology in [1], [2].

Let \( B^\infty_{\mathbb{F}_T}([0, T], L^2(\Omega, \mathbb{R}^n)) \) denote the space of \( \mathbb{F}_T \)-adapted \( \mathbb{R}^n \) valued second order random processes endowed with the norm topology \( \| \cdot \| \) defined by

\[ \| x \|^2 \triangleq \sup_{t \in [0, T]} \mathbb{E}|x(t)|^2_{\mathbb{R}^n}. \]

The existence of strong solution is based on the following assumptions.

**Assumptions 1. (Main assumptions)** The coefficients of the state and observation equations (6), (7) are Borel measurable maps:

\[ f : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathbb{R}^n, \quad \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{A}^{(N)} \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), \]

\[ h^i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^{k_i}, \quad \forall i \in \mathbb{Z}_N. \]

These satisfy the following basic conditions.

There exists a \( K > 0 \) such that

(A1) \( |f(t, x, u) - f(t, z, u)|_{\mathbb{R}^n} \leq K|x - z|_{\mathbb{R}^n} \) uniformly in \( u \in \mathbb{A}^{(N)}; \)

(A2) \( |f(t, x, u) - f(t, x, v)|_{\mathbb{R}^n} \leq K|u - v|_{\mathbb{R}^n} \) uniformly in \( x \in \mathbb{R}^n; \)
Let $\{u^i\}_{i=1}^N$ be any pair of DM strategies from $\mathcal{A}^i$. The following lemma proves the existence of solutions and their continuous dependence on the decision variables.

**Lemma 1.** Suppose Assumptions $\mathcal{A}$ hold. Then for any $\mathbb{F}_{0,0}$-measurable initial state $x_0$ having finite second moment, and any $u \in \mathbb{U}^{(N)} \cdot y^u [0, T]$, the following hold.

1. **System** $(6), (7)$ has a unique solution $(x, y) \in B_{\mathbb{F}_T}^{\infty} ([0, T], L^2(\Omega, \mathbb{R}^{n+k}))$ having a continuous modification, that is, $(x, y) \in C([0, T], \mathbb{R}^{n+k}), \mathbb{P}$-a.s, $\forall i \in \mathbb{Z}_N$.

2. The solution of system $(6), (7)$ is continuously dependent on the control, in the sense that, as $u^{i,\alpha} \longrightarrow u^{i,0}$ in $\mathbb{U}^{(N)} \cdot y^u [0, T], \forall i \in \mathbb{Z}_N, (x^\alpha, y^\alpha) \longrightarrow (x^0, y^0)$ in $B_{\mathbb{F}_T}^{\infty} ([0, T], L^2(\Omega, \mathbb{R}^{n+k})), \forall i \in \mathbb{Z}_N$.

Similarly for $u \in \mathbb{U}^{(N)} \cdot y^u [0, T]$.

**Proof:** (1) Consider the augmented system $X \triangleq (x, y)$ and the associated stochastic differential equation of $X$. The proof for the first part of the lemma is classical and hence omitted.

(2) Next, we consider the second part asserting the continuity of $u$ to solution map $u \longrightarrow (x, y)$. Let $\{u^{i,\alpha} : i = 1, 2, \ldots, N\}, u^0$ be any pair of DM strategies from $\mathbb{U}^{(N)} \cdot y^u [0, T] \times \mathbb{U}^{(N)} \cdot y^u [0, T]$ and $\{x^\alpha, y^\alpha, x^0, y^0\}$ denote the corresponding pair of solutions of the system $(6), (7)$. Let $u^{i,\alpha} \longrightarrow u^{i,0}, i = 1, 2, \ldots, N$. We must show that $(x^\alpha, y^\alpha) \longrightarrow (x^0, y^0)$ in $B_{\mathbb{F}_T}^{\infty} ([0, T], L^2(\Omega, \mathbb{R}^{n+k}))$. By
the definition of solution to (6), it can be verified that

\[ x^\alpha(t) - x^\circ(t) = \int_0^t \left\{ f(s, x^\alpha(s), u_s^\alpha) - f(s, x^\circ(s), u_s^\circ) \right\} ds + \int_0^t \left\{ \sigma(s, x^\alpha(s), u_s^\alpha) - \sigma(s, x^\circ(s), u_s^\circ) \right\} dW(s) + e_1^\alpha(t) + e_2^\alpha(t), \quad t \in [0, T], \]

(12)

where

\[ e_1^\alpha(t) = \int_0^t \left\{ f(s, x^\circ(s), u_s^\circ) - f(s, x^\circ(s), u_s^\circ) \right\} ds \]

(13)

\[ e_2^\alpha(t) = \int_0^t \left\{ \sigma(s, x^\circ(s), u_s^\circ) - \sigma(s, x^\circ(s), u_s^\circ) \right\} dW(s). \]

(14)

Using the standard martingale inequality into (12), it follows from it and (A5) that there exist constants \( C_1, C_2 > 0 \) such that

\[ \mathbb{E}|x^\alpha(t) - x^\circ(t)|^2 \leq C_1 \int_0^t K^2 \mathbb{E}|x^\alpha(s) - x^\circ(s)|^2 + C_2 \left( \mathbb{E}|e_1^\alpha(t)|^2 + \mathbb{E}|e_2^\alpha(t)|^2 \right). \]

(15)

Clearly, by the Cauchy-Schwartz inequality, and martingale inequality, it follows from (A2), (A5) that

\[ \mathbb{E}|e_1^\alpha(t)|^2 \leq T \mathbb{E} \int_0^t |f(s, x^\circ(s), u_s^\circ) - f(s, x^\circ(s), u_s^\circ)|^2 ds \leq T \mathbb{E} \int_0^t K^2 |u_s^\circ - u_s^\circ|_2^2 ds, \]

(16)

\[ \mathbb{E}|e_2^\alpha(t)|^2 \leq 4 \mathbb{E} \int_0^t |\sigma(s, x^\circ(s), u_s^\circ) - \sigma(s, x^\circ(s), u_s^\circ)|^2 ds \leq 4 \mathbb{E} \int_0^t K^2 |u_s^\circ - u_s^\circ|_2^2 ds. \]

(17)

Similarly, by (A8)

\[ \mathbb{E}|y^\alpha(t) - y^\circ(t)|^2 \leq T \int_0^t K^2 \mathbb{E} \left( |x^\alpha(s) - x^\circ(s)|^2 + |y^\alpha(s) - y^\circ(s)|^2 \right) ds. \]

(18)

The integrands in the right side of inequalities (16), (17) converge to zero for almost all \( s \in [0, T] \), \( \mathbb{P} \)-a.s. Moreover, these integrands are dominated by integrable functions. Hence, by Lebesgue dominated convergence theorem the terms \( \{e_1^\alpha, e_2^\alpha\} \) converge to zero uniformly on \( [0, T] \). Define \( \rho^\alpha(t) \triangleq \mathbb{E} \left( |x^\alpha(s) - x^\circ(s)|^2 + |y^\alpha(s) - y^\circ(s)|^2 \right) \). Then by Gronwall inequality applied to \( \rho^\alpha \), it can be shown that \( \rho^\alpha \to 0 \) as \( u^{i,\alpha} \to u^{i,\circ} \) in \( \mathbb{U}^{i,\circ}(0, T) \), \( \forall i \in \mathbb{Z}_N \). The above analysis holds for innovations information structures. This completes the derivation.

Throughout the paper we assume existence of a minimizer \( u^\circ \in \mathbb{U}^{(N)}y^\circ(0, T) \) for Problem 1. For randomize (relaxed) strategies existence can be shown as in [2].
Next, we prepare to introduce the variational equation of the augmented system \((x, y)\).

Define the augmented vectors

\[
X \triangleq (x, y) \in \mathbb{R}^n \times \mathbb{R}^k, \quad B \triangleq (B^1, B^2, \ldots, B^N) \in \mathbb{R}^k, \quad y \triangleq (y^1, y^2, \ldots, y^N) \in \mathbb{R}^k,
\]

and the augmented drift and diffusion coefficients, and terms in the pay-off associated with them by

\[
F(t, X, u) \triangleq \begin{bmatrix} f(t, x, u) \\ h(t, x, y) \end{bmatrix}, \quad G(t, X, u) \triangleq \begin{bmatrix} \sigma(t, x, u) & 0 \\ 0 & D^x(t) \end{bmatrix},
\]

\[
h(t, x, y) \triangleq \begin{bmatrix} h^1(t, x, y) \\ \vdots \\ h^N(t, x, y) \end{bmatrix}, \quad D^x(t) \triangleq \text{diag}\{D^{x, 1}(t), \ldots, D^{x, N}(t)\}
\]

\[
L(t, X, u) \triangleq \ell(t, x, u), \quad \Phi(X) \triangleq \varphi(x).
\]

Then the augmented system is expressed in compact form by

\[
dX(t) = F(t, X(t), u_t) dt + G(t, X(t), u_t) \begin{bmatrix} dW(t) \\ dB(t) \end{bmatrix}, \quad X(0) = X_0, \quad t \in (0, T].
\]  \(19\)

For strategies \(\mathcal{U}^{(N), y_u}[0, T]\), since the state of the augmented system is \(X = (x, y)\), when considering variations of the state trajectory \(X\), due to variation of \(u\), there will be derivatives of \(u(\cdot)\) with respect to \(y\). To avoid this technicality we introduce the following assumptions.

**Assumptions 2.** The diffusion coefficients \(\sigma\) is restricted to the Borel measurable map \(\sigma : [0, T] \times \mathbb{R}^n \times A^{(N)} \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\) (e.g., it is independent of \(u\)) and

(A11) \(\sigma(\cdot, \cdot)\) and \(\sigma^{-1}(\cdot, \cdot)\) are bounded.

Under the additional Assumptions 2 we can show the following Lemma.

**Lemma 2.** Consider Problem 1 under Assumptions 1, 2 hold. Define the \(\sigma\)–algebras

\[
\mathcal{F}^{x(0), W, B}_{0,t} \triangleq \sigma\{ (x(0), W(s), B(s)) : 0 \leq s \leq t \}, \quad \mathcal{F}^{x, y_u}_{0,t} \triangleq \sigma\{ (x(s), y_u(s)) : 0 \leq s \leq t \}, \quad \forall t \in [0, T].
\]

If \(u \in \mathbb{U}^{(N), y_u}[0, T]\) then \(\mathcal{F}^{x(0), W, B}_{0,t} = \mathcal{F}^{x, y_u}_{0,t}, \forall t \in [0, T].\)

**Proof:** This follows directly from Assumptions 2 and the invertibility of \(D^i(\cdot), \forall i \in \mathbb{Z}_N\).
Recall that \( \{ x(t), y(t) : t \in [0, T] \} \) are the strong \( \mathbb{F}_T \)-adapted solutions of the state and observation equations. Under the conditions of Lemma 2 for any \( u^i \in \mathbb{V}^{i,u} [0, T] \) which is \( \mathcal{G}^{i,u}_T \)-adapted there exists a function \( \phi^i(\cdot) \) measurable with respect to a sub-\( \sigma \)-algebra of \( \mathcal{F}_{0,t} \subset \mathcal{F}_{0,T} \) such that \( u^i_t(\omega) = \phi^i(t, x(0), B(\cdot \wedge t, \omega), W(\cdot \wedge t, \omega)), \mathbb{P} - a.s. \omega \in \Omega, \forall t \in [0, T], i = 1, \ldots, N. \)

Define all such adapted nonanticipative functions by

\[
\mathbb{U}^i_{na}(0, T) = \left\{ u^i \in L^2_{\mathcal{F}_T}([0, T], \mathbb{R}^{d_i}) : \exists \; u^i \in \mathbb{A}^i \subset \mathbb{R}^{d_i}, \text{ a.e. } t \in [0, T], \mathbb{P} - a.s. \right\}, \quad \forall i \in \mathbb{Z}_N.
\]

Next, we introduce the following additional assumptions.

**Assumptions 3.** \( \mathbb{V}^{i,u} [0, T] \) is dense in \( \mathbb{U}^i_{na} [0, T], \forall i \in \mathbb{Z}_N. \)

Under Assumptions 3 we can show the following theorem.

**Theorem 1.** Consider Problem 1 under Assumptions 1, 3. Further, assume \( \ell \) is Borel measurable, continuously differentiable with respect to \((x, u)\), and \( \varphi \) is continuously differentiable with respect to \(x\), and there exist \( K_1, K_2 > 0 \) such that

\[
|\ell_x(t, x, u)|_{\mathbb{R}^n} + |\ell_u(t, x, u)|_{\mathbb{R}^d} \leq K_1 \left(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}\right), \quad |\varphi_x(T, x)|_{\mathbb{R}} \leq K_2 \left(1 + |x|_{\mathbb{R}^n}\right).
\]

Then

\[
\inf_{u^i \in \times_{i=1}^N \mathbb{U}^i_{na}[0, T]} J(u) = \inf_{u^i \in \times_{i=1}^N \mathbb{V}^{i,u}[0, T]} J(u).
\]

**Proof:** Since Assumptions 3 holds, it is sufficient to show that as \( u^{i,\alpha} \longrightarrow u^i \) in \( \mathbb{U}^i_{na}[0, T], \forall i \in \mathbb{Z}_N \), then \( J(u^{\alpha}) \longrightarrow J(u) \). From the derivation of Lemma 1 we can show that \( \mathbb{E} \sup_{s \in [0,t]} |x^{\alpha}(s) - x(s)|_{\mathbb{R}^n} \) converges to zero as \( \alpha \longrightarrow \infty \), hence it is sufficient to show that \( |J(u^{\alpha}) - J(u)| \) also converges to zero, as \( \alpha \longrightarrow \infty \). By the assumptions on \( \{\ell, \varphi\} \), and by the mean value theorem we have the following inequality.

\[
|J(u^{\alpha}) - J(u)| \leq K_1 \mathbb{E} \left\{ \int_{[0,t]} \left( |x^{\alpha}(t)|_{\mathbb{R}^n} + |u^{\alpha}_t|_{\mathbb{R}^d} + |x(t)|_{\mathbb{R}^n} + |u_t|_{\mathbb{R}^d} + 1 \right) \right. \\
\left. \left( |x^{\alpha}(t) - x(t)|_{\mathbb{R}^n} + |u^{\alpha}_t - u_t|_{\mathbb{R}^d} \right) dt \right\} \\
+ K_2 \mathbb{E} \left\{ \left( |x^{\alpha}(T)|_{\mathbb{R}^n} + |x(T)|_{\mathbb{R}^n} + 1 \right) |x^{\alpha}(T) - x(T)|_{\mathbb{R}^n} \right\}.
\]

Since \( \mathbb{E} \sup_{s \in [0,t]} |x^{\alpha}(s) - x(s)|_{\mathbb{R}^n} \longrightarrow 0 \) as \( \alpha \longrightarrow \infty \), then \( |J(u^{\alpha}) - J(u)| \) also converges to zero, as \( \alpha \longrightarrow \infty \).
The point to be made regarding Theorem 1 is that if \( u \in \mathcal{U}^{(N),y^n}[0,T] \) achieves the infimum of \( J(u) \) then it is also optimal with respect to some measurable functionals of subsets \( \{(x(0), (W(s), B(s)) : 0 \leq s \leq T\} \). Consequently, the necessary conditions for \( u \in \mathcal{U}^{(N),y^n}[0,T] \) to be optimal are those for which \( u \in \times_{i=1}^{N} \mathcal{U}^{i}[0,T] \) is optimal.

**Remark 2.** Strategies adapted to the innovations process \( u^i \in \mathcal{G}^{i,u}_T, \forall i \in \mathbb{Z}_N \) are often utilized to derive the separation theorem of partially observed stochastic control problems. It is well known that \( \{I^{i,u}(t) : t \in [0, T]\} \) is an \( \mathcal{G}^{i,u}_T \)-adapted Wiener process. Moreover, if the innovations process and observation process generate the same \( \sigma \)-algebra, \( \mathcal{G}^{i,u}_{0,s} = \mathcal{G}^{i,u}_{0,t} \), and the innovations process is independent of \( u \), then \( I^{i,u}(t) = I^{i,0}(t), \forall t \in [0, T] \). Define the \( \sigma \)-algebra \( \mathcal{G}^{i,0}_{0,t} \triangleq \sigma\{I^{i,0}(s) : 0 \leq s \leq t\}, t \in [0, T] \). Under these conditions the necessary conditions for \( u \in \mathcal{U}^{(N),y^n}[0,T] \) to be optimal are those for which \( u \in \mathcal{U}^{(N),y^n}[0,T] \), defined by

\[
\mathcal{U}^{(N),y^n}[0,T] \triangleq \left\{ \mathcal{U}^{(N),y^n} : u^i_t \text{ is } \mathcal{G}^{i,0}_{0,t} - \text{ adapted}, \forall i \in \mathbb{Z}_N \right\}. \tag{23}
\]

Note that Assumptions 3 are not required for distributed filtering applications because the decentralized information structures \( \mathcal{G}^{i}_T \) are independent of \( u \), and hence it is not very difficult to show \( \mathcal{G}^{i}_T = \mathcal{G}^{i,0}_T, \forall i \in \mathbb{Z}_N \).

After deriving the necessary conditions we also show that under certain convexity conditions that these are also sufficient. Consequently, for the sufficient part we do not require Assumptions 3.

For the derivation of stochastic minimum principle of optimality we shall require stronger regularity conditions on the maps \( \{f, \sigma, h\} \), as well as, for the running and terminal pay-offs functions \( \{\ell, \varphi\} \). These are given below.

**Assumptions 4.** \( \mathbb{E}|x(0)|_{\mathbb{R}^n} < \infty \) and the maps of \( \{f, \sigma, \ell, \varphi\} \) satisfy the following conditions.

- **(B1)** The map \( f : [0, T] \times \mathbb{R}^n \times \mathcal{A}^{(N)} \rightarrow \mathbb{R}^n \) is continuous in \( (t, x, u) \) and continuously differentiable with respect to \( (x, u) \);

- **(B2)** The map \( \sigma : [0, T] \times \mathbb{R}^n \times \mathcal{A}^{(N)} \rightarrow \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \) is continuous in \( (t, x, u) \) and continuously differentiable with respect to \( (x, u) \);

- **(B3)** The map \( h : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^k \) is continuous in \( (t, x, y) \) and continuously differentiable with respect to \( (x, y) \);

- **(B4)** The first derivatives \( \{f_x, f_u, \sigma_x, \sigma_u\} \) are bounded uniformly on \( [0, T] \times \mathbb{R}^n \times \mathcal{A}^{(N)} \).
(B5) The first derivative \( \{h_x, h_y\} \) are bounded uniformly on \([0, T] \times \mathbb{R}^n \times \mathbb{R}^k\);

(B6) The maps \( \ell : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow (-\infty, \infty) \) is Borel measurable, continuously differentiable with respect to \((x, u)\), the map \( \varphi : [0, T] \times \mathbb{R}^n \rightarrow (-\infty, \infty) \) is continuously differentiable with respect to \(x\), and there exist \(K_1, K_2 > 0\) such that

\[
|\ell_x(t, x, u)|_{\mathbb{R}^n} + |\ell_u(t, x, u)|_{\mathbb{R}^d} \leq K_1(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}), \quad |\varphi_x(t, x)|_{\mathbb{R}^n} \leq K_2(1 + |x|_{\mathbb{R}^n})
\]

(B7) Conditions \((A9), (A10)\) of Assumptions \([7]\) hold.

Consider the Gateaux derivative of \( G \) with respect to the variable at the point \((t, z, v) \in [0, T] \times \mathbb{R}^{n+k} \times A^{(N)}\) in the direction \(\eta \in \mathbb{R}^{n+k}\) defined by

\[
G_X(t, z, v; \eta) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ G(t, z + \varepsilon \eta, v) - G(t, z, v) \right\}, \quad t \in [0, T].
\]

Note that the map \(\eta \rightarrow G_X(t, z, v; \eta)\) is linear, and it follows from Assumptions \([4]\) \((B3), (B5)\) that there exists a finite positive number \(\beta > 0\) such that

\[
|G_X(t, z, v; \eta)|_{\mathcal{L}(\mathbb{R}^{n+k}, \mathbb{R}^{n+k})} \leq \beta |\eta|_{\mathbb{R}^{n+k}}, \quad t \in [0, T].
\]

In order to present the necessary conditions of optimality we need the so called variational equation. Suppose \(u^o \triangleq (u^{1,o}, u^{2,o}, \ldots, u^{N,o}) \in \mathcal{U}^{(N), I^u}[0, T]\) denotes the optimal decision and \(u \triangleq (u^1, u^2, \ldots, u^n) \in \mathcal{U}^{(N), I^u}[0, T]\) any other decision. Since \(\mathcal{U}^{I^u}[0, T]\) is convex \(\forall i \in \mathbb{Z}_N\), it is clear that for any \(\varepsilon \in [0, 1]\),

\[
u_t^{i, \varepsilon} \triangleq u_t^{i,o} + \varepsilon (u_t^i - u_t^{i,o}) \in \mathcal{U}^{I^u}[0, T], \quad \forall i \in \mathbb{Z}_N.
\]

Let \(X^\varepsilon(\cdot) \equiv X^\varepsilon(\cdot; u^\varepsilon(\cdot))\) and \(X^o(\cdot) \equiv X^o(\cdot; u^o(\cdot)) \in B^\infty_{F_T}([0, T], L^2(\Omega, \mathbb{R}^{n+k}))\) denote the solutions of the system equation \((19)\) corresponding to \(u^\varepsilon(\cdot)\) and \(u^o(\cdot)\), respectively. Consider the limit

\[
Z(t) \triangleq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ X^\varepsilon(t) - X^o(t) \right\}, \quad t \in [0, T].
\]

We have the following result characterizing the the variational process \(\{Z(t) : t \in [0, T]\}\).

**Lemma 3.** Suppose Assumptions \([4]\) hold. For strategies \(\mathcal{U}^{(N), I^u}[0, T]\) the process \(\{Z(t) : t \in [0, T]\}\) is an element of the Banach space \(B^\infty_{F_T}([0, T], L^2(\Omega, \mathbb{R}^{n+k}))\) and it is the unique solution
of the variational stochastic differential equation
\[
dZ(t) = F_X(t, X^o(t), u_t^o)Z(t)dt + G_X(t, X^o(t), u_t^o; Z(t)) \begin{bmatrix} dW(t) \\ dB(t) \end{bmatrix} \\
+ \sum_{i=1}^{N} F_{u^i}(t, X^o(t), u_t^{i^o})(u_t^i - u_t^{i^o})dt + \sum_{i=1}^{N} G_{u^i}(t, X^o(t), u_t^{i^o}; u_t^i - u_t^{i^o}) \begin{bmatrix} dW(t) \\ dB(t) \end{bmatrix}, \quad Z(0) = 0.
\]

having a continuous modification.

Under the addition Assumptions 3 the above statements hold for strategies \( \mathcal{U}(N).y^u[0, T] \).

Moreover, (24) is the variational equation for distributed filtering applications (without imposing Assumptions 3).

Proof: This follows from [1] by considering the augmented system.

Using the variation equation of Lemma 3 we note that the results given in [1] for nonrandomized strategies are directly applicable to the augmented system (19). In fact one can also consider randomized strategies as in [1].

IV. OPTIMALITY CONDITIONS FOR NOISY INFORMATION STRUCTURES

In this section we derive necessary and sufficient optimality conditions for the team game of Problem 1. In view of the results obtained in the previous section, specifically, Lemma 3, the stochastic minimum principle of optimality for Problem 1 described in terms of the augmented system (19), follows directly from the results in [1].

Before we introduce the optimality conditions we define the Hamiltonian system of equations. To this end, define the Hamiltonian

\[
\mathcal{H} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n} \times \mathcal{L}(\mathbb{R}^{m}, \mathbb{R}^{n}) \times \mathbb{A}^{(N)} \rightarrow \mathbb{R}
\]

by

\[
\mathcal{H}(t, x, \psi, q_{11}, u) \triangleq \langle f(t, x, u), \psi \rangle + tr(q_{11}^{*} \sigma(t, x)) + \ell(t, x, u), \quad t \in [0, T].
\]

For any \( u \in \mathcal{U}(N).y^u[0, T], \mathcal{U}(N).I^u[0, T] \), the adjoint process is

\[
(\psi, q_{11}, q_{12}) \in L^2_{\mathbb{F}^T}([0, T], \mathbb{R}^{n}) \times L^2_{\mathbb{F}^T}([0, T], \mathcal{L}(\mathbb{R}^{m}, \mathbb{R}^{n})) \times L^2_{\mathbb{F}^T}([0, T], \mathcal{L}(\mathbb{R}^{k}, \mathbb{R}^{n}))
\]
and satisfies the following backward stochastic differential equation

\[ d\psi(t) = - f_x^t(t, x(t), u_t)\psi(t)dt - V_{q_{11}}(t)dt - \ell_x(t, x(t), u_t)dt \]

\[ + q_{11}(t)dW(t) + q_{12}(t)dB(t), \quad t \in [0, T), \]

\[ = - \mathbb{H}_x(t, x(t), \psi(t), q_{11}(t), u_t)dt + q_{11}(t)dW(t) + q_{12}(t)dB(t), \quad t \in [0, T), \] \hspace{1cm} (26)

\[ \psi(T) = \varphi_x(x(T)), \] \hspace{1cm} (27)

where \( V_{q_{11}} \in L^2_{F_T}([0, T], \mathbb{R}^n) \) is given by \( \langle V_{q_{11}}, \zeta \rangle = tr(q_{11}^*(t)\sigma_x(t, x(t); \zeta)), t \in [0, T] \) (e.g., \( V_{q_{11}}(t) = \sum_{k=1}^n \left( \sigma_x^{(k)}(t, x(t)) \right)^t q_{11}^{(k)}(t), \quad t \in [0, T], \sigma^{(k)} \) is the \( k \)th column of \( \sigma, \sigma_x^{(k)} \) is the derivative of \( \sigma^{(k)} \) with respect to the state, \( q_{11}^{(k)} \) is the \( k \)th column of \( q_{11}, \) for \( k = 1, 2, \ldots, m). \)

The state process satisfies the stochastic differential equation

\[ dx(t) = f(t, x(t), u_t)dt + \sigma(t, x(t))dW(t), \quad t \in (0, T], \]

\[ = \mathbb{H}\psi(t, x(t), \psi(t), q_{11}(t), u_t)dt + \sigma(t, x(t))dW(t), \quad t \in (0, T], \] \hspace{1cm} (28)

\[ x(0) = x_0 \] \hspace{1cm} (29)

The above Hamiltonian system of equations is expressed in terms of the original distributed system of equations (6, 7), and it is obtained by first deriving the Hamiltonian system of equations for the augmented system (19) (we shall clarify this step in the next section).

A. Necessary Conditions of Optimality

We now prepare to derive the necessary conditions for team optimality. Specifically, given that \( u^o \in \mathbb{U}^{(N)}y^a[0, T] \) or \( u^o \in \mathbb{U}^{(N)}t^a[0, T] \) is team optimal the question we address is whether it satisfies the Hamiltonian system of equations (25)-(29).

By utilizing [1] we have following necessary conditions.

\textbf{Theorem 2.} (Necessary conditions for team optimality) Consider Problem [7] under Assumptions 4 and \( \mathbb{H}^s \) a closed, bounded and convex subset of \( \mathbb{R}^k, i = 1, \ldots N. \)

For an element \( u^o \in \mathbb{U}^{(N)}t^a[0, T] \) with the corresponding solution \( x^o \in B^\infty_{F_T}([0, T], L^2(\Omega, \mathbb{R}^n)) \) to be team optimal, it is necessary that the following hold.

\begin{enumerate}
  \item There exists a square integrable semi martingale \( m^o \) with the intensity process \( (\psi^o, q_{11}^o, q_{12}) \in L^2_{F_T}([0, T], \mathbb{R}^n) \times L^2_{F_T}([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \times L^2_{F_T}([0, T], \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)). \)
\end{enumerate}
(2) The variational inequality is satisfied:

\[ \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \langle \mathbb{H}_{ui}(t, x^o(t), \psi^o(t), q_{i1}(t), u_i^o), u_i - u_i^o \rangle \, dt \geq 0, \quad \forall u \in \mathbb{U}^{(N)} u [0, T]. \quad (30) \]

(3) The process \((\psi^o, q_{i1}^o, q_{i2}^o) \in L^2_{F_T}([0, T], \mathbb{R}^n) \times L^2_{F_T}([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \times L^2_{F_T}([0, T], \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n))\) is a unique solution of the backward stochastic differential equation (26), (27) such that \(u^o \in \mathbb{U}^{(N), I^u} [0, T]\) satisfies the point wise almost sure inequalities with respect to the \(\sigma\)-algebras \(\mathcal{G}_{0,t}^{I^u} \subset \mathbb{F}_{0,t}, \ t \in [0, T], i = 1, 2, \ldots, N:\)

\[ \mathbb{E}\left\{ \mathbb{H}_{ui}(t, x^o(t), \psi^o(t), q_{i1}(t), u_t^o)|\mathcal{G}_{0,t}^{I^u}\right\}, u_i - u_i^o \geq 0, \quad \forall u_i \in \mathbb{A}_i, \ a.e. t \in [0, T], \mathbb{P}|_{\mathcal{G}_{0,t}^{I^u}} - a.s., i = 1, 2, \ldots, N. \quad (31) \]

Under the additional Assumptions 3 the results also hold for strategies \(\mathbb{U}^{(N), y^u} [0, T]\) with conditional expectation taken with respect to \(\mathcal{G}_{0,t}^{y^u}\).

For distributed filtering strategies are \(u \in \mathbb{U}^{(N), y} [0, T]\) with conditional expectation taken with respect to \(\mathcal{G}_{0,t}^{y^u}\).

Proof: The derivation consists of two steps. The first step utilizes [1] to derive the optimality conditions for the augmented system (19). Hence, by direct application of [1] we have the following.

Define the Hamiltonian of the augmented system (19)

\[ \mathcal{H} : [0, T] \times \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \times \mathcal{L}(\mathbb{R}^{n+k}, \mathbb{R}^{n+k}) \times \mathbb{A}^{(N)} \rightarrow \mathbb{R} \]

by

\[ \mathcal{H}(t, X, \Psi, M, u) \triangleq \langle F(t, X, u), \zeta \rangle + tr(M^*G(t, X)) + L(t, X, u), \quad t \in [0, T]. \quad (32) \]

Then the result of [1] for nonrandomized strategies apply to the system (19), hence for any \(u \in \times_{i=1}^{N} \mathbb{U}^{y^i,u} [0, T]\) or \(u \in \times_{i=1}^{N} \mathbb{U}^{I^i,u} [0, T]\) the adjoint process of the augmented system exists
and satisfies the following backward stochastic differential equation.

\[ d\Psi(t) = -F^*(t, X(t), u_t)\Psi(t)dt - V_Q(t)dt - L_X(t, X(t), u_t)dt + Q(t) \begin{bmatrix} dW(t) \\ dB(t) \end{bmatrix}, \quad t \in [0, T), \]

\[ = -\mathcal{H}_X(t, X(t), \Psi(t), Q(t), u_t)dt + Q(t) \begin{bmatrix} dW(t) \\ dB(t) \end{bmatrix}, \quad \Psi(T) = \Phi_X(X(T)), t \in [0, T), \]

where \( V_Q \) is given by \( \langle V_Q(t), \zeta \rangle = tr(Q^*(t)G_X(t, X(t); \zeta)), t \in [0, T]. \)

The second step translates the necessary conditions of the augmented system to the original system (6), (7). To this end, we introduce the following decompositions which will lead to a simplified Hamiltonian system of equations.

\[
\Psi \triangleq \begin{bmatrix} \psi \\ \zeta \end{bmatrix}, \quad Q \triangleq \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}. \tag{34}
\]

By utilizing this decomposition it can be shown that \( \psi \) satisfies (26), (27). The second component of \( \Psi \) in (34) satisfies the following equation

\[ d\zeta(t) = q_{21}(t)dW(t) + q_{22}(t)dB(t), \quad \zeta(T) = 0, \quad t \in [0, T). \tag{35} \]

Since this equation has terminal condition \( \zeta(T) = 0 \), and its right hand side martingale terms are orthogonal, then necessarily, \( q_{21}(t) = 0, q_{22}(t) = 0, \forall t \in [0, T], \text{a.s.} \), which imply \( \zeta(t) = 0, \forall t \in [0, T], \text{a.s.} \). Finally, statements (1)-(3) are obtained from equivalent statements of the augmented system (1).

It is interesting to note that the necessary conditions for a \( u^o \in \mathbb{U}^{(N)}y^u[0, T] \) or \( u^o \in \mathbb{U}^{(N)}y^u[0, T] \) to be a person-by-person optimal can be derived following the methodology of Theorem 2 and that these necessary conditions are the same as the necessary conditions for the team optimal strategy. These results are stated as a Corollary.

**Corollary 1.** (*Necessary conditions for person-by-person optimality*) Consider Problem 2 under the assumptions of Theorem 2. For an element \( u^o \in \mathbb{U}^{(N)}y^u[0, T] \) with the corresponding solution \( x^o \in B^\infty_T([0, T], L^2(\Omega, \mathbb{R}^n)) \) to be a person-by-person optimal strategy, it is necessary that the statements of Theorem 2 (1), (3) hold and statement (2) is replaced by

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The variational inequalities are satisfied:
\[
\mathbb{E} \int_0^T \langle H_i(t, x^o(t), \psi^o(t), q_{11}(t), u_i^o), u_i - u_i^o \rangle dt \geq 0, \quad \forall u_i \in \mathbb{U}_i^{I_i, u_0}, \forall i \in \mathbb{Z}_N.
\] (36)

Under the additional Assumptions 3, the results also hold for strategies \( U^{(N), y^u}[0, T] \) with conditional expectation taken with respect to \( G_{0,t}^{y^u} \).

For distributed filtering strategies are \( u \in U^{(N), y}[0, T] \) with conditional expectation taken with respect to \( G_{0,t}^{y^u} \).

**Proof:** The derivation is based on the procedure of Theorem 2, which is completely described in [1].

The following remark helps identifying the martingale term in the adjoint process.

**Remark 3.** According to [1], the Riesz representation theorem for Hilbert space martingales, determine the martingale term of the adjoint process \( M_t = \int_0^t \Psi_X(s)G(s, X^o(s)) \begin{bmatrix} dW(s) \\ dB(s) \end{bmatrix} \), dual to the first martingale term in the variational equation (27), hence \( Q \) in the adjoint equation, is identified as \( Q(t) = \Psi_X(t)G(t, X(t)) \). By translating this to the original system then \( q_{11} = \psi_x \sigma, q_{12} = \psi_y D^{1/2} \), provided the derivatives \( \psi_x, \psi_y \) exist.

**B. Sufficient Conditions of Optimality**

In this section, we show that the necessary condition of optimality (31) is also a sufficient condition for optimality, under a convexity conditions on the Hamiltonians and the terminal condition.

**Theorem 3.** (Sufficient conditions for team optimality) Consider Problem 1 with strategies from \( U^{(N), y^u}[0, T] \) (respectively \( U^{(N), y}[0, T] \)), under Assumptions 4 and \( A_i \) a closed, bounded and convex subset of \( \mathbb{R}^{k_i}, i = 1, \ldots, N \). Let \( (x^o(\cdot), u^o(\cdot)) \) denote an admissible state and decision pair and let \( \psi^o(\cdot) \) the corresponding adjoint processes.

Suppose the following conditions hold.

(C4) \( H(t, \cdot, \psi, q_{11}, \cdot), t \in [0, T] \) is convex in \( (x, u) \in \mathbb{R}^n \times A^{(N)} \);

(C5) \( \varphi(\cdot) \) is convex in \( x \in \mathbb{R}^n \).
Then \((x^o(\cdot), u^o(\cdot))\) is a team optimal pair if it satisfies (31) (respectively (37) with conditional expectation taken in terms of \(G_{0,t}^{u_i^o,x^o_i}, i = 1, \ldots, N\)).

**Proof:** Let \(u^o \in U^{(N)}y^* [0, T]\) denote a candidate for the optimal team decision and \(u \in U^{(N)}y^* [0, T]\) any other decision. Then

\[
J(u^o) - J(u) = \mathbb{E} \left\{ \int_0^T \left( \ell(t, x^o(t), u^o_t) - \ell(t, x(t), u_t) \right) dt + \left( \varphi(x^o(T)) - \varphi(x(T)) \right) \right\}.
\]

(37)

By the convexity of \(\varphi(\cdot)\) then

\[
\varphi(x(T)) - \varphi(x^o(T)) \geq \langle \varphi_x(x^o(T)), x(T) - x^o(T) \rangle.
\]

(38)

Substituting (38) into (37) yields

\[
J(u^o) - J(u) \leq \mathbb{E} \left\{ \langle \varphi_x(x^o(T)), x^o(T) - x(T) \rangle \right\}
+ \mathbb{E} \left\{ \int_0^T \left( \ell(t, x^o(t), u^o_t) - \ell(t, x(t), u_t) \right) dt \right\}.
\]

(39)

Applying the Ito differential rule to \(\langle \psi^o, x - x^o \rangle\) on the interval \([0, T]\) and then taking expecation we obtain the following equation.

\[
\mathbb{E} \left\{ \langle \psi^o(T), x(T) - x^o(T) \rangle \right\} = \mathbb{E} \left\{ \langle \psi^o(0), x(0) - x^o(0) \rangle \right\}
+ \mathbb{E} \left\{ \int_0^T \langle -f^*_x(t, x^o(t), u^o_t)\psi^o(t) dt - Vq^o_{11}(t) - \ell_x(t, x^o(t), u^o_t), x(t) - x^o(t) \rangle dt \right\}
+ \mathbb{E} \left\{ \int_0^T \langle \psi^o(t), f(t, x(t), u_t) - f(t, x^o(t), u^o_t) \rangle dt \right\}
+ \mathbb{E} \left\{ \int_0^T tr(q^o_{11}(t)\sigma(t, x(t)) - q^o_{11}(t)\sigma(t, x^o(t))) dt \right\}
= -\mathbb{E} \left\{ \int_0^T \langle \mathbb{H}_x(t, x^o(t), \psi^o(t), q^o_{11}(t), u^o_t), x(t) - x^o(t) \rangle dt \right\}
+ \mathbb{E} \left\{ \int_0^T \langle \psi^o(t), f(t, x(t), u_t) - f(t, x^o(t), u^o_t) \rangle dt \right\}
+ \mathbb{E} \left\{ \int_0^T tr(q^o_{11}(t)\sigma(t, x(t)) - q^o_{11}(t)\sigma(t, x^o(t))) dt \right\}
\]

(40)

Note that \(\psi^o(T) = \varphi_x(x^o(T))\). Substituting (40) into (39) we obtain

\[
J(u^o) - J(u) \leq \mathbb{E} \left\{ \int_0^T \left[ \mathbb{H}(t, x^o(t), \psi^o(t), q^o_{11}(t), u^o_t) - \mathbb{H}(t, x(t), \psi^o(t), q^o_{11}(t), u_t) \right] dt \right\}
- \mathbb{E} \left\{ \int_0^T \langle \mathbb{H}_x(t, x^o(t), \psi^o(t), q^o_{11}(t), u^o_t), x^o(t) - x(t) \rangle dt \right\}.
\]

(41)
Since by hypothesis $H$ is convex in $(x,u) \in \mathbb{R}^n \times A^{(N)}$, then
\[
H(t,x(t),\psi^o(t),q^o_{11}(t),u_t) - H(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^o)
\geq \sum_{i=1}^{N} \langle H_{u_i}(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^i), u_t^i - u_t^{i,o} \rangle
+ \langle H_{u}(t,x^o(t),\psi^o(t),Q^o(t),u_t^o), x(t) - x^o(t) \rangle, \quad t \in [0,T]
\]  
(42)

Substituting (42) into (41) yields
\[
J(u^o) - J(u) \leq -\mathbb{E}\left\{ \sum_{i=1}^{N} \int_{0}^{T} \langle H_{u_i}(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^i), u_t^i - u_t^{i,o} \rangle dt \right\}.
\]  
(43)

By (31) and by definition of conditional expectation we have
\[
\mathbb{E}\left\{ I_{A^i}(\omega) \langle H_{u_i}(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^i), u_t^i - u_t^{i,o} \rangle \right\} = \mathbb{E}\left\{ I_{A^i}(\omega) \mathbb{E}\left\{ \langle H_{u_i}(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^i), u_t^i - u_t^{i,o} \rangle | G_{0,t}^{y^i,u} \right\} \right\} \geq 0, \quad \forall A^i_t \in G_{0,t}^{y^i,u}, \quad \forall i \in \mathbb{Z}_N.
\]  
(44)

Hence, \( \langle H_{u_i}(t,x^o(t),\psi^o(t),q^o_{11}(t),u_t^i), u_t^i - u_t^{i,o} \rangle \geq 0, \forall u_t^i \in A^i, a.e.t \in [0,T], \mathbb{P} - a.s., \) \( i = 1,2,\ldots,N \). Substituting the this inequality into (43) gives
\[
J(u^o) \leq J(u), \quad \forall u \in \mathbb{U}^{(N)\cdot y^o}[0,T].
\]

Hence, sufficiency of (31) with conditional expectation taken in terms of \( G_{0,t}^{y^i,u}, i = 1,\ldots,N \) is shown. For \( \mathbb{U}^{(N)\cdot y^o}[0,T] \) the derivation is identical.

Since the necessary conditions for team optimal and person-by-person optimal are equivalent (this follows from Theorem [2] Corollary [1]), then one can go one step further to show that under the conditions of Theorem [3] that any person-by-person optimal strategy is also a team optimal strategy.

We conclude our discussion on team and person-by-person game optimality conditions for distributed stochastic differential systems with decentralized noisy information structures, by stating once again that the results derived are also applicable to distributed estimation problems (see Remark [1] with strategies taken from \( \mathbb{U}^{(N)\cdot y^o}[0,T] \).
V. APPLICATIONS IN COMMUNICATION, FILTERING AND CONTROL

In this section we investigate various applications of the optimality conditions to communication, filtering and control applications. For most applications we give explicit optimal team strategies, when the dynamics and the reward have the structures defined below. Throughout, we assume validity of the convexity conditions of Theorem 3 (C4), (C5), why imply sufficiency of (31) with conditional expectation taken in terms of $G^{y_i,u_i}_0,t$, $i = 1, \ldots, N$, and strategies taken from $U^{(N),y}[0,T]$.

Definition 1. (Team games with special structures) We define the following classes of team games.

(NF): Nonlinear Form. The team game is said to have "nonlinear form" if

$$f(t,x,u) \triangleq b(t,x) + g(t,x)u,$$

$$g(t,x)u \triangleq \sum_{j=1}^{N} g^{(j)}(t,x)u^j,$$  \hspace{1cm} (45)

$$\sigma(t,x) \triangleq \left[ \begin{array}{cccc} \sigma^{(1)}(t,x) & \sigma^{(2)}(t,x) & \ldots & \sigma^{(N)}(t,x) \end{array} \right],$$  \hspace{1cm} (46)

$$\ell(t,x,u) \triangleq \frac{1}{2} \langle u, R(t,x)u \rangle + \frac{1}{2} \lambda(t,x) + \langle u, \eta(t,x) \rangle,$$  \hspace{1cm} (47)

where

$$\langle u, R(t,x)u \rangle \triangleq \sum_{i=1}^{N} \sum_{j=1}^{N} u^{i,*}R_{ij}(t,x)u^j, \quad \langle u, \eta(t,x) \rangle \triangleq \sum_{i=1}^{N} u^{i,*}\eta^i(t,x),$$

and $\sigma^{(i)}(\cdot, \cdot)$ is the $i$th column of an $n \times m$ matrix $\sigma(\cdot, \cdot)$, for $i = 1, 2, \ldots, m$, $R(\cdot, \cdot)$ is symmetric uniformly positive definite, and $\lambda(\cdot, \cdot)$ is uniformly positive semidefinite.

(LQF): Linear-Quadratic Form. A team game is said to have "linear-quadratic form" if

$$f(t,x,u) = A(t)x + B(t)u,$$

$$\sigma(t,x,u) = G(t),$$  \hspace{1cm} (48)

$$\ell(t,x) = \frac{1}{2} \langle u, R(t)u \rangle + \frac{1}{2} \langle x, H(t)x \rangle + \langle x, F(t) \rangle + \langle u, E(t)x \rangle + \langle u, m(t) \rangle,$$  \hspace{1cm} (49)

and $R(\cdot)$ is symmetric uniformly positive definite and $H(\cdot)$ is symmetric uniformly positive semidefinite.

Below we compute the optimal strategies for the two cases of Definition I. First, we introduce the following definitions.

$$\bar{u}^{i,j,o}(t) \triangleq \mathbb{E} \left( u^{i,o}_t | \mathcal{G}^{y_i,u_i}_0,t \right), \quad \bar{u}^{i,o}(t) \triangleq \text{Vector} \{ \bar{u}^{1,i,o}(t), \ldots, \bar{u}^{N,i,o}(t) \}, i, j = 1, \ldots, N, \ldots,N,$$
For a team game is of normal form then from the previous optimal strategies one obtains coefficients. We discuss this below.

The last equation can be written in the form of a fixed point matrix equation with random

Utilizing the definition of Hamiltonian of Theorem 2, its derivative is given by

\[
H_u(t,x,\psi,q_{11},u) = g^*(t,x)\psi + R(t,x)u + \eta(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n.
\] (50)

The explicit expression for \( u_t^{i,o} \) is given by

\[
u_t^{i,o} = - \left\{ \mathbb{E} \left( R_{ii}(t,x^o(t))|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) \right\}^{-1} \left\{ \mathbb{E} \left( \eta^i(t,x^o(t))|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) + \sum_{j=1,j \neq i}^N \mathbb{E} \left( R_{ij}(t,x^o(t))u_t^{j,o}|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) - \mathbb{E} \left( g^{(i)*}(t,x^o(t))\psi^o(t)|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) \right\}, \quad \mathbb{P}|_{\mathcal{G}_{0,t}^{y^{i,u^o}}} - a.s., \quad i = 1,2,\ldots,N.
\] (51)

**Special Case.** Suppose \( R(t,x) = \overline{R}(t) \), e.g., independent of \( x \). Since both sides of (51) are \( \mathcal{G}_{0,t}^{y^{i,u^o}} \)–measurable taking conditional expectations of both side with respect to \( \mathcal{G}_{0,t}^{y^{i,u^o}} \) gives the expression

\[
\tilde{u}^{i,o}(t) = - \left\{ \Pi_{ii}(t) \right\}^{-1} \left\{ \mathbb{E} \left( \eta^i(t,x^o(t))|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) + \sum_{j=1,j \neq i}^N \Pi_{ij}(t)\tilde{u}^{j,o}(t) - \mathbb{E} \left( g^{(i)*}(t,x^o(t))\psi^o(t)|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) \right\}, \quad \mathbb{P}|_{\mathcal{G}_{0,t}^{y^{i,u^o}}} - a.s., \quad i = 1,2,\ldots,N.
\] (52)

The last equation can be written in the form of a fixed point matrix equation with random

**Case LQF.**

For a team game is of normal form then from the previous optimal strategies one obtains

\[
u_t^{i,o} = - R_{ii}^{-1}(t) \left\{ m^i(t) + \sum_{j=1}^N E_{ij}(t)\mathbb{E} \left( x^{j,o}(t)|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) + \sum_{j=1,j \neq i}^N R_{ij}(t)\mathbb{E} \left( u_t^{j,o}|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) - B^{(i)*}(t)\mathbb{E} \left( \psi^o(t)|\mathcal{G}_{0,t}^{y^{i,u^o}} \right) \right\}, \quad \mathbb{P}|_{\mathcal{G}_{0,t}^{y^{i,u^o}}} - a.s., \quad i = 1,2,\ldots,N.
\] (53)
Similarly as above, (53) can be put in the form of fixed point matrix equation as follows:

\[
\begin{align*}
\text{diag}\{R^{[1]}(t), \ldots, R^{[N]}(t)\}u^{\circ}(t) + \text{diag}\{E^{[1]}(t), \ldots, E^{[N]}(t)\}v^{\circ}(t) \\
+ \text{diag}\{B^{[1],*}(t), \ldots, B^{[N],*}(t)\}w^{\circ}(t) + m(t) &= 0.
\end{align*}
\]

(54)

Therefore, (54) can be solved via fixed point methods. One can proceed further to determine the adjoint processes and the explicit optimal team strategy. This is done in the next subsection.

\[\text{A. Communication Channels with Memory and Feedback}\]

In this section we discuss applications of team games to communication channels with feedback and memory. We consider applications in which the state process is a RV, and decentralized information structures with feedback and/or correlation among them. Consider a filtered probability space \((\Omega, \mathbb{F}, \mathbb{F}_T, \mathbb{P})\) on which the following are defined.

A Gaussian RV \(\theta \overset{\Delta}{=} \text{Vector}\{\theta^1, \ldots, \theta^N\} : \Omega \longrightarrow \mathbb{R}^n, \theta^i \in \mathbb{R}^{n_i}, (\mathbb{E}(\theta), \text{Cov}(\theta)) = (\bar{\theta}, P_0),\)

Mutual Independent Brownian motions \(B^i : [0, T] \times \Omega \rightarrow \mathbb{R}^{k_i}, i = 1, \ldots, N,\) independent of \(\theta.\)

The information structure of each DM \(u^i\) is \(\mathcal{G}^i_{0,t} = \sigma\{y^i(s) : 0 \leq s \leq t\}, t \in [0, T], i = 1, \ldots, N,\) which is defined by a communication channel with memory feedback via the stochastic differential equation

\[
y^i(t) = \int_0^t C^{i}(s,y^i(s))\theta ds + \int_0^t D^{i}_2(s)dB^i(s), \quad t \in [0, T], \quad i = 1, 2, \ldots, N, \tag{55}
\]

where \(C^{i} : [0, T] \times \mathbb{R}^{k_i} \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{k_i}).\) The communication channel (55) models a Gaussian Broadcast channel in which there is a single transmitter and multiple receivers, \(i = 1, \ldots, N.\) The transmitter wishes to send linear combinations of messages \(\{\theta^1, \ldots, \theta^N\}\) to receivers \(\{y^1, \ldots, y^N\}.\)

The objective is to reconstruct at each receiver \(y^i\) the intended linear combination of the messages denoted by \(L^i\theta,\) where \(L^i\) is an appropriately chosen matrix. A reasonable pay-off for reconstructing the intended linear combination \(L^i\theta\) at receiver \(i\) by \(u^i\) is the average weighted estimation error \(\mathbb{E} \int_{[0,T]} (u_t - \text{diag}\{L^1, \ldots, L^N\}\theta, R(t)(u_t - \text{diag}\{L^1, \ldots, L^N\}\theta))dt.\) A more general pay-off which also incorporates any power constraints at the transmitter is the
quadratic pay-off defined by

\[ J(u^1, \ldots, u^N) \triangleq \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \langle u_t, R(t)u \rangle + \langle \theta, H(t)\theta \rangle + \langle \theta, F(t) \rangle 
\right. 
\left. + \langle u_t, E(t)\theta \rangle + \langle u_t, m(t) \rangle \right) dt. \]  

(56)

Noticed that the information structures (55) are defined via channels with feedback since

\[ \mathbb{P} \left\{ y^i(t) \in A_i | \{ y^j(s) : 0 \leq s \leq t - \epsilon \}, \theta \right\} \neq \mathbb{P} \left\{ y^i(t) \in A_i | \theta \right\}, \ A_i \in \mathcal{B}(\mathbb{R}^k), \epsilon > 0, i \in \mathbb{Z}_N. \]  

(57)

The previous communication model can be easily generalized to other network communication channels.

Since minimizing (56) over feedback information structures subject to (55) is a team problem, then we will apply the optimality conditions of Theorem 2.

First, note that the stochastic differential equation (55) has a continuous strong solution which is unique. Since the state is a RV (static state), then \( \psi^o = 0 \), hence the optimal strategies are given component wise by

\[ u^{i,o}_t = -R_{ii}^{-1}(t) \left\{ \sum_{j=1}^N E_{ij}(t) \mathbb{E} \left( \theta^j | \mathcal{G}^y_{0,t} \right) + m^i(t) + \sum_{j=1, j \neq i}^N R_{ij}(t) \mathbb{E} \left( u^{j,o}_t | \mathcal{G}^y_{0,t} \right) \right\}, \quad \mathbb{P}_{| \mathcal{G}^y_{0,t}} \text{ a.s., } i \in \mathbb{Z}_N. \]  

(58)

Define the filter version of \( \theta \) by

\[ \hat{\theta}^i(t) \triangleq \mathbb{E} \left( \theta | \mathcal{G}^y_{0,t} \right), \quad t \in [0, T], i \in \mathbb{Z}_N. \]  

Then these bank of filters satisfy the following stochastic differential equations

\[ d\hat{\theta}^i(t) = P^i(t, y^i) C_{ii}^*(t, y^i(t)) D_{ii}^{-1}(t) \left( dy^i(t) - C_{ii}(t, y^i(t)) \hat{\theta}^i(t) dt \right), \quad \hat{\theta}^i(0) = \overline{\theta}, \quad t \in (0, T], \ i \in \mathbb{Z}_N \]  

(59)

\[ \dot{\bar{P}}^i(t, y^i) = -P^i(t, y^i) C_{ii}^*(t, y^i(t)) D_{ii}^{-1}(t) C_{ii}(t, y^i(t)) P^i(t, y^i), \quad \bar{P}(0) = \overline{P}_0, \quad t \in (0, T], \ i \in \mathbb{Z}_N. \]  

(60)

Define the innovations process and the \( \sigma \)-algebra generated by it as follows.

\[ I^i(t) \triangleq \int_0^t D_{ii}^{-1}(s) \left( y^i(s) - C_{ii}(s, y^i(s)) \hat{\theta}^i(s) ds \right), \quad t \in [0, T], \ i \in \mathbb{Z}_N, \]  

(61)

\[ \mathcal{G}^i_{0,t} \triangleq \sigma \left\{ I^i(s) : 0 \leq s \leq t \right\}, \quad t \in [0, T], \ i \in \mathbb{Z}_N. \]  

(62)

Then \( \{ I^i(t) : 0 \leq t \leq T \} \) is an \( \left( \mathcal{G}^y_T, \mathbb{P} \right) \)-adapted Brownian motion \( \forall i \in \mathbb{Z}_N \), and for \( i \neq j \), the innovations \( I^i(\cdot), I^j(\cdot) \) are independent (in view of independence of \( B^i(\cdot), B^j(\cdot) \) for \( i \neq j \)).
Moreover, the processes $\{\theta(t), P_i(t, y^i), y^i(t) : 0 \leq t \leq T\}$ are weak solutions \[38\] of the system
\[
d\hat{\theta}(t) = P_i(t, y^i)C_{ii}(t, y^i(t))N_{ii}^{-1}(t)dI(t), \quad \hat{\theta}(0) = \bar{\theta}, \quad t \in (0, T], \quad i \in \mathbb{Z}_N \tag{63}
\]
\[
dy^i(t) = C_{ii}(t, y^i(t))\hat{\theta}(t)dt + D_{ii}^{1/2}(t)d\Gamma(t), \quad t \in [0, T], \quad i \in \mathbb{Z}_N, \tag{64}
\]
\[
\dot{P}_i(t, y^i) = -P_i(t, y^i)C_{ii}(t, y^i(t))D_{ii}^{-1}(t)C_{ii}(t, y^i(t))P_i(t, y^i), \quad P(0) = \bar{P}_0, \quad t \in (0, T], \quad i \in \mathbb{Z}_N. \tag{65}
\]

Next, we establish existence of strong solutions to the system \[63\]-\[65\] which will imply that $\mathcal{G}^u_i$, $\forall i \in \mathbb{Z}_N$ generate the same information. Under assumption that $C_{ii}(t, y^i)$ satisfy the Lipschitz and linear growth conditions, uniformly in $t \in [0, T]$, the system \[63\], \[64\] has a unique $\mathcal{G}^i_T$-adapted continuous solution \[38\], hence $\tilde{\theta}(\cdot)$ is $\mathcal{G}^i_{0,t}$-measurable, $\forall t \in [0, T], \forall i \in \mathbb{Z}_N$. Thus, $\mathcal{G}^I_{0,t} \subseteq \mathcal{G}^i_{0,t}, \forall t \in [0, T], \forall i \in \mathbb{Z}_N$. The reverse $\mathcal{G}^I_{0,t} \subseteq \mathcal{G}^u_i, \forall t \in [0, T], \forall i \in \mathbb{Z}_N$ follows from the construction of innovations processes \[61\]. Hence, $\mathcal{G}^I_{i} = \mathcal{G}^u_i, \forall i \in \mathbb{Z}_N$. Since each DM $u^i$ is $\mathcal{G}^u_i = \mathcal{G}^I_i$-adapted $\forall i \in \mathbb{Z}_N$, and the innovations sigma algebras $\mathcal{G}^I_{i}$ are independent for $\forall i, j \in \mathbb{Z}_N, i \neq j$ then the optimal strategies \[58\] are given by
\[
u_t^{i,o} = -R_{ii}^{-1}(t)\left\{E[i](t)\tilde{\theta}(t) + m^i(t) + \sum_{j=1,j \neq i}^{N} R_{ij}(t)\mathbb{E}\left[u_t^j,o\right] \right\}, \quad \mathbb{P}|_{\mathcal{G}^i_{0,t}} \text{ a.s., } i \in \mathbb{Z}_N. \tag{66}
\]

Next, we determine the vector by $u^o \overset{\Delta}{=} \text{Vector}\{\mathbb{E}(u^{1,o}), \mathbb{E}(u^{2,o}), \ldots, \mathbb{E}(u^{N,o})\}$. Taking expectation of both sides of \[66\] gives the following linear system of equations.
\[
\bar{u}_t^{i,o}(t) = -R_{ii}^{-1}(t)\left\{E[i](t)\mathbb{E}\left[\theta\right] + m^i(t) + \sum_{j=1,j \neq i}^{N} R_{ij}(t)\bar{u}_t^{j,o}(t) \right\}, \quad i \in \mathbb{Z}_N. \tag{67}
\]

The last equation can be put into a fixed point form. Define
\[
M(t) \overset{\Delta}{=} \begin{bmatrix}
-R_{11}^{-1}(t)E[1](t)\mathbb{E}\left[\theta\right] \\
-R_{22}^{-1}(t)E[2](t)\mathbb{E}\left[\theta\right] \\
\vdots \\
-R_{NN}^{-1}(t)E[N](t)\mathbb{E}\left[\theta\right]
\end{bmatrix}, \quad K(t) \overset{\Delta}{=} \begin{bmatrix}
-R_{11}^{-1}(t)m^1(t) \\
-R_{22}^{-1}(t)m^2(t) \\
\vdots \\
-R_{NN}^{-1}(t)m^N(t)
\end{bmatrix}, \tag{68}
\]
From (67), we have
\[
\Lambda(t) \triangleq \begin{bmatrix}
I & R_{11}^{-1}(t)R_{12}(t) & R_{11}^{-1}(t)R_{13}(t) & \cdots & R_{11}^{-1}(t)R_{1N}(t) \\
R_{22}^{-1}(t)R_{21}(t) & I & R_{22}^{-1}(t)R_{23}(t) & \cdots & R_{22}^{-1}(t)R_{2N}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{NN}^{-1}(t)R_{N1}(t) & R_{NN}^{-1}(t)R_{N2}(t) & R_{NN}^{-1}(t)R_{N3}(t) & \cdots & I
\end{bmatrix}\]
(69)

Finally, the optimal strategies are given by (66) and (70).

The previous calculations can be generalized to other channel models. Moreover, \( \theta \) can be extended to a Random process described by Itô stochastic differential equations. For linear dynamics this generalization is a straightforward repetition of the previous calculations, hence it is omitted.

**B. Linear-Quadratic Form and Linear Stochastic Differential Dynamics**

In this section we invoke the minimum principle to compute the optimal strategies, with respect to a quadratic pay-off, for distributed stochastic dynamical decision systems consisting of an interconnection of two subsystems, each governed by a linear stochastic differential equation with coupling.
Subsystem Dynamics 1:

\[ dx^1(t) = A_{11}(t)x^1(t)dt + B_{11}(t)u^1_t dt + G_{11}(t)dW^1(t) \]
\[ + A_{12}(t)x^2(t)dt + B_{12}(t)u^2_t dt, \quad x^1(0) = x^1_0, \quad t \in (0, T], \] (71)

Subsystem Dynamics 2:

\[ dx^2(t) = A_{22}(t)x^2(t)dt + B_{22}(t)u^2_t dt + G_{22}(t)dW^2(t) \]
\[ + A_{21}(t)x^1(t)dt + B_{21}(t)u^1_t dt, \quad x^2(0) = x^2_0, \quad t \in (0, T] \] (72)

For any \( t \in [0, T] \) the feedback information structure of \( u^1_t \) of subsystem 1 is the \( \sigma \)-algebra \( G_{0,t}^{y^1,u} \triangleq \sigma \{ y^1(s) : 0 \leq s \leq t \} \), and the feedback information structure of \( u^2_t \) of subsystem 2 is the \( \sigma \)-algebra \( G_{0,t}^{y^2,u} \triangleq \sigma \{ y^2(s) : 0 \leq s \leq t \} \). These information structures are defined by the following linear observation equations.

Information structure of Local Control \( u^1 \):

\[ y^1(t) = \int_0^t C_{11}(s)x^1(s)ds + \int_0^t D_{11}^1(s)dB^1(s), \quad t \in [0, T]. \] (73)

Information structure of Local Control \( u^2 \):

\[ y^2(t) = \int_0^t C_{22}(s)x^2(s)ds + \int_0^t D_{22}^1(s)dB^2(s), \quad t \in [0, T]. \] (74)

We may also assume the DMs strategies \( u^1 \) and \( u^2 \) are functionals of the innovations information structures \( G_{0,t}^{I^1,u} \triangleq \sigma \{ I^1(s) : 0 \leq s \leq t \} \), \( G_{0,t}^{I^2,u} \triangleq \sigma \{ I^2(s) : 0 \leq s \leq t \} \) defined by the innovations processes of (73), (74), respectively.

The pay-off or reward is quadratic in \((x^1, x^2, u^1, u^2)\).
Pay-off Functional:

\[ J(u^1, u^2) = \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left\langle \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix}, H(t) \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u^1_t \\ u^2_t \end{pmatrix}, R(t) \begin{pmatrix} u^1_t \\ u^2_t \end{pmatrix} \right\rangle \right\} dt \]

\[ + \left\langle \begin{pmatrix} x^1(T) \\ x^2(T) \end{pmatrix}, M(T) \begin{pmatrix} x^1(T) \\ x^2(T) \end{pmatrix} \right\rangle \} \right\}. \]  

(75)

We assume that the initial condition \( x(0) \), the system Brownian motion \( \{W(t) : t \in [0,T]\} \), and the observations Brownian motion \( \{B^1(t) : t \in [0,T]\} \), and \( \{B^2(t) : t \in [0,T]\} \) are mutually independent and \( x(0) \) is Gaussian \((\mathbb{E}(x(0)), \text{Cov}(x(0))) = (\bar{x}_0, P_0)\).

For decentralized filtering we set \( u^1 = 0, u^2 = 0 \) in the right hand sides of (71), (72), but we should take as pay-off (56).

Define the augmented variables by

\[ x \triangleq \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad y \triangleq \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \quad u \triangleq \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \psi \triangleq \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad q_{11} \triangleq \begin{pmatrix} q^1_{11} \\ q^2_{11} \end{pmatrix}, \]

\[ W \triangleq \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}, \quad B \triangleq \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}, \]

and matrices by

\[ A \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C \triangleq \begin{bmatrix} C^1 & 0 \\ 0 & C^2 \end{bmatrix}, \]

\[ B^{(1)} \triangleq \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \quad B^{(2)} \triangleq \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}, \quad C^{[1]} \triangleq \begin{bmatrix} C_{11} & 0 \end{bmatrix}, \quad C^{[2]} \triangleq \begin{bmatrix} 0 & C_{22} \end{bmatrix}, \]

\[ G \triangleq \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}, \quad D^{\frac{1}{2}} \triangleq \begin{bmatrix} D_{11}^{\frac{1}{2}} & 0 \\ 0 & D_{22}^{\frac{1}{2}} \end{bmatrix}. \]

The distributed system is described in compact form by

\[ dx(t) = A(t)x(t)dt + B(t)u(t)dt + G(t)dW(t), \quad x(0) = x_0, \quad t \in [0,T], \]  

(76)

\[ dy(t) = C(t)x(t)dt + D^{\frac{1}{2}}(t)dB(t), \quad t \in [0,T]. \]  

(77)
while the pay-off is expressed by
\begin{equation}
J(u^1, u^2) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[ \langle x(t), H(t)x(t) \rangle + \langle u_t, R(t)u_t \rangle \right] dt + \langle x(T), M(T)x(T) \rangle \right\}.
\end{equation}
(78)

By Theorem 2, the Hamiltonian is given by
\begin{equation}
\mathcal{H}(t, x, \psi, q_{11}, u) = \langle A(t)x + Bu, \psi \rangle + tr(q_{11}^* G(t)) + \frac{1}{2} \langle x, H(t)x \rangle + \frac{1}{2} \langle u, R(t)u \rangle.
\end{equation}
(79)

The optimal decision \( \{u_t^o = (u^1_t, u^2_t) : 0 \leq t \leq T \} \) is obtained from (80), (81) by using the information structure available to each DM.

Let \((x^o(\cdot), \psi^o(\cdot), q_{11}^o(\cdot), q_{12}^o(\cdot)) \) denote the solutions of the Hamiltonian system, corresponding to the optimal control \( u^o \), then
\begin{equation}
dx^o(t) = A(t)x^o(t)dt + B(t)u^o_t dt + G(t)dW(t), \quad x^o(0) = x_0
\end{equation}
(82)
\begin{equation}
d\psi^o(t) = - A^*(t)\psi^o(t)dt - H(t)x^o(t)dt - V_{q_{11}}^o(t)dt
+ q_{11}^o(t)dW(t) + q_{12}^o(t)dB(t), \quad \psi^o(T) = M(T)x^o(T)
\end{equation}
(83)

Next, we identify the martingale term in (83). Let \( \{\Phi(t, s) : 0 \leq s \leq t \leq T \} \) denote the transition operator of \( A(\cdot) \) and \( \Phi^*(\cdot, \cdot) \) that of the adjoint \( A^*(\cdot) \) of \( A(\cdot) \). Then \( \{\psi^o(t) : t \in [0, T] \} \) is given by
\begin{equation}
\psi^o(t) = \Phi^*(T, t)M(T)x^o(T) + \int_t^T \Phi^*(s, t) \left\{ \begin{array}{c} H(s)x^o(s)ds + V_{q_{11}}^o(s)ds \\
- q_{12}^o(s)dW(s) - q_{11}^o(s)dB(s) \end{array} \right\}.
\end{equation}
(84)

By using the using the identity \( \frac{\partial}{\partial s} \Phi^*(t, s) = -A^*(s)\Phi^*(t, s), 0 \leq s \leq t \leq T \) one can verify by differentiation that (84) is a solution of \((\psi^o(\cdot), q_{11}^o(\cdot), q_{12}^o(\cdot)) \) governed by (83). Since for any control policy, \( \{x^o(s) : 0 \leq t \leq s \leq T \} \) is uniquely determined from (82) and its current value \( x^o(t) \), then (84) can be expressed via
\begin{equation}
\psi^o(t) = \Sigma(t)x^o(t) + \beta^o(t), \quad t \in [0, T],
\end{equation}
(85)
where $\Sigma(\cdot), \beta^o(\cdot)$ determine the operators to the one expressed via (84).

Next, we determine the operators $(\Sigma(\cdot), \beta^o(\cdot))$. Applying the Itô differential rule to both sides of (85), and then using (82), (83) we obtain

$$-A^*(t)\psi^o(t)dt - H(t)x^o(t)dt - V_{q_1^{11}}(t)dt + q_{11}^o(t)dW(t) + q_{12}^o(t)dB(t)$$

$$= \dot{\Sigma}(t)x^o(t)dt + \Sigma(t)\left\{ A(t)x^o(t)dt + B(t)u^o_tdt + G(t)dW(t) \right\} + d\beta^o(t). \quad (86)$$

Substituting the claimed relation (85) into (86) we obtained the identity

$$\left\{ - A^*(t)\Sigma(t) - \Sigma(t)A(t) - H(t) - \dot{\Sigma}(t) \right\}x^o(t)dt - V_{q_1^{11}}(t)dt + q_{11}^o(t)dW(t) + q_{12}^o(t)dB(t)$$

$$= A^*(t)\beta^o(t)dt + \Sigma(t)B(t)u^o_tdt + \Sigma(t)G(t)dW(t) + d\beta^o(t). \quad (87)$$

Since $\sigma(t, x) = G(t)$, then $V_{q_{11}^1}(t) = 0, \forall t \in [0, T]$. By matching the intensity of the martingale terms $\{\cdot\}dW(t)$ in (87), and the rest of the terms we obtain the following equations.

$$V_{q_1^{11}}(t) = 0, \quad \forall t \in [0, T], \quad (88)$$

$$q_{11}^o(t) = \Sigma(t)G(t), \quad t \in [0, T], \quad (89)$$

$$\dot{\Sigma}(t) + A^*(t)\Sigma(t) + \Sigma(t)A(t) + H(t) = 0, \quad t \in [0, T], \quad \Sigma(T) = M(T), \quad (90)$$

$$d\beta^o(t) + A^*(t)\beta^o(t)dt + \Sigma(t)B(t)u^o_tdt - q_{12}^o(t)dB(t) = 0, \quad t \in [0, T], \quad \beta^o(T) = 0. \quad (91)$$

Notice that $q_{12}^o$ is also obtained by Remark 3 since $q_{12}^o(t) = \psi^o(t)G(t) = \Sigma(t)G(t), \forall t \in [0, T]$.

By Theorem 2, $\{(u_1^{1,0}, u_2^{2,0}) : 0 \leq t \leq T\}$ obtained from (80) and (81), are given by

$$\mathbb{E}\left\{ \mathcal{H}_{u_1}(t, x^{1,0}(t), x^{2,0}(t), \psi^{1,0}(t), \psi^{2,0}(t), q_{11}^{1,0}(t), q_{11}^{2,0}(t), u_1^{1,0}, u_2^{2,0})|G_{0,t}^{1, u^o} \right\} = 0,$$

$$a.e.t \in [0, T], \quad \mathbb{P}|_{G_{0,t}^{1, u^o}} - a.s. \quad (92)$$

$$\mathbb{E}\left\{ \mathcal{H}_{u_2}(t, x^{1,0}(t), x^{2,0}(t), \psi^{1,0}(t), \psi^{2,0}(t), q_{11}^{1,0}(t), q_{11}^{2,0}(t), u_1^{1,0}, u_2^{2,0})|G_{0,t}^{2, u^o} \right\} = 0,$$

$$a.e.t \in [0, T], \quad \mathbb{P}|_{G_{0,t}^{2, u^o}} - a.s. \quad (93)$$

where $(x^o(\cdot), \psi^o(\cdot), q_{11}^{1,0}(\cdot)) \equiv (x^{1,0}(\cdot), x^{2,0}(\cdot), \psi^{1,0}(\cdot), \psi^{2,0}(\cdot), q_{11}^{1,0}(\cdot), q_{11}^{2,0}(\cdot))$ are solutions of the Hamiltonian system (82), (83) corresponding to $u^o$. From (92), (93) the optimal decisions are

$$u_1^{1,0} = -R_{11}^{-1}(t)B^{(1)*}(t)\mathbb{E}\left\{ \psi^o(t)|G_{0,t}^{1, u^o} \right\} - R_{11}^{-1}(t)R_{12}(t)\mathbb{E}\left\{ u_2^{2,o} |G_{0,t}^{2, u^o} \right\}, \quad t \in [0, T]. \quad (94)$$
\[ u_{t}^{2,o} = -R^{-1}_{22}(t)B^{(2)\circ}(t)\mathbb{E}\left\{ \psi^o(t)\mid\mathcal{G}_{0,t}^{2,o}\right\} - R^{-1}_{22}(t)R_{21}(t)\mathbb{E}\left\{ u_{t}^{1,o}\mid\mathcal{G}_{0,t}^{2,o}\right\}. \quad t \in [0,T]. \]  

Clearly, the previous equations illustrate the coupling between the two subsystems, since \( u_{t}^{1,o} \) is estimating the optimal decision of the other subsystem \( u_{t}^{2,o} \) as well as the adjoint processes \( \psi^o \) from its own observations, and vice-versa.

Let \( \phi(t) \) be any square integrable and \( \mathbb{F}_T \)-adapted matrix-valued process or scalar-valued processes, and define its filtered and predictor versions by

\[ \pi^i(\Phi)(t) \triangleq \mathbb{E}\left\{ \phi(t)\mid\mathcal{G}_{0,t}^{i}\right\}, \quad \pi^i(\Phi)(s,t) \triangleq \mathbb{E}\left\{ \phi(s,t)\mid\mathcal{G}_{0,t}^{i}\right\}, \quad t \in [0,T], \quad s \geq t, \quad i = 1,2. \]

For any admissible decision \( u \) and corresponding \( (x(t), \psi(t)) \) define their filter versions with respect to \( \mathcal{G}_{0,t}^{i} \) for \( i = 1,2 \), by

\[ \pi^i(u)(t) \triangleq \mathbb{E}\left\{ u_{t}^{i}\mid\mathcal{G}_{0,t}^{i} \right\}, \quad t \in [0,T], \quad i = 1,2, \]

and their predictor versions by

\[ \pi^i(u)(s,t) \triangleq \mathbb{E}\left\{ u_{s}^{i}\mid\mathcal{G}_{0,t}^{i} \right\}, \quad t \in [0,T], \quad s \geq t, \quad i = 1,2. \]

From (94), (95) the optimal decisions are

\[ u_{t}^{1,o} \equiv -R^{-1}_{11}(t)B^{(1)\circ}(t)\pi^1(\psi^o)(t) - R^{-1}_{11}(t)R_{12}(t)\mathbb{E}\left\{ u_{t}^{2,o}\mid\mathcal{G}_{0,t}^{1,o}\right\}, \quad t \in [0,T], \]  

\[ u_{t}^{2,o} \equiv -R^{-1}_{22}(t)B^{(2)\circ}(t)\pi^2(\psi^o)(t) - R^{-1}_{22}(t)R_{21}(t)\mathbb{E}\left\{ u_{t}^{1,o}\mid\mathcal{G}_{0,t}^{2,o}\right\}, \quad t \in [0,T]. \]

The previous optimal decisions require the conditional estimates

\[ \{ (\pi^1(\psi^o)(t), \pi^2(\psi^o)(t)) : 0 \leq t \leq T \}. \] These are obtained by taking conditional expectations of \( (84) \) giving

\[ \pi^i(\psi^o)(t) = \Phi^s(T,t)M(T)\pi^i(x^o)(T,t) + \int_{t}^{T} \Phi^s(s,t)H(s)\pi^i(x^o)(s,t)ds, \quad t \in [0,T], \quad i = 1,2. \]
For any admissible decision, the filtered versions of \( x(\cdot) \) are given by the following stochastic differential equations [38].

\[
d\pi^1(x)(t) = A(t)\pi^1(x)(t)dt + B^{(1)}(t)u^1(t)dt + B^{(2)}(t)\pi^1(u^2)(t)dt + \left\{ \pi^1(x^*)(t) \right\}C^{[1]}D^{1\pi}_{11}(dt) - C^{[1]}(t)\pi^1(x)(t)dt, \quad \pi^1(x)(0) = \bar{x}_0, \tag{99}
\]

\[
d\pi^2(x)(t) = A(t)\pi^2(x)(t)dt + B^{(2)}(t)u^2(t)dt + B^{(1)}(t)\pi^2(u^1)(t)dt + \left\{ \pi^2(x^*)(t) \right\}C^{[2]}D^{2\pi}_{22}(dt) - C^{[2]}(t)\pi^2(x)(t)dt, \quad \pi^2(x)(0) = \bar{x}_0. \tag{100}
\]

From the previous filtered versions of \( x(\cdot) \) it is clear that subsystem 1 estimates the actions of subsystem 2 based on its own observations, namely, \( \pi^1(u^2)(\cdot) \) and subsystem 2 estimates the actions of subsystem 1 based on its own observations, namely, \( \pi^2(u^1)(\cdot) \).

For any admissible decision \((u^1, u^2) \in \mathbb{U}^{(2)}_{reg} [0,T]\) define the innovation processes associated with \( \{G_{0,t}^u : t \in [0,T], i = 1,2\} \) as follows

\[
I_i(t) \triangleq y^i(t) - \int_0^t C^{[i]}(s)\pi^i(x)(s)ds, \quad G_{0,t}^{i,u} \triangleq \sigma \left\{ I_i(s) : 0 \leq s \leq t \right\}, \quad t \in [0,T], \quad i = 1,2 \tag{101}
\]

Let \( I_i^o(t) \) the innovations processes corresponding to where \((x^o, u^o), i = 1,2\). Then by (101), \( \{I_i^o(t) : t \in [0,T]\} \) is adapted Brownian motion, \( I_i^o(t) \) has covariance \( \text{Cov}(I_i^o(t)) \triangleq \int_0^t D_{ii}(s)ds \), for \( i = 1,2 \) and \( \{I_1(t) : t \in [0,T]\}, \{I_2(t) : t \in [0,T]\} \) are independent.

For any admissible decision \( u \) the predicted versions of \( x(\cdot) \) are obtained from (99) and (100) as follows. Utilizing the identity \( \pi^i(x)(s,t) = \mathbb{E} \left\{ \mathbb{E} \left\{ x(s) | G_{0,s}^{y^i,u} \right\} | G_{0,t}^{y^i,u} \right\} \right\} = \mathbb{E} \left\{ \pi^i(x)(s) | G_{0,t}^{y^i,u} \right\} \), for \( 0 \leq t \leq s \leq T \) then

\[
d\pi^1(x)(s,t) = A(s)\pi^1(x)(s,t)ds + B^{(1)}(s)\pi^1(u^1)(s,t)ds + B^{(2)}(s)\pi^1(u^2)(s,t)ds, \quad t < s \leq T, \tag{102}
\]

\[
\pi^1(x)(t,t) = \pi^1(x)(t), \quad t \in [0,T), \tag{103}
\]

\[
d\pi^2(x)(s,t) = A(s)\pi^2(x)(s,t)ds + B^{(2)}(s)\pi^2(u^1)(s,t)ds + B^{(1)}(s)\pi^2(u^1)(s,t)ds, \quad t < s \leq T, \tag{104}
\]

\[
\pi^2(x)(t,t) = \pi^2(x)(t), \quad t \in [0,T). \tag{105}
\]

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Since for a given admissible policy and observation paths, \( \{ \pi^1(x)(s,t) : 0 \leq t \leq s \leq T \} \) is determined from (102) and its current value \( \pi^1(x^o)(t,t) = \pi^1(x)(t) \), and \( \{ \pi^2(x)(s,t) : 0 \leq t \leq s \leq T \} \) is determined from (104), and its current value \( \pi^2(x)(t,t) = \pi^2(x)(t) \), then (98) can be expressed via
\[
\pi^i(t)(t) = K^i(t)\pi^i(x^o)(t) + r^i(t), \quad t \in [0,T], \quad i = 1,2.
\]

where \( K^i(\cdot), r^i(\cdot) \) determines the operators to the one expressed via (98), for \( i = 1,2 \). Utilizing (106) into (96) and (97) then
\[
u^1_t = R^{-1}_1(t)B^{(1),\ast}(t)\left\{ K^1(t)\pi^1(x^o)(t) + r^1(t) \right\} - R^{-1}_1(t)R_{12}(t)\pi^1(u^2_0)(t), \quad t \in [0,T],
\]
\[
u^2_t = R^{-1}_2(t)B^{(2),\ast}(t)\left\{ K^2(t)\pi^2(x^o)(t) + r^2(t) \right\} - R^{-1}_2(t)R_{21}(t)\pi^2(u^1_0)(t), \quad t \in [0,T].
\]

Let \( \{ \Psi_{K^i}(t,s) : 0 \leq s \leq t \leq T \} \) denote the transition operator of \( A_{K^i}(t) \equiv \left( A(t) - B^{(i)}(t)R^{-1}_{ii}(t)B^{(i),\ast}(t)K^i(t) \right) \), for \( i = 1,2 \).

Next, we determine \( K^i(\cdot), r^i(\cdot), i = 1,2 \). Substituting the previous equations into (102), (103) and (104), (105) then
\[
\pi^1(x^o)(s,t) = \Psi_{K^1}(s,t)\pi^1(x^o)(t) - \int_t^s \Psi_{K^1}(s,\tau)B^{(1)}(\tau)R_{11}^{-1}(\tau)B^{(1),\ast}(\tau)r^1(\tau)d\tau
\]
\[
- \int_t^s \Psi_{K^1}(s,\tau)B^{(1)}(\tau)R_{11}^{-1}(\tau)R_{12}(\tau)\pi^1(u^2_0)(\tau,t)d\tau
\]
\[
+ \int_t^s \Psi_{K^1}(s,\tau)B^{(2)}(\tau)\pi^1(u^2_0)(\tau,t)d\tau, \quad t \leq s \leq T,
\]
\[
\pi^2(x^o)(s,t) = \Psi_{K^2}(s,t)\pi^2(x^o)(t) - \int_t^s \Psi_{K^2}(s,\tau)B^{(2)}(\tau)R_{22}^{-1}(\tau)B^{(2),\ast}(\tau)r^2(\tau)d\tau
\]
\[
- \int_t^s \Psi_{K^2}(s,\tau)B^{(2)}(\tau)R_{22}^{-1}(\tau)R_{21}(\tau)\pi^2(u^1_0)(\tau,t)d\tau
\]
\[
+ \int_t^s \Psi_{K^2}(s,\tau)B^{(1)}(\tau)\pi^2(u^1_0)(\tau,t)d\tau, \quad t \leq s \leq T.
\]

We now introduce the following assumption regarding the measurability of admissible decisions \( \mathbb{U}^{y_{i,u}}[0,T], i = 1,2 \).
Assumptions 5. Any admissible decentralized feedback information structure $u^i \in \mathbb{U}^{i\pi} [0, T]$ is adapted to $\left\{ G_{0,t}^{i,u} : t \in [0, T] \right\}$, $i = 1, 2$, and $\left[ 99 \right]$ has a strong $\left\{ G_{0,t}^{i,u} : t \in [0, T] \right\}$—adapted solution $\pi^{1}(x)(\cdot)$ and $\left[ 100 \right]$ has a strong $\left\{ G_{0,t}^{i,u} : t \in [0, T] \right\}$—adapted solution $\pi^{2}(x)(\cdot)$.

Now, we can state the first main result.

Theorem 4. Under the conditions of Assumptions 5 the optimal decisions $(u^{1,\pi}, u^{2,\pi})$ are given

$$u^{1,\pi}_t = -R^{-1}_{11}(t)B^{(1),*}(t) \left\{ K^{1}(t)\pi^{1}(x^{\pi})(t) + r^{1}(t) \right\} - R^{-1}_{11}(t)R_{12}(t)u^{2,\sigma}(t), \quad t \in [0, T],$$

$$u^{2,\pi}_t = -R^{-1}_{22}(t)B^{(2),*}(t) \left\{ K^{2}(t)\pi^{2}(x^{\pi})(t) + r^{2}(t) \right\} - R^{-1}_{22}(t)R_{21}(t)u^{1,\sigma}(t), \quad t \in [0, T],$$

where $\pi^{i}(x^{\pi})(\cdot)$, $i = 1, 2$ satisfy the filter equations $\left[ 99 \right]$, $\left[ 100 \right]$, and $\left( K^{i}(\cdot), r^{i}(\cdot), x^{\pi}(\cdot), u^{i,\sigma}(\cdot) \right)$, $i = 1, 2$ are solutions of the ordinary differential equations $\left[ 113 \right]$, $\left[ 114 \right]$, $\left[ 115 \right]$, $\left[ 116 \right]$, $\left[ 117 \right]$, $\left[ 118 \right]$.

$$K^{i}(t) + A^{i}(t)K^{i}(t) + K^{i}(t)A(t) - K^{i}(t)B^{(i)}(t)R^{-1}_{ii}(t)B^{(i),*}(t)K^{i}(t) + H(t) = 0, \quad t \in [0, T], \quad i = 1, 2,$$

$$K^{i}(T) = M(T), \quad i = 1, 2,$$

$$r^{1}(t) = \left\{ -A^{*}(t) + \Phi^{*}(T, t)M(T)\Psi_{K^{1}}(T, t)B^{(1)}(t)R^{-1}_{11}(t)B^{(1),*}(t) \right.$$  

$$+ \left( \int_{t}^{T} \Phi^{*}(s, t)H(s)\Psi_{K^{1}}(s, t)ds \right)B^{(1)}(t)R^{-1}_{11}(t)B^{(1),*}(t) \right\} r^{1}(t)$$  

$$- \left( \int_{t}^{T} \Phi^{*}(s, t)H(s)\Psi_{K^{1}}(s, t)ds \right) \left( B^{(2)}(t) - B^{(1)}(t)R^{-1}_{11}(t)R_{12}(t) \right)u^{2,\sigma}(t),$$

$$- \Phi^{*}(T, t)M(T)\Psi_{K^{1}}(T, t) \left( B^{(2)}(t) - B^{(1)}(t)R^{-1}_{11}(t)R_{12}(t) \right)u^{2,\sigma}(t) \right\} t \in [0, T], \quad r^{1}(T) = 0,$$

$$r^{2}(t) = \left\{ -A^{*}(t) + \Phi^{*}(T, t)M(T)\Psi_{K^{2}}(T, t)B^{(2)}(t)R^{-1}_{22}(t)B^{(2),*}(t) \right.$$  

$$+ \left( \int_{t}^{T} \Phi^{*}(s, t)H(s)\Psi_{K^{2}}(s, t)ds \right)B^{(2)}(t)R^{-1}_{22}(t)B^{(2),*}(t) \right\} r^{2}(t)$$  

$$- \left( \int_{t}^{T} \Phi^{*}(s, t)H(s)\Psi_{K^{2}}(s, t)ds \right) \left( B^{(1)}(t) - B^{(2)}(t)R^{-1}_{22}(t)R_{21}(t) \right)u^{1,\sigma}(t),$$

$$- \Phi^{*}(T, t)M(T)\Psi_{K^{2}}(T, t) \left( B^{(1)}(t) - B^{(2)}(t)R^{-1}_{22}(t)R_{21}(t) \right)u^{1,\sigma}(t) \right\} t \in [0, T], \quad r^{2}(T) = 0.$$
Substituting (119), (120) into (109), (110), and then (109), (110) into (98) we have
the optimality conditions of Theorem 2 are valid. Utilizing the independence of the innovations

\[ \pi^1 (u^2, s, t) = \mathbb{E} \left( u_s^2 \mid G_{0,t}^y \right) = \mathbb{E} \left( u_s^2 \right) = \bar{u}^2 (s), \quad t \leq s \leq T, \]
\[ \pi^2 (u^1, s, t) = \mathbb{E} \left( u_s^1 \mid G_{0,t}^y \right) = \mathbb{E} \left( u_s^1 \right) = \bar{u}^1 (s), \quad t \leq s \leq T. \]

Proof: By invoking Assumptions since \( y^i (t) = \int_0^t C^{[i]} (s) \pi (x) (s) ds + I^i (t) \), and \( \pi^i (x) (\cdot) \)
is a strong solution then \( G_{0,t}^{y^{i,u}} \subseteq G_{0,t}^{l^{i,u}}, \forall t \in [0, T] \) and thus \( G_{0,t}^{y^{i,u}} = G_{0,t}^{l^{i,u}} \), \( i = 1, 2 \). Hence, the optimality conditions of Theorem 2 are valid. Utilizing the independence of the innovations processes \( I^1(\cdot) \) and \( I^2(\cdot) \) then

\[ \pi^1 (\psi^o) (t) = \left\{ \Phi^* (T, t) M (T) \Psi_{K^1} (T, t) + \int_t^T \Phi^* (s, t) H (s) \Psi_{K^1} (s, t) ds \right\} \pi^1 (x^o) (t) \]
\[ + \Phi^* (T, t) M (T) \int_t^T \Psi_{K^1} (T, \tau) \left( B^{(2)} (\tau) - B^{(1)} (\tau) R_{11}^{-1} (\tau) \right) \bar{u}^2 (\tau) d\tau \]
\[ + \int_t^T \Phi^* (s, t) H (s) \int_s^t \Psi_{K^1} (s, \tau) \left( B^{(2)} (\tau) - B^{(1)} (\tau) R_{11}^{-1} (\tau) \right) \bar{u}^2 (\tau) d\tau ds \]
\[ - \Phi^* (T, t) M (T) \int_t^T \Psi_{K^1} (T, \tau) B^{(1)} (\tau) R_{11}^{-1} (\tau) B^{(1)} (\tau) r^{1} (\tau) d\tau d\tau, \]

\[ \pi^2 (\psi^o) (t) = \left\{ \Phi^* (T, t) M (T) \Psi_{K^2} (T, t) + \int_t^T \Phi^* (s, t) H (s) \Psi_{K^2} (s, t) ds \right\} \pi^2 (x^o) (t) \]
\[ + \Phi^* (T, t) M (T) \int_t^T \Psi_{K^2} (T, \tau) \left( B^{(1)} (\tau) - B^{(2)} (\tau) R_{22}^{-1} (\tau) \right) \bar{u}^1 (\tau) d\tau \]
\[ + \int_t^T \Phi^* (s, t) H (s) \int_s^t \Psi_{K^2} (s, \tau) \left( B^{(1)} (\tau) - B^{(2)} (\tau) R_{22}^{-1} (\tau) \right) \bar{u}^1 (\tau) d\tau ds \]
\[ - \Phi^* (T, t) M (T) \int_t^T \Psi_{K^2} (T, \tau) B^{(2)} (\tau) R_{22}^{-1} (\tau) B^{(2)} (\tau) r^{2} (\tau) d\tau d\tau. \]
Comparing (106) with the previous two equations then \( K^i(\cdot), i = 1, 2 \) are identified by the operators
\[
K^i(t) = \Phi^*(T, t)M(T)\Psi_{K^i}(T, t) + \int_t^T \Phi^*(s, t)H(s)\Psi_{K^i}(s, t)ds, \quad t \in [0, T], \quad i = 1, 2,
\]
and \( r^i(\cdot), i = 1, 2 \) by the processes
\[
\begin{align*}
\hat{r}^1(t) &= \Phi^*(T, t)M(T) \int_t^T \Psi_{K^1}(T, \tau) \left( B^{(2)}(\tau) - B^{(1)}(\tau)R_{11}^{-1}(\tau)R_{12}(\tau) \right) \overline{u^{1, \omega}(\tau)}d\tau \\
&+ \int_t^T \Phi^*(s, t)H(s) \int_t^s \Psi_{K^1}(s, \tau) \left( B^{(2)}(\tau) - B^{(1)}(\tau)R_{11}^{-1}(\tau)R_{12}(\tau) \right) \overline{u^{2, \omega}(\tau)}d\tau ds \\
&- \Phi^*(T, t)M(T) \int_t^T \Psi_{K^1}(T, \tau)B^{(1)}(\tau)R_{11}^{-1}(\tau)B^{(1), *}(\tau)r^1(\tau)d\tau \\
&- \int_t^T \Phi^*(s, t)H(s) \int_t^s \Psi_{K^1}(s, \tau)B^{(1)}(\tau)R_{11}^{-1}(\tau)B^{(1), *}(\tau)r^1(\tau)d\tau ds, \\
\hat{r}^2(t) &= \Phi^*(T, t)M(T) \int_t^T \Psi_{K^2}(T, \tau) \left( B^{(1)}(\tau) - B^{(2)}(\tau)R_{22}^{-1}(\tau)R_{21}(\tau) \right) \overline{u^{1, \omega}(\tau)}d\tau \\
&+ \int_t^T \Phi^*(s, t)H(s) \int_t^s \Psi_{K^2}(s, \tau) \left( B^{(1)}(\tau) - B^{(2)}(\tau)R_{22}^{-1}(\tau)R_{21}(\tau) \right) \overline{u^{1, \omega}(\tau)}d\tau ds \\
&- \Phi^*(T, t)M(T) \int_t^T \Psi_{K^2}(T, \tau)B^{(2)}(\tau)R_{22}^{-1}(\tau)B^{(2), *}(\tau)r^2(\tau)d\tau \\
&- \int_t^T \Phi^*(s, t)H(s) \int_t^s \Psi_{K^2}(s, \tau)B^{(2)}(\tau)R_{22}^{-1}(\tau)B^{(2), *}(\tau)r^2(\tau)d\tau ds.
\end{align*}
\]
Differentiating both sides of (123) the operators \( K^i(\cdot), i = 1, 2 \) satisfy the following matrix differential equations (113), (114). Differentiating both sides of (124), (125) the processes \( r^i(\cdot), i = 1, 2 \) satisfy the differential equations (115), (116). Utilizing (119), (120) we obtain the optimal strategies (111), (112). Next, we determine \( \overline{u^{i, \omega}} \) for \( i = 1, 2 \) from (111), (112).
Define the averages
\[
\overline{x^i}(t) \triangleq \mathbb{E}\left\{ x(t) \right\} = \mathbb{E}\left\{ \pi^i(x)(t) \right\}, \quad i = 1, 2.
\]
Then \( \overline{x^i}(\cdot) \) satisfies the ordinary differential equation (117). Taking the expectation of both sides of (111), (112) we deduce the corresponding equations
\[
\begin{align*}
\overline{u^{1, \omega}}(t) &= -R_{11}^{-1}(t)B^{(1), *}(t)\left\{ K^1(t)\overline{x^i}(t) + r^1(t) \right\} - R_{11}^{-1}(t)R_{12}(t)\overline{u^{2, \omega}}(t), \quad t \in [0, T], \\
\overline{u^{2, \omega}}(t) &= -R_{22}^{-1}(t)B^{(2), *}(t)\left\{ K^2(t)\overline{x^i}(t) + r^2(t) \right\} - R_{22}^{-1}(t)R_{21}(t)\overline{u^{1, \omega}}(t), \quad t \in [0, T].
\end{align*}
\]
The last two equations can be written in matrix form (118). This completes the derivation. 

Hence, the optimal strategies are computed from (111), (112), where the filter equations for \( \pi^i(x^o)(\cdot), i = 1, 2 \) satisfy (99), (100), while \( (R^i(\cdot), r^i(\cdot), u^{i,o'}(\cdot), \bar{x}^o(\cdot)) \), \( i = 1, 2 \) are computed off-line utilizing the ordinary differential equations (113), (114), (115), (116), (117), (118).

It is important to make the following observations.

(O1): The optimal strategies or laws (111), (112) are precisely the optimal strategies obtained in [39] for noiseless decentralized information structures. This property is analogous to that of optimal centralized strategies of fully and partially observed Linear-Quadratic-Gaussian systems.

(O2): The filter equations for \( \pi^i(x^o)(\cdot), i = 1, 2 \) given by (99), (100) are nonlinear and may require higher order moments, leading to the so-called moment closure problem of nonlinear filtering. Further analysis is required to determine whether the conditional error covariance in (99), (100) have the Kalman filter form.

Next, we state analogous results for distributed filtering problems.

**Corollary 2.** Consider distributed filter dynamics (71), (72) with \( B^{(i)} = 0, i = 1, 2 \) and LQF pay-off (49).

Then the optimal strategies \( (u^{1,o}, u^{2,o}) \) are given by

\[
\begin{align*}
    u^{i,o}_t &= -R^{-1}_{ii}(t) \left\{ m^i(t) + \sum_{j=1}^{2} E_{ij}(t) \mathbb{E} \left( x^j(t)|G_{0,t} \right) + \sum_{j=1,j \neq i}^{2} R_{ij}(t) \mathbb{E} \left( u^{j,o}_t \right) \right\} \}, \mathbb{P}|_{G_{0,t}} - a.s., \forall i \in \mathbb{Z}_2,
\end{align*}
\]

(129)

where \( \hat{x}^i(t) = \text{Vector} \left\{ \mathbb{E} \{ x^1(t)|G_{0,t} \}, \mathbb{E} \{ x^2(t)|G_{0,t} \} \right\}, \ i = 1, 2 \) satisfy the linear Kalman filter equations

\[
\begin{align*}
    d\hat{x}^1(t) &= A(t)\hat{x}^1(t)dt + P^1(t)C^{[1],[i]}D^{-1}_{11}(t) \left( dy^1(t) - C^{[1]}(t)\hat{x}^1(t)dt \right), \ \hat{x}(0) = \bar{x}_0, \quad (130) \\
    d\hat{x}^2(t) &= A(t)\hat{x}^2(t)dt + P^2(t)C^{[2],[i]}D^{-1}_{22}(t) \left( dy^2(t) - C^{[2]}(t)\hat{x}^2(t)dt \right), \ \hat{x}^o(0) = \bar{x}_0, \quad (131) \\
    \dot{P}^i(t) &= A(t)P^i(t) + P^i(t)A(t) - P^2(t)C^{[i],[i]*}(t)D^{-1}_{ii}(t)C^{[i]}(t)P^2(t) \\
    &+ G(t)G^*(t), \quad \dot{P}^i(0) = P^i_0, \quad i = 1, 2. \quad (132)
\end{align*}
\]

and \( \bar{\pi}(t) \triangleq \text{Vector} \left\{ \mathbb{E} \{ u^1_t \}, \mathbb{E} \{ u^2_t \} \right\} \) satisfy the equations

\[
R(t)\bar{u}^o(t) + E(t)\bar{x}(t) + m(t) = 0, \quad \frac{d}{dt}\bar{x}(t) = A\bar{x}(t), \quad \bar{x}(0) = \bar{x}_0. \quad (133)
\]
Proof: (129) is obtained from (53) by setting $B^{(i)} = 0, N = 2$, and the discussion in Section V-A (for filtering problems the observations and innovations generate the same filtrations). The filters (130)-(132) are follow from the linear and Gaussian nature of the state and observation equations. Taking expectation of both sides of (129) yields (133).

Finally, we state a remark describing extensions of the previous examples.

**Remark 4.** Theorem 4 is easily generalized to the following arbitrary coupled dynamics

$$dx^i(t) = A_{ii}(t)x^i(t)dt + B^{(i)}u^i_t dt + G_{ii}dW^i(t)$$

$$+ \sum_{j=1, j\neq i}^N A_{ij}x^j(t)dt + \sum_{j=1, j\neq i}^N B^{(j)}u^j_t dt, \ x^i(0) = x^i_0, \ t \in (0, T], \ i \in \mathbb{Z}_N$$

and information structures generated by observation equations with feedback

$$y^i = \int_0^t C_{ii}(s, y^i(s))x(s)ds + \int_0^t D_{ii}^s(s)dB^i(s), \ t \in [0, T], \ i \in \mathbb{Z}_N. \quad (134)$$

The optimal strategies are extensions of the ones given in Theorem 4. Similarly, one can generalize the filtering results of Corollary 2 to the above models with $B^{(i)} = 0$.

**VI. Conclusions and Future Work**

In this paper we have considered team games for distributed stochastic differential decision systems, with decentralized noisy information patterns for each DM, and we derived necessary and sufficient optimality conditions with respect to team optimality and person-by-person optimality criteria, based on Stochastic Pontryagin’s minimum principle.

However, several additional issues remain to be investigated. Below, we provide a short list.

(F1) In the derivation of optimality conditions we can relax some of the assumptions by considering spike or needle variations instead of strong variations of the decision strategies (or use relaxed strategies as in [2]). Moreover, for team games with non-convex action spaces $A^i, i = 1, 2, \ldots, N$ and diffusion coefficients which depend on the decision variables it is necessary to derive optimality conditions based on second-order variations.

(F2) The derivation of optimality conditions can be used in other type of games such as Nash-equilibrium games with decentralized noisy information structures for each DM, and minimax games.
(F3) It will be interesting determine whether (99) and (100) are given by the Kalman filter equations.

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