Generalization of the Sherman–Morrison–Woodbury formula involving the Schur complement

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Abstract
Let $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{n \times n}$ be nonsingular matrices, and let $N \in \mathbb{C}^{m \times n}$. Explicit expressions for the Moore–Penrose inverses of $M = XNY$ and a two-by-two block matrix, under appropriate conditions, have been established by Castro-González et al. [Linear Algebra Appl. 471 (2015) 353–368]. Based on these results, we derive a novel expression for the Moore–Penrose inverse of $A + UV^*$ under suitable conditions, where $A \in \mathbb{C}^{m \times n}$, $U \in \mathbb{C}^{m \times r}$, and $V \in \mathbb{C}^{n \times r}$. In particular, if both $A$ and $I + V^*A^{-1}U$ are nonsingular matrices, our expression reduces to the celebrated Sherman–Morrison–Woodbury formula. Moreover, we extend our results to the bounded linear operators case.

Keywords: Sherman–Morrison–Woodbury formula, Moore–Penrose inverse, Schur complement

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1. Introduction
Let $\mathbb{C}^{m \times n}$ be the set of all $m \times n$ matrices over complex field $\mathbb{C}$. The identity matrix of order $n$ is denoted by $I_n$ or $I$ when its size is clear in the context. For any $A \in \mathbb{C}^{m \times n}$, let $A^*$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the conjugate transpose, the range, and the null space of $A$, respectively. The Moore–Penrose (MP) inverse of $A \in \mathbb{C}^{m \times n}$ is denoted by $A^\dagger$, which is defined as the unique matrix $Z \in \mathbb{C}^{n \times m}$ satisfying the following equations:

$$(1) \ AZA = A, \quad (2) \ ZAZ = Z, \quad (3) \ (AZ)^* = AZ, \quad (4) \ (ZA)^* = ZA.$$ 

Clearly, the MP inverse $A^\dagger$ coincides with the usual inverse $A^{-1}$ when $A$ is nonsingular. The symbols $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$ denote the orthogonal projectors onto $\mathcal{N}(A^*)$ and $\mathcal{N}(A)$, respectively. A matrix $Z \in \mathbb{C}^{n \times m}$ is referred to as a $\{1\}$-inverse of $A$ if it satisfies the equality (1); see, e.g., [1, Chapter 1, Definition 1].

Let $A \in \mathbb{C}^{n \times n}$, $U \in \mathbb{C}^{n \times r}$, and $V \in \mathbb{C}^{n \times r}$. If both $A$ and $I + V^*A^{-1}U$ are nonsingular, then $A + UV^*$ is also nonsingular and

$$(A + UV^*)^{-1} = A^{-1} - A^{-1}U(I + V^*A^{-1}U)^{-1}V^*A^{-1},$$

which is the celebrated Sherman–Morrison–Woodbury (SMW) formula (see [2–4]). Assume that $A^{-1}$ has been precomputed. If $r$ is much smaller than $n$, then $I + V^*A^{-1}U$ is much easier to invert than $A + UV^*$. Hence, the formula (1.1) provides an effective way to compute $(A + UV^*)^{-1}$. 

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The SMW formula is widely used in many fields, such as statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations; see, e.g., [5]. Here we only mention two specific applications of the SMW formula. Using the SMW formula, Malyshev and Sadkane [6] obtained a fast numerical algorithm for solving systems of linear equations with tridiagonal block Toeplitz matrices. Lai and Venuri [7] applied the SMW formula to solve the surface smoothing problem. Using finite element methods to discretize the variational formulation of the surface smoothing problem may yield a linear system. The SMW formula can convert the problem of solving the original linear system to solving a Lyapunov matrix equation or a cascade of two Lyapunov matrix equations. The simplified problem can be solved efficiently using the ADI method and the bi-conjugate-gradient technique.

However, the SMW formula (1.1) is invalid when $A$ is not square or the assumption is not satisfied. Suppose that $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$ (here $\mathbb{C}^m = \mathbb{C}^{m \times 1}$). It is well-known that the minimum $\ell_2$-norm solution of the Least Squares Problem (LSP)

$$\min_{x \in \mathbb{C}^n} \|Ax - b\|_2$$

is $x = A^\dagger b$. This solution vector can be statistically interpreted as providing an optimal estimator among all linear unbiased estimators, and it can be geometrically interpreted as providing an orthogonal projection of $b$ onto $\mathcal{R}(A)$. Because of the relationship with the LSP, the MP inverse may be the most important of all other generalized inverses. The perturbation theory of the MP inverse is a classical topic in matrix analysis and numerical linear algebra; see, e.g., [8–11].

In general, the perturbations of a matrix can be divided into two categories: additive type and multiplicative type. In this paper, we derive an explicit expression for the MP inverse of the additive perturbations of a matrix using the results in [12, Theorems 2.2 and 3.2], which generalizes the classical SMW formula (1.1). Let $A \in \mathbb{C}^{m \times n}$, $U \in \mathbb{C}^{m \times r}$, and $V \in \mathbb{C}^{n \times r}$. If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(U^*) \subseteq \mathcal{R}(S_A)$, and $\mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*)$, then we have

$$(A + UV^*)^\dagger = (I + A^\dagger UFE_{S_A}U^*(A^\dagger))^\dagger \left( (A^\dagger - A^\dagger UFS_A^*V^*A^\dagger)^\dagger \right)^{-1}$$

where $S_A = I + V^*A^\dagger U$. Note that, if both $A$ and $I + V^*A^{-1}U$ are nonsingular, the conditions are automatically satisfied and our expression (1.2) reduces to (1.1).

The rest of this paper is organized as follows. In Section 2, we present two useful lemmas, which play an important role in our subsequent derivations. In Section 3, we give an explicit expression for $(A + UV^*)^\dagger$, provided that $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(U^*) \subseteq \mathcal{R}(S_A)$, and $\mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*)$. In Section 4, we extend the results established in Theorems 3.2 and 3.7 below to the bounded linear operators case.

2. Preliminaries

In order to prove the expression (1.2), we need the following two lemmas (see [12, Theorems 2.2 and 3.2]), which play a key role in our subsequent derivations.
Let \( X \in \mathbb{C}^{m \times m} \) and \( Y \in \mathbb{C}^{n \times n} \) be nonsingular matrices, and let \( N \in \mathbb{C}^{m \times n} \). The following lemma gives an explicit expression for \((XNY)^\dagger\), provided that \( XE_N = E_N \) and \( F_NY = F_N \).

**Lemma 2.1.** Let \( N \in \mathbb{C}^{m \times n}, X \in \mathbb{C}^{m \times m}, Y \in \mathbb{C}^{n \times n}, \) and \( M = XNY \). If both \( X \) and \( Y \) are nonsingular, \( XE_N = E_N \), and \( F_NY = F_N \), then

\[
M^\dagger = (I + L^*)(I + LL^*)^{-1}Y^{-1}N^\dagger X^{-1}(I + R^*R)^{-1}(I + R^*),
\]

where \( R = E_N(I - X^{-1}) \) and \( L = (I - Y^{-1})F_N \).

Generalized inverses of partitioned matrices possess some important and interesting properties; see, e.g., [13–16]. Now, let \( M \) be the following two-by-two block matrix

\[
M = \begin{pmatrix} A & C \\ B & D \end{pmatrix},
\]

where \( A \in \mathbb{C}^{p \times q}, B \in \mathbb{C}^{r \times q}, C \in \mathbb{C}^{p \times s}, \) and \( D \in \mathbb{C}^{r \times s} \) (\( p, q, r, \) and \( s \) are all positive integers) are the corresponding submatrices of \( M \). The matrix \( S_A = D - BA^\dagger C \) is called the *generalized Schur complement* of \( A \) in \( M \). If \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \) and \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \), then we have the following expression for \( M^\dagger \).

**Lemma 2.2.** Let \( M \) be a block matrix of the form (2.2). If \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \) and \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \), then

\[
M^\dagger = \begin{pmatrix} \Sigma & \Sigma H^*E_{S_A} - \Psi K S_A^\dagger \\ F_{S_A} K^* \Sigma - S_A^\dagger H \Phi & S_A^\dagger - S_A^\dagger H \Phi H^* E_{S_A} - F_{S_A} K^* \Psi K S_A^\dagger + F_{S_A} K^* \Sigma H^* E_{S_A} \end{pmatrix},
\]

where

\[
H = BA^\dagger, \ K = A^\dagger C, \ \Phi = (I + H^*E_{S_A}H)^{-1}, \ \Psi = (I + KF_{S_A}K^*)^{-1}, \ \Sigma = \Psi(A^\dagger + KS_A^\dagger H)\Phi.
\]

3. Main results

In order to prove our main formula in Theorem 3.2 below, we first give an important lemma.

**Lemma 3.1.** Let \( A \in \mathbb{C}^{m \times n}, U \in \mathbb{C}^{m \times r}, V \in \mathbb{C}^{n \times r}, X = \begin{pmatrix} I & -U \\ 0 & I \end{pmatrix}, N = \begin{pmatrix} A & U \\ -V^* & I \end{pmatrix}, \) and \( Y = \begin{pmatrix} I & 0 \\ V^* & I \end{pmatrix} \). If \( \mathcal{R}(U) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(V) \subseteq \mathcal{R}(A^*) \), then

\[
XE_N = E_N \iff \mathcal{R}(U^*) \subseteq \mathcal{R}(S_A), \quad F_NY = F_N \iff \mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*)
\]

where \( S_A = I + V^*A^\dagger U \).

**Proof.** (i) Due to \( \mathcal{R}(U) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(V) \subseteq \mathcal{R}(A^*) \), it follows from Lemma 2.2 that

\[
N^\dagger = \begin{pmatrix} N_1 & N_3 \\ N_2 & N_4 \end{pmatrix},
\]

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where
\[ N_1 = \Sigma, \quad (3.1a) \]
\[ N_2 = F_{S_A} K^* \Sigma - S_A^\dagger H \Phi, \quad (3.1b) \]
\[ N_3 = \Sigma H^* E_{S_A} - \Phi K S_A^\dagger, \quad (3.1c) \]
\[ N_4 = S_A^\dagger - S_A^\dagger H \Phi H^* E_{S_A} - F_{S_A} K^* \Psi K S_A^\dagger + F_{S_A} K^* \Sigma H^* E_{S_A}. \quad (3.1d) \]

And here
\[ S_A = I + V^* A^\dagger U, \quad H = -V^* A^\dagger, \quad K = A^\dagger U, \quad (3.2a) \]
\[ \Phi = (I + H^* E_{S_A} H)^{-1}, \quad \Psi = (I + K F_{S_A} K^*)^{-1}, \quad \Sigma = \Psi (A^\dagger + K S_A^\dagger H) \Phi. \quad (3.2b) \]

Let \( E_N = \begin{pmatrix} E_1 & E_3 \\ E_2 & E_4 \end{pmatrix} \), \( X E_N = \begin{pmatrix} \tilde{E}_1 & \tilde{E}_3 \\ \tilde{E}_2 & \tilde{E}_4 \end{pmatrix} \), \( F_N = \begin{pmatrix} F_1 & F_3 \\ F_2 & F_4 \end{pmatrix} \), and \( F_N Y = \begin{pmatrix} \tilde{F}_1 & \tilde{F}_3 \\ \tilde{F}_2 & \tilde{F}_4 \end{pmatrix} \). Direct computations yield
\[ E_1 = I - AN_1 - UN_2, \quad E_2 = V^* N_1 - N_2, \quad E_3 = -AN_3 - UN_4, \quad E_4 = I + V^* N_3 - N_4, \quad (3.3a) \]
\[ \tilde{E}_1 = I - (A + U V^*) N_1, \quad \tilde{E}_2 = E_2, \quad \tilde{E}_3 = -U - (A + U V^*) N_3, \quad \tilde{E}_4 = E_4. \quad (3.3b) \]
\[ F_1 = I - N_1 A + N_4 V^*, \quad F_2 = -N_2 A + N_4 V^*, \quad F_3 = -N_1 U - N_3, \quad F_4 = I - N_2 U - N_4. \quad (3.3c) \]
\[ \tilde{F}_1 = I - N_1 (A + U V^*), \quad \tilde{F}_2 = V^* - N_2 (A + U V^*), \quad \tilde{F}_3 = F_3, \quad \tilde{F}_4 = F_4. \quad (3.3d) \]

(ii) We claim that the following equalities hold:
\[ E_1 - \tilde{E}_1 = U E_{S_A} V^* A^\dagger \Phi, \quad E_3 - \tilde{E}_3 = U E_{S_A} (I + HH^* E_{S_A})^{-1}, \quad (3.4a) \]
\[ F_1 - \tilde{F}_1 = \Psi A^\dagger U F_{S_A} V^*, \quad F_2 - \tilde{F}_2 = -(I + F_{S_A} K^* K)^{-1} F_{S_A} V^*. \quad (3.4b) \]

In fact,
\[ E_1 - \tilde{E}_1 = U (V^* N_1 - N_2) \]
\[ = U [V^* \Psi (A^\dagger + K S_A^\dagger H) - F_{S_A} K^* \Psi (A^\dagger + K S_A^\dagger H) + S_A^\dagger H] \Phi \]
\[ = U [V^* \Psi A^\dagger + V^* \Psi K S_A^\dagger H - F_{S_A} K^* \Psi A^\dagger - F_{S_A} K^* \Psi K S_A^\dagger H + S_A^\dagger H] \Phi \]
\[ = U [V^* \Psi A^\dagger + V^* \Psi K S_A^\dagger H - F_{S_A} K^* \Psi A^\dagger + (I - F_{S_A} K^* \Psi K) S_A^\dagger H] \Phi \]
\[ = U [V^* \Psi A^\dagger + V^* \Psi K S_A^\dagger H - F_{S_A} K^* \Psi A^\dagger + (I + F_{S_A} K^* K)^{-1} S_A^\dagger H] \Phi, \quad (3.5) \]

where we used the fact that \( I - F_{S_A} K^* \Psi K = (I + F_{S_A} K^* K)^{-1} \). By formula (1.1), we have
\[ \Psi = (I + K F_{S_A} K^*)^{-1} = I - K (I + F_{S_A} K^* K)^{-1} F_{S_A} K^*, \]
which yields
\[ V^* \Psi A^\dagger = V^* A^\dagger - V^* K (I + F_{S_A} K^* K)^{-1} F_{S_A} K^* A^\dagger, \quad (3.6) \]
\[ V^* \Psi K S_A^\dagger H = V^* K S_A^\dagger H - V^* K (I + F_{S_A} K^* K)^{-1} F_{S_A} K^* K S_A^\dagger H, \quad (3.7) \]
\[ F_{S_A} K^* \Psi A^\dagger = F_{S_A} K^* A^\dagger - F_{S_A} K^* K (I + F_{S_A} K^* K)^{-1} F_{S_A} K^* A^\dagger. \quad (3.8) \]
Note that $V^*K = S_A - I$. By substituting $V^*K = S_A - I$ into (3.6) and (3.7), we obtain

$$V^*\Psi A^\dagger = V^*A^\dagger - (S_A - I)(I + F_{SA}K^*K)^{-1}F_{SA}K^*A^\dagger,$$

$$V^*\Psi K S_A^\dagger H = (S_A - I)S_A^\dagger H - (S_A - I)(I + F_{SA}K^*K)^{-1}F_{SA}K^*K S_A^\dagger H.$$ 

Due to $S_A F_{SA} = S_A(I - S_A^\dagger S_A) = 0$, it follows that

$$-(S_A - I)(I + F_{SA}K^*K)^{-1}F_{SA} = (I + F_{SA}K^*K)^{-1}F_{SA}.$$

Hence,

$$V^*\Psi A^\dagger = V^*A^\dagger + (I + F_{SA}K^*K)^{-1}F_{SA}K^*A^\dagger, \quad (3.9)$$

$$V^*\Psi K S_A^\dagger H = (S_A - I)S_A^\dagger H + (I + F_{SA}K^*K)^{-1}F_{SA}K^*K S_A^\dagger H. \quad (3.10)$$

From (3.8) and (3.9), we have

$$V^*\Psi A^\dagger - F_{SA}K^*\Psi A^\dagger = V^*A^\dagger - F_{SA}K^*A^\dagger + F_{SA}K^*A^\dagger = V^*A^\dagger. \quad (3.11)$$

From (3.10), we have

$$V^*\Psi K S_A^\dagger H + (I + F_{SA}K^*K)^{-1}S_A^\dagger H = (S_A - I)S_A^\dagger H + S_A^\dagger H = S_A S_A^\dagger H. \quad (3.12)$$

Inserting (3.11) and (3.12) into (3.5) gives

$$E_1 - \tilde{E}_1 = U(V^*A^\dagger + S_A S_A^\dagger H)\Phi = U(V^*A^\dagger - S_A S_A^\dagger V^*A^\dagger)\Phi = U E_{SA} V^*A^\dagger \Phi.$$ 

We next verify the second equality in (3.4a). On account of (1.1), (3.1c), (3.1d), (3.2a), and (3.2b), we obtain

$$N_4 = S_A^\dagger - S_A^\dagger H \Phi H^*E_{SA} - F_{SA}K^*\Psi K S_A^\dagger + F_{SA}K^*\Sigma H^*E_{SA}$$

$$= S_A^\dagger(I + HH^*E_{SA})^{-1} - F_{SA}K^*\Psi K S_A^\dagger + F_{SA}K^*\Psi (A^\dagger + KS_A^\dagger H)\Phi H^*E_{SA}$$

$$= S_A^\dagger(I + HH^*E_{SA})^{-1} + F_{SA}K^*\Psi A^\dagger \Phi H^*E_{SA} - F_{SA}K^*\Psi K S_A^\dagger(I + HH^*E_{SA})^{-1}$$

$$= (I + F_{SA}K^*K)^{-1}S_A^\dagger(I + HH^*E_{SA})^{-1} + F_{SA}K^*\Psi A^\dagger \Phi H^*E_{SA}, \quad (3.13)$$

$$V^*N_3 = V^*\Sigma H^*E_{SA} - V^*\Psi K S_A^\dagger$$

$$= V^*\Psi(A^\dagger + KS_A^\dagger H)\Phi H^*E_{SA} - V^*\Psi K S_A^\dagger$$

$$= V^*\Psi A^\dagger \Phi H^*E_{SA} - V^*\Psi K S_A^\dagger(I + HH^*E_{SA})^{-1}, \quad (3.14)$$

$$\Phi H^*E_{SA} = (I + H^*E_{SA}H)^{-1}H^*E_{SA}$$

$$= H^*E_{SA} - H^*E_{SA}(I + HH^*E_{SA})^{-1}H^*E_{SA}$$

$$= H^*E_{SA}(I + HH^*E_{SA})^{-1}. \quad (3.15)$$

In view of (3.3a), (3.3b), (3.13), (3.14), and (3.15), we derive

$$E_3 - \tilde{E}_3 = U(I - N_4 + V^*N_3) = U\tilde{E}_3(I + HH^*E_{SA})^{-1}, \quad (3.16)$$

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where
\[ \tilde{E}_3 = I + HH^*E_{SA} - (I + FS_AK^*K)^{-1}S_A^\dagger + (V^*\Psi - FS_AK^*\Psi)A^\dagger H^*E_{SA} - V^*\Psi KS_A^\dagger. \]

Using \( \Psi = I - K(I + FS_AK^*K)^{-1}FS_AK^* \), \( V^*K = S_A - I \), and \( S_AFSA = 0 \), we obtain
\[ V^*\Psi = V^* - (S_A - I)(I + FS_AK^*K)^{-1}FS_AK^* = V^* + (I + FS_AK^*K)^{-1}FS_AK^*, \]
\[ FS_AK^*\Psi = FS_AK^* - FS_AK^*(I + FS_AK^*K)^{-1}FS_AK^* = (I + FS_AK^*K)^{-1}FS_AK^*, \]
\[ V^*\Psi KS_A^\dagger = (S_A - I)S_A^\dagger + (I + FS_AK^*K)^{-1}FS_AK^*KS_A^\dagger = S_AS_A^\dagger - (I + FS_AK^*K)^{-1}S_A^\dagger. \]

Hence,
\[ \tilde{E}_3 = I + HH^*E_{SA} - (I + FS_AK^*K)^{-1}S_A^\dagger - HH^*E_{SA} - S_AS_A^\dagger + (I + FS_AK^*K)^{-1}S_A^\dagger = E_{SA}. \]  
(3.17)

By plugging (3.17) back into (3.16), we get that \( E_3 - \tilde{E}_3 = UE_{SA}(I + HH^*E_{SA})^{-1} \).

Analogously, we can show that the equalities in (3.4b) also hold (note that \( HU = I - S_A \) and \( E_{SA}S_A = 0 \)).

(iii) If \( XE_N = E_N \), then \( UE_{SA} = 0 \), or equivalently, \( \mathcal{R}(U^*) \subseteq \mathcal{R}(S_A) \). Conversely, if \( \mathcal{R}(U^*) \subseteq \mathcal{R}(S_A) \), then we conclude that \( XE_N = E_N \) due to (3.4a). Similarly, we can verify that \( F_N Y = F_N \) if and only if \( \mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*) \). This completes the proof. \( \square \)

Based on Lemmas 2.1 and 3.1, we can prove the following result, which provides an explicit expression for \((A + UV^*)^\dagger\) under suitable conditions.

**Theorem 3.2.** Let \( A \in \mathbb{C}^{m \times n} \), \( U \in \mathbb{C}^{m \times r} \), and \( V \in \mathbb{C}^{n \times r} \). If \( \mathcal{R}(U) \subseteq \mathcal{R}(A) \), \( \mathcal{R}(V) \subseteq \mathcal{R}(A^*) \), \( \mathcal{R}(U^*) \subseteq \mathcal{R}(S_A) \), and \( \mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*) \), then the MP inverse of \( A + UV^* \) is given by
\[ (A + UV^*)^\dagger = (I + A^\dagger UFS_AU^*(A^\dagger)^*)^{-1}(A^\dagger - A^\dagger U S_A^\dagger V^*A^\dagger)(I + (A^\dagger)^*V E_S A V^* A^\dagger)^{-1}, \]
(3.18)

where \( S_A = I + V^*A^\dagger U \).

**Proof.** (i) We first note that
\[ \mathcal{R}(U) \subseteq \mathcal{R}(A) \iff E_A U = 0, \quad \mathcal{R}(V) \subseteq \mathcal{R}(A^*) \iff V^* F_A = 0, \]
\[ \mathcal{R}(U^*) \subseteq \mathcal{R}(S_A) \iff U E_{SA} = 0, \quad \mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*) \iff F_S V^* = 0. \]

Let \( X, N, \) and \( Y \) be defined as in Lemma 3.1, and let \( M = XNY \). The assumption of this theorem implies that \( XE_N = E_N \) and \( F_N Y = F_N \) due to Lemma 3.1. Therefore, we can apply formula (2.1) to compute \( M^\dagger \).

We now calculate some quantities involved in (2.1). Straightforward computations yield
\[ R = E_N (I - X^{-1}) = \begin{pmatrix} 0 & -E_1 U \\ 0 & -E_2 U \end{pmatrix}, \quad L = (I - Y^{-1}) F_N = \begin{pmatrix} 0 & 0 \\ V^* F_1 & V^* F_3 \end{pmatrix}. \]
Hence,

\[
I + L^* = \begin{pmatrix} I & F_1^*V \\ 0 & I + F_3^*V \end{pmatrix}, \quad I + R^* = \begin{pmatrix} I & 0 \\ -U^*E_1^* & I - U^*E_2^* \end{pmatrix},
\]

\[
(I + LL^*)^{-1} = \begin{pmatrix} I & 0 \\ 0 & (I + V^*F_1F_1^*V + V^*F_3F_3^*V)^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \tilde{F} \end{pmatrix},
\]

\[
(I + R^*R)^{-1} = \begin{pmatrix} I & 0 \\ 0 & (I + U^*E_1^*E_1U + U^*E_2^*E_2U)^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \tilde{E} \end{pmatrix},
\]

where

\[
\tilde{F} := (I + V^*F_1F_1^*V + V^*F_3F_3^*V)^{-1}, \quad \tilde{E} := (I + U^*E_1^*E_1U + U^*E_2^*E_2U)^{-1}.
\]

We then have

\[
(I + L^*)(I + LL^*)^{-1}Y^{-1} = \begin{pmatrix} L_1 & L_3 \\ L_2 & L_4 \end{pmatrix}, \quad X^{-1}(I + R^*R)^{-1}(I + R^*) = \begin{pmatrix} R_1 & R_3 \\ R_2 & R_4 \end{pmatrix},
\]

where

\[
L_1 = I - F_1^*V\tilde{F}V^*, \quad L_2 = -(I + F_3^*V)\tilde{F}V^*, \quad L_3 = F_1^*V\tilde{F}, \quad L_4 = (I + F_3^*V)\tilde{F},
\]

\[
R_1 = I - U\tilde{E}U^*E_1^*, \quad R_2 = -\tilde{E}U^*E_1^*, \quad R_3 = U\tilde{E}(I - U^*E_2^*), \quad R_4 = \tilde{E}(I - U^*E_2^*).
\]

An application of Lemma 2.1 yields

\[
M^\dagger = \begin{pmatrix} (L_1N_1 + L_3N_2)R_1 + (L_1N_3 + L_3N_4)R_2 & (L_1N_1 + L_3N_2)R_3 + (L_1N_3 + L_3N_4)R_4 \\ (L_2N_1 + L_4N_2)R_1 + (L_2N_3 + L_4N_4)R_2 & (L_2N_1 + L_4N_2)R_3 + (L_2N_3 + L_4N_4)R_4 \end{pmatrix}.
\]

Because

\[
M = \begin{pmatrix} I & -U \\ 0 & I \end{pmatrix} \begin{pmatrix} A & U \\ -V^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ V^* & I \end{pmatrix} = \begin{pmatrix} A + UV^* & 0 \\ 0 & I \end{pmatrix},
\]

we immediately have

\[
M^\dagger = \begin{pmatrix} A + UV^* & 0 \\ 0 & I \end{pmatrix}^\dagger = \begin{pmatrix} (A + UV^*)^\dagger & 0 \\ 0 & I \end{pmatrix}.
\]

Due to \((A + UV^*)^\dagger \in \mathbb{C}^{n \times m}\) and \((L_1N_1 + L_3N_2)R_1 + (L_1N_3 + L_3N_4)R_2 \in \mathbb{C}^{n \times m}\), it follows that

\[
(A + UV^*)^\dagger = (L_1N_1 + L_3N_2)R_1 + (L_1N_3 + L_3N_4)R_2.
\]

(ii) Nevertheless, the expression (3.19) is not very legible. We next devote to simplifying (3.19). From \(E_NN = 0\) and \(NF_N = 0\), we have

\[
E_1U + E_3 = 0, \quad E_2U + E_4 = 0, \quad -V^*F_1 + F_2 = 0, \quad -V^*F_3 + F_4 = 0.
\]
Since both \( E_N \) and \( F_N \) are orthogonal projectors, it follows that

\[
E_1^* = E_1, \quad E_3^* = E_2, \quad E_4^* = E_4, \quad F_1^* = F_1, \quad F_2^* = F_3, \quad F_4^* = F_4,
\]

\[
E_2E_3 + E_4^2 = E_4, \quad F_2F_3 + F_4^2 = F_4.
\]

Hence,

\[
\hat{F} = (I + V^*F_1F_1^*V + V^*F_3F_3^*V)^{-1} = (I + F_2F_2^* + F_4F_4^*)^{-1}
\]

\[
= (I + F_2F_3 + F_4^2)^{-1} = (I + F_4)^{-1},
\]

\[
\hat{E} = (I + U^*E_1E_1U + U^*E_2E_2U)^{-1} = (I + E_3E_3 + E_4E_4)^{-1}
\]

\[
= (I + E_2E_3 + E_4^2)^{-1} = (I + E_4)^{-1}.
\]

By (3.3a), (3.16), and (3.17), we have

\[
E_4 = I + V^*N_3 - N_4 = E_S(I + HH^*E_S)^{-1},
\]

which gives

\[
\hat{E} = (I + HH^*E_S)(I + E_S + HH^*E_S)^{-1}
\]

\[
= I - E_S(I + E_S + HH^*E_S)^{-1}
\]

\[
= I - (I + E_S + HH^*)^{-1}E_S.
\]

By (3.3c), (3.3d), and (3.4b), we have

\[
F_2 - \hat{F}_2 = -(I - N_4 - N_2U)V^* = -(I + F_SA K^*K)^{-1}F_SA V^*,
\]

which yields

\[
F_4 = I - N_2U - N_4 = (I + F_SA K^*K)^{-1}F_SA.
\]

Thus,

\[
\hat{F} = (I + F_SA + F_SA K^*K)^{-1}(I + F_SA K^*K)
\]

\[
= I - (I + F_SA + F_SA K^*K)^{-1}F_SA
\]

\[
= I - F_SA(I + F_SA + K^*KF_SA)^{-1}.
\]

Owing to \( E_SA U^* = 0 \) and \( V F_SA = 0 \), by (3.20) and (3.21), we have \( \hat{E} U^* = U^* \) and \( V \hat{F} = V \). Hence,

\[
L_1 = I - F_1^* V \hat{F} V^* = I - F_1 V V^* = I - F_2^* V^* = I - F_3 V^*, \quad (3.22a)
\]

\[
L_3 = F_1^* V \hat{F} = F_1 V = F_2^* = F_3, \quad (3.22b)
\]

\[
R_1 = I - U \hat{E} U^* E_1^* = I - U U^* E_1 = I + U E_3 = I + U E_2, \quad (3.22c)
\]

\[
R_2 = -\hat{E} U^* E_1^* = -U^* E_1 = E_3^* = E_2. \quad (3.22d)
\]
due to $F_1^* = F_1$, $F_2^* = F_3$, $E_1^* = E_1$, $E_2^* = E_2$, $V^*F_1 = F_2$, and $E_1U = -E_3$. Using $N^\dagger E_N = 0$ and $F_N N^\dagger = 0$, we obtain

$$N_1 E_1 + N_3 E_2 = 0, \quad F_1 N_1 + F_3 N_2 = 0, \quad F_1 N_3 + F_3 N_4 = 0. \quad (3.23)$$

By substituting (3.22a)–(3.22d) into (3.19), we obtain from (3.23) that

$$(A + UV^*)^\dagger = [(I - F_3 V^*) N_1 + F_3 N_2] (I + UE_2) + [(I - F_3 V^*) N_3 + F_3 N_4] E_2$$

$$= [(I - F_3 V^*) N_1 - F_3 N_1] (I + UE_2) + [(I - F_3 V^*) N_3 - F_3 N_3] E_2$$

$$= (I - F_3 V^* - F_1) N_1 (I + UE_2) + (I - F_3 V^* - F_1) N_3 E_2$$

$$= (I - F_3 V^* - F_1) [N_1 (I + UE_2) - N_1 E_1]$$

$$= (I - F_3 V^* - F_1) N_1 (I + UE_2 - E_1)$$

$$= (N_1 UV^* + N_1 A) N_1 (UV^* N_1 + A N_1)$$

$$= N_1 (A + UV^*) N_1 (A + UV^*) N_1, \quad (3.24)$$

where

$$N_1 = (I + A^\dagger U S_A U^* (A^\dagger)^*)^{-1} (A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger) (I + (A^\dagger)^* V E S_A V^* A^\dagger)^{-1}.$$  

(iii) Subsequently, we further simplify the expression (3.24). Because $V^* F_A = 0$ and $E S_A S_A = 0$, it follows that

$$(I + (A^\dagger)^* V E S_A V^* A^\dagger) (A + UV^*) = A + UV^* + (A^\dagger)^* V E S_A V^* A^\dagger A + (A^\dagger)^* V E S_A V^* A^\dagger UV^*$$

$$= A + UV^* - (A^\dagger)^* V E S_A V^* F_A + (A^\dagger)^* V E S_A S_A V^*$$

$$= A + UV^*.$$

Similarly, by $E_A U = 0$ and $S_A F_S = 0$, we can derive

$$(A + UV^*) (I + A^\dagger U S_A U^* (A^\dagger)^*) = A + UV^*.$$

Therefore,

$$(I + (A^\dagger)^* V E S_A V^* A^\dagger)^{-1} (A + UV^*) = A + UV^*, \quad (3.25)$$

$$(A + UV^*) (I + A^\dagger U S_A U^* (A^\dagger)^*)^{-1} = A + UV^*. \quad (3.26)$$

Furthermore, by $E_A U = 0$ and $U E S_A = 0$, we have

$$(A + UV^*) (A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger) = AA^\dagger + UV^* A^\dagger - AA^\dagger U S_A^\dagger V^* A^\dagger - UV^* A^\dagger U S_A^\dagger V^* A^\dagger$$

$$= AA^\dagger + UV^* A^\dagger + E_A U S_A^\dagger V^* A^\dagger - U S_A S_A^\dagger V^* A^\dagger$$

$$= AA^\dagger + U E S_A V^* A^\dagger + E_A U S_A^\dagger V^* A^\dagger$$

$$= AA^\dagger. \quad (3.27)$$
Analogously, using $V^*F_A = 0$ and $F_A V^* = 0$, we obtain

\[(A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger)(A + UV^*) = A^\dagger A. \tag{3.28}\]

Consequently, using (3.24)–(3.28), we arrive at

\[(A + UV^*)^\dagger = (I + A^\dagger U F_S A^* U^*)^{-1}(A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger)(I + (A^\dagger)^* V E_S A^* U^*)^{-1}. \]

This completes the proof.

\[\square\]

**Remark 3.3.** Due to $A^\dagger U F_S A^* U^*$ and $(A^\dagger)^* V E_S A^* U^*$ are (Hermitian) positive semidefinite, it follows that $I + A^\dagger U F_S A^* U^*$ and $I + (A^\dagger)^* V E_S A^* U^*$ are (Hermitian) positive definite and hence they are nonsingular. Moreover, the factors $(I + A^\dagger U F_S A^* U^*)^{-1}$ and $(I + (A^\dagger)^* V E_S A^* U^*)^{-1}$ can be computed via the SMW formula (1.1), that is,

\[
\begin{align*}
(I + A^\dagger U F_S A^* U^*)^{-1} & = I - A^\dagger U (I + F_S A^* A^U)^{-1} F_S A^* U^*, \\
(I + (A^\dagger)^* V E_S A^* U^*)^{-1} & = I - (A^\dagger)^* V (I + E_S A^* A^*(A^\dagger)^* V)^{-1} E_S A^* U^*.
\end{align*}
\]

For a given matrix $A \in \mathbb{C}^{m \times n}$, we assume that $A^\dagger$ has been precomputed. For a varied perturbation $UV^*$ (here $U \in \mathbb{C}^{m \times r}$ and $V \in \mathbb{C}^{n \times r}$), we can compute $(A + UV^*)^\dagger$ via the singular value decomposition of $A + UV^*$. However, this approach is expensive, especially when $m$ and $n$ are large. Fortunately, if $r$ is much smaller than $m$ and $n$, $S_A^\dagger = (I + V^* A^\dagger U)^\dagger$ is much easier to compute than $(A + UV^*)^\dagger$. Therefore, the formula (3.18) provides an effective method to compute $(A + UV^*)^\dagger$, provided that the conditions of Theorem 3.2 are satisfied.

On the basis of Theorem 3.2, we can derive some simple expressions for $(A + UV^*)^\dagger$ under several special conditions.

**Corollary 3.4.** If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(U^*) \subseteq \mathcal{R}(S_A) \cap \mathcal{R}(S_A^*)$, and $\mathcal{R}(V^*) \subseteq \mathcal{R}(S_A) \cap \mathcal{R}(S_A^*)$, then

\[(A + UV^*)^\dagger = A^\dagger - A^\dagger U (I + V^* A^\dagger U)^\dagger V^* A^\dagger. \]

**Corollary 3.5.** If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, and $S_A = I + V^* A^\dagger U$ is nonsingular, then

\[(A + UV^*)^\dagger = A^\dagger - A^\dagger U (I + V^* A^\dagger U)^{-1} V^* A^\dagger. \]

**Corollary 3.6.** Let $A \in \mathbb{C}^{n \times n}$, $U \in \mathbb{C}^{n \times r}$, and $V \in \mathbb{C}^{n \times r}$. If $A$ and $S_A = I + V^* A^{-1} U$ are nonsingular, then

\[(A + UV^*)^\dagger = A^{-1} - A^{-1} U (I + V^* A^{-1} U)^{-1} V^* A^{-1}. \]

Indeed, $A + UV^*$ is also nonsingular and $(A + UV^*)^{-1} = A^{-1} - A^{-1} U (I + V^* A^{-1} U)^{-1} V^* A^{-1}$, which is exactly the SMW formula (1.1).
Under the conditions $\mathcal{R}(U) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, we next give a necessary and sufficient condition to validate (3.18).

**Theorem 3.7.** Let $A \in \mathbb{C}^{m \times n}$, $U \in \mathbb{C}^{m \times r}$, and $V \in \mathbb{C}^{n \times r}$. If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, then (3.18) holds if and only if both $UE_{S_A}V^*A^\dagger$ and $A^\dagger UF_{S_A}V^*$ are Hermitian and $A^\dagger UF_{S_A}E_{S_A}V^*A^\dagger = 0$, where $S_A = I + V^*A^\dagger U$.

**Proof.** According to the derivations of (3.25) and (3.26), by $\mathcal{R}(U) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, we obtain that (3.25) and (3.26) hold.

(i) Let $\tilde{A} = (I + A^\dagger UF_{S_A}U^*(A^\dagger)^*)^{-1}(A^\dagger - A^\dagger US_A^V*A^\dagger)(I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1}$. Because $E_AU = 0$, $V^*F_A = 0$, and $E_{S_A}S_A = 0$, it follows that

\[
(A + UV^*)\tilde{A}(A + UV^*) = (A + UV^*)(A^\dagger - A^\dagger US_A^V*A^\dagger)(A + UV^*)
\]
\[
= (AA^\dagger + UE_{S_A}V^*A^\dagger)(A + UV^*)
\]
\[
= A + AA^\dagger UV^* + UE_{S_A}V^*A^\dagger A + UE_{S_A}V^*A^\dagger UV^*
\]
\[
= A + UV^* + UE_{S_A}V^* + UE_{S_A}(S_A - I)V^*
\]
\[
= A + UV^*,
\]
where we have used the equalities (3.25) and (3.26).

(ii) By (3.25) and (3.26), we can easily get that

\[
\tilde{A}(A + UV^*)\tilde{A} = (I + A^\dagger UF_{S_A}U^*(A^\dagger)^*)^{-1}\tilde{A}(I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1},
\]

where

\[
\tilde{A} = (A^\dagger - A^\dagger US_A^V*A^\dagger)(A + UV^*)(A^\dagger - A^\dagger US_A^V*A^\dagger).
\]

Due to $E_AU = 0$ and $S_A^\dagger E_{S_A} = 0$, it follows that

\[
\tilde{A} = (A^\dagger - A^\dagger US_A^V*A^\dagger)(AA^\dagger + UE_{S_A}V^*A^\dagger)
\]
\[
= A^\dagger - A^\dagger US_A^V*A^\dagger + (A^\dagger - A^\dagger US_A^V*A^\dagger)UE_{S_A}V^*A^\dagger
\]
\[
= A^\dagger - A^\dagger US_A^V*A^\dagger + (A^\dagger UE_{S_A}V^*A^\dagger - A^\dagger US_A^V(S_A - I)E_{S_A}V^*A^\dagger)
\]
\[
= A^\dagger - A^\dagger US_A^V*A^\dagger + A^\dagger UF_{S_A}E_{S_A}V^*A^\dagger.
\]

Hence, $\tilde{A}(A + UV^*)\tilde{A} = \tilde{A}$ if and only if $A^\dagger UF_{S_A}E_{S_A}V^*A^\dagger = 0$.

(iii) Using (3.25), (3.26), $E_AU = 0$, and $V^*F_A = 0$, we obtain

\[
(A + UV^*)\tilde{A} = (AA^\dagger + UE_{S_A}V^*A^\dagger)(I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1},
\]
\[
\tilde{A}(A + UV^*) = (I + A^\dagger UF_{S_A}U^*(A^\dagger)^*)^{-1}(A^\dagger A + A^\dagger UF_{S_A}V^*).
\]

Then we have

\[
((A + UV^*)\tilde{A})^* = (I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1}(AA^\dagger + (UE_{S_A}V^*A^\dagger)^*),
\]
\[
(\tilde{A}(A + UV^*))^* = (A^\dagger A + (A^\dagger UF_{S_A}V^*))^*(I + A^\dagger UF_{S_A}U^*(A^\dagger)^*)^{-1}.
\]
Note that

\[
(I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1}AA^\dagger = AA^\dagger (I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1},
\]

\[
(I + A^\dagger UFS_A U^* (A^\dagger)^*)^{-1} A^\dagger A = A^\dagger A(I + A^\dagger UFS_A U^* (A^\dagger)^*)^{-1}.
\]

Hence, \((A + UV^*)^\dagger \hat{A}\) is Hermitian if and only if \(VE_{S_A}V^*A^\dagger (I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)^{-1}\) is Hermitian, and \(\hat{A}(A + UV^*)\) is Hermitian if and only if \((I + A^\dagger UFS_A U^* (A^\dagger)^*)^{-1} A^\dagger UFS_A V^*\) is Hermitian, that is,

\[
(I + (A^\dagger)^*VE_{S_A}V^*A^\dagger)VE_{S_A}V^*A^\dagger = (VE_{S_A}V^*A^\dagger)^* (I + (A^\dagger)^*VE_{S_A}V^*A^\dagger),
\]

\[
A^\dagger UFS_A V^* (I + A^\dagger UFS_A U^* (A^\dagger)^*) = (I + A^\dagger UFS_A U^* (A^\dagger)^*) (A^\dagger UFS_A V^*).
\]

From (3.30) and (3.31), we obtain that \((A + UV^*)^\dagger \hat{A}\) is Hermitian if and only if \(VE_{S_A}V^*A^\dagger\) is Hermitian, and \(\hat{A}(A + UV^*)\) is Hermitian if and only if \(A^\dagger UFS_A V^*\) is Hermitian.

According to the definition of the MP inverse of a matrix, it follows that the statement in Theorem 3.7 holds.

**Remark 3.8.** Under the conditions \(\mathcal{R}(U) \subseteq \mathcal{R}(A)\) and \(\mathcal{R}(V) \subseteq \mathcal{R}(A^\dagger)\), by comparing Theorem 3.2 with Theorem 3.7, we can readily observe that \(\mathcal{R}(U^\dagger) \subseteq \mathcal{R}(S_A)\) and \(\mathcal{R}(V^\dagger) \subseteq \mathcal{R}(S_A^\ast)\) are only sufficient conditions to validate (3.18). In addition, from the proof of (3.29), we find that

\[
(I + A^\dagger UFS_A U^* (A^\dagger)^*)^{-1} (A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger) (I + (A^\dagger)^* V E_{S_A} V^* A^\dagger)^{-1}
\]

is a \(\{1\}\)-inverse of \(A + UV^*\), provided that \(\mathcal{R}(U) \subseteq \mathcal{R}(A)\) and \(\mathcal{R}(V) \subseteq \mathcal{R}(A^\dagger)\).

### 4. Extensions

In this section, we devote to extending the results stated in Theorems 3.2 and 3.7 to the bounded linear operators case.

Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be two Hilbert spaces over the same field. The set of all bounded linear operators from \(\mathcal{H}_1\) into \(\mathcal{H}_2\) is denoted by \(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)\). For any \(T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)\), let \(T^*, \mathcal{R}(T),\) and \(\mathcal{N}(T)\) denote the adjoint, the range, and the null space of \(T\), respectively. The Moore–Penrose inverse of \(T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)\) is denoted by \(T^\dagger \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)\) (if it exists), which is defined as the unique operator \(Z \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)\) satisfying the four equations

\[
TZT = T, \quad ZTZ = Z, \quad (TZ)^* = TZ, \quad (ZT)^* = ZT.
\]

Unlike the matrices case, \(T^\dagger\) is not always existent. Indeed, an operator \(T\) has the MP inverse if and only if \(\mathcal{R}(T)\) is closed [17].

In view of Theorems 3.2 and 3.7, we can establish the following more general results whose detailed proofs are omitted, because we can directly check that the operator

\[
(I + A^\dagger UFS_A U^* (A^\dagger)^*)^{-1} (A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger) (I + (A^\dagger)^* V E_{S_A} V^* A^\dagger)^{-1}
\]
satisfies the four equations in (4.1).

**Theorem 4.1.** Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $U \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$, and $V \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$, where $\mathcal{H}_i$ $(i = 1, 2, 3)$ are Hilbert spaces over the same field. Assume that $\mathcal{R}(A)$ and $\mathcal{R}(S_A)$ are closed. If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(U^*) \subseteq \mathcal{R}(S_A)$, and $\mathcal{R}(V^*) \subseteq \mathcal{R}(S_A^*)$, then $\mathcal{R}(A + UV^*)$ is closed and

$$
(A + UV^*)^\dagger = (I + A^\dagger UF_{S_A}U^* (A^\dagger)^*)^{-1} (A^\dagger - A^\dagger U S_A^\dagger V^* A^\dagger) (I + (A^\dagger)^* V E_{S_A} V^* A^\dagger)^{-1},
$$

(4.2)

where $S_A = I + V^* A^\dagger U$, $E_{S_A} = I - S_A S_A^\dagger$, and $F_{S_A} = I - S_A^\dagger S_A$.

**Theorem 4.2.** Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $U \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$, and $V \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$, where $\mathcal{H}_i$ $(i = 1, 2, 3)$ are Hilbert spaces over the same field. Assume that $\mathcal{R}(A)$, $\mathcal{R}(S_A)$, and $\mathcal{R}(A + UV^*)$ are closed. If $\mathcal{R}(U) \subseteq \mathcal{R}(A)$, $\mathcal{R}(V) \subseteq \mathcal{R}(A^*)$, then (4.2) holds if and only if both $UE_{S_A} V^* A^\dagger$ and $A^\dagger UF_{S_A} V^*$ are self-adjoint and $A^\dagger UF_{S_A} E_{S_A} V^* A^\dagger = 0$, where $S_A = I + V^* A^\dagger U$.

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