New exact and numerical solutions for the KdV system arising in physical applications

M. B. Almatrafi\textsuperscript{a} \& A. R. Alharbi\textsuperscript{a} and Mahmoud A. E. Abdelrahmana,b
\textsuperscript{a}Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia; \textsuperscript{b}Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

\textbf{ABSTRACT}

The Kortweg–de Vries (KdV) equation is more appropriate to simulate some natural phenomena and gives more accurate results for some physical systems such as the movement of water waves. In this work, novel analytical traveling wave solutions for a nonlinear KdV system are explored using the sech method. The exact solutions are presented in the form of hyperbolic functions. These solutions show the propagation of wave waves on the surface. We also implement the numerical adaptive moving technique (MMPDEs) to construct the relevant system’s numerical solutions. A detailed comparison between the numerical and analytical solutions is also presented to confirm the accuracy of the numerical technique. We present some new 2D and 3D figures to illustrate the behaviour of the exact and numerical solutions. The obtained numerical solutions verify the accuracy of the considered methods qualitatively and quantitatively. The stability of the obtained solutions is investigated using the Hamiltonian system. The achieved results can be applied for some new observations in the ocean, coastal water, macroscopic phenomena, and processes. The proposed techniques are potent tools to solve many other non-linear partial differential equations in applied mathematics.

\textbf{1. Introduction}

The Kortweg–de Vries (KdV) equation is generally recognized as a model for representing weakly nonlinear long waves in various branches of fluid mechanics, physics and applied sciences. This equation depicts the development of waves under the competing influences of weak nonlinearity and weak dispersion. Moreover, this model is encountered in many unrelated phenomena such as aerodynamics, continuum mechanics, and fluids as a model for solitons, shock wave formation, turbulence, and mass transport. These applications are widely formed from some common physical models which are described by the KdV system by considering a characteristic limit of the physical model. Ultimately, integrable equations have a significant and considerable effect in hydrodynamics and nonlinear optics since they emerge in several vital approximations to the underlying problems. The nonlinear Schrödinger equation is used to investigate the rogue waves in deep water, while the KdV model depicts the behaviour in shallow water. Furthermore, the KdV system deals with the ion-acoustic solutions in plasma physics (Gao & Tian, 2001).

A third-order KdV equation (Dubard, Gaillard, Klein, & Matveev, 2010) reads as

\[ u_t + 6uu_x + u_{xxx} = 0. \] (1)

Here, \( u_t \) represents the time evolution, \( uu_x \) is nonlinear and \( u_{xxx} \) is a dispersion. Equation (1) is specifically considered as an extremely exquisite model of a nonlinear solvable equation whose solutions can be exactly and accurately determined. Furthermore, the analysis of the KdV equation has mainly attracted a massive number of active scientists since it is a tremendously interesting topic.

This work aims to highlight the solutions of the KdV system shown as follows:

\[ U_t + \alpha U_{xxx} + 3 \alpha (U^2)_x - \beta (G^2)_x = 0, \]
\[ G_t + G_{xxx} + 3 U G_x = 0, \] (2)

subject to

\[ U(a, t) = f_l(a, t), \quad G(a, t) = g_l(a, t), \]
\[ U(b, t) = f_h(b, t), \quad G(b, t) = g_h(b, t), \quad \forall t \geq 0. \] (3)

Here, the parameters \( \alpha \) and \( \beta \) are nonzero, \( x \in [a, b], \ f_l, f_h, g_l \) and \( g_h \) are functions of \( t \). Hirota and Satsuma (Hirota & Satsuma, 1981) deduced system...
To characterize the iterations of water waves with various dispersion relations.

Several researchers have extensively concentrated on studying the exact solutions of system (2) by applying numerous approaches. For instance, Cao, Yan, and Zhang (2002) used an effective and explicit function transform technique depended on the concept of the homogeneous balance approach to solve system (2). Zhang, Wang, Wang, and Fang (2006) applied the F-expansion method to deal with system (2) and derived the periodic wave solutions based on the Jacobi elliptic functions. Ganji and Rafei (2006) utilized the homotopy perturbation approach for investigating the exact solutions of system (2). For more details about other methods used to solve system (2), we refer to (Abdel-Aty et al., 2020a; Abdel-Aty et al., 2020b; Assas, 2008; Ali, Khan, Khater, Mousa, & Attia, 2021; Chu, Khater, & Hamed, 2021; Gokdogan, Yildirim, & Merdan, 2012; Jaradat, Syam, & Alquran, 2017; Khater, Attia, Park, & Lu, 2020; Khater, Attia, Alodhaibi, & Lu, 2020; Khater, Park, & Lu, 2020; Khater, Lu, Attia, & Inc, 2019; Qian, Attia, Qiu, & Khater, 2019; Yue et al., 2020; Yue, Khater, Attia, & Lu, 2020; Yue, Khater, Inc, Attia, & Lu, 2020).

Investigating both the analytical and numerical results for a given nonlinear partial differential equations (NLPDEs) performs a remarkable role in the explanation of complex phenomena in biology, economy, engineering, mathematical, signal processing, ocean engineering, optics, fluid mechanics, plasma physics, and chemical physics (Alharbi & Almatrafi, 2020a; Alharbi & Almatrafi, 2020b; Alharbi, Almatrafi, & Abdelrahman, 2020; Seadawy, Iqbal, & Lu, 2019). Thus, various vital approaches for determining the solutions of PDEs have been vastly suggested and presented. Among them, we mention the tanh–sech technique (Wazwaz, 2004), the Jacobi elliptic function approach (Dai & Zhang, 2006), the exp-function approach (Aminikhad, Moosaei, & Hajipour, 2009), the sine–cosine approach (Wazwaz, 2005), the homogeneous balance technique (Fan & Zhang, 1998), the F-expansion approach (Zhang, Wang, Wang, & Fang, 2006), the extended tanh-method (Yang, Deng, & Wei, 2015), the Riccati–Bernoulli sub-ODE technique (Alharbi & Almatrafi, 2020c; Huang & Russell, 2011), the r-adaptive moving mesh strategy (Ahmad & Khan, 2019; Budd, Huang, & Russell, 2009), the variational Iteration algorithm-I (Ahmad, Khan, & Cesarano, 2019; Ahmad, Seadawy, & Khan, 2020), the variational iteration algorithm-II (Ahmad, Seadawy, Khan, & Thounthong, 2020; Bazighifan, Ahmad, & Yao, 2020), the Riccatti transformation method (Inc et al., 2020) and the meshless techniques (Ganjii & Abbodlahzadeh, 2008).

The principal objective of this work is to apply the sech technique (Seadawy, Lu, & Yue, 2017) and the MMPDEs method on the considered problem to extract its exact and numerical solutions, respectively. The stability of the obtained solutions is examined using the Hamiltonian system (Russell, Huang, & Ren, 1994). The obtained numerical solutions are successfully compared with the analytical solution. We also present some 2D and 3D figures to show the behaviour of the obtained solutions. The obtained new solutions may be applicable to verify and confirm the wave observations and wave progress in plasma physics. Indeed, the proposed two methods can be also applied for further models arising in applied sciences.

This paper is organized as follows. In Section 2, we discuss the exact solution of system (2). Section 3 presents the numerical solution of system (2) and shows some relevant figures. Section 4 discusses the most important results that we achieve in this work. In Section 5, we conclude this article.
2. Exact solution

This part is devoted to discover the exact solutions of system (2). Applying the transformation

\[ U(x, t) = u(\xi), \quad G(x, t) = v(\xi), \quad \xi = x + wt, \] (4)

where \( w \) is the wave speed, system (2) is converted into the following ODEs:

\[ \begin{align*}
    w u_{\xi} + \alpha u_{\xi\xi} + 3 \alpha (u^2)_\xi - \beta (v^2)_\xi &= 0, \\
    w v_{\xi} + v_{\xi\xi} + 3 u v_\xi &= 0.
\end{align*} \] (5)

We now integrate each equation in system (5) once with respect to \( \xi \). Then, we equate the constant of integrals to zero to end up with

\[ \begin{align*}
    w u + \alpha u_{\xi\xi} + 3 \alpha u^2 - \beta v^2 &= 0, \\
    w v + v_{\xi\xi} + 3 u v_\xi &= 0.
\end{align*} \] (6)

Balancing \( u_{\xi\xi} \) with \( u^2 \) appearing in the first equation of system (6) gives \( N = 2 \). Similarly, we balance \( v_{\xi\xi} \) with \( u v_\xi \) in the second equation of system (6) to have \( M = 2 \). As a result, the solutions take the following forms:

\[ \begin{align*}
    u(\xi) &= a_0 + a_1 \text{sech} ^2 \xi + a_2 \text{sech} \xi, \\
    v(\xi) &= b_0 + b_1 \text{sech} ^2 \xi + b_2 \text{sech} \xi.
\end{align*} \] (7) (8)

Substituting Eqs. (7) and (8) into system (6) and equating the whole coefficients of \( \text{sech} ^2 \xi \) to zero give nonlinear algebraic equations. Obtaining the exact solutions of these equations leads to different cases explained as follows.

- **Case 1**

  \[ \begin{align*}
    a_0 &= -\frac{4+w}{3}, \quad a_1 = 0, \quad a_2 = 4, \\
    b_0 &= \pm \frac{2\sqrt{6}\alpha + 4\sqrt{6}\beta - 6w}{6\sqrt{6}\sqrt{\beta}}, \quad b_1 = 0, \quad b_2 = \pm \frac{2\sqrt{6}\sqrt{\alpha}}{\sqrt{\beta}}.
  \end{align*} \] (9)

Hence, the exact traveling wave solutions are given by

\[ U(x, t) = -\frac{4+w}{3} + 4 \text{sech} ^2(x + x_0 + wt), \] (10)

\[ G(x, t) = \frac{2\sqrt{6}\alpha + 4\sqrt{6}\beta - 6w}{6\sqrt{6}\sqrt{\beta}} + \frac{2\sqrt{6}\sqrt{\alpha}}{\sqrt{\beta}} \text{sech} ^2(x + x_0 + wt). \] (11)

- **Case 2**

  \[ \begin{align*}
    a_0 &= -1 + \frac{w}{3}, \quad a_1 = 0, \quad a_2 = 2, \\
    b_0 &= 0, \quad b_1 = \pm \frac{4\alpha + 4\beta + 2w}{\beta}, \quad b_2 = 0.
  \end{align*} \] (12)

Therefore, the exact solutions are shown as follows:

\[ \begin{align*}
    U(x, t) &= -\frac{1+w}{3} + 2 \text{sech} ^2(x + x_0 + wt), \\
    G(x, t) &= \sqrt{\frac{4\alpha + 4\beta + 2w}{\beta}} \text{sech}(x + x_0 + wt), \\
    w &< \frac{2\alpha}{2\alpha - 1}.
  \end{align*} \] (13) (14)

It is worth illustrating that the boundary conditions are originally initiated from the long behaviour of the exact solutions. It can be observed from Figure 2 (left) that the travelling wave solution \( u(x, t) \) approaches \(-0.5\) at the boundaries of the domain while Figure 3 (left) demonstrates that \( G(x, t) \) tends to zero at the boundaries. The parameters are fixed by \( \alpha = 1.2, \beta = 0.2, w = 0.5 \) and \( x_0 = -12 \). Thus, it can be concluded that

\[ U(x, t) \to -0.5, \quad \text{and} \quad G(x, t) \to 0, \quad \text{as} \quad x \to \pm \infty. \] (15)

In order to establish a numerical solution for a given PDE, we discretize the physical domain into subintervals. In other words, the proposed PDE is...
discretized on a uniform or non-uniform mesh. In a uniform mesh, the used domain is equally partitioned into subintervals so that $\Delta x$ is fixed for the entire domain. The resultant error is commonly reduced by using small enough $\Delta x$. Consequently, this strategy is intensive. However, when we distribute the mesh so that the region where the solution has rapidly changed takes more points and fewer points where the solution is not changed, the resultant error is diminished. Hence, the employed approach plays a principal role in decreasing the error in the results by sending more points to the regions with high errors. Another method named the mesh methods randomly distribute the nodes and send more points to the regions with high error to reduce the error occurred in the solution. These methods produce reliable, appropriate, and acceptable results. Therefore, we apply the MMPDE to find the numerical results for the considered PDEs.

3. Numerical results

In this article, we utilize a vital method called the MMPDE (Huang & Russell, 1998) to find the numerical solution of the considered problem. This approach was developed in Huang and Russell (1998) to obtain the numerical solutions of one-dimensional PDE while it was clearly described in Budd and Williams (2009) to solve multi-dimensional PDEs. The solution of the proposed PDE is determined by using novel meshes called gradient flow equations. The vital purpose of using such technique is to initiate these meshes and to reduce the resultant error in the results by sending more points to the regions with high errors. Another method named the Parabolic Monge-Ampere (PMA) (Alharbi & Naire, 2017) is employed to some equations to approximate the general solutions of multi-dimensional PDEs. The first approach is utilized here to discover the numerical solutions of system (2) which is rewritten as:

$$\begin{align*}
U_t + \alpha U_{xxx} + 3 \alpha (U^2)_x - \beta (G^2)_x &= 0, \\
G_t + G_{xxx} + 3 U G_x &= 0, \\
(x, t) &\in [a, b] \times (0, T_e),
\end{align*}$$

(16)

subject to

$$
U(x_l, t) = -0.5, \quad U(x_R, t) = -0.5,
$$

(17)

and the initial conditions

$$
U(x, 0) = -\frac{1 + w}{3} + 2 \text{sech}^2(x + x_0),
$$

(18)

$$
G(x, 0) = \sqrt{\frac{4 \alpha w + 4 \alpha + 2 w \text{sech} (x + x_0)}{\beta}},
$$

$$
w > \frac{2 \alpha}{2 \alpha - 1},
$$

(19)

where $x_l = a$, $x_R = b$ and $T_e$ is an appropriate time. For the non-uniform mesh, we utilize the coordinate transform $x = x(\eta, t) : [0, 1] \rightarrow [x_L, x_R]$, where $x \in [x_L, x_R]$ and $\eta \in [0, 1]$ present the space and computational coordinates, respectively. The discretisation of the spatial and temporal derivatives, using a chain rule, is given by

$$
\begin{align*}
U_t - U_x x_t &= -\alpha U_{xxx} - 3 \alpha (U^2)_x + \beta (G^2)_x, \\
G_t - G_x x_t &= -G_{xxx} - 3 U G_x.
\end{align*}
$$

(20)

Equation (20) can be also written as

$$
\begin{align*}
U_t - \left(\frac{U_x}{x_\eta}\right) x_t &= -\alpha \frac{1}{x_\eta} \left(\frac{1}{x_\eta} \left(\frac{U_x}{x_\eta}\right)\right) - 3 \alpha \frac{U_x^2}{x_\eta} + \beta \frac{G^2_x}{x_\eta}, \\
G_t - \left(\frac{G_x}{x_\eta}\right) x_t &= -\frac{1}{x_\eta} \left(\frac{1}{x_\eta} \left(\frac{G_x}{x_\eta}\right)\right) - 3 U \frac{G_x}{x_\eta}.
\end{align*}
$$

(21)

In order to establish $x(\eta, t)$, we utilize the following equation (Alharbi & Naire, 2019; Alharbi, 2016; Ceniceros & Hou, 2001):
MMPDE7 : \[ \tau \left( 1 - \lambda \frac{\partial}{\partial \eta} \right) U_t = \frac{1}{F} \left( F U \right)_x, \] (22)

with the boundary conditions
\[ x_0 = x_l, \quad x_{N+1} = x_R, \] (23)

and the initial condition
\[ x_{i, t=0} = (i-1)(x_R - x_L)/N + x_L, \] (24)

Here, \( F(U, G, x) \) describes the mesh density function, \( \tau \) indicates a relaxation parameter and \( \tau \left( 1 - \lambda \frac{\partial}{\partial \eta} \right) \) is called the smoothing operator developed by Ceniceros and Hou (2001). The movement of the mesh is mainly controlled by the function \( F(U, G, x) \). According to Alharbi (Ceniceros & Hou, 2001), the proposed technique gives appropriate results when a suitable function \( F \) is selected. Cook (2016) chose a modified density function given by
\[ F(U, G, x) = \sqrt{1 + a_0 \left( \frac{U_x}{x} \right)^2 + \left( \frac{G_x}{x} \right)^2}, \] (25)

where \( a_0 \) presents a non-negative constant related to the average quantity of \( (U_x)^2 + (G_x)^2 \) and \( (U^2 + G^2) \), respectively. It is worth mentioning that the spatial derivative is semi-discretised while the temporal one is kept continuous. Achieving this, the KdV problem becomes a system of \( (N+1) \) ODEs which can be simply solved by using line methods. The MATLAB ODE solver (ode15i) is utilized to integrate the resultant system numerically. We begin by separating the physical domain as follows:
\[ x_i < x_{i+1}, \quad i = 0, 1, 2, ..., N. \] (26)

The computational coordinates can be written as
\[ \eta_j = j \frac{1}{N}, \quad j = 0, 1, 2, ..., N. \] (27)

Moreover, the boundary mesh is fixed by \( x_0 = x_L \) and \( x_{N+1} = x_R \), while the interior points are approximated by solving MMPDE7 (Eq. 22) subject to the ODE boundary conditions \( x_{t, 0} = x_{t, N} = 0 \). Therefore, the discretisation of Eq. (21) is established by employing finite differences as follows:
\[
\begin{align*}
U_{x_i} &= \frac{U_{i+1} - U_{i-1}}{x_{i+1} - x_{i-1}}, \\
G_{x_i} &= \frac{G_{i+1} - G_{i-1}}{x_{i+1} - x_{i-1}}.
\end{align*}
\] (28)
4. Results and discussion

Here, we discuss the acquired results presented in this study. The sech technique and the MMPDEs method have been worthily applied to construct vital solutions. It has been noted that new structures for the KdV system have been clearly reported and all of these solutions may be serviceable in fluid dynamics. However, the physical situations in which the KdV system emerge tend to be highly idealized because the assumption of constant coefficients. Figures 1–7 illustrate the profile pictures of the presented solutions for the KdV system. These solutions provide an excellent performance to distinguish the kinds of solitary solutions according to the physical parameters. Namely, the behaviour of the solutions being solitons, super-solitons, dissipative or periodic and so on, is an indication for the values of the physical parameters. The adaptive moving method is an appropriate technique for dealing with most partial differential equations. Specifically, this scheme is an effective tool for solving KdV system. This technique approximates the numerical results and reduces the error perfectly. The comparison between the exact and numerical solutions shows that the MMPDEs scheme works efficiently. Furthermore, the presented 2D graphs compare the achieved numerical results to the exact ones. As can be seen in the above figures, both solutions are nearly coincided and almost have the same behaviours. Therefore, the resultant error is very small and can be neglected.

\[
G_{x,x,i} = \frac{2}{x_{i+1} - x_{i-1}} \left[ \frac{G_{i+1} - G_i}{x_{i+1} - x_i} - \frac{G_i - G_{i-1}}{x_i - x_{i-1}} \right], \quad U_{x,x,i} = \frac{2}{x_{i+1} - x_{i-1}} \left[ \frac{U_{i+1} - U_i}{x_{i+1} - x_i} - \frac{U_i - U_{i-1}}{x_i - x_{i-1}} \right],
\]

\[
U_{i+1} = \frac{U_{i+1} + U_i}{2},
\]

(29)

Figure 6. (a) shows the time evolution for a single wave solution of \( U(x, t) \) while (b) presents the time evolution for one wave solution of \( G(x, t) \) at \( 0 \leq t \leq 10 \). (c) illustrates the corresponding mesh \( x = x(\eta, t) \).
Finally, the proposed two methods can be applied to many other nonlinear models arising in new physics.

5. Conclusions

In this article, the sech method and the MMPDEs technique have been applied to extract new soliton solutions of the KdV system. Some novel hyperbolic solutions have been successfully presented. In particular, the exact solutions are presented in the form of the sech function. The derived solutions are stable inside the interval $0 \leq x \leq 20$. The development of the mesh is shown in (c).

Finally, the proposed two methods can be applied to many other nonlinear models arising in new physics.

**References**

Abdel-Aty, A.-H., Khater, M. M. A., Baleanu, D., Abo-Dahab, S. M., Bouslimi, J., & Omri, M. (2020a). Oblique explicit wave solutions of the fractional biological population (BP) and equal width (EW) models. *Advances in Difference Equations*, 2020(1), 1–17. doi:10.1186/s13662-020-03005-0

Abdel-Aty, A.-H., Khater, M. M. A., Baleanu, D., Khalil, E. M., Bouslimi, J., & Omri, M. (2020b). Abundant distinct types of solutions for the nervous biological fractional FitzHugh-Nagumo equation via three different sorts of schemes. *Advances in Difference Equations*, 2020(1), 1–17. doi:10.1186/s13662-020-02852-1

Ahmad, H., & Khan, T. A. (2019). Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations. *Advances in Difference Equations*, 38(3–4), 1113–1124. doi:10.1186/s13662-019-1948-2

Ahmad, H., Seadawy, A. R., & Khan, T. A. (2020). Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm. *Mathematics and Computers in Simulation*, 177, 13–23. doi:10.1016/j.matcom.2020.03.005

Ahmad, H., Seadawy, A. R., Khan, T. A., & Thounthong, P. (2020). Analytic approximate solutions for some nonlinear parabolic dynamical wave equations. *Journal of Taibah University for Science*, 14(1), 346–358. doi:10.1080/16583655.2020.1741943

Alharbi, A. R. (2016). *Numerical solution of thin-film flow equations using adaptive moving mesh methods* (Ph.D. thesis). Keele University.

Alharbi, A. R., & Almatrafi, M. B. (2020a). Analytical and numerical solutions for the variant Boussinesq equations. *Journal of Taibah University for Science*, 14(1), 454–462. doi:10.1080/16583655.2020.1746575

Alharbi, A. R., & Almatrafi, M. B. (2020b). Numerical investigation of the dispersive long wave equation using an adaptive moving mesh method and its stability. *Results in Physics*, 16, 102870. doi:10.1016/j.rinp.2019.102870

Alharbi, A. R., & Almatrafi, M. B. (2020c). Riccati-Bernoulli Sub-ODE approach on the partial differential equations and applications. *International Journal of Mathematical Computer Science*, 15, 367–388.

Alharbi, A. R., & Naire, S. (2017). An adaptive moving mesh method for thin film flow equations with surface tension. *The Journal of Computational and Applied Mathematics*, 319, 365–384. doi:10.1016/j.cam.2017.01.019
Alharbi, A. R., & Naire, S. (2019). An adaptive moving mesh method for two-dimensional thin film flow equations with surface tension. *The Journal of Computational and Applied Mathematics.*, 356, 219–230. doi:10.1016/j.cam.2019.02.010

Alharbi, A., Almatrafi, M. B., & Abdelrahman, M. A. E. (2020). Analytical and numerical investigation for Kadomtsev-Petviashvili equation arising in plasma physics. *Physica Scripta*, 95(4), 045215. doi:10.1088/1402-4896/ab6ce4

Ali, U., Khan, M. A., Khter, M. M. A., Mousa, A. A., & Attila, R. A. M. (2021). A new numerical approach for solving 1D fractional diffusion-wave equation. *Journal of Function Spaces*, 2021, 1–7. Article ID 7 pages. doi:10.1155/2021/6638597

Aminikhad, H., Moosaei, H., & Hajipour, M. (2009). Exact solutions for nonlinear partial differential equations via Exp-function method. *Numerical Methods Partial Differential Equation*, 26, 1427–1433.

Assas, L. M. B. (2008). Variational iteration method for solving coupled-KdV equations. *Chaos, Solitons & Fractals*, 38(4), 1225–1228. doi:10.1016/j.chaos.2007.02.012

Bazighifan, O., Ahmad, H., & Yao, S. W. (2020). New Oscillation criteria for advanced differential equations of fourth order. *Mathematics*, 8(5), 728. doi:10.3390/math8050728

Budd, C. J., & Williams, G. F. (2009). Moving mesh generation using the parabolic Monge-Ampere equation. *SIAM Journal on Scientific Computing.*, 31(5), 3438–3465. doi:10.1137/080716773

Budd, C. J., Huang, W., & Russell, R. D. (2009). Adaptivity with moving grids. *Acta Numerica*, 18, 111–241. doi:10.1017/S0962492909000015

Cao, D. B., Yan, J. R., & Zhang, Y. (2002). Exact solutions for a new coupled MKdV equations and a coupled KdV equations. *Physics Letters A*, 297(1–2), 68–74. doi:10.1016/S0375-9601(02)00376-6

Ceniceros, H. D., & Hou, T. Y. (2001). An efficient dynamically adaptive mesh for potentially singular solutions. *Journal of Computational Physics.*, 172(2), 609–639. doi:10.1006/jcph.2001.6844

Chu, Y., Khter, M. M. A., & Hamed, Y. S. (2021). Diverse novel analytical and semi-analytical wave solutions of the generalized (2 + 1)-dimensional shallow water waves model. *AIP Advances*, 11(1), 015223. doi:10.1063/5.0036261

Cook, S. P. (2016). *Adaptive Mesh Methods for Numerical Weather Prediction* Uni Bath, PhD thesis.

Dal, C. Q., & Zhang, J. F. (2006). Jacobian elliptic function method for nonlinear differential difference equations. *Chaos Solutions Fractals*, 27(4), 1042–1049. doi:10.1016/j.chaos.2005.04.071

Dubard, P., Gaillard, P., Klein, C., & Matveev, V. B. (2010). On multi-rogue wave solutions of the NLS equation and position solutions of the KdV equation. *The European Physical Journal Special Topics*, 185(1), 247–258. doi:10.1140/epjst/e2010-01252-9

Fan, E., & Zhang, H. (1998). A note on the homogeneous balance method. *Physics Letters A*, 246(5), 403–406. doi:10.1016/S0375-9601(98)00547-7

Ganji, D. D., & Abdollahzadeh, M. (2008). Exact travelling solutions for the Lax’s seventhorder KdV equation by sech method and rational exp-function method. *Journal of Applied Mathematics and Computing.*, 206, 438–444.

Ganji, D. D., & Rafei, M. (2006). Solitary wave solutions for a generalized Hirota-Satsuma coupled-KdV equation by homotopy perturbation method. *Physics Letters A*, 356(2), 131–137. doi:10.1016/j.physleta.2006.03.039

Gao, Y. T., & Tian, B. (2001). Ion-acoustic shocks in space and laboratory dusty plasmas: Two-dimensional and non-traveling-wave observable effects. *Physics of Plasmas*, 8(7), 3146–3149. doi:10.1063/1.1379589

Gokdogan, A., Yıldırım, A., & Merdan, M. (2012). Solving coupled KdV equations by differential transformation method. *World Applied Science Journal*, 19, 1823–1828.

Hirotta, R., & Satsuma, J. (1981). Soliton solutions of a coupled Korteweg-de Vries equation. *Physics Letters A*, 85(8–9), 407–408. doi:10.1016/0375-9601(81)90423-0

Huang, W., & Russell, R. D. (1998). A high dimensional moving mesh strategy. *Applied Numerical Mathematics.*, 26(1–2), 63–76. doi:10.1016/S0168-9274(97)00082-2

Huang, W., & Russell, R. D. (2011). *The adaptive moving mesh method*. Springer.

Inc, M., Khan, M. N., Ahmad, I., Yao, S. W., Ahmad, H., & Thounthong, P. (2020). Analysing time-fractional exotic options via efficient local meshless method. *Results in Physics.*, 19, 103385. doi:10.1016/j.rinp.2020.103385

Jaradat, H. M., Syam, M., & Alquran, M. (2017). A two-mode coupled Korteweg–de Vries: Multiple-soliton solutions and other exact solutions. *Nonlinear Dynamics*, 90(1), 371–377. doi:10.1007/s11071-017-3668-x

Khter, M. M. A., Attia, R. A. M., Alodhaibi, S. S., & Lu, D. (2020). Novel soliton waves of two fluid nonlinear evolution models in the view of computational scheme. *International Journal of Modern Physics B*, 34(10), 2050096. doi:10.1142/S0217979220500964

Khter, M. M. A., Lu, D.-C., Attia, R. A. M., & Inc, M. (2019). Analytical and approximate solutions for complex nonlinear Schrödinger equation via generalized auxiliary equation and numerical schemes. *Communications in Theoretical Physics.*, 71(11), 1267. doi:10.1088/0253-6102/71/11/1267

Khter, M. M. A., Park, C., & Lu, D. (2020). Two effective computational schemes for a prototype of an excitable system. *AIP Advances*, 10(10), 105120. doi:10.1063/5.0024417

Khter, M. M. A., Attia, R. A. M., Park, C., & Lu, D. (2020). On the numerical investigation of the interaction in plasma between (high & low) frequency of (Langmuir & ion-acoustic) waves. *Results in Physics.*, 18, 103317. doi:10.1016/j.rinp.2020.103317

Qian, L., Attia, R. A. M., Qiu, Y., Lu, D., & Khter, M. M. A. (2019). The shock peakon wave solutions of the general Degasperis-Procesi equation. *International Journal of Modern Physics B*, 33(29), 1950351. doi:10.1142/S021797921950351X

Russell, R. D., Huang, W., & Ren, Y. (1994). Moving Mesh Partial Differential Equations (MMPDES) Based on the Equidistribution Principle. *SIAM Journal on Numerical Analysis*, 31(3), 709–730. doi:10.1137/0731038

Seadawy, A. R., Iqbal, M., & Lu, D. (2019). Nonlinear wave solutions of the Kudryashov-Sinelshchikov dynamical equation in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity. *Journal of Taibah University for Science*, 13(1), 1060–1072. doi:10.1080/16583655.2019.1680170

Seadawy, A. R., Lu, D., & Yue, C. (2017). Travelling wave solutions of the generalized nonlinear fifth-order KdV water wave equations and its stability. *Journal of Taibah University for Science*, 11(4), 623–633. doi:10.1016/j.jtusci.2016.06.002
Wazwaz, A. M. (2004). The tanh method for travelling wave solutions of nonlinear equations. Journal of Applied Mathematics and Computing, 154, 714–723.

Wazwaz, A. M. (2005). Exact solutions to the double sinh-Gordon equation by the tanh method and a variable separated ODE method. Computers & Mathematics with Applications, 50(10–12), 1685–1696. doi:10.1016/j.camwa.2005.05.010

Wazwaz, A. M. (2007). The extended tanh method for abundant solitary wave solutions of nonlinear wave equations. Journal of Applied Mathematics and Computing, 187(2), 1131–1142. doi:10.1016/j.amc.2006.09.013

Yang, X. F., Deng, Z. C., & Wei, Y. (2015). A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application. Advances in Difference Equations, 1, 117–133.

Yue, C. H. E. N., Lu, D., Khater, M. M. A., Abdel-Aty, A.-H., Alharbi, W., & Attia, R. A. M. (2020). On explicit wave solutions of the fractional nonlinear DSW system via the modified Khater method. Fractals, 28(08), 2040034. 10 pages. doi:10.1142/S0218348X20400344

Yue, C., Khater, M. M. A., Attia, R. A. M., & Lu, D. (2020). Computational simulations of the couple Boiti-Leon-Pempinelli (BLP) system and the (3 + 1)-dimensional Kadomtsev-Petviashvili (KP) equation. AIP Advances, 10(4), 045216. doi:10.1063/1.5142796

Yue, C., Khater, M. M. A., Inc, M., Attia, R. A. M., & Lu, D. (2020). Abundant analytical solutions of the fractional nonlinear (2 + 1)-dimensional BLMP equation arising in incompressible fluid. International Journal of Modern Physics B, 34(09), 2050084. doi:10.1142/S0217979220500848

Zhang, J.-L., Wang, M.-L., Wang, Y.-M., & Fang, Z.-D. (2006). The improved F-expansion method and its applications. Physics Letters A, 350(1–2), 103–109. doi:10.1016/j.physleta.2005.10.099

Zhang, S., Tong, J. L., & Wang, W. (2008). A generalized $G'/G$-expansion method for the mKdv equation with variable coefficients. Physics Letters A, 372, 2254–2257.