HOMOGENIZATION FOR SOME NONLOCAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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We establish homogenization principles for nonlocal stochastic partial differential equations with oscillating coefficients, by a martingale approach. This is motivated by data assimilations with non-Gaussian observations. The nonlocal operators in these stochastic partial differential equations are the generators of non-Gaussian Lévy processes of either integrable or non-integrable jump kernels. In particular, this work leads to homogenization of nonlocal Zakai equations, in the context of data assimilation with α-stable Lévy fluctuations. The homogenized systems are shown to approximate the original systems in the sense of probabilistic distribution.

1. Introduction. Homogenization has become a useful tool in effective descriptions of multiscale systems, especially when the systems are also under random fluctuations. These multiscale systems arise in various fields in applied sciences such as flow of fluids in porous media, composites materials [7] in advanced technologies, and data assimilation [4, 35].

The homogenization of stochastic partial differential equations has attracted a lot of attention recently [6, 3, 19, 21], due to its importance in mathematical modeling and simulation. The homogenization results for certain stochastic partial differential equations can be proved by probabilistic tools such as the ergodic theorems and functional central limit theorems. Buckdahn and Hu [10] studied the homogenization of stochastic partial differential equations, using techniques involving the forward-backward

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stochastic differential equations. Cerrai and Freidlin [18] studied the averaging of stochastic partial differential equations by an invariant measure. On the other hand, analytic approaches also play an important role for homogenization of stochastic multiscale systems. Ichihara [24] established a homogenization principle by employing analytic tools like variational formula and Fredholm alternatives. Hairer and Pardoux [22] proved a homogenization theorem for a degenerate system.

However, there are few papers dealing with the homogenization for non-local stochastic partial differential equations. In this present paper, we will show that a nonlocal stochastic partial differential equation has an effective, homogenized system that is a local stochastic partial differential equation. This further leads to the homogenization of a nonlocal Zakai equation, for a nonlinear filtering system under non-Gaussian Lévy fluctuations. This implies that the nonlocal Zakai equation can be effectively approximated by a local Zakai equation, with benefits for the simulation and analysis of a class of non-Gaussian data assimilation systems.

The paper is divided into two parts. In Part I, the linear operator in stochastic partial differential equation is the generator of the Lévy process with integrable jump kernel. In Part II, the linear operator is the generator of a α-stable Lévy process, with non-integrable jump kernel.

In Part I, we consider the homogenization for the following nonlocal stochastic partial differential equation (heterogeneous system) with a small positive scale parameter $\epsilon$:

$$
\begin{cases}
  du^\epsilon(t, x) = A^\epsilon u^\epsilon(t, x)dt + B^\epsilon u^\epsilon(t, x)dt + \sigma(\frac{x}{\epsilon})u^\epsilon(t, x)dW_t, \\
  u^\epsilon(0, x) = u_0(x),
\end{cases}
$$

(1.1)

where the initial datum $u_0$ is in $L^2(\mathbb{R})$, $W = (W(t))_{t \in [0,T]}$ is a Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$, $A^\epsilon$ and $B^\epsilon$ are linear operators of the forms:

$$
(A^\epsilon u)(x) = a(\frac{x}{\epsilon})u''(x) + \frac{1}{\epsilon}b(\frac{x}{\epsilon})u'(x),
$$

$$
(B^\epsilon u)(x) = \frac{1}{\epsilon^3}\lambda(\frac{x}{\epsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\epsilon}) (u(y) - u(x)) dy.
$$

(1.2)

Here $a(\cdot), b(\cdot), \sigma(\cdot), \lambda(\cdot)$ are known functions of period 1, $\lambda(\cdot)$ is also bounded and positive, and $c(z)$ is the jump kernel being a positive integrable function with symmetry property $c(z) = c(-z)$. That is, $\int_{\mathbb{R}} c(\xi) d\xi < \infty$. In what follows we identify periodic functions of period 1 with function defined on
the $[0,1]$, and denote the space of these functions by $\mathbb{T}$. For convenience, we also define the linear operator $T^\epsilon u = A^\epsilon u + B^\epsilon u$.

Our purpose is to examine the convergence of the solution $u^\epsilon$ of (1.1) in some probabilistic sense, as $\epsilon \to 0$, and to specify the limit $u$. We will see that the limit process $u$ satisfies the following local stochastic partial differential equation (homogenized system):

\begin{equation}
\begin{aligned}
du(t,x) &= T^0 u(t,x) dt + M^0 u(t,x) dW_t, \\
u(0,x) &= u_0(x),
\end{aligned}
\end{equation}

where

\begin{equation}
(T^0 u)(x) = Qu''(x), \quad (M^0 u)(x) = u(x) \int_\mathbb{T} \sigma(y)m(y)dy.
\end{equation}

Here the coefficient $Q$ is determined by

\begin{equation}
Q = \int_\mathbb{T} a(y)m(y)(\chi'(y) + 1)^2 dy + \frac{1}{2} \int_\mathbb{T} \int_\mathbb{R} c(y - q)\lambda(q)m(q)[(y - q) + (\chi(y) - \chi(q))]^2 dy dq,
\end{equation}

and the functions $\chi$ and $m$ are the unique solutions of the following deterministic partial differential equations, respectively:

\begin{equation}
\begin{aligned}
\tilde{T}\chi(y) + b(y) &= 0, \quad y \in \mathbb{T}, \\
\int_0^1 \chi(y)m(y)dy &= 0,
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\tilde{T}^* m(y) &= 0, \quad y \in \mathbb{T}, \\
\int_0^1 m(y)dy &= 1,
\end{aligned}
\end{equation}

with the linear operator $\tilde{T}$ being defined by:

\begin{equation}
(\tilde{T}v)(y) := a(y)v''(y) + b(y)v'(y) + \lambda(y) \int_\mathbb{R} c(x - y)(v(x) - v(y))dx, y \in \mathbb{T}.
\end{equation}

Note that $\tilde{T}^*$ is the adjoint operator of $\tilde{T}$ in $L^2(\mathbb{R})$. For convenience, we denote

\begin{equation}
(\tilde{A}v)(y) = a(y)v''(y) + b(y)v'(y),
\end{equation}

\begin{equation}
(\tilde{B}v)(y) = \lambda(y) \int_\mathbb{R} c(x - y)(v(x) - v(y))dx.
\end{equation}
We will apply the proceeding homogenization result to obtain the homogenized equation for the following nonlocal Zakai equation:

\begin{equation}
\begin{cases}
\frac{du}{\varepsilon}(t, x) = (T^\varepsilon)^* u(t, x) dt + u(t, x) \sigma(\frac{x}{\varepsilon}) dt + u(t, x) \sigma(\frac{x}{\varepsilon}) dW_t, \\
u^\varepsilon(0, x) = u_0(x).
\end{cases}
\end{equation}

This is the Zakai equation for the conditional probability density of the following nonlinear filtering system (Qiao and Duan [32]):

\begin{equation}
\begin{cases}
dx_t = \frac{1}{\varepsilon} b(\frac{x}{\varepsilon}) dt + \sigma_1(\frac{x}{\varepsilon}) dW_t^\varepsilon + dL_t^\varepsilon, \\
dy_t = \sigma(\frac{x}{\varepsilon}) dt + dW_t,
\end{cases}
\end{equation}

where \( x_t \) is system (or signal) state, \( y_t \) is the observation, and \( \varepsilon \)\( W_t^\varepsilon, W_t \) are mutually independent Brownian motions. Moreover, \( L_t^\varepsilon \) is a Lévy process with the generator

\[ (B^\varepsilon u)(x) = \frac{1}{\varepsilon^3} \lambda(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c\left(\frac{x-y}{\varepsilon}\right)(u(y) - u(x)) dy. \]

Let us set \( H = L^2(\mathbb{R}) \), and \( H^1 = H^1(\mathbb{R}) \), the usual Sobolev space of order 1, that is, the completion of \( C^\infty_c(\mathbb{R}) \), the set of smooth functions with compact support.

We fix a smooth and positive function \( \theta \) on \( \mathbb{R} \) such that \( \theta(x) = |x| \) for all \( |x| \geq 1 \). Then for \( n = 0 \) or \( 1 \), \( \lambda \in [0, \infty) \), we define the weighted Sobolev space with norm \( \| \cdot \|_{H^n_\lambda} \) :

\[ H^n_\lambda = \{ v \mid ve^{\lambda \theta} \in H^n \}, \quad \| v \|_{H^n_\lambda} = \| ve^{\lambda \theta} \|_{H^n}. \]

Let \( (H^n_\lambda)' \) be the dual space of \( H^n_\lambda \). Then

\[ (H^n_\lambda)' = H^{-n}_{-\lambda} = \{ v \mid ve^{-\lambda \theta} \in H^{-n} \}. \]

We also denote \( H_{-\lambda} = H^0_{-\lambda} \). Introduce a function space

\[ K = C(0, T; H^{-1}_{-\lambda}) \cap L^2(0, T; H_{-\lambda}). \]

In \( C(0, T; H^{-1}_{-\lambda}) \) we take the uniform convergence topology \( T_1 \), and in \( L^2(0, T; H_{-\lambda}) \) we choose the topology induced by \( L^2 \)-norm and denote it by \( T_2 \).

In Part II, we consider the homogenization for the following nonlocal stochastic partial differential equation (heterogeneous system) with a small positive scale parameter \( \varepsilon \):

\begin{equation}
\begin{cases}
dv^\varepsilon(t, x) = F^\varepsilon v^\varepsilon(t, x) dt + L^\varepsilon v^\varepsilon(t, x) dt + \sigma(\frac{x}{\varepsilon}) v^\varepsilon(t, x) dW_t, \\
v^\varepsilon(0, x) = v_0(x),
\end{cases}
\end{equation}
When we take \( \gamma \) is the operator whose action on \( D \) \( \Delta \) is given by

\[
(F^\epsilon u)(x) = g(\frac{x}{\epsilon})u'(x) + \left[ -\frac{1}{\epsilon^\alpha}e(\frac{x}{\epsilon}) + f(\frac{x}{\epsilon}) \right]u(x),
\]

\[
(L^\epsilon u)(x) = \int_{\mathbb{R}\setminus\{0\}} (u(x + \delta(\frac{x}{\epsilon})y) - u(x))\nu^\alpha(dy) + \frac{1}{\epsilon^{\alpha-1}}d(\frac{x}{\epsilon})u'(x),
\]

where the integral is in the sense of Cauchy principal value. Here \( d(\cdot), g(\cdot), e(\cdot), f(\cdot), \delta(\cdot) \) are known functions of period 1, and the jump measure \( \nu^\alpha(dy) = |y|^{-(1+\alpha)}dy \). Note that this jump kernel is non-integrable: \( \int_{\mathbb{R}} |y|^{-(1+\alpha)}dy = \infty \). In what follows we identify periodic functions of period 1 with function defined on the \([0,1]\), and denote \( V^\epsilon u = F^\epsilon u + L^\epsilon u \).

Recall that the nonlocal or fractional Laplacian is defined as [14, 12]

\[
(-\Delta)^{\alpha/2}u(x) = \int_{\mathbb{R}\setminus\{0\}} \frac{u(x) - u(y)}{|y-x|^{1+\alpha}}dy,
\]

where the integral is in the sense of Cauchy principal value. Note that \((-\Delta)^{\alpha/2}\) is the generator for a symmetric \(\alpha\)-stable Lévy motion \(L^\alpha_t\). Thus \(L^\epsilon u(x) = \delta^\alpha(\frac{x}{\epsilon})(-\Delta)^{\alpha/2}u(x) + \frac{1}{\epsilon^\alpha}d(\frac{x}{\epsilon})u'(x)\).

Given the mapping \( \beta(x,y), \gamma(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with \( \gamma \) antisymmetric, the action of the nonlocal divergence \( D \) on \( \beta \) is defined as

\[
D(\beta)(x) := \int_{\mathbb{R}} (\beta(x,y) + \beta(y,x)) \cdot \gamma(x,y)dy \quad \text{for } x \in \mathbb{R}.
\]

Given the mapping \( u(x) : \mathbb{R} \to \mathbb{R} \), the adjoint operator \( D^* \) corresponding to \( D \) is the operator whose action on \( u \) is given by

\[
D^*(u)(x,y) = -(u(y) - u(x))\gamma(x,y) \quad \text{for } x, y \in \mathbb{R}.
\]

When we take \( \gamma(x,y) = (y-x)\frac{1}{|y-x|^{1+\alpha}} \), then

\[
DD^* = -\frac{1}{2}(-\Delta)^{\alpha/2}
\]

We will examine the convergence of the solution \( v^\epsilon \) of (1.10) in some probabilistic sense, as \( \epsilon \to 0 \), and to specify the limit \( v \). We will see that the limit process \( w \) satisfies the following local stochastic partial differential equation (homogenized system):

\[
\begin{align*}
\frac{dv(t,x)}{dt} &= V^0v(t,x)dt + M^0v(t,x)dW_t, \\
v(0,x) &= v_0(x),
\end{align*}
\]

where the initial datum \( v_0 \) is in \( L^2(\mathbb{R}) \), \( W = (W(t))_{t \in [0,T]} \) is a Brownian motion in a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \( F^\epsilon \) and \( L^\epsilon \) are linear operators of the forms:
where

\[(M^0 u)(x) = u(x) \int_{\mathbb{T}} \sigma(y)m(y)dy,\]

\[(V^0 u)(x) = \int_{\mathbb{T}} \delta^\alpha(y)m_1(y)dy \cdot (-(-\Delta)^{\alpha/2})u(x) + u'(x) \int_{\mathbb{T}} g(y)m_1(y)dy + u(x) \int_{\mathbb{T}} f(y)m_1(y)dy.\]

Here the functions \(m_1\) is the unique solution of the following deterministic partial differential equation, respectively:

\[
\begin{align*}
\tilde{L}^* m_1(y) &= 0, \quad y \in \mathbb{T}, \\
\int_0^1 m_1(y)dy &= 1,
\end{align*}
\]

with the linear operator \(\tilde{L}\) being defined by:

\[
(\tilde{L}u)(y) := -\delta^\alpha(y)(-\Delta)^{\alpha/2}u(y) + d(y)u'(y), \quad y \in \mathbb{T}.
\]

Note that \(\tilde{L}^*\) is the adjoint operator of \(\tilde{L}\) in \(L^2(\mathbb{R})\).

We will also apply the homogenization result to obtain the homogenized equation for the following nonlocal Zakai equation:

\[
\begin{align*}
\frac{d\nu(t,x)}{dt} &= (L^\epsilon)^* \nu(t,x) dt + \nu(t,x)(\bar{\sigma}^2)dW_t + \nu(t,x)\sigma(x_{\epsilon})dW_t, \\
\nu(0,x) &= \nu_0(x).
\end{align*}
\]

This is the Zakai equation for the conditional probability density of the following nonlinear filtering system

\[
\begin{align*}
\frac{dx(t)}{dt} &= \frac{1}{\epsilon} d\frac{x_t}{\epsilon} dt + \delta\frac{x_t}{\epsilon} dL^\alpha_t, \\
\frac{dy(t)}{dt} &= \sigma\frac{x_t}{\epsilon} dt + dW_t,
\end{align*}
\]

where \(x_t\) is system (or signal) state, \(y_t\) is the observation, and \(W_t\) are mutually independent Brownian motions. Moreover, \(L^\alpha_t\) is a \(\alpha\)-stable Lévy process.

For any \(\alpha \in (0,2)\), we define \(H^{\alpha/2}(\mathbb{R})\) as follows

\[H^{\alpha/2}(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : \frac{|u(x) - u(y)|}{|x-y|^{(1+\alpha)/2}} \in L^2(\mathbb{R} \times \mathbb{R})\};\]

i.e., an intermediary Banach space between \(L^2(\mathbb{R})\) and \(H^1(\mathbb{R})\), endowed with the natural norm

\[
||u||^2_{H^{\alpha/2}} = ||u||^2_{L^2} + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} dxdy.
\]
Then, we can do as before, for \( \alpha \in (0, 2) \), \( \lambda \in [0, \infty) \), we define the weighted Sobolev space \( H^{\alpha/2}_\lambda \), \( H^{-\alpha/2}_\lambda \) and \( K_1 = C(0, T; H^{-\alpha/2}_\lambda) \cap L^2(0, T; H_{-\lambda}) \).

We will establish a homogenization principle for (1.1) and (1.10) by a martingale approach. We show that the laws of the solutions of (1.1) and (1.10) converge weakly to the laws of the solution of the effective, homogenized equation (1.3) and (1.11).

The major difference between (1.1) and (1.10) is the different scaling in the generator of the Lévy process. In Part I, the operator in stochastic partial differential equation (1.1) is the generator of the Lévy process with special jump kernel which excludes the big jumps. There is a \( \epsilon^{-3} \) coefficient in the generator. In this case, the operator in homogenized system (1.3) is just a generator of Brownian motion. On the other hand, the operator in stochastic partial differential equation (1.10) is the generator of a multiplicative \( \alpha \)-stable Lévy process which includes big jumps. There is no explicit \( \epsilon \) coefficient in the generator. Then we can find the operator in homogenized system (1.11) is also a generator of a \( \alpha \)-stable Lévy process.

**Theorem 1.** (Homogenization)
Let \( u^\epsilon \) be the solution of the heterogeneous equation (1.1). We denote \( \pi^{m, \epsilon} \), \( \pi^\epsilon \) the laws of \( m^\epsilon u^\epsilon \) and \( u^\epsilon \) respectively, where we have set \( m^\epsilon(x) = m(\frac{x}{\epsilon}) \), \( (m^\epsilon u^\epsilon)(x) = m(\frac{x}{\epsilon})u^\epsilon(x) \), and \( m \) is the solution of the equation (1.7). Then, we have
\[
\pi^{m, \epsilon} \Rightarrow \pi \quad \text{in} \quad (K, T_1), \quad \pi^\epsilon \Rightarrow \pi \quad \text{in} \quad (K, T_2)
\]
as \( \epsilon \to 0 \), where \( \pi \) is the probability law on \( K \) induced by the solution of equation (1.3).

**Theorem 2.** (Homogenization)
Let \( v^\epsilon \) be the solution of the heterogeneous equation (1.10). We denote \( \pi^{m_1, \epsilon} \), \( \pi_1^\epsilon \) the laws of \( m_1^\epsilon v^\epsilon \) and \( v^\epsilon \) respectively, where we have set \( m_1^\epsilon(x) = m_1(\frac{x}{\epsilon}) \), \( (m_1^\epsilon v^\epsilon)(x) = m_1(\frac{x}{\epsilon})v^\epsilon(x) \), and \( m_1 \) is the solution of the equation (1.12). Then, we have
\[
\pi^{m_1, \epsilon} \Rightarrow \pi_1 \quad \text{in} \quad (K_1, T_1), \quad \pi_1^\epsilon \Rightarrow \pi_1 \quad \text{in} \quad (K_1, T_2)
\]
as \( \epsilon \to 0 \), where \( \pi_1 \) is the probability law on \( K_1 \) induced by the solution of equation (1.11).

We will prove **Theorem 1** and **Theorem 2** in **Part I** and **Part II**, respectively.

For Part I, we make the following **assumptions**.
(i) The coefficient \( a(\cdot), b(\cdot) \in C^3(\mathbb{T}), \sigma(\cdot) \in C^3_b(\mathbb{T}) \), where \( C^3_b \) stands for the
set of function of class $C^3$ whose partial derivatives of order less than or equal to 3 are bounded.

(ii) For all $y \in \mathbb{T}$, there exists $\kappa > 0$ such that

$$\kappa \leq a(y) \leq \kappa^{-1}.$$  

(iii) The function $\lambda(x)$ is periodic and bounded (with bounds $\alpha_1, \alpha_2$): $0 < \alpha_1 \leq \lambda(x) \leq \alpha_2 < \infty$.

(iv) The kernel function $c(z) \geq 0; c(-z) = c(z)$, and

$$\|c\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} c(z) dz = a_1 > 0, \int_{\mathbb{R}} |z|^2 c(z) dz < \infty.$$  

(v) The function $b(\cdot)$ satisfies the “centering condition”:

$$\int_{\mathbb{T}} b(y)m(y) = 0,$$

where $m$ is the solution of (1.7).

For Part II, we make the following assumptions.

(a) The coefficient $d(\cdot), g(\cdot), e(\cdot), f(\cdot), \delta(\cdot), \sigma(\cdot) \in C^2(\mathbb{T})$.

(b) Function $d(\cdot)$ satisfies the “centering condition”:

$$\int_{\mathbb{T}} d(y)m_1(y) = 0,$$

where $m_1$ is the solution of (1.12).

This paper is organized as follows. For Part I, in Section 2, we present some preliminary results. In Section 3, we will prove Theorem 1. In Section 4, we apply our homogenization theorem to the Zakai equation (1.8). For Part II, we will prove Theorem 2.

**PART I**

We now prove Theorem 1.

2. Well-posedness of heterogeneous and homogenized equations.  

We now discuss the well-posedness for the heterogeneous equation (1.1) and the homogenized equation (1.3).
Lemma 1. (Well-posedness for heterogeneous equation)
There exists a unique mild solution \( u^\epsilon(t) \in C(0,T;H) \cap L^2(0,T;H^1) \) of the heterogeneous equation (1.1).

Proof. Note that \( A^\epsilon \) is the infinitesimal generator of a \( C_0 \) semigroup \( S(t)(Eberle [36]) \). Moreover,

\[
\int_0^T ||B^\epsilon u^\epsilon(s,x)||_{L^2} + ||(u^\epsilon(s,x))\sigma(\frac{x}{\epsilon})||^2_{L^2} ds \\
\leq \alpha_2 \int_0^T \left( \int_{\mathbb{R}} c(\frac{x-y}{\epsilon}) u^\epsilon(s,y) dy \right) ||u^\epsilon(s,x)||_{L^2} ds \\
+ \alpha_2 \int_0^T ||u^\epsilon(s,x)||_{L^2} ds \int_{\mathbb{R}} c(\frac{x-y}{\epsilon}) dy ||u^\epsilon(s,x)||_{L^2} \\
+ \int_0^T ||(u^\epsilon(s,x))^2 \sigma^2(\frac{x}{\epsilon})^2||_{L^2} ds.
\]  

(2.1)

We thus conclude that

\[
\left( \int_{\mathbb{R}} c(\frac{x-y}{\epsilon}) u^\epsilon(s,y) dy \right)^2 \leq \int_{\mathbb{R}} c(y) dy \int_{\mathbb{R}} c(q) dq \int_{\mathbb{R}} u^\epsilon(s,x+\epsilon y) u^\epsilon(s,x+\epsilon q) dx \\
\leq a_1^2 ||u^\epsilon||^2_{L^2} < \infty.
\]

So the right hand side of (2.1) is finite. Hence the equation (1.1) has a solution given by

\[
u^\epsilon(t,x) = S(t)u_0(x) + \int_0^t S(t-s)B^\epsilon u^\epsilon(s,x) ds + \int_0^t S(t-s)u^\epsilon(s,x)\sigma(\frac{x}{\epsilon})dW_s.
\]

Suppose

\[
a_1 \sup_{x \in \mathbb{R}} ||\lambda^\epsilon(x)|| \leq \beta \text{ for all } \epsilon > 0,
\]

where \( \beta \) is a positive constant independence of \( \epsilon \). Suppose further that \( T \) verifies

\[
\beta T < 1.
\]

Then, the solution is unique in the following sense: \( \mathbb{P}(u^\epsilon_t = v^\epsilon_t \text{ in } H^{-1}, \forall t \in [0,T]) = 1 \), for every \( u^\epsilon, v^\epsilon \) satisfying the equation. \( \square \)

The existence and uniqueness of the solution for the homogenized equation (1.3) is well known (Pardoux [27]).

Lemma 2. (Well-posedness for homogenized equation)
There exists a mild solution \( u \in L^2([0,T];H^1) \) for the homogenized equation (1.3). It is unique in the sense: \( \mathbb{P}(u = v \text{ in } H^{-1}, \forall t \in [0,T]) = 1 \), for \( u, v \) satisfy the equation.
3. Proof of the Theorem 1. We now prove Theorem 1 by a martingale approach. The proof is divided into two parts: the tightness and the limit law.

3.1. Tightness. We denote $\pi^{m, \epsilon}$ the probability measure induced by $m^\epsilon u^\epsilon$. We will show the tightness of $\{\pi^{m, \epsilon} : \epsilon > 0\}$ in $(K, \mathcal{T}_1)$.

**Lemma 3.** If $u^\epsilon$ is a solution of heterogeneous equation (1.1), and $\pi^{m, \epsilon}$ is the probability measure induced by $m^\epsilon u^\epsilon$. Then, $\{\pi^{m, \epsilon} : \epsilon > 0\}$ is tight in $(K, \mathcal{T}_1)$.

To prove Lemma 3, we begin with the following two lemma.

**Lemma 4.** Let $u^\epsilon$ be a solution of heterogeneous equation (1.1) with initial value $u_0 \in H$. Then there exists a positive constant $C$, independent of $\epsilon$, such that

$$
\sup_{\epsilon} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u_t^\epsilon\|_{L^2}^4 \right] + \sup_{\epsilon} \mathbb{E} \left[ \int_0^T \|u_t^\epsilon\|_{H^1}^2 dt \right] \leq C \left( 1 + \|u_0\|_{L^2}^4 \right).
$$

**Proof.** First we assume that $u_0$ is in $C^\infty_c(\mathbb{R})$, and choose a version of $u^\epsilon$ such that $u^\epsilon(t, x) \in C^2_0([0, T] \times \mathbb{R}), \mathbb{P}$-almost surely. This is possible under the assumptions. By Itô’s formula, we have

$$
u_t^\epsilon(x)^2 = u_0(x)^2 + 2 \int_0^t (T^\epsilon u^\epsilon)(s, x)u^\epsilon(s, x)ds + \int_0^t u^\epsilon(s, x)^2 \sigma(x)\frac{x}{\epsilon} dW_s.
$$

This is equivalent to

$$
m_t^\epsilon(x)u_t^\epsilon(t, x)^2 = m_t^\epsilon(x)u_0(x)^2 + 2 \int_0^t (T^\epsilon_m u^\epsilon)(s, x)u^\epsilon(s, x)ds + \int_0^t m_t^\epsilon(x)u_t^\epsilon(s, x)^2 \sigma(x)\frac{x}{\epsilon} ds + 2 \int_0^t m_t^\epsilon(x)u_t^\epsilon(s, x)^2 \sigma(x)\frac{x}{\epsilon} dW_s,
$$

(3.2)
where \(m^\varepsilon(x) = m(\frac{x}{\varepsilon})\), and \(T_m^\varepsilon = m^\varepsilon T^\varepsilon\) is defined by:

\[
(T_m^\varepsilon u)(x) = a^m(\frac{x}{\varepsilon})u''(x) + \frac{1}{\varepsilon}b^m(\frac{x}{\varepsilon})u'(x) \\
+ \frac{1}{\varepsilon^3}\lambda_m^m(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\varepsilon})(u(y) - u(x))dy
\]

\[
= (a^m(\frac{x}{\varepsilon})u'(x))^t + \frac{1}{\varepsilon}\beta_m^m(\frac{x}{\varepsilon})u'(x) \\
+ \frac{1}{\varepsilon^3}\lambda_m^m(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\varepsilon})(u(y) - u(x))dy.
\]

We introduce notations: \(a^m(y) = a(y)m(y), b^m(y) = m(y)b(y), \lambda^m(y) = \lambda(y) \cdot m(y), \) and

\[
\beta^m(y) = b^m(y) - a^m(y').
\]

Then

\[
(B^\varepsilon_m u + \frac{1}{\varepsilon}\beta^m(\frac{x}{\varepsilon})u', u) \\
= (-\frac{1}{2\varepsilon} (\beta^m(\frac{x}{\varepsilon}'), u^2) + \frac{1}{2\varepsilon^3}\lambda_m^m(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\varepsilon})(u^2(y) - u^2(x))dy) \\
+ (B^\varepsilon_m u, u) - \frac{1}{2\varepsilon^3}(\lambda_m^m(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\varepsilon})(u(y) - u(x))u(x)dydx) \\
- \frac{1}{2\varepsilon^3}2 \int_{\mathbb{R}} \lambda_m^m(\frac{x}{\varepsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\varepsilon})(u^2(y) - u^2(x))dydx \\
= -\frac{1}{2\varepsilon^3} \int_{\mathbb{R}} \lambda_m^m(\frac{x}{\varepsilon})c(\frac{x-y}{\varepsilon})(u(y) - u(x))^2dydx \leq 0,
\]

where \((\tilde{T})^m(y) = 0\). For convenience, we denote \(a^{m,\varepsilon}(x) = a^m(\frac{x}{\varepsilon}), u^\varepsilon(t, x) = u^\varepsilon_t, \) for \(t \in [0, T]\). Integrating both sides of (3.2) with respect to \(x\), we then have

\[
(m^\varepsilon u^\varepsilon_t, u^\varepsilon_t) = (m^\varepsilon u_0, u_0) - 2\int_0^t (a^{m,\varepsilon}(u^\varepsilon_s)'', (u^\varepsilon_s)')ds \\
+ \int_0^t (m^\varepsilon u^\varepsilon_s\sigma^\varepsilon, u^\varepsilon_s\sigma^\varepsilon)ds + 2\int_0^t (m^\varepsilon u^\varepsilon_s\sigma^\varepsilon, u^\varepsilon_s)dW_s \\
- \frac{1}{2\varepsilon^3} \int_0^t \int_{\mathbb{R}} \lambda_m^m(\frac{x}{\varepsilon})c(\frac{x-y}{\varepsilon})(u(y) - u(x))^2dydxds,
\]

where \(m\) satisfies \(\delta < m < \delta^{-1}\) for some \(\delta > 0, \) and \(\sigma(\cdot)\) is bounded. From now on, we will denote by \(C_i, i = 1, 2, \cdots, \) the constants which may depend
on $\kappa, \delta, T$. Then we have

$$
\delta \|u^\epsilon_s\|_{L^2}^2 + 2\kappa \delta \int_0^t \|\nabla u^\epsilon_s\|_{L^2}^2 ds \leq \delta^{-1} \|u_0\|_{L^2}^2 + C_2 \int_0^t \|u^\epsilon_s\|_{L^2}^2 ds + 2 \int_0^t (m^\epsilon u^\epsilon_s \sigma^\epsilon, u^\epsilon_s) dW_s.
$$

(3.6)

By Gronwall's inequality, we obtain

$$
\sup_{0 \leq t \leq T} \sup_{\epsilon} \mathbb{E}[\|u^\epsilon_t\|_{L^2}^4] \leq C_3 (1 + \|u_0\|_{L^2}^2).
$$

Applying Ito’s formula to equation (3.5) we can see

$$
(m^\epsilon u^\epsilon_t, u^\epsilon_t)^2 + 4 \int_0^t (a^{m, \epsilon}(u^\epsilon_s), (u^\epsilon_s)')(m^\epsilon u^\epsilon_s, u^\epsilon_s) ds
\leq (m^\epsilon u_0, u_0)^2 + 2 \int_0^t (m^\epsilon u^\epsilon_s \sigma^\epsilon, u^\epsilon_s \sigma^\epsilon)(m^\epsilon u^\epsilon_s, u^\epsilon_s) ds
+ 4 \int_0^t (m^\epsilon u^\epsilon_s \sigma^\epsilon, u^\epsilon_s)(m^\epsilon u^\epsilon_s, u^\epsilon_s) dW_s + 4 \int_0^t (m^\epsilon u^\epsilon_s \sigma^\epsilon, u^\epsilon_s)^2 ds.
$$

Now we consider the expectation after taking the supremum with respect to $t$ of both sides, we deduce that

$$
C_4 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|u^\epsilon_s\|_{L^2}^4 \right] + C_4 \mathbb{E} \left[ \int_0^t \|\nabla u^\epsilon_s\|_{L^2}^2 \|u^\epsilon_s\|_{L^2}^2 ds \right]
\leq \|u_0\|_{L^2}^4 + \int_0^t \mathbb{E}[\|u^\epsilon_s\|_{L^2}^4] ds + \int_0^t \mathbb{E}[\sup_{0 \leq r \leq s}\|u^\epsilon_r\|_{L^2}^4] ds
+ \mathbb{E}[\sup_{0 \leq s \leq t} \int_0^s \sigma^\epsilon(m^\epsilon u^\epsilon_s, u^\epsilon_s)^2 W_r].
$$

By Burkholder-Davis-Gundy inequality [18], the third term of the right hand side can be estimated as

$$
\mathbb{E}[\sup_{0 \leq s \leq t} \int_0^s \sigma^\epsilon(m^\epsilon u^\epsilon_s, u^\epsilon_s)^2 W_r]
\leq C_5 \mathbb{E}[\left( \int_0^t (1 + \|u^\epsilon_s\|_{L^2}^2) \|u^\epsilon_s\|_{L^2}^2 ds \right)^{\frac{3}{2}}]
\leq \frac{C_5}{2} \mathbb{E}[\sup_{0 \leq s \leq t} \|u^\epsilon_s\|_{L^2}^4] + C_6 \mathbb{E}[\int_0^t (1 + \|u^\epsilon_s\|_{L^2}^2) \|u^\epsilon_s\|_{L^2}^2 ds].
$$

Thus, we conclude that

$$
\sup_{0 \leq t \leq T} \mathbb{E}[\|u^\epsilon_t\|_{L^2}^4] \leq C_7 (1 + \|u_0\|_{L^2}^4).
$$
Moreover, from (3.6)
\[
\mathbb{E}\left[\left(\int_0^T ||\nabla u_\epsilon^s||_{L^2}^2 ds\right)^2\right] \leq C_8 (1 + ||u_0||_{L^2}^4) + C_8 \mathbb{E}\left[\left(\int_0^T ||u_\epsilon^s||_{L^2}^2 ds\right)^2\right] + C_8 \mathbb{E}\left[\left(\int_0^T (m^\epsilon \sigma^\epsilon u_\epsilon^s, u_\epsilon^s) dW_s\right)^2\right] \leq C_9 (1 + ||u_0||_{L^2}^4).
\]

We can further verify that (3.1) holds for every \(u_0 \in H\), by a density argument.

Next, we shall show the equicontinuity of \(\{(m^\epsilon u^\epsilon, \varphi)\}_{\epsilon > 0}\) for each \(\varphi \in C_\infty_c(\mathbb{R})\).

**Lemma 5.** Let \(u^\epsilon\) be the solution of the heterogeneous equation (1.1) with initial value \(u_0 \in H\). Then, for every \(\varphi \in C_\infty_c(\mathbb{R})\), there exists a positive constant \(C\) such that
\[
\sup_\epsilon \mathbb{E}[|m^\epsilon u_\epsilon^t - m^\epsilon u_\epsilon^s, \varphi|^4] \leq C|t - s|^2 (1 + ||u_0||_{L^2}^4),
\]
for all \(s, t \in [0, T]\).

**Proof.** From the definition of \(u^\epsilon\), we infer that
\[
(m^\epsilon u_\epsilon^t - m^\epsilon u_\epsilon^s, \varphi)^4 \leq 8\left(\int_s^t (T_m^\epsilon u_\epsilon^r, \varphi) dr\right)^4 + 8\left(\int_s^t (m^\epsilon \sigma^\epsilon u_\epsilon^r, \varphi) dW_r\right)^4.
\]

Note that there exists a function \(\gamma\) in \(H^1(\mathbb{T})\), such that
\[
\beta^m(y) = \gamma'(y), y \in \mathbb{T}.
\]
Consider the following partial differential equation on \(\mathbb{T}\):
\[
\begin{cases}
(\bar{A}_m)^* \varphi - \beta^m = 0, \\
\int_0^1 \varphi(y) dy = 0,
\end{cases}
\]
where \(\bar{A}_m = m\bar{A}\). Then
\[
\gamma(y) = a^m(y) \varphi'(y) - \beta^m \varphi(y).
\]
We denote $(\Gamma u)(x) = \int_{\mathbb{R}} c\left(\frac{x-y}{\epsilon}\right)(u(y) - u(x))dy$ and $\beta^m, \gamma^\epsilon = \gamma(x^\epsilon)$. It follows that $\Gamma$ is a symmetric operator on $L^2(\mathbb{R})$. Moreover, 

\[
<T_\epsilon^m u, \varphi> = (a^m, \epsilon u', \varphi') - \epsilon^{-1}(\beta^m, \epsilon u, \varphi') \\
\quad + \frac{1}{2}\int_{\mathbb{R}} \lambda^m, \epsilon(x)\int_{\mathbb{R}} c\left(\frac{x-y}{\epsilon}\right)(u(y) - u(x)) (\varphi(y) - \varphi(x)) dy \ dx \\
\quad + \varphi(y)(u(x) - u(y))|dydx \\
\leq (a^m, \epsilon u', \varphi') - \epsilon^{-1}(\beta^m, \epsilon u, \varphi') + \frac{\alpha_2}{\delta} (\Gamma u(x), \varphi(x)),
\]

where $||\Gamma u||^2_{L^2} \leq a_1^2 ||u||^2_{L^2}$. Hence

\[
<T_\epsilon^m u, \varphi \leq (a^m, \epsilon u', \varphi') + (\gamma^\epsilon u', \varphi') + (\gamma^\epsilon u, \varphi') + C_0 ||u||_{L^2}||\varphi||_{L^2} \\
\leq C_1 ||\varphi||_{H^2} ||u||_{H^1},
\]

for some $C_0, C_1 > 0$. Then we deduce that

\[
\mathbb{E}[(m^\epsilon u_t^\epsilon - m^\epsilon u_{s}^\epsilon, \varphi)^4] \leq C_2 ||\varphi||^4_{H^2} |t-s|^2 \{\mathbb{E}[\int_s^t ||u_t^\epsilon||^2_{H^1} dr]^2 + 1 \\
\quad + \mathbb{E}[\sup_{0 \leq t \leq T} ||u_t^\epsilon||^4_{L^2}] \leq C_3 ||\varphi||^4_{H^2} |t-s|^2 (1 + ||u_0||^4_{L^2})\}.
\]

The proof is complete. 

As a result of Lemma 4 and Lemma 5, we will show the Lemma 3.

**Proof.** It is obvious that

\[
\sup_{\epsilon} \mathbb{E}[\sup_{0 \leq t \leq T} ||m^\epsilon u_t^\epsilon||^4_{L^2}] < \infty.
\]

By Kolmogorov’s tightness criterion [9], we can obtain the tightness of real valued processes \(\{m^\epsilon u_t^\epsilon, \varphi; \epsilon > 0\}\) for every $\varphi \in C^\infty_c(\mathbb{R})$. Since the injection $H \hookrightarrow H^{-1}_{-\lambda}$ is compact, \(\{\pi^{m, \epsilon}; \epsilon > 0\}\) is tight in $(K, \mathcal{T}_1)$ (Section 4 of [31]).

From the tightness for \(\{\pi^{m, \epsilon}; \epsilon > 0\}\), we conclude with the following result.

**Lemma 6.** There exist a subsequence $\epsilon_k \to 0$ as $k \to \infty$, and a probability measure $\bar{\pi}$ on $K$ such that

\[
\pi^{m, \epsilon_k} \Rightarrow \bar{\pi} \quad \text{in} \quad (K, \mathcal{T}_1).
\]

Next, we will show that $\bar{\pi}$ is also the limit measure of $\pi^{\epsilon_k}$ in $(K, \mathcal{T}_2)$.
**Lemma 7.** Let $\epsilon_k$ be the subsequence in Lemma 6 and $\pi^{\epsilon_k}$ be the probability measure induced by $u^{\epsilon_k}$. Then

$$\pi^{\epsilon_k} \Rightarrow \tilde{\pi} \quad \text{in} \quad (K, \mathcal{T}_2),$$

as $k \to \infty$.

**Proof.** From Skorokhod’s theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $K$-valued random variables $\tilde{u}_{m,\epsilon_k}^\ast, \tilde{u}^\ast$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that $\pi_{m,\epsilon_k}$ and $\tilde{\pi}$ are the laws of $\tilde{u}_{m,\epsilon_k}^\ast$ and $\tilde{u}^\ast$, respectively. Moreover,

$$\tilde{u}_{m,\epsilon_k}^\ast \to \tilde{u} \quad \text{in} \quad (K, \mathcal{T}_1), \tilde{\mathbb{P}}\text{-almost surely.}$$

Now we want to show that

$$\tilde{\mathbb{E}}\left[ \int_0^T \left| \left( (m^{\epsilon_k})^{-1} \tilde{u}_{m,\epsilon_k}^\ast - \tilde{u}^\ast \right) \right|_{H_{-\lambda}}^2 \right] \to 0,$$

as $k \to \infty$, where $\tilde{\mathbb{E}}$ stands for the expectation with respect to $\tilde{\mathbb{P}}$. First, we have

$$\tilde{\mathbb{E}}\left[ \int_0^T \left| \left( (m^{\epsilon_k})^{-1} \tilde{u}_{m,\epsilon_k}^\ast - \tilde{u}^\ast \right) \right|_{H_{-\lambda}}^2 dt \right]$$

$$\leq 2\tilde{\mathbb{E}}\left[ \int_0^T \left| (m^{\epsilon_k})^{-1} \tilde{u}_{m,\epsilon_k}^\ast (1 - m^{\epsilon_k}) \right|_{H_{-\lambda}}^2 dt \right]$$

$$+ 2\tilde{\mathbb{E}}\int_0^T \left| \tilde{u}_{m,\epsilon_k}^\ast - \tilde{u}^\ast \right|_{H_{-\lambda}}^2 dt$$

$$= 2\tilde{\mathbb{E}}\left[ \int_0^T \left| u_{m,\epsilon_k}^\ast (1 - m^{\epsilon_k}) \right|_{H_{-\lambda}}^2 dt \right] + 2\tilde{\mathbb{E}}\int_0^T \left| \tilde{u}_{m,\epsilon_k}^\ast - \tilde{u}^\ast \right|_{H_{-\lambda}}^2 dt.$$

Let $\zeta(y)$ be a solution of the following partial differential equation:

$$\begin{aligned}
\Delta \zeta(y) &= 1 - m(y), \quad y \in \mathbb{T} \\
\int_0^1 \zeta(y) dy &= 0.
\end{aligned}$$

Then, we have

$$\left| u_{m,\epsilon_k}^\ast (1 - m^{\epsilon_k}) \right|_{H_{-\lambda}} = \left| u_{m,\epsilon_k}^\ast \Delta \zeta^{\epsilon_k} \right|_{H_{-\lambda}} \leq \sup_{v \neq 0} \frac{(u_{m,\epsilon_k}^\ast \Delta \zeta^{\epsilon_k}, v)}{\left| v \right|_{H^1}} \leq C\epsilon_k \left| u_{m,\epsilon_k}^\ast \right|_{H^1},$$
for some positive constant $C$, independent of $\epsilon_k$. Therefore, we conclude that

$$
\mathbb{E}\left[ \int_0^T \left\| (m^{\epsilon_k})^{-1}u^{\epsilon_k}_t - \tilde{u}_t \right\|^2_{H^{-1}_{-\lambda}} dt \right] \leq 2\epsilon_k^2 C^2 \mathbb{E}\left[ \int_0^T \left\| u^{\epsilon_k}_t \right\|^2_{H^1} dt \right] + 2\mathbb{E}\left[ \int_0^T \left\| \tilde{u}_t^{m^{\epsilon_k}} - \tilde{u}_t \right\|^2_{H^{-1}_{-\lambda}} dt \right] \to 0,
$$

as $k \to \infty$. Since $H^1, H_{-\lambda}, H^{-1}_{-\lambda}$ are reflective Banach space, and the inclusion $H^1 \hookrightarrow H_{-\lambda}$ is compact, $H_{-\lambda} \hookrightarrow H^{-1}_{-\lambda}$ is continuous. We know that, for every $\rho > 0$, there exists a constant $C(\rho) > 0$ such that

$$
\| \psi \|_{H^{-1}_{-\lambda}} \leq \rho \| \psi \|_{H^1} + C(\rho) \| \psi \|_{H^{-1}_{-\lambda}}, \quad \psi \in H^1.
$$

Hence the conclusion in this lemma follows. \hfill \square

3.2. Identification of the limit law. In this section, we will verify that $\tilde{\pi}$ coincides with the law induced by the solution of the homogenized equation (1.3).

Let $X = (X_t)$ be the canonical process, that is, $X_t(\omega) = \omega_t$ for $\omega \in K$, and $D_t$ the canonical filtration on $K$. We define a $\sigma$-field $\mathcal{D} = \bigvee_{0 \leq t \leq T} D_t$.

**Definition 1.** A probability measure $\mu$ on $(K, \mathcal{D})$ is called a solution of martingale problem for the homogenized equation (1.3) if $\mu$ satisfies the following conditions:

(i) The probability $\mu(X_0(\cdot) = u_0) = 1$; and

(ii) For every $\phi \in C^\infty_c(\mathbb{R})$ and $\xi \in C^\infty_c(\mathbb{R})$, the stochastic process $H_{\phi, \xi}(t)$ defined by

$$
H_{\phi, \xi}(t) = H_{\phi, \xi}^{T_0, M_0}(t) = \phi((X_t, \xi)) - \phi((X_0, \xi)) - \int_0^t \phi'((X_s, \xi))(X_s, (T^0)^*\xi) ds - \frac{1}{2} \int_0^1 \phi''((X_s, \xi))(M^0(X_s), \xi)^2 ds,
$$

is a continuous local martingale under $\mu$. That is $\mathbb{E}(H_{\phi, \xi}(t)|D_s) = H_{\phi, \xi}(s)$, $0 \leq s \leq t \leq T$.

**Lemma 8.** The homogenized equation (1.3) has at most one martingale solution on $(K, \mathcal{D})$.

**Proof.** For a solution $u_t$ of the homogenized equation (1.3), we set $\tilde{u}_t = u_t e^{-\lambda t}$.

Similar to Lemma 2, we have
\( \mathbb{P}(\tilde{u}_t = \tilde{v}_t, \text{ in } H^{-1}, \ \forall t \in [0, T]) = 1, \)

for two stochastic processes \( u, v \) satisfying the homogenized equation \((1.3)\). Hence we obtain the pathwise uniqueness of stochastic partial differential equations \((1.3)\) on \( K \).

We now show that \( \tilde{\pi} \) is the martingale solution for the homogenized equation \((1.3)\).

The stochastic process \( H_{\phi, \xi}(t) \) is a continuous \( D_t - \)martingale under the probability measure \( \tilde{\pi} \), that is, for every bounded, \( D_s - \)measurable function \( \Phi \) on \( K \),

\[
\mathbb{E}^{\tilde{\pi}}[\Phi(H_{\phi, \xi}(t) - H_{\phi, \xi}(s))] = 0, \ 0 \leq s \leq t \leq T.
\]

Due to the uniqueness for the homogenized equation \((1.3)\), we can use \( \epsilon \) in place of \( \epsilon_k \), the subsequence in Lemma 6. Without lost of generality, we assume that \( \Phi \) is continuous with respect to the supremum topology of \( T_1 \) and \( T_2 \).

Define \( H^{\epsilon}_{\phi, \xi}(t) \) on \( (K, D) \) by

\[
H^{\epsilon}_{\phi, \xi}(t) = \phi((m^\epsilon X_t, \xi)) - \phi((m^\epsilon X_0, \xi)) - \int_0^t \phi'((m^\epsilon X_s, \xi))(X_s, (T^0)^* \xi)ds
- \frac{1}{2} \int_0^1 \phi''((m^\epsilon X_s, \xi))(M^0(X_s), \xi)^2 ds.
\]

From Lemma 7, we can extract a subsequence \( \{\epsilon_l\}_{l \geq 1} \) such that

\[
(m^\epsilon)^{-1} \tilde{u}^{m, \epsilon_l} \rightarrow \tilde{u} \ \text{ in } (K, T_2), \ \tilde{\mathbb{P}} - \text{ almost surely}
\]
as \( l \rightarrow \infty \). Therefore, we have

\[
\mathbb{E}^{\pi^{\epsilon_l}}[\Phi(H^{\epsilon_l}_{\phi, \xi}(t) - H^{\epsilon_l}_{\phi, \xi}(s))] = \mathbb{E}[\Phi((m^\epsilon)^{-1} \tilde{u}^{m, \epsilon_l})(H^{\epsilon_l}_{\phi, \xi}(t) - H^{\epsilon_l}_{\phi, \xi}(s))]
\]

as \( l \rightarrow \infty \) in view of bounded convergence theorem. Then we just need to show that

\[
\mathbb{E}^{\pi^{\epsilon_l}}[\Phi(H^{\epsilon_l}_{\phi, \xi}(t) - H^{\epsilon_l}_{\phi, \xi}(s))] \rightarrow 0
\]
as \( \epsilon \to 0 \). For \( \xi_\epsilon \in C^\infty_c(\mathbb{R}) \), we define another function \( H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(t) \) on \( S \) by

\[
H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(t) = \phi((X_t, \xi_\epsilon)) - \phi((X_0, \xi_\epsilon)) - \int_0^t \phi'(X_s, \xi_\epsilon))(X_s, (T^\epsilon)^s \xi_\epsilon)ds - \frac{1}{2} \int_0^t \phi''((X_s, \xi_\epsilon))(\sigma^\epsilon(x)X_s, \xi_\epsilon)^2 ds.
\]

It follows that

\[
\mathbb{E}^\epsilon\mathbb{E}_\epsilon[T\{H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(t) - H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(s)\}] = 0, \quad 0 \leq s \leq t \leq T.
\]

So, we consider

\[
\mathbb{E}^\epsilon\mathbb{E}_\epsilon[T\{H^{T_\epsilon}_{\phi, \xi_\epsilon}(t) - H^{T_\epsilon}_{\phi, \xi_\epsilon}(s) - H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(t) + H^{T_\epsilon, M_\epsilon}_{\phi, \xi_\epsilon}(s)\}]
\]

which is equal to

\[
(3.7) \quad \mathbb{E}\{\Phi(u^\epsilon)\{I_1(u^\epsilon) - I_2(u^\epsilon) - I_3(u^\epsilon) - I_4(u^\epsilon) - I_5(u^\epsilon)\}\},
\]

where

\[
I_1 = \phi(m^\epsilon X_t, \xi) - \phi((X_t, \xi_\epsilon)) - \phi((m^\epsilon X_s, \xi)) + \phi((X_s, \xi_\epsilon)),
\]

\[
I_2 = \int_s^t \{\phi'(m^\epsilon X_r, \xi) - \phi'(X_r, \xi_\epsilon)\}(X_r, (T^0)^s \xi)dr,
\]

\[
I_3 = \int_s^t \phi'(X_r, \xi_\epsilon)(X_r, (T^0)^s \xi - (T^0)^s \xi_\epsilon))dr,
\]

\[
I_4 = \frac{1}{2} \int_s^t \{\phi''(m^\epsilon X_r, \xi) - \phi''(X_r, \xi_\epsilon)\}(m^0(X_r), \xi)^2 dr,
\]

\[
I_5 = \frac{1}{2} \int_s^t \phi''(X_r, \xi_\epsilon)\{(m^0(X_r), \xi)^2 - (\sigma^\epsilon(x)X_r, \xi_\epsilon)^2\}dr.
\]

We will construct a family of test functions \( \xi^\epsilon \in C^\infty_c(\mathbb{R}) \) such that (3.7) goes to 0 as \( \epsilon \to 0 \). So we define \( \xi^\epsilon \in C^\infty_c(\mathbb{R}) \) as follows:

\[
(3.8) \quad \xi^\epsilon(x) = m(x^\epsilon)(\xi(x) + \epsilon h_1(x^\epsilon)\xi'(x) + \epsilon^2 h_2(x^\epsilon)\xi''(x)),
\]

where the periodic function \( h_1, h_2 \) are in \( L^2(\mathbb{T}) \). Then, we have

\[
(T^\epsilon)^s(\xi^\epsilon)(x) = \frac{1}{\epsilon^3} \int_{\mathbb{R}} c(\frac{x-y}{\epsilon})\{\lambda(\frac{y}{\epsilon})m(\frac{y}{\epsilon})(\xi(y)+\epsilon h_1(\frac{y}{\epsilon})\xi'(y) + \epsilon^2 h_2(\frac{y}{\epsilon})\xi''(y))
\]

\[
- \lambda(\frac{x}{\epsilon})m(\frac{x}{\epsilon})(\xi(x) + \epsilon h_1(\frac{x}{\epsilon})\xi'(x) + \epsilon^2 h_2(\frac{x}{\epsilon})\xi''(x))
\]

\[
+ \{a^m(\frac{x}{\epsilon})(\xi(x)+\epsilon h_1(\frac{x}{\epsilon})\xi'(x) + \epsilon^2 h_2(\frac{x}{\epsilon})\xi''(x))\}''
\]

\[
- \frac{1}{\epsilon}(b^m(\frac{x}{\epsilon})(\xi(x)+\epsilon h_1(\frac{x}{\epsilon})\xi'(x) + \epsilon^2 h_2(\frac{x}{\epsilon})\xi''(x)))dy.
\]
We consider the \((T^*)^*(\xi)\) separately,

\[
(B^*)^*(\xi) = \frac{1}{\epsilon^2} \int c(z)dz \{ \lambda(x + z)\xi(x) - \lambda(x)\xi(x) + \epsilon \int c(z)dz \{ \lambda(x + z)\xi(x) - \lambda(x)\xi(x) \}
\]

Using the following identities based on the integral form of remainder term in the Taylor expansion

\[
\xi(y) = \xi(x) + \int_0^1 \frac{\partial}{\partial t} \xi(x + (y - x)t)dt = \xi(x) + \int_0^1 \xi'(x + (y - x)t) \cdot (y - x)dt,
\]

and

\[
\xi(y) = \xi(x) + \xi'(x)(y - x) + \int_0^1 \xi''(x + (y - x)t)(y - x) \cdot (y - x)(1 - t)dt,
\]

which is valid for each \(x, y \in \mathbb{R}\), we conclude that

\[
(B^*)^*(\xi) = \frac{1}{\epsilon^2} \int c(z)dz \{ \lambda(x + z)\xi(x) - \lambda(x)\xi(x) + \epsilon \int c(z)dz \{ \lambda(x + z)\xi(x) - \lambda(x)\xi(x) \}
\]

Collecting the equal power terms with \((A^*)^*\xi\), we obtain

\[
(T^*)^*(\xi) = \frac{1}{\epsilon^2} \xi(x) \{ \int c(z)\lambda(y - z)m(y - z) - \lambda(y)\xi(y)dz + (a(y)m(y))' - b(y)m(y)
\]

\[
- (b(y)m(y))' + \frac{1}{\epsilon} \xi'(x) \{ \int c(z)[(-z + h_1(y - z))\lambda(y - z)m(y - z) - \lambda(y)m(y)h_1(y)]dz + 2(a(y)m(y))' + (a(y)m(y)h_1(y))'' - b(y)m(y)
\]

\[
- \lambda(y)m(y)h_1(y)]dz + 2(a(y)m(y)h_1(y))'' - b(y)m(y)
\]

\[
+ h_2(y - z)) - \lambda(y)m(y)h_2(y)]dz + a(y)m(y) + 2(a(y)m(y)h_1(y))' + (a(y)m(y)h_2(y))'' - b(y)m(y)h_2(y))'' - h_1(y)b(y)m(y) + \phi(x),
\]
with
\[
\phi_\epsilon(x) = \frac{1}{\epsilon^2} \int_{\mathbb{R}} dz c(z) \{ \epsilon^2 \int_0^1 \lambda(y-z)m(y-z)\xi''(x-\epsilon z)t z^2(1-t)dt
- \frac{\epsilon^2}{2} \lambda(y-z)m(y-z)\xi''(x)z^2 + \epsilon^3 h_1(y-z) \int_0^1 \xi'''(x-\epsilon z)t z^2(1-t)dt
- \epsilon^3 h_2(y-z) \int_0^1 \xi'''(x-\epsilon z)t dt \}.
\]

Our next two steps are to show that \( \| \phi_\epsilon(x) \|_{L^2} \) is vanishing and construct the corrector \( h_1, h_2 \) to ensure that \( (T_\epsilon^*) (\xi_\epsilon) \to Q_\xi'' \) as \( \epsilon \to 0 \).

Choose the term of order \( \epsilon^0 \), and denote it by \( \phi_\epsilon^{(1)}(x) \). For an arbitrary positive constant \( M \), we infer that
\[
\phi_\epsilon^{(1)}(x) = \frac{1}{\epsilon^2} \int_{\{|z| \leq M \cup |z| > M\}} dz c(z) \epsilon^2 \lambda(y-z)m(y-z) \int_0^1 (\xi''(x-\epsilon z) - \xi''(x))z^2(1-t)dt := \phi_\epsilon^{(2)}(x) + \phi_\epsilon^{(3)}(x).
\]

Then
\[
\| \phi_\epsilon^{(2)} \|_{L^2} \leq \frac{\alpha_2}{2\delta} \sup_{|z| \leq M} \| \xi''(x-\epsilon z) - \xi''(x) \|_{L^2} \int_{\mathbb{R}} z^2 c(z) dz,
\]
\[
\| \phi_\epsilon^{(3)} \|_{L^2} \leq \frac{2\alpha_2}{\delta} \| \xi''(x) \|_{L^2} \int_{\mathbb{R}} z^2 c(z) dz.
\]

If we take \( M = \frac{1}{\sqrt{\epsilon}} \), then
\[
\| \phi_\epsilon^{(2)} \|_{L^2} \to 0 \quad \text{and} \quad \| \phi_\epsilon^{(3)} \|_{L^2} \to 0, \quad \text{as} \quad \epsilon \to 0.
\]

This implies that
\[
\| \phi_\epsilon^{(1)} \|_{L^2} \to 0, \quad \epsilon \to 0.
\]

For the second part of \( \phi_\epsilon(x) \),
\[
\phi_\epsilon^{(4)}(x) = \epsilon \int_{\mathbb{R}} c(z) dz h_1 \left( \frac{x}{\epsilon} - z \right) \cdot \int_0^1 \xi'''(x-\epsilon z)t z^2(1-t)dt,
\]
we have
\[
(3.9) \quad \| \phi_\epsilon^{(4)}(x) \|_{L^2} \leq \frac{\epsilon \alpha_2}{2\delta} \sup_{z,q \in \mathbb{R}} \| h_1 \left( \frac{x}{\epsilon} - z \right) \xi'''(x-\epsilon z + q) \|_{L^2} \int_{\mathbb{R}} z^2 c(z) dz.
\]

Next, we estimate
\[
\sup_{z,q \in \mathbb{R}} \| h_1 \left( \frac{x}{\epsilon} - z \right) \xi'''(x-\epsilon z + q) \|_{L^2}.
\]
Taking \( y = x - \epsilon z \), we deduce that

\[
\sup_{q \in \mathbb{R}} \| h_1 \left( \frac{y}{\epsilon} \right) \xi'''(y + q) \|_{L^2} = \sup_{q \in \mathbb{T}} \| h_1 \left( \frac{y}{\epsilon} \right) \xi'''(y + q) \|_{L^2} \\
\leq \sup_{q \in \mathbb{T}} \sum_{k \in \mathbb{Z}} \int_{k+\epsilon}^{k+\epsilon+\epsilon} [h_1 \left( \frac{y}{\epsilon} \right)]^2 \xi'''(y + q)^2 dy \\
\leq \| h_1 \|_{L^2}^2 \sum_{k \in \mathbb{Z}} \max_{y \in [k+\epsilon, k+\epsilon+\epsilon], q \in \mathbb{T}} [\xi'''(y + q)]^2 dy \\
\to \| h_1 \|_{L^2}^2 \| \xi''' \|_{L^2}^2,
\]

as \( \epsilon \to 0 \). Thus from (3.9), it follows that \( \| \phi_1^{(4)} \|_{L^2} \to 0 \), as \( \epsilon \to 0 \).

Similarly, for the third term, we have

\[
\| \phi_1^{(5)}(x) \|_{L^2} = \left\| \epsilon \int_{\mathbb{R}} dz c(z) h_2(y - z) \int_{0}^{1} \xi'''(x - \epsilon z t) z dt \right\|_{L^2} \\
\to 0.
\]

In summary, we have \( \phi_1^{(1)}(x) \to 0 \), as \( \epsilon \to 0 \).

Our next step is to construct the correctors \( h_1 \) and \( h_2 \). First, we collect all the terms of the order \( \epsilon^{-2} \) and equate them to 0.

\[
(3.10) \quad \int_{\mathbb{R}} c(z) ([\lambda(y - z)m(y - z) - \lambda(y)m(y)]) dz' \\
+ (a(y)m(y))'' - (b(y)m(y)) = (\hat{T})^*m(y) = 0.
\]

Similarly, for the terms of order \( \epsilon^{-1} \),

\[
(3.11) \quad 0 = \int_{\mathbb{R}} c(z)[-z + h_1(y - z)]\lambda(y - z)m(y - z) \\
- \lambda(y)m(y)h_1(y) dz + 2(a(y)m(y))' \\
+ (a(y)m(y)h_1(y))'' - b(y)m(y) - (b(y)m(y)h_1(y))'.
\]

We denote \( \hat{T}_m = m \hat{T} \). Then

\[
(3.12) \quad (\hat{T}_m)^*(h_1) = \int_{\mathbb{R}} zc(z)m(y - z)\lambda(y - z) dz + b(y)m(y) \\
- 2(a(y)m(y))' = l(y),
\]

where \( l(y) \) is bounded for every \( y \in \mathbb{T} \),

\[
| \int_{\mathbb{R}} zc(z)m(y - z)\lambda(y - z) dz | \leq C \int_{\mathbb{R}} c(z)|z| dz < \infty.
\]
For $y - z = q$, define the bilinear form:

$$a[u, v] = \int_T \int_\mathbb{R} c(y - q)(\lambda^m(q)u(q) - \lambda^m(y)u(y))\,d\eta(y)\,dy$$

$$+ \int_T (a^m(y)u(y))^\prime v(y)\,dy - \int_T (b^m(y)u(y))'v(y)\,dy.$$ 

Next, we should consider the existence and uniqueness of the equation (3.12). First, we give a lemma to verify conditions of the Fredholm alternative theorem.

**Lemma 9.** If the assumptions (i) – (v) are satisfied, then there exist constants $\alpha, \mu, \nu$, such that:

$$|a[u, v]| \leq \nu \|u\|_{H^1} \|v\|_{H^1},$$

and

$$\frac{\alpha}{2} \|u\|^2_{H^1} \leq a[u, u] + \mu \|u\|^2_{L^2},$$

for every $u, v \in H^1(T)$.

**Proof.** Note that

$$|a[u, v]| \leq |\int_T \int_\mathbb{R} c(y - \eta)(\lambda^m(\eta)u(\eta) - \lambda^m(y)u(y))\,d\eta(y)\,dy|$$

$$+ |\int_T (a^m(y)u(y))^\prime v(y)\,dy| + |\int_T (b^m(y)u(y))'v(y)\,dy|.$$  

(3.13)

For the first term of (3.13),

$$\int_T \int_\mathbb{R} c(y - \eta)\lambda^m(\eta)u(\eta)d\eta\,dy \leq \int_T \int_\mathbb{R} c(y - \eta)\lambda^m(y)u(y)d\eta\,dy$$

$$\leq \int_T (\int_\mathbb{R} c(y - \eta)\lambda^m(\eta)d\eta)^2\,dy \leq a_1 a_2 \int_T u(y)v(y)\,dy.$$ 

In fact,

$$\int_T (\int_{\mathbb{R}} c(y - \eta)\lambda^m(\eta)u(\eta)d\eta)^2\,dy$$

(3.14) 

$$= \int_T (\int_\mathbb{R} c(y - \eta)\lambda^m(\eta)u(\eta)d\eta)(\int_\mathbb{R} c(y - q)\lambda^m(q)u(q)dq)\,dy$$

$$\leq \frac{\alpha^2}{\delta^2} \int_\mathbb{R} c(\eta)d\eta \int_\mathbb{R} c(q)dq \int_T u(y + \eta)u(y + \eta)\,dy \leq \frac{\alpha^2 \alpha^2}{\delta^2} \|u\|^2_{L^2}.$$
Combining with (3.13), we conclude that
\[ |a[u,v]| \leq C_3 \|u\|_{L^2} \|v\|_{L^2} + C_4 (\|u\|_{L^2} \|v'\|_{L^2} + \|u'\|_{L^2} \|v'\|_{L^2}) + C_5 \|u\|_{L^2} \|v'\|_{L^2} \leq \nu \|u\|_{H^1} \|v\|_{H^1}. \]

We now use the assumptions to infer that
\[ \alpha \|u'\|_{L^2} \leq - \int_T (a(y)u(y))''u(y)dy = a[u,u] - \int_T (b(y)u(y))'u(y)dy + \int \int_R c(y-\eta)(\lambda(\eta)u(\eta) - \lambda(y)u(y))d\eta u(y)dy \leq a[u,u] + \int \|b\|_{\infty} |u'||u| + C_7 |u|^2 dy. \]

Using this in the second term on the right hand side of (3.15), we obtain
\[ \int_T |u'||u|dy \leq \delta \|u'\|_{L^2}^2 + \frac{1}{4\delta} \|u\|_{L^2}^2. \]

We choose \( \delta \), so that
\[ \alpha - \|b\|_{\infty} \delta = \frac{\alpha}{2}. \]

Thus
\[ \frac{\alpha}{2} \|u'\|_{L^2}^2 \leq a[u,u] + \frac{1}{4\delta} \|b\|_{\infty} \|u\|_{L^2}^2 + C_7 \|u\|_{L^2}^2. \]

We now add \( \frac{\alpha}{2} \|u\|_{L^2}^2 \) on both sides of the preceding inequality to obtain
\[ \frac{\alpha}{2} \|u\|_{H^1}^2 \leq a[u,u] + \mu \|u\|_{L^2}^2, \]

with
\[ \mu = \frac{1}{4\delta} \|b\|_{\infty} + C_7 + \frac{\alpha}{2}. \]

This proves the lemma. \( \square \)

Now we consider the resolvent operator
\[ R(\tilde{T}_m)^*(\lambda) = (\tilde{T}_m)^* + \lambda I)^{-1}, \]
where $I$ stands for the identity operator and $\lambda > 0$. Note that this operator is compact. For $\lambda$ sufficiently large, consequently, Fredholm theorem can be used. It is easy to see that $\text{Ker}(\tilde{T}_m)^* = \{C\}$, where $C$ is a constant. Then the solvability condition for (3.12) takes the form:

(3.16) \[ \int_T l(y)dy = 0. \]

Next we will confirm the validity of condition (3.16). We take $z = y - q$.

Noting the fact that

\[
\int_{\mathbb{R}} \int_Tqc(y-q)m(y)\lambda(y)dqdy = \int_{\mathbb{R}} \int_Tyc(y-q)m(q)\lambda(q)dqdy,
\]

we infer that

\[
\int_T l(y)dy = \int_{\mathbb{R}} \int_T qc(y-q)(m(y)\lambda(y) - m(q)\lambda(q))dqdy
+ \int_T b(y)m(y)dq - \int_T 2(a(y)m(y))'dy
= \int_T y \int_{\mathbb{R}} c(q-y)(m(q)\lambda(q) - m(y)\lambda(y))dqdy
+ \int_T b(y)m(y)dy - \int_T 2(a(y)m(y))'dy
= \int_T y[-(a(y)m(y))'' + (b(y)m(y))']dy
+ \int_T b(y)m(y)dy - \int_T 2(a(y)m(y))'dy
= 0.
\]

Thus, the solution $h_1$ of equation (3.12) exists and is unique to a constant. In order to fix the choice of this constant, we assume that the average of each component of $h_1$ over the period is equal to 0.

As the next step we collect the term of the order $\varepsilon^0$. Our goal is to find the function $h_2$, such that the sum of these terms will be equal to $Qu''_0$ with $Q > 0$. Due to

(3.17) \[
(\tilde{T}_m)^*(h_2(y)) = -Q + \int_{\mathbb{R}} dzc(z)\lambda(y-z)m(y-z)(\frac{1}{2}z^2 - zh_1(y-z)) + a(y)m(y)
+ 2(a(y)m(y)h_1(y))' - b(y)m(y)h_1(y),
\]
we see that $Q$ is determined from the following solvability condition for equation (3.16)

$$Q = \int_T \int_R c(z)\lambda(y-z)m(y-z)(\frac{1}{2}z^2 - zh(y-z))dz + \int_T a(y)m(y)dy - \int_T b(y)m(y)h_1(y)dy.$$ 

From

$$\int_T \int_R c(z)\lambda(y-z)m(y-z)zh(y-z)dz = \int_T \int_R (y-q)c(y-q)\lambda(q)m(q)h_1(q)dqdy$$

$$= \int_T \int_R c(y-q)\lambda(y)m(y)h_1(y)dqdy$$

$$= \int_T \int_R [c(y-q)(q-y)\lambda(y)m(y)h_1(y)dqdy$$

$$= 0,$$

$$\int_T b(y)h_1(y)dy = -\int_T \tilde{T}_m\chi(y)h_1(y)dy = \int_T \chi(y)(\tilde{T}_m)^*h_1(y)$$

$$= \int_T \chi(y)\int_R yc(z)m(y-z)\lambda(y-z)dz + b(y) - 2(a'(y))'dy,$$

and

$$\int_T \chi(y)b(y)dy = -\int_T \chi(y)\tilde{T}_m\chi(y)dy$$

$$= \int_T a(y)(\chi'(y))^2 + \frac{1}{2} \int_T \int_R \lambda(y)c(z)(\chi(y-z) - \chi(y))^2dzdy$$

$$= \int_T a(y)(\chi'(y))^2 + \frac{1}{2} \int_T \int_R \lambda(y)c(y-q)(\chi(q) - \chi(y))^2dqdy,$$

we conclude that

$$Q = \int_T a(y)m(y)(\chi'(y)+1)^2dy + \frac{1}{2} \int_T \int_R c(y-q)\lambda(q)m(q)((y-q) + (\chi(y) - \chi(q))^2 dyq.$$ 

In a word, we have constructed a family of test function $\xi^\varepsilon$ such that $(T^\varepsilon)^*\xi^\varepsilon \to (T^0)^*\xi$, as $\varepsilon$ goes to 0. It may be summarized with the following lemma.
Finally, we will show that we obtain that $E$ is the martingale solution for the homogenized equation (1.5), such that for the function $\xi^\epsilon$ defined by (3.8), we have

$$(T^\epsilon)^*(\xi^\epsilon) = Q\xi^\epsilon_0 + \phi^\epsilon,$$

where

$$\lim_{\epsilon \to 0} \|\phi^\epsilon\|_{L^2} = 0.$$ 

At last, we can show that $E[|I_i(u^\epsilon)|] \to 0$, as $\epsilon$ goes to 0. That is to say $\tilde{\pi}$ is the martingale solution for the homogenized equation (1.3).

**Lemma 10.** Assume that $f \in S(\mathbb{R})$, which is the space of rapidly decaying functions. Then there exist functions $h_1, h_2 \in L^2(\mathbb{T})$ and a positive constant $Q$ defined in (1.5), such that for the function $\xi^\epsilon$ defined by (3.8), we have

$$(T^\epsilon)^*(\xi^\epsilon) = Q\xi^\epsilon_0 + \phi^\epsilon,$$

where

$$\lim_{\epsilon \to 0} \|\phi^\epsilon\|_{L^2} = 0.$$ 

**Proof.** We just need to prove that $E[|I_i(u^\epsilon)|] \to 0$ in (3.7), for each $i = 1, 2, \ldots, 5$. From

$$|m^\epsilon u^\epsilon_i, \xi - (u^\epsilon_i, \xi)| \leq \|m^\epsilon u^\epsilon_i\|_{L^2} \|\xi - (m^\epsilon)^{-1}\xi\|_{L^2} \to 0,$$

we obtain that $E[|I_1(u^\epsilon)|] \to 0$. Similarly, for $i = 2, 4$, we can show that $E[|I_i(u^\epsilon)|] \to 0$. For $i = 3$, we have

$$E[|I_3(u^\epsilon)|] \to 0 \leq C\|T^0)^*(\xi) - (T^\epsilon)^*(\xi)\|_{H^{-1}}E[\int_0^T \|u^\epsilon_r\|_{H^1} dr] \to 0.$$ 

Finally, we will show $E[|I_5(u^\epsilon)|] \to 0$ as $\epsilon \to 0$. Since $u^\epsilon$ and $(m^\epsilon)^{-1}\tilde{u}^{m,\epsilon}$ have the same law on $K$, we deduce that

$$(m^\epsilon)^{-1}\tilde{u}^{m,\epsilon}(x, \xi) - (M^0((m^\epsilon)^{-1}\tilde{u}^{m,\epsilon}), \xi)$$

$$= (m^\epsilon)^{-1}\tilde{u}_r\sigma^\epsilon(x) - \tilde{u}^{m,\epsilon}_r\sigma^\epsilon(x, \xi) + (M^0(\tilde{u}_r) - M^0((m^\epsilon)^{-1}\tilde{u}^{m,\epsilon}), \xi)$$

$$+ (m^\epsilon\tilde{u}_r\sigma^\epsilon(x), (m^\epsilon)^{-1}\xi) - (m^\epsilon\tilde{u}_r\sigma^\epsilon(x) - M^0(\tilde{u}_r), \xi) \to 0,$$

as $\epsilon \to 0$ by using the convergence

$$((m^\epsilon u^\epsilon) - M^0(u), \xi) \to 0,$$

for each $u \in H_{-\lambda}$. Hence $E[|I_5(u^\epsilon)|] \to 0$. The proof is complete. \qed
Proof of Theorem 1

Since the uniqueness of the martingale solution for the homogenized equation (1.3) and the conclusion in Lemma 6, we know that $\pi^{m,\epsilon} \Rightarrow \pi$ in $(K, T_1)$. On the other hand, $\pi^\epsilon$, the law of the solution of the heterogeneous equation (1.1), goes to $\pi$ in $(K, T_2)$ in Lemma 7. At the same time, by Lemma 11, we know that $\pi$ is the martingale solution for the homogenized equation (1.3). In conclusion Theorem 1 is then proved.

4. Application to Data Assimilation. In this section, we present an application of Theorem 1. We consider the following nonlinear filtering problem

(4.1) \[
\begin{align*}
\dot{x}_t &= \frac{1}{\epsilon} b(x_\epsilon) dt + \sigma_1(x_\epsilon) dw_t + dL^\epsilon_t, \\
\dot{y}_t &= \sigma(x_\epsilon) dt + dW_t,
\end{align*}
\]

where $x_t$ is the system state (or signal) and $y_t$ is the observation. Here $w^\epsilon_t, W^\epsilon_t$ are mutually independent Brownian motions, and $L^\epsilon_t$ is a Lévy process with generator

$$
(B^\epsilon u)(x) = \frac{1}{\epsilon^3} \lambda(\frac{x}{\epsilon}) \int_{\mathbb{R}} c(\frac{x-y}{\epsilon})(u(y) - u(x)) dy.
$$

Then the Zakai equation for the conditional probability density function $u$ of the filtering system (4.1) is the following nonlocal stochastic partial differential equation

(4.2) \[
\begin{align*}
\dot{u}^\epsilon(t, x) &= (T^\epsilon)^* u^\epsilon(t, x) dt + u^\epsilon(t, x) \sigma^2(x_\epsilon) dt + u^\epsilon(t, x) \sigma(x_\epsilon) dW_t, \\
u^\epsilon(0, x) &= u^0(x).
\end{align*}
\]

By Theorem 1, we can deduced that: The family of laws $\{\pi^\epsilon : \epsilon > 0\}$ for the solution $u^\epsilon$ of the nonlocal Zakai equation (4.2) converges in $(K, T_1)$ to the law of the solution $u$ for the following effective, homogenized local equation

(4.3) \[
\begin{align*}
\dot{u}(t, x) &= \tilde{T}^0 u(t, x) dt + \int_{\mathbb{R}} m(y) \sigma^2(y) dy \cdot u(t, x) dt + M^0 u(t, x) dW_t, \\
u(0, x) &= u^0(x),
\end{align*}
\]

where $M^0$ is the same operator as (1.4), and

$$
(\tilde{T}^0 u)(x) = Q_0 u''(x),
$$

where $Q_0$ is the same constant as (1.5).
In fact, set \( \tilde{u}^\varepsilon = (m^\varepsilon)^{-1} u^\varepsilon \). Then the system (4.2) can be rewritten as

\[
\begin{aligned}
&\left\{ 
\begin{array}{l}
d\tilde{u}^\varepsilon(t, x) = \tilde{T}^\varepsilon \tilde{u}^\varepsilon(t, x) dt + \tilde{u}^\varepsilon(t, x) \sigma^2(\varepsilon x) dt + \sigma(\varepsilon x) dW_t, \\
\tilde{u}^\varepsilon(0) = (m^\varepsilon)^{-1} u_0,
\end{array}
\right.
\end{aligned}
\]

where

\[
(\tilde{T}^\varepsilon u)(x) = (m^\varepsilon)^{-1}(A^\varepsilon)^* u(x) + \varepsilon^{-3}(m^\varepsilon)^{-1} \int_\mathbb{R} c\left(\frac{x-y}{\varepsilon}\right) (\lambda m^\varepsilon(y) \frac{y}{\varepsilon}) u(y) dy
\]

\[
- \frac{1}{\varepsilon} \left[ b(x) \right] u \frac{y}{\varepsilon} \frac{\sigma \left( \frac{x-y}{\varepsilon} \right) u(y)}{\varepsilon} dy.
\]

In fact, we have

\[
(\varepsilon^{-3} m^\varepsilon)^{-1} \int_\mathbb{R} c\left(\frac{x-y}{\varepsilon}\right) (\lambda m^\varepsilon(y) \frac{y}{\varepsilon}) \tilde{u}(y) - \lambda m^\varepsilon(x) \tilde{u}(y) dy, \varphi(x) \right)_{L^2(\mathbb{R})} = 0.
\]

As \( \varepsilon \) goes to 0, we have

\[
(\varepsilon^{-3} m^\varepsilon)^{-1} \int_\mathbb{R} c\left(\frac{x-y}{\varepsilon}\right) (\lambda m^\varepsilon(y) \frac{y}{\varepsilon}) \tilde{u}(y) - \lambda m^\varepsilon(x) \tilde{u}(y) dy, \varphi(x) \right)_{L^2(\mathbb{R})} = (B^\varepsilon \tilde{u}(x), \varphi).
\]

In a word, we deduced that \( \tilde{T}^\varepsilon u(x) = (m^\varepsilon)^{-1}(A^\varepsilon)^* u(x) + B^\varepsilon u(x) \) in \( L^2(\mathbb{R}) \).

Then \( \tilde{T}^\varepsilon \) satisfy the Assumptions (i) – (v). Using Theorem 1, we infer that the family of laws induced by \( u^\varepsilon = m^\varepsilon \tilde{u}^\varepsilon \) converges weakly in \( (K, T_1) \) to the law \( \pi \) induced by the solution of the following homogenized Zakai equation:

\[
\begin{aligned}
&\left\{ 
\begin{array}{l}
du(t, x) = \tilde{T}^0 u(t, x) dt + M^0 u(t, x) dW_t, \\
u(0, x) = u_0(x),
\end{array}
\right.
\end{aligned}
\]

where

\[
(\tilde{T}^0 u)(x) = Q_1 u''(x) + u(x) \int_T m(y) \sigma^2(y) dy,
\]

\[
Q_1 = \int_T a(y)m(y)(\lambda y(y) + 1)^2 dy + \frac{1}{2} \int_R \int_T c(y-q)\lambda(q)m(q)(\chi_1(y) - \chi_1(q)) dy dq,
\]

and \( \chi_1 \) is the solution of

\[
\begin{aligned}
&\begin{cases}
\tilde{T}\lambda_1(y) - (b(y) - \frac{2}{m(y)}(a(y)m(y))) = 0, \\
\int_T \lambda_1(y)m(y) dy = 0,
\end{cases}
\end{aligned}
\]
where

\[
(\tilde{T}v)(y) := a(y)v''(y) + (b(y) - \frac{2}{m(y)}(a(y)m(y))')v'(y)
\]
\[+ \lambda(y) \int_{\mathbb{R}} c(x-y)(v(x) - v(y))dx, y \in T.
\]

In fact, it is easy to check that \( Q_1 = Q_0 \).

**PART II**

5. **Well-posedness of heterogeneous and homogenized equations.**

We now discuss the well-posedness for (1.10) and (1.11).

Let us denote by \( \{S_\alpha(t), t \geq 0\} \) the analytic semigroup generated by \(-\delta^\alpha(\mathbb{I})(-\Delta)^{\alpha/2}\), moreover, we have \( H^\alpha \hookrightarrow H_{-\lambda} \) is compact. then, \(-\delta^\alpha(\mathbb{I})(-\Delta)^{\alpha/2}\) is the generator of compact semigroup \( S_\alpha(t) \) on \( H_\lambda \). Then, for the equation (1.10) and (1.11), we can obtain the existence and uniqueness of the mild solutions naturally.

6. **Proof of the Theorem 2.** We now prove Theorem 2 by a martingale approach. The proof is also divided into two parts: the tightness and the limit law.

6.1. **Tightness.** We denote \( \pi_1^{m_1, \epsilon} \) the probability measure induced by \( m_1^\epsilon v^\epsilon \). We can infer the next lemma.

**Lemma 12.** If \( v^\epsilon \) is a solution of heterogeneous equation (1.10), and \( \pi_1^{m_1, \epsilon} \) is the probability measure induced by \( m_1^\epsilon v^\epsilon \). Then, \( \{\pi_1^{m_1, \epsilon} : \epsilon > 0\} \) is tight in \( (K_1, T_1) \).

Thanks to the Lemma 4, we can obtain several uniform estimates concerning the solution \( v^\epsilon \) for the original heterogeneous system (1.11).

**Lemma 13.** Let \( v^\epsilon \) be a solution of heterogeneous equation (1.10) with initial value \( v_0 \in H^{\alpha/2} \). Then there exists a positive constant \( C \), independent of \( \epsilon \), such that

\[
\sup_{\epsilon} \mathbb{E}[\sup_{0 \leq t \leq T} ||v^\epsilon_t||^4_{L^2}] + \sup_{\epsilon} \mathbb{E}[\int_0^T ||v^\epsilon_t||^2_{H^{\alpha/2}} dt]^2 \leq C ||v_0||^4_{L^2}.
\]
Next, we shall show the equicontinuity of \( \{ (m_1^\epsilon, v^\epsilon, \varphi) \}_{\epsilon > 0} \) for each \( \varphi \in C_c^\infty(\mathbb{R}) \).

**Lemma 14.** Let \( v^\epsilon \) be the solution of the heterogeneous equation (1.10) with initial value \( v_0 \in H \). Then, for every \( \varphi \in C_c^\infty(\mathbb{R}) \), there exists a positive constant \( C \) such that

\[
\sup_{\epsilon} \mathbb{E}[(m_1^\epsilon v_t^\epsilon - m_1^\epsilon v_s^\epsilon, \varphi)^4] \leq C|t-s|^2||v_0||^4_{L^2},
\]

for all \( s, t \in [0, T] \).

**Proof.** By the definition of \( v^\epsilon \), we can see

\[
(m_1^\epsilon v_t^\epsilon - m_1^\epsilon v_s^\epsilon, \varphi)^4 \leq 27\left( \int_s^t (F_{m_1^\epsilon} v^\epsilon, \varphi) \, dr \right)^4 + 27\left( \int_s^t (L_{m_1^\epsilon} v^\epsilon, \varphi) \, dr \right)^4 + 27\left( \int_s^t (m_1^\epsilon v^\epsilon \sigma(x, \epsilon), \varphi) \, dW_r \right)^4.
\]

Then, for the operators \( F_{m_1^\epsilon} \) and \( L_{m_1^\epsilon} \), we have

\[
(F_{m_1^\epsilon} v^\epsilon, \varphi) \leq C_0||v^\epsilon||_{L^2}||\varphi||_{H^1},
\]

\[
(L_{m_1^\epsilon} v, v) = -\frac{1}{2}(D^* v(x, y), \delta^{\alpha, \epsilon}(x)m_1^\epsilon(x)D^* v(x, y))_{L^2(\mathbb{R}^2)} \leq 0.
\]

We use \( \lambda^\epsilon \) to denote the eigenvalues of \( L_{m_1^\epsilon} \). From the fact that the function \( \delta, m \) are bounded, we know that there exists a positive constant \( C_1, C_2, C_3 \) such that

\[
(\lambda^\epsilon v, v) \geq -C_1||v||_{H^{\alpha/2}}^2.
\]

Moreover, we have

\[
\mathbb{E}\left( \int_0^t (L_{m_1^\epsilon} v_r^\epsilon, v_r^\epsilon) ds \right)^2 \leq \mathbb{E}\left( \int_0^t \lambda^\epsilon ||v_r^\epsilon||_{L^2}^2 ds \right)^2 \leq C_2||v_0||_{L^2}^4.
\]

Then,

\[
\mathbb{E}\left( \int_s^t (L_{m_1^\epsilon} v_r^\epsilon, \varphi) \, dr \right)^4 \leq \mathbb{E}\left( \int_s^t -\lambda^\epsilon ||v_r^\epsilon||_{L^2}||\varphi||_{L^2} \, dr \right)^4 \leq (\lambda^\epsilon)^4 ||\varphi||_{L^2}^4 \mathbb{E}\left( \int_s^t ||v_r^\epsilon||_{L^2} \, dr \right)^4 \leq (\lambda^\epsilon)^2 C_1^2 ||\varphi||_{L^2}^4 \mathbb{E}\left( \int_s^t ||v_r^\epsilon||_{H^{\alpha/2}} \, dr \right)^4 \leq C_3||\varphi||_{L^2}^4 |t-s|^2 ||v_0||_{L^2}^4.
\]
Then, we can conclude that

\[ \sup_\epsilon \mathbb{E}[(m_1^1 v_t^\epsilon - m_1^1 v_s^\epsilon, \varphi)^4] \leq C|t - s|^2 \|v_0\|_{L^2}^4. \]

\[ \square \]

We can also infer the next three lemmas in the same way we used in Lemma 6, 7, 8.

**Lemma 15.** There exist a subsequence \( \epsilon_k \to 0 \) as \( k \to \infty \), and a probability measure \( \tilde{\pi}_1 \) on \( K_1 \) such that

\[ \pi_{m_1, \epsilon_k} \Rightarrow \tilde{\pi}_1 \quad \text{in} \quad (K_1, \mathcal{F}_1). \]

**Lemma 16.** Let \( \epsilon_k \) be the subsequence in Lemma 6 and \( \pi_{\epsilon_k}^1 \) be the probability measure induced by \( v^{\epsilon_k} \). Then

\[ \pi_{\epsilon_k}^1 \Rightarrow \tilde{\pi}_1 \quad \text{in} \quad (K_1, \mathcal{F}_2), \]

as \( k \to \infty \).

**Lemma 17.** The homogenized equation (1.10) has at most one martingale solution on \( (K_1, \mathcal{D}) \).

### 6.2. Identification of the limit law.

Recall the calculation in subsection 3.2, in order to obtain that \( \tilde{\pi}_1 \) is the martingale solution for the homogenized equation (1.10), we only need to show, as \( \epsilon \) goes to 0,

\[ ((V^{\epsilon})^\ast \xi^{\epsilon}, \psi)_{L^2} \to \left( \int_T \delta^\alpha(y)m_1(y)dy \cdot (-(-\Delta)^{\alpha/2})^2 \xi(x) + \xi'(x) \int_T g(y)m_1(y)dy \right) 
\]

\[ + \xi(x) \int_T f(y)m_1(y)dy, \psi). \]

We will construct a family of test functions \( \xi^{\epsilon} \in C^\infty_c(\mathbb{R}) \) as follows:

\[ \xi^{\epsilon}(x) = m_1(x) (\xi(x) + \epsilon h_3(x)) \xi'(x), \]

where the function \( h_3 \) is the solution of the following equations:

\[ (\tilde{L}_m)^* h_3(y) = d(y)m_1(y), \quad y \in \mathbb{T}, \]

\[ \int_0^1 h_3(y)dy = 1. \]
First, for functions \( f, g, \psi \in H^{\alpha/2} \),
\[
(-\Delta)^{\alpha/2}(f \cdot g)(x), \psi(x))_{L^2} = \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x)g(x) - f(y)g(y))\psi(x)\gamma^2(x,y)dx\,dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x)g(x) - f(y)g(y))(\psi(x) - \psi(y))\gamma^2(x,y)dx\,dy
\]
\[
= \frac{1}{2}(D^* \psi(x,y), D^* \psi(x,y))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} [(f(x) - f(y))g(x) + f(x)(g(x) - g(y))](\psi(x) - \psi(y))\gamma^2(x,y)dx\,dy
\]
\[
= \frac{1}{2}(D^*(f)(x,y)g(x) + D^*(g)(x,y)f(y)), D^* \psi(x,y))_{L^2(\mathbb{R} \times \mathbb{R})}.
\]

Next, we denote \( \delta_1(x) = \delta^\alpha(x) \). For the operator \( \mathcal{L}^\epsilon \), we have
\[
((\mathcal{L}^\epsilon)^* \xi^\epsilon, \psi) = ((-\Delta)^{\alpha/2}(\delta_1^{\alpha/2})(x), \psi(x))_{L^2} - \frac{1}{\epsilon^\alpha-1}((\mathcal{L}^\epsilon(x)\xi^\epsilon(x))', \psi(x))
\]
\[
= \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})} - \frac{1}{\epsilon^\alpha-1}((\mathcal{L}^\epsilon(x)\xi^\epsilon)(x), \psi(x))
\]
\[
= \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})} + \frac{\epsilon}{2}(D^* (\delta_1^{\alpha/2} h_3)(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
+ (\epsilon^\alpha (dm_1))' + \epsilon^\alpha (dm_1) + \epsilon^\alpha (d(\delta_1^{\alpha/2} h_3)(x,y))' \xi' + \epsilon^\alpha (d(\delta_1^{\alpha/2} h_3)(x,y))'' \xi'' (x,\psi(x))
\]
\[
= \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})} + \frac{\epsilon}{2}(D^* (\delta_1^{\alpha/2} h_3)(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
+ (\epsilon^\alpha (dm_1))' + \epsilon^\alpha (dm_1) + \epsilon^\alpha (d(\delta_1^{\alpha/2} h_3))' \xi' + \epsilon^\alpha (d(\delta_1^{\alpha/2} h_3))'' \xi'' (x,\psi(x)) + (\phi_1^\epsilon, \psi(x))
\]
where \( (\phi_1^\epsilon, \psi(x)) \to 0 \), as \( \epsilon \) goes to 0. Furthermore, we can show that,
\[
G_1 = \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
= \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y)\xi(x) + D^* (\xi)(x,y)\delta_1^{\alpha/2}(y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
= \frac{1}{2}(D^* (\delta_1^{\alpha/2})(x,y), D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})}
\]
\[
+ \frac{1}{2}(D^* (\xi)(x,y), \delta_1^{\alpha/2}(y)D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})} = (\langle -\Delta\rangle^{\alpha/2}(\delta_1^{\alpha/2})(x), \psi(x)\xi(x))
\]
\[
+ \frac{1}{2}(D^* (\xi)(x,y), \delta_1^{\alpha/2}(y)D^*(\psi(x,y)))_{L^2(\mathbb{R} \times \mathbb{R})} := I_1 + I_2.
\]
For $I_2$, we deduce that,

$$I_2 = \frac{1}{2}(\mathcal{D}^*(\xi)(x,y)\delta_{m_1,\epsilon}(y)\mathcal{D}^*\psi(x,y) - \mathcal{D}^*(\delta_{m_1,\epsilon}(x,y)\psi(y)))_{L^2(\mathbb{R} \times \mathbb{R})}$$

$$= \frac{1}{2}(\mathcal{D}^*(\xi)(x,y)\mathcal{D}^*(\delta_{m_1,\epsilon}(x,y)\psi(x,y) + \psi(y)))_{L^2(\mathbb{R} \times \mathbb{R})}$$

$$= (-(-\Delta)^{\alpha/2}\xi(x),\delta_{m_1,\epsilon}(x)\psi(x))_{L^2(\mathbb{R})} - \frac{1}{2}(\mathcal{D}^*(\xi)(x,y),\mathcal{D}^*(\delta_{m_1,\epsilon}(x,y).$$

where we have $I_3 \to 0$. In fact,

$$I_3 = -\frac{1}{2}(\mathcal{D}^*(\xi)(x,y),\mathcal{D}^*(\delta_{m_1,\epsilon}(x,y)(\psi(x) + \psi(y))))_{L^2(\mathbb{R} \times \mathbb{R})}$$

$$= -\frac{1}{2}\int_\mathbb{R} \int_\mathbb{R} (\xi(x) - \xi(y))(\delta_{m_1,\epsilon}(x) - \delta_{m_1,\epsilon}(y))(\psi(x) + \psi(y))\gamma^2(x,y)dxdy$$

$$= -\int_\mathbb{R} \int_\mathbb{R} (\xi(x) - \xi(y))\delta_{m_1,\epsilon}(x)(\psi(x) + \psi(y))\gamma^2(x,y)dxdy$$

$$= -\int_\mathbb{R} \int_\mathbb{R} (\xi(x) - \xi(y))(\psi(x) + \psi(y))\gamma^2(x,y)dxdy, \delta_{m_1,\epsilon}(x)_{L^2} \to 0.$$  

From the calculation above, we can obtain that,

$$G_2 = \epsilon(-(-\Delta)^{\alpha/2}(m_1^2 h_3^2)(x),\psi(x)\xi'(x)) + (\phi_2^3,\psi(x)\xi'(x)),$$

where $(\phi_2^3,\psi(x)\xi'(x)) \to 0$, as $\epsilon$ goes to 0. Then, we have,

$$((L^\epsilon)^*\xi',\psi)_{L^2} = (-(-\Delta)^{\alpha/2}\delta_{m_1,\epsilon}(x),\psi(x)\xi(x))_{L^2(\mathbb{R})} + (-(-\Delta)^{\alpha/2}\xi(x),\delta_{m_1,\epsilon}(x)\psi(x)_{L^2} + \epsilon(-(-\Delta)^{\alpha/2}(m_1^2 h_3^2)(x),\psi(x)\xi'(x)) + \phi_2^3 + (\epsilon^{-\alpha}(dm_1')^2 + (1-\alpha)d_{m_1}\epsilon + e^{-\alpha}(dm_1 h_3')\xi',\psi(x)),$$

where $\phi_2^3$ goes to 0. Using the equation (1.12), (6.3), as $\epsilon \to 0$, we have

$$((L^\epsilon)^*\xi',\psi)_{L^2} \to \int_\mathbb{T} \delta^\alpha(y)m_1(y)dy \cdot (-(-\Delta)^{\alpha/2}\xi(x),\psi(x))_{L^2}.$$  

For the term $\epsilon^{-\alpha}e(\xi')$, we assume that, there is a function $e_1$ satisfies the following equation:

$$(6.4) \begin{cases} Le_1(y) + e(y) = 0, \\ \int_\mathbb{T} e_1(y)m_1(y) = 0. \end{cases}$$
Then, we have
\[-\delta_1^{m_1,\epsilon}(-\Delta)^{\alpha/2} e_1(x) + \epsilon^{-\alpha} a^{m_1,\epsilon}(x) e_1'(x) = \epsilon^{-\alpha} e^{m_1}(x).\]

Next, we will show
\[((F^\epsilon)^* \xi^\epsilon, \psi) \to (\xi'(x) \int_T g(y) m_1(y) dy + \xi(x) \int_T f(y) m_1(y) dy, \psi),\]
as \(\epsilon \to 0\). So, we just need to show
\[(\epsilon^{-\alpha} e^{m_1}(x), \psi) \to 0,\]
as \(\epsilon \to 0\). In fact,
\[(\epsilon^{-\alpha} e^{m_1}(x), \psi) = (-\delta_1^{m_1,\epsilon}(-\Delta)^{\alpha/2} e_1(x), \psi) + (\epsilon^{-\alpha} a^{m_1,\epsilon}(x) e_1'(x), \psi).\]

For the first term,
\[((-\Delta)^{\alpha/2} e_1(x), \psi) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} e_1'(x) - e_1'(y) \cdot \frac{\psi(x) - \psi(y)}{|x-y|^{1+\alpha/2}} dx dy \cdot \psi(x) - \psi(y) \frac{e_1'(x) - e_1'(y)}{|x-y|^{1+\alpha/2}} dx dy,\]
where \(M\) is large enough to make sure
\[(\int_{-\infty}^{-M} + \int_{M}^{\infty}) \frac{e_1'(x) - e_1'(y)}{|x-y|^{1+\alpha/2}} \frac{\psi(x) - \psi(y)}{|x-y|^{1+\alpha/2}} dx dy < \epsilon.\]

Then, as \(\epsilon\) goes to 0, we have
\[((-\Delta)^{\alpha/2} e_1(x), \psi) \to \frac{1}{2} \int_{\mathbb{R}} \int_{T} \int_{T} e_1(x') - e_1(y') \cdot \frac{\psi(x) - \psi(y)}{|x-y|^{1+\alpha/2}} dx' dy' dx dy = 0.\]

Just like equation (6.4), For the second term, we also assume that there is a function \(d_1\) satisfies the following equation:
\[(6.5) \quad \begin{cases} \hat{L} d_1(y) + d(y) = 0, \\ \int_T d_1(y) m_1(y) = 0. \end{cases} \]

Then we obtain
\[(\epsilon^{-\alpha} a^{m_1,\epsilon}(x) e_1'(x), \psi) \to 0,\]
as \(\epsilon\) goes to 0.

In summary, we can infer that
\[((V^\epsilon)^* \xi^\epsilon, \psi)_{L^2} \to (\int_T \delta^\alpha(y) m_1(y) + (-\Delta)^{\alpha/2} \xi(x) + \xi'(x) \int_T g(y) m_1(y) dy + \xi(x) \int_T f(y) m_1(y) dy, \psi).\]

Moreover, the theorem 2 is proved through the way in Lemma 11.
7. Application to Data Assimilation. In this section, we present an application of Theorem 2. We consider the nonlinear filtering problem (1.14)

\[
\begin{align*}
    dx_t &= \frac{1}{\varepsilon} d\left(\frac{x_t}{\varepsilon}\right) dt + \delta\left(\frac{x_t}{\varepsilon}\right) dL_t^\alpha, \\
    dy_t &= \sigma\left(\frac{x_t}{\varepsilon}\right) dt + dW_t,
\end{align*}
\]

then the nonlocal Zakai equation for the conditional probability density function \(v\) of the filtering system (1.14) is the following:

\[
\begin{align*}
    dv^\varepsilon(t, x) &= (L^\varepsilon)^* v^\varepsilon(t, x) dt + v^\varepsilon(t, x) \sigma^2\left(\frac{x}{\varepsilon}\right) dt + v^\varepsilon(t, x) \sigma\left(\frac{x}{\varepsilon}\right) dW_t, \\
    v^\varepsilon(0, x) &= v_0(x).
\end{align*}
\]

We assume that \( g\left(\frac{x}{\varepsilon}\right) = e\left(\frac{x}{\varepsilon}\right) = 0, f\left(\frac{x}{\varepsilon}\right) = \sigma^2\left(\frac{x}{\varepsilon}\right)\), in by Theorem 1, we can deduced that: The family of laws \(\{\pi^\varepsilon_1 : \varepsilon > 0\}\) for the solution \(v^\varepsilon\) of the nonlocal Zakai equation (1.14) converges in \((K_1, T_1)\) to the law of the solution \(v\) for the following effective, homogenized local equation

\[
\begin{align*}
    dv(t, x) &= \hat{L}^0 v(t, x) dt + \int_T m_1(y) \sigma^2(y) dy \cdot v(t, x) dt + M^0 v(t, x) dW_t, \\
    v(0, x) &= v_0(x),
\end{align*}
\]

where \(M^0\) is the same operator as (1.4), and

\[
(\hat{L}^0 v)(x) = \int_T \delta^\alpha(y) m_1(y) dy \cdot (-\Delta)^{\alpha/2} v(x).
\]

In fact, set \(\hat{v}^\varepsilon = (m_1^\varepsilon)^{-1} v^\varepsilon\). Then the system (1.14) can be rewritten as

\[
\begin{align*}
    d\hat{v}^\varepsilon(t, x) &= \hat{L}^\varepsilon \hat{v}^\varepsilon(t, x) dt + \hat{v}^\varepsilon(t, x) \sigma^2\left(\frac{x}{\varepsilon}\right) dt + \sigma\left(\frac{x}{\varepsilon}\right) \hat{v}^\varepsilon(t, x) dW_t, \\
    \hat{v}^\varepsilon(0) &= (m_1^\varepsilon)^{-1} v_0,
\end{align*}
\]

where

\[
(\hat{L}^\varepsilon v)(x) = \frac{1}{\varepsilon} d\left(\frac{x}{\varepsilon}\right) v^\varepsilon + (m_1^\varepsilon)^{-1} \int_\mathbb{R} c\left(\frac{x - y}{\varepsilon}\right)(\delta^m_1(y) v(y) - \delta^m_1\left(\frac{x}{\varepsilon}\right) v(x)) dy.
\]

Similar to the calculation in the fourth section, we deduced that \(\hat{L}^\varepsilon u(x) = -\frac{1}{\varepsilon} d\left(\frac{x}{\varepsilon}\right) u^\varepsilon - \delta^\alpha\left(\frac{x}{\varepsilon}\right)(-\Delta)^{\alpha/2} u(x)\) in \(L^2(\mathbb{R})\).

Then \(\hat{L}^\varepsilon\) satisfy the Assumptions (vi), (vii). Using Theorem 2, we infer that the family of laws induced by \(v^\varepsilon = m_1^\varepsilon \hat{v}^\varepsilon\) converges weakly in \((K_1, T_1)\) to the law \(\pi_1\) induced by the solution of the following homogenized Zakai equation:

\[
\begin{align*}
    dv(t, x) &= V^0 v(t, x) dt + M^0 v(t, x) dW_t, \\
    v(0, x) &= v_0(x),
\end{align*}
\]

where

\[
V^0 v(t, x) = \hat{L}^0 v(t, x) + \int_T m_1(y) \sigma^2(y) dy \cdot v(t, x).
\]
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