GL Flatness of $OSp(1|2n)$ and Higher Spin Field Theory from Dynamics in Tensorial Spaces

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Abstract

A main purpose of this paper is to explain how the theory of higher spin fields in the flat $D = 4$ space and in $AdS_4$ emerges as a result of the quantization of a superparticle propagating in so called tensorial superspaces which have the property of a ‘generalized conformal’ or simply General Linear (GL) flatness.

1 Introduction

We present some results of the development of particle dynamics and field theory in tensorial spaces and explain their relation to Higher Spin Field Theory. The plan of the paper is as follows. We shall first provide a motivation why it is interesting to consider the dynamics of particles, strings, etc. and field theory in tensorial superspaces. In Section 2 we shall give the definition of tensorial supermanifolds and discuss their implication to the theory of higher spin fields. In Section 3 we shall introduce the notion of $GL(2n)$ flatness of manifolds and give examples of tensorial superspaces which possess this property, actually, the only known non–trivial example being $OSp(1|2n)$ supergroup manifolds. We shall then consider the dynamics of a particle in flat tensorial space and on $Sp(4)$, and analyze constraints and physical degrees of freedom of this object. Finally, in Sections 4 and 5 we shall quantize this system using GL flatness, find the general solutions of the field equations of the quantum system and demonstrate that its quantum spectrum consists of an infinite tower of massless integer and half–integer higher spin states which obey so called unfolded higher spin field equations in flat $D = 4$ space and/or in $AdS_4$ in the formulation of M. Vasiliev [1].

2 Tensorial (super)spaces and higher spins

We call a space tensorial if its points are parametrized by symmetric $2n \times 2n$ matrix coordinates $x^{\alpha \beta} = x^{\beta \alpha} \ (\alpha, \beta = 1, \ldots, 2n)$ linearly transformed by the group $Sp(2n)$ $^\dagger$.

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$^\dagger$Generally speaking the group of linear transformations of $x^{\alpha \beta}$ is $GL(2n)$, but in what follows we shall always restrict it to its subgroup $Sp(2n)$ to keep the simplectic and related spinorial structure manifest.
The bosonic tensorial space can be extended to a tensorial superspace by adding Grass- 
mann odd directions parametrized by $N$ fermionic spinor coordinates $\theta^\alpha_i$ $(i = 1, \cdots, N)$. Then a group of linear transformations of $(x^{\alpha\beta}, \theta^\alpha_i)$ becomes $Sp(2n) \times O(N)$.

Examples of tensorial superspaces are

- $OSp(N|2n)$ supergroup manifolds and, in particular, $OSp(1|4)$ to be considered in detail below,

- flat tensorial superspaces invariant under centrally extended super Poincare translations, e.g. under the $N = 1, D = 4$ super Poincare group transformations

$$\delta \theta^\alpha = \epsilon^\alpha, \quad \delta x^{\alpha\beta} = \frac{i}{2}(\theta^\alpha \epsilon^\beta + \theta^\beta \epsilon^\alpha) = i\theta^{(\alpha} \epsilon^{\beta)}, \quad \alpha, \beta = 1, 2, 3, 4. \quad (1)$$

The algebra of these transformations contains in addition to $D = 4$ translations $P_m$ ($m = 0, 1, 2, 3$) also six tensorial charge generators $Z_{mn} = -Z_{nm}$

$$\{Q_\alpha, Q_\beta\} = 4P_{\alpha\beta} = 2P_m \gamma^m_{\alpha\beta} + Z_{mn} \gamma^{mn}_{\alpha\beta}, \quad (2)$$

where in the r.h.s. we have made the decomposition of the momentum $P_{\alpha\beta} = P_{\beta\alpha}$ conjugate to $x^{\alpha\beta}$ in a basis of $D = 4 \gamma$–matrices.

Probably, the first who suggested a physical application of tensorial spaces was C. Fronsdal.

### 2.1 Fronsdal’s proposal of ’85 – alternative to Kaluza & Klein

In his Essay of 1985 \[2\] Fronsdal conjectured that four–dimensional higher spin field theory can be realized as a field theory on a ten–dimensional tensorial manifold parametrized by the coordinates

$$x^{\alpha\beta} = \frac{1}{2} x^m \gamma^m_{\alpha\beta} + \frac{1}{4} y^{mn} \gamma^{\alpha\beta}_{mn}, \quad m, n = 0, 1, 2, 3; \quad \alpha, \beta = 1, 2, 3, 4, \quad (3)$$

where $x^m$ are associated with four coordinates of the conventional $D = 4$ space–time and $y^{mn} = -y^{mn}$ describe spinning degrees of freedom.

The assumption was that by analogy with, for example, D=10 or D=11 supergravities, which are relatively simple theories but whose dimensional reduction to four dimensions produces very complicated extended supergravities, there may exist a theory in ten–dimensional tensorial space whose alternative Kaluza–Klein reduction may lead in $D = 4$ to an infinite tower of fields with increasing spins instead of the infinite tower of Kaluza–Klein particles of increasing mass. The assertion was based on the argument that the symmetry group of the theory should be $OSp(1|8) \supset SU(2, 2)$, which contains the $D = 4$ conformal group as a subgroup such that an irreducible (oscillator) representation of $OSp(1|8)$ contains each and every massless higher spin representation of $SU(2, 2)$ only once. So the idea was that using a single representation of $OSp(1|8)$ in the ten-dimensional tensorial space one could describe an infinite tower of higher spin fields in $D = 4$ space–time in a simpler way. Fronsdal regarded the tensorial space as a space transforming homogeneously under the transformations of $Sp(8)$. Ten is the minimal dimension of such a space which can contain $D=4$ space–time as a subspace. For some reason Fronsdal
gave only a general definition and did not identify this ten–dimensional space with any conventional manifolds, like the ones mentioned above.

In his Essay Fronsdal also stressed the importance of $OSp(1|2n)$ supergroups for the description of theories with superconformal symmetry. In the same period and later on different people studied $OSp(1|2n)$ supergroups in various physical contexts. For instance, $OSp(1|32)$ and $OSp(1|64)$ have been assumed to be underlying superconformal symmetries of string- and M-theory.

### 2.2 Particle dynamics in tensorial superspace

Without relation to Higher Spins, in 1998 I. Bandos and J. Lukierski [3] proposed an $OSp(1|4n)$–invariant exotic BPS superparticle in a flat tensorial superspace preserving all but one supersymmetry of the target superspace, for instance, 3/4 SUSY in $N = 1, D = 4$ superspace. One of the motivations for Bandos and Lukierski was a generalization of the Penrose twistor program to tensorial superspaces and associated superalgebras with tensorial charges. But it happened that this model turned out to be the first dynamical realization of the Fronsdal proposal.

Quantum states of the tensorial superparticle was shown to form an infinite series of massless higher spin states in $D = 4$ and first quantized field equations for wave functions in tensorial superspace have been obtained [5]. In [5] quantum superparticle dynamics on $OSp(1|4)$ was assumed to describe higher spin theory in $N = 1$ super $AdS_4$.

In [5] it was shown explicitly how the alternative Kaluza–Klein compactification produces higher spin fields. It turns out that in the tensorial superparticle model, in contrast to the conventional Kaluza–Klein theory, the compactification occurs in the momentum space and not in the coordinate space. The coordinates conjugate to the compactified momenta take discrete (integer and half integer values) and describe spin degrees of freedom of the quantized states of the superparticle in conventional space–time.

In [7] M. Vasiliev has extensively developed this subject by having shown that the first–quantized field equations in tensorial superspace of a bosonic dimension $n(2n + 1)$ and of a fermionic dimension $2nN$ are $OSp(N|4n)$ invariant, and for $n = 2$ correspond to so called unfolded higher spin field equations in $D = 4$. It has also been shown [8] that the theory possesses properties of causality and locality.

An alternative derivation that the quantized dynamics of the superparticle in flat tensorial superspace and on $OSp(1|4)$ reproduces, respectively, the unfolded higher spin field dynamics in flat $D = 4$ superspace and in $N = 1$ $AdS_4$ has been given in [9], where the $GL(2n)$ flatness of $OSp(1|2n)$ manifolds has been observed and used to quantize the $OSp(1|4)$ model.

### 3 GL flatness versus conformal flatness

Before introducing the notion of GL flatness let us remind what the (super)conformal flatness of supermanifolds is.
A supermanifold is called superconformally flat if its supervielbeins differ from the flat supervielbeins by a conformal factor $e^\rho(x,\theta)$ (possibly, up to Lorentz rotations). The vector supervielbein has the following form
\[ E^a = e^\rho(x,\theta) (dx^a - i d\bar{\theta} \gamma^a \theta) \equiv e^\rho(x,\theta) \Pi^a, \quad a = 0, 1, \cdots, D - 1 \] (4)
and the spinor supervielbein is
\[ E^\alpha = e^{-\rho/2} (d\theta^\alpha + i \frac{1}{2} \Pi^a \gamma^\alpha \gamma_a \partial_\rho), \] (5)
where $D_\beta = \partial_\beta + i \bar{\theta} \gamma^a \partial_a$ is the flat supercovariant derivative.

An example of the superconformally flat manifolds is $N = 1$ super $AdS_4$ which is a coset superspace $OSp(1|4)_{SO(1,3)}$. A detailed analysis, along with a criteria for supermanifolds to be superconformally flat, and a list of such superspaces has been given in [10].

Recall that the bosonic AdS metric is conformally flat, i.e. can be presented in the form $ds = e^{2\rho(x)} dx^m dx^n \eta_{mn}$.

### 3.1 GL(2n) flatness of tensorial supermanifolds

We shall call a tensorial supermanifold GL flat if its supervielbeins differ from the supervielbeins of flat tensorial superspace by a GL-group rotation
\[ \Omega^{\alpha\beta} = (dx^{\gamma\delta} - \frac{i}{2} d\theta^\gamma \theta^\delta - \frac{i}{2} d\theta^\delta \theta^\gamma) \mathcal{G}_{\gamma}^\alpha(x,\theta) \mathcal{G}_{\delta}^\beta(x,\theta) \] (6)
\[ E^\alpha = e^{\rho(x,\theta)} (D\theta^\alpha - \theta^a D\rho), \]
where $dx^{\gamma\delta} - \frac{i}{2} d\theta^\gamma \theta^\delta - \frac{i}{2} d\theta^\delta \theta^\gamma$ is a flat tensorial space supervielbein, $\mathcal{G}_{\gamma}^\alpha(x,\theta)$ is a general linear matrix and $D$ is a covariant differential.

Apart from the flat space, the only example of GL-flat tensorial superspaces known to us is the example of supergroup manifolds $OSp(1|2n)$.

### 3.2 $OSp(1|2n)$ supergroup manifolds

A group element $\mathcal{O}(x,\theta)$ of $OSp(1|2n)$ is parametrized by bosonic symmetric $2n \times 2n$ matrices $x^{\alpha\beta}$ and fermionic variables $\theta^\alpha$. The Cartan forms are defined as usual
\[ \mathcal{O}^{-1} d\mathcal{O} = \Omega^{\alpha\beta}(x,\theta) M_{\alpha\beta}^\gamma E^\gamma Q_\alpha + E^\alpha Q_\alpha \] (7)
and take values in the $OSp(1|2n)$ algebra formed by $Sp(2n)$ generators $M_{\alpha\beta} = M_{\beta\alpha}$ and by their supersymmetric partners $Q_\alpha$, such that $\{Q_\alpha, Q_\beta\} = M_{\alpha\beta}$.

The Cartan forms satisfy the Maurer–Cartan equations
\[ d\Omega^{\alpha\beta} + \frac{\varsigma}{2} \Omega^{\alpha\gamma} \wedge \Omega^{\gamma \beta} = - E^\alpha \wedge E^\beta, \quad dE^\alpha + \frac{\varsigma}{2} E^\gamma \wedge \Omega^{\gamma \alpha} = 0, \] (8)
where $\varsigma$ is a dimensional parameter which in the case of $OSp(1|4)$ can be associated with the inverse radius of $N = 1 AdS_4$, or equivalently with the square root of the cosmological constant absolute value. When $\varsigma \to 0$ the tensorial superspace becomes flat. So $\varsigma$ plays the role of a contraction parameter. From a physical point of view $\varsigma$ appeared in the commutation relations of the $OSp(1|2n)$ superalgebra because we would like to endow
the coordinates \( x^{\alpha\beta} \) with a conventional dimension of length \( \varsigma^{-1} \). Then the generators \( M_{\alpha\beta} \) have a dimension of \( \varsigma \) and \( Q_\alpha \) that of \( \varsigma^{1/2} \).

It turns out that it is possible to choose such a parametrization of \( OSp(1|2n) \) that its Cartan forms become GL-flat [9] as in (6), with

\[
G_\beta^\alpha(x, \theta) = G_\beta^\alpha(x) - \frac{i\varsigma}{8} (\Theta_\beta - 2G_\gamma^\gamma(x)\Theta_\gamma) \Theta^\alpha, \quad G_\beta^{-1\alpha}(x) = \delta_\beta^\alpha + \frac{\varsigma}{4} x_\beta^\alpha, \quad (9)
\]

\( \Theta^\alpha(\theta) \) being defined as the inverse function of \( \theta^\alpha = \Theta_\beta G_\gamma^{-1\alpha}(x)e^{-\rho(\Theta)} \) and \( e^{\rho(\Theta)} = \sqrt{1 + \frac{i\varsigma}{8}\Theta^\beta\Theta_\beta} \).

The GL-flatness of bosonic manifolds and supermanifolds seems novel and is quite interesting also from the mathematical point of view, and this should be appreciated yet. So far this property has found a physical application to the quantization of a superparticle on the group manifold \( OSp(1|4) \) (and in general on \( OSp(1|2n) \)), and helped to find an explicit solution of the tensorial field equations and to demonstrate its relation to higher spin field theory in \( AdS_4 \). For the sake of simplicity we will henceforth consider only a non-supersymmetric particle model in tensorial spaces associated with four-dimensional space-time. The quantization of this model [5, 9] gives rise to a free higher spin field theory in four-dimensional flat and AdS spaces [1].

### 4 Twistor-like particle dynamics in tensorial spaces

The action proposed in [3] to describe a particle propagating in a ten-dimensional tensorial space is

\[
S = \int \Omega^{\alpha\beta}(x(\tau)) \lambda_\alpha \lambda_\beta, \quad \alpha = 1, 2, 3, 4 \quad (10)
\]

where \( \lambda_\alpha(\tau) \) is an auxiliary commuting Majorana spinor variable and \( \Omega^{\alpha\beta}(x(\tau)) \) is the pullback on the particle worldline of the tensorial space vielbein.

When \( \Omega^{\alpha\beta}(x(\tau)) = d\tau \partial_\tau x^{\alpha\beta} = dx^{\alpha\beta}(\tau) \) we deal with flat tensorial space, and when \( \Omega^{\alpha\beta}(x(\tau)) = dx^{\gamma\delta}(\tau) G_\gamma^{\alpha}(x) G_\delta^{\beta}(x) \) with inverse of \( G_\gamma^{\alpha}(x) \) defined in eq. (9), the particle propagates on the group manifold \( Sp(4) \).

It is now easy to realize that because of the GL flatness of the \( Sp(4) \) manifold, particle dynamics in flat tensorial space and in \( Sp(4) \) are related to each other by a simple redefinition of \( \lambda_\alpha \to \tilde{\lambda}_\alpha = G_\alpha^{\beta}(x)\lambda_\beta \) and hence are classically equivalent

\[
S = \int \Omega^{\alpha\beta}(x(\tau)) \lambda_\alpha \lambda_\beta = \int dx^{\gamma\delta}(\tau) G_\gamma^{\alpha}(x) G_\delta^{\beta}(x) \lambda_\alpha \lambda_\beta = \int dx^{\alpha\beta}(\tau) \tilde{\lambda}_\alpha \tilde{\lambda}_\beta. \quad (11)
\]

Without going into details which the reader may find in [3, 9], let us note that the action (11) is invariant under \( Sp(8) \) transformations acting non-linearly on \( x^{\alpha\beta} \) and \( \lambda_\alpha \), i.e. possesses the symmetry which Fronsdal considered to be an underlying symmetry of higher spin field theory in \( D = 4 \) [2]. A group theoretical reason behind the \( Sp(8) \) invariance of (11) is that the flat tensorial space and \( Sp(4) \) are different realizations of a coset space \( \frac{Sp(8)}{GL(4) \times K} \), where \( K_{mn} \) are tensorial analogs of conformal boosts [4].
4.1 Hamiltonian analysis and dynamical properties

Because of GL flatness and classical equivalence of particle dynamics in flat tensorial space and in $Sp(4)$ we will first perform the Hamiltonian analysis and the quantization of the both cases in the “flat” basis, which will allow us to understand the physical content of the model in the simplest way.

The particle momenta conjugate to the tensorial coordinates $x^{\alpha \beta} = \frac{1}{2} x^{m \gamma}_{\alpha \beta} + \frac{1}{4} y^{mn \gamma}_{\alpha \beta}$ are constrained to be bilinear in $\lambda_{\alpha}$

$$\frac{\delta S}{\delta x^{\alpha \beta}} = P_{\alpha \beta} = \frac{1}{2} P_{m \alpha}^{m} + \frac{1}{4} Z_{mn}^{mn \gamma \alpha \beta} = \lambda_{\alpha} \lambda_{\beta}. \quad (12)$$

As a consequence of (12) the $D = 4$ part $P_{m}$ of the momenta is expressed via the Cartan–Penrose (twistor) relation and therefore is light–like in virtue of gamma–matrix Fierz identities in $D = 3, 4, 6$ and 10

$$P_{m} = \frac{1}{2} \lambda \gamma_{m} \lambda \quad \Rightarrow \quad P_{m} P^{m} = 0. \quad (13)$$

Hence, from the perspective of $D = 4$ space–time the particle is massless.

The constraints which restrict the dynamics of the particle are

$$D_{\alpha \beta} = P_{\alpha \beta} - \lambda_{\alpha} \lambda_{\beta} = 0, \quad y^{\alpha} = 0, \quad (14)$$

where $y^{\alpha}$ is the momentum conjugate to $\lambda_{\alpha}$. It is zero because in (10) $\lambda_{\alpha}$ does not have the kinetic term.

In view of the canonical Poisson brackets

$$[P_{\alpha \beta}, x^{\gamma \delta}] = \frac{1}{2} (\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} + \delta_{\beta}^{\gamma} \delta_{\alpha}^{\delta}), \quad [y^{\alpha}, \lambda_{\beta}] = -\delta_{\beta}^{\alpha}, \quad (15)$$

the constraints obey the following Poisson brackets

$$[D_{\alpha \beta}, D_{\gamma \delta}]_{PB} = 0, \quad [y^{\alpha}, y^{\beta}]_{PB} = 0, \quad [y^{\alpha}, D_{\beta \gamma}]_{PB} = \delta_{\beta}^{\gamma} \lambda_{\alpha} + \delta_{\gamma}^{\alpha} \lambda_{\beta} \neq 0. \quad (16)$$

From (16) we conclude that the constraints (14) are a mixture of the first and second class constraints. To quantize the theory it is easier to work with systems which have only first class constraints.

Note that the change of variables performed in the $Sp(4)$ action (11) to pass to the flat basis corresponds to the following canonical transformation of the Hamiltonian variables which does not change the canonical Poisson brackets (15) of the new variables

$$x^{\alpha \beta} = \bar{x}^{\alpha \beta}, \quad P_{\alpha \beta} = \bar{P}_{\alpha \beta} - \frac{\zeta}{8} (\bar{\lambda}_{\alpha} y_{\beta} + \bar{\lambda}_{\beta} y_{\alpha}), \quad \lambda_{\alpha} = G^{-1}_{\alpha} \bar{\lambda}_{\beta}, \quad y^{\alpha} = \bar{y}^{\beta} G_{\beta}^{\alpha} (x). \quad (17)$$

Note also that the transformed momentum $\bar{P}_{\alpha \beta}$ coincides with the initial one up to the terms which are proportional to the constraint $y_{\alpha} = 0$ and hence are weakly equal to zero.

To pass to a system with only first class constraints which is physically equivalent to the original one we should make a conversion of the constraints (14) into the first class in such a way that the number of physical degrees of freedom remains the same. In our case the

\footnote{We have called this momentum $y^{\alpha}$ to indicate that this variable is related to one which appears in the Vasiliev unfolded formulation of higher spin fields [1, 7, 8].}
conversion procedure is very simple. One just promotes \( y^\alpha \) to an unconstrained dynamical variable. Then the remaining constraints \( D_{\alpha\beta} \) are of the first class and generate local worldvolume symmetries of the action (10), while the condition \( y^\alpha = 0 \), when imposed, is regarded as gauge fixing of a part of these local symmetries. The conversion procedure described above is equivalent to adding to the action (10) the first–order kinetic term \( \int d\lambda_\alpha(\tau) y^\alpha(\tau) \) for \( \lambda_\alpha \) (see e.g. [7]).

4.2 Quantization and field equations in flat tensorial space

Upon the conversion the quantization of particle dynamics is straightforward. One should promote the dynamical variables to operators, to replace the Poisson brackets with commutators and to impose the first class constraints on the particle wave function.

One can consider the particle wave function in different (momentum and/or coordinate) representations related to each other by the Fourier transform.

For instance, in the representation considered in [3], which we shall call the \( \lambda \)–representation, the wave function \( \Phi(x, \lambda) \) is assumed to depend on \( x^{\alpha\beta} \) and \( \lambda_\alpha \), while \( P_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}} \) and \( y^\alpha = i \frac{\partial}{\partial \lambda_\alpha} \) are realized as differential operators. The wave function satisfies the equation, which is the quantum counterpart of the first class constraints [5]

\[
D_{\alpha\beta} \Phi(x, \lambda) = \left( \frac{\partial}{\partial x^{\alpha\beta}} - i \lambda_\alpha \lambda_\beta \right) \Phi(x, \lambda) = 0.
\]

The general solution of (18) is very simple

\[
\Phi(x, \lambda) = e^{ix^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda),
\]

where \( \varphi(\lambda) \) is a generic function of \( \lambda_\alpha \).

We can now make the Fourier transform of (19) to another representation to be called \( y \)–representation

\[
C(x, y) = \int d^4 \lambda e^{-iy^{\alpha} \lambda_\alpha} \Phi(x, \lambda) = \int d^4 \lambda e^{-iy^{\alpha} \lambda_\alpha + ix^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda).
\]

The wave function \( C(x, y) \) satisfies the Fourier transformed eq. (18)

\[
\left( \frac{\partial}{\partial x^{\alpha\beta}} + i \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) C(x, y) = 0.
\]

This equation has been analyzed in [7] for the wave functions which are polynomials in \( y^\alpha \)

\[
C(x, y) = \sum_{n=0}^{\infty} C_{\alpha_1 \cdots \alpha_n} (x) y^{\alpha_1} \cdots y^{\alpha_n} = b(x) + f_\alpha(x) y^\alpha + \cdots.
\]

Substituting (22) into (21) one finds that the scalar field \( b(x) \) and the spinor field \( f_\alpha(x) \) satisfy the following equations

\[
(\partial_{\alpha\beta} \partial_{\gamma\delta} - \partial_{\alpha\gamma} \partial_{\beta\delta}) b(x) = 0, \quad \partial_{\alpha\beta} f_\gamma(x) - \partial_{\alpha\gamma} f_\beta(x) = 0,
\]

so these fields are dynamical, while all higher components in the expansion (22) are expressed in terms of (higher) derivatives of \( b(x) \) and \( f_\alpha(x) \) and, hence, are auxiliary fields.
Thus the quantum dynamics of the particle in tensorial spaces is described by only two dynamical fields, which can be obtained from the wave function in the \( \lambda \)-representation by integrating the latter over \( \lambda_\alpha \) as follows

\[
b(x) = \int d^4 \lambda \Phi(x, \lambda), \quad f_\alpha(x) = -i \int d^4 \lambda \lambda_\alpha \Phi(x, \lambda).
\]  

(24)

In virtue of the GL flatness of \( Sp(4) \) the consideration above is applicable both to flat space and to \( Sp(4) \), though in the latter case, because of the field redefinition \( \{Y_{\alpha}, \tilde{\lambda}_\alpha \} \), \( x^{\alpha\beta} \) and \( \tilde{\lambda}_\alpha \) transform under \( Sp(4) \) in a highly non–linear way. So, \( Sp(4) \) symmetry is not manifest. We are in a similar situation to that of a ‘free-fermion’ model with a non–linearly realized \( SU(n|1) \) symmetry considered in \[\text{[11]}\]. To restore manifest \( Sp(4) \) invariance we should return to original variables at the expense of the loss of the ‘free’ character of dynamics.

### 4.3 Particle dynamics on \( Sp(4) \)

The Hamiltonian constraints which follow from \[\text{[11]}\] (without doing the GL rotation) have the form

\[
y^{\alpha} = 0, \quad D_{\alpha\beta} = G_{\alpha}^{-1\gamma}(x)G_{\beta}^{-1\lambda}(x)P_{\gamma\delta} - \lambda_\alpha \lambda_\beta = \nabla_{\alpha\beta} - \lambda_\alpha \lambda_\beta = 0,
\]

(25)

where \( \nabla_{\alpha\beta} = G_{\alpha}^{-1\gamma}(x)G_{\beta}^{-1\lambda}(x)P_{\gamma\delta} \) generate the \( Sp(4) \) algebra \( [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}]_{PB} = -\frac{\imath}{2}C_{\alpha\gamma\beta\delta} - \frac{\imath}{2}C_{\beta\gamma\alpha\delta} - \frac{\imath}{2}C_{\alpha\gamma\delta\alpha} \), and \( C_{\alpha\beta} \) is a simplectic metric. (Remember that \( G_{\alpha}^{-1\gamma}(x) = \delta^\gamma_\alpha + \frac{\imath}{2}x_\alpha \gamma \) and \( G_{\alpha}^{-1\gamma}(x)G_{\beta}^{-1\lambda}(x) \) is inverse of the Cartan form matrix \[\text{[11]}\], i.e. the inverse vielbein of the group manifold \( Sp(4) \)).

Because of the non–commutativity of \( \nabla_{\alpha\beta} \) the constraints \( D_{\alpha\beta} \) do not commute even in the weak Dirac sense, i.e. \( [D_{\alpha\beta}, D_{\gamma\delta}]_{PB} \neq 0 \). This is in contrast to what we had in the flat case \[\text{[16]}\]. However the weak commutativity can be restored if we modify \( D_{\alpha\beta} \) by adding to them terms linear and quadratic in the constraint \( y^{\alpha} = 0 \) as follows

\[
D_{\alpha\beta} = \nabla_{\alpha\beta} - (\lambda_\alpha + \frac{\imath}{8}y_\alpha)(\lambda_\beta + \frac{\imath}{8}y_\beta) = \nabla_{\alpha\beta} - Y_\alpha Y_\beta = 0.
\]

(26)

The constraints \[\text{[26]}\] can be obtained from the flat constraints \[\text{[14]}\] performing the canonical transformations \[\text{[17]}\] and adding to \[\text{[14]}\] appropriate terms linear and quadratic in the constraint \( y^{\alpha} \), which can always be done. At the classical level the addition of these terms is just another choice of constraints, however at the quantum level this changes background geometry (in our case from flat tensorial space to \( Sp(4) \)). It occurs in the following way.

Since \( Y_\alpha \equiv \lambda_\alpha + \frac{\imath}{8}y_\alpha \) do not commute and \( [Y_\alpha, Y_\beta]_{PB} = \frac{\imath}{4}C_{\alpha\beta} \), the constraints \( D_{\alpha\beta} \), like \( \nabla_{\alpha\beta} \), generate the \( Sp(4) \) algebra \( [D_{\alpha\beta}, D_{\gamma\delta}]_{PB} = -\frac{\imath}{2}C_{\alpha\gamma\beta\delta} - \frac{\imath}{2}C_{\beta\gamma\alpha\delta} - \frac{\imath}{2}C_{\alpha\gamma\delta\alpha} \) and hence weakly commute. Thus, the constraints \( D_{\alpha\beta} \) reflect the \( Sp(4) \) structure of the tensorial space where the particle propagates.

As in the flat case \[\text{[16]}\] \( y^{\alpha} \) do not commute with \( D_{\alpha\beta} \) and the whole system of the constraints is again a mixture of the first and second class ones. As before we convert it into the first class by regarding \( y^{\alpha} \) to be unconstrained.

Now the quantization of the system is performed as in the previous Subsection. The first–class constraints become operators which annihilate physical states of the particle on \( Sp(4) \). In the \( \lambda \)-representation the first quantized wave function satisfies the equation

\[
D_{\alpha\beta}\Phi(x, \lambda) = \left[ \nabla_{\alpha\beta} - \frac{i}{2}(Y_\alpha Y_\beta + Y_\beta Y_\alpha) \right] \Phi(x, \lambda) = 0, \quad Y_\alpha \equiv \lambda_\alpha + \frac{\imath}{8} \frac{\partial}{\partial \lambda_\alpha},
\]

(27)
and in the $y$–representation

$$D_{\alpha \beta} C(x, y) = \left[ \nabla_{\alpha \beta} - \frac{i}{2} (Y_{\alpha} Y_{\beta} + Y_{\beta} Y_{\alpha}) \right] C(x, y) = 0 \quad Y_{\alpha} \equiv i \frac{\partial}{\partial y^\alpha} + \frac{\varsigma}{8} y_{\alpha}, \quad (28)$$

where $\nabla_{\alpha \beta}$ is a covariant derivative on $Sp(4)$.

We see that on $Sp(4)$ the two representations are completely equivalent, or dual, to each other with respect to the exchange of $\lambda_{\alpha}$ and $\frac{\varsigma}{8} y_{\alpha}$, which is reflected in the form of the general solutions of these equations.

Symmetries and solutions of eq. (28) have been studied in [12]. The GL–flat realization of the $Sp(4)$ Cartan forms (11) and of the covariant derivatives (25) allows us to find the general solutions of these equations. To find (29) and (30) we have used that

$$\Phi(x^{\alpha \beta}, \lambda) = \int d^4 y \sqrt{\det G^{-1}(x)} e^{i x^{\alpha \beta} (\lambda_{\alpha} + \frac{\varsigma}{8} y_{\alpha}) (\lambda_{\beta} + \frac{\varsigma}{8} y_{\beta}) + i \lambda_{\alpha} y^{\alpha}} \varphi(y), \quad (29)$$

$$C(x^{\alpha \beta}, y) = \int d^4 \lambda \sqrt{\det G^{-1}(x)} e^{i x^{\alpha \beta} (\lambda_{\alpha} + \frac{\varsigma}{8} y_{\alpha}) (\lambda_{\beta} + \frac{\varsigma}{8} y_{\beta}) - i \lambda_{\alpha} y^{\alpha}} \varphi(\lambda). \quad (30)$$

To find (29) and (30) we have used that

$$G^{-1}_{\alpha^\beta}(x) = \delta_{\alpha^\beta} + \frac{\varsigma}{8} x_{\alpha^\beta}, \quad \nabla_{\alpha \beta} \det G^{-1} = G^{-1}_{\alpha \beta} \frac{\partial \det G^{-1}}{\partial x^{\alpha \beta}} = \frac{\varsigma^2}{16} x_{\alpha \beta} \det G^{-1}, \quad (31)$$

$$\nabla_{\alpha \beta} G_{\gamma^\delta} = \frac{\varsigma}{4} (\delta^\gamma_{\{\alpha} + 2 G^\gamma_{\{\alpha} \delta^\delta_{\beta\}}).$$

For completeness let us also present the explicit form of the $\det G^{-1}$:

$$\det G^{-1} = 1 - \frac{1}{2} \left( \frac{\varsigma}{4} \right)^2 x_{\alpha} x_{\beta} + \frac{1}{8} \left( \frac{\varsigma}{4} \right)^4 (x_{\alpha} x_{\beta})^2 - \frac{1}{4} \left( \frac{\varsigma}{4} \right)^4 x_{\alpha} x_{\beta} x_{\gamma} x_{\delta} x_{\gamma} x_{\delta}. \quad (32)$$

One can wonder what is the $Sp(4)$ analog of the equations (23) of the dynamical fields $b(x)$ and $f_{\alpha}(x)$ entering the polynomial wave function (22). Upon some algebra we arrive at the following system of equations

$$\nabla_{\alpha \beta} \nabla_{\gamma} b(x) = \frac{\varsigma}{16} \left( C_{\alpha \beta \gamma \delta} - C_{\delta \gamma \alpha \beta} + 2 C_{\beta \gamma} \nabla_{\alpha \delta} b(x) + \frac{\varsigma^2}{64} (2 C_{\alpha \delta} C_{\beta \gamma} - C_{\alpha \beta} C_{\gamma \delta}) b(x), \quad (33)$$

$$\nabla_{\alpha \beta} f_{\gamma}(x) = -\frac{\varsigma}{4} \left( C_{\alpha \beta \gamma} f_{\beta}(x) + 2 C_{\beta \gamma} f_{\alpha}(x) \right). \quad (34)$$

These $Sp(4)$ equations can be regarded as tensorial counterparts of the equations of motion of a massless scalar and spinor field in $AdS_3$.

We should stress that the consideration and the formulas presented in Section 4 with the example of particle dynamics on $Sp(4)$ remain the same for a generic case of the group manifold $Sp(2n)$. In particular, in $Sp(2n)$ the wave equations (27), (28), (33) and (34) and their solutions have the same form as above. When $n = 1$ the group manifold $Sp(2) \sim SO(1, 2)$ is isomorphic to $AdS_3 = \frac{SO(1, 2)}{SO(1, 1)}$, and the equations (33) and (34) reduce to the well–known equations of motion of a massless scalar and spinor field in $AdS_3$. 

9
5 Higher spin fields in ordinary $D = 4$ space–time

Let us now demonstrate how higher spin fields and their unfolded field equations emerge in the ordinary four–dimensional subspace of the tensorial space. For this we should just formally rewrite the wave function (20) in a different way, namely

\[ C(x^{\alpha\beta}, y^{\alpha}) = \int d^4 \lambda e^{-iy^\alpha \lambda_\alpha + i x^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda) = \int d^4 \lambda e^{-iy^\alpha \lambda_\alpha + \frac{i}{2} x^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda) = C(x^m, \tilde{y}^{\alpha}), \]

where we have used the coordinate decomposition (3) and have hidden the six tensorial coordinates $y^{mn}$ into redefined $\tilde{y}^{\alpha} = y^{\alpha} - \frac{1}{4} y^{mn} \gamma^{\alpha\beta}_{mn} \lambda_\beta$. In other words, what we have actually done is we have taken $C(x^{\alpha\beta}, y^{\alpha})$ at $y^{mn} = 0$.

By construction the wave function $C(x^m, \tilde{y}^{\alpha})$ satisfies the field equations in $D = 4$ space–time

\[ \left( \frac{\partial}{\partial x^m} + \frac{i}{2} \gamma^{\alpha\beta}_{mn} \frac{\partial^2}{\partial y^\alpha \partial \tilde{y}^\beta} \right) C(x^m, \tilde{y}^{\alpha}) = 0. \]

(36)

These are unfolded field equations of Vasiliev [11, 17] which produce the flat $D = 4$ equations of motion of the field strengths of the higher spin ($s = n/2$) fields. The field strengths are the components $C_{\alpha_1 \ldots \alpha_n}(x^m)$ of the polynomial expansion of $C(x^m, \tilde{y}^{\alpha})$ (called the generating function)

\[ C(x^m, \tilde{y}^{\alpha}) = \sum_{n=0}^{\infty} C_{\alpha_1 \ldots \alpha_n}(x^m) \tilde{y}^{\alpha_1} \ldots \tilde{y}^{\alpha_n}. \]

(37)

To obtain the generating function of higher spin fields in $AdS_4$ we should just take the solution (30) at $y^{mn} = 0$, which then takes the form

\[ C(x^m, y^{\alpha}) = \int d^4 \lambda \left( 1 - \frac{S^2}{32} x^m x_m \right) e^{\frac{i}{2} x^m \gamma_{\alpha\beta}(\lambda_\alpha + \frac{1}{4} y^{ab} \gamma_{\alpha\beta} - i \lambda_\alpha y^a) \varphi(\lambda), \]

(38)

where now $G^{-1\alpha\beta}(x^m) = \delta_\alpha^\beta + \frac{1}{8} x^m \gamma_{\alpha\beta}$ and $\det G^{-1}(x^m) = (1 - \frac{S^2}{64} x^m x_m)^2$.

To get the unfolded $AdS_4$ equations satisfied by (38) we should multiply equations (27) or (28) by $\frac{1}{2} G_\delta^{\alpha} \gamma_{\alpha\beta} G_\beta^{\beta}$ and then take $G_\delta^{\alpha}(x^m)$ at $y^{mn} = 0$, we thus get

\[ \left[ \frac{\partial}{\partial x^m} + i \gamma_{\alpha\beta}^{\alpha\beta}(x) Y_\alpha Y_\beta \right] C(x^m, y^{\alpha}) = 0, \]

(39)

where

\[ \Omega^{\alpha\beta}(x^m) = dx^m \Omega^{\alpha\beta}(x^m) = \frac{1}{2} dx^m \gamma_{\alpha\beta} G_\alpha^{\alpha}(x) G_\beta^{\beta}(x) = \frac{1}{4} dx^m \omega^{ab}(x) \gamma_{ab}^{\alpha\beta} + \frac{1}{2} dx^m e_m^{\alpha}(x) \gamma_{\alpha\beta} \]

(40)

is the generalized $AdS_4$ connection satisfying the zero curvature condition $d\Omega + \frac{1}{2} \Omega \wedge \Omega = 0$. It is composed of the $AdS_4$ spin connection $\omega^{ab}(x)$ and the $AdS_4$ vielbein $e_m^{\alpha}(x)$, and hence takes values in $Sp(4)$.

$C(x^m, y^{\alpha})$ of (38) is the generating function of the field strengths of higher spin fields in $AdS_4$. Its form is different from that considered in [13, 21] because the latter was written in the $AdS_4$ parametrization in which the $AdS$ metric is conformally flat, while eq. (38) is in the $GL$–flat basis of the $AdS$ isometry group $Sp(4)$.

\footnote{Note that $x^{\alpha\beta}$ can be completely absorbed by a redefined $y^{\alpha}$, then we recover a twistor–like transform of the tensorial space [3, 5].}
6 Conclusion and discussion

Having based upon results of \[5, 6, 7, 9\] we have demonstrated how free higher spin field theory in \(D = 4\) flat space-time and in \(AdS_4\) emerges upon the quantization of a simple particle model \[3\], respectively, in flat tensorial space and on the group manifold \(Sp(4)\) generating isometries of \(AdS_4\).

To analyze the model we have used the property of these tensorial spaces to be \(GL(4)\)-flat, which is a tensorial analog of the conformal flatness of the Minkowski and \(AdS\) spaces.

As a generalization and development of these results, higher dimensional and supersymmetric \(OSp(N|2n)\) extensions of tensorial particle dynamics and corresponding first-quantized higher-spin field theories have been studied in \[3\]–\[9\], \[12\]–\[15\].

The dynamics of extended relativistic objects in tensorial superspaces has been analyzed as well. For instance tensionless (null) superbranes in tensorial spaces have been considered in \[16\]–\[18\] and fully fledged superstrings in \[19\]–\[20\]. Group-theoretical aspects of brane dynamics involving tensorial charges have been discussed in \[21\].

In conclusion let us note that in spite of a progress in understanding the subject considered above, as in most of the theoretical fields, many questions are still to be answered, for instance

- Do there exist other \(GL\)-flat (super)manifolds in addition to \(OSp(1|2n)\), e.g. \(OSp(N|2n)\) with \(N > 1\)?

- Do field equations \(23\), \(33\) and \(34\) in tensorial spaces admit a Lagrangian interpretation, i.e. whether they can follow from an action principle? Note that the equations are rather simple but highly degenerate.

- Whether particle and field dynamics in curved tensorial spaces may be helpful in solving the Interaction Problem of Higher Spin Field Theory? A step in this direction was made in \[22\] where cubic interactions of fields in tensorial spaces were analyzed.

- Is there any relation of field theory in tensorial spaces to higher spin field theory produced by strings \[23\]–\[25\]?

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