Hypergraph model for assembly sequence problem

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Abstract. The article deals with the assembly sequence problem of complex technical products. It is shown that geometric coordination of parts in a product is a variable arity relation. In general, this relation cannot be described by graph means. The hypernetwork and hypergraph models are proposed. It is shown that the mathematical description of coherent and sequential assembly operations is a normal contraction of edges of the hypergraph. The notion of an s-hypergraphs is introduced. It is shown that s-hypergraphs correctly describe structures of technical systems. A theorem on necessary conditions for contractibility of s-hypergraphs is proved.

1. Introduction
The Assembly Sequence Problem (ASP) is an important and actual problem of modern design and manufacturing. The assembly sequence affects the quality and reliability of products. The duration of production and its cost largely depend on this design decision.

In current research on computer aided assembly planning the basic structural model of technical objects is a so-called connection graph (liaison diagram, liaison graph, parts liaison graph, product liaison graph, attributed liaison graph, part mating graph, connective relation graph, component mating graph, hierarchical relation graph, component graph, adjacency graph, weighted undirected connected graph and etc.). In most articles the model is defined as an undirected graph whose vertices describe parts and the edges are mechanical contacts between the parts. The connection graph was first proposed by A. Bourjault as the main source of topological information about products [1]. In some publications for these purposes a matrix of contacts between parts is used [2]. This model is, in fact, an adjacency matrix of the connection graph. Therefore we consider matrix and graph models to be equivalent.

The connection graph allows formulating the general conditions for the existence of design solutions. These are coherence of assembly operations and connectivity of assembly units. The coherence means that any assembly operation can be described as a union of vertices of the graph that are connected by at least one edge. The connectivity of assembly units is the connectivity of the subgraph representing this constructive fragment [2–4].

The coherence and connectivity are weak necessary conditions. They serve only to reject deliberately the unusable design decisions. To select rational alternatives, it is necessary to use additional design and technological information. To this end various additions and extensions of the connection graph have been proposed. For example, a weighted graph is used in which the weights of vertices and edges encode relevant design data [5]. On the basis of the connection graph a semantic network is constructed that describes the actual technological context [6, 7]. Topological information of the connection graph is used together with heuristics and the decision maker [8].
A regular ASP solution method is described in [3]. It is based on cutting the connection graph into two connected subgraphs, which are checked for geometric obstacles.

In [9] two graphs are used to solve ASP: the face adjacency graph and the component mating graph. The first model is used to describe necessary geometric conditions for disassembly of products. The second graph is used to model the coherence of assembly operations.

An original method for designing assembly sequence and structural analysis of products was proposed in [7]. To describe products the authors offer two main models: the structural model – a knowledge assembly liaison graph (KALG) and the geometric model – a geometric liaison model (GLM). The KALG is an extension of the connection graph in which two vertices are connected by an arc if one vertex must be installed before the other. This precedence may be caused by any reasons: constructive, industrial or economic. The GLM is a complexly organized tensor that stores information about possible contacts and geometric obstacles.

Building the connection graph manually is a time-consuming and unreliable operation. Many works are devoted to methods of synthesis of the connection graph on three-dimensional models of products. A method for generating the connection graph from the 3D model created in the I-DEAS system was proposed in [10]. An algorithm for generating the connection graph using the SolidWorks API is discussed in [11]. Extraction of structural information from the STEP file and construction of the connection graph are considered in [12]. A method for constructing an attributed liaison graph on the 3D model of a product is proposed in [13]. In this graph every edge is assigned properties of connection between two parts of the product. A method of generating the connection graph according to the relational model of a product is described in [14]. An overview of methods of computer-aided generation is given in [15].

In general, geometrical certainty of parts during the assembly of products can be achieved with the help of several (two, three or more) mechanical connections. The connection graph and its numerous modifications are binary mathematical objects. For this reason, these models cannot adequately describe the multi-dimensional relation of spatial coordination. This is the main drawback of the works in which graphs are the main tool for modeling structures of assembled products.

2. Hypernetwork and hypergraph models of products
In this paper we propose the hypernetwork and hypergraph models of the mechanical structure of a technical system. They correctly describe the spatial relationship of parts. They can be used to solve ASP and generate product decompositions into assembly units [16].

**Definition 1.** The hypernetwork is vector \( HS = (X,V,R,F,W) \) in which:
- \( X = \{x_1,\ldots,x_n\} \) – a set of vertices;
- \( V = \{v_1,\ldots,v_q\} \) – a set of edges;
- \( R = \{r_1,\ldots,r_m\} \) – a set of hyperedges;
- \( P \) is a mapping \( P : V \to 2^X \) that assigns a set \( P(v) \subseteq X \) of its vertices to each element \( v \in V \). Thus, \( P \) defines a hypergraph \( PS = (X,V,P) \) on the set of vertices \( X \).
- \( F \) is a mapping \( F : R \to 2^V \) that assigns a set \( F(r) \subseteq V \) of edges to each element \( r \in R \). Every set \( F(r) \) is a connected subgraph of the hypergraph \( PS \). The mapping \( F \) defines a hypergraph \( FS = (V,R,F) \)
- \( W \) is a mapping \( \forall r \in R : W : R \to 2^{P(F(r))} \) that assigns a subset of its vertices \( W(r) \subseteq P(F(r)) \subseteq X \) to each element \( r \in R \), where \( P(F(r)) \) is the set of vertices in \( PS \) incident to the edges \( F(r) \subseteq V \). Thus the mapping \( W \) defines a hypergraph \( WS = (X,R,W) \). The hypergraph \( PS \) is called the primary network, and the hypergraph \( WS \) is called the secondary network of the hypernetwork \( HS \) [17].

The structure of a technical system is represented as the hypernetwork \( HS = (X,V,R,P,F,W) \), in which:
- The set of vertices \( X = \{x_i\}_{i=1}^n \) describes parts;
The set of edges $V = \{v_i\}_{i=1}^q$ – mechanical contacts between parts;

A hyperedge $r \in R = \{r_i\}_{i=1}^m$ corresponds to a minimum subset of parts, which has the property of mutual coordination, achieved due to internal mechanical connections;

The mapping $P : V \rightarrow 2^X$ assigns each edge two parts, between which there is a mechanical contact, i.e. $\forall v \in V \mid |P(v)| = 2$. Therefore, the primary network $PS = (X,V,P)$ is an undirected graph;

The mapping $F : R \rightarrow 2^V_{PS}$ assigns a set of constraints (edges) that form the minimal coordinated subset of the parts to each hyperedge $r \in R = \{r_i\}_{i=1}^m$. This mapping describes the hyperedge as a subset of edges (mechanical contacts);

The mapping $W : R \rightarrow 2^{p(F(r))}$ associates each hyperedge $r$ with the subset of vertices $P(F(r)) \subseteq X$. This mapping describes hyperedges as a subset of vertices (parts).

The primary network $PS = (X,V,P)$ of the hypernetwork $HS$ describes the structure of mechanical connections and is an undirected graph. The secondary network $WS = (X,R,W)$ models the spatial coordination of parts, which is achieved by means of mechanical contacts. Each hyperedge of the secondary network $WS$ represents a minimum geometrically defined subset of parts. Such subsets may have different cardinality, and therefore, in general, $WS$ is a hypergraph.

In figure 1 a mechanical system drawing is shown, and in figure 2 – the hypernetwork of this product. In this figure the hyperedges of the secondary network are shown in dotted lines.

**Figure. 1.** A mechanical system.
The hypernetwork model describes connections and mutual coordination of the parts, which is achieved through these connections. This is an important property which is necessary for many design and manufacturing operations: dimensioning, decomposition into assembly units, assembly, disassembly, testing etc.

The primary network $PS$ of the hypernetwork $HS$ is a connection graph for which methods of computer-aided synthesis have been developed [10–15]. Creation of the secondary network $WS$, correctly describing the spatial coordination of parts, is a much more complicated task. A mechanical connection is the contact between a pair of parts. Mechanical contacts can be determined by the regular means of modern CAD systems. The identification of the geometric coordination is an informal design procedure that cannot be performed without the participation of a decision maker.

Analysis of a large number of designs has shown that in most cases the hyperedges of the secondary network correspond to cliques of the primary network (figure 2). Therefore, the procedure for synthesizing the secondary network can be organized on the basis of the expert verification of the cliques of the primary network. This procedure will certainly be more efficient than any algorithm based on the brute force mechanism.

3. Mathematical description of the assembly sequence

**Definition 2.** An assembly operation is called $n$-handed, if it requires $n$ hands (executive agents) that perform independent movements of parts [2].

An example of a construction, which is assembled with the help of three hands, is shown in figure 3.

**Definition 3.** A two-handed assembly operation is called sequential [15].

It is assumed in the article that the vast majority of products do not require $n$-handed, $n > 2$, assembly operations. This article covers only sequential assembly operations.

**Definition 4.** An assembly operation is called coherent if it realizes at least one mechanical connection.

A construction, for assembly of which an incoherent assembly operation is used, is shown in figure 4. Obviously the parts A and B must be coordinated relative to each other before being installed in C. This coordination is achieved without mechanical connections, so this operation is incoherent. The assembly operations with such properties are out of scope of this article.

**Figure 2.** The hypernetwork of the mechanical system: the vertices represent the parts; the solid lines – the mechanical contacts; the dotted lines – the minimum subsets of parts, which have the property of mutual geometric coordination.

**Figure 3.** A 3-handed assembly operation.

**Figure 4.** An incoherent assembly operation.

During the assembly of products the mechanical connections between parts are realized, and every assembled fragment is an integrity in which each part occupies a geometrically determined position.
Definition 5. The contraction of a hyperedge of degree 2 of the hypergraph $WS$ is called normal.

Assertion. The assembly process of the product $X = \{x_j\}_{j=1}^n$ is represented as a sequence of the contractions $Cnt = (WS_1, ..., WS_N)$ of the hypergraph $WS = (X, R, W)$, and for $Cnt$ the following conditions must be fulfilled:

1. $WS_1 = WS$;
2. $WS_N$ represents the single-vertex hypergraph without loops and isolated edges;
3. Each contraction is normal;
4. For all $WS_j = (X_j, R_j, W_j)$ and $WS_{j+1} = (X_{j+1}, R_{j+1}, W_{j+1})$, $|R_j| = |R_{j+1}| + 1$, $j = 1, N - 1$.

The first condition means that the assembly begins with the start state, in which no connections is realized. This state is represented as the secondary network $WS$. The second condition formalizes the final state of the assembly process when the product is assembled and all connections are made. The third condition means that each assembly operation is sequential and coherent. The fourth condition describes an assembly operation that is performed without overbasing (overconstraint).

Overbasing is a situation of excessive coordination, when the spatial position of a part is determined with the help of an excessive number of bases [18]. The simplest example of overbasing is the installation of a prism key in a keyway with zero clearance on all four side faces. If the part is an absolutely rigid body, the overbasing is unacceptable, since it generates insoluble design dimensional chains and entails radical changes in technological instructions. Deformable parts simply do not need redundant bases. The excessive coordination of parts is consequence of design errors. It is important to note that the modern CAD systems have no means to identify this effect automatically.

The primary network $PS$ is an auxiliary model. It serves to identify gross design errors and generate the secondary network $WS$. The main source of project information about products is the hypergraph $WS$.

Definition 6. A vertex $y \in X_j$ of the hypergraph $WS_j = (X_j, R_j, W_j)$ is called an s-vertex, if it is obtained by identifying two vertices of the hypergraph $WS_{j+1} = (X_{j+1}, R_{j+1}, W_{j+1})$ as a result of the normal contraction of an edge $r \in R_{j+1}$, $j = 2, N$.

Definition 7. A hypergraph for which there exists a sequence of contractions $Cnt$ that takes it to the one-vertex hypergraph without loops and isolated edges (a point) is called an s-hypergraph.

S-hypergraphs serve as mathematical models of the product structures that can be assembled using a sequence of the sequential and coherent assembly operations.

Theorem. If the hypergraph $WS = (X, R, W)$ contracts to the point, then:

1. There is at least one edge of degree 2 among the edges of $WS$;
2. The hypergraph $WS$ is connected;
3. The number of vertices $|X|$ and the number of edges $|R|$ of the hypergraph $WS$ satisfy the relation $|X| = |R| + 1$.

Proof. The necessity of the first condition is proved by the first operation of normal contraction which is applied to vertices of the original hypergraph $WS$. The connectivity of the hypergraph $WS$ follows from the existence of the sequence of normal contractions that takes the hypergraph to the point. The third condition is proved by induction on the number of vertices of the s-hypergraph $WS$.

For $n = 2$ there is only one contractible hypergraph. This is the two-vertex hypergraph with one edge, for which all the conditions of the theorem are trivial.

Suppose that condition 3 is valid for all s-hypergraphs with $k$ vertices; $k < n, n > 2$ (the inductive hypothesis). We show that condition 3 holds for any $n$-vertex s-hypergraph. We consider the last operation of the normal contraction of the hypergraph $WS$. Denote by $y, z$ the vertices that are identified in this operation, and $R_{yz}$ is the edge of degree two that connects these vertices.

relative to other parts. Therefore, any assembly operation can be described as a contraction of the hyperedge of the hypergraph $WS$. The assembly process can be represented as a sequence of contractions.
We consider the case where \( y \) is a vertex and \( z \) is a \( s \)-vertex (definition 6). The vertex \( z \) is the image of the contractible hypergraph \( WS_y = (Z, R_y) \), \( Z \subset X \), \( R_y \subset R \). The number of vertices of the hypergraph \( WS_z \) is \(|Z| = |X| - 1\), i.e. \(|Z| < n\). By the inductive hypothesis the equality \(|Z| = |R_z| + 1\) is valid. The set of the edges of the hypergraph \( WS \) consists of the set of the edges \( R_z \) and the edge \( R_y \). This implies the equality \(|R| = |R_z| + 1\). We calculate the number of the vertices of the original hypergraph \( WS \) by the formula \(|X| = |Z| + 1 = |R_z| + 1 + 1 = |R| + 1\). That proves the validity of condition 3.

We consider the case where the vertices \( y \) and \( z \) are \( s \)-vertices. We also denote \( WS_y = (Y, R_y) \) and \( WS_z = (Z, R_z) \) the \( s \)-subgraphs of the hypergraph \( WS \), which are respectively contracted to the vertices \( y \) and \( z \). For the sets of vertices and edges of these hypergraphs the following relations are valid \( Z \cap Y = \emptyset \), \( Z \cup Y = X \) and \( R = R_y \cup R_z \cup \{R_y\} \). It follows that \(|R| = |R_y| + |R_z| + 1\) and \(|Z|, |Y| < n\). Therefore, the hypergraphs \( WS_y \) and \( WS_z \) satisfy the inductive hypothesis and the following equalities \(|R_y| = |R_y| + 1\) and \(|Z| = |R_z| + 1\) hold for them. Adding these equalities, we obtain \(|R_y| + |Z| = |R_y| + |R_z| + 1 + 1 = |R| + 1\). Because \(|R_y| + |Z| = |X|\), then \(|X| = |R| + 1\). The theorem is proved.

The necessary conditions are not sufficient. Two examples of non-contractible hypergraphs are shown in figure 5. They satisfy all the conditions of the theorem. It is easy to see that after the first normal contraction these hypergraphs become «double triangle», the contraction of which is impossible.

![Figure 5. The examples of non-contractible hypergraphs: (a) – the non-contractible hypergraphs; (b) – the hypergraphs after contracting the second-degree edge.](image)

Any constructive fragment of the product that has the property of the internal coordination can be described using a \( s \)-subgraph of the hypergraph \( WS \). This makes the hypergraph model very informative and promising. In particular, it can be used to synthesize a product decomposition into assembly units [18]. Obviously, this problem can be formulated as a cutting the hypergraph \( WS \) into \( s \)-subgraphs that satisfies certain additional criteria. These criteria can formalize various properties of the production system or economic indicators, for example the duration of production cycle, the production cost, etc.

### 4. Conclusion

The hypernetwork and hypergraph models describing the assembly structure of complex technical systems are proposed. It is shown that the hypergraph model correctly describes the coordination of the parts in assembled products, which is achieved by means of the mechanical connections. It is proposed the mathematical description of the assembly process which consists of the coherent and sequential operations. It is shown that the assembly sequences of the product can be represented as an ordered set of normal contractions of the hypergraph edges. The \( s \)-hypergraph concept is introduced and the theorem on the necessary contractibility conditions is proved.

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