Entanglement and Quantum Gate Operations with Spin-Qubits in Quantum Dots

John Schliemann\textsuperscript{1} and Daniel Loss\textsuperscript{2}

\textsuperscript{1}Department of Physics, The University of Texas, Austin, TX 78712
\textsuperscript{2}Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 84, CH-4056 Basel, Switzerland

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Abstract

We give an elementary introduction to the notion of quantum entanglement between distinguishable parties and review a recent proposal about solid state quantum computation with spin-qubits in quantum dots. The indistinguishable character of the electrons whose spins realize the qubits gives rise to further entanglement-like quantum correlations. We summarize recent results concerning this type of quantum correlations of indistinguishable particles.

1 Introduction

Quantum entanglement is one of the most intriguing features of quantum mechanics\cite{Einstein1935, Schrodinger1935, Bell1964, Hardy1993}. In the beginning of modern quantum theory, the notion of entanglement was first noted by Einstein, Podolsky, and Rosen\cite{Einstein1935}, and by Schrödinger\cite{Schrodinger1935}. While in those days quantum entanglement and its predicted physical consequences were (at least partially) considered as an unphysical property of the formalism (a "paradox"), the modern perspective on this issue is very different. Nowadays quantum entanglement is to be seen as an experimentally verified property of nature providing a resource for a vast variety of novel phenomena and concepts such as quantum computation, quantum cryptography, or quantum teleportation.

While the basic notion of entanglement in pure quantum states of bipartite systems (Alice and Bob) is theoretically well understood, fundamental questions are open concerning entanglement in mixed states (described by a proper density matrix)\cite{Bennett1996, Horodecki1996, Wootters1998, Peres1996}, or entanglement of more than two parties\cite{Bennett1996, Horodecki1996, Wootters1998, Peres1996, Kiefer1982}. The most elementary example for entanglement in a pure quantum state is given by a spin singlet composed from two spin-$\frac{1}{2}$ objects (qubits) owned by A(lice) and B(ob), respectively,

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B) . \]

For such a state, the state of the combined system cannot be described by specifying the state of Alice’s and Bob’s qubit separately. It is a standard result of quantum information

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B) . \]

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theory \[1\] that this property does not depend on the basis chosen in Alice’s or Bob’s Hilbert space. As we shall see below, the entanglement of such a quantum state (quantified by an appropriate measure) is invariant under (independent) changes of basis in both spaces.

Physically measurable consequences of quantum entanglement of the above kind arise typically (but not exclusively) in terms of two-body correlations between the subsystems. In this case the effects of entanglement can typically be cast in terms of so-called Bell inequalities \[18\] whose violation manifests the presence entanglement in a given quantum state. Using this formal approach the physical existence of quantum entanglement (as opposed to classical correlations) has unambiguously been verified for the polarization states of photons by Aspect and coworkers \[19\]. Moreover, quantum entanglement is an essential ingredient of algorithms for quantum computation \[4, 3\], in particular for Shor’s algorithm for decomposing large numbers into their prime factors \[20\]. This problem is intimately related to public key cryptography systems such as RSA encoding which is widely used in today’s electronic communication.

Among the many proposals for experimental realizations of quantum information processing solid state systems have the advantage of offering the perspective to integrate a large number of quantum gates into a quantum computer once the single gates and qubits are established. Recently, a proposal has been put forward involving qubits formed by the spins of electrons living on semiconductor quantum dots \[21, 22, 23, 24, 25\]. In this scenario, the indistinguishable character of the electrons leads to entanglement-like quantum correlations which require a description different from the usual entanglement between distinguishable parties (Alice, Bob, ...) in bipartite (or multipartite) systems. In such a case the proper statistics of the indistinguishable particles has to be taken into account.

In this article we give an elementary introduction to the notion of quantum entanglement between distinguishable parties and review the aforementioned proposal for quantum computation with spin-qubits in quantum dots. The indistinguishable character of the electrons whose spins realize the qubits gives rise to further entanglement-like quantum correlations. We summarize recent results on the characterization and quantification of these quantum correlations which are analogues of quantum entanglement between distinguishable parties \[24, 26, 27, 28, 29, 30\].

### 2 Quantum Entanglement between distinguishable parties

We now give an introduction to basic concepts of characterizing and quantifying entanglement between distinguishable parties. We concentrate on pure states (i.e. elements of the joint Hilbert space) of bipartite systems. We then comment only briefly on the case of mixed states (described by a proper density operator), and entanglement in multipartite systems.

One of the most prominent examples of an entangled state was already given in the previous section, namely a spin singlet built up from two qubits. More generally, if Alice and Bob own Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimensions $m$ and $n$, respectively, a state $|\psi\rangle$ is called nonentangled if it can be written as a product state,

$$|\psi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B$$

with $|\alpha\rangle_A \in \mathcal{H}_A$, $|\beta\rangle_B \in \mathcal{H}_B$. Otherwise $|\psi\rangle$ is entangled. The question arises whether a given state $|\psi\rangle$, expressed in some arbitrary basis of the joint Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, is entangled or not, i.e. whether there are states $|\alpha\rangle_A$ and $|\beta\rangle_B$ fulfilling (3). Moreover, one would like to quantify the entanglement contained in a state vector.

An important tool to investigate such questions for bipartite systems is the biorthogonal Schmidt decomposition \[1\]. It states that for any state vector $|\psi\rangle \in \mathcal{H}$ there exist bases of
\[ \mathcal{H}_A \text{ and } \mathcal{H}_B \text{ such that} \]
\[ |\psi\rangle = \sum_{i=1}^{r} z_i \left( |a_i\rangle \otimes |b_i\rangle \right) \quad (3) \]

with coefficients \( z_i \neq 0 \) and the basis states fulfilling \( \langle a_i|a_j \rangle = \langle b_i|b_j \rangle = \delta_{ij} \). Thus, each vector in both bases for \( \mathcal{H}_A \) and \( \mathcal{H}_B \) enters at most only one product vector in the above expansion. As a usual convention, the phases of the basis vectors involved in (3) can be chosen such that all \( z_i \) are positive. The expression (3) is an expansion of the state \( |\psi\rangle \) into a basis of product vectors \( |a\rangle \otimes |b\rangle \) with a minimum number \( r \) of nonzero terms. This number ranges from one to \( \min\{m, n\} \) and is called the Schmidt rank of \( |\psi\rangle \).

With respect to arbitrary bases in \( \mathcal{H}_A \) and \( \mathcal{H}_B \) a given state vector reads
\[ |\psi\rangle = \sum_{a,b} M_{ab} |a\rangle \otimes |b\rangle \quad (4) \]

with an \( m \times n \) coefficient matrix \( M \). Under unitary transformations \( U_A \) and \( U_B \) in \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, \( M \) transforms as
\[ M \mapsto M' = U_A M U_B^T \quad (5) \]

with \( U_B^T \) being the transpose of \( U_B \). The fact that there are always bases in \( \mathcal{H}_A \) and \( \mathcal{H}_B \) providing a biorthogonal Schmidt decomposition of \( |\psi\rangle \) is equivalent to stating that there are matrices \( U_A \) and \( U_B \) such that the resulting matrix \( M' \) consists of a diagonal block with only nonnegative entries while the rest of the matrix contains only zeroes. For the case of equal dimensions of Alice’s and Bob’s space, \( m = n \), this is also a well-known theorem of matrix algebra [31].

Obviously, \( |\psi\rangle \) is nonentangled, i.e. a simple product state, if and only if its Schmidt rank is one. More generally, the Schmidt rank of a pure state can be viewed as a rough characterization for its entanglement. However, since the Schmidt rank is by construction a discrete quantity it does not provide a proper quantification of entanglement. Therefore finer entanglement measures are desirable. For the case of two distinguishable parties, a useful measure of entanglement is the von Neumann-entropy of partial density matrices constructed from the pure-state density matrix \( \rho = |\psi\rangle \langle \psi| \) [22]:
\[ E(|\psi\rangle) = -\text{tr}_A \left( \rho_A \log_2 \rho_A \right) = -\text{tr}_B \left( \rho_B \log_2 \rho_B \right), \quad (6) \]

where the partial density matrices are obtained by tracing out one of the subsystems, \( \rho_{A/B} = \text{tr}_{A/B} \rho \). With the help of the biorthogonal Schmidt decomposition of \( |\psi\rangle \) one shows that both partial density matrices have the same spectrum and therefore the same entropy, as stated in Eq. (5). In particular, the Schmidt rank of \( |\psi\rangle \) equals the algebraic rank of the partial density matrices. \( |\psi\rangle \) is nonentangled if and only if the partial density matrices of the pure state \( \rho = |\psi\rangle \langle \psi| \) are also pure states, and \( |\psi\rangle \) is maximally entangled if its partial density matrix are “maximally mixed”, i.e. if they have only one non-zero eigenvalue with a multiplicity of \( \min\{m, n\} \).

It is important to observe that the entanglement measure (1) of a given state \( |\psi\rangle \) does not depend on the bases used in Alice’s and Bob’s Hilbert space to express this state. This is because the trace operations in the definition of \( E(|\psi\rangle) \) are invariant under an eventual change of bases (performed, in general, independently in both spaces). Therefore, entanglement in bipartite systems is a basis-independent quantity.

Thus, the problem of characterizing and quantifying quantum entanglement for pure states in bipartite systems can been seen as completely solved. Unfortunately, the situation is much less clear for mixed states [4] [5] [8] [9], and for multipartite entanglement. The main obstacle in the latter issue is the fact that the biorthogonal Schmidt decomposition in bipartite systems does not have a true analogue in the multipartite case. For details we refer the reader to the research literature; a nonexhaustive collection of recent papers includes [8, 11, 13, 14, 15, 16, 17].
3 Quantum computing with electron spins in quantum dots

We will now illustrate the phenomenon of quantum entanglement on the example of a specific (possible) realization of a quantum information processing system [21]. The proposal discussed below deals with qubits realized by the spins of electrons residing on semiconductor quantum dots. As we shall see in this and the following section, the indistinguishable character of the electrons gives rise to quantum correlations which are beyond entanglement between distinguishable parties.

An array of coupled quantum dots, see Fig. 1, each dot containing a top most spin $1/2$, was found to be a promising candidate for a scalable quantum computer [21] where the quantum bit (qubit) is defined by the spin $1/2$ on the dot. Quantum algorithms can then be implemented using local single-spin rotations and the exchange coupling between nearby spins, see Fig. 1. This proposal is supported by experiments where, e.g., Coulomb blockade effects, tunneling between neighboring dots, and magnetization have been observed as well as the formation of a delocalized single-particle state in coupled dots [36]. For a detailed review of quantum computing with electron spins in quantum dots see Ref. [23].

The charge of the electron can further be used to transport a spin-qubit along conducting wires [37]. This allows one to use spin-entangled electrons as Einstein-Podolsky-Rosen (EPR) pairs, which can be created (e.g. in coupled quantum dots or near a superconductor-normal interface), transported, and detected in transport and noise measurements [38]. Such EPR pairs represent the fundamental resources for quantum communication [10]. The electron spin is a natural candidate for a qubit since its spin state in a given direction, $| \uparrow \rangle$ or $| \downarrow \rangle$, can be identified with the classical bits $| 0 \rangle$ and $| 1 \rangle$, while an arbitrary superposition $\alpha | \uparrow \rangle + \beta | \downarrow \rangle$ defines a qubit. In principle, any quantum two-level system can be used to define a qubit. However, one must be able to control coherent superpositions of the basis states of the quantum computer, i.e. no transition from quantum to classical behavior should occur. Thus, the coupling of the environment to the qubit should be small, resulting in a sufficiently large decoherence time $T_2$ (the time over which the phase of a superposition of $| 0 \rangle$ and $| 1 \rangle$ is well-defined). Assuming weak spin-orbit effects, the spin decoherence time $T_2$ can be completely different from the charge decoherence time (a few nanoseconds), and in fact it is known [11] that $T_2$ can be orders of magnitude longer than nanoseconds. Time-resolved optical measurements were used to determine $T_2^*$, the decoherence time of an ensemble of spins, with $T_2^*$ exceeding 100 ns in bulk GaAs [11]. More recently, the single spin relaxation time $T_1$ (generally $T_1 \geq T_2$) of a single quantum dot attached to leads was measured via transport to be longer than a few $\mu$s [12], consistent with calculations [13].

Let us now consider a system of two laterally tunnel-coupled dots having one electron each. Using an appropriate model [24], theoretical calculations have demonstrated the possibility of performing two-qubit quantum gate operations in such a system by varying the tunnel barrier between the dots. An important point to observe here is the fact that the electrons whose spins realize the qubits are indistinguishable particles [24]. Differently from the usual scenario of distinguishable parties (Alice, Bob, ...) the proper quantum statistics has to be taken into account when a finite tunneling between the dots is inferred [25, 26, 27].

In the following section we give an elementary introduction to the theory of “entanglement-like” quantum correlations in systems of indistinguishable particles. We concentrate on the fermionic case and illustrate our findings on the above example of coupled quantum dots.
Figure 1: Quantum dot array, controlled by electrical gating. The electrodes (dark gray) define quantum dots (circles) by confining electrons. The spin 1/2 ground state (arrow) of the dot represents the qubit. These electrons can be moved by electrical gating into the magnetized or high-$g$ layer, producing locally different Zeeman splittings. Alternatively, magnetic field gradients can be applied, as e.g. produced by a current wire (indicated on the left of the dot-array). Then, since every dot-spin is subjected to a different Zeeman splitting, the spins can be addressed individually, e.g. through ESR pulses of an additional in-plane magnetic ac field with the corresponding Larmor frequency $\omega_L = g\mu_B B / h$. Such mechanisms can be used for single-spin rotations and the initialization step. The exchange coupling between the quantum dots can be controlled by lowering the tunnel barrier between the dots. In this figure, the two rightmost dots are drawn schematically as tunnel-coupled. Such an exchange mechanism can be used for the XOR gate operation involving two nearest neighbor qubits. The XOR operation between distant qubits is achieved by swapping (via exchange) the qubits first to a nearest neighbor position. The read-out of the spin state can be achieved via spin-dependent tunneling and SET devices [21], or via a transport current passing the dot [44]. Note that all spin operations, single and two spin operations, and spin read-out, are controlled electrically via the charge of the electron and not via the magnetic moment of the spin. Thus, no control of local magnetic fields is required, and the spin is only used for storing the information. This spin-to-charge conversion is based on the Pauli principle and Coulomb interaction and allows for very fast switching times (typically picoseconds). A further advantage of this scheme is its scalability into an array of arbitrary size.

4 Quantum Correlations between indistinguishable particles

For indistinguishable particles a pure quantum state must be formulated in terms of Slater determinants or Slater permanents for fermions and bosons, respectively. Generically, a Slater determinant contains correlations due to the exchange statistics of the indistinguishable fermions. As the simplest possible example consider a wavefunction of two (spinless) fermions,

$$\Psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} [\phi(\vec{r}_1)\chi(\vec{r}_2) - \phi(\vec{r}_2)\chi(\vec{r}_1)]$$

with two orthonormalized single-particle wavefunctions $\phi(\vec{r})$, $\chi(\vec{r})$. Operator matrix elements between such single Slater determinants contain terms due to the antisymmetrization of coordinates ("exchange contributions" in the language of Hartree-Fock theory). However, if the moduli of $\phi(\vec{r})$, $\chi(\vec{r})$ have only vanishingly small overlap, these exchange
correlations will also tend to zero for any physically meaningful operator. This situation is generically realized if the supports of the single-particle wavefunctions are essentially centered around locations being sufficiently apart from each other, or the particles are separated by a sufficiently large energy barrier. In this case the antisymmetrization present in Eq. (7) has no physical effect.

Such observations clearly justify the treatment of indistinguishable particles separated by macroscopic distances as effectively distinguishable objects. So far, research in Quantum Information Theory has concentrated on this case, where the exchange statistics of particles forming quantum registers could be neglected, or was not specified at all.

The situation is different if the particles constituting, say, qubits are close together and possibly coupled in some computational process. This the case for all proposals of quantum information processing based on quantum dots technology [21, 22, 23, 24, 25]. Here qubits are realized by the spins of electrons living in a system of quantum dots. The electrons have the possibility of tunneling eventually from one dot to the other with a probability which can be modified by varying external parameters such as gate voltages and magnetic field. In such a situation the fermionic statistics of electrons and the associated Pauli principle are clearly essential.

Additional correlations in many-fermion-systems arise if more than one Slater determinant is involved, i.e. if there is no single-particle basis such that a given state of \( N \) indistinguishable fermions can be represented as an elementary Slater determinant (i.e. fully antisymmetric combination of \( N \) orthogonal single-particle states). These correlations are the analog of quantum entanglement in separated systems and are essential for quantum information processing in non-separated systems.

As an example consider a “swap” process exchanging the spin states of electrons on coupled quantum dots by gating the tunneling amplitude between them [22, 24]. Before the gate is turned on, the two electrons in the neighboring quantum dots are in a state represented by a simple Slater determinant, and can be regarded as distinguishable since they are separated by a large energy barrier. When the barrier is lowered, more complex correlations between the electrons due to the dynamics arise. Interestingly, as shown in Refs. [22, 24], during such a process the system must necessarily enter a highly correlated state that cannot be represented by a single Slater determinant. The final state of the gate operation, however, is, similarly as the initial one, essentially given by a single Slater determinant. Moreover, by adjusting the gating time appropriately one can also perform a “square root of a swap” which turns a single Slater determinant into a “maximally” correlated state in much the same way [24]. Illustrative details of these processes will be given below. In the end of such a process the electrons can again be viewed as effectively distinguishable, but are in a maximally entangled state in the usual sense of distinguishable separated particles. In this sense the highly correlated intermediate state can be viewed as a resource for the production of entangled states.

In the following we give an elementary introduction to recent results in the theory of quantum correlations in systems of indistinguishable particles [24, 27, 28, 29, 26]. These correlations are analogues of entanglement between distinguishable parties. However, to avoid confusion with the existing literature and in accordance with Refs. [26, 27, 29], we shall reserve in the following the term “entanglement” for separated systems and characterize the analogous quantum correlation phenomenon in non-separated systems in terms of the Slater rank and the correlation measure to be defined below.

For the purposes of this article we shall concentrate on elementary results for the case of pure states of two identical fermions. Results for mixed states and more than two fermions can be found in [21, 22]. Results for the case of identical bosons can be found in [28, 29, 27]. We consider the case of two identical fermions sharing an \( n \)-dimensional single-particle space \( \mathcal{H}_n \) resulting in a total Hilbert space \( \mathcal{A}(\mathcal{H}_n \otimes \mathcal{H}_n) \) with \( \mathcal{A} \) denoting the antisym-
A general state vector can be written as

$$|w⟩ = \sum_{a,b=1}^{n} w_{ab} f_a^+ f_b^+ |0⟩ \quad (8)$$

with fermionic creation operators $f_a^+$ acting on the vacuum $|0⟩$. The antisymmetric coefficient matrix $w_{ab}$ fulfills the normalization condition

$$\text{tr} (\bar{w} w) = -\frac{1}{2}, \quad (9)$$

where the bar stands for complex conjugation. Under a unitary transformation of the single-particle space,

$$f_a^+ \mapsto U f_a^+ U = U_{ba} f_b^+, \quad (10)$$

$w$ transforms as

$$w \mapsto U w U^T, \quad (11)$$

where $U^T$ is the transpose (not the adjoint) of $U$. For any complex antisymmetric matrix $n \times n$ matrix $w$ there is a unitary transformation $U$ such that $w' = U w U^T$ has nonzero entries only in $2 \times 2$ blocks along the diagonal $[26, 31]$. That is,

$$w' = \text{diag} [Z_1, \ldots, Z_r, Z_0] \quad \text{with} \quad Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix}, \quad (12)$$

$z_i \neq 0$ for $i \in \{1, \ldots, r\}$, and $Z_0$ being the $(n-2r) \times (n-2r)$ null matrix. Each $2 \times 2$ block $Z_i$ corresponds to an elementary Slater determinant in the state $|w⟩$. Such elementary Slater determinants are the analogues of product states in systems consisting of distinguishable parties. Thus, when expressed in such a basis, the state $|w⟩$ is sum of elementary Slater determinants where each single-particle basis state enters not more than one term. This property is analogous to the biorthogonality of the Schmidt decomposition discussed above. The matrix (12) represents an expansion of $|w⟩$ into a basis of elementary Slater determinants with a minimum number $r$ of non-vanishing terms. This number is analogous to the Schmidt rank for the distinguishable case. Therefore we shall call it the (fermionic) *Slater rank* of $|w⟩$ [20], and an expansion of the above form a *Slater decomposition* of $|w⟩$.

We now turn to the case of two fermions in a four-dimensional single-particle space. This case is realized in a system of two coupled quantum dots hosting in total two electrons which are restricted to the lowest orbital state on each dot. In such a system, a simple correlation measure can be defined as follows [24, 26]: For a given state (8) with a coefficient matrix $\omega_{ab}$ one defines a dual state $|\tilde{w}⟩$ characterized by the dual matrix

$$\tilde{w}_{ab} = \frac{1}{2} \sum_{c,d=1}^{4} \varepsilon_{abcd} \tilde{w}_{cd}, \quad (13)$$

with $\varepsilon_{abcd}$ being the usual totally antisymmetric unit tensor. Then the correlation measure $\eta(|w⟩)$ can be defined as

$$\eta(|w⟩) = |⟨\tilde{w} |w⟩| = \left| \sum_{a,b,c,d=1}^{4} \varepsilon_{abcd} \tilde{w}_{ab} w_{cd} \right| = \left| 8 (w_{12} w_{34} + w_{13} w_{42} + w_{14} w_{23}) \right|. \quad (14)$$

Obviously, $\eta(|w⟩)$ ranges from zero to one. Importantly it vanishes if and only if the state $|w⟩$ has the fermionic Slater rank one, i.e. $\eta(|w⟩)$ is an elementary Slater determinant.
This statement was proved first in Ref. [24]; an alternative proof can be given using the Slater decomposition of $|w\rangle$ and observing that

$$\det w = \left(\frac{1}{8} \langle \tilde{w}|w\rangle\right)^2. \quad (15)$$

The quantity $\eta(|w\rangle)$ measures quantum correlation contained in the two-fermion state $|w\rangle$ which are beyond simple antisymmetrization effects. This correlation measure in under many aspects analogous to the entanglement measure “concurrence” used in systems of two distinguishable qubits [45]. These analogies are discussed in detail in [27] including also the case of indistinguishable bosons. An important difference between just two qubits, i.e. two distinguishable two-level systems, and the present case of two electrons in a two-dot system is that in latter system both electrons can eventually occupy the same dot while the other is empty. Therefore the total Hilbert space is larger than in the two-qubit system, and a generalized correlation measure becomes necessary. Furthermore, similar as in the two-qubit case, the correlation measure $\eta$ defined here for pure states of two fermions has a natural extension to mixed fermionic and bosonic states [26, 27].

The expansion of the form (12) for a two-fermion system has an analogue in two-boson systems which was presented very recently in Refs. [28, 29]. Moreover the fermionic analogue (12) of the biorthogonal Schmidt decomposition of bipartite systems was also used earlier in studies of electron correlations in Rydberg atoms [46].

We note that the aforementioned double occupancies have temporarily given rise to some controversy about the principle suitability of such systems as quantum gates; these concerns were eliminated in a recent theoretical study, see [24] and references therein.

Let us now have a closer look at a specific quantum gate operation, namely a swap process outlined already before. This operation interchanges the contents of the qubits on two dots, e.g.,

$$| \uparrow \downarrow \rangle \rightarrow | \downarrow \uparrow \rangle, \quad (16)$$

where obvious notation has been used for the spin state on each dot. As we shall see below, the “square root” of such a swap operation provides an efficient way to generate entangled states. Moreover, the “square root of a swap” can be combined with further single-qubit operations to an exclusive-OR (XOR, or controlled-NOT) gate, which has been shown to be sufficient for the implementation of any quantum algorithm [47].

Both the initial and the final state in the above example of a quantum gate operation are single Slater determinants. In the beginning and the end of the operation the tunneling amplitude between the dots is exponentially small. The swap process is performed by temporarily gating the tunneling with a pulse-shaped time dependence as shown in Fig. 2. In the presence of a finite tunneling amplitude, i.e. during the swap operation, a finite probability for both electrons being on the same dot necessarily occurs. However, this double occupancy probability can be suppressed very efficiently in the final swapped state provided that the dynamics of the system is sufficiently close to its adiabatic limit. In fact, as shown in Ref. [24] this quasi-adiabatic regime is remarkably large. As a result, a clean swap process can be performed even if the tunneling pulse is switched on and off on a time scale close to the natural time scale of the problem given by $\hbar/U_H$ where $U_H$ is an effective repulsion between electrons on the same dot. In the middle of the swap process the system is in an highly correlated quantum state with the correlation measure $\eta$ being close to its maximum.

Next let us look at the “square root of a swap”, which is obtained from the situation of Fig. 2 by halving the pulse duration $T$. The probability of double occupancies is again strongly suppressed after the tunneling pulse. As shown in Fig. 3, the resulting state is a maximally correlated ($\eta = 1$) complex linear combination of the incoming state $| \uparrow \downarrow \rangle$ and the outgoing state $| \downarrow \uparrow \rangle$ of the full swap with both states having the same weight. The
quantum mechanical weights of the latter states are plotted as thick solid lines. After the tunneling amplitude is switched off again to exponentially small values, both dots carry one electron each. As explained above, in this situation, due to the high tunneling barrier between the dots, the two electrons can be considered as effectively distinguishable. In this sense the resulting state in Fig. 3 can be seen as a usual entangled state for distinguishable parties. However, during the gate operation such a view is not possible, since there is necessarily a finite amplitude for doubly occupied dots. Since the amplitude of such spin singlet states contributes to the correlation measure \( \eta(t) \), the intermediate state during the gate operations shown in Figs. 2, 3 can, loosely speaking, be interpreted to contain spin as well as orbital entanglement. In the end of the square root of the swap, however, the correlations are purely due to the spin degree of freedom. As a result, the double dot two-qubit system is also an efficient entangler.

5 Summary

We have given an elementary introduction to the notion of quantum entanglement between distinguishable parties. Entanglement phenomena can be illustrated on the example of the recently proposed spin-qubits in quantum dots [21]. As long as each dots carries one (valence) electron only with high barriers to the neighboring dots, the particles constituting the qubits can be seen as effectively distinguishable, and the usual concept and theory of quantum entanglement applies. However, two-qubit quantum gate operations in such systems are performed by temporarily lowering the tunneling barriers. In such a situation, the indistinguishable character and proper statistics of the electrons have to be taken into account. Therefore, the question arises how to describe “entanglement” (or, more precisely, quantum correlations analogous to entanglement between distinguishable parties) in systems of indistinguishable particles. In this article we have provided a simple introduction to this kind of questions and reported on some elementary results. Interesting
Figure 3: A square root of a swap, which is obtained from the situation of figure 2 by halfing the pulse duration $T$. The probability of double occupancies is again strongly suppressed after the tunneling pulse. The resulting state is a fully correlated complex linear combination of the incoming state $|\uparrow\downarrow\rangle$ and the outgoing state $|\downarrow\uparrow\rangle$ of the full swap. The quantum mechanical weights of the latter states are plotted as thick solid lines.

Questions for further research include experimental manifestations of entanglement-like quantum correlations between fermions using the full antisymmetrized (or, in the case of bosons, symmetrized) Fock space. In particular, possible generalizations of Bell inequalities to the case of indistinguishable particles might complement the approach outlined in this article and suggest experimental studies and applications.

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References

[1] A. Peres, "Quantum Theory: Concepts and Methods," (Kluver Academic Publishers, The Netherlands, 1995).
[2] A. Steane, Rep. Prog. Phys. 61, 117 (1998).
[3] A. Ekert, P. Hayden, and H. Inamori. quant-ph/0106067.
[4] R. F. Werner, quant-ph/0101063.
[5] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[6] E. Schrödinger, Naturwiss. 23, 807 (1935).
[7] For a recent review see M. Lewenstein, D. Bruß, J. I. Cirac, M. Kus, J. Samsonowicz, A. Sanpera, and R. Tarrach, J. Mod. Opt. 77, 2481 (2000).
[8] For a recent review see B. M. Terhal, quant-ph/0101032.
[9] A. Peres, Phys. Rev. Lett. 76, 1413 (1996).
[10] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1, (1996).
[11] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61 052306 (2000).
[12] A. Acín, A. Adrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000); A. Acín, E. Jane, W. Dür, and G. Vidal, Phys. Rev. Lett. 85, 4811 (2000); A. Acín, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001).
[13] H. A. Carteret, A. Higuchi, and A. Sudbery, J. Phys. A: Math. Gen. 41, 7932 (2000).
[14] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[15] W. K. Wootters, quant-ph/0001114.
[16] K. M. O’Connor and W. K. Wootters, Phys. Rev. A 63 052302 (2001).
[17] A.V. Thapliyal, Phys. Rev. A 59, 3336 (1998).
[18] J. S. Dell, Physics 1, 195 (1964).
[19] A. Aspect, J. Dalibard, and C. Roger, Phys. Rev. Lett. 49, 1804 (1982); A. Aspect, Europhys. News 22, 73 (1991).
[20] P. W. Shor, Proc. 35th Annual Symp. on Foundations of Computer Science, IEEE Computer Society Press; A. Ekert, R. Josza, Rev. Mod. Phys. 68, 733 (1996).
[21] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57, 120 (1998).
[22] G. Burkard, D. Loss, and D. P. DiVincenzo, Phys. Rev. B 59, 2070 (1999).
[23] G. Burkard, H.-A. Engel, and D. Loss, Fortschir. Phys. 48, 965 (2000).
[24] J. Schliemann, D. Loss, and A. H. MacDonald, Phys. Rev. B 63, 085311 (2001).
[25] X. Hu, R. de Sousa, and S. Das Sarma, cond-mat/0108339, to appear in the proceedings of the ISQM-Tokyo’01 conference.
[26] J. Schliemann, J. I. Cirac, M. Kus, M. Lewenstein, and D. Loss, Phys. Rev. A 64, 022303 (2001).
[27] K. Eckert, J. Schliemann, D. Bruß, M. Lewenstein, and D. Loss, in preparation.
[28] Y. S. Li, B. Zeng, X. S. Liu, and G. L. Long, quant-ph/0101410.
[29] R. Paskauskas and L. You, quant-ph/0106117, to appear in Phys. Rev. A.
[30] P. Zanardi, quant-ph/0104114.
[31] M. L. Mehta, “Elements of Matrix Theory”, Hindustan Publishing Corporation, Delhi (1977).
[32] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
[33] F. R. Waugh, M. J. Berry, D. J. Mar, R. M. Westervelt, K. L. Campman, and A. C. Gossard, Phys. Rev. Lett. 75, 705 (1995); C. Livermore, C. H. Crouch, R. M. Westervelt, K. L. Campman, and A. C. Gossard, Science 274, 1332 (1996).
[34] L.P. Kouwenhoven, G. Schön, and L. L. Sohn (Eds.), Mesoscopic Electron Transport, NATO ASI Series E, Vol. 345, (Kluwer Academic Publishers, Dordrecht, 1997).
[35] T. H. Oosterkamp, S. F. Godijn, M. J. Uilenreef, Y.V. Nazarov, N. C. van der Vaart, and L. P. Kouwenhoven, Phys. Rev. Lett. 80, 4951 (1998).
[36] R. H. Blick, D. Pfannkuche, R. J. Haug, K. v. Klitzing, and K. Eberl, Phys. Rev. Lett. 80, 4032 (1998); R. H. Blick, D. W. van der Weide, R. J. Haug, and K. Eberl, Phys. Rev. Lett. 81, 689 (1998); T. H. Oosterkamp, T. Fujisawa, W. G. van der Wiel, K. Ishibashi, R. V. Hijman, S. Tarucha, and L. P. Kouwenhoven, Nature 395, 873 (1998); I. J. Maasila and V. J. Goldman, Phys. Rev. Lett. 84, 1776 (2000).
[37] G. Burkard, D. Loss, and E.V. Sukhorukov, Phys. Rev. B 61, R16303 (2000).
[38] P. Recher, E.V. Sukhorukov, and D. Loss, Phys. Rev. B 63, 165314 (2001).
[39] D. Loss and E.V. Sukhorukov, Phys. Rev. Lett. 84, 1035 (2000).
[40] C. H. Bennett and D. P. DiVincenzo, Nature 404, 247 (2000).
[41] J.M. Kikkawa, I.P. Smorchkova, N. Samarth, and D.D. Awschalom, Science 277, 1284 (1997); J.M. Kikkawa and D.D. Awschalom, Phys. Rev. Lett. 80, 4313 (1998); D.D. Awschalom and J.M. Kikkawa, Physics Today 52(6), 33 (1999).
[42] T. Fujisawa, Y. Tokura, and Y. Hirayama, Phys. Rev. B 63, 081304 (2001).
[43] A.V. Khaetskii and Y.V. Nazarov, Phys. Rev. B 61, 12639 (2000).
[44] H.-A. Engel and D. Loss, Phys. Rev. Lett. 86, 4648 (2001); H.-A. Engel and D. Loss, cond-mat/0109470.
[45] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78; W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[46] R. Grobe, K. Rzazwski, and J. H. Eberly, J. Phys. B 27, L503 (1994); M. Y. Ivanov, D. Bitouk, K. Rzazwski, and S. Kotochigova, Phys. Rev. A 52, 149 (1995).
[47] D. P. DiVincenzo, Phys. Rev. A 51, 1015 (1995).