A NONLOCAL TRANSPORT EQUATION MODELING COMPLEX ROOTS OF POLYNOMIALS UNDER DIFFERENTIATION

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Abstract. Let $p_n : \mathbb{C} \to \mathbb{C}$ be a random complex polynomial whose roots are sampled i.i.d. from a radial distribution $u(r)rdr$ in the complex plane. A natural question is how the distribution of roots evolves under repeated (say $n/2$-times) differentiation of the polynomial. We derive a mean-field expansion for the evolution of $\psi(s) = u(s)s$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{x} \int_0^x \psi(s)ds \right)^{-1} \psi(x).$$

The evolution of $\psi(s) \equiv 1$ corresponds to the evolution of random Taylor polynomials

$$p_n(z) = \sum_{k=0}^{n} \gamma_k \frac{z^k}{k!} \text{ where } \gamma_n \sim \mathcal{N}_C(0,1).$$

We discuss some numerical examples suggesting that this particular solution may be stable. We prove that the solution is linearly stable. The linear stability analysis reduces to the classical Hardy integral inequality. Many open problems are discussed.

1. Introduction

1.1. Introduction. We ask a very simple question; a simple special case is as follows: suppose $p_n : \mathbb{C} \to \mathbb{C}$ is a random polynomial given by

$$p_n(z) = \prod_{k=1}^{n} (z - z_k) \text{ where the roots } z_k \sim \mathcal{N}_C(0,1)$$

are independently and identically distributed following a standard (complex) Gaussian. What can be said about the roots of the polynomial $p_n^{(n/2)}$? Are these roots still distributed like a Gaussian or do they follow another distribution? What if we replace the Gaussian by another distribution? To the best of our knowledge, there is only one example where rigorous results are available: define random Taylor polynomials via

$$p_n(z) = \sum_{k=0}^{n} \gamma_k \frac{z^k}{k!} \text{ where } \gamma_n \sim \mathcal{N}_C(0,1).$$

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Clearly, the $k$–th derivative of a random Taylor polynomial is a random Taylor polynomial of degree $n - k$. In particular, if their roots happen to have a nice limiting distribution, then this would give us an insight into the possible evolution. A result of Kabluchko & Zaporozhets [35] shows that the roots are asymptotically contained in a disk of size $\sim n$ and the renormalized roots converge in distribution

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{2n^{-1}} \to \frac{\chi_{|z| \leq 1}}{2\pi|z|} \quad \text{as } n \to \infty.$$ 

Rescaling shows that the roots of

$$p_n(z) = \sum_{k=0}^{n} \gamma_k \frac{z^k}{k!}$$

follow the same distribution (up to dilation) in the disk $\{z \in \mathbb{C} : |z| \leq n\}$. The purpose of this short paper is to pose the question and to derive a mean field approximation that might answer the question (and is intrinsically interesting).

![Figure 1. Roots of a Random Taylor polynomial of degree 1000 in the complex plane (left). Distances of the roots to the origin as a plot (right); it is approximately linear as predicted by the limiting measure $1/(2\pi|z|)$.](image)

1.2. Related results. The detailed study of the distribution of roots of $p'_n$ depending on $p_n$ is an active field [4] [5] [13] [17] [27] [30] [44] [40] [41] [45] [49] [50] [54] [57] [58] [59]. There are less results for the case of repeated differentiation. If $p_n$ is a polynomial of degree $n$ having $n$ distinct roots on the real line, then the $k$–th derivative has all of its $n - k$ roots also on the real line and one could wonder about their evolution. A result commonly attributed to Riesz [56] implies that the minimum gap between consecutive roots of $p'_n$ is bigger than that of $p_n$: zeroes even out and become more regular. We refer to results of Farmer & Rhoades [24], Farmer & Yerrington [25], Feng & Yao [26] and Pemantle & Subramnian [49]. Our result is inspired by a one-dimensional investigation due to the second author [55]:


if the roots of \( p_n \) are all real and follow a nice distribution \( u(0, x) \), what can be said about the distribution \( u(t, x) \) of the \((t \cdot n)\)–th derivative of \( p_n \) where \( 0 < t < 1 \)? In [55] it is proposed that the limiting dynamics exists and is given by the partial differential equation

\[
  u_t + \frac{1}{\pi} \left( \arctan \left( \frac{H u}{u} \right) \right)_x = 0
\]

where the equation is valid on the support \( \text{supp} u = \{ x : u(x) > 0 \} \) and \( H \) is the Hilbert transform. The equation has been shown to give the correct predictions for Hermite polynomials and Laguerre polynomials. One interesting aspect is that the equation has similarities to one-dimensional transport equations that are studied in fluid dynamics, see e.g. [9, 10, 11, 12, 14, 18, 20, 38, 39, 51]. Granero-Belinchon [29] studied an analogue of the equation on the one-dimensional torus.

2. Results

2.1. The Equation. Let us assume that \( p_n : \mathbb{C} \to \mathbb{C} \) is a polynomial of degree \( n \) whose roots are distributed according to a radial density function \( u(|z|)dz \). Then the density of roots at distance \( r \) from the origin is given by \( \psi(r) = u(r)rdr \). We will, throughout the paper, understand \( u \) as the probability density of the measure of roots at time \( t = 0 \). Assuming radial structure, understanding \( u \) and its evolution in time, is equivalent to understanding \( \psi(r) = u(r)rdr \) and its evolution in time (one can be understood as the representation of the other in polar coordinates). Again, we understand \( \psi(r) \) as the initial distribution and write \( \psi(t, r) \) as the distribution of roots after we differentiated \( t \cdot n \)-times.

We are now interested in studying the process of differentiation and its effect on the distribution of the roots. We derive a mean field equation that models the density of roots at distance \( x \) at time \( t \) via the nonlocal transport equation

\[
  \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{x} \int_0^x \psi(y)dy \right)^{-1} \psi(x).
\] (1)

We believe that this equation may be interesting in its own right and will give some supporting evidence of this belief. There is a scaling symmetry for \( \lambda > 0 \)

\[
  u(t, x) \to \lambda \cdot u(t, \lambda x).
\]

This scaling symmetry is a reflection of the chain rule: if we rescale the roots of the polynomial, then all the derivatives obey the same rescaling as well since

\[
  \frac{d}{dz^k} p_n(\lambda z) = \lambda^k \left( \frac{d}{dz^k} p_n \right)(\lambda z).
\]

Needless to say, the factor \( \lambda^k \) does not impact the presence or absence of roots and we recover a scale-invariance of the system. There is one more property that can be predicted from the behavior of polynomials: the \((t \cdot n)\)–th derivative of a polynomial \( p_n \) has \((1 - t)n\) roots. In particular, what one would then expect from any such equation is that there is a constant loss of mass that is independent of the function and independent of time. This loss would also imply that the function vanishes at time \( t = 1 \). This is indeed the case: if we assume the solutions are
continuous, there is a constant loss of mass on $(0, \infty)$:

$$\frac{\partial}{\partial t} \int_0^\infty \psi(t, x) dx = \int_0^\infty \frac{\partial}{\partial t} \psi(t, x) dx$$

$$= \int_0^\infty \frac{\partial}{\partial x} \left( \frac{1}{x} \int_0^x \psi(y) dy \right)^{-1} \psi(x) dx$$

$$= - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \psi(y) dy$$

$$= -1.$$

2.2. **Linear Stability.** We note that there is an explicit solution (when properly interpreted) given by

$$\psi(t, x) = \chi_{0 \leq x \leq 1-t}$$

which corresponds to the random Taylor polynomials and respects their evolution (as implied by the result of Kabluchko & Zaporozhets [35]).

![Figure 2](image_url)

**Figure 2.** Roots of a random polynomial $p_{1000}(z) = \sum_{k=1}^{1000} \gamma_k \cdot z^k (k!)^{-1/4}$ (left) and the roots of the polynomial arising from differentiating 250 times (right).

One natural question is now whether the dynamics that can be observed for random Taylor polynomials is in any way universal. We refer to Figure 2 for an example: after differentiating a random polynomial of a particular type 250 times, we observe a clustering of roots around the origin that is quite similar to that of random Taylor polynomials (or, in any case, seems to have a similar scaling).

Our main result deals with linear stability for perturbations around the solution for random Taylor polynomials. More precisely, we will consider a small perturbation around the constant function $\psi(t, x) = 1 + w(t, x)$, where we assume that
\[ \|w(t,x)\|_{L^\infty} \ll 1. \] The linearized evolution can be derived from
\[
\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left( \left( \frac{1}{x} \int_0^x \psi(y)dy \right)^{-1} \psi(x) \right)
= \frac{\partial}{\partial x} \left( \left( 1 + \frac{1}{x} \int_0^x w(y)dy \right)^{-1} (1 + w(x)) \right)
= \frac{\partial}{\partial x} \left( \left( 1 - \frac{1}{x} \int_0^x w(y)dy \right) (1 + w(x)) \right) + \text{l.o.t.}
= \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \frac{1}{x} \int_0^x w(y)dy + \text{l.o.t.}
\]
and is thus given by
\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \frac{1}{x} \int_0^x w(y)dy.
\tag{2}
\]
Our main result is that the constant solution
\[
\psi(t,x) = \chi_{0 \leq x \leq 1-t}
\]
(the dynamics of random Taylor polynomials) has linear stability for small perturbations close to the origin.

**Theorem.** If the perturbation \(w(0,x)\) is compactly supported and has mean value 0, then the linearized evolution (2) is an \(L^2\)-contraction, i.e.
\[
\frac{\partial}{\partial t} \int_0^\infty w(t,x)^2 dx \leq 0.
\]
One interesting aspect of this result is that it follows from a variation of a classical Hardy inequality: if \(f : (0,\infty) \to \mathbb{R}\) is measurable, then
\[
\int_0^\infty \frac{f(x)}{x^2} \left( \int_0^x f(y)dy \right) dx \leq \int_0^\infty \frac{f(x)^2}{x} dx,
\]
where the inequality is strict unless \(f \equiv 0\). (We interpret the inequality in the usual sense: if the right-hand side is finite, then so is the left-hand side and it is smaller.) Inequalities of this flavor have been studied intensively in their own right.

2.3. **Open Problems.** There are many open problems, we mention a few.

1. **Rigorous derivation.** Our derivation is not fully rigorous insofar as we assume the existence of an underlying limiting dynamics. This is also true for the one-dimensional limiting equation derived in [55]. A rigorous derivation requires a more detailed understanding of the local behavior in time. In the one-dimensional case, this is an old problem: do the roots of polynomials (all of whose roots are on the real line) become more regularly distributed under differentiation? In particular, do they obey a regular spacing at a local scale? We refer to [24, 25, 26, 49] and references therein.

2. **Qualitative behavior.** Certainly one of the most appealing aspects of having a partial differential equation describing the limiting behavior is the ability to understand and analyze the asymptotic behavior of polynomials with a large number of roots. What can be said about typical solutions? Is there a way to move between the partial differential equation and families of polynomials in a way where they
mutually inform each other?

3. Breakdown of symmetry. Something that we intrinsically assume in our derivation is that radial distributions stay radial under differentiation; however, slight deviations from radial symmetry stemming from the fact that we only have finitely many points might get amplified. Radial stability is a question that might already be interesting and accessible from a mean field perspective.

4. Special cases. Are there families of polynomials with radial roots whose roots exhibit particularly interesting dynamical behavior under differentiation? We explicitly mention the random Taylor polynomials: are there others?

5. Mixed Real/Complex Behavior. One question one could ask is to understand the behavior of the real roots in the presence of complex roots (which, naturally, also evolve). Is the corresponding equation simpler?

6. Attractors. Do generic distributions converge to the behavior of random Taylor polynomials in some weak sense (say, in a neighborhood of the origin)? Can the stability analysis of Random Taylor polynomials be further refined?

We emphasize the question 7. the Non-Radial Case. The ‘Lagrangian perspective’ (following an individual fluid particle, here that would be a single root) is easy enough to describe: if we are given a density of roots $\mu$, then locally at a point $z$, the roots can be thought of as moving in a direction given by the

$$\text{Cauchy-Stieltjes transform} \quad -\left( \int_{\mathbb{C}} \frac{d\mu(y)}{z-y} \right)^{-1}.$$

Our equation can be understood as a simplification of the Cauchy-Stieltjes transform in the radial setting. Can this intuition be made precise? What can be said about the dynamics of this more general equation? We conclude with an interesting question that becomes only relevant in the non-radial setting.

8. Which roots vanishes? Suppose we are given a polynomial $p_n : \mathbb{C} \rightarrow \mathbb{C}$ of degree $n$. It has $n$ roots, its derivative $p'_n$ has $n-1$ roots. We like to think of the roots of the derivative $p'_n$ has the roots of $p_n$ being moved a little bit in a direction prescribed by the Cauchy-Stieltjes transform. This line of reasoning has been pursued and made rigorous by Williams and the first author [46 47] who showed various types of pairing results between the roots of $p_n$ and $p'_n$. However, one of the roots ‘vanishes’ (or, put differently, will be unpaired). It has been empirically observed that the root that vanishes seems to be close to a root of the Cauchy-Stieltjes transform. The converse is fairly easy to establish: if the Cauchy-Stieltjes transform is large in a point where we have an isolated root, then the derivative has a root nearby (at scale roughly $\sim n^{-1}$). In our radial setting, the Cauchy-Stieltjes transform vanishes at the origin which is consistent with numerics. However, a rigorous argument in that direction is missing.

3. Derivation of the Equation

In this section we will derive the mean-field limit equation

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} \left( \left( \frac{1}{x} \int_{0}^{x} \psi(y) dy \right)^{-1} \psi(x) \right)$$
under the assumptions that the limiting evolution exists and is continuous. Another assumption is that radial measures remain radial under the evolution (this is clearly the case in the limit but not necessarily clear for large values of $n$, we refer to §2.3). The key ingredient is the identity

$$\frac{p_n'(z)}{p_n(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k},$$

where $z_1, \ldots, z_n$ are the roots of $p_n(z)$. We will now fix the root $z_\ell$ and are interested whether there is a root of $p_n'$ nearby and whether that nearby root can be written in terms of $z_\ell$. In case of a nice limiting distribution, we can assume that the $n$ roots are spread out over area $\sim 1$, this means that the distance from a root and its nearby neighbors is $\sim n^{-1/2}$. Rotational symmetry allows us to assume $z_\ell \in \mathbb{R}$. Roots of $p_n'$ nearby satisfy the equation

$$z_\ell - z = \left( \sum_{\substack{k=1 \ \text{k}\neq\ell}}^{n} \frac{1}{z - z_k} \right)^{-1}.$$

It remains to estimate the size of the sum under the assumption that the roots are distributed according to a density $\psi(t,x)$. To this end, we consider the following integral for two parameters $r, s > 0$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r - se^{it}} dt = \begin{cases} 0 & \text{if } r < s \\ 1/r & \text{if } r > s. \end{cases}$$

If the number of roots at distance $x$ from the origin is given by $\psi(t,x)$, then this integral (see Fig. 2) suggests that

$$z_\ell - z \sim \left( \sum_{\substack{k=1 \ \text{k}\neq\ell}}^{n} \frac{1}{z_\ell - z_k} \right)^{-1} \sim n^{-1} \left( \int_0^{x} \frac{\psi(t,x)}{z_\ell} dx \right)^{-1}.$$

This suggests that the root moves to the left by a factor only determined by this integral over roots with smaller norm.

Figure 3. Only the inner circles contribute to the integral.
Phrased differently, roots at distance $x$ from the origin (whose total density is given by $\psi(t, x)$) move to slightly smaller distance with a speed determined by

$$\text{the nonlocal vectorfield } \sim -\left(\frac{1}{x} \int_0^x \psi(t, y) dy\right)^{-1}$$

and this results in the desired equation.

### 4. Proof of Linear Stability

We conclude the paper by establishing linear stability. We start with a dual version of the generalized Hardy inequality. Given the substantial literature, it is presumably stated somewhere in the literature, however, it is also rather easy to deduce from the generalized form of the Hardy inequality.

**Lemma (A Hardy-type inequality).** Let $f : (0, \infty) \to \mathbb{R}$ be measurable. Then

$$\int_0^\infty \frac{f(x)}{x^2} \left( \int_0^x f(y) dy \right) dx \leq \int_0^\infty \frac{f(x)^2}{x} dx,$$  \hspace{1cm} (3)

where the inequality is strict unless $f \equiv 0$.

We understand the inequality in the usual sense (i.e. if the right-hand side is finite, then so is the left-hand side which is then also bounded by the right-hand side).

**Proof.** We start with an application of Cauchy-Schwarz

$$\int_0^\infty \frac{f(x)}{x^2} \left( \int_0^x f(y) dy \right) dx = \int_0^\infty \frac{f(x)}{\sqrt{x}} \frac{1}{x^{3/2}} \left( \int_0^x f(y) dy \right) dx \leq \left( \int_0^\infty \frac{f(x)^2}{x} dx \right)^{1/2} \cdot \left( \int_0^\infty \frac{1}{x} \left( \frac{1}{x} \int_0^x f(y) dy \right)^2 dx \right)^{1/2}.$$

It thus suffices to prove the inequality

$$\int_0^\infty \frac{1}{x} \left( \frac{1}{x} \int_0^x f(y) dy \right)^2 dx \leq \int_0^\infty \frac{f(x)^2}{x} dx.$$

This inequality, however, is known as the generalized Hardy inequality (see [2]). Indeed, a more general result is given by Theorem 330 in the book of Hardy-Littlewood-Polya [33] implying for $p > 1$ and $r > 1$ that

$$\int_0^\infty x^{-r} \left( \int_0^x f(y) dy \right)^p dx \leq \left( \frac{p}{r - 1} \right)^p \int_0^\infty x^{-r}(xf(x))^p dx.$$

Our case is merely $(p, r) = (2, 3)$. \hfill \Box

**Proof of the Stability Statement.** The linearized equation is given by

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \frac{1}{x} \int_0^x w(y) dy.$$
We try to understand the evolution of
\[
\frac{\partial}{\partial t} \frac{1}{2} \int_0^\infty w(t,x)^2 \, dx = \int_0^\infty w(t,x) \frac{\partial w(t,x)}{\partial x} \, dx - w(t,x) \frac{\partial}{\partial x} \frac{1}{2} \int_0^x w(t,y) \, dy \, dx
\]
\[
= \frac{1}{2} \int_0^\infty \left( \frac{\partial}{\partial x} w(t,x)^2 \right) \, dx - \int_0^\infty w(t,x) \left( \frac{\partial}{\partial x} \frac{1}{2} \int_0^x w(t,y) \, dy \right) \, dx.
\]
However, since \(w\) is compactly supported, the first term reduces to
\[
\frac{1}{2} \int_0^\infty \left( \frac{\partial}{\partial x} w(t,x)^2 \right) \, dx = -\frac{w(t,0)^2}{2} \leq 0.
\]
We will disregard this quantity and focus on the second term. We differentiate in \(x\) and obtain
\[
\frac{\partial}{\partial x} \frac{1}{2} \int_0^x w(t,y) \, dy = -\frac{1}{x^2} \int_0^x w(t,y) \, dy + \frac{w(t,x)}{x}.
\]
It thus remains to understand the sign of the integral which, due to the Hardy inequality proven above, is given by
\[
\int_0^\infty \left( \frac{w(t,x)^2}{x} - \frac{w(t,x)}{x^2} \int_0^x w(t,y) \, dy \right) \, dx \geq 0.
\]

\[\square\]

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