PULLBACK ATTRACTORS FOR BI-SPATIAL CONTINUOUS RANDOM DYNAMICAL SYSTEMS AND APPLICATION TO STOCHASTIC FRACTIONAL POWER DISSIPATIVE EQUATION ON AN UNBOUNDED DOMAIN

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Abstract. In this article, a notion of bi-spatial continuous random dynamical system is introduced between two completely separable metric spaces. It is show that roughly speaking, if such a random dynamical system is asymptotically compact and random absorbing in the initial space, then it admits a bi-spatial pullback attractor which is measurable in two spaces. The measurability of pullback attractor in the regular spaces is completely solved theoretically. As applications, we study the dynamical behaviour of solutions of the non-autonomous stochastic fractional power dissipative equation on \( \mathbb{R}^N \) with additive white noise and a polynomial-like growth nonlinearity of order \( p \), \( p \geq 2 \). We prove that this equation generates a bi-spatial \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))\)-continuous random dynamical system, and the random dynamics for this system is captured by a bi-spatial pullback attractor which is compact and attracting in \( H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), where \( H^s(\mathbb{R}^N) \) is a fractional Sobolev space with \( s \in (0, 1) \). Especially, the measurability of pullback attractor is individually derived by proving the the continuity of solutions in \( H^s(\mathbb{R}^N) \) and \( L^p(\mathbb{R}^N) \) with respect to the sample. A difference estimates approach, rather than the usual truncation estimate and spectral decomposition technique, is employed to overcome the loss of Sobolev compact embedding in \( H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), \( s \in (0, 1) \), \( N \geq 1 \).

1. Introduction. This article is concerned with the random dynamics of solutions of the non-autonomous stochastic dissipative equation involving the fractional power Laplacian on the whole space \( \mathbb{R}^N \)

\[
\frac{du}{dt} + \lambda u + (-\Delta)^s u + f(t, x, u) = g(t, x) + \phi(x)\dot{W}(t), \quad t > \tau,
\]

where \( s \in (0, 1), \lambda > 0, \tau \in \mathbb{R}, g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^N)), W(t) \) is a two-sided real-valued Brownian motion defined on a probability space \((\Omega, \mathcal{F}, P)\) specified later, and \( \phi \) is a given function on \( \mathbb{R}^N \). The equation is understood in the sense of Stratonovich integration.

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The operator \((-\Delta)^s\) with \(s \in (0, 1)\) is called a fractional power Laplacian, whose limit as \(s \uparrow 1\) is the classic Laplacian \(\Delta\), see [17]. It is a nonlocal generalization of the classic Laplacian that is often used to model the diffusive processes, see [17, 27, 40] and the references therein. As for the long-time dynamics, Hu et al [32] discussed the existence of a random attractor in \(L^2(\mathbb{R}^N)\) for the fractional power dissipative stochastic equation with additive noises. Wang [46] discussed the existence and uniqueness of random attractor for the same equation with multiplicative noise on bounded domain. The dynamics of 3D Ginzburg-Landau equation involving fractional power Laplacian were studied in [33, 34]. Most recently, Gu et al [21] obtained the regularity of the random attractor for problem (1) in \(H^s(\mathbb{R}^N)\) in the case of multiplicative noise, where the authors employed the tail estimate and spectral decomposition technique to overcome the loss of compactness on unbounded domains. But little is known about the continuity and measurability of its solutions in \(H^s(\mathbb{R}^N)\) and \(L^p(\mathbb{R}^N)\) for \(s \in (0, 1)\) and \(p \geq 2\). In this paper, we prove the existence, regularity and measurability of pullback attractors for the stochastic fractional power dissipative equation (1) defined on the whole space \(\mathbb{R}^N\).

Most of the stochastic partial differential equations (SPDEs) present a regular solution, namely, if the initial data belongs to a separable Banach space, called an initial space, the corresponding solution may enter into another space, called a target space or regular space in the terminology as [31]. Thus the random dynamical system always possesses a bi-spatial characteristic property. In general, the continuity and measurability of a solution of an SPDE in the initial space are easy to check, and therefore to derive a (measurable) random attractor in such a space we only need to look for the compact condition and the absorption property, which has been extensively studied ever since [14, 15, 37] first introduced the concept of random attractor. For example, we may refer to [8, 9, 10, 19] for the autonomous case and [6, 43, 44, 45, 48] for the non-autonomous case. However, it is challenging to prove the continuity and measurability of solution of an SPDE in the corresponding regular spaces, see the statements in the introductions of the literature [7, 8, 31]. Therefore, the measurability of a random attractor in the regular space is not explicitly studied in the literature, although the existence of bi-spatial random attractor has been obtained for a large volume of stochastic partial equations (cf. [29, 30, 31, 41, 49, 50, 53, 54, 57]).

Firstly, we introduce a new notion of a bi-spatial continuous random dynamical system (briefly, bi-spatial continuous RDS) between two completely separable metric spaces \(X\) and \(Y\), and establish the existence criterion of its bi-spatial (random) pullback attractor. By definition, a bi-spatial continuous RDS is continuous from \(X\) to \(X\), and in addition is continuous from \(X\) to \(Y\) with respect to the initial data. It is showed that roughly speaking, if such a random dynamical system is asymptotically compact and absorbing in the initial space \(X\), then it admits automatically a bi-spatial “true random” pullback attractor. The expression “true random” means that this bi-spatial pullback attractor (as a set-valued mapping) is not only compact and attracting but also measurable in these spaces, which is different from the parallel concept of bi-spatial random attractor developed in [31, 56], where the measurability of attractors in the regular space \(Y\) is unknown. In a sense, the automatical presentation of a bi-spatial “true random” pullback attractor for such a RDS is attractive and interesting. In other word, the measurability of pullback attractor in the regular space is completely solved without requesting any embedding relations between \(X\) and \(Y\).
Secondly, we derive the continuity of solutions to problem (1) in the regular spaces $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$, respectively, when the initial datum is given in $L^2(\mathbb{R}^N)$, where $H^s(\mathbb{R}^N)$ is a fractional Sobolev space with $s \in (0, 1)$. By means of the equation satisfied by the difference of two solutions at the common initial time, and using some technical iteration, we obtain that the difference $(t - \tau)\bar{u}(t)$ in $L^p(\mathbb{R}^N)$ is not larger than the difference of the initial data which belongs to $L^2(\mathbb{R}^N)$, for any $t > \tau$ and the non-autonomous force $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$. On the other hand, we overcome the difficulty caused by the fractional power Laplacian to obtain a semi-intervals estimate of the bound of the solution in $L^{2p-2}_{loc}(\mathbb{R}, L^{2p-2}\mathbb{R}^N)$, and then an appropriate multiplier $(r - \frac{L + \tau}{2})^{\frac{2p}{p-2}}$ is chosen to prove the continuity of solutions in $H^s(\mathbb{R}^N)$. This approach was used to prove the asymptotical compactness of solutions of reaction-diffusion equation, p-Laplacian equation and Fitzhugh-Nagumo system on bounded or unbounded domain, see [7, 53, 54, 55, 59].

Thirdly, we consider the measurability of solutions. It is well known that the existence theorem of pullback attractor for the deterministic dynamical system can be applied to analyse the long-time dynamics of random dynamical system with some modifications. However, in the case of random (stochastic) differential equations, we need additional measurability properties to assure that the solutions are well-posed [8]. This leads to an additional and great difference in comparison with the non-random case. We here prove the measurability of solution as a single valued mapping from $\Omega$ to $L^p(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. This is achieved by proving the continuity of solution with respect to the sample $\omega \in \Omega_k$, where $\Omega_k, k \in \mathbb{N}$ is a countable decomposition of the sample space $\Omega$. Note that $\Omega_k$ is a Polish space, possessing some nice analytical properties to serve us. In particular, the properties of the Ornstein-Uhlenbeck process over $\Omega_k$ are substantially employed. The reader is referred to [8, 9, 10, 47] for the idea to restrict $\Omega$ to $\Omega_k$.

Finally, as direct results of our theoretic criterion of section 2, we derive the existence of a measurable pullback attractor (as a set-valued mapping) in $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N), p \geq 2$ and $s \in (0, 1)$, see Theorem 7.1 and Theorem 7.2. The asymptotical compactness of solution in the initial space $L^2(\mathbb{R}^N)$ is proved by estimate of the far-field values of solutions and some locally compact embedding on bounded domain.

In comparison with the results recently published in the literature, the novelty and difficulties of this work are in three aspects: (i) We study the dynamics by proving of the bi-spatial continuity of solution with respect to the initial data to obtain the asymptotical compactness, which is substantially different from [21, 57] by tails estimate and spectral approaches. (ii) We give a detailed analysis of the measurability of solution and the measurability of pullback attractor in the regular spaces without embedding relation for the considered spaces, e.g. $L^p(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ for $p > 2$, with a special interest in the fractional Sobolev space $H^s(\mathbb{R}^N), s \in (0, 1)$. (iii) In the case of additive noise, the structure of the equation is varied and some new term emerges. With these problems, some new estimates involving the fractional power Laplacian and the nonlinear function are developed here. It is also pointed out that the approaches utilized in this paper are applicable for a variety of stochastic differential equations, such as stochastic fractional Burgers equation, stochastic fractional Ginzburg-Landau equation and stochastic fractional $p$-Laplacian equation and etc.
This article is arranged as follows. In section 2, we introduce the concepts of bi-spatial continuous RDS and bi-spatial pullback attractor. A criterion on the existence of bi-spatial pullback attractor for the bi-spatial continuous RDS is demonstrated. In section 3, we first introduce the fractional power Laplacian and its related fractional power Sobolev space. Then we transform the stochastic equation into a deterministic one by the well-known O-U transformation. A random cocycle through the solution of problem (1) is defined, especially the domain of attraction is given. In section 4, we prove the existence of random absorbing set and the asymptotical compactness in $L^2(\mathbb{R}^N)$. In section 5, we prove the the bi-spatial continuity of solutions in the $p$-times integrable functions spaces $L^p(\mathbb{R}^N)$ and the fractional Sobolev spaces $H^s(\mathbb{R}^N)$ with respect to the initial data belonging to $L^2(\mathbb{R}^N)$, for any $s \in (0, 1)$ and $p \geq 2$. In section 6, we prove the continuity and measurability of solutions in the spaces $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$ with respect to the sample $\omega$. The final section is concerning the existence of bi-spatial (measurable) pullback attractor in the spaces $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$.

2. Abstract results on the bi-spatial continuous RDS. In this section, we present some concepts related to the bi-spatial RDS, and prove a theoretic results on the existence of a bi-spatial pullback attractor for a bi-spatial continuous RDS. For a comprehensive survey of the theory for the RDS, the readers may refer to [1, 13] and the deterministic dynamical system to [11, 36, 42].

2.1. Preliminaries. We introduce two Polish spaces (completely separable metric space) $(X, d_X)$ and $(Y, d_Y)$ with the $\sigma$-algebras of Borel subsets $B(X)$ and $B(Y)$, respectively, where $X$ serves as the initial space, and $Y$ as the target space or regular space, whose intersection is nontrivial. We assume that each of them is continuously imbedded in a Hausdorff topological vector space $X$. This implies that for every sequence $x_n \in X \cap Y, n = 1, 2, ..., $ such that $x_n \to x$ in $X$ and $x_n \to y$ in $Y$ respectively, then we have $x = y \in X \cap Y$. For example, the usual Lebesgue spaces $L^p(\mathbb{R}^N)(p > 2)$ and $L^\infty(\mathbb{R}^N)$ are continuously embedded into the separately topological space $\mathcal{C}_c(\mathbb{R}^N)$ (the space of all continuous linear functionals on the space $C_0^\infty(\mathbb{R}^N)$ equipped with a particular topology as in [51, p28]). Thus it makes sense to consider the common topology in these targeted spaces, see [56].

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which there is a $((\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping $\vartheta_t : \mathbb{R} \times \Omega \to \Omega$ such that $\vartheta_0 = I, \vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ and $\vartheta_t P = P$ for all $t, \tau \in \mathbb{R}$. In particular, the mapping $\vartheta_t, t \in \mathbb{R}$ is called a driving system on $\Omega$, whereas the sample space of the probability space $(\Omega, \mathcal{F}, P)$ is called a base space of the driving system $\vartheta_t$, see [16]. The quadruple $(\Omega, \mathcal{F}, P, \{\vartheta_t : t \in \mathbb{R}\})$ is called a metric (or measurable) dynamical system (briefly, MDS $\vartheta$), which is of importance to deal with the white noises.

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ and $2^X$ be the collection of all subsets of $X$.

**Definition 2.1.** A family of single-valued mappings $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X, (t, \tau, \omega, x) \mapsto \varphi(t, \tau, \omega, x)$ is called a random cocycle on $X$ over an MDS $\vartheta$ if for all $s, t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the following statements are satisfied:

- $\varphi(., \tau, ., .) : \mathbb{R}^+ \times X \times \Omega \to \mathbb{R}^+ \times X \times \Omega \to X$ is $(\mathcal{B}(\mathbb{R}^+)) \times \mathcal{F} \times \mathcal{B}(X) \times \mathcal{B}(X))$-measurable;
- $\varphi(0, \tau, \omega, .)$ is the identity on $X$;
- $\varphi(t+s, \tau, \omega, .) = \varphi(t, \tau, \omega, \varphi(s, \tau, \omega, .))$.

A random cocycle $\varphi$ on $X$ is said to take its values into the target space $Y$ if $\varphi(t, \tau, \omega, .)$ maps $X$ into $Y$ for every $t > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$. 
Definition 2.2. Let $\varphi$ be a random cocycle on $X$ over an MDS $\vartheta$, taking its values into $Y$. A random cocycle $\varphi$ is said to be $(X,X)$-continuous (briefly, continuous in $X$) if the mapping $\varphi(t,\tau,\omega,.) : X \to X$ is continuous for each $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. A random cocycle $\varphi$ is said to be $(X,Y)$-continuous, and in addition the mapping $\varphi(t,\tau,\omega,.) : X \to Y$ is continuous for each $t > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

The universe of sets in the initial space $X$ is denoted by $\mathcal{D}$, which is a collection of some families of nonempty subsets of the initial space $X$, parameterized by the initial time $\tau$ and the sample $\omega$ such that $\mathcal{D} = \{D = \{\emptyset \neq D(\tau,\omega) \in 2^X : \tau \in \mathbb{R}, \omega \in \Omega\}\}$. The universe $\mathcal{D}$ will serve as the domain of attraction (in the terminology in [38]). In practice, the element of $\mathcal{D}$ has particular growth property $t \mapsto D(\tau - t, \vartheta_{-t}\omega)$ for each fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

The universe $\mathcal{D}$ is called inclusion closed if for every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ with every $D'(\tau,\omega) \subset D(\tau,\omega)$ for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then the family $D' = \{D'(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Note that the inclusion closed universe $\mathcal{D}$ is also called for short an IC system which was first introduced in [2, 19, 38]. In this paper, it suffices to define the universe $\mathcal{D}$ in the initial space $X$, not necessary in the target space $Y$.

Definition 2.3. Let $D : \mathbb{R} \times \Omega \to 2^X \backslash \emptyset ; D : (\tau,\omega) \mapsto D(\tau,\omega) \in 2^Y$ be a set-valued mapping. We say $D : (\tau,\omega) \mapsto D(\tau,\omega)$ is measurable with respect to $\mathcal{F}$ (briefly, measurable) in $X$ if for every fixed $x \in X$ and $\tau \in \mathbb{R}$, the mapping

$$\omega \mapsto \inf_{z \in D(\tau,\omega)} d_X(x,z)$$

is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable. If the set-valued mapping $D$ is measurable, then the family of its images $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is also called a random set. If for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the image $D(\tau,\omega)$ is closed (resp. compact) in $X$, then the family $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a closed (resp. compact) random set in $X$.

It is well known that a set-valued mapping $D$ is measurable $X$ if and only if for every open subset $O \subset X$, the inverse image $\{\omega : D(\tau,\omega) \cap O \neq \emptyset\} \in \mathcal{F}$ for every fixed $\tau \in \mathbb{R}$, see [3, 12, 22]. Recall that a single-valued mapping $f : \Omega \to X$ is called measurable (precisely, $(\mathcal{F},\mathcal{B}(X))$-measurable) if for every open subset $O \subset X$, the inverse image $f^{-1}(O) \in \mathcal{F}$ (or equivalently for every closed set $C \subset X$, $f^{-1}(C) \in \mathcal{F}$) (cf [3, p307]).

Definition 2.4. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of the initial space $X$. $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set in $X$ if for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists an absorbing time $T = T(\tau,\omega,D) > 0$ such that

$$\varphi(t,\tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)) \subseteq K(\tau,\omega) \quad \text{for all} \quad t \geq T.$$  

Furthermore, if $K$ is a random set, the family $K$ is called a $\mathcal{D}$-pullback random absorbing set in $X$.

Definition 2.5. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of the initial space $X$. Let $\varphi$ be a random cocycle on $X$ over an MDS $\vartheta$.

(i) A family of sets $\mathcal{A} = \{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $(X,X)$-pullback attractor for $\varphi$ if the next three statements hold:

- $\mathcal{A}$ is a random set in $X$ and $A(\tau,\omega)$ is compact in $X$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
• $\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\varphi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \vartheta(\omega), \forall t \geq 0$.

• $\mathcal{A}$ is attracting in $X$, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathcal{D}$,

$$\lim_{t \to \infty} \text{dist}_X(\varphi(t, \tau - t, \vartheta_{-t, \omega}, D(\tau - t, \vartheta_{-t, \omega})), \mathcal{A}(\tau, \omega)) = 0.$$

(ii) Suppose further that $\varphi$ takes its values into $Y$. Then a family of sets $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $(X,Y)$-pullback attractor for $\varphi$ if $\mathcal{A}$ is a $(X,X)$-pullback attractor, and in addition they hold

• $\mathcal{A}$ is a random set in $Y$, $\mathcal{A}(\tau, \omega)$ is compact in $Y$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$;

• $\mathcal{A}$ is attracting in $Y$, namely, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and $D \in \mathcal{D}$,

$$\lim_{t \to \infty} \text{dist}_Y(\varphi(t, \tau - t, \vartheta_{-t, \omega}, D(\tau - t, \vartheta_{-t, \omega})), \mathcal{A}(\tau, \omega)) = 0,$$

where $\text{dist}_Y$ is the Hausdorff semi-metric in $2^Y$ with $\text{dist}_Y(A, B) = \sup_{x \in A} \inf_{y \in B} d_Y(x, y)$.

The concept of attractors in different spaces was first introduced by Babin and Bishik [4, p119]. However, here we emphasize that the bi-spatial $(X,Y)$-attractor is a pullback attractor in both $X$ and $Y$, which is stronger than the one in [4]. Moreover in the random setting, just as the above definition, the measurability in the regular space $Y$ is included, which is also largely different from the bi-spatial notion in [31, 49, 50, 52].

We also note that the attracting in Lemma 2.5 is called pullback attraction, which implies the forward attraction:

$$\lim_{t \to \infty} \text{dist}_Y(\varphi(t, \tau, \omega, D(\tau, \omega)), \mathcal{A}(\tau + t, \vartheta_{t, \omega})) = 0 \text{ in probability.} \quad (2)$$

As for the attractor with forward attraction of point-wise convergence is considered by Kloeden [16, 23, 24, 25]. However, the considered convergence in this paper is in the sense of pullback attraction, which is stronger than the forward attraction in probability, defined by (2).

**Definition 2.6.** Let $\mathcal{D}$ be a collection of some families of nonempty subsets of the initial space $X$. Let $\varphi$ be a random cocycle $\varphi$ on $X$ over an MDS $\vartheta$.

(i) The random cocycle $\varphi$ is said to be $(X,X)$-pullback asymptotically compact if for every $\tau \in \mathbb{R}, \omega \in \Omega, D \in \mathcal{D}$ and whenever $t_n \to \infty$, $x_n \in D(\tau - t_n, \vartheta_{-t_n, \omega})$, the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n, \omega}, x_n)\}_{n=1}^{\infty} \text{ is precompact in } X.$$

(ii) Suppose further that $\varphi$ takes its values into $Y$. Then $\varphi$ is called $(X,Y)$-pullback asymptotically compact if $\varphi$ is $(X,X)$-pullback asymptotically compact and in addition the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n, \omega}, x_n)\}_{n=1}^{\infty} \text{ is precompact in } Y,$$

for every $\tau \in \mathbb{R}, \omega \in \Omega, D \in \mathcal{D}$ and whenever $t_n \to \infty$ and $x_n \in D(\tau - t_n, \vartheta_{-t_n, \omega})$.

2.2. **Pullback attractors for the random dynamical systems with the bi-spatial continuity.** The existence theorem of bi-spatial random attractor seems to be established in [31] for the autonomous case and [52] for the non-autonomous case. However, the measurability of a bi-spatial random attractor is assumed to hold on the initial space $X$, not considering the measurability in the regular space $Y$. In fact, the measurability of a bi-spatial random attractor can be proved in the regular space $Y$ if we impose some measurability of the cocycle on $\omega$ and the continuity of cocycle in $Y$ with respect to the initial values belonging to $X$. We will
prove this in this subsection. We first give an interesting lemma which claims that the bi-spatial continuity implies the bi-spatial asymptotical compactness.

**Lemma 2.7.** Let \( \varphi \) be a random cocycle on \( X \) over an MDS \( \theta \), taking its values into \( Y \). Suppose that \( \varphi \) is \( (X, X) \)-pullback asymptotically compact, and in addition \( \varphi \) is \( (X, Y) \)-continuous. Then \( \varphi \) is \( (X, Y) \)-pullback asymptotically compact.

**Proof.** Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let \( \{ \varphi(t_n, \tau - t_n, \vartheta_{-t_n}, \omega, u_n) \}_{n \in \mathbb{N}} \) be a sequence and \( D \in \mathcal{D} \). Since \( \varphi \) is \( (X, X) \)-pullback asymptotically compact, then there exists a point \( u_0 \in X \) such that, up to a subsequence,

\[
\varphi(t_n - 1, \tau - t_n, \vartheta_{-t_n}, \omega, u_n) \rightarrow u_0 \quad \text{in} \ X,
\]

whenever \( t_n \rightarrow \infty \) and \( u_n \in D(\tau - t_n, \vartheta_{-t_n}, \omega) \). By the \( (X, Y) \)-continuity of \( \varphi \), we know that \( \varphi(t, \tau, \omega, u) : X \rightarrow Y \) is continuous for every \( t > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

Therefore by (3), we deduce that

\[
\varphi(1, \tau - 1, \vartheta_{-1}, \omega, \varphi(t_n - 1, \tau - t_n, \vartheta_{-t_n}, \omega, u_n)) \rightarrow \varphi(1, \tau - 1, \vartheta_{-\tau}, u_0) \quad \text{in} \ Y.
\]

Then the final proof is as a result of the cocycle property of \( \varphi \). \( \square \)

The following lemma is concerning the properties for a random set, which will be needed in the following arguments. See [8, 12, 22].

**Lemma 2.8.** (i) For any random set \( E = \{ E(\omega), \omega \in \Omega \} \subset Y \) with nonempty closed images, there exist a sequence \( \{ f_n(\omega) \}_{n=1}^{\infty} \) of \( (\mathcal{F}, \mathcal{B}(Y)) \)-measurable variables such that \( f_n(\omega) \in E(\omega) \) for every \( \omega \in \Omega \), and moreover,

\[
E(\omega) = \bigcup_{n \in \mathbb{N}} f_n(\omega)^Y.
\]

(ii) Let \( E_n : \Omega \rightarrow 2^Y \) be a random set with nonempty closed images with \( n \in \mathbb{N} \). Then

\[
\Omega \ni \omega \rightarrow \bigcup_{n \in \mathbb{N}} E_n(\omega)^Y
\]

is measurable in \( Y \) with nonempty closed images.

(iii) Let \( E_n : \Omega \rightarrow 2^Y \) be a random set with nonempty closed image with \( n \in \mathbb{N} \). Suppose that for each fixed \( \omega \in \Omega \), every sequence \( \{ y_n \}_{n=1}^{\infty} \) with \( y_n \in E_n(\omega) \) has a convergence subsequence in \( Y \), and in addition \( \{ E_n \}_{n=1}^{\infty} \) is a decreasing sequence. Then

\[
\Omega \ni \omega \rightarrow \cap_{n=1}^{\infty} E_n(\omega)
\]

is measurable in \( Y \) with nonempty closed images.

The following lemma is adapted from Lemma 8.2.3 in [3].

**Lemma 2.9.** ([3]) Let the mapping \( \Psi : \Omega \times X \rightarrow Y \). Suppose that for every \( x \), \( \Psi(\cdot, x) \) is \( (\mathcal{F}, \mathcal{B}(Y)) \)-measurable and for every \( \omega \in \Omega \), \( \Psi(\omega, \cdot) \) is continuous. If \( \chi : \Omega \rightarrow X \) is \( (\mathcal{F}, \mathcal{B}(X)) \)-measurable, then the mapping \( \omega \rightarrow \Phi(\omega, \chi(\omega)) \) is \( (\mathcal{F}, \mathcal{B}(Y)) \)-measurable.

We are now at the point to present a new result on the existence of a bi-spatial pullback attractor for a random dynamical system with the bi-spatial continuity. Note that we do not impose any inclusion relations between \( X \) and \( Y \).

**Theorem 2.10.** Let \( \varphi \) be a random cocycle on \( X \) (over an MDS \( \theta \)) which is \( (X, Y) \)-continuous, and \( \mathcal{D} \) be inclusion closed universe in \( X \). Suppose that

(i) \( \varphi \) has a closed \( \mathcal{D} \)-pullback random absorbing set \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) in \( X \);
\( (ii) \) \( \varphi \) is \( (X,X) \)-pullback asymptotically compact in \( X \);
\( (iii) \) For every fixed \( t > 0, \tau \in \mathbb{R}, x \in X \), the mapping \( \varphi(t,\tau,.,x) : \Omega \to Y \) is \( (\mathcal{F},\mathcal{B}(Y)) \)-measurable.

Then the random cocycle \( \varphi \) possesses a unique \( (X,Y) \)-pullback attractor \( A = \{ A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}, \) where
\[
A(\tau,\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))^X.
\]

Furthermore, it can also be structured by the \( Y \)-metric, namely,
\[
A(\tau,\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))^Y.
\]

Proof. By (i) and (ii), it follows that \( \varphi \) admits a \( (X,X) \)-pullback attractor \( A = \{ A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) in \( X \), where
\[
A(\tau,\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))^X.
\]

For fixed \( \tau \in \mathbb{R} \), the measurability of \( A(\tau,.) \) in \( X \) is proved by Wang [44]. We here mainly consider the measurability, compactness and attracting properties of \( A \) in the target space \( Y \). By Lemma 2.7, \( \varphi \) is \( (X,Y) \)-asymptotic compact. Therefore by [52], for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the omega-limits \( A(\tau,\omega) \) of \( K \),
\[
A(\tau,\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))^X \subset Y, \tag{4}
\]
is an invariant subset which is compact, attracting in \( Y \). Furthermore, by the \( (X,Y) \)-asymptotic compactness, \( A(\tau,\omega) \) can also be reformulated in \( Y \)-metric, namely,
\[
A(\tau,\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))^Y, \tag{5}
\]
under the condition that \( X \) and \( Y \) are continuous embedding into a Hausdorff topological vector space \( X \).

It remains to prove the \( A \) is a random set. To this end, we first prove for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the right hand side of (5) can be reformulated as a discrete form, namely,
\[
A(\tau,\omega) = \bigcap_{n \in \mathbb{N}} \bigcup_{m=n} \varphi(m,\tau-m,\vartheta_{-m}\omega,K(\tau-m,\vartheta_{-m}\omega))^Y. \tag{6}
\]

Obviously, by (5), we have
\[
A(\tau,\omega) \supseteq \bigcap_{n \in \mathbb{N}} \bigcup_{m=n} \varphi(m,\tau-m,\vartheta_{-m}\omega,K(\tau-m,\vartheta_{-m}\omega))^Y. \tag{7}
\]

We prove the inverse inclusion relation. First, by the definition \( A(\tau,\omega) \) in (4) and the absorption of \( K \), it is easy to see that \( A(\tau,\omega) \subset K(\tau,\omega) \) for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). Therefore by the invariance of \( A \), we have
\[
A(\tau,\omega) = \varphi(m,\tau-m,\vartheta_{-m}\omega,A(\tau-m,\vartheta_{-m}\omega)) \subset \varphi(m,\tau-m,\vartheta_{-m}\omega,K(\tau-m,\vartheta_{-m}\omega)), \tag{8}
\]
for every fixed \( m \in \mathbb{N} \). Note that \( A(\tau,\omega) \subset Y \) and \( \varphi(t,\tau,.,.) \) maps \( X \) into \( Y \) for all \( t > 0 \). Then we have
\[
A(\tau,\omega) = \bigcap_{n \in \mathbb{N}} \bigcup_{m=n} \varphi(m,\tau-m,\vartheta_{-m}\omega,K(\tau-m,\vartheta_{-m}\omega))^Y. \tag{9}
\]
Thus we have proved (6). For a fixed \( \tau \in \mathbb{R} \) and each \( n \in \mathbb{N} \), denote by
\[
E_n(\omega) = \bigcup_{m=n} \varphi(m,\tau-m,\vartheta_{-m}\omega,K(\tau-m,\vartheta_{-m}\omega))^Y. \tag{10}
\]
By Lemma 2.8 (iii), we need to prove that for each \( n \in \mathbb{N} \) and \( \tau \), \( E_n = \{ E_n(\omega) : \omega \in \Omega \} \) is a random set in \( Y \) with nonempty closed images. Since \( K \) is a random set with nonempty closed images, then by Lemma 2.8 (i), there exists a countable family \( \{ f_i(\omega) \}_{i=1}^{\infty} \) of \( (\mathcal{F}, \mathcal{B}(X)) \)-measurable variables, such that \( \{ f_i(\omega) \}_{i=1}^{\infty} \) is dense in \( K(\tau - m, \vartheta_{-m}\omega) \) for each fixed \( m \in \mathbb{N} \) and \( \tau \). On the other hand, by our assumption (iii) and the continuity of \( \varphi(t, \tau, \omega, \cdot) : X \to Y \), using Lemma 2.9, it follows that every fixed \( m \in \mathbb{N} \) and \( \tau \in \mathbb{R} \), the mapping \( \omega \to \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \) is \( (\mathcal{F}, \mathcal{B}(Y)) \)-measurable. Therefore by Lemma 2.8 (ii), it follows that

\[
\Omega \ni \omega \mapsto \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y ,
\]

is measurable in \( Y \) for fixed \( m \in \mathbb{N} \) and \( \tau \in \mathbb{R} \). We now claim that, for every fixed \( m \in \mathbb{N} \) and \( \tau \in \mathbb{R} \),

\[
\varphi(m, \tau - m, \vartheta_{-m}\omega, K(\tau - m, \vartheta_{-m}\omega)) = \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y .
\]

(12)

In fact, since \( K(\tau - m, \vartheta_{-m}\omega) \) is closed and \( \varphi(t, \tau, \omega, \cdot) \) continuous from \( X \) to \( Y \), the left side hand is closed in \( Y \). It is easy to say, for every fixed \( m \in \mathbb{N} \),

\[
\varphi(m, \tau - m, \vartheta_{-m}\omega, K(\tau - m, \vartheta_{-m}\omega)) \supseteq \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y .
\]

(13)

For the inverse inclusion, let \( y \in \varphi(m, \tau - m, \vartheta_{-m}\omega, K(\tau - m, \vartheta_{-m}\omega)) \). Then there exists a point \( u \in K(\tau - m, \vartheta_{-m}\omega) \) such that \( y = \varphi(m, \tau - m, \vartheta_{-m}\omega, u) \). Note that \( \{ f_i(\omega) \}_{i=1}^{\infty} \) is dense in \( K(\tau - m, \vartheta_{-m}\omega) \). Therefore there is a sequence \( \{ f_{ik}(\omega) \}_{k=1}^{\infty} \supseteq \{ f_i(\omega) \}_{i=1}^{\infty} \) with \( f_{ik}(\omega) \to u \) in \( X \) as \( k \to \infty \). By the the continuity of \( \varphi(t, \tau, \omega, \cdot) : X \to Y \), it follows that

\[
\varphi(m, \tau - m, \vartheta_{-m}\omega, f_{ik}(\omega)) \to \varphi(m, \tau - m, \vartheta_{-m}\omega, u) \text{ in } Y.
\]

That is to say, we find a sequence

\[
y_k = \varphi(m, \tau - m, \vartheta_{-m}\omega, f_{ik}(\omega)) \in \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}
\]

with \( y_k \to y \) in \( Y \). Therefore \( y \in \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y \). Then the inverse inclusion of (13) is proved. Thus (12) holds true. By (12), we rewrite (10) as

\[
E_n(\omega) = \bigcup_{m=n}^{\infty} \varphi(m, \tau - m, \vartheta_{-m}\omega, K(\tau - m, \vartheta_{-m}\omega)) \subseteq \bigcup_{m=n}^{\infty} \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y .
\]

(14)

Thanks to (11) and using Lemma 2.8 (ii) again, we know that

\[
\Omega \ni \omega \mapsto \bigcup_{n=m}^{\infty} \bigcup_{i \in \mathbb{N}} \{ \varphi(m, \tau - m, \vartheta_{-m}\omega, f_i(\omega)) \}^Y
\]

is measurable in \( Y \) for fixed \( \tau \in \mathbb{R} \). Then by (14) and (15) we get \( \Omega \ni \omega \mapsto E_n(\omega) \) is measurable in \( Y \) for fixed \( \tau \in \mathbb{R} \).

On the other hand, for each \( n \in \mathbb{N} \), taking a \( y_n \in E_n(\omega) \), by Lemma 2.8 (iii), we need to prove that \( \{ y_n \}_{n=1}^{\infty} \) has a convergence subsequence in \( Y \). It seems that this can not be derived directly from the asymptotical compactness of \( \varphi \) in \( Y \). We now give the detained proof. Indeed, for every fixed \( \tau \in \mathbb{R}, \omega \in \Omega \), by (10), for each \( n \in \mathbb{N} \), there exist \( \mathbb{N} \ni n_k \geq n \) and \( z_m^n \in K(\tau - m_k, \vartheta_{-m_k}\omega) \) such that

\[
d_Y(\varphi(m_k^n, \tau - m_k^n, \vartheta_{-m_k^n}\omega, z_m^n), y_n) \to 0 ,
\]

(16)
as \( k \to \infty \). We now choose a sequence \( \{j_n\}_{n=1}^{\infty} \subset \mathbb{N} \) by the following rules.

\[
j_1 = m_{k_1}^1 \geq 1 \quad \text{if} \quad d_Y(\varphi(m_{k_1}^1, \tau - m_{k_1}^1, \vartheta_{-m_{k_1}^1}, z_{m_{k_1}^1}), y_1) \leq \frac{1}{2};
\]

\[
j_2 = m_{k_2}^2 \geq \max\{j_1, 2\} \quad \text{if} \quad d_Y(\varphi(m_{k_2}^2, \tau - m_{k_2}^2, \vartheta_{-m_{k_2}^2}, z_{m_{k_2}^2}), y_2) \leq \frac{1}{2^2};
\]

\[\vdots\]

\[
j_n = m_{k_n}^n \geq \max\{j_{n-1}, n\} \quad \text{if} \quad d_Y(\varphi(m_{k_n}^n, \tau - m_{k_n}^n, \vartheta_{-m_{k_n}^n}, z_{m_{k_n}^n}), y_n) \leq \frac{1}{2^n};
\]

\[\vdots\]

where \( \{k_i\}_{i=1}^{\infty} \) is the subsequence of \( \{k\}_{k=1}^{\infty} \). Therefore we get a sequence \( \{\varphi(j_n, \tau - j_n, \vartheta_{-j_n}, z_{j_n})\}_{n=1}^{\infty} \) satisfying that \( j_n \) is increasing in \( n \), \( z_{j_n} \in K(\tau - z_{j_n}, \vartheta_{-z_{j_n}}, \omega) \) and

\[
d_Y(\varphi(j_n, \tau - j_n, \vartheta_{-j_n}, z_{j_n}), y_n) \leq \frac{1}{2^n}; \tag{17}\]

since as \( n \to \infty \), we have \( j_n \to \infty \), then by the \( (X,Y) \)-asymptotical compactness of \( \varphi \), there exists a point \( y \in Y \) such that, up to a subsequence (denoted by itself for simplicity),

\[
d_Y(\varphi(j_n, \tau - j_n, \vartheta_{-j_n}, z_{j_n}), y) \to 0 \tag{18}\]

as \( n \to \infty \). By the triangle inequality, along with (17) and (18) we have

\[
d_Y(y_n, y) \leq d_Y(\varphi(j_n, \tau - j_n, \vartheta_{-j_n}, z_{j_n}), y_n) + d_Y(\varphi(j_n, \tau - j_n, \vartheta_{-j_n}, z_{j_n}), y) \to 0 \]

as \( n \to \infty \), and consequently the sequence \( \{y_n\}_{n=1}^{\infty} \) converges in \( Y \) (thus \( E_n(\omega) \) is compact in \( Y \)). So by Lemma 2.8 (iii) and (10) it follows that

\[
\Omega \ni \omega \mapsto A(\tau, \omega) = \cap_{n \in \mathbb{N}} E_n(\omega). \tag{19}\]

is measurable in \( Y \) for fixed \( \tau \in \mathbb{R} \), and therefore by Definition 2.3, the family \( A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \) is a random set in \( Y \). Since \( A \in \mathcal{D} \), then the uniqueness is proved by the fact that \( A \) attracts itself. The proof is completed. \( \Box \)

3. Fractional power dissipative equation on \( \mathbb{R}^N \) with additive noise. In this section, we first present the concepts of fractional power Laplacian and fractional Sobolev space. Then we give the condition on the nonlinearity and make a parameter transformation of the fractional power dissipative equation. Finally the random dynamical system generated by the solution is defined.

3.1. Fractional power Laplacian and fractional Sobolev space. The fractional power Laplacian, as a positive power of the classical Laplacian, has been introduced for a long time from several areas of mathematics and real-world applications, such as potential theory [27], fractional calculus [39], probability [20, 35] and etc. However, until the past decade this operator became a very popular object in the field of partial differential equations. There are two ways to define the fractional power Laplacian in the literature. One is the spectral fractional operator (cf.[28, 58] and the references therein), the other is the integral fractional operator (cf.[17]), which is one of main topics in this paper.

Let \( \mathcal{J} \) be the Schwartz space of rapidly decaying \( C^\infty \) functions on \( \mathbb{R}^N \). For \( s \in (0, 1) \), the integral fractional power Laplacian \((-\Delta)^s\) is defined as, for \( u \in \mathcal{J} \),
\((-\Delta)^s u(x) = C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} dy \)
\[-\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{N + 2s}} dy, \quad x \in \mathbb{R}^N, \quad (20)\]

where \(P.V.\) means the principal value of the integral, and \(C(N, s)\) is an explicit positive constant depend only on \(N\) and \(s\), see [17].

Define
\[H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy < \infty \}. \quad (21)\]

Then \(H^s(\mathbb{R}^N)\) is a Hilbert space equipped with the inner product, for \(u, v \in H^s(\mathbb{R}^N)\),
\[(u, v)_{H^s(\mathbb{R}^N)} = (u, v) + \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} dxdy \]
and the norm
\[\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2 + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy, \quad (22)\]

where and later the notations \((\cdot, \cdot)\) and \(\|\cdot\|\) denote the inner product and norm in \(L^2(\mathbb{R}^N)\), respectively. From Proposition 3.6 in [17], we have
\[\|(-\Delta)^{\frac{s}{2}} u\|^2 = \frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy, \]

namely, the norm in (22) is rewritten as
\[\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2 + \frac{2}{C(N, s)} \|(-\Delta)^{\frac{s}{2}} u\|^2, \quad u \in H^s(\mathbb{R}^N), \]

and consequently \(\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2 + \|(-\Delta)^{\frac{s}{2}} u\|^2\) is an equivalent norm of \(H^s(\mathbb{R}^N)\).

3.2. Fractional power dissipative stochastic equation. In this subsection, we present some settings about the the non-autonomous fractional power dissipative stochastic equation on the whole space \(\mathbb{R}^N\), driven by an additive noise:
\[\frac{du}{dt} + \lambda u + (-\Delta)^s u + f(t, x, u) = g(t, x) + \phi(x)\tilde{W}(t), \quad t > \tau, \tau \in \mathbb{R}, \quad (23)\]
with the initial condition
\[u(\tau, x) = u_\tau \in L^2(\mathbb{R}^N), \quad (24)\]

where \(s \in (0, 1)\) and \(\lambda > 0\). The nonlinear function \(f\) is continuous on \(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\) satisfying the following conditions, for all \(t, u_1, u_2 \in \mathbb{R}\) and \(x \in \mathbb{R}^N\), \(f(t, x, 0) = 0\) and
\[|f(t, x, u_1) - f(t, x, u_2)| \leq \psi_2(t, x)|u_1 - u_2|, \quad (25)\]
\[|f(t, x, u_1) - f(t, x, u_2)| \leq \psi_1(t, x)|u_1 - u_2|(1 + |u_1|^p + |u_2|^p), \quad (26)\]
where \(\gamma > 0, p \geq 2\) and \(\psi_1, \psi_2 \in L^\infty(\mathbb{R}^{N+1})\). For example, if \(f(t, x, u) = |u|^{p-2}u - u, u \in \mathbb{R}\), then \(f\) satisfies (25) and (26).
We further assume that
\[ \lambda > \mu = 2(\|\psi_1\|_\infty + \|\psi_2\|_\infty), \] (27)
where \( \psi_1 \) and \( \psi_2 \) are given in (25) and (26). Here \( \| \cdot \|_\infty \) is the \( \ell^\infty \) norm. The non-autonomous term \( g \in L_{loc}^2(\mathbb{R}, L^2(\mathbb{R}^N)) \) satisfies, for every \( \tau \in \mathbb{R} \),
\[ \int_{-\infty}^\tau e^{\alpha_0 r} \| g(r, \cdot) \|^2 dr < \infty, \] (28)
where \( \alpha_0 = \lambda - \mu > 0 \). For the noise coefficient, we assume that \( \phi \) satisfies that, for any \( s \in (0, 1) \) and \( p \geq 2 \),
\[ \phi \in H^s(\mathbb{R}^N) \cap L^{2p-2}(\mathbb{R}^N), \quad (-\Delta)^s \phi \in L^2(\mathbb{R}^N). \] (29)
Then by the Sobolev interpolation, we have \( \phi \in L^p(\mathbb{R}^N) \).

To model the random noise in (23), we introduce the classic Wiener probability space \((\Omega, \mathcal{F}, P)\), where \( \Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \) with the compact-open topology, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, \( P \) is the Wiener measure on \((\Omega, \mathcal{F})\). Then the Brown motion is identified as \( W(t) = W(t, \omega) = \omega(t) \). The driving system is a shift operator \( \vartheta_t \) on \( \Omega \) by
\[ \vartheta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad \forall \ t \in \mathbb{R}. \]
It is known that the Wiener measure \( P \) is ergodic and invariant under the drift \( \vartheta_t \), and the quadruple form \((\Omega, \mathcal{F}, P, \{ \vartheta_t \}_{t \in \mathbb{R}})\) is a metric dynamical system, see Arnold [1]. By the law of the iterated logarithm (cf.[18]), there exists a \( \vartheta_t \)-invariant set \( \Omega_1 \subset \Omega \) with \( P(\tilde{\Omega}) = 1 \), and in addition,
\[ \frac{|\omega(t)|}{t} \to 0, \text{ as } |t| \to \infty, \quad \omega \in \Omega_1. \] (30)

Thanks to the Brownian motion \( W(t) = \omega(t) \) is not differentiable in \( t \), the stochastic equation need to be transformed into an equation with random coefficient, usually by an Ornstein-Uhlenbeck process over \((\Omega, P, F, \{ \vartheta_t \}_{t \in \mathbb{R}})\), which is a stationary process
\[ y(\vartheta_t \omega) = -\lambda t \int_{-\infty}^0 e^{\lambda r} \vartheta_t \omega(r) dr, \quad t \in \mathbb{R}, \] (31)
satisfying the stochastic differential equation:
\[ dy(\vartheta_t \omega) + \lambda y(\vartheta_t \omega) dt = d\omega(t). \]
In addition, it follows that from [1], that there exits a \( \vartheta_t \)-invariant set \( \Omega_2 \subset \Omega \) of full measure such that \( y(\vartheta_t \omega) \) is pathwise continuous in \( t \) for every fixed \( \omega \in \Omega_2 \), and
\[ \lim_{t \to \pm \infty} \frac{|y(\vartheta_t \omega)|}{|t|} = 0, \text{ and } \lim_{t \to \pm \infty} \frac{1}{|t|} \int_0^t y(\vartheta_r \omega) dr = 0, \quad \omega \in \Omega_2. \] (32)

Let \( \tilde{\Omega} = \Omega_1 \cap \Omega_2 \). Then \( \tilde{\Omega} \subset \Omega \) is a \( \vartheta_t \)-invariant set of full measure, and in addition, for every \( \omega \in \tilde{\Omega} \), (30) and (32) hold. In the sequel, all arguments are understood to hold on this \( \tilde{\Omega} \), but we keep the notation \( \Omega \) for \( \tilde{\Omega} \).
Put
\[ J(\varphi_t \omega) = |y(\varphi_t \omega)|^2 + |y(\varphi_t \omega)|^p + |y(\varphi_t \omega)|^2p^{-2}. \] (33)

Then by Proposition 4.3.3 in [1] there exists a random variable \( \rho(\omega) \) such that
\[ J(\varphi_t \omega) \leq \rho(\varphi_t \omega) \leq \rho(\omega)e^{\frac{\mu}{2}|t|}, \quad \forall \omega \in \Omega, \] (34)
where \( \alpha_0 = \lambda - \mu > 0 \) with \( \mu \) as in (27).

Set \( z(\varphi_{t} \omega) = \phi(x)y(\varphi_t \omega) \) and let \( v(t) = u(t) - z(\varphi_t \omega) \), where \( u(t) \) satisfies the equations (23) and (24). Then the stochastic equations (23) and (24) are transformed into a pathwise deterministic one, namely, \( v(t) \) satisfies
\[ \frac{dv}{dt} + \lambda v + (-\Delta)^p v + f(t, x, v + z(\varphi_t \omega)) = g(t, x) - (-\Delta)^p z(\varphi_t \omega), \] (35)
with the initial condition
\[ v(t, x) = v_r = u_r - z(\varphi_r \omega). \] (36)

Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_r \in L^2(\mathbb{R}^N) \), define
\[ \varphi(t, \tau, \omega, u_r) = u(t + \tau, \tau, \varphi_{-\tau} \omega, u_r) = v(t + \tau, \tau, \varphi_{-\tau} \omega, v_r) + z(\varphi_t \omega), \] (37)
where \( u_r = v_r + z(\varphi_r \omega) \). In the later sections, we will show that the random cocycle \( \varphi(t, \tau, \omega, u_r) \) is \( (\mathcal{F}, B(Z)) \)-measurable, and in addition the mapping \( \varphi(t, \tau, \omega, \cdot) : L^2(\mathbb{R}^N) \to Z \) is continuous with respect to the initial data, where \( Z = L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \) and \( H^s(\mathbb{R}^N) \). Then \( \varphi \) is a measurable bi-spatially continuous RDS.

For the universe of sets, we consider a family \( D = \{D(\tau, \omega) \subset L^2(\mathbb{R}^N) : \tau \in \mathbb{R}, \omega \in \Omega \} \) such that for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[ \lim_{t \to \infty} e^{-\alpha_0 t} \|D(\tau - t, \varphi_{-t} \omega)\|^2 = 0, \] (38)
where \( \alpha_0 = \lambda - \mu > 0 \) and \( \|D\| = \sup\{\|u\| : u \in D\} \). Such a set \( D(\tau, \omega) \) satisfying (38) is called tempered set. Denote by \( \mathcal{D} \) the collection of all tempered families of nonempty bounded subsets of \( L^2(\mathbb{R}^N) \). Then it is obvious that \( \mathcal{D} \) is inclusion closed.

4. Absorbing set and asymptotical compactness in \( L^2(\mathbb{R}^N) \). In this section, we prove the existence of a random pullback absorbing set and the pullback asymptotical compactness of \( \varphi \) defined by (37) in \( L^2(\mathbb{R}^N) \).

4.1. Existence of random absorbing set.

Lemma 4.1. Let (25)-(29) hold. Let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \). Then there exists a constant \( T = T(\tau, \omega, D) \geq 1 \) such that for all \( t \geq T \) and \( u_{\tau-t} \in D(\tau-t, \varphi_{-\tau} \omega) \), the solution \( u \) of equations (23) and (24) satisfies
\[ \|u(\sigma, \tau - t, \varphi_{-\tau} \omega, u_{\tau-t})\|^2 \leq c(1 + C(\tau, \omega)), \quad \sigma \in [\tau - 1, \tau]; \] (39)
\[ \int_{\tau-t}^{\tau} e^{\alpha_0(r-t)}(\|v(r)\|_{H^s(\mathbb{R}^N)}^2 + \|v(r)\|_{L^p}^p)dr \leq c(1 + C(\tau, \omega)), \] (40)
where \( c > 0 \) is a deterministic generic constant and \( C(\tau, \omega) = \int_{-\infty}^{0} e^{\mu r} \|g(r + \tau, \cdot)\|^2 dr + \rho(\omega) \). Define
\[ K(\tau, \omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \leq c(1 + C(\tau, \omega))\}. \] (41)
Then $K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$ is a closed $\mathcal{D}$-pullback random absorbing set for the random cocycle $\varphi$ in $L^2(\mathbb{R}^N)$.

Proof. Taking the inner product of (35) in $L^2(\mathbb{R}^N)$ with $v$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \lambda \|v\|^2 + \|(-\Delta)^{\frac{1}{2}}v\|^2 + \int_{\mathbb{R}^N} f(t, x, u)vdx = \int_{\mathbb{R}^N} g(t, x)vdx
\]
where we used the inequality $|a + b|^r \geq 2^{1-r} |a|^r - |b|^r$ for any $r \geq 1$ in the first term of the last inequality on the right hand side of (43). By $\phi \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and $\psi_1, \psi_2 \in L^\infty(\mathbb{R}^{N+1})$, along with (27), we deduce from (43) that
\[
\int_{\mathbb{R}^N} f(t, x, u)vdx \geq \gamma_2 \|v\|^p_p - \mu \|v\|^2 - c|z(\varphi_{t\omega})|^{p} = \gamma_2 \|v\|^p_p - \mu \|v\|^2 - cJ(\varphi_{t\omega}),
\]
where $J(\cdot)$ is given as in (33). On the other hand, by integration by parts and Young inequality, along with $\phi \in H^s(\mathbb{R}^N)$, we have
\[
\left|(-\Delta)^{\frac{1}{2}}z(\varphi_{t\omega}), v\right| = \frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{y(\varphi_{t\omega})(\phi(x) - \phi(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy
\]
\[
\leq \frac{C(N, s)}{4} \int_{\mathbb{R}^{2N}} \frac{|v(x)|^2}{|x - y|^{N+2s}} dxdy + \frac{C(N, s)}{4} |y(\varphi_{t\omega})|^2 \int_{\mathbb{R}^{2N}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dxdy
\]
\[
\leq \frac{1}{2} \|(-\Delta)^{\frac{1}{2}}v\|^2 + c|y(\varphi_{t\omega})|^2,
\]
where $c > 0$ is a deterministic constant depending on $n, s$ and $\phi$. In addition,
\[
\left|\int_{\mathbb{R}^N} g(t, x)vdx\right| \leq \frac{\lambda - \mu}{4} \|v\|^2 + c\|g(t, \cdot, \cdot)\|^2,
\]
where $c$ is determined by $\lambda - \mu > 0$. By a combination of (44)-(46) into (42), it follows that

$$
\frac{d}{dt} ||v||^2 + \alpha_0 ||v||^2 + \frac{\alpha_0}{2} ||v||^2 + \|(-\Delta)^{\frac{1}{2}} v\|^2 + \frac{\gamma}{2^p} ||v||^p_p \\
\leq c||g(t,.)||^2 + cJ(\partial_t \omega),
$$

(47)

where $\alpha_0 = \lambda - \mu$ as in (28). Apply Gronwall lemma to (47) over the interval $[\tau - t, \sigma]$ for $\sigma \in [\tau - 1, \tau]$ and $t \geq 1$, meanwhile, using $\partial_{-t} \omega$ to replace $\omega$, to find that

$$
||v(\sigma)||^2 + \int_{\tau - t}^\sigma e^{\alpha_0 (r - \sigma)} (\frac{\alpha_0}{2} ||v(r)||^2 + \|(-\Delta)^{\frac{1}{2}} v(r)\|^2 + \frac{\alpha}{2^p} ||v(r)||^p_p) dr
\\
\leq e^{\alpha_0} e^{-\alpha_0 t} ||v_{\tau - t}||^2 + c e^{\alpha_0} \int_{\tau - t}^\sigma e^{\alpha_0 (r - \tau)} ||g(r,.)||^2 dr
\\
+ c e^{\alpha_0} \sigma \int_{\tau - t}^\sigma e^{\alpha_0 (r - \tau)} e^{\frac{\lambda}{2} |r - \tau|} dr
\\
\leq 2e^{\alpha_0} e^{-\alpha_0 t} (||u_{\tau - t}||^2 + ||\phi||^2 |y(\partial_{-t} \omega)|^2)
\\
+ c \int_{-\infty}^0 e^{\alpha_0 \tau} ||g(r + \tau,.)||^2 dr + c\phi(\omega),
$$

(48)

where we used (34) in the last term of the first inequality. By noting that $u_{\tau - t} \in D(\tau - t, \partial_{-t} \omega)$ and $|y(\omega)|$ is tempered, we have

$$
2e^{\alpha_0} e^{-\alpha_0 t} (||u_{\tau - t}||^2 + ||\phi||^2 |y(\partial_{-t} \omega)|^2) \to 0,
$$

(49)

as $t \to \infty$. By (28), we have $\int_{-\infty}^0 e^{\alpha_0 \tau} ||g(r + \tau,.)||^2 dr < \infty$. Therefore, it follows from (48) and (49) that, for every fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and all $u_{\tau - t} \in D(\tau - t, \partial_{-t} \omega)$, there exists a $T = T(\tau, \omega, D) \geq 1$ such that for all $t \geq T$,

$$
||v(\sigma)||^2 + \int_{\tau - t}^\sigma e^{\alpha_0 (r - \tau)} (||v(r)||^2 + ||(-\Delta)^{\frac{1}{2}} v(r)||^2 + ||v(r)||^p_p) dr
\\
\leq c(1 + \int_{-\infty}^0 e^{\alpha_0 \tau} ||g(r + \tau,.)||^2 dr + \phi(\omega)).
$$

(50)

Note that $\mu(t) = v(t) + z(\partial_t \omega)$. Then by (34) it follows that, for $\sigma \in [\sigma - 1, \sigma]$, 

$$
||u(\sigma)||^2 \leq 2 ||v(\sigma)||^2 + 2 ||\phi||^2 |y(\partial_{\sigma - t} \omega)|^2
\\
\leq 2 ||v(\sigma)||^2 + 2 ||\phi||^2 \phi(\omega) e^{\frac{\lambda}{2} |\sigma - \tau|}
\\
\leq 2 ||v(\sigma)||^2 + 2 ||\phi||^2 \phi(\omega) e^{\frac{\lambda}{2}}.
$$

(51)

Therefore, by (50) and (51), we get (39) and (40). On the other hand, for any $x \in L^2(\mathbb{R}^N)$ and a fixed $\tau \in \mathbb{R}$, we have

$$
\inf_{\nu \in K(\tau) \omega} d_{L^2(\mathbb{R}^N)}(x, K(\tau) \omega) = ||x|| + c(1 + C(\tau, \omega))^{\frac{1}{2}}
$$
is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable since \(C(\tau, \omega) = \int_{-\infty}^{0} e^{\alpha \tau t} \|g(r + \tau, \cdot)\|^2 dr + \phi(\omega)\) as a random variable is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable for fixed \(\tau \in \mathbb{R}\). Then the set-valued mapping \(K\) defined by (41) is a closed random set in \(L^2(\mathbb{R}^N)\), which completes the proof. \(\square\)

4.2. **Estimate of the tail of solution.** To prove the asymptotical compactness of \(\varphi\) in \(L^2(\mathbb{R}^N)\), we need to prove that the far-field value of solution to (35) vanishes when the time and radius of ball are large enough. Let \(\xi(\cdot): \mathbb{R}^+ \rightarrow [0, 1]\) be a smooth function satisfying

\[
\xi(r) = \begin{cases} 
0, & \text{if } r \in [0, 1]; \\
1, & \text{if } r \in [2, \infty). 
\end{cases}
\] (52)

Then there exists a positive constant \(c\) such that \(|\xi'(\cdot)| \leq c\).

We first give a lemma about the cutoff function.

**Lemma 4.2.** Let \(\xi\) be defined by (52). Then for any \(y \in \mathbb{R}^N\) and \(k \in \mathbb{N}\), we have

\[
\int_{\mathbb{R}^N} \frac{\left|\xi\left(\frac{|x|}{k}\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{N+2s}} \, dx \leq \frac{c}{k^{2s}},
\] (53)

where the constant \(c > 0\) is independent of \(k\) and \(y\).

**Proof.** Make a variable change \(x - y = kz\). Then \(dx = k^N dz\), and hence we have

\[
\int_{\mathbb{R}^N} \frac{\left|\xi\left(\frac{|x|}{k}\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{N+2s}} \, dx = \frac{1}{k^{2s}} \int_{\mathbb{R}^N} \frac{\left|\xi\left(|z + \frac{y}{k}|\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|z|^{N+2s}} \, dz
\]

\[
= \frac{1}{k^{2s}} \int_{|z| \leq 1} \frac{\left|\xi\left(|z + \frac{y}{k}|\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|z|^{N+2s}} \, dz
\]

\[
+ \frac{1}{k^{2s}} \int_{|z| > 1} \frac{\left|\xi\left(|z + \frac{y}{k}|\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|z|^{N+2s}} \, dz.
\] (54)

By the Lagrange mean theorem and \(|\xi'(\cdot)| \leq c\), we get

\[
\frac{1}{k^{2s}} \int_{|z| \leq 1} \frac{\left|\xi\left(|z + \frac{y}{k}|\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|z|^{N+2s}} \, dz \leq \frac{1}{k^{2s}} \int_{|z| \leq 1} \frac{\left|\xi'(\cdot)|\right|z + \frac{y}{k} - \frac{|y|}{k}\right|^2}{|z|^{N+2s}} \, dz
\]

\[
\leq \frac{c}{k^{2s}} \int_{|z| \leq 1} \frac{1}{|z|^{N+2s-2}} \, dz < \infty,
\] (55)

where we used the fact that \(\frac{1}{|z|^{N+2s-2}}\) is integrable for \(N + 2s - 2 < N\). On the other hand, since \(|\xi(\cdot)| \leq 1\), we have

\[
\frac{1}{k^{2s}} \int_{|z| > 1} \frac{\left|\xi\left(|z + \frac{y}{k}|\right) - \xi\left(\frac{|y|}{k}\right)\right|^2}{|z|^{N+2s}} \, dz \leq \frac{4}{k^{2s}} \int_{|z| > 1} \frac{1}{|z|^{N+2s}} \, dz < \infty,
\] (56)
where we used the fact that $\frac{1}{|x|^{N+2s}}$ is integrable for $N + 2s > N$. Then combine (54)-(56) to get the desired. \hfill \Box

We now present the tail estimate of solutions, which demonstrates that the far-field value of solution vanishes in $L^2(\mathbb{R}^N)$ as the time and the radius larger enough.

**Lemma 4.3.** Let (25)-(29) hold. Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for any $\eta > 0$, there exist constants $C_0 = C_0(\omega), R = R(\tau, \omega, D, \eta) \geq 1$ and $T = T(\tau, \omega, D, \eta) \geq 1$ such that the solution $u$ of equations (23) and (24) satisfies, for all $\sigma \in [\tau - 1, \tau]$ and $t \geq T$,

$$\int_{Q_R^c} |u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau - t})|^2 dx \leq C_0(\omega)\eta,$$

where $Q_R^c = \mathbb{R}^N - Q_R$ and $Q_R$ is the ball of $\mathbb{R}^N$ centred at zero with radius $R$.

**Proof.** Let $\xi_k = \xi(\frac{|x|}{k})$ for $x \in \mathbb{R}^N$ and $k \in \mathbb{N}$, where $\xi(\cdot)$ is given by (52). Take the inner product of (35) in $L^2(\mathbb{R}^N)$ with $\xi_k v$ to find

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \xi_k |v|^2 dx + \lambda \int_{\mathbb{R}^N} \xi_k |v|^2 dx = -((-\Delta)^s v, \xi_k v) - \int_{\mathbb{R}^N} f(t, x, u)\xi_k v dx$$

$$+ \int_{\mathbb{R}^N} g(t, x)\xi_k v dx - ((-\Delta)^s (\theta_t \omega), \xi_k v). \quad (57)$$

We now estimate every terms on the right hand side of (57). By some calculations, the first term on the right hand side of (57) is rewritten as

$$-((-\Delta)^s v, \xi_k v) = -\frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(\xi(\frac{|x|}{k})v(x) - \xi(\frac{|y|}{k})v(y))}{|x - y|^{N+2s}} dx dy$$

$$= -\frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2 \xi(\frac{|x|}{k})}{|x - y|^{N+2s}} dx dy$$

$$- \frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(\xi(\frac{|x|}{k}) - \xi(\frac{|y|}{k}))v(y)}{|x - y|^{N+2s}} dx dy. \quad (58)$$

By Hölder inequality and Lemma 4.2, the second term on the right hand side of (58) is estimated as

$$-\frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(\xi(\frac{|x|}{k}) - \xi(\frac{|y|}{k}))v(y)}{|x - y|^{N+2s}} dx dy$$

$$\leq \frac{1}{2} C(N, s) \|v\| \left\{ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\xi(\frac{|x|}{k}) - \xi(\frac{|y|}{k}))}{|x - y|^{N+2s}} dx \right)^2 dy \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{2} C(N, s) \|v\| \left\{ \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx \left( \int_{\mathbb{R}^N} \frac{|\xi(\frac{|x|}{k}) - \xi(\frac{|y|}{k})|^2}{|x - y|^{N+2s}} dx \right) \right\}^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{c}}{2k^s} C(N, s) \|v\| \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \leq \frac{\sqrt{c}}{2k^s} C(N, s) \|v\|^2_{H^s(\mathbb{R}^N)}.$$
which, along with (58), gives

\[-((-\Delta)^s v, \xi_k v) \leq -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2 \xi\left(\frac{|x|}{k}\right) - v(y)\xi\left(\frac{|y|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[+ \frac{\sqrt{c_2}}{2k^n} C(N, s) \|v\|_{H^s(\mathbb{R}^N)}^2. \tag{59}\]

For the last term on the right hand side of (57), we rewrite as

\[-((-\Delta)^s z(\partial_t \omega), \xi_k v)\]

\[= -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(v(x)\xi\left(\frac{|x|}{k}\right) - v(y)\xi\left(\frac{|y|}{k}\right))}{|x - y|^{N+2s}} dxdy\]

\[= -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(v(x) - v(y))\xi\left(\frac{|x|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[-\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(\xi\left(\frac{|x|}{k}\right) - \xi\left(\frac{|y|}{k}\right))v(y)}{|x - y|^{N+2s}} dxdy. \tag{60}\]

The first term on the right hand side of (60) is bounded by

\[-\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(v(x) - v(y))\xi\left(\frac{|x|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[\leq \frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2 \xi\left(\frac{|x|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[+ \frac{1}{2} C(N, s) |y(\partial_t \omega)|^2 \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2 \xi\left(\frac{|x|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[\leq \frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2 \xi\left(\frac{|x|}{k}\right)}{|x - y|^{N+2s}} dxdy\]

\[+ \frac{1}{2} C(N, s) |y(\partial_t \omega)|^2 \int_{|x| \geq k} \left( \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy \right) dx. \tag{61}\]

By Lemma 4.2, along with (29), the second term on the right hand side of (60) is bounded by

\[-\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(\xi\left(\frac{|x|}{k}\right) - \xi\left(\frac{|y|}{k}\right))v(y)}{|x - y|^{N+2s}} dxdy\]

\[\leq \frac{1}{2} C(N, s) \|v\|_{L^2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{y(\partial_t \omega)(\phi(x) - \phi(y))(\xi\left(\frac{|x|}{k}\right) - \xi\left(\frac{|y|}{k}\right))}{|x - y|^{N+2s}} dxdy \right)^2 dy \right\}^{\frac{1}{2}}\]

\[\leq \frac{1}{2} C(N, s) \|v\|_{L^2(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|y(\partial_t \omega)|^2}{|x - y|^{N+2s}} dxdy \right) \|v\|_{L^2(\mathbb{R}^N)}^2 dx.\]
On the other hand, we have
\[
\left| \int f(t,x,u)\xi_k v dx \right| \leq \frac{c}{2k^s} \int_{\mathbb{R}^N} \left| v(y(\vartheta t \omega)) \right| \left( \int_{\mathbb{R}^N} \left( \frac{|f(x)|^2}{|x-y|^{N+2s}} dx \right) \right)^{\frac{1}{2}} dy
\]
\[
\leq \frac{c}{2k^s} \int_{\mathbb{R}^N} \left| v(y(\vartheta t \omega)) \right| \int_{\mathbb{R}^N} \left( \frac{|f(x)|^2}{|x-y|^{N+2s}} dx \right)^{\frac{1}{2}} dy
\]
\[
\leq \frac{c}{2k^s} \left( \|v\|^2 + |y(\vartheta t \omega)|^2 \right).
\]  
\[(62)\]

Incorporate (61) and (62) into (60) to yield
\[
-((-\Delta)^s z(\vartheta t \omega), \xi_k v) \leq \frac{c}{2k^s} \left( \|v\|^2 + |y(\vartheta t \omega)|^2 \right) + \frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dx dy
\]
\[
+ \frac{1}{2} C(N, s) |y(\vartheta t \omega)|^2 \int_{|x| \geq k} \left( \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \right) dx.
\]  
\[(63)\]

For the nonlinearity term in (57), by (25),(26) and (27), we deduce that
\[
\int f(t,x,u)\xi_k v dx \geq \frac{1}{2} \int_{\mathbb{R}^N} \xi_k |u|^p dx - \mu \int_{\mathbb{R}^N} \xi_k |v|^2 dx
\]
\[
- c \int_{\mathbb{R}^N} \xi_k |\phi z(\vartheta t \omega)|^p dx - c \int_{\mathbb{R}^N} \xi_k |\phi z(\vartheta t \omega)|^2 dx.
\]  
\[(64)\]

On the other hand, we have
\[
\int_{\mathbb{R}^N} \xi_k g(t,x) v dx \leq \frac{\lambda - \mu}{2} \int_{\mathbb{R}^N} \xi_k |v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \xi_k |g(t,x)|^2 dx.
\]  
\[(65)\]

Consequently, combine (57) with (59),(63)-(65) to yield
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \xi_k |v|^2 dx + \alpha_0 \int_{\mathbb{R}^N} \xi_k |v|^2 dx \leq \frac{c}{k^s} \left( \|v\|^2_{H^s(\mathbb{R}^N)} + |y(\vartheta t \omega)|^2 \right)
\]
\[
+ c \int_{\mathbb{R}^N} \xi_k (|\phi(x)|^p z(\vartheta t \omega)|^p + |\phi(x)|^2 z(\vartheta t \omega)|^2) dx + c \int_{\mathbb{R}^N} \xi_k |g(t,x)|^2 dx
\]
\[
+ c |y(\vartheta t \omega)|^2 \int_{|x| \geq k} \left( \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \right) dx,
\]  
\[(66)\]

where \(\alpha_0 = \lambda - \mu\) as in (28). Since \(\phi \in L^p(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)\), then for every \(\eta > 0\) there exists a \(R_1 = R_1(\eta) \geq 1\) such that for all \(k \geq R_1\),
\[
c \int_{\mathbb{R}^N} \xi_k (|\phi(x)|^p z(\vartheta t \omega)|^p + |\phi(x)|^2 z(\vartheta t \omega)|^2) dx \leq c \int_{|x| \geq k} \left( (|\phi(x)|^p z(\vartheta t \omega)|^p + |\phi(x)|^2 z(\vartheta t \omega)|^2) dx
\]
\[
\leq \frac{\eta}{2} J(\vartheta t \omega), \quad (67)
\]
\[ c|y(\partial_t \omega)|^2 \int_{|x| \geq k} \left( \int_{|x-y| \geq 2t} \left| \phi(x) - \phi(y) \right|^2 dy \right) dx \leq \frac{\eta}{2} |y(\partial_t \omega)|^2. \] (68)

Therefore, it follows from (66)-(68) that, for all \( k \geq R_1 \),
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \xi_k |v|^2 dx + \alpha_0 \int_{\mathbb{R}^N} \xi_k |v|^2 dx \leq \frac{c}{k^s} \left( \|v\|^2_{H^s(\mathbb{R}^N)} + |y(\partial_t \omega)|^2 \right)
+ c \int_{|x| \geq k} |g(t, x)|^2 dx + \eta J(\partial_t \omega). \] (69)

Apply Gronwall lemma to (69) over the interval \([\tau - t, \sigma]\) with \( \sigma \in [\tau - 1, \tau] \) and \( t \geq 1 \), replacing \( \omega \) by \( \partial_{-T} \omega \), to find that, for all \( k \geq R_1 \),
\[
\int_{\mathbb{R}^N} \xi_k |v(\sigma)|^2 dx \leq e^{\alpha_0} e^{-\alpha_0 t} \int_{\mathbb{R}^N} \xi_k |v_{\tau-t}|^2 dx
+ \frac{c e^{\alpha_0}}{k^s} \int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} \left( \|v(r)\|^2_{H^s(\mathbb{R}^N)} + |y(\partial_{-T} \omega)|^2 \right) dr
+ c e^{\alpha_0} \int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} \int_{|x| \geq k} |g(r, x)|^2 dx dr
+ c e^{\alpha_0} \eta \int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} J(\partial_{-T} \omega) dr. \] (70)

By (38), for every \( \eta > 0 \) and \( u_{\tau-t} \in D(\tau-t, \partial_{-T} \omega) \), there exists a \( T_1 = T_1(\tau, \omega, D, \eta) \geq 1 \) such that for all \( t \geq T_1 \),
\[
e^{\alpha_0} e^{-\alpha_0 t} \int_{\mathbb{R}^N} \xi_k |v_{\tau-t}|^2 dx \leq 2e^{\alpha_0} e^{-\alpha_0 t} (\|u_{\tau-t}\|^2 + \|\phi z(\partial_{-T} \omega)\|^2) \leq \frac{\eta}{4}. \] (71)

By (40), there is a \( T_2 = T_2(\tau, \omega, D) \geq T_1 \) such that for all \( t \geq T_2 \),
\[
\int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} \|v(r)\|^2_{H^s(\mathbb{R}^N)} dr \leq c(1 + C(\tau, \omega)).
\]

Then there exists a \( R_2 = R_2(\tau, \omega, D, \eta) \geq R_1 \) such that for all \( t \geq T_2 \) and \( k \geq R_2 \),
\[
\frac{c e^{\alpha_0}}{k^s} \int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} \|v(r)\|^2_{H^s(\mathbb{R}^N)} dr \leq \frac{c(1 + C(\tau, \omega))}{k^s} \leq \frac{\eta}{4}. \] (72)

By (34), we have
\[
\int_{\tau-t}^{\tau} e^{\alpha_0 (r-t)} |y(\partial_{-T} \omega)|^2 dr = \int_{-t}^{0} e^{\alpha_0 r} |y(\partial_r \omega)|^2 dr
\]
If the initial value sequence is bounded in L^{\infty}, we first prove a convergence result of the solution to equations (23) and (24) corresponding the initial value x_n.

4.3. Asymptotical compactness. In order to prove the asymptotical compactness, we first prove a convergence result of the solution to equations (23) and (24) if the initial value sequence is bounded in L^2(\mathbb{R}^N), which is the idea from [26].

Lemma 4.4. Given R > 0, let \( Q_R = \{ x \in \mathbb{R}^N : |x| \leq R, R > 0 \} \) and \( \{ x_n \}_{n=1}^{\infty} \subset L^2(\mathbb{R}^N) \) be an initial values sequence with sup \( \| x_n \| \leq M \) for some fixed \( M > 0 \).

Let \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). Then for almost every \( \sigma \in [\tau - 1, \tau] \), there exits a fixed point \( u_\sigma \in L^2(\mathbb{R}^N) \) such that the sequence \( \{ u_n(\sigma, \tau - 1, \omega, x_n) \}_{n=1}^{\infty} \) has a convergent subsequence with \( u_n(\sigma, \tau - 1, \omega, x_n) \to u_\sigma \) in \( L^2(Q_R) \), where \( u_n(\sigma) \) is a solution to (23) and (24) corresponding the initial value \( x_n \).

Proof. Integrate (47) from \( \tau - 1 \) to \( \sigma \) for \( \sigma \in [\tau - 1, \tau] \) to find

\[
\| v_n(\sigma, \tau - 1, \omega, x_n - \phi(\theta_{\tau - 1}\omega)) \|^2
\]
Lemma 4.5. Let (25)-(29) hold and \( \mathfrak{D} \) be defined by (38). Then the random cocycle \( \varphi \) defined by (37) is \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\)-pullback asymptotically compact, namely, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the sequence \( \{\varphi(t_n, \tau-t_n, \vartheta_{-t_n} \omega, u_{0,n})\}_{n=1}^{\infty} \) has a convergent subsequence in \( L^2(\mathbb{R}^N) \) whenever \( t_n \to \infty \) and \( u_{0,n} \in D(\tau-t_n, \vartheta_{-t_n} \omega) \) with \( D \in \mathfrak{D} \).

Consider that \( g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N)), \phi \in L^2(\mathbb{R}^N) \cap H^s(\mathbb{R}^N), J(\vartheta, \omega) \) is continuous in \( r \in \mathbb{R} \) and \( u_n(t) = v_n(t) + \psi(\vartheta \omega) \). Then by (76), there exists a positive random constant \( C_0(\tau, \omega) \) (independent of \( n \)) such that, for every \( \sigma \in [\tau-1, \tau] \),

\[
\|u_n(\sigma, \tau-1, \omega, x_n)\| \leq C_0(\tau, \omega),
\]

and

\[
\int_{\tau-1}^{\tau} \|u_n(r, \tau-1, \omega, x_n)\|^2_{H^s(\mathbb{R}^N)} dr \leq C_0(\tau, \omega).
\]

By (77), for every \( \sigma \in [\tau-1, \tau] \), there exist \( u_{\sigma} \in L^2(\mathbb{R}^N) \) and a subsequence (for simplicity denoted by itself) by a diagonal process such that

\[
u_n(\sigma, \tau-1, \omega, x_n) \to u_{\sigma} \quad \text{weakly in } L^2(\mathbb{R}^N).
\]

On the other hand, by (78) we deduce that

\[
u_n(\cdot, \tau-1, \omega, x_n) \text{ is uniformly bounded with respect to } n \text{ in } L^2(\tau-1, \tau; H^s(\mathbb{R}^N)).
\]

We now need to define a new space, for \( s \in (0, 1) \),

\[
V = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } Q_0\}.
\]

Then \( V \) is a Hilbert space, and by Theorem 7.1 in [17], the embedding of \( V \hookrightarrow L^2(Q_R) \) is compact. From (80), we have

\[
u_n(\cdot, \tau-1, \omega, x_n) \text{ is uniformly bounded with respect to } n \text{ in } L^2(\tau-1, \tau; V).
\]

Thus by the compact embedding, there is a subsequence (for simplicity labeled by itself) by a diagonal process such that

\[
\{u_n(\tau-1, \omega, x_n)\}_{n=1}^{\infty} \text{ strongly converges in } L^2(\tau-1, \tau; L^2(Q_R)).
\]

Thanks to (79), it follows that, for all most every \( \sigma \in [\tau-1, \tau] \),

\[
u_n(\sigma, \tau-1, \omega, x_n) \to u_{\sigma} \quad \text{strongly in } L^2(Q_R).
\]

In addition, it is obvious that \( u_{\sigma} \) is independent of the choice of \( R \), and consequently (81) holds for any \( R > 0 \).

The following lemma is concerning the asymptotical compactness of random cocycle \( \varphi \) in \( L^2(\mathbb{R}^N) \).

Lemma 4.5. Let (25)-(29) hold and \( \mathfrak{D} \) be defined by (38). Then the random cocycle \( \varphi \) defined by (37) is \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\)-pullback asymptotically compact, namely, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the sequence \( \{\varphi(t_n, \tau-t_n, \vartheta_{-t_n} \omega, u_{0,n})\}_{n=1}^{\infty} \) has a convergent subsequence in \( L^2(\mathbb{R}^N) \) whenever \( t_n \to \infty \) and \( u_{0,n} \in D(\tau-t_n, \vartheta_{-t_n} \omega) \) with \( D \in \mathfrak{D} \).
Proof. Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_{0,n} \in D(\tau - t_n, \vartheta_{-t_n} \omega)$. By Lemma 4.1, there exists a $N \ni N_1 = N_1(\tau, \omega, D) \geq 1$ such that, for all $n \geq N_1$,

$$
\|u(\tau - 1, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n})\|^2 \leq c(1 + C(\tau, \omega)).
$$

(82)

Now replacing $x_n$ by $u(\tau - 1, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n})$ in Lemma 4.4, we find that for any fixed $R > 0$, there is a certain real number $\sigma \in [\tau - 1, \tau], u_\sigma \in L^2(\mathbb{R}^N)$ and up to a subsequence (for simplicity labeled by itself) such that

$$
\|u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u_\sigma\| \leq \eta, \quad \text{in } L^2(\mathbb{Q}_R),
$$

(83)

where $u_\sigma$ is independent of $R > 0$. By the cocycle property of $\varphi$ and (83), it follows that

$$
u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) \to u_\sigma \text{ in } L^2(\mathbb{Q}_R),
$$

(84)

for every $R > 0$.

Let $\eta > 0$ be arbitrary. Then by Lemma 4.3, there exist $N \ni N_2 = N_2(\tau, \omega, D, \eta) \geq N_1$ and $R_1 = R_1(\tau, \omega, D, \eta) \geq N_1$ such that for all $n \geq N_2$ and $R \geq R_1$,

$$
ed^{\mu_n} \int_{|x| \geq R} |u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n})|^2 dx \leq \frac{\eta}{6},
$$

(85)

for this fixed $\sigma \in [\tau - 1, \tau]$, where $N_1$ is as in (82). Since $u_\sigma \in L^2(\mathbb{R}^N)$, there is a number $R_2 = R_2(\eta) \geq R_1$ such that for all $R \geq R_2$,

$$
ed^{\mu_n} \int_{|x| \geq R} |u_\sigma|^2 dx \leq \frac{\eta}{6}.
$$

(86)

By (84), there exists a $N \ni N_3 = N_3(\tau, \omega, D, \eta) \geq N_2$ such that for all $n \geq N_3$ and $R \geq R_2$,

$$
ed^{\mu_n} \int_{|x| \leq R} \|u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u_\sigma\|^2 dx \leq \frac{\eta}{3}.
$$

(87)

Consider that

$$
u(\tau, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) = u(\tau, \sigma, \vartheta_{-\tau} \omega, u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n})).
$$

Then by (94) in the following section, we deduce that, for this fixed $\sigma \in [\tau - 1, \tau]$,

$$
\|u(\tau, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_\sigma)\|^2
\leq e^{(\mu_n - \sigma)} \|u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u_\sigma\|^2
\leq e^{\mu_n} \int_{|x| \leq R} |u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u_\sigma|^2 dx
+ e^{\mu_n} \int_{|x| \geq R} |u(\sigma, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u_\sigma|^2 dx,
$$

(88)

where $R \geq R_2$ and $n \geq N_3$. Consequently, by a combination of (85)-(88), it follows that, for all $n \geq N_3$,

$$
\|u(\tau, \tau - t_n, \vartheta_{-\tau} \omega, u_{0,n}) - u(\sigma, \vartheta_{-\tau} \omega, u_\sigma)\|^2 \leq \eta,
$$

which and (37) end the proof.
5. Continuity of solutions.

5.1. Continuity w.r.t. the initial data in $L^p(\mathbb{R}^N)$. To consider the continuity of solutions, we turn to the difference equation. Let $u_i$ be the solution of problem (23) and (24) corresponding to the initial value $u_{r,i}, i = 1, 2$. Given $\bar{u} = u_1 - u_2$, $\bar{u}$ is the solution to the following difference equation:

$$\frac{d\bar{u}}{dt} + \lambda \bar{u} + (-\Delta)^s \bar{u} + f(t, x, u_1) - f(t, x, u_2) = 0$$

(89)

with the initial value $\bar{u}_r = u_{r,1} - u_{r,2}$ and $0 < s < 1$.

**Lemma 5.1.** (See Bartsch [5], Lemma 3.6.) Assume that $x, y \in \mathbb{R}^N$. Then for any $d \geq 2$, there exists a constant $c_{N,d} > 0$ such that

$$(|x|^{d-2}x - |y|^{d-2}y). (x - y) \geq c_{N,d}|x - y|^d,$$

where $.$ is the inner product in $\mathbb{R}^N$.

For the fractional power Laplacian $(-\Delta)^s \bar{u}$ and the nonlinear term $f$ in (89), we have the following characteristic properties, which is of importance for us to cope with the subsequential proofs.

**Lemma 5.2.** For any $d \geq 2$,

$$( (-\Delta)^s \bar{u}, |\bar{u}|^{d-2}\bar{u}) \geq 0.$$

**Proof.** Note that, by Lemma 5.1 for the case $N = 1$, we have

$$\int_{\mathbb{R}^N} \frac{(\bar{u}(t)(x) - \bar{u}(t)(y))(|\bar{u}(t)(x)|^{d-2}\bar{u}(t)(x) - |\bar{u}(t)(y)|^{d-2}\bar{u}(t)(y))}{|x - y|^{N+2s}} dxdy$$

$$\geq c_{1,d} \int_{\mathbb{R}^N} |\bar{u}(t)(x) - \bar{u}(t)(y)|^d |x - y|^{N+2s} dxdy.$$ 

Therefore we have

$$( (-\Delta)^s \bar{u}, |\bar{u}|^{d-2}\bar{u}) \geq 0,$$

which is the desired. \hfill \Box

**Lemma 5.3.** If $f$ satisfies (25), then for any $d \geq 2$

$$\int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2))|\bar{u}|^{d-2}\bar{u} dx \geq \gamma \|\bar{u}\|_d^{d+p-2} - \|\psi\|_\infty \|\bar{u}\|_d^d.$$ 

(90)

**Proof.** It is clear by (25). \hfill \Box

We are now at the point to prove the continuity of solution to problem (23) and (24) in $L^p(\mathbb{R}^N)$ with respect to the initial value belonging to $L^2(\mathbb{R}^N)$. By Lemma 2.7, this implies the asymptotical compactness of solution in $L^p(\mathbb{R}^N)$.

**Lemma 5.4.** Let (25) hold. Let $u_i(t) = u_i(t, \tau, \omega, u_{r,i})$ be the solution of equations (23) and (24) with respect to the initial value $u_{r,i}, i = 1, 2$, respectively. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > \tau$, then there exists a positive deterministic constant $C_{T-\tau}$ depending only on $\gamma, p$ and $\mu_0$ such that

$$(t - \tau)\| (t - \tau) \tilde{u}(t) \|_p^p \leq C_{T-\tau} \| \bar{u}_r \|^2, \quad t \in (\tau, T),$$
and
\[
\int_\tau^T (r - \tau)^2 \|\pi(r)\|^2_{2p-2} dr \leq C_{T-\tau} \|\bar{u}\|^2,
\]
where \(\pi(t) = u^1(t) - u^2(t)\).

**Proof.** Take the inner product of (89) in \(L^2(\mathbb{R}^N)\) with \(\bar{u}\) to get
\[
\frac{1}{2} \frac{d}{dt} \|\bar{u}\|^2 + \|(\Delta)^{\frac{1}{2}} \bar{u}\|^2 + \int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) \bar{u} dx \leq 0.
\]
(91)

By Lemma 5.3 for the case \(d = 2\), we have
\[
\int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) \bar{u} dx \geq \gamma \|\bar{u}\|^p - \|\psi_1\|_\infty \|\bar{u}\|^2.
\]
(92)

By a combination of (91)-(92), it gives
\[
\frac{d}{dt} \|\bar{u}\|^2 + \|(\Delta)^{\frac{1}{2}} \bar{u}\|^2 + \gamma \|\bar{u}\|^p \leq \mu_0 \|\bar{u}\|^2,
\]
(93)

where \(\mu_0 = 2\|\psi_1\|_\infty\). By using Gronwall lemma in (93) over the interval \((\tau, t)\), we see that
\[
\|\bar{u}(t)\|^2 \leq e^{\mu_0(t-\tau)} \|\bar{u}(\tau)\|^2,
\]
(94)

Integrate (93) over the interval \((\tau, T)\) and then use (94) to find
\[
\int_\tau^T \|\bar{u}(\tau)\|^p d\tau \leq \frac{1}{\gamma} (1 + e^{\mu_0(T-\tau)}) \|\pi_\tau\|^2.
\]
(95)

Then, we get
\[
\int_\tau^T (r - \tau)^p \|\bar{u}(r)\|^p d\tau \leq \frac{1}{\gamma} (T - \tau)^p (1 + e^{\mu_0(T-\tau)}) \|\pi_\tau\|^2 =: C_{T-\tau} \|\pi_\tau\|^2,
\]
(96)

where \(C_{T-\tau} > 0\) is a generic constant, which is increasing with respect to \(T - \tau\). Now, taking the inner product of (89) in \(L^2(\mathbb{R}^N)\) with \(|\bar{u}|^{p-2} \bar{u}|\), using Lemma 5.2, we obtain
\[
\frac{1}{p} \frac{d}{dt} \|\bar{u}\|^p_p + \int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) |\bar{u}|^{p-2} \bar{u} dx \leq 0.
\]
(97)

Here we let \(d = p\) in Lemma 5.3. Combining (97) and (90), we get
\[
\frac{d}{dt} \|\bar{u}\|^p_p + \gamma \|\bar{u}\|^2_{2p-2} \leq \mu_0 \|\bar{u}\|^p_p.
\]
(98)

Multiply (98) by \((t - \tau)^p\) to yield
\[
(t - \tau)^p \frac{d}{dt} \|\bar{u}\|^p_p + \gamma \|(t - \tau)^{\frac{p}{2p-2}} \bar{u}\|^2_{2p-2} \leq \mu_0 p (t - \tau)^p \|\bar{u}\|^p_p.
\]
(99)

Since
\[
(t - \tau)^p \frac{d}{dt} \|\bar{u}\|^p_p = \frac{d}{dt} ||(t - \tau)^{p-1} \bar{u}\|^p_p - p(t - \tau)^{p-1} \|\bar{u}\|^p_p,
\]
(100)
then by (99) and (100) we get, for \( t > \tau \),
\[
\frac{d}{dt} \| (t - \tau) \bar{u} \|_p^p + \gamma \| (t - \tau) \bar{u} \|_{2p-2}^{2p-2} \leq \mu_0 (t - \tau)^p \| \bar{u} \|_p^p + (t - \tau)^{p-1} \| \bar{u} \|_p^p \\
\leq p \left( \mu_0 + \frac{1}{t - \tau} \right) \| (t - \tau) \bar{u} \|_p^p. \tag*{(101)}
\]

Obviously, it follows that
\[
(t - \tau) \frac{d}{dt} \| (t - \tau) \bar{u} \|_p^p \leq C_{T - \tau} \| (t - \tau) \bar{u} \|_p^p, \quad t \in [\tau, T]. \tag*{(102)}
\]

Through integrating (102) over the interval \((\tau, t)\) for every \( t \in (\tau, T) \), by integration by parts for the left term, we get
\[
\int_{\tau}^{t} (r - \tau) \frac{d}{dr} \| (r - \tau) \bar{u}(r) \|_p^p \, dr = (t - \tau) \| (t - \tau) \bar{u} \|_p^p - \int_{\tau}^{t} \| (r - \tau) \bar{u}(r) \|_p^p \, dr \\
\leq C_{T - \tau} \int_{\tau}^{t} \| (r - \tau) \bar{u}(r) \|_p^p \, dr. \tag*{(103)}
\]

Therefore by (96) and (103), it follows that, for any \( t \in (\tau, T) \),
\[
(t - \tau) \| (t - \tau) \bar{u} \|_p^p \leq (C_{T - \tau} + 1) \int_{\tau}^{t} \| (r - \tau) \bar{u}(r) \|_p^p \, dr \leq C_{T - \tau} \| \bar{u}_r \|^2, \tag*{(104)}
\]

which shows that \((A)\) holds true. Multiply (101) by \((t - \tau)^p\) to yield, in conjunction with (104),
\[
(t - \tau)^p \frac{d}{dt} \| (t - \tau) \bar{u}(t) \|_p^p + \gamma \| (t - \tau)^{2p-2} \bar{u}(t) \|_{2p-2}^{2p-2} \leq p \left( \mu_0 (t - \tau)^p + (t - \tau)^{p-1} \right) \| (t - \tau) \bar{u}(t) \|_p^p \\
\leq p \left( \mu_0 (T - \tau)^{p-1} + (T - \tau)^{p-2} \right) (t - \tau) \| (t - \tau) \bar{u}(t) \|_p^p \\
\leq C_{T - \tau} \| \bar{u}_r \|^2, \tag*{(105)}
\]

for all \( t \in (\tau, T) \) and \( p \geq 2 \) (noting that \( p - 2 \geq 0 \)). Note that by integration by parts, we have
\[
\int_{\tau}^{T} (r - \tau)^p \frac{d}{dr} \| (r - \tau) \bar{u}(r) \|_p^p \, dr = (T - \tau)^p \| (T - \tau) \bar{u}(T) \|_p^p \\
- p \int_{\tau}^{T} (r - \tau)^{p-1} \| (r - \tau) \bar{u}(r) \|_p^p \, dr. \tag*{(106)}
\]

Therefore, integrating the last inequality of (105) over the interval \((\tau, T)\), along with (106) and (104), it follows that
\[
\gamma \int_{\tau}^{T} \| (r - \tau)^b \bar{u}(r) \|_{2p-2}^{2p-2} \, dr \leq C_{T - \tau} (T - \tau) \| \bar{u}_r \|^2 + p \int_{\tau}^{T} (r - \tau)^{p-1} \| (r - \tau) \bar{u}(r) \|_p^p \, dr
\]
\[
\leq (T - \tau)\|\bar{u}_r\|^2 + pC_{T - \tau}\|\bar{u}_r\|^2 \int_{\tau}^{T} (r - \tau)^{p - 2} dr
\]
\[
\leq C_{T - \tau}\|\bar{u}_r\|^2,
\] (107)

where \( b = \frac{2p}{2p - 2} \). Then we conclude the proof. \( \square \)

According to Lemma 5.4 and (94), we immediately get Theorem 5.5. Under the conditions (25)-(29), the solution \( u \) of problem (23) and (24) is bi-spatial \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-continuous with respect to the initial data. Namely, for each fixed \( \tau \in \mathbb{R}, \omega \in \Omega \), the solution \( u(t, \tau, \omega, \cdot) \) is continuous from \( L^2(\mathbb{R}^N) \) to \( L^2(\mathbb{R}^N) \) for all \( t \geq \tau \), and in addition \( u(t, \tau, \omega, \cdot) \) is continuous from \( L^2(\mathbb{R}^N) \) to \( L^p(\mathbb{R}^N) \) for all \( t > \tau \).

5.2. Continuity w.r.t. the initial data in \( H^s(\mathbb{R}^N) \). We first present a lemma in [55].

Lemma 5.6. Let \( y, g \) and \( h \) be three nonnegative and locally integrable functions on \( \mathbb{R} \) such that \( \frac{dy}{dt} \) is also locally integrable and
\[
\frac{dy(t)}{dt} + \nu y(t) + g(t) \leq h(t), \quad t \in \mathbb{R},
\]
for some constant \( \nu \geq 0 \). Then

(i) for arbitrary \( r > 0 \) and \( \tau \in \mathbb{R} \),
\[
y(\tau) \leq \frac{1}{\nu} \int_{\tau - r}^{\tau} e^{\nu(t - \tau)} y(t) dt + \int_{\tau - r}^{\tau} e^{\nu(t - \tau)} h(t) dt.
\]

(ii) for arbitrary \( r, \varepsilon > 0 \) and \( \sigma \in [\tau - r, \tau] \),
\[
y(\sigma) + \int_{\tau - r}^{\tau} e^{\nu(t - \tau)} g(t) dt \leq \frac{e^{\nu r} + 1}{\varepsilon} \int_{\tau - r - \varepsilon}^{\tau} e^{\nu(t - \tau)} y(t) dt + (e^{\nu r} + 2) \int_{\tau - r - \varepsilon}^{\tau} e^{\nu(t - \tau)} h(t) dt.
\]

In particular, (i) and (ii) hold for \( \nu = 0 \).

Lemma 5.7. Let (25) and (26) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the solution of problem (23) and (24) satisfies
\[
\|v(t, \tau, \omega, v_\tau)\|_p^p \leq G(t, \tau, \omega, u_\tau),
\] (108)
and
\[
\int_{\tau}^{t} \|u(r, \tau, \omega, u_\tau)\|_{2p - 2}^{2p - 2} dr \leq G(t, \tau, \omega, u_\tau),
\] (109)
where
\[
G(t, \tau, \omega, u_\tau) \leq \frac{c}{t - \tau} \left( \int_{\tau}^{t} \|g(r, \cdot)\|^2 dr + \int_{\tau}^{t} J(\vartheta, \omega) dr + \|u_\tau\|^2 + c|y(\vartheta, \omega)|^2 \right)
\]
\[ + c \int_{\tau}^{t} \|g(r, \cdot, \cdot)\|^2 dr + c \int_{\tau}^{t} J(\vartheta, \omega) dr. \]  

(110)

**Proof.** By (47), we have
\[ \frac{d}{dt} \|v\|^2 + c\|v\|^p \leq c\|g(t, \cdot, \cdot)\|^2 + cJ(\vartheta, \omega), \]  
where \(J(\vartheta, \omega)\) is as in (33). Integrating (111) from \(\tau\) to \(t\), we get
\[ \int_{\tau}^{t} \|v(r)\|^p dr \leq c \int_{\tau}^{t} (\|g(r, \cdot, \cdot)\|^2 + J(\vartheta, \omega)) dr + \|v_\tau\|^2 \]
\[ \leq c \int_{\tau}^{t} \|g(r, \cdot, \cdot)\|^2 dr + \int_{\tau}^{t} J(\vartheta, \omega) dr + \|y(\vartheta, \omega)\|^2 + \|u_r\|^2). \]  

(112)

Take the inner product of (36) in \(L^2(\mathbb{R}^N)\) with \(|v|^{p-2}v\), we obtain
\[ \frac{1}{p} \frac{d}{dt} \|v\|^p_p + \lambda\|v\|^p_p + ((-\Delta)^s v, |v|^{p-2}v) + \int_{\mathbb{R}^N} f(t, x, u)|v|^{p-2}v dx \]
\[ = \int_{\mathbb{R}^N} g(t, x)|v|^{p-2}v dx - (-\Delta)^s z(\vartheta, \omega), |v|^{p-2}v. \]  

(113)

By Lemma 5.1, it gives
\[ ((-\Delta)^s v, |v|^{p-2}v) \]
\[ = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(|v(x)|^{p-2}v(x) - |v(y)|^{p-2}v(y))}{|x - y|^{N+2s}} dxdy \]
\[ \geq c_{1,p} C(N, s) \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+2s}} dxdy \geq 0. \]  

(114)

For the last term on the right hand side of (113), we have
\[ ((-\Delta)^s z(\vartheta, \omega), |v|^{p-2}v) \]
\[ = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{y(\vartheta, \omega)(\phi(x) - \phi(y))(|v(x)|^{p-2}v(x) - |v(y)|^{p-2}v(y))}{|x - y|^{N+2s}} dxdy \]
\[ = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{y(\vartheta, \omega)(\phi(x) - \phi(y))|v(x)|^{p-2}v(x)}{|x - y|^{N+2s}} dxdy \]
\[ - \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{y(\vartheta, \omega)(\phi(x) - \phi(y))|v(y)|^{p-2}v(y)}{|x - y|^{N+2s}} dxdy. \]  

(115)

Tanks to \((-\Delta)^s \phi \in L^2(\mathbb{R}^N),\) by Hölder inequality, we have
\[ \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{y(\vartheta, \omega)(\phi(x) - \phi(y))|v(t)(x)|^{p-2}v(t)(x)}{|x - y|^{N+2s}} dxdy \]
On the other hand, we have

\[
\int f(t, x, u)|v|^{r-2}v \geq \frac{\gamma}{2p} \|v\|^{2p-2} - 2 \int (|\psi_1(t, x)| + |\psi_2(t, x)|)|v|^p dx
- \int 2|\psi_1(t, x)| + 3|\psi_2(t, x)| |\phi(x)y(\vartheta_t\omega)|^2|v|^{r-2} dx
- \int (c|\psi_2(t, x)|^p + \frac{\gamma}{2})|\phi(x)y(\vartheta_t\omega)|^p|v|^{r-2} dx
\geq \frac{\gamma}{2p+1}\|v\|^{2p-2} - c\|v\|^p - c|\phi(\vartheta_t\omega)|^p - c|y(\vartheta_t\omega)|^{2p-2}. 
\]

Then it follows from (115) and (116) that

\[
((\Delta)^*z(\vartheta_t\omega), |v|^{r-2}v) \leq \frac{\gamma}{2p+1}\|v\|^{2p-2} + c|y(\vartheta_t\omega)|^2. 
\]

By (43), and using Young inequality repeatedly, we infer that, along with \( \phi \in L^2(\mathbb{R}^N) \cap L^{2p-2}(\mathbb{R}^N) \),

\[
\int g(t, x)|v|^{r-2}v dx \leq \frac{\gamma}{2p+1}\|v\|^{2p-2} + c\|g(t, .)\|^2. 
\]

Incorporate (114) and (117)-(119) into (113) to yield

\[
\frac{d}{dt}\|v\|_p^p + c\|v\|_p^{2p-2} \leq c(\|v\|_p^p + \|g(t, .)\|^2) + cJ(\vartheta_t\omega). 
\]

By applying Lemma 5.6 (ii) for the cases \( \nu = 0 \) and \( \varepsilon = \frac{t-\tau}{\tau} \) to (120), we get

\[
\|v(t)\|_p^p + \int_{\frac{t-\tau}{\tau}}^{t} \|v(r)\|_p^{2p-2} dr \leq \frac{c}{t-\tau} \int_{\tau}^{t} \|v(r)\|_p^p dr + \int_{\tau}^{t} \|g(r, .)\|^2 \quad + c \int_{\tau}^{t} J(\vartheta_t\omega) dr, 
\]

which implies that

\[
\|v(t)\|_p^p + \int_{\frac{t-\tau}{\tau}}^{t} \|u(r)\|_p^{2p-2} dr \leq \frac{c}{t-\tau} \int_{\tau}^{t} \|v(r)\|_p^p dr + \int_{\tau}^{t} \|g(r, .)\|^2 \quad + c \int_{\tau}^{t} J(\vartheta_t\omega) dr, 
\]

which and (112) together give

\[
\|v(t)\|_p^p + \int_{\frac{t-\tau}{\tau}}^{t} \|u(r)\|_p^{2p-2} dr 
\]
\[
\frac{c}{t-\tau} \left( \int_{\tau}^{t} \|g(r,\cdot)\|^2 dr + \int_{\tau}^{t} J(\vartheta,\omega) dr + \|u_{\tau}\|^2 + c|y(\vartheta,\omega)|^2 \right) \\
+ c \int_{\tau}^{t} \|g(r,\cdot)\|^2 dr + c \int_{\tau}^{t} J(\vartheta,\omega) dr,
\]
which completes the proof. \(\square\)

**Remark 1.** Lemma 5.7 manifests that for every \(t > 0, \tau \in \mathbb{R}\) and \(\omega \in \Omega\), the mapping \(\varphi(t,\tau,\omega,) = u(t+\tau,\tau,\omega,.)\) maps \(L^2(\mathbb{R}^N)\) into \(L^p(\mathbb{R}^N)\), namely, it makes sense to investigate the dynamics of the random cocyle \(\varphi\) on \(L^p(\mathbb{R}^N)\).

**Lemma 5.8.** Let (25) and (26) hold. Let \(u_i(t) = u_i(t,\tau,\omega, u_{\tau,i})\) be the solution of problem (23) and (24) with respect to the initial value \(u_{\tau,i}, i = 1, 2\). Given \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), there exist a positive deterministic constant \(C_{i-\tau}^1\) depending on \(p, \lambda\) and \(\mu_0\), and a random constant \(C_{i-\tau}^2\) depending on \(p, \lambda, \mu_0\) and \(G(t,\tau,\omega, u_{\tau,i})\) such that, for all \(t > \tau\)

\[
\|(-\Delta)^{1/2} \bar{u}(t,\tau,\omega, \bar{u}_{\tau})\| \leq C_{i-\tau}^1 \|ar{u}_{\tau}\|^2 + C_{i-\tau}^2 \|ar{u}_{\tau}\|^{\frac{2}{p-1}},
\]
where \(\bar{u}(t) = u_1(t) - u_2(t)\) and \(G(t,\tau,\omega, u_{\tau,i})\) is as in (110).

**Proof.** Taking the inner product of (89) in \(L^2(\mathbb{R}^N)\) with \(\bar{u}_i\), we get

\[
\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{1/2} \bar{u}_i\|^2 + \|ar{u}_i\|^2 = -\lambda \int_{\mathbb{R}^N} \bar{u}_i \bar{u}_i dx - \int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) \bar{u}_i dx
\]

\[
\leq \frac{1}{4} \|ar{u}_i\|^2 + c\|ar{u}\|^2 + \int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) \bar{u}_i dx. \tag{122}
\]

By employing first (26), then Young inequality and Hölder inequality, it follows that

\[
\left| \int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2)) \bar{u}_i dx \right| \leq c \int_{\mathbb{R}^N} (1 + |u_1|^{p-2} + |u_2|^{p-2}) |\bar{u}| |\bar{u}_i| dx
\]

\[
\leq \frac{1}{4} \|ar{u}_i\|^2 + c\|ar{u}\|^2 + c \int_{\mathbb{R}^N} (|u_1|^{2p-4} + |u_2|^{2p-4}) |\bar{u}|^2 dx
\]

\[
\leq \frac{1}{4} \|ar{u}_i\|^2 + c\|ar{u}\|^2 + c(\|u_1\|^{2p-4}_{2p-2} + \|u_2\|^{2p-4}_{2p-2}) \|ar{u}\|_{2p-2}^2. \tag{123}
\]

Combine (122) and (123) to find that, replacing \(t\) by \(r\),

\[
\frac{d}{dr} \|(-\Delta)^{1/2} \bar{u}(r)\|^2 \leq c \|ar{u}(r)\|^2 + c(\|u_1(r)\|^{2p-4}_{2p-2} + \|u_2(r)\|^{2p-4}_{2p-2}) \|ar{u}(r)\|_{2p-2}^2. \tag{124}
\]

Multiplying (124) by \((r - \frac{t+\tau}{2})^{\frac{2p}{p+1}}\) with \(r \in [\frac{t+\tau}{2}, t]\), we get

\[
(r - \frac{t+\tau}{2})^{\frac{2p}{p+1}} \frac{d}{dr} \|(-\Delta)^{1/2} \bar{u}(r)\|^2 \leq c(r - \frac{t+\tau}{2})^{\frac{2p}{p+1}} \|ar{u}(r)\|^2
\]

\[
+ c(r - \frac{t+\tau}{2})^{\frac{2p}{p+1}} (\|u_1(r)\|^{2p-4}_{2p-2} + \|u_2(r)\|^{2p-4}_{2p-2}) \|ar{u}(r)\|_{2p-2}^2
\]

\[
\leq c(t - \tau)^{\frac{2p}{p+1}} \|ar{u}(r)\|^2
\]

\[
+ c(t - \tau)^{\frac{2p}{p+1}} \|ar{u}(r)\|^2 \|u_1(r)\|_{2p-2}^2 + \|u_2(r)\|_{2p-2}^2. \tag{125}
\]
Note that by integration by parts, we have

$$
\int_{t-\tau}^{t} \left( r - \frac{t + \tau}{2} \right)^{\frac{2p}{p-1}} \frac{d}{dr} \|(-\Delta)^{\frac{p}{2}} u(r)\|^{2} dr = \left( \frac{t - \tau}{2} \right)^{\frac{2p}{p-1}} \|(-\Delta)^{\frac{p}{2}} u(r)\|^{2}
$$

\[ - \frac{2p}{p-1} \int_{t-\tau}^{t} \left( r - \frac{t + \tau}{2} \right)^{\frac{2p}{p-1}} \|(-\Delta)^{\frac{p}{2}} u(r)\|^{2} dr. \tag{126} \]

Therefore integrating (125) over the interval \([\frac{t-\tau}{2}, t]\), we obtain, along with (126),

$$
\left( \frac{t - \tau}{2} \right)^{\frac{2p}{p-1}} \|(-\Delta)^{\frac{p}{2}} u(t)\|^{2}
\leq \frac{2p}{p-1} \left( \frac{t - \tau}{2} \right)^{\frac{2p}{p-1}} \int_{\frac{t-\tau}{2}}^{t} \left\| (-\Delta)^{\frac{p}{2}} u(r) \right\|^{2} ds + c(t - \tau)^{\frac{2p}{p-1}} \int_{\frac{t-\tau}{2}}^{t} \|u(r)\|^{2} dr
$$

\[ + c \int_{\frac{t-\tau}{2}}^{t} \left( r - \tau \right)^{\frac{2p}{p-1}} \left\| \bar{u}(r) \right\|_{2p-2} \left( \|u_{1}(r)\|^{2p-4}_{2p-2} + \|u_{2}(r)\|^{2p-4}_{2p-2} \right) dr, \tag{127} \]

where we use \((r - \tau)^{\frac{2p}{p-1}} \geq (r - \frac{t - \tau}{2})^{\frac{2p}{p-1}}\) for any \(r \in [\frac{t+\tau}{2}, t]\). Integrating (93) from \(\tau\) to \(t\), and in conjunction with (94), we get

$$
\int_{\tau}^{t} \left\| (-\Delta)^{\frac{p}{2}} u(r) \right\|^{2} dr \leq \|\bar{u}(\tau)\|^{2} + \mu_{0} \int_{\tau}^{t} \|\bar{u}(r)\|^{2} dr \leq (1 + e^{\mu_{0}(t-\tau)}) \|\bar{u}_{\tau}\|^{2},
$$

which gives, for all \(t > \tau\),

$$
\frac{2p}{p-1} \left( \frac{t - \tau}{2} \right)^{\frac{2p}{p-1}} \int_{\frac{t-\tau}{2}}^{t} \left\| (-\Delta)^{\frac{p}{2}} u(r) \right\|^{2} dr \leq c(t - \tau)^{\frac{2p}{p-1}} (1 + e^{\mu_{0}(t-\tau)}) \|\bar{u}_{\tau}\|^{2}. \tag{128} \]

In terms of (94), we also have

$$
c(t - \tau)^{\frac{2p}{p-1}} \int_{\frac{t-\tau}{2}}^{t} \|\bar{u}(r)\|^{2} dr \leq c(t - \tau)^{\frac{2p}{p-1}} e^{\mu_{0}(t-\tau)} \|\bar{u}_{\tau}\|^{2}, \quad t > \tau. \tag{129} \]

Note that by Hölder inequality, it follows that

$$
\int_{\frac{t-\tau}{2}}^{t} \left( r - \tau \right)^{\frac{2p}{p-1}} \|\bar{u}(r)\|_{2p-2} \left( \|u_{1}(r)\|^{2p-4}_{2p-2} + \|u_{2}(r)\|^{2p-4}_{2p-2} \right) dr
$$

\[ \leq c \left( \int_{\frac{t-\tau}{2}}^{t} \left( \|u_{1}(r)\|^{2p-2}_{2p-2} + \|u_{2}(r)\|^{2p-2}_{2p-2} \right) dr \right)^{\frac{p}{p-2}}
$$

\[ \times \left( \int_{\tau}^{t} \left( r - \tau \right)^{2p} \|\bar{u}(r)\|_{2p-2} dr \right)^{\frac{1}{p-1}}. \tag{130} \]

By Lemma 5.7, we have
\[
\int_{r}^{t} (\|u_1(r)\|^{2p-2}_2 + \|u_2(r)\|^{2p-2}_2)dr \leq G(t, \tau, \omega, u_{r,1}) + G(t, \tau, \omega, u_{r,2}),
\]
where \(G(t, \tau, \omega, u_{r,i})(i = 1, 2)\) is given by (110). By Lemma 5.4, we find that, there exist a deterministic constant \(C_{t-r}\) such that
\[
\int_{r}^{t} (r - \tau)^{2p}\|\bar{u}(r)\|^{2p-2}_2 dr \leq C_{t-r}\|\bar{u}_r\|^2_2,
\]
(132)

By a combination of (130) to (132), it follows that
\[
\int_{r}^{t} (r - \tau)^{2p}\|\bar{u}(r)\|^{2p-2}_2 (\|u_1(r)\|^{2p-4}_2 + \|u_2(r)\|^{2p-4}_2)dr
\leq c(G(t, \tau, \omega, u_{r,1}) + G(t, \tau, \omega, u_{r,2})) \frac{\bar{p}}{2} (C_{t-r}) \frac{1}{p-1} \|\bar{u}_r\|^2_2.
\]
(133)

Incorporate (128), (129) and (133) into (127) to find
\[
\left(\frac{t - \tau}{2}\right) \frac{\bar{p}}{2} \|(-\Delta)^{2p}\bar{u}(t)\|^2 \leq c\left(\frac{t - \tau}{2}\right) \frac{\bar{p}}{2} (1 + \mu_0 e^{\mu_0(t-\tau)}) + c (t - \tau)^{2p} e^{\mu_0(t-\tau)} \|\bar{u}_r\|^2 + c(G(t, \tau, \omega, u_{r,1}) + G(t, \tau, \omega, u_{r,2})) \frac{\bar{p}}{2} (C_{t-r}) \frac{1}{p-1} \|\bar{u}_r\|^2_2,
\]
and therefore we have
\[
\|(-\Delta)^{2p}\bar{u}(t)\|^2 \leq \frac{c}{t - \tau} (1 + e^{\mu_0(t-\tau)}) \|\bar{u}_r\|^2 + \frac{c e^{\mu_0(t-\tau)}}{t - \tau} \|\bar{u}_r\|^2 + c(G(t, \tau, \omega, u_{r,1}) + G(t, \tau, \omega, u_{r,2})) \frac{\bar{p}}{2} (C_{t-r}) \frac{1}{p-1} \|\bar{u}_r\|^2_2,
\]
which is the desired result.

In terms of Lemma 5.8 and (94), we have the continuity of solutions in the fractional \(H^s(\mathbb{R}^N)\) for \(s \in (0, 1)\).

**Theorem 5.9.** Under the conditions (25)-(29), the solution \(u\) of problem (23) and (24) is bi-uniform \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\)-continuous with respect to the initial data.

6. **Measurability of solution in the sample \(\omega\).** In this section, we study the measurability of solution with respect to the sample \(\omega\), which is a crucial condition to prove the measurability of pullback attractor in the regular space \(L^p(\mathbb{R}^N)\) and \(H^s(\mathbb{R}^N)\). To this end, the topological property and structure of the sample space \(\Omega\) are analysed and furthermore the exponential growth property of the Ornstein-Uhlenbeck process defined by (31) is substantially employed.

6.1. **Properties of the sample space \(\Omega\).** For \(k \in \mathbb{N}\), define the set
\[
\Omega_k = \{\omega \in \Omega : |\omega(t)| \leq ke^{|t|}, t \in \mathbb{R}\}, \quad \forall k \in \mathbb{N},
\]
(134)
where \(0 < \zeta(2p-2) < \lambda\). It is easy to check that \(\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k\) and \(\Omega_k \in \mathcal{F}\).

Let \(\mathcal{F}_{\Omega_k}\) be the trace \(\sigma\)-algebra of \(\mathcal{F}\) with respect to \(\Omega_k\), and let \(B_{\Omega_k}(\omega_0, r), \omega_0 \in \Omega_k, r > 0\) be a ball in \(\Omega_k\). This ball can be generated by \(B_{\Omega}(\omega_0, r) \cap \Omega_k\), where \(B_{\Omega}(\omega_0, r)\) is a ball in \(\Omega\), namely, \(B_{\Omega_k}(\omega_0, r) = B_{\Omega}(\omega_0, r) \cap \Omega_k\). The same is true for
all open sets in $\Omega_k$. Therefore $F_{\Omega_k}$ is just the Borel $\sigma$-algebra of $\Omega_k$. Furthermore, since $\Omega_k \in F$, then we have $F_{\Omega_k} \subset F$.

The following is concerning some important properties about $\Omega_k, k \in \mathbb{N}$, see [9, 10].

**Lemma 6.1.** (i) For every $k \in \mathbb{N}$, the mapping $\Omega_k \ni \omega \to y(\theta_t \omega)$ is uniformly continuous on any $t$-bounded intervals, where $y(\theta_t \omega)$ is the Ornstein-Uhlenbeck process defined by (31).

(ii) For every $k \in \mathbb{N}$, the space $(\Omega_k, \rho)$ is a Polish space, namely, a completely separable metric space, where $\rho$ is the Fréchet metric defined by

$$
\rho(\omega_1, \omega_2) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|\omega_1 - \omega_2\|_n}{1 + \|\omega_1 - \omega_2\|_n},
$$

where $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$, see [9, 10].

To prove the continuity of solutions in $\omega$, we also need the following lemma about the property of $J(\theta_t \omega)$, where $J(\theta_t \omega)$ is as in (33). The proof is similar to [10].

**Lemma 6.2.** For every $k \in \mathbb{N}$, there exists a constant $L > 0$ depending only on $p$ and $k$ such that

$$
J(\theta_t \omega) = |y(\theta_t \omega)|^2 + |y(\theta_t \omega)|^p + |y(\theta_t \omega)|^{2p-2} \leq Le^{\lambda|t|}, \quad t \in \mathbb{R},
$$

for any $\omega \in \Omega_k$.

Let $\Lambda$ be a separable Banach space. Then the measurability of a mapping $\varphi$ from $\Omega$ to $\Lambda$ is determined by the sample space $\Omega_k$. In particular, we have

**Lemma 6.3.** Let $\varphi$ be a mapping from $\Omega$ to $\Lambda$ such that $\varphi : \Omega_k \to \Lambda$ is $(F_{\Omega_k}, B(\Lambda))$-measurable for every fixed $k \in \mathbb{N}$, then $\varphi : \Omega \to \Lambda$ is $(F, B(\Lambda))$-measurable.

**Proof.** Note that $\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k$. Then for any open set $O \in B(\Lambda)$, we have

$$
\varphi^{-1}(O) = \{\omega \in \Omega : \varphi(\omega) \in O\} = \bigcup_{k \in \mathbb{N}} \{\omega \in \Omega_k : \varphi(\omega) \in O\}.
$$

Let $D_k = \{\omega \in \Omega_k : \varphi(\omega) \in O\}$. Since $\varphi$ is $(F_{\Omega_k}, B(\Lambda))$-measurable, then $D_k \in F_{\Omega_k}$. Consider that $F_{\Omega_k} \subset F$. Then $\varphi^{-1}(O) = \bigcup_{k \in \mathbb{N}} D_k \in F$. Thus $\varphi : \Omega \to \Lambda$ is $(F, B(\Lambda))$-measurable.

### 6.2. Continuity w.r.t. the sample $\omega$ in $L^2(\mathbb{R}^N)$.

In order to obtain a measurable attractor in $L^2(\mathbb{R}^N)$, it is necessary to prove the measurability of $\varphi(t, \tau, x)$ with respect to $F$ in the target space, for every fixed $t > 0, \tau \in \mathbb{R}, x \in L^2(\mathbb{R}^N)$. Nevertheless, this should be achieved by the continuous dependence of solutions on $\omega \in \Omega_k$ in the initial space $L^2(\mathbb{R}^N), k \in \mathbb{N}$.

Given $\omega_1, \omega_2 \in \Omega_k$, for any fixed $t \geq \tau, \tau \in \mathbb{R}$ and $v_1, v_2 \in L^2(\mathbb{R}^N)$, denote by $v_1(t, \tau, \omega_1, v_1)$ and $v_2(t, \tau, \omega_2, v_2)$ the solutions of problem (23) and (24) at the samples $\omega_1$ and $\omega_2$, respectively. Let $V = v_1 - v_2$. Then $V$ is the solution to the following difference equation:

$$
\frac{dV}{dt} + \lambda V + (-\Delta)^s V + f(t, x, v_1 + z(\theta_t \omega_1)) - f(t, x, v_2 + z(\theta_t \omega_2)) = -(-\Delta)^s(z(\theta_t \omega_1) - z(\theta_t \omega_2))
$$

with the initial value $V_0 = 0$.

For convenience, we write $\tilde{y} = y(\theta_t \omega_1) - y(\theta_t \omega_2)$ and $\tilde{U} = u_1(t, \tau, \omega_1, v_1) - u_2(t, \tau, \omega_2, v_2)$. Then we have
Lemma 6.4. Let $t \geq \tau, \tau \in \mathbb{R}$ and $u_{\tau} \in L^2(\mathbb{R}^N)$. Suppose that $u(t, \tau, \omega, u_{\tau})$ is the solution to problems (23) and (24). Then the mapping $\omega \rightarrow u(t, \tau, \omega, u_{\tau})$ is continuous from $(\Omega_k, \rho)$ to $L^2(\mathbb{R}^N)$ for each $k \in \mathbb{N}$.

Proof. By (136), we have

$$\frac{1}{2} \frac{d}{dt} \|V\|^2 + \lambda\|V\|^2 + \|(-\Delta)^{\frac{3}{2}} V\|^2$$

$$+ \int_{\mathbb{R}^N} \left( f(t, x, v_1 + z(\partial_1 \omega_1)) - f(t, x, v_2 + z(\partial_2 \omega_2)) \right) V \, dx$$

$$= -\left( (-\Delta)^{\frac{3}{2}}(z(\partial_1 \omega_1) - z(\partial_2 \omega_2)), V \right).$$

(137)

For the nonlinearity, by (25) and (26), we have

$$(f(t, x, u_1) - f(t, x, u_2))V$$

$$= (f(t, x, u_1) - f(t, x, u_2))\bar{U} - (f(t, x, u_1) - f(t, x, u_2))\bar{\phi}(x)$$

$$\geq \gamma|\bar{U}|^p - |\psi_1(t, x)|\bar{U}|^2 - |\psi_2(t, x)|\bar{U}(1 + |u_1|^{p-2} + |u_2|^{p-2})\bar{\phi}(x)|$$

$$\geq \gamma|\bar{U}|^p - (\psi_1(t, x) + \psi_2(t, x))|\bar{U}|^2 - \psi_2(t, x)|\bar{\phi}(x)|$$

$$- \psi_2(t, x)\bar{U}(|u_1|^{p-2} + |u_2|^{p-2})\bar{\phi}(x)|,$$

(138)

where $u_i = v_i + z(\partial_i \omega_i)$, $i = 1, 2$ and $\bar{U} = \bar{V} + \bar{\phi}$. By the generalized Young inequality:

$$|abc| \leq |a|^p + |b|^q + |c|^r$$

$$\text{if } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \text{ for any } p, q, r > 1,$$

(139)

we deduce that

$$\bar{U}(|u_1|^{p-2} + |u_2|^{p-2})\bar{\phi}(x)$$

$$\leq (|u_1|^{p-1} + |u_2|^{p-1} + |u_1||u_2|^{p-2} + |u_2||u_1|^{p-2})\bar{\phi}(x)|\bar{\phi}(x)|$$

$$\leq c(|u_1|^{p} + |u_2|^{p} + |\bar{\phi}|^{p})\bar{\phi}.$$  

(140)

Therefore by (138) and (140), it follows that

$$(f(t, x, u_1) - f(t, x, u_2))\bar{V} \geq \gamma|\bar{U}|^p - (\psi_1(t, x) + \psi_2(t, x))|\bar{U}|^2 - \psi_2(t, x)|\bar{\phi}(x)|$$

$$- c\psi_2(t, x)(|u_1|^{p} + |u_2|^{p} + |\bar{\phi}|^{p})\bar{\phi},$$

(141)

from which and $\phi \in L^p(\mathbb{R}^N), \psi_i \in L^\infty(\mathbb{R}^{N+1}), i = 1, 2$, we derive

$$\int_{\mathbb{R}^N} (f(t, x, u_1) - f(t, x, u_2))\bar{V} \, dx \geq \gamma\|\bar{U}\|^p - c\|\bar{V}\|^2 - c|\bar{\phi}|^2$$

$$- c(\|u_1\|_p^p + \|u_2\|_p^p + \|\bar{\phi}\|_p^p)|\bar{\phi}|.$$  

(142)

For the term on the right hand side of (137), we have

$$\left| -((-\Delta)^{\frac{3}{2}}(z(\partial_1 \omega_1) - z(\partial_2 \omega_2)), \bar{V}) \right|$$

$$= \left| \frac{1}{2} C(N, s) \int_{\mathbb{R}^{2N}} \frac{\bar{\phi}(x) - \bar{\phi}(y))|\bar{V}(t)(x) - \bar{V}(t)(y)|}{|x - y|^{N+2s}} \, dx \, dy \right|$$

$$\leq \frac{1}{4} C(N, s) \int_{\mathbb{R}^{2N}} \frac{|\bar{V}(t)(x) - \bar{V}(t)(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.$$
By a combination of (137), (142) and (143), it follows that

\[
\frac{1}{4} C(N, s) |\bar{y}|^2 \int_{\mathbb{R}^2 \times N} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy 
\leq \frac{1}{2} (-\Delta) \bar{z} \bar{V}^2 + c|\bar{y}|^2. \tag{143}
\]

By a combination of (143), (142) and (143), it follows that

\[
d \frac{d}{dt} \|V\|^2 + \lambda \|V\|^2 + \|(-\Delta) \bar{z} \bar{V}\|^2 + \gamma \|\bar{U}\|^p_p \leq c \|\bar{V}\|^2 + c|\bar{y}|^2 + c(\|u_1\|^p_p + \|u_2\|^p_p)|\bar{y}|. \tag{144}
\]

By using Gronwall lemma to (144) over the intervals \([\tau, t]\), we get

\[
\|V(t)\|^2 + e^{ct} \int_{\tau}^{t} e^{-c\tau} (\|(-\Delta) \bar{z} \bar{V}(r)\|^2 + \lambda \|\bar{V}(r)\|^2) \, dr + \gamma e^{ct} \int_{\tau}^{t} e^{-c\tau} \|\bar{U}(r)\|^p_p \, dr 
\leq ce^{ct} \int_{\tau}^{t} e^{-c\tau} |\bar{y}(r)|^2 \, dr + ce^{ct} \int_{\tau}^{t} e^{-c\tau} (\|u_1(r)\|^p_p + \|u_2(r)\|^p_p)|\bar{y}(r)| \, dr. \tag{145}
\]

Due to the definition of \((\Omega_k, \rho)\), we know that if \(\rho(\omega_2, \omega_1) \to 0\), we have \(\omega_2 \to \omega_1\) on any \(t\)-bounded intervals. And hence by Lemma 6.1, the mapping \(y(\bar{z}, \omega) : \Omega_k \ni \omega \to \mathbb{R}\) is uniformly continuous on the arbitrary bounded intervals \([\tau, t]\), for every \(k \in \mathbb{N}\), and therefore we deduce that \(ce^{ct} \int_{\tau}^{t} e^{-c\tau} |\bar{y}(r)|^2 \, dr\) converges to zero as \(\omega_2 \to \omega_1\) in \((\Omega_k, \rho)\). On the other hand, in terms of (112), in conjunction with Lemma 6.2, we find that, for \(\omega_i \in \Omega_k, i = 1, 2\),

\[
\int_{\tau}^{t} e^{-c\tau} \|u_1(r)\|^p_p \, dr \leq e^{-c\tau} \int_{\tau}^{t} \|u_1(r)\|^p_p \, dr 
\leq ce^{-c\tau} (\int_{\tau}^{t} \|g(r, \cdot)\|^2 \, dr + \int_{\tau}^{t} J(\bar{z}, \omega_i) \, dr + \|u_i\|^2 + c|y(\bar{z}, \omega_i)|^2)
\leq ce^{-c\tau} (\int_{\tau}^{t} \|g(r, \cdot)\|^2 \, dr + L \int_{\tau}^{t} e^{|\tau|} \, dr + \|u_i\|^2 + cLe^{|\tau|})
=: M(t, \tau), \tag{146}
\]

which is independent of \(\omega_i, i = 1, 2\). In view of (146) and by Lebesgue dominated convergence, we have

\[
ce^{ct} \int_{\tau}^{t} e^{-c\tau} (\|u_1(r)\|^p_p + \|u_2(r)\|^p_p)|\bar{y}(r)| \, dr \to 0,
\]
as \(\omega_2 \to \omega_1\) in \((\Omega_k, \rho)\). Consequently, from (145) we get

\[
\|\bar{V}(t)\|^2 + e^{ct} \int_{\tau}^{t} e^{-c\tau} (||(-\Delta) \bar{z} \bar{V}(r)||^2 + \lambda \|\bar{V}(r)\|^2) \, dr
\]
Lemma 6.5. For every \( \omega_1, \omega_2 \in \Omega_k \) in \((\Omega_k, \rho)\). Note that \( u_i = v_i + z(\vartheta_i \omega_1) \), \( i = 1, 2 \). Then we have
\[
\|\bar{U}(t)\| = \|u_1(t) - u_2(t)\| \leq 2\|\bar{V}(t)\|^2 + 2\|\phi\|^2|\bar{g}|^2,
\]
which and (147) together give the desired result.

From the convergence in the above (147), we formulate a lemma which is useful in the sequel arguments.

Lemma 6.5. For every \( \omega_1, \omega_2 \in \Omega_k \) for every fixed \( k \in \mathbb{N} \). Let \( \bar{V}(t) \) be the solution to (136). Then we have
\[
\int_\tau^t \left( \|(-\Delta)^{\frac{1}{2}}\bar{V}(r)\|^2 + \|\bar{V}(r)\|^2 \right) dr \to 0,
\]
as \( \omega_2 \to \omega_1 \).

6.3. Continuity w.r.t. the sample \( \omega \) in \( \mathcal{L}^p(\mathbb{R}^N) \). In this subsection, we prove the continuous of solutions to problem (23) and (24) in \( \mathcal{L}^p(\mathbb{R}^N) \) with respect to \( \omega \in \Omega_k \) for every \( k \in \mathbb{N} \), which is crucial for us to derive the measurability of pullback attractor in \( \mathcal{L}^p(\mathbb{R}^N) \). The proof is technical and involved.

Lemma 6.6. Let \( t > \tau, \tau \in \mathbb{R} \) and \( u_\tau \in \mathcal{L}^2(\mathbb{R}^N) \). Suppose that \( u(t, \tau, \omega, u_\tau) \) is the solution to problem (23) and (24). Then the mapping \( \omega \to u(t, \tau, \omega, u_\tau) \) is continuous from \((\Omega_k, \rho)\) to \( \mathcal{L}^p(\mathbb{R}^N) \), \( p \geq 2 \), for each \( k \in \mathbb{N} \).

Proof. By (136), we have
\[
\frac{1}{p} \frac{d}{dt} \|\bar{V}\|_p^p + \lambda\|\bar{V}\|_p^2 + \int_{\mathbb{R}^N} (f(t, x, v_1 + z(\vartheta_1 \omega_1)) - f(t, x, v_2 + z(\vartheta_1 \omega_2))) |\bar{V}|^{p-2}\bar{V} dx = -((-\Delta)^{s}\bar{V}, |\bar{V}|^{p-2}\bar{V}) - ((-\Delta)^{s}(z(\vartheta_1 \omega_1) - z(\vartheta_1 \omega_2)), |\bar{V}|^{p-2}\bar{V}).
\]
We need to estimate the terms in (148). First, from (141) and the relation \( |\bar{U}|^p \geq 2^{1-p}|\bar{V}|^p - |\bar{g}|^p|\phi|^p \) it follows that
\[
(f(t, x, u_1) - f(t, x, u_2))\bar{V} \geq \gamma |\bar{U}|^p - (\psi_1(t, x) + \psi_2(t, x))|\bar{U}|^2 - \psi_2(t, x)|\bar{g}|^2|\phi(x)|^2 - c\psi_2(t, x)(|u_1|^p + |u_2|^p + |\phi|^p)|\bar{g}|
\geq \frac{\gamma}{2^{p-1}} |\bar{V}|^p - |\bar{g}|^p|\phi|^p - (\psi_1(t, x) + \psi_2(t, x))|\bar{U}|^2 - \psi_2(t, x)|\bar{g}|^2|\phi(x)|^2 - c\psi_2(t, x)(|u_1|^p + |u_2|^p + |\phi|^p)|\bar{g}|
\]
which gives
\[
(f(t, x, u_1) - f(t, x, u_2))|\bar{V}|^{p-2}\bar{V} \geq \frac{\gamma}{2^{p-1}} |\bar{V}|^{2p-2} - |\bar{g}|^p|\phi|^p|\bar{V}|^{p-2}
- (\psi_1(t, x) + \psi_2(t, x))|\bar{U}|^2|\bar{V}|^{p-2} - \psi_2(t, x)|\bar{g}|^2|\phi(x)|^2|\bar{V}|^{p-2}
- c\psi_2(t, x)(|u_1|^p + |u_2|^p + |\phi|^p)|\bar{V}|^{p-2}|\bar{g}|.
\]
The terms on the right hand side of (150) can be estimated by Young inequality, namely, we have
\[
\begin{align*}
&\left\{
|\bar{y}|^p |\phi|^p |\bar{V}|^{p-2} \leq \frac{\gamma}{2^{p+1}} |\bar{V}|^{2p-2} + c|\phi|^{2p-2} |\bar{y}|^{2p-2}; \\
&\psi_i(t,x)|\bar{U}|^2 |\bar{V}|^{p-2} \leq \frac{\gamma}{2} |\bar{V}|^p + |\psi_i|^2 |\bar{U}|^p, \quad i = 1, 2; \\
&\psi_2(t,x)|\bar{y}|^2 |\phi(x)|^2 |\bar{V}|^{p-2} \leq \frac{\gamma}{4} |\bar{V}|^p + |\psi_2|^2 |\phi| |\bar{y}|^p; \\
&c\psi_2(t,x)(|u_1|^p + |u_2|^p + |\phi|^p)|\bar{V}|^{p-2}|\bar{y}| \leq \frac{\gamma}{2^{p+1}} |\bar{V}|^{2p-2} \\
&\quad + c|\psi_2(t,x)|^{\frac{2p-2}{p}} (|u_1|^{2p-2} + |u_2|^{2p-2} + |\phi|^{2p-2}) |\bar{y}|^{\frac{2p-2}{p}},
\end{align*}
\]
from which and (150), the nonlinearity in (148) is estimated as
\[
\int_{\mathbb{R}^N} (f(t,x,u_1) - f(t,x,u_2)) |\bar{V}|^{p-2} \bar{V} \, dx \geq \frac{\gamma}{2p} |\bar{V}|^{2p-2} - \frac{\lambda}{2} |\bar{V}|^p \\
- c(||\bar{U}||^p_p + |\bar{y}|^p + |\bar{y}|^{2p-2} + |\bar{y}|^{\frac{2p-2}{p}}) \\
- c(||u_1|^{2p-2} + ||u_2||^{2p-2} - |\phi|^{2p-2}) |\bar{y}|^{\frac{2p-2}{p}}. \quad (151)
\]
For the first term on the right hand of (148), we have
\[
((-\Delta)^s \bar{V}, |\bar{V}|^{p-2} \bar{V}) \\
= \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{(\bar{V}(x) - \bar{V}(y))(||V(x)||^{p-2} \bar{V}(x) - ||V(y)||^{p-2} \bar{V}(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\geq \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{|\bar{V}(x) - \bar{V}(y)|^p}{|x - y|^{N+2s}} \, dx \, dy \geq 0. \quad (152)
\]
On the other hand, for the second term on the right hand of (148), by a similar argument as in (117), we deduce
\[
- ((-\Delta)^s (z(\partial_1 \omega_1) - z(\partial_1 \omega_2)), |\bar{V}|^{p-2} \bar{V}) \\
= \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{\bar{y}(\phi(x) - \phi(y))(||V(x)||^{p-2} \bar{V}(x) - ||V(y)||^{p-2} \bar{V}(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\leq \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{\bar{y}(\phi(x) - \phi(y))|V(x)|^{p-2} \bar{V}(x)}{|x - y|^{N+2s}} \, dx \, dy \\
+ \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{\bar{y}(\phi(x) - \phi(y))|V(y)|^{p-2} \bar{V}(y)}{|x - y|^{N+2s}} \, dx \, dy \\
\leq C(N,s) |\bar{y}| ||V||^{p-1}_{L^p(\mathbb{R}^N)} ((-\Delta)^s \phi) \leq \frac{\gamma}{2^{p+1}} ||V||^{2p-2}_{L^p(\mathbb{R}^N)} + c|\bar{y}|^2. \quad (153)
\]
Therefore incorporation (151)- (153) into (148) to find
\[
\frac{d}{dt} ||V||^p_p + \frac{\gamma}{2^{p+1}} ||V||^{2p-2}_{L^p(\mathbb{R}^N)} \leq c||V||^p_p + c(|\bar{y}|^2 + |\bar{y}|^p + |\bar{y}|^{2p-2} + |\bar{y}|^{\frac{2p-2}{p}}) \\
+ c(||u_1||^{2p-2}_{L^p(\mathbb{R}^N)} + ||u_2||^{2p-2}_{L^p(\mathbb{R}^N)} + |\phi|^{2p-2}) |\bar{y}|^{\frac{2p-2}{p}}. \quad (154)
\]
Employing Gronwall lemma to (154) over the intervals \((\varsigma, t)\) with \(\varsigma \in [\frac{t}{2} + \tau, t]\), we get
\[
\|\hat{V}(t)\|_p^p + \frac{\gamma}{2p+1} e^{ct} \int_\varsigma^t e^{-cr} \|\hat{V}(r)\|_{2p-2}^{2p-2} dr
\]
\[
\leq e^{ct} e^{-c\varsigma} \|\hat{V}(\varsigma)\|_p^p + ce^{ct} \int_\tau^t e^{-cr} (|\tilde{g}(r)|^2 + |\tilde{g}(r)|^p + |\tilde{g}|^{2p-2} + |\tilde{g}(r)|^{\frac{2p-2}{p}}) dr
\]
\[
+ ce^{ct} \int_\tau^t e^{-cr} (\|u_1(r)\|_{2p-2}^p + \|u_2(r)\|_{2p-2}^p) |\tilde{g}(r)|^{\frac{2p-2}{p}} dr.
\]  
\tag{155}

Integrate the last inequality of (155) with respect to \(\varsigma\) over the intervals \((\frac{t}{2} + \tau, t)\) to yield
\[
\|\hat{V}(t)\|_p^p + \frac{\gamma}{2p+1} \frac{2e^{ct} t - \tau}{t - \tau} \int_\tau^t e^{-cr} \|\hat{V}(r)\|_{2p-2}^{2p-2} dr d\varsigma \leq \frac{2}{t - \tau} e^{ct} \int_\tau^t e^{-cr} \|\hat{V}(r)\|_p^p dr
\]
\[
+ ce^{ct} \int_\tau^t e^{-cr} (|\tilde{g}(r)|^2 + |\tilde{g}(r)|^p + |\tilde{g}|^{2p-2} + |\tilde{g}(r)|^{\frac{2p-2}{p}}) dr
\]
\[
+ ce^{ct} \int_\tau^t e^{-cr} (\|u_1(r)\|_{2p-2}^p + \|u_2(r)\|_{2p-2}^p) |\tilde{g}(r)|^{\frac{2p-2}{p}} dr
\]
\[
= I_1 + I_2 + I_3.
\]  
\tag{156}

By (147), we deduce that, for any \(t > \tau\),
\[
I_1 = \frac{2}{t - \tau} e^{ct} \int_\tau^t e^{-cr} \|\hat{V}(r)\|_p^p dr \to 0,
\]  
\tag{157}

as \(\omega_2 \to \omega_1\). By the Lebesgue dominated convergence theorem, it follows that, for any \(t > \tau\),
\[
I_2 = ce^{ct} \int_\tau^t e^{-cr} (|\tilde{g}(r)|^2 + |\tilde{g}(r)|^p + |\tilde{g}|^{2p-2} + |\tilde{g}(r)|^{\frac{2p-2}{p}}) dr \to 0,
\]  
\tag{158}

as \(\omega_2 \to \omega_1\). On the other hand, in terms of (110), in conjunction with Lemma 6.2, we find that, for \(\omega_1 \in \Omega_k, i = 1, 2\),
\[
G(t, \tau, \omega_i, u_r) \leq \frac{c}{t - \tau} (\int_\tau^t \|g(r,.)\|^2 dr + \int_\tau^t \|J(\vartheta, \omega_i) dr + \|u_r\|^2 + c|g(\vartheta, \omega_i)|^2)
\]
\[
+ c \int_\tau^t \|g(r, .)\|^2 dr + c \int_\tau^t \|J(\vartheta, \omega_i) dr.
\]
from which and (162) we derive Lemma 6.7.

Let \( \omega \) as discussions.

\[ \omega \text{ is the solution to problem (23) and (24) at the sample} \]

\[ \omega \text{ and Lebesgue dominate convergence theorem, we get} \]

\[ \omega \] as \( \omega \) is independent of \( \omega_i, i = 1, 2 \). So by (109) and (159), we find, for any \( \omega_i \in \Omega_k, i = 1, 2, \)

\[ \int_{\frac{t}{2}}^{t} e^{-cr} \| u_i(r) \|_{2p-2}^2 dr \leq e^{-ct} \int_{\frac{t}{2}}^{t} \| u_i(r) \|_{2p-2}^2 dr \leq e^{-ct} N(t, \tau). \] (160)

Therefore, the Lebesgue dominated convergence theorem can be applied to yield

\[ I_3 = ce^{-ct} \int_{\frac{t}{2}}^{t} e^{-cr} (\| u_1(r) \|_{2p-2}^2 + \| u_2(r) \|_{2p-2}^2) \| \tilde{y}(r) \|_{\infty}^{2p-2} dr \to 0, \] (161)

as \( \omega_2 \to \omega_1 \). This achieves the proof. \( \square \)

From the above proof, we get the following lemma which is useful for our sequel discussions.

**Lemma 6.7.** Let \( t > \tau, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\mathbb{R}^N) \). Suppose that \( u(t, \tau, \omega_i, u_\tau) \) is the solution to problem (23) and (24) at the sample \( \omega_i \in \Omega_k \) for each fixed \( k \in \mathbb{N}, i = 1, 2 \). Then

\[ \int_{\frac{t}{2}}^{t} (r - \frac{t + \tau}{2}) \| \tilde{U}(r) \|_{2p-2}^2 dr \to 0, \]

as \( \omega_2 \to \omega_1 \), where \( \tilde{U}(r) = u(r, \tau, \omega_1, u_\tau) - u(r, \tau, \omega_2, u_\tau) \).

**Proof.** From (156), it follow that

\[ \int_{\frac{t}{2}}^{t} \int_{\frac{t}{2}}^{t} e^{-cr} \| \tilde{V}(r) \|_{2p-2}^2 dr d\xi \to 0, \] (162)

as \( \omega_2 \to \omega_1 \). By exchanging the integral order of the integral in (162), we write

\[ \int_{\frac{t}{2}}^{t} \int_{\frac{t}{2}}^{t} e^{-cr} \| \tilde{V}(r) \|_{2p-2}^2 dr d\xi = \int_{\frac{t}{2}}^{t} e^{-cr} (r - \frac{t + \tau}{2}) \| \tilde{V}(r) \|_{2p-2}^2 dr, \]

from which and (162) we derive

\[ \int_{\frac{t}{2}}^{t} e^{-cr} (r - \frac{t + \tau}{2}) \| \tilde{V}(r) \|_{2p-2}^2 dr \to 0, \] (163)

as \( \omega_2 \to \omega_1 \). Note that \( \tilde{U}(t) = \tilde{V}(t) + \phi \tilde{y}(t) \) and \( \phi \in L^{2p-2}(\mathbb{R}^N) \). Then by (163) and Lebesgue dominate convergence theorem, we get
6.4. Continuity w.r.t. the sample \( \omega \) in \( H^s(\mathbb{R}^N) \). To consider the measurability of solution in \( H^s(\mathbb{R}^N), 0 < s < 1 \), we first prove the continuity of solution with respect to sample \( \omega \) for \( \omega \in (\Omega_k, \rho) \).

**Lemma 6.8.** Let \( t > \tau, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\mathbb{R}^N) \). Suppose that \( u(t, \tau, \omega, u_\tau) \) is the solution to problem (23) and (24). Then the mapping \( \omega \to u(t, \tau, \omega, u_\tau) \) is continuous from \( (\Omega_k, \rho) \) to \( H^s(\mathbb{R}^N), 0 < s < 1 \), for each \( k \in \mathbb{N} \).

**Proof.** We begin with by proving some priori estimates. First, by (26), we infer that

\[
\int_{t-\frac{\tau}{2}}^{t} e^{-c(r - \frac{t + \tau}{2})} \left\| \bar{U}(r)^{2p-2} dr \right\| \leq c \int_{t-rac{\tau}{2}}^{t} e^{-c(r - \frac{t + \tau}{2})} \left\| \bar{V}(r)^{2p-2} dr \right\| + c \int_{t-rac{\tau}{2}}^{t} e^{-c(r - \frac{t + \tau}{2})} \left\| \bar{g}(r)^{2p-2} dr \right\| \to 0, \quad (164)
\]
as \( \omega_2 \to \omega_1 \). This obviously implies that

\[
\int_{t-\frac{\tau}{2}}^{t} (r - \frac{t + \tau}{2}) \left\| \bar{U}(r)^{2p-2} dr \right\| \to 0,
\]
as \( \omega_2 \to \omega_1 \), which completes the proof. \( \square \)

where \( \bar{V}_t \) is the derivative of \( \bar{V} \) with respect to the time \( t \). By Hölder inequality, Young inequality and \( (-\Delta)^s \phi \in L^2(\mathbb{R}^N) \), it follows that

\[
\left| -(-\Delta)^s(z(\partial_1 \omega_1) - z(\partial_1 \omega_2), \bar{V}_t) \right| = \frac{1}{2} C(N, s) \left| \int_{\mathbb{R}^{2N}} \frac{\bar{g}(\phi(x) - \phi(y))(\bar{V}_t(x) - \bar{V}_t(y))}{|x-y|^{N+2s}} dxdy \right|
\]

\[
\leq \frac{1}{2} C(N, s) \left| \int_{\mathbb{R}^{2N}} \frac{\bar{g}(\phi(x) - \phi(y))\bar{V}_t(x)}{|x-y|^{N+2s}} dxdy \right|
\]

\[
+ \frac{1}{2} C(N, s) \left| \int_{\mathbb{R}^{2N}} \frac{\bar{g}(\phi(x) - \phi(y))\bar{V}_t(y)}{|x-y|^{N+2s}} dxdy \right|
\]

\[
\leq C(N, s) \|\bar{g}\| \|\bar{V}_t\| \|(-\Delta)^s \phi\| \leq \frac{1}{4} \|\bar{V}_t\|^2 + c|\bar{g}|^2. \quad (166)
\]
Taking the inner product of (136) in $L^2(\mathbb{R}^N)$ with $\tilde{V}$, from (165) and (166) we deduce that, with $t$ being replaced by $r$,
\[
\frac{d}{dr} \|(-\Delta)^{\frac{1}{2}} \tilde{V}(r)\|^2 \leq c \|\tilde{V}(r)\|^2 + c |\bar{y}(r)|^2 \\
+ c(\|u_1(r)\|_{2p-4}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2.
\] (167)

Multiply (167) by $r - \frac{t + \tau}{2}$ with $r \in (\frac{t + \tau}{2}, t)$ to yield
\[
(r - \frac{t + \tau}{2}) \frac{d}{dt} \|(-\Delta)^{\frac{1}{2}} \tilde{V}(r)\|^2 \leq c(r - \frac{t + \tau}{2}) \|\tilde{V}(r)\|^2 + c(r - \frac{t + \tau}{2}) |\bar{y}(r)|^2 \\
+ c(r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2 \\
\leq c(t - \tau) \|\tilde{V}(r)\|^2 + c(t - \tau) |\bar{y}(r)|^2 \\
+ c(r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2.
\] (168)

By integrating (168) over the intervals $(\frac{t + \tau}{2}, t)$, we obtain
\[
\frac{t - \tau}{2} \|(-\Delta)^{\frac{1}{2}} \tilde{V}(t)\|^2 \leq \int_{\frac{t + \tau}{2}}^t \|(-\Delta)^{\frac{1}{2}} \tilde{V}(r)\|^2 dr + c(t - \tau) \int_{\frac{t + \tau}{2}}^t \|\tilde{V}(r)\|^2 + |\bar{y}(r)|^2 dr \\
+ c \int_{\frac{t + \tau}{2}}^t (r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2 dr.
\] (169)

We now estimate every terms on the right hand side of (169). By Lemma 6.5, we have
\[
\int_{\frac{t + \tau}{2}}^t \|(-\Delta)^{\frac{1}{2}} \tilde{V}(r)\|^2 dr + c(t - \tau) \int_{\frac{t + \tau}{2}}^t \|\tilde{V}(r)\|^2 dr \to 0,
\] (170)
as $\omega_2 \to \omega_1$. For the last term on the right hand side of (169), by using Hölder inequality, we deduce that
\[
\int_{\frac{t + \tau}{2}}^t (r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2 dr \\
\leq \left( \int_{\frac{t + \tau}{2}}^t (r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) dr \right)^{\frac{p-2}{2p-4}} \\
\times \left( \int_{\frac{t + \tau}{2}}^t (r - \frac{t + \tau}{2}) \|\bar{U}(r)\|_{2p-2}^2 dr \right)^{\frac{2p-4}{2p-2},}
\]
so it follows from (160) and Lemma 6.7 that
\[
\int_{\frac{t + \tau}{2}}^t (r - \frac{t + \tau}{2})(\|u_1(r)\|_{2p-2}^{2p-4} + \|u_2(r)\|_{2p-2}^{2p-4}) \|\bar{U}(r)\|_{2p-2}^2 dr \to 0,
\] (171)
as \( \omega_2 \to \omega_1 \). Consequently in the end, incorporate (170) and (171) into (169) to produce
\[
\| (-\Delta) \hat{V}(t) \| \to 0,
\]
as \( \omega_2 \to \omega_1 \). This and Lemma 6.4 together conclude the proof.

7. Existence of bi-spatial measurable pullback attractor. In this section, we are at the point to present the bi-spatial \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\)-pullback attractor and \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-pullback attractor, respectively. Especially, it is pointed out that this attractor is measurable (as a set-valued mapping) in the regular spaces \( H^s(\mathbb{R}^N) \) and \( L^p(\mathbb{R}^N) \), where \( s \in (0,1) \) and \( p \geq 2 \).

**Theorem 7.1.** Suppose that (25)-(29) hold. Then the random cocycle \( \varphi \) defined by (37) admits a unique bi-spatial \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\)-pullback attractor \( \mathcal{A}_{H^s} = \{ \mathcal{A}_{H^s}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \), in the sense of Definition 2.5.

**Proof.** By Theorem 5.9, the random cocycle \( \varphi \) is bi-spatial \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\)-continuous with respect to the initial data. By Lemma 4.1, \( \varphi \) has a closed and pullback random absorbing set in the initial space \( L^2(\mathbb{R}^N) \). By Lemma 4.5, \( \varphi \) is \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\)-asymptotically compact. In view of Theorem 2.10, it remains to check that the mapping \( \varphi(t, \tau, x) : \Omega \ni \omega \to H^s(\mathbb{R}^N) \) is \((\mathcal{F}, \mathcal{B}(H^s(\mathbb{R}^N)))\)-measurable for every fixed \( t, \tau, x \) and \( \omega \). Therefore by Lemma 6.3, the mapping \( \varphi(t, \tau, x) : \Omega \ni \omega \to H^s(\mathbb{R}^N) \) is \((\mathcal{F}, \mathcal{B}(H^s(\mathbb{R}^N)))\)-measurable. This concludes the proof.

**Theorem 7.2.** Suppose that (25)-(29) hold. Then the random cocycle \( \varphi \) defined by (37) admits a unique bi-spatial \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-pullback attractor \( \mathcal{A}_{L^p} = \{ \mathcal{A}_{L^p}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \), in the sense of Definition 2.5. Furthermore, \( \mathcal{A}_{L^p} = \mathcal{A}_{H^s}, 0 < s < 1 \) and \( p \geq 2 \).

**Proof.** By Theorem 5.5, Lemma 6.6, Lemma 4.1, Lemma 4.5 and Lemma 6.3. The result is followed from Theorem 2.10 by a similar argument as Theorem 7.1.

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