Approximate amenability of tensor products of Banach algebras

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Abstract

Examples constructed by the first author and Charles Read make it clear that many of the hereditary properties of amenability no longer hold for approximate amenability. These and earlier results of the authors also show that the presence of a bounded approximate identity often facilitates positive results. Here we show that the tensor product of approximately amenable algebras need not be approximately amenable, and investigate conditions under which $A$ and $B$ being approximately amenable implies, or is implied by, $A \hat{\otimes} B$ or $A^* \hat{\otimes} B^*$ being approximately amenable. Once again, the role of having a bounded approximate identity comes to the fore. Our methods also enable us to prove that if $A \hat{\otimes} B$ is amenable, then so are $A$ and $B$, a result proved by Barry Johnson in 1996 under an additional assumption.

Keywords: Approximately amenable Banach algebra, amenable Banach algebra, tensor product, approximate identity

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In memoriam Charles John Read – mathematician, gentleman and friend

1. Introduction

The concept of amenability for a Banach algebra, introduced by Johnson in 1972 [9], has proved to be of enormous importance in Banach algebra theory. In [9], and subsequently in [14], several modifications of this notion were introduced, in particular that of approximate amenability; and much work has been

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done in the last 10 years or so, for example. See also for a recent survey. In this paper the focus is on these newer notions for tensor products. In particular, we investigate relations between the approximate amenability of $A$ and $B$ and that of $A \hat{\otimes} B$ or $A^\# \hat{\otimes} B^\#$.

Let $A$ be an algebra, and let $X$ be an $A$-bimodule. A derivation is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a$, $A \to X$. Then $ad_x$ is a derivation; these are the inner derivations.

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. A continuous derivation $D : A \to X$ is approximately inner if there is a net $(x_\alpha)$ in $X$ such that

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

so that $D = \lim_{\alpha} ad_{x_\alpha}$ in the strong-operator topology of $B(A)$.

**Definition 1.1.** Let $A$ be a Banach algebra. Then $A$ is approximately amenable (respectively approximately contractible) if, for each Banach $A$-bimodule $X$, every continuous derivation $D : A \to X^*$ (respectively $D : A \to X$), is approximately inner. If it is always possible to choose the approximating net $(ad_{x_\alpha})$ to be bounded (with the bound dependent only on $D$) then $A$ is boundedly approximately amenable (respectively, boundedly approximately contractible).

Of course $A$ is amenable (respectively, contractible) if every continuous derivation $D : A \to X^*$ (respectively $D : A \to X$), is inner.

Of these various notions, amenability, contractibility, approximately amenable, boundedly approximately amenable and boundedly approximately contractible are all distinct, approximately contractible and approximately amenable coincide, also, requiring the approximating net of derivations to converge weak$^*$ is the same as approximately amenable, $\Box$. This latter notion will arise naturally below.

### 2. Some observations

Recall the result of Barry Johnson [17, Proposition 5.4]:

**Proposition 2.1.** Let $A$ and $B$ be amenable Banach algebras. Then $A \hat{\otimes} B$ is amenable. $\Box$

A version of this for the approximately amenable case was stated in [8, Proposition 2.3], but the argument there is incomplete. The matter is clarified in [8, Proposition 4.1], from which we state:

**Theorem 2.2.** Suppose that $A$ is a boundedly approximately amenable Banach algebra with a bounded approximate identity, and that $B$ is an amenable Banach algebra. Then $A \hat{\otimes} B$ is boundedly approximately amenable. $\Box$
In [3], the question is raised whether the tensor product of two (boundedly) approximately amenable Banach algebras is itself (boundedly) approximately amenable. We begin by answering this question in the negative.

**Theorem 2.3.** The tensor product of two boundedly approximately amenable Banach algebras need not be approximately amenable.

**Proof.** Let $A$ be the Banach algebra constructed in [10] such that $A$ is boundedly approximately amenable yet $A \oplus A^\text{op}$ is not approximately amenable. For convenience, set $B = A^\text{op}$. Adjoin identities $1_A$ to $A$ and $1_B$ to $B$, and set $A = A^\# \otimes B^\#$. Then we have the decomposition into closed subspaces:

$$A = (C_1A \otimes 1_B) + (1_A \otimes B) + (A \otimes 1_B) + (A \hat{\otimes} B).$$

Now $A \hat{\otimes} B$ is a closed two-sided ideal in $A$, so if $A$ is approximately amenable, the quotient algebra $(C_1A \otimes 1_B) + (1_A \otimes B) \oplus (A \otimes 1_B) \simeq (1_A \otimes B) \oplus (A \otimes 1_B)$ [9, Proposition 2.4]. The map

$$(1_A \otimes B) \oplus (A \otimes 1_B) \to B \oplus A : (1_A \otimes b) + (a \otimes 1_B) \mapsto (b, a)$$

is an isometric surjective algebra isomorphism, so that $A \oplus B$ is approximately amenable. But this contradicts the specific choice of $A$ and $B$. Thus $A$ cannot be approximately amenable. 

Note that the argument sheds no light on whether in this case the smaller $A \hat{\otimes} B$ is approximately amenable.

**Remark 2.4.** The same example from [10] also answers Question 1 raised in [8, §9]. Namely $A \oplus B$ is not approximately amenable, yet the ideal $A$ is boundedly approximately amenable, as is the quotient $B$.

We now build on this example to give a slightly sharper result.

**Theorem 2.5.** There exists a unital boundedly approximately amenable Banach algebra $A$ such that $A \hat{\otimes} A$ is not approximately amenable.

**Proof.** Let $A$ and $B$ be the boundedly approximately amenable algebras as above, and set $A = A^\# \oplus B^\#$, boundedly approximately amenable by [14, Proposition 6.1]. We have

$$A \hat{\otimes} A = (A^\# \hat{\otimes} A^\#) \oplus (B^\# \hat{\otimes} B^\#) \oplus (B^\# \hat{\otimes} A^\#) \oplus (A^\# \hat{\otimes} B^\#),$$

and

$$I = (A^\# \hat{\otimes} A^\#) \oplus (B^\# \hat{\otimes} B^\#) \oplus (B^\# \hat{\otimes} A^\#)$$

is a closed two-sided ideal. The quotient $A \hat{\otimes} A/I = A^\# \hat{\otimes} B^\#$, which is not approximately amenable from Theorem 2.3, so that $A \hat{\otimes} A$ is not approximately amenable. 

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In comparison, note that since boundedly approximately contractible algebras have bounded approximate identities [4, Theorem 2.5], so the direct sum of boundedly approximately contractible algebras is boundedly approximately contractible by a variant of [9, Proposition 2.7].

There is a special situation when approximate amenability of the tensor product comes for free.

**Proposition 2.6.** Let $A$ and $B$ be Banach function algebras on their respective carrier spaces, and suppose that $A$ and $B$ have bounded approximate identities consisting of elements of finite support. Then $A \hat{\otimes} B$ is approximately amenable.

**Proof.** That $A$ and $B$ are approximately amenable follows from [12, Proposition 4.2]. Now $A$ and $B$ have the bounded approximation property, so by [19, §3.2.18] $A \hat{\otimes} B$ is semisimple, and so is again a Banach function algebra. It also has a bounded approximate identity of elements of finite support, built from those of $A$ and $B$, and once more [12, Proposition 4.2] applies.

The same style of argument as above using compositions can also give some positive results.

**Theorem 2.7.** Suppose that $A^\# \hat{\otimes} B^\#$ is approximately amenable. Then $A$, $B$ and $A \oplus B$ are approximately amenable.

**Proof.** The algebra $A^\#$ admits a character $\varphi$, and $a \hat{\otimes} b \mapsto \varphi(a)b$ defines an epimorphism $A^\# \hat{\otimes} B^\# \to B^\#$ so that $B^\#$, and hence $B$, is approximately amenable. Similarly for $A$.

We have the decomposition into closed subalgebras,

$$A^\# \hat{\otimes} B^\# = (C1_A \otimes 1_B) + (1_A \otimes A) + (A \otimes 1_B) + (A \hat{\otimes} B).$$

Here $A \hat{\otimes} B$ is a closed ideal, with approximately amenable quotient

$$(C1_A \otimes 1_B) + (1_A \otimes A) + (B \otimes 1_B)$$

having zero product between the second and third summands. But this latter is isomorphic to the unitization of $A \oplus B$.

The obvious omission here is whether $A \hat{\otimes} B$ is approximately amenable. This is certainly the case under an additional hypothesis.

**Theorem 2.8.** Suppose that $A^\# \hat{\otimes} B^\#$ is (boundedly) approximately amenable and that $A$ and $B$ have bounded approximate identities. Then $A \hat{\otimes} B$ is (boundedly) approximately amenable.

**Proof.** Since $A \hat{\otimes} B$ has a bounded approximate identity, it suffices to show that for every essential Banach $A \hat{\otimes} B$-bimodule $X$, continuous derivations from $A \hat{\otimes} B$ into $X^*$ are approximately inner.

Let $D : A \hat{\otimes} B \to X^*$ be a continuous derivation. Then $D$ extends uniquely to a derivation $\hat{D} : \Delta(A \hat{\otimes} B) \to X^*$, where $\Delta(A \hat{\otimes} B)$ is the double centralizer algebra of $A \hat{\otimes} B$, [16, 17]. Then restrict $\hat{D}$ to $A^\# \hat{\otimes} B^#$. By hypothesis this restriction is approximately inner, a fortiori, so is $D$. 

\[\square\]
Remark 2.9. An alternate proof would be to use the argument of [9, Corollary 2.3].

Remark 2.10. 1. A possibly related question is whether \( c_0(A) \) is approximately amenable given that \( A \) is approximately amenable. The argument of [9, Example 6.1] shows that \( c_0(A^\#) \) will be approximately amenable. For the algebra \( A \) of [10, c.0 \( c_0(A) \) will be approximately amenable. For the algebra \( A \) of [10, c.0 \( c_0(A) \) is again of the specified form of [10, Theorem 3.1], and so is approximately amenable. The more general question as to whether \( c_0(A_n) \) is approximately amenable, where the \( (A_n) \) are approximately amenable, has a negative answer in general, as shown by the example \( A \oplus A \) of [10].

2. Note that \( c_0 \hat{\otimes} A \) will be approximately amenable if \( A \) is boundedly approximately amenable and has a bounded approximate identity (Proposition 2.2). For more general \( A \) the question is open. Of course there is a natural homomorphism \( c_0 \hat{\otimes} A \to c_0(A) \) determined by \( (\alpha_n) \otimes x \mapsto (\alpha_n x) \). Since elements of the range are summable sequences of elements of \( A \), the homomorphism has properly dense range. Supposing that \( c_0 \hat{\otimes} A \) is approximately amenable it is not known whether \( c_0(A) \) must be approximately amenable. However the epimorphism \( c_0 \hat{\otimes} A \to A \oplus A \) determined by \( (\alpha_n) \otimes x \mapsto (\alpha_1 x, \alpha_2 x) \) shows that \( A \oplus A \) would be.

3. Semi-inner derivations

We first introduce a new notion which will come up in later arguments of this section. The concept itself is not new, but the variant we wish to use seems to be.

Definition 3.1. Let \( A \) be a algebra, \( X \) an \( A \)-bimodule. A derivation \( D : A \to X \) is semi-inner\(^2\) if there are \( m, n \in X \) such that

\[
D(a) = a \cdot m - n \cdot a \quad (a \in A).
\]

More generally, for \( A \) a Banach algebra, \( X \) a Banach \( A \)-bimodule, a continuous derivation \( D : A \to X \) is approximately semi-inner if there are nets \( (m_i), (n_i) \) in \( X \) with

\[
D(a) = \lim_i (a \cdot m_i - n_i \cdot a) \quad (a \in A).
\]

Here \( m \) and \( n \) may be distinct but are highly constrained. The derivation identity shows that if \( D \) is a semi-inner derivation then

\[
a \cdot (m - n) \cdot b = 0 \quad (a, b \in A).
\]

\(^2\)Such maps, without the derivation condition, are called generalized inner, or ‘generalized inner derivations’ in the literature [2, 1, 5]. We require the approximate version, and view ‘approximately generalized’ as an oxymoron, and so will use ‘semi-inner’, but only for derivations.
Thus in the Banach case, with $D : A \rightarrow X^*$, then $m = n$ if $X$ is neo-unital, and we have an inner derivation. In the approximately semi-inner case
\[
\lim_i (a \cdot (m_i - n_i) \cdot b) = 0 \quad (a, b \in A),
\]
and for $X$ neo-unital this latter gives $m_i - n_i \rightarrow 0$ weak*, so that $D$ is in fact weak* approximately inner.

**Example 3.2.** The algebra $\ell^2$ under pointwise operations is not approximately amenable. It is, however, approximately semi-amenable. For let $D : \ell^2 \rightarrow X$ be a continuous derivation into a bimodule. Set $(E_n)$ to be the standard (unbounded) approximate identity of $\ell^2$. Then $D_n : E_n \ell^2 \rightarrow X$ is a derivation from a finite-dimensional semisimple algebra and hence is inner, say implemented by $\xi_n \in X$. Thus for $a \in \ell^2$,
\[
D(a) = \lim_n D(E_n a) = \lim_n (E_n a \cdot \xi_n - \xi_n \cdot E_n a) \\
= \lim_n (a \cdot (E_n \cdot \xi_n) - (\xi_n \cdot E_n) \cdot a),
\]
and so $D$ is approximately semi-inner.

**Theorem 3.3.** Suppose that $A \hat{\otimes} B$ is one of
(i) approximately amenable,
(ii) boundedly approximately amenable,
(iii) boundedly approximately contractible.
Then for any continuous derivation $D$ from $A$ or $B$

- into any bimodule is approximately semi-inner in case (i),
- into any dual bimodule is boundedly approximately semi-inner in case (ii),
- into any bimodule is boundedly approximately semi-inner in case (iii).

**Proof.** Given a Banach $A$-bimodule $X$, we make $X \hat{\otimes} B$ into a Banach $A \hat{\otimes} B$-bimodule as follows: for $a \in A, b_1 \in B, b_2 \in B, x \in X$,
\[
(a \otimes b_1) \cdot (x \otimes b_2) = a \cdot x \otimes b_1 b_2, \quad (x \otimes b_2) \cdot (a \otimes b_1) = x \cdot a \otimes b_2 b_1.
\]
Given a continuous derivation $D : A \rightarrow X$, we define $\Delta : A \hat{\otimes} B \rightarrow X \hat{\otimes} B$ by setting
\[
\Delta(a \otimes b) = D(a) \otimes b \quad (a \in A, b \in B).
\]
Then
\[
\Delta((a_1 \otimes b_1)(a_2 \otimes b_2)) = \Delta(a_1 a_2 \otimes b_1 b_2) \\
= (D(a_1) \cdot a_2 + a_1 \cdot D(a_2)) \otimes (b_1 b_2) \\
= ((D(a_1) \cdot a_2) \otimes b_1 b_2) + ((a_1 \cdot D(a_2)) \otimes b_1 b_2) \\
= ((D(a_1) \otimes b_1) \cdot (a_2 \otimes b_2)) + ((a_1 \otimes b_1) \cdot (D(a_2) \otimes b_2)).
\]
so that $\Delta$ is a derivation.

In clause (i), since approximate amenability and approximate contractibility coincide, [14, Proposition 2.1], there is a net $(m_i)$ in $X \hat{\otimes} B$ such that for all $a \in A, b \in B$,

$$\Delta(a \otimes b) = \lim_i \left( (a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) \right).$$

Let

$$m_i = \sum_{k=1}^{\infty} x_{k,i} \otimes b_{k,i},$$

where $x_{k,i} \in X, b_{k,i} \in B$. Then

$$D(a) \otimes b = \Delta(a \otimes b) = \lim_i \left( \sum_k (a \cdot x_{k,i}) \otimes bb_{k,i} - \sum_k (x_{k,i} \cdot a) \otimes b_{k,i}b \right).$$

Fix $b_0 \in B$ non-zero, and take $b_0^* \in B^*$ with $\langle b_0^*, b_0 \rangle = 1$. Applying the operator $T : X \hat{\otimes} B \to X$ specified by $T(a \otimes b) = \langle b_0^*, b \rangle x$ to both sides of (2) yields

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a),$$

where $m_i' = \sum_k \langle b_0^*, b_0 b_{k,i} \rangle x_{k,i}, n_i' = \sum_k \langle b_0^*, b_{k,i} b_0 \rangle x_{k,i}$.

In clause (iii), the same argument with the extra condition that

$$|| (a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) || \leq K ||a|| ||b||,$$

yields

$$|| a \cdot m_i' - n_i' \cdot a || \leq K' ||a||.$$

For clause (ii), let $D : A \to X^*$ be a continuous derivation into a dual bimodule. Since $X^* \hat{\otimes} B$ is unlikely to be a dual space, let alone a dual module, view the derivation $\Delta$ as mapping into $(X^* \hat{\otimes} B)^{**}$. Then there is a net $(m_i)$ in $(X^* \hat{\otimes} B)^{**}$ and a constant $K > 0$ such that for all $a \in A, b \in B$,

$$D(a) \otimes b = \Delta(a \otimes b) = \lim_i \left( (a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) \right),$$

and

$$|| (a \otimes b) \cdot m_i - m_i \cdot (a \otimes b) || \leq K ||a|| \||b||.$$

Fix $b_0 \in B$ of unit norm and take $b_0^* \in B^*$ with $\langle b_0^*, b_0 \rangle = 1$. Let $S : X \to (X^* \hat{\otimes} B)^*$ be specified by

$$\langle S(x), x^* \otimes b \rangle = \langle x^*, x \rangle \langle b_0^*, b \rangle,$$

and set $T = S^* : (X^* \hat{\otimes} B)^{**} \to X^*$. Now take $m \in (X^* \hat{\otimes} B)^{**}, a \in A, b \in B,$ and $x \in X$. Then

$$\langle T((a \otimes b) \cdot m), x \rangle = \langle (a \otimes b) \cdot m, S(x) \rangle = \langle m, S(x) \cdot (a \otimes b) \rangle.$$
For \( x^* \in X^* \) and \( c \in B \),
\[
\langle S(x) \cdot (a \otimes b_0), x^* \otimes c \rangle = \langle S(x), (a \otimes b_0) \cdot (x^* \otimes c) \rangle = \langle S(x), a \cdot x^* \otimes b_0 \rangle \tag{6}
\]
\[
= \langle S(x), a \cdot x^* \otimes b_0 \rangle = \langle a \cdot x^*, x \rangle \langle b_0^*, b_0 \rangle. \tag{7}
\]
Thus, setting \( m = \sum_k x_k^* \otimes b_k \), and \( x^*(m) = \sum_k \langle b_0^*, b_0 b_k \rangle x_k^* \),
\[
T((a \otimes b_0) \cdot m) = \sum_k \langle b_0^*, b_0 b_k \rangle a \cdot x_k^* = a \cdot x^*(m),
\]
where we have the estimate
\[
\|x^*(m)\| \leq \|b_0\| \|b_0^*\| \|m\|.
\]
A general \( m \in (X^* \hat{\otimes} B)^{**} \) is the weak*-limit of a net \( (\mu_n) \subset X^* \hat{\otimes} B \), bounded by \( \|m\| \), and as an adjoint \( T \) is weak*-to weak*-continuous. It follows that the associated net \( (x^*(\mu_n)) \) is bounded and so has a weak* limit point \( \xi^* \in X^* \) (depending on \( m \)) which satisfies
\[
T((a \otimes b_0) \cdot m) = a \cdot \xi^* \quad (a \in A). \tag{8}
\]
Similarly, there is \( \eta^* \in X^* \) with
\[
T(m \cdot (a \otimes b_0)) = \eta^* \cdot a \quad (a \in A). \tag{9}
\]
Applying \( T \) to \( \xi \) and \( \xi \) with \( b = b_0 \), gives nets \( (m_i^*) \) and \( (n_i^*) \) in \( X^* \) with
\[
D(a) = \lim_i (a \cdot m_i^* - n_i^* \cdot a) \quad (a \in A), \tag{10}
\]
\[
\|a \cdot m_i^* - n_i^* \cdot a\| \leq K \|T\| \|a\| \quad (a \in A). \tag{11}
\]

To get beyond semi-inner we first observe that if
\[
D(a) = \lim_i (a \cdot m_i^* - n_i^* \cdot a) \quad (a \in A), \tag{12}
\]
and \( D \) is a continuous derivation, then for \( a_1, a_2 \in A \),
\[
D(a_1 a_2) = D(a_1) a_2 + a_1 D(a_2)
= \lim_i \left[ (a_1 \cdot m_i^* - n_i^* \cdot a_1) a_2 + a_1 (a_2 \cdot m_i^* - n_i^* \cdot a_2) \right] \tag{13}
\]
and
\[
D(a_1 a_2) = \lim_i (a_1 a_2 \cdot m_i^* - n_i^* \cdot a_1 a_2). \tag{14}
\]
Comparing \( \text{(13)} \) and \( \text{(14)} \) yields
\[
\lim_i (a_1 \cdot (m_i^* - n_i^*) \cdot a_2) = 0. \tag{15}
\]
Moreover, in the “bounded” case, we have
\[
\|a_1 \cdot (m_i^* - n_i^*) \cdot a_2\| \leq 3K \|a_1\| \cdot \|a_2\|. \tag{16}
\]
We can now look at conditions that enable us show that \( m_i^* = n_i^* \), or at least \( m_i^* - n_i^* \to 0 \).
Theorem 3.4. Suppose that $A \hat{\otimes} B$ is approximately amenable (respectively boundedly approximately amenable, boundedly approximately contractible). If $B$ has an element $b_0$ with $b_0 \notin \{b_0b - bb_0 : b \in B\}$, then $A$ is approximately amenable (respectively boundedly approximately amenable, boundedly approximately contractible).

Proof. Choose the functional $b_0^*$ in the proof of Theorem 3.3 to vanish on $\{b_0b - bb_0 : b \in B\}$. Then the resulting nets $(m'_i)$ and $(n'_i)$ are the same. Hence the result.

Natural conditions on $B$ which are sufficient for the above condition are listed in [18]. Note that there is unfortunately no conclusion about approximate amenability of $B$. Of course in special situations more can be said.

Throughout the next theorem $G$ is a locally compact group and $L^1(G)$ is the usual group algebra of $G$.

Theorem 3.5. Suppose that $L^1(G) \hat{\otimes} A$ is approximately amenable (respectively (boundedly) approximately amenable). Then $G$ is amenable and $A$ is approximately amenable (respectively boundedly approximately amenable). Conversely, if $G$ is amenable and $A$ is boundedly approximately amenable with a bounded approximate identity, then $L^1(G) \hat{\otimes} A$ is boundedly approximately amenable.

Proof. Let $\Lambda : f \mapsto \int_G f$ be the augmentation character on $L^1(G)$. Then $T : f \otimes a \mapsto \Lambda(f)a$ gives a continuous epimorphism of $L^1(G) \hat{\otimes} A$ onto $A$. Thus $A$ is approximately amenable (respectively boundedly approximately amenable).

Let $I_0 = \text{Ker} \Lambda$. Since $I_0 \hat{\otimes} A$ is a complemented ideal in $L^1(G) \hat{\otimes} A$, by [8, Corollary 2.4] it has a left approximate identity. Hence $I_0$ has a left approximate identity [8, Theorem 8.2], and so $G$ is amenable by [20, Theorem 5.2].

For the partial converse, $G$ amenable implies $L^1(G)$ amenable, now apply Theorem 2.2.

Note that if $\Lambda(f_0) = 1$ then $L^1(G) \to I_0 : f \mapsto f - \Lambda(f)f_0$ is a bounded projection onto $I_0$, whence it follows that the norm on $I_0 \hat{\otimes} A$ is equivalent to that inherited from $L^1(G) \hat{\otimes} A$. Hence the complementation fact.

Theorem 3.6. Suppose that $A \hat{\otimes} B$ is boundedly approximately contractible (respectively boundedly approximately amenable). Suppose that one of $A$ or $B$ has an identity. Then $A$ and $B$ are boundedly approximately contractible (respectively boundedly approximately amenable).

Proof. Suppose that $B$ has an identity $e$. Then, by Theorem 3.4, $A$ is boundedly approximately contractible (respectively boundedly approximately amenable).

Now let $X$ be a Banach $B$-bimodule. Then

$$X = e \cdot X \cdot e + (1 - e) \cdot X \cdot e + e \cdot X \cdot (1 - e) + (1 - e) \cdot X \cdot (1 - e)$$

is a decomposition into submodules. Given a continuous derivation $D : B \to X$, it decomposes into the sum of 4 derivations into $e \cdot X \cdot e$, $(1 - e) \cdot X \cdot e$ etc. The
last three of these have trivial module action by $B$ on at least one side, so the corresponding derivations are inner. Thus we may suppose that $e \cdot x = x = x \cdot e$ for $x \in X$.

Let $D : B \to X^*$ be a continuous derivation, and consider the nets given by Theorem 3.3. For the boundedly approximately contractible situation, use clause (iii), for boundedly approximately amenable use clause (ii). Putting $a_1 = a_2 = e$ in \(\ref{15}\) we have $m_i - n_i \to 0$, so that \(\ref{10}\) and \(\ref{11}\) give $D$ is boundedly approximately inner.

\[\text{Lemma 3.7.} \quad \text{Let } A \text{ have a bounded approximate identity. Suppose that any continuous derivation from } A \text{ into the dual of a neo-unital bimodule is boundedly weak}^*\text{-approximately inner. Then } A \text{ is boundedly weak}^*\text{-approximately amenable, and so approximately amenable.} \]

\[\text{Proof.} \quad \text{Let } X \text{ be a general } A\text{-bimodule, } D : A \to X^* \text{ a continuous derivation. Let } (e_\alpha) \text{ be a bounded approximate identity for } A. \text{ By Cohen’s factorization theorem, } X_{\text{ess}} = A \cdot X \cdot A \text{ is a neo-unital } A\text{-bimodule. Let } E \text{ be a weak}^*\text{-limit point of the left multiplication operators on } X^* \text{ by the elements } e_\alpha, F \text{ similarly for right multiplication. Then } E \text{ and } F \text{ are commuting projections on } X^*, \text{ and give a decomposition}
\]

\[X^* = EFX^* \oplus E(I - F)X^* \oplus (I - E)X^*. \quad \text{(17)}\]

Correspondingly, set

\[D_1 = EFD, D_2 = E(I - F)D, D_3 = (I - E)D.\]

These are easily seen to be derivations into the corresponding summands in \(\text{(17)}\). Now let $\varphi \in (X_{\text{ess}})^*$, and extend it by Hahn-Banach to $\tilde{\varphi} \in X^*$. Then

$\theta(\varphi) = E\tilde{\varphi}$ is easily seen to be a well-defined $A$-bimodule monomorphism of $(X_{\text{ess}})^*$ into $EFX^*$. It is surjective since for $x^* \in X^*$, $\theta(x^*|_{X_{\text{ess}}}) = EFx^*$. Thus $EFX^*$ is isomorphic to $(X_{\text{ess}})^*$, whence $D_1$ is boundedly weak$^*$-approximately inner. Now this weak$^*$-topology is $\sigma((X_{\text{ess}})^*, X_{\text{ess}})$, which is clearly weaker than the restriction of $\sigma(X^*, X)$. The unit ball in $(X_{\text{ess}})^*$ is compact under both topologies by Banach-Alaoglu, and so the two topologies coincide on bounded sets in $(X_{\text{ess}})^*$. Thus $D_1$ is boundedly weak$^*$-approximately inner considered as mapping into $X^*$.

Now the actions of $A$ on the right of $E(I - F)X^*$ and on the left of $(I - E)X^*$ are trivial, and since $A$ has a bounded approximate identity, $D_2$ and $D_3$ are boundedly approximately inner. It follows that $D$ is boundedly weak$^*$-approximately inner.

That $A$ is approximately amenable now follows from \(\text{[14, Proposition 2.1]}\).

\[\text{Remark 3.8.} \quad 1. \text{ The hypothesis here of the derivations being boundedly weak}^*$\text{-approximately inner is used to get equality of two weak}^*\text{-topologies. Subsequently, the boundedness is lost with the appeal to \(\text{[14, Proposition 2.1]}\). It is not known whether } A \text{ must be boundedly approximately amenable.} \]
2. In [14, Proposition 2.1], the argument loses control over boundedness as Goldstine is invoked on the implementing elements, which in general will be unbounded. Indeed, since boundedly approximately contractible gives a bounded approximate identity [4, Corollary 3.4], which approximately amenable algebras need not have [10, Corollary 3.2], the implication (2) ⇒ (1) of [14, Proposition 2.1] fails with the qualifier ‘bounded’. It is not known whether (3) ⇒ (2) fails.

3. Note that by Banach-Steinhaus sequentially weak$^*$-approximately inner implies boundedly weak$^*$-approximately inner.

**Theorem 3.9.** Suppose that $A\hat{\otimes}B$ is boundedly approximately amenable and that $A$ has a bounded approximate identity. Then $A$ is approximately amenable.

**Proof.** Let $D : A \to X^*$ be a continuous derivation into the dual of a neo-unital bimodule $X$. From Theorem 3.3(ii), we have nets $(m_i')$ and $(n_i')$ in $X^*$ such that

$$D(a) = \lim_i (a \cdot m_i' - n_i' \cdot a) \quad (a \in A),$$

and $\|a \cdot m_i' - n_i' \cdot a\| \leq K\|a\|$, where from (15) and (16)

$$\lim_i (a_1 \cdot (m_i' - n_i') \cdot a_2) = 0, \quad \|a_1 \cdot (m_i' - n_i') \cdot a_2\| \leq 3K\|a_1\| \cdot \|a_2\|$$

for $a_1, a_2 \in A$.

In particular, for a given $x \in X$, and $a_1, a_2 \in A$,

$$\langle m_i' - n_i', a_2xa_1 \rangle \to 0, \quad \|\langle m_i' - n_i', a_2xa_1 \rangle\| \leq 3K\|a_1\| \cdot \|a_2\| \cdot \|x\|.$$

Since $X$ is neo-unital, it follows that

$$\langle m_i' - n_i', x \rangle \to 0,$$

and letting $a_1, a_2$ range over an approximate identity with bound $M$,

$$\|m_i' - n_i'\| \leq 3KM^2.$$  

Thus for $a \in A$,

$$D(a) = \text{weak}^* - \lim_i (a \cdot m_i' - m_i' \cdot a), \quad \|a \cdot m_i' - m_i' \cdot a\| \leq 4K\|a\|.$$  

So we have that derivations into duals of neo-unital bimodules are boundedly weak$^*$-approximately inner, and the result follows from Lemma 3.7.

The unwanted ‘bounded’ assumption of Lemma 3.7 and Theorem 3.9 can be removed at the expense of a stronger hypothesis on the bounded approximate identity. However, with this assumption comes a bonus to the conclusion of Theorem 3.9.

**Theorem 3.10.** Suppose that $A\hat{\otimes}B$ is approximately amenable and that one of $A$ or $B$ has a central bounded approximate identity. Then $A$ and $B$ are approximately amenable.
Proof. Suppose that \((e_\alpha)\) is a central bounded approximate identity in \(B\). Let \(D : B \to X^*\) be a continuous derivation into the dual of a bimodule \(X\). From Theorem 3.3(i), we have a nets \((m'_i)\) and \((n'_i)\) in \(X^*\) such that

\[
D(b) = \lim_i (b \cdot m'_i - n'_i \cdot b) \quad (b \in B),
\]

and

\[
\lim_i (b_1 \cdot (m'_i - n'_i) \cdot b_2) = 0 \quad (b_1, b_2 \in B).
\]

Now follow Lemma 3.7 to get \(D_1, D_2\) and \(D_3\). Then for \(b \in B\),

\[
D_1(b) = (w^* - \lim_\alpha (w^* - \lim_\beta e_\alpha D(b)e_\beta) \lim_\beta [e_\alpha (b \cdot m'_i - n'_i \cdot b)e_\beta].
\]

Then (22) and (23) give, using centrality of the bounded approximate identity,

\[
D_1(b) = (w^* - \lim_\alpha (w^* - \lim_\beta \lim_i [e_\alpha (b \cdot m'_i - n'_i \cdot b)e_\beta].
\]

Thus the standard method of considering finite subsets of \(B\) and \(X\), gives a net \((x^*_\gamma) \subset X^*\) such that

\[
D_1(b) = w^* - \lim_\gamma (b \cdot x^*_\gamma - x^*_\gamma \cdot b), \quad (b \in B).
\]

Since \(D_2\) and \(D_3\) are approximately inner we finally deduce that \(D\) is \(w^*\)-approximately inner. Thus \(B\) is approximately amenable.

That \(A\) is approximately amenable is now a consequence of Theorem 3.11.

Finally, an application of our method that improves on the result \([18, \text{Proposition 3.5}]\).

**Theorem 3.11.** Suppose that \(A \hat{\otimes} B\) is amenable. Then \(A\) and \(B\) are amenable.

Proof. Amenability of \(A \hat{\otimes} B\) implies it has a bounded approximate identity, whence so do \(A\) and \(B\), \([8, \text{Theorem 8.2}]\). Now arguing as in Theorem 3.3 until at (1) and using the necessary part of \([13, \text{Proposition 1}]\) we obtain a bounded net \((m_i)\). Then proceed as before to (20) where the net \((m'_i)\) is now bounded. Now use second part of Lemma 3.7 to see that derivations from \(A\) into a dual module are \(w^*\)-approximately inner, with a bounded net of implementing elements. The argument of \([14, \text{Proposition 2.1}]\) now shows that any continuous derivation into any \(A\)-bimodule is approximately inner with a bounded net of implementing elements, that is, \(A\) is amenable by the sufficient part of \([13, \text{Proposition 1}]\).

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