Dark soliton-like magnetic domain walls in a two-dimensional ferromagnetic superfluid

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We report a magnetic domain wall in a uniform ferromagnetic spin-1 condensate, a stable topological excitation characterized by the magnetization having a dark soliton profile with nonvanishing superfluid density. In the absence of magnetic fields, this domain wall relates various distinct solitary excitations in binary condensates through SO(3) spin rotations, which otherwise are unconnected. We find an exact solution for a particular ratio of interaction parameters, and develop an accurate analytic solution applicable to the whole ferromagnetic phase. Studying the dynamics of a quasi-two-dimensional (quasi-2D) system we show that standing wave excitations of the domain wall oscillate without decay, being stable against the snake instability. The domain wall is dynamically unstable to modes that cause the magnetization to twist. However, dynamics in the presence of noise reveals that this “spin twist” instability does not destroy the topological structure of the magnetic domain wall.

Introduction— A spin-1 Bose-Einstein condensate (BEC) is a coherent matter wave field composed of atoms occupying hyperfine spin levels \( |F = 1, m = +1, 0, -1 \rangle \). Spin-mixing collisions allow the atoms to redistribute between the spin levels, and this system exhibits both superfluid and magnetic order \([1–5]\), providing an alternative platform of exploring quantum magnetism. The rotational SO(3) symmetry plays a central role in the spin-1 system, supporting magnetic topological excitations absent in scalar and binary BECs.

A wide class of topological excitations associated with \( \mathbb{Z}_2 \) symmetry breaking are unstable to the so-called snake instability in higher dimensional systems \((d > 1)\). Examples include dark solitons \([6–8]\), phase domain walls \([9, 10]\), and magnetic solitons \([11–13]\), in scalar, coherently coupled, and binary BECs, respectively. In ferromagnetic spin-1 BECs, magnetization serves as the local order parameter quantifying the spontaneous rotational symmetry breaking of the spin degrees of freedom, and magnetic domain walls are interfaces separating regions magnetized in opposite directions. Most work in ferromagnetic spin-1 BECs has instead focused on spin of textures and their the nonequilibrium dynamics \((e.g. [14–22])\). The recent experimental observation of a long-lived nematic domain-wall-vortex composite in an anti-ferromagnetic spin-1 BEC \([23]\) brings a renewed interest in domain walls in magnetic superfluids. The stability of this composite structure has been attributed to the domain wall possessing a soliton-like structure \([23, 24]\). However, in ferromagnetic superfluids very little is known about stable magnetic domain walls in high dimensions and potential connections to vector solitons \([25–27]\).

In this Letter we study a novel magnetic domain wall in a quasi-2D spin-1 ferromagnetic BEC. The corresponding magnetization \( \mathbf{F} \) has the typical profile of a dark soliton \((F, F\rangle\) a \( \pi \) phase (direction of \( \mathbf{F} \)) jump crossing the domain wall and \( \mathbf{F} = \mathbf{0} \) at the centre. In the absence of magnetic fields, SO(3) spin rotations relate a family of degenerate solutions, and we show that for particular rotations the underlying component wavefunction can map onto a range of solitons and domain walls proposed for binary condensates. Thus a distinct set of unrelated non-linear excitations are found to be contained within our solution, unified by its symmetries. We study the quasi-2D dynamics of standing waves on these domain walls and find that periodic oscillations persist without decay. The system has a linear dynamic instability driven by modes localized near the domain wall core that twist the magnetization.
(perpendicular to the unperturbed magnetization). The resulting spin texture corresponds to a chain of spin vortex antivortex pairs along the domain wall. Real-time dynamics in the presence of white noise shows that the magnetic domain wall survives against such unstable modes.

**Formalism for a spin-1 BEC—** The Hamiltonian density of a quasi-2D spin-1 BEC [28] reads

$$\mathcal{H} = \hbar^2 |\nabla \psi|^2 / 2M + g_n |\psi|^2 + g_z S^z \psi^2 + g_F S^x \psi^2,$$

(1)

where the three component wavefunction $\psi = (\psi_1, \psi_0, \psi_{-1})^T$ describes the condensate amplitude in the $m = +1, 0, -1$ sublevels, respectively. Here $M$ is the atomic mass, $g_n > 0$ is the density interaction strength, $g_z$ is the spin-dependent interaction strength, $S = (S_x, S_y, S_z)$ with $S_{x,y,z}$ being the spin-1 matrices [29], and $q$ denotes the quadratic Zeeman energy.

The spin-dependent interaction term allows for spin-mixing collisions in which two $m = 0$ atoms collide and convert into $m = +1$ and $-1$ atoms, and the reverse process.

The dynamics of the field $\psi$ is given by the Gross-Pitaevskii equations (GPEs) $i \hbar \partial \psi / \partial t = \delta \mathcal{H} / \delta (\overline{\partial} \psi^\dagger) \equiv \mathcal{L}_{\text{GP}} \psi$, which in component form is

$$i \hbar \partial \psi_{m+1} / \partial t = [H_0 + g_z (n_0 + n_{m+1} - n_{m+1}) + q] \psi_{m+1} + g_F \psi_{m}^2 \psi_{m+1},$$

(2a)

$$i \hbar \partial \psi_{m} / \partial t = [H_0 + g_z (n_1 + n_{m-1})] \psi_{m} + 2g_F \psi_{m-1} \psi_{m-1},$$

(2b)

where $H_0 = -\hbar^2 \nabla^2 / 2M + g_n n$, with $n = \sum_m n_m$ and $n_m = |\psi_m|^2$ being the total and component densities, respectively. Spin-1 BECs exhibit magnetic order, e.g., the magnetization $\mathbf{F} = \psi \mathbf{S} \psi$ [30] is the order parameter of ferromagnetic phases $|\mathbf{F}| > 0$ for $g_z > 0$ (87Rb or 7Li). In contrast anti-ferromagnetic phases with $g_z > 0$ (23Na) have $\mathbf{F} = 0$. In the absence of magnetic fields, i.e. $q = 0$, $\mathcal{H}$ is invariant under SO(3) spinrotations and the total magnetization $\int d^2 r \mathbf{F}$ is conserved.

**Dark soliton-like magnetic domain walls —** For a uniform ferromagnetic system with total density $n_b$ and at $q = 0$, the energy density $\mathcal{H} = g_n n_b^2 / 2 + g_F |\nabla n_b|^2 / 2$ is minimized for states with $|\mathbf{F}| = n_b$. The chemical potential is $\mu = (g_n + g_z)n_b$. We search for a straight line domain wall connecting the two distinct magnetic ground states characterized by $\mathbf{F} = \pm n_b \hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector along an arbitrary direction [see Fig. 1(a)]. Taking the domain wall along the y-axis, i.e. the core located at $x = 0$, we find a solution of the general form

$$\mathbf{F} \approx n(x) \tanh (x / \xi) \hat{\mathbf{e}},$$

(3)

where $\xi = \hbar / \sqrt{4 |g_F| M n_b}$. This domain wall is of the Ising type, rather than the Bloch or Néel sign, signified by $\mathbf{F}$ vanishing at the core and changing its sign (undergoing a $\pi$ phase jump) across the core. The solution (3) has the characteristic profile of dark soliton and we refer to it as dark soliton-like magnetic domain wall (MDW). Note that this domain wall is in magnetic order but not in the superfluid order, i.e. the superfluid density $n(x)$ does not vanish, but has a dip at the core. When the width of the density dip coincides with $\xi$, occurring at $g_z = -g_n / 2$, Eq. (2) admits an exact solution [31]

$$\mathbf{F}(x) = n_b \tanh \left( \frac{x}{2 \xi} \right) \hat{\mathbf{e}}, \quad n(x) = n_b \left[ 1 - \frac{1}{2} \sech^2 \left( \frac{x}{2 \xi} \right) \right].$$

(4)

This system has a SO(3) symmetry which relates a continuous family of degenerate MDW solutions connected by U(1) gauge and spin $U(\alpha, \beta, \tau) = e^{-i \alpha S^z} e^{-i \beta S^x} e^{-i \tau S^y}$; rotations, where $(\alpha, \beta, \tau)$ are the Euler angles. We illustrate three members of this family in Table I: (i) For the case of an $F_2$ domain wall [i.e. $\hat{\mathbf{e}} = \hat{\mathbf{x}}$, the underlying wavefunctions can have two distinct vector soliton profiles, and the corresponding stationary GPE can be mapped onto that of a miscible binary BEC. (ii) A Sine-Gordon type soliton (SGS) of the phase difference $\theta_d \equiv \theta_{+1} - \theta_{0}$, where $\psi_m = |\psi_m|^2 e^{i \theta_m}$. A SGS also occurs in a coherently-coupled binary BEC [32]. Here the SGS can be produced by a spin rotation of the vector soliton in Table I and the nonlinear spin-mixing interaction provides the necessary couplings between the component phases. (iii) For an $F_2$ domain wall, the corresponding wavefunction coincides with a (density) domain wall of an immiscible binary BEC [33].

In the context of binary BECs, the vector solitons, the SGS and the density domain wall are unrelated. In a spin-1 BEC, these distinct nonlinear excitations are unified by spin rotations of our MDW solution. With inadequate degrees of freedom and symmetries, such connection can not be made within the binary BEC [4]. However it is important to note that the dynamics and stability properties of the MDW reveal the spin-1 nature and exhibit distinct behaviors from related excitations in binary systems (see below).

Away from the exactly solvable point we develop a self-consistent asymptotic analysis of the stationary GPEs at $x \gg \xi$, combined with an account of the local core structure [35], and we find an accurate approximate form for the density

$$n(x) / n_b = \begin{cases} 
\frac{\cosh (x / \lambda \ell)}{a_1 \cosh (x / \lambda \ell) + a_2 b_1} + 1 - \frac{1}{a_1}, & g_z < -\frac{g_n}{2} \xi, \\
1 + \frac{4 b_1 g_z}{2 (g_n + g_z) \cosh^2 (x / \lambda \ell) + g_z}, & g_z < g_n / 2 \lambda
\end{cases}$$

(5a)

(5b)

where $\lambda = -g_z / (g_n + g_z)$, $a_1 = -(2 g_z^2 + 2 g_n g_z + g_z^2) / (3 g_z (g_n + g_z))$, $b_1 = 3 (g_n + g_z) / (2 g_n + g_z)$, and $g_1 = 2 b_1 (2 g_n - 5 g_z) / 3$. Here $\mathbf{F}(x)$ is assumed to take the same form as at the exactly solvable point. The strong spin interaction regime (5a) has an effective density length scale $\ell_d \equiv \lambda \xi$. This regime includes the exactly solvable point, $g_z = -g_n / 2$ with $\lambda = 1$, where $\ell_d = \lambda$, and a single length scale describes the spin and density character of the MDW [here (5a) reduces to Eq. (4)]. The crossover to the weak spin interaction regime (5b) occurs at $g_n + 5 g_z = 0$ where $\lambda = 1 / 2$, given by matching the density widths $\ell / 2$ and $\ell_d$ (see [35]). The density width diverges when $g_z \rightarrow 0$ and $g_z \rightarrow -g_n$. For comparison we calculate numerical MDW results using a gradient flow method [36, 37]. The analytic and numerical results in Figs. 1(b) and (c) show excellent agreement.

Since the magnetization vanishes at the core, there is no spin current across the MDW. However, the nematic tensor
Finite magnetic fields—A magnetic field along the $z$-axis breaks the SO(3) symmetry and the degeneracy of states presented in Table I is lifted. For $q > 0$ the ground state magnetization prefers to be transverse realizing an easy-plane ferromagnetic phase [4, 5]. Here the type-I vector soliton is energetically favored, and exists, with some modifications, in the whole easy-plane phase. At $g_s = -g_n/2$, the exact solution is

$$F_s = \sqrt{1 - \tilde{q}^2} \tan \left( \frac{x}{2\ell_\theta} \right), \quad n(x) = 1 - \frac{1 - \tilde{q}}{2} \text{sech}^2 \left( \frac{x}{2\ell_\theta} \right),$$

where $\tilde{q} = -q/2g_s n_b$, and $\ell_\theta = \ell / \sqrt{1 - \tilde{q}}$. An example of a $q \neq 0$ result is shown in Fig. 1. Note that for $q \neq 0$, the SGS and the binary domain wall are no longer stationary solutions.

Standing waves—A notable feature of the MDW is that it is stable against transverse deformations. We consider easy-plane domains with $F$ along the $x$-axis and two static MDW geometries in the $x$-$y$ plane for $q = 0$: closed circle and open straight line with endpoints attached on the boundaries (see Fig. 2). We excite standing waves on these static MDWs by deforming them transversely. The subsequent time evolution shown in Figs. 2(a)-(c) is periodic and resembles harmonic modes vibrations. Our simulations [40] also show that the standing waves persist without decay [35], indicating that the motion of the localized MDW is well-decoupled from the other degrees of freedom in the system. Since $F_x$ and $F_y$ remain zero, the magnetization conservation manifests itself as a geometrical constraint of the domain wall motion: the area enclosed by the domain wall keeps unchanged. There is no spin current crossing the MDW. The enclosed regions form magnetic bubbles, inside which the magnetization $F_z$ has the opposite orientation from the outer one and such feature remains in the presence of noise (see Fig. 2 and Fig. 4 in [35]). Consequently, propagating open MDWs and expanding/shrinking closed MDWs are prohibited, becoming possible when applying magnetic fields along the $z$-axis.

Dynamical instability—Here we systematically study the stability of the MDW by means of Bogoliubov-de Gennes equations (BdGs). Let us consider a straight infinite long MDW along $y$-axis located in the middle of a slab of width $L_x \gg \ell$. Denoting the stationary MDW as $\psi_s$, we consider

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$(\alpha, \beta, \gamma)$ & type-I: 0 & type-II: $(\pi/2, -\pi/2, -\pi/2)$ & $(-\pi/2, -\pi/4, -\pi/2)$ & $(\pi/2, \pi/2, 0)$ \\
\hline
U(1) & type-I: $e^{x^2/2}$ & type-II: $e^{x^2/2}$ & $e^{x^2/4}$ & $e^{x^2/2}$ \\
\hline
$\psi_{s1}$ & $\psi_{s1} = \sqrt{m_{12}} \tan \left( \frac{x}{2\ell_\theta} \right)$ & $\psi_{s1} = \sqrt{m_{12} e^{x^2/4}}$, $\psi_0 = \sqrt{m_{12} e^{x^2/4}}$ & $\psi_{s1} = \sqrt{m_{12} e^{x^2/4}} [1 + \tanh \left( \frac{x}{2\ell_\theta} \right)]$ & $\psi_{s1} = 0$ \\
\hline
$\theta_{s}(x)$ & $\theta_{s}(x) = 2 \arctan e^{x^2/2}$, $2n_{s1} = n = n/2$ & $\theta_{s}(x) = 2 \arctan e^{x^2/4}$, $2n_{s1} = n = n/2$ & $\theta_{s}(x) = 2 \arctan e^{x^2/2}$, $2n_{s1} = n = n/2$ & $\theta_{s}(x) = 2 \arctan e^{x^2/4}$, $2n_{s1} = n = n/2$ \\
\hline
F & $F_s = n_b \tan \left( \frac{x}{2\ell_\theta} \right)$ & $F_s = n_b \tan \left( \frac{x}{2\ell_\theta} \right)$ & $F_s = n_b \tan \left( \frac{x}{2\ell_\theta} \right)$ & $F_s = n_b \tan \left( \frac{x}{2\ell_\theta} \right)$ \\
\hline
GPE & $0 = [H' + 2g_s n_1 + (g_s + 2g_n)n_0] \psi_{s1}$ & $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ & $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ & $0 = [H' + 2g_s n_1 + (g_s + 2g_n)n_{s1}] \psi_{s1}$ \\
& $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ & $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ & $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ & $0 = [H' + g_s n_0 + 2(g_s + 2g_n)n_{s1}] \psi_{s1}$ \\
& $H' = -\frac{n}{2} \sum \delta^2 - \mu$ & $H' = -\frac{n}{2} \sum \delta^2 - \mu$ & $H' = -\frac{n}{2} \sum \delta^2 - \mu$ & $H' = -\frac{n}{2} \sum \delta^2 - \mu$ \\
\hline
Related system & Vector soliton of a three-component BEC and a miscible binary BEC & Sine-Gordon type soliton, also realized in a coherently-coupled binary BEC & Density domain wall of an immiscible binary BEC & Density domain wall of an immiscible binary BEC \\
\hline
\end{tabular}
\caption{Component representation of the MDW after various spin rotations. Vector soliton sector: type-I vector soliton is chosen as a reference point. In this presentation, the reduced GPEs are related to a miscible binary system and becomes decoupled at $g_s = -g_n/2$, allowing the exact solution. SGS: $\theta_{s}$ satisfies the Sine-Gordon equation. Binary domain wall sector: the reduced GPEs describe an immiscible binary system and the corresponding exact solution coincides with a solution discussed in a different context [34].}
\end{table}
a perturbation $\delta \psi = u(\mathbf{r})e^{-i\omega t} + v(\mathbf{r})e^{i\omega t}$. Linearizing about $\psi = \psi_s + \delta \psi$ in Eq. (2) yields the BdG equations
\[
\begin{pmatrix}
\omega & \mu \\
-\mu & \omega - \Delta\end{pmatrix}
\begin{pmatrix}
u \\
\mu \end{pmatrix} = \begin{pmatrix}
u \\
0 \end{pmatrix},
\]
where the stationary wavefunction satisfies $\mathcal{L}_G \psi_s = \mu \psi_s$, $X = g_s \sum_n S_n \psi_s \psi_s^T S_n + g_s \psi_s \psi_s^T$, and $\Delta = g_s \psi_s \psi_s^T + g_s \sum_n (S_n \psi_s)(S_n \psi_s)^T$. The translational symmetry along y allows us to parameterize the perturbations with the wave-vector $\mathbf{k}_\parallel$ as $u(\mathbf{r}) = u(x)e^{i\mathbf{k}_\parallel \cdot \mathbf{r}}$ and $v(\mathbf{r}) = v(x)e^{i\mathbf{k}_\parallel \cdot \mathbf{r}}$. We numerically solve Eq. (7) with Neumann boundary conditions at x-axis boundaries [42], and find two modes with an imaginary energy [Fig. 3(a)], marking a dynamical instability in the system (a mode that grows exponentially with time).

Different from the snake instability [7], the imaginary part of the excitation energy $\text{Im}(\omega)$ does not vanish as $k_y \to 0$, but instead approaches a finite value [Fig. 3(a)], implying that this instability also exists in 1D. $F_\perp$ is unchanged as the unstable mode grows, however it causes the unmagnetized core of the MDW to develop a $F_\perp$-texture of wavelength $\pi/k_y$. This corresponds to the formation of a chain of “magnetic vortex” cores [43] at the nodes of this texture [Fig. 3(b)].

The $k_y$ range of unstable modes and the magnitude of the imaginary energy is largest at intermediate values of $g_s/g_n$, and increases with increasing $q$ [see inset to Fig. 3(a)]. Based on the magnetic texture created by the unstable mode, we refer to it as spin-twist instability. In dynamics the growth of this instability leads to spin waves of $F_\perp$ and $F_\parallel$, while the topological structure of the MDW in $F_\parallel$ remains unchanged, consistent with the noisy dynamics observed in Fig. 2(d).

Conclusion and outlook—We have presented a novel dark soliton-like magnetic domain wall in a quasi-2D ferromagnetic spin-1 BEC that is stable against the snake instability and supports long-lived standing waves. Through the underlying symmetries of the spin-1 system, we have shown that various distinct nonlinear structures such as the Sine-Gordon soliton (phase domain wall), vector solitons and an immiscible binary density domain wall occurring in unrelated binary systems are the different faces of our stationary magnetic domain wall. For large deformations of the magnetic domain wall from its equilibrium position the motion is periodic but is not harmonic, opening the possibility of exploring rich phenomena of nonlinear dynamics. Our findings might be important in the problem of a coarsening dynamics involving both spin order and superfluid order [22, 44] and could also play a role of connecting stretched polar-core vortices [45, 46].

It will be feasible to observe magnetic domain walls in current experiments with ferromagnetic spinor BECs. The necessary techniques for manipulating the spin degrees of freedom of a spin-1 BEC [13], and for non-destructively measuring its spin dynamics [15] have already been demonstrated. Coupled with a planar or flat-bottom optical trap (e.g. [47, 48]) opens the possibility for investigating of domain wall dynamics. Most work with ferromagnetic spin-1 BECs to date has been conducted with $^{87}$Rb which has $-g_s/g_n \sim 10^{-2}$ and is in the weakly spin-interacting regime. However, recently a $^3$Li spin-1 BEC has been prepared with $-g_s/g_n \sim 0.5$ [49], thus in the strong spin interacting regime close to the exactly solvable point.

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FIG. 3. Unstable spin-twist modes. (a) Spectrum of the unstable modes for two values of $g_s/g_n$. For $g_s/g_n = -1/2$ the bifurcation point ($\omega = 0$) occurs exactly at $k_y \xi_n = 1/\sqrt{2}$. Insets shows the magnitude of the long wavelength instability as $g_s/g_n$ varies. The spin-texture created by the unstable mode [41] at $k_y \xi_n \approx 0.445$ where the maximum imaginary frequency is reached for $g_s/g_n = -1/2$. White circles with + and − indicating positive and negative circulation spin-vortices, respectively. The red arrows and the background color represent transverse magnetization field ($F_\parallel$, $F_\perp$) and longitudinal magnetization $F_z$ respectively.
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The size of the system in $z$-direction is much smaller than the spin healing length $\xi_z = \hbar/\sqrt{2g_{21}M_{n_0}}$.

Generators of the rotational group SO(3):

\[ S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

\[ F_x = (\psi_0^* \psi_{-1} + \psi_0 \psi_{+1})/\sqrt{2}; \quad F_y = [\psi_0^* (\psi_{-1} - \psi_{+1}) - h.c.]/\sqrt{2}; \quad F_z = n_{+1} - n_{-1}. \]

This exact solutions is not in the Manakov regime [50] which corresponds to $g_1 = 0$ in spin-1 BECs, or in the magnetic soliton regime where the total number density is constant [11].

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The BdG equation is $G(u,v)^T = E(u,v)^T$, where $G$ is the BdG operator. The symmetry $\sigma_x G \sigma_x = -G^*$, with $\sigma_x$ the $x$ Pauli matrix, ensures that $(v', u')^T$ is an eigenstate with eigenvalue $-E'$. For the unstable modes, $\text{Re}(E)=0$, hence $-E' = E$. Then any linear combination of $(u,v)^T$ and $(v', u')^T$ is an eigenstate of $G$. Here the unstable modes are chosen to be real, in which case the two unstable modes are identical.

The unstable modes must be localized in space and are insensitive to boundary conditions as long as $L_x \gg \ell$. In 2D vector fields with three components, winding number is not well defined and hence the terminology vortex is quoted.

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SUPPLEMENTAL MATERIALS

APPROXIMATE SOLUTIONS FOR THE MDW

Here we provide a detailed analysis of the stationary magnetic domain wall solution. Specializing to the SGS (see Table 1) we work with the hydrodynamical variables \( \{ n, n_d, \theta_d, \theta_s \} \) where \( n_d = 2n_{s1} - n_0 \) and \( \theta_d = \theta_{s1} + \theta_0 \). Note consistent with the SGS form of the solution we assume that \( \theta_{s1} = \theta_{s1} \) and \( n_{s1} = n_{s1} \). In terms of these variables the stationary GPE (2) for \( q = 0 \) becomes

\[
0 = -\frac{\hbar^2}{2M} \nabla \cdot (n \nabla \theta_s - n_d \nabla \theta_d),
\]

\[
0 = \frac{\hbar^2}{2M} \nabla \cdot (n \nabla \theta_d - n_d \nabla \theta_s) + g_s (n^2 - n_d^2) \sin(2\theta_d),
\]

\[
0 = -\frac{\hbar^2}{2M} \left[ \frac{1}{\sqrt{n + n_d}} \nabla^2 \sqrt{n + n_d} + \frac{1}{\sqrt{n - n_d}} \nabla^2 \sqrt{n - n_d} - \frac{1}{2} (|\nabla \theta_s|^2 + |\nabla \theta_d|^2) + g_s n \cos(2\theta_d) + (g_s + 2g_n)n - 2\mu. \right.
\]

\[
0 = -\frac{\hbar^2}{2M} \left[ \frac{1}{\sqrt{n - n_d}} \nabla^2 \sqrt{n - n_d} - \frac{1}{\sqrt{n + n_d}} \nabla^2 \sqrt{n + n_d} - \nabla \theta_s \cdot \nabla \theta_d \right] - g_s n \cos(2\theta_d) - g_s n_d.
\]

We look for a straight line MDW along the y-axis with a core at \( x = 0 \). For a stationary state, the total number current

\[
J_n = \frac{\hbar^2}{2M} (n \nabla \theta_s - n_d \nabla \theta_d)
\]

should vanish, implying that

\[
n \partial_x \theta_s - n_d \partial_x \theta_d = 0.
\]

Apparently \( \theta_s = 0 \) and \( n_d = 0 \) solve Eq. (10), and for this case Eq. (8) reduces to

\[
0 = \frac{\hbar^2}{2M} \partial_x (n \partial_x \theta_d) + g_s n^2 \sin (2\theta_d),
\]

\[
\mu = \frac{\hbar^2}{4M} \left( \frac{\partial_x \theta_d}{2} - \frac{2 \partial_x^2 \sqrt{n}}{\sqrt{n}} \right) + g_s n \cos^2 \theta_d + g_s n.
\]

The exact solution at \( g_n + 2g_s = 0 \) is

\[
\theta_d(x) = 2 \arctan e^{\kappa x}, \quad n(x) = \frac{\cosh(x/\ell)}{1 + \cosh(x/\ell)} n_b,
\]

where \( \ell = \hbar / \sqrt{|4g_s| M n_b} \), as introduced earlier. The corresponding wavefunction and the transverse magnetization read

\[
\psi_{\pm 1} = \sqrt{\frac{n_b}{8}} e^{-\kappa x/2} \frac{1 + e^{i \kappa x/2}}{\sqrt{1 + \cosh(x/\ell)}}, \quad \psi_0 = \sqrt{2} \psi_{+1};
\]

\[
F_x = n(x) \cos \theta_d(x) = -\frac{\cosh(x/\ell)n_b}{1 + \cosh(x/\ell)} \tanh (x/\ell).
\]

Away from the exactly solvable point we assume that the expression of \( \theta_d(x) \) in Eq. (12) remains a good approximation. The reason for this will become clear later.

Let us examine the asymptotic form of Eq. (11b) far away from the core \( x = 0 \). Assuming that \( g(x) \equiv [n(x) - n_b]/4n_b \) decays slower than \( (\partial_x \theta_d)^2 \sim e^{-2\xi x/\ell} \) for large \( x \gg \ell \) (there is no solution for \( g(x) \) decaying faster than \( e^{-2\xi x/\ell} \)), in the large \( x \) limit, the dominant part of Eq. (11b) reads \( g_n + g_s \) \( g(x) + \theta_0(x) \) in Eq. (12) serves a good approximation in the whole
parameter range. First of all, it captures the main feature of the domain wall in the strongly interacting regime where the exact solution Eq. (12) is found. On the other hand, in the weak interaction limit (|\(g_s/g_n| \ll 1\)) the density \(n\) can be approximated as a constant and the energy density becomes

\[
\mathcal{H} = -\frac{\hbar^2 n}{8M} \nabla \theta_d^2 + \frac{g_n}{2} n^2 b - \frac{1}{2} g_s n^2 b \cos^2 \theta_d. \tag{17}
\]

A local minimum of the energy density, determined by \(\delta \mathcal{H}/\delta \theta_d = 0\), leads to the elliptic sine-Gordon equation

\[
\frac{\hbar^2}{2M} \nabla^2 (2\theta_d) + 2n_b g_s \sin(2\theta_d) = 0, \tag{18}
\]

having the solution \(\theta_d = 2 \arctan e^{x/t}\).

**REAL TIME EVOLUTION**

Here we present further evidence of the MDW stability during dynamics. Figure 4 shows that the topological nature of the MDW, i.e. the \(\pi\) phase jump across the core, is well preserved during the domain wall motion. This is can be also seen in Fig. 5 (a) which shows the profile of the transverse magnetization \(F_x(x, y = 0)\) at different times.

The standing wave excitation on the magnetic domain wall can last a long time without decay. In order to quantify this property, we introduce an overlap function

\[
F_{oi}(t) \equiv \left( \int d^2 \mathbf{r} |F_x(\mathbf{r}, t) - F_x(\mathbf{r}, 0)|^2 \right)^{1/2} \tag{19}
\]

that measures the overlap of transverse magnetization profile at time \(t\) with its initial profile. Figure 5(b) shows \(F_{oi}(t)\) for the open domain wall [Fig. 2(a)], revealing that it exhibits periodic oscillations over a long period of time.

**FIG. 4.** The transverse magnetization \(F_x\) during the time evolution. (c'1)-(c'5) and (d'1)-(d'5) correspond to Fig. 2 (c1)-(c5) and (d1)-(d5) in the main text, respectively.

**FIG. 5.** (a) shows the spin-density cross section \(F_x(x, y = 0)\) of the open MDW shown in Fig. 2(a) at different times. The soliton-like profile of the magnetization is preserved during the motion. (b) shows the periodic behavior of the overlap function for the open domain wall configuration [Fig. 2(a)], demonstrating that the standing wave on the MDW persists without decay over long time periods.