A NOTE ON TRANSVERSE DIVERGENCE AND TAUT RIEmannIAN FoliATIONS

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Abstract. In this note we give a characterization of taut Riemannian foliations using the transverse divergence. This result turns out to be a convenient tool in the case of some standard examples. Furthermore, we show that a classical tautness result of Haefliger can be obtained in our particular setting as a straightforward consequence. In the final part of the paper we obtain a tautness characterization for transversally oriented foliations with dense leaves and investigate the case of a submanifold of lower dimension which is closed and orthogonal to the leaves.

Key Words: Riemannian foliations; transverse divergence; taut foliations.

Mathematics Subject Classification (2010): 57R30, 55R10, 58J50.

1. Introduction

Throughout this paper we consider a $C^\infty$ foliated manifold $(M, \mathcal{F})$, endowed with a Riemannian metric $g$. A particular class of foliations with remarkable dynamical and geometric properties is represented by the taut foliations. A foliation is called taut if there exist a metric on $M$ such that every leaf of $\mathcal{F}$ is a minimal submanifold of the ambient manifold $M$. The work of Sullivan [24] and later of Haefliger [8] on compact foliated manifolds offered a characterization of these aspects, showing that tautness is basically related to the transverse geometry of the foliation.

An usual additional condition for a foliation is the existence of a bundle-like metric; with respect to this metric the foliation can be locally identified with a Riemannian submersion [20]. This particular class of foliations are called Riemannian foliations. Examples of this type occur for instance in the theory of locally conformal Kähler manifolds and their submanifolds [5, 27], in the case of Sasakian manifolds [3], while the characteristic leafwise holonomy invariance is relevant from the point of view of Yang-Mills equations (see e.g. [9] for further references related to particle physics).

For Riemannian foliations, the tautness properties has been intensively studied [1, 4, 7, 11, 15, 21]; for a chronological and more detailed presentation we refer to [17, 21]. Remarkably, the tautness depends on the topology of the manifold, and a characterization in terms of the basic cohomology is possible. More exactly, if the manifold $M$ is closed (i.e. compact and without boundary), then it can be shown that the foliation is taut if and only if a certain basic cohomology class $\xi(\mathcal{F})$-called Álvarez class, is trivial [1]. If, furthermore, the foliation is transversally orientable, then it is taut if and only if the top dimensional cohomology group does not vanish [1, 11, 15].
As a consequence, in many papers the study of taut foliations is based on the topological and cohomological aspects. Thus, when investigating particular examples, beside a direct study of the basic cohomology groups (see e.g. [4]), a tautness result via some adapted vanishing techniques can be expected at most in the case of foliations with positive defined transverse curvature operator (see [1, 14]).

In the present note we take a different approach, establishing a new characterization of taut Riemannian foliations using the transverse divergence operator. From the dynamical point of view, a foliation can be regarded as a generalization of a differential manifold (the classical setting is obtained in the absolute case when the manifold is foliated by points). On a closed Riemannian manifold we cannot construct vector fields with positive, non-vanishing divergence, as a trivial consequence of the Green theorem. Considering the transverse geometry of the foliations as a generalization for the geometry of the manifolds and the transverse divergence as an extension of the classical differential operator, we prove in this note that taut Riemannian foliations are precisely the extended framework of Riemannian manifolds when the above classical fact still holds.

As this result does not rely on basic cohomology and Bochner technique, we show that it can be applied on classical examples with non-positive transverse curvature operator.

It also turns out that some classical statements from the theory of taut Riemannian foliations can be derived as straightforward consequences of our analytical characterization. In particular, a classical tautness result of Haefliger [8], asserting that the existence of a closed transverse submanifold ensures the tautness is proved for our particular framework, revealing an interesting link between this result and the Green theorem. Also, an important extension to the finite Riemannian covering spaces of the tautness properties proved in [1] can be alternatively derived using our arguments in a simpler manner.

We also give a tautness characterization for transversely oriented Riemannian foliation with dense leaves using the transverse volume form. In the final part of the paper we investigate the foliations considering transverse manifolds with lower dimension.

The paper is organized as follows: in the second section we define the main geometric objects we use in this note and briefly state the relevant previous results. The central result of the note is proved in the third section, while in the last part we present the above straightforward applications.

2. Preliminaries

We start out this section considering a differential manifold $M$ endowed with a Riemannian metric structure $g$ and with a non-necessary integrable distribution $D$ of dimension $d$. In the following we use the classical musical isomorphisms $\flat$ and $\sharp$ [19]. Let $\{E_i\}_{1 \leq i \leq d}$ and $\{E_\flat^i\}_{1 \leq i \leq d}$ be a local orthonormal frame and coframe for $D$ with respect to the metric $g$.

The action of the divergence operator associated to $D$ on a vector field $v \in \Gamma (TM)$ is defined using the Levi-Civita connection $\nabla$

\[
\text{div}^D v := \sum_{i=1}^{d} g(\nabla_{E_i} v, E_i).
\]
In the following, we briefly present the interplay between the divergence operator and a (locally defined) volume form \( \nu_D \) of \( D \) induced in the canonical way by the metric \( g \). According to [2] (see also [26]), if \( \mathcal{L}_v \) is the Lie derivative along the vector field \( v \), we consider the restriction \( \theta_v := \mathcal{L}_v|_D \); for instance, if \( 1 \leq i, j \leq d \), then

\[
\mathcal{L}_v E_i^\nu(E_j) = \left( v(E_i^\nu(E_j)) - E_i^\nu([v, E_j]) \right)
= -g([v, E_j], E_i),
\]

and, consequently

\[
\theta_v E_i^\nu = - \sum_{j=1}^{d} g([v, E_j], E_i) E_j^\nu.
\]

Now, taking \( \nu_D = E_1^\nu \wedge \ldots \wedge E_d^\nu \), as the Levi-Civita connection has vanishing torsion, we get

\[
\theta_v \nu_D = \theta_v E_1^\nu \wedge \ldots \wedge E_d^\nu + \ldots + E_1^\nu \wedge \ldots \wedge \theta_v E_d^\nu
= - \sum_{j=1}^{d} g([v, E_j], E_j) \nu_D
= \text{div}^D v \cdot \nu_D.
\]

**Remark 1.** In the classical case when \( D \equiv TM \), we get the well-known equivalent definition for the divergence operator (see e.g. [13])

\[
\mathcal{L}_v \nu_{TM} = \text{div}^TM v \cdot \nu_{TM}.
\]

Considering the complementary distribution \( D^\perp \) with respect to the metric \( g \) and taking the canonical projections \( \pi_D \) and \( \pi_{D^\perp} \), if \( v \in D^\perp \), then it is easy to see that

\[
\theta_v \nu_D = - g(v, \pi_{D^\perp} (\sum_{j=1}^{d} \nabla E_j E_j)) \nu_D.
\]

Using the musical isomorphism \( \sharp \) we can define the mean curvature vector field \( \kappa_D^\sharp := \pi_{D^\perp} (\sum_{j=1}^{d} \nabla E_j E_j) \), the mean curvature form \( \kappa_D \) being then subject to the condition \( \kappa_D(U) = g(\kappa_D^\sharp, U) \), for any vector field \( U \) (see e.g. [28]).

After we made the above considerations in a larger setting, from now on the framework of this paper will be represented by a smooth, closed Riemannian foliation \( (M, \mathcal{F}, g) \). The leafwise distribution tangent to leaves will be denoted by \( T\mathcal{F} \); a corresponding transverse distribution \( Q = T\mathcal{F}^\perp \cong TM/T\mathcal{F} \) is obtained. Let us assume \( \dim M = n \), \( \dim T\mathcal{F} = p \) and \( \dim Q = q \), with \( p + q = n \).

A first consequence is the splitting of the tangent and the cotangent vector bundles associated with \( M \)

\[
TM = Q \oplus T\mathcal{F},
TM^* = Q^* \oplus T\mathcal{F}^*.
\]

For local investigation of the transverse geometry of the foliated manifold we will use local orthonormal basis \( \{E_i, F_a\} \) defined on a neighborhood of a point \( p \in M \); \( \{E_i\}_{1 \leq i \leq q} \) will span the distribution \( Q \) while \( \{F_a\}_{1 \leq a \leq p} \) will span the distribution \( T\mathcal{F} \). It is also convenient to employ basic (projectable) local vector fields \( \{E_i\}_{1 \leq i \leq q} \), defined by the condition [26]

\[
\pi_Q ([U, E_i]) = 0
\]
for any $U \in \Gamma(T\mathcal{F})$.

Another important transverse geometric object is represented by the basic de Rham complex. It is defined as a restriction of the set of classical differential forms $\Omega(M)$ [26]

$$\Omega_b(M) := \{ \omega \in \Omega(M) \mid \iota_U \omega = \mathcal{L}_U \omega = 0 \} ,$$

where $U$ is again an arbitrary leafwise vector field, while $\iota$ stands for interior product. The corresponding basic de Rham derivative $d_b$ comes as a restriction of the classical de Rham derivative, $d_b := d\mid_{\Omega_b(M)}$. It is also possible to define as well the adjoint operator, namely the basic co-derivative $\delta_b$ (see e.g. [1]). The basic cohomology groups of the basic de Rham complex $(\Omega_b(M), d_b)$ are defined in a classical way [26] and are denoted by $H^i_b(F)$, with $1 \leq i \leq q$.

One differential form of particular importance (which may not be necessarily a basic differential form) is the mean curvature form associated to the leafwise distribution $T\mathcal{F}$. We denote it by $\kappa$, in accordance with [1, 26]. As above, it is subject to the relation $\kappa^\sharp := \pi_Q \left( \sum_{a=1}^p \nabla F_a F_a \right)$.

Returning now to the relation (2) and (3), in the case of a Riemannian foliation with oriented leaves, if $D \equiv T\mathcal{F}$ and $\chi_F$ is the leafwise volume form, for $v \in \Gamma(Q)$ we obtain the well known relation between $\kappa$ and $\chi_F$ [22, 26].

$$\theta_v \chi_F = \text{div}^T_F v \cdot \chi_F = -\kappa(v) \cdot \chi_F .$$

If the distribution $D \equiv Q$, then using [1] we obtain the transverse divergence operator $\text{div}^Q$ [26]. We state here the following useful relation

$$\text{div}^Q v = \text{div} v + g(v, \kappa^\sharp) ,$$

which holds for any vector field $v \in \Gamma(TM)$.

A deeper investigation of the mean curvature form on Riemannian foliations is presented in [1]; the author show that we have the orthogonal decomposition $\Omega(M) = \Omega_b(M) \oplus \Omega_b(M)^\perp$, with respect to the $C^\infty$-Frechet topology. Thus, on any Riemannian foliation, the mean curvature form can be decomposed as the sum $\kappa = \kappa_b + \kappa_o$, where $\kappa_b \in \Omega_b(M)$ is the basic component of the mean curvature, $\kappa_o$ being the orthogonal complement. From now on we denote $\tau := \kappa^\sharp_b$.

Remark 2. A fundamental property of $\kappa_b$ is that it is a closed differential basic form, the corresponding basic cohomology class being denoted by $\xi(F)$; this cohomology class (called Álvarez class [16, 17]) is independent on the metric $g$.

In [1] the author also initiates a study of two types of metric transformations which leave the transverse metric on the normal bundle intact. The first is based on a change of the transversal sub-bundle $Q$, while the second is a conformal change of the leafwise metric. It turns out that these are in fact the fundamental metric changes needed in order to study the basic component of the mean curvature. With respect to the above metric the basic component of the mean curvature $\kappa_b$ varies, but remains inside $\xi(F)$. 
Consequently, a first remarkable characterization of the taut Riemannian foliation using basic cohomology is obtained [1].

**Theorem 1** (Álvarez López [1]). A Riemannian foliation $F$ on a compact manifold is taut if and only if $\xi(F) = 0$. Moreover, when $F$ is transversely oriented, the foliation is taut if and only if $H^b_0(F) \neq 0$.

For results of this type in the non-compact case see [21].

Furthermore, concerning the above metric changes, a relevant result in this direction is due to Domínguez.

**Theorem 2** (Domínguez [6]). The bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) such that the new bundle-like metric has basic mean curvature.

In order to obtain the above result the author uses a composition of these metric changes that ensures the vanishing of the orthogonal part $\kappa_o$ of the mean curvature form $\kappa$, while the basic part $\kappa_b$ holds. We point out that quite recently a partial generalization for non-compact case was presented [17].

Finally, we state the following important result of Mason, which is achieved employing a conformal change of the leafwise metric which arises from the theory of stochastic flows.

**Theorem 3** (Mason [12]). Furthermore, the above bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) into a metric with basic-harmonic mean curvature.

### 3. A characterization of taut Riemannian foliations

In the following, we show that using the above changes of the bundle-like metric we can derive a characterization of the non-taut Riemannian foliation of arbitrary codimension using the transverse divergence.

**Lemma 1.** Let us consider a Riemannian foliation with bundle-like metric $(M, g, F)$. If $v$ is a basic vector field such that $\text{div}^Q v \geq 0$ at any point and $\text{div}^Q v > 0$ at some point on $M$, then the foliation is non-taut.

**Proof.** As the metric $g$ is bundle-like, it is easy to see that $\text{div}^Q v = \text{div}^Q \pi_Q(v)$, so we can assume that $v$ is also perpendicular to the leaves at any point. Let us consider the transverse Green formula (see e.g. [20]):

$$\int_M \text{div}^Q v \, d\mu = \int_M g(v, k^:) \, d\mu,$$

the integrals being taken with respect to the measure $d\mu$ induced by the metric $g$.

We point out that the above formula arises from the relation (5) and the theorem of Green on closed, non-necessarily orientable Riemannian manifold (see e.g. [19]).

Let us assume that it is possible to change the bundle-like metric $g$ in order to obtain a new metric $g'$ in such a way that the new mean curvature 1-form vanishes, $\kappa'_b = 0$, the transverse part of the metric remaining intact. We consider the relation

$$\text{div}^Q v = \sum_{i=1}^q \langle E_i, \nabla_{E_i} v^: \rangle,$$
the musical isomorphism being considered with respect to the initial metric \( g \). It turns out that \( v^\flat \) is a basic 1-form on \( F \). Now, taking the basic local vector fields \( \{ E'_i \}_{1 \leq i \leq q} \) as a transverse orthonormal frame with respect to the new metric \( g' \) spanning the new orthogonal complement \( Q' \) of \( T_F \), as the transverse part of the metric is invariant, we see that

\[
\sum_{i=1}^{q} \iota_{E'_i} \nabla_{E'_i} v^\flat = \sum_{i=1}^{q} \iota_{E'_i} \nabla'_{E'_i} v^\flat ,
\]

and

\[
\text{div}^Q v = \text{div}^{Q'} v' ,
\]

where we denote \( v' := (v^\flat)' \), this time the operator \( \sharp' \) being constructed with respect to the bundle-like metric \( g' \); note that \( v' \) may not equal \( v \), as \( Q' \) may not coincide with \( Q \). In fact, regarding locally the Riemannian foliation as a Riemannian submersion, if we consider a local transverse manifold \( T \), as all geometrical objects are projectable, then both operators equal \(-\delta_T\), where \( \delta_T \) is the classical de Rham coderivative on \( T \) (see e.g. [1]). Integrating the real function \( \text{div}^{Q} v \) over the closed manifold \( M \) with respect to the modified metric, we have

\[
\int_M \text{div}^{Q} v \, d\mu' = \int_M \text{div}^{Q'} v' \, d\mu' = \int_M g'(v', \kappa'^\sharp') \, d\mu' .
\]

As we assumed \( \kappa' = 0 \), we get also \( \kappa'^\sharp' = 0 \), and \( \int_M \text{div}^{Q} v \, d\mu' = 0 \). Now the contradiction comes from the positivity of \( \text{div}^{Q} v \). \( \square \)

In the following we study the converse statement.

**Lemma 2.** If \((M, g, F)\) is non-taut, then we can construct a basic vector field \( v \) such that \( \text{div}^{Q} v \geq 0 \) and \( \text{div}^{Q} v > 0 \) at some point on \( M \).

**Proof.** Now let \((M, g, F)\) be a non-taut Riemannian foliation. According to [6][11], we can change the metric \( g \), keeping the transversal part intact, such that, with respect to the new metric \( g' \), the mean curvature \( k' \) is basic and harmonic. Then, in accordance with [1],

\[
\delta_b k' = -\sum_{i=1}^{q} \iota_{E'_i} \nabla'_{E'_i} k' + \iota_{\tau'} k' = 0 ,
\]

where \( \iota \) stands for interior multiplication and \( \tau' \) is defined like \( \tau \) for \((M, g', F)\).

On the other side,

\[
\sum_{i=1}^{q} \iota_{E'_i} \nabla'_{E'_i} k' = \text{div}^{Q} \tau' \text{ and } \iota_{\tau'} k' = \| \tau' \|^2 ,
\]

and thus we get

\[
\text{div}^{Q} \tau' = \| \tau' \|^2 .
\]
As the foliation is non-taut, then \( \tau' \neq 0 \), so \( \text{div}^Q\tau' \geq 0 \) and \( \text{div}^Q\tau' > 0 \) at some point. Now, we take

\[
\tau_{\xi(F),g} := \left( (\tau')\gamma \right)^\sharp = (\kappa')^\sharp.
\]

Let us remark that the transverse vector field \( \tau_{\xi(F),g} \) is determined only by the basic cohomology class \( \xi(F) \) of the foliation and the initial bundle-like metric \( g \). As before, we obtain

\[
\text{div}^Q\tau_{\xi(F),g} = \text{div}^Q\tau'.
\]

We notice that \( \tau' \) and \( \tau_{\xi(F),g} \) are basic vector fields, while \( \text{div}^Q\tau_{\xi(F),g} \) is a basic real function, which finishes the proof. \( \Box \)

**Remark 3.** As \( \tau_{\xi(F),g} \) is the image of the exact differential form \( \kappa' \) from the basic cohomology class \( \xi(F) \) through the musical isomorphism \( \sharp \), we get that

\[
\|\tau_{\xi(F),g}\| = \|\kappa'\|,
\]

where the norm is defined in the classical way taking integrals on the closed Riemannian manifold \( M \). Thus, considering the basic Hodge-de Rham theory based on heat equation method (see e.g. [2, 26]) if we define \( \xi(F)^\sharp := \{ \omega^\sharp \mid \omega \in \xi(F) \} \), then \( \tau_{\xi(F),g} \in \xi(F)^\sharp \) can be characterized as the vector field for which \( \min_{v \in \xi(F)^\sharp} \|v\| \) is attained. A more refined study of \( \tau_{\xi(F),g} \) can be carried out using [11].

Summing up, we obtain a characterization of non-taut foliations in terms of the transverse divergence.

**Theorem 4.** Let \((M, g, F)\) be a Riemannian foliation defined on a closed manifold \( M \). Then the foliation is non-taut if and only if there is a basic vector field \( v \) on \( M \) such that \( \text{div}^Qv \geq 0 \) and \( \text{div}^Qv > 0 \) at some point, where \( \text{div}^Q \) is the transverse divergence operator associated to the metric \( g \).

**Corollary 1.** From Theorem 4 we can obtain equivalently a characterization of Riemannian taut foliations; i.e. a Riemannian foliation on a closed manifold is taut if and only if for any basic vector field \( v \) we have \( \text{div}^Qv = 0 \), or there are at least two points \( p_1 \) and \( p_2 \) such that \( \text{div}^Q_{p_1}v \geq 0 \) and \( \text{div}^Q_{p_2}v \leq 0 \).

**Remark 4.** If we assume the transverse orientation, then another result that holds only on taut Riemannian foliation is represented by the Poicaré duality (see e.g. [11]). This, together with the above result lead us to regard taut foliations as the maximal extension of closed Riemannian manifolds when some classical features are still valid.

We end this section indicating [18] for the interplay between divergence operator and taut, codimension-one foliations with arbitrary Riemannian metric.

4. **Several applications**

In the following we test the above tautness criterion on some standard examples of Riemannian foliations.

**Example 1.** Let us start out by considering a classical Riemannian flow [4, 12, 25, 26]. We take a matrix \( A \in \text{SL}(2, \mathbb{Z}) \), with trace condition \( \text{Tr}A > 2 \); for instance
we can choose \( A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \). If \( \lambda_1, \lambda_2 \) are the eigenvalues, it is easy to see that
\[
0 < \lambda_1 \cdot \lambda_2 = 1, \lambda_i \neq 1, \lambda_i > 0,
\]
for \( 1 \leq i \leq 2 \). Let \( V_1, V_2 \) be the corresponding orthonormal eigenvectors. Then, one can define a semi-direct product \( H = \mathbb{R} \times \mathbb{R}^2 \), where \( \{ \varphi_t \}_{t \in \mathbb{R}} \) is defined as
\[
\varphi_t(x) = A^t x, \quad \forall x \in \mathbb{R}^2.
\]

We get a Lie group \( H \), such that if \( p_1, p_2 \in H, \), \( p_1 = (t_1, x_1) \), and \( p_2 = (t_2, x_2) \), with \( t_1, t_2 \in \mathbb{R}, x_1, x_2 \in \mathbb{R}^2 \), then the multiplication will be defined in the following way:
\[
p_1 \cdot p_2 = (t_1 + t_2, A^{t_1} x_2 + x_1).
\]

Now, if \( x_1 = \alpha_1 V_1 + \beta_1 V_2, x_2 = \alpha_2 V_1 + \beta_2 V_2, \) as \( \mathbb{R} \times \mathbb{R} \mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^3 \), we can consider \( p_1 = (t_1, \alpha_1, \beta_1), p_2 = (t_2, \alpha_2, \beta_2) \), and the above multiplication can be written as:
\[
p_1 \cdot p_2 = (t_1 + t_2, \lambda_1^t \alpha_2 + \alpha_1, \lambda_2^t \beta_2 + \beta_1).
\]

As \( \mathbb{R} \times \mathbb{R} \mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2 \times \mathbb{R} \), it turns out that the first term becomes isomorphic with the orientation preserving affine group \( GA \), defined as
\[
GA := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}_{a \in \mathbb{R}^+, b \in \mathbb{R}},
\]
the isomorphism being expressed by the correspondence \([4]\):
\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto (\log \lambda_1, a, b).
\]

Is it now easy to see that \( \Gamma := \mathbb{Z} \times \mathbb{Z} \mathbb{Z} \) represents in fact a discrete and uniform subgroup.

Let us now introduce a left invariant metric on \( H \). With respect to the above identification of \( \mathbb{R}^3 \), at \( e = (0, 0, 0) \) we take the vectors \( \{(E_1)_e, (E_2)_e, (E_3)_e\} \), with \( (E_1)_e = (1, 0, 0), (E_2)_e = (0, 1, 0) \) and \( (E_3)_e = (0, 0, 1) \).

For any \( p = (t_p, \alpha_p, \beta_p) \in H \), let \( t_p \) be the left multiplication by \( p \), and \( L_p := dl_p \) the differential application. If we consider the curve \( c : \mathbb{R} \rightarrow H \) given by \( c(s) := s \ (E_1)_e \), then \( (E_1)_e = \frac{dc}{ds} \mid_{s=0} \) and, as a consequence, we define:
\[
(E_1)_p = L_p((E_1)_e)
= \frac{dc}{ds} \mid_{s=0} \left(t_p + s, \lambda_1^{r_p} 0 + \alpha_p, \lambda_2^{r_p} 0 + \beta_p\right)
= (1, 0, 0)_p.
\]

Analogously, we consider:
\[
(E_2)_p = \lambda_1^{r_p}(0, 1, 0)_p ,
(E_3)_p = \lambda_2^{r_p}(0, 0, 1)_p .
\]

Now the metric tensor \( g_H \) will be chosen such that \( \{(E_1)_p, (E_2)_p, (E_3)_p\} \) will be an orthonormal basis at each point \( p = (t_p, \alpha_p, \beta_p) \in H \). The resulting metric tensor will be a warped metric tensor, and being a left invariant metric, \( g_H \) will be also \( \Gamma \)-invariant. Consequently, we can consider the metric \( g \) induced on the Lie group \( T^3_A = \Gamma \setminus H \), which becomes a \( GA \)-foliated manifold when considering the flow induced by \( E_3 \) \([4]\).
We calculate the Lie brackets
\[ [E_1, E_2] = \ln \lambda_1 E_2, \]
\[ [E_1, E_3] = \ln \lambda_2 E_3, \]
\[ [E_2, E_3] = 0. \]

If we consider the Cartan and Christoffel coefficients
\[
C^k_{ij} E_k = [E_i, E_j], \\
\Gamma^k_{ij} = g(\nabla E_i E_j, E_k)
\]
for \(1 \leq i, j, k \leq 3\), using the Koszul formula we get
\[
\Gamma^k_{ij} = \frac{1}{2} \left( C^k_{ij} + C^j_{ki} + C^i_{kj} \right).
\]
We can calculate now the necessary coefficients
\[
\Gamma^1_{11} = \Gamma^2_{33} = \Gamma^1_{12} = 0, \\
\Gamma^1_{33} = \ln \lambda_2, \Gamma^2_{21} = -\ln \lambda_1.
\]
As a consequence we obtain the mean curvature vector field
\[
\tau = \Gamma^1_{33} E_1 + \Gamma^2_{33} E_2 = \ln \lambda_2 E_1.
\]
Now, we take \(v := \tau\), and obtain
\[
\text{div}^Q \tau = \ln \lambda_2 \Gamma^1_{11} + \ln \lambda_2 \Gamma^2_{21} = -\lambda_2 \cdot \ln \lambda_1 = (\ln \lambda_1)^2 > 0.
\]

From Theorem 4 we obtain that the flow \(T^A_\lambda\) is non-taut.

**Remark 5.** The same conclusion can be reached using a direct study of the basic Rham cohomology \([4]\). On the other side, it is also easy to see that for the above GA-foliated manifold the transverse curvature operator is non-positive, so the basic cohomology cannot be investigated using vanishing results or other classical analytical method (see e.g. \([14, 23]\)).

**Remark 6.** A natural question is if on Riemannian non-taut foliations all globally defined basic vector fields share the above property regarding the transverse divergence. Let us notice that if \(v\) is basic and \(v \perp \kappa^\sharp\) at any point, then from \([2]\) we obtain that \(\text{div}^Q v = \text{div} v\), and we get either \(\text{div}^Q v \equiv 0\), or this function has positive and negative values on the manifold \(M\). As a direct consequence \(\tau_{\xi(F),g}\) is not perpendicular on \(\kappa^\sharp\) at any point. For instance, in the above example, considering the basic vector field \(E_2\), we can calculate \(\text{div}^Q E_2 = \Gamma^1_{12} = 0\).

**Example 2.** We can obtain another example increasing the transversal dimension of the foliations; namely, we consider a matrix \(A \in \text{SL}(3, \Z)\), for instance
\[
A = \begin{pmatrix}
2 & 0 & -1 \\
0 & 3 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\]
If \( \{\lambda_i\}_{1 \leq i \leq 3} \) are the eigenvalues and \( \{V_i\}_{1 \leq i \leq 3} \) the corresponding eigenvectors, then calculating the sign of the characteristic polynomial of \( A \) we obtain that

\[
0 < \lambda_1 < 1, \ 2 < \lambda_2 < 3, \ 3 < \lambda_2 < 4,
\]
and, as in the previous example

\[
\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1, \ \lambda_i \neq 1, \ \lambda_i > 0,
\]
for \( 1 \leq i \leq 3 \). Introducing the bundle-like metric in a similar manner as above, for \( v := \tau \) we finally end up with

\[
\text{div}^Q \tau = - \ln \lambda_2 (\ln \lambda_1 + \ln \lambda_3) = (\ln \lambda_2)^2 > 0,
\]
which show us that the Riemannian foliation is non-taut.

**Remark 7.** For a proof using cohomological arguments see [6].

Finally, we give an application of Corollary 1.

**Example 3.** We may consider the torus \( T^2 := \mathbb{R}^2/\mathbb{Z}^2 \) with the metric \( g = e^{2f(y)} dx^2 + dy^2 \), for some periodic function \( f \); as consequence, \( \{\partial_y, e^{-f(y)} \partial_x\} \) represent an orthonormal basis at any point, \( Q = \text{span}\{\partial_y\} \), \( T_F = \text{span}\{e^{-f(y)} \partial_x\} \). Then any \( C^\infty \) transversal vector field \( v \) will be written as

\[
v = \varphi(y) \partial_y,
\]
\( \varphi(y) \) being again a periodic \( C^\infty \) function. Then, the transverse divergence can be calculated

\[
\text{div}^Q v = g (\nabla \varphi(y) \partial_y, \partial_y) = \frac{\partial \varphi(y)}{\partial y}.
\]

Now, we have

\[
\varphi(0) = \varphi(1).
\]

If \( \varphi(y) \) is constant with respect to \( y \), then \( \text{div}^Q v = 0 \); otherwise we will have \( \frac{\partial \varphi(y)}{\partial y} > 0 \) at some points and \( \frac{\partial \varphi(y)}{\partial y} < 0 \) at other points. The tautness of the foliation can be obtained from Corollary [1].

As pointed out in the introductory section, from the characterization theorem we can obtain a simpler proof of some well-known results.

In the following, let us first of all refer to the classical tautness result of Haefliger concerning foliation with arbitrary metric [3]. Precisely, the author proves that tautness is a transverse invariant; in fact, a characterization of tautness is given using the holonomy pseudogroup, and this characterization is trivially satisfied when there is a representative of the holonomy pseudogroup defined on a compact manifold. Consequently, tautness holds when there is a closed transversal which cuts all leaves defined on our foliation [3, Corollary 2].

In the particular case of foliations with bundle-like metric, following Remark 4 we give a short proof of this result applying the Green theorem.

Let us consider a closed transverse manifold \( T \) and suppose by abstract nonsense that the foliation is non-taut. If at some point \( p \in M \) the manifold \( T \) meets a leaf, then at that point it generates a subspace \( Q_p \) of the tangent space \( T_p M \), which is a complement of \( T_p F \). It is easy to see that \( Q_p \) can be determined by a
unique retraction $r_p : T_p M \to T_p \mathcal{F}$, with $\ker r_p = Q_p$. Considering now the vector bundle constructed over $M$ using these mathematical objects, we can extend the complementary distribution $Q$ from the closed subset $T$ to the whole manifold $M$ as a classical extension of a section in a fibre bundle (see e.g. [10, Theorem 5.7]).

In a similar way as in [1, 6], requesting that $Q_p \perp T_p \mathcal{F}$ at each $p$, we can obtain a new metric which keeps the transverse and the leafwise part of the initial metric, but this time $T$ is a transverse orthogonal manifold. The foliation is still non-taut.

According to the characterization, there will be some basic vector field $v$ which can be also made perpendicular to the leaves, such that $\text{div}^Q v \geq 0$ and $\text{div}^Q v > 0$ at some point. At any point where $T$ meets a leaf $\text{div}^Q v$ in fact equals the classical divergence operator $\text{div}^T v$ on $T$; as $T$ meets all the leaves and the function $\text{div}^Q v$ is constant along leaves, we will have $\text{div}^T v \geq 0$ and $\text{div}^T v > 0$ at some point on $T$. This fact is impossible considering the classical Green theorem, this time on the closed manifold $T$.

Next, let us stress the fact that the orientation in the leaves represents an interesting aspect from the point of view of taut Riemannian foliations. In [1, Lemma 6.3] the author proves that the foliation $\mathcal{F}$ on $M$ is taut if and only if the foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$ is taut, where $\pi : \tilde{M} \to M$ is a Riemannian finite covering and $\tilde{\mathcal{F}} = \pi^* (\mathcal{F})$. Consequently, the condition regarding orientation in the leaves can be removed from previous results in [11, 15], and the second part of Theorem 1 can be obtained.

As pointed out in [1], in order to achieve the result we cannot take the sum of metrics in some averaging process to obtain a minimizing metric; consequently the proof was obtained using Sullivan purification [24] and Rummelr formula [22]. In turn, with respect to a chosen metric, a finite sum of basic vector fields will always preserve the positivity of transverse divergence. We present a swift proof of the result below.

If $\mathcal{F}$ is non-taut, then a basic vector field $v$ with $\text{div}^Q v \geq 0$ and $\text{div}^Q v > 0$ at some point exists on $M$. Using the local isometry $\pi$ we can consider on $\tilde{M}$ the pull-back $\pi^* (v)$ which will verify the positivity of transverse divergence, so $\tilde{\mathcal{F}}$ will be non-taut. Conversely, assume that $\tilde{\mathcal{F}}$ is non-taut and $\tilde{v}$ is a basic vector field on $\tilde{M}$ such that $\text{div}^Q \tilde{v} \geq 0$ and $\text{div}^Q \tilde{v} > 0$ at some point. Let $\Gamma$ be the group of deck transformation of $\pi$. Then the finite sum $\sum_{\gamma \in \Gamma} \gamma^* (\tilde{v})$ represent a basic vector field on $\tilde{M}$ which also verify the divergence property and it can be projected on $M$ to obtain a vector $v$ with the same features; consequently, $\mathcal{F}$ is non-taut.

In the final part of the paper we briefly present some new consequences of the main result of Section 3. First, we use Corollary [1] to obtain a characterization of the Riemannian foliations transversally oriented and with dense leaves.

**Proposition 1.** If the foliation is transversally oriented and has dense leaves, then it is taut if and only if any globally defined basic vector field $v$ preserves the transverse volume form $\nu_Q$, i.e.

$$L_v \nu_Q = 0. \tag{6}$$

**Proof.** From the defining property of the basic vector field $v$ it is easy to see that the differential operators $L_v$ and $\theta_v$ agree on $\Omega_b (M)$ for $v$ basic. As the foliation has dense leaves, the smooth basic function $\text{div}^Q v$ is constant, then in accordance
with Corollary 1 \( \text{div}^Q v \) vanishes on \( M \). The conclusion comes now from the relation (2), written for the transverse distribution \( Q \).

\[\blacksquare\]

**Remark 8.** Consequently, the image of any projectable vector field on a local transversal \( T \) is solenoidal and volume preserving (see e.g. [13]).

Finally, we investigate the case of a closed manifold of lower dimension, perpendicular to the leaves of the foliations. Starting with (2) and (3), we need some additional hypothesis concerning the divergence operator relative to some sub-distribution of the transverse distribution \( Q \).

More precisely, we prove the following result.

**Proposition 2.** Let us consider a foliated manifold with bundle-like metric \((M, F, g)\) and assume that \( \tau_{\xi(F), g} \) (the vector field for which \( \min_{v \in \xi(F)} \|v\| \) is attained) is known to be in \( \Gamma(Q^1) \), where \( Q^1 \subset Q \) is a sub-distribution of \( Q \) of dimension \( q_1 \leq q \). We assume also that \( \pi_{Q^1}(\sum_{j=1}^{q-q_1} \nabla_{E_j} E_j) = 0 \), where \( \{E_i\}_{1 \leq i \leq q-q_1} \) is an orthonormal local frame for \( Q^1_\perp \), with \( Q = Q^1 \oplus Q^1_\perp \). Under these circumstances, if for any leaf \( L \) of \( F \) we can construct a closed transversal \( T \), integral with respect to the distribution \( Q_1 \) such that \( T \) intersects \( L \), then the foliation \((M, F, g)\) is taut.

**Proof.** Let us assume that the foliation is non-taut; from the above consideration we obtain that \( \text{div}^Q \tau_{\xi(F), g} \geq 0 \) and \( \text{div}^Q \tau_{\xi(F), g} > 0 \) at some point \( p \in M \). If \( L \) is a leaf, \( p \in L \), let \( T \) be a closed manifold, integral with respect to distribution \( Q_1 \), with \( p' = T \cap L \). Then, from (2) and (3) we see that

\[
\text{div}^Q \tau_{\xi(F), g} = \text{div}^{Q^1} \tau_{\xi(F), g}.
\]

From the assumption, we have that \( \text{div}^{Q^1} \tau_{\xi(F), g} \geq 0 \) and \( \text{div}^{Q^1} \tau_{\xi(F), g} > 0 \) at \( p' \). As above, we see that on \( T \) we can identify \( \text{div}^{Q^1} \tau_{\xi(F), g} \) with \( \text{div}^T \tau_{\xi(F), g} \), and the contradiction comes from the theorem of divergence on \( T \). Consequently, the foliation \((M, F, g)\) is taut. \[\blacksquare\]

**Remark 9.** For the case \( Q_1 \equiv Q \), the dimension of the closed manifold perpendicular to the leaves is \( q \), and we obtain the above classical case.

**Acknowledgements**

The author would like to thank J. A. Álvarez López for helpful conversations.

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