GATEAUX DERIVATIVE OF $C^*$ NORM

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ABSTRACT. We find an expression for Gateaux derivative of the $C^*$-algebra norm. This gives us alternative proofs or generalizations of various known results on the closely related notions of subdifferential sets, smooth points and Birkhoff-James orthogonality for spaces $\mathcal{B}(H)$ and $C_0(\Omega)$. We also obtain an expression for subdifferential sets of the norm function at $A \in \mathcal{B}(H)$ and a characterization of orthogonality of an operator $A \in \mathcal{B}(H,K)$ to a subspace, under the condition $\operatorname{dist}(A, \mathcal{K}(H)) < \|A\|$ and $\operatorname{dist}(A, \mathcal{K}(H,K)) < \|A\|$ respectively.

1. INTRODUCTION

Let $\mathbb{F}$ stand for $\mathbb{R}$ or $\mathbb{C}$. Let $A$ be a $C^*$-algebra over $\mathbb{F}$. Let $C_0(\Omega)$ be a $C^*$-algebra of bounded $\mathbb{F}$-valued functions on a locally compact Hausdorff space $\Omega$ and $C_0(\Omega) \subseteq C_0(\Omega)$ be the space of functions vanishing at infinity. It is well known that every commutative $C^*$-algebra is isomorphic to $C_0(\Omega)$. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces over $\mathbb{F}$ with the inner product $\langle \cdot | \cdot \rangle$. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ be the normed spaces of bounded and compact $\mathbb{F}$-linear operators from $\mathcal{H}$ to $\mathcal{K}$ with operator norm, respectively. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ stand for $\mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathcal{K}(\mathcal{H},\mathcal{H})$, respectively. It is also well known that any $C^*$-algebra is isomorphic to a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Let $(\mathcal{V}, \| \cdot \|)$ be a normed space over $\mathbb{F}$. Let $v \in \mathcal{V}$ and let $W$ be a subspace of $\mathcal{V}$. Let $\operatorname{dist}(v, W)$ denotes $\inf \{ \| v - w \| : w \in W \}$. The Gateaux derivative of $\| \cdot \|$ at $v$ in direction of $u$ is defined as $\lim_{t \to 0} \frac{\| v + tu \| - \| v \|}{t}$. We say $\| \cdot \|$ is Gateaux differentiable at $0 \neq v$ if and only if for all $u \in \mathcal{V}$, $\lim_{t \to 0} \frac{\| v + tu \| - \| v \|}{t}$ exists. A concept related to the Gateaux derivative of norm function is the subdifferential set of norm function (see [9]). The subdifferential set of norm function at $v \in \mathcal{V}$, denoted by $\partial \| v \|$, is the set of bounded linear functionals $f \in \mathcal{V}^*$ satisfying $\| v \| - \| v \| \geq \Re f(u - v)$ for all $u \in \mathcal{V}$. It is easy to prove that for $0 \neq u, v \in \mathcal{V}$, $\partial \| v \| = \{ f \in \mathcal{V}^* : f(v) = \| v \|, \| f \| = 1 \}$ and

$$
\lim_{t \to 0^+} \frac{\| v + tu \| - \| v \|}{t} = \max \{ \Re f(u) : f \in \partial \| v \| \}.
$$

There is a notion to define orthogonality in a normed space, called Birkhoff-James orthogonality which is closely related to the Gateaux derivative of norm function (see [11]). We say $v$ is said to be Birkhoff-James orthogonal to $W$ if $\| v \| \leq \| v + w \|$ for all $w \in W$. Let $u, v \in \mathcal{V}$, then $v$ is said to be Birkhoff-James orthogonal to $u$ if and only if $v$ is Birkhoff-James orthogonal to $\mathcal{F}u$. In [14], the concept of $\phi$-Gateaux derivative was introduced which also gives a characterization of Birkhoff-James orthogonality. The number $D_{v,v}(u) = \lim_{t \to 0^+} \frac{\| v + te^{i\phi}u \| - \| v \|}{t}$ is called the $\phi$-Gateaux derivative of $\| \cdot \|$ at $v$, in the $u$ and $\phi$ directions.

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\end{itemize}
Proposition 1.1. [12, Proposition 1.5] The limit $D_{\phi,v}(u)$ always exists for $u, v \in V$ and $\phi \in [0, 2\pi)$. And $v$ is Birkhoff-James orthogonal to $u$ if and only if $\inf_{\phi} D_{\phi,v}(u) \geq 0$.

In [12], an expression for the $\phi$-Gateaux derivative of $B(H)$ norm was obtained and as an application, a proof of a known characterization of smooth points and Birkhoff-James orthogonality of an operator to another operator in $B(H)$ was given. In this paper, an expression for $\phi$-Gateaux derivative of the norm of $A$ in terms of states is obtained and various applications are discussed. Few definitions are in order. A positive element in $A$ is of the form $a^*a$ for some $a \in A$. A positive functional on $A$ is a linear functional which takes positive elements of $A$ to non-negative real numbers. For $F = \mathbb{C}$, a state $\psi$ on $A$ is positive functional of norm one. For $F = \mathbb{R}$, an additional requirement for $\psi$ to be a state is that $\psi(a^*) = \psi(a)$ for all $a \in A$. Let $S_A$ be the set of states on $A$. The main theorem of this paper is as below.

Theorem 1.2. Let $0 \neq a, b \in A$. Then
\[
\lim_{t \to 0^+} \frac{||a + tb|| - ||a||}{t} = \frac{1}{||a||} \max \{ Re \psi(a^*b) : \psi \in S_A, \psi(a^*a) = ||a||^2 \}.
\]

It is well known that if $I$ is a two-sided closed ideal of $A$, then any $\psi \in S_I$ has a unique extension $\tilde{\psi} \in S_A$. We will be using the same notation $\psi$ for the extension $\tilde{\psi}$ throughout the article. As an application of the above theorem, we get the following corollary.

Corollary 1.3. Let $I$ be a two-sided closed ideal of $A$. Let $0 \neq a \in A$ such that $\text{dist}(a, I) < ||a||$. Then for any $0 \neq b \in A$, we have
\[
\lim_{t \to 0^+} \frac{||a + tb|| - ||a||}{t} = \frac{1}{||a||} \max \{ Re \psi(a^*b) : \psi \in S_I, ||\psi|| = 1, \psi(a^*a) = ||a||^2 \}.
\]

Another concept which is closely related to the Gateaux derivative of norm function and Birkhoff-James orthogonality is that of the smooth points of the unit ball in a normed space. We say that a vector $v$ of norm one is a smooth point of the unit ball of $V$ if there exists a unique functional $F_v$, called the support functional, such that $||F_v|| = 1$ and $F_v(v) = 1$. It is a general fact that $v$ is a smooth point of the unit ball of $V$ if and only if the norm is Gateaux differentiable at $v$. And in that case, $\lim_{t \to 0} \frac{||v + tu|| - ||v||}{t} = Re F_v(u)$. Further, $v$ is Birkhoff-James orthogonal to $u$ if and only if $F_v(u) = 0$ (see [1] Theorem 2.1 and [12, Proposition 1.3]). Along the lines of proof of Corollary 2.2 of [13], we get that the smooth points of the unit ball of $C_0(\Omega)$ are those $f \in C_0(\Omega)$ such that $||f||_\infty = 1$ such that $|f|$ is equal to 1 at exactly one point. For a characterization of smooth points of the unit ball of $C_0(\Omega)$ for $\Omega$ a normal space, see [13, Corollary 3.2].

In Section 2, we give the proof of Theorem 1.2 and Corollary 1.3. In Section 3, we obtain different proofs for several known results and their generalizations. In Corollary 3.6, a proof for a characterization of Birkhoff-James orthogonality of an element to a subspace in a $C^*$-algebra is obtained. In Corollary 3.6, we obtain an expression for Gateaux derivative of norm function of $B(H)$ at $A$ in direction of $B$ when $\text{dist}(A, K(H)) < ||A||$. Using Corollary 3.6, an alternate proof for a known characterization of smooth points of the unit ball of $B(H)$ is given in Corollary 3.7. In Corollary 3.8, we obtain the subdifferential set $\partial ||A||$, when $\text{dist}(A, K(H)) < ||A||$.

2. Proofs

To prove theorem 1.2, we need the following lemma.
Lemma 2.1. Let \( a, b \in A \). Then
\[
\lim_{t \to 0^+} \frac{\|a + tb\| - \|a\|}{t} = \frac{1}{\|a\|} \lim_{t \to 0^+} \frac{|a^*a + ta^*b| - |a^*a|}{t}.
\]

Proof. We have
\[
\frac{1}{\|a\|} \lim_{t \to 0^+} \frac{|a^*a + ta^*b| - |a^*a|}{t} \leq \frac{1}{\|a\|} \lim_{t \to 0^+} \frac{|a^*a| \left(\|a + tb\| - \|a\|\right)}{t} = \lim_{t \to 0^+} \frac{|a + tb| - \|a\|}{t}.
\]
Now we need to prove
\[
\lim_{t \to 0^+} \frac{|a + tb| - \|a\|}{t} \leq \frac{1}{\|a\|} \lim_{t \to 0^+} \frac{|a^*a + ta^*b| - |a^*a|}{t}.
\]
\[
\lim_{t \to 0^+} \frac{|a + tb| - \|a\|}{t} = \lim_{t \to 0^+} \frac{|a + tb|^2 - \|a\|^2}{t} = \frac{1}{2\|a\|} \lim_{t \to 0^+} \frac{|a^*a + t(a^*b + b^*a) + t^2b^*b| - |a^*a|}{t} = \frac{1}{2\|a\|} \lim_{t \to 0^+} \frac{|a^*a + t(a^*b + b^*a)| - |a^*a|}{t} \leq \frac{1}{2\|a\|} \lim_{t \to 0^+} \frac{|a^*a + 2ta^*b| + |a^*a + 2tb^*a| - 2|a^*a|}{2t} = \frac{1}{2\|a\|} \lim_{t \to 0^+} \frac{|a^*a + ta^*b| - |a^*a|}{t}.
\]

\[\square\]

Lemma 2.2. Let \( f \in A^* \) of norm one such that \( f(a^*a) = \|a^*a\| \) for some \( a \neq 0 \). Then \( f \in S(A) \).

Proof. We extend \( f \) to a functional \( \tilde{f} \) of norm one on \( A^+ \), where \( A^+ \) is unitization of \( A \). Using Theorem II.6.3.4 of [3], we have \( \tilde{f} = (f_1 - f_2) + \iota(f_3 - f_4) \) where \( f_i \) is positive linear functional with \( \|f_1 - f_2\| = \|f_1\| + \|f_2\| \) and \( \|f_3 - f_4\| = \|f_3\| + \|f_4\| \). Now \( \tilde{f}(a^*a) = \|a^*a\| \) implies \( f_1(a^*a) = \|a^*a\| \) and \( \|f_1\| = 1 \). This gives \( f_1(1_A) = 1 \) and thus \( \tilde{f}(1_A) = 1 \). And using Theorem II.6.2.5(ii), \( \tilde{f} \) is a positive functional, hence \( f \) is a positive functional and thus a state.

\[\square\]

Proof of Theorem 1.2 We have \( \partial \|a^*a\| = \{ f \in A^* : \|f\| = 1, f(a^*a) = \|a^*a\| \} \). Now any linear functional \( f \) of norm one such that \( f(a^*a) = \|a^*a\| \) for some \( a \neq 0 \) is a state by Lemma 2.2. Hence we get \( \partial \|a^*a\| = \{ \psi \in S_A : \psi(a^*a) = \|a^*a\| \} \). Now using Lemma 2.1 and equation (1.1), we get
\[
\lim_{t \to 0^+} \frac{|a + tb| - \|a\|}{t} = \frac{1}{\|a\|} \lim_{t \to 0^+} \frac{|a^*a + ta^*b| - |a^*a|}{t} = \frac{1}{\|a\|} \max\{\Re \psi(a^*b) : \psi \in \partial \|a^*a\|\}.
\]

Before proving Corollary 1.3, we note that Theorem 3.1 of [5] follows as a corollary of Lemma 2.1 and Proposition 1.1.

Corollary 2.3. [5, Theorem 3.1] Let \( a \in A \). Let \( B \) be a subspace of \( A \). Then \( a \) is Birkhoff-James orthogonal to \( B \) if and only if \( a^*a \) is Birkhoff-James orthogonal to \( a^*B \).
We will denote the closed convex hull of a set $A$ by $\text{conv}(A)$. And for a convex set $C$, $\text{ext}(C)$ will denote the set of extreme points of $C$. We now prove Corollary 1.3 by imitating the proof of Lemma 3.1 of [20].

Proof of Corollary 1.3 We prove that if $\psi \in A^*$ such that $\psi(a^*a) = \|a\|^2$, then $\|\psi\| = 1$. The result then follows from Theorem 1.2. Now any two-sided ideal is an $M$-ideal in $A$, by Theorem V.4.4 of [6]. So $A^* = I^# \oplus_j I^\perp$, by Proposition I.1.12 of [6], where $I^# = \{f \in A^* : \|f\| = \|f\|_1\}$. Now let $K = \{f \in A^* : \|f\| = 1, f(a^*a) = \|a\|^2\}$. Then $K$ is a weak*-compact convex subset of $A^*$. By Krein-Milman theorem, $K = \text{conv}(\text{ext}(K))$. Since $K$ is a face of $A^*$, $\text{ext}(K) \subseteq \text{ext}(I^#) \cup \text{ext}(I^\perp)$. If we prove $\text{ext}(K) \cap I^\perp = \emptyset$, then $\text{ext}(K) \subseteq \text{ext}(I^#)$, and hence $K \subseteq I^#$. Let $f \in I^\perp$ and $\|f\| = 1$. We show $f \notin K$ by proving $f(a^*a) \neq \|a\|^2$. As an application of Corollary 2.3, we get that $\text{dist}(a, I) < \|a\|$ implies $\text{dist}(a^*a, I) < \|a\|^2$. Now $\text{dist}(a^*a, I) < \|a\|^2$ implies there exists $b \in I$ such that $\|a^*a - b\| < \|a\|^2$. Now $f(a^*a) = f(a^*a - b) \leq \|a^*a - b\| < \|a\|^2$. □

3. Applications

Using the Hahn-Banach theorem with Corollary 2.3 we get the following generalization of Theorem 3.1 of [5].

Corollary 3.1. Let $I$ be a two-sided closed ideal of $A$. Let $a \in A$ such that $\text{dist}(a, I) < \|a\|$. Let $B$ be a subspace of $A$. Then $a$ is Birkhoff-James orthogonal to $B$ if and only if there exists $\psi \in S_I$ such that $\psi(a^*a) = \|a\|^2$ with $\psi(a^*b) = 0$ for all $b \in I$.

Using the Riesz Representation Theorem with Corollary 3.1 we get the following generalization of Corollary 3.2 of [5].

Corollary 3.2. Let $f \in C_b(\Omega)$ such that $\text{dist}(f, C_0(\Omega)) < \|f\|_\infty$. Let $B$ be a subspace of $C_b(\Omega)$. Then $f$ is Birkhoff-James orthogonal to $B$ if and only if there exists a regular Borel probability measure $\mu$ on $\Omega$ such that $\text{supp}(\mu)$ is contained in $\{x \in \Omega : |f(x)| = \|f\|_\infty\}$ and $\int f \, d\mu = 0$ for all $h \in B$. If $f$ attains its norm at only one point $x_0$, then $f$ is orthogonal to $B$ if and only if $h(x_0) = 0$ for all $h \in B$.

Next we obtain a characterization of orthogonality in $\mathcal{B}(\mathcal{H}, K)$ by using Corollary 3.1. To do so, we require the following lemma, proved in [5]. For $u, v \in \mathcal{H}$, $u \overset{\circ}{\otimes} v$ denotes the finite rank operator of rank one on $\mathcal{H}$ defined as $u \overset{\circ}{\otimes} v(w) = \langle v|w \rangle u$ for all $w \in \mathcal{H}$. Let $\mathcal{G}_1(\mathcal{H})$ denotes the space of trace class on $\mathcal{H}$, with the trace norm $\|\cdot\|_1$.

Lemma 3.3. ([5, Lemma 3.8]) Let $A \in \mathcal{B}(\mathcal{H})$. Let $T \in \mathcal{G}_1(\mathcal{H})$ be a positive element of $\mathcal{B}(\mathcal{H})$ with $\|T\|_1 = 1$ and $\text{tr}(AT) = \|A\|$. Then there is an atmost countable index set $J$, a set of positive numbers $\{s_j : j \in J\}$ and an orthonormal set $\{u_j : j \in J\} \subseteq \text{Ker}(T)^\perp$ such that

(i) $\sum_{j \in J} s_j = 1$,
(ii) $Au_j = \|A\| u_j$ for each $j \in J$,
and
(iii) $T = \sum_{j \in J} s_j u_j \overset{\circ}{\otimes} u_j$.

We know that $\mathcal{K}(\mathcal{H})^*$ is isomorphic to $\mathcal{G}_1(\mathcal{H})$ and an isomorphism is given by $\mathcal{F} : \mathcal{G}_1(\mathcal{H}) \to \mathcal{K}(\mathcal{H})^*$ by $\mathcal{F}(T)(B) = \text{tr}(BT)$ for all $B \in \mathcal{K}(\mathcal{H})$. And $\mathcal{F}(T) \in S_{K(\mathcal{H})}$ if and only if $T$ is a
positive element of $\mathcal{K}(\mathcal{H})$ with $\|T\|_1 = 1$. We will be using this identification throughout this paper.

**Corollary 3.4.** Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $\text{dist}(A, \mathcal{K}(\mathcal{H}, \mathcal{K})) < \|A\|$. Let $B \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a subspace. Then $A$ is Birkhoff-James orthogonal to $B$ if and only if there exists at most countable set $J$, a set of positive numbers $\{s_j : j \in J\}$ and an orthonormal set $\{u_j \in \mathcal{H} : j \in J\}$ such that

(i) $\sum_{j \in J} s_j = 1$,
(ii) $A^*Au_j = \|A\|^2u_j$ for each $j \in J$, and
(iii) $\sum_{j \in J} s_j\langle Au_j|Bu_j\rangle = 0$ for all $B \in B$.

**Proof.** We can embed $\mathcal{B}(\mathcal{H}, \mathcal{K})$ into $\mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K})$ isometrically as $T \rightarrow \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$. Then using Corollary 2.3 for $\mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K})$, we get $A$ is Birkhoff-James orthogonal to $B$ if and only if $A^*A$ is Birkhoff-James orthogonal to $A^*B$. Since $\text{dist}(A, \mathcal{K}(\mathcal{H}, \mathcal{K})) < \|A\|$, then $\text{dist}(A^*A, \mathcal{K}(\mathcal{H})) < \|A^*A\|$. Hence using Corollary 3.1, we get that there exists $T \in \mathcal{C}_1(\mathcal{H})$ such that $\text{tr}(A^*AT) = \|A^*A\|$ and $\text{tr}(A^*BT) = 0$ for all $B \in B$. And now the result follows using Lemma 3.3.

Using Corollary 3.4 and Hausdorff-Toeplitz theorem, we get the following generalizations of Theorem 1.1 of [2].

**Corollary 3.5.** Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\text{dist}(A, \mathcal{K}(\mathcal{H}, \mathcal{K})) < \|A\|$. Let $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $A$ is Birkhoff-James orthogonal to $B$ if and only if there exists a sequence of unit vectors $u_n \in \mathcal{H}$ such that $\|Au_n\| = \|A\|$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \langle Au_n|Bu_n\rangle = 0$.

It is worth mentioning that $A \in \mathcal{B}(\mathcal{H})$ such that the set $H_0 = \{u \in \mathcal{B}(\mathcal{H}) : \|Au\| = \|A\|\}$ is finite dimensional and $\|A\|_{H_0^*} < \|A\|$, then $\text{dist}(A, \mathcal{K}(\mathcal{H})) < \|A\|$. And now Theorem 2.8 of [15] Theorem 3.1 of [17] follows from Corollary 3.4.

In Theorem 2.6 of [14], Keckic proved that if $\mathcal{H}$ is a separable Hilbert space and $A, B \in \mathcal{K}(\mathcal{H})$, then $\lim_{t \rightarrow 0^+} \frac{\|A + tB\| - \|A\|}{t} = \max_{\|u\| = 1, \|u\| = 1} \text{Re}\langle u|U^*Bu\rangle$, where $A = U|A|$ is polar decomposition. The next result gives the generalization of this, for any Hilbert space $\mathcal{H}$ and for $A, B \in \mathcal{B}(\mathcal{H})$ with $\text{dist}(A, \mathcal{K}(\mathcal{H})) < \|A\|$.

**Corollary 3.6.** Let $0 \neq A \in \mathcal{B}(\mathcal{H})$ be such that $\text{dist}(A, \mathcal{K}(\mathcal{H})) < \|A\|$, then for any $0 \neq B \in \mathcal{B}(\mathcal{H})$

$$\lim_{t \rightarrow 0^+} \frac{\|A + tB\| - \|A\|}{t} = \max_{\|u\| = 1, \|u\| = 1} \text{Re}\langle v|Bu\rangle$$

\begin{equation}
(3.1)
\end{equation}

**Proof.** Using Theorem 1.2, we have

$$\lim_{t \rightarrow 0^+} \frac{\|A + tB\| - \|A\|}{t} = \frac{1}{\|A\|} \max \left\{ \text{Re} \, \text{tr}(A^*BT) : T \text{ is a positive element of } \mathcal{K}(\mathcal{H}), \|T\|_1 = 1, \text{tr}(A^*AT) = \|A\|^2 \right\}.$$
Now using Lemma 3.3, we get
\[
\lim_{t \to 0^+} \frac{\|A + tB\| - \|A\|}{t} = \frac{1}{\|A\|} \max \left\{ \sum_{j \in J} s_j \langle A^* Bu_j \rangle : s_j > 0, \sum_{j} s_j = 1, \|u_j\| = 1, A^* Au_j = \|A\|^2 u_j, J \text{ is a countable set} \right\}.
\]

This gives the required result. \qed

The immediate consequence of Corollary 3.6 is a characterization of smooth points of unit ball of \(B(H)\), which was first given in [1] Theorem 3.1 (for \(F = \mathbb{R}\)) and then an alternative proof was given in [12] Corollary 3.3 (for \(F = \mathbb{R}\) or \(\mathbb{C}\)). The proof we have given is simpler.

**Corollary 3.7.** An operator \(A\) is a smooth point of the unit ball of \(B(H)\) if and only if \(A\) attains its norm at \(\pm h\) with \(\|h\| = 1\) and \(\sup_{x \perp h, \|x\|=1} \|Ax\| < \|A\|\). In that case, \(\lim_{t \to 0} \frac{\|A + tB\| - \|A\|}{t} = \text{Re} \langle Ah|Bh \rangle\).

**Proof.** It was shown in Theorem 1 of [15], if \(\text{dist}(A, \mathcal{K}(H)) = \|A\|\), then \(A\) is not a smooth point of the unit ball of \(B(H)\). It is well known fact that if \(A\) is a smooth point, then it can't attain its norm at more than one point (see [3] Theorem 2.1). Hence we only need to prove that if \(A\) attains its norm at \(\pm h\) with \(\|h\| = 1\) and \(\sup_{x \perp h, \|x\|=1} \|Ax\| < \|A\|\), then \(A\) is a smooth point. In this case, using Corollary 3.6
\[
\lim_{t \to 0^+} \frac{\|A + tB\| - \|A\|}{t} = \frac{1}{\|A\|} \text{Re} \langle Ah|Bh \rangle.
\]

And we also have
\[
\lim_{t \to 0^-} \frac{\|A + tB\| - \|A\|}{t} = \lim_{t \to 0^-} \frac{\|A - t(-B)\| - \|A\|}{t} = -\lim_{t \to 0^+} \frac{\|A + t(-B)\| - \|A\|}{t} = -\frac{1}{\|A\|} \text{Re} \langle Ah - Bh \rangle = \frac{1}{\|A\|} \text{Re} \langle Ah|Bh \rangle.
\]

Hence \(\lim_{t \to 0} \frac{\|A + tB\| - \|A\|}{t}\) exists and equal to \(\text{Re} \langle Ah|Bh \rangle\). Since norm function is Gateaux differentiable at \(A\), \(A\) is a smooth point of the unit ball of \(B(H)\). \qed

The special case of the above theorem in the setting of \(\mathcal{K}(H)\) for separable Hilbert space was first proved in [10] Theorem 3.3. Using the above expression of Gateaux derivative and the fact that \(\mathcal{C}_1(H)\) is the dual of \(\mathcal{K}(H)\), we get that rank one operators are extreme points of \(\mathcal{C}_1(H)\), along the lines of proof of [1] Corollary 3.3. Characterization of the extreme points of \(\mathcal{C}_1(H)\) was first proved in [10] Theorem 3.1. Study of extreme points and smooth points has been a subject of interest for many authors, see [1] [7] [10] [15] [13]. Subdifferential sets in space of matrices \(M_n(F)\) equipped with various norms have also been an interest to many authors, see [21] for a brief description. Using Corollary 3.6 we now give the following expression for the subdifferential set \(\partial \|A\|\) when \(\text{dist}(A, \mathcal{K}(H)) < \|A\|\), with the idea used in [19] Theorem 2.

**Corollary 3.8.** Let \(0 \neq A \in B(H)\) be such that \(\text{dist}(A, \mathcal{K}(H)) < \|A\|\), then
\[
\partial \|A\| = \text{conv} \{u \otimes v : \|u\| = \|v\| = 1, Au = \|A\| v\}.
\]
Proof: Clearly \( \text{conv}\{u \otimes v : \|u\| = \|v\| = 1, Au = \|A\| v\} \subseteq \partial \|A\| \). If equality doesn’t occur, then there exists \( T \in \partial \|A\| \subsetneq C \) such that \( T \notin \text{conv}\{u \otimes v : \|u\| = \|v\| = 1, Au = \|A\| v\} \).

Then by the Hahn-Banach separation theorem, there exists \( B \in \mathcal{X}(\mathcal{H}) \) such that \( \text{tr}(TB) = 1 \) and \( \text{tr}(B(u \otimes v)) = \langle u|Bv \rangle = 0 \) for all \( u, v \in \mathcal{H} \) such that \( \|u\| = \|v\| = 1, Au = \|A\| v \).

Hence
\[
\max_{\|u\| = \|v\| = 1, Au = \|A\| v} \Re \langle u|Bv \rangle < \text{tr}(BT) \leq \max\{\text{tr}(BT) : T \in \partial \|A\|\} = \lim_{t \to 0^+} \frac{\|A + tB\| - \|A\|}{t},
\]
which contradicts Corollary 3.6.

4. Remarks

1. Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Let \( H_\varepsilon \) stands for \( E_{\varepsilon \cdot A}(\|A\| - \varepsilon, \|A\|^2) \), and \( E_{\varepsilon \cdot A} \) stands for the spectral measure of operator \( \varepsilon \cdot A \). It was proved in [12, Theorem 2.4] that
\[
(\text{4.1}) \quad \lim_{t \to 0^+} \frac{\|A + tB\| - \|A\|}{t} = \frac{1}{\|A\|} \inf_{\|v\| = 1} \sup_{h \in H_\|v\|} \Re(Ah|Bh).
\]

We will give an alternative proof of Theorem 1.2 as an application of (4.1). Let \( \alpha = \|A\| \lim_{t \to 0^+} \frac{\|A + tB\| - \|A\|}{t} \). Then we get a sequence \( \{h_n\} \) such that \( \|h_n\| = 1, h_n \in H_{1/n} \text{ and } \Re(h_n|A^*Bh_n) \to \alpha \). Now \( h_n \in H_{1/n} \) implies that \( \|Ah_n\|^2 = (\|A\| - 1/n)^2, \|A\|^2) \). We can choose a subsequence, if necessary, so that \( \|Ah_n\| \to \|A\| \).

Now define \( \psi_n(T) = (h_n|Th_n) \). Then \( \psi_n \in S_{\mathcal{B}(\mathcal{H})} \) such that \( \psi_n(A^*A) \to \|A\|^2 \) and \( \psi_n(A^*B) \to \alpha \). Now using weak*-compactness of \( S_{\mathcal{B}(\mathcal{H})} \), there exists \( \psi \in S_{\mathcal{B}(\mathcal{H})} \) such that \( \psi(A^*A) = \|A\|^2 \) and \( \psi(A^*B) = \alpha \). By using the GNS construction, we get the proof of Theorem 1.2.

2. Using the Riesz Representation Theorem with Corollary 1.3 we get that if \( f \in C_b(\Omega) \) such that \( \text{dist}(f, C_b(\Omega)) < \|f\|_\infty \), then for any \( g \in C_b(\Omega) \), we have
\[
(\text{4.2}) \quad \lim_{t \to 0^+} \frac{\|f + tg\|_\infty - \|f\|_\infty}{t} = \sup \{\Re(e^{-\arg f(x)}g(x)) : x \in \Omega, \|f(x)\| = \|f\|_\infty\}.
\]

This is special case of [13, Theorem 3.1] where it was proved that if \( f, g \in C_b(\Omega) \), then
\[
(\text{4.3}) \quad \lim_{t \to 0^+} \frac{\|f + tg\| - \|f\|}{t} = \inf_{\delta > 0} \sup_{x \in E_\delta} \Re(e^{-\arg f(x)}g(x)),
\]
where \( E_\delta = \{x \in \Omega : \|f(x)\|_\infty > \|f\|_\infty - \delta\} \). Note that (4.2) is a special case of (4.3). Similarly formula for Gateaux derivative obtained in (3.1) is a special case of (4.1).

Now equation 4.2 and formula 3.1, were obtained as the applications of Corollary 1.3. It raises question what can be analogous of Corollary 1.3 without the condition \( \text{dist}(f, C_b(\Omega)) < \|f\|_\infty \), that will be generalization of formulas (4.1) and (4.3)?

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