A non radical based approach to study of associative algebras

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Abstract

We study pairs of associative algebras and linear functionals. New results together with corrected proofs for previously published material are presented. In particular, we prove the identity \( \text{ind} \, \text{Mat}_n \otimes \mathfrak{A} = n \cdot \text{ind} \, \mathfrak{A} \) for finite-dimensional unital associative algebra \( \mathfrak{A} \) with index 1.

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1 Introduction

In this paper we describe our progress in the study of interaction between associative algebras and linear functionals defined on them. This aspect of associative algebras is related to the following classical concepts:

- The Orbit method in the theory of Lie algebras. As any associative algebra can be converted into a Lie algebra one hopes the additional structure present in associative algebras will expose new phenomena.

- The classical notion of multiplicative functionals. As will be shown later multiplicative functionals are exactly the functionals whose associated bilinear form has rank 1.

- Hopf algebras. The pair (associative algebra, functional) can be considered as intermediate concept between associative algebras and Hopf algebras.

This investigation has been prompted by the observation (see [1]) that for a class of subalgebras of matrix algebra the index in Lie algebra sense (i.e. the dimension of the kernel of Kirillov’s form $B_F$ in generic functional $F$) of a tensor product of $\text{Mat}_n$ with the algebra $\mathfrak{A}$ was exactly $n$ (which is the index of $\text{Mat}_n$) times the index of the algebra $\mathfrak{A}$.

This identity does not readily generalize to an arbitrary Lie algebra nor to an arbitrary pair of associative algebras. The last theorem
in this paper establishes that the identity does hold for two type 1 associative algebras provided that the pair satisfies some additional conditions. These conditions are fulfilled by Mat$_n$ and unital associative algebra of index 1.

The question of expressing index of a tensor product of two associative algebras via known invariants of these algebras is still open, even if one restricts consideration to type 1 algebras.

The method developed for proving this result has several interesting properties:

• the decomposition obtained can be considered an exponentiated version of root spaces decomposition of Lie algebras (and is, in fact, exactly so for Mat$_n$, with Cartan subalgebra being Stab$_F(1)$). This decomposition is defined for any associative algebra and has proved very convenient in analyzing coadjoint representation.

• besides the Kirillov’s form defined on coadjoint representation of associative algebras one obtains a quadratic form on the stabilizer of coadjoint action. The non-degeneracy of the latter corresponds to type 1 algebras. For regular functionals, non-degeneracy of this quadratic form implies that Stab$_F(1)$ is a Frobenius algebra.

• characteristic polynomial presents an easy way to obtain invariants of coadjoint action.

Some of the material has appeared in our earlier preprints [2], [3] and [4] that mark the progress of our study. Besides corrections to proofs, this paper refines notation for characteristic polynomial and spaces Stab$_F(\alpha)$ and $V(\alpha)$. We also introduce the definitions of $\alpha(F)$-regular and $\alpha$-precise functionals.

Also new are the definitions of three types of algebras. The most studied ones are type 1 with many of results having generalizations to type 2 algebras.

Type 3 algebras do not allow decomposition into direct sum of spaces $V(\alpha)$ even after factoring by Nil$_F$, but possess an interesting property of having non-empty Stab$_F(\alpha)$ with non-trivial multiplication table for any $\alpha$. This awaits further study.
2 Definitions, characteristic polynomial

Before proceeding further let us introduce some definitions.

Let \( \mathfrak{A} \) be an associative algebra. Unless specially noted we will assume \( \mathfrak{A} \) to be a finite-dimensional algebra over complex field. We denote by \( \mathfrak{A}^* \) the dual space of linear functionals on \( \mathfrak{A} \).

The multiplication law \( \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A} \) can be considered as an \( \mathfrak{A} \)-valued bilinear form \( A \) on \( \mathfrak{A} \). If one picks a basis \( \{e_k\} \) in the algebra \( \mathfrak{A} \) then \( A \) can be represented by a \( \mathfrak{A} \)-valued matrix \( (e_ie_j) \) in this basis. Usually we will abuse notation by denoting this matrix by the same letter \( A \).

For a linear functional \( F \in \mathfrak{A}^* \) we denote by \( A_F \) the bilinear form \( F(A(\cdot, \cdot)). \)

We denote by \( \text{Stab}_F(0) = \ker A_F \) and by \( \text{Stab}_F(\infty) = \ker A_TF \). Let \( \text{Nil}_F = \text{Stab}_F(0) \cap \text{Stab}_F(\infty). \)

We also introduce the characteristic polynomial

\[
\chi_V(\lambda, \mu, F) = \det_V \left( \lambda A_F + \mu A_TF \right)
\]

where determinant is evaluated in some basis of vector space \( V \subset \mathfrak{A} \) - this polynomial is thus defined up to a constant multiple.

In places where the functional \( F \) is fixed we will use the notation \( \chi_{F,V}(\lambda, \mu) \) - this saves a little space in formulas. Lastly, we omit \( V \) when \( V = \mathfrak{A} \).

We will distinguish three configurations \((\mathfrak{A}, F)\):

**Type 1:** The characteristic polynomial of the entire algebra \( \chi_{F,\mathfrak{A}} \) does not vanish. This implies \( \dim \text{Nil}_F = 0. \)

**Type 2:** \( \text{Nil}_F \) has positive dimension and the characteristic polynomial of the subspace of maximal dimension that is transversal to \( \text{Nil}_F \) does not vanish.

**Type 3:** The characteristic polynomial of the vector space of maximal dimension that is transversal to \( \text{Nil}_F \) does vanish.

We will consider an associative algebra \( \mathfrak{A} \) to be of type \( N \) if its dual space \( \mathfrak{A}^* \) possesses non-empty open subset (in either Zariski or Euclid topology) of functionals \( F \) that form a pair of type \( N \) with \( \mathfrak{A} \).

We will now establish correctness of these definitions.

**Definition 1** Fix a certain topology on the space of linear functionals on \( \mathfrak{A} \). We will call a condition on linear functionals \( F \) generic if, for
a given associative algebra, it is either not satisfied for any functional or is satisfied for an open dense set of functionals.

**Theorem 1** Fix a subspace $V$ inside each finite-dimensional associative algebra over $\mathbb{C}$. Then the condition that $\chi_{F,V}(\lambda, \mu)$ does not vanish as polynomial in $\lambda$ and $\mu$ is generic in both Euclid and Zariski topologies.

**Proof** First of all let us note that for a fixed subspace $V \in \mathfrak{A}$ the existence of non-empty set of functionals $F$ for which $\chi_{F,V}(\lambda, \mu)$ does not vanish is equivalent to non-vanishing of the polynomial $\chi_V(\lambda, \mu, F)$ in all three variables.

Secondly, the condition that $\chi_{F,V}(\lambda, \mu)$ vanishes can be written as a system of polynomial equations in $F$ by equating coefficients at $\lambda$ and $\mu$ in $\chi_V(\lambda, \mu, F)$ to zero. If there is a point $F$ when at least one of these coefficients is non-zero then, by continuity, there exists an open neighbourhood (in either Zariski or Euclid topology) in which this coefficient does not vanish and thus $\chi_{F,V}(\lambda, \mu)$ does not vanish at all in this neighbourhood.

Lastly if there were an open neighbourhood such that for all $F$ in it the coefficients vanish this would imply that the coefficients vanish identically in $F$ as they are polynomial.

Thus either the set of functionals $F$ for which $\chi_{F,V}(\lambda, \mu)$ does not vanish is empty or it is open and dense.

**Theorem 2** The condition that a functional $F$ has the smallest $\dim \text{Nil}_F$ is generic in both Euclid and Zariski topologies.

**Proof** Indeed, consider any functional $F_0$ that has the property that $\dim \text{Nil}_{F_0}$ is minimal (it exists as the $\dim \text{Nil}_F$ is a non-negative integer). The coefficients of the linear system that defines $\text{Nil}_F$ are themselves linear in $F$. Therefore the minors of this system are polynomials in $F$. There must be a minor of dimension $\dim \mathfrak{A} - \dim \text{Nil}_{F_0}$ that does not vanish for a dense open set (in either Zariski or Euclid topologies) of functionals $F$ that includes $F_0$. However, all of these functionals must have $\dim \text{Nil}_F$ not less than $\dim \text{Nil}_{F_0}$. Therefore, the set of all functionals $F$ with minimal $\dim \text{Nil}_F$ is a union of dense open sets and thus is itself open and dense.

This condition is always satisfied by some functionals as $\dim \text{Nil}_{F_0}$ is a non-negative integer.
**Theorem 3**  A finite-dimensional associative algebra over \( \mathbb{C} \) is either type 1, 2 or 3. The definition does not change whether one considers Zariski or Euclid topologies.

**Proof**  As proved above, the condition that \( \chi_{F, \mathfrak{A}}(\lambda, \mu) \) does not vanish identically is generic and implies \( \dim \text{Nil}_F = 0 \). Therefore, type 1 algebras are mutually exclusive with type 2 or 3.

Consider now the case of associative algebra with \( \dim \text{Nil}_F = 0 \). Either \( \chi_{\mathfrak{A}}(\lambda, \mu, V) \) does not vanish, in which case it is type 1, or it does vanish, in which case it is type 3. Thus, finite-dimensional associative algebra with \( \dim \text{Nil}_F = 0 \) is either type 1 or type 3.

Now we will concentrate on the situation where the minimal dimension of \( \text{Nil}_F \) is positive.

For each associative algebra pick \( F_0 \) such that \( \dim \text{Nil}_{F_0} \) is minimal and pick a subspace \( V_0 \) of dimension \( \dim \mathfrak{A} - \dim \text{Nil}_{F_0} \) that is transversal to \( \text{Nil}_{F_0} \).

We know that the condition that \( \chi_{F, V_0}(\lambda, \mu) \) does not vanish is generic. Therefore, for each associative algebra, either there exists a dense open set of functionals which possess a subspace of maximal dimension (i.e. \( V_0 \)) with non-vanishing characteristic polynomial or there is a dense open set of functionals which possess a subspace of maximal dimension (i.e. \( V_0 \)) on which characteristic polynomial vanishes.

We will now prove that, for a given functional \( F \), the characteristic polynomial \( \chi_{F, V}(\lambda, \mu) \) either vanishes for all subspaces \( V \) of maximal dimension that are transversal to \( \text{Nil}_F \) or does not vanish for any such \( V \).

Consider a basis of \( \mathfrak{A} \) subordinate to the direct sum \( \mathfrak{A} = \text{Nil}_F \oplus V \). In this basis linear automorphisms of \( \mathfrak{A} \) (as a vector space) that preserve \( \text{Nil}_F \) have the following block structure:

| \begin{array}{cc} \text{Nil}_F & V \\ \hline \text{Nil}_F & T_{NN} & 0 \\ V & T_{NV} & T_{VV} \end{array} |

These automorphisms act transitively on the set of maximal subspaces \( V \) that are transversal to \( \text{Nil}_F \).

The multiplication table written in this basis has zeros in all rows and columns corresponding to basis vectors from \( \text{Nil}_F \). When acted on by linear transformation that preserved \( \text{Nil}_F \) the multiplication table will still have zeros in all rows and columns corresponding to basis
vectors from $\text{Nil}_F$. Furthermore, the entries corresponding to two basis vectors from $V$ will only depend on $T_{VV}$ - an inner automorphism of $V$. Therefore the property that the minor formed by restriction of $A_F$ to $V$ is zero or not is independent of the choice of subspace $V$.

Thus, an algebra with positive minimal dim $\text{Nil}_F$ either possesses a dense open set of functionals $F$ that have a maximal subspace transversal to $\text{Nil}_F$ with non-vanishing characteristic polynomial (and so it is type 2) or there is a dense open set of functionals $F$ (which includes those with minimal dim $\text{Nil}_F$) for which the characteristic polynomial vanishes on any maximal subspace transversal to $\text{Nil}_F$ (and so it is type 3).

$\blacksquare$

3 Examples

We will now present examples of associative algebras of all three types.

We will use the following notation: when writing multiplication tables letters denote basis elements in $\mathfrak{A}$ and when writing characteristic polynomial we will use the same letters to denote value of generic functional $F$ on this element.

In other words, we are considering $\mathfrak{A}$ as linear functions on $\mathfrak{A}^*$ and we compute characteristic polynomials by using multiplication of $S(\mathfrak{A})$ (i.e. we multiply them as polynomials over $\mathfrak{A}^*$), not the multiplication of the associative algebra itself.

3.1 Type 1

Example 1 [Mat2] Let $a,b,c,d$ denote the matrix units $E_{1,1},E_{1,2},E_{2,1}$ and $E_{2,2}$ correspondingly. Then the multiplication table $A$ is

|     | a   | b   | c   | d   |
|-----|-----|-----|-----|-----|
| a   | a   | b   | 0   | 0   |
| b   | 0   | 0   | a   | b   |
| c   | c   | d   | 0   | 0   |
| d   | 0   | 0   | c   | d   |

The characteristic polynomial is equal to

$$
\chi(\lambda, \mu, F) = \det(\lambda A + \mu A^T) = \\
= -(\lambda + \mu)^2(ad - bc)((\lambda - \mu)^2(ad - bc) + \lambda \mu (a + d)^2)
$$
There are plenty of functionals $F$ for which the above expression does not vanish.

**Example 2 [Seaweed 12x21]** Let $\mathfrak{A}$ be the following subalgebra of $\text{Mat}_3(\mathbb{C})$:

$$
\begin{pmatrix}
  a & b & 0 \\
  0 & c & 0 \\
  0 & d & e
\end{pmatrix}
$$

The multiplication table $A$ of $\mathfrak{A}$ is

|   | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|-----|-----|-----|-----|-----|
| $a$ | $a$ | $b$ | 0   | 0   | 0   |
| $b$ | 0   | 0   | $b$  | 0   | 0   |
| $c$ | 0   | 0   | $c$  | 0   | 0   |
| $d$ | 0   | 0   | $d$  | 0   | 0   |
| $e$ | 0   | 0   | 0   | $d$ | $e$ |

The characteristic polynomial is equal to

$$
\chi(\lambda, \mu, F) = \lambda^2 \mu^2 (\lambda + \mu) b^2 d^2 (a + c + e)
$$

As in the previous example the set of functionals $F$ for which characteristic polynomial of the entire algebra does not vanish is Zariski open.

**Example 3 [Mat$_n$]**

**Theorem 4** The characteristic polynomial of the entire algebra is quasi-invariant under coadjoint action. That is

$$
\chi(\lambda, \mu, \text{Ad}^*_g F) = (\det \text{Ad}_g)^{-2} \chi(\lambda, \mu, F)
$$

**Proof** Indeed, the matrix element $(i, j)$ of $A_F$ is given by the expression $F(e_i e_j)$. Since

$$
(\text{Ad}^*_g F) (e_i e_j) = F(g^{-1} e_i e_j g) = F((g^{-1} e_i g)(g^{-1} e_j g))
$$

the substitution $F \rightarrow \text{Ad}^*_g F$ is equivalent to the change of basis induced by the matrix $\text{Ad}_g^{-1}$.  


Definition 2 [Generalized resultant] Let \( p(x) \) and \( q(x) \) be two polynomials over an algebraically closed field. We define generalized resultant of \( p(x) \) and \( q(x) \) to be

\[
R(\lambda, \mu) = \prod_{i,j} (\lambda \alpha_i + \mu \beta_j)
\]

where \( \{\alpha_i\} \) and \( \{\beta_j\} \) are roots of polynomials \( p(x) \) and \( q(x) \) respectively.

Generalized resultant is a polynomial in two variables. It is easy to show that its coefficients are polynomials in coefficients of \( p(x) \) and \( q(x) \) so the condition on the base field to be algebraically closed can be omitted.

Theorem 5 The characteristic polynomial \( \chi(\lambda, \mu, F) \) for algebra Mat\(_n\) in point \( F \in \text{Mat}^*_n \) over the entire algebra Mat\(_n\) in basis of matrix units is equal to the generalized resultant of characteristic polynomial of \( F \) (as a matrix) with itself times \((-1)^{\frac{n(n-1)}{2}}\). That is

\[
\chi(\lambda, \mu, F) = (-1)^{\frac{n(n-1)}{2}} \det(F)(\lambda + \mu)^n \prod_{i \neq j} (\lambda \alpha_i + \mu \beta_j)
\]

where \( \alpha_i \) are eigenvalues of \( F \) (this formulation assumes that the base field is algebraically closed).

Proof We will make use of theorem 4. The coadjoint action on \( \text{Mat}^*_n \) is simply conjugation by invertible matrices. The generic orbit consists of diagonalizable matrices. Thus we can compute \( \chi(\lambda, \mu) \) by assuming first that \( F \) is diagonal and then extrapolating the resulting polynomial to the case of all \( F \).

Assume the base field to be \( \mathbb{C} \). Let \( F = \text{diag}(\alpha_1, ..., \alpha_n) \). We choose a basis \( \{E_{i,j}\} \) of matrix units in the algebra Mat\(_n\). The only case when \( F(E_{i,j}E_{k,l}) \) is non-zero is when \( i = l \) and \( j = k \). Thus the multiplication table \( A \) (restricted to the subspace of diagonal matrices
in $\text{Mat}_n^*$ is

\[
\begin{array}{ccc}
E_{i,i} & E_{i,j}^+ & E_{i,j}^- \\
\alpha_1 & 0 & 0 \\
\vdots & 0 & \alpha_n \\
0 & \alpha_j' & 0 \\
E_{i,j}^+ & 0 & \alpha_j'' \\
0 & \alpha_j' & 0 \\
E_{i,j}^- & 0 & \alpha_j''
\end{array}
\]

here $E_{i,j}^+$ denotes elements $E_{i,j}$ with $i > j$ and $E_{i,j}^-$ denotes elements $E_{i,j}$ with $i < j$. The matrix $\lambda A + \mu A^T$ will have $(\lambda + \mu)\alpha_i$ in the $E_{i,i} \times E_{i,i}$ block, and the pair $(E_{i,j}^+, E_{i,j}^-)$ will produce a $2 \times 2$ matrix

\[
\begin{pmatrix}
0 & \lambda \alpha_j + \mu \alpha_i \\
\lambda \alpha_i + \mu \alpha_j & 0
\end{pmatrix}
\]

Computing the determinant yields

\[
(-1)^{n(n-1)/2} (\lambda + \mu)^n \prod_i \alpha_i \prod_{i \neq j} (\lambda \alpha_i + \mu \alpha_j) = (-1)^{n(n-1)/2} \prod_{i,j} (\lambda \alpha_i + \mu \alpha_j)
\]

thus proving the theorem for the case when $F$ is diagonal. But characteristic polynomial $\det(F - x)$ is invariant under coadjoint action. Thus this expression is true for all $F$ up to a possibly missing factor depending only on $F$ (but not $\lambda$ or $\mu$) which is quasi-invariant under coadjoint action. However, in view of the fact that this multiple must be a polynomial in $F$ and that the degree of the expression above in $F$ is exactly $n^2$ this multiple must be trivial.

The case of an arbitrary field is proved by observing that both sides of the equality are polynomials with integral coefficients and thus if equality holds over $\mathbb{C}$ it should hold over any field.

One easily observes that for any $F$ with all distinct, non-zero eigenvalues (as a matrix) the characteristic polynomial does not vanish.
3.2 Type 2

The easiest example of a type 2 algebra is given by direct sum of a type 1 algebra with an algebra with trivial (identically 0) multiplication law.

A more interesting example is given by the following construction:

**Example 4** Let \( V \) be a vector space of dimension \( k \) and let \( B : V \times V \to W \) be a bilinear map of cross product \( V \times V \) into vector space \( W \) of dimension \( m \).

We define algebra \( \mathfrak{A}(B) \) by the following multiplication table:

|     | \( V \) | \( W \) |
|-----|--------|--------|
| \( V \) |     |       |
| \( W \) |     |       |

The algebra \( \mathfrak{A}(B) \) possesses a remarkable property - the product of any three elements is always zero. Thus it is always associative, no matter what \( B \) is.

It is straightforward to see that for any \( F \in \mathfrak{A}(B)^* \) we have \( W \subset \text{Nil}_F \). If one chooses \( B \) and \( F \in \mathfrak{A}(B)^* \) in such a way that \( \det(F(B)) \) is non-zero we obtain an example of a type 2 pair \((\mathfrak{A}(B), F)\).

Since the inequality \( \det(F(B)) \neq 0 \) defines a Zariski open set of functionals \( F \) any algebra \( \mathfrak{A}(B) \) that possesses \( F \) of type 2 is a type 2 algebra.

3.3 Type 3

Let us consider a special case of the example with \( \dim W = 1 \).

In this situation the matrix \( A_F \) depends only on the value of \( F \) on the single basis vector of \( W \) and \( B \). Let us select \( 0 \neq w \in W \) and any \( F \) such that \( F(w) = 1 \).

By manipulating \( B \) we can thus set \( A_F \) to anything we like with only restriction that the last row and column are identically 0. This provides a lot of examples of type 3 algebras, in particular the following \( B \) will do just fine:

\[
B = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

For this \( B \) the space \( \text{Stab}_F(0) = \ker A_F \) is spanned by \( v_1, v_2 \) and \( w \) and \( \text{Stab}_F(\infty) = \ker A_F^T \) is spanned by \( v_1 - v_2, v_3 \) and \( w \). Thus \( \text{Nil}_F = W \) but \( \det \left( \lambda B + \mu B^T \right) = 0 \).
4 Spaces $\text{Stab}_F(\alpha)$

**Definition 3** Let $\mathfrak{A}$ be an associative algebra and $F$ be a linear functional on it. We define

$$\text{Stab}_F(\alpha) := \{a \in \mathfrak{A} : \forall x \in \mathfrak{A} \Rightarrow F(ax) - \alpha F(xa) = 0\}$$

In other words

$$\text{Stab}_F(\alpha) = \ker (A_F - \alpha A_F^T)$$

If one considers Lie algebra $\mathfrak{A}$ with bracket $[a, b] = ab - ba$ then $\text{Stab}_F(1) = \text{Stab}_F$ in the conventional definition of stabilizer of a linear functional on a Lie algebra.

**Example 5** Returning to the example 3 we see that for $i \neq j$

$$\text{Stab}_F\left(\alpha_i \over \alpha_j\right) = \mathbb{C} \cdot e_{ij}$$

and

$$\text{Stab}_F(1) = \text{span} < e_{11}, \ldots, e_{nn}>$$

**Theorem 6**

$$\text{Stab}_F(\alpha) \cdot \text{Stab}_F(\beta) \subset \text{Stab}_F(\alpha\beta)$$

$$\text{Stab}_F(0) \cdot \text{Stab}_F(\infty) \subset \text{Nil}_F$$

$$\text{Stab}_F(0) \cdot \mathfrak{A} \subset \text{Stab}_F(0)$$

$$\mathfrak{A} \cdot \text{Stab}_F(\infty) \subset \text{Stab}_F(\infty)$$

$$\dim \text{Stab}_F(\alpha) = \dim \text{Stab}_F(1/\alpha)$$

**Proof** Let $a \in \text{Stab}_F(\alpha)$ and $b \in \text{Stab}_F(\beta)$. Then for all $x$

$$F((ab)x) = F(abx) = aF(bxa) \neq \alpha \beta F(x(ab))$$

For $a \in \text{Stab}_F(\infty)$ and $b \in \text{Stab}_F(\beta)$, $\beta \neq 0$ and any $x$ we have:

$$F(x(ab)) = F(xab) = \frac{1}{\beta} F(bxa) = 0$$

and

$$F(x(ba)) = F(xba) = F((xb)a) = 0$$

Secondly, for $a \in \text{Stab}_F(0)$ and $b \in \text{Stab}_F(\infty)$ and any $x$ we have

$$F((ab)x) = F(a(bx)) = 0$$

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and

\[ F(x(ab)) = F((xa)b) = 0 \]

Thus \( \text{Stab}_F(0) \) is a right ideal in \( \mathfrak{A} \) and \( \text{Stab}_F(\infty) \) is a left ideal in \( \mathfrak{A} \). The product of \( \text{Stab}_F(0) \) and \( \text{Stab}_F(\infty) \) must be in \( \text{Stab}_F(0) \cap \text{Stab}_F(\infty) = \text{Nil}_F \).

Lastly from linear algebra we know that for any matrix \( R \)

\[ \dim \ker R = \dim \ker R^T \]

Thus, from definition of \( \text{Stab}_F(\alpha) \), it follows that \( \dim \text{Stab}_F(\alpha) = \dim \text{Stab}_F(1/\alpha) \).

\[ \square \]

**Corollary:** \( \text{Stab}_F(1), \text{Stab}_F(0) \) and \( \text{Stab}_F(\infty) \) are subalgebras of \( \mathfrak{A} \).

**Theorem 7** Let \( \mathfrak{A} \) be a unital associative algebra and \( F \) a linear functional on it. Then for all \( \alpha \neq 1 \) we have

\[ F(\text{Stab}_F(\alpha)) = \{0\} \]

**Proof** Indeed, consider first the case when \( \alpha \) is finite. By definition for any element \( a \in \text{Stab}_F(\alpha) \) we have

\[ 0 = F(a \cdot 1) - \alpha F(1 \cdot a) = (1 - \alpha)F(a) \]

and thus \( F(a) = 0 \).

Similarly, for \( \alpha = \infty \) we must have

\[ 0 = F(1 \cdot a) = F(a) \]

\[ \square \]

### 5 Multiplicative functionals

Multiplicative functionals play an important role in classical representation theory and in the study of commutative algebras.

**Proposition 8** Let \( F \) be a multiplicative functional on unital associative algebra \( \mathfrak{A} \). Then \( A_F \) has rank 1.

**Proof** Indeed, from definition

\[ A_F = (F(e_i e_j))_{i,j=1}^n = (F(e_i)F(e_j))_{i,j=1}^n = (F(e_i))_{i=1}^n (F(e_j))_{j=1}^n \]

Since \( F(1) = 1 \) the matrix \( A_F \) cannot be 0.

\[ \square \]
Lemma 9 Let \( \mathfrak{A} \) be an associative algebra with linear functional \( F \) such that \( \text{Stab}_F(0) = \text{Stab}_F(\infty) = \text{Nil}_F \). Then \( \text{Nil}_F \) is an ideal.

Proof This is a direct consequence of the corollary to theorem 6.

Theorem 10 Let \( F \) be a linear functional on unital associative algebra \( \mathfrak{A} \) such that the matrix \( A_F \) has rank 1 and \( F(1) = 1 \). Then \( F \) is multiplicative.

Proof Since rank \( A_F \) is 1 it must be that codim \( \text{Stab}_F(0) = \text{Stab}_F(\infty) = 1 \). On the other hand

\[
F(x) = F(1 \cdot x) = F(x \cdot 1)
\]

and thus

\[
F(\text{Stab}_F(0)) = F(\text{Stab}_F(\infty)) = 0
\]

But codim ker \( F = 1 \).

Thus \( \text{Stab}_F(0) = \text{Stab}_F(\infty) = \ker F = \text{Nil}_F \). From the previous lemma we know that \( \ker F = \text{Nil}_F \) is an ideal.

Thus for any \( a, b \in \mathfrak{A} \):

\[
F(ab) = F((F(a) + (a - F(a)))(F(b) + (b - F(b))) = \\
= F(F(a)F(b)) + F(F(a)(b - F(b))) + F((a - F(a))F(b)) + \\
+ F((a - F(a))(b - F(b)))
\]

The first term is exactly \( F(a)F(b) \). The second and third terms vanish because \( x - F(x) \) belongs to \( \ker F \). The last term also vanishes because \( \ker F \) is an ideal. Thus \( F(ab) = F(a)F(b) \) and \( F \) is multiplicative.

It is interesting to note that this proof applies to both finite- and infinite-dimensional associative algebras. If the algebra has been endowed with topology than we can restrict out attention to only continuous functionals.

6 Regular functionals

The multiplicative functionals correspond to the situation when \( A_F \) (or \( \lambda A_F + \mu A_T^F \)) has the smallest rank possible. The case when \( A_F \) (or \( \lambda A_F + \mu A_T^F \)) has the maximum possible rank is described by regular functionals.

Following we first prove the following lemma:
Lemma 11 Let $W$ be a vector subspace of $\mathfrak{A}$. Fix $\lambda$ and $\mu$. Then, the set $R$ of all $F \in \mathfrak{A}^*$ such that $W \cap \ker (\lambda A_F + \mu A_F^T) \neq \{0\}$ is closed in $\mathfrak{A}^*$.

Proof Let $e_1, \ldots, e_n$ be the basis of $\mathfrak{A}$ such that the first $p$ vectors form the basis of $W$. The system of equations

$$
(\lambda A_F + \mu A_F^T) \sum_{i=1}^{p} \epsilon_i e_i = 0
$$

is equivalent to

$$
\sum_{i=1}^{p} \epsilon_i (\lambda F(e_k e_i) + \mu F(e_i e_k)) = 0
$$

where $k$ ranges from 1 to $n$. The matrix of the latter system has entries that are linear in $F$. The existence of the non-zero solution is equivalent to the requirement that all $p$-minors vanish. This proves that the set $R$ can be defined as a solution to the system of polynomial equations. Hence it is closed.

This argument can also be generalized to the case of $\lambda$ and $\mu$ varying with $F$. However, one would have to specify the degree of regularity of $\lambda$ and $\mu$. We prefer to state it with the assumption of continuity:

Lemma 12 Let $W$ be a vector subspace of $\mathfrak{A}$. Fix two continuous functions $\lambda(F)$ and $\mu(F)$. The set $R$ of all $F \in \mathfrak{A}^*$ such that

$$
W \cap \ker (\lambda(F) A_F + \mu(F) A_F^T) \neq \{0\}
$$

is closed in $\mathfrak{A}^*$.

Theorem 13 Let $S$ be a subspace of $\mathfrak{A}^*$. Let $F$ be such that for a fixed pair of continuously differentiable functions $\lambda(F), \mu(F)$ the dimension of the space

$$
\ker (\lambda(F) A_F + \mu(F) A_F^T)
$$

is the smallest among functionals in small neighbourhood of $F$ inside affine set $F + S$.

Then for any $G \in S$, any $x \in \ker (\lambda(F) A_F + \mu(F) A_F^T)$ and any $y \in \ker (\mu(F) A_F + \lambda(F) A_F^T)$ we have

$$
G(\lambda(F)xy + \mu(F)y) + (L_G \lambda)(F)(xy) + (L_G \mu)(F)(yx) = 0
$$

Here $L_G$ denotes directional derivative.
and using notation of Lie derivative we get: $Q_w$ is of the type $p$ of it must be $\epsilon$

After setting $w$ the system remain non-vanishing for $\epsilon$ $p$ ∩ $\ker$ $\mathcal{L}$

The homogeneous part of this system is the linear system that defines $\ker$ $\mathcal{L}$ $\mathcal{W}$ remains transversal to $W$ is open by lemma 11 and thus contains a small neighbourhood of 0. Let $\{e_i\}$ be the basis of $\mathcal{W}$ with first $p$ vectors forming the basis of $W$.

Pick $x \in \ker (\lambda(F_e)A_{F_e} + \mu(F_e)A^T_{F_e})$. As $W$ and $\ker (\lambda(F_e)A_{F_e} + \mu(F_e)A^T_{F_e})$ form a decomposition of $\mathcal{W}$ there exists $w \in W$ such that

$$x - w \in \ker (\lambda(F_e)A_{F_e} + \mu(F_e)A^T_{F_e})$$

I.e. for all $i$

$$\lambda(F_e)F_e((x-w)e_i) + \mu(F_e)F_e(e_i(x-w)) = 0$$

Hence

$$\lambda(F_e)F_e(we_i) + \mu(F_e)F_e(e_iw) = \lambda(F_e)F_e(xe_i) + \mu(F_e)F_e(e_ix)$$

The homogeneous part of this system is the linear system that defines $W \cap \ker (\lambda(F_e)A_{F_e} + \mu(F_e)A^T_{F_e})$. As this intersection is trivial the rank of it must be $p$ for all $\epsilon$ in a small neighbourhood of 0. Noticing that the right hand side can be derived by substituting $w = x$ (i.e. our system is of the type $Qw = Qx$) we conclude that there is a unique solution $w_e$. Recall that $F_e$ is linear in $\epsilon$. When $\lambda(F)$ and $\mu(F)$ are constant $w_e$ is a rational function in $\epsilon$. When $\lambda(F)$ and $\mu(F)$ are continuously differentiable the non-vanishing $p$-minors of the homogeneous part of the system remain non-vanishing for $\epsilon$ in a small neighbourhood of 0 and thus $w_e$ is continuously differentiable as well.

Pick now any $y \in \ker (\lambda(F)A_F + \mu(F)A^T_F)$. We have:

$$\lambda(F_e)F_e((x-w_e)y) + \mu(F_e)F_e(y(x-w_e)) = 0$$

Differentiating with respect to $\epsilon$ produces:

$$\lambda(F_e)G_e((x-w_e)y) + \mu(F_e)G_e(y(x-w_e)) - (\lambda(F_e)F_e(w'_ey) + \mu(F_e)F_e(yw'_e)) +$$

$$+ \lambda(F_e)'F_e((x-w_e)y) + \mu(F_e)'F_e(y(x-w_e)) = 0$$

After setting $\epsilon = 0$ we have:

$$\lambda(F)G(xy) + \mu(F)G(yx) + \lambda(F_e)'|_{\epsilon=0}F(xy) + \mu(F_e)'|_{\epsilon=0}F(yx) = 0$$

and using notation of Lie derivative we get:

$$\lambda(F)G(xy) + \mu(F)G(yx) + (L_G\lambda)(F)F(xy) + (L_G\mu)(F)F(yx) = 0$$
Corollary 1. For the case of constant $\lambda$ and $\mu$ we obtain a simpler expression:
$$\lambda(F)G(xy) + \mu(F)G(yx) = 0$$

Corollary 2. Let $F$ be a functional such that $\text{Stab}_F(\alpha(F))$ has the smallest dimension in a neighbourhood of $F$. $(\alpha(\cdot)$ is fixed, finite and non-zero in $F)$. Then for all $x \in \text{Stab}_F(\alpha(F))$ and $y \in \text{Stab}_F(1/\alpha(F))$ we have:
$$xy - \alpha yx = 0$$

Corollary 3. Let $F$ be a functional with the smallest dimension of $\text{Stab}_F(1)$. Then $\text{Stab}_F(1)$ is a commutative subalgebra of $\mathfrak{A}$.

Corollary 4. Let $F$ be a functional with the smallest dimension of $\text{Stab}_F(0)$. Then
$$\text{Stab}_F(0) \cdot \text{Stab}_F(\infty) = \{0\}$$
In this case $\text{Nil}_F$ is a subalgebra of $\mathfrak{A}$ with trivial (identically 0) multiplication law.

Corollary 5. Let $F$ be a functional such that $\text{Stab}_F(\alpha(F))$ has the smallest dimension in a neighbourhood of $F$. $(\alpha(\cdot)$ is fixed, finite and non-zero). Then the set
$$[\text{Stab}_F(\alpha(F)), \text{Stab}_F(1/\alpha(F))]_{\alpha(F)} := \{xy - \alpha(F)yx : x \in \text{Stab}_F(\alpha(F)) \text{ and } y \in \text{Stab}_F(1/\alpha(F))\}$$
has at most dimension 1.

Proof Let $\lambda = 1$ and $\mu = -\alpha(F)$. From the theorem we know that
$$\lambda G(xy) + \mu G(yx) + (L_G\lambda)(F)G(xy) + (L_G\mu)(F)G(yx) = 0$$
Substituting we get
$$G(xy) - \alpha(F)G(yx) - (L_G\alpha)(F)G(yx) = 0$$
There exists an element $z \in \mathfrak{A}$ such that for all $G \in \mathfrak{A}^*$ we have
$$(L_G\alpha)(F) = G(z)$$
Thus for all $G \in \mathfrak{A}^*$
$$G(xy) - \alpha(F)G(yx) - G(z)F(yx) = 0$$
and
$$G(xy - \alpha(F)yx - zF(yx)) = 0$$
which implies
$$xy - \alpha(F)yx - zF(yx) = 0$$
and thus $\dim [\text{Stab}_F(\alpha(F)), \text{Stab}_F(1/\alpha(F))]_{\alpha(F)}$ is at most 1. \hfill \blacksquare
Definition 4 Let $\mathfrak{A}$ be an associative algebra and $F$ a linear functional. Let $S$ be a subspace of linear functionals on $\mathfrak{A}$. Let $\alpha$ be either a constant or a continuously differentiable function.

We will call $F$ $(\alpha, S)$-regular if the space $\text{Stab}_F(\alpha(F))$ has the smallest dimension among functionals in the neighbourhood of the affine set $F + S$.

In the case $S = \mathfrak{A}^*$ we will simply call $F$ $\alpha$-regular.

Definition 5 Let $\mathfrak{A}$ be an associative algebra and $F$ be a 1-regular linear functional on it.
We define $\text{ind}\mathfrak{A} := \dim \text{Stab}_F(1)$

Note: the index defined above is the same as the index of Lie algebra $\mathfrak{A}^{\text{Lie}}$ obtained from $\mathfrak{A}$ by defining the bracket operation as $[a, b] = ab - ba$.

7 Type 1 algebras

We will now study type 1 algebras in more detail.

We will establish a criteria for recognizing type 1 algebras, analyze their characteristic polynomial, obtain decomposition into subspaces $V(\alpha)$ and study tensor products of type 1 algebras.

In the end we will prove the identity

$$\text{ind Mat}_n \otimes \mathfrak{B} = n$$

for any finite dimensional unital associative algebra $\mathfrak{B}$ over complex numbers that has index 1.

7.1 Recognizing type 1 algebras

Definition 6 For an associative algebra $\mathfrak{A}$ and a functional $F$ we define the skew-symmetric form $B_F$ as

$$B_F(a, b) = F(ab - ba)$$

and the symmetric form $Q_F$ as

$$Q_F(a, b) = F(ab + ba)$$

Theorem 14 Let $\mathfrak{A}$ be an associative algebra and $F$ a linear functional on it. Suppose that the restriction of the form $Q_F$ on the space $\text{Stab}_F(1)$ is a non-degenerate symmetric form. Then $F$ is of type 1.
Proof. By definition $F$ is of type 1 if and only if $\det(\lambda A_F + \mu A_F^T) \neq 0$. Let $\epsilon = (\lambda + \mu)/2$ and $\sigma = (\lambda - \mu)/2$. Let $k$ denote $\dim \text{Stab}_F(1)$. We compute:

$$
\det(\lambda A_F + \mu A_F^T) = \det((\epsilon + \sigma)A_F + (\epsilon - \sigma)A_F^T) = \\
= \det(\epsilon(A_F + A_F^T) + \sigma(A_F - A_F^T)) = \\
= \det(\epsilon Q_F + \sigma B_F) = \\
= \epsilon^k \det(\left. Q_F \right|_{\text{Stab}_F(1)}) \det \left( \sigma B_F \right|_{\text{Stab}_F(1)^\perp} \right) + o(\epsilon^k)
$$

Since $B_F$ is non-degenerate on $\text{Stab}_F(1)$-transversal subspace of $\mathfrak{A}$ of complimentary dimension we must have $\det(\lambda A_F + \mu A_F^T) \neq 0$.

The following is an easy consequence of theorem 14.

**Theorem 15** Let $\mathfrak{A}$ be a unital associative algebra of index 1. (i.e. $\text{ind} \mathfrak{A} = 1$). Then $\mathfrak{A}$ is type 1.

**Proof** Indeed, consider the set of 1-regular functionals $F$ that do not vanish on unity. The form $Q_F$ is scalar and equal to $F(1)$. Thus all such functionals $F$ are type 1.

We are now in position to formulate sufficient condition for an algebra to be type 1:

**Theorem 16** Let $\mathfrak{A}$ be an associative algebra and let $F$ be a linear functional on it such that $\text{Stab}_F(1)$ is commutative and the restriction of $Q_F$ on $\text{Stab}_F(1)$ is non-degenerate. Then $F$ is 1-regular and $\mathfrak{A}$ is a type 1 algebra.

**Proof** If $F$ is 1-regular then the fact that $\mathfrak{A}$ is type 1 follows from theorem 14.

We now concentrate on proving that $F$ is 1-regular.

Recall the following facts from the theory of Lie algebras:

The map $\mathfrak{A} \to \mathfrak{A}^*$ defined as

$$\text{ad}^*(a) : F(x) \mapsto F(ax - xa)$$

vanishes exactly on $\text{Stab}_F(1)$. The images in each functional $F$ form a locally integrable distribution.

Under the map $\text{ad}^*$ the form $B_F$ is mapped to Kirillov’s form on the image of $\text{ad}^*$.
Kirillov’s form provides symplectic structure on the leaves of this distribution. Of particular note to us is the fact that for any two functionals $F$ from the same leaf the dimension of $\text{Stab}_F(1)$ is the same.

Thus if we were to construct a section in $F$ - that is a manifold of dimension complimentary to the dimension of the leaf that $F$ belongs to, containing $F$ and transversal to the leaf containing $F$ - and prove that in a small neighbourhood in the section around $F$ the dimension of $\text{Stab}_F(1)$ does not vary we would be able to parameterize all points in the neighbourhood of $F$ by the leaf that point belongs to and the intersection of that leaf with the section in $F$. Since in a small neighbourhood of $F$ inside the section the dimension of $\text{Stab}_F(1)$ does not vary we would obtain that $F$ is 1-regular.

Consider the following map of $\text{Stab}_F(1)$ into $\mathfrak{A}^*$:

$$\rho : a \mapsto \tilde{F}(x) := F(x) + F(xa) + F(ax) = F(x) + Q_F(a, x)$$

This map possesses the following properties:

- the image of $\rho$ is a linear submanifold of $\mathfrak{A}^*$
- map $\rho$ is a bijection between $\text{Stab}_F(1)$ and $\text{Im} \rho$. Indeed, assume this is not true and there are two $a$ and $b$ that mapped into the same functional $\tilde{F}$. Then, for all $x$:

$$F(x) + Q_F(a, x) = F(x) + Q_F(b, x)$$

Thus

$$Q_F(a - b, x) = 0$$

Since $Q_F$ is non-degenerate on $\text{Stab}_F(1)$ we must have $a - b = 0$ and thus $a$ and $b$ coincide.

- the tangent space of $\text{Im} \rho$ in point $F$ is transversal to the tangent space of the leaf passing through $F$. Indeed, assume this is not so and there exists $a \in \text{Stab}_F(1)$ and $b \in \mathfrak{A}$ such that

$$Q_F(a, x) = F(bx - xb)$$

We note that $a$ cannot be zero as otherwise the tangent vector itself must be 0. Since $Q_F$ is non-degenerate there exists $c \in \text{Stab}_F(1)$ such that $Q_F(a, c) \neq 0$. By definition of $\text{Stab}_F(1)$ we have $F(bc - cb) = 0$ - a contradiction. Therefore these spaces are transversal.
• for all \( \tilde{F} \) we have \( \text{Stab}_F(1) \subset \text{Stab}_{\tilde{F}}(1) \). Indeed, let \( x \) be an arbitrary element of \( \mathfrak{A} \), \( b \) an arbitrary element of \( \text{Stab}_F(1) \) and \( a \) be an element of \( \text{Stab}_F(1) \) that defines \( \tilde{F} \).

By the assumption of the theorem \( \text{Stab}_F(1) \) is commutative, thus

\[
\tilde{F}(bx) = F(bx) + F(bxa) + F(abx) = F(xb) + F(xba) + F(axb) = \tilde{F}(xb)
\]

• there exists a small neighbourhood of \( F \) inside \( \text{Im} \rho \) where the dimension of \( \text{Stab}_F(1) \) does not vary. Indeed, the set of all functionals \( G \in \text{Im} \rho \) such that \( \dim \text{Stab}_G(1) \) is greater than \( \dim \text{Stab}_F(1) \) is defined by a set of polynomial equations in \( G \) (namely that the minors of the order \( \dim \mathfrak{A} - \dim \text{Stab}_F(1) \) of the matrix \( G(e_i e_j - e_j e_i) \) all vanish) and thus is Zariski closed.

The complement is a Zariski open set - and \( F \) already belongs to it. Since we have already proven that all \( \tilde{F} \) in \( \text{Im} \rho \) have \( \dim \text{Stab}_{\tilde{F}}(1) \geq \dim \text{Stab}_F(1) \), the set of all \( \tilde{F} \) such that \( \dim \text{Stab}_{\tilde{F}}(1) = \dim \text{Stab}_F(1) \) is Zariski open.

Therefore, \( \text{Im} \rho \) is the section we desire and thus \( F \) is 1-regular.

\[\blacksquare\]

7.2 Characteristic polynomial of type 1 algebras

**Theorem 17** \( \chi_F(\lambda, \mu) \) is divisible by \( (\lambda - \mu/\alpha)^{\dim \text{Stab}_F(\alpha)} \) \( \mu^{\dim \text{Stab}_F(0)} \) for the case \( \alpha = 0 \).

**Proof** Let \( \epsilon = \lambda - \mu/\alpha \) and \( \sigma = \mu \). Let \( k \) denote \( \dim \text{Stab}_F(\alpha) \).

We will choose two bases in \( \mathfrak{A} \): one such that the first \( k \) vectors form \( \text{Stab}_F(\alpha) \) and second so that the first \( k \) vectors form \( \text{Stab}_F(1/\alpha) \). We compute:

\[
det(\lambda A_F + \mu A_F^T) = det((\epsilon - \sigma/\alpha)A_F + (\sigma A_F^T) = det(\epsilon A_F + \sigma(A_F - \alpha A_F^T)/\alpha) =
\]

\[
= \epsilon^k \det \left( A_F|_{\text{Stab}_F(\alpha) \times \text{Stab}_F(1/\alpha)} \right) \det \left( \frac{\sigma(A_F - \alpha A_F^T)}{\alpha} |_{\text{Stab}_F(\alpha) \times \text{Stab}_F(1/\alpha)} \right) + o(\epsilon^k)
\]

Thus \( \det(\lambda A_F + \mu A_F^T) \) is divisible by at least \( \epsilon^k \).
The cases $\alpha = 0$ and $\alpha = \infty$ are resolved in a similar manner:

$$\det(\lambda A_F + \mu A_F^T) =$$

$$= \lambda^{\dim \ker A_F^T} \det \left( A_F \mid_{\text{Stab}_F(\alpha) \times \text{Stab}_F(0)} \right) \det \left( A_F^T \mid_{\text{Stab}_F(\infty) \times \text{Stab}_F(0)} \right) +$$

$$+ \mathcal{O}(\epsilon^{\dim \ker A_F^T}) =$$

$$= \mu^{\dim \ker A_F} \det \left( A_F^T \mid_{\text{Stab}_F(0) \times \text{Stab}_F(\infty)} \right) \det \left( A_F \mid_{\text{Stab}_F(\infty) \times \text{Stab}_F(0)} \right) +$$

$$+ \mathcal{O}(\epsilon^{\dim \ker A_F})$$

\[ \blacksquare \]

**Corollary 1.** We see from the proof that $\dim \text{Stab}_F(\alpha)$ coincides with the highest degree $k$ such that $\chi_F(\lambda, \mu)$ is divisible by $(\lambda - \mu/\alpha)^k$ if and only if the restriction of the form $A_F$ ($A_F^T$ for $\alpha = 0$) on the space $\text{Stab}_F(\alpha) \times \text{Stab}_F(1/\alpha)$ is non-degenerate.

**Corollary 2.** $\dim \text{Stab}_F(1)$ is equal to the highest power of $\lambda - \mu$ that divides $\chi_F(\lambda, \mu)$ if and only if the restriction of the form $Q_F$ to $\text{Stab}_F(1)$ is non-degenerate.

**Definition 7** We will call a functional $F$ on $\mathfrak{A}$ $\alpha$-precise if the dimension of $\text{Stab}_F(\alpha)$ is equal exactly to the highest power of $(\lambda - \mu/\alpha)$ that divides $\chi_F(\lambda, \mu)$.

**Definition 8** Fix a continuous function $\alpha(F)$ defined on an open dense subset of $\mathfrak{A}^*$. We call an associative algebra $\mathfrak{A}$ $\alpha(F)$-precise if there exists an open dense subset of functionals $F$ that are $\alpha(F)$-precise.

**Example 6** Let us compute characteristic polynomial for $(2, 1; 1, 2)$ seaweed algebra\(^1\) with multiplication table:

|   | a   | b   | c   | d   | e   |
|---|-----|-----|-----|-----|-----|
| a | a   | b   | 0   | 0   | 0   |
| b | 0   | 0   | b   | 0   | 0   |
| c | 0   | 0   | c   | 0   | 0   |
| d | 0   | 0   | d   | 0   | 0   |
| e | 0   | 0   | 0   | d   | e   |

We compute

$$\det(\lambda A_F + \mu A_F^T) = \lambda^2 \mu^2 b^2 d^2 (\lambda + \mu)(a + c + e)$$

\(^1\)For definition of seaweed algebras see [1]
The space $\text{Stab}_F(1)$ is generated by unity $a+c+e$, the space $\text{Stab}_F(0)$ is spanned by $F(a)b - F(b)a$ and $F(e)d - F(d)e$ and the space $\text{Stab}_F(\infty)$ is spanned by $F(b)c - F(c)b$ and $F(b)d - F(d)b$.

In this case we see that the dimensions of the spaces $\text{Stab}_F(\alpha)$ match exactly the order of zero of polynomial $\chi_F(1, -x)$ in point $1/\alpha$.

Note that in this case $\mathfrak{A} = \text{Stab}_F(0) \oplus \text{Stab}_F(1) \oplus \text{Stab}_F(\infty)$.

### 7.3 Decomposition of type 1 algebras

The previous example of $(2, 1; 1, 2)$ seaweed algebra and the example of $n \times n$ matrices investigated in section 3.1 have the property that, for generic $F$, the algebra $\mathfrak{A}$ is a direct sum of spaces $\text{Stab}_F(\alpha)$. While it is true that, for generic $F$, the spaces $\text{Stab}_F(\alpha)$ do form a direct sum for any type 1 algebra one can construct examples when their sum is not the entire algebra.

**Example 7** The simplest way is to consider an algebra $\mathfrak{A}(B)$ from example 4 with $\dim W = 1$ and extend it with unity:

|   | 1 | V | w |
|---|---|---|---|
| 1 | 1 | V | w |
| V | V | Bw | 0 |
| w | w | 0 | 0 |

The characteristic polynomial of such an algebra is equal to

$$\chi_F(\lambda, \mu) = (\lambda + \mu)^2 \det (\lambda B + \mu B^T) w^{\dim \mathfrak{A}}$$

For this algebra 1 and $w$ are always within $\text{Stab}_F(1)$.

Without loss of generality we can assume that $F(1) = 1$ and $F(w) \neq 0$. For $\alpha \neq 1$ the element $x = a \cdot 1 + v + b \cdot w$ belongs to $\text{Stab}_F(\alpha)$ if and only if

$$\begin{cases} 
    a + F(v) + bF(w) & = 0 \\
    a(1 - \alpha)F(v) + F(w)(B(v, \cdot) - \alpha B(\cdot, v)) & = 0 \\
    a(1 - \alpha)F(w) & = 0
\end{cases}$$

Thus $a = 0$ and $b = F(v)/F(w)$ and

$$(B - \alpha B^T)v = 0$$

Let us restrict our attention to those $B$ with $\det B \neq 0$. In this case the previous equation reduces to $(1 - \alpha B^{-1}B^T)v = 0$ and the question
of whether \( \mathfrak{A} \) is the direct sum of spaces \( \text{Stab}_F(\alpha) \) is equivalent to the question of whether \( B^{-1}B^T \) is diagonalizable.

Now consider the following family of matrices \( B \):

\[
B = \begin{pmatrix} 0 & B_0 \\ B_1 & 0 \end{pmatrix}
\]

Then

\[
B^{-1}B^T = \begin{pmatrix} 0 & B_0^{-1} \\ B_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & B_1^T \\ B_1^T & 0 \end{pmatrix} = \begin{pmatrix} B_1^{-1}B_0^T & 0 \\ 0 & B_0^{-1}B_1^T \end{pmatrix}
\]

There are plenty of choices for \( B_0 \) and \( B_1 \) that yield non-diagonalizable \( B^{-1}B^T \).

However, one can extend the theory of Jordan decomposition of matrices to this case. The following is a rather technical presentation of such.

### 7.3.1 Definition of spaces \( V_k(\alpha) \)

**Definition 9** We define \( V_0(\alpha) = \{0\} \) and \( V_1(\alpha) = \text{Stab}_F(\alpha) \).

**Definition 10** Let \( \mathfrak{A} \) be an associative algebra. Fix \( \alpha_0 \neq \alpha \). We define \( V_k(\alpha) \) - a space of "Jordan vectors" - as

\[
V_{k+1}(\alpha) := \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow \\forall x \in \mathfrak{A} \Rightarrow F(bx) - \alpha F(xb) = F(ax) - \alpha_0 F(xa) \}
\]

or in terms of multiplication table of \( \mathfrak{A} \):

\[
V_{k+1}(\alpha) := \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow A_F b - \alpha A_F^T b = A_F a - \alpha_0 A_F^T a \}
\]

For \( \alpha = \infty \) we define

\[
V_{k+1}(\infty) := \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow (\forall x \in \mathfrak{A} \Rightarrow F(xb) = F(ax) - \alpha_0 F(xa)) \}
\]

or

\[
V_{k+1}(\infty) := \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow A_F^T b = A_F a - \alpha_0 A_F^T a \}
\]

**Lemma 18** The spaces \( V_k(\alpha) \) do not depend on the choice of \( \alpha_0 \).
Proof Indeed this is so by definition for $k \leq 1$. Assume that for this statement holds for all $k \leq n$.

Let $b$ be an element of $V_{n+1}(\alpha)$ constructed using $\alpha_0$. By definition, this implies existence of $a \in V_n(\alpha)$ such that

$$F(bx) - \alpha F(xb) = F(ax) - \alpha_0 F(xa)$$

We will show existence of element $a'' \in V_n(\alpha)$ such that $F(ax) - \alpha_0 F(xa) = F(a''x) - \alpha_1 F(xa'')$.

$$F(ax) - \alpha_0 F(xa) = \begin{align*}
&= (1 - \delta)F(ax) + \delta a F(xa) + \delta (F(a'x) - \alpha_1 F(xa')) - \alpha_0 F(xa) \\
&= (1 - \delta)F(ax) + (\delta \alpha - \alpha_0)F(xa) + \delta (F(a'x) - \alpha_1 F(xa')) \\
&= F((1 - \delta)a + \delta a')x - \alpha_1 F(xa))
\end{align*}$$

Here $a' \in V_{n-2}(\alpha)$ satisfies $F(ax) - \alpha F(xa) = F(a'x) - \alpha_1 F(xa')$ by assumption of induction.

Choosing $\delta = \frac{\alpha_0 - \alpha_1}{\alpha - \alpha_1}$ and $a'' = (1 - \delta)a + \delta a'$ we obtain $F(ax) - \alpha_1 F(xa) = F(a''x) - \alpha_0 F(xa'')$, where $a''$ is also in $V_{n-1}(\alpha)$.

Thus the space $V_{n+1}(\alpha)$ constructed using $\alpha_0$ is a subset of space $V_{n+1}(\alpha)$ constructed using any $\alpha_1 \neq \alpha$. Therefore for any $\alpha_0$ and $\alpha_1$, different from $\alpha$, the spaces $V_{n+1}(\alpha)$ are identical.

A similar argument can be used to prove the case $\alpha = \infty$. However, there is another way. We can observe that when we introduce a new "transposed" multiplication law $a \ast b := ba$ the parameter $\alpha$ is transformed into its inverse, i.e. space $\text{Stab}_F(\alpha)$ become $\text{Stab}_F(1/\alpha)$ and spaces $V_k(\alpha)$ become spaces $V_k(1/\alpha)$. Since we already proven the case $\alpha = 0$ we must conclude that the case $\alpha = \infty$ is true as well.

Note. We emphasize that this definition is valid for any associative algebra, not necessarily finite dimensional or type 1.

For the case of finite dimensional type 1 algebras there is an equivalent way of defining $V_k(\alpha)$ that exposes their nature as Jordan spaces of an operator. Choose $\alpha_0$ so that $\chi_F(\alpha_0) = \det (A_F - \alpha_0 A_F^T) \neq 0$. 

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From definition,
\[ V_{k+1}(\alpha) := \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow A_F b - \alpha A_F^T b = A_F a - \alpha_0 A_F^T a \} = \]
\[ = \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow (A_F - \alpha_0 A_F^T)^{-1} (A_F - \alpha A_F^T) b = a \} = \]
\[ = \left( \left( (A_F - \alpha_0 A_F^T)^{-1} (A_F - \alpha A_F^T) \right)^{k+1} \right)^{-1} V_0(\alpha) = \]
\[ = \left( \left( 1 - (\alpha - \alpha_0) (A_F - \alpha_0 A_F^T)^{-1} A_F^T \right)^{k+1} \right)^{-1} V_0(\alpha) = \]
\[ = \left( \left( (A_F - \alpha_0 A_F^T)^{-1} A_F^T - \frac{1}{\alpha - \alpha_0} \right)^{k+1} \right)^{-1} V_0(\alpha) \]

We observe that \( V_k(\alpha) \) is exactly the \( k \)-th level Jordan space of operator \( (A_F - \alpha_0 A_F^T)^{-1} A_F^T \) corresponding to eigenvalue \( \frac{1}{\alpha - \alpha_0} \).

Thus

**Theorem 19** Let \( \mathfrak{A} \) be a type 1 algebra. Then
\[ \mathfrak{A} = \bigoplus_{\alpha} \bigcup_k V_k(\alpha) \]

**Remark.** For type 2 algebras there is no \( \alpha_0 \) such that \( A_F - \alpha_0 A_F^T \) is invertible. However, by considering \( \mathfrak{A}/\text{Nil}_F \) instead of \( \mathfrak{A} \) we notice that the induced form \( (A_F - \alpha_0 A_F^T)_{\text{Nil}_F} \) is non-degenerate for most \( \alpha_0 \). Therefore, \( \mathfrak{A}/\text{Nil}_F = \bigoplus_{\alpha} (V_k(\alpha)/\text{Nil}_F) \). For type 3 algebras one can construct an example where the spaces \( V_k(\alpha) \) are pairwise transversal for different values of \( \alpha \), but do not form a direct sum.

**7.3.2 Properties of spaces \( V_k(\alpha) \)**

**Theorem 20** The spaces \( V_k(\alpha) \) possess the following properties:

1. \( V_k(\alpha) \subset V_{k+1}(\alpha) \)
2. For \( \alpha, \beta \notin \{0, \infty\} \) we have \( V_k(\alpha) \cdot V_m(\beta) \subset V_{k+m-1}(\alpha \beta) \)
3. For \( \alpha \neq 0 \) we have \( V_k(\alpha) \cdot V_m(\infty) \subset V_{k+m-1}(\infty) \)
4. For \( \alpha \neq 0 \) we have \( V_k(\infty) \cdot V_m(\alpha) \subset V_{k+m-1}(\infty) \)
5. For \( \alpha \neq \infty \) we have \( V_k(\alpha) \cdot V_m(0) \subset V_{k+m-1}(0) \)
6. For \( \alpha \neq \infty \) we have \( V_k(0) \cdot V_m(\alpha) \subset V_{k+m-1}(0) \)

**Proof** Property 1 follows by induction from the fact that \( V_{-1}(\alpha) \subset V_0(\alpha) \).
To prove property 2 we make induction on the parameter $N = k + m$. The base of induction follows immediately from properties of $\text{Stab}_F(\alpha)$ (theorem 10). Assume that the statement is true for all $k$ and $m$ such that $k + m < N + 1$. For a given $\alpha$ and $\beta$ we pick $\alpha_0 = 0$ as this value is different from both $\alpha$ and $\beta$. Let $b_1 \in V_k(\alpha)$, $b_2 \in V_m(\beta)$, where $k + m = N + 1$. Let $a_1 \in V_{k-1}(\alpha)$ be an element corresponding to $b_1$ according to definition 10 and $a_2 \in V_{m-1}(\beta)$ be the element corresponding to $b_2$. Let $x$ be an arbitrary element of $\mathfrak{A}$. Then:

$$F(b_1 b_2 x) - \alpha \beta F(x b_1 b_2) = \alpha F(b_2 x b_1) - \alpha \beta F(x b_1 b_2) + F(a_1 b_2 x) =\alpha F(a_2 x b_1) + F(a_1 b_2 x) =\alpha F(b_1 a_2 x) + F(a_1 a_2 x)) + F(a_1 b_2 x) =\alpha F((b_1 a_2 + a_1 a_2 + a_1 b_2) x)$$

Now by assumption of induction we have

$$b_1 a_2 + a_1 a_2 + a_1 b_2 \in V_{k+m-2}(\alpha \beta)$$

and thus $b_1 b_2$ is an element of $V_{k+m-1}(\alpha \beta)$.

Property 3. We perform induction the same way as in proof of property 2. For the same reasons we choose $\alpha_0 = 0$. Let $b_1 \in V_k(\alpha)$, $b_2 \in V_m(\infty)$, where $k + m = N + 1$. Let $a_1 \in V_{k-1}(\alpha)$ be an element corresponding to $b_1$ according to definition 10 and $a_2 \in V_{m-1}(\infty)$ be the element corresponding to $b_2$.

We compute:

$$F(x b_1 b_2) = F(a_2 x b_1) = \frac{1}{\alpha} (F(b_1 a_2 x) + F(a_1 a_2 x)) = F \left( \frac{b_1 a_2 + a_1 a_2 x}{\alpha} \right)$$

By assumption of induction we have

$$\frac{b_1 a_2 + a_1 a_2}{\alpha} \in V_{k+m-2}(\infty)$$

and thus $b_1 b_2$ is an element of $V_{k+m-1}(\infty)$.

Property 4 is proved almost identically to property 3. We will write down the computation of $F(x b_1 b_2)$:

$$F(x b_1 b_2) = F(\frac{b_2 x b_1}{\alpha} + F(a_2 x b_1)) = F(\frac{a_2 x b_1 + F(a_1 a_2 x)}{\alpha}) = F \left( \frac{a_2 b_1 + a_1 a_2}{\alpha} \right)$$

Properties 5 and 6 can be proven by similar computation (it might be useful to use $\alpha = \infty$), however we will simply refer to the correspondence $V_k(\alpha) \leftrightarrow V_k(1/\alpha)$ that occurs when one considers a transposed algebra $\mathfrak{A}$ with multiplication $a \ast b := a \cdot b$.

$\blacksquare$

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Lemma 21  Let $\mathfrak{A}$ be an associative algebra with unity. Then $F(V_k(\alpha)) = 0$ for all $\alpha \neq 1$.

Proof  Case $k = 1$, $\alpha \neq \infty$: for all $b \in V_0(\alpha)$ we must have

$$F(bx) = \alpha F(xb)$$

Setting $x = 1$ we get $F(b) = \alpha F(b)$, hence $F(b) = 0$.

Case $k = 1$, $\alpha = \infty$: we have $F(xb) = 0$. Again setting $x = 1$ yields $F(b) = 0$.

For arbitrary $k$ and $\alpha \neq \infty$ we proceed by induction. Again let us set $x = 1$ in the definition. We get

$$F(b) - \alpha F(b) = F(a)$$

But we already know that $F(a) = 0$, thus $F(b) = 0$ as well.

For arbitrary $k$ and $\alpha = \infty$: from the definition we get

$$F(b) = F(a)$$

And thus $F(b) = 0$.  

Lemma 22  Let $\alpha \notin \{0, \infty\}$. Let $K_1$ be the bilinear form on $V_k(\alpha) \times \mathfrak{A}$ defined by $K_1(x, y) = F(xy)$. Let $K_2$ be the bilinear form on $V_k(\alpha) \times \mathfrak{A}$ defined by $K_2(x, y) = F(yx)$. Then there exists an operator $C : V_k(\alpha) \to V_k(\alpha)$ which has a unique eigenvalue $\alpha$ such that

$$K_1(x, y) = K_2(Cx, y)$$

Proof  We proceed by induction on $k$.

For $k = 1$ the operator $C$ is $\alpha \cdot 1$.

Assume that the lemma is true for all $m \leq k$. Consider the case $k + 1$. Let $C_k$ be the operator constructed for the space $V_k(\alpha)$.

Pick a basis in $V_{k+1}(\alpha)$ such that the first $r$ vectors belong to $V_k(\alpha)$. Let $s = \dim V_{k+1}(\alpha)$. For each vector $v_{r+1} \ldots v_s$ pick an element $a_i \in V_k(\alpha)$ using the definition of the space $V_{k+1}(\alpha)$ with $\alpha_0 = 0$:

$$F(v_iy) - \alpha F(yv_i) = F(a_iy)$$

For $y \in \mathfrak{A}$ we have

$$K_1(v_i, y) - \alpha K_2(v_i, y) = K_1(a_i, y) = K_2(Cka_i, y)$$
Thus
\[ K_1(v_i, y) = K_2(\alpha v_i + C_k a_i, y) \]

We now define \( C_{k+1} \) by its action on basis vectors of \( V_{k+1}(\alpha) \):
\[
C_{k+1} v_i = \begin{cases} 
1 \leq i \leq r & C_k v_i \\
(r + 1) \leq i \leq s & \alpha v_i + C_k a_i 
\end{cases}
\]

We see that \( C_{k+1} \) has indeed only one eigenvalue \( \alpha \). Also for any basis vector \( v_i \) and any \( y \in A \) we have
\[ K_1(v_i, y) = K_2(C_{k+1} v_i, y) \]

Since \( \{v_i\} \) form the basis of \( V_{k+1}(\alpha) \) the above equality holds for any element of \( V_{k+1}(\alpha) \).

\[ \square \]

**Lemma 23** For \( \alpha \neq \infty \) we have
\[ V_{k+1}(\alpha) = \{ b \in \mathfrak{A} : \exists a \in V_k(\alpha) \Rightarrow (\forall x \in \mathfrak{A} \Rightarrow F(b x) - \alpha F(x b) = F(x a)) \} \]

In other words, we can set \( \alpha_0 = \infty \) in the definition of the spaces \( V_k(\alpha) \) with \( \alpha \neq \infty \).

**Proof** Pick \( \alpha_0 \neq \alpha \). Let us proceed by induction. For \( k = 1 \) the lemma is true because the right hand side of the equation in definition of \( V_1(\alpha) \) is 0. Assume the lemma holds for all \( k \leq n \). From definition of \( V_{n+1}(\alpha) \) we have
\[
V_{n+1}(\alpha) = \{ b \in \mathfrak{A} : \exists a \in V_n(\alpha) \Rightarrow (\forall x \in \mathfrak{A} \Rightarrow F(b x) - \alpha F(x b) = F(x a) - \alpha_0 F(x a)) \} = \\
= \{ b \in \mathfrak{A} : \exists a \in V_n(\alpha) \exists a' \in V_{n-1}(\alpha) \Rightarrow (\forall x \in \mathfrak{A} \Rightarrow F(b x) - \alpha F(x b) = \alpha F(x a) + F(x a') - \alpha_0 F(x a)) \} = \\
= \{ b \in \mathfrak{A} : \exists a \in V_n(\alpha) \exists a' \in V_{n-1}(\alpha) \Rightarrow (\forall x \in \mathfrak{A} \Rightarrow F(b x) - \alpha F(x b) = F(x ((\alpha - \alpha_0) a + a'))) \}
\]

We observe that \( (\alpha - \alpha_0) a + a' \) is an element of \( V_n(\alpha) \). Thus the lemma holds for \( V_{n+1}(\alpha) \) as well. \( \square \)

**Note.** We observe that theorem 20 and lemmas 21, 22 and 23 hold for any associative algebra \( \mathfrak{A} \).
Lemma 24 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two associative algebras. Let $F$ and $G$ be linear functionals on algebras $\mathfrak{A}$ and $\mathfrak{B}$ correspondingly. Let $\alpha$ and $\beta$ be such that $\{\alpha, \beta\} \neq \{0, \infty\}$.

Then $$V_{k}^{\mathfrak{A}}(\alpha) \otimes V_{m}^{\mathfrak{B}}(\beta) \subset V_{k+m-1}^{\mathfrak{A} \otimes \mathfrak{B}}(\alpha \beta)$$ where the latter space was constructed using functional $F \otimes G$.

Proof First of all, let us note that because the algebra with transposed multiplication law numerates spaces $V_{k}$ with $1/\alpha$ it is sufficient to prove this lemma in the case of finite $\alpha$ and $\beta$.

Let $b_{1} \in V_{k}^{\mathfrak{A}}(\alpha)$, $b_{2} \in V_{m}^{\mathfrak{B}}(\beta)$, $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$. We have:

$$(F \otimes G)((b_{1} \otimes b_{2}) \cdot (x \otimes y)) := F(b_{1}x)G(b_{2}y) = (\alpha F(xb_{1}) + F(xa_{1})) (\beta G(yb_{2}) + G(ya_{2})) = \alpha \beta F(xb_{1})G(yb_{2}) + \alpha F(xb_{1})G(ya_{2}) + \beta F(xa_{1})G(yb_{2}) + F(xa_{1})G(ya_{2}) = (F \otimes G)((x \otimes y) \cdot (b_{1} \otimes b_{2}) + (x \otimes y) \cdot (b_{1} \otimes a_{2} + a_{1} \otimes b_{2} + a_{1} \otimes a_{2}))$$

Therefore the lemma holds for $k = m = 1$ as in this case $a_{1} = a_{2} = 0$. Also the computation above serves as an induction step in $n = k + m$.

Lemma 25 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two associative algebras. Let $F$ and $G$ be two linear functionals on algebras $\mathfrak{A}$ and $\mathfrak{B}$ correspondingly. Then

$$\text{Stab}_{F}^{\mathfrak{A}}(0) \otimes \text{Stab}_{G}^{\mathfrak{B}}(\infty) + \text{Stab}_{F}^{\mathfrak{A}}(\infty) \otimes \text{Stab}_{G}^{\mathfrak{B}}(0) \subset \text{Nil}_{F \otimes G}^{\mathfrak{A} \otimes \mathfrak{B}}$$

Proof Again because of the argument that algebra with transposed multiplication law numerates spaces $V_{k}$ with $1/\alpha$ it is sufficient to establish that $\text{Stab}_{F}^{\mathfrak{A}}(0) \otimes \text{Stab}_{G}^{\mathfrak{B}}(\infty)$ is a subset of $\text{Nil}_{F \otimes G}^{\mathfrak{A} \otimes \mathfrak{B}}$.

Let $b_{1} \in \text{Stab}_{F}^{\mathfrak{A}}(0)$ and $b_{2} \in \text{Stab}_{G}^{\mathfrak{B}}(\infty)$. We have

$$(F \otimes G)((b_{1} \otimes b_{2}) \cdot (x \otimes y)) := F(b_{1}x)G(b_{2}y) = 0 \cdot G(b_{2}y) = 0$$

Also

$$(F \otimes G)((x \otimes y) \cdot (b_{1} \otimes b_{2})) := F(xb_{1})G(yb_{2}) = F(xb_{1}) \cdot 0 = 0$$

Thus $\text{Stab}_{F}^{\mathfrak{A}}(0) \otimes \text{Stab}_{G}^{\mathfrak{B}}(\infty) \subset \text{Nil}_{F \otimes G}^{\mathfrak{A} \otimes \mathfrak{B}}$. 

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Lemma 26 Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two associative algebras. Let \( F \) and \( G \) be two linear functionals on algebras \( \mathfrak{A} \) and \( \mathfrak{B} \) correspondingly. Then

\[
\text{Stab}_{\mathfrak{A}} F(0) \otimes \mathfrak{B} + \mathfrak{A} \otimes \text{Stab}_{\mathfrak{B}} G(0) \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(0)
\]

\[
\text{Stab}_{\mathfrak{A}} F(\infty) \otimes \mathfrak{B} + \mathfrak{A} \otimes \text{Stab}_{\mathfrak{B}} G(\infty) \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(\infty)
\]

**Proof** Consider first the case \( \text{Stab}_{\mathfrak{A}} F(0) \otimes \mathfrak{B} \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(0) \).

Using definition of \( \text{Stab}_{\mathfrak{A}} F(0) \) we derive:

\[
\text{Stab}_{\mathfrak{A}} F(0) \otimes \mathfrak{B} = \{ b_1 \in \mathfrak{A} : \forall x \in \mathfrak{A} \Rightarrow F(b_1 x) = 0 \} \otimes \mathfrak{B} = \text{span} \{ b_1 \otimes b_2 : b_1 \in \mathfrak{A}, b_2 \in \mathfrak{B}, \forall x \in \mathfrak{A} \forall y \in \mathfrak{B} \Rightarrow \} \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(0)
\]

Next, observe that by argument of symmetry \( \mathfrak{A} \otimes \mathfrak{B} \leftrightarrow \mathfrak{B} \otimes \mathfrak{A} \) and symmetry \( \mathfrak{A} \leftrightarrow (\mathfrak{A} \text{ with transposed multiplication}) \), \( \alpha \leftrightarrow 1/\alpha \) we must have as well

\[
\mathfrak{A} \otimes \text{Stab}_{\mathfrak{B}} G(0) \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(0)
\]

\[
\text{Stab}_{\mathfrak{A}} F(\infty) \otimes \mathfrak{B} \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(\infty)
\]

\[
\mathfrak{A} \otimes \text{Stab}_{\mathfrak{B}} G(\infty) \subset \text{Stab}_{\mathfrak{A} \otimes \mathfrak{B}} F \otimes G(\infty)
\]

which concludes the proof of this lemma.

7.4 Tensor products of type 1 algebras

In the previous section we have seen that the spaces \( V_k(\alpha) \) satisfy some remarkable properties with respect to tensor products of associative algebras. A natural question is whether this reflects on the characteristic polynomial of a tensor product of associative algebras.

7.4.1 Tensor products of matrices

Since the definition of characteristic polynomial involves determinant of matrices we first turn our attention to tensor products of matrices.

**Definition 11** [Tensor product of matrices]

Let \( A \) and \( B \) be two matrices with coefficients in rings \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) respectively. Let commutative ring \( \mathcal{R} \) be a subring of both \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). The tensor product \( A \otimes_{\mathcal{R}} B \) is defined as a block matrix with each block \( (i,j) \) having dimensions of matrix \( B \) and equal to \( A_{i,j} \otimes_{\mathcal{R}} B \), that is the matrix obtained from \( B \) by taking tensor products of a certain element of \( A \) with entries of \( B \). Thus \( A \otimes_{\mathcal{R}} B \) has coefficients in \( \mathcal{R}_1 \otimes_{\mathcal{R}} \mathcal{R}_2 \).
Proposition 27 The tensor product of matrices has the following properties:

1. distributive w.r.t. addition
2. \((A \otimes R B) \cdot (C \otimes R D) = (AC) \otimes R (BD)\)
3. \((A \otimes R B)^{-1} = (A^{-1}) \otimes R (B^{-1})\)

Theorem 28 Let \(A\) and \(B\) be square matrices of dimensions \(k\) and \(n\) respectively, with coefficients in commutative rings \(R_1\) and \(R_2\). Let ring \(R\) have the property that \(R \subset R_1\) and \(R \subset R_2\). Then

\[
\det (A \otimes R B) = (\det A)^n \otimes R (\det B)^k
\]

Proof

1. If \(A\) and \(B\) are diagonal the statement is proved by a simple computation.

2. Let \(R_1 = R_2 = R = \mathbb{C}\). Let \(A = C_1 D_1 C_1^{-1}\) and \(B = C_2 D_2 C_2^{-1}\) where \(D_1\) and \(D_2\) are diagonal. Then

\[
A \otimes R B = (C_1 D_1 C_1^{-1}) \otimes R (C_2 D_2 C_2^{-1}) = \\
= (C_1 \otimes R C_2) (D_1 \otimes R D_2) (C_1^{-1} \otimes R C_2^{-1}) = \\
= (C_1 \otimes R C_2) (D_1 \otimes R D_2) (C_1 \otimes R C_2)^{-1}
\]

and

\[
\det (A \otimes R B) = \\
= \det \left((C_1 \otimes R C_2) (D_1 \otimes R D_2) (C_1 \otimes R C_2)^{-1}\right) = \\
= \det (D_1 \otimes R D_2) = \\
= (\det D_1)^n (\det D_2)^k = \\
= (\det A)^n (\det B)^k
\]

3. Since both sides of the equation \(\det (A \otimes R B) = (\det A)^n \otimes R (\det B)^k\) are polynomials in elements of \(A\) and \(B\) with integral coefficients and we know that over \(\mathbb{C}\) all generic \(A\) and \(B\) satisfy the equation we must have that the polynomials are identical. This concludes the proof of the theorem.

Theorem 29 Let \(A\) and \(B\) be two matrices with coefficients in commutative rings \(R_1\) and \(R_2\) respectively. Let \(R\) be a unital subring of
both $R_1$ and $R_2$. Then there exists invertible matrix $U$ with coefficients in $R$ that depends only on dimensions of $A$ and $B$ such that

$$A \otimes_{R} B = U (B \otimes_{R} A) U^{-1}$$

**Proof** Let $k$ denote the size of $A$ and $m$ denote the size of $B$. Consider $A$ and $B$ as operators acting in $k$-dimensional space $V$ with basis $\{f_i\}_{i=0}^{k-1}$ and $m$-dimensional space $W$ with basis $\{g_i\}_{i=0}^{m-1}$ correspondingly.

Then the tensor product $A \otimes_{R} B$ is uniquely defined as an operator in $V \otimes_{R} W$ (which we consider to be a vector space over $R_1 \otimes_{R} R_2$). The matrix representation of operator $A \otimes_{R} B$ depends on the choice of basis $\{e_i\}_{i=0}^{km-1}$ in $V \otimes_{R} W$.

If we use the formula

$$e_i = f_{[i/m]} \otimes g_{i-[i/m]-m}$$

we obtain the definition of matrix $A \otimes_{R} B$ (Here $[x]$ denotes the integral part of $x$).

If we use the formula

$$e_i = f_{i-[i/k]-k} \otimes g_{[i/k]}$$

we obtain the definition of matrix $B \otimes_{R} A$.

Thus $A \otimes_{R} B$ and $B \otimes_{R} A$ are simply two different matrix representations of the same operator and the matrix $U$ is the transformation matrix from one basis to the other. 

**Theorem 30** [Extended Cayley theorem] Let $A$ and $B$ be two $n \times n$ matrices over an algebraically closed field $k$ and $C$ and $D$ be two $m \times m$ matrices over the same field $k$. Define $\chi(\lambda, \mu) = \det(\lambda A + \mu B)$. Then

$$\det(\lambda A \otimes C + \mu B \otimes D) = \det(\chi(\lambda C, \mu D))$$

Before proceeding with the proof we must explain in what sense we consider $\chi(\lambda C, \mu D)$. Indeed, matrices $C$ and $D$ might not commute making $\chi(\lambda C, \mu D)$ ambiguous. In our situation the right definition is as follows:

First, we notice that $\chi(\lambda, \mu)$ is homogeneous, thus it can be decomposed into a product of linear forms ($k$ is algebraically closed):

$$\chi(\lambda, \mu) = \prod_i (\lambda \alpha_i + \mu \beta_j)$$

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Then we define

$$\chi(\lambda C, \mu D) = \prod_i (\lambda \alpha_i C + \mu \beta_j D)$$

There is still some ambiguity about the order in which we multiply linear combinations of $C$ and $D$ but it does not affect the value of $\det(\chi(\lambda C, \mu D))$.

**Proof Step 1.** Let $A$ and $C$ be identity matrices of sizes $n \times n$ and $m \times m$ respectively.

Then

$$\chi(\lambda, \mu) = \prod_i (\lambda + \mu \gamma_i)$$

where $\gamma_i$ are eigenvalues of $B$.

$$\det(\chi(\lambda, \mu D)) = \det \left( \prod_i (\lambda + \mu \gamma_i D) \right)$$

$$= \prod_i \det (\lambda + \mu \gamma_i D) = \prod_{i,j} (\lambda + \mu \gamma_i \epsilon_j) \quad (1)$$

where $\epsilon_j$ are eigenvalues of $D$.

On the other hand one easily derives that eigenvalues of $B \otimes D$ are $\gamma_i \epsilon_j$ and thus

$$\det(\lambda + \mu B \otimes D) = \prod_{i,j} (\lambda + \mu \gamma_i \epsilon_j) = \det(\chi(\lambda, \mu D))$$

**Step 2.** Let us assume now only that matrices $A$ and $C$ are invertible.

We have

$$\chi(\lambda, \mu) = \det(\lambda A + \mu B) = \det(A) \det(\lambda + \mu A^{-1} B)$$

Denote $\chi'(\lambda, \mu) = \det(\lambda + \mu A^{-1} B) = \prod_i (\lambda + \mu \gamma_i)$, where $\gamma_i$ are eigenvalues of $A^{-1} B$.

$$\det(\chi(\lambda C, \mu D)) = \det \left( \det(A) \prod_i (\lambda C + \mu \gamma_i D) \right)$$

$$= \det(A)^m \prod_i \det(\lambda C + \mu \gamma_i D) =$$

$$= \det(A)^m \prod_i (\det(C) \det(\lambda + \mu \gamma_i C^{-1} D)) =$$
By step 1 we have
\[ \det(\lambda + \mu (A^{-1}B) \otimes (C^{-1}D)) = \det(\chi'(\lambda, \mu C^{-1}D)) \]
Observing also that \( \det(A \otimes C) = \det(A)^m \det(C)^n \) we obtain
\[ \det(\chi(\lambda C, \mu D)) = \det(A \otimes C) \det(\lambda + \mu (A^{-1}B) \otimes (C^{-1}D)) = \]
\[ = \det(\lambda A \otimes C + \mu B \otimes D) \]
which is the desired formula.

Step 3. Assume now that \( \chi(\lambda, \mu) \neq 0 \). Observe that the right side of the formula involves only polynomials in entries of matrices \( C \) and \( D \). Furthermore, it only involves polynomials in coefficients of \( \chi(\lambda, \mu) \) computed for \( C \) and \( D \).

We observe that the left side is polynomial in entries of matrices \( A, B, C \) and \( D \).

A natural question is what happens under the symmetry \( A \leftrightarrow C, B \leftrightarrow D \).

The left part is unchanged:
\[ \det (\lambda A \otimes C + \mu B \otimes D) = \det (\lambda U (C \otimes A) U^{-1} + \mu U (D \otimes B) U^{-1}) = \]
\[ = \det (\lambda C \otimes A + \mu D \otimes B) \]
Thus the right hand part is polynomial in coefficients of polynomials \( \chi(\lambda, \mu) \) computed for pair \( A \) and \( B \) and pair \( C \) and \( D \). This polynomial can be viewed as another generalization of the resultant of a pair of polynomials.

Since the restriction that \( A \) and \( C \) be invertible selects a Zariski open subset, the formula should hold for all \( A, B, C, D \) by continuity.

7.4.2 Tensor products of algebras

We are now in position to prove several results about tensor products of type 1 algebras.

Theorem 31 Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two associative algebras. Let \( F \) and \( G \) be linear functionals on \( \mathfrak{A} \) and \( \mathfrak{B} \) correspondingly. Let \( V_1 \subset \mathfrak{A} \) and \( V_2 \subset \mathfrak{B} \) be two linear subspaces. Then
\[ \chi^{\mathfrak{A} \otimes \mathfrak{B}}_{F \otimes G, V_1 \otimes V_2}(\lambda, \mu) = \det \chi^{\mathfrak{A}}_{F, V_1}(\lambda B_G, \mu B_G) \]
Here \( B_G \) is the multiplication table of \( V_2 \) evaluated in \( G \).
Proof Let us pick a basis \( \{ f_i \}_{i=0}^{k-1} \) in \( V_1 \) and a basis \( \{ g_i \}_{i=0}^{m-1} \) in \( V_2 \). We use \( A_F \) to denote the matrix \( \| F(f_i f_j) \| \) and \( B_G \) to denote the matrix \( \| G(g_i g_j) \| \).

We define the basis \( \{ e_i \}_{i=0}^{km-1} \) in \( V_1 \otimes V_2 \) as following:

\[
e_i = f_{[i/m]} \otimes g_{i-[i/m]m}
\]

Let \( C \) be the matrix \( \| (F \otimes G)(e_i e_j) \| = \| F \left( f_{[i/m]} f_{[j/m]} \right) G \left( g_{i-[i/m]m} g_{j-[j/m]m} \right) \|. \) Then

\[
C = A_F \otimes B_G
\]

Using theorem 30 we compute

\[
\chi^{A \otimes B}_{F \otimes G, V_1 \otimes V_2} (\lambda, \mu) := \det (\lambda C + \mu C^T) = \det \chi^A_{F, V_1} (\lambda B_G, \mu B_G^T)
\]

Theorem 32 Let \( A \) and \( B \) be two type 1 associative algebras which are 1-precise. Suppose that characteristic polynomial of \( A \) has all its roots except 1 depend non-trivially on \( F \).

Then \( \text{ind} A \otimes B = \text{ind} A \cdot \text{ind} B \).

Proof Because characteristic polynomial of \( A \) has all its roots \( \alpha \neq 1 \) depend non-trivially on \( F \) it is possible to find an open subset of functionals \( F \) in which this polynomial is not divisible by \( \lambda \) or \( \mu \), i.e. \( \text{Stab}_F(0) = \text{Stab}_F(\infty) = \{0\} \).

For both algebras \( A \) and \( B \) there exists an open set of functionals \( F \) that are 1-regular and 1-precise. Therefore the spaces \( \text{Stab}_F(1) \) are commutative algebras and the symmetric form \( Q_F \) is non-degenerate.

Consider the intersection of open sets obtained in first and second paragraphs of this proof. Let \( F \) be an element of it.

Let \( G \) be a regular 1-precise functional on \( B \).

Consider the functional \( F \otimes G \) on the algebra \( A \otimes B \). We can compute the value of characteristic polynomial of \( A \otimes B \) in \( F \otimes G \) by using theorem 31.

Since the characteristic polynomial of \( A \) in \( F \) is not divisible by either \( \lambda \) or \( \mu \) we must have that the characteristic polynomial of \( A \otimes B \) in \( F \otimes G \) is non-zero - and thus the pair \( (A \otimes B, F \otimes G) \) is type 1.

All the roots except unity of characteristic polynomial of \( A \) depend non-trivially on \( F \). Thus we can vary \( F \) to make sure that the only time the product \( \alpha \beta \) (where \( \alpha \) is the root of characteristic polynomial
of $A$ in $F$ and $\beta$ is the root of characteristic polynomial of $B$ in $G$) is equal to 1 is when $\alpha = \beta = 1$. Therefore, the highest power of $\lambda - \mu$ that divides the characteristic polynomial of $A \otimes B$ in $F \otimes G$ is equal to $\text{ind } A \cdot \text{ind } B$.

By lemma 24, $\text{Stab}_F(1) \otimes \text{Stab}_G(1) \subset \text{Stab}_{F \otimes G}(1)$. Considering the dimensions it must be $\text{Stab}_F(1) \otimes \text{Stab}_G(1) = \text{Stab}_{F \otimes G}(1)$. Thus $F \otimes G$ is 1-precise.

Since both $\text{Stab}_F(1)$ and $\text{Stab}_G(1)$ are commutative $\text{Stab}_{F \otimes G}(1)$ is commutative as well. Since symmetric form $Q_F$ ($Q_G$) on $\text{Stab}_F(1)$ (respectively $\text{Stab}_G(1)$) is non-degenerate it must be that the symmetric form $Q_{F \otimes G}$ on $\text{Stab}_{F \otimes G}(1)$ is non-degenerate as well. Therefore, by theorem 16, $F \otimes G$ is 1-regular.

We have proven that $F \otimes G$ is both 1-precise and 1-regular and that $\text{Stab}_{F \otimes G}(1) = \text{Stab}_F(1) \otimes \text{Stab}_G(1)$. Therefore, $\text{ind } A \otimes B = \text{ind } A \cdot \text{ind } B$.

**Corollary 1.** The algebra $A \otimes B$ is type 1 and 1-precise.

**Corollary 2.** Since the algebra of $n \times n$ matrices $\text{Mat}_n$ satisfies conditions on algebra $A$ and any algebra $B$ with unity and index 1 satisfies conditions on algebra $B$ we must have for such algebras

$$\text{ind } \text{Mat}_n \otimes B = n$$

### 8 Interaction with radical methods

A classical method of studying associative algebras is to introduce a notion of radical (see [9, 11]). It is reasonable to ask whether the method presented in this paper provides anything beyond and, in particular, whether Jacobson’s radical structure can be analyzed with this method. Also, one may wonder whether the spectrum $\{\alpha(F)\}$ of the algebra will reflect anything more than the structure of the semisimple factor.

We would like to note that according to the definition in [9] the $P$-radical is an ideal that is intrinsic to the algebra. However, we know that, for a unital algebra, the functional $F$ is identically zero on all spaces $\text{Stab}_F(\alpha)$ and $\text{Nil}_F$ except for $\text{Stab}_F(1)$. Furthermore, we know, that for a 1-regular $F$, $\text{Stab}_F(1)$ is commutative.

Therefore, if one chooses 1-regular $F$ so that it does not vanish on at least one element of whatever radical we are interested in, the
radical will either intersect the spaces \( \text{Stab}_F(\alpha) \) in a non-trivial way, or be contained entirely within \( \text{Stab}_F(1) \) and thus be commutative.

We would like to augment this point with the following theorem:

**Theorem 33** Let \( S = \{e_i\} \) be a countable collection of non-zero elements of \( \mathfrak{A} \). Then the set of functionals \( F \) that does not vanish on all elements of \( S \) is Baire category 2 (for definition of Baire categories see [10]). We consider \( \mathfrak{A}^* \) in Euclidian topology.

**Proof** Indeed the set of functionals \( F \) that does not vanish on a single element \( e_i \) is open and dense in \( \mathfrak{A}^* \). Therefore the set of all functionals \( F \) that do not vanish on \( S \) is an intersection of open dense sets in a complete space with Euclidian metric and thus is Baire category 2.

Thus, even if we have selected a countable family of non-trivial ideals, we can choose 1-regular \( F \) so that it does not vanish on any of them.

In regards to the question of whether the spectrum \( \{\alpha(F)\} \) provides anything beyound characteristics of the semisimple factor, we would like to note that the spectrum of \( \text{Mat}_n \) consists of functions \( \alpha(F) \) that, for a generic \( F \), depend non-trivially on it and do not vanish in any points. As there are plenty of examples where the spectrum includes constants \( \alpha \), in particular 0 and \( \infty \), this cannot be only due to the semisimple factor.

On the other hand, if we want to discount contribution of a particular ideal \( \mathfrak{I} \) within an algebra, we can either study its factor, or, equivalently, study functionals \( F \) pulled back from the factor - this will result in \( \mathfrak{I} \subset \text{Nil}_F \) for all such functionals.

### 9 Open questions

The identity \( \text{ind} \text{Mat}_n \otimes \mathfrak{B} = n \) does not generalize to any pair of type 1 algebras as shown by the following example:

**Example 8** The index of the algebra \( UT(2) \) of upper triangular \( 2 \times 2 \) matrices is 1. The index of \( UT(2) \otimes UT(2) \) is 3.
Proof  Indeed, the multiplication table of $UT(2)$ is

\[
\begin{array}{ccc}
  & a & b & c \\
  a & a & b & 0 \\
b & 0 & 0 & b \\
c & 0 & 0 & c \\
\end{array}
\]

The characteristic polynomial if $UT(2)$ is

\[
\chi(\lambda, \mu, F) = -\lambda\mu b^2(\lambda + \mu)(a + c)
\]

And thus $\dim Stab_F(0) = \dim Stab_F(1) = \dim Stab_F(\infty) = 1$

For the tensor product $UT(2) \otimes UT(2)$ the multiplication table is

\[
\begin{array}{ccccccccccc}
  & a & b & c & d & e & f & g & h & p \\
a & a & b & c & d & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & b & d & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & c & d & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
e & 0 & 0 & 0 & 0 & e & f & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \\
g & 0 & 0 & 0 & 0 & 0 & 0 & g & h & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p \\
\end{array}
\]

The characteristic polynomial of $UT(2) \otimes UT(2)$ is

\[
\chi(\lambda, \mu, F) = -\lambda^3\mu^3d^3(\lambda + \mu)^3(ch - dg)(fb - ed)(fb + ch + da + dp)
\]

Therefore the only non-zero subspaces $Stab_F(\alpha)$ are $Stab_F(0)$, $Stab_F(1)$ and $Stab_F(\infty)$ with dimensions that could be anywhere between 1 and 3.

Direct computation yields:

\[
\begin{align*}
Stab_F(0) &= \{ \left(-\frac{d}{a}, 0, 0, 1, 0, 0, 0, 0, 0, 0\right), \\
&\quad \left(-\frac{c}{a}, 0, 1, 0, 0, 0, 0, 0, 0, 0\right), \\
&\quad \left(-\frac{b}{a}, 1, 0, 0, 0, 0, 0, 0, 0, 0\right) \} \\
Stab_F(1) &= \{ \left(\frac{d}{h}, 1, -\frac{dch + fbd}{h^2d^2}, 0, \frac{b}{h}, 0, \frac{c}{h}, \frac{d}{h}\right), \\
&\quad \left(0, -\frac{f}{d}, 0, \frac{fb}{h}, 1, -\frac{b}{h}, 0, 0, 0, 0\right), \\
&\quad \left(1, \frac{f}{d}, 0, -\frac{fb}{h}, 0, \frac{b}{h}, 1, 0, 1\right) \} \\
Stab_F(\infty) &= \{ \left(0, 0, 0, -\frac{d}{a}, 0, 0, 0, 0, 0, 1\right), \\
&\quad \left(0, 0, 0, -\frac{b}{a}, 0, 0, 0, 1, 0, 0\right), \\
&\quad \left(0, 0, 0, -\frac{c}{a}, 0, 1, 0, 0, 0, 0\right) \} \\
\end{align*}
\]

and thus $\text{ind} UT(2) \otimes UT(2) = 3$.  ■
A plausible generalization of the index formula to this case is

**Conjecture:** Let $\mathfrak{A}$ and $\mathfrak{B}$ be two type 1 associative algebras which resonant spectral values are precise in regular functionals. Then their product is also a type 1 algebra and the index is given by this formula:

$$\text{ind}\mathfrak{A} \otimes \mathfrak{B} = \text{ind}\mathfrak{A} \otimes \mathfrak{B} + \sum_{1 \neq \alpha \in \text{Spec}\mathfrak{A} \cap \text{Spec}\mathfrak{B}} \dim \text{Stab}_F^\mathfrak{A}(\alpha) \otimes \dim \text{Stab}_G^\mathfrak{B}(1/\alpha)$$

Here $\text{Spec}\mathfrak{A} \cap \text{Spec}\mathfrak{B}$ denotes resonant spectral values - namely those constants $\alpha$ for which $\text{Stab}_F^\mathfrak{A}(\alpha)$ and $\text{Stab}_G^\mathfrak{B}(\alpha)$ are non-zero for $\alpha$-regular functionals.

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