Norm–Free Radon–Nikodym Approach to Machine Learning

Vladislav Gennadievich Malyshkin

Ioffe Institute, Politekhnicheskaya 26, St Petersburg, 194021, Russia

(Dated: December, 10, 2015)

$Id: MLVector.tex,v 1.85 2015/12/15 19:00:05 mal Exp$

For Machine Learning (ML) classification problem, where a vector of x–observations (values of attributes) is mapped to a single y value (class label), a generalized Radon–Nikodym type of solution is proposed. Quantum–mechanics –like probability states $\psi^2(x)$ are considered and “Cluster Centers”, corresponding to the extremums of $\frac{< y\psi^2(x) >}{< \psi^2(x) >}$, are found from generalized eigenvalues problem. The eigenvalues give possible $y^{[i]}$ outcomes and corresponding to them eigenvectors $\psi^{[i]}(x)$ define “Cluster Centers”. The projection of a $\psi$ state, localized at given x to classify, on these eigenvectors define the probability of $y^{[i]}$ outcome, thus avoiding using a norm ($L^2$ or other types), required for “quality criteria” in a typical Machine Learning technique. A coverage of each ‘Cluster Center’ is calculated, what potentially allows to separate system properties (described by $y^{[i]}$ outcomes) and system testing conditions (described by $C^{[i]}$ coverage). As an example of such application $y$ distribution estimator is proposed in a form of pairs $(y^{[i]}, C^{[i]})$, that can be considered as Gauss quadratures generalization. This estimator allows to perform $y$ probability distribution estimation in a strongly non–Gaussian case.

*malyshki@ton.ioffe.ru*
I. INTRODUCTION

Machine Learning (ML) explores the study and construction of algorithms that can learn from and make predictions on data. The key four elements \[1\] of any ML model is (1) Attribute selection. (2) Knowledge Representation. (3) Quality Criteria. (4) Search algorithm. The first three elements are the most important in practice, but search algorithms often attract most attention of ML researchers. In this work we will try to address the first three elements. The main idea of this work is to find observations “Cluster Centers” as corresponding to matrix spectrum of class label, and then project the state to classify on these “Cluster Centers”, thus receive probabilities directly and avoid using a norm for quality criteria.

II. GENERALIZED RADON–NIKODYM SOLUTION

Consider the following ML problem where attributes vector of \(d_x\) components is mapped to a single outcome (class label) observation \(y\) for \(l = [1..M]\).

\[
(x_0, x_1, \ldots, x_m, \ldots, x_{d_x - 1})^{(l)} \rightarrow y^{(l)}
\]  

(1)

A number of other problems can be converted to this problem, e.g. distribution regression problem \[2\], can be converted by using bag’s distribution moments as \(x_m\) vector components. A lot of ML theories are of interpolatory type, where the (1) is piecewise interpolated by regression coefficients, propositional rules, decision trees or Neural Networks minimizing some norm–like quality criteria, see \[3\] for excellent review and implemented algorithms. But we are going to treat the (1) not in terms of some error minimization, but probabilistically. Consider the \(\psi(x)\) state

\[
\psi(x) = \sum_{m=0}^{d_x - 1} \psi_m x_m
\]  

(2)

that is defined by \(\psi_m; m = [0..d_x - 1]\) vector. Any more complex forms of \(\psi\) (e.g. some functions \(f(x_m)\)) is equivalent to adding \(f(x_m)\) terms to \(x\) vector (1), and the form (2) pose no limitation, because any of such can be avoided by adding more \(x\)–components.

Introduce the measure \(\sum_{l=1}^{M}\) and treat the \(\psi^2\) as “probability density”. Consider corre-
sponding $y_\psi$

$$y_\psi = \frac{\sum_{l=1}^{M} y^{(l)} \psi^2(x^{(l)})}{\sum_{l=1}^{M} \psi^2(x^{(l)})}$$  \hspace{1cm} (3)$$

$$< f > = \sum_{l=1}^{M} f^{(l)}$$  \hspace{1cm} (4)$$

$$(G)_{qr} = \langle x_q x_r \rangle = \sum_{l=1}^{M} x_q^{(l)} x_r^{(l)}$$  \hspace{1cm} (5)$$

$$(yG)_{qr} = \langle y x_q x_r \rangle = \sum_{l=1}^{M} y^{(l)} x_q^{(l)} x_r^{(l)}$$  \hspace{1cm} (6)$$

$$y_\psi = \frac{\sum_{q,r=0}^{d_x-1} \psi_q (yG)_{qr} \psi_r}{\sum_{q,r=0}^{d_x-1} \psi_q (G)_{qr} \psi_r}$$  \hspace{1cm} (7)$$

The $\psi^{[i]}$ states, corresponding to the extremums of $y_\psi$, can be found from generalized eigenvectors problem

$$\sum_{r=0}^{d_x-1} (yG)_{qr} \psi^{[i]}_r = y^{[i]} \sum_{r=0}^{d_x-1} (G)_{qr} \psi^{[i]}_r$$  \hspace{1cm} (8)$$

The $y^{[i]}$ give possible outcomes and

$$\psi^{[i]}(x) = \sum_{m=0}^{d_x-1} \psi^{[i]}_m x_m$$  \hspace{1cm} (9)$$

define “Cluster Centers”, corresponding to $y^{[i]}$. The value of $(\psi^{[i]}(x^{(l)}))^2$ is typically large only for the $l$’s at which $y^{(l)}$ value is close to the eigenvalue $y^{[i]}$. Note that

$$\sum_{q,r=0}^{d_x-1} \psi^{[j]}_r (yG)_{qr} \psi^{[i]}_r = y^{[i]} \delta_{ji}$$  \hspace{1cm} (10)$$

$$\sum_{q,r=0}^{d_x-1} \psi^{[j]}_r (G)_{qr} \psi^{[i]}_r = \sum_{l=1}^{M} \psi^{[j]}(x^{(l)}) \psi^{[i]}(x^{(l)}) = \delta_{ji}$$  \hspace{1cm} (11)$$

The (11) allows to treat the $(\psi^{[i]}(x^{(l)}))^2$ as the value proportional to the probability of $l$–th learning observation from (10) to have the $y^{[i]}$ outcome. Similarly for two given $x^A$ and $x^B$ their projection to each other

$$< x^A | x^B >_x = \sum_{q,r=0}^{d_x-1} x^A_q (G)_{qr}^{-1} x^B_r$$  \hspace{1cm} (12)$$
The probabilities, calculated by projecting the given \( x \) to “Cluster Centers” are:

\[
\begin{align*}
\Pr[i](x) &= \frac{w[i](x)}{\sum_{r=0}^{d_x-1} w[r](x)} \\
\Pi[i](x) &= \left( \sum_{r=0}^{d_x-1} x_r \psi[i][r] \right)^2
\end{align*}
\]

This is the solution to classification problem: for a given \( x \) the eigenvalues \( y[i] \) from (8) provide possible outcomes and \( \Pi[i](x) \) from (14) provide each outcome probability. This answer is much more general than, say, regression type of answer, in which only \( y \) estimate can be given and probability distribution can be estimated from standard deviation only for Gaussian type of random variables. The (8) does not use second \( y \) moment at all, so the answer can be successfully applied to non–Gaussian samples, e.g. the ones with infinite standard deviation of \( y \).

If \( y \) estimate for a given \( x \) is required two answers can be provided, see Appendix D, Least Squares \( A_{LS} \) and Radon–Nikodym \( A_{RN} \). The answers are:

\[
\begin{align*}
Y_q &= \sum_{l=1}^{M} y[l][q] x[l] \\
A_{LS}(x) &= \sum_{q,r=0}^{d_x-1} x_q (G)_{qr}^{-1} Y_r \\
A_{RN}(x) &= \frac{\sum_{q,r,s,t=0}^{d_x-1} x_q (G)_{qr}^{-1} (yG)_{rs} (G)_{st}^{-1} x_t}{\sum_{q,r=0}^{d_x-1} x_q (G)_{qr}^{-1} x_r}
\end{align*}
\]

The (16) is least squares answer to \( y \) estimation given \( x \). The (17) is Radon–Nikodym answer to \( y \) estimation given \( x \). These answers can be considered as an extension of least squares and Radon–Nikodym type of answers to vector input. In case \( x_m \) components in (11) are the moments of some random variable the \( A_{LS} \) and \( A_{RN} \) are reduced exactly to the problem of learning from random distribution we considered in Ref. [2]. Note, that the \( A_{LS}(x) \) answer not necessary preserve \( y \) sign, but \( A_{RN}(x) \) always preserve \( y \) sign, same as we have in our earlier works.

One more issue we want to discuss is coverage estimation for each “Cluster Center” \( \psi[i](x) \). The states \( \psi[l](x) \), corresponding to specific \( x[l] \) is normalized projection (12) with
\[ x = x^A \text{ and } x^{(l)} = x^B, \]

\[
\psi^{(l)}(x) = \frac{\sum_{q=0}^{d_x-1} x_q (G)_{qr}^{-1} x^{(l)}_r}{\sqrt{\sum_{q,r=0}^{d_x-1} x_q (G)_{qr}^{-1} x^{(l)}_r}}
\]

\[
\omega^{(l)}_{[i;j]} = \frac{\sum_{q=0}^{d_x-1} \psi^{[i]}_q x^{(l)}_q \sum_{q=0}^{d_x-1} \psi^{[j]}_q x^{(l)}_q}{\sum_{q,r=0}^{d_x-1} x_q (G)_{qr}^{-1} x^{(l)}_r}
\]

\[
= < \psi^{[i]}(x) \mid \psi^{(l)}(x) > < \psi^{(l)}(x) \mid \psi^{[j]}(x) >
\]

\[
C^{[i]} = \sum_{l=1}^{M} \omega^{(l)}_{[i;i]}
\]

\[
D^{[i]} = \sum_{l=1}^{M} \omega^{(l)}_{[i;i]} \left( 1 - \omega^{(l)}_{[i;i]} \right) = \sum_{l=1}^{M} \sum_{j=0; j \neq i}^{d_x-1} (\omega^{(l)}_{[i;j]})^2
\]

This approach produce \( d_x \) clusters, and for each \( i = [0..d_x-1] \) eigenvalue \( y^{[i]} \), coverage \( C^{[i]} \) and localization measure \( D^{[i]} \) are obtained. The question about selection of “true” clusters arise. Among the \( x_m; m = [0..d_x-1] \) components of (1) many may be irrelevant (will be weeded out, not a problem) or redundant (lead to \( G_{qr} \) matrix (5) degeneracy and require regularization of the problem). A criteria can be applied, that select only some best \( d \leq d_x \) components as linear combination of \( d_x \) components, initially available in (1). The power of our approach is that the generalized eigenvalues equation (8) can be used not only for possible \( y \) outcomes estimation, but also for components selection using various criteria. As illustration let us find the \( d \leq d_x \) components, providing maximal coverage \( C \). Consider a (2) state, then using the (20) definition, the coverage \( C_{\psi} \), corresponding to a \( \psi(x) \) state, can
be calculated as

\[ n^{(l)} = \frac{1}{\sum_{k,m=0}^{d_x-1} x^{(l)}_k (G)^{-1}_{km} x^{(l)}_m} \]  \hspace{1cm} (22)

\[ (CG)_{qr} = \sum_{l=1}^{M} n^{(l)} x^{(l)}_q x^{(l)}_r \]  \hspace{1cm} (23)

\[ C_\psi = \frac{\sum_{q,r=0}^{d_x-1} \psi_q (CG)_{qr} \psi_r}{\sum_{q,r=0}^{d_x-1} \psi_q (G)_{qr} \psi_r} \]  \hspace{1cm} (24)

\[ \sum_{r=0}^{d_x-1} (CG)_{qr} \psi_r^{[j]} = C^{[j]} \sum_{r=0}^{d_x-1} (G)_{qr} \psi_r^{[j]} \]  \hspace{1cm} (25)

\[ M = \sum_{j=0}^{d_x-1} C^{[j]} \]  \hspace{1cm} (26)

The Eq. (25) is exactly (8) but, instead of matrix \((yG)^{qr}\), the matrix \((CG)_{qr}\) is used, and eigenvectors \(\psi^{[j]}_r\) from (25) correspond to the states having coverage extremums. Selecting \(d \leq d_x\) \(\psi^{[j]}_r\) states, corresponding to maximal \(C^{[j]}\) would give the required \(d\) states with maximal coverage.

Similar to coverage, a number of observations falling within given \(y\) interval, can be obtained by modification of (22): take the \(l\)-th observation only when \(y^{(l)}\) fall within the given \(y\) interval. This way an entropy of \(y\) distribution for a given \(\psi(x)\) state, can be easily obtained. However, the expression for the entropy is not a plain ratio of two quadratic form on \(\psi(x)\) components, so the problem (8) cannot be directly applied to entropy calculation in general case. But in the case of a discrete \(y^{(l)}\), taking only two class values, a classifier, maximizing the difference in outcomes number between the two classes can be readily obtained. Modify the (22) by taking the terms with \(y^{(l)}\) in class 1 using positive sign and with \(y^{(l)}\) in class 2 using negative sign. Then the (24) provide the difference in observations number between class 1 and class 2 as a ratio of two quadratic forms on \(\psi(x)\) components. Solve (25), the eigenvalues now the strength of class prediction and eigenvectors provide the classifiers. Usage, instead of \(y^{(l)}\), that take two class values, the difference in observations number between class 1 and class 2 allows to overcome \(y\)-spectrum degeneracy, arising from only two \(y^{(l)}\) outcomes in a discrete case of the problem.

One more important advantage of using generalized eigenvalues problem (8) is that the solution is stable with respect to \(y^{(l)}\) outlier observations. Such observations give, for some
limited number of $i$ values, “Cluster Centers” with outlier eigenvalue $y[i]$ and small coverage $C[i]$. These “Cluster Centers” with small coverage can be either treated as outliers and disregarded in case of measurement errors presence, or, typically more reasonable approach, used for fat tails estimation of $y$ probability distribution. Numerical experiments show, that few $y(l)$ outliers typically skew spectrum and coverage only for a single $i$ value, leaving the rest of $(y[i], C[i])$ pairs intact. This is drastically different from $y$–norm based approaches, where a single $y(l)$ outlier can easily skew a $L^2$ norm, like standard deviation.

A. Quantum Mechanics Analogy

Quantum Mechanics interpretation can be given to this ML approach. Consider some time–independent quantum system with Hamiltonian $H$, described by a state with $d_x$ components $x = \{x_m\}; m = [0..d_x - 1]$ of unknown nature. Assume they are unknown, but stable combinations of coordinate, momentum and angular momentum and quantum system occupy a quantum state, that change from one classical measurement to another. The problem: From a number of measurement experiments recover information about quantum system and experimental conditions.

Every classical measurement for such system is a set of pairs (state,energy): $(x(l), E(l))$, index $l = [1..M]$ label classical measurement experiment. It is clear that any kind of piecewise interpolation of $E(x)$ make no sense, because of possible degeneracy of parameters $x$ and having a different mixed quantum state in each measurement experiment. However, the $E(l)$ correspond to quantum system Hamiltonian $H$. The idea is: given large enough observations number $M$ select the experiments that are close to “pure quantum state”. Introduce a wavefunction $\psi(x) = \sum_{m=0}^{d_x-1} \psi_m x_m$. The entire observation set $l = [1..M]$ can be considered as a quantum mechanics mixed state described by the density matrix

$$
\rho(x^A, x^B) = \sum_{q,r=0}^{d_x-1} x_q^A (G)_{qr}^{-1} x_r^B
$$

normalized for convenience on the number of components $\langle \rho(x, x) \rangle = Tr(\rho) = \sum_{q,r=0}^{d_x-1} (G)_{qr} (G)_{rq}^{-1} = d_x$, not to 1, like regular quantum mechanics density matrix. Note, that convenient in applications matrix averages, see Ref. [4] Appendix E, are the averages, calculated on mixed
state with the density matrix \( \rho \) \((27)\), e.g.

\[
y_{\rho} = \sum_{q,r=0}^{d_x-1} (y G)_{qr} (G)^{-1}_{rq} / d_x = \frac{\langle y \rho(x,x) \rangle}{\langle \rho(x,x) \rangle} \quad (28)
\]

Reproducing Kernel \((12)\) is plain density matrix \( \rho(x^A,x^B) = \langle x^A | x^B \rangle \). The energy, corresponding to a pure state \( \psi(x) \), is 

\[
E_{\psi} = \sum_{l=1}^{M} E^{(l)} |\psi^{(l)}(x)|^2 / \sum_{l=1}^{M} |\psi^{(l)}(x)|^2.
\]

With a replacement of \( E \) by \( y \) we receive exactly the problem \((3)\). Were we only know the \( y_{\rho} \) from \((28)\), corresponding to the mixed state \((27)\), no information about system pure states can be obtained. However, the problem \((1)\) have the \( y^{(l)} \) outcome available for each measurement experiment and generalized eigenvalues problem \((8)\) now allows to estimate Hamiltonian spectrum, then a projection \((31)\) of a given state \( \psi^{(c)}(x) \) to eigenvectors of \((8)\) allows to estimate \( j \)-th Hamiltonian state contribution:

\[
\psi[i](x) = \sum_{q=0}^{d_x-1} x_q \psi[i]_q \quad (29)
\]

\[
\psi^{(c)}(x) = \sqrt{\frac{\sum_{q=0}^{d_x-1} x_q (G)^{-1}_{qr} x^{(c)}_r}{\sum_{q,r=0}^{d_x-1} x_q^2 (G)^{-1}_{qr} x^{(c)}_r}} \quad (30)
\]

\[
\langle \psi^{(c)} | \psi[i] \rangle = \sqrt{\frac{\sum_{q=0}^{d_x-1} x_q^{(c)} \psi[i]_q}{\sum_{q,r=0}^{d_x-1} x_q^{(c)} (G)^{-1}_{qr} x^{(c)}_r}} \quad (31)
\]

\[
\psi^{(c)}(x) = \sum_{q=0}^{d_x-1} \langle \psi^{(c)} | \psi[i] \rangle \psi[i](x) \quad (32)
\]

If one put \( x^{(c)}_m = x^{(l)}_m \) to \((30)\) then \((32)\) give \( l \)-th experiment wavefunction expanded over the Hamiltonian states \( \psi[i](x) \) and the \((\langle \psi^{(l)} | \psi[i] \rangle)^2 = \omega^{(l)}_{[i]}\) give \( i \)-th outcome probability for \( l \)-th experiment. The Hamiltonian spectrum \( y[i] \) eigenvalues are the characteristics of quantum system itself, but the coverage \((20)\) provide information how often the \( \psi[i] \) pure state give substantial contribution to mixed quantum state \((27)\) of classical experiment, i.e. about experiment conditions, not about quantum system itself.

The simplest ML application of this approach can be just to put into \((30)\) as a \( x^{(c)}_m \) not the state \( x^{(l)}_m \), corresponding to \( l \)-th training datapoint, but the state we want to classify (this state is exactly \((30)\), with \( x^{(c)}_m \) components equal to the components of \( x \) vector we want to classify) in a hope to receive a reasonable prediction.
But such a direct approach, very much typical for ML applications, looks more like of interpolatory type. However, there is a much deeper application of this technique. Obtaining the spectrum $y[i]$ and coverage $C[i]$ (along with cluster localization $D[i]$ from (21)) we actually managed to separate the properties of quantum system itself and experimental conditions under which the system was tested. This possibility of separation is the key element of the approach, because typical ML technique does not separate them and build a model combining system properties and experimental conditions together.

**B. $y$–Distribution Estimation. Gauss Quadratures Generalization.**

Considered in previous section idea of system property and experimental conditions separation is applicable to a variety of problems. Consider the simplest one: estimate $y$ distribution from (11) sample. Evidently, that the pairs $(y[i], C[i])$ can serve as distribution estimator, the states $y[i]$ describe properties of the system itself and coverages $C[i]$ describe “experimental conditions” during $l = [1..M]$ system observations.

$$P(y[i]) = C[i]$$

$$\sum_{i=0}^{d_y-1} P(y[i]) = M$$

(33)

(34)

The (33) can be considered as Gauss quadratures generalization[5, 6]. If one put to (11) $x_m^{(l)} = Q_m(y^{(l)})$, then in (33) the $y[i]$ would be exactly quadrature nodes and $C[i]$ would be quadrature weights. Regular Gauss $n$–points quadrature distribution estimate is using $[0..2n-1]$ moments of $y$ as input, so its application is limited to distributions with finite $[0..2n-1]$ moments. The (33) is much more general in this sense, it uses some other, $x_q$ random variables moments $<x_qx_r>$ and $<yx_qx_r>$ as input, and only first $y$ moment enter the (8). This make these new quadratures much better applicable to ML, because dependent ($y$) and independent ($x$) variables are now separated in left– and right– hand sides of (8). In addition to that, because only first $y$ moment enter the (8) the results can be applied to prediction of a value out of a non–Gaussian distributions, e.g. the ones with infinite second $y$ moment.

Another straightforward application of distribution estimation (33) may be information recovery. In Ref. [7] a basis, obtained as eigenvectors of (8), was introduced (it is called there not “Cluster Centers”, but “natural basis”), and image reconstruction was performed in full
basis with all 10000 elements \((d_x = 100; d_y = 100)\) in maximal case. However, selection as partial basis the states with maximal coverage \(33\) (or cluster relative localization \(D[i]/C[i]\) in some cases) can be a good choice for applications like lossy compression methods and partial information recovery. This type of application uses the “experimental conditions”, the coverage \(C[i]\) and cluster localization \(D[i]\) as a selection rule for basis components.

III. NUMERICAL ALGORITHM

Numerical instability similar to the ones we have in Multiple Instance Learning [2] can also arise here. The solutions of (8), (14) and (17) are invariant with respect to arbitrary linear transform of \(x\) components, but numerical stability of calculations is drastically different because the condition number of \(G_{qr}\) from (5) depend strongly on basis choice[8]. While in [2], where the moments \(<Q_k>\) were used as vector components of \(x\), the answer for stable basis choice was rather trivial: for numerical stability use polynomials \(Q_k(x)\) orthogonal with respect to some measure, e.g. Chebyshev or Legendre polynomials (see [9], java implementation of Chebyshev, Legendre, Laguerre and Hermite bases and library description in Appendix A of [4]). Now, when the \(x\) components can be of different nature the question of finding linear transform of \(x\), that give \(G_{qr}\) with a good enough condition number becomes more complicated, see [8] as a good starting point. However, in this work, we will limit the number of elements in basis \(d_x\) by 20, what make specific basis choice not that important, compared to say [7] work, where a problem with \(d_x = d_y = 100\), i.e. 10000 elements in basis have been successfully considered. Another important stability issue is degenerate components presence in \(x\), when \(x_m^{(l)} = x_q^{(l)}\) for \(m \neq q\) and all \(l\). Such components make Gramm matrix (5) degenerate and special treatment, like Tikhonov regularization[10], subspace selection by mutual information, or similar regularization methods may be required.

The algorithm for \(y\) estimators of (16) or (17) is this: Calculate (5) and (6) matrices and (15) vector using (1) input data. Inverse matrix \((G)_{qr}\) from (5), this matrix is similar to Gramm matrix, but is build from the components of \(x^{(l)}\) vector. Finally put all these to (16) for least squares \(y(x)\) estimation or to (17) for Radon–Nikodym \(y(x)\) estimation. The (16) is a linear function of \(x\) components that posses all the problems typical for least squares–type answers. The (17) is a ratio of two quadratic forms of \(x\) components. It was shown in Ref. [11] that in multi–dimensional signal processing stable estimators can be only of two
quadratic forms ratio and the (17) is exactly of this form.

If $y$– distribution is required then solve generalized eigenvalues problem (8), obtain $y^{[i]}$ as possible $y$–outcomes, that describe the system itself, and $C^{[i]}$ from (20), that describe the testing conditions of the system, (they both do not depend on vector $x$ to classify). Then calculate $x$–dependent probabilities (14), these are squared projection coefficient of a state with given $x$ to $\psi^{[i]}$ eigenvector.

To show an application of this approach consider a problem, that can be reduced to Multiple Instance Learning problem of previous publication (the case $N = 1$ of Multiple Instance Learning example problem from [2]). 1) For $l = [1..M]$ take random $x$ out of $[-1; 1]$ interval. 2) Calculate $y = f(x)$, take this $y$ as $y^{(l)}$. 3) Calculate $x^* = x + R\epsilon$, (where $R$ is a parameter, $\epsilon$ is uniformly $[-1; 1]$ distributed and $Q_k(x)$ is a polynomial of $k$–th degree), then take $Q_m(x^*); m = [0..d_x - 1]$ as the components of input vector $x^{(l)}$ in (1). As a function $f$ we take the same three examples from [2]):

$$f(x) = x$$

$$f(x) = \frac{1}{1 + 25x^2}$$

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

In Figs. 1, 2, 3, the (16) and (17) the answers are presented for $f(x)$ from (35), (36) and (37) respectively for $R = \{0.1, 0.3\}$ and $d_x = \{10, 20\}$. The $x$ range is specially taken slightly wider that $[-1; 1]$ interval to see possible divergence outside of measure support. In most cases Radon–Nikodym answer is superior, and in addition to that it preserves the sign of $y$. Least squares approximation is good for special case $f(x) = x$ and typically diverges for $x$ outside of measure support. The figures are similar to the ones from [2]), because same data sample was used.

An example of numerical estimation of probabilities $P^{[i]}(x)$ is presented in Fig. 4 for simplistic case (35). See Ref. [9], file com/polytechnik/ algorithms/ ExampleInterpolation-VectorML.scala for algorithm implementation.
IV. DISCUSSION

In this work a generalized eigenvectors approach is applied to ML problem (1). The interpolatory–type results of least squares (16) and Radon–Nikodym (17) y value estimator for a given x are obtain. In addition to that distribution estimator of y is obtained:
FIG. 3. The $y(x)$ estimation for $f(x)$ from (37).

FIG. 4. Probabilities for all $d_x = 10$ outcomes of $y^{[i]}$ as a function of $x$ for $f(x)$ from (35).
Computer code implementing the algorithms is available[9].

[1] Andrey Gorodetsky Vladislav Malyshtkin, Ray Bakhramov, “A Massive Local Rules Search Approach to the Classification Problem,” eprint arXiv:cs/0609007 (2001). cs/0609007.

[2] Vladislav Gennadievich Malyshtkin, “Multiple-Instance Learning: Radon-Nikodym Approach to Distribution Regression Problem,” ArXiv e-prints (2015). arXiv:1511.09058 [cs.LG].

[3] Ian H Witten, Eibe Frank, and Mark A. Hall, Data Mining: Practical machine learning tools and techniques.

[4] Vladislav Gennadievich Malyshtkin and Ray Bakhramov, “Mathematical Foundations of Real-time Equity Trading. Liquidity Deficit and Market Dynamics. Automated Trading Machines. http://arxiv.org/abs/1510.05510” ArXiv e-prints (2015) arXiv:1510.05510 [q-fin.CP].

[5] Paul G Nevai, “Géza Freud, Orthogonal Polynomials. Christoffel Functions. A Case Study,” Journal Of Approximation Theory 48, 3–167 (1986).

[6] Vilmos Totik, “Orthogonal polynomials,” Surveys in Approximation Theory 1, 70–125 (11 Nov. 2005).

[7] Vladislav Gennadievich Malyshtkin, “Radon–Nikodym approximation in application to image analysis. http://arxiv.org/abs/1511.01887” ArXiv e-prints (2015) arXiv:1511.01887 [cs.CV].

[8] Bernhard Beckermann, On the numerical condition of polynomial bases: estimates for the condition number of Vandermonde, Krylov and Hankel matrices. Ph.D. thesis. Habilitationsschrift, Universität Hannover (1996).

[9] Vladislav Gennadievich Malyshtkin, (2014), the code for polynomials calculation, http://www.ioffe.ru/LNEPS/malyshkin/code.html.

[10] Andrei Nikolaevich Tikhonov and Vasilii Yakovlevich Arsenin, Solutions of ill-posed problems (Winston & Sons, 1977).

[11] Gennadii Stepanovich Malyshtkin, Optimal and Adaptive Methods of Hydroacoustic Signal Processing. Vol 1. Optimal methods. (in Russian). (Elektropribor Publishing, 2009).