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LOCALLY PATH-LIKE GRAPHS

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Summary. If $G$ is a graph and $x$ its vertex then $N_G(x)$ is the subgraph of $G$ induced by the set of all vertices adjacent to $x$ in $G$. A graph $G$ is said to be locally path-like if $N_G(x)$ is a path for each vertex $x$ of $G$.

In the paper the upper bound of the number of edges of locally path-like graphs is determined.

Keywords: Neighbourhood of a vertex, local properties of graphs.

AMS Classification: 05C35.

Let $G$ be a finite graph without loops and multiple edges. The neighbourhood of a vertex $x$ in the graph $G$ is understood to be the subgraph of $G$ induced by all vertices adjacent to $x$.

A. Zykov [4] suggested a problem concerning the characterization of graphs with a given neighbourhood. Denote by $N_G(x)$ the neighbourhood of $x$ in $G$. If for each vertex $x$, $N_G(x)$ is isomorphic to a given graph $H$ then $H$ is called the realizable graph and $G$ is called the realization of $H$. The set of all realizations of $H$ will be denoted by $\mathcal{R}(H)$.

B. Zelinka [3] studied the class $\mathcal{R}(T)$ of the locally tree-like graphs where $T$ is any tree. He proposed the following problem: to find the upper bound of the number of edges of a finite connected locally tree-like graph with $n$ vertices.

In this paper we study a certain subclass of $\mathcal{R}(T)$. The graph $G$ is called locally path-like if $N_G(x)$ is a path for each vertex $x$ of $G$. Denote the class of all locally path-like graphs by $\mathcal{R}(P)$.

We will also give the upper bound for the number of edges of the locally path-like graphs. Zelinka [2] has shown that the maximal number of edges of the polytopic locally path-like graph (called locally snake-like graph) with $n$ vertices is $\lceil 11n/4 - 6 \rceil$. He also constructed the maximal graphs of this class.

Lemma 1. Let $G$ be a locally path-like graph. Then every edge of $G$ belongs to at least one triangle of $G$.

Proof. Let an edge $e = x_1x_2$ belong to no triangle of $G$. Then $x_2$ is an isolated vertex in $N_G(x_1)$ and $N_G(x_1)$ is not a path, which is a contradiction.
Lemma 2. Every edge of a locally path-like graph $G$ belongs to at most two triangles of $G$.

Proof. Let an edge $e = x_1x_2$ belong to the triangles $T_1, T_2, \ldots, T_k$ with vertex sets $V(T_i) = \{x_1, x_2, y_i\}$ for $i = 1, 2, \ldots, k$ ($k \geq 3$). Then $N_G(x_1)$ contains the subgraph $K_{1,k}$ with the vertices $x_2, y_1, y_2, \ldots, y_k$, which is a contradiction.

If an edge $e$ belongs to exactly one triangle of $G$ then we call this edge a boundary edge of $G$.

Lemma 3. Let $G$ be a locally path-like graph with $n$ vertices. Then $G$ contains exactly $n$ boundary edges.

Proof. Let $x$ be any vertex of $G$. Then $N_G(x) \cong P_k$ with the vertex set $\{y_1, y_2, \ldots, y_k\}$ ($k \geq 2$). If $y_1$ and $y_2$ are the end vertices of $P_k$ then the edge $e_1 = xy_1$ belongs to exactly one triangle $T_1$ and $e_k = xy_k$ belongs to exactly one triangle $T_{k-1}$. The other edges $e_i = xy_i$ ($i = 2, 3, \ldots, k - 1$) belong to two triangles $T_{i-1}$ and $T_i$. Hence every vertex $x \in V(G)$ is incident to exactly two boundary edges and thus $G$ contains exactly $n$ boundary edges.

It is evident that every boundary edge belongs to exactly one circuit which consists of boundary edges only.

The following theorem is a corollary of the above lemmas.

Theorem 1. Let $G$ be a locally path-like graph with $n$ vertices. Then every edge of $G$ is either a boundary edge or belongs to exactly two triangles and the number of boundary edges is $n$.

Zelinka [3] proved that the minimal number of edges of a connected locally tree-like graph with $n$ vertices is $2n - 3$. Now we determine the upper bound for the number of edges of the locally path-like graphs.

The following simple lemma is proved in [1].

Lemma 4. Let $G$ be a graph with $n$ vertices and let $d_i$ be the degree of a vertex $x_i$.

Let

$$(1) \quad \sum_{i=1}^{n} d_i = nk.$$ \hspace{1cm} (1)

Then

$$(2) \quad \sum_{i=1}^{n} d_i^2 \geq nk^2.$$ \hspace{1cm} (2)

Theorem 2 (Zelinka [3]). Let $G$ be a locally tree-like graph $n$ vertices, $m$ edges and $t$ triangles. Then

$$(3) \quad t = \frac{2m - n}{3}.$$ \hspace{1cm} (3)
As $\mathcal{A}(P) \subset \mathcal{A}(T)$, it is evident that the assertion mentioned above holds also for locally path-like graphs.

Lemma 5. Let $G$ be a locally path-like graph with $n$ vertices and $m$ edges. Let

\begin{equation}
2m = nk.
\end{equation}

Then exactly one of the following assertions holds:

(i) $G$ contains a triangle $T$ with vertices $x_1, x_2, x_3$ for which

\begin{equation}
d_1 + d_2 + d_3 > 3k;
\end{equation}

(ii) for each triangle $T_j$ ($j = 1, 2, \ldots, t$) we have

\begin{equation}
\sum_{x_i \in T_j} d_i = 3k.
\end{equation}

Proof. Let $\sum_{x_i \in T_j} d_i = 3k + r_j$ for $j = 1, 2, \ldots, t$. Then

\[
\sum_{j=1}^{t} \sum_{x_i \in T_j} d_i = 3kt + \sum_{j=1}^{t} r_j.
\]

It follows from (3) and (4) that $t = (nk - n)/3$ and hence

\begin{equation}
\sum_{j=1}^{t} \sum_{x_i \in T_j} d_i = nk^2 - nk + \sum_{j=1}^{t} r_j.
\end{equation}

As each vertex $x_i$ belongs to $d_i - 1$ triangles then the degree $d_i$ of $x_i$ is in (7) included $(d_i - 1)$ times and thus, in order to obtain $\sum_{i=1}^{n} d_i$ on the left-hand side of (7), we have to subtract the expression $d_i(d_i - 2)$ from the right-hand side of (7).

Hence

\[
\sum_{i=1}^{n} d_i = nk^2 - nk + \sum_{j=1}^{t} r_j - \sum_{i=1}^{n} d_i(d_i - 2) = nk^2 - nk + \sum_{j=1}^{t} r_j - \sum_{i=1}^{n} d_i^2 + 2\sum_{i=1}^{n} d_i.
\]

This yields

\[
\sum_{i=1}^{n} d_i^2 - nk^2 = -nk + \sum_{j=1}^{t} r_j + \sum_{i=1}^{n} d_i.
\]

Using the inequality (2) from Lemma 4 we get

\[
0 \leq -nk + \sum_{i=1}^{n} d_i + \sum_{j=1}^{t} r_j.
\]

As we assumed that $\sum_{i=1}^{n} d_i = nk$, we can see that

\[
\sum_{j=1}^{t} r_j \geq 0.
\]
If \( r_j = 0 \) for each \( j \in \{1, 2, \ldots, t\} \) then \( \sum_{x \in T_j} d_i = 3k \). In the opposite case at least one \( r_j \) is positive and \( T_j \) is the triangle \( T \) from the assertion of our lemma.

Now we can determine the upper bound of the number of edges of locally path-like graphs.

**Theorem 3.** Let \( G \) be a locally path-like graph with \( n \) vertices and \( m \) edges. Then

\[
m \leq \frac{n(n + 6)}{6}.
\]

**Proof.** Suppose that \( m > n(n + 6)/6 \). If we substitute \( k = (n + 6)/3 \) in (4), Lemma 5 implies that \( G \) contains a triangle \( T \) with vertices \( x_1, x_2, x_3 \) which satisfy

\[
\sum_{i=1}^{3} d_i > n + 6.
\]

As there exists no vertex \( x \) adjacent to all the vertices \( x_1, x_2, x_3 \) (in the opposite case \( N_G(x) \) would contain \( C_3 \) thus there exist at least two vertices \( x_4, x_5 \) adjacent to both end vertices of an edge \( e \in T \). Without loss of generality we can suppose that it is the edge \( x_1x_2 \). Then \( x_2 \) is of degree 3 in the graph \( N_G(x_1) \), which is a contradiction.

References

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Souhrn

**LOKÁLNĚ CESTOVITÉ GRAFY**

**DALIBOR FRONČEK**

Jestliže \( G \) je graf a \( x \) jeho vrchol, potom \( N_G(x) \) je podgraf grafu \( G \) indukovaný na množině všech vrcholů \( G \), sousedních s \( x \). Graf \( G \) nazveme lokálně cestovitým, je-li \( M_G(x) \) cesta pro každý vrchol \( x \) z \( G \).

V článku je stanovena horní hranice počtu hran lokálně cestovitých grafů.
Резюме

**ЛОКАЛЬНО ЦЕПНЫЕ ГРАФЫ**

**DALIBOR FRONČEK**

Для графа $G$ и его вершины $x$ пусть $N_G(x)$ обозначает подграф графа $G$, порожденный множеством всех вершин смежных с $x$ в $G$. В статье анализируются графы, для которых $N_G(x)$ является простой цепью для всех $x$ из $G$, и найдена верхняя граница числа ребер таких графов.

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