Cubic Hodge integrals and integrable hierarchies of Volterra type

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Abstract
A tau function of the 2D Toda hierarchy can be obtained from a generating function of the two-partition cubic Hodge integrals. The associated Lax operators turn out to satisfy an algebraic relation. This algebraic relation can be used to identify a reduced system of the 2D Toda hierarchy that emerges when the parameter $\tau$ of the cubic Hodge integrals takes a special value. Integrable hierarchies of the Volterra type are shown to be such reduced systems. They can be derived for positive rational values of $\tau$. In particular, the discrete series $\tau = 1, 2, \ldots$ correspond to the Volterra lattice and its hungry generalizations. This provides a new explanation to the integrable structures of the cubic Hodge integrals observed by Dubrovin et al. in the perspectives of tau-symmetric integrable Hamiltonian PDEs.

1 Introduction

The Hodge integrals
\[ \int_{\mathcal{M}_{g,n}} \lambda_1^{j_1} \cdots \lambda_g^{j_g} \psi_1^{k_1} \cdots \psi_n^{k_n}, \quad j_1, \ldots, j_g, k_1, \ldots, k_n \geq 0, \]
are intersection numbers of two kinds of tautological cohomology classes $\psi_1, \ldots, \psi_n$ (the $\psi$-classes) and $\lambda_1, \ldots, \lambda_g$ (the Hodge classes) on the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of connected complex stable curves. Special linear combinations of these integrals of the form
\[ \int_{\mathcal{M}_{g,n}} \prod_{i=1}^d \Lambda^\vee_g(a_i), \quad \Lambda^\vee_g(u) = u^g - u^{g-1} \lambda_1 + \cdots + (-1)^g \lambda_g, \]  
(1)

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appear in computation of Gromov-Witten invariants of a $d$-dimensional manifold by the method of localization \[1\].

A combinatorial expression of the cubic (i.e., $d = 3$) special Hodge integrals \[1\] was proposed by the Gokakumar-Mariño-Vafa conjecture \[2, 3\]. The Hodge integrals in this formula are organized into an all-genus generating function $G_\mu(a_1, a_2, a_3)$ that depends on an integer partition $\mu$, hence called the one-partition Hodge integrals. The parameters $a_i$ are required to satisfy the so called Calabi-Yau condition $a_1 + a_2 + a_3 = 0$. Moreover, since the generating function enjoys scale invariance, $a_i$’s can be effectively parametrized by a single parameter $\tau \neq 0, 1$ as

\[ a_1 = 1, \quad a_2 = \tau, \quad a_3 = -\tau - 1. \]

Let $G_\mu(\tau)$ denote $G_\mu(a_1, a_2, a_3)$ in this parametrization. Liu, Liu and Zhou proved the Gopakumar-Mariño-Vara conjecture \[4\] and extended it to a two-partition version $G_{\mu \bar{\mu}}(\tau)$ \[6\]. Moreover, Zhou pointed out that the KP and 2D Toda hierarchies underlie these combinatorial expressions of the cubic Hodge integrals \[7\].

Some ten years after the work of Liu et al., Dubrovin, Zhang and their collaborators addressed the cubic Hodge integrals in the perspectives of integrable Hamiltonian PDEs and quasi-Miura transformations \[8\]. They observed that integrable hierarchies of the Volterra type, typically the Volterra lattice \[9\] (referred to as the discrete KdV hierarchy there), emerge in the cubic Hodge integrals for particular values of the parameters \[10, 11, 12\].

In this paper, we return to Zhou’s work \[7\] and elucidate an origin of the integrable structures of the Volterra type observed by Dubrovin et al. The two-partition Hodge integrals are organized to a generating function of two copies of the power sum variables $p_i, \bar{p}_i$, $i = 1, 2, \ldots$. Zhou used a fermionic expression of the generating function to show that the generating function (slightly modified to depend on the lattice coordinate $s$ \[3\]) becomes a tau function of the 2D Toda hierarchy. We consider the associated two Lax operators $L, \bar{L}$ of the 2D Toda hierarchy. These Lax operators turn out to satisfy an algebraic relation (Theorem \[4\]). This implies the existence of an auxiliary Lax operator $\mathcal{L}$ (Corollary \[1\]).

The operator $\mathcal{L}$ plays a central role to identify an underlying integrable structure. When $\tau$ is a positive integer, $\mathcal{L}$ coincides with the Lax operator of a generalized Volterra lattice called the Bogoyavlensky-Itoh-Narita (aka the hungry Lotka-Volterra) system \[13, 14, 15\]. The case of $\tau = 1$ corresponds to the the Volterra lattice as proved by Dubrovin et al. \[11\] by a different

\[1\] Okounkov and Pandharipande presented another proof \[5\].

\[2\] The meaning of $s$ in the context of Hodge integrals is obscure.
method. When $\tau$ is a positive rational number, $\mathcal{L}$ becomes the Lax operator of a generalization of the Bogoyavlensky-Itoh-Narita system. This amounts to the rational Volterra hierarchy introduced by Liu et al. [12]. We can thus explain the origin of integrable hierarchies of the Volterra type in a unified way. On the other hand, when $\tau$ is a negative rational number, we encounter a different kind of integrable structures, namely, a lattice version of the Gelfand-Dickey hierarchy [16] and further reductions thereof.

We thus find the following integrable structures as reductions of the master 2D Toda hierarchy (or one of its lattice KP sectors).

- $\tau = N, N = 1, 2, \ldots$: The relevant integrable structure is the $N+1$-step Bogoyavlensky-Itoh-Narita system. The first member of this series is the Volterra lattice.

- $\tau = b/a$, $a$ and $b$ are positive coprime integers: The relevant integrable structure is a generalization of the Bogoyavlensky-Itoh-Narita system.

- $\tau = -b/a$, $a$ and $b$ are positive coprime integers: A lattice version of the Gelfand-Dickey hierarchy emerges in a further reduced form. A particularly interesting subset is the discrete series $\tau = -N/(N+1)$, $N = 1, 2, \ldots$.

To prove the algebraic relation of the Lax operators, we use a method developed in our previous work on the melting crystal model and a family of topological string theory [17, 18, 19]. This method is based on a factorization problem that characterizes the dressing operators behind the Lax operators. We use it to compute the initial values of the Lax operators. As it turns out, these initial values of the Lax operators satisfy the algebraic relation in question. We can then conclude, by a simple reasoning, that the algebraic relation is also satisfied throughout the time evolutions.

This paper is organized as follows. Section 2 reviews the combinatorial expression of the two-partition Hodge integrals. Relevant geometric and combinatorial notions are introduced here. Section 3 recalls the construction of the tau function from the two-partition Hodge integrals. A 2D complex free fermion system is a fundamental tool of this section. Section 4 presents the key theorem and its proof. Fermionic expressions of building blocks of the tau function are translated to the language of difference operators. Those difference operators play a central role in this and the next sections. Section 5 is devoted to various integrable structures that emerge when $\tau$ takes special values.
2 Two-partition Hodge integrals

2.1 Geometric definition

Let $G_{g,\mu\bar{\mu}}(\tau)$ denote the following cubic Hodge integral \cite{6} that depend on a parameter $\tau \neq 0$ and two partitions\footnote{All partitions in this paper are understood to have no restriction on the length. Such a partition $\mu$ is represented by a non-increasing sequence $(\mu_i)_{i=1}^{\infty}$, $\mu_1 \geq \mu_2 \geq \cdots$, of non-negative integers such that $\mu_i = 0$ for all $i$'s greater than a bound $n$. The minimum of the bound $n$ is the length of $\mu$, and denoted by $l(\mu)$.} $\mu = (\mu_i)_{i=1}^{\infty}$, $\bar{\mu} = (\bar{\mu}_i)_{i=1}^{\infty}$, $(\mu, \bar{\mu}) \neq (\emptyset, \emptyset)$:

$$G_{g,\mu\bar{\mu}}(\tau) = c_{\mu\bar{\mu}}(\tau) \int_{\overline{\mathcal{M}}_{g,2l(\mu)+l(\bar{\mu})}} \prod_{i=1}^{l(\mu)} \frac{1}{\mu_i} \prod_{i=1}^{l(\bar{\mu})} \frac{1}{\bar{\mu}_i} \frac{\psi_i}{\tau - \psi_i(\mu+i)} \prod_{i=1}^{l(\mu)} (1 - \psi_i) \prod_{i=1}^{l(\bar{\mu})} (\bar{\mu}_i - 1).$$

$c_{\mu\bar{\mu}}(\tau)$ is a combinatorial factor of the form

$$c_{\mu\bar{\mu}}(\tau) = \frac{\sqrt{-1}^{l(\mu)+l(\bar{\mu})}}{z(\mu)z(\bar{\mu})} \frac{\tau(\tau + 1)^{l(\mu)+l(\bar{\mu})-1}}{\prod_{i=1}^{l(\mu)} (\mu_i + 1) \prod_{i=1}^{l(\bar{\mu})} (\bar{\mu}_i - 1)},$$

where $(a)_k$ is the Pochhammer symbol, $(a)_k = a(a+1)\cdots(a+k-1)$, and $z(\mu)$ and $z(\bar{\mu})$ are defined as

$$z(\mu) = \prod_{i=1}^{\infty} i^{m_i} m_i!, \quad z(\bar{\mu}) = \prod_{i=1}^{\infty} i^{\bar{m}_i} \bar{m}_i!$$

in terms of the cycle type, $\mu = (1^{m_1} 2^{m_2} \cdots)$ and $\bar{\mu} = (1^{\bar{m}_1} 2^{\bar{m}_2} \cdots)$, of $\mu$ and $\bar{\mu}$.

As briefly explained in Introduction, $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ of connected smooth complex algebraic curve $C$ of genus $g$ with $n$ marked points $x_1, \ldots, x_n$. The $i$-th $\psi$-class $\psi_i$ is the first Chern class $c_1(L_i)$ of a line bundle $L_i$. The fiber of $L_i$ at $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ is the cotangent space $T^*_x C$ of $C$ at $x_i$. $\Lambda^\psi_g(u)$ is the same special linear combination of 1 and the Hodge classes $\lambda_1, \ldots, \lambda_g$ as used in (1). The $k$-th Hodge class $\lambda_k$ is the $k$-th Chern class $c_k(E_g)$ of the Hodge bundle $E_g$. In this sense, $\Lambda^\psi_g(u)$ is the characteristic polynomial in the definition of the Chern classes in terms of the curvature form of $E_g$. The fiber of $E_g$ at $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ is the $g$-dimensional linear space of holomorphic 1-forms on $C$. 


2.2 Generating functions

Several generating functions are constructed from these cubic Hodge integrals. Firstly, the all-genus generating function $G_{\mu\bar{\mu}}(\tau)$ is the power series

$$G_{\mu\bar{\mu}}(\tau) = \sum_{g=0}^{\infty} \hbar^{2g-2+l(\mu)+l(\bar{\mu})} G_{g\mu\bar{\mu}}(\tau)$$

of a new variable $\hbar$. Secondly, two sets of variables $p = (p_k)_{k=1}^{\infty}$, $\bar{p} = (\bar{p}_k)_{k=1}^{\infty}$ are introduced to make a generating function with respect to the $p$ partitions $\mu$, $\bar{\mu}$ as

$$G(\tau, p, \bar{p}) = \sum_{\mu, \bar{\mu} \in P, (\mu, \bar{\mu}) \neq (\emptyset, \emptyset)} G_{\mu\bar{\mu}}(\tau) p_{\mu} \bar{p}_{\bar{\mu}},$$

where $P$ denote the set of all partitions, and $p_{\mu}$ and $\bar{p}_{\bar{\mu}}$ are the monomials

$$p_{\mu} = \prod_{i=1}^{l(\mu)} p_{\mu_i}, \quad \bar{p}_{\bar{\mu}} = \prod_{i=1}^{l(\bar{\mu})} \bar{p}_{\bar{\mu}_i}.$$ 

Lastly, this generating function is exponentiated to

$$G^*(\tau, p, \bar{p}) = \exp G(\tau, p, \bar{p}).$$

2.3 Schur functions

The $p$-variables originate in the so called power sums that relate the Schur functions $s_\mu(x)$ in the sense of Macdonald’s book [20] to a set of polynomials $S_\mu(p)$. $S_\mu(p)$’s can be directly defined by the determinant formula

$$S_\mu(p) = \det(S_{\mu_i-i+j}(p))_{i,j=1}^{\infty},$$

$$\sum_{m=0}^{\infty} S_m(p) z^m = \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{k} z^k \right).$$

The right hand side of this formula is understood to be an $n \times n$ determinant $\det(S_{\mu_i-i+j}(p))_{i,j=1}^{n}$, $n \geq l(\mu)$, which is independent of $n$. $s_\mu(x)$ is obtained from $S_\mu(p)$ by substituting the power sums

$$p_k = \sum_{i \geq 1} x_i^k, \quad k = 1, 2, \ldots$$

of the components $x_i$ of $x$. In the following, we need an infinite-variate version of the Schur functions $s_\mu(x)$, $x = (x_i)_{i=1}^{\infty}$. The approach to $s_\mu(x)$ from $S_\mu(p)$ is particularly convenient for that purpose.

The skew Schur functions $s_{\mu/\nu}(x)$ and $S_{\mu/\nu}(p)$ can be treated in the same way. The determinant formula of $S_{\mu/\nu}(p)$ can be generalized to $S_{\mu/\nu}(p)$ as

$$S_{\mu/\nu}(p) = (S_{\mu_i-\nu_j-i+j}(p))_{i,j=1}^{\infty}.$$
2.4 Combinatorial expression

Liu, Liu and Zhou [6] proved the following combinatorial expression of the generating function $G^{*}(\tau, p, \bar{p})$.

**Theorem 1.**

$$G^{*}(\tau, p, \bar{p}) = R^{*}(\tau, p, \bar{p}),$$

(6)

where $R^{*}(\tau, p, \bar{p})$ is defined as

$$R^{*}(\tau, p, \bar{p}) = \sum_{\nu, \bar{\nu} \in P} q^{(\kappa(\nu) \tau + \kappa(\bar{\nu}) \tau^{-1})/2} \mathcal{W}_{\nu \bar{\nu}}(q) S_{\nu}(p) S_{\bar{\nu}}(\bar{p})$$

(7)

with the combinatorial building blocks

$$\mathcal{W}_{\nu \bar{\nu}}(q) = s_{\nu}(q^\rho) s_{\bar{\nu}}(q^{\nu^+ \rho}), \quad q = e^{\sqrt{-1} \hbar},$$

$$q^\rho = (q^{-i+1/2})_{i=1}^{\infty}, \quad q^{\nu^+ \rho} = (q^{\nu_i - i + 1/2})_{i=1}^{\infty},$$

$$\kappa(\nu) = \sum_{i=1}^{\infty} \nu_i (\nu_i - 2i + 1), \quad \kappa(\bar{\nu}) = \sum_{i=1}^{\infty} \bar{\nu}_i (\bar{\nu}_i - 2i + 1).$$

The main building block of $R^{*}(\tau, p, \bar{p})$ is the two-leg topological vertex $\mathcal{W}_{\nu \bar{\nu}}(q)$ [21]. This is a rational function of $q^{1/2}$ that enjoys the non-trivial symmetries [22]

$$\mathcal{W}_{\nu \bar{\nu}}(q) = \mathcal{W}_{\bar{\nu} \nu}(q) = (-1)^{|\nu| + |\bar{\nu}|} \mathcal{W}_{\nu \bar{\nu}}(q^{-1}),$$

(8)

where $\nu^+$ and $\nu^-$ stand for the conjugate partitions of $\nu$ and $\bar{\nu}$ that amount to the transpose of the associated Young diagrams. The symmetry under the inversion $q \mapsto q^{-1}$ is a consequence of the relation (as rational functions of $q^{1/2}$)

$$p_k(q^{\nu^+ \rho}) = -p_k(q^{-\nu^- \rho})$$

(9)

between the values of the power sums $p_k(x) = \sum_{i=1}^{\infty} x^k$ at two special points (cf. the proof of Lemma 1 in our previous work [23]) and the identity

$$S_{\mu}(-p) = (-1)^{|\mu|} S_{t \mu}(p)$$

(10)

of the Schur functions.

Let us mention that the last identity of the Schur functions can be generalized to the skew Schur functions as

$$S_{\mu/\nu}(-p) = (-1)^{|\mu| + |\nu|} S_{t \mu/\nu}(p).$$

(11)

Accordingly, the inversion symmetry [3] of $\mathcal{W}(q)$ can be extended to the three-leg topological vertex $C_{\lambda \mu \nu}(q)$ [21] as

$$C_{\lambda \mu \nu}(q) = (-1)^{|\lambda| + |\mu| + |\nu|} C_{t \lambda^t \mu^t \nu^t}(q^{-1}).$$

(12)
3 Lift to tau function

3.1 Fermionic language

The goal of this section is to convert (or, rather, lift) $R^*(\tau,\mathbf{p},\mathbf{p})$ to a tau function of the 2D Toda hierarchy. To this end, we use the language of complex fermions. The following (partially somewhat unusual) formulation of fermionic operators and Fock spaces is the same as our previous work \cite{17,18,19,23}.

Let $\psi_n$ and $\psi_n^*$, $n \in \mathbb{Z}$, denote the Fourier modes\footnote{Do not confuse them with the $\psi$-classes on $\mathcal{M}_{g,n}$. Moreover, as opposed to the usual formulation, we label these operators with integers rather than half integers.} of the 2D free fermion fields

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n-1}$$

that satisfy the anti-commutation relations

$$\psi_m \psi_n^* + \psi_n \psi_m^* = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0, \quad \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0.$$ 

The (bra- and ket-) Fock spaces are decomposed to the charge-$s$ sectors for $s \in \mathbb{Z}$. The charge-$s$ sectors are spanned by the vectors

$$\langle \mu, s \rangle = \langle -\infty | \cdots | \psi_{\mu_{i+1} + s}^* \cdots | \psi_{\mu_1 + s}^* | -\infty \rangle,$$

$$| s \rangle = \psi_{-\mu_1 - s} \psi_{-\mu_2 - 1 - s} \cdots | -\infty \rangle$$

labelled by partitions $\mu \in \mathcal{P}$. Their pairing is defined as

$$\langle \mu, r \rangle | s \rangle = \delta_{rs} \delta_{\mu\nu}.$$ 

The vectors $\langle 0, s \rangle$ and $| 0, s \rangle$ are called the ground states of the charge-$s$ sector, and abbreviated as $\langle s \rangle$ and $| s \rangle$. In particular, $\langle 0 \rangle$ and $| 0 \rangle$ represent the vacuum states of the whole fermion system. Moreover, let $\langle \mu \rangle$ and $| \mu \rangle$ denote the vectors $\langle \mu, 0 \rangle$ and $| \mu, 0 \rangle$ in the charge-0 sector.

Let $L_0$, $K$, and $J_m$, $m \in \mathbb{Z}$, denote the special fermion bilinears

$$L_0 = \sum_{n \in \mathbb{Z}} n :\psi_{-n} \psi_n^*:,$$

$$K = \sum_{n \in \mathbb{Z}} (n - 1/2)^2 :\psi_{-n} \psi_n^*:,$$

$$J_m = \sum_{n \in \mathbb{Z}} :\psi_{-n} \psi_{n+m}^*:,$$

$s \in \mathbb{Z},$

where $:\psi_{-m} \psi_n^*:,$'s are the normal ordered product:

$$:\psi_{-m} \psi_n^*: = \psi_{-m} \psi_n^* - \langle 0 | \psi_{-m} \psi_n^* | 0 \rangle,$$

$$\langle 0 | \psi_{-m} \psi_n^* | 0 \rangle = \begin{cases} 1 & \text{if } m = n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$
$J_m$’s are used to construct the vertex operators \[21, 25\]

\[
\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right),
\]

\[
\Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right),
\]

and the multi-variate extensions

\[
\Gamma_\pm(x) = \prod_{i=1}^{\infty} \Gamma_\pm(x_i), \quad \Gamma'_\pm(x) = \prod_{i \geq 1} \Gamma'_\pm(x_i).
\]

The action of these operators preserves the charge. The matrix elements in the charge-$s$ sector take the following form:

\[
\langle \mu, s|L_0|\nu, s \rangle = \delta_{\mu \nu} (|\mu| + s(s + 1)/2), \quad (13)
\]

\[
\langle \mu, s|K|\nu, s \rangle = \delta_{\mu \nu} (\kappa(\mu) + 2s|\mu| + (4s^3 - s)/12), \quad (14)
\]

\[
\langle \mu, s|\Gamma_-(x)|\nu, s \rangle = \langle \nu, s|\Gamma_+(x)|\mu, s \rangle = s_{\mu/\nu}(x), \quad (15)
\]

\[
\langle \lambda, s|\Gamma'_+(x)|\mu, s \rangle = \langle \mu, s|\Gamma'_-(x)|\lambda, s \rangle = s_{\lambda/\mu}(x). \quad (16)
\]

The matrix elements of the vertex operators are thus independent of $s$, and yield the fermionic expression

\[
s_{\mu/\nu}(x) = \langle \mu|\Gamma_-(x)|\nu \rangle = \langle \nu|\Gamma_+(x)|\mu \rangle \quad (17)
\]

of the skew Schur functions of the $x$-variables. (Recall that $|\mu|$, etc. are abbreviations of $|\mu, 0|$, etc.) This expression can be derived from the fermionic expression \[26\] 

\[
S_{\mu/\nu}(p) = \langle \mu|\exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k} J_{-k}\right)|\nu \rangle = \langle \nu|\exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k} J_{k}\right)|\mu \rangle \quad (18)
\]

of the skew Schur functions of the $p$-variables.

### 3.2 Fermionic expression of $W_{\nu\bar{\nu}}(q)$

The two-leg topological vertex has yet another expression \[21, 22\]:

\[
W_{\nu\bar{\nu}}(q) = q^{\kappa(\nu) + \kappa(\bar{\nu})}/2 \sum_{\eta \in P} s_{\nu/\eta}(q^\theta) s_{\bar{\nu}/\eta}(q^\theta). \quad (19)
\]

The right hand side of expression is actually a finite sum over all $\eta$’s with $\eta \subseteq ^t\nu$ and $\eta \subseteq ^t\bar{\nu}$, where $\subseteq$ stands for the inclusion relation of the associated Young diagrams. We can translate this expression to the language of fermions as follows:
Lemma 1. 

\[ \mathcal{W}_{\nu \bar{\nu}}(q) = \langle \nu | q^{-K/2} \Gamma_{-} (q^{\rho}) \Gamma_{+} (q^{\rho}) q^{-K/2} | \nu \rangle. \] 

(20)

Here (and in the rest of this paper) the \( q \)-exponential \( q^A \) of an operator \( A \) stands for \( \exp(\log q^A) \).

Proof. By the fermionic formula (17) of the skew Schur functions and the partition of unity \( 1 = \sum_{\eta \in P} |\eta \rangle \langle \eta | \) in the charge-0 sector, we can express the sum on the right hand side of (19) as

\[ \sum_{\eta \in P} s_{\nu \eta} (q^{\rho}) s_{\bar{\nu} \eta} (q^{\rho}) = \sum_{\eta \in P} \langle \nu | \Gamma_{-} (q^{\rho}) | \eta \rangle \langle \eta | \Gamma_{+} (q^{\rho}) | \bar{\nu} \rangle = \langle \nu | \Gamma_{-} (q^{\rho}) \Gamma_{+} (q^{\rho}) | \bar{\nu} \rangle. \]

To move the remaining \( q \)-factors inside \( \langle \nu | \cdots | \bar{\nu} \rangle \), we use the relation 
\[ \kappa(\nu) = -\kappa(\nu) \]

and the formula (14) of the matrix elements of \( K \) as

\[ q^{\kappa(\nu)/2 + \kappa(\bar{\nu})/2} \langle \nu | \cdots | \bar{\nu} \rangle q^{-\kappa(\nu)/2} = \langle \nu | q^{-K/2} \cdots q^{-K/2} | \bar{\nu} \rangle. \]

(20) is an expression of the rational function \( \mathcal{W}_{\nu \bar{\nu}}(q) \) in the region \(|q| > 1 \).

The vertex operators in this expression can be computed as

\[ \Gamma_{\pm} (q^{\rho}) = \exp \left( \sum_{k, i=1}^{\infty} \frac{q^{-(i-1/2)k}}{k} J_{\pm k} \right) = \exp \left( \sum_{k=1}^{\infty} \frac{q^{-k/2}}{k(1 - q^{-k})} J_{\pm k} \right), \]

and this computation is valid only in the region \(|q| > 1 \).

If we can now start from the last operator (whose matrix elements are rational functions of \( q^{1/2} \)) and rewrite it as

\[ \exp \left( \sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1 - q^k)} J_{\pm k} \right) = \exp \left( - \sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1 - q^k)} J_{\pm k} \right), \]

we can proceed in an opposite direction as

\[ \exp \left( - \sum_{k=1}^{\infty} \frac{q^{i-1/2)k}}{k(1 - q^{-k})} J_{\pm k} \right) = \exp \left( - \sum_{k, i=1}^{\infty} \frac{q^{(i-1/2)k}}{k} J_{\pm k} \right) = \Gamma'_{\pm} (-q^{-\rho}). \]
Note that this computation is valid in the region $|q| < 1$.

These considerations show that the operator $\Gamma_\pm(q^\rho)$ in the region $|q| > 1$ and the operator $\Gamma'_\pm(-q^{-\rho})$ are analytic continuation of each other. We are thus led to the following expression of $W_{\nu\bar{\nu}}(q)$ that is valid in the region $|q| < 1$:

\[
W_{\nu\bar{\nu}}(q) = \langle t\nu | q^{-K/2}\Gamma'_-(-q^{-\rho})\Gamma'_+(-q^{-\rho})q^{-K/2}|t\bar{\nu} \rangle. \tag{21}
\]

Actually, this is an intermediate stage. We rewrite it further as follows.

**Lemma 2.**

\[
W_{\nu\bar{\nu}}(q) = (-1)^{|\nu|+|\bar{\nu}|}\langle \nu | q^{K/2}\Gamma_-(-q^{-\rho})\Gamma_+(q^{-\rho})q^{K/2}|\bar{\nu} \rangle. \tag{22}
\]

**Proof.** Let us rewrite the right hand side of (21) as

\[
\langle t\nu | q^{-K/2}\Gamma'_-(-q^{-\rho})\Gamma'_+(-q^{-\rho})q^{-K/2}|t\bar{\nu} \rangle = q^{\kappa(\nu)+\kappa(\bar{\nu})/2}\langle t\nu | \Gamma'_-(-q^{-\rho})\Gamma'_+(q^{-\rho})|t\bar{\nu} \rangle.
\]

Since $L_0$ and $J_k$’s satisfy the commutation relations

\[
[L_0, J_k] = -kJ_k,
\]

the negative sign in front of $q^{-\rho}$ can be eliminated by the adjoint action of $(-1)^{L_0}$ as

\[
\Gamma'_\pm(-q^{-\rho}) = (-1)^{L_0}\Gamma'_\pm(q^{-\rho})(-1)^{L_0}.
\]

Consequently,

\[
\langle t\nu | \Gamma'_-(-q^{-\rho})\Gamma'_+(q^{-\rho})|t\bar{\nu} \rangle = (-1)^{|\nu|+|\bar{\nu}|}\langle t\nu | \Gamma'_-(-q^{-\rho})\Gamma'_+(q^{-\rho})|t\bar{\nu} \rangle.
\]

Since the matrix elements (15) and (16) of the two types of vertex operators correspond to each other by transposing the partitions, we have the identity

\[
\langle t\nu | \Gamma'_-(-q^{-\rho})\Gamma'_+(q^{-\rho})|t\bar{\nu} \rangle = \langle \nu | \Gamma_-(-q^{-\rho})\Gamma_+(q^{-\rho})|\bar{\nu} \rangle.
\]

Collecting these formulae yields (22)

We use the last fermionic expression (22) to convert $R^*(\tau, p, \bar{p})$ to a tau function of the 2D Toda hierarchy. As mentioned above, this expression itself is valid in the region $|q| < 1$. It is easy to see that the first expression (20) is connected with this expression by the inversion $q \rightarrow q^{-1}$:

\[
W_{\nu\bar{\nu}}(q) = (-1)^{|\nu|+|\bar{\nu}|}W_{\nu\bar{\nu}}(q^{-1}). \tag{23}
\]

This is exactly the inversion relation (8) mentioned in the end of the previous section. We have derived it from a slightly different route.
3.3 Lifting $R^\bullet(\tau, p, \bar{p})$ to tau function

All building blocks of the definition \((7)\) of $R^\bullet(\tau, p, \bar{p})$ are now translated to the language of fermions. This leads to the following fermionic expression of $R^\bullet(\tau, p, \bar{p})$.

**Theorem 2.**

$$R^\bullet(\tau, p, \bar{p}) = \langle 0 | \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k p_k}{k} J_k \right) \hbar \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k \bar{p}_k}{k} J_{-k} \right) | 0 \rangle, \quad (24)$$

where

$$h = q^{(r+1)K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{(r-1)K/2}.$$  \((25)\)

**Proof.** Let us apply the fermionic formulae \((14), (18)\) and \((22)\) to the building blocks of \((7)\). We want to achieve the summation over $\nu, \bar{\nu} \in \mathcal{P}$ with the aid of the partition of unity

$$1 = \sum_{\nu \in \mathcal{P}} | \nu \rangle \langle \nu | = \sum_{\bar{\nu} \in \mathcal{P}} | \bar{\nu} \rangle \langle \bar{\nu} |$$

in the charge-0 sector. An obstacle is the sign factor $(-1)^{|\nu|+|\bar{\nu}|}$. This factor can be absorbed by the Schur functions as

$$(-1)^{|\nu|+|\bar{\nu}|} S_\nu(p) S_{\bar{\nu}}(\bar{p}) = S_\nu(\ldots, (-1)^k p_k, \ldots) S_{\bar{\nu}}(\ldots, (-1)^k \bar{p}_k, \ldots).$$

We can thus use the partition of unity to obtain \((24)\). \(\square\)

\((24)\) is very close to a general fermionic expression of tau functions of the 2D Toda hierarchy \([27, 28]\) (see also ref. \([29]\), Section 3.3). A tau function $\mathcal{T}(s, t, \bar{t})$ of the lattice coordinate $s$ and the time variables $t = (t_k)_{k=1}^{\infty}$ and $\bar{t} = (\bar{t}_k)_{k=1}^{\infty}$ can be obtained by replacing

$$\frac{(-1)^k p_k}{k} \to t_k, \quad \frac{(-1)^k \bar{p}_k}{k} \to -\bar{t}_k,$$

$$\langle 0 | \to \langle s |, \quad | 0 \rangle \to | s \rangle, \quad s \in \mathbb{Z}.$$

as

$$\mathcal{T}(s, t, \bar{t}) = \langle s | \exp \left( \sum_{k=1}^{\infty} t_k J_k \right) \hbar \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle. \quad (26)$$

This is the tau function constructed by Zhou \([7]\).
### 3.4 $s$-dependence of $\mathcal{T}(s, t, \bar{t})$

By twice inserting the partition of unity as we have done in the proof of Theorem 2, we can expand $\mathcal{T}(s, t, \bar{t})$ into a double sum over partitions:

\[
\mathcal{T}(s, t, \bar{t}) = \sum_{\nu, \rho \in \mathcal{P}} q^{(\tau + 1)(\kappa(\nu)/2 + s|\nu| + (4s^3 - s))/24} \times q^{(\tau - 1)(\kappa(\bar{\nu})/2 + s|\bar{\nu}| + (4s^3 - s))/24} \times S_{\nu}(t)S_{\bar{\rho}}(\bar{t}).
\]

(27)

$s$ and $\bar{t}$ are the Schur functions $S_{\nu}(p)$ and $S_{\bar{\rho}}(\bar{p})$ regarded as functions of $t$ and $\bar{t}$ by the relation

\[
p_k = kt_k, \quad \bar{p}_k = k\bar{t}_k, \quad k = 1, 2, \ldots.
\]

The $s$-dependent factors come from the matrix elements (14) of $K$.

Although $s$ is originally a lattice coordinate, hence a discrete variable, (27) hints that $s$ may be thought of as a continuous variable. The following fact shows that this point of view is meaningful enough.

**Theorem 3.** For any constant $c$, the function $\mathcal{T}(s + c, t, \bar{t})$ restricted to $s \in \mathbb{Z}$ is a tau function of the 2D Toda hierarchy.

**Proof.** When $s$ is shifted to $s + c$, the exponents of the two exponential factors in (27) vary as

\[
\kappa(\nu)/2 + s|\nu| + (4s^3 - s)/24 \rightarrow \kappa(\nu)/2 + (s + c)|\nu| + (4(s + c)^3 - (s + c))/24
\]

\[
= \langle \nu, s | (K/2 + cL_0 + (c^2 - c)J_0/2) | \nu, s \rangle + (4c^3 - c)/24
\]

and

\[
\kappa(\bar{\nu})/2 + s|\bar{\nu}| + (4s^3 - s)/24 \rightarrow \kappa(\bar{\nu})/2 + (s + c)|\bar{\nu}| + (4(s + c)^3 - (s + c))/24
\]

\[
= \langle \bar{\nu}, s | (K/2 + cL_0 + (c^2 - c)J_0/2) | \bar{\nu}, s \rangle + (4c^3 - c)/24.
\]

This implies that the shifted tau function $\mathcal{T}(s + c, t, \bar{t})$ can be expressed in the fermionic form (26) with $h$ replaced by

\[
h(c) = q^{(\tau + \tau^{-1} + 2)(4c^3 - c)/24} q^{(\tau + 1)(K/2 + cL_0 + (c^2 - c)J_0/2)}
\]

\[
\times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\bar{\rho}}) q^{(\tau - 1)(K/2 + cL_0 + (c^2 - c)J_0/2)}.
\]

The tau function $\mathcal{T}(s, t, \bar{t})$ thus yields a solution of the Toda hierarchy on any integral sublattice $\mathbb{Z} + c \subset \mathbb{R}$. In the following section, we consider this solution in the Lax formalism.
4 Perspectives in Lax formalism

4.1 Fractional powers of Lax operators

Let $\Lambda$ denote the shift operator in the variable $s$:

$$\Lambda = e^{\partial_s}, \quad \Lambda^n f(s) = f(s + n).$$

The Lax formalism of the 2D Toda hierarchy uses two (so to speak, pseudo-difference) operators $L, \bar{L}$ of the form

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}, \quad \bar{L} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^n,$$

$$u_n = u_n(s, t, \bar{t}), \quad \bar{u}_n = \bar{u}_n(s, t, \bar{t}),$$

that satisfy the Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$

$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{< 0},$$

where $(\ldots)_{\geq 0}$ and $(\ldots)_{< 0}$ denote the projection onto the non-negative and negative power parts of difference operators:

$$\left( \sum_{n \in \mathbb{Z}} a_n \Lambda^n \right)_{\geq 0} = \sum_{n \geq 0} a_n \Lambda^n, \quad \left( \sum_{n \in \mathbb{Z}} a_n \Lambda^n \right)_{< 0} = \sum_{n < 0} a_n \Lambda^n.$$

By the standard procedure [30, 31, 29], the tau function $\mathcal{T}(s, t, \bar{t})$ yields the Lax operators via two dressing operators

$$W = 1 + \sum_{n=1}^{\infty} w_n e^{-n \partial_s}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n e^{n \partial_s}, \quad \bar{w}_0 \neq 0.$$
of quotients of two shifted tau functions at $z = \infty$ and $z = 0$, respectively. The Lax operators are thereby expressed as

$$L = W\Lambda W^{-1}, \quad L^{-1} = \bar{W}\Lambda^{-1}\bar{W}^{-1}.$$ 

Since $s$ is now interpreted to be a continuous variable, we can define the logarithm and the fractional powers of $L$ and $\bar{L}$ as

$$\log L = W\log \Lambda W^{-1}, \quad \log \bar{L} = \bar{W}\log \Lambda\bar{W}^{-1},$$

$$L^\alpha = W\Lambda^\alpha W^{-1}, \quad \bar{L}^\alpha = \bar{W}\Lambda^\alpha\bar{W}^{-1}.$$ 

Note that $\log \Lambda$ and $\Lambda^\alpha$ are differential and shift operators in the variable $s$:

$$\log \Lambda = \partial_s, \quad \Lambda^\alpha = e^{\alpha}\partial_s.$$ 

Consequently, $\log L$, $\log \bar{L}$, $L^\alpha$ and $\bar{L}^\alpha$ can be expressed as

$$\log L = \partial_s - \frac{\partial W}{\partial s}W^{-1}, \quad \log \bar{L} = \partial_s - \frac{\partial \bar{W}}{\partial s}\bar{W}^{-1},$$

$$L^\alpha = W\cdot W^{-1}|_{s \to s+\alpha} \cdot e^{\alpha}\partial_s = (1 + p_1\Lambda^{-1} + \cdots)e^{\alpha}\partial_s,$$

$$\bar{L}^\alpha = \bar{W}\cdot \bar{W}^{-1}|_{s \to s+\alpha} \cdot e^{\alpha}\partial_s = (\bar{p}_0 + \bar{p}_1\Lambda + \cdots)e^{\alpha}\partial_s,$$

where $p_1, p_2, \ldots$ and $\bar{p}_0, \bar{p}_1, \ldots$ are determined by the coefficients of $W$ and $\bar{W}$, e.g.,

$$p_1 = w_1(s, t, \bar{t}) - w_1(s + \alpha, t, \bar{t}), \quad \bar{p}_0 = \frac{\bar{w}_0(s, t, \bar{t})}{\bar{w}_0(s + \alpha, t, \bar{t})}.$$ 

Calculus of this kind of fractional difference operators is discussed in the paper of Liu, Zhang and Zhou [12].

### 4.2 Algebraic relations of Lax operators

We can now state the key theorem:

**Theorem 4.** The Lax operators obtained from the tau function $\mathcal{T}(s, t, \bar{t})$ of (26) satisfy the algebraic relation

$$L^{1/(\tau+1)} = -\bar{L}^{-1/(\tau^{-1}+1)}.$$ 

Implications of this algebraic relation are discussed in the next section. The rest of this section is devoted to proving this theorem. Let us note here the following immediate consequence.
Corollary 1. There is a function $u = u(s, t, \bar{t})$ such that
\[ L^{1/(\tau+1)} = -\bar{L}^{-1/(\tau^{-1}+1)} = (1 - u\Lambda^{-1})\Lambda^{1/(\tau+1)} \] (31)

Proof. The both sides of (30) are fractional difference operators of the following form:
\[ L^{1/(\tau+1)} = (1 + p_1\Lambda^{-1} + p_2\Lambda^{-2} + \cdots)\Lambda^{1/(\tau+1)} \]
\[ = \Lambda^{1/(\tau+1)} + p_1\Lambda^{-\tau/(\tau+1)} + \cdots, \]
\[ -\bar{L}^{-1/(\tau^{-1}+1)} = -(\bar{p}_0 + \bar{p}_1\Lambda + \bar{p}_2\Lambda^2 + \cdots)\Lambda^{-\tau/(\tau+1)} \]
\[ = -\bar{p}_0\Lambda^{-\tau/(\tau+1)} - \bar{p}_1\Lambda^{1/(\tau+1)} + \cdots. \]
Therefore only two terms survive:
\[ L^{1/(\tau+1)} = -\bar{L}^{-1/(\tau^{-1}+1)} = \Lambda^{1/(\tau+1)} + p_1\Lambda^{-\tau/(\tau+1)}, \]
\[ \bar{p}_1 = -1, \quad \bar{p}_0 = -p_1. \]
\[ \square \]

4.3 Factorization problem

The proof of Theorem 4 borrows a main idea from our previous work [17, 18, 19]. The idea is to use the factorization problem
\[ \exp\left(\sum_{k=1}^{\infty} t_k\Lambda^k\right)U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k\Lambda^{-k}\right) = W^{-1}\bar{W} \] (32)
at a particular point of the $(t, \bar{t})$-space where the factors $W, \bar{W}$ can be obtained explicitly.

In the usual setting [30, 31, 32], (32) is an equation for $\mathbb{Z} \times \mathbb{Z}$ matrices. Namely, given an invertible (typically, exponential) matrix $U$, the problem is to find a lower-triangular matrix $W$ and an upper-triangular matrix $\bar{W}$ satisfying (32). Arbitrariness of $W$ and $\bar{W}$ (i.e., gauge freedom $W \to DW$, $\bar{W} \to D\bar{W}$ by a diagonal matrix $D$) disappears under the normalization condition that all diagonal elements of $W$ are equal to 1. The Lax operators defined by $W$ and $\bar{W}$ as
\[ L = W\Lambda W^{-1}, \quad \bar{L}^{-1} = \bar{W}\Lambda\bar{W}^{-1} \]
give a solution of the 2D Toda hierarchy.
Moreover, this $\mathbb{Z} \times \mathbb{Z}$ matrix formalism is directly related to the fermionic expression of tau functions like (26) [27, 28]. If $h$ takes the exponential form $h = e^A$ with a fermion bilinear

$$\hat{A} = \sum_{i,j \in \mathbb{Z}} a_{ij} \psi_i \psi_j^*,$$

$U$ becomes the exponential matrix $e^A$ with

$$A = (a_{ij})_{i,j \in \mathbb{Z}}.$$ 

If $h$ is a product of such exponential operators, $U$ is a product of the associated exponential matrices. In this sense, $K$, $L_0$ and $J_k$ correspond to diagonal and shift matrices as

$$K \leftrightarrow (\Delta - 1/2)^2, \quad L_0 \leftrightarrow \Delta, \quad J_k \leftrightarrow \Lambda^k,$$

where

$$1 = (\delta_{ij})_{i,j \in \mathbb{Z}}, \quad \Delta = (i \delta_{ij})_{i,j \in \mathbb{Z}}, \quad \Lambda^k = (\delta_{i+k,j})_{i,j \in \mathbb{Z}}.$$ 

The single-variate vertex operators

$$\Gamma_{\pm}(z) = \exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right)$$

amount to the matrix

$$\exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} \Lambda_{\pm k} \right) = \exp \left( - \log(1 - z \Lambda_{\pm 1}) \right) = (1 - z \Lambda_{\pm 1})^{-1},$$

hence

$$\Gamma_{\pm}(q^{-\rho}) = \prod_{i=1}^{\infty} \Gamma_{\pm}(q^{i-1/2}) \longleftrightarrow \prod_{i=1}^{\infty} (1 - q^{i-1/2} \Lambda_{\pm 1})^{-1}.$$ 

Thus the operator $h$ of (25) corresponds to the following matrix:

$$U = q^{(r+1)(\Delta-1/2)^2/2} \cdot \prod_{k=1}^{\infty} (1 - q^{i-1/2} \Lambda)^{-1} \cdot q^{(r-1+1)(\Delta-1/2)^2/2} \cdot \prod_{k=1}^{\infty} (1 - q^{i-1/2} \Lambda)^{-1} \cdot q^{(r-1+1)(\Delta-1/2)^2/2}.$$ 

\[\text{To simplify the notations, we use } \Lambda^k \text{ for both the matrix and the shift operator.}\]
In the present setting, \( s \) is considered to be a continuous variable. Therefore we interpret (32) as equations for operators on the continuous space \( \mathbb{R} \).

Accordingly, \( K \), \( L_0 \) and \( \Lambda^k \) now correspond to multiplication and difference operators as

\[
K \leftrightarrow (s - 1/2)^2, \quad L_0 \leftrightarrow s, \quad J_k \leftrightarrow \Lambda^k = e^{k\partial_s}.
\]

Thus the matrix \( U \) of (34) is replaced by the operator

\[
U = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{k=1}^{\infty} (1 - q^{i-1/2} \Lambda^{-1})^{-1} \cdot \prod_{k=1}^{\infty} (1 - q^{i-1/2} \Lambda)^{-1} \cdot q^{(\tau-1+1)(s-1/2)^2/2}.
\]

Note that this interpretation is consistent with the computation in the proof of Theorem 3. We have seen therein that the operator \( K \) in \( h \) turns into \( K + cL_0 + c(c-1)J_0/2 \) as \( s \) is shifted to \( s + c \). This exactly corresponds to the variation

\[
(s - 1/2)^2 \rightarrow (s + c - 1/2)^2 = (s - 1/2)^2 + cs + c(c-1)/2
\]

of the associated multiplication operator.

### 4.4 Initial values of operators

We use the factorization problem (32) to compute the initial values of \( L^{1/(\tau+1)} \) and \( \bar{L}^{1/(\tau-1+1)} \) at \( t = \bar{t} = 0 \). The first step toward this end is to find the initial values of the dressing operators.

When \( t = \bar{t} = 0 \), the factorization problem takes the simpler form

\[
U = W_0^{-1} \bar{W}_0.
\]

where

\[
W_0 = W|_{t=\bar{t}=0}, \quad \bar{W}_0 = \bar{W}_{t=\bar{t}=0}.
\]

\( W_0 \) and \( \bar{W}_0 \) are the initial values of the dressing operators, and thus characterized by the simpler factorization problem (37).

Unlike the original factorization problem (32), this factorization problem can be solved easily, because the operator \( U \) of (36) is already factorized in an almost final form. The only thing to do is to adjust the factors by gauge
freedom so that the leading coefficient of the $\Lambda^{-1}$-expansion of the first factor is equal to 1. We thus obtain the following expression of $W_0$ and $\bar{W}_0$:

$$W_0 = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{i=1}^{\infty} (1 - q^{-1/2} \Lambda^{-1}) \cdot q^{-(\tau+1)(s-1/2)^2/2}; \quad (38)$$

$$\bar{W}_0 = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{i=1}^{\infty} (1 - q^{-1/2} \Lambda^{-1}) \cdot q^{(\tau+1)(s-1/2)^2/2}. \quad (39)$$

This enables us to compute the initial values

$$L_0^{1/(\tau+1)} = L_0^{1/(\tau+1)}|_{t=\bar{t}=0}, \quad \bar{L}_0^{-1/(\tau-1+1)} = \bar{L}_0^{-1/(\tau-1+1)}|_{t=\bar{t}=0}$$

of $L^{1/(\tau+1)}$ and $\bar{L}^{-1/(\tau-1+1)}$ from $W_0$ and $\bar{W}_0$ as

$$L_0^{1/(\tau+1)} = W_0 \Lambda^{1/(\tau+1)} W_0^{-1}, \quad \bar{L}_0^{-1/(\tau-1+1)} = \bar{W}_0 \Lambda^{-1/(\tau-1+1)} \bar{W}_0^{-1}.$$ 

For convenience, we introduce the auxiliary operators

$$M_0 = W_0 s W_0^{-1}, \quad \bar{M}_0 = \bar{W}_0 \bar{s} \bar{W}_0^{-1},$$

which are the initial values of the Orlov-Schulman operators

$$M = W \left( s + \sum_{k=1}^{\infty} k t_k \Lambda^k \right) W^{-1},$$

$$\bar{M} = \bar{W} \left( s - \sum_{k=1}^{\infty} k \bar{t}_k \Lambda^{-k} \right) \bar{W}^{-1}.$$ 

**Lemma 3.**

$$L_0^{1/(\tau+1)} = q^M_0 q^{-s} \Lambda^{1/(\tau+1)}, \quad q^M_0 = q^s(1 - q^{(\tau+1)s-\tau-3/2} \Lambda^{-1}). \quad (40)$$

**Proof.** Let us split the expression of $W_0$ into two parts:

$$W_0 = V q^{-\tau+1)(s-1/2)^2/2}, \quad V = q^{(\tau+1)(s-1/2)^2/2} \prod_{i=1}^{\infty} (1 - q^{-1/2} \Lambda^{-1}),$$

and compute $L_0^{1/(\tau+1)} = W_0 \Lambda^{1/(\tau+1)} W_0^{-1}$ step by step. The first step is to use the general formula

$$\Lambda^{1/(\tau+1)} f(s) = f(s + 1/(\tau + 1)) \Lambda^{1/(\tau+1)}$$
Lemma 4.

Let \( L_0^{-1/(\tau+1)} = q^{\tilde{M}_0} q^{\tau s - \tau - 1/2 \Lambda^{-1}/(\tau+1)} \), \( q^{\tilde{M}_0} = q^s (1 - q^{-1/2}) \). (41)

Proof. These expressions can be derived in much the same way as the proof of the previous lemma. We omit the detail. \( \square \)
4.5 End of proof of Theorem 4

We can see from (40) and (41), by straightforward computation, that

\[ L_0^{1/(\tau+1)} = (1 - q^{(\tau+1)s - \tau - 1/2} \Lambda^{-1}) A^{1/(\tau+1)} = -\bar{L}_0^{-1/(\tau-1+1)}. \]  

This means that the algebraic relation (30) is satisfied at the initial time \( t = \bar{t} = 0 \).

This is enough to conclude that (30) itself is satisfied at all time. Note that both sides of (30) satisfy the same Lax equations of the form

\[ \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad k = 1, 2, \ldots \]  

(43)

Therefore, by the uniqueness of solution in the initial value problem of these equations, the both sides of (30) with the same initial value should be equal throughout the time evolutions. This completes the proof of Theorem 4.

Remark 1. The Lax and Orlov-Schulman operators turn out to satisfy the algebraic relation

\[ (q^{-M} L_0^{1/(\tau+1)})^{-\tau} = q^{(\tau+1)/2} q^{-\bar{M}} \bar{L}^{-1/(\tau-1+1)} \]  

(44)

that supplements (30). These two relations form a pair of conditions that single out a solution of the 2D Toda hierarchy [31]. (44) can be derived by the same logic as the foregoing derivation of (30) as follows. Let us rewrite (40) and (41) as

\[ q^{-M_0} L_0^{1/(\tau+1)} = q^{-s} \Lambda^{1/(\tau+1)}, \quad q^{-\bar{M}_0} \bar{L}_0^{-1/(\tau-1+1)} = q^{\tau s - \tau - 1/2} \Lambda^{-1/(\tau-1+1)}. \]

Since the right hand side of these equations satisfy the operator identity

\[ (q^{-s} \Lambda^{1/(\tau+1)})^{-\tau} = q^{(\tau+1)/2} q^{-s - \tau - 1/2} \Lambda^{-1/(\tau-1+1)}, \]

we obtain the algebraic relation

\[ (q^{-M_0} L_0^{1/(\tau+1)})^{-\tau} = q^{(\tau+1)/2} q^{-\bar{M}_0} \bar{L}_0^{-1/(\tau-1+1)}. \]

This means that (44) is satisfied at \( t = \bar{t} = 0 \). Since the both sides of (44) satisfy the same Lax equations, (44) holds at all time.

Remark 2. (44) can be reduced to the linear relation

\[ \log L - (\tau + 1)(M - 1/2) \log q = \log \bar{L} = (\tau^{-1} + 1)(\bar{M} - 1/2) \log q \quad (45) \]
among the four operators $\log L$, $\log \bar{L}$, $M$, $\bar{M}$. Let us note that these operators satisfy the canonical commutation relations

$$[\log L, M] = [\log \bar{L}, \bar{M}] = 1.$$  

We can thereby use the Baker-Campbell-Hausdorff formula to rewrite $q^{-M} L^{1/(\tau+1)}$ and $q^{-\bar{M}} \bar{L}^{-1/(\tau^{-1}+1)}$ as

\begin{align*}
q^{-M} L^{1/(\tau+1)} &= \exp \left(-M \log q + \frac{\log L}{\tau+1} + \frac{\log q}{2(\tau+1)}\right), \\
q^{-\bar{M}} \bar{L}^{-1/(\tau^{-1}+1)} &= \exp \left(-\bar{M} \log q - \frac{\log \bar{L}}{\tau^{-1}+1} - \frac{\log q}{2(\tau^{-1}+1)}\right).
\end{align*}

Plugging these expressions into (44) yields the exponentiated form

$$\exp (\log L - (\tau + 1)(M - 1/2) \log q)$$

$$= \exp (\log \bar{L} + (\tau^{-1} + 1)(\bar{M} - 1/2) \log q)$$

of (45). Thus (45) implies (44). Actually, we can derive (45) directly by the same method as the derivation of (44). The fact that (45) is satisfied at the initial time $t = \bar{t} = 0$ is a consequence of the relations

\begin{align*}
\log L_0 &= (\tau + 1)(M_0 - 1/2) \log q + \log \Lambda - (\tau + 1)(s - 1/2) \log q, \\
\log \bar{L}_0 &= -(\tau^{-1} + 1)(\bar{M}_0 - 1/2) + \log \Lambda - (\tau + 1)(s - 1/2) \log q.
\end{align*}

These relations are obtained by computing $W_0 \Lambda^\epsilon W_0^{-1}$ and $\bar{W}_0 \Lambda^\epsilon \bar{W}_0$ explicitly with the aid of (38) and (39) and taking the derivative at $\epsilon = 0$.

5 Integrable structures in cubic Hodge integrals

5.1 Reduced system of Lax equations

Let $\mathcal{L}$ denote the operator (31) that emerges as a consequence of the key algebraic relation (30). This operator satisfies the Lax equations (43). Actually, each equation can be rewritten in two different forms as

\begin{align*}
\frac{\partial \mathcal{L}}{\partial t_k} &= [(L^k)_{\geq 0}, \mathcal{L}] = -[(L^k)_{< 0}, \mathcal{L}], \\
\frac{\partial \mathcal{L}}{\partial \bar{t}_k} &= [(\bar{L}^{-k})_{< 0}, \mathcal{L}] = -[(\bar{L}^{-k})_{\geq 0}, \mathcal{L}].
\end{align*}
This implies that only a $\Lambda^{-\tau/(\tau+1)}$-term survives on the right hand side. Thus the Lax equations can be reduced to equations of the form

$$\frac{\partial u}{\partial t_k} = F_k, \quad \frac{\partial u}{\partial \bar{t}_k} = \bar{F}_k,$$

(48)

where

$$F_k = u \left( (L^k)_{0} - (L^k)_{0|s\rightarrow s-\tau/(\tau+1)} \right) = (L^k)_{-1} - (L^k)_{-1|s\rightarrow s+1/(\tau+1)}$$

and

$$\bar{F}_k = \left( \bar{L}^{-k} \right)_{-1|s\rightarrow s+1/(\tau+1)} - (\bar{L}^{-k})_{-1} = u \left( (\bar{L}^{-k})_{0|s\rightarrow s-\tau/(\tau+1)} - (\bar{L}^{-k})_{0} \right).$$

$(A)_n$ denotes the coefficient of $\Lambda^n$ in the difference operator $A$.

If $B_k$’s and $\bar{B}_k$’s have local expressions with respect to $u$, namely, depend on a finite number of the shifted $u$’s $u(s), u(s \pm 1), u(s \pm 2), \ldots$, so do $F_k$’s and $\bar{F}_k$’s. In such a case, (48) is a system of evolution equations for $u$ in a genuine sense. (43) becomes a Lax representation thereof. The problem of nonlocality arises, e.g., when we attempt to construct fractional powers of a difference operator directly, namely, without recourse to the use of a dressing operator (cf. Carlet’s construction of the bigraded Toda hierarchy [33]).

We shall not pursue the problem of locality, and develop our consideration rather formally. As we show below, various integrable hierarchies emerge when $\tau$ takes rational values of particular forms.

### 5.2 When $\tau$ is a positive integer

Let us consider the case where $\tau$ is equal to a positive integer $N$. In this case, $L$ is a fractional difference operator of the form

$$L = \Lambda^{1/(N+1)} - u\Lambda^{-N/(N+1)}.$$  

(49)

This is exactly the Lax operator of the Bogoyavlensky-Itoh-Narita system [13] realized on the fractional lattice $(N+1)^{-1}\mathbb{Z} \subset \mathbb{Z}$. The case of $N = 1$ amounts to the usual Volterra lattice. Thus, as conjectured and partially proved by Dubrovin et al. [10] [11] [12], integrable hierarchies of the Volterra type underlie the cubic Hodge integrals when $\tau$ is a positive integer.

The $t$-flows generated by

$$B_k = (L^{(N+1)k})_{\geq 0}, \quad k = 1, 2, \ldots,$$

can be identified with the genuine time evolutions of the Bogoyavlensky-Itoh-Narita system. Moreover, the $(N + 1)$-st power of $L$ is a difference operator of the form

$$L^{N+1} = L = (-1)^{N+1} \bar{L}^{-N} = \Lambda + p_1 + \cdots + p_{N+1}\Lambda^{-N},$$

(50)
and can be identified with the Lax operator of the bigraded Toda hierarchy of the type \((1, N)\). Since
\[
(L^{(N+1)k})_{<0} = (-1)^{(N+1)k}\tilde{B}_N,
\]
the \(t\)-flows coincide with part of the \(\bar{t}\)-flows up to sign factors. The other negative flows \(\bar{t}_k, k \not\equiv 0 \mod N\), are nonlocal.

The problem of nonlocality can be avoided if we leave the 2D Toda hierarchy and treat the Bogoyavlensky-Itoh-Narita system as a reduction of the lattice KP hierarchy (aka the discrete KP hierarchy [34])
\[
\frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \ldots
\]
The lattice KP hierarchy is simply a subset of the 2D Toda hierarchy that consists of the same Lax operator \(L\) and the Lax equations with respect to \(t\). \(B_k\)'s are defined by \(L^n\) \(\geq 0\), hence local. The tau function \(T(s, t, 0)\) restricted to \(\bar{t} = 0\), which is a generating function of the one-partition Hodge integrals, becomes a tau function of the lattice KP hierarchy.

In a limit as \(N \to \infty\), the Bogoyavlensky-Itoh-Narita hierarchy turns into a continuous version [35] [39]. The reduced Lax operator (49) is replaced therein by a difference-differential operator of the form
\[
L = \log \Lambda - u\Lambda^{-1} = \partial_s - ue^{-\partial_s}.
\]
In the same limit, the cubic Hodge integrals become the linear Hodge integrals that are related to the Hurwitz numbers of \(\mathbb{CP}^1\) (the ELSV formula) [37]. We can thus reconfirm our recent result [38] that the continuous Bogoyavlensky-Itoh hierarchy underlies the Hurwitz numbers.

Let us mention that almost the same difference-differential operator as (52) is used in Buryak and Rossi’s new Lax representation of the intermediate long wave hierarchy [39]. This fact is extremely significant, because the intermediate long wave hierarchy was proposed by Buryak as an integrable structure of the linear Hodge integrals [40] [41]. Buryak’s approach is based on the Dubrovin-Zhang theory of integrable Hamiltonian PDEs. Buryak and Ross’s Lax representation is derived along the same line.

### 5.3 When \(\tau\) is a positive rational number

Let us turn to the more general case where \(\tau\) is a positive rational number, i.e., \(\tau = b/a\) where \(a\) and \(b\) are positive coprime integers. In this case, \(L\) becomes the following generalization of (49):
\[
L = \Lambda^{a/(a+b)} - u\Lambda^{-b/(a+b)}.
\]

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This is a fractional difference operator defined on the lattice \((a + b)^{-1} \mathbb{Z} \subset \mathbb{R}\). Its \((a + b)\)-th power is a difference operator of the form

\[
\mathcal{L}^{a+b} = \mathcal{L}^a = (-1)^{a+b} \mathcal{L}^{-b} = \Lambda^a + p_1 \Lambda^{a-1} + \cdots + p_{a+b} \Lambda^{-b},
\]

and can be identified with the Lax operator of the bigraded Toda hierarchy of the type \((a, b)\).

Thus, just like the relation between the Volterra lattice and the Toda lattice \([9]\), the generalized Bogoyavlensky-Itoh-Narita system sits over the bigraded Toda hierarchy. The flows of these systems are generated by the common generators \(B_k, \bar{B}_k, \ k = 1, 2, \ldots\) (apart from the problem of locality).

Let us mention that these reduced systems exhibit the duality under the exchange

\[
a \leftrightarrow b, \quad t \leftrightarrow \bar{t}, \quad L \leftrightarrow \bar{L}.
\]

This duality stems from the symmetry of the Hodge integrals \(G_{g\mu}(\tau)\) under the exchange

\[
\tau \leftrightarrow \tau^{-1}, \quad \mu \leftrightarrow \bar{\mu}.
\]

### 5.4 When \(\tau\) is a negative rational number

The situation changes qualitatively when \(\tau\) is a negative rational number \(-b/a\). (Just like the previous case, \(a\) and \(b\) are assumed to be positive coprime integers.) This case is divided to two cases, namely, \(a > b\) and \(a < b\). Since these cases can be interchanged by the aforementioned duality, let us focus our consideration on the first case.

If \(\tau = -b/a\) and \(a > b\), \(\mathcal{L}\) comprises only positive powers of \(\Lambda\):

\[
\mathcal{L} = \Lambda^{a/(a-b)} - u \Lambda^{b/(a-b)}.
\]

Its \((a - b)\)-th power, too, contains only positive powers:

\[
\mathcal{L}^{a-b} = \mathcal{L}^a = (-1)^{a-b} \mathcal{L}^{-b} = \Lambda^a + p_1 \Lambda^{a-1} + \cdots + p_{a-b} \Lambda^b.
\]

Consequently, every \(a\)-th flow in the \(t\)-space are stationary:

\[
\frac{\partial \mathcal{L}}{\partial t_ka} = [B_{ka}, \mathcal{L}] = [\mathcal{L}^{k(a-b)}, \mathcal{L}] = 0, \quad k = 1, 2, \ldots.
\]

This is reminiscent of a periodic reduction of the 2D Toda hierarchy \([30]\), but there is no periodicity in the present setting (as far as \(u \neq 0\)). The dressing operator \(W\) is \(p\)-periodic, i.e., \([W, \Lambda^p] = 0\), if and only if \(L^p = \Lambda^p\). The same equivalence holds for \(\bar{W}\) and \(\bar{L}\).
In a sense, this case may be thought of as the bigraded Toda hierarchy of “the type \((a, -b)\)”. It is, however, also possible to forget \(\bar{L}\) and \(\bar{t}\) and to consider the reduced system within the lattice KP hierarchy (51).

(56) shows that the reduced system is a lattice version of the Gelfand-Dickey hierarchy [10]. Speaking more precisely, this is slightly different from the usual lattice Gelfand-Dickey hierarchy in the sense that the \(i\)-th powers of \(\Lambda\) for \(i < b\) are missing in (56). The usual \(a\)-th Gelfand-Dickey reduction of the lattice KP hierarchy is characterized by the condition that the \(a\)-th power of \(L\) contains no negative powers of \(\Lambda\):

\[
L^a = \Lambda^a + p_1 \Lambda^{a-1} + \cdots + p_a.
\]

Truncating the \(\Lambda^i\)-terms for \(i < b\) is consistent with the Lax equations of the lattice KP hierarchy, hence yields a further reduction of the system. Detailed properties of these reduced systems of the lattice KP hierarchy remain to be studied. We shall return to this issue elsewhere.

Lastly, let us examine the discrete series

\[
\tau = -N/(N + 1), \quad a = N + 1, \quad b = N, \quad N = 1, 2, \ldots.
\]

The one- and two-partition Hodge integrals of this type are hidden in our recent work on the three-partition Hodge integrals [42]. This explains why we encountered the Gelfand-Dickey hierarchy in the usual sense (namely, a reduction of the usual KP hierarchy) therein. It is well known [34] that the Gelfand-Dickey reduction of the lattice KP hierarchy in the \(s\)-space is accompanied by the usual Gelfand-Dickey hierarchy in the \(t\)-space.

For these values of \(\tau\), \(L\) comprises two positive integral powers of \(\Lambda\):

\[
\mathcal{L} = \Lambda^{N+1} - u \Lambda^N.
\]

Since \(\mathcal{L} = L^{N+1} = -\bar{L}^N\), this case is the closest to a periodic reduction \((L^{N+1} = \bar{L}^{N+1} = \Lambda^{N+1})\) of the 2D Toda hierarchy. In many aspects, this case is situated at the opposite end of the case where \(\tau\) is a positive integer. Unlike the Bogoyavlensky-Itoh-Narita system, the main degrees of freedom is contained in \(L\) and \(\bar{L}\), and \(u\) is rather an auxiliary field.

\footnote{Frenkel’s formulation [16] uses the \(q\)-shift operator \(\Lambda = q^{x\partial_x}\) rather than the shift operator \(\Lambda = e^{x\partial_x}\), but this is not an essential difference. It should be stressed that the discrete KdV hierarchy in the sense of Dubrovin et al. [8, 11] is distinct from the lattice KdV hierarchy in the present context. The discrete KdV hierarchy considered therein is an alias of the Volterra hierarchy. Its Lax operator comprises both positive and negative powers of \(\Lambda\).}
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