Modulus $p^2$ congruences involving harmonic numbers

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Abstract: The harmonic number $H_k = \sum_{j=1}^{k} 1/j (k = 1, 2, 3 \ldots)$ play an important role in mathematics. Let $p > 3$ be a prime. In this paper, we establish a number of congruences with the form $\sum_{k=1}^{p-1} k^m H_k^n (mod p^2)$ for $m = 1, 2 \ldots, p - 2$ and $n = 1, 2, 3$.

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1 Introduction

For $\alpha \in \mathbb{N}$, the generalized harmonic numbers are defined by

$$H_0^{(\alpha)} = 0 \quad \text{and} \quad H_n^{(\alpha)} = \sum_{i=1}^{n} \frac{1}{i^\alpha}, \quad \text{for} \ n \in \mathbb{N},$$

when $\alpha = 1$, they reduce to the well-known harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad \text{for} \ n \in \mathbb{N}.$$
In 1862, Wolstenholme proved that if $p > 3$ is a prime, then
\[ H_p - 1 \equiv 0 \pmod{p^2} \quad \text{and} \quad H_p^{(2)} - 1 \equiv 0 \pmod{p}. \] (1.1)

There are many results in related to the Wolstenholme theorem (see, e.g., [1], [2], [3], [8], [9], [10]). Recently Z.-W. Sun gave
\[ H_p - k \equiv H_k - 1 \pmod{p} \]
and
\[ (-1)^k \binom{p - 1}{k} \equiv 1 - pH_k + \frac{p^2}{2} \left( H_k^2 - H_k^{(2)} \right) \pmod{p^3} \]
for every $k = 1, 2, \ldots, p - 1$. Using these congruences, Z.-W. Sun obtained a series of modulus $p$ congruences involving harmonic numbers. For example,
\[ \sum_{k=1}^{p-1} k^2 H_k^2 \equiv -\frac{4}{9} \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} H_k^2 \equiv 2p - 2 \pmod{p^2}. \]

In this paper, we investigate the properties of $\sum_{k=1}^{p-1} k^m H_k^n \pmod{p^2}$ $(m = 1, 2, \ldots, p-2, n = 1, 2, 3)$ and establish a series of modulus $p^2$ congruences involving harmonic numbers, some of which are generalizations of partial results in [8]. For example, we establish the congruences:
\[ \sum_{k=1}^{p-1} k^2 H_k^2 \equiv \frac{79}{108} p - \frac{4}{9} \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} kH_k^3 \equiv \frac{27}{8} p - \frac{1}{6} pB_{p-3} - 3 \pmod{p^2}, \] (1.2)
where $B_0, B_1, B_2, \cdots$, are Bernoulli numbers given by
\[ B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0(n = 1, 2, 3, \cdots). \]
2 Lemmas

In this paper, we set
\[ S(m) = \sum_{r=0}^{m} \binom{m+1}{r} B_r B_{m-r}. \]

In this section, we first state some basic facts which will be used very often. For any prime \( p \) and integer \( n \), provided \( p - 1 \nmid n \), it is well-known that
\[ H_p(n)^{p-1} = \sum_{k=1}^{p-1} \frac{1}{k^n} \equiv 0 \pmod{p}, \]
which implies
\[ H_p(n)^{p-1} - \sum_{j=1}^{k-1} \frac{1}{(p-j)^n} \equiv (-1)^{n+1} H_k(n) \pmod{p}. \]

Lemma 2.1 Let \( m, p \) be positive integers. Then
\[
\sum_{k=1}^{p-1} k^m H_k = \frac{H_{p-1}}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r p^{m+1-r} - (p-1)B_m \\
- \frac{1}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-r} \binom{m+1}{r} \binom{m+1-r}{\lambda} B_r B_{\lambda} p^{m+1-r-\lambda}. \tag{2.1}
\]

Proof. For \( m, p \in \mathbb{N} \), we have
\[
\sum_{k=1}^{p-1} k^m H_k = \sum_{k=1}^{p-1} k^m \sum_{j=1}^{k} \frac{1}{j} = \sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=j}^{p-1} k^m = \sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=1}^{p-1} k^m - \sum_{s=1}^{j-1} k_s^m.
\]
It is well-known that
\[
\sum_{k=1}^{p-1} k^m = \frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r p^{m+1-r} \quad \text{for any } m, p \in \mathbb{Z}^+. \tag{2.2}
\]
Thus
\[ \sum_{k=1}^{p-1} k^m H_k = \frac{H_{p-1}}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r p^{m+1-r} - \frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p-1} j^{m-r} \]
\[ = \frac{H_{p-1}}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r p^{m+1-r} - (p-1) B_m - \frac{1}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r \sum_{j=1}^{p-1} j^{m-r}. \]

Using (2.2) again, we finally obtain (2.1).

**Lemma 2.2** Let \( m > 0 \) be an odd integer and \( m \neq 3 \). Then
\[ \frac{2}{m+1} \sum_{r=0}^{m} \sum_{\lambda=0}^{m-r} \frac{\binom{m+1-r}{r} \binom{m+1-r}{\lambda}}{m+1-r} B_r B_{\lambda} B_{m-r-\lambda} + \frac{1}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} \]
\[ = - \frac{m+2}{2m} \sum_{r=0}^{m-1} \binom{m}{r} B_r B_{m-1-r} + \frac{1}{4} \sum_{r=0}^{m-1} B_r B_{m-1-r}. \quad (2.3) \]

**Proof.** It is clear for the case of \( m = 1 \). When \( m > 3 \), set
\[ \Sigma = \sum_{r=0}^{m} \sum_{\lambda=0}^{m-r} \frac{\binom{m+1-r}{r} \binom{m+1-r}{\lambda}}{m+1-r} B_r B_{\lambda} B_{m-r-\lambda} \]
\[ = \sum_{r=0}^{m} \sum_{\lambda=0}^{m-r} \frac{\binom{m+1-r}{r} \binom{m+1-r}{\lambda}}{m+1-r} B_r B_{\lambda} B_{m-r-\lambda} \]
\[ + \sum_{r=0}^{m} \sum_{\lambda=0}^{m-r} \frac{\binom{m+1-r}{r} \binom{m+1-r}{\lambda}}{m+1-r} B_r B_{\lambda} B_{m-r-\lambda} \]
\[ + \sum_{r=0}^{m} \sum_{\lambda=0}^{m-r} \frac{\binom{m+1-r}{r} \binom{m+1-r}{\lambda}}{m+1-r} B_r B_{\lambda} B_{m-r-\lambda}. \]
Recall that $B_m = 0$ for $m = 3, 5, 7, \ldots$. Then

$$
\Sigma = \sum_{\lambda=0}^{m-1} \frac{m+1}{m} \binom{m}{\lambda} B_1 B_{\lambda} B_{m-1-\lambda} + \sum_{r=0}^{m-1} \binom{m+1}{r} B_1 B_r B_{m-1-r}
$$

$$
+ \sum_{r=0}^{m-1} \frac{m-r}{2} \binom{m+1}{r} B_1 B_r B_{m-1-r}.
$$

When $m > 3$ and $m, r$ are odd integers, we have $B_r B_{m-1-r} = 0$. Thus

$$
\Sigma = \sum_{r=0}^{m-1} \frac{m+1}{m} \binom{m}{r} B_1 B_r B_{m-1-r} + \sum_{r=0}^{m-1} \binom{m+1}{r} B_1 B_r B_{m-1-r}
$$

$$
+ \sum_{r=0}^{m-1} \frac{m-r}{2} \binom{m+1}{r} B_1 B_r B_{m-1-r}.
$$

(2.4)

Therefore, we get

$$
\frac{2}{m+1} \Sigma + \frac{1}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r}
$$

$$
= -\frac{m+2}{2m} \sum_{r=0}^{m-1} \binom{m}{r} B_r B_{m-1-r} + \frac{1}{2(m+1)} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r},
$$

(2.5)

with the help of (2.4). Observe that

$$
\sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} = \sum_{r=2}^{m-3} \binom{m+1}{r} B_r B_{m-1-r} + \frac{m^2 + m + 2}{2} B_{m-1},
$$

since $m$ is an odd integer. Recall that for $n$ is an even integer(cf. [4] (4.12))

$$
(n+2) \sum_{k=2}^{n-2} B_k B_{n-k} = 2 \sum_{k=2}^{n-2} \binom{n+2}{k} B_k B_{n-k} + n(n+1)B_n,
$$

which implies

$$
\sum_{r=2}^{m-3} \binom{m+1}{r} B_r B_{m-1-r} = \frac{m+1}{2} \sum_{r=2}^{m-3} B_r B_{m-1-r} - \frac{m(m-1)}{2} B_{m-1}.
$$
Hence
\[ \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} = \frac{m+1}{2} \sum_{r=0}^{m-1} B_r B_{m-1-r}. \tag{2.6} \]

In view of (2.5) and (2.6), we finally get
\[ \frac{2}{m+1} \sum + \frac{1}{m+1} m \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} \]
\[ = -\frac{m+2}{2m} \sum_{r=0}^{m-1} \binom{m}{r} B_r B_{m-1-r} + \frac{1}{4} \sum_{r=0}^{m-1} B_r B_{m-1-r}. \]

The proof of (2.3) is completed.

**Lemma 2.3** Let \( p \) be a prime and \( 0 < m < p - 1 \) be an even integer. Then
\[ \sum_{k=1}^{p-1} \frac{H^2_k}{k^m} \equiv -\sum_{j=0}^{p-1-m} B_j B_{p-1-m-j} - \sum_{j=p-m}^{p-2} B_j B_{2p-2-m-j} \pmod{p}. \tag{2.7} \]

**Proof.** Observe that
\[ \sum_{k=1}^{p-1} \frac{H^2_k}{k^m} = \sum_{k=1}^{p-1} \frac{H^2_{p-k}}{(p-k)^m} = \sum_{k=1}^{p-1} \frac{H^2_{k-1}}{k^m} = \sum_{k=1}^{p-1} \frac{\left( \sum_{j=1}^{p-1-j} j^2 \right)^2}{k^m} \pmod{p}, \]
with the help of Fermat’s little theorem. In view of (2.2), we obtain
\[ \sum_{k=1}^{p-1} \frac{H^2_k}{k^m} \equiv \sum_{k=1}^{p-1} \frac{1}{k^m} \left( \sum_{j=0}^{p-2} \binom{p-1}{j} B_j k^{p-1-j} \right)^2 \equiv \sum_{k=1}^{p-1} \frac{1}{k^m} \left( \sum_{j=0}^{p-2} B_j k^{p-1-j} \right)^2 \pmod{p}, \]
since \( (p-1) B_j \equiv B_j \pmod{p} \). Thus,
\[ \sum_{k=1}^{p-1} \frac{H^2_k}{k^m} \equiv \sum_{k=1}^{p-1} \frac{1}{k^m} \left( \sum_{j=0}^{p-2} B_j k^{p-1-j} \right) \left( \sum_{i=0}^{p-2} B_{p-2-i} k^{p-1-i} \right) \]
\[ = \sum_{k=1}^{p-1} \frac{1}{k^m} \sum_{j=0}^{p-2} B_j B_{p-2-j-1} k^{p-1-j}\]
= \sum_{i,j=0}^{p-2} B_j B_{p-2-i} \sum_{k=1}^{p-1} k^{p-j-m+i} (\text{mod } p).

It is well-known that for each $t \in \mathbb{Z}$, we have
\[
\sum_{k=1}^{p-1} k^t = \begin{cases} 
-1 \pmod{p} & \text{if } p-1 \mid t, \\
0 \pmod{p} & \text{otherwise.} 
\end{cases} 
\tag{2.8}
\]
(see, e.g., [6, p.235].) Thus, we arrive at
\[
\sum_{k=1}^{p-1} B_{k^m} \equiv - \sum_{j=0}^{p-1-m} B_j B_{p-1-m-j} - \sum_{j=p-m}^{p-2} B_j B_{2p-2-m-j} \pmod{p}.
\]

3 Main results

Theorem 3.1 Let $p > 3$ be a prime and $0 < m < p - 1$ be an integer. Then
\[
\sum_{k=1}^{p-1} k^m H_k \equiv B_m - \frac{p}{m+1} S(m) \pmod{p^2}. \tag{3.1}
\]
Proof. In view of (2.1), we have
\[
\sum_{k=1}^{p-1} k^m H_k \equiv (1-p)B_m - \frac{1}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-r} \frac{(m+1)(m+1-r)(\lambda+1)}{m+1-r} B_r B_{m-r} B_{m-r} \pmod{p^2},
\]
since $H_{p-1} \equiv 0 \pmod{p^2}$. It is clear that
\[
\sum_{k=1}^{p-1} k^m H_k \equiv (1-p)B_m - \frac{p}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-r} \equiv B_m - \frac{p}{m+1} S(m) \pmod{p^2}.
\]
The proof of the theorem is completed.

Theorem 3.2 Let $p > 3$ be a prime and $0 < m < p - 1$ be an integer. Then
\[
\sum_{k=1}^{p-1} k^2 H_k^2 \equiv \frac{79}{108} p - \frac{4}{9} \pmod{p^2} . \tag{3.2}
\]
\[\sum_{k=1}^{p-1} k^3 H_k^2 = -\frac{59}{144}p + \frac{1}{6} (\text{mod } p^2). \] (3.3)

and

\[\sum_{k=1}^{p-1} k^m H_k^2 \equiv \begin{cases} B_{m-1} + p\mu_m \pmod{p^2} & \text{if } 2 \nmid m \text{ and } m \neq 3 \\
\nu_m + p\lambda_m \pmod{p^2} & \text{if } 2 \mid m \text{ and } m \neq 2 \end{cases}, \] (3.4)

where

\[
\mu_m = -\frac{m+2}{2m} S(m-1) + \frac{1}{4} \sum_{r=1}^{m-1} B_r B_{m-1-r}, \quad \nu_m = -\frac{2}{m+1} S(m),
\]

\[
\lambda_m = \frac{2}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r S(m-r) - \frac{m^2 - m + 6}{12} B_{m-2}.
\]

**Proof.** When \(m \in \{1, 2, \ldots, p-1\}\), we have

\[
\sum_{k=1}^{p-1} k^m H_k^2 = \sum_{k=1}^{p-1} k^m H_k \sum_{j=1}^{k} \frac{1}{j} = \sum_{j=1}^{p-1} \sum_{k=j}^{p-1} k^m H_k
\]

\[
= \sum_{j=1}^{p-1} \frac{1}{j} \left( \sum_{k=1}^{p-1} k^m H_k - \sum_{s=1}^{j-1} s^m H_s \right)
\]

\[
= -\sum_{j=1}^{p-1} \sum_{s=1}^{j-1} s^m H_s \pmod{p^2},
\]

since \(H_{p-1} \equiv 0 \pmod{p^2}\). In view of (2.1), we have

\[
\sum_{k=1}^{p-1} k^m H_k^2 = -\sum_{j=1}^{p-1} \frac{1}{j} \left\{ \frac{H_{j\lambda} - \sum_{r=1}^{m} \binom{m+1}{r} B_r H_{j\lambda}^{m+1-r}}{m+1} - (j-1) B_m \right. \\
- \frac{1}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-r} \binom{m+1-r}{\lambda} B_r B_{j\lambda} H_{j\lambda}^{m+1-r-\lambda} \left. \right\}
\]

\[
= -\frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p-1} j^{m-r} H_{j-1} + (p-1) B_m - H_{p-1} B_m
\]

\[
+ \frac{1}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-r} \binom{m+1-r}{\lambda} B_r B_{j\lambda} \sum_{j=1}^{p-1} j^{m-r-\lambda} \pmod{p^2}.
\]

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With the help of (2.2), we can get

\[
p^{-1} \sum_{k=1}^{p^m} H_k^2 \equiv -\frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p^{-1}} j^{m-r} H_{j-1} + \frac{p-1}{m+1} S(m)
\]

\[+ \frac{p}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-1-r} \binom{m+1}{r} \binom{m+1-r}{\lambda} B_r B_{m-r-\lambda} \pmod{p^2}.
\]

Note that

\[
p^{-1} \sum_{j=1}^{p^{-1}} j^{m-r} H_{j-1} = \sum_{j=1}^{p^{-1}} j^{m-r} H_j - \sum_{j=1}^{p^{-1}} j^{m-1-r}.
\]

In view of (3.1), we have

\[
p^{-1} \sum_{k=1}^{p^m} H_k^2 \equiv -\frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \left( B_{m-r} - \frac{p}{m+1-r} S(m-r) \right)
\]

\[+ \frac{p}{m+1} S(m) + \frac{p}{m+1} \sum_{r=0}^{m-1} \sum_{\lambda=0}^{m-1-r} \binom{m+1}{r} \binom{m+1-r}{\lambda} B_r B_{m-r-\lambda}
\]

\[+ \frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p^{-1}} j^{m-1-r}
\]

\[= -\frac{2}{m+1} S(m) + \frac{2p}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r S(m-r)
\]

\[+ \frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p^{-1}} j^{m-1-r} \pmod{p^2}.
\]

Observe that

\[
\frac{1}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_r \sum_{j=1}^{p^{-1}} j^{m-1-r}
\]

\[= H_{p-1} B_m + \frac{(p-1)m}{2} B_{m-1} + \frac{1}{m+1} \sum_{r=0}^{m-2} \binom{m+1}{r} B_r \sum_{j=1}^{p^{-1}} j^{m-1-r}
\]

\[\equiv \frac{(p-1)m}{2} B_{m-1} + \frac{p}{m+1} \sum_{r=0}^{m-2} \binom{m+1}{r} B_r B_{m-1-r}
\]

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\[
- \frac{m}{2} B_{m-1} + \frac{p}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} \pmod{p^2}.
\]

Therefore
\[
\sum_{k=1}^{p-1} k^m H_k^2 \equiv \frac{2p}{m+1} \sum_{r=0}^{m} \left( \binom{m+1}{r} B_r S(m-r) + \frac{p}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} \right) - \frac{2}{m+1} S(m) - \frac{m}{2} B_{m-1} \pmod{p^2}.
\] (3.5)

Note that \( S(1) = -3/2, S(2) = 17/12, S(3) = -5/6 \). Taking \( m = 2, 3 \) in (3.5), we can obtain (3.2) and (3.3) respectively.

When \( m > 0 \) is an odd integer and \( m \neq 3 \), we have
\[
- \frac{2}{m+1} S(m) = - \frac{2}{m+1} \sum_{r=0}^{m} \left( \binom{m+1}{r} B_r B_{m-1-r} = B_{m-1} + \frac{m}{2} B_{m-1}. \right. (3.6)
\]

Combining (2.3), (3.5) and (3.6), we obtain
\[
\sum_{k=1}^{p-1} k^m H_k^2 \equiv B_{m-1} - p \left( \frac{m+2}{2m} S(m-1) - \frac{1}{4} \sum_{r=0}^{m-1} B_r B_{m-1-r} \right) \pmod{p^2}.
\] (3.7)

When \( m > 2 \) is an even integer, we have \( B_{m-1} = 0 \) and
\[
\frac{p}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} B_r B_{m-1-r} = - \frac{m^2 - m + 6}{12} p B_{m-2}.
\]

Therefore,
\[
\sum_{k=1}^{p-1} k^m H_k^2 \equiv \nu_m + p \lambda_m \pmod{p^2},
\]
where
\[
\nu_m = - \frac{2}{m+1} S(m),
\]
\[
\lambda_m = \frac{2}{m+1} \sum_{r=0}^{m} \left( \binom{m+1}{r} B_r S(m-r) - \frac{m^2 - m + 6}{12} B_{m-2} \right).
\]

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We are done.

**Remark 1** The congruences (3.2) and (3.3) are the generalized forms of [8 (1.2)] and the third congruences of [8 Corollary 1.1] respectively.

Taking \( m = 1 \) in (3.7), we have

**Corollary 3.3** Let \( p > 3 \) be a prime. Then
\[
\sum_{k=1}^{p-1} kH_k^2 \equiv -\frac{5}{4}p + 1 \pmod{p^2}.
\]
(3.8)

**Remark 2** The congruences (3.8) is the generalized form of the first congruences of [8 Corollary 1.1].

Noticing \( S(4) = 7/90 \) and taking \( m = 5 \) in (3.7), we have

**Corollary 3.4** Let \( p > 5 \) be a prime. Then
\[
\sum_{k=1}^{p-1} k^5H_k^2 \equiv -\frac{77}{1200}p - \frac{1}{30} \pmod{p^2}.
\]
(3.9)

**Corollary 3.5** Let \( p > 3 \) be a prime and \( 0 < m < p - 1 \) be an odd integer. Then
\[
\sum_{k=1}^{p-1} k^mH_k^2 \equiv B_{m-1}(\pmod{p}).
\]

**Proof.** In view of (3.5), we have
\[
\sum_{j=1}^{p-1} k^mH_k^2 \equiv -\frac{2}{m+1} \sum_{r=0}^{m} \binom{m+1}{r} B_rB_{m-r} - \frac{m}{2}B_{m-1} = B_{m-1}(\pmod{p}),
\]

since \( m \) is an odd integer.

**Remark 3** Taking \( m = 1 \) and \( m = 3 \), we also can get the first and third congruences of [8 Corollary 1.1] respectively.
**Corollary 3.6** Let $p > 3$ be a prime and $0 < m < p - 1$ be an odd integer. Then

$$
\sum_{k=1}^{p-1} \frac{H_k^2}{k^m} \equiv B_{p-2-m} \pmod{p}.
$$

(3.10)

**Proof.** Taking $m \rightarrow p - 1 - m$ in Corollary 3.5, we have

$$
\sum_{k=1}^{p-1} k^{p-1-m} H_k^2 \equiv B_{p-2-m} \pmod{p},
$$

by Fermat’s little theorem, we obtain

$$
\sum_{k=1}^{p-1} \frac{H_k^2}{k^m} \equiv B_{p-2-m} \pmod{p}
$$

as desired.

Taking $m = 4$ in (3.4), we obtain

**Corollary 3.7** Let $p > 5$ be a prime. Then

$$
\sum_{k=1}^{p-1} k^4 H_k^2 \equiv \frac{5743}{27000} p - \frac{7}{229} \pmod{p^2}.
$$

(3.11)

**Corollary 3.8** Let $p > 5$ be a prime and $0 < m \leq p - 5$ is an even integer. Then

$$
\sum_{k=1}^{p-1} k^{p-m} H_k^2 \equiv B_{p-1-m} - p \left( \frac{m-1}{4} \sum_{k=1}^{p-1} \frac{H_k^2}{k^m} + \frac{1}{4} \sum_{r=p-m}^{p-2} B_r B_{p-2-m-r} \right) \pmod{p^2}.
$$

(3.12)

**Proof.** In view of (3.3), we have

$$
\sum_{k=1}^{p-1} k^{p-m} H_k^2 \equiv B_{p-1-m} + p \left( \frac{-m + 2}{2(p-m)} S(p-1-m) + \frac{1}{4} \sum_{r=0}^{p-1-m} B_r B_{p-1-m-r} \right).
$$

$$
\equiv B_{p-1-m} + p \left( \frac{-m + 2}{2m} S(p-1-m) + \frac{1}{4} \sum_{r=0}^{p-1-m} B_r B_{p-1-m-r} \right) \pmod{p^2}
$$
and
\[ \sum_{k=1}^{p-1} \frac{H_k^2}{k^m} \equiv \sum_{k=1}^{p-1} k^{p-1-m} H_k^2 \equiv -\frac{2}{p-m} S(p-1-m) \equiv \frac{2}{m} S(p-1-m) \pmod{p}. \]

Combining (2.7) and the above, we can obtain
\[ \sum_{k=1}^{p-1} k^{p-1-m} H_k^2 \equiv B_{p-1-m} - p \left( \frac{m-1}{4} \sum_{k=1}^{p-1} \frac{H_k^2}{k^m} + \frac{1}{4} \sum_{r=p-m}^{p-2} B_r B_{2p-2m-r} \right) \pmod{p^2}, \]
we are done.

Taking \( m = 2 \) in (3.12), we have

**Corollary 3.9** Let \( p > 5 \) be a prime. Then
\[ \sum_{k=1}^{p-1} k^{p-2} H_k^2 \equiv B_{p-3} \pmod{p^2}. \]

**Proof.** Observe that
\[ \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}. \]
(cf. [8, (1.5)].) Thus, we obtain (3.13) from (3.12).

Taking \( m = 4 \) in (3.12), we have

**Corollary 3.10** Let \( p > 7 \) be a prime. Then
\[ \sum_{k=1}^{p-1} k^{p-4} H_k^2 \equiv B_{p-5} \pmod{p^2}. \]

**Proof.** Observe that
\[ \sum_{1 \leq i \leq j \leq k \leq p-1} \frac{1}{ijk^j} \equiv \frac{1}{3} B_{p-3}^2 \pmod{p}, \text{ (cf. [5, Theorem 7.2])} \]
and

\[ \sum_{1 \leq i \leq j \leq k \leq p-1} \frac{1}{ij^4k} = \sum_{j=1}^{p-1} \frac{H_j(H_{p-1} - H_{j-1})}{j^4} \equiv -\sum_{j=1}^{p-1} \frac{H_jH_{j-1}}{j^4} \]
\[ \equiv -\sum_{j=1}^{p-1} \frac{H_j^2}{j^4} + \sum_{j=1}^{p-1} \frac{H_j}{j^3} \pmod{p}. \]

With the help of Fermat’s little theorem and (3.1), we get

\[ \sum_{k=1}^{p-1} \frac{H_k}{k^3} \equiv \sum_{k=1}^{p-1} k^{p-6}H_k \equiv B_{p-6} = 0 \pmod{p}. \]

Combining (3.12) and the above, we finally obtain

\[ \sum_{k=1}^{p-1} k^{p-4}H_k^2 \equiv B_{p-5} \pmod{p^2}. \]

The proof of Corollary 3.10 is completed.

4 Further results

It is not difficult to find the following recurrence relation to \( \sum_{k=1}^{p-1} k^mH_k^3 \pmod{p^2} \).

For \( m = 1, 2, \ldots, p-2 \), set \( a_m \equiv \sum_{k=1}^{p-1} k^mH_k^3 \pmod{p^2} \). Then

\[ a_{m-1} \equiv \frac{1}{m} \left( -\sum_{i=2}^{m} \binom{m}{i} a_{m-i} - 3 \sum_{k=1}^{p-1} k^{m-1}H_k^2 + 3 \sum_{k=1}^{p-1} k^{m-2}H_k - \sum_{k=1}^{p-1} k^{m-3} \right) \pmod{p^2}, \quad (4.1) \]

which can be deduced by

\[ \sum_{k=1}^{p-1} (k+1)^mH_k^n \equiv \sum_{k=1}^{p-1} k^mH_k^n \pmod{p^2}, (n \in \mathbb{N}) \quad (4.2) \]

Using the Theorem 3.1, Theorem 3.2 and (4.1), we can calculate the congruences \( \sum_{k=1}^{p-1} k^mH_k^3 \pmod{p^2} \) for \( m = 0, 1, 2, \ldots, p-2 \). As the examples, we show the cases of \( m = 0, 1, 2, 3 \).
Theorem 4.1 Let $p > 3$ be a prime. Then

\[ \sum_{k=1}^{p-1} H_k^3 \equiv \frac{1}{3} p B_{p-3} - 6p + 6 \pmod{p^2}, \quad (4.3) \]
\[ \sum_{k=1}^{p-1} k H_k^3 \equiv \frac{27}{8} p - \frac{1}{6} p B_{p-3} - 3 \pmod{p^2}, \quad (4.4) \]
\[ \sum_{k=1}^{p-1} k^2 H_k^3 \equiv -\frac{365}{216} p + \frac{1}{18} p B_{p-3} + \frac{23}{18} \pmod{p^2}, \quad (4.5) \]
\[ \sum_{k=1}^{p-1} k^3 H_k^3 \equiv \frac{425}{576} p - \frac{5}{12} \pmod{p^2}. \quad (4.6) \]

Proof of (4.3). Taking $m = 1$ in (4.1), we obtain

\[ \sum_{k=1}^{p-1} H_k^3 \equiv -3 \sum_{k=1}^{p-1} H_k^2 + 3 \sum_{k=1}^{p-1} H_k - H_p^{(2)} \pmod{p^2}. \]

In view of (4.2), we can get

\[ \sum_{k=1}^{p-1} H_k^2 \equiv -2 \sum_{k=1}^{p-1} H_k \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} H_k \equiv 1 - p \pmod{p^2}. \quad (4.7) \]

It is easy to find that

\[ H_{p-1}^2 = 2 \sum_{k=1}^{p-1} \frac{H_k}{k} - H_p^{(2)}. \]

Thus,

\[ \sum_{k=1}^{p-1} \frac{H_k}{k} = \frac{1}{2} \left( H_{p-1}^2 + H_p^{(2)} \right) \equiv \frac{1}{2} H_p^{(2)} \pmod{p^2}. \]

Combining the above, we obtain

\[ \sum_{k=1}^{p-1} H_k^3 \equiv \frac{1}{3} H_{p-1}^2 - 6p + 6 \equiv \frac{1}{3} p B_{p-3} - 6p + 6 \pmod{p^2}, \]

since $H_{p-1}^2 \equiv 2/3 p B_{p-3} \pmod{p^2}$ by ([7 Corollaries 5.1]).
Proof of (4.4). Taking \( m = 2 \) in (4.1), we obtain
\[
\sum_{k=1}^{p-1} kH_k^3 \equiv \frac{1}{2} \left( -\sum_{k=1}^{p-1} H_k^3 - 3 \sum_{k=1}^{p-1} kH_k^2 + 3 \sum_{k=1}^{p-1} H_k - H_{p-1} \right) \pmod{p^2}.
\]
In view of Corollary 3.3 and (4.3), we have
\[
\sum_{k=1}^{p-1} kH_k^3 \equiv \frac{27}{8} - \frac{1}{6} pB_{p-3} - 3 \pmod{p^2},
\]
from which we obtain
\[
\sum_{k=1}^{p-1} kH_k \equiv 1 - p \pmod{p^2}.
\]

Remark 4 The congruences (4.4) is the generalized form of the second congruences of [8, Corollary 1.1].

Proof of (4.5). Taking \( m = 3 \) in (4.1), we obtain
\[
\sum_{k=1}^{p-1} k^2H_k^3 \equiv \frac{1}{3} \left( -3 \sum_{k=1}^{p-1} kH_k^3 - \sum_{k=1}^{p-1} H_k^3 - 3 \sum_{k=1}^{p-1} k^2H_k^2 + 3 \sum_{k=1}^{p-1} kH_k - p + 1 \right) \pmod{p^2}.
\]
In view of (3.1), we can obtain
\[
\sum_{k=1}^{p-1} kH_k \equiv \frac{3}{4} - \frac{1}{2} \pmod{p^2}.
\]
Hence
\[
\sum_{k=1}^{p-1} k^2H_k^3 \equiv \frac{365}{216} p + \frac{1}{18} pB_{p-3} + \frac{23}{18} \pmod{p^2},
\]
with the help of (3.2), (4.3) and (4.4).

Proof of (4.6). Taking \( m = 4 \) in (4.1), we obtain
\[
\sum_{k=1}^{p-1} k^3H_k^3 \equiv \frac{1}{4} \left( -6 \sum_{k=1}^{p-1} k^2H_k^3 - 4 \sum_{k=1}^{p-1} kH_k^3 - \sum_{k=1}^{p-1} H_k^3 - 3 \sum_{k=1}^{p-1} k^3H_k^2 + 3 \sum_{k=1}^{p-1} k^2H_k - \sum_{k=1}^{p-1} k \right) \pmod{p^2}.
\]
In view of (3.1), we can obtain
\[
\sum_{k=1}^{p-1} k^2 H_k \equiv -\frac{17}{36} p + \frac{1}{6} \pmod{p^2}.
\]
Hence
\[
\sum_{k=1}^{p-1} k^3 H_k^3 \equiv \frac{425}{576} p - \frac{5}{12} \pmod{p^2},
\]
with the help of (3.3), (4.3), (4.4) and (4.5).

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