Regularity of the inverse mapping in Banach function spaces

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Funding information
European Union’s Horizon 2020, Grant/Award Number: 847693

Abstract
We study the regularity properties of the inverse of a bilipschitz mapping $f$ belonging to $W^m X_{loc}$, where $X$ is an arbitrary Banach function space. Namely, we prove that the inverse mapping $f^{-1}$ is also in $W^m X_{loc}$. Furthermore, the paper shows that the class of bilipschitz mappings in $W^m X_{loc}$ is closed with respect to composition and multiplication.

KEYWORDS
Banach function space, bilipschitz mapping, inverse mapping theorem

MSC (2020)
Primary: 46E30, 46E35; Secondary: 26B10

1 | INTRODUCTION

Sufficient conditions, concerning the derivatives, for a $C^k$-smooth mapping in $\mathbb{R}^n$ to be invertible, are provided by the well-known inverse function theorem. This subject has attracted the attention of many researchers due to a large number of relevant applications. There are two main lines of research. The first one, motivated by control theory, deals with the theorem for mappings in general metric spaces regarding a variational or alternative formalism, that provides a better fit to practical problems. For more information on this topic, we refer the interested reader to the research of Frankowska [15], see also [11, 17, 30], as well as many others not explicitly mentioned here. The second question appears in connection with PDEs and goes back to Arnold’s paper on hydrodynamics [3]. The technique proposed there rests on an analysis of geodesics belonging to the group of volume-preserving diffeomorphisms of an (orientated) Riemannian manifold. It requires an investigation of the regularity properties other than $C^k$ of the inverse mapping, as well as of the composition of two mappings. At the same time, concerning continuum mechanics, the study of function spaces, different from the ones of smooth or Sobolev mappings, is of great interest. In particular, there are advantages in using Sobolev–Orlicz spaces for nonlinear elasticity [4], Lorentz spaces for the Shrödinger equation [6] and for the $p$-Laplace system [1], grand Sobolev spaces for $p$-harmonic operators [10, 18]. Thoroughly studied, has been the question of the regularity of derivatives of the inverse mapping. Thus, we refer the reader to [20] for Sobolev $W^{1,p}$-regularity in the planar case, to [9, 19, 21, 34, 35] for $BV$- and $W^{1,p}$-regularity in spatial case. Also, the articles [7, 22] deal with the regularity of the inverse mapping and the composition of diffeomorphic or bilipschitz $W^{m,p}$-Sobolev mappings.

In this paper, instead of studying the inverse mapping problem for all the classes of function spaces separately, we take a concept that covers all these options at once. More precisely, we prove a result for the general rearrangement invariant...
Banach function spaces. This approach, developed in [5], has recently been very fruitful and many authors have considered issues such as Sobolev embeddings, the regularity of solutions to given PDEs and so on in this general setting, see, for example, [1].

The inspiration for our research is a result in classical Sobolev spaces from [7], the proof there builds on the classical Sobolev–Gagliardo–Nirenberg inequality. This inequality appears in a much more general form in [13], and this allows us to derive the results which follow. In the following text, $\beta_X$ stands for the upper Boyd index of a Banach function space $X$ (see Definition 2.5). In what follows, we prove the following three theorems.

**Theorem 1.1.** Let $m, n \in \mathbb{N}$, let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets, and let $X$ be a rearrangement invariant Banach function space such that $\beta_X < 1$. Also, let $f : \Omega \to \Omega'$ be a locally bilipschitz homeomorphism with $f \in W^m X_{\text{loc}}(\Omega, \mathbb{R}^n)$. Then

$$f^{-1} \in W^m X_{\text{loc}}(\Omega', \mathbb{R}^n).$$

**Theorem 1.2.** Let $m, n \in \mathbb{N}$, let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets, and let $X$ be a rearrangement invariant Banach function space such that $\beta_X < 1$. Also, let $f : \Omega \to \mathbb{R}^n$ be a locally Lipschitz mapping with $f \in W^m X_{\text{loc}}(\Omega, \mathbb{R}^n)$, and let $g : \Omega' \to \Omega$ be locally bilipschitz with $g \in W^m X_{\text{loc}}(\Omega', \Omega)$. Then

$$f \circ g \in W^m X_{\text{loc}}(\Omega', \mathbb{R}^n).$$

**Theorem 1.3.** Let $m, n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be an open set, and let $X$ be a rearrangement invariant Banach function space such that $\beta_X < 1$. Also, let $f, g : \Omega \to \mathbb{R}$ be locally Lipschitz mappings such that $f, g \in W^m X_{\text{loc}}(\Omega, \mathbb{R})$. Then $fg \in W^m X_{\text{loc}}(\Omega, \mathbb{R})$ and $fg$ is a locally Lipschitz mapping.

**Remark 1.4.** The result for a product of $f$ and $g$ can be even generalized for $f, g : \Omega \to \mathbb{R}^n$ being mappings and not just functions, then we understand the product $f \cdot g$ as a scalar product and the proof can be done in the same way with the arguments repeated for all coordinates.

In particular, these theorems are valid for Lorentz and Orlicz spaces. Since these spaces are of special interest in applications, we provide an explicit formulation for the reader’s convenience.

**Corollary 1.5.** Let $m, n \in \mathbb{N}$, let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets, and let $p > 1$ and $q \geq 1$. Also, let

$$f \in W^m L^{p,q}_{\text{loc}}(\Omega, \mathbb{R}^n), \quad f : \Omega \to \Omega', \text{ be a locally bilipschitz homeomorphism,}$$

$$u \in W^m L^{p,q}_{\text{loc}}(\Omega, \mathbb{R}^n) \text{ be a locally Lipschitz mapping,}$$

$$\varphi \in W^m L^{p,q}_{\text{loc}}(\Omega', \mathbb{R}^n), \varphi : \Omega' \to \Omega, \text{ be a locally bilipschitz mapping, and}$$

$$g, h \in W^m L^{p,q}_{\text{loc}}(\Omega, \mathbb{R}) \text{ be locally Lipschitz mappings.}$$

Then it follows that

$$f^{-1} \in W^m L^{p,q}_{\text{loc}}(\Omega', \mathbb{R}^n),$$

$$u \circ \varphi \in W^m L^{p,q}_{\text{loc}}(\Omega', \mathbb{R}^n), \text{ and}$$

$$gh \in W^m L^{p,q}_{\text{loc}}(\Omega, \mathbb{R}).$$

**Remark 1.6.** It is well known that in the case $p > 1$ and $q \geq 1$ the upper Boyd index $\beta_{L^{p,q}} < 1$. However, for any $q > 1$ the Lorentz space $L^{1,q}$ is not a Banach function space. In fact, it cannot be even equivalently renormed. Thus, it needs a different approach, and we leave the case of $L^{1,q}$ open.

**Corollary 1.7.** Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets, let $A$ be a Young function such that there exists a positive constant $c$ for which

$$\int_0^t \frac{A(s)}{s^2} \, ds \leq \frac{A(ct)}{t} \quad (1.1)$$
holds for all $t > 0$. Also, let

$$f \in W^m L^A_{\text{loc}}(\Omega, \mathbb{R}^n), f : \Omega \to \Omega', \text{ be a locally bilipschitz homeomorphism},$$

$$u \in W^m L^A_{\text{loc}}(\Omega, \mathbb{R}^n) \text{ be a locally Lipschitz mapping},$$

$$\varphi \in W^m L^A_{\text{loc}}(\Omega', \mathbb{R}^n), \varphi : \Omega' \to \Omega, \text{ be a locally bilipschitz mapping, and}$$

$$g, h \in W^m L^A_{\text{loc}}(\Omega, \mathbb{R}) \text{ be locally Lipschitz mappings.}$$

Then

$$f^{-1} \in W^m L^A_{\text{loc}}(\Omega', \mathbb{R}^n),$$

$$u \circ \varphi \in W^m L^A_{\text{loc}}(\Omega', \mathbb{R}^n), \text{ and}$$

$$gh \in W^m L^A_{\text{loc}}(\Omega, \mathbb{R}).$$

Remark 1.8. The inequality (1.1) is an equivalent condition to the boundedness of maximal operator and is in fact equivalent to $\beta_{L^A} < 1$, see [23] and Remark 2.6.

2 | PRELIMINARIES

We use the notation $| \cdot |$ for three different operations on three exclusive types of argument. If the argument is of real value, we consider the symbol to be an absolute value. If the argument is a matrix or a linear operator, we understand the operator norm. If the argument is a set in $\mathbb{R}^n$, we understand $n$-dimensional Lebesgue measure of this set.

In the following text $\Omega$ and $\Omega'$ stand for open subsets of $\mathbb{R}^n$ with finite Lebesgue measure. We denote a scaling parameter by

$$\eta := \frac{|\Omega'|}{|\Omega|}. \quad (2.1)$$

We write $A(\xi) \lesssim B(\xi)$ if there exists a constant $C > 0$ independent of the parameter $\xi$ such that $A(\xi) \leq CB(\xi)$.

2.1 | Banach function spaces

Let us first remind some notions from the theory of Banach function spaces (later in the text it will referred just as BFS and r.i. BFS if the space is also rearrangement invariant). We refer the reader to [5] and [31] for the theory of BFS.

**Definition 2.1.** Given a BFS $X$ and a real number $\alpha > 0$, the space $X^\alpha$ consists of all measurable mappings $u$ such that

$$\|u\|_{X^\alpha} := (\|u\|_X^\alpha)^{1/\alpha} < \infty.$$ 

We use the convention

$$X^\infty = L^\infty. \quad (2.2)$$

If $\alpha \geq 1$, then $\| \cdot \|_{X^\alpha}$ is a Banach function norm (see [25, §1.d] and [26]). In this case the space $X^\alpha$ is often referred in the literature as an $\alpha$-convexification of $X$.

Consider numbers $p_i \in [1, \infty], i = 1, \ldots, k$, such that

$$\sum_{i=1}^{k} \frac{1}{p_i} = 1,$$
and locally integrable functions $f_i, i = 1, \ldots, k$, then the following Hölder inequality
\[ \left\| \prod_{i=1}^{k} f_i \right\|_X \leq \prod_{i=1}^{k} \left\| f_i \right\|_{X^{p_i}} \] (2.3)
follows from [25, Proposition 1.d.2] and the induction by $k$. Let us remind the classical Hardy–Littlewood–Polya principle [5, Corollary II.4.7]. For an open set $\Omega \subset \mathbb{R}^n$ and a r.i. BFS $X(\Omega)$ the following holds. If
\[ \int_0^t f^*(s) \, ds \leq \int_0^t g^*(s) \, ds \quad \text{holds for all } 0 < t < |\Omega|, \] (2.4)
then
\[ \|f\|_{X(\Omega)} \leq \|g\|_{X(\Omega)}. \]

Here $u^*$ is the non-increasing rearrangement of a measurable function $u$,
\[ u^*(s) := \inf \{ \lambda : |\{ |u| > \lambda \}| \leq s \}. \]
We also define $u^{**}$ for a measurable function $u$ as
\[ u^{**}(s) := \frac{1}{s} \int_0^s u^*(t) \, dt. \]

The Luxemburg representation theorem [5, Theorem II.4.10] states:

For every r.i. BFS $X(\Omega)$ there exists a r.i. BFS $X(0, |\Omega|)$, referred as a representation space, such that
\[ \|f\|_{X(\Omega)} = \|f^*\|_{X(0, |\Omega|)}. \] (2.5)

For our purposes we need a more general form of the Hardy–Littlewood–Polya principle, applicable when the underlying measure space is variable.

**Definition 2.2.** Let $s, a \in (0, \infty)$. The dilation operator $E_s$ is defined on the space of measurable functions on $(0, a)$ by
\[ E_s f(x) := \begin{cases} f(sx), & \text{for } sx < a, \\ 0, & \text{otherwise}, \end{cases} \]
for all $x > 0$.

Note that for any Banach function space $X$ one has
\[ \|E_s f\|_X \leq \max \{ s^{-1}, 1 \} \|f\|_X. \] (2.6)
Indeed, it follows from the fact that
\[ \|E_s f\|_{L^1} \leq s^{-1} \|f\|_{L^1}, \quad \|E_s f\|_{L^\infty} \leq \|f\|_{L^\infty} \]
and [5, Theorem 2.2, p. 106].

**Definition 2.3.** Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets of finite measure, and let $X(\Omega), Y(\Omega')$ be a pair of r.i. BFS such that
\[ \eta \|f\|_{X(\Omega)} = \|g\|_{Y(\Omega')} \]
holds providing
\[ E_\eta(g^*) = f^* \]
with respect to the notation (2.1). Such spaces are called similar spaces. To unify the notation of all spaces similar to each other, we use the same name for the space independent of the domains, i.e. we denote \( X(\Omega') := Y(\Omega') \).

**Lemma 2.4 (Hardy–Littlewood–Polya principle for different measure spaces).** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open sets of finite measure, let \( f \) and \( g \) be measurable functions on \( \Omega \) and \( \Omega' \), correspondingly. Let \( X(\Omega), X(\Omega') \) be similar r.i. Banach function spaces. If
\[
\int_0^t E_\eta(g^*)(s) \, ds \leq \int_0^t f^*(s) \, ds \quad \text{holds for all } t \in (0, |\Omega|),
\]
then this implies that
\[
\|g\|_{X(\Omega')} \leq \max \{ \eta^{-1}, 1 \} \|f\|_{X(\Omega)}. \]

**Proof.** By the Luxemburg representation theorem (2.5), estimate (2.6) and the classical Hardy–Littlewood–Polya principle (2.4) we obtain
\[
\|g\|_{X(\Omega')} = \|g^*\|_{X[0,|\Omega'|]} \\
\leq \max \{ \eta^{-1}, 1 \} \|E_\eta g^*\|_{X[0,|\Omega|]} \\
\leq \max \{ \eta^{-1}, 1 \} \|f^*\|_{X[0,|\Omega|]} \\
= \max \{ \eta^{-1}, 1 \} \|f\|_{X(\Omega)}. \quad \square
\]

**Definition 2.5 (Upper Boyd index).** The upper Boyd index of a r.i. BFS \( X \) is defined by
\[
\beta_X := \lim_{t \to \infty} \frac{\log \left( \|E_{1/t}\|_{\tilde{X} \to \tilde{X}} \right)}{\log t}.
\]

**Remark 2.6.** Remind that the maximal operator \( M \) is bounded on \( X \) if and only if the upper Boyd index \( \beta_X < 1 \), see [32, Theorem 1, p. 3], which is a sufficient condition for Theorem 2.10 being valid. The formulas for calculating the Boyd indices of classical function spaces may be found in literature see, for example, [12].

### 2.2 Some estimates for weak derivatives

We refer the reader to the classical book [28] for the theory of Sobolev spaces. Let \( u : \Omega \to \mathbb{R}^n \) be a \( k \)-times weakly differentiable mapping. Let us remind, that for almost every fixed \( x_0 \in \Omega \), the \( k \)-th weak derivative \( D^k u(x_0) \) is a \( k \)-linear mapping. It can be represented by a multidimensional matrix or tensor consisting of all weak partial derivatives of \( u \) of order \( k \).

Let \( X(\Omega) \) be a BFS. The *Sobolev space* \( V^k X(\Omega) \) is defined to be the space of \( k \)-times weakly differentiable mappings \( u \) such that \( D^k u \in X(\Omega) \). This space is equipped with the semi-norm
\[
\|u\|_{V^k X(\Omega)} := \|D^k u\|_{X(\Omega)} < \infty.
\]

The space \( W^k X(\Omega) \) is defined to be the space of all \( k \)-times weakly differentiable mappings \( u \) such that
\[
\|u\|_{W^k X(\Omega)} := \sum_{i=0}^k \|D^i u\|_{X(\Omega)} < \infty.
\]
We also use the notation
\[ W^k X_{\text{loc}}(\Omega) := \{ u \in W^k X(G) : \text{for all } G \text{ open and } G \Subset \Omega \}. \]

Here and further \( G \Subset \Omega \) means that the closure of \( G \) is a compact subset of \( \Omega \).

**Remark 2.7.** For any BFS \( X(\Omega) \) one has \( X(\Omega) \subset L^1(\Omega) \) provided that \( |\Omega| < \infty \), which implies that \( V^k X_{\text{loc}}(\Omega) \subset V^k L^1_{\text{loc}}(\Omega) \), for arbitrary \( k \in \mathbb{N} \) and an open \( \Omega \subset \mathbb{R}^n \) of finite measure.

A mapping \( f : \Omega \to \mathbb{R}^n \) is said to be *locally bilipschitz* if for every ball \( B(x_0, \delta) \Subset \Omega \) centered at \( x_0 \) with radius \( \delta \) there exists \( L > 0 \) such that
\[ L^{-1} |x - y| < |f(x) - f(y)| < L|x - y| \]
holds for all \( x, y \in B(x_0, \delta) \).

**Lemma 2.8 ([2, Corollary 3.19]).** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open and let \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m) \). Suppose that the mapping \( g : \Omega' \to \Omega \) is a bilipschitz homeomorphism. Then \( f \circ g \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^m) \) and
\[ D(f \circ g(y)) = Df(g(y))Dg(y) \quad \text{for almost all } y \in \Omega'. \]

The crucial part of this paper is the Sobolev–Gagliardo–Nirenberg interpolation inequality, which enables estimates to be made of lower order derivatives of the function in terms of higher-order ones and the function itself. Namely, the inequality
\[ \| D^j u \|_X \lesssim \| (D^k u) \|_Y^{j/k} \| u \|_{Z}^{1 - j/k}, \]
which was originally stated by Gagliardo [16] and Nirenberg [29] in case of \( X, Y, Z \) being Lebesgue spaces. For our purposes, the particular case of the inequality for BFS recently proved in [14] is needed. For the reader’s convenience, let us state the theorem here.

**Theorem 2.9 (Gagliardo–Nirenberg inequality for r.i. BFS).** If \( j, k \in \mathbb{N} \), \( 1 \leq j < k \), and if \( X, Y \) are rearrangement invariant Banach function spaces over \( \mathbb{R}^n \) such that
\[ Y^{k/j} \overset{\text{loc}}{\to} X, \]
then the estimate
\[ \| D^j u \|_X \lesssim \| (D^k u) \|_Y^{j/k} \| u \|_{Z}^{1 - j/k} \tag{2.7} \]
holds for all \( k \)-times weakly differentiable functions \( u \) with a constant independent of \( u \), where \( Z = \left( \left( Y^{k/j} \right)^X \right)^{1 - j/k} \).

As a corollary we obtain the following theorem once we realise that \( X^Y = L^\infty \).

**Theorem 2.10.** Let \( 1 \leq j < k \) be natural numbers, and let \( Y \) be a r.i. BFS with upper Boyd index \( \beta_Y < 1 \). Then the estimate
\[ \| D^j u \|_{Y^{k/j}} \lesssim \| D^k u \|_{Y^j} \| u \|_{L^\infty}^{1 - j/k} \tag{2.8} \]
is valid for all \( k \)-times weakly differentiable functions \( u \).
Remark 2.11. In the following proof we use the notation for BFS \( Z = X^Y \), which means that \( Z \) is an optimal space such that the Hölder-type inequality \( \|fg\|_X \leq \|f\|_Y \|g\|_Z \) holds (see [14, Lemma 2.2]). This tool maybe called the space of Hölder multipliers, see [24] for more details.

Proof of Theorem 2.10. Let us set \( X := Y^k/j \). Note that the assumptions of Theorem 2.9 are satisfied since

\[
Y^k/j = X \Rightarrow Y^k/j \subset X
\]

holds and thus, from the Hölder inequality (2.3), one has

\[
L^\infty = \left( \left( Y^k/j \right)^{X} \right)^{1-j/k}.
\]

By using Theorem 2.9 and the convention (2.2), we derive

\[
\|D^j u\|_{Y^k/j} \leq \|(D^k u)^{**}\|_{Y}^{j/k} \|u^{**}\|_{L^\infty}^{1-j/k}.
\]

The boundedness of the maximal operator on \( Y \) is guaranteed by the assumption on the Boyd index in \( Y \), it implies \( \|(D^k u)^{**}\|_{Y} \leq \|D^k u\|_{Y} \), the similar property \( \beta L^\infty < 1 \) results in \( \|u^{**}\|_{L^\infty} \leq \|u\|_{L^\infty} \). From this we deduce (2.8).

Remark 2.12. In the case of \( X = L^\infty \), Theorem 2.10 coincides with the classic case known as the Kolmogorov–Stein inequality.

To get the local version of the theorem above we need an extension operator \( \mathcal{E} \), for the construction of which see [33, Theorem 5, p. 181]. Moreover, the boundedness of the extension operator in the case of classical Sobolev spaces \( V^kL^p \) was proven there. The next theorem for the Sobolev space \( V^kX \) follows from the general version [8, Theorem 4.1].

Theorem 2.13 (On the extension operator). Let \( B \subset \mathbb{R}^n \) be a ball and let \( k \in \mathbb{N} \). Then there exists a linear operator such that for every r.i. BFS \( X \) it follows that

(i) \( \mathcal{E} : V^kX(B) \to V^kX(\mathbb{R}^n) \),

(ii) \( \mathcal{E}u|_B = u \).

We can now formulate a local Sobolev–Gagliardo–Nirenberg type theorem.

Theorem 2.14. Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( 1 \leq j < k \) be natural numbers. Then for the r.i. BFS \( X \), with \( \beta_X < 1 \), it follows that

\[
\left( V^kX_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \right) \subset V^jX^k/j_{loc}(\Omega).
\]

Proof. For \( x \in \Omega \) choose a ball \( B = B(x, r) \subset \Omega \). Theorem 2.13 implies that the extension \( \mathcal{E}u \) belongs to \( V^kX(\mathbb{R}^n) \). From Theorem 2.10 we derive

\[
\|D^j u\|_{X^k/j(B)} \leq \|D^j(\mathcal{E}u)\|_{X^k/j(\mathbb{R}^n)} \\
\leq \|D^k(\mathcal{E}u)\|_{X(\mathbb{R}^n)}^{j/k} \|\mathcal{E}u\|_{L^\infty(\mathbb{R}^n)}^{1-j/k} \\
\leq \|D^k u\|_{X(B)}^{j/k} \|u\|_{L^\infty(B)}^{1-j/k}
\]

(2.9)
The last inequality is valid due to the extension operator and can be chosen such that

\[ E : L^\infty(B) \to L^\infty(\mathbb{R}^n) \quad \text{and} \quad E : V^kX(B) \to V^kX(\mathbb{R}^n), \]

see [8] for details.

\[ \square \]

2.3 | High-order derivatives

We refer the reader to [37, §10] for the basic properties of multi-linear mappings and differential calculus, which is useful to deal with high-order derivatives.

The critical tool of the paper is the chain rule. Formally, for normed vector spaces \( S, T, R \), and the mappings \( f : S \to T, \) \( g : R \to S \) and \( r \in R \) we compute

\[ D(f \circ g)(r)(h) = Df(g(r))(Dg(r)(h)) \quad \text{for all} \ h \in R, \]

which can be written in a matrix form as

\[ D(f \circ g) = (Df \circ g) \cdot Dg. \]

For the second-order derivative we obtain

\[ D^2(f \circ g)(r)(h_1, h_2) = D^2f(g(r))(Dg(r)(h_1), Dg(r)(h_2)) + Df(g(r))(D^2g(r)(h_1, h_2)) \]

for all \( h_1, h_2 \in R \), which can be expressed in short as

\[ D^2(f \circ g) = (D^2f \circ g) \cdot Dg \otimes Dg + (Df \circ g) \cdot D^2g, \]

where \( \cdot \) is used to express the composition of (multi-)linear mappings and \( \otimes \) is a tensor product which makes a bilinear mapping from two linear ones, so that composition has sense. Further,

\[ D^3(f \circ g) = (D^3f \circ g) \cdot Dg \otimes Dg \otimes Dg + (D^2f \circ g) \cdot D^2g \otimes Dg + 2(D^2f \circ g) \cdot Dg \otimes D^2g + (Df \circ g) \cdot D^3g. \]

Direct calculations show that \( D^m(f \circ g) \) is made up from terms of the form

\[ (D^{k_0}f \circ g) \cdot \bigotimes_{i=1}^{k_0} D^{k_i}g \]

with some coefficients, where \( k_0 \geq 1, k_i = 0 \) if and only if \( i > k_0 \) and \( \sum_{i=1}^{m} k_i = m \).

Moreover, for multi-linear mappings \( A \in \mathcal{L}(S^{l+n}, T), \ B \in \mathcal{L}(R^k, S^l), \) and \( C \in \mathcal{L}(R^m, S^n) \) it follows that \( A \cdot (B \otimes C) \in \mathcal{L}(R^{k+m}, T) \) is a \( (k + m) \)-linear mapping and we can estimate a norm as

\[ |A \cdot (B \otimes C)| \leq |A||B| |C|. \]

For more details of this topic, we refer the curious reader to [27], and to [36] for the tensor calculus. The corresponding coordinate representation of the high-order chain rule is described in the best possible way in [7, §2.2]. For the sake of simplicity, we will omit \( \cdot \) and \( \otimes \) further in the text, when it can be done without ambiguity.

3 | PROOF OF THEOREM 1.1. THE CASE \( M = 2 \)

To start an induction process, we need to investigate the regularity of the second derivative of the inverse mapping. We start with the Sobolev regularity case.
Theorem 3.1 (Theorem 1.3 of [19]). Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open, let $p \geq 1$ and suppose that $f : \Omega \to \Omega'$ is a bilipschitz mapping. If $Df \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, then $Df^{-1} \in W^{1,p}_{\text{loc}}(\Omega', \mathbb{R}^n)$.

We provide a more general case involving BFS-regularity.

Theorem 3.2. Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open and suppose that $f : \Omega \to \Omega'$ is a bilipschitz homeomorphism. Let $X$ be a rearrangement invariant Banach function space. If $Df \in W^{1,X}_{\text{loc}}(\Omega, \mathbb{R}^n)$, then $Df^{-1} \in W^{1,X}_{\text{loc}}(\Omega', \mathbb{R}^n)$.

Proof of Theorem 3.2. Since $Df \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$, by Theorem 3.1 we know that $Df^{-1} \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^n)$. Then, following the proof of [19, Theorem 1.3], we use Lemma 2.8 to differentiate the identity $f \circ f^{-1} = id$ twice to obtain the equation

$$D^2f(f^{-1}(y))(Df^{-1}(y))^2 + Df(f^{-1}(y))D^2f^{-1}(y) = 0.$$ 

Since $f$ is bilipschitz we also know that there exists a positive constant $L$ such that for almost every $y \in \Omega'$ it holds

$$\left| \left( Df(f^{-1}(y)) \right)^{-1} \right| \leq L, \quad |Df^{-1}(y)| \leq L, \quad \text{and} \quad |J_f^{-1}(y)| \geq L^{-n},$$

and from the previous arguments we derive an estimate

$$|D^2f(f^{-1}(y))||Df^{-1}(y)|^2 = |Df(f^{-1}(y))||D^2f^{-1}(y)|,$$

$$|D^2f^{-1}(y)| = |D^2f(f^{-1}(y))||Df(f^{-1}(y))|^{-1}|(Df^{-1}(y))^2| \leq L^3|D^2f(f^{-1}(y))|. \quad (3.1)$$

Note that $|D^2f|$ is a measure absolutely continuous with respect to Lebesgue measure (since $|D^2f| \in L^1_{\text{loc}}$). Then for chosen $\varepsilon > 0$, there exists $\delta > 0$ such that $|E| < \delta$ implies

$$\int_E |D^2f(x)| \ dx < \varepsilon.$$

Let $A \subset K \subset \Omega'$ be measurable and let $G \subset K$ be an open set such that $A \subset G$ and

$$|G \setminus A| < L^{-n}\delta.$$

By (3.1) we get, up to multiple of $L$, the following estimate

$$\int_A |D^2f^{-1}(y)| \ dy \leq \int_G |D^2f^{-1}(y)| \ dy \leq \int_G |D^2f(f^{-1}(y))||J_{f^{-1}}(y)| \ dy.$$

By the change-of-variable formula for Lipschitz functions we obtain

$$\int_G |D^2f(f^{-1}(y))||J_{f^{-1}}(y)| \ dy = \int_{f^{-1}(G)} |D^2f(x)| \ dx$$

$$= \int_{f^{-1}(A)} |D^2f(x)| \ dx + \int_{f^{-1}(G \setminus A)} |D^2f(x)| \ dx.$$  

Since $f^{-1}(G \setminus A) < L^nL^{-n}\delta$, by the Lipschitz property of $f^{-1}$, the second term can be estimated and, therefore,

$$\int_A |D^2f^{-1}(y)| \ dy \leq \int_{f^{-1}(A)} |D^2f(x)| \ dx + \varepsilon.$$
For the next calculation, set \( \gamma_t := \min \left\{ L^n t, |f(\Omega)| \right\} \). Recall that

\[
\int_0^t h^*(s) \, ds = \sup_{|A|=t} \int_A |h(x)| \, dx,
\]

where the supremum is taken over all measurable sets \( A \) with \( |A| = t \). Then,

\[
\int_0^t |D^2 f^{-1}|^*(s) \, ds = \sup_{|A|=t} \int_A |D^2 f^{-1}(y)| \, dy
\]

\[
\leq \sup_{|A'|=\gamma} \int_{f^{-1}(A)} |D^2 f(x)| \, dx + \varepsilon
\]

\[
\leq \sup_{|A'|=\gamma} \int_{A'} |D^2 f(x)| \, dx + \varepsilon
\]

\[
= \int_0^\gamma |D^2 f|^*(s) \, ds + \varepsilon
\]

\[
\leq L^{-n} \int_0^\gamma E_\eta |D^2 f|^*(s) \, ds + \varepsilon,
\]

where \( \eta \) is given by (2.1).

Here, the constant \( \varepsilon > 0 \) can be chosen as small as we wish. Hence,

\[
\int_0^t |D^2 f^{-1}|^*(s) \, ds \lesssim \int_0^t s^m E_\eta |D^2 f|^*(s) \, ds,
\]

which implies that

\[
\int_0^t E_{\gamma-1} |D^2 f^{-1}|^*(s) \, ds \lesssim \int_0^t |D^2 f|^*(s) \, ds
\]

holds for all \( t > 0 \). Then Lemma 2.4 guarantees that \( \|D^2 f^{-1}\|_{X(f(\Omega))} \lesssim \|D^2 f\|_{X(\Omega)} \). \qed

4 | PROOF OF THEOREM 1.1. THE CASE \( M \geq 3 \)

The basic idea of the proof follows [7] and is to differentiate the identity \( f \circ f^{-1} = id \) to obtain a representation of the second derivative of the inverse mapping \( D^2 f^{-1} \) through the second derivative \( D^2 f \) and the first derivatives \( Df \) and \( Df^{-1} \). Further, using the Leibniz and chain rules, we represent \( D^k f^{-1} \) as a product of lower order derivatives of \( f \) and \( f^{-1} \). Then the Sobolev–Gagliardo–Nirenberg and Hölder inequalities give us a desirable regularity.

**Lemma 4.1 (Lemma 3.1 of [7]).** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open. Let \( f : \Omega \to \Omega' \) be a bilipschitz homeomorphism such that \( f \in W^{2,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \). Then \( f^{-1} \in W^{2,1}_{\text{loc}}(\Omega', \mathbb{R}^n) \) and

\[
D^2 f^{-1}(y) = -Df^{-1}(y) \cdot D^2 f \left( f^{-1}(y) \right) \cdot (Df^{-1}(y) \otimes Df^{-1}(y))
\]

for almost all \( y \in \Omega' \).

**Lemma 4.2 (Lemma 3.3 of [7]).** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open. Let \( f : \Omega \to \Omega' \) be a bilipschitz homeomorphism such that \( f \in W^{m,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \). Then

\[
|D(D^{m-1} f (f^{-1})))| \in L^1_{\text{loc}}(\Omega')
\]
and
\[
D(D^{m-1}f(f^{-1}(y))) = D^m f(f^{-1}(y)) \cdot Df^{-1}(y)
\]
for almost all \( y \in \Omega' \).

**Remark 4.3.** Formula (4.2) basically means that
\[
D(D^{m-1}f(f^{-1})) = D^m f(f^{-1}) \cdot (Df^{-1} \otimes I \otimes \cdots \otimes I),
\]
where \( I \) is the identity mapping.

Since \( f^{-1} \) is bilipschitz, from [7, Lemma 3.3] it is easy to obtain:

**Lemma 4.4.** Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open. Let \( X \) be a r.i. BFS with Boyd index \( \beta_X < 1 \). Let \( f : \Omega \to \Omega' \) be a locally bilipschitz homeomorphism such that \( f \in W^{m,X}_{\text{loc}}(\Omega, \mathbb{R}^n) \). Then
\[
|D(D^{m-1}f(f^{-1}))| \in X_{\text{loc}}(\Omega')
\]
and (4.2) holds for almost all \( y \in \Omega' \).

**Proof of Theorem 1.1.** We will prove the statement by using induction on \( m \). The case \( m = 1 \) follows from the fact that \( f \) is bilipschitz. Theorem 3.2 ensures the case \( m = 2 \).

Now, consider the general case \( m \geq 3 \). Assume that \( |D^k f^{-1}| \in X_{\text{loc}}^{k-1}(\Omega') \) results from \( |D^k f| \in X_{\text{loc}}^{k-1}(\Omega) \) for all \( 1 \leq k \leq m-1 \) and any BFS \( X \) with \( \beta_X < 1 \).

Again, as in the proof of [7, Theorem 1.1] we differentiate (4.1) \( m-2 \) times. We claim that \( D^m f^{-1}(y) \) is composed of
\[
D^{k-1} f^{-1}(y) \cdot D^k f^{-1}(y) \cdot \bigotimes_{i=1}^{k_0} D^{k_i} f^{-1}(y)
\]
for almost all \( y \in \Omega' \). Here \( k_{-1} \geq 1, k_0 \geq 2, k_i = 0 \) if and only if \( i > k_0 \), and \( k_{-1} + \sum_{i=1}^{k_0} k_i = m + 1 \).

Since \( k_{-1}, k_i \leq m-1 \) for all \( i \geq 1 \), from Theorem 2.14 with \( u = Df, k = m-1, j = k_i - 1 \) we derive that
\[
|D^k f| \in X_{\text{loc}}^{k_i-1}(\Omega),
\]
and hence by the induction assumption we have
\[
|D^k f^{-1}| \in X_{\text{loc}}^{k_i-1}(\Omega').
\]
Now, calculate
\[
\frac{k_{-1} - 1}{m-1} + \frac{k_0 - 1}{m-1} + \sum_{i=1}^{k_0} \frac{k_i - 1}{m-1} = \frac{1}{m-1} \left( k_{-1} - 1 + k_0 - 1 + \sum_{i=1}^{k_0} (k_i - 1) \right) = 1.
\]
By using this equality as indices in inequality (2.3) we have
\[
\|D^m f^{-1}\|_X \lesssim \|D^{k_0} f\|_{X^{k_0-1}} \cdot \prod_{i=1}^{k_0} \|D^{k_i} f^{-1}\|_{X^{k_i-1}},
\]
which implies \( |D^m f^{-1}| \in X_{\text{loc}}^{m-1}(\Omega') \). \( \square \)
5 PROOF OF THEOREMS 1.2 AND 1.3

We need the next generalization of [7, Lemma 4.1].

Lemma 5.1. Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be open. Let \( g : \Omega' \to \Omega \) be a bilipschitz mapping such that \( g \in W^k_{1, \text{loc}}(\Omega', \mathbb{R}^n) \), and \( f \in W^k_{1, \text{loc}}(\Omega, \mathbb{R}^n) \). Then

\[
|D(D^{k-1}f(g))| \in X_{\text{loc}}(\Omega')
\]

and

\[
D(D^{k-1}f(g(y))) = D^k f(g(y)) \cdot Dg(y).
\]

Proof of Lemma 5.1. The proof of the pointwise equality can be carried out in the very same way as in [7] since \( W^k_{1, \text{loc}}(\Omega) \subset W^{k, 1}_{\text{loc}}(\Omega) \) and we can use [7, Lemma 4.1]. In order to do so it is enough to realize that \( g \) is bilipschitz and thus \( Dg \) is bounded. The rest follows from the pointwise equality. \( \square \)

Proof of Theorem 1.2. Due to the fact that \( g \) is bilipschitz, Lemma 2.8 (applied on \( f = u, g = F^{-1} \)) provides with the case \( m = 1 \). Lemma 5.1 gives

\[
D^2 f \circ g = D^2 f(g) \cdot (Dg \otimes Dg) + Df(g) \cdot D^2 g
\]

with \( |Df(g)| \) and \( |Dg| \) bounded a.e. and \( |D^2 f(g)| \) and \( |D^2 g| \) belonging to \( X_{\text{loc}}(\Omega') \).

Following the proof of [7, Theorem 1.2], within Lemmata 5.1, 2.8 and the Leibniz rule, we obtain that \( D^m(f \circ g)(y) \) is composed of

\[
D^k_0 f(g(y)) \bigotimes_{i=1}^{k_0} D^{k_i} g(y)
\]

a.e. with \( k_0 \geq 1 \), \( k_i = 0 \) if and only if \( i > k_0 \), and \( \sum_{i=1}^m k_i = m \). Following the same calculations and estimates as for (4.3) we ensure that \( D^m(f \circ g) \in X_{\text{loc}}(\Omega') \). \( \square \)

Proof of Theorem 1.3. The Leibniz rule yields

\[
D^m(fg) = \sum_{0 \leq j \leq m} \binom{m}{j} D^j f \otimes D^{m-j} g.
\]

Therefore, it is enough to show that \( |D^j f \otimes D^{m-j} g| \in X_{\text{loc}} \) for all \( j \). We may exclude the case \( j = 0 \) or \( j = m \) since these terms are a product of Lipschitz function and function belonging to \( X_{\text{loc}} \). For now, we exclude the case \( j = 1 \) or \( j = m - 1 \). By the Hölder inequality (2.3) and the Sobolev–Gagliardo–Nirenberg type estimate (2.9) for both \( Df \) and \( Dg \) for any ball \( B \Subset \Omega \) we obtain

\[
\|D^j f \otimes D^{m-j} g\|_{X(B)} \leq \|D^j f\|_{X^{m-j}(B)} \|D^{m-j} g\|_{X^{m-j}(B)}
\]

\[
\leq \|D^j f\|_{X(B)} \|Df\|_{L^{\infty}(B)}^{1-j} \|D^{m-j} g\|_{X(B)}^{1-j} \|Dg\|_{L^{\infty}(B)}^{1-j} \|D^{m-j} g\|_{X(B)}^{1-j} \|X^{m-j}(B)} \cdot \tag{5.1}
\]

Three out of four terms are finite by assumptions, we estimate the remaining term \( \|D^{m-j} g\|_{X(B)} \) by (2.3) and (2.9) as before

\[
\|D^{m-j} g\|_{X(B)}^\frac{m-j}{m-1} \leq \|D^{m-j} g\|_{X^{m-j}(B)} \|1\|_{X^{m-j-1}(B)}^\frac{m-j}{m-1} \|1\|_{X^{m-j-1}(B)}
\]

\[
\leq \|D^m g\|_{X(B)}^\frac{m-2}{m-1} \|Dg\|_{L^{\infty}(B)}^{1-m-2} \|1\|_{X^{m-1}(B)}.
\]
All four terms of (5.1) are finite so all the items $D^j f \otimes D^{m-j} g$ belong to the space. In case $j = 1$ we estimate the term by (2.3) and (2.9) as follows
\[
\|D^1 f \otimes D^{m-1} g\|_{X(B)} \leq \|D^1 f\|_{X(\infty(B))} \|D^{m-1} g\|_{X^1(\infty(B))}.
\]
The first term can be estimated by $X^\infty = L^\infty$. The second case can be considered as the previous one. The case $j = m - 1$ is analogous.

Note that the boundedness of the maximal operator is needed due the application of the Gagliardo–Nirenberg inequality, which was proved so far only in the case of spaces on which the operator is bounded. The case without the boundedness of the maximal operator is still open.

ACKNOWLEDGEMENTS

The first named author was supported by the Austrian Science Fund (FWF) project M 2670, and by the European Union’s Horizon 2020 research and innovation programme under the Marie Składowska-Curie grant agreement No. 847693, the second named author was supported by the grant GAČR 18-00960Y, and the third named author was supported by EF–IGS2017–Soudsky–IGS07P1.

REFERENCES

[1] A. Alberico, A. Cianchi, and C. Sbordone, Continuity properties of solutions to the p-Laplace system, Adv. Calc. Var. 10 (2017), no. 1, 1–24.
[2] L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Math. Monogr., Oxford Univ. Press, New York, 2000.
[3] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966), 319–361. (French).
[4] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977), 337–403.
[5] C. Bennett and R. Sharpley, Interpolation of operators, Pure Appl. Math. (Amst.), vol. 129, Academic Press, Inc., Boston, MA, 1988.
[6] H. Brézis and T. Gallouet, Nonlinear Schrödinger evolution equations, Nonlinear Anal. 4 (1980), 677–681.
[7] D. Campbell, S. Hencl, and F. Konopecký, The weak inverse mapping theorem, Z. Anal. Anwendung 34 (2015), no. 3, 321–342.
[8] A. Cianchi and M. Randolfi, On the modulus of continuity of weakly differentiable functions, Indiana Univ. Math. J. 60 (2011), 1939–1973.
[9] M. Csörnyei, S. Hencl, and J. Malý, Homeomorphism in the Sobolev space $W^{1,\infty}$, J. Reine Angew. Math. 644 (2010), 221–235.
[10] L. D’Onofrio, C. Sbordone, and R. Schiattarella, Grand Sobolev spaces and their applications in geometric function theory and PDEs, J. Fixed Point Theory Appl. 13 (2013), no. 2, 309–340.
[11] H. Fattorini and S. Sritharan, Optimal control problems with state constraints in fluid mechanics and combustion, Appl. Math. Optim. 38 (1998), 159–192.
[12] A. Fiorenza and M. Krbec, A formula for the Boyd indices in Orlicz spaces, Funct. Approx. Comment. Math. 26 (1998), 173–179.
[13] A. Fiorenza et al., Detailed proof of classical Gagliardo–Nirenberg interpolation inequality with historical remarks, Z. Anal. Anwendung 40 (2021), no. 2, 217–236.
[14] A. Fiorenza et al. Gagliardo–Nirenberg inequality for rearrangement-invariant Banach function spaces, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 30 (2019), no. 4, 847–864.
[15] H. Frankowska, Some inverse mapping theorems, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), no. 3, 183–234.
[16] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, Ric. Mat. 8 (1959), 24–51. (Italian).
[17] M. H. Gaydu, M. Geoffroy, and C. Jean-Alexis, An inverse mapping theorem for $h$-differentiable set-valued maps, J. Math. Anal. Appl. 421 (2015), no. 1, 298–313.
[18] L. Greco, T. Iwaniec, and C. Sbordone, Inverting the $p$-harmonic operator, Manuscripta Math. 92 (1997), 249–258.
[19] S. Henc, Bilinear mappings with derivatives of bounded variation, Publ. Mat. 52 (2008), no. 1, 91–99.
[20] S. Henc and P. Koskela, Regularity of the inverse of a planar Sobolev homeomorphism, Arch. Ration. Mech. Anal. 180 (2006), 75–95.
[21] S. Henc, P. Koskela, and J. Onninen, Homeomorphism of bounded variations, Arch. Ration. Mech. Anal. 186 (2007), 351–360.
[22] H. Inci, P. Kappeler, and P. Topalov, On the regularity of the composition of diffeomorphisms, Mem. Amer. Math. Soc. 226 (2013), no. 1062, 1–60.
[23] H. Inci, On Hardy–Littlewood maximal functions in Orlicz spaces, Math. Nachr. 183 (1997), no. 1, 135–155.
[24] P. Kolwicz, K. Lešnik, and L. Maligranda, Pointwise products of some Banach function spaces and factorization, J. Funct. Anal. 266 (2014), no. 2, 616–659.
[25] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II. Function spaces, Ergeb. Math. Grenzgeb., vol. 97, Springer-Verlag, Berlin–New York, 1979.
[26] G. Lozanovskii, On topologically reflexive KB-spaces, Dokl. Akad. Nauk SSSR 158 (1964), 516–519.
[27] J. Manton, Differential calculus, tensor products and the importance of notation, arXiv:1208.0197.
[28] V. Maz’ya, Sobolev spaces, Grundlehren Math. Wissen., vol. 342, Springer-Verlag, Berlin–Heidelberg, 2011.
[29] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) 13 (1959), no. 2, 115–162.
[30] Z. Páles, Inverse and implicit function theorems for nonsmooth maps in Banach spaces, J. Math. Anal. Appl. 209 (1997), no. 1, 202–220.
[31] L. Pick et al., Function spaces. Vol. 1, extended ed., De Gruyter Ser. Nonlinear Anal. Appl., vol. 14, Walter de Gruyter & Co., Berlin, 2013.
[32] T. Shimogaki, Hardy–Littlewood majorants in function spaces, J. Math. Soc. Japan 17 (1965), 365–373.
[33] E. Stein, Singular integrals and differentiability properties of functions, Princeton Math. Ser., No. 30, Princeton University Press, Princeton, N.J., 1970.
[34] S. Vodop’yanov, On regularity of mappings inverse to Sobolev mappings, Dokl. Math. 78 (2008), no. 3, 891–895.
[35] S. Vodop’yanov, Regularity of mappings inverse to Sobolev mappings, Sb. Math. 203 (2012), no. 10, 1383–1410.
[36] T. Yokonuma, Tensor spaces and exterior algebra, Transl. Math. Monogr., vol. 108, Amer. Math. Soc., Providence, RI, 1992.
[37] V. Zorich, Mathematical analysis II, Springer-Verlag, Berlin–Heidelberg, 2004.

How to cite this article: Molchanova A, Roskovec T, Soudský F. Regularity of the inverse mapping in Banach function spaces. Mathematische Nachrichten. 2021;294:2382–2395. https://doi.org/10.1002/mana.201900374