Small time heat expansion of the laplacian on an analytic hypersurface with an isolated singularity

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Abstract

I prove the existence of small time heat expansion for the Laplace operator on an analytic hypersurface with an isolated singularity. First we obtain a local parametrization of the hypersurface near the singularity. We introduce the notion of quasihomogeneous tangent cone. Then perturb the parametrization of the cone employing a Newton scheme and obtain a parametrization with functions of specific form. These allow us to obtain local models for the Laplace operator near the singularity. These are operators with irregular singularities. We derive the estimates required by singular asymptotics.

1 Introduction

The small time heat expansion of a positive second order elliptic operator $H$ defined on a smooth manifold $M$ is used in index calculations of operators that contain geometric information about the manifold $M$.

Specifically, let $L$ be a first order differential operator of geometric interest, $L^\dagger$ its adjoint then the celebrated McKean-Singer formula states that

$$\text{ind}(L) = \text{tr} \left( e^{-tL^\dagger L} - e^{-tLL^\dagger} \right) = \lim_{t \to 0^+} \text{tr} \left( e^{-tL^\dagger L} - e^{-tLL^\dagger} \right)$$

The small time expansion of the self adjoint second order differential operators operators $L^\dagger L, LL^\dagger$ arise naturally. In particular the asymptotic coefficients are spectral invariants of $L^\dagger L, LL^\dagger$.

Cheeger in his resolution of the Ray-Singer conjecture, in the course of topological operations encountered spaces with conical singularities, cones over smooth manifolds. However the classical heat expansions are no longer valid and are modified with additional logarithmic terms. Callias-Taubes departing from quantum field theory came across with such singular heat expansions. Namely the computed determinants that arise in the case of fermions in instanton fields. The main issue is the asymptotics of integrals that are of power-log form. These type of expansions were already common in algebraic geometry and their existence is asserted through the celebrated Atiyah-Bernstein theorem. The direct appeal to the Atiyah-Bernstein theorem is not always possible, a simple case...
is achieved in [P]. Therefore direct analytic methods for the treatment of these integrals were developed and are based either singualar asymptotic lemma ([C0],[C2],[CU]) or Melrose’s [Me1] push forward fomula.

These methods were efficient for singular spaces with cone like singularities. In these cases the operators that appear contain regular singular points and their resolvents are expressed through Bessel functions. Callias obtained the small time expansion for operators with irregular singular points [C2]. Here we built on his results and prove the existence of heat expansion for the laplacian on an analytic hypersurface with an isolated singularity. Indeed the laplacian on a real analytic hypersurface with an isolated singularity leads to an operator with an irregular singularity. This requires a parametrization of the hypersurface near the singularity. In the case of complex algebraic curves this is given by the classical Puiseux expansion and leads again to conical singularities. This was achieved in [BL]. Moreover in the case of varieties defined by quasihomogeneous polynomials there is also such a parametrization, given by the quasihomogeneous blow up. Toric varieties possess also a direct parametrization with monomial maps but is inefficient for our purposes. In the general case of an analytic hypersurface with isolated singularity (abbreviated here as AHIS) it is neccessary to construct a parametrisation and we perform this here. This parametrization is a perturbation of the quasihomogeneous blow up. Actually we reduce an AHIS to its quasihomogeneous tangent cones with the reasonable introduction of quasihomeneous scaling. Then we introduce appropriate function spaces inspired from the Puiseux series and incorporate a suitable Newton method. This construction allow me to calculate the model operator and employ the methods form [C0],[C1],[C2] for the existence of the asymptotic expansion of the distributional trace of the heat operator. The singularity invariants defined through the Newton diagramm appear in the exponents of the exponents of the expansion.

2 Notation and results

Let $H \subset \mathbb{R}^{n+1}$ be the germ of an analytic hypersurface at the origin, having an isolated singularity at the origin. In order to avoid weird situations that are not interesting at the moment we assume that the hypersurface is irreducible and Zariski dense in its complexification. Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the germ at origin of an analytic function that defines $H$ for $\delta > 0$:

$$H \cap B_\delta(0) = \{ |x| < \delta : f(x) = 0 \} \quad (1)$$

Then the Newton polytope

$$N(f) = \{ \alpha \in \mathbb{N}^{n+1} / f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}(0) \neq 0 \} + \mathbb{N}^{n+1} \quad (2)$$

with Newton diagram $\Delta(f)$ i.e. the union of the compact faces of $N(f)$. If $\Gamma_j$ is a compact face of $N(f)$ then

$$\Delta(f) = \bigcup_{j=1}^N \Gamma_j$$

The face $\Gamma$ defines a quasihomogeneous polynomial of type $(\sigma_1, \ldots, \sigma_{n+1}, m_\Gamma) \in \mathbb{Q}^{n+2}$ and call $\sigma_1, \ldots, \sigma_{n+1}$ quasihomogeneity exponents

$$f_\Gamma(x) = \sum_{\alpha \in \Gamma} f_\alpha x^\alpha \quad (3)$$
since
\[ \gamma = (\gamma_1, \ldots, \gamma_{n+1}) \in \Gamma : \gamma_1 \sigma_1 + \cdots + \gamma_{n+1} \sigma_{n+1} = m \Gamma \]
and
\[ f_{\Gamma}(\lambda^{\sigma_1}x_1, \ldots, \lambda^{\sigma_{n+1}}x_{n+1}) = \lambda^{m_{\Gamma}} f_{\Gamma}(x_1, \ldots, x_{n+1}) \]
Its zero set \( \mathcal{H}_\Gamma \). Let also
\[ E_{\Gamma}(\sum_{j=1}^{n+1} \sigma_j x_j) = \frac{\partial}{\partial x_j} \]
be the corresponding Euler vector fields,
\[ E_{\Gamma}(f_{\Gamma}) = m_{\Gamma} f_{\Gamma} \]
Introduce the function:
\[ \phi : \mathbb{R}^{n+1} \to \mathbb{R} : \phi_{\Gamma}(x) = x_{\sigma_1}^2 + \cdots + x_{\sigma_{n+1}}^2 \]
that is obviously quasihomogeneous of type \((\sigma_1, \ldots, \sigma_{n+1}, 2m)\) Consider further the sets that we call real Brieskorn spheres \( s_{\Gamma}(\epsilon) \)
\[ S_{\epsilon}^\Gamma(0) = \{ x \in \mathbb{R}^{n+1} : \phi_{\Gamma}(x) = \epsilon^{2m} \} \]
Write also
\[ f(x) = f_{\Gamma}(x) + R_{\gamma}(x) \]
We will see that the hypersurface \( \mathcal{H} \) decomposes near the origin into semianalytic pieces \( L_{\gamma} \) that reveal the approximation by the quasihomogeneous tangent cones that we define and provide the local parametrization of the hypersurface. Recall that the degree of the Gauss map of the link is the Milnor number \( \mu \) of the singularity and gives a bound for the number of branches of the hypersurface that emanate from the origin.
We introduce also the space of functions \( P(\epsilon, \Omega) \) generalizing Puiseux series and we specify later. Denote also \( C_{\epsilon}(\Omega) = [0, \epsilon] \times \Omega \)

**Theorem 2.1** Let \( \epsilon_j > 0, j = 1, \ldots, N \). Let \( \Omega_j \subset \mathbb{R}^{n-1} \) then there is a collection of maps for:
\[ \Phi_j : C_{\epsilon_j}(\Omega_j) \to \mathbb{R}^{n+1}, \quad \Phi_j (C_{\epsilon_j}(\Omega_j)) \subset \mathcal{H} \]
such that \( \Phi_j \in P(\epsilon, \Omega)^{n+1} \) with a sequence of rational exponents \( \{ q_{j,\ell,k}, j = 1, \ldots, N, \ell = 1, \ldots, n + 1, k = 1, 2, \ldots \} \)
determined by the Newton digramma.
These maps are essentially a perturbation of the quasihomogeneous blow-up. Their construction is based on the Newton method. The essence of this method is that it could be generalized to arbitrary codimension and complicated singularities; this is under preparation at the moment.
Having obtained this parametrization we have also model operators in hand that allow us to prove the existence of the small time heat expansion for the Laplace-Beltrami \( \Delta_{\mathcal{H}} \) on \( \mathcal{H} \) treating its Friedrichs self-adjoint extension. Let
\[ U = B_{\delta}(0) \]
and split this as

$$U = \bigcup_{j=1}^{N} U_j$$

Indeed we prove the following

**Theorem 2.1** Let $\chi_j \equiv 1$ in $U_j$ and $\chi \equiv 0$ outside $U_j$. Then for $\chi = \sum_j \chi_j$ we have that as $t \to 0^+$:

$$\text{tr}(\chi e^{-t\Delta n}) \sim t^{-\frac{n}{2}} \sum_{k=0}^{\infty} c_k t^k + \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} u_{l,i}[\chi]^{\alpha_i} \log^l t$$

where $\{\alpha_i\}$ is an increasing sequence of rational numbers determined explicitly by the sequences $\{\mu_{j,l,k}\}$. The map $S : \mathbb{N} \to \mathbb{N}$ is also determined by the Newton diagramm of the singularity. The $c_k$’s are usual heat coefficients while the $u_{l,i}$’s are distributions with supp$(u_{l,k}) = \{0\}$.

### 3 The quasihomogeneous case

Let $\mathcal{H}$ be an AHIS with Newton polytope $\Delta(f)$. We assume that $f$ is chosen so that no coordinate plane is contained in $\mathcal{H}$. This possible after a suitable rotation. We will consider first the case of $\Delta(f)$ consisting of a single face $\Gamma$. Then we deal with a *quasihomogeneous hypersurface singularity*, upere. Then we have the vector $(\sigma_1, \ldots, \sigma_{n+1}) \in \mathbb{N}^{n+1}$ and $m$ such that for all $\gamma \in \Gamma$

$$\gamma_1 \sigma_1 + \cdots + \gamma_{n+1} \sigma_{n+1} = m$$

Then consider the intersection with real Brieskow spheres, $\epsilon > 0$:

$$L(\epsilon) = S^\Gamma(0) \cap \mathcal{H}$$

is smooth for all $\epsilon > 0$ due to the fact that on $L(\epsilon)$

$$E_r(f) = 0 \quad E_r(\phi_{\Gamma}) = 2mr^2m > 0$$

and hence $\mathcal{H}$ and $S^\Gamma(0)\epsilon$ meet transversely.

Then write look for $\xi$ such that

$$f(\xi) = 0 \quad \phi_{\gamma}(\xi) = 1$$

and due to trasversality and the implicit function theorem there exist $\Omega \subset \mathbb{R}^{n-1}$ and maps such that after a possible renaming of the variables such that for $\xi' = (\xi_3, \ldots, \xi_{n+1})$

$$\Psi_1, \Psi_2 : \Omega \to \mathbb{R}, \quad \xi_1 = \Psi_1(\xi'), \xi_2 = \Psi_2(\xi')$$

Therefore we have that for a mapping $\zeta : \Omega \to \mathbb{R}^{n+1}, \Omega \subset \mathbb{R}^{n-1}$ if we denote $\eta = \xi'$

$$x_j = r^{\sigma_j} \zeta_{\eta_j}$$

Then the parametrization is given if we perform $r \mapsto r^\lambda$ so that we make $\min\{\sigma_1, \ldots, \sigma_{n+1}\} = 1$

$$x_1 = r^{\sigma_1} \Psi_1(\eta), \quad x_2 = r^{\sigma_1} \Psi_2(\eta)$$
and for \( i > 2 \) we have that
\[
x_i = r^{\sigma_i} \eta_i^{-2}
\]

Then the metric induced on \( \mathcal{H} \) takes the form
\[
g_{\mathcal{H}} = \chi(r, \eta) dr^2 + \beta(r, \eta) dr + \gamma_L(r, \eta)
\]

where
\[
\chi(r, \eta) = \sum_{j=1}^{n+1} \alpha_j^2 r^{2(\alpha_j - 1)} \zeta_j
\]
\[
\beta(r, \eta) = 2 \sum_{j,k} \alpha_j r \zeta_j \zeta_{j,k} d\eta_k
\]
\[
\gamma(r, \eta) = \sum_{i,j=1}^{n-1} \gamma_{k\ell}(r, \xi) d\eta_k d\eta_{\ell}
\]
\[
\gamma_{k\ell}(r, \eta) = r^2 \sum_{j=1}^{n+1} r^{\alpha_j + \alpha_k - 2} \zeta_{j,k} \zeta_{j,\ell}
\]

Notice that introducing the \( (n+1) \times (n-1) \) matrix
\[
\Lambda_{jk} = \zeta_{j,k}
\]
as well as the \( (n+1) \times (n+1) \) diagonal matrix
\[
D_{jj} = \alpha_j r^{\alpha_j - 1}, \quad d\eta = (d\eta_1, \ldots, d\eta_{n-1})
\]

we have that
\[
\chi(r, \eta) = ||D\zeta||^2
\]
\[
\beta(r, \eta) = 2(\zeta, D\Lambda d\eta)
\]
\[
\gamma(r, \eta) = r^2 \Lambda^T D^2 \Lambda
\]

### 4 Puiseux functions

Let \( \Omega = [-\delta, \delta]^{n} \subset \mathbb{R}^n, \delta > 0 \) be a cube and \( \epsilon > 0 \) and form the cylinder \( C_\epsilon(\Omega) = [0, \epsilon] \times \Omega \subset \mathbb{R}^{n+1} \).

Let \( \{q_n\} \subset \mathbb{Q}^+ \) be an increasing sequence of nonnegative rational numbers and
\[
f_n : \Omega \to \mathbb{R}, \Omega \ni \xi \mapsto f(\xi)
\]
is a sequence of analytic function such that
\[
\sum_{n=1}^\infty \sup_{\Omega} ||f_n|| \epsilon^{q_n} < \infty
\]

where an analytic function in \( \Omega \)
\[
f(\xi) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} x^\alpha
\]
the norm is

$$||f||_\Omega = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \eta^{\alpha}$$

We call the function

$$f(r, \xi) = \sum_{n=1}^{\infty} f_n(\xi) r^{q_n}$$

a Puiseux function and denote their space $P(\Omega, \epsilon)$. It is elementary to check that the sum and products of Puiseux functions are also Puiseux functions by merging the sequences of exponents. We also define the norm in $P(\Omega, \epsilon)$ as

$$||f||_{P(\omega, \epsilon)} = \sum_{n=1}^{\infty} ||f_n||_{\Omega} \epsilon^{q_n}$$

and $P(\Omega, \epsilon)$ is a Banach space. Let $\Omega' \subset \mathbb{R}^k, \Omega \subset \mathbb{R}^n$ domains and $g : \Omega' \to \mathbb{R}$ be a real analytic function and $f_1, \ldots, f_k \in P(\Omega, \epsilon)$. Then for $F = (f_1, \ldots, f_k)$ the function $g \circ F : C_\epsilon(\Omega) \to \mathbb{R}$ is in $P(\Omega, \epsilon)$. Indeed let the partial sums

$$g_N(x) = \sum_{n=0}^{N} \sum_{|\alpha| \leq k} g_\alpha x^\alpha$$

that converges in the polydisc $Q(\eta) = [-\eta_1, \eta_1] \times [-\eta_k, \eta_k]$. We have also

$$x_j^{N_j} = \sum_{s=1}^{N_j} f_{m_j^s}(\xi) r^{q_j^s}$$

Moreover assume that after possible shrinking $\Omega'$:

$$\sup_{\xi \in \Omega} |f_{m_j^s}| r^{q_j^s} \leq \eta_j \quad \Omega' \subset Q(\eta)$$

Set also $s = (s_1, \ldots, s_k)$ and

$$Q : \mathbb{N}^k \times \mathbb{Q}^k_+ \to \mathbb{Q}$$

$$Q(\alpha, s) = a_1 q_{s_1}^1 + \cdots + a_k q_{s_k}^k$$

We form an increasing sequence that we denote as $\{Q_m\}$ and set as $Q = \max\{q_1, s_1 \leq N_1, \cdots, s_k \leq N_k\}, Q_{N'} = Q$. Furthermore

$$S_r = \{ (\alpha, s) : q(\alpha, s) = Q_r \}$$

Then we compute

$$g_N(r, \xi) = g_N(r, \xi) = \sum_{r=1}^{N'} G_r(\xi) r^{Q_r}$$

where

$$G_r(\xi) = \sum_{(\alpha, s) \in S_r} \prod_{j=1}^{k} \sum_{\beta_j = \alpha_j} \frac{g_{\alpha}}{(\beta_1 j \cdots \beta_{N_j})} f_{m_j^s}^{\beta_1 j} \cdots f_{m_j^s}^{\beta_{N_j}}$$

6
Hence
\[ \sup_{\xi \in \Omega} |G(\xi)| \leq \sum_{(\alpha,s) \in S_r} |g_{\alpha}| \prod_{j=1}^{k} \left( \sum_{\beta_1^{j_1} + \cdots + \beta_{N_j}^{j}} \left( \beta_1^{j_1} \cdots \beta_{N_j}^{j} \right) \sup_{\xi \in \Omega} |f_{m_1}^{j_1} \cdots f_{m_{N_j}}^{j} | \beta_1^{j_1} \cdots \beta_{N_j}^{j} \right) \]

Then
\[ \sum_{r=1}^{\infty} \sup_{\xi \in \Omega} |G(\xi)| e^{rq} \leq \sum_{\ell=0}^{\infty} \sum_{|\alpha| = \ell} |g_{\alpha}| |\eta|^\alpha < \infty \]

5 Quasihomogeneous tangent cones

For a face of the Newton diagramm of the singularity \( \Gamma \) with vector \( \sigma = (\sigma_1, \ldots, \sigma_{n+1}) \):
\[ \sigma_1 \gamma_1 + \cdots + \sigma_{n+1} \gamma_{n+1} = m_{\Gamma} \]
for all \( \gamma = (\gamma_1, \ldots, \gamma_{n+1}) \in \Gamma \). Then we introduce the scaling operator for \( t > 0 \):
\[ S_{t,\sigma}(x) = t^{\sigma_1} x_1, \ldots, t^{\sigma_{n+1}} x_{n+1} \]
Then we introduce as quasihomogeneous tangent cone \( K_{\Gamma}(H) \) of \( H \) at the singular point at 0 as:
\[ K_{\Gamma}(H) = \{ \eta \in \mathbb{R}^{n+1} : \eta = \lim_{S_{t,\alpha}\rightarrow +\infty} t x_j, x_j \in H \} \]
We show that \( K_{\Gamma}(H) \) is analytic set: for \( \eta \in K_{\Gamma}(H) \) then
\[ f_{\Gamma}(\eta) = 0 \]
Write
\[ f(x) = f_{\Gamma}(x) + R_{\Gamma}(x) \]
Then for the sequence \( \{x_j\} \subset H \) we have that
\[ f(x_j) = 0 \Rightarrow f_{\Gamma}(x_j) + R_{\Gamma}(x_j) = 0 \]
Then since as \( t_j \rightarrow \infty \):
\[ x_j = \eta_j + \rho_j, \eta_j = S_{1/t_j,\alpha} \eta, \rho_j = S_{1/t_j,\alpha} \rho, \quad |\rho| = o(1) \]
we have that due to the analyticity of \( f \) and Taylor series
\[ \frac{1}{t_j^{m_{\Gamma}}} (f_{\Gamma}(\eta) + \rho) + \frac{1}{t_j^{m_{\Gamma}}} (R_{\Gamma}(\eta) + \rho) = 0 \Rightarrow f_{\Gamma}(\eta) + o(1) + \frac{1}{t_j^{m_{\Gamma} - m_{\Gamma}}} (R_{\Gamma}(\eta) + o(1)) = 0 \]
Since by convexity of the Newton diagramm \( m_{\Gamma} > m_{\Gamma} \) taking the limit as
\[ f_{\Gamma}(\eta) = 0 \]
and consequently as well
\[ f_{\Gamma}(\eta_j) = 0 \]
Also we have that for $\eta \in K^\Gamma(H)$, $x_k = \eta_k + \rho_k$

$$|f(\eta)| \leq \frac{c_1}{k^{m'-m}} \quad |f(\eta_k)| \leq \frac{c_1}{k}$$

and selecting $\epsilon > 0$

$$\frac{c_1}{k^{m'-m}} < \epsilon$$

Also we have that

$$|f(\eta)| \leq c_1 \epsilon \quad |f(\eta_k)| \leq c_1 \epsilon$$

Similarly

$$\nabla f(\eta) = \nabla f\Gamma(\eta) + v(\eta) \quad (1)$$

for $c_2 > 0$:

$$|v(\eta)| \leq c_2 \epsilon$$

Therefore denoting $g_j = \frac{\partial g}{\partial x_j}$:

$$\langle \nabla f(x_k), \nabla f\Gamma(x_k) \rangle^2 = |\nabla f(x_k)|^2 |\nabla f\Gamma(x_k)|^2 - \sum_{i<j} (f_i(\eta) f_i, j(x_k) - f_j(x_k) f_i, i(x_k))^2$$

Now for

$$n_{\Gamma, i} = \frac{f_{\Gamma, i}}{|\nabla f\Gamma|}, \quad r_{\Gamma, i} = \frac{R_{\Gamma, i}}{|\nabla f|}$$

we have through (1):

$$\sum_{i<j} (f_i(x_k) f_{\Gamma, j}(x_k) - f_j(x_k) f_{\Gamma, i}(x_k))^2 = |\nabla f\Gamma|^2 |\nabla f|^2 \sum_{i<j} (r_i(x_k) n_{\Gamma, j}(x_k) - r_j(x_k) n_{\Gamma, i}(x_k))^2$$

Since

$$|r_{\Gamma, j}(x_k)| \leq \frac{c}{k^{m'-m}}$$

that

$$| \langle \nabla f(x_k), \nabla f\Gamma(x_k) \rangle | \geq (1 - \epsilon^2) |\nabla f(x_k)||\nabla f\Gamma(x_k)|$$

Continuity of $f, f\Gamma$ allows us to conclude that for $\epsilon > 0$ there is $\delta$ such that for $x \in \mathbb{R}^{n+1}, |f\Gamma(x)| \leq \delta$ that

$$| \langle \nabla f(x), \nabla f\Gamma(x) \rangle | \geq (1 - \epsilon^2) |\nabla f(x)||\nabla f\Gamma(x)|$$

Notice that the analytic set $K^\Gamma(H)$ does not have neccessarily isolated singularities! However the semianalytic set defined by

$$K^\epsilon, \delta(H) = K^\Gamma(H) \cap \{ x \in \mathbb{R}^{n+1} : |f(x)| < \delta \} \cap \{ x \in \mathbb{R}^{n+1} : \phi(x) \leq \epsilon \}^{2m}$$

has an isolated singularity at the origin. We refer to this set as the quasihomogeneous tangent cone of height $\delta, d_\epsilon$ close to $H$, where $d_\epsilon = O(\epsilon^{m\Gamma})$. Similarly we have that

$$H^\epsilon, \delta = H \cap \{ x \in \mathbb{R}^{n+1} : |f\Gamma(x)| < \delta \} \cap \{ x \in \mathbb{R}^{n+1} : \phi(x) \leq \epsilon \}^{2m}$$
6 The Newton scheme

We set up a Newton scheme that provides the parametrization of the analytic hypersurface near its singular point. Specifically we will obtain a perturbation of the parametrization of the quasi-homogeneous singularity. The perturbation lies in $P(\epsilon, \Omega)$ for suitable $\epsilon, \Omega$. Let $\eta \in K^{\epsilon, \delta}_{\Gamma}(\mathcal{H})$ then we set

$$\eta = S_{r, \alpha}(\zeta), \quad \zeta \in L^{\epsilon, \delta}_{\Gamma} = K^{\epsilon, \delta}_{\Gamma}(\mathcal{H}) \cap S^{\Gamma}_{r}(0)$$

which is a smooth analytic hypersurface with an analytic parametrization for $\Omega \subset \mathbb{R}^{n-1}$:

$$\zeta : \Omega \to \mathbb{R}^{n+1} : \xi \mapsto \zeta(\xi), \quad f(\zeta(\xi)) = \phi_{\Gamma}(\zeta(\xi)) = \delta^{2m}$$

Then we introduce the map

$$N(t, \eta) = t - \frac{f(\eta + t n_{\Gamma}(\eta))}{f'(\eta + t n_{\Gamma}(\eta))} f''(\eta + t n_{\Gamma}(\eta))$$

for

$$n_{\Gamma}(\eta) = \frac{\nabla f_{\Gamma}(\eta)}{|\nabla f_{\Gamma}(\eta)|}$$

Notice that since

$$f_{\Gamma, j}(S_{r, \alpha} \xi) = r^{m-\alpha_j} f_{\Gamma, j}(\xi)$$

then

$$n_{\Gamma}(\eta)$$

is a regular function in $C_{\epsilon}(\Omega)$. Moreover the map is well defined in $P(\epsilon, \Omega)$.

$$N : P(\epsilon, \Omega) \to P(\epsilon, \Omega)$$

We establish that $N$ is a contraction in $P(\epsilon, \Omega)$. We assume that uniformly in $\xi$:

$$t = O(r^\mu), \quad \mu = |\alpha|_{\infty}$$

We compute

$$N'(t) = \frac{f(t)f''(t)}{(f'(t))^2}$$

where

$$f'(t) = \nabla f(\eta + t n_{\Gamma}(\eta)) \cdot n_{\Gamma}(\eta)$$

$$f''(t) = \sum_{i,j=1}^{n+1} f_{ij}(\eta + t n_{\Gamma}(\eta)) n_{\Gamma, i}(\eta) n_{\Gamma, j}(\eta)$$

Then by Cauchy-Schwarz and the estimate introduced above

$$|N(t)| \leq \frac{|f(t)|}{(1 - \epsilon^2)|\nabla f(\eta + t n_{\Gamma})|^2}$$

Notice that this again regular since for $t = O(r^\mu)$

$$|H(f)| \leq C r^{m-2\mu}$$

Then since for suitable choice of $\epsilon, \delta$ for the link $L^{\epsilon, \delta}_{\Gamma}$

$$|f(\eta + t n_{\Gamma}(\eta))| \leq r^m \left( |f_{\Gamma}(\eta)| + r^{m'-m} R_{\Gamma}(\eta) \right) \leq \delta r^m$$
for $\delta < 1$. We look up for function with exponents given by linear combinations of the form for the quasihomogeneity exponents $\sigma_1, \ldots, \sigma_{n+1}$

$$
\alpha_1 \sigma_1 + \cdots + \alpha_{n+1} \sigma_{n+1} \quad \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{Z}_{\geq 0}
$$

$$
T_n(r, \xi) = \sum_{\ell=1}^{\infty} T_{n,\ell}(\xi) r^{q_n(\ell)}
$$

then

$$
T_{n+1}(r, \xi) - T_n(r, \xi) = \phi'(T_n) T_n(r, \xi) - T_{n-1}(r, \xi) + \frac{1}{2} \phi''(\xi_n) (T_n - T_{n-1})^2
$$

Then setting $\zeta_n(r, \xi) = T_n(r, \xi) - T_{n-1}(r, \xi)$ we have that

$$
\zeta_{n+1} = \phi'(T_{n-1}) \zeta_n + \frac{1}{2} \phi''(\xi_n) \zeta_n^2
$$

Substituting and collecting terms we obtain for $c < 1$ and the uniform norm:

$$
\|\zeta_{n+1,1}\| \leq c \|\zeta_{n,1}\|
$$

and for $\ell > 1$:

$$
\|\zeta_{n+1,\ell}\| \leq c \|\zeta_{n,\ell}\| + F(\|\zeta_{n,1}\|, \ldots, \|\zeta_{n,\ell-1}\|)
$$

This leads to recursive inequalities of the form

$$
\|\zeta_n\| \leq c \|\zeta_{n-1}\| + \kappa_1 n e^n
$$

that leads to

$$
\|\zeta_n\| \leq \kappa_2 n^2 e^n
$$

and hence that for $n > m$:

$$
\|T_n - T_m\| \leq \kappa_2 \sum_{k=m}^{\infty} k e^k \to 0
$$

Furthermore we differentiate (IR) and obtain for any $\alpha \in \mathbb{N}^{n-1}$

$$
D^\alpha \zeta_{n+1} = \left( \phi'(T_{n-1}) - \frac{1}{2} \phi''(\xi_n) \right) D^\alpha \zeta_n + F(\zeta_n, D\zeta_n, \ldots, D^{\alpha-1} \zeta_n)
$$

This in turn leads to the following series of inequalities for the uniform norm

$$
\|D^\alpha \zeta_{n+1}\| \leq c_2 \|D^\alpha \zeta_n\| + F(\|\zeta_n\|, \|D\zeta_n\|, \ldots, \|D^{\alpha-1} \zeta_n\|)
$$

and get for $k$ depending on $\alpha$

$$
\|D^\alpha \zeta_{n+1}\| \leq c_\alpha \|D^\alpha \zeta_n\| + \kappa_\alpha n^k
$$

Then we get the desired convergence of $D^\alpha T_n$ and therefore the limit belongs indeed in $P(\epsilon, \Omega)$. Therefore we have that $T_n$ converges in $P(\epsilon, \Omega)$.

We have shown that if $x \in \mathcal{H}_{\Gamma}^{\epsilon, \delta}$ there exist $\xi, T(r, \xi)$ such that

$$
x = \chi(r, \xi), \quad \chi(r, \xi) = \zeta(\xi) + T(r, S_{r,\alpha} \zeta(\xi)) n_{\Gamma}(\zeta(\xi))
$$
Then the parametrization is derived setting

\[ \alpha_i = \min \{\alpha_1, \ldots, \alpha_n, 1\} \]

and

\[ \nu_j = \begin{cases} \frac{\alpha_j}{\alpha_i}, & \text{for } j \neq i \\ 1, & \text{for } j = i \end{cases} \]

and

\[ x_k = r^{\nu_k} \chi_k(r, \eta) \]

where \( \chi_k \in P(\epsilon, \Omega) \). Moreover we have that for \( f_{r,j} \neq 0 \) we have that for \( m \neq j \)

\[ \chi_m(0, \eta) = \eta_m \]

7 The Laplacian operator

7.1 The metric model

Given the parametrization obtained above

\[ C_\epsilon(\Omega) \to \mathcal{H} \]

that has the form with \( \zeta_j \in P(\epsilon, \Omega) \):

\[ x_j = r^{\nu_j} \zeta_j(r, \eta) \quad j = 1, \ldots, n + 1 \]

and uniformly in \( \eta \) as \( r \to 0^+ \):

\[ \zeta_j(r, \eta) = \tilde{\zeta}_j + r_j(r, \eta) \quad r_j(r, \eta) = o(1) \]

we compute

\[ dx_j = \nu_j r^{\nu_j-1} \chi_j dr + \sum_{k=1}^{n-1} r^{\nu_k} \zeta_{j,k} d\eta_k, \]

where

\[ \chi_j(r, \eta) = \zeta_j(r, \eta) + \frac{1}{\nu_j} r^{\tilde{\zeta}_j,r} \]

The induced metric on the hypersurface is then

\[ g = \omega dr^2 + \sum_{i=1}^{n-1} \beta_i d\eta_i dr + \Sigma \]

where

\[ A = \text{diag}(\nu_1, \ldots, \nu_{n+1}) \]

\[ R = \text{diag}(r^{\nu_1-1}, \ldots, r^{\nu_{n+1}-1}) \]

\[ \zeta = (\zeta_1, \ldots, \zeta_{n+1}) \]

\[ \chi = (\chi_1, \ldots, \chi_{n+1}) \]

\[ \beta = (\beta_1, \ldots, \beta_{n-1}) \]

\[ \Lambda_{k,j} = \zeta_{k,j}, k = 1, \ldots, n + 1, j = 1, \ldots, n - 1 \]

\[ \omega(r, \eta) = ||AR\chi||^2 \]

\[ \beta(r, \eta) = r \Lambda(r, \eta)^T AR^2 \chi \]

\[ \Sigma(r, \eta) = r^2 A^T R^2 \Lambda \]
Notice that $\lambda > 0$ for $r > 0$. We will modify further the metric removing the cross term. Therefore solving the system in $\mathcal{P}$ asking for the flow out of the link through the initial conditions

$$\eta_i(\epsilon) = \theta_i$$

as

$$\eta_j = \phi_j(r, \theta)$$

This leads to the singular non-linear system:

$$\eta' = -\Sigma^{-1}R\beta, \quad \eta' = \frac{dp}{dr}$$

if we set $M = R\Lambda, \Sigma = M^TM$. Actually it is an elementary fact that for $\mu = \max\{\nu_1, \ldots, \nu_{n+1}\} - 1$

where $p, q \geq 1, q = \frac{p}{p-1}$.

This system has a unique solution for $r > 0$ and we will show that the solution extends to $r = 0$.

Introduce the function

$$\phi(\zeta) = \zeta_1^{2m/\nu_1} + \ldots + \zeta_{n+1}^{2m/\nu_{n+1}}$$

Notice that for suitable choice of $\epsilon > 0$ and $r \in [0, \epsilon]$ we have that

$$\phi(\zeta) > c||\eta||_2^{2m}$$

Moreover we compute

$$\phi' = 2m \left[ \frac{1}{\nu_1} \zeta_1^{2m/\nu_1-1} \zeta_1' + \ldots + \frac{1}{\nu_{n+1}} \zeta_{n+1}^{2m/\nu_{n+1}-1} \zeta_{n+1}' \right]$$

We have that

$$\zeta' = \frac{\partial \zeta}{\partial t} = -\Lambda\Sigma^{-1}R\beta$$

Noting that

$$\left| \frac{\partial \zeta}{\partial t} \right| \leq C|\Lambda\Sigma^{-1}R\beta|$$

we invert time as $t = \epsilon - r$ and consider the equation

$$\dot{\zeta} = \frac{\partial \zeta}{\partial t} + \Lambda\Sigma^{-1}R\beta \quad \dot{\zeta} = \frac{d\zeta}{dt}$$

Apply Hölder inequality and arrive at

$$\left| \left[ \frac{1}{\nu_1} \zeta_1^{2m/\nu_1-1} \zeta_1' + \ldots + \frac{1}{\nu_{n+1}} \zeta_{n+1}^{2m/\nu_{n+1}-1} \zeta_{n+1}' \right] \right| \leq 2m \left[ \zeta_1^{(2\lambda_1-1)p} + \ldots + \zeta_{n+1}^{(2\lambda_{n+1}-1)p} \right]^{1/p} \||\zeta'||_q$$

for

$$\lambda_j = \frac{m}{\nu_j}$$

Now for suitable choice of $\delta > 0$ depending on $\epsilon$ we have that

$$||\zeta||_p \leq C||\Lambda\Sigma^{-1}R\beta|| \leq \delta(\epsilon - t)^{-2\mu+2}||R\beta||_p \leq C\delta(\epsilon - t)^{-2\mu+4}||\zeta||_p$$
Then we select for suitable $e > 0$

$$p = \frac{2\Lambda}{2\Lambda - 1} + e \quad \Lambda = \frac{m}{\nu}, \quad \nu = \max\{\nu_1, \ldots, \nu_{n+1}\}$$

or

$$p = 1 + e + \kappa, \quad \kappa = \frac{1}{2\Lambda - 1} q = 1 + \frac{1}{\kappa + e}$$

and implies

$$(2\lambda_j - 1)p > 2\lambda_j + e(2\lambda_j - 1)$$

This in turn for $\zeta^{2\lambda_j} < \phi$

$$\left[\zeta^{(2\lambda_j-1)p} + \cdots + \zeta^{(2\lambda_{n+1}-1)p}\right]^{1/p} \leq \phi^{\frac{1+m}{2m}}$$

Also we have that since $\frac{\eta}{2\lambda_j} > \frac{q}{2m}$ and some constant $c > 0$

$$\left[|\zeta_1|^q + \cdots + |\zeta_{n+1}|^q\right]^{1/q} = \left[\zeta_1^{\frac{2\lambda_1 q}{\lambda_1}} + \cdots + \zeta_{n+1}^{\frac{2\lambda_{n+1} q}{\lambda_{n+1}}}\right]^{1/q} \leq c \phi^{1/2m}$$

Therefore we end up for $C$ an $\delta > 0$ to be chosen

$$\dot{\phi} \leq C\delta(\epsilon - t)^{-2\mu + \frac{1}{2}} \phi^{1+\tau}$$

where for suitable $e$

$$\tau = \frac{1}{2m} - \frac{\kappa}{1 + \kappa + e} > 0$$

Then we have that for suitable $C, \alpha > 0$ we have that

$$\chi(r) = Cr^\alpha \quad \chi(t) = \chi(\epsilon - t), \quad \dot{\chi} = \frac{\kappa}{(\epsilon - t)^2\mu - 2}\lambda^{1+e}$$

Then employ the elementary estimate for

$$\psi(t) = \frac{\phi(t)}{\chi(t)}$$

that satisfies the inequality

$$\dot{\psi}(t) \leq \frac{\kappa}{(\epsilon - t)^2\mu - 2} \cdot \psi \cdot (\psi^e - 1)$$

Then if we arrange $\epsilon$ so that $\psi(0) < 1$ then we have that

$$\psi(t) < 1$$

and therefore we have the desired inequality

$$\phi(\zeta) \leq \delta r^\alpha$$

and hence that the limits exist and vanishes. In conclusion we have that the metric admits the expression

$$g = \omega(r, \theta) dr^2 + \Sigma(r, \theta)$$
The hypersurface does not contain any coordinate axis and hence
\[ \omega(r, \theta) > 0 \]
for suitable small \( \epsilon \) and \( 0 \leq r \leq \epsilon \). Therefore we consider the metric
\[ \tilde{g} = dr^2 + \tilde{\Sigma}(r, \theta) \]
for
\[ \tilde{\Sigma} = \frac{\Sigma}{\omega} \]

### 7.2 The operator model

Employing now the metric near the singularity we obtain the model operators near each sector defined by the quasihomogeneous cones where the constants \( \kappa, \mu, \alpha \) have no reference to the constants introduced in the preceding sections while \( \alpha \) depends on the sector
\[ \Delta_g = \frac{1}{\sqrt{\sigma(r, \theta)}} \partial_r (\sqrt{\sigma(r, \theta)} \partial_r + \Delta_{\tilde{\Sigma}}) \]
where
\[ \sigma = \det(\tilde{\Sigma}) \]

Then we will use the model operator model
\[ H_\alpha = H_{0, \alpha, \kappa} + Q_\alpha \]
where for \( k \geq \alpha > 2 \)
\[
H_{0, \alpha} = \partial_r^2 + \frac{\Delta_k}{r^{n-1}} \\
Q = \frac{L}{r^{n-1}} + V(r, \theta) \\
V(r, \theta) = O(r^{-2}) \\
\Delta_k = \sum_{i=1}^{k} \partial_{\theta_i}^2 \\
\Delta_{n-1} = \sum_{i=1}^{n-1} \partial_{\theta_i}^2 \\
L = L_0 - \Delta_k + L_r \\
L_0(\phi) = \frac{1}{\sqrt{\sigma}} \sum_{i,j=1}^{k} \partial_{\theta_i} (\sqrt{\sigma} \Sigma_{ij} \partial_{\theta_j} \phi) \\
L_r(\phi) = \frac{1}{\sqrt{\sigma}} \sum_{i,j=k+1}^{n-1} \partial_{\theta_i} (\sqrt{\sigma} \Sigma_{ij} \partial_{\theta_j} \phi) \\
R_{0, \alpha, \lambda} = (H_{0, \alpha} - \lambda)^{-1}
\]
We start with freezing the coefficients of $L$ in the $\theta$ variables constructing a partition of unity $\{\chi_\ell\}_{\ell=1}^N$ subordinate to the cover $\{U_\ell\}_{\ell=1}^N$:

$$\Omega = \bigcup_{\ell=1}^N U_\ell$$

such that for $\Delta_k = \sum_{i=1}^k \partial_{\theta_k}^2$

$$||L_0 - \Delta_k|| \leq \frac{c}{|\lambda|}$$

This is obtained using standard estimates for the metric in terms of geodesic normal coordinated in the $\theta$-variables and uniform in $r$. We denote

$$H_\alpha = -\partial_r^2 - \Delta_{r^\alpha}$$

8 The heat expansion

8.1 Operator domain and estimates

The operator domain $D_{0,\alpha} \subset (L^2(C_c(\Omega)))$ that is specified through the following inequality:

**Lemma 8.1** There exist constants $c, B > 0$ such that for all $\phi \in C_0^\infty(C_c(\Omega))$:

$$||H_\alpha \phi||^2 \geq ||\partial_r^2 \phi||^2 + c||\Delta_{r^\alpha} \phi||^2 - B||\phi||^2$$

We follow the lines of Lemma A.1 in [C2] and compute since $[\Delta, \partial_r] = 0$:

$$(H_\alpha)^2 = \partial_r^4 + 2\Delta_{r^\alpha/2}(-\partial_r^2)r^{-\alpha/2} - \frac{\Delta_{\alpha/2}^2}{2r^\alpha + 2} + \frac{\Delta^2}{r^{2\alpha}}$$

Now use the inequalities $\phi \in C_0^\infty(C_c(\Omega))$:

$$\int_R r^{-2-\alpha} \phi^2 \leq C(\alpha) \left( \int_R \partial_r \left( r^{-\alpha/2} \phi \right) \right)^2$$

$$\int_\Omega \phi^2 \leq C \int_\Omega |\nabla \phi|^2 \quad \int_\Omega |\nabla \phi|^2 \leq C \int_\Omega |\Delta \phi|^2$$

and since $\alpha > 2$

$$r^{-\alpha-2} \leq \varepsilon r^{-2\alpha} + B$$

and the inequality follows.

This defines the unique self adjoint extension of $H_\alpha$ as an unbounded operator in $L^2(C_c(\omega))$. Then along the same lines of Proposition A.2 we have that for $R_\alpha(\lambda) = (\lambda - H_\alpha)^{-1}$

$$B_{\beta,d}(\lambda) = r^{-\beta} \Delta_{r^\alpha} \partial_r^d R_\alpha(\lambda)$$

and we have that

$$||B_{\beta,d}(\lambda)|| \leq C_{c}\varepsilon|\lambda|^{-1+\frac{\beta}{\alpha}+\frac{d}{2}}$$

for $\Re(\lambda) < \varepsilon \Im(\lambda) - \varepsilon$ and

$$\beta \geq 0 \quad \frac{\beta}{\alpha} + \frac{d}{2} \leq 1$$

These lead to the estimate

$$P_\alpha = QR_{\beta,\alpha}, \quad ||P_\alpha|| \leq c|\lambda|^{-\kappa}$$

where

$$\kappa = 1 - \frac{\beta}{\alpha}, \quad \beta = \max\{\nu_{ij}\}$$
8.2 Neumann series

We have the standard formula for the heat kernel for $p$ chosen so that the operator inside the integral is trace class:

$$e^{-tH} = \int e^{-t\lambda} t^{-p} \partial^p \lambda R_\lambda$$

for $R_\lambda = R_\alpha(H_\alpha)$. We write

$$R_\lambda = R_\alpha(\lambda) [1 + Q \cdot R_\alpha(\lambda)]^{-1}$$

The preceding estimates allow us to use Neumann series and write for

$$R_\lambda = R_\alpha(\lambda) [1 + PR_\alpha(\lambda)]^{-1} = \sum_{j=0}^{\infty} R^{(j)}_\lambda$$

where

$$R^{(j)}_\lambda = R_\alpha(\lambda) (PR_\alpha(\lambda))^j$$

Hence

$$\partial^p \lambda R_\lambda = \sum_{j=0}^{\infty} \partial^p \lambda R^{(j)}_\lambda$$

Now for $c(p, m_0, \ldots, m_j)$ being the multinomial coefficient appearing in Leibniz formula for the $p$-th derivative of a product

$$\partial^p \lambda R^{(j)}_\lambda = \sum_{m_0 + \cdots + m_j = p+j+1} c(p, m_0, \ldots, m_j) \Pi_{(m_0, \ldots, m_j)}$$

and

$$\Pi_{m_0, \ldots, m_j} = R_\alpha(\lambda)^{m_0} Q \cdots R_\alpha(\lambda)^{m_j-1} Q R_\alpha(\lambda)^{m_j}$$

Consequently $\Pi_{m_0, \ldots, m_j}$ is written as sum of terms of the form that we denote by $\Pi^{(j)}_{\mu, \beta}$ and following $[C1],[C2]$ we call them *resolvent products*:

$$R_\alpha(\lambda)^{m_0} \cdot V_1 \cdot R_\alpha(\lambda)^{m_2} \cdots V_j R_\alpha(\lambda)^{m_j}$$

where

$$V_j = r^{\mu_j} f_j(\theta) \partial^\beta_j$$

with $\mu_j \in \mathbb{Q}$ while $\beta_j$ is a polyindex with $|\beta_j| \leq 2$ and $f_j$ an analytic function in the $\theta$ variables. We denote such a term as

$$\Pi^{(j)}_{m, \mu, \beta}$$

with

$$m^j = (m_0, \ldots, m_j), \quad \mu^j = (\mu_1, \ldots, \mu_j) \quad m^j = (\beta_1, \ldots, \beta_j)$$

and in concise notation

$$\Pi^{(j)}_{m^j, \mu^j, \beta^j}$$

We denote as

$$F_j(t, r, \theta) = t^{2-p} \int_C \frac{d\lambda}{2\pi i} e^{-t\lambda} \Pi^{(j)}_{\lambda}(r, \theta)$$

For each of these integrals we prove the following theorem
Theorem 8.1 Let

\[ f_j(\xi, \eta, \theta) \]

be defined for \( \xi >, \eta > 0, \theta \in \Omega \) then for

\[ F_j(\xi, \eta, \theta) = f_j(\eta \xi^\alpha, \eta, \theta) \]

Then

\[ F_j \in \Gamma^{S_1, S_2}(R_+ \times R_+|\Omega) \]

for

\[ S_1(\ell \alpha) = 1 = S_2 \left( \ell \left( \frac{(3p-1) + 2j}{2} \right) - 1 \right) \]

\( \ell = k, k = 0, 1, 2 \)

while also \( S_1(z) = 1 \) for \( z \) non-negative integer linear combinations of the \( \mu_1, \ldots, \mu_j \). Furthermore

\[ P^\alpha_2[S_2]F_j(\xi, \eta) \leq C(\eta)^R \]

8.3 Proof of the asymptotic expansion

We proceed to the asymptotic expansion of the generic term appearing in the Neumann series and for a constant \( q > \) that we will select later

\[ F_j(t, r, \theta) = t^{\frac{n}{2} + p} \int \frac{d\lambda}{2\pi i} e^{-t\lambda \Pi^{(j)}(\lambda, \theta)} \]

and obtain the estimates required by the Singular asymptotics lemma \([C0],[C2]\) and we recall in the appendix along with the necessary notions. Following \([C2]\) we introduce the variables

\[ \xi = \frac{t^{1/\alpha}}{r}, \quad \eta = \frac{r}{\lambda} \]

and

\[ \eta \partial_\eta = \alpha t \partial_t + r \partial_r \quad \xi \partial_\xi = \alpha t \partial_t \]

Then we have for

\[ F_j(\xi, \eta, \theta) = F\left(\frac{t^{1/\alpha}}{r}, r, \theta\right) \]

Then we apply

\[ \eta \partial_\eta F_j(\xi, \eta) = (\alpha t \partial_t + r \partial_r) \left( F_j\left(\frac{t^{1/\alpha}}{r}, r, \theta\right) \right) = t^{\frac{n}{2} + p} \int \frac{d\lambda}{2\pi i} e^{-t\lambda \Pi^{(j)}(\lambda, \theta)} \]

for

\[ D = \alpha \left( \frac{n}{2} + q - p \right) - \alpha (\lambda \partial_\lambda + 1) + r \partial_r \]

Then

\[ D \Pi^{(j)}(\lambda, \theta) = \sum_{k=0}^{j} \Pi^{(k)}_{\alpha, \mu, \beta} \left[ \alpha \left( \frac{n}{2} + q - p \right) + m_k [-\alpha \lambda \partial_\lambda R_0(\lambda) + r \partial_r R_0(\lambda)] R_0(\lambda) \right] \Pi^{(j+k)}_{\alpha, \mu, \beta} + (\mu_1 + \cdots + \mu_j) \Pi^{(j)} + E_j \]
where the error is computed through the commutator

\[ [r, R_\alpha(\lambda)] = 2R_\alpha(\lambda)\partial_r R_\alpha(\lambda) \]

It introduces a factor with norm decaying faster by a factor of \(|\lambda|^{-\frac{1}{2}}\) as \(\lambda \to \infty\). The important identity from [C2] is modified trivially:

\[ H_{0,\alpha} = -\frac{1}{2}[r\partial_r, H_{0,\alpha}] + \frac{\alpha - 2\Delta}{r^\alpha} \]

and

\[ \lambda \partial_\lambda R_\alpha(\lambda) = -R_\alpha(\lambda) - R_\alpha(\lambda)H_{0,\alpha}R_\alpha(\lambda) \]

These combine to the formula

\[ \frac{\alpha}{2} R_\alpha(\lambda) = \alpha(\lambda \partial_\lambda R_\alpha(\lambda) + R_\alpha(\lambda)) + r[\partial_r, R_\alpha(\lambda)] = \]

\[ = \left(\frac{\alpha}{2} - 1\right) R_\alpha(\lambda) - [r\partial_r, R_\alpha(\lambda)] - R_\alpha(\lambda) \frac{\alpha \Delta}{r^\alpha} + R_\alpha(\lambda) \]

We obtain then that

\[ m_k \left[ -\alpha \left( \lambda \partial_\lambda R_\alpha(\lambda) + R_\alpha(\lambda) \right) + r[\partial_r, R_\alpha(\lambda)] \right] = \]

\[ m_k \left[ \frac{\alpha}{2} R_\alpha(\lambda) - \alpha \left( \lambda \partial_\lambda R_\alpha(\lambda) + R_\alpha(\lambda) \right) + r[\partial_r, R_\alpha(\lambda)] - \frac{m_k \alpha}{2} R_\alpha(\lambda) \right] = \]

\[ = m_k \alpha \left( \frac{\alpha}{2} - 1 \right) R_\alpha(\lambda) \partial_r R_\alpha(\lambda) \frac{(-\Delta)}{r^\alpha + 1} R_\alpha(\lambda) - \frac{m_k \alpha}{2} R_\alpha(\lambda) \]

Then we have that

\[ D\Pi^{(j)}(r, \theta) = \alpha \left( \frac{n}{2} + q - p - \frac{p}{2} \right) \Pi^{(j)}(r, \theta) + \sum_{k=0}^{j} m_k \Pi^{(k)} \left[ \alpha \left( \frac{\alpha}{2} - 1 \right) R_\alpha(\lambda) \partial_r R_\alpha(\lambda) \frac{(-\Delta)}{r^\alpha + 1} R_\alpha(\lambda) \right] \]

\[ + (\mu_1 + \cdots + \mu_j) \Pi^{(j)} + E_j \]

At this point we select \( p = 2\ell, q = 3\ell - \frac{p}{2} \) and then employing commutators introducing factors of with strictly smaller norms we have that for a function \( \phi \in L^1, \phi = \phi_1 \phi_2, \phi_2 \in L^2 \) we have that

\[ \| \phi \Pi^{(k)}_{\alpha, \mu, \beta} \left[ R_\alpha(\lambda) \partial_r R_\alpha(\lambda) \frac{(-\Delta)}{r^\alpha + 1} R_\alpha(\lambda) \right] \Pi^{(j+1-k)}_{\alpha, \mu, \beta} \|_1 \leq C \| \phi R_\alpha(\lambda) \partial_\lambda R_\alpha(\lambda) \frac{(-\Delta)}{r^\alpha + 1} R_\alpha(\lambda) \leq -1 \| V_1 \cdots V_j R_\alpha(\lambda) \| \]

\[ \leq c \| \phi \| |\lambda|^{-\ell + \frac{p}{2} + j} \]

This allows the estimate by duality

\[ \| f \|_\infty = \sup_{g \in L^1} \left| \int f g \right| \]

Then we conclude that

\[ |\eta \partial_\eta f(\xi, \eta)| \leq C|\xi\eta|^z \]

for \( z = \left( \frac{3(p-1)}{2} + j \right) \alpha - 1 \).
Higher $\eta \partial_\eta$-derivatives These are derived using the commutator technique as is given in [C2]

$\xi \partial_\xi$-derivatives The $\xi \partial_\xi$ reduces to $at \partial_t$ and hence the $\xi \rightarrow 0$ asymptotics to small time asymptotics of $F_j$ that are reduced to classical expansions.
9 Appendix

9.1 Resolvent estimates

**Resolvent comparison formula** Following the lines of [C] we obtain the resolvent comparison formula: the resolvent of the irregular laplacian to the free laplacian. Let \( \Delta_n \) be the laplacian in euclidean space and \( R_0(\lambda) \) its resolvent. Also let \( \Omega \) be a domain in \( \mathbb{R}^{n-1} \)

\[
\Pi : L^2(\mathbb{R}^n) \to L^2((0, \infty) \times \Omega)
\]

Define

\[
R_\lambda^0 = \Pi \cdot R_0(\lambda) \cdot \Pi
\]

and

\[
Q_{k,\alpha,\lambda} = \frac{\Delta}{r^\alpha} \cdot R_\alpha(\lambda)
\]

Then we have the formula

\[
R_\alpha(\lambda) = R_\lambda^0 + R_\lambda^0 \cdot Q_{k,\alpha,\lambda}
\]

**Resolvent factors** We encounter terms of the form

\[
P_{\beta,\gamma,d,\alpha} = r^\beta \cdot \partial_\gamma^d \cdot R_\alpha(\lambda)^m(\lambda)
\]

and its index as in [C2]:

\[
\text{ind}(P_{\beta,\gamma,d,\alpha}) = \begin{cases} 
  m - \frac{\beta}{2} + \frac{d}{2}, & \text{if } \beta \leq 0 \\
  m - \frac{d}{2}, & \text{if } \beta > 0 
\end{cases}
\]

and its degree \( \deg_+ \) for \( \beta > 0 \) as

\[
\deg_+ = \beta - d + 1
\]

Then we have the following estimates that are modified versions of the corresponding ones from [C1]:

- if \( \beta \leq 0 \) then
  \[
  ||\phi P_{\beta,\gamma,d,\alpha}||_k \leq c ||\phi||_2 |\lambda|^{-\text{ind}(P_{\beta,\gamma,d,\alpha})+\frac{1}{2}}
  \]

- if \( r \in (0, \sqrt{\lambda}) \) then
  \[
  ||\phi P_{\beta,\gamma,d,\alpha}||_k \leq c ||\phi||_2 |\lambda|^{-\text{ind}(P_{\beta,\gamma,d,\alpha})+\frac{1}{2}-\frac{d}{2}}
  \]

We recall here the following estimates form [C1] for \( 0 \leq \alpha \leq 1 \)

\[
||r^{-\alpha} R_0(\lambda)^p||_k \leq c_k |\lambda|^{-p - \frac{\alpha}{2} + \frac{1}{p}}
\]

Employing the preceding estimates and the comparison formula and operator norm estimate and derive the estimate

\[
||\phi r^{-\alpha} R_\alpha(\lambda)^p||_k \leq c_k |\lambda|^{-p - \frac{\alpha}{2} + \frac{1}{p}}
\]
9.2 Singular Asymptotics

We recall here the basic definitions necessary for the singular asymptotic expansions. Let \( S : \mathbb{C} \to \mathbb{Z}_+ \) with \( \sum_{\Re(z) < k} S(z) < \infty \) for all \( k \in \mathbb{C} \) and its degree \( m = m(S) = \sum_{z \in \mathbb{C}} S(z) \). Then we introduce the operators:

\[
P^x_z[S] = \prod_{\Re(z') \leq \Re(z), z' \neq z} (x \partial_x - z')^{S(z')}
\]

Similarly we introduce the space of functions that possess power-log expansions at \( 0^+ \). We call this space \( \Gamma^S(0, \infty) \) and a function \( f : (0, \infty) \to \mathbb{C} \) if for \( m = m(S) \) \( f = C^m(0, \infty) \) and

\[
\partial_s^t f(t) = \sum_{\Re(z) < \Re(k) \sum_{0 \leq j < S(z')}} f_{zz} \partial_t^z [t^z \log^j t] + O(t^{k+\delta_k-s})
\]

Define similarly the asymptotics for functions of \( n \) variables as we approach corners. Then we have the following basic facts:

- If \( m(S) = \infty \) then \( f \in \Gamma^S(0, \infty) \) if \( f \in C^\infty(0, \infty) \) and for all \( z \in \mathbb{C} \)

\[
P^x_z[S] f(t) = O(t^{\epsilon})
\]

for \( \epsilon > 0 \).

- Let \( f \in \Gamma^{S_1, S_2} (\mathbb{R}_+ \times \mathbb{R}_+) \), \( m(S_1) = m(S_2) = \infty \) and for all \( k \in \mathbb{C} \) and \( \epsilon > 0 \)

\[
|P^x_k(S_i) f(x, y)| \leq (xy)^{\Re(k)-\epsilon} h_{k, \epsilon}(y) \int_1^\infty h_{k, \epsilon}(t) \frac{dt}{t}
\]

Moreover let

\[
F(t) = \int_0^\infty f \left( x, \frac{t}{x} \right) \frac{dx}{x}
\]

Then

\[
F \in \Gamma^{S_1+S_2} (\mathbb{R}_+)
\]
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