Sampling Theorem and Discrete Fourier Transform on the Hyperboloid

M. Calixto, J. Guerrero and J.C. Sánchez-Monreal

Abstract

Using Coherent-State (CS) techniques, we prove a sampling theorem for holomorphic functions on the hyperboloid (or its stereographic projection onto the open unit disk \( \mathbb{D}_1 \)), seen as a homogeneous space of the pseudo-unitary group \( SU(1,1) \). We provide a reconstruction formula for bandlimited functions, through a sinc-type kernel, and a discrete Fourier transform from \( N \) samples properly chosen. We also study the case of undersampling of “quasi-bandlimited” functions and the conditions under which a partial reconstruction from \( N \) samples is still possible and the accuracy of the approximation.

1 Introduction

In a previous article [1], we proved sampling theorems and provided discrete Fourier transforms for holomorphic functions on the Riemann sphere, using the machinery of Spin CS related to the special unitary group \( SU(2) \), which is the double cover of the group \( SO(3) \) of motions of the sphere \( S^2 \). Here we study similar discretization problems for its noncompact counterpart \( SO(2,1) \) (the group of motions of the Lobachevsky plane) or, more precisely, for its double cover \( SU(1,1) \). Both, \( SU(2) \) and \( SU(1,1) \), appear as the underlying symmetry groups of many physical systems for which they constitute a powerful computational and classification tool. In fact, Angular Momentum Theory proves to be essential when studying systems exhibiting rotational invariance (isotropy). In the same manner, the representation theory of \( SU(1,1) \) or \( SL(2,\mathbb{R}) \) is useful when dealing with systems bearing conformal invariance, specially in two dimensions, where this finite-dimensional symmetry can be promoted to an infinite-dimensional one (the Virasoro group). In particular, the group \( SL(2,\mathbb{R}) \) was used in [2] to define wavelets on the circle and the real line in a unified way. Furthermore, \( SU(2) \) and \( SU(1,1) \) CS, generalizing canonical CS of the Heisenberg-Weyl group (Gabor frames), find a great variety of applications, mainly in the study of quantum mechanical systems and their classical limit (see e.g. [3, 4, 5]). For example, ground states of superconductors and superfluids (like Bose-Einstein condensates) are coherent states.

Also, standard Continuous Wavelet Theory (see e.g. [6]) can be seen as a chapter of CS on the group of affine transformations (translations and dilations). Here the discretization process turns out to be essential for computational applications in, for example, signal processing. These results revived interest in the question of discretization and we hope that the establishment of new sampling theorems for harmonic analysis on non-Abelian groups and their homogeneous
spaces will be of importance for numerical study and simulation of those physical systems bearing
that symmetries. Actually, there are some important general results about sampling and
efficient computation of Fourier transforms for compact groups (see e.g. [7, 8]). However, a
comprehensive study of the non-compact case is far more complicated, although some results in
this direction have been obtained for specific groups (see e.g. [9] for a survey). For instance,
we would like to point out Ref. [10] for the motion group and its engineering applications [11]
(namely in robotics [12]) and [13] for discrete frames of the Poincaré group and its potential
applications to Relativity Theory.

This article intends to be a further step in this direction. Completeness criteria for CS
subsystems related to discrete subgroups of $SU(1,1)$ have been proved using the theory of
Automorphic Forms (see e.g. [3]). Here we shall follow a different approach. Working in the
open unit disk $\mathbb{D}_1 = SU(1,1)/U(1)$ (as an appropriate realization of the Lobachevsky plane or
the hyperboloid), we shall choose as sampling points for analytic functions inside $\mathbb{D}_1$ (carrying a
unitary irreducible representation of $SU(1,1)$ of Bargmann index $s$) a set of $N$ equally distributed
points on a circumference of radius $r < 1$. For bandlimited holomorphic functions on $\mathbb{D}_1$ of
bandlimit $M < N$ and index $s$, the resolution operator $A$ is diagonal, providing a reconstruction
formula by means of a (left) pseudoinverse. The Fourier coefficient can be obtained by means of the (rescaled) Fourier transform of the data, allowing for a straightforward fast extension
of the reconstruction algorithm. The reconstruction of arbitrary (band-unlimited) functions is
not exact for a finite number $N$ of samples. However, for fast-decaying, or “quasi-bandlimited”,
functions it is still possible to give partial reconstruction formulas and to analyze the accuracy of
the approximation in terms of $N$, the radius $r$ and the index $s$, this time through the sampled CS
overlap (or reproducing kernel) $B$ (see later on Sec. 2 for definitions), which exhibits a “circulant”
structure and can be easily inverted using the properties of the Rectangular Fourier Matrices
(RFM) and the theory of Circulant Matrices [14]. This helps us to provide a reconstruction
formula accomplished through an eigen-decomposition $B = FDF^{-1}$ of $B$, where $F$ turns out to
be the standard discrete Fourier transform matrix.

The plan of the article is as follows. In order to keep the article as self-contained as possible,
we shall introduce in the next section general definitions and results about CS and frames based
on a group $G$. The standard construction of CS related to the discrete series of representations
of $G = SU(1,1)$ is briefly sketched in Sec. 3. We refer the reader to Refs. [3, 4, 9, 15] for
more information. In Section 4 we provide sampling theorems, discrete Fourier transforms and
reconstruction formulas for bandlimited holomorphic functions on $\mathbb{D}_1$ of bandlimit $M$ and index
$s$, and discuss the effect of over- and under-sampling. For quasi-bandlimited functions these
reconstruction formulas are not exact and we analyze the error committed in terms of $N, r$ and
$s$. Finally, Sec. 5 is devoted to conclusions and outlook.

2 Coherent States, Frames and Discretization

Let us consider a unitary representation $U$ of a Lie group $G$ on a Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$. Consider also the space $L^2(G,dg)$ of square-integrable complex functions $\Psi$ on $G$, where $dg = d(g'g)$, $\forall g' \in G$, stands for the left-invariant Haar measure, which defines the scalar product

\[
\langle \Psi | \Phi \rangle = \int_G \Psi(g)\Phi(g)dg.
\]
A non-zero function $\gamma \in \mathcal{H}$ is called admissible (or a fiducial vector) if $\Gamma(g) \equiv \langle U(g)\gamma | \gamma \rangle \in L^2(G, dg)$, that is, if

$$c_\gamma = \int_G \overline{\Gamma(g)} \Gamma(g) dg = \int_G |\langle U(g)\gamma | \gamma \rangle|^2 dg < \infty. \tag{2}$$

Let us assume that the representation $U$ is irreducible, and that there exists a function $\gamma$ admissible, then a system of coherent states (CS) of $\mathcal{H}$ associated to (or indexed by) $G$ is defined as the set of functions in the orbit of $\gamma$ under $G$

$$\gamma_g = U(g)\gamma, \quad g \in G. \tag{3}$$

We can also restrict ourselves to a suitable homogeneous space $Q = G/H$, for some closed subgroup $H$. Then, the non-zero function $\gamma$ is said to be admissible mod($H, \sigma$) (with $\sigma : Q \rightarrow G$ a Borel section), and the representation $U$ square integrable mod($H, \sigma$), if the condition

$$\int_Q |\langle U(\sigma(q))\gamma | \psi \rangle|^2 dq < \infty, \quad \forall \psi \in \mathcal{H} \tag{4}$$

holds, where $dq$ is a measure on $Q$ “projected” from the left-invariant measure $dg$ on the whole $G$. The coherent states indexed by $q$ are defined as $\gamma_{\sigma(q)} = U(\sigma(q))\gamma, q \in Q$, and they form an overcomplete set in $\mathcal{H}$.

The condition (4) could also be written as an “expectation value”

$$0 < \int_Q |\langle U(\sigma(q))\gamma | \psi \rangle|^2 dq = \langle \psi | A_\sigma | \psi \rangle < \infty, \quad \forall \psi \in \mathcal{H}, \tag{5}$$

where $A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle \langle \gamma_{\sigma(q)}| dq$ is a positive, bounded, invertible operator.

If the operator $A_\sigma^{-1}$ is also bounded, then the set $S_\sigma = \{|\gamma_{\sigma(q)}\rangle, q \in Q\}$ is called a frame (see [16] for details on frames), and a tight frame if $A_\sigma$ is a positive multiple of the identity, $A_\sigma = \lambda I, \lambda > 0$.

To avoid domain problems in the following, let us assume that $\gamma$ generates a frame (i.e., that $A_\sigma^{-1}$ is bounded). The CS map is defined as the linear map

$$T_\gamma : \mathcal{H} \longrightarrow L^2(Q, dq)$$

$$\psi \longmapsto \Psi_\gamma(q) = [T_\gamma \psi](q) = \frac{\langle \gamma_{\sigma(q)} | \psi \rangle}{\sqrt{\sigma(q)}}, \tag{6}$$

Its range $L^2_\gamma(Q, dq) \equiv T_\gamma(\mathcal{H})$ is complete with respect to the scalar product $(\Phi | \Psi)_\gamma \equiv \langle \Phi | T_\gamma A_\sigma^{-1} T_\gamma^{-1} \Psi \rangle_Q$ and $T_\gamma$ is unitary from $\mathcal{H}$ onto $L^2_\gamma(Q, dq)$. Thus, the inverse map $T_\gamma^{-1}$ yields the reconstruction formula

$$\psi = T_\gamma^{-1} \Psi_\gamma = \int_Q \Psi_\gamma(q) A_\sigma^{-1} \gamma_{\sigma(q)} dq, \quad \Psi_\gamma \in L^2_\gamma(Q, dq), \tag{7}$$

which expands the signal $\psi$ in terms of CS $A_\sigma^{-1} \gamma_{\sigma(q)}$ with wavelet coefficients $\Psi_\gamma(q) = [T_\gamma \psi](q)$. These formulas acquire a simpler form when $A_\sigma$ is a multiple of the identity, as is for the case considered in this article.

*In this paper we shall extensively use the Dirac notation in terms of “bra” and “kets” (see e.g. [2, 9]). The Dirac notation is justified by the Riesz Representation Theorem, and is valid in more general settings than Hilbert spaces of square integrable functions.*
When it comes to numerical calculations, the integral \( A_\sigma = \int_\mathcal{Q} |\gamma_{\sigma(q)}\rangle\langle \gamma_{\sigma(q)}| dq \) has to be discretized, which means to restrict oneself to a discrete subset \( \mathcal{Q} \subset \mathcal{Q} \). The question is whether this restriction will imply a loss of information, that is, whether the set \( \mathcal{S} = \{ |q_k\rangle \equiv |\gamma_{\sigma(q_k)}\rangle, q_k \in \mathcal{Q} \} \) constitutes a discrete frame itself, with resolution operator

\[
\mathcal{A} = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle q_k|.
\]  

(8)

The operator \( \mathcal{A} \) need not coincide with the original \( A_\sigma \). In fact, a continuous tight frame might contain discrete non-tight frames, as happens in our case (see later on Sec. 4).

Let us assume that \( \mathcal{S} \) generates a discrete frame, that is, there are two positive constants \( 0 < b < B < \infty \) (frame bounds) such that the admissibility condition

\[
b||\psi||^2 \leq \sum_{q_k \in \mathcal{Q}} |\langle q_k|\psi\rangle|^2 \leq B||\psi||^2
\]  

(9)

holds \( \forall \psi \in \mathcal{H} \). To discuss the properties of a frame, it is convenient to define the frame (or sampling) operator \( T : \mathcal{H} \rightarrow \ell^2 \) given by \( T(\psi) = \{ \langle q_k|\psi\rangle, q_k \in \mathcal{Q} \} \). Then we can write \( \mathcal{A} = T^*T \), and the admissibility condition (9) now adopts the form

\[
bI \leq T^*T \leq BI,
\]  

(10)

where \( I \) denotes the identity operator in \( \mathcal{H} \). This implies that \( \mathcal{A} \) is invertible. If we define the dual frame \( \{ |q\rangle \equiv \mathcal{A}^{-1}|q\rangle \} \), one can easily prove that the expansion (reconstruction formula)

\[
|\psi\rangle = \sum_{q_k \in \mathcal{Q}} \langle q_k|\psi\rangle |\tilde{q}_k\rangle
\]  

(11)

converges strongly in \( \mathcal{H} \), that is, the expression

\[
T_t^+T = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle\langle q_k| = T^* (T_t^+)^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle \tilde{q}_k| = I
\]  

(12)

provides a resolution of the identity, where \( T_t^+ \equiv (T^*T)^{-1}T^* \) is the (left) pseudoinverse (see, for instance, [17]) of \( T \) (see e.g. [15, 9] for a proof, where they introduce the dual frame operator \( \tilde{T} = (T_t^+)^* \) instead).

It is interesting to note that the operator \( P = TT_t^+ \) acting on \( \ell^2 \) is an orthogonal projector onto the range of \( T \).

We shall be mainly interested in cases where there are not enough points to completely reconstruct the signal, i.e., undersampling, but a partial reconstruction is still possible. In these cases \( \mathcal{S} \) does not generate a discrete frame, and the resolution operator \( \mathcal{A} \) would not be invertible. But we can construct another operator from \( T \), \( \mathcal{B} = TT^* \), acting on \( \ell^2 \).

The matrix elements of \( \mathcal{B} \) are

\[
\mathcal{B}_{kl} = \langle q_k|q_l\rangle,
\]  

(13)

therefore \( \mathcal{B} \) is the discrete reproducing kernel operator, see eq. (37). If the set \( \mathcal{S} \) is linearly independent, the operator \( \mathcal{B} \) will be invertible and a (right) pseudoinverse can be constructed for \( T \), \( T_r^+ \equiv T^*(TT^*)^{-1} \), in such a way that \( TT_r^+ = I_\ell^2 \). As in the previous case there is
another operator, $P_S = T_r^+ T$ acting on $\mathcal{H}$ which is an orthogonal projector onto the subspace $\mathcal{H}_S$ spanned by $\mathcal{S}$. A pseudo-dual frame can be defined as

$$\langle \tilde{q}_k \rangle = \sum_{q_l \in \mathcal{Q}} (B^{-1})_{lk} |q_l\rangle$$

providing a resolution of the projector $P_S$,

$$T_r^+ T = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle \langle q_k| = T^* (T_r^+)^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle \langle \tilde{q}_k| = P_S$$

Using this, a partial reconstruction (an “alias”) $\hat{\psi}(q)$ of the signal $\psi(q)$ is obtained,

$$\hat{\psi}(q) = \langle q| \hat{\psi} \rangle = \sum_{q_k \in \mathcal{Q}} \hat{\Xi}_k(q) \Psi(q_k),$$

from its samples $\Psi(q_k) = \langle q_k| \psi \rangle$, through some “sinc-type” kernel (or “Lagrange-like” interpolating functions)

$$\hat{\Xi}_k(q) = \langle q| \tilde{q}_k \rangle$$

fulfilling $\hat{\Xi}_k(q_l) = \delta_{kl}$. The alias $\hat{\psi}$ is the orthogonal projection of $\psi$ onto the subspace $\mathcal{H}_S$, that is, $|\psi\rangle = P_S |\psi\rangle$. The distance from the exact $\psi$ to the reconstructed $\hat{\psi}$ signal is given by the error function:

$$E_\psi(S) = \|\psi - \hat{\psi}\| = \sqrt{\langle \psi| I - P_S |\psi\rangle}. \quad (18)$$

The two operators $A$ and $B$ are intertwined by the frame operator $T$, $T A = B T$. If $T$ is invertible, then both $A$ and $B$ are invertible and $T_r^+ = T_l^+ = T^{-1}$. This case corresponds to critical sampling, where both operators $A$ and $B$ can be used to fully reconstruct the signal.

It should be noted that in the case in which there is a finite number $N$ of sampling points $q_k$, the space $\ell^2$ should be substituted by $\mathbb{C}^N$, and the operator $B$ can be identified with its matrix once a basis has been chosen.

### 3 Representations of $SU(1, 1)$: Discrete Series

Discrete series representations of $SU(1, 1)$ can be found in the literature [3, 4]. Here we shall try to summarize what is important for our purposes, in order to keep the article as self-contained as possible.

#### 3.1 Coordinate Systems and Generators

The group $SU(1, 1)$ consists of all unimodular $2 \times 2$ matrices leaving invariant the pseudo-Euclidean metric $\eta = \text{diag}(1, -1)$ and can be parametrized as

$$SU(1, 1) = \{U(\zeta) = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \bar{\zeta}_2 & \bar{\zeta}_1 \end{pmatrix}, \, \zeta_1, \zeta_2 \in \mathbb{C} : \det(U) = |\zeta_1|^2 - |\zeta_2|^2 = 1 \}. \quad (19)$$

The group $SU(1, 1)$ is locally isomorphic to the three-dimensional Lorentz group $SO(2, 1)$ (the group of “rotations” of the three-dimensional pseudo-Euclidean space). More precisely
\[ SO(2,1) = SU(1,1)/\mathbb{Z}_2, \] where \( \mathbb{Z}_2 = \{ I, -I \} \) (\( I \) is the \( 2 \times 2 \) identity matrix) is the cyclic group with two elements. The group \( SU(1,1) \) acts on \( \mathbb{C} \) as

\[ U(\zeta) : \mathbb{C} \to \mathbb{C}, \ z \mapsto z' = \frac{\zeta_1 z + \bar{\zeta}_2}{\zeta_2 z + \bar{\zeta}_1}. \] \hspace{1cm} (20)

This action is not transitive, so that \( \mathbb{C} \) is foliated into three orbits:

\[ \mathbb{D}_1 = \{ z \in \mathbb{C} : |z| < 1 \}, \ \mathbb{C} - \overline{\mathbb{D}_1} = \{ z \in \mathbb{C} : |z| > 1 \}, \ \mathbb{S}_1 = \{ z \in \mathbb{C} : |z| = 1 \}. \] \hspace{1cm} (21)

The open unit disk \( \mathbb{D}_1 \) may be considered as the stereographical projection of the upper sheet of the two-sheet hyperboloid \( \mathbb{H}^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_0^2 - x_1^2 - x_2^2 = 1\} \) onto the complex plane. The hyperboloid \( \mathbb{H}^2 \) may be identified with the set of elements of \( SU(1,1) \) with \( \zeta_1 = x_0 = \cosh(\tau/2) \) and \( \zeta_2 = x_1 + i x_2 = \sinh(\tau/2)e^{i \alpha}, \tau > 0, \alpha \in [0, 2\pi[ \) the stereographical projection being given by \( z = \frac{\zeta_2}{\zeta_1} = \tanh(\tau/2) e^{i \alpha} \in \mathbb{D}_1 \). Thus, we could also identify \( \mathbb{D}_1 \) with the coset \( SU(1,1)/U(1) \), where \( U(1) \) is the (diagonal) subgroup of the phase \( e^{i \varphi} = \zeta_1/|\zeta_1| \).

The infinitesimal generators of the action (20) of \( SU(1,1) \) on \( \mathbb{C} \) can be written in terms of the Lie algebra \( su(1,1) \) basis elements:

\[ K_+ = z^2 \frac{d}{dz}, \ K_- = z \frac{d}{dz}, \ K_0 = \frac{d}{dz}. \] \hspace{1cm} (22)

They close the following Lie algebra commutation relations:

\[ [K_0, K_\pm] = \pm K_\pm, \ [K_-, K_+] = 2K_0. \] \hspace{1cm} (23)

It is not difficult to verify that the quadratic (Casimir) operator

\[ C = K_0^2 - (K_+ K_- + K_- K_+)/2 \] \hspace{1cm} (24)

commutes with every \( K \).

### 3.2 Unitary irreducible representations: \( SU(1,1) \) coherent states

We are seeking for unitary irreducible representations of \( SU(1,1) \). By Schur’s lemma, for any irreducible representation of \( su(1,1) \), the Casimir operator \( C \) must be a multiple of the identity \( I \), which we shall set \( C = s(s - 1)I \). Thus, an irreducible representation of \( SU(1,1) \) is labelled by a single number \( s \) (which we shall refer to as the symplectic spin or just “sympling”). We shall restrict ourselves to discrete series of representations, for which \( s \) is half-integer \( s = 1, 3/2, 2, 5/2, \ldots \). We shall take the orthonormal basis vectors \( |s, n \rangle \) in the carrier (Hilbert) space \( \mathcal{H}_s \) to be eigenvectors of \( K_0 \):

\[ K_0 |s, n \rangle = (n + s) |s, n \rangle. \] \hspace{1cm} (25)

From the commutation relations (23), we observe that \( K_\pm \) act as raising and lowering ladder operators, respectively, whose action on the basis vectors proves to be

\[ K_+ |s, n \rangle = \sqrt{(n+1)(2s+n)} |s, n+1 \rangle, \ K_- |s, n \rangle = \sqrt{n(2s+n-1)} |s, n-1 \rangle. \] \hspace{1cm} (26)

Indeed, it can be easily checked that the action (25,26) preserves the commutation relations (23); for example:

\[ [K_+, K_-] |s, n \rangle = \cdots = 2(n+s) |s, n \rangle = 2K_0 |s, n \rangle, \] \hspace{1cm} (27)
and so on. From the expression \(26\) we deduce that the spectrum of \(K_0\) is unbounded from above \(n = 0, 1, 2, \ldots\), that is, the Hilbert space \(\mathcal{H}_s\) is infinite-dimensional.

Any group element \(U(\zeta) \in SU(1, 1)\) can also be written through the exponential map

\[
U(z, \bar{z}, \varphi) = e^{\xi K_+ - \xi K_-} e^{i\varphi K_0}, \quad \xi = |\xi| e^{i\beta}, \quad z = \tanh |\xi| e^{i\beta}.
\]

Note that the structure subgroup \(U(1) \subset SU(1, 1)\), generated by \(K_0\), stabilizes any basis vector up to an overall multiplicative phase factor (a character of \(U(1)\)), i.e., \(e^{i\varphi K_0}|s, m\rangle = e^{i(m + s)\varphi}|s, m\rangle\). Thus, according to the general prescription explained in Sec. 2 letting \(Q = SU(1, 1)/U(1) = \mathbb{D}_1\) and taking the Borel section \(\sigma : Q \to G\) with \(\sigma(z, \bar{z}) = (z, \bar{z}, 0)\), we shall define, from now on, families of covariant coherent states mod\((U(1), \sigma)\) (see \(9\)). In simple words, we shall set \(\varphi = 0\) and drop it from the vectors \(U(z, \bar{z}, \varphi)|s, m\rangle\).

For any choice of fiducial vector \(|\gamma\rangle = |s, m\rangle\) the set of coherent states \(|z, m\rangle \equiv U(z, \bar{z})|\gamma\rangle\) is overcomplete (for any \(m\) in \(\mathcal{H}_s\). We shall use \(|\gamma\rangle = |s, 0\rangle\) as fiducial vector (i.e., the lowest weight vector), so that \(K_-|\gamma\rangle = 0\) and the coherent states

\[
|z\rangle \equiv U(z, \bar{z})|\gamma\rangle = e^{\xi K_+ - \xi K_-}|s, 0\rangle = N_s(z, \bar{z}) e^{z K_+}|s, 0\rangle,
\]

are holomorphic (only a function of \(z\)), apart from the normalization factor \(N_s\) which can be determined as follows. Exponentiating the relations \(26\) gives

\[
e^{z K_+}|s, 0\rangle = |s, 0\rangle + z \sqrt{2s}|s, 1\rangle + \frac{1}{2} z^2 \sqrt{2s} \sqrt{2(2s + 1)}|s, 2\rangle + \ldots
\]

\[
= \sum_{n=0}^{\infty} \left(2s + n - 1\right)^{1/2} \frac{1}{n!} z^n |s, n\rangle \equiv N_s(z, \bar{z})^{-1} |z\rangle.
\]

Then, imposing unitarity, i.e., \(\langle z | z \rangle = 1\), we arrive at \(N_s(z, \bar{z}) = (1 - |z|^2)^s\).

The frame \(\{|z\rangle, \ z \in \mathbb{C}\}\) is tight in \(\mathcal{H}_s\), with resolution of unity

\[
I = \frac{2s - 1}{\pi} \int_{D_1} |z\rangle \langle z| \frac{d^2 z}{(1 - z \bar{z})^2},
\]

where we denote \(d^2 z = d\text{Re}(z) d\text{Im}(z)\). Indeed, using \(30\) we have that

\[
\frac{2s - 1}{\pi} \int_{D_1} |z\rangle \langle z| \frac{d^2 z}{(1 - z \bar{z})^2} = \frac{2s - 1}{\pi} \int_{D_1} N_s(z, \bar{z})^2 \sum_{n,m=0}^{\infty} \left(\frac{2s + n - 1}{n}\right) \left(\frac{2s + m - 1}{m}\right) z^n \bar{z}^m |s, 0\rangle \langle s, 0| \frac{d\text{Re}(z) d\text{Im}(z)}{(1 - z \bar{z})^2}
\]

\[
= 2(2s - 1) \sum_{n=0}^{\infty} \left(\frac{2s + n - 1}{n}\right) |s, n\rangle \langle s, n| \int_0^1 (1 - r^2)^{2s - 2n + 1} dr
\]

\[
= \sum_{n=0}^{\infty} |s, n\rangle \langle s, n| = I,
\]

where polar coordinates were used at intermediate stage.
Using (30), the decomposition of the coherent state $|z\rangle$ over the orthonormal basis \{|$s,m\rangle$\} gives the irreducible matrix coefficients

$$\langle z|s,m \rangle = \langle s,0|U(z,\bar{z})^*|s,m \rangle = \left(\frac{2s+m-1}{m}\right)^{1/2}(1-\bar{z}z)^{s}z^{m}$$

$$\equiv U^s_m(z).$$

(33)

A general sympling $s$ state

$$|\psi\rangle = \sum_{m=0}^{\infty} a_m|s,m\rangle$$

(34)

is represented in the present complex characterization by

$$\Psi(z) \equiv \langle z|\psi\rangle = \sum_{m=0}^{\infty} a_m U^s_m(z),$$

(35)

which is an anti-holomorphic function of $z$. The Fourier coefficients $a_n$ can be calculated through the following integral formula:

$$a_n = \langle s,n|\psi\rangle = \frac{2s-1}{\pi} \int_{\mathbb{D}_1} \Psi(z) \bar{U}_n^s(z) \frac{d^2z}{(1-\bar{z}z)^{2s}}.$$  

(36)

Note that the set of CS \{|$z\rangle$\} is not orthogonal. The CS overlap (or Reproducing Kernel) turns out to be

$$C(z,z') = \langle z|z'\rangle = \frac{(1-\bar{z}z)^s(1-\bar{z}'z')^s}{(1-\bar{z}z)^{2s}}.$$  

(37)

This quantity will be essential in our sampling procedure on the disk $\mathbb{D}_1$.

There are other pictures of CS for $SU(1,1)$ corresponding to other parameterizations like, for example, the one that takes $\mathbb{D}_1$ to the upper complex plane, but we shall not discuss them here.

4 Sampling Theorem and DFT on $\mathbb{D}_1$

Sampling techniques consist in the evaluation of a continuous function (“signal”) on a discrete set of points and later (fully or partially) recovering the original signal without losing essential information in the process, and the criteria to that effect are given by various forms of Sampling Theorems. Basically, the density of sampling points must be high enough to ensure the reconstruction of the function in arbitrary points with reasonable accuracy. We shall concentrate ourselves on sympling $s$ holomorphic functions and sample them at appropriate points.

In our case, there is a convenient way to select the sampling points in such a way that the resolution operator $A$ and/or the reproducing kernel operator $B$ are invertible and explicit formulas for their inverses are available. These are given by the set of $N$ points uniformly distributed on a circumference of radius $r$:

$$Q = \{q_k = z_k = re^{2\pi ik/N}, k = 0,1,...,N-1, r \in (0,1)\},$$

(38)

1Here we abuse notation when representing the non-analytic functions $U^s_m$ and $\Psi$ simply as $U^s_m(z)$ and $\Psi(z)$, which are indeed (anti-)holomorphic up to the normalizing, non-analytic (real), pre-factor $N_s = (1-\bar{z}z)^s$. Usually, this pre-factor is absorbed into the integration measure in (31).
which is a discrete subset of the homogeneous space $Q = SU(1,1)/U(1) = \mathbb{D}_1$, made of the $N$th roots of $r^N$, with $0 < r < 1$. Denote by $S = \{|z_k\rangle, k = 0, 1, \ldots, N - 1\}$ the subset of coherent states associated with the points in $Q$ and by

$$\mathcal{H}_s^S \equiv \text{Span}(|z_0\rangle, |z_1\rangle, \ldots, |z_{N-1}\rangle)$$ (39)

the subspace of $\mathcal{H}_s$ spanned by $S$. For finite $N$ we have $\mathcal{H}_s^S \neq \mathcal{H}_s$, so that we cannot reconstruct exactly every function $\psi \in \mathcal{H}_s$ from $N$ of its samples $\Psi(z_k) = \langle z_k | \psi \rangle$, but we shall proof that for bandlimited functions

$$|\psi\rangle \in \mathcal{H}_s^M \equiv \text{Span}(|s, 0\rangle, |s, 1\rangle, \ldots, |s, M\rangle)$$ (40)

of bandlimit $M < \infty$ we can always provide an exact reconstruction formula.

### 4.1 Bandlimited Functions

**Theorem 4.1.** Given a bandlimited function $\psi \in \mathcal{H}_s^M$ on the disk $\mathbb{D}_1$, of band limit $M$, with a finite expansion

$$|\psi\rangle = \sum_{m=0}^{M} a_m |s, m\rangle,$$ (41)

there exists a reconstruction formula (11) of $\psi$

$$\psi(z) = \sum_{k=0}^{N-1} \Xi_k(z)\Psi(z_k),$$ (42)

from $N > M$ of its samples $\Psi(z_k)$ taken at the sampling points in (38), through a “sinc-type” kernel (or “Lagrange-like” interpolating function) given by

$$\Xi_k(z) = \frac{1}{N} \left( \frac{1 - z\bar{z}}{1 - \bar{z}_kz_k} \right)^s \sum_{m=0}^{M} (\bar{z}z - \bar{z}_kz_k)^m.$$ (43)

Firstly, we shall introduce some notation and prove some previous lemmas.

**Lemma 4.2.** The frame operator $T : \mathcal{H}_s^M \to \mathbb{C}^N$ given by $T(\psi) = \{|z_k|\psi\rangle, z_k \in Q\}$ [remember the construction after Eq. (9)] is such that the resolution operator $A = T^*T$ is diagonal, $A = \text{diag}(\lambda_0, \ldots, \lambda_M)$, in the basis (40) of $\mathcal{H}_s^M$, with $\lambda_m \equiv N(1 - r^2)^2s \binom{2s + m - 1}{m} r^{2m}$, $m = 0, \ldots, M$. (44)

Hence, $A$ is invertible in $\mathcal{H}_s^M$. Therefore, denoting $|\tilde{z}_k\rangle \equiv A^{-1}|z_k\rangle$, the dual frame, the expression

$$I_M = \sum_{k=0}^{N-1} |z_k\rangle \langle \tilde{z}_k | = \sum_{k=0}^{N-1} |\tilde{z}_k\rangle \langle z_k |$$ (45)

The quantities $\lambda_m$ are well defined for $m \in \mathbb{N} \cup \{0\}$ and they will be used in the case of band-unlimited functions.
provides a resolution of the identity in $H_s^M$.

**Proof.** Using [33], the matrix elements of $T$ can be written as:

$$T_{kn} = (z_k | s, n) = \left(\frac{2s + n - 1}{r^2 n}\right)^{1/2} (1 - r^2)^s r^n e^{-2\pi i kn/N} = \lambda_n^{1/2} F_{kn},$$

(46)

where $F$ denotes the Rectangular Fourier matrix (see [1]) given by $F_{kn} = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i kn}{N}}$, $k = 0, \ldots, N - 1, n = 0, \ldots, M$. Then, the matrix elements of the resolution operator turn out to be

$$A_{nm} = \sum_{k=0}^{N-1} (T_{kn})^* T_{km} = \lambda_n^{1/2} \lambda_n^{1/2} \sum_{k=0}^{N-1} F_{nk}^* F_{km} = \lambda_n \delta_{nm},$$

(47)

where we have used the well-known orthogonality relation for Rectangular Fourier Matrices (RFM) (see e.g. Appendix A of [1] for a discussion of some of their properties):

$$\sum_{k=0}^{N-1} \left( e^{2\pi i (n-m)/N} \right)^k = \left\{ \begin{array}{ll} N, & \text{if } n = m \mod N \\ 0, & \text{if } n \neq m \mod N \end{array} \right\} = N \delta_{nm} \mod N,$$

(48)

and the fact that $N > M$. Since $A$ is diagonal with non-zero diagonal elements $\lambda_n$, then it is invertible and a dual frame and a (left) pseudoinverse for $T$ can be constructed, $T_T^+ = A^{-1} T^*$, providing, according to eq. (12), a resolution of the identity. ■

**Proof of theorem 4.1.** From the resolution of the identity (45), any $\psi \in H_s^M$ can be written as $|\psi\rangle = \sum_{k=0}^{N-1} \Psi(z_k) |z_k\rangle$, and therefore $\Psi(z) = \langle z | \psi \rangle = \sum_{k=0}^{N-1} \Psi(z_k) \langle z | z_k \rangle$. Using that $|z_k\rangle = A^{-1} |z_k\rangle$, we derive that

$$\langle z | \tilde{z}_k \rangle = \sum_{m=0}^{M} \langle z | s, m \rangle (A^{-1})_{nm} T^*_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{M} \lambda_n^{-1/2} e^{2\pi i km/N} \langle z | s, m \rangle,$$

(49)

which coincides with $\Xi_k(z)$ given in eq. (45) when eq. (33) is used. ■

**Remark 4.3.** It is interesting to note that eq. (12) can be interpreted as a Lagrange-type interpolation formula, where the role of Lagrange polynomials is played by the functions $L_k(z) = \Xi_k(z)$, satisfying the “orthogonality relations” $\Xi_k(z_l) = \delta_{lk}$, where the operator $P = TT_T^+$ is an orthogonal projector onto a $M$-dimensional subspace of $\mathbb{C}^N$, the range of $T$. In the case of critical sampling, $N = M + 1$, the usual result $\Xi_k(z_l) = \delta_{lk}$ is recovered, but for the strict oversampling case, $N > M + 1$, a projector is obtained to account for the fact that an arbitrary set of overcomplete data $\Psi(z_k), k = 0, \ldots, N - 1$, can be incompatible with $|\psi\rangle \in H_s^M$.

A reconstruction in terms of the Fourier coefficients can be directly obtained by means of the (left) pseudoinverse of the frame operator $T$:

**Corollary 4.4.** The Fourier coefficients $a_m$ of the expansion $|\psi\rangle = \sum_{m=0}^{M} a_m |s, m\rangle$ of any $\psi \in H_s^M$ can be determined in terms of the data $\Psi(z_k) = \langle z_k | \psi \rangle$ as

$$a_m = \frac{1}{\sqrt{N \lambda_n}} \sum_{k=0}^{N-1} e^{2\pi i km/N} \Psi(z_k), m = 0, \ldots, M.$$

(50)

---

5For the sake of briefness, we shall use here the same notation for Rectangular Fourier Matrices as for the square ones, namely $F$, in the hope that no confusion arises (see Appendix A of [1] for a more precise distinction between both cases).
Proof. Taking the scalar product with \( \langle z_k | \) in the expression (41) of \( | \psi \rangle \), we arrive at the over-determined system of equations

\[
\sum_{m=0}^{M} T_{km} a_m = \Psi(z_k), \quad T_{km} = \langle z_k | s, m \rangle,
\]

which can be solved by left multiplying it by the (left) pseudoinverse of \( T \), \( T^+ = (T^* T)^{-1} T^* = A^{-1} T^* \). Using the expressions of \( A^{-1} = \text{diag}(\lambda_0^{-1}, \lambda_1^{-1}, \ldots, \lambda_M^{-1}) \), given in Lemma 4.2 and the matrix elements \( T_{kn} \), given by the formula (33), we arrive at the desired result.

Remark 4.5. Note that the Fourier coefficients \( a_m \) are obtained as a (rectangular) Fourier transform of the data \( \Psi(z_k) \) followed by a rescaling (which can be seen as a filter) by \( A^{-1/2} \).

4.2 Band-Unlimited Functions and Undersampling

When the reconstruction of a band-unlimited function \( | \psi \rangle = \sum_{n=0}^{\infty} a_n | s, n \rangle \) from a finite number \( N \) of samples is required, we cannot use the results of the previous section since the resolution operator \( A \) is no longer invertible. However, the overlapping kernel operator \( B \) turns out to be invertible, and a partial reconstruction can be done following the guidelines of the end of Sec. 2 (undersampling).

Contrary to the case of the sphere [1], where the Hilbert space of functions of spin \( s \), \( \mathcal{H}_s \), is finite-dimensional, here \( \mathcal{H}_s \) is infinite-dimensional, and therefore in the partial reconstruction of an arbitrary state \( | \psi \rangle = \sum_{n=0}^{\infty} a_n | s, n \rangle \) a considerable error will be committed unless further assumptions on the Fourier coefficients \( a_n \) are made. Since \( | \psi \rangle \) is normalizable, the Fourier coefficients should decrease to zero, thus even if \( | \psi \rangle \) is not bandlimited, if \( a_n \) decrease to zero fast enough, it will be “approximately” band limited if the norm of \( | \psi_M \rangle \equiv \sum_{n=M+1}^{\infty} a_n | s, n \rangle \) is small compared to the norm of \( | \psi \rangle \), for an appropriately chosen \( M \). Let us formally state these ideas.

Definition 4.6. Let us define by

\[
P_M = \sum_{m=0}^{M} | s, m \rangle \langle s, m |
\]

the projector onto the subspace \( \mathcal{H}_s^M \) of bandlimited functions of bandlimit \( M \). We shall say that a function \( \psi \in \mathcal{H}_s \) on the disk \( \mathbb{D}_1 \), with an infinite expansion

\[
| \psi \rangle = \sum_{n=0}^{\infty} a_n | s, n \rangle,
\]

is a quasi-bandlimited function of bandlimit \( M \) and error \( \epsilon_M > 0 \) iff

\[
\frac{\langle \psi | 1 - P_M | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\sum_{n=M+1}^{\infty} |a_n|^2}{\sum_{n=0}^{\infty} |a_n|^2} \leq \epsilon_M^2.
\]
Theorem 4.7. Given a quasi-bandlimited function of bandlimit $M$ and error $\epsilon_M$, $\psi \in \mathcal{H}_s$, there exists a partial reconstruction of $\psi$,

$$\hat{\psi}(z) = \sum_{k=0}^{N-1} \hat{\Xi}_k(z)\Psi(z_k),$$

(55)

from $N$ of its samples $\Psi(z_k)$, taken at the sampling points in (38), up to an error (18)

$$\frac{E_{\psi}^2(r,N)}{\langle \psi|\psi \rangle} < \epsilon_M^2 + \sqrt{1 - \epsilon_M^2} \epsilon_M \sqrt{N \left(2s + N - 1\right)}^{-1/2} r^{N + O(r^2N)}$$

(56)

provided that $N > M$. The sinc-type kernel (17) now adopts the following form:

$$\hat{\Xi}_k(z) = \frac{1}{N} \left(1 - \frac{z z_k}{1 - z z_k}\right)^s \sum_{j=0}^{N-1} \lambda_j^{-1} \sum_{q=0}^{\infty} \lambda_{j+qN} \left(z z_k^{-1}\right)^{j+qN},$$

(57)

where

$$\lambda_j = \sum_{q=0}^{\infty} \lambda_{j+qN}, \ j = 0, \ldots, N - 1,$$

(58)

are the eigenvalues of the discrete reproducing kernel operator $\mathcal{B} = TT^*$ [defined in (13) with matrix elements $\mathcal{B}_{k,l} = \langle z_k|z_l \rangle$] and $\lambda_n$ is given by (44), but now for $n = 0, 1, 2, \ldots$.

Note that the quadratic error (56) approaches $\epsilon_M^2$ as $r \to 0$ and/or $N \to \infty$. Before tackling the proof of this theorem, we shall introduce some notation and prove some previous lemmas.

Lemma 4.8. The pseudo-frame operator $T : \mathcal{H}_s \to \mathbb{C}^N$ given by $T(\psi) = \{\langle z_k|\psi \rangle, z_k \in Q\}$ [remember the construction after Eq. (9)] is such that the overlapping kernel operator $\mathcal{B} = TT^*$ is an $N \times N$ Hermitian positive definite invertible matrix, admitting the eigen-decomposition $\mathcal{B} = FDF^*$, where $D = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$ is a diagonal matrix with $\lambda_j$ given by (58) and $F$ is the standard Fourier matrix.

Proof. Let us see that $\mathcal{B}$ is diagonalizable and its eigenvalues $\lambda_k$ are given by the expression (58), which actually shows that all are strictly positive and hence $\mathcal{B}$ is invertible. This can be done by taking advantage of the circulant structure of $\mathcal{B}$ (see e.g. Appendix B in [1]). Actually, using the expression of the CS overlap (37), we have:

$$\mathcal{B}_{k,l} = \langle z_k|z_l \rangle = \left(1 - r^2 \frac{1}{1 - r^2 e^{2\pi i(l-k)/N}}\right)^{2s} \equiv \mathcal{C}_{l-k},$$

(59)

where the circulant structure becomes apparent. The eigenvalues of $\mathcal{B}$ are easily computed by the formula:

$$\lambda_k = \tilde{D}_{kk} = (F^*\mathcal{B}F)_{kk} = \frac{1}{N} \sum_{n,m=0}^{N-1} e^{i2\pi kn/N} \mathcal{C}_{m-n} e^{-i2\pi nk/N}.$$  

(60)

Expanding

$$\mathcal{C}_l = (1 - r^2)^{2s} \sum_{q=0}^{\infty} \left(\frac{2s + q - 1}{q} + 2q e^{2\pi i q/N}\right),$$

(61)

and using the general orthogonality relation for Rectangular Fourier Matrices (18), we arrive at (58). It is evident that $\lambda_k > 0, \forall k = 0, 1, \ldots, N - 1$, so that $\mathcal{B}$ is invertible. ■

Following the general guidelines of Sec. 2, we now introduce the projector $P_S$:
Lemma 4.9. Under the conditions of the previous Lemma, the set \( \{ | \bar{z}_k \rangle = \sum_{l=0}^{N-1} (B^{-1})_{lk} | z_l \rangle, k = 0, \ldots, N-1 \} \) constitutes a dual pseudo-frame for \( \mathcal{S} \), the operator \( P_S = T_r^+ T \) is an orthogonal projector onto the subspace \( \mathcal{H}_S \), where \( T_r^+ = T^* B^{-1} \) is a (right) pseudoinverse for \( T \), and

\[
\sum_{k=0}^{N-1} | \bar{z}_k \rangle \langle z_k | = P_S
\]

provides a resolution of the projector \( P_S \), whose matrix elements in the orthonormal base \( | n \rangle \) of \( \mathcal{H}_S \) exhibit a structure of diagonal \( N \times N \) blocks:

\[
\langle s, m | P_S | s, n \rangle \equiv P_{mn}(r, N) = (\lambda_m \lambda_n)^{1/2} \hat{\lambda}_n^{-1} \delta_{n, m} \delta_{r, m} \delta_{r, n}, \quad m, n = 0, \ldots, M,
\]

with \( \lambda_n \) and \( \hat{\lambda}_n \) given by (58).

Proof. If we define \( T_r^+ = T^* B^{-1} \) it is easy to check that \( TT_r^+ = I_N \) is the identity in \( \mathbb{C}^N \). In the same way, \( P_S = T_r^+ T \) is a projector since \( P_S^2 = T_r^+ TT_r^+ = T_r^+ T = P_S \) and it is orthogonal \( P_S^* = (T^* B^{-1} T)^* = T^* B^{-1} T = P_S \) since \( B \) is self-adjoint. The resolution of the projector is provided by Eq. (15). Its matrix elements can be calculated through:

\[
P_{mn}(r, N) = \sum_{k,l=0}^{N-1} T_{ml}(B^{-1})_{lk} T_{kn}.
\]

The inverse of \( B \) can be obtained through the eigen-decomposition:

\[
(B^{-1})_{lk} = (\mathcal{F} \hat{D}^{-1} \mathcal{F}^*)_{lk} = \frac{1}{N} \sum_{j=0}^{N-1} \hat{\lambda}_j^{-1} e^{2\pi i j (k-l)/N}.
\]

Combining this expression with (64), and using the general orthogonality relation (48) for RFM, we finally arrive at (63). \( \blacksquare \)

The matrix elements (63) will be useful when computing the error function (18) for a general quasi-bandlimited function (53-54). We are interested in their asymptotic behavior for large \( N \) (large number of samples) and small \( r \).

Lemma 4.10. Denoting \( m = j + pN \) and \( n = j + qN \), with \( j = 0, \ldots, N-1 \) and \( p, q = 0, 1, \ldots \), we have the following asymptotic behaviour for the matrix elements (63) of the projector \( P_S \):

\[
P_{mn}(r, N) = O(r^{(p+q)N}).
\]

Proof. Let’s define:

\[
\varepsilon_n(r, N) = \frac{\hat{\lambda}_n - \lambda_n}{\lambda_n} = \sum_{u=1}^{\infty} \left( \frac{2s-1+n+uN}{n+uN} \right) \left( \frac{2s-1+n}{n} \right) r_{2uN} = \left( \frac{2s-1+n+N}{n} \right) r_{2N} + O(r^{4N}),
\]

Using (63) we have:

\[
P_{mn}(r, N) = \left( \frac{\lambda_{j+pN} \lambda_{j+qN}}{\sum_{u=1}^{\infty} \lambda_{j+uN}} \right)^{1/2} \frac{\left(2s+j+pN-1\right)^{1/2} \left(2s+j+qN-1\right)^{1/2}}{\left(2s-j\right)^{1/2}} \frac{1}{1 + \varepsilon_j(r, N)}
\]

\[
= \frac{\left(2s+j+pN-1\right)^{1/2} \left(2s+j+qN-1\right)^{1/2}}{\left(2s-j\right)^{1/2}} r_{(p+q)N} \left(1 - O(r^{2N})\right),
\]

which gives the announced asymptotic behaviour (66) for small \( r \) and/or large \( N \). \( \blacksquare \)
Lemma 4.11. The functions (67) are decreasing sequences of \( n \), that is:

\[
\varepsilon_n(r, N) < \varepsilon_m(r, N) \iff n > m, \quad n, m = 0, \ldots, N - 1.
\] (69)

Proof. The sequence \( \varepsilon_n(r, N) \) is decreasing in \( n \) since the quotient of binomial coefficients 
\[
\binom{2s-1+n+uN}{n+uN}/\binom{2s-1+n}{n}
\] in (67) is decreasing in \( n \) for any \( u \in \mathbb{N} \), as can be seen by direct computation. 

Now we are in conditions to prove our main theorem in this section.

Proof of Theorem (4.7)
Introducing the expression (65) in (14) (with \( q_k = z_k \)) and noting that
\[
\langle z|z_l \rangle = \sum_{n=0}^{\infty} \langle z|s,n \rangle \langle s,n|z_l \rangle,
\] (70)
we arrive at (57) after using (17), (33) and the orthogonality relation (48).

Now it remains to prove the asymptotic behaviour (56) for the error. Decomposing \( |\psi\rangle \) in terms of \( |\psi_M\rangle \equiv P_M|\psi\rangle \) and \( |\psi_M^\perp\rangle \equiv (I - P_M)|\psi\rangle \), we can write
\[
E_\psi^2(r, N) = \langle \psi_M|\psi_M\rangle + \langle \psi_M^\perp|\psi_M^\perp\rangle - \langle \psi_M|P_S|\psi_M\rangle - \langle \psi_M^\perp|P_S|\psi_M^\perp\rangle - 2\text{Re}\langle \psi_M|P_S|\psi_M^\perp\rangle
\] (71)

For simplicity, let us assume that \( N = M + 1 \). Using that \( \psi \) is a quasi-band limited function and the results of Lemma 4.10 and 4.11, we have that
\[
E_\psi^2(r, N) < \varepsilon_M^2 \langle \psi|\psi\rangle + (1 - \varepsilon_M^2) \frac{\varepsilon_0(r, N)}{1 + \varepsilon_0(r, N)} \langle \psi|\psi\rangle - \langle \psi_M^\perp|P_S|\psi_M^\perp\rangle - 2\text{Re}\langle \psi_M|P_S|\psi_M^\perp\rangle
\] (72)

The third term to the r.h.s. is positive and small (of \( O(r^{2N}) \), according to Lemma 4.10), and can therefore be neglected. The last term has no definite sign, and can be seen to be of \( O(r^N) \), so it must be taken into account. Bounding it in absolute value and using repeatedly the Cauchy-Schwarz inequality we have that:
\[
\frac{E_\psi^2(r, N)}{\langle \psi|\psi\rangle} < \varepsilon_M^2 + (1 - \varepsilon_M^2) \frac{\varepsilon_0(r, N)}{1 + \varepsilon_0(r, N)} + 2\sqrt{1 - \varepsilon_M^2 \varepsilon_N(1 + \varepsilon_N(r, N))}
\] (73)

Using (67) we arrive to the desired result. ■

Remark 4.12. Given \( \epsilon > \epsilon_M \), an upper bound for \( r \) and a lower bound for \( N \) can be given in order to have \( E_\psi^2(r, N) < \epsilon^2 \), by solving numerically (73) \( < \epsilon^2 \). Equation (56) can be used to obtain an analytic estimate of an upper bound for \( r \), which turns to be:
\[
r \lesssim \left( \frac{\epsilon^2 - \epsilon_M^2}{2\sqrt{(1 - \epsilon_M^2)} N \epsilon_M (2s+N-1)} \right)^{\frac{1}{N}}.
\] (74)
Corollary 4.13. (Discrete Fourier Transform) The Fourier coefficients \( a_n \) of the expansion \([41]\) can be approximated by the discrete Fourier transform on the hyperboloid:

\[
\hat{a}_n = \frac{\lambda_n^{1/2}}{\lambda_{n \mod N}} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i nk/N} \Psi(z_k),
\]

(75)

Proof.

\[
\hat{a}_n = \langle s, n | \hat{\psi} \rangle = \langle s, n | P_S | \hat{\psi} \rangle = \sum_{k=0}^{N-1} \langle s, n | \hat{z}_k \rangle \Psi(z_k) = \sum_{k,l=0}^{N-1} \langle s, n | z_l \rangle (B^{-1})_{kl} \Psi(z_k)
\]

(76)

and this provides the desired result once \([46], [65]\) and the orthogonality relation \([48]\) are used.

\[\blacksquare\]

Remark 4.14. The discrete Fourier transform on the hyperboloid resembles the traditional discrete Fourier transform up to a normalizing factor \( \frac{\lambda_n^{1/2}}{\lambda_{n \mod N}} \), which can be seen as a filter.

Remark 4.15. The definition of the Fourier coefficients \( \hat{a}_n \) entails a kind of “periodization” of the original \( a_n \) in the sense that

\[
\hat{a}_n = \sum_{q=0}^{\infty} \lambda_n^{1/2} \lambda_j^{-1/2} \lambda_{j+qN} a_{j+qN}, \quad j = n \mod N,
\]

(77)

which implies

\[
\lambda_n^{-1/2} \hat{a}_n = \lambda_{n+pN}^{-1/2} \hat{a}_{n+pN} \Rightarrow \hat{a}_{n+pN} = \sqrt{\frac{\lambda_n}{\lambda_{n+pN}}} \hat{a}_n.
\]

(78)

We could think that, for the case \( \epsilon_M = 0 \), we should recover the results of Section \([41]\) but we shall see that this is not the case. Before, a process of truncation and rescaling of \( |\hat{\psi}\rangle \) in \([55]\) is necessary to recover the reconstruction formula \([42]\) for strict bandlimited functions \([41]\). Indeed, the truncation operation

\[
|\hat{\psi}_M\rangle \equiv P_M |\hat{\psi}\rangle = \sum_{m=0}^{M} \hat{a}_n |s, n\rangle
\]

(79)

followed by a rescaling of the Fourier coefficients

\[
|\hat{\psi}_M^R\rangle \equiv R |\hat{\psi}_M\rangle = \sum_{m=0}^{M} \frac{\hat{\lambda}_n}{\lambda_n} \hat{a}_n |s, n\rangle
\]

(80)

renders the reconstruction formula for \( \hat{\psi}_M^R (z) = \langle z | R P_M | \hat{\psi} \rangle \) to the expression \([42]\).

This fact suggests us an alternative approach to the sampling of quasi-bandlimited functions \( \psi \) for small \( \epsilon_M \), which will turn out more convenient in a certain limit. Actually, for \( \epsilon_M \ll 1 \) we have

\[
||\psi - P_M \psi||^2 = \epsilon_M^2 ||\psi||^2 \ll ||\psi||^2,
\]

(81)

so that, the reconstruction formula \([42]\) for \( \psi_M = P_M \psi \) would give a good approximation of \( \psi \). The problem is that, in general, the data \( \Psi(z_k) = \langle z_k | \psi \rangle \) for \( \psi \) and the data \( \Psi_M(z_k) = \langle z_k | P_M | \psi \rangle \) for \( \psi_M \) are different unless \( \langle z_k | P_M = \langle z_k | \forall k = 0, \ldots, N-1 \), which is equivalent to \( \langle z_k | P_M | z_k \rangle = 1, \forall k = 0, \ldots, N-1 \). The following proposition studies the conditions under which such requirement is satisfied.
Proposition 4.16. For large sympling $s \to \infty$ and large band limit $M \to \infty$, the diagonal matrix elements of $P_M$ in $H^S$ have the following asymptotic behaviour:

$$P^s_M(r) \equiv \langle z_k | P_M | z_k \rangle \simeq \begin{cases} 1 & \text{if } 0 \leq r < r_c \\ 0 & \text{if } r_c < r < 1 \end{cases} \quad (82)$$

where

$$r_c = (1 + \frac{2s-1}{M})^{-1/2} \quad (83)$$

denotes a critical radius. For $M \gg 2s \gg 1$ we have $r_c \lesssim 1$.

Proof. Using the expression (46) we have

$$P^s_M(r) \equiv \langle z_k | P_M | z_k \rangle = \sum_{m=0}^{M} T_{km} T_{mk}^* = (1 - r^2)^{2s} \sum_{m=0}^{M} \binom{2s + m - 1}{m} r^{2m}. \quad (84)$$

Denoting by $p = r^2$ we can compute

$$\frac{\partial P^s_M(\sqrt{p})}{\partial p} = (2s + M) \left( \frac{2s + M - 1}{M} \right) (1 - p)^{2s-1} p^M. \quad (85)$$

We identify here the Binomial distribution $B(2s - 1 + M, p)$ (up to a factor $2s + M$), which has a maximum (as a function of $p$) for $p_c = 1/(1 + \frac{2s-1}{M})$. Using the Central Limit Theorem for $2s - 1 + M \to \infty$ and the representation of the Dirac delta function as the limit of a normal distribution, we identify (82) as a Heaviside-type function, concluding the desired result.

5 Conclusions and Outlook

We have proved sampling theorems and provided DFT for holomorphic functions on $\mathbb{D}_1$ carrying a unitary irreducible representation of $SU(1, 1)$ of sympling $s$. To accomplish our objective, we used the machinery of Coherent States and discrete frames, and benefit from the theory of Circulant Matrices and Rectangular Fourier Matrices to explicitly invert resolution and reproducing kernel operators. We also paved the way for more general coset spaces $Q = G/H$ and their discretizations.

Heisenberg-Weyl (and Newton-Hooke) CS could be seen as a zero curvature limit $\kappa \to 0$ (and large $s$) of $SU(2)$ (positive curvature) and $SU(1, 1)$ (negative curvature) CS, a unified treatment of sampling for the three cases being possible. This is left for future work [18].

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