Abstract: We investigate the tail asymptotic behavior of the sojourn time for a large class of centered Gaussian processes $X$, in both continuous- and discrete-time framework. All results obtained here are new for the discrete-time case. In the continuous-time case, we complement the investigations of [1, 2] for non-stationary $X$. A by-product of our investigation is a new representation of Pickands constant which is important for Monte-Carlo simulations and yields a sharp lower bound for Pickands constant.

Key Words: sojourn time; occupation time; exact asymptotics; Gaussian process; locally stationary processes.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Let $X(t), t \in \mathbb{R}$ be a centered Gaussian process with variance function $\sigma^2$, correlation function $\rho$ and continuous trajectories. By

$$L_u[a, b] := \int_a^b \mathbb{1}_u (X(t)) \, dt$$

we define the sojourn time spent above a fixed level $u$ by the process $X$ on the interval $[a, b]$, where $\mathbb{1}_u(x) := \mathbb{1}(x > u)$.

In a series of papers culminating in [3], S. Berman derived results on the tail asymptotic behaviour of $L_u[a, b]$, as $u \to \infty$. The sojourn time approach to tackle this problem consists in finding explicitly an appropriate scaling function $v(u)$ such that for some function $C(x) > 0, x \geq 0$

$$(1.1) \quad \mathbb{P} \{L_u[a, b] > x/v(u)\} \sim C(x)\mathbb{P} \{L_u[a, b] > 0\} = C(x)\mathbb{P} \left\{ \sup_{t \in [a,b]} X(t) > u \right\}, \quad u \to \infty$$

for any $x \geq 0$ a continuity point of $C(\cdot)$. In our notation $\sim$ stands for asymptotic equivalence of two functions as the argument tends to 0 or $\infty$. Additional inside of this approach is the explicit calculation of the exact asymptotics of $\mathbb{P} \{\sup_{t \in [a,b]} X(t) > u\}$ as $u \to \infty$. For example, as shown in several works of Berman and Pickands (see e.g., [3–5]) for $X$ a centered stationary Gaussian process the asymptotic tail behaviour of $v(u)L_u([a, b])$ and that of $\sup_{t \in [a, b]} X(t)$ can be studied.
under appropriate assumptions on the correlation function \( \rho \). Pickands’ assumption for \( X \) stationary with unit variance function reads

\[
1 - \rho(t) \sim |t|^{\alpha}, \quad t \to 0 \quad \text{and} \quad \rho(t) < 1, \quad \forall \ t \neq 0,
\]

where \( \alpha \in (0, 2] \). Under (1.2) in view of [5] (see also [6]) taking the scaling function \( v(u) = u^{2/\alpha} \) we have (consider for simplicity \([a, b] = [0, T], T > 0\))

\[
P\left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim T H_{\alpha} v(u) P\{X(0) > u\}, \quad u \to \infty,
\]

where \( H_{\alpha} \) is the Pickands constant given by

\[
H_{\alpha} = \lim_{S \to \infty} S^{-1} H_{\alpha}([0, S]) \in (0, \infty),
\]

with

\[
H_{\alpha}([0, S]) = \mathbb{E}\left\{ \sup_{t \in [0, S]} e^{W_{\alpha}(t)} \right\}, \quad W_{\alpha}(t) := \sqrt{2} B_{\alpha}(t) - |t|^\alpha
\]

and \( B_{\alpha} \) is a standard fractional Brownian motion (fBm) with Hurst index \( \alpha/2 \in (0, 1] \). A refinement of (1.3) is given in [3][Theorem 3.3.1]. Namely, (1.1) holds with

\[
C(x) = \tilde{B}_{\alpha}(x)/H_{\alpha}
\]

for any \( x > 0 \) a continuity point of \( \tilde{B}_{\alpha}(\cdot) \). Here \( \tilde{B}_{\alpha}(x) = \int_{x}^{\infty} \frac{1}{y} dG_{\alpha}(y) \in (0, \infty) \), with

\[
G_{\alpha}(x) = \mathbb{P}\left\{ \int_{\mathbb{R}} I_{0}(W_{\alpha}(s) + \mathcal{E}) ds \leq x \right\},
\]

where \( \mathcal{E} \) is a unit exponential random variable independent of \( W_{\alpha} \). Furthermore, as shown in [3][Theorem 10.5.1]

\[
\tilde{B}_{\alpha}(0) = \lim_{x \downarrow 0} \int_{x}^{\infty} \frac{1}{y} dG_{\alpha}(y) = H_{\alpha}.
\]

We note that the only known values of Pickands constants are \( H_1 = 1 \) and \( H_2 = \frac{1}{\sqrt{\pi}} \) and both (1.4) and (1.6) are not tractable for simulations. In Theorem 1.1 we present an interesting formula for \( H_{\alpha} \), which is a consequence of Berman’s theory on extremes of random processes. We believe that this new formula is of particular interest for simulations, since it is given as an expectation, see [7–11] for alternative formulas. Another advantage of this new formula is that it implies the uniformly (with respect to \( \alpha \)) sharpest lower bound for the Pickands constant available in the literature so far. Next, let \( \Gamma(\cdot) \) stands for Euler Gamma function.

**Theorem 1.1.** For \( \alpha \in (0, 2] \) we have

\[
H_{\alpha} = \mathbb{E}\left\{ \frac{1}{\int_{\mathbb{R}} I_{0}(W_{\alpha}(s) + \mathcal{E}) ds} \right\} \geq \frac{\Gamma(1/\alpha)}{4\Gamma(2/\alpha)}.
\]
Interestingly, the same lower bound for $\mathcal{H}_\alpha$ as derived in Theorem 1.1 was obtained heuristically in [12][J20a,J20b]. The above finding uniformly improves the result of [13] (see also [14]):

$$\mathcal{H}_\alpha \geq \frac{4^{-\frac{1}{\alpha}}}{\Gamma(1/\alpha + 1)};$$

see Fig. 1. We refer to [15] for the proof that $\mathcal{H}_\alpha \geq \frac{(1.1527)^{1/\alpha}}{\Gamma(1/\alpha)}$ for $\alpha$ sufficiently close to 0, which subverted an opened for long time hypothesis that $\mathcal{H}_\alpha = \frac{1}{\Gamma(1/\alpha)}$. Other estimates for Pickands constants can be found in e.g., [16] and [7].

The main interest of this contribution is the investigation of the tail asymptotics of $L_u[a, b]$ and its discrete counterpart.

Our method here is completely different from that of Berman. Namely, in this paper we developed the uniform double-sum method for the sojourn time functional. Interestingly, this approach leads to a new representation of Berman’s constants $\tilde{B}_\alpha(x)$; see Section 2 where the asymptotics for the tail distribution of sojourns of locally-stationary Gaussian processes was derived and compared with the classical results of Berman.

Our main findings in this paper can be summarized as follows: for both locally-stationary Gaussian processes and general non-stationary Gaussian processes with variance maximal at some unique point, we show that (1.1) holds for almost all $x$ and moreover, we calculate explicitly $C(x)$ and give the appropriate scaling function $v$. Our results are new for non-stationary Gaussian processes, and agree with those of Berman for the locally stationary ones. In particular, all results are new for the discrete setup introduced in the next section.
Brief organisation of the rest of the paper: In Section 2 we derive the tail asymptotics of sojourn time for locally stationary Gaussian processes. Corresponding results for general non-stationary Gaussian processes are then presented in Section 3. All the proofs are displayed in Section 4 whereas few technical results are included in Section 5.

2. Sojourns of Locally Stationary Gaussian Processes

In this section we analyze sojourns for the class of *locally stationary* Gaussian processes, introduced by Berman in [3], see also [17–22]. Specifically, let \( X(t), t \in [0, T] \) be a centered Gaussian process with unit variance and correlation function \( \rho \) satisfying

\[
\lim_{\varepsilon \to 0} \sup_{t, t+s \in [0, T], |s| < \varepsilon} \left| \frac{1 - \rho(t, t+s)}{K(|s|)} - H(t) \right| = 0,
\]

where \( H \) is a continuous positive function on \([0, T]\) and \( K \) is a regularly varying function at 0 with index \( \alpha \in (0, 2) \). In the following let \( v \) be the asymptotically unique function (which exists, see [3]) such that \( \lim_{u \to \infty} v(u) = \infty \) and

\[
\lim_{u \to \infty} u^2 K(1/v(u)) = 1.
\]

We shall investigate the tail asymptotics of \( L^*_u[0, T] := v(u) L_u[0, T] \). Given some \( \eta > 0 \) we define the discrete counterpart of \( L^*_u[0, T] \) as

\[
L^*_\eta,u[a, b] := v(u) \int_a^b \mathbb{I}_u(X(t)) \mu_\eta(dt) = \eta \sum_{t \in (\eta \mathbb{Z}) \cap [a,b]} \mathbb{I}_u(X(t)) ,
\]

where \( \eta = \eta/v(u) \) and \( \mu_c(dt)/c \) denotes the counting measure on \( c\mathbb{Z} \). In the sequel we interpret \( 0\mathbb{Z} \) as \( \mathbb{R} \) and \( \mu_0 \) as the Lebesgue measure on \( \mathbb{R} \). Since \( \mu_c \) converges to the Lebesgue measure \( \mu_0 \) on \( \mathbb{R} \) as \( c \to 0 \), with this convention we set

\[
L^*_0,u[a, b] := L^*_u[a, b] = v(u) L_u[a, b].
\]

In order to state our first result, define for any \( \lambda > 0, \eta \geq 0, x \in [0, \mu_\eta([0, S])) \)

\[
\mathcal{B}^\eta_{\alpha, \lambda}(S, x) := \int_\mathbb{R} \mathbb{P} \left\{ \int_0^S \mathbb{I}_0 \left( W_\alpha(\lambda^{1/\alpha} s) + z \right) \mu_\eta(ds) > x \right\} e^{-z}dz ,
\]

where \( W_\alpha \) is defined in (1.4). Further, for any \( x \geq 0 \) set

\[
\mathcal{B}^{\eta,H}_{\alpha}(x) := \lim_{S \to \infty} \frac{\int_0^T \mathcal{B}^\eta_{\alpha, H(t)}(S, x)dt}{S}, \quad \mathcal{B}^{\eta}_{\alpha, 1}(S, x) := \lim_{S \to \infty} S^{-1} \mathcal{B}^{\eta}_{\alpha, 1}(S, x).
\]

Hereafter, when we mention that \( x \) is a continuity point for some function \( f \) we also assume that \( f(x) > 0 \).

Next we state our first result. The case \( \eta > 0 \) is new, whereas for the case \( \eta = 0 \) we retrieve the result of Berman, however the asymptotic constant (pre-factor) is given in a different form than in the original Berman’s result, see e.g. [4], which is due to a different technique applied here.
Theorem 2.1. Let \( X(t), t \in [0, T] \) be a centered, sample path continuous Gaussian process with unit variance and correlation function satisfying assumption (2.1). If further \( \rho(s, t) < 1 \) for all \( s, t \in [0, T] \), \( s \neq t \), then for any \( x > 0 \) a continuity point of \( B_{\alpha}^{0,H}(\cdot) \) and for \( x = 0 \) we have

\[
P \{ L_{\eta,u}^{*}[0,T] > x \} \sim B_{\alpha}^{0,H}(x) v(u) P \{ X(0) > u \}, \quad u \rightarrow \infty,
\]

where \( v(u) \) is given in (2.2) and \( B_{\alpha}^{0,H}(\cdot) \) defined in (2.4) is positive and finite for any \( x, \eta \geq 0 \).

Remark 2.2. i) If \( X(t), t \in [0, T] \) is a centered, stationary, sample path continuous Gaussian process with unit variance function and its correlation function \( \rho \) satisfies Pickands condition (1.2), then \( X \) is locally stationary with function \( H(t) \equiv 1 \), \( t \in [0, T] \). For such \( H \) we have that \( B_{\alpha}^{0,H}(x) = TB_{\alpha}^{0}(x) \).

ii) For \( \eta = 0 \), by [3][Theorem 3.3.1] and (2.5) we have

\[
B_{0}^{0}(x) = \tilde{B}_{\alpha}(x)
\]

for all continuity points of \( \tilde{B}_{\alpha}(\cdot) \) (since both \( B_{\alpha}^{0}(\cdot) \) and \( \tilde{B}_{\alpha}(\cdot) \) are monotone non-increasing).

3. Sojourns of Non-Stationary Gaussian Processes

In this section we analyze sojourns of non-stationary centered Gaussian processes. Suppose that \( X(t), t \in [-T, T] \) is a centered Gaussian process with continuous sample paths. Tractable assumptions on both variance \( \sigma^{2}(t) = \text{Var}(X(t)) \) and correlation function \( \rho(s, t) \), adopted from a vast literature on the asymptotic analysis of supremum of non-stationary Gaussian processes, see e.g., [1, 2, 6, 19, 23–28], are as follows:

A0: For some \( T > 0 \)

\[
t_0 = \arg\max_{t \in [-T, T]} \sigma(t)
\]

is unique. For notational simplicity we assume further that \( t_0 = 0 \) and \( \sigma(t_0) = 1 \).

A1: For some \( \alpha \in (0, 2] \) we have

\[
1 - \rho(s, t) \sim |t - s|^{\alpha}, \quad s, t \rightarrow t_0.
\]

A2: For some positive constants \( b, \beta \)

\[
1 - \sigma(t) \sim b |t|^{\beta}, \quad t \rightarrow t_0.
\]

Under the assumptions A0-A1, if further

\[
\lim_{s,t \rightarrow 0, s \neq t} \frac{|\sigma(s) - \sigma(t)|}{\mathbb{E} \{ (X(s) - X(t))^{2} \}} = 0
\]

in view of [1][Theorem 6.1] we have

\[
\lim_{u \rightarrow \infty} \frac{\int_{0}^{x} y d\mathbb{P} \{ L_{\eta,u}^{*}[-T,T] \leq y \}}{\mathbb{E} \{ L_{\eta,u}^{*}[-T,T] \}} = G_{\alpha}(x)
\]
for any continuity point \( x \) of \( \mathcal{G}_\alpha \) defined in (1.5).

See also [2] for another result shown under A0, A2 assuming further that

\[
\lim_{t \to 0} \frac{\mathbb{E} \{(X(t) - X(0))^2\}}{1 - \sigma(t)} = 0.
\]

We present next the main result of this section. Under the assumptions A0-A2 we shall derive the tail asymptotics of \( L_{\eta,u}^*[T,T] \), where we chose the scaling function \( v(u) \) as follows

(3.1) \[ v(u) = u^{2/\min(\alpha,\beta)}. \]

As in the case of Piterbarg’s result for sup \( t \in [-T,T] X(t) \) (see [6]), if \( \alpha = \beta \) in the asymptotic results a new constant \( b_{\alpha-\eta} \) appears, which is defined for any \( b > 0 \) by

(3.2) \[ b_{\alpha-\eta}(x) := \int_{\mathbb{R}} \mathbb{P} \left\{ \int_{\mathbb{R}} \mathbb{I}_0 (W_\alpha(s) - b |s|^\alpha + z) \mu_\eta(ds) > x \right\} e^{-z}dz. \]

Additionally, for \( \eta > 0 \) we set

(3.3) \[ \mathcal{T}^b_{\eta}(x) = \begin{cases} 1 & \text{if } x \in [0, \eta) \\ e^{-b(\eta)} & \text{if } x \in [(2k - 1)\eta, (2k + 1)\eta), k \in \mathbb{N} \end{cases} \]

and for \( \eta = 0, x \geq 0 \) let \( \mathcal{T}^b_{\eta}(x) = e^{-b(x)^\beta} \).

We present next the main result of this section.

**Theorem 3.1.** Let \( X \) be a centered Gaussian process satisfying A0-A2, \( \eta \geq 0 \) and \( v(u) = u^{2/\min(\alpha,\beta)}. \)

i) If \( \alpha < \beta \), then for any \( x > 0 \) a continuity point of \( \mathcal{B}_\alpha(x) \) and \( x = 0 \)

(3.4) \[ \mathbb{P} \left\{ L_{\eta,u}^*[T,T] > x \right\} \sim 2b^{-1/\beta} \Gamma(1/\beta + 1) \mathcal{B}^\eta_{\alpha}(x) u^{2-2/\beta} \mathbb{P} \{ X(0) > u \}, \quad u \to \infty. \]

ii) If \( \alpha = \beta \), then for any \( x > 0 \) a continuity point of \( \mathcal{B}^\eta_{\alpha}(x) \) and \( x = 0 \)

(3.5) \[ \mathbb{P} \left\{ L_{\eta,u}^*[T,T] > x \right\} \sim \mathcal{B}^\eta_{\alpha}(x) \mathbb{P} \{ X(0) > u \}, \quad u \to \infty. \]

iii) If \( \beta < \alpha \), then for any \( x > 0 \) a continuity point of \( \mathcal{T}^b_{\eta}(x) \) and \( x = 0 \)

(3.6) \[ \mathbb{P} \left\{ L_{\eta,u}^*[T,T] > x \right\} \sim \mathcal{T}^b_{\eta}(x) \mathbb{P} \{ X(0) > u \}, \quad u \to \infty. \]

**Remark 3.2.** i) If \( \eta = x = 0 \), then \( \mathcal{B}^b_{\alpha}(0) = \mathcal{B}^b_{\alpha}. \) Indeed, for any \( b > 0 \) we have

\[
\mathcal{B}^b_{\alpha}(0) = \int_{\mathbb{R}} \mathbb{P} \left\{ \int_{s \in \mathbb{R}} \mathbb{I}_0 (W_\alpha(s) - b |s|^\alpha + z) ds > 0 \right\} e^{-z}dz = \mathbb{E} \left\{ \sup_{s \in \mathbb{R}} e^{W_\alpha(s) - b |s|^\alpha} \right\} =: \mathcal{B}^b_{\alpha}.
\]

In the literature, \( \mathcal{B}^b_{\alpha} \) is referred to as Piterbarg constant, see [24, 29-31] for related constants and basic properties.
ii) If \( t_0 \in \{-T, T\} \) in A0, then Theorem 3.1 still holds subject to appropriate change of the constants in the asymptotics. Specifically, if \( \alpha < \beta \), (3.4) holds with the constant 2 removed from the expression. If \( \alpha = \beta \), then \( \mathcal{P}^{\alpha,\eta}_\beta(x) \) in (3.5) has to be changed to

\[
\int_\mathbb{R} \mathbb{P} \left\{ \int_0^{\infty} \mathbb{I}_0 (W_\alpha(s) - b|s|^{\alpha} + z) \mu_\eta(ds) > x \right\} e^{-z}dz.
\]

If \( \alpha > \beta \), then in (3.6) \( T^{\alpha,\eta}_{\beta}(x) \) has to be substituted by \( e^{-bx^{\beta}} \), \( x \geq 0 \) for \( \eta = 0 \), and \( e^{-b(k\eta)^{\beta}} \) if \( x \in [k\eta, (k+1)\eta) \), \( k \in \{0\} \cup \mathbb{N} \) for \( \eta > 0 \).

4. Proofs

Below \([x]\) stands for the integer part of \( x \) and \([x]\) is the smallest integer not less than \( x \). Further \( \Psi \) is the survival function of an \( N(0,1) \) random variable.

**Proof of Theorem 1.1** Since \( W_\alpha \) has almost surely continuous trajectories with \( W_\alpha(0) = 0 \) and \( \mathcal{E} > 0 \) almost surely, then \( I_\alpha = \int_\mathbb{R} \mathbb{I}_0 (W_\alpha(s) + \mathcal{E}) ds > 0 \) almost surely. Consequently, by the definition of Pickands constant in (1.6) and the monotone convergence theorem we obtain

\[
\mathcal{H}_\alpha = \lim_{x \downarrow 0} \int_x^{\infty} \frac{1}{y} d\mathcal{G}_\alpha(y) = \mathbb{E} \left\{ \frac{1}{I_\alpha} \right\} \in (0, \infty).
\]

Hence by Jensen’s inequality we have

\[
\mathbb{E} \left\{ \frac{1}{I_\alpha} \right\} \geq \frac{1}{\mathbb{E} \{I_\alpha\}}.
\]

Further, we have

\[
\mathbb{E} \{I_\alpha\} = \int_\mathbb{R} \mathbb{P} \{W_\alpha(t) + \mathcal{E} > 0\} dt
\]
\[ \begin{align*}
&= 2 \int_0^\infty \mathbb{P} \{ W_\alpha(t) + \mathcal{E} > 0 \} \, dt \\
&= 2 \int_0^\infty \mathbb{P} \left\{ \sqrt{2t^\alpha} B_\alpha(1) - t^\alpha + \mathcal{E} > 0 \right\} \, dt \\
&= 4 \int_0^\infty \mathbb{P} \left\{ B_\alpha(1) > \sqrt{\frac{t^\alpha}{2}} \right\} \, dt \\
&= 4 \int_0^\infty \mathbb{P} \left\{ \max(0, B_\alpha(1)) > \sqrt{\frac{t^\alpha}{2}} \right\} \, dt \\
&= 4 \mathbb{E} \left\{ 2^{1/\alpha} (\max(0, B_\alpha(1)))^{2/\alpha} \right\} \\
&= \frac{4^{1/\alpha + 1/2}}{\sqrt{\pi}} \Gamma(1/\alpha + 1/2),
\end{align*}\]

where in (4.1) we used Lemma 5.4 from Appendix. Thus the proof is complete. \( \Box \)

**Proof of Theorem 2.1** Let \( S > 1 \) be a positive constant. Define \( S_0 = S, S_\eta = \eta[S] \) for \( \eta > 0 \) and set further
\[ \Delta_k = [kS_\eta, u, (k + 1)S_\eta], \quad k = 0, \ldots, N_u, \]
where \( S_\eta, u = S_\eta/v(u) \) and \( N_u = \left\lfloor T/S_\eta, u \right\rfloor \). We have for all \( u \) positive and \( x \geq 0 \)
\[ (4.2) \]
\[ I_1(u) \leq \mathbb{P} \left\{ L^*_u[0, T] > x \right\} \leq I_2(u), \]
where
\[ I_1(u) = \sum_{k=0}^{N_u-1} \mathbb{P} \left\{ L^*_u \Delta_k > x \right\} - \sum_{0 \leq i < k \leq N_u-1} q_{i,k}(u), \]
\[ I_2(u) = \sum_{k=0}^{N_u} \mathbb{P} \left\{ L^*_u \Delta_k > x \right\} + \sum_{0 \leq i < k \leq N_u} q_{i,k}(u), \]
with
\[ q_{i,k}(u) = \mathbb{P} \left\{ \sup_{t \in \Delta_i} X(t) > u, \sup_{t \in \Delta_k} X(t) > u \right\}. \]

We first show that, as \( u \to \infty \) and then \( S \to \infty \), the first sum in \( I_1 \) is asymptotically equivalent to \( v(u)\Psi(u) \) and the double sum is negligible with respect to the former one.

For any \( x \geq 0 \) and \( t \in [0, T] \), put
\[ F_u(t, x) = \Psi^{-1}(u) \mathbb{P} \left\{ v(u) \int_0^{S_\eta, u} \mathbb{I}_u(X(t + s)) \mu_{\eta, u}(ds) > x \right\}. \]

According to (2.1), we choose \( \varepsilon \) small enough such that
\[ (4.3) \quad 1 - \rho(s, t) \leq 2\bar{h}K(|t - s|) \leq 1, \quad \forall s, t \in [0, T], |t - s| \leq \varepsilon \]
with $\bar{H} = \max_{t \in [0,T]} H(t)$. Let $Y$ be a centered stationary Gaussian process with continuous trajectories, unit variance function and covariance function satisfying
\[
1 - \text{Cov}(Y(t), Y(t+s)) \sim 4\bar{H}K(|s|), \quad s \to 0.
\]
The existence of such a Gaussian process is guaranteed by the Assertion in [32][p.265] and follows from [33, 34]. Consequently, by Slepian lemma and [30][Lemma 5.1] for any $\eta \geq 0$ and sufficiently large $u$
\[
\sup_{t \in [0,T]} F_\eta(t, x) \leq \Psi^{-1}(u) \sup_{t \in [0,T]} \mathbb{P}\left\{ \sup_{s \in [0,S_{\eta,u}]} X(t + s) > u \right\} \leq \Psi^{-1}(u) \mathbb{P}\left\{ \sup_{s \in [0,S_{\eta,u}]} Y(s) > u \right\} \leq 2\left(4\bar{H}1/\alpha S_\eta\right) \mathcal{H}_\alpha([0,1]),
\]
where $\mathcal{H}_\alpha(\cdot)$ is defined in (1.4). Therefore,
\[
\left| \frac{1}{S_{\eta}} \int_0^T \frac{1}{\Psi(u)} \mathbb{P}\left\{ L_{\eta,u}^*[t, t + S_{\eta,u}] > x \right\} \mu_{S_{\eta,u}}(dt) - \frac{1}{v(u)} \sum_{k=0}^{N_{u}-1} \mathbb{P}\left\{ L_{\eta,u}^* \Delta_k > x \right\} \right| \leq \frac{1}{v(u)} \sup_{t \in [0,T]} F_\eta(t, x) \to 0
\]
as $u \to \infty$ implying
\[
\frac{1}{\Psi(u)} \sum_{k=0}^{N_{u}-1} \mathbb{P}\left\{ L_{\eta,u}^* \Delta_k > x \right\} \sim \frac{1}{S_{\eta}} \int_0^T F_\eta(t, x) \mu_{S_{\eta,u}}(dt).
\]
Let $x_0 \in (0, \mu_\eta([0, S_\eta]))$ be a continuity point of $\int_0^T \mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x) dt$, then
\[
\lim_{\varepsilon_0 \to 0} \int_0^T \mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x_0 + \varepsilon_0) - \mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x_0 - \varepsilon_0) dt = 0.
\]
Application of the dominated convergence theorem with Lemma 5.2 in Appendix yields
\[
\int_0^T \mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x_0 + ) - \mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x_0 - ) dt = 0.
\]
Since $\mathcal{B}_{\alpha,H(t)}^\eta(S_\eta, x)$ is monotone in $x$ for each $t \in [0, T]$, it follows that $x_0$ is a continuity point for any $t \in [0, T] \setminus B$, where $B$ is some subset of $[0, T]$ with Lebesgue measure 0. Next, by Lemma 5.1-i), for any $t_u$ such that $\lim_{u \to \infty} t_u = t_0 \in [0, T] \setminus B$
\[
(4.5) \quad \lim_{u \to \infty} F_\eta(t_u, x_0) = \mathcal{B}_{\alpha,H(t_0)}^\eta(S_\eta, x_0).
\]
By (4.4) for sufficiently large $u_0$, $F_\eta(\cdot, x_0), u \geq u_0$ is uniformly bounded on $[0, T]$. Consequently, [35][Lemma 9.3] implies
\[
\lim_{u \to \infty} \frac{1}{\Psi(u) v(u)} \sum_{k=0}^{N_{u}-1} \mathbb{P}\left\{ L_{\eta,u}^* \Delta_k > x_0 \right\} = \lim_{u \to \infty} \frac{1}{S_{\eta}} \int_0^T F_\eta(t, x_0) \mu_{S_{\eta,u}}(dt)
\]
Further, by Lemma 5.1-i), (4.5) is also valid for \( x_0 = 0 \). Therefore, (4.6) holds for \( x_0 = 0 \) and for any \( x_0 \in (0, \mu_\eta([0, S_\eta])) \) a continuity point of \( \int_0^T B_{\alpha,H(t)}^\eta(S_\eta, x)dt \).

Define

\[
A_\varepsilon = \{(i, k) : 1 \leq i + 1 < k \leq N_u - 1, k + 1 - \varepsilon \leq \lfloor \varepsilon/S_{\eta,u} \rfloor \},
\]

\[
B_\varepsilon = \{(i, k) : 1 \leq i + 1 < k \leq N_u - 1, k + 1 - \varepsilon > \lfloor \varepsilon/S_{\eta,u} \rfloor \}.
\]

Then

\[
\sum_{0 \leq i < k \leq N_u - 1} q_{i,k}(u) \leq \sum_{0 \leq i \leq N_u - 1} q_{i,i+1}(u) + \sum_{(i,k) \in A_\varepsilon} q_{i,k}(u) + \sum_{(i,k) \in B_\varepsilon} q_{i,k}(u).
\]

Since \( \rho(s, t) < 1 \) for all \( s, t \in [0, T], s \neq t \), with a similar argument as used in the proof of Theorem 4 in [13]

\[
\limsup_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{(i,k) \in B_\varepsilon} q_{i,k}(u) = 0
\]

holds. In view of Lemma 5.3, for large enough \( u \)

\[
\sum_{(i,k) \in A_\varepsilon} q_{i,k}(u) \leq 2\left[(16h^{1/\alpha})^2 \left| S_\eta \right|^2 H_\alpha^2([0,1])\Psi(u)N_u \sum_{k=1}^\infty \exp\left(-\frac{1}{16}h \left| kS_\eta \right|^\alpha/2 \right) \right].
\]

Since \( \lim_{S \to \infty} e^S \sum_{k=1}^\infty e^{-Sk^{\alpha/2}} < 2 \), then for sufficiently large \( S \)

\[
\limsup_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{(i,k) \in A_\varepsilon} q_{i,k}(u) \leq 4\left[(16h^{1/\alpha})^2 H_\alpha^2([0,1])TS_\eta \exp\left(-\frac{1}{16}hS_\eta^{\alpha/2} \right) \right].
\]

Further, choosing large \( S \) such that \( S_\eta > 1 \), for large \( u \) and each \( i < N_u \) we have

\[
\frac{q_{i,i+1}(u)}{\Psi(u)} \leq \frac{1}{\Psi(u)} \mathbb{P} \left\{ \sup_{t \in [(i+1)S_\eta,(i+1)S_\eta+\sqrt{S_\eta}/v(u)]} X(t) > u \right\}
\]

\[
+ \frac{1}{\Psi(u)} \mathbb{P} \left\{ \sup_{t \in [(i+1)S_\eta+\sqrt{S_\eta}/v(u)]} X(t) > u, \sup_{t \in [(i+1)S_\eta+\sqrt{S_\eta}/v(u)]} X(t) > u \right\}
\]

\[
\leq 2\left[(4h)^{1/\alpha} \sqrt{S_\eta} \right] H_\alpha([0,1]) + 2\left[(16h^{1/\alpha})^2 \left| S_\eta \right|^2 H_\alpha^2([0,1]) \exp\left(-\frac{1}{16}h \left| S_\eta \right|^\alpha/4 \right) \right],
\]

where in the last inequality we have used (4.4) and Lemma 5.3. Therefore,

\[
\limsup_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{i=0}^{N_u-1} q_{i,i+1}(u) \leq \frac{2TH_\alpha([0,1])}{S_\eta} \left( \left[(4h)^{1/\alpha} \sqrt{S_\eta} \right] + \left[(16h^{1/\alpha})^2 \left| S_\eta \right|^2 H_\alpha([0,1]) \exp\left(-\frac{1}{16}h \left| S_\eta \right|^\alpha/4 \right) \right) \right).
\]
Substituting \((4.8)-(4.10)\) into \((4.7)\) yields
\[
(4.11) \quad \lim_{S \to \infty} \limsup_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{0 \leq i < k \leq N_u - 1} q_{i,k}(u) = 0.
\]

Next, take \(S = n\) for \(n = 2, 3, \ldots\) and denote by \(E_n\) the set of discontinuity points of \(\int_0^T \mathcal{B}^n_{\alpha,H(t)}(n_\eta, x)dt\) on \((0, \mu_\eta([0, n_\eta]))\). For each \(n \geq 2\), \(E_n\) has measure 0 since \(\mathcal{B}^n_{\alpha,H(t)}(n_\eta, \cdot)\) is monotone in \(x\) and uniformly bounded for \(t \in [0, T]\) by Lemma 5.2. Thus, in view of \((4.2)\), combing \((4.6)\) with \((4.11)\) we get
\[
\limsup_{n \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(n_\eta, x)dt}{n_\eta} \leq \liminf_{u \to \infty} \frac{\mathbb{P}\{L^*_{n,u}[0,T] > x\}}{v(u)\Psi(u)} \leq \limsup_{u \to \infty} \frac{\mathbb{P}\{L^*_{n,u}[0,T] > x\}}{v(u)\Psi(u)} \leq \liminf_{n \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(n_\eta, x)dt}{n_\eta}
\]
for any \(x \in \{0\} \cup E^c\), where
\[
(4.12) \quad E^c := \mathbb{R}^+ \setminus \bigcup_{n=2}^{\infty} E_n.
\]

Further, for all \(t \in [0, T], x \geq 0\) and any \(S > 1\)
\[
\frac{\mathcal{B}^n_{\alpha,H(t)}(\lfloor S \rfloor_\eta, x)}{S_\eta} \leq \frac{\mathcal{B}^n_{\alpha,H(t)}(S_\eta, x)}{S_\eta} \leq \frac{\mathcal{B}^n_{\alpha,H(t)}(\lceil S \rceil_\eta, x)}{\lceil S \rceil_\eta},
\]
which implies that for any \(x \in \{0\} \cup E^c\)
\[
\lim_{u \to \infty} \frac{\mathbb{P}\{L^*_{n,u}[0,T] > x\}}{v(u)\Psi(u)} = \lim_{n \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(n_\eta, x)dt}{n_\eta} = \lim_{S \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(S_\eta, x)dt}{S_\eta} = \lim_{S \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(S, x)dt}{S} := \mathcal{B}^n_{\alpha,H}(x).
\]
We determine \(\mathcal{B}^n_{\alpha,H}(x)\) by the right limit for each \(x \in \bigcup_{n=2}^{\infty} E_n\). Hence, by monotonicity, \(\mathcal{B}^n_{\alpha,H}(x)\) is well-defined for any \(x \geq 0\). Let \(x_0 > 0\) be any continuity point of \(\mathcal{B}^n_{\alpha,H}(\cdot)\). Since \(E^c\) is dense in \(\mathbb{R}^+\) we can choose two sequences of points \(\{y_n, z_n, n \in \mathbb{N}\}\) from \(E^c\) such that \(y_n \nearrow x_0\) and \(z_n \searrow x_0\). By the monotonicity again
\[
\mathcal{B}^n_{\alpha,H}(z_n) = \liminf_{S \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(S, z_n)dt}{S} \leq \liminf_{S \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(S, x_0)dt}{S} \leq \limsup_{S \to \infty} \frac{\int_0^T \mathcal{B}^n_{\alpha,H(t)}(S, x_0)dt}{S}
\]
\[ \limsup_{S \to \infty} \frac{\int_0^T B_{\alpha,H(t)}^n(S, y_n) dt}{S} = B_{\alpha,H}^n(y_n), \]

and similarly
\[ B_{\alpha,H}^n(z_n) = \liminf_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^*[0, T] > z_n\}}{v(u)\Psi(u)} \leq \liminf_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^*[0, T] > x_0\}}{v(u)\Psi(u)} \leq \limsup_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^*[0, T] > y_n\}}{v(u)\Psi(u)} = B_{\alpha,H}^n(y_n). \]

Letting \( n \to \infty \) in the above inequalities implies that (4.13) holds also for any \( x > 0 \) continuity point of \( B_{\alpha,H}^n(\cdot) \).

Next we show that \( B_{\alpha,H}^n(\cdot) \) is finite and positive. The finiteness follows from Lemma 5.2 in Appendix.

In order to prove positivity of \( B_{\alpha,H}^n(\cdot) \), we note that by Bonferroni inequality
\[ \mathbb{P}\{L_{\eta,u}^*[0, T] > x\} \geq \mathbb{P}\left\{ \bigcup_{k=0}^{[N_u/2]-1} \{L_{\eta,u}^* \Delta_{2k} > x\} \right\} \geq \sum_{k=0}^{[N_u/2]-1} \mathbb{P}\{L_{\eta,u}^* \Delta_{2k} > x\} - \sum_{0 \leq i < k \leq [N_u/2]-1} q_{2i,2k}(u). \]

Let \( x \in (0, \mu_\eta([0, S_\eta])) \) be a continuity point of \( \int_0^T B_{\alpha,H(t)}^n(S_\eta, t) dt \), then by a similar argument as used in (4.6)
\[ \lim_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{k=0}^{[N_u/2]-1} \mathbb{P}\{L_{\eta,u}^* \Delta_{2k} > x\} = \lim_{u \to \infty} \frac{1}{2 \mu_\eta} \int_0^T F_u(t, x) \mu_{2S_\eta,u}(dt) = \frac{1}{2 \mu_\eta} \int_0^T B_{\alpha,H(t)}^n(S_\eta, t) dt. \]

Further, as shown in (4.7)-(4.9)
\[ \limsup_{u \to \infty} \frac{1}{v(u)\Psi(u)} \sum_{0 \leq i < k \leq [N_u/2]-1} q_{2i,2k}(u) \leq 2[(16\bar{h})^{1/\alpha}]^2 H_\alpha^2([0, 1]) T S_\eta \exp\left(-\frac{1}{16} h(S_\eta)^{\alpha/2}\right). \]

Consequently,
\[ \liminf_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^*[0, T] > x\}}{v(u)\Psi(u)} \geq \frac{1}{2 \mu_\eta} \left( \int_0^T B_{\alpha,H(t)}^n(S_\eta, t) dt - 4[(16\bar{h})^{1/\alpha}]^2 H_\alpha^2([0, 1]) T S_\eta \exp\left(-\frac{1}{16} h(S_\eta)^{\alpha/2}\right) \right), \]

hence the proof follows. \( \square \)

**Proof of Theorem 3.1** First note that for any \( \eta, x \geq 0 \)
\[ \mathbb{P}\{L_{\eta,u}^* \Lambda_u > x\} \leq \mathbb{P}\{L_{\eta,u}^*(-T, T] > x\} \leq \mathbb{P}\{L_{\eta,u}^* \Lambda_u > x\} + \mathbb{P}\left\{ \sup_{t \in [-T, T] \setminus \Lambda_u} X(t) > u \right\}, \]
where
\[
\delta(u) = \left(\ln u / u\right)^{2/\beta} \quad \text{and} \quad \Lambda_u = [-\delta(u), \delta(u)].
\]
By A0-A2, for arbitrary \( \varepsilon_1 > 0 \) there exist \( \varepsilon \in (0, T) \) such that
\[
\mathbb{E} \left\{ (X(t)/\sigma(t) - X(s)/\sigma(s))^2 \right\} \leq 3 |t - s|^\alpha, \quad \forall s, t \in [-\varepsilon, \varepsilon],
\]
\[
\sigma(t) \leq 1 - (1 - \varepsilon_1)b |t|^{\beta}, \quad \forall t \in [-\varepsilon, \varepsilon],
\]
\[
\sigma(t) \leq 1 - (1 - \varepsilon_1)b\varepsilon^{\beta}, \quad \forall t \in [-T, T] \setminus [-\varepsilon, \varepsilon].
\]
Consequently, by Piterbarg inequality (see e.g., [24][Theorem 8.1]) for large enough \( u \) and some positive \( C \)
\[
\mathbb{P}\left\{ \sup_{t \in [-\varepsilon, \varepsilon] \setminus \Lambda_u} X(t) > u \right\} \leq 2C\varepsilon u^{2/\alpha - 1} \exp\left(\frac{u^2}{2(1 - (1 - \varepsilon_1)b\varepsilon^{\beta}(u))^2}\right).
\]
By Borell-TIS inequality (see Theorem 2.1.1 in [36]) for some positive \( C_1 \)
\[
\mathbb{P}\left\{ \sup_{t \in [-T, T] \setminus [-\varepsilon, \varepsilon]} X(t) > u \right\} \leq \exp\left(\frac{u - C_1^2}{2(1 - (1 - \varepsilon_1)b\varepsilon^{\beta})^2}\right).
\]
Combining (4.15) with (4.16) we get
\[
\mathbb{P}\left\{ \sup_{t \in [-T, T] \setminus \Lambda_u} X(t) > u \right\} = o\left(\Psi(u)\right)
\]
as \( u \to \infty \). Hence
\[
\mathbb{P}\left\{ L_{\eta,u}^* [-T, T] > x \right\} \sim \mathbb{P}\left\{ L_{\eta,u}^* \Lambda_u > x \right\}, \quad u \to \infty,
\]
if the latter is asymptotically equivalent to \( \Psi(u) \), and thus we need to investigate the asymptotics of
\[\mathbb{P}\left\{ L_{\eta,u}^* \Lambda_u > x \right\}.
\]
Ad 1) We use the same notation as introduced in the proof of Theorem 2.1. Let
\[\Delta_k = [kS_{\eta,u}, (k + 1)S_{\eta,u}], \quad k = 0, \pm 1, \ldots, \pm N_u',\]
where \( N_u' = [\delta(u)/S_{\eta,u}] \). By Bonferroni inequality
\[
I_1'(u) = \mathbb{P}\left\{ L_{\eta,u}^* \Lambda_u > x \right\} \leq I_2'(u)
\]
holds for any \( x \geq 0 \), where
\[
I_1'(u) = \sum_{k=-N_u'}^{N_u'-1} \mathbb{P}\left\{ L_{\eta,u}^* \Delta_k > x \right\} - \sum_{-N_u' \leq i < k \leq N_u' - 1} q_{i,k}(u),
\]
\[
I_2'(u) = \sum_{k=-N_u'-1}^{N_u'} \mathbb{P}\left\{ L_{\eta,u}^* \Delta_k > x \right\} + \sum_{-N_u' - 1 \leq i < k \leq N_u'} q_{i,k}(u).
\]
Next, set
\[
\xi_{u,k}(t) = \frac{X(kS_{\eta,u} + t/v(u))}{\sigma(kS_{\eta,u} + t/v(u))}, \quad t \in [0, S_{\eta}],
\]
and
\[
g_k(u) = \begin{cases} 
  u(1 + (1 - \varepsilon_1)b|kS_{\eta,u}|^\beta), & k \in K_u, k \geq 0, \\
  u(1 + (1 - \varepsilon_1)b|(k + 1)S_{\eta,u}|^\beta), & k \in K_u, k < 0,
\end{cases}
\]
where \( K_u = \{-N_u',-1,\ldots,0,\ldots,N_u'\} \). It follows that \( g_k(u) \) converges as \( u \to \infty \) to infinity uniformly for \( k \in K_u \). Moreover, assumptions C1-C3 in Theorem 5.1 are fulfilled by the family of Gaussian processes \( \{\xi_{u,k}(t), t \in [0,S_n], k \in K_u\} \) given above. Specifically, \( h(t) = t^{\alpha} \) and \( \zeta(t) = B_\alpha(t) \) for \( t \in [0,S_\eta] \), and \( \nu \) required in C3 as shown by (4.14) is equal to \( \alpha \). Therefore, by the uniform convergence as stated in Theorem 5.1, we have
\[
\sum_{k=-N_u'-1}^{N_u'} \mathbb{P}\{L_{\eta,u}^* \Delta_k > x\} \leq \sum_{k=-N_u'-1}^{N_u'} \mathbb{P}\left\{ \int_{[0,S_n]} \mathbb{I}_0 (g_k(u)(\xi_{u,k}(t) - g_k(u))) \mu_\eta(dt) > x \right\}
\]
\[
\sim \mathcal{B}_\alpha^n (S_\eta, x) \sum_{k=-N_u'-1}^{N_u'} \Psi(g_k(u)), \quad u \to \infty
\]
at \( x = 0 \) and \( x \in (0, \mu_\eta([0,S_\eta])) \) a continuity point of \( \mathcal{B}_\alpha^n (S_\eta, x) \), where \( \mathcal{B}_\alpha^n (S_\eta, x) = \mathcal{B}_\alpha^{n,1} (S_\eta, x) \) with the latter defined in (2.3). Further, as \( u \to \infty \),
\[
\sum_{k=-N_u'-1}^{N_u'} \Psi(g_k(u)) \sim \frac{2}{\sqrt{2\pi u}} \exp\left( -\frac{u^2(1 + (1 - \varepsilon_1)b|kS_{\eta,u}|^\beta)^2}{2} \right)
\]
\[
\sim \frac{2\Psi(u)}{S_\eta} \int_0^{\delta(u)} \exp\left( -b(1 - \varepsilon_1)u^2t^\beta \right) dt
\]
\[
\sim \frac{2(b(1 - \varepsilon_1))^{-1/\beta}}{S_\eta} \Gamma(1/\beta + 1) u^{2/\alpha - 2/\beta} \Psi(u)
\]
and thus
\[
(4.19) \quad \limsup_{u \to \infty} \frac{\sum_{k=-N_u'-1}^{N_u'} \mathbb{P}\{L_{\eta,u}^* \Delta_k > x\}}{2b^{-1/\beta} \Gamma(1/\beta + 1) u^{2/\alpha - 2/\beta} \Psi(u)} \leq (1 - \varepsilon_1)^{-1/\beta} \frac{\mathcal{B}_\alpha^n (S_\eta, x)}{S_\eta}
\]
at \( x = 0 \) and all continuity points \( x \in (0, \mu_\eta([0,S_\eta])) \). Moreover, as shown in [6] (see p. 22 therein)
\[
(4.20) \quad \lim_{S \to \infty} \limsup_{u \to \infty} \frac{1}{u^{2/\alpha - 2/\beta} \Psi(u)} \sum_{-N_u'-1 \leq k \leq N_u'} q_i,k(u) = 0,
\]
Consequently, substituting (4.19) and (4.20) into (4.18), then taking \( S = n \) for \( n = 2, 3, \ldots \) yields
\[
\limsup_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^* \Lambda_u > x\}}{2b^{-1/\beta} \Gamma(1/\beta + 1) u^{2/\alpha - 2/\beta} \Psi(u)} \leq (1 - \varepsilon_1)^{-1/\beta} \liminf_{n \to \infty} \frac{\mathcal{B}_\alpha^n (n_\eta, x)}{n_\eta}
\]
at any \( x \in \{0\} \cup E^c \), with \( E^c \) as defined in (4.12). Here \( E_n \) denotes the set of discontinuity points of \( \mathcal{B}_\alpha^n (n_\eta, x) \) on \( (0, \mu_\eta([0,n_\eta])) \).
Similarly, for any \( x \in \{0\} \cup E^c \)
\[
\liminf_{u \to \infty} \frac{\mathbb{P}\{L_{\eta,u}^* \Lambda_u > x\}}{2b^{-1/\beta} \Gamma(1/\beta + 1) u^{2/\alpha - 2/\beta} \Psi(u)} \geq (1 + \varepsilon_1)^{-1/\beta} \limsup_{n \to \infty} \frac{\mathcal{B}_\alpha^n (n_\eta, x)}{n_\eta}.
\]
Since $\varepsilon_1$ is arbitrary, then by the same argument as used in the proof of Theorem 2.1 we have
\[
\lim_{u \to \infty} \mathbb{P} \left\{ L_{n,u}^* \Lambda_u > x \right\} = \lim_{n \to \infty} \frac{B^n_\alpha(n\eta, x)}{n\eta} = \lim_{S \to \infty} \frac{B^n_\alpha(S, x)}{S} := B^n_\alpha(x)
\]
at $x = 0$ and any $x > 0$ a continuity point of $B^n_\alpha(\cdot)$. This together with (4.17) validates the claim (3.4).

Ad ii) Set for large $S$

\begin{equation}
\Delta_S = [-S/v(u), S/v(u)]
\end{equation}

and then for arbitrary $x \geq 0$

\[
\mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\} \leq \mathbb{P} \left\{ L_{n,u}^* \Lambda_u > x \right\} \leq \mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\} + \mathbb{P} \left\{ \sup_{t \in \Lambda_u \setminus \Delta_S} X(t) > u \right\}.
\]

It follows from Lemma 5.1-ii) that
\[
\lim_{u \to \infty} \frac{\mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\}}{\Psi(u)} = \mathcal{P}_\alpha^{b,\eta}(S, x)
\]
at $x = 0$ and all continuity points $x \in (0, \mu([-S,S]))$ of $\mathcal{P}_\alpha^{b,\eta}(S, x)$ defined in (5.9). Further, as shown in [6] (see p. 22 therein),

\[
\mathbb{P} \left\{ \sup_{t \in \Lambda_u \setminus \Delta_S} X(t) > u \right\} = O \left( e^{-cS^{\alpha}} \right) \Psi(u)(1 + o(1)), \quad u \to \infty
\]

holds for some $c > 0$. Then, with similar arguments as in the proof of Theorem 2.1, we obtain
\[
\lim_{u \to \infty} \frac{\mathbb{P} \left\{ L_{n,u}^* \Lambda_u > x \right\}}{\Psi(u)} = \lim_{S \to \infty} \mathcal{P}_\alpha^{b,\eta}(S, x) = \mathcal{P}_\alpha^{b,\eta}(x) \in (0, \infty)
\]
at $x = 0$ and any $x > 0$ a continuity point of $\mathcal{P}_\alpha^{b,\eta}(\cdot)$. The finiteness of $\mathcal{P}_\alpha^{b,\eta}(\cdot)$ follows from the fact that $\mathcal{P}_\alpha^{b,\eta}(x) \leq \mathcal{P}_\alpha^b$. Using further (4.17) establishes (3.5).

Ad iii) For large $S$ define $\Delta_S$ as in (4.21). Note that $v(u) = u^{2/\beta}$ since $\alpha > \beta$. For any $\varepsilon > 0$ and all large $u$, we have $\delta(u) < \varepsilon u^{-2/\alpha}$. Hence for any $x \geq 0$

\[
\mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\} \leq \mathbb{P} \left\{ L_{n,u}^* \Lambda_u > x \right\} \leq \mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\} + \mathbb{P} \left\{ \sup_{t \in [-\varepsilon u^{-2/\alpha}, S/v(u)] \setminus \Delta_S} X(t) > u \right\}.
\]

In view of Lemma 5.1-ii) we have
\[
\lim_{u \to \infty} \frac{\mathbb{P} \left\{ L_{n,u}^* \Delta_S > x \right\}}{\Psi(u)} = \mathcal{P}_\beta^{b,\eta}(S, x)
\]
at $x = 0$ and all continuity points $x \in (0, \mu([-S,S]))$ of $\mathcal{P}_\beta^{b,\eta}(S, x)$ defined in (5.10). Further, Lemma 5.1 in [37] implies
\[
\mathbb{P} \left\{ \sup_{t \in [-\varepsilon u^{-2/\alpha}, S/v(u)] \setminus \Delta_S} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [-\varepsilon u^{-2/\alpha}, S/v(u)] \setminus \Delta_S} \frac{X(t)}{\sigma(t)} > u \left( 1 + (1 - \varepsilon)b \left( S/v(u) \right)^{\beta} \right) \right\}
\]
\[ \leq \mathbb{E} \left\{ \sup_{s \in [-\varepsilon, \varepsilon]} e^{\sqrt{2}B_s(s) - \varepsilon^\alpha} \right\} e^{-b(1-\varepsilon)S^\alpha} \Psi(u)(1 + o(1)), \quad u \to \infty. \]

Following the same argument as in case ii), we obtain
\[ \lim_{u \to \infty} \frac{\mathbb{P}\{L_{n,u}^*[T,T] > x\}}{\Psi(u)} = \lim_{S \to \infty} T_{\beta,\eta}^b(S, x) := T_{\beta,\eta}^b(x) \]

at \( x = 0 \) and all positive continuity points of \( T_{\beta,\eta}^b(\cdot) \), where for \( \eta = 0 \), \( T_{\beta,\eta}^b(x) = e^{-b(x)} \) if \( x \geq 0 \) and for \( \eta > 0 \), \( T_{\beta,\eta}^b(x) = 1 \) if \( x \in [0, \eta) \) and \( T_{\beta,\eta}^b(x) = e^{-b(x)} \) if \( x \in [(2k - 1)\eta, (2k + 1)\eta), k \in \mathbb{N} \).

This completes the proof. \( \square \)

5. Appendix

Let \( K_u \) be an index function of \( u \), \( D \) be a compact set in \( \mathbb{R}^n \) and suppose without loss of generality that \( 0 \in D \). Further, let \( \{\xi_{u,k}(t), t \in D, k \in K_u\} \) be a family of centered Gaussian random fields with a.s. continuous sample paths and variance function \( \sigma_{\xi_{u,k}}^2 \). For \( t \) such that \( \sigma_{\xi_{u,k}}^2(t) > 0 \) define the standardised process
\[ \tilde{\xi}_{u,k}(t) := \frac{\xi_{u,k}(t)}{\sigma_{\xi_{u,k}}(t)}, \quad t \in D. \]

Suppose that:

**C0:** \( \{g_k(u), k \in K_u\} \) is a sequence of deterministic functions of \( u \) satisfying
\[ \lim_{u \to \infty} \inf_{k \in K_u} g_k(u) = \infty. \]

**C1:** \( \sigma_{\xi_{u,k}}(0) = 1 \) for all large \( u \) and any \( k \in K_u \), and there exists some bounded continuous function \( h \) on \( D \) such that
\[ \lim_{u \to \infty} \sup_{t \in D, k \in K_u} \left| g_k^2(u)(1 - \mathbb{E}\{\xi_{u,k}(t)\xi_{u,k}(0)\}) - h(t)\right| = 0. \]

**C2:** There exists a centered Gaussian random field \( \zeta(t), t \in \mathbb{R}^n \) with a.s. continuous trajectories such that for any \( s, t \in D \)
\[ \lim_{u \to \infty} \sup_{k \in K_u} \left| g_k^2(u)(\text{Var}(\tilde{\xi}_{u,k}(t) - \tilde{\xi}_{u,k}(s))) - 2\text{Var}(\zeta(t) - \zeta(s))\right| = 0. \]

**C3:** There exist positive constants \( C, \nu, u_0 \) such that
\[ \sup_{k \in K_u} g_k^2(u) \mathbb{E}\{(\xi_{u,k}(t) - \xi_{u,k}(s))^2\} \leq C\|s - t\|^{\nu} \]
holds for all \( s, t \in D, u \geq u_0 \), where \( \|t\|^{\nu} = \sum_{i=1}^n |t_i|^\nu \).

We present below an extension of Theorem 2.1 in [38]. Hereafter, \( C_i, i \in \mathbb{N} \) are positive constants which might be different from line to line. We recall that \( \mu_\eta(dt)/\eta^n \) denotes the counting measure on \( \eta\mathbb{Z}^n, \eta > 0 \) and \( \mu_0 \) is the Lebesgue measure on \( \mathbb{R}^n \).
Theorem 5.1. Let \( h, g_k(u), \xi_{u,k}(t), t \in D, k \in K_u \) and \( \zeta \) be such that C0-C3 hold. Then, for \( \eta \geq 0 \)

\[
(5.1) \quad \lim_{u \to \infty} \sup_{c \in K_u} \left| \mathbb{P} \left\{ \int_D 1_0 (g_k(u)(\xi_{u,k}(t) - g_k(u))) \mu_\eta(dt) > x \right\} - B^h_\zeta(\eta, D, x) \right| = 0
\]
at \( x = 0 \) and all \( x \in (0, \mu_\eta(D)) \) continuity points of \( B^h_\zeta(\eta, D, x) \), where

\[
B^h_\zeta(\eta, D, x) = \int_{\mathbb{R}} \mathbb{P} \left\{ \int_D 1_0 (\sqrt{2} \zeta(t) - h(t) + z) \mu_\eta(dt) > x \right\} e^{-z^2} \, dz.
\]

Proof of Theorem 5.1 Suppose that C0-C3 are satisfied. We begin from the observation that

\[
(5.2) \quad \lim_{u \to \infty} \sup_{c \in K_u} g_k^2(u) \mathbb{E} \left\{ (\tilde{\xi}_{u,k}(t) - \tilde{\xi}_{u,k}(s))^2 \right\} \leq C_1 \| s - t \|^\nu, \quad \forall s, t \in D,
\]

where \( C_1, \nu \) are positive constants. Indeed, note that

\[
1 - \sigma^2_{\tilde{\xi}_{u,k}}(t) = 2 \left( 1 - \mathbb{E} \{ \xi_{u,k}(t) \xi_{u,k}(0) \} \right) - \mathbb{E} \{ (\xi_{u,k}(t) - \xi_{u,k}(0))^2 \},
\]

which together with C1 and C3 implies

\[
(5.3) \quad \lim_{u \to \infty} \sup_{t \in D, k \in K_u} \left| \sigma^2_{\tilde{\xi}_{u,k}}(t) - 1 \right| = 0.
\]

Consequently, for sufficiently large \( u \)

\[
g_k^2(u) \mathbb{E} \left\{ (\tilde{\xi}_{u,k}(t) - \tilde{\xi}_{u,k}(s))^2 \right\} = \frac{g_k^2(u) 2\sigma_{\xi_{u,k}}(t)\sigma_{\xi_{u,k}}(s) - 2 \mathbb{E} \{ \xi_{u,k}(t) \xi_{u,k}(s) \}}{\sigma_{\xi_{u,k}}(t)\sigma_{\xi_{u,k}}(s)} 
\leq g_k^2(u) \frac{\mathbb{E} \{ (\xi_{u,k}(t) - \xi_{u,k}(s))^2 \}}{\inf_{t \in D} \sigma^2_{\xi_{u,k}}(t)} \leq 2C \| s - t \|^\nu, \quad \forall k \in K_u, s, t \in D.
\]

Next, for notational simplicity denote by \( R_{u,k} \) and \( \rho_{u,k} \) the covariance and the correlation function of \( \xi_{u,k} \). Further set

\[
\chi_{u,k}(t) := g_k(u)(\tilde{\xi}_{u,k}(t) - \rho_{u,k}(t, 0)\tilde{\xi}_{u,k}(0)), \quad t \in D
\]

and

\[
f_{u,k}(t, z) := zR_{u,k}(t, 0) - g_k^2(u) \left( 1 - R_{u,k}(t, 0) \right), \quad t \in D, z \in \mathbb{R}.
\]

Conditioning on \( \xi_{u,k}(0) \) and using that \( \xi_{u,k}(0) \) and \( \xi_{u,k}(t) - R_{u,k}(t, 0)\xi_{u,k}(0) \) are mutually independent for large \( u \), we obtain

\[
\mathbb{P} \left\{ \int_D 1_0 (g_k(u)(\xi_{u,k}(t) - g_k(u))) \mu_\eta(dt) > x \right\} = \exp \left( -\frac{z^2}{2g_k^2(u)} \right) \int_{\mathbb{R}} \mathbb{P} \left\{ \int_D 1_0 (g_k(u)(\xi_{u,k}(t) - g_k(u))) \mu_\eta(dt) > x | \xi_{u,k}(0) = g_k(u) + zg_k^{-1}(u) \right\} \, dz
\]

\[
= \exp \left( -\frac{z^2}{2g_k^2(u)} \right) \int_{\mathbb{R}} \mathbb{P} \left\{ \int_D 1_0 (g_k(u)(\xi_{u,k}(t) + f_{u,k}(t, z)) \mu_\eta(dt) > x \right\} \, dz.
\]
Let
\[ I_{u,k}(x; z) := \mathbb{P} \left\{ \int_D \mathbb{I}_0 \left( \sigma_{u,k}(t) \chi_{u,k}(t) + f_{u,k}(t, z) \right) \mu_\eta(dt) > x \right\}. \]

Consequently, in order to show the claim it suffices to prove that
\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \int_{\mathbb{R}} \exp \left( -z - \frac{z^2}{2g_k^2(u)} \right) I_{u,k}(x; z)dz - B_{\zeta}^{h,\eta}(D, x) \right| = 0
\]

at \( x = 0 \) and all \( x \in (0, \mu_\eta(D)) \) positive continuity points of \( B_{\zeta}^{h,\eta}(D, x) \). Since for all \( x \geq 0 \) and any large \( M \)
\[
\sup_{k \in K_u} e^{-z} I_{u,k}(x; z) \leq e^{-z}, \quad z \geq -M
\]
and by Piterbarg inequality for all large \( u \) and \( M \)
\[
\sup_{k \in K_u} e^{-z} I_{u,k}(x; z) \leq \sup_{k \in K_u} \mathbb{P} \left\{ \sup_{t \in D} \{ \sigma_{u,k}(t) \chi_{u,k}(t) + f_{u,k}(t, z) \} > 0 \right\} e^{-z}
\leq \sup_{k \in K_u} \mathbb{P} \left\{ \sup_{t \in D} \chi_{u,k}(t) > C_2 |z| - C_3 \right\} e^{-z}
\leq C_4 |z|^{2n/\nu - 1} e^{-C_5z^2-C_6z^2}, \quad z < -M,
\]
then by the dominated convergence theorem and assumption C0
\[
\sup_{k \in K_u} \left| \int_{\mathbb{R}} \exp \left( -z - \frac{z^2}{2g_k^2(u)} \right) I_{u,k}(x; z)dz - \int_{\mathbb{R}} e^{-z} I_{u,k}(x; z)dz \right|
\leq \int_{\mathbb{R}} \sup_{k \in K_u} \left( e^{-z} I_{u,k}(x; z) \right) \left| 1 - e^{-z^2/(2g_k^2(u))} \right| dz \to 0, \quad u \to \infty.
\]
Therefore, in order to prove the convergence in (5.4) it suffices to show that
\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \int_{\mathbb{R}} e^{-z} I_{u,k}(x; z)dz - B_{\zeta}^{h,\eta}(D, x) \right| = 0
\]
at \( x = 0 \) and all continuity points \( x \in (0, \mu_\eta(D)) \).

Let \( C(D) \) denote the Banach space of all continuous functions on \( D \) equipped with sup-norm. For any \( s, t \in D \), from C2 and (5.2) we have
\[
Var(\chi_{u,k}(t) - \chi_{u,k}(s)) = g_k^2(u) \left\{ (\xi_{u,k}(t) - \xi_{u,k}(s))^2 - (\rho_{\xi_{u,k}(t, 0) - \rho_{\xi_{u,k}(s, 0)}})^2 \right\}
\to 2Var(\zeta(t) - \zeta(s))
\]
uniformly with respect to \( k \in K_u \) as \( u \to \infty \). Hence, the finite-dimensional distributions of \( \chi_{u,k} \) converge to that of \( \sqrt{2}\zeta(t), t \in D \) uniformly with respect to \( k \in K_u \). In view of (5.2), we know that the measures on \( C(D) \) induced by \( \{ \chi_{u,k}(t), t \in D, k \in K_u \} \) are uniformly tight for large \( u \), and by (5.3) \( \sigma_{\xi_{u,k}(t)} \) converges to 1 uniformly for \( t \in D \) and \( k \in K_u \) as \( u \to \infty \). Therefore,
\{\sigma_{u,k}(t)\chi_{u,k}(t), t \in D\} converge weakly to \{\sqrt{2}\zeta(t), t \in D\} as \(u \to \infty\) uniformly with respect to \(k \in K_u\). Further, by C0-C1 for each \(z \in \mathbb{Z}\)

\[
\lim_{u \to \infty} \sup_{k \in K_u, t \in D} |f_{u,k}(t, z) - z + h(t)| = 0
\]

implying that for each \(z \in \mathbb{Z}\), the probability measures on \(C(D)\) induced by \(\{\chi_{u,k}(t, z), t \in D\}\), where

\[
\chi_{u,k}(t, z) := \sigma_{u,k}(t)\chi_{u,k}(t) + f_{u,k}(t, z) \quad \text{and} \quad \zeta_h(t) := \sqrt{2}\zeta(t) - h(t),
\]

converge weakly, as \(u \to \infty\), to that induced by \(\{\zeta_h(t) + z, t \in D\}\) uniformly with respect to \(k \in K_u\). Consequently, for any \(\eta > 0, z \in \mathbb{Z}\)

\[
(5.7) \quad \lim_{u \to \infty} \sup_{k \in K_u} |\mathcal{I}_{u,k}(x, z) - \mathcal{I}(x, z)| = 0
\]

holds at all continuity points \(x \in (0, \mu_\eta(D))\) (depending on \(z\)) of \(\mathcal{I}(x, z)\) defined by

\[
\mathcal{I}(x; z) := \mathbb{P}\left\{ \int_D \|I_0(\zeta_h(t) + z)\mu_\eta(dt) > x \right\}
\]

For \(\eta = 0\), by \([39]\)[Lemma 4.2] the set of discontinuity points of

\[
\int_D \|I_0(f(t))\| dt, \quad f \in C(D)
\]

is of measure 0 under the probability measure induced by \(\{\zeta_h(t) + z, t \in D\}\). Consequently, by the continuous mapping theorem we also have (5.7). Next, we borrow an argument from \([3]\)[Theorem 1.3.1] to verify (5.6) for all positive continuity points. Let \(x \in (0, \mu_\eta(D))\) be such a continuity point, i.e.,

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} (\mathcal{I}(x_0 + \varepsilon; z) - \mathcal{I}(x_0 - \varepsilon; z)) e^{-z} dz = 0.
\]

Since for large \(M\) and all \(x \geq 0\) by Borell-TIS inequality

\[
(5.8) \quad e^{-z}\mathcal{I}(x; z) \leq C_7e^{-C_8z^2-C_9z}, \quad z < -M
\]

it follows from the dominated convergence theorem that

\[
\int_{\mathbb{R}} (\mathcal{I}(x_0^+; z) - \mathcal{I}(x_0^-; z)) e^{-z} dz = 0,
\]

and thus by the monotonicity of \(\mathcal{I}(x; z)\) in \(x\) for each fixed \(z, x_0\) is a continuity point of \(\mathcal{I}(x; z)\) for almost all \(z \in \mathbb{R}\). Hence by (5.7) for almost all \(z \in \mathbb{R}\)

\[
\lim_{u \to \infty} \sup_{k \in K_u} |\mathcal{I}_{u,k}(x_0, z) - \mathcal{I}(x_0; z)| = 0.
\]

As shown in (5.5) and (5.8) it follows from the dominated convergence that

\[
\sup_{k \in K_u} \left| \int_{\mathbb{R}} e^{-z}\mathcal{I}_{u,k}(x_0; z) dz - \int_{\mathbb{R}} e^{-z}\mathcal{I}(x_0; z) dz \right| \leq \int_{\mathbb{R}} \sup_{k \in K_u} |\mathcal{I}_{u,k}(x_0; z) - \mathcal{I}(x_0; z)| e^{-z} dz \to 0,
\]
as \( u \to \infty \), establishing the proof for all continuity points \( x \in (0, \mu_\eta(D)) \).

The case \( x = 0, \eta = 0 \) is shown in [38]. Since the case \( x = 0, \eta > 0 \) can be established by arguments similar to the presented above, we omit the details. This completes the proof.

Let for any \( \eta \geq 0, S > 0, x \in [0, \mu_\eta([-S, S])] \)

\[
\mathcal{P}_\alpha^{b, \eta}(S, x) := \int_\mathbb{R} \mathbb{P} \left\{ \int_{-S}^S \mathbb{I}_0 (W_\alpha(s) - b|s|^{\alpha} + z) \mu_\eta(ds) > x \right\} e^{-z} dz
\]

and

\[
\mathcal{T}_\beta^{b, \eta}(S, x) := \int_0^\infty \mathbb{P} \left\{ \int_{-S}^S \mathbb{I}_0 (-b|s|^\beta + z) \mu_\eta(ds) > x \right\} e^{-z} dz.
\]

**Lemma 5.1.** i) Let \( X \) be as in Theorem 2.1 and let \( v(u) \) be as in (2.2). For any \( \eta \geq 0, S > \eta \) and \( t_u, u > 0 \) such that \( \lim_{u \to \infty} t_u = t_0 \in [0, T] \), we have

\[
\lim_{u \to \infty} \Psi^{-1}(u) \mathbb{P} \left\{ v(u) \int_0^{S/v(u)} \mathbb{I}_u (X(t_u + s)) \mu_\eta(ds) > x \right\} = \mathcal{B}_{\alpha, H(t_0)}^\eta(S, x)
\]

at \( x = 0 \) and any \( x \in (0, \mu_\eta([0, S])) \) continuity point of \( \mathcal{B}_{\alpha, H(t_0)}^\eta(S, x) \).

ii) Let \( X \) be as in Theorem 3.1 and \( v(u) \) be defined in (3.1). Then for any \( \eta \geq 0, S > 0 \)

\[
\mathbb{P} \left\{ L_{\eta, u}[-S/v(u), S/v(u)] > x \right\} \sim \Psi(u) \times \begin{cases} 
\mathcal{P}_\alpha^{b, \eta}(S, x) & \text{if } \alpha = \beta \\
\mathcal{T}_\beta^{b, \eta}(S, x) & \text{if } \alpha > \beta,
\end{cases}
\]

as \( u \to \infty \), for \( x = 0 \) and \( x \in (0, \mu_\eta([-S, S])) \) a continuity point of \( \mathcal{P}_\alpha^{b, \eta}(S, x) \) or \( \mathcal{T}_\beta^{b, \eta}(S, x) \) respectively.

**Proof of Lemma 5.1** i) For any \( x \geq 0 \)

\[
\mathbb{P} \left\{ v(u) \int_0^{S/v(u)} \mathbb{I}_u (X(t_u + s)) \mu_\eta(ds) > x \right\} = \mathbb{P} \left\{ \int_0^S \mathbb{I}_0 (u(X(t_u + t/v(u)) - u)) \mu_\eta(dt) > x \right\}.
\]

Set \( D = [0, S], g_k(u) = u, K_u = 1 \) and \( \xi_{u,k}(t) = X(t_u + t/v(u)) \). By the Uniform Convergence Theorem and Potter’s Theorem (see e.g., [40][Theorem 1.5.2 and Theorem 1.5.3]) it follows that \( \xi_{u,k} \) satisfies the assumptions C1-C3 with

\[
h(t) = H(t_0) |t|^{\alpha}, \quad \zeta(t) = \sqrt{H(t_0)B_\alpha(t)} \text{ for } t \in [0, S], \quad C = C_\alpha \text{ and } \nu = \alpha/2.
\]

Hence the claim follows by Theorem 5.1 with \( \mathcal{B}_\zeta^{b, \eta}(D, x) = \mathcal{B}_{\alpha, H(t_0)}^\eta(S, x) \) and the claim in ii) follows with similar arguments.

**Lemma 5.2.** If \( h \) and \( \zeta \) given in C1-C2 satisfy \( h(t) = \text{Var}\zeta(t) \) and \( \zeta(t) = \sum_{i=1}^n \sqrt{\lambda_i} B^{(i)}_\alpha(t_i) \) for some positive constants \( \lambda_1, \ldots, \lambda_n \), where \( B^{(i)}_\alpha \)'s are independent fBm’s with Hurst index \( \alpha/2 \), then for any \( x, \eta \geq 0 \) and \( D = \prod_{i=1}^n [0, T_i] \) we have

\[
\mathcal{B}_\zeta^{b, \eta}(D, x) \leq \prod_{i=1}^n [\lambda_i^{1/\alpha}] [T_i] H^{\eta}_\alpha([0, 1]).
\]
Proof of Lemma 5.2 Let $\xi$ be a mean zero homogeneous Gaussian field with covariance function $c(s + t, s) = r(t) = \exp(-\sum_{i=1}^{n} \lambda_i |t_i|^\alpha)$. Taking $K_u = 1$, $g_k(u) = u$ and $\xi_{u,k}(t) = \xi(u^{-2/\alpha}t)$ Theorem 5.1 yields for $\eta \geq 0$, $x = 0$ and $x \in (0, \mu_\eta(D))$ a continuity point of the constant below

$$
\lim_{u \to \infty} \Psi^{-1}(u)P\left\{ \int_D \mathbb{I}_0 \left( u(\xi(u^{-2/\alpha}t) - u) \right) \mu_\eta(dt) > x \right\} = \mathcal{B}^{k,\eta}(D, x).
$$

By the homogeneity of $\xi$, we have further

$$
\Psi^{-1}(u)P\left\{ \int_{\prod_{i=1}^{n} [0, T_i]} \mathbb{I}_0 \left( u(\xi(u^{-2/\alpha}t) - u) \right) \mu_\eta(dt) > x \right\} 
\leq \Psi^{-1}(u)P\left\{ \sup_{t \in \prod_{i=1}^{n} [0, T_i]} \xi(u^{-2/\alpha}t) > u \right\} 
\leq \Psi^{-1}(u) \prod_{i=1}^{n} [T_i] \mathbb{E}\left\{ \sup_{t \in [0, 1]^n} e^{\sqrt{\mathbb{V}(t) - h(t)}} \right\} 
\to \prod_{i=1}^{n} [T_i] \mathbb{E}\left\{ \sup_{t_i \in [0, \lambda_i^{1/\alpha}]} e^{\sqrt{\mathbb{V}(t_i) - |t_i|^\alpha}} \right\} 
\leq \prod_{i=1}^{n} [\lambda_i^{1/\alpha} [T_i] H_{\alpha}^2([0, 1])
$$

as $u \to \infty$, where the last inequality follows from the fact $H_{\alpha}([0, T]) \leq [T] H_{\alpha}([0, 1])$. 

Lemma 5.3. If $X$ is a centered Gaussian process fulfilling (2.1), $v(u)$ and $\varepsilon$ are defined in (2.2) and in (4.3), respectively, then for $0 \leq S_1 < S_2 < T_1 < T_2 < \infty$, $T_1 - S_2 \geq 1$ and $u$ large enough such that

$$(5.12) \quad (T_2 - S_1)/v(u) \leq \varepsilon,$$

we have

$$
P\left\{ \sup_{t \in [S_1, S_2]/v(u)} X(t) > u, \sup_{t \in [T_1, T_2]/v(u)} X(t) > u \right\} \leq C(\alpha, S_1, S_2, T_1, T_2) \Psi(u),
$$

where

$$
C(\alpha, S_1, S_2, T_1, T_2) = 2 \left[ (16\overline{h})^{1/\alpha} \right]^2 [S_2 - S_1] [T_2 - T_1] H_{\alpha}^2([0, 1]) \exp\left( -\frac{1}{16} \frac{h}{T_1 - S_2} |T_1 - S_2|^{\alpha/2} \right),
$$

with $\underline{h} = \inf_{t \in [0, T]} H(t) > 0$ and $\overline{h} = \sup_{t \in [0, T]} H(t) < \infty$.

Proof of Lemma 5.3 We borrow some arguments from the proof of [13][Lemma 5]. Define next

$$
A_u = [S_1, S_2]/v(u), \quad B_u = [T_1, T_2]/v(u),
$$

$$
Y(s, t) = X(s) + X(t), \quad \sigma^2(s, t) = \text{Var}(Y(s, t)).
$$
By (2.1), for sufficiently close \(s, t \in [0, T]\)
\[
\frac{1}{2} hK(|s - t|) \leq 1 - \rho(s, t) \leq 2hK(|s - t|).
\]
Consequently, for sufficiently large \(u\) such that (5.12) holds, by (4.3)
\[
\inf_{(s,t) \in A_u \times B_u} \sigma^2(s, t) \geq 4 - 4h \sup_{(s,t) \in A_u \times B_u} K(|s - t|) > 2
\]
and
\[
\sup_{(s,t) \in A_u \times B_u} \sigma^2(s, t) \leq 4 - \frac{h}{2} \inf_{(s,t) \in A_u \times B_u} K(|s - t|),
\]
implies that
\[
\mathbb{P}\left\{ \sup_{t \in A_u} X(t) > u, \sup_{t \in B_u} X(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in A_u \times B_u} Y(s,t) > 2u \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in A_u \times B_u} Y^*(s,t) > u^* \right\},
\]
(5.13)
where \(Y^*(s, t) = Y(s, t)/\sigma(s, t)\) and
\[
u^* = \frac{2u}{\sqrt{4 - h \inf_{(s,t) \in A_u \times B_u} K(|s - t|)}}.
\]
As in [13], we have \((s_1, t_1) \in A_u \times B_u, (s_2, t_2) \in A_u \times B_u\) and \(u\) sufficiently large
\[
\text{Cov}(Y^*(s_1, t_1), Y^*(s_2, t_2)) \geq 1 - 8hK(|s_2 - s_1|) - 8hK(|t_2 - t_1|).
\]
Let \(Z(s, t) := \frac{1}{\sqrt{2}} (\vartheta_1(s) + \vartheta_2(t))\), where \(\vartheta_i, i = 1, 2\) are mutually independent copies of a mean zero stationary Gaussian process \(\vartheta\) with unit variance and covariance function satisfying
\[
1 - \text{Cov}(\vartheta(s), \vartheta(t)) \sim 32hK(|s - t|), |t - s| \to 0.
\]
As mentioned in the proof of Theorem 2.1, the existence of such a Gaussian process is guaranteed by the Assertion in [32][p.265]. Hence by Slepian inequality, for sufficiently large \(u\) we have
\[
\mathbb{P}\left\{ \sup_{(s,t) \in A_u \times B_u} Y^*(s,t) > u^* \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in A_u \times B_u} Z(s,t) > u^* \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in (A'_u \times B'_u)} Z(s,t) > u^* \right\},
\]
(5.14)
where the last inequality follows from stationarity with \(A'_u = [0, (S_2 - S_1)]/\nu(u)\) and \(B'_u = [0, (T_2 - T_1)]/\nu(u)\).
Next, set \(D = [0, S_2 - S_1] \times [0, T_2 - T_1], g_k(u) = u^*\) and \(\xi_{u,k}(s, t) = Z(s/\nu(u), t/\nu(u))\). It is straightforward to check that assumptions C0-C2 are fulfilled with
\[
h(s, t) = 16h(|s|^{\alpha} + |t|^{\alpha}), \quad \zeta(s, t) = 4\sqrt{h}(B_1^2(s) + B_2^2(t)),
\]
where $B_{a_i}^{(i)}, i = 1, 2$ are two independent fBm’s with Hurst index $\alpha/2$. Further, by Potter’s Theorem, assumption C3 holds for $v = \alpha/2$ and some constant $C$ depending on the sides length of $D$. Thus, by Theorem 5.1 and Lemma 5.2, for $u$ sufficiently large

$$
(5.15) \quad \mathbb{P}\left\{ \sup_{(s,t) \in A_u \times B_u} Z(s,t) > u^* \right\} \leq 2\left[ (16h)^{1/\alpha} \right]^2 [S_2 - S_1][T_2 - T_1] H^2_\alpha([0,1]) \Psi(u^*).
$$

Moreover, since $T_1 - S_2 \geq 1$, then by Potter’s Theorem again, we have for sufficiently large $u$, that

$$
\left( u^* \right)^2 \inf_{(s,t) \in A_u \times B_u} K(|s - t|) \geq \frac{1}{2} |T_1 - S_2|^{\alpha/2},
$$

which implies that

$$
(u^*)^2 \frac{4u^2}{4 - h \inf_{(s,t) \in A_u \times B_u} K(|s - t|)} \geq u^2(1 + \frac{1}{4} \frac{h}{u^2} \inf_{(s,t) \in A_u \times B_u} K(|s - t|)) \geq u^2 + \frac{1}{8} h |T_1 - S_2|^{\alpha/2}.
$$

Consequently, the claim follows by (5.13)-(5.15) and the fact that $\sqrt{2/\pi} \Psi(u) \leq u^{-1} e^{-\frac{1}{2}u^2}$ for $u > 0$.

□

**Lemma 5.4.** Let $W$ be an $N(0,1)$ random variable independent of $Z$ which is exponentially distributed with parameter 1. For any $c > 0$ we have

$$
\mathbb{P}\{cW - c^2/2 + Z > 0\} = 2\mathbb{P}\{W > c/2\}.
$$

**Proof of Lemma 5.4** Since $Z > 0$ almost surely, then

$$
\mathbb{P}\{cW - c^2/2 + Z > 0, cW - c^2/2 \geq 0\} = \mathbb{P}\{W > c/2\}.
$$

Let the random variable $V$ be such that

$$
\mathbb{P}\{V \leq x\} = \mathbb{E}\left\{e^{cW - c^2/2}I(cW - c^2/2 \leq x)\right\}, \quad x \in \mathbb{R}.
$$

It is well-known, see e.g., [35][Lemma 7.1] that $V$ has an $N(c^2/2, c^2)$ distribution. Hence by the independence of $Z$ and $W$

$$
\mathbb{P}\{cW - c^2/2 + Z > 0, cW - c^2/2 \leq 0\} = \mathbb{E}\left\{e^{cW - c^2/2}I(cW - c^2/2 \leq 0)\right\}
$$

$$
= \mathbb{P}\{V \leq 0\} = \mathbb{P}\{W \leq -c/2\}
$$

establishing the proof.

□

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