Abstract:

We discuss in this Chapter a series of theoretical developments which motivate the introduction of a quantum evolution equation for which the eikonal approximation results in the geodesics of a four dimensional manifold. This geodesic motion can be put into correspondence with general relativity. The well-known problem of the self-interaction of a relativistic charged particle is studied from the point of view of a manifestly Lorentz covariant classical theory admitting the particle mass as a dynamical variable, i.e., the theory is intrinsically off-shell. The evolution parameter, $\tau$, is an invariant parameter that can be identified with Newton’s time (sometimes called, as in Schwinger’s formulation, “proper time”). Gauge invariance requires the definition of five gauge fields; the fifth field has for its source the matter density. Together with the results of Gupta and Padmanabhan, showing a connection between the radiation reaction force and geometry, this structure motivates an investigation into the connection between these dynamical equations and gravity. We show that the fifth gauge field can indeed be absorbed into a conformal metric in the kinematic terms, which then results in a geodesic equation generated by the conformal metric and the standard Lorentz force. We then go on to show that the generalized radiation field passing through an optical medium with non-trivial dielectric tensor results in an analog gravity for the eikonal approximation for an arbitrary metric. A mathematically simpler system for which the eikonal approximation provides the geodesic motion on a four dimensional pseudo-Riemannian manifold is that of the Stueckelberg-Schrödinger equation with a spacetime dependent tensor $g_{\mu\nu}$ (of the form of the Einstein metric tensor), somewhat analogously to a gauge field, coupling to the kinetic terms. This theory can be realized as a quantum theory in a flat spacetime, obeying the rules of the standard quantum theory in Lorentz covariant form. Since the geodesics predicted by the eikonal approximation, with appropriate choice of $g_{\mu\nu}$, can be those of general relativity, this theory provides a quantum theory which underlies classical gravitation, and coincides with it in this classical ray approximation. This result is the principal content of this work.

In order to understand the possible origin of the structure of this form of the Stueckelberg-Schrödinger equation, we appeal to the approach of Nelson in constructing a Schrödinger equation from the properties of Brownian motion. Extending the notion of
Brownian motion to spacetime in a covariant way, we show that such an equation follows from correlations between spacetime dimensions in the stochastic process.
I. Introduction

We shall deal with the problem of constructing a quantum evolution equation which describes, in the eikonal approximation, the geodesics of general relativity, on essentially three levels which include the basic motivations arising from the self-interaction problem of a relativistic charged particle in the framework of a generalized electromagnetic theory, an analysis of the conformal structure that is associated with this theory, and the discussion of the eikonal approximation of wave equations that lead to our basic result. To understand the structure of the relativistically covariant quantum equation, we study a relativistic generalization of Brownian motion, and follow Nelson’s construction of the Schrödinger equation \[1\] for which local spacetime correlations result in a modification of the kinematic terms by introducing coupling to a Lorentz tensor \(g_{\mu\nu}\) somewhat analogous to a gauge field.

It is quite remarkable that Gupta and Padmanabhan \[2\], using essentially geometrical arguments (solving the static problem in the frame of the accelerating particle with a curved background metric), have shown that the description of the motion of an accelerating charged particle must include the radiation terms of the Abraham-Lorentz-Dirac equation \[3\],

\[
m\frac{d^2x^\mu}{ds^2} = F^\mu_\nu \frac{dx^\mu}{ds} + \Gamma^\mu, \tag{1.1}
\]

where \(m\) is the electron mass, including electromagnetic correction, \(s\) is the proper time along the trajectory \(x^\mu(s)\) in spacetime, \(F^\mu_\nu\) is the covariant form of the electromagnetic force tensor, \(e\) is the electron charge, and

\[
\Gamma^\mu = \frac{2e^2}{3c^3} \left( \frac{d^3x^\mu}{ds^3} - \frac{d^2x^\nu}{ds^2} \frac{dx^\mu}{ds} \right) \tag{1.2}
\]

Here, the indices \(\mu, \nu\), running over 0, 1, 2, 3, label the spacetime variables that represent the action of the Lorentz group; the index raising and lowering Lorentz invariant tensor \(\eta_{\mu\nu}\) is of the form \(\text{diag}(-1, +1, +1, +1)\). The expression for \(\Gamma^\mu\) was originally found by Abraham in 1905 \[3\], shortly after the discovery of special relativity, and is known as the Abraham four-vector of radiation reaction. Dirac’s derivation \[3\] was based on a direct application of the Green’s functions for the Maxwell fields, obtaining the form (1.1).

Recognizing that the electron’s acceleration precludes the use of a sequence of “instantaneous” inertial frames to describe the action of the forces on the electron \[4\], they carry out a Fermi-Walker transformation \[5\], going to an accelerating frame (assuming constant acceleration) in which the electron is actually inertial, and there solve the Coulomb problem in the curved coordinates provided by the Fermi-Walker transformation. Transforming back to laboratory coordinates, they find the Abraham-Lorentz-Dirac equation without the direct use of the Maxwell Green’s functions for the radiation field. This result, suggesting the relevance of curvature in the spacetime manifold, such as that generated by sources in general relativity, along with other, more elementary manifestations of mass renormalization (such as the contribution to the mass due to electromagnetic interactions and the identification of the Green’s function singularity contribution with part of the electron mass), carries an implication that the electron mass may play an important dynamical role.
Stueckelberg, in 1941 [6], proposed a manifestly covariant form of classical and quantum mechanics in which space and time become dynamical observables. They are therefore represented in quantum theory by operators on a Hilbert space on square integrable functions in space and time. The dynamical development of the state is controlled by an invariant parameter $\tau$, which one might call the world time, coinciding with the time on the (on mass shell) freely falling clocks of general relativity. Stueckelberg [6] started his analysis by considering a classical world-line, and argued that under the action of forces, the world line would not be straight, and in fact could be curved back in time. He identified the branch of the curve running backward in time with the antiparticle, a view taken also by Feynman in his perturbative formulation of quantum electrodynamics in 1948 [6]. Realizing that such a curve could not be parametrized by $t$ (for some values of $t$ there are two values of the space variables), Stueckelberg introduced the parameter $\tau$ along the trajectory.

This parameter is not necessarily identical to proper time, even for inertial motion for which proper time is a meaningful concept. Stueckelberg postulated the existence of an invariant “Hamiltonian” $K$, which would generate Hamilton equations for the canonical variables $x^\mu$ and $p^\mu$ of the form

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu},$$

(1.3)

and

$$\dot{p}^\mu = -\frac{\partial K}{\partial x_\mu},$$

(1.4)

where the dot indicates differentiation with respect to $\tau$. Taking, for example, the model

$$K_0 = \frac{p^\mu p_\mu}{2M},$$

(1.5)

we see that the Hamilton equations imply that

$$\dot{x}^\mu = \frac{p^\mu}{M},$$

(1.6)

It then follows that

$$\frac{dx}{dt} = \frac{P}{E},$$

(1.7)

where $p^0 \equiv E$, where we set the velocity of light $c = 1$; this is the correct definition for the velocity of a free relativistic particle. It follows, moreover, that

$$\dot{x}^\mu \dot{x}_\mu = \frac{dx^\mu dx_\mu}{d\tau^2} = \frac{p^\mu p_\mu}{M^2}.$$

(1.8)

With our choice of metric, $dx^\mu dx_\mu = -ds^2$, and $p^\mu p_\mu = -m^2$, where $m$ is the classical experimentally measured mass of the particle (at a given instant of $\tau$). We see from this that

$$\frac{ds^2}{d\tau^2} = \frac{m^2}{M^2}.$$

(1.9)
and hence the proper time is not identical to the evolution parameter $\tau$. In the case that $m^2 = M^2$, it follows that $ds = d\tau$, and we say that the particle is “on shell”.

For example, in the case of an external potential $V(x)$, where we write $x \equiv x^\mu$, the Hamiltonian becomes

$$K = \frac{p^\mu p_\mu}{2M} + V(x) \tag{1.10}$$

so that, since $K$ is a constant of the motion, $m^2$ varies from point to point with the variations of $V(x)$. It is important to recognize from this discussion that the observable particle mass depends on the state of the system (in the quantum theory, the expectation value of the operator $p^\mu p_\mu$ provides the expected value of the mass squared).

One may see, alternatively, that phenomenologically the mass of a nucleon, such as the neutron, clearly depends on the state of the system. The free neutron is not stable, but decays spontaneously into a proton, electron and antineutrino, since it is heavier than the proton. However, bound in a nucleus, it may be stable (in the nucleus, the proton may decay into neutron, positron and neutrino, since the proton may be sufficiently heavier than the neutron). The mass of the bound electron (in interaction with the electromagnetic field), as computed in quantum electrodynamics, is different from that of the free electron, and the difference contributes to the Lamb shift. This implies that, if one wishes to construct a covariant quantum theory, the variables $E$ (energy) and $p$ should be independent, and not constrained by the relation $E^2 = p^2 + m^2$, where $m$ is a fixed constant. This relation implies, moreover, that $m^2$ is a dynamical variable. It then follows, quantum mechanically, through the Fourier relation between the energy-momentum representation of a wave function and the spacetime representation, that the variable $t$, along with the variable $x$ is a dynamical variable. Classically, $t$ and $x$ are recognized as variables of the phase space through the Hamilton equations.

Since, in nature, particles appear with fairly sharp mass values (not necessarily with zero spread), we may assume the existence of some mechanism which will drive the particle’s mass back to its original mass-shell value (after the source responsible for the mass change ceases to act) so that the particle’s mass shell is defined. We shall not take such a mechanism into account explicitly here in developing the dynamical equations. We shall assume that this mechanism is working, and it is a relatively smooth function (for example, a minimum in free energy which is broad enough for our off-shell driving force to work fairly freely)$^1$.

In an application of statistical mechanics to this theory [8], it has been found that a high temperature phase transition can be responsible for the restriction of the particle’s mass (on the average, in equilibrium). In the classical theory, the non-linear equations induced by radiation reaction may have a similar effect [9].

$^1$ A relativistic Lee model has been worked out which describes a physical mass shell as a resonance, and therefore a stability point on the spectrum [7], but at this point it is not clear to us how this mechanism works in general. It has been suggested by T. Jordan [personal communication] that the definition of the physical mass shell could follow from the interaction of the particle with fields (a type of “self-interaction”); this mechanism could provide for perhaps more than one mass state for a particle, such as the electron and muon and the various types of neutrinos, but no detailed model has been so far studied.
The Stueckelberg formulation implies the existence of a fifth “electromagnetic” potential, through the requirement of gauge invariance, and there is a generalized Lorentz force which contains a term that drives the particle off-shell, whereas the terms corresponding to the electric and magnetic parts of the usual Maxwell fields do not (for the nonrelativistic case, the electric field may change the energy of a charged particle, but not the magnetic field; the electromagnetic field tensor in our case is analogous to the usual magnetic field, and the new field strengths, derived from the $\tau$ dependence of the fields and the additional gauge field, are analogous to the usual electric field, as we shall see). The second quantization of this gauge theory has been carried out as well \[\text{[10]}\].

In the following, we give the structure of the field equations, and show that the standard Maxwell theory is properly contained in this more general framework. Applying the Green’s functions to the current source provided by the relativistic particle, and the generalized Lorentz force, we obtain equations of motion for the relativistic particle which is, in general off-shell. As in Dirac’s result, these equations are of third order in the evolution parameter, and therefore are highly unstable. However, the equations are very nonlinear, and give rise to chaotic behavior \[\text{[9]}\].

Our results exhibit what appears to be a strange attractor in the phase space of the autonomous equation for the off-mass shell deviation. This attractor may stabilize the electron’s mass in some neighborhood. We conjecture that it stabilizes the orbits macroscopically as well, but a detailed analysis awaits the application of more powerful computing facilities and procedures.

We then show that the fifth (scalar) field can be eliminated through the introduction of a conformal metric on the spacetime manifold \[\text{[11]}\]. The geodesic equation associated with this metric coincides with the Lorentz force, and is therefore dynamically equivalent. Since the generalized Maxwell equations for the five dimensional fields provide an equation relating the fifth field with the spacetime density of events, one can derive the spacetime event density associated with the Friedmann-Robertson-Walker solution of the Einstein equations. The resulting density, in the conformal coordinate space, is isotropic and homogeneous, decreasing as the square of the Robertson-Walker scale factor. Using the Einstein equations, one see that both for the static and matter dominated models, the conformal time slice in which the events which generate the world lines are contained becomes progressively thinner as the inverse square of the scale factor, establishing a simple correspondence between the configurations predicted by the underlying Friedmann-Robertson-Walker dynamical model and the configurations in the conformal coordinates.

The conformal metric is not, however, even locally equivalent to a Schwarzschild metric. To achieve a more general framework for achieving an underlying model for gravity, we study the eikonal approximation of the (generalized) electromagnetic equations in a medium with non-trivial dielectric tensor.

It has been known for many years that the Hamilton-Jacobi equation of classical mechanics defines a function which appears to be the eikonal of a wave equation, and therefore that classical mechanics appears to be a ray approximation to some wave theory \[\text{[12]}\]. The propagation of rays of waves in inhomogeneous media appears, from this point of view (as a result of the application of Fermat’s principle), to correspond to geodesic motion in a metric derived from the properties of the medium \[\text{[13]}\]. This geometrical
interpretation has been exploited recently by several authors to construct models which exhibit three dimensional analogs of general relativity by studying the wave equations of light in an inhomogeneous medium [14], and, to achieve four dimensional analogs, sound waves and electromagnetic propagation in inhomogeneously moving materials [15]. Visser et al [16] have pointed out that condensed matter systems such as acoustics in flowing fluids, light in moving dielectrics, and quasiparticles in a moving superfluid can be used to mimic kinematical aspects of general relativity. Leonhardt and Piwnicki [17] and Lorenci and Klippert [18], for example, have discussed the case of electromagnetic propagation in moving media. In order to achieve four dimensional geodesic flows, it has been necessary to introduce a motion of the medium. 2 There is considerable interest in extending these analog models for the kinematical aspects of gravity to include dynamical aspects, i.e. considering gravity as an emergent phenomenon [16].

The manifestly covariant classical and quantum mechanics introduced by Stueckelberg [6] in 1941 has the structure of the Hamilton dynamics with the Euclidean three dimensional space replaced by four-dimensional Minkowski space (since all four of the components of energy-momentum are kinematically independent, the theory is intrinsically “off-shell”). This theory leads to five dimensional wave equations for the associated gauge fields (in fact, for wave phenomena in general, such as acoustic or hydrodynamic waves), which in the eikonal approximation in the presence of an inhomogeneous medium, provides a basis for geodesic motion in four dimensional spacetime.

This theory has been used to account for the known bound state spectra of the (spinless) two-body problem in potential theories formulated in a manifestly covariant way [21] (as well as to study the classical relativistic Kepler orbits [6,22]). As we have remarked above, in order for the Stueckelberg-Schrödinger equation of the quantum form of this theory to be gauge invariant, it is necessary to introduce a fifth gauge field, compensating for the derivative of the invariant evolution parameter $\tau$ [10]. Generalized gauge invariant field strengths, $f_{q,p}$, with $q, p = 0, 1, 2, 3, 5$ occurring in the Lagrangian to second order generate second order field equations analogous to the usual Maxwell equations, with source given by the four-vector matter field current and an additional Lorentz scalar density. Taking the Fourier transform of these equations over the invariant parameter, as we demonstrate below, one sees that the zero frequency component (zero mode) of the equations coincides with the standard Maxwell theory. The field equations go over in the same way; the equation for the fifth field decouples. The Maxwell theory is therefore properly contained in the five-dimensional generalization as the zero frequency component. In the quantum case the four-currents are given by bilinears in the wave function containing first derivatives and the fifth source is the (Lorentz) scalar probability density. The symmetry of the homogeneous equations, which can be $O(3,2)$ or $O(4,1)$, depending on the sign chosen for raising and lowering the fifth index, is therefore not realized in the inhomogeneous equations. Hence, without augmenting the symmetry of the matter fields beyond $O(3,1)$, 2

2 We remark that Obukhov and Hehl [19] have shown that a conformal class of metrics for spacetime can be derived by imposing constrained linear constitutive relations between the electromagnetic fields $(E,B)$ and the excitations $(D,H)$, using Urbantke’s formulas [20], developed to define locally integrable parallel transport orbits in Yang-Mills theories (on tangent 2-plane elements on which the Yang-Mills curvature vanishes).
the fifth field, whose source in the Maxwell-like equations is the probability density (or, classically, the matter density) in spacetime, can play a special role. There appears to be no kinematic basis for choosing one or the other of these signatures; atomic radiative decay, for example, contains points in phase space (for radiation of off-shell photons) for either type. We note, however, that the homogeneous equations corresponding to the $O(4,1)$ signature appear, under Fourier transform of the $\tau$ variable, as Klein-Gordon type wave equations with positive mass-squared (physical particles), but for the $O(3,2)$ choice of signature, these equations have the “wrong” sign (tachyonic) for interpreting the additive term as a mass-squared. We do not find an objective criterion for choosing one or the other of these possibilities, and therefore leave open the question of a definitive choice of the signature for the five dimensional radiation field at this stage. It is possible that both may play a dynamical role.

The structure of the gauge theory obtained from the non-relativistic Schrödinger equation is precisely analogous. The four dimensional gauge invariant field strengths obey second order equations, for which the sources are the vector currents and the scalar probability density (or, classically, the matter density in three dimensional space). No linear transformation can connect the Schrödinger probability density with the vector currents, so that the scalar density and the fourth gauge field can play a special role.

Since the eikonal approximation naturally lowers the dimension of the differential equations describing the fields by one, the eikonal approximation to the five-dimensional field equations results in four dimensional differential equations.

In the presence of a non-trivial dielectric structure of the medium, the four dimensional field equations resulting from the eikonal approximation can describe geodesic motion in four dimensional spacetime without the necessity of adding motion to the medium. We emphasize that the underlying manifold, on which the fields are defined, is a flat Cartesian space, but that the dynamically induced trajectories move on the geodesics of a pseudo-Riemannian manifold. This result forms our basic motivation for studying a generalized dynamics of Stueckelberg [6] form in this context.

In Section 5, we study the eikonal structure of waves in a 5D inhomogeneous medium, in which Minkowski spacetime is embedded, both for second order wave equations and a wave equation of Schrödinger type, and show that the resulting rays have a precise analog with the results of Kline and Kay [13], but in a 4D spacetime manifold with a pseudo-Riemannian structure. Kline and Kay [13] show that the rays are geodesic in the metric associated with the anisotropic inhomogeneous medium. As for the case of Kline and Kay [13], it follows from the existence of a Hamiltonian that the corresponding Lagrangian obeys an extremum condition which describes the rays as geodesics. We show that there is mass flow along these rays, and that the flow is controlled by generating functions of Hamiltonian type, establishing a relation between geodesic flow and a particle mechanics of symplectic form. We stress that this construction does not necessary imply that the equation is written in a curved spacetime, as for the wave equations of Birrell and Davies [23], for example. The quantum theory associated with this equation is defined on a flat Minkowski spacetime; the wave functions satisfy Euclidean normalization conditions, and four-momentum remains the generator of spacetime translations. It is only in the eikonal approximation that the rays emerge as geodesics of a curved space.
This result is the principal content of this work.

We close our study here by considering a basis for the structure of the Steuckelberg-Schrödinger equation with a second rank Lorentz tensor applied to the kinematic terms.

The ideas of stochastic processes originated in the second half of the 19th century in thermodynamics, through the manifestation of the kinetic theory of gases. In 1905 A. Einstein [24] in his paper on Brownian motion provided a decisive breakthrough in the understanding of the phenomena. Moreover, it was a proof convincing physicists of the reality of atoms and molecules, the motivation for Einstein’s work. It is interesting to note that Einstein predicted the so called Brownian motion of suspended microscopic particles not knowing that R. Brown first discovered it in 1827 [24]. The resemblance of the Schrödinger equation to the diffusion equation had lead physicists (including Einstein and Schrödinger) to attempt to connect quantum mechanics with an underlying stochastic process, the Brownian process.

Nelson [1], in 1966, constructed the Schrödinger equation from an analysis of Brownian motion by identifying the forward and backward average velocities of a Brownian particle with the real and imaginary parts of a wave function. He pointed out that the basic process involved is defined non-relativistically, and can be used if relativistic effects can be “safely” neglected. The development of a relativistically covariant formulation of Brownian motion could therefore provide some insight into the structure of a relativistic quantum theory.

Nelson pointed out that the formulation of his stochastic mechanics in the context of general relativity is an important open question [1]. The Riemannian metric spaces one can achieve, in principle, which arise due to nontrivial correlations between fluctuations in spatial directions, could, in the framework of a covariant theory of Brownian motion, lead to spacetime pseudo-Riemannian metrics in the structure of diffusion and Schrödinger equations. Morato and Viola [25] have recently constructed a relativistic quantum equation for a free scalar field. They assumed the existence of a 3D (spatial) diffusion in a comoving frame, a non-inertial frame in which the average velocity field of the Brownian particle (current velocity) is zero. In this frame the location of the Brownian particle in space experiences Brownian fluctuations parametrized by the proper time of the comoving observer. They interpreted possible negative 0-component current velocities with what they called ‘rare events’, which are time reversed Brownian processes (a peculiarity arising in the relativistic treatment). The equation they achieved this way is approximately the Klein-Gordon equation. It is important to note that in the inertial frame they do not obtain a normal diffusion. This is due to the fact that their process is stochastic only in three degrees of freedom and therefore is not covariant. In this paper we shall study a manifestly covariant form of Brownian motion. par In a previous work [27] we introduced a new approach to the formulation of relativistic Brownian motion in 1+1 dimensions. The process we formulate is a straightforward generalization of the standard one dimensional diffusion to 1+1 dimensions (where the actual random process is thought of as a ‘diffusion’ in the time direction as well as in space), in an inertial frame. The equation achieved is an exact Klein-Gordon equation. It is a relativistic generalization of Nelson’s Brownian process, the Newtonian diffusion. In this work we review the relativistic Brownian process in 1+1 dimensions [27] where the inclusion of both spacelike and timelike motion for the Brownian particle (event) is considered; if the timelike motion is considered as “physical”
the “unphysical” spacelike motion is represented (through analytic continuation) by imaginary quantities. We extend the treatment to 3+1 dimensions using appropriate weights for the imaginary representations. The complete formalism then can be used to construct relativistic general covariant diffusion and Schrödinger equations with pseudo-Riemannian metrics which follow from the existence of nontrivial correlations between the coordinate random variables.

Finally, we discuss the possible implications of the process we consider (i.e. a relativistic stochastic process with Markov property which preserves macroscopic Lorentz covariance) on the entangled system, where we claim that though fluctuations which exceed the velocity of light occur, the macroscopic behavior dictated by the resulting Fokker-Planck equation is local.

II. Equations of Motion and Self-Interaction

The Stueckelberg-Schrödinger equation which governs the evolution of a quantum state over the manifold of spacetime was postulated by Stueckelberg [6] to be, for the free particle,

$$i \frac{\partial \psi}{\partial \tau} = \frac{p^\mu p_\mu}{2M} \psi$$

where, on functions of spacetime, $p_\mu$ is represented by $-i \partial / \partial x^\mu \equiv -i \partial_\mu$.

Taking into account full $U(1)$ gauge invariance, corresponding to the requirement that the theory maintain its form under the replacement of $\psi$ by $e^{i \varepsilon_0 \Lambda} \psi$, the Stueckelberg-Schrödinger equation (including a compensation field for the $\tau$-derivative of $\Lambda$) is [10]

$$(i \frac{\partial}{\partial \tau} + e_0 \gamma_5(x, \tau)) \psi_\tau(x) = \frac{(p^\mu - e_0 a^\mu(x, \tau))(p_\mu - e_0 a_\mu(x, \tau))}{2M} \psi_\tau(x),$$

where the gauge fields satisfy

$$a'_\alpha = a_\alpha + \partial_\alpha \Lambda$$

under gauge transformations generated by

$$\psi' = e^{i \varepsilon_0 \Lambda} \psi,$$

for $\Lambda$ a differentiable function of $\{x^\mu, \tau\} \equiv x^\alpha$ ($\alpha = 0, 1, 2, 3, 5$), which may depend on $\tau$, and $e_0$ is a coupling constant which we shall see has the dimension $\ell^{-1}$. The corresponding classical Hamiltonian then has the form

$$K = \frac{(p^\mu - e_0 a^\mu(x, \tau))(p_\mu - e_0 a_\mu(x, \tau))}{2M} - e_0 \gamma_5(x, \tau),$$

in place of (2.1). Stueckelberg [6] did not take into account this full gauge invariance requirement, working in the analog of what is known in the nonrelativistic case as the Hamilton gauge (where the gauge function $\Lambda$ is restricted to be independent of time). The equations of motion for the field variables are given (for both the classical and quantum theories) by [10]

$$\lambda \partial_\alpha f^{\beta \alpha}(x, \tau) = e_0 j^{\beta}(x, \tau),$$

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where $\alpha, \beta = 0, 1, 2, 3, 5$, the last corresponding to the $\tau$ index, $j^5 \equiv \rho$ is the density of events in spacetime, and $\lambda$, of dimension $\ell^{-1}$, is a factor on the terms $f^{\alpha\beta} f_{\alpha\beta}$ in the Lagrangian associated with (2.2) (with, in addition, degrees of freedom of the fields) required by dimensionality. The field strengths are

$$f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha,$$  \hfill (2.5)

and the current satisfies the conservation law [10]

$$\partial_\alpha j^\alpha (x, \tau) = 0; \hfill (2.6)$$

integrating over $\tau$ on $(-\infty, \infty)$, and assuming that $j^5(x, \tau)$ vanishes (pointwise) at $|\tau| \to \infty$, one finds that

$$\partial_\mu J^\mu (x) = 0,$$

where (for some dimensionless $\eta$) [9,11]

$$J^\mu (x) = \eta \int_{-\infty}^{\infty} d\tau j^\mu (x, \tau). \hfill (2.7)$$

We identify this $J^\mu (x)$ with the Maxwell conserved current. In ref. [28], for example, this expression occurs with

$$j^\mu (x, \tau) = \dot{x}^\mu (\tau) \delta^4 (x - x(\tau)), \hfill (2.8)$$

and $\tau$ is identified with the proper time of the particle (an identification which can be made for the motion of a free particle). The conservation of the integrated current then follows from the fact that

$$\partial_\mu j^\mu = \dot{x}^\mu (\tau) \partial_\mu \delta^4 (x - x(\tau)) = -\frac{d}{d\tau} \delta^4 (x - x(\tau)),$$

a total derivative; we assume that the world line runs to infinity (at least in the time dimension) and therefore the $\delta$-function vanishes at the end points[6,28], in accordance with the discussion above.

As for the Maxwell case, one can write the current formally in five-dimensional form

$$j^\alpha = \dot{x}^\alpha \delta^4 (x(\tau) - x). \hfill (2.9)$$

For $\alpha = 5$, the factor $\dot{x}^5$ is unity, and this component therefore represents the event density in spacetime.

Integrating the $\mu$-components of Eq. (2.4) over $\tau$ (assuming $f^{\mu5}(x, \tau) \to 0$ (pointwise) for $\tau \to \pm \infty$), we obtain the Maxwell equations with the Maxwell charge $e = e_0 / \eta$ and the Maxwell fields given by

$$A^\mu (x) = \lambda \int_{-\infty}^{\infty} a^\mu (x, \tau) d\tau. \hfill (2.10)$$

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A Hamiltonian of the form (2.3) without \( \tau \) dependence of the fields, and without the \( a_5 \) terms, as written by Stueckelberg [6], can be recovered in the limit of the zero mode of the fields (with \( a_5 = 0 \)) in a physical state for which this limit is a good approximation i.e., when the Fourier transform of the fields, defined by

\[
a^\mu(x, \tau) = \int ds \hat{a}^\mu(x, s)e^{-is\tau},
\]

has support only in a small neighborhood \( \Delta s \) of \( s = 0 \). The vector potential then takes on the form \( a^\mu(x, \tau) \sim \Delta s \hat{a}^\mu(x, 0) = (\Delta s/2\pi \lambda) A^\mu(x) \), and we identify \( e = (\Delta s/2\pi \lambda) e_0 \). The zero mode therefore emerges when the inverse correlation length of the field \( \Delta s \) is sufficiently small, and then \( \eta = 2\pi \lambda/\Delta s \). We remark that in this limit, the fifth equation obtained from (2.4) decouples. The Lorentz force obtained from this Hamiltonian, using the Hamilton equations, coincides with the usual Lorentz force, and, as we have seen, the generalized Maxwell equation reduce to the usual Maxwell equations. The theory therefore contains the usual Maxwell Lorentz theory in the limit of the zero mode; for this reason we have called this generalized theory the “pre-Maxwell” theory.

If such a pre-Maxwell theory really underlies the standard Maxwell theory, then there should be some physical mechanism which restricts most observations in the laboratory to be close to the zero mode. For example, in a metal there is a frequency, the plasma frequency, below which there is no transmission of electromagnetic waves. In this case, if the physical universe is imbedded in a medium which does not allow high “frequencies” to pass, the pre-Maxwell theory reduces to the Maxwell theory. Some study has been carried out, for a quite different purpose (of achieving a form of analog gravity), of the properties of the generalized fields in a medium with general dielectric tensor [29]; we discuss this study in detail in a later section. Moreover, as we describe below [9] the high level of nonlinearity of the theory of the electric charge in interaction with itself may generate an approximate effective reduction to Maxwell-Lorentz theory, with high frequency chaotic behavior providing the regularization achieved by models of the type discussed by Rohrlich [30].

We remark that integration over \( \tau \) does not bring the generalized Lorentz force into the form of the standard Lorentz force, since it is nonlinear, and a convolution remains. If the resulting convolution is trivial, i.e., in the zero mode, the two theories then coincide. Hence, we expect to see dynamical effects in the generalized theory which are not present in the standard Maxwell-Lorentz theory. In the following, we describe results we have obtained in a study of the self-interaction of a classical relativistic charged particle.

Writing the Hamilton equations

\[
\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}; \quad \dot{p}^\mu = \frac{dp^\mu}{d\tau} = -\frac{\partial K}{dx_\mu} ;
\]

for the Hamiltonian (2.3), we find the generalized Lorentz force [9,10]

\[
M \ddot{x}^\mu = e_0 f^\mu_\nu \dot{x}^\nu + f^\mu_5 .
\]

Multiplying this equation by \( \dot{x}_\mu \), one obtains

\[
M \dot{x}_\mu \ddot{x}^\mu = e_0 \dot{x}_\mu f^\mu_5 ;
\]
The coefficients are defined as follows
\[
\varepsilon = 1 + \hat{x}^\mu \hat{x}_\mu = 1 - \frac{ds^2}{d\tau^2},
\]
(2.15)
where \(ds^2 = dt^2 - dx^2\) is the square of the proper time. Since \(\hat{x}^\mu = (p^\mu - e_0 a^\mu)/M\), if we interpret \(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu) = -m^2\), the gauge invariant particle mass [10], then
\[
\varepsilon = 1 - \frac{m^2}{M^2}
\]
(2.17)
measures the deviation from “mass shell” (on mass shell, \(ds^2 = d\tau^2\)).

We see that the \(a_5\) field is strongly associated with the mass distribution; its source is the event density (mass density in spacetime), i.e. (in generalized Lorentz gauge \(\partial_\alpha a^\alpha = 0\),
\[-\partial_\alpha \partial^\alpha a^5 = e j^5 \equiv \rho.\]

We carry out a power series expansion of the Green’s function in the neighborhood of the parameter \(\tau\) locating the particle on its worldline, and compute the field strengths entering into the generalized Lorentz force. This results in the system of equations [9] (using a cutoff \(\mu\), which we estimate to be of the order of \(10^{-23}\) seconds [31] to avoid explicit singularities)
\[
M(\varepsilon, \dot{\varepsilon}) \hat{x}^\mu = -\frac{1}{2} M(\varepsilon, \dot{\varepsilon}) \hat{\varepsilon} \hat{x}^\mu + F(\varepsilon) e^2 \left\{ \hat{x}^\mu + \frac{1}{1 - \varepsilon} \hat{x}_\nu \hat{x}^{\nu} \hat{x}^\mu \right\}
+ e_0 \int_{ext} \frac{\mu}{\nu} \hat{x}^\mu + e_0 \left( \hat{x}^\mu \hat{x}_\nu + \delta^\mu_{\nu} \right) \int_{ext} \nu.
\]
(2.18)
for the orbits in spacetime. We moreover obtain an autonomous equation for the off-shell deviation,
\[
(1 + 2 \frac{F_5}{F_3}) \varepsilon - A \dot{\varepsilon} + B \varepsilon^2 + C \ddot{\varepsilon} - D + E \dot{\varepsilon}^3 + I \dot{\varepsilon} \ddot{\varepsilon} = 0.
\]
(2.19)
The coefficients are defined as follows
\[
A = \frac{2}{F_3} \left( \frac{M}{2e^2} + F_2 \right) + \frac{2M(\varepsilon, \dot{\varepsilon})}{e^2 F(\varepsilon)} - \frac{4M(\varepsilon, \dot{\varepsilon}) F_5}{2e^2 F(\varepsilon)}.
\]
\[
B = \frac{2F_3'}{F_3^2} (F_2 - \frac{M}{2e^2}) - \frac{2F_3'}{F_3} + \frac{1}{1 - \varepsilon} \frac{M(\varepsilon, \dot{\varepsilon})}{F_3} - \frac{2F_1}{1 - \varepsilon},
\]
\[
C = \frac{4M(\varepsilon, \dot{\varepsilon})}{e^2 F(\varepsilon)} \frac{1}{F_3} \left( \frac{M}{2e^2} + F_2 \right) - \frac{2}{F_3} F_1 F_3' - \frac{2F_1}{(1 - \varepsilon) F_3} + \frac{2}{F_3} F_1',
\]
\[
D = \frac{4M(\varepsilon, \dot{\varepsilon})}{e^2 F(\varepsilon)} F_1
\]
\[
E = \frac{2F_4}{(1 - \varepsilon) F_3} - 2 \left( \frac{F_1 F_3'}{F_3^2} - \frac{F_1'}{F_3} \right),
\]
\[
I = 2 \left( \frac{F_5'}{F_3} + 2 \frac{F_4}{F_3} - \frac{F_3 F_3'}{F_3^2} \right) - \frac{2F_5}{(1 - \varepsilon) F_3}.
\]
(2.20)
We have retained the function $M(\varepsilon, \dot{\varepsilon})$ in some of the coefficients for convenience, although it contains a part linear in $\dot{\varepsilon}$, and would therefore modify the definitions of the coefficients somewhat (the form of (4.6) would remain) if we were to redefine them to make all the derivative of $\varepsilon$ terms explicit.

The basic functions in terms of which the $F_i$ above are defined, extracted directly from the Green’s function, are

\[
\begin{align*}
 f_1(\varepsilon) &= \frac{3}{2} \ln\left|\frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}\right| - \frac{3}{\varepsilon^2(1 - \varepsilon)} + \frac{2}{\varepsilon(1 - \varepsilon)^2} \\
 f_2(\varepsilon) &= \frac{3}{2} \ln\left|\frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}\right| - \frac{1}{\varepsilon^2} - \frac{2 - \varepsilon}{\varepsilon^2(1 - \varepsilon)} \\
 f_3(\varepsilon) &= -\frac{1}{3} \ln\left|\frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}\right| + \frac{1}{\varepsilon(1 - \varepsilon)}.
\end{align*}
\] (2.21)

For either sign of $\varepsilon$, when $\varepsilon \sim 0$,

\[
\begin{align*}
 f_1(\varepsilon) &\sim \frac{8}{5} + \frac{24}{7} \varepsilon + \frac{16}{3} \varepsilon^2 + O(\varepsilon^3), \\
 f_2(\varepsilon) &\sim -\frac{2}{5} - \frac{4}{7} \varepsilon - \frac{2}{3} \varepsilon^2 + O(\varepsilon^3), \\
 f_3(\varepsilon) &\sim \frac{2}{3} + \frac{4}{5} \varepsilon + \frac{6}{7} \varepsilon^2 + O(\varepsilon^3).
\end{align*}
\] (2.22)

and therefore the functions defined in (2.21) are smooth near $\varepsilon = 0$. We then define the auxiliary functions

\[
\begin{align*}
 g_1 &= f_1 - f_2 - 3f_3, \\
 g_2 &= \frac{1}{2}f_1 - f_2 - 2f_3, \\
 g_3 &= \frac{1}{6}f_1 - \frac{1}{2}f_2 - \frac{1}{2}f_3, \\
 h_1 &= \frac{1}{2}f_1' - \frac{1}{2}f_2' - f_3', \\
 h_2 &= \frac{1}{4}f_1' - \frac{1}{2}f_2' - \frac{1}{2}f_3', \\
 h_3 &= (f_1' - f_2' - f_3'), \\
 h_4 &= f_1'' - f_2'' - f_3''.
\end{align*}
\] (2.23)

Finally,

\[
\begin{align*}
 F_1(\varepsilon) &= \frac{g_1(\varepsilon - 1)}{3\mu^2} \\
 F_2(\varepsilon) &= \frac{g_2 - 2(\varepsilon - 1)h_1}{4\mu} \\
 F_3(\varepsilon) &= g_3 + \frac{1}{12}(\varepsilon - 1)h_3 \\
 F_4(\varepsilon) &= \frac{1}{2}h_2 + \frac{1}{8}(\varepsilon - 1)h_4 \\
 F_5(\varepsilon) &= \frac{1}{8}(1 - \varepsilon)h_3.
\end{align*}
\] (2.24)
and
\[ F(\varepsilon) = \frac{f_1}{3} (1 - \varepsilon) + g_3. \] (2.25)

Here, the coefficients of $\ddot{x}^\mu$ have been grouped formally into a renormalized (off-shell) mass term, defined (as done in the standard radiation reaction problem) as
\[ M(\varepsilon, \dot{\varepsilon}) = M + \frac{e^2}{2\mu} \left[ \frac{f_1 (1 - \varepsilon)}{2} + g_2 \right] - e^2 \left[ \frac{1}{4} f_1' (1 - \varepsilon) + h_2 \right] \dot{\varepsilon}, \] (2.26)

where [9]
\[ e^2 = \frac{2e_0^2}{\lambda (2\pi)^3 \mu} \] (2.27)
can be identified with the Maxwell charge by studying the on-shell limit (compare with our discussion above; one concludes that in this case $\Delta s^2 \simeq \lambda / 2\pi \mu$). If $\varepsilon$ varies slowly, this “renormalized mass” term can have some significance as a measurable mass, but if $\varepsilon$ is rapidly varying, the identification is only formal; its connection with a measurable mass is only in the sense of some average over local variations.

We remark that when one multiplies Eq.(2.18) by $\dot{x}_\mu$, it becomes an identity (all of the terms except for $e_0 f_{ext}^\mu \dot{x}^\nu$ may be grouped to be proportional to $(\dot{x}_\mu \dot{x}_\nu + \delta^\mu_\nu$); one must use Eq.(2.19) to compute the off-shell mass shift $\varepsilon$ corresponding to the longitudinal degree of freedom in the direction of the four velocity of the particle. Eq.(2.18) determines the motion orthogonal to the four velocity. Equations (2.18) and (2.19) are the fundamental dynamical equations governing the off-shell orbit.

It can be shown [9] that Eq.(2.18) reduces to the ordinary (Abraham-Lorentz-Dirac) radiation reaction formula for small, slowly changing $\varepsilon$ and that that no instability, no radiation, and no acceleration of the electron occurs when it is precisely on shell. There is therefore no “runaway solution” for the exact mass shell limit of this theory. This result indicates that in the mass shell limit, the theory is fundamentally different than the “standard” theory. The unstable Dirac result is approximate for $\varepsilon$ close to, but not precisely zero. The Dirac instabilities are therefore necessarily associated with at least small deviations from mass shell.

The results of this calculation demonstrate the strong connection between the $a_5$ field, whose source is the mass distribution, and the off-shell mass variations of the particle. In the next section we give an explicit geometrical interpretation, constructing a conformal metric which replaces the fifth gauge field (as in standard electromagnetism, the scalar gauge field cannot be removed in the presence of sources).

**III. Conformal Equivalence for the Fifth Potential**

In this section, we study the replacement of the fifth potential in a flat space picture in terms of a new Hamiltonian which contains only a 4-potential, and takes into account the fifth potential with a metric coefficient [11]. The generator of motion is (we assume no explicit $\tau$ dependence in the fields in this section, so that $a_\mu(x)$ is a zero-mode field)
\[ K_\tau = g^{\mu\nu} \frac{(p_\mu - e a_\mu(x))(p_\nu - e a_\nu(x))}{2M} \] (3.1)

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The conformal structure of $g^{\mu\nu}$ follows from the fact that in (2.3), the contraction of 
\[
\frac{(p_\mu - ea_\mu(x))(p_\nu - ea_\nu(x))}{2M}
\] with the Minkowski metric is just $K + ea_5$, and hence the conformal factor $\Phi$ is given by the expression preceding (3.5).

We now verify that this functional of $(p_\sigma, x^\sigma)$ gives Hamilton equations which are equivalent dynamically to those of (2.3). We find
\[
\frac{dx^\sigma}{d\tau} \equiv \dot{x}^\sigma = \frac{\partial K}{\partial p_\sigma} = g^{\sigma\nu} \dot{\xi}_\nu, \tag{3.2}
\]
where we have defined the auxiliary variable $\xi$ by
\[
\dot{\xi}_\nu = p_\nu - ea_\nu.
\]

Note that the covariant Poisson bracket
\[
M\{\dot{\xi}_\nu, x^\lambda\} = \{p_\nu, x^\lambda\} = \delta_\nu^\lambda,
\]
so that
\[
\{x^\sigma, x^\lambda\} = g^{\sigma\lambda}\{\dot{\xi}_\nu, x^\lambda\} = \frac{1}{M}g^{\sigma\lambda}.
\]

This type of commutation relation, a generalization [10] of the assumption made originally by Feynman, maintains the interpretation of $p_\nu$ as the generator of translations in $x^\lambda$ (in the corresponding quantum theory, $p_\nu \to -i\partial/\partial x^\nu$). We shall return to this point in a more general context later.

This equation gives us the transformation law $dx^\sigma = g^{\sigma\nu}d\xi_\nu$ and $d\xi_\sigma = g_{\sigma\mu}dx^\mu$. The second Hamilton equation gives
\[
\frac{dp_\sigma}{d\tau} = -\frac{\partial K_r}{\partial x^\sigma} = -\frac{M}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma} \dot{\xi}_\mu \dot{\xi}_\nu + g^{\mu\nu} \frac{\partial a_\mu}{\partial x^\sigma} \frac{(p_\nu - ea_\nu)}{M}, \tag{3.3}
\]

We now replace $x$ with $\xi$ using the transformation law and substitute for $\dot{p}_\sigma$ to obtain
\[
e\dot{\xi}_\mu \frac{\partial a_\mu}{\partial \xi_\sigma} + e\frac{\partial a_5}{\partial \xi_\sigma} = -\frac{M}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma} g_{\alpha\sigma} \dot{\xi}_\mu \dot{\xi}_\nu + g^{\mu\nu} \frac{\partial a_\mu}{\partial \xi_\alpha} g_{\alpha\sigma} \dot{\xi}_\nu \tag{3.4}
\]

The functionals $K, K_r$ are different; however, on the physical trajectories they take the same numerical value, $K$. We now show that choosing a conformal metric $g^{\mu\nu} = \Phi(x)\eta^{\mu\nu}$, the Lorentz force derived from $K_r$ is the same as the one derived from $K$. In this case we have
\[
g_{\mu\nu} = \frac{1}{\Phi(x)}\eta_{\mu\nu}, \quad \Phi(x) = \frac{1}{1 + e\frac{a_5(x)}{K}}.
\]

and Eq.(2.8) gives
\[
e\frac{\partial a_5}{\partial \xi_\sigma} = -\frac{M}{2}\frac{\partial \Phi}{\partial \xi_\sigma} \eta^{\mu\nu} \dot{\xi}_\mu \dot{\xi}_\nu. \tag{3.5}
\]

Using $\frac{M}{2}\eta^{\mu\nu} \dot{\xi}_\mu \dot{\xi}_\nu = \frac{K}{\Phi}$ we find that Eq. (2.9) is indeed satisfied. This shows that the Hamilton equations for the two generators are identical.
It is now interesting to examine the geodesic motion of this dynamical system, assuming the fields are static in \( \tau \). The Lagrangian in this case is

\[
L = p_\mu \dot{x}^\mu - H = \frac{M}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\mu a_\mu. \tag{3.6}
\]

We now make a small variation in \( x^\mu \)

\[
x^\mu \rightarrow x^\mu + \delta x^\mu
\]

\[
\delta S = \int d\tau \left[ \frac{M}{2} \left( \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \dot{x}^\mu \dot{x}^\nu \delta x^\sigma + 2 g_{\mu\nu} \dot{x}^\mu \frac{d\delta x^\nu}{d\tau} \right) + e a_\mu \frac{d\delta x^\mu}{d\tau} + \frac{\partial a_\mu}{\partial x^\sigma} \dot{x}^\mu \delta x^\sigma \right]
\]

From the minimal action principal we obtain, by integration by parts of the \( \tau \) derivatives (\( \tau \) independence of the field implies \( \frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\sigma} \))

\[
0 = \frac{1}{2} \left( \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \dot{x}^\mu \dot{x}^\nu - 2 g_{\sigma\nu} \dot{x}^\mu \dot{x}^\sigma - 2 g_{\sigma\nu} \ddot{x}^\nu \right) + \frac{e}{M} \left( -\dot{x}^\mu \frac{\partial a_\sigma}{\partial x^\mu} + \frac{\partial a_\mu}{\partial x^\sigma} \dot{x}^\mu \right);
\]

multiplying by \( g^{\lambda\sigma} \) we finally get

\[
\ddot{x}^\lambda = -\Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{e}{M} \dot{x}^\mu f^\lambda_{\mu}. \tag{3.7}
\]

where \( f^\lambda_{\mu} = g^{\lambda\sigma} f_{\sigma\mu} \) and \( f_{\sigma\mu} = \frac{\partial a_{\sigma}}{\partial x^\mu} - \frac{\partial a_{\mu}}{\partial x^\sigma} \).

As an example of the application of our result for the conformal metric, let us consider the Friedmann-Robertson-Walker universe.

In the “flat space” Robertson-Walker model (see, e.g., [32]), for the spatial geometry characterized by \( k=0 \), the metric

\[
ds^2 = d\tau^2 - \Phi^2(\tau)(dx^2 + dy^2 + dz^2). \tag{3.8}
\]

can be brought to the form

\[
ds^2 = \Phi^2(t)(dt^2 - dx^2 - dy^2 - dz^2). \tag{3.9}
\]

by using the transformation

\[
t = \int \frac{d\tau}{\Phi(\tau)}. \tag{3.10}
\]

\( \tau \) is the time coordinate of a freely-falling object and therefore coincides with our notion of universal the \( \tau \). The function \( \Phi(\tau) \) is often designated by \( R \) or \( a \) and is the (dimensionless) spatial scale of the expanding universe. In the conformal coordinates the time-coordinate is therefore related to \( \tau \), according to the transformation above, through

\[
\frac{dt}{d\tau} = \frac{1}{\Phi}. \tag{3.11}
\]
It is interesting to use the Lorentz force in order to achieve the same result. Let us assume that $a_5$ depends on $t$ alone. In this case, the force is

$$\ddot{t} = \frac{e}{M} f^0_5 = -e \frac{da_5}{dt}$$

(3.12)

The relation

$$\Phi^2 = \frac{1}{1 + \frac{K}{a^5}}$$

(3.13)

then implies

$$2 \frac{d\Phi}{dt} = -\Phi^4 e \frac{da_5}{dt},$$

i.e.,

$$\frac{da_5}{dt} = \frac{K}{e} \frac{d}{dt} \left( \frac{1}{\Phi^2} \right)$$

We substitute this in the force equation and multiply by $2\dot{t}$ to obtain

$$\frac{d\dot{t}^2}{d\tau} = -2 \frac{K}{M} \frac{d}{d\tau} \left( \frac{1}{\Phi^2} \right).$$

(3.14)

Finally, putting $K = -\frac{M}{2}$ we arrive at the remarkable result

$$\frac{dt}{d\tau} = \frac{1}{\Phi},$$

which coincides with the transformation (3.11) from the time on the freely falling clock $\tau$ to the redshifted $t$ in the conformal form of the Robertson-Walker metric. We see that this $t$ corresponds to the Einstein time satisfying the dynamical Hamilton equations, and the conformal factor of the Robertson-Walker metric coincides with the conformal factor of the curved space embedding.

In this construction we have assumed the $a_5$ field to depend on $t$ alone. The generalized Maxwell equations then provide a simple connection between the Robertson-Walker scale and the event density.

The generalized Maxwell equations [10] are

$$\partial_\alpha f^{\beta\alpha} = e j^\beta,$$

(3.15)

where $j^\beta = (j^\mu, \rho)$ satisfies $\partial_\beta j^\beta = \partial_\mu j^\mu + \partial_5 \rho = 0$, and $\rho$ is the event density. In the generalized Lorentz gauge $\partial_\alpha a^\alpha = 0$, we have

$$-\partial_\alpha \partial^\alpha a_5 = e j^5 = e \rho.$$  

(3.16)

Since $a_5$ depends on $t$ alone (3.16) becomes

$$\partial_t^2 a_5 = e \rho.$$  

(3.17)
From (3.13),
\[ a_5 = \frac{K}{e} \left( \frac{1}{\Phi^2} - 1 \right) \]
so that from (3.17)
\[ \rho = -\frac{2K}{e^2} \left[ \frac{\Phi_{tt}}{\Phi} - \frac{3}{\Phi^4} \right] \tag{3.18} \]

The space-time geometry is related to the density of matter \( \rho_M \) through the Einstein equations
\[ G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}, \tag{3.19} \]
where \( R^{\mu\nu} \) is the Ricci tensor, \( R \) is the scalar curvature and \( T^{\mu\nu} \) is the energy-momentum tensor. For the perfect fluid model (isotropy implies the \( T^{\mu\nu} \) is diagonal)
\[ T^{\mu\nu} = \rho u^\mu u^\nu + P (g^{\mu\nu} + u^\mu u^\nu). \tag{3.20} \]
The (0, 0) component (referring to \( \tau \)) is then
\[ T^{\tau\tau} = 8\pi G \rho_M. \tag{3.21} \]
using the affine connection derived from the metric (3.8) one finds
\[ G^{\tau\tau} = 3 \frac{\dot{\Phi}^2}{\Phi^2} = 8\pi G \rho_M \tag{3.22} \]
and the (equal) diagonal space-space components are (for example, we write the \( x, x \) component
\[ G^{xx} = -\frac{1}{\Phi^2} \left[ \frac{2}{\Phi^2} \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a^2} \right] = 8\pi G T^{xx} = \frac{8\pi GP}{\Phi^2}. \tag{3.23} \]
Since \( T^{\tau\tau} \) is the (0, 0) component of a tensor, it follows from (3.11) and (3.21) that the matter density in the conformal coordinates is given by
\[ \rho_M' = \frac{1}{\Phi^2} \rho_M. \tag{3.24} \]
To establish a connection between the density of events in spacetime \( \rho \) and the density of matter(particles) \( \rho_M' \), in space at a given time \( t \), we assume that,
\[ \rho_M' = \rho \Delta t \tag{3.25} \]
where \( \Delta t \) is the time interval (in the conformal coordinates associated with the Stueckelberg evolution) in which the events generating the particle world lines are uniformly spread. It then follows from (3.25) that
\[ \Delta t = \frac{\rho M}{\rho \Phi^2}. \tag{3.26} \]
We now consider two examples. For the static universe, for $\rho_M$ constant, it follows from Eq. (3.22) that $\Phi$ is given by an exponential; it then follows from (3.18) that $\rho$ is constant, so that

$$\Delta t \propto \Phi^{-2}.$$  \hspace{1cm} (3.27)

For the matter dominated universe, where the pressure is negligible [32], one sees from (3.23) that

$$2\Phi \Phi_{tt} = \Phi_t^2,$$

and substituting in (3.18), one finds after changing $\tau$ derivatives to $t$ derivatives in (3.22) that $\frac{\rho}{\rho_M}$ is constant. It then follows that $\Delta t \propto \Phi^{-2}$ in this case as well.

This result implies that, at any given stage of development of the universe, i.e., for a given $\tau$, the events generating the world lines lie in an interval of the conformal time $t$ which becomes smaller as $\Phi$ becomes large in the order of $\Phi^{-2}$ With the relation (3.11), this corresponds, on the other hand, to a narrowing distribution, of order $\Phi^{-1}$ in $\tau$, contributing to a set of events observed at a given value of the conformal time $t$. In general, if one observes the configuration of a system at a given $t$, the events detected may have their origin at widely different values of the world time $\tau$ parametrizing the trajectories (world lines) of the spacetime events. It would be generally difficult to relate such configurations to the configurations in spacetime (at a given $\tau$, instead of at a given $t$) predicted by a dynamical theory. However, in this case, we see that the spreading is narrowed for large $\Phi$, so that the set of events occurring at a given $\tau$ is essentially the same as the set of particles occurring at a given $t$. The observed configurations therefore become very close to those predicted by the underlying dynamical model. In the general case, the relation between $\rho(\tau)$ and $\rho_M(t)$ could be very complicated, and it may be difficult to see in the observed configurations a simple relation to the dynamical model evolving according to the world time. In the static and matter dominated Friedmann-Robertson-Walker model, the correspondence between the dynamical theory and observed configurations becomes more clear as $\Phi$ becomes large.

We have shown that the fifth potential of the generalized Maxwell theory, obtained through the requirement of gauge invariance of the Stueckelberg-Schrödinger equation, can be eliminated in the function generating evolution of the classical system by replacing the Minkowski metric in the kinetic term by a conformal metric. The Hamilton equations resulting from this function coincide with the geodesic associated with this metric, and with the Hamilton equations of the original form, i.e., the geodesic equations of the conformal metric describe orbits that coincide with solutions of the original Hamilton equations, as found in previous work which studied the replacement of an invariant (action-at-a-distance) potential by a conformal metric [33]. In this case the geodesic equations are those obtained from the conformal geodesic with the addition of a Lorentz force in standard form.

The Robertson-Walker metric can be put into conformal form. The conformal factor of the Robertson-Walker metric can then be put into correspondence with the $a^5$ field of the generalized Maxwell theory and therefore, through its $t$ derivatives (we assume no explicit $\tau$ dependence) with the event density. In both the static and the matter dominated models, the set of events generating the world lines of the expanding universe condense into progressively thinner slices of the conformal time.
We now consider a framework in which one can construct an analog model for gravity of much greater generality.

### IV. Eikonal Approximation to Wave Equations

In this section, we derive the relation between the eikonal approximation to the five dimensional generalization of Maxwell theory required by the gauge invariance of Stueckelberg’s covariant classical and quantum dynamics to demonstrate, in this approximation, the existence of geodesic motion for the flow of mass in a four dimensional pseudo-Riemannian manifold. These results provide a foundation for the geometrical optics of the five dimensional radiation theory and establish a model in which there is mass flow along geodesics.

We apply this method as well to the interesting case of relativistic quantum theory with a general second rank tensor coefficient, as in the conformal case studied above. In this case the eikonal approximation to the relativistic quantum mechanical current coincides with the geodesic flow governed by the pseudo-Riemannian metric obtained from the eikonal approximation to solutions of the Stueckelberg-Schrödinger equation. This construction provides a model in which there is an underlying quantum mechanical structure for classical dynamical motion along geodesics on a pseudo-Riemannian manifold. The locally symplectic structure which emerges is that of Stueckelberg’s covariant mechanics on this manifold, and provides the principal result of this work. We apply a technique similar to that used by Kline and Kay [13] in the study of four dimensional (Maxwell) wave equations to study the structure of wave equations in five dimensions [34] which follow as a consequence of gauge invariance of the covariant classical and quantum mechanics of Stueckelberg [6]. Since the eikonal approximation results in the loss of one dimension (a high frequency limit), the eikonal method applied to four dimensional wave equations in a medium with non-trivial dielectric tensor results in a ray approximation on a manifold in three dimensions (Riemannian). In the case of the five dimensional radiation theory, one finds a ray approximation corresponding to geodesic flow in a four dimensional pseudo-Riemannian manifold. We show that there is a Hamiltonian form for the generation of the rays.

We shall use, in this section, Greek letters for space time indices ($\mu = 0, 1, 2, 3$) and Latin letters to include a fifth index representing the Poincaré invariant $\tau$ parameter in addition to the usual 4 spacetime coordinates (e.g., $q = 0, 1, 2, 3, 5$). The analysis proceeds by a generalization of the method [29] of replacing the electric and magnetic vector fields by the electromagnetic tensor fields ($E, B$) and the excitation tensor fields ($D, H$). The generalized electromagnetic field tensor is written

\[
f_{q_1q_2} \equiv \partial_{q_1} a_{q_2} - \partial_{q_2} a_{q_1},
\]

where $a_q$ are the so-called pre-Maxwell electromagnetic potentials (as pointed out above, the fifth gauge potential $a_5$ is required for gauge compensation of $i\partial_5$, generating the evolution of the Stueckelberg wave function [10]).

We introduce the dual (third rank) tensor

\[
k^{l_1l_2l_3} = \varepsilon^{l_1l_2l_3q_1q_2} f_{q_1q_2},
\]
where $\varepsilon_{l_1 l_2 l_3 q_1 q_2}$ is the antisymmetric fifth rank Levi-Civita tensor density. The homogeneous pre-Maxwell equations are then given by

$$\partial_{l_3} k^{l_1 l_2 l_3} = 0,$$  \hspace{1cm} (4.1)

or, more explicitly ($\partial_5 = \pm \partial/\partial \tau$, according to the signature of the $\tau$ variable, i.e., corresponding, as we have discussed above, to $O(4,1)$ or $O(3,2)$ symmetry of the homogeneous field equations),

$$\partial_5 \varepsilon_{l_1 l_2} f_{q_1 q_2} + \partial_3 \varepsilon_{l_1 l_2 q_1 q_2} f_{q_1 q_2} = 0.$$  \hspace{1cm} (4.2)

We now divide Eq.(4.2) into two cases. In the first, the indices $l_1, l_2$ correspond only to space-time indices:

$$\partial_{5} \varepsilon^{\mu \nu} f_{\lambda \sigma} + 2 \partial_{\sigma} \varepsilon^{\mu \nu} f_{\lambda \sigma} = 0 \rightarrow \partial_{5} \varepsilon^{\mu \nu} f_{\lambda \sigma} + 2 \partial_{\sigma} \varepsilon^{\mu \nu} f_{\lambda \sigma} = 0,$$  \hspace{1cm} (4.3)

where $\varepsilon^{\mu \nu \lambda \sigma}$ is the four dimensional Levi-Civita tensor density. This equation, on the 0-mode ($\tau$ independent Fourier components) does not involve any of the usual Maxwell fields but only the fifth (Lorentz scalar) electromagnetic field. The second set from Eq.(4.2) corresponds to $l_1$ or $l_2 = 5$. It is clear then that all the other 4-remaining indices must be space-time indices and we obtain

$$\epsilon^{5 \mu \sigma \delta \nu} \partial_\sigma f_{\delta \nu} = 0 \rightarrow \epsilon^{\mu \sigma \delta \nu} \partial_\sigma f_{\delta \nu} = 0.$$  \hspace{1cm} (4.4)

It is this equation that reduces on integration over all $\tau$, to the two usual homogeneous Maxwell equations. The operation of integrating over all $\tau$ extracts the zero frequency components (in $\tau$), which we call the 0-mode. This has the effect of reducing the pre-Maxwell system of equations, as we discuss below, to the usual Maxwell equations. With appropriate identification of the integrated quantities, the zero mode of the pre-Maxwell equations coincides with the Maxwell theory (it is for this reason that the five dimensional gauge fields associated with the Stueckelberg theory are called “pre-Maxwell” fields).

We now turn to the current dependent pre-Maxwell equations. These can be written as

$$\partial_{l_3} n^{l_1 l_2} = -j^{l_2},$$  \hspace{1cm} (4.5)

where $n^{l_1 l_2}$ are the matter induced (excitation) fields (corresponding to $H, D$ in the 4D theory). We remark that, restricting our attention to the spacetime components of Eq. (4.5), which then reads

$$\partial_5 n^{\mu 5} + \partial_\nu n^{\mu \nu} = -j^\mu,$$  \hspace{1cm} (4.6)

we may extract the 0-mode by integrating over all $\tau$. Since $j^k$ satisfies the five dimensional conservation law $\partial_k j^k = 0$, its integral over $\tau$ (assuming $j^5 \rightarrow 0$ for $\tau \rightarrow \pm \infty$) reduces to the four dimensional conservation law $\partial_\mu J^\mu = 0$, where $J^\mu = \int j^\mu(x, \tau) d\tau$ is the 0-mode part of $j^\mu$ (this formula for the conserved $J^\mu$ is given in Jackson [28]). The first term of the left side of (4.6) vanishes (assuming that $n^{\mu 5} \rightarrow 0$ for $\tau \rightarrow \pm \infty$), and one obtains the form

$$\partial^\nu F^{\mu \nu} = J^\mu,$$
where we may identify the zero mode fields $F^{\mu\nu}$, as above, with the Maxwell fields $H, D$. With the zero mode of (4.4), we see that the Maxwell theory is properly contained in the five dimensional generalization we are studying here.

We assume the existence of linear constitutive equations in the dynamical structure of the 5D fields in a medium which connects the $n$ tensor-field to the $k$ tensor-fields using a fifth rank tensor $\mathcal{E}$ which is a generalization of the fourth rank covariant permeability-dielectric tensor [29] which relates the $E, B$ fields to the excitation fields $D, H$ in the usual Maxwell electrodynamics. The constitutive equations have the form

$$n^{l_1 l_2} = \mathcal{E}^{l_1 l_2 q_1 q_2 q_3 k_{q_1 q_2 q_3}},$$

(4.7)

antisymmetric in $l_1 l_2$ as well as $q_1 q_2 q_3$ (the indices of $k$ have been lowered with the Minkowski metric tensor; we shall treat other tensors in the same way in the following). It is useful at this point to distinguish between the space-time elements $f_{\mu 5}$ and the elements $f_{\mu 5}$, and assume that the tensor introduced in Eq. (4.7) does not mix these fields (for $n^{\mu 5}$, if $\varepsilon^{\mu 5 q_1 q_2 q_3}$ has $q_1 q_2 q_3 = \alpha \beta \gamma$, then the components of $k$ that enter are of the form $k_{\alpha \beta \gamma} = \mathcal{E}_{\alpha \beta \gamma \mu 5} f^{\mu 5}$ only; similarly, for $n_{\mu \nu}$, only the components $\mathcal{E}_{\mu \nu \alpha \beta 5}$ can occur, and $k^{\alpha \beta 5}$ connects only to the components $f^{\lambda \sigma}$ of the field tensor). As we have pointed out above, the vector $f_{\mu 5}$ is physically distinguished from the antisymmetric tensor $f_{\mu \nu}$ in the inhomogeneous field equations, since the source terms break the higher symmetry of the homogeneous field equations. The assumption that the constitutive equations do not couple these components results in a simpler system to analyze, although (as for Hall type effects in the non-relativistic theory) it is conceivable that the more general case could occur.

We introduce the new set of fields:

$$b^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} f^{\lambda \sigma},$$

(4.8)

so that

$$n^{\lambda \sigma} = 2 \mathcal{E}^{\lambda \sigma \alpha \beta 5} b_{\alpha \beta}.$$  

(4.9)

On the zero mode, the fields $b^{\mu \nu}$ correspond to the dual Maxwell fields; in this theory they play a role analogous to the $B$ fields in the Maxwell theory. In a similar way, the $f_{\mu 5}$ fields are analogous to $E$. The part of the tensor $\mathcal{E}^{l_1 l_2 q_1 q_2 q_3}$ connecting the $\mu 5$ fields is discussed below.

Working with these fields enables us to construct the equations in a form which, as we shall show, generalizes the Maxwell theory to a form where the invariant time $\tau$ plays the role of $t$ and spacetime plays the role of space. This analogy helps to interpret the physics and it distinguishes between the familiar physical quantities $f_{\mu \nu}$ and the new fields $f_{\mu 5}$. Substituting these fields in (4.3) and (4.4), we find

$$\partial_5 b^{\mu \nu} + \partial_\sigma \varepsilon^{\mu \nu \sigma \lambda} f_{\lambda 5} = 0,$$

(4.10)

and

$$\partial_\sigma b^{\mu \sigma} = 0.$$  

(4.11)
For the spacetime excitation fields we define
\[ h_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} n^{\lambda\sigma} \]
and we get from (4.5), for \( l_2 = \mu \),
\[ \partial_5 n^{\mu 5} - \frac{1}{2} \varepsilon^{\mu\sigma\lambda\nu} \partial_{\sigma} h_{\lambda\nu} = - j^{\mu}, \tag{4.12} \]
where we have used
\[ \varepsilon_{\alpha\beta\gamma\delta} \varepsilon^{\eta\delta\gamma\mu} = -2 (\delta^{\gamma}_{\alpha} \delta^{\mu}_{\beta} - \delta^{\mu}_{\alpha} \delta^{\gamma}_{\beta}). \tag{4.13} \]
To complete the set of equations, we note that for \( l_2 = 5 \), we get from (4.5)
\[ \partial_\sigma n^{5\sigma} = - j^5. \tag{4.14} \]

To obtain a mass-energy conservation law for the fields, we multiply (4.12) by \( f_{\mu 5} \) and (4.10) by \( h_{\mu\nu} \), and then combine them, obtaining
\[ \left[ f_{\mu 5} \partial_5 n^{\mu 5} + \frac{1}{2} h_{\mu\nu} \partial_{\tau} b^{\mu\nu} \right] + \frac{1}{2} \varepsilon^{\sigma\mu\lambda\nu} \partial_{\sigma} (f_{\mu 5} h_{\lambda\nu}) = - j^{\mu} f_{\mu 5}. \tag{4.15} \]
Assuming the dielectric tensor reduced into the \( \mu 5 \) and \( \mu\nu \) subspaces is symmetric (the relations of \( n^{\mu 5} \) to \( f_{\mu 5} \) and \( n^{\mu\nu} \) to \( f_{\mu\nu} \) go by the contraction \( \mathcal{E} \varepsilon \); the exclusive property of indices of \( \varepsilon \) then imply simple conditions on \( \mathcal{E} \) for the symmetry of these forms) we can write (4.15) as
\[ \frac{1}{2} \partial_5 \left[ f_{\mu 5} n^{\mu 5} + \frac{1}{2} h_{\mu\nu} b^{\mu\nu} \right] + \frac{1}{2} \varepsilon^{\sigma\mu\lambda\nu} \partial_{\sigma} (f_{\mu 5} h_{\lambda\nu}) = - j^{\mu} f_{\mu 5}. \tag{4.16} \]

From the Stueckelberg Hamiltonian [10]
\[ K = \frac{1}{2M} (p^{\mu} - e_0 a^{\mu}) (p^{\mu} - e_0 a^{\mu}) - e_0 a_5, \tag{4.17} \]
it follows from the relativistic Lorentz force [9,10]
\[ M \ddot{x}^{\mu} = e_0 f^{\mu}_{\nu} \dot{x}^{\nu} + e_0 f^{\mu 5}, \tag{4.18} \]
as discussed above, that
\[ M \dot{x}_\mu \ddot{x}^{\mu} = M \frac{d}{d\tau} \left( \dot{x}_\mu \dot{x}^{\mu} \right) = \dot{x}_\mu f^{\mu 5}. \tag{4.19} \]

Since, in the Stueckelberg theory, the Hamilton equations imply that
\[ \dot{x}^{\mu} = \frac{1}{M} (p^{\mu} - e_0 a^{\mu}), \]
and hence the $\tau$ derivative in the central equality of (4.19) corresponds to the change in the mass-squared $(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu)$ of the particle. It therefore follows that $j^{\mu} f_{\mu5}$ is the rate of mass change of the system. We therefore identify $s^\sigma = \frac{1}{2} \varepsilon^{\sigma\mu\lambda\nu} f_{\mu5} h_{\lambda\nu}$ as the analogue of the Maxwell Poynting vector. This Poynting 4-vector is the mass radiation of the field. We see, furthermore, that $\frac{1}{2} [f_{\mu5} n^{\mu5} + \frac{1}{2} h_{\mu\nu} b^{\mu\nu}]$ is the scalar mass density of the field (its four integral is the dynamical generator of evolution of the non-interacting field \[10\]).

We now introduce the eikonal approximation, i.e., set $f_{l_1l_2}(x,\tau) = f_{l_1l_2}(x) \exp i \kappa (\tau - \Psi(x))$ for large $\kappa$. In the absence of sources the 5D-Maxwell equations (4.10), (4.11), (4.12), (4.14) take the form (for large $\kappa$)

\begin{align}
 b^{\mu\nu} - \varepsilon^{\mu\nu\sigma\lambda} p_\sigma f_{\lambda5} &= 0, \quad (4.20) \\
 p_\sigma b^{\mu\sigma} &= 0, \quad (4.21) \\
 n^{\mu5} + \frac{1}{2} \varepsilon^{\mu\sigma\lambda\nu} p_\sigma h_{\lambda\nu} &= 0, \quad (4.22) \\
 p_\sigma n^{5\sigma} &= 0, \quad (4.23)
\end{align}

where $p_\sigma = \partial_\sigma \Psi$.

We now relate the direction of $p_\mu$ to the polarization of the fields. We write the “cross product” of $n$ and $b$ (analogous to the cross product of $D$ and $B$ in Maxwell’s theory):

\[ \varepsilon_{\mu\nu\sigma\lambda} n^{\nu5} b^{\sigma\lambda} = 2 n^{\nu5} f_{\nu5} p_\mu, \]

or

\[ p_\mu = \frac{1}{2 n^{\alpha5} f_{\alpha5}} \varepsilon_{\mu\nu\sigma\lambda} n^{\nu5} b^{\sigma\lambda}, \]

where we have used (4.20) and (4.23). It is clear that since $p_\mu$ and the Poynting four-vector are cross products of tensors which are not necessarily aligned in the same four-directions, they are in general not parallel to each other (in space-time) due to the anisotropy of the medium, i.e., the wave normal and radiation flow directions are not, in general, the same.

The relations (4.20) – (4.23), along with the constitutive relations relating $n^{\mu\nu}$, $f_{\sigma\lambda}$, and $n^{\mu5}$, $f_{\sigma5}$, provide relations analogous to those of the standard Maxwell theory characterizing the possible field strengths of the eikonal approximation in terms of properties of the medium. We shall not treat these relations here, but discuss the mass-radiation flows, along the rays, on spacetime geodesics in the interesting special case where $h^{\mu\nu} = b^{\mu\nu}$, which is analogous to the case of materials with $\mu = 1$ in Maxwell’s electromagnetism. This case is interesting since, although the space is empty in the usual sense (i.e. $E = D$, $B = H$), the dielectric effect involving the $f_{\mu5}$ components can drive the radiation on curved trajectories, i.e., the corresponding spacetime can have a non-trivial metric structure.

We multiply (37) by $\frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} p_\beta$. We then use (39) and (30) to obtain

\[ n_\alpha^5 - p_\alpha p^\beta f_{\beta5} + p_\beta p^\beta f_{\alpha5} = 0 \]

(4.25)
Defining the reduced dielectric tensor $\mathcal{E}_\alpha^\beta$ as the part of the general dielectric tensor which connects only the $\alpha 5$ components of the fields, i.e.,

$$n_{\alpha 5} = \mathcal{E}_\alpha^\beta f_\beta^5,$$

the condition (40) then implies that $\mathcal{E}_\alpha^\beta f_\beta^5$ cannot be in the direction of $p^\alpha$ (unless it is lightlike). We obtain from Eq. (42)

$$(\mathcal{E}_\alpha^\beta - p_\sigma p^\sigma \delta_\alpha^\beta + p_\alpha p^\beta)f_\beta^5 = 0,$$ (4.27)

where we have chosen the negative sign for the signature of the fifth index, $n_{\alpha 5} = -n_{\alpha 5}$ [with this choice the flat space limit, for which $\mathcal{E}_\alpha^\beta = \delta_\alpha^\beta$, Eq. (4.27), with (4.23), admits only spacelike $p_\alpha$; for positive signature of the fifth index, in this limit, $p_\alpha$ would be timelike].

Eq. (4.24) has a solution only if the determinant of the coefficients vanishes (a similar calculation in which the field strengths $f_{\mu \nu}$ enter in place of $f_{\mu 5}$ results in the same condition on these coefficients, as it must). It is somewhat simpler to work with the eigenvalue equation (4.27). Assuming as before that this dielectric tensor is symmetric, we can work in a Lorentz frame in which it is diagonal. In this frame we have (for the transformed fields)

$$f_{\alpha 5} = -\frac{p_\alpha}{(\mathcal{E}_\alpha^\beta - p^2)}(p^\beta f_\beta^5).$$ (4.28)

Note that in the isotropic case for which all of the $\mathcal{E}_\alpha^\alpha$ are equal, one obtains $p_\beta f_\beta^5 = 0$, and the metric becomes conformal, i.e., one obtains the condition

$$\mathcal{E}^{-1} \eta^{\mu \nu} p_\mu p_\nu = -1,$$

where $\eta^{\mu \nu}$ is the flat space Minkowski metric $(-1, 1, 1, 1)$.

Multiplying the equation (4.28) on both sides by $p^\mu$, and summing over $\alpha$, one obtains the condition ($p^2 \equiv p_\mu p^\mu$),

$$0 = K = \frac{p_1^2}{\mathcal{E}_1 - p^2} + \frac{p_2^2}{\mathcal{E}_2 - p^2} + \frac{p_3^2}{\mathcal{E}_3 - p^2} - \frac{p_0^2}{\mathcal{E}_0 - p^2} + 1.$$ (4.29)

This condition determines, in this case, the Fresnel surface of the wave fronts.

It then follows that

$$\frac{\partial K}{\partial p_\mu} = \frac{2p^\mu}{\mathcal{E}^\mu - p^2} + 2p^\mu \frac{\partial K}{\partial p^2}. \quad (4.30)$$

Calculating the scalar product of (4.28) and (4.30) one then obtains

$$f_{\mu 5} \frac{\partial K}{\partial p_\mu} =$$

$$= -2(p_\nu f^{\nu 5}) \left\{ \sum_{i=1,2,3} \frac{(p^i)^2}{(\mathcal{E}^i - p^2)^2} - \frac{p_0^2}{(\mathcal{E}^0 - p^2)^2} - \frac{\partial K}{\partial p^2} \right\} = 0. \quad (4.31)$$

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Multiplying the expression (4.20) for $b_{\mu\nu}(h_{\mu\nu})$ by (4.30), the contribution of the second term of (4.30) vanishes since the Levi-Civita tensor is antisymmetric; the first term, according to (4.28), is proportional to $f^{\alpha 5}$, and vanishes for the same reason. It therefore follows that
\[
\frac{\partial K}{\partial p^{\mu}} h^{\mu\nu} = 0. \tag{4.32}
\]
Since the scalar product of $\frac{\partial K}{\partial p^{\mu}}$ with both $h^{\mu\nu}$ and $f^{\mu 5}$ is zero, it is proportional to their “cross product” i.e., it is parallel to the Poynting vector. To make the proof explicit, it is convenient to define $V^\mu = \frac{\partial K}{\partial p^{\mu}}$, $H_i = -h_{0j}$, $F_i = f_{i5}$, and $D^i = \varepsilon^{ijk} h_{jk}$ (the space index may be raised or lowered without changing sign in our Minkowski metric). In this case, the conditions $V^\mu h_{\mu\nu} = 0$ and $V^\mu f_{\mu 5}$ become
\[
\begin{align*}
V^0 H - V \times D &= 0 \\
-V^0 f^{05} + V \cdot f &= 0 \\
V \cdot H &= 0,
\end{align*}
\tag{4.33}
\]
where we have used boldface to represent the space components of the vector. In these terms, the Poynting vector is given by
\[
\begin{align*}
S^0 &= D \cdot f \\
S &= f^{05} D + f \times H \tag{4.34}
\end{align*}
\]
Taking the cross product of $f$ with the first of (4.33), one obtains
\[
V^0 (f \times H) = V (f \cdot D) - D(f \cdot V).
\]
For $V^0 \neq 0$, one may substitute this into the second of (4.34). The $f^{05} D$ term, with the help of the second of (4.33), cancels, and we are left with
\[
S = \frac{S^0}{V^0} V.
\]
It then follows that $S^\mu = \frac{S^0}{V^0} V^\mu <$, i.e., $V^\mu$ is proportional to the Poynting vector. For the case $V^0 = 0$, the second and third of (4.33) imply that
\[
V \cdot f = V \cdot H = 0,
\]
i.e., if $V \neq 0$ (the case $V^\mu = 0$ is exceptional in the eikonal approximation), it must be proportional to $f \times H$. From the first of (4.33), we see that $V \times D = 0$, and if $D \neq 0$, it must be proportional to $V$. The space part of $S^\mu$, from the second of (51) is then proportional to $V$. Under these conditions, the time part of $S^\mu$ vanishes, and therefore we again obtain the result that $V^\mu$ is proportional to $S^\mu$. If $D = 0$, then $S^0 = 0$ and, since $V$ is proportional to $f \times H$, it again follows that $V^\mu$ is proportional to $S^\mu$.  

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From this point, one may follow the same procedure used in the case of Maxwell’s electromagnetism [13] to obtain the Hamiltonian flow corresponding to the admissible modes. The Lagrangian associated with the Hamiltonian (4.29) satisfies a minimal principle, from which it follows that the Hamiltonian flow is geodesic on this manifold. Replacing $f^{\beta 5}$ in (4.27) by $(E^{-1})^\beta \gamma n^\gamma 5$, one obtains

$$\{ \delta^\alpha_\gamma - M^\alpha_\gamma_{\mu \nu} n^\mu_\gamma 5 = 0, \quad (4.35)$$

where

$$M^\alpha_\gamma_{\mu \nu} = \delta^\alpha_\mu (E^{-1})^\nu_\gamma - \delta_\mu_\nu (E^{-1})^\alpha_\gamma. \quad (4.36)$$

The condition (4.23) implies that the solutions can lie only in the hyperplane orthogonal to $p^\sigma$. The projection of the matrix $M^\alpha_\gamma_{\mu \nu} n^\mu_\gamma 5$ is symmetric, and can therefore be diagonalized by an orthogonal (or pseudo-orthogonal) transformation in three dimensions. In fact, the Gauss law and a gauge condition restrict the polarization degrees of freedom to three [10] (the eikonal approximation is far from the zero mode, which corresponds to the Maxwell limit, for which only two polarizations survive) and hence one finds three geodesics.

For $p^\sigma$ timelike, one can choose a Lorentz frame in which the eigenvalue condition has the form

$$\{ \delta^\alpha_\gamma - M^\alpha_\gamma_{0 0} (p^0)^2 \} n^\gamma_5 5 = 0, \quad (4.37)$$

and for $p^\sigma$ spacelike,

$$\{ \delta^\alpha_\gamma - M^\alpha_\gamma_{3 3} (p^3)^2 \} n^\gamma_5 5 = 0. \quad (4.38)$$

In each of these cases, the matrix can be diagonalized under the little group acting in the space orthogonal to $p^\sigma$ (leaving it invariant). For the lightlike case, up to a rotation, $p^\sigma$ has the form $(p, 0, 0, p)$. The remaining matrix may then be diagonalized under SO(2) rotations, to obtain just two geodesics, corresponding to the polarization states of a massless Maxwell-like theory. This special limiting case will be investigated in detail elsewhere.

With the same procedure as applied to the Maxwell case [13] one finds the symplectic structure of the flow of matter in space time.

It has been shown by Kline and Kay [13], that for the three dimensional Maxwell case, the Hamilton equations resulting from the eikonal coincide with the geodesic flow generated by the resulting metric (recall that the direction of momentum associated with both eigenstates is the same); a similar proof can be applied to the 4D cases we have studied here.

The study of the five dimensional wave equations we have carried out above provide an interesting example of the construction of an analog gravity. We have emphasized the important special case where the dielectric tensor in the Maxwell part of the constitutive equations is trivial, and sufficient curvature is introduced through coupling to the fifth component alone. We now turn to a study of a quantum equation which results in gravitational physics in the ray approximation, providing a quantum theory which may underlie the observed classical gravitational fields. In a later section, we give a mechanism, following Nelson[1], based on correlations in relativistic Brownian motion, providing an interpretation for the structure of this equation.
One could think of applying the eikonal method to a Schrödinger equation in a medium which is not isotropic, for example, in a crystal with shear forces [35], with locally varying band structure (as in a crystal under nonuniform stress, or near the boundaries or impurities); the second order derivatives in the Schrödinger equation then appear multiplied by a “mass matrix.”

The rays are directly associated with the (probability) flow of particles. The eikonal eigenvalue condition is one dimensional in this case, since the field is scalar. For an analog of this structure (corresponding, for example, to a distribution of events periodic in both space and time) in four dimensions described by a relativistically covariant equation of Stueckelberg-Schrödinger type, the metric one obtains is a spacetime metric, and the geodesic flow is that of the quantum probability for the spacetime events (matter) described by the Stueckelberg wave function. We propose a study of the equation

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = -\frac{1}{2M} \partial^\mu g_{\mu\nu}(x) \partial^\nu \psi_\tau(x),$$  \hspace{1cm} (4.39)

where $g_{\mu\nu}(x)$ is assumed to be symmetric, and is somewhat analogous to a gauge field. We shall refer to it as a tensor gauge field. We assume no explicit $	au$ dependence in $g_{\mu\nu}(x)$ in this work. The Schrödinger current (satisfying the five dimensional conservation law (2.6)) is then

$$j^\nu_\tau(x) = -\frac{i}{2M} \left( \psi^*_\tau g_{\mu\nu} \partial^\mu \psi_\tau - \psi_\tau g_{\mu\nu} \partial^\mu \psi^*_\tau \right).$$  \hspace{1cm} (4.40)

In the eikonal approximation, for which the frequency associated with $\tau$ (essentially the total mass of the system [21]) is large, we assume a form for the solution

$$\psi \sim e^{-i(\kappa \tau - \sqrt{\kappa} S)},$$

where $S$ is the eikonal phase.

One obtains the condition, for $\kappa \to \infty$,

$$\frac{1}{2M} g^{\mu\nu} p_\mu p_\nu = 1,$$  \hspace{1cm} (4.41)

where

$$p_\mu = \partial_\mu S$$

analogous to the Fresnel surface condition (4.29) for the optical case.

We define

$$K = \frac{p_\mu p_\nu}{2M} g^{\mu\nu}$$

as the generator of motion in the corresponding classical dynamics. It is clear that $\partial K / \partial p_\mu$ is in the direction of $j^\mu_\tau$. This implies that $K$ is the operator of evolution for the dynamical flow of particles, corresponding to the rays. It follows from the Hamilton equations that the flow is geodesic, where $g_{\mu\nu}$ is the metric for this manifold.

As in the discussion of the conformal case, we see that

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = g^{\mu\nu} p_\nu / M,$$
so that

\[ p_\nu = Mg_{\mu\nu}\dot{x}^\mu. \]

It then follows that

\[ K = \frac{1}{2}M\dot{x}^\lambda\dot{x}^\sigma g_{\lambda\sigma}. \]

If we assume that in the flat asymptotic limit the particle is on-shell, since \( K \) conserved, with the value \( K = -\frac{M}{2} \) everywhere, we would obtain

\[ d\tau^2 = -g_{\lambda\sigma}dx^\lambda dx^\sigma. \]

If we identify the world time interval \( d\tau \) with a measure of length (as in free fall), then we can understand \( g_{\mu\nu} \) as a metric on a pseudo-Riemannian manifold. As for the conformal case, however, the function \( p_\nu \) induces canonical translations on \( x^\nu \).

We emphasize that (4.39) is an equation on flat Minkowski space, and the tensor gauge function \( g_{\mu\nu} \) is given as a second rank tensor under the Lorentz group. It is only in the eikonal approximation that one finds geodesic flow governed by \( g_{\mu\nu} \) and its affine associated connection with which the manifold is naturally lifted to a curved space, and the structure may become covariant under local diffeomorphisms (general covariance). The equivalence principle then appears as a characterization of the tangent space of this new manifold.

With this equation, one can study a quantum theory which underlies classical gravity and, in particular study the quantum behavior in the neighborhood of singularities of the metric for which smooth eikonaals may not exist. In the case that the eikonal approximation is valid, the ray approximation provides the geodesics of the corresponding gravitational field, and assures that the probability flow is along the geodesic.

V. An Interpretation Based on Relativistic Brownian Motion

In this section, we shall discuss relativistic Brownian motion, and follow Nelson for the construction of the Stueckelberg-Schrödinger equation with tensor gauge coupling. In this framework, the tensor coupling arises from correlations between spacetime dimensions in the underlying Brownian processes. In the following, we pose and solve some of the difficulties in achieving a definition of relativistic Brownian motion.

Brownian motion, thought of as a series of “jumps” of a particle along its path, necessarily involves an ordered sequence. In the nonrelativistic theory, this ordering is naturally provided by the Newtonian time parameter. In a relativistic framework, the Einstein time \( t \) does not provide a suitable parameter. If we contemplate jumps in spacetime, to accommodate a covariant formulation, a possible spacelike interval between two jumps may appear in two orderings in different Lorentz frames. The introduction of proper time as a parameter for the relativistic Brownian process (RBP) is not possible since the second order correlations in the simplest case (i.e. for an isotropic homogeneous process with a diffusion constant \( \sigma^2 \) ) have the form, for each \( \mu \),

\[ E(\Delta x^\mu \Delta x^\mu) = 2\sigma^2\Delta s \quad (5.1) \]
for each $\mu$; however, summing over $\mu$,

$$E(\Delta x_\mu \Delta x^\mu) \equiv \Delta s^2 \propto \Delta s,$$

(5.2)

where the first equality is by the definition of proper time and the second equality is due to the Brownian property expressed in Eq.(5.1). There is an obvious contradiction. We therefore adopt the invariant parameter $\tau$ as the dynamical variable for the Brownian process, first suggested by Stueckelberg [6].

The interpretation of an event going backwards in time as the antiparticle was given first by Stueckelberg and was later used by Feynman; it is now an accepted concept. In Feynman’s perturbative formulation of quantum electrodynamics [6], pair annihilation and creation occurs at points which are sharp vertices at the transitions. However, on a smooth worldline describing the pair annihilation process, as described by Stueckelberg [6], there are segments in which the event goes faster than light speed (either forward or backward in $t$). In the formulation of our relativistic Brownian process, such sectors appear to play an important role; the occurrence of such states of motion is dictated by the demand of achieving a Lorentz invariant operator (more explicitly, the d’Alembertian) in the relativistic diffusion equation.

A second fundamental difficulty in formulating a covariant theory of Brownian motion lies in the form of the correlation function of the random variables of spacetime. The correlation function for the usual isotropic non-relativistic Wiener (i.e. [1]) process is given by

$$dw_i(t)dw_j(t') = \sigma^2 \delta^{ij} \delta(t-t'),$$

for $i, j = 1, 2, 3$, where $dw_i$ corresponds to the Brownian random part of the Langevin evolution ($\beta^i$ corresponds to a smooth drift)

$$dx^i = \beta^i dt + dw^i$$

(5.3)

A straightforward covariant generalization to the relativistic case is

$$dw_\mu(\tau)dw_\nu(\tau') = \alpha^2 \eta_{\mu\nu} \delta(\tau-\tau')$$

(5.4)

where $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. It then follows that $dw_0(\tau)dw_0(\tau') < 0$, which is impossible. Let us consider, however, a process which is physically restricted to only to spacelike or timelike jumps. One may argue that Brownian motion in spacetime should be a generalization of the non-relativistic problem, constructed by observing the non-relativistic process from a moving frame according to the transformation laws of special relativity (one could, alternatively, argue that as an idealization of a collision model, the “jumps” should be timelike; we shall consider both possibilities in the following). Hence the process taking place in space in the non-relativistic theory would be replaced by a spacetime process in which the Brownian jumps are spacelike. The pure time (negative) self-correlation does therefore not occur. In order to meet this requirement, we shall use a coordinatization in terms of generalized polar coordinates which assures that all jumps are spacelike. A corresponding distribution for such a relativistic Brownian probability density
could be, for example, of the form $e^{-\frac{\mu^2}{2\lambda^2}}$, where $\mu$ is the invariant spacelike interval of the jump. This is a straightforward generalization of the standard Brownian process in 3D, which is generated by a probability density of the form $e^{\frac{-r^2}{2\alpha^2}}$, where $r$ is the rotation invariant (i.e. the vector length) and $\alpha$ is proportional to the diffusion constant. We shall refer to this function as the relativistic Gaussian.

As we shall see, a Brownian motion based on purely spacelike jumps does not, however, yield the correct form for an invariant diffusion process. We must therefore consider the possibility as well that, in the framework of relativistic dynamics, there are timelike jumps. The corresponding distribution would be expected to be of the form $e^{-\frac{\sigma^2}{4b^2}}$, where $\sigma$ is the invariant interval for the timelike jumps, and $b$ is some constant. By suitably weighting the occurrence of the spacelike process (which we take for the present discussion to be “physical”, since its nonrelativistic limit coincides with the usual Brownian motion) and an analytic continuation of the timelike process, we show that one indeed obtains a Lorentz invariant Fokker-Planck equation in which the d’Alembert operator appears in place of the Laplace operator of the 3D Fokker-Planck equation\(^3\). One may, alternatively, consider the timelike process as “physical” (as might emerge from a microscopic model with scattering) and analytically continue the spacelike (“unphysical”) process to achieve a d’Alembert operator with opposite sign.

5.1 Brownian motion in 1+1 dimensions

We consider a Brownian path in 1+1 dimensions generated by a stochastic differential (analogue to the Langevin equation and Smoluchowsky process \([36]\)), of the form

$$dx^\mu(\tau) = \beta^\mu(x(\tau))d\tau + dw^\mu(\tau),$$

(5.5)

where $dw$ is a random process which is a relativistic generalization of the Wiener process, and $\beta^\mu$ is a smooth deterministic field (the drift).

We start by considering the second order term in the series expansion of a function of position of the particle on the world line, $f(x^\mu(\tau) + \Delta x^\mu)$, involving the operator

$$\mathcal{O} = \Delta x^\mu \Delta x^\nu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}.$$  \hspace{1cm} (5.6)

We have remarked that one of the difficulties in describing Brownian motion in spacetime is the possible occurrence of a negative value for the second moment of some component of the Lorentz four vector random variable. If the Brownian jump is timelike, or spacelike, however, the components of the four vector are not independent, but must satisfy the timelike or spacelike constraint. Such constraints can be realized by using parameterizations for the jumps in which they are restricted geometrically to be timelike or spacelike. We

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\(^3\) The d’Alembert operator alone does not, as an evolution kernel, assure that an initial positive density remains positive; additional conditions must be imposed. We thank Phillip Pearle for a discussion of this point.
now separate the random jumps into space-like jumps and time-like jumps accordingly, i.e., for the spacelike jumps,

\[ \Delta w^1 = \pm \mu \cosh \alpha , \quad \Delta w^0 = \mu \sinh \alpha \]  \hspace{1cm} (5.7)

and for the timelike jumps,

\[ \Delta w^1 = \sigma \sinh \alpha , \quad \Delta w^0 = \pm \sigma \cosh \alpha \]  \hspace{1cm} (5.8)

Here we assume that the two sectors have the same distribution on the hyperbolic variable. We furthermore assume that \( \mu, \sigma \) are generated by a relativistic Gaussian distribution, working in a Lorentz frame where the \( \alpha \) distribution is assumed to be independent of \( \mu, \sigma \) and is uniformly distributed on the restricted interval \([-L, L]\) (see discussion below) where \( L \) is arbitrary large. Therefore, in this frame \( \langle \Delta w^\mu \rangle = 0 \) (this is true in all frames; see discussion in Section 5) and we pick a normalization such that (for any component) \( \langle \Delta w^n \rangle \propto \Delta \tau_n^\mu \) so to first order in \( \Delta \tau \) the contribution to \( \langle O \rangle \) comes only from \( \langle \Delta w^\mu \Delta w^\nu \rangle \).

For a particle experiencing space-like jumps only, the operator \( O \) takes the following form:

\[ O_{\text{spacelike}} = \mu^2 [\cosh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \sinh^2 \alpha \frac{\partial^2}{\partial t^2}] \]  \hspace{1cm} (5.9)

If the particle undergoes time-like jumps only the operator \( O \) takes the form:

\[ O_{\text{timelike}} = \sigma^2 [\sinh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \cosh^2 \alpha \frac{\partial^2}{\partial t^2}] \]  \hspace{1cm} (5.10)

Since \( \mu, \sigma \) and \( \alpha \) are random processes, the average value of the operator \( O \) is the sum of the two averages of Eq.(5.9) and Eq.(5.10). A difference between these two averages, leading to the d’Alembert operator can only be obtained by considering the analytic continuation of the timelike process to the spacelike domain, choosing \( \mu^2 = -\sigma^2 \).

This procedure is analogous to the effect, well-known in relativistic quantum scattering theory, of a physical process in the crossed \( (t) \) channel on the observed process in the direct \( (s) \) channel. For example, in the LSZ formulation of relativistic scattering in quantum field theory (e.g.,[37]), a creation operator in the “in” state may be moved to the left in the vacuum expectation value expression for the \( S \)-matrix, and an annihilation operator for the “out” state may be moved to the right. The resulting amplitude, identical to the original one in value, represents a process that is unphysical; its total “energy” (the sum of four-momenta squared) now has the wrong sign. Assuming that the \( S \)-matrix is an analytic function, one may then analytically continue the energy-momentum variables to obtain the correct sign for the physical process in the new channel. Although we are dealing with an apparently classical process, as Nelson has shown, the Brownian motion problem gives rise to a Schrödinger equation, and therefore contains properties of the differential equations of the quantum theory. We thus see the remarkable fact that one must take into account the physical effect of the analytic continuation of processes occurring in a non-physical, in this case timelike, domain, on the total observed behavior of the system.
In the timelike case, the velocity of the particle $\Delta w^1/\Delta w^0 \leq 1$. We shall here use the dynamical association of coordinate increments with energy and momentum

$$E = M \frac{\Delta w^0}{\Delta \tau}, \quad p = M \frac{\Delta w^1}{\Delta \tau}, \quad (5.11)$$

so that

$$\sigma^2 = \left(\frac{\Delta \tau}{M}\right)^2 (E^2 - p^2), \quad (5.12)$$

where $M$ is a parameter of dimension mass associated with the Brownian particle. It then follows that $E^2 - p^2 = \left(\frac{M}{\Delta \tau}\right)^2 \sigma^2 > 0$. For the spacelike case, where $p/E > 1$, we may consider the transformation to an imaginary representation $E \rightarrow iE'$ and $p \rightarrow ip'$, for $E', p'$ real (this transformation is similar to the continuation $p \rightarrow ip'$ in nonrelativistic tunnelling, for which the analytic continuation appears as an instanton), but $E^2 - p^2 \rightarrow p'^2 - E'^2 > 0$. In this case, we take the analytic continuation such that the magnitude of $\sigma^2$ remains unchanged, but can be called $-\mu^2$, so that $E'^2 - p'^2 = \mu^2$ with $\mu$ imaginary. The spacelike contributions are therefore obtained in this mapping by $E, p \rightarrow iE, ip$ and $\sigma \rightarrow i\mu$, assuring the formation of the d’Alembert operator when the timelike and spacelike fluctuations are added with equal weight (this equality is consistent with the natural assumption, in this case, of an equal distribution between spacelike and timelike contributions). The preservation of the magnitude of the interval reflects the conservation of a mass-like property which remains, as an intrinsic property of the particle, for both spacelike and timelike jumps.

As mentioned before, one recalls the role of analytic continuation in quantum field theory; for the well known Wick rotation (e.g.,[38]), however, in that case, only the 0-component is analytically continued and no clear direct physical idea or quantity is associated with it. In the RBP the identification of the imaginary 4-momentum is dynamical in origin. It is due to the Lorentz structure of spacetime, which distinguishes the transitions $\Delta x^\mu > 1$ from those with $\Delta x^\mu < 1$. Though one may object to the association of $\Delta x^\mu$ with a dynamical momentum (since the instantaneous derivative $\frac{dx^\mu}{d\tau}$ is not defined for a Brownian process) the Brownian motion could be understood as an approximation to a microscopic process, just as it appears in Einstein’s work in 1905 [24], where it is assumed that the Brownian motion is produced by collisions. The effective conservation of $E^2 - p^2$ as a real quantity in both timelike and spacelike processes suggests that it is a physical property which preserves its meaning in both sectors.

With these assumptions, the cross-term in hyperbolic functions cancels in the sum, which now takes the form

$$\mathcal{O} = \mu^2 \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right]$$

(5.13)

Taking into account the drift term in Eq.(6.2.1), one then finds the relativistic Fokker-Planck equation

$$\frac{\partial \rho(x, \tau)}{\partial \tau} = \left\{ -\frac{\partial}{\partial x^\mu} \beta^\mu + \langle \mu^2 \rangle \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} \right\} \rho(x, \tau), \quad (5.14)$$

where $\partial/\partial x^\mu$ operates on both $\beta^\mu$ and $\rho$. 34
We see that the procedure we have followed, identifying $\sigma^2 = -\mu^2$ and assuming equal weight, permits us to construct the Lorentz invariant d’Alembertian operator, as required for obtaining a relativistically covariant diffusion equation.

To see this process in terms of a higher symmetry, let us define the invariant $\kappa^2 \equiv E_t^2 - p_t^2 \geq 0$ for the timelike case; our requirement is then that $E_s^2 - p_s^2 = -\kappa^2$ for the spacelike case. In the framework of a larger group that includes $\kappa$ as part of a three vector $(E, \kappa, p)$, the relation for the timelike case can be considered in terms of the invariant of the subgroup $O(1,2)$, i.e., $E^2 - \kappa^2 - p^2$. The change in sign for the spacelike case yields the invariant $E^2 + \kappa^2 - p^2$; we designate the corresponding symmetry (keeping the order of $E$ and $p$) as $O(2,1)$. These two groups may be thought of as subgroups of $O(2,2)$, where there exists a transformation which changes the sign of the metric of the subgroups holding the quantity $\kappa^2$ constant. The kinematic constraints we have imposed correspond to setting these invariants to zero (the zero interval in the $2 + 1$ and $1 + 2$ spaces).

The constraint we have placed on the relation of the timelike and spacelike invariants derives from the properties of the distribution function and the requirement of obtaining the d’Alembert operator, i.e, Lorentz covariance of the diffusion equation. It appears that in order for the Brownian motion to result in a covariant diffusion equation, the distribution function has a higher symmetry reflecting the necessary constraints. The transformations $E \rightarrow iE'$ and $p \rightarrow ip'$ used above would then correspond to analytic continuations from one (subgroup) sector to another. We shall see a similar structure in the $3 + 1$ case, where the groups involved can be identified with the symmetries of the $U(1)$ gauge fields associated with the quantum Stueckelberg-Schrödinger equation.

We now investigate the Lorentz invariance of the process and the correlation functions. The averaging operations are summations with weights (probability) assigned to each quantity in the sum. The sums in the continuum are, of course, expressed by integrals. If we wish to assign a relativistic Gaussian distribution function then the hyperbolic angle integration is infinite unless we introduce a cutoff. The question then arises whether our process is invariant or not.

We will show that we can use an arbitrary non-invariant (scalar) probability distribution (for example, a cutoff on the hyperbolic angle) and still obtain Lorentz invariant averages, using the imaginary representations of the ‘unphysical jumps’. For example, $<\Delta w^{\mu}>$ stands for a summation with a scalar weight (given by the density) over all the vectors $\Delta w^{\mu}$, in the domain. It is therefore a vector. Moreover under the imaginary representation of spacelike increments relative to the timelike ones (here we assume the timelike jumps physical), $\Delta w^{\mu}$ is a simple vector function over all spacetime which has the following form:

$$\Delta w^{\mu} = \begin{cases} \Delta w^{\prime \mu} & \text{timelike} \\ i\Delta w^{\prime \mu} & \text{spacelike} \end{cases} \ . \quad (5.15)$$

where the $\Delta w^{\prime \mu}$ are real. The quantity $<\Delta w^{\mu}>$ (formally written as a discrete sum) is given therefore by

$$<\Delta w^{\mu}> = \sum_{\text{timelike}} P(\Delta w)\Delta w^{\mu} = \sum_{\text{timelike}} P'(\Delta w')\Delta w'^{\mu} + i \sum_{\text{spacelike}} P'(\Delta w')\Delta w'^{\mu}$$

$$= <\Delta w'^{\mu}>_{\text{timelike}} + i <\Delta w'^{\mu}>_{\text{spacelike}} \quad (5.16)$$
where \( P(\Delta w) \) (or \( P'(\Delta w') \)) is the probability(weight) of having the vector \( \Delta w \) (or \( \Delta w' \)). The two vectors in the last equality in Eq.(5.16) are just normal Lorentz vectors. If we now pick a distribution in a given frame for which the average of each of them (independent of the other) is zero then \( <\Delta w^\mu> = 0 \) is true in all frames since the 0-vector is Lorentz invariant.

Constructing the second correlation, with the assumption of no correlation between spacelike jumps and timelike jumps, we find

\[
<\Delta w^\mu \Delta w'^\nu> = \sum_{\text{timelike}} P(\Delta w) \Delta w^\mu \Delta w'^\nu + \sum_{\text{spacelike}} P'(\Delta w') \Delta w'^\mu \Delta w'^\nu
\]

\[+ i^2 \sum_{\text{spacelike}} P'(\Delta w') \Delta w'^\mu \Delta w'^\nu = \sigma_{\text{timelike}} - \sigma_{\text{spacelike}} \]

\[\propto \eta^{\mu\nu} D \Delta \tau,
\]

(5.17)

where \( \sigma^{\mu\nu} \) is the correlation tensor in each case. From the definition of \( \Delta w'^\mu \) (a four vector) it follows that \( \sigma^{\mu\nu} \) are real Lorentz tensors. The last equality in Eq.(5.17) is a demand that could be achieved for the general \( 1 + n \) case, by assuming that in a given frame there is an invariant Gaussian distribution where the distribution is uniform in all angles and that there is a cutoff in the hyperbolic angle. The sum of the two covariant tensors (each a result of summation on different sectors) is a Lorentz invariant tensor. The higher correlation functions do not interest us since they are of higher order in \( \rho \) and therefore in \( \Delta \tau \) and do not contribute to the Fokker-Plank equation.

The mapping given in Eq.(5.15) leads necessarily to a deviation from the standard mathematical formulation of Brownian motion. There the probability that a particle starting at \( x \) at time \( \tau \) ending at \( x' \) at time \( \tau' \) is equal to the probability that the particle starts at \( x \) at time \( \tau \) passing through any possible intermediate point \( x'' \) at time \( \tau'' < \tau \) and going from there to the point \( x' \) at time \( \tau' \). This property is expressed in the Chapman-Kolmogorov equation (e.g. [36]),

\[
p(x, \tau, x', \Delta \tau') = \int_R p(x, \tau, x'', \tau'') p(x'', \tau'', x', \tau') d^4 x'',
\]

(5.18)

In the relativistic formulation the vector \( \Delta w' = x - x' \) could be a timelike vector therefore resulting in a real valued vector \( \Delta w \) according to the mapping in Eq.(5.15) However, the two intermediate vectors \( \Delta w'_1 = x - x'' \) and \( \Delta w'_2 = x'' - x \) could be spacelike, and take the event out of the real manifold into a complex valued coordinate. In this case the Chapman-Kolmogorov equation does not hold, and the event may be found outside of the real manifold. In order to build a consistent process one must adopt the concept of ‘Brownian jumps’ which could be a result for example of a process in which the event (similar to Einstein’s original construction) undergoes collisions and for each collision, or ‘jump’ the mapping in Eq.(5.15) holds. Therefore at each point in the physical manifold the event may take any increment spacelike or timelike (with a possible complex valued contribution to the averages). However, although the vector leading from the initial point, say \( O \), to
the end point, $A$, may be spacelike and therefore be represented as an imaginary vector it is understood that the event arrives at the real spacetime point $A$, never physically leaving the real spacetime. This structure separates the two manifolds, spacetime which is real and represents the physical coordinates of the event and a complex space representing the processes the event undergoes (virtually) going from one point to another. This structure differs in that sense from the mathematical formulation due to Wiener and others, but still it can be shown that the process is invariant on the average under decomposition into shorter time subprocesses. In other words, we consider the event starting at some arbitrary point, and going for some time $\Delta \tau$. We next decompose the time interval into $M$ intervals, so that:

$$\sum_{i=1}^{M} \Delta \tau_i = \Delta \tau$$

We consider then the expression appearing in the Fokker-Plank equation

$$< \Delta x^\mu \Delta x^\nu >= \left( \sum_{i=1}^{M} \Delta x_i^\mu \right) \left( \sum_{j=1}^{M} \Delta x_i^\nu \right)$$

Since we assume that any two non-equal time jumps are not correlated, i.e. $< \Delta x_i \Delta x_j = 0 >$ for $i \neq j$, which leaves only the equal time averages in the sum,

$$< \Delta x^\mu \Delta x^\nu >= \sum_{i=1}^{M} < \Delta x_i^\mu \Delta x_i^\nu > = \sigma^2 \eta^\mu\nu \sum_{i=1}^{M} \Delta \tau_i = \sigma^2 \eta^\mu\nu \Delta \tau$$

where $\sigma^2$ is the diffusion constant and we used Eq.(5.17) going from the second piece in the equality to the third.

The notion of “jumps” suggests the consideration of discrete processes, which can also be formulated within the relativistic framework and leads, under certain assumptions, to a covariant Fokker-Plank equation. For example let us assume a physical process in which the “jumps” occur in a very ordered way every $\tau_J$ seconds with a very small time spread (i.e. a very small probability that a collision occurs within a time different significantly from $\tau_J$). Then, averaging the ‘jumps’ over a period $\tau >> \tau_J$ leads to

$$< \Delta w^\mu \Delta w^\nu > \approx N \sigma^2 \eta^\mu\nu \tau_J N \tau_J < \tau < (N+1) \tau_J$$

This result is due to the fact that under our assumptions during the time $\tau$, $N$ single ‘jumps’ within separation of each other of $\tau_J$ occurred. The average in Eq.(5.17) does not change when $\tau$ changes in less then $\tau_J$; however if $\tau_J$ is small then one can replace Eq.(5.17) with

$$< \Delta w^\mu \Delta w^\nu > \approx \sigma^2 \eta^\mu\nu \tau$$

Therefore we recover the standard result for Brownian motion. However there is one very important difference which is the fact that $\tau$ can be taken to be finitely small where in the standard Brownian process $\tau$ can be actually taken to zero. This implies that higher order
derivative terms enter into the resulting ‘diffusion’ equation. For example for an isotropic homogeneous Gaussian distribution there will be additional even order derivative operators beyond the second order (d’Alembert) with coefficients \( \sigma^n \tau_J (2n - 1) \) where \( n \) is the (even) order of the differential operator. Since both \( \tau_J \) and \( \sigma^2 \) are small these operators could be neglected in general, though there might be special configurations in which their effect may be significant. In the following we assume that the \( \tau_J \) are very small compared with the macroscopic scale and that the ‘jumps’ are practically ordered with zero spread, thus the approximation in Eq.(5.17) is valid and no higher order terms are considered.

5.2 Brownian motion in \( 3 + 1 \) dimensions

In the \( 3 + 1 \) case, we again separate the jumps into timelike and spacelike types. The spacelike jumps may be parameterized, in a given frame, by

\[
\begin{align*}
\Delta w^0 &= \mu \sinh \alpha \\
\Delta w^1 &= \mu \cosh \alpha \cos \phi \sin \vartheta \\
\Delta w^2 &= \mu \cosh \alpha \sin \phi \sin \vartheta \\
\Delta w^3 &= \mu \cosh \alpha \cos \vartheta
\end{align*}
\]

(5.19)

We assume the four variables \( \mu, \alpha, \vartheta, \phi \) are independent random variables. In addition we demand in this frame that \( \vartheta \) and \( \phi \) are uniformly distributed in their ranges \((0, \pi)\) and \((0, 2\pi)\), respectively. In this case, we may average over the trigonometric angles, i.e., \( \vartheta \) and \( \phi \) and find that:

\[
\begin{align*}
< \Delta w^{12} >_{\varphi, \vartheta} &= < \Delta w^{22} >_{\varphi, \vartheta} = < \Delta w^{32} >_{\varphi, \vartheta} = \frac{\mu^2}{3} \cosh^2 \alpha \\
< \Delta w^{02} >_{\varphi, \vartheta} &= \mu^2 \sinh^2 \alpha
\end{align*}
\]

(5.20)

We may obtain the averages over the trigonometric angles of the timelike jumps by replacing everywhere in Eq.(5.20)

\[
\begin{align*}
\cosh^2 \alpha &\rightarrow \sinh^2 \alpha, \quad \mu^2 \rightarrow \sigma^2
\end{align*}
\]

to obtain

\[
\begin{align*}
< \Delta w^{12} >_{\varphi, \vartheta} &= < \Delta w^{22} >_{\varphi, \vartheta} = < \Delta w^{32} >_{\varphi, \vartheta} = \frac{\sigma^2}{3} \sinh^2 \alpha \\
< \Delta w^{02} >_{\varphi, \vartheta} &= \sigma^2 \cosh^2 \alpha
\end{align*}
\]

(5.21)

where \( \sigma \) is a real random variable, the invariant timelike interval. Assuming, as in the \( 1 + 1 \) case, that the likelihood of the jumps being in either the spacelike or (virtual) timelike phases are equal, and making an analytic continuation for which \( \sigma^2 \rightarrow -\lambda^2 \), the total average of the operator \( O \), including the contributions of the remaining degrees of freedom \( \mu, \lambda \) and \( \alpha \) is

\[
< O > = ( < \mu^2 > < \sinh^2 \alpha > - < \lambda^2 > < \cosh^2 \alpha > ) \frac{\partial^2}{\partial t^2} + \frac{1}{3} ( < \mu^2 > < \cosh^2 \alpha > - < \lambda^2 > < \sinh^2 \alpha > ) \Delta
\]

(5.22)
If we now insist that the operator $\langle \mathcal{O} \rangle$ is invariant under Lorentz transformations (i.e. the d’Alembertian) we impose the condition

$$<\mu^2><\sinh^2\alpha>-<\lambda^2><\cosh^2\alpha> = -\frac{1}{3}(<\mu^2><\cosh^2\alpha>-<\lambda^2><\sinh^2\alpha>)$$

(5.23)

Using the fact that

$$<\cosh^2\alpha>-<\sinh^2\alpha>=1$$

, and defining $\gamma \equiv <\sinh^2\alpha>$, we find that

$$<\lambda^2> = \frac{1+4\gamma}{3+4\gamma} <\mu^2>$$

(5.24)

The Fokker-Planck equation then takes on the same form as in the $1+1$ case, i.e., the form Eq.(5.14). We remark that for the $1+1$ case, one finds in the corresponding expression that the 3 in the denominator is replaced by unity, and the coefficients 4 are replaced by 2; in this case the requirement reduces to $<\mu^2> = <\lambda^2>$ and there is no $\gamma$ dependence.

We see that in the limit of a uniform distribution in $\alpha$, for which $\gamma \to \infty$,

$$<\lambda^2> \to <\mu^2> .$$

In this case, the relativistic generalization of nonrelativistic Gaussian distribution of the form $e^{-\frac{r^2}{\pi}}$ is $e^{-\frac{\mu^2}{\pi\tau}}$, which is Lorentz invariant.

The limiting case $\gamma \to 0$ corresponds to a stochastic process in which in the spacelike case there are no fluctuations in time, i.e., the process is that of a nonrelativistic Brownian motion. For the timelike case (recall that we have assumed the same distribution function over the hyperbolic variable) this limit implies that the fluctuations are entirely in the time direction. The limit $\gamma \to \infty$ is Lorentz invariant, but the limit $\gamma \to 0$ can clearly be true only in a particular frame.

5.3 The Markov Relation and the 4D Gaussian Process

In developing the previous ideas leading to the formulation of a RBP, we assumed that the probability distribution is consistent with the Markov property expressed in the Chapman-Kolmogorov equation. However, for the relativistic Gaussian it is not clear whether Eq.(5.18) holds. Therefore we now consider an alternative process, using the ideas developed above, resulting eventually in the Klein-Gordon equation. Let us consider a 2D Gaussian process generated by a distribution of the form:

$$p(w, d\tau) = \frac{1}{2\pi D d\tau} exp\left(-\frac{\Delta w_0^2 - \Delta w_1^2}{2D d\tau}\right)$$

(5.25)

This distribution corresponds to a Markov process, a standard normalized Wiener process, where $D$ is the diffusion constant. We now use the coordinate representation given in
Eq.(5.7) and Eq.(5.8) for the timelike and spacelike sectors to transform the distribution function in Eq.(5.25) in both sectors to (use $\mu^2$ in both cases):

$$\frac{1}{2\pi Dd\tau} e^x(\frac{-\mu^2 \cosh 2\alpha}{2Dd\tau})$$  \hspace{1cm} (5.26)

where timelike 'jumps' are physical and the measure for both sectors is $\mu d\mu d\alpha$. Then, using Eq.(5.15), we get for the combination of the timelike and spacelike contributions (with the appropriate sign) of the averages, say, $\Delta w_0^2$:

$$<\Delta w_0^2> = \frac{1}{2\pi Dd\tau} \int_{0}^{\infty} \int_{-\infty}^{\infty} \mu^3 e^x(\frac{-\mu^2 \cosh 2\alpha}{2Dd\tau}) d\mu d\alpha = \frac{1}{\pi} Dd\tau$$  \hspace{1cm} (5.27)

where we integrated over $\mu$ first, using

$$\int_{0}^{\infty} \mu^n e^x(-a\mu^2) = \frac{\Gamma(n+1/2)}{2^n (n+1/2)^{1/2}}$$  \hspace{1cm} (5.28)

and then integrated over $\alpha$ using

$$I_2 = \int_{-\infty}^{\infty} \frac{d\alpha}{\cosh^2 2\alpha} = 1$$  \hspace{1cm} (5.29)

In a similar way one finds that (using Eq.(5.15)) leading to the negative sign)

$$<\Delta w_1^2> = -\frac{1}{\pi} Dd\tau$$  \hspace{1cm} (5.30)

Since the probability distribution Eq.(6.5.2) is symmetric in $\Delta w_i$ in each sector $<\Delta w_0 \Delta w_1 > = 0$ as well as the first moments. Therefore we get in this particular frame a d’Alembertian. However, following the methods used above, we see that it is an invariant result in all Lorentz frames (though in other frames the distribution may not appear to be Gaussian).

Next we consider the application of the 4D form of Eq.(5.25)

$$p(w, d\tau) = \frac{1}{4\pi^2 D^2 (d\tau)^2} e^x(\frac{-\Delta w_0^2 - \Delta w_1^2 - \Delta w_2^2 - \Delta w_3^2}{2Dd\tau})$$  \hspace{1cm} (5.31)

with measure $\mu^3 d\mu \cosh^2 \alpha \sin \theta d\theta d\alpha d\phi$ for the spacelike sector and $\mu^3 d\mu \sinh^2 \alpha \sin \theta d\theta d\alpha d\phi$ for the timelike sector.

However, now calculating $<\Delta w_0^2>$ for the timelike case, after averaging over the spatial angles $\theta$ and $\phi$ we find, using Eq.(5.19),

$$<\Delta w_0^2> = \frac{1}{\pi D^2 (d\tau)^2} \int_{0}^{\infty} \int_{-\infty}^{\infty} \mu^5 e^x(\frac{-\mu^2 \cosh 2\alpha}{2Dd\tau}) \cosh^2 \alpha \sinh^2 \alpha d\mu d\alpha$$  \hspace{1cm} (5.32)
and for the spacelike case we get the same result since the spacelike parametrization of $\Delta w_0^2$ is proportional to $\sinh^2 \alpha$ and the spacelike volume element is proportional to $\cosh^2 \alpha$. Therefore if we use Eq.(5.15), adding the contribution of the two sectors one obtains a complete cancellation to zero. In order to avoid this we extend Eq.(5.15) to the form

\[
\begin{align*}
\Delta w^\mu &= \Delta w'^\mu, \quad \Delta w'^\mu \text{ timelike} \\
\Delta w^\mu &= i\lambda \Delta w'^\mu, \quad \Delta w'^\mu \text{ spacelike}
\end{align*}
\] (5.33)

Before completing the calculation, we discuss the inclusion of the factor $\lambda$, in Eq.(5.33). Let us consider a classical (i.e. non-stochastic) event with a given value $m^2 \equiv \Delta w_\mu \Delta w^\mu$, moving in a timelike direction; according to Eq.(5.33) $m^2$ changes into $\lambda^2 m^2$. Moreover, though the event may move according to a Gaussian distribution which makes no distinction between timelike and spacelike motions, the outcome of this motion as represented by the $\Delta w$, in Eq.(5.15); Eq.(5.33) does distinguish the two phases of motion. We shall see that a specific value of $\lambda$ is required for the realization of the Fokker-Planck equation.

The $w$ manifold is complex and it is a function of the motion on the real manifold $w'$. Our macroscopic (physical) equations are written on the real plane of the $w$ manifold. One can then visualize the flow of an event in spacetime similar to a motion of a particle in a cloud chamber. There as the particle moves the gas condenses, therefore the particle leaves a track. The track itself is not the particle but a result of the actual motion of the particle and its interaction with the gas in the cloud chamber. The track in the cloud chamber is analogous to the complex representation we use for the ‘jumps’.

We calculate first the expectation $<\Delta w_0^2>$ which is the total expectation, summed over the timelike and spacelike sectors. Averaging over the spherical angles $\theta, \phi$ we get using Eq.(5.32) and Eq.(5.33),

\[
<\Delta w_0^2> = \frac{1 - \lambda^2}{\pi} D^2 (d\tau)^2 \int_0^\infty \int_{-\infty}^\infty \mu^5 \exp(-\frac{\mu^2 \cosh 2\alpha}{2Dd\tau}) \cosh^2 \alpha \sinh^2 \alpha d\mu d\alpha
\]

\[
= \frac{8(1 - \lambda^2)}{\pi D d\tau} \int_{-\infty}^\infty \frac{\cosh^2 \alpha \sinh^2 \alpha}{\cosh^3 2\alpha} d\alpha
\]

(5.34)

where Eq.(5.28) was used in the $\mu$ integration leading to the last equality in Eq.(5.34).

Using

\[
\cosh^2 \alpha = \frac{1}{2} (\cosh 2\alpha + 1) \sinh^2 \alpha = \frac{1}{2} (\cosh 2\alpha - 1)
\]

(6.5.12)

in Eq.(5.34) and integrating over $\alpha$ we get

\[
<\Delta w_0^2> = \frac{(1 - \lambda^2)}{\pi} D d\tau \frac{\pi}{2}
\]

(5.36)

where we used

\[
I_1 \equiv \int_{-\infty}^\infty \frac{d\alpha}{\cosh 2\alpha} = \frac{\pi}{2}
\]

\[
I_3 \equiv \int_{-\infty}^\infty \frac{d\alpha}{\cosh^3 2\alpha} = \frac{\pi}{4}
\]

(5.37)
We now calculate the expectation of $<\Delta w_1^2>$. Averaging over the spherical angles $\theta, \varphi$ we get, using Eq.(5.32) and Eq.(5.33),

$$<\Delta w_1^2> = \frac{1}{3\pi} D^2 (d\tau)^2 \int_0^\infty \int_{-\infty}^\infty \mu^5 \exp\left(-\frac{\mu^2 \cosh 2\alpha}{2Dd\tau}\right) (\sinh^4 \alpha - \lambda^2 \cosh^4 \alpha) d\mu d\alpha =$$

$$= \frac{8}{3\pi} D d\tau \int_{-\infty}^\infty \frac{(\sinh^4 \alpha - \lambda^2 \cosh^4 \alpha)}{\cosh^2 2\alpha} d\alpha$$

Using Eqs.(5.35), (5.29) and (5.37), we get after integration over $\alpha$,

$$<\Delta w_1^2> = \frac{D d\tau}{\pi} \frac{1}{3} \left[(1 - \lambda^2) \frac{3\pi}{2} - 4(1 + \lambda^2)\right]$$

In order to obtain the d’Alembertian we insist that $<\Delta w_1^2> = -<\Delta w_0^2>$, which leads to

$$\lambda^2 = \frac{3\pi - 4}{3\pi + 4}$$

(5.39)

Finally, substituting (for example) Eq.(5.39) in Eq.(5.36) we find that

$$<\Delta w^\mu \Delta w^\nu> = \eta^{\mu\nu} \frac{4D}{3\pi + 4} d\tau = \eta^{\mu\nu} \tilde{D} d\tau$$

(5.40)

where $\tilde{D}$ is the actual effective diffusion constant defined by

$$\tilde{D} \equiv \frac{4D}{3\pi + 4}$$

(5.41)

VI. Discussion and Conclusions

We have discussed in the first section the properties of a classical relativistic charged particle in the framework of a consistent relativistically covariant classical dynamics. Such a theory, without a constraint relating the particle variables $E$ and $p$ admits variations of the particle mass from the “mass-shell” (for which an interval of proper time is equal to the corresponding interval of the universal world-time); this mass variation plays an important role in governing the evolution of the system, and may in fact, macroscopically stabilize the spacetime orbit in the presence of the highly singular self-interaction. The work of Gupta and Padmanabhan shows that the radiation reaction terms in the Lorentz force of the self-interacting system have a geometrical interpretation. In view of this result, one is motivated to study a possible connection between such dynamical systems and gravity.

In the next section, we showed that the $a_5$ field, primarily responsible for driving the particle off-shell, can indeed be absorbed in an effective conformal metric structure for the motion on a manifold. We showed that one can compare this form to the Robertson-Walker-Friedmann model, and, through the field equations, obtain a relation for the matter density and understand how, in such a world, the Einstein time (in conformal coordinates) and $\tau$ dependence can be constrained to be similar.
Following this idea further, we find that the eikonal approximation to wave equations for the (generalized) propagation of electromagnetic waves in a medium with non-trivial dielectric tensor can result in an analog gravity.

A mathematically simpler system with this property is that of a Stueckelberg-Schrödinger equation with a “tensor gauge” coupling. This equation permits one to study a quantum theory on a flat space, for which the eikonal approximation yields a system of rays which correspond to the flow of probability along the geodesics of a manifold with metric determined by the tensor gauge coupling. Such a system can be lifted by general covariance to give the complete structure of general relativity. One can therefore study, in this way, the properties of a well-defined canonical quantum theory for which the eikonal approximation is classical gravity. The quantum properties of such a system in the neighborhood of singularities, such as the black hole horizon, may yield interesting information on questions such as Bekenstein’s conjecture about the spectrum of black hole radiation.

To understand the structure of a Stueckelberg-Schrödinger equation of this form, we appealed to Nelson’s construction of the Schrödinger equation from the process of Brownian motion. To do this, it was necessary to establish a relativistic form of Brownian motion, and we described and solved the difficulties in achieving such a result.

In particular, we constructed a relativistic generalization of Brownian motion, using the invariant world-time, $\tau$, to order the Brownian fluctuations, and separated consideration of spacelike and timelike jumps to avoid the problems of negative second moments which might otherwise follow from the Minkowski signature. Associating the Brownian fluctuations with an underlying dynamical process, one may think of $\gamma$ discussed in the 3+1 case as an order parameter, where the distribution function (over $\alpha$), associated with the velocities, is determined by the temperature of the underlying dynamical system (the result for the 1+1 case is independent of the distribution on the hyperbolic variable). More generally it is suggestive to consider the possible thermodynamical effects of the ‘medium’ generating the relativistic Brownian fluctuations, following similar steps taken by Einstein [Einstein] in his famous work and verify whether any physical effect can be predicted.

At equilibrium, where $\partial \rho / \partial \tau = 0$, the resulting diffusion equation turns into a classical wave equation which, in the absence of a drift term $K^\mu$, is the wave equation for a massless field. An exponentially decreasing distribution in $\tau$ of the form $\exp(-\kappa \tau)$ would correspond to a Klein-Gordon equation for a particle in a tachyonic state (mass squared $-\kappa$), for physical spacelike motion and for physical timelike motion to a particle with mass squared $\kappa$.

Choosing a cutoff in the hyperbolic angle, one finds a covariant moment and, therefore, covariant differential operators. However the underlying process is not invariant; thus one can think of a special frame in which the hyperangular distribution is uniformly distributed around 0. Boosting breaks the symmetry of the hyperangular distribution, but since the the averages are tensor quantities the invariance properties are conserved, and therefore the Fokker-Plank equation (leading to the quantum equation) is invariant. This property is also used to construct the 4D Gaussian process.

It was shown that a (Euclidian) Gaussian process with an appropriate (weighted) complex representation for the timelike and spacelike random motions can be used to achieve the covariant quantum equation, with the assurance that it is Markovian (it is
a relativistic generalization of the Wiener process). This leads to a ‘cloud chamber-like’ picture in which the event as it evolves leaves a track (carries a real or an imaginary phase), which is a representation of the actual motion, distinguishing the timelike and spacelike motion.

In the classical Stueckelberg theory the timelike (forward or backward) propagation is associated with the standard particle or antiparticle interpretation, where spacelike propagation is needed whenever one discusses classical pair creation or annihilation (with continuous passage from forward to backward motion in time). This suggests that the spacelike process may be associated with the annihilation and creation of pairs. Moreover, though the resulting macroscopic equation (i.e. on the level of the Fokker-Planck equation) is local and causal in the spacetime variables, the underlying microscopic process (i.e., on the level of the Brownian fluctuations) is not. It is however local and causal in τ even at the microscopic level.

Nelson has shown that non-relativistic Brownian motion can be associated with a Schrödinger equation. Equipped with the procedures we presented here, constructing relativistic Brownian motion, Nelson’s methods can be generalized. One then can construct relativistic equations of Schrödinger (Schrödinger-Stueckelberg) type. The eigenvalue equations for these relativistic forms are also Klein-Gordon type equations. Moreover one can also generalize the case where the fluctuations are not correlated in different directions into the case where correlations exist, as discussed by Nelson for three dimensional Riemannian spaces. In this case the resulting equation will be a quantum equation with a local tensor coupling; as we have pointed out, the eikonal approximation to the solutions of such an equation contains the geodesic motion of classical general relativity. The medium supporting the Brownian motion may be identified with an ‘ether’ for which the problem of local Lorentz symmetry is solved. This study opens up several tracks of possible research. Nelson [1], discussing the E.P.R. system confronted the fact that such a system may be either described by a non-local Markov process or a local non-Markov process. The Markov process is simple to implement but Nelson was disturbed by the introduction of non-local interaction. However, the non-Markov process is very difficult to apply. The RPB developed here may bridge the two possibilities since an ordered (causal) Markov process in τ may appear to be a non-Markovian (or possibly non-local and certainly non causal) process in t. For example for the Gaussian process, looking for the probability of finding the event changing its spatial position Δx after Δt has passed, one may integrate Eq.(5.25) over all τ. This results however, in \( \frac{1}{\Delta x^2 + \Delta t^2} \) which is not integrable and therefore cannot be normalized. This however is not surprising, since the probability of finding the event in Δx after Δt is not well defined (there may be several values of Δx for a given Δt). For example the number of particles, as in Stueckelberg’s original construction [6] depends on the trajectory through which the point is reached. Defining an appropriate one particle probability resulting from the initial process occurring in τ demands a restriction of the sample space before integrating over τ i.e., using the conditional probability restricted to processes for which the event’s t coordinate is monotonic in τ (no pairs are created).

Finally we would like to point out that generating a covariant quantum equation through an RBP leads to a possible relation between quantum mechanics and gravitation. In the context of this work, the metric of gravity can appear as an anisotropy in the
correlations that lead to quantum equations for which the ray, or eikonal approximation, corresponds to the classical geodesic flow of general relativity. It furthermore appears interesting to generalize Einstein’s famous work on this process introducing thermodynamic concepts to the resulting geometrical structure of the theory.

It emerges that the spacetime metric associated with the eikonal geodesic flow, in such a theory, would have its origin in correlations in the underlying Brownian process.

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