General stability criterion for inviscid parallel flow

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Abstract
Arnol’d’s second stability theorem is approached from an elementary point of view. First, a sufficient criterion for stability is found analytically as either

\[-\mu_1 < \frac{U''}{V''} < 0 \text{ or } \frac{U''}{V''} < 0\]

in the flow, where \( U_1 \) is the velocity at the inflection point, and \( \mu_1 \) is the eigenvalue of Poincaré’s problem. Second, this criterion is generalized to barotropic geophysical flows in the \( \beta \) plane. And the connections between present criteria and Arnol’d’s nonlinear criteria are also discussed. The proofs are completely elementary and so could be used to teach undergraduate students.

1. Introduction
The instability due to shear in the flow is one of the fundamental and the most attractive problems in many fields, such as fluid dynamics, astrophysical fluid dynamics, oceanography, meteorology, etc. More generally, shear instability is also referred to as barotropic instability in geophysical flows, where the gravitational and buoyancy effects are ignored. Shear instability has been intensively investigated, which greatly helps understanding of other instability mechanisms in complex shear flows. Rayleigh investigated the growth of linear disturbances by means of normal mode expansion, which leads to Rayleigh’s equation [11]. Using this equation, Rayleigh first proved a necessary criterion for instability, i.e., the inflection point theorem, which is also called the Rayleigh–Kuo theorem (e.g. [5]) for Kuo’s generalization to barotropic geophysical flows in the \( \beta \) plane [10]. Then, Fjørtoft found a stronger necessary criterion for instability [8]. Besides, Tollmien gave a heuristic result that the criteria are also sufficient for instability if the velocity profiles are symmetrical or monotonic [13]. These criteria are well known and have been widely used in various applications (e.g. [5, 7, 9]).
On the other hand, Arnol’d considered the shear instability in a totally different way [1, 2, 4]. He studied the conservation law of the inviscid flow via Euler’s equations and found two nonlinear stability theorems by means of the variational principle. Arnol’d’s first stability theorem corresponds to Fjørtoft’s criterion [6, 7]. However, Arnol’d’s second nonlinear stability theorem has no such corresponding linear criterion. Though Arnol’d’s second nonlinear theorem is more useful in geophysical flows [6], it is seldom used by scientists in other fields. Dowling suggested that Arnol’d’s idea should be added to the general fluid-dynamics curriculum [6]. Yet his suggestion has not been followed until now (e.g. [5, 7, 9, 14], since the proofs of Arnol’d’s theorems are very advanced and mathematically complex.

The aim of this paper is to find elementary proofs for Arnol’d’s theorems, which could be used to teach undergraduate students as the variational method is too advanced and complex for undergraduate students. The new proofs are obtained in a totally different way, where the linear stability problem is considered by using Rayleigh’s equation.

2. Stable criterion

To find the criteria, Rayleigh’s equation for an inviscid parallel flow is employed [5, 7, 9, 11, 12], which is the vorticity equation of the disturbance [7, 9]. For a parallel flow with mean velocity $U(y)$ in figure 1, the vorticity is conserved along pathlines. The amplitude of a disturbed flow streamfunction, namely $\phi$, satisfies

\[
(\phi'' - k^2\phi) - \frac{U''}{U - c} \phi = 0,
\]

where $k$ is the non-negative real wavenumber and $c = c_r + ic_i$ is the complex phase speed and the double prime $''$ denotes $d^2/dy^2$. The real part of complex phase speed $c_r$ is the wave phase speed, and $\omega_i = kc_i$ is the growth rate of the wave. This equation is to be solved subject to homogeneous boundary conditions

\[
\phi = 0 \quad \text{at} \quad y = a, b.
\]
There are three main categories of boundaries: (i) open channels with both $a$ and $b$ being finite, (ii) boundary layers with either $a$ or $b$ being infinite and (iii) free shear flows with both $a$ and $b$ being infinite.

It is obvious that the criterion for stability is $o_1 = 0$ ($c_1 = 0$), for which the complex conjugate quantities $\phi^*$ and $c^*$ are also a physical solution of equations (1) and (2).

Multiplying equation (1) by the complex conjugate $\phi^*$ and integrating over the domain $a \leq y \leq b$, we get the following equations:

$$\int_a^b \left( (\|\phi\|^2 + k^2\|\phi^2\|) + \frac{U''(U - c)}{\|U - c\|^2} \|\phi\|^2 \right) dy = 0, \quad (3)$$

and

$$c_1 \int_a^b \frac{U''}{\|U - c\|^2} \|\phi\|^2 dy = 0. \quad (4)$$

Rayleigh used only equation (4) to prove his theorem, i.e., a necessary condition for instability is $U''(y_j) = 0$, where $y_j$ is the inflection point and $U_j = U(y_j)$ is the velocity at $y_j$. Fjortoft noted that equation (3) should also be satisfied, then he obtained his necessary criterion. To find a more restrictive criterion, we shall investigate the conditions for $c_1 = 0$. Unlike the former investigations, we consider this problem in a totally different way: if the velocity profile is stable ($c_1 = 0$), then the hypothesis $c_1 \neq 0$ should result in contradictions in some cases. Following this, some more restrictive criteria can be obtained.

To find a stronger criterion, we need to estimate the ratio of $\int_a^b \|\phi\|^2 dy$ to $\int_a^b \|\phi\|^2 dy$. This is known as Poincaré’s problem:

$$\int_a^b \|\phi\|^2 dy = \mu \int_a^b \|\phi\|^2 dy, \quad (5)$$

where the eigenvalue $\mu$ is positive definite for any $\phi \neq 0$. The smallest eigenvalue value, namely $\mu_1$, can be estimated as $\mu_1 > \left( \frac{r}{b-a} \right)^2$, as Tollmien has done [13].

Then using Poincaré’s relation equation (5), a new stability criterion may be found: the parallel flow is stable if $-\mu_1 < \frac{U''}{U - U_s} < 0$ everywhere.

To get this criterion, we introduce an auxiliary function $f(y) = \frac{U''}{U - U_s}$, where $f(y)$ is finite at the inflection point. We will prove the criterion by two steps. First, we will prove proposition 1: if the velocity profile is subject to $-\mu_1 < f(y) < 0$, then $c_1 \neq U_s$.

**Proof.** Since $-\mu_1 < f(y) < 0$, then

$$-\mu_1 < \frac{U''}{U - U_s} = \frac{U''(U - U_s)}{(U - U_s)^2} \leq \frac{U''(U - U_s)}{(U - U_s)^2 + c_1^2}. \quad (6)$$

Substitution of $c_1 = U_s$ and equation (6) into equation (3) results in

$$\int_a^b \left[ \|\phi\|^2 + k^2\|\phi^2\| + \frac{U''(U - U_s)}{\|U - c\|^2} \|\phi\|^2 \right] dy > 0. \quad (7)$$

This contradicts equation (3). So proposition 1 is proved. □

Then, we will prove proposition 2: if $-\mu_1 < f(y) < 0$ and $c_1 \neq U_s$, there must be $c_1^2 = 0$.

**Proof.** If $c_1^2 \neq 0$, then multiplying equation (4) by $(c_1 - U_s)/c_1$, where the arbitrary real constant $U_t$ does not depend on $y$, and adding the result to equation (3), yields

$$\int_a^b \left[ (\|\phi\|^2 + k^2\|\phi^2\|) + \frac{U''(U - U_s)}{\|U - c\|^2} \|\phi\|^2 \right] dy = 0. \quad (8)$$
But the above equation (8) does not hold for some special $U_t$. For example, if $U_t = 2c_t - U_s$, then there is $(U - U_t)(U - U_t) < \|U - c\|^2$, and
\[
\frac{U''(U - U_t)}{\|U - c\|^2} = f(y) \frac{(U - U_t)(U - U_t)}{\|U - c\|^2} > -\mu_1. \tag{9}
\]
This yields
\[
\int_a^b \left\{ \|\phi'\|^2 + \left[ k^2 + \frac{U''(U - U_t)}{\|U - c\|^2} \right] \|\phi\|^2 \right\} dy > 0, \tag{10}
\]
which also contradicts equation (8). So proposition 2 is also proved. □

Using ‘proposition 1: if $-\mu_1 < f(y) < 0$ then $c_t \neq U_t$’ and ‘proposition 2: if $-\mu_1 < f(y) < 0$ and $c_t \neq U_t$ then $c_t = 0$’, we find a stability criterion. If the velocity profile satisfies $-\mu_1 < \frac{U''(U - U_t)}{U - U_t} < 0$ everywhere, the parallel flow is stable. Moreover, the above proof is still valid for $0 < f(y)$, which is equivalent to Fjørtoft’s criterion. Thus we have the following theorem.

**Theorem 1.** If the velocity profile satisfies either $-\mu_1 < \frac{U''(U - U_t)}{U - U_t} < 0$ or $0 < \frac{U''(U - U_t)}{U - U_t}$, the flow is stable.

This criterion covers Rayleigh’s and Fjørtoft’s criteria. And the proofs here are very simple and easy to understand compared to Arnol’d’s proofs. As mentioned above, we have investigated the stable criterion via Rayleigh’s equation, while Arnol’d [3] considered the hydrodynamic stability in a totally different way. Is there any relationship between these proofs? Two points are outlined here. First, this criterion is essentially the same as Arnol’d’s second stability theorem and is more restrictive than Fjørtoft’s criterion. Second, the proofs here are similar to Arnol’d’s variational principle method. For the arbitrary real number $U_t$, which is like a Lagrange multiplier in the variational principle method, plays a key role in the proofs.

### 3. Discussion

One may note that the above criterion is something different from Fjørtoft’s criterion. Why are the functions of $U''/(U - U_t)$ used in Arnol’d’s theorems and present theorems, unlike $U''/(U - U_t)$ in Fjørtoft’s theorem? This is due to the property of Rayleigh’s equation. It can be seen from equation (1) that the stability of profile $U(y)$ is not only Galilean invariant, but also independent of the magnitude of $U(y)$ due to the linearity. So the stability of $U(y)$ is the same as that of $AU(y) + B$, where $A$ and $B$ are any arbitrary nonzero real numbers. As the value of $U''/(U - U_t)$ is only Galilean invariant but not magnitude free, it satisfies only part of the Rayleigh equation’s properties. On the other hand the value of $U''/(U - U_t)$ satisfies both conditions; this is the reason why the criteria in both Arnol’d’s theorems and present theorems are the functions of $U''/(U - U_t)$. Since the stability of inviscid parallel flow depends only on the velocity profile’s geometry shape, namely $f(y)$, and the magnitude of the velocity profile can be free, the instability of inviscid parallel flow could be called ‘geometry shape instability’ of the velocity profile. This distinguishes from the viscous instability associated with the magnitude of the velocity profile.

Moreover, the above theorem is essentially associated with vorticity distribution in the flow field. As known from Fjørtoft’s criterion, the necessary condition for instability is that the base vorticity profile $\xi = -U'$ has a local maximum. Note that $U''/(U - U_t) \approx \frac{\xi''}{\xi_t}$ near the inflection point, where $\xi_t$ is the vorticity at the inflection point, which means that
the base vorticity $\xi$ must be convex enough near the local maximum for instability, i.e., the vorticity should be concentrated somewhere in the flow for instability. Otherwise, the flow is stable if the vorticity distribution is smooth enough near the inflection point at $y_s$. A simple example can be obtained by following Tollmien’s method [13]. Figure 2 depicts three vorticity profiles within the interval $-1 \leq y \leq 1$, which have local maximal at $y=0$. Profile 2 ($U = -2\sin(y)$) is neutrally stable, while profile 1 ($U = -\sin(y)$) and profile 3 ($U = -\sin(2y)$) are stable and unstable, respectively.

To show the advantage of the criteria obtained above, we consider the stability of velocity profile $U = \tanh(\alpha y)$ within the interval $-1 \leq y \leq 1$, where $\alpha$ is a constant. This velocity profile is a classical model of mixing layer, and has been investigated by many researchers (see [5, 9, 12] and references therein). Since $U''(U - U_s) = -2\alpha^2 \tanh^2(\alpha y) / \cosh^2(\alpha y) < 0$ for $-1 \leq y \leq 1$, it might be unstable for any $\alpha$ according to both Rayleigh’s and Fjørtoft’s criteria. But it can be derived from theorem 1 that the flow is stable for $\alpha^2 < \pi^2/8$. For example, we choose $\alpha_1 = 1.1$ and $\alpha_2 = 1.3$ for velocity profiles $U_1(y)$ and $U_2(y)$. The growth rate of the profiles can be obtained by the Chebyshev spectral collocation method [12] with 100 collocation points, as shown in figure 3. It is obvious that $c_1 = 0$ for $U_1$ and $c_1 > 0$ for $U_2$, which agrees well with the criteria obtained above. This is also a counterexample that Fjørtoft’s criterion is not sufficient for instability. So this new criterion for stability is more useful in real applications.

Then, recalling the proof of theorem 1, we will find that the following Rayleigh’s quotient $I(f)$ plays a key role in determining the stability of parallel flows:

$$I(f) = \frac{\int_a^b \left[ \phi''^2 + f(y) \phi^2 \right] \, dy}{\int_a^b \phi^2 \, dy}. \quad (11)$$

Note that the proof of theorem 1 is still valid in the case of $I(f) > 0$. We have such a result: parallel flows are stable if $I(f) > 0$. Although this criterion is more restrictive than that in theorem 1, it is inconvenient for the real applications due to the unknown Rayleigh’s quotient $I(f)$. Theorem 1 is more convenient for real applications in different research fields.
Finally, the stable criterion for the parallel inviscid flows can be applied to the barotropic geophysical flows in a differentially rotating system. Considering the barotropic plane flows in a rotating frame, which are the approximations of barotropic geophysical flows [5, 6, 10], equation (1) changes to

\[(\phi'' - k^2 \phi) - \frac{U'' - \beta}{U - c} \phi = 0, \quad (12)\]

where \(\beta\) is the gradient of the Coriolis parameter with respect to \(y\). Equation (12) is a generalized Rayleigh’s equation, and there is a generalized stability criterion for these flows.

**Theorem 2.** The flow is stable, if the velocity profile satisfies either \(\mu_1 < \frac{U'' - \beta}{U - U_s} < 0\) or \(0 < \frac{U'' - \beta}{U - U_s}\), where \(U_s\) is the velocity at the point \(U''(y_s) = \beta\).

4. Conclusion

In summary, Arnol’d’s second stability theorem for inviscid parallel flow is obtained from an elementary point of view. Both the criteria and the proofs are remarkably simple and easy to understand, compared to Arnol’d’s nonlinear theorems. The new criteria extend the former theorems proved by Rayleigh, Tollmien and Fjortoft. The proofs are completely elementary and so could be used to teach undergraduate students.

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