AUSLANDER THEOREM FOR PI ARTIN-SCHelter REGULAR ALGEBRAS

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Abstract. We prove a version of a theorem of Auslander for finite group actions or coactions on noetherian polynomial identity Artin-Schelter regular algebra.

Introduction

Throughout, \( k \) is an algebraically closed field of characteristic zero. All vector spaces, algebras and Hopf algebras are over \( k \).

A classical theorem of Maurice Auslander states that if \( G \) is a finite subgroup of \( \text{SL}_n(k) \), acting linearly on the commutative polynomial ring \( R = k[x_1, \ldots, x_n] \) with invariant subring \( R^G \), then the natural map from the skew group algebra \( R \# G \) to \( \text{End}_{R^G}(R) \) is an isomorphism of graded algebras. The Auslander theorem is a fundamental result in the study of the McKay correspondence. Recently, some researchers have studied the Auslander theorem and McKay correspondence in the noncommutative setting, see [5, 6, 9–11, 13, 22]. One of the main open questions concerning a noncommutative version of Auslander theorem is the following conjecture that was stated in [9, Conjecture 0.2].

Conjecture 0.1. Let \( R \) be a noetherian connected graded Artin-Schelter regular algebra [3] and \( H \) be a semisimple Hopf algebra acting homogeneously and inner faithfully on \( R \). If the homological determinant of the \( H \)-action on \( R \) is trivial, then the Auslander map

\[
\varphi : R \# H \rightarrow \text{End}_{R^H}(R), \quad r \# h \mapsto (x \mapsto r(h \cdot x))
\]

is an isomorphism of graded algebras.

It is worth pointing out that the Hopf action with trivial homological determinant generalizes the group action by a subgroup of \( \text{SL}_n(k) \).

The above conjecture holds when \( R \) has global dimension two [9, Theorem 0.3]. Bao, He and Zhang proved that the Auslander map is related to an invariant of the \( H \)-action on \( R \) known as the pertinency. The pertinency of the \( H \)-action on \( R \) is defined to be

\[
p(R, H) = \text{GKdim } R - \text{GKdim } R \# H / (1 \# t)
\]

where \( t \) is a nonzero integral of \( H \), see [5, Definition 0.1].

Theorem 0.2. [5, Theorem 0.3] Let \( R \) be a noetherian, connected graded Artin-Schelter regular, Cohen-Macaulay algebra of Gelfand-Kirillov dimension at least two. Let \( H \) be a semisimple Hopf algebra acting on \( R \) homogeneously and inner faithfully. Then the Auslander map is an isomorphism if and only if \( p(R, H) \geq 2 \).

Some other interesting partial results concerning Auslander’s Theorem have been proved in [5,6,11,13] by using the pertinency as a major tool.

The goal of this paper is to verify the conjecture for finite group actions and coactions on polynomial identity (PI) Artin-Schelter regular algebras. The ideal of the proof is to use a result given by Buchweitz [8, Proposition 2.9].

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Theorem 0.3. (Theorem 1.27) Let $R$ be a noetherian PI Artin-Schelter regular algebra, and $H$ be a semisimple Hopf algebra acting on $R$ homogeneously and inner faithfully. Suppose that $H = \mathbb{k} G$ or $(\mathbb{k} G)^*$ where $G$ is a finite group. If the homological determinant of the $H$-action on $R$ is trivial, then the Auslander map $\varphi : R\#H \to \text{End}_{R^H}(R)$ is bijective.

In the paper [24], Qin, Wang and Zhang shows that whenever Auslander Theorem holds one can view $R\#H$ as a noncommutative quasi-resolution (NQR for short, which is a generalization of noncommutative crepant resolution (NCCR) in the sense of Van den Bergh [32]) of $R^H$, and when $R^H$ is a central subalgebra of $R$, $R\#H$ is a NCCR of $R^H$.

Applying Theorem 0.3, we have the following result concerning whenever the center of a noetherian PI Artin-Schelter regular algebra has a NCCR.

Theorem 0.4 (Theorem 2.5). Let $R$ be a noetherian PI connected graded Calabi-Yau algebra with the center $Z$. If $R^{\text{GrAut}_Z(R)} = Z$, then $\text{End}_Z(R)$ is a NCCR of $Z$.

1. Noncommutative Auslander Theorem

1.1. The Auslander map for $(B,e)$. Firstly, we recall some definitions.

Definition 1.1. Let $R$ be a (left and right) noetherian algebra.

1. The grade of a left or right $R$-module $M$ is

$$j_R(M) = j(M) := \inf\{j \mid \text{Ext}_R^j(M, R) \neq 0\}.$$  

2. $R$ is called to satisfy the Auslander condition if, for every finitely generated left or right $R$-module $M$ and for all $i \geq 0$, $j_R(N) \geq i$ for all right or left submodules $N \subseteq \text{Ext}_R^i(M, R)$.

3. $R$ is called Auslander Gorenstein if it satisfies the Auslander condition, and has finite left and right injective dimensions. If further, $R$ has finite global dimension, then $R$ is called Auslander regular.

4. $R$ is called Cohen-Macaulay (or CM for short) with respect to Gelfand-Kirillov (or GK for short) dimension and Krull dimension if for any finitely generated $R$-module $M$

$$j(M) + \text{GKdim} M = \text{GKdim} R < +\infty \quad \text{and} \quad j(M) + \text{Kdim} M = \text{Kdim} R < +\infty,$$

respectively. See [20] for the definitions of GK-dimension and Krull dimension.

Lemma 1.2. Let $R$ be a noetherian Auslander Gorenstein ring. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated $R$-modules, then $j(M) = \min\{j(M'), j(M'')\}$.

An algebra $R$ is called connected graded if $R = \oplus_{i \in \mathbb{N}} R_i$, $1 \in R_0 = \mathbb{k}$, and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. A polynomial identity algebra (or PI algebra for short), is an algebra satisfying a polynomial identity. We refer to [20, Chapter 13] for some basic materials about PI algebras.

Lemma 1.3. [30, Lemma 6.1] Let $R$ be a connected graded PI ring $R$ and $M$ be a finitely generated graded $R$-module. Then $\text{GKdim} M = \text{Kdim} M \in \mathbb{N}$.

Throughout this section, let $B$ be a ring and $e \in B$ be a nonzero idempotent. The Auslander map is the natural map

$$\varphi : B \to \text{End}_{eBe}(Be), \quad \text{which is defined by } \varphi(b)(x) = bx \text{ for all } b \in B, x \in Be.$$  

In order to show our main theorem, we need the following results proved by Buchweitz.

Proposition 1.4. [8, Proposition 2.9]

1. $\varphi$ is injective if and only if $j(B/BeB) \geq 1$,

2. $\varphi$ is bijective if and only if $j(B/BeB) \geq 2$.  

Let $A$ be a $k$-algebra and $M$ be a right $A$-module. Then there is a morphism
\[ \gamma_M : M \otimes_A \text{Hom}_A(M, A) \rightarrow \text{End}_A(M), \quad x \otimes f \mapsto (y \mapsto xf(y)). \]
Set $A = eBe$. Then there is an $A$-$B$-bimodule morphism
\[ \nu : eB \rightarrow \text{Hom}_A(Be, A), \quad y \mapsto (x \mapsto yx). \]

Now we have a useful commutative diagram of $B$-$B$-bimodule morphisms
\[ \begin{array}{c}
Be \otimes_A eB \\
\downarrow \text{id} \otimes \nu \\
Be \otimes_A \text{Hom}_A(Be, A) \end{array} \xrightarrow{\mu} B \xrightarrow{\varphi} \begin{array}{c}
\text{End}_A(Be) \\
\gamma = \gamma_{Be} \end{array}. \tag{1.1} \]
where $\mu(x \otimes y) = xy$ for any $x \in Be$ and $y \in eB$.

**Lemma 1.5.** If the Auslander map $\varphi$ is injective, then the morphism $\nu$ is also injective.

**Proof.** The conclusion follows essentially from the commutativity of the following diagram
\[ \begin{array}{ccc}
eB & \xrightarrow{\nu} & B \\
\downarrow & & \downarrow \varphi \\
\text{Hom}_A(Be, A) & \xrightarrow{j} & \text{End}_A(Be). \end{array} \]

**Proposition 1.6.** If the Auslander map $\varphi : B \rightarrow \text{End}_A(Be)$ is bijective, then
(a) $\varphi$ is injective,
(b) $\nu$ is bijective,
(c) $j_B(\text{Coker } \gamma) \geq 2$.

**Proof.** (a) It is obvious.
(b) By Lemma 1.5, we only need to show that $\nu$ is surjective. For any $f \in \text{Hom}_A(Be, A)$, there exists an element $b \in B$ such that $\varphi(b) = f$ because $\varphi$ is bijective. Since $\nu(\nu(b)) = \varphi(\nu(b)) = e\varphi(b) = ef = f$, it follows that $\nu$ is surjective as required.
(c) According to the commutativity of the diagram (1.1), $\text{Coker } \gamma \cong \text{Coker } \mu = B/BeB$.
Hence by Proposition 1.4, $j_B(\text{Coker } \gamma) \geq 2$. \qed

By using the Auslander property, we show that the necessary condition for the bijectivity of the Auslander map in Proposition 1.6, is also sufficient.

**Theorem 1.7.** Suppose that $B$ is a noetherian Auslander Gorenstein ring. Then $\varphi$ is bijective if and only if the condition (a), (b) and (c) in Proposition 1.6 hold.

**Proof.** It suffices to show that $\varphi$ is bijective when the condition (a), (b) and (c) in Proposition 1.6 hold. Since $\nu$ is bijective and $\varphi$ is injective, it follows from the commutative diagram (1.1) that the sequence
\[ 0 \rightarrow \text{Coker } \mu = B/BeB \rightarrow \text{Coker } \gamma \rightarrow \text{Coker } \varphi \rightarrow 0 \]
is exact. According to Lemma 1.2, $j_B(B/BeB) \geq j_B(\text{Coker } \gamma) \geq 2$. Hence $\varphi$ is bijective by Proposition 1.4. \qed
1.2. Smash products and the morphism $\tilde{t}$. In the following, let’s consider the Hopf action. Let $H$ stand for a Hopf algebra $(H, \Delta, \varepsilon)$ with the bijective antipode $S$. We use the Sweedler notation $\Delta(h) = \sum(h) h_1 \otimes h_2$ for all $h \in H$. We recommend [21] as a basic reference for the theory of Hopf algebras and their actions on algebras.

Let $H$ be a finite dimensional semisimple Hopf algebra, and $t$ be a nonzero integral of $H$ with $\varepsilon(t) = 1$. Let $R$ be a left $H$-module algebra, i.e., $H$ acts on $R$. The $H$-action on $R$ is said inner faithful if $I \rightarrow R \neq 0$ for every nonzero Hopf ideal $I$ of $H$.

Now assume that $B$ is the smash product algebra $R \# H$, and that $e$ is the idempotent $1 \# t$. Let $R^H$ be the invariant subring of $R$ under the $H$-action. Then as $k$-algebras

$$A = eBe = (1 \# t)(R \# H)(1 \# t) = R^H \# t \cong R^H,$$

and $Be = R \# t \cong R$ as $B$-$A$-bimodules. Thus we can rewrite the Auslander map as

$$\varphi : R^H \rightarrow \text{End}_{R^H}(R), \quad r \# h \mapsto (x \mapsto r(h \cdot x)).$$

Lemma 1.8. There is a right $R$-module isomorphism $\lambda : R \rightarrow eB$, $r \mapsto \sum_{(t)}(t_1 \rightarrow r) \# t_2$.

Proof. Let $H^*$ be the dual Hopf algebra of $H$, and $\alpha$ be an integral of $H^*$ with $\langle \alpha, t \rangle = 1$. We claim that the map

$$\lambda' : eB \rightarrow R, \quad \sum_i r_i \# h_i \mapsto \sum_i r_i \langle \alpha, h_i \rangle$$

is the inverse of $\lambda$. Note that $\sum_{(t)} t_1 \langle \alpha, t_2 \rangle = 1$. For any $r \in R$,

$$\lambda' \lambda(r) = \sum_{(t)} t_1 \rightarrow r \langle \alpha, t_2 \rangle = \langle \alpha, t \rangle r = r.$$

For any $\sum_i r_i \# h_i \in B$,

$$\lambda' (e(\sum_i r_i \# h_i)) = \lambda' (\sum t_1 \rightarrow r_i \# t_2 h_i) = \lambda' (\sum_{i,(t)} (t_1 S^{-1} h_i) \rightarrow r_i \# t_2)$$

$$= \lambda (\sum_{i,(t)} (t_1 S^{-1} h_i) \rightarrow r_i \langle \alpha, t_2 \rangle) = \lambda (\sum_i S^{-1} h_i \rightarrow r_i)$$

$$= \sum_{i,(t)} t_1 S^{-1} h_i \rightarrow r_i \# t_2 = \sum_{i,(t)} t_1 \rightarrow r_i \# t_2 h_i$$

$$= e(\sum_i r_i \# h_i).$$

Hence $\lambda$ is an isomorphism. \hfill \Box

Let’s consider a map $\tilde{t} : R \rightarrow \text{Hom}_{R^H}(R, R^H)$ which is defined by $\tilde{t}(r)(x) = t \cdot (rx)$.

Lemma 1.9. Then $\nu$ is bijective if and only if $\tilde{t}$ is bijective.

Proof. The conclusion follows from the commutative diagram

$$\begin{array}{ccc}
R & \xrightarrow{\lambda} & eB \\
\downarrow \tilde{t} & & \downarrow \nu \\
\text{Hom}_{R^H}(R, R^H) & \xrightarrow{\cong} & \text{Hom}_A(Be, A).
\end{array}$$

\hfill \Box
1.3. **Homological determinant and the bijectivity of ν.** In the following, let’s recall firstly the definition of Artin-Schelter regular algebras [3].

**Definition 1.10.** A noetherian connected graded algebra $R$ is called Artin-Schelter Gorenstein (or AS Gorenstein, for short) of dimension $d$, if the following conditions hold:

1. $R$ has finite injective dimension $d$ on the left and on the right,
2. $\text{Ext}^i_R(\mathbb{k}, R) \cong \text{Ext}^i_R(\mathbb{k}, R) \cong \begin{cases} 0, & i \neq d \\ \mathbb{k}(l), & i = d \end{cases}$ for some integer $l$, where $\mathbb{k} := R/ \oplus_{i \geq 0} R_i$.

Here $l$ is called the **AS index** of $R$.

If in addition, $R$ has finite global dimension and finite GK dimension, then $R$ is called Artin-Schelter regular (or AS regular, for short) of dimension $d$.

The homological determinant has been an important tool for understanding the theory of Hopf algebra actions on connected graded AS Gorenstein algebras. Let $R$ be a connected graded AS Gorenstein algebra. Assume that $R$ is a graded left $H$-module algebra, i.e., $H$ acts on $R$ homogeneously. The homological determinant of $H$-action on $R$ is an algebra homomorphism $\text{hdet} : H \to \mathbb{k}$ (see [15, Definition 3.3] for the definition). If $\text{hdet} = \varepsilon$, then we say that the homological determinant $\text{hdet}$ is trivial. The graded automorphism group of $R$ is denoted by $\text{GrAut}(R)$, and the special linear automorphism group $\text{SL}(R)$ is the group of graded automorphisms of $R$ with homological determinant $1$:

$$\text{SL}(R) := \{g \in \text{GrAut}(R) \mid \text{hdet}(g) = 1\}.$$ 

The dualizing complexes over noncommutative rings were introduced by Yekutieli in [35]. It was studied further by Van den Bergh [31] and Yekutieli-Zhang [36].

**Definition 1.11.** Let $R$ be a noetherian connected graded $\mathbb{N}$-graded algebra. Let $\mathcal{D}(\text{GrMod } R^e)$ be the derived category of complexes of left graded $R^e$-modules. A bounded complex $\Omega \in \mathcal{D}(\text{GrMod } R^e)$ is called a dualizing complex over $R$ if it satisfies the following conditions:

1. $\Omega$ has finite injective dimension on both sides,
2. each cohomology $H^i(\Omega)$ is noetherian over $R$ and over $R^{op}$ for every $i$,
3. the canonical morphisms $R \to \text{RHom}_R(\Omega, \Omega)$ and $R \to \text{RHom}_{R^{op}}(\Omega, \Omega)$ are isomorphisms in $\mathcal{D}(\text{GrMod } R^e)$.

Let $R$ be a noetherian connected graded algebra and $m$ be the maximal graded ideal of $R$. For any graded right $R$-module $M$, the $m$-torsion functor $\Gamma_m$ is defined to be

$$\Gamma_m(M) = \{x \in M \mid x m^n = 0, \exists n \geq 0\}.$$

For any graded left $R$-module $M$, the $m^{op}$-torsion functor $\Gamma_{m^{op}}$ is defined to be

$$\Gamma_{m^{op}}(M) = \{x \in M \mid m^n x = 0, \exists n \geq 0\}.$$

The derived functor $\text{R}\Gamma_m$ (respectively, $\text{R}\Gamma_{m^{op}}$) is defined on the derived category $\mathcal{D}^+(\text{GrMod } R)$ (respectively, $\mathcal{D}^+(\text{GrMod } R^{op})$) of bounded below complexes of graded right (respectively, left) $R$-modules.

See [4] for more details.

**Definition 1.12.** A dualizing complex $\Omega$ over a noetherian connected graded algebra $R$ is called balanced if there are isomorphisms

$$\text{R}\Gamma_m(\Omega) \cong \text{R}\Gamma_{m^{op}}(\Omega) \cong \bigoplus_{n \in \mathbb{N}} \text{Hom}(R_n, \mathbb{k})$$

in $\mathcal{D}(\text{GrMod } R^e)$.

For any $R$-$R$-bimodule $M$ and automorphism $\sigma : R \to R$, $^\sigma M$ is the $R$-$R$-bimodule with the same ground vector space $M$ and the bimodule structure is given by $x \cdot m \cdot y = \sigma(x)my$. 

Lemma 1.13. [15, Lemma 1.6] Let $R$ be a noetherian connected graded algebra. Then $R$ is AS Gorenstein if and only if $R$ admits a balanced dualizing complex of the form $\mu R(-l)[d]$ where $\mu$ is a graded algebra automorphism of $R$ and where $d$ and $l$ are the integers given in Definition 1.10.

If $R$ admits a balanced dualizing complex $\mu R(-l)[d]$, then $\mu$ is called the Nakayama automorphism of $R$. It is clear that $\mu$ is unique up to inner automorphisms of $R$.

We need the following results about the balanced dualizing complex.

Lemma 1.14. Let $R$ be a noetherian connected graded AS regular algebra of dimension $d$ with AS index $l$, and $H$ be a semisimple Hopf algebra. Suppose that $H$ acts homogeneously and inner faithfully on $R$ such that the homological determinant is trivial.

1. [15, Section 3] Then $R^H$ is AS Gorenstein of dimension $d$ with a balanced dualizing complex $\Omega(R^H) := \mu(R^H)(-l)[d]$, where $\mu$ is the Nakayama automorphism of $R^H$.

2. [36, Corollary 4.17] Then $\Omega(R) \cong R\text{Hom}_{R^H}(R, \Omega(R^H))$ in $\mathcal{D}(\text{GrMod } R)$, where $\Omega(R)$ is the balanced dualizing complex of $R$.

Lemma 1.15. Let $H$ be a semisimple Hopf algebra, and $R$ be a noetherian connected graded AS regular algebra. Suppose that $H$ acts homogeneously and inner faithfully on $R$. If the homological determinant is trivial, then the map $\nu$ is bijective.

Proof. According to Lemma 1.14, $\text{Hom}_{R^H}(R, R^H) \cong R$ as graded right $R$-modules. Then the nonzero graded $R$-module structure $\tilde{t} : R \rightarrow \text{Hom}_{R^H}(R, R^H)$ is bijective. Thus $\nu$ is bijective by Lemma 1.9.

1.4. The injectivity of the Auslander map. Under some mild assumptions, the primeness of $R\#H$ is equivalent to the faithfulness of the action of $R\#H$ on $R$, see [7, Theorem 3.1] and [5, Lemma 3.10] for example. Obviously, the Auslander map is injective if and only if $R$ is a faithful $R\#H$-module.

The next lemma follows immediately from [28, Lemma 4.2] and [7, Corollary 3.4].

Lemma 1.16. Let $H$ be a semisimple Hopf algebra, and $R$ be a domain which is a noetherian $H$-module algebra. Then $R\#H$ is prime if and only if the Auslander map is injective.

Let’s recall some results about Galois theory of division rings. See [21] for the definition of Hopf Galois extension.

Theorem 1.17. [12, Theorem 3.1] Let $D$ be a division algebra, and $G$ be a finite subgroup of $\text{Aut}(D)$. Then the extension $D^G \subseteq D$ is a $(kG)^*$-Galois extension.

Lemma 1.18. [23, Theorem A.I.4.2] Let $G$ be a finite group and $D = \bigoplus_{g \in G} D_g$ be a $G$-graded division ring. Suppose that $(kG)^*$-action on $D$ is inner faithful, that is, $D_g \neq 0$ for all $g \in G$. Then $D$ is strongly $G$-graded, that is, $D_1 \subseteq D$ is a $kG$-Galois extension.

Now we can prove the following result concerning the injectivity of the Auslander map for group actions and coactions.

Proposition 1.19. Let $R$ be a noetherian domain, and $H$ be a semisimple Hopf algebra acting on $R$ inner faithfully. Suppose that $H = kG$ or $(kG)^*$ where $G$ is a finite group. Then the Auslander map $\varphi : R\#H \rightarrow \text{End}_{R^H}(R)$ is injective.

Proof. Let $Q(R)$ be the quotient division ring of $R$. Note that the $H$-module structure on $R$ has a unique extension to $Q(R)$ with respect to which $Q(R)$ becomes a left $H$-module algebra [27, Theorem 2.2]. Since $R\#H$ is semiprime by [27, Theorem 0.5], $R\#H$ has a semisimple quotient ring $Q(R\#H)$ which is isomorphic to $Q(R)\#H$. By Theorem 1.17 and Lemma 1.18, $Q(R)^H \subseteq Q(R)$ is a Galois extension. Hence the quotient ring $Q(R\#H)$ is simple by [21, Theorem 8.3.7]. Then $R\#H$ is prime, and $\varphi$ is injective by Lemma 1.16. □
1.5. Auslander theorem for PI AS regular algebras. In this subsection, our main results are stated and proved. To prove Theorem 1.27, we need several lemmas.

Lemma 1.20. [21] Let $H$ be a semisimple Hopf algebra, $R$ be a noetherian $H$-module algebra. Then $R^H$ is also noetherian, and $R$ is a finitely generated $R^H$-module.

Lemma 1.21. Let $H$ be a semisimple Hopf algebra, and $R$ be a left $H$-module algebra with center $Z(R)$. Suppose that $Z(R)$ is noetherian and $R$ is a finitely generated $Z(R)$-module. If $H = \mathbb{k}G$ or $(\mathbb{k}G)^*$ where $G$ is a finite group, then $R^H$ is a finitely generated module over $Z = Z(R) \cap R^H$.

Proof. If $H = \mathbb{k}G$, then $Z(R) \cap R^G = Z(R)^G$. Hence $Z(R)$ is a finitely generated $Z$-module by Lemma 1.20.

Now assume that $H = (\mathbb{k}G)^*$. Hence $R = \bigoplus_{g \in G} R_g$ is a $G$-graded algebra. By our assumption, $Z(R)R_1$ is also a finitely generated $Z(R)$-module. Let $x_1, \ldots, x_n \in R_1$ such that $Z(R)R_1 = \sum_{i=1}^n Z(R)x_i$. For any $x \in R_1$, there exists $z_i \in Z(R)$, $u_i \in R_1$ and $v_i \in \bigoplus_{g \notin 1} R_g$ such that $x = \sum_{i=1}^n z_i x_i$ and $z_i = u_i + v_i$. Since $R$ is $G$-graded and $x$ is a homogeneous element, $x = \sum_{i=1}^n u_i x_i$. For all $g \in G$ and $r \in R_g$, $ru_i = ur$ because $ru_i + rv_i = rz_i = z_i r = u_i r + v_i r$. This implies that $u_i \in Z(R)$ for all $i$. So $R^H = R_1 = \sum_{i=1}^n Z x_i$. Hence $R^H$ is finitely generated as a $Z$-module.

Let $R$ be a noetherian prime ring with artinian simple quotient ring $Q$. Then $R$ is called an order in $Q$ and an order $S$ in $Q$ is said to be equivalent to $R$ if there exist units $a, b, c, d \in Q$ such that $aRb \subseteq S$ and $cSd \subseteq R$. And $R$ is called a maximal order if it is maximal within its equivalence class, that is, if $S$ is an order in $Q$ equivalent to $R$ and containing $R$, then $S$ must be equal to $R$.

The following result is due to [20, Theorem 5.3.13 and Proposition 13.6.11].

Lemma 1.22. Let $R$ be a prime noetherian ring which is a finitely generated module over its center $Z(R)$. If $R$ is a maximal order (in its quotient ring), then $Z(R)$ a noetherian normal domain.

Lemma 1.23. [20, Theorem 5.3.16] Let $R$ be a noetherian prime ring which is a finitely generated module over its center $Z(R)$. Suppose that $R$ is a maximal order. Then $R$ is a hereditary ring if its center $Z(R)$ is a Dedekind domain.

Lemma 1.24. Let $R$ be a prime noetherian algebra which is a finitely generated module over its center $Z(R)$. Let $H$ be a semisimple Hopf algebra acting on $R$ inner faithfully. Suppose that $H = \mathbb{k}G$ or $(\mathbb{k}G)^*$ where $G$ is a finite group. If $R$ is a maximal order, then $\text{Kdim}_R \text{Coker } \gamma \leq \text{Kdim } R - 2$.

Proof. Let $A = R^H$, $Z = Z(R) \cap A$ and $p$ be a height one prime ideal of $Z$. Since $R$ is a maximal order over its center $Z(R)$, it follows that $Z(R)$ is a noetherian normal domain by Lemma 1.22. Since $Z$ is a subring of $Z(R)$ such that $Z(R)$ is a finitely generated $Z$-module by Lemma 1.21, the localization $Z(R)_p := T^{-1}Z(R)$ of $Z(R)$ is a Dedekind domain, where $T = Z \setminus p$. Let $R_p := T^{-1}R$. Since $R_p$ is also a maximal order by [25, Theorem 11.1], $\text{gldim } R_p \leq 1$ by Lemma 1.23.

According to [17, Theorem 1.1], $\text{gldim } R_p \#H = \text{gldim } R_p \leq 1$. Since $A_p = T^{-1}A = (1 \# t)(R_p \# H)(1 \# t)$, the global dimension of $A_p$ is no more than one by [20, Proposition 7.8.9]. Since $R_p$ is a reflexive module over $A_p$, $R_p$ is a finitely generated projective $A_p$-module by [24, Corollary 2.13]. Notice that the localization map $(\gamma_R)_p = T^{-1} \gamma_R$ of $\gamma_R$ can be seen canonically as

$$
\gamma_R : R_p \otimes_{A_p} \text{Hom}_{A_p}(R_p, A_p) \to \text{End}_{A_p}(R_p), \quad x \otimes f \mapsto (y \mapsto xf(y)).
$$
So \((\gamma_R)_p\) is bijective for any \(p \in \text{Spec } Z\) with \(\text{ht } p \leq 1\). It follows that
\[
K\dim_Z \text{Coker } \gamma \leq K\dim Z - 2.
\]
By [20, Corollary 6.4.13], we know that
\[
K\dim_R \text{Coker } \gamma = K\dim_Z \text{Coker } \gamma, \quad K\dim Z = K\dim R.
\]
This completes the proof. \(\square\)

The CM and Auslander properties of the AS regular algebras have been studied in [1,30,36,37].

**Theorem 1.25.** [30, Theorem 1.1 and Corollary 1.2] Let \(R\) be a noetherian connected graded PI ring. If \(R\) has finite injective dimension \(d\), then \(R\) is Auslander-Gorenstein and CM with \(G\text{Kdim } R = \text{Kdim } R = d\). If \(R\) has finite global dimension, then

1. \(R\) is a domain and a maximal order in its quotient division ring,
2. \(R\) is finitely generated as a module over its center \(Z(R)\).

As well known, the smash product \(B(:= R\# H)\) is a Frobenius extension of \(R\), that is, \(B\) is a finitely generated projective \(R\)-module and \(B \cong \text{Hom}_R(B, R)\) as \(R\)-\(B\)-bimodules.

**Lemma 1.26.** [2, Section 5.4] Let \(R \subseteq B\) be a Frobenius extension. Then \(j^R(M) = j^B(M)\) for any finitely generated \(B\)-module \(M\). If \(R\) is noetherian Auslander Gorenstein, then so is \(B\).

**Theorem 1.27.** Let \(R\) be a noetherian AS regular PI algebra, and \(H\) be a semisimple Hopf algebra acting on \(R\) homogeneously and inner faithfully. Suppose that \(H = \mathbb{k} G\) or \((\mathbb{k} G)^*\) where \(G\) is a finite group. If the homological determinant of the \(H\)-action on \(R\) is trivial, then the Auslander map \(\varphi : R\# H \to \text{End}_{R^u}(R)\) is bijective.

**Proof.** In order to apply Theorem 1.7 we need to verify the conditions (a), (b) and (c) in Proposition 1.6.

(a) It follows from Corollary 1.19.
(b) It follows from Lemma 1.15.
(c) Since \(R\) is a maximal order which is a finitely generated module over its center by Theorem 1.25, it follows that \(K\dim_R \text{Coker } \gamma \leq K\dim R - 2\) by Lemma 1.24. Hence \(j^R(\text{Coker } \gamma) \geq 2\) because \(R\) is CM. Therefore, \(j^R(\text{Coker } \gamma) = j^R(\text{Coker } \gamma) \geq 2\) by Lemma 1.26.

Then the conclusion follows from Theorem 1.7. \(\square\)

We now give a few examples.

**Example 1.28.** Assume that \(R\) is the \((-1)\)-skew polynomial ring \(\mathbb{k}_{-1}[x_1, \ldots, x_d] := \mathbb{k}(x_1, \ldots, x_d)/(x_i x_j + x_j x_i \mid i < j)\).

Gaddis, Kirkman, Moore and Won proved that the Auslander map is an isomorphism for any finite subgroup of the symmetric group \(S_d\) [13, Theorem 3]. As we know, the automorphism group of \(R\) is \((\mathbb{k}^\times)^d \rtimes S_d\) and \(\text{hdet} : \text{GrAut}(R) \to \mathbb{k}^\times\) is given by \(\text{hdet}(\lambda_1, \ldots, \lambda_d; \sigma) = \prod_{i=1}^d \lambda_i\) (see [16, Lemma 1.12]). Hence

\[
\text{SL}(R) = \{(\lambda_1, \ldots, \lambda_d; \sigma) \mid \prod_{i=1}^d \lambda_i = 1, \lambda_i \in \mathbb{k}, \sigma \in S_d\} \supseteq S_d,
\]

and the Auslander map is an isomorphism for any finite subgroup of \(\text{SL}(R)\) by Theorem 1.27.

**Example 1.29.** For any \(\alpha \in \mathbb{k}\) and \(\beta \in \mathbb{k}^\times\), the algebra \(A(\alpha, \beta)\) over \(\mathbb{k}\) with generators \(d, u\) and the defining relations
\[
d^2 u = \alpha du d + \beta ud^2, \quad du^2 = \alpha u d + \beta u^2 d
\]
is a down-up algebra. Bao, He and Zhang proved that the Auslander map is an isomorphism for any finite group of graded automorphisms when $\beta \neq -1$, and also for the case $A(2, -1)$ [6, Theorem 0.6]. Let $\Delta = \sqrt{a^2 + 4\beta}$, $\omega_1 = \frac{a + \Delta}{2}$, and $\omega_2 = \frac{a - \Delta}{2}$. Then $A(\alpha, \beta)$ is a PI algebra if and only if $\Delta \neq 0$, and $\omega_1, \omega_2$ are roots of unity [38, Theorem 1.3]. According to [14, Proposition 1.1], the graded automorphism group of $A(\alpha, -1)$ is

$$\left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \mid a_{ij} \in \mathbb{k}^\times \right\}.$$

By some computations, we can see that

$$\text{SL}(A(\alpha, -1)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \amu & 0 \end{pmatrix} \mid a \in \mathbb{k}^\times \right\}.$$

Now assume that $\omega(\neq \pm 1)$ is a root of unity. Then $A(\omega + \omega^{-1}, -1)$ is a PI AS regular algebra. By Theorem 1.27, we can see that the Auslander map is an isomorphism for any finite subgroup of $\text{SL}(A(\alpha, \beta))$ when $\alpha = \omega + \omega^{-1}$ and $\beta = -1$.

2. Applications to Noncommutative Resolutions

In this section, some applications to noncommutative crepant (or quasi-) resolutions are indicated. Let’s recall the definitions of noncommutative resolutions.

Definition 2.1. [32] Let $A$ be a noetherian Gorenstein normal domain. A noncommutative crepant resolution (or NCCR for short) of $A$ is an algebra $\text{End}_A(M)$ where $M$ is a reflexive $A$-module and where $\text{End}_A(M)$ has finite global dimension and is a maximal CM $A$-module.

Let $n$ be a nonnegative integer, $A$ and $B$ be two $\mathbb{N}$-graded algebras. Two $\mathbb{Z}$-graded $B$-$A$-bimodules $X, Y$ are called $n$-isomorphic, denoted by $X \cong_n Y$, if there exists a $\mathbb{Z}$-graded $B$-$A$-bimodules $P$ and $\mathbb{Z}$-graded bimodule morphisms $f : X \rightarrow P$ and $g : Y \rightarrow P$ such that both the kernel and cokernel of $f$ and $g$ have GK-dimension no more than $n$.

Definition 2.2. [24, Definition 0.5] Let $A$ be a noetherian $\mathbb{N}$-graded algebra with $\text{GKdim}(A) = d(\geq 2) \in \mathbb{N}$. If there exists a noetherian locally finite $\mathbb{N}$-graded Auslander regular CM algebra $B$ with $\text{GKdim}(B) = d$ and two $\mathbb{Z}$-graded bimodules $B_M A$ and $A_N B$, finitely generated on both sides, such that

$$M \otimes_A N \cong_{d-2} B, \text{ and } N \otimes_B M \cong_{d-2} A$$

as $\mathbb{Z}$-graded bimodules, then the triple $(B, M, N)$ or simply the algebra $B$ is called a noncommutative quasi-resolution (or NQR for short) of $A$.

Many examples of NQRs are produced by the Auslander theorem.

Theorem 2.3. [24, Proposition 8.3 and Example 8.5] Let $R$ be a noetherian connected graded Auslander regular CM algebra with $\text{GKdim}(R) = d \geq 2$, and $H$ be a semisimple Hopf algebra acting on $R$ homogeneously and inner faithfully with integral $t$ such that $\varepsilon(t) = 1$. If the Auslander map is bijective, then $(B, Be, eB)$ is a NQR of $R^H$, where $B = R\#H$ and $e = 1\#t$.

A connected graded AS regular algebra $R$ is called Calabi-Yau if the Nakayama automorphism of $R$ is the identity map.

Lemma 2.4. [34] Let $R$ be a noetherian connected graded Calabi-Yau algebra. Suppose that there exists a normal regular element $z \in R$ and $\sigma \in \text{Aut}(R)$ such that $zx = \sigma(x)z$ for all $x \in R$. Then $\text{hdet}(\sigma) = 1$.

By abuse of notation, the smash product of $R$ by a group algebra $\mathbb{k} G$ is denoted by $R\#G$.

Theorem 2.5. Let $R$ be a noetherian AS regular PI algebra.
(1) Let $G$ be a finite subgroup of $\text{SL}(R)$. Then $R^G$ is a NQR of $R^G$. If further, $R^G$ is commutative, then $R^G$ is a NCCR of $R^G$.

(2) Let $Z$ be the center of $R$. Suppose that $R$ is a finite subgroup of $\text{SL}(R)$.

(i) Then $\text{GrAut}_Z(R)$ is a finite subgroup of $\text{SL}(R)$.

(ii) If $R^{\text{GrAut}_Z(R)} = Z$, then $\text{End}_Z(R)$ is a NCCR of $Z$.

Proof. (1) The first assertion follows from that Theorem 1.27 and 2.3. By Lemma 1.14, $R^G$ is a Gorenstein domain such that $R$ is a maximal CM $R^G$-module. Recall that $R^G \cong \text{End}_{R^G}(R)$ has finite global dimension by [17, Theorem 1.1], and that $R^G = Z(\text{End}_{R^G}(R))$ is normal by [29, Lemma 2.2]. Since $R \cong \text{Hom}_{R^G}(R, R^G)$ as $R^G$-modules, $R$ is a reflexive $R^G$-module. Thus $\text{End}_{R^G}(R)$ is a maximal order. So $R^G$ is a NCCR of $R^G$ by the definition.

(2) (i) Let $K$ be the fraction field of $Z$. Then $Q := R \otimes Z K$ is a central simple $K$-algebra by Posner’s theorem [20, Theorem 13.6.5]. Let $\sigma \in \text{GrAut}_Z(R)$. Since $\sigma$ extends to a $K$-algebra automorphism of $Q$, it follows that $\sigma$ is inner, that is, there exists a nonzero element $z \in Q$ such that $\sigma(q) = zqz^{-1}$ for all $q \in Q$. Without loss of generality, we will assume that $z \in R$. Clearly, $z$ is a normal regular element of $R$. Then by Proposition 2.4, $\text{idet}(\sigma) = 1$. Hence $\text{GrAut}_Z(R) \subseteq \text{SL}(R)$.

For any $\sigma \in \text{GrAut}_Z(R)$, we claim that $\sigma^m = \text{id}_R$ where $m = \dim_K(Q)$. As proved above, there exists a normal regular element $z \in R$ such that $\sigma(r)z = rz$ for all $r \in R$. Since $z$ is integral over $Z$, there exists an integer $k \leq n$ and $a_i \in Z$ such that

\[
z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0 = 0.
\]

Without loss of generality, we will assume that $k$ is minimum. Since $R$ is a noetherian connected graded ring, $R_m$ is finite dimensional over $k$ for all $m \geq 0$. Recall that $k$ is an algebraically closed field of characteristic zero. For any $m \geq 0$, if $R_m$ is not a semisimple $k(\sigma)$-module, then there exist $x, y \in R_m \setminus \{0\}$ such that $\sigma(y) = \lambda y + x$ and $\sigma(x) = \lambda x$ for some $\lambda \in k \setminus \{0\}$. Notice that for any $i \geq 1$,

\[
0 = (z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0)x^i = x'((\lambda^i z)^k + a_{k-1}(\lambda^i z)^{k-1} + \cdots + a_1(\lambda^i z) + a_0).
\]

Hence $\lambda^i z$ is also a solution of the equation

\[
X^k + a_{k-1}X^{k-1} + \cdots + a_1X + a_0 = 0.
\]

Since $Z[z]$ is commutative subring of a domain $R$, the equation (2.1) has at most $k$ distinct roots in $Z[z]$. Hence there exists an integer $l \leq k$ such that $\lambda^l = 1$. Let $y' = x'^{-1}y$ and $x' = \lambda^{-l}x$. Then $\sigma(y') = y' + x'$ and $\sigma(x') = x'$. Thus we have

\[
0 = (z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0)y' = (y' + kx')z^k + (y' + (k - 1)x')a_{k-1}z^{k-1} + \cdots + (y' + x')a_1z + y' a_0 = y'(z^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0) + x'(kz^k + (k - 1)a_{k-1}z^{k-1} + \cdots + a_1z) = x'(kz^{k-1} + (k - 1)a_{k-1}z^{k-1} + \cdots + a_1z).
\]

which is a contradiction since $R$ is a domain. Hence $R_m$ is a semisimple $k(\sigma)$-module for any $m \geq 0$. Therefore, for any $x \in R_m$, there exist $\lambda_1, \ldots, \lambda_s \in k \setminus \{0\}$ and $x_1, \ldots, x_s \in R_m$ such that

\[
x = x_1 + \cdots + x_s, \text{ and } \sigma(x_i) = \lambda_i x_i \text{ for all } i.
\]

By the above proof, there exist positive integers $l_1, \ldots, l_s \leq n$ such that $\lambda_i^{l_i} = 1$ for all $i$. It follows that $\sigma^{m} = \text{id}_R$.

Since $G = \text{GrAut}_Z(R)$ is a subgroup of $\text{Aut}_K(Q)(\subseteq \text{GL}_n(R))$ with finite exponent, it follows that $G$ is a finite group by Burnside theorem [26, 8.1.11].

(ii) The conclusion follows immediately from (1).
According to Theorem 2.5 (2) (ii), we establish a criterion for the center of a noetherian PI connected graded Calabi-Yau algebra to have a NCCR. Now let’s consider the following example.

Example 2.6. Let \( \{p_{ij} \in \mathbb{K}^x \mid 1 \leq i < j \leq d \} \) be a set of roots of unity, and set \( p_{ji} = p_{ij}^{-1} \) and \( p_{ii} = p_{ij} \) for all \( i < j \). The skew polynomial ring is defined to be the algebra generated by \( x_1, \ldots, x_d \) subject to the relations \( x_j x_i = p_{ij} x_i x_j \) for all \( i < j \), and is denoted by \( \mathbb{K}_{p_{ij}}[x_1, \ldots, x_d] \).

Assume that \( \prod_{j=1}^d p_{j} = 1 \) for all \( i = 1, \ldots, d \). Hence the Nakayama automorphism is the identity map by [18, Proposition 4.1]. Let \( Z \) be the center of \( \mathbb{K}_{p_{ij}}[x_1, \ldots, x_d] \). It is easy to see that \( G := \text{GrAut}_Z(\mathbb{K}_{p_{ij}}[x_1, \ldots, x_d]) \) is generated by \( \sigma_1, \cdots, \sigma_d \), where \( \sigma_i \) is defined by \( \sigma_i(x_j) = p_{ij} x_j \). Since \( Z = \mathbb{K}_{p_{ij}}[x_1, \ldots, x_d]^G \), it follows that \( Z \) has a NCCR by Theorem 2.5. In fact, \( G \) can be seen as a subgroup of the automorphism group of polynomial ring \( \mathbb{K}[x_1, \ldots, x_d] \). It is well known that the skew group algebra \( \mathbb{K}[x_1, \ldots, x_d] \# G \) is a NCCR of the invariant subring \( \mathbb{K}[x_1, \ldots, x_d]^G \) (see [32, Example 1.1]).

Obviously, not all of PI connected graded Calabi-Yau algebra satisfies the assumption of Theorem 2.5 (2) (ii). See the following examples.

Example 2.7. Let \( R \) be the down-up algebra \( A(0, -1) \). By [19, E 1.5.6], the Nakayama automorphism of \( R \) is the identity map. The center of \( R \) is

\[
Z = \mathbb{K}[d^4, u^4, dudu + udud, (du + \sqrt{-4}ud)^4, (du - \sqrt{-4}ud)^4].
\]

It is not difficult to see that

\[
\text{GrAut}_Z(R) = \left\{ \begin{pmatrix} (\sqrt{-4})^i & 0 \\ 0 & (\sqrt{-4})^{4-i} \end{pmatrix} \right\},
\]

and the invariant subring \( \text{GrAut}_Z(R)^{(R)} = \mathbb{K}[d^4, u^4, d^2u^2, dudu, udud] \neq Z \) is a commutative algebra. Then \( \text{GrAut}_Z(R)^{(R)} \) has a NCCR by Theorem 2.5.

There also exists a connected graded Calabi-Yau algebra whose center doesn’t have a NCCR (see [33, Example 9.3]).

Example 2.8. With \( P_{m,n} \) we denote the graded polynomial algebra

\[
\mathbb{K}[x_{ij}(l) \mid 1 \leq i, j \leq n; 1 \leq l \leq m],
\]

with \( \text{deg}(x_{ij}(l)) = 1 \). The \( \mathbb{K} \)-subalgebra of the matrix ring \( M_n(P_{m,n}) \) generated by the matrices

\[
X_l = (x_{ij}(l))_{i,j}, \quad \text{where } 1 \leq l \leq m,
\]

is called the ring of \( m \) generic \( n \times n \) matrices \( \mathcal{G}_{m,n} \). The \( \mathbb{K} \)-subalgebra of \( M_n(P_{m,n}) \) generated by \( \mathcal{G}_{m,n} \) and \( \text{Tr}(\mathcal{G}_{m,n}) \) is the trace ring of \( m \) generic \( n \times n \) matrices and is denoted by \( \mathcal{T}_{m,n} \). Note that both \( \mathcal{G}_{m,n} \) and \( \mathcal{T}_{m,n} \) are connected graded subalgebra of \( M_n(P_{m,n}) \).

By [33, Example 9.3], \( \mathcal{T}_{3,2} \) is a twisted NCCR of its center \( Z_{3,2} \), but \( Z_{3,2} \) doesn’t have a NCCR. Hence By [33, Example 9.3], the center \( Z_{3,2} \) of the Calabi-Yau algebra \( \mathcal{T}_{3,2} \) doesn’t have a NCCR. Hence

\[
\mathcal{T}_{3,2}^{\text{GrAut}_{Z_{3,2}}(\mathcal{T}_{3,2})} \neq Z_{3,2}.
\]

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