BERNSTEIN THEOREM FOR TRANSLATING SOLITONS OF HYPERSURFACES

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ABSTRACT. In this paper, we prove some Bernstein type results for translating solitons of hypersurfaces in $\mathbb{R}^{n+1}$, giving some conditions under which a translating soliton is a hyperplane. We also show a gap theorem for the translating solitons of hypersurfaces in $\mathbb{R}^{n+k}$, namely, if the $L^n$ norm of the second fundamental form of the soliton is small enough, then it is a hyperplane.

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1. INTRODUCTION

We study the translating solitons of hypersurfaces $F = F(x,t) \subset \mathbb{R}^{n+1}$, $x \in M^n, 0 \leq t < T$, evolving under the mean curvature flow defined by

$$(\partial_t F) ^\perp = \bar{H}(F),$$

where $\bar{H}(F)$ is the mean curvature vector of the hypersurface $F = F(x,t)$ at time $t$ and $M \subset \mathbb{R}^{n+1}$ is a fixed hypersurface. These solitons are characterized by the soliton equation

$$(1) \quad H = < \nu, \omega >$$

where $\nu$ is a unit vector field normal to the fixed hypersurface $M \subset \mathbb{R}^{n+1}$, $\bar{H}(F) = H\nu$, and $\omega$ is a fixed unit vector in $\mathbb{R}^{n+1}$. In this case the flow is given by

$$F(x,t) := F(x) + t\omega, \quad \text{or} \quad \partial_t F = F_s(\partial_t) = \omega,$$

with the right side $F : M \rightarrow \mathbb{R}^{n+1}$ being a fixed hypersurface in $\mathbb{R}^{n+1}$.

As type II singularity models of mean curvature flow, the properties of translating solitons may be of importance to study. In particular, the Bernstein type theorems of translating solitons are important.

There are relatively few results about the translating solitons. Let us just mention a few. In [10], X.-J.Wang studies symmetric properties of

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the convex graphic solitons; in [8], L. Ma studies the stability of the Grim Reaper, which is a translating soliton to the curve shortening flow in the plane; in the series of papers [12, 13, 14] X. H. Nguyen constructs new examples of translating solitons; in a very recent paper [10], F. Martin, A. Savas.Halilaj and K. Smoczyk give, among others, some rigidity theorems for the hyperplanes and the Grim-Reaper planes and topological obstructions to their existence.

Let $S: \mathbb{R}^{n+1} \to \mathbb{R}$ be the function defined by $S(x) = <x, \omega>$. Then the soliton equation (1) can be written as

\begin{equation}
H = D_\nu S \text{ in } M,
\end{equation}

where $D$ denotes the usual directional derivative in $\mathbb{R}^{n+1}$.

Let $g$ be the metric induced on $M$ by the standard euclidean metric $< , >$ on $\mathbb{R}^{n+1}$, and $dv_g$ the volume form induced by the metric $g$ on $M$. By $\nabla$ we shall denote the Levi-Civita connection induced on $M$ by its metric $g$, and also the gradient and the differential of a function $f: M \to \mathbb{R}$.

With the exception of section 6 from now on we will suppose that $(M, g)$ is connected and complete.

Our first contributions about the properties of translating solitons are the following two easy observations:

**Proposition 1.** Assume that $|\nabla S(x)| \in L^1(M, dv_g)$, where $(M, g)$ is a translating soliton in $\mathbb{R}^{n+1}$. Then $M$ is a hyperplane.

**Proposition 2.** Assume that $n = 2$, $(M^2, g)$ is a simply connected translating soliton and $S(x) \leq 0$ for every $x \in M$. Then $S$ is constant on $M$ and $M$ is a plane in $\mathbb{R}^3$.

In the next proposition we shall use the notation $D_\nu H$, where $\nu$ is normal to the submanifold $M$ and $H$ is a function on $M$. This has no sense in general, but in case of translating solitons, because they are a temporal slice of a flow, we can consider the foliation given on an open set if $\mathbb{R}^{n+1}$ by the union of the temporal slices of the flow, which gives an extension of $H$ to a open set of $\mathbb{R}^{n+1}$ and gives sense to $D_\nu H$.

**Proposition 3.** Assume that $n = 2$, $(M^2, g)$ is a simply connected mean convex translating soliton and $D_\nu H \leq 0$. Then $M$ is a plane in $\mathbb{R}^3$.

These propositions are our motivation to look for other conditions giving Bernstein type theorems. For most of those theorems we shall use on $M$ the measure

\[ d\mu = e^{S(x)} dv_g. \]

With this measure, the same ideas used to prove the propositions above will give the following result.

**Proposition 4.** Assume that $H \geq 0$ and $|\nabla H| \in L^1(M, d\mu)$, where $(M, g)$ is a translating soliton in $\mathbb{R}^{n+1}$. Then $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is a minimal hypersurface in $\mathbb{R}^n$. 

Generally speaking the condition that $|\nabla H| \in L^1(M, d\mu)$ is very restrictive and one may try to find other conditions weaker but still related to the second fundamental form of the translating solitons.

Before to state our other results, we shall declare our notation. $h$ will denote the second fundamental form of $M$ associated to the election of $\nu$ and defined as $h(X, Y) = g(D_X Y, \nu) = -g(D_X \nu, Y)$, and $AX = -D_X \nu$ will denote the associated shape operator.

With this notation we have $\vec{H}(F) = H \nu$, $H = \text{tr} A$ is the mean curvature function on the surface $F(x)$.

Using ideas from the paper [15], we shall use these facts to prove the following result.

**Theorem 5.** Assume that $M^n \subset \mathbb{R}^{n+1}$ is a mean convex (that is $H \geq 0$) translating soliton of dimension $n \leq 6$ which satisfies the conditions

1. For every $x \in M$, there is an orthonormal basis $\{e_i\}$ of $T_x M$ formed by principal vectors $e_i$ of $M$ at $x$ such that
   \[
   \sum_j \nabla_j h_{jj} \nabla_j H \leq \frac{n+1}{2n} |\nabla H|^2, \quad \text{and}
   \]
2. If $B_R$ denotes a ball of radius $R$ in $\mathbb{R}^{n+1}$, $\mu(M \cap B_R) \leq CR^5$ for some uniform constant $C > 0$ and every $R > 0$. Then $M$ is a hyperplane.

We shall comment how strong is condition (1) in section 4.

The following two results are proved following the same lines that in the proof of the above theorem or as corollaries of it.

**Theorem 6.** Assume that $M^n \subset \mathbb{R}^{n+1}$ is a mean convex translating soliton which satisfies (3). Assume further that

\[
\int_M |\hat{A}|^2 d\mu < \infty. \quad \text{(where $\hat{A}$ is the traceless part of $A$)}
\]

Then $M$ is a hyperplane.

**Theorem 7.** Assume that $M^n \subset \mathbb{R}^{n+1}$ is a mean convex translating soliton which satisfies (3). Assume further that $M$ is the graph of a function $u = u(x_1, ..., x_n)$, where the coordinate $x_{n+1}$ is in the direction of $\omega$, $n \leq 5$ and $|u| \leq C$ for some uniform constant $C > 0$. Then $M$ is a hyperplane.

Finally we state a theorem which is true also in higher codimension. In this case the equation (1) for the mean curvature vector of a soliton becomes

\[
\vec{H} = \omega^\perp
\]

where $\omega^\perp$ is the projection on the normal bundle of the submanifold $M$ of a unit vector $\omega \in \mathbb{R}^{n+k}$. In this case, instead of the scalar second fundamental form $h$, we shall use the vectorial second fundamental form $\alpha$ defined by $\alpha(X, Y) = (D_X Y)^\perp$. 


**Theorem 8.** Let $M^n \to \mathbb{R}^{n+k}$ be a translating soliton, $n \geq 2$. There exists a constant $\epsilon_0 > 0$ such that if

$$
\int_M |\alpha|^n dv_g \leq \epsilon_0,
$$

then $M^n$ is a $n$-dimensional plane.

Here is the plan of the paper. In next section we propose some elementary properties of the translating solitons and prove propositions 1 to 3. Sections 3 and 4 are dedicated to prove technical lemmas that will be used to prove Theorem 4 in section 5, where we also prove theorems 5 and 6. Section 6 is dedicated to the proof of Theorem 8 and related results on the compactness of the space of translating solitons with bounded total curvature and a decay result.

2. **Elementary properties of translating solitons**

From (1) it is easy to compute the Hessian of $S$ in $M$. If $X, Y$ are vector fields on $M$ such that $\nabla X Y(x) = 0$,

$$
\nabla^2 S(X, Y)(x) = \langle D_X Y, \omega \rangle = H \langle \nu, \omega \rangle = H h(X, Y).
$$

Then, if $\Delta$ denotes the Laplacian on $M$,

$$
\Delta S(x) = H \langle \nu, \omega \rangle = H^2.
$$

From the definitions of $S$ and $\nabla$, and [2], follows that $DS(X) = S(X) = \langle \omega, X \rangle$ for every vector $X$ in $\mathbb{R}^{n+1}$ and

$$
|\nabla S|^2 = |\omega|^2, \quad |DS|^2 = H^2 + |\nabla S|^2 = |\omega|^2 = 1.
$$

Then know that $M$ is non-compact. For otherwise, assuming $M$ is compact, we know that $S(x)$ attains its maximum at some point $x_0 \in M$, where $\nabla S(x) = 0$, which implies that $|H(x_0)| = 1$, and $H^2(x_0) = \Delta S(x_0) \leq 0$ and then we have $H(x_0) = 0$, a contradiction. Hence, $M$ is non-compact.

From (8) one immediately gets Proposition 4. In fact,

$$
\int_{B_R \cap M} H^2 dv_g = \int_{\partial B_R \cap M} \langle \nabla S(x), \nu \rangle \leq \int_{\partial B_R \cap M} |\nabla S|,
$$

but, because $|\nabla S(x)| \in L^1(M, dv_g)$, for suitable $R \to \infty$, $\int_{\partial B_R \cap M} |\nabla S| \to 0$. Then $H = 0$, that implies that $S(x) = \langle x, \omega \rangle$ is a harmonic function in $L^1(M)$. Then $M$ must be an hyperplane.

By the uniformization theorem (cf. [5], is $M$ is simply connected, it is conformal to the euclidean plane. Since there is no non-positive subharmonic function on $R^2$, we immediately obtain Proposition 2 from the equation (8).

Now, we shall give the
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Proof. of Proposition 3. Since $M$ is the 0-slice of the flow $\partial_t F = H\nu$, one has, having into account the evolution of $H$ under mean curvature flow (cf. [4]), that

$$0 \geq D_\nu H = (1/H)\partial_t H = (1/H)(\Delta H + |A|^2 H).$$

Then, since $H \geq 0$, either $H = 0$ or $\Delta H \leq -|A|^2 H < 0$, that is, $H$ is a positive superharmonic function, which is impossible, because, by the uniformization theorem, $M$ is conformal to the plane, and the existence of a positive superharmonic function on $M$ would give a positive superharmonic function on the euclidean plane. Then $<\omega, N> = H = 0$, which implies that $M$ is a ruled surface containing lines of the form $p + t\omega$, which are parallel in $\mathbb{R}^3$, that is, is a generalized cylinder $C \times \mathbb{R}$ which is minimal in $\mathbb{R}^3$, then $C$ must be a geodesic in $\mathbb{R}^2$ and $M$ must be a plane. □

Proof. of Proposition 4. Recall from [4] that for the flow $F(x,t)$, the mean curvature functions $H(x,t)$ satisfy

$$\partial_t H = \Delta H + |A|^2 H$$

where $\Delta$ is the induced Laplacian operator on the hypersurface $F(x,t)$.

Since $F(x,t) = F_0(x) + t\omega$ on the translating soliton, we have $\partial_t H(x,0) = -\nabla_{\omega^\top} H(x)$ for every $x \in M$, where $\omega^\top$ is the component of $\omega$ tangent to $M$. We then derive from (10) the following elliptic equation for the mean curvature function $H$ on $M$:

$$\Delta H + \nabla_{\omega^\top} H + |A|^2 H = 0.$$

which can be written as

$$LH = -|A|^2 H, \quad \text{with } Lf = \Delta f + \nabla_{\omega^\top} f$$

The advantage of using the operator $L$ is that

$$\text{div}(e^S \nabla f) = Lf \ e^S,$$

which allows us to use the divergence theorem under the form:

$$\int_{\Omega} Lf \ d\mu = -\int_{\partial\Omega} <N, \nabla f> \ d\mu \text{ and}$$

$$\int_{M} g \ Lf \mu = -\int_{M} <\nabla f, \nabla g> \ d\mu \text{ for } g \in C_o(M).$$

By (12) and (14) one immediately gets that

$$\int_{BR \cap M} |A|^2 H d\mu = \int_{\partial BR \cap M} <\nabla H(x), N> \leq \int_{\partial BR \cap M} |\nabla S| \to 0$$

for suitable $R \to \infty$, and, as in the proof of Proposition 4, $0 = H = <\omega, \nu>$. This implies that $M$ is a ruled surface containing lines of the form $p + t\omega$, which are parallel in $\mathbb{R}^{n+1}$, that is, is a generalized cylinder $\Sigma \times \mathbb{R}$ which is minimal in $\mathbb{R}^{n+1}$, then $\Sigma$ must be minimal in $\mathbb{R}^n$. □
3. A Basic lemma

In this section we give a basic analytic lemma which will be used to give a Bernstein type theorem.

Lemma 9. Assume that $M^n \subset \mathbb{R}^{n+1}$ is a hypersurface such that there are positive functions $u > 0$ and $B > 0$ on $M$ such that
\begin{equation}
ul u + Bu^2 \geq c_0 |\nabla u|^2
\end{equation}
for some uniform constant $c_0 > 0$ and with the “stability” condition
\begin{equation}
\int_M (|\nabla \phi|^2 - B\phi^2) \, d\mu \geq 0
\end{equation}
for any $\phi \in C_0^2(M)$. Assume that $B \geq b u^2$ for some constant $b > 0$. Then, for any $\eta \in C_0^2(M)$, there is a small $\varepsilon > 0$ such that for every $p \in \left[4 - 2\sqrt{\frac{c_0}{1 + \varepsilon}}, 4 + 2\sqrt{\frac{c_0}{1 + \varepsilon}}\right]$, there is a constant $C(n, p)$ such that
\begin{equation}
\int_M u^p \eta^p \, d\mu \leq C(n, p) \int_M (|\nabla \eta|^p) \, d\mu
\end{equation}

Proof. We shall follow the arguments in [15]. In below, all the integrals are along $M$ and with respect to the measure $d\mu$. Moreover we shall use the divergence formulas (14).

Let us take $\phi = u^{1+2q} \eta$ in the stability inequality (16). We get that
\begin{equation}
\int_M Bu^{2+2q}\eta^2 \leq \int (1 + q)^2 u^{2q}|\nabla u|^2 \eta^2 + \int u^{2+2q}|\nabla \eta|^2
\end{equation}
\begin{equation}
+ 2(1 + q) \int u^{1+2q} \eta < \nabla u, \nabla \eta >.
\end{equation}
\begin{equation}
= \int u^{2q}(1 + q)\eta \nabla u + u \nabla \eta |^2
\end{equation}

Multiplying (15) by $\eta^2 u^{2q}$ and integrating over $M$ using the divergence formula, having into account that $\eta$ has compact support, we get that
\begin{equation}
\int c_0 u^{2q} \eta^2 |\nabla u|^2 \leq -(1 + 2q) \int u^{2q}|\nabla u|^2 \eta^2 + \int Bu^{2+2q} \eta^2
\end{equation}
\begin{equation}
- 2 \int u^{1+2q} \eta < \nabla u, \nabla \eta >.
\end{equation}

Applying (18) to the above inequality we have
\begin{equation}
\int c_0 u^{2q} \eta^2 |\nabla u|^2 \leq q^2 \int u^{2q}|\nabla u|^2 \eta^2 + 2q \int u^{1+2q} \eta < \nabla u, \nabla \eta > + \int u^{2q}|\eta \nabla u + u \nabla \eta|^2
\end{equation}

Using the Cauchy-Schwartz, and Young’s inequalities
\begin{equation}
2 \int u^{1+2q} \eta < \nabla u, \nabla \eta > \leq \varepsilon q^2 \int \eta^2 u^{2q}|\nabla u|^2 + \varepsilon^{-1} \int u^{2+2q}|\nabla \eta|^2,
\end{equation}
we get that
\begin{equation}
[c_0 - (1 + \epsilon)q^2] \int u^{2q} |\nabla u|^2 \eta \leq (1 + \epsilon^{-1}) \int u^{2q+2} |\nabla \eta|^2.
\end{equation}

Let us take \( p = 2q + 4 > 0 \) (which implies \( p > 4 \) in order \( q > 0 \)) in (18) and \( q^2 < c_0 \). Choose \( \epsilon > 0 \) small such that \( c_0 - (1 + \epsilon)q^2 > 0 \). Then
\begin{equation}
\int u^{p-4} |\nabla u|^2 \eta^2 \leq C \int u^{p-2} |\nabla \eta|^2
\end{equation}
for some constant \( C > 0 \).

Using the fact \( B \geq bu^2 \), (19) and the inequality \(|x + y|^2 \leq 2|x|^2 + 2|y|^2\),
\begin{equation}
b \int u^p \eta^2 \leq \int u^{p-2} B \eta^2 \leq u^{p-4}(1 + q)\eta \nabla u + u \nabla \eta^2
\leq \int 2u^{p-4} ((1 + q)^2 \eta^2 |\nabla u|^2 + u^2 |\nabla \eta|^2)
\end{equation}
Applying now (21), we obtain
\begin{equation}
b \int u^p \eta^2 \leq 2 C ((1 + q)^2 + 1) \int u^{p-2} \nabla \eta^2
\end{equation}
Now we use the the Young’s inequality
\begin{equation}
u^{p-2} |\nabla \eta|^2 = u^{p-2} \eta^{2(p-2)/p} \frac{|\nabla \eta|^2}{\eta^{2(p-2)/p}} \leq \delta u^p \eta^2 + C \frac{|\nabla \eta|^p}{\eta^{p-2}},
\end{equation}
and the substitution of this inequality in (22) gives
\begin{equation}
\int u^p \eta^2 \leq C_1 \int (|\nabla \eta|^p \eta^{2-p})
\end{equation}
for some constant \( C_1 \). Replacing \( \eta \) by \( \eta^{p/2} \) we then get (17). \( \square \)

4. Kato type inequality and a consequence

We are following, for translating solitons, the ideas in [15] for minimal surfaces. In that paper, a basic tool in the study of minimal hypersurfaces is Kato’s inequality \(|\nabla A|^2 \geq (1 + \frac{2}{n}) |\nabla |A||^2\). In this section we shall obtain a similar inequality for the traceless part \( \tilde{A} = A - \frac{1}{n} \text{tr} A I \) of \( A \), under the hypothesis (3).

**Lemma 10.** Let \( M^n \) be a hypersurface of \( \mathbb{R}^{n+1} \) satisfying (3), then
\begin{equation}
|\nabla \tilde{A}|^2 \geq (1 + \frac{1}{n}) |\nabla |\tilde{A}||^2.
\end{equation}

**Proof.** We shall identify \( M \) and \( F(M) \). For any \( p \in M \), we choose a local orthonormal frame \( \{e_1, ..., e_n\} \) such that, at \( p \), \( \nabla e_i e_j(p) = 0 \) and \( \{e_1(p), ..., e_n(p)\} \) are eigenvectors of the shape operator \( A \), that is, \( ae_i = k_i e_i \).
Let us remark that this implies that \( \tilde{A} e_i = \left(k_i - \frac{H}{n}\right) e_i \), that is, \( \{e_i\} \) are also eigenvectors of \( \tilde{A} \).
We can choose the \( \{e_i\} \) such that, at \( p \), coincide with those in the hypothesis (3).

At \( p \in M, \nabla h^0 \) is in the vector space of 3-covariant tensors \( T \) on \( T_p M \) satisfying \( \sum_i T(X, e_i, e_i) = 0 \) for every \( X \in T_p M \). If we consider on this space the natural metric induced by \( <,> \), we can choose an orthonormal basis of it formed by the tensors

\[
\theta^i \otimes \frac{h^0}{|h|}, \quad \theta^i \otimes S^{jk}, \quad \theta^i \otimes T^\ell, \quad 1 \leq i \leq n, 1 \leq j \leq k \leq n, 2 \leq \ell \leq n - 1,
\]

where \( \{\theta^i\} \) is the dual basis of \( \{e_i\} \), \( S^{jk} \) are the 2-covariant symmetric tensors whose matrix in the basis \( \{e_i\} \) is

\[
S^{jk}_{rs} = \frac{1}{\sqrt{2}} \left( \delta^j_r \delta^k_s + \delta^j_s \delta^k_r \right)_{1 \leq r, s \leq n},
\]

and \( T^\ell \) are other 2-covariant tensors necessary to complete the basis. Using this basis we can write

\[
|\nabla h^0|_2 = |\nabla A|_2
\]

\[
= \sum_i <\nabla h^0, \frac{h^0}{|h|}>^2 + \sum_{i<j<k} \frac{1}{2} (\nabla_i h^0 j k)^2 + \sum_{i,\ell} <\nabla_i h^0, T^\ell>^2
\]

But we have the following expressions for the first and second summands in the last term of the above equalities

\[
\sum_i <\nabla h^0, \frac{h^0}{|h|}>^2 = |\nabla |h^0||_2 = |\nabla |A||_2
\]

\[
\sum_{i<j<k} \frac{1}{2} (\nabla_i h^0 j k)^2 = \sum_{i,j \neq k} (\nabla_i h^0 j k)^2 \geq \sum_{j \neq k} (\nabla_j h^0 j k)^2
\]

\[
= \sum_{j \neq k} (\nabla_j h^0 j k - \frac{1}{n} \nabla_j H \delta_j k)^2 = \sum_{j \neq k} (\nabla_j h^0 j j)^2
\]

\[
= \sum_{j \neq k} (\nabla_j h^0 j j + \frac{1}{n} \nabla_j H g_{jj})^2
\]

\[
= \sum_{j \neq k} (\nabla_j h^0 j j)^2 + \frac{n-1}{n^2} |\nabla H|^2 + \frac{2}{n} \sum_{j \neq k} \nabla_j H \nabla_k h^0 j j
\]

\[
= \sum_{j \neq k} (\nabla_j h^0 j j)^2 + \frac{n-1}{n^2} |\nabla H|^2 + \frac{2}{n} |\nabla H|^2 - \frac{2}{n} \sum_j \nabla_j H \nabla_j h^0 j j
\]

(25)

where we have used the Codazzi equation for the third equality. Now, let us observe that (3) is equivalent to

\[
\sum_j \nabla_j H \nabla_j h^0 j j \leq \frac{n-1}{2n} |\nabla H|^2,
\]
then, by substitution of this inequality in (26), we obtain
\[
\sum_{i,j<k} \frac{1}{2} (\nabla_i h_{jk})^2 = \sum_{i,j<k} (\nabla_i h_{jk})^2 \geq \sum_{j\neq k} (\nabla_j h_{jk})^2 \geq \sum_{j\neq k} (\nabla_j h_{jj})^2 + 2n|\nabla H|^2 \geq \sum_{j\neq k} (\nabla_j h_{jj})^2 \tag{27}
\]
Moreover, since \( \hat{h} \) diagonalizes in the basis \( \{e_i\} \),
\[
|\nabla h| = \sum_k < \nabla_k \hat{h}, \hat{h}> = \sum_k \frac{1}{|\hat{h}|^2} (\sum_j (\nabla_k \hat{h}_{jj})^2) \leq \sum_k (\nabla_k \hat{h}_{jj})^2 = \sum_{k\neq j} (\nabla_k \hat{h}_{jj})^2 + \sum_j (\nabla_k \hat{h}_{jj})^2 \leq \sum_{k\neq j} (\nabla_k \hat{h}_{jj})^2 + \sum_j (n-1) \sum_{i\neq j} (\nabla_i \hat{h}_{jj})^2 = n \sum_{k\neq j} (\nabla_k \hat{h}_{jj})^2 \tag{28}
\]
From (24), (27) and (28), we obtain (23). \( \square \)

As a corollary, we obtain

**Lemma 11.** Let \( M^n \) be a translating soliton of \( \mathbb{R}^{n+1} \) satisfying (3), then
\[
|\hat{A}|L|\hat{A}| + |A|^2|\hat{A}|^2 \geq \frac{1}{n} |\nabla |\hat{A}|^2 |. \tag{29}
\]

**Proof.** The evolution formula for the second fundamental form \( h \) under mean curvature flow is (see corollary 3.5 in [4])
\[
\partial_t |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4
\]
Since \( F(x,t) = F_0(x) + t \omega \) on the translating soliton, we have \( \partial_t |A|^2(x,0) = -\nabla_{\omega^\dagger} |A|^2(x) \) for every \( x \in M \). Hence we have
\[
\Delta |A|^2 + \nabla_{\omega^\dagger} |A|^2 = 2|\nabla A|^2 - 2|A|^4 \tag{30}
\]
For the laplacian of
\[
|\hat{A}|^2 = |A - \frac{1}{n} H I|^2 = |A|^2 - \frac{1}{n} H^2,
\]
using (12) and (30), we obtain the following formula
\[
\Delta |\hat{A}|^2 + \nabla_{\omega^\dagger} |\hat{A}|^2 = 2|\nabla \hat{A}|^2 - 2|A|^2|\hat{A}|^2 \tag{31}
\]
Then \(^{(29)}\) follows from this equality and inequality \(^{(23)}\). \(\Box\)

Some remarks on condition \(^{(3)}\). This condition appears as a necessary technical condition to have the Kato’s type inequality \(^{(23)}\) for the tensor \(\hat{A}\).

Here we want to grasp a little this condition by looking at its meaning in two simple families of hypersurfaces.

The first are surfaces \(\Gamma \times \mathbb{R}^{n-1}\), with \(\Gamma\) a curve in \(\mathbb{R}^2\). If we denote by \(e_1\) the unit vector tangent to \(\Gamma\), any orthonormal basis used to write condition \(^{(3)}\) contains \(e_1\), and \(H = k\), the curvature of the curve \(\Gamma\) in \(\mathbb{R}^2\).

Then condition \(^{(3)}\) just states \(|D_{e_1}k|^2 \leq \frac{n+1}{2n}|D_{e_1}k|^2\), which is true if and only if \(n = 1\) or \(\Gamma\) is a circle or a line.

The second family to consider are revolution surfaces obtained by the rotation of a curve \(c(s) = (x_1(s), x_{n+1}(s))\), parametrized respect its arc length \(s\), in the plane \(\{x_1, x_{n+1}\}\) around the axis \(X_{n+1}\). In this case we are forced to take as one the the vectors (say \(e_1\)) of the orthonormal basis the unit vector tangent to \(c(s)\) (or those obtained by rotation of it), because this curve is a curvature line. For such a revolution surface, \(h_{11} = k\), the curvature of \(c\), and, for \(j \neq 1\), \(h_{jj} = \frac{1}{x(s)} < (-1, 0), (-x'_n + 1(s), x'_1(s)) >= \frac{x'_n + 1(s)}{x(s)}.\) Both curvatures have derivative 0 in the directions \(e_j, j \geq 2\), then condition \(^{(3)}\) means that \((D_{e_1}k)D_{e_1}(k + (n - 1)\frac{x'_n + 1(s)}{x(s)}) \leq \frac{n + 1}{2n}(D_{e_1}(k + (n - 1)\frac{x'_n + 1(s)}{x(s)}))^2\), that is \(\frac{n - 1}{2n}k'(s)^2 \leq \frac{n - 1}{n}k(s)\left(\frac{x'_n + 1}{x}\right)'(s) + \frac{n + 1}{2n}\left(\frac{x'_n + 1}{x}\right)'(s)^2.\) Obviously this inequality is satisfied when \(c\) is a line or a circle. Explicit computations with concrete functions show that it is satisfied, for instance, when \(c\) is the graph of the functions \(x_{n+1}(x_1) = \sqrt{x_1}, x_1^2\) for \(n \leq 7, \sinh(x_1), \cosh(x_1)\).

5. Proof of Theorem \(^{5}\) and related

Proof. of Theorem \(^{5}\). We plan to show that under the assumption \(H \geq 0\), we must have \(\hat{A} = 0\), which implies that \(M\) is umbilical everywhere on \(M\) and, because \(M\) is complete and non compact, this tells that \(M\) is a hyperplane.

Translating solitons are critical points of the volume functional \(V(\Omega) = \int_\Omega d\mu, \Omega \subset M.\) A computation of the second variation formula similar to which is done for minimal surfaces shows that a translating soliton \(M\) is stable for the functional \(V\) if and only if, for any \(\phi \in C^2(\Omega),\)

\[
\int_M (|\nabla \phi|^2 - |A|^2 \phi^2) \, d\mu \geq 0.
\]

We remark that for any nontrivial mean convex soliton, that is, \(H \geq 0\), by maximum principle applied to \(^{11}\) we must have \(H > 0\). This then implies the stability condition \(^{(32)}\), as follows by an standard argument (for instance, see the argument in pages 46-47 of \(^{3}\) for minimal submanifolds).
Then, the function $B = |A|^2$ satisfies (16). Moreover, by the hypothesis (3) and Lemma 11, $u = \overset{\circ}{A}$ satisfies (29), that is, satisfies (15) with $c_0 = 1/n$, and $B \geq bu^2$ with $b=1$. Then we can apply the Basic Lemma 9 to conclude (17).

Let $r(x)$ denote the distance from $x$ to the origin in $\mathbb{R}^{n+1}$. In (17), choose $\eta = \eta(r(x))$ to be the cut-off function defined by

$$\eta(r) = 1, \text{ if } r \leq \theta R; \quad \eta(r) = 0, \text{ for } r \geq R, \text{ and linear for } r \in [\theta R, R]$$

where $\theta \in (0, 1)$ is a any fixed constant. Take $p = 4 + 2\sqrt{2/n}$, its almost maximal value in Lemma 5. For $n \leq 6$, $6 > p > 5$. Substitution of these values in (17) gives

$$\int_{B_p(\theta R)} |\overset{\circ}{A}|^p d\mu \leq \frac{C}{(1 - \theta)^p R^p} \mu(B_p(R))$$

From the second hypothesis of Theorem 5, $R^{-p} \mu(B_p(R)) \to 0$ as $R \to \infty$. We then have $\int |\overset{\circ}{A}|^p d\mu = 0$. This implies that $\overset{\circ}{A} = 0$. \hfill \Box

**Proof. of Theorem 6.** The proof is quite similar to the one given above. Since we still have condition (3) and mean convexity, we can apply again the the computations in the proof of Lemma 9, with the function $\eta$ defined in (33) and $q = 0$ in (20) to obtain

$$\int_{B_{R/2}} |\nabla |\overset{\circ}{A}||^2 d\mu \leq cR^{-2} \int_M |\overset{\circ}{A}|^2 d\mu$$

which goes to 0 as $R \to \infty$ because of the hypothesis (5). Hence we have

$$\nabla |\overset{\circ}{A}| = 0.$$ 

Then $|\overset{\circ}{A}|$ is constant. This and hypothesis (5) tell us that that either $\overset{\circ}{A} = 0$ or $M$ has finite $S$-volume. But last possibility implies that also the last hypothesis in Theorem 5 is satisfied, and we can obtain (without restrictions on the dimension $n$) the inequality (34) for $p = 4$, and $|\overset{\circ}{A}| = 0$ as in the proof of that theorem. Then $\overset{\circ}{A} = 0$ and $M$ is a hyperplane. \hfill \Box

**Proof. of Theorem 7.** It is a consequence of Theorem 5 and the following lemma..

**Lemma 12.** Assume that the coordinate $x_{n+1}$ is in the direction of $\omega$ and $M$ is a $n$-dimensional translating soliton is given as the graph of the graph of a function $u(x_1, ..., x_n)$, and $|u| \leq C$ for some constant $C$. Then

$$\mu(M \cap B_R) \leq C_n e^C R^n$$

where $C_n$ is the euclidean volume of the ball of radius 1 in $\mathbb{R}^n$. 


Proof. Since we are considering on $M$ the measure $\mu$ defined by the measure element $d\mu = e^S dv$, it is convenient to use on $\mathbb{R}^{n+1}$ the measure $\overline{\mu}$ defined by the measure element $d\overline{\mu} = e^S dv$, where $dv$ is the standard euclidean volume element. Both measures are related, as their corresponding volume elements, by $d\mu = \iota_*d\overline{\mu}$. Since $M$ is the graph of a function $u$, we can extend $d\mu$ over all $\mathbb{R}^{n+1}$ by $d\mu(x_1,\ldots,x_n,x_{n+1}) = d\mu(x_1,\ldots,x_n,u(x_1,\ldots,x_n))$. This has as a consequence that

$$d(d\mu)(x_1,\ldots,x_n,x_{n+1}) = d(d\mu)(x_1,\ldots,x_n,u(x_1,\ldots,x_n)) = d(\iota_*d\overline{\mu}) = \text{div}_S \nu \overline{\mu} = (H + \langle \nu, \omega \rangle)\overline{\mu} = 2 < \nu, \omega > \overline{\mu}.$$

Given $R > 0$ and a ball $B_R(p)$, let us choose the coordinate system centred at $p$. Let $D_R$ be the ball of radius $R$ in $\mathbb{R}^n$. Let $V$ be the domain in $\mathbb{R}^{n+1}$ bounded by $M$, $\partial D_R \times R$ and $D_R \times \{-C\}$. The application of the Stokes theorem to the integration of the extended form $d\mu$ over the boundary $\partial V$ of $V$ gives

$$\mu(M \cap D_R \times R) = \int_{M \cap \partial V} d\mu = -\int_{\partial V - M} d\mu + \int_V 2 < \nu, \omega > \overline{\mu}.$$

Note that $V$ is contained in $D_R \times [-C,C]$, $\partial V - M \subset D_R \times \{-C\} \cup \partial D_R \times [-C,C]$, and $d\mu$ restricted to $D_R \times \{-C\} \cup \partial D_R \times [-C,C]$ is lower or equal than $e^C|dv|$, where $dv$ is the standard volume element on that hypersurface. Then we have that the right side of above equality can be bounded by

$$e^C (\text{vol}(\partial D_R \times [-C,C]) + \text{vol}(D_R \times \{-C\}) + \text{vol}(D_R \times [-C,C]))$$

$$\leq e^C (2\omega_{n-1}R^{n-1}C + \omega_n R^n + 4\omega_n R^n C)$$

$$= e^C (2\omega_{n-1}C + \omega_n (1 + 4C)R) R^{n-1}.$$

This completes the proof of Lemma 12. \qed

Combining this result with Theorem 12 we have Theorem 7. \qed

6. COMPACTNESS AND GAP RESULTS

The purpose of this section is to prove Theorem 8. This result is the analog, for translating solitons, of Corollary 2.3 in [1] for minimal submanifolds of $\mathbb{R}^n$. The proof of it and the necessary preliminary results follow the same arguments than in that paper and, some parts, in [2]. Then we’ll indicate only the points where the condition of being minimal is used in [1] or [2] and how things still work when $\hat{H} = 0$ is changed by condition [4].

The proof of Theorem 8 relies on the following sequence of lemmas, where $N = n + k$.

Lemma 13 (Compactness Theorem). Let $\{M_j^n\}$ be a sequence of connected translating solitons in $B^N(1)$ such that $\partial M_j^n \cap B^N(1) = \emptyset$. Suppose that there is an uniform constant $C > 0$ such that $\sup |\alpha_j(x)| \leq C$ for all $j$. Then there is a subsequence of $(M_j)$, still denoted by $(M_j)$, that converges in the
$C^\infty_{\text{loc}}$ norm to a smooth translating soliton $M_\infty$ in $B^N(1)$ with $\sup |\alpha_\infty|(x) \leq C$.

This Lemma is stated and proved for minimal submanifolds (in different situations) in [1], [2], and [6]. The proof of [1] and [2] works also for translating solitons. In fact, the condition $H = 0$ is used to state that, locally, $M_j$ can be written as a graph of a function $f_j$ satisfying the elliptic equations system $\mathcal{M}(f_j) = 0$, where $\mathcal{M}$ is the operator giving the mean curvature of the graph of $f_j$. For translating solitons we only need to change this equation by $\mathcal{M}(f_j) = \omega_j^\perp$, which is still an elliptic system, with $|\omega_j^\perp| \leq 1$, and the rest of the arguments is like in the quoted papers.

We can get the following two assertions. The first one is the Heinz type estimate.

**Lemma 14.** There is a constant $\epsilon_0 > 0$ such that if $F : M \to \mathbb{R}^N$ is a translating soliton with $D_1(p) \cap \partial M = \emptyset$ for some $p \in M$ and with

$$\int_{D_1(p)} |\alpha|^p \, dv_g \leq \epsilon_0,$$

then

$$\sup_{s \in [0,1]} \left[ s^2 \sup_{D_{1-s}(p)} |\alpha|^2 \right] \leq 4.$$

The proof follows exactly the arguments of the proof Assertion (S') in [1]. Minimality condition in that argument is used to apply compactness theorems of a sequence of immersions $\tilde{F}_i : M_i \to \mathbb{R}^N$ which are rescalings $\tilde{F}_i = |\alpha|(y_i)F_i$ of minimal immersions $F_i$. In our case, if the $F_i$ are translating solitons, the mean curvature $\tilde{H}_i$ of the rescaled immersion will satisfy $\tilde{H}_i = \omega_i^\perp$. If $|\alpha|(y_i) \geq 1$, then $|\omega_i^\perp| \leq 1$, and the compactness theorem [14] is still true on this family. If $|\alpha|(y_i) \leq 1$ for some $i$, we can still have a family satisfying the hypothesis of the compactness theorem taking $\tilde{F}_i = F_i$ for this $i$. With this small change in the definition of the $\tilde{F}_i$, the argument in [1] works also here.

**Lemma 15.** For any small constant $\epsilon_0 > 0$ there is a constant $\delta > 0$ such that if $F : M \to \mathbb{R}^N$ is a translating soliton with $D_1(p) \cap \partial M = \emptyset$ for some $p \in M$ and with

$$\int_{D_1(p)} |\alpha|^p \, dv_g \leq \epsilon_0,$$

then

$$\sup_{D_{1/2}(p)} |\alpha|^2 \leq \delta.$$

This Lemmas can be proved from the above one like in [1] with no change. Also the proof of Theorem 8 follows from Lemma [?] like in [1] with no change.
Once we know these lemmas, the following results are proved following exactly the same arguments that in [1]

**Proposition 16.** Let \( M^n \rightarrow \mathbb{R}^N \) be a complete translating soliton, \( n \geq 2 \) with \( \int_M |\alpha|^n < \infty \). Fix \( 0 \in M \). Then there is a uniform constant \( R_0 > 0 \) such that

\[
\sup_{x \in \partial B_R(0)} |\alpha|^2(x) \leq R^{-2}\lambda(\int_{B(R/2, 2R)} |\alpha|^n dv_g)
\]

for all \( R \geq R_0 \), where \( \lambda(\epsilon) \to 0 \) as \( \epsilon \to 0 \) and \( B(R/2, 2R) = B_2R(0) - B_R(0) \).

We remark that using the Sobolev inequality [7] we have the following gap result.

**Theorem 17.** Let \( M^n \rightarrow \mathbb{R}^N \) be a complete translating soliton, \( n \geq 2 \). There is a small positive constant \( \epsilon(n) \) such that if for some \( p \in (1, n) \),

\[
\int_M |\alpha|^p dv_g \leq \epsilon(n),
\]

then \( \alpha = 0 \), that is, \( M \) is a \( n \)-plane.

The proof of above result is similar to that of Theorem [8] We only give an outline of the proof. In fact, one can use Moser’s iteration argument and the Sobolev inequality [7] to conclude that \( |\alpha| \) is uniformly bounded, saying \( |\alpha| \leq C \) for some uniform constant \( C > 0 \). Then we have

\[
\int_M |A|^n dv \leq C^{n-p} \int_M |A|^p dv \leq C(n).
\]

Then we can apply Theorem [8] to get the conclusion of Theorem [17].

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