Inefficiency of Data Augmentation for Large Sample Imbalanced Data

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Abstract

Many modern applications collect large sample size and highly imbalanced categorical data, with some categories being relatively rare. Bayesian hierarchical models are well motivated in such settings in providing an approach to borrow information to combat data sparsity, while quantifying uncertainty in estimation. However, a fundamental problem is scaling up posterior computation to massive sample sizes. In categorical data models, posterior computation commonly relies on data augmentation Gibbs sampling. In this article, we study computational efficiency of such algorithms in a large sample imbalanced regime, showing that mixing is extremely poor, with a spectral gap that converges to zero at a rate proportional to the square root of sample size or faster. This theoretical result is verified with empirical performance in simulations and an application to a computational advertising data set. In contrast, algorithms that bypass data augmentation show rapid mixing on the same dataset.

Keywords: Bayesian; Gibbs sampling; large sample; Markov chain Monte Carlo; mixing; Pólya-gamma; rare events; sparse data.

1 Introduction

Highly imbalanced categorical data with moderate to large sample sizes are commonplace in modern statistical applications. For example, in genetic epidemiology, it is standard to encounter a rare disease state as an outcome and rare genetic variants as predictors, with perhaps tens of thousands of subjects (Visscher et al. (2012)). In industry applications, it is routine to record instances of rare events for enormous numbers of customers. Examples include insurance fraud (Sithic and Balasubramanian (2013)), occurrence and duration of hospitalization (Xie et al. (2016)), incidents of fraud in e-commerce and consumer credit (Bolton and Hand (2002); Phua et al. (2010)), and occurrences of consecutive visits to pairs of websites in quantitative advertising – the motivating example considered in this article.

Traditionally, large sample sizes are associated with small parameter uncertainty (Van der Vaart (2000)). However, for highly imbalanced categorical data, such as result from recording rare events, the effective sample size for estimating a particular parameter can be a vanishingly small fraction of the full data sample.
Large-sample mixing of latent variable Gibbs size. Hence, even with immense total sample sizes, there is often substantial uncertainty about key parameters. This provides a compelling motivation for Bayesian hierarchical modeling, providing a framework for borrowing of information in parameter estimation and uncertainty quantification (Gelman et al. (2013)). In such settings, current deterministic approximations to posterior distributions lack sufficient accuracy, and Markov chain Monte Carlo algorithms are commonly used (Gilks (2005); Gelfand and Smith (1990); Robert and Casella (2013)). In particular, data augmentation Gibbs samplers remain the default choice for posterior computation in categorical data applications (Albert and Chib (1993); Polson et al. (2013)).

In this article, our focus is on studying the performance of these algorithms for large sample imbalanced categorical data. We consider an asymptotic regime in which the number of successes is fixed but sample size $n$ goes to infinity, referred to as infinitely imbalanced by Owen (2007). Owen (2007) focused on the limiting behavior of the coefficient estimates in logistic regression. In contrast, we are the first to study computational complexity in this important large sample regime. In particular, our focus is on providing strong guarantees for how mixing rates scale with $n$. We obtain a surprising result that data augmentation Gibbs sampling algorithms have a spectral gap that converges to zero at a rate proportional to $\sqrt{n}$ or faster. This result suggests that data augmentation algorithms should be avoided in large sample imbalanced settings. Indeed, empirical results in simulations and for our motivating computational advertising application show that data augmentation Gibbs sampling performs much worse in practice than Metropolis-Hastings algorithms that avoid augmentation.

2 Computational Complexity Theory

2.1 Preliminaries

Letting $y$ denote the observed data and $\theta$ parameters in the likelihood function, common data augmentation Gibbs samplers introduce latent variables $\omega$ in such a way that the full conditional posterior distributions of the latent data $\omega$ and parameters $\theta$ have forms that can be sampled from easily. In particular, assuming a multivariate Gaussian prior for $\theta$,

\[
\begin{align*}
ω &\mid θ,y ∼ p(ω \mid θ,y) \\
θ &\mid ω,y ∼ N\{μ(ω),Σ(ω)\}.
\end{align*}
\]

For probit models, the data augmentation scheme of Albert and Chib (1993) induces a truncated Gaussian form for $p(ω \mid θ,y)$. Polson et al. (2013) developed a similar approach for logistic regression, which samples the latent variables from Pólya-Gamma distributions. The popularity of these algorithms cannot be overstated; indeed, they are used routinely in Bayesian categorical data analysis including in much more complex settings than described above.

The computational efficiency of Markov chain Monte Carlo algorithms depends on the computing time per iteration and the mixing rate, which can be measured by the spectral gap. In studying computational complexity, we focus on the simple case in which $y$ is the number of events in $n$ trials and there are no covariates. In particular,

\[
\begin{align*}
y &\sim \text{Binomial}(n,p), \quad p = \varphi(\theta), \quad θ ∼ N(b,B),
\end{align*}
\]

where $\varphi : \mathbb{R} \to (0,1)$. For probit models $\varphi(θ) = Φ(θ)$ is the standard Gaussian cumulative distribution function, while for logistic regression $\varphi(θ) = \{1 + \exp(-θ)\}^{-1}$. Model (2) is an intercept only binary response generalized linear model with $\varphi$ the link function. Extensions to include covariates, hierarchical
structure and other complications increase computational complexity. However, we will show that even in the simple case \(2\), complexity becomes critically large as \(n\) grows.

Running a Markov chain with transition kernel \(P\) corresponding to the update rule in \(1\) has \(\theta\)-marginal invariant measure the posterior \(\Pi(\theta | y)\) under the model in \(2\). These samples are used to estimate posterior summaries of interest based on ergodic averages. The number of samples needed to keep the Monte Carlo error in these estimates below a desired level is approximately proportional to \(1/\delta_n(P)\), with \(\delta_n(P)\) the spectral gap of \(P\) with sample size \(n\). As the complexity of each evaluation of \(P\) is linear in \(n\), the overall computational complexity of data augmentation Gibbs sampling in our simple setting is approximately \(n/\delta_n(P)\). A precise and mathematically rigorous discussion of the relationship between \(\delta_n(P)\) and computational complexity is provided in the Supplementary Materials. In the next subsection, we provide upper bounds on \(\delta_n(P)\) for the data sequence \((n, y_n)\), where \(y_n = 1\) for every \(n\) following the infinitely-imbalanced setup of Owen (2007). These upper bounds can be converted into approximate lower bounds on computational complexity.

2.2 Main results

We provide bounds on the spectral gap for the Pólya-Gamma (Polson et al. (2013)) and Albert and Chib (1993) data augmentation algorithms in Theorems 2.1 and 2.2, respectively. Details of these samplers are provided in the Supplementary Materials.

**Theorem 2.1.** Let \(P\) be the transition kernel of the Pólya-Gamma sampler in the case that \(y_n = 1\) for every \(n\). Then \(P\) has spectral gap

\[
\delta_n(P) = \mathcal{O}\left\{ \frac{(\log n)^{5.5}}{\sqrt{n}} \right\},
\]

where \(g(n) = \mathcal{O}\{f(n)\}\) means there exists \(C, n_0 < \infty\) such that \(n > n_0\) implies \(g(n) < Cf(n)\). The proof of Theorem 2.1 is deferred to the Appendix.

**Theorem 2.2.** Let \(P\) be the transition kernel of the Albert and Chib sampler in the case that \(y_n = 1\) for every \(n\). Then \(P\) has spectral gap

\[
\delta_n(P) = \mathcal{O}\left\{ \frac{(\log n)^{2.5}}{\sqrt{n}} \right\}.
\]

The proof of Theorem 2.2 uses a similar strategy to that of Theorem 2.1 and is provided in the Supplementary Materials.

Since the overall computational complexity of either algorithm is roughly \(n/\delta_n(P)\), Theorems 2.1 and 2.2 imply computational complexity of at least \(n^{1.5}\) neglecting logarithmic factors. The effect of \(\delta_n(P)\) is observed in the mixing properties of the chain. When \(\delta_n(P)\) is close to zero, the chain will mix slowly. In the following section, we provide intuition for why the spectral gap converges to zero at these rates and how this results in poor mixing.

2.3 Intuition

The root cause of slow mixing in the infinitely imbalanced setting is a discrepancy between the rate at which the posterior concentrates as \(n \to \infty\) and the rate at which the step sizes for these algorithms converge to zero. This is illustrated by the graphic in Fig. 1. When \(\varphi\) is either the logit or probit link in \(2\) and \(y_n/n \to 0\),
the width of the high probability region of $\Pi(\theta \mid y)$ is at least $(\log n)^{-1}$. This is depicted by the large region in the graphic where the posterior density is non-trivial. However, the steps in the data augmentation samplers depend on the mean of $n$ independent random variables, so the typical move size is proportional to $n^{-1/2}$. A graphic depiction of such a move is provided in Fig. 1 in the region indicated by $\theta_t \to \theta_{t+1}$. Thus, the algorithm needs order $(\log n)^{-1}\sqrt{n}$ or more moves to traverse the high-probability region of the posterior. These rough computations omit some logarithmic factors that appear in the rigorous bounds, but this is irrelevant to the basic intuition. This reasoning applies to any data augmentation sampler with step sizes that depend on the mean of independent random variables. If the high probability region of the posterior is contracting at a rate slower than $n^{1/2}$, then the algorithm will mix poorly in large samples. Even more generally, any time the step sizes converge to zero at a different rate than the width of the high probability region of the posterior, the algorithm will mix slowly as $n \to \infty$.

![Figure 1: Cartoon comparing the high posterior density region and typical move size.](image)

3 Synthetic Data Examples

3.1 Theoretical motivation and focus

Our empirical analyses focus on understanding the contribution of mixing to computational complexity via the variance of ergodic averages. Under conditions given in Supplementary Materials and satisfied by the Albert and Chib and Pólya-Gamma samplers, ergodic averages of a function $f$ obey a central limit theorem with asymptotic variance $\sigma_f^2$. If we could collect independent Monte Carlo samples $\theta_1, \ldots, \theta_T$ from $\Pi(\theta \mid y)$, then $\text{var} \{T^{-1} \sum_{t=1}^{T} f(\theta_t)\} = \text{var}_{\Pi}(f)/T$. In Markov chain Monte Carlo, this variance is approximately $\sigma_f^2/T$ for large $T$, with $\sigma_f^2 \geq \text{var}_{\Pi}(f)$. This leads to a common definition of the effective sample size as

$$T_{\text{eff}} = \frac{\text{var}_{\Pi}(f)T}{\sigma_f^2},$$

which is the number of independent Monte Carlo samples necessary to achieve the same variance as an ergodic average from the Markov Chain of length $T$.

Thus, the rate at which the path length required to achieve any desired estimation accuracy scales with $n$ is approximately the rate at which $\sigma_f^2$ scales with $n$. The asymptotic variance is related to autocorrelations.
and the spectral gap by

$$\frac{\sigma^2_f}{\text{var}_n(f)} = \eta_1 + 2 \sum_{t=2}^{\infty} \eta_t \leq \frac{2}{\delta_n(\mathcal{P})},$$

(6)

where \( \eta_t \) is the lag-\( t \) autocorrelation \( \text{cor}\{f(\theta_1), f(\theta_t)\} \) with \( \{\theta_t\} \) evolving according to \( \mathcal{P} \). Equation (6) motivates our theoretical focus on the spectral gap and our argument that computational complexity is inversely proportional to \( \delta_n(\mathcal{P}) \).

In the remainder of this section, we perform example computations to assess the practical performance of the Albert and Chib and Pólya-Gamma data augmentation algorithms. These algorithms are compared to a Hamiltonian Monte Carlo algorithm that does not rely on data augmentation. We also consider related samplers for multinomial logit or probit in the situation where one or more of the observed cell counts is 1. As the spectral gap is hard to estimate empirically, we focus on estimates of autocorrelations and of effective sample size for the identity function \( f(\theta) = \theta \) based on truncations of the sum in (6).

### 3.2 Binomial logit and probit

In the first set of examples, we consider the model in (2) with probit and logit link. We set \( y = 1 \) and vary \( n \) between 10 and 10,000. We perform computation using the Albert and Chib algorithm for the probit link and the Pólya-Gamma algorithm for the logit link, then estimate autocorrelations and effective sample sizes. For the probit, we use a prior of \( b = 0, B = 49 \), whereas for the logit we use a prior of \( b = 0, B = 100 \), reflecting the lighter tails of the probit function. For the probit, we initialize the sampler at the maximum likelihood estimate and collect no burn-in; for the logit, we initialize the sampler at zero and collect burn-in of 5,000 iterations. Similar results were achieved using the opposite choice for each algorithm. For probit, we wrote code for the Gibbs sampler in R. For logit, we use the package BayesLogit for computation. We also estimate each model using Hamiltonian Monte Carlo, implemented with the Stan environment and Rstan package.

![Figure 2: Estimated autocorrelations at lags 1-100 for the two data augmentation samplers for binomial probit and logit as well as for logit with computation by Hamiltonian Monte Carlo, using a different vertical scale for readability. Four different values of \( n \) are shown.](image)

Fig. 2 shows the autocorrelation function, estimated using the coda package in R, for lags 1-100 and each of the alternative samplers. The autocorrelations increase with \( n \) for both the data augmentation Gibbs samplers, but are essentially identical for all \( n \) for Hamiltonian Monte Carlo, which additionally shows near zero values after lag ten. Table 1 shows \( T_{\text{eff}}/T \), also computed using coda, for the same three algorithms with \( y = 1 \) and increasing values of \( n \). Also shown for comparison is the same calculation for random walk.
Table 1: Estimated value of effective sample size divided by path length for probit and logit models using Albert and Chib and Pólya-Gamma samplers and for the logit model with computation by Hamiltonian Monte Carlo and random walk Metropolis. Here y = 1 in each case and n varies between 10 and 10,000.

| n   | Albert and Chib | Pólya-Gamma | Hamiltonian Monte Carlo | Metropolis |
|-----|----------------|-------------|-------------------------|------------|
| 10  | 0.1827         | 0.2615      | 0.2286                  | 0.0947     |
| 50  | 0.0516         | 0.0636      | 0.2133                  | 0.0973     |
| 100 | 0.0258         | 0.0344      | 0.2051                  | 0.0961     |
| 500 | 0.0115         | 0.0147      | 0.1993                  | 0.0853     |
| 1000| 0.0026         | 0.0062      | 0.1975                  | 0.0684     |
| 5000| 0.0014         | 0.0020      | 0.2184                  | 0.0653     |
| 10000| 0.0017       | 0.0018       | 0.1976                  | 0.0759     |

Metropolis. The effective sample size is anemic for the data augmentation Gibbs samplers but about 0.2T for Hamiltonian Monte Carlo and about 0.1T for random walk Metropolis for all values of n.

Figure 3: Estimated autocorrelation functions for synthetic data examples that vary y and n

Table 2: Values of $T_{eff}/T$ for synthetic data examples that vary y and n

| n      | y   | Albert and Chib | Pólya-Gamma |
|--------|-----|----------------|-------------|
| 10000  | 1   | 0.0014         | 0.0015      |
| 50000  | 5   | 0.0023         | 0.0020      |
| 100000 | 10  | 0.0036         | 0.0034      |

Although the theoretical results in [2] consider the case where y = 1 and n is increasing, empirically we observe poor mixing whenever y/n is small. To demonstrate this, we perform another set of computational examples where y and n both vary in such a way that y/n is constant. Specifically, we consider n = 10, 000, n = 50, 000, and n = 100, 000 with y = 1, 5, 10. Computation is performed for the two data augmentation Gibbs samplers as above, and effective sample sizes and autocorrelation functions estimated. Fig. 3 shows estimated autocorrelations, which are similarly near 1 at lag 1 and decay slowly. Table 2 shows values of $T_{eff}/T$ for T = 5,000 for the two algorithms. Neither measure of computational efficiency shows a meaningful effect of increasing y when y/n remains constant.
3.3 Empirical analysis of computational complexity

In this section, we estimate empirically the computational complexity of the two data augmentation Gibbs samplers via finite path estimates of $\sigma^2_f$ in (6). These empirical estimates can be compared to our theoretical bounds of $n^{1.5}$ to assess tightness.

Let $\sigma^2_f(n)$ be the asymptotic variance when the sample size is $n$, and assume $\sigma^2_f(n) \approx Cn^k$ for large $n$, so that $\log \{ \sigma^2_f(n) \} = \log(C) + k \log(n)$. This suggests first estimating $\sigma^2_f(n)$ for different values of $n$, and then estimating $k$ by regression of $\log \{ \sigma^2_f(n) \}$ on $\log(n)$.

We estimate $\sigma^2_f$ based on autocorrelations via a truncation of (6),

$$\hat{\rho}_f = \hat{\eta}_t + 2 \sum_{t=2}^S \hat{\eta}_t,$$

(7)

where $\hat{\eta}_t$ is a point estimate of $\eta_t$. It is important to choose $S \gg \{ \delta_n(\mathcal{P}) \}^{-1}$. The lower bounds derived in §2 have $\{ \delta_n(\mathcal{P}) \}^{-1} \geq n^{1/2}$ up to a log factor and a universal constant, so we use $S = n$ to compute the sum in (7). To further improve the estimates, we use multiple chains to compute $\hat{\eta}_t$ and run all of the chains for $10^6$ iterations.

Fig. 4 shows $\log(n)$ versus $\log(\hat{\rho}_f)$ for values of $n$ between 10 and 10,000 for the Pólya-Gamma and Albert and Chib sampler. The relationships are linear, and the least squares estimate of the slope is 0.86 for Pólya-Gamma and 0.84 for Albert and Chib, so the effective sample size scales approximately as $n^{-0.85}$, giving overall computational complexity of about $n^{1.85}$. This is close to our theoretical lower bound of $n^{1.5}$.

3.4 Data augmentation algorithms for multinomial likelihoods

So far we have considered data augmentation algorithms for binomial likelihoods. Similar algorithms exist for multinomial logit and probit models. Specifically, let

$$y \sim \text{Multinomial}(n, \pi), \quad \pi = g^{-1}(\theta), \quad \theta \sim N(0, B),$$

(8)

where $y$ is a length $d$ vector of nonnegative integers whose sum is $n$, $\pi = (\pi_1, \ldots, \pi_d)^T$ is a probability vector, and $g^{-1}(\theta)$ is a multinomial logit or probit link function. Posterior computation under (8) is commonly
performed using data augmentation algorithms of the form in (1). [Polson et al. (2013)] describe a Pólya-Gamma sampler for the multinomial logit, which is implemented in BayesLogit, while [Imai and van Dyk (2005)] propose a data augmentation Gibbs sampler for the multinomial probit, which is implemented in package MNP.

We study a synthetic data example where \( y \) is a \( 4 \times 1 \) count vector with entries adding to \( n \). The first three entries of \( y \) are always 1, the final entry is \( n - 3 \), and a series of values of \( n \) between \( n = 10 \) and \( n = 10,000 \) are considered. Estimated values of \( T_{eff}/T \) for the first three entries of \( \theta \) for both algorithms are shown in Table 3. The results are similar to those for the binomial logit and probit, and are consistent across the different entries of \( \theta \). It is exceedingly common for contingency tables to have many cells with small or zero entries. Our results suggest that data augmentation algorithms should be avoided in such settings.

Table 3: Estimated values of \( T_{eff}/T \) for the three entries of \( \theta \) for multinomial logit (Pólya-Gamma) and probit (Imai and van Dyk) data augmentation for increasing values of \( n \) with data \( y = (1, 1, 1, n - 3) \). Results are based on 5,000 samples gathered after discarding 5,000 samples as burn-in.

|         | Pólya-Gamma | Imai and van Dyk |
|---------|-------------|------------------|
| \( n = 10 \) | 0.1069 0.0921 0.1509 | 0.2479 0.2303 0.2645 |
| \( n = 50 \) | 0.0250 0.0293 0.0528 | 0.0593 0.0635 0.0635 |
| \( n = 100 \) | 0.0111 0.0153 0.0250 | 0.0337 0.0376 0.0388 |
| \( n = 500 \) | 0.0017 0.0023 0.0039 | 0.0115 0.0080 0.0137 |
| \( n = 1000 \) | 0.0031 0.0036 0.0066 | 0.0072 0.0068 0.0042 |
| \( n = 5000 \) | 0.0006 0.0006 0.0031 | 0.0021 0.0011 0.0017 |
| \( n = 10000 \) | 0.0010 0.0009 0.0023 | 0.0008 0.0015 0.0013 |

4 Application to quantitative advertising

In quantitative advertising, it is important to accurately estimate the probability that users view two websites within a specified time window. In particular, advertisers are interested in the “organic” probability that a user views a client’s website during the same browsing session that the user also visits one of thousands of high traffic sites. Here, “organic” means the user views the client’s site without clicking on a link in an advertisement. Small differences in these organic transition probabilities often translate to commercially significant differences in the effectiveness of ads. Ultimately, the goal is to develop a list of potential high-traffic sites to serve ads, ranked in order of the rate at which users organically view the client site.

These transitions are rare events, and in most cases it is necessary to obtain data on tens or hundreds of thousands of users to view even a single organic transition, so the data are imbalanced, with only a few transitions observed for most of the high traffic sites. Thus, it makes sense to borrow information across the different high-traffic sites to obtain lower risk point estimates of the transition probabilities. Moreover, estimates of uncertainty are useful, since a site with a relatively high transition probability that is estimated precisely may be a better target than one that has an even higher estimated transition probability with relatively large credible bands.

A simple approach to borrowing information and inducing shrinkage is to use a hierarchical prior structure. For example

\[
y_i \sim \text{Binomial}(n_i, p_i), \quad p_i = \frac{e^{\theta_i}}{1 + e^{\theta_i}}, \quad i = 1, \ldots, n,
\]

(9)
\( \theta_i \sim N(\theta_0, \tau^2), \quad (\theta_0, \tau^2) \sim f(\theta_0, \tau^2). \)  

(10)

Here, \( i \) indexes the high-traffic sites, and \( p_i \) is the probability of visiting the client site in the same browsing session. A more refined model can also be considered, which includes multiple client sites and adds more layers of hierarchy, perhaps grouping the sites according to content or topic. However, our interest is in computational efficiency of Markov chain Monte Carlo algorithms for these models; if mixing is critically poor for the simple model in (9)-(10), then there is little hope of the algorithm being useful in more complex settings.

We use data on transitions from 59,317 high traffic sites to a client site to do computation for the model in (9)-(10) by Pólya-Gamma data augmentation. We use an improper uniform prior on \( \tau \) and a \( N(\theta_{00}, \tau_0^2) \) prior on \( \theta_0 \), with \( \theta_{00} = -12 \) and \( \tau_0^2 = 49 \). The prior on \( \theta_0 \) is weakly informative on the logit scale and consistent with information solicited from experts in quantitative advertising. As an alternative, we use a hybrid Metropolis-within-Gibbs algorithm with Student’s \( t \) proposals centered on the conditional models for \( \theta_i \), as described in the Supplemental Materials. We also performed computation using Hamiltonian Monte Carlo implemented in the Stan language with the Rstan package.

Fig. 5 shows the autocorrelations and estimated values of \( T_{\text{eff}}/T \) for the Pólya-Gamma data augmentation algorithm, Hamiltonian Monte Carlo, and the hybrid algorithm. The lag-1 autocorrelation for the data augmentation Gibbs sampler is \( \approx 1 \), and the autocorrelation function decays very slowly. In contrast, Hamiltonian Monte Carlo has autocorrelations near zero, while the hybrid algorithm has an autocorrelation function that decays to below 0.25 for the majority of the parameters by lag 20. The values of \( T_{\text{eff}}/T \) are consistent with what is expected given the autocorrelation structure for each Markov chain. These results demonstrate a practical applied problem in which the data augmentation Gibbs sampler is not useful, while Hamiltonian Monte Carlo provides excellent mixing.
5 Discussion

For several decades, there has been substantial interest in easy to implement and reliable algorithms for posterior computation in generalized linear models. Data augmentation Gibbs sampling, particularly for probit and logit links, has received much of this attention. A series of data augmentation schemes ([Frühwirth-Schnatter and Frühwirth (2010); Holmes and Held (2006); O’Brien and Dunson (2004)], of which Polson et al. (2013) is the most recent, have steadily improved the accessibility of Gibbs sampling for logistic regression. This is a specific case of the larger focus of Bayesian computation on Gibbs samplers, in algorithm development, routine use, and theoretical analysis. The appeal of Gibbs samplers is largely due to their conceptual simplicity, minimal tuning, and widespread familiarity. In addition, there is a common misconception that Gibbs samplers are more efficient than alternative Metropolis-Hastings algorithms.

The literature studying theoretical efficiency of Gibbs samplers has largely focused on showing uniform or geometric ergodicity; for example, refer to [Choi et al. (2013)] for uniform ergodicity results for the PolyGa-Gamma sampler. In contrast, we focus on obtaining lower bounds on computational complexity, which is more in line with the theoretical computer science literature. However, in this literature, it is typically thought that a polynomial-time algorithm has good efficiency. In contrast to this view, we show that data augmentation Gibbs samplers having complexity of approximately $n^{1.85}$ are critically inefficient. The underlying cause of this poor behavior is a step size in the transition kernel that converges to zero at a different rate than the posterior. In contrast to Gibbs sampling, Metropolis-Hastings algorithms have the flexibility to adjust for this mismatch, with Hamiltonian Monte Carlo applied without data augmentation providing a promising general strategy. Our approach for studying computational efficiency in large sparse data regimes should be broadly useful beyond our motivating setting.

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Appendix

A Proofs

A.1 Proof of Corollary A.1

The following Corollary to Theorem S1.2 in Supplementary Materials is useful in our setting.

\textbf{Corollary A.1.} Let \((\theta, \omega)\) be a data augmentation Markov chain on state space \(\Omega_1 \times \Omega_2 \subset \mathbb{R} \times \mathbb{R}^n\). Denote by \(P = P_1 P_2\) the transition kernel of this chain, where \(P_1[(\theta, \omega), \Omega_1 \times \{\omega\}] = P_2[(\theta, \omega), \{\theta\} \times \Omega_2] = 1\) for all \((\theta, \omega) \in \Omega_1 \times \Omega_2\). Denote by \(\Pi\) the stationary measure of \(P\), and denote by \(\Pi_1\) and \(\Pi_2\) the marginals of this stationary measure on \(\Omega_1\) and \(\Omega_2\); denote by \(\mu, \mu_1\) and \(\mu_2\) their densities. Assume that there exists an interval \(I = (a, b) \subset \Omega_1\) that satisfies

\begin{equation}
\Pi_1(I) \geq 1 - \epsilon 
\end{equation}

\begin{equation}
c^* \leq \inf_{\theta \in I} \mu_1(\theta) \leq \sup_{\theta \in I} \mu_1(\theta) \leq C^* \quad \text{for all} \quad (r < 1), \quad \text{trivial for} \quad r < 1.
\end{equation}

\begin{equation}
\sup_{\theta \in I, z \in \Omega_2} \Pr \{(\theta_{x+1} - \theta_x)^2 > \gamma \mid (\theta_x, \omega_x) = (\theta, z)\} \leq r^{-2} + \gamma
\end{equation}

for some \(\epsilon, \zeta > 0\), \(0 \leq \gamma < \infty\) and \(0 < c^* < C^* < \infty\), and for all \(0 \leq r < (1 - \epsilon) / (4c^*)\). Since (13) is trivial for \(r < 1\), this is equivalent to (13) holding for \(1 \leq r \leq (1 - \epsilon) / (4c^*)\). Assume that \(\zeta \leq \frac{1}{4c^*}\). Then

\begin{equation}
\delta(P) \leq 16C^*\zeta + \frac{2C^*\gamma}{(1 - \epsilon)^2} + \frac{2c^*\epsilon}{(1 - \epsilon)}.
\end{equation}

\textit{Proof.} Let \(m = \inf \left\{ x > a : \int_a^x \mu_1(y) dy \geq \frac{\Pi_1(I)}{2} \right\} \geq a + \frac{1 - \epsilon}{2c^*}\) be the median of the restriction of \(\Pi_1\) to \(I\) and let \(S = (a, m] \times \Omega_2\). By inequality (12),

\begin{equation}
\frac{1 - \epsilon}{2c^*} \geq m - a \geq \frac{1 - \epsilon}{2C^*}.
\end{equation}

We now bound the conductance \(\kappa\) – see Definition S1 in Supplementary Materials – by showing an upper bound on \(\kappa(S)\)

\begin{equation}
\kappa(S) = \frac{\int_{(x,y) \in S} P\{ (x,y), S^c \} \mu(x,y) dxdy}{\Pi(S) \{1 - \Pi(S)\}}
\end{equation}

\begin{equation}
\leq \frac{4}{(1 - \epsilon)^2} \int_{(x,y) \in S} P\{ (x,y), S^c \} \mu(x,y) dxdy
\end{equation}

\begin{equation}
\leq \frac{4}{(1 - \epsilon)^2} \int_a^m C^* \left[ \min \left\{ 1, \frac{\zeta^2}{\min(x - a, m - x)^2} + \gamma \right\} \right] dx
\end{equation}

where in the last step we applied (13) with \(r \leq \max(x - a, m - x) \leq (1 - \epsilon) / (4c^*)\) on \([a, m]\). Continuing

\begin{equation}
\kappa(S) \leq \frac{8C^*}{(1 - \epsilon)^2} \left\{ \int_0^\zeta (1 + \gamma) dx + \int_0^{\frac{m-a}{2}} \left( \frac{\zeta^2}{x^2} + \gamma \right) dx \right\}
\end{equation}

\begin{equation}
= \frac{8C^*}{(1 - \epsilon)^2} \left\{ \zeta + \zeta^2 \left( \zeta^{-1} - \frac{2}{m-a} \right) + \gamma \frac{m-a}{2} \right\}
\end{equation}
\[
\leq \frac{16C^*\zeta}{(1-\epsilon)^2} + \frac{8C^*\gamma}{(1-\epsilon)^2} \cdot 4c^*
\]
\[
= \frac{16C^*\zeta}{(1-\epsilon)^2} + \frac{2C^*\gamma}{c^*(1-\epsilon)}.
\]

The result now follows immediately from an application of Theorem S1.2 in Supplementary Materials. \qed

A.2 Verifying Corollary A.1

We briefly outline the strategy for showing the three conditions in Corollary A.1. To show (11), the existence of an interval \(I(n)\) satisfying \(\pi_1\{I(n)\} \geq 1 - \epsilon\), we first find an interval \(I(n)\) containing the posterior mode for large enough \(n\) on which the posterior density ratio is bounded below by a constant. Then, we find a second interval \(I'(n)\) outside of which the posterior integrates to \(o(1)\) and that satisfies \(I(n) \subset I'(n)\). By lower bounding the width of \(I(n)\) and upper bounding the width of \(I'(n)\), we obtain a lower bound on \(\pi_1\{I(n)\}\) and bounds on \(c^*(n)\) and \(C^*(n)\), sequences of constants corresponding to \(C^*\) and \(c^*\) in (12). To show (13), we study the dynamics of the chain on \(I(n)\) and use concentration inequalities.

A.3 Proof of Theorem 2.1

We begin by proving (3) with an application of Corollary A.1. The proof consists of verifying the three conditions given by inequalities (11), (12), and (13). The proof proceeds in three parts:

(a) Showing an interval \(I(n)\) on which the posterior density ratio is bounded by a constant and lower bounding its width;

(b) Showing an interval \(I'(n) \supset I(n)\) outside of which the posterior integrates to \(o(1)\) and upper bounding its width; and,

(c) Showing a concentration result of the form (13) on an interval containing \(I(n)\).

Part (a): showing the posterior is almost constant on an interval \(I(n)\) containing the mode and lower bounding its width. First, we provide bounds of the form (11) and (12). Recall that the posterior density of \(\theta\) is

\[
p(\theta|y = 1) = \frac{n}{(2\pi)^{1/2}B} \cdot (1 + e^\hat{\theta})^{-n} e^\theta e^{-\frac{\theta^2}{2}}.
\]

We begin by showing that \(p(\theta|y = 1)\) is near-constant on a small region around the mode \(\hat{\theta} = \text{argmax}_\theta p(\theta|y = 1)\) of width \(\Omega\{\log n\}^{-1}\) given by \(I(n) = [\hat{\theta} - \{\log(n)\}^{-1}, \hat{\theta} + \{\log(n)\}^{-1}]\). By straightforward calculus, \(\hat{\theta}\) satisfies

\[
\frac{\hat{\theta}}{B} + n \frac{e^\hat{\theta}}{1 + e^\hat{\theta}} = 1,
\]

and so

\[
\hat{\theta} = -\log(n) + O\{\log(\log n)\}.
\]

Therefore, there exists an \(A < \infty\) such that \(\hat{\theta} \in [-\log(n) - A \log(\log n), -\log(n) + A \log(\log n)]\) for all \(n > N_0\), where \(N_0\) depends only on \(A\).
Consider pairs $\theta_1, \theta_2$ that satisfy $|\theta_1 - \theta_2| \leq (\log n)^{-1}$ and also $|\theta_1 + \log(n)|, |\theta_2 + \log(n)| \leq A \log(\log n)$. Define $\zeta_1, \zeta_2$ by $\theta_1 = - \log(n) + \zeta_1, \theta_2 = - \log(n) + \zeta_2$. Then we calculate

\[
p(\theta_1 | y = 1) = \frac{e^{\theta_1 - \theta_2} \left( 1 + e^{\theta_2} \right)^n e^{\frac{n}{2} \left( \theta_2^2 - \theta_1^2 \right)}}{1 + e^{\theta_1}} \quad \text{(15)}
\]

Thus,

\[
p(\theta_2 | y = 1) = e^{\zeta_1 - \zeta_2} \left( 1 + \frac{1}{n} e^{\zeta_2} \right)^n e^{\frac{n}{2} \left( \zeta_1 - \zeta_2 \right)^2 (2 \log(n) - \zeta_1 + \zeta_2)} \geq (e^{-2}) (2e^{-2}) (e^{-2}, B).
\]

Since this holds for any pair of points satisfying $|\theta_1 - \theta_2| \leq (\log n)^{-1}$ inside the interval $- \log(n) \pm A \log(\log n)$, and $\hat{\theta}$ is inside this interval for $n > N_0$, we conclude the posterior density ratio is bounded below by a constant on an interval $I(n)$ of width $O((\log n)^{-1})$ centered at $\hat{\theta}$ for all $n > N_0$. Since the posterior density must integrate to 1, this shows that $\mu_1(\theta) = O(\log n)$ in (12), so we can take $C^*(n) = O(\log n)$ in (12).

**Part (b): showing the posterior is negligible outside an interval $I'(n) \supset I(n)$ and upper bounding its width.** Next, we show that $p(\theta | y = 1)$ is negligible outside of the interval $I'(n) = (-5 \log(n), 3 \log(n))$. This interval clearly contains $I(n)$ for all large $n$, since $I(n)$ is an interval of width $O(\log(\log n))$ containing $- \log(n)$. If $\theta = - \log(n) + C \log(n)$ for some $C \geq 4$,

\[
p(\theta | y = 1) \leq \frac{n}{(2\pi)^{1/2} B} C^{-1} (1 + n^{-1} C^{-1})^n e^{-\left(C^{-1} \frac{1}{2} (\log(n)) \right)^2}
\]

Thus,

\[
\int_{3 \log(n)}^{\infty} p(\theta | y = 1) d\theta \leq \sum_{C=4}^{\infty} \log(n) \times \frac{1}{(2\pi)^{1/2} B} n C^{-n(n) - \left(C^{-1} \frac{1}{2} (\log(n)) \right)^2} = o(1). \quad \text{(16)}
\]

If $\theta = - \log(n) - C \log(n)$ for some $C \geq 4$, then

\[
p(\theta | y = 1) \leq \frac{n}{(2\pi)^{1/2} B} n^{-C-1} (1 + n^{-1} C^{-1})^n e^{-\left(C^{-1} \frac{1}{2} (\log(n)) \right)^2}
\]

Thus,

\[
\int_{-5 \log(n)}^{-\infty} p(\theta | y = 1) d\theta \leq \sum_{C=4}^{\infty} \log(n) \times \frac{2}{(2\pi)^{1/2} B} n^{C-\left(C^{-1} \frac{1}{2} (\log(n)) \right)^2} = o(1). \quad \text{(17)}
\]

Combining inequalities (16) and (17) gives

\[
\int_{I'(n)} p(\theta | y = 1) d\theta = o(1). \quad \text{(18)}
\]
Therefore, since the posterior is negligible outside a region of width $\mathcal{O}(\log n)$, and the density is unimodal and smooth, we can take $c = \Omega((\log n)^{-1})$ in (12). This also shows that $\pi_1 \{ I(n) \} = \Omega((\log n)^{-2})$, so we can take $1 - c(n) = \Omega((\log n)^{-2})$ in (11).

**Part (c): Showing** (13) **on an interval containing the mode.** Fix a constant $0 < C < 1$ and consider the interval

$$I^*(n) = [\log(n)(1 + C), -\log(n)(1 - C)].$$

This interval contains $\hat{\theta} \in -\log(n) \pm \Theta \{ \log(\log n) \}$ for sufficiently large $n$. We will show (13) on $I^*(n)$. We can write values of $\hat{\theta}$ inside this interval as $\hat{\theta} = \log(n)(1 + a_i)$ for $|a_i| \leq C$. Recall that we are considering the Polya-Gamma sampler with an update rule consisting of sampling $\omega_{t+1} | \hat{\theta}_t, n$ and then sampling $\hat{\theta}_{t+1} | \omega_{t+1}, y, n$. We first obtain bounds on the conditional expectation and variance of $\omega_{t+1} | \hat{\theta}_t, n$ for $\hat{\theta}_t$ inside of $I^*(n)$, which will be used to show concentration. We have

$$E(\omega_{t+1} | \hat{\theta}_t, n) = \frac{n}{2\hat{\theta}_t} \tanh(\hat{\theta}_t/2)$$

$$= \frac{n}{2\hat{\theta}_t} \frac{1 - e^{\log(n)(1 + a_i)}}{-2\log(n)(1 + a_i) + e^{\log(n)(1 + a_i)}}$$

$$= \frac{n}{2\log(n)(1 + a_i)} \frac{1 - n^{1+a_i}}{1 + n^{1+a_i}}$$

$$= \frac{n}{2\log(n)(1 + a_i)} [1 - 2n^{-1-a_i} + o(1)]$$  \hspace{1cm} (20)

and

$$\text{var}(\omega_{t+1} | \hat{\theta}_t, n) = \frac{n}{4\hat{\theta}_t} \{\sinh(\hat{\theta}_t) - \hat{\theta}_t\} \text{sech}^2 \left( \frac{\hat{\theta}_t}{2} \right)$$

$$= \frac{n}{4(1 + a_i)^3 \log(n)^2} \left[ \frac{1 - e^{2(1+a_i)\log(n)}}{2e^{(1+a_i)\log(n)}} + (1 + a_i) \log(n) \right] \left[ \frac{2e^{(1+a_i)\log(n)}}{1 + e^{(1+a_i)\log(n)}} \right]^2$$

$$= \frac{n}{4(1 + a_i)^3 \log(n)^2} \left[ \frac{1}{2} n^{1+a_i} + o(1) \right] \left[ \frac{4}{n^{1+a_i}} + o(1) \right]$$

$$= \frac{n}{2(1 + a_i)^3 \log(n)^3} (1 + o(1)).$$  \hspace{1cm} (21)

Define $\zeta(n) = n^{1/2} (\log n)^{-1.5}$. Combining (20) and (21), we have by Chebyshev’s inequality that

$$\Pr \left( \left| \omega_{t+1} - \frac{n}{2\log(n)(1 + a_i)} \right| > r \frac{n^{1/2}}{\log(n)^{1.5}} \mid \hat{\theta}_t \right) = \mathcal{O}(r^{-2})$$  \hspace{1cm} (22)

for any $r > 0$, $\hat{\theta}_t \in I^*(n)$. Next, we bound $|\theta_{t+1} - \theta_t|$ for $\theta_t \in I^*(n)$. Recall

$$\theta_{t+1} | \omega_{t+1} \sim N \left\{ \sigma_{\omega_{t+1}}^{-1} (y - n/2), \sigma_{\omega_{t+1}}^{-1} \right\}, \quad \sigma_{\omega_{t+1}}^{-1} = (\omega_{t+1} + B^{-1})^{-1}.$$  

Define $r_t$ by $\omega_{t+1} = n \{2 \log(n)(1 + a_i) \}^{-1} + r_t n^{1/2} \log(n)^{-1.5}$. In the following, we condition on $r_t \{4 \log(n) \}^{1/2} n^{-1/2} \leq 1/8$ and $4B^{-1} \log(n)n^{-1} \leq 1/8$ to obtain concentration results. Clearly, the
second condition holds for fixed \(0 < B < \infty\) for all sufficiently large \(n\). To show that the second condition holds for the relevant values of \(r_t\), recall that we need to show (13) only for \(1 \leq r \leq (1 - \epsilon)/(4c^*)\). Since \(1 - \epsilon(n) \leq 1\) and \(c^*(n) = \Omega\{\log(n)^{-1}\}, \, r \leq \{1 - \epsilon(n)/(4c^*(n)\} \) gives \(r = \mathcal{O}\{\log(n)\}, \) so \(r_t\{4\log(n)\}^{1/2}n^{-1/2} = o(1)\), as required.

Conditional on \(r_t\{4\log(n)\}^{1/2}n^{-1/2} \leq 1/8\) and \(4B^{-1}\log(n)n^{-1} \leq 1/8\), we have

\[
\sigma^2_{\omega_{t+1}} = \left\{ \frac{n}{2\log(n)(1 + a_t)} + r_t\frac{n^{1/2}}{(\log n)^{1.5}} + B^{-1} \right\}^{-1}
= \frac{2\log(n)(1 + a_t)}{n} \left\{ 1 + B^{-1}\frac{2\log(n)(1 + a_t)}{n} + r_t\frac{2(1 + a_t)}{(n\log n)^{1/2}} \right\}^{-1}
= \frac{2\log(n)(1 + a_t)}{n} \left\{ 1 - \mathcal{O}\left\{ \frac{B^{-1}\log(n)(1 + a_t)}{n} + r_t\frac{2(1 + a_t)}{(n\log n)^{1/2}} \right\} \right\}^{-1}
= \frac{2\log(n)(1 + a_t)}{n} \left\{ 1 - \mathcal{O}\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\} \right\}. 
\]

Thus, still conditional on \(r_t\{4\log(n)\}^{1/2}n^{-1/2} \leq 1/8\) and \(4B^{-1}\log(n)n^{-1} \leq 1/8\),

\[
\theta_{t+1}|\omega_{t+1} \sim N\left\{ \sigma^2_{\omega_{t+1}}(y - n/2), \sigma^2_{\omega_{t+1}} \right\}
= N\left\{ \frac{2 - n}{\log(n)(1 + a_t)} \left\{ 1 + \mathcal{O}\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\} \right\}, \log(n)(1 + a_t) \left\{ 1 + \mathcal{O}\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\} \right\} \right\}
= N\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\}, \log(n)(1 + a_t) \left\{ 1 + \mathcal{O}\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\} \right\}
= N\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\}, \frac{\log(n)(1 + a_t)}{n} \left\{ 1 + \mathcal{O}\left\{ \frac{r_t + 1}{(n\log n)^{1/2}} \right\} \right\}. 
\]

Applying this bound along with Chebyshev’s inequality to the second term, and applying inequality (22) to the first term, we conclude that

\[
\text{pr}\left\{ |\theta_{t+1} - \theta_t| > 2r\left(\frac{\log(n)}{n}\right) \right\} \leq \text{pr}\left\{ \left| \omega_{t+1} - \frac{n}{2\log(n)(1 + a_t)} \right| > r\frac{n^{1/2}}{(\log n)^{1.5}} \right\} \quad (23)
+ \text{pr}\left\{ |\theta_{t+1} - \theta_t| > 2r\left(\frac{\log(n)}{n}\right)^{1/2} \right\} \left| \omega_{t+1} - \frac{n}{2\log(n)(1 + a_t)} \right| \leq r\frac{n^{1/2}}{(\log n)^{1.5}} \right\}
= \mathcal{O}\left\{ \frac{\log(n)}{n} \right\}^{1/2} \right\}. 
\]

Thus, inequality (13) is satisfied for two sequences of constants \(\zeta = \zeta(n)\) and \(\gamma = \gamma(n)\) that satisfy

\[
\zeta(n) = \mathcal{O}\left\{ \left(\frac{\log(n)}{n}\right)^{1/2} \right\}, \quad \gamma(n) = \mathcal{O}\left\{ \left(\frac{\log(n)}{n}\right)^{1/2} \right\} \quad (24)
\]
on any sequence of sets \(I^* = I^*(n)\) satisfying \(I^*(n) \subset (-\log(n)(1 + C), -\log(n)(1 - C))\) and fixed \(0 < C < 1\).
By inequalities (15) and (18), the Inequalities (11) and (12) are satisfied with $\epsilon = \epsilon(n)$, $c = c(n)$ and $C = C(n)$ satisfying
\[
\{1 - \epsilon(n)\}^{-1} = O\left(\{\log(n)\}^2\right), \quad c(n) = \Omega\{\log(n)^{-1}\}, \quad C^*(n) = O(\log n)
\] (25)
and a set $I(n) \subset (-\log(n) - (\log(n))^{-1} - \eta_n, -\log(n) + (\log(n))^{-1} - \eta_n)$, where by Inequality (14) we have $\eta_n = O\{\log(\log(n))\}$. Combining this with (24) and Corollary A.1 completes the proof of Equation (3).

Finally, Equality (34) follows immediately from inequalities (15) and (18). This completes the proof of the Theorem.
Supplementary Materials

S1 Background

The purpose of this section is to provide a mathematically rigorous justification of the connection between the spectral gap \( \delta(P) \) and computational complexity. As in the main text, we focus on autocorrelations and effective sample size \( T_{\text{eff}} \). We also consider the \( \epsilon \)-mixing time is given by

\[
 t_\epsilon = \inf \left\{ t : \sup_S | \Pr(\theta_{t+1} \in S) - \Pi(S) | < \epsilon \right\},
\]

where \( \theta_0 \sim \nu \) for some initial state distribution \( \nu \), \( \Pi \) is the stationary measure, and the supremum is taken over \( \Pi \)-measurable sets \( S \), giving the total variation metric. Thus, the \( \epsilon \) mixing time is related to the length of the path necessary before the bias of ergodic averages becomes small. The rate at which \( t_\epsilon \) and \( T_{\text{eff}} \) scale with the sample size \( n \) will determine how the required path length scales with \( n \). We now describe the relationship between these quantities and the spectral gap.

The convergence behavior of Markov chain Monte Carlo is often studied by showing that the associated kernel \( P \) obeys a general ergodicity condition, such as the popular geometric ergodicity condition, see Roberts and Rosenthal (1997) and Meyn and Tweedie (1993):

**Geometric ergodicity** A Markov chain on a countably generated state space \( \Theta \) evolving according to a transition kernel \( P(\theta; \cdot) \) with invariant measure \( \Pi \) is \( V \)-geometrically ergodic if there exists constants \( \rho \in (0, 1) \) and \( B < \infty \) and a \( \Pi \)-almost everywhere-finite measurable function \( V : \Theta \to [1, \infty) \) such that

\[
 ||P^t(\theta_0; \cdot) - \Pi||_V \leq BV(\theta_0)\rho^t,
\]

where for a probability measure \( \mu(\cdot) \), \( ||\mu(\cdot)||_V = \sup_{f \leq V} |\mu(f)| \), with \( \mu(f) = \int f \, d\mu \).

The constants \( \rho \) and \( B \) do not directly lead to reasonable estimates of either the mixing time \( t_\epsilon \) or the effective sample size \( T_{\text{eff}} \) discussed in the main text. Both of these quantities can, however, be bounded in terms of the spectral gap of the associated chain. The following central limit theorem, quoted from Jones (2004), gives an estimate of the asymptotic variance of a Markov chain Monte Carlo estimate for a chain started at stationarity, which is related to the effective sample size via (5):

**Theorem S1.1** (Markov chain Monte Carlo Central Limit Theorem). Let \( \{\theta_t\}_{t \in \mathbb{N}} \) be a Markov chain evolving according to a transition kernel \( P(\theta; \cdot) \) with invariant measure \( \Pi \) satisfies the condition in Definition 57, for some function \( V \). Let \( f : \Theta \to [1, \infty) \) satisfy \( f^2 \leq V \), assume \( \theta_1 \) is distributed according to the unique stationary measure \( \Pi \), and define

\[
 \sigma_f^2 = \text{Var}\{f(\theta_1)\} + 2 \sum_{t=2}^\infty \text{Cov}\{f(\theta_1), f(\theta_t)\}.
\]

Then \( \sigma_f^2 \in [0, \infty) \). Furthermore, if \( \sigma_f^2 > 0 \), then

\[
 \lim_{T \to \infty} T^{1/2} \left\{ \frac{1}{T} \sum_{t=1}^T f(\theta_t) - \Pi f \right\} \overset{d}{=} \text{No}(0, \sigma_f^2),
\]

where \( \Pi f = \mathbb{E}_\Pi(f) \). [28] holds for any initial distribution on \( \theta_1 \).
The variance $\sigma_f^2$ is controlled by the spectral gap:

**Spectral Gap** Let $P(\theta; \cdot)$ be the transition kernel of a Markov chain with unique stationary distribution $\Pi$. The spectrum of $P$ is

$$S = \{ \lambda \in \mathbb{C} \setminus \{0\} : (\lambda I - P)^{-1} \text{ is not a bounded linear operator on } L^2(\Pi) \},$$

where $L^2(\Pi)$ is the space of $\Pi$-square integrable functions. The spectral gap of $P$ is given by

$$\delta(P) = 1 - \sup \{ |\lambda| : \lambda \in S, \lambda \neq 1 \}$$

when the eigenvalue 1 has multiplicity 1, and $\delta(P) = 0$ otherwise.

The asymptotic variance $\sigma_f^2$ always satisfies

$$\sigma_f^2 \leq \frac{2\var_{\Pi}(f)}{\delta(P)}, \quad (29)$$

which goes to infinity linearly in $\delta(P)^{-1}$. This bound is sharp for worst case functions when $P$ has no residual spectrum, which holds, for example, for reversible Markov chains on discrete state spaces. Thus, the spectral gap generally controls the asymptotic effective sample size via (29). Under stronger assumptions that are satisfied by the Markov chains studied in this paper — see Theorem S1.3 and Corollaries S3.1 and S3.2 — the spectral gap also controls the burn-in time. To state this carefully, we relate the spectral gap to the conductance of a Markov chain:

**Conductance** Let $P(\theta; \cdot)$ be the transition kernel of a Markov chain with invariant measure $\Pi$. For $\Pi$-measurable sets $S \subset \Theta$ with $0 < \Pi(S) < 1$, define

$$\kappa(S) = \frac{\int_{\theta \in S} P(\theta, S^c)\Pi(ds)}{\Pi(S)\{1 - \Pi(S)\}}$$

and the Cheeger constant or conductance

$$\kappa = \inf_{0 < \Pi(S) < 1} \kappa(S).$$

Theorem 2.1 of Lawler and Sokal (1988) relates conductance to the spectral gap:

**Theorem S1.2.** The spectral gap $\delta(P) = 1 - \lambda_1(P)$ of $P$ satisfies

$$\frac{\kappa^2}{8} \leq \delta(P) \leq \kappa.$$

The conductance, and thus the spectral gap, of a chain may be related to the required burn-in time by the following Theorem from Lovász and Simonovits (1993), which bounds the bias of a Markov chain Monte Carlo estimate after any number of steps:

**Theorem S1.3 (Warm Start Bound).** Let $P(\theta; \cdot)$ be the transition kernel of a Markov chain $\{\theta_t\}_{t \in \mathbb{N}}$ with invariant measure $\Pi$ and conductance $\kappa$. Then for all measurable sets $S \subset \Theta$,

$$|\text{pr}(\theta_{t+1} \in S) - \Pi(S)| \leq M^{1/2} \left(1 - \frac{\kappa^2}{2}\right)^{t}, \quad (30)$$

where $M = \sup_{A \subset \Theta} \frac{\text{pr}(\theta_0 \in A)}{\Pi(A)}$. 

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Let $\kappa_{\text{sup}}$ be an upper bound on $\kappa$. Then, using the definition in (26), we obtain the following upper bound on $t_c$:

$$t_c \leq \left[ \frac{\log \left( \frac{1}{2} \right)}{\log \left( 1 - \frac{\kappa_{\text{sup}}^2}{2} \right)} \right] \asymp -\frac{\kappa_{\text{sup}}^2}{2} \log \left( \frac{e}{M^{1/2}} \right).$$

(31)

As indicated by the $\asymp$ relation in (31), this quantity behaves like $\kappa^2$ for $\kappa^2$ near zero; since our results in §2 for the Pólya-Gamma and Albert and Chib samplers show the conductance converging to zero and also describe simple warm starts, this is the relevant asymptotic regime.

In light of these results, it is clear that the spectral gap gives an effective estimate of both the asymptotic effective sample size and required burn-in period. This justifies our emphasis on estimating the spectral gaps of Markov chains throughout the main article, since the effective sample size and burn-in time effectively control the length of a sample path necessary to obtain reasonable estimates of posterior expectations of functions based on pathwise ergodic averages.

### S2 Details of Data Augmentation samplers

We provide more detail on the two data augmentation samplers considered in the main article. [Polson et al. (2013)] introduce a data augmentation Gibbs sampler for posterior computation when $\varphi$ in (2) is the logit link $\ell(\cdot)$. The sampler has update rule given by

$$\omega \mid \theta \sim \text{PG}(n, \theta)$$

$$\theta \mid \omega \sim N \left( \left( \omega + B^{-1} \right)^{-1} \alpha, \left( \omega + B^{-1} \right)^{-1} \right),$$

(32a, 32b)

where $\alpha = y - n/2$ and $\text{PG}(a, c)$ is the Pólya-Gamma distribution with parameters $a$ and $c$. The transition kernel $P(\theta \mid \cdot)$ given by this update has $\theta$-marginal invariant measure the posterior $\Pi(\theta \mid y)$ for the model in (2). Choi et al. (2013) show that this sampler is uniformly ergodic.

A similar data augmentation scheme exists for the case where $\varphi$ is the standard Gaussian distribution function $\Phi(\cdot)$. Initially proposed by Albert and Chib (1993), the sampler has update rule

$$\omega \mid \theta = \sum_{i=1}^y z_i + \sum_{i=1}^{n-y} u_i, \quad z_i \sim \text{TN}(\theta, 1; 0, \infty), \quad u_i \sim \text{TN}(\theta, 1; -\infty, 0)$$

$$\theta \mid \omega \sim N \left( \left( n + B^{-1} \right)^{-1} \omega, \left( n + B^{-1} \right)^{-1} \right),$$

(33a, 33b)

where $\text{TN}(\mu, \tau^2; a, b)$ is the normal distribution with parameters $\mu$ and $\tau^2$ truncated to the interval $(a, b)$. The transition kernel $P(\theta \mid \cdot)$ for $\theta$ defined by this update has $\theta$-marginal invariant distribution $\Pi(\theta \mid y)$ for the model in (2) when $\varphi = \Phi$. It is clear from (33a) that the computational complexity per iteration scales linearly in $n$ for this algorithm. Although a recent manuscript proposes some more efficient samplers, the samplers in [Polson et al. (2013)] for $\text{PG}(n, \theta)$ also scale linearly in $n$. These observations will factor into our analysis of the overall run-time for these algorithms.

### S3 Corollaries of main results and implications for computational complexity

The following corollaries of Theorems 2.1 and 2.2 show that it is straightforward to begin from a “warm start” for either algorithm. This implies that the spectral gap bounds of Theorems 2.1 and 2.2 also give bounds on
Corollary S3.1 (Warm start for Pólya-Gamma). Let $\theta_{\text{max}} \equiv \arg\max_{\theta} p(\theta \mid y = 1)$ be the mode of $\Pi$ for the model in (2) with $\varphi$ the logit link. Then the distribution $\mu_n = \text{Unif}(\theta_{\text{max}} + \log(n), \theta_{\text{max}}) = O \left( \log(n) \right)^2$ (34) and thus provides a ‘warm start’ distribution for the Pólya-Gamma sampler. Note that $\theta_{\text{max}}$ is the unique solution to $\theta_{\text{max}} B + n e^{\theta_{\text{max}}} = 1$.

Corollary S3.2 (Warm start for Albert and Chib). Let $\Phi$ be the standard Gaussian cumulative distribution function. The distribution $\mu_n = \text{Unif}(\Phi^{-1} \left( \sum \frac{B_i + 2}{n+2} \right), \Phi^{-1} (\frac{C(B+2)}{Bn+2} ) )$ satisfies $\sup_{A \subset \mathbb{R}} \mu_n(A) = \mathcal{O}\left( \log(n) \right)$ (35) and thus provides a ‘warm start’ distribution for the Pólya-Gamma sampler.

The proofs of Corollaries S3.1 and S3.2 are contained within the proofs of Theorems 2.1 and 2.2.

S4 Metropolis within Gibbs steps for Application

1. Compute the conditional mode of $\hat{\theta}_i = p(\theta_i \mid \theta_0, y_i, n_i)$ for each $i = 1, \ldots, N$ using Newton-Raphson. This consists of $N$ independent univariate convex optimization problems for which both the first and second derivatives with respect to $\theta_i$ are available. This step is thus quite fast.

2. Propose $\theta_i^*$ from $t_5(\hat{\theta}_i, v)$, where $t_\nu(m, v)$ is a $t$ distribution with $\nu$ degrees of freedom and scale $v$ centered at $m$. Tune $v$ to give 30-40 percent acceptance rates.

3. Compute the acceptance probability

$$\log(q^*) = Y \log \left( \frac{p_i^*}{p_i} \right) + (N - Y) \log \left( \frac{1 - p_i^*}{1 - p_i} \right) - \frac{1}{2} \frac{(\theta_0 - \theta_i^*)^2}{\tau^2} + \frac{1}{2} \frac{(\theta_0 - \theta_i)^2}{\tau^2}$$

where $p_i = e^{\theta_i}/(1 + e^{\theta_i})$ and perform a Metropolis step.

4. Sample $\theta_0$ and $\tau^2$ from

$$\theta_0 \mid \theta_i, \tau^2 \sim N \left( sm, s \right), \quad s = (N/\tau^2 + 1/\tau_0^2)^{-1}, \quad m = \left( \sum_i \theta_i/\tau^2 + \theta_0/\tau_0^2 \right)$$

$$\tau^{-2} \mid \theta_i \sim \text{Ga} \left( \frac{(N - 1)}{2}, \sum_i (\theta_0 - \theta_i)^2/2 \right).$$
S5  Proof of Theorem 2.2

First we give a lemma that is used in the main proof to bound \( \Phi^{-1}(x) \) and \( \{\Phi^{-1}(x)\}^2 \).

**Lemma S5.1.** Let \( \Phi(\cdot) \) be the standard normal distribution function and fix \( x > 0 \). Then, as \( n \to \infty \),

\[
\Phi^{-1}\left( \frac{x}{n} \right) = -\{2 \log(n/x)\}^{1/2} \left( 1 - \frac{\log \left\{ \frac{4}{n} \log(n/x) \right\}}{2 \log(n/x)} + O\left( \frac{1}{\log(n/x)^{1.5}} \right) \right)^{1/2}
\]

(36)

Furthermore,

\[
\{\Phi^{-1}\left( \frac{x}{n} \right)\}^2 = 2 \log \left( \frac{n}{x} \right) - \log \left\{ 2 \log \left( \frac{n}{x} \right) \right\} + \log (2/\pi) + O\left( \frac{1}{\log(n/x)} \right).
\]

(37)

**Proof.** From equations 7.8.1 and 7.8.2 of [DLMF](2015), we have for \( x \geq 0 \)

\[
\frac{1}{x/\sqrt{2} + (x^2/2 + 2)^{1/2}} < e^{x^2/2} \int_{x/\sqrt{2}}^{\infty} e^{-t^2} dt \leq \frac{1}{x/\sqrt{2} + (x^2/2 + 4/\pi)^{1/2}}
\]

(38)

\[
\frac{1}{x + (x^2 + 4)^{1/2}} < (\pi/2)^{1/2} e^{x^2/2} \{1 - \Phi(x)\} \leq \frac{1}{x + (x^2 + 8/\pi)^{1/2}}
\]

Thus, we can write

\[
(\pi/2)^{1/2} e^{x^2/2} \{1 - \Phi(x)\} = \frac{1}{x + \{x^2 + h(x)\}^{1/2}}
\]

for some function \( h(x) \) that satisfies \( 8/\pi \leq h(x) \leq 4 \) for all \( x \geq 0 \), giving

\[
1 - \Phi(x) = (2/\pi)^{1/2} e^{-x^2/2} \frac{1}{x + \{x^2 + h(x)\}^{1/2}}.
\]

Writing \( y = \Phi(x) \), so that \( y > 1/2 \) from the original condition for the inequality, and inverting gives

\[
1 - y = (2/\pi)^{1/2} e^{-x^2/2} \left[ x + \{x^2 + h(x)\}^{1/2} \right]^{-1}
\]

(39)

\[
\log(1 - y) = \log(2/\pi)/2 - x^2/2 - \log \left[ x + \{x^2 + h(x)\}^{1/2} \right]
\]

\[
x^2 = -2 \log(1 - y) + \log(2/\pi) - 2 \log \left[ x + \{x^2 + h(x)\}^{1/2} \right].
\]

We now claim that for any fixed \( \epsilon > 0 \) and any sufficiently large \( x > X(\epsilon) \), we have \( \{-2 - \epsilon \log(1 - y)\}^{1/2} < x < \{-2 + \epsilon \log(1 - y)\}^{1/2} \). To see this, recall that by Inequality (38), for any fixed \( \epsilon > 0 \),

\[
(2/\pi)^{1/2} e^{-(1+\epsilon)x^2/2} \leq 1 - \Phi(x) \leq (2/\pi)^{1/2} e^{-(1-\epsilon)x^2/2}
\]

for all sufficiently large \( x \). Substituting this bound into (39) we obtain

\[
x^2 = -2 \log(1 - y) + \log(2/\pi) - \log\{-2 \log(1 - y)\} + O \left( \frac{1}{-\log(1 - y)} \right),
\]

22
for $1 - y < 1/2$ which gives

$$x = \pm \left[ -2 \log(1 - y) + \log(2/\pi) - \log\{-2 \log(1 - y)\} + O\left\{ \frac{1}{-\log(1 - y)} \right\} \right]^{1/2}.$$  

To get the result for arguments $1 - y < 1/2$, we take the negative solution, giving

$$x = -\left\{ -2 \log(1 - y) \right\}^{1/2} \left( 1 + \frac{\log\{(4/\pi) \log(1 - y)\}}{2 \log(1 - y)} + O\left\{ \frac{1}{\left\{ \log(1 - y) \right\}^{1.5}} \right\} \right)^{1/2}.$$  

Also since $(1 + o(1))^{1/2} = 1 + [(1 + o(1))^{1/2} - 1] = 1 + o(1),$

$$x = -\left\{ -2 \log(1 - y) \right\}^{1/2} \{1 + o(1)\}.$$  

Setting $1 - y = x/n$ for $x/n < 1/2$ – the region where $\Phi^{-1}(x/n) < 0$ – we have

$$\{\Phi^{-1}(x/n)\}^2 = 2 \log(n/x) + \log(2/\pi) - \log\{2 \log(n/x)\} + O\left\{ \frac{1}{\log(n/x)} \right\}$$  

and

$$\Phi^{-1}(x/n) = -\left\{ 2 \log(n/x) \right\}^{1/2} \left( 1 - \frac{\log\{(4/\pi) \log(n/x)\}}{2 \log(n/x)} + O\left\{ \frac{1}{\left\{ \log(n/x) \right\}^{1.5}} \right\} \right)^{1/2}$$  

$$= -\left\{ 2 \log(n/x) \right\}^{1/2} \{1 + o(1)\},$$

completing the proof. □

**Proof of main result**

The main result is proved in four steps; the rationale for each step is outlined in §A.2

(a) Obtain bounds on quantities that will appear in steps (b) through (d);

(b) Find an interval $I'(n)$ outside of which the posterior is negligible, in the sense of integrating to $o(1)$, and find an upper bound its width;

(c) Find an interval $I(n) \subset I'(n)$ containing the posterior mode $\hat{\theta}$ on which the posterior density ratio is bounded below by a constant and show a lower bound on its width; and

(d) Show a concentration inequality for $|\theta_t - \theta_{t+1}|$ when $\theta_t \in I(n)$.

**Part (a) : obtaining additional bounds**

Recall that the Albert and Chib sampler has the update rule given by sampling $\omega_{t+1} \mid y, n, \theta_t$ then sampling $\theta_{t+1} \mid \omega_{t+1}$ from a Gaussian. $\omega_{t+1}$ is the sum of $n - y$ independent Gaussians truncated below by zero and $y$ independent Gaussians truncated above by zero; here we always have $y = 1$. Then, the expectation and variance of $\omega_{t+1}$ given $\theta_t$ are

$$E(\omega_{t+1} \mid \theta_t, n, y) = (n - 1) \left\{ \theta_t - \frac{\phi(\theta_t)}{1 - \Phi(\theta_t)} \right\} + \left\{ \theta_t + \frac{\phi(\theta_t)}{\Phi(\theta_t)} \right\}$$

$$= n \theta_t - (n - 1) \frac{\phi(\theta_t)}{1 - \Phi(\theta_t)} + \frac{\phi(\theta_t)}{\Phi(\theta_t)}$$

$$\text{var}(\omega_{t+1} \mid \theta_t, n, y) = v_t = n + (n - 1) \left[ \theta_t \frac{\phi(\theta_t)}{1 - \Phi(\theta_t)} - \left\{ \frac{\phi(\theta_t)}{1 - \Phi(\theta_t)} \right\}^2 \right] - \theta_t^2 \frac{\phi(\theta_t)}{\Phi(\theta_t)} - \frac{\phi(\theta_t)^2}{\Phi(\theta_t)^2}.$$
We now compute the posterior mode \( \hat{\theta} \). We begin by reparameterizing our problem by the one-to-one transformation \( \hat{\theta} = \Phi^{-1}(x/n) \). We will compute \( \hat{x} \), the posterior mode under this transformation, and then use this to compute the mode \( \hat{\theta} \) on the original scale by the equation \( \hat{\theta} = \Phi^{-1}(\hat{x}/n) \).

We will require an approximation to \( \phi_B(\Phi^{-1}(x/n)) \), where \( \phi_B \) is the density of \( N(0, B) \). Using (37),

\[
\phi_B \left\{ \Phi^{-1} \left( \frac{x}{n} \right) \right\} = \frac{1}{(2\pi B)^{1/2}} \exp \left[ -\frac{2 \log(n/x)}{2B} + \frac{1}{2B} \log \left\{ 2 \log \left( \frac{n}{x} \right) \right\} - \frac{1}{2B} \log \left( 2/\pi \right) + o(1) \right]
\]

where

\[
= \frac{1}{(2\pi B)^{1/2}} \left( \frac{x}{n} \right)^{1/B} \left\{ \left( 2 \log(n/x) \right)^{1/2} \right\}^{1/B} \left( \pi/2 \right)^{1/2B} \exp(o(1)). \tag{40}
\]

The posterior density when \( y = 1 \) is proportional to

\[
p(\theta|n, y) \propto n\Phi(\theta) \{1 - \Phi(\theta)\}^{n-1} \phi_B(\theta).
\]

Under our reparameterization,

\[
\log p(x \mid n, y) \propto \log x + (n - 1) \log \left( 1 - \frac{x}{n} \right) - \frac{\Phi^{-1}(x/n)^2}{2B}.
\]

Differentiating to find the mode,

\[
\frac{\partial}{\partial x} \log p(x \mid n, y) = \frac{1}{x} - \frac{n - 1}{n - x} = \frac{(2\pi)^{1/2}}{Bn} \exp \left\{ -\frac{\Phi^{-1}(x/n)^2}{2B} \right\} \Phi^{-1}(x/n).
\]

Using (37) and (36), we have

\[
\frac{\partial}{\partial x} \log p(x \mid n, y) = \frac{1}{x} - \frac{n - 1}{n - x} = \frac{(2\pi)^{1/2}}{Bn} \left\{ 2 \log(n/x) \right\}^{1/2} (2/\pi)^{1/2}
\]

\[
\times \exp(o(1)) \left\{ -\{2 \log(n/x)\}^{1/2}(1 + o(1)) \right\}
\]

\[
= \frac{1}{x} - \frac{n - 1}{n - x} + \frac{2}{Bn} \left\{ 1 + o(1) \right\},
\]

so in the limit as \( n \to \infty \) the posterior mode is

\[
\hat{x} = \frac{n \{ B + 2 + o(1) \}}{Bn + 2}. \tag{41}
\]

In particular, for large enough \( n \), \( \hat{x}/n \) is in the interval

\[
\frac{\hat{x}}{n} \in \left[ \frac{B + 2}{2(Bn + 2)}, \frac{2(B + 2)}{Bn + 2} \right].
\]

**Part (b) : find an interval \( I'(n) \) outside of which the posterior is negligible** We now implement the first part of the approach to showing conditions (11) and (12). As described in §A.2, we show an interval \( I'(n) \) outside of which the posterior is negligible, that is, integrates to \( o(1) \). Fix \( C > 2 \) and consider the interval \( I'(n) = [\Phi^{-1}(n^{-C^2}), \Phi^{-1}(1 - n^{-C^2})] \). First we bound the width of this interval and the size of the increments \( |\Phi^{-1}(1 - n^{-C^2}) - \Phi^{-1}(1 - n^{-C^2})| \). Bounding the width from above is necessary for showing
condition (11), and bounding the size of the increments is necessary to show that the posterior integrates to \( o(1) \) outside this interval. From (36):

\[
\Phi^{-1}(1 - n^{-C^2}) = -\Phi^{-1}(n^{-C^2}) = \{2 \log(n^{C^2})\}^{1/2} \{1 + o(1)\}
\]

So then

\[
|\Phi^{-1}(n^{-C^2}) - \Phi^{-1}(1 - n^{-C^2})| = 2C\{2 \log(n)\}^{1/2} \{1 + o(1)\}
\]  

(42)

and

\[
|\Phi^{-1}(n^{-(C+1)^2}) - \Phi^{-1}(1 - n^{-C^2})| = \{2 \log(n)\}^{1/2} \{1 + o(1)\}.
\]

Now we bound the posterior density \( \Phi^{-1}(1 - n^{-C^2}) \), which will be used to bound the integral of the posterior on the complement of \( I'(n) \)

\[
p(\theta \mid y = 1) = n(1 - n^{-C^2}) \left\{1 - (1 - n^{-C^2})\right\}^{n-1} \phi_B\{\Phi^{-1}(1 - n^{-C^2})\}
\]

\[
\leq n^{-C^2} n^{C^2+1} (2\pi B)^{-1/2}.
\]

We have with \( p_1(\theta) = p(\theta \mid y = 1) \)

\[
\int_{\Phi^{-1}(1 - n^{-C^2})}^{\Phi^{-1}(n^{-C^2})} p_1(\theta) d\theta \leq (2\pi B)^{-1/2} \sum_{\phi=2}^{\infty} \{2 \log(n)\}^{1/2} \{1 + o(1)\} n^{C^2+1-C^2 n} = o(1).
\]

So the posterior measure of the part of \( \{I'(n)\}^c \) that contains values of \( \theta \) greater than those in \( I'(n) \) is \( o(1) \).

Now we take the same approach to show this for the part of \( \{I'(n)\}^c \) consisting of values of \( \theta \) less than those in \( I'(n) \). We have that \( p(\theta \mid y = 1) \) for \( \theta = \Phi^{-1}(n^{-C^2}) \) satisfies

\[
p(\theta \mid y = 1) = n(n^{-C^2})(1 - n^{-C^2})^{n-1} (2\pi B)^{-1/2}
\]

\[
\leq n^{1-C^2} e^{-n^{-(C^2-1)}} (1 - n^{-C^2})^{-1} (2\pi B)^{-1/2}
\]

\[
\leq n^{1-C^2} e^{-1(4/3)/(2\pi B)^{-1/2}}
\]

when \( n \geq 2 \). So then

\[
\int_{-\infty}^{\Phi^{-1}(n^{-C^2})} p_0(\theta)d\theta \leq (4/3)(2\pi B)^{-1/2} \sum_{\phi=2}^{\infty} \{2 \log(n)\}^{1/2} \{1 + o(1)\} n^{1-C^2} = o(1).
\]

We conclude

\[
\int_{\{I'(n)\}^c} p(\theta \mid y = 1) d\theta = o(1)
\]  

(43)

for \( I'(n) = [\Phi^{-1}(n^{-C^2}), \Phi^{-1}(1 - n^{-C^2})] \) with \( C > 2 \), so the posterior is negligible outside an interval of length \( O(\{\log(n)\}^{1/2}) \) based on (42).
Part (c): Find an interval $I(n) \subset I'(n)$ containing the mode on which the posterior is almost constant

We now do the second step outlined in Part (c): Find an interval $I(n)$ containing the posterior mode on which the posterior is bounded below by a constant for all large $n$. Again fix a constant $2 < C < \infty$. We now show that for $n > N(C)$ sufficiently large, where the function $N(C)$ depends only on $C$, the posterior is almost constant on the interval

$$I(n) = \left[ \Phi^{-1} \left\{ \frac{B+2}{C(Bn+2)} \right\}, \Phi^{-1} \left\{ \frac{C(B+2)}{Bn+2} \right\} \right].$$

As shown in Equality (41), this interval includes the posterior mode for all large enough $n$. This interval has width $\Omega \{(\log n)^{-1/2}\}$. To see this, put $q(n) = (B+2)^{-1}(Bn+2)$, then

$$|I(n)| = 2 \log\{Cq(n)\} \left\{ 1 - \frac{\log[4/\pi \log C q(n)]}{2 \log\{Cq(n)\}} + \mathcal{O}\left(\log\{q(n)\}\right)^{-1.5} \right\}^{1/2}$$

$$- 2 \log\{q(n)/C\} \left\{ 1 - \frac{\log[4/\pi \log \{q(n)/C\}]}{2 \log\{q(n)/C\}} + \mathcal{O}\left(\log\{q(n)\}\right)^{-1.5} \right\}^{1/2}$$

$$\equiv f_1(n,C) - f_2(n,C),$$

where $|I(n)|$ is the width of $I(n)$. Now multiply the right side by $f_1(n,C) + f_2(n,C)$ to get

$$2 \log\{Cq(n)\} - \log[4/\pi \log C q(n)] + \mathcal{O}\left(\log\{q(n)\}\right)^{-1/2}$$

$$- 2 \log\{q(n)/C\} + \log[4/\pi \log \{q(n)/C\}] + \mathcal{O}\left(\log\{q(n)\}\right)^{-1/2}$$

$$= 4 \log(C) - 2 \log\{\log(C)\} + o(1).$$

Since

$$f_1(n,C) + f_2(n,C) = \mathcal{O}\left\{ (\log n)^{1/2}\right\}$$

we get that

$$|I(n)| = \Omega \{ (\log n)^{-1/2}\}. \quad (44)$$

Recall the posterior mode is $\hat{\theta} = \Phi^{-1}\{[B + 2 + o(1)]/(Bn + 2)\}$, which is contained in $I(n)$ for sufficiently large $n$. Set $\theta_0 = \Phi^{-1}\{C(B + 2)/(Bn + 2)\}$. We will bound the ratio of the posterior densities on the interval $I_1(n) = [\hat{\theta}, \theta_0]$, which is a subset of our interval $I(n)$. Repeatedly applying Lemma [S5.1] we have

$$\frac{p(y = 1 | \hat{\theta})}{p(y = 1 | \theta_0)} = \frac{n \left\{ \frac{B+2+o(1)}{Bn+2} \right\} \left\{ 1 - \frac{B+2+o(1)}{Bn+2} \right\}^{n-1} \Phi_B \left\{ \frac{B+2+o(1)}{Bn+2} \right\}}{n \left\{ \frac{B+2}{Bn+2} \right\} \left\{ 1 - \frac{B+2}{Bn+2} \right\}^{n-1} \Phi_B \left\{ \frac{B+2}{Bn+2} \right\}}$$

$$= \frac{n \left\{ \frac{B+2}{Bn+2} \right\} \left\{ 1 - \frac{B+2}{Bn+2} \right\}^{n-1} \Phi_B \left\{ \frac{B+2}{Bn+2} \right\}}{n \left\{ \frac{C(B+2)}{Bn+2} \right\} \left\{ 1 - \frac{C(B+2)}{Bn+2} \right\}^{n-1} \Phi_B \left\{ \frac{C(B+2)}{Bn+2} \right\}} + o(1)$$
Then clearly, there exists $J.E. \text{ Johndrow}, \text{ A. Smith}, \text{ N. Pillai}, \text{ D. Dunson}$

Now define $I$ so in particular, since the posterior is unimodal and $\theta_0$ is the endpoint of $I_1(n)$, there exists $N_0(C) < \infty$ such that $n > N_0(C)$ implies

$$
\inf_{\theta_0 \in I_1(n)} \frac{p(y = 1 | \theta)}{p(y = 1 | \theta_0)} > (1/2) \left( 1/C \right)^{1 + 1/B} e^{(B+2)(C-1)/B}.
$$

Now define $I_2(n) = [\theta_1, \tilde{\theta}]$ with $\theta_1 = \Phi^{-1}[(B + 2)/\{C(Bn + 2)\}] = \Phi^{-1}[(1/C)(B + 2)/(Bn + 2)]$.

Then clearly, there exists $N_1(C) < \infty$ such that $n > N_1(C)$ such that

$$
\inf_{\theta_0 \in I_2(n)} \frac{p(y = 1 | \theta)}{p(y = 1 | \theta_0)} > (1/2) \left( 1/C \right)^{1 + 1/B} e^{(B+2)(C-1)/B}.
$$

Put $N(C) = \max\{N_0(C), N_1(C)\}$. Then since $I(n) = I_1(n) \cup I_2(n), n > N(C)$ implies

$$
\inf_{\theta_0 \in I(n)} \frac{p(y = 1 | \theta)}{p(y = 1 | \theta_0)} > (1/2) \min \left\{ \left( 1/C \right)^{1 + 1/B} e^{(B+2)(C-1)/B}, C^{1 + 1/B} e^{(B+2)(1/C-1)/B} \right\}. \quad (45)
$$
Combining with (44), this implies the posterior density is bounded below by a constant on an interval of width \( \Omega \left( \{ \log(n) \}^{-1/2} \right) \). Parts (b) and (c) together then give \( c^*(n) = \Omega\{ \log(n)^{-1/2} \} \), \( C^*(n) = \mathcal{O} \{ \log(n)^{1/2} \} \), and \( 1 - \epsilon(n) = \Omega \{ \log(n)^{-1} \} \).

**Part (d):** show a concentration result for \( |\theta_{t+1} - \theta_t| \) inside \( I(n) \). We now show a concentration inequality for \( |\theta_{t} - \theta_{t+1}| \) for \( \theta_t \) inside the interval \( I(n) \). Fix a constant \( 2 < C < \infty \). When we have \( \theta_t \) inside the interval

\[
I(n) = \left[ \Phi^{-1} \left\{ \frac{B + 2}{C(Bn + 2)} \right\}, \Phi^{-1} \left\{ \frac{C(B + 2)}{Bn + 2} \right\} \right],
\]

we can write \( \theta_t = \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 2} \right\} \) for \( a_t \in [C^{-1}, C] \), which by (41) contains the posterior mode for large enough \( n \). The term \( \phi \left( \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 2} \right\} \right) \) will appear often. We have that

\[
\phi \left( \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 2} \right\} \right) = \mathcal{O} \left\{ \frac{2 \log(Bn + 2)^{1/2}}{Bn + 2} \right\}
\]

by (50).

The conditional mean of \( \omega_{t+1} | \omega_{t+1} \) will be approximately \( \omega_{t+1}/n \) for large \( n \), so we calculate the first two moments of \( \omega_{t+1}/n \) for use in the concentration argument that follows. For \( \theta_t \in I(n) \) as above we have

\[
E \left( \omega_{t+1}/n \mid \theta_t, y, n \right) = \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\} - \phi \left[ \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\} \right]
\]

\[
\times \left( \frac{n - 1}{n} - \frac{1}{n \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\}} \right)
\]

\[
= \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\} - \phi \left[ \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\} \right]
\]

\[
\times \left( 1 - \frac{1}{n \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 1} \right\}} \right)
\]

\[
= \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 2} \right\} - \mathcal{O} \left\{ \frac{2 \log(Bn + 2)^{1/2}}{Bn + 2} \right\}
\]

\[
= \Phi^{-1} \left\{ \frac{a_t(B + 2)}{Bn + 2} \right\} + \mathcal{O} \left\{ n^{-1}(\log n)^{1/2} \right\},
\]

and

\[
\text{var}(\omega_{t+1}/n \mid \theta_t, y, n) = \frac{1}{n} + \frac{n - 1}{n^2} \left[ \theta_t \phi(\theta_t) - \left( \frac{\phi(\theta_t)}{1 - \Phi(\theta_t)} \right)^2 \right] - \theta_t \phi(\theta_t)
\]

\[
= \frac{1}{n} + \theta_t \phi(\theta_t) \left[ \frac{n - 1}{n^2} - \frac{1}{1 - \Phi(\theta_t)} \right]
\]

\[
- \phi(\theta_t)^2 \left[ \frac{n - 1}{n^2 (1 - \Phi(\theta_t))^2} - \frac{1}{n^2 \Phi(\theta_t)^2} \right]
\]

\[
= \frac{1}{n} + \theta_t \phi(\theta_t) \left[ \frac{n - 1}{n^2} \frac{Bn + 2}{Bn + 2 - a_t(B + 2)} - \frac{1}{n^2 a_t(B + 2)} \right]
\]
applying (47) and (46)

By (47) and (46), there exists a constant $\zeta$

\[ \text{pr} \left( \frac{\theta - \theta_0}{\sqrt{n}} \geq r \zeta \right) \]

\[ = \frac{1}{n} + \phi(\theta_0) \mathcal{O}(1) - \phi(\theta_0)^2 \mathcal{O}(1) \]

\[ = \frac{1}{n} + \phi(\theta_0) \mathcal{O}(1) \]

\[ = \frac{1}{n} + \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right) \mathcal{O} \left( \frac{2 \log(B + 2)}{B + 2} \right) \]

\[ = \frac{1}{n} + \left( \frac{2 \log \left( \frac{B + 2}{a_t(B + 2)} \right)}{B + 2} \right)^{1/2} + \mathcal{O} \left( \frac{2 \log(B + 2)}{(B + 2)^2} \right) \]

\[ \times \mathcal{O} \left( \frac{2 \log(B + 2)}{B + 2} \right) + \mathcal{O} \left( \frac{\log \{2 \log(B + 2)\}}{B + 2} \right) + \mathcal{O} \left( \frac{2 \log(B + 2)}{(B + 2)^2} \right) \]

\[ = \mathcal{O} \left( \frac{\log n}{n} \right) \quad (47) \]

Next, for $\theta_t = \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right)$ – equivalently, $\theta_t \in I(n)$ – we want to show a uniform upper bound on $\text{pr}(|\theta_t - \theta_{t+1}| > r \zeta)$. Our strategy is to show a uniform lower bound on $\text{pr}(|\theta_t - \theta_{t+1}| < r \zeta)$ for $\zeta > 0, r \geq 1$. By the triangle inequality,

\[ |\theta_t - \theta_{t+1}| < \left| \theta_t - \frac{\omega_{t+1}}{n} \right| + \left| \omega_{t+1}/n - \theta_{t+1} \right|. \]

It follows that,

\[ \text{pr}(|\theta_t - \theta_{t+1}| < r \zeta) \geq \text{pr} \left( \left| \theta_t - \frac{\omega_{t+1}}{n} \right| < \frac{r \zeta}{2} |\omega_{t+1}/n - \theta_{t+1}| < \frac{r \zeta}{2} \right) \]

\[ \geq \text{pr} \left( \left| \theta_t - \frac{\omega_{t+1}}{n} \right| < \frac{r \zeta}{2} \right) \text{pr} \left( |\omega_{t+1}/n - \theta_{t+1}| < \frac{r \zeta}{2} \right). \]

Since $\theta_t = \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right)$, the first term on the right side is

\[ \text{pr} \left( \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right) - \frac{\omega_{t+1}}{n} < \frac{r \zeta}{2} \right). \]

By (47) and (46), there exists a constant $1 < A < \infty$ and an $N_0 < \infty$ such that $n > N_0$ implies $\text{var}(\omega_{t+1}/n | \theta_t, y, n) < A^2 (\log n)^{n^{-1}}$, and

\[ \delta_\omega(n) = \left| E \left( \omega_{t+1}/n | \theta_t, y, n \right) - \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right) \right| < 2A(\log n)^{1/2}n^{-1/2} \]

Putting $\zeta = 8An^{-1/2}(\log n)^{1/2}$ and recognizing that the distribution of $\omega_{t+1} | \theta_t$ is sub-Gaussian, we have, applying (47) and (46)

\[ \text{pr} \left( \left| \frac{\omega_{t+1}}{n} - \Phi^{-1} \left( \frac{a_t(B + 2)}{B + 2} \right) \right| > \frac{r \zeta}{2} \right) < e^{-2n^2} \quad (48) \]
For the second term, recall
\[
\theta_{t+1} \mid \omega_{t+1}, n \sim \text{No}\{(n + B^{-1})^{-1} \omega_{t+1}, (n + B^{-1})^{-1}\}
\]
\[\sim \text{No}\left\{\frac{n}{(n + B^{-1})}, (n + B^{-1})^{-1}\right\}.
\]
So then there exists \(N_1 < \infty\) depending only on \(A, B\) such that for all \(n > N_1\), the following holds using a Gaussian tail bound,
\[
\text{pr}\left(\frac{\omega_{t+1}}{n} - \Phi^{-1}\left\{\frac{a_t(B + 2)}{Bn + 1}\right\} > \frac{r8A(\log n)^{1/2}}{2n^{1/2}}\right) \leq e^{-r^2}
\]
Conditional on \(\Phi^{-1}\left\{\frac{a_t(B + 2)}{Bn + 2}\right\} - \frac{\omega_{t+1}}{n} < \frac{r8A(\log n)^{1/2}}{2n^{1/2}}\) we have
\[
\frac{\omega_{t+1}}{n} < \left[2\log \left\{\frac{Bn + 2}{a_t(B + 2)}\right\}\right]^{1/2}\{1 + o(1)\} + \frac{r8A(\log n)^{1/2}}{2n^{1/2}},
\]
\[
< \left[2\log \left\{\frac{C(Bn + 2)}{(B + 2)}\right\}\right]^{1/2}\{1 + o(1)\} + \frac{r8A(\log n)^{1/2}}{2n^{1/2}},
\]
where the second line followed since \(a_t \in [1/C, C]\). So there exists \(C_0 < \infty\) and a function \(N_0(r)\) depending only on \(r\) and \(A < \infty\) such that for every \(r \geq 1, n > N_0(r)\) implies
\[
\frac{\omega_{t+1}}{Bn^2 + n} < \frac{C_0(\log n)^{1/2}}{Bn}.
\]
So then for any \(r\) there exists \(N_1(r) < \infty\) depending on \(r\) and \(B\) such that \(n > \max\{N_1(r), N_0(r)\} = N_{\text{max}}(r)\) implies
\[
\frac{\omega_{t+1}}{Bn^2 + n} < \frac{r8A(\log n)^{1/2}}{4n^{1/2}}.
\]
Since \(r8A(\log n)^{1/2}(4n^{1/2})^{-1}\) is increasing in \(r\) and we have \(r \geq 1\), choose \(N^{\text{max}} = N_{\text{max}}(1)\), so that \(n > N_{\text{max}}\) implies \(\frac{\omega_{t+1}}{Bn^2 + n} < \frac{r8A(\log n)^{1/2}}{4n^{1/2}}\) uniformly over \(r\). Then for all \(n > \max\{N_{\text{max}}, N_1\}\), we have
\[
\text{pr}\left(\frac{\omega_{t+1}}{n} - \theta_{t+1} > \frac{r8A(\log n)^{1/2}}{2n^{1/2}}\right|\theta_t - \omega_{t+1}/n < \frac{1}{2}) \leq e^{-r^2}\log(n).
\]
Putting together (48) and (50) we have for all 
\[ n > \max\{N_{\max}, N_1, N_0\}, \]
that \( \theta_t \in I(n) \) implies

\[
\Pr \left( |\theta_t - \theta_{t+1}| < \frac{8A(\log n)^{1/2}}{n^{1/2}} \right) \geq \left[ 1 - e^{-r^2\log(n)} \right] \left[ 1 - e^{-2r^2} \right]
\]

so

\[
\Pr \left( |\theta_{t+1} - \theta_t| > \frac{8A(\log n)^{1/2}}{n^{1/2}} \right) \leq 1 - \left\{ 1 - e^{-r^2\log(n)} \right\} \left\{ 1 - e^{-2r^2} \right\}
\]

\[
\leq e^{-r^2\log(n)} + e^{-2r^2} - e^{-r^2\log(n) + 2r^2}
\]

\[
\leq e^{-r^2\log(n)} \left[ 1 - e^{-2r^2} \right] + e^{-2r^2}.
\]

For \( r \geq 1 \), the term \( 1 - e^{-2r^2} \) is bounded above by 1, and \( e^{-2r^2} < r^{-2} \). So then

\[
\Pr \left( |\theta_{t+1} - \theta_t| > \frac{r8A(\log n)^{1/2}}{n^{1/2}} \right) \leq O(n^{-1}) + r^{-2},
\]

uniformly over \( r \), since \( e^{-r^2\log n} = O(n^{-1}) \) for \( r \geq 1 \).

Since the posterior is negligible outside a region of width \( O\{(\log n)^{1/2}\} \) by (42) and is almost constant on an interval of width \( \Omega\{(\log n)^{-1/2}\} \) by (44) and (45), we have \( 1 - \epsilon(n) = \Omega\{(\log n)^{-1}\} \); \( \zeta(n) = O\{(\log n)^{-1/2}\} \) from (51), \( \gamma = O(n^{-1}) \) from (51) and \( c^*(n) = \Omega\{(\log n)^{-1/2}\} \). This gives

\[
\delta(\mathcal{P}) = O\{(\log n)^{-2.5} n^{-1/2}\} \leq O\{(\log n)^{-1}\}.
\]

Finally, we prove Inequality (35). Combining inequalities (41) and (45) with Lemma S5.1, we have shown that the mode is contained within an interval of length \( \Omega(\log n)^{-1/2} \) for which the density is \( \Omega\{(\log n)^{-1/2}\} \). Combining inequality (43) with Lemma S5.1, we have shown that the posterior distribution is negligible outside of an interval of length \( \Omega\{(\log n)^{1/2}\} \). Inequality (35) follows immediately.