Abstract

The main result is that a finite dimensional normed space embeds isometrically in \( \ell_p \) if and only if it has a discrete Levy \( p \)-representation. This provides an alternative answer to a question raised by Pietch, and as a corollary, a simple proof of the fact that unless \( p \) is an even integer, the two-dimensional Hilbert space \( \ell_2^2 \) is not isometric to a subspace of \( \ell_p \). The situation for \( \ell_q^2 \) with \( q \neq 2 \) turns out to be much more restrictive. The main result combined with a result of Dor provides a proof of the fact that if \( q \neq 2 \) then \( \ell_q^2 \) is not isometric to a subspace of \( \ell_p \) unless \( q = p \). Further applications concerning restrictions on the degree of smoothness of finite dimensional subspaces of \( \ell_p \) are included as well.
On a question of Pietch

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1 Introduction

Which finite-dimensional subspaces of $L_p$ are isometric to subspaces of $\ell_p$? This question, according to [1], was asked by A. Pietch. The results of [1] revealed an answer that depends on whether $p$ is an even integer ($p \in 2\mathbb{N}$) or not. It is summarized by the following theorem.

**Theorem 1.** 1. If $p \in 2\mathbb{N}$, then every finite dimensional subspace of $L_p$ is isometric to a subspace of $\ell_p$. ([1], Theorem B).

2. If $p \not\in 2\mathbb{N}$, then a closed linear subspace $X$ of $L_p$ is isometric to a subspace of $\ell_p$ if and only if every (equivalently, some) unital subspace (i.e. a subspace containing the constant functions) which is isometric to $X$, consists of functions with discrete distribution. ([1], Theorem A).

Since every subspace of $L_p$ is isometric to a unital subspace of $L_p$. ([1], Lemma 1.2), it follows that if $p \not\in 2\mathbb{N}$ then a subspace $X$ of $L_p$ is isometric to a subspace of $\ell_p$ if and only if every two-dimensional subspace of $X$ is isometric to a subspace of $\ell_p$. ([1], Corollary 1.7).

Since $\ell_2$ is not isomorphic to a subspace of $\ell_p$ unless $p = 2$, it follows that ([1], Corollary 1.8):

**Proposition 1.** If $p \not\in 2\mathbb{N}$, then the two-dimensional Hilbert space $\ell_2^2$ is not isometric to a subspace of $\ell_p$.

Proposition 1 was apparently unknown until A. Pełczyński announced the results of ([1]) at the Convex Geometry meeting at Oberwolfach, 1997. A weaker result, with $\ell_p$ replaced by $\ell^2_n$, had been proved earlier by Y. Lyubich ([3]). Proposition 1 was mentioned in [6], and more recently in [5]. Theorem 2 below, which provides an alternative answer to Pietch’s question, and captures another aspect of the connection between discrete distributions and isometric embeddings into $\ell_p$, also provides a direct, short proof of Proposition 1.

2 Preliminaries

2.1 Notation

Throughout this paper, $p$ denotes a positive, real number. For a measure space $(X, \Sigma, \mu)$ with $\mu$ positive, let $V_p = V_p(X, \Sigma, \mu)$ denote the vector space of all $\Sigma$-measurable functions
such that \( \int_X |f(x)|^p \, d\mu(x) < \infty \), and \( Z_p \subset V_p \) the subspace of all functions vanishing \( \mu \)-almost everywhere. The quotient space \( V_p/Z_p \) is denoted by \( L_p(X, \Sigma, \mu) \). If \( X = \mathbb{N} \), the set of all subsets of \( \mathbb{N} \) is \( \Sigma \) and \( \mu \) is the counting measure, then the corresponding \( L_p \) space is denoted by \( \ell_p \). In what follows, \( L_p \) will denote the space \( L_p([0,1], \Sigma, \lambda) \), where \( \Sigma \) is the Borel \( \sigma \)-algebra, and \( \lambda \) — the Lebesgue measure. The standard inner product in \( \mathbb{R}^n \) is denoted by \( \langle \cdot, \cdot \rangle \), and the Euclidean norm by \( \| \cdot \|_2 \). The unit-sphere in \( \mathbb{R}^n \) is \( S^{n-1} = \{ u \in \mathbb{R}^n : \| u \|_2 = 1 \} \), and the normalized spherical measure on \( S^{n-1} \) is denoted by \( \sigma_{n-1} \). All isometries in this note are between vector spaces, and are linear, i.e., isometric isomorphisms. All vector spaces are over the field of the real numbers.

### 2.2 The Levy \( p \)-representation

**Definition 1.** 1. A finite dimensional normed space \((F, \| \cdot \|)\) is said to have a Levy \( p \)-representation if there exists a linear isomorphism \( J : F \to \mathbb{R}^n \) \((n = \dim F)\) and an even measure \( \xi \) on \( S^{n-1} \) such that for every \( x \in F \)

\[
\|x\|^p = \int_{S^{n-1}} |\langle Jx, v \rangle|^p \, d\xi(v)
\]

2. The representation (1) is said to be discrete if its support is discrete, i.e., if there exists a sequence of positive numbers \( \{a_j\}_{j=1}^\infty \) and a sequence of points \( \{v_j\}_{j=1}^\infty \) in \( S^{n-1} \) such that \( \xi = \sum_{j=1}^\infty a_j (\delta_{v_j} + \delta_{-v_j}) \), where \( \delta_v \) denotes the unit-mass measure concentrated at \( v \in S^{n-1} \).

It is well known that every finite dimensional subspace of \( L_p \) has a Levy \( p \)-representation. For a proof, see ([7], Lemma 6.4). Moreover, if \( p \notin 2\mathbb{N} \), then the even measure \( \xi \) in (1) is uniquely determined by the norm, because in that case, the linear span of the functions \( |\langle y, \cdot \rangle|^p \), with \( y \in \mathbb{R}^n \), is dense in \( C_e(S^{n-1}) \), the Banach space of even, continuous functions on \( S^{n-1} \). ([10], Th. 2.1). If \( p \in 2\mathbb{N} \), then the linear span of the functions \( |\langle y, \cdot \rangle|^p \) coincides with the set of homogeneous polynomials of degree \( p \), a set not large enough to uniquely determine the measure in (1).

Different choices of an isomorphism \( J \) in Definition 1 lead to different measures in the Levy \( p \)-representations. Nevertheless, a simple change-of-density argument shows that the property of a f.d. space, of having a discrete Levy \( p \)-representation, (i.e., a representation (1) with a discrete measure \( \xi \)), is invariant under isometries of \( F \). Note that in general this does not imply that having one discrete Levy \( p \)-representation forces all Levy \( p \)-representations to be discrete. If \( p \in 2\mathbb{N} \), then a finite dimensional subspace of \( L_p \) may have a discrete Levy \( p \)-representation as well as a non-discrete one. In fact, if \( p \in 2\mathbb{N} \) then every f.d subspace of \( L_p \) must have a discrete Levy \( p \)-representation. This is Corollary 1 below. However, it is easy to construct concrete examples, the simplest of which is a discrete Levy 2-representation of \( \ell_2 \).

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1However, the uniqueness of \( \xi \) in (1) for \( p \notin 2\mathbb{N} \) was proved earlier in ([6], Corollary 1)).
Example 1. For every $p > 0$ and $n \geq 1$, the invariance of $\sigma_{n-1}$ under orthogonal transformations shows that for every $u \in \mathbb{R}^n$

$$\|u\|_2^p = c_{p,n} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle|^p \, d\sigma_{n-1}(v)$$

(2)

where $c_{p,n} = \int_{\mathbb{S}^{n-1}} |v_1|^p \, d\sigma_{n-1}(v)$, with $v_1$ denoting the first coordinate of $v$ with respect to the standard basis. Take $p = 2$, and let $\{v_i\}_{i=1}^n$ be any orthonormal basis in $\mathbb{R}^n$. Let $\mu$ be the discrete measure on $\mathbb{S}^{n-1}$ concentrated at $\{\pm v_i\}_{i=1}^n$, assigning mass $\frac{1}{2}$ to each one of the $2n$ points. Then for every $u \in \mathbb{R}^n$

$$\|u\|_2^2 = \sum_{i=1}^n |\langle u, v_i \rangle|^2 = \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle|^2 \, d\mu(v)$$

Thus the discrete measure $\mu$ and the invariant measure $c_{2,n}\sigma_{n-1}$ both provide a Levy $2$-representation of $\ell_2^n$.

3 An alternative answer to Pietch’s question

Theorem 2. A finite dimensional normed space is isometric to a subspace of $\ell_p$ if and only if it has a discrete Levy $p$-representation.

Proof. Fix an $n$-dimensional normed space $F$. If $F$ has a discrete Levy $p$-representation, then there exists an isomorphism $J : F \to \mathbb{R}^n$, a sequence of points $\{v_j\}_{j=1}^\infty$ in $\mathbb{S}^{n-1}$ and a sequence of positive numbers $\{a_j\}_{j=1}^\infty$ such that for every $x \in F$

$$\|x\|^p = \int_{\mathbb{S}^{n-1}} |\langle J(x), v \rangle|^p \, d\mu(v) = \sum_{j=1}^\infty a_j |\langle J(x), v_j \rangle|^p$$

The map $T : F \to \ell_p$ defined by

$$T(x) := \sum_{j=1}^\infty a_j^{1/p} |\langle J(x), v_j \rangle| e_j, \quad (\forall x \in F)$$

is an isometry from $F$ into $\ell_p$.

Conversely, assume there exists an isometry $T : F \to \ell_p$. Let $J : F \to \mathbb{R}^n$ be an isomorphism. Put $K = J(B_F)$, where $B_F$ is the unit-ball of $F$. The map $S = T \circ J^{-1}$ is then an isometry from $(\mathbb{R}^n, \| \cdot \|_K)$ into $\ell_p$. Let $f_1, \ldots, f_n$ denote the standard basis in $\mathbb{R}^n$, and $\{e_j\}_{j=1}^\infty$ the standard basis in $\ell_p$. For each $1 \leq k \leq n$, there exists a sequence $\{x_{k,j}\}_{j=1}^\infty \in \ell_p$ such that $S(f_k) = \sum_{j=1}^\infty x_{k,j} e_j$. Put $v_j = (x_{1,j}, x_{2,j}, \ldots, x_{n,j})$. Then for every $u \in \mathbb{R}^n$,

$$S(u) = S \left( \sum_{k=1}^n \langle u, f_k \rangle f_k \right) = \sum_{k=1}^n \langle u, f_k \rangle \sum_{j=1}^\infty x_{k,j} e_j = \sum_{j=1}^\infty \langle u, v_j \rangle e_j$$
Hence, since $S$ is a linear isometry,

$$
\|u\|_K^p = \|S(u)\|_p^p = \sum_j |\langle u, v_j \rangle|^p \quad (\forall u \in \mathbb{R}^n)
$$

(3)

Put $I = \{ j : v_j \neq 0 \}$. Define an even measure $\xi$ on $S^{n-1}$ that assigns mass $\frac{1}{2}||v_j||^p_2$ to the points $\pm \frac{1}{||v_j||}$ if $j \in I$, and zero otherwise. Then (3) can be rewritten as

$$
\|u\|_K^p = \int_{S^{n-1}} |\langle u, v \rangle|^p \, d\xi(v) \quad (\forall u \in \mathbb{R}^n)
$$

Therefore, for every $x \in F$, as $J : F \to (\mathbb{R}^n, \| \cdot \|_K)$ is an isometry,

$$
\|x\|_p^p = \|J(x)\|_K^p = \int_{S^{n-1}} |\langle J(x), v \rangle|^p \, d\xi(v)
$$

Hence $F$ has a discrete Levy $p$-representation.

\[ \square \quad \square \]

Remark 1 Theorem 2 provides a simple answer to Pietch’s question in terms of the measures appearing in the Levy $p$-representation of finite dimensional subspaces of $L_p$. It should be noted that the deep part of Theorem 1 deals with infinite-dimensional subspaces of $L_p$, of which Theorem 2 remains silent. However, Theorem 2 provides a quick way to prove Proposition 1, which follows as an immediate corollary, because for $p \notin 2\mathbb{N}$ the unique Levy $p$-representation of the finite-dimensional Euclidean norm is not discrete.

Another corollary, already mentioned above, follows immediately from Theorem 1 part 1 and Theorem 2.

Corollary 1. If $p \in 2\mathbb{N}$, then every finite dimensional subspace of $L_p$ has a discrete Levy $p$-representation.

4 Applications

Throughout this section, it will be convenient to denote the statement ”$X$ isometrically embeds in $Y$” by the notation $X \hookrightarrow Y$, and the negation of the same statement by $X \not\hookrightarrow Y$.

In [2], Dor gave a complete answer to the question — for which $p, q$ is it true that $\ell^n_q \hookrightarrow L_p$? Here is his answer ([2], Theorem 2.1).

Theorem 3. (Dor’s Theorem) Assume $n \geq 2$, $p \geq 1$ and $q \geq 1$. Then $\ell^n_q \hookrightarrow L_p$ only in one of the following situations:

(a) $p < q \leq 2$.

(b) $q = 2$.

(c) $p = q$.  

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(d) \( n = 2, p = 1 \), and any \( q \in [1, \infty] \).

As an application of Theorem 2 above, we have an \( \ell_p \) version" of Dor’s theorem, which turns out to be much more restrictive.

**Theorem 4.** Assume \( n \geq 2 \), \( p \geq 1 \) and \( q \geq 1 \).

(a) If \( q \neq 2 \), then \( \ell_q^n \ncon \ell_p \) unless \( p = q \).

(b) \( \ell_2^n \ncon \ell_p \) unless \( p \in 2\mathbb{N} \).

The case of \( p = 1 \) is rather special, because there is much more information regarding the Levy 1-representation compared with Levy \( p \)-representations where \( p > 1 \). For \( p = 1 \), a striking difference between Dor’s Theorem and Theorem 4 emerges: while every two-dimensional normed space embeds isometrically into \( L_1 \), many two-dimensional norms do not embed isometrically in \( \ell_1 \). We will need the next Lemma in the proof of Theorem 4.

**Lemma 1.** Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^2 \) such that the function

\[
G(\theta) = \| (\cos \theta, \sin \theta) \|, \quad 0 \leq \theta \leq 2\pi
\]

is of class \( C^2 \). Then \( (\mathbb{R}^2, \| \cdot \|) \) is not isometric to a subspace of \( \ell_1 \).

*Proof.* We can write for every \( 0 \leq \varphi \leq 2\pi \),

\[
G(\varphi) = \frac{1}{4} \int_0^{2\pi} |\cos(\varphi - \theta)| \left[ G(\theta - \pi/2) + G''(\theta - \pi/2) \right] d\theta,
\]

as can be verified by partial integration. ([11], p.195). Observe that this is nothing but the unique Levy 1-representation of \( (\mathbb{R}^2, \| \cdot \|) \) (with respect to a fixed basis), with an even measure on \( S^1 \) whose density with respect to \( d\theta/4 \) is the continuous, nonnegative function \((G + G'')(\theta - \pi/2)\). Hence if \( G \) is of class \( C^2 \), then \( (\mathbb{R}^2, \| \cdot \|) \) does not have a discrete Levy 1-representation. By Theorem 2, it is does not embed isometrically in \( \ell_1 \).

*Proof. of Theorem 4.*

Part (b) is not new, and is added for the sake of completeness. It is an immediate consequence of part 1 of Theorem 4 and Proposition 4. Let us prove part (a).

Assume \( q \neq 2 \). The cases \( q > 2 \) and \( q < 2 \) are treated separately.

(i) \( q > 2 \). Assume \( \ell_q^2 \cong \ell_p \). Since \( \ell_p \cong L_p \), Dor’s theorem implies that \( p = 1 \), or \( p = q \). But \( p = 1 \) contradicts Lemma 4, because the \( \ell_q^2 \)-norm is of class \( C^2 \) for \( q > 2 \). Therefore, if \( q > 2 \) then \( \ell_q^n \ncon \ell_p \) unless \( p = q \).
(ii) $q < 2$. Excluding the trivial case $p = q$, Dor’s theorem implies we need only consider $1 \leq p < q$. It is well known that for $1 \leq p < q < 2$, the space $\ell^n_q$ has a Levy $p$-representation obtained by means of the standard $n$-dimensional $q$-stable measure $\mu_q$, whose density is the Fourier transform of the function $\exp(-\|x\|_q^q)$. In fact, one has for every $x \in \mathbb{R}^n$,

$$
\|x\|_q^p = \frac{1}{\int_\mathbb{R} |t|^p \, d\nu_q(t)} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p \, d\mu_q(\xi) \tag{5}
$$

where $\nu_q$ is the 1-dimensional $q$-stable measure. That the one-dimensional integral appearing in (5) is finite for $1 \leq p < q < 2$ is a classical fact. An exact calculation of that integral appeared in [8] (ibid. p. 762; an alternative proof, without calculating the exact value, can be found in [12], Proposition 15, p. 93). A Levy $p$-representation of $\ell^n_q$ (in the sense of definition (1) above) is readily derived from (5) by projecting $\mu_q$ from $\mathbb{R}^n$ onto the sphere $S^{n-1}$. More precisely, for each Borel subset $B \subset S^{n-1}$, let $\tilde{B} \subset \mathbb{R}^n$ denote the union of all lines in $\mathbb{R}^n$ passing through $B$, that is, $\tilde{B} = \{ tu : t \in \mathbb{R}, u \in B \}$, and define

$$
\nu_q(B) = \frac{1}{2} \int_{\tilde{B}} \|x\|_2^p \, d\mu_q(x), \tag{6}
$$

where $\|x\|_2$ is the Euclidean norm of $x$. Then one has

$$
\int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p \, d\mu_q(\xi) = \int_{S^{n-1}} |\langle x, u \rangle|^p \, d\nu_q(u), \tag{7}
$$

which, combined with (5) yields a Levy $p$-representation of the space $\ell^n_q$. In [8], the measure $\nu_q$ in (6) is called the $p$-projection of $\mu_q$. (ibid. p. 760). Since the standard $n$-dimensional $q$-stable measure $\mu_q$ has a smooth density, namely, the Fourier transform of $\exp(-\|x\|_q^q)$, the integral in (5) must vanish along every one-dimensional line in $\mathbb{R}^n$, and in particular, its $p$-projection $\nu_q$ cannot be discrete; for if $u_0 \in S^{n-1}$ is a point to which $\nu_q$ assigns positive mass, then the integral in (6), taken over the line $\{ tu_0 : t \in \mathbb{R} \}$ would be positive. Since $1 \leq p < 2$, the Levy $p$-representation of the $\ell^n_q$-norm is unique. Hence the $\ell^n_q$-norm does not have a discrete Levy $p$-representation. It follows from Theorem 2 that $\ell^n_q \not\subseteq \ell_p$.

Summarizing the above, we find that if $q \neq 2$ then $\ell^n_q \not\subseteq \ell_p$ unless $p = q$, and $\ell^n_2 \not\subseteq \ell_p$ unless $p$ is an even integer. This completes the proof of Theorem 4.

Utilizing the main theme of this section we can prove an additional result, which places restrictions on the smoothness of finite dimensional subspaces of $\ell_p$. 

\[\square\]
Proposition 2. Fix a number $p \geq 1$, $p \notin 2\mathbb{N}$. Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$ whose restriction to $S^{n-1}$ is of class $C^{2r}$, where $r \in \mathbb{N}$ and $2r > n + p$. Then $(\mathbb{R}^n, \| \cdot \|)$ does not embed isometrically in $\ell_p$.

Proof. Since $\min_{u \in S^{n-1}} \| u \| > 0$, the function $x \to \| x \|^p$ is also of class $C^{2r}$ on $S^{n-1}$. Our proposition is then an immediate corollary of a result by Koldobsky, asserting that under the assumptions of the proposition, there exists a real-valued continuous function $f$ defined on $S^{n-1}$ such that

$$\| x \|^p = \int_{S^{n-1}} |\langle x, u \rangle|^p f(u) \, du \quad (x \in S^{n-1})$$

See [3], Theorem 1. The conclusion follows now from the uniqueness of the Levy $p$-representation for $p \notin 2\mathbb{N}$, combined with Theorem 2.

Remark 2 For the special case $p = 1$, Lemma 1 is stronger than the previous proposition, which for $n = 2, p = 1$ requires the norm to be at least of class $C^4$. With a little more work and a few tools from differential and convex geometry, the $C^2$ assumption in Lemma 1 can be even further relaxed to the class $C^1$. Here is a sketch of a proof. The equation (4) can be generalized without any regularity assumptions on $G$, where $G''$ is treated as a distribution in the sense of Schwartz. $G$ being convex guarantees that $G + G''$ is a positive measure that must coincide with the 1-Levy representing measure, by uniqueness of the latter (we are assuming $p = 1$). This measure can contain atoms only if the first derivative $G'$ is not a function, but a distribution with a jump-discontinuity. However, such a state of affairs is excluded if $G$ is assumed to be continuously differentiable.

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