Quantization of systems with time-dependent constraints. Example of relativistic particle in plane wave

S.P.Gavrilov
Department of General Physics
Tomsk State Pedagogical Institute
Tomsk 634041
Russia

D.M.Gitman
Instituto de Física
Departamento de Física Matemática
Universidade de São Paulo, Caixa Postal 20516
01498-São Paulo, S.P.
Brazil

January 5, 2022

Abstract

A modification of the canonical quantization procedure for systems with time-dependent second-class constraints is discussed and applied to the quantization of the relativistic particle in a plane wave. The time dependence of constraints appears in the problem in two ways. The Lagrangian depends on time explicitly by origin, and a special time-dependent gauge is used. Two possible approaches to the quantization are demonstrated in this case. One is to solve directly a system of operator equations, proposed by Tyutin and one of the authors (Gitman) as a generalization of Dirac canonical quantization in
nonstationary case, and another to find first a canonical transformation, which makes it possible to describe the dynamics in the physical sector by means of some effective Hamiltonian. Quantum mechanics constructed in both cases proves to be equivalent to Klein-Gordon theory of the relativistic particle in a plane wave. The general conditions of unitarity of the dynamics in the physical sector are discussed.

1 Introduction

In the relativistic particle theories, string theories, theories of gravity and so on, these appear sometimes constraints which depend on time manifestly. The causes are both an explicit time dependence of a Lagrangian, theories in external nonstationary fields may be an example of that, and necessity sometimes to impose gauge conditions which depend on time. The rules of canonical quantization of systems with second-class constraints \cite{1}, which do not depend on time, have to be changed in that case. The corresponding modification was proposed in \cite{2}. The canonical operator quantization implies in this case the solution of both commutation relations with a Dirac bracket, equations of constraints and, besides, some differential equations, which determine the time evolution of operators due to the time-dependence of constraints. This procedure is briefly described in Sect.2. In general case the whole time-evolution is not unitary one and can not be reduced to the solution of the Schrödinger equation only with some Hamiltonian. However, there exist cases when one can make constraints time-independent by means of a classical canonical transformation and then use the ordinary rules of quantization. Particularly, in such a way an operator quantization of the relativistic particle in the constant magnetic field and a chronological gauge was fulfilled \cite{3}.

In this paper (Sect.3) we consider the quantization of the relativistic spinless particle in the external field of a plane wave, which is instructive as a demonstration how does the general theory works and at the same time is new and interesting problem itself. The time dependence of constraints appears here in two possible ways, the Lagrangian depends on time by origin and we use a special time-dependent gauge condition. We demonstrate here both of above mentioned possibilities in the quantization of systems with time-dependent constraints. It turns out that in spite of the dynamics is not
unitary one in the whole, it is unitary in the physical sector. In Sect.4 we consider such a situation in general terms and discuss the problem of the canonical quantization of a relativistic particle in different backgrounds.

2 Canonical quantization of theories with time-dependent second-class constraints

Here, we briefly describe the modification of the Dirac brackets method of quantization for time-dependent second class constraints \([2]\).

Let we have a theory in Hamiltonian formulation with second-class constraints \(\Phi(\eta, t) = 0, \eta = (q, p)\), which can explicitly depend on time \(t\). Then the equation of motion of such a system may be written in the usual form, if one formally introduces a momentum \(\epsilon\) conjugated to the time \(t\), and defines the Poisson bracket in the extended space of canonical variables \((q, p; t, \epsilon) = (\eta; t, \epsilon)\),

\[
\dot{\eta} = \{\eta, H + \epsilon\}_D(\Phi), \, \Phi(\eta, t) = 0, \tag{1}
\]

where \(H\) is a Hamiltonian of the system, and \(\{A, B\}_D(\Phi)\) is the denotation for the Dirac bracket with respect to a system of second-class constraints \(\Phi\). The Poisson bracket, wherever encountered, is henceforth understood as one in such above mentioned extended space. The total derivative of an arbitrary function \(A(\eta, t)\), with allowance made for the equations (1), has the form

\[
\frac{dA}{dt} = \{A, H + \epsilon\}_D(\Phi). \tag{2}
\]

The quantization procedure in "Schrödinger" picture can be formulated in that case as follows. The variables \(\eta\) of the theory are assigned the operators \(\hat{\eta}\), which satisfy the equal-time commutation relations (\([, ]\) is a denotation for the generalized commutator, commutator or anticommutator depending on parities of variables),

\[
[\hat{\eta}, \hat{\eta}'] = i\{\eta, \eta'\}_D(\Phi)|_{\eta = \hat{\eta}}, \tag{2}
\]

the constraints equations

\[
\Phi(\hat{\eta}, t) = 0, \tag{3}
\]
and equations of evolution (We disregard problems connected with operators ordering),

\[
\dot{\eta} = \{\eta, \epsilon\}_D(\Phi) |_{\eta = \hat{\eta}} = -\{\eta, \Phi^l\} \{\Phi, \Phi^l\}_D^{-1} \frac{\partial \Phi^l}{\partial t} |_{\eta = \hat{\eta}}. \tag{4}
\]

To each physical quantity \(A\) given in the Hamiltonian formalism by the function \(A(\eta, t)\), we assign a ”Schrödinger” operator \(\hat{A}\) by the rule \(\hat{A} = A(\hat{\eta}, t)\); in the same manner we construct the quantum Hamiltonian \(\hat{H}\), according to the classical Hamiltonian \(H(\eta, t)\). The time evolution of the state vector \(\Psi\) in the”Schrödinger” picture is determined by the Schrödinger equation

\[
i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi, \quad \hat{H} = H(\hat{\eta}, t). \tag{5}
\]

From (4) it follows, in particular, that

\[
\frac{d\hat{A}}{dt} = \{A, \epsilon\}_D(\Phi) |_{\eta = \hat{\eta}}. \tag{6}
\]

and, as consequence of (2), for arbitrary ”Schrödinger” operators \(\hat{A}, \hat{B}\), we have

\[
[\hat{A}, \hat{B}] = i\{A, B\}_D(\Phi) |_{\eta = \hat{\eta}}. \tag{7}
\]

It is possible to see that quantum theories, which correspond to different initial data for the equation (4), are equivalent.

One can to adduce some arguments in favor of the proposed quantization procedure. For instance, to check that the correspondence principle between classical and quantum equations of motion holds true in this procedure. To this end we pass over to the Heisenberg representation, whose operators \(\hat{\eta}\) are related to the operators \(\hat{\eta}\) as \(\hat{\eta} = U^{-1}\hat{\eta}U\), where \(U\) is the operator of the evolution of the Schrödinger equation, \(i\partial U/\partial t = \hat{H} U, \quad U|_{t=0} = 1\). Heisenberg operators \(\hat{A}\) of an arbitrary physical quantity \(A\) are constructed from the corresponding ”Schrödinger” operator \(\hat{A}\) in the same manner \(\hat{A} = U^{-1}\hat{A}U\). One can find the total time derivative of the Heisenberg operator \(\hat{A}\). Making use of (3), we have
\[
\frac{\text{d} \hat{A}}{\text{d}t} = U^{-1} \left[ -i[\hat{A}, \hat{H}] + \{A, \epsilon\}_D(\Phi)|_{\eta = \hat{\eta}} \right] U. \quad (8)
\]

Taking into account eq. (7) and the connection between the operators \(\hat{\eta}\) and \(\hat{\eta}\), we derive the equation for the Heisenberg operator \(\hat{A}\):

\[
\frac{\text{d} \hat{A}}{\text{d}t} = \{A, H + \epsilon\}_D(\Phi)|_{\eta = \hat{\eta}}, \quad (9)
\]

which coincides in form with the classical equation of motion.

It follows from (9) that the Heisenberg operators \(\hat{\eta}\) also satisfy the equation

\[
\dot{\hat{\eta}} = \{\eta, H + \epsilon\}_D(\Phi)|_{\eta = \hat{\eta}}. \quad (10)
\]

Besides, one can easily verify that the equal-time relations hold for these operators,

\[
[\hat{\eta}, \hat{\eta}'] = i\{\eta, \eta'\}_D(\Phi)|_{\eta = \hat{\eta}}, \quad \Phi(\hat{\eta}, t) = 0 \quad (11)
\]

The relations, together with (10), may be regarded as a prescription of the quantization in the Heisenberg picture for theories with time-dependent second-class constraints.

Note, that time dependence of Heisenberg operators in the theories, being considered, is not unitary in general case. In other words, no such ("Hamiltonian") operator exists, whose commutator with a physical quantity would give its total time derivative. This is explained by the existence of two factors which determine the time evolution of the Heisenberg operator. The first one is a unitary evolution of the state vector in the "Schrödinger" picture, while the second one is the time variation of "Schrödinger" operators \(\hat{\eta}\), which in the general case is of non-unitary character. The existence of these two factors is connected with the division of the right-hand side of (8) into two summands. Physically, this is explained by the fact that the dynamics develops on a surface which itself changes with time; in the general case, not in an unitary way.
Quantization of the relativistic particle in a plane wave

Let us consider a spinless particle in the external electromagnetic field of a plane wave directed along the axis $x^3$. The reparametrization invariant action has in that case the form

$$S = -\int \left(m\sqrt{\dot{x}^2 + e\dot{x}A}\right) d\tau$$

$$= -\int \left[m\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2} + e\dot{x}^a A_a(x_-)\right] d\tau,$$

$$A_\mu = (0, A^a, 0), \quad x_\pm = (x^0 \pm x^3/\sqrt{2}, x_\perp = (x^a), \quad a = 1, 2.$$  

Introducing the momenta

$$\pi_\pm = \frac{\partial L}{\partial \dot{x}_\pm} = -\frac{m\dot{x}_\pm}{\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2}},$$

$$\pi_a = \frac{\partial L}{\partial \dot{x}^a} = \frac{m\dot{x}^a}{\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2}} - eA_a(x_-),$$

one can see that a primary constraint exists,

$$\Phi^{(1)} = \pi_- - \frac{[\pi_a + eA_a(x_-)]^2 + m^2}{2\pi_+} = 0, \quad \pi_\pm \neq 0.$$  

Going over to the Hamiltonian formulation, one has to express velocities via momenta, by means of eq. (13). In the case of consideration, which is singular one, the velocities $\dot{x}_+$ and $\dot{x}_\perp$ can only be expressed in such a way:

$$\dot{x}_+ = \frac{\pi_-}{\pi_+} \dot{x}_-, \quad \dot{x}_a = -\frac{(\pi_a + eA_a)}{\pi_+} \dot{x}_-.$$  

Here, the velocity $\dot{x}_-$ is primary unexpressible one. Nevertheless, it follows from the eq.(13) that

$$\text{sign} \dot{x}_- = -\text{sign} \pi_+ = \zeta.$$  

6
Thus, in fact, the unexpressible is only the modulus of the velocity $\dot{x}_-$. In keeping with general recipes \([1, 2]\), we construct a Hamiltonian $H^{(1)}$ by substituting in the expression $\pi_+ \dot{x}_+ + \pi_- \dot{x}_- + \pi_a \dot{x}^a - L$, the velocities $\dot{x}_+ , \dot{x}_a$ and sign $\dot{x}_-$, according to (15), (16). As a result, we obtain

$$H^{(1)} = 2\lambda \zeta \Phi^{(1)}, \lambda = |\dot{x}_-|. \quad (17)$$

One can readily make sure that the Hamiltonian equations with the Hamiltonian (17) and primary constraint (14) are equivalent to the Lagrange equations of motion. In particular,

$$\dot{\pi}_+ = \dot{\pi}_a = 0, \dot{\pi}_- = -e \dot{x}^a A'_a, A'_a = \frac{\partial A_a}{\partial x_-}. \quad (18)$$

The Hamiltonian (17) is equal to zero on the constraints surface, because of the reparametrization invariance of the action. There are not more constraints here, so we have only one first-class constraint (14). The theory is degenerate, it is impossible to find $\lambda$ in the Dirac procedure. We are following the canonical way and impose the gauge condition similar to the one of the work \([3]\),

$$\Phi^G = x_- - \zeta \tau = 0. \quad (19)$$

Considering the consequences of the conservation of this gauge in the time $\tau$ on the constraints surface,

$$\frac{d\Phi^G}{d\tau} = \frac{\partial \Phi^G}{\partial \tau} + \{\Phi^G, H^{(1)}\} = \zeta (2\lambda - 1) = 0,$$

we obtain $\lambda = \frac{1}{2}$. No other constrains arise. The complete set of constraints, in the gauge of consideration, $\Phi = (\Phi^G, \Phi^{(1)})$, is of second-class one, and depends on time $\tau$.

We consider here two ways of quantization. On the first way (item a) we strictly follow the procedure described in the Sect. 2. On the second one (item b) we use a canonical transformation, which, in combination with the first method, allows one to reach the result more simple.

a) In the case of consideration, when the Hamiltonian $H$ equals zero, the “Schrödinger” quantization (2) - (4) coincide with the Heisenberg one (10), (11). The state vectors do not depend on time $\tau$. All the time dependence is
due to the one of operators, according the equation (2) - (4) or (10), (11). We will denote operators which correspond to the variables \( \eta = (x_\pm, \pi_\pm, x^a, \pi_a) \) as Heisenberg operators \( \hat{\eta} \). The denotation \( \hat{\eta} \) we will use in this item for operators in some special representation.

To write the equations of quantization (2)-(4), we have to take into account the following Dirac brackets:

\[
\{ x^+, \pi^+ \}_D(\Phi) = 1, \quad \{ x^a, \pi_b \}_D(\Phi) = \delta^a_b, \tag{20}
\]

\[
\{ x^+, \pi^- \}_D(\Phi) = -\{ x_+, \Phi_2 \}, \quad \{ x^a, \pi_- \}_D(\Phi) = -\{ x^a, \Phi_2 \}. \tag{21}
\]

The Dirac brackets between the rest canonical variables are zero. The matrix, inverse to the one \( \| \{ \Phi_l, \Phi_l' \} \| \), is

\[
\| \{ \Phi, \Phi \} \|^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

So we can write the equal-time commutation relations (2) for the operators of the independent canonical variables \( x^+, \pi^+, x^a, \pi_a \),

\[
[\hat{x}^+, \hat{\pi}^+] = i, \quad [\hat{x}^a, \hat{\pi}_b] = i\delta^a_b, \tag{22}
\]

\[
[\hat{x}^+, \hat{\pi}^-] = i\{ x_+, \pi^- \}_D(\Phi)_{|\eta=\hat{\eta}} = -i \{ x_+, \Phi_2 \}_{|\eta=\hat{\eta}}, \quad [\hat{x}^a, \hat{\pi}^-] = i \{ x^a, \pi_- \}_D(\Phi)_{|\eta=\hat{\eta}} = -i \{ x^a, \Phi_2 \}_{|\eta=\hat{\eta}}. \tag{23}
\]

The operator equations of constraints (3) are

\[
\hat{\Phi}_1 = \hat{x}^- - \zeta \tau = 0, \quad \hat{\Phi}_2 = \hat{\pi}^- - \frac{[\hat{\pi}_a + eA_a(\hat{x}_-)]^2 + m^2}{2\hat{\pi}^+}. \tag{24}
\]

And the equations of evolution in time (4) look so

\[
\begin{align*}
\dot{\hat{x}}^+ &= \{ x_+, \Phi_2 \}_{|\eta=\hat{\eta}}, \quad \dot{\hat{x}}^+ = 0, \\
\dot{\hat{x}}^a &= \{ x^a, \Phi_2 \}_{|\eta=\hat{\eta}}, \quad \dot{\hat{x}}^a = 0, \\
\dot{\hat{x}}^- &= \{ x_-, \Phi_2 \}_{|\eta=\hat{\eta}} = \zeta, \\
\dot{\hat{\pi}}^- &= \{ \pi_-, \Phi_2 \}_{|\eta=\hat{\eta}} = \frac{\partial}{\partial x_-} \left[ \frac{(\pi_a + eA_a(x_-))^2 + m^2}{2\pi_+} \right] \zeta_{|\eta=\hat{\eta}} \\
&= \frac{d}{d\tau} \left[ \frac{(\pi_a + eA_a(\hat{x}_-))^2 + m^2}{2\hat{\pi}_+} \right].
\end{align*}
\]
The two last equations of (25) are, merely, the sequences of the eq. (24) of constraints. Taking into account eq. (23) and (24), we can rewrite the equations of the evolution for the independent variables in the form

\[ \dot{x}_+ = i[x_+, \pi_- \zeta] = -i[x_+, \tilde{H}], \]
\[ \dot{x}^a = i[x^a, \pi_- \zeta] = -i[x^a, \tilde{H}], \]
\[ \dot{\pi}_+ = -i[\pi_+, \tilde{H}] = 0, \quad \dot{\pi}_a = -i[\pi_a, \tilde{H}] = 0, \]

where

\[ \tilde{H} = \frac{[\pi_a + eA_a(\zeta \tau)]^2 + m^2}{|\pi_+|}. \]

We suppose that \( \pi_+ = -\zeta|\pi_+| \), like in classical theory. The difficulties with definition of such kind operators as \( \zeta = -\text{sign } \pi_+ \), \( |\pi_+| \) and \( |\pi_+|^{-1} \), we are going to avoid, working in the representation of eigen functions of the operator \( \pi_+ \).

As we mentioned in Sect.2, the time evolution of Heisenberg operators is not unitary in general case. In the case of consideration, it is unitary, but only for the operators \( x_+, x^a, \pi_+, \pi_a \), i.e. in the physical sector.

Let us go over from the picture in question, where state vectors do not depend on time, and the operators depend on time according the equations (25),(26), to another one (the corresponding operators will be denoted as \( \hat{\eta} \)) connected with the former by quantum canonical transformation \( \hat{\eta} = U^{-1}(\tau)\hat{\eta}U(\tau), \quad U^+(\tau) = U^{-1}(\tau), \quad U(0) = 1 \), so that, the operators of the independent variables \( \hat{x}_+, \hat{x}^a, \hat{\pi}_+, \hat{\pi}_a \), \( \hat{\pi}_a \) in new representation do not depend on time. To this end we have to choose the operator \( U(\tau) \) as the solution of the equation \( i\partial U/\partial \tau = \tilde{H}U, \quad U(0) = 1 \) where

\[ \tilde{H} = \frac{[\pi_a + eA_a(\zeta \tau)]^2 + m^2}{|\pi_+|}. \]

In the new representation, state vectors obey the equation of motion

\[ i\frac{\partial \Psi}{\partial \tau} = \tilde{H}\Psi, \]
and the operators $\hat{x}_+, \hat{\pi}_+, \hat{x}^a, \hat{\pi}_a$ do not depend on time and have, according to (22), (26), the canonical equal-time commutation relations

$$[\hat{x}_+, \hat{\pi}_+] = i, \ [\hat{x}^a, \hat{\pi}_b] = i\delta^a_b. \tag{29}$$

We realize the commutation relations (29) in the space of function, which depend on variables $\pi_+, x_\perp = (x^a)$, as

$$\hat{\pi}_+ = \pi_+, \ \hat{x}_+ = i\frac{\partial}{\partial \pi_+}, \ \hat{x}^a = x^a, \ \hat{\pi}_a = -i\frac{\partial}{\partial x^a}, \tag{30}$$

(in particular, $\hat{\zeta} = -\text{sign} \pi_+, |\hat{\pi}_+| = |\pi_+|$) and by analogy with the classical theory, introduce the variable $x_-$, which is defined as $x_- = \zeta \tau$ for eigenfunctions of the operator $\hat{\zeta}$ with the eigenvalue $\zeta$. Then the eq. (28) with the Hamiltonian (27) takes the form

$$i\frac{\partial \Psi}{\partial x_-} = -\left[ -i\frac{\partial}{\partial x^a} - eA^a(x_-) \right]^2 + m^2 \frac{2\pi_-}{\pi_+} \Psi. \tag{31}$$

It is easy to verify that (31) is the Klein-Gordon equation. Indeed, the Klein-Gordon equation

$$(\mathcal{P}^2 - m^2)\psi(x) = 0, \ \mathcal{P}_\mu = i\partial_\mu - eA_\mu(x), \tag{32}$$

in the external field of a plane wave (12), being written in the light cone variables $(x_+, x_\perp = (x^a))$, has the form

$$\left(2\frac{\partial^2}{\partial x_-\partial x_+} + \left[ -i\frac{\partial}{\partial x^a} - eA^a(x_-) \right]^2 + m^2 \right) \psi = 0. \tag{33}$$

Decomposing the $\psi$ - function from (33) in the Fourier integral with respect to the variable $x_+$,

$$\psi(x_-, x_+, x_\perp) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\pi x_+} \Psi(x_-, \pi_+, x_\perp) d\pi_+, \tag{33}$$

we get from (33) the equation (31) for the function $\Psi(x_-, \pi_+, x_\perp)$.

The variable $\zeta$ is discrete and describes, as in case of the relativistic particle in the magnetic field $[3]$, particle $\zeta = +$ and antiparticle $\zeta = -$. The demonstration of the statement on the classical level is quite similar to that
of the above mentioned work. To do that on the quantum level, we have to interpret a function $\Psi(x_-, \pi_+, x_\perp)$ with $\pi_+ < 0$, $\zeta = +$, as a wave function of a particle, and a function $\Psi^*(x_-, \pi_+, x_\perp)$ with $\pi_+ > 0$, $\zeta = -$, as the wave function of an antiparticle (see also [4]).

$$\Psi_+ = \Psi(x_-, \pi_+, x_\perp), \quad \zeta = +,$$
$$\Psi_- = \Psi^*(x_-, \pi_+, x_\perp), \quad \zeta = -.\quad (34)$$

Then, it follows from eq. (31),

$$i \frac{\partial \Psi_\zeta}{\partial x_-} = \left[ \hat{p}_\perp - g \hat{A}_\perp(x_-) \right]^2 + m^2 \Psi_\zeta, \quad (34)$$
$$\hat{p}_\perp = -i \frac{\partial}{\partial x_-}, \quad \hat{A}_\perp(x_-) = (A^a(x_-)), \quad g = \zeta e, \quad \zeta = \pm.$$

Thus, the eq. (34) can be interpreted as two Schrödinger (in time $x_-$) equations with positive defined Hamiltonians, one for particle with charge $e$, another for antiparticle with charge $-e$.

One can also prove, by analogy with the work [3], that the quantum mechanics constructed is fully equivalent to the one-particle sector of the quantum theory of the charged scalar field in the external electromagnetic field of a plane wave. This statement can be also confirmed if one constructs the causal propagator $D^\zeta(x, y)$ in Feynman spirit [5] as the sum over a complete set of the solutions of the equations (34), with $\zeta = +$ at $x^0 > y^0$, and with $\zeta = -$ at $x^0 < y^0$. Such a calculation, made in connection with some other problems, one can find in [6, 7]. Solutions describing particles and antiparticles, were classified in that calculation namely according the quantum number $\zeta$, which we have used in the course of the quantization.

b) Here we point out an alternative way of quantization. Namely, already on the classical level, one can make the canonical transformation from the variables $\eta$ to the new ones $\eta'$. The corresponding generating function has the form

$$W = x_+ \pi'_+ + x_- \pi'_- + x^0 \pi'_a + |\pi'_-| \tau, \quad (35)$$

so that
\[ x'_- = x_- + \text{sign} \pi_- \tau, \quad x'_+ = x_+, \quad x'^a = x^a, \]  
(36)
\[ \pi'_\pm = \pi_\pm, \quad \pi'_a = \pi_a. \]

We change, in fact, only the coordinate \( x_- \), and therefore the primes at the other variables are omitted below. In the new variables the constraints surface is described by equations

\[ \Phi_1 = \Phi^G = x'_- = 0, \quad \Phi_2 = \Phi^{(1)} = \pi_- - \frac{[\pi_a + e A_a (\zeta \tau)]^2 + m^2}{\pi^+_a} = 0. \]  
(37)

The Hamiltonian \( H^{(1)} \), arising as a result of the canonical transformation (36), has the form

\[ H^{(1)} = H^{(1)} + \frac{\partial W}{\partial \tau} = H^{(1)} + |\pi_-|, \]

and on the constraint surface reduces to

\[ H = |\pi_-| = \frac{[\pi_a + e A_a (\zeta \tau)]^2 + m^2}{|\pi^+_a|}. \]  
(38)

Because of one constraint depends on \( \tau \), we have to solve all the equations of quantization (3) - (5) or (10), (11). The Hamiltonian (38) does not equal zero, so the "Schrödinger" picture and Heisenberg picture differ one another. Let we quantize in the "Schrödinger" picture. The nonzero Dirac brackets are the same as in (20), (21). Thus, the time-equal commutation relations for the operators \( (\hat{x}_\pm, \hat{x}^a, \hat{\pi}_a) \) look like (29). On account of only the constraint \( \Phi_2 \) depends on time explicitly, and the constraint \( \Phi_1 \) commutes with all independent variables and does not only commute with \( \pi_- \), no one independent operator depend on time, according the eq. (4). The time evolution of the operator \( \hat{\pi}_- \) is consequence of the equation of constraint \( \Phi_2 = 0 \). The Schrödinger equation has to be written in the form (28). Starting that moment one can repeat all arguments of the previous (after eq. (28) consideration, to demonstrate that, as a result of quantization, we get the Klein-Gordon theory for a particle in a plane wave.
4 Discussion

First of all, we would like to underline the principle character of the quantization rules (2)-(5) in the case of time-dependent constraints. Indeed, in that case the Schrödinger equation alone can not determine the time evolution of a quantum system. That evolution is determined both by the equation (4) for the "Schrödinger" operators and by the Schrödinger equation (5) itself. If the Hamiltonian $H$ is equal to zero on the constraints surface, like, for instance, in the relativistic particle theory or in Gravity, whole the time dependence is due to constraints, intrinsic ones and gauge conditions. In this case the eq. (4) coincide with eq.(10) and fully determines the dynamics. This equation shows us what a freedom we possess in forcing upon of the dynamics by means of time-dependent gauge conditions.

As we said, in general case the dynamics is determined by both eq.(4) and the Schrödinger equation and is not unitary one. Nevertheless, there exist particular cases (particular time-dependence of constraints) when whole the dynamics, or the dynamics in the physical sector only, is in fact unitary and can be described in the frame of the Schrödinger equation only with some effective Hamiltonian. The simplest one is the case when one can fulfill a canonical transformation to new variables in which the constraints do not already depend on time. That means the time evolution due to the constraints changing is a canonical transformation. As an example we can refer to the problem of quantization of a relativistic particle in the constant magnetic field in the time-dependent gauge of the form $x^0 + \tau \text{sign} \pi_0 = 0$, where $\pi_0$ is the momentum conjugated to $x^0$ (see [3]). The example considered in the present paper is more complicated because of the above mentioned canonical transformation does not exist. But, as we have demonstrated in item b) of Sect.3, there exists such a canonical transformation, that in new variables the dynamics in the physical sector (here - the sector of independent, unconstrained variables) is unitary one and can be only described by the Schrödinger equation with some effective Hamiltonian. One can formulate conditions for constraints in general terms, which correspond to the latter important case. So, let we have the theory with the equations of motion (1), and let $\eta'$ are some new variables connected with initial ones by means of a canonical transformation. We denote the constraints in the new variables as $\Phi'(\eta', t) = 0$ and the corresponding Hamiltonian as $H'(\eta', t)$. The constraints, as any independent set of second-class constraints [2], can always be solved.
explicitly with respect to a part of the variables \( \eta' \),

\[ \eta' = f(\tilde{\eta}', t), \eta' = (\eta, \tilde{\eta}), \quad (39) \]

so that \( \eta' \) and \( \tilde{\eta}' \) are sets of pairs of canonically conjugated variables, \( \eta' = (q', p'), \tilde{\eta}' = (\bar{q}', \bar{p}') \). The equations of motion (1) for the new variables can be written in terms of the equivalent set of constraints \( \bar{\Phi}' \) and the reduced Hamiltonian \( \bar{H}' \) as follow [2] :

\[ \dot{\tilde{\eta}}' = \{ \tilde{\eta}' , \bar{\Phi}' \} _{D(\bar{\Phi}' , \bar{\Phi}')} , \quad \bar{\Phi}' = \eta' - f(\tilde{\eta}', t) = 0 , \quad (40) \]

\[ \bar{H}' = H'|_{\bar{\Phi}' = f(\tilde{\eta}', t)} . \]

If for any \( k \) and \( m \) the equation holds,

\[ \{ \tilde{\eta}'_k , f^l \} \{ \bar{\Phi}' , \bar{\Phi}' \} _{D(\bar{\Phi}' , \bar{\Phi}')} ^{-1} \{ f^t , \tilde{\eta}'_m \} = 0 , \quad (41) \]

then the Dirac bracket in (40) reduces to the Poisson bracket and, because of \( \{ \tilde{\eta}' , \epsilon \} = 0 \), the system (40) appears to be an ordinary system of Hamiltonian equations for the independent variables \( \tilde{\eta}' \),

\[ \tilde{\eta}' = \{ \tilde{\eta}' , \bar{H}' \} . \]

In this case the quantization in the physical sector can be done in the usual way by means of Poisson bracket, all the corresponding operators being time-independent in the Schrödinger picture. The dynamics is unitary one and is controled by the Schrödinger equation with the Hamiltonian \( \bar{H}' \). That can also be seen from the eq. (4), which reduces in that particular case to \( \dot{\eta}' = 0 \). One can point out the possible structure of the functions \( f(\tilde{\eta}', t) \) from eq. (39), which obey the condition (11). Namely, if for each pair of conjugated variables \( \eta^l = (q^l , p^l) , \quad l = (\zeta , n) , \quad \zeta = 1, 2 \), at least one of the function \( f^{1n} , f^{2n} \) in (39) be equal to zero (so called constraints of special form [2]), then the condition (11) holds. In the example considered in the Sect.3, the constrains (37) have just the above mentioned special form.

Finally, in the connection with the concrete problem of the quantization of the relativistic particle in the presence of various backgrounds, one ought to say the follow. If one uses the method of quantization in which all first-class constraints are considered in the sense of restrictions on the
state vectors, and the commutation relations are built by means of the Dirac bracket taken with respect to second-class constraints, without the adding of any gauge conditions on classical level, then the Klein-Gordon equation or Dirac equation formally appear immediately, after the transition from classical variables to the corresponding operators, because of some of first-class constraints have just the form of these equations. Unfortunately, in this approach the problem of the Hilbert space definition stays unsolved in general case. In the frame of canonical operator quantization, which is, in fact, quantization in the physical sector, and which we have applied here and in [3], the problem looks technically more complicated, but all the steps are consistent and grounded. Nevertheless, on such a way each concrete background needs to be analysed especially. As to electromagnetic backgrounds, they can be divided in three principal classes, within each of them the quantization problem looks similar to our mind. The simplest, in that sense, is the class of constant electromagnetic fields, which, at the same time, does not violate the vacuum stability (does not create pairs from vacuum) from the point of view of the QED with an external field [7]. We have demonstrated the quantization in this case on the example of a relativistic particle in the constant magnetic field [3]. In all such fields there exists a discrete integral of motion, which gives the classification in particles and antiparticles. Besides, there exists such a chronological gauge in which all the dynamics is unitary one. To the second class belong electromagnetic fields, which can depend on time, but do not violate the vacuum stability. The typical configuration here is a plane wave field, the quantization in this background was considered in the present paper. In all fields, belonging to that class, as before, one can find a discrete integral of motion (in our case $\zeta$), which gives the classification in particles and antiparticles. As to dynamics, one can choose such a chronological gauge in which the dynamics in physical sector stays unitary. To the third class belong external electromagnetic fields, which do violate the vacuum stability. The typical representative here is an electric field. The difficulties which appear in this case are connected with the lack of the above mentioned integral of motion, which gives the classification in particles and antiparticles. The same situation exists in an external gravitational field. We believe the solution of the problem in the case of an electromagnetic field violating the vacuum stability can give the key to the canonical quantization of the relativistic particle in gravitational background.
References

[1] P.A.M.Dirac, *Lectures on Quantum Mechanics* (Befer Graduate School of Science, Yeshiva University, New York 1964)

[2] D.M.Gitman, I.V.Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, 1990)

[3] D.M.Gitman, I.V.Tyutin, Class. Quantum Grav. **7** (1990) 2131

[4] V.G.Bagrov, D.M.Gitman, *Exact Solutions of the Relativistic Wave Equations* (Kluwer, 1990)

[5] R.P.Feynman, Phys.Rev. **76** (1949) 749

[6] S.P.Gavrilov, D.M.Gitman, Sh.M.Shvartsman, Yadern. Fiz. **29** (1979) 1392

[7] E.S.Fradkin, D.M.Gitman, Sh.M.Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, 1991)