MAXIMUM-LIKELIHOOD NON-DECREASING RESPONSE ESTIMATES

L. THOMAS RAMSEY

Abstract. Let \( x_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n \), be observations from a doubly-indexed sequence \( \{X_{i,j}\} \) of independent random variables (all of them discrete, or all of them absolutely continuous). Suppose that each \( X_{i,j} \) has the PDF \( f(x | \theta_i) \) from a one-parameter family of PDFs \( f(x | \theta) \). Mild assumptions are described under which there is a unique compound estimate \( \phi = \langle \phi_1, \ldots, \phi_m \rangle \) of \( \theta = \langle \theta_1, \ldots, \theta_m \rangle \) such that

(i) For integers \( i < k \) in \([1, m]\), \( \phi_i \leq \phi_k \) (\( \phi \) is non-decreasing with respect to the index \( i \)).

(ii) Among all non-decreasing vectors of parameters \( \lambda = \langle \lambda_1, \ldots, \lambda_m \rangle \), \( \lambda \neq \phi \Rightarrow \ell(x | \lambda) < \ell(x | \phi) \)

where

(a) \( x \) is the (doubly-indexed) vector of observations

(b) \( \ell \) is the compound likelihood function:

\[
\ell(\lambda | x) = \prod_{i=1}^{m} \prod_{j=1}^{n} f(x_{i,j} | \lambda_i)
\]

An efficient algorithm is described to compute \( \phi \). The notation \([a \ldots b]\) denotes the integers in the real interval \([a, b]\). For \( J \subset [1 \ldots m] \) let

\[
\mu(J) = \frac{\sum_{i \in J} \sum_{j=1}^{n} x_{i,j}}{\sum_{i=a}^{b} n_i}
\]

That is, \( \mu(J) \) is the sample mean of observations \( x_{i,j} \) with \( i \in J \). Here is the theorem that justifies the algorithm:

(i) Let \( \tau_1 < \tau_2 < \ldots < \tau_s \) be a complete list of the distinct components of \( \phi \).

Let \( A_r = \{ i \in [1 \ldots m] : \phi_i = \tau_r \} \). There are integers \( a_r \leq b_r \) such that

\( A_r = [a_r \ldots b_r] \). Also, \( \tau_r = \mu(A_r) \).

(ii) For integers \( r \in [1 \ldots s] \), set

\[
\kappa_r = \min\{ \mu[a_r \ldots k] : k \in [a_r \ldots m] \}
\]

Then,

\[
b_r = \max\{ k \in [a_r \ldots m] : \mu[a_r \ldots k] = \kappa_r \}
\]

1. Introduction and Formal Context

Let \( x_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n \), be observations from a doubly-indexed sequence \( \{X_{i,j}\} \) of independent random real variables. Suppose
that each $X_{i,j}$ has the PDF $f(x \mid \theta_i)$ from a one-parameter family of PDFs $\{ f(x \mid \theta) : \theta \in \Theta \}$.

In this note, we make compound maximum likelihood estimations $\hat{\theta}$ of $\theta = \langle \theta_1, \ldots, \theta_m \rangle$, subject only to a non-decreasing constraint:

$$i \leq k \implies \hat{\theta}_i \leq \hat{\theta}_k$$

Compound estimates $\hat{\theta} \in \Theta^m$ that meet this constraint will be called non-decreasing.

Think of each value of the index $i$ specifying consecutive levels of an explanatory variable, and $\theta_i$ is the response to that $i$-th level of the explanatory variable. The goal of this note is to specify the maximum likelihood non-decreasing response estimate (existence and uniqueness) and provide an algorithm for its efficient computation.

Of course, the same technology gives also the maximum likelihood non-increasing response function. One simply reverses the ordering of the levels of the explanatory variables, and applies the same theory and algorithm to the reversely ordered data.

1.1. Formal Context (Assumptions). Let $x_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n_i$, be observations from a doubly-indexed sequence $\{X_{i,j}\}$ of independent random real variables. Suppose that each $X_{i,j}$ has the PDF $f(x \mid \theta_i)$ from a one-parameter family $\mathcal{H}$ of PDFs $\{ f(x \mid \theta) : \theta \in \Theta \}$. Set

$$\mathcal{D} = \{ x \in \mathbb{R} : (\exists \theta \in \Theta)(f(x \mid \theta) > 0) \}$$

A real number $x$ is called observable if and only if $x \in \mathcal{D}$.

**Assumption 1.** One of two cases are assumed:

(i) The PDFs in $\mathcal{H}$ are for discrete real random variables.

(ii) The PDFs in $\mathcal{H}$ are for absolutely continuous real random variables.

**Assumption 2.** We assume that $\mathcal{D} \subset \Theta$ and $\Theta$ is a real interval of positive length. Consequently, $\Theta$ includes the arithmetic means of finite sequences of observable real numbers.

**Assumption 3.** Let $y_s$, $1 \leq s \leq t$ be observations from the independent random variables $Y_s$, $1 \leq s \leq t$, with PDFs in the given family. As in [2] and on pages 337–341 of [1], the likelihood function is

$$L(\theta \mid y) = \prod_{s=1}^{t} f(y_s \mid \theta)$$

where $y = \langle y_1 \ldots y_t \rangle$. We assume that, if each $y_i$ is observable and $\overline{y}$ is the arithmetic mean of $y_1, \ldots, y_t$, then $L(\theta \mid y)$ is strictly increasing for $\theta \leq \overline{y}$ and strictly decreasing for $\theta \geq \overline{y}$. 
The next lemma is an immediate consequence of these assumptions. It will be applied frequently to subsets of the observations under discussion.

**Lemma 1.** Let \( y_s \in \mathcal{D}, 1 \leq s \leq t, \) be observations from the independent random variables \( Y_s, 1 \leq s \leq t, \) with PDFs in the given family. Then \( \overline{y} \in \Theta \) and, for all \( \eta \in \Theta \) such that \( \eta \neq \overline{y} \)

\[
L(\overline{y} \mid y) > L(\eta \mid y)
\]

Also, \( L(\overline{y} \mid y) > 0. \)

**Proof.** by Assumption 2, \( \overline{y} \in \Theta. \) Equation 1 follows directly from Assumption 3. Since \( \Theta \) is a real interval of positive length, there is some \( \eta \neq \overline{y} \) in \( \Theta. \) Therefore

\[
L(\overline{y} \mid y) > L(\eta \mid y) \geq 0
\]

\[ \square \]

1.2. **Applicability of the Formal Context.** The formal context applies to many of the common one-parameter distributions, if one parameterizes them by their means and takes care to include some boundary distributions. Here are some examples, verified in an appendix:

(i) Bernoulli random variables with parameter \( p \in [0, 1]. \) Note that we include the boundary cases 0 and 1 because they could be sample means. This follows the usual practice for maximum likelihood estimators (see page 318 of [2]).

(ii) Poisson random variables parameterized by their means, with \( \Theta = [0, \infty) \) (see page 644 of [1]). Note the inclusion of the boundary value 0.

(iii) Let \( \Theta = [0, \infty) \) and \( F \) the family of geometric random variables (here including one constant random variable), parameterized by their means (see page 644 of [1]). Given \( \theta \in \Theta, \) set \( p = 1/(1 + \theta). \) For \( \theta > 0, \) the PDF with parameter \( \theta \) is defined as follows:

\[
f(x \mid \theta) = \begin{cases} 
(1 - p)^x p, & \text{if } x \geq 0 \text{ and an integer} \\
0 & \text{otherwise}
\end{cases}
\]

For \( \theta = 0, f(x \mid 0) = 0 \) for all \( x \) except that \( f(0 \mid 0) = 1. \)

(iv) Normal distributions \( N(\theta, \sigma) \) with \( \sigma > 0 \) fixed and \( \theta \in \mathbb{R}. \) It is noted on page 317 of [2] that \( L(\theta \mid y) \) has a unique maximum at \( \theta = \overline{y}. \) To verify the slightly stronger assumptions of this note, one slightly modifies the argument given in [2].
(v) Exponential random variables parameterized by their means (page 638 of [1]). Let \( \Theta = (0, \infty) \). Given \( \theta \in \Theta \), the PDF with parameter \( \theta \) is defined as follows:

\[
f(x \mid \theta) = \begin{cases} 
\frac{1}{\theta} \exp(-x/\theta), & \text{for } x > 0 \\
0, & \text{for } x \leq 0
\end{cases}
\]

The motivating example of the next section has binomial distributions. The example is an application of a corresponding Bernoulli random variable case. The compound likelihood functions of the two situations differ by a non-zero factor \( C \) that does not depend on any of the parameters \( p_i \). \( C \) has the form

\[
\prod_{i=1}^{m} \binom{n_i}{d_i}
\]

where \( d_i \) is the number of successes (1s) in the Bernoulli observations \( x_{i,1}, \ldots, x_{i,n_i} \).

1.3. The Algorithm. Within this formal context, and with each \( x_{i,j} \) observable, it will be proved that there is a unique non-decreasing compound estimate \( \phi = (\phi_1, \ldots, \phi_m) \) of \( \theta = (\theta_1, \ldots, \theta_m) \) such that, among all non-decreasing \( \lambda \in \Theta^m \),

\[
\lambda \neq \phi \Rightarrow \ell(x \mid \lambda) < \ell(x \mid \phi)
\]

where

(i) \( x \) is the (doubly-indexed) vector of observations

(ii) \( \ell \) is the compound likelihood function:

\[
\ell(\lambda \mid x) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f(x_{i,j} \mid \lambda_i)
\]

There is an efficient algorithm to compute \( \phi \). The notation \([a \ldots b]\) denotes the integers in the \([a, b]\). For \( J \subset [1 \ldots m] \) let

\[
\mu(J) = \mu(J) = \frac{\sum_{i \in J} \sum_{j=1}^{n_i} x_{i,j}}{\sum_{i=a}^{b} n_i}
\]

That is, \( \mu(J) \) is the sample mean of observations \( x_{i,j} \) with \( i \in J \).

Let \( \tau_1 < \tau_2 < \ldots < \tau_s \) be a complete list of the distinct components of \( \phi \). Let \( A_r = \{ i \in [1 \ldots m] : \phi_i = \tau_r \} \). By a theorem, there are integers \( a_r \leq b_r \) such that \( A_r = [a_r \ldots b_r] \). Also, \( \tau_r = \mu(A_r) \).

For integers \( r \in [1 \ldots s] \), set

\[
\kappa_r = \min\{ \mu[a_r \ldots k] : k \in [a_r \ldots m] \}
\]
It will be proved that
\[ b_r = \max \{ k \in [a_r \ldots m] : \mu[a_r \ldots k] = \kappa_r \} \]

Here is the algorithm:

(i) Set \( a_1 \) equal to 1. Compute \( \kappa_1 \) according to Equation 2 Then compute \( b_1 \) according to Equation 3 For \( i \in [a_1 \ldots b_1] \), set \( \phi_i \) equal to \( \mu[a_1 \ldots b_1] \).

(ii) Proceed recursively. Suppose that \( \{a_t\}_{t=1}^r \) and \( \{b_t\}_{t=1}^r \) satisfy Equation 3 with \( a_1 = 1 \), and \( a_t = b_{t-1} + 1 \) for \( 1 < t \leq r \). If \( b_r = m \), the algorithm stops. If \( b_r < m \), set \( a_{r+1} = b_r + 1 \). Then compute \( \kappa_{r+1} \) according to Equation 2 Then compute \( b_{r+1} \) according to Equation 3 For \( i \in [a_{r+1} \ldots b_{r+1}] \), set \( \phi_i \) equal to \( \mu[a_{r+1} \ldots b_{r+1}] \).
2. A Motivating Example

For a particular mathematics course, can one use various SAT scores to predict performance on the final examination? A significant subset of the students failed to take the final examination, in some cases because they withdrew from the course (with a record of W) or they simply received a letter grade of F. In the Data Appendix, Table 2 tabulates the no-show counts by SAT-R levels. The data are noisy.

Table 1 gives the maximum likelihood non-decreasing response estimate. In Figure 1 are plotted both the observed no-show rates for each SAT-R score, and the maximum-likelihood non-decreasing response estimate.

On the other hand, the maximum likelihood non-increasing response estimate is the constant $\frac{26}{152}$ for every SAT-R level.

To estimate the significance of the sharp difference between the maximum-likelihood non-decreasing and non-increasing response estimates, 10,000 data tables with the same structure as Table 2 were generated under a null hypothesis of a constant no-show rate of $\frac{26}{152}$ regardless of SAT-R level.

- For each simulated table, the maximum-likelihood non-decreasing response estimate $f$ and the maximum-likelihood non-increasing response estimate $g$ were computed. This test statistic was tabulated:
  \[ \Delta = [f(800) - f(330)] - [g(330) - g(800)] \]
- The simulated quantile ranking is 0.9899 for the same statistic for the actual table (namely 1). That is, 101 of the simulated tables had $\Delta = 1$ (the maximum logically possible).

There are easier and more routine ways to reject the null hypothesis, but the non-decreasing response estimate itself has proved useful for directing resources to students who have a greater risk of failure.

The observed table is broadly consistent with an alternative hypothesis of the maximum-likelihood non-decreasing response estimate; its simulated loglikelihood rank (based on 10,000 simulated tables generated under this alternative hypothesis) is 0.3476.
3. Existence of Likelihood-Maximizing Non-Decreasing Response Estimates

Every $\theta \in \Theta^m$ defines a partition $S(\theta)$ of $[1 \ldots m]$ as follows. Let $\tau_1, \tau_2, \ldots, \tau_s$ be a listing of the distinct components of $\theta$ without the repetition of any component. Let

$$S(\theta) = \{A_1, \ldots, A_s\} \quad \text{where} \quad A_t = \{i \in [1 \ldots m] : \theta_i = \tau_t\}$$

Let $I(\theta)$, the index of $\theta$, be equal to the number of $t \in [1 \ldots s]$ such that

$$\tau_t \neq \mu(A_t)$$

**Lemma 2.** Let $x = \{x_{i,j}\}_{j=1}^{m_{i=1}}$ have each $x_{i,j} \in D$. Suppose that $\theta \in \Theta^m$ is non-decreasing and has $I(\theta) > 0$. Then there is some non-decreasing $\tilde{\theta} \in \Theta^m$ such that

(i) $S(\tilde{\theta})$ has fewer members than $S(\theta)$ or, $S(\theta) = S(\tilde{\theta})$ and $I(\tilde{\theta}) < I(\theta)$. 

---

**Table 1.** ML Non-Decreasing Response Estimate of No-Show Rates

| SAT-R Scores | Total Count | No-Show Count | No-Show Rate |
|-------------|-------------|---------------|--------------|
| 330-430     | 15          | 0             | 0%           |
| 440-530     | 69          | 9             | 13.0%        |
| 540-580     | 35          | 5             | 14.3%        |
| 590         | 5           | 1             | 20%          |
| 600         | 3           | 1             | 33.3%        |
| 610-660     | 19          | 7             | 36.8%        |
| 680-750     | 5           | 2             | 40%          |
| 800         | 1           | 1             | 100%         |

Overall: 152 26 17.1%
(ii) For the compound likelihood function $\ell(\cdot \mid x)$,
\[
\ell(\theta \mid x) < \ell(\tilde{\theta} \mid x)
\]

**Proof.** Suppose first that $\ell(\theta \mid x) = 0$. Let $\bar{x} = \mu[1\ldots m]$, the arithmetic mean of all the observations. By Lemma 1 because every $x_{i,j}$ is observable, we have $\bar{x} \in \Theta$. Let $\tilde{\theta} \in \Theta^m$ be the constant vector with components equal to $\bar{x}$. Clearly $\tilde{\theta}$ is non-decreasing, $S(\tilde{\theta}) = \{[1\ldots m]\}$, and for all $i$ we have
\[
\tilde{\theta}_i = \bar{x} = \mu[1\ldots m]
\]
Thus $I(\theta) = 0$. By Lemma 1
\[
\ell(\tilde{\theta} \mid x) = L(\bar{x} \mid x) > 0 = \ell(\theta \mid x)
\]
For the rest of the proof we assume that $\ell(\theta \mid x) > 0$.

Let $\tau_1 < \tau_2 \ldots < \tau_s$ be a complete list of the distinct components of $\theta$. Set
\[
A_t = \{ i \in [1\ldots m] : \theta_i = \tau_t \}
\]
Then $S(\theta) = \{A_1, \ldots, A_s\}$.

Because $I(\theta) > 0$, there is at least one $T \in [1\ldots s]$ such that $\tau_T \neq \mu(A_T)$.

Because $\theta$ is non-decreasing, there are integers $a \leq b$ in $[1\ldots m]$ such that $A_T = [a\ldots b]$.

Let $y$ be the vector with $n_a + \ldots + n_b$ components $\{(x_{i,j})_{j=1}^{n_i}\}_{i=a}^b$. Then,
\[
\ell(\theta \mid x) = L(\tau_T \mid y) \cdot C
\]
where
\[
L(w \mid y) = \prod_{i=a}^b \prod_{j=1}^{n_i} f(x_{i,j} \mid w)
\]
and
\[
C = \prod_{t \in [1\ldots s] \setminus \{T\}} \prod_{i \in A_t} \prod_{j=1}^{n_i} f(x_{i,j} \mid \theta_i)
\]
Since $\ell(\theta \mid x) > 0$ and $C \geq 0$, we have $C > 0$.

For $t \in [1\ldots s] \setminus \{T\}$ and for all $i \in A_t$, we set
\[
\tilde{\theta}_i = \tau_t = \theta_i
\]
For $t = T$, we will select some $\tilde{\tau}_T$ and for all $i \in A_T$ we will set $\tilde{\theta}_i = \tilde{\tau}_T$. Note that we will then have
\[
\ell(\tilde{\theta} \mid x) = L(\tilde{\tau}_T \mid y) \cdot C
\]
We break the selection of $\tilde{\tau}_T$ into 6 cases:

(i) $\tilde{\tau}_T = \mu(A_1)$ if $s = 1$. 

(ii) \( \tilde{\tau}_T = \mu(A_1) \) if \( s > 1, \ T = 1 \) and \( \mu(A_1) < \tau_2 \).

(iii) \( \tilde{\tau}_T = \mu(A_T) \) if \( s > 1, \ T = s \), and \( \mu(A_s) > \tau_{s-1} \).

(iv) \( \tilde{\tau}_T = \tau_{T+1} \) if \( s > 1, \ 1 \leq T < s \) and \( \mu(A_T) \geq \tau_{T+1} \).

(v) \( \tilde{\tau}_T = \tau_{T-1} \) if \( s > 1, \ 1 < T < s \), and \( \mu(A_T) \leq \tau_{T-1} \).

(vi) \( \tilde{\tau}_T = \mu(A_T) \) if \( s > 1, \ 1 < T < s \), and

\[ \tau_{T-1} < \mu(A_T) < \tau_{T+1} \]

Note that in every case, \( \tilde{\tau}_T \in \Theta \) and thus \( \tilde{\theta} \in \Theta^m \):

- In cases 1, 2, 3 and 6, \( \tilde{\tau}_T \) is the arithmetic mean of observable real numbers. By Lemma 1, \( \tilde{\tau}_T \) is in \( \Theta \).

- In cases 4 and 5, \( \tilde{\tau}_T \) is equal to a component of \( \theta \). Since \( \theta \in \Theta^m \), we have \( \tilde{\tau}_T \) in \( \Theta \).

Next, please note that in every case \( \tilde{\tau}_T \) has been carefully selected to make \( \tilde{\theta} \) be non-decreasing.

It will be argued shortly that the change from \( \theta_i = \tau_T \) to \( \tilde{\theta}_i = \tilde{\tau}_T \) for \( i \in A_T \) strictly increases \( L(\cdot \mid y) \). Since \( C > 0 \), that will give us

\[ \ell(\tilde{\theta} \mid x) = L(\tilde{\tau}_T \mid y) \cdot C > L(\tau_T \mid y) \cdot C = \ell(\theta \mid x) \]

- In cases 1, 2, 3 and 6, the change is from \( \tau_T \neq \mu(A_T) \) to \( \mu(A_T) \). By Lemma 1,

\[ L(\mu(A_T) \mid y) > L(\tau_T \mid y) \]

- In case 4, we have \( \tau_T < \tau_{T+1} \leq \mu(A_T) \). By Assumption 3, \( L(w \mid y) \) is strictly increasing for \( w \leq \mu(A_T) \). Therefore

\[ L(\tau_{T+1} \mid y) > L(\tau_T \mid y) \]

In case 4, \( \tilde{\tau}_T = \tau_{T+1} \) and thus

\[ L(\tilde{\tau}_T \mid y) > L(\tau_T \mid y) \]

- In case 5, we have \( \mu(A_T) \leq \tau_{T-1} < \tau_T \). By Assumption 3, \( L(w \mid y) \) is strictly decreasing for \( w \geq \mu(A_T) \). Therefore,

\[ L(\tau_{T-1} \mid y) > L(\tau_T \mid y) \]

In case 5, \( \tilde{\tau}_T = \tau_{T-1} \) and thus

\[ L(\tilde{\tau}_T \mid y) > L(\tau_T \mid y) \]

Lastly, consider Item (i) of the Lemma.

- In case 1, \( I(\tilde{\theta}) = 0 \). Also \( S(\tilde{\theta}) \) has one member. If \( S(\theta) \) has more than one member, Item (i) is satisfied. If \( S(\theta) \) has one member, then \( S(\theta) = S(\tilde{\theta}) = \{ [1 \ldots m] \} \). Since \( I(\theta) > 0 \) by hypothesis, we have \( I(\tilde{\theta}) < I(\theta) \). That, too, satisfies Item (i).
• In cases 2, 3 and 6, \( S(\theta) = S(\tilde{\theta}) \). Also, for \( r \neq T \), there is no change in whether \( \tau_r \) is equal to \( \mu(A_r) \). However, for \( i \in A_T \), by changing for \( \bar{\theta}_i = \tau_T \neq \mu(A_T) \) to \( \bar{\theta}_i = \mu(A_T) \), we have made \( I(\tilde{\theta}) = I(\theta) - 1 \).

• In case 4, \( S(\tilde{\theta}) \) has one fewer partition member than \( S(\theta) \), because

\[
S(\tilde{\theta}) = (S(\theta) \setminus \{A_T, A_{T+1}\}) \cup \{\tilde{A}_T\}
\]

where

\[
\tilde{A}_T = A_T \cup A_{T+1}
\]

• In case 5, again \( S(\tilde{\theta}) \) has one fewer partition member than \( S(\theta) \), because

\[
S(\tilde{\theta}) = (S(\theta) \setminus \{A_{T-1}, A_T\}) \cup \{\tilde{A}_{T-1}\}
\]

where

\[
\tilde{A}_{T-1} = A_{T-1} \cup A_T
\]

\[\square\]

Lemma 3. Let \( \mathbf{x} = \{\{x_{i,j}\}_{j=1}^{n_i}\}_{i=1}^{m} \) have each \( x_{i,j} \in \mathcal{D} \). Suppose that \( \theta \in \Theta^m \) is non-decreasing with \( I(\theta) > 0 \). Then there is some non-decreasing \( \tilde{\theta} \in \Theta^m \) with \( I(\theta) = 0 \) such that

\[
\ell(\theta \mid \mathbf{x}) < \ell(\tilde{\theta} \mid \mathbf{x})
\]

Proof. Apply Lemma 2 to \( \theta^{(0)} = \theta \) to get \( \theta^{(1)} \), and continue recursively to apply Lemma 2 to \( \theta^{(k)} \) to get \( \theta^{(k+1)} \), but stop if \( I(\theta^{(k)}) = 0 \).

In Lemma 2, \( \theta^{(k)} \) has the role of \( \theta \) (provided that \( I(\theta^{(k)}) > 0 \)) and \( \theta^{(k+1)} \) has the role of \( \tilde{\theta} \).

With each recursion

\[
\ell(\theta^{(k)} \mid \mathbf{x}) < \ell(\theta^{(k+1)} \mid \mathbf{x})
\]

and either \( S(\theta^{(k+1)}) \) has fewer members than \( S(\theta^{(k)}) \), or

\[
S(\theta^{(k+1)}) = S(\theta^{(k)}) \quad \text{and} \quad I(\theta^{(k+1)}) < I(\theta^{(k)})
\]

Because \( S(\theta^{(0)}) \) is finite, we can reduce the size of \( S(\theta^{(k)}) \) at most finitely many times. Let \( K \) be the last time that \( S(\theta^{(K)}) \) is smaller than \( S(\theta^{(K-1)}) \). Then, after \( M = I(\theta^{(K)}) \) more steps of the recursion, \( I(\theta^{(K+M)}) = 0 \).

\[\square\]

Theorem 1. There is a non-decreasing response estimate \( \theta \in \Theta^m \) such that \( I(\theta) = 0 \), \( \ell(\theta \mid \mathbf{x}) > 0 \) and, for all non-decreasing \( \lambda \in \Theta^m \),

\[
\ell(\lambda \mid \mathbf{x}) \leq \ell(\theta \mid \mathbf{x})
\]
Proof. By Lemma 3, for any non-decreasing \( \lambda \in \Theta^m \), there is some non-decreasing \( \theta \in \Theta^m \) with \( I(\theta) = 0 \) such that
\[
\ell(\lambda | x) \leq \ell(\theta | x)
\]
Note that, when \( I(\theta) = 0 \), the partition \( S(\theta) \) induced by \( \theta \) determines \( \theta \):
\[
i \in A \in S(\theta) \rightarrow \theta_i = \mu(A)
\]
Because \( [1 \ldots m] \) is finite, there are finitely many distinct partitions of it. With the condition \( I(\theta) = 0 \) imposed, that gives at most finitely many members of \( \Theta^m \) that can play the role of \( \tilde{\theta} \) in Lemma 3. Among these finitely many possibilities for \( \tilde{\theta} \) in Lemma 3, choose some \( \hat{\theta} \) with \( \ell(\hat{\theta} | x) \) being largest.

In particular, consider the \( \lambda \in \Theta^m \) that is constant with components equal to \( \bar{\pi} = \mu[1 \ldots m] \). Because all \( x_{i,j} \) are observable, by Lemma 1 we have
\[
\ell(\lambda | x) = \ell(\bar{\pi} | x) > 0
\]
It follows that
\[
\ell(\hat{\theta} | x) \geq \ell(\lambda | x) > 0
\]
\( \square \)

4. Algorithm and Uniqueness

Lemma 4. Suppose that all \( x_{i,j} \in D \). Let \( \theta \) satisfy the conclusions of Theorem 2. Let \( \tau_1 < \tau_2 < \ldots < \tau_s \) be a complete listing of the distinct components of \( \theta \). For \( r \in [1 \ldots s] \), set
\[
A_r = \{ i \in [1 \ldots m] : \theta_i = \tau_r \}
\]
Then
(i) There are integers \( a_r \leq b_r \) in \([1 \ldots m]\) such that \( A_r = [a_r \ldots b_r] \).
(ii) For all \( t \in [(a_r + 1) \ldots b_r] \),
\[
\mu[t \ldots b_r] \leq \mu(A_r)
\]
Proof. Because \( \theta \) is non-decreasing, Item (i) is immediate.

We will prove Item (ii) by contradiction. Suppose there is \( r \) and \( t \in [(a_r + 1) \ldots b_r] \) such that
\[
\mu[t \ldots b_r] > \mu(A_r)
\]
Because \( I(\theta) = 0 \), we have \( \mu(A_r) = \tau_r \).

For some real \( \eta \) to be chosen soon, we define \( \tilde{\theta} \) as follows: for \( i \in [1 \ldots m] \) let
\[
\tilde{\theta}_i = \begin{cases} 
\theta_i & \text{if } i < t \text{ or } i > b_r \\
\eta & \text{if } i \in [t \ldots b_r]
\end{cases}
\]
If \( r = s \), let \( \eta = \mu[t \ldots b_r] \). If \( r < s \), select
\[
\eta \in (\tau_r, \min\{\mu[t \ldots b_r], \tau_{r+1}\})
\]
Please note that this selection makes \( \tilde{\theta} \) be non-decreasing. Also, \( \eta \in \Theta \):
\begin{enumerate}
  \item Because every \( x_{i,j} \) is an observable real number, \( \eta = \mu[t \ldots b_r] \) puts \( \eta \in \Theta \) by Lemma 1.
  \item Because \( \Theta \) is a real interval and both \( \tau_r \) and \( \tau_{r+1} \) are in \( \Theta \), \( \eta \in (\tau_r, \tau_{r+1}) \) puts \( \eta \in \Theta \).
\end{enumerate}
Therefore \( \tilde{\theta} \) is a non-decreasing member of \( \Theta^m \).

Let
\[
C = \left( \prod_{i=1}^{t-1} n_i \prod_{j=1}^{n_i} f(x_{i,j} \mid \theta_i) \right) \cdot \left( \prod_{i=b_r+1}^{m} n_i \prod_{j=1}^{n_i} f(x_{i,j} \mid \theta_i) \right)
\]
(with empty products set equal to 1). Since \( C \geq 0 \) and a factor of \( \ell(\theta \mid x) > 0 \), we have \( C > 0 \).

By Assumption 3, with \( \tau_r < \eta \leq \mu[t \ldots b_r] \),
\[
\prod_{i=t}^{b_r} \prod_{j=1}^{n_i} f(x_{i,j} \mid \eta) > \prod_{i=t}^{b_r} \prod_{j=1}^{n_i} f(x_{i,j} \mid \tau_r)
\]
\[
= \prod_{i=t}^{b_r} \prod_{j=1}^{n_i} f(x_{i,j} \mid \theta_i)
\]

Therefore,
\[
\ell(\tilde{\theta} \mid x) = C \cdot \prod_{i=t}^{b_r} \prod_{j=1}^{n_i} f(x_{i,j} \mid \eta) > C \cdot \prod_{i=t}^{b_r} \prod_{j=1}^{n_i} f(x_{i,j} \mid \theta_i) = \ell(\theta \mid x)
\]
This contradicts the likelihood maximizing property of \( \theta \) in Theorem 1. \( \square \)

**Theorem 2.** Suppose that all \( x_{i,j} \in D \). Let \( \theta \) satisfy the conclusions of Theorem 1. Let \( \tau_1 < \tau_2 < \ldots < \tau_s \) be a complete listing of the distinct components of \( \theta \). For \( r \in [1 \ldots s] \), set
\[
A_r = \{ i \in [1 \ldots m] : \theta_i = \tau_r \}
\]
Then
\begin{enumerate}
  \item There are integers \( a_r \leq b_r \) in \( [1 \ldots m] \) such that \( A_r = [a_r \ldots b_r] \).
  \item Let \( \kappa_r = \min\{ \mu[a_r \ldots k] : k \in [a_r \ldots m] \} \)

and set
\[
t_r = \max\{ k \in [a_r \ldots m] : \mu[a_r \ldots k] = \kappa_r \}
\]
Then \( t_r = b_r \).

**Proof.** Item (i) is the same as Item(i) in Lemma 4, and is repeated here to establish the notation.

Because \( I(\theta) = 0 \), we have \( \tau_r = \mu(A_r) \) for all \( r \).

Suppose first there is some \( r \) such that \( t_r = b_q \) for some \( q > r \). Then \( \mu[a_r \ldots t_r] \) is a convex combination with positive coefficients of \( \tau_h \) for \( h \in [r \ldots q] \). For \( h > r \), we have \( \tau_h > \tau_r \). By the definition of \( \kappa_r \), we have

\[
\tau_r = \mu(A_r) = \mu[a_r \ldots b_r] \geq \kappa_r
\]

Therefore, \( \kappa_r > \kappa_r \). This contradiction proves that \( t_r \neq b_q \) for all \( q \in [r + 1 \ldots s] \).

Next suppose that \( t_r \in [a_q \ldots (b_q - 1)] \) for some \( q \in [r \ldots s] \). By the definition of \( \kappa_r \) and of \( t_r \),

- \( \mu[a_r \ldots b_q] > \kappa_r \)
- When \( q > r \),

\[
h \in [r \ldots (q - 1)] \quad \Rightarrow \quad \mu[a_r \ldots b_h] \geq \kappa_r
\]

In particular, \( \mu[a_r \ldots b_{q-1}] \geq \kappa_r \).

We now show that \( \mu[a_q \ldots t_r] \leq \kappa_r \). If \( q = r \) this is immediate from the definitions of \( \kappa_r \) and \( t_r \). Suppose that \( q > r \). There are positive integers \( e \) and \( f \) such that

\[
\mu[a_r \ldots t_r] = \frac{e \mu[a_r \ldots b_{q-1}] + f \mu[a_q \ldots t_r]}{e + f}
\]

Therefore

\[
\mu[a_q \ldots t_r] = \frac{(e + f) \mu[a_r \ldots t_r] - e \mu[a_r \ldots b_{q-1}]}{f}
\]
\[
= \frac{(e + f) \kappa_r - e \mu[a_r \ldots b_{q-1}]}{f}
\]
\[
= \kappa_r + \left( \frac{e}{f} \right) \{ \kappa_r - \mu[a_r \ldots b_{q-1}] \}
\]
\[
\leq \kappa_r
\]

because \( \mu[a_r \ldots b_{q-1}] \geq \kappa_r \).

Next we show that \( \mu[(t_r + 1) \ldots b_q] > \kappa_r \). There are positive integers \( e \) and \( f \) such that

\[
\mu[a_r \ldots b_q] = \frac{e \mu[a_r \ldots t_r] + f \mu[(t_r + 1) \ldots b_q]}{e + f}
\]
Consequently
\[
\mu[(t_r + 1) \ldots b_q] = \frac{(e + f)\mu[a_r \ldots b_q] - e\mu[a_r \ldots t_r]}{f} = \frac{(e + f)\mu[a_r \ldots b_q] - e\kappa_r}{f} = \mu[a_r \ldots b_q] + (e/f) \cdot \{\mu[a_r \ldots b_q] - \kappa_r\} > \mu[a_r \ldots b_q] > \kappa_r
\]

because \(\mu[a_r \ldots b_q] > \kappa_r\).

Third, we argue that \(\mu[(t_r + 1) \ldots b_q] > \tau_q\). We’ve shown already that
\[
\mu[a_q \ldots t_r] \leq \kappa_r < \mu[(t_r + 1) \ldots b_q]
\]
There are positive integers \(e\) and \(f\) such that
\[
\tau_q = \mu[a_q \ldots b_q] = \frac{e\mu[a_q \ldots t_r] + f\mu[(t_r + 1) \ldots b_q]}{e + f}
\]
It follows that
\[
\tau_q < \frac{e\mu[(t_r + 1) \ldots b_q] + f\mu[(t_r + 1) \ldots b_q]}{e + f} = \mu[(t_r + 1) \ldots b_q]
\]
However, this contradicts Lemma 4 (since \(\theta\) satisfies the conclusions of Theorem 1).

The only possibility left for \(t_r\) is to be equal to \(b_r\) as desired. \(\square\)

**Corollary 1.** Suppose that all \(x_{i,j} \in D\). There is a unique \(\theta \in \Theta^n\) that is non-decreasing and, for all non-decreasing \(\lambda \in \Theta^m\),
\[
(4) \quad \ell(\lambda | \mathbf{x}) \leq \ell(\theta | \mathbf{x})
\]
where \(\ell\) is the compound likelihood function.

**Proof.** By Theorem 1, there is some \(\theta\) such that satisfies the conclusions of that theorem. In particular, \(\theta\) satisfies Equation 4 for all non-decreasing \(\lambda \in \Theta^m\).

Suppose that \(\hat{\theta}\) also satisfies Equation 4 for all non-decreasing \(\lambda \in \Theta^m\). By Lemma 3 if \(I(\hat{\theta}) > 0\), there would be some non-decreasing \(\tilde{\theta} \in \Theta^m\) such that
\[
\ell(\tilde{\theta} | \mathbf{x}) > \ell(\hat{\theta} | \mathbf{x})
\]
So we must have \(I(\hat{\theta}) = 0\).

Because each \(x_{i,j}\) is observable, their arithmetic mean \(\mu[1 \ldots m]\) is in \(\Theta\). Let \(\lambda \in \Theta^m\) which has the constant component \(\mu([1 \ldots m])\). By Assumption 3,
\[
\ell(\lambda | \mathbf{x}) > 0
\]
Therefore, $\ell(\hat{\theta} \mid x) > 0$.

So $\hat{\theta}$ satisfies the conclusions of Theorem 1.

Note that Theorem 2 specifies $S(\theta) = S(\hat{\theta})$ uniquely. First, the theorem determines $A_1 = [1 \ldots b_1]$ as $b_1$ must equal $t_1$. Once $b_r$ is determined, if $b_r < m$ the theorem then determines $b_{r+1} = t_{r+1}$. Of course $a_{r+1} = b_r + 1$, and thus $A_{r+1}$ is specified.

However, since $I(\theta) = I(\hat{\theta}) = 0$, we have for $i \in A_r$

$$\theta_i = \hat{\theta}_i = \mu(A_r)$$

Thus $\theta = \hat{\theta}$. □

5. Examples of the Formal Context

Throughout this section, let $T$ be a positive integer, $y : [1 \ldots T] \to \mathcal{D}$, and we think of each $y_i$ as an observation of a random variable $Y_i$, with \{\{Y_i\}_{i=1}^{T}\} independent with PDFs from the given family.

Example 1. Let $\Theta = [0, 1]$ and $\mathcal{F}$ be the family of Bernoulli random variables (here including two constant random variables). Given $\theta \in \Theta$, the PDF with parameter $\theta$ is defined as follows:

$$f(x \mid \theta) = \begin{cases} \theta & \text{if } x = 1 \\ 1 - \theta & \text{if } x = 0 \\ 0 & \text{if } x \in \mathbb{R} \setminus \{0, 1\} \end{cases}$$

Here $\mathcal{D} = \{0, 1\}$. We show that Assumption 3 holds.

Let $y$ have $r$ ones and $s$ zeros. Thus $r + s = T$ and $\overline{y} = r/(r + s)$. For any $\lambda \in [0, 1]$,

$$h(\lambda) := L(\lambda \mid y) = \lambda^r (1 - \lambda)^s$$

**Case 1:** Suppose $r = 0$. So $s = T > 0$. Then

$$h'(\lambda) = s(1 - \lambda)^{s-1}(-1)$$

For $\lambda \in [0, 1)$, this derivative is negative. Because $h(\lambda)$ is continuous in $\lambda$, the function $h$ is strictly decreasing on $[0, 1]$. Since $y$ has all zeros, $\overline{y} = 0$. So Assumption 3 holds: $h$ is strictly decreasing on $[0, 1] = [\overline{y}, 1]$ and strictly increasing (trivially) on $[0, 0]$.

**Case 2:** Suppose $s = 0$. Then $r = T > 0$ and

$$h'(\lambda) = r \lambda^{r-1}$$

For $\lambda \in (0, 1)$, this derivative is positive. Because $h(\lambda)$ is continuous in $\lambda$, the function $h$ is strictly increasing on $[0, 1]$. Since $y$ has all ones, $\overline{y} = 1$. Consequently, Assumption 3 holds: $h$ is strictly increasing on $[0, \overline{y}] = [0, 1]$ and $h$ is strictly decreasing (trivially) on $[1, 1]$. 
Case 3: Suppose $r > 0$ and $s > 0$. Then
\[
    h'(\lambda) = r\lambda^{r-1}(1 - \lambda)^s + \lambda^r \cdot s(1 - \lambda)^{s-1}(-1) \\
    = \lambda^{r-1}(1 - \lambda)^{s-1}[r(1 - \lambda) - s\lambda] \\
    = \lambda^{r-1}(1 - \lambda)^{s-1}(r + s) \left[ \frac{r}{r + s} - \lambda \right]
\]
For $\lambda \in (0, 1)$, this derivative is positive for $\lambda < r/(r + s)$ and negative for $\lambda > r/(r + s)$. Because $h(\lambda)$ is continuous in $\lambda$, it follows that $h$ is strictly increasing on $[0, r/(r + s)]$ and strictly decreasing on $[r/(r + s), 1]$. Since $\overline{y} = r/(r + s)$, Assumption 3 holds.

Example 2. Let $\Theta = [0, \infty)$ and $\mathcal{F}$ be the family of Poisson random variables (here including one constant random variable). Given $\theta \in \Theta$, the PDF with parameter $\theta$ is defined as follows:
\[
    f(x \mid \theta) = \begin{cases} 
        e^{-\theta x} \frac{x^{|x|}}{x!}, & \text{if } x \geq 0 \text{ and an integer} \\
        0 & \text{otherwise}
    \end{cases}
\]
Here $\mathcal{D}$ is the set of non-negative integers. We show that Assumption 3 holds.
For any $\lambda \in [0, \infty)$, let
\[
    h(\lambda) := L(\lambda \mid y) = \prod_{i=1}^{T} \left( e^{-\lambda \frac{y_i}{\theta}} \right) = Ke^{-T \lambda \theta y} 
\]
where $K > 0$ is a factor that does not depend on $\lambda$.

Case 1. Suppose that $\overline{y} = 0$. Then $h(\lambda) = Ke^{-T \lambda}$ and hence
\[
    h'(\lambda) = -KT e^{-T \lambda} 
\]
Note that $h'$ is negative for all $\lambda \in \Theta$. Thus $h$ is strictly decreasing on $[0, \infty) = [\overline{y}, \infty)$ and strictly increasing (trivially) on $[0, 0]$.

Case 2. Suppose that $\overline{y} > 0$. Then
\[
    h'(\lambda) = (-KT)e^{-T \lambda \theta y} + Ke^{-T \lambda} \cdot T \overline{y} \cdot \lambda^{T \overline{y}-1} \\
    = KT e^{-T \lambda} \lambda^{T \overline{y}-1} \left[ -\lambda + \overline{y} \right]
\]
Note that $h'$ is positive on $(0, \overline{y})$ and $h'$ is negative on $(\overline{y}, \infty)$. Since $\overline{y} > 0$, we have $h$ continuous on $[0, \infty) = \Theta$. Therefore, $h$ is strictly increasing on $[0, \overline{y}]$ and $h$ is strictly decreasing on $[\overline{y}, \infty)$.

Example 3. Let $\Theta = [0, \infty)$ and $\mathcal{F}$ be the family of geometric random variables (here including one constant random variable), parameterized
by their means. Given \( \theta \in \Theta \), set \( p = 1/(1 + \theta) \). For \( \theta > 0 \), the PDF with parameter \( \theta \) is defined as follows:

\[
f(x \mid \theta) = \begin{cases} 
(1 - p)^x p, & \text{if } x \geq 0 \text{ and an integer} \\
0 & \text{otherwise}
\end{cases}
\]

For \( \theta = 0 \), and thus \( p = 1 \), let \( f(x \mid 0) = 0 \) for all \( x \) except that \( f(0 \mid 0) = 1 \).

Here \( D \) is the set of non-negative integers.

For any \( \lambda \in [0, \infty) \), set \( p = 1/(1 + \lambda) \). Then for \( \lambda > 0 \)

\[
h(\lambda) := L(\lambda \mid y) = \prod_{i=1}^{T} ((1 - p)^{y_i} p) = p^T (1 - p)^T y
\]

For \( \lambda = 0 \), \( h(0) = 1 \) if \( y \) is a vector of zeros and 0 otherwise.

Please note that \( h \) is continuous on \([0, \infty)\):

- Suppose \( \overline{y} = 0 \). Then \( y \) is a vector of zeros and \( h(0) = 1 \). For \( \lambda > 0 \), we have

\[
h(\lambda) = p^T (1 - p)^T 0 = p^T = (1 + \lambda)^{-T}
\]

Clearly \( h \) is continuous on \((0, \infty)\); it is also continuous at 0 because \( \lim_{\lambda \downarrow 0} h(\lambda) = 1 = h(0) \).

- Suppose \( \overline{y} > 0 \). Then \( y \) has at least one non-zero component and thus \( h(0) = 0 \). For \( \lambda > 0 \),

\[
h(\lambda) = \lambda^{\overline{y}} (1 + \lambda)^{-\overline{y} - T}
\]

Clearly \( h \) is continuous on \((0, \infty)\). Since \( T \overline{y} > 0 \), we have \( \lim_{\lambda \downarrow 0} h(\lambda) = 0 = h(0) \). Thus \( h \) is continuous at 0 as well.

**Case 1.** Suppose that \( \overline{y} = 0 \). Then \( y \) is a vector of zeros and \( h(0) = 1 \). For \( \lambda > 0 \),

\[
h(\lambda) = p^T = (1 + \lambda)^{-T} \quad \text{and thus} \quad h'(\lambda) = (-T)(1 + \lambda)^{-T-1} < 0
\]

Since \( h \) is continuous on \([0, \infty)\) and has a negative derivative on \((0, \infty)\), we know that \( h \) is strictly decreasing on \([0, \infty)\). Thus \( h \) is strictly decreasing on \([0, \infty) = [\overline{y}, \infty)\) and strictly increasing (trivially) on \([0, 0)\).
Case 2. Suppose that $\overline{y} > 0$. For $\lambda > 0$,

\[
h'(\lambda) = \frac{dh}{dp} \cdot \frac{dp}{d\lambda} = \left\{ Tp^{T-1}(1-p)^{T\overline{y}} + p^T \cdot T\overline{y} \cdot (1-p)^{T\overline{y}-1} \right\} \cdot (-1)(1 + \lambda)^{-2}
\]

\[
= -(1 + \lambda)^{-2} Tp^{T-1}(1-p)^{T\overline{y}-1} [(1-p) - \overline{y}p]
\]

\[
= (1 + \lambda)^{-2} Tp^{T-1}(1-p)^{T\overline{y}-1} \left[ \frac{1 + \overline{y}}{1 + \lambda} - 1 \right]
\]

\[
= (1 + \lambda)^{-2} Tp^{T-1}(1-p)^{T\overline{y}-1} \left[ \frac{\overline{y} - \lambda}{(1 + \lambda)(1 + \overline{y})} \right]
\]

For $\lambda > 0$, all the factors in the previous line are positive except for the factor $\overline{y} - \lambda$. Thus for $\lambda \in (0, \overline{y})$, we have $h'(\lambda) > 0$ and for $\lambda \in (\overline{y}, \infty)$ we have $h'(\lambda) < 0$. Since $h$ is continuous on $[0, \infty)$, we have $h$ strictly increasing on $[0, \overline{y}]$ and strictly decreasing on $(\overline{y}, \infty)$.

Example 4. Let $\Theta = \mathbb{R}$ and, for a fixed $\sigma > 0$, let $F$ be the family of normal random variables with standard deviation $\sigma$. Given $\theta \in \Theta$, the PDF with parameter $\theta$ is defined as follows: for all real $x$

\[
f(x \mid \theta) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x - \theta)^2}{2\sigma^2} \right\}
\]

Here $D = \mathbb{R}$. We'll argue that Assumption 3 holds.

For all real $\lambda$, let

\[
h(\lambda) := L(\lambda \mid y) = \prod_{i=1}^{T} \left( \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(y_i - \theta)^2}{2\sigma^2} \right\} \right)
\]

\[
= K \exp \left\{ -\frac{\sum_{i=1}^{T} (y_i - \lambda)^2}{2\sigma^2} \right\}
\]

where $K$ is a positive factor that does not depend on $\lambda$. Then

\[
h'(\lambda) = K \exp \left\{ -\frac{\sum_{i=1}^{T} (y_i - \lambda)^2}{2\sigma^2} \right\} \cdot \left( \frac{\sum_{i=1}^{T} 2(y_i - \lambda)}{2\sigma^2} \right)
\]

\[
= \frac{K}{\sigma^2} \cdot \exp \left\{ -\frac{\sum_{i=1}^{T} (y_i - \lambda)^2}{2\sigma^2} \right\} \cdot \left( -T\lambda + \sum_{i=1}^{T} y_i \right)
\]

\[
= \frac{TK}{\sigma^2} \cdot \exp \left\{ -\frac{\sum_{i=1}^{T} (y_i - \lambda)^2}{2\sigma^2} \right\} \cdot (-\lambda + \overline{y})
\]
All factors immediately above for $h'(\lambda)$, except the last one, are positive for all $\lambda$. So $h'$ is positive for $\lambda$ in $(-\infty, \overline{y})$ and $h'$ is negative for $\lambda$ in $(\overline{y}, \infty)$. Since $h$ is continuous on $\mathbb{R}$, we have $h$ is strictly increasing on $(-\infty, \overline{y})$ and strictly decreasing on $[\overline{y}, \infty)$.

**Example 5.** Let $\Theta = (0, \infty)$ and $\mathcal{F}$ be the family of exponential random variables parameterized by their means. Given $\theta \in \Theta$, set $\tau = 1/\theta$. In terms of $\tau$, the PDF with parameter $\theta$ is defined as follows:

$$f(x \mid \theta) = \begin{cases} \tau e^{-\tau x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$$

Here $\mathcal{D} = (0, \infty)$. We’ll argue that Assumption 3 holds. Note that, since $\mathcal{D}$ has only positive numbers, $\overline{y} > 0$. For any real $\theta > 0$ and with $\tau = 1/\theta$, let

$$h(\theta) := L(\theta \mid y) = \prod_{i=1}^{T} (\tau e^{-\tau y_i}) = \tau^{T} e^{-\tau T \overline{y}}$$

Then

$$h'(\theta) = \frac{dh}{d\tau} \cdot \frac{d\tau}{d\theta}$$

$$= \left\{ T \tau^{T-1} e^{-\tau T \overline{y}} + \tau^{T} \cdot (-T \overline{y}) \cdot e^{-\tau T \overline{y}} \right\} \cdot (-\theta^{-2})$$

$$= \frac{-T \tau^{T-1} e^{-\tau T \overline{y}}}{\theta^2} \cdot \{1 - \tau \overline{y}\}$$

$$= \frac{T \tau^{T-1} e^{-\tau T \overline{y}}}{\theta^2} \left\{ \frac{\overline{y} - \theta}{\theta} \right\}$$

It follows that $h'(\theta) > 0$ if if $\theta < \overline{y}$ and $h'(\theta) < 0$ if $\theta > \overline{y}$. Because $h(\theta)$ is continuous on $(0, \infty)$, we have $h$ strictly increasing on $(0, \overline{y}]$ and strictly decreasing on $[\overline{y}, \infty)$. 
| SAT-R Scores | Total Count | No-Show Count |
|--------------|-------------|---------------|
| 330          | 1           | 0             |
| 390          | 2           | 0             |
| 400          | 1           | 0             |
| 410          | 2           | 0             |
| 420          | 5           | 0             |
| 430          | 4           | 0             |
| 440          | 4           | 1             |
| 450          | 3           | 2             |
| 460          | 2           | 0             |
| 470          | 8           | 1             |
| 480          | 11          | 3             |
| 490          | 9           | 0             |
| 500          | 4           | 1             |
| 510          | 11          | 0             |
| 520          | 9           | 0             |
| 530          | 8           | 1             |
| 540          | 11          | 4             |
| 550          | 6           | 1             |
| 560          | 5           | 0             |
| 570          | 6           | 0             |
| 580          | 7           | 0             |
| 590          | 5           | 1             |
| 600          | 3           | 1             |
| 610          | 5           | 3             |
| 620          | 4           | 2             |
| 630          | 1           | 0             |
| 640          | 7           | 2             |
| 650          | 1           | 0             |
| 660          | 1           | 0             |
| 680          | 1           | 1             |
| 690          | 1           | 1             |
| 700          | 1           | 0             |
| 710          | 1           | 0             |
| 750          | 1           | 0             |
| 800          | 1           | 1             |

*Table 2. No-Show Counts for SAT-R Levels*

6. **Data Appendix**

Table 2 has the no-show counts, tabulated by SAT-R scores.
References

[1] Klugman, Stuart A., Panjer, Harry H., and Willmot, Gordon E. *Loss Models, Second Edition*, John Wiley and Sons, Inc., Hoboken, New Jersey, 2004, pages 337 — 341 and pages 627 — 645.

[2] Casella, George and Berger, Roger L. *Statistical Inference, Second Edition* Duxbury Press, Pacific Grover, California, 2002, pages 315 — 318.

Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822
E-mail address: ramsey@math.hawaii.edu
URL: http://www.math.hawaii.edu/~ramsey