Solvability of the system of equations of one-dimensional movement of a viscous liquid in a deformable viscous porous medium

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Abstract. The initial-boundary value problem for the system of one-dimensional motion of viscous liquid in deformable viscous porous medium is considered. Local theorem of existence and uniqueness of problem is proved in case of compressible liquid. In case of incompressible liquid the theorem of global solvability in time is proved in Hölder classes.

1. Introduction
The problem of fluid filtration in a deformable porous medium is relevant for solving important applied problems [1–3]. In particular, it is the problems of hemodynamics, as well as the movement of other physiological fluids in the muscle tissue [4]. It is necessary to take into account the compressibility of the fluid and the skeleton when solving problems of the motion of gases or carbonated liquids in porous media. The purpose of this study is to take into account the compressibility of a porous skeleton and justify the filtration model of a viscous compressible fluid in a deformable porous medium. The numerical studies of various initial boundary-value problems for systems of equations describing process of fluid filtration through deformable viscoelastic medium were carried out in [2, 5]. The local solvability of the problem of filtration of compressible fluid in poroelastic medium was established in [6]. Structurally similar systems of equations was considered in [7, 9]. In these studies, based on a number of simplifying assumptions, the original system were reduced to one higher order equation. The local solvability of the Cauchy problem in Sobolev spaces was established in [7]. Travelling wave solutions have been studied in [8, 9].

2. Problem statement
The system of equations describing the one-dimensional unsteady motion of a compressible fluid in a viscous porous medium in the domain \((x, t) \in Q_T = \Omega \times (0, T), \Omega = (0, 1)\), is as follows [5, 10]:

\[
\frac{\partial (1 - \phi) \rho_s}{\partial t} + \frac{\partial}{\partial x} ((1 - \phi) \rho_s v_s) = 0, \quad \frac{\partial (\rho_f \phi)}{\partial t} + \frac{\partial}{\partial x} (\rho_f \phi v_f) = 0, \quad (1)
\]

\[
\phi (v_f - v_s) = -k(\phi) \left( \frac{\partial p_f}{\partial x} - \rho_f g \right), \quad (2)
\]

\[
\frac{\partial v_s}{\partial x} = -\frac{1}{\xi(\phi)} p_c, \quad p_c = p_{tot} - p_f, p_{tot} = \phi p_f + (1 - \phi) p_s, \quad (3)
\]
\[
\frac{\partial}{\partial x} \left( 2\eta(1 - \phi) \frac{\partial \varphi}{\partial x} \right) = \rho_{tot} g + \frac{\partial p_{tot}}{\partial x}, \quad \rho_{tot} = \phi \rho_f + (1 - \phi) \rho_s.
\]

The problem is written in the Eulerian coordinates \((x, t)\). The real density of the solid particles \(\rho_s\) are assumed constant, and clapeyron dependence is taken for fluid \(p_f = R \rho_f, R = const > 0\).

At the boundary of the region \(\Omega\), the velocities of the phases \(\varphi_s, \varphi_f\) are set, and at the initial moment of time - the density \(\rho_f^0(x)\) and porosity \(\phi^0(x)\). The following conditions are considered for the system (1)–(4):

\[
v_s|_{x=0.1} = v_f|_{x=0.1} = 0, \quad \rho_f|_{t=0} = \rho^0(x), \quad \phi|_{t=0} = \phi^0(x).
\]

After passing to Lagrange variables by velocity of solid phase we convert the system of equation and get following system of equations for porosity and density of fluid [6]:

\[
\frac{\partial}{\partial t}(a(\phi)\rho_f) - \frac{\partial}{\partial x}(K(\phi)b(\rho_f)\frac{\partial \rho_f}{\partial x} - \frac{K(\phi)}{1 - \phi^2}g) = 0,
\]

\[
\frac{\partial G}{\partial t} = p_f(\rho_f) - \rho^0 + \int_0^x \frac{\rho_{tot} g}{1 - \phi} d\xi,
\]

where

\[
a(\phi) = \frac{\phi}{1 - \phi}, \quad K(\phi) = k(\phi)(1 - \phi), \quad b(\rho_f) = \rho_f \frac{\partial p_f(\rho_f)}{\partial \rho_f}, \quad \frac{\partial G}{\partial \phi} = 1 + \frac{\xi(\phi)}{1 - \phi}.
\]

For system (6)–(7) conditions (5) could be written as

\[(1 - \phi) \frac{\partial p_f(\rho_f)}{\partial x} - \rho_f g \big|_{x=0, x=1} = 0, \quad \rho_f \big|_{t=0} = \rho^0(x), \quad \phi \big|_{t=0} = \phi^0(x),
\]

where

\[
p^0 = \left( \int_0^1 \frac{1}{(1 - \phi)(\xi(\phi) + 1 - \phi)} \left( p_f(\rho_f) + \int_0^x \frac{\rho_{tot} g}{1 - \phi} d\xi \right) dx \right)^{-1} \equiv P^0(\phi, \rho_f).
\]

3. Compressible fluid

In the notation of function spaces we follow [11]: \(C^{k,a,m+\beta}(Q_T)\) – Hölder’s space, where \(k, m\) are natural numbers, \((\alpha, \beta) \in (0, 1]\), with the norm \(\|f\|_{C^{k,a,m+\beta}(Q_T)}\).

**Definition 1.** The solution of problem (6)–(8) is the set of functions \(\phi, \rho_f \in C^{2+\alpha,1+\beta}(Q_T)\), such that \(0 < \phi < 1, \rho_f > 0\). These functions satisfy the equations (6)–(7) and the initial and boundary conditions (8) and regarded as continuous functions in \(Q_T\).

**Theorem 1.** Suppose that the data of problem (6)–(8) satisfies the following conditions:

1. the functions \(k(\phi), \xi(\phi)\) and their derivatives up to the second order are continuous for \(\phi \in (0, 1), \rho_f > 0\), and satisfy the conditions \(k_0^{-1} \phi^{q_1}(1 - \phi)^{q_2} \leq k(\phi) \leq k_0 \phi^{q_2}(1 - \phi)^{q_4}, 1/\xi(\phi) = a_0(\phi)\phi^{q_1}(1 - \phi)^{q_2-1} \leq a_0(\phi) \leq R_2\), where \(k_0, a_0, R_1, R_2, q_1, \ldots, q_4\) are fixed real parameters,
2. the initial functions \(\rho^0, \rho^0\) and function \(g\) satisfy the following smoothness conditions: \(\phi^0 \in C^{2+\alpha}(\Omega), \rho^0 \in C^{2+\alpha}(\Omega), g \in C^{1+\alpha,1+\beta}(\Omega)\), and the matching conditions \((1 - \phi^0) \frac{\partial p_f(\rho^0)}{\partial x} - \rho^0 g(x, 0) \big|_{x=0, x=1} = 0\), as well as satisfy the inequalities \(0 < m_0 \leq \phi^0(x) \leq M_0 < 1, 0 < m_1 \leq \rho^0(x) \leq M_1 < \infty, 0 < g(x, t) \leq g_0 < \infty, x \in \Omega\), where
We have an estimate $0 < \rho(x,t) < 1$, $\rho(x,t) > 0$ in $\overline{Q}_t_0$.

The solvability of problem (6)-(8) is established by using the Tikhonov-Schauder fixed-point theorem [12].

Since the function $\psi = G(\phi)$ is strictly monotone, at $\phi \in (0,1)$, that the inverse function is exist: $\phi = G^{-1}(\psi)$. Assuming that $\rho(x,t) = \rho_f(x,t) - \rho^0(x)$, $\omega(x,t) = G(\phi) - G(\phi^0)$. We represent the equations (6)-(7) in the form

$$\frac{\partial}{\partial t} \left( a(\omega)(\rho + \rho^0) \right) = \frac{\partial}{\partial x} \left( K(\omega)b(\rho + \rho^0) \frac{\partial(\rho + \rho^0)}{\partial x} - \frac{K(\omega)}{1 - \phi(\omega)}(\rho + \rho^0)^2g \right),$$  \hfill (9)

$$\frac{\partial \phi}{\partial t} = p_f(\rho + \rho^0) - \rho^0 + \int_0^x \frac{\rho_0 g}{1 - \phi(\omega)} d\xi, \hfill (10)$$

Here $a(\omega) = \phi(\omega)/(1 - \phi(\omega))$, $K(\omega) = k(\phi(\omega))(1 - \phi(\omega))$, $\phi(\omega) = G^{-1}(\omega + G(\phi^0))$. Moreover,

$$\rho |_{t_0} = \omega |_{t_0} = ((1 - \phi(\omega)) \frac{\partial(\rho + \rho^0)}{\partial x} - (\rho + \rho^0)g) |_{x=0,x=1} = 0.$$

For the Banach space, we choose the space $C^{2+\beta,1+\beta/2}(\overline{Q}_t_0)$, where $\beta$ is any number from the interval $(0, \alpha)$, $\alpha \in [0,1)$. Let $V = \{ \bar{\rho}(x,t), \bar{\omega}(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_t_0)| \bar{\rho} |_{t=0} = \bar{\omega} |_{t=0} = ((1 - \phi(\omega)) \frac{\partial(\bar{\rho} + \rho^0)}{\partial x} - (\bar{\rho} + \rho^0)g) |_{x=0,x=1} = 0, \bar{m}_1 - \rho^0(x) \leq \bar{\rho}(x,t) \leq M_1 - \rho^0(x) < \infty, \bar{m}_1 = m_1/2(1+g_0/(R(1-M_0)))^{-1}, G(m_0/2) - G(0) \leq \bar{\omega}(x,t) \leq G((M_0+1)/2) - G(0) \leq \infty, (x,t) \in Q_t_0, M_1 = 2M_1(1+g_0/(R(1-M_0)))$. $\bar{\omega} |_{1+a,(1+a)/2,Q_0} \leq K_1, (\bar{\omega}|_{2+\alpha,(2+\alpha)/2,Q_0}, \bar{\rho}|_{2+\alpha,(2+\alpha)/2,Q_0}) \leq K_1 + K_2$, where $K_1$ is an arbitrary positive constant, while the positive constant $K_2$ will be given later. We note that on the set $V$ following inequalities hold: $0 < m_0/2 \leq \phi(\omega) \leq (M_0+1)/2 < 1$, $\alpha(\omega) > 0$, $K(\omega) > 0$.

Let us construct an operator $\Lambda$ mapping $V$ in $V$. Suppose that $\bar{\omega}, \bar{\phi} \in V$. Using (10), we define the function $\omega$ by the equality

$$\omega = \int_0^t \left( R \left( \bar{\rho}(x,\tau) + \rho^0(x) \right) - P^0(\phi(\omega), \bar{\rho}) + \int_0^x g(\rho_s + (\bar{\rho} + \rho^0(\xi))/1 - \phi(\omega)) d\xi \right) d\tau. \hfill (11)$$

From the representation (11) it follows that smoothness $\omega$ is determined by the smoothness of functions $\bar{\rho}, \bar{\omega}, \rho^0, \rho^0$ and $g$. Therefore there exists a value $t_1 = t_1(m_0, M_0, m_1, M_1)$, such that for all $t_0 \leq t_1$ the following inequality holds $0 < m_0/2 \leq \phi(x,t) \leq (M_0+1)/2$, $(x,t) \in Q_t_0$. In particular, we have an estimate

$$|\omega|_{2+\alpha,(2+\alpha)/2,Q_0} \leq C_0(m_0, M_0, m_1, M_1, K_1, T, |g|_{1+a,\Omega}, |\rho^0|_{2+a,\Omega}, |\rho^0|_{a/2,0,T})(1 + t_0)|\bar{\rho}|_{2+\alpha,\omega/2,\Omega}.$$

We also have the estimate for function $\omega(x,t)$: $G(m_0/2) \leq \omega(x,t) + G(0) \leq G((M_0+1)/2)$. Using (9), $\bar{\rho}$ and $\omega(x,t)$, we find the function $\rho(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):

$$\frac{\partial}{\partial t} (a(\omega)(\rho + \rho^0)) = \frac{\partial}{\partial x} \left( K(\omega)b(\bar{\rho}) \frac{\partial(\bar{\rho} + \rho^0)}{\partial x} - \frac{K(\omega)}{1 - \phi(\omega)}(\bar{\rho} + \rho^0)(\rho + \rho^0)g \right),$$  \hfill (12)

$$\rho |_{t=0} = 0, \ (1 - \phi(\omega))R \frac{\partial(\rho + \rho^0)}{\partial x} - (\rho + \rho^0)g |_{x=0,x=1} = 0.$$
The equation for $\rho(x,t)$ is uniformly parabolic. In view of the properties of $\vec{\omega}(x,t)$ and $\rho^0(x)$ problem (12) has a classical solution [13]. In addition, we have the following estimate:

$$\left| \frac{1}{a(\vec{\omega})} \frac{\partial a(\vec{\omega})}{\partial t} \right| \leq C_1(m_0, M_0, m_1, M_1, \max_{0 \leq t \leq T} \mid \rho^0(t) \mid).$$

Under the additional condition smallness for the value of the time interval the following statement holds.

**Lemma 1.** There exists such a $t_2$, that when $t_0 \leq \min(t_1, t_2)$, the classical solution of problem (12) satisfies the following inequality in $Q_{t_0}$: $0 < \bar{m}_1 \leq \rho(x, t) + \rho^0(x) \leq M_1 < \infty$.

**Proof.** Further, setting $U(x, t) = \rho(x, t) + \rho^0(x)$, we can express problem (13) in the form

$$\frac{\partial}{\partial t}(a(\vec{\omega})U) = \frac{\partial}{\partial x} \left( K(\vec{\omega})b(\vec{\rho}) \frac{\partial U}{\partial x} - K(\vec{\omega}) \left( \frac{\partial \rho}{\partial x} + \rho^0 \right) U g \right), \left( \frac{\partial U}{\partial x} - \bar{d} U \right) \mid_{x=0, x=1} = 0, U \mid_{t=0} = \rho^0,$$

where $\bar{d} = g/((1 - \phi(\vec{\omega}))R)$. First, we show that $U(x, t) \geq 0$, $(x, t) \in Q_{t_0}$. Following [13], we get estimate

$$\int_0^1 a(\vec{\omega}) \mid_{x=0}^t \int_0^1 \left| \frac{\partial a}{\partial \tau} \right| dx \mid_{x=0}^t + e^{1/2} \int_0^1 a \mid_{t=0}^T dx = 0.$$ 

Passing to the limit as $\varepsilon \to 0$, we find that $z(0) = 0$, i.e. $U \geq 0$.

We can express problem (13) in the form:

$$U_t - \bar{a}_{11} U_{xx} + \bar{a}_1 U_x + \bar{a} U = 0, \quad (U_x - \bar{d} U) \mid_{x=0,1} = 0,$$

where

$$\bar{a}_{11} = \frac{Kb}{a}, \quad \bar{a}_1 = \frac{d - (Kb)_x}{a}, \quad \bar{a} = \frac{a_t + d_x}{a}, \quad \bar{d} = \frac{g}{(1 - \phi)R}.$$

Following [13], we move from function $U(x, t)$ to a new function $w(x, t)$ related to it by equality $w(x, t) = e^{-\lambda t} \varphi(x) U(x, t)$, where

$$\varphi = -mx^2 + mx + 1 > 0, m \equiv 2 \max_{Q_t} \mid \bar{d} \mid = \frac{4g_0}{(1 - M_0)R},$$

and number $\lambda$ will be indicated later.

Function $w$ because of (14) is the solution of the equation

$$w_t - \bar{a}_{11} w_{xx} + (\bar{a}_1 + \frac{2\bar{a}_{11} \varphi_x}{\varphi}) w_x + (-2\bar{a}_{11} \frac{\varphi_x^2}{\varphi} + \bar{a}_{11} \frac{\varphi_{xx}}{\varphi} - \bar{a}_1 \frac{\varphi_x}{\varphi} + \bar{a} + \lambda) w = 0,$$

$$w_x \mid_{x=0,1} = (\frac{\varphi_x}{\varphi} + \bar{d}) w \mid_{x=0,1},$$

where $w \mid_{x=0} \geq 0, \quad w \mid_{x=1} \leq 0$, because $\varphi \mid_{x=0,1} = 1, \quad \varphi \mid_{x=0} = m > 0, \quad \varphi \mid_{x=1} = -m \equiv -2 \max \bar{d} < 0$. Choosing

$$\lambda > \max_{Q_t} [2\bar{a}_{11} \frac{\varphi_x^2}{\varphi} - \bar{a}_{11} \frac{\varphi_{xx}}{\varphi} + \bar{a}_1 \frac{\varphi_x}{\varphi} - \bar{a}],$$

the function $w$ reaches a positive maximum at $t = 0$, such

$$U(x, t)e^{-\lambda t} \varphi(x) \leq \max_{Q_t} (U(x, t)e^{-\lambda t} \varphi(x)) = \max_{Q_t} w(x, t) \leq \max_{x} w \mid_{t=0} = \max_{x} (U(x, t)e^{-\lambda t} \varphi(x)) \mid_{t=0}.$$
Therefore, we obtain upper bound for $U : U \leq e^{M_1}(1 + g_0/(1 - M_0)R))$. Then there exists a value $\hat{t}_2 = \ln 2^{1/\lambda_1}$, that for all $t \leq \hat{t}_2$ we have the estimate for $p$ from above from Lemma 1. To obtain a lower estimate we represent equation (14) in the form $(z(x, t) = 1/U(x, t))$

$$z_t - \tilde{a}_{11}z_{xx} + \frac{2\tilde{a}_{11}}{x}(z_x^2) + \tilde{a}_1z_x - \tilde{a}_2 = 0.$$ 

Then in the same way we obtain the estimate

$$U(x, t) \geq m_1e^{-\lambda t}(1 + \frac{g_0}{R(1 - M_0)})^{-1}, \lambda_1 > \max_{\tilde{q}_t}[2\tilde{a}_{11} \frac{\varphi^2}{\varphi^2} - \tilde{a}_{11} \frac{\varphi_{xx}}{\varphi} + \tilde{a}_1 \varphi_x - 2\tilde{a}_{11} \frac{\varphi^2}{\varphi^2} + \tilde{a}].$$

Then there exists a value $\hat{t}_3 = \ln 2^{1/\lambda_1}$, that for all $t \leq \hat{t}_3$ we have the estimate for $p$ from above from Lemma 1. Choosing $t \leq t_2 = \min\{\hat{t}_2, \hat{t}_3\}$, we proof Lemma 1.

Following [6], we obtain the necessary estimates. Thus, the operator $\Lambda$ maps the set $V$ into itself for sufficiently small values of $t_0$. Using the obtained estimates, we can easily show the continuity of the operator $\Lambda$ in the norm of the space $C^{2+\beta,1+\beta/2}(Q_{t_0})$. By the Tikhonov-Schauder theorem, there exists a fixed point $(\rho, \omega) \in V$ of the operator $\Lambda$. Uniqueness is established in the standard way [6]. Theorem 1 is proved.

4. The case of an incompressible media

If the density of the liquid phase is constant $(\rho_f = \text{const})$ the system of equations (6)–(7) can be reduced to one equation for the porosity of $\phi$ in Lagrange variables:

$$\frac{\partial}{\partial t} \phi ((1 - \phi) \frac{\partial G(\phi)}{\partial x} - g(\rho_{tot} + \rho_f)), \tag{15}$$

This equation is supplemented by the following initial-boundary conditions:

$$\phi|_{t=0} = \phi^0, (k(\phi)((1 - \phi) \frac{\partial G(\phi)}{\partial x} - g(\rho_{tot} + \rho_f)))|_{x=0,1} = 0. \tag{16}$$

**Definition 2.** The solution of problem (15)–(16) is the function $(\phi_t, \phi_{xxt}) \in C^{\alpha,\beta}(Q_T)$, such that $0 < \phi < 1$. This functions satisfy the equation (15) and the initial and boundary conditions (16) and regarded as continuous functions in $Q_T$.

**Theorem 2.** Suppose that the data of problem (15)–(16) satisfies the following conditions: 1) the functions $k(\phi), \xi(\phi)$ and their derivatives up to the second order are continuous for $\phi \leq (0, 1)$ and satisfy the conditions $k_0^{1}\phi^{a_1}(1 - \phi)^{b_1} \leq k(\phi) \leq k_0\phi^{a_1}(1 - \phi)^{b_1}, 1/\xi(\phi) = a_0(\phi)\phi^{a_1}(1 - \phi)^{a_2}, 0 < R_1 \leq a_0(\phi) \leq R_2 < \infty$, where $k_0, a_i, R_i, i = 1, 2$ are positive constants, $q_1, ..., q_4$ are fixed real parameters. 2) the function $g$ and the initial function $\phi^0$ satisfy the following smoothness conditions: $g \in C^{1+a,1+b}(Q_T), \phi^0 \in C^{2+a}(Q_T)$, and inequalities $0 < m_0 \leq \phi^0(x) \leq M_0 < 1, |g(x, t)| \leq g_0 < \infty, x \in \Omega$, where $m_0, M_0, g_0$ are given positive constants. Then problem (15)–(16) has a local solution, i.e., there exists a value of $t_0$ such that $(\phi_t, \phi_{xxt}) \in C^{\alpha,\beta}(Q_{t_0})$. Moreover $0 < \phi(x, t) < 1$ in $Q_{t_0}$.

**Theorem 3.** Let, in addition to the conditions of Theorem 2, the functions $k(\phi), \xi(\phi)$ satisfy the conditions

$$k(\phi) = \frac{k}{\mu} \phi^n, \xi(\phi) = \eta \phi^{-m}, n \geq 1, m \geq 1,$$

where $k, \mu, \eta$ are given positive constants. Then for all $t \in [0, T], T < \infty$ uniqueness solution of problem (15)–(16) exists, and there are numbers $0 < m_1 < M_1 < 1$ such that $m_1 \leq \phi(x, t) \leq M_1, (x, t) \in Q_T.$
In the case of incompressible liquid the local solvability of the problem (15)–(16) is established in the same way as in [14]. Let \(0 < \phi_1 \leq \phi_2 < 1\). From the definition of the functions \(G(\phi)\) and \(\xi(\phi)\) we have

\[
0 < \Delta G \equiv G(\phi_2) - G(\phi_1) = \int_{\phi_1}^{\phi_2} (1 + \frac{\xi(s)}{1 - s}) ds \geq (1 + \frac{1}{R_2})(\phi_2 - \phi_1).
\]

We have an estimate of the form

\[
(1 + R_2)|G(\phi_1) - G(\phi_2)| \geq |\phi_1 - \phi_2|.
\]

After the obtained estimates for \(\omega\) the property \(0 < m_0/2 \leq \phi(x,t) \leq (M_0 + 1)/2\) is easily obtained. Further proof is similar [14]. Theorem 2 is proved.

5. Global solvability

Proof of Theorem 3. By virtue of Theorem 2, we assume that a solution to the problem (15)–(16) exists on the interval \([0,t_0]\), with \(0 < \phi(x,t) < 1\), \(x \in \Omega\), \(t \in [0,t_0]\). After obtaining the necessary a priori estimates that are independent of the value of \(t_0\), the local solution can be extended to the entire segment \([0,T]\).

Let \(s(x,t) = \phi/(1 - \phi)\) in (15). Then the function \(s\) satisfies the next problem

\[
\frac{\partial s}{\partial t} - \frac{\partial}{\partial x} \left( \bar{k}(s) \left( \frac{1}{1 + s} \frac{\partial^2 G(s)}{\partial x \partial t} - \bar{g}(\phi(s)) \right) \right) = 0,
\]

\[
s|_{t=0} = s^0, \quad \left( \frac{1}{1 + s} \frac{\partial^2 G(s)}{\partial x \partial t} - \bar{g}(\phi(s)) \right) |_{x=0,x=1} = 0,
\]

where

\[
\bar{k}(s) = k(\phi(s)), \quad \bar{g}(s) = g\left( \frac{1}{1 + s} \rho_a + \frac{1 + 2s}{1 + s} \rho_f \right), \quad s^0 = \frac{\phi^0}{1 - \phi^0}.
\]

The following Lemma holds.

**Lemma 2.** Let \(s(x,t)\) is the solution of the problem (17)–(18). Then \(\int_0^1 s(x,t)dx = \int_0^1 s^0(x)dx \equiv \lambda\), and there exists such a point \(a(t) \in [0,1]\), that \(s(a(t),t) = \lambda > 0\).

The proof fully follows [12].

Further consider the function \(\psi(s)\), that satisfies the relation

\[
\frac{d^2 \psi(s)}{ds^2} = \bar{k}^{-1}(1 + s) \frac{dG(s)}{ds}.
\]

**Lemma 3.** Let \(s(x,t)\) is the solution of the problem (17)–(18), \(n,m \geq 1\). Then function \(\psi\) satisfies the following inequality \(\psi \geq P(1 + s)^2/2\), where \(P = (\mu\eta)/k = \text{const} > 0\).

Proof. For the function \(G\) we have for \(s > 0\)

\[
\frac{dG}{ds} = 1 + \eta s^{-m}(1 + s)^{-m-1} = 1 + \eta \left(1 + \frac{1}{s}\right)^{m-1} \frac{1}{s} \geq \frac{1}{\eta s}.
\]
Then
\[ \frac{d^2 \psi}{ds^2} = \tilde{k}^{-1}(s)(1 + s)^{dG/ds} = \bar{P} s^{1 + \bar{m}} + \bar{P}(1 + s)^{\bar{m}}, \]
where \( \bar{p} = n + 1 \geq 2, p = m + n \geq 2, \bar{P} = \mu/k. \)

If \( \bar{p}, p \) are integers numbers, then function \( \psi \) satisfy equality
\[ \psi = \bar{P} \left( \frac{\bar{p}s^2}{2} + s(c_3 - \frac{\bar{p}(\bar{p} - 1)}{2}) + c_4 + \frac{\bar{p}(\bar{p} - 1)}{2} L + s^3 \frac{3^{\bar{p} - 1}}{6} + \sum_{i=1}^{\bar{p}-3} C_i^3 \left( \frac{1}{i + 1} \frac{1}{s^i} \right) \right) + 
+ \bar{P} \left( \frac{s^2}{2} + s(c_1 - p) + c_2 + pL + \sum_{i=3}^{p} C_p^i \frac{1}{(i - 1)(i - 2)} \frac{1}{s^{i-2}} \right), \quad C_p^i = \frac{p!}{i!(p - i)!}, \]
where \( c_1, c_2, c_3, c_4 \) are arbitrary constants,
\[ \bar{L} = \left( s - \frac{(\bar{p} - 2)}{3} \right) \text{ln}s, \quad L = \left( s - \frac{(p - 1)}{2} \right) \text{ln}s. \]

It is enough to check the non-negativity of functions \( \bar{L}, L \). Lemma 3 is proved.

**Lemma 4.** Let \( s(x, t) \) is the solution of the problem (17)–(18), \( n, m \geq 1. \) Then there exists positive constants \( A \) and \( B \) such that \( 0 < A \leq s \leq B < \infty \) for all \( t \in [0, T], \) where
\[ A = \lambda e^{-1/\eta N_1^{1/2}}, B = \lambda e^{1/\eta N_1^{1/2}}, N_1 = e^{T N_2} \int_0^1 \left( \psi(s^0) + C_2^2(s^0) \right) dx, N_2 = \frac{1}{2} \max \{ 1, \frac{4\bar{g}_0^2}{\bar{P}} (\bar{\rho}_x^2 + 4\bar{\rho}_y^2) \}. \]

Proof. Proof fully follows by [14].

Further we use the theory of elliptic equations for the function \( \partial G/\partial t \) set forth in [11]. Theorem 3 is proved.

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