INTERSECTION COHOMOLOGY AND PERVERSE EIGENSPACES OF THE MONODROMY

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ABSTRACT. We describe the relationship between intersection cohomology with twisted coefficients and the perverse sheaves which play the role of the eigenspaces for the Milnor monodromy of an affine hypersurface.

1. Introduction

The Monodromy Theorem ([4], [10], [15], [16]) tells us that the classical Milnor monodromy for an affine hypersurface is quasi-unipotent; over $\mathbb{C}$, this means that the eigenvalues of the monodromy of the Milnor fiber at each point in the hypersurface are roots of unity. We wish to look at this in the category of perverse sheaves and see what it has to do with intersection cohomology, but first we need to describe our set-up; general references for this background are [2], [7], [13], [5], [17], and [25].

Suppose that $U$ is a non-empty open subset of $\mathbb{C}^{n+1}$, where $n \geq 1$, and that $f : U \to \mathbb{C}$ is a reduced, nowhere locally constant, complex analytic function. Then $X := V(f) = f^{-1}(0)$ is a hypersurface in $U$ of pure dimension $n$. Furthermore, since $f$ is reduced, the singular set $\Sigma$ of $X$ is equal to the intersection of the hypersurface with the critical locus, $\Sigma f$, of $f$.

If we use $\mathbb{C}$ for our base ring for perverse sheaves, then the shifted constant sheaf $\mathbb{C}^\bullet_U[n+1]$ is perverse on $U$, as is the shifted constant sheaf $\mathbb{C}^\bullet_X[n]$ on $X$. Furthermore, on $X$, there are the perverse sheaves of nearby and vanishing cycles of $\mathbb{C}^\bullet_U[n+1]$ along $f$, denoted, respectively, as $\psi_f[-1]\mathbb{C}^\bullet_U[n+1]$ and $\phi_f[-1]\mathbb{C}^\bullet_U[n+1]$. The stalk cohomology of these complexes at a point $x \in X$ yield the ordinary cohomology and reduced cohomology of the Milnor fiber $F_{f,x}$ of $f$ at $x$ with a shift.

There are monodromy automorphisms $T_f$ and $\tilde{T}_f$ on the nearby and vanishing cycles, respectively. On stalk cohomology, these yield the ordinary Milnor monodromy automorphisms. For each root of unity $\xi$ (actually, for any complex number), we may consider the perverse eigenspaces of these automorphisms, namely $\ker\{\xi \text{id} - T_f\}$ and $\ker\{\xi \text{id} - \tilde{T}_f\}$. It is important to note that the stalk cohomology of these perverse kernels need not be isomorphic to the eigenspaces of the monodromy on the stalk cohomology.

When $\xi = 1$, we can relate $\ker\{\xi \text{id} - \tilde{T}_f\}$ to the intersection cohomology $I_X^\bullet$, with constant coefficients, on $X$; in fact, we can do this over $\mathbb{Z}$, not just over $\mathbb{C}$. We show:

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Theorem 1.1. (Proposition 3.1 and Theorem 3.3) Using $\mathbb{Z}$ as our base ring, there is an isomorphism $\mathbb{Z}_X[n] \cong \ker\{\id - T_f\}$, and the kernel of the canonical perverse surjection $\mathbb{Z}_X[n] \to \mathbf{I}_X^*$ is isomorphic to $\ker\{\id - T_f\}$.

The analogous statements over $\mathbb{C}$ are also true.

For $\xi \neq 1$, we must use intersection cohomology $\mathbf{I}_X^*(\xi)$ on $\mathcal{U}$ with a twist by $\xi$ around the hypersurface (we shall make this precise later). Letting $j : X \hookrightarrow \mathcal{U}$ be the inclusion, we show:

**Theorem 1.2.** (Theorem 4.1) Using $\mathbb{C}$ as our base ring, and supposing the $\xi \neq 0$ or 1, there is an isomorphism of perverse sheaves

$$j^*[1] \mathbf{I}_X^*(\xi) \cong \ker\{\xi^{-1}\id - T_f\} \cong \ker\{\xi^{-1}\id - T_f\}.$$

We begin with the case of $\xi = 1$; it is, in fact, the complicated case. First, in Section 2, we look at some basic results about the kernel of the canonical perverse surjection $\mathbb{Z}_X[n] \to \mathbf{I}_X^*$. Then we prove the main theorem in the $\xi = 1$ case in Section 3. We prove the main theorem for $\xi \neq 1$ in Section 4.

In Section 5, we will give some down-to-Earth applications of these results, applications which do not, when possible, use the language of the derived category.

As a final comment in this introduction, we should mention that the flavor of our techniques and result are reminiscent of the work of Beilinson in [1] (see, also, [24]); however, we do not see how to use those results to obtain our results.

2. The Comparison Complex

We continue with $\mathcal{U}$, $f$, and $X$ as in the introduction. We denote the respective inclusions as follows: $j : X \hookrightarrow \mathcal{U}$, $i : \mathcal{U}\setminus X \hookrightarrow \mathcal{U}$, $m : \Sigma \hookrightarrow X$, and $l : X\setminus \Sigma \hookrightarrow X$. Furthermore, we let $m := j \circ m$ be the inclusion of $\Sigma$ into $\mathcal{U}$. In this section, our base ring is $\mathbb{Z}$ (though all statements hold with base ring $\mathbb{C}$).

In this setting, there (at least) three canonical perverse sheaves on $X$: the shifted constant sheaf $\mathbb{Z}_X^*[n] \cong j^*[1] \mathbb{Z}_U^*[n + 1]$, the Verdier dual sheaf $j^!\mathbb{Z}_U^*[n + 1]$, and the intersection cohomology complex $\mathbf{I}_X^*$ (with constant $\mathbb{Z}$ coefficients), which is the intermediate extension to all of $X$ of $\mathbb{Z}_X^*[\Sigma\setminus n]$. In addition, there is a canonical surjection

$$\mathbb{Z}_X^*[n] \to \mathbf{I}_X^*$$

in the Abelian category $\text{Perv}(X)$ of perverse sheaves (with middle perversity) on $X$ (the surjectivity follows from the fact that $\tau_X$ is an isomorphism on $X\setminus \Sigma$ and that the intermediate extension $\mathbf{I}_X^*$ has no non-trivial quotients with support contained in $\Sigma$). There is also a dual injection of $\mathbf{I}_X^*$ into $j^!\mathbb{Z}_U^*[n + 1]$.

We shall focus on the surjection $\tau_X : \mathbb{Z}_X^*[n] \to \mathbf{I}_X^*$; one can dualize the results there to obtain results for the dual injection.

Our fundamental definition is:

**Definition 2.1.** The comparison complex, $\mathbf{N}_X^*$, on $X$ is the kernel (in $\text{Perv}(X)$) of $\tau_X$. Hence, by definition, the support of $\mathbf{N}_X^*$ is contained in $\Sigma$ and there is a short exact sequence

$$0 \to \mathbf{N}_X^* \to \mathbb{Z}_X^*[n] \to \mathbf{I}_X^* \to 0.$$
Remark 2.2. While $N^*_X$ is interesting as a perverse sheaf, on the level of stalks, $N^*_X$ merely gives the shifted, reduced intersection cohomology of $X$. To be precise, the long exact sequence on stalk cohomology at $x \in X$ yields a short exact sequence

$$0 \to \mathbb{Z} \to H^{-n}(\mathcal{I}_X)_x \to H^{-n+1}(N^*_X)_x \to 0$$

and, for all $k \neq -n+1$, isomorphisms

$$H^{k-1}(\mathcal{I}_X)_x \cong H^k(N^*_X)_x.$$ 

If we let $\tilde{IH}$ denote reduced intersection cohomology, with topological indexing (as is used for intersection homology in [6]), then we have

$$H^k(N^*_X)_x \cong \tilde{IH}^{-n+k-1}(B^\epsilon_\gamma(x) \cap X; \mathbb{Z}),$$

where $B^\epsilon_\gamma(x)$ denotes a small open ball, of radius $\epsilon$, centered at $x$.

We wish to recall some basic definitions and results on intersection cohomology, real links, and complex links.

Let $B_\epsilon(x)$ denote the closed ball of radius $\epsilon$, centered at $x$, in $\mathbb{C}^{n+1}$, so that its boundary $\partial B_\epsilon(x)$ is a $(2n+1)$-dimensional sphere. We continue to denote the open ball by $B^\epsilon_\gamma(x)$.

The real link of $X$ at $x$ is (the homeomorphism-type of)

$$K_{X,x} := X \cap \partial B_\epsilon(x)$$

for sufficiently small $\epsilon > 0$, which is homotopy-equivalent to $X \cap (B^\epsilon_\gamma(x) \setminus \{x\})$. See, for instance, [23] and [8].

The support and cosupport conditions tell us that, for all $x \in X$, the stalk cohomology of $\mathcal{I}_X^*$ has the following properties:

$$H^k(\mathcal{I}_X^*)_x \cong \begin{cases} \mathbb{H}^k(K_{X,x}; \mathbb{I}_X^*), & \text{if } k \leq -1; \\ 0, & \text{if } k \geq 0. \end{cases}$$

In particular, we have:

**Proposition 2.3.** If $x$ is an isolated singular point of $X$, then $N^*_X$ is a perverse sheaf which has $x$ as an isolated point in its support, and so $H^k(N^*_X)_x = 0$ if $k \neq 0$, and

$$H^0(N^*_X)_x \cong \tilde{H}^{-n-1}(K_{X,x}; \mathbb{Z}).$$

The complex link of $X$ at $x$ is (the homeomorphism-type of)

$$\mathbb{L}_{X,x} := B^\epsilon_\gamma(x) \cap X \cap L^{-1}(\gamma),$$

where $L$ is a generic affine linear form such that $L(x) = 0$ and $0 < |\gamma| \ll \epsilon \ll 1$ (see, for instance, [8] and [17]). As $X$ is a hypersurface (and so, a local complete intersection), $\mathbb{L}_{X,x}$ has the homotopy-type of a bouquet of a finite number of $(n-1)$-dimensional spheres. The number of spheres in this bouquet equals $\operatorname{rank} \tilde{H}^{-n-1}(\mathbb{L}_{X,x}; \mathbb{Z})$, which equals the rank of $H^0(\phi_L[-1]Z^*_X[n])_x$. Furthermore, it is known that this rank is equal to an intersection number,

$$\operatorname{rank} H^0(\phi_L[-1]Z^*_X[n])_x = (\Gamma^1_{f,L} \cdot V(L))_x,$$

where $\Gamma^1_{f,L}$ is the relative polar curve of $f$ with respect to $L$; see Corollary 2.6 of [18] (though this was known earlier by Hamm, Lê, Siersma, and Teissier).
We can prove a non-trivial proposition merely using the defining short exact sequence in Definition 2.1, and so this result really follows just from the surjectivity of $\tau_x$. Compare with Proposition 6.1.22 of [5].

**Proposition 2.4.** Suppose that $x$ is an isolated singular point of $X$. Then,

$$\text{rank } H^{n-1}(L_{X,x}; \mathbb{Z}) - \text{rank } H^{n-1}(K_{X,x}; \mathbb{Z}) = \text{rank } H^0(\phi_L[-1]I^*_x) \geq 0,$$

where $L$ is a generic affine linear form such that $L(x) = 0$.

**Proof.** Applying $\phi_L[-1]$ to the defining short exact sequence for $N^*_x$, we obtain the short exact sequence

$$0 \to \phi_L[-1]N^*_X \to \phi_L[-1]Z^*_X[n] \to \phi_L[-1]I^*_X \to 0,$$

in which each term has support contained in $\{x\}$ locally; this implies that the long exact sequence on stalk cohomology at $x$ yields a short exact sequence

$$0 \to H^0(\phi_L[-1]N^*_X)_x \to H^0(\phi_L[-1]Z^*_X[n])_x \to H^0(\phi_L[-1]I^*_X)_x \to 0.$$

However, since $x$ is an isolated singular point of $X$, $x$ is an isolated point in the support of $N^*_X$; hence,

$$H^0(\phi_L[-1]N^*_X)_x \cong H^0(N^*_X)_x \cong \tilde{H}^{n-1}(K_{X,x}; \mathbb{Z}).$$

Therefore, from the previous short exact sequence, we obtain

$$\text{rank } \tilde{H}^{n-1}(L_{X,x}; \mathbb{Z}) - \text{rank } \tilde{H}^{n-1}(K_{X,x}; \mathbb{Z}) = \text{rank } H^0(\phi_L[-1]Z^*_X[n])_x - \text{rank } H^0(\phi_L[-1]I^*_X)_x.$$

Finally, the difference in the ranks of the reduced cohomologies equals the difference without the reductions. □

**Remark 2.5.** Suppose that $X$ is a pure-dimensional (i.e., connected) local complete intersection of dimension $d$. Then, the result of Lê in [17] implies that $Z^*_X[d]$ is perverse and, again, the canonical map from $Z^*_X[d]$ to the intersection cohomology on $X$ is a surjection. If $X$ has a isolated singular point at $x$, then the result of Proposition 2.4 remains true if one replaces $n$ with $d$.

### 3. The Main Theorem for $\xi = 1$

We continue with the notation from the previous section and the introduction, and we continue to use $\mathbb{Z}$ as our base ring.

Recall that

$$T_f : \psi_f[-1]Z^*_\mathcal{U}[n + 1] \xrightarrow{\cong} \psi_f[-1]Z^*_\mathcal{U}[n + 1]$$

and

$$\overline{T}_f : \phi_f[-1]Z^*_\mathcal{U}[n + 1] \xrightarrow{\cong} \phi_f[-1]Z^*_\mathcal{U}[n + 1]$$

denote the monodromy automorphisms on the nearby and vanishing cycles along $f$, respectively.

Throughout this section and the remainder of the paper, we will use the notation from the introduction. In addition, we let $s := \dim \Sigma$ (or, when we focus on the germ at $x \in \Sigma$, we will let $s := \dim_x \Sigma$).

Recall that there are two canonical short exact sequences in $\text{Perv}(X)$:

$$0 \to j^*[n + 1] \to \psi_f[-1]Z^*_\mathcal{U}[n + 1] \xrightarrow{\text{can}} \phi_f[-1]Z^*_\mathcal{U}[n + 1] \to 0$$
Proposition 3.1. There is an isomorphism of perverse sheaves
\[ Z_X^\bullet[n] \cong \ker \{ \text{id} - T_f \}. \]

Dually, there is an isomorphism of perverse sheaves
\[ j^!|1]Z^\bullet_\varphi[n + 1] \cong \coker \{ \text{id} - T_f \}. \]

As we mentioned in the introduction, this proposition is very unsatisfying, since we get \( Z_X^\bullet[n] \) for the kernel, regardless of whether or not 1 is an eigenvalue of the Milnor monodromy in some degree \( > 0 \). However, as we shall see, the vanishing cycle analog, \( \ker \{ \text{id} - T_f \} \), is more interesting.

First, we need a lemma. We use \( \mu H^k(A^\bullet) \) to denote the degree \( k \) perverse cohomology of a complex (with middle perversity). See Section 10.3 of [13] or Sections 5.1 and 5.2 of [5].

Lemma 3.2. There is an isomorphism of perverse sheaves
\[ N_X^\bullet \cong \mu H^0(m_*m^*Z_X^\bullet[n]). \]

Proof. The intersection cohomology complex \( I_X^\bullet \) on \( X \) is the intermediate extension of the shifted constant on \( X\setminus \Sigma \), i.e., the image (in \( \text{Perv}(X) \)) of the canonical morphism
\[ \mu H^0(l_*Z_{X\setminus \Sigma}^\bullet[n]) \to \mu H^0(l_*Z_{X\setminus \Sigma}^\bullet[n]). \]

The morphism \( \alpha \) factors through the perverse sheaf \( Z_X^\bullet[n] \); \( \alpha \) is the composition of the canonical maps
\[ \mu H^0(l_*Z_{X\setminus \Sigma}^\bullet[n]) \cong \mu H^0(l_*l^!Z_X^\bullet[n]) \to \mu H^0(l_*Z_X^\bullet[n]) \cong \mu H^0(l_*Z_{X\setminus \Sigma}^\bullet[n]). \]

We claim that \( \beta \) is a surjection and, hence, \( \text{im} \gamma \cong I_X^\bullet \). To see this, take the canonical distinguished triangle
\[ \to l!l^!Z_X^\bullet[n] \to Z_X^\bullet[n] \to m_*m^*Z_X^\bullet[n] \to 0, \]

and consider a portion of the long exact sequence in \( \text{Perv}(X) \) obtained by applying perverse cohomology:
\[ \to \mu H^0(l!l^!Z_X^\bullet[n]) \to Z_X^\bullet[n] \to \mu H^0(m_*m^*Z_X^\bullet[n]) \to . \]

We want to show that \( \mu H^0(m_*m^*Z_X^\bullet[n]) = 0 \).

We have
\[ \mu H^0(m_*m^*Z_X^\bullet[n]) \cong \mu H^0(m_*Z_X^\bullet[n]). \]

Then it is trivial that the complex \( Z_X^\bullet[s] \) satisfies the support condition, and so \( \mu H^k(Z_X^\bullet[s]) = 0 \) for \( k \geq 1 \). But
\[ \mu H^0(m_*Z_X^\bullet[n]) \cong m_*\mu H^{n-s}(Z_X^\bullet[s]), \]

which equals 0 since \( n - s \geq 1 \). Therefore, \( \beta \) is a surjection and \( \text{im} \gamma \cong I_X^\bullet \).
Now take the canonical distinguished triangle
\[ \rightarrow \text{m} \text{m}'Z^\bullet_X[n] \rightarrow Z^\bullet_X[n] \rightarrow l_! l^*Z^\bullet_X[n] \xrightarrow{[1]} \]
and consider a portion of the long exact sequence in \( \text{Perv}(X) \) obtained by applying perverse cohomology:
\[ \rightarrow \mu H^{-1}(l_! l^*Z^\bullet_X[n]) \rightarrow \mu H^0(\text{m} \text{m}'Z^\bullet_X[n]) \rightarrow Z^\bullet_X[n] \xrightarrow{\sim} \mu H^0(l_! l^*Z^\bullet_X[n]) \rightarrow . \]

We claim that \( \mu H^{-1}(l_! l^*Z^\bullet_X[n]) = 0. \) This is easy; as \( Z^\bullet_X[n] \) is perverse, \( l^*Z^\bullet_X[n] \) is perverse (since it is the restriction to an open subset) and, in particular, satisfies the co-support condition. By 10.3.3.iv of \([13], l_! l^*Z^\bullet_X[n] \) also satisfies the co-support condition. Thus, \( \mu H^{-1}(l_! l^*Z^\bullet_X[n]) = 0. \)

Therefore, \( \mu H^0(\text{m} \text{m}'Z^\bullet_X[n]) \) is the kernel of the map \( \gamma \), whose image is \( I_X^\bullet \), i.e., this kernel is how we defined \( N^\bullet_X \), and we are finished. \( \square \)

Now we can prove the first theorem that we stated in the introduction, Theorem 1.1.

**Theorem 3.3.** In \( \text{Perv}(X) \), there is an isomorphism
\[ N^\bullet_X \cong \ker \{ \text{id} - \tilde{T}_f \}. \]

Dually, there is an isomorphism between coker \( \{ \text{id} - \tilde{T}_f \} \) and the cokernel of the canonical injection \( I_X^\bullet \hookrightarrow j'[1]Z^\bullet_d[n + 1] \).

**Proof.** Consider the two nearby-vanishing short exact sequences from the beginning of the section:
\[ 0 \rightarrow j^*[-1]Z_d^\bullet[n + 1] \rightarrow \psi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{can}} \phi_f[-1]Z_d^\bullet[n + 1] \rightarrow 0 \]
and
\[ 0 \rightarrow \phi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{var}} \psi_f[-1]Z_d^\bullet[n + 1] \rightarrow j'[1]Z_d^\bullet[n + 1] \rightarrow 0 \]
and apply \( \text{m} \text{m}' \) to obtain two distinguished triangles:
\[ \rightarrow \text{m} \text{m}'j^*[-1]Z_d^\bullet[n + 1] \rightarrow \text{m} \text{m}'\psi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{m} \text{m}'\text{can}} \text{m} \text{m}'\phi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{[1]} \]
and
\[ \rightarrow \text{m} \text{m}'\phi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{m} \text{m}'\text{var}} \text{m} \text{m}'\psi_f[-1]Z_d^\bullet[n + 1] \rightarrow \text{m} \text{m}'j'[1]Z_d^\bullet[n + 1] \xrightarrow{[1]} . \]

As \( j^*[-1]Z_d^\bullet[n + 1] \cong Z_X^\bullet[n] \) and the support of \( \phi_f[-1]Z_d^\bullet[n + 1] \) is \( \Sigma \), these distinguished triangles become
\[ (\dagger) \rightarrow \text{m} \text{m}'Z^\bullet_X[n] \rightarrow \text{m} \text{m}'\psi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{m} \text{m}'\text{can}} \phi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{[1]} \]
and
\[ (\ddagger) \rightarrow \phi_f[-1]Z_d^\bullet[n + 1] \xrightarrow{\text{m} \text{m}'\text{var}} \text{m} \text{m}'\psi_f[-1]Z_d^\bullet[n + 1] \rightarrow \text{m} \text{m}'j'[1]Z_d^\bullet[n + 1] \xrightarrow{[1]} . \]

By applying perverse cohomology to \( (\dagger) \), using that \( \phi_f[-1]Z_d^\bullet[n + 1] \) is perverse, and using the lemma, we immediately conclude that
\[ N^\bullet_X \cong \mu H^0(\text{m} \text{m}'Z^\bullet_X[n]) \cong \ker \{ \mu H^0(\text{m} \text{m}'). \text{can} \}. \]
Now, it is an easy exercise to verify that $\hat{m}^!\mathbb{Z}^*[2n + 2 - s]$ satisfies the cosupport condition. This implies that $H^k(\hat{m}^!\mathbb{Z}^*[2n + 2 - s]) = 0$ for all $k \leq -1$, i.e.,

$$H^k(m^!\hat{m}^![1]\mathbb{Z}^*[n + 1]) \cong m^!H^k(\hat{m}^![1]\mathbb{Z}^*[n + 1]) = 0,$$ for $k \leq -1 + (n - s)$.

As $n - s \geq 1$, $H^k(m^!\hat{m}^![1]\mathbb{Z}^*[n + 1])$ is zero for $k \leq 0$. Therefore, applying perverse cohomology to (†), yields that

$$\phi_f[-1]\mathbb{Z}^*[n + 1] \xrightarrow{\mu^H m^!m^! \var} H^0(m^!\hat{m}^![1]\mathbb{Z}^*[n + 1])$$

is an isomorphism.

Finally, we have

$$\ker \{ \text{id} - \tilde{T}_f \} \cong \ker \{ \mu^H m^! \text{(id} - \tilde{T}_f) \} \cong \ker \{ \mu^H m^!m^! \text{can} \} \cong \ker \{ \mu^H m^!m^! \text{var} \} \cong \ker \{ \mu^H m^! \text{can} \} \cong N_X^\bullet.$$

The dual statement follows by dualizing all of the above arguments.

**Remark 3.4.** While we have proved Theorem 3.3 with integral coefficients, the proof of the analogous statement with coefficients in an arbitrary field $K$ is precisely the same. That is, if $N_X^\bullet(\mathbb{R})$ is defined as the kernel of the canonical surjection from the constant sheaf $\mathbb{R}^\bullet[n]$ to the intersection cohomology with constant $\mathbb{R}$ coefficients $I^\bullet_X$, then $N_X^\bullet(\mathbb{R})$ is isomorphic to the kernel of

$$\text{id} - \tilde{T}_f : \phi_f[-1]\mathbb{R}^\bullet[n + 1] \to \phi_f[-1]\mathbb{R}^\bullet[n + 1].$$

With field coefficients, we proved a simple corollary of Theorem 3.3 in Theorem 3.2 of [21], where we showed that $\ker \{ \text{id} - \tilde{T}_f \} = 0$ if and only if $\mathbb{R}^\bullet[n]$ is isomorphic to $I^\bullet_X(\mathbb{R})$.

As an immediate, somewhat surprising, corollary of the fact that intersection cohomology is a topological invariant, we have:

**Corollary 3.5.** The perverse sheaf $\ker \{ \text{id} - \tilde{T}_f \}$ is an invariant of the (non-embedded) topological-type of the hypersurface $V(f)$.

Precisely, suppose that $f : \mathcal{U} \to \mathbb{C}$ and $g : \mathcal{U} \to \mathbb{C}$ are reduced complex analytic functions, let $X := V(f)$ and $Y := V(g)$, and suppose that $H : X \to Y$ is a homeomorphism. Then,

$$H\left( \text{supp} \left( \ker \{ \text{id} - \tilde{T}_f \} \right) \right) = \text{supp} \left( \ker \{ \text{id} - \tilde{T}_g \} \right)$$

and

$$H_*\left( \ker \{ \text{id} - \tilde{T}_f \} \right) \cong \ker \{ \text{id} - \tilde{T}_g \}.$$

In the next section, we look at the other eigenspaces of the monodromy. As we shall see, that case is much simpler than the $\xi = 1$ case.
4. THE MAIN THEOREM FOR $\xi \neq 0, 1$

Throughout this section, we use $\mathbb{C}$ as our base ring.

We need to look at the germ of the situation at a point $p \in X$. So, fix a $p \in X$, and choose real $\epsilon$ and $\delta$, $0 < \delta \ll \epsilon \ll 1$, such that $B^*_\epsilon(p) \subseteq \mathcal{U}$ and

$$B^*_\epsilon(p) \cap f^{-1}(\mathbb{D}^*_\delta \setminus \{0\}) \xrightarrow{\hat{f}} \mathbb{D}^*_\delta \setminus \{0\}$$

is a locally trivial fibration, whose fiber is the Milnor fiber of $f$ at $p$, where $\hat{f}$ denotes the restriction of $f$. That this can be done is a standard result on the Milnor fibration inside a ball. Replace the open set $\mathcal{U}$ with the open set $B^*_\epsilon(p) \cap f^{-1}(\mathbb{D}^*_\delta \setminus \{0\})$. On the open dense subset $B^*_\epsilon(p) \cap f^{-1}(\mathbb{D}^*_\delta \setminus \{0\})$, we take the local system $\mathcal{L}_\xi$, which is in degree $-(n + 1)$, has stalk cohomology $\mathbb{C}$ in degree $-(n + 1)$ and is given by the representation

$$\pi_1(B^*_\epsilon(p) \cap f^{-1}(\mathbb{D}^*_\delta \setminus \{0\})) \xrightarrow{\hat{f}} \pi_1(\mathbb{D}^*_\delta \setminus \{0\}) \cong \mathbb{Z} \to \text{Aut}(\mathbb{C}),$$

where $\mathbb{D}^*_\delta \setminus \{0\}$ is oriented counterclockwise and $h$ is the homomorphism which takes the generator $< 1 >$ to multiplication by $\xi$. Thus, $\mathcal{L}_\xi$ is the rank 1 local system, in degree $-(n + 1)$, which multiplies by $\xi$ as one goes once around a counterclockwise meridian around the hypersurface $X$.

As in the introduction, we let $I^*_\mathcal{U}(\xi)$ denote intersection cohomology using the local system $\mathcal{L}_\xi$: this is the intermediate extension of $\mathcal{L}_\xi$ to all of $\mathcal{U}$. It is a simple object in the category of perverse sheaves; see [2]. Then, it is well-known that $j^*[-1]I^*_\mathcal{U}(\xi)$ and $j^*[1]I^*_\mathcal{U}(\xi)$ are perverse, but, for lack of a convenient reference, we give a quick argument for this.

In the derived category, consider the two canonical distinguished triangles:

$$j_*j^*[-1]I^*_\mathcal{U}(\xi) \to i_!i^*I^*_\mathcal{U}(\xi) \to I^*_\mathcal{U}(\xi) \xrightarrow{[1]}$$

and

$$I^*_\mathcal{U}(\xi) \to i_*i^*I^*_\mathcal{U}(\xi) \to j_*j^*[1]I^*_\mathcal{U}(\xi) \xrightarrow{[1]}. $$

Now, $i_!i^*I^*_\mathcal{U}(\xi) \cong i^*I^*_\mathcal{U}(\xi) \cong \mathcal{L}_\xi$, and $i_*\mathcal{L}_\xi$ and $i_*\mathcal{L}_\xi$ are well-known to be perverse (by Proposition 10.3.3 and 10.3.17 of [13], combined with Theorem 5 of Section 5.1 of [9]). It follows immediately from this, and the simplicity of $I^*_\mathcal{U}(\xi)$, that $j_*j^*[-1]I^*_\mathcal{U}(\xi)$ and $j_*j^*[1]I^*_\mathcal{U}(\xi)$ are perverse. As $j_* \cong j_!$ is simply extension by zero, we conclude that $j^*[-1]I^*_\mathcal{U}(\xi)$ and $j^*[1]I^*_\mathcal{U}(\xi)$ are perverse.

Now we can prove the main theorem for $\xi \neq 0$ or 1:

**Theorem 4.1.** Using $\mathbb{C}$ as our base ring, and supposing the $\xi \neq 0$ or 1, there is an isomorphism of perverse sheaves

$$j^*[-1]I^*_\mathcal{U}(\xi) \cong \ker\{\xi^{-1} \text{id} - T_f\} \cong \ker\{\xi^{-1} \text{id} - \overline{T_f}\}.$$ 

Dually,

$$j^*[1]I^*_\mathcal{U}(\xi) \cong \text{coker}\{\xi^{-1} \text{id} - T_f\} \cong \text{coker}\{\xi^{-1} \text{id} - \overline{T_f}\}.$$ 

**Proof.** We prove the first statement, and leave the dual argument to the reader.

As the nearby cycles along $f$ are determined by perturbing $f$ in a radial direction in $\mathbb{D}^*_\delta \setminus \{0\}$, it follows that there is an isomorphism

$$Q : \psi_f[-1]C^*_\mathcal{U}[n + 1] \to \psi_f[-1]I^*_\mathcal{U}(\xi);$$
moreover, by construction, this isomorphism identifies the monodromy $T_f(\xi)$ on $\psi_f[-1]I^*_U(\xi)$ with $\xi \cdot T_f$ on $\psi_f[-1]C^*_U[n+1]$, i.e.,
\[ \xi \cdot T_f = Q^{-1} \circ T_f(\xi) \circ Q. \]
Therefore,
\[ \ker\{\id - T_f(\xi)\} \cong \ker\{\id - \xi \cdot T_f\} \cong \ker\{\xi^{-1} \id - T_f\}. \]
Consider again the two nearby-vanishing short exact sequences:
\[ 0 \to j^*[\omega]I^*_U(\xi) \to \psi_f[-1]I^*_U(\xi) \xrightarrow{\can} \phi_f[-1]I^*_U(\xi) \to 0 \]
and
\[ 0 \to \phi_f[-1]I^*_U(\xi) \xrightarrow{\var} \psi_f[-1]I^*_U(\xi) \xrightarrow{j^!} j^*[\omega]I^*_U(\xi) \to 0, \]
where $\var \circ \can \cong \id - T_f(\xi)$. We immediately conclude that
\[ j^*[\omega]I^*_U(\xi) \cong \ker\{\id - T_f(\xi)\} \cong \ker\{\xi^{-1} \id - T_f\}. \]
Finally, consider the triple of endomorphisms
\[ (\xi^{-1} \id - \id, \xi^{-1} \id - T_f, \xi^{-1} \id - \tilde{T}_f) \]
which acts on the short exact sequence
\[ 0 \to j^*[\omega]C^*_U[n+1] \to \psi_f[-1]C^*_U[n+1] \xrightarrow{\can} \phi_f[-1]C^*_U[n+1] \to 0 \]
and commutes with its maps. Then we have the associated long exact sequence
\[ 0 \to \ker\{\xi^{-1} \id - T_f\} \to \ker\{\xi^{-1} \id - \tilde{T}_f\} \to \ker\{\xi^{-1} \id - \tilde{T}_f\} \to \ker\{\xi^{-1} \id - T_f\} \to \coker\{\xi^{-1} \id - T_f\} \to \cdots. \]
As $\ker\{\xi^{-1} \id - T_f\}$ and $\ker\{\xi^{-1} \id - \tilde{T}_f\}$ are zero, we conclude the final isomorphism of the theorem, that
\[ \ker\{\xi^{-1} \id - T_f\} \cong \ker\{\xi^{-1} \id - \tilde{T}_f\}. \]
\[ \square \]

**Corollary 4.2.** If $\xi \neq 0, 1$, then there is a short exact sequence in $\Perv(U)$
\[ 0 \to j_* \ker\{\xi^{-1} \id - \tilde{T}_f\} \to i_! \mathcal{L}_\xi \xrightarrow{\omega_\xi} I^*_U(\xi) \to 0, \]
where $\omega_\xi$ is the canonical surjection.
In particular, for $x \in X$, for all $k$, there are isomorphisms on stalk cohomology
\[ H^k(\ker\{\xi^{-1} \id - \tilde{T}_f\})_x \cong H^{k+1}(I^*_U(\xi))_x. \]

5. Applications

What we have proved in Theorem 3.3 and Theorem 4.1 is that there is a short exact sequence in $\Perv(X)$

\[(*) \quad 0 \to \ker\{\id - \tilde{T}_f\} \to \mathbb{Z}[n] \xrightarrow{\tau_X} I^*_X \to 0, \]
where $\tau_X$ is the canonical surjection, and, if $\xi \neq 0, 1$, a short exact sequence in $\Perv(U)$

\[(**\text{)} \quad 0 \to j_* \ker\{\xi^{-1} \id - \tilde{T}_f\} \to i_! \mathcal{L}_\xi \xrightarrow{\omega_\xi} I^*_U(\xi) \to 0, \]
where $\omega_\xi$ is the canonical surjection.
The reader should naturally ask: what does this tell us about the topology of hypersurface singularities without the language of the derived category and Perv(\(X\))?

It is not true, in general, that cohomology of the stalk of the kernel is isomorphic to the kernel of the cohomology on the stalks, i.e., there may exist \(x \in \Sigma\) and a degree \(k\) such that

\[
H^k(\ker \left\{ \text{id} - \tilde{T}_f \right\})_x \not\cong \ker \left\{ \text{id} - (\tilde{T}_f)_x^k \right\},
\]

where \((\tilde{T}_f)_x^k\) is the induced map on \(H^k(\phi_f[-1]\mathbb{Z}_U[n+1])_x\). This makes the relationship with the topological theory complicated, since it is \((\tilde{T}_f)_x\) which is the classical Milnor monodromy on the reduced cohomology of the Milnor fiber of \(f\) at \(x\).

However, we have the following known, easy lemma, which we prove for lack of a convenient reference.

**Lemma 5.1.** Suppose that \(G : A^\bullet \to B^\bullet\) is a morphism of perverse sheaves on an analytic space \(Y\). Let \(y \in Y\), and let \(d := \dim_y Y\). Then, there is an isomorphism

\[
H^{-d}(\ker G)_y \cong \ker \left\{ G_y^{-d} : H^{-d}(A^\bullet)_y \to H^{-d}(B^\bullet)_y \right\}.
\]

**Proof.** Consider the canonical short exact sequences in Perv(\(Y\)):

\[
0 \to \ker G \to A^\bullet \xrightarrow{\alpha} \text{im} G \to 0
\]

and

\[
0 \to \text{im} G \xrightarrow{\beta} B^\bullet \to \text{coker} G \to 0,
\]

where \(\beta \circ \alpha = G\).

Since the stalk cohomology at \(y\) of all of these perverse sheaves is zero below degree \(-d\), the non-zero terms of the associated long exact sequences on stalk cohomology begin as follows:

\[
0 \to H^{-d}(\ker G)_y \to H^{-d}(A^\bullet)_y \xrightarrow{\alpha_y^{-d}} H^{-d}(\text{im} G)_y \to \ldots
\]

and

\[
0 \to H^{-d}(\text{im} G)_y \xrightarrow{\beta_y^{-d}} H^{-d}(B^\bullet)_y \to \ldots.
\]

The conclusion is immediate, since \(\beta_y^{-d} \circ \alpha_y^{-d} = G_y^{-d}\). \(\square\)

**Remark 5.2.** Of course we intend to apply Lemma 5.1 to the case where \(G = \xi^{-1} \text{id} - \tilde{T}_f\) (including the case where \(\xi\) may be 1). While \(\phi_f[-1]\mathbb{Z}_U[n+1]\) (or with \(\mathbb{Z}\) coefficients) is a perverse sheaf on all of \(X\), its support is \(\Sigma\), and we may (and will) apply Lemma 5.1 after restricting, without further comment, to \(\Sigma\).

**Remark 5.3.** The result of the lemma should be contrasted with our result in Theorem 3.1 of [22], where we looked at an endomorphism of perverse sheaves, \(T : P^\bullet \to P^\bullet\), with a field as the base ring. However, there, we looked at the kernel in degree \(-e\), where \(e := \dim \text{supp}(\ker)\), which could be strictly less than what we consider in the lemma, which is essentially \(\dim \text{supp} P^\bullet\).

Below, we will, as in the introduction, use topological indexing for the reduced intersection cohomology \(IH\).

From Theorem 3.3 and the lemma, we conclude:
Proposition 5.4. Let \( x \in \Sigma \) and let \( s := \dim_x \Sigma \). Let
\[
H^{n-s}(F_{f,x}; \mathbb{Z}) \xrightarrow{\partial_f} H^{n-s}(F_{f,x}; \mathbb{Z})
\]
be the standard Milnor monodromy on the degree \((n-s)\) cohomology of the Milnor fiber of \( f \) at \( x \).

Then, there are isomorphisms
\[
\ker \{ id - (\partial_f)^{-s} \} \cong \mathbb{Z}^\omega \cong \tilde{IH}^{n-s-1}(B^s_\omega(x) \cap X; \mathbb{Z}),
\]
where \( 0 < \epsilon \ll 1 \) and
\[
\omega := \begin{cases} 
\text{rank } H^{n+s}(K_{X,x}; \mathbb{Z}), & \text{if } s \neq n - 1; \\
-1 + \text{rank } H^{n+s}(K_{X,x}; \mathbb{Z}), & \text{if } s = n - 1.
\end{cases}
\]

Proof. Consider a portion of the long exact sequence on stalk cohomology which comes from the short exact sequence \((s)\), and apply Lemma 5.1; we obtain:
\[
0 \to H^{n-s-1}(\mathbb{Z}_x^*) \to H^{-s-1}(I^*_x) \to \ker \{ id - (\partial_f)^{-s} \} \to 0.
\]
Note that \( H^{n-s-1}(\mathbb{Z}_x^*) \) is free Abelian is classical; it follows from the connectivity result of Kato and Matsumoto [14], the Hurewicz Theorem, and the Universal Coefficient Theorem. Thus, \( \ker \{ id - (\partial_f)^{-s} \} \) is free Abelian.

Since our remaining claim is just about the value of \( \omega \), it suffices for us to work over a base ring that is the field \( \mathbb{C} \), rather than \( \mathbb{Z} \). With field coefficients, \( I_X^* \) is self-(Verdier) dual, i.e., \( D^! I_X^* \cong I_X^* \).

Let \( j_x \) denote the inclusion of \( \{ x \} \) into \( X \). Then,
\[
H^{-s-1}(I^*_x) = H^{-s-1}(j_x^* I_X^*) \cong H^{-s-1}(D^! j_x^! D^! I_X^*) \cong H^{s+1}(j_x^! I_X^*) \cong H^{s+1}(j_x^! C_X^*[n]),
\]
where the last isomorphism follows by applying \( j_x^! \) to the short exact sequence defining \( N_X^* \) and taking the associated long exact sequence.

Now,
\[
H^{s+1}(j_x^! C_X^*[n]) \cong H^{n+s+1}(B^s_\omega(x), B^s_\omega(x) \setminus \{ x \}; \mathbb{C}) \cong H^{n+s}(B^s_\omega(x) \setminus \{ x \}; \mathbb{C}) \cong H^{n+s}(K_{X,x}; \mathbb{C}).
\]
This completes the proof. \( \square \)

Remark 5.5. We should point out that one does not need Theorem 3.3 to prove the ordinary cohomology statement in Proposition 5.4. Milnor’s work in Section 8 of [23] tells us how the homology/cohomology of the complement of the real link \( K_{X,x} \) inside \( S^{2n+1} \) relates to the kernel of \( \partial_f \). Then, using Alexander Duality, one can recover the first isomorphism in Proposition 5.4.

Also, related to the case where \( s = n - 1 \), it is well-known that the rank of \( H^{2n-1}(K_{X,x}; \mathbb{Z}) \) is equal to the rank of \( IH^0(B^s_\omega(x) \cap X; \mathbb{Z}) \), which is equal to the number of irreducible components of \( X \) at \( x \).
The proposition with $\xi \neq 0, 1$ which corresponds to Proposition 5.4 is essentially impossible to state without at least using intersection cohomology with twisted coefficients. Still, we find it interesting that one immediately concludes from Corollary 4.2 and Lemma 5.1 that:

**Proposition 5.6.** Let $x \in \Sigma$ and let $s := \dim_x \Sigma$. Let

$$H^{n-s}(F_{f,x}; \mathbb{C}) \to H^{n-s}(F_{f,x}; \mathbb{C})$$

be the standard Milnor monodromy on the degree $(n-s)$ cohomology of the Milnor fiber of $f$ at $x$, and let $\xi \neq 0, 1$.

Then, there is an isomorphism

$$\ker \{ \xi^{-1} \text{id} - (\widetilde{T}_f)_{x}^{-1} \} \cong H^{-s-1}(I^*_x(\xi))_x.$$

Another application of the short exact sequence $(\ast)$ is that, for each $x \in X$, we can apply the vanishing cycle functor, $\phi_L[-1]$, which is exact on $\text{Perv}(X)$, where $L$ is the restriction to $X$ of a generic affine linear form such that $L(x) = 0$. We then obtain a short exact sequence of $\mathbb{Z}$-modules:

$$(\bigcirc) \quad 0 \to \ker \{ \text{id} - (\phi_L[-1]\widetilde{T}_f)_x^0 \} \to H^0(\phi_L[-1]Z^\bullet_X[n])_x \to H^0(\phi_L[-1]I^*_X)_x \to 0,$$

where $(\phi_L[-1]\widetilde{T}_f)_x^0$ is the automorphism induced by the $f$-monodromy on

$$H^0(\phi_L[-1]\phi_f[-1]Z^\bullet_{\mathcal{U}}[n+1])_x \cong \mathbb{Z}_{\lambda_{f,L}^0}(x),$$

where $\lambda_{f,L}^0(x)$ is the 0-th Lê number of $f$ with respect to $L$ at $x$ (see [19]).

Now, $\text{rank } H^0(\phi_L[-1]I^*_X)_x$ is the coefficient of $\{x\}$ in the characteristic cycle of intersection cohomology; this is a great importance in some settings (see, for instance, [3] for a discussion). (There are different shifting/sign conventions on the characteristic cycle; we are using a convention that, for a perverse sheaf, gives us that all of the coefficients are non-negative.) However, without the language of the derived category and using topological indexing, $H^0(\phi_L[-1]I^*_X)_x$ is isomorphic to

$$\text{coker} \left\{ \int H^{n-1}(B^\epsilon(x) \cap X; \mathbb{Z}) \xrightarrow{r_{X,x}} \int H^{n-1}(B^\epsilon(x) \cap X \cap L^{-1}(\gamma); \mathbb{Z}) \right\},$$

where $r_{X,x}$ is induced by the restriction and, as before, $L$ is a generic affine linear form such that $L(x) = 0$ and $0 < |\gamma| \ll \epsilon \ll 1$.

Furthermore, as the result of Lê in [17] tells us that the complex link $\mathbb{L}_{X,x}$ of $X$ at $x$ has the homotopy-type of a bouquet of $(n-1)$-spheres, the number of spheres in this homotopy-type is precisely $\text{rank } H^0(\phi_L[-1]Z^\bullet_X[n])_x$, which is known to equal the intersection number $(\Gamma^1_{f,L} \cdot V(L))_x$, where $\Gamma^1_{f,L}$ is the relative polar curve of $f$ with respect to $L$; see Corollary 2.6 of [18] (though this was known earlier by Hamm, Lê, Siersma, and Teissier).

Now, without the language of the derived category, $\ker \{ \text{id} - \phi_L[-1](\widetilde{T}_f)_x \}$ would be the kernel of an endomorphism on the cohomology of a pair of pairs of spaces; this, again, is a complicated object. However, the dual argument to that appearing near the end of the proof of Theorem 3.3 tells us that

$$\mu H^0(m_*m^*\psi_f[-1]Z^\bullet_{\mathcal{U}}[n+1]) \xrightarrow{\text{can}} \phi_f[-1]Z^\bullet_{\mathcal{U}}[n+1]$$

is an isomorphism.
As $\phi_L[-1]$ is $t$-exact (and as $L$ is generic), we have
\[
\phi_L[-1]|\phi_f[-1]Z_u^*[n + 1] \cong \phi_L[-1]^{\mu}H^0(m_*m^*\psi_f[-1]Z_u^*[n + 1]) \cong
\mu H^0(\phi_L[-1](m_*m^*\psi_f[-1]Z_u^*[n + 1])),
\]
which are perverse sheaves with $x$ as an isolated point in their support. Thus,
\[
H^0(\phi_L[-1]|\phi_f[-1]Z_u^*[n + 1])_x \cong H^0((\phi_L[-1]m_*m^*\psi_f[-1]Z_u^*[n + 1])_x),
\]
by an isomorphism which takes $\text{id} - (\phi_L[-1]|\overline{T}_f)_{\ast x}$ to $\text{id} - (\phi_L[-1]m_*T_f)_{\ast x}$.

Finally, we conclude that the short exact sequence (**) tells us:

**Corollary 5.7.** There is an equality
\[
\text{rank ker}\{r_{x,x}\} = (\Gamma^1_{f,L} \cdot V(L))_x - \text{rank ker}\{\text{id} - \overline{T}_f\},
\]
where
\[
r_{x,x} : \overline{H}^{n-1}(B_\varepsilon(x) \cap X; \mathbb{Z}) \to \overline{H}^{n-1}(B_\varepsilon(x) \cap X \cap L^{-1}(\gamma); \mathbb{Z}),
\]
is induced by restriction, and where $\overline{T}_f$ is the automorphism induced by the Milnor monodromy of $f$ on the relative cohomology
\[
H^n(B_\varepsilon(x) \cap f^{-1}(\eta); D^\delta_\varepsilon \cap f^{-1}(\eta); \mathbb{Z}),
\]
where $0 < |\eta| \ll \delta \ll |\gamma| \ll \epsilon \ll 1$ and $D^\delta_\varepsilon$ is the union of all the open balls of radius $\delta$ centered at each of the points in $B_\varepsilon(x) \cap \Sigma \cap L^{-1}(\gamma)$.

In the case where $s = 0$, Corollary 5.7 reduces to Proposition 2.4. The case where $s = 1$ simplifies enough that it is worth stating separately.

**Corollary 5.8.** Suppose that $\dim_x \Sigma = 1$ and that $\dim_x \Sigma(f_{\mid v(L)}) = 0$. Then, there is an equality
\[
\text{rank ker}\{r_{x,x}\} = (\Gamma^1_{f,L} \cdot V(L))_x - \text{rank ker}\{\text{id} - \overline{T}_f\},
\]
where
\[
r_{x,x} : H^{n-1}(B_\varepsilon(x) \cap X; \mathbb{Z}) \to H^{n-1}(B_\varepsilon(x) \cap X \cap L^{-1}(\gamma); \mathbb{Z}),
\]
is induced by restriction, and where $\overline{T}_f$ is the automorphism induced by the Milnor monodromy of $f$ on the relative cohomology
\[
H^n(F_{x,x} \cup F_{f,x}; \mathbb{Z}),
\]
where the union is over $x_i \in B_\varepsilon(x) \cap \Sigma \cap L^{-1}(\gamma)$, where $0 < |\gamma| \ll \epsilon \ll 1$, and $F_{f,x}$ and $F_{x,x}$ denote the Milnor fibers of $f$ at the respective points.

Now, we want to address the $\xi \neq 0, 1$ case. For each $x \in X$, we can also apply to the short exact sequence (**) the vanishing cycle functor, $\phi_L[-1]$, where $L$ is a generic affine linear form on $U$ such that $L(x) = 0$. We then obtain a short exact sequence of $\mathbb{C}$-vector spaces:
\[
0 \to \ker\{\zeta^{-1} \text{id} - (\phi_L[-1]|\overline{T}_f)_{\ast x}\} \to H^0(\phi_L[-1]|i_!\mathcal{L}_\xi)_x \to H^0(\phi_L[-1]|i_!^\mathbb{C}(\xi))_x \to 0,
\]
where $(\phi_L[-1]|\overline{T}_f)_{\ast x}$ is the automorphism induced by the $f$-monodromy on
\[
H^0(\phi_L[-1]|\phi_f[-1]|\overline{U}_n[1 + 1])_x \cong \mathbb{C}^{N^0_{f,L}(x)}.
\]
Now, it follows from Theorem 4.2.B of [20] that
\[
\dim H^0(\phi_L[-1]|_L \xi) = (\Gamma^1_f \cdot V(L))_x.
\]

Thus, we obtain:

**Corollary 5.9.** There is an equality
\[
(\Gamma^1_f \cdot V(L))_x - \dim \ker \{ \xi^{-1} \operatorname{id} - (\phi_L[-1]|_L \xi)^0 \} = \dim H^0(\phi_L[-1]|_L \xi)_x \geq 0.
\]

In particular, if \( x \) is an isolated singular point of \( X \), then
\[
(\Gamma^1_f \cdot V(L))_x - \dim \ker \{ \xi^{-1} \operatorname{id} - (\ov{T_f})^0 \} = \dim H^0(\phi_L[-1]|_L \xi)_x \geq 0.
\]

**Remark 5.10.** If \( x \) is an isolated singular point of \( X \), then Corollary 5.9 and Corollary 5.7 combine to tell us the curious fact that \((\Gamma^1_f \cdot V(L))_x\) is an upper bound for the dimension/rank of all of the eigenspaces of the Milnor monodromy in the one non-trivial degree, degree \( n \).

As a final application of Theorem 3.3, we now consider the setting of [12] and [11], in which \( X = V(f) \) has a smooth normalization \( M \). In this case, the normalization map \( F : M \to X \) is a parameterization of \( X \), a finite, surjective, analytic map which is an analytic isomorphism over \( X \setminus \Sigma \). This necessarily requires that \( \Sigma \) is pure of codimension 1 inside of \( X \) (this vacuously includes the case where \( \Sigma = \emptyset \)). It follows from the support and cosupport characterization of intersection cohomology that \( I^\bullet_X \cong F_* \mathbb{Z}_M^\bullet[n] \). For all \( x \in X \), we let \( m(x) := |F^{-1}(x)| \) be the number of points in the fiber over \( x \) (not counted with any sort of algebraic multiplicity).

Via this isomorphism, on the stalk cohomology at a point \( x \in X \), the canonical map \( \tau_x : Z^\bullet_X[n] \to I^\bullet_X \) induces the diagonal map in the only non-zero degree:
\[
\tau_{x,x} : H^{-n}(Z^\bullet_X[n])_x \cong \mathbb{Z} \to \mathbb{Z}^{m(x)} \cong \bigoplus_{y \in F^{-1}(x)} H^{-n}(Z^\bullet_M[n])_y.
\]

It follows that the stalk cohomology of \( N^\bullet_X \) is given by
\[
H^k(N^\bullet_X)_x \cong \begin{cases} 
\mathbb{Z}^{m(x)-1}, & \text{if } k = -n + 1; \\
0, & \text{if } k \neq -n + 1.
\end{cases}
\]

In this context, we have defined \( N^\bullet_X \) to be the **multiple-point complex** of \( F \).

Using our results and notation above, we quickly conclude:

**Corollary 5.11.** Suppose that \( X = V(f) \) is 2-dimensional with a smooth normalization, and let \( x \in \Sigma \), so that \( \dim_x \Sigma = 1 \).

Let
\[
H^1(\ov{F}_{f,x}; \mathbb{Z}) \xrightarrow{\ov{\tau}_f} H^1(\ov{F}_{f,x}; \mathbb{Z})
\]
be the Milnor monodromy on the cohomology of the Milnor fiber of \( f \) at \( x \).

Then, \( m(x) = \operatorname{rank} H^1(K_{X,x}; \mathbb{Z}) \), which equals the number of irreducible components of \( X \) at \( x \), and there are isomorphisms
\[
\ker \{ \operatorname{id} - (\ov{T_f})_{x}^{-1} \} \cong \mathbb{Z}^\omega \cong \ov{\omega} H^0_\omega(B^\circ_\Sigma(x) \cap X; \mathbb{Z}),
\]
where \( 0 < \epsilon \ll 1 \) and \( \omega = m(x) - 1 \).
Furthermore, suppose that $\dim_x \Sigma(f|_{V(L)}) = 0$. Then, there is an equality
\[ \text{rank ker } \{ \text{id} - \hat{T}_{f,L} \} = (\Gamma^1_{f,L} \cdot V(L))_x - \text{rank coker } r_{x,x} = -m(x) + \sum_i m(x_i), \]
where $\hat{T}_{f,L}$ is the automorphism induced by the Milnor monodromy of $f$ on the relative cohomology
\[ H^2(F_{f,x}, \bigcup_i F_{f,x_i}; \mathbb{Z}), \]
the summation and union are over $x_i \in B^o(x) \cap \Sigma \cap L^{-1}(\gamma)$, $0 < |\gamma| \ll \epsilon \ll 1$, and $F_{f,x}$ and $F_{f,x_i}$ denote the Milnor fibers of $f$ at the respective points and, finally,
\[ r_{x,x} : IH^1(B^o(x) \cap X; \mathbb{Z}) \to IH^1(B^o(x) \cap X \cap L^{-1}(\gamma); \mathbb{Z}), \]
is induced by restriction.

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