HAUSDORFF DIMENSION AND PROJECTIONS RELATED TO INTERSECTIONS

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Abstract. For \( S_g(x,y) = x - g(y), x, y \in \mathbb{R}^n, g \in O(n) \), we investigate the Lebesgue measure and Hausdorff dimension of \( S_g(A) \) given the dimension of \( A \), both for general Borel subsets of \( \mathbb{R}^{2n} \) and for product sets.

1. INTRODUCTION

Let \( A \) and \( B \) be Borel subsets of \( \mathbb{R}^n \). Under which conditions on the Hausdorff dimensions \( \dim A \) and \( \dim B \) do we have \( A \cap (g(B) + z) \neq \emptyset \) for positively many, in the sense of Lebesgue measure, \( z \in \mathbb{R}^n \) for almost all \( g \in O(n) \)? Defining \( S_g(x,y) = x - g(y) \), \( A \cap (g(B) + z) \neq \emptyset \) for positively many \( z \in \mathbb{R}^n \) is equivalent to \( \mathcal{L}^n(S_g(A \times B)) > 0 \). Or we can also ask when \( A \cap (g(B) + z) \neq \emptyset \) for \( z \) and \( g \) in sets of Hausdorff dimensions of certain size. This reduces to estimating the dimension of \( S_g(A) \) and the dimension of the corresponding exceptional set of orthogonal transformations.

In this paper we study more generally the Lebesgue measure and Hausdorff dimension of \( S_g(A) \) for \( A \subset \mathbb{R}^{2n} \). In Theorem 3.4 we shall show for a Borel set \( A \subset \mathbb{R}^{2n} \) that for almost all \( g \in O(n) \), \( \mathcal{L}^n(S_g(A)) > 0 \), if \( \dim A > n + 1 \), \( \dim S_g(A) \geq \dim A - 1 \), if \( n - 1 \leq \dim A \leq n + 1 \), and \( \dim S_g(A) \geq \dim A \), if \( \dim A \leq n - 1 \). In all cases we also derive Hausdorff dimension estimates for the sets of exceptional \( g \in O(n) \). In Theorems 4.2 and 4.3 we show that these estimates can be improved for product sets. We shall also comment on some relations to Falconer’s distance set problem.

Instead of asking \( A \cap (g(B) + z) \) to be non-empty, we could ask on the Hausdorff dimension of these intersections. This problem was studied in \([K],[M1],[M2],[M3],[M4],[M5]\) and \([M7]\). I shall make comments on it at the end of the paper. Unfortunately this study has not lead to any improvements on earlier results. I expect the following to be true: if \( A \) and \( B \) are Borel subsets of \( \mathbb{R}^n \) with \( \dim A + \dim B > n \), then for almost all \( g \in O(n) \), \( \dim A \cap (g(B) + z) \geq \dim A + \dim B - n \) for positively many \( z \in \mathbb{R}^n \). This is only known if one of the sets has dimension bigger than \((n+1)/2\). Part of Theorem 4.2(1) is a special case of this. In fact, all the statements in part (1) of Theorems 4.2 and 4.3 are essentially special cases of earlier intersection theorems, but we give here more direct proofs.

The family \( S_g, g \in O(n) \), is a restricted family of orthogonal projections onto \( n \)-planes in \( \mathbb{R}^{2n} \); it is only \( n(n-1)/2 \) dimensional while the full family of orthogonal projections has dimension \( n^2 \). Similar questions for other restricted families of orthogonal projections have been studied by many people, see \([JJ1],[JJ2],[FO],[Or],[O],[OO],[KO],[OV]\). There also are discussions on these in \([M5]\) and \([M6]\).
Hausdorff dimension results for projections have their origin in Marstrand’s projection theorem [M]: for a Borel set $A \subset \mathbb{R}^2$, for almost all orthogonal projections $p$ onto lines, $\mathcal{L}^1(p(A)) > 0$, if $\dim A > 1$, and $\dim p(A) = \dim A$, if $\dim A \leq 1$. The study of exceptions was started by Kaufman [K] who showed that in the second statement the dimension of the set of the exceptional projections is at most $\dim A$, and continued by Falconer [F1] who showed that in the first statement the set of the exceptions has dimension at most $2 - \dim A$. Discussion and further references can be found for example in [M5].

2. Preliminaries

We denote by $\mathcal{L}^n$ the Lebesgue measure in the Euclidean $n$-space $\mathbb{R}^n$, $n \geq 2$, and by $\sigma^{n-1}$ the surface measure on the unit sphere $S^{n-1}$. The orthogonal group of $\mathbb{R}^n$ is $O(n)$ and its Haar probability measure is $\theta_n$. For $A \subset \mathbb{R}^n$ (or $A \subset O(n)$) we denote by $M(A)$ the set of non-zero Radon measures $\mu$ on $\mathbb{R}^n$ with compact support $\text{spt} \mu \subset A$. The Fourier transform of $\mu$ is defined by

$$\hat{\mu}(x) = \int e^{-2\pi ix \cdot y} d\mu y, \ x \in \mathbb{R}^n.$$ 

We shall also use $\mathcal{F}$ to denote the Fourier transform.

For $0 < s < n$ the $s$-energy of $\mu \in M(\mathbb{R}^n)$ is

$$(2.1) \quad I_s(\mu) = \iint |x - y|^{-s} d\mu x d\mu y = c(n, s) \int |\hat{\mu}(x)|^2 |x|^{s-n} dx.$$ 

The second equality is a consequence of Parseval’s formula and the fact that the distributional Fourier transform of the Riesz kernel $k_s, k_s(x) = |x|^{-s}$, is a constant multiple of $k_{n-s}$, see, for example, [M4], Lemma 12.12, or [M5], Theorem 3.10. These books contain most of the background material needed in this paper.

Notice that if $\mu$ satisfies the Frostman condition $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^n, r > 0$, then $I_t(\mu) < \infty$ for all $t < s$. We have for any Borel set $A \subset \mathbb{R}^n$ with $\dim A > 0$, cf. Theorem 8.9 in [M4],

$$(2.2) \quad \dim A = \sup \{s: \exists \mu \in M(A) \text{ such that } \mu(B(x, r)) \leq r^s \text{ for all } x \in \mathbb{R}^n, r > 0\} = \sup \{s: \exists \mu \in M(A) \text{ such that } I_s(\mu) < \infty\}.$$ 

We shall denote by $f_#\lambda$ the push-forward of a measure $\lambda$ under a map $f : f_#\lambda(A) = \lambda(f^{-1}(A))$.

By the notation $M \lesssim N$ we mean that $M \leq CN$ for some constant $C$. The dependence of $C$ should be clear from the context. The notation $M \approx N$ means that $M \lesssim N$ and $N \lesssim M$. By $c$ we mean positive constants with obvious dependence on the related parameters. The closed ball with centre $x$ and radius $r$ will be denoted by $B(x, r)$.

3. Projections of general sets

For $g \in O(n), t \in \mathbb{R}$, define

$$S_g, \pi_t : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, S_g(x, y) = x - g(y), \pi_t(x, y) = x - ty.$$
Both of these can be realized as families of orthogonal projections. The family $S_g$ has curvature (in any natural sense) while $\pi_t$ does not have. See [FO], [OT] and [KOV] for the role of curvature in projection theorems.

More precisely, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Set $u_i = \frac{1}{\sqrt{2}}(e_i, -g^{-1}(e_i))$, $i = 1, \ldots, n$. Then $\{u_1, \ldots, u_n\}$ is an orthonormal basis for an $n$-plane $V_g \subset \mathbb{R}^{2n}$. The orthogonal complement of $V_g$, spanned by $\frac{1}{\sqrt{2}}(e_i, g^{-1}(e_i))$, $i = 1, \ldots, n$, is the kernel of $S_g$. Since $\frac{1}{\sqrt{2}}S_g(u_i) = e_i$, $\frac{1}{\sqrt{2}}S_g$ is essentially the orthogonal projection onto $V_g$.

When $n = 2$ we have in complex notation, $g$ identified with the angle $\phi$: $S_g(x, y) = x - e^{i\phi}y$.

Some relations between the projections $\pi_t$ and the Kakeya problem are discussed in [M6].

Recall the following lemma from [M7], Lemma 2.1. In [M7] only the first bound was proven, but the second is easily reduced to the first. Notice that the term $(n-1)(n-2)/2$ is needed there: the subgroup $(x, t)$ ∩ $(g(x), t, x \in \mathbb{R}^{n-1}, t \in \mathbb{R}, g \in O(n-1)$, has dimension $(n-1)(n-2)/2$ and $(g(0), 1) = (0, 1)$ for all $g \in O(n-1)$.

**Lemma 3.1.** Let $\theta \in \mathcal{M}(O(n))$, $\alpha > (n-1)(n-2)/2$ and $\beta = \alpha - (n-1)(n-2)/2$. If $\theta(B(g, r)) \leq r^\alpha$ for all $g \in O(n)$ and $r > 0$, then for $x, z \in \mathbb{R}^n \setminus \{0\}$, $r > 0$, $(3.1)$

$$\theta(\{g : |x - g(z)| < r\}) \lesssim \min\{r/|z|^\beta, (r/|x|)^\beta\}.$$

This will be applied via the following proposition, as in Chapter 5 of [M5] and in many other places:

**Proposition 3.2.** Let $A \subset \mathbb{R}^n$ be a Borel set and $\beta > 0, \gamma > 0$. Suppose that for any $\theta \in \mathcal{M}(O(n))$ such that (3.1) holds, $\mathcal{L}^\alpha(S_g(A)) > 0$ (or $\dim S_g(A) \geq \gamma$) for $\theta$ almost all $g \in O(n)$. Then there is a Borel set $E \subset O(n)$ such that $\dim E \leq \beta + (n-1)(n-2)/2$ and $\mathcal{L}^\alpha(S_g(A)) > 0$ (or $\dim S_g(A) \geq \gamma$) for $g \in O(n) \setminus E$.

**Proof.** I skip the easy measurability arguments. If this fails, the set $G$ of $g \in O(n)$ for which $\mathcal{L}^\alpha(S_g(A)) = 0$ has dimension greater than $\alpha = \beta + (n-1)(n-2)/2$. Then by (2.2) there is $\theta \in \mathcal{M}(G)$ such that $\theta(B(g, r)) \leq r^\alpha$ for all $g \in O(n)$ and $r > 0$, so that (3.1) holds by Lemma 3.1. By assumption, $\mathcal{L}^\alpha(S_g(A)) > 0$ for $\theta$ almost all $g \in O(n)$, which contradicts the definition $G$ and that $\theta \in \mathcal{M}(G)$.

The following theorem for $\pi_t$ essentially is a special case of Oberlin’s results in [O]. It was not explicitly stated there, but (1) and (2) follow by his arguments, see in particular the proof of Lemma 3.1 in [O]. The proof of (3) is a standard argument of Kaufman from [K9], see the proof of Theorem 3.3. The proof of Theorem 3.4 also gives Theorem 3.3 changing $g(x)$ to $tx$.

**Theorem 3.3.** Let $A \subset \mathbb{R}^{2n}$ be a Borel set.

(1) If $\dim A > 2n - 1$, then $\mathcal{L}^\alpha(\pi_t(A)) > 0$ for $\mathcal{L}^1$ almost all $t \in \mathbb{R}$. Moreover, there is $E \subset \mathbb{R}$ such that $\dim E \leq 2n - \dim A$ and $\mathcal{L}^\alpha(\pi_t(A)) > 0$ for $t \in \mathbb{R} \setminus E$.

(2) If $n \leq \dim A \leq 2n - 1$, then $\dim \pi_t(A) \geq \dim A - n + 1$ for $\mathcal{L}^1$ almost all $t \in \mathbb{R}$. Moreover, for $\dim A - n \leq u \leq \dim A - n + 1$ there is $E \subset \mathbb{R}$ such that $\dim E \leq u + n - \dim A$ and $\dim \pi_t(A) \geq u$ for $t \in \mathbb{R} \setminus E$.
(3) If \( \dim A \leq n \), then \( \dim \pi_t(A) \geq \min\{\dim A, 1\} \) for \( \mathcal{L}^1 \) almost all \( t \in \mathbb{R} \). Moreover, for \( 0 < u \leq \min\{\dim A, 1\} \) there is \( E \subset \mathbb{R} \) such that \( \dim E \leq u \) and \( \dim \pi_t(A) \geq u \) for \( t \in \mathbb{R} \setminus E \).

(4) For all \( t \in \mathbb{R} \), \( \dim \pi_t(A) \geq \dim A - n \).

Notice that the last statement is trivial, because \( A \subset \pi_t(A) \times \pi_t^{-1}(0) \) and \( \dim(\pi_t(A) \times \pi_t^{-1}(0)) = \dim \pi_t(A) + n \).

This theorem is valid also when \( n = 1 \); it is Marstrand’s projection with Kaufman’s and Falconer’s exceptional set estimates.

We have a similar result for \( S_g \). Observe also there that (4) is trivial. The proof below for (1) and (2) is a modification of Oberlin’s proof. The proof of (3) again is Kaufman’s argument.

**Theorem 3.4.** Let \( A \subset \mathbb{R}^{2n} \) be a Borel set.

(1) If \( \dim A > n + 1 \), then \( \mathcal{L}^n(S_g(A)) > 0 \) for \( \theta_n \) almost all \( g \in O(n) \). Moreover, there is \( E \subset O(n) \) such that \( \dim E \leq 2n - \dim A + (n - 1)(n - 2)/2 \) and \( \mathcal{L}^n(S_g(A)) > 0 \) for \( g \in O(n) \) for all \( E \).

(2) If \( n - 1 \leq \dim A \leq n + 1 \), then \( \dim S_g(A) \geq \dim A - 1 \) for \( \theta_n \) almost all \( g \in O(n) \). Moreover, for any \( \dim A - n \leq u \leq \dim A - 1 \) there is \( E \subset O(n) \) such that \( \dim E \leq u + n - \dim A + (n - 1)(n - 2)/2 \) and \( \dim S_g(A) \geq u \) for \( g \in O(n) \) for all \( E \).

(3) If \( \dim A \leq n - 1 \), then \( \dim S_g(A) \geq \dim A \) for \( \theta_n \) almost all \( g \in O(n) \). Moreover, for \( 0 < u \leq \dim A \) there is \( E \subset O(n) \) such that \( \dim E \leq u + (n - 1)(n - 2)/2 \) and \( \dim S_g(A) \geq u \) for \( g \in O(n) \) for all \( E \).

(4) For all \( g \in O(n) \), \( \dim S_g(A) \geq \dim A - n \)

**Proof.** Let \( 0 < s < \dim A \) and \( \mu \in \mathcal{M}(A) \) with \( I_s(\mu) < \infty \).

Let \( \mu_g \in \mathcal{M}(S_g(A)) \) be the push-forward of \( \mu \) under \( S_g \). Then for \( \xi \in \mathbb{R}^n \),

\[
\hat{\mu}_g(\xi) = \int e^{-2\pi i \xi \cdot S_g(x,y)} \, d\mu(x,y) = \int e^{-2\pi i (\xi - g^{-1}(\xi)) \cdot (x,y)} \, d\mu(x,y) = \hat{\mu}(\xi, -g^{-1}(\xi)).
\]

Let \( 0 < \beta \leq n - 1 \) and \( \theta \in \mathcal{M}(O(n)) \) be such that for \( x, z \in \mathbb{R}^n \setminus \{0\}, r > 0 \),

\[
\theta(\{g \in O(n) : |x - g(z)| < r\}) \leq \min\{r/|z|^\beta, (r/|x|)^\beta\}.
\]

To prove (1) and (2) we shall show that for \( R > 1 \),

\[
\iint_{R \leq |\xi| \leq 2R} |\hat{\mu}(\xi, -g^{-1}(\xi))|^2 \, d\xi \, dg \lesssim R^{2n-s-\beta}.
\]

This is applied to the dyadic annuli, \( R = 2^k, k = 1, 2, \ldots \). The sum converges if \( s > 2n - \beta \), and we can choose \( \mu \) with such \( s \) if \( \dim A > 2n - \beta \). This gives \( \iint |\hat{\mu}_g(\xi)|^2 \, d\xi \, dg < \infty \).

Hence for \( \theta \) almost all \( g \in O(n) \), \( \mu_g \) is absolutely continuous with \( L^2 \) density, and so \( \mathcal{L}^n(S_g(A)) > 0 \). Taking \( \beta = n - 1 \) and \( \theta = \theta_n \), we get the first part of (1). The second follows with general \( \beta \) and \( \theta \) using Proposition 3.2.

To prove part (2) let \( 0 < u < s + \beta - n \) and \( \mu \) as above. Then (3.3) yields

\[
\iint |\hat{\mu}_g(\xi)|^2 |\xi|^{u-n} \, d\xi \, dg < \infty,
\]
so by (2.1) and (2.2), \( \dim S_y(A) \geq u \) for \( \theta \) almost all \( g \in O(n) \) and thus (2) follows with the same argument as above.

From (3.2) we get for \( \xi, y \in \mathbb{R}^n, R \leq |\xi| \leq 2R, M > \beta, \)

\[
(3.4) \quad \int (1 + |\xi + g(y)|)^{-M} d\theta g \lesssim R^{-\beta},
\]

because

\[
\int (1 + |\xi + g(y)|)^{-M} d\theta g \\
\leq \theta(\{g \in O(n) : |\xi + g(y)| \leq 1\}) + \int_{\{g : |\xi + g(y)| > 1\}} (1 + |\xi + g(y)|)^{-M} d\theta g \\
\lesssim R^{-\beta} + \sum_{j=0}^{\infty} 2^{-Mj} \theta(\{g \in O(n) : 2^j \leq |\xi + g(y)| < 2^{j+1}\}) \\
\lesssim R^{-\beta} + \sum_{j=0}^{\infty} 2^{-Mj} (2^j / |\xi|)^{\beta} \lesssim R^{-\beta}.
\]

To prove (3.3), choose a smooth function \( \phi \) with compact support which equals 1 on the support of \( \mu \). Then \( \hat{\mu} = \hat{\phi} \mu = \hat{\phi} \ast \hat{\mu} \) and the integral in (3.3) equals

\[
I_R := \int_{R \leq |\xi| \leq 2R} |\hat{\phi}(\xi, -g^{-1}(\xi))|^2 d\xi d\theta g = \int_{R \leq |\xi| \leq 2R} \left| \int_{R \leq |\xi| \leq 2R} \hat{\phi}(\xi, -g^{-1}(\xi) - y) \hat{\mu}(y) dy \right|^2 d\xi d\theta g.
\]

By the Schwartz inequality,

\[
I_R \leq \int_{R \leq |\xi| \leq 2R} \left| \int_{R \leq |\xi| \leq 2R} \hat{\phi}(\xi, -g^{-1}(\xi) - y) |\hat{\phi}(\xi, -g^{-1}(\xi) - y)| \hat{\mu}(y) dy \right|^2 d\xi d\theta g \\
\lesssim \int_{R \leq |\xi| \leq 2R} \left| \int_{R \leq |\xi| \leq 2R} \hat{\phi}(\xi, -g^{-1}(\xi) - y) |\hat{\mu}(y)|^2 dy \right| d\xi d\theta g \\
\lesssim \int_{R \leq |\xi| \leq 2R} (1 + |(\xi, -g^{-1}(\xi)) - y|)^{-3M} |\hat{\mu}(y)|^2 dy d\xi d\theta g,
\]

by the fast decay of \( \hat{\phi} \), where \( M > 2n \). Clearly, with \( y = (y_1, y_2), y_1, y_2 \in \mathbb{R}^n, \)

\[
|((\xi, -g^{-1}(\xi)) - y)| \geq \max\{|\xi - y_1|, |\xi + g(y_2)|\}.
\]

Moreover, \( |(\xi, -g^{-1}(\xi)) - y)| \approx |y| \), when \( R \leq |\xi| \leq 2R \) and \( |y| > 5R \). Hence

\[
I_R \lesssim \int_{|y| \leq 5R} \int_{R \leq |\xi| \leq 2R} \left( 1 + |\xi + g(y_2)| \right)^{-M} d\theta g (1 + |\xi - y_1|)^{-M} d\xi |\hat{\mu}(y)|^2 dy \\
+ \int_{|y| > 5R} \int_{R \leq |\xi| \leq 2R} \left( 1 + |\xi + g(y_2)| \right)^{-M} d\theta g (1 + |\xi - y_1|)^{-M} d\xi |y|^{-M} dy.
\]

We have by (3.4)

\[
\int (1 + |\xi + g(y_2)|)^{-M} d\theta g \lesssim R^{-\beta}.
\]
Since $\int (1 + |\xi - y_1|)^{-M} d\xi$ is bounded, we obtain
\[
I_R \lesssim R^{-\beta} \left( \int_{|y| \leq 5R} |\hat{\mu}(y)|^2 dy + \int_{|y| > 5R} |y|^{-M} dy \right).
\]
The second integral is bounded and for the first we have by [M5], Section 3.8,
\[
(3.5) \quad \int_{|y| \leq 5R} |\hat{\mu}(y)|^2 dy \lesssim R^{2n-s},
\]
which imply $I_R \lesssim R^{2n-s-\beta}$ as required.

Suppose now $0 < u < \dim A \leq n - 1, \mu \in \mathcal{M}(A)$ with $I_u(\mu) < \infty$ and let $\theta$ and $\beta$ be as in (3.2) with $\beta > u$. It suffices to show that $\dim S_g(A) \geq u$ for $\theta$ almost all $g \in O(n)$. Using (3.2) this follows from
\[
\int I_u(\mu_g) d\theta g = \int \int_{x \in \mathbb{R}^n} |x - y|^{-u} dS_g(x) dS_g(y) d\theta g
= \int \int_{w \in \mathbb{R}^n} |S_g(w - z)|^{-u} d\mu w d\mu z d\theta g
= \int \int_{0}^{\infty} \theta(\{g : |S_g(w - z)|^{-u} > r\}) dr d\mu w d\mu z
= \int \int_{0}^{\infty} \theta(\{g : |S_g(w - z)| < r^{-1/u}\}) dr d\mu w d\mu z
\lesssim \int \int_{0}^{\infty} dr d\mu w d\mu z + \int \int_{0}^{\infty} (r^{-1/u}/|w - z|)^{\beta} dr d\mu w d\mu z
\approx I_u(\mu) < \infty.
\]

3.1. Sharpness. The bounds in the $L^1$ almost all statements of Theorem 3.3 are sharp when $n = 2$. To see this let $0 \leq s \leq 1, C_s \subset \mathbb{R}$ with $\dim C_s = s$, and $A_s = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 \in C_s, y_1 = 0\}$. Then $\dim A_s = 2 + s, \pi_t(A_s) = C_s \times \mathbb{R}$ and $\dim \pi_t(A_s) = 1 + s$. This shows that (2) is sharp. For (1) we can choose $C_1$ with $L^1(C_1) = 0$, then $\dim A_1 = 3$ and $L^2(\pi_t(A)) = 0$. If $1 \leq \dim A \leq 2$ we can only say that $\dim \pi_t(A) \geq 1$ for almost all $t \in \mathbb{R}$ since $\pi_t(\mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}) = \mathbb{R}$. Hence (3) also is sharp. Probably the bounds for the dimensions of the exceptional sets are sharp, too. Perhaps this could be seen using similar examples as in [KM], see also Example 5.13 in [M5], but I haven’t checked it.

When $n \geq 3$ a similar argument shows that the $L^1$ almost all statements of Theorem 3.3 are sharp when $\dim A \geq 2n - 2$ or $\dim A \leq 2$. Probably it is not sharp in the remaining ranges.

I don’t know if the bounds are sharp for $S_g$. In the next section we shall see that some of them can be improved for product sets. By the above examples this is not possible for $\pi_t$. I illustrate the role of $\dim O(n - 1) = (n - 1)(n - 2)/2$ with two simple examples: For $0 < s \leq 1$ choose a compact set $C_s \subset \mathbb{R}$ such that $\dim C_s = \dim(C_s - C_s) = s, \dim(C_s \times C_s) = 2s$ and $L^1(C_1 - C_1) = 0$. Such sets are easy to construct. If $A_s =$
$\mathbb{R}^{n-1} \times C_s \times \mathbb{R}^{n-1} \times C_s$, then $\dim A_s = 2s + 2(n - 1), S_g(A) = \mathbb{R}^{n-1} \times (C_s - C_s)$ and $\dim S_g(A_s) = s + n - 1$ for $g \in O(n - 1)$ (identified with $(x, t) \mapsto (g(x), t)$). In particular, $\dim A_1 = 2n$ and $\mathcal{L}^n(S_g(A_1)) = 0$ for $g \in O(n - 1)$. Next, take $B_s = \{0\} \times C_s \times \{0\} \times C_s$. Then $\dim B_s = 2s, S_g(A) = \{0\} \times (C_s - C_s)$ and $\dim S_g(B_s) = s$ for $g \in O(n - 1)$.

3.2. An alternative argument. Here is another simple argument for the statement 'If $\dim A > n + 1$, then $\mathcal{L}^n(S_g(A)) > 0$ for $\theta_n$ almost all $g \in O(n)$':

Let $\mu \in \mathcal{M}(A)$ with $I_{n+1}(\mu) < \infty$. Consider for $r > 0$,

$$I_r = r^{-n} \int \int S_g \# \mu(B(z, r)) \, dS_g \# \mu z \, d\theta_n g,$$

$$= r^{-n} \int \theta_n(\{g : |x - u - g(y - v)| \leq r\}) \, d\mu(u, v) \, d\mu(x, y),$$

$$\lesssim r^{-1} \int \{ (x, y) : |x - u| - |y - v| \leq r \} \, |y - v|^{1-n} \, d\mu(u, v) \, d\mu(x, y).$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(y) = 1$ when $(x, y) \in \text{spt} \mu$ for some $x$, and let

$$\psi_r(x, y) = \chi_{\{ (x, y) : |x - |y|| \leq r \}}(x, y) |y|^{1-n} \phi(y).$$

Then

$$I_r \lesssim r^{-1} \int \psi_r * \mu = r^{-1} \int \widehat{\psi_r} |\mu|^2.$$

Let $\sigma_r$ be the surface measure on $\{ x \in \mathbb{R}^n : |x| = r \}$. Then for any $u, y \in \mathbb{R}^n$, $\sigma_{[y]}(u) = |y|^{n-1} |u|^{1-n} \sigma_{[u]}(y)$. Thus for small $r$,

$$|r^{-1} \widehat{\psi_r}(u, v)| = \left| r^{-1} \int \int |y|^{1-n} \phi(y) e^{-2\pi i(u \cdot x + v \cdot y)} \, dx \, dy \right|$$

$$\approx \left| \int |y|^{1-n} \phi(y) \sigma_{[y]}(u) e^{-2\pi i v \cdot y} \, dy \right|$$

$$= \left| \int |y|^{1-n} \phi(y) |y|^{n-1} |u|^{1-n} \sigma_{[u]}(y) e^{-2\pi i v \cdot y} \, dy \right|$$

$$= |u|^{1-n} F(\phi \sigma_{[u]})(v)\right| = \left| |u|^{1-n} \int \hat{\phi}(y - v) \, d\sigma_{[u]}|y| \right|$$

$$\lesssim |u|^{1-n} (1 + |u| - |v|)^{1-n} \lesssim |(u, v)|^{1-n},$$

the second to last by the fast decay of $\hat{\phi}$. Hence

$$I_r \lesssim \int \int |(u, v)|^{1-n} |\hat{\mu}(u, v)|^2 \, d(u, v) \approx I_{n+1}(\mu).$$

Define the lower derivative, with $\alpha(n) = \mathcal{L}^n(B(0, 1))$,

$$D(S_g \# \mu)(z) = \liminf_{r \to 0} \alpha(n)^{-1} r^{-n} S_g \# \mu(B(z, r)).$$
Letting $r \to 0$ and using Fatou’s lemma we then see that
\begin{equation}
\int \int D(S_g^\#)(z)^2 \, dz \, d\theta_n g = \int \int D(S_g^\#)(z) \, dS_g^\# \mu z \, d\theta_n g < \infty,
\end{equation}
which implies that $S_g^\# \mu << \mathcal{L}^n$ with $L^2$ density, see e.g. [M4], Theorem 2.12, for $\theta_n$ almost all $g$, from which the claim follows.

### 3.3. Averages over a cone.

When $\theta = \theta_n$ we have for the integral in (3.3)
\begin{equation}
\int \int R_{|\xi| \leq 2R} \, |\hat{\mu}(\xi, g(\xi))|^2 \, d\xi \, d\theta_n g = R^n \int_{1 \leq |x| = |y| \leq 2} \, |\hat{\mu}(Rx, Ry)|^2 \, d\gamma(x, y),
\end{equation}
where the integration on the right side is with respect to a suitably normalized surface measure $\gamma$ on the conical surface $\Gamma = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : 1 \leq |x| = |y| \leq 2 \}$. Let $\phi$ be a smooth non-negative function with compact support in $\{ y : 1/2 < |y| < 3 \}$ and with $\phi(y) = 1$ when $1 \leq |y| \leq 2$. Define the measure $\lambda$ by
\begin{equation}
\int f \, d\lambda = \int \int f(x, y) \, d\sigma_{|y|} x \phi(y) \, dy.
\end{equation}
Then $\gamma \lesssim \lambda$.

The Fourier transform of $\lambda$ has the estimate
\begin{equation}
|\hat{\lambda}(\xi)| \lesssim |\xi|^{1-n},
\end{equation}
because
\begin{align*}
|\hat{\lambda}(u, v)| &= \int e^{-2\pi i (u \cdot x + v \cdot y)} \, d\lambda(x, y) \\
&= c \int e^{-2\pi i y \cdot \sigma_{|y|}(u) \phi(y)} \, dy \\
&= c \int e^{-2\pi i y \cdot |y|^{n-1} |u|^{1-n} \sigma_{|u|}(y) \phi(y)} \, dy \\
&= c |u|^{1-n} \mathcal{F}(|y|^{n-1} \phi(y) \sigma_{|u|}(y))(v) \\
&= c |u|^{1-n} \int \mathcal{F}(|y|^{n-1} \phi(y))(v - x) \, d\sigma_{|u|} x \approx |(u, v)|^{1-n},
\end{align*}
where the last estimate follows as for the Fourier transform of $\psi_r$ above. Let $\mu \in \mathcal{M}(\mathbb{R}^{2n})$ with $I_s(\mu) < \infty$. Then using a general theorem of Erdoğan, Theorem 1 in [E], we obtain for $R > 1$,
\[
\begin{align*}
\int \int_{1 \leq |x| = |y| \leq 2} |\hat{\mu}(Rx, Ry)|^2 \, d\gamma(x, y) & \lesssim \int \int |\hat{\mu}(Rx, Ry)|^2 \, d\lambda(x, y) \\
& \lesssim R^{1-s}, \\
& \lesssim R^{-s}, \text{ if } 0 < s \leq n - 1.
\end{align*}
\]

(3.8)

These do not improve Theorem 3.4; they give another proof for the almost all statements with respect to \(\theta_n\), but they don’t give the exceptional set estimates. In fact, the last estimate is the same as (3.3) with \(\theta = \theta_n, \beta = n-1\). Better decay estimates for (3.8) might lead to improvements for Theorem 3.4. In particular, any improvement of the exponent \(1-s\) in the range \(n < s < n + 1\) would lead to an improvement of the first statement of Theorem 3.4(1). I am not aware of such results. However, in addition to the spherical averages (see the next section) which have been studied for a long time, there are recent estimates for cones and hyperboloids, see [CHL], [H] and [BEH]. We shall see in the next section that the estimates (3.8) can be improved for product measures.

4. Product sets

For product sets we can improve Theorem 3.4 for \(S_g\), but not for \(\pi_t\), as the previous examples show. Let \(\theta \in \mathcal{M}(S^{n-1})\) and \(0 < \beta \leq n - 1\). Suppose that for \(x, z \in \mathbb{R}^n, r > 1,\)

\[
\theta(\{g : |x - g(z)| < r\}) \lesssim (r/|z|)^\beta.
\]

(4.1)

Let \(\mu \in \mathcal{M}(\mathbb{R}^n)\) and set for \(r > 1\) and \(\xi \in \mathbb{R}^n,\)

\[
\sigma(\mu)(r) = \int_{S^{n-1}} |\hat{\mu}(rv)|^2 \, d\sigma^{n-1}v, \\
\sigma_\theta(\mu)(\xi) = \int |\hat{\mu}(g^{-1}(\xi))|^2 \, d\theta g.
\]

Then \(\sigma_{\theta_n}(\mu)(\xi) = c\sigma(\mu)(|\xi|)\).

The decay estimates for \(\sigma(\mu)(r)\) have been studied by many people, a discussion can be found in [M5]. The best known estimates, due to Wolff, [W], when \(n = 2\), and to Du and Zhang, [DZ], in the general case, are the following: Let \(\mu \in \mathcal{M}(\mathbb{R}^n)\) with \(\mu(B(x, r)) \leq r^s\) for \(x \in \mathbb{R}^n, r > 0\). Then for all \(\epsilon > 0, r > 1,\)

\[
\sigma(\mu)(r) \lesssim r^{-(n-1)s/n+\epsilon}, \\
\lesssim r^{-s+\epsilon} \text{ if } 0 < s \leq (n-1)/2.
\]

(4.2)

For \(r > 1\), let

\[A_r = \{x \in \mathbb{R}^n : r-1 < |x| < r+1\}.\]

It is easy to see that for large \(r, \sigma(\mu)(r) \lesssim r^{-\alpha+\epsilon}\) for all \(\epsilon > 0\) if and only if \(r^{1-n} \int_{A_r} |\hat{\mu}(x)|^2 \, dx \lesssim r^{-\alpha+\epsilon}\) for all \(\epsilon > 0\). Indeed, the implication from left to right is trivial. The opposite implication is Proposition 16.2 in [M5]. This and the above estimates for \(\sigma(\mu)(r)\) yield the following lemma:
Lemma 4.1. If \( \mu(B(x,r)) \leq r^s \) for \( x \in \mathbb{R}^n, r > 0 \), then for every \( \xi \in \mathbb{R}^n \) with \( |\xi| > 1 \) and for every \( \epsilon > 0 \),

\[
\sigma_\theta(\mu)(\xi) \lesssim |\xi|^{-(n-1)s/n+n-1-\beta+\epsilon},
\]

\[
\lesssim |\xi|^{-s+n-1-\beta+\epsilon}, \text{ if } 0 < s \leq (n-1)/2.
\]

Proof. Using the above estimate for \( \sigma(\mu)(r) \) and the above mentioned relation to the estimates over the annuli \( A_r \) and for every \( \epsilon > 0 \), we have

\[
r^{1-n} \int_{A_r} |\tilde{\mu}(x)|^2 \, dx \lesssim r^{-(n-1)s/n+\epsilon}
\]

for all \( \epsilon > 0 \). The proof of Proposition 16.2 in [M5] works for \( \theta \) in place of \( \theta_n \) as such yielding the first estimate. The second follows in the same way. \( \square \)

Theorem 4.2. Let \( A, B \subset \mathbb{R}^n \) be Borel sets.

1. Suppose \( \dim A + \dim B > n \). If \( \dim A + (n-1) \dim B/n > n \) or \( \dim A > (n+1)/2 \), then \( \mathcal{L}^2(S_g(A \times B)) > 0 \) for \( \theta_n \) almost all \( g \in O(n) \).

2. Suppose \( \dim A + \dim B \leq n \). Then \( \dim S_g(A \times B) \geq \dim A + (n-1) \dim B/n \) for \( \theta_n \) almost all \( g \in O(n) \).

Moreover, \( \dim S_g(A \times B) \geq \dim A + \dim B \) for \( \theta_n \) almost all \( g \in O(n) \) if \( \dim B \leq (n-1)/2 \).

We have the following exceptional set estimates:

Theorem 4.3. Let \( A, B \subset \mathbb{R}^n \) be Borel sets.

1. Suppose \( \dim A + \dim B > n \). Then there is \( E \subset O(n) \) such that \( \mathcal{L}^2(S_g(A \times B)) > 0 \) for \( g \in O(n) \setminus E \) and

\[
\dim E \leq 2n - 1 - \dim A - (n-1) \dim B/n + (n-1)(n-2)/2.
\]

Moreover, \( \dim E \leq 2n - 1 - \dim A - \dim B + (n-1)(n-2)/2 \), if \( \dim B \leq (n-1)/2 \).

2. Suppose \( \dim A + \dim B \leq n \) and let \( \alpha > 0 \). Then there is \( E \subset O(n) \) such that

\[
\dim S_g(A \times B) \geq \alpha \text{ for } g \in O(n) \setminus E \text{ and }
\]

\[
\dim E \leq \alpha + n - 1 - \dim A - (n-1) \dim B/n + (n-1)(n-2)/2.
\]

Moreover, \( \dim E \leq \alpha + n - 1 - \dim A - \dim B + (n-1)(n-2)/2 \), if \( \dim B \leq (n-1)/2 \).

Notice that in some cases the upper for \( \dim E \) is bigger than \( n-1 + (n-1)(n-2)/2 = n(n-1)/2 = \dim O(n) \). Then we can take \( E = O(n) \) and the statement is empty.

Proofs of Theorem 4.2 and 4.3. The case \( \dim A > (n+1)/2 \) in the first part of (1) of Theorem 4.2 follows from Lemma 13.9 in [M4] and from Lemma 7.1 in [M5]. I don’t know any exceptional set estimates under the condition \( \dim A > (n+1)/2 \).

Let \( 0 < s < \dim A \) and \( 0 < t < \dim B \) and let \( \mu \in \mathcal{M}(A), \nu \in \mathcal{M}(B) \) with \( \mu(B(x,r)) \leq r^s, \nu(B(x,r)) \leq r^t \) for some \( s' > s, t' > t \) and for \( x \in \mathbb{R}^n, r > 0 \). Let \( \lambda_g = S_{g\#}(\mu \times \nu) \in \mathcal{M}(A \times B) \)
If \( \alpha \leq n \) we have by Lemma 4.1
\[
\int \int |\hat{\lambda}(\xi)|^2 |\xi|^{\alpha-n} \, d\xi \, d\theta g
\]

(4.3)
\[
= \int \sigma(\nu)(-\xi)|\hat{\mu}(\xi)|^2 |\xi|^{\alpha-n} \, d\xi
\]
\[
\leq \int |\hat{\mu}(\xi)|^2 |\xi|^{\alpha-1-(n-1)t/n-\beta} \, d\xi
\]
\[
= cI_{\alpha+n-1-(n-1)t/n-\beta}(\mu) \lesssim I_s(\mu) < \infty,
\]
if \( \beta \geq \alpha + n - 1 - (n-1)t/n - s \).

Similarly, if \( t \leq (n-1)/2 \),
\[
\int \int |\hat{\lambda}(\xi)|^2 |\xi|^{\alpha-n} \, d\xi \, d\theta g \lesssim I_{\alpha+n-1-t-\beta}(\nu) \lesssim I_s(\mu) < \infty,
\]
if \( \beta \geq \alpha + n - 1 - s - t \).

To get (1) of Theorems 4.2 and 4.3 we take \( \alpha = n \). If \( \beta \geq 2n - 1 - (n-1)t/n - s \), we have \( S'_{g#}(\mu \times \nu) \ll \mathcal{L}^n \), and so \( \mathcal{L}^n(S_g(A \times B)) > 0 \), for \( \theta \) almost all \( g \in O(n) \). In the case \( \text{dim } A + (n-1) \text{ dim } B/n > n \) we can choose \( s \) and \( t \) so that \( n - 1 \geq 2n - 1 - (n-1)t/n - s \).

Then we can take \( \theta = \theta_n \) to get (1) of Theorem 4.2 in this case. For Theorem 4.3(1) we have \( \mathcal{L}^n(S_g(A \times B)) > 0 \) for \( \theta \) almost all \( g \in O(n) \) provided \( \beta \geq 2n - 1 - (n-1)t/n - s \). Using Proposition 3.2 we see from this that the set of \( g \in O(n) \) for which \( \mathcal{L}^n(S_g(A \times B)) \) has dimension at most \( 2n - 1 - \text{dim } A - (n-1) \text{ dim } B/n \). The case \( \text{dim } B \leq (n-1)/2 \) follows in the same way using (4.4).

For any \( 0 < \alpha \leq n \) we have that if \( \beta \geq \alpha + n - 1 - (n-1)t/n - s \), then by (4.3) \( I_\alpha(S'_{g#}(\mu \times \nu)) < \infty \), and so \( \text{dim } S_g(A \times B) \geq \alpha \), for \( \theta \) almost all \( g \in O(n) \). To get the first statement of (2) of Theorem 4.2 we take \( \alpha = (n - 1)t/n + s \) and \( \beta = n - 1 \). The case \( \text{dim } B \leq (n - 1)/2 \) of Theorem 4.2(2) follows in the same way. For Theorem 4.3(2) we use Proposition 3.2 as before.

\[\square\]

### 4.1. Distance sets and measures.
There are some connections of this topic to Falconer’s distance set problem. For general discussion and references, see for example [M5].

Falconer showed in [F2] that for a Borel set \( A \subset \mathbb{R}^n \) the distance set \( \{ |x-y| : x, y \in A \} \) has positive Lebesgue measure if \( \text{dim } A > (n+1)/2 \). We had the same condition in Theorem 4.2 and it appeared in the intersection results of [M2]. When \( n = 2 \) Wolff [W] improved 3/2 to 4/3. Observe that when \( \text{dim } A = \text{dim } B \), the assumption \( \text{dim } A + \text{dim } B/2 > 2 \) in Theorem 4.2 becomes \( \text{dim } A > 4/3 \) and is the same as Wolff’s. For the most recent, and so far the best known, distance set results, see [GIOW] and [DGOWZ].

The proofs of distance set results often involve the distance measure \( \delta(\mu) \) of a measure \( \mu \) defined by
\[
\delta(\mu)(B) = \mu \times \mu(\{ x, y : |x-y| \in B \}), \quad B \subset \mathbb{R}.
\]

For example, Wolff showed that \( \delta(\mu) \in L^1(\mathbb{R}) \), if \( I_s(\mu) < \infty \) for some \( s > 4/3 \). To do this he used decay estimates for the spherical averages \( \sigma(\mu)(r) \) and proved (1.12) for \( n = 2 \).
From the argument in subsection 3.2 we see that when $\mu$ is replaced by $\mu \times \nu$ we have

$$\int \int D(Sg\#(\mu \times \nu))(z)^2 \, dz \, d\theta_n g$$

$$\leq \liminf_{r \to 0} \int \int \alpha(n)^{-1} r^{-n} Sg\#(\mu \times \nu)(B(z, r)) dSg\#(\mu \times \nu) z \, d\theta_n g$$

$$= \liminf_{r \to 0} \int \int \alpha(n)^{-1} r^{-n} \theta_n(\{g : |x - g(y) - (u - g(v))| \leq r\}) d(\mu \times \nu)(x, y) d(\mu \times \nu)(u, v)$$

$$\leq \liminf_{r \to 0} c \int r^{-1} \mu(\{x, u\} : ||x - u| - |y - v|| \leq r\})|y - v|^{1-n} \, d(\mu \times \nu)(y, v)$$

$$= \liminf_{\delta \to 0} c \int r^{-1} \delta(\mu)(B(t, r)) t^{1-n} \, d\delta(\nu) t$$

$$= c \int \delta(\mu)(t) \delta(\nu)(t) t^{1-n} \, dt,$$

provided the distance measures $\delta(\mu)$ and $\delta(\nu)$ are $L^2$ functions, and even a bit better so that we can move lim inf inside the integral. In fact, we have equality everywhere in the above argument if $\mu$ and $\nu$ are smooth functions with compact support. Since by an example in [GIOW], when $n = 2$, for any $s < 4/3$, $I_s(\mu) < \infty$ is not enough for $\delta(\mu)$ to be in $L^2$, probably it is not enough for $Sg\#(\mu \times \mu)$ to be in $L^2$. But in [GIOW] it was shown that if $I_s(\mu) < \infty$ for some $s > 5/4$, there is a modification of $\mu$ with good $L^2$ behaviour. Maybe this method could be used to show, for instance, that if $n = 2$ and $\dim A = \dim B > 5/4$, then $L^2(Sg(A \times B)) > 0$ for almost all $g \in O(2)$. One problem is that for distance sets one can split the measure to two parts with positive distance and only consider distances between points in the different supports, so one need not consider arbitrarily small distances, and the authors of [GIOW] seem to use this essentially. Here such reduction may not be possible.

### 4.2. Product measures and conical averages.

The estimates of the subsection 3.3 of the quadratic averages over the cone $\Gamma$ can trivially be improved for product measures. We just plug in the spherical estimates from (4.2). Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ be such that for some $0 < s \leq n$ and $0 < t \leq n$ we have $\mu(B(x, r)) \leq r^s$ and $\nu(B(x, r)) \leq r^t$ for $x \in \mathbb{R}^n, r > 0$. Then for all $\epsilon > 0, R > 1$,

$$\int \int_{1 \leq |x| = |y| \leq 2} |\mu \times \nu(Rx, Ry)|^2 \, d\gamma(x, y) \lesssim R^{-s-(n-1)t/n+\epsilon},$$

$$\lesssim R^{-s-t+\epsilon} \text{ if } 0 < t \leq (n - 1)/2,$$
To see this note that $(\hat{\mu} \times \hat{\nu})(x, y) = \hat{\mu}(x) \hat{\nu}(y)$. Then if $\sigma(\nu)(r) \lesssim r^{-\alpha}$, we get by (3.5)

$$\int_{1 \leq |x| = |y| \leq 2} |\mu \times \nu(Rx, Ry)|^2 d\gamma(x, y) = c \int_{1 \leq |x| \leq 2} \sigma(\nu)(R|x|)|\hat{\mu}(Rx)|^2 dx \lesssim R^{-\alpha} \int_{1 \leq |x| \leq 2} |\hat{\mu}(Rx)|^2 dx \lesssim R^{-\alpha-s},$$

and the claims follow from (4.2).

### 4.3. Hausdorff dimension of intersections.

One motivation for this study is hope to shed light on intersection problems. The main question is: what conditions on $\dim A$ and $\dim B$ guarantee that for almost all $g \in O(n)$, $\dim A \cap (g(B) + z) \geq \dim A + \dim B - n$ for positively many $z \in \mathbb{R}^n$. I expect that $\dim A + \dim B > n$ should be enough. This is only known when one of the sets has dimension bigger than $(n + 1)/2$. A necessary condition of course is that $L^n(S^1(A \times B)) > 0$ for almost all $g \in O(n)$. By Theorem 1.2 we have this when $\dim A + (n - 1) \dim B/n > n$, but even then I only know the estimate $\dim A \cap (g(B) + z) \geq \dim A + (n - 1) \dim B/n - n$, which follows from [M3] and (4.2). Since $A \cap (g(B) + z)$ is the projection on the first factor of $(A \times B) \cap S_g^{-1}(z)$, the problem is equivalent to getting dimension estimates for the sections $(A \times B) \cap S_g^{-1}(z)$.

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