MULTI-DIRECTIONAL AND SATURATED CHAOTIC
ATTRACTORS WITH MANY SCROLLS FOR FRACTIONAL
DYNAMICAL SYSTEMS

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ABSTRACT. Chaotic dynamical attractors are themselves very captivating in
Science and Engineering, but systems with multi-directional and saturated
chaotic attractors with many scrolls are even more fascinating for their multi-
directional features. In this paper, the dynamics of a Caputo three-dimensional
saturated system is successfully investigated by means of numerical techniques.
The continuity property for the saturated function series involved in the model
preludes suitable analytical conditions for existence and stability of the solution
to the model. The Haar wavelet numerical method is applied to the satu-
rated system and its convergence is shown thanks to error analysis. Therefore,
the performance of numerical approximations clearly reveals that the Caputo
model and its general initial conditions display some chaotic features with
many directions. Such a chaos shows attractors with many scrolls and many
directions. Then, the saturated Caputo system is indeed chaotic in the stan-
dard integer case (Caputo derivative order α = 1) and this chaos remains in
the fractional case (α = 0.9). Moreover the dynamics of the system change
depending on the parameter α, leading to an important observation that the
saturated system is likely to be regulated or controlled via such a parameter.

1. Introduction to the saturated model. Dynamical systems capable of gener-
ating multi-wing and chaotic strange attractors are very useful in Applied Technol-
ogy and Engineering, especially in areas like data encryption, power-systems pro-
tection, flow dynamics or security in communications and biomedical [8, 15, 13, 17].
Many authors have developed innovative methods through marvelous dynamical
models able to generate strange and chaotic attractors with many scrolls [30, 1, 8,
15, 13]. Some authors used simple circuits (like Chua’s circuit) to generate those
attractors. Other exploited functions like the sine-function and a system capable of
producing attractors with up to ten wings [31, 30]. Moreover, after the authors in
the articles [34, 24] managed to show that a hysteresis circuit can generate three-
dimensional multi-wing chaotic attractors, the question that worries the scientific
community was about the generation of the same types of three-dimensional multi-
wing chaotic attractors, by a saturated circuit. That question was successfully

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answered in [23]. Recall that the saturated circuit is one of the basic piecewise-linear circuit together with a hysteresis circuit. It is well known that fractional differentiation offers alternative approaches and additional features to those chaotic systems [15, 13, 17]. The question that rises now is to know whether or not a Caputo fractional and saturated circuit can generate three-dimensional multi-wing chaotic attractors. This paper aims at responding to that question. For that we consider the following three-dimensional saturated system.

\[
D^\alpha_t X(t) = MX + NY(PX)
\]  

where \(X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\) represents the state vector and

\[
M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -\frac{e_2}{e_1} & 0 \\ 0 & 0 & -\frac{e_3}{e_1} \\ e_1 & e_2 & e_3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and \(D^\alpha_t\) is the Caputo operator, comprehensively defined in the Section 1.1 below.

The saturated property of system (1) comes from the three-dimensional term

\[
\Upsilon(PX) = \begin{pmatrix} g(x; k_x, h_x, p_x, q_x) \\ g(y; k_y, h_y, p_y, q_y) \\ g(z; k_z, h_z, p_z, q_z) \end{pmatrix}
\]

where \(g(x; k_x, h_x, p_x, q_x)\) is the saturated function series given by

\[
g(x; k_x, h_x, p_x, q_x) =
\begin{cases}
  k_x(2q_x + 1) & \text{for } q_x h_x + 1 < x \\
  k_x(x - j h_x) + 2j k_x & \text{for } |x - j h_x| \leq 1 \text{ and } -p_x \leq j \leq q_x \\
  k_x(2j + 1) & \text{for } 1 < x - j h_x < h_x - 1 \text{ and } -p_x \leq j \leq q_x - 1 \\
  -k_x(2p_x + 1) & \text{for } -p_x h_x - 1 > x.
\end{cases}
\]

A similar definition holds for the functions

\(g(y; k_y, h_y, p_y, q_y)\) and \(g(z; k_z, h_z, p_z, q_z)\).

To have a better insight into this concept of saturated function series, we can have a look at a more explicit case, namely the case of a one-variable function. Hence, we can state the following remark [23]:

**Remark 1.**

- A one-variable saturated function, as shown in Fig 1 (a), takes the form

\[
g_0(x; k) = \begin{cases}
  k & \text{for } 1 < x \\
  kx & \text{for } |x| \leq 1 \\
  -k & \text{for } -1 > x.
\end{cases}
\]

where the so-called saturated plateaus, represented by \(\{g_0(x; k) = -k/x \leq -1\}\) and \(\{g_0(x; k) = k/x \geq 1\}\) are respectively the lower and the upper radial. The segment located between the two saturated plateaus is mathematically expressed as \(\{g_0(x; k) = k/x| x| \leq 1\}\) and called the saturated slope. The slope of the middle segment is given by \(k > 0\).

- We call saturated function series the piecewise linear function taking the general form

\[
g(x; k, h, p, q) = \sum_{j=-p}^{q} g_j(x; k, h),
\]

(5)
where $p, q \in \mathbb{N}^*$, with $k > 0$ and $h > 2$ representing respectively the slope and saturated time delay of the function series. Moreover,

$$
g_j(x; k, h) = \begin{cases} 
2k & \text{for } 1 + jh < x \\
2k + k(x - jh) & \text{for } |x - jh| \leq 1 \\
0 & \text{for } jh - 1 > x,
\end{cases}
$$

(6)

$$
g_{-j}(x; k, h) = \begin{cases} 
0 & \text{for } -(jh - 1) < x \\
-k + k(x + jh) & \text{for } |x + jh| \leq 1 \\
-2k & \text{for } -(1 + jh) > x,
\end{cases}
$$

(7)

• From the series (5), the definitions (6) and (7), it is easy to understand how the saturated function series (3) was remodeled. An illustration of the phase portrait of that saturated function series is depicted in Fig 1 (b) for the parameters $h = 4$, $k = 1$. We can clearly see that the continuity property holds for the saturated function series (3) which therefore exhibits suitable analytical properties related to existence and stability of the solution.

We aim to solve the model (1) and evaluate its dynamics by means of Haar wavelet method and numerical simulations. Note that like Legendre wavelet method [33], Haar wavelet method is another effective and promising method used to address the solvability of fractional differential equations.

**Figure 1.** Representation of the phase portrait of saturated functions. In (a), the one-variable saturated function (4). In (b), the phase portrait of saturated function series (5) with $k = 1$, $h = 4$.

1.1. **Brief view on fractional derivatives and variants.** In several Applied Sciences domains like for instance, Engineering and Computer Sciences, applications using derivatives with non-integer orders (real or complex) have been successful [2, 20, 4, 6, 7, 18, 15, 25, 21, 26, 27, 29, 11, 36, 10], raising in the same time a huge interest for fractional calculus among the scientific community. In fractional differentiation, there are many different definitions of fractional derivatives with several examples and that include the Liouville, Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Marchaud, Hadamard Chen, Davidson-Essex, Coimbra, Canavati, Jumarie, Reisz, Cossar, New Riemann-Liouville, the two-parameter fractional derivative, the fractional derivatives with non local or non singular kernel. Only a few of these are commonly used in the current study of fractional calculus and these are listed below together with their definitions [5, 16, 14, 21, 28, 35, 12].
Grünwald-Letnikov:

The Grünwald-Letnikov fractional derivative with fractional order \( \alpha \) if \( x \in C^m [0, T], \ (T > 0) \), is defined as follows:

\[
GLD_{0,t}^\alpha x(t) = \sum_{k=0}^{m-1} \frac{x^{(k)}(0)}{\Gamma(-\alpha + k + 1)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \tag{8}
\]

with \( t \in [0, T], \ m - 1 \leq \alpha < m \in \mathbb{Z}^+ \).

The original expression is by a limit function, but this definition is not particularly useable for analysis.

Riemann-Liouville:

The Riemann-Liouville fractional derivative with fractional order \( \alpha \) of \( x(t) \) is defined as follows:

\[
RLD_{0,t}^\alpha x(t) = \frac{d^m}{dt^m} D_0^{\alpha-m} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \tag{9}
\]

with \( t \in [0, T], \ m - 1 \leq \alpha < m \in \mathbb{Z}^+ \).

Caputo:

The Caputo fractional derivative with fractional order \( \alpha \) of \( x(t) \) is defined as follows:

\[
CD_{0,t}^\alpha x(t) = D_{0,t}^{\alpha-m} \frac{d^m}{dt^m} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} x(\tau) d\tau, \tag{10}
\]

with \( t \in [0, T], \ m - 1 \leq \alpha < m \in \mathbb{Z}^+ \). Both Riemann-Liouville and Caputo fractional derivatives have as anti-derivatives the following fractional integral of order \( \alpha \) :

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{11}
\]

which is obviously defined for \( m = 1 \) and was based on Euler transform for analytic function and Cauchy’s formula for iterated integrals. We use the Caputo fractional derivative in the rest of this paper and for the sake of simplicity, we set \( m = 1 \) and \( CD_{0,t}^\alpha x(t) = D_{t}^\alpha x(t) \). Hence, the operator \( D_t^\alpha \) given in (1) is defined as

\[
D_t^\alpha X(t) = \left( \begin{array}{c}
D_t^\alpha x(t) \\
D_t^\alpha y(t) \\
D_t^\alpha z(t)
\end{array} \right).
\]

There exist number of other definitions of fractional derivatives and the reader may consult the references mentioned here above to get more details about them.

1.2. Haar wavelet scheme for the saturated model. The following definition is done according to the works in [3, 22]. The Haar wavelet denotes the following real line

\[
h(t) = \begin{cases}
1, & 0 \leq t < 1/2; \\
-1, & 1/2 \leq t < 1; \\
0, & \text{elsewhere},
\end{cases}
\]

taking its values in \( \mathbb{R} \). When we take each \( i = 0, 1, 2, \cdots \), we can always rewrite it as \( i = 2^j + l \) with \( j = 0, 1, 2, \cdots \), and \( l = 0, 1, 2, \cdots , 2^j - 1 \). Let us now consider for each \( i = 0, 1, 2, 3, \cdots \), the family

\[
\delta_i(t) = \begin{cases}
2^{j}h(2^j t - l), & \text{for } i = 1, 2, \cdots , \\
1, & \text{for } i = 0,
\end{cases} \tag{12}
\]
for \( t \in [0, 1) \). Therefore, it is well-known that the system \( \{ \mathcal{H}_i(t) \}_{i=0}^{\infty} \) is a complete orthonormal system in \( L^2[0, 1) \) [9, 22]. If the function \( \mathbf{x} \in C[0, 1) \), hence, the series 
\[
\sum_{i=0}^{\infty} \langle \mathbf{x}, \mathcal{H}_i \rangle \mathcal{H}_i
\]
converges uniformly to \( \mathbf{x} \) with \( \langle \mathbf{x}, \mathcal{H}_i \rangle = \int_{0}^{\infty} \mathbf{x}(t) \mathcal{H}_i(t) \, dt \). this allows us to decompose \( \mathbf{x} \) as
\[
\mathbf{x}(t) = \sum_{i=0}^{\infty} d_i \mathcal{H}_i(t),
\]
with \( d_i = \langle \mathbf{x}, \mathcal{H}_i \rangle \). This leads to the approximated solution that is
\[
\mathbf{x}(t) \approx \mathbf{x}_l(t) = \sum_{i=0}^{l-1} d_i \mathcal{H}_i(t)
\]
where \( l \in \{2^j \mid j = 0, 1, 2, \cdots \} \).

With \( r \in \mathbb{N} \), we can use on the interval \([0, r)\), the translation of the Haar function in order to define
\[
\mathcal{H}_{n,i}(t) = \mathcal{H}_i(t - n + 1) \quad n = 1, 2, \cdots, r \quad \text{and} \quad i = 0, 1, 2, \cdots,
\]
where \( \mathcal{H}_i \) is given by (12) and obviously satisfies the same properties like \( \mathcal{H}_{n,i} \). Thus, the family \( \{ \mathcal{H}_{n,i}(t) \}_{i=0}^{\infty}, n = 1, 2, \cdots, r, \) is also a complete orthonormal system in \( L^2[0, 1) \). We are now lead to the following family of basis functions
\[
d_{n,i} = \langle \mathbf{x}, \mathcal{H}_{n,i} \rangle = \int_{0}^{\infty} \mathbf{x}(t) \mathcal{H}_{n,i}(t) \, dt,
\]
which is called Haar basis functions and also form an orthonormal system. We can use those Haar basis functions to transform the solution \( \mathbf{x} \in L^2[0, r) \) into the series
\[
\mathbf{x}(t) = \sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \mathcal{H}_{n,i}(t).
\]
In a similar manner, we consider the approximation
\[
\mathbf{x}(t) \approx \mathbf{x}_l(t) = \sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \mathcal{H}_{n,i}(t),
\]
with \( l \in \{2^j \mid j = 0, 1, 2, \cdots \} \).

**Remark 2.** For the sake of notation simplicity in our analysis, it is crucial to mention that (15) can be rewritten into the compact form
\[
\mathbf{x}(t) \approx \mathbf{x}_l(t) = ^T \mathbf{O}_{r \times 1} \mathbf{h}_{r \times 1}
\]
with \(^T \mathbf{O}_{r \times 1} \) meaning the transpose vector of
\[
\mathbf{O}_{r \times 1} =
\begin{pmatrix}
  d_{1,0} \\
  \vdots \\
  d_{1,l-1} \\
  d_{2,0} \\
  \vdots \\
  d_{2,l-1} \\
  \vdots \\
  d_{r,0} \\
  \vdots \\
  d_{r,l-1}
\end{pmatrix}
\quad \text{and} \quad
\mathbf{h}_{r \times 1} =
\begin{pmatrix}
  \mathcal{H}_{1,0} \\
  \vdots \\
  \mathcal{H}_{1,l-1} \\
  \mathcal{H}_{2,0} \\
  \vdots \\
  \mathcal{H}_{2,l-1} \\
  \vdots \\
  \mathcal{H}_{r,0} \\
  \vdots \\
  \mathcal{H}_{r,l-1}
\end{pmatrix}.
\]
2. Numerical and solvability results for the three-dimensional saturated system (1). The three-dimensional saturated system (1) that is solved in this section reads

$$D_t^\alpha X(t) = MX + NY(PX).$$

To proceed, we assume that it satisfies initial conditions

$$X(0) = (x(0), y(0), z(0)) = G_0(x, y, z).$$

with the matrices $M$, $N$, $P$ and $Y(PX)$ defined in (1) and (2) and where the left-hand-side is the Caputo derivative vector $D_t^\alpha X(t)$ given as

$$D_t^\alpha X(t) = \begin{pmatrix} D_t^\alpha x(t) \\ D_t^\alpha y(t) \\ D_t^\alpha z(t) \end{pmatrix}$$

and satisfying the initial conditions

$$G_0(x, y, z) = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$ 

System (17) can equivalently reads as

$$\begin{pmatrix} D_t^\alpha x(t) \\ D_t^\alpha y(t) \\ D_t^\alpha z(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 & -\frac{e_2}{\alpha} & 0 \\ 0 & 0 & -\frac{e_2}{\alpha} \\ e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} g(x; k_x, h_x, p_x, q_x) \\ g(y; k_y, h_y, p_y, q_y) \\ g(z; k_z, h_z, p_z, q_z) \end{pmatrix}$$

or

$$D_t^\alpha x(t) = \mathcal{M}_1(X(t), t, g)$$
$$D_t^\alpha y(t) = \mathcal{M}_2(X(t), t, g)$$
$$D_t^\alpha z(t) = \mathcal{M}_3(X(t), t, g)$$

and satisfying the initial conditions

$$x(0) = u, \ y(0) = v, \ z(0) = w,$$

where

$$\mathcal{M}_1(X(t), t, g) = y - \frac{e_2}{\alpha} g(y; k_y, h_y, p_y, q_y)$$
$$\mathcal{M}_2(X(t), t, g) = z - \frac{e_2}{\alpha} g(z; k_z, h_z, p_z, q_z)$$
$$\mathcal{M}_3(X(t), t, g) = -ax - by + cz + e_1 g(x; k_x, h_x, p_x, q_x) + e_2 g(y; k_y, h_y, p_y, q_y) + e_3 g(z; k_z, h_z, p_z, q_z).$$

For the sake of numerical solvability, it is important to recall the explicit representations of the saturated series functions $g(x; k_x, h_x, p_x, q_x)$, $g(y; k_y, h_y, p_y, q_y)$ and $g(z; k_z, h_z, p_z, q_z)$ respectively given by

$$g(x; k_x, h_x, p_x, q_x) = \begin{cases} k_x(2q_x + 1) & \text{for} \ q_xh_x + 1 < x \\ k_x(x - jh_x) + 2jk_x & |x - jh_x| \leq 1 \text{ and } -p_x \leq j \leq q_x \\ k_x(2j + 1) & 1 < x - jh_x < h_x - 1 \text{ and } -p_x \leq j \leq q_x - 1 \\ -k_x(2p_x + 1) & \text{for } -p_xh_x - 1 > x. \end{cases}$$

$$g(y; k_y, h_y, p_y, q_y) = \begin{cases} k_y(2q_y + 1) & \text{for} \ q_yh_y + 1 < y \\ k_y(y - jh_y) + 2jk_y & |y - jh_y| \leq 1 \text{ and } -p_y \leq j \leq q_y \\ k_y(2j + 1) & 1 < y - jh_y < h_y - 1 \text{ and } -p_y \leq j \leq q_y - 1 \\ -k_y(2p_y + 1) & \text{for } -p_yh_y - 1 > y. \end{cases}$$
and
\[
\begin{align*}
g(z, k_z, h_z, p_z, q_z) &= \\
k_z(2q_z + 1) &\quad \text{for } q_z h_z + 1 < z \\
k_z(z - j h_z) + 2 j k_z &\quad \text{for } |z - j h_z| \leq 1 \text{ and } -p_z \leq j \leq q_z \\
k_z(2j + 1) &\quad \text{for } 1 < z - j h_z < h_z - 1 \text{ and } -p_z \leq j \leq q_z - 1 \\
-k_z(2p_z + 1) &\quad \text{for } -p_z h_z - 1 > z.
\end{align*}
\]

(23)

The model (19) (with (20)) that is expressed by means of Caputo derivative can be Haar wavelet approximated using (16) and we obtain
\[
\begin{align*}
D_x^\alpha x(t) &= M_1(X(t), t, g) = D_x^\alpha x(t) = T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r} \\
D_y^\beta y(t) &= M_2(X(t), t, g) = D_y^\beta y(t) = T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r} \\
D_z^\gamma z(t) &= M_3(X(t), t, g) = D_z^\gamma z(t) = T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r}.
\end{align*}
\]

(24)

Hence,
\[
\begin{align*}
x(t) - u &\approx D_x^\alpha x(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r} \\
y(t) - v &\approx D_y^\beta y(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r} \\
z(t) - w &\approx D_z^\gamma z(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r},
\end{align*}
\]

(25)

that is the same as
\[
\begin{align*}
x(t) &\approx x_1(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r} + u \\
y(t) &\approx y_1(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r} + v \\
z(t) &\approx z_1(t) = T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r} + w,
\end{align*}
\]

(26)

where we have applied the Caputo fractional integral (11) to the left- and right-hand side of (24) and where \(W_{r|x \times r}^\alpha\) fractional matrix in the sense of Haar wavelets [3, 9].

The technique of Galerkin applied to collocation points is used to address the solvability of (17)-(18), which allows the substitution of the approximation models (24) and (26) into (17) to cause the residual errors that are expressed as follows:
\[
\begin{align*}
Err_1(\beta^1, \beta^2, \beta^3, t) &= T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r} \\
- M_1\left(T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|x \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, t\right); \\
Err_2(\beta^1, \beta^2, \beta^3, t) &= T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r} \\
- M_2\left(T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|x \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, t\right); \\
Err_3(\beta^1, \beta^2, \beta^3, t) &= T O_{r|t \times 1^r} \mathbf{h}_{r|t \times 1^r} \\
- M_3\left(T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|t \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, T O_{r|x \times 1^r} W_{r|x \times r} \mathbf{h}_{r|t \times 1^r}, t\right),
\end{align*}
\]

(27)

with
\[
\begin{align*}
\beta^1 &= d^1_{1,0}, \ldots, d^1_{1,t-1}, \ldots, d^1_{r,0}, \ldots, d^1_{r,t-1}; \\
\beta^2 &= d^2_{1,0}, \ldots, d^2_{1,t-1}, \ldots, d^2_{r,0}, \ldots, d^2_{r,t-1}; \\
\beta^3 &= d^3_{1,0}, \ldots, d^3_{1,t-1}, \ldots, d^3_{r,0}, \ldots, d^3_{r,t-1};
\end{align*}
\]

and where \(d^i\) is the components of \(T O_{x}^i\).

With the assumption that
\[
\begin{align*}
Err_1(\beta^1, \beta^2, \beta^3, t_n, i) &= 0; \\
Err_2(\beta^1, \beta^2, \beta^3, t_n, i) &= 0; \\
Err_3(\beta^1, \beta^2, \beta^3, t_n, i) &= 0,
\end{align*}
\]
where
\[ t_{n,i} = \frac{2i - 1}{2l} + n - 1, \quad n = 1, 2, \ldots, r; \quad i = 1, 2, \ldots, l \]
are number \( rl \) of collocation points, then, the following system of \( 3rl \) equations, with \( 3rl \) unknowns is obtained and reads as
\[
\begin{align*}
&d_{1,0}^1, \ldots, d_{1,l-1}^1, \ldots, d_{r,0}^1, \ldots, d_{r,l-1}^1; \\
&d_{1,0}^2, \ldots, d_{1,l-1}^2, \ldots, d_{r,0}^2, \ldots, d_{r,l-1}^2; \\
&d_{1,0}^3, \ldots, d_{1,l-1}^3, \ldots, d_{r,0}^3, \ldots, d_{r,l-1}^3.
\end{align*}
\]
At this stage, we just need to substitute these unknowns into (26) and the desired approximated solution is found as
\[
x(t) \approx \begin{pmatrix} x_k(t) \\ y_k(t) \\ z_k(t) \end{pmatrix}.
\]

3. Convergence of the method through error analysis. Let us now assess the exact error bounds used in the application of the aforementioned numerical method to solve the Caputo saturated system (17)-(18). To achieve it, we use the error analysis method. Because of \( x \in L^2[0, r) \), hence we also have \( x \in L^2[0, r) \), \( y \in L^2[0, r) \) and \( z \in L^2[0, r) \) and set
\[
\|x\|_2 = \left( \|x\|_{L^2}^2 + \|y\|_{L^2}^2 + \|z\|_{L^2}^2 \right)^{1/2},
\] (28)
where
\[
\|x\|_{L^2} = \left( \int_0^r |x(t)|^2 \, dt \right)^{1/2}, \quad \|y\|_{L^2} = \left( \int_0^r |y(t)|^2 \, dt \right)^{1/2}, \quad \|z\|_{L^2} = \left( \int_0^r |z(t)|^2 \, dt \right)^{1/2}.
\]
\( \|x\|_2 \) is obviously a norm. From (15) and (16), it can be assumed that similar to (26), the Caputo operator \( D_t^\alpha x_K(t) \) is an approximation of the operator \( D_t^\alpha x(t) \) that reads as
\[
D_t^\alpha x(t) \approx D_t^\alpha x_K(t) = \sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \delta_{n,i}(t),
\]
and that is also the same as
\[
\begin{pmatrix}
D_t^\alpha x_i(t) \\
D_t^\alpha y_i(t) \\
D_t^\alpha z_i(t)
\end{pmatrix} = D_t^\alpha x_i(t) = \sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \delta_{n,i}(t) = \begin{pmatrix}
\sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \delta_{n,i}(t) \\
\sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \delta_{n,i}(t) \\
\sum_{n=1}^{r} \sum_{i=0}^{l-1} d_{n,i} \delta_{n,i}(t)
\end{pmatrix}
\]
where \( l \in \{2^j : j = 0, 1, 2, \ldots \} \) and \( d_{n,i} = \langle D_t^\alpha x_i, \delta_{n,i} \rangle_r = \int_0^r D_t^\alpha x_i(t) \delta_{n,i}(t) \, dt \),
\[
\begin{align*}
d_{n,i}^1 &= \langle D_t^\alpha x_i, \delta_{n,i} \rangle_r = \int_0^r D_t^\alpha x_i(t) \delta_{n,i}(t) \, dt, \\
d_{n,i}^2 &= \langle D_t^\alpha y_i, \delta_{n,i} \rangle_r = \int_0^r D_t^\alpha y_i(t) \delta_{n,i}(t) \, dt, \\
d_{n,i}^3 &= \langle D_t^\alpha z_i, \delta_{n,i} \rangle_r = \int_0^r D_t^\alpha z_i(t) \delta_{n,i}(t) \, dt.
\end{align*}
\]
Hence,

\[ D^\alpha_t x(t) - D^\alpha_t x_l(t) = \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \]

\[ = \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \quad j = 0, 1, 2, \ldots \]  

(30)

Exploiting the norm (28), we formulate the following convergence observations, that are also true for functions \( x, y \) and \( z \) belonging to the Sobolev space \( H^1[0, r) \).

**Proposition 1.** Let \( 0 \leq \alpha \leq 1 \) and assume that \( x \in H^1[0, r), \ y \in H^1[0, r) \) and \( z \in H^1[0, r) \). If the Caputo functions \( D^\alpha_t x_l(t) \) are obtained thanks to the Haar wavelet approximations of \( D^\alpha_t x(t) \) then, the exact upper bound used in such a numerical approximation reads as follows:

\[
\| D^\alpha_t x(t) - D^\alpha_t x_l(t) \|_2 \leq \frac{1}{\Gamma(1-\alpha)} (r \varpi)^{-1/2}.
\]

Here \( \varpi \in \mathbb{R}^+ \) and \( \varpi_\alpha = \frac{1-\alpha}{2\alpha} \left( \frac{3\alpha}{2\alpha-2} (1-\Gamma(1-\alpha)) + \frac{3\alpha}{2\alpha-2} (1-\Gamma(2-2\alpha)) \right)^{-1/2} \).

**Proof.** From (28) and using (30) we get

\[
\| D^\alpha_t x(t) - D^\alpha_t x_l(t) \|_2
\]

\[
= \left( \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \right)^2 + \| D^\alpha_t y - D^\alpha_t y_l \|_2^2 + \| D^\alpha_t z - D^\alpha_t z_l \|_2^2 \right)^{1/2}
\]

\[
= \left( \int_0^r \left( \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \right)^2 dt + \int_0^r \left( \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \right)^2 dt \right)^{1/2}
\]

\[
\leq \left( \sum_{n=1}^{\infty} \sum_{i=1}^{r} d_{n,i} \delta_{n,i}(t) \right)^2 \quad \text{for all } t \in [0, r].
\]

Recalling that \( \{ \delta_{n,i}(t) \}_{n=1}^{\infty} \) is a complete orthonormal family on \([0, r]\), meaning that \( \int_0^r h_{n,i}(t) \delta_{n,i}(t) dt = \delta_{n,i} \), identity matrix, hence Fubini-Tonelli theorem for positive functions \([32, 19]\) yields

\[
\| D^\alpha_t x(t) - D^\alpha_t x_l(t) \|_2 \leq \frac{1}{\Gamma(1-\alpha)} (r \varpi)^{-1/2}.
\]

(32)

equivalent to

\[
\| D^\alpha_t x(t) - D^\alpha_t x_l(t) \|_2 \leq \left( \sum_{n=1}^{\infty} \sum_{j=0}^{2^l+1} \int_0^r d_{n,j}^2 dt + \sum_{n=1}^{\infty} \sum_{j=0}^{2^l+1} \int_0^r d_{n,j}^2 dt + \sum_{n=1}^{\infty} \sum_{j=0}^{2^l+1} \int_0^r d_{n,j}^2 dt \right)^{1/2},
\]

where \( d_{n,j}^2 \), \( q = 1, 2, 3 \) are given in (29) and where \( l \) is assumed to take the number 2'powers \(( l \in \{2^j : j = 0, 1, 2, \ldots \})\). Calculating each \( d_{n,j}^2 \) by means of (29), the
definitions (12) and (13) of $\mathcal{S}_{n,l}$ imply
\[
d^{1}_{n,i} = (\sqrt{2})^{j} \left[ \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} x(t) dt - \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} x(t) dt \right];
\]
\[
d^{2}_{n,i} = (\sqrt{2})^{j} \left[ \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} y(t) dt - \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} y(t) dt \right];
\]
\[
d^{3}_{n,i} = (\sqrt{2})^{j} \left[ \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} z(t) dt - \int_{\frac{l+\frac{1}{2j}}{2j} - 1 + n}^{\frac{l+\frac{1}{2j}}{2j} - 1 + n} D^{\alpha}_{l} z(t) dt \right].
\]
The Mean value theorem for definite integrals implies that there exist $t_{1x}$ and $t_{2x}$, two times such that $t_{1x} \in \left(\frac{l+\frac{1}{2j}}{2j} - 1 + n, \frac{l+\frac{1}{2j}}{2j} - 1 + n\right)$ and $t_{2x} \in \left(\frac{l+\frac{1}{2j}}{2j} - 1 + n, \frac{l+\frac{1}{2j}}{2j} - 1 + n\right)$ satisfying
\[
d^{1}_{n,i} = (\sqrt{2})^{j} \left( \frac{1}{2j+1} D^{\alpha}_{l} x(t_{1x}) dt - \frac{1}{2j+1} D^{\alpha}_{l} x(t_{2x}) dt \right) = 2^{-\left(\frac{j+1}{2}\right)} (D^{\alpha}_{l} x(t_{1x}) dt - D^{\alpha}_{l} x(t_{2x}) dt).
\]
The definition of the Caputo fractional derivative (10) yields
\[
|d^{1}_{n,i}| = 2^{-\left(\frac{j+1}{2}\right)} |D^{\alpha}_{l} x(t_{1x}) dt - D^{\alpha}_{l} x(t_{2x}) dt|
\]
\[
= 2^{-\left(\frac{j+1}{2}\right)} \frac{1}{\Gamma(1-\alpha)} \left| \int_{0}^{t_{1x}} (t_{1x} - \zeta)^{-\alpha} d\zeta \right| d\zeta - \int_{0}^{t_{2x}} (t_{2x} - \zeta)^{-\alpha} d\zeta \right| d\zeta.
\]
Knowing that $x \in H^{1}[0,r]$, there exists $C_{x}$, a non-negative constant so that $||\dot{x}(\zeta)|| \leq C_{x}$ for all $\zeta \in (0,t_{1x})$ and $\zeta \in (0,t_{2x})$. Whence,
\[
|d^{1}_{n,i}| \leq C_{x} 2^{-\left(\frac{j+1}{2}\right)} \frac{1}{\Gamma(1-\alpha)} \left| \int_{0}^{t_{1x}} (t_{1x} - \zeta)^{-\alpha} d\zeta \right| d\zeta - \int_{0}^{t_{2x}} (t_{2x} - \zeta)^{-\alpha} d\zeta \right| d\zeta.
\]
Integration followed by simplification give
\[
|d^{1}_{n,i}| \leq \frac{C_{x} 2^{-\left(\frac{j+1}{2}\right)}}{(1-\alpha)\Gamma(1-\alpha)} \left| t_{1x}^{(1-\alpha)} - t_{2x}^{(1-\alpha)} \right| \leq \frac{C_{x} 2^{-\left(\frac{j+1}{2}\right)}}{(1-\alpha)\Gamma(1-\alpha)} 2^{j(1-\alpha)},
\]
where the facts that
\[
0 \leq \alpha \leq 1, \ t_{1x} \in \left(\frac{l+\frac{1}{2j}}{2j} - 1 + n, \frac{l+\frac{1}{2j}}{2j} - 1 + n\right)
\]
and
\[
\ t_{2x} \in \left(\frac{l+\frac{1}{2j}}{2j} - 1 + n, \frac{l+\frac{1}{2j}}{2j} - 1 + n\right)
\]
have been used.
The exact same steps are followed to show that there exist \( C_y \) and \( C_z \), two non-negative constants such that

\[
|d^2_{n,i}| \leq \frac{C_y 2^{-\left(\frac{4}{3}+1\right)}}{\left(1-\alpha\right)\Gamma\left(1-\alpha\right)} 2^{j(1-\alpha)}
\]

and

\[
|d^2_{n,i}| \leq \frac{C_z 2^{-\left(\frac{4}{3}+1\right)}}{\left(1-\alpha\right)\Gamma\left(1-\alpha\right)} 2^{j(1-\alpha)}.
\]

Define \( \omega = \max(C_x, C_y, C_z) \). The substitution of (35), (36) and (37) into (32) gives

\[
\|D^\alpha_t x(t) - D^\alpha_t x_i(t)\|_2 \leq \sqrt{\frac{3r\omega^2}{4(\Gamma(1-\alpha))^2(1-\alpha)^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \sum_{l=2}^{2j+1} 2^{2l}(1-\alpha) 2^l}
\]

\[
\leq \sqrt{\frac{3r\omega^2}{4(\Gamma(1-\alpha))^2(1-\alpha)^2} \left(\frac{2^{2\alpha} - 2^{2\alpha} l(1-\alpha)}{2^{2\alpha} - 2} + \frac{2^{2\alpha} - 2^{2\alpha} l(2-2\alpha)}{2^{2\alpha} - 4}\right)}
\]

\[
\leq \frac{\sqrt{\omega}}{2(\Gamma(1-\alpha))(1-\alpha)} \left(\frac{2^{2\alpha} (1 - l(1-\alpha))}{2^{2\alpha} - 2} + \frac{2^{2\alpha} (1 - l(2-2\alpha))}{2^{2\alpha} - 4}\right)^{1/2},
\]

and the proof is done. \( \square \)

**Remark 3.** For the states functions \( x, y \) and \( z \) that are not taken from \( H^1(0,r) \), is no longer possible to formulate the Proposition 1 with assumptions \( x \in L^2(0,r) \), \( y \in L^2(0,r) \) and \( z \in L^2(0,r) \) only. The reason is that \([0, r)\) is not a closed interval and the state functions \( x, y, z \) and their time derivatives may not be bounded or may not attain their bounds on \([0, r)\).

This leads to the proof of following corollary:

**Corollary 1.** Let \( 0 \leq \alpha \leq 1 \), \( x \in L^2(0,r) \), \( y \in L^2(0,r) \), \( z \in L^2(0,r) \) and assume that their time derivatives \( x', y' \) and \( z' \) are continuous and bounded on \([0, r)\). If the Caputo functions \( D^\alpha_t x_i(t) \) are obtained thanks to the Haar wavelet approximations of \( D^\alpha_t x(t) \) then, the exact upper bound in such a numerical approximation reads as follows:

\[
\|D^\alpha_t x(t) - D^\alpha_t x_i(t)\|_2 \leq \frac{1}{\Gamma(1-\alpha)} \cdot (\omega \omega^{-1}).
\]

Here \( \omega \in \mathbb{R}^+ \) and \( \omega^{-1} = \frac{1}{2^{1-\alpha}} \left(\frac{3r^2}{2^{2\alpha-2}} \left(1 - l(1-\alpha)\right) + \frac{3r^2}{2^{2\alpha-4}} \left(1 - l(2-2\alpha)\right)\right)^{-1/2}.\)

4. **Simulations and interpretation.** During the analysis in the previous sections, we have noticed that the continuity property for the saturated function series involved in our saturated system was setting appropriate analytical conditions for the solution to the model to exist and be stable. After that, we have successfully proved that the error committed when applying Haar wavelet is reasonable and inconsequential. This allows us in this section to provide some numerical simulations for the solutions to the saturated system (1). Those simulations reveal that the Caputo saturated model whose solvability was addressed via the Haar wavelet scheme, is indeed characterized by chaotic dynamics. The numerical plots are depicted in Fig. 2 and...
Fig. 3 and they clearly show that the system (1) is not only a saturated chaotic system, but can also generate multi-dimensional chaotic attractors with many scrolls. Two cases are considered here for (1): The regular case (Fig. 2) where \( \alpha = 1 \) and the fractional case (Fig. 3) where \( \alpha = 0.9 \). In Fig. 2 (a), depicted for the parameter values \( a = b = 3/4, c = e_1 = 4/5, p_x = 2, q_x = 2, k_x = 19/2, h_x = 19 \), we have a projection on the \((y, x)\)-plane representing the one-dimensional saturated chaotic attractor with six scrolls. In Fig. 2 (b), depicted for the parameter values \( a = b = 3/4, c = e_1 = e_2 = 4/5, p_x = q_x = p_y = q_y = 2, k_x = k_y = 211/4, h_x = h_y = 211/2 \), we have a projection on the \((y, x)\)-plane representing the two-dimensional saturated chaotic attractor in a grid of \(6 \times 6\) scrolls. Lastly in Fig. 2 (c), depicted for the parameter values \( a = e_x = 3/4, b = c = e_y = e_z = 4/5, p_x = p_y = p_z = q_x = q_y = q_z = 2, k_x = 211/2, h_x = 211, k_y = k_z = 83/2, h_y = h_z = 83 \), we have a projection on the \((z, x)\)-plane representing the three-dimensional saturated chaotic attractor in a grid of \(6 \times 6 \times 6\) scrolls. Similar dynamics are observed in the fractional case as shown Fig. 2. It appears that the dynamics of the saturated model (17) can change with the parameter \( \alpha \) but the fractional system conserves the same type of multi-scroll attractors, likely to be \( \alpha \)-controlled.

5. Conclusion. In this paper, the dynamics of a Caputo three-dimensional saturated system have been investigated by means of a numerical method. The continuity property noticed in the saturated function series involved in the Caputo saturated system (Fig. 1) was a prelude for suitable analytical conditions for existence and stability of it solution. The whole analysis consisted on applying the Haar wavelet numerical method to the saturated system and assessing the convergence by means of error analysis, which was done successfully. This made possible the performance of some numerical approximations which clearly revealed that the model (17), satisfying the initial conditions (18), displays chaotic features with many directions. Such a chaos, depicted by Fig. 2 and Fig. 3 appears to be characterized by attractors with many scrolls and dimensions. Hence, the saturated Caputo system (17) is chaotic in the standard case (\( \alpha = 1 \)) and this chaos remains even when the Caputo derivative order is purely fractional (\( \alpha = 0.9 \)). Moreover the dynamics of the system change depending on the parameter \( \alpha \) as shown in Fig. 2 and Fig. 3. This observation is very important because it proves that the saturated system (17) is a likely to be regulated or controlled via the parameter \( \alpha \).

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REFERENCES

[1] P. Arena, S. Baglio, L. Fortuna and G. Manganaro, Generation of n-double scrolls via cellular neural networks, International Journal of Circuit Theory and Applications, 24 (1996), 241–252.
[2] A. Atangana, Derivative with a New Parameter: Theory, Methods and Applications, Academic Press, 2016.
[3] E. Babolian and A. Shahsavaran, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, Journal of Computational and Applied Mathematics, 225 (2009), 87–95.
[4] E. G. Bazhlekova, Subordination principle for fractional evolution equations, Fractional Calculus and Applied Analysis, 3 (2000), 213–230.
[5] D. Brockmann and L. Hufnagel, Front propagation in reaction-superdiffusion dynamics: Taming Lévy flights with fluctuations, Physical Review Letters, 98 (2007), 178–301.
Figure 2. Plot representing multi-dimensional simulations of the saturated system (1) with $\alpha = 1$ (conventional case). In (a), projection of a one-dimensional saturated chaotic attractor with six scrolls. In (b), a projection of a two-dimensional saturated chaotic attractor with a grid of $6 \times 6$ scrolls. In (c), a projection of a three-dimensional saturated chaotic attractor with a grid of $6 \times 6 \times 6$ scrolls.

[6] M. Caputo, Linear models of dissipation whose Q is almost frequency independent–II, Geophysical Journal International, 13 (1967), 529–539.
[7] M. Caputo and M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl, 1 (2015), 1–13.
[8] G. Chen and J. Lü, Dynamics of the Lorenz system family: Analysis, control and synchronization, SciencePress, Beijing.
[9] Y. Chen, M. Yi and C. Yu, Error analysis for numerical solution of fractional differential equation by Haar wavelets method, Journal of Computational Science, 3 (2012), 367–373.
[10] A. Coronel-Escamilla, J. Gómez-Aguilar, L. Torres and R. Escobar-Jiménez, A numerical solution for a variable-order reaction-diffusion model by using fractional derivatives with non-local and non-singular kernel, Physica A: Statistical Mechanics and its Applications, 491 (2018), 406–424.
[11] A. Coronel-Escamilla, J. Gómez-Aguilar, L. Torres, R. Escobar-Jiménez and M. Valtierra-Rodríguez, Synchronization of chaotic systems involving fractional operators of Liouville–Caputo type with variable-order, Physica A: Statistical Mechanics and its Applications, 487 (2017), 1–21.
[12] A. Coronel-Escamilla, J. F. Gómez-Aguilar, D. Baleanu, T. Córdova-Fraga, R. F. Escobar-Jiménez, V. H. Olivares-Peregrino and M. M. A. Qurashi, Bateman–feshbach tikhochinsky and caldirola–kanai oscillators with new fractional differentiation, Entropy, 19 (2017), 55.
Figure 3. Plot representing multi-dimensional simulations of the saturated system (1) with $\alpha = 0.9$ (fractional case). The same dynamics as Fig. 2 are shown with a one-dimensional chaotic attractor with six scrolls in (a), two-dimensional saturated chaotic attractor with a grid of $6 \times 6$ scrolls in (b) and a three-dimensional saturated chaotic attractor with a grid of $6 \times 6 \times 6$ scrolls in (c).

[13] E. F. Doungmo Goufo and J. J. Nieto, Attractors for fractional differential problems of transition to turbulent flows, *Journal of Computational and Applied Mathematics*, 339 (2018), 329–342.

[14] E. F. Doungmo Goufo, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Bergers equation, *Mathematical Modelling and Analysis*, 21 (2016), 188–198.

[15] E. F. Doungmo Goufo, Chaotic processes using the two-parameter derivative with nonsingular and non-local kernel: Basic theory and applications, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26 (2016), 084305, 10pp.

[16] E. F. Doungmo Goufo, Stability and convergence analysis of a variable order replicator–mutator process in a moving medium, *Journal of Theoretical Biology*, 403 (2016), 178–187.

[17] E. F. Doungmo Goufo, Solvability of chaotic fractional systems with 3D four-scroll attractors, *Chaos, Solitons & Fractals*, 104 (2017), 443–451.

[18] E. F. Doungmo Goufo and A. Atangana, Analytical and numerical schemes for a derivative with filtering property and no singular kernel with applications to diffusion, *The European Physical Journal Plus*, 131 (2016), 269.

[19] G. Fubini, Opere scelte. II, Cremonese, Roma, 1958.
[20] J. Gómez-Aguilar and A. Atangana, New insight in fractional differentiation: Power, exponential decay and Mittag-Leffler laws and applications, The European Physical Journal Plus, 132 (2017), 13.

[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, Elsevier Science Limited, 2006.

[22] U. Lepik and H. Hein, Haar Wavelets: With Applications, Springer Science & Business Media, 2014.

[23] J. Lu, G. Chen, X. Yu and H. Leung, Design and analysis of multiscroll chaotic attractors from saturated function series, IEEE Transactions on Circuits and Systems I: Regular Papers, 51 (2004), 2476–2490.

[24] J. Lü, F. Han, X. Yu and G. Chen, Generating 3-D multi-scroll chaotic attractors: A hysteresis series switching method, Automatica, 40 (2004), 1677–1687.

[25] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, 1st edition, John Wiley & Sons, Inc, 1993, 1993.

[26] K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, 1st edition, Academic Press, Inc, 1974.

[27] I. Podlubny, The laplace transform method for linear differential equations of the fractional order, Website arxiv.org/pdf/func-an/9710005.pdf, 1997.

[28] S. Pooseh, H. S. Rodrigues and D. F. Torres, Fractional derivatives in dengue epidemics, in AIP Conference Proceedings, 1389 (2011), 739–742.

[29] J. Prüss, Evolutionary Integral Equations and Applications, vol. 87, Birkhäuser, 2013.

[30] J. A. Suykens and J. Vandewalle, Generation of n-double scrolls (n= 1, 2, 3, 4,...), IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 40 (1993), 861–867.

[31] W. K. Tang, G. Zhong, G. Chen and K. Man, Generation of n-scroll attractors via sine function, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 48 (2001), 1369–1372.

[32] L. Tonelli, Sull’integrazione per parti, Rend. Acc. Naz. Lincei, 5 (1909), 246–253.

[33] M. ur Rehman and R. A. Khan, The Legendre wavelet method for solving fractional differential equations, Communications in Nonlinear Science and Numerical Simulation, 16 (2011), 4163–4173.

[34] M. E. YALÇİN, J. A. Suykens, J. Vandewalle and S. Özoğuz, Families of scroll grid attractors, International Journal of Bifurcation and Chaos, 12 (2002), 23–41.

[35] C. Zuñiga-Aguilar, J. Gómez-Aguilar, R. Escobar-Jiménez and H. Romero-Ugalde, Robust control for fractional variable-order chaotic systems with non-singular kernel, The European Physical Journal Plus, 133 (2018), 13.

[36] C. Zuñiga-Aguilar, H. Romero-Ugalde, J. Gómez-Aguilar, R. Escobar-Jiménez and M. Valtierra-Rodríguez, Solving fractional differential equations of variable-order involving operators with Mittag-Leffler kernel using artificial neural networks, Chaos, Solitons & Fractals, 103 (2017), 382–403.

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