Weighted Hurwitz numbers and hypergeometric \( \tau \)-functions: an overview\(^\ast\)

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Abstract

This is a survey of recent developments in the use of 2D Toda \( \tau \)-functions of hypergeometric type as generating functions for multiparametric families of weighted Hurwitz numbers. Such \( \tau \)-functions are obtained by using diagonal group elements in their expression as fermionic vacuum expectation values, implying that their expansion in a basis of tensor products of Schur functions is diagonal. A corresponding abelian group action on the center of the \( S_n \) group algebra is defined by evaluating symmetric functions formed multiplicatively from a weight generating function \( G(z) \) on the Jucys-Murphy elements of the group algebra. The resulting central elements act diagonally in the basis of orthogonal idempotents, with eigenvalues \( r^G(z) \) that coincide with the coefficients in the double Schur function expansion. The group action is represented in the basis of cycle sums by matrices whose elements, expanded as power series in \( z \), are the weighted double Hurwitz numbers. Both their geometrical meaning, as weighted sums over \( n \)-sheeted branched coverings, and combinatorial one, as weighted enumeration of paths in the Cayley graph of \( S_n \) generated by transpositions, follow from expanding the Cauchy-Littlewood generating functions over dual pairs of bases of the algebra of symmetric functions and evaluating on the Jucys-Murphy elements. It follows that the coefficients in the expansion of the \( \tau \)-function in the basis of products of power sum symmetric functions are the weighted Hurwitz numbers. All previously studied cases are obtained by making suitable choices for \( G(z) \). Expansion in powers of some of the parameters determining the weighting provide generating series for multi-species weighted Hurwitz numbers. Replacement of the Cauchy-Littlewood generating function by the one for Macdonald polynomials provides \((q,t)\)-deformations that are generating functions for quantum weighted Hurwitz numbers.

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## Contents

1. **Hurwitz numbers**
   1.1 Enumerative geometrical Hurwitz numbers ............................... 6
   1.2 Combinatorial Hurwitz numbers ............................................ 7

2. **mKP and 2D Toda \( \tau \)-functions**
   2.1 Fermionic Fock space ...................................................... 7
   2.2 Abelian group actions, mKP and 2D Toda lattice \( \tau \)-functions and Hirota relations ....................................................... 9
   2.3 Bose-Fermi equivalence and Schur function expansions ................. 10
   2.4 Hypergeometric \( \tau \)-functions and convolution symmetries .......... 11

3. **The center \( Z(C[S_n]) \) of the \( S_n \) group algebra and symmetric functions**
   3.1 The \( \{C_\mu\} \) and \( \{F_\lambda\} \) bases .................................. 12
   3.2 The characteristic map ..................................................... 13
   3.3 Combinatorics of Hurwitz numbers and the Frobenius-Schur formula ... 14
   3.4 Jucys-Murphy elements, central elements and weight generating functions .................................................. 14
   3.5 Bose-Fermi equivalence and \( \oplus_{n \in \mathbb{N}} Z(C[S_n]) \) .................. 16

4. **Hypergeometric \( \tau \)-functions as generating functions for weighted Hurwitz numbers**
   4.1 The Cauchy-Littlewood formula and dual bases for \( \Lambda \) ............. 17
   4.2 Multiplication by \( m_\lambda(J) \) and \( e_\lambda(J) \) in the \( C_\mu \) basis .................. 20
   4.3 Weighted double Hurwitz numbers: enumerative geometric and combinatorial .................................................. 21
   4.4 Hypergeometric 2D Toda \( \tau \)-functions as generating functions for weighted Hurwitz number .................................................. 22

5. **Examples of weighted double Hurwitz numbers**
   5.1 Double Hurwitz numbers for simple branchings; enumeration of \( d \)-step paths in the Cayley graph with equal weight \([42,44]\) .................. 24
   5.2 Coverings with three branch points (Belyi curves): strongly monotonic paths \([2,18,26,33,52]\) .................................................. 25
   5.3 Fixed number of branch points and genus: multimonomonic paths \([26]\) ........ 26
   5.4 Signed Hurwitz numbers at fixed genus: weakly monotonic paths \([12,13,17]\) . 28
   5.5 Quantum weighted branched coverings and paths \([18]\) .................. 30

6. **Multispecies weighted Hurwitz numbers**
   6.1 Hybrid signed Hurwitz numbers at fixed genus: hybrid monotonic paths \([17,18,26]\) .................................................. 33
1 Hurwitz numbers

The study of Hurwitz numbers, which enumerate branched covers of the Riemann sphere with specified ramification profiles, began with the pioneering work of Hurwitz [27, 28]. Their relation to enumerative factorization problems in the symmetric group and irreducible characters was developed by Frobenius [19, 20] and Schur [46]. In recent years, following the discovery by Pandharipande [44] and Okounkov [42] that certain KP and 2D Toda $\tau$-functions [49–51], fundamental to the modern theory of integrable systems [45, 47], could serve as generating functions for weighted Hurwitz numbers, there has been a flurry of activity [1–4, 12, 13, 17, 18, 22, 23, 26, 33, 40, 41, 52] concerned with finding new classes of $\tau$-functions that can similarly serve as generating functions for various types of weighted Hurwitz numbers.

Two closely related interpretations of these weighted Hurwitz numbers exist. The enumerative geometrical one consists of weighted sums of Hurwitz numbers for $n$-sheeted branched coverings of the Riemann sphere. The other consists of weighted enumeration of factorizations of elements of the symmetric group $S_n$ in which the factors are in specified conjugacy classes. This may equivalently be interpreted as a weighted counting of paths in the Cayley graph generated by transpositions, starting and ending in specified classes. The two approaches are related by the monodromy representation of the fundamental group of the sphere punctured at the branch points obtained by lifting closed paths to the covering surface. Variants of this also exist for branched coverings of higher genus surfaces [35] and other groups.

Some generating functions of enumerative invariants are known to also have representations as matrix integrals [2–4, 12, 13, 17, 40, 41]. These include, in particular, the well-known Harish-Chandra-Itzykson-Zuber (HCIZ) integral [21, 29], which plays a fundamental rôle both in representation theory and in coupled matrix models. In [12, 13, 17], it was shown...
that when the Toda flow parameters are equated to the trace invariants of a pair of $N \times N$ hermitian matrices, and the expansion parameter is equated to $-1/N$, this gives the generating function for the enumeration of weakly monotonic paths in the Cayley graph with a fixed number of steps while, geometrically, it coincides with signed enumeration of branched coverings of fixed genus and variable numbers of branch points \cite{18, 26}. Other matrix integrals give “hybrid” paths consisting of both weakly and strongly monotonic segments or, equivalently, enumeration of coverings with multispecies “coloured” branch points \cite{17, 26}. Certain of these may also be shown to satisfy differential constraints, the so-called \textit{Virasoro constraints} \cite{33, 37, 52}, due to reparametrization invariance, and \textit{loop equations} \cite{2–4, 6} following from the structure of the underlying matrix integrals.

These, and other generating functions for various enumerative, topological, combinatorial and geometrical invariants related to Riemann surfaces, such as intersection numbers \cite{34}, higher Gromov-Witten invariants, Hodge numbers \cite{31, 32}, knot invariants \cite{5, 38}, and a growing number of other cases, can be placed into the \textit{topological recursion} scheme \cite{8–10, 15}, which aims at determining the generating functions through algorithmic recursion sequences stemming from an underlying \textit{spectral curve} \cite{4, 33}. This has turned out to be a very effective approach to a broad class of examples.

However not all such generating functions are known to be $\tau$-functions in the usual sense of integrable systems, nor partition functions or correlaters for matrix models. It remains something of a mystery exactly which class of invariants is amenable to such a representation. A further remarkable fact is that, in some cases, different generating functions corresponding to distinct enumerative problems, such as Hurwitz numbers and Hodge integrals, may be $\tau$-functions that are related through algebraic transformations that themselves involve the spectral curve \cite{31, 32}.

The present work is concerned solely with the case of Hurwitz numbers, but in the generalized sense, allowing infinite parametric families of weightings. It provides a unified approach encompassing all cases of weighted Hurwitz numbers that have appeared to date, interpreting these as special cases of an infinite parametric family of weighting functions determining mKP or 2D-Toda $\tau$-functions of generalized hypergeometric type. The parameters serve to specify the particular weighting used when summing over the various configurations. Their values are determined by a “weight generating” function $G(z)$, and define the weighting by evaluation of the standard bases $(e_\lambda, h_\lambda, m_\lambda, f_\lambda)$, for the space $\Lambda$ of symmetric functions in an infinite number of indeterminates \cite{36} consisting of \textit{elementary, complete, monomial} and \textit{forgotten} symmetric functions, respectively, at the given set of parameters $(c_1, c_2, \ldots)$ determined by $G(z)$. The other two standard bases, the Schur functions $\{s_\lambda\}$ and the power sum symmetric functions $\{p_\mu\}$, serve as bases for expansions of the $\tau$-function, in which the coefficients in the first are diagonal and of \textit{content product} form, guaranteeing that the Hirota bilinear equations of the integrable hierarchy are satisfied, while those in the second provide the weighted Hurwitz numbers.
Besides the various “classically weighted” cases, arising through different choices of the parameters $(c_1, c_2, \ldots)$, there are also “quantum deformations”, depending on an additional pair $(q, t)$ that are closely linked to the MacDonald symmetric functions [36]. This leads to the notion of ‘quantum weighted” Hurwitz numbers, of various types [18, 23], which may depend both on the infinity of classical weighting parameters $(c_1, c_2, \ldots)$, and the further pair $(q, t)$, in a specific way, involving $q$-deformations. Another generalization consists of introducing multiple expansion parameters $(z_1, z_2, \ldots)$, leading to generating functions for weighted “multispecies” weighted Hurwitz numbers [22], which are counted with different weighting factors, depending on the species type, or “colour”.

In Section 2, a quick review is given of the fermionic approach to $\tau$-functions for the KP hierarchy and modified KP sequence of $\tau$-functions as introduced by Sato, [45, 47] as well as the 2D Toda case introduced in [49–51]. Section 3 recalls basic notions regarding the $S_n$ group algebra, including the commuting Jucys-Murphy elements [30, 39], Frobenius’ characteristic map from the center $\mathbb{Z}(C[S_n])$ to the algebra $\Lambda$ of symmetric functions, and the abelian group within $\mathbb{Z}(C[S_n])$ that is generated through a combination of these. Section 4 gives a summary of the new approach to the construction of $\tau$-functions of hypergeometric type interpretable as generating functions for infinite parametric families of weighted Hurwitz numbers developed in [17, 18, 22, 23, 26]. The weightings are interpreted both geometrically, as weighted enumeration of $n$-sheeted branched covers of the Riemann sphere, and combinatorially, as weighted enumeration of paths in the Cayley graph of $S_n$ generated by transpositions. The relation between these is easily seen algebraically through the Cauchy-Littlewood generating functions for dual pairs of bases for $\Lambda$.

Section 5 is devoted to the various examples that have so far been considered in the literature. These include: the original case of single and double Hurwitz numbers, generated by the special KP and 2D Toda $\tau$-functions studied by Pandharipande and Okounkov [42, 44]; the case of the HCIZ integral [12, 13, 17, 21, 29], which is known to have the combinatorial interpretation of counting weakly monotonically increasing paths of transpositions in the Cayley graph, to which is added the geometrical one of signed enumeration of branched coverings with an arbitrary number of branch points with arbitrary branching profiles, at fixed genus; another case [17], which counts strongly monotonic such paths, and can be related to the special case of counting Belyi curves [2, 3, 33, 52] (with three branch points) or “Dessins d’enfants”; and a hybrid case [17], which combines the two, and counts branching configurations of multiple “colour” type and, moreover also has a matrix model representation. More general “multispecies” branched coverings, with their associated combinatorial equivalents [22, 26], and other, more general parametric families of weighted Hurwitz numbers are considered in Section 6.

Already in the “classical” setting, it is possible to select the parameters $(c_1, c_2, \ldots)$ appearing in the associated weight generating functions in such a way that the resulting weightings, both for branched coverings and for paths, involve what may be interpreted as a
quantum deformation parameter $q$. When suitably interpreted in terms of Planck’s constant $\hbar$ and temperature, the resulting distributions can be related to the energy distribution law for a Bose gas with linear energy spectrum. In Section 7, we extend the family of weight generating functions by introducing a further pair $(q,t)$ of deformation parameters that play the same role as those appearing in the MacDonald symmetric functions [36], with the Cauchy-Littlewood generating functions replaced by the corresponding one for Macdonald functions [23]. The resulting weighted Hurwitz numbers are interpretable as multispecies quantum Hurwitz numbers, whose distributions are again related to those for a Bosonic gas. Various specializations are obtained by choosing specific values for the parameters $q$ and $t$, or relations between them, or various limits. Besides recovering the “classical” weighting, for $q = t$, this leads to various other specializations, such as weightings involving the quantum analog of the elementary and complete symmetric functions, the Hall-Littlewood polynomials and the Jack polynomials.

### 1.1 Enumerative geometrical Hurwitz numbers

For any set of partitions $\{\mu^{(1)}, \ldots, \mu^{(k)}\}$ of $n \in \mathbb{N}^+$, we define the geometrical Hurwitz number $H(\mu^{(1)}, \ldots, \mu^{(k)})$ to be the number of $n$-sheeted branched coverings of the Riemann sphere having no more than $k$ branch points $\{q_1, \ldots, q_k\}$, with ramification profiles of type $\{\mu^{(i)}\}$, weighted by the inverse of the order of their automorphism groups. The Frobenius-Schur formula [19,20,35,46] expresses these in terms of the irreducible characters $\chi_\lambda(\mu^{(i)})$ of the symmetric group $S_n$

$$H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda, |\lambda|=n} h_\lambda^{k-2} \prod_{i=1}^{k} z_{\mu^{(i)}}^{-1} \chi_\lambda(\mu^{(i)}) \quad (1.1)$$

where $\lambda$ is the partition corresponding to the irreducible representation with Young symmetrizer of type $\lambda$, and the parts of the partitions $\{\mu^{(i)}\}$ are the cycle lengths defining the ramifications profiles that determine the conjugacy classes $\text{cyc}(\mu^{(i)})$ on which $\chi_\lambda$ is evaluated. Here

$$z_\mu = \prod_{i=1}^{\ell(\mu)} i^{m_i(\mu)} (m_i(\mu))! \quad (1.2)$$

is the order of the stabilizer of any element in $\text{cyc}(\mu)$ under conjugation, where $m_i(\mu)$ is the number of parts of $\mu$ equal to $i$ and

$$h_\lambda := \det \left( \frac{1}{(\lambda_i - i + j)!} \right)^{-1} \quad (1.3)$$

is the product of the hook lengths of the partition $\lambda$. 


1.2 Combinatorial Hurwitz numbers

The combinatorial definition of the Hurwitz number, denoted \(F(\mu^{(1)}, \ldots, \mu^{(k)})\) (and perhaps more aptly called the \textit{Frobenius number}, although the two turn out to be equal!) is the following: \(n!F(\mu^{(1)}, \ldots, \mu^{(k)})\) is the number of ways the identity element \(I \in S_n\) may be factorized into a product

\[ I = g_1 \cdots g_k, \]  

(1.4)
in which the \(i\)th factor \(g_i \in S_n\) is in the conjugacy class \(\text{cyc}(\mu^{(i)})\). The equality of these two quantities

\[ F(\mu^{(1)}, \ldots, \mu^{(k)}) = H(\mu^{(1)}, \ldots, \mu^{(k)}) \]  

(1.5)
follows from the monodromy representation of the fundamental group \(\pi_1(\mathbb{CP}^1/\{q_1, \ldots, q_k\})\) of the punctured sphere with the branch points removed [35, Appendix A].

As shown in Section 3.5, relation (1.1) follows from (1.5) and the Frobenius character formula. Avatars of this equality will be seen to recur repeatedly in the various versions of weighted Hurwitz numbers studied below.

2 mKP and 2D Toda \(\tau\)-functions

2.1 Fermionic Fock space

The fermionic Fock space \(\mathcal{F}\) is defined [47] as the semi-infinite wedge product space

\[ \mathcal{F} := \Lambda^\infty/2\mathcal{H} \]  

(2.1)
constructed from a separable Hilbert space \(\mathcal{H}\) with orthonormal basis \(\{e_i\}_{i \in \mathbb{Z}}\), that is split into an orthogonal direct sum of two subspaces

\[ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \]  

(2.2)
where

\[ \mathcal{H}_- = \text{span}\{e_i\}_{i \in \mathbb{N}}, \quad \mathcal{H}_+ = \text{span}\{e_{-i}\}_{i \in \mathbb{N}^+}. \]  

(2.3)
and \(\{e_i\}_{i \in \mathbb{Z}}\) is an orthonormal basis.

Remark 2.1. The curious convention of using negative \(i\)'s to label the basis for \(\mathcal{H}_+\) and positive ones for \(\mathcal{H}_-\) stems from the notion of the “Dirac sea”, in which all negative energy levels are filled and all positive ones empty, where the integer lattice is identified with the energies. If we take Segal and Wilson’s [48] model for \(\mathcal{H}\)

\[ \mathcal{H} := L^2(S^1) = \text{span}\{z^i\}_{i \in \mathbb{Z}} \]  

(2.4)
with \(e_i := z^{-i-1}\),

we may view \(\mathcal{H}_+\) and \(\mathcal{H}_-\) either as the subspaces of positive and negative Fourier series on the circle \(S^1\) or, equivalently, the Hardy spaces of square integrable functions admitting a holographic extension to inside and outside the unit circle, with the latter vanishing at \(z = \infty\).
\( \mathcal{F} \) is the graded sum

\[
\mathcal{F} = \bigoplus_{N \in \mathbb{Z}} \mathcal{F}_N \tag{2.5}
\]

of the subspaces \( \mathcal{F}_N \) with fermionic charge \( N \in \mathbb{Z} \). An orthonormal basis \( \{|\lambda; N\rangle\} \) for these is provided by the semi-infinite wedge product states

\[
|\lambda; N\rangle := e_{\ell_1} \wedge e_{\ell_2} \wedge \cdots \tag{2.6}
\]

labeled by pairs of partitions \( \lambda \) and integers \( N \in \mathbb{Z} \), where

\[
\{\ell_i := \lambda_i - i + N\} \tag{2.7}
\]

are the “particle coordinates”, indicating the occupied points on the integer lattice, corresponding to the parts of the partition \( \lambda \), with the usual convention that, for \( i \) greater than the length \( \ell(\lambda) \) of the partition, \( \lambda_i := 0 \). The vacuum state in the charge \( N \) sector \( \mathcal{F}_N \) of the Fock space is denoted

\[
|N\rangle := |0; N\rangle. \tag{2.8}
\]

In Segal and Wilson’s [48] sense, the image \( \mathcal{P}(W) \), under the Plücker map

\[
\mathcal{P} : Gr_{\mathcal{H}^+} (\mathcal{H}) \to \mathcal{P}(\mathcal{F})
\]

\[
\mathcal{P} : W \mapsto \mathcal{P}(W)
\]

\[
\mathcal{P} : \text{span}\{w_i \in \mathcal{H}\}_{i \in \mathbb{N}^+} \mapsto [w_1 \wedge w_2 \wedge \cdots], \tag{2.9}
\]

of an element \( W \in Gr_{\mathcal{H}^+} (\mathcal{H}) \) of the infinite Grassmannian modeled on \( \mathcal{H}^+ \subset \mathcal{H} \), having virtual dimension \( N \) (i.e., such that the Fredholm index of the orthogonal projection map \( \pi^\perp : W \to \mathcal{H}_+ \) is \( N \)) is in the charge \( N \) sector \( \mathcal{P}(W) \in \mathcal{F}_N \subset \mathcal{F} \), and the entire image consists of all decomposable elements of \( \mathcal{F} \). In particular, \( \mathcal{H}^+ \) is mapped to the projectivization of the vacuum element

\[
\mathcal{P} : \mathcal{H}^+ \mapsto [|0\rangle] := [|0; 0\rangle] = [e_{-1} \wedge e_{-2} \wedge \cdots]. \tag{2.10}
\]

The Fermi creation and annihilation operators \( \psi_i, \psi^\dagger_i \) are defined as exterior multiplication by the basis element \( e_i \) and interior multiplication by the dual basis element \( \bar{e}^i \), respectively.

\[
\psi := e_i \wedge \quad \psi^\dagger := i(\bar{e}^i). \tag{2.11}
\]

These satisfy the usual anticommutation relations

\[
[\psi_i, \psi^\dagger_j]_+ = \delta_{ij} \tag{2.12}
\]

defining the corresponding Clifford algebra on \( \mathcal{H} + \mathcal{H}^* \) with respect to the natural quadratic form in which both \( \mathcal{H} \) and \( \mathcal{H}^* \) are totally isotropic.
The infinite general linear algebra $\mathfrak{gl}(\mathcal{H}) \subset \Lambda^2(\mathcal{H} + \mathcal{H}^*)$, in the standard Clifford representation, is spanned by the elements $\psi_i \psi_j^\dagger$, with the usual convention for normal ordering:

$$\psi_i \psi_j^\dagger := \langle \psi_i \psi_j \rangle,$$

(2.13)

where $\langle O \rangle$ denotes the vacuum expectation value

$$\langle O \rangle := \langle 0 | O | 0 \rangle.$$

(2.14)

The corresponding group $\mathfrak{Gl}(\mathcal{H})$ consists of invertible endomorphisms, having well defined determinants. (See \[45, 47, 48\] for more detailed definitions.)

A typical exponentiated element in the Clifford representation is of the form

$$\hat{g} = e^{\sum_{ij \in \mathbb{Z}} A_{ij} \psi_i \psi_j^\dagger},$$

(2.15)

where the doubly infinite square matrix with elements $A_{ij}$ satisfies suitable convergence conditions \[45, 45, 47, 48\] that will not be detailed here.

### 2.2 Abelian group actions, mKP and 2D Toda lattice $\tau$-functions and Hirota relations

The KP and 2D-Toda flows are generated by the multiplicative action on $\mathcal{H}$ of the two infinite abelian subgroups $\Gamma_{\pm} \subset \mathfrak{Gl}_0(\mathcal{H})$ of the identity component $\mathfrak{Gl}_0(\mathcal{H})$ of the general linear group $\mathfrak{Gl}(\mathcal{H})$, defined by:

$$\Gamma_{\pm} := \{ \gamma_{\pm}(t) := e^{\sum_{i=1}^{\infty} t_i z^i} \}, \quad \text{and} \quad \Gamma := \{ \gamma_{\pm}(s) := e^{\sum_{i=1}^{\infty} s_i z^{-i}} \},$$

(2.16)

where $t = (t_1, t_2, \ldots)$ is an infinite sequence of (complex) flow parameters corresponding to one-parameter subgroups, and $s = (s_1, s_2, \ldots)$ is a second such sequence. These in turn have the following Clifford group representations on $\mathcal{F}$

$$\hat{\Gamma}_{\pm} := \{ \hat{\gamma}_{\pm}(t) := e^{\sum_{i=1}^{\infty} t_i J_i} \}, \quad \text{and} \quad \hat{\Gamma} := \{ \hat{\gamma}_{\pm}(s) := e^{\sum_{i=1}^{\infty} s_i J_i^{-1}} \},$$

(2.17)

where

$$J_i := \sum_{k \in \mathbb{Z}} : \psi_i \psi_{i+k}^\dagger : \quad \pm i \in \mathbb{N}^+$$

(2.18)

are referred to as the “current components”. In this infinite dimensional setting, whereas the abelian groups $\Gamma_{\pm}$ commute, their Clifford representations $\hat{\Gamma}_{\pm}$ involve a central extension, so that

$$\hat{\gamma}_{\pm}(t) \hat{\gamma}_{\pm}(s) = \hat{\gamma}_{\pm}(s) \hat{\gamma}_{\pm}(t) e^{\sum_{i \in \mathbb{Z}} i t_i s_i}.$$
The mKP-chain and 2D-Toda $\tau$-functions corresponding to the element $g \in \mathfrak{gl}_0(\mathcal{H})$ are given, within a nonzero multiplicative constant, by the vacuum expectations values (VEV’s)  

$$
\tau^m_{g}(N, t) := \langle N | \hat{\gamma}_+(t) \hat{g} | N \rangle , \\
\tau^{(2\text{Toda})}_{g}(N, t, s) := \langle N | \hat{\gamma}_+(t) \hat{\gamma}_-(s) | N \rangle .
$$

(2.20)

(2.21)

If the group element $g \in \mathfrak{gl}_0(\mathcal{H})$ is interpreted, relative to the $\{ e_i \}_{i \in \mathbb{Z}}$ basis, as a matrix exponential $g = e^A$, where the algebra element $A \in \mathfrak{gl}(\mathcal{H})$ is represented by the infinite matrix with elements $\{ A_{ij} \}_{i, j \in \mathbb{Z}}$, then the corresponding representation of $GL(\mathcal{H})$ on $\mathcal{F}$ is given by

$$
\hat{g} := e^{\sum_{i, j \in \mathbb{Z}} A_{ij} \psi_i \psi_j^\dagger} .
$$

(2.22)

These satisfy the Hirota bilinear relations

$$
\oint_{z = \infty} z^{N'-N} e^{-\xi(\delta t, z)} \tau_{g}^m(\mathcal{H}) (N, t + \delta t + [z^{-1}]) \tau_{g}^m(N', t - [z^{-1}]) = 0
$$

(2.23)

understood to hold identically in $\delta t = (\delta t_1, \delta t_2, \ldots)$, $\delta s := (\delta s_1, \delta s_2, \ldots)$, where  

$$
[z]_i := \frac{1}{i^2} .
$$

(2.25)

### 2.3 Bose-Fermi equivalence and Schur function expansions

It follows from the identities [45, 47]  

$$
\langle N | \hat{\gamma}_+(t) | \lambda ; N \rangle = \langle \lambda ; N | \hat{\gamma}_-(t) | N \rangle = s_\lambda(t),
$$

(2.26)

where $s_\lambda$ is the Schur function corresponding to partition $\lambda$, viewed as function of the parameters  

$$
t_i := \frac{p_i}{t} ,
$$

(2.27)

where the $p_i$’s are the power sums, that the $\tau$-functions may be expressed, at least formally, as single and double Schur functions expansions

$$
\tau_{g}^m(N, t) = \sum_{\lambda} \pi_{\lambda}(N, g) s_\lambda(t) \\
\tau_{g}^{(2\text{Toda})}(N, t, s) = \sum_{\lambda} \sum_{\mu} B_{\lambda\mu}(N, g) s_\lambda(t) s_\mu(t)
$$

(2.28)

(2.29)

where  

$$
\pi_{\lambda}(N, g) := \langle \lambda ; N | \hat{g} | N \rangle , \quad B_{\lambda\mu}(N, g) := \langle \lambda ; N | \hat{g} | \mu ; N \rangle ,
$$

(2.30)
are the Plücker coordinates of the elements \( \hat{g} | N \) and \( \hat{g} | \mu; N \) when \( g \in \text{GL}_0(\mathcal{H}) \) is in the identity component of \( \text{GL}(\mathcal{H}) \). The Hirota bilinear relations (2.23), (2.24) are then equivalent to the Plücker relations satisfied by these coefficients.

The “Bose-Fermi equivalence” gives an isomorphism between a completion \( \mathcal{B}_0 \) of the space of symmetric functions \( \Lambda \) of an infinite number of “bosonic” variables \( \{x_i\}_{i \in \mathbb{N}} \), labelled by the natural numbers and the \( N = 0 \) (zero charge) sector of the Fermionic Fock space \( \mathcal{F}_0 \subset \mathcal{F} \) which identifies the basis states \( \{|\lambda; 0\rangle\} \) with the basis of Schur functions \( \{s_\lambda \in \Lambda\} \) through the “bosonization” map:

\[
\mathcal{B} : \mathcal{F}_0 \to \mathcal{B}_0 \quad \mathcal{B} : |v\rangle \mapsto \langle 0 | \hat{\gamma}_+ |v\rangle \\
\mathcal{B} : |\lambda; 0\rangle \mapsto s_\lambda.
\] (2.31)

More generally, this can be extended to the full (graded) fermionic Fock space \( \mathcal{F} = \bigoplus_{N \in \mathbb{Z}} \mathcal{F}_N \) by adding a parameter \( \zeta \) to the Bosonic Fock space, taking formal Laurent expansions in this

\[
\mathcal{B} := \mathcal{B}_0[[\zeta]],
\] (2.32)

and defining

\[
\mathcal{B} : \mathcal{F}_N \to \mathcal{B}_N \\
\mathcal{B} : |v\rangle \mapsto \langle N | \hat{\gamma}_+ |v\rangle \zeta^N
\] (2.33)

Using \( \mathcal{B} \) as an intertwining map, this defines identifications between operators in \( \text{End}(\mathcal{F}) \) and those in \( \text{End}(\mathcal{B}) \). However, what appears in the Fermionic representation as a “locally” defined element of the Clifford algebra or group is in general a nonlocal operator in the Bosonic representation (involving exponentiated differential operators in terms of the \( t \) co-ordinates), as is the case, e.g., for the Bosonic representations of the operators \( \psi, \psi^\dagger \), which are special types of “vertex operators”. In particular, the Bosonization of fermionic states of the type \( \hat{g} \hat{\gamma}_- |0\rangle \) is given by application of nonlocal operators of the type that were interpreted in [14] as “cut-and-join” operators, to the gauge transform of the vacuum state, defined by \( \hat{\gamma}_- |0\rangle \)

### 2.4 Hypergeometric \( \tau \)-functions and convolution symmetries

A special subfamily of the above consists of those \( \tau \)-functions for which the group element \( \hat{g} \) is diagonal

\[
\hat{g} = e^{\sum_{i \in \mathbb{Z}} T_i \psi_i \psi_i^\dagger}, \quad A_{ij} = T_i \delta_{ij}
\] (2.34)

in the basis \( |\lambda; N\rangle \). These were named \textit{convolutions symmetries} in [25], since in the Segal-Wilson representations of \( \text{GL}(\mathcal{H}) \) they may be interpreted as (generalized) convolution products on \( \mathcal{H} \sim L^2(S_1) \). Their eigenvalues \( r_\lambda(N, g) \) in the basis \( |\lambda; N\rangle \)

\[
e^{\sum_{i \in \mathbb{Z}} T_i \psi_i \psi_i^\dagger} |\lambda; N\rangle = r_\lambda(N, g) |\lambda; N\rangle
\] (2.35)
can be written in the form of a content product [25, 43]:

$$r_\lambda(N, g) := r_0(N, g) \prod_{(i, j) \in \lambda} r_{N+j-i}(g), \quad r_i(g) := e^{T_i - T_{i-1}}$$

(2.36)

where

$$r_0(N, g) := \begin{cases} 
\prod_{i=0}^{N-1} e^{T_i} & \text{if } N > 0 \\
1 & \text{if } N = 0 \\
\prod_{i=N}^{-1} e^{-T_i} & \text{if } N < 0.
\end{cases}$$

(2.37)

The double Schur function expansion (2.21) in this case reduces to the diagonal form

$$\tau^{2\text{Toda}}_g(N, t, s) = \sum_\lambda r_\lambda(N, g)s_{\lambda}(t)s_{\lambda}(s).$$

(2.38)

If we view the second set of parameters \((c_1, c_2, \ldots)\) as fixed, and consider only the first set \((1, 1_2, \ldots)\) as KP flow parameters, we may interpret (2.38) as defining a chain of mKP \(\tau\) functions. A specific value of special interest is \((c_1, c_2, \ldots) = (1, 0, 0, \ldots)\), for which the Schur function evaluates to

$$s_{\lambda}(1, 0, \ldots) = h^{-1}_{\lambda}$$

(2.39)

and (2.38) reduces to

$$\tau^{mKP}_g(N, t, s) = \sum_\lambda r_\lambda(N, g)h^{-1}_{\lambda}s_{\lambda}(t).$$

(2.40)

In the following, only such hypergeometric \(\tau\)-functions will be needed. By defining suitable parametric families of the latter, and expanding these in powers of some auxiliary parameters, while leaving the others to define the weightings, It will be seen that we can interpret them as generating functions for finite or infinite parametric families of weighted Hurwitz numbers, both classical and quantum, obtaining both a natural enumerative geometric and combinatorial interpretation in all cases.

3 The center \(Z(C[S_n])\) of the \(S_n\) group algebra and symmetric functions

3.1 The \(\{C_\mu\}\) and \(\{F_\lambda\}\) bases

There are two natural bases for the center \(Z(C[S_n])\) of the group algebra of the symmetric group \(S_n\), both labelled by partitions of \(n\). The first is the basis of cycle sums \(\{C_\mu\}_{|\mu|=n}\), defined by

$$C_\mu := \sum_{h \in \text{cyc}(\mu)} h.$$ 

(3.1)
The second is the basis of orthogonal idempotents \( \{ F_\lambda \}_{|\lambda|=n} \), which project onto the irreducible representations of type \( \lambda \) and satisfy

\[
F_\lambda F_\mu = F_\lambda \delta_{\lambda\mu}. \tag{3.2}
\]

These are related by

\[
F_\lambda = h_\lambda^{-1} \sum_{\mu, |\mu|=|\lambda|=n} \chi_\lambda(\mu)C_\mu \tag{3.3}
\]

\[
C_\mu = z_\mu^{-1} \sum_{\lambda, |\lambda|=|\mu|=n} h_\lambda \chi_\lambda(\mu)F_\lambda \tag{3.4}
\]

which is equivalent to the Frobenius character formula (see below). The main property of the \( \{ F_\lambda \} \) basis is that multiplication by any element of the center \( Z(\mathbb{C}[S_n]) \) is diagonal in this basis (as follows immediately from (3.2)).

### 3.2 The characteristic map

Frobenius’ characteristic map defines a linear isomorphism between the characters of \( S_n \) and the characters of tensor representations of \( GL(k) \), of total tensor weight \( n \), for \( k \) sufficiently large. It maps the irreducible character \( \chi_\lambda \) to the Schur function \( s_\lambda \), viewed as the corresponding \( GL(k) \) character through the Weyl character formula for any \( k \geq \ell(\lambda) \). Equivalently, it defines a linear endomorphism

\[
\text{ch} : Z(\mathbb{C}[S_n]) \rightarrow \Lambda
\]

\[
\text{ch} : F_\lambda \mapsto \frac{s_\lambda}{h_\lambda} \tag{3.5}
\]

from the center \( Z(\mathbb{C}[S_n]) \) of the group algebra to the algebra \( \Lambda \) of symmetric functions [36]. The change of basis formulae (3.3), (3.4), together with the Frobenius character formula

\[
s_\lambda = \sum_{\mu, |\mu|=|\lambda|=n} z_\mu^{-1} \chi_\lambda(\mu)p_\mu, \tag{3.6}
\]

where

\[
p_\mu := \prod_{i=1}^{\ell(\mu)} p_{\mu_i} \tag{3.7}
\]

is the power sum symmetric function, then imply that the characteristic map takes the cycle sum basis into the \( \{ p_\mu \} \) basis for \( \Lambda \)

\[
\text{ch} : C_\mu \mapsto \frac{p_\mu}{z_\mu}. \tag{3.8}
\]
3.3 Combinatorics of Hurwitz numbers and the Frobenius-Schur formula

The two bases \( \{ C_\mu \} \), \( \{ F_\lambda \} \) can be used to deduce the Frobenius-Schur formula (1.1), expressing \( H(\mu^{(1)}, \ldots, \mu^{(k)}) \) in terms of the irreducible group characters \( \chi_\lambda(\mu) \). The product \( \prod_{i=1}^{k} C_{\mu^{(i)}} \) of elements of the cycle sum basis is central and hence can be expressed relative to the same basis:

\[
\prod_{i=1}^{k} C_{\mu^{(i)}} = \sum_{\nu, |\nu| = n} H(\mu^{(1)}, \ldots, \mu^{(i)}, \nu) z_\nu C_\nu,
\]

and, in particular, the coefficient of the identity class, for which \( \mu = (1)^n \) is \( n! \) times the Hurwitz number

\[
[I = C_{(1)^n}] \prod_{i=1}^{k} C_{\mu^{(i)}} = n! H(\mu^{(1)}, \ldots, \mu^{(k)}),
\]

giving the number of factorizations of the identity element into a product of \( k \) elements within the conjugacy classes \( \{ \text{cyc}(\mu^{(i)}) \}_{i=1}^{k} \).

Substituting the change of basis formula (3.4) into (3.10), applying both sides to the basis element \( \{ F_\lambda \} \) and equating the eigenvalues that result gives the Frobenius-Schur formula:

\[
H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda, |\lambda| = |\mu| = n} \chi_\lambda(\mu^{(i)}) \prod_{i=1}^{k} \frac{1}{z_\mu^{(i)}}.
\]

3.4 Jucys-Murphy elements, central elements and weight generating functions

We now recall the special commuting elements \( \{ J_1, \ldots, J_n \} \) of the group algebra \( \mathbb{C}[S_n] \) introduced by Jucys [30] and Murphy [39]. (See also [7]). These are defined by

\[
J_b := \sum_{a=1}^{n-1} (ab) \quad \text{for } b > 1, \quad \text{and } J_1 := 0.
\]

where \( (ab) \in S_n \) is the transposition that interchanges \( a \) with \( b \). Although these are not central elements, they have two remarkable properties: Any symmetric function \( f(J_1, \ldots, J_n) \), \( f \in \Lambda_n \) formed from them is central, and this central element has eigenvalues in the \( F_\lambda \) basis that are equal to the evaluation on the content of the partition \( \lambda \); i.e. the set of number \( j - i \), where \( \{ (i, j) \in \lambda \} \) are the set of positions (in the English convention) in the Young diagram of \( \lambda \):

\[
f(J_1, \ldots, J_n) F_\lambda = f(\{ j - i \}_{(i,j) \in \lambda}) F_\lambda.
\]
A particular case of symmetric functions of $n$ variables consists of taking a single generating function $G(z)$, expressed formally either as an infinite product

$$G(z) = \prod_{i=1}^{\infty} (1 + c_i z) \quad (3.14)$$

or an infinite sum

$$G(z) = 1 + \sum_{i=1}^{\infty} G_i z^i \quad (3.15)$$

or some limit thereof, and defining the central element as a product

$$G_n(z, \mathcal{J}) := \prod_{b=1}^{n} G(z J_a). \quad (3.16)$$

(For the present, we are not concerned with whether $G(z)$ is polynomial, rational, a convergent series, in some field extension or just a formal infinite series or infinite product; the considerations that follow are mainly algebraic, but are easily extended to include either convergent series, through suitable completions, or formal series and products, as in the generating functions for symmetric functions.) When applied multiplicatively to the $\{F_\lambda\}$ basis, the central element $G_n(z, \mathcal{J})$ has eigenvalues that are expressible as content products

$$G_n(z, \mathcal{J}) F_\lambda = \prod_{(ij) \in \lambda} G(z(j - i)) F_\lambda, \quad |\lambda| = n. \quad (3.17)$$

We also consider the “dual” generating function:

$$\tilde{G}(z) := \frac{1}{G(-z)} = \prod_{i=1}^{\infty} (1 - c_i z)^{-1} \quad (3.18)$$

and associated central element

$$\tilde{G}_n(z, \mathcal{J}) := \prod_{b=1}^{n} \tilde{G}(z J_a), \quad (3.19)$$

which similarly satisfies

$$\tilde{G}_n(z, \mathcal{J}) F_\lambda = \prod_{(ij) \in \lambda} \tilde{G}(z(j - i)) F_\lambda, \quad |\lambda| = n. \quad (3.20)$$

This suggests comparison with the “convolution symmetry” elements in the fermionic representation of the group $\mathfrak{sl}(\mathcal{H})$ and an extension of the Bose-Fermi equivalence, using the characteristic map, to a correspondence between the direct sum $\oplus_{n \in \mathbb{N}} \mathbb{Z}(C[S_n])$ and the $N = 0$ sector $\mathcal{F}_0 \subset \mathcal{F}$ of the fermonic Fock space.
3.5 Bose-Fermi equivalence and $\oplus_{n \in \mathbb{N}} Z(C[S_n])$

Composing the characteristic map with the Bose-Fermi equivalence we obtain an endomorphism $\mathcal{E}$ from the direct sum $\oplus_{n \in \mathbb{N}} Z(C[S_n])$ of the centers of the group algebras to the zero charge sector $\mathcal{F}_0$ in the Fermionic Fock space

$$\mathcal{E} : \oplus_{n \in \mathbb{N}} Z(C[S_n]) \rightarrow \mathcal{F}_0$$

$$\mathcal{E} : F_\lambda \mapsto h_\lambda^{-1} |\lambda; 0\rangle$$ (3.21)

This provides an intertwining map between the central elements in the completion of the group algebra formed from products of functions of a single variable, acting by multiplications, and the convolution symmetries discussed in Section 2.4.

Choosing the parameters $T_j$ in (2.34) as

$$T^G(z)_j = \sum_{k=1}^{j} \ln G(zk), \quad T^{G(z)}_0(z) = 0, \quad T^{-G(z)}_j(z) = -\sum_{k=0}^{j-1} \ln G(-zk) \quad \text{for } j > 0. \quad (3.22)$$

so that

$$\hat{g} = \hat{C}_G := e^{\sum_{i \in \mathbb{Z}} T^G(z)_i \psi_i \psi_i^\dagger}, \quad (3.23)$$

it follows that

$$r_j(g) := r_j^{G(z)} = G(jz) \quad (3.24)$$

and

$$\hat{C}_G|\lambda; N\rangle = r_\lambda^{G(z)}(N)|\lambda; N\rangle \quad (3.25)$$

with eigenvalues

$$r_\lambda^{G(z)}(N) := r_0^{G(z)}(N) \prod_{(i,j) \in \lambda} G(z(N + j - i)), \quad (3.26)$$

where

$$r_0^{G(z)}(N) = \prod_{j=1}^{N-1} G((N - j)z)^j, \quad r_0(0) = 1, \quad r_0^{G(z)}(-N) = \prod_{j=1}^{N} G((j - N)z)^{-j}, \quad N > 1 \quad (3.27)$$

The map $\mathcal{E}$ defined in (3.21) therefore intertwines the action of $\oplus_{n \in \mathbb{N}} G_n(z, J)$ on $\oplus_{n \in \mathbb{N}} Z(C[S_n])$ with that of $\hat{C}_G$ on $\mathcal{F}_0$. The same applies to the dual generating functions $\tilde{G}(z)$, for which we obtain the corresponding content product formula expression

$$\tilde{r}_\lambda^{\tilde{G}(z)}(N) := r_0^{\tilde{G}(z)}(N) \prod_{(i,j) \in \lambda} \tilde{G}(z(N + j - i)). \quad (3.28)$$
For the following, we only have need of the $N = 0$ case, for which we simplify the notation for the content product coefficients to

$$r^G_\lambda := r^G_\lambda(0) = \prod_{(i,j) \in \lambda} G(z(j - i)), \quad (3.29)$$

$$r^{\tilde{G}}_\lambda := r^{\tilde{G}}_\lambda(0) = \prod_{(i,j) \in \lambda} \tilde{G}(z(j - i)), \quad (3.30)$$

4 Hypergeometric $\tau$-functions as generating functions for weighted Hurwitz numbers

We are now ready to state the main results, which show that the KP and 2D Toda $\tau$-functions of hypergeometric type

$$\tau^G(z)(t) = \sum_\lambda r^G_\lambda h^{-1}_\lambda s_\lambda(t), \quad (4.1)$$

$$\tau^G(z)(t, s) = \sum_\lambda r^G_\lambda s_\lambda(t)s_\lambda(s), \quad (4.2)$$

when expanded in bases of (products of) the power sum symmetric functions $\{p_\mu\}$, are interpretable as generating functions for suitably defined infinite parametric weighted Hurwitz numbers, both in the enumerative geometric and the combinatorial sense. The details and proofs may be found in [17, 18, 22, 23, 26].

4.1 The Cauchy-Littlewood formula and dual bases for $\Lambda$

We have already encountered the two bases consisting of Schur functions $\{s_\lambda\}$ and power sum symmetric functions $\{p_\lambda\}$ for the ring $\Lambda$ of symmetric functions in an arbitrary number of indeterminates [36]. In addition to these, there are four other useful bases, consisting of the products of the elementary symmetric functions

$$e_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}, \quad (4.3)$$

the complete symmetric functions

$$h_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}, \quad (4.4)$$

with generating functions

$$E(z) = \prod_{ij}(1 + zx_i) = \sum_{i=0}^{\infty} e_i z^i, \quad H(z) = \prod_{ij}(1 - z x_i)^{-1} = \sum_{i=0}^{\infty} h_i z^i, \quad (4.5)$$
the monomial sum symmetric functions

\[ m_\lambda(x) := \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} x_{i_\sigma(1)}^{\lambda_1} \cdots x_{i_\sigma(k)}^{\lambda_k}, \]  

and the “forgotten” symmetric functions

\[ f_\lambda(x) := (\frac{-1}{|\text{aut}(\lambda)|})^{\ell(\lambda)} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 \leq \cdots \leq i_k} x_{i_\sigma(1)}^{\lambda_1} \cdots x_{i_\sigma(k)}^{\lambda_k}, \]  

where

\[ x := (x_1, x_2, \ldots) \] is an infinite sequence of indeterminates, and defining \( m_i(\lambda) \) to be the number of parts of \( \lambda \) equal to \( i \),

\[ |\text{aut}(\lambda)| := \prod_{i=1}^{\ell(\lambda)} (m(\lambda_i))! \]  

is the order of the automorphism group of the conjugacy class of type \( \lambda \) under conjugation.

These bases have the following duality and orthogonality properties with respect to the standard scalar product \((,\)) in which the Schur functions are orthonormal [36]:

\[ (s_\lambda, s_\mu) = \delta_{\mu\nu}, \quad (p_\lambda, p_\mu) = z_\mu \delta_{\mu\nu}, \quad (e_\lambda, m_\mu) = \delta_{\mu\nu}, \quad (f_\lambda, h_\mu) = \delta_{\mu\nu}. \]  

It follows [36] that the Cauchy-Littlewood formula is expressible bilinearly in terms of these dually paired bases

\[ \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 - x_iy_j)^{-1} = \sum_{\lambda} s_\lambda(x)s_\lambda(y) \]  

\[ = \sum_{\lambda} z_\mu^{-1} p_\lambda(x)p_\lambda(y) \]  

\[ = \sum_{\lambda} e_\lambda(x)m_\lambda(y) = \sum_{\lambda} e_\lambda(y)m_\lambda(x) \]  

\[ = \sum_{\lambda} f_\lambda(x)h_\lambda(y) = \sum_{\lambda} f_\lambda(y)h_\lambda(x). \]  

The dual Cauchy-Littlewood generating function is similarly expressed in terms these in a
dual way [36]:

\[
\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda} s_{\lambda}(x)s_{\lambda'}(y) = \sum_{\lambda} (-1)^{\ell^*(\lambda)} \mu_1 p_{\lambda}(x)p_{\lambda}(y) = \sum_{\lambda} e_{\lambda}(x)f_{\lambda}(y) = \sum_{\lambda} h_{\lambda}(x)m_{\lambda}(y),
\]

(4.15)

(4.16)

(4.17)

(4.18)

where \( \lambda' \) is the partition whose Young diagram is the transpose of that for \( \lambda \) and

\[
\ell^*(\lambda) = |\lambda| - \ell(\lambda)
\]

(4.19)

is the colength of \( \lambda \) (i.e., the complement of its length).

We make use of these formulae in the following way: for the indeterminates \((x_1, x_2, \ldots)\) we substitute the parameters \((c_1, c_2, \ldots)\) defining the weight generating function \(G(z)\) as an infinite product, or its dual \(\tilde{G}(z)\), while for the indeterminates \((y_1, y_2, \ldots)\), we substitute the \(x\) times the Jucys-Murphy elements \((z\mathcal{J}_1, z\mathcal{J}_2, \ldots)\) up to a finite number \(n\) of these, and 0 for the rest, to obtain a finite sum in \(j\).

This gives the following central elements, expressed as sums of products of these bases, either evaluated on the contents \((c_1, c_2, \ldots)\) or the commuting elements \((\mathcal{J}_1, \mathcal{J}_2, \ldots)\):

\[
G_n(z, \mathcal{J}) = \prod_{i=1}^{n} \prod_{a=1}^{n} (1 + z c_i \mathcal{J}_a)
\]

(4.20)

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} e_{\lambda}(c)m_{\lambda}(\mathcal{J})
\]

(4.21)

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} m_{\lambda}(c)e_{\lambda}(\mathcal{J})
\]

(4.22)

and

\[
\tilde{G}_n(z, \mathcal{J}) = \prod_{i=1}^{n} \prod_{a=1}^{n} (1 - z c_i \mathcal{J}_a)^{-1}
\]

(4.23)

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} h_{\lambda}(c)m_{\lambda}(\mathcal{J})
\]

(4.24)

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} f_{\lambda}(c)e_{\lambda}(\mathcal{J}).
\]

(4.25)
Recall that the elements $G_n(z, J), \tilde{G}_n(z, J) \in \mathbb{Z}(\mathbb{C}[S_n])$ are diagonal in the basis of orthogonal idempotents, with the contact product coefficients as eigenvalues

$$G_n(z, J) F_\lambda = r_\lambda^{G(z)} F_\lambda, \quad \tilde{G}_n(z, J) F_\lambda = r_\lambda^{\tilde{G}(z)} F_\lambda$$  \tag{4.26}

where

$$r_\lambda^{G(z)} = \prod_{(ij) \in \lambda} \prod_{k=1}^{\infty} (1 + z c_k (j - i))$$  \tag{4.27}

$$r_\lambda^{\tilde{G}(z)} = \prod_{(ij) \in \lambda} \prod_{k=1}^{\infty} (1 - z c_k (j - i))^{-1}.$$  \tag{4.28}

4.2 Multiplication by $m_\lambda(J)$ and $e_\lambda(J)$ in the $C_\mu$ basis

In order to proceed further, we need to compute the effect of multiplication by $G_n(z, J)$ and $\tilde{G}_n(z, J)$ in the basis $\{C_\mu\}$ of $\mathbb{Z}(\mathbb{C}[S_n])$ consisting of cycle sums. Combinatorially, this requires the notion of the signature of a path in the Cayley graph of $S_n$ generated by the transpositions.

Definition 4.1. Given a $d$-step path in the Cayley graph of $S_n$ generated by transpositions $(ab), a, b \in \{1, \ldots, n\}, a < b$, consisting of the sequence:

$$h, \rightarrow (a_1 b_1) h \rightarrow (a_2 b_2) (a_1 b_1) h \rightarrow \ldots \rightarrow (a_d b_d) \cdots (a_1 b_1) h,$$  \tag{4.29}

its signature $\lambda$ is the partition of weight $|\lambda| = d$ whose length $\ell(\lambda)$ equals the number of distinct second elements appearing in the sequence, and whose parts $\{\lambda_i\}_{i=1, \ldots, d}$ consist of the number of times each given second element occurs.

The effect of multiplication of $C_\mu$ by the central element $m_\lambda(J) \in \mathbb{Z}(\mathbb{C}[S_n])$ is given by the following easily proved lemma [18]:

**Lemma 4.1.** Multiplication by $m_\lambda(J)$ defines an endomorphism of $\mathbb{Z}(\mathbb{C}[S_n])$ which, expressed in the $\{C_\mu\}$ basis, is given by

$$m_\lambda(J) C_\mu = \frac{1}{|\mu|!} \sum_{\nu, |\nu| = |\mu|} m^\lambda_{\mu \nu} z_\nu C_\nu,$$  \tag{4.30}

where $m^\lambda_{\mu \nu}$ is the number of monotonic $|\lambda|$-step paths in the Cayley graph of $S_n$ generated by all transpositions, starting from an element $h$ in the conjugacy class $\text{cyc}(\nu)$ to $\text{cyc}(\mu)$ with signature $\lambda$. Equivalently,

$$m^\lambda_{\mu \nu} := \prod_{i=1}^{\ell(\lambda)} \frac{\lambda_i!}{|\lambda|!} \tilde{m}^\lambda_{\mu \nu}$$  \tag{4.31}

where $\tilde{m}^\lambda_{\mu \nu}$ is the number of $|\lambda|$ step paths of signature $\lambda$ in the Cayley graph of $S_n$ generated by transpositions, starting at the conjugacy class $\text{cyc} \mu$ and ending in the class $\text{cyc} \nu$.  


On the other hand, the effect of multiplication of $C_{\mu}$ by the central element $e_\lambda(\mathcal{J}) \in Z(C[S_n])$ is given by the following [18]:

**Lemma 4.2.** Multiplication by $e_\lambda(\mathcal{J})$ defines an endomorphism of $Z(C[S_n])$ which, expressed in the $\{C_{\mu}\}$ basis, is given by

$$
e_\lambda(\mathcal{J})C_{\mu} = \sum_{\mu^{(1)},\ldots,\mu^{(k)} \in \mathcal{S}_n} \left( \prod_{i=1}^{k} C_{\mu^{(i)}} \right) C_{\mu}$$

(4.32)

$$= \sum_{\mu^{(1)},\ldots,\mu^{(k)} \in \mathcal{S}_n} H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu)z_\nu C_{\nu},$$

(4.33)

(where the identity (3.9) has been used in the second line).

### 4.3 Weighted double Hurwitz numbers: enumerative geometric and combinatorial

We now proceed to the enumerative geometrical definition of weighted Hurwitz numbers. For a fixed pair of branch points, say at $(0, \infty)$, with ramification profiles $(\mu,\nu)$ and an additional set of $k$ branch points $(q_1,\ldots,q_k)$ with ramification profiles $(\mu^{(1)},\ldots,\mu^{(k)})$, we define the weights to be given by the evaluation of the monomial sum and “forgotten” symmetric functions at the parameter values $c = (c_1,c_2,\ldots)$ for the two cases corresponding to the dual weight generating functions $G(z)$ and $\tilde{G}(z)$.

$$W_G(\mu^{(1)},\ldots,\mu^{(k)}) := m_\lambda(c)$$

(4.34)

$$W_\tilde{G}(\mu^{(1)},\ldots,\mu^{(k)}) := f_\lambda(c).$$

(4.35)

The weighted geometrical Hurwitz numbers $H^d_G(\mu,\nu)$, giving the weighted count of such $n$-sheeted branched coverings of the Riemann sphere, having a pair of specified branch points with ramification profiles $\mu$ and $\nu$ and any number $k$ of further branch points, with arbitrary ramification profiles $(\mu^{(1)},\ldots,\mu^{(k)})$, but fixed genus, are defined to be the weighted sum

$$H^d_G(\mu,\nu) := \sum_{k=0}^{\infty} \sum_{\mu^{(1)},\ldots,\mu^{(k)}}' W_G(\mu^{(1)},\ldots,\mu^{(k)})H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu)$$

(4.36)

$$= \sum_{k=0}^{\infty} \sum_{\mu^{(1)},\ldots,\mu^{(k)}}' W_\tilde{G}(\mu^{(1)},\ldots,\mu^{(k)})H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu),$$

(4.37)
where $\sum'$ denotes the sum over all partitions other than the cycle type of the identity element. The genus $g$ of the covering cover is given by the Riemann-Hurwitz formula [35],

$$2 - 2g = \ell(\mu) + \ell(\nu) - d. \quad (4.38)$$

where

$$d := \sum_{i=1}^{k} \ell^*(\mu^{(i)}). \quad (4.39)$$

The weighted combinatorial Hurwitz numbers $F^d_G(\mu, \nu)$ give weighted enumerations of the paths in the Cayley graph of $S_n$ generated by transpositions. Expanding the weight generating functions $G(z)$ as a Taylor series

$$G(z) = 1 + \sum_{i=1}^{\infty} G_i z^i, \quad \tilde{G}(z) = 1 + \sum_{i=1}^{\infty} \tilde{G}_i z^i, \quad (4.40)$$

the weight for a given path depends only upon the signature $\lambda$, and is chosen to be the product of the coefficients of the Taylor series of the generating functions $G(z)$ (or $\tilde{G}(z)$) corresponding to the parts $\lambda_i$

$$G_\lambda := \prod_{i=1}^{\ell(\lambda)} G_{\lambda_i} = e_\lambda(c) \quad (4.41)$$

$$\tilde{G}_\lambda := \prod_{i=1}^{\ell(\lambda)} \tilde{G}_{\lambda_i} = h_\lambda(c). \quad (4.42)$$

The path weights for signature $\lambda$, are thus chosen to be either the products $e_\lambda(c)$ of the elementary symmetric functions evaluated at the weighting parameters $c = (c_1, c_2, \ldots)$ entering in the infinite product representation of $G(z)$ or, in the dual case $\tilde{G}(z)$, the products $h_\lambda(c)$ of the complete symmetric functions.

The weighted combinatorial Hurwitz numbers $F^d_G(z, \mu, \nu)$ and $F^d_{\tilde{G}}(z, \mu, \nu)$ are defined to be the weighted number of $d$-step paths, starting in the conjugacy class $\text{cyc}(\mu)$ and ending in $\text{cyc}(\nu)$

$$F^d_G(\mu, \nu) := \frac{1}{n!} \sum_{\lambda, \|\lambda\| = d} e_\lambda(c) m^\lambda_{\mu\nu} \quad (4.43)$$

$$F^d_{\tilde{G}}(\mu, \nu) := \frac{1}{n!} \sum_{\lambda, \|\lambda\| = d} f_\lambda(c) m^\lambda_{\mu\nu}. \quad (4.44)$$

4.4 Hypergeometric 2D Toda $\tau$-functions as generating functions for weighted Hurwitz number

Applying the central elements $G_n(z, \mathcal{J})$, $\tilde{G}_n(z, \mathcal{J})$ to the cycle sums $C_\mu$, $|\mu| = n$, and using (4.21), (4.22), (4.24), (4.25), (4.32) and (4.33), gives [18]
Proposition 4.3.

\[ G_n(z, J) C_\mu = \sum_{d=1}^{\infty} z^d \sum_{|\nu|=|\mu|=n} F^d_G(\mu, \nu) C_\nu = \sum_{d=1}^{\infty} z^d \sum_{|\nu|=|\mu|=n} H^d_G(\mu, \nu) C_\nu \]  \hspace{1cm} (4.45)

\[ \tilde{G}_n(z, J) C_\mu = \sum_{d=1}^{\infty} z^d \sum_{|\nu|=|\mu|=n} F^d_G(\mu, \nu) C_\nu = \sum_{d=1}^{\infty} z^d \sum_{|\nu|=|\mu|=n} H^d_G(\mu, \nu) C_\nu \]  \hspace{1cm} (4.46)

This implies, in particular, that the two definitions of weighted Hurwitz numbers coincide:

Corollary 4.4.

\[ F^d_G(\mu, \nu) = H^d_G(\mu, \nu). \]  \hspace{1cm} (4.47)

Since

\[ G_n(z, J) F_\lambda = \tau^G(z) F_\lambda, \quad |\lambda| = n, \]  \hspace{1cm} (4.48)

\[ \tilde{G}_n(z, J) F_\lambda = \tau^G(z) F_\lambda, \]  \hspace{1cm} (4.49)

the change of basis formulae (3.3), (3.4) together with Proposition 4.3 imply that

\[ \sum_{d=0}^{\infty} z^d H^d_G(\mu, \nu) = \sum_{d=0}^{\infty} z^d F^d_G(\mu, \nu) = \sum_{|\lambda|=|\mu|=|\nu|} \tau^G(z) z^{-1}_\mu^1 \chi_\lambda(\mu) \chi_\nu(\nu) \]  \hspace{1cm} (4.50)

This leads to our main result [18]

Theorem 4.5. The 2D Toda \(\tau\)-functions \(\tau^G(z)(t, s)\), \(\tau^{\tilde{G}}(z)(t, s)\) can be expressed as

\[ \tau^G(z)(t, s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} H^d_G(\mu, \nu)p_\mu(t)p_\nu(s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} F^d_G(\mu, \nu)p_\mu(t)p_\nu(s) \]  \hspace{1cm} (4.51)

\[ \tau^{\tilde{G}}(z)(t, s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} H^d_G(\mu, \nu)p_\nu(t)p_\mu(s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} F^d_G(\mu, \nu)p_\nu(t)p_\mu(s), \]  \hspace{1cm} (4.52)

and hence are generating functions for the weighted Hurwitz numbers \(H^d_G(\mu, \nu)\), \(F^d_G(\mu, \nu)\), \(H^d_G(\mu, \nu)\), \(F^d_G(\mu, \nu)\).

5 Examples of weighted double Hurwitz numbers

We now consider several examples of different types of weighted Hurwitz numbers that are special cases of this approach. All these have appeared in the recent literature on the subject [1–3, 12, 13, 17, 18, 22, 23, 26, 33, 42, 44, 52], and new examples are easily constructed. Further details for all cases may be found in [17, 18, 22, 23, 26].
5.1 Double Hurwitz numbers for simple branchings; enumeration of $d$-step paths in the Cayley graph with equal weight \[42, 44\]

This was the original case studied by Okounkov \[42\], extending an earlier result of Pandharipande \[44\]. The weight generating function in this case is just the exponential $G = \exp $,

$$ G(z) = \tilde{G}(z) = e^z. \quad (5.1) $$

The central element $G(z, \mathcal{J}) \in \mathbb{Z}(\mathbb{C}[S_n])$ is therefore

$$ \exp_n(z, \mathcal{J}) = e^{z \sum_{b=1}^n \mathcal{J}_b} \quad (5.2) $$

and the content product formula and fermionic exponent coefficients are given by

$$ r_j^{\exp}(z) = e^{jz}, \quad r_\lambda^{\exp}(z) = e^{z \sum_{i=1}^{\ell(\lambda)} \lambda_i (\lambda_i - 2i + 1)} \quad (5.3) $$

$$ T_j^{\exp(z)} = \frac{1}{2} j(j + 1)z. \quad (5.4) $$

The generating hypergeometric 2D Toda $\tau$-function is thus

$$ \tau^{\exp(z)}(t) = \sum_\lambda e^{z \sum_{i=1}^{\ell(\lambda)} \lambda_i (\lambda_i - 2i + 1)} s_\lambda(t) s_\lambda(s), \quad (5.5) $$

For this case, the infinite product form (3.14) of the generating function must be interpreted as a limit

$$ e^z = \lim_{m \to \infty} \left(1 + \frac{z}{m}\right)^m, \quad (5.6) $$

and the expression (4.34) for the geometrical weighting becomes:

$$ W^{\exp}(\mu^{(1)}, \ldots, \mu^{(k)}) = \prod_{i=1}^k \delta_{(\tau^{(\mu^{(i)})}, 1)} \quad (5.7) $$

(since we require $\ell^*(\mu^{(i)}) \geq 1$, $\forall i$). Theorem 4.5 therefore gives the generating function

$$ \tau^{\exp(z)}(t) = \sum_{d=0}^\infty z^d \sum_{|\mu|, |\nu|} H^d_{\exp}(\mu, \nu) p_\mu(t) p_\nu(s) \quad (5.8) $$

function for the (weighted) numbers

$$ H^d_{\exp}(\mu, \nu) := H((2, (1)^{n-2}), \ldots, (2, (1)^{n-2}), \mu, \nu) \quad (5.9) $$

of $n$-sheeted branched coverings of the Riemann sphere having $d$ branch points with simple ramification (i.e., profile $(2, (1)^{n-2})$ and two more (say, at 0 and $\infty$) with profiles $\mu$ and $\nu$ (weighted, as usual, by the inverse of the automorphism group).
The combinatorial definition of the weighted Hurwitz number \( (4.43) \) gives
\[
F^d_{\exp}(\mu, \nu) = \frac{1}{n!} \sum_{\lambda, |\lambda|=d} \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} m^\lambda_{\mu, \nu} \\
= \frac{1}{d! n!} \times (\# \ d\text{-step paths from an element } h \in \text{cyc}(\mu) \text{ to } \text{cyc}(\nu)). \quad (5.10)
\]

5.2 Coverings with three branch points (Belyi curves): strongly monotonic paths \([2, 18, 26, 33, 52]\)

In this case the weight generating function is
\[
G(z) = E(z) := 1 + z, \quad (5.11)
\]
so
\[
c_1 = 1, \quad c_i = 0, \quad i > 1. \quad (5.12)
\]
Therefore the central element \( E_n(z, J) \in \mathbb{Z}(\mathbb{C}[S_n]) \) is
\[
E_n(z, J) = \prod_{a=1}^{n} (1 + zJ_a), \quad (5.13)
\]
\[
r_j^{E(z)} = 1 + zj, \quad r_\lambda^{E(z)} = \prod_{(i,j) \in \lambda} (1 + z(j - i)) = z^{\ell(\lambda)} (1/z)^\lambda, \quad (5.14)
\]
\[
T_j^{E(z)} = \sum_{i=1}^{j} \ln(1 + iz), \quad T^{E(z)}_{-j} = -\sum_{i=1}^{j-1} \ln(1 - iz), \quad j > 0, \quad (5.15)
\]
where
\[
(u)_\lambda := \prod_{i=1}^{\ell(\lambda)} (u - i + 1)_{\lambda_i} \quad (5.16)
\]
is the multiple Pochhammer symbol corresponding to the partition \( \lambda \).

The generating \( \tau \)-function is thus \([17, 26]\)
\[
\tau^{E(z)}(t, s) = \sum_{\lambda} z^{\ell(\lambda)} (1/z)^\lambda s_\lambda(t)s_\mu(s) \\
= \sum_{d=0}^{\infty} z^d \sum_{|\mu| = |\nu|} H^d_E(\mu, \nu)p_\mu(t)p_\nu(s), \quad (5.17)
\]
where
\[
H^d_E(\mu, \nu) = \sum_{\mu^{(1)}}, \ell^*(\mu^{(1)}) = d \quad (5.18)
\]
is the number of \( n = |\mu| = |\nu| = |\mu^{(1)}| \) sheeted branched covers with branch points of ramification type \((\mu, \nu)\) at \((0, \infty)\), and one further branch point, with colength \( \ell^*(\mu^{(1)}) = d; \)
These are the double Hurwitz numbers for Belyi curves \([2, 18, 26, 33, 52]\), which enumerate \(n\)-sheeted branched coverings of the Riemann sphere having three ramification points, with ramification profile type \(\mu\) and \(\nu\) at 0 and \(\infty\), and a single additional branch point, with ramification profile \(\mu^{(1)}\) of colength

\[
\ell^{*}(\mu^{(1)}) := n - \ell(\mu^{(1)}) = d,
\]

i.e., with \(n - d\) preimages. The genus is again given by the Riemann-Hurwitz formula (4.38).

Combinatorially, we have the weight

\[
e_{\lambda}(c) = \delta_{\lambda,(1)|\lambda} \quad (5.20)
\]

and therefore

\[
\sum_{\lambda, |\lambda| = d} e_{\lambda}(c) m_{\lambda}(J) = \sum_{b_1 < \cdots < b_d} J_{b_1} \cdots J_{b_d}. \quad (5.21)
\]

The coefficient \(F_{E}^{d}(\mu, \nu)\) is thus

\[
F_{E}^{d}(\mu, \nu) = m_{\mu^{(1)} \nu}^{d}, \quad (5.22)
\]

which enumerates all \(d\)-step paths in the Cayley graph of \(S_n\) starting at an element in the conjugacy class of cycle type \(\nu\) and ending in the class of type \(\mu\), that are strictly monotonically increasing in their second elements \([18, 26]\).

### 5.3 Fixed number of branch points and genus: multimonotonic paths [26]

In this case the weight generating function is :

\[
G(z) = E^{k}(z) := (1 + z)^k, \quad (5.23)
\]

and hence

\[
c_i = 1, \quad 1 \leq i \leq k, \quad c_i = 0, \quad \forall i > k. \quad (5.24)
\]

Therefore the central element is

\[
E_{n}(z, J)^k = \prod_{a=1}^{n} (1 + z J_a)^k, \quad (5.25)
\]

and

\[
r_{j}^{E^{k}(z)} = (1 + z j)^k, \quad r_{\lambda}^{E^{k}(z)} = \prod_{(i, j) \in \lambda} (1 + z (j - i))^k = z^{k|\lambda|}(1/z)^{k \lambda} \quad (5.26)
\]

\[
T_{j}^{E^{k}(z)} = k \sum_{i=1}^{j} \ln(1 + iz), \quad T_{-j}^{E^{k}(z)} = -k \sum_{i=1}^{j-1} \ln(1 - iz), \quad j > 0. \quad (5.27)
\]
The generating $\tau$-function is

$$\tau^{E^k(z)}(t,s) = \sum_\lambda^\infty z^{\vert\lambda\vert} (1/z)^{\lambda} s_\lambda(t)s_\mu(s)$$

$$= \sum_{d=0}^\infty z^d \sum_{\mu,\nu, \vert\mu\vert = \vert\nu\vert} H^d_{E^k}(\mu,\nu) p_\mu(t)p_\nu(s),$$  \hspace{1cm} (5.28)

where

$$H^d_{E^k}(\mu,\nu) = \sum_{\mu^{(1)},\ldots,\mu^{(k)} \atop \sum_{i=1}^k \ell^*(\mu_i) = d} H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu)$$  \hspace{1cm} (5.29)

is the number of $n = \vert\mu\vert = \vert\nu\vert = \vert\mu^{(i)}\vert$ sheeted branched covers with branch points of ramification type $(\mu,\nu)$ at $(0,\infty)$, and (at most) $k$ further branch points, such that the sum of the colengths of their ramification profile type (i.e., the “defect” in the Riemann Hurwitz formula (4.38)) is equal to $d$:

$$\sum_{i=1}^k \ell^*(\mu^{(i)}) = kn - \sum_{i=1}^k \ell(\mu^{(i)}) = d.$$  \hspace{1cm} (5.30)

This amounts to counting covers with the genus fixed by (4.38) and the number of additional branch points fixed at $k$, but no restriction on their simplicity.

The combinatorial weighting for paths of signature $\lambda$ is

$$e_\lambda(c) = \prod_{i=1}^{\ell(\lambda)} \binom{k}{\lambda_i}$$  \hspace{1cm} (5.31)

and hence,

$$\sum_{\lambda, \vert\lambda\vert = d} \left( \prod_{i=1}^{\ell(\lambda)} \binom{k}{\lambda_i} \right) m_\lambda(J) = [z^d] \prod_{a=1}^n (1 + zJ_a)^k$$  \hspace{1cm} (5.32)

where $[z^d]$ means the coefficient of $z^d$ in the polynomial.

The weighted combinatorial Hurwitz number

$$F^d_{E^k}(\mu,\nu) = \sum_{\lambda, \vert\lambda\vert = k} \left( \prod_{i=1}^{\ell(\lambda)} \binom{k}{\lambda_i} \right) m^\lambda_{\mu\nu}$$  \hspace{1cm} (5.33)

is thus the number of $(d+1)$-term products $(a_1 b_1)\cdot\ldots\cdot(a_d b_d)h$ such that $h \in \text{cyc}(\mu)$, while $(a_1 b_1)\cdot\ldots\cdot(a_d b_d)h \in \text{cyc}(\nu)$, which consist of a product of $k$ consecutive subsequences, each of which is strictly monotonically increasing in their second elements $[17,26]$. 

27
5.4 Signed Hurwitz numbers at fixed genus: weakly monotonic paths \([12, 13, 17]\)

This case was studied from the combinatorial viewpoint, and related to the HCIZ internal in \([12, 13, 17]\). It is the dual \(\tilde{E}\) of the weight generating function of Section 5.2.

\[
\tilde{E}(z) := H(z) := \frac{1}{1 - z}
\]

and hence we have

\[
c_i = 1, \ 1 \leq i \leq k, \quad c_i = 0, \ \forall i > k
\]

as before, but the relevant combinatorial weighting factor is

\[
h_\lambda(c) = 1 \ \forall \lambda.
\]

The corresponding central element is

\[
H_n(z, \mathcal{J}) = \prod_{a=1}^{n} (1 - z\mathcal{J}_a)^{-1},
\]

and therefore

\[
T^H_j(z) = (1 - zj)^{-1}, \quad T^H_\lambda(z) = \prod_{(i,j) \in \lambda} (1 - z(j - i))^{-1} = (-z)^{-|\lambda|}((-1/z)_\lambda)^{-1},
\]

\[
T^H_{-j}(z) = -\sum_{i=1}^{j} \ln(1 - iz), \quad T^H_{-j}(z) = \sum_{i=1}^{j-1} \ln(1 + iz), \quad j > 0.
\]

The generating \(\tau\)-function for this case is \([17, 26]\)

\[
\tau^H(z)(t, s) = \sum_{\lambda} (-z)^{-|\lambda|} (-1/z)_\lambda^{-1} s_\lambda(t) s_\nu(s)
\]

\[
= \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu|=|\nu|} \sum_{\mu^{(1)} \ldots \mu^{(k)} : \sum_{i=1}^{k} \ell(\mu_i) = d} H^d_H(\mu, \nu) p_\mu(t) p_\nu(s)
\]

where

\[
H^d_H(\mu, \nu) = (-1)^{n+d} \sum_{k=1}^{\infty} (-1)^k \sum_{\mu^{(1)} \ldots \mu^{(k)} : \sum_{i=1}^{k} \ell(\mu_i) = d} H(\mu^{(1)}, \ldots \mu^{(k)}, \mu, \nu)
\]

is the signed enumeration of \(n = |\mu| = |\nu|\) sheeted branched covers with branch points of ramification type \((\mu, \nu)\) at \((0, \infty)\), and any number further branch points, the sum of whose colengths is \(d\), with sign determined by the parity of the number of branch points \([26]\).
These are thus double Hurwitz numbers for \( n \)-sheeted branched coverings of the Riemann sphere with branch points having ramification profile type \((\mu, \nu)\) at \((0, \infty)\) and an arbitrary number of further branch points, such that the sum of the colengths of their ramification profile lengths is again equal to \(d\)

\[
\sum_{i=1}^{k} \ell^*(\mu^{(i)}) = kn - \sum_{i=1}^{k} \ell(\mu^{(i)}) = d. \tag{5.42}
\]

The latter are counted with a sign, which is \((-1)^{n+d}\) times the parity of the number of branch points [26]. The genus is again given by (4.38).

The combinatorial Hurwitz number \( F^d_H(\mu, \nu) \), derived from

\[
\sum_{\lambda, |\lambda| = k} h_\lambda(\mathbf{c})m_\lambda(J) = \sum_{b_1 \leq \cdots \leq b_d} J_{b_1} \cdots J_{b_d}. \tag{5.43}
\]

is therefore is given by

\[
F^d_H(\mu, \nu) = \sum_{\lambda, |\lambda| = k} m^\lambda_{\mu \nu}, \tag{5.44}
\]

which is the number of products of the form \((a_1 b_1) \cdots (a_d b_d)\) for \(g \in \text{cyc}(\mu)\) that are weakly monotonically increasing, such that \((a_1 b_1) \cdots (a_d b_d)h \in \text{cyc}(\nu)\). These thus enumerate the \(d\)-step paths in the Cayley graph of \(S_n\) from an element in the conjugacy class of cycle type \(\mu\) to the class cycle type \(\nu\), that are weakly monotonically increasing in their second elements [17].

Equivalently, they are double Hurwitz numbers for \(n\)-sheeted branched coverings of the Riemann sphere with branch points at 0 and \(\infty\) having ramification profile type \(\mu\) and \(\nu\), and an arbitrary number of further branch points, such that the sum of the colengths of their ramification profile lengths is again equal to \(d\)

\[
\sum_{i=1}^{k} \ell^*(\mu^{(i)}) = kn - \sum_{i=1}^{k} \ell(\mu^{(i)}) = d. \tag{5.45}
\]

The latter are counted with a sign, which is \((-1)^{n+d}\) times the parity of the number of branch points [26]. The genus is again given by (4.38).

This case is known to have a matrix model representation [12, 13] when the flow parameters \(t\) and \(s\) are restricted to be trace invariants of a pair of \(N \times N\) normal matrices \(A, B\):

\[
t_i = \frac{1}{i} \text{tr}(A^i), \quad s_i = \frac{1}{i} \text{tr}(B^i), \tag{5.46}
\]

Within a normalization, setting

\[
z = -\frac{1}{N} \tag{5.47}
\]
as the expansion parameter, we have equality with the HCIZ double matrix integral
\[
\tau^{H(-\frac{1}{N})}(t, s) = I_N(A, B) := \int_{U \in U(N)} e^{tr(U^T A U B)} d\mu(U) = \left( \prod_{k=0}^{N-1} k! \right) \frac{\det(e^{a_i b_j})_{1 \leq i,j \leq N}}{\Delta(a) \Delta(b)},
\]
where \( d\mu(U) \) is the Haar measure on \( U(N) \), \( a = (a_1, \ldots, a_N) \), \( b = (b_1, \ldots, b_N) \) are the eigenvalues of \( A \) and \( B \) respectively, and \( \Delta(a) \), \( \Delta(b) \) are the Vandermonde determinants.

The identification (5.47), however, gives rise to a cutoff in the expansion (5.40), giving a sum only over partitions \( \lambda \) of length \( \ell(\lambda) \leq N \):
\[
\tau^{H(-\frac{1}{N})}(t, s) = \sum_{\lambda, \ell(\lambda) \leq N} \frac{N^{\ell(\lambda)}}{(N)_{\lambda}} s_{\lambda}(t) s_{\mu}(s).
\]

### 5.5 Quantum weighted branched coverings and paths [18]

In [18] three variants of quantum Hurwitz numbers were studied, with weight generating functions denoted \( E(q, z) \), \( H(q, z) \) and \( E'(q, z) \). We only consider the case \( E'(q, z) \), which has the most interesting interpretation in relation to Bosonic gases. The other two are developed in detail in [18] and may also be obtained as special cases of the MacDonald polynomial approach to quantum Hurwitz numbers developed in [23] which is summarized in Section 7 below.

The weight generating function is
\[
E'(q, z) := \prod_{i=1}^{\infty} (1 + q^i z) = 1 + \sum_{i=0}^{\infty} E'_i(q) z^i,
\]
\[
E'_i(q) := \frac{q^{\frac{i(i+1)}{2}}}{\prod_{j=1}^{i} (1 - q^j)}, \quad i \geq 1,
\]
where \( q \) is viewed as a quantum deformation parameter that may interpreted (see below) in terms of the energy distribution of Bosonic gases with a linear energy spectrum. This is related to the quantum dilogarithm function by
\[
(1 + z)E'(q, z) = e^{-Li_2(q, -z)}, \quad Li_2(q, z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)}.
\]

We thus have
\[
c_i = q^i, \quad i \geq 1, \quad e_\lambda(c) =: E'_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \frac{q^{\frac{i(i+1)}{2}} \lambda_i (\lambda_i + 1)}{\prod_{j=1}^{\lambda_i} (1 - q^j)}.
\]

The central element \( E'_n(q, xJ) \in \mathbb{Z}(\mathbb{C}[S_n]) \) is given by
\[
E'_n(q, xJ) = \prod_{a=1}^{n} \prod_{k=1}^{\infty} (1 + q^k xJ_a),
\]
and hence that content product coefficient is
\[
r_{j}^{E'(q,z)} = \prod_{k=1}^{\infty} (1 + q^k z^j),
\]
(5.55)

\[
\tau_{\lambda}^{E'(q,z)}(z) = \prod_{k=1}^{\infty} \prod_{(i,j) \in \lambda} (1 + q^k z(j - i)) = \prod_{k=1}^{\infty} (zq^k)^{|\lambda|} (1/(zq^k))_{\lambda}.
\]
(5.56)

The generating τ-function is therefore [18]
\[
\tau^{E'(q,z)}(t, s) = \sum_{\lambda} \left( \prod_{k=1}^{\infty} (zq^k)^{|\lambda|} (1/(zq^k))_{\lambda} \right) \cdot s_{\lambda}(t)s_{\mu}(s)
\]
\[
= \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu| = |\nu|} H_{E'(q)}^{d}(\mu, \nu)p_{\mu}(t)p_{\nu}(s)
\]
(5.57)

where
\[
H_{E'(q)}^{d}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)})H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu)
\]
(5.58)
is the quantum weighted enumeration of \( n = |\mu| = |\nu| \) sheeted branched coverings with genus \( g \) given by (4.38) and weight \( W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) for branched coverings of type \( (\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \) given by
\[
W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{|\operatorname{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} q^{1i_1}t^{\epsilon*_{\mu^{(i_1)}}} \cdots q^{1i_k}t^{\epsilon*_{\mu^{(i_k)}}}
\]
\[
= \frac{1}{|\operatorname{aut}(\lambda)|} \sum_{\sigma \in S_k} \frac{q^{k\epsilon*_{\mu^{(i_1)}}} \cdots q^{\epsilon*_{\mu^{(i_k)}}}}{(1 - q^{\epsilon*_{\mu^{(i_1)}}}) \cdots (1 - q^{\epsilon*_{\mu^{(i_k)}}})}
\]
\[
= \frac{1}{|\operatorname{aut}(\lambda)|} \sum_{\sigma \in S_k} \frac{1}{(q^{-\epsilon*_{\mu^{(i_1)}}} - 1) \cdots (q^{-\epsilon*_{\mu^{(i_k)}}} - 1)}
\]
(5.59)

where \( \lambda \) is the partition with parts \( \{\ell^*(\mu^{(i)})\}_{i=1, \ldots, k} \)

The combinatorial Hurwitz number \( F_{E'(q)}^{d}(\mu, \nu) \) giving the weighted enumeration of paths is
\[
F_{E'(q)}^{d}(\mu, \nu) = \sum_{\lambda : |\lambda| = d} \frac{q^{\frac{1}{2}i(i+1)}}{\prod_{j=1}^{i}(1 - q^j)} m_{\mu, \nu}^{\lambda}.
\]
(5.60)

and we have the usual equality
\[
H_{E'(q)}^{d}(\mu, \nu) = F_{E'(q)}^{d}(\mu, \nu).
\]
(5.61)
Remark 5.1. Relation to Bosonic gas distribution. If we identify
\[
q := e^{-\beta \hbar \omega_0}, \quad \beta = k_B T, \tag{5.62}
\]
where \( \hbar \omega_0 \) is the lowest energy state in a gas of identical Bosonic particles, assume the energy spectrum to consist of integer multiples of \( \hbar \omega_0 \)
\[
\epsilon_k = k \hbar \omega_0, \tag{5.63}
\]
and assign the energy
\[
\epsilon(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{i=1}^{k} \epsilon^{*}(\mu^{(i)}) \tag{5.64}
\]
to a configuration with branching profiles \((\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu)\), the distribution function for Bosonic gases gives the weight
\[
W(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{e^{\beta \epsilon(\mu^{(1)}, \ldots, \mu^{(k)})} - 1}. \tag{5.65}
\]
The weighting factor \(W_E(q)(\mu^{(1)}, \ldots, \mu^{(k)})\) in eq. (5.59) is thus the symmetrized product
\[
W_E(q)(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} W(\mu^{(\sigma(1))}) \cdots W(\mu^{(\sigma(k))}) \tag{5.66}
\]
of that for each subconfiguration.

In [18], a dual pair of similar weight generating functions \(E(q, z), H(q, z)\) were introduced, which correspond to two slightly different definitions of quantum Hurwitz numbers. These are the \(q\)-analsogs of what, when extended to the Cauchy-Littlewood formula, become the generating functions of the elementary and the complete symmetric functions:
\[
E(q, z) := \prod_{k=0}^{\infty} (1 + z q^k) \tag{5.67}
\]
\[
H(q, z) := \prod_{k=0}^{\infty} (1 - z q^k)^{-1} \tag{5.68}
\]
The corresponding weights for branched covers with ramification profiles \((\mu^{(1)}, \ldots, \mu^{(k)})\) and \((\nu^{(1)}, \ldots, \nu^{(k)})\) at the branch points are:
\[
W_E(q)(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{0 \leq i_1 < \cdots < i_k} q^{i_1 \ell^{*}(\mu^{(\sigma(1))})} \cdots q^{i_k \ell^{*}(\mu^{(\sigma(k))})}
\]
\[
= \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \frac{q^{(k-1)\ell^{*}(\mu^{(\sigma(1))})} \cdots q^{\ell^{*}(\mu^{(\sigma(k-1))})}}{(1 - q^{\ell^{*}(\mu^{(\sigma(1))})}) \cdots (1 - q^{\ell^{*}(\mu^{(\sigma(1))})}) q^{\ell^{*}(\mu^{(\sigma(k))})}} \tag{5.69}
\]
where \( \lambda \) is the partition with parts \((\ell^*(\mu(1)), \ldots, \ell^*(\mu(\tilde{k})))\), and

\[
W_{H(q)}(\nu(1), \ldots, \nu(\tilde{k})) := (-1)^{\ell^*(\lambda)} \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{0 \leq i_1 \leq \cdots \leq i_k} q^{i_1 \ell^*(\nu(\sigma(1)))} \cdots q^{i_k \ell^*(\nu(\sigma(\tilde{k})))},
\]

(5.70)

where \( \lambda \) is the partition with parts \((\ell^*(\mu(1)), \ldots, \ell^*(\mu(\tilde{k})))\).

The associated hypergeometric \( \tau \)-functions \( \tau_{E}^{q,z}(t, s) \), and \( \tau_{H}^{q,z}(t, s) \) are defined similarly to \( \tau_{E}^{q,z}(\mu, \nu) \) and are generating functions for the correspondingly modified Hurwitz numbers \( F_{dE}(q) = H_{dE}(q) \) and \( F_{dH}(q) = H_{dH}(q) \). (See \([18]\) for further details.)

For later use, we denote the product of these

\[
W_{Q(q)}(\mu(1), \ldots, \mu(k); \nu(1), \ldots, \nu(\tilde{k})) := W_{E(q)}(\mu(1), \ldots, \mu(k))W_{H(q)}(\nu(1), \ldots, \nu(\tilde{k})).
\]

(5.71)

6 Multispecies weighted Hurwitz numbers

6.1 Hybrid signed Hurwitz numbers at fixed genus: hybrid monotonic paths \([17, 18, 26]\)

This case is just a hybrid product of the cases of Section 5.2 and Section 5.4. We choose as generating function

\[
Q(w, z) := \frac{1 + w}{1 - z}
\]

(6.1)

taking power series in both parameters \((w, z)\). The associated central element is

\[
Q(w, z, \mathcal{J}) = E_n(w, \mathcal{J})H_n(z, \mathcal{J}) = \prod_{a=1}^{n} \frac{1 + w \mathcal{J}_a}{1 - z \mathcal{J}_a},
\]

(6.2)

and therefore

\[
r_j^Q(w, z) = \frac{1 + jw}{1 - jz},
\]

(6.3)

\[
r_{\lambda}^Q(w, z) = \prod_{i,j} \frac{1 + (j - i)w}{1 - (j - i)z} = (-w/z)^{|\lambda|} \frac{(1/w)^{\lambda}}{(-1/z)^{\lambda}},
\]

(6.4)

\[
T_j^Q(w, z) = \sum_{i=1}^{j} \ln \frac{1 + iw}{1 - iz}, \quad T_{-j}^Q(w, z) = \sum_{b=1}^{j-1} \ln \frac{1 + iz}{1 - iw}, \quad j > 0,
\]

(6.5)
The generating τ-function is thus \([17, 26]\)

\[
\tau^Q(w, z)(t, s) = \sum_{\lambda} (-w/z)^{|\lambda|} \left(1/w\right)_\lambda \left(-1/z\right)_\lambda s_\lambda(t)s_\mu(s)
\]

\[
= \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} w^c z^d \sum_{\mu, \nu, |\mu|=|\nu|} H^c_d(\mu, \nu)p_\mu(t)p_\nu(s), \tag{6.6}
\]

where

\[
H^c_d(\mu, \nu) = \sum_{k=0}^{\infty} \sum_{\mu^{(1)}=c} \sum_{\nu^{(1)}, \ldots, \nu^{(k)}} (-1)^{k+d} H^{c, \mu^{(1)}, \nu^{(1)}, \ldots, \nu^{(k)}, \mu, \nu} \tag{6.7}
\]

is the number of \(n = |\mu| = |\nu| = |\mu^{(1)}| = |\nu^{(1)}| = \cdots = |\nu^{(k)}|\) sheeted branched covers with branch points of ramification type \((\mu, \nu)\) at \((0, \infty)\), one further branch point, of “first class”, with colength \(\ell^*(\mu^{(1)}) = c\) and \(k\) further branch points, \((\nu^{(1)}, \ldots, \nu^{(k)})\) of “second class” with total colength equal to \(d\),

\[
\sum_{i=1}^{k} \ell^*(\mu^{(i)}) = d \tag{6.8}
\]

counted with sign \((-1)^{k+d}\) determined by the parity of \(k\). Note that the sum over \(k\) is actually finite, because of the constraint (6.8). As usual, the Riemann-Hurwitz formula

\[
2 - 2g = \ell(\mu) + \ell(\nu) - d \tag{6.9}
\]
determines the genus \(g\) of the covering surface.

The meaning of the combinatorial Hurwitz number \(F^c_d(\mu, \nu) = H^c_d(\mu, \nu)\) in this case is clear from combining its meaning for the cases considered in Section 5.2 and Section 5.4; it is the number of \(c + d\) step paths in the Cayley graph of \(S_n\) starting at an element \(h \in \text{cyc}(\mu)\) in the conjugacy class of type \(\text{cyc}(\mu)\) and ending in \(\text{cyc}(\nu)\) such that the first \(c\) steps are strictly monotonic and the next \(d\) steps are weakly monotonic.

This case also has a matrix integral representation, analogous to the HCIZ integral when the flow parameters \(t\) and \(s\) are again restricted to equal the trace invariants of a pair \(A, B\) of normal matrices as in (5.46), and the expansion parameters are equated to

\[
w = \frac{1}{N - \alpha}, \quad z = -\frac{1}{N} \tag{6.10}
\]

for some parameter \(\alpha\).

\[
\tau^{Q(1/(N-\alpha), 1/N)}(t, s) = \int_{U \in U(N)} \det(I - \zeta UAU^\dagger B)^{\alpha-N} d\mu(U) \tag{6.11}
\]

\[
= \left(\prod_{k=0}^{N-1} \frac{k!}{(1 - \alpha)_k}\right) \frac{\Delta(1 - \zeta a_ib_j)^{\alpha-1}}{\Delta(a)\Delta(b)} \tag{6.12}
\]

34
where
\[ \zeta := \frac{N}{N - \alpha}. \] (6.13)

The identification (5.47) again gives rise to a cutoff in the expansion (5.40), restricting the sum to partitions \( \lambda \) of length \( \ell(\lambda) \leq N \):
\[ \tau^{Q(1/(N-\alpha),-1/N)}(t, s) = \sum_{\lambda, \ell(\lambda) \leq N} (1 - \frac{\alpha}{N})^{\ell(\lambda)} \frac{(N - \alpha)_{\lambda}}{(N)_\lambda} s_{\lambda}(t)s_{\mu}(s). \] (6.14)

### 6.2 Signed multispecies Hurwitz numbers: hybrid multimonomonic paths [26]

Now consider the multiparametric generalization of the previous example. We introduce \( l+m \) expansion parameters
\[ w := (w_1, \ldots, w_l), \quad z = (z_1, \ldots, z_m). \] (6.15)

The weight generating functions \( G \) is chosen to be products of those for the previous case:
\[ Q^{(l,m)}(w, z) := \prod_{\alpha=1}^{l} E(w_\alpha) \prod_{\beta=1}^{m} H(z_\beta) = \frac{\prod_{\alpha=1}^{l} (1 + w_\alpha)}{\prod_{\beta=1}^{m} (1 - z_\beta)}. \] (6.16)

The corresponding element of the center \( Z(C[S_n]) \) is
\[ Q^{(l,m)}_n(w, z, \mathcal{J}) = \prod_{a=1}^{n} Q^{(l,m)}(w_\mathcal{J}_a, z_\mathcal{J}_a), \] (6.17)
and therefore the eigenvalues of \( Q^{(l,m)}_n(w, z) \) are
\[ r^{Q^{(l,m)}(w, z)}_{\lambda} = \prod_{(i,j) \in \lambda} \frac{\prod_{\alpha=1}^{l} (1 + (j - i)w_\alpha)}{\prod_{\beta=1}^{m} (1 - (j - i)z_\beta)} = \frac{\prod_{\alpha=1}^{l} (w_\alpha)^{|\lambda|} (1/w_\alpha)^{\lambda}}{\prod_{\beta=1}^{m} (-z_\beta)^{|\lambda|} (-1/z_\beta)^{\lambda}}, \] (6.18)

while the diagonal exponential fermionic coefficients are
\[ T^{Q^{(l,m)}(w, z)}_{j} = \sum_{i=1}^{j} \ln \frac{\prod_{\alpha=1}^{l} (1 + iw_\alpha)}{\prod_{\beta=1}^{m} (1 - iz_\beta)}, \quad T^{Q^{(l,m)}(w, z)}_{-j} = -\sum_{i=0}^{j-1} \ln \frac{\prod_{\alpha=1}^{l} (1 - iw_\alpha)}{\prod_{\beta=1}^{m} (1 + iz_\beta)}, \quad j > 0. \] (6.19)

The generating \( \tau \)-function is thus [26]
\[ \tau^{Q^{(l,m)}(w, z)}(t, s) = \sum_{\lambda} \frac{\prod_{\alpha=1}^{l} (w_\alpha)^{|\lambda|} (1/w_\alpha)^{\lambda}}{\prod_{\beta=1}^{m} (-z_\beta)^{|\lambda|} (-1/z_\beta)^{\lambda}} s_{\lambda}(t)s_{\mu}(s). \]
\[
\sum_{\mathbf{d} \in \mathbb{N}^l} \sum_{\tilde{\mathbf{d}} \in \mathbb{N}^m} w^d z^{\tilde{d}} \sum_{\mu, \nu, |\mu| = |\nu|} H_{Q(l,m)}^{(\mathbf{d}, \tilde{\mathbf{d}})}(\mu, \nu)p_\mu(t)p_\nu(s),
\]

(6.20)

where multi-index notation has been used:

\[
w^d := \prod_{\alpha=1}^l w^d_\alpha, \quad z^{\tilde{d}} := \prod_{\beta=1}^m z^{\tilde{d}}_\beta
\]

(6.21)

with

\[
d := (d_1, \ldots, d_l), \quad \tilde{\mathbf{d}} := (\tilde{d}_1, \ldots, \tilde{d}_m), \quad d_\alpha, \tilde{d}_\beta \in \mathbb{N}.
\]

(6.22)

Here

\[
H_{Q(l,m)}^{(\mathbf{d}, \tilde{\mathbf{d}})}(\mu, \nu) = (-1)^D \sum_{\{k_\beta\}_{\beta=1}^m} \sum_{\{\mu^{(\alpha)}\}} \sum_{\{\nu^{(\beta,i_\beta)}\}} (-1)^C H(\{\mu^{(\alpha)}\}, \{\nu^{(\beta,i_\beta)}\}_{i_\beta=1}^{k_\beta}, \mu, \nu),
\]

(6.23)

is the signed total number of branched coverings, weighted by the inverses of their automorphism groups, with branch points at \((0, \infty)\) having ramification profiles \((\mu, \nu)\), and further branch points divided into two types: \(l\) “plain” branch points \(\{\mu^{(\alpha)}\}_{\alpha=1}^l\) with colengths

\[
\ell^*(\mu^{(\alpha)}) = d_\alpha
\]

(6.24)

and

\[
C = \sum_{\beta=1}^m k_\beta,
\]

(6.25)

“coloured” branch points with colours labeled by \(\beta = 1, \ldots, m\) and ramification profiles \(\{\nu^{(\beta,i_\beta)}\}_{\beta=1, \ldots, m; i_\beta=1}^{k_\beta}\), of total ramification type colengths

\[
\sum_{i_\beta=1}^{k_\beta} \ell^*(\nu^{(\beta,i_\beta)}) = \tilde{d}_\beta
\]

(6.26)

in each colour group, and

\[
D = \sum_{\beta=1}^m \tilde{d}_\beta
\]

(6.27)

is the sum of these colengths over all colours. The genus \(g\) of the covering surface is determined by the Riemann-Hurwitz formula:

\[
2 - 2g = \ell(\mu) + \ell(\nu) - \sum_{\alpha=1}^l d_\alpha - D.
\]

(6.28)

The combinatorial significance of the weighted Hurwitz number \(F_{Q(l,m)}^{(\mathbf{d}, \tilde{\mathbf{d}})}(\mu, \nu)\) in this case is given (see [26]) by:
Theorem 6.1. The coefficients $H^{(d,d̃)}_{Q(t,m)}(\mu, \nu) = F^{(d,d̃)}_{Q(t,m)}(\mu, \nu)$ in the expansion (6.20) are equal to the number of paths in the Cayley graph of $S_n$ generated by transpositions $(a \, b)$, $a < b$, starting at an element in the conjugacy class with cycle type given by the partition $\mu$ and ending in the conjugacy class with cycle type given by partition $\nu$, such that the paths consist of a sequence of

$$k := \sum_{\alpha=1}^{l} d_\alpha + \sum_{\beta=1}^{m} d_\beta$$  \hfill (6.29)

transpositions $(a_1 b_1) \cdots (a_k b_k)$, divided into $l + m$ subsequences, the first $l$ of which consist of $\{d_1, \ldots, d_l\}$ transpositions that are strictly monotonically increasing (i.e. $b_i < b_{i+1}$ for each neighbouring pair of transpositions within the subsequence), followed by $\{d̃_1, \ldots, d̃_m\}$ subsequences within each of which the transpositions are weakly monotonically increasing (i.e. $b_i \leq b_{i+1}$ for each neighbouring pair).

6.3 General weighted multispecies Hurwitz numbers [22, 23]

We may extend the multispecies signed Hurwitz numbers considered in the preceding section to general multispecies weighting [22, 23] by replacing the factors $E(w_\alpha)$ and $H(z_\beta)$ in the above by arbitrary weight generating functions of type $G^\alpha(w_\alpha)$ and dual type $\tilde{G}^\beta(z_\beta)$.

The partitions are divided into two classes: those corresponding to the weight factors of type $G^\alpha$, labelled $\{\mu^{(\alpha,u_\alpha)}\}$, and those corresponding to dual type $\tilde{G}(z)$, labelled $\{\nu^{(\beta,v_\beta)}\}$. These are further subdivided into $l$ “colours”, or “species” for the first class, denoted by the label $\alpha = 1, \ldots, l$ and $m$ in the second, denoted by $\beta = 1, \ldots, m$. Any given configuration $\{(\mu^{(\alpha,u_\alpha)})_{1 \leq u_\alpha \leq k_\alpha}, (\nu^{(\beta,v_\beta)})_{1 \leq v_\beta \leq \tilde{k}_\beta}\}$ has $k_\alpha$ elements of colour $\alpha$ in the first class and $\tilde{k}_\beta$ elements of colour $\beta$ in the second class, for a total of

$$k = \sum_{\alpha=1}^{l} k_\alpha + \sum_{\beta=1}^{m} \tilde{k}_\beta$$  \hfill (6.30)

partitions.

Denoting the $l + m$ expansion parameters again as

$$w = (w_1, \ldots, w_l), \quad z = (z_1, \ldots, z_m),$$  \hfill (6.31)

the multispecies weight generating function is formed from the product

$$G^{(l,m)}(w, z) := \prod_{\alpha=1}^{l} G^\alpha(w_\alpha) \prod_{\beta=1}^{m} \tilde{G}^\beta(z_\beta),$$  \hfill (6.32)
where each factor has an infinite product representation that is of one of the two types

\[ G^\alpha(w) = \prod_{i=1}^{\infty} (1 + c^\alpha_i w), \ \alpha = 1, \ldots, l \]  
\[ \tilde{G}^\beta(w) = \prod_{i=1}^{\infty} (1 - \tilde{c}^\beta_i w), \ \beta = 1, \ldots, m. \]  

for \( l + m \) infinite sequences of parameters

\[ c^\alpha = (c^\alpha_1, c^\alpha_2, \ldots), \ \alpha = 1, \ldots, l \]  
\[ \tilde{c}^\alpha = (\tilde{c}^\alpha_1, \tilde{c}^\alpha_2, \ldots), \ \beta = 1, \ldots, m. \]  

The corresponding central element, denoted

\[ G^{l,m}_n(w, z, J) := \prod_{\alpha=1}^{l} \left( \prod_{\alpha=1}^{l} G^\alpha(w_{\alpha}J_{\alpha}) \right) \left( \prod_{\beta=1}^{m} \tilde{G}^\beta(z_{\beta}J_{\beta}) \right) \]  

has eigenvalues

\[ r^G_{\lambda}(w, z) := \prod_{\alpha} G(\alpha_{\alpha}(j - i)), \ \tilde{r}^G_{\lambda}(z, J) := \prod_{\beta} \tilde{G}(\beta_{\beta}(j - i)). \]  

The diagonal exponential fermionic coefficients are

\[ T_{G^{l,m}}(w, z) = \sum_{\lambda} \prod_{\alpha} G^\alpha(\alpha_{\alpha}(j)), \ \tilde{T}_{G^{l,m}}(w, z) = \prod_{\beta} \tilde{G}(\beta_{\beta}(j)). \]  

The generating hypergeometric \( \tau \)-function is \[ \tau_{G^{l,m}}(t, s) = \sum_{\lambda} \prod_{\alpha} G^\alpha(\alpha_{\alpha}(j)), \ \tilde{T}_{G^{l,m}}(w, z) = \prod_{\beta} \tilde{G}(\beta_{\beta}(j)). \]  

\[ H^{(d, d)}_{G^{l,m}}(\mu, \nu) p_{\mu}(t) p_{\nu}(s), \]  

38
and parts equal to the colengths $\ell = \ell(\mu, \nu)$.

\[ H_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu) := \sum_{k_1, \ldots, k_l} \sum_{k_{l+1}, \ldots, k_m} \sum_{\{\lambda(a)\}} \sum_{\{\nu(\beta)\}} \sum_{\{\mu(\alpha)\}} \sum_{\{\nu(\beta)\}} W_{G(l,m)}(\{\mu(\alpha)\}, \{\nu(\beta)\}) H(\{\mu(\alpha)\}, \{\nu(\beta)\}, \mu, \nu) \]

is the geometrical multispecies Hurwitz number giving the weighted enumeration of $n$-sheeted branched coverings with $l + m$ branch points of type $\{\{\mu(\alpha)\}_{1 \leq a \leq k_\alpha}, \{\nu(\beta)\}_{1 \leq \nu \leq \tilde{k}_\beta}\}$ and $(\mu, \nu)$ at $(0, \infty)$, with weighting factor equal to the product of those for single species

\[ W_{G(l,m)}(\{\mu(\alpha)\}, \{\nu(\beta)\}) = \prod_{\alpha=1}^{l} m_\lambda(\alpha) (c^{\alpha}) \prod_{\beta=1}^{m} m_{\tilde{\lambda}}(\beta)(\tilde{c}^{\beta}), \quad (6.43) \]

Here the partitions $\{\lambda(a)\}_{a=1, \ldots, l}$ and $\{\tilde{\lambda}(\beta)\}_{\beta=1, \ldots, m}$ have lengths

\[ \ell(\lambda(a)) = k_\alpha, \quad \ell(\tilde{\lambda}(\beta)) = \tilde{k}_\beta, \quad (6.44) \]

weights

\[ |\lambda(a)| = d_\alpha, \quad |\tilde{\lambda}(\beta)| = \tilde{d}_\beta, \quad (6.45) \]

and parts equal to the colengths $\ell(\mu(\alpha))$ and $\ell(\nu(\beta))$ for $\lambda(a)$ and $\tilde{\lambda}(\beta)$ respectively.

The combinatorial multispecies Hurwitz number $F_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu)$ is determined as follows [22]. Let $D_n$ denote the number of partitions of weight $n$. For each generating function $G_\alpha^a(w_\alpha)$ or $\tilde{G}_\beta^\beta(z_\beta)$, let $F_{G_\alpha}^{d_\alpha}$ and $F_{\tilde{G}_\beta}^{\tilde{d}_\beta}$ denote the $D_n \times D_n$ matrices whose elements are $F_{G_\alpha}^{d_\alpha}(\mu, \nu)$ and $F_{\tilde{G}_\beta}^{\tilde{d}_\beta}(\mu, \nu)$, respectively, as defined in (4.43), (4.44). From the fact that the central elements $\{G_\alpha^a(w_\alpha, J), \tilde{G}_\beta^\beta(z_\beta, J)\}$ all commute, it follows that so do the matrices $\{F_{G_\alpha}^{d_\alpha}, F_{\tilde{G}_\beta}^{\tilde{d}_\beta}\}$. Denoting the product of these in any order,

\[ F_{G(l,m)}^{(d,\tilde{d})} := \prod_{\alpha=1}^{l} F_{G_\alpha}^{d_\alpha} \prod_{\beta=1}^{m} F_{\tilde{G}_\beta}^{\tilde{d}_\beta}, \quad (6.46) \]

the $(\mu, \nu)$ matrix element $F_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu)$ is the combinatorial multispecies weighted Hurwitz number, and is equal to the geometrically defined one.

\[ F_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu) = H_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu) \quad (6.47) \]

The combinatorial meaning of $F_{G(l,m)}^{(d,\tilde{d})}(\mu, \nu)$ is as follows, Let

\[ d := \sum_{\alpha=1}^{l} d_\alpha + \sum_{\beta=1}^{m} \tilde{d}_\beta. \quad (6.48) \]
Then $F_{G_G([t,m])}^{(d,\bar{d})}(\mu,\nu)$, may be interpreted as the weighted sum over all sequences of $d$ step paths in the Cayley graph from an element $h \in \text{cyc}(\mu)$ in the conjugacy class of cycle type $\mu$ to one $(a_d b_d) \cdots (a_1 b_1) h$ of cycle type $\nu$, in which the transpositions appearing are subdivided into subsets consisting of $(d_1, \ldots, d_l, \bar{d}_1, \ldots, \bar{d}_m)$ transpositions in all $\frac{d_l!}{(\prod_{\alpha=1}^l d_{\alpha})!} \frac{\bar{d}_m!}{(\prod_{\beta=1}^m \bar{d}_{\beta})!}$ possible ways, and to each of these, if the signatures are $(\lambda^{(1)}, \ldots, \lambda^{(l)}, \bar{\lambda}^{(1)}, \ldots, \bar{\lambda}^{(m)})$, a weight is given that is equal to the product

$$\prod_{\alpha=1}^l e_{\lambda^{(\alpha)}}(c^\alpha) \prod_{\beta=1}^m h_{\bar{\lambda}^{(\beta)}}(\bar{c}^\beta).$$

6.49

Theorem 6.2.

$$F_{G_G([t,m])}^{(d,\bar{d})}(\mu,\nu) = H_{G_G([t,m])}^{(d,\bar{d})}(\mu,\nu).$$

6.50

For proofs of these results, see [22].

7 Quantum weighted Hurwitz numbers and Macdonald polynomials [23]

Only a summary of the results will be given here; for details see [23].

7.1 Generating functions for Macdonald polynomials

Following [36], for two infinite sets of indeterminates $x := (x_1, x_2, \ldots), y := (y_1, y_2, \ldots)$, we define the generating function

$$\Pi(x, y, q, t) := \prod_{a=1}^\infty \prod_{b=1}^\infty \frac{(tx_a y_b; q)_\infty}{(x_a y_b; q)_\infty},$$

7.1

where

$$(t; q)_\infty := \prod_{i=0}^\infty (1 - t q^i).$$

7.2

is the infinite $q$-Pochhammer symbol. $\Pi(q, t, x, y)$ can be expanded in a number of ways in terms of products of dual bases for the algebra $\Lambda$ of symmetric functions

$$\Pi(x, y, q, t) = \sum_{\lambda} P_\lambda(x, q, t) P_\lambda(y, q, t)$$

7.3

$$= \sum_{\lambda} z_{\mu}^{-1}(q, t) p_\lambda(x) p_\lambda(y)$$

7.4

$$= \sum_{\lambda} g_\lambda(x, q, t) m_\lambda(y)$$

7.5

$$= \sum_{\lambda} g_\lambda(y, q, t) m_\lambda(x)$$

7.6
where

\[ z_{\mu}(q, t) := z_{\mu} n_{\mu}(q, t), \quad n_{\mu} := \prod_{i=1}^{\ell(\mu)} \frac{1 - q_i^\mu}{1 - t_i^\mu}. \]  \tag{7.7}

Here \( \{P_{\lambda}(q, t, x)\} \) are the MacDonnell symmetric functions, which are orthogonal

\[ (P_{\lambda}(q, t), P_{\mu}(q, t))(q, t) = 0, \quad \lambda \neq \mu \]  \tag{7.8}

with respect to the inner product \(( , )_{(q, t)}\) in which the power sum symmetric functions \( \{p_{\lambda}\} \) satisfy

\[ (p_{\lambda}, p_{\mu})_{(q, t)} = z_{-1}^\mu(q, t) \delta_{\lambda\mu}, \]  \tag{7.9}

\( \{m_{\lambda}\} \) are the basis of monomial sum symmetric functions and

\[ g_{\lambda}(x, q, t) := \prod_{i=1}^{\ell(\lambda)} g_{\lambda_i}(x, q, t), \quad g_j(x, q, t) := (P_j, P_j)^{-1} P_j(q, t, x), \]  \tag{7.10}

is the \((q, t)\) analog of the interpolating function between the elementary \( e_{\lambda}(x) \) and complete \( h_{\lambda}(x) \) symmetric function bases.

### 7.2 Quantum families of central elements and weight generating functions

We now consider an extended infinite parametric family of generating functions \( M(q, t, c, z) \), depending on the infinite set of “classical” parameters \( c \) appearing in the infinite product representations as in (3.14), (3.18) as well as the further pair of “quantum deformation” parameters \((q, t)\) appearing in the MacDonnell polynomials [36].

For a dual pair of “classical” generating function \( G(z), \tilde{G}(z) \), with infinite product representations (3.14), (3.18) we introduce a \((q, t)\) deformed parametric family of weight generating functions \( M(q, t, c, z) \) as follows

\[ M(q, t, c, z) := \prod_{k=0}^{\infty} G(tq^k z) \tilde{G}(q^k z) = \prod_{k=0}^{\infty} \prod_{i=1}^{\infty} \frac{1 - tzq^k c_i}{1 - zq^k c_i}. \]  \tag{7.11}

The associated central element \( M_n(q, t, c, z, \mathcal{J}) \in \mathbb{Z}(\mathbb{C}[S_n]) \) is defined as

\[ M_n(q, t, c, z, \mathcal{J}) := \prod_{a=1}^{b} M(q, t, c, z, \mathcal{J}_a) = \Pi(c, z, \mathcal{J}, q, t). \]  \tag{7.12}

The eigenvectors are the orthogonal idempotents:

\[ M_{\lambda}(q, t, c, z) F_\lambda = t^{M(q, t, c, z)} F_\lambda \]  \tag{7.13}
and the eigenvalues $r^M_{\lambda}(q, t, c, z)$ have the usual content product form

$$
r^M_{\lambda}(q, t, c, z) = \prod_{(i, j) \in \lambda} M(q, t, c, z(j - i)) = \prod_{i = 1}^{\infty} \frac{(ztc_i; q)_{\lambda}}{(zc_i; q)_{\lambda}} \tag{7.14}
$$

where, for a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell(\lambda) > 0)$, the $q$-Pochhammer symbol $(t; q)_{\lambda}$ is defined as

$$
(t; q)_{\lambda} := \prod_{i=1}^{\ell(\lambda)} (tq^{i-1}; q)_{\lambda_i}, \quad (t; q)_{l} := \prod_{k=0}^{l-1} (1 - tq^k). \tag{7.15}
$$

### 7.3 MacDonald family of quantum weighted Hurwitz numbers

Using the content product formula (7.14), we define the associated 2D Toda $\tau$-function for $N = 0$ as

$$
\tau^M(q, t, c, z)(t, s) := \sum_{\lambda} r^M_{\lambda}(q, t, c, z) s_{\lambda}(t)s_{\lambda}(s) \tag{7.16}
$$

Substitution of the parameters $c$ and the Jucys-Murphy elements $(J_1, \ldots, J_n)$ for the indeterminates $x$ and $y$ in (7.5), (7.6) gives

$$
M_n(q, t, c, zJ) = \sum_{\lambda} g_{\lambda}(q, t, c) m_{\lambda}(J) \tag{7.17}
$$

$$
= \sum_{\lambda} g_{\lambda}(q, t, J) m_{\lambda}(c). \tag{7.18}
$$

Applying $M_n(q, t, c, zJ)$ to $C_\mu$ and using (7.17) and Theorem 4.1, gives

$$
M_n(q, t, c, zJ)C_\mu = \sum_{d=0}^{\infty} z^d \sum_{\nu, \vert \nu \vert = \vert \mu \vert} F^d_{M(q, t, c)}(\mu, \nu) z_\nu C_\nu \tag{7.19}
$$

where we define, as before, the combinatorial quantum weighted Hurwitz numbers $F^d_{M(q, t, c)}(\mu, \nu)$ associated to the weight generating function $M(q, t, c, z)$ as

$$
F^d_{M(q, t, c)}(\mu, \nu) := \frac{1}{\vert \mu \vert!} \sum_{\lambda, \vert \lambda \vert = d} g_{\lambda}(c, q, t) m_{\mu\nu}^\lambda. \tag{7.20}
$$

It follows as before that when the $\tau$-function $\tau^M(q, t, c, z)(t, s)$ is expanded in the basis of products of power sum symmetric functions, the coefficients are the quantum weighted Hurwitz combinatorial numbers $F^d_{M(q, t, c)}(\mu, \nu)$:

$$
\tau^M(q, t, c, z)(t, s) = \sum_{d=0}^{\infty} \sum_{\vert \mu, \nu \vert} z^d F^d_{M(q, t, c)}(\mu, \nu) p_\mu(t)p_\nu(s). \tag{7.21}
$$
The corresponding geometrically defined quantum weighted Hurwitz numbers are somewhat more intricate. Let \( \{\mu^{(i,u)}\}_{u=1,\ldots,k_i}, \{\nu^{(i,v)}\}_{v=1,\ldots,\tilde{k}_i}, \mu, \nu \) denote the branching profiles of an \( n \)-sheeted covering of the Riemann sphere, with two specified branch points of ramification profile types \((\mu, \nu)\), at \((0, \infty)\), and the rest divided into two classes I and II, denoted \( \{\mu^{(i,u)}\}_{u=1,\ldots,k_i} \) and \( \{\nu^{(i,v)}\}_{v=1,\ldots,\tilde{k}_i} \), respectively. These are further subdivided into \( l \) species, or “colours”, labelled by \( i = 1, \ldots, l \), the elements within each colour group distinguished by the labels \((u_i = 1, \ldots, k_i)\) and \((v_i = 1, \ldots, \tilde{k}_i)\). To such a grouping, we assign a partition \( \lambda \) of length

\[
\ell(\lambda) =: l
\]

and weight

\[
d := |\lambda| = \sum_{i=1}^{l} \left( \sum_{u=1}^{k_i} \ell^* (\mu^{(i,u)}) + \sum_{v=1}^{\tilde{k}_i} \ell^* (\nu^{(i,v)}) \right) = \sum_{i=1}^{l} d_i,
\]

whose parts \( (\lambda_1 \geq \cdots \geq \lambda_l > 0) \) are equal the total co-lengths

\[
d_i := \sum_{u=1}^{k_i} \ell^* (\mu^{(i,u)}) + \sum_{v=1}^{\tilde{k}_i} \ell^* (\nu^{(i,v)})
\]

in weakly decreasing order. By the Riemann-Hurwitz formula, the genus \( g \) of the covering curve is given by

\[
2 - 2g = \ell(\mu) + \ell(\nu) - d.
\]

We now assign a weight \( W_{Q(q)}(\{\mu^{(i,u)}\}, c) \) to each such covering as in (5.71), consisting of the product of all the weights \( W_{E(q)}(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i}), W_{H(q)}(\{\nu^{(i,v)}\}_{v=1,\ldots,\tilde{k}_i}) \) for the subsets of different colour and class with the weight \( m_\lambda(c) \) given by the monomial symmetric functions evaluated at the parameters \( c \)

\[
W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}), c) := W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}))m_\lambda(c)
\]

where

\[
W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)})) = \prod_{i=1}^{l} W_{E(q)}(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i} W_{H(q)}(\{\nu^{(i,v)}\}_{v=1,\ldots,\tilde{k}_i})
\]

Using these weights, for every pair \((d,e)\) of non-negative integers and \((\mu, \nu)\) of partitions of \( n \), we define the geometrical quantum weighted Hurwitz numbers \( H_{(d,e)}(\mu, \nu) \) as the sum

\[
H_{(d,e)}(\mu, \nu) := z_\nu \sum_{l=0}^{d} \sum'_{k_1, \ldots, \tilde{k}_l} W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}), c) H(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i}, \nu^{(i,v)}), c) \mid_{i=1,\ldots,l}
\]

We then have the following theorem, which is proved in [23]:

\[
H_{(d,e)}(\mu, \nu) := z_\nu \sum_{l=0}^{d} \sum'_{k_1, \ldots, \tilde{k}_l} W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}), c) H(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i}, \nu^{(i,v)}), c) \mid_{i=1,\ldots,l}
\]

\[
(7.22)
\]

\[
\ell(\lambda) =: l
\]

\[
d := |\lambda| = \sum_{i=1}^{l} \left( \sum_{u=1}^{k_i} \ell^* (\mu^{(i,u)}) + \sum_{v=1}^{\tilde{k}_i} \ell^* (\nu^{(i,v)}) \right) = \sum_{i=1}^{l} d_i,
\]

\[
(7.23)
\]

\[
2 - 2g = \ell(\mu) + \ell(\nu) - d.
\]

\[
(7.25)
\]

\[
W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}), c) := W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}))m_\lambda(c)
\]

\[
(7.26)
\]

\[
W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)})) = \prod_{i=1}^{l} W_{E(q)}(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i} W_{H(q)}(\{\nu^{(i,v)}\}_{v=1,\ldots,\tilde{k}_i})
\]

\[
(7.27)
\]

\[
H_{(d,e)}(\mu, \nu) := z_\nu \sum_{l=0}^{d} \sum'_{k_1, \ldots, \tilde{k}_l} W_{Q(q)}(\{\mu^{(i,u)}\}, \nu^{(i,v)}), c) H(\{\mu^{(i,u)}\}_{u=1,\ldots,k_i}, \nu^{(i,v)}), c) \mid_{i=1,\ldots,l}
\]

\[
(7.28)
\]
Theorem 7.1. The combinatorial Hurwitz numbers $F^d_{M(q,t,c)}(\mu, \nu)$ are degree $d$ polynomials in $t$, whose coefficients are equal to the geometrical quantum weighted Hurwitz numbers $H^{(d,e)}_{(c,q)}(\mu, \nu)\) :

$$F^d_{M(q,t,c)}(\mu, \nu) = \sum_{e=0}^{d} H^{(d,e)}_{(c,q)}(\mu, \nu)t^e. \quad (7.29)$$

Hence $\tau^M(q,t,c,z)(t, s)$, when expanded in the basis of products of power sum symmetric functions and power series in $z$ and $t$ is the generating function for the $H^{(d,e)}_{(c,q)}(\mu, \nu)'s:

$$\tau^M(q,t,c,z)(t, s) = \sum_{d=0}^{\infty} \sum_{e=0}^{d} z^d t^e H^{(d,e)}_{(c,q)}(\mu, \nu)p_{\mu}(t)p_{\nu}(s). \quad (7.30)$$

7.4 Examples

We now give several examples of special classes of weighted Hurwitz numbers that arise through restrictions or limits involving the parameters $(q, t, z)$. The details for all these examples are provided in [23]; we limit ourselves to specifying the restrictions and limits involved, giving only the generating functions, $\tau$-functions and quantum weighted Hurwitz formulae for each case.

7.4.1 Elementary quantum weighted Hurwitz numbers

In this case, we take the limits $z \to 0, t \to \infty$, but keeping the value of $-tz$ fixed at a finite value, that is renamed $z$. The resulting weight generating function is

$$E(q, c, z) := \prod_{k=0}^{\infty} \prod_{i=1}^{\infty} (1 + zq^k c_i) = \prod_{i=1}^{\infty} (-zc_i; q)_\infty =: \sum_{j=0}^{\infty} e_j(q, c)z^j \quad (7.31)$$

Here $e_j(q, c)$ is the quantum deformation of the elementary symmetric function $e_j(c)$. The corresponding central element $E_n(q, c, z, J) \in Z(C[S_n])$ is:

$$E_n(q, c, zJ) := \prod_{a=1}^{n} E(q, c, zJ_a) = \sum_{\lambda} z^{\lambda} e_{\lambda}(q, c) m_{\lambda}(J) = \sum_{\lambda} z^{\lambda} m_{\lambda}(J) e_{\lambda}(q, c), \quad (7.32)$$

where

$$e_{\lambda}(q, c) := \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(c). \quad (7.33)$$

Applying $E_n(q, c, zJ)$ to the orthogonal idempotents $\{F_{\lambda}\}$ to obtain the content product coefficients and to the cycle sums $\{C_{\mu}\}$ to obtain the Hurwitz numbers, the resulting hyper-
geometric 2D Toda $\tau$-function is

\[
\tau^{E(q,c,z)}(t, s) = \sum_{\lambda} E^{E(q,c,z)}_{\lambda}(t) s_{\lambda}(s)
\]

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda} E^{d}_{\lambda}(\mu, \nu) p_{\mu}(t) p_{\nu}(s),
\]

where

\[
E^{E(q,c,z)}_{\lambda} := \prod_{(ij) \in \lambda} (-zc_k; q)_\infty
\]

is the content product coefficient and

\[
E^{d}_{\lambda}(\mu, \nu) := \sum_{|\lambda| = d} e_{\lambda}(q, c) m_{\mu\nu}^\lambda
\]

is the weighted number of $d$-step paths in the Cayley graph of $S_n$ generated by transpositions, starting at the conjugacy class $\text{cyc}(\mu)$ and ending at $\text{cyc}(\nu)$, with weight $e_{\lambda}(q, c)$ for a path of signature $\lambda$.

Now consider again $n$-fold branched coverings of $\mathbb{CP}^1$ with a fixed pair of branch points at $(0, \infty)$ with ramification profiles $(\mu, \nu)$ and a further $\sum_{i=1}^{l} k_i$ branch points $\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}$ of $l$ different species (or “colours”), labelled by $i = 1, \ldots, l$, with non trivial ramifications profiles. The weight $W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}, c)$ for such a covering is defined by

\[
W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}, c) := W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}, m_{\lambda}(c))
\]

where

\[
W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}) := \prod_{i=1}^{l} W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}).
\]

It follows from the general result that

\[
E^{d}_{E(q,c)}(\mu, \nu) = H^{d}_{E(q,c)}(\mu, \nu),
\]

where

\[
H^{d}_{E(q,c)}(\mu, \nu) := z^d \sum_{l=0}^{d} \sum_{\{\mu^{(i,u_i)}\}, k_i \geq 1,}^{d^l} W^{E}(q)(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}, c) H(\{\mu^{(i,u_i)}\}_{u_i=1,\ldots,k_i}, \mu, \nu)
\]

is the geometrically defined quantum weighted Hurwitz number for this case.
7.4.2 Complete quantum weighted Hurwitz numbers

This is the dual of the preceding case, obtained by setting \( t = 0 \). The weight generating function becomes

\[
H(q, c, z) := \prod_{k=0}^{\infty} \prod_{i=1}^{\infty} (1 - zq^k c_i) = \prod_{i=1}^{\infty} (zq c_i; q)_\infty =: \sum_{j=0}^{\infty} h_j(q, c) z^j, \tag{7.42}
\]

where \( h_j(q, c) \) is the quantum deformation of the complete symmetric function \( h_j(c) \). The corresponding central element \( H_n(q, c, z, \mathcal{J}) \in \mathbb{Z}[C[S_n]] \) is:

\[
H_n(q, c, z, \mathcal{J}) := \prod_{a=1}^{n} H(q, c, \mathcal{J}_a) = \sum_{\lambda} z^{\lambda} h_\lambda(q, c) m_\lambda(\mathcal{J}) = \sum_{\lambda} z^{\lambda} m_\lambda(c) h_\lambda(q, \mathcal{J}), \tag{7.43}
\]

where

\[
h_\lambda(q, c) := \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}(q, c). \tag{7.44}
\]

By specializing the general case by setting \( t = 0 \), the resulting hypergeometric 2D Toda \( \tau \)-function is

\[
\tau^{H(q, c, z)}(t, s) = \sum_{\lambda} r_\lambda^{H(q, c, z)} s_\lambda(t) s_\lambda(s) \tag{7.45}
\]

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda} F_{d}^{H(q, c)}(\mu, \nu) p_{\lambda}(t) p_{\lambda}(s), \tag{7.46}
\]

where

\[
r_\lambda^{H(q, c, z)} := \prod_{(ij) \in \lambda} \prod_{k=0}^{\infty} (z(j-i)c_k; q)_\infty \tag{7.47}
\]

and

\[
F_{d}^{H(q, c)}(\mu, \nu) := \sum_{|\lambda|=d} h_\lambda(q, c) m_{\lambda}^{\mu, \nu} \tag{7.48}
\]

is the weighted number of paths in the Cayley graph of \( S_n \) generated by transpositions, starting at the conjugacy class \( \text{cyc}(\mu) \) and ending at \( \text{cyc}(\nu) \), with weight \( h_\lambda(q, c) \) for a path of signature \( \lambda \).

Consider again the \( n \)-fold branched covering of \( \mathbb{CP}^1 \), with a fixed pair of branch points at \((0, \infty)\) with ramification profiles \((\mu, \nu)\) and a further \( \sum_{i=1}^{l} \tilde{k}_i \) branch points \( \{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i} \) of \( l \) different species (or “colours”), labelled by \( i = 1, \ldots, l \), with nontrivial ramifications profiles. The weight \( W_{H_l(q)}(\{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i}, c) \) for such a covering is defined by

\[
W_{H_l(q)}(\{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i}, c) := W_{H_l(q)}(\{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i}) m_\lambda(c) \tag{7.49}
\]

\[
W_{H_l(q)}(\{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i}) := \prod_{i=1}^{l} W_{H(q)}(\{\nu^{(i, \nu_i)}\}_{\nu_i=1,...,\tilde{k}_i}). \tag{7.50}
\]
We again have
\[ F^d_{H(q,c)}(\mu, \nu) = H^d_{H(q,c)}(\mu, \nu), \]  
where
\[ H^d_{H(c,q)}(\mu, \nu) := z_{\nu} \sum_{l=0}^{d} \sum' W_{H(q)}(\{\nu^{(i,v_i)}\}_{i=1,\ldots,l}, c) \cdot H(\{\nu^{(i,v_i)}\}_{i=1,\ldots,l}, \mu, \nu). \]

\[ (7.52) \]

\subsection*{7.4.3 Hall-Littlewood function weighted Hurwitz numbers}

The generating function for Hall-Littlewood polynomials \( P_\lambda(x, t) \), is obtained by setting \( q = 0 \) in eq. (7.11). The orthogonality relations [36] become
\[ (P_\lambda, P_\mu)_t = \delta_{\lambda\mu}(b_\lambda(t))^{-1}, \quad b_\lambda(t) := \prod_{i \geq 1} \prod_{k=1}^{m_i(\lambda)} \frac{1}{1 - t^k}. \]

(7.53)

with respect to the scalar product \((\ ,\ ,\ )_t\) defined by
\[ (p_\lambda, p_\mu)_t = \delta_{\lambda\mu} z_{\lambda\nu} n_\lambda(t), \quad n_\lambda := \prod_{i=1}^{\ell(\lambda)} \frac{1}{1 - t^i}. \]

(7.54)

Defining, as in [36],
\[ q_\lambda(x, t) := b_\lambda(t) \prod_{i=1}^{\ell(\lambda)} P_j(x, t) \]

(7.55)

we obtain the following expansion
\[ L(t, x, y) = \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \sum_{\lambda} q_\lambda(x, t) m_\lambda(y) = \sum_{\lambda} q_\lambda(y, t) m_\lambda(x). \]

(7.56)

The corresponding central element of \( C[S_n] \) is then
\[ L(t, c, z\mathcal{J}) := \prod_{i=1}^{n} \prod_{a=1}^{n} \frac{1 - tc_i z\mathcal{J}_a}{1 - c_i z\mathcal{J}_a} = \sum_{\lambda} z^{|\lambda|} c^{|\lambda|} \lambda \omega(z\mathcal{J} \cdot t) = \sum_{\lambda} z^{|\lambda|} c^{|\lambda|} \lambda \omega(z\mathcal{J}, t). \]

(7.57)

The hypergeometric 2D Toda \( \tau \)-function then reduces to
\[ \tau^{L(t,c,z)}(s, t) := \sum_{\lambda} \tau^{L(t,c,z)}(s) \lambda(t) \lambda(s) = \sum_{d=0}^{\infty} \sum_{\lambda} z^d F^d_{L(t,c)}(\mu, \nu)p_\mu(t)p_\nu(s). \]

(7.58)

(7.59)

(7.60)
where the content product coefficient \( r^L_{(t,c,z)}(\lambda) \) is

\[
r^L_{(t,c,z)} := \prod_{(ij) \in \lambda} \prod_{k=1}^{\infty} \frac{1 - tz(j - i)c_k}{1 - z(j - i)c_k} = \prod_{k=1}^{\infty} (-t)^{|\lambda|} \frac{\left(-1/(tzc_k)\right)_\lambda}{\left(-1/(zc_k)\right)_\lambda},
\]

and

\[
F^d_{L(t,c)}(\mu, \nu) := \sum_{|\lambda| = d} q_\lambda(c, t) m^\lambda_{\mu\nu}
\]

is the weighted number of paths in the Cayley graph with the weight \( q_\lambda(c, t) \) for a path of signature \( \lambda \).

As in the general case, we also have

\[
F^d_{L(t,c)}(\mu, \nu) = \sum_{e=0}^{d} H^d_{(c,e)}(\mu, \nu) t^e
\]

where, denoting the total number of branch points,

\[
K := \sum_{i=1}^{l} (k_i + \tilde{k}_i)
\]

the weighted generalization of the multispecies hybrid signed Hurwitz numbers studied in [25] is

\[
H^d_{L(c)}(\mu, \nu) := z_{\nu} \sum_{l=0}^{d} \sum_{\ell^*} (-1)^{K + d - e} H\left(\left\{\mu^{(i,u_i)}\right\}_{i=1,...,l}, \left\{\nu^{(i,v_i)}\right\}_{i=1,...,l}, \mu, \nu\right).
\]

Its interpretation in terms of weighted enumerations of multispecies Hurwitz numbers of two classes is the same as in the general Macdonald case, with the general quantum weighting factor reducing to a sign times the standard classical one \( m_\lambda(c) \).

### 7.4.4 Jack function weighted Hurwitz numbers

The Jack polynomials \( P_\lambda^{(\alpha)} \) are obtained by setting \( t = q^\alpha \) and taking the limit \( q \to 1 \). These satisfy the orthogonality relations [36]

\[
\langle P_\lambda^{(\alpha)}, P_\mu^{(\beta)} \rangle_\alpha = \delta_{\lambda\mu} z_\lambda b_\lambda(t), \quad b_\lambda(t) := \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} (1 - t^j)
\]
where the scalar product $\langle \cdot , \cdot \rangle_\alpha$ is defined by [36]

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda \mu} z_\lambda \alpha^{\ell(\lambda)}. \quad (7.67)$$

The corresponding family of weight generating functions becomes

$$J(\alpha, c, z) := \prod_{k=1}^\infty (1 - z c_k)^{-1/\alpha} \quad (7.68)$$

and the central elements

$$J(\alpha, z, J) := \prod_{i=1}^\infty \prod_{a=1}^n (1 - z c, J_a)^{-1/\alpha} = \sum_{\lambda} z^{\ell(\lambda)} g_\alpha(\lambda) m_\lambda(c) = \sum_{\lambda} z^{\ell(\lambda)} g_\alpha(c) m_\lambda(J), \quad (7.69)$$

where the symmetric functions $g_\alpha(x)$ are the analogs of the $e_\lambda(x)$ or $h_\lambda(x)$ bases formed from products of elementary or complete symmetric functions [36]

$$g_\alpha(x) = \alpha^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} P_{(\lambda_i)}(x). \quad (7.70)$$

The associated hypergeometric 2D Toda $\tau$-function is

$$\tau^{J(\alpha, c, z)}(t, s) = \sum_{\lambda} r^{J(\alpha, c, z)}(\lambda) s_\lambda(t) s_\lambda(s) \quad (7.71)$$

where the content product coefficients are

$$r^{J(\alpha, c, z)}(\lambda) := \prod_{(ij) \in \lambda} \prod_{k=0}^\infty (1 - z(j - i)c_k)^{-1/\alpha} = \prod_{k=1}^\infty (1 - z c_k)^{-1/\alpha} \quad (7.72)$$

Expanding over products of power sum symmetric functions gives

$$\tau^{J(\alpha, c, z)}(t, s) = \sum_{d=0}^\infty \sum_{|\mu| = |\nu| = n} z^d F^{d}_{J(\alpha, c)}(\mu, \nu) p_\mu(t) p_\nu(s) \quad (7.73)$$

where

$$F^{d}_{J(\alpha, c)}(\mu, \nu) = \sum_{\lambda} g_\alpha(c) m^\lambda_{\mu\nu} \quad (7.74)$$

is the combinatorial Hurwitz number giving the weighted number of $d$-step paths of signature $\lambda$ in the Cayley graph of $S_n$, starting in the conjugacy class cyc($\mu$) and ending in cyc($\nu$), with weight $g_\alpha(c)$. We again have equality with the weighted geometrical Hurwitz number

$$F^{d}_{J(\alpha, c)}(\mu, \nu) = H^{d}_{J(\alpha, c)}(\mu, \nu), \quad (7.75)$$
where
\[ H^d_{\alpha,\omega}(\mu, \nu) := \sum_{k=0}^{\infty} \left( -\frac{1}{\alpha} \right)^k \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} \lambda_{\mu}(c) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \] (7.76)
with the sum is over partitions \( \lambda \) of length \( k \), and weight \( d \) whose parts are \( \{\ell^*(\mu^{(1)}), \ldots, \ell^*(\mu^{(k)})\} \).

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