BRST COHOMOLOGY AND HODGE DECOMPOSITION THEOREM
IN ABELIAN GAUGE THEORY

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Abstract

We discuss the Becchi-Rouet-Stora-Tyutin (BRST) cohomology and Hodge decomposition theorem for the two dimensional free $U(1)$ gauge theory. In addition to the usual BRST charge, we derive a local, conserved and nilpotent co(dual)-BRST charge under which the gauge-fixing term remains invariant. We express the Hodge decomposition theorem in terms of these charges and the Laplacian operator. We take a single photon state in the quantum Hilbert space and demonstrate the notion of gauge invariance, no-(anti)ghost theorem, transversality of photon and establish the topological nature of this theory by exploiting the concepts of BRST cohomology and Hodge decomposition theorem. In fact, the topological nature of this theory is encoded in the vanishing of the Laplacian operator when equations of motion are exploited. On the two dimensional compact manifold, we derive two sets of topological invariants with respect to the conserved and nilpotent BRST- and co-BRST charges and express the Lagrangian density of the theory as the sum of terms that are BRST- and co-BRST invariants. Mathematically, this theory captures together some of the key features of both Witten- and Schwarz type of topological field theories.

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1 Introduction

The principles of local gauge invariance have played a very significant role in the development of modern theoretical high energy physics up to the energy scale of the order of grand unification. One of the key features of these theories is the existence of first-class constraints on them. The most natural and handy framework for the covariant canonical quantization of such a class of theories is the BRST formalism [1,2]. In this scheme, we enlarge the phase space of the original gauge system by incorporating the gauge-fixing and the Faddeev-Popov ghost terms in the starting Lagrangian density. The ensuing theory is unitary and turns out to be invariant under a new, supersymmetric type and nilpotent BRST symmetry which incorporates the local gauge symmetry of the original Lagrangian density in a subtle way. This symmetry is generated by a conserved and nilpotent BRST charge $Q_B$. The foot-prints of the original gauge theory are encoded in the BRST charge because the requirement of the physical state condition $Q_B |\text{phys} > = 0$ leads to the annihilation of the physical states in the quantum Hilbert space by the first-class constraints of the original gauge theory (see, e.g., equation (5.3) below) [3-6]. The nilpotency of the BRST charge ($Q_B^2 = 0$) and the physical state condition ($Q_B |\text{phys} > = 0$) are the two key properties which are intimately connected with the differential geometry and its application to cohomology [5,6,9-12]. For instance, two BRST closed states ($Q_B |\text{phys} > = 0$, $Q_B |\text{phys} >' = 0$) in the quantum Hilbert space are said to be cohomologically equivalent if they differ by a BRST exact state (i.e., $|\text{phys} >' = |\text{phys} > + Q_B |\chi >$ for any nonzero $|\chi >$ in the Hilbert space). This property is analogous to the property of the exterior derivative of differential geometry where two closed forms (e.g., $df = 0$, $df' = 0$), defined on a compact manifold, are cohomologically equivalent if they differ by an exact form (i.e., $f' = f + dg$). One of the key theorems in the de Rham cohomology is the celebrated Hodge decomposition theorem defined on a compact manifold. This theorem states that on this manifold any $p$-form $f_p$ can be decomposed into a harmonic form $\omega_p$ ($\Delta \omega_p = 0$, $d \omega_p = 0$, $\delta \omega_p = 0$) an exact form $dg_{p-1}$ and a co-exact form $\delta h_{p+1}$ as follows:

$$ f_p = \omega_p + dg_{p-1} + \delta h_{p+1}, \quad (1.1) $$

where $\delta (= \pm \ast d \ast; \delta^2 = 0)$ is the Hodge dual of $d$ and the Laplacian $\Delta$ is defined as $\Delta = (d + \delta)^2 = d\delta + \delta d$ [9-12]. So far, the analogue of $d$ has been found out as the conserved and nilpotent BRST charge $Q_B$ which generates a nilpotent BRST symmetry for a locally gauge invariant Lagrangian density in any arbitrary dimension of spacetime. It will be, therefore, an interesting endeavour to express $\delta$ and $\Delta$ in terms of the local conserved charges corresponding to some specific symmetry properties of a given BRST invariant Lagrangian density in any particular dimension of spacetime.

The purpose of the present paper is to shed some light on the analogues of $\delta$ and $\Delta$ in the language of the nilpotent (for $\delta$), local, covariant and continuous symmetry properties.

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\[ ^\dagger \text{Some attempts have also been made to discuss the second class constraints in the framework of BRST formalism (see, e.g., Refs. [7,8] and references therein).} \]
of the free $U(1)$ gauge theory described by a BRST invariant Lagrangian density in two (1 + 1) dimensions (2D) of spacetime. Some attempts have been made towards this goal in any arbitrary dimension of spacetime for the Abelian as well as non-Abelian gauge theories. However, the symmetry transformations turn out to be nonlocal and noncovariant [13-16]. In the covariant formulation, the corresponding symmetries become even non-nilpotent and the nilpotency is restored only when certain specific restrictions are imposed [17]. We will demonstrate that for the 2D BRST invariant $U(1)$ gauge theory, a local, conserved and nilpotent (co)dual BRST charge $Q_D$ can be defined that generates a new local, covariant and nilpotent symmetry transformation under which the gauge-fixing term $\delta A = (\partial \cdot A)$ remains invariant. This property should be compared and contrasted with the usual BRST transformation under which the two-form $F = dA$ remains invariant in the $U(1)$ gauge theory. Further, we show that the anticommutator of both these charges $W = \{Q_B, Q_D\}$ is the analogue of the Laplacian operator $\Delta$ and it turns out to be the Casimir operator for the extended BRST algebra. It is, however, the topological nature of the 2D free $U(1)$ gauge theory that $W \to 0$ when equations of motion are exploited and all the fields are assumed to fall off rapidly at $x \to \pm \infty$. To be more specific, we demonstrate that, for a single photon state in the quantum Hilbert space, the BRST- and co-BRST symmetries are good enough to gauge away both the degrees of freedom of photon and the free 2D $U(1)$ gauge theory becomes topological in nature (see, e.g., Ref. [18]). In the framework of BRST cohomology and Hodge decomposition theorem, the topological nature of this theory is encoded in the vanishing of the Laplacian operator when equations of motion are exploited. In fact, the on-shell expression for the Laplacian operator ($W$) encompasses the left-over degrees of freedom in the theory. We derive two sets of topological invariants on the 2D compact manifold w.r.t. BRST- and co-BRST charges and express the Lagrangian density as well as energy momentum tensor as the sum of BRST- and co-BRST invariant parts. These properties, together with symmetry considerations, are essential to establish the topological nature of the 2D free $U(1)$ gauge theory.

The material of our work is organized as follows. In Sec. 2, we set up the notations and give the bare essentials of the BRST formalism for the $U(1)$ gauge theory in any arbitrary dimension of spacetime. In Sec. 3, we discuss various kinds of dualities in two dimensional free $U(1)$ gauge theory and derive expressions for the co-BRST charge and the Casimir operator. Sec. 4 is devoted to the derivation of the extended BRST algebra. The constraints on the physical states of the total Hilbert space are obtained in Sec. 5. We take a single photon state as the harmonic state of the Hodge decomposition theorem and demonstrate the strength of BRST cohomology in the analysis of transversality of photon, gauge invariance, no-ghost theorem, etc., and give a concise proof of the topological nature of this theory. We derive two sets of topological invariants on the 2D compact manifold w.r.t. BRST- and co-BRST charges in Sec. 6. Finally, we discuss our main results, make

\[ \text{Here the vector potential } A_{\mu} \text{ is defined through the one-form } A = A_{\mu} \, dx^{\mu}. \text{ Furthermore, it can be easily seen that the gauge-fixing term } (\partial \cdot A) = \delta A \text{ is the Hodge dual of the two form } F = dA \text{ for the Abelian gauge theory in any arbitrary dimension of spacetime (see, e.g., Ref. [11] for details).} \]
some concluding remarks and propose some speculative ideas for future investigations.

2 Preliminary: BRST Formalism

Let us begin with the BRST invariant Lagrangian density ($L_b$) for the D-dimensional $U(1)$ gauge theory in the Feynman gauge (see, e.g., [4-6])

$$L_b = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + B (\partial \cdot A) + \frac{1}{2} B^2 + i \bar{C} \Box C,$$

where $\Box = \partial_{\mu} \partial^\mu$ ($\mu = 0, 1, 2, ..., D - 1$), $(\partial \cdot A) = \partial_{\mu} A^{\mu}$, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, $B$ is the Nakanishi-Lautrup auxiliary field and $\bar{C}(C)$ are (anti)ghost fields with $\bar{C}^2 = C^2 = 0$. The local gauge symmetry of the starting Maxwell Lagrangian density ($-\frac{1}{4} F_{\mu\nu} F_{\mu\nu}$) is now traded with the off-shell nilpotent ($\delta_2^b = 0$) BRST symmetry transformations

$$\begin{align*}
\delta_b A_{\mu} &= \eta \partial_{\mu} C, & \delta_b F_{\mu\nu} &= 0, & \delta_b C &= 0, \\
\delta_b \bar{C} &= i \eta B, & \delta_b B &= 0, & \delta_b (\partial \cdot A) &= \eta \Box C,
\end{align*}$$

(2.2)

where $\eta$ is an anticommuting ($\eta C = -C \eta$, $\eta \bar{C} = -\bar{C} \eta$) spacetime independent transformation parameter. The following Lagrangian density obtained from (2.1) (with $B = - (\partial \cdot A)$)

$$L_B = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2 + i \bar{C} \Box C,$$

(2.3)

remains invariant under the on-shell ($\Box C = 0$) nilpotent ($\delta_2^B = 0$) BRST transformations

$$\begin{align*}
\delta_B A_{\mu} &= \eta \partial_{\mu} C, & \delta_B F_{\mu\nu} &= 0, & \delta_B C &= 0, \\
\delta_B \bar{C} &= -i \eta (\partial \cdot A), & \delta_B (\partial \cdot A) &= \eta \Box C,
\end{align*}$$

(2.4)

The conserved ($\dot{Q}_{b,B} = \delta_0 Q_{b,B} = 0$) and nilpotent ($Q_{b,B}^2 = 0$) BRST charge ($Q_{b,B}$)

$$Q_{b,B} = \int d^{(D-1)}x \ (B \dot{C} - \dot{B} C) \equiv \int d^{D-1}x \ [ \partial_0 (\partial \cdot A) C - (\partial \cdot A) \dot{C} ],$$

(2.5)

is the generator of the transformations (2.2) and (2.4) as the following equation

$$\delta_k \Phi = -i \eta [ \Phi, Q_k ]_\pm, \quad k = b, B,$$

(2.6)

where $(+)-$ stands for the (anti)commutator (depending on whether the generic field $\Phi$ is (fermionic)bosonic), generates the above transformations if one exploits the covariant canonical (BRST) quantization of the Lagrangian density (2.3)

$$\begin{align*}
\{ A_0(x, t), (\partial \cdot A)(y, t) \} &= -i \delta^{(D-1)}(x - y), \\
\{ A_i(x, t), E_j(y, t) \} &= i \delta_{ij} \delta^{(D-1)}(x - y), \\
\{ C(x, t), \dot{C}(y, t) \} &= \delta^{(D-1)}(x - y), \\
\{ \dot{C}(x, t), \dot{C}(y, t) \} &= -\delta^{(D-1)}(x - y),
\end{align*}$$

(2.7)

and all the rest of the (anti)commutators are zero.
The invariance of the ghost action $I_{F,P} = i \int d^D x \ C \square C$ under the global scale transformations: $C \rightarrow e^\lambda C, \ \bar{C} \rightarrow e^{-\lambda} \bar{C}$ leads to the derivation of a conserved charge ($Q_g$)

$$Q_g = -i \int d^{(D-1)}x \ (C \ \dot{\bar{C}} + \bar{C} \ \dot{C}). \quad (2.8)$$

Furthermore, the discrete symmetry: $C \rightarrow \pm i \bar{C}, \ \bar{C} \rightarrow \pm i C$ invariance of $I_{F,P}$ leads to the existence of a nilpotent and conserved anti-BRST charge $Q_{AB}$ whose expression as well as the symmetry transformations it generates, can be obtained from equations (2.5), (2.2) and (2.4) by the substitution: $C \rightarrow \pm i \bar{C}$. The following BRST algebra

$$\{Q_B, Q_B\} = \{Q_{AB}, Q_{AB}\} = 0,$$

$$\{Q_B, Q_{AB}\} = Q_B Q_{AB} + Q_{AB} Q_B = 0,$$

$$i[Q_g, Q_B] = +Q_B, \quad i[Q_g, Q_{AB}] = -Q_{AB}. \quad (2.9)$$

states that the ghost number is $+1$ for $Q_B$ and $-1$ for $Q_{AB}$.

At this stage, it is important to pin-point some of the salient features which will be relevant for our further discussions. First of all, the statements that have been made above, are valid in any arbitrary dimension of spacetime. Secondly, it can be checked that the transformations generated by $Q_B$ and $Q_{AB}$ anticommute ($\delta_B \delta_{AB} + \delta_{AB} \delta_B = 0$) when they act on any field. Thirdly, under both the transformations, it is the two-form $F = dA$ (or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) that remains invariant and not the gauge-fixing term $\delta A$ (or $\partial \cdot A$) which is dual to it. Finally, it is obvious that the anti-BRST charge $Q_{AB}$ is not the analogue of the dual exterior derivative $\delta = \pm * d *$ in the discussion of the cohomological aspects of BRST formalism. As a consequence, in any arbitrary spacetime dimension, there are no analogues of the dual exterior derivative $\delta$ and the Laplacian $\Delta$ in the language of the nilpotent, local, covariant and continuous symmetry properties of the BRST invariant Lagrangian (2.3) or (2.1). Hence, the Hodge decomposition theorem defined on a compact manifold, can not be implemented in the quantum Hilbert space of such theories. In 2D of spacetime, however, we shall demonstrate that symmetries of the BRST invariant Lagrangian density are such that there is one-to-one correspondence with the (dual)exterior derivative and the Laplacian operator of differential geometry and the (dual)BRST charge and the Casimir operator of the extended BRST algebra.

3 BRST-Type Symmetries and Dualities in 2D

In addition to the symmetries: $C \rightarrow \pm i \bar{C}, \ \bar{C} \rightarrow \pm i C$, the ghost action $i \int d^2 x \ \bar{C} \square C$ in 2D has another symmetry; namely, \(^5\)

$$\partial_\mu \rightarrow \pm i \varepsilon_{\mu\nu} \partial^\nu, \quad \varepsilon_{\mu\nu} \varepsilon^{\mu\lambda} = -\delta^\lambda_\nu, \quad (3.1)$$

\(^5\) We adopt here the notations in which the 2D flat Minkowski metric is: $\eta_{\mu\nu} = \text{diag}(+1, -1)$ and $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_0 \partial_0 - \partial_1 \partial_1, F_{01} = \partial_0 A_1 - \partial_1 A_0 = E = F^{10}, \varepsilon_{01} = \varepsilon^{10} = +1, (\partial \cdot A) = \partial_0 A_0 - \partial_1 A_1.$
under which the D'Alembertian \( \Box \) remains invariant. As a consequence, an analogue of the symmetry (2.4) for the Lagrangian density (2.3), can be obtained due to the additional symmetry property of the 2D ghost term. These two symmetries are juxtaposed as

\[
\begin{align*}
\delta_B A_\mu &= \eta \partial_\mu C, & \delta_D A_\mu &= -\eta \varepsilon_{\mu\nu} \partial^\nu \bar{C}, \\
\delta_B C &= 0, & \delta_D \bar{C} &= 0, \\
\delta_B \bar{C} &= -i\eta (\partial \cdot A), & \delta_D C &= -i\eta E, \\
\delta_B E &= 0, & \delta_D (\partial \cdot A) &= 0, \\
\delta_B (\partial \cdot A) &= \eta \Box C, & \delta_D E &= \eta \Box \bar{C},
\end{align*}
\]

(3.2)

where we have taken: \( C \rightarrow +i\bar{C}, \partial_\mu \rightarrow +i\varepsilon_{\mu\nu} \partial^\nu \) in deriving symmetry transformations \( \delta_D \) from the BRST symmetries \( \delta_B \). It can be checked that both these symmetry transformations are on-shell nilpotent for the Lagrangian (2.3). Furthermore, under the above transformations, it can be checked that the 2D BRST invariant Lagrangian density

\[
\mathcal{L}_B = \frac{1}{2}E^2 - \frac{1}{2}(\partial \cdot A)^2 + i \bar{C} \Box C,
\]

(3.3)

transforms to itself modulo some total derivative terms. We christen the \( \delta_D \) transformations in (3.2) as the dual-BRST transformations because in contrast to \( \delta_B \) transformations where electric field \( E \) is invariant, in the case of \( \delta_D \), it is the gauge-fixing term \( (\partial \cdot A) \) that remains invariant \( \ddagger \). Moreover, it is interesting to see that under \( \partial_\mu \rightarrow \pm i\varepsilon_{\mu\nu} \partial^\nu \), the gauge-fixing term and the kinetic energy term in the Lagrangian density (3.3), transform to each other. This mutual exchange can be obtained even with the transformation \( A_\mu \rightarrow \pm i\varepsilon_{\mu\nu} A^\nu \). However, the latter transformation is not the symmetry of the 2D ghost action with which the BRST-type symmetries are connected. Thus, we shall call the duality transformations as the ones in which \( C \rightarrow \pm i\bar{C}, \bar{C} \rightarrow \pm iC, \partial_\mu \rightarrow \pm i\varepsilon_{\mu\nu} \partial^\nu \). This will be the analogue of the Hodge dual operation \( * \) in the differential geometry connected with the cohomological aspects of differential forms defined on a compact manifold.

In 2D, the field strength tensor \( F_{\mu\nu} \) has only one independent component which corresponds to the electric field \( E \) and there is no magnetic field \( B \). Thus, the usual duality invariance of the Maxwell equations under \( E_i \rightarrow B_i, B_i \rightarrow -E_i \) in 4D cannot be obtained here in 2D. However, the gauge-fixing term \( \partial \cdot A = \partial_\mu A^\mu \) and the electric field \( E = -\varepsilon_{\mu\nu} \partial^\mu A^\nu \) are like scalar and pseudoscalar in 2D. Thus, a duality between gauge-fixing and the electric field can be defined. For instance, it can be seen that the following Maxwell equations

\[
\partial_\mu F^{\mu\nu} + \partial^\nu (\partial \cdot A) = 0,
\]

(3.4)

lead to two independent equations

\[
\begin{align*}
\partial_0 E + \partial_1 (\partial \cdot A) &= 0, \\
\partial_1 E + \partial_0 (\partial \cdot A) &= 0,
\end{align*}
\]

(3.5)

\( \ddagger \) Here and in what follows, we shall take only the (+) sign in the transformations: \( C \rightarrow \pm i\bar{C}, \bar{C} \rightarrow \pm iC, \partial_\mu \rightarrow \pm i\varepsilon_{\mu\nu} \partial^\nu \). However, analogous statements will be valid if we take (−) sign.

\( \ddagger \) As per our definition in the introduction, the gauge-fixing term \( \delta A = (\partial \cdot A) \) with \( \delta = \pm *d* \) is the Hodge dual of the two-form \( F = dA \) which is the electric field \( E \) here in 2D.
which remain invariant under the exchange of gauge field \( E \) and gauge-fixing term \((\partial \cdot A)\). The same symmetry can be achieved if the derivatives \( \partial_0 \) and \( \partial_1 \) are exchanged with each other. Furthermore, if we define \( \tilde{E} = E + i(\partial \cdot A) \), \( \tilde{B} = E - i(\partial \cdot A) \), the equation (3.5) can be re-expressed as: \( \partial_0 \tilde{E} + i\partial_1 \tilde{B} = 0; \ \partial_0 \tilde{B} - i\partial_1 \tilde{E} = 0 \), which respects duality-type invariance: \( \tilde{E} \rightarrow \tilde{B}, \tilde{B} \rightarrow -\tilde{E} \). Now we take the Hodge decomposition of the one-form \( A \) in terms of the analogues of a scalar field \( \phi \) and a pseudoscalar field \( \lambda \) in 2D

\[
A_\mu \, dx^\mu = \partial_\mu \phi \, dx^\mu + \varepsilon_{\mu \nu} \partial^\nu \lambda \, dx^\mu, \tag{3.6}
\]

and define \( \kappa = \phi + i\lambda \) and \( \xi = \phi - i\lambda \), the Maxwell equations (3.5) can be recast into

\[
\Box (\partial_0 \kappa - i\partial_1 \xi) = 0, \\
\Box (\partial_0 \xi + i\partial_1 \kappa) = 0, \tag{3.7}
\]

which have \( \kappa \rightarrow \xi, \ \xi \rightarrow -\kappa \) symmetry\(^*\) that resembles the usual duality symmetry of the Maxwell equations in more than 2D. In fact, due to the decomposition (3.6), the Lagrangian density (2.3) can be rewritten in the following form

\[
\mathcal{L}_B = \frac{1}{2}(\Box \lambda)^2 - \frac{1}{2}(\Box \phi)^2 + iC \Box C, \tag{3.8}
\]

which respects the symmetry transformations \( \partial_\mu \rightarrow \pm i\varepsilon_{\mu \nu} \partial^\nu, C \rightarrow \pm i\bar{C}, \bar{C} \rightarrow \pm iC, \lambda \rightarrow \pm i\phi, \phi \rightarrow \pm i\lambda \). For this Lagrangian density, the symmetry transformations \( \delta_B \) and \( \delta_D \) are

\[
\delta_B \phi = \eta \, C, \quad \delta_D \phi = 0, \\
\delta_B \lambda = 0, \quad \delta_D \lambda = -\eta \bar{C}, \\
\delta_B C = 0, \quad \delta_D C = i\eta \Box \lambda, \\
\delta_B \bar{C} = -i\eta \Box \phi, \quad \delta_D \bar{C} = 0. \tag{3.9}
\]

As symmetry transformations (3.2) and (3.9) are fermionic in nature, it is straightforward to check that their anticommutator \( \{\delta_B, \delta_D\} \) would also lead to a symmetry transformation for the Lagrangian (2.3). Such a symmetry transformation \( \delta_W = \{\delta_B, \delta_D\} \) with the transformation parameter \( \zeta = -i \eta \eta' \) is

\[
\delta_W C = \delta_W \bar{C} = 0, \\
\delta_W A_0 = \zeta (\partial_0 E + \partial_1 (\partial \cdot A)), \\
\delta_W A_1 = \zeta (\partial_1 E + \partial_0 (\partial \cdot A)), \tag{3.10}
\]

where \( \eta \) and \( \eta' \) are the anticommuting transformation parameters for the transformations \( \delta_B \) and \( \delta_D \). For the transformations \( \delta_W \), it is the ghost term of the Lagrangian density (2.3) which remains invariant and the gauge-fixing as well as the electric fields transform. These transformations can be recast in terms of \( \phi \) and \( \lambda \) fields of the Lagrangian density (3.8) as

\[
\delta_W \phi = -\zeta \Box \lambda, \quad \delta_W C = 0, \\
\delta_W \lambda = -\zeta \Box \phi, \quad \delta_W \bar{C} = 0. \tag{3.11}
\]

\(^*\) Notice that even though we denote \( \kappa, \xi \) and \( \tilde{E}, \tilde{B} \) by different symbols, they are complex conjugate to each other in each pair because \( E, (\partial \cdot A), \phi \) and \( \lambda \) are taken to be real.
Under all the above three symmetries, the action $S = \int d^2x \mathcal{L}_B$ remains invariant because $\delta_k S = 0$ for $k = B, D, W$ if all the fields fall off rapidly at $x \to \pm \infty$.

4 Extended BRST Algebra and Conserved Quantities

The additional two local, continuous and covariant symmetries $\delta_D$ (with $\delta_D^2 = 0$) and $\delta_W$ of the BRST invariant Lagrangian density for the $U(1)$ gauge theory in 2D lead to the following conserved charges due to the Noether theorem

$$Q_D = \int dx \left[ E \dot{\bar{C}} - \dot{E} \bar{C} \right], \quad W = \int dx \left[ \partial_0 (\partial \cdot A) E - (\partial \cdot A) \dot{E} \right],$$

which are in addition to the three charges $Q_B, Q_{AB}$ and $Q_g$ of Sec. 2 (cf. equations (2.5) and (2.8)). However, the presence of the symmetry $C \to \pm i \bar{C}$ and $\bar{C} \to \pm i C$, leads to the existence of the analogues of transformations (3.2), (3.9–3.11) under which the Lagrangian density (2.3) remains invariant. The set of all these charges for 2D case is

$$Q_B = \int dx \left[ \partial_0 (\partial \cdot A) C - (\partial \cdot A) \dot{C} \right], \quad Q_{AB} = i \int dx \left[ \partial_0 (\partial \cdot A) \bar{C} - (\partial \cdot A) \dot{\bar{C}} \right],$$

$$Q_D = \int dx \left[ E \dot{\bar{C}} - \dot{E} \bar{C} \right], \quad Q_{AD} = i \int dx \left[ E \dot{\bar{C}} - \dot{E} C \right],$$

$$W = \int dx \left[ \partial_0 (\partial \cdot A) E - \dot{E} (\partial \cdot A) \right], \quad Q_g = -i \int dx \left[ C \dot{\bar{C}} + \bar{C} \dot{C} \right].$$

If we exploit the covariant canonical (anti)commutators of equation (2.7), these conserved charges obey the following algebra

$$[W, Q_k] = 0, k = B, D, AB, AD, g,$n$$

$$Q_B^2 = Q_{AB}^2 = Q_D^2 = Q_{AD}^2 = 0,$n$$

$$\{Q_B, Q_D\} = \{Q_{AB}, Q_{AD}\} = W,$n$$

$$\{Q_B, Q_B\} = +Q_B, \quad \{Q_B, Q_{AB}\} = -Q_{AB},$$

$$\{Q_B, Q_D\} = -Q_D, \quad \{Q_B, Q_{AD}\} = +Q_{AD},$$

and all the rest of the (anti)commutators turn out to be zero. A few remarks are in order. First of all, we see that the operator $W$ is the Casimir operator for the whole algebra and its ghost number is zero. The ghost number of $Q_B$ and $Q_{AD}$ is $+1$ and that of $Q_D$ and $Q_{AB}$ is $-1$. There exist four nilpotent and conserved charges which generate (anti)BRST and (anti)dual BRST transformations. Now given a state $|\psi\rangle$ in the quantum Hilbert space with ghost number $n$ (i.e., $iQ_g |\psi\rangle = n |\psi\rangle$), it is straightforward to see that:

$$iQ_g Q_B |\psi\rangle = (n + 1) Q_B |\psi\rangle,$$n$$

$$iQ_g Q_D |\psi\rangle = (n - 1) Q_D |\psi\rangle,$$n$$

$$iQ_g W |\psi\rangle = n W |\psi\rangle,$$n

which demonstrate that, whereas $W$ keeps the ghost number of a state intact and unaltered, the operator $Q_B$ increases the ghost number by one and $Q_D$ reduces this number by one. This property is similar to the operation of a Laplacian, an exterior derivative and a dual
exterior derivative on a $n$-form defined on a compact manifold. Thus, we see that the
degree of the differential form is analogous to the ghost number in the Hilbert space, the
differential form itself is analogous to the quantum state in the Hilbert space, a compact
manifold has an analogy with the quantum Hilbert space and $d$, $\delta$ and $\Delta = d\delta + \delta d$ are
$Q_B$, $Q_D$ and $W$ respectively. It is a notable point that $d$ and $\delta$ can be also identified
with $Q_{AB}$ and $Q_{AD}$ because of the fact that the gauge parameter of the original local
gauge symmetry can be replaced either by a ghost field or by an antighost field due to the
symmetry properties of the ghost action.

There are other conserved quantities in the theory due to the form of the equations of
motion in 2D $U(1)$ gauge theory. For instance, it can be seen, from the equation of motion
(3.5), that the gauge-fixing term $(\partial \cdot A)$ and the electric field $E$ are conserved quantities.
Since these quantities are conjugate momenta w.r.t. $A_0$ and $A_1$ fields, they commute with
each other. In fact, one can construct infinite number of commuting conserved quantities
with $E$ and $(\partial \cdot A)$ because of the fact that the linear equation (3.5) is integrable (see, e.g.,
Ref. [19] for the nonlinear PDE like Boussinesq). Some of these quantities are:

\[
I_0 = \int dx \, E, \quad I_1 = \int dx \, (\partial \cdot A), \quad I_2 = \int dx \, E(\partial \cdot A),
I_3 = \int dx \, [E^2 + (\partial \cdot A)^2], \quad I_4 = \int dx \, [\partial_0(\partial \cdot A) \, E - (\partial \cdot A)\dot{E}],
I_k = \int dx \, Z_k, \quad k = 5, 6, 7, 8,
I_9 = \frac{1}{2} \int dx \, [Z_5^2 + Z_6^2], \quad I_{10} = \frac{1}{2} \int dx [Z_7^2 + Z_8^2],
\]

(4.5)

where the quantities $Z_k$ are functions of $(\partial \cdot A)$ and $E$ given by:

\[
Z_5 = \cos(\partial \cdot A) \cos(E), \quad Z_6 = \sin(\partial \cdot A) \sin(E),
Z_7 = \cos(\partial \cdot A) \sin(E), \quad Z_8 = \sin(\partial \cdot A) \cos(E).
\]

(4.6)

It can be checked that the $Z_k's$ obey the following equations

\[
\frac{\partial Z_k}{\partial t} = \frac{\partial Z_k'}{\partial x}, \quad \Box Z_k = 0, \quad (k = 5, 6, 7, 8),
\]

(4.7)

if we assume the validity of the on-shell conditions (3.5) for the $U(1)$ gauge fields $A_\mu$. Thus,
it will be noticed that even without taking recourse to the (anti)ghost fields, the conserved
quantity $I_4$, which is the Casimir operator $W$ of the algebra (4.3), can be constructed
just by looking at the equations of motion for the $U(1)$ gauge field $A_\mu$. In the algebra
(4.3), $W$ emerges due to the anticommutation relation between two charges which contain
(anti)ghost fields. It is not clear whether all the other conserved charges in (4.5) can be
expressed as the anticommutator of two fermionic charges which contain (anti)ghosts.

The equations of motion in the ghost sector $\Box C = \Box \dot{C} = 0$ are such that the following
quantities (as functions of (anti)ghost fields and their conjugate momenta)

\[
G_0 = -i \int dx \, (C \, \dot{C} + C \, \dot{C}),
G_1 = -i \int dx \, \dot{C}(x, t), \quad G_{-1} = i \int dx \, \dot{C}(x, t),
\]

(4.8)
are conserved on-shell if we assume that all the fields fall off rapidly at $x \to \pm \infty$ and there is no nontrivial topology at the boundary of the manifold. It can be seen that $G_0$ is the expression of the ghost charge $Q_g$. These charges obey the following algebra if we exploit the anticommutators of (2.7)

$$G_1^2 = \frac{1}{2} \{ G_1, G_1 \} = 0, \quad G_{-1}^2 = \frac{1}{2} \{ G_{-1}, G_{-1} \} = 0, \quad \{ G_1, G_{-1} \} = 0, \quad i\{ G_0, G_1 \} = +G_1, \quad i\{ G_0, G_{-1} \} = -G_{-1}. \quad (4.9)$$

The above algebra shows that the ghost number of $G_{\pm 1}$ is $\pm 1$ and they generate trivial symmetry transformations for (anti)ghost fields where these fields transform by a constant.

With the help of (4.7), one can construct other conserved charges which are analogous to $Q_B$ and $Q_D$ and contain $Z'_k$s and (anti)ghost fields; namely,

$$J_s = \int dx \left[ (\partial_\theta Z_s) C - Z_s \dot{C} \right],$$

$$J'_r = \int dx \left[ Z'_r \dot{C} - (\partial_\theta Z'_r) C \right], \quad (4.10)$$

where $r, s = 5, 6, 7, 8$ of (4.6) and (4.7). It is clear that the anticommutator of $J_s$ and $J'_r$ will also produce analogue of $I_4$. All the conserved quantities in (4.7), (4.8) and (4.10) generate certain symmetries. However, the conserved quantity $W = I_4$ is singled out from all the rest of the conserved quantities in (4.5) and (4.10) because of its very special nature. It turns out that, without resorting to the on-shell condition (3.5), the Lagrangian density (3.3) transforms to a total derivative under (3.10) which is generated by $I_4 = W$. The rest of the conserved quantities generate nontrivial symmetry of the Lagrangian density (3.3) only when the on-shell conditions (3.5) and $\Box C = \Box \bar{C} = 0$ are exploited. For instance, $I_2$ generates the transformations: $\delta_2 C = \delta_2 \bar{C} = 0$, $\delta_2 A_0 = -\rho \ E$, $\delta_2 A_1 = \rho \ (\partial \cdot A)$ where $\rho$ is a constant parameter. This transformation is a symmetry transformation of (3.3) only if the on-shell equations (3.5) are utilized. Thus, the Casimir operator $W = I_4$ is unique in some sense.

5 BRST Cohomology and Physical States

It is obvious from the algebra (4.3) and the consideration of the ghost number of states $(Q_B|\psi >, Q_D|\psi >$ and $W|\psi >$ in (4.4)) that one can now implement the Hodge decomposition theorem in the language of the BRST and dual-BRST charges

$$|\psi >_n = |\omega >_n + Q_B| \theta >_{n-1} + Q_D|\chi >_{n+1}, \quad (5.1)$$

by which, any state $|\psi >_n$ in the quantum Hilbert space with ghost number $n$ can be decomposed into a harmonic state $|\omega >_n$, a BRST exact state $Q_B|\theta >_{n-1}$ and a dual-BRST
exact state $Q_D |\chi >_{n+1}$. To refine the BRST cohomology, however, we have to choose a representative state from the total states of (5.1) as a physical state. We take here the physical state as the harmonic state: $|\text{phys} > = |\omega >$. By definition, such a state would satisfy the following conditions:

$$Q_B |\text{phys} > = 0, \quad Q_D |\text{phys} > = 0, \quad W |\text{phys} > = 0,$$

(5.2)

which leads to the following constraints on the physical states:

$$-\Pi^0 = (\partial \cdot A) |\text{phys} > = 0,$$
$$-\partial_1 E = \partial_0 (\partial \cdot A) |\text{phys} > = 0,$$
$$-\varepsilon_{\mu\nu}\partial^\mu A^\nu = E |\text{phys} > = 0,$$
$$-\partial_1 (\partial \cdot A) = \dot{E} |\text{phys} > = 0.$$

(5.3)

At this juncture, it is worthwhile to point out that the latter pair of constraints are related to the former ones by duality transformations $(\partial \cdot A) \rightarrow \pm i E, \ E \rightarrow \pm i (\partial \cdot A)$. This demonstrates that mathematically different looking theories in 2D, with different constraints (cf. (5.3)), are same theories because they are related to each-other by the duality transformations between gauge-fixing term $(\partial \cdot A)$ and classical gauge field $E$. The first pair of constraints are obtained by the requirement that $Q_B |\text{phys} > = 0$ which demonstrate that the first class constraints $\Pi^0 \approx 0$ (momentum w.r.t. $A_0$ field) and the Gauss law constraint $(\partial_1 E \approx 0)$ annihilate the physical state. The latter constraints are also interesting as they lead to the proof of topological nature of 2D free $U(1)$ gauge theory (see, e.g. Sect. 6). In fact, they restrict the physical (harmonic) states to a sector carrying zero electric flux. As a result, even the electric field turns out to be physically superfluous (in some sense) because of the presence of the new co-BRST symmetry. This happens due to the fact that there are no propagating degrees of freedom in the theory. Furthermore, the above restrictions imply the masslessness $(\Box A_\mu = 0)$ of the photon from the BRST- and co-BRST symmetries alone. In the normal formulation with a single charge $(Q_B)$, the topological nature of the 2D photon is proven by transversality requirement $(\partial \cdot A |\text{phys} > = 0)$ which emerges from symmetry considerations $(Q_B |\text{phys} > = 0)$ and masslessness condition which emerges from equation of motion $(\Box A_\mu = 0)$.

We shall dwell a bit more on the constraints (5.3) in the phase space representation and demonstrate that the physical state conditions (5.2) contain a lot of information about the gauge theory. Because of the simple form of the equations of motion $\Box A_\mu = 0, \Box C = 0$ and $\Box \bar{C} = 0$, it is very convenient to express the fields $A_\mu, C$ and $\bar{C}$ in terms of the normal mode expansion [21]

$$A_\mu(x,t) = \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ a_\mu(k)e^{-ik \cdot x} + a_\mu^\dagger(k)e^{ik \cdot x} \right],$$

$$C(x,t) = \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ c(k)e^{-ik \cdot x} + c^\dagger(k)e^{ik \cdot x} \right],$$

$$\bar{C}(x,t) = \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ b(k)e^{-ik \cdot x} + b^\dagger(k)e^{ik \cdot x} \right],$$

(5.4)
where $k_\mu$ is the 2D momenta with the components $(k_0, k = k_1)$. The symmetry transformations (3.2) can now be exploited to obtain the (anti)commutation relations. These are:

$$
\begin{align*}
{[Q_B, a^\dagger_\mu(k)]} &= -k_\mu c^\dagger(k), & {[Q_D, a^\dagger_\mu(k)]} &= \varepsilon_{\mu\nu} k^\nu b^\dagger(k), \\
{[Q_B, a_\mu(k)]} &= k_\mu c(k), & {[Q_D, a_\mu(k)]} &= -\varepsilon_{\mu\nu} k^\nu b(k), \\
{\{Q_B, c^\dagger(k)\}} &= 0, & {\{Q_D, c^\dagger(k)\}} &= i\varepsilon_{\mu\nu} k_\mu a^\dagger_\nu, \\
{\{Q_B, c(k)\}} &= 0, & {\{Q_D, c(k)\}} &= -i\varepsilon_{\mu\nu} k_\mu a_\nu, \\
{\{Q_B, b^\dagger(k)\}} &= -i k^\mu a^\dagger_\mu, & {\{Q_D, b^\dagger(k)\}} &= 0, \\
{\{Q_B, b(k)\}} &= +i k^\mu a_\mu, & {\{Q_D, b(k)\}} &= 0,
\end{align*}
$$

where we have inserted the normal mode expansion (5.4) in the symmetry transformation for the fields. Similarly, the Casimir operator $W$ generates the following commutation relations with creation and annihilation operators:

$$
\begin{align*}
{[W, a^\dagger_\mu(k)]} &= +i k^2 \varepsilon_{\mu\nu} (a^\nu)^\dagger, & {[W, a_\mu(k)]} &= -i k^2 \varepsilon_{\mu\nu} a^\nu, \\
{[W, c(k)]} &= [W, c^\dagger(k)] = [W, b(k)] = [W, b^\dagger(k)] = 0.
\end{align*}
$$

It is clear that if we exploit the on-shell condition (i.e., $k^2 = 0$) for the fields, the commutation relations generated by $W$ will be trivial.

Let us define the physical vacuum $|\text{vac}>$ of the theory as

$$
\begin{align*}
Q_B |\text{vac}> &= Q_D |\text{vac}> = W |\text{vac}> = 0, \\
\alpha_\mu(k) |\text{vac}> &= c(k) |\text{vac}> = b(k) |\text{vac}> = 0.
\end{align*}
$$

Now a single photon state with polarization $e_\mu$ can be created from the physical vacuum by the application of a creation operator $a^\dagger_\mu$; namely, $e_\mu a^\dagger_\mu |\text{vac}>$. We denote this state by $|e, \text{vac}>$. Similarly, a single photon state with momentum $k_\mu$ can be represented as $|k, \text{vac}> = k^\mu a^\dagger_\mu |\text{vac}>$. Exploiting the commutation relations of (5.5) and the physical state condition (5.7), this state can be written as $|k, \text{vac}> = Q_B (ib^\dagger(k)|\text{vac}>).$ Thus, the normal gauge transformation (with any arbitrary constant $\alpha$) can be expressed as

$$
|e + \alpha k, \text{vac}> = |e, \text{vac}> + Q_B(i \alpha b^\dagger(k))|\text{vac}>.
$$

According to the de Rham cohomology [9-12], all the BRST exact states are trivial. Thus, state $|e + \alpha k, \text{vac}>$ is equivalent to state $|e, \text{vac}>$ which demonstrates the gauge invariance in the theory. Now let us concentrate on the physicality criteria on one photon state with polarization $e_\mu$. Using the commutation relations from (5.5), it is clear that

$$
Q_B |e, \text{vac}> = -(k \cdot e) c^\dagger(k) |\text{vac}> = 0,
$$

which proves the transversality $k \cdot e = 0$ of photon because $c^\dagger(k)|\text{vac}>$ is not a null state. For the longitudinal or scalar photon for which $k \cdot e \neq 0$, we find that

$$
c^\dagger(k)|\text{vac}> = -\frac{1}{k \cdot e} Q_B |e, \text{vac}>,
$$

which is the statement of no-ghost theorem in the language of BRST cohomology. In fact, in the proof of unitarity of the non-Abelian gauge theory, it is well known [22] that
the ghost contributions cancel the contributions coming from the longitudinal or scalar gluons. Hence, ghosts are believed to exist in the virtual processes where there is a gluon loop contribution. The meaning of equation (5.10) can be stated in a different way. It says that any state with momentum \( k \), that is created by the application of \( \bar{c} \) on the physical vacuum, is a BRST exact state if corresponding gauge photon of momentum \( k \) is longitudinal or scalar. Thus, states corresponding to the longitudinal or scalar photon are trivial states from the point of view of BRST cohomology.

Now let us consider a dual state to the state \( |k, \text{vac}> \). This state can be written as: \( \varepsilon_{\mu\nu} k\mu(a\nu)^{\dagger}|\text{vac}>= iQ_D\bar{c}^{\dagger}(k)|\text{vac}> \). This shows that the dual state is a BRST co-exact state. Now the physicality criterion on the one photon state \( |e, \text{vac}> \) w.r.t. \( Q_D \) implies that \( \varepsilon_{\mu\nu}e^\mu k^\nu = 0 \). This is same as the transversality condition on photon if we take into account the masslessness \( (k^2 = 0) \) condition. The precise expression is

\[
Q_D |e, \text{vac}>= \varepsilon_{\mu\nu} e^\mu k^\nu b^{\dagger}(k) |\text{vac}>= 0.
\]

If photons are not transverse then the above equation implies the no-antighost theorem because the \( b^{\dagger}(k)|\text{vac} > \) state turns out to be BRST co-exact state. This, in turn, implies that the \( b^{\dagger}(k)|\text{vac} > \) state is not a physical state as far as the full BRST cohomology on the physical harmonic state is concerned. Similarly, we can apply the Casimir operator \( W \) on a single photon state \( |e, \text{vac}> \) and require the physicality criterion which ultimately leads to the masslessness condition \( k^2 = 0 \) of the photon. All these results are:

\[
\begin{align*}
Q_B |e, \text{vac}>& = 0 \Rightarrow k\cdot e = 0, \\
Q_D |e, \text{vac}>& = 0 \Rightarrow \varepsilon_{\mu\nu} e^\mu k^\nu = 0, \\
W |e, \text{vac}>& = 0 \Rightarrow k^2 = 0.
\end{align*}
\]

For the free \( U(1) \) gauge theory, the criteria of physicality condition lead to the relations (5.12) which are consistent with one-another. In fact, the top two basic relations imply the third one. Furthermore, the relations \( k\cdot e = 0 \) and \( \varepsilon_{\mu\nu} e^\mu k^\nu = 0 \) of (5.12) respect gauge invariance under transformations: \( e_\mu \to e_\mu + \alpha k_\mu, e_\mu \to e_\mu + \beta \varepsilon_{\mu\nu} k^\nu \) if \( k^2 = 0 \). Here \( \alpha \) and \( \beta \) are c-number constants. Normally, one defines a harmonic (a single photon) state as the one which is annihilated by the Laplacian operator \( (W|e, \text{vac}>) = 0 \). This condition, in turn, implies \( Q_B|e, \text{vac}>= 0, Q_D|e, \text{vac}>= 0 \). It can be readily seen that the condition \( k^2 = 0 \) finds its solution in the form of conditions \( k\cdot e = 0 \) (i.e., \( k_0 e_0 = k e_1 \)) and \( \varepsilon_{\mu\nu} e^\mu k^\nu = 0 \) (i.e., \( k_0 e_1 = k e_0 \)) which are the relations among the components of the momentum vector \( k_\mu \) and polarization vector \( e_\mu \). These relations emerge due to the conditions: \( Q_B|e, \text{vac}>= 0, Q_D|e, \text{vac}>= 0 \) respectively. Now comparing the expression for \( e_0 \) (i.e., \( e_0 = \frac{k e_1}{k_0} = \frac{k e_1}{k} \)), it can be seen that the masslessness condition emerges very naturally. Thus, these relations on a single photon state do satisfy the fact that \( W|e, \text{vac}>= 0 \) implies \( Q_B|e, \text{vac}>= 0, Q_D|e, \text{vac}>= 0 \) in a subtle way.

\(*\) A dual state can be obtained from a state by the transformations \( k_\mu \to \pm i \varepsilon_\mu k^\nu, C \to \pm i \bar{C}, \bar{C} \to \pm i C \). This is the analogue of the Hodge \( * \) operation of cohomology in the language of symmetry transformations. It can be seen explicitly that \( Q_B|\psi>= - * Q_D *|\psi> \), where \( |\psi> \), in general, may depend on \( k, C, \bar{C} \).
It is obvious that the states \( c^\dagger(k)|\text{vac}\rangle, b^\dagger(k)|\text{vac}\rangle \) are BRST exact and co-exact respectively (\textit{c.f.} (5.10) and (5.11)). Thus, these are not the physical states. It is interesting to note that a single photon state (\( a^\dagger_\mu(k)|\text{vac}\rangle \)) in 2D can also be written as the sum of BRST- and co-BRST exact states. This happens here because of the topological nature of the theory. The existence of BRST- and co-BRST symmetries enables one to decompose both the degrees of freedom of a single 2D photon into a component parallel to the momentum vector (BRST exact state) and the other component parallel to the polarization vector (co-BRST exact state) (see, \textit{e.g.} eqns. (3.6), (5.8), (5.11)). Thus, a 2D photon is also not a physical state \textit{per se}. This statement is in conformity with its topological nature.

6 Topological Invariants

It is evident from equation (5.2) that for a single physical photon state, we obtain conditions (5.12) due to the BRST cohomology. These are mutually consistent with one-another. In other words, the validity of any two of them implies the third condition. Thus, if basic symmetries are the guiding principles, the operation of \( W \) on a single physical photon state is superfluous because the symmetry generated by \( W \) can be obtained from the ones generated by \( Q_B \) and \( Q_D \). In fact, the presence of the basic BRST- and dual BRST symmetries are good enough to gauge away both the physical degrees of freedom of photon in 2D. Further, as a consequence of the presence of these two basic symmetries (\( \partial \cdot A = 0, \varepsilon_{\mu\nu}\partial^\mu A^\nu = 0 \)), the 2D physical photon is forced to propagate on its mass-shell as well as on-shell (\( \Box A_\mu = 0 \)). Thus, free \( U(1) \) gauge theory becomes topological in nature [18]. In the framework of the BRST cohomology and the Hodge decomposition theorem, this fact is encoded in the vanishing of the Laplacian operator \( W \)

\[
W = \int dx \frac{d}{dx} \left[ \frac{1}{2} (\partial \cdot A)^2 - \frac{1}{2} E^2 \right] \to 0 \quad \text{as} \quad x \to \pm \infty, \quad (6.1)
\]

when the equation of motion \( \partial_\mu E - \varepsilon_{\mu\nu}\partial^\nu(\partial \cdot A) = 0 \) is exploited. Physically, the on-shell expression for the Laplacian operator encompasses the physical degrees of freedom left-over in the theory after some (or all) of them have been gauged away by BRST- and co-BRST symmetries. Thus, expression (6.1) justifies the topological nature of the 2D free \( U(1) \) gauge theory. This situation should be contrasted with the interacting \( U(1) \) gauge theory where the gauge field couples with the Dirac fields in 2D. As it turns out, the on-shell expression of the Laplacian(Casimir) operator \( W \) contains only the fermionic degrees of freedom (present in the theory) and it does not go to zero on the on-shell [23].

The topological nature of this theory is confirmed by the existence of two sets of topological invariants w.r.t. conserved and on-shell (\( \Box C = \Box \bar{C} = 0 \)) nilpotent (\( Q_B^2 = 0, Q_D^2 = 0 \)) BRST- and co-BRST charges. For the 2D compact manifold, these are

\[
I_k = \oint_{C_k} V_k, \quad J_k = \oint_{C_k} W_k, \quad (k = 0, 1, 2), \quad (6.2)
\]

where \( C_k \) are the k-dimensional homology cycles in the 2D manifold and \( V_k \) and \( W_k \) are
where, as the above expression shows, \( d \) and \( k \) are not completely independent.

All practical purposes, it can be seen that one can take \( \delta \) and \( V \) to be the following relations (see, e.g., \( \square C = \square \bar{C} = 0 \)) because \( \delta B(\partial \cdot A) = \eta \square C \) and \( \delta_D E = \eta \square \bar{C} \). Thus, for all practical purposes, it can be seen that one can take \( \delta B(\partial \cdot A) = 0 \) and \( \delta_D E = 0 \) on the on-shell. The above BRST- and co-BRST invariants (cf. (6.2) and (6.3)) can be also written for the off-shell nilpotent BRST- and co-BRST transformations as the Lagrangian density (3.3) can be re-expressed as:

\[
\mathcal{L}_B = BE - \frac{1}{2} B^2 + B(\partial \cdot A) + \frac{1}{2} B^2 + i\bar{C}\square C,
\]

by introducing two auxiliary fields \( \mathcal{B} \) and \( B \). The off-shell nilpotent BRST- and co-BRST symmetry transformations for the above Lagrangian density (6.4) are same as the transformations (3.2) except for the ghost- and auxiliary fields. These additional transformations are:

\[
\begin{align*}
\delta_B \bar{C} &= i\eta B, \\
\delta_D C &= -i\eta \mathcal{B},
\end{align*}
\]

Now the BRST- and co-BRST invariants (w.r.t. off-shell nilpotent BRST- and co-BRST transformations) can be expressed in terms of the auxiliary fields \( B \) and \( B \) by using the equations of motion \( \mathcal{B} = E \) and \( B = -(\partial \cdot A) \) (cf. (6.2) and (6.3)). Furthermore, it can be checked that \( V_2 \) and \( W_2 \) are closed \( (dV_2 = 0) \) and co-closed \( (\delta W_2 = 0) \) respectively. The ghost number for \( V_k \) and \( W_k \) are \((+1,0,-1)\) and \((-1,0,+1)\) respectively as can be seen from the following commutation relations:

\[
\begin{align*}
i[Q_g, V_k] &= (-1)^{1-k} (k-1) V_k, \\
i[Q_g, W_k] &= (-1)^{1-k} (1-k) W_k,
\end{align*}
\]

where \( k = 0, 1, 2 \) and \( Q_g \) is the ghost charge. The above BRST- and co-BRST invariants obey the following relations (see, e.g., [24,25])

\[
\begin{align*}
\delta_B V_k &= \eta \, d \, V_{k-1}, \\
\delta_D W_k &= \eta \delta \, W_{k-1},
\end{align*}
\]

where, as the above expression shows, \( d \) and \( \delta \) are exterior- and dual exterior derivatives respectively on the 2D compact manifold. Both these sets of topological invariants are connected with each-other by the duality transformations: \((\partial \cdot A) \rightarrow iE, \bar{C} \rightarrow i\bar{C}, \partial_k \rightarrow i\varepsilon_{\mu\nu}(B \rightarrow -iB)\) as \( I_k \rightarrow J_k \). Thus, these invariants are not entirely independent of each-other as, in some sense, the exterior derivative \( d \) and the dual exterior derivative \( \delta(= \pm \ast d \ast) \) are not completely independent.
It is interesting to verify that the Lagrangian density (3.3), modulo some total derivatives, can be written as the sum of anticommutators with BRST- and co-BRST charges:

$$\mathcal{L}_B = \{Q_D, S_1\} + \{Q_B, S_2\}, \tag{6.8}$$

where $S_1 = \frac{1}{2} EC, S_2 = -\frac{1}{2}(\partial \cdot A)\bar{C}$. Using the fact that $Q_r (r = B, D)$ is the generator of the transformations $\delta_r \Phi = -i\eta [\Phi, Q_r]$, where $(+)$—stands for the (anti)commutator depending on the generic field $\Phi$ being (fermionic)bosonic, it can be checked that: $\eta \mathcal{L}_B = \frac{1}{2} \delta_D [iEC] - \frac{1}{2} \delta_B [(\partial \cdot A)\bar{C}]$. Mathematically, this observation shows that the 2D free $U(1)$ gauge theory is similar in outlook as the Witten type topological field theories [25] but quite different from the Schwarz type topological theories [26]. To be very precise, there is a bit of difference with the Witten type theories as well. This is primarily because of the fact that in our discussion there are two nilpotent charges w.r.t. which topological invariants and the BRST cohomology are defined whereas in the Witten type theories there is only one nilpotent BRST charge [25] which is obtained by combining a topological shift symmetry with the local gauge symmetry. It is clear, however, that there are no shift symmetries in our whole discussions. Thus, from symmetry point of view, the free 2D $U(1)$ gauge theory is more like Schwarz type topological theories where only local gauge symmetries exist. As a consequence, this 2D free theory is a new type of topological field theory which captures together some of the salient features of both Witten- and Schwarz type topological theories. For both (i.e., Witten as well as Schwarz) type of theories the energy-momentum tensor $(T_{\alpha\beta})$ is always a BRST (anti)commutator. It can be seen that for the free 2D $U(1)$ gauge theory, the expression for the symmetric $T_{\alpha\beta}$ is:

$$T_{\alpha\beta} = -\frac{1}{2} \left[ \varepsilon_{\alpha\rho} E + \eta_{\alpha\rho}(\partial \cdot A) \right] \partial_{\beta}A^\rho - \frac{1}{2} \left[ \varepsilon_{\beta\rho} E + \eta_{\beta\rho}(\partial \cdot A) \right] \partial_{\alpha}A^\rho$$

$$- i \partial_{\alpha}C \partial_{\beta}C - i \partial_{\beta}C \partial_{\alpha}C - \eta_{\alpha\beta} \mathcal{L}_B, \tag{6.9}$$

where $\mathcal{L}_B$ is the Lagrangian density of equation (3.3) (or (6.8)). This energy momentum tensor, modulo some total derivatives, can be re-expressed as:

$$\eta T_{\alpha\beta} = \frac{i}{2} \delta_B \left[ \partial_{\alpha}C A_{\beta} + \partial_{\beta}C A_{\alpha} + \eta_{\alpha\beta}(\partial \cdot A) \bar{C} \right]$$

$$+ \frac{i}{2} \delta_D \left[ \partial_{\alpha}C \varepsilon_{\beta\rho} A^\rho + \partial_{\beta}C \varepsilon_{\alpha\rho} A^\rho - \eta_{\alpha\beta} E C \right], \tag{6.10}$$

where $\delta_B$ and $\delta_D$ correspond to transformations in (3.2). In terms of BRST- and dual BRST charges, we can write $T_{\alpha\beta}$ as:

$$T_{\alpha\beta} = \{Q_B, V^{(1)}_{\alpha\beta}\} + \{Q_D, V^{(2)}_{\alpha\beta}\}, \tag{6.11}$$

where

$$V^{(1)}_{\alpha\beta} = \frac{1}{2} \left[ \partial_{\alpha}C A_{\beta} + \partial_{\beta}C A_{\alpha} + \eta_{\alpha\beta}(\partial \cdot A)\bar{C} \right]$$

$$V^{(2)}_{\alpha\beta} = \frac{1}{2} \left[ \partial_{\alpha}C \varepsilon_{\beta\rho} A^\rho + \partial_{\beta}C \varepsilon_{\alpha\rho} A^\rho - \eta_{\alpha\beta} E C \right]. \tag{6.12}$$

Thus, for the theory under discussion, the form of energy-momentum tensor is just like Witten as well as Schwarz type of topological theories. It is now straightforward to argue
that the partition functions as well as the expectation values of the BRST invariants, co-
BRST invariants and the topological invariants are metric independent. The key point to
show this fact in the framework of the BRST cohomology and the Hodge decomposition
theorem is the requirement that $Q_B|\text{phys}>=0, Q_D|\text{phys}>=0$ (see, e.g., Ref. [18] for
details) and the metric independence of the path integral measure (see, e.g., Ref [24] for
details). We have taken here only the flat Minkowski metric. However, our arguments and
discussions are valid even if we take a nontrivial metric. In Ref. [24], it has been argued
that the fermionic-bosonic symmetry of the BRST formalism is good enough to show that
the path integral measure will be independent of the choice of the metric.

7 Discussion

It is clear that the usual nilpotent BRST transformations correspond to a symmetry in
which the two-form $F = dA$ (e.g., electric field $E$ in 2D) of the U(1) gauge theory remains
invariant. The nilpotent dual-BRST charge is the generator of a transformation in which
the gauge-fixing term ($(\partial \cdot A) = \delta A$) remains invariant. The anticommutator of these
two transformations corresponds to a symmetry that is generated by the Casimir operator
for the whole algebra. Under this conserved operator, it is the ghost fields that remain
invariant. We see from algebra (4.3) and the ghost number considerations in (4.4) that
the generators $Q_B, Q_D$ and $W$ of symmetry transformations correspond to the geometrical
quantities $d, \delta$ and $\Delta$ of differential geometry which describe the de Rham cohomology
of differential forms on a compact manifold. It is, however, the peculiarity of the BRST
formalism that these geometrical quantities can be identified with two conserved charges.
For instance, in addition to the previous identifications, $d$ and $\delta$ could also be identified with
$Q_{AB}$ and $Q_{AD}$. Thus, $W$ and $\Delta$ can be expressed in two different ways : $W = \{Q_B, Q_D\} =
\{Q_{AB}, Q_{AD}\}$ and $\Delta = d\delta + \delta d = \delta\delta + \delta\delta$. This shows that the compact manifold, on which
$d, \delta$ and $\Delta$ are defined, should be a complex manifold as far as the analogy with BRST
cohomology of physical states in the quantum Hilbert space is concerned.

We know that the classical (e.g., electric and magnetic) fields correspond to the two-
form $F = dA$ which turn out to be gauge- and BRST invariant because of the structure
of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the specific tranformation on $A_\mu$.
We know from the celebrated Aharonov-Bohm effect that it is the vector potential (i.e.
one-form $A = A_\mu dx^\mu$) that plays decisive role at the quantum level and not the classical
electric or magnetic fields $F = dA$. Thus, symmetry transformation that leaves the gauge-
fixing term ($\delta A$) invariant is a quantum mechanical symmetry (right from the outset) by
its very nature. It is precisely due to this reason that the physical state condition with
dual-BRST charge ($Q_D |\text{phys}>= 0$) leads to a quantum mechanical restriction on the
physical state when $U(1)$ gauge field is coupled to the Dirac fields. In fact, it has been
shown in Ref. [23] that the dual-BRST transformation $\delta_D A_\mu = -\eta \varepsilon_{\mu\nu} \partial^\nu \bar{C}$ corresponds
to the chiral transformation on the Dirac fields for fermions in 2D. It is obvious that the
condition \(Q_D |phys >= 0\) would shed some light on the 2D anomaly term \((E \sim \varepsilon_{\mu\nu} F^{\mu\nu})\) in QED. Thus, full strength of the BRST cohomology might provide a clue to the well known result that in 2D, the “anomalous” gauge theory is consistent, unitary and amenable to particle interpretation [27,20]. To begin with, the theory under discussion (Ref. [23]) is not a chiral gauge theory in 2D. The chiral symmetry in our discussion is respected at the quantum level as it corresponds to the dual BRST symmetry.

For the free \(U(1)\) gauge theory, all the conditions in equation (5.12) are consistent with one-another and physically they imply the masslessness and transversality of photon. In fact, the existence of gauge symmetries: \(e_\mu \to e_\mu + \alpha k_\mu, e_\mu \to e_\mu + \beta \varepsilon_{\mu\nu} k_\nu\) (for \(\alpha\) and \(\beta\) being arbitrary constants), imply that both the degrees of freedom of photon \(A_\mu\) can be gauged away in two dimensions of spacetime by symmetry considerations alone and masslessness condition \(k^2 = 0\) emerges due to these symmetries. Thus, this theory becomes topological in nature as there are no propagating degrees of freedom left in the theory [18]. Normally, it is the masslessness \(k^2 = 0\) and transversality \(k \cdot e = 0\) criteria that are sufficient to get rid of both the degrees of freedom of photon in 2D. The former comes out from the equation of motion \((\Box A_\mu = 0)\) and the latter is a gauge-fixing constraint imposed on the theory \((\partial \cdot A = 0)\). In our discussion, however, the transversality of the photon \((k \cdot e = 0)\) and the relation \((\varepsilon_{\mu\nu} e^\nu k^\nu = 0)\) between the polarization vector \(e_\mu\) and momentum vector \(k_\mu\), emerge because of the presence of BRST- and co-BRST symmetries. In fact, these two basic (BRST and co-BRST) symmetries gauge away both the physical degrees of freedom of photon and, in a subtle way, these are the solutions to the masslessness condition \(k^2 = 0\) which emerges due to \(W|e, vac >= 0\) (see, e.g., eqn. (5.12)). However, it due to the topological nature of this theory that \(W \to 0\) when equations of motion are exploited. The existence of the topological invariants on 2D compact manifold confirms the topological nature of this theory in a cogent way. Such arguments have also been provided for the proof of topological nature of 2D free non-Abelian gauge theory (having no interaction with matter fields) [28]. It will be interesting to understand the BRST cohomology and Hodge decomposition theorem for the interacting theory in two- and four dimensions of spacetime where there is a coupling between the (non)Abelian gauge field and matter fields [29]. Some of these results have been obtained for the interacting \(U(1)\) gauge theory in Ref. [23] where the dual BRST transformation on the gauge field has been shown to correspond to the chiral transformation on the Dirac fields in 2D spacetime.

The upshot of our whole discussion is to capitalize on the insight gained in the case of 2D free \(U(1)\) gauge theory and generalize these ideas to 4D. In fact, it is already known that the duality of the Maxwell equations and the chirality of the massless fermions are very intimately connected even in 4D (see, e.g., Ref. [30]). It will be interesting to study the implication of the dual BRST symmetry in four dimensional spacetime when two-potentials are present in the field strength tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon_{\mu\nu\lambda\xi} \partial^\lambda V^\xi\) where \(V_\mu\) is an axial-vector [30]. This understanding might shed some light on the axial-vector anomaly in 4D and the Dirac quantization condition in QED. The latter is connected to the duality
in electric-magnetic couplings due to presence of two potentials and their coupling with matter (Dirac) fields. In the case of 4D non-Abelian gauge theories, the complete understanding of $Q_B, Q_D$ and $W$ might provide some insight into the problem of confinement of quarks and gluons in the language of BRST formalism [31,32]. These are some of the issues for future investigations [29].

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References

1. C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42, 127 (1975); Ann. Phys. (N.Y.) 98, 287 (1976).
2. I. V. Tyutin, Lebedev Preprint FIAN report no 39, (1975) (unpublished).
3. K. Sundermeyer, Constrained Dynamics: Lecture Notes in Physics Vol. 169, (Springer-Verlag, Berlin, New York, 1982).
4. K. Nishijima, in Progress in Quantum Field Theory eds. H. Ezawa and S. Kamefuchi, (North-Holland, Amsterdam, 1986) p.99.
5. M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, Princeton, 1992).
6. N. Nakanishi and I. Ojima, Covariant Operator Formalism of Gauge Theories and Quantum Gravity, (World Scientific, Singapore, 1990).
7. I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. A6, 3255 (1991).
8. I. A. Batalin, S. L. Lyakhovich and I. V. Tyutin, Mod. Phys. Lett. A7, 1931 (1992); Int. J. Mod. Phys. A10, 1917 (1995).
9. T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. 66, 213 (1980).
10. J. W. van Holten, Phys. Rev. Lett. 64, 2863 (1990); Nucl. Phys. B339, 158 (1990).
11. S. Mukhi and N. Mukunda, Introduction to Topology, Differential Geometry and Group Theory for Physicists, (Wiley Eastern Ltd., New Delhi, 1990).
12. H. Aratyn, J. Math. Phys. 31, 1240 (1990).
13. D. McMullan and M. Lavelle, Phys. Rev. Lett. 71, 3758 (1993); ibid. 75, 4151 (1995).
14. V. O. Rivelles, Phys. Rev. Lett. 75, 4150 (1995); Phys. Rev. D 53, 3257 (1996).
15. H. S. Yang and B. -H. Lee, J. Math. Phys. 37, 6106 (1996).
16. R. Marnelius, *Nucl. Phys.* **B494**, 346 (1997).
17. T. Zhong and D. Finkelstein, *Phys. Rev. Lett.* **73**, 3055 (1994); *ibid.* **75**, 4152 (1995).
18. D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* **209**, 129 (1991).
19. R. P. Malik, *Int. J. Mod. Phys.* **A12**, 231 (1997), see also, *Commuting conserved quantities in nonlinear realizations of W₃*, JINR- report no. **E2-96-120** (1996).
20. R. P. Malik, *Phys. Lett.* **212B**, 445 (1988).
21. S. Weinberg, *The Quantum Theory of Fields: Modern Applications V.2* (Cambridge University Press, Cambridge, 1996).
22. I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics: A Practical Introduction* (Adam Hilger, Bristol, 1982).
23. R. P. Malik, *Dual BRST Symmetry in QED*; [hep-th/ 9711056](http://arxiv.org/abs/hep-th/9711056).
24. R. K. Kaul and R. Rajaraman, *Phys. Lett.* **B265**, 335 (1991); *ibid.* **B249**, 433 (1990).
25. E. Witten, *Commun. Math. Phys.* **121**, 351 (1988).
26. A. S. Schwarz, *Lett. Math. Phys.* **2**, 217 (1978).
27. R. Jackiw and R. Rajaraman, *Phys. Rev. Lett.* **54**, 1219 (1985).
28. R. P. Malik, *Mod. Phys. Lett.* **A14**, 1937 (1999).
29. R. P. Malik, in preparation.
30. R. P. Malik and T. Pradhan, *Z. Phys.* **C 28**, 525 (1985).
31. K. Nishijima, *Int. J. Mod. Phys.* **A9**, 3799 (1994); *ibid.* **A10**, 3155 (1995).
32. T. Kugo and I. Ojima, *Prog. Theor. Phys. (Suppl)* **66**, 1 (1979); *Phys. Lett.* **73B**, 459 (1978).