ON THE SUPPORT OF POLLICOTT-RUELLE RESONANT STATES FOR ANOSOV FLOWS

TOBIAS WEICH

Abstract. We show that all generalized Pollicott-Ruelle resonant states of a topologically transitive $C^\infty$ Anosov flow with an arbitrary $C^\infty$ potential, have full support.

1. Introduction

Let $(M, g)$ be a smooth, compact Riemannian manifold without boundary. Let $X \in C^\infty(M, TM)$ be a smooth Anosov vector field and let us denote by $\varphi_t$ the flow on $M$ generated by $X$ and let $V \in C^\infty(M; \mathbb{C})$ be a smooth potential function. Then we can define the following differential operator

$$P := \frac{1}{i}X + V : C^\infty(M) \to C^\infty(M).$$

It is a well established approach to study the dynamical properties of Anosov flows by the discrete spectrum of the operator $P$, the so called Pollicott-Ruelle resonances. The fact that for volume preserving flows and real valued potentials, the operator $P$ is an unbounded, essentially self adjoint operator on $L^2(M)$ might suggest, that $P$ has good spectral properties on $L^2(M)$. However, due to the lack of ellipticity $P$ has mainly continuous spectrum which carries little information on the dynamics of the flow. A very important progress was thus to construct Banach spaces \cite{Liv04, BL07} or Hilbert spaces \cite{FS11, DZ13} for Anosov flows in which the operator $P$ has discrete spectrum in a sufficiently large region (see also \cite{Rug92, Kit99, BKL02, GL06, BT07} for analogous results for Anosov diffeomorphism). More precisely, it has been shown that there is a family of Hilbert spaces $H_{sG}$ parametrized by $s > 0$ such that for any $C_0 > 0$ and for sufficiently large $s$ the operator $P$ acting on $H_{sG}$ has discrete spectrum in the region $\{\text{Im} \lambda > -C_0\}$. This discrete spectrum is known to be intrinsic to the Anosov flow together with the potential function and does not depend on the sufficiently large parameter $s$ (see Section \ref{sec:support} for a more precise statement). Accordingly we call $\lambda_0 \in \mathbb{C}$ a Pollicott-Ruelle resonance if it is an eigenvalue of $P$ on $H_{sG}$ for sufficiently large $s$ and we call $\text{ker}_{H_{sG}}(P - \lambda_0)$ the space of Pollicott-Ruelle resonant states. As $P$ is not anymore a normal operator on $H_{sG}$, there might exist also finite dimensional Jordan blocks and given a Ruelle resonance $\lambda_0$ we denote by $J(\lambda_0)$ the maximal size of a corresponding Jordan block and we call $\text{ker}_{H_{sG}}(P - \lambda_0)^{J(\lambda_0)}$ the space of generalized Pollicott-Ruelle resonant states.

The interest in these Pollicott-Ruelle resonances and their resonant states arises from the fact that they govern the decay of correlations. In order to illustrate this in more detail, let us for the moment assume that the Anosov flow is contact, which includes for example all geodesic flows on compact manifolds with negative sectional curvature. If we denote by $dw$ the invariant contact volume form and take two smooth functions $f, g \in C^\infty(M)$, then one is interested in the behavior of the correlation function

$$C_{f,g}(t) := \int_M f(m) \varphi_t^* g(m) dw.$$
In this case the following expansion of the correlation function has been established in [Tsu10, Corollary 1.2] (see also [NZ13, Corollary 5])

\[ C_{f,g}(t) = \int_M f \, dw \int_M g \, dw + \sum_{j=1}^J \sum_{k=1}^{K_j} t^k e^{-it\mu_j} u_{j,k}(f) v_{j,k}(g) + \mathcal{O}(e^{-\gamma t}). \]

Here \( \mu_j \) are the Pollicott-Ruelle resonances for a vanishing potential \((V = 0)\) which are lying in a strip \( 0 > \text{Im}(\mu) > -\gamma \). Furthermore \( u_{j,1}, v_{j,1} \) are the corresponding left and right generalized eigenstates and the sum over \( k \) arises from the possible existence of Jordan blocks of size \( K_j \). As \( \text{Im}(\mu_j) < 0 \) all terms except the first one vanish exponentially in the limit of large times. This effect is interpreted as an exponential convergence towards equilibrium, known as exponential mixing and has first been established in the seminal works of Dolgopyat [Dol98] andLiverani [Liv04]. The way towards equilibrium is governed by the sub leading resonances and resonant states. While the resonances determine the possible decay modes, the resonant states determine the coefficients. If for example there would be a simple Pollicott-Ruelle resonance \( \mu_j \) with a resonant state, that vanishes on an open set \( U \subset M \), then for all observables \( f \in C^\infty_c(U) \) this resonance would not appear in the correlation expansion (1). The main theorem of this article states, that this can not be the case.

**Theorem 1.** Let \( \varphi_t \) be a topologically transitive \( C^\infty \) Anosov flow. Let \( \lambda_0 \) be a generalized Pollicott-Ruelle resonance and \( u \in \ker_{H_{\varphi_t}}(P - \lambda_0)^{f(\lambda_0)} \setminus \{0\} \) then \( \text{supp} \, u = M \).

**Remark 1.** Note that contrary to the correlation expansion (1) which we mentioned as a motivation, our result for the resonant states holds for a general \( C^\infty \) Anosov flow with no assumption on contact structures, smooth invariant measures, or even mixing properties. Taking a suspension of a topological transitive Anosov diffeomorphism, the theorem directly implies an analogous result for the resonant states of the Anosov diffeomorphism.

**Remark 2.** To our knowledge there has up to now little been known about the structure of Pollicott-Ruelle eigenstates. However in the proof of [GL08, Theorem 5.1] Gouëzel and Liverani show an analogous result for the peripheral spectrum of topologically mixing hyperbolic maps. In this case, there is however only one simple peripheral eigenvalue so the statement applies only to this one resonant state which corresponds to the equilibrium measure. Furthermore for the particular case of a geodesic flows on manifolds of constant negative curvature, Dyatlov, Faure and Guillarmou [DFG15] proved an explicit relation between Pollicott-Ruelle resonant states and Laplace eigenstates. Using this relation allows to transfer well established support properties of Laplace eigenstates to support properties of Pollicott-Ruelle eigenstates.

The article is organized as follows: In Section 2 we introduce the basic definitions and recall some known facts about Anosov flows and the microlocal approach to Pollicott-Ruelle resonances. Section 3 is devoted to the proof of the main Theorem 1.

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composition of each tangent space

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2. Notation and Preliminaries

2.1. Anosov flows. Let \((M, g)\) be a smooth, compact Riemannian manifold without boundary and let us denote the Riemannian distance between two points \(m, \tilde{m} \in M\) by \(d(m, \tilde{m})\). A smooth vector field \(X \in C^\infty(M, TM)\), respectively the flow \(\varphi_t \) on \(M\) generated by \(X\), are called Anosov if there is a direct sum decomposition of each tangent space

\[
T_m M = E_0(m) \oplus E_s(m) \oplus E_u(m)
\]

which depends continuously on the base point \(m\) and which is invariant under \(d\varphi_t\). Furthermore the decomposition has to fulfill \(E_0(m) = \mathbb{R}X(m)\) and there has to exist some fixed \(C\) and \(\theta > 0\) with

\[
\|d\varphi_t(m)v\|_{T_{\varphi_t(m)}M} \leq Ce^{-\theta t}\|v\|_{T_m M} \quad \forall v \in E_u(m), \quad t < 0
\]

\[
\|d\varphi_t(m)v\|_{T_{\varphi_t(m)}M} \leq Ce^{-\theta t}\|v\|_{T_m M} \quad \forall v \in E_s(m), \quad t > 0.
\]

An Anosov flow is said to be topological transitive if there exists a dense orbit.

It is known, that the bundles \(E_0 \oplus E_s\) and \(E_0 \oplus E_u\) are uniquely integrable \(\text{[Ano67, HPS70]}\) and given a point \(m \in M\) we denote by \(W^{wu}(m)\) and \(W^{us}(m)\) the integral manifolds through the point \(m\) which are smooth immersed submanifolds of \(M\). They are called weak stable and weak unstable manifolds through \(m\). Also the bundles \(E_s\) and \(E_u\) are uniquely integrable. The corresponding integral manifolds will be denoted by \(W^s(m), W^u(m)\) and will be called strong stable and strong unstable manifolds.

For further reference let us denote by \(n_{us}, n_{wu}, n_s, n_u \in \mathbb{N}\) the dimension of the manifolds \(W^{us}(m), W^{wu}(m), W^s(m)\) and \(W^u(m)\), respectively. Let us recall the following well established result on the continuous dependence of the invariant manifolds on the base point.

**Theorem 2** (c.f. \[HP70\] Theorem 3.2]). \(W^{us}\) is a family of \(n_{us}\)-dimensional immersed \(C^\infty\)-submanifolds that depends continuously on the base point w.r.t. the \(C^\infty\)-topology. More precisely, this means that for any \(m\) there is a neighborhood \(m \in U \subset M\) as well as a continuous map \(g : U \to C^\infty(D^{us}, M)\) such that for any \(\tilde{m} \in U\), \(g(\tilde{m})\) is a \(C^\infty\)-diffeomorphism of the \(n_{us}\) dimensional disk \(D^{us}\) onto a neighborhood of \(\tilde{m}\) in \(W^{us}(\tilde{m})\).

The analogous result holds for \(W^{wu}, W^s, W^u\).

**Remark 3.** The stable and unstable foliations are known to be even Hölder continuous in general, and under certain assumptions such as curvature pinching even regularity \(C^1\) or better can be obtained (c.f. \[Has02 Section 2.3]\). We will, however, not explicitly need these refined regularity estimates in the sequel.

Given two points \(y, z \in W^{us}(m)\) we denote by \(d^{us}(y, z)\) the metric distance on \(W^{us}(m)\) coming from the metric inherited from the Riemannian metric on \(M\). In the same way we can define \(d^{wu}, d^s\) and \(d^u\). We can now define the following different open balls of radius \(r > 0\).

\[
B_r (m) := \{ \tilde{m} \in M : d(m, \tilde{m}) < r \}
\]

\[
B_r^{wu}(m) := \{ \tilde{m} \in W^{wu}(m) : d^{wu}(m, \tilde{m}) < r \}
\]

\[
B_r^{wu}(m) := \{ \tilde{m} \in W^{us}(m) : d^{wu}(m, \tilde{m}) < r \}
\]

\[
B_r^s(m) := \{ \tilde{m} \in W^s(m) : d^s(m, \tilde{m}) < r \}
\]

\[
B_r^u(m) := \{ \tilde{m} \in W^u(m) : d^u(m, \tilde{m}) < r \}.
\]
Theorem 3 (Product Neighborhood Theorem). There exists $\delta_0 > 0$ independent of $m \in M$ such that for any $m \in M$ and $\delta \leq \delta_0$ the following maps

\begin{align*}
H^{w,u,s}_m : & \left\{ B^s_\delta(m) \times B^s_\delta(m) \right\} \rightarrow M \\
& (x, y) \mapsto B^u_{2\delta}(x) \cap B^u_{2\delta}(y) \\
H^{w,u,s}_m : & \left\{ B^u_\delta(m) \times B^u_\delta(m) \right\} \rightarrow M \\
& (x, y) \mapsto B^s_{2\delta}(x) \cap B^s_{2\delta}(y)
\end{align*}

are unambiguously defined, injective and homeomorphisms onto their images.

For a proof see [PS70] (see also [Pla72] for a more detailed statement). Given a point $m \in M$ and $0 < \alpha, \beta < \delta_0$ we define the open product neighborhoods of $m$

\[ \mathcal{P}N^{w,u,s}_{\alpha,\beta}(m) := H^{w,u,s}_m(B^{w,s}_\alpha(m) \times B^{w,s}_\beta(m)) \subset M \]

and

\[ \mathcal{P}N^{w,u,s}_{\alpha,\beta}(m) := H^{w,u,s}_m(B^{w,u}_\alpha(m) \times B^{w,u}_\beta(m)) \subset M. \]

We will call such product neighborhoods also rectangular neighborhoods. These product neighborhoods have the important property that the defining homeomorphisms imply foliations by local invariant manifolds. In the sequel we will only need the foliations of $(w,u,s)$-rectangles which are given by

\[ \mathcal{P}N^{w,u,s}_{\alpha,\beta}(m) = \bigcup_{x \in B^{w,s}_\alpha(m)} S_x = \bigcup_{y \in B^{w,u}_\beta(m)} U_y \]

where $S_x := H^{w,u,s}_m(\{x\} \times B^s_\beta(m))$ are the local strong stable leaves and $U_y := H^{w,u,s}_m(B^{w,u}_\alpha(m) \times \{y\})$ the local weak unstable leaves. Note that from Theorem 2 as well the strong stable leaves $S_x$ as the weak unstable leaves $U_y$ are smooth submanifolds of the product neighborhood. Still the foliations are no smooth foliations as the trivialization map $H^{w,u,s}_m$ is only continuous, which reflects the fact from Theorem 2 that the local invariant manifolds $S_x$ ($U_y$ respectively) depend only continuously on their basepoints $x$ ($y$ respectively). As $M$ is a Riemannian manifold, the leaves $S_x$ ($U_y$ respectively) inherit a Riemannian metric and thus also a Lebesgue measure which we denote by $dm_{S_x}$ ($dm_{U_y}$ respectively).

One way to introduce Sinai-Ruelle-Bowen (SRB) measures is to demand, that they are absolute continuous w.r.t. the weak unstable foliation.

Definition 1 (SRB measure). A measure $\mu$ on $M$ which is invariant under the Anosov flow $\varphi_t$ is called a SRB-measure if for any product neighborhood $\mathcal{P}N^{w,u,s}_{\alpha,\beta}(m)$ and for all $y \in B^s_\beta(m)$ there are positive, measurable functions $\tau_y$ on $U_y$ as well as a measure $d\sigma$ on $B^s_\beta(m)$ such that the restriction of $\mu$ to $\mathcal{P}N^{w,u,s}_{\alpha,\beta}(m)$ is given by

\[ \mu|_{\mathcal{P}N^{w,u,s}_{\alpha,\beta}(m)} = \int_{B^s_\beta(m)} (\tau_y d\mu_{U_y}) d\sigma(y). \]

More precisely this equality means, that for any $f \in C_c(\mathcal{P}N^{w,u,s}_{\alpha,\beta}(m))$

\[ \int f d\mu = \int_{B^s_\beta(m)} \left( \int_{U_y} f_{U_y}(z) \tau_y(z) d\mu_{U_y}(z) \right) d\sigma(y). \]

Remark 4. There are other equivalent possibilities to define SRB measure in terms of metric entropy and ergodic averages, see e.g. [You02] for an overview.

Theorem 4. For any Anosov flow there exists a SRB measure. If the Anosov flow is topological transitive, the SRB measure is unique and the Anosov flow is ergodic w.r.t. the SRB measure.

For an original proof of this result in the context of Axiom A diffeomorphism see [Bow70] and for generalizations to flows [Bow73].
2.2. Microlocal approach to Pollicott-Ruelle resonances for Anosov flows.

Let us collect some facts about the spectral theory of Pollicott-Ruelle resonances. Suitable Banach spaces, in which the differential operator \( P \), has discrete spectrum, have first been introduced by Liverani [Liv14]. In this article we will use the microlocal approach to transfer operators, which has been introduced by Faure, Roy and Sjöstrand in a series of papers [FR06, FRS08, FS11]. The following result for Anosov flows has originally been shown by Faure and Sjöstrand in a series of papers [FS11, Theorem 1.4]. The reader might also be interested in [DZ13, Proposition 3.2] where an alternative proof, using different microlocal techniques, is given. While these two publications only mention the case of zero potential, a more general statement including arbitrary smooth potentials can be found in [DG14].

**Theorem 5.** For any \( C_0 > 0 \) there is an \( s > 0 \) and Hilbert spaces \( H_{sG} \) and \( D_{sG} \) such that in the region \( \{ \text{Im} \lambda > -C_0 \} \) the operator \( P - \lambda : D_{sG} \to H_{sG} \) is Fredholm of index 0.

Here the Hilbert space \( H_{sG} \) is an anisotropic Sobolev space that fulfills the relations

\[
C^\infty(M) \subset H^s(M) \subset H_{sG} \subset H^{-s}(M) \subset \mathcal{D}'(M),
\]

where \( H^s(M) \) denotes the ordinary Sobolev space. The Hilbert spaces \( D_{sG} \) can be defined by considering the operator \( P \) acting on \( \mathcal{D}'(M) \) and we set

\[
D_{sG} := \{ u \in H_{sG} : Pu \in H_{sG} \}
\]

which is a Hilbert space with the norm \( \|u\|_{D_{sG}} := \|u\|_{H_{sG}} + \|Pu\|_{H_{sG}} \).

Since \( (P - \lambda) \) is invertible for \( \text{Re}(\lambda) \) large enough (see e.g. [FS11, Lemma 3.3] or [DZ13 Prop 3.1]), analytic Fredholm theory implies that the resolvent

\[
R(\lambda) := (P - \lambda)^{-1} : H_{sG} \to H_{sG}
\]

has a meromorphic continuation to \( \{ \text{Im} \lambda > -C_0 \} \) and the Pollicott-Ruelle resonances are defined to be the poles of the meromorphic continuation. Given a Pollicott-Ruelle resonance \( \lambda_0 \) the space \( \ker_{H_{sG}}(P - \lambda_0)^k \) is nonempty and its elements are called the corresponding Pollicott-Ruelle resonant states. Furthermore there is an integer \( J(\lambda_0) \geq 1 \) determining the maximal size of a Jordan block with spectral value \( \lambda_0 \), i.e. we have for all \( k \geq J(\lambda_0) \) \( \ker_{H_{sG}}(P - \lambda_0)^k = \ker_{H_{sG}}(P - \lambda_0)^{J(\lambda_0)} \) and we call the space \( \ker_{H_{sG}}(P - \lambda_0)^{J(\lambda_0)} \) the space of generalized Pollicott-Ruelle resonant states. Note that the fact that \( C^\infty(M) \) is densely contained in all \( H_{sG} \) together with the uniqueness of meromorphic continuation imply that neither the position of the resonance \( \lambda_0 \) nor the spaces \( \ker_{H_{sG}}(P - \lambda_0)^k \) and \( \ker_{H_{sG}}(P - \lambda_0)^{J(\lambda_0)} \) depend on the parameter \( s \), so they are intrinsic objects of the Anosov flow (c.f. [FS11, Theorem 1.5]).

A central ingredient in the proof of Theorem 5 and the construction of the anisotropic Sobolev spaces is to study the symplectic lift of the flow action to \( T^* M \). Faure and Sjöstrand therefore introduce the decomposition

\[
T^*_m M = E^*_0(m) \oplus E^*_s(m) \oplus E^*_w(m)
\]

which is defined such that

\[
E^*_0(m)(E_s(m) \oplus E_u(m)) = 0,
E^*_s(m)(E_0(m) \oplus E_u(m)) = 0,
E^*_w(m)(E_0(m) \oplus E_s(m)) = 0.
\]

Note that if one identifies \( T^*_m M \) with \( T^*_m M \) via the Riemannian metric this definition seems counterintuitive, as \( E^*_w(m) \) is identified with \( E_u(m) \) and vice versa. The reason for this choice in [FS11] is that it is natural from the point of view of the symplectic lift of \( \varphi_t \) to \( T^* M \) (c.f. [FS11 eq. (1.13)]). We will not need
further details on the construction of the anisotropic Sobolev spaces and refer to the excellent expositions in [FS11, DZ13]. However, we will crucially use the following consequence of these constructions on the microlocal regularity of the Pollicott-Ruelle resonant states [FS11, Theorem 1.7]

\[ \ker_{\mathcal{H}^r}(\mathbf{P} - \lambda_0) \subset \mathcal{D}'_{E^r}(M), \]

where \( \mathcal{D}'_{E^r}(M) \) denotes the space of distributions on \( M \) whose wavefront sets are contained in some closed cone \( \Gamma \subset T^*M \) (see [Hör03, Lemma 8.2.1] for further details).

3. On the support of Pollicott-Ruelle eigenstates

In this section, we will prove Theorem 1. Let us start by giving a brief outline of the proof: As a first step, we will introduce suitable charts that are adapted to the foliation of \( M \) into stable and unstable manifolds. Then we study some particular distributions, denoted by \( \rho_A \), which are defined in these charts and which correspond to characteristic functions of tubes along the strong stable foliation. Proposition 6 then gives estimates on the decay of the Fourier transformation of \( \rho_A \). As a consequence, one deduces that the wavefront set of the distributions \( \rho_A \) are bounded away from \( E^r \), thus they can be paired (or multiplied) with Pollicott-Ruelle resonant states. Using the Fourier estimates in Proposition 6 we then show, that any Pollicott-Ruelle resonant state that does not have full support, has to be identically zero. Therefore we only use two properties of the resonant states: the flow invariance of their support and their microlocal regularity. A major technical difficulty in the proof comes from the fact that the stable and unstable foliations for general Anosov flows are only Hölder regular.

We call a finite open cover \( M = \bigcup_{i=1}^{N} R_i \) a cover of product neighborhoods if there are \( m_i \in M \), \( r_i^{wu}, r_i^s > 0 \) such that \( R_i = \mathcal{P}_r^{r_i^{wu}, r_i^s} (m_i) \) (c.f. Theorem 3). In order to simplify the notation we write \( X_i := B_{r_i^{wu}}^{wu}(m_i) \), \( Y_i := B_{r_i^s}^s(m_i) \) and denote the homeomorphism of Theorem 3 by \( h_i : \mathcal{H}_i^{wu,s} : X_i \times Y_i \to R_i \) recall that \( h_i(X_i \times \{ m_i \}) = B_{r_i^{wu}}^{wu}(m_i) \subset R_i \) is a smooth submanifold.

Now let us construct a smooth atlas of \( M \) by choosing \( C^\infty \)-charts \( \kappa_i : R_i \to V_i \subset \mathbb{R}^n = \mathbb{R}^{n_{wu}} \times \mathbb{R}^{n_s} \) for our open cover of product neighborhoods such that

\[
\begin{align*}
\kappa_i(m_i) &= 0 \in \mathbb{R}^n, \\
\kappa_i(B_{r_i^{wu}}^{wu}(m_i)) &= (\mathbb{R}^{n_{wu}} \times \{0\}) \cap V_i, \\
\kappa_i(B_{r_i^s}^s(m_i)) &= (\{0\} \times \mathbb{R}^{n_s}) \cap V_i.
\end{align*}
\]

We can then define the open set \( V_i^{wu} := \{ x \in \mathbb{R}^{n_{wu}} : (x,0) \in V_i \} \) and directly see that

\[ h_i^{wu} : x \in X_i \mapsto \kappa_i \circ h_i(x, m_i) \in V_i^{wu} \times \{0\} \cong V_i^{wu} \]
defines a homeomorphism between \( X_i \) and \( V_{i}^{wu} \). Now for any Borel set \( A \subset V_{i}^{wu} \) we define

\[
S_A := \kappa_i \circ h_i((h_i^{wu})^{-1}(A) \times Y_i) \subset V_i
\]

which is again a Borel subset. Note that for a point \( x \in V_{i}^{wu} \) the sets \( S_x := S_{\{x\}} \) are exactly the image of the local strong stable manifolds under the coordinate chart \( \kappa_i \)

\[
S_x = \kappa_i(B_{\epsilon_i}^x((h_i^{wu})^{-1}(x))).
\]

According to Theorem 2 the \( S_x \subset V_i \) are \( C^\infty \)-submanifolds depending continuously on the basepoint \( x \) w.r.t. the \( C^\infty \) topology. Furthermore, the definition of \( S_A \) allows us to associate to any Borel set \( A \subset V_{i}^{wu} \) the distribution \( \rho_A \in \mathcal{D}'(V_i) \) by setting for \( \psi \in C_c^\infty(V_i) \)

\[
\rho_A[\psi] := \int_{V_i} 1_{S_A}(z) \psi(z) dz.
\]

Given a chart \( \kappa_i \), we write \( T^*V_i = V_i \times \mathbb{R}^n \) and we define \( \Gamma_1, \Gamma_2 \subset \mathbb{R}^n \) to be the minimal closed cones such that

\[
\forall \ m \in R_i, \eta \in E^*_u(m) : \ (\kappa_i^{-1})^*(\eta) \in V_i \times T^*V_i
\]

and

\[
\forall \ x \in V_{i}^{wu}, (s, \xi) \in N^*S_x \subset V_i \times \mathbb{R}^n : \ \xi \in \Gamma_2.
\]

Note that for \( \eta \in E^*_u(m_1) \) one has \((\kappa_i^{-1})^*(\eta) \in \{0\} \times (\{0\} \times \mathbb{R}^n)\) and that on the other side \( N^*S_0 = \{0\} \times (\mathbb{R}^{n_{wu}} \times \{0\}) \) are complementary cones. Thus from the continuity of the foliations and after a possible refinement of the charts we can assume that

\[
\Gamma_1 \cap \Gamma_2 = \{0\}
\]

We now want to study the Fourier transform of \( \rho_A \):

**Proposition 6.** Let \( \chi \in C_c^\infty(V_i) \) be an arbitrary cutoff function, then there is a constant \( C > 0 \) such that for any Borel set \( A \subset V_{i}^{wu} \),

\[
|\hat{\chi} \cdot \rho_A(\xi)| \leq C \text{vol}_{n_{wu}}(A)
\]

uniformly in \( \xi \in \mathbb{R}^n \). Here \( \text{vol}_{n_{wu}}(A) \) is the \( n_{wu} \)-dimensional euclidean volume of \( A \subset V_{i}^{wu} \subset \mathbb{R}^{n_{wu}} \).

Furthermore, fix a closed cone \( \Omega \subset \mathbb{R}^n \) such that \( \Omega \cap \Gamma_2 = \{0\} \). Then for all \( N \in \mathbb{N} \) there is a \( C_N > 0 \) such that for all Borel sets \( A \subset V_{i}^{wu} \)

\[
|\hat{\chi} \cdot \rho_A(\xi)| \leq C_N \text{vol}_{n_{wu}}(A) \xi^{-N}
\]

uniformly for \( \xi \in \Omega \).

**Proof.** In order to prove this theorem we use the definition of the Fourier transform for compactly supported distributions and get

\[
|\hat{\chi} \cdot \rho_A(\xi)| = |\rho_A[\chi(x)e^{-ix\xi}]| = \left| \int_{V_i} 1_{S_A}(z) \chi(z)e^{-iz\xi} dz \right|.
\]

In order to obtain the estimates for this integral let us recall the following fact which is a consequence of the absolute continuity of the strong stable foliation.

**Theorem 7.** For any \( V_i \subset \mathbb{R}^n \) defined as above and any \( x \in V_{i}^{wu} \) there is a smooth function \( \delta_x \in C^\infty(S_x) \) such that for any \( \psi \in C_c^\infty(V_i) \)

\[
\int_{V_i} \psi(x, y) dx dy = \int_{V_{i}^{wu}} \left( \int_{S_x} \psi(s) \cdot \delta_x(s) dm_{S_x}(s) \right) dx.
\]
Here $dx, dy$ are the euclidean measures on $\mathbb{R}^{n_u}, \mathbb{R}^{n_s}$, $\psi|_{S_x} \in C^\infty(S_x)$ is the restriction to the smooth submanifold $S_x$ and $dm_{S_x}$ the induced measure on the submanifold $S_x$.

Furthermore, their dependence on $x$ on $V_{i}^{wu}$ is continuous in the following sense: If $T^1(S_x)$ is the set of normalized tangent vectors in $s \in S_x$, then for any $k \in \mathbb{N}$ the following $k$-norm

$$\|\delta_x\|_k := \sup_{s \in S_x} \sup_{X_1, \ldots, X_k \in T^1(S_x)} |X_1 \ldots X_k \delta_x(s)|$$

is finite and for all $k \in \mathbb{N}$ the map $x \mapsto \|\delta_x\|_k$ is continuous.

Proof. The existence of the conditional measures can be derived by a standard argument from the absolute continuity of the strong stable foliation (see Appendix A respectively standard dynamical systems literature, e.g. [Boc13, Section 6.2]). In order to obtain the smoothness of the conditional measures as well as the continuous dependence on the base point $x$ one additionally needs the smoothness of the Jacobians of holonomy maps along the strong stable leaves. This statement can for example be found in [GLPT13, Appendix E]. □

First we use the Fubini formula for the strong stable foliation (12) in order to estimate the integral in (11) we obtain

$$\left| \int_A \left( \int_{S_x} \chi(s) e^{-is\xi} \delta_x(s) dm_{S_x}(s) \right) dx \right|.$$

Now the uniform bounds on $\|\delta_x\|_0$ directly imply (9).

In order to see (10) we use the boundedness of $\|\delta_x\|_k$ together with the following partial integration argument: For $\xi_0 \in \Omega$ with $|\xi_0| = 1$ and $x \in V_{i}^{wu}$ we define the tangent vector field

$$\chi_{\xi_0,x} : \begin{cases} T S_x & \mapsto T S_x \\ s & \mapsto \text{Pr}_{T_x S_x}(\xi_0). \end{cases}$$

Here $\text{Pr}_{T_x S_x}$ is the orthogonal projection onto $T_x S_x$ which makes sense after having identified $TV_i = V_i \times \mathbb{R}^n$. From the definition of $S_x$ and $\Gamma_2$ we conclude that for any $x \in V_{i}^{wu}$ and $s \in S_x$ we have have $(s, \xi_0) \notin N^* S_x$. Thus there is $C_{\xi_0} > 0$ such that

$$\langle \chi_{\xi_0,x}(s), \xi_0 \rangle \geq C_{\xi_0}$$

uniformly in $s \in S_x$ and $x \in V_{i}^{wu}$. By continuity we find an open neighborhood $W_{\xi_0} \subset \mathbb{R}^n$ of $\xi_0$ such that

$$\langle \chi_{\xi_0,x}(s), \xi \rangle \geq \frac{1}{2} C_{\xi_0}$$

uniformly in $\xi \in W_{\xi_0}$, $s \in S_x$ and $x \in V_{i}^{wu}$.

Now we use the closedness of $\Omega$ and a compactness argument to conclude that there are $\xi_1, \ldots, \xi_K$ as well as a constant $C_0 > 0$ such that for any $\xi \in \Omega$ there is $1 \leq l \leq K$, such that

$$\langle \chi_{\xi_l,x}, \xi \rangle \geq C_0 |\xi|$$

uniformly in $x \in V_{i}^{wu}, s \in S_x$. Furthermore Theorem 2 implies that $s \mapsto \langle \chi_{\xi_l,x}(s), \xi \rangle \in C^\infty(S_x)$. Even if this function is not compactly supported, the fact, that the $S_x$ are precompact, and that these functions can always be smoothly extended to a slightly larger stable leaf implies that for all $k \in \mathbb{N}$ the norm $||\langle \chi_{\xi_l,x}(s), \xi \rangle||_k$ is finite and depends continuously on $x \in V_{i}^{wu}$.

Now consider

$$\left| \int_{S_x} \chi(s) e^{-is\xi} \delta_x(s) dm_{S_x}(s) \right|.$$
Performing $N$ times partial integration w.r.t. the differential operator

$$L_i := \frac{1}{-i(\chi_{\ell_i,x}(s), \xi)} \chi_{\ell_i,x}$$

we get

$$\left| \int_{S_x} \chi(s) e^{-is\delta_x(s)} dm_{S_i}(s) \right|$$

$$\leq \int_{S_x} \left( \chi_{\ell_i,x} \frac{1}{-i(\chi_{\ell_i,x}(s), \xi)} \right)^N \delta_x(s) \chi(s) \left| dm_{S_i}(s) \right|$$

$$\leq C_N(\xi)^{-N},$$

where $C_N$ is independent of $x \in V_i^{wu}$. Note that for this independence we crucially use that $x \mapsto \|\delta_x\|_\ell$ as well as $x \mapsto \|\chi_{\ell_i,x}(s), \xi\|_k$ are continuous.

Recall that we can choose $\Omega$ to be an arbitrary cone with $\Omega \cap \Gamma_2 = \{0\}$. Thus [10] implies that for any Borel set $A \subset V_i^{wu}$ we have $\rho_A \in \mathcal{D}'_\Gamma(V_i)$, i.e. the wavefront set of $\rho_A$ is contained in $\Gamma_2$. Multiplication of $\rho_A$ with an arbitrary compactly supported distribution $v \in \mathcal{E}_\Gamma(V_i)$ is therefore well defined and yields a compactly supported distribution $v \cdot \rho_A \in \mathcal{E}(V_i)$.

**Lemma 8.** For each distribution $v \in \mathcal{E}'_\Gamma(V_i)$ there is a constant $C_v > 0$ such that for all Borel sets $A \subset V_i^{wu}$ we have

$$|(v \cdot \rho_A)[1]| \leq C_v \text{vol}_{swu}(A).$$

**Proof.** As $v$ has compact support in $V_i$ let us choose a cutoff function $\chi \in C_c(\mathbb{R}^n)$ with $\chi = 1$ in a neighborhood of $\text{supp} v$. Then we calculate

$$(v \cdot \rho_A)[1] = (v \cdot (\chi \rho_A))[1] = v \cdot (\chi \cdot \rho_A)(0) = (\hat{v} \cdot \hat{\chi} \cdot \rho_A)(0)$$

$$= \int_{\mathbb{R}^n} \hat{v}(\xi) \hat{\chi} \cdot \rho_A(\xi) \, d\xi$$

and note that the last integral converges as $\Gamma_1 \cap \Gamma_2 = \{0\}$.

Now let us fix a closed cone $\Omega \subset \mathbb{R}^n$ fulfilling $\Gamma_1 \setminus \{0\} \subset \text{Int}\Omega$ and $\Omega \cap \Gamma_2 = \{0\}$. As $\text{WF}(v) \subset \Gamma_1$ we get for any $N \in \mathbb{N}$ a constant $C_N$ such that

$$|\hat{v}(\xi)| \leq C_{v,N} \langle \xi \rangle^{-N}$$

uniformly for $\xi \in \mathbb{R}^n \setminus \Omega$. As any distribution is of finite order we in addition get the existence of an integer $K_v$ and a constant $C_{v,2}$ such that

$$|\hat{v}(\xi)| \leq C_{v,2} \langle \xi \rangle^{K_v}$$

uniformly for all $\xi \in \mathbb{R}^n$. Splitting the integral over $\xi$ we thus obtain

$$|(v \cdot \rho_A)[1]| \leq C_{v,N} \int_{\mathbb{R}^n \setminus \Omega} \langle \xi \rangle^{-N} \left| \hat{\chi} \cdot \rho_A(\xi) \right| \, d\xi + C_{v,2} \int_{\Omega} \langle \xi \rangle^{K_v} \left| \hat{\chi} \cdot \rho_A(\xi) \right| \, d\xi$$

for any given $N \in \mathbb{N}$. Then we can use [9] in the first integral and [10] with $N = K_v + n + 1$ in the second integral which leads to the desired estimate [14].

Next we study the local structure of $\text{supp} u \subset M$ for a resonant state $u \in \ker_{H_{\alpha}}(P - \lambda_0)$. Therefore note that $\text{supp} u \subset M$ is a $\varphi_t$-invariant closed subset. This is the only property that we use in the following lemma.

**Lemma 9.** Let $\Sigma \subset M$ be a closed subset invariant under the topological transitive Anosov flow. If $\Sigma \neq M$, then there exists a finite cover of product neighborhoods $M = \bigcup_{i=1}^N R_i$ as well as for any $i = 1, \ldots, N$ an open subset $N_i \subset X_i$ such that

$$\text{vol}_{X_i}(X_i \setminus N_i) = 0 \text{ and } h_i(N_i \times Y_i) \cap \Sigma = 0.$$
where $\text{vol}_{X_i}$ is the Lebesgue measure on $X_i$.

Proof. As $M \setminus \Sigma$ is nonempty and open, there exists $\varepsilon_s, \varepsilon_{wu} > 0$ and $m_0 \in M$ such that $\mathcal{P}N_{\varepsilon_{wu},s}(m_0) \cap \Sigma = \emptyset$. Let us introduce the open sets

$$U_0 := \mathcal{P}N_{\varepsilon_{wu},\varepsilon_s/2}(m_0), \quad U_1 := \mathcal{P}N_{\varepsilon_{wu},s}(m_0)$$

Now take a cover of product neighborhoods $M = \bigcup_{i=1}^{N} R_i$ with $r_i^s < \varepsilon_s/4$. We now want to consider the backwards iterates $\varphi^{-t}(U_0)$ for $t > 0$. Firstly, the flow invariance of $\Sigma$ assures that $\varphi^{-t}(U_0) \cap \Sigma = \emptyset$. Secondly the dynamical properties of the flow assures that the sets $\varphi^{-t}(U_0)$ are stretched into the strong stable direction and thirdly, if $\mu$ denotes the SRB measure, then ergodicity implies that

$$\mu(\bigcup_{t>0} \varphi^{-t}(U_0)) = \mu(M).$$

We will subsequently use these three facts in order to construct the sets $N_i$.

Therefore we first consider for an arbitrary index $i$ and $t > 0$ the connected components of $\varphi^{-t}(U_0) \cap R_i$. If a connected component can be written in the form $h_i(A \times Y_i)$ with $A \subset X_i$ open, we call it a complete empty tube. We call them “empty” as these sets have no intersection with $\Sigma$ and we intend to build our set $N_i$ from the sets $A_i$. Note that for large $t$ the stretching and folding will imply that $\varphi^{-t}(U_0) \cap R_i$ has many connected components. But as we started with a product neighborhood $U_0$ and as the strong stable foliation is preserved by the flow, there are maximally two connected components which are not complete empty tubes. They correspond to the two ends of the product neighborhood (c.f. Figure 2). We can however make them complete by considering the connected components $\varphi^{-t}(U_1) \cap R_i$. As $U_1 \cap \Sigma = \emptyset$, these connected components are also empty. Furthermore the fact that $r_i^s < \varepsilon_s/4$ together with the fact, that the product neighborhoods are stretched along the strong stable direction for negative times implies the following statement: Any connected component of $\varphi^{-t}(U_1) \cap R_i$ that contains a connected component of $\varphi^{-t}(U_1) \cap R_i$ is a complete tube. Thus we can define an open set $A_{i,t} \subset X_i$ such that the union of connected components of $\varphi^{-t}(U_1) \cap R_i$ that contain a connected component of $\varphi^{-t}(U_1) \cap R_i$ is given by $h_i(A_{i,t} \times Y_i)$ with this definition we obviously have

$$\varphi^{-t}(U_0) \cap R_i \subset h_i(A_{i,t} \times Y_i) \subset \varphi^{-t}(U_1) \cap R_i \subset M \setminus \Sigma.$$ 

We now can define the opens set $N_i := \bigcup_{t>0} A_{i,t} \subset X_i$ and obtain

$$\bigcup_{t>0} \varphi^{-t}(U_0) \cap R_i \subset h_i(N_i \times Y_i) \subset M \setminus \Sigma.$$

As $h_{i}^{wu} : X_i \rightarrow V_{i}^{wu}$ is a diffeomorphism, it only remains to prove that $\text{vol}_{n=\varepsilon_i}^{A}(h_{i}^{wu}(N_i)) = \text{vol}_{n=\varepsilon_i}^{A}(V_{i}^{wu})$. This, however, is an immediate consequence of

$$\mu(h_i(N_i \times Y_i)) \geq \mu \left( \bigcup_{t>0} \varphi^{-t}(U_0) \cap R_i \right) = \mu(R_i)$$

(which uses the ergodicity w.r.t. the SRB measure) as well as the following Lemma.

Lemma 10. Let $\mu$ be the SRB measure, and $A \subset X_i$ with $\text{vol}_{n=\varepsilon_i}^{A}(h_{i}^{wu}(A)) > 0$ then $\mu(h_i(A \times Y_i)) > 0$.

Proof. Using the fact that the SRB measure is absolutely continuous w.r.t. the weak unstable foliation we can write

$$\mu(h_i(A \times Y_i)) = \int_{B_{r_i}^{x_i}(m)} \left( \int_{h_i(A \times \{y\})} \tau_y \ d\mu_y \right) d\sigma(y).$$

□
Now, let us analyze the function $g(y)$. For $y = 0$ we known from the definition of our charts, that $U_0$ is diffeomorphic to $V_i^{\text{wu}}$. As $dm_{U_0}$ comes from a Riemannian metric this induced measure is equivalent to Lebesgue measure on $V_i^{\text{wu}}$ and thus the assumption $\text{vol}^{\text{wu}}(h_i(A)) > 0$ implies $m_{U_0}(h_i(A \times \{0\})) > 0$. As $\rho_0$ is a positive density this implies $g(0) > 0$. Now let us consider an arbitrary other point $y \neq 0$. We want to show, that also the sets $h_i(A \times \{y\}) \subset U_y$ are of positive measure w.r.t. $m_{U_y}$. This however is a direct consequence of the fact that $h_i(A \times \{0\})$ has positive measure together with the fact, that the strong stable holonomy maps are absolutely continuous (see e.g. [BP07, Section 8.6.2]). Again the positivity of the density $\rho_y$ implies $g(y) > 0$. $g$ is thus a strictly positive measurable function and consequently $\mu(h_i(A \times Y)) = \int_{B_{\Sigma(m)}} g(y) d\sigma(y) > 0$. □

We are now able to prove Theorem 1.

**Proof.** Let $\lambda_0$ be a Pollicott-Ruelle resonance, $\tilde{u} \in \ker_{H,sG}(P - \lambda_0)^J(\lambda_0) \setminus \{0\}$ an associated generalized resonant state and suppose that $\text{supp} \tilde{u} \neq M$. Let $k = 0, \ldots, J(\lambda_0) - 1$ be the unique integer, such that $u := (P - \lambda_0)^k \tilde{u} \in \ker_{H,sG}(P - \lambda_0) \setminus \{0\}$.

As the differential operator $P$ is local we have $\text{supp} u \subset \text{supp} \tilde{u} \subset M$. Furthermore because $\varphi_\ast^\sigma u = e^{-ij\lambda_0}u$, the support $\text{supp} u$ is invariant by the Anosov flow so we can apply Lemma 9 to $\Sigma = \text{supp}(u)$. Thus we obtain a cover $M = \bigcup_{i=1}^N R_i$ of product neighborhoods as well as open subsets $N_i \subset X_i$ fulfilling (16)

$$h_i(N_i \times Y_i) \cap \text{supp} u = \emptyset.$$ 

After a possible further refinement we can also guarantee the existence of $C^\infty$-charts $\kappa_i : R_i \to V_i \subset \mathbb{R}^n$ as well as closed cones $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ fulfilling (8). Now let us choose a smooth partition of unity

$$\sum_{i=1}^N \chi_i = 1 \in C^\infty(M) \text{ with } \chi_i \in C_c^\infty(R_i)$$

and define $u_i := (\kappa_i^{-1})^\ast(\chi_i \cdot u) \in \mathcal{E}'(V_i)$. Recall from (8) that $u \in \ker_{H,sG}(P - \lambda_0)$ implies $\text{WF}(u) \subset E_n^\ast$ and by (8) this implies that $\text{WF}(u_i) \subset E_n^\ast(V_i)$. Now, using Lemma 8 we obtain for an arbitrary Borel set $A \subset V_i^{\text{wu}}$ the estimate $|(u_i \cdot \rho_A)[1]| \leq$
$C_{u_i \cdot \text{vol}_{n_{w_u}}} (A)$. We now want to use this estimate in order to conclude, that $u_i = 0$ for all $i = 1, \ldots, N$.

Now for any $\varepsilon > 0$ let us introduce the set

$$N_{i, \varepsilon} := \{ x \in N_i : \{ y \in \mathbb{R}^{n_{w_u}} : |x - y| < \varepsilon \} \subset N_i \} \subset V_{i}^{wu}.$$ 

As $N_i \subset V_i^{wu}$ are open, also the family $N_{i, \varepsilon}$ is a family of open sets, that increases monotonously and converges to $N_i$ for $\varepsilon \to 0$. Considering the complement $N_{i, \varepsilon}^c := V_i^{wu} \setminus N_{i, \varepsilon}$ we obtain a monotonously decreasing family of closed sets converging to the nullset $V_i^{wu} \setminus N_i$. Furthermore, from [10] we see that the distribution $\rho_{N_{i, \varepsilon}^c}$ defined in [5] is equal to 1 on an open neighborhood of supp($u_i$). Taking an arbitrary $\psi \in C^\infty (V_i)$ and any $\varepsilon > 0$ we can use this fact to calculate

$$u_i[\psi] = (u_i \cdot \psi)[1] = ((u_i \cdot \psi) \cdot \rho_{N_{i, \varepsilon}^c})[1].$$

But as the multiplication by a smooth function doesn’t increase the wavefront set we can apply Lemma 8 to $(u_i \cdot \psi) \in \mathcal{E}_V^r (V_{i}^{wu})$ and obtain for any $\varepsilon$

$$|u_i[\psi]| \leq C_{u_i \cdot \text{vol}_{n_{w_u}}} (N_{i, \varepsilon}).$$

Finally, as the constants $C_{u_i \cdot \psi}$ are independent of $\varepsilon$, the right hand side converges to zero as $\varepsilon \to 0$, so we have shown $u_i[\psi] = 0$. As $\psi \in C^\infty (V_i)$ was an arbitrary test function this implies $u_i = 0$ for any $i$ and we conclude that $u = 0$ which is a contradiction. This finishes the proof of Theorem 1.

\begin{appendix}
\section{Smoothness of the conditionnal measures}

In the proof of the support property of the Pollicott-Ruelle resonant states, the smoothness of the conditional measures (Theorem 7) plays a central role. While the result is well known to experts in dynamical system theory it is difficult to find a precise statement or a proof in the existing literature. This is why I wanted to provide some more details in the arxiv-version of this article, on how to connect the smoothness of the conditional densities to the smoothness of holonomy maps, by standard dynamical systems arguments:

Let $V_i \subset \mathbb{R}^n$ be the range of a coordinate chart $\kappa_i$ as in Theorem 2. As we only work in one fixed chart, we drop the index $i$ in order to simplify the notation and write $V := V_i$. Recall that we have already defined $V_{i}^{wu} = \{ x \in \mathbb{R}^{n_{w_u}} : (x, 0) \in V \}$ and analogously we define $V^s := \{ y \in \mathbb{R}^{n_u} : (0, y) \in V \}$. Recall that the strong stable manifolds define a foliation of $V$ by the leaves $S_x$ which are parametrized by $x \in V^{wu}$. Note that $S_0 = V \cap (\{0\} \times \mathbb{R}^{n_s})$ so it can be identified with $V^s$. Now for any $y \in V^s$ we define $T_y := V \cap (\mathbb{R}^{n_{w_u}} \times \{y\})$ which defines a smooth foliation transversal to the foliation $S_x$. Recall that by the construction of the coordinate charts 4, $T_0$ coincides with a weak unstable leaf, but in general the other leaves can not coincide with the weak unstable leaves unless the particular case, that the weak unstable foliation is smooth. Let us now reduce the set $V$ such that it has a product structure w.r.t. the foliations $S_x$ and $T_y$, i.e. such that for all $x \in V^{wu}$ and $y \in V^s$ one has $S_x \cap T_y := \mathcal{H}(x, y)$ defines a unique point contained in $V$. Note that this poses not problem for proving the theorem which claims the existences of the conditional measures in some product neighborhood of the strong stable and weak unstable foliation as such a set is always contained in the reduced set $V$.

Now with this product structure of $S_x$ and $T_y$ we can define two holonomy maps

$$\mathcal{H}^S_y : \{ \begin{array}{ll}
T_0 = V^{wu} & \rightarrow T_y \\
 x & \mapsto \mathcal{H}(x, y) \end{array}$$

and

$$\mathcal{H}^T_x : \{ \begin{array}{ll}
S_0 = V^s & \rightarrow S_x \\
y & \mapsto \mathcal{H}(x, y) \end{array}.$$

\end{appendix}
Let us first discuss the second one, belonging to the smooth transversal foliation. According to Theorem 2 $\mathcal{H}_x^T$ is a $C^\infty$ diffeomorphism and depends continuously on $x$ in the $C^\infty$-topology. Note that the euclidean metric on $V$ induces metrics and thus measures on the submanifolds $S_x$ which we call $dm_{S_x}(s)$ and as $S_0$ is simply the $y$-axis $dm_{S_0}$ is simply the euclidean measure $dy$ on $V^s$ after identifying $S_0$ and $V^s$. The fact that $\mathcal{H}_x^T$ are diffeomorphisms gives us a Jacobian function $J_x^T \in C(S_0)$ such that for a any function $\psi \in C_c(S_x)$

$$\int_{S_x} \psi(s)dm_{S_x}(s) = \int_{S_0} \psi \circ \mathcal{H}_x^T(y)J_x^T(y)dm_{S_0}(y).$$

The theorem of change of variables gives us, the precise form of $J_x^T(y)$: For any $y \in S_0$ the differential of the holonomy maps is a linear isomorphism $D\mathcal{H}_x^T : T_yS_0 \subset \mathbb{R}^n \to T_yS_0 \subset \mathbb{R}^n$.

Choosing two orthonormal basis of the $n_s$-dimensional subspaces $T_yS_0 \subset \mathbb{R}^n$ and $T_yS_0 \subset \mathbb{R}^n$ (w.r.t. to the euclidean metric on $\mathbb{R}^n$) the differential can be expressed as an $n_s \times n_s$ matrix and the Jacobian is simply the absolute value of the determinant.

Now the fact, that $\mathcal{H}_x^T$ is $C^\infty$ and depends continuously on $x$ translates to the fact that $J_x^T \in C^\infty(S_0)$ and depends continuously on $x$ w.r.t. the $C^\infty$-topology.

Let us now turn to the holonomy maps of the strong stable foliation. Here the situation is infinitely more complicated, as for a general Anosov flow the maps $\mathcal{H}_y^S$ are only Hölder continuous homeomorphisms. However it is known that they are absolutely continuous and that they possess a Jacobian $J_y^S$ such that for any function $\psi \in C_c(T_0)$

$$\int_{T_0} \psi(x)d\text{Leb}_{T_0}(x) = \int_{T_0} \psi \circ \mathcal{H}_y^S(x)J_y^S(x)d\text{Leb}_{T_0}(x).$$

The existence of such Jacobians for holonomies in dynamical systems can be found in various textbooks (see e.g. [BS02 Section 6.2]. It is however more difficult to find sharp statements on their regularity in $x$ and $y$. The following sufficiently strong statement for our purpose can for example be found in [GLPT13 Appendix E]

**Proposition 11.** For any $x$, the map $y \in S_0 \mapsto J_y^S(x) \in \mathbb{R}_{>0}$ is in $C^\infty(S_0)$ and the map $x \in V^u \mapsto J_x^S(x) \in C^\infty(S_0)$ is continuous.

Let us now combine all the ingredients in order to calculate the conditional densities. Given a function $\psi \in C_c(V)$ we calculate

$$\int_V \psi(x,y)dx\,dy = \int_{V^s} \left[ \int_{T_y} f(x,y)d\text{Leb}_{T_0}(x) \right] dy$$

$$= \int_V \left[ \int_{T_y} \psi \circ \mathcal{H}_y^S(x)J_y^S(x)d\text{Leb}_{T_0}(x) \right] dy$$

$$= \int_{T_0} \left[ \int_{S_0} \psi(H(x,y))J_y^S(x)dm_{S_0}(y) \right] d\text{Leb}_{T_0}(x)$$

$$= \int_{T_0} \left[ \int_{S_0} \psi(s)J_x^S((\mathcal{H}_x^T)^{-1}(s))dm_{S_0}(s) \right] d\text{Leb}_{T_0}(x).$$

1Note that in [GLPT13 Appendix E] they state the result for the strong unstable foliation. Furthermore they give even more precise information on the Hölder regularity of the map $x \mapsto J_x^S(x)$ which is not relevant for our purpose.
We have thus shown that the conditional densities take the form
\[ \delta_x(s) = \frac{J_{(H^T_x)^{-1}}(s)}{J_{H^T_x}((H^T_x)^{-1}(s))}. \]
Collecting the statements on the regularity of the Jacobians and holonomy maps we conclude, that for any \( x \) they are smooth and that the map
\[ x \mapsto \delta_x(H^T_x(\bullet)) \in C^\infty(S_0) \]
depends continuously on \( x \) w.r.t. the \( C^\infty \) topology. This statement assures, the uniform bounds on \( \|\delta_x\|_k \).

**Remark 5.** In the case of \( C^\infty \)-foliations we could take coordinate charts \( \kappa_i \) such that \( S_x = (\{x\} \times \mathbb{R}^n) \cap V_i \). Then (12) would reduce to the ordinary Fubini theorem. For a general foliation such generalizations of Fubini theorems are a nontrivial task and there are counterexamples of foliations that are not absolutely continuous (see e.g. [BP07, Section 8.6]).

**Remark 6.** For Anosov maps, an analogous statement to Theorem 7 can be found in [Che02, Remark 3.4, p.377].

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Institut für Mathematik, Universität Paderborn, Paderborn, Germany

E-mail address: weich@math.uni-paderborn.de