WELL-POSEDNESS AND CONVERGENCE OF A VARIATIONAL DISCRETIZATION OF THE CAMASSA–HOLM EQUATION

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Abstract. We show how the two-component Camassa–Holm (2CH) system, for which the Camassa–Holm equation is a special case, can be derived from variational principles by introducing a potential energy. The variation is done in Lagrangian coordinates, but the Euler–Lagrange equations have a pure Eulerian formulation. After discretizing the kinetic and potential energies, we use the same variational principles to derive a semi-discrete system of equations as an approximation of the 2CH system. In the discrete case, the Euler–Lagrange equations are only available in Lagrangian variables. In this derivation there naturally appears a discrete Sturm–Liouville operator whose coefficients depend on the initial data. We show the existence of a decaying Green’s function for this operator. Then, we prove that the semi-discrete system admits globally defined unique solutions, and preserves discrete versions of the energy and momentum for the 2CH system. Finally, we show how the solutions of the semi-discrete system can be used to construct a sequence of functions which converges to the solution of the 2CH system.

1. Introduction

The Camassa–Holm (CH) equation

\[ u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \]

is first known to have appeared as a special case in a hierarchy of completely integrable partial differential equations presented in \[\text{[16, Eqs. (26e) and (30)]}\], although written in an alternative form. However, it gained prominence after it was derived in \[\text{[7]}\] as a limiting case in the shallow water regime of the Green–Naghdi equations from hydrodynamics. In \([\text{[11]}]\), \(u = u(t,x)\) is the fluid velocity in position \(x\) at time \(t\), and the subscripts denote partial derivatives with respect to these variables.

The Camassa–Holm equation has been widely studied due to its rich mathematical structure and interesting properties, for instance it is bi-Hamiltonian, completely integrable and its solutions may develop singularities in finite time even for smooth initial data.

In this paper, we follow the geometrical point of view where the CH equation appears as a geodesic equation in the group of diffeomorphisms, see \([\text{13, 11, 12]}\].

Key words and phrases. Camassa–Holm equation, two-component Camassa–Holm system, calculus of variations, Lagrangian coordinates, energy preserving discretizations, discrete Green’s functions, discrete Sturm–Liouville operators.
The evolution of the particle positions is parameterized by a curve in the group of diffeomorphisms: We have \( \xi \rightarrow y(t, \xi) \) for a particle labeled by \( \xi \in \mathbb{R} \), while the Lagrangian velocity is given by \( U(t, \xi) = y_t(t, \xi) \). The Eulerian velocity is given in the observer frame,

\[
u(t, x) = U(t, y(t, \cdot)^{-1}(x)).
\]

The equations of motion are given by the least action principle, and can also be interpreted as a geodesic if we identify the measure of distance with the energy of the system. Between two given configurations \( y(t_0, \xi) = y_0(\xi) \) and \( y(t_1, \xi) = y_1(\xi) \), the equation of motion is given by the path \( y(t, \xi) \) that minimizes a distance which is set as the energy. In practice, the equation is thus derived by variational principles. For the CH equation, the energy is given by the \( H^1 \)-norm of the velocity in Eulerian coordinates. More precisely, we have

\[
E = \frac{1}{2} \| u \|_{H^1}^2 = \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_x^2) \, dx.
\]

To derive a discrete approximation of the CH equations, we propose to follow the two steps of this variational approach. First, we discretize the path functions \( y(t, \xi) \) by piecewise linear functions, \( y_i(t) = y(t, \xi_i) \) for \( \xi_i = i \Delta \xi, \, i \in \mathbb{Z} \) and \( \Delta \xi > 0 \). Then, we approximate the \( H^1 \)-norm expressed in Lagrangian coordinates using these discretized variables. We obtain finally the governing equation for the discretized system using variational principles.

When the energy is given only in terms of the Eulerian velocity – as is the case here – it admits the whole group of diffeomorphisms as a symmetry group. Indeed, we can check directly that, for any diffeomorphism \( \psi \), the transformation \( \tilde{y}(t, \xi) = y(t, \psi(\xi)) \) leaves the Eulerian velocity and therefore the energy invariant. This symmetry is natural for systems where the particles are equivalent in the sense that no physical property can distinguish them from each other. In a fluid, each point can be seen as a particle and the transformation \( y \mapsto \tilde{y} \) corresponds to a relabeling of the particles. Since the particles are all equivalent, it is expected that the label of the particle should not play any role for the evolution of the system. To make this point clearer, it helps to consider a system composed of two non-miscible fluids with different properties which affect the motion. In general, the dynamics is dependent on the position of each fluid. We do not have labeling invariance as the two fluids are by assumption not interchangeable, and a pure Eulerian description of the evolution of the system is not available.

The group of diffeomorphisms is of infinite dimension which explains why we obtain infinitely many invariants: Indeed, for every \( \xi \in \mathbb{R} \), we have the conserved quantity

\[
\frac{d}{dt} \left( y_\xi(t, \xi)^2 (u - u_{xx}) \circ y(t, \xi) \right) = 0.
\]

Here, \( y \) is reconstructed from the velocity field \( u \), that is \( y_t = u(t, y) \). By analogy to the rigid body motion, Arnold and Khesin denote these invariants as angular momentum \( \mathbb{I} \). Another result from the same authors is that for systems with such symmetry, labeling invariance, the governing equation can be expressed in terms of the Eulerian variables only. This result may be expected, but since the variation is taken in Lagrangian variables, i.e. we consider infinitesimal perturbations of the particle path, it is not clear from the beginning that we will necessarily end up with
a purely Eulerian formulation. In the discretization process we do not carry over the group structure, as the composition rule is not defined in the discrete setting, and so we do not obtain an energy with relabeling symmetry. In practice, it implies that our discretized equation will not have a purely Eulerian form and should be solved in Lagrangian variables. We retain two symmetries though, the time and space translation invariance. As a result, we obtain the conservation of the integrals
\[ \int_{\mathbb{R}} u^2 + u_x^2 \, dx \quad \text{and} \quad \int_{\mathbb{R}} u \, dx, \]
in their discrete form, see Section 2.

Let us then consider (1.1) in Lagrangian variables, and we start by rewriting it as
\[ (1.3) \quad u_t + uu_x = -P_x \]
where \( P \) is defined through the inverse of the Helmholtz operator \( \text{Id} - \partial_{xx} \), as
\[ (1.4) \quad P - P_{xx} = u^2 + \frac{1}{2} u_x^2. \]
We rewrite (1.4) as a system of first-order equations,
\[ (1.5) \quad \begin{bmatrix} -\partial_x & 1 \\ 1 & -\partial_x \end{bmatrix} \circ \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}, \]
for \( Q = P \). Now, we need the Lagrangian transformation of \( P \), namely \( \bar{P}(\xi) = P(y(\xi)) \), and note that we omit the time dependence for the moment. Using the chain rule, we obtain a Sturm–Liouville operator given by
\[ (1.6) \quad \begin{bmatrix} -\partial_\xi & y_\xi \\ y_\xi & -\partial_\xi \end{bmatrix} \circ \begin{bmatrix} \bar{Q} \\ \bar{P} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{f} \end{bmatrix}, \]
for \( \bar{f} = f \circ y \). In the expression above, the operator denoted \( y_\xi \) corresponds to pointwise multiplication. Thus, to compute \( \bar{P} \) we need to invert a time-dependent operator, since \( y_\xi \) evolves in time. This is in contrast to the Eulerian case (1.5), where the operator is constant. This difficulty can be overcome because we can invert the Sturm–Liouville operator using the kernel \( G(x) = \frac{1}{2} e^{-|x|} \) of the Helmholtz operator as follows,
\[ P(x) = \int_{\mathbb{R}} G(x - z) f(x) \, dz. \]
Hence, \( \bar{P} \) is given by
\[ \bar{P}(\xi) = (P \circ y)(\xi) = \int_{\mathbb{R}} G(y(\xi) - y(\eta)) \bar{f}(\eta) y_\xi(\eta) \, d\eta. \]

In the discrete case, the equation which corresponds to (1.6) is given by a discrete Sturm–Liouville equation
\[ (1.7) \quad \begin{bmatrix} -D_- & D_+ y_i \\ D_+ y_i & -D_- \end{bmatrix} \circ \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{f} \end{bmatrix}, \]
where \( D_{\pm} \) denotes forward and backward difference operators, see Section 2.2 for their definitions. Since the discrete structure is not equipped with a composition law, we cannot use the Eulerian reference frame to compute the solution of (1.7) as we do in the continuous case (1.6). Consequently, we have to solve (1.7) directly, and this is achieved by finding the Green’s function for this second-order difference system. In \([8, 23]\) where (1.1) is discretized in Eulerian coordinates, the construction
of the corresponding Green’s function boils down to solving a difference equation with constant coefficients, and it is relatively straightforward to derive the Green’s function explicitly. On the other hand, in Lagrangian coordinates the coefficients in the difference equation are nonconstant, and we have to solve a form of Jacobi difference equation, see [28]. However, the coefficients in this equation have $D_+ y_i$ in the denominator which makes the equation difficult to work with, especially as we want to allow $D_+ y_i = 0$. This motivates a change of variables which remove this explicit dependence. For the resulting system we invoke results generalizing Poincaré–Perron theory on difference equations found in [15, 27].

We address this non-trivial task in Section 3.2, where the difficulty is to show the existence of solutions with the required decay at infinity. Then, it also becomes clear that the method we use to find the Green’s function, if it can be applied initially, will be challenging to obtain the Lipschitz estimates we need for $t > 0$.

Lipschitz estimates are required to establish the short-time existence of solutions to the semi-discrete system. Therefore, instead of solving (1.7) for each time $t > 0$ to compute $\bar{P}$, we rather propagate the fundamental solutions in time, as outlined below. The fundamental solution is given by the discrete operators $(g, k, \gamma, \kappa)$ which satisfy

$$\begin{pmatrix} -D_- & D_+ y \\ D_+ y & -D_+ \end{pmatrix} \circ \begin{pmatrix} \gamma & k \\ g & \kappa \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},$$

where we refer to (4.27) for the detailed definitions of the variables entering this identity. We use (1.8) to derive a system of ordinary differential equations (ODEs) for the operators $(g, k, \gamma, \kappa)$. Then, we add these operators as auxiliary variables and complement our governing equations with the ODEs we have obtained. This approach can in fact also be used in the continuous case, and we describe this in detail in the first part of Section 4.

The CH equation can blow up in finite time – even for smooth initial data. The blow-up scenario has been described in [9, 10, 14] and consists of a singularity where $\lim_{t \to t_c} u_x(t, x_c) = -\infty$ for some critical time $t_c$ and location $x_c$. However, since the $H^1$-norm of the solution is preserved, the solution remains continuous. In fact, the solution can be prolonged in two consistent ways: Conservative solutions will recover the total energy after the singularity, while dissipative solutions remove the energy that has been trapped in the singularity, see [4, 25, 21, 20, 5, 26, 22].

In Eulerian coordinates, assuming the solution is smooth, the governing equations (1.3) and (1.4) yield the following energy conservation law,

$$\frac{1}{2}(u^2 + u_x^2)_t + (u \frac{1}{2}(u^2 + u_x^2))_x = -(uR)_x,$$

for $R = P - \frac{1}{2}U^2$. This equation enables us to compute the evolution of the cumulative energy

$$H(t, \xi) = \frac{1}{2} \int_{-\infty}^{\xi(t, \xi)} (u^2 + u_x^2) \, dx$$
in Lagrangian coordinates, and we obtain

\[ \frac{dH}{dt} = -uR. \]

Since the right-hand side in (1.10) can be shown to be continuous, this ODE gives us a control on the energy distribution. Furthermore, it can be observed that blow-up of a solution corresponds to concentration of energy in sets of measure zero, more precisely the energy density \( \frac{1}{2}(u^2 + u_x^2) \) becomes a measure with a singular part. However, Equation (1.10) enables us to track the energy, and the further dynamics contained in the governing equations impose that the concentrated energy is immediately redistributed in the system which evolves away from this singularity, see [25, Lemma 2.7]. The peakon-antipeakon solution is a good illustration of the dynamics of such collisions, see for example [24, 19]. The variable \( R \) was not used in previous works, but appeared in a natural way when we constructed the discrete system, and with it (1.3) takes the form

\[ u_t + (u^2)_x = -R_x. \]

Let us present the equivalent system for the continuous case. We introduce the operators \( \mathcal{G} \) and \( \mathcal{K} \) that defines the fundamental solutions

\[ \left[ \begin{array}{cc} -\partial_{\xi} & y_{\xi} \\ y_{\xi} & -\partial_{\xi} \end{array} \right] \circ \left[ \begin{array}{cc} \mathcal{K} & \mathcal{G} \\ \mathcal{G} & \mathcal{K} \end{array} \right] = \left[ \begin{array}{cc} \delta & 0 \\ 0 & \delta \end{array} \right]. \]

By definition of the characteristics, we have \( y_t = U \). We can then obtain the equivalent system in the continuous case,

\[ \begin{align*}
y_t &= U, \\
U_t &= -Q, \\
H_t &= -UR, \\
\partial_t \mathcal{G} &= [U, \mathcal{K}], \\
\partial_t \mathcal{K} &= [U, \mathcal{G}],
\end{align*} \]

(1.11)

where

\[ \left[ \begin{array}{c} R \\ Q \end{array} \right] = \left[ \begin{array}{cc} \mathcal{K} & \mathcal{G} \\ \mathcal{G} & \mathcal{K} \end{array} \right] \circ \left[ \begin{array}{c} \frac{1}{2}U^2 \\ H \end{array} \right] \]

and \([U, \mathcal{K}]\) denotes the commutator of \( U \) and \( \mathcal{K} \). The analysis of (1.11) can be carried out in the same way as in [25] but, in addition to the fact that it looks simpler, this new form also simplifies significantly the analysis of the discrete system. The equivalent system in the discrete case, which is given in 4.42, takes indeed a form similar to (1.11). The short-time existence relies on Lipschitz estimates. At this stage, one of the main ingredients in the proofs is the Young-type estimate for discrete operators presented in Proposition 3.2. For the global existence, we have to adapt the argument of the continuous case and complement it with \textit{a priori} estimates of the fundamental solutions \((g, k, \gamma, \kappa)\). These estimates follow from the monotonicity properties of these operators which are preserved by the evolution of the system, see Lemma 4.1.
In this paper, we cover the two-component Camassa–Holm (2CH) system given by

\begin{align}
&\begin{cases}
  u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho \rho_x = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\
  \rho_t + (\rho u)_x = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) = \rho_0(x), & x \in \mathbb{R},
\end{cases}
\end{align}

where \( \rho \) is called the density. We can see that the density is transported by the velocity field \( u \). The dynamics of the system is modified to include an elastic or potential energy \( E_{\text{pot}} \) that brings repulsive forces when particle concentrates (so that the density increases),

\begin{align}
E_{\text{pot}} = \frac{1}{2} \int_{\mathbb{R}} (\rho - \rho_\infty)^2 \, dx.
\end{align}

The scalar \( \rho_\infty \) gives the value of \( \rho \) as \( x \) tends to \( \pm \infty \). The density regularizes the solution by penalizing the concentration of the particles, see [21]. We denote \( r(t, \xi) = \rho(y(t, \xi)) y(t, \xi) \), the density in Lagrangian coordinates and it fulfills \( \frac{\partial r}{\partial t} = 0 \). In Section 2.2 we show how (1.12) can be derived as the Euler–Lagrange equation for the kinetic energy \( E_{\text{kin}} \) defined as in (1.2) and the potential energy \( E_{\text{pot}} \).

For a given step \( \Delta \xi \), We consider piecewise-linear discretization of \( y, U \) and \( H \) and piecewise-constant discretization of \( r \), which we denote \( y_i, U_i, H_i \) and \( r_i \), respectively. We set \( X = (y_i, U_i, H_i, r_i, g_{ij}, k_{ij}, \gamma_{ij}, \kappa_{ij}) \). The discrete system is then also obtained as a Euler–Lagrange equation for the discrete Lagrangian

\[ L_{\text{dis}} = E_{\text{kin}}^{\text{dis}} - E_{\text{pot}}^{\text{dis}}, \]

where these quantities appear are natural discretization of \( E_{\text{kin}} \) and \( E_{\text{pot}} \), see (2.4) and (2.8). The discretized equation takes the form of an system of ordinary differential equation in a Banach space,

\begin{align}
X_t = F(X),
\end{align}

where \( F \) is defined in (4.42). The main results of the paper consists of a theorem on the construction of initial data, given in a short form as

**Theorem 1.1 (Initial data).** Consider (1.1) with initial datum \( u_0 \in H^1 \), where we allow for the initial energy to contain a singular part. Then, we may use the energy distribution to define initial discrete initial datum for (1.14),

where we refer to Theorem 5.7 for details, and a convergence theorem,

**Theorem 1.2 (Convergence).** The semi-discrete CH system given by (1.14) has a unique, globally defined solution, and preserves its own discrete versions of energy and momentum. Given \( T > 0 \) the discrete solutions can be used to construct a sequence of functions that converges to the solution of (1.14).

For details and proof of the unique, globally defined solutions in Theorem 1.2 we refer to Theorem 5.6 while the convergence proof is addressed in Section 6.
2. Derivation of the semi-discrete CH system using a variational approach

The 2CH system can be derived as the Euler–Lagrange equation for the kinetic and potential energy given by (1.2) and (1.13) respectively. The variation is done with respect to the particle path, which we denote $y(t, \xi)$, so that we rewrite the energies in Lagrangian variables. We obtain

\begin{equation}
E^{\text{kin}}(t) = \frac{1}{2} \int_{\mathbb{R}} \left( y_t^2 y_\xi + \frac{y_{\xi}^2}{y_\xi} \right) (t, \xi) \, d\xi,
\end{equation}

while the potential energy can be written as

\begin{equation}
E^{\text{pot}}(t) = \frac{1}{2} \int_{\mathbb{R}} \left( \rho_0(y(0, \xi)) \frac{y_\xi(0, \xi)}{y_\xi(t, \xi)} - \rho_\infty \right)^2 y_\xi(t, \xi) \, d\xi.
\end{equation}

Indeed, since the Lagrangian density defined as $r(t, \xi) = \rho(t, y(t, \xi)) y_\xi(t, \xi)$ is invariant in time, by definition of the density, we have the identity

$$\rho(t, y(t, \xi)) y_\xi(t, \xi) = \rho(0, y(0, \xi)) y_\xi(0, \xi).$$

In the remaining of this section, we will assume without loss of generality that $y(0, \xi) = \xi$, which is a natural choice to take in the case of smooth initial data. For general initial data, we need the general form, see Section 5.2.

2.1. Variational derivation of the continuous system. Before we introduce the discretized functionals, we derive the 2CH system (1.12) from variational principles, as this will be mimicked when we derive our discrete system. In the following formal calculations we assume the functions involved to be regular enough for our manipulations to hold.

First we form the associated Lagrangian by taking the difference between the kinetic and potential energies $\mathcal{L}(t) = E^{\text{kin}}(t) - E^{\text{pot}}(t)$, thus we may write the Lagrangian as a sum of a kinetic part

$$\mathcal{L}^{\text{kin}}(y, y_t) := \frac{1}{2} \int_{\mathbb{R}} \left( y_t^2 y_\xi + \frac{y_{\xi}^2}{y_\xi} \right) (t, \xi) \, d\xi,$$

and a potential part

$$\mathcal{L}^{\text{pot}}(y, y_t) := -\frac{1}{2} \int_{\mathbb{R}} \left( \frac{\rho_0}{y_\xi} - \rho_\infty \right)^2 y_\xi(t, \xi) \, d\xi.$$

We obtain the equations of motion by taking variations of the functional

$$\delta J := \int_0^T \mathcal{L}(y, y_t) \, dt$$

for some $T > 0$. The variation reads

$$\delta J = \int_0^T \left( \left\langle \frac{\delta \mathcal{L}}{\delta y_t}, \delta y_t \right\rangle + \left\langle \frac{\delta \mathcal{L}}{\delta y}, \delta y \right\rangle \right) dt,$$
where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2}$ denotes the standard $L^2(\mathbb{R})$-inner product. The governing equation is then given by the Euler–Lagrange equation

\begin{equation}
\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0.
\end{equation}

We compute the variation with respect to $\delta y$,

\[
\left\langle \frac{\delta L^{\text{kin}}}{\delta y}, \delta y \right\rangle := \frac{1}{2} \int_{\mathbb{R}} \left( y_t^2 \delta y_\xi - \frac{y_t y_\xi}{y_\xi} \delta y_\xi \right) d\xi
\]

\[
= \int_{\mathbb{R}} \left( -y_t y_\xi + \frac{1}{2} \left( \frac{y_\xi}{y_\xi} \right)^2 \right) \delta y d\xi,
\]

which gives

\[
\frac{\delta L^{\text{kin}}}{\delta y} (y, y_t) = -y_t y_\xi + \frac{1}{2} \left( \frac{y_\xi}{y_\xi} \right)^2 \xi.
\]

The variation with respect to $\delta y_t$ is computed as

\[
\left\langle \frac{\delta L^{\text{kin}}}{\delta y_t}, y_t \right\rangle := \int_{\mathbb{R}} \left( y_t \delta y_t y_\xi + \frac{y_\xi y_t}{y_\xi} \right) d\xi
\]

\[
= \int_{\mathbb{R}} \left( y_t y_\xi - \left( \frac{y_\xi}{y_\xi} \right) \right) \delta y_t d\xi,
\]

and yields

\[
\frac{\delta L^{\text{kin}}}{\delta y_t} (y, y_t) = y_t y_\xi - \left( \frac{y_\xi}{y_\xi} \right) \xi.
\]

For the potential energy, we have

\[
\left\langle \frac{\delta L^{\text{pot}}}{\delta y}, \delta y \right\rangle := -\frac{1}{2} \int_{\mathbb{R}} \left( 2 \left( \frac{\rho_0}{y_\xi} - \rho_\infty \right) - \rho_0 \frac{\delta y_\xi}{y_\xi} + \left( \frac{\rho_0}{y_\xi} - \rho_\infty \right)^2 \delta y_\xi \right) d\xi
\]

\[
= \frac{1}{2} \int_{\mathbb{R}} \left( \frac{\rho_0}{y_\xi} - \rho_\infty \right) \left( \frac{2 \rho_0}{y_\xi} - \rho_\infty \right) \left( \frac{\rho_0}{y_\xi} - \rho_\infty \right) \delta y_\xi d\xi
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}} \left( \left( \frac{\rho_0}{y_\xi} \right)^2 - \rho_\infty^2 \right) \delta y d\xi,
\]

leading to

\[
\frac{\delta L^{\text{pot}}}{\delta y} (y, y_t) = -\frac{1}{2} \left( \left( \frac{\rho_0}{y_\xi} \right)^2 - \rho_\infty^2 \right) \xi.
\]
Now, from the relations \( (\rho \circ y) y_\xi = \rho_0 \), \( y_t = u \circ y \), \( y_\xi = (u_x \circ y) y_\xi \), and \( y_{tt} = (u_t + uu_x) \circ y \) which implies \( y_{tt\xi} = ((u_t + uu_x)_x \circ y) y_\xi \), we can write

\[
\frac{\partial}{\partial t} \frac{\delta L_{\text{kin}}}{\delta y_t} (y, y_t) = (y_t y_\xi)_t - \left( \frac{y_\xi}{y_\xi} \right)_{tt}
\]

\[
= y_{tt} y_\xi + y_t y_\xi - \left( \frac{y_{tt\xi}}{y_\xi} - \frac{y_{tt}}{y_\xi^2} \right) \xi
\]

\[
= ((u_t + 2uu_x) \circ y) y_\xi - (((u_t + uu_x)_x - u_x^2) \circ y) y_\xi
\]

\[
= ((u_t + 2uu_x - ((u_t + uu_x)_x - u_x^2)_x) \circ y) y_\xi
\]

\[
= ((u_t - u_{txx} + 2uu_x - uu_{xx} - uu_{xxx}) \circ y) y_\xi.
\]

\[
\frac{\delta L_{\text{kin}}}{\delta y} (y, y_t) = ((-uu_x + u_x u_{xx}) \circ y) y_\xi,
\]

\[
\frac{\delta L_{\text{pot}}}{\delta y} (y, y_t) = -\frac{1}{2} \left( (\rho \circ y)^2 - \rho_{\infty}^2 \right) \xi
\]

\[
= -(\rho p_x \circ y) y_\xi.
\]

Inserting the above identities in the Euler–Lagrange equation (2.3) we get

\[
((u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho \rho_x) \circ y) y_\xi = 0
\]

which is exactly the 2CH system (1.12) when \( y_\xi \neq 0 \). Note that here the mass conservation \( \rho_t + (\rho u)_x = 0 \) is given implicitly by the relation \((\rho \circ y) y_\xi = \rho_0 \) used in the derivation.

Following the theory of classical mechanics we define the Hamiltonian \( \mathcal{H} \) as the Legendre transform of the Lagrangian \( L \), that is

\[
\mathcal{H} = \left\langle \frac{\delta L}{\delta y_t}, y_t \right\rangle - L.
\]

In Eulerian coordinates, the Hamiltonian corresponds to

\[
\mathcal{H} = \frac{1}{2} \int_\mathbb{R} \left( u^2 + u_x^2 + (\rho - \rho_{\infty})^2 \right) dx.
\]

The invariance of the Lagrangian \( L \) with respect to time translation implies that \( \mathcal{H} \) is preserved by time. We check it directly in this case. We have

\[
\frac{d}{dt} L(y, y_t) = \left\langle \frac{\delta L}{\delta y_t}, y_{tt} \right\rangle + \left\langle \frac{\delta L}{\delta y}, y_t \right\rangle
\]

Inserting the Euler–Lagrange equation in the above yields

\[
\frac{d}{dt} L(y, y_t) = \left\langle \frac{\delta L}{\delta y_t}, y_{tt} \right\rangle + \left\langle \frac{\partial}{\partial t} \frac{\delta L}{\delta y_t}, y_t \right\rangle
\]

\[
= \frac{d}{dt} \left\langle \frac{\delta L}{\delta y_t}, y_t \right\rangle,
\]

leading to

\[
\frac{d\mathcal{H}}{dt} = 0.
\]
2.2. Derivation of the semi-discrete system. When deriving our discrete system we follow the steps of the continuous case after introducing a suitable discretization of the energies. We divide the line into an equi-sized grid by defining

$$\xi_j := j \Delta \xi$$

for some $\Delta \xi > 0$ and $j \in \mathbb{Z}$, and then introducing a countable family of characteristics $y_j(t) := y(t, \xi_j)$ which will approximate

$$\dot{y}_j(t) = u(t, y(t, \xi_j)), \quad y_j(0) = \xi_j, \quad j \in \mathbb{Z}.$$  

Note how we use the dot notation for differentiation with respect to time of grid functions, as to avoid overloading the subscript denoting the position in the grid.

Next we approximate the derivatives in the definitions of the energies (2.1) and (2.2) by finite differences. The discrete kinetic energy is consequently defined as

$$E^{\text{kin}}_{\text{dis}} := \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} \left( (\dot{y}_j)^2 D_+ y_j + \frac{(D_+ \dot{y}_j)^2}{D_+ y_j} \right),$$

where $D_+$ is the forward difference operator defined by

$$D_+ y_j := \frac{y_{j+1} - y_j}{\Delta \xi}.$$  

Similarly, we define the backward difference

$$D_- y_j := \frac{y_j - y_{j-1}}{\Delta \xi},$$

and note that for the above differences we have the product rule

$$D_{\pm} (v_j w_j) = (D_{\pm} v_j) w_{j \pm 1} + v_j (D_{\pm} w_j).$$

When we later encounter operators, which correspond to sequences with several indices, such as $g_{i,j}$ for $i, j \in \mathbb{Z}$, we will indicate partial differences by including the index in the difference operator, for instance $D_{i+} g_{i,j} = (g_{i,j+1} - g_{i,j})/\Delta \xi$.

We introduce the velocity $U_i = \dot{y}_i$ such that (2.4) reads

$$E^{\text{kin}}_{\text{dis}} = \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} \left( U_j^2 D_+ y_j + \frac{(D_+ U_j)^2}{D_+ y_j} \right).$$

The discrete potential energy is similarly defined as

$$E^{\text{pot}}_{\text{dis}} := \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} \left( \rho_{0,j} D_+ y_j - \rho_\infty \right)^2 (D_+ y_j),$$

where $\rho_{0,j} := \rho_0(\xi_j)$. Let us also introduce the Lagrangian velocity $U_j := \dot{y}_j$ and the discrete $L^2$-inner product

$$\langle v, w \rangle_{L^2} := \Delta \xi \sum_{j \in \mathbb{Z}} v_j w_j, \quad v, w \in L^2.$$  

In the following we define the adjoint, or transpose as we are working with real functions, of the difference operators $(D_{\pm})^\top : \ell^2 \rightarrow \ell^2$ by the relation

$$\sum_{j \in \mathbb{Z}} ((D_{\pm})^\top v_j) w_j = \sum_{j \in \mathbb{Z}} v_j (D_{\pm} w_j), \quad v, w \in \ell^2,$$

where $v = \{v_j\}_{j \in \mathbb{Z}}$, $w = \{w_j\}_{j \in \mathbb{Z}}$. Summing by parts in the right-hand side of (2.9) we find the relevant identities for our setting, $(D_{\pm})^\top = -D_{\mp}$. 

Now we define the discrete Lagrangian

\[ L_{\text{dis}} = L_{\text{kin}}^{\text{dis}} + L_{\text{pot}}^{\text{dis}}, \]

where \( L_{\text{kin}}^{\text{dis}} := E_{\text{dis}}^{\text{kin}} \) and \( L_{\text{pot}}^{\text{dis}} := -E_{\text{dis}}^{\text{pot}} \). We compute the variation with respect to \( \delta y \) and \( \delta U = \delta y_t \), for which we obtain

\[
\delta L_{\text{kin}}^{\text{dis}} = \Delta \xi \sum_{j \in Z} \left( U_j (\delta U)_j (D_+ y_j) + \frac{D_+ U_j}{D_+ y_j} (D_+ (\delta y)_j) \right) \\
+ \frac{1}{2} \Delta \xi \sum_{j \in Z} \left( (U_j)^2 D_+ (\delta y)_j - \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 D_+ (\delta y)_j \right) \\
= \Delta \xi \sum_{j \in Z} \left( U_j (D_+ y_j) - D_+ \left( \frac{D_+ U_j}{D_+ y_j} \right) \right) (\delta U)_j \\
- \Delta \xi \sum_{j \in Z} \frac{1}{2} D_- \left( (U_j)^2 - \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 \right) (\delta y)_j,
\]

having used (2.9) in the final identity. This leads to

\[
\left( \frac{\delta L_{\text{kin}}^{\text{dis}}}{\delta y} \right)_j = -\frac{1}{2} D_- \left( (U_j)^2 - \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 \right)
\]

and

\[
(2.10) \quad \left( \frac{\delta L_{\text{kin}}^{\text{dis}}}{\delta U} \right)_j = U_j (D_+ y_j) - D_- \left( \frac{D_+ U_j}{D_+ y_j} \right) = \left( \frac{\delta L_{\text{dis}}}{\delta U} \right)_j,
\]

where the rightmost equality in (2.10) is a consequence of \( L_{\text{pot}}^{\text{dis}} \) being independent of \( U \). For the potential term we find

\[
\delta L_{\text{pot}}^{\text{dis}} = -\frac{\Delta \xi}{2} \sum_{j \in Z} \left( 2 \left( \frac{\rho_{0,j}}{D_+ y_j} - \rho_\infty \right) \frac{-\rho_{0,j} D_+ (\delta y)_j}{D_+ y_j} + \left( \frac{\rho_{0,j}}{D_+ y_j} - \rho_\infty \right)^2 D_+ (\delta y)_j \right) \\
= -\Delta \xi \sum_{j \in Z} \frac{1}{2} D_- \left( \left( \frac{\rho_{0,j}}{D_+ y_j} \right)^2 - \rho_\infty^2 \right) \delta y_j,
\]

which gives the variation

\[
\left( \frac{\delta L_{\text{pot}}^{\text{dis}}}{\delta y} \right)_j = -\frac{1}{2} D_- \left( \left( \frac{\rho_{0,j}}{D_+ y_j} \right)^2 - \rho_\infty^2 \right).
\]

The discrete Euler–Lagrange equation is

\[
(2.11) \quad \frac{\partial}{\partial t} \frac{\delta L_{\text{dis}}}{\delta U} = \frac{\delta L_{\text{dis}}}{\delta y},
\]
and from the kinetic part of the discrete Lagrangian we get
\[
\frac{\partial}{\partial t} \left( \frac{\delta L_{\text{dis}}}{\delta U} \right)_j = \frac{\partial}{\partial t} \left( U_j(D_+ y_j) - D_\omega \left( \frac{D_+ U_j}{D_+ y_j} \right) \right)
= \dot{U}_j(D_+ y_j) + U_j(D_+ y_j) - D_\omega \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 (D_+ y_j)
= \dot{U}_j(D_+ y_j) - D_\omega \left( \frac{D_+ U_j}{D_+ y_j} \right) + U_j(D_+ U_j) + D_\omega \left( \frac{D_+ U_j}{D_+ y_j} \right)^2.
\]
Using this when writing out (2.11) we obtain the following system of equations
\[
\dot{y}_j = U_j,
(D_+ y_j) \dot{U}_j - D_\omega \left( \frac{D_+ U_j}{D_+ y_j} \right) = -U_j(D_+ U_j)
- \frac{1}{2} D_\omega \left( (U_j)^2 + \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 + \left( \frac{\rho_0, j}{D_+ y_j} \right)^2 \right)
\]
for \( j \in \mathbb{Z} \). Note that we have omitted \( \rho_\infty^2 \) on the right hand side in (2.12) as \( D_- \) maps constants to zero, and that for \( \rho_0 \) not identically zero we must have \( y_j(0) = \xi_j \) in (2.12).

The discrete two-component Camassa–Holm system is a Hamiltonian system in its own right, and we will show that it leaves a discrete Hamiltonian time-invariant. Appealing once more to classical mechanics, we use the Legendre transform to obtain discrete Hamiltonian
\[
H_{\text{dis}} = \left\langle \frac{\delta L_{\text{dis}}}{\delta U}, U \right\rangle_{\ell^2} - L_{\text{dis}}.
\]
Writing out the above Hamiltonian explicitly we have
\[
H_{\text{dis}} = \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} \left( (U_j)^2 + \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 + \left( \frac{\rho_0, j}{D_+ y_j} - \rho_\infty \right)^2 \right) (D_+ y_j).
\]
Like its continuous counterpart, \( H_{\text{dis}} \) is time-invariant,
\[
\frac{dH_{\text{dis}}}{dt} = 0,
\]
because the Lagrangian is invariant with respect to time translation. There exists an other invariant for the discrete system which can be obtained from Noether’s theorem. We denote by \( \psi : \ell^2 \times \mathbb{R} \to \ell^2 \) the transformation given by the uniform translation,
\[
(\psi(y, \varepsilon))_j = y_j + \varepsilon.
\]
For simplicity, we write \( y^\varepsilon(t) = \psi(y(t), \varepsilon) \). From the definition of \( \psi \), we have
\[
\dot{y}^\varepsilon(t) = \dot{y}(t) \quad \text{and} \quad D_+ y^\varepsilon(t) = D_+ y(t).
\]
Hence, the Lagrangian \( L_{\text{dis}} \) is invariant with respect to the transformation \( \psi \). Then, Noether’s theorem gives us that the quantity
\[
\left\langle \frac{\delta L_{\text{dis}}}{\delta U}, \frac{\delta y^\varepsilon}{\delta \varepsilon} \right\rangle_{\ell^2}
\]
is preserved by the flow, where in our case \( \left( \frac{\delta U_j}{\delta y_j} \right)_j = 1 \) and

\[
\left( \frac{\delta L_{\text{dis}}}{\delta U} \right)_j = U_j(D_+ y_j) - D_+ \left( \frac{D_+ U_j}{D_+ y_j} \right),
\]

see (2.10). We obtain that the quantity

\[
I = \sum_{j \in \mathbb{Z}} U_j(D_+ y_j) - D_+ \left( \frac{D_+ U_j}{D_+ y_j} \right),
\]

which simplifies to

\[
I = \sum_{j \in \mathbb{Z}} U_j(D_+ y_j),
\]

is preserved. Note that \( I \) corresponds to a discretization of

\[
\int_{\mathbb{R}} (u - u_{xx}) \, dx = \int_{\mathbb{R}} u \, dx,
\]

in Eulerian coordinates, which is preserved by the Camassa–Holm equation.

Let us return to (2.12), and in particular the left-hand side. For a given sequence \( a = \{a_j\}_{j \in \mathbb{Z}} \in \ell^\infty \) and an arbitrary sequence \( w = \{w_j\}_{j \in \mathbb{Z}} \in \ell^2 \) we define the operator \( A[a] : \ell^2 \to \ell^2 \) by

\[
(A[a]w)_j := a_j w_j - D_+ \left( \frac{D_+ w_j}{a_j} \right), \quad j \in \mathbb{Z},
\]

which has the form of a Sturm–Liouville operator with differences replacing the usual derivatives. Note that when \( a \) is the constant sequence \( \{1\}_{i \in \mathbb{Z}} \) we obtain a discrete version of the Helmholtz operator \( \text{Id} - \partial_{xx} \). The operator \( A[a] \) is symmetric and positive definite for sequences \( a \) such that \( a_j > 0 \), as can be seen from

\[
\Delta \xi \sum_{j \in \mathbb{Z}} v_j (A[a]w)_j = \Delta \xi \sum_{j \in \mathbb{Z}} \left( a_j w_j v_j + \frac{1}{a_j} (D_+ w_j)(D_+ v_j) \right),
\]

where we have used (2.9). When \( A[D_+ y] \) is positive definite it is invertible and we may formally write (2.12) as a system of first order ordinary differential equations,

\[
\begin{cases}
\dot{y}_j = U_j, \\
\dot{U}_j = -A[D_+ y]^{-1} \left( U_j(D_+ U_j) + \frac{1}{2} D_- \left( (U_j)^2 + \left( \frac{D_+ U_j}{D_+ y_j} \right)^2 \right) \right),
\end{cases}
\]

where for a sequence \( a \) we have \( D_+ a = \{D_+ a_j\}_{j \in \mathbb{Z}} \), and we emphasize once more that \( \rho_0 \) not identically zero forces us to have \( y_j(0) = \xi_j \) in (2.17). When solving the above system, we obtain approximations of the fluid velocity and density in Lagrangian variables, \( U_j(t) \approx u(t,y(t,\xi_j)) \) and \( \rho_0/y_j(D_+ y_j(t)) \approx \rho(t,y(t,\xi_j)) \).

### 2.3. The Hamiltonian structure

Note that if we had used (2.11) directly together with the characteristics equation \( \dot{y}_j = U_j \) we would have obtained the equations of motion for a Hamiltonian system with generalized position and momentum.
variables $q$ and $p$, and Hamiltonian $\mathcal{H}(q, p)$,

\begin{align}
\dot{q} &= \frac{\delta \mathcal{H}}{\delta p}, \\
\dot{p} &= -\frac{\delta \mathcal{H}}{\delta q}.
\end{align}

(2.18)

Let us set $\rho \equiv 0$ to simplify the equations, which amounts to discretizing (1.1). As $y_j$ is a position coordinate, we identify it with $q_j$, while we introduce the momentum $p_j = (A[D_+ y]U)_j$ as found in (2.15). Then we formally have $U_j = (A[D_+ y]^{-1}p)_j$, and after a summation by parts we may write our conserved Hamiltonian (2.14) as

$$
\mathcal{H} = \frac{1}{2}\Delta \xi \sum_{j \in \mathbb{Z}} p_j (A[D_+ q]^{-1}p)_j.
$$

Taking variations of this Hamiltonian with respect to $q$ and $p$ and plugging into (2.18) we obtain exactly the characteristics and the discrete Euler–Lagrange equations from before. Moreover, if we introduce the Green’s function $g_{i,j}$ for the operator $A[D_+ q]$ which will appear in Section 4, the Hamiltonian can be rewritten as

$$
\mathcal{H} = \frac{1}{2}\Delta \xi \sum_{j \in \mathbb{Z}} p_j \Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} p_i = \frac{1}{2} \sum_{i,j \in \mathbb{Z}} (\Delta \xi p_i)(\Delta \xi p_j)g_{i,j},
$$

where the final form is analogous to the $N$-soliton Hamiltonian

$$
\frac{1}{2} \sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|}
$$

given in [7], where it is pointed out that Hamiltonians of this form describe geodesic motion. In this setting, our Green’s function $g_{i,j}$ plays the role of an inverse metric tensor for the motion of the particles $q_j$, and our discrete system can, as pointed out in the introduction for the CH equation (1.1), be interpreted as geodesic motion. We refer to [6] for a description of how characteristics can be used to reformulate the CH equation as a Hamiltonian system, and a discrete particle method further developed in [24].

3. Construction of the fundamental solutions of the discrete Sturm-Liouville operator

Here we present some auxiliary results to be used later in our analysis of the semi-discrete 2CH-system.

3.1. Sequence spaces and operators on sequence spaces. Given a grid parameter $\Delta \xi > 0$, let $\ell$ be the space of all bilaterally infinite sequences $a = \{a_j\}_{j \in \mathbb{Z}}$ defined on the lattice $\Delta \xi \mathbb{Z} := \{j \Delta \xi \mid j \in \mathbb{Z}\}$. The spaces $\ell^\infty$ and $\ell^p$ are defined by
the norms
\[ \ell^\infty : \|a\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |a_j|, \]
\[ \ell^p : \|a\|_{\ell^p} := \left( \Delta \xi \sum_{j \in \mathbb{Z}} |a_j|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \]
(3.1)

Let us also define a discrete analogue of the \(H^1(\mathbb{R})\)-inner product,
\[ (a, b)_{h^1} := \Delta \xi \sum_{j \in \mathbb{Z}} [a_j b_j + (D_+ a_j)(D_+ b_j)], \]
which induces a norm in the usual manner. Finally, we introduce the subspace of \(\ell^\infty\) defined as
\[ v := \{ a \in \ell^\infty \mid D_+ a \in \ell^2 \}, \quad \|a\|_v := \|a\|_{\ell^\infty} + \|D_+ a\|_{\ell^2}. \]
(3.3)

After establishing these norms, we are set to present some results for sequences which will be useful to us.

**Proposition 3.1** (Useful results for sequences). We list some useful results for sequences \(a : \mathbb{Z} \to \mathbb{R}, b : \mathbb{Z} \to \mathbb{R}, \) and \(f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}, \) where we use the convention \(q/\infty = 0 \) for \(q < \infty, \) and \(\infty/\infty = 1.\)

The results are the inverse inequalities
\[ \|a\|_{\ell^\infty} \leq \frac{1}{\sqrt{\Delta \xi}} \|a\|_{\ell^2} \leq \frac{1}{\Delta \xi} \|a\|_{\ell^1}, \]
(3.4)

the discrete Sobolev-type inequality
\[ \|a\|_{\ell^\infty} \leq \frac{1}{\sqrt{2}} \|a\|_{h^1}, \]
(3.5)

the summation by parts formula
\[ \Delta \xi \sum_{j=m}^{n} (D_+ a_j)b_j + \Delta \xi \sum_{j=m}^{n} a_j(D_+ b_j) = a_{n+1}b_n - a_m b_{m-1}, \]
(3.6)

and a discrete generalized Hölder inequality
\[ \left\| \prod_{k=1}^{n} a_k \right\|_{\ell^q} \leq \prod_{k=1}^{n} \left\| a_k \right\|_{\ell^{p_k}} \quad \text{for} \quad \sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{q}, \quad q, p_k \in [1, \infty], \]
where in (3.7) the \(j\)-th component of a product of sequences is interpreted as
\[ \left( \prod_{k=1}^{n} a_k \right)_j = \prod_{k=1}^{n} (a_k)_j. \]

Furthermore, any sequence \(a\) such that \(D_+ a \in \ell^2\) is bounded in the “discrete Hölder seminorm”
\[ \sup_{j, k \in \mathbb{N}, j \neq k} \frac{|a_j - a_k|}{(\Delta \xi |j - k|)^{1/2}} \leq \|D_+ a\|_{\ell^2}. \]
(3.8)

In addition, such sequences satisfy the asymptotic relation
\[ \lim_{j \to \pm \infty} \sqrt{\Delta \xi} |D_+ a_j| = 0, \]
(3.9)
which in particular implies
\[
\lim_{j \to \pm \infty} |a_{j+1} - a_j| = \lim_{j \to \pm \infty} \Delta \xi |D_x a_j| = 0 \quad \text{for } a \in \mathbf{v}.
\]

Next we consider linear operators on sequences, and note that such an operator takes the form of a double-indexed sequence. For an operator \( g_{i,j} \), we denote by \((g \ast f)_j\) its action on a sequence \( f \), which is defined as
\[
(g \ast f)_j = \Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} f_i,
\]
namely summing their elementwise product over the left index of \( g \). We use the following norms for the operators
\[
\|g\|_{\ell^q} = \sup_i \|g_i\|_{\ell^q} = \sup_i \left( \Delta \xi \sum_{j \in \mathbb{Z}} |g_{i,j}|^p \right)^{\frac{1}{p}},
\]
\[
\|g\|_{\ell^\infty} = \sup_i \left( \sup_j |g_{i,j}| \right).
\]
The transpose \( g^\top \) of \( g \) is given by \((g^\top)_{i,j} = g_{j,i}\). Then we can prove a result reminiscent of Young’s convolution inequality.

**Proposition 3.2 (Young’s inequality for general operators).**
\[
\|g \ast f\|_{\ell^r} \leq \|g\|_{\ell^q} \|g^\top\|_{\ell^p} \|f\|_{\ell^r},
\]
for
\[
1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad p, q, r \in [1, \infty].
\]

Above, we again use the convention \( q/\infty = 0 \) for \( q < \infty \), and \( \infty/\infty = 1 \). Note that the standard Young’s inequality is usually given for a translation invariant kernel where \( g \) takes the form \( g_{i,j} = \hat{g}_{i-j} \) for some sequence \( \hat{g} \). For an operator of this form, we can check that \( g^\top = \tau \circ g \circ \tau \), where the operator \( \tau \) inverts the indexing, that is \( \tau(f)_j = f_{-j} \). Since the operator \( \tau \) is an isometry in all \( \ell^q \)-spaces, the expression \( (3.12) \) simplifies to
\[
\|g \ast f\|_{\ell^r} \leq \|g\|_{\ell^q} \|f\|_{\ell^r}.
\]

For the proof of Propositions 3.1 and 3.2 we refer to Appendix A.

3.2. Construction of Green’s function for the operator \( A \). Note that when \( D_+ y \) coincides with the constant sequence \( 1 = \{1\}_{j \in \mathbb{Z}} \), as is the case for our initial conditions for nonzero \( \rho_0 \), and we use forward differences \( D = D_+ \), we have from \( (2.16) \) that \( A[1] = \text{Id} - D_- D_+ \). As the operator in this case is defined...
by a recurrence relation with constant coefficients, the authors are able to find an explicit Green’s function \( g \) which in our setting can be stated as

\[
g_j = \frac{1}{\sqrt{4 + \Delta \xi^2}} \left( 1 + \frac{\Delta \xi^2}{2} + \frac{\Delta \xi}{2} \sqrt{4 + \Delta \xi^2} \right)^{-|j|}
\]

fulfilling \((\text{Id} - D_- D_+)g = \delta_0\). Here \( \delta_0 = \{ \delta_{0,j} \}_{j \in \mathbb{Z}} \) for the Kronecker delta \( \delta_{i,j} \), equal to one when the indices coincide and zero otherwise. In our case, the coefficients appearing in the definition of \( A[D_+ y] \) are varying with the grid index \( j \), which leads to greater difficulty in establishing the existence of a Green’s function.

Let us consider the operator \( A[a] \) from (2.16) and the equation \((A[a]g)_j = f_j\), for which we would like to prove that there exists a solution which decreases exponentially as \( j \to \pm \infty \). To this end, we want to find a Green’s function for the operator \( A[a] \), and the first step is to realize that the homogeneous operator equation \((A[a]g)_j = 0\) can be written as

\[
\frac{D_+ g_j}{a_j} = \Delta \xi a_j g_j + \frac{D_+ g_{j-1}}{a_{j-1}}.
\]

This can again be restated as a Jacobi difference equation, see [28, Eq. (1.19)],

\[
-\frac{1}{a_j} g_{j+1} + \left( \frac{1}{a_j} + \frac{1}{a_{j-1}} + a_j (\Delta \xi)^2 \right) g_j - \frac{1}{a_{j-1}} g_{j-1} = 0,
\]

or equivalently in matrix form

\[
\begin{bmatrix}
g_{j+1} \\
g_j
\end{bmatrix} = \begin{bmatrix}
0 & 1 + \left( a_j (\Delta \xi)^2 \right) \\
-\frac{a_j}{a_{j-1}} & 1 + \frac{a_j}{a_{j-1}} + (a_j \Delta \xi)^2
\end{bmatrix} \begin{bmatrix}
g_{j-1} \\
g_j
\end{bmatrix} =: \tilde{A}_j \begin{bmatrix}
g_{j-1} \\
g_j
\end{bmatrix}.
\]

Observe that \( \tilde{A}_j \) is not symmetric and always contains positive, negative and zero entries under the assumption \( a_j > 0 \). Moreover, we run into trouble defining \( \tilde{A}_j \) and its determinant \( a_j/a_{j-1} \) whenever \( a_{j-1} = 0 \). This case is of importance to us, as it will correspond to “wave-breaking” for our discrete system, and we want to be able to define the Green’s function also in this scenario.

However, if we instead go back to the first restatement of the operator equation and introduce the variable

\[
\gamma_j := \frac{D_+ g_j}{a_j} = \frac{g_{j+1} - g_j}{a_j \Delta \xi}
\]

we get the following characterization of the homogeneous problem

\[
\begin{bmatrix}
-D_+ & a_j \\
a_j & -D_-
\end{bmatrix} \begin{bmatrix}
g_j \\
\gamma_j
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

or equivalently

\[
\begin{bmatrix}
g_{j+1} \\
\gamma_{j+1}
\end{bmatrix} = \begin{bmatrix}
1 + (a_j \Delta \xi)^2 & a_j \Delta \xi \\
a_j \Delta \xi & 1
\end{bmatrix} \begin{bmatrix}
g_j \\
\gamma_{j-1}
\end{bmatrix} =: A_j \begin{bmatrix}
g_j \\
\gamma_{j-1}
\end{bmatrix}.
\]

Here \( A_j \) is a symmetric matrix with positive entries whenever \( a_j > 0 \), and it reduces to the identity matrix when \( a_j = 0 \). As symmetric matrices have more desirable properties than asymmetric ones, especially when it comes to eigenvalues and diagonalization, we will use (3.17) to construct our Green’s function. This will prove worthwhile when analyzing the asymptotic behavior of the solutions.
Lemma 3.3 (Properties of matrix $A_j$). Consider $A_j$ from (3.17) and assume $a_j = 1 + D_j b_j > 0$ where $D_j, b_j \in \mathcal{E}^2$. Then $\det A_j = 1$ and there exist $M_b > m_b > 0$ depending on $\|D_j b_j\|_{\mathcal{E}^2}$ and $\Delta \xi$ such that the eigenvalues $\lambda_j^{\pm}$ of $A_j$ satisfy

$$m_b \leq \lambda_j^- < 1 < \lambda_j^+ \leq M_b$$

when $a_j > 0$, and $\lambda_j^\pm = 1$ when $a_j = 0$. Asymptotically we have $\lim_{j \to \pm \infty} A_j = A$, where $A$ is given by $A_j$ after setting $a_j = 1$, and the eigenvalues $\lambda^\pm$ of $A$ satisfy

$$m \leq \lambda^- < 1 < \lambda^+ \leq M$$

for $M > m > 0$ depending only on $\Delta \xi$. Moreover, as the eigenvalues are strictly positive it follows that the spectral radius of $A_j$, $\text{spr}(A_j) := \max\{|\lambda_j^+|, |\lambda_j^-|\}$ satisfies $\|A_j\| = \text{spr}(A_j) = \lambda^+_j$, $\|A\| = \text{spr}(A) = \lambda^+$, and both matrices can be diagonalized: $A_j = R_j \Lambda_j R_j^\top$, $A = R \Lambda R^\top$.

Proof of Lemma 3.3. To see that $\det A_j = 1$ one can compute it directly, or see it from the eigenvalues

$$\lambda_j^\pm := \frac{1}{2} \left[ 2 + (a_j \Delta \xi)^2 \pm \sqrt{(2 + (a_j \Delta \xi)^2)^2 - 4} \right]$$

$$= 1 + \frac{(a_j \Delta \xi)^2}{2} \pm \frac{a_j \Delta \xi}{2} \sqrt{4 + (a_j \Delta \xi)^2}$$

$$= \frac{1}{4} \left( \sqrt{4 + (a_j \Delta \xi)^2} \pm a_j \Delta \xi \right)^2,$$

which shows that $A_j$ is invertible irrespective of the value of $a_j$. As $A_j$ is real and symmetric it can be diagonalized with orthonormal eigenvectors $r_j^\pm$ as follows

$$A_j = R_j \Lambda_j R_j^\top, \quad \Lambda_j = \begin{bmatrix} \lambda_j^- & 0 \\ 0 & \lambda_j^+ \end{bmatrix}, \quad R_j = \begin{bmatrix} 1 & 1 \\ \sqrt{1 + \lambda_j^-} & \sqrt{1 + \lambda_j^+} \\ \sqrt{1 + \lambda_j^-} & \sqrt{1 + \lambda_j^+} \end{bmatrix}.$$

As $D_j b_j \in \mathcal{E}^2$, we have from (3.8)

$$0 \leq a_j \Delta \xi \leq \Delta \xi + \sqrt{\Delta \xi} \|D_j b_j\|_{\mathcal{E}^2} =: K_b$$

so that $a_j$ is bounded from above and below. This yields the bounds

$$0 < \left( \frac{\sqrt{4 + K_b^2} - K_b}{2} \right)^2 \leq \lambda_j^- \leq 1 \leq \lambda_j^+ \leq \left( \frac{\sqrt{4 + K_b^2} + K_b}{2} \right)^2 < (1 + K_b)^2$$

corresponding to (3.18). Furthermore, by (3.10), we have $\lim_{j \to \pm \infty} a_j \Delta \xi = \Delta \xi$. We denote by $A, \Lambda, R, \text{ and } \lambda^{\pm}$ the matrices and eigenvalues given $A_j$, $\Lambda_j$, $R_j$, and $\lambda_j^{\pm}$ after replacing $a_j$ by 1. From the preceding limit, (3.20) and (3.21) we obtain

$$\lim_{j \to \pm \infty} (A_j, \Lambda_j, R_j) = (A, \Lambda, R).$$

Bounds for $\lambda^{\pm}$ are obtained similarly to the bounds for $\lambda_j^{\pm}$. As $A_j, A$ are symmetric and hence normal, their norms coincide with the spectral radius $\text{spr}(\cdot)$ which here coincides with the largest eigenvalue. \qed
Note that (3.17) corresponds to a transition from \((g_j, \gamma_{j-1})\) to \((g_{j+1}, \gamma_j)\), and so \(A_j\) can be considered as a transfer matrix from a current state to a new state. Thus, solving the homogeneous operator equation \((A[a]g)_j = 0\) bears clear resemblance to propagating a discrete dynamical system, and this is also the idea employed in the analysis of Jacobi difference equations in [28, Eq. (1.28)]. However, in making our variable change to obtain (3.17) we lose symmetricity, and so the results in [28] are no longer directly applicable. On the other hand, our system can be regarded as a more general Poincaré difference system, and our idea is then to apply the results [15, Thm. 1.1] and [27, Thm. 1] to the matrix product

\[
\Phi_{k,j} := \begin{cases} 
A_{k-1} \ldots A_j, & k > j, \\
I, & k = j, \\
(A_k)^{-1} \ldots (A_{j-1})^{-1}, & k < j
\end{cases}
\]

which is the transition matrix from \((g_j, \gamma_{j-1})\) to \((g_k, \gamma_{k-1})\).

**Lemma 3.4** (Exponentially decaying solutions). The matrix equation

\[
v_n = (\Phi_{n,0}) v_0,
\]

coming from (3.17) with \(\Phi_{n,0}\) as defined in (3.23), has initial vectors \(v_0 = v_0^\pm\) such that the corresponding solutions \(v_n^\pm\) satisfy

\[
\lim_{n \to \pm \infty} \sqrt{\|v_n^\pm\|} = \lambda^-.
\]

That is, solutions \(v_n\) with exponential decay in either direction, owing to the Lyapunov exponent \(\lambda^- < 1\). Moreover, the initial vectors are unique up to a constant factor.

**Proof of Lemma 3.4** We begin with the case of increasing \(n\), and we want to apply [15, Thm. 1.2] which states that for sequences of positive matrices \(\{A_n\}\) satisfying

\[
\lim_{n \to +\infty} A_n = A
\]

for some positive matrix \(A\) we have

\[
\lim_{n \to +\infty} \frac{A_n A_{n-1} \ldots A_1 A_0}{\|A_n A_{n-1} \ldots A_1 A_0\|} = vw^T
\]

for some vectors \(v\) and \(w\) with positive entries such that \(Av = \text{spr}(A)v\). As mentioned in [2, Rem. 4], there is in general no easy way of determining the vector \(w\) explicitly.

We recall that our \(A_n\) has positive entries, unless \(a_n = 0\) in which case we have \(A_n = I\). Because of (3.22), there can only be finitely many \(n \geq 0\) for which \(A_n\) reduces to the identity. If we instead consider the sequence of positive matrices consisting of our \(\{A_n\}\) where we have omitted the finitely many identity matrices, they clearly still satisfy (3.22) and so (3.26) holds with \(\text{spr}(A) = \lambda^+\) and \(v = r^+\) from (3.20) and (3.21). However, as the matrices we omitted were identities, it is clear that the limit in (3.26) for both sequences coincide. Hence, [15, Thm. 1.1] holds for our nonnegative sequence as well.

Now, as \(A_n \geq I\) entrywise it follows that the entries of \(\Phi_{n,0}\) are nondecreasing for \(n \geq 0\), which means that \(\|\Phi_{n,0}\|\) also is nondecreasing for such \(n\). Therefore, by
we have that any initial vector \( v_0 \) such that \( w^\top v_0 \neq 0 \) leads to a solution \( v_n \) with nondecreasing norm, and which then by [27, Thm. 1] must satisfy

\[
\varrho = \lim_{n \to +\infty} \sqrt[n]{\|v_n\|}
\]

with \( \varrho = \lambda^+ > 1 \), i.e. an asymptotically exponentially increasing solution. Indeed, the nondecreasing norm rules out the possibility of \( v_n = 0 \) for \( n \) large enough. It follows that (3.27) holds for \( \varrho = \lambda^- \), but it if were \( \lambda^- < 1 \), then \( \|v_n\| \) could not be nondecreasing. However, choosing instead a nonzero \( v_0 \) such that \( w^\top v_0 = 0 \), we obtain an asymptotically exponentially decreasing solution \( v_n \) satisfying (3.27) with \( \varrho = \lambda^- < 1 \). This follows by once more excluding the scenario of \( v_n = 0 \) for large enough \( n \) since \( v_0 \) is nonzero and each \( A_n \) has full rank. Then the only remaining possibility is \( v_n \) satisfying (3.27) with \( \varrho = \lambda^- \). An obvious choice of \( v_0 \) given \( w = [w_1, w_2]^\top \) is then \( v_0 = [w_2 - w_1]^\top \).

For decreasing \( n \), we will be able to reuse the arguments from above. From (3.23) we find that \( \Phi_{n,0} \) is a product of inverses of \( A_n \) for \( n < 0 \), and by (3.17) we have

\[
\begin{bmatrix}
  g_j \\
  \gamma_{j-1}
\end{bmatrix} = (A_j)^{-1} \begin{bmatrix}
  g_{j+1} \\
  \gamma_j
\end{bmatrix} = \begin{bmatrix}
  1 & -a_j \Delta \xi \\
  -a_j \Delta \xi & 1 + (a_j \Delta \xi)^2
\end{bmatrix} \begin{bmatrix}
  g_{j+1} \\
  \gamma_j
\end{bmatrix}.
\]

Since \( (A_n)^{-1} \) contains entries of opposite sign, it would appear that we may not be able to use our previous argument. However, a change of variables will do the trick for us. First recall (3.15) which shows that \( \gamma_j \) corresponds to a rescaled forward difference for \( g_j \), hence its sign indicates whether \( g \) is increasing or decreasing at index \( j \). For an increasing solution in the direction of increasing \( n \) it is then necessary for \( g_n \) and \( \gamma_{n-1} \) to share the same sign as \( n \to +\infty \). On the other hand, for an increasing solution in the direction of decreasing \( n \), the forward difference for \( \gamma_{n-1} \) should have the opposite sign of \( g_n \) as \( n \to -\infty \). Therefore, a change of variables allows us to rewrite the previous equations as

\[
\begin{bmatrix}
  g_j \\
  -\gamma_{j-1}
\end{bmatrix} = \begin{bmatrix}
  1 & a_j \Delta \xi \\
  a_j \Delta \xi & 1 + (a_j \Delta \xi)^2
\end{bmatrix} \begin{bmatrix}
  g_{j+1} \\
  -\gamma_j
\end{bmatrix} =: B_j \begin{bmatrix}
  g_{j+1} \\
  -\gamma_j
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
  g_n \\
  -\gamma_{n-1}
\end{bmatrix} = B_n \ldots B_1 \begin{bmatrix}
  g_0 \\
  -\gamma_{-1}
\end{bmatrix}, \quad n < 0
\]

and for this system we may use the positive matrix technique from before. The eigenvalues of \( B_j \) in (3.28) are the same as those of \( A_j \), but they switch positions in the corresponding eigenvectors \( \vec{r}_j^{\pm} \) compared to \( r_j^{\pm} \) of \( A_j \):

\[
\begin{bmatrix}
  1 \\
  \pm \frac{\sqrt{1 + \lambda_j^{+}}}{\sqrt{1 + \lambda_j^{-}}}
\end{bmatrix}, \quad r_j^{\pm} = \begin{bmatrix}
  1 \\
  \pm \frac{\sqrt{1 + \lambda_j^{-}}}{\sqrt{1 + \lambda_j^{+}}}
\end{bmatrix}.
\]

The same argument as in the case of increasing \( n \) then proves the existence of \( v_0 \) giving exponentially decreasing solutions as \( n \to -\infty \). The uniqueness follows from the uniqueness of limits in (3.26), which for a given eigenvector \( v \) of \( A \) means that \( w \) is unique up to a constant factor. But then again, since we are in \( \mathbb{R}^2 \), the vector orthogonal to \( w \) is unique up to a constant factor. \( \square \)
Remark 3.5 (Signs of initial vectors). Here we underline that the form of $\Phi_{n,0}$ implies that the entries of $v_0^\pm$ in Lemma 3.4 must be nonzero, with opposite signs for $v_0^-$ and same sign for $v_0^+$. Indeed, by (3.26) and (3.25) we have

$$\lim_{n \to +\infty} \| (\Phi_{n,0}) v_0^- \| = 0.$$ 

Let us then assume $v_0^- \neq 0$ with nonnegative entries of the same sign, namely $v_0^- \geq 0$ ($v_0^- \leq 0$) understood entrywise. From the definition (3.23) and $A_n \geq I$, it is clear that $(\Phi_{n,0}) v_0^- \geq v_0^- \cdot (\Phi_{n,0}) v_0^- \leq v_0^-$ for $n \geq 0$, and so it is impossible for the norm to tend to zero. Hence, the entries of $v_0^-$ must be nonzero and of opposite sign. For $n \to -\infty$, we can use (3.28) and the same argument to arrive at the same conclusion for $[g_0 - \gamma_{-1}]^\top$, implying that $v_0^+ = [g_0 - \gamma_{-1}]^\top$ has nonzero entries of equal sign.

Theorem 3.6 (Existence of a discrete Green’s function). Assume $a$ to be a non-negative sequence such that $a_j = 1 + D_+ b_j$ with $D_+ b \in \mathcal{E}^2$. Then, for any given index $i$, there exists a unique sequence $g_i = \{g_{i,j}\}_{j \in \mathbb{Z}}$ such that

$$\tag{3.29} (A[a] g_i)_j = \frac{\delta_{i,j}}{\Delta \xi}.$$ 

Proof. Our strategy follows the standard approach for constructing Green’s functions. That is, taking solutions of the homogeneous version of (3.29) with exponential decay, and then “gluing” them together at one point to obtain a delta function. We start by constructing $g_{0,j}$ centered at $i = 0$.

Choosing $v_0^\pm$ from Lemma 3.4 we set

$$\tag{3.30} \begin{bmatrix} g_0^- \\ \gamma_{-1} \end{bmatrix} := v_0^- , \quad \begin{bmatrix} g_0^+ \\ \gamma_{-1} \end{bmatrix} := v_0^+ ,$$

and define the sequences

$$\tag{3.31} \begin{bmatrix} g_n^- \\ \gamma_{-1} \end{bmatrix} := \Phi_{n,0} \begin{bmatrix} g_0^- \\ \gamma_{-1} \end{bmatrix} , \quad \begin{bmatrix} g_n^+ \\ \gamma_{-1} \end{bmatrix} := \Phi_{n,0} \begin{bmatrix} g_0^+ \\ \gamma_{-1} \end{bmatrix} , \quad n \in \mathbb{Z},$$

where by construction $g^\pm, \gamma^\pm$ are exponentially decreasing for $n \to \pm \infty$. Then, applying the operator $A[a]$ to $g^\pm$ we find

$$\begin{cases} (A[a] g^\pm)_j = a_j g_j^\pm - D_- \gamma_j^\pm = 0, & j \in \mathbb{Z} \end{cases}$$

by construction of $g^\pm$ and $\gamma^\pm$. Let us then define

$$\tag{3.32} g_{0,j} := C \begin{cases} g_j^- g_0^+ , & j \geq 0 , \\ g_j^+ g_0^- , & j < 0 \end{cases} , \quad \gamma_{0,j} := C \begin{cases} \gamma_j^- g_0^+ , & j \geq 0 , \\ \gamma_j^+ g_0^- , & j < 0 \end{cases}$$

for some hitherto unspecified constant $C$, and observe from the homogeneous equation that $a_j g_{0,j} - D_- \gamma_{0,j} = 0$ for $j \neq 0$. Moreover, we have $D_+ g_{0,j} = a_j \gamma_{0,j}$ for all $j$ by construction. Now we would like to show that the constant $C$ can be chosen to obtain $A[a] g_{0,0} = 1/\Delta \xi$. We recall (3.6), which shows that we may write

$$\tag{3.33} \Delta \xi \sum_{j=m}^{n} g_j^+ (A[a] g^-)_j - \Delta \xi \sum_{j=m}^{n} (A[a] g^+)_j g_j^+ = W_n (g^-, g^+) - W_{n-1} (g^-, g^+),$$

where

$$W_n (g^-, g^+) := \sum_{j=m}^{n} (\gamma_j^- g_j^+ - \gamma_j^+ g_j^-) ,$$

and

$$W_n (g^-, g^+), W_{n-1} (g^-, g^+) := \sum_{j=m}^{n} (g_j^- g_{j+1}^+ - g_j^+ g_{j+1}^-).$$
where we in the spirit of [28 Eq. (1.21)] have defined a discrete Wronskian
\begin{equation}
W_n(g^-, g^+) := g_{n+1}^- \gamma_n_+ - g_{n+1}^+ \gamma_n^- = g_n^- \gamma_n_+ - g_n^+ \gamma_n^-,
\end{equation}
and the last equality follows from the identity \( g_{n+1}^\pm = g_{n}^\pm + \Delta \xi g_{n}^\mp \). Since
the left-hand side of \((3.33)\) vanishes by definition of \( g^\pm \), we have
\( W_n(g^-, g^+) = W_{n-1}(g^-, g^+) \) for any \( n, m \in \mathbb{Z} \). That is, the Wronskian \( W_n(g^-, g^+) \) is a constant
\( W(g^-, g^+) \) for the constructed sequences \( g^+ \) and \( g^- \). Another way to see this can
be found in Remark \( 3.7 \).

Next, we want to show that the Wronskian is nonzero. Considering
\[
W(g^-, g^+) = W_{-1}(g^-, g^+) = g_0^- \gamma_{-1}^+ - g_0^+ \gamma_{-1}^- = g_0^- \gamma_{-1}^+ + g_0^- (-\gamma_{-1}^-)
\] and the definition \((3.30)\), we use the sign properties stated in Remark \( 3.5 \) to conclude
that the two terms in the final sum are always nonzero and of the same sign,
implicating \( W(g^-, g^+) \neq 0 \). Finally, we will identify the constant \( C \) by considering
the backward difference
\[
D_{j-\gamma_0,0} = C \frac{\gamma_0^- g_0^+ - \gamma_{-1}^- g_0^-}{\Delta \xi} = C \frac{\gamma_0^- g_0^+ - \gamma_{-1}^- g_0^- + \gamma_{-1}^+ g_0^+ - \gamma_{-1}^+ g_0^-}{\Delta \xi} = C g_0^+ a_0 g_0^- - C \frac{W_{-1}(g^-, g^+)}{\Delta \xi} = a_0 g_0,0 - C \frac{W(g^-, g^+)}{\Delta \xi},
\]
which leads to
\[
(A[a]g_0) = a_0 g_{0,0} - D_{-\gamma_0,0} = C \frac{W(g^-, g^+)}{\Delta \xi}.
\]
Consequently, setting \( C^{-1} = W(g^-, g^+) \) in \((3.32)\) gives the desired Green’s function.

Note that there is nothing special about the index \( i = 0 \) where we centered the
Green’s function. We can simply use the sequences \((3.31)\) from before and define
\begin{equation}
g_{i,j} = \frac{1}{W(g^-, g^+)} \begin{cases} g_j^+ g_i^- , & j \geq i, \\ g_j^- g_i^+ , & j < i, \end{cases} \quad \gamma_{i,j} = \frac{1}{W(g^-, g^+)} \begin{cases} \gamma_j^+ g_i^- , & j \geq i, \\ \gamma_j^- g_i^+ , & j < i \end{cases}
\end{equation}
to obtain a Green’s function \( g_{i,j} \) centered at an arbitrary \( i \).

The uniqueness of \( g_{i,j} \) follows from the vectors \( v_0^\pm \) in Lemma \( 3.4 \) being uniquely
defined up to constant factors. Indeed, when constructing the Green’s function in
\((3.35)\) these factors disappear since we are dividing by the Wronskian \( W(g^-, g^+) \),
and so we have no degrees of freedom left in our construction of \( g_{i,j} \), hence it is
unique.

**Remark 3.7.** The constancy of the Wronskian \((3.34)\) can be derived in an alternative
way using only \((3.17)\). Observe that
\[
W_{n-1}(g^-, g^+) = \begin{bmatrix} g_n^+ & \gamma_{n-1}^+ \end{bmatrix} \begin{bmatrix} -\gamma_{n-1}^- \\ g_n^- \end{bmatrix}.
\]
Without loss of generality we may assume \( n \geq k \), and transposing \((3.17)\) we find
\[
\begin{bmatrix} g_n^+ & \gamma_{n-1}^+ \end{bmatrix} = \begin{bmatrix} g_k^+ & \gamma_{k-1}^+ \end{bmatrix} (\Phi_{n,k})^T.
\]
On the other hand, by interchanging rows (3.17) can be written
\[
\begin{bmatrix}
-\gamma_{n-1} \\
g_{n}
\end{bmatrix} = \begin{bmatrix}
1 & -a_{n-1}\Delta\xi \\
-a_{n-1}\Delta\xi & 1 + (a_{n-1}\Delta\xi)^2
\end{bmatrix} \begin{bmatrix}
-\gamma_{n-2} \\
g_{n-1}
\end{bmatrix} = (A_{n-1})^{-1} \begin{bmatrix}
-\gamma_{n-2} \\
g_{n-1}
\end{bmatrix},
\]
which leads to
\[
\begin{bmatrix}
-\gamma_{n-1} \\
g_{n}
\end{bmatrix} = (\Phi_{n,k})^{-\top} \begin{bmatrix}
-\gamma_{k-1} \\
g_{k}
\end{bmatrix}.
\]
It is then clear that
\[
W_{n-1}(g^-, g^+) = [g_k^+ \gamma_{k-1}^+ (\Phi_{n,k})^{-\top} \begin{bmatrix}
-\gamma_{k-1} \\
g_{k}
\end{bmatrix} = W_{k-1}(g^-, g^+),
\]
which is what we claimed.

Note that \(A[a]\) is not the only way to discretize the operator
\[
a(\xi) \text{Id} - \partial \frac{1}{\partial \xi} a(\xi) \partial \frac{1}{\partial \xi}
\]
with first order differences, we may also consider
\[
(B[a]k)_j := a_j k_j - D_+ \left( \frac{D_{-k}}{a_j} \right).
\]
In fact, we will need the Green’s function for this operator as well to close our upcoming system of differential equations. Fortunately, the existence of a such a function follows from the considerations already made in Theorem 3.6.

**Corollary 3.8.** Under the same assumptions on \(a\) as in Theorem 3.6, for any given index \(i\) there exists a unique sequence \(k_i = \{k_{i,j}\}_{j \in \mathbb{Z}}\) such that
\[
\begin{equation}
(B[a]k)_j = \delta_{i,j} \Delta\xi.
\end{equation}
\]

**Proof of Corollary 3.8** Manipulating the homogeneous version of (3.37) we find it to be equivalent to
\[
\frac{D_{-k_{j+1}}}{a_{j+1}} = \Delta\xi a_j k_j + \frac{D_{-k_j}}{a_j}.
\]
Introducing
\[
\kappa_j = \frac{D_{-k_j}}{a_j} = \frac{k_j - k_{j-1}}{a_j \Delta\xi},
\]
the previous equation can be written as
\[
\begin{bmatrix}
\kappa_{j+1} \\
k_j
\end{bmatrix} = \begin{bmatrix}
1 + (a_j \Delta\xi)^2 & a_j \Delta\xi \\
a_j \Delta\xi & 1
\end{bmatrix} \begin{bmatrix}
k_j \\
k_{j-1}
\end{bmatrix} = A_j \begin{bmatrix}
k_j \\
k_{j-1}
\end{bmatrix},
\]
where we recognize the matrix \(A_j\) from (3.17). Going backward we find
\[
\begin{bmatrix}
k_j \\
k_{j-1}
\end{bmatrix} = \begin{bmatrix}
1 & -a_j \Delta\xi \\
-a_j \Delta\xi & 1 + (a_j \Delta\xi)^2
\end{bmatrix} \begin{bmatrix}
\kappa_{j+1} \\
k_j
\end{bmatrix},
\]
or equivalently
\[
\begin{bmatrix}
-\kappa_j \\
k_{j-1}
\end{bmatrix} = \begin{bmatrix}
1 & a_j \Delta\xi \\
a_j \Delta\xi & 1 + (a_j \Delta\xi)^2
\end{bmatrix} \begin{bmatrix}
-\kappa_{j+1} \\
k_j
\end{bmatrix} = B_j \begin{bmatrix}
-\kappa_{j+1} \\
k_j
\end{bmatrix}.
with $B_j$ from (3.28). Hence, we get the solution for free from (3.6). Indeed, choosing

$$
\begin{bmatrix}
\kappa_n^- \\
\kappa_{n-1}^-
\end{bmatrix} = 
\begin{bmatrix}
g_n \\
\gamma_{n-1}
\end{bmatrix},
\begin{bmatrix}
-\kappa_n^+ \\
\kappa_{n-1}^+
\end{bmatrix} = 
\begin{bmatrix}
g_n^+ \\
-\gamma_{n-1}^+
\end{bmatrix}
$$

we know that these sequences have the correct decay at infinity. Defining

$$
\begin{aligned}
k_{i,j} &= \frac{1}{W(g^-,g^+)} \begin{cases} 
\gamma_j^- k_i^+, & j > i, \\
-\gamma_j^- k_i^-, & j \leq i,
\end{cases} \\
\kappa_{i,j} &= \frac{1}{W(g^-,g^+)} \begin{cases} 
\gamma_j^+ k_i^+, & j > i, \\
-\gamma_j^+ k_i^-, & j \leq i,
\end{cases}
\end{aligned}

(3.39)
$$

it follows from (3.16) that $(B[a]k)_j = a_j k_{i,j} - D_j \kappa_{i,j} = 0$ for $j \neq i$. Moreover, by the constancy of (3.34) we find $(B[a]g)_i = 1/\Delta \xi$ in the same way as for $(A[a]g)_i$.

**Remark 3.9.** Note that we may observe directly from (3.35) and (3.39) that $g_{i,j} = g_{j,i}$, $k_{i,j} = k_{j,i}$, and $\kappa_{i,j} = -\gamma_{j,i}$. Moreover, the eigenvalues

$$
\lambda^\pm = \frac{1}{2} \left( 2 + \Delta \xi^2 \mp \Delta \xi \sqrt{4 + \Delta \xi^2} \right)
$$

are exactly those found in (3.13) for the operator $Id - D_- D_+$. In fact, for $a_j \equiv 1$ the sequences $g$ and $k$ coincide since $D_- D_+ = D_+ D_-$, and their explicit expression (3.13) can be recovered from the columns of $\Lambda R^{-1}$ in the diagonalization $A^n = \Lambda R^{-1}$. In particular, this leads to the monotonicity properties

$$
g_{i,j} > 0 \text{ and } k_{i,j} > 0 \text{ for } j \in \mathbb{Z},
$$

where the arrows denote monotone decrease.
Figure 1. Sketch of $g_{i,n}$, $\gamma_{i,n}$, $k_{i,n}$, and $\kappa_{i,n}$ for $\Delta \xi = 0.2$, $i = 0, 4$ and $a_n = a(n\Delta \xi)$ for $a(\xi)$ defined in (3.42). Note the jump of size $-1 + O(\Delta \xi)$ at $n = i$ for both $\gamma$ and $\kappa$.

In Figure 1 we have included a sketch of $g_{i,n}$, $\gamma_{i,n}$, $k_{i,n}$, and $\kappa_{i,n}$ for $\Delta \xi = 0.2$, $i = 0, 4$ and $a_n = a(n\Delta \xi)$ given by

$$(3.42) \quad a(\xi) = \begin{cases} 2, & -1 < \xi \leq 0.5, \\ 0, & 0.5 < \xi \leq 1, \\ 4, & 1 < \xi \leq 1.5 \\ 1, & \text{otherwise} \end{cases}$$

We say sketch, as they have been computed on a finite grid $n \in \{-20, \ldots, 20\}$ with boundary conditions $\gamma_{i,-21} = g_{i,21} = 0$ and $k_{i,-21} = \kappa_{i,21} = 0$, and consequently we find that neither of $g_{i,-20}$, $\gamma_{i,20}$, $\kappa_{i,-20}$ or $k_{i,20}$ are exactly zero. However, the exponential decay makes them very small, in this case of order $O(\Delta \xi^{2.5})$, and the qualitative behavior indicated in Lemma 3.10 is still the same. Note how $a(\xi)$ being zero on the interval $(0.5, 1]$ leads to constant kernel values in that neighborhood, even at the peaks for the kernels centered at $\xi_4 = 0.8$.

Proof of Lemma 3.10. We prove this only for $g$ and $\gamma$ as the proof for $k$ and $\kappa$ is similar. The proof relies on the reasoning in Remark 3.5.

As a first step we want to show that the properties (i) and (ii) hold for $g_{i,i}$, $g_{i,i+1}$, $\gamma_{i,i-1}$, and $\gamma_{i,i}$. To this end, we recall from the proof of Theorem 3.6 that since $g_{i,j}$ and $\gamma_{i,j}$ satisfy (3.40), they must also satisfy

$$\begin{bmatrix} g_{i,j} \\ -\gamma_{i,j-1} \end{bmatrix} = \left( \prod_{k=j}^{i-1} B_k \right) \begin{bmatrix} g_{i,i} \\ -\gamma_{i,i-1} \end{bmatrix}, \quad j \leq i - 1$$

and

$$\begin{bmatrix} g_{i,j} \\ \gamma_{i,j-1} \end{bmatrix} = \left( \prod_{k=i+1}^{j-1} A_k \right) \begin{bmatrix} g_{i,i+1} \\ \gamma_{i,i} \end{bmatrix}, \quad j \geq i + 2,$$

with $A_k$ and $B_k$ as defined in (3.17) and (3.28). By our assumptions, the Green’s functions must tend to zero asymptotically, and we recall from Remark 3.5 that
a necessary condition for this is for the vectors \([g_{i,i}, \gamma_{i,i-1}]^\top\) and \([g_{i,i+1}, \gamma_{i,i}]^\top\) to have entries of opposite sign. Hence, \(g_{i,i}\gamma_{i,i-1} > 0\) and \(g_{i,i+1}\gamma_{i,i} < 0\), where we stress the importance of \(a_j \geq 0\) for this argument to hold. Using only \((3.40)\) we calculate

\[
0 > g_{i,i+1}g_{i,i} - g_{i,i}\gamma_{i,i-1} = \Delta \xi \left( g_{i,i+1} - g_{i,i} \right) \gamma_{i,i} + g_{i,i} \left( \gamma_{i,i} - \gamma_{i,i-1} \right)
\]

\[
= \Delta \xi \left( a_i \gamma_{i,i} \right) + g_{i,i} \left( \gamma_{i,i} - \gamma_{i,i-1} \right)
\]

\[
= \Delta \xi a_i \left( (g_{i,i})^2 + (\gamma_{i,i})^2 \right) - g_{i,i}.
\]

Since \(a_j \geq 0\), it follows that \(g_{i,i} \geq 0\). Recalling that \(g_{i,i}\) must be nonzero according to the sign requirements, we necessarily have \(g_{i,i} > 0\), and then \(\gamma_{i,i-1} > 0\) follows. Moreover, multiplying the identity \(g_{i,i+1} - \Delta \xi a_i \gamma_{i,i} = g_{i,i}\) by \(g_{i,i+1}\) and using \(a_i \geq 0\), \(g_{i,i} > 0\) and \(g_{i,i+1}\gamma_{i,i} < 0\), we must have \(g_{i,i+1} > 0\), which then implies \(\gamma_{i,i} < 0\).

Next we must prove that \([i]\) and \([ii]\) hold for the remaining values of \(j\), and this will be achieved with a contradiction argument. We define the vectors

\[
v_j^+ := \begin{bmatrix} g_{i,j} \\ \gamma_{i,j-1} \end{bmatrix}, \quad v_j^- := \begin{bmatrix} g_{i,j+1} \\ \gamma_{i,j} \end{bmatrix}
\]

such that \(v_{i+1}^+\) and \(v_{i-1}^-\) both have positive first component and negative second component, and satisfy

\[
v_{j+1}^+ := A_j v_j^+ \quad \text{for } j \geq i + 1, \quad v_{j-1}^- := B_j v_j^- \quad \text{for } j \leq i - 1.
\]

If we can prove that they retain the sign property under the above propagation, then we are done. Let us consider

\[
v_{j+1}^+ := A_j v_j^+, \quad j \geq i + 1.
\]

Assume that \(v_j^+\) does not retain the sign property, then there is some \(k \geq i + 1\) which is the first index such that \(v_{k+1}^+\) does not have a positive first component and negative second component. We consider the two possible cases.

The first case is \(v_{k+1}^+ \geq 0\) \((v_{k+1}^- \leq 0)\) considered elementwise. First of all, \(v_{k+1}^+\) cannot be the zero vector as \(A_k\) has full rank, since then \(v_k^+\) would also have to be zero which contradicts \(k + 1\) being the first problematic index. Otherwise, the entrywise inequality \(A_{k+1} \geq I\) leads to \(v_{k+2}^+ = A_{k+1} v_{k+1}^+ \geq v_{k+1}^+\) \((v_{k+2}^- \leq v_{k+1}^-)\), and thus \(\lim_{n \to +\infty} v_n^+ \geq v_{k+1}^+\) \((\lim_{n \to +\infty} v_n^- \leq v_{k+1}^-)\). This is however impossible, as it contradicts the assumed decay of the Green’s functions.

The remaining case is that the entries interchange sign from \(v_k^+\) to \(v_{k+1}^+\). However, then we would have

\[
v_k^+ = (A_k)^{-1} v_{k+1}^+ = \left[ \begin{array}{cc} 1 & -a_k \Delta \xi \\ -a_k \Delta \xi & 1 + (a_k \Delta \xi)^2 \end{array} \right] v_{k+1}^+,
\]

and since \(a_k \geq 0\), then \(v_k^+\) would also have negative first component and positive second component, contradicting \(k + 1\) being the first problematic index. Hence,
$v_j^+$ always has positive first component and negative second component for $j > i$, thus for $j \geq i$ it follows that $g_{i,j}$ is always positive, while $\gamma_{i,j}$ is always negative which shows that $g_{i,j}$ is decreasing in this direction.

A similar argument holds in the other direction when considering $v_j^−$ and $B_j$. Then $-\gamma_{i,j}$ is always negative for $j < i$, which means that $g_{i,j}$ is increasing with $j$ for these indices. Thus, [i] and [ii] hold for $\{g_{i,j}\}_{j \in \mathbb{Z}}$ and $\{\gamma_{i,j}\}_{j \in \mathbb{Z}}$. □

4. Analysis of the semi-discrete system

We now return to the initial value problem for the full line (1.12). We follow earlier works on the continuous equation and highlight the properties we want to retain in our semi-discrete system.

4.1. Reformulation of the continuous problem using operator propagation. The 2CH system can be written as

\begin{align*}
  u_t + uu_x + P_x &= 0, \\
  \rho_t + (u\rho)_x &= 0
\end{align*}

for $P$ implicitly defined as

\begin{equation}
  P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\bar{\rho}^2.
\end{equation}

Let us introduce $\bar{\rho} := \rho - \rho_\infty \in \mathbb{L}^2$ to ease notation. Note that most expressions simplify when we consider $\rho_\infty = 0$. We have chosen here to cover the case where $\rho_\infty$ is arbitrary to be able to include the initial condition $\rho(0,x) = \varepsilon$, for any $\varepsilon > 0$, which leads to solutions without blow-up, see [18]. In the case of the 2CH system, the conservation law for the energy is given by

\begin{align*}
  \left(\frac{1}{2}(u^2 + u_x^2 + \bar{\rho}^2)\right)_t + (u\frac{1}{2}(u^2 + u_x^2 + \bar{\rho}^2))_x + (uR)_x &= 0,
\end{align*}

with

\begin{equation}
  R = P - \frac{1}{2}u^2 - \frac{1}{2}\rho_\infty^2.
\end{equation}

We can check that, for the solution $(R, Q)$ to

\begin{equation}
  \begin{bmatrix}
    -\partial_x & 1 \\
    1 & -\partial_x
  \end{bmatrix} \circ \begin{bmatrix} R \\ Q \end{bmatrix} = \begin{bmatrix} u u_x \\ \frac{1}{2}(u^2 + u_x^2 + \bar{\rho}^2) + \rho_\infty \bar{\rho} \end{bmatrix},
\end{equation}

after defining $P$ through (4.3), we obtain (4.1). Hence,

\begin{align*}
  (4.5a) & \quad u_t + uu_x + Q = 0, \\
  (4.5b) & \quad \rho_t + (u\rho)_x = 0
\end{align*}

with (4.4) is yet another form of the 2CH system.

We introduce as before the Lagrangian variables $y(t, \xi)$ and $U(t, \xi)$, which denotes the Lagrangian velocity. Moreover, we define the Lagrangian density $r(t, \xi) :=
\[\rho(t, y(t, \xi)) y_\xi(t, \xi) \text{ and the cumulative energy } H \text{ given by}
\]
\[
H(t, \xi) = \frac{1}{2} \int_{-\infty}^{\xi} (u^2 + u_x^2 + \bar{\rho}^2)(t, x) \, dx
\]
\[
= \frac{1}{2} \int_{-\infty}^{\xi} ((u^2 + u_x^2 + \bar{\rho}^2) \circ y) y_\xi(t, \eta) \, d\eta,
\]

as well as the Lagrangian variables \(\bar{Q} = Q \circ y\) and \(\bar{R} = R \circ y\). From (4.5), we get
\[(4.7) \quad U_t = -\bar{Q} \quad \text{and} \quad r_t = 0,
\]
while the equation of conservation of energy (4.2) yields
\[
H_t = U R.
\]

Finally, we rewrite also the system (4.4) in terms of the Lagrangian variables. To simplify the notation, we replace \(\bar{Q}\) by \(Q\), and similarly for \(\bar{R}\). The equivalent system in Lagrangian variable is thus given by
\[(4.8a) \quad y_t = U,
\]
\[(4.8b) \quad U_t = -Q,
\]
\[(4.8c) \quad H_t = -UR,
\]
\[(4.8d) \quad r_t = 0,
\]
with
\[(4.9) \quad \begin{bmatrix} -\partial_\xi & y_\xi \\ y_\xi & -\partial_\xi \end{bmatrix} \circ \begin{bmatrix} R \\ Q \end{bmatrix} = \begin{bmatrix} U U_\xi \\ H_\xi + \rho_\infty (r - \rho_\infty y_\xi) \end{bmatrix}.
\]

In the expression above, we use the same notation for the variable \(y_\xi\) and the point-wise multiplication operator by \(y_\xi\). We will use this convention for the rest of the paper. The equivalence between (4.4) and (4.9) holds only assuming the that \(y_\xi \geq 0\) and all the functions are smooth enough to do the manipulation.

Let us define the operators \(G\) and \(K\) as the fundamental solutions to the operator in (4.9), meaning that they satisfy
\[(4.10) \quad \begin{bmatrix} -\partial_\xi & y_\xi \\ y_\xi & -\partial_\xi \end{bmatrix} \circ \begin{bmatrix} K \\ G \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \delta \end{bmatrix}.
\]

As we mentioned in the introduction, the operators \(K\) and \(G\) can be computed explicitly using the fundamental solution of the Helmholtz operators in Eulerian coordinates. If we define
\[(4.11a) \quad g(\eta, \xi) = \frac{1}{2} e^{-|y(\xi) - y(\eta)|}
\]
and
\[(4.11b) \quad \kappa(\eta, \xi) = -\frac{1}{2} \text{sgn}(\xi - \eta) e^{-|y(\xi) - y(\eta)|},
\]
then we can check that the operators defined as \(G(f) = \int_{\mathbb{R}} g(\eta, \xi) f(\eta) \, d\eta\) and \(K(f) = \int_{\mathbb{R}} \kappa(\eta, \xi) f(\eta) \, d\eta\) are solutions to (4.10), again assuming \(y\) is monotone. It means that we can obtain explicit expressions for \(R\) and \(Q\) given by
\[(4.12a) \quad R = \int_{\mathbb{R}} \kappa(\eta, \xi) U(\eta) U_\xi(\eta) \, d\eta + \int_{\mathbb{R}} g(\eta, \xi)(H_\xi(\eta) + \rho_\infty (r(\eta) - \rho_\infty y_\xi(\eta))) \, d\eta
\]
Given a function
\[ \partial (4.16) \]
Then, we can rewrite the last equality as
\[ (4.12b) \quad Q = \int_{\mathbb{R}} g(\eta, \xi) U(\eta) U_\xi(\eta) \, d\eta + \int_{\mathbb{R}} \kappa(\eta, \xi)(H_\xi(\eta) + \rho_\infty(r(\eta) - \rho_\infty y_\xi(\eta))) \, d\eta \]

We need to decompose the variables \( y \) and \( r \) to give them the right decay which enables us to define them in a proper functional setting. We define \( \zeta \) and \( \bar{r} \) as
\[ y(t, \xi) = \zeta(t, \xi) + \xi \quad \text{and} \quad r(t, \xi) = \bar{r}(t, \xi) + \rho_\infty y_\xi(t, \xi). \]
The Banach space which contains \( \zeta \) and \( H \) is the subspace of bounded and continuous functions with derivative in \( L^2 \),
\[ (4.13) \quad \mathbf{V} := \{ f \in C_b(\mathbb{R}) \mid f_\xi \in L^2(\mathbb{R}) \}, \]
endowed with the norm \( \| f \|_\mathbf{V} := \| f \|_{L^\infty} + \| f_\xi \|_{L^2} \). Then we let
\[ (4.14) \quad \mathbf{E} := \mathbf{V} \times H^1 \times \mathbf{V} \times L^2 \]
be a Banach space tailored for the tuple \( X = (\zeta, U, H, \bar{r}) \) with norm
\[ (4.15) \quad \| X \|_\mathbf{E} := \| \zeta \|_\mathbf{V} + \| U \|_{H^1} + \| H \|_\mathbf{V} + \| \bar{r} \|_{L^2}. \]

In \cite{23 21}, the authors prove that the right-hand side of their respective versions of \( (4.8) \) is locally Lipschitz, and consecutive contraction arguments yields the existence of a unique short-time solution. In the same manner, we would like to prove that there exists a unique short-time solution for our semi-discrete system, but the explicit forms for \( R \) and \( Q \) in \( (4.9) \) are not available in the discrete setting. As a remedy, we propagate the kernel operators corresponding to \( \mathcal{K} \) and \( \mathcal{G} \) by incorporating them in the governing equations. Given the evolution of \( y \), that is \( y_t = U \), we can derive a differential equation for the evolution of \( \mathcal{G} \) and \( \mathcal{K} \). Let us see how this can be done in the continuous case before dealing with the discrete case. Formally we have
\[ \frac{\partial}{\partial t} \mathcal{G}(f) = \frac{1}{2} \int_{\mathbb{R}} \frac{\partial}{\partial t} e^{-|y(t, \xi) - y(t, \eta)|} f(\eta) \, d\eta = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(y(t, \xi) - y(t, \eta))(y_t(t, \xi) - y_t(t, \eta)) e^{-|y(t, \xi) - y(t, \eta)|} f(\eta) \, d\eta \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(y(t, \xi) - y(t, \eta))(U(t, \xi) - U(t, \eta)) e^{-|y(t, \xi) - y(t, \eta)|} f(\eta) \, d\eta. \]

Here again, we assume that we know a priori that \( y \) remains a monotone function. Then, we can rewrite the last equality as
\[ (4.16) \quad \frac{\partial}{\partial t} \mathcal{G}(f) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(\xi - \eta)(U(t, \xi) - U(t, \eta)) e^{-|y(t, \xi) - y(t, \eta)|} f(\eta) \, d\eta. \]

Given a function \( U \), we can associate the point-wise multiplication operator, which we denote \( \mathcal{U} \), defined as \( \mathcal{U}(f)(\xi) = U(\xi) f(\xi) \) for any function \( f \) and any point \( \xi \). The kernel of the operator would be singular and equal to \( U(\xi) \delta(\xi - \eta) \). Using these notations, we can rewrite \( (4.16) \) as
\[ \frac{\partial}{\partial t} \mathcal{G}(f) = (\mathcal{U} \circ \mathcal{K})(f) - (\mathcal{K} \circ \mathcal{U})(f), \]
or
\[ (4.17) \quad \frac{\partial}{\partial t} \mathcal{G} = [\mathcal{U}, \mathcal{K}]. \]
Similarly, we show that formally we have

\[
\frac{\partial}{\partial t} K = [U, G].
\]

We can use directly the evolution equation we have just derived to define a new equivalent system, which does not assume a priori the particular form of the kernels \( g \) and \( k \), as defined in (4.11). Those identities will nevertheless hold, but only because they are propagated by the system, as we are going to see. An equivalent system of equations for the 2CH system is then given by

\[
\begin{align*}
y_t &= U, \\
U_t &= -Q, \\
H_t &= UR, \\
r_t &= 0, \\
\frac{\partial}{\partial t} G &= [U, K], \\
\frac{\partial}{\partial t} K &= [U, G],
\end{align*}
\]

with \( R \) and \( Q \) given as

\[
\frac{\partial}{\partial t} \begin{bmatrix} R \\ Q \end{bmatrix} = \begin{bmatrix} K & G \\ G & K \end{bmatrix} \circ \begin{bmatrix} U \xi \\ H_\xi + \rho_\infty (r - \rho_\infty y_\xi) \end{bmatrix}
\]

For well-chosen initial conditions, the new system (4.19) and (4.20) gives rise to the same solutions as the one given by (4.8), (4.11) and (4.12). Indeed, if we choose the initial data for the kernels of the operators \( G \) and \( K \) as

\[
g[0](\eta, \xi) = \frac{1}{2} e^{-|y(0, \xi) - y(0, \eta)|}
\]

and

\[
\kappa[0](\eta, \xi) = -\frac{1}{2} \text{sgn}(\xi - \eta) e^{-|y(0, \xi) - y(0, \eta)|},
\]

without changing the initial data for \( y, U \) and \( H \), we claim that the solution to (4.19) and (4.20) provides a solution to (4.8), (4.11) and (4.12). We consider a solution \((y, U, H)\) to (4.8), (4.11) and (4.12) and the system of differential equations

\[
\frac{\partial}{\partial t} \tilde{G} = [U, \tilde{K}], \quad \frac{\partial}{\partial t} \tilde{K} = [U, \tilde{G}],
\]

for the unknown operators \( \tilde{G} \) and \( \tilde{K} \). Since the system (4.21) is linear and bounded, it admits a unique solution. By bounded, we mean that \( \| [U, \tilde{G}] \| \leq C \| \tilde{G} \| \) for some constant \( C \). The computation which led to (4.16) shows that the kernels \( \tilde{g}[t](\xi, \eta) = \frac{1}{2} e^{-|y(t, \xi) - y(t, \eta)|} \) and \( \tilde{\kappa}[t](\xi, \eta) = -\frac{1}{2} \text{sgn}(\xi - \eta) e^{-|y(t, \xi) - y(t, \eta)|} \) are solutions to (4.21). Hence, \((y, U, h, \tilde{G}, \tilde{K})\) is a solution to (4.19) and (4.20) and this proves our claim.

We note that the evolution equation for \( G \) and \( K \) can be obtained directly from the product identity (4.10). Indeed, after differentiation with respect to time, we get

\[
\begin{bmatrix} 0 & U_\xi \\ U_\xi & 0 \end{bmatrix} \circ \begin{bmatrix} K & G \\ G & K \end{bmatrix} + \begin{bmatrix} -\partial_\xi & y_\xi \\ y_\xi & -\partial_\xi \end{bmatrix} \circ \begin{bmatrix} \tilde{K} & \tilde{G} \\ \tilde{G} & \tilde{K} \end{bmatrix} = 0,
\]
(4.22) \[
\begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix} = - \begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix} \circ \begin{bmatrix}
0 & U_\xi \\
U_\xi & 0
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix}.
\]

This expression, which corresponds to the general form of an inverse for a matrix, namely \(\frac{dM^{-1}}{dt} = -M^{-1}\frac{dM}{dt}M^{-1}\), can be simplified to (4.21) using integration by parts. It follows then by the construction of (4.22) that the identity (4.10) is preserved by the evolution equation.

We remark that the operator \(\mathcal{K}\) is antisymmetric and \(\mathcal{G}\) is symmetric. Here, we want to show that this is a consequence of the special form of the operator we are inverting, and not due to the form of the kernels given in (4.11). First, we note the block structure of the matrices in (4.10). This block structure is preserved by composition and (4.10) is equivalent to

\[
\begin{bmatrix}
-\partial_\xi, & y_\xi \\
y_\xi, & -\partial_\xi
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix} = \delta, \quad \begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi, & y_\xi \\
y_\xi, & -\partial_\xi
\end{bmatrix} = 0.
\]

We apply the transpose

\[
\begin{bmatrix}
\mathcal{K}^T & \mathcal{G}^T
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi^T \\
y_\xi^T
\end{bmatrix} = \delta, \quad \begin{bmatrix}
\mathcal{K}^T & \mathcal{G}^T
\end{bmatrix} \circ \begin{bmatrix}
-y_\xi^T \\
-\partial_\xi^T
\end{bmatrix} = 0.
\]

The point-wise multiplication operator by \(y_\xi\) is symmetric and \(-\partial_\xi\) is antisymmetric. Hence, the two identities above can be rewritten as

(4.23) \[
\begin{bmatrix}
-\mathcal{K}^T, & \mathcal{G}^T
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi \\
y_\xi
\end{bmatrix} = \delta, \quad \begin{bmatrix}
\mathcal{K}^T, & -\mathcal{G}^T
\end{bmatrix} \circ \begin{bmatrix}
y_\xi \\
-\partial_\xi
\end{bmatrix} = 0.
\]

We use the fact that an operator commutes with its inverse so that we have

\[
\begin{bmatrix}
\mathcal{K} & \mathcal{G} \\
\mathcal{G} & \mathcal{K}
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi & y_\xi \\
y_\xi & -\partial_\xi
\end{bmatrix} = \begin{bmatrix}
\delta & 0 \\
0 & \delta
\end{bmatrix},
\]

which can be rewritten as

(4.24) \[
\begin{bmatrix}
\mathcal{K}, & \mathcal{G}
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi \\
y_\xi
\end{bmatrix} = \delta, \quad \begin{bmatrix}
\mathcal{K}, & \mathcal{G}
\end{bmatrix} \circ \begin{bmatrix}
y_\xi \\
-\partial_\xi
\end{bmatrix} = 0.
\]

We combine (4.23) and (4.24) to obtain

\[
\begin{bmatrix}
\mathcal{K} + \mathcal{K}^T & \mathcal{G} - \mathcal{G}^T \\
\mathcal{G} - \mathcal{G}^T & \mathcal{K} + \mathcal{K}^T
\end{bmatrix} \circ \begin{bmatrix}
-\partial_\xi & y_\xi \\
y_\xi & -\partial_\xi
\end{bmatrix} = 0,
\]

which implies \(\mathcal{K}^T = -\mathcal{K}\) and \(\mathcal{G}^T = \mathcal{G}\), because the matrix operator on the right is invertible.

In (4.11a) and (4.11b) we have in fact explicit formulas for the solutions of (4.10). From those we find \(|g(\eta, \xi)| = |\kappa(\eta, \xi)|\) almost everywhere and that these functions are bounded above by \(\frac{1}{2}\). However, we do not need these explicit formulas to deduce these estimates. In fact, it suffices to know that they solve the system (4.10) and satisfy

\[
y_\xi(\xi) \geq 0, \ g(\eta, \xi) \geq 0, \ \text{sgn}(\kappa(\eta, \xi)) = \text{sgn}(\eta - \xi).
\]

The proof of this claim essentially follows a common proof of the Sobolev inequality

(4.25) \[
\|f\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|f\|_{H^1},
\]
for $f \in H^1$. This observation will be useful for us in the discrete case, since there, we do not have explicit formulas for the kernels. We start by observing

$$
(g(\eta, \xi))^2 = \frac{1}{2} \left[ \int_{-\infty}^{\xi} (g(\eta, s))^2 \, ds - \int_{\xi}^{+\infty} (g(\eta, s))^2 \, ds \right]
$$

$$
= \int_{-\infty}^{\xi} g(\eta, s)g_s(\eta, s) \, ds - \int_{\xi}^{+\infty} g(\eta, s)g_s(\eta, s) \, ds,
$$

Then, we have

$$
(g(\eta, \eta))^2 = \int_{-\infty}^{\eta} g(\eta, s)g_s(\eta, s) \, ds - \int_{\eta}^{+\infty} g(\eta, s)g_s(\eta, s) \, ds
$$

$$
= \int_{-\infty}^{\eta} y(\eta, s)g(\eta, s)\kappa(\eta, s) \, ds - \int_{\eta}^{+\infty} y(\eta, s)g(\eta, s)\kappa(\eta, s) \, ds,
$$

where we use $\partial_s g = y\kappa$ from (4.10). It follows that

$$
(g(\eta, \eta))^2 = \int_{-\infty}^{+\infty} y(\eta, s)g(\eta, s)\kappa(\eta, s) \, ds
$$

$$
\leq \frac{1}{2} \int_{-\infty}^{+\infty} y_s(s) [g(\eta, s)^2 + \kappa(\eta, s)^2] \, ds
$$

$$
= \frac{1}{2} \int_{-\infty}^{+\infty} g(\eta, s) [y(\eta, s)g(\eta, s) - (\kappa(\eta, s))_s] \, ds.
$$

Hence, by $y\kappa = \partial_s g$ from (4.10), we find

$$
(4.26) \quad (g(\eta, \eta))^2 \leq \frac{1}{2} \int_{-\infty}^{+\infty} g(\eta, s)\delta(\eta - s) \, ds = \frac{1}{2} g(\eta, \eta),
$$

and the result $g(\eta, \xi) \leq \frac{1}{2}$ follows. Next we find that for $\xi < \eta$ we can write

$$
(\kappa(\eta, \xi))^2 = 2 \int_{-\infty}^{\xi} \kappa(\eta, s)k_s(\eta, s) \, ds
$$

$$
= 2 \int_{-\infty}^{\xi} \kappa(\eta, s)y(\eta, s)g(\eta, s) \, ds
$$

$$
= 2 \int_{-\infty}^{\xi} g_s(\eta, s)g(\eta, s) \, ds
$$

$$
= (g(\eta, \xi))^2,
$$

and consequently $\kappa(\eta, \xi) = g(\eta, \xi)$ for $\xi < \eta$, where we have used the sign properties of $g$ and $\kappa$. In a similar way we find $\kappa(\eta, \xi) = -g(\eta, \xi)$ for $\xi > \eta$, obtaining

$$
\lim_{\xi \to \eta^\pm} \kappa(\eta, \xi) = \pm g(\eta, \eta)
$$

entirely without knowing the explicit formulas for $g$ and $\kappa$. Thus, we have

$$
\sup \limits_\xi |\kappa(\eta, \xi)| = \lim_{\xi \to \eta^\pm} |\kappa(\eta, \xi)| \leq \frac{1}{2}.
$$

In fact, we find that we must have equality, as the above limits show

$$
\lim_{\xi \to \eta^+} \kappa(\eta, \xi) - \lim_{\xi \to \eta^-} \kappa(\eta, \xi) \geq 1,
$$
but the jump condition for $\kappa$ which is required at $\eta = \xi$ to obtain a delta implies that the difference above is exactly one, which can only happen if
\[
\lim_{\xi \to \eta \mp} \pm \kappa(\eta, \xi) = g(\eta, \eta) = \frac{1}{2}.
\]

4.2. Equivalent formulation of the semi-discrete system. Turning back to the formal expression (2.17), we use the the Green’s functions from Theorem 3.6 and Corollary 3.8 to write out the right-hand side explicitly. Considering (3.40) where we now have $a_j = D_+ y_j$, we observe that they correspond to the discrete versions of (4.10). Indeed, we have the following identity
\[
\left[-D_j - \begin{pmatrix} D_+ y_j \\ D_+ y_j \end{pmatrix} \right] \circ \begin{bmatrix} \gamma_{i,j} & k_{i,j} \\ g_{i,j} & \kappa_{i,j} \end{bmatrix} = \frac{1}{\Delta \xi} \begin{bmatrix} \delta_{i,j} & 0 \\ 0 & \delta_{i,j} \end{bmatrix}
\]
which has to be compared with (4.10) in the continuous case. Thus, (2.17) can be rewritten as
\[
\dot{U}_j = -\Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} \left( U_i(D_+ U_i) + D_+ \left( \frac{h_i}{D_+ y_i} + \rho_\infty \bar{r}_i \right) \right),
\]
where we have defined
\[
\bar{r}_i := \rho_{0,i} - \rho_\infty(D_+ y_{i})
\]
and
\[
h_i := \frac{1}{2} (U_i)^2(D_+ y_i) + \frac{1}{2} (D_+ U_i)^2 + \frac{1}{2} \bar{r}_i^2
\]
From the expressions in (4.28) and (4.30), it seems that, if $D_+ y_i$ goes to zero for some index $i$ and time $t$, then $U_j$ and $h_i$ blow up. It turns out that these quantities remain bounded, which allows to have a global in time solution. This corresponds to the blow-up of the Camassa-Holm equation, as illustrated by the peakon-antipeakon collision, see for example [24, 19]. To obtain a well-defined system, we need to remove explicit dependence in the system with respect to $1/D_+ y_i$. To do so, we will, among other things, derive an explicit evolution for $h_i$.

With the kernels $g, k, \gamma, \kappa$, we are able to express $A[D_+ y]^{-1}$ in (2.17) to obtain (4.28), but since we do not know their explicit form as functions of $D_+ y_i$, it is not possible to obtain explicit forms for $g, k, \gamma, \kappa$ as we have for $G$ and $K$ in (4.11). Instead we derive a system analogous to (4.19) by introducing the kernels as variables in our system, and for that we need their time evolution. We repeat the procedure from the continuous case to obtain (4.19) and (4.18). By differentiating (4.27) and using the fact that $\hat{y}_i = U_i$, we get
\[
\begin{bmatrix} \dot{\gamma} \\ \dot{k} \\ \dot{g} \\ \dot{\kappa} \end{bmatrix} = -\begin{bmatrix} \gamma \\ k \\ g \\ \kappa \end{bmatrix} * \begin{bmatrix} 0 & D_+ U & \gamma \\ D_+ U & 0 & k \\ \gamma & k & g \\ g & k & \kappa \end{bmatrix}
\]
which in explicit form yields
\[
\dot{\gamma}_{i,j} = -\kappa_{m,j} * ((D_+ U_m) \gamma_{i,m}) - g_{m,j} * ((D_+ U_m) g_{i,m}),
\]
\[
\dot{k}_{i,j} = -\kappa_{m,j} * ((D_+ U_m) \kappa_{i,m}) - \gamma_{m,j} * ((D_+ U_m) \gamma_{i,m}),
\]
\[ k_{i,j} = -k_{m,j} \ast ((D_+ U_m)k_{i,m}) - \gamma_{m,j} \ast ((D_+ U_m)\kappa_{i,m}), \]
\[ \dot{k}_{i,j} = -k_{m,j} \ast ((D_+ U_m)k_{i,m}) - g_{m,j} \ast ((D_+ U_m)\kappa_{i,m}). \]

We use the time evolution of the operators to establish the following result. In particular, the index-wise multiplication operator by \( D_+ y_j \) is symmetric, while we have \( D_- = -D_+ \). It can be derived, in a similar way as in the continuous case, that the operators \( g \) and \( k \) are symmetric while we have \( \kappa^+ = -\gamma \). We prove this result directly in the following Lemma.

**Lemma 4.1.** Let \( T > 0 \), and assume that, for \( t \in [0,T] \), \((D_+ y_j(t))_t = D_+ U_j(t)\) for \( j \in \mathbb{Z} \), and that \( g, k, \gamma, \kappa \) and \( D_+ U \) are bounded in \( \mathcal{E}^2 \)-norm in \([0,T] \). Then, for \( t \in [0,T] \) the sequences \( g_{i,j}(t), k_{i,j}(t), \gamma_{i,j}(t), \kappa_{i,j}(t) \) satisfy the following identities:

(i) The Green’s function identities ([4.27]).
(ii) The symmetry identities
\[ g_{j,i} = g_{i,j} \quad \text{and} \quad k_{j,i} = k_{i,j}, \]
and the antisymmetry identity
\[ \gamma_{j,i} = -\kappa_{i,j}. \]

**Proof of Lemma 4.1.** Recall from Remark 3.9 that these identities are satisfied for \( t = 0 \), by construction. The rest of the proof then relies on Grönwall’s inequality. (i): We introduce the four operators \( z_l \) for \( l = 1, \ldots, 4 \) defined as
\[ z_{1,i,j} = (D_+ y_j)g_{i,j} - D_j - \gamma_{i,j} - \frac{\delta_{i,j}}{\Delta t}, \]
\[ z_{2,i,j} = (D_+ y_j)k_{i,j} - D_j + \kappa_{i,j} - \frac{\delta_{i,j}}{\Delta t}, \]
\[ z_{3,i,j} = (D_+ y_j)\gamma_{i,j} - D_j + g_{i,j}, \]
\[ z_{4,i,j} = (D_+ y_j)\kappa_{i,j} - D_j - k_{i,j}. \]

Using \((D_+ y_j(t))_t = D_+ U_j(t)\) and ([4.31]) we find that
\[ (z_{1,i,j})_t = (D_+ y_j)g_{i,j} + (D_+ y_j)\dot{g}_{i,j} - D_j - \dot{\gamma}_{i,j} \]
\[ = (D_+ U_j)g_{i,j} - (D_+ y_j)\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)(g_{i,m}g_{m,j} + \gamma_{i,m}\kappa_{m,j}) \]
\[ + D_j - \Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)(g_{i,m}\gamma_{m,j} + \gamma_{i,m}k_{m,j}) \]
\[ = (D_+ U_j)g_{i,j} - \Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)g_{i,m}((D_+ y_j)g_{m,j} - D_j - \gamma_{m,j}) \]
\[ - \Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)\gamma_{i,m}((D_+ y_j)\kappa_{m,j} - D_j - k_{m,j}) \]
\[ = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)(g_{i,m}z_{1,m,j} + \gamma_{i,m}z_{4,m,j}). \]

Similarly, one shows that
\[ (z_{2,i,j})_t = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)(k_{i,m}z_{2,m,j} + \kappa_{i,m}z_{3,m,j}), \]
\[ (z_{3,i,j})_t = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m)(g_{i,m}z_{3,m,j} + \gamma_{i,m}z_{2,m,j}). \]
and

\[ (z_{4,i,j})_t = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_m U_m) (k_{i,m} z_{4,m,j} + \kappa_{i,m} z_{1,m,j}). \]

Integrating the first of these, taking absolute values, applying Hölder’s inequality and taking supremum over \( i \) we obtain

\[
\sup_i |z_{1,i,j}(t)| \leq \sup_i |z_{1,i,j}(0)| + \int_0^t \left( \|D_m U_m\|_\ell^2 \left( \|g(s)\|_\ell^2 \sup_m |z_{1,m,j}(s)| + \|\gamma(s)\|_\ell^2 \sup_m |z_{4,m,j}(s)| \right) \right) ds
\]

Treating the three other relations similarly and defining

\[ Z(t) = \sum_{l=1}^4 \|z(t)\|_\ell^\infty, \]

we may add the four inequalities to obtain an inequality of the form

\[ Z(t) \leq Z(0) + \int_0^t C(s) Z(s) ds, \]

where

\[ C(s) = 2 \|D_m U_m\|_\ell^2 \left( \|g\|_\ell^2 + \|k\|_\ell^2 + \|\gamma\|_\ell^2 \right) (s) \]

is bounded by assumption. Since \( Z(0) = 0 \), Grönwall’s inequality yields \( Z(t) = 0 \) for \( t \in [0,T] \), which proves the result.

(ii): We prove the symmetry of \( g \). Since, from (4.27), we have \( (D_m y_m) g_{i,m} - D_{m-\gamma_{i,m}} = \frac{\Delta \xi}{\Delta^2} \delta_{i,m} \), we have

\[ g_{j,i} = \Delta \xi \sum_{m \in \mathbb{Z}} [(D_m y_m) g_{i,m} - D_{m-\gamma_{i,m}}] g_{j,m} \]

\[ = \Delta \xi \sum_{m \in \mathbb{Z}} [(D_m y_m) g_{i,m} g_{j,m} + \gamma_{i,m} D_m g_{j,m}]. \]

Then, we use twice the identity \( D_m g_{j,m} = (D_m y_m) \gamma_{j,m} \), also from (4.27), first for \( j \) and then for \( i \), to obtain

\[ g_{j,i} = \Delta \xi \sum_{m \in \mathbb{Z}} [(D_m y_m) g_{i,m} g_{j,m} + \gamma_{i,m} (D_m y_m) \gamma_{j,m}] \]

\[ = \Delta \xi \sum_{m \in \mathbb{Z}} [(D_m y_m) g_{i,m} g_{j,m} + (D_m g_{i,m}) \gamma_{j,m}]. \]

After summation by parts, we end up with

\[ g_{j,i} = \Delta \xi \sum_{m \in \mathbb{Z}} g_{i,m} [(D_m y_m) g_{j,m} + D_m \gamma_{j,m}] = g_{i,j}, \]

because \( (D_m y_m) g_{j,m} - D_{m-\gamma_{j,m}} = \delta_{i,m} \) and the symmetry of \( g \) is proved. A similar procedure shows the symmetry of \( k \). For the antisymmetry we also use (4.27) to compute

\[ \gamma_{j,i} = \Delta \xi \sum_{m \in \mathbb{Z}} [(D_m y_m) k_{i,m} - D_m \kappa_{i,m}] \gamma_{j,m} \]

\[ = \Delta \xi \sum_{m \in \mathbb{Z}} [k_{i,m} D_m + g_{j,m} + \kappa_{i,m} D_m - \gamma_{j,m}] \]
\[ \begin{align*}
&= -\Delta \xi \sum_{m \in \mathbb{Z}} [(D_m - k_{i,m})g_{j,m} - \kappa_{i,m}D_m - \gamma_{j,m}] \\
&= -\Delta \xi \sum_{m \in \mathbb{Z}} \kappa_{i,m} [(D_m + y_m)g_{j,m} - D_m - \gamma_{j,m}] \\
&= -\kappa_{i,j}.
\end{align*} \]

Note that with the variables (4.29) and (4.30), the discrete Hamiltonian (2.14) can be written

\[ H_{\text{dis}} = \Delta \xi \sum_{i \in \mathbb{Z}} h_i. \]

Using summation by parts we will show that (4.28) can be seen as a direct discretization of (4.12b). We have for the second term in the parenthesis of (4.28)

\[ \begin{align*}
&= -\Delta \xi \sum_{i \in \mathbb{Z}} \gamma_{j,i} (h_i + \rho_\infty \bar{r}_i) \\
&= -\Delta \xi \sum_{i \in \mathbb{Z}} \kappa_{i,j} (h_i + \rho_\infty \bar{r}_i),
\end{align*} \]

where we have used (4.27) and (4.34). We define

(4.35a) \( (Q_1)_j := \Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} U_i (D_i U_i), \)

(4.35b) \( (Q_2)_j := \Delta \xi \sum_{i \in \mathbb{Z}} \kappa_{i,j} (h_i + \rho_\infty \bar{r}_i), \)

(4.35c) \( Q_j := (Q_1)_j + (Q_2)_j. \)

Then, the evolution of \( U \) is given by

(4.36) \( \dot{U}_j = -Q_j \)

The form of (4.35a) and (4.35b) motivates the definitions

(4.37a) \( (R_1)_j := \Delta \xi \sum_{i \in \mathbb{Z}} \gamma_{i,j} U_i (D_i U_i), \)

(4.37b) \( (R_2)_j := \Delta \xi \sum_{i \in \mathbb{Z}} \kappa_{i,j} (h_i + \rho_\infty \bar{r}_i), \)

(4.37c) \( R_j := (R_1)_j + (R_2)_j. \)

We then have

\[ \begin{bmatrix}
(R_1)_j \\
(Q_1)_j
\end{bmatrix}
\begin{bmatrix}
(R_2)_j \\
(Q_2)_j
\end{bmatrix}
= \begin{bmatrix}
\gamma & k \\
g & \kappa
\end{bmatrix}
\begin{bmatrix}
U_j (D_i U_i) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
h_j + \rho_\infty \bar{r}_i
\end{bmatrix}, \]

and combining these we obtain

\[ \begin{bmatrix}
R \\
Q
\end{bmatrix}
= \begin{bmatrix}
\gamma & k \\
g & \kappa
\end{bmatrix}
\begin{bmatrix}
U_j (D_i U) \\
h + \rho_\infty \bar{r}_i
\end{bmatrix}. \]
meaning $R$ and $Q$ satisfies
\begin{equation}
(4.38) \begin{bmatrix}
-D_j - (D_+ y_j) \\
(D_+ y_j) - D_{j+}
\end{bmatrix}
\begin{bmatrix}
R_j \\
Q_j
\end{bmatrix} = \begin{bmatrix}
U_j (D_+ U_j) \\
h_j + \rho \infty \bar{r}_j
\end{bmatrix}.
\end{equation}
We recognize the discrete version of (4.20).

The relation \( \dot{U}_j = -Q_j \) shows that we have a differential equation for $U$ in the variables $y$, $U$, $H$, $\bar{r}$, $g$, and $\kappa$. From (4.29) we obtain
\begin{equation}
(4.39) \dot{\bar{r}}_j = \bar{r}_j - \rho \infty D_+ \bar{y}_j = -\rho \infty D_+ U_j.
\end{equation}
Then we need a discrete cumulative energy $H_j$ with a corresponding evolution equation. We choose to set $h_j = D_+ H_j$ such that
\begin{equation}
(4.40) H_j = \Delta \xi \sum_{i=-\infty}^{j-1} h_i,
\end{equation}
where the summand is defined in (4.30).

Next, we multiply (4.30) by $D_+ y_i$ to simplify and then differentiate with respect to time to obtain
\[
\frac{d}{dt}((D_+ y_i) h_i) = \frac{d}{dt} \frac{1}{2} [U_i^2 (D_+ y_i)^2 + (D_+ U_i)^2 + \bar{r}_i^2]
= -U_i Q_i (D_+ y_i)^2 + U_i^2 (D_+ y_i) (D_+ U_i)
- (D_+ U_i) (D_+ Q_i) - \rho \infty \bar{r}_i (D_+ U_i),
\]
after using the new governing equations given by (4.36) and (4.39). Then we use the relation between $Q$ and $R$ given in (4.27) to obtain
\[
\frac{d}{dt}((D_+ y_i) h_i) = -U_i (D_+ y_i) [U_i (D_+ U_i) + D_- R_i] + U_i^2 (D_+ y_i) (D_+ U_i)
+ (D_+ U_i) [h_i + \rho \infty \bar{r} - (D_+ y_i) R_i] - \rho \infty \bar{r}_i (D_+ U_i)
= (D_+ U_i) h_i - (D_+ y_i) [U_i (D_- R_i) + R_i (D_+ U_i)],
\]
where we have used (4.36), (4.39) and (4.38). This is again equivalent to
\[
\dot{h}_i (D_+ y_i) = - [U_i (D_- R_i) + R_i (D_+ U_i)] (D_+ y_i).
\]

After simplifying by $D_+ y_i$, we obtain an evolution equation for $h$,
\[
\dot{h}_i = - [U_i (D_- R_i) + R_i (D_+ U_i)].
\]

This leads to
\begin{equation}
(4.41) \dot{H}_j = -\Delta \xi \sum_{i=-\infty}^{j-1} [U_i (D_- R_i) + R_i (D_+ U_i)] = U_j R_{j-1},
\end{equation}
where in the last equality we have used the decay at infinity together with (3.6).

Collecting all the equations and applying the relations (4.33) and (4.34) we obtain the closed system
\begin{align}
(4.42a) & \quad \dot{\bar{r}}_j = U_j, \\
(4.42b) & \quad \dot{H}_j = -Q_j \\
(4.42c) & \quad H_j = U_j R_{j-1}, \\
(4.42d) & \quad \dot{r}_j = -\rho \infty D_+ U_j,
\end{align}
\[
\dot{g}_{i,j} = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m) (g_{i,m} g_{m,j} + \gamma_{i,m} \kappa_{m,j}), \\
\dot{k}_{i,j} = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m) (k_{i,m} k_{m,j} + \kappa_{i,m} \gamma_{m,j}), \\
\dot{\gamma}_{i,j} = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m) (\gamma_{i,m} k_{m,j} + g_{i,m} \gamma_{m,j}), \\
\dot{\kappa}_{i,j} = -\Delta \xi \sum_{m \in \mathbb{Z}} (D_+ U_m) (\kappa_{i,m} g_{m,j} + k_{i,m} \kappa_{m,j}),
\]

where \( y_j = j \Delta \xi + \zeta_j \), and we recall

\[
R_j = \Delta \xi \sum_{i \in \mathbb{Z}} \gamma_{i,j} U_i (D_+ U_i) + \Delta \xi \sum_{i \in \mathbb{Z}} k_{i,j} (h_i + \rho_\infty \bar{r}_i), \\
Q_j = \Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} U_i (D_+ U_i) + \Delta \xi \sum_{i \in \mathbb{Z}} \kappa_{i,j} (h_i + \rho_\infty \bar{r}_i).
\]

In Section 5, we prove that this system admits a unique, global solution, and we discuss how to construct initial data.

5. Existence, uniqueness and initial data for the semi-discrete system

In this section, we show that the semi-discrete system (4.42) has a unique, globally defined solution. Furthermore, we will consider the regularity of initial data allowed for (4.42). We emphasize that in this section, when writing \( \zeta, U \), etc. we mean the sequences \( \{\zeta_j\}_{j \in \mathbb{Z}} \) and \( \{U_j\}_{j \in \mathbb{Z}} \) and not the solutions of the continuous problem. First we introduce discrete versions of the spaces used in the continuous setting, namely

\[
e := v \times h^1 \times v \times \ell^2,
\]

with norm

\[
||| (\zeta, U, H, \bar{r}) |||_e := \| \zeta \|_v + \| U \|_{h^1} + \| H \|_v + \| \bar{r} \|_{\ell^2}.
\]

Since we have included the operator kernels as solution variables in (4.42), we will consider solution tuples of the form

\[
X = (\zeta, U, H, \bar{r}, g, k, \gamma, \kappa) \in e \times (\ell^*)^4 =: e_{\text{ker}},
\]

where \( e_{\text{ker}} \) denotes the space \( e \) augmented with the kernel space.

5.1. Existence and uniqueness of the solution to the semi-discrete system.

To prove the short-time existence of (4.42), we consider the following auxiliary system, which is the same as (4.42) except that we have decoupled \( \zeta, U \) and \( H \) from their discrete derivatives \( D_+ \zeta, D_+ U \) and \( D_+ H \) by introducing the sequences \( \alpha, \beta \) and \( h \). The reason for this is that we cannot take for granted that the kernels satisfy (4.27) for \( t > 0 \), and then we cannot use (4.38) when estimating for instance the \( h^1 \)-norm of \( U \) in (4.42). Once the short-time existence of the solution to the auxiliary system is established, we will prove that the coupling between \( y, U, H \) and their discrete derivatives is preserved if it holds initially and, in this way, the
short-time existence of solution to (4.42) will be established. The auxiliary system reads

\[
\begin{align*}
\dot{\zeta}_j &= U_j, \\
\dot{U}_j &= -Q_j, \\
\dot{H}_j &= -U_j R_{j-1}, \\
\dot{r}_j &= -\rho_\infty \beta_j, \\
\dot{\alpha}_j &= \beta_j, \\
\dot{\beta}_j &= -R_j (1 + \alpha_j) + h_j + \rho_\infty r_j, \\
\dot{h}_j &= ((U_j)^2 - R_j) \beta_j - U_j Q_j (1 + \alpha_j), \\
\dot{g}_{i,j} &= -\Delta \xi \sum_{m \in \mathbb{Z}} \beta_m (g_{i,m} g_{j,m} - \gamma_{i,m} \gamma_{j,m}), \\
\dot{k}_{i,j} &= -\Delta \xi \sum_{m \in \mathbb{Z}} \beta_m (k_{i,m} k_{j,m} - \kappa_{i,m} \kappa_{j,m}), \\
\dot{\gamma}_{i,j} &= -\Delta \xi \sum_{m \in \mathbb{Z}} \beta_m (\gamma_{i,m} k_{j,m} - g_{i,m} \kappa_{j,m}), \\
\dot{\kappa}_{i,j} &= -\Delta \xi \sum_{m \in \mathbb{Z}} \beta_m (\kappa_{i,m} g_{j,m} - k_{i,m} \gamma_{j,m}),
\end{align*}
\]

where we have momentarily redefined \( R \) and \( Q \) as

\[
\begin{bmatrix} R \\ Q \end{bmatrix} = \begin{bmatrix} \gamma & k \\ g & \kappa \end{bmatrix}^* \begin{bmatrix} U \beta \\ h + \rho_\infty \bar{r} \end{bmatrix}.
\]

The evolution equations (5.1e), (5.1f) and (5.1g) have been obtained formally by applying \( D_+ \) to (4.42a), (4.42b) and (4.42c) respectively, in combination with (4.38). We collect all the variables and auxiliary variables in a tuple

\[
Y = (\zeta, U, H, r, \alpha, \beta, h, g, k, \gamma, \kappa) \in \ell^\infty \times (\ell^2 \cap \ell^\infty) \times \ell^\infty \times (\ell^2)^4 \times (\ell^*)^4 =: e_{aux}
\]

and introduce the corresponding norm

\[
\|Y\|_{e_{aux}} := \|\zeta\|_{\ell^\infty} + \|U\|_{\ell^2} + \|H\|_{\ell^\infty} + \|r\|_{\ell^2} + \|\alpha\|_{\ell^2} + \|\beta\|_{\ell^2} + \|h\|_{\ell^2} + \|g\|_{\ell^2} + \|k\|_{\ell^2} + \|\gamma\|_{\ell^2} + \|\kappa\|_{\ell^2}.
\]

Note how we require \( U \in \ell^\infty \) to account for the fact that the decoupling of \( U \) and \( D_+ U \) deprives us of the relation

\[
U \in h^1 \Rightarrow U \in \ell^\infty
\]

coming from (3.5).

**Lemma 5.1** (Short-time solution for (5.1)). Let \( Y_0 \in e_{aux} \) be such that \( 1 + \alpha_j \geq 0 \) for all \( j \), and with initial auxiliary variables \( g_0, k_0, \gamma_0, \kappa_0 \) constructed according to Theorem 3.6 and Corollary 3.8 with \( 1 + \alpha_j \) playing the role of \( a_j \). Then, there exists a time \( T > 0 \) depending only on \( \|Y_0\|_{e_{aux}} \) such that (5.1) has a unique solution \( Y \in C^1([0, T], e_{aux}) \) with initial data \( Y_0 \).

**Proof of Lemma 5.1.** We are going to use the symmetry and anti-symmetry identities (4.33) and (4.34) in our estimates and we explain now why it can be done. First, we note that, in the construction of the initial auxiliary variables we derived
(3.35) and (3.39), which show that the symmetry identities hold initially. Then, by looking at the evolution equations (5.1h)–(5.1k), we can check that the symmetry identities are preserved by the Picard fixed-point operator, which we use here to prove the short-time existence of (5.1) by establishing local Lipschitz regularity of the right-hand side. Thus, we can prove the existence of a short-time solution in the closed subset of $e_{aux}$ where the symmetry and anti-symmetry conditions (4.33) and (4.34) hold.

Let us then consider two functions in $e_{aux}$, $Y = (\zeta, U, H, r, \alpha, \beta, h, g, k, \gamma, \kappa)$ and $\tilde{Y} = (\tilde{\zeta}, \tilde{U}, \tilde{H}, \tilde{r}, \tilde{\alpha}, \tilde{\beta}, \tilde{h}, \tilde{g}, \tilde{k}, \tilde{\gamma}, \tilde{\kappa})$.

For the Lipschitz estimates, we first treat the right-hand sides of (5.1h)–(5.1k). We only provide details for (5.1h) as (5.1i)–(5.1k) can be treated similarly.

We start by considering the $L^\infty$-norm with the following splitting,

$$
-\Delta \xi \sum_{m \in \mathbb{Z}} \beta_m (g_{j,m}^i g_{i,m}^j - g_{i,m}^i g_{j,m}^j)
+ \Delta \xi \sum_{m \in \mathbb{Z}} \tilde{\beta}_m (\tilde{g}_{j,m}^i \tilde{g}_{i,m}^j - \tilde{g}_{i,m}^i \tilde{g}_{j,m}^j)
\leq \Delta \xi \sum_{m \in \mathbb{Z}} \beta_m g_{j,m}^i g_{i,m}^j - \Delta \xi \sum_{m \in \mathbb{Z}} \tilde{\beta}_m \tilde{g}_{j,m}^i \tilde{g}_{i,m}^j
+ \Delta \xi \sum_{m \in \mathbb{Z}} \beta_m g_{i,m}^i g_{j,m}^j - \Delta \xi \sum_{m \in \mathbb{Z}} \tilde{\beta}_m \tilde{g}_{i,m}^i \tilde{g}_{j,m}^j
$$

Now we consider only the first term, as the second can be treated similarly,

$$
\left| \Delta \xi \sum_{m \in \mathbb{Z}} \beta_m g_{j,m}^i g_{i,m}^j - \Delta \xi \sum_{m \in \mathbb{Z}} \tilde{\beta}_m \tilde{g}_{j,m}^i \tilde{g}_{i,m}^j \right|
\leq \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m - \tilde{\beta}_m| g_{j,m}^i g_{i,m}^j + \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m| g_{j,m}^i g_{j,m}^j - \tilde{g}_{j,m}^i g_{i,m}^j
+ \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m| g_{i,m}^i g_{i,m}^j - \tilde{g}_{i,m}^i g_{i,m}^j
\leq \left( \sup_{m} |g_{i,m}^i| \right) \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m - \tilde{\beta}_m| g_{j,m}^j
+ \left( \sup_{m} |g_{i,m}^i| \right) \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m| g_{j,m}^i - \tilde{g}_{j,m}^i
+ \left( \sup_{m} |g_{i,m}^i - \tilde{g}_{i,m}^i| \right) \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m| \tilde{g}_{j,m}^i
\leq \|g\|_{L^\infty} \|g\|_{L^2} \|\beta - \tilde{\beta}\|_{L^2} + \|g\|_{L^\infty} \|\beta\|_{L^2} \|g - \tilde{g}\|_{L^2}
+ \|\beta\|_{L^2} \|\tilde{g}\|_{L^2} \|g - \tilde{g}\|_{L^\infty}
$$

Then we consider the $\ell^1$-norm, use the same splitting and consider again only the first term. We make use of the symmetry properties of the kernel operators, as given in Lemma (4.1), to switch between indices and obtain

$$
\Delta \xi \sum_{i \in \mathbb{Z}} \Delta \xi \sum_{m \in \mathbb{Z}} \beta_m g_{j,m} g_{i,m} - \Delta \xi \sum_{m \in \mathbb{Z}} \beta_m \tilde{g}_{j,m} \tilde{g}_{i,m} \\
\leq \Delta \xi \sum_{i \in \mathbb{Z}} \Delta \xi \sum_{m \in \mathbb{Z}} |\beta_m - \tilde{\beta}_m| |g_{j,m}| |g_{i,m}| \\
+ \Delta \xi \sum_{i \in \mathbb{Z}} \Delta \xi \sum_{m \in \mathbb{Z}} |\tilde{\beta}_m| |g_{j,m} - \tilde{g}_{j,m}| |g_{i,m}| \\
+ \Delta \xi \sum_{i \in \mathbb{Z}} \Delta \xi \sum_{m \in \mathbb{Z}} |\tilde{\beta}_m| |\tilde{g}_{j,m}| |g_{i,m} - \tilde{g}_{i,m}| \\
= \Delta \xi \sum_{m \in \mathbb{Z}} |g_{j,m}| |\beta_m - \tilde{\beta}_m| \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,m}| \\
+ \Delta \xi \sum_{m \in \mathbb{Z}} |\tilde{\beta}_m| |g_{j,m} - \tilde{g}_{j,m}| \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,m}| \\
+ \Delta \xi \sum_{m \in \mathbb{Z}} |\tilde{\beta}_m| |\tilde{g}_{j,m}| \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,m} - \tilde{g}_{i,m}| \\
\leq \|g\|_{\ell^1} \|g\|_{\ell^1} |\beta - \tilde{\beta}|_{\ell^2} + \|g\|_{\ell^1} \|\tilde{\beta}\|_{\ell^2} \|g - \tilde{g}\|_{\ell^2} \\
+ \|\tilde{\beta}\|_{\ell^2} \|\tilde{g}\|_{\ell^2} \|g - \tilde{g}\|_{\ell^2},
$$

From the Young’s inequality (3.12), we get $\|g - \tilde{g}\|_{\ell^2} \leq \|g\|_{\ell^1}$ and therefore we can conclude that the right-hand side in (5.1b) is Lipschitz-continuous with respect to the $e_{aux}$-norm.

Let us consider Lipschitz properties of $R$ and $Q$. We have $Q_2 = \kappa * f$ for $f = h + \rho_\infty r$ so that

$$
\|f\|_{\ell^2} = \|h + \rho_\infty r\|_{\ell^2} \leq \|h\|_{\ell^2} + \rho_\infty \|r\|_{\ell^2}.
$$

Starting with $Q_2$, we have

$$
\|Q_2 - \tilde{Q}_2\|_{\ell^2} = \|\kappa * f - \tilde{\kappa} * \tilde{f}\|_{\ell^2} \leq \|\kappa - \tilde{\kappa}\| \|f\|_{\ell^2} + \|\tilde{\kappa} * (f - \tilde{f})\|_{\ell^2}
$$

For the first term above, applying the Young’s inequality (3.12) with $r = p = 2$ and $q = 1$, we get

$$
\|\kappa - \tilde{\kappa}\| \|f\|_{\ell^2} \leq \|\kappa - \tilde{\kappa}\|_{\ell^1} \|\kappa - \tilde{\kappa}\|_{\ell^1} \|f\|_{\ell^2}
$$

Using the antisymmetry property (4.34) of $\kappa$ and $\tilde{\kappa}$, namely $\kappa^\top = -\gamma$ and $\tilde{\kappa}^\top = -\tilde{\gamma}$, we get

$$
\|\kappa - \tilde{\kappa}\| \|f\|_{\ell^2} \leq \|\gamma - \tilde{\gamma}\| \|f\|_{\ell^2} + \|\kappa - \tilde{\kappa}\|_{\ell^1} \|f\|_{\ell^2}
$$

Hence, we obtain the following estimate in $\ell^2$-norm,

$$
\|Q_2 - \tilde{Q}_2\|_{\ell^2} \leq \|\gamma - \tilde{\gamma}\| \|f\|_{\ell^2} + \|\kappa - \tilde{\kappa}\|_{\ell^1} \|f\|_{\ell^2} + \|\kappa - \tilde{\kappa}\|_{\ell^1} \|f - \tilde{f}\|_{\ell^2}.
$$

For the $\ell^\infty$-norm, we use the same splitting

$$
\|Q_2 - \tilde{Q}_2\|_{\ell^\infty} \leq \|\kappa - \tilde{\kappa}\| f_{\ell^\infty} + \|\kappa * (f - \tilde{f})\|_{\ell^\infty}.
$$
Applying Young’s inequality (3.12), for \( r = \infty \) and \( p = q = 2 \), and the symmetry property of \( \kappa \), we obtain in a similar way as before that
\[
\|Q_2 - \tilde{Q}_2\|_e^\infty \leq \|\gamma - \tilde{\gamma}\|_e \|f\|_e + \|\tilde{\gamma}\|_e \|f - \tilde{f}\|_e.
\]
In a similar fashion as for \( Q_2 \) we find
\[
\|R_2 - \tilde{R}_2\|_e \leq \|k - \tilde{k}\|_e \|f\|_e + \|\tilde{k}\|_e \|f - \tilde{f}\|_e,
\]
\[
\|R_2 - \tilde{R}_2\|_e^\infty \leq \|k - \tilde{k}\|_e \|f\|_e + \|\tilde{k}\|_e \|f - \tilde{f}\|_e.
\]
Furthermore, analogous applications of (3.12) and (4.34) produce
\[
\|Q_1 - \tilde{Q}_1\|_e \leq \|g - \tilde{g}\|_e \|U\beta\|_e + \|\tilde{g}\|_e \|U\beta - \tilde{U}\beta\|_e,
\]
\[
\|Q_1 - \tilde{Q}_1\|_e^\infty \leq \|g - \tilde{g}\|_e \|U\beta\|_e + \|\tilde{g}\|_e \|U\beta - \tilde{U}\beta\|_e,
\]
\[
\|R_1 - \tilde{R}_1\|_e \leq \|\gamma - \tilde{\gamma}\|_e \|U\beta\|_e + \|\tilde{\gamma}\|_e \|U\beta - \tilde{U}\beta\|_e,
\]
\[
\|R_1 - \tilde{R}_1\|_e^\infty \leq \|\gamma - \tilde{\gamma}\|_e \|U\beta\|_e + \|\tilde{\gamma}\|_e \|U\beta - \tilde{U}\beta\|_e.
\]
For the \( \ell^1 \)-norms above we then apply the Cauchy–Schwarz inequality to obtain
\[
\|U\beta\|_e \leq \|U\|_e \|\beta\|_e,
\]
\[
\|U\beta - \tilde{U}\beta\|_e \leq \|U\|_e \|\beta - \tilde{\beta}\|_e + \|\tilde{U}\|_e \|\beta - \tilde{\beta}\|_e,
\]
which contain the relevant norms.

From the preceding estimates on \( Q_1 \) and \( Q_2 \) the local Lipschitz property of the right-hand side of (5.1b) in the \( \ell^1 \cap \ell^\infty \)-norm is clear. Furthermore, since \( U \in \ell^\infty \), the previous \( \ell^\infty \)-estimates on \( R \) and \( Q \) also show that the right-hand sides of (5.1f) and (5.1g) are locally Lipschitz in the \( \ell^1 \)-norm. For (5.1e), we introduce the right-shift operator \((\tau R)_{j} = R_{j-1}\) and we have
\[
\|U(\tau R) - \tilde{U}(\tau \tilde{R})\|_{e^\infty} \leq \|U - \tilde{U}\|_{e^\infty} \|\tau R\|_{e^\infty} + \|\tilde{U}\|_{e^\infty} \|\tau(R - \tilde{R})\|_{e^\infty}
\]
\[
\leq \|U - \tilde{U}\|_{e^\infty} \|R\|_{e^\infty} + \|\tilde{U}\|_{e^\infty} \|R - \tilde{R}\|_{e^\infty}.
\]
The remaining right-hand sides of (5.1a), (5.1e) and (5.1d) are linear in the solution variables and thus Lipschitz in their respective norms. Hence, for (5.1) written as \( \tilde{Y} = \tilde{F}(Y) \) we have
\[
\|\tilde{F}(Y) - \tilde{F}(\tilde{Y})\|_{e_{aux}} \leq C(\|Y\|_{e_{aux}}, \|\tilde{Y}\|_{e_{aux}})\|Y - \tilde{Y}\|_{e_{aux}},
\]
which is what we set out to prove. \[\square\]

Let us now show, if we have initial data for the auxiliary system (5.1) that satisfy
\[
(5.3a) \quad \begin{bmatrix} -D_- & (1 + \alpha_j) \\ (1 + \alpha_j) & -D_+ \end{bmatrix} \circ \begin{bmatrix} \gamma_{i,j} & k_{i,j} \\ g_{i,j} & \kappa_{i,j} \end{bmatrix} = \begin{bmatrix} \delta_{i,j} & 0 \\ 0 & \delta_{i,j} \end{bmatrix} \frac{1}{\Delta \xi} \begin{bmatrix} \delta_{i,j} & 0 \\ 0 & \delta_{i,j} \end{bmatrix}
\]
\[
(5.3b) \quad \alpha = D_+ \zeta, \quad \beta = D_+ U, \quad \text{and} \quad h = D_+ H,
\]
then the solution preserve these identities in time. The result for (5.3a) has been proved in Lemma 4.1, as it only depends on the identity \((D_+ y_j(t))_t = D_+ U_j(t)\) which is replaced here by \(\alpha = \beta\). From (5.3a), we infer from (4.38) that
\[
(5.4) \quad \begin{bmatrix} -D_- & (1 + \alpha_j) \\ (1 + \alpha_j) & -D_+ \end{bmatrix} \circ \begin{bmatrix} R_j \\ Q_j \end{bmatrix} = \begin{bmatrix} U_j \beta_j \\ h_j + \rho_\infty \tilde{r}_j \end{bmatrix}.
\]
From the definition of the auxiliary system, we get

\[(5.5a) \quad \frac{d}{dt}(\alpha_j - D_+ \zeta_j) = \beta_j - D_+ U_j,\]

the expression for \(D_+ Q_j\) from (5.4) yields

\[(5.5b) \quad \frac{d}{dt}(\beta_j - D_+ U_j) = 0,\]

while from the expression for \(D_- R_j\) we obtain

\[(5.5c) \quad \frac{d}{dt}(h_j - D_+ H_j) = -R_j(\beta_j - D_+ U_j)\]

Hence, the equations (5.5) give us that (5.3b) holds for all time if it holds initially.

Then we obtain the following theorem.

**Theorem 5.2** (Short-time solution for (4.42)). Given \(X_0 \in \ker\) such that \(1 + D_+ \zeta_j \geq 0\) and \(g_0, k_0, \gamma_0, \) and \(\kappa_0\) are constructed according to Theorem 3.6 and Corollary 3.8 with \(1 + D_+ \zeta_j\) replacing \(a_j\). Then there exists a time \(T\) depending only on \(\|X_0\|_{\ker}\) such that (4.42) has a unique solution \(X \in C^1([0, T], \ker)\) with initial data \(X_0\).

Note that in the rest of the paper we will often use \(h_j\) and \(D_+ H_j\) interchangeably for the energy density. The next step is to prove that there exists a subset of the Banach space where we have we have established the existence of short-time solution, which is preserved by the evolution equation, and such that the solutions exist globally in time when they belong to they belong to this set. The set of interest is the following.

**Definition 5.3.** The set \(B\) is composed of all \((\zeta, U, H, \bar{r}, g, k, \gamma, \kappa)\) \(\in \ker\) such that

(a) \(g, k, \gamma, \kappa\) satisfy the properties listed in Lemma 4.1 for \(a = D_+ y,\)
(b) \((D_+ y, D_+ U, D_+ H, \bar{r}) \in (\ell^\infty)^4,\)
(c) \(2(D_+ y_j)(D_+ H_j) = (U_j)^2(D_+ y_j)^2 + (D_+ U_j)^2 + \bar{r}_j^2\) for all \(j,\)
(d) \(D_+ y_j \geq 0, D_+ H_j \geq 0, D_+ y_j + D_+ H_j > 0\) for all \(j.\)

**Lemma 5.4** (Properties preserved by the flow). Given initial data \(X_0 \in B, \) let \(X(t) \in C^1([0, T], \ker)\) be the corresponding short-time solution given by Theorem 5.2. Then \(X(t) \in B\) for all \(t \in [0, T].\)

**Proof of Lemma 5.4.** Property (a) follows from Lemma 4.1 since the solution variables in \(X(t)\) satisfy \(D_+ y_j = D_+ U_j\) and \(D_+ U \in \ell^2,\) where we as usual have \(D_+ y_j = 1 + D_+ \zeta_j.\)

The proof of property (b) essentially follows [21, Lemma 3.3], which again is based on [23, Lemma 2.4], and the argument is as follows. Consider \(U, R, Q\) defined in (4.35a), (4.35b), (4.37a) and (4.37b) as given functions for \(t \in [0, T]\) based on the solution variables in \(X(t).\) Then we can read off from (5.1) that the variables

\[(5.5a) \quad \frac{d}{dt}(\alpha_j - D_+ \zeta_j) = \beta_j - D_+ U_j,\]

\[(5.5b) \quad \frac{d}{dt}(\beta_j - D_+ U_j) = 0,\]

while from the expression for \(D_- R_j\) we obtain

\[(5.5c) \quad \frac{d}{dt}(h_j - D_+ H_j) = -R_j(\beta_j - D_+ U_j)\]

Hence, the equations (5.5) give us that (5.3b) holds for all time if it holds initially.
(D_y, D_U, D_H, \bar{r})(t) coming from X(t) satisfy the following affine system,
\begin{equation}
\begin{aligned}
\dot{\alpha}_j &= \beta_j \\
\dot{\beta}_j &= -R_j \alpha_j + h_j + \rho_\infty r_j \\
\dot{h}_j &= (U_j)^2 - R_j \beta_j - U_j Q_j \alpha_j, \\
\dot{r}_j &= -\rho_\infty \beta_j,
\end{aligned}
\end{equation}
(5.6)
in the respective variables ($\alpha, \beta, h, r$). We know that $U, R,$ and $Q$ are bounded in the $L^\infty$-norm, and so the affine system (5.6) has bounded coefficients. Then we may take any norm we like, in particular the $L^\infty$-norm, and right-hand side of (5.6) will be locally Lipschitz in that norm. Hence, the result follows from a standard contraction argument.

To prove (c) we simply differentiate the identity with respect to time while applying (4.42), or (5.1) if you will, to find
\[\frac{d}{dt} \left[ (D+y_j)h_j - \frac{1}{2}(U_j)^2(D+y_j)^2 - \frac{1}{2}(D+U_j)^2 - \frac{1}{2}(\bar{r}_j)^2 \right] = (D+y_j)\dot{h}_j + (D+y_j)h_j - U_j \dot{U}_j(D+y_j)^2 - (U_j)^2(D+y_j)(D+y_j)
- (D+U_j)(D+U_j) - \bar{r}_j \dot{\bar{r}}_j
= (D+y_j) \left[ (U_j)^2 - R_j \right] (D+U_j) - U_j Q_j (D+y_j) + (D+U_j)h_j + U_j Q_j (D+y_j)^2
- (U_j)^2(D+y_j)(D+U_j) + (D+U_j) \left[ R_j (D+y_j) - h_j - \rho_\infty \bar{r}_j \right] + \bar{r}_j \rho_\infty (D+U_j)
= 0,
\]
where we identify $h_j$ and $D_u H_j$. Consequently, if (c) holds for $t = 0$, then it will hold for all $t \in (0, T]$.

To prove (d) we fix $j \in \mathbb{Z}$ and define
\[t^* := \sup \{ t \in [0, T] : D+y_j(t') \geq 0, t' \in [0, t] \},\]
and assume $t^* < T$. Since $D+y_j$ is continuous with respect to time we have $D+y_j(t^*) = 0$, which by (4.1) and $h \in \ell^\infty$ from (3.3) implies
\[D+y_j(t^*) = D+U_j(t^*) = \bar{r}_j(t^*) = 0.
\]
From (4.38) and (4.42) we get
\[D+y_j = -D+S_j = -R_j(D+y_j) + h_j + \rho_\infty \bar{r}_j,
\]
implying $D+y_j(t^*) = h_j(t^*)$. Assume first $h_j(t^*) = 0$ which implies
\[(D+y_j, D+U_j, h_j, \bar{r}_j)(t^*) = (0, 0, 0, 0).
\]
Uniqueness of solutions for (5.6) then yields
\[(D+y_j, D+U_j, h_j, \bar{r}_j)(0) = (0, 0, 0, 0),\]
which contradicts $X_0 \in \mathcal{B}$. Assume then $h_j(t^*) < 0$. This contradicts the definition of $t^*$ as there would then be a neighborhood of $t^*$ where $D+y_j < 0$. Therefore we must have $h_j(t^*) > 0$ and so there must be a neighborhood of $t^*$ where $D+y_j > 0$ contradicting the definition of $t^*$. Hence $t^* = T$ and we have proved $D+y_j(t) \geq 0$ for $t \in [0, T]$. When $D+y_j > 0$ it follows from (4.39) that $h_j \geq 0$. On the other hand, if $D+y_j(t) = 0$ we have just seen that $h_j(t) < 0$ would imply that $D+y_j < 0$ in
a punctured neighborhood of $t$, which is impossible. Thus we must have $h_j(t) \geq 0$ for $t \in [0,T]$. For the last inequality, assume that $D_+ y_j + h_j > 0$ does not hold for $t \in [0,T]$. Then by continuity there is a $t$ such that $(D_+ y_j + h_j)(t) = 0$, but this would again by uniqueness of solutions for (5.6) mean that $(D_+ y_j + h_j)(0) = 0$ which contradicts $X_0 \in B$. □

For the rest of the paper we will only consider $X \in B \cap e_{\ker}$, as such solutions will turn out to be sufficient for our purposes. Lemma 5.4, in particular the preservation of the identity

$$ (5.7) \quad 2(D_+ y_j) h_j = U_j^2 (D_+ y_j)^2 + (D_+ U_j)^2 + \bar{r}_j^2, $$

(c) and (d) for solutions in the set $B$ from Definition 5.3 allows us to prove the two following useful inequalities. The first one is

$$ (5.8) \quad \Delta \xi \sum_{j \in \mathbb{Z}} |U_j| |D_+ U_j| \leq H_{\infty}(t), $$

where $H_{\infty}(t) = \lim_{n \to +\infty} H_n$ is the total energy of the discrete system. This quantity corresponds to $H_{\text{dis}}$ in (2.13) and, indeed, due to the energy preserving structure of our discrete scheme, we have that for $t \in [0,T]$, the Hamiltonian (2.14) is conserved,

$$ H_{\infty}(t) = H_{\infty}(0) < \infty, \quad t \in [0,T]. $$

We denote the preserved total energy $H_{\infty}(t)$ by $H_{\infty}$. The preservation of $H_{\infty}(t)$ can be seen directly from (4.41), as we have

$$ \frac{d}{dt} H_{\infty}(t) = \frac{d}{dt} \lim_{n \to +\infty} H_n(t) = \lim_{n \to +\infty} -U_j R_{j-1}(t) = 0, $$

where the final identity is due to the fact that $U, R \in h^1$.

Turning back to the inequality (5.8), it can be proved as follows,

$$ \Delta \xi \sum_{j \in \mathbb{Z}} |U_j| |D_+ U_j| \leq \Delta \xi \sum_{j \in \mathbb{Z}} |U_j| \sqrt{(D_+ y_j)[2h_j - U_j^2 (D_+ y_j)]} $$

$$ \leq \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} U_j^2 (D_+ y_j) + \frac{1}{2} \Delta \xi \sum_{j \in \mathbb{Z}} [2h_j - U_j^2 (D_+ y_j)] $$

$$ = H_{\infty}, $$

where in the first inequality we have used (5.7) and in the second inequality we have used the $D_+ y_j \geq 0$ together with the Cauchy–Schwarz inequality. An immediate consequence of (5.8) is that $\|U\|_{\ell^{\infty}}$ can be uniformly bounded by a constant depending only on $H_{\infty}$. Indeed, by adding and subtracting in (2.7) we have the identity

$$ D_{\pm}(U_i)^2 = 2U_i (D_{\pm} U_i) \pm \Delta \xi (D_{\pm} U_i)^2. $$

Taking advantage of the decay of $U$ at infinity we may then write

$$ (U_j)^2 = -2 \Delta \xi \sum_{i=j}^{\infty} U_i (D_+ U_i) - (\Delta \xi)^2 \sum_{i=j}^{\infty} (D_+ U_i)^2 $$

$$ \leq 2 \Delta \xi \sum_{i=j}^{\infty} |U_i| |D_+ U_i| $$
\[
\Delta \xi \sum_{j \in \mathbb{Z}} |D_j + g_i, j| = 2 \|g\|_{\ell^\infty},
\]
from which the bound
\[
(5.9) \quad \sup_{0 \leq t \leq T} \|U(t)\|_{\ell^\infty} \leq \sqrt{2H_{\infty}}.
\]
follows. From \((5.9)\) and \((4.42a)\) in integral form, \(\zeta_j(t) = \zeta_j(0) + \int_0^t U_j(s) \, ds\), we then have the estimate
\[
(5.10) \quad \|\zeta(t)\|_{\ell^\infty} \leq \|\zeta(0)\|_{\ell^\infty} + \sqrt{2H_{\infty}} t.
\]
Yet another useful estimate coming from \((5.7)\) is
\[
(5.11) \quad |\bar{r}_j| \leq \sqrt{2(D_j + y_j) h_j},
\]
Now that Lemma 5.4 has established \(D_j + y_j(t) \geq 0\) in the short-time solution for \(t \in [0, T]\), we can apply Lemma 3.10 with \(a_j = D_j + y_j\) because the sequences \(g, \gamma, k, \) and \(\kappa\) solve \((4.27)\) and belong to \(\ell^\infty\) for \(t \in [0, T]\), so that they correspond to the unique decaying solution. These properties contained in Lemmas 3.10 and 4.1 are essential to establish the \textit{a priori} estimates contained in the next lemma.

**Lemma 5.5** (A priori relations and inequalities for the kernels). As a consequence of establishing the preservation of the summation kernels and their sign properties over time, we have the identities
\[
\Delta \xi \sum_{j \in \mathbb{Z}} |D_j + g_{i, j}| = \Delta \xi \sum_{j \in \mathbb{Z}} |D_j + g_{i, j}| = 2 \|g\|_{\ell^\infty},
\]
and the bounds
\[
(5.14) \quad \|g\|_{\ell^\infty}, \|k\|_{\ell^\infty}, \|\gamma\|_{\ell^\infty}, \|\kappa\|_{\ell^\infty} \leq 1,
\]
and for \(g, \kappa, \gamma, \) and \(\kappa, \gamma, \) respectively.

Proof of Lemma 5.5 To prove \((5.12)\) we use \(D_j + y_j \geq 0\) and \((4.27)\) for the leftmost equalities, while for the rightmost equalities we use the monotonicity properties of \((3.41)\) to write
\[
\Delta \xi \sum_{j \in \mathbb{Z}} |D_j + g_{i, j}| = \Delta \xi \sum_{j = -\infty}^{i-1} D_j + g_{i, j} - \Delta \xi \sum_{j = i} \infty D_j + g_{i, j} = 2g_{i, j} = 2 \|g\|_{\ell^\infty},
\]
\[ \Delta \xi \sum_{j \in \mathbb{Z}} \{ D_j - k_{i,j} \} = \Delta \xi \sum_{j = -\infty}^{i} D_j - k_{i,j} - \Delta \xi \sum_{j = i+1}^{\infty} D_j - k_{i,j} = 2k_{i,i} = 2 \| k \|_{\ell^\infty}. \]

To obtain (5.13), we use the definitions of the operators A in (2.16) and B in (3.36), and apply telescopic cancellation to the differences \( D_{j + \gamma_{i,j}} \) and \( D_{j - \kappa_{i,j}} \) in the identities (4.27).

In the same manner, telescopic cancellation applied to (4.27) yields

\[ \gamma_{i,j} = \begin{cases} \Delta \xi \sum_{j = -\infty}^{i} (D_j + y_m) g_{i,m}, & j \leq i - 1, \\ -\Delta \xi \sum_{m = j+1}^{\infty} (D_j + y_m) g_{i,m}, & j \geq i, \end{cases} \]

and

\[ \kappa_{i,j} = \begin{cases} \Delta \xi \sum_{m = -\infty}^{j-1} (D_j + y_m) k_{i,m}, & j \leq i, \\ -\Delta \xi \sum_{m = j}^{\infty} (D_j + y_m) k_{i,m}, & j \geq i + 1. \end{cases} \]

Using the fact that \( D_j + y_j \), \( g_{i,j} \), \( k_{i,j} \) \( \geq 0 \), the triangle inequality and (5.13) yield (5.14) for \( \gamma \) and \( \kappa \). For \( g \) and \( k \), observe that, using (4.27), we can rewrite them as

\[ g_{i,j} = \sum_{m \in \mathbb{Z}} g_{i,m} \delta_{j,m} \]

\[ = \Delta \xi \sum_{m \in \mathbb{Z}} [ (D_j + y_m) g_{j,m} - D_m - \gamma_{j,m} ] \]

\[ = \Delta \xi \sum_{m \in \mathbb{Z}} (D_j + y_m) [ g_{i,m} g_{j,m} + \gamma_{i,m} \gamma_{j,m} ], \]

and similarly

\[ k_{i,j} = \Delta \xi \sum_{m \in \mathbb{Z}} (D_j + y_m) [ k_{i,m} k_{j,m} + \kappa_{i,m} \kappa_{j,m} ]. \]

Using the decay at infinity we can then write

\[ (g_{i,i})^2 = \sum_{m = i}^{+\infty} [(g_{i,m+1})^2 - (g_{i,m})^2] \]

\[ = \Delta \xi \sum_{m = i}^{+\infty} [g_{i,m+1} + g_{i,m}] D_m + g_{i,m} \]

\[ = \Delta \xi \sum_{m = i}^{+\infty} [g_{i,m+1} + g_{i,m}] (D_j + y_m) |\gamma_{i,m}| \]

\[ \leq 2 \Delta \xi \sum_{m = i}^{+\infty} g_{i,m} (D_j + y_m) |\gamma_{i,m}| \]

\[ \leq \Delta \xi \sum_{m = i}^{+\infty} (D_j + y_m) [(g_{i,m})^2 + (\gamma_{i,m})^2] \]

\[ \leq \Delta \xi \sum_{m \in \mathbb{Z}} (D_j + y_m) [(g_{i,m})^2 + (\gamma_{i,m})^2] \]

\[ = g_{i,i}, \]
where we have used (3.41) for the first inequality, and (5.16) for the final identity. The bound \( g_{i,i} \leq 1 \) follows, and note how the above estimates align nicely with those that lead to (4.26). We then use \( 0 \leq g_{i,j} \leq g_{i,i} \), see (3.41), to conclude. A similar procedure using (3.41) and (5.17) shows
\[
(k_{i,i})^2 = \sum_{m=-\infty}^{i} [ (k_{i,m})^2 - (k_{i,m-1})^2 ] \leq k_{i,i},
\]
which implies \( k_{i,i} \leq 1 \), and once more Lemma 3.10 yields the desired result. Furthermore, we have
\[
\Delta \xi \sum_{j \in \mathbb{Z}} g_{i,j} = \Delta \xi \sum_{j \in \mathbb{Z}} [D_+ y_j - D_+ \zeta_j] g_{i,j}
\]
\[
= 1 + \Delta \xi \sum_{j \in \mathbb{Z}} \zeta_{j+1}(D_j + g_{i,j}), \text{ from (5.13),}
\]
\[
= 1 + \Delta \xi \sum_{j \in \mathbb{Z}} \zeta_{j+1}(D_j + y_j) \gamma_{i,j}, \text{ from (4.27),}
\]
\[
\leq 1 + \| \xi \|_\infty \Delta \xi \sum_{j \in \mathbb{Z}} (D_j + y_j) |\gamma_{i,j}|
\]
and the result on the \( \ell^1 \) bound of \( g \) follows from (5.12) and (5.14). A similar procedure proves the bound on \( \| k \|_\ell^1 \). For the bound on \( \| \gamma \|_\ell^1 \) we find
\[
\Delta \xi \sum_{j \in \mathbb{Z}} |\gamma_{i,j}| = \Delta \xi \sum_{j \in \mathbb{Z}} [D_+ y_j - D_+ \zeta_j] |\gamma_{i,j}|
\]
\[
= 2g_{i,i} - \Delta \xi \sum_{j=-\infty}^{i-1} (D_+ \zeta_j) \gamma_{i,j} + \Delta \xi \sum_{j=i}^{+\infty} (D_+ \zeta_j) \gamma_{i,j}
\]
\[
= 2\| g \|_\infty - 2\zeta_i \gamma_{i,i-1} + \Delta \xi \sum_{j=-\infty}^{i-1} \zeta_j (D_j - \gamma_{i,j}) - \Delta \xi \sum_{j=i}^{+\infty} \zeta_j (D_j - \gamma_{i,j})
\]
\[
= 2\| g \|_\infty + (1 - 2\gamma_{i,i-1}) \zeta_i + \Delta \xi \sum_{j \in \mathbb{Z}} \text{sgn} (i - j - \frac{1}{2}) \zeta_j (D_j + y_j) g_{i,j}
\]
\[
\leq 2\| g \|_\infty + \| \zeta \|_\infty \left[ |1 - 2\gamma_{i,i-1}| + \Delta \xi \sum_{j \in \mathbb{Z}} (D_j + y_j) g_{i,j} \right],
\]
where in the second equality we use Lemma 3.10, the third equality uses summation by parts (3.6), and the fourth is due to the kernel definition property (4.27). Then the result follows from (5.13), (5.14), and \( 0 \leq \gamma_{i,i-1} \leq 1 \). A similar procedure proves the bound on \( \| \kappa \|_\ell^1 \).

A direct consequence of (5.14) is that the \( \ell^\infty \)-norms of the kernels remain bounded by 1 for all time. Moreover, combining (5.15) with (5.10) we find that the \( \ell^1 \)-norms remain bounded for any finite \( t \), namely
\[
\| g(t) \|_\ell^1, \| k(t) \|_\ell^1 \leq 1 + 2 \left[ \| \zeta(0) \|_\ell^\infty + \sqrt{2H_\infty} t \right],
\]
\[
\| \gamma(t) \|_\ell^1, \| \kappa(t) \|_\ell^1 \leq 2 \left[ 1 + \| \zeta(0) \|_\ell^\infty + \sqrt{2H_\infty} t \right]
\]
(5.18)
Furthermore, Lemma 5.5 allows us to find a bound similar to (5.9) for $\|R\|_{\ell^\infty}$ and $\|Q\|_{\ell^\infty}$. Indeed, considering $Q$ we find

$$
\|Q\|_{\ell^\infty} \leq \|g\|_{\ell^\infty} \|U(D_+ U)\|_{\ell^\infty} + \|\kappa \ast (h + \rho_\infty \bar{r})\|_{\ell^\infty}
$$

(5.19)

Using (5.11) and the Cauchy–Schwarz inequality, we have

$$
\rho_\infty \|\kappa \ast |\bar{r}|\|_{\ell^\infty} \leq \frac{1}{2} \rho_\infty^2 \|\kappa\|_{\ell^\infty} \|D_+ y\|_{\ell^\infty} + \frac{1}{2} \|\kappa\|_{\ell^\infty} \|2h\|_{\ell^\infty}
$$

which, using (5.12) and (5.14), simplifies to

$$
\rho_\infty \|\kappa \ast |\bar{r}|\|_{\ell^\infty} \leq \frac{1}{2} \rho_\infty^2 (2\|k\|_{\ell^\infty}) + \|\kappa\|_{\ell^\infty} \|h\|_{\ell^2} \leq \rho_\infty^2 + H_\infty
$$

Using (5.8), we get $\|UD_+ U\|_{\ell^1} \leq H_\infty$. Hence, from (5.19), we get

$$
\|Q\|_{\ell^\infty} \leq 3H_\infty + \rho_\infty^2.
$$

An analogous estimate for $R$ can be obtained so that we can conclude with the bounds

$$
\sup_{0 \leq t \leq T} \|R(t)\|_{\ell^\infty} \leq 3H_\infty + \frac{1}{2} \rho_\infty^2, \quad \sup_{0 \leq t \leq T} \|Q(t)\|_{\ell^\infty} \leq 3H_\infty + \rho_\infty^2.
$$

(5.20)

Now we are set to prove global existence for solutions of (4.42), as summarized in the following theorem.

**Theorem 5.6 (Global existence).** Given initial data $X_0$ in the set $B$ from Definition 5.3, the system (4.42) admits a unique global solution $X \in C^1([0, \infty], e)$, such that $X \in B$ for all times. In particular, for $t > 0$, the norm $\|X(t)\|_e$ is bounded by $C \|X(0)\|_e$ for a constant $C$ depending only on $t, H_\infty, \rho_\infty$, and $\|\zeta(0)\|_{\ell^\infty}$.

**Proof.** The solution has a finite time of existence $T$ only if

$$
\|X\|_e = \|\zeta\|_{\ell^1} + \|U\|_{h^1} + \|H\|_{\ell^2} + \|\bar{r}\|_{\ell^2}
$$

blows up as $t$ approaches $T$. Otherwise the solution can be prolonged by a small time interval by Theorem 5.2. To this end, let $X$ be the short-time solution given by (4.42) for initial data $X_0$. We will prove that $\sup_{0 \leq t \leq T} \|X\|_e < \infty$.

We begin with the easy part; from the definition of the $h^1$-norm and (4.5) we find the right-hand side of (4.42a) to be bounded in $v$-norm by $\frac{2 + \sqrt{2}}{4} \|U\|_{h^1}$, while the right-hand side of (4.42d) is bounded in $\ell^2$-norm by $\rho_\infty \|U\|_{h^1}$.

Next we estimate the right-hand side of (4.42b),

$$
\|Q\|_{h^1} \leq \|Q\|_{\ell^2} + \|D_+ Q\|_{\ell^2} \leq \|Q\|_{\ell^2} + \|\bar{r}\|_{\ell^2} + \|R(1 + D_+ \zeta) - h - \rho_\infty \bar{r}\|_{\ell^2} \leq \|Q\|_{\ell^2} + \|R\|_{\ell^2} + \|R\|_{\ell^\infty} \|D_+ \zeta\|_{\ell^2} + \|h + \rho_\infty \bar{r}\|_{\ell^2},
$$

where we have used the definition of the $h^1$-norm, (4.38) and the decomposition $D_+ y_j = 1 + D_+ \zeta_j$. Then, recalling the definitions (4.35a) and (4.35b) and applying
Then, applying (5.8), (5.9), (5.18), (5.20) and the definitions of the Young inequality (3.12) to the final expression above we see that it is bounded by
\[ \left\| g \right\|_{\mathcal{L}^1} \left\| U(D_+U) \right\|_{\mathcal{L}^2} + \left| \gamma \right| \left\| \kappa \right\|_{\mathcal{L}^1} \left\| h +\rho_\infty \bar{r} \right\|_{\mathcal{L}^2} + \left| \gamma \right| \left\| \kappa \right\|_{\mathcal{L}^1} \left\| U(D_+U) \right\|_{\mathcal{L}^2} + \left| k \right| \left\| h \right\|_{\mathcal{L}^1} + \left| \rho_\infty \bar{r} \right\|_{\mathcal{L}^2} \]
\[ + \left| k \right| \left\| h \right\|_{\mathcal{L}^1} + \left| \rho_\infty \bar{r} \right\|_{\mathcal{L}^2}. \]

Finally, the v-norm of the right-hand side of (4.42c can be estimated as
\[ \left\| U(\tau R) \right\|_v = \left\| U(\tau R) \right\|_{\mathcal{L}^\infty} + \left\| U^2 - R(D_+U) - UQ[1 + D_+\bar{\zeta}] \right\|_{\mathcal{L}^2} \]
\[ \leq \left\| R \right\|_{\mathcal{L}^\infty} \left\| U \right\|_{\mathcal{L}^\infty} + \left\| U^2 \right\|_{\mathcal{L}^\infty} + \left\| R \right\|_{\mathcal{L}^\infty} \left\| D_+U \right\|_{\mathcal{L}^2} \]
\[ + \left\| Q \right\|_{\mathcal{L}^\infty} \left\| U \right\|_{\mathcal{L}^\infty} + \left\| Q \right\|_{\mathcal{L}^\infty} \left\| D_+\bar{\zeta} \right\|_{\mathcal{L}^2} \]
\[ \leq \left( \frac{2 + \sqrt{2}}{2} (3H_\infty + \rho_\infty^2) + 2H_\infty \right) \left\| U \right\|_{\mathcal{L}^1} + \sqrt{2H_\infty} \left( 3H_\infty + \frac{1}{2} \rho_\infty^2 \right) \left\| \bar{\zeta} \right\|_v, \]
where we again use the notation (\( \tau R \))_j = R_{j-1}. In the first identity above we have employed (4.38), while in the final line we have used the definitions of the v- and h\(^1\)-norms together with (3.5), (5.9) and (5.20).

Gathering all the above estimates of the right-hand sides, writing (4.42) in integral form, and taking norms we obtain the following inequality for \( X(t) = (\zeta, U, H, \bar{r})(t), \)
\[ \left\| X(t) \right\|_{\mathcal{L}^1} \leq \left\| X(0) \right\|_{\mathcal{L}^1} + C(H_\infty, \left\| \zeta(0) \right\|_{\mathcal{L}^\infty}, \rho_\infty) \int_0^t (1 + s) \left\| X(s) \right\|_{\mathcal{L}^1} ds, \quad t \in [0, T], \]
for some constant \( C(H_\infty, \left\| \zeta(0) \right\|_{\mathcal{L}^\infty}, \rho_\infty) \) depending only on \( H_\infty, \left\| \zeta(0) \right\|_{\mathcal{L}^\infty} \) and \( \rho_\infty. \)
Grönwall’s inequality then yields
\[ \left\| X(t) \right\|_{\mathcal{L}^1} \leq \left\| X(0) \right\|_{\mathcal{L}^1} \exp \left\{ C(H_\infty, \left\| \zeta(0) \right\|_{\mathcal{L}^\infty}, \rho_\infty) \left[ t + \frac{1}{2} t^2 \right]\right\}, \quad t \in [0, T], \]
which shows that \( \left\| X(T) \right\|_{\mathcal{L}^1} \) is bounded, and we may according to Theorem 5.2 extend our solution indefinitely.

In retrospect, with the estimates (5.9) and (5.20) at hand, we see that in the proof of the preservation of property (\( b \)) in Lemma 5.4 a Grönwall estimate shows that the \( \mathcal{L}^\infty \)-norm of \( D_+y, D_+U, h \) and \( \bar{r} \) at time \( t \in [0, T] \) is bounded by their \( \mathcal{L}^\infty \)-norm at time \( t = 0 \) times a factor \( \exp\{C(H_\infty, \rho_\infty)T\} \), where the constant \( C(H_\infty, \rho_\infty) \) depends only on \( H_\infty \) and \( \rho_\infty. \)

Thus, Theorem 5.6 shows that (4.42) has unique, globally defined solutions, as indicated in Theorem 1.2.
5.2. The choice of initial data. In this subsection we will elaborate upon the choice of initial data for (4.42). Let us first consider (1.12) with nonzero $\rho_0$, where we in addition to $u_0 \in H^1$, assume $u_{0,x}, \rho_0 - \rho_\infty \in L^\infty \cap L^\infty$ and $\rho_0 \geq \varepsilon > 0$. This is enough for the solutions of (1.12) not to blow up, see [21] [18], and justifies our choice of $y_j(0) = \xi_j$ in the derivation of the discrete system. As a consequence, $\xi_j(0) = 0$ and the initial conditions for (4.42) can be chosen as $U_j(0) = (U_0)_j$ and $\rho_j(0) = (\rho_0)_j$. Then we define initial values for the auxiliary variables through

$$
\bar{r}_j(0) = (\rho_0)_j - \rho_\infty,
$$

(5.21)

$$
H_j(0) = \Delta \xi \sum_{m=-\infty}^{j-1} \left[ \left( (U_0)_m + (D_+(U_0)_m)^2 + (\bar{r}_m(0))^2 \right) \right],
$$

and $g_{i,j}(0)$, $k_{i,j}(0)$ being Green’s functions for $A[1] = B[1] = \text{Id} - D_-D_+$, with $\gamma_{i,j}(0) = D_+g_{i,j}(0)$, $k_{i,j} = D_j-g_{i,j}(0)$. Indeed, for $D_+y_j(0) = 1$ we have

$$
g_{i,j}(0) = k_{i,j}(0) = \frac{(\lambda^+)^{-|i-j|}}{\sqrt{4 + \Delta \xi^2}},
$$

with $\lambda^+$ as in (3.13). Thus, initially we have the Eulerian Green’s sequences.

Note that it is the sufficiently regular initial data which enables us to initially choose $y_j = \xi_j$, and then again find explicit expressions for the initial Green’s functions as above. However, in our construction of the Green’s functions we have allowed for $D_+y_j = 0$, and so our discretization should be able to handle singular initial data as well, specifically for the CH equation (1.1). Indeed, let us in the usual manner place the equispaced grid on $\mathbb{R}$, and let the initial velocity $u_0 = u_0(x)$ and cumulative energy $\mu_0(( -\infty, x))$ be given. Inspired by works on conservative solutions of (1.1) in Lagrangian coordinates, we then define

$$
y_j(0) := \sup \{ x \ : \ \mu_0(( -\infty, x)) + x < \xi_j \}.
$$

This is of course the same as interpolating the function

$$
y_0(\xi) := \sup \{ x \ : \ \mu_0(( -\infty, x)) + x < \xi \}
$$

in the gridpoints $\xi_j$. In fact, (5.22) is given in [25, Eq. (3.21a)], and we can use the results therein to show that our choice of initial datum will satisfy Definition 5.3. We also adopt their definition of $U_0$, $U_0(\xi) = u_0 \circ y_0(\xi)$, in [25, Eq. (3.21)], but we will have to modify the definition of $H$ to satisfy our discrete identity (5.1).

In our endeavor we will use that their continuous-setting variables $(y_0, U_0, H_0) \in V \times H^1 \times V$. From [25] we have $|y_0(\xi) - \xi| \leq \mu_0(\mathbb{R})$, and since the total energy $\mu_0(\mathbb{R})$ is bounded, we have $\|y_0 - \text{Id}\|_L^\infty \leq \mu_0(\mathbb{R})$. Since $y_j(0) = y_0(\xi_j)$, this carries directly over to our setting, $|y_j(0) - \xi_j| \leq \mu_0(\mathbb{R})$, meaning $\|\xi(0)\|_L^\infty \leq \mu_0(\mathbb{R})$. Moreover, they prove $\xi \mapsto y_0(\xi)$ to be 1-Lipschitz, which yields

$$
|y_0(\xi_{j+1}) - y_0(\xi_{j})| \leq |\xi_{j+1} - \xi_{j}| = \Delta \xi \implies |D_+y_j(0)| \leq 1,
$$

thus $D_+y(0) \in L^\infty$. They also prove $\xi \mapsto \int_{-\infty}^{y(\xi)} u_0^2(x)dx$ to be 1-Lipschitz, which gives

$$
|U_0(\xi_{j+1}) - U_0(\xi_{j})| = \left| \int_{y(\xi_{j})}^{y(\xi_{j+1})} u(\xi) \, dx \right| \leq \sqrt{y(\xi_{j+1}) - y(\xi_{j})} \sqrt{\int_{y(\xi_{j})}^{y(\xi_{j+1})} u_0^2(x)\, dx}.
$$

(5.23)
Using that both factors in the final expression of (5.23) are 1-Lipschitz we obtain $|U_{j+1}(0) - U_j(0)| \leq \Delta \xi$, implying $|D_+ U_j(0)| \leq 1$ and $D_+ U(0) \in \ell^\infty$. In addition, as $u_0 \in L^\infty$ it is clear from $U_j(0) = u(y_j(0))$ that $\|U(0)\|_{\ell^\infty} \leq \|u_0\|_{L^\infty}$.

Now we need to choose $H_j$ in such a way as to satisfy property (ii) in Definition 5.3 and we will separate two possible cases. If $D_+ y_j(0) > 0$ we define $h_j(0) \geq 0$ such that it satisfies $2h_j(D_+ y_j) = (U_j)^2(D_+ y_j)^2 + (D_+ U_j)^2$. On the other hand, if $D_+ y_j(0) = 0$ we set $h_j(0) = \frac{1}{2}$. Then we define $H_j(0) = \Delta \xi \sum_{m = -\infty}^{j-1} h_m(0)$. Let us estimate $h_j(0)$ in the case $D_+ y_j(0) > 0$, where we note that another possible takeaway from (5.23) is $|U_{j+1}(0) - U_j(0)| \leq \sqrt{\Delta \xi} \sqrt{y_{j+1}(0) - y_j(0)}$, or equivalently $|D_+ U_j(0)| \leq \sqrt{D_+ y_j(0)}$. Using this and $D_+ y_j(0) \leq 1$ together with property (iii) we find

$$2h_j = U_j^2 D_+ y_j + \frac{(D_+ U_j)^2}{D_+ y_j} \leq U_j^2 + 1 \leq \|u_0\|_{L^\infty}^2 + 1.$$  

Thus, $h(0) \in \ell^\infty$. Finally,

$$h_j(0) + D_+ y_j(0) = \begin{cases} \frac{1}{2} > 0, & D_+ y_j(0) = 0, \\ h_j(0) + D_+ y_j(0) > 0, & D_+ y_j(0) > 0, \end{cases}$$

ans so all requirements for the initial datum to be in $\mathcal{B}$ are satisfied.

It remains to verify $(y(0), U(0), H(0), 0) \in \mathfrak{e}$. We already know that $\zeta(0) \in \ell^\infty$. We know $y_0$ is continuous, so it follows that $\zeta_0 = y_0 - \text{Id}$ is bounded and continuous, and we may write

$$|\zeta_0(\xi_{j+1}) - \zeta_0(\xi_j)|^2 = \left| \int_{\xi_j}^{\xi_{j+1}} (\zeta_0)_{\xi}(\xi) \, d\xi \right|^2 \leq \Delta \xi \int_{\xi_j}^{\xi_{j+1}} |(\zeta_0)_{\xi}(\xi)|^2 \, d\xi,$$

or equivalently

$$\Delta \xi |D_+ \zeta_0(0)|^2 \leq \int_{\xi_j}^{\xi_{j+1}} |(\zeta_0)_{\xi}(\xi)|^2 \, d\xi.$$

Summing over $j$ in the above equation we obtain $\|D_+ \zeta(0)\|_{\ell^2} \leq \|(\zeta_0)_{\xi}\|_{L^2}$, and so $\zeta(0) \in \mathfrak{v}$. A completely analogous procedure shows $\|D_+ U(0)\|_{\ell^2} \leq \|(U_0)_{\xi}\|_{L^2}$. For the $L^2$-norm of $U$ we estimate

$$\Delta \xi \sum_{j \in \mathbb{Z}} |U_j|^2 = \sum_{j \in \mathbb{Z}} \int_{\xi_j}^{\xi_{j+1}} \left[ U_0(\xi) - \int_{\xi_j}^{\xi} (U_0)(\xi) \, d\xi \right] d\xi$$

$$\leq 2 \sum_{j \in \mathbb{Z}} \int_{\xi_j}^{\xi_{j+1}} |U_0(\xi)|^2 \, d\xi + \sum_{j \in \mathbb{Z}} \int_{\xi_j}^{\xi_{j+1}} \left( \int_{\xi_j}^{\xi_{j+1}} |(U_0)(\xi)| \, d\xi \right)^2 \, d\xi$$

$$\leq 2 \|U_0\|^2_{L^2} + \sum_{j \in \mathbb{Z}} \Delta \xi^2 \int_{\xi_j}^{\xi_{j+1}} |(U_0)(\xi)|^2 \, d\xi,$$

which translates into $\|U(0)\|^2_{\ell^2} \leq 2 \|U_0\|^2_{L^2} + 2\Delta \xi^2 \|U(0)\|_{L^2}$, and so $U(0) \in \mathfrak{h}^1$.

Then it remains to check that $H(0) \in \mathfrak{v}$, and from (5.7) we estimate

$$2h_j = U_j^2 D_+ y_j + (D_+ U_j)^2 \leq U_j^2 + (D_+ U_j)^2 + 2h_j |D_+ \zeta_j|$$
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\[ \leq U_j^2 + (D_+U_j)^2 + h_j + h_j |D_+\zeta_j|^2. \]

Now, summing over \( j \) we find
\[ \|h(0)\|_{\ell^1} \leq \|U(0)\|_{h^1} + \|h(0)\|_{\ell^\infty} \|D_+\zeta(0)\|_{\ell^2}, \]
where the right-hand side is bounded by our previous estimates. Since \( h_j(0) > 0 \), it follows from our definition of \( H_j(0) \) that \( H_j(0) < H_{j+1}(0) \) and \( H_j < \|h(0)\|_{\ell^1} \), which yields \( \|H(0)\|_{\ell^\infty} = \|h(0)\|_{\ell^1} \). Finally, we have \( \|h(0)\|_{2^2} \leq \|h(0)\|_{\ell^\infty} \|h(0)\|_{\ell^1} \), so \( H(0) \in \mathbf{v} \).

In conclusion we have the following theorem, which contains the details of Theorem 1.1, and where the functions involved should be compared to Definition 3.1 in [25].

**Theorem 5.7.** We consider general initial data of the Camassa–Holm equation, which are given by a pair \( u_0 \in H^1 \) and \( \mu_0 \), where \( \mu_0 \) is a positive finite Radon measure whose absolutely continuous part satisfies
\[ \mu_{0,ac} = (u^2 + u_x^2) \, dx. \]
Then, we can construct sequences of initial data of the type \((\zeta_0, U_0, H_0, g_0, k_0, \gamma_0, \kappa_0) \in \mathbf{v} \times h^1 \times \mathbf{v} \times (\ell^\ast)^4\) for the semi-discrete system (4.42), which belongs to the set \( \mathcal{B} \) in Definition 5.3 (for \( \bar{r} \equiv 0 \)) and such that the corresponding solutions give us convergent approximations of the conservative solution to the Camassa–Holm equation with initial data \((u_0, \mu_0)\).

The downside of not having \( y_j = \xi_j \) is that we do not have an explicit expression for the initial Green’s functions. However, Theorem 3.6 guarantees their existence, so the semi-discrete can be used in all cases. In an upcoming paper we have implemented (4.42) as a numerical method for the periodic version of (1.12). As the problem then is finite-dimensional, computing the Green’s functions amounts to inverting a matrix and we are able to find them for any \( D_+y_j \geq 0 \). An interesting feature is then that we may allow for singular initial data from the very beginning in our numerical experiments. For instance, we could let \( \mu_0((\infty, x)) \) be a pure step-function, meaning that all initial energy is concentrated in separated points on the domain, and then our scheme would yield the conservative solutions for this system.

6. Convergence of the approximation to the solution of the 2CH system

We will now use the sequences obtained in the previous section to build interpolated functions on the real line which in the limit \( \Delta \xi \to 0 \) satisfy (4.8) together with (4.9). A Grönwall estimate will show that for any fixed \( T > 0 \), by decreasing \( \Delta \xi > 0 \) we can make our interpolated functions arbitrarily close to the solutions of (4.8).

Let us in this section use \( Y_{\Delta \xi} \) to denote the tuple of grid functions obtained in Theorem 5.6

\[ (6.1) \quad Y_{\Delta \xi}(t) = (\zeta, U, H, \bar{r})(t) \in \mathbf{v} \times \mathbf{h} \times \mathbf{v} \times \ell^2, \]
for \( t \in [0, T] \). In order to ease notation below we will write \( \|Y_{\Delta \xi}\| \) for \( \sup_{0 \leq t \leq T} \|Y_{\Delta \xi}(t)\|_e \).

Then we define
\[
\zeta_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \left[ \zeta_j(t) + (\xi - \xi_j)(D_+ \zeta_j(t)) \right] \chi_j(\xi),
\]
\[
U_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \left[ U_j(t) + (\xi - \xi_j)(D_+ U_j(t)) \right] \chi_j(\xi),
\]
\[
H_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \left[ H_j(t) + (\xi - \xi_j)(D_+ H_j) \right] \chi_j(\xi),
\]
\[
\bar{r}_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \bar{r}_j(t) \chi_j(\xi),
\]
\[
R_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \left[ R_j(t) + (\xi - \xi_{j+1})(D_- R_j(t)) \right] \chi_j(\xi)
\]
\[
S_{\Delta}(t, \xi) = \sum_{j \in \mathbb{Z}} \left[ S_j(t) + (\xi - \xi_j)(D_+ S_j(t)) \right] \chi_j(\xi),
\]
\[
(6.2)
\]

where \( \chi_j(\xi) \) denotes the indicator function for the interval \([\xi_j, \xi_{j+1}]\). Moreover, we introduce the functions
\[
y_{\Delta}(t, \xi) := \xi + \zeta_{\Delta}(t, \xi), \quad r_{\Delta \xi}(t, \xi) := \bar{r}_{\Delta}(t, \xi) + \rho \frac{\partial y_{\Delta}(t, \xi)}{\partial \xi}.
\]

Observe that the functions in \((6.2)\) are piecewise linear and continuous, except for \( \bar{r}_{\Delta \xi} \) which is piecewise constant. In particular we note the identity
\[
R_j + (\xi - \xi_{j+1})(D_- R_j) = R_{j-1} + (\xi - \xi_j)(D_- R_j), \quad \xi \in [\xi_j, \xi_{j+1}],
\]
which shows \( R_{\Delta \xi}(t, \xi_j) = R_{j-1} \). Another useful observation is that the support of \( \chi_i \) and \( \chi_j \) are disjoint for \( i \neq j \), which means that multiplying two such piecewise defined functions amounts to multiplying the coefficients of \( \chi_j \):
\[
\left( \sum_{j \in \mathbb{Z}} a_j \chi_j(\xi) \right) \left( \sum_{j \in \mathbb{Z}} b_j \chi_j(\xi) \right) = \sum_{j \in \mathbb{Z}} a_j b_j \chi_j(\xi).
\]

We recall the definition of the space
\[
E := V \times H^1 \times V \times L^2.
\]

A consequence of Theorem 5.6 is that the interpolated functions \((6.2)\) constructed from the solutions given by the theorem
\[
X_{\Delta}(t) := (\zeta_{\Delta}(t, \cdot), U_{\Delta}(t, \cdot), H_{\Delta}(t, \cdot), \bar{r}_{\Delta}(t, \cdot))
\]
satisfy \( X_{\Delta}(t) \in C^1([0, T], E) \) for any fixed \( T > 0 \) and \( \Delta \xi > 0 \).

The aim of this section is to show that our interpolated functions \((y_{\Delta}, U_{\Delta}, H_{\Delta}, r_{\Delta})\) satisfy \((4.8)\) and \((4.9)\), where we allow for a small error of order \( O(\Delta \xi) \). For \((4.8a), (4.8b), \) and \((4.8d)\), we observe that, by construction, we have
\[
\frac{\partial y_{\Delta}}{\partial t} = U_{\Delta}, \quad \frac{\partial U_{\Delta}}{\partial t} = -Q_{\Delta}, \quad \frac{\partial r_{\Delta}}{\partial t} = 0
\]
This identity then implies

$$\frac{\partial H_\Delta}{\partial t} = - \sum_{j \in Z} [U_j R_{j-1} + (\xi - \xi_j)[\mu_j (D_- R_j) + R_j (D_+ U_j)] \chi_j$$

$$= - \sum_{j \in Z} [U_j [R_{j-1} + (\xi - \xi_j)(D_- R_j)] + R_j (\xi - \xi_j)(D_+ U_j)] \chi_j$$

$$= - \sum_{j \in Z} [U_j + (\xi - \xi_j)(D_+ U_j)] [R_{j-1} + (\xi - \xi_j)(D_- R_j)] \chi_j$$

$$+ \sum_{j \in Z} (\xi - \xi_j)(\xi - \xi_{j+1})(D_+ U_j)(D_- R_j) \chi_j$$

$$= -U_\Delta R_\Delta + \sum_{j \in Z} (\xi - \xi_j)(\xi - \xi_{j+1})(D_+ U_j)(D_- R_j) \chi_j.$$ 

This identity then implies

$$\left( \frac{\partial H_\Delta}{\partial t} + U_\Delta R_\Delta \right) = \sum_{j \in Z} (2\xi - \xi_j - \xi_{j+1})(D_+ U_j)(D_- R_j) \chi_j,$$

almost everywhere. Combining the above identities we can estimate the error in the $V$-norm as follows,

$$\left\| \frac{\partial H_\Delta}{\partial t} + U_\Delta R_\Delta \right\|_V$$

$$\leq \Delta \xi^2 \sum_{j \in Z} \|D_+ U_j\|_2 \|D_- R_j\|_2 + \left( \Delta \xi \sum_{j \in Z} \Delta \xi^2 \|D_+ U_j\|_2^2 \|D_- R_j\|_2^2 \right)^{1/2}$$

$$\leq \Delta \xi \|D_+ U\|_2 \|D_- R\|_2 + \Delta \xi \|D_+ U\|_\infty \|D_- R\|_\infty$$

$$\leq \Delta \xi \left( \|D_+ U\|_2 + \|D_+ U\|_\infty \right) \left( \|D_+ y\|_2 \|Q\|_2 \|U\|_\infty + \|U\|_\infty \right).$$

Now, for the relations (4.9) the correct norm to measure the error in is the $L^2$-norm, as the functions $R$ and $Q$ are supposed to be in $H^1$. From (4.38) we obtain the relation

$$\frac{\partial y_\Delta}{\partial \xi} Q_\Delta - \frac{\partial R_\Delta}{\partial \xi} - U_\Delta \frac{\partial U_\Delta}{\partial \xi} = \sum_{j \in Z} (\xi - \xi_j) [(D_+ y_j)(D_+ Q_j) - (D_+ U_j)^2] \chi_j,$$

and take $L^2$-norm to find

$$\left\| \frac{\partial y_\Delta}{\partial \xi} Q_\Delta - \frac{\partial R_\Delta}{\partial \xi} - U_\Delta \frac{\partial U_\Delta}{\partial \xi} \right\|_{L^2}$$

$$\leq \Delta \xi \left( \|D_+ y\|_2 \|D_+ Q\|_2 + \|D_+ U\|_\infty \right).$$

Finally, using (4.38) once more we have

$$\frac{\partial y_\Delta}{\partial \xi} R_\Delta - \frac{\partial S_\Delta}{\partial \xi} - \frac{\partial H_\Delta}{\partial \xi} - \rho_{\infty} R_\Delta = \sum_{j \in Z} (\xi - \xi_{j+1})(D_- R_j)(D_+ y_j) \chi_j.$$
which can be estimated as
\[
\left\| \frac{\partial y_\Delta}{\partial \xi} R_\Delta - \frac{\partial S_\Delta}{\partial \xi} - \frac{\partial H_\Delta}{\partial \xi} - \rho_\infty \bar{r}_\Delta \right\|_{L^2} \leq \Delta \xi \| D_Y \|_{\ell^*} \| D_R \|_{\ell^*}.
\]

The estimate (6.4) is exactly as we want it, (4.8c) is satisfied in the appropriate norm up to some small remainder. However, the estimates (6.5) and (6.6) require some more work, as we shall see next.

Let us estimate the \( E \)-norm of the difference between \( X_\Delta(T) \) and the exact solution \( X(T) := (\zeta, U, H, \bar{r})(T) \). From the above estimates and (4.8), we find
\[
\|(\zeta_\Delta - \zeta)(T, \cdot)\|_V \leq \|(\zeta_\Delta - \zeta)(0, \cdot)\|_V + \int_0^T \|(U_\Delta - U)(t, \cdot)\|_V dt
\]
\[
\|(U_\Delta - U)(T, \cdot)\|_{H^1} \leq \|(U_\Delta - U)(0, \cdot)\|_{H^1} + \int_0^T \|(Q_\Delta - Q)(t, \cdot)\|_{H^1} dt
\]
\[
(6.7) \quad \|(H_\Delta - H)(T, \cdot)\|_V \leq \|(H_\Delta - H)(0, \cdot)\|_V + \int_0^T \|(U_\Delta R_\Delta - U R)(t, \cdot)\|_V dt + \Delta \xi C_H(\|Y_\Delta\|) T^2,
\]
\[
\|(\bar{r}_\Delta - \bar{r})(T, \cdot)\|_{L^2} \leq \|(\bar{r}_\Delta - \bar{r})(0, \cdot)\|_{L^2} + \rho_\infty \int_0^T \left\| \frac{\partial (U_\Delta - U)(t, \cdot)}{\partial \xi} \right\|_{L^2} dt,
\]
where we have used that the final expression in (6.4) can be bounded by \( \Delta \xi C_H(\|Y_\Delta\|) \) for some constant \( C_H \) depending only on \( \|Y_\Delta\| \).

From (6.7), it is clear that we need estimates on \( \|Q_\Delta - Q\|_{H^1}, \|R_\Delta - R\|_{L^\infty}, \) and \( \|R_\Delta - R\|_{L^2} \) in terms of
\[
\|X_\Delta - X\|_E = \|(\zeta_\Delta - \zeta)\|_V + \|(U_\Delta - U)\|_{H^1} + \|(H_\Delta - H)\|_V + \|(\bar{r}_\Delta - \bar{r})\|_{L^2},
\]
and by definition of the \( H^1 \)-norm and the Sobolev inequality (4.25) it will be sufficient to bound \( \|Q_\Delta - Q\|_{H^1} \) and \( \|R_\Delta - R\|_{H^1} \). To this end, we note that by the estimates (6.5) and (6.6) it follows
\[
(6.8) \quad \left[ \begin{array}{c}
-\frac{\partial \zeta}{(y_\Delta)^\xi} \\
\frac{\partial x}{(y_\Delta)^\xi}
\end{array} \right] \circ \left[ \begin{array}{c}
R_\Delta \\
Q_\Delta
\end{array} \right] = \left[ \begin{array}{c}
U_\Delta(U_\Delta)_{\xi} \\
(H_\Delta)_{\xi} + \rho_\infty \bar{r}_\Delta
\end{array} \right] + \Delta \xi \left[ \begin{array}{c}
v_\Delta \\
w_\Delta
\end{array} \right]
\]
for some functions \( v_\Delta, w_\Delta \in L^2 \) which are bounded by a constant depending only on the norm \( \|Y_\Delta\| \) of (6.1). Recalling (4.12) and the operators defined in (4.10), we know that \( R(t, \xi) \) and \( Q(t, \xi) \) can be written as
\[
R(t, \xi) = \int_R \kappa[t](\eta, \xi) U U_{\xi}(t, \eta) \, d\eta + \int_R g[t](\eta, \xi)[H_\xi + \rho_\infty \bar{r}] (t, \eta) \, d\eta
\]
\[
= \mathcal{K}(U U_{\xi}) + \mathcal{G}(H_\xi + \rho_\infty \bar{r}),
\]
\[
Q(t, \xi) = \int_R g[t](\eta, \xi) U U_{\xi}(t, \eta) \, d\eta + \int_R \kappa[t](\eta, \xi)[H_\xi + \rho_\infty \bar{r}] (t, \eta) \, d\eta
\]
\[
= \mathcal{G}(U U_{\xi}) + \mathcal{K}(H_\xi + \rho_\infty \bar{r})
\]
with kernels
\[
g[t](\eta, \xi) := \frac{1}{2} e^{-|y(t, \xi) - y(t, \eta)|}, \quad \kappa[t](\eta, \xi) := -\text{sgn}(\xi - \eta) g[t](\eta, \xi).
\]
Due to the obvious similarities between (6.8) and (4.20) we would like to generalize the operator identity (4.10) by replacing \( y(t, \xi) \) with any function \( b(t, \xi) \) such that \( b(t, \cdot) - \text{Id} \in \mathbf{V} \) and \( b_\xi(t, \xi) \geq 0 \), in particular this holds for our \( y_\Delta(t, \xi) \) in (6.3) by virtue of Lemma 3.4. This is can be done, and the unique \( \mathbf{H}^1 \)-solution of

\[
\begin{bmatrix}
-\partial_\xi & b_\xi(t, \xi) \\
 b_\xi(t, \xi) & -\partial_\xi
\end{bmatrix}
\begin{bmatrix}
\phi(t, \xi) \\
\psi(t, \xi)
\end{bmatrix}
= \begin{bmatrix}
\phi(t, \xi) \\
\psi(t, \xi)
\end{bmatrix}
\]

for \( \phi(t, \cdot), \psi(t, \cdot) \in \mathbf{L}^2 \) is then

\[
\phi(t, \xi) = \int_\mathbb{R} \frac{1}{2} e^{-|b(t, \xi) - b(t, \eta)|} |w(t, \eta) - \text{sgn}(\xi - \eta) v(t, \eta)| \, d\eta,
\]

\[
\psi(t, \xi) = \int_\mathbb{R} \frac{1}{2} e^{-|b(t, \xi) - b(t, \eta)|} |v(t, \eta) - \text{sgn}(\xi - \eta) w(t, \eta)| \, d\eta.
\]

Consequently, we can generalize \( \mathcal{G} \) and \( \mathcal{K} \) from (4.10) to be operators from \( \mathbf{V} \times \mathbf{L}^2 \) to \( \mathbf{H}^1 \) as follows,

\[
\mathcal{G}[t, \xi](b - \text{Id}, f) := \int_\mathbb{R} \frac{1}{2} e^{-|b(t, \xi) - b(t, \eta)|} f(\eta) \, d\eta,
\]

\[
\mathcal{K}[t, \xi](b - \text{Id}, f) := -\int_\mathbb{R} \text{sgn}(\xi - \eta) \frac{1}{2} e^{-|b(t, \xi) - b(t, \eta)|} f(\eta) \, d\eta.
\]

Using these operators, we may write the general solutions \( \phi(t, \xi), \psi(t, \xi) \) as

\[
\phi(t, \xi) = \mathcal{K}[t, \xi](b - \text{Id}, v) + \mathcal{G}[t, \xi](b - \text{Id}, w),
\]

\[
\psi(t, \xi) = \mathcal{G}[t, \xi](b - \text{Id}, v) + \mathcal{K}[t, \xi](b - \text{Id}, w).
\]

An argument analogous to [21] Lemma 3.1 then proves that the operators

\[
\mathcal{R}_1[t, \cdot] : (\zeta, U, H, \bar{r}) \mapsto \mathcal{K}[t, \cdot](\zeta, UU_\xi) + \mathcal{G}[t, \cdot](\zeta, H_\xi + \rho_\infty \bar{r})
\]

and

\[
\mathcal{R}_2[t, \cdot] : (\zeta, v, w) \mapsto \mathcal{K}[t, \cdot](\zeta, v) + \mathcal{G}[t, \cdot](\zeta, w)
\]

are locally Lipschitz as operators from \( \mathbf{E} \to \mathbf{H}^1 \) and \( \mathbf{V} \times (\mathbf{L}^2)^2 \to \mathbf{H}^1 \) respectively, and the same is true for

\[
\mathcal{Q}_1[t, \cdot] : (\zeta, U, H, \bar{r}) \mapsto \mathcal{G}[t, \cdot](\zeta, UU_\xi) + \mathcal{K}[t, \cdot](\zeta, H_\xi + \rho_\infty \bar{r})
\]

and

\[
\mathcal{Q}_2[t, \cdot] : (\zeta, v, w) \mapsto \mathcal{G}[t, \cdot](\zeta, v) + \mathcal{K}[t, \cdot](\zeta, w)
\]

Finally turning back to the functions we are interested in, we note that since our interpolants \( R_\Delta \) and \( Q_\Delta \) are solutions of (6.8), they can be written as

\[
R_\Delta(t, \xi) = \mathcal{R}_1[t, \xi](\zeta_\Delta, U_\Delta, H_\Delta, \bar{r}_\Delta) + \Delta \mathcal{R}_2[t, \xi](\zeta_\Delta, v_\Delta, w_\Delta),
\]

\[
Q_\Delta(t, \xi) = \mathcal{Q}_1[t, \xi](\zeta_\Delta, U_\Delta, H_\Delta, \bar{r}_\Delta) + \Delta \mathcal{Q}_2[t, \xi](\zeta_\Delta, v_\Delta, w_\Delta).
\]

These should then be compared to \( R \) and \( Q \) for the exact solution, which now can be restated as

\[
R(t, \xi) = \mathcal{R}_1[t, \xi](\zeta, U, H, \bar{r}),
\]

\[
Q(t, \xi) = \mathcal{Q}_1[t, \xi](\zeta, U, H, \bar{r}).
\]

Since we then may write

\[
Q_\Delta(t, \xi) - Q(t, \xi) = \mathcal{Q}_1(\zeta_\Delta, U_\Delta, H_\Delta, \bar{r}_\Delta) - \mathcal{Q}_1(\zeta, U, H, \bar{r})
\]
it follows from the Lipschitz property that
\[ \|Q(t, \cdot \) - Q(t, \cdot \)\|_{H^1} \leq C_{Q,1}(\|X(\cdot \)\|_{E}, \|X(\cdot \)\|_{E}) \|X(t) - X(t)\|_{E} + \Delta \xi C_{Q,2}(\|Y_{\Delta \xi}\|) \]

for constants \( C_{Q,1}, C_{Q,2} \), and an analogous estimate holds for \( \|R(t, \cdot \) - R(t, \cdot \)\|_{H^1}. \)

Then, from the above estimates, the obvious inequality \( \|f_\xi\|_{L^2} \leq \|f\|_{H^1} \), and \( \|f\|_{V} \leq \frac{2\sqrt{2}}{\sqrt{T}} \|f\|_{H^1} \) coming from (4.25), we may add the equations in (6.7) to obtain
\[
\|X(\cdot \) - X(\cdot \)\|_{E} \leq \|X(0) - X(0)\|_{E} + \Delta \xi C_{1}(\|Y_{\Delta \xi}\|)T + C_{2}(\|Y_{\Delta \xi}\|, \|X\|) \int_{0}^{T} \|X(t) - X(t)\|_{E} dt,
\]

where we have used \( \|X\| := \sup_{0 \leq t \leq T} \|X(t)\|_{E} \) and \( \|X(\cdot \)\|_{E} \leq C(\|Y_{\Delta \xi}\|) \) are bounded by constants depending on \( T \) and the \( E \)-norm of their initial data. In particular, by Theorem 5.6 we know \( \|Y_{\Delta \xi}\| \) is bounded by a constant depending only on \( T, H_\infty, \|\zeta(0)\|_{\ell\infty}, \) \( \) and \( \rho_\infty. \) Grönwall’s inequality then yields the estimate
\[
\|X(\cdot \) - X(\cdot \)\|_{E} \leq C_{3}(\|Y_{\Delta \xi}\|, \|X\|) \[ \|X(0) - X(0)\|_{E} + \Delta \xi C_{1}(\|Y_{\Delta \xi}\|)T \].
\]

This shows that if we choose a projection \( \pi_{\Delta \xi} : E \to e \) such that the corresponding \( X(0) \) given by (6.2) satisfies
\[
\|X(0) - X(0)\|_{E} \xrightarrow{\Delta \xi \to 0} 0,
\]
which must hold for any reasonable discretization of the initial datum, then
\[
\|X(\cdot \) - X(\cdot \)\|_{E} \xrightarrow{\Delta \xi \to 0} 0.
\]

Hence, our interpolants converge to the unique solution of (4.8), as indicated in Theorem 1.2.

Since convergence in Lagrangian coordinates implies convergence in the corresponding Eulerian coordinates, see [17] for details, this shows that interpolated solutions of the discrete two-component Camassa–Holm system can be used to obtain conservative solutions of the 2CH system (1.12). In particular, as conservative solutions of (1.1) are unique according to [3], our discretization of the CH equation corresponds to the unique conservative solution of the CH equation.

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Appendix A. Proof of Proposition 3.1

Proof of (3.4). This is a consequence of the following inequalities,
\[ |a_j|^2 = \frac{1}{\Delta \xi} \Delta \xi |a_j|^2 \leq \frac{1}{\Delta \xi} \Delta \xi \sum_{j \in \mathbb{Z}} |a_j|^2, \]
and
\[ \Delta \xi \sum_{j \in \mathbb{Z}} |a_j|^2 \leq \frac{1}{\Delta \xi} \Delta \xi \sum_{j \in \mathbb{Z}} |a_j| \Delta \xi \sum_{i \in \mathbb{Z}} |a_i| = \frac{1}{\Delta \xi} \left( \Delta \xi \sum_{j \in \mathbb{Z}} |a_j| \right)^2, \]
where we have used the definitions (3.1). \hfill \Box

Proof of (3.5). We rewrite \( (a_j)^2 \) as
\[
(a_j)^2 = \frac{1}{2} \sum_{i=-\infty}^{j-1} ( (a_{i+1})^2 - (a_i)^2 ) - \frac{1}{2} \sum_{i=j}^{\infty} ( (a_{i+1})^2 - (a_i)^2 )
\]
\[
= \frac{\Delta \xi}{2} \left[ \sum_{i=-\infty}^{j-1} (a_{i+1} + a_i)D_+a_i - \sum_{i=j}^{\infty} (a_{i+1} + a_i)D_+a_i \right]
\]
\[
\leq \frac{\Delta \xi}{4} \sum_{i \in \mathbb{Z}} \left( |a_{i+1}|^2 + |a_i|^2 + 2|D_+a_i|^2 \right)
\]
\[
= \frac{1}{2} \|a\|_{\mathbb{Z}_1}^2,
\]
where we have applied (3.2). \hfill \Box

Proof of (3.6). Telescopic cancellations yield
\[
\Delta \xi \sum_{j=m}^{n} (D_+a_j) b_j = \sum_{j=m}^{n} (a_{j+1} - a_j) b_j
\]
\[
= \sum_{j=m}^{n} a_{j+1} b_j - \sum_{j=m}^{n} a_j b_{j-1} - \sum_{j=m}^{n} a_j(b_j - b_{j-1})
\]
\[
= a_{n+1} b_n - a_m b_{m-1} - \Delta \xi \sum_{j=m}^{n} a_j(D_-b_j).
\]
\hfill \Box

A proof of (3.7) follows from the standard Hölder inequality by induction.

Proof of (3.12). Define
\[
h_j := \Delta \xi \sum_{i \in \mathbb{Z}} g_{i,j} f_i
\]
Note that $r < \infty \implies p, q < \infty$, which shows that some configurations are impossible and can be excluded. We deal with the three remaining cases:

(i) $r < \infty$: From the generalized Hölder inequality we obtain

$$|h_j| \leq \Delta \xi \sum_{i \in \mathbb{Z}} \left( |f_i|^{\frac{r}{p}} |g_{i,j}|^{\frac{r}{q}} \right)^\frac{1}{r} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |f_i|^{1 - \frac{r}{p}} |g_{i,j}|^{1 - \frac{r}{q}} \right)^\frac{r}{r - \frac{r}{p}} \times \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^{1 - \frac{r}{q}} \right)^\frac{1}{r - \frac{r}{q}}$$

$$= \left[ \Delta \xi \sum_{i \in \mathbb{Z}} |f_i|^p |g_{i,j}|^q \right]^{\frac{1}{p}} \left[ \Delta \xi \sum_{i \in \mathbb{Z}} |f_i|^p \right]^{\frac{r - q}{rp}} \left[ \left( \sup_{j \in \mathbb{Z}} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right) \right) \right]^{\frac{1}{q}}$$

which implies

$$\Delta \xi \sum_{j \in \mathbb{Z}} |h_j|^r \leq \|f\|_{L^p} \left[ \sup_{j \in \mathbb{Z}} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right) \right]^{\frac{1}{p}} \left( \Delta \xi \sum_{j \in \mathbb{Z}} |g_{i,j}|^q \right)^{\frac{r - q}{rp}} \Delta \xi \sum_{i \in \mathbb{Z}} \left| f_i \right|^p |g_{i,j}|^q$$

$$\leq \|f\|_{L^p} \left[ \sup_{j \in \mathbb{Z}} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right) \right]^{\frac{1}{p}} \left( \Delta \xi \sum_{j \in \mathbb{Z}} |g_{i,j}|^q \right)^{\frac{r - q}{rp}} \Delta \xi \sum_{i \in \mathbb{Z}} \left| f_i \right|^p \Delta \xi \sum_{j \in \mathbb{Z}} |g_{i,j}|^q$$

$$\leq \|f\|_{L^p} \left[ \sup_{j \in \mathbb{Z}} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right) \right]^{\frac{1}{p}} \left( \sup_{j \in \mathbb{Z}} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right) \right)^{\frac{1}{q}} \sup_{i \in \mathbb{Z}} \left( \Delta \xi \sum_{j \in \mathbb{Z}} |g_{i,j}|^q \right)^{\frac{1}{q}}$$

where we have used Fubini’s theorem in the second inequality. Taking $r$-th roots we obtain the result.

(ii) $r = \infty$, $p < \infty$: We find

$$|h_j| \leq \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}| \lesssim \|f\|_{L^p} \left( \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}|^q \right)^{\frac{1}{q}}$$

and taking supremum over $j$ this corresponds to (3.12) where $q/\infty = 0$.

(iii) $r = q = \infty$: We find

$$|h_j| \leq \Delta \xi \sum_{i \in \mathbb{Z}} |g_{i,j}| \|f_i\| \leq \Delta \xi \sum_{i \in \mathbb{Z}} |f_i| \left( \sup_{j \in \mathbb{Z}} |g_{i,j}| \right) \leq \sup_{i \in \mathbb{Z}} \left( \sup_{j \in \mathbb{Z}} |g_{i,j}| \right) \Delta \xi \sum_{i \in \mathbb{Z}} |f_i|,$$

and taking supremum over $j$ this corresponds to (3.12) where $\infty/\infty = 1$. □
Proof of (3.8). Without loss of generality we may assume $k \leq j$ and compute

$$|a_j - a_k| = \left| \Delta \xi \sum_{m=k}^{j-1} D_+ a_m \right|$$

$$\leq \left( \Delta \xi \sum_{m=k}^{j-1} |D_+ a_m|^2 \right)^{1/2} \left( \Delta \xi \sum_{m=k}^{j-1} 1 \right)^{1/2}$$

$$\leq \|D_+ a\|_{l_2} |\Delta \xi (j - k)|^{1/2}.$$  

The result follows from taking supremum over $j$ and $k$.  

Proof of (3.9). We first note

$$\|D_+ a\|_{l_2}^2 = \Delta \xi \sum_{|j| < n} |D_+ a_j|^2 + \Delta \xi \sum_{|j| \geq n} |D_+ a_j|^2 =: l_n + u_n, \quad n \in \mathbb{N},$$

where $l_n \nearrow \|D_+ a\|_{l_2}^2$ and $u_n \searrow 0$ as $n \to +\infty$ by Bolzano–Weierstraß. Furthermore,

$$\sqrt{\Delta \xi} |D_+ a_j| = \left( \Delta \xi |D_+ a_j|^2 \right)^{1/2}$$

$$\leq \min \left\{ \left( \Delta \xi \sum_{k=j}^{+\infty} |D_+ a_k|^2 \right)^{1/2}, \left( \Delta \xi \sum_{k=-\infty}^{j-1} |D_+ a_k|^2 \right)^{1/2} \right\}$$

so that

$$\lim_{j \to \pm \infty} \sqrt{\Delta \xi} |D_+ a_j| \leq \lim_{j \to \pm \infty} \left( \Delta \xi \sum_{|k| \geq |j|} |D_+ a_k|^2 \right)^{1/2} = \lim_{j \to \pm \infty} (u_{|j|})^{1/2} = 0.$$

References

[1] V. I. Arnold and B. A. Khesin. Topological methods in hydrodynamics, volume 125 of Applied Mathematical Sciences. Springer-Verlag, New York, 1998.

[2] J. Borcea, S. Friedland, and B. Shapiro. Parametric Poincaré–Perron theorem with applications. J. Anal. Math., 113:197–225, 2011.

[3] A. Bressan, G. Chen, and Q. Zhang. Uniqueness of conservative solutions to the Camassa-Holm equation via characteristics. Discrete Contin. Dyn. Syst., 35(1):25–42, 2015.

[4] A. Bressan and A. Constantin. Global conservative solutions of the Camassa-Holm equation. Arch. Ration. Mech. Anal., 183(2):215–239, 2007.

[5] A. Bressan and A. Constantin. Global dissipative solutions of the Camassa-Holm equation. Anal. Appl. (Singap.), 5(1):1–27, 2007.

[6] R. Camassa. Characteristics and the initial value problem of a completely integrable shallow water equation. Discrete Contin. Dyn. Syst. Ser. B, 3(1):115–139, 2003.

[7] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. Phys. Rev. Lett., 71(11):1661–1664, 1993.

[8] G. M. Coclite, K. H. Karlsen, and N. H. Risebro. A convergent finite difference scheme for the Camassa-Holm equation with general $H^1$ initial data. SIAM J. Numer. Anal., 46(3):1554–1579, 2008.
[9] A. Constantin and J. Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(2):303–328, 1998.

[10] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181(2):229–243, 1998.

[11] A. Constantin and B. Kolev. Least action principle for an integrable shallow water equation. *J. Nonlinear Math. Phys.*, 8(4):471–474, 2001.

[12] A. Constantin and B. Kolev. On the geometric approach to the motion of inertial mechanical systems. *J. Phys. A*, 35(32):R51–R79, 2002.

[13] A. Constantin and B. Kolev. Geodesic flow on the diffeomorphism group of the circle. *Comment. Math. Helv.*, 78(4):787–804, 2003.

[14] A. Constantin and L. Molinet. Global weak solutions for a shallow water equation. *Comm. Math. Phys.*, 211(1):45–61, 2000.

[15] S. Friedland. Convergence of products of matrices in projective spaces. *Linear Algebra Appl.*, 413(2-3):247–263, 2006.

[16] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D.*, 4(1):47–66, 1981.

[17] M. Grasmair, K. Grunert, and H. Holden. On the equivalence of Eulerian and Lagrangian variables for the two-component Camassa-Holm system. In *Current research in nonlinear analysis*, volume 135 of *Springer Optim. Appl.*, pages 157–201. Springer, Cham, 2018.

[18] K. Grunert. Blow-up for the two-component Camassa-Holm system. *Discrete Contin. Dyn. Syst.*, 35(5):2041–2051, 2015.

[19] K. Grunert and H. Holden. The general peakon-antipeakon solution for the Camassa-Holm equation. *J. Hyperbolic Differ. Equ.*, 13(2):353–380, 2016.

[20] K. Grunert, H. Holden, and X. Raynaud. Global conservative solutions to the Camassa-Holm equation for initial data with nonvanishing asymptotics. *Discrete Contin. Dyn. Syst.*, 32(12):4209–4227, 2012.

[21] K. Grunert, H. Holden, and X. Raynaud. Global solutions for the two-component Camassa–Holm system. *Comm. Partial Differential Equations*, 37(12):2245–2271, 2012.

[22] K. Grunert, H. Holden, and X. Raynaud. Global dissipative solutions of the two-component Camassa-Holm system for initial data with nonvanishing asymptotics. *Nonlinear Anal. Real World Appl.*, 17:203–244, 2014.

[23] H. Holden and X. Raynaud. Convergence of a finite difference scheme for the Camassa-Holm equation. *SIAM J. Numer. Anal.*, 44(4):1655–1680, 2006.

[24] H. Holden and X. Raynaud. Global conservative multi-peakon solutions of the Camassa-Holm equation. *J. Hyperbolic Differ. Equ.*, 4(1):39–64, 2007.

[25] H. Holden and X. Raynaud. Global conservative solutions of the Camassa-Holm equation—a Lagrangian point of view. *Comm. Partial Differential Equations*, 32(10-12):1511–1549, 2007.

[26] H. Holden and X. Raynaud. Dissipative solutions for the Camassa-Holm equation. *Discrete Contin. Dyn. Syst.*, 24(4):1047–1112, 2009.

[27] M. Pituk. More on Poincaré’s and Perron’s theorems for difference equations. *J. Difference Equ. Appl.*, 8(3):201–216, 2002.

[28] G. Teschl. *Jacobi operators and completely integrable nonlinear lattices*, volume 72 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.