LONG ABELIAN IDEALS

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Let \( \mathfrak{b} \) be a Borel subalgebra of a simple Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{A} \mathfrak{b} \) denote the set of all Abelian ideals of \( \mathfrak{b} \). It is easily seen that any \( \alpha \in \mathfrak{A} \mathfrak{b} \) is actually contained in the nilpotent radical of \( \mathfrak{b} \). Therefore \( \alpha \) is determined by the corresponding set of roots. More precisely, let \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g} \) lying in \( \mathfrak{b} \) and let \( \Delta \) be the root system of the pair \( (\mathfrak{g};\mathfrak{t}) \). Choose the system of positive roots, \( \Delta^+ \), so that the roots of \( \mathfrak{b} \) are positive. Then \( \alpha = \gamma_2 I_{\mathfrak{g} \gamma} \), where \( I \) is a suitable subset of \( \Delta^+ \) and \( \gamma_2 \) is the root space for \( \gamma \Delta^+ \).

A nice result of D. Peterson says that the cardinality of \( \mathfrak{A} \mathfrak{b} \) is \( 2^{\text{rk} \mathfrak{g}} \). His approach uses a one-to-one correspondence between the Abelian ideals and the so-called ‘minuscule’ elements of the affine Weyl group \( \hat{\mathfrak{W}} \) (see Section 1 for precise definitions). An exposition of Peterson’s results is found in [3]. Peterson’s work appeared to be the point of departure for active recent investigations of Abelian ideals, and related problems of representation theory and combinatorics [1], [2], [3], [7], [8]. Our definition of minuscule elements follows Kostant’s paper [3], so that \( w_2 \hat{\mathfrak{W}} \) is minuscule in our sense if and only if \( w_1 \) is minuscule in the sense of Cellini–Papi [1], [2]. An elegant proof of Peterson’s theorem is given in [1]. Let \( \mathcal{A} \) be the fundamental alcove of \( \hat{\mathfrak{W}} \). Cellini and Papi show that \( w_2 \hat{\mathfrak{W}} \) is minuscule if and only if \( w_1 \mathcal{A} = 2\mathcal{A} \). Since \( 2\mathcal{A} \) consists of \( 2^{\text{rk} \mathfrak{g}} \) alcoves and \( \hat{\mathfrak{W}} \) acts simply transitively on the set of alcoves, Peterson’s theorem follows.

In this paper, we first show that methods of [1] can be adapted to solving the following problems:

Suppose \( \mathfrak{g} \) has two root lengths.

- describe (enumerate) the Abelian ideals such that the corresponding set \( I \) consists only of long roots (such Abelian ideals are said to be long);
- give a characterization of the corresponding (= long) minuscule elements.

Write \( \mathfrak{A} \mathfrak{b} l \) (resp. \( \mathfrak{M} l \)) for the set of long Abelian ideal (resp. long minuscule elements). We give a uniform answer to both problems. Let \( d > 1 \) be the ratio of squares of root lengths and \( a \) the number of long simple roots. Then

\[
\# (\mathfrak{A} \mathfrak{b} l) = d^a ;
\]

To obtain a characterisation of the long minuscule elements, we consider a certain simplex \( \mathcal{A}_s \) that lies between \( \mathcal{A} \) and \( 2\mathcal{A} \). Then the answer is that \( w_2 \mathfrak{M}_l \) if and only if \( w_1 \mathcal{A} \prec \mathcal{A}_s \).

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and only if any reduced decomposition of $w$ contains no short simple reflections (we say that a reflection is short, if it corresponds to a short root). We also describe the rootlets of long Abelian ideals.

Second, we show that the theory of long Abelian ideals is closely related to describing of commutative subalgebras in the isotropy representations of some symmetric spaces. Let $\theta$ be the highest root and $\theta_s$ the short dominant root of $\Delta^+$. Write $\overline{V}(\lambda)$ for the $g$-module with highest weight $\lambda$. Assume that $\theta_j^2 = \theta_j = 2$, i.e., we exclude the case of $G_2$. Then $\overline{g} = g \overline{V}(\theta_s)$ has a structure of a $Z_2$-graded Lie algebra, so that it makes sense to speak about commutative subalgebras of $\overline{V}(\theta_s)$. In Section 3 we prove that there is a natural one-to-one correspondence between the $b$-stable commutative subalgebras of $\overline{V}(\theta_s)$ and the long Abelian $b_-$-ideals in $g^-$. Here $g^-$ is the Langlands dual Lie algebra for $g$.

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1. Notation and other preliminaries

(1.1) Main notation. $\Delta$ is the root system of $(g; t)$ and $W$ is the usual Weyl group. For $\alpha \in \Delta$, $g_\alpha$ is the corresponding root space in $g$.

$\Delta^+$ is the set of positive roots and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

$\Pi = \{ \alpha_1 ; \ldots ; \alpha_p \} \uparrow$ is the set of simple roots in $\Delta^+$.

$\Phi_1 ; \ldots ; \Phi_p$ are the fundamental weights corresponding to $\Pi$. We set $V := t_e = \bigwedge_{i=1}^p R \alpha_i$ and denote by $(;)$ a $W$-invariant inner product on $V$. As usual, $\mu^\perp = 2\mu - \mu \mu$ is the coroot for $\mu \in \Delta$. Letting $\overline{V} = V \otimes \delta \otimes R \lambda$, we extend the inner product $(;)$ on $\overline{V}$ so that $(\delta_i ; V) = (\delta_i ; V) = (\delta_i ; \delta) = (\lambda ; \lambda) = 0$ and $(\delta ; \lambda) = 1$.

$\overline{H} = \frac{\Delta + k \delta}{j} / Z g$ is the set of affine real roots and $\overline{V}$ is the affine Weyl group. Then $\overline{H}^+ = \Delta^+ / \{ \Delta^+ + k \delta \}$ is the set of positive affine roots and $\overline{H} = \Pi / \{ \delta_0 \}$ is the corresponding set of affine simple roots. Here $\delta_0 = \delta - \theta$, where $\theta$ is the highest root in $\Delta^+$. The inner product $(;)$ on $\overline{V}$ is $\overline{H}$-invariant.

For $\alpha_0 (0 \ i \ p)$, we let $s_i$ denote the corresponding simple reflection in $\overline{V}$. The length function on $\overline{V}$ with respect to $s_0 ; s_1 ; \ldots ; s_p$ is denoted by $\prime$. We consider two different actions of $\overline{V}$: the linear action on $\overline{V}$ and the affine-linear action on $V$. To distinguish these two, we use dot ‘.’ for denoting the second action. For any $w \in \overline{V}$, we set $\overline{V}(w) = \delta_0 \ 2 \ \overline{H}^+ \ jw (\alpha) 2 \ \overline{H}^+ + g$.

Our convention concerning $\overline{H}(w)$ is the same as in [3], [7] but is opposite to that in [1], [2].

(1.2) Abelian ideals. Let $b$ be an Abelian ideal. It is easily seen that $a \supseteq b$. Therefore $a = \bigcup_{\alpha \in I} g_\alpha$ for a subset $I \subseteq \Delta^+$, which is called the set of roots of $a$. In what follows, an Abelian ideal will be identified with the respective set of roots. That is, $I$ is said to be an Abelian ideal, too. We also say that $a$ is a geometric Abelian ideal, while $I$ is a combinatorial
Abelian ideal. In the combinatorial context, the definition of an Abelian ideal can be stated as follows.

I $\Delta^+$ is an Abelian ideal, if the following two conditions are satisfied:

(a) for any $\mu, \nu \in I$, we have $\mu + \nu \in \Delta$;

(b) if $\gamma \in I$, $\nu \in \Delta^+$, and $\gamma + \nu \in \Delta$, then $\gamma + \nu \in I$.

Following D. Peterson, an element $w \in \Psi$ is said to be minuscule, if $\Psi (w)$ is of the form $f^\delta \gamma \Psi (w)g$, where $I$ is a subset of $\Delta^+$. It was shown by Peterson that such an $I$ is a combinatorial Abelian ideal and, conversely, each Abelian ideal occurs in this way, see [1, Prop. 2.8], [3]. Hence one obtains a one-to-one correspondence between the Abelian ideals of $\mathfrak{b}$ and the minuscule elements of $\Psi$. The set of minuscule elements in $\Psi$ is denoted by $M$. If $w \in M$, then we write $I_w$ (resp. $a_w$) for the corresponding combinatorial (resp. geometric) Abelian ideal. That is,

$I_w = f^\gamma 2 \Delta^+ \gamma ^{\delta} \gamma 2 \Psi (w)g$ and $a_w = \alpha 2 I_w \alpha :$

Conversely, given $I \in \mathfrak{b}$, we write $wI$ for the respective minuscule element. Notice that

$\dim a_w = \# (I_w) = l(w) :$

Throughout the paper, $I$ or $I_w$ stands for a combinatorial Abelian ideal.

2. LONG MINUSCULE ELEMENTS AND LONG ABELIAN IDEALS

From now on, we assume that $\Delta^+$ has two root lengths. To distinguish different objects related to long and short roots, we use the subscripts ‘$l$’ and ‘$s$’. For instance, $\Pi_l$ is the set of long simple roots and $\Delta^+_l$ is the set of long positive roots. Accordingly, each simple reflection $s_i$ is either short or long. Since $\theta$ is long, the reflection $s_0$ is regarded as long. Write $\theta_s$ for the unique short dominant root in $\Delta^+_s$.

We say that $I \in \mathfrak{b}$ is long, if $I \subseteq \Delta^+_l$. Write $\mathfrak{b}_l$ for the set of all long Abelian ideals. The corresponding subset of $M$ is denoted by $M_l$. Notice that the analogous notion of a short Abelian ideal does not make sense, for any non-empty Abelian ideal contains the highest root $\theta$, which is long.

2.1 Proposition. Suppose $w \in M$. Then $w \in M_l$ if and only if any reduced decomposition of $w$ does not contain short simple reflections.

Proof. ‘$
\gamma$
’ Suppose $w = w_{\alpha_1}^0 w_{\alpha_2}^0 $, where $\alpha_i$ is a short simple root. As was noticed in [2, Sect. 2], every right substring in a reduced decomposition of $w$ is again a minuscule element and the corresponding (combinatorial) ideal is a subset of $I_w$. In particular, we have $s_i w_{\alpha_1}^0 w_{\alpha_2}^0 \in M$. Furthermore,

$I_{s_i w_{\alpha_1}^0 w_{\alpha_2}^0} = f^\gamma g [ I_{w_{\alpha_1}^0 w_{\alpha_2}^0} ;$

where $w_{\alpha_1}^0 \gamma = $ $\alpha_i$. It follows that $\gamma$ is a short root lying in $I_{s_i w_{\alpha_1}^0 w_{\alpha_2}^0}$. Since $\gamma$ is long, we conclude that $w \in M_l$.

‘$
\gamma$
’ Agree by induction on the length of $w$. If $w_{\alpha_1}^0 2 M_l$, $w \in M$, and $w = s_i w_{\alpha_2}^0$ for a long reflection $s_i$, then $I_w = f^\gamma g [ I_{w_{\alpha_2}^0}$ where $w_{\alpha_1}^0 \gamma = $ $\alpha_i$. Since $\gamma$ is long, we conclude that $w \in M_l$.
Consider the collection of affine hyperplanes in \( V \)
\[
H_{\mu, k} = \text{fix } 2 \; V \; \langle \mu; x \rangle = k g ;
\]
where \( \mu \in \Delta^+ \) and \( k \in \mathbb{Z} \). The connected components of \( V \) \( n \) \( H_{\mu, k} \) are called alcoves. It is well-known that \( \mathcal{A} \), as group of affine transformations of \( V \), acts simply-transitively on the set of alcoves. Recall that the fundamental alcove of \( \mathcal{A} \) is
\[
\mathcal{A} = \text{fix } 2 \; V \; \langle \alpha; x \rangle > 0 \; \text{and} \; (\alpha; \theta) < 1 g :
\]
Set
\[
\mathcal{A}_s = \text{fix } 2 \; V \; \langle \alpha; x \rangle > 0 \; \text{and} \; (\alpha; \theta_s) < 1 g :
\]
It is clear that \( \mathcal{A}_s \) is a union of (finitely many) alcoves and it contains \( \mathcal{A} \).

2.2 Proposition. Let \( w \in \mathcal{W} \) be a minuscule element. We have \( I_w 2 \mathcal{A}_\theta \) if and only if \( w^{-1} \mathcal{A} = \mathcal{A}_s \).

Proof. We use the following result proved in [1, 1.1]:
\[
\mu + k \delta 2 N (w) , \quad H_{\mu, k} \text{ separates } \mathcal{A} \text{ and } w^{-1} \mathcal{A} \quad (k > 0)
\]
\[
\mu + k \delta 2 N (w) , \quad H_{\mu, k} \text{ separates } \mathcal{A} \text{ and } w^{-1} \mathcal{A} \quad (k < 0) ;
\]
We also know that \( w^{-1} \mathcal{A} = 2 \mathcal{A}_s \), since \( w \in \mathcal{M} \).

(a) If \( w^{-1} \mathcal{A} \neq \mathcal{A}_s \) then the hyperplane \( H_{\theta, \mu} \) separates \( \mathcal{A} \) and \( w^{-1} \mathcal{A} \). Hence \( \delta \theta 2 N (w) \), i.e., \( \theta_s 2 I_w \).

(b) Conversely, if \( w^{-1} \mathcal{A} = \mathcal{A}_s \) then any hyperplane \( H_{\mu, k} \) separating \( \mathcal{A} \) and \( w^{-1} \mathcal{A} \) must meet \( \mathcal{A}_s \). If \( x \mathcal{A}_s \setminus H_{\mu, k} \) and \( \mu \in \Delta^+ \), then \( 0 < (x; \mu) = (\alpha; \theta_s) < 1 \), which is impossible. Hence the hyperplanes \( H_{\mu, \pm} \), with \( \mu \) short, do not separate \( \mathcal{A} \) and \( w^{-1} \mathcal{A} \). Thus, \( \mathcal{A} \setminus \Delta^+ = \emptyset \).

It follows that the number of long Abelian ideals is equal to the number of alcoves that fit in \( \mathcal{A}_s \). In other words,
\[
\# (\mathcal{A}_s) = \text{vol } (\mathcal{A}_s) = \text{vol } (\mathcal{A}) : \tag{2.3}
\]

2.4 Theorem.
\[
\frac{\text{vol } (\mathcal{A}_s)}{\text{vol } (\mathcal{A})} = \frac{\Theta_s^2}{\Theta_s^2} \prod_{i} \frac{m_i}{\Theta_s} .
\]

Proof. Write \( \theta = \sum_{i=1}^{p} m_i \alpha_i \) and \( \theta_s = \sum_{i=1}^{p} c_i \alpha_i \). Then
\[
\frac{\text{vol } (\mathcal{A}_s)}{\text{vol } (\mathcal{A})} = \prod_{i=1}^{p} \frac{\phi_i; \theta_s}{(\phi_i; \theta_s)} = \prod_{i=1}^{p} \frac{m_i}{c_i} ;
\]
Consider the dual root system \( \Delta^- \). We have \( f \alpha^- 2 \Pi g \) is a set of simple roots, \( \theta^-_s \) is the highest root in \( \Delta^- \), and
\[
\theta^-_s = \frac{2}{\theta_s; \theta_s} \sum_{i=1}^{p} c_i \alpha_i = \sum_{\alpha_i 2 \Pi_s} c_i \alpha_i^- + \sum_{\alpha_i 2 \Pi_s} c_i \frac{\Theta_s^2}{\Theta_s^2} \alpha_i^- ;
\]
On the other hand, the collection of the coefficients of the highest root in \( \Delta^- \) is the same as in \( \Delta \). Hence
\[
\prod_{i=1}^{p} m_i = \prod_{i=1}^{p} c_i \left( \frac{\theta_i^2}{\theta_i^2} \right)^{\#I_i};
\]
and we are done.

2.5 Corollary. The number of long Abelian ideals equals:
- \( 2^{p-1} \) for \( \mathfrak{so}_{2p+1} \);
- \( 2 \) for \( \mathfrak{sp}_{2p} \);
- \( 4 \) for \( \mathfrak{F}_4 \);
- \( 3 \) for \( \mathfrak{G}_2 \).

In [7], we introduced the notion of the rootlet of an Abelian ideal. Let \( I_w, w \in \mathcal{M} \), be a non-trivial Abelian \( \mathfrak{b} \)-ideal. Then \( \tau(\theta_w) = w(\alpha_0) + \delta \) is a long positive root [7, 2.4], which is called the rootlet of \( I_w \). It is natural to inquire as to what the rootlets of long Abelian ideals are.

2.6 Proposition. Suppose \( I \) is a nontrivial Abelian ideal. Then \( I \neq \mathfrak{A} \mathfrak{b} \mathfrak{n} \mathfrak{f} ? \mathfrak{g} ! \Delta \) if and only if \( \theta \neq \tau(\theta) \) is a certain sum of only long simple roots, i.e., \( \theta \) and \( \tau(\theta) \) have equal coefficients of all short simple roots.

Proof. ‘(’) By Proposition 2.1, a reduced decomposition of \( w = w_1 \mathcal{H}_I \) contains only reflections \( s_0 \) and \( s_i, i > 0 \), for \( \alpha_i \neq I \). We argue by induction on ‘(\( w \) ).

(a) If ‘(\( w \)) = 1, then \( w = s_0 \) and \( s_0(\alpha_0) + \delta = \theta \).

(b) Suppose \( w \neq \mathcal{M}_I \) and \( w(\alpha_0) + \delta = \gamma \Delta^+ \). If \( s_i w \neq \mathcal{M} \), then \( s_i(\theta - \delta) + \delta = s_i(\gamma) \). Consider two possibilities for \( s_i \). If \( s_i \) corresponds to \( \alpha_i \neq I \), then \( \gamma \) and \( s_i(\gamma) \) have equal coefficients of the short simple roots. If \( s_i = s_0 \), then
\[
s_0(\gamma) = \begin{cases} 
\gamma & \text{if } (\gamma;\theta) = 0 \\
\delta & \text{if } (\gamma;\theta) \neq 0.
\end{cases}
\]

As the last case is impossible in the minuscule situation (for, we would obtain a root which is not in \( \Delta^+ \)), the induction step is complete.

‘(‘ Assume that a reduced decomposition of \( w \) contains a short reflection. Then the first occurrence of short reflections will certainly reduce some short simple root coefficients of the current rootlet. It is conceivable that a consequent occurrence of short reflections would restore the previous coefficients of the short simple roots. However, it is not possible in view of [7, Corollary 3.3]. Thus, the reduced decompositions of \( w \) cannot contain short simple reflections.

In [7], we studied the poset structure of the fibres of the mapping \( \tau : \mathfrak{A} \mathfrak{b} \mathfrak{n} \mathfrak{f} ? \mathfrak{g} ! \Delta^+ \) that takes a non-trivial Abelian ideal to its rootlet. By the previous proposition, we have the set of non-trivial long Abelian ideals is the union of fibres of this mapping.

2.7 Examples. We use the notation and numbering of roots as in [7, Tables].

1. \( \mathfrak{g} = \mathfrak{sp}_{2p} \). Here \( \theta = 2\varepsilon_1 = 2\varphi_1 \) and \( \theta_s = \varepsilon_1 + \varepsilon_2 = \varphi_2 \). Normalize the bilinear form \( (,;) \) so that \( \varepsilon_1(\varepsilon_1) = 1 \). Then \( \mathcal{A} \) has the vertices \( \varepsilon_1 \varepsilon_1 + \varepsilon_2 ; \ldots ; \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_p \), whereas \( \mathcal{A}_s \) has the vertices \( 2\varepsilon_1 \varepsilon_1 + \varepsilon_2 ; \ldots ; \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_p \). The unique non-trivial long Abelian ideal is \( \mathfrak{f} \theta \mathfrak{g} \).
2. $g = \mathfrak{so}_{2p+1}$. Here $\theta = \varepsilon_1 + \varepsilon_2 = \varphi_2$ and $\theta_s = \varepsilon_1 = \varphi_1$. The unique maximal long Abelian ideal consists of the roots $\varepsilon_i + \varepsilon_j$ for $i < j$. The only generator is $\varepsilon_{p+1} + \varepsilon_p$. The corresponding rootlet is also $\varepsilon_{p+1} + \varepsilon_p = \alpha_{p+1} + 2\alpha_p$.

3. $g = \mathbf{F}_4$. Here $\theta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = \varphi_4$ and $\theta_s = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = \varphi_1$. The non-trivial long Abelian ideals appear in the first three rows of Table 1 in [7]. The unique maximal long Abelian ideal is $\ell \theta; \theta, \alpha_4 \theta, \alpha_4, \alpha_3 \varphi$.

4. $g = \mathbf{G}_2$. The non-trivial long Abelian ideals are $\ell 3\alpha_1 + 2\alpha_2 \varphi, \ell 3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2 \varphi$.

### 3. Long Abelian Ideals and Little Adjoint Representations

In this section, we elaborate on a relationship between long Abelian ideals and commutative subalgebras in the isotropy representations of some symmetric spaces.

We still assume that $g$ has two root lengths. Then the $g$-representation with highest weight $\theta$, is said to be little adjoint. The corresponding $g$-module is denoted by $V(\theta_s)$. The properties of the $g$-module $V(\theta_s)$ are similar to that of $g = V(\theta)$. For instance, the set of nonzero weights of $V(\theta_s)$ is $\Delta_s$; the dimension of the zero-weight space is $\#(\Pi_s)$ and each weight space corresponding to $\mu \in \Delta_s^+$ is one-dimensional, see [6, 2.8].

From now on, we stick to the case, where $\frac{\theta_{s-}}{\theta_s} = 2$. Then $V(\theta_s)$ is the isotropy representation of a symmetric space; more precisely,

$$\tilde{\gamma} = g \ V(\theta_s)$$

has a natural structure of $\mathbb{Z}_2$-graded simple Lie algebra. The following table shows all possibilities for $\theta_s$ and $\tilde{\gamma}$.

| $g$     | $\mathfrak{so}_{2p+1}$ | $\mathfrak{sp}_{2p}$ | $\mathbf{F}_4$ |
|---------|------------------------|-----------------------|---------------|
| $\theta_s$ | $\varphi_1$ | $\varphi_2$ | $\varphi_4$ |
| dim $V(\theta_s)$ | $2p + 1$ | $2p^2$ | $p$ | $1$ | $26$ |
| $\tilde{\gamma}$ | $\mathfrak{so}_{2p+2}$ | $\mathfrak{sp}_{2p}$ | $\mathbf{E}_6$ |

Hence, given a subspace $a \ V(\theta_s)$, it makes sense to say that it can be a commutative subalgebra (of $\tilde{\gamma}$).

The problem of describing Abelian $\mathfrak{b}$-ideals can be generalized to the setting of $\mathbb{Z}_2$-graded Lie algebras as follows:

*Suppose $\mathfrak{h} = h \ V$ is a semisimple $\mathbb{Z}_2$-graded Lie algebra, i.e., $V$ is an $h$-module and $\mathfrak{h} \ V \mathfrak{h}$. Let $\mathfrak{b}(\mathfrak{h})$ be a Borel subalgebra of $\mathfrak{h}$. Describe (enumerate) the $\mathfrak{b}(\mathfrak{h})$-stable commutative subalgebras in $\mathfrak{h}$.*

The interest of commutative subalgebras in $\mathfrak{h}$ is explained by the fact that if $\dim a = k$, then the corresponding decomposable $k$-vector in $^k\mathfrak{h}V$ is an eigenvector a Casimir element of $\mathfrak{h}$ whose eigenvalue is maximal possible (and equal $k=2$); the converse is also true, see [5, Sect.4]. I do not think that an answer to the above problem can be given in a
uniform way (see also examples below). But, for $\mathbb{Z}_2$-gradings connected with little adjoint representations, there is a reasonably nice description.

Let $\mathfrak{g}^-$ denote the Langlands dual Lie algebra for $\mathfrak{g}$, i.e., the root system of $\mathfrak{g}^-$ is $\Delta^-$. We have $(\Delta^+_1)^- = (\Delta^-_1)^+$. Notice that $(\Delta^+_1)^- = (\Delta^-_1)^+$ and $\mathcal{V}(\Delta^-)$ is the little adjoint module for $\mathfrak{g}^-$. Given a set $S$ of short roots in $\Delta^-$, we write $a(S)$ for the respective subspace of $\mathcal{V}(\Delta^-)$.

3.1 Lemma. If $a$ is a commutative $\mathfrak{b}^-$-stable subalgebra of $\mathcal{V}(\Delta^-)$, then $a = a(S)$ for some $S$ $(\Delta^+_1)^-$.  

**Proof.** It follows from [5, Prop. 4.9] that $a$ has no zero weight and $a = a(S)$ for some $S$ lying in an open halfspace of $\mathcal{V}$. However, if $\mu - 2 S$ for some $\mu 2 (\Delta^+_1)^-$, then the $\mathfrak{b}^-$-invariance of $a$ implies that it has a nonzero component in the zero-weight space of $\mathcal{V}(\Delta^-)$.

The following result asserts that there is a bijection between the long Abelian $\mathfrak{b}$-ideals in $\mathfrak{g}$ and the $\mathfrak{b}^-$-stable commutative subalgebras of $\mathcal{V}(\Delta^-)$.

3.2 Theorem. 

$I$ $\Delta^+_1$ is an Abelian ideal , $a(\Delta^-)$ is a $\mathfrak{b}^-$-stable commutative subalgebra in $\mathcal{V}(\Delta^-)$.

**Proof.** 

1. Suppose $\mu 2 I$, $\beta 2 \Delta^+$, and $\mu - \beta - 2 \Delta^-$. 

- If $\beta$ is long, then $\mu - \beta = (\mu + \beta) - 2 I^-$. 

- If $\beta$ is short, then $\mu + \beta 2 I$ is a short root, which is impossible. 

Thus, the space $a(\Delta^-)$ is $\mathfrak{b}^-$-stable.

2. Let $\mu, \beta 2 I$ and assume that $\mu + \beta - 2 \Delta^-$. Since $I$ is Abelian, we have $\mu + \beta \in \Delta$. Then $\mu + \beta = 2\gamma$, where $\gamma 2 \Delta$. Therefore $\mu(\beta) = 0$ and $\mu(\gamma) = (\beta, \gamma) > 0$. Hence $\gamma \mu(\beta) = 2 \Delta$ and $\gamma + (\beta, \beta) = 0$. Because one of these roots is positive, we conclude that $\gamma 2 I \setminus \Delta$. This contradiction proves that $\mu + \beta \notin \Delta$, i.e., $a(\Delta^-)$ is a commutative subalgebra.

The argument is similar. One has only use the following property of this $\mathbb{Z}_2$-grading:

Suppose $\nu, \gamma 2 (\Delta^-)$, and $\nu, \gamma$ are corresponding weight vectors in $\mathcal{V}(\Delta^-)$. If $\nu + \gamma 2 \Delta^-$, then $0 \notin \nu, \gamma \gamma 2 \mathfrak{g}^-$. 

Since $(\mathfrak{sp}_{2p+1})^- = \mathfrak{sp}_{2p}$ and $\mathfrak{F}_4 = \mathfrak{F}_4$, we see that the number of $\mathfrak{b}$-stable commutative subalgebras is equal to 

- $2$ for the $\mathfrak{sp}_{2p}$-module $\mathcal{V}(\Phi_2)$; 
- $2$ for the $\mathfrak{sp}_{2p+1}$-module $\mathcal{V}(\Phi_1)$; 
- $2$ for the $\mathfrak{F}_4$-module $\mathcal{V}(\Phi_4)$. 

One should not think, however, that if $\mathfrak{h} = \mathfrak{h}$ $\mathcal{V}$ is a $\mathbb{Z}_2$-grading, then the number of $\mathfrak{b}(\mathfrak{h})$-stable commutative subalgebras of $\mathcal{V}$ is always a power of 2.
Examples. Straightforward computations give us the following.

1. If $\hat{h} = g_1 h_n$ and $h = g_1 h r$, then there are \( n + 1 \) \( n \) commutative \( b(h) \)-stable subalgebras in $V$.

2. If $\hat{h} = s_2 p + 1$ and $h = s_2 p + 1$, then the number of such commutative subalgebras is $2^{p+1}$.

3.3 Remark. In the previous exposition the case of $g = G_2$ is omitted, and the reason is that the little adjoint $G_2$-module is associated with a certain $\mathbb{Z}_3$-grading. Namely, there is an automorphism $\sigma$ of order 3 of $so_8$ such that the fixed-point subalgebra is $G_2$ and two other eigenspaces of $\sigma$ are little adjoint $G_2$-modules. That is,

$$so_8 = G_2 \circ \mathfrak{w} \circ \mathfrak{w}^0;$$

where $\mathfrak{w} \circ \mathfrak{w}^0 \circ \mathfrak{w} (\theta_3)$ and $[\mathfrak{w} \circ \mathfrak{w}] \circ \mathfrak{w}^0, [\mathfrak{w}^0 \circ \mathfrak{w}] \circ \mathfrak{w}$. Still, it makes sense to speak about $b$-stable commutative subalgebras of $\mathfrak{w}$. We have $G_2 \circ \mathfrak{g}_2$. Then completely the same argument shows that the $b$-stable commutative subalgebras in $\mathfrak{w}$ are in one-to-one correspondence with the long Abelian $b$-ideals in $G_2$.

REFERENCES

[1] P. Cellini and P. Papi. Ad-nilpotent ideals of a Borel subalgebra, J. Algebra, 225(2000), 130–141.
[2] P. Cellini and P. Papi. Ad-nilpotent ideals of a Borel subalgebra II, J. Algebra, 258(2002), 112–121.
[3] B. Kostant. The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations, Intern. Math. Res. Notices, no. 5, (1998), 225–252.
[4] C. Krattenthaler, L. Orsina and P. Papi. Enumeration of ad-nilpotent $b$-ideals for simple Lie algebras, Adv. in Appl. Math. 28(2002), 478–522.
[5] D. Panyushev. Isotropy representations, eigenvalues of a Casimir element, and commutative Lie subalgebras, J. London Math. Soc. 61, Part 1 (2001), 61–80.
[6] D. Panyushev. The exterior algebra and “spin” of an orthogonal $g$-module, Transform. Groups 6, no. 4 (2001), 371–396.
[7] D. Panyushev. Abelian ideals of a Borel subalgebra and long positive roots, Preprint arXiv:math.RT/0210428
[8] D. Panyushev and G. Röhrle. Spherical orbits and Abelian ideals, Adv. Math. 159(2001), 229–246.
[9] B. Vinberg, A. L. Onishchik. Seminar po grupam Li i algebraicheskim grupam. Moskva: "Nauka" 1988 (Russian). English translation: A. L. Onishchik and E. B. Vinberg. “Lie groups and algebraic groups”, Berlin: Springer, 1990.