The leader operators of the \((d+1)\)-dimensional relativistic rotating oscillators

Ion I. Cotăescu, Ion I. Cotăescu Jr. and Nicolina Pop

West University of Timisoara
V. Pârvan Ave. 4, RO-300223 Timişoara, Romania

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Abstract

The main pairs of leader operators of the quantum models of relativistic rotating oscillators in arbitrary dimensions are derived. To this end one exploits the fact that these models generate Pöschl-Teller radial problems with remarkable properties of supersymmetry and shape invariance.

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1 Introduction

In general relativity one can build geometric models of pure harmonic or even rotating oscillators constituted by test particles freely moving in suitable curved spacetimes. In these models the classical and quantum geodesic motions represent either harmonic oscillations in $(1 + 1)$ dimensions [1]-[4] or oscillations that can be accompanied by supplemental uniform rotations in three space dimensions [5] as well as in the general case of any $(1 + d)$ dimensions [6, 7].

The backgrounds of these models have static central charts with pure or deformed anti-de Sitter metrics. For $d > 1$ all the models are of rotating oscillations apart from the anti-de Sitter ones where the rotation effects disappear [5, 6]. In these central charts, the Klein-Gordon equations, describing the quantum scalar test particle, can be analytically solved leading to standard Pöschl-Teller radial problems [8, 9]. This means that these models can be studied using the method of supersymmetry and shape invariance [10] with minimal changes requested by the specific form of the Klein-Gordon equation [3, 7]. In this way we have shown that all the $(1+1)$-dimensional Pöschl-Teller models have the same dynamical algebra, $so(1,2)$, formed by a pair of leader operators and the operator of the number of oscillation quanta [2, 3, 4]. In the case of any dimensions we have studied the supersymmetry of the relativistic Pöschl-Teller radial problems but without to write down the leader operators [7].

In the present letter we should like to continue the study of the supersymmetry of the radial relativistic Pöschl-Teller problems in arbitrary dimensions. Our purpose is to derive the principal pairs of leader operators able to increase or decrease the quantum numbers of the radial modes of these models.

In section 2 we briefly present the radial problems of the models of (rotating) oscillators in $(d + 1)$ dimensions showing that these are Pöschl-Teller problems with a very simple parametrization since all the physical parameters concentrate into only one parameter which characterize the model. The section 3 is devoted to the supersymmetry and the shape invariance of the radial potentials that lead to the Rodrigues formula of the normalized radial functions and some useful identities given in the next section. These results help us to identify in section 5 the principal pairs of leader operators, including a pair which change with $\pm 1$ the value of the parameter of these models.
The final results are presented in the last section.

2 The radial Pöschl-Teller problems

Let us consider the models of relativistic rotating oscillators of the de Sitter type \[7\] with the static central charts of coordinates \((t, \mathbf{x})\), where \(t = x^0\) is the time while \(\mathbf{x} = (x^1, x^2, ..., x^d)\) represent the Cartesian space coordinates. In these models the oscillating test particle is described by a scalar quantum field \(\Phi\) of mass \(M\), minimally coupled with the gravitational field. The quantum scalar modes are given by the particular solutions of the Klein-Gordon equation

\[
\frac{1}{\sqrt{g}} \partial_{\mu}(\sqrt{g}g^{\mu\nu}\partial_{\nu}\Phi) + M^2\Phi = 0, \quad g = |\det(g_{\mu\nu})|.
\]

The spherical variables can be separated using generalized spherical harmonics, \(Y_{l(\lambda)}^{d-1}(\mathbf{x}/r)\), where \(r = |\mathbf{x}|\). These are normalized eigenfunctions of the angular Laplace operator \[11\],

\[
-\Delta_s Y_{l(\lambda)}^{d-1}(\mathbf{x}/r) = l(l + d - 2)Y_{l(\lambda)}^{d-1}(\mathbf{x}/r),
\]

corresponding to eigenvalues depending on the angular quantum number \(l\) which takes only integer values, \(l = 0, 1, 2, ...,\) selected by the boundary conditions on the sphere \(S^{d-1}\) \[11\]. The particular solutions of energy \(E\) (and positive frequency) \[7\],

\[
\Phi_{E,l(\lambda)}^{(+)}(t, \mathbf{x}) = \frac{1}{\sqrt{2E}}(\omega \cot \omega r)^{d-3}R_{E,l}(r)Y_{l(\lambda)}^{d-1}(\mathbf{x}/r)e^{-iEt},
\]

involve the radial wave function \(R_{E,l} \propto R_{k,l,n}\) which satisfy the radial equation

\[
\left[-\frac{1}{\omega^2}\frac{d^2}{dr^2} + \frac{2s(2s-1)}{\sin^2 \omega r} + \frac{2p(2p-1)}{\cos^2 \omega r}\right]R_{k,l,n} = \nu^2 R_{k,l,n},
\]

where \(\nu\) depends on \(E\) as in Ref. \[7\] and we have

\[
k = \sqrt{\frac{M^2}{\omega^2} + \frac{d^2}{4} + \frac{d}{2}}, \quad 2s = l + \delta, \quad 2p = k - \delta, \quad \delta = \frac{d-1}{2}.
\]
In this parametrization the specific parameter $k$ concentrates all the other ones, playing thus the role of the principal parameter of our models. For this reason we denote the model by $[k]$ understanding that it generates the radial problems $(k,l)$ with different values of $l$, each one having its own sequence of radial wave functions, $R_{k,l,n_r}(r)$. These are labeled by the radial quantum number $n_r = 0, 1, 2, ...$ giving the quantization condition $\nu = 2(n_r + s + p)$ [6, 7]. In this way one obtains a typical Pöschl-Teller problem with square integrable radial functions with respect to the radial scalar product

$$\langle R, R' \rangle = \int_{D_r} dr R^*(r)R'(r).$$

(6)

### 3 The radial supersymmetry

We have shown [7] that the supersymmetric formalism of the radial problems $(k,l)$ can be constructed in the same manner as in the case of the relativistic $(1 + 1)$-dimensional Pöschl-Teller models [3]. The first step is to introduce the operator

$$\{\Delta[V]R\}(r) = \left[-\frac{1}{\omega^2} \frac{d^2}{dr^2} + V(r)\right]R(r),$$

(7)

which should play the same role as the Hamiltonian of the one-dimensional nonrelativistic problems. Furthermore, one observes that from Eq. (2) we can derive the normalized ground-state radial functions

$$R_{k,l,0}(r) = \sqrt{2\omega} \left[\frac{\Gamma(k + l + 1)}{\Gamma(l + d^2)\Gamma(k + 1 - d^2)}\right]^{\frac{1}{2}} \sin^{2s}\omega r \cos^{2p}\omega r,$$

(8)

producing the radial superpotentials

$$W_{k,l}(r) = -\frac{1}{\omega R_{k,l,0}(r)} \frac{dR_{k,l,0}(r)}{dr} = 2p \tan \omega r - 2s \cot \omega r.$$

(9)

In its turn, each superpotential $W_{k,l}$ give rise to the pair of superpartner radial potentials,

$$V_{\pm}(k, l, r) = \pm \frac{dW_{k,l}(r)}{dr} + W_{k,l}(r)^2 = \frac{2s(2s \pm 1)}{\sin^2\omega r} + \frac{2p(2p \pm 1)}{\cos^2\omega r} - (2s + 2p)^2.$$
Now Eq. (2) can be rewritten as
\[
\Delta [V_-(k, l)] R_{k, l, n_r} = d_{k, l, n_r} R_{k, l, n_r},
\]
(11)
where
\[
d_{k, l, n_r} = 4 n_r (n_r + k + l).
\]
(12)

According to the standard procedure [10, 7], we define the pair of adjoint operators, \(A_{k, l}\) and \(A_{k, l}^\dagger\), having the action
\[
(A_{k, l} R)(r) = \left[ \frac{1}{\omega} \frac{d}{dr} + W_{k, l}(r) \right] R(r),
\]
(13)
\[
(A_{k, l}^\dagger R)(r) = \left[ -\frac{1}{\omega} \frac{d}{dr} + W_{k, l}(r) \right] R(r).
\]
(14)

These are the supersymmetry generators that allow us to write
\[
\Delta [V_-(k, l)] = A_{k, l}^\dagger A_{k, l}, \quad \Delta [V_+(k, l)] = A_{k, l} A_{k, l}^\dagger.
\]
(15)

The crucial point of this approach is that the radial potentials \(V_-(k, l)\) and \(V_+(k, l)\) are shape invariant since
\[
V_+(k, l, r) = V_-(k + 1, l + 1, r) + 4(k + l + 1).
\]
(16)

Consequently, we can verify that
\[
\Delta [V_+(k, l)] R_{k+1, l+1, n_r-1} = d_{k, l, n_r} R_{k+1, l+1, n_r-1}, \quad n_r = 1, 2, \ldots,
\]
(17)
which means that the spectrum of the operator \(\Delta [V_+(k, l)]\) coincides with that of \(\Delta [V_-(k, l)]\), apart from the lowest eigenvalue \(d_{k, l, 0} = 0\). From Eqs. (15) combined with Eq. (11) it results that the normalized radial functions satisfy
\[
A_{k, l} R_{k, l, n_r} = \sqrt{d_{k, l, n_r}} R_{k+1, l+1, n_r-1},
\]
(18)
\[
A_{k, l}^\dagger R_{k+1, l+1, n_r-1} = \sqrt{d_{k, l, n_r}} R_{k, l, n_r}.
\]
(19)

Hence, the superpartner radial problems given by the superpartner radial potentials \(V_\pm(k, l, r)\) are \((k, l)\) and, respectively, \((k + 1, l + 1)\). Since both the parameters of these radial problems increase simultaneously with one unit, we can say that the parameter \(k\) simulates the behavior of a quantum number playing a similar role as the orbital quantum number \(l\).
4 The normalized radial functions

The above presented properties lead to the operator version of the Rodrigues formula of the normalized radial functions [7],

\[ R_{k,l,n_r} = \frac{1}{2^n_r} \left[ \frac{\Gamma(n_r + k + l)}{n_r! \Gamma(2n_r + k + l)} \right]^{\frac{1}{2}} A^\dagger_{k,l} A^\dagger_{k+1,l+1} \cdots \cdot \cdots A^\dagger_{k+n_r-1,l+n_r-1} R_{k+n_r,l+n_r,0} . \]  

(20)

This expression can be brought in the usual standard form observing that in the new variable \( u = \cos 2\omega r \) the operators (13) and (14) read

\[ A_{k,l} = -2(1 - u)^{s+\frac{1}{2}}(1 + u)^{p+\frac{1}{2}} \frac{d}{du} (1 - u)^{-s}(1 + u)^{-p}, \]  

(21)

\[ A^\dagger_{k,l} = 2(1 - u)^{-s+\frac{1}{2}}(1 + u)^{-p+\frac{1}{2}} \frac{d}{du} (1 - u)^{s}(1 + u)^{p}. \]  

(22)

Then Eq.(20) yields the following normalized radial functions:

\[ R_{k,l,n_r}(u) = N_{k,l,n_r} \chi_{k,l,n_r}(u), \]  

(23)

where

\[ N_{k,l,n_r} = \frac{1}{2^{n_r+k+1}} \left[ \frac{2\omega(2n_r + k + l)\Gamma(n_r + k + l)}{n_r! \Gamma(n_r + l + \frac{d}{2}) \Gamma(n_r + k + 1 - \frac{d}{2})} \right]^{\frac{1}{2}}, \]  

(24)

\[ \chi_{k,l,n_r}(u) = (1 - u)^{-s+\frac{1}{2}}(1 + u)^{-p+\frac{1}{2}} \times \frac{d^{n_r}}{du^{n_r}} (1 - u)^{2s+n_r+\frac{d}{2}}(1 + u)^{2p+n_r-\frac{d}{2}} . \]  

(25)

This expression is suitable for deriving some basic identities that will help us to write down the action of the leader operators. First, we consider the commutation relation

\[ \left[ 1 - u^2, \frac{d^{n+1}}{du^{n+1}} \right] = 2u(n + 1) \frac{d^n}{du^n} + n(n + 1) \frac{d^{n-1}}{du^{n-1}} . \]  

(26)
and using Eq. (25) we obtain the basic identity

\[
(1 - u^2) \frac{d\chi_{k,l,n_r}}{du} = \left[ u \left( 2n_r + 1 + \frac{k + l}{2} \right) + \frac{k - l}{2} + \delta \right] \chi_{k,l,n_r} \\
+ \chi_{k,l,n_r+1} + n_r(n_r + 1)\sqrt{1 - u^2} \chi_{k+1,l+1,n_r-1}
\]

Furthermore, from Eq. (25) we deduce other two useful auxiliary identities,

\[
\chi_{k,l,n_r+1} + 2(n_r + l + \delta + \frac{1}{2})\chi_{k,l,n_r} \\
- (2n_r + k + l + 1)\sqrt{1 - u} \chi_{k,l+1,n_r} = 0,
\]  

\[
\chi_{k,l,n_r+1} - 2(n_r + k - \delta + \frac{1}{2})\chi_{k,l,n_r} \\
+ (2n_r + k + l + 1)\sqrt{1 + u} \chi_{k+1,l,n_r} = 0,
\]

which will help us to calculate the action of the first order operators.

5 Pairs of leader operators

The next step is to identify the pairs of leader operators shifting not only the genuine quantum numbers \(l\) and \(n_r\) but also the parameter \(k\) which simulates the behavior of a quantum number. To this end it is convenient to work only with the functions \(\chi\) giving up the normalized factors the presence of which could lead to some useless difficulties.

We observe that the action of the supersymmetry generators on the functions \(\chi_{k,l,n_r}\) reads

\[
\frac{1}{2}A_{k,l} \chi_{k,l,n_r} = n_r(n_r + k + l)\chi_{k+1,l+1,n_r-1},
\]

\[
\frac{1}{2}A_{k,l}^{\dagger} \chi_{k+1,l+1,n_r-1} = \chi_{k,l,n_r},
\]

and we look for other pairs of operators with similar effects. Assuming that the pairs of leader operators depend linearly on the supersymmetry generators, we define the general form of the leader operators as

\[
\tilde{A}_{k,l,n_r}^{(-)} = \sqrt{\phi} \left( \frac{1}{2}A_{k,l} + H_{k,l,n_r} \right),
\]

\[
\tilde{A}_{k,l,n_r}^{(+)} = \phi \left( \frac{1}{2}A_{k,l}^{\dagger} + H_{k,l,n_r} \right) \frac{1}{\sqrt{\phi}},
\]
where \( H_{k,l,n_r} \) and \( \phi \) are functions of \( u \). These operators may have different actions

\[
\tilde{A}^{(-)}_{k,l,n_r} \chi_{k,l,n_r} = c^{(-)}_{k,l,n_r} \chi_{k+\Delta k,l+\Delta l,n_r+\Delta n_r},
\]
\[
\tilde{A}^{(+)}_{k,l,n_r} \chi_{k+\Delta k,l+\Delta l,n_r+\Delta n_r} = c^{(+)}_{k,l,n_r} \chi_{k,l,n_r},
\]

depending on the values of the shifts \( \Delta k, \Delta l \) and \( \Delta n_r \) we choose. In addition, any pair of such operators must obey the identity

\[
\tilde{A}^{(+)}_{k,l,n_r} \tilde{A}^{(-)}_{k,l,n_r} \chi_{k,l,n_r} = \phi \left[ n_r (n_r + k + l) + H^2_{k,l,n_r} \right] + \sqrt{1 - u^2} \frac{dH_{k,l,n_r}}{du} + W_{k,l} H_{k,l,n_r} \chi_{k,l,n_r} = c^{(+)}_{k,l,n_r} c^{(-)}_{k,l,n_r} \chi_{k,l,n_r},
\]

resulted from Eqs. (11), (12) and (15).

Thus we have formulated a problem that may be solved finding the concrete form of the functions \( H_{k,l,n_r} \) and \( \phi \). First of all, we observe that some previous results \([9, 12]\) suggest us to consider functions of the form

\[
H_{k,l,n_r}(u) = \alpha_{k,l,n_r} \sqrt{1 - u} + \beta_{k,l,n_r} \sqrt{1 + u},
\]

(37)

Furthermore, we have to use the Eqs. (21), (22) and (36) for finding the constants \( \alpha_{k,l,n_r}, \beta_{k,l,n_r} \) and \( c^{(\pm)}_{k,l,n_r} \). The shifting action will be deduced from Eqs. (30), (31) combined with the identities (27), (28) and (29).

In this way we find many solutions which will be labeled by an index preceding the symbols. The first three pairs of operators, \( m \tilde{A}^{(\pm)}_{k,l,n_r} \), with \( m = 1, 2, 3 \), have their defining elements as given in the table below

| Nr. | symbol | \( \Delta k \) | \( \Delta l \) | \( \Delta n_r \) | \( \phi(u) \) | \( \alpha_{k,l,n_r} \) | \( \beta_{k,l,n_r} \) |
|-----|--------|-------------|-------------|-------------|-----------|--------------|--------------|
| 1   | \( \tilde{A}^{(\pm)}_{k,l,n_r} \) | 1           | 0           | 0           | 1 - \( u \) | \( n_r + k + l \) |
| 2   | \( \tilde{A}^{(\pm)}_{k,l,n_r} \) | 0           | 1           | 0           | 1 + \( u \) | \( n_r + k + l \) |
| 3   | \( \tilde{A}^{(\pm)}_{k,l,n_r} \) | 0           | 1           | -1          | 1 + \( u \) | \( n_r \)    |

while their action is defined by the constants listed in the next table

| Nr. | \( c^{(-)}_{k,l,n_r} \) | \( c^{(+)}_{k,l,n_r} \) |
|-----|-----------------|-----------------|
| 1   | \( n_r + k + l \) | \( 2(n_r + k - \frac{1}{2} d + 1) \) |
| 2   | \( n_r + k + l \) | \( 2(n_r + l + \frac{1}{2} d) \) |
| 3   | \( 2n_r(n_r + k - \frac{1}{2} d) \) | 1 |
In addition, there is a fourth solution which is more complicated corresponding to $\Delta k = \Delta l = 0$ and $\Delta n_r = 1$. In this case we find that the pair of operators $\tilde{A}^{(\pm)}_{k,l,n_r}$ is defined by $\phi(u) = 1 - u^2$ and the function $4H_{k,l,n_r}$ of the form (37) with the coefficients

$$4\alpha_{k,l,n_r} = \frac{n_r(n_r + l + \frac{1}{2}d - 1)}{2n_r + k + l - 1}, \quad 4\beta_{k,l,n_r} = -\frac{n_r(n_r + k - \frac{1}{2}d)}{2n_r + k + l - 1}. \quad (38)$$

The respective action constants read

$$4\alpha_{k,l,n_r}^{(-)} = \frac{4n_r(n_r + k - \frac{1}{2}d)(n_r + l + \frac{1}{2}d - 1)}{2n_r + k + l - 1}, \quad (39)$$

$$4\alpha_{k,l,n_r}^{(+)} = \frac{n_r + k + l - 1}{2n_r + k + l - 1}. \quad (40)$$

We note that for each pair producing the shifts $\Delta k$, $\Delta l$, $\Delta n_r$ one can define another pair performing the inverse shifts $-\Delta k$, $-\Delta l$, $-\Delta n_r$. As a matter of fact this means to change among themselves the roles of the operators $\tilde{A}^{(-)}$ and $\tilde{A}^{(+)}$ in a suitable parametrization.

## 6 Final results

Using the above results we can write the action of the pairs of leader operators upon the normalized wave functions $R_{k,l,n_r}$ as

$$m\tilde{A}^{(-)}_{k,l,n_r} R_{k,l,n_r} = mC_{k,l,n_r}^{(-)} R_{k+\Delta k,l+\Delta l,n_r+\Delta n_r}, \quad (41)$$

$$m\tilde{A}^{(+)}_{k,l,n_r} R_{k+\Delta k,l+\Delta l,n_r+\Delta n_r} = mC_{k,l,n_r}^{(+)} R_{k,l,n_r}, \quad (42)$$

where

$$mC_{k,l,n_r}^{(\pm)} = mC_{k,l,n_r}^{(\pm)} \left[ \frac{N_{k+\Delta k,l+\Delta l,n_r+\Delta n_r}}{N_{k,l,n_r}} \right]^{\pm 1}. \quad (43)$$

The operators of the first pair ($m = 1$) are no genuine leader operators since they produce the variation with one unit of the parameter $k$ which is not a quantum number. Nevertheless, these operators are useful in combination with the supersymmetry generators which increase or decrease the value of $k$ in a similar way.
The second and third pairs are important because these represent the leader operators shifting the principal quantum number \( n = 2n_r + l \). In terms of the quantum numbers \((n, l)\), the second pair performs the shifts \( \Delta n = 1 \) and \( \Delta l = 1 \) while the shifts given by the third one are \( \Delta n = -1 \) and \( \Delta l = 1 \). We observe that these operators can not be defined in a global manner, independent on the values of the parameters \((l, n_r)\) or \((n, l)\), since the dependence on \( l \) can not be removed as long as a differential operator with eigenvalues \( l \) does not exist.

The fourth solution \((m = 4)\) yields the unique pair of leader operators which can be put in a suitable closed form, depending only on the principal parameter of the model, \( k \). Indeed, if we proceed as in Ref. [2] introducing the non-differential operator \( \hat{N} \) defined as

\[
\hat{N} R_{k,n,l} = nR_{k,n,l},
\]

then we observe that the operators \(2 \hat{A}_{k,l,n_r}^{(\pm)}\) produce the same radial effects as the operators

\[
A_{k}^{(\pm)} = \pm 2(1 - u^2) \frac{d}{du} - u(\hat{N} + k) + \frac{\Delta_s + k(k - d) + 2d - 1}{N + k + 1},
\]

which act in the space of the normalized wave functions as

\[
A_{k}^{(\pm)} R_{k,n,l} Y_{l(\lambda)}^{d-1} = C_{k,n,l}^{(\pm)} R_{k,n\pm 2} Y_{l(\lambda)}^{d-1},
\]

where

\[
C_{k,n,l}^{(-)} = \left[ \frac{(n + k)(n - l)}{n + k - 2} \right]^{1/2} \frac{[(n + 2k + l - 2)(n + 2k - l - d)(n + l + d - 2)]^{1/2}}{n + k - 1},
\]

\[
C_{k,n,l}^{(-)} = \left[ \frac{(n + k - 2)(n - l)}{n + k} \right]^{1/2} \frac{[(n + 2k + l - 2)(n + 2k - l - d)(n + l + d - 2)]^{1/2}}{n + k - 1}.
\]

We note that it is not clear if these operators are adjoint to each other with respect to the scalar product (6). This is because it is not certain if the
coordinate operator commutes with \( \hat{N} \). We hope that further investigations should enlighten this delicate point.

Concluding we can say that the results presented here are in accordance with our previous results we have obtained for the relativistic (1 + 1)-dimensional Pöschl-Teller problems [2, 3]. One can convince ourselves that the operators shifting the quantum number \( n \) (of the pairs with \( m = 2, 3 \)) have similar forms to those of the leader operators of the models with \( d = 1 \) [2]. The difference is that for \( d > 1 \) there are many pairs of leader operators which could lead to a more complicated dynamical algebra as indicated in [12].

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