A Simple and Fast Linear-Time Algorithm for Proportional Apportionment

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Cheng and Eppstein\cite{CE14} describe a linear-time algorithm for computing highest-average allocations in proportional apportionment scenarios, for instance assigning seats to parties in parliament so that the distribution of seats resembles the vote tally as well as possible.

We propose another linear-time algorithm that consists of only one call to a rank selection algorithm after elementary preprocessing. Our algorithm is conceptually simpler and faster in practice than the one by Cheng and Eppstein.

1. Introduction

The problem of proportional apportionment arises whenever we have a finite supply of $k$ indivisible, identical resource units which we have to distribute across $n$ parties fairly, that is according to the proportional share of publicly known and agreed-upon values $v_1, \ldots, v_n$ (of the sum $V = \sum v_i$ of these values). We elaborate in this section on applications of and solutions for this problem.

We continue to fix formal notation in Section 2 which we use to state a simplistic algorithm for solving apportionment problems in Section 3. We present improvements and the final algorithm in Sections 4 and 5 respectively.

We close with a comparison of our algorithm to the best known algorithm (by Cheng and Eppstein, to the best of our knowledge) in Section 6. Additional material includes a notation index in Appendix D.

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1. Introduction

Apportionment scenarios abound in practice. Here are two prominent examples from politics:

- In a proportional-representation electoral system we have to assign seats in parliament to political parties according to their share of all votes. The resources are seats, and the values are vote counts.

- In federal states the number of representatives from each component state often reflects the population of that state, even though there will typically be at least one representative for any state no matter how small it is. Resources are again seats, values are the numbers of residents.

In order to use consistent language throughout this article, we will stick to the first metaphor. That is, we assign $k$ seats to parties proportionally to their respective votes $v_i$, and we call $k$ the house size.

A fair allocation should assign $v_i/V$ seats to party $i$, where $V = v_1 + \cdots + v_n$ is the total vote count of all parties. As resources are indivisible, this is only possible if, by chance, all $v_i/V$ are integers; otherwise we have to come up with some rounding scheme. This is where apportionment methods come into play.

Mathematically speaking, an apportionment method is a function $f : \mathbb{R}_{>0}^n \times \mathbb{N} \rightarrow \mathbb{N}_0^n$ that maps vote counts $v = (v_1, \ldots, v_n)$ and house size $k$ to a seat allocation $s = (s_1, \ldots, s_n) := f(v, k)$ so that $s_1 + \cdots + s_n = k$. We interpret $s$ as party $i$ getting $s_i$ seats.

There are many conceivable such methods, but there are three natural properties one would like apportionment systems to have:

(P1) **Pairwise vote monotonicity:** When votes change, $f$ should not take away seats from a party that has gained votes while at the same time awarding seats to one that has lost votes.

(P2) **House monotonicity:** $f$ should not take seats away from any party when the house grows (in number of seats) but votes do not change.

(P3) **Quota rule:** The number of seats of each party should be its proportional share, rounded either up or down.

Balinski and Young [BY10] show that, unfortunately, no apportionment system can satisfy all three properties. If we have to violate any one of these properties we should arguably prefer to sometimes violate (P3). This essentially leaves us with the highest averages methods [BY77], a.k.a. divisor or Huntington methods. They are defined as iterative processes that assume a fixed increasing divisor sequence $d = (d_j)_{i=0}^\infty$.

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1 Many electoral systems first exclude parties below a certain threshold of overall votes from seat allocation altogether. We assume they have already been removed from our list of $n$ parties.

2 Not having Property (P1) means violating the principle of “one-person, one-vote” which is a constitutional requirement in many countries.
1. Introduction

Algorithm 1: HIGHEST AVERAGES$_d(v, k)$:

Step 1 Initialize $s = 0^n$.

Step 2 While $k > 0$,

Step 2.1 Determine $\arg \max_{i=1}^n v_i / d_{s_i}$.

Step 2.2 Update $s_i \leftarrow s_i + 1$ and $k \leftarrow k - 1$.

Step 3 Return $s$.

The name stems from the ratio used in step 2.1 for $d_j = j$, we select the party with highest current average number of votes per seat. Other sequences (cf. Table 1) skew the averages in order to achieve higher (perceived) fairness; clearly, a good scheme will result in the ratios being close to each other in the end.

It is not per se clear which divisor sequence is the best; there still seems to be active discussion, e.g., for the U.S. House of Representatives. One reason is that no-one has yet been able to propose a convincing, universally agreed-upon mathematical criterion that would single out one method as superior to the others. In fact there are competing notions of fairness, each favoring a different highest averages method [Cam07]. A reasonable approach is therefore to run computer simulations of different methods and compare their outcomes empirically, for example w.r.t. the distribution of final average votes per seat $v_i / s_i$. For this purpose, many apportionments have to be computed, so efficient algorithms become an issue.

We thus study in this article the problem of computing the final seat allocation by highest averages methods (given by divisor sequences) according to vote counts and house size. Cheng and Eppstein [CE14] show that this problem can be solved in time $O(n)$ if the divisor sequence is close to linear, which applies to all electoral systems in use (cf. Table 1). Cheng and Eppstein also discuss complexities of related problems.

While its runtime is asymptotically optimal, the algorithm by Cheng and Eppstein [CE14] is rather complex and difficult to implement (cf. Section 6.1), motivating the search for other approaches.

In this article, we describe another linear-time algorithm which is based on a generalization of our solution for the envy-free stick-division problem [RW15b] (which corresponds to $d_j = j + 1$). It turns out to be both conceptually simpler and faster than the competition. See Section 6 for detail.
2. Notation

### 2.1. Divisor Sequences

Let $d = (d_j)_{j=0}^\infty$ be an arbitrary nonnegative, (strictly) increasing and unbounded sequence of real numbers. We formally set $d_{-1} := -\infty$.

We assume that there is a smooth continuation of $d$ on the reals which is easy to invert. That is, we assume a function $\delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq d_0}$ with

i) $\delta$ is continuous and strictly increasing,

ii) $\delta^{-1}(x)$ for $x \geq d_0$ can be computed with a constant number of arithmetic operations, and

iii) $\delta(j) = d_j$ (and thus $\delta^{-1}(d_j) = j$) for all $j \in \mathbb{N}_0$.

All the divisor sequences used in practice fulfill these requirements; cf. Table 1. For convenience, we continue $\delta^{-1}$ on the complete real line requiring

iv) $\delta^{-1}(x) \in [-1, 0)$ for $x < d_0$.

**Corollary 1:** Assuming [i] to [iv], $\delta^{-1}(x)$ is continuous and strictly increasing on $\mathbb{R}_{\geq d_0}$. Furthermore, it is the inverse of $j \mapsto d_j$ in the sense that

$$[\delta^{-1}(x)] = \max\{j \in \mathbb{Z}_{\geq -1} \mid d_j \leq x\}$$

for all $x \in \mathbb{R}$.

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### Table 1: Commonly used highest averages methods [CE14 Table 1].

| Method               | Divisor Sequence | $\delta(x)$ | Sandwich |
|----------------------|------------------|-------------|----------|
| Smallest divisors    | 0, 1, 2, 3, ...  | $x$         | —        |
| Greatest divisors    | 1, 2, 3, 4, ...  | $x + 1$     | —        |
| Sainte-Lagué        | 1, 3, 5, 7, ...  | $2x + 1$    | —        |
| Modified Sainte-Lagué| 1.4, 3, 5, 7, ...| $\{\frac{2x+1}{1.6x+1.4} \text{ if } x \geq 1\}$ | $2x + \frac{6}{5} \pm \frac{1}{5}$ |
| Equal Proportions    | 0, $\sqrt{2}$, $\sqrt{6}$, $\sqrt{12}$, ... | $\sqrt{x(x+1)}$ | $x + \frac{1}{2} \pm \frac{1}{2}$ |
| Imperiali            | 2, 3, 4, 5, ...  | $x + 2$     | —        |
| Danish               | 1, 4, 7, 10, ... | $3x + 1$    | —        |

*Table 1: Commonly used highest averages methods [CE14 Table 1]. For each of the methods, we give a possible continuation $\delta$ of the respective divisor sequence (cf. Section 2) as well as linear sandwich bounds on $\delta$, if non-trivial (cf. Lemma 5).*
2. Notation

In particular, \(\lfloor \delta^{-1}(x) \rfloor = j\) for \(d_j \leq x < d_{j+1}\) so the floored \(\delta^{-1}\) is the (zero-based) rank function for the set of all \(d_j\) as long as \(x \geq d_0\).

2.2. Highest Averages Apportionment

Following the notation of Cheng and Eppstein \[CE14\], we consider for given votes \(v = (v_1, \ldots, v_n) \in \mathbb{Q}_{>0}^n\) the sets

\[
A_i := \{a_{i,j} \mid j = 0, 1, 2, \ldots\} \quad \text{with} \quad a_{i,j} := \frac{d_j}{v_i}
\]

and their multiset union

\[
\mathcal{A} := \bigcup_{i=1}^n A_i.
\]

Note that we work with \(d_j/v_i\) instead of \(v_i/d_j\) in Algorithm 1; we prefer the reciprocals because the case \(d_0 = 0\) then handles gracefully and without special treatment.

As we will see in Section 3 we can solve the apportionment problem by finding the \(k\)th smallest element in \(\mathcal{A}\) which we denote with \(\mathcal{A}(k)\). We thus introduce the rank function \(r(x, \mathcal{A})\) which denotes the number of elements in multiset \(\mathcal{A}\) that are no larger than \(x\), that is

\[
r(x, \mathcal{A}) := |\mathcal{A} \cap (-\infty, x]| = \sum_{i=1}^n |\{a_{i,j} \mid a_{i,j} \leq x\}|.
\]  

(1)

We write \(r(x)\) instead of \(r(x, \mathcal{A})\) when \(\mathcal{A}\) is clear from context.

Unsurprisingly, the rank function reduces to \(\delta^{-1}\); moreover, if we are only interested in ranks of values \(x\) less than some given bound \(\bar{x}\) we may be able to simplify the computation of \(r(x)\).

**Lemma 2:** For rank function \(r(x, \mathcal{A})\),

\[
r(x, \mathcal{A}) = \sum_{i=1}^n \lfloor \delta^{-1}(v_i \cdot x) \rfloor + 1.
\]

Moreover, for \(x < \bar{x}\) we have

\[
r(x, \mathcal{A}) = \sum_{i \in I_{\bar{x}}} \lfloor \delta^{-1}(v_i \cdot x) \rfloor + 1
\]

with \(I_{\bar{x}} = \{i \in \{1, \ldots, n\} \mid v_i > d_0/\bar{x}\}\).
3. Apportionment by Rank Selection

The proof proceeds by expanding \( r \) via Corollary 1 into a sum over parties \( i \), expressing each summands in terms of \( \delta^{-1} \) and fixing corner cases with condition iv) on \( \delta \). Find the details in Appendix A.1.

We need one more convenient shorthand: We denote with

\[
A^{\pi} := \bigcup_{i \in I_{\pi}} \left\{ \frac{d_j}{v_i} \middle| \frac{d_j}{v_i} < \pi \right\} = \bigcup_{i=1}^{n} \left\{ \frac{d_j}{v_i} \middle| \frac{d_j}{v_i} < \pi \right\} = A \cap (-\infty, \pi)
\]

the multiset of elements from sequences from \( I_{\pi} \) that are smaller than \( \pi \).

3. Apportionment by Rank Selection

Even though \textsc{HighestAverages} is an iterative process, it solves what is in essence a static problem.

Since divisor sequence \( d \) is strictly increasing, the ratios \( v_i/d_j \) are strictly decreasing in \( j \) for every \( i \). Therefore, the sequence of maximal \( v_i/d_s \) is decreasing as well. As a consequence, if the \( r \)th seat is allocated to party \( i \), the corresponding (maximum) ratio \( v_i/d_j \) is the \( r \)th largest ratio overall. In terms of \( A \), \( a_{i,j} = (v_i/d_j)^{-1} = A(r) \), i.e. the \( r \)th smallest element of \( A \). \textsc{HighestAverages} thus assigns seats to the respective \( i \) of \( A(1) \) down to \( A(k) \).

Conversely, if we know the value \( a^* := A(k) \) of the \( k \)th smallest element in \( A \), we can determine (in time \( O(n) \)) for each party \( i \) how many seats it should receive, namely \( s_i = r(a^*, A_i) = \lfloor \delta^{-1}(v_i \cdot a^*) \rfloor + 1 \).

We have thus shown that running \textsc{HighestAverages} is equivalent\(^3\) to solving the following problem.

**Problem 1:** Given a divisor sequence \( (d_j)_{j=0}^{\infty} \), vote counts \( v \in \mathbb{Q}_{>0}^n \) and house size \( k \), determine \( a^* = A(k) \).

Note that even though \( A \) is infinite, \( A(k) \) always exists because the terms \( a_{i,j} = d_j/v_i \) are strictly increasing in \( j \) for all \( i \in \{1, \ldots, n\} \). Using rank function \( r(x, A) \), we can equivalently state the problem as finding

\[
a^* = \min \{ a \in A \mid r(a, A) \geq k \}.
\]

(3)

Borrowing terminology from the field of mathematical optimization, we call \( a \) a feasible if \( r(a) \geq k \), otherwise it is infeasible. Feasible \( a \neq a^* \) are called suboptimal.

Now since \( d \) is unbounded, setting any upper bound \( \pi \) on \( a_{i,j} \) yields a finite search space \( A^{\pi} \). By choosing any such bound that leaves \( |A^{\pi}| \geq k \), we retain the property that \( a^* \) is

\(^3\)Note that both algorithms need tie-breaking rules. Where the maximum in \textsc{HighestAverages} is not unique, we may have \( \sum_{i=1}^{n} r(a^*, A_i) > k \) if \( a^* \) occurs more than once in \( A \), that is we might assign too many seats in total.
the $k$th smallest element under consideration. The following algorithm does so crudely by ensuring that the party with the most votes contributes at least $k$ values.

**Algorithm 2:** \texttt{SelectAstarNaive}(v, $k$) :

\begin{enumerate}
  \item \textbf{Step 1} Find $v^{(1)} = \max\{v_1, \ldots, v_n\}$.
  \item \textbf{Step 2} Set $\pi := d_{k-1}/v^{(1)} + \varepsilon$ for suitable constant $\varepsilon > 0$.
  \item \textbf{Step 3} Compute $I_\pi$ as per Lemma 2.
  \item \textbf{Step 4} Construct multiset $\mathcal{A}^\pi$.
  \item \textbf{Step 5} Select and return $\mathcal{A}^\pi_{(k)}$.
\end{enumerate}

**Theorem 3:** \texttt{SelectAstarNaive}(v, $k$) = $a^*$.

**Proof:** We have $\pi > a^* = \mathcal{A}^\pi$ as already $r(\pi - \varepsilon) = r(d_{k-1}/v^{(1)}) \geq k$; at least the $k$ elements $d_0/v^{(1)}, \ldots, d_{k-1}/v^{(1)} \in \mathcal{A}$ are no larger than $d_{k-1}/v^{(1)}$. We thus never need to consider elements $a \geq \pi$, and in particular $\mathcal{A}_{(k)} = \mathcal{A}^\pi_{(k)}$ as $\mathcal{A}^\pi = \mathcal{A} \cap (-\infty, \pi]$.

So far, we have needed no additional restriction on $\varepsilon$ in \textbf{step 2} we only need it to be positive so we do not discard $a^*$ by accident if it is exactly $d_{k-1}/v^{(1)}$. However, the size of $\mathcal{A}^\pi$ can be arbitrarily large – depending on the input values $v_i$ which we do not want. Therefore, we require
\begin{equation}
0 < \varepsilon < \frac{d_k - d_{k-1}}{v^{(1)}}, \quad (4)
\end{equation}
such exists because $d$ is strictly increasing. Note how then $\pi < d_k/v^{(1)}$ so we do not keep any additional suboptimal values. \hfill \Box

While \texttt{Algorithm 2} (with above mentioned postprocessing) solves the proportional apportionment problem, its running time is far from satisfactory. While \textbf{Steps 1} to \textbf{3} all take time $O(n)$, both \textbf{step 4} and \textbf{step 5} need time proportional to $|\mathcal{A}|$. Since $|\mathcal{A}| = nk$ in the worst case, for instance if the values are all equal, the overall running time of \texttt{SelectAstarNaive} is $\Theta(kn)$ – that is actually worse than \texttt{HighestAverages} which can be implemented to run in time $\Theta(k \log n)!$

We cannot improve our upper bound $\pi$; it is tight for the case that $v^{(1)} \gg v^{(2)}$. We can, however, exclude many elements in $\mathcal{A}$ because they are too small to be feasible.
4. Sandwiching the Rank Function and $a^*$

As a direct consequence of Lemma 2 together with the fundamental bounds $y - 1 < |y| \leq y$ on floors, we find that

$$\sum_{i \in I_x} \delta^{-1}(v_i \cdot x) < r(x, A) \leq \sum_{i \in I_x} (\delta^{-1}(v_i \cdot x) + 1) = |I_x| + \sum_{i \in I_x} \delta^{-1}(v_i \cdot x) \quad (5)$$

for any $\pi$ and all $x < \pi$. We can therewith pin down the value of $r$ to an interval of width $|I_x|$ using only $\delta^{-1}$. We can use this to derive upper and lower bounds on $a^*$.

**Lemma 4:** Let $\pi > a^*$ and assume $\overline{a}$ and $\underline{a}$ are chosen so that they fulfill

$$\sum_{i \in I_x} \delta^{-1}(v_i \cdot \underline{a}) \leq k - |I_x| \quad \text{and} \quad \sum_{i \in I_x} \delta^{-1}(v_i \cdot \overline{a}) \geq k.$$

Then, $\underline{a} \leq a^* \leq \overline{a}$.

The lemma follows more or less directly; one uses the sandwich bounds on $r$ to show that $\underline{a} < a$ are infeasible, i.e., $r(a) < k$, and that $\overline{a}$ is feasible, and thus all $a > \overline{a}$ are suboptimal by (3). Find the complete proof in Appendix A.2.

Depending on $\delta^{-1}$, the equations given in Lemma 4 may be hard to solve analytically. However, we can explicitly compute suitable bounds for divisor sequences which behave roughly linear, including all those from Table 1.

**Lemma 5:** Assume the continuation $\delta$ of divisor sequence $d$ fulfills

$$\alpha x + \beta \leq \delta(x) \leq \alpha x + \tilde{\beta}$$

for all $x \in \mathbb{R}_{\geq 0}$ with $\alpha > 0$ and $\beta, \tilde{\beta} \geq 0$. Let further some $\overline{x} > a^*$ be given. Then, the pair $(\underline{a}, \overline{a})$ defined by

$$\underline{a} := \max \left\{ 0, \frac{\alpha k - \alpha \cdot |I_x|}{V_x} \right\} \quad \text{and} \quad \overline{a} := \frac{\alpha k + \beta \cdot |I_x|}{V_x}$$

with $V_x := \sum_{i \in I_x} v_i$ fulfills the conditions of Lemma 4, that is $\underline{a} \leq a^* \leq \overline{a}$. Moreover,

$$|A \cap [\underline{a}, \overline{a}]| \leq 2(1 + \beta/\alpha) \cdot |I_x|.$$

The proof consists mostly of rote calculation towards applying Lemma 4; see Appendix A.3 for the details.

Note that the lemma applies in particular to all linear divisor sequences, that is such with $d_j = \alpha j + \beta$ as long as $\alpha > 0$ and $\beta \geq 0$. 
5. Linear-Time Algorithm

We have now derived our main improvement over the work by Cheng and Eppstein [CE14]; where they have only a one-sided bound on $a^*$ and thus have to employ an involved search on $A$, we have sandwiched $a^*$ from both sides, and so tightly that the remaining search space is small enough for a simple rank selection to be efficient.

Building on the bounds from Lemma 5, we can improve \texttt{SELECT$A$STAR$\text{NAIVE}$} by excluding also small elements from $A$ which are for sure not $a^*$. This means that we have to modify the rank we select, too; we will see that our bounds are chosen so that we can use $\delta^{-1}$ to \textit{count} the number of elements we discard \textit{exactly}.

Recall that we assume a fixed apportionment scheme, that is fixed $d$ with known $\alpha$ and $\beta$ as per Lemma 5.

\textbf{Algorithm 3:} \texttt{SELECT$A$STAR}(v, k) :

\begin{enumerate}
  \item Find the $v^{(1)} = \max\{v_1, \ldots, v_n\}$.
  \item Set $\pi := d_{k-1}/v^{(1)} + \varepsilon$ for suitable\footnote{Neither correctness nor $\Theta$-runtime is affected by the choice of $\varepsilon$ here since it affects only the size of $I_{\pi}$, which is bounded by $n$ in any case. In particular, the size of $\hat{A}$ is affected only up to a constant factor. For tweaking performance in practice, see the proof of Theorem 3.} constant $\varepsilon > 0$.
  \item Compute $I_{\pi}$ as per Lemma 2.
  \item Compute $\bar{a}$ and $\bar{\alpha}$ as per Lemma 5.
  \item Initialize $\hat{A} := \emptyset$ and $\hat{k} := k$.
  \item For all $i \in I_{\pi}$, do:
    \begin{enumerate}
      \item Compute $\tilde{j} := \max\{0, \lceil\delta^{-1}(v_i \cdot \bar{a})\rceil\}$ and $\bar{j} := \lfloor\delta^{-1}(v_i \cdot \bar{\alpha})\rfloor$.
      \item Add all $d_j/v_i$ to $\hat{A}$ for which $\tilde{j} \leq j \leq \bar{j}$.
      \item Update $\hat{k} \leftarrow \hat{k} - \tilde{j}$.
    \end{enumerate}
  \item Select and return $\hat{A}_{(\hat{k})}$.
\end{enumerate}

\textbf{Theorem 6:} \texttt{Algorithm 3} computes $a^*$ in time $O(n)$ for any divisor sequence $d$ that fulfills the requirements of Lemma 5.
6. Comparison of Algorithms

**Proof:** The first steps are the same as for Select\textsuperscript{AstNaive}. We then construct multiset \( \hat{A} \subseteq A \) as the subsequent union of \( A_i \cap [a, \pi] \), that is

\[
\hat{A} = \bigcup_{i \in I_x} \left\{ \frac{d_j}{v_i} \mid j(i) \leq j \leq \bar{j}(i) \right\}
\]

\[
= \bigcup_{i \in I_x} \left\{ \frac{d_j}{v_i} \in A \mid \delta^{-1}(v_i \cdot a) \leq j \leq \delta^{-1}(v_i \cdot \pi) \right\}
\]

\[
= \bigcup_{i \in I_x} \left\{ \frac{d_j}{v_i} \in A \mid v_i \cdot a \leq d_j \leq v_i \cdot \pi \right\}
\]

\[
= \bigcup_{i \in I_x} \left\{ \frac{d_j}{v_i} \in A \mid a \leq d_j \leq a \right\}
\]

\[
= A \cap [a, \pi].
\]

In particular, the last step follows from (2) with \( \pi > a^* \) (cf. the proof of Theorem 3).

By Lemma 5, we know that \( a \leq a^* \leq \pi \) for the bounds computed in step 4, so we get in particular that \( a^* \in \hat{A} \).

It remains to show that we calculate \( \hat{k} \) correctly. Clearly, we discard with \( (a_{i,0}, \ldots, a_{i,j-1}) \) exactly \( j \) elements in step 6.2 that is \( |A_i \cap (-\infty, a]| = \bar{j}(i) \). Therefore, we compute with

\[
\hat{k} = k - \sum_{i \in I_x} |A_i \cap (-\infty, a]| = r(a^*, A) - |A \cap (-\infty, a]| = r(a^*, \hat{A})
\]

the correct rank of \( a^* \) in \( \hat{A} \).

For the running time, we observe that the computations in steps 1 to 5 are easily done in \( O(n) \) time. The loop in step 6 and therewith steps 6.1 and 6.3 are executed \( |I_x| \leq n \) times. The overall number of set operations in step 6.2 is \( |A| \in O(|I_x|) \subseteq O(n) \) (cf. Lemma 5). Finally, step 7 runs in time \( O(|\hat{A}|) \subseteq O(n) \) when using a (worst-case) linear-time rank selection algorithm (e.g., the median-of-medians algorithm [Blu+73]). \( \Box \)

6. Comparison of Algorithms

Our final algorithm Select\textsuperscript{Ast} is conceptually simple in the sense that there is little hidden complexity. We need exactly one call to a rank selection algorithm on a linear-size list which takes five additional linear-time operations to come up with: finding the maximal value \( v^{(1)} \), constructing index set \( I_\pi \), computing \( V_\pi \), constructing multiset \( \hat{A} \) and computing \( \hat{k} \). These are all quite elementary tasks in that they use one for-loop each which run for at most \( n \) iterations with only few operations in each.

On the other hand, the linear-time algorithm by Cheng and Eppstein [CE14] (we call it Ast\textsuperscript{ChengEppstein}, cf. Appendix B) computes a linear number of medians and
requires a linear number of evaluations of rank function $r(x, A)$; the (multi)sets in the
calls are geometrically shrinking – otherwise the algorithm would not run in linear time –
but these selection calls may be quite costly in practice.

Therefore, we suspect that $\text{SELECT\!A\!STAR}$ is more efficient in terms of runtime in seconds
than $\text{A\!STAR\!C\!H\!E\!N\!G\!E\!P\!P\!STE\!IN}$. We will investigate that claim in the sequel.

6. The Programmer’s Perspective

We have implemented both algorithms in Java [RW15a] with a focus on performance
rather than neat API design. The number of lines of code alone – even though this
is admittedly a simplistic measure – shows that $\text{SELECT\!A\!STAR}$ is less complex than
$\text{A\!STAR\!C\!H\!E\!N\!G\!E\!P\!P\!STE\!IN}$. Our Java classes have 61 resp. 264 lines of code. The interested
reader may want to inspect the sources and decide for themselves which algorithm is
simpler w.r.t. their measure of choice.

It took us some hours to come up with implementations of both algorithms that return
the same, correct result. For $\text{SELECT\!A\!STAR}$, the delicate part was to get the bounds on $j$
(cf. step 6.1) right. We use floor and ceiling functions on real numbers, so rounding errors
that occur fixed-precision floating-point arithmetic can cause harm. However, we can
safely add (subtract) a conservatively large constant to the mantissa of the floats before
taking floors (ceilings). If this constant is larger than necessary for covering rounding
errors we will slightly degrade performance but not affect correctness.

For $\text{A\!STAR\!C\!H\!E\!N\!G\!E\!P\!P\!STE\!IN}$, we also have to compute such floors and ceilings, but here
we have to be exact since we otherwise compute a wrong result. We evaluate $r(x, A)$
several times by computing terms of the form $\lfloor \delta^{-1}(\_\_\_\_) \rfloor$ (cf. Lemma 2). The problem is
that the result of $\delta^{-1}(\_\_\_\_)$ is non-integral in general, but is integral when the argument
evaluates exactly to a $d_j$. With the usual floating-point arithmetic the result might
be slightly smaller, though, which we then erroneously round down to the next smaller
integer – a critical error! We may apply the same “hack” as above in practice, that is add
a small constant to the mantissa before taking the floor. This constant has to be chosen
large enough to cover potential rounding errors, but here also small enough as to not
change other results. This is a very delicate requirement! The hack furthermore requires
that numbers in the input are not too close. We do not have a satisfactory solution to
this problem other than switching to much slower arbitrary-precision arithmetic.

6.2. Running Time

We compare $\text{SELECT\!A\!STAR}$ and $\text{A\!STAR\!C\!H\!E\!N\!G\!E\!P\!P\!STE\!IN}$ by running both on artificially
created apportionment instances. In order to get a feeling for the potential impact the
structure of vote counts can have, we consider two random models. For given $n$, we draw
the vote counts i.i.d. either (a) uniformly from $\{1, \ldots, 100\}$, or (b) from an exponential
distribution with parameter $\lambda = 10$. We also choose house size $k$ at random, namely
6. Comparison of Algorithms

Figure 1: This figure shows running times of \texttt{SELECTASTAR} and \texttt{ASTARCHENGEPSTEIN} normalized by the number of vote counts $n$. The inputs are random apportionment instances across several orders of magnitude of $n$, with vote counts drawn i.i.d. uniformly from $\{1, \ldots, 100\}$ (left plot) respectively with exponential distribution of rate 10 (right plot). In both cases, $k$ is drawn uniformly at random from $\{n, \ldots, 10n\}$. The lines show averages excluding $n = 10^6$ and $n = 10^7$.

For both algorithms, the number of arithmetic operations is linear in $n$ and independent of $k$, so we use $n$ as the free variable in our tests.

We only consider the time needed to compute $a^*$, the optimal cutoff value; we have discussed in Section 3 how to determine the actual seat allocation from this value. Since this procedure is clearly independent of how $a^*$ itself was found, we can disregard it in our comparison.

We describe the machine configuration used in the running time tests and further details of the setup in Appendix C.

Figure 1 shows the results of our running time tests; it is obvious that \texttt{SELECTASTAR} is indeed faster than \texttt{ASTARCHENGEPSTEIN}. To quantify the difference, we normalized all results by $n$, so that one would expect to see a constant line for each algorithm in Figure 1; however, both algorithms slow down significantly from $n \geq 10^6$ upwards. We conjecture that for small sizes, all data still fits in fast levels of the memory hierarchy and from $n = 10^6$ on, caching effects kick in. Further studies are needed to investigate this issue, though. For this article, we ignore the large instances; the lines in Figure 1 show the average time per party for $n \leq 10^5$. According to this measure, \texttt{SELECTASTAR} is about 6.5 times faster than \texttt{ASTARCHENGEPSTEIN}.

Both algorithms show very similar behavior for our two random models; we take this as evidence for their stability w.r.t. the structure of the vote counts.

In practice, we usually have rather tiny instances compared to the $n$ used in Figure 1 so one might question the necessity of intricate, asymptotically efficient algorithms al-

\footnote{This would not be true were we to compare with \texttt{HIGHESTAVERAGES} as well.}
7. Conclusion

together. Therefore, we also compare the two linear-time algorithms with a simple implementation of \texttt{SelectAstarNaive} for uniformly random instances with $n = 50$ (the number of states to apportion for the U.S. House of Representatives) and $k$ drawn uniformly at random from \{50, \ldots, 500\} (compare to $k = 435$ for the House). In this setting, \texttt{SelectAstarNaive} needed on average $\approx 1.08$ ms to compute $a^*$, \texttt{AstarChengEppstein} required $\approx 0.436$ ms, and \texttt{SelectAstar} took $\approx 0.062$ ms.

7. Conclusion

According to our experiments, \texttt{AstarChengEppstein} \cite{CE14} is six to seven times slower than our \texttt{SelectAstar} (cf. 9), and a naive algorithm is two orders of magnitude slower even for small input sizes. Whenever a large number of simulations is needed – say for statistical investigation of different divisor sequences – a speed-up by such a factor certainly pays off. Our algorithm \texttt{SelectAstar} as introduced in this paper is not only reasonably simple to implement, but also the currently most efficient method in practice to compute proportional apportionments (to the best of our knowledge).

Acknowledgments

We thank Chao Xu for pointing us towards the work by Cheng and Eppstein \cite{CE14} and noting that the problem of envy-free stick-division \cite{RW15b} is related to proportional apportionment as discussed there. He also observed that our approach for cutting sticks – the core ideas of which turned out to carry over to this article – could be improved to run in linear time.

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A. Proofs of Lemmata

A.1. Proof of Lemma 2

By eq.(1) on page 5, it suffices to show that
\[
\left| \left\{ a_{i,j} \mid a_{i,j} \leq x \right\} \right| = \left\lfloor \delta^{-1}(v_i \cdot x) \right\rfloor + 1
\]
for each \( i \in \{1, \ldots, n\} \). Now, if \( x \geq a_{i,j} = d_j/v_i \) for some \( j \), then \( v_i \cdot x \geq d_j \), and so \( \left\lfloor \delta^{-1}(v_i \cdot x) \right\rfloor \) is the largest index \( j' \) for which \( a_{i,j'} = d_j'/v_i \leq x \). As \( d_j \) is zero-based, there are \( j' + 1 \geq 1 \) such elements \( a_{i,j} \leq x \) and the equation follows.

Otherwise, that is \( a_{i,j} > x \) for all \( j \), we have \( j' = \left\lfloor \delta^{-1}(v_i \cdot x) \right\rfloor = -1 \) by iv) and Corollary 1 and the equality holds with 0 on both sides.

For the second equality, we only have to show that the omitted summands are zero. So let \( i \notin I_x \) be given, that is \( v_i \leq d_0/\bar{x} \). For \( x < \bar{x} \), we have
\[
v_i \cdot x \leq \frac{d_0}{\bar{x}} \cdot x < \frac{d_0}{\bar{x}} \cdot \bar{x} = d_0,
\]
and hence \( \left\lfloor \delta^{-1}(v_i \cdot x) \right\rfloor = -1 \) by iv).

A.2. Proof of Lemma 4

We show that smaller \( a \) are infeasible and larger \( a \) are clearly suboptimal, so the optimal \( a^* \) must lie in between. Let us first consider \( a < \underline{a} \). There are two cases: if there is a \( v_i \), such that \( v_i a \geq d_0 \), we get by strict monotonicity of \( \delta^{-1} \)
\[
r(a) < \left| I_\bar{x} \right| + \sum_{i \in I_{\bar{x}}} \delta^{-1}(v_i \cdot a)
\]
\[
< \left| I_{\bar{x}} \right| + \sum_{i \in I_{\bar{x}}} \delta^{-1}(v_i \cdot \underline{a})
\]
\[
\leq k
\]
and \( a \) is infeasible. If otherwise \( v_i a < d_0 \), i.e., \( a < d_0/v_i \), for all \( i \), \( a \) must clearly have rank \( r(a) = 0 \) as it is smaller than any element \( a_{i,j} \in A \). In both cases we found that \( a < \underline{a} \) has rank \( r(a) < k \).

Now consider the upper bound, i.e., we have \( a > \bar{a} \). In case \( \bar{a} \geq \bar{x} \), we have \( a > \bar{x} > a^* \) by assumption and any such \( a \) cannot be optimal. Otherwise, for \( \bar{a} < \bar{x} \), we have
\[
r(\bar{a}) \geq \sum_{i \in I_{\bar{x}}} \delta^{-1}(v_i \cdot \bar{a}) \geq k,
\]
so \( \bar{a} \) is feasible. Any element \( a > \bar{a} \) can thus not be the optimal solution \( a^* \), which is the minimal \( a \) with \( r(a) \geq k \), see eq.(3) on page 6.
A. Proofs of Lemmata

A.3. Proof of Lemma 5

We consider the linear divisor sequence continuations \( \delta(j) = \alpha j + \tilde{\beta} \) and \( \delta(j) = \alpha j + \beta \) for all \( j \in \mathbb{R}_{\geq 0} \) and start by noting that the inverses are \( \delta^{-1}(x) = \frac{x}{\alpha} - \tilde{\beta}/\alpha \) and \( \delta^{-1}(x) = \frac{x}{\alpha} - \beta/\alpha \) for \( x \geq \tilde{\delta}(0) = \tilde{\beta} \) resp. \( x \geq \delta(0) = \beta \). For smaller \( x \), we are free to choose the value of the continuation from \([-1, 0)\), see iv); noting that \( \frac{x}{\alpha} - \beta/\alpha < 0 \) for \( x < \beta \), a choice that will turn out convenient is

\[
\delta^{-1}(x) := \max \left\{ \frac{x}{\alpha} - \frac{\tilde{\beta}}{\alpha}, -1 \right\} \quad \text{resp.} \quad \tilde{\delta}^{-1}(x) := \max \left\{ \frac{x}{\alpha} - \frac{\beta}{\alpha}, -1 \right\}. \tag{6}
\]

We state the following simple property for reference; it follows from \( \delta(j) \leq \delta(j) \leq \delta(j) \) and the definition of the inverses:

\[
\frac{x}{\alpha} - \frac{\beta}{\alpha} \leq \delta^{-1}(x) \leq \delta^{-1}(x) \leq \frac{x}{\alpha}, \quad \text{for } x \geq 0. \tag{7}
\]

Equipped with these preliminaries, we compute

\[
\bar{a} = \frac{\alpha k + \beta |I_{\tau}|}{V_{\tau}},
\]

\[
\iff \frac{\bar{a}}{\alpha} \sum_{i \in I_{\tau}} v_i = k + \frac{\beta}{\alpha} |I_{\tau}|,
\]

\[
\iff k = \sum_{i \in I_{\tau}} \left( \frac{v_i \cdot \bar{a}}{\alpha} - \frac{\beta}{\alpha} \right) \leq \sum_{i \in I_{\tau}} \delta^{-1}(v_i \cdot \bar{a}),
\]

so \( \bar{a} \) satisfies the condition of Lemma 4. Similarly, we find

\[
\bar{a} = \frac{\alpha k - \alpha |I_{\tau}|}{V_{\tau}},
\]

\[
\iff \frac{\bar{a}}{\alpha} V_{\tau} = k - |I_{\tau}|,
\]

\[
\iff k = |I_{\tau}| + \sum_{i \in I_{\tau}} \frac{v_i \cdot \bar{a}}{\alpha} \geq |I_{\tau}| + \sum_{i \in I_{\tau}} \delta^{-1}(v_i \cdot \bar{a}),
\]

i.e., \( \bar{a} \) also fulfills the conditions of Lemma 4.

For the bound on the number of elements falling between \( \bar{a} \) and \( \bar{\pi} \), we compute

\[
|A \cap [\bar{a}, \bar{\pi}]| = \sum_{i \in I_{\tau}} |A_i \cap [\bar{a}, \bar{\pi}]| = \sum_{i \in I_{\tau}} \left| \left\{ j \in \mathbb{N}_0 \mid a \leq \frac{d_j}{v_i} \leq \bar{\pi} \right\} \right| = \sum_{i \in I_{\tau}} \left| \left\{ j \in \mathbb{N}_0 \mid v_i \cdot \bar{a} \leq d_j \leq v_i \cdot \bar{\pi} \right\} \right|
\]

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\[ \sum_{i \in I} \left| \left\{ j \in \mathbb{N}_0 \mid \delta^{-1}(v_i \cdot a) \leq j \leq \delta^{-1}(v_i \cdot \overline{a}) \right\} \right| \]
\[ \leq \sum_{i \in I} \left( \delta^{-1}(v_i \cdot \overline{a}) - \delta^{-1}(v_i \cdot a) + 1 \right) \]
\[ \leq \sum_{i \in I} \left( \frac{v_i \cdot \overline{a} - v_i \cdot a - \beta}{\alpha} + 1 \right) \]
\[ = \sum_{i \in I} \left( \frac{v_i \cdot \overline{a} - v_i \cdot a}{\alpha} + (1 + \beta/\alpha) \right) \]
\[ = \left( 1 + \frac{\beta}{\alpha} \right) |I| + \frac{(\overline{a} - a)V}{\alpha} \]
\[ = \left( 1 + \frac{\beta}{\alpha} \right) |I| + \frac{(\alpha + \beta)|I|V}{\alpha} \]
\[ = 2 \left( 1 + \frac{\beta}{\alpha} \right) |I|. \]

B. The Algorithm of Cheng and Eppstein

Cheng and Eppstein [CE14] do not give pseudocode for the main procedure of their algorithm which would combine the individual steps to compute \( A(k) \). For the reader’s convenience and for clarity concerning our running-time comparisons we give this top-level procedure as we have inferred it.

Algorithm 4: *AstarChengEppstein*(\( v, k \)):

Step 1 Compute a suitable finite representation of \( A \).

Step 2 \( C := \text{FindContributingSequences}(A, k) \).

Step 3 \( \xi := s^{-1}(k) \) [CE14, (3)].

Step 4 If \( r(\xi, A) \geq k \) then
\[ \xi := \text{LowerRankCoarseSolution}(A, k, \xi). \]

Step 5 Return \( \text{CoarseToExact}(A, k, \xi) \).

The subroutines are given in sufficient detail by Cheng and Eppstein [CE14] in Algorithms 1 to 3, respectively.

We have implemented the algorithm in detail for our runtime study [RW15a]. We have taken care not to render the algorithm uselessly inefficient in order to perform a fair comparison on runtimes; the result is to the best of our abilities conditioned on a limited time budget. In particular, all of our implementations have been refined on the programming level to the same degree.
Note that we have (hopefully) fixed an off-by-one mistake in the text. The definition of rank $r(x, A)$ is, “the number of elements of $A$ less than or equal to $x$”; that is, the rank of $A(j)$ is $j + 1$ since $A$ is zero-based (the first element is $A(0)$). However, the authors continue to say that $r(x, A)$ “is the index $j$ such that $A(j) \leq x < A(j+1)$.”

C. Experimental Setup

We have run the experiments with Java 7 on Ubuntu 12.04 LTS running kernel 3.2.0-80-generic x86_64 GNU/Linux. The hardware platform is a ThinkPad X201 Tablet with the following core parameters according to lshw.

**CPU:** Intel® Core™ i7 CPU L 620 @ 2.00GHz

**Cache:** L1 32KiB, L2 256KiB, L3 4MiB

**RAM:** 2+4GiB SODIMM DDR3 Synchronous 1334 MHz (0.7 ns)

For all running time tests, we use divisor sequence $d_j = 2^{j} + 1$ (Sainte-Laguë). As our code is written in Java, we include a warm-up phase to trigger just-in-time compilation of our methods. All times are measured using the built-in method `System.nanoTime()`.

We use the same set of inputs for all algorithms. For smallish $n$, we repeat the execution of the algorithm on each input several times and measure the total time to increase accuracy; we then consider the average time. The Java library does not contain a rank selection algorithm, therefore we use the randomized Quicksort-based implementation by Sedgewick and Wayne [SW11].

For reproducing our running time experiments, run

```
java RunningTimeMain rw,ce 50 100000 10 42424242 exponential 2 1
java RunningTimeMain rw,ce 50 100000 10 42424242 uniform 2 1
java RunningTimeMain rw,ce 100,1000 10000 10 42424242 uniform 2 1
java RunningTimeMain rw,ce 100,1000 10000 10 42424242 exponential 2 1
java RunningTimeMain rw,ce 10000,100000 100 100 42424242 uniform 2 1
java RunningTimeMain rw,ce 10000,100000 100 100 42424242 exponential 2 1
java RunningTimeMain rw,ce 1000000 100 100 42424242 uniform 2 1
java RunningTimeMain rw,ce 1000000 100 100 42424242 exponential 2 1
```

for the data represented in Figure 1.
D. Index of Used Notation

In this section, we collect the notation used in this paper. Some might be seen as “standard”, but we think including them here hurts less than a potential misunderstanding caused by omitting them.

Generic Mathematical Notation

\[ [x], [x] \] floor and ceiling functions, as used in [GKP94].

\( M(k) \) The \( k \)th smallest element of (multi)set/vector \( M \) (assuming it exists); if the elements of \( M \) can be written in non-decreasing order, \( M \) is given by \( M(1) \leq M(2) \leq M(3) \leq \cdots \).

Example: For \( M = \{5, 8, 8, 10, 10\} \), we have \( M(1) = 5 \), \( M(2) = M(3) = M(4) = 8 \), and \( M(5) = M(6) = 10 \).

\( M^{(k)} \) Similar to \( M(k) \), but \( M^{(k)} \) denotes the \( k \)th largest element.

\( x = (x_1, \ldots, x_d) \) to emphasize that \( x \) is a vector, it is written in bold; components of the vector are written in regular type.

\( M \) to emphasize that \( M \) is a multiset, it is written in bold calligraphic type.

\( M_1 \sqcup M_2 \) multiset union; multiplicities add up.

Notation Specific to the Problem

count, chamber size  
Parties are assigned seats (in parliament), so that the number of seats \( s_i \) that party \( i \) is assigned is (roughly) proportional to that party’s vote count \( v_i \) and the overall number of assigned seats equals the chamber size \( k \).

d = (d_j)_{j=0}^\infty \) the divisor sequence used in the highest averages method; \( d \) must be a nonnegative, (strictly) increasing and unbounded sequence.

\( \delta, \delta^{-1} \) a continuation of \( j \mapsto d_j \) on the reals and its inverse, both of which can be evaluated in constant time.

\( n \) number of parties in the input.

\( v, v_i \) vote counts of the parties in the input.

\( V \) the sum \( v_1 + \cdots + v_n \) of all vote counts.

\( k \) \( k \in \mathbb{N} \), the number of seats to be assigned; also called house size.

\( s, s_i \) the number of seats assigned to the respective parties; the result.

\( a_{i,j} \) \( a_{i,j} := d_j / v_i \), the reciprocal of the ratio used in \textsc{HighestAverages}.
D. Index of Used Notation

is the party, \( j \) is the number of seats \( i \) has already been assigned.

\( A_i \) . . . . . . . . . . For party \( i \), \( A_i := \{a_{i,0}, a_{i,1}, a_{i,2}, \ldots \} \) is the list of (reciprocals of) party \( i \)'s ratios.

\( a \) . . . . . . . . . . . We use \( a \) as a free variable when an arbitrary \( a_{i,j} \) is meant.

\( A \) . . . . . . . . . . . \( A := A_1 \uplus \cdots \uplus A_n \) is the multiset of all averages.

\( r(x, A) \) . . . . . . . . . . . the rank of \( x \) in \( A \), that is the number of elements in multiset \( A \) that are no larger than \( x \); \( r(x) \) for short if \( A \) is clear from context.

\( a^* \) . . . . . . . . . . . the ratio \( a^* = a_{i,j} \), selected for assigning the last (i.e. the \( k \)th) seat; corresponds to \( s \) by \( s_i = r(a^*, A_i) \); \( a^* = A_{(k)} \) (cf. Section 3).

\( \overline{x} \) . . . . . . . . . . . an upper bound \( \overline{x} > a^* \); we use \( \overline{x} = d - 1/v_1 + \varepsilon \), where \( \varepsilon > 0 \) is a suitable constant.

\( I_\overline{x} \) . . . . . . . . . . . \( I_\overline{x} := \{i \mid v_i > d / \overline{x} \} \); the set of parties \( i \) whose vote count is large enough, so that \( a_{i,0} < \overline{x} \), i.e. so that they contribute to the rank of \( \overline{x} \) in \( A \).

\( V_\overline{x} \) . . . . . . . . . . . the sum of the vote counts of all parties in \( I_\overline{x} \).

\( A^\overline{x} \) . . . . . . . . . . . the elements in \( A \) that are smaller than \( \overline{x} \), i.e., \( A \cap (-\infty, \overline{x}) \).

\( a, \overline{x} \) . . . . . . . . . . . lower and upper bounds on candidates \( a \leq a \leq \overline{x} \) such that still \( a^* \in A \cap [a, \overline{x}] \).