Riemannian geometry on contact Lie groups

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1 Introduction–summary

A contact Lie group is a Lie group $G$, say of dimension $2n + 1$, having a differential 1-form $\eta^+$ which is invariant under left translations by the group elements (left invariant, for short) and which satisfies $(d\eta^+)^n \wedge \eta^+ \neq 0$ pointwise over $G$, where $d\eta^+$ is the de Rham differential of $\eta^+$. If $\mathcal{G}$ is the Lie algebra of $G$ and $\eta := \eta^+_\epsilon$ the value of $\eta^+$ at the unit $\epsilon$ of $G$, then $(\mathcal{G}, \eta)$ is called a contact Lie algebra.

Such Lie groups may somehow be seen as playing the odd dimensional version of Lie groups admitting a left invariant symplectic structure (symplectic Lie groups). Historically, these latter have been largely studied by many authors, in line with numerous interesting problems in geometry and physics, see e.g. [4–6,10–12,18,19,25]. But although contact topology and geometry are increasingly acquiring a popular interest due to their numerous applications, contact Lie groups have not been so widely explored (see e.g. [9]).

Amongst other results in [9], we gave a method to construct contact Lie algebras of dimension $2n + 1$ with a trivial centre, unlike the ordinary contactization (central extension) which
only produces contact Lie algebras with center of dimension 1. We discussed some applications of such a construction. It turns out that the Lie algebra \( G_1 \) of any exact symplectic Lie group \((G_1, d\alpha^+)\) can be embedded as a codimension 1 subalgebra of many non-isomorphic Lie algebras all of which having a family of contact forms whose restriction (pullback) to \( G_1 \) coincide with the 1-form \( \alpha \). We also gave a full classification of contact Lie algebras of dimension 5 and exhibited an infinite family of non-isomorphic solvable contact Lie algebras of dimension 7.

Here we consider contact Lie groups \( G \) which display an additional structure, namely a left invariant Riemannian or pseudo-Riemannian metric with specific properties such as being bi-invariant, flat, negatively curved, Einstein, etc. The reason for considering these additional structures lies, on the one hand, in the interest in (pseudo-) Riemannian geometry. On the other hand, this can also be motivated by the fact that the relationship between the contact and the algebraic structures of Lie groups does not, a priori, show to be strong enough to ensure certain general consequences or to affect certain invariants of Lie groups. So one has to consider specific families of Lie groups. Indeed, the main obstructions to the existence of left invariant contact structures on odd dimensional Lie groups so far known to the author, are the non-degeneracy of their Killing form (Theorem 5 of [3]) and the dimension of their centre (it is readily checked that the centre should have dimension \( \leq 1 \)).

In this paper, using some known results from Riemannian Geometry, we classify contact Lie groups (via their Lie algebras) having some given Riemannian or pseudo-Riemannian structure, give properties and derive some obstructions to the existence of left invariant contact structures on Lie groups. We carry out a comparison with the symplectic case, whenever we find it interesting. Contact Lie groups turn out to exhibit some behaviour different from that of symplectic ones: in presence of some ‘nice’ (pseudo-) Riemannian structures, contact structures can exist in abundance where under the same assumptions in even dimensions symplectic ones would be rather rare and vice versa (see Remarks 1, 2, 3, 5).

For the present purposes, we only need to use the presence of left invariant contact and some given Riemannian or pseudo-Riemannian structures on the same Lie group. The actual relationship of such structures between one another as in [2], will be discussed in a subsequent paper. Below, we quote some of our main results.

A Riemannian or pseudo-Riemannian structure on a Lie group is said to be bi-invariant if it is invariant under both left and right translations, see [20,21]. The Killing forms of semi-simple Lie groups are examples of such bi-invariant structures. In Theorem 5 of [3], W.M. Boothby and H.C. Wang proved, by generalizing a result from J.W. Gray [13], that the only semi-simple Lie groups that carry a left invariant contact structure are those which are locally isomorphic to \( SL(2) \) or to \( SO(3) \). We extend such a result to all Lie groups with bi-invariant Riemannian or pseudo-Riemannian structures.

**Theorem 1** Let \( G \) be a Lie group. Suppose (i) \( G \) admits a bi-invariant Riemannian or pseudo-Riemannian metric and (ii) \( G \) admits a left invariant contact structure. Then \( G \) is locally isomorphic to \( SL(2, \mathbb{R}) \) or to \( SU(2) \).

This contrasts with the symplectic case, see Theorem 2 and Remarks 1, 2.

In his main result of [1] (see also [2]), D.E. Blair proved that a flat Riemannian metric in a contact manifold \( M \) of dimension \( \geq 5 \), cannot be a contact metric structure (see Sect. 2 for the definition). We prove that in the case of contact Lie groups of dimension \( \geq 5 \), there is no flat left invariant Riemannian metric at all, even if such a metric has nothing to do with the given contact structure.