ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS TO A
CLASS OF HEAT EQUATIONS WITH
GRADIENT NONLINEARITY

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(Communicated by Michael Winkler)

Abstract. The paper is devoted to investigating a semilinear parabolic equation with a nonlinear gradient source term:
\[ u_t = u_{xx} + x^m |u_x|^p, \quad t > 0, \quad 0 < x < 1, \]
where \( p > m + 2, \ m \geq 0 \). Zhang and Hu [Discrete Contin. Dyn. Syst. 26 (2010) 767-779] showed that finite time gradient blowup occurs at the boundary and the accurate blowup rate is also obtained for super-critical boundary value. Throughout this paper, we present a complete large time behavior of a classical solution \( u \): \( u \) is global and converges to the unique stationary solution in \( C^1 \) norm for subcritical boundary value, and \( u_x \) blows up in infinite time for critical boundary value. Gradient growup rate is also established by the method of matched asymptotic expansions. In addition, gradient estimate of solutions is obtained by the Bernstein-type arguments.

1. Introduction. In this paper, we consider the following problem

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = u_{xx} + x^m |u_x|^p, \quad t > 0, \quad 0 < x < 1, \\
&u(t, 0) = 0, \quad u(t, 1) = M, \quad t > 0, \\
&u(0, x) = u_0(x), \quad 0 < x < 1,
\end{aligned}
\]

where \( p > m + 2, \ m \geq 0 \) and \( M \geq 0 \). The initial data \( u_0 \) satisfies:
\[ u_0 \in X := \{ v \in C^1([0, 1]): \ v(0) = 0, \ v(1) = M \} \]
and
\[ u_0(x) \leq U(x), \quad x \in [0, 1], \]

2020 Mathematics Subject Classification. 35A01, 35B40, 35B44, 35K20.

Key words and phrases. Asymptotic behavior, boundedness, gradient nonlinearity, gradient growup rate, Lyapunov functional, matched asymptotic expansions.

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where \( U(x) \) is the singular steady state of (1) for critical boundary value \( M = M_c \).
Problem (1) admits a unique maximal classical solution \( u = u(t, x) \), whose existence time will be denoted by \( T^* = T^*(u_0) \in (0, \infty) \). Note that we make no restriction on the sign of \( u \) or \( u_x \).

The equation (1) arises from several interesting mathematical and physical fields. The Kardar-Parisi-Zhang(KPZ) equation

\[
\partial_t h = \nu \Delta h + \lambda |\nabla h|^2 + \eta(t, x)
\]
was first proposed in [24], and it can be considered as the fundamental nonlinear Langevin equation. Here \( h(t, x) \) describes the height of the surface at time \( t \) above the point \( x \) in the reference plane. Let \( \nu \Delta h \) be diffusional relaxation with diffusion coefficient \( \nu \) related to the surface tension. The growth term \( \lambda |\nabla h|^2 \) corresponds to the deposition of new particles on the surface, and \( \lambda \) accounts for the growth speed and modifies the universal scaling properties of the model if \( \lambda \neq 0 \). The noise term \( \eta(t, x) \), which relates with molecular collision, is irrelevant.

To account for further surface growth effects, Krug-Spohn [26] considered KPZ equation with more general nonlinearity:

\[
\partial_t h = \nu \Delta h + \lambda |\nabla h|^{\beta}
\]
where \( \beta \geq 1 \). The large-scale properties of the surface are governed by scaling exponents \( \beta \) in the stochastic case. Some properties of solutions to this equation have been studied numerically in[2, 32]. In this paper, we consider the case \( \lambda = x^m \), which means that, there is a power-like degeneracy at \( x = 0 \), and we clarify clearly the effects of the degeneracy on the long time behavior of global solutions.

During the past several decades, an abundance of works regrading existence and asymptotic behavior of gradient blowup solutions to the equation

\[
u_t = \Delta u + F(t, x, u, \nabla u)
\]
(4)
have emerged in the literatures. Gradient blowup is the phenomenon which \( u_x \) blows up in \( L^\infty \) norm as \( t \to T^* \), whereas \( u \) remains uniformly bounded (see [37] and references therein). Several problems like blowup conditions, blowup time, blowup rates, blowup locations and blowup profiles have attracted a lot of attentions of many scholars. In [8], Fila and Lieberman studied the case \( F = F(u_x) \), \( 0 < x < L \), \( F \in C^2(\mathbb{R}) \), they proved that for any \( u_0 \in C^1([0, L]) \), there exists a \( T \in (0, \infty) \) such that \( \limsup_{t \to T} u_x(t, 0) = \infty \). Considering the case \( F = F(u, \nabla u) \) with general boundary conditions, Souplet [37] showed that gradient blowup occurs for suitable large initial data, which solutions are global and bounded in \( C^1 \) norm if the initial data are small enough. Many mathematicians pay special attention to (4) in the case \( F = F(\nabla u) = |\nabla u|^p \), a vast literatures exists [28, 40, 17, 39, 3]. It was proved in [40] that gradient blowup can only occur on the boundary and an upper estimate of blowup profile of \( |\nabla u| \) is given. Li and Souplet [28] provided the precise location of gradient blow-up points within the boundary under suitable assumptions on \( \Omega \subset \mathbb{R}^2 \) and the initial data \( u_0 \). Guo and Hu [17] obtained that \( (T - t)^{-\frac{1}{p-2}} \) is the exact blowup rate in the one-dimensional case for time-increasing solutions. The self-similar rate is not the only rate, there are other rates which is usually refereed to as Type II rate, for which we refer the readers to the survey works [10, 15, 18, 19, 20, 21].

We take into account the problem (1), corresponding to the case \( F = F(x, u_x) = x^m|u_x|^p \) under special boundary conditions. When \( m = 0 \), \( M > M_c = \frac{(p-1)^{\frac{m-2}{2}}}{p-2} \),
Alikakos et al. [1] proved that \( u_x \) blows up in a finite time. When \( m > 0 \) for the case \( M > M_c = \frac{(m + 1)^{\frac{1}{p-1}}(p-1)^{\frac{p-2}{p-1}}}{p-m-2} \) to (1), Zhang and Hu [42] proved that gradient blowup occurs exactly at the single boundary point \( x = 0 \) in finite time (see [42, Lemma 3.1]). Meanwhile, the authors derived the blowup rate lower and upper bound estimates: \( u_x(t,0) \geq c_0(T-t)^{-\frac{m+1}{p-1}} \) and \( \sup_{0 \leq \tau \leq T, x \in [0,1]} u_x(\tau, x) \leq C_0(T-t)^{-\frac{m+1}{p-1}} \) (see [42, Theorems 4.1 and 5.2]. In [42], conditions are added to \( u_0 \) such that \( u_t \geq 0 \) and \( u_x \geq 0 \), which is not required in the present paper.

As we all know, steady-state solutions are the possible limits as \( t \to \infty \) of the corresponding time-dependent solutions if they exist globally. As for the steady states, there are three cases according to the value of \( M \).

(i) if \( 0 \leq M < M_c \), the problem (1) admits a unique stationary solution \( V_M = V_M(x) \in C^2([0,1]) \), an explicit formula will be given in Section 2. \( M_c \) is the critical value

\[
M_c = \frac{(m + 1)^{\frac{1}{p-1}}(p-1)^{\frac{p-2}{p-1}}}{p-m-2}.
\]

Further, the steady state \( V_M \) increases monotonically with respect to \( M \).

(ii) if \( M = M_c \), the problem (1) admits a steady state solution \( V_{M_c} = U \), with

\[
U(x) = M_c x^{\frac{n-m-2}{p-1}}.
\]

Clearly, \( U(x) \in C([0,1]) \cap C^2((0,1]) \) and \( U_x(0) = \infty \), so it is singular at \( x = 0 \).

(iii) if \( M > M_c \), there is no steady state.

Next, a fundamental result is that the solution of (1) satisfies the maximum principle:

\[
\min \left\{ \min_{[0,1]} u_0, 0 \right\} \leq u(t,x) \leq \max \left\{ \max_{[0,1]} u_0, M \right\}, \quad 0 \leq t < T^*, \quad 0 \leq x \leq 1. \tag{5}
\]

Since the problem (1) is well-posed in \( C^1 \), so only three possibilities can occur:

(I) \( u \) is global in time and is bounded in \( C^1 \) norm:

\[
T^* = \infty \text{ and } \sup_{t \geq 0} \| u_x(t, \cdot) \|_\infty < \infty.
\]

(II) \( u \) blows up in finite time in \( C^1 \) norm:

\[
T^* < \infty \text{ and } \lim_{t \to T^*} \| u_x(t, \cdot) \|_\infty = \infty.
\]

(III) \( u \) is global in time but is unbounded in \( C^1 \):

\[
T^* = \infty \text{ and } \lim_{t \to \infty} \| u_x(t, \cdot) \|_\infty = \infty.
\]

For results on case (I), we refer to [3, 4, 40, 44, 29]. For (1), assuming \( 0 \leq M < M_c \) and \( m = 0 \), Arrieta et al. [3] derived that \( u \) is global and converges in \( C^1 \) norm to the unique stationary solution. Recently, Attouchi [4] modified the method used in [3] and extended results to the one-dimensional degenerate Hamilton-Jacobi equation:

\[
\begin{cases}
u_t = \left( |u_x|^{p-2} u_x \right)_x + |u_x|^q, & t > 0, \ 0 < x < 1, \\
u(t,0) = 0, \ u(t,1) = M, & t > 0, \\
u(0,x) = u_0(x), & 0 < x < 1,
\end{cases}
\]

where \( q > p > 2 \). It was showed that \( u \) is global and converges in \( W^{1,\infty}(0,1) \) to the stationary solution. The paper [4] faces a number of additional technical difficulties,
caused by the lack of regularity. For higher dimensional case, Li [29] focused on the following problem:
\[
\begin{align*}
  u_t &= \Delta u + |\nabla u|^p, \quad t > 0, \ x \in B_{r,R}, \\
  u(t,x) &= 0, \quad t > 0, \ x \in \partial B_r, \\
  u(t,x) &= M, \quad t > 0, \ x \in \partial B_R, \\
  u(0,x) &= u_0(x), \quad x \in B_{r,R},
\end{align*}
\]
where \( B_{r,R} = \{ x \in \mathbb{R}^n; r < |x| < R \}, \partial B_r = \{ x \in \mathbb{R}^n; |x| = r \} \) and \( p > 2 \). Li presented that the global solution converges in \( C^1(B_{r,R}) \) to the unique regular steady state, and obtained uniform exponential convergence rate for subcritical boundary value. Motivated by the results of the works [3, 4, 29], the present paper considers (1) with the weighted nonlinear gradient term, we get similar results (see Theorem 1.1). Due to the accurate estimates (cf. Section 2), classical Lyapunov argument (1) with the weighted nonlinear gradient term, we get similar results (see Theorem 1.1). Due to the accurate estimates (cf. Section 2), classical Lyapunov argument and regularity results (see Propositions 1 and 2), we proceed by contradiction by showing that any \( C^1 \) unbounded global solution would converge to a singular steady state, which does not exist.

For the case (III), Souplet and Zhang [40] studied inhomogeneous Hamilton-Jacobi equation:
\[
\begin{align*}
  u_t - \Delta u &= |\nabla u|^p + \lambda h(x), \quad t > 0, \ x \in B_R, \\
  u(t,x) &= 0, \quad t > 0, \ x \in \partial B_R, \\
  u(0,x) &= u_0(x), \quad x \in B_R,
\end{align*}
\]
where \( p > 2, \ h \in C^1(B_R) \) with \( h \geq 0, \ h \not\equiv 0 \) and \( h \) is radially symmetric. They showed that there exists \( \lambda^* \in (0, \infty) \) such that \( u \) has gradient blowup in infinite time for \( \lambda = \lambda^* \). Further, some mathematicians focus on descriptions of infinite growup rate of gradient, and it can be obtained by matching of inner and outer asymptotic expansions. Souplet and Vázquez [39] devoted to studying a heat equation with a nonlinearity depending on the first-order spatial derivatives of \( u \):
\[
\begin{align*}
  u_t &= u_{xx} + |u_x|^p, \quad t > 0, \ 0 < x < 1, \\
  u(t,0) &= 0, \ u(t,1) = M, \quad t > 0, \\
  u(0,x) &= u_0(x), \quad 0 < x < 1,
\end{align*}
\]
where \( p > 2, \ M \geq 0 \). The rate of divergence of \( u_x(t,0) \) and the asymptotic behavior of \( u(t,x) \) as \( t \to \infty \) are precisely obtained for critical boundary value. The paper seems to be the first example of gradient blowup in the infinite time for a semilinear parabolic equation. For higher dimensional case, Li [29] extended results to (6) and obtained the growup estimate:
\[
\lim_{t \to \infty} \frac{1}{t} \log |u_x(t,x)| = \frac{\lambda_1}{p-2}, \quad x \in \partial B_r,
\]
where \( \nu \) is any normal vector field, and \( \lambda_1 \) is the first eigenvalue of an associated linearized problem. Moreover, the present paper considers the equation (1) with weighted nonlinearity, i.e. \( x^m|u_x|^p \), where \( p > m + 2 \) and \( m \geq 0 \). The weighted term \( x^m \) brings many obstacles, which will be explained later.

The matched asymptotic method can also be used to describe blowup profiles of solutions. We refer the readers to the survey papers [16, 6, 10, 31, 12, 7, 13, 11, 23, 25]. Let us now mention some results already known for the semilinear equation
\[
\begin{align*}
  u_t &= \Delta u + w^p, \quad \text{in} \ \mathbb{R}_+ \times B_R, \\
  u &= 0, \quad \text{on} \ \mathbb{R}_+ \times \partial B_R, \\
  u(0,x) &= u_0(x), \quad \text{in} \ B_R.
\end{align*}
\]
Form Galaktionov and King [12], we know that if \( p = \frac{N+2}{N-2} \) and \( u_0 \) is positive and symmetric, then as \( t \to \infty \),

\[
\log \|u(t, x)\|_\infty \sim \frac{\pi t^2}{4}, \quad \text{for } N = 3.
\]

\[
\log \|u(t, x)\|_\infty \sim 2\sqrt{t}, \quad \text{for } N = 4.
\]

\[
\log \|u(t, x)\|_\infty \sim \gamma_0 t^{\frac{2}{N}}, \quad \text{for } N = 5,
\]

where \( \gamma_0 = \gamma_0(N) > 0 \) is a constant independent of initial data. Related results for the case \( p \geq p_u = \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}, \; N > 10 \) were obtained in Galaktionov et al. [7]. It was proved that if \( u_0 \) belows the singular steady state \( U_s(x) \), then as \( t \to \infty \),

\[
\log \|u(t, x)\|_\infty = \alpha_0 t(1 + o(1)) \quad \text{with } \alpha_0 = C_0(N, p) > 0.
\]

Meanwhile [7] also focused on the semilinear Frank-Kamenetskii equation

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + e^u, & \text{in } \mathbb{R}_+ \times B_R, \\
\frac{\partial u}{\partial t} &= 0, & \text{on } \mathbb{R}_+ \times \partial B_R, \\
\frac{\partial u}{\partial t} &= u_0(x), & \text{in } B_R.
\end{aligned}
\]

(9)

It showed that if \( N > 10 \) and \( u_0 \) belows the singular steady state \( U_s(x) \), then

\[
\|u(t, x)\|_\infty = \alpha_0 t + O(1), \quad t \to \infty,
\]

where \( \alpha_0 = \alpha_0(N) > 0 \). Next, Galaktionov and King [13] considered the phenomenon of critical dimension \( N = 10 \), they derived that

\[
\|u(t, 0)\|_\infty = \alpha_0 t + O(\ln t), \quad t \to \infty,
\]

where \( \alpha_0 \) is given by the first eigenvalue of the associated linear differential operator.

The method of matched asymptotic expansions can also be applied to other PDE models, see previous studies [23, 25, 35]. Ju et al. [23] studied the quasi-neutral limit for the two-fluid isentropic Euler-Poisson system in multi-dimension. For the well-prepared initial data, they formally derived the compressible Euler equations, and this formal limit is very different from the unipolar case for which the limit is the incompressible type Euler equations. Kavallaris and Souplet [25] considered a special case of the Patlak-Keller-Segel system in a disc, which arises in the modeling of chemotaxis phenomenon. For a critical value of the total mass, the solutions are global but density is unbounded, leading to a phenomenon of mass-concentration in infinite time. It is worth noting that the maximum of density behaves like \( e^{\sqrt{t}} \), unlike the conclusions mentioned above, and the grow-up rate is neither polynomial nor exponential.

For (1), the case \( M > M_c \), which corresponds to (II), was studied by Zhang and Hu in [42]. The cases \( 0 \leq M < M_c \) and \( M = M_c \), which correspond to (I) and (III) respectively, will be considered in the present paper. Clearly, the three possibilities of the solution for (1) are all taken into account.

Our main results are stated as follows.

**Theorem 1.1.** Assume \( 0 \leq M < M_c \) and the condition (2) holds. Then all global solutions of (1) are bounded in \( C^1 \). Moreover, they converge in \( C^1 \) norm to \( V_M \).

**Theorem 1.2.** Assume \( M = M_c \) and the conditions (2) and (3) hold. Then the solution \( u \) of (1) exists globally and satisfies

\[
\lim_{t \to \infty} \|U(x) - u(t, x)\|_\infty = e^{-\lambda t}
\]
and
\[ \lim_{t \to \infty} u_x(t,0) = e^{\mu t}, \]
where \( \lambda = \lambda(p,m) > 0 \) is defined in Proposition 6 as the first eigenvalue of an associated linearized problem, and \( \mu = \frac{(m+1)\lambda}{p-m-2} \).

For (1), the works [3] and [39], which had discussed the case \( m = 0 \) throughly, are significant references for us to solve problems in the case \( m > 0 \). The presence of \( x^m \) leads some difficulties as follows.

First, for the case \( m > 0 \), the stationary solution presents a complicated formula in the integral form (the explicit formula appears at the beginning of Section 2), which poses two major difficulties. One is the convergence proof between steady states satisfying different boundary conditions (see \( U_{\mu} \to U \) as \( \mu \to \infty \) in Section 4.2). We apply monotone convergence theorem to prove it, avoiding the complicated calculation caused by integral form. The other is in process of matching the inner and outer regions (see the proof of Theorem 4.4). We introduce a sufficiently small parameter \( \epsilon \) to characterize the boundedness of \( x \) in the inner region. Finally, a complicated computation can present an inequality with \( \epsilon \), letting \( \epsilon \to 0 \), we can get the asymptotic behavior of \( u_x \). Moreover, the remainder term of Taylor expansion is also given by analyzing series convergence. When \( m = 0 \), it is easy to calculate because the steady state solution is given in polynomial form.

Second, the presence of \( x^m \) leads to the failure of the maximum principle. When \( m > 0 \), by differentiating (1) with respect to \( x \), the function \( h = u_x \) satisfies
\[ h_t = h_{xx} + b(t,x)h_x + mx^{m-1}|h|^p, \]  
where \( b(t,x) = px^m|h|^{p-2}h \). The maximum principle is obviously false by the presence of the term \( mx^{m-1}|h|^p \). The failure of maximum principle leads to two major difficulties. One is to rule out infinite time gradient blowup at boundary point \( x = 1 \) (see Lemmas 3.1 and 3.2). The work [3] (see Lemma 2.6) illustrated the situation with the fact that \( u_x \) satisfies the maximum principle and attains the maximum on the boundary in the case \( m = 0 \). The present paper needs to use more delicate and mathematical techniques than those which suffice for the case \( m = 0 \). We construct a Lyapunov functional to obtain the convergence result of steady state \( W(x) \) (see Proposition 2). Combining with the fact that \( W(x) \) satisfies the maximum principle for
\[ \left\{ \begin{array}{ll} W_{xx} + x^m|W_x|^p = 0, & 0 < x < 1, \\
W(0) = 0, \ W(1) = M, \end{array} \right. \]
we can get the conclusion, avoiding the fact that \( u_x \) can not attain the maximum on the boundary. The other is that the time-dependent perturbation vanishes as \( t \to \infty \) (see Lemma 4.2), the perturbation appears after transformation for (1). We construct appropriate auxiliary function \( z \) and use the Hopf lemma to obtain the decaying property of perturbation.

Third, the presence of \( x^m \) affects the range of \( k \), which is defined as the index of weighed term in an associated linearized problem (51). We perfect the conclusion of eigenvalue problem in case \( k \in [1,3) \) (see [3, Proposition 5.1]) to the case \( k \in [1, \infty) \) (see Remark 4 (ii)), so that we can obtain asymptotic behavior of \( u \) in outer region by constructing supersolution and subsolution.

The paper is organized as follows. In Section 2, we give some preliminary estimates and gradient estimate. In Section 3, we consider the case \( 0 \leq M < M_c \) and
prove Theorem 1.1. In Section 4, we study the case $M = M_c$ and prove Theorem 1.2.

2. Preliminary estimates and gradient estimate. Let us recall that the solutions $v \in C([0,1]) \cap C^2((0,1))$ of
\[
\begin{cases}
v_{xx} + x^m |v_x|^p = 0, & 0 < x < 1, \\
v(0) = 0
\end{cases}
\]
are given by $v = 0$ or $v = v_k(x) = \int_0^x \frac{p-1}{(m+1)s^{m+1}} + (p-1)k \frac{1}{s^p} ds, \ k \in [0,\infty)$. For given $M \geq 0$, (1) admits a unique steady state in the integral form. In special case $m = 0$, a unique stationary solution of (1) is given in the polynomial form, i.e. $v = 0$ or $v = v_k(x) = M_c \left[ (x + k) \frac{p-1}{(m+1)} - k \frac{p-1}{p} \right], \ k \in [0,\infty)$.

In this section, we will give the following useful lemmas which are similar to the ones (see Lemmas 2.1, 2.2 and 2.4) in [3], the proofs are omitted here.

**Lemma 2.1.** Let $u$ be a maximal solution of (1). For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that
\[
|u_t| \leq C_1, \ t_0 \leq t < T^*, \ 0 \leq x \leq 1.
\]

By using the upper bound of $u_t$, the following lemma provides the upper and lower bounds on $u_x$, in particular, it shows that $u_x$ is bounded away from the boundary.

**Lemma 2.2.** Let $u$ be a maximal solution of (1). For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that, for all $t_0 \leq t < T^*$ and $0 \leq x \leq 1$,
\[
u_x(t, x) \leq C_1 x + \left[ \frac{p-1}{m+1} x^{m+1} \right]^{-\frac{1}{p-1}} \tag{11}
\]
and
\[
u_x(t, 1-x) \geq -C_1 x - \left[ \frac{p-1}{m+1} x^{m+1} \right]^{-\frac{1}{p-1}}. \tag{12}
\]

The following lemma provides a lower bound on the blowup profile of $u_x$ in case gradient blowup occurs in infinite time near $x = 0$ and $x = 1$.

**Lemma 2.3.** Let $u$ be a maximal solution of (1). For all $t_0 \in (0, T^*)$, there exists $C_2 > 0$ such that, for all $t_0 \leq t < T^*$ and $0 \leq x \leq 1$,
\[
[u_x^+(t, x) + C_2]^{1-p} \leq [u_x^+(t, 0) + C_2]^{1-p} + \frac{p-1}{m+1} x^{m+1} \tag{13}
\]
and
\[
[-u_x^+(t, 1-x) + C_2]^{1-p} \leq [-(u_x^+(t, 1-x) + C_2]^{1-p} + \frac{p-1}{m+1} x^{m+1}. \tag{14}
\]

The following lemma will give gradient estimate on the basis of the Bernstein-type arguments and maximum principle, see [40, 44, 28] for further details.

**Lemma 2.4.** Let $u$ be the maximal, classical solution of (1), then
\[
|u_x| \leq \left[ C_3 x^{\frac{2p(m+1)}{p-1}} + C_4 x^{-2m} \right]^\frac{1}{p} + C_5 \text{ in } [0, T^*) \times (0,1), \tag{15}
\]
where $C_3 = C_3(p, m)$, $C_4 = C_4(||u_0||_{C^1}, p, m)$, $C_5 = C_5(||u_0||_{C^1}, p, m) > 0$ and $\| \cdot \|_{C^1}$ denotes the $C^1$ norm in $[0,1]$. 

Proof. Let \( x_0 \in (0,1) \) be fixed and choose \( R = \text{dist}(x_0,0) = x_0 \). Put \( v = u \), and \( w = |v_x|^2 \). Then \( v \) satisfies
\[
 v_t = v_{xx} + x^m |v_x|^p.
\] (16)
Differentiating the equation (16) with respect to \( x \) yields that:
\[
 v_{tx} = v_{xxxx} + mx^{-1} v_x |v_x|^p + px^m |v_x|^{p-2} v_x v_{xx}.
\] (17)
Multiplying (17) by \( 2v_x \), we have
\[
 w_t = w_{xx} - 2v_{xx}^2 + 2mx^{-1} w_x^p v_x + px^m w^{p-2} v_x w_x,
\]
here we use the facts that \( w_1 = 2v_x v_{xt} \), \( w_x = 2v_x v_{xx} \), and \( w_{xx} = 2v_{xx}^2 + 2v_x v_{xxx} \).
Define
\[
 \bar{b} = -px^m w^{p-2} v_x,
\]
\[
 Nw = 2mx^{-1} w_x^p v_x,
\]
\[
 \mathcal{L}w = w_t - w_{xx} + \bar{b} w_x.
\]
Then it holds \( \mathcal{L}w = -2v_x^2 + Nw \). Setting \( R' = \frac{3}{4} R \) and \( y = \min(x_0 + R', 1) \), we can select a cut-off function \( \eta \in C^2([x_0 - R', y]) \) which satisfies
\[
 0 \leq \eta \leq 1 \text{ and } \eta = 0 \text{ for } |x - x_0| = R'
\]
and
\[
 |\eta_x| \leq CR^{-1} \eta^\alpha, \quad |\eta_{xx}| + |\eta_x| |\eta_x|^2 \leq CR^{-2} \eta^\alpha, \quad \text{for } |x - x_0| < R',
\] (18)
where \( \alpha \in (0,1) \) is chosen below, \( C = C(\alpha) > 0 \). Let \( z = \eta w \), then it holds
\[
 \mathcal{L}z = \eta \mathcal{L}w + w \mathcal{L}\eta - 2\eta_x w_x.
\]
Using Young’s inequality, we have
\[
 2|\eta_x w_x| \leq 4\eta^{-1} |\eta_x|^2 w + \eta \eta_x^2,
\]
then
\[
 \mathcal{L}z + \eta \eta_x^2 \leq \eta N w + (\mathcal{L} \eta + 4\eta^{-1} |\eta_x|^2) w.
\] (19)
Next, we start to estimate each term appearing in the inequality (19).
By the inequality in (18), we obtain
\[
 |\bar{b} \eta_x w| = |px^m w^{p-2} v_x \eta_x w| \leq Cp R^{-1} x^m w^{\frac{p+1}{2}} \eta^\alpha
\] (20)
and
\[
 [-\eta_{xx} + 4\eta^{-1} |\eta_x|^2] w \leq C' R^{-2} w \eta^\alpha,
\] (21)
where \( C' = 4C \). Besides,
\[
 |\eta N w| = 2mx^{-1} w_x^p v_x \eta \leq 2mx^{-1} w^{\frac{p+1}{2}} \eta.
\] (22)
Combining (19)-(22), it suffices to show that
\[
 \mathcal{L}z + \eta \eta_x^2 \leq 2mx^{-1} w^{\frac{p+1}{2}} \eta + C' R^{-2} w \eta^\alpha + Cp R^{-1} x^m w^{\frac{p+1}{2}} \eta^\alpha.
\]
Using the fact \( |x^m | u_x |^p - u_t | = |u_{xx}| \), we have \( \frac{1}{2} x^{2m} |u_x|^{2p} \leq |u_t|^2 + |u_{xx}|^2 \). Hence
\[
 \mathcal{L}z + \frac{1}{2} x^{2m} w^p \eta \leq 2mx^{-1} w^{\frac{p+1}{2}} \eta + C' R^{-2} w \eta^\alpha + Cp R^{-1} x^m w^{\frac{p+1}{2}} \eta^\alpha + \eta |u_t|^2.
\] (23)
In the following, let us estimate each term appearing on the right side of (23). By the local theory [29], there exists \( t_0 \in (0, T) \) such that
\[
 ||u(t)||_{C^2} + \sup_{t \in [0, t_0]} ||u(t)||_{C^1} \leq C(||u_0||_{C^1}),
\] (24)
where $\| \cdot \|_{C^k}$ stands for the $C^k$ norm in $[0,1]$. By (1) and the maximum principle for $u_t$ (see [40, Proposition 2.4]), we know
\[
|u_t| \leq \|u_{xx}(t_0)\|_{\infty} + \|x^m|u_x(t_0)|^p\|_{\infty} \leq C(\|u_0\|_{C^1}, p, m) \text{ in } [t_0, T^*) \times (0, 1). \tag{25}
\]
Note that $x \geq x_0 - R' = R - \frac{\pi}{2}R = \frac{1}{2}R$, so $x^{-\beta} \leq C(\beta)R^{-\beta}$ for any $\beta > 0$. From Young’s inequality and $\eta$, we get
\[
w^{-\eta} \leq C(\epsilon, p, m)R^{-\frac{2(m+p)}{p-1}} \tag{26}
\]and
\[
R^{-2}\alpha \leq \epsilon \left[ x^{-\frac{m+1}{p}} w^{\frac{m+1}{p}} \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} + C(\epsilon) \left[ 2m \left( \frac{m+1}{p} \right) \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} \leq \epsilon \left[ x^{-\frac{m+1}{p}} w^{\frac{m+1}{p}} \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} \tag{27}
\]and
\[
p R^{-1} w^{\frac{m+1}{p}} \eta^{\alpha} \leq \epsilon \left[ x^{-\frac{m+1}{p} \times \frac{m+1}{p}} w^{\frac{m+1}{p} \times \frac{m+1}{p}} \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} + C(\epsilon) \left[ p R^{\frac{1}{2}} \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} \leq \epsilon \left[ x^{-\frac{m+1}{p}} w^{\frac{m+1}{p}} \eta^{\frac{m+1}{2p}} \right]^{\frac{2p}{m+1}} \tag{28}
\]
where $\frac{p+1}{2p} \leq \alpha < 1$, $\epsilon > 0$. Letting $\epsilon = \frac{1}{12}$ and combining inequality (23), (25)-(28), we get
\[
Lz \leq \frac{1}{4} x^{2m} z^p + B \text{ in } Q = (t_0, T^*) \times (x_0 - R', y)
\]with
\[
B = C(p, m)R^{-\frac{2(m+p)}{p-1}} + C(\|u_0\|_{C^1}, p, m)
\]
and $y = \min(x_0 + R', 1)$. It follows from the maximum principle (see [40, Proposition 2.2]) that
\[
\sup_Q z \leq \max \left( \max_{x \in [x_0 - R', y]} z(t_0, x), \left[ \frac{4B}{x^{2m}} \right]^{\frac{1}{p}} \right)
\]
Hence we have
\[
|u_x(t, x_0)| \leq \sup_Q z^{\frac{1}{2}} \leq \left[ C_3 R^{-\frac{2(m+1)}{p-1}} + C_4 R^{-2m} \right]^{\frac{1}{2p}} + C_5, \quad t_0 < t < T^*.
\]
where $C_3 = C_3(p, m)$, $C_4 = C_4(\|u_0\|_{C^1}, p, m)$, and $C_5 = C_5(\|u_0\|_{C^1}, p, m)$. We complete the proof.

**Remark 1.** (i) In the proof of Lemma 2.4, we set $R = x_0 = \text{dist}(x_0, 0)$. Assuming that $R = 1 - x_0 = \text{dist}(x_0, 1)$, we derive
\[
|u_x| \leq \left\{ C_3 R^{-\frac{2p}{p-1}} + C_4 \left[ 1 - \frac{7}{4} R \right]^{-\frac{2m}{p}} \right\}^{\frac{1}{p}} + C_5 \text{ in } [0, T^*) \times (0, 1).
\]
For $R \to 0$, it holds
\[
|u_x| \leq \left[ C_3 R^{-\frac{2p}{p-1}} + C_4 \right]^{\frac{1}{p}} + C_5 \text{ in } [0, T^*) \times (0, 1).
\]
Clearly, the upper estimate of blowup profile of $u_x$ has nothing to do with $m$, and this makes sense from (1). In fact, as $R \to 0$ i.e. $x \to 1$, (1) becomes $u_t = u_{xx} + |u_x|^p$, so it is easy to see $m$ does not work.
(ii) In [42], Zhang and Hu showed that $x = 0$ is the only blow-up point of (1) and provided the following gradient estimate:

$$u_x(t, x) \leq \left[ \frac{p-1}{m+1} x^{m+1} \right]^{-\frac{1}{p-1}} + Cx.$$

Obviously, the estimate in Lemma 2.4 is more precise than that in [42].

(iii) The cut-off function $\eta(x)$ had been constructed in [40]:

$$\eta(x) = \left[ 1 - \frac{(x-x_0)^2}{R^2} \right]^k, \quad k \geq \frac{2}{1-\alpha}, \quad 0 < \alpha < 1.$$  

3. The case $0 \leq M < M_c$. To explain the proof, we construct a Lyapunov functional, which exists for one-dimensional uniformly parabolic equations. A Lyapunov functional was possessed by Zelenyak in a classical paper [41], and it plays a vital role in the proof of the convergence to steady states.

**Proposition 1.** Fix any constant $K > 0$ and let $D_K = \mathbb{R} \times [-K, K]$. There exist a pair of functions $\phi \in C^1(D_K; \mathbb{R})$ and $\psi \in C(D_K; (0, \infty))$ with the following property.

For any solution $u$ of (1) with $|u| \leq K$ with the definition

$$\mathcal{L}(u(t)) := \int_0^1 \phi(u(t, x), u_x(t, x))dx,$$

it holds

$$\frac{d}{dt} \mathcal{L}(u(t)) = -\int_0^1 \psi(u(t, x), u_x(t, x))u_x^2(t, x)dx, \quad 0 < t < T^*. \quad (29)$$

Moreover, we have

$$\phi \geq 0. \quad (30)$$

The above proposition can be proved by slightly modifying the Proposition 3.1 in [3], so we omit the details here. According to Proposition 1, Lemmas 2.1 and 2.2, we would obtain the following convergence conclusion.

**Proposition 2.** Let $u$ be a global solution of (1). Then $u(t)$ converges in $C([0, 1])$ to a steady state $W$ of (1) as $t \to \infty$. The function $W \in C([0, 1]) \cap C^2((0, 1))$ is the solution of

$$\begin{cases} W_{xx} + x^m|W_x|^p = 0, & 0 < x < 1, \\ W(0) = 0, \quad W(1) = M. \end{cases} \quad (31)$$

Moreover, the convergence also holds in $C^1([\epsilon, 1-\epsilon])$ for all $\epsilon > 0$.

**Proof.** Fix a sequence $\{t_n\}_{n \in \mathbb{N}}, \quad t_n \to \infty$ and let $u_n(t, x) = u(t + t_n, x)$. It follows from (5) and Lemma 2.1 that

$$|u| + |u_x| \leq C \quad \text{in} \quad [1, \infty) \times [0, 1]. \quad (32)$$

Using (11), (12) and $p > m + 2$, we obtain

$$\|\partial_x u_n\|_{L^\infty([1, \infty); L^1(0, 1))} \leq C. \quad (33)$$

Thus, by the Arzelà-Ascoli theorem, a diagonal procedure, (32) and (33), there exists a subsequence $\{u_n\}_{n \in \mathbb{N}}$ such that $\{u_n\}_{n \in \mathbb{N}}$ is relatively compact in $C([0, T] \times [0, 1])$ for each $T > 0$.

On the other hand, using (11) and (12), we know that

$$|u_x| \leq C(\epsilon) \quad \text{in} \quad [1, \infty) \times (\epsilon, 1-\epsilon). \quad (34)$$
It follows from (34) and Lemma 2.1 that

$$|u_{xx}| = |u_t - x^m u_x|^p \leq C(\epsilon, p, m) \text{ in } [1, \infty) \times (\epsilon, 1 - \epsilon).$$  \hfill (35)

Let $w = u_x$ satisfy

$$w_t = w_{xx} + px^m |u_x|^{p-2} u_x w_x + m x^{m-1} |w|^p.$$  \hfill (36)

Applying parabolic regularity estimates (see [22, P. 24, Theorem 3.10]), we have for each $q > 1$,

$$|w(t + t_n, x)|_{L^q(0, T \times (\epsilon, 1 - \epsilon))} \leq C(\epsilon, T, q), \quad T > 0.$$  \hfill (36)

By the Arzelà-Ascoli theorem, a diagonal procedure, (34), (35) and (36), there exists a subsequence $\{\partial_x u_n\}_{n \in \mathbb{N}}$ such that $\{\partial_x u_n\}_{n \in \mathbb{N}}$ is relatively compact in $C([0, T] \times [\epsilon, 1 - \epsilon])$ for each $\epsilon, T > 0$.

Thus, for any $\epsilon$ and $T > 0$, some subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ converges to a function $W \in C([0, \infty) \times [0, 1])$, with $W_0 \in C([0, \infty) \times [\epsilon, 1 - \epsilon])$, which satisfies

$$\begin{cases}
W_t = W_{xx} + x^m |W_x|^p, & \text{in } [0, \infty) \times (0, 1),
W(t, 0) = 0, W(t, 1) = M, & t \geq 0.
\end{cases}$$

Next, we claim that $W_t \equiv 0$. By (5), there exists $K > 0$ such that

$$|u| \leq K, \quad \text{in } [0, \infty) \times [0, 1].$$

Since $\psi \in C(D_K; (0, \infty))$, given by Proposition 1, we have

$$\pi(K, R) := \inf \{\psi(u, v); |u| \leq K, |v| \leq R\} > 0, \quad \text{for all } R > 0.$$  \hfill (37)

Fix any $\epsilon \in (0, 1)$. By (34), (37), (29) and (30), we obtain for all $T > 1$,

$$\pi(K, C(\epsilon)) \int_1^T \int_1^{1-\epsilon} u_t^n(t, x)dxdt \leq \int_1^T \int_0^1 \psi(u, u_x) u_t^n(t, x)dxdt = L(u(1)) - L(u(T)) \leq L(u(1)).$$

This implies that $\int_1^T \int_1^{1-\epsilon} u_t^n(t, x)dxdt < \infty$. Introducing the transformation $s = t + t_n$, it follows from Lemma 2.1 that

$$\int_0^\infty \int_\epsilon^{1-\epsilon} (\partial_s u_{n_k})^2(t, x)dxdt = \int_{t_n}^\infty \int_\epsilon^{1-\epsilon} u_s^2(s, x)dxds \to 0, \quad k \to \infty.$$  \hfill (36)

Since $\partial_s u_{n_k} \to W_t$ as $t \to \infty$ and $\epsilon$ is arbitrary, we know that $W_t \equiv 0$. Thus, $W$ satisfies (31). Since the sequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ is arbitrary and the stationary solution (for given $M$) is unique, these imply that the whole solution $u(t, x)$ converges to $W(x)$. \hfill \Box

The following lemma is a direct conclusion of the convergence of $u$ to the steady state.

**Lemma 3.1.** Let $u$ be a global solution of (1). Then it holds

$$\lim_{t \to \infty} \left[ \max_{x \in [0, 1]} u(t, x) \right] = M.$$

**Proof.** It follows from the equation (31) and the maximum principle that $W(x) \leq M$. Using $u(t, x) \rightarrow W(x)$ in $C([0, \infty) \times [0, 1])$, it holds that for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that if $t > t_\epsilon$ then

$$u(t, x) \leq W(x) + \epsilon \leq M + \epsilon, \quad \text{in } [t_\epsilon, \infty) \times [0, 1].$$

\hfill (38)
So, \( \max_{x \in [0,1]} u(t, x) \leq M + \epsilon \). Letting \( \epsilon \to 0 \), we get \( \max_{x \in [0,1]} u(t, x) \leq M \). Moreover, the boundary condition \( u(t, 1) = M \) can imply that \( \max_{x \in [0,1]} u(t, x) \geq M \). The proof is complete.

Thanks to (12), we can obtain that infinite time gradient may blow up towards \(-\infty\) for \( x = 1 \), the following lemma can enable us to rule out this case.

**Lemma 3.2.** Let \( u \) be a global solution of (1). Then
\[
\inf_{[0,\infty) \times [0,1]} u_x > -\infty.
\]

**Proof.** Assume that the lemma is false. By (12), there exists a sequence \( t_n \to +\infty \) such that \( u_x(t_n, 1) \to -\infty \). Fix \( \epsilon > 0 \), for \( n \geq n_0(\epsilon) \) large enough, it holds that \( (-u_x)^+(t_n, 1) \to +\infty \). Using (14) and \( p > m + 2 \geq 2 \), we deduce for \( 0 \leq x \leq \epsilon \)
\[
\left[ (-u_x)^+(t_n, 1 - x) + C_2 \right]^{1-p} \leq \left[ (-u_x)^+(t_n, 1) + C_2 \right]^{1-p} + \frac{p-1}{m+1} x^{m+1}
\]
\[
\leq \epsilon + \frac{p-1}{m+1} \epsilon^{m+1}
\leq \frac{m+p}{m+1} \epsilon.
\]

This implies that
\[
(-u_x)^+(t_n, 1 - x) \geq \left[ \frac{m+p}{m+1} \epsilon \right]^{-\frac{1}{p-1}} - C_2, \quad 0 \leq x \leq \epsilon.
\]
Choosing \( \epsilon = \epsilon(p, m, C_2) > 0 \) small enough, we deduce that \( (-u_x)^+(t_n, 1 - x) \geq 1 \) on \([0, \epsilon] \), so \( u_x(t_n, 1 - x) \leq -1 \) on \([0, \epsilon] \). It is clear that
\[
\int_0^x u_y(t_n, 1 - y) dy \leq \int_0^x -1 dy, \quad x \in [0, \epsilon],
\]
then
\[
u(t_n, 1 - x) \geq x + u(t_n, 1) = x + M, \quad x \in [0, \epsilon].
\]
This contradicts to Lemma 3.1.

Thanks to (11), we can obtain that infinite time gradient may blow up towards \(+\infty\) for \( x = 0 \). In what follows, we are ready to rule out this case.

**Proof of Theorem 1.1.** We proceed by contradiction. Assume that \( u \) is a global solution of (1) which is unbounded in \( C^1 \). From Proposition 2, we have that as \( t \to \infty \), \( u(t) \) converges to \( W = V_M \) in \( C([0,1]) \) and in \( C^1([\epsilon, 1-\epsilon]) \) for all \( \epsilon > 0 \).

It follows from Lemmas 2.2 and 3.2 that, if \( u \) is unbounded in \( C^1 \), then there exists a sequence \( t_n \to \infty \) such that
\[
u_x(t_n, 0) \to \infty.
\]
Using this and (13), we deduce
\[
\left[ W_x^+(x) + C_2 \right]^{1-p} \leq \frac{p-1}{m+1} x^{m+1}, \quad x \in (0,1),
\]
then
\[
W_x(x) \geq \left[ \frac{p-1}{m+1} x^{m+1} \right]^{-\frac{1}{p-1}} - C_2, \quad x \in (0,1).
\]
This can imply that \( W_x(0) = \infty \). But \( W = V_M \in C^2([0,1]) \) for the case \( 0 \leq M < M_0 \) leads to a contradiction. The proof is complete.
4. The case $M = M_c$.

4.1. Global existence and preliminary estimates. From the conditions (2), (3) and the maximum principle, it is clear that a classical solution of (1) satisfies $-C \leq u(t,x) \leq U(x)$ as long as it exists. Next, we will give some propositions which can be proved by slightly modifying the Propositions 2.1, 3.1 and 3.2 in [39], so we omit details here.

**Proposition 3.** Let the assumptions of Theorem 1.2 be in force, the solutions are global in time, i.e. $t^* = \infty$.

**Proposition 4.** We have the following estimate:

$$\int_0^1 x [U(x) - u(t,x)]^2 dx \leq Ce^{-2t}, \quad t \geq 0.$$ 

Next in view of Proposition 4, an exponential lower bound of $u_x(t,0)$ has been provided in following proposition. It plays a key role in the inner region analysis.

**Proposition 5.** There exists $\tau = \tau(m,p) = \frac{m+1}{2p-m-3} > 0$ such that

$$u_x(t,0) \geq Ce^{\tau t}, \quad t \gg 1.$$ 

4.2. Inner-region analysis. This section provides the asymptotic behavior of $u$ in a small region near $x = 0$ for large $t$. This is called inner region (boundary layer) analysis. Following the ideas in [39, 7], the solution $u(t,x)$ is given by a quasi-steady problem and it is close to a stationary solution.

Consider the re-normalized stationary equation

$$\begin{cases}
V_{xx} + x^m |V_x|^p = 0, & x > 0, \\
V(0) = 0, \quad V_x(0) = 1,
\end{cases}$$

the solution $U_1(x)$ is given in the form $U_1(x) = \int_0^x [1 + \frac{p-1}{m+1}s^{m+1}]^{\frac{1}{p-1}} ds$. Clearly, $U_1(x) > 0$ and $U_1'(x) > 0$ for all $x > 0$. Given a fixed $\mu \in \mathbb{R}$, $U_\mu$ is the unique solution of

$$\begin{cases}
V_{xx} + x^m |V_x|^p = 0, & x > 0, \\
V(0) = 0, \quad V_x(0) = \mu^{(m+1)/p}.
\end{cases} \tag{39}$$

It has the form $U_\mu(x) = \mu^* U_1(x\mu^3)$ with $\beta > 0$, $\gamma = \frac{m+2-p}{p-1} \beta < 0$.

Moreover, we can select special range for $\gamma$, which is mainly used to obtain gradient growth rate (see the proof of Theorem 4.4). We assume $\gamma - \frac{\lambda - \sqrt{\lambda^2 + 4\lambda E}}{2E} < 0$, where $E = \frac{(p-1)(p-3)}{2(p-m-3)} > 0$ and $\lambda = \lambda(p,m) > 0$ is defined in Proposition 6 as the first eigenvalue of an associated linearized problem.

We claim that $U_\mu(x)$ increases monotonically to $U(x)$ as $\mu \to \infty$. In fact,

$$\frac{\partial U}{\partial \mu}(x) = \gamma U_1(x\mu^3)\mu^{\gamma-1} + \beta U_\mu'(x\mu^3)x\mu^\gamma < 0, \quad \mu \to \infty,$$

which is deduced from $U_\mu'(x) > 0$ for all $x > 0$. This implies that $U_\mu(x)$ increases monotonically with respect to $\mu$ for $\mu$ large enough. In addition, it follows from $U(x) = M_c x^\frac{p-m-2}{p-1}$ that $U_\mu(0) = \infty$. In other words, $U(x)$ is the solution of (39) corresponding to $\mu = \infty$, which completes the proof of the claim.

**Remark 2.** For the case $m = 0$ of problem (1), Souplet and Vázquez [39] deduced

$$U_1(x) = \frac{1}{p-2} \left\{ [1 + (p-1)x]^\frac{p}{p-1} - 1 \right\}, \quad x > 0.$$
For each $\mu > 0$, $U_\mu(x) = \mu^{2-p}U_1(x\mu^{p-1})$ is the unique solution of

$$\begin{align*}
V_{xx} + |V_x|^p &= 0, \quad x > 0, \\
V(0) &= 0, \\
V_x(0) &= \mu.
\end{align*}$$

Clearly, $U_\mu(x)$ is given in polynomial form, the result $U_\mu(x) \to U(x)$ as $\mu \to \infty$ can be obtained through detailed calculation. In the present paper, $U_\mu(x)$ has the integral form, which leads to some computational difficulties. So we apply the monotone convergence theorem to illustrate the situation.

The present paper would introduce the rescaling parameter

$$\alpha(t) = [u_x(t, 0)]^\frac{p-1}{m+1}\beta$$

that diverges to $\infty$ as $t \to \infty$, as we already know from Proposition 5. Furthermore, the fact that $\alpha(t)$ increases monotonically with respect to $t$ can be proved in Theorem 4.1(i).

Next, we would provide a quasi-stationary behavior:

$$u(t, x) \sim U_{\alpha(t)}(x), \quad t \to \infty$$

in a suitable inner region near $x = 0$ (see Remark 5). More precisely, we obtain:

**Theorem 4.1.** (i) For large $t$, $\alpha(t)$ is (strictly) increasing and

$$u(t, \cdot) > U_{\alpha(t)} \text{ in } (0, 1]$$

(ii) We have

$$u(t, x) = U_{\alpha(t)}(x) + o(\alpha^\gamma(t)) = \alpha^\gamma(t)[U_1(x\alpha^\beta(t)) + o(1)],$$

in the sense that

$$\lim_{t \to \infty} \frac{u(t, ya^{-\beta}(t))}{\alpha^\gamma(t)} = U_1(y),$$

and

$$\lim_{t \to \infty} \frac{u_x(t, ya^{-\beta}(t))}{\alpha^\gamma+\beta(t)} = U_{1y}(y).$$

uniformly for $y = xa^\beta(t) \geq 0$ in bounded sets.

The proof of Theorem 4.1 (i) is similar to the proof in [39, Theorem 4.1(i)], so we omit it here. In order to prove Theorem 4.1(ii), we introduce the rescaled function $\theta$. Let

$$u(t, x) = \alpha^\gamma(t)\theta(\tau, y), \quad y = x\alpha^\beta(t),$$

with $\tau = \tau(t)$ to be defined. We compute

$$u_x(t, x) = \alpha^{\beta+\gamma}(t)\theta_y(\tau, y), \quad u_{xx}(t, x) = \alpha^{2\beta+\gamma}(t)\theta_{yy}(\tau, y),$$

and

$$u_t = \gamma\alpha'\alpha^{\gamma-1}\theta + \alpha^\gamma\theta_{\tau} \frac{d\tau}{dt} + \alpha'\alpha^{\gamma-1}\beta y\theta_y.$$}

Hence, equation (1) reads

$$\alpha^\gamma \frac{d\tau}{dt} = \alpha^{2\beta}(t)$$

and the new domain

$$D = \{(\tau, y) : \tau > 0, 0 < y < R(\tau)\},$$

where

$$\begin{align*}
\alpha^{2\beta}(t) &= \gamma\alpha'\alpha^{\gamma-1}\theta + \alpha^\gamma\theta_{\tau} \frac{d\tau}{dt} + \alpha'\alpha^{\gamma-1}\beta y\theta_y, \\
\alpha^{2\beta+\gamma}(t) &= \gamma\alpha'\alpha^{\gamma+1}(t)\theta + \beta \alpha'\alpha^{\gamma}(t)\theta_{\tau} \frac{d\tau}{dt} - \alpha^{2\beta+\gamma}(t)(\theta_{yy} + y^m|\theta_y|^p) = 0.
\end{align*}$$
where $R(\tau) = \alpha^\beta(t)$. As a result of Proposition 5, we have that $\tau(t) \to \infty$ and $R(\tau) \to \infty$ as $t \to \infty$. Dividing (43) by $\alpha^{2\beta+\gamma}$, we have that

\[
\begin{align*}
\theta_x = \theta_{yy} + \gamma^m|\theta_y|^p + g(\tau)(\theta + \frac{\beta}{\gamma}y\theta_y), & \quad \text{in } D, \\
\theta(\tau, 0) = 0, & \quad \theta_y(\tau, 0) = 1,
\end{align*}
\]

where

\[g(\tau) = -\gamma\alpha'(t)\alpha^{-1-2\beta}(t)\]

is nonnegative for $t$ large by Theorem 4.1(i) and Proposition 5. The equation (44) looks like a time-dependent perturbation of equation (1). It is clear that (41) is equivalent to show that $\theta(\tau, y)$ converges to $U_1(y)$ as $\tau \to \infty$. We note that

\[
\int_0^\infty g(\tau)d\tau = \int_0^\infty g(\tau(t))\frac{d\tau}{dt}dt = -\gamma \int_0^\infty \alpha^{-1}dt = \infty,
\]

so that the perturbation is not integrable in time. However, the following lemma can guarantee that $g(\tau) \to 0$ as $\tau \to \infty$.

**Lemma 4.2.** It holds

\[\lim_{\tau \to \infty} g(\tau) = 0.\]

**Proof.** Let $z = u_t - C(1-e^{\rho u^2})$, where the constants $C$, $\rho$ will be determined later. We claim that

\[z_x(t, 0) \leq 0, \quad \text{for } t > 0. \quad (45)\]

Differentiating the equation (1) with respect to $t$:

\[u_{tt} = u_{xx}t + px^m|u_x|^{p-2}u_xu_{xt}. \quad (46)\]

A direct calculation shows that

\[z_t = u_{tt} + 2C\rho uu_t e^{\rho u^2}, \quad z_x = u_{tx} + 2C\rho uu_x e^{\rho u^2},\]

and

\[z_{xx} = u_{xxx} + 2C\rho uu_{xx} e^{\rho u^2} + 4C\rho^2 u_x^2 e^{\rho u^2} + 2C\rho^2 u_{xx} e^{\rho u^2}.\]

It follows from above equalities, (46) and (1) that

\[
\begin{align*}
z_t &= z_{xx} + px^m|u_x|^{p-2}u_xz_x + 2C\rho e^{\rho u^2}|u_x|^2 \left\{-2u^2\rho^2 - [(p-1)x^m|u_x|^{p-2}u + 1]\right\} \\
&= z_{xx} + px^m|u_x|^{p-2}u_xz_x + 2C\rho e^{\rho u^2}|u_x|^2 F(\rho) \\
&< z_{xx} + px^m|u_x|^{p-2}u_xz_x,
\end{align*}
\]

(47)

where $C > 0$ and $\rho < 0$ is small enough. The last inequality holds due to $F(\rho) < 0$. In fact, $F(\rho)$ could be seen as a quadratic function of $\rho$. The fact, which $u(t, x) \to U(x) = M_x x^{\frac{p-2}{p-1}}$ as $t \to \infty$, indicates that $x^m|u_x|^{p-2}u \sim 1$ as $t \to \infty$, so the coefficient of $\rho$ is bounded in the term $F(\rho)$. Moreover, the result that the coefficient of $\rho^2$ is bounded can be provided by (2) and (5). Hence, $F(\rho) < 0$ is obviously true for $\rho < 0$ small enough.

From (2), (5) and $\rho < 0$, we derive that $0 < 1 - e^{\rho u^2} \leq C'$, where $C' > 0$. In view of Lemma 2.1, there exists a constant $C > 0$ that

\[u_t \leq C(1 - e^{\rho u^2}), \quad \text{i.e. } z(t, x) \leq 0 \quad \text{for } 0 < x < 1. \quad (48)\]

Using (47), (48), $z(t, 0) = 0$ and the Hopf lemma, we obtain that $z_x(t, 0) < 0$. It follows from (5) that

\[u_{tx}(t, 0) < -2C\rho e^{\rho u^2}u_x(t, 0) < C_uz(t, 0),\]
Remark 3. In addition, the auxiliary function $z$ can be given by

$$z = u_t + \sum_{i=0}^{\infty} C_i u^i - C,$$

where $C_i$, $C > 0$. The term $\sum_{i=0}^{\infty} C_i u^i$ can be regarded as Taylor expansion of $e^{\rho u^2}$. Further, the order of expansion also has corresponding requirements.

Proof of Theorem 4.1(ii). We first give some estimates on $\theta$ and $\theta_y$. By Lemma 2.1, there exists $C_1 > 0$ such that $|u_t| \leq C_1$ on $[t_0, \infty) \times [0, 1]$ for $t_0$ large enough. Then

$$u_{xx} = u_t - x^m |u_x|^p \leq u_t \leq C_1.$$  

It follows from Lemma 3.2 that

$$-C \leq u_x(t, x) \leq u_x(t, 0) + C_1 x \leq u_x(t, 0) + C_1.$$  

From this and Proposition 5, we know $|u_x(t, x)| \leq 2u_x(t, 0)$ on $[t_0, \infty) \times [0, 1]$ for $t_0$ large enough. Hence,

$$|\theta_y(\tau, y)| = |\alpha^{-\gamma - \beta} u_x(t, x)| \leq \left| u_x(t, 0) \left( \frac{p-1}{(m+1)\beta} \right)^2 2u_x(t, 0) \right| = 2 \quad (49)$$  

and

$$|\theta(\tau, y)| \leq 2y \quad (50)$$  

for $0 \leq y \leq R(\tau)$ and $\tau \geq \tau_0$ large. So, the fact that $|\theta(\tau, y)| \leq C$ can be obtained for $y = x\alpha^\beta(t) \geq 0$ in bounded sets.

Next, fix a sequence $\{\tau_n\}_{n \in \mathbb{N}}$, $\tau_n \to \infty$, and let $\theta_n = \theta(\tau_n + \cdot, \cdot)$ and $Q = [0, \infty) \times [0, \infty)$. Applying Lemma 4.2, (44), (49), (50) and parabolic estimates, we have for each $1 < q < \infty$, the sequences $\theta_{n,\tau}$ and $\theta_{n,yy}$ are bounded in $L^q_{loc}(Q)$.

Differentiating the equation (44) with respect to $y$, we deduce the fact that $\theta_{n,y\tau}$ is bounded in $L^q_{loc}(Q)$ for each $1 < q < \infty$ by parabolic estimates. Therefore, $\theta_n$ and $\theta_{ny}$ are bounded in $C_{loc}^{\beta}(Q)$ for some $\beta \in (0, 1)$. From the regularity, we deduce that $\theta(\tau_n + \tau, y) \to \theta(\tau, y)$ uniformly on compact subsets of $Q$, where $\hat{\theta}(\tau, y)$ satisfies the limit equation of (44):

$$\begin{cases}
\hat{\theta}_\tau = \hat{\theta}_{yy} + y^m |\hat{\theta}_y|^p, & \tau > 0, \ y > 0, \\
\hat{\theta}(\tau, 0) = 0, \ \hat{\theta}_y(\tau, 0) = 1, & \tau > 0.
\end{cases}$$

Moreover, we have $\hat{\theta} \geq U_1$ due to (40). By $\hat{\theta}(\tau, 0) = U_1(0) = 0$ and the Hopf lemma, we derive that $\hat{\theta}_y(\tau, 0) \geq U_{1y}(0)$. It follows from $\hat{\theta}_y(\tau, 0) = U_{1y}(0) = 1$ that $\hat{\theta} \equiv U_1$. The conclusion follows.
4.3. **Outer-region analysis.** This section provides the asymptotic behavior of $u$ in outer region away from $x = 0$ for large $t$. In connection with the linearized analysis, we need to concern the singular eigenvalue problem

$$ -(x^k \varphi_x)_x = \lambda x^k \varphi, \quad 0 < x < 1, \quad \varphi(1) = 0. \quad (51) $$

For a given real $k > 0$, we introduce the Hilbert space

$$ H = \left\{ v \in H^1_{\text{loc}}((0, 1]); \int_0^1 x^k v_x^2 dx < \infty, \: v(1) = 0 \right\}. $$

**Proposition 6.** (See [39, Proposition 5.1]) Let $k > 0$ and define

$$ \lambda = \inf \left\{ \int_0^1 x^k v_x^2 dx; \: v \in H, \int_0^1 x^k v^2 dx = 1 \right\}. \quad (52) $$

Then $\lambda$ is well-defined, $\lambda \in (0, \infty)$ and there exists $\varphi \in H$ which solves the minimization problem (52) and enjoys the following properties.

(i) $\varphi \in C^2((0, 1]), \: \varphi > 0$ in $(0, 1)$;
(ii) $\varphi$ is a solution of (51);
(iii) if $k \geq 1$, then $\varphi_x < 0$ in $(0, 1)$; if $1 \leq k < 3$, then $\varphi \in C^1([0, 1]), \: \varphi_x(0) = 0$ and $\varphi(0) > 0$.

**Remark 4.** (i) $\lambda$ takes different values in the case $m = 0$ and the case $m \in [0, \infty)$. Moreover, there is a certain relationship between them. For $k > 0$, we show the following weighted Poincaré inequality with best constant:

$$ \int_0^1 x^q v^2 dx \leq \frac{4}{(q + 1)^2} \int_0^1 x^k v_x^2 dx, \: v \in H, \: q > -1, \: q \geq k - 2. \quad (53) $$

It suffices to show that $\lambda = \frac{(k + 1)^2}{4} \lambda_*$ by (52) in the case $q = k$. Clearly, the value of $\lambda$ depends on $k$. Souplet and Vázquez [39] considered the case $m = 0$ and deduced that $k = \frac{p}{p - 1}$, then $\lambda_* := \lambda = \frac{1}{4} \frac{(2p - 1)^2}{(p - 1)^2}$. In this paper, we consider the case $m \in [0, \infty)$ and obtain $k = \frac{m(2p - 1)}{2p - 1}$ (see the proof of Theorem 4.3), then

$$ \lambda = \frac{1}{4} \frac{(mp + 2p - 1)^2}{2p - 1}. $$

(ii) We perfect the conclusion of Proposition 6 (iii) and conclude that: if $k \geq 1$, then $\varphi \in C^1([0, 1]), \: \varphi_x(0) = 0$ and $\varphi(0) > 0$, proven by referring to the proof of Proposition 5.1 in [39]. Integrating on both sides of (51), we know

$$ x^k \varphi_x - \lim_{x \to 0} x^k \varphi_x = -\lambda \int_0^x s^k \varphi(s) ds. $$

Combining with $\lim_{x \to 0} x^k \varphi_x = 0$ (see [39, Proposition 5.1]), Young’s inequality and (4.15), we derive

$$ |\varphi_x(x)| = \left| \lambda \int_0^x \left( \frac{s}{x} \right)^k \varphi(s) ds \right| = \left| \lambda \int_0^x s^k \frac{\varphi(s)}{x^k} ds \right|. $$
we obtain
\[
\psi \quad \text{and} \quad \lambda \left( \int_0^1 s^k \varphi^2(s) ds \right)^{\frac{1}{2}} \leq \left( \frac{2\lambda + 1}{\zeta} \right) \left( \int_0^1 s^k \varphi^2(s) ds \right)^{\frac{1}{2}} \left( \int_0^1 \frac{s^{2k-\zeta}}{x^{2k}} ds \right)^{\frac{1}{2}}
\]
for a given \( \zeta \in (-1, 0) \). Clearly, if \( k \geq 1 \), then \( \varphi \in C^1([0,1]) \) and \( \varphi(x) = 0 \). Moreover, \( \varphi(0) \) is finite and positive.

In order to prove the main result, we need to introduce the following regular eigenvalue problem, as an approximation to a singular problem (51):
\[
\begin{align*}
-x^{-k}(x^k \varphi_{xx})_x &= \lambda \varphi, & \epsilon < x < 1, \\
\varphi(x) &= 0, & x > 1
\end{align*}
\]
for each \( \epsilon \in (0, 1) \). We denote \( \lambda_\epsilon > 0, \varphi_\epsilon > 0 \) to be the first eigenvalue and the first eigenfunction of problem (54), respectively.

Next, the exponential convergence rate is established by the accurate constant \( \lambda \) just defined in Proposition 6.

**Theorem 4.3.** Let \( \lambda \) be defined by (52) with \( k = \frac{p(m+1)}{p-1} \). It satisfies
\[
U(x) - u(t, x) \leq C(1 - x)e^{-\lambda t}, \quad t > 0, \quad 0 < x < 1
\]
and for each \( \epsilon > 0 \),
\[
U(x) - u(t, x) \geq c_\epsilon (1 - x)e^{-(\lambda + \eta(\epsilon)) t}, \quad t > 0, \quad \epsilon < x < 1
\]
with \( \eta(\epsilon) \to 0 \) as \( \epsilon \to 0 \) and \( c_\epsilon > 0 \). In particular, it holds
\[
e^{-\lambda t} \leq \|U - u(t)\|_\infty \leq Ce^{-\lambda t}, \quad t \to \infty.
\]

**Proof.** Let \( w = U - u \). We have
\[
w_t - w_{xx} = x^m |U_x|^p - x^m |U_x - w_x|^p = px^m U_x^{p-1} w_x - x^m F(x, w_x),
\]
where
\[
F(x, w_x) = \frac{p(p-1)}{2} |U_x - \theta(x, w_x)| w_x |w_x|^2 \geq 0,
\]
with \( \theta \in (0, 1) \). Hence
\[
\begin{align*}
w_t - w_{xx} &= k \frac{w_x}{w_x} - x^m F(x, w_x), & t > 0, \quad 0 < x < 1, \\
w(t, 0) &= 0, \quad w(t, 1) = 0, & t > 0,
\end{align*}
\]
where \( k = \frac{p(m+1)}{p-1} \in (m + 1, m + 2) \).

First, one can see that \( \overline{w}(t, x) = Ce^{-\lambda t} \varphi(x) \) is a supersolution of (55), where \( \lambda \) and \( \varphi \) are given by Proposition 6. A direct computation shows that
\[
\overline{w}_t - \overline{w}_{xx} \geq k \frac{w_x}{w_x} - x^m F(x, \overline{w_x}).
\]
It follows from Proposition 6 and Remark 4 (ii) that \( \varphi(1) < 0 \) and \( \varphi(0) > 0 \). So, we have \( w(0, x) = U(x) - u_0(x) \leq C \varphi(x) = \overline{w}(0, x) \) for \( C > 0 \) large, \( \overline{w}(t, 0) = Ce^{-\lambda t} \varphi(0) > 0 \) and \( \overline{w}(t, 1) = Ce^{-\lambda t} \varphi(1) = 0 \). Therefore,
\[
w \leq Ce^{-\lambda t} \varphi.
\]

Second, let us give the lower estimate. Fix \( \epsilon \in (0, 1) \). By Lemmas 2.2 and 3.2, we obtain \( u_x \) is bounded on \([0, \infty) \times [\epsilon, 1] \). Therefore,
\[
w_t - w_{xx} \geq k \frac{w_x}{w_x} - K u_x^2, \quad \text{in} \ (0, \infty) \times [\epsilon, 1],
\]
for some $K = K(\epsilon, p, m) > 0$. Setting the Cole-Hopf trick $z = 1 - e^{-Kw}$, we obtain
\[
\begin{cases}
x^k z_t \geq (x^k z_x)_x, & t > 0, \; \epsilon < x < 1, \\
z(t, \epsilon) \geq 0, \; z(t, 1) = 0, & t > 0.
\end{cases}
\] (57)

It follows from the comparison principle that $\tilde{z}(t, x) = c_\epsilon e^{-\lambda t} \varphi_\epsilon(x) \leq z(t, x)$. Actually, a direct computation shows that
\[x^k z_t = (x^k z_x)_x, \; t > 0, \; \epsilon < x < 1\]
and $\tilde{z}(t, 1) = \tilde{z}(t, \epsilon) = 0$. Moreover, $z(0, x) = 1 - e^{-Kw(0, x)} \geq c_\epsilon \varphi_\epsilon(x) = \tilde{z}(0, x)$ on $[\epsilon, 1]$ for some small $c_\epsilon > 0$.

Finally, it is sufficient to show that $Kw \geq z \geq \tilde{z}$ on $[0, \infty) \times [\epsilon, 1]$. Therefore,
\[w(t, x) \geq c_\epsilon(1 - x)e^{-\lambda t}, \; \text{in } [0, \infty) \times [\epsilon, 1].\]

By $\lambda_\epsilon \to \lambda$ as $\epsilon \to 0$ (See [39, Lemma 5.1]), the lower estimate follows. \qed

4.4. Matching. Combining Sections 4.2 and 4.3, the following result summarizes the asymptotic behavior of $u_x(t, 0)$ in inner and outer regions.

**Theorem 4.4.** Assume $M = M_\epsilon$ and the conditions (2) and (3) hold. Then
\[Ce^{\mu t} \leq u_x(t, 0) \leq e^{(\mu + o(1))t} \left(1 + o(e^{(\lambda + \gamma E)t})\right), \; t \to \infty,
\] (58)
where $\lambda$ is defined by (52), $\mu = \frac{(m+1)\lambda}{p-m-2}$, $E = \frac{p-1}{2(p-m-3)\beta} > 0$ and $C = C(u_0, m, p)$. There exists $\gamma < -\frac{\lambda - \sqrt{\lambda^2 + 4AE}}{2E}$ such that $\lambda + \gamma E < 0$. Furthermore, the asymptotic behavior of the inner region is given by
\[
\lim_{t \to \infty} \frac{u_x(t, yo^{-\beta}(t))}{\alpha^{\gamma+\beta}(t)} = \left[1 + \frac{p-1}{m+1} y^{m+1}\right]^{\frac{1}{\sqrt{\beta}}}, \; \alpha(t) = [u_x(t, 0)]^{\frac{p-1}{m+1} \beta},
\]
uniformly for $y \geq 0$ in bounded sets. Besides, $u_x$ converges uniformly to $U_x$ in the outer region.

**Proof.** It follows from (56) that
\[u(t, x) \geq U(x) - \tilde{C} e^{-\lambda t}, \; t > 0, \; 0 < x < \eta,
\]
where $\eta$ is small enough and $\tilde{C} > 0$. Hence
\[\frac{u(t, x)}{x} \geq M_\epsilon x^{-\frac{m+1}{p-1}} - \tilde{C} e^{-\lambda t}, \; t > 0, \; 0 < x < \eta.
\] (59)

By $u(t, x) \to U(x) = M_\epsilon x^{\frac{p-m-2}{p-1}}$ as $t \to \infty$, we derive that $x^m |u_x|^p \sim \left[\frac{m+1}{p-1}\right]^{\frac{p}{m+1}} x^{\frac{m+1}{p-1}}$ as $t \to \infty$. So, $x^m |u_x|^p \to +\infty$ for $\eta$ small enough. Combining with Lemma 2.1, we get that
\[u_{xx} = u_t - x^m |u_x|^p \leq C_1 - x^m |u_x|^p < 0, \; t \gg 1, \; 0 < x < \eta.
\]

Clearly, $u_x$ is decreasing in $x$. Letting $\tilde{\alpha} = \frac{p-m-2}{p-1} > 0$ and $x = x_1(t) = \tilde{C} e^{-\tilde{\alpha} t} > 0$, we conclude that $x_1(t) \to 0$ as $t \to \infty$. For $t$ large enough, by the monotonicity of $u_x$ and (59), we can deduce that
\[u_x(t, 0) > u_x(t, x_1(t)) = \lim_{t \to \infty} \frac{u(t, x_1(t)) - u(t, 0)}{x_1(t)} \geq \frac{u(t, x_1(t))}{x_1(t)} \geq [M_\epsilon \tilde{C}^{-\frac{m+1}{p-1}} - 1] e^{(\frac{m+1)\lambda}{p-m-2}) t}, := C e^{\mu t},
\]
which is the lower estimate in (58).
Let us turn to the upper estimate. If there is no confusion, we note \( \alpha := \alpha(t) \). By Theorem 4.1(i) and Taylor expansion, for \( t \gg 1 \), we obtain

\[
w(t, x) = U(t, x) - u(t, x) \leq U(t, x) - U_\alpha(x)
\]

By Theorem 4.3, we obtain

\[
\text{due to } \gamma < 0 \text{ and } \alpha(t) \to \infty \text{ as } t \to \infty, \text{ each term on the right of (61) converges to 0 as } t \to \infty. \text{ Moreover,}
\]

\[
\frac{\alpha^{\gamma - \epsilon}}{\alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} = \alpha^{-\epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \to 0, \quad t \to \infty.
\]

Then, for (61), every term after the second term converges to 0 faster than the first term. Hence

\[
w(t, x) \leq M_e \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} + o \left( \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \right).
\]

By Theorem 4.3, we obtain

\[
e^{-\lambda + o(1)t} \leq M_e \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} + o \left( \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \right).
\]

It follows from Proposition 5 and \( \alpha(t) = [u_\alpha(t, 0)]^{\frac{p-1}{p-m+n}} \) that

\[
\alpha \geq C e^{\frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} t}.
\]

According to above two inequalities and Taylor expansion, we derive

\[
\alpha \leq \left[ \frac{1}{M_e} e^{-\lambda + o(1)t} + o \left( \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \right) \right]^{-\frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}}}
\]

\[
\leq \left[ \frac{1}{M_e} \right]^{-\frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} t}
\]

\[
+ \sum_{k=1}^{\infty} \frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \left[ \frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} - 1 \right] \cdots \left[ \frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} - k + 1 \right] \left[ \frac{1}{M_e} \right]^{-\frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} - k}
\]

\[
e^{-\lambda + o(1)} \left[ \frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} - k \right] t \left[ o \left( \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \right) \right]^{k}
\]

\[
\leq \left[ \frac{1}{M_e} \right]^{-\frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} t}
\]

\[
e^{-\lambda + o(1)} \left[ \frac{1}{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} - k \right] t \left[ o \left( \alpha^{\gamma - \epsilon} \frac{\alpha^{n-m-2}}{x^{n-m-2}} \right) \right]^{k}
\]
The last equal sign holds due to (64) and
\[ \sum_{k=1}^{\infty} C e^{-(\lambda + o(1)) t} = \frac{1}{M_c} \sum_{k=1}^{\infty} e^{-(\lambda + o(1)) t} + o \left( \frac{1}{k^2} \right) \]
\[ = \frac{1}{M_c} \sum_{k=1}^{\infty} e^{-(\lambda + o(1)) t} + o \left( \frac{1}{k^2} \right) \sum_{k=1}^{\infty} \frac{1}{k} \]
\[ = \frac{1}{M_c} \sum_{k=1}^{\infty} e^{-(\lambda + o(1)) t} + o \left( e^{(1-\frac{1}{2}) (\lambda + o(1)) + DE} t \sum_{k=1}^{\infty} \frac{1}{k^2} \right) \]
\[ \qquad \quad \text{as } t \to \infty. \]

where \( D = \gamma - \epsilon \frac{p-m-2}{p-1} \) \( \leq 0 \) and \( E = \frac{p-1}{2(p-m-3)\beta} \) \( > 0 \). Next, we could find a suitable function \( Q(t) \) which satisfies
\[ e^{[\lambda + o(1)) + DE] t} \leq \frac{Q(t)}{k^2}. \]

Let \( f(k) = Q(t) - k^2 e^{[\lambda + o(1)) + DE] t} \). Since \( \gamma - \frac{\lambda + \alpha + \lambda E}{2E} < 0 \), for any \( \epsilon \geq 0 \), we have
\[ \left( 1 - \frac{1}{D} \right) (\lambda + o(1)) + DE < 0. \]

It follows that \( \lambda + o(1) + DE < \lambda + o(1) + (\frac{1}{2} - 1)(\lambda + o(1)) = \frac{1}{2}(\lambda + o(1)) < 0 \). Obviously, \( f(k) \) is increasing with respect to \( k \) as \( t \to \infty \), so \( f(k) \geq f(1) = Q(t) - e^{[\lambda + o(1)) + DE] t} \). The inequality (63) can be obtained by \( Q(t) \geq e^{[\lambda + o(1)) + DE] t} \).

In particular, we set \( Q(t) = e^{[\lambda + o(1)) + DE] t} \). From (62) and (63), we derive
\[ \alpha \leq \frac{1}{M_c} \sum_{k=1}^{\infty} e^{-(\lambda + o(1)) t} + o \left( \frac{1}{k^2} \right) \sum_{k=1}^{\infty} \frac{1}{k^2} \]
\[ \leq \left( \frac{1}{M_c} \right)^{\frac{1}{2}} e^{-(\lambda + o(1)) t} + o \left( e^{(1-\frac{1}{2}) (\lambda + o(1)) + DE} t \sum_{k=1}^{\infty} \frac{1}{k^2} \right) \]
\[ \qquad \quad \text{as } t \to \infty. \]

The last equal sign holds due to (64) and \( \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \). It follows from (65) and Taylor expansion that
\[ u_x(t, 0) = \alpha \frac{(m+1)\beta}{p-1} \]
\[ \leq \left( \frac{1}{M_c} \right)^{\frac{1}{2}} e^{-(\lambda + o(1)) t} + o \left( e^{(1-\frac{1}{2}) (\lambda + o(1)) + DE} t \right) \frac{(m+1)\beta}{p-1}. \]
The width of the inner layer, is estimated as

\[ O \]

Remark 5. Combining estimates in the inner and outer regions, we could derive

\[ \text{REFERENCES} \]

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions. Chang and Zhang are supported by the National Natural Science Foundation of China (No. 11371286) and the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2019JM-165). Ju is supported by the NSFC of China (Nos. 11571046, 11671225), the ISF-NSFC joint research program of China (No. 11761141008) and the BJNSF of China (No. 1182004)

\[ = \left[ \frac{1}{M_c} \right] \frac{(m+1)\beta}{p-1} e^{-\frac{(\lambda+\alpha(1))}{p-1}t} \]

\[ + \sum_{k=1}^{\infty} \frac{(m+1)\beta}{p-1} \frac{(m+1)\beta}{p-1} \frac{1}{k!} - k \right] \frac{1}{M_c} \frac{(m+1)\beta}{p-1} - k \]

\[ \cdot e^{-\frac{(\lambda+\alpha(1))}{p-1} \frac{(m+1)\beta}{p-1} - k} \left( e^{[1 - \frac{1}{p}(\lambda+\alpha(1)) + DE]kt} \right) \]

\[ \left[ \frac{1}{M_c} \right] \frac{(m+1)\beta}{p-1} e^{-\frac{(\lambda+\alpha(1))}{p-1} t} \left( 1 + o \left( \sum_{k=1}^{\infty} e^{[\lambda+\alpha(1) + DE]kt} \right) \right) \]

\[ \leq \left[ \frac{1}{M_c} \right] \frac{(m+1)\beta}{p-1} e^{-\frac{(\lambda+\alpha(1))}{p-1} t} \left( 1 + o \left( e^{[\lambda+\alpha(1) + DE]t} \right) \right) \]

as \( t \to \infty \).

The calculation of the last inequality is similar to (65). Letting \( \epsilon \to 0 \), we derive

\[ u_x(t, 0) \leq C(m, p) e^{\frac{m+1}{p-1}(\lambda+\alpha(1))t} \left( 1 + o(e^{\lambda+\gamma E}) \right) \]

\[ = C(m, p) e^{[\mu+\alpha(1)]t} \left( 1 + o(e^{\lambda+\gamma E}) \right) , \]

which is the upper estimate in (58).

Remark 5. Combining estimates in the inner and outer regions, we could derive the width of the inner layer, is estimated as \( O \left( e^{-\frac{m+1}{p-1} \mu t} \right) \). In fact, since \( y = x^\alpha(t) \) is in bounded sets, \( x = O \left( \alpha^{-\beta}(t) \right) \). It follows from (58) that \( u_x(t, 0) = O(e^{\mu t}) \).

Hence

\[ x = O \left( \alpha^{-\beta}(t) \right) = O \left( \left( u_x(t, 0) \right) \frac{1}{m+1\beta} \right) = O \left( e^{-\frac{m+1}{p-1} \mu t} \right) . \]
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Received March 2020; revised May 2020.

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