EXPONENTIAL DECAY AND SYMMETRY OF SOLITARY WAVES TO THE DEGASPERIS-PROCESI EQUATION

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Abstract. We prove that solitary waves to the steady Degasperis-Procesi equation, if they exist, decay as $e^{-|x|}$ for large $|x|$ and are symmetric with respect to the unique symmetry axis located at the only crest (although only peaked solitary waves are allowed). Moreover, the solitary waves are strictly monotone on each side of the crest and decay as good as the kernel function corresponding to the dispersive operator in the nonlocal formulation. The argument here assumes a priori the existence of solitary waves and cannot distinguish their type, but starts with little information about regularity of solitary waves and can be adapted to treat general solitary waves to other dispersive equations which may not be integrable. Finally, we give an alternative, more intuitive proof for the fact that all classical solutions with a priori spatial symmetry to the Degasperis-Procesi equation must be traveling waves.

1. Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{xxx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0$$

is a unidirectional model for shallow water waves and has the nonlocal formulation

$$\partial_t u + u \partial_x u + \partial_x L(\frac{3}{2}u^2) = 0,$$

where the dispersive operator $L = (1-\partial_x^2)^{-1}$ corresponds to the Fourier symbol $m(\xi) = (1+\xi^2)^{-1}$. Being completely integrable and having bi-hamiltonian structure [17], this equation together with the KdV and Camassa-Holm were three well-known representatives in both integrable system theory and water wave problems. Although firstly put forward from the perspective of integrability, this model was later rigorously derived as a model for shallow water waves and proved to have the same accuracy as the Camassa-Holm equation [8]. The Degasperis-Procesi equation is locally well-posed in the classical Sobolev space $H^s$, $s > \frac{3}{2}$, in both periodic and non-periodic settings [21], and it allows global weak and classical solutions [22, 20] while the latter may blow up in the form of wave-breaking [13].

As an important feature in soliton theory and water wave problems, traveling wave solutions (both periodic and solitary) to (1.1) were found in [19]. These waves can be peakons, lump-like, and loop-like. Later, a complete classification of traveling waves to DP was made by Lenells [14] so that all possible traveling solutions are found. Very recently, Arnesen [2] worked on the non-local formulation (1.2) and proved that differentiable, even traveling waves with uniform bound have the wave speed as the upper bound, and are smooth waves if wave height strictly smaller than wave speed, and periodic waves will form a peak if the height reaches the wave speed. This result indicates that DP does not allow for cusp-like periodic traveling waves. It is worth to mention that DP also allow for peakon and
multipeakon solutions, which are not traveling wave solutions \cite{10}. This indicates the rich structure of this equation due to the delicate balance of dispersive and nonlinear effect.

It is clear from \cite{14} that solitary waves to the DP equation must be peaked waves, while a peak will form if the wave height reaches the speed $c$ by \cite{2}. However, we hope to give a method here, which requires as little as possible the information of solitary waves to the DP equation, but still gives good estimate on the decay rate of solitary waves and determine their symmetry structure. Our method for decay and symmetry of solitary waves is not designed to distinguish the type of solitary waves, but guarantees that a general solitary wave to the DP equation, if it exists (no matter it is smooth, peaked or cusped), decay at least as fast as $e^{-|x|}$ as $|x| \to \infty$, although only peaked solitary waves are allowed to the DP equation. We hope our proofs here can be adapted to other equations, for which integrability is not clear and smooth, peaked or cusped solitary waves may exist. So, in the following we pretend that steady DP allows for both smooth and peaked solitary waves and give proof details for deriving the decay estimate and symmetry analysis (the decay estimate works for cusped wave as well, see Sect. 3.9, so we skip the proof).

In this paper, we first confirms that solitary waves to the DP equation decays as $e^{-|x|}$ for large $|x|$, which is as good as the decay of the kernel corresponding to the operator $L$ and as the peakon and multipeakon solutions constructed in \cite{7,15}. In addition, we prove that solitary waves are symmetry and has a unique crest, which in particular implies that a peaked solitary wave of the form $\phi(x - ct)$ only has one peak$^1$. It seems that there exists no argument for confirming the symmetry of a solitary wave with wave speed as its height up to now, so we put forward an argument here to treat such waves and expect it can be adapted to solve similar problems for other equations, like the Whitham in \cite{5}. Finally, traveling waves solutions are studied by a priori assuming that they are even or symmetric, and it rises the question whether there exists asymmetric traveling waves. For this question, we confirm that solitary waves to the DP must be symmetric with only one symmetry axis, and give another proof for the fact that classical symmetric solutions must be traveling waves (see \cite{4,11} for earlier proofs). This result is expected for weaker solutions, and therefore it is quite reasonable to study traveling waves with a priori spatial symmetry.

Traveling solutions have fixed shape and speed, and can be imposed with the ansatz $u(t, x) = \phi(x - ct)$ for a constant $c$ denoting the wave speed. Inserting this ansatz into (1.2), we arrive at the following steady form Degasperis-Procesi equation

$$\frac{\phi}{3}(2c - \phi) = L\phi^2 + a$$

where $a$ is the integration constant. Direct calculation shows that it is not possible to remove the constant $a$ by Galiean translation. However, it can be calculated by Fourier analysis that $L$ corresponds to a convolution kernel $K(x) = e^{-|x|}$, which then make $a$ simply vanish if $\phi(\cdot)$ decays (no matter how fast it decays) at infinity. Moreover, the smoothing effect of $L$ guarantees that all $L^\infty$ bounded solutions to (1.3) actually are continuous. Therefore, in the following, by solitary waves we mean $L^\infty$-bounded solutions with decay at infinity to the following steady Degasperis-Procesi equation

$$\frac{\phi}{3}(2c - \phi) = K * \phi^2.$$ 

and $K$ denotes the convolution kernel corresponding to $L$. For a general convolution kernel $K(\cdot)$ and other constant coefficients on the left side, (1.4) actually is the general form for$^1$

\footnote{This does not contradict with multi-peakon solutions where peaks are of different speed.}
SYMMETRY AND DECA Y OF TRAVELING WAVES

a large class of dispersive equations after proper reformulation. In particular, it covers the steady Whitham equation for which the decay and symmetry of solitary waves are studied in [5] based on the argument in [3, 16] and the method of moving planes, respectively. For the decay result in [5], the authors firstly prove that the kernel decays exponentially fast, and then solitary waves decays faster than polynomial decay in $L^p$ and $L^\infty$ spaces, namely $|x|^n\phi(x) \in L^\infty$ for any $n \in \mathbb{N}$. Then the careful analysis of the decay rate of $|x|^n\phi(x)$ for each $n$ guarantees that the sum of these solitary waves with weight converges so that the solitary waves decay exponentially.

In this paper, we firstly improve the convolution estimate of polynomial type in [3, 5] to exponential type. This improvement allows to remove the analysis of decay rate of $|x|^n\phi(x)$ for each $n$ in [3, 5] so that the proof of exponential decay of solitary waves can be considerably simplified. In addition, this new convolution estimate applies to general convolution equations (not limited to the form of (1.4)), where a kernel with exponential decay is convoluted with a nonlinearity on solitary waves. The exponential decay of solitary waves then helps to remove the commonly used one-side monotonicity assumption when one applies the method of moving planes (see [6, 9, 1, 18]) to prove the symmetry of such waves. For the symmetry of solitary waves with height smaller than the wave speed $c$, our proof follows the idea in [5] for the Whitham equation, where an important observation is that the nonlocal operator $L$ behaves as an elliptic operator and that there exists a touching lemma on half-plane, which plays the role as the maximum principle for elliptic equations. For solitary waves of the maximal height (see [12]), i.e., $\sup_{x \in \mathbb{R}} \phi(x) = c$, the argument in [5] unfortunately fails to confirm the symmetry. We here put forward an argument which relies on a careful analysis of the local structure of the solitary wave near the crest and confirms the symmetry of the highest solitary wave to the Degasperis-Procesi equation. It is expected that the argument here can be modified to finally prove the symmetry of solitary waves of the maximal height to the Whitham equation, where the kernel function is more complicated.

The spatial symmetry is more than a byproduct property of traveling waves as shown above. In fact, as shown in Principle (P1) in [4], all classical solutions to the Degasperis-Procesi equation are traveling wave solutions. It is not clear how to prove this result for the local formulation (1.1), but it is easy to see that its equivalent nonlocal formulation (1.2) satisfies Principle (P1), which also indicates the advantage of the nonlocal formulation of an equation which is previously known in the local form. In this paper, we give another proof for the fact that classical solutions with a priori spatial symmetry must be traveling waves. Classical symmetric solutions with spatial symmetry satisfy two constrained equations, and the proof in [4] studied one of them and prove the steadiness of such solutions by a quite constructive argument. Here, we find each of the two constrained equation clearly determines one structural feature of symmetric waves: the one not discussed in [4] guarantees that these waves must remain the form of the initial data in the later evolution, while the other guarantees that these waves must travel with constant speed.

We now state the structure of this paper. Section 2 starts with an estimate where the kernel $K(\cdot)$ is convoluted with exponential type functions. Based on this lemma, we prove that the solitary solutions decay exponentially fast at infinity and the decay rate is at least as good as the decay rate of the kernel $K(\cdot)$. Section 3 focuses on the symmetry of solitary waves. In particular, we treat solitary waves with height smaller than the wave speed in section 3.1, while waves with height equal to wave speed are treated in section 3.2. Finally, we give a new proof in section 4 for the fact classical symmetry solutions to the Degasperis-Procesi equation must be traveling waves.
2. Exponential decay of solitary waves at infinity

For a traveling wave solution \( u(t, x) = \phi(x - ct) \) with speed \( c \), the sign of \( c \) distinguishes only the direction of the propagation of the wave. So, we will only work with \( c > 0 \) in the following. As mentioned above, direct calculation by fourier analysis gives that

\[
\mathcal{F}[Lf](\xi) = \frac{1}{1 + \xi^2} \mathcal{F}[f](\xi) = \mathcal{F}[K * f](\xi)
\]  

(2.1)

where \( \mathcal{F} \) denotes the usual Fourier transform and \( K(x) = \frac{1}{2} e^{-|x|} \) denotes the convolution kernel of \( L \). By definition, \( L \) lifts a \( L^\infty \) bounded function to a continuous function (see [2] for detail), we will then work with continuous solutions in the following. We first prove that the integration constant \( a = 0 \) for solitary waves and solitary solutions are uniformly bounded.

**Lemma 2.1.** The integration constant \( a \) vanishes in (1.3) for solitary waves and all non-trivial solitary waves to (1.4) satisfies

\[
0 < \phi \leq \sup_{x \in \mathbb{R}} \phi < 2c
\]  

(2.2)

**Proof.** Assume that \( \phi(x) \rightarrow 0 \) as \( |x| \rightarrow 0 \). In view of (1.4), it suffices to prove that \( L\phi^2 \) approaches 0 as \( \phi \rightarrow 0 \). Note that

\[
L\phi^2(x) = \int_{\mathbb{R}} K(x - y)\phi^2(y)dy = \int_{|x - y| < N} K(y)\phi^2(x - y)dy + \int_{|x - y| > N} K(y)\phi^2(x - y)dy
\]  

(2.3)

For any \( \epsilon > 0 \), we can choose \( N \in \mathbb{N} \) large enough such that \( \phi(x) < \epsilon \) if \( |x| > N \). Then by definition of \( K(\cdot) \) we have

\[
\int_{|x - y| > N} K(y)\phi^2(x - y)dy < \epsilon^2 \int_{\mathbb{R}} K(y)dy = \epsilon^2
\]  

(2.4)

Fix the \( N \) and \( \epsilon \) chosen above, then there exists \( M_1 > 0 \) large enough such that \( K(y) < \frac{\epsilon}{2N\|\phi\|_{L^\infty}} \) for any \( |y| > M_1 \). Note that for any \( y \in \{y||x - y| < N\} \), we have

\[
|x| - N < |y| < |x| + N
\]  

(2.5)

Define \( \Sigma = \{y||x - y| < N, |x| > M_1 + N\} \). Then we have

\[
\int_{|x - y| < N} K(y)\phi^2(x - y)dy < N\|\phi\|_{L^\infty} \sup_{y \in \Sigma} K(y) < \frac{\epsilon}{2}
\]  

(2.6)

Combining (2.6) and (2.4), we have \( L\phi^2(x) < \epsilon \) for any small \( \epsilon > 0 \) if \( |x| \) is sufficiently large, which implies that \( a \) vanishes.

To prove (2.2), we first notice from the property of \( K(x) \) that \( L \) is a monotone operator on continuous bounded functions, i.e., \( Lf > Lg \) if \( f \geq g \) but \( f \neq g \). In addition, it holds that \( LC = C \) for any constant \( C \). Therefore, we derive from (1.2) that

\[
\phi^2 - 2c\phi = -3L\phi^2 < 0
\]  

(2.7)

which implies that \( \phi \in (0, 2c) \). The decay of \( \phi \) then implies that \( \sup_{x \in \mathbb{R}} \phi \) must be reached at a finite \( x_0 \in \mathbb{R} \) and (2.2) follows. \( \square \)

As mentioned earlier, we hope to simplify the procedure of the proof for exponential decay of solitary solutions in [3, 5], and the key for such simplification is the following lemma which generalizes the polynomial type estimates in [3] to exponential type.
Lemma 2.2 (Convolution estimate of exponential type). For $0 < l < m$ and any $\sigma > 0$, the following inequality holds

$$\int_{\mathbb{R}} \frac{e^{l|x|}}{(1 + \sigma e^{x})^{m} e^{m|x-y|}} dx \leq B \frac{e^{l|y|}}{(1 + \sigma e^{y})^{m}}, \quad y \in \mathbb{R},$$

where $B = (\min\{l, m - l\})^{-1}$.

Proof. By symmetry of the structure in (2.8), it suffices to prove for the case $x, y > 0$. Note that

$$\int_{0}^{\infty} \frac{e^{l|x|}}{(1 + \sigma e^{x})^{m} e^{m|x-y|}} dx = \left( \int_{0}^{y} + \int_{y}^{\infty} \right) \frac{e^{lx}}{(1 + \sigma e^{x})^{m} e^{m|x-y|}} dx =: I_{1} + I_{2}$$

For $I_{1}$, we have

$$I_{1} = \int_{0}^{y} \frac{e^{lx}}{(1 + \sigma e^{x})^{m} e^{m(y-x)}} dx \leq \frac{e^{ly} - 1}{e^{my} (\sigma + e^{-y})^{m} l} \leq \frac{e^{ly}}{l (1 + \sigma e^{y})^{m}}$$

For $I_{2}$, we have

$$I_{2} = \int_{y}^{\infty} \frac{e^{lx}}{(1 + \sigma e^{x})^{m} e^{m(x-y)}} dx \leq \frac{e^{my}}{(1 + \sigma e^{y})^{m}} \int_{y}^{\infty} e^{l(m-l)x} dx \leq \frac{(m-l)^{-1} e^{ly}}{(1 + \sigma e^{x})^{m}}.$$

The inequality (2.8) and hence this lemma follow directly. \hfill \Box

We now prove that solitary solutions to the Degasperis-Procesi equation decay exponentially fast at infinity. For convenience, we introduce the notation $M := \sup_{x \in \mathbb{R}} \phi$.

Theorem 2.3. The map $x \mapsto e^{l|x|} \phi(x) \in L^{\infty}(\mathbb{R}, \mathbb{R})$.

Proof. We first prove that

$$e^{\alpha l|x|} \phi(\cdot) \in L^{q}(\mathbb{R})$$

for any $\alpha \in (0, 1)$ and $q > 1$. Note that we have $e^{\alpha l|x|} K(x) \in L^{p}(\mathbb{R})$ for any $\alpha \in (0, 1)$. Define

$$C_{\alpha,p} := 3(2c - M)^{-1} \| e^{\alpha l|x|} K(\cdot) \|_{L^{p}(\mathbb{R})},$$

and let $q$ be the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. By (1.4) and Hölder’s inequality, we have

$$\phi = \frac{3}{(2c - \phi)} \int_{\mathbb{R}} K(x - y) e^{\alpha l|x-y|} \frac{\phi^{2}(y)}{e^{\alpha l|x-y|}} dy \leq C_{\alpha,p} \left( \int_{\mathbb{R}} \frac{|\phi^{2}(y)|^{q}}{e^{\alpha l|x-y|}} dy \right)^{\frac{1}{q}}. \quad (2.10)$$

Let $l \in [0, \alpha)$ and define

$$h_{\varepsilon}(x) := \frac{e^{l|x|}}{(1 + e|x|)^{\alpha}} \phi(x)$$

for small $\varepsilon \in (0, 1)$. For each $\varepsilon \in (0, 1)$ fixed, the function $h_{\varepsilon}$ is bounded in $L^{q}(\mathbb{R})$, by the choice of $l$ and $\phi$ being bounded. The aim is to prove that \{ $h_{\varepsilon} \mid \varepsilon \in (0, 1)$ \} is uniformly bounded in $L^{q}(\mathbb{R})$, seeing that this implies that $\lim_{\varepsilon \to 0} h_{\varepsilon} = e^{l|x|} \phi$ belongs to $L^{q}(\mathbb{R})$, by dominated convergence.

Since $\phi$ tends to zero as $|x| \to \infty$, the quadratic nonlinearity provides that for every $\alpha > 0$ there exists a constant $R_{\alpha} > 1$ such that

$$|\phi^{2}(x)| \leq \delta |\phi(x)| \quad \text{for} \quad |x| \geq R_{\delta}.$$

Since

$$\|h_{\varepsilon}\|_{L^{q}(\mathbb{R})} = \int_{\mathbb{R}} |h_{\varepsilon}(x)|^{q} dx \leq C + \int_{|x| \geq R_{\delta}} |h_{\varepsilon}(x)|^{q} dx, \quad (2.12)$$

Symmetry and Decay of Traveling Waves
where $C = C(R_\delta) > 0$ is a constant independent of $\varepsilon$, we are left to study the last integral on the right-hand side of (2.12).

Let $r \in (0, q)$. By (2.10) and Hölder’s inequality we have

$$\int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \leq \int_{|x| \geq R_\delta} |h_\varepsilon(x)|^{q-r} \left( \frac{e^{l|x|}}{(1 + e^{l|x|})^\alpha} \right)^r |\phi(x)|^r \, dx$$

$$\leq \int_{|x| \geq R_\delta} |h_\varepsilon(x)|^{q-r} \left( \frac{e^{l|x|}}{(1 + e^{l|x|})^\alpha} \right)^r C_{\alpha,p}^r \left( \int_R |\phi^2(y)|^q \, dy \right)^{\frac{r}{q}} \, dx$$

$$\leq C_{\alpha,p}^r \left[ \int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \right]^{-\frac{q-r}{q}} \left[ \int_{|x| \geq R_\delta} \frac{e^{l|x|}}{(1 + e^{l|x|})^\alpha} \left( \int_R |\phi^2(y)|^q \, dy \right) \, dx \right]^{\frac{r}{q}}.$$

Dividing both sides of the inequality by $\left[ \int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \right]^{-\frac{q-r}{q}}$, we find that

$$\int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \leq K_{\alpha,p}^q \int_{|x| \geq R_\delta} \frac{e^{l|x|}}{(1 + e^{l|x|})^\alpha} \left( \int_R |\phi^2(y)|^q \, dy \right) \, dx =: K_{\alpha,b}^q T. \quad (2.13)$$

By Fubini’s theorem and lemma 2.2, we obtain that

$$T = \int_R |\phi^2(y)|^q \left[ \int_{|x| \geq R_\delta} \frac{e^{l|x|}}{(1 + e^{l|x|})^\alpha e^{\alpha q|x-y|}} \, dx \right] \, dy$$

$$\leq \int_{|x| \geq R_\delta} |\phi^2(y)|^q \left( \int_{|x| \geq R_\delta} \frac{Be^{lq|y|}}{(1 + e^{l|x|})^\alpha e^{\alpha q|x-y|}} \, dx \right) \, dy \quad (2.14)$$

$$+ \int_{|y| < R_\delta} |\phi^2(y)|^q \left( \int_{|x| \geq R_\delta} \frac{e^{lq|x|}}{(1 + e^{l|x|})^\alpha e^{\alpha q|x-y|}} \, dx \right) \, dy,$$

where $B = B(l, q, \alpha) > 0$ does not depend on $\varepsilon$. Since $0 < l < \alpha$, the last integral in (2.14) is bounded by a constant $C_1$ which depends on $l, \alpha, q, \|\phi\|_\infty$ and $R_\delta$ but is independent of $\varepsilon$. Combining (2.13), (2.14) and recalling that $|\phi^2(y)| < \alpha |\phi(y)|$ for all $|y| \geq R_\delta$, we deduce that

$$\int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \leq K_{\alpha,p}^p \left[ \alpha^q B \int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx + C \right]. \quad (2.15)$$

Choosing $\alpha$ small enough so that $K_{\alpha,p}^p \alpha^q B < \frac{1}{2}$, (2.15) implies that

$$\int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \leq C_2,$$

where $C_2 = C_2(\alpha, p, \|\phi\|_\infty, R_\delta) > 0$ is a constant which does not rely on $\varepsilon$.

Hence, we have shown that

$$\int_{|x| \geq R_\delta} |h_\varepsilon(x)|^q \, dx \lesssim 1.$$

Letting $\varepsilon \to 0$, dominated convergence ensures that

$$\int_{\mathbb{R}} e^{lq|x|} |\phi(x)|^q \, dx \lesssim 1,$$

Note that the term we are dividing by vanishes if and only if $\phi = 0$ everywhere in $\{|x| \geq R_\delta\}$, in which case the lemma is obviously true.
which implies in particular \( x \mapsto e^{\|x\|} f(x) \in L_q(\mathbb{R}) \) for \( q = \frac{p}{p-1} \) and \( l \in [0, \alpha) \). Then, by (1.4), (2.9) and Young’s inequality, we have

\[
e^{\alpha \|x\|} \phi(x) \lesssim \frac{3}{2c-M} \left[ \left( e^{\alpha \|K\|} \right) * \left( e^{\alpha \|\phi^2\|} \right) \right] (x) \in L_\infty(\mathbb{R})
\]

for any \( \eta \in (0, 1) \).

Now we use the structure of (1.2) to improve the decay rate of \( \phi \) and prove that \( \phi \) decays at least as good as the kernel \( K \). In fact, by Young’s inequality

\[
e^{x\|x\|} \phi(x) \leq \frac{1}{2c-M} \int_{\mathbb{R}} K(x-y) e^{\|x-y\|} \left( \phi(y) e^{\frac{|y|}{2}} \right)^2 dy \leq \|e^{\|\cdot\|} \cdot |K(\cdot)| \|_{L\infty} \|\phi e^{\frac{|\cdot|}{2}}\|_{L^2}^2 < \infty.
\]

\[\square\]

3. SYMMETRY AND ONE-CREST STRUCTURE OF SOLITARY WAVES

3.1. Solitary waves below the maximal height. We first introduce the notion of supersolution and subsolution of the steady DP equation. A solution \( \phi \) to the steady Degasperis-Procesi equation (1.4) is called a **supersolution** if

\[
\frac{\phi}{3} (2c - \phi) \geq K * \phi^2
\]

and a **subsolution** if the inequality above is replaced by \( \leq \).

**Lemma 3.1.** Let \( \phi_1 \) and \( \phi_2 \) be a super– and a subsolution of the steady Degasperis-Procesi equation (1.4) on a subset \( [\lambda, \infty) \subset \mathbb{R} \), respectively, satisfying \( \phi_1 \geq \phi_2 \) on \( [\lambda, \infty) \) and \( \phi_1^2 - \phi_2^2 \) being odd with respect to \( \lambda \), that is \( (\phi_1^2 - \phi_2^2)(x) = - (\phi_1^2 - \phi_2^2)(2\lambda - x) \). Then either

- \( \phi_1 = \phi_2 \) in \( \lambda, \infty \), or
- \( \phi_1 > \phi_2 \) with \( \phi_1 + \phi_2 < 2c \) in \( \lambda, \infty \).

**Proof.** The symmetry and monotonocity of \( K \) allow to deduce that \( K \) acts as a positive convolution operator on odd functions with respect to \( \lambda \) on the half line \( \lambda, \infty \). Let \( f \geq 0 \) on \( \lambda, \infty \), \( f(x) = -f(2\lambda - x) \) and \( x \geq \lambda \). Then

\[
K * f(x) = \int_{\lambda}^{\infty} K(y) f(x-y) dy + \int_{-\infty}^{\lambda} K(x-y) f(y) dy
\]

\[
= \int_{\lambda}^{\infty} K(x-y) f(y) dy + \int_{\lambda}^{\infty} K(x+y-2\lambda) f(2\lambda-y) dy
\]

\[
= \int_{\lambda}^{\infty} (K(x-y) - K(x+y-2\lambda)) f(y) dy,
\]

where the last equality holds thanks to \( f \) being odd with respect to \( \lambda \). In view of \( K \) being symmetric and monotonically decreasing on \( (0, \infty) \), we obtain that

\[
K * f(x) \geq 0 \quad \text{for all } \quad x \geq \lambda.
\]

In particular, \( K * f > 0 \) or \( f = 0 \) on \( \lambda, \infty \). Assume that \( \phi_1 \) and \( \phi_2 \) are super– and subsolutions to the steady DP equation, respectively, \( \phi_1 \geq \phi_2 \) for all \( x \geq \lambda \) and \( \phi_1^2 - \phi_2^2 \) is odd with respect to \( \lambda \), that is \( \phi_1^2 - \phi_2^2 \) plays the role of \( f \) above. Then,

\[
(2c - (\phi_1 + \phi_2))(\phi_1 - \phi_2) \geq 3K * (\phi_1^2 - \phi_2^2) > 0
\]

for all \( x > \lambda \) unless \( \phi_1 = \phi_2 \) on \( \lambda, \infty \). \[\square\]
We will use the method of moving planes to prove the symmetry and one-crest structure of the wave profile. In view of the above result on decay of solitary solutions, we first prove that we can find a sufficiently negative \( \lambda \) such that the reflected wave profile always stays above the wave profile itself on the left side of the reflection axis \( x = \lambda \). To proceed, we define the open sets

\[
\Sigma_\lambda := \{ x \in \mathbb{R} \mid x > \lambda \} \quad \text{and} \quad \Sigma^-_\lambda := \{ x \in \Sigma_\lambda \mid \phi(x) < \phi_\lambda(x) \},
\]

where \( \phi_\lambda(\cdot) := \phi(2\lambda - \cdot) \) is the reflection of \( \phi \) about the axis \( x = \lambda \).

**Theorem 3.2.** There exists a \( N > 0 \) sufficiently large such that

\[
\phi(x) < \phi_\lambda(x), \quad x < \lambda,
\]

for any \( \lambda \leq -N \). In other words, \( \Sigma^-_\lambda = \emptyset \) for any \( \lambda \leq -N \).

**Proof.** Note that

\[
2c(\phi_\lambda(x) - \phi(x)) - 3 \left( \int_{\Sigma_\lambda \setminus \Sigma^-_\lambda} (K(x - y) - K(2\lambda - x - y))(\phi_\lambda^2(y) - \phi^2(y))dy \right. \\
\left. + \phi_\lambda^2(x) - \phi^2(x) \right).
\]

Since

\[
|x - y| - |2\lambda - x - y| = \min\{2(x - \lambda), 2(y - \lambda)\} > 0, \quad x, y \in \Sigma_\lambda,
\]

we deduce from the symmetry and complete monotonicity of \( K \) that

\[
K(x - y) - K(2\lambda - x - y) > 0
\]

for \( x, y \in \Sigma_\lambda \). Then, the integral over \( \Sigma_\lambda \setminus \Sigma^-_\lambda \) on the right side of (3.2) is negative and for \( x \in \Sigma^-_\lambda \) in (3.2) we have

\[
2c(\phi_\lambda(x) - \phi(x)) - 3 \int_{\Sigma^-_\lambda} (K(x - y) - K(2\lambda - x - y))(\phi_\lambda^2(y) - \phi^2(y))dy + \phi_\lambda^2(x) - \phi^2(x).
\]

Moreover, Theorem 2.3 implies that for any small \( \epsilon > 0 \), we can choose sufficiently large \( N \) such that

\[
\phi(x) < \phi_\lambda(x) < \epsilon, \quad x \in \Sigma^-_\lambda
\]

for any \( \lambda \leq -N \). Then by taking the \( L^\infty \)-norm on both side of (3.2) over \( \Sigma^-_\lambda \) and Lemma 3.1, we have

\[
\|\phi_\lambda - \phi\|_{L^\infty(\Sigma^-_\lambda)} \leq \frac{3}{2c} \|\phi + \phi_\lambda\|_{L^\infty(\Sigma^-_\lambda)} \left( \|K\|_{L^1(\mathbb{R})} + 1 \right) \|\phi_\lambda - \phi\|_{L^\infty(\Sigma^-_\lambda)}
\]

\[
\leq \frac{3\epsilon}{c} \left( \|K\|_{L^1(\mathbb{R})} + 1 \right) \|\phi_\lambda - \phi\|_{L^\infty(\Sigma^-_\lambda)}
\]

where \( (\Sigma^-_\lambda)^* \) is the reflection of \( \Sigma^-_\lambda \) about the plane \( x = \lambda \). By choosing \( \epsilon < \frac{c}{6\|K\|_{L^1(\mathbb{R})} + 1} \), we get a contradiction in (3.6) unless \( \|\phi - \phi_\lambda\|_{L^\infty(\Sigma^-_\lambda)} = 0 \) for \( \lambda \leq -N \). As a consequence \( \Sigma^-_\lambda \) must be of measure zero. Since \( \Sigma^-_\lambda \) is open, we deduce that \( \Sigma^-_\lambda \) is empty for \( \lambda \leq -N \). \( \Box \)

We are now ready to prove that solitary waves solutions are symmetric and have exactly one crest at the symmetric axis. Note that the method in [5, 4] for proving the symmetry of solitary waves to the Whitham equation is restricted to waves whose height is smaller than
maximal height $\frac{c}{2}$. We put forward an argument to treat the highest waves to Degasperis-Procesi equation here, and expect that it can be adapted for the more complicated case in the Whitham equation.

**Theorem 3.3.** Let $\phi$ be a supercritical solution to the steady Degasperis-Procesi equation. Then, there exists a unique $\lambda \in \mathbb{R}$ such that $\phi$ is symmetric about the axis $x = \lambda$ and $\phi$ is strictly monotonic on each side of the symmetric axis.

**Proof.** Clearly, there can not be any crest at a point $x \leq -N$, since $\Sigma^-_\lambda$ is empty for all $\lambda < -N$. We now move the axis $x = \lambda$ from $\lambda = -N$ to the right and it is clear that $\Sigma^-_\lambda = \emptyset$ unless it reaches a local maximum or the reflection $\phi_\lambda(x)$, $x < \lambda$, or touches the wave profile at some point on the right side of $x = \lambda$. However, the latter could not happen. In fact, if we assume that the moving plane stops at a point $x = \lambda_0$, where $\phi(x) \geq \phi_{\lambda_0}(x)$, but $\phi(x)$ does not match $\phi_{\lambda_0}(x)$ exactly for all $x \in \Sigma_{\lambda_0}$. By taking $\phi$ and $\phi_\lambda$ as the super- and subsolution, respectively, we get from Lemma 3.1 that $\phi(x) > \phi_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$ so that $\Sigma_{\lambda_0}$ has measure zero and $\phi(x)$ can not touch $\phi_{\lambda_0}(x)$ at some point on the right side of $x = \lambda_0$. So, the above process stops at $x = \lambda_0$ where $\phi$ reaches its local maximum.

We now assume that $\phi(x)$ does not match $\phi_{\lambda_0}(x)$ for all $x \in \Sigma_\lambda$ and show that this leads to a contradiction. First of all, the touching lemma excludes the case $\phi(x) \equiv \phi(\lambda_0)$ on the interval $[\lambda_0, \lambda + \epsilon]$ for any small $\epsilon > 0$. Also, it is clear that the above process guarantees that $\phi$ is monotonically increasing on $(-\infty, \lambda_0)$. Then by the continuity of $\phi$, the touching lemma, the set $\Sigma^-_\lambda$ will be simply connected and its size can be made very small for $\lambda \in (\lambda_0, \lambda_0 + \delta)$ if we choose $\delta > 0$ sufficiently small. For a fixed $\lambda \in (\lambda_0, \lambda_0 + \delta)$, we denote $\Sigma^-_\lambda$ by $[\lambda, b]$ and it is clear that $2\lambda - \lambda_0 \in (\lambda, b)$. Without loss of generality, we assume that the value of $\phi$ at the crest $x = \lambda_0$ but not reach the maximal height $c$, that is

$$\phi(x) \leq \phi(\lambda_0) < c$$

so that

$$c_1 := \sup_{x \in \Sigma^-_\lambda} (\phi + \phi_\lambda)(x) < 2\phi(\lambda_0) < 2c$$

for some small $\epsilon_1 > 0$.

Then, we use (3.2) on $\Sigma^-_\lambda$ and get

$$(2c - c_1)(\phi_\lambda - \phi)(x) \leq 3 \int_{\Sigma^-_\lambda} (K(x - y) - K(2\lambda - x - y))(\phi_\lambda^2(y) - \phi^2(y))dy$$

for $x \in \Sigma^-_\lambda$. We now take the $L^\infty(\Sigma^-_\lambda)$ norm on both side of (3.1), and use Young’s inequality and the Hölder’s inequality, we get

$$\|\phi_\lambda - \phi\|_{L^\infty(\Sigma^-_\lambda)} < \frac{3\delta}{2c - c_1} \|K\|_{L^\infty(\Sigma^-_\lambda)} \|\phi_\lambda + \phi\|_{L^\infty(\Sigma^-_\lambda)} \|\phi_\lambda - \phi\|_{L^\infty(\Sigma^-_\lambda)}$$

which leads a contradiction if we choose $\delta < \frac{2c - c_1}{6c_1}$. Therefore $\phi(x)$ matches $\phi_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$, i.e., $\phi$ is symmetric with respect to $x = \lambda_0$. In addition, the above process of moving the $x = \lambda$ from far left to $x = \lambda_0$ also guarantees that $\phi$ has a unique crest located at $x = \lambda_0$ and is monotonic on each side of this symmetry axis. $\square$
3.2. Solitary waves of the maximal height. Although there is no proof, it is expected that a solitary wave with the maximal height $\phi = c$ exist based on the observation on solitary waves to the relevant Whitham equation. It is proved in [2] that if an even solitary wave, which is monotonic on each side of its unique crest, will be exactly lipschitz so that a peak appears. Therefore, it is reasonable to assume that a peak will appear when the solitary wave reach the maximal height at some $x_0 \in \mathbb{R}$, namely $\phi(x_0) = c$. Here we assume this peak has the following general structure

$$c - \phi(x) \in [C_1|x|^\alpha, C_2|x|^\alpha]$$

(3.9)

for some constants $C_1, C_2 > 0$ where $\alpha \in (0, 1]$, $x$ is close to $x_0$. In this way, our the argument below can be adapted to treat the symmetry of steady solutions to evolutionary equations with other angles at the peak$^3$.

**Theorem 3.4.** There exists a finite $\lambda \in \mathbb{R}$ such that the solitary solution $\phi$ to the steady Degasperis-Procesi equation is symmetric about $x = \lambda$ and $\phi$ is strictly monotonic on each side of the symmetric axis.

**Proof.** We can prove similarly as in Theorem 3.4 that $\lambda$ stops at some $\lambda_0$, and we assume that that $\phi(x)$ does not match $\phi\lambda_0(x)$ for all $x \in \Sigma_\lambda$. Again, the set $\Sigma_\lambda$ will be simply connected and its size can be made very small for $\lambda \in (\lambda_0, \lambda_0 + \delta)$ if we choose $\delta > 0$ sufficiently small. For a fixed $\lambda \in (\lambda_0, \lambda_0 + \delta)$, we denote $\Sigma_\lambda^-$ by $[\lambda, \lambda_1]$ and define

$$\delta_1 := \lambda - \lambda_0, \quad \delta_2 := \lambda_1 - \lambda_0, \quad \lambda_3 = 2\lambda - \lambda_0.$$  

(3.10)

A key observation is that $c > \phi_\lambda > \phi$ on $\Sigma_\lambda^-$ and it

$$2c - (\phi + \phi_\lambda) = (c - \phi) + (c - \phi_\lambda) \geq 2C_1|x - \lambda_0|^\alpha = 2^{1+\alpha}C_1\delta_2^\alpha$$

(3.11)

for some bounded constant $C_1$ which is determined by the peak. Also, from (3.3), we have

$$0 < H(x - y) - H(2\lambda - x - y) \leq \delta_2, \quad x, y \in \Sigma_\lambda^-.$$  

(3.12)

By inserting the above two equations into (3.2) we have

$$\phi_\lambda(x) - \phi(x) = \frac{3}{2c - (\phi + \phi_\lambda)} \int_{\Sigma_\lambda^-} (K(x - y) - K(2\lambda - x - y))(\phi_\lambda^2(y) - \phi^2(y))dy$$

$$\leq \frac{3}{2(1+\alpha)C_1}\delta_2^{-\alpha} (\delta_2|\Sigma_\lambda^-|) \|\phi + \phi_\lambda\|_{L^\infty_{\Sigma_\lambda^-}} \|\phi_\lambda - \phi\|_{L^\infty_{\Sigma_\lambda^-}}$$

$$< 6C\delta_2^{-2-\alpha} \|\phi_\lambda - \phi\|_{L^\infty_{\Sigma_\lambda^-}}$$

where we used $|\Sigma_\lambda^-| < \delta_2$. As mentioned above, by choosing $\delta_1$ sufficiently small, we can make $\delta_2 < (3C^{-1})^{-\alpha}$ and therefore $6C\delta_2^{-2-\alpha} < \frac{1}{2}$. Then we get a contradiction by taking the $L^\infty_{\Sigma_\lambda^-}$ on the left side of the above equation. \hfill $\square$

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$^3$It is generally believed that the type of the peak for the solution will be the same as that of the peak of the convolutional kernel $K(\cdot)$ (see [12]).
4. Solutions with a priori spatial symmetry are traveling waves

The method in [11] for local equations and in [5, 4, 11] for the DP equation in local or nonlocal form is a constructive method based on the uniqueness of solution to the Cauchy problem. Here, we give a more intuitive argument which relies on an observation of the structure of solutions with a priori spatial symmetry.

**Theorem 4.1.** Solutions to the Degasperis-Procesi equation with a priori spatial symmetry are steady solutions.

**Proof.** Assume \( u(t, x) \) is a solution to the Degasperis-Procesi equation with symmetric axis \( x = \lambda(t) \) for some function \( \lambda(\cdot) \in C^1(\mathbb{R}) \), i.e.,

\[
u(t, x) = u(t, 2\lambda(t) - x) \tag{4.1}
\]

Then,

\[
u_t|_{(t,x)} = (u_t + 2\dot{\lambda}u_x)|_{(t,2\lambda-x)}, \quad u_x|_{(t,x)} = -u_x|_{(t,2\lambda-x)} \tag{4.2}
\]

and

\[
\frac{1}{2}\partial_x L(u^2)|_{(t,x)} = -\int_{\mathbb{R}} k(y)[uu_x](t,2\lambda-x+y)dy = -L(uu_x)|_{(t,2\lambda-x)} \tag{4.3}
\]

where in the second equality we used the evenness of the kernel \( k(\cdot) \). Inserting (4.1)-(4.3) into (1.2) and in view of the arbitrariness of \( t \) and \( x \), we find that \( u \) satisfies the following equation

\[
u_t + 2\dot{\lambda}u_x - uu_x - 3L(uu_x) = 0 \tag{4.4}
\]

The comparison between (4.4) and (1.2) then leads to the following system

\[
u_t + \dot{\lambda}u_x = 0, \tag{4.5}
\]

\[
-\dot{\lambda}u_x + uu_x + 3L(uu_x) = 0. \tag{4.6}
\]

A key observation is that (4.5) is a linear PDE of first order with variable coefficients so that \( u(t, x) \) must take the form

\[
u(t, x) = g(x - \lambda(t)) \tag{4.7}
\]

for some function \( g(\cdot) \). Inserting (4.7) into (4.6), we get the following differential equation

\[
\left[-\dot{\lambda}(t)g' + gg' + 3L(gg')\right]|_{x-\lambda(t)} = 0. \tag{4.8}
\]

Choose arbitrarily two pairs \((t_1, x_1), (t_2, x_2) \in \mathbb{R}^+ \times \mathbb{R}\) (for which the solution exists) such that

\[
x_1 - \lambda(t_1) = x_2 - \lambda(t_2) =: X \tag{4.9}
\]

Evaluating (4.8) at these two pairs gives

\[
(\dot{\lambda}(t_1) - \dot{\lambda}(t_2))g'(X) = 0
\]

which implies that \( \dot{\lambda} \) is a constant and therefore \( u(t, x) \) given by (4.7) is a traveling wave solution. \(\square\)

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