The boundedness of commutators of rough $p$-adic fractional Hardy type operators on Herz-type spaces

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Abstract

In this paper, we obtain some inequalities about commutators of a rough $p$-adic fractional Hardy-type operator on Herz-type spaces when the symbol functions belong to two different function spaces.

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1 Introduction

During the last several decades, the $p$-adic analysis has cemented its role in the field of mathematical physics (see, for example, [1, 22, 32, 33]). That stimulates researchers to pay attention to harmonic analysis on $p$-adic fields [18–21, 24, 30, 31, 35], which has direct implications in the stochastic process [2, 3], theoretical biology [6], and $p$-adic pseudo-differential equations [23, 34]. In continuation of the ongoing research, the present paper considers an extension of the investigation of $p$-adic Hardy-type operators discussed in [19–21, 25, 36, 37].

For every non-zero rational number $x$ there is a unique $\gamma = \gamma(x) \in \mathbb{Z}$ such that $x = p^{\gamma} m/n$, where $p \geq 2$ is a fixed prime number which is coprime to $m, n \in \mathbb{Z}$. We define a mapping $\| \cdot \|_p : \mathbb{Q} \rightarrow \mathbb{R}$, as follows:

\[
| x |_p = \begin{cases} 
 p^{-\gamma} & \text{if } x \neq 0, \\
 0 & \text{if } x = 0.
\end{cases}
\]

(1.1)

The $p$-adic absolute value $| \cdot |_p$ has many properties of the usual real norm $| \cdot |$ with an additional non-Archimedean property,

\[
| x + y |_p \leq \max \{ | x |_p, | y |_p \}.
\]

The field of $p$-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of rational numbers with respect to the $p$-adic absolute value $| \cdot |_p$. A $p$-adic number $x \in \mathbb{Q}_p$ can be written in the...
formal power series as [34]:

$$x = p^\gamma \left( \beta_0 + \beta_1 p + \beta_2 p^2 + \cdots \right), \quad (1.2)$$

where $\gamma \in \mathbb{Z}$ and $\beta_i \in \{0, 1, \ldots, p - 1\}$, $i = 0, 1, 2, \ldots$. The $p$-adic absolute value ensures the convergence of series (1.2) in $\mathbb{Q}_p$, because the inequality $|p^\gamma \beta_i p^i|_p \leq p^{-\gamma - i}$ holds for all $\gamma \in \mathbb{Z}$ and $i \in \mathbb{N}$.

The $n$-dimensional vector space $\mathbb{Q}_p^n$, $n \geq 1$, consists of the vectors $x = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$ and $j = 1, 2, \ldots, n$, with the following absolute value:

$$|x|_p = \max_{1 \leq k \leq n} |x_k|_p. \quad (1.3)$$

For $\gamma \in \mathbb{Z}$ and $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Q}_p^n$, we denote by

$$B_\gamma(a) = \{ x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma \}$$

the closed ball with the center $a$ and radius $p^\gamma$ and by

$$S_\gamma(a) = \{ x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma \}$$

the corresponding sphere. For $a = 0$, we write $B_\gamma(0) = B_\gamma$, and $S_\gamma(0) = S_\gamma$. It is easy to see that the equalities

$$a_0 + B_\gamma = B_\gamma(a_0) \quad \text{and} \quad a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0)$$

hold for all $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a unique Haar measure $dx$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0} dx = |B_0|_h = 1,$$

where $|B|_h$ denotes the Haar measure of measurable subset $B$ of $\mathbb{Q}_p^n$. Furthermore, a simple calculation shows that

$$|B_\gamma(a)|_h = p^{n\gamma} \quad \text{and} \quad |S_\gamma(a)|_h = p^{n\gamma} (1 - p^{-n})$$

hold for all $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

The one-dimensional Hardy operator

$$\mathcal{H}f(x) = \frac{1}{x} \int_0^x f(y) \, dy, \quad x > 0,$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable functions, was introduced by Hardy in [13]. This operator satisfies the inequality:

$$\| \mathcal{H}f \|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q - 1} \| f \|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty, \quad (1.4)$$
where the constant $q/(q-1)$ is sharp. In [7], Faris proposed an extension of the operator $H$ on higher dimensional Euclidean space $\mathbb{R}^n$ which is given by

$$Hf(x) = \frac{1}{|x|^n} \int_{|y| \leq |x|} f(y) \, dy,$$

(1.5)

for $x = (x_1, \ldots, x_n)$. In addition, Christ and Grafakos [4] obtained the exact value of the norm of operator $H$ defined by (1.5). For boundedness results for these operators on function spaces we refer to some recent publications including [8, 10, 16, 17, 28, 29, 38].

On the other hand, the $n$-dimensional fractional $p$-adic Hardy operator

$$H_p^\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y| \leq |x|} \Omega(|y|) f(y) \, dy,$$

(1.6)

was defined and studied for $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ and $0 \leq \alpha < n$ in [36]. When $\alpha = 0$, the operator $H_p^0$ transfers to the $p$-adic Hardy-type operator (see [10] for more details). Fu et al. in [9], fixed the optimal bounds of $p$-adic Hardy operator on $L^q(\mathbb{Q}_p^n)$. On the central Morrey space the $p$-adic Hardy-type operators and their commutators were discussed in [37]. In this connection see also [19, 21, 25].

There is still zero attention towards the rough Hardy operators on the $p$-adic linear spaces. Motivated by papers cited above and results of Fu et al. in [8], we define the special kind of $p$-adic rough fractional Hardy operator $H_{\Omega, a}^p$ and its commutators as follows.

**Definition 1.1** Let $f: \mathbb{Q}_p^n \to \mathbb{R}$, $b: \mathbb{Q}_p^n \to \mathbb{R}$ be measurable mappings and let $0 < \alpha < n$. Then, for $x \in \mathbb{Q}_p^n \setminus \{0\}$, we define the rough $p$-adic fractional Hardy operator $H_{\Omega, a}^p$ by

$$H_{\Omega, a}^p f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y| \leq |x|} \Omega(|y|) f(y) \, dy,$$

(1.6)

and its commutator $H_{\Omega, a}^{p, b}$ by

$$H_{\Omega, a}^{p, b} f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y| \leq |x|} (b(x) - b(y)) \Omega(|y|) f(y) \, dy,$$

(1.7)

whenever

$$\int_{|y| \leq |x|} |\Omega(|y|) f(y)| \, dy < \infty$$

(1.8)

and

$$\int_{|y| \leq |x|} |b(y)\Omega(|y|) f(y)| \, dy < \infty,$$

(1.9)

where $\Omega \in L^s(S_0(0))$, $1 \leq s < \infty$.

**Remark 1.2** Obviously

$$\{|y|: y \in \mathbb{Q}_p^n\} = \{p^\gamma: \gamma \in \mathbb{Z}\} \cup \{0\}$$
holds for every integer \( n \geq 1 \) and prime \( p \geq 2 \). Since the inclusion
\[
\{0\} \cup \{p^r : y \in \mathbb{Z}\} \subseteq \mathbb{Q}_p
\]
holds and \( \mathbb{Q}_p^n \) is a linear space over field \( \mathbb{Q}_p \), the product \( |y|_p y \) is well defined. Moreover, if a non-zero \( y \in \mathbb{Q}_p^n \) has the form \( y = (y_1, \ldots, y_n) \) and
\[
y_i = p^{n_i}(\beta_{0,i} + \beta_{1,i}p + \beta_{2,i}p^2 + \cdots), \quad i = 1, \ldots, n
\]
(see (1.2)), then there is \( i_0 \in \{1, \ldots, n\} \) such that
\[
|y_{i_0}|_p = p^{-\gamma_{i_0}} \geq p^{-\gamma_i} = |y_i|_p
\]
(1.11) whenever \( y_i \neq 0 \). Using (1.3) we obtain \( |y|_p = p^{-\gamma_0} \). Now from (1.10) and (1.11) it follows that
\[
|y|_p \gamma = \max_{1 \leq i \leq n} |y_i|_p = \max_{1 \leq i \leq n} |y_i|_p = p^{\gamma_0} = 1.
\]
Thus, for every non-zero \( y \in \mathbb{Q}_p^n \), the vector \( |y|_p y \) belongs to the sphere
\[
S_0(0) = \{y \in \mathbb{Q}_p^n : |y|_p = 1\}.
\]
From (1.8) it directly follows that \( H^{p}_{\Omega} \in \mathbb{R} \) for every non-zero \( x \in \mathbb{Q}_p^n \) and using (1.8), (1.9) we have
\[
|H^{p,b}_{\Omega}f(x)| \leq \frac{|b(x)|}{|x|_p^{n-a}} \int_{|y|_p \leq |x|_p} |\Omega(|y|_p y)f(y)| dy
+ \frac{1}{|x|_p^{n-a}} \int_{|y|_p \leq |x|_p} |b(y)\Omega(|y|_p y)f(y)| dy < \infty
\]
for every \( x \in \mathbb{Q}_p^n \setminus \{0\} \). Consequently, the operators \( H^{p}_{\Omega} \) and \( H^{p,b}_{\Omega} \) are well defined.

The aim of the current paper is to study the boundedness of \( H^{p,b}_{\Omega} \) on \( p \)-adic Herz-type spaces by considering the symbol function \( b \) belonging to the \( p \)-adic \( \text{CMO} \) and Lipschitz spaces. In Euclidean space \( \mathbb{R}^n \), Herz spaces and Morrey–Herz spaces were firstly introduced in [14] and [26], respectively. For more recent developments in the said spaces we mention the articles [15, 27, 39] and the references therein. Also, some operators with rough kernels defined on Euclidian space were recently studied on function spaces; see for example [11, 12]. Before turning to our main results, let us recall the definitions of \( p \)-adic function spaces first.

**Definition 1.3** ([97]) Suppose \( 1 < q < \infty \). The \( p \)-adic central bounded mean oscillation (CBMO) space \( \text{CMO}^q(\mathbb{Q}_p^n) \) is the set of all measurable functions \( f: \mathbb{Q}_p^n \to \mathbb{R} \) which satisfy
\[
\|f\|_{\text{CMO}^q(\mathbb{Q}_p^n)} = \sup_{y \in \mathbb{Z}} \left( \frac{1}{|B_y|_p} \int_{B_y} |f(x) - f_{B_y}|^q dx \right)^{1/q} < \infty,
\]
(1.12)
where \( f_{B_y} = \frac{1}{|B_y|_p} \int_{B_y} f(x) dx \), \( |B_y|_p \) is the Haar measure of \( B_y \).
Definition 1.4 ([9]) Suppose $0 < r < \infty$, $0 < q < \infty$ and $\beta \in \mathbb{R}$. The homogeneous $p$-adic Herz space $\dot{K}_{q}^{\beta,r}(\mathbb{Q}_{p}^{n})$ is defined by

$$\dot{K}_{q}^{\beta,r}(\mathbb{Q}_{p}^{n}) = \{f \in L^{q}(\mathbb{Q}_{p}^{n}) : \|f\|_{\dot{K}_{q}^{\beta,r}(\mathbb{Q}_{p}^{n})} < \infty\},$$

where

$$\|f\|_{\dot{K}_{q}^{\beta,r}(\mathbb{Q}_{p}^{n})} = \left(\sum_{k=-\infty}^{\infty} p^{k\beta r}\|f \chi_{k}\|_{L^{q}(\mathbb{Q}_{p}^{n})}^{r}\right)^{1/r},$$

and $\chi_k$ is the characteristic function of $S_k$.

Obviously, the equalities $\dot{K}_{q}^{0,q}(\mathbb{Q}_{p}^{n}) = L^{q}(\mathbb{Q}_{p}^{n})$ and $\dot{K}_{q}^{\beta/q,q}(\mathbb{Q}_{p}^{n}) = L^{q}(|x|^\beta p)$ hold.

Definition 1.5 ([5]) Suppose $0 < r < \infty$, $0 < q < \infty$, $\beta \in \mathbb{R}$ and $\lambda \geq 0$. The homogeneous $p$-adic Morrey–Herz space is defined by

$$M\dot{K}_{q}^{\beta,\lambda}(\mathbb{Q}_{p}^{n}) = \{f \in L^{q}_{\text{loc}}(\mathbb{Q}_{p}^{n}) \setminus \{0\}) : \|f\|_{M\dot{K}_{q}^{\beta,\lambda}(\mathbb{Q}_{p}^{n})} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{q}^{\beta,\lambda}(\mathbb{Q}_{p}^{n})} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=\infty}^{k_0} p^{k\beta r}\|f \chi_{k}\|_{L^{q}(\mathbb{Q}_{p}^{n})}^{r}\right)^{1/r}.$$

It is evident that $M\dot{K}_{q}^{\beta,0}(\mathbb{Q}_{p}^{n}) = \dot{K}_{q}^{\beta,r}(\mathbb{Q}_{p}^{n})$ and $M\dot{K}_{q}^{\beta/q,0}(\mathbb{Q}_{p}^{n}) = L^{q}(|x|^\beta p)$.

Definition 1.6 ([5]) Suppose $\delta$ is a positive real number. The Lipschitz space $\Lambda_{\delta}(\mathbb{Q}_{p}^{n})$ is defined to be the space of all measurable function $f$ on $\mathbb{Q}_{p}^{n}$ such that

$$\|f\|_{\Lambda_{\delta}(\mathbb{Q}_{p}^{n})} = \sup_{x,h \in \mathbb{Q}_{p}^{n}, h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\delta_{p}} < \infty.$$

2 CBMO estimates for commutators of $p$-adic rough fractional Hardy operator

The present section discusses the boundedness of $p$-adic rough fractional Hardy operator on $p$-adic Herz-type spaces. We begin this section with the following useful lemma.

Lemma 2.1 ([36]) Suppose $b$ is a $\text{CMO}^1(\mathbb{Q}_{p}^{n})$ function and suppose $i,j \in \mathbb{Z}$. Then the inequality

$$|b(y) - b_{B_i}| \leq |b(y) - b_{B_i}| + p^{|i-j|}\|b\|_{\text{CMO}^1(\mathbb{Q}_{p}^{n})},$$

holds.

Remark 2.2 From now on the letter $C$ indicates a positive constant which may vary from line to line.
Theorem 2.3 Let $0 < r_1 \leq r_2 < \infty$, $1 \leq q_1, q_2 < \infty$. Also, let $\frac{1}{q_1} - \frac{1}{q_2} = \frac{a}{n}, q_2 < s < \infty, \frac{1}{q_1} - \frac{1}{s} = \frac{1}{s}$. If $\beta < \frac{n}{q_2}$, then the inequality

$$
\|H_{\Omega, \alpha}^\beta f\|_{K^{\beta, q_2}_1(Q^p_\beta)} \leq C\|f\|_{K^{\beta, q_1}_1(Q^p_\beta)},
$$

holds for all $\Omega \in L^4(S_0(0))$, $b \in \text{CMO}^{\text{max}(q_2)}(Q^p_\beta)$, and $f \in L^{q_1}_\text{loc}(Q^p_\beta)$.

Proof of Theorem 2.3 For the sake of brevity, we write

$$
\sum_{j = -\infty}^{\infty} f(x) \chi_j(x) = \sum_{j = -\infty}^{\infty} f_j(x).
$$

Since

$$
\|H_{\Omega, \alpha}^\beta f\|_{L^{q_2}(Q^p_\beta)} = \int_{S_k} |x|^{-2(q_1 - a)} \left| \int_{|y| = |x|} \Omega(|y|) f(y) (b(x) - b(y)) \, dy \right|^{q_2} \, dx
$$

$$
\leq C p^{-kq_2(n-a)} \int_{S_k} \left( \int_{|y| \leq |x|} |\Omega(|y|) f(y) (b(x) - b(y))| \, dy \right)^{q_2} \, dx
$$

$$
= C p^{-kq_2(n-a)} \int_{S_k} \left( \sum_{j = -\infty}^{\infty} \int_{S_j} |f(y)\Omega(p'y)(b(x) - b(y))| \, dy \right)^{q_2} \, dx
$$

$$
\leq C p^{-kq_2(n-a)} \int_{S_k} \left( \sum_{j = -\infty}^{\infty} \int_{S_j} |f(y)\Omega(p'y)(b(x) - b_B)| \, dy \right)^{q_2} \, dx
$$

$$
+ C p^{-kq_2(n-a)} \int_{S_k} \left( \sum_{j = -\infty}^{\infty} \int_{S_j} |f(y)\Omega(p'y)(b(y) - b_B)| \, dy \right)^{q_2} \, dx
$$

$$
= I + II. \tag{2.1}
$$

For $j, k \in \mathbb{Z}$ with $j \leq k$, we get

$$
\int_{S_j} |\Omega(p'y)|^s \, dy = \int_{|x| = 1} |\Omega(z)|^p \, dz \leq C p^m. \tag{2.2}
$$

Note that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{a}{n}$ and $\frac{1}{q_1} + \frac{1}{s} + \frac{1}{t} = 1$, where $\frac{1}{t} = \frac{1}{q_1} - \frac{1}{s}$. Applying Hölder’s inequality we have

$$
I \leq C p^{-kq_2(n-a)} \int_{B_k} |b(x) - b_B|^{q_2} \times \left\{ \sum_{j = -\infty}^{k} \left( \int_{S_j} |f(y)|^{q_1} \, dy \right)^{1/q_1} \left( \int_{S_j} |\Omega(p'y)|^s \, dy \right)^{1/s} p^{m(1/q_1 - 1/s)} \right\}^{q_2} \, dx
$$

$$
\leq C \|b\|_{\text{CMO}^{\text{max}(q_2)}(Q^p_\beta)}^{q_2} p^{-kq_2(n-a)} \left\{ \sum_{j = -\infty}^{k} p^{m(1/q_1 - 1/s)} \|f\|_{L^{q_1}(Q^p_\beta)} \right\}^{q_2}
$$

$$
= C \|b\|_{\text{CMO}^{\text{max}(q_2)}(Q^p_\beta)}^{q_2} \left\{ \sum_{j = -\infty}^{k} p^{(s-k)m(1/q_1 - 1/s)} \|f\|_{L^{q_1}(Q^p_\beta)} \right\}^{q_2}. \tag{2.3}
$$
Lemma 2.1 will be helpful for estimating $I$. Thus

$$I \leq C \| b \|_{\text{CMO}^2(Q_p)}^2 P^{kq_2(n-\alpha)} \int_{S_k} \left( \sum_{j=\infty}^{k} \int_{S_j} |f(y)\Omega(p'y)(b(y) - b_{B_j})| \, dy \right)^{q_2} \, dx$$

$$+ C \| b \|_{\text{CMO}^2(Q_p)}^2 P^{kq_2(n-\alpha)} \int_{S_k} \left( \sum_{j=\infty}^{k} (k-j) \int_{S_j} |f(y)\Omega(p'y)| \, dy \right)^{q_2} \, dx$$

$$= I_1 + I_2. \tag{2.4}$$

We use Hölder’s inequality to estimate $I_1$. We have

$$I_1 \leq C \| b \|_{\text{CMO}^2(Q_p)}^2 \int_{S_k} \left\{ \sum_{j=\infty}^{k} \left( \int_{S_j} |b(y) - b_{B_j}|^t \, dy \right)^{1/t} \right\}^{q_2} \, dx$$

$$\times \left( \int_{S_j} |\Omega(p'y)|^s \, dy \right)^{1/s} \left( \int_{S_j} |f(y)|^{q_1} \, dy \right)^{1/q_1} \right\}^{q_2} \, dx$$

$$\leq \| b \|_{\text{CMO}^2(Q_p)}^2 \sum_{j=\infty}^{k} \left\{ p^{-\delta(k-q_2/2)} p^{\alpha(j)} \left( \frac{1}{|B_j|} \int_{|B_j|} |b(y) - b_{B_j}| \, dy \right)^{1/q_1} \right\}^{q_2}$$

$$= C \| b \|_{\text{CMO}^2(Q_p)}^2 \left\{ \sum_{j=\infty}^{k} p^{(j-k)(1/q_1 - 1/\alpha)} \| f_j \|^2_{L^{q_1}(Q_p)} \right\}^{q_2}.$$ \tag{2.5}

In a similar fashion we can estimate $I_2$. Using Hölder’s inequality we have

$$I_2 \leq C \| b \|_{\text{CMO}^2(Q_p)}^2 \int_{S_k} \left( \sum_{j=\infty}^{k} (k-j) \left( \int_{S_j} |f(y)\Omega(p'y)|^s \, dy \right)^{1/s} \left( \int_{S_j} |f(y)|^{q_1} \, dy \right)^{1/q_1} \right\}^{q_2} \, dx$$

$$\leq \| b \|_{\text{CMO}^2(Q_p)}^2 \left\{ \sum_{j=\infty}^{k} p^{-\delta(k-q_2/2)} p^{\alpha(j)} \left( \frac{1}{|B_j|} \int_{|B_j|} |b(y) - b_{B_j}| \, dy \right)^{1/q_1} \right\}^{q_2}$$

$$= C \| b \|_{\text{CMO}^2(Q_p)}^2 \left\{ \sum_{j=\infty}^{k} (k-j)p^{(j-k)(1/q_1 - 1/\alpha)} \| f_j \|^2_{L^{q_1}(Q_p)} \right\}^{q_2}.$$ \tag{2.6}

From (2.3), (2.5) and (2.6) together with the Jensen inequality, we have

$$\| H_{\Omega,\beta}^B \|_{\mathcal{K}_{q_2}^{p_{q_2}}(Q_p)}$$

$$= \left( \sum_{k=\infty}^{\infty} P^{k\beta} \| H_{\Omega,\beta}^B \|_{L^{q_2}(Q_p)} \right)^{1/r_2}$$

$$\leq \left( \sum_{k=\infty}^{\infty} P^{k\beta} \| H_{\Omega,\beta}^B \|_{L^{q_1}(Q_p)} \right)^{1/r_1}$$

$$\leq C \| b \|_{\text{CMO}^2(Q_p)} \left( \sum_{k=\infty}^{\infty} P^{k\beta} \left( \sum_{j=\infty}^{k} p^{p(j-k)n/2} \| f_j \|^2_{L^{q_1}(Q_p)} \right)^{r_1} \right)^{1/r_1}$$

$$+ C \| b \|_{\text{CMO}^2(Q_p)} \left( \sum_{k=\infty}^{\infty} P^{k\beta} \left( \sum_{j=\infty}^{k} p^{p(j-k)n/2} \| f_j \|^2_{L^{q_1}(Q_p)} \right)^{r_1} \right)^{1/r_1}.$$
\[ + C\|b\|_{CMO^{1}(Q_{\Omega})} \left( \sum_{k=0}^{\infty} p_{k}^{\beta} \left( \sum_{j=0}^{k} (k-j)p_{j}^{(j-k)n/(n-t)} \|f\|_{L^{1}(Q_{\Omega})} \right)^{r_{1}/r_{1}} \right) \]

\[ = J. \]

For brevity, we may choose \(\|b\|_{CMO^{max}(Q_{2},t)}(Q_{\Omega}) = 1\). Consequently,

\[ J \leq C \left( \sum_{k=0}^{\infty} p_{k}^{\beta} \left( \sum_{j=0}^{k} (k-j)p_{j}^{(j-k)n/(n-t)} \|f\|_{L^{1}(Q_{\Omega})} \right)^{r_{1}/r_{1}} \right). \]

Case 1: When \(0 < r_{1} \leq 1\), we have

\[ J^{r_{1}} = C \sum_{k=0}^{\infty} p_{k}^{\beta} \left( \sum_{j=0}^{k} (k-j)p_{j}^{(j-k)n/(n-t)} \|f\|_{L^{1}(Q_{\Omega})} \right)^{r_{1}/r_{1}} \]

\[ = C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} p_{j}^{\beta} \|f\|_{L^{1}(Q_{\Omega})} (k-j)p_{j}^{(j-k)(n/(n-t)-1)} \right)^{r_{1}/r_{1}} \]

\[ \leq C \sum_{k=0}^{\infty} \sum_{j=0}^{k} p_{j}^{\beta} \|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}} (k-j)^{r_{1}/r_{1}} p_{j}^{(j-k)(n/(n-t)-1)r_{1}/r_{1}} \]

\[ = C \sum_{k=0}^{\infty} p_{k}^{\beta} \|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}} \sum_{j=0}^{k} (k-j)^{r_{1}/r_{1}} p_{j}^{(j-k)(n/(n-t)-1)r_{1}/r_{1}} \]

\[ = C\|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}}. \]

Case 2: When \(r_{1} > 1\), applying Hölder’s inequality we get

\[ J^{r_{1}} = C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} p_{j}^{\beta} \|f\|_{L^{1}(Q_{\Omega})} (k-j)p_{j}^{(j-k)(n/(n-t)-1)} \right)^{r_{1}/r_{1}} \]

\[ \leq C \sum_{k=0}^{\infty} \sum_{j=0}^{k} p_{j}^{\beta} \|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}} (k-j)^{r_{1}/r_{1}} p_{j}^{(j-k)(n/(n-t)-1)r_{1}/r_{1}} \]

\[ \times \left( \sum_{j=0}^{k} (k-j)^{r_{1}/r_{1}} p_{j}^{(j-k)(n/(n-t)-1)r_{1}/r_{1}} \right)^{1/r_{1}} \]

\[ = C \sum_{k=0}^{\infty} p_{k}^{\beta} \|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}} \sum_{j=0}^{k} (k-j)^{r_{1}/r_{1}} p_{j}^{(j-k)(n/(n-t)-1)r_{1}/r_{1}} \]

\[ = C\|f\|_{L^{1}(Q_{\Omega})}^{r_{1}/r_{1}}. \]

The proof of Theorem 2.3 is thus completed. \( \square \)

**Theorem 2.4** Let \(0 < r_{1} \leq r_{2} < \infty\), \(1 \leq q_{1}, q_{2} < \infty\). Also, let \(\frac{1}{q_{1}} - \frac{1}{q_{2}} = \frac{\alpha}{n}, q_{1}', s < \infty, \frac{1}{q_{1}'} - \frac{1}{t} = \frac{1}{r},\) and \(\lambda > 0\). If \(\beta < \frac{\alpha}{n} + \lambda\), then the inequality

\[ \|H_{\Omega}^{b,\alpha}\|_{M_{1}^{r_{1},r_{2}}(Q_{\Omega})} \leq C\|f\|_{M_{1}^{r_{1},r_{2}}(Q_{\Omega})} \]

holds for all \(\Omega \in L^{s}(S_{0}(0))\), \(b \in CMO^{max}(Q_{2},t)(Q_{\Omega})\) and \(f \in L^{r_{1}}_{loc}(Q_{\Omega}).\)
\textbf{Proof of Theorem 2.4} From the proof of Theorem 2.3 and 
\[ \| (H_{\Omega, \alpha}^{p,h}) \chi_k \|_{L^2(Q_p^n)} \leq C \sum_{j=\infty}^k (k-j) p^{i-j} \| f \|_{L^1(Q_p^n)}, \]
together with the definition of a Morrey–Herz space, the Jensen inequality, \( \beta < n/t + \lambda \), \( \lambda > 0 \) and \( 1 < r_1 < \infty \), it follows that 
\[ \| H_{\Omega, \alpha}^{p,h} \|_{M^\beta_{r_2-1,q_2}(Q_p^n)} \]
\[ \leq C \sup_{k \in \mathbb{Z}} p^{-k} \left( \sum_{j=\infty}^k \left( \sum_{j=\infty}^k (k-j) p^{-j} \| f \|_{L^1(Q_p^n)} \right)^{r_1} \right) \]
\[ \leq C \sup_{k \in \mathbb{Z}} p^{-k} \left( \sum_{j=\infty}^k \left( \sum_{j=\infty}^k (k-j) p^{-j} \right)^{r_1} \right) \]
\[ \leq C \| f \|_{M^\beta_{r_2-1,q_2}(Q_p^n)}. \]

\[ \Box \]

3 \textbf{Lipschitz estimates for commutators of } p \textbf{-adic rough fractional Hardy operator}

The current section deals with the boundedness for the commutators of \( p \)-adic rough fractional Hardy operator on homogeneous \( p \)-adic Herz-type spaces by considering the symbol function from Lipschitz space. We open the discussion for this section from the following lemma.

\textbf{Lemma 3.1} \textit{Suppose } \( f \in \Lambda_{\delta}(Q_p^n) \) \textit{and } \( 0 < \delta < 1 \), \textit{then}
\[ |f(x) - f(y)| \leq |x - y|^\delta \| f \|_{\Lambda_{\delta}(Q_p^n)}. \]

\textit{Proof} \textit{Proof immediately follows from Definition 1.6.} \[ \Box \]

\textbf{Theorem 3.2} \textit{Let } \( 1 \leq q_1, q_2 < \infty \), \( 0 < r_1 \leq r_2 < \infty \). \textit{Also, let } \( \frac{1}{q_1} - \frac{1}{q_2} = \frac{1}{n} \), \( q_1' < s < \infty \), \( \frac{1}{q_1} - \frac{1}{s} = \frac{1}{3} \), \textit{and } \( 0 < \delta < 1 \). \textit{If } \( \beta < n(\frac{1}{q_1} - \frac{1}{3}) \), \textit{then the inequality}
\[ \| H_{\Omega, \alpha}^{p,h} \|_{L^2(Q_p^n)} \leq C \| f \|_{L^2(Q_p^n)} \]
\[ \| f \|_{L^1(Q_p^n)} \]

\textit{holds for all } \( \Omega \in L^1(S_0(0)), b \in \Lambda_{\delta}(Q_p^n) \), \textit{and } \( f \in L^1_{\text{loc}}(Q_p^n) \).
Proof of Theorem 3.2 By Hölder’s inequality along with Lemma 3.1, we have

$$\left\| (H_{\Omega,}\alpha f)X_k \right\|_{L^q(G_\alpha^p)}^{q_2} = \int_{S_k} |x| \left| \int_{|y| \leq |x|} \Omega((y)_{\alpha}f)(y)(b(x) - b(y)) \, dy \right|^q \, dx$$

$$\leq C p^{-d_2(n-\alpha)} \int_{S_k} \left( \int_{|y| \leq |x|} \Omega((y)_{\alpha}f)(y) \, dy \right)^{q_2} \, dx$$

$$= C \int_{S_k} |x| \left( \int_{|y| \leq |x|} \Omega((y)_{\alpha}f)(y) \, dy \right) \, dx$$

By virtue of (2.2), inequality (3.1) takes the following form:

$$I \leq C \left\| b \right\|_{L^q(G_\alpha^p)}^{q_2} p^{-d_2(n-\alpha)} \left( \sum_{j=\infty}^{k} p^{kn/2} \left\| f \right\|_{L^1(G_\alpha^p)} \right)^q$$

$$\leq C \left\| b \right\|_{L^q(G_\alpha^p)}^{q_2} \left( \sum_{j=\infty}^{k} p^{q_{j}q_{j-1}} \left\| f \right\|_{L^1(G_\alpha^p)} \right)^q_2.$$

For the sake of brevity, we take \( \left\| b \right\|_{L^q(G_\alpha^p)}^{q_2} = 1 \). Now, by definition of Herz spaces and the Jensen inequality, it follows that

$$\left\| H_{\Omega,}\alpha f \right\|_{L^q(G_\alpha^p)}^{q_{j}q_{j-1}} = \left( \sum_{k=\infty}^{\infty} p^{kq_{j}q_{j-1}} \left\| (H_{\Omega,}\alpha f)X_k \right\|_{L^q(G_\alpha^p)}^{q_{j}q_{j-1}} \right)^{1/q_{j}q_{j-1}}$$

$$\leq C \left( \sum_{k=\infty}^{\infty} p^{kq_{j}q_{j-1}} \left\| (H_{\Omega,}\alpha f)X_k \right\|_{L^q(G_\alpha^p)}^{q_{j}q_{j-1}} \right)^{1/q_{j}q_{j-1}}$$

$$= C \left( \sum_{k=\infty}^{\infty} p^{kq_{j}q_{j-1}} \left\| (H_{\Omega,}\alpha f)X_k \right\|_{L^q(G_\alpha^p)}^{q_{j}q_{j-1}} \right)^{1/q_{j}q_{j-1}}.$$
Case 1: If $0 < r_1 \leq 1$, then

$$\|H_{\Omega,2,\alpha}^{r_1} f\|_{K^p_{\alpha, r_2} (Q^n_p)} \leq C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} p^{j} p^{j(k)(q_{1} - n/s - \beta)r_1} \|f\|_{K^p_{\alpha, r_2} (Q^n_p)}$$

$$= C \sum_{j=-\infty}^{\infty} p^{j} \|f\|_{K^p_{\alpha, r_2} (Q^n_p)} \sum_{k=-\infty}^{\infty} p^{j(k)(q_{1} - n/s - \beta)r_1}$$

$$\leq C \|f\|_{K^p_{\alpha, r_2} (Q^n_p)}.$$

Case 2: When $r_1 > 1$, applying Hölder’s inequality, we have

$$\|H_{\Omega,2,\alpha}^{r_1} f\|_{K^p_{\alpha, r_2} (Q^n_p)} \leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} p^{j} p^{j(k)(q_{1} - n/s - \beta)} \|f\|_{K^p_{\alpha, r_2} (Q^n_p)} \right)^{r_1}$$

$$\leq C \sum_{j=-\infty}^{\infty} p^{j} \|f\|_{K^p_{\alpha, r_2} (Q^n_p)} \sum_{k=-\infty}^{\infty} p^{j(k)(q_{1} - n/s - \beta)r_1/2}$$

$$\times \left( \sum_{j=-\infty}^{\infty} p^{j(k)(q_{1} - n/s - \beta)r_1/2} \right)^{r_1/r_1'}$$

$$\leq C \sum_{j=-\infty}^{\infty} p^{j} \|f\|_{K^p_{\alpha, r_2} (Q^n_p)} \sum_{k=-\infty}^{\infty} p^{j(k)(q_{1} - n/s - \beta)r_1/2}$$

$$\leq C \|f\|_{K^p_{\alpha, r_2} (Q^n_p)}.$$

**Theorem 3.3** Let $1 \leq q_1, q_2 < \infty$, $0 < r_1 \leq r_2 < \infty$. Also, let $\frac{1}{q_1} - \frac{1}{q_2} = \frac{s + \alpha}{n}$, $s > q_{1}', \frac{1}{q_2} + \frac{1}{r_1'} = 1$, $\lambda \geq 0$ and $0 < \delta < 1$. If $n(\frac{1}{q_1} - \frac{1}{2}) + \lambda > \beta$, then the inequality

$$\|H_{\Omega,2,\alpha}^{r_1} f\|_{MK^K_{\alpha, r_2} (Q^n_p)} \leq C \|f\|_{MK^K_{\alpha, r_2} (Q^n_p)}$$

holds for all $\Omega \in L^1(S_0(0))$, $b \in \Lambda_\delta (Q^n_p)$, and $f \in L^{q_1}_\text{loc} (Q^n_p)$.

*Proof of Theorem 3.3* The proof follows from standard analysis performed in our previous theorems. So, we omit the details.

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**Authors’ contributions**
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32. Varadarajan, V.S.: Path integrals for a class of $p$-adic Schrödinger equations. Lett. Math. Phys. 39(2), 97–106 (1997)
33. Vladimirov, V.S.: Tables of integrals of complex valued functions of $p$-adic arguments. Proc. Steklov Inst. Math. 284, 1–59 (2014)
34. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: $p$-Adic Analysis and Mathematical Physics. World Scientific, Singapore (1994)
35. Volosivets, S.S.: Weak and strong estimates for rough Hausdorff type operator defined on $p$-adic linear space. P-Adic Numb. Ultrametr. Anal. Appl. 9(3), 236–241 (2017)
36. Wu, Q.Y.: Boundedness for commutators of fractional $p$-adic Hardy operator. J. Inequal. Appl. 2012, 293 (2012)
37. Wu, Q.Y., Mi, L., Fu, Z.W.: Boundedness of $p$-adic Hardy operators and their commutators on $p$-adic central Morrey and BMO spaces. J. Funct. Spaces Appl. 2013, Article ID 359193 (2013)
38. Xiao, J.: $L^p$ and $BMO$ bounds of weighted Hardy–Littlewood averages. J. Math. Anal. Appl. 262, 660–666 (2001)
39. Yee, T.L., Ho, K.-P.: Hardy’s inequalities and integral operators on Herz–Morrey spaces. Open Math. 18, 106–121 (2020)