On the critical exponent $p_c$ of the 3D quasilinear wave equation

$$-\left(1 + (\partial_t \phi)^p\right) \partial_t^2 \phi + \Delta \phi = 0$$

with short pulse initial data. II, shock formation

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Abstract

In the previous paper [Ding Bingbing, Lu Yu, Yin Huicheng, On the critical exponent $p_c$ of the 3D quasilinear wave equation $-\left(1 + (\partial_t \phi)^p\right) \partial_t^2 \phi + \Delta \phi = 0$ with short pulse initial data. I, global existence, Preprint, 2022], for the 3D quasilinear wave equation $-\left(1 + (\partial_t \phi)^p\right) \partial_t^2 \phi + \Delta \phi = 0$ with short pulse initial data $(\phi, \partial_t \phi) (1, x) = (\delta^{2-\epsilon_0} \phi_0 (\frac{r-1}{2}, \omega), \delta^{1-\epsilon_0} \phi_1 (\frac{r-1}{2}, \omega))$, where $p \in \mathbb{N}$, $0 < \epsilon_0 < 1$, under the outgoing constraint condition $(\partial_t + \partial_r)^k \phi (1, x) = O (\delta^{2-\epsilon_0 - k \max\{0, 1-(1-\epsilon_0)p\}})$ for $k = 1, 2$, the authors establish the global existence of smooth large solution $\phi$ when $p > p_c$ with $p_c = \frac{1}{1-\epsilon_0}$. In the present paper, under the same outgoing constraint condition, when $1 \leq p \leq p_c$, we will show that the smooth solution $\phi$ blows up and further the outgoing shock is formed in finite time.

Keywords: Short pulse initial data, critical exponent, inverse foliation density, shock formation

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1 Introduction

As a continuation of [10], motivated by [16], we continue to study the following 3D quasilinear wave equation

\[-(1+(\partial_t\phi)^p)\partial_t^2\phi + \Delta \phi = 0\]  \hspace{1cm} (1.1)

with the short pulse initial data

\[(\phi, \partial_t\phi)(1, x) = (\delta^{2-\varepsilon_0}\phi_0(\frac{r-1}{\delta}, \omega), \delta^{1-\varepsilon_0}\phi_1(\frac{r-1}{\delta}, \omega)),\]  \hspace{1cm} (1.2)

where \(p \in \mathbb{N}, 0 < \varepsilon_0 < 1, \delta > 0, x = (x^1, x^2, x^3) \in \mathbb{R}^3, t \geq 1, r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2},\)

\(\omega = (\omega_1, \omega_2, \omega_3) = \xi \in \mathbb{S}^2,\) and \((\phi_0, \phi_1)(s, \omega) \in C^\infty_0((-1,0) \times \mathbb{S}^2).\)

In [10], under such an outgoing constraint condition

\[(\partial_t + \partial_r)^k\phi(1, x) = O(\delta^{2-\varepsilon_0-k}\max(0,1-(1-\varepsilon_0)p)), \quad k = 1, 2,\]  \hspace{1cm} (1.3)

the authors have established the global existence of smooth solution \(\phi\) to (1.1)-(1.2) when \(p > p_c = \frac{1}{2-\varepsilon_0}\).

As explained in [10], (1.3) is very necessary in order to keep the local (or global) smallness of \(\partial_t\phi\) and the strict hyperbolicity of (1.1). On the other hand, when (1.2) is given, \((\partial_t + \partial_r)^k\phi(1, x) = O(\delta^{2-\varepsilon_0-k})\) generally holds from (1.1). This will lead to the over-determination of (1.3) for arbitrary \((\phi_0, \phi_1)\). Thus the choice of \((\phi_0, \phi_1)\) in (1.2) together with (1.3) will be somewhat restricted (see Appendix A below).

For \(p = 2\) and \(\varepsilon_0 = \frac{1}{2}\) in (1.1)-(1.2), when the following incoming constraint condition is posed

\[(\partial_t - \partial_r)^k\phi(1, x) = O(\delta^{\frac{k}{2}}), \quad k = 1, 2,\]  \hspace{1cm} (1.4)

then under the suitable assumption of \((\phi_0, \phi_1)\), it is shown in [16] that the incoming shock will be formed before the time \(t = 2\). The short pulse initial data (1.2) with \(\varepsilon_0 = \frac{1}{2}\) are firstly introduced by D. Christodoulou in [6]. For such short pulse data and by the short pulse method, the authors in monumental papers [6] and [14] showed the formation of black holes in vacuum spacetime for the 3D Einstein general relativity equations.

Our main result in the paper is

**Theorem 1.1.** Assume the condition (1.3) holds. When \(1 \leq p \leq p_c = \frac{1}{1-\varepsilon_0}\), for small \(\delta > 0\), the smooth solution \(\phi\) of (1.1) with (1.2) will blow up and further form the outgoing shock before the time \(t^* = 1 + \delta^{1-(1-\varepsilon_0)p}\) under the following assumption of \((\phi_0, \phi_1)\): there exists a point \((s_0, \omega_0) \in (-1,0) \times \mathbb{S}^2\) such that

\[
\phi_1^{p-1}(s_0, \omega_0)\partial_s \phi_1(s_0, \omega_0) > \frac{2}{p} \quad \text{for} \ 1 \leq p < p_c, \\
\phi_1^{p-1}(s_0, \omega_0)\partial_s \phi_1(s_0, \omega_0) > \frac{(p-1)2^p}{(2^{p-1}-1)p} \quad \text{for} \ p = p_c.
\]  \hspace{1cm} (1.5)
Remark 1.1. The assumption (1.5) can be fulfilled due to the arbitrariness of \( \phi_0 \) and the suitable choice of \( \phi_1 \) (see (A.1) below).

Remark 1.2. When equation (1.1) admits the small initial data
\[
\phi(1, x) = \delta \phi_0(x), \quad \partial_t \phi(1, x) = \delta \phi_1(x),
\]
where \((\phi_0(x), \phi_1(x)) \in C_0^\infty(\mathbb{R}^3), \delta > 0\) is sufficiently small, (1.1) with (1.6) has a global smooth solution \( \phi \) for \( p \geq 2 \) (see [12] or Chapter 6 of [17]). If \( p = 1 \) holds in (1.1), then the smooth solution \( \phi \) of (1.1) with (1.6) will blow up in finite time as long as \((\phi_0(x), \phi_1(x)) \neq 0\) (see [1], [2] and [7]).

Remark 1.3. Due to the special forms of (1.1) and (1.2), then (1.3) is actually equivalent to
\[
(\partial_t + c\partial_x)^k \Omega^\alpha \partial^\beta \phi(1, x) = O(\delta^{2-\varepsilon_0-k\max\{0, 1-(1-\varepsilon_0)p\}-|\beta|}), \quad k = 1, 2, \alpha \in \mathbb{N}_0^3, \beta \in \mathbb{N}_0^4,
\]
where \( \Omega \in \{e^i \partial_j - x^j \partial_i : 1 \leq i < j \leq 3\} \) stands for the derivatives on \( \mathbb{S}^2 \), \( \partial \in \{\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}\} = \{\partial_t, \partial_1, \partial_2, \partial_3\} \), and \( c = (1 + (\partial_t \phi)^p)^{-\frac{1}{2}} \) is the wave speed with \( c \sim 1 \) (see (3.2)).

Remark 1.4. Consider the semilinear symmetric hyperbolic system
\[
\partial_t u + \sum_{j=1}^n A_j(t, x) \partial_j u = G(t, x, u)
\]
with the short pulse initial data
\[
u(0, x) = \delta^p u_0(x, \frac{\phi(x)}{\delta}),
\]
where \( u = (u_1, \ldots, u_m)^T, A_j(t, x) \) are smooth \( m \times m \) symmetric matrix, \( G(t, x, u) \) is a smooth \( 1 \times m \) column vector, \( p \in \mathbb{N}_0, \nabla_{t,x} \phi \neq 0, u_0 \in C^\infty \) is a rapidly decreasing function on its arguments. The authors in [3]-[6] establish the local existence of (1.8) and further give the corresponding correctors by the techniques of asymptotic analysis. On the other hand, for the quasilinear form of (1.8) with (1.9), the authors in [12] and [15] have also discussed some formal asymptotic correctors. We point out that there are no blow-up results of (1.8) with (1.9) shown in the references above.

We now comment on the proof of Theorem 1.1. As in [16] or [10], strongly motivated by the geometric methods of D. Christodoulou [5], we will introduce the inverse foliation density \( \mu = -\frac{1}{(1 + (\partial_t \phi)^p)\partial_t u} \), where the optical function \( u \) satisfies \(-\left(1 + (\partial_t \phi)^p\right)(\partial_t u)^2 + \sum_{i=1}^3 (\partial_i u)^2 = 0\) with the initial data \( u(1, x) = 1 - r \) and \( \partial_t u > 0 \). Under the suitably weighted bootstrap assumptions of \( \partial \phi \) with the precise smallness powers of \( \delta \), it follows from involved analysis and computations that the first order transport equation
\[
L \mu = -\frac{p}{2} \delta^{-\left[1-(1-\varepsilon_0)p\right]} \phi_1^{-p-1}(s, \omega) \partial_s \phi_1(s, \omega) + O(\delta^{1-(1-\varepsilon_0)p-\left[1-(1-\varepsilon_0)p\right]} t)
\]
is eventually obtained, where \( L \) is a vector field approximating \( \partial_t + \partial_x \). By the integration from \( t = 1 \) to \( t < t^* = 1 + \delta^{-1} \) along the characteristics of \( L \), one gets \( \mu = 1 - \frac{p}{2} \delta^{-\left[1-(1-\varepsilon_0)p\right]} \phi_1^{-p-1}(s, \omega) \partial_s \phi_1(s, \omega) \frac{1}{p-1} \left(1 - \frac{1}{p-1}\right) + O(\delta^{(1-\varepsilon_0)p}) \) for \( 1 < p \leq p_c \) and \( \mu = 1 - \frac{p}{2} \delta^{-\left[1-(1-\varepsilon_0)p\right]} \phi_1^{-p-1}(s, \omega) \partial_s \phi_1(s, \omega) \ln t + O(\delta^{(1-\varepsilon_0)p}) \) for \( p = 1 \), which derives \( \mu \to 0 \) and further the shock forms before \( t^* \) under the assumption (1.5). To prove the bootstrap assumptions of \( \partial \phi \), we will use the energy method as in [6], [16] and [17]. However, there are still some new features in the present paper as follows.
• Through analyzing the special structures of short pulse initial data and equation (1.1), we can choose a class of \((\phi_0, \phi_1)\) to fulfill (1.3) (see Appendix A). Moreover, for \(1 \leq p < p_c\), the loss of smallness order \(\delta^{1-(1-\epsilon_0)p}\) of \(\phi(1, x)\) along the directional derivative \(\partial_t + \partial_r\) is naturally derived (see (1.3)). Note that in [16], the authors look for \((\phi_0, \phi_1)\) to satisfy (1.4) by solving the resulting ordinary differential equations (see Lemma 1.1 of [16]).

• We directly treat the linearized equation of (1.1) under the Lorentzian spacetime metric \(g = (g_{\alpha\beta}) = \text{diag}(-c^2, 1, 1, 1)\) with \(c = (1 + (\partial_t \phi)^p)^{-\frac{1}{2}}\), while the authors in [16] introduce another new metric \(\tilde{g}\) to homogenize the resulting linearized equation and this leads to more involved computations. On the other hand, due to our direct treatment methods for the linearized equation of (1.1) to obtain the uniform positive constant \(C\) (see Appendix A). Moreover, for \(\phi\) to satisfy (1.4) by solving the resulting ordinary differential equations with short pulse initial data can be investigated by the analogous manners (one can also see [8]-[9]).

• Due to the over-determination of (1.3) and further the loss of smallness with respect to the orders of \(\delta\) along \(\partial_t + \partial_r\) when \(1 \leq p < p_c\) (this is slightly different from the no loss condition of smallness in (1.4) of [16] because of the corresponding \(p = 2, \varepsilon_0 = \frac{1}{2}\) and \(\max\{0, 1 - (1 - \varepsilon_0)p\} = 0\) in (1.3)), we require to deliberately choose the weighted bootstrap assumptions of \(\partial \phi\) and estimate the optimal powers of \(\delta\) for all the related quantities so that the shock formation of equation (1.1) can be shown when \(1 \leq p \leq p_c\).

• To close the bootstrap assumptions of \(\partial \phi\) up to the \(N\)-order derivatives \((N \in \mathbb{N}\) is a suitably large integer), we need to obtain the uniform positive constant \(C\) in bootstrap assumptions (see (3.1)). For this purpose, our ingredient is: at first, acting the second order operator \(L_L\) \((L\) is a first order differential operator approximate to \(\partial_t - \partial_r\) on the corresponding quantities and make full use of the special form of (1.1)) to obtain the uniform positive constant \(C\) (independent of \(M\)) in the bootstrap assumptions up to \((N - 2)\)-order derivatives of \(\partial \phi\). Secondly, based on this, the bootstrap assumptions for the \((N - 1)\)-order and \(N\)-order derivatives of \(\partial \phi\) can be closed by the energy estimates since the related constants only depend on \(C\) rather than \(M\). In [16], the bootstrap assumptions up to \(N\)-orders are treated together. If we take the same methods in [16], it will cause difficulties for us to close the bootstrap assumptions because some estimated constants in the energy estimates will depend on the given a priori constant \(M\) and one can not get the uniform constant \(C\) (independent of \(M\)) in the bootstrap assumptions.

Our paper is organized as follows. In Section 2, we give some preliminaries as in [10]. In Section 3 the bootstrap assumptions are listed. Meanwhile, the lower order \(L^\infty\) estimates of some quantities and the behavior of \(\mu\) near blow-up time are derived. In Section 4 under the bootstrap assumptions, we will establish the higher order derivative \(L^\infty\) estimates and close the partial bootstrap assumptions. In Section 5 we first take the energy estimates for the linearized covariant wave equation and define some suitable higher order weighted energies and fluxes as in [6] and [16]-[17]. In addition, the commuted covariant wave equations are derived. In Section 6 we first establish the non-top order \(L^2\) estimates without derivative loss. Then the top order \(L^2\) estimates for \(\partial \chi\) and \(\partial^2 \mu\) are investigated, where \(\chi\) is the second fundamental form of metric \(g\) with respect to \(L\). In Section 7 we treat the estimates of the error terms and derive the top order energy estimates on the resulting covariant wave equations. In Section 8 collecting all the estimates in the previous sections, we complete the bootstrap argument. From this, the mechanism of shock formation is shown and the proof of the main theorem is also completed.

As in [10], throughout the whole paper, without special mentions, the following conventions are used:

• Greek letters \(\{\alpha, \beta, \gamma, \cdots\}\) corresponding to the spacetime coordinates are chosen in \(\{0,1,2,3\}\); Latin letters \(\{i, j, k, \cdots\}\) corresponding to the spatial coordinates are chosen in \(\{1,2,3\}\); Capital letters \(\{A, B, C, \cdots\}\) corresponding to the sphere coordinates are chosen in \(\{1,2\}\).

• We use the Einstein summation convention to sum over the repeated upper and lower indices.
• The convention \( f \lesssim h \) means that there exists a generic positive constant \( C \) independent of the parameter \( \delta \) and the variables \((t, x)\) such that \( f \leq Ch \).

• If \( \xi \) is a \((0, 2)\)-type spacetime tensor, \( \Lambda \) is a one-form, \( U \) and \( V \) are vector fields, then the contraction of \( \xi \) with respect to \( U \) and \( V \) is defined as \( \xi_{UV} = \xi_{\alpha\beta}U^\alpha V^\beta \), and the contraction of \( \Lambda \) with respect to \( U \) is defined as \( \Lambda_U = \Lambda_\alpha U^\alpha \).

• The restriction of quantity \( \xi \) (including the metric \( g \) or any \((m, n)\)-type spacetime tensor field) on the sphere is represented by \( \xi \). But if \( \xi \) is already defined on the sphere, it is still represented by \( \xi \).

• \( L_V \xi \) stands for the Lie derivative of \( \xi \) with respect to vector field \( V \), and \( L_V \xi \) is the restriction of \( L_V \xi \) on the sphere.

• \( c = (1 + (\partial_t \phi)^p)^{\frac{1}{2}} \) is the wave speed.

• \( \varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) = (\partial_t \phi, \partial_x \phi, \partial_x \phi, \partial_x \phi) \).

• \( t^* = 1 + \delta^{1-(1-\varepsilon_0)p} \).

• For \( p = 1 \), define \( \frac{1}{p-1} (1 - \frac{1}{p^p-1}) \lim_{p \to 1} \frac{1}{p-1} (1 - \frac{1}{p^p-1}) = \ln t \) for \( t > 1 \).

## 2 Some preliminaries

In this section, we give some preliminaries on the Lorentzian geometry as in [10], which have been also illustrated in [5], [16] and [17]. However, for readers’ convenience, we still give the details.

By (1.1), it is natural to introduce such an inverse spacetime metric

\[ g^{-1} = (g^{\alpha\beta}) = \text{diag}(-c^{-2}, 1, 1, 1), \quad (2.1) \]

and the corresponding spacetime metric

\[ g = (g_{\alpha\beta}) = \text{diag}(-c^2, 1, 1, 1). \quad (2.2) \]

In this case, (1.1) can be rewritten as

\[ g^{\alpha\beta}\partial^{2}_{\alpha\beta}\phi = 0. \quad (2.3) \]

The related Christoffel symbols of \( g \) are defined by

\[ \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\alpha\gamma}), \]

\[ \Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\lambda} \Gamma_{\alpha\lambda\beta}, \]

\[ \Gamma_{\gamma} = g^{\alpha\beta} \Gamma^{\gamma}_{\alpha\beta}. \quad (2.4) \]

One can introduce the following optical function as in [5].

**Definition 2.1.** A \( C^1 \) function \( u(t, x) \) is called the optical function of (2.3), if \( u(t, x) \) satisfies the eikonal equation

\[ g^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0 \quad (2.5) \]

with the initial data \( u(1, x) = 1 - r \) and the condition \( \partial_t u > 0 \).
According to the definition of optical function, define the inverse foliation density $\mu$ as in \[5\] and \[17\]:

$$
\mu = \frac{1}{g^{\alpha\beta} \partial_\alpha u \partial_\beta t} \left( = \frac{1}{c^{-2} \partial_t u} \right).
$$

(2.6)

In fact, $\mu^{-1}$ measures the foliation density of the outgoing characteristic surfaces and indicates that the shock will be formed when $\mu \to 0^+$ with the development of time $t$.

Note that

$$
\hat{L} = -\nabla u = -g^{\alpha\beta} \partial_\alpha u \partial_\beta
$$

(2.7)

is a tangent vector field of the outgoing light cone \( \{ u = C \} \) and $\hat{L}t = \mu^{-1}$ holds. One naturally rescale $\hat{L}$ as

$$
L = \mu \hat{L}.
$$

(2.8)

To obtain a tangent vector field $L$ of the incoming light cone, let

$$
\hat{T} = c^{-1} (\partial_t - L).
$$

(2.9)

Set

$$
T = c^{-1} \mu \hat{T},
$$

(2.10)

$$
\underline{L} = c^{-2} \mu L + 2T.
$$

(2.11)

Then $L$ and $\underline{L}$ are two vector fields in the null frame. About the other vector fields $\{X_1, X_2\}$ in the null frame, we utilize the vector field $L$ to construct them. To this end, one extends the local coordinates $\{\vartheta^1, \vartheta^2\}$ on $S^2$ as follows

$$
L \vartheta^A = 0, \quad \vartheta^A|_{t=1} = \theta^A,
$$

here and below $A = 1, 2$. Subsequently, let

$$
X_1 = \frac{\partial}{\partial \vartheta^1}, \quad X_2 = \frac{\partial}{\partial \vartheta^2}.
$$

(2.12)

A direct computation yields

**Lemma 2.1.** $\{L, \underline{L}, X_1, X_2\}$ constitutes a null frame with respect to the metric $g$ in \[2.2\], and admits the following identities

$$
g(L, L) = g(\underline{L}, \underline{L}) = g(L, X_A) = g(\underline{L}, X_A) = 0, \quad g(L, \underline{L}) = -2\mu.
$$

In addition,

$$
g(L, T) = -\mu, \quad g(T, T) = c^{-2} \mu^2.
$$

Moreover,

$$
Lt = 1, \quad Lu = 0, \quad Tt = 0, \quad Tu = 1, \quad \underline{L}t = c^{-2} \mu, \quad \underline{L}u = 2, \quad X_A t = X_A u = 0.
$$

**Remark 2.1.** On $t = 1$, by $u = 1 - r$, then $T = -\partial_r$. Therefore, in terms of \[2.9\], we have $L = \partial_t + c\partial_r$, for $t = 1$. 
Lemma 2.2. According to [17, Lemma 3.47], the components of \( \mathfrak{g} \) satisfy
\[
\mathfrak{g}^{\alpha\beta} = \mathfrak{g}^{\alpha\beta} + \frac{1}{2} \mu^{-1} (L^\alpha L^\beta + L^\alpha L^\beta), \\
\mathfrak{g}^{\alpha\beta} = \mathfrak{g}^{AB} X^\alpha_A X^\beta_B,
\]
and for any smooth function \( \Psi \),
\[
\mathfrak{g}^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi = |d \Psi|^2 := \mathfrak{g}^{AB} \partial_A \Psi \partial_B \Psi,
\]
where \( \partial \) stands for the restriction of differential operator \( d \) on the sphere and \( \partial_A f = X_A f \) for any smooth function \( f \).

We now perform the change of coordinates as in [10]: \( (t, x^1, x^2, x^3) \rightarrow (t, u, \vartheta^1, \vartheta^2) \) with
\[
\begin{aligned}
t &= t, \\
u &= u(t, x), \\
\vartheta^1 &= \vartheta^1(t, x), \\
\vartheta^2 &= \vartheta^2(t, x).
\end{aligned}
\]
(2.13)

For notational convenience, the following domains are introduced

Definition 2.2.
\[
\begin{align*}
\Sigma_t &= \{(t', u', \vartheta) : t' = t\}, \\
\Sigma^u_t &= \{(t', u', \vartheta) : t' = t, 0 \leq u' \leq u\}, \\
C_u &= \{(t', u', \vartheta) : t' \geq 1, u' = u\}, \\
C^t_u &= \{(t', u', \vartheta) : 1 \leq t' \leq t, u' = u\}, \\
S_{t,u} &= \Sigma_t \cap C_u, \\
D^{t,u} &= \{(t', u', \vartheta) : 1 \leq t' < t, 0 \leq u' \leq u\}.
\end{align*}
\]

Note that \( \vartheta = (\vartheta^1, \vartheta^2) \) are the coordinates on sphere \( S_{t,u} \), then under the new coordinate system \( (t, u, \vartheta^1, \vartheta^2) \), one has \( L = \frac{\partial}{\partial t}, T = \frac{\partial}{\partial u} - \Xi \) with \( \Xi = X^A X_A \). In addition, it follows from direct computations that

Lemma 2.3. In \( D^{t,u} \), the Jacobian determinant of map \( (t, u, \vartheta^1, \vartheta^2) \rightarrow (t, x^1, x^2, x^3) \) is
\[
\det \frac{\partial (t, x^1, x^2, x^3)}{\partial (t, u, \vartheta^1, \vartheta^2)} = c^{-1} \mu \sqrt{\det \mathfrak{g}}.
\]

Remark 2.2. Since the metric \( \mathfrak{g} \) are locally regular, that is, \( \det \mathfrak{g} > 0 \) (see Remark 2.3 below), it is clear that the coordinate transformation will fail to be a diffeomorphism when \( \mu \to 0^+ \).

We now give some integrations and \( L^2 \)-norms, which will be utilized repeatedly in subsequent sections.

Definition 2.3. For any continuous function \( f \), set
\[
\int_{S_{t,u}} f = \int_{S_{t,u}} f(t, u, \vartheta) \sqrt{\det \mathfrak{g}(t, u, \vartheta)} d\vartheta, \quad \|f\|_{L^2(S_{t,u})}^2 = \int_{S_{t,u}} |f|^2.
\]
Lemma 2.5. The components of the second fundamental forms and torsion one-forms satisfy the following relation

\[
\begin{align*}
\chi_{AB} &= g(D_Al, X_B), \quad \chi_{AB} = g(D_L\bar{L}, X_B), \quad \sigma_{AB} = g(D_A T, X_B). \\
\zeta_A &= g(D_A L, T), \quad \eta_A = -g(D_A T, L).
\end{align*}
\]

As in [16], direct computations yield

\[\chi_{AB} = c^{-2} \mu \chi_{AB}, \quad \sigma_{AB} = -c^{-1} \chi_{AB}, \quad \zeta_A = -c^{-1} \mu \zeta_A, \quad \eta_A = c \zeta_A (c^{-1} \mu).\]

**Lemma 2.4.** The components of the second fundamental forms and torsion one-forms satisfy the following relation

\[
\begin{align*}
\chi_{AB} &= -c^{-2} \mu \chi_{AB}, \quad \sigma_{AB} = -c^{-1} \chi_{AB}, \\
\zeta_A &= -c^{-1} \mu \zeta_A, \quad \eta_A = c \zeta_A (c^{-1} \mu).
\end{align*}
\]

**Lemma 2.5.** For the connection coefficients of the related frames, it holds that

\[
\begin{align*}
D_L L &= \mu^{-1} \mu L, \quad D_T L = \eta^A X_A - c^{-1} L(c^{-1} \mu)L, \quad D_A L = -\mu^{-1} \zeta_A L + \chi_A B X_B, \\
D_L T &= -\zeta^A X_A - c^{-1} L(c^{-1} \mu)L, \quad D_A T = \mu^{-1} \eta_A T - c^{-1} \mu \chi_A X_B, \\
D_T T &= c^{-3} \mu [Tc + L(c^{-1} \mu)L + \{c^{-1} Tc + L(c^{-1} \mu)\} + T \ln(c^{-1} \mu)]T - c^{-1} \mu \zeta_A (c^{-1} \mu) X_A, \\
D_L X_A &= D_A L, \quad D_T X_A = D_A T, \quad D_A X_B = \bar{\nabla} A X_B + \mu^{-1} \chi_A B T, \quad D_{\bar{\nabla} A} L = \mu^{-1} \eta_A L - c^{-1} \mu \chi_A B X_B, \\
D_{\bar{\nabla} L} L &= -L(c^{-2} \mu)L + 2 \eta^A X_A, \quad D_{\bar{\nabla} T} L = -2 \zeta^A X_A, \quad D_{\bar{\nabla} L} L = [\mu^{-1} L \mu + L(c^{-2} \mu)L] L - 2 \mu \zeta_A (c^{-2} \mu) X_A,
\end{align*}
\]

where \(\zeta^A = g^{AB} \zeta_B, \eta^A = g^{AB} \eta_B\) and \(\chi^A_B = g^{BC} \chi_{AC}\).

On \(t = 1\), one has \(L^0 = 1\), \(L^i = \frac{x^i}{\rho} + O(\delta^{(1-\varepsilon_0)p})\), \(\bar{\nabla}^i = -\frac{x^i}{\rho}\) and \(\chi_{AB} = \bar{\nabla}^i \eta_{AB}\) with \(r = 1 - u\). Therefore, for \(t \geq 1\), we define the “error vectors” with the components being

\[
\begin{align*}
\bar{L}^0 &= 0, \\
\bar{L}^i &= L^i - \frac{x^i}{\rho}, \\
\bar{\nabla}^i &= \bar{\nabla}^i + \frac{x^i}{\rho}, \\
\bar{\chi}_{AB} &= \chi_{AB} - \frac{1}{\rho} \eta_{AB},
\end{align*}
\]
here and below \( \rho = t - u \), tr\( g \chi = g^{AB} \chi_{AB} \). We point out that these error vectors will admit good smallness orders of \( \delta \) in subsequent estimates. In addition, it follows from direct computation that the error vectors satisfy

\[
\begin{align*}
\tilde{L}^i &= -ct^i + \rho^{-1}(c - 1)x^i, \\
\text{tr}_g \tilde{\chi} &= \text{tr}_g \chi - 2\rho^{-1}, \\
|\tilde{\chi}|^2 &= |\chi|^2 - 2\rho^{-1}\text{tr}_g \chi + 2\rho^{-2}.
\end{align*}
\] (2.16)

Let

\[v_i = g(\Omega_i, \tilde{T}) = \epsilon_{ijk}x^j\hat{\tilde{T}}^k,\] (2.17)

then

\[R_i = \Omega_i - v_i\tilde{T} \] (2.18)

are the rotation vector fields of \( S_{wu} \), where \( \Omega_i = \epsilon_{ijk}x^j\partial_k \), \( i = 1, 2, 3 \), \( \epsilon_{ijk} = 1 \) when \( ijk \) is 123’s odd permutation, and \( \epsilon_{ij}^k = \epsilon_{ijk} = 1 \) when \( ijk \) is 123’s even permutation.

The Riemann curvature tensor \( \mathcal{R} \) of \( g \) is defined as

\[\mathcal{R}_{WXYZ} = -g(\mathcal{D}_W \mathcal{D}_X Y - \mathcal{D}_X \mathcal{D}_W Y - \mathcal{D}_{[W,X]}Y, Z).\] (2.19)

Due to

\[\mathcal{R}_{\mu\nu \alpha \beta} = \partial_\nu \Gamma_{\mu \beta \alpha} - \partial_\mu \Gamma_{\nu \beta \alpha} + g^{\kappa \lambda} (\Gamma_{\nu \kappa \alpha} \Gamma_{\mu \lambda \beta} - \Gamma_{\mu \kappa \alpha} \Gamma_{\nu \lambda \beta}),\]

the only non-vanishing component of \( \mathcal{R} \) with respect to \( g \) in (2.2) is

\[\mathcal{R}_{\mu \nu \alpha \beta} = c\partial_\mu \partial_\nu c.\] (2.20)

The only non-vanishing component under the frame \( \{L, T, X_1, X_2\} \) is

\[\mathcal{R}_{LALB} = -cT c^{-1} \chi_{AB} + \mathcal{R}_{LALB},\] (2.21)

where

\[\mathcal{R}_{LALB} = c\Psi_{AB}^2 c = -\frac{1}{2} pc^4 \varphi_0^{-1} \Psi_{AB}^2 \varphi_0 - \frac{3}{2} pc^3 \varphi_0^{-1} \mathcal{D}_A c \mathcal{D}_B \varphi_0 - \frac{1}{2} p(p - 1) c^4 \varphi_0^{-2} \mathcal{D}_A \varphi_0 \mathcal{D}_B \varphi_0.\] (2.22)

Here we point out that \( \mathcal{R}_{LALB} \) will admit the higher smallness orders of \( \delta \) than \( \mathcal{R}_{LALB} \).

For any smooth function \( \Psi \), one can denote its energy-momentum tensor with respect to \( g \) by

\[Q_{\alpha \beta} = Q_{\alpha \beta}[\Psi] = \partial_\alpha \Psi \partial_\beta \Psi - \frac{1}{2} g_{\alpha \beta} g^{\kappa \lambda} \partial_\kappa \Psi \partial_\lambda \Psi.\] (2.23)

**Lemma 2.6.** The components of energy-momentum tensor in terms of the null frame \( \{L, T, X_1, X_2\} \) can be computed as follows:

\[Q_{LL} = (L \Psi)^2, \quad Q_{LL} = (L \Psi)^2, \quad Q_{LL} = \mu |\hat{\Psi}|^2,\]

\[Q_{LA} = L \Psi \mathcal{D}_A \Psi, \quad Q_{LA} = L \Psi \mathcal{D}_A \Psi, \quad Q_{AB} = \mathcal{D}_A \Psi \mathcal{D}_B \Psi - \frac{1}{2} g_{AB}( |\hat{\Psi}|^2 - \mu^{-1} L \Psi \mathcal{L} \Psi).\] (2.24)

For any vector field \( V \), denote its deformation tensor with respect to \( g \) by

\[\mathcal{V}^{(V)} \pi_{\alpha \beta} = g(\mathcal{D}_V \alpha, \partial_\beta) + g(\mathcal{D}_\beta V, \partial_\alpha).\] (2.25)

Moreover, for any two vector fields \( X \) and \( Y \), one has

\[\mathcal{V}^{(V)} \pi_{XY} = \mathcal{V}^{(V)} \pi_{\alpha \beta} X^\alpha Y^\beta = g(\mathcal{D}_X V, Y) + g(\mathcal{D}_Y V, X).\]
Lemma 2.7. The components of the deformation tensor under the corresponding frames and the metric $g$ in (2.2) can be derived as follows

(1) when $V = L$,

\begin{align*}
^{(L)}\pi_{LL} &= 0, 
^{(L)}\pi_{LT} = -L\mu, 
^{(L)}\pi_{TT} = 2c^{-1}\mu L(c^{-2}\mu), \\
^{(L)}\#_{LA} &= 0, 
^{(L)}\#_{TA} = c^2\delta_A(c^{-2}\mu), 
^{(L)}\#_{AB} = 2\chi_{AB}. \\
^{(L)}\#_{LL} &= -2L\mu, 
^{(L)}\pi_{LL} = 4\mu L(c^{-2}\mu), 
^{(L)}\#_{LA} = 2c^2\delta_A(c^{-2}\mu). 
\end{align*} 

(2.26)

(2) when $V = \rho L$,

\begin{align*}
^{(\rho L)}\pi_{LL} &= 0, 
^{(\rho L)}\pi_{LL} = 4\rho L(c^{-2}\mu) - 4c^{-2}L^2 + 8\mu, 
^{(\rho L)}\pi_{LL} = -2\rho L\mu - 2\mu, \\
^{(\rho L)}\#_{LA} &= 0, 
^{(\rho L)}\#_{LA} = 2\rho c^2\delta_A(c^{-2}\mu), 
^{(\rho L)}\#_{AB} = 2\rho\chi_{AB}. 
\end{align*} 

(2.27)

(3) when $V = L$,

\begin{align*}
^{(L)}\pi_{LL} &= 0, 
^{(L)}\pi_{LL} = 2L\mu - 2\mu L(c^{-2}\mu), \\
^{(L)}\#_{LA} &= -2c^2\delta_A(c^{-2}\mu), 
^{(L)}\#_{LA} = -2\mu\delta_A(c^{-2}\mu), 
^{(L)}\#_{AB} = -2c^{-2}\mu\chi_{AB}. 
\end{align*} 

(2.28)

(4) when $V = T$,

\begin{align*}
^{(T)}\pi_{LL} &= 0, 
^{(T)}\pi_{LT} = -T\mu, 
^{(T)}\pi_{TT} = T(c^{-2}\mu^2), \\
^{(T)}\#_{LA} &= -c^2\delta_A(c^{-2}\mu), 
^{(T)}\#_{TA} = 0, 
^{(T)}\#_{AB} = -2c^{-2}\mu\chi_{AB}. \\
^{(T)}\pi_{LL} &= -2T\mu, 
^{(T)}\pi_{LL} = 4\mu T(c^{-2}\mu), 
^{(T)}\#_{LA} = -\mu\delta_A(c^{-2}\mu). 
\end{align*} 

(2.29)

(5) when $V = R_i$,

\begin{align*}
^{(R_i)}\pi_{LL} &= 0, 
^{(R_i)}\pi_{LT} = -R_i\mu, 
^{(R_i)}\pi_{TT} = 2c^{-1}\mu R_i(c^{-1}\mu), \\
^{(R_i)}\#_{LA} &= -R_i^B\chi_{AB} + \epsilon_{ijk}L^j\delta_Ax^k - \nu_i\delta_Ac, \\
^{(R_i)}\#_{TA} &= c^{-2}\mu R_i \chi_{AB} - \rho^{-1}c^{-2}(c-1)\mu\delta_{AB} R_i^B + c^{-1}\mu\epsilon_{ijk}T^j \delta_Ax^k + \nu_i\delta_A(c^{-1}\mu), \\
^{(R_i)}\#_{AB} &= 2c^{-1}\nu_i\chi_{AB}, 
^{(R_i)}\pi_{LL} = -2R_i\mu, 
^{(R_i)}\pi_{LL} = 4\mu R_i(c^{-2}\mu), \\
^{(R_i)}\#_{LA} &= c^{-2}\mu R_i^B \chi_{AB} - 2\rho^{-1}c^{-2}(c-1)\mu\delta_{AB} R_i^B - 3c^{-2}\mu\nu_i\delta_Ac + 2c^{-1}\nu_i\delta_A\mu \\
&+ c^{-2}\mu\epsilon_{ijk}L^j\delta_Ax^k + 2c^{-1}\mu\epsilon_{ijk}T^j \delta_Ax^k. 
\end{align*} 

(2.30)

Lemma 2.8. For any tensor field $\xi$, according to Lemmas 7.5, 8.6 and Corollary 4.13 in [17], then

(1) $X[T(Y_1, \cdots, Y_p)] = (\mathcal{L}_X T)(Y_1, \cdots, Y_p) + \sum_{i=1}^p T(Y_1, \cdots, \mathcal{L}_X Y_i, \cdots, Y_p).$

(2) $\xi \#_{AB} = \left(\#_{AB}\right)\xi, \xi \#_{AB} = -\left(\#_{AB}\right)\xi, Z \in \{L, \rho L, T, L, R_1, R_2, R_3\}.$

(3) $\xi \#_A f = \#_A f, Z \in \{L, \rho L, T, R_1, R_2, R_3\}, \xi \#_A f = T_A f + [L, X_A] f = \#_A f - \phi_A(c^{-2}\mu)L f.$

(4) $[\xi_X, \xi_Y] = \xi_{[X,Y]}.$
Lemma 2.9. According to Lemmas 8.9, 8.11 in [17], then

(1) For any vector field $Z \in \{L, \rho L, T, R_1, R_2, R_3\}$ and symmetric $(0, 2)$-type tensor field $\xi$ on $S_{t,u}$,

\[
([\nabla_A, \xi_Z] \xi)_{BC} = (\tilde{\nabla}_A (Z)^B_C \xi_B^D + (\tilde{\nabla}_A (Z)^C_D) \xi_{BD},
\]

where

\[
\tilde{\nabla}_A (Z)^B_C = \frac{1}{2} (\nabla_A (Z)^B_C + \nabla_B (Z)^A_C - \nabla_C (Z)^A_B).
\]

(2) For any vector field $Z \in \{L, \rho L, T, R_1, R_2, R_3\}$ and smooth function $f$,

\[
([\nabla^2, \xi_Z] f)_{AB} = (\tilde{\nabla}_A (Z)^B_C f_B^D + \tilde{\nabla}_A (Z)^C_D f_{AB},
\]

and

\[
[\Delta, Z] f = (Z)^{AB} \nabla^2 f + \tilde{\nabla}_A (Z)^{AB} f_B f,
\]

in particular,

\[
[L, \Delta] f = -2 \chi^{AB} \nabla^2 f - 2 \chi^{-1} \Delta f - 2 \tilde{\nabla}_A \chi^{AB} f_B f,
\]

\[
[T, \Delta] f = -2 c^{-2} \mu \chi^{AB} \nabla^2 f - 2 \chi^{-1} \mu \Delta f - 2 \tilde{\nabla}_A (c^{-2} \mu \chi^{AB}) f_B f.
\]

(3) The following commutator relations hold from Lemma 2.5 that

\[
[L, R_i] = (R_i)^{AB} \nabla^2 f - 2 \mu \chi^{AB} \Delta f - 2 \tilde{\nabla}_A (c^{-2} \mu \chi^{AB}) f_B f,
\]

\[
[L, T] = (T)^{AB} \nabla^2 f - 2 \mu \chi^{AB} \Delta f - 2 \tilde{\nabla}_A (c^{-2} \mu \chi^{AB}) f_B f.
\]

Lemma 2.10. As in [16], for any smooth function $f$, one has

\[
|R_i f| \sim |f| \sim |\nabla f|.
\]

Moreover, for any $S_{t,u}$-tangential one-from $\xi$ (or trace-free symmetric $S_{t,u}$-tangential $(0, 2)$-type tensor field), we have

\[
|\nabla R_i \xi| \sim |\xi| + |\nabla \xi|.
\]

Remark 2.3. As in [16], the formula

\[
L (\ln \sqrt{\det g}) = tr \chi
\]

illustrates that $\sqrt{\det g}$ is bounded and never vanishes along each null generator when the bounded estimate of $\chi$ is established.

We now look for the equation of $\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) = (\partial_t \phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi)$ under the action of the covariant wave operator $\square_g = g^{\alpha \beta} \partial_\alpha^2 \partial_\beta$ with the help of metrics and Christoffel symbols. Taking the derivative on two sides of (2.3) with the help of the variable $x^\gamma$ derives

\[
-c^{-2} \partial^2_\gamma \varphi + \Delta \varphi = p \varphi_0^{-1} \partial_t \varphi_0 \partial_t \varphi.
\]

Then it follows from direct computation that

\[
\mu \square_g \varphi = F_\gamma,
\]

(2.36)
where

\[ F_\gamma = \frac{1}{2} p \mu \varphi_0^{p-1} L \varphi_0 L \varphi_\gamma + \frac{1}{2} p \varphi_0^{p-1} L \varphi_0 T \varphi_\gamma + \frac{1}{2} p \varphi_0^{p-1} T \varphi_0 L \varphi_\gamma - \frac{1}{2} p^2 \mu \varphi_0^{p-1} d_A \varphi_0 d^A \varphi_\gamma. \] (2.37)

On the other hand, due to

\[ \mu \Box g \varphi_\gamma = - (L + \frac{1}{2} \text{tr}_g \chi) L \varphi_\gamma + 2c^{-1} \mu d_A c d^A \varphi_\gamma + \mu \Delta \varphi_\gamma + \frac{1}{2} c^{-2} \mu \text{tr}_g \chi L \varphi_\gamma, \]

this yields

\[ (L + \frac{1}{2} \text{tr}_g \chi) L \varphi_\gamma = \mu \Delta \varphi_\gamma + H_\gamma - F_\gamma, \] (2.38)

where

\[ H_\gamma = \frac{1}{2} c^{-2} \mu \text{tr}_g \chi L \varphi_\gamma + 2c^{-1} \mu d_A c d^A \varphi_\gamma. \] (2.39)

For convenience, define

\[ \hat{H}_\gamma = 2c^{-1} \mu d_A c d^A \varphi_\gamma. \] (2.40)

At the last of this section, we give the structure equations of \( \mu, \chi \) and \( L^i \) as in [10] and [16].

**Lemma 2.11.** It holds that

\[ L_\mu = c^{-1} Lc \mu - cTc, \] (2.41)

\[ \mathcal{L}_{L \chi_{AB}} = c^{-1} Lc \chi_{AB} + \chi_A c \chi_{BC} - \tilde{R}_{LALB}, \] (2.42)

\[ \mathcal{L}_{T \chi_{AB}} = c^{-1} Tc \chi_{AB} - c^2 \mu \chi_A c \chi_{BC} + c \nabla_{AB} c e^{-1} \mu, \] (2.43)

\[ (\text{div}_g \chi)_A = d_A \text{tr}_g \chi + c^{-1} A^B \chi_{AB} c_A c_B - c^{-1} A^C \chi_{CA} c_B, \] (2.44)

\[ LL^i = c^{-1} Lc L^i - c A c d^A x^i, \] (2.45)

\[ TL^i = c^{-1} Tc L^i + c A c (c^{-1} \mu) d^A x^i, \] (2.46)

\[ d_A L^i = \chi_{AB} d^B x^i. \] (2.47)

### 3 Bootstrap assumptions and \( L^\infty \) estimates of lower order derivatives

By virtue of the initial data (1.2) and Remarks 1.3, 2.1, we make the following bootstrap assumptions in \( D^{1,u} \)

\[ \delta^{1+S[1-(1-\varepsilon_0)]p} \| Z^\alpha \varphi_\gamma \|_{L^\infty(\Sigma_0^u)} \leq M \delta^{1-\varepsilon_0}, \alpha = 0, 1, 2, 3, \] (3.1)

where \( 1 \leq t \leq t^*, |\alpha| \leq N, N \) is a large positive integer, \( M \) is some positive number which is suitably chosen (at least double bounds of the corresponding quantities on \( t = 1 \)), \( Z \in \{ \rho L, T, R_1, R_2, R_3 \}, l \)

or \( s \) is the number of \( T \) (or \( \rho L \) included in \( Z^\alpha \) with \( s \leq 2 \).

In terms of the definition of \( c \) and the Leibnizian rule, we have from (3.1) that

\[ \| c - 1 \|_{L^\infty(\Sigma_0^u)} + \delta^{1+S[1-(1-\varepsilon_0)]p} \| Z^\alpha c \|_{L^\infty(\Sigma_0^u)} \lesssim M^p \delta^{1-\varepsilon_0} p, \] (3.2)

where \( 1 \leq |\alpha| \leq N \) and \( 1 \leq t \leq t^* \).

Next, we derive \( L^\infty \) estimates of some lower order derivatives.
Proposition 3.1. Under the assumptions (3.1), for any vector field $Z \in \{ \rho L, T, R_1, R_2, R_3 \}$, when $\delta > 0$ is small, it holds that for $|\alpha| \leq 1$ and $1 \leq t \leq t^*$,

$$\delta^{4 + s[1-(1-\epsilon_0)p]} \|\tilde{L}_Z \tilde{e}_j^\alpha \tilde{x}\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\epsilon_0)p},$$

$$\delta^{4 + s[1-(1-\epsilon_0)p]} \|Z \tilde{e}_j^\alpha \tilde{x}\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\epsilon_0)p},$$

$$\delta^{4 + s[1-(1-\epsilon_0)p]} \left( \|Z \tilde{e}_j^\alpha (\tilde{T}^l) \|_{L^\infty(\Sigma^u_t)} + \|Z \tilde{e}_j^\alpha \tilde{R}_j \|_{L^\infty(\Sigma^u_t)} + \|Z \tilde{e}_j^\alpha \tilde{v}_j \|_{L^\infty(\Sigma^u_t)} \right) \lesssim M^p \delta^{(1-\epsilon_0)p},$$

$$\delta^{4 + s[1-(1-\epsilon_0)p]} \left( \|Z \tilde{e}_j^\alpha (\tilde{T}^l) \|_{L^\infty(\Sigma^u_t)} + \|Z \tilde{e}_j^\alpha \tilde{R}_j \|_{L^\infty(\Sigma^u_t)} + \|Z \tilde{e}_j^\alpha \tilde{v}_j \|_{L^\infty(\Sigma^u_t)} \right) \lesssim M^p \delta^{(1-\epsilon_0)p},$$

where $l$ (or $s$) is the number of $T$ (or $\rho L$) appearing in the string of $Z$ with $s \leq 2$.

Proof. Taking the $j$-th component on both sides of $\partial_j = \tilde{T}^j \tilde{T} + \tilde{d}_j A_j X_A$ yields $1 = |\tilde{T}^j|^2 + |\tilde{d}_j x^j|^2$, which immediately gives $|\tilde{d}_j x^j| \lesssim 1$ and hence $|\tilde{d}_j| \lesssim 1$ holds in terms of $\tilde{d}_{AB} = g_{ij} \tilde{d}_A x^i \tilde{d}_B x^j$.

Part 1. Estimate of $\tilde{\chi}$

To estimate $\tilde{\chi}$, by substituting $\chi_{AB} = \tilde{\chi}_{AB} + \rho^{-1} \tilde{d}_{AB}$ into (2.42) and in view of $\tilde{L}_L \tilde{d}_{AB} = 2 \chi_{AB}$, one has

$$\tilde{L}_L \tilde{\chi}_{AB} = c^{-1} L \tilde{\chi}_{AB} + \tilde{C}_A \tilde{\chi}_{AC} + \rho^{-1} c^{-1} \tilde{L}_c \tilde{d}_{AB} - \tilde{\Phi}_{LALB} \tag{3.4}$$

and hence,

$$L(\rho^4 |\tilde{\chi}|^2) = 2 \rho^4 \left( L c^4 |\tilde{\chi}|^2 - \tilde{\chi}^A \tilde{\chi}^B X_A + \rho^{-1} c^{-1} L \tilde{\chi} - \tilde{\chi}^A \tilde{\Phi}_{LALB} \right). \tag{3.5}$$

Then by (2.22) and (3.2), we obtain

$$L(\rho^2 |\tilde{\chi}|) \lesssim M^p \delta^{(1-\epsilon_0)p-[1-(1-\epsilon_0)p]} \rho^2 |\tilde{\chi}| + \rho^2 |\tilde{\chi}|^2 + M^p \delta^{(1-\epsilon_0)p-[1-(1-\epsilon_0)p]}.$$

Due to

$$\tilde{\chi}_{AB} |_{t=1} = (c-1) \tilde{d}_{AB} |_{t=1} \lesssim M^p \delta^{(1-\epsilon_0)p}, \tag{3.7}$$

then integrating (3.6) along integral curves of $L$ from 1 to $t \leq t^*$ yields

$$|\tilde{\chi}| \lesssim M^p \delta^{(1-\epsilon_0)p}. \tag{3.8}$$

Part 2. Estimates of $\mu$ and $Z \mu$

To estimate $\mu$, integrating the transport equation (2.41) along integral curves of $L$ from 1 to $t$ yields

$$\mu(t) = e^{t \rho^{-1} L c^{-1} L \tilde{\chi}^i \tilde{d}^i} \mu(1) - \int_1^t e^{t \rho^{-1} L c^{-1} L \tilde{\chi}^i \tilde{d}^i} c T c(t') dt'. \tag{3.8}$$

Since $\mu = c$ on $t = 1$, together with (3.2) and $t \leq t^*$, we have

$$|\mu| \lesssim M^p,$$

which also implies the analogous estimate of $L \mu$ in terms of (2.41).
To estimate $R_{ij}$ and $T$, we will act $\mathcal{L}$ and $T$ to the transport equation (2.41) respectively. For $R_{ij}$, one has
\[
\mathcal{L} T \mu = \partial T \mu = c^{-1} Lc \mu + d(c^{-1} Lc) \mu - d(Tc).
\]
(3.9)

By (3.2) and the estimate of $\mu$, we have
\[
|\mathcal{L} T \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-1} |\mu| + M^p \delta^{(1-\varepsilon_0)p-1}.
\]
(3.10)

Then integrating (3.10) along integral curves of $L$ from 1 to $t$ yields
\[
|\mu| \lesssim M^p
\]
and hence $|R_{ij}| \lesssim M^p$. For $T$, one has from (2.35) that
\[
LT \mu = TL \mu + [L, T] \mu = c^{-1} Lc T \mu + T(c Lc \mu - T(c Tc) - c^2 A(c^{-2} \mu) \mathcal{A} \mu).
\]
(3.11)

By (3.2) and the estimate of $\mu$, we have
\[
|LT \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-1} |T \mu| + M^p \delta^{(1-\varepsilon_0)p-2}.
\]
(3.12)

Then integrating (3.12) along integral curves of $L$ from 1 to $t$ yields
\[
|T \mu| \lesssim M^p \delta^{-1}.
\]

**Part 3. Estimates of $\tilde{L}^j$ and $Z \tilde{L}^j$**

To estimate $\tilde{L}^j$, by substituting $L^j = \tilde{L}^j + \rho^{-1} x^j$ into (2.45), one has
\[
L \tilde{L}^j = (c^{-1} Lc - \rho^{-1}) \tilde{L}^j + \rho^{-1} c^{-1} Lc x^j - c \mathcal{A} c \mathcal{A} x^j
\]
(3.13)

and hence,
\[
L(\rho \tilde{L}^j) = \rho c^{-1} Lc \tilde{L}^j + \rho^{-1} c^{-1} Lc x^j - \rho c \mathcal{A} c \mathcal{A} x^j.
\]
(3.14)

Together with (3.2), this yields
\[
|L(\rho \tilde{L}^j)| \lesssim M^p \delta^{(1-\varepsilon_0)p-1} |\rho \tilde{L}^j| + M^p \delta^{(1-\varepsilon_0)p-1}.
\]
(3.15)

Then integrating (3.15) along integral curves of $L$ from 1 to $t$ yields
\[
|\tilde{L}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p},
\]

which also implies the analogous estimate of $L \tilde{L}^j$ in terms of (3.13).

To estimate $R_{ij} \tilde{L}^j$ and $T \tilde{L}^j$, by substituting $L^j = \tilde{L}^j + \rho^{-1} x^j$ into (2.47) and (2.46), we arrive at
\[
\mathcal{A} \tilde{L}^j = \tilde{\chi} \mathcal{A} \mathcal{B} x^j,
\]
(3.16)

\[
T \tilde{L}^j = (c^{-1} Tc + \rho^{-1} c^{-2}) \tilde{L}^j + (\rho^{-1} c^{-1} Tc + \rho^{-2} c^{-2} \mu - \rho^{-2}) x^j + c \mathcal{A} (c^{-1} \mu) \mathcal{A} x^j.
\]
(3.17)

which immediately give the estimates of $R_{ij} \tilde{L}^j$ and $T \tilde{L}^j$.

**Part 4. Estimates of the other quantities**

The estimates of $\tilde{v}^j$, $v^j$, $Z \tilde{v}^j$ and $Z v^j$ follow immediately from the first identity in (2.16) and (2.17).

Then the estimates of $\mathcal{L}_Z \partial x^j$ can be obtained in view of $\mathcal{L}_Z \partial x^j = \partial Z^j$ and (2.18). In addition, $R_j$ and $\mathcal{L} R_j$ are bounded due to $R_j^A := g(R_j, X^A) = \sum_{k=1}^{3} R_j^k \mathcal{A} x^k$ and (2.18).

On the other hand, the first order derivatives of $\tilde{\chi}$ and the second order derivatives of $\mu$, $\tilde{L}^j$ are bounded by acting the related derivatives on (3.4), (2.41) and (3.13) as in Part 1,2,3. Analogously, the second order derivatives of $\tilde{L}^j$, $v^j$, $\partial x^j$ and $R_j$ can be bounded.

Finally, the estimates of the deformation tensors and their first order derivatives are bounded by (2.30) and (2.29), since the corresponding quantities have been estimated in the above. \(\square\)
We now close bootstrap assumptions (3.1) for $\varphi_\gamma$ and $Z\varphi_\gamma$.

**Proposition 3.2.** For sufficiently small $\delta > 0$, it holds that

$$\|\varphi_\gamma\|_{L^\infty(\Sigma^u_t)} + \|R_i \varphi_\gamma\|_{L^\infty(\Sigma^u_t)} + \delta^{1-(1-\varepsilon_0)p} \|L \varphi_\gamma\|_{L^\infty(\Sigma^u_t)} + \delta \|T \varphi_\gamma\|_{L^\infty(\Sigma^u_t)} \lesssim \delta^{1-\varepsilon_0}. \tag{3.18}$$

**Proof.**

**Part 1. Estimates of $T \varphi_\gamma$ and $\varphi_\gamma$**

Recall (2.37) and (2.39), it follows from (3.1) that

$$|F_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p - [1-(1-\varepsilon_0)p], \quad |H_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p.$$

Then by (2.38), one has

$$|LL \varphi_\gamma + \frac{1}{2} \text{tr}_g \chi L \varphi_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p - [1-(1-\varepsilon_0)p].$$

Therefore, by virtue of (2.16) and the estimate of $\tilde{\chi}$ in Proposition 3.1, we obtain

$$|L(p L \varphi_\gamma)| \lesssim M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p - [1-(1-\varepsilon_0)p]. \tag{3.19}$$

Integrating (3.19) along integral curves of $L$ from 1 to $t$ yields that for small $\delta > 0$,

$$|L \varphi_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p \lesssim \delta^{-\varepsilon_0}$$

and hence,

$$|T \varphi_\gamma| \lesssim \delta^{-\varepsilon_0}.$$

Then integrating along integral curves of $T$ from 0 to $u$ yields

$$|\varphi_\gamma| \lesssim \delta^{1-\varepsilon_0}.$$

**Part 2. Estimate of $L \varphi_\gamma$**

To estimate $L \varphi_\gamma$, by (2.38) and (2.35), one has

$$LL \varphi_\gamma = L(L \varphi_\gamma) - [L, L] \varphi_\gamma$$

$$= -\frac{1}{2} \text{tr}_g \chi L \varphi_\gamma + \mu \Delta \varphi_\gamma + H_\gamma - F_\gamma - L(c^{-2} \mu) L \varphi_\gamma + 2c^2 d^A (c^{-2} \mu) d_A \varphi_\gamma.$$

Then by Proposition 3.1, we have

$$|LL \varphi_\gamma| \lesssim \delta^{-\varepsilon_0} + M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p - [1-(1-\varepsilon_0)p]$$

and hence,

$$|TL \varphi_\gamma| \lesssim \delta^{-\varepsilon_0} + M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p - [1-(1-\varepsilon_0)p].$$

Integrating along integral curves of $T$ from 0 to $u$ yields that for small $\delta > 0$,

$$|L \varphi_\gamma| \lesssim \delta^{1-\varepsilon_0} + M^{p+1} \delta^{-\varepsilon_0} + (1-\varepsilon_0)p \lesssim \delta^{1-\varepsilon_0} - [1-(1-\varepsilon_0)p].$$

**Part 3. Estimate of $R_i \varphi_\gamma$**
To estimate $R_i \varphi_\gamma$, by acting $R_i$ on (2.38), one has

$$LLR_i \varphi_\gamma = R_i LL \varphi_\gamma + [LL, R_i] \varphi_\gamma = -\frac{1}{2} \text{tr}_g \chi LLR_i \varphi_\gamma + \frac{1}{2} \text{tr}_g \chi [LL, R_i] \varphi_\gamma - \frac{1}{2} R_i \text{tr}_g \chi L \varphi_\gamma + R_i (\mu \Delta \varphi_\gamma + H_\gamma - F_\gamma) + [LL, R_i] \varphi_\gamma.$$  

Then by (2.35) and Proposition 3.1, we have

$$|LLR_i \varphi_\gamma| \leq |LR_i \varphi_\gamma| + Mp^{1} \delta^1 - (1 - \epsilon_0)p.$$  

Integrating along integral curves of $L$ from 1 to $t$ yields

$$|LR_i \varphi_\gamma| \leq Mp^{1} \delta^1 - (1 - \epsilon_0)p \lesssim \delta^1 - \epsilon_0.$$  

Then integrating along integral curves of $T$ from 0 to $u$ yields

$$|R_i \varphi_\gamma| \lesssim \delta^{1 - \epsilon_0}.$$  

At the end of this section, we give the key estimates of $\mu$ and its derivatives, which will imply the basic behavior of $\mu$ especially near the blow-up time as in [16].

**Proposition 3.3.** For sufficiently small $\delta > 0$, it holds that

$$\mu(t, u, \vartheta) = 1 - \frac{1}{p - 1} (\frac{1}{t^{p-1}} - 1) L\mu(1, u, \vartheta) + O(M^{p}\delta^{1 - \epsilon_0}p).$$  

Moreover, when $\mu(t, u, \vartheta) \leq \frac{1}{10}$, then

$$L\mu(t, u, \vartheta) \lesssim -\delta^{1 - (1 - \epsilon_0)p}$$  

and

$$\left(\frac{\mu^{-1}T\mu}{\mu} \right)(t, u, \vartheta) \lesssim \frac{1}{\sqrt{t^* - t}} M^{\frac{p}{2}} \delta^{1 - \frac{1}{2}(1 - \epsilon_0)p}.$$  

**Proof.** Part 1. Estimate of $\mu$

According to (2.41), (3.19) and Newton-Leibniz formula, one has

$$t^p L\mu(t, u, \vartheta)$$  

$$= \frac{1}{2} \text{det}(t \varphi_0)^{p-1} t T \varphi_0(t, u, \vartheta) + O(M^{p}\delta^{1 - \epsilon_0}p - [1 - (1 - \epsilon_0)p])$$  

$$= \frac{1}{2} t^p \left[1 + O(\delta^{1 - \epsilon_0)p}) \right] \left[\varphi_0(1, u, \vartheta) + O(M^{p+1}\delta^{1 - \epsilon_0 + (1 - \epsilon_0)p}) \right]^{-p-1} \left[t \varphi_0(1, u, \vartheta) + O(M^{p+1}\delta^{1 - \epsilon_0 + (1 - \epsilon_0)p}) \right]$$  

$$+ O(M^{p}\delta^{1 - \epsilon_0}p - [1 - (1 - \epsilon_0)p])$$  

$$= L\mu(1, u, \vartheta) + O(M^{p}\delta^{1 - \epsilon_0}p - [1 - (1 - \epsilon_0)p]).$$  

(3.23)
Then integrating along integral curves of \( L \) from 1 to \( t \) yields
\[
\mu(t, u, \vartheta) - \mu(1, u, \vartheta) = \int_1^t L \mu(t, u, \vartheta) \, dt' + \int_1^t \frac{O(M^p \delta^{1-\varepsilon_0} p^{-1} - 1)}{t^p} \, dt'
\]
\[
= -\frac{1}{p-1} \left( \frac{1}{t^p-1} - 1 \right) L \mu(1, u, \vartheta) + O(M^p \delta^{1-\varepsilon_0}) \cdot \delta^{1-\varepsilon_0}
\]
\[
= -\frac{1}{p-1} \left( \frac{1}{t^p-1} - 1 \right) L \mu(1, u, \vartheta) + O(M^p \delta^{1-\varepsilon_0}).
\]

**Part 2. Estimate of \( L\mu \)**

If \( \mu(t, u, \vartheta) < \frac{1}{10} \), in view of (3.20), we claim that
\[
\frac{1}{p-1} \left( \frac{1}{t^p-1} - 1 \right) L \mu(1, u, \vartheta) \geq \frac{1}{2}.
\]
Otherwise, for sufficiently small \( \delta > 0 \), we would have \( \mu(t, u, \vartheta) > 1 - \frac{1}{2} + O(M^p \delta^{1-\varepsilon_0}) \), which is a contradiction. Therefore,
\[
L \mu(1, u, \vartheta) \leq -\frac{p-1}{2} \frac{t^p-1}{t^p-1}.
\]

Then by (3.23), one has
\[
t^p L \mu(t, u, \vartheta) \leq -\frac{p-1}{2} \frac{t^p-1}{t^p-1} + O(M^p \delta^{1-\varepsilon_0})
\]
\[
\leq -\frac{p-1}{2} \frac{1}{1 + 2 + \cdots + 2^{p-2}} \delta^{1-\varepsilon_0} \delta^{1-\varepsilon_0} + O(M^p \delta^{1-\varepsilon_0})
\]
\[
\approx -\delta^{1-\varepsilon_0}.
\]

**Part 3. Estimate of \( T\mu \)**

By the methods in [16], we start to estimate \( T \mu \). Suppose \( u^* \) is a maximum point of \( T(\ln \mu)(u) \) on \([0, \delta]\), then \( T^2(\ln \mu)(u^*) = 0 \). Therefore, at the point \((t, u^*, \vartheta)\), we have \( \mu^{-1} T^2 \mu - \mu^{-2} (T \mu)^2 = 0 \). Hence,
\[
\left( \mu^{-1} T \mu \right)_+ (t, u, \vartheta) \leq \frac{\|T^2 \mu\|_{L^\infty(S^u)}}{\inf_{u \in [0, \delta]} \mu(u)}.
\]

(3.24)

For \( T^2 \mu \), by Proposition 3.1 one has
\[
\|T^2 \mu\|_{L^\infty(S^u)} \lesssim M^p \delta^{-2}.
\]

For \( \inf \mu \), by (3.21), integrating along integral curves of \( L \) from \( t \) to \( t^* \) yields
\[
\mu(t, u, \vartheta) = \mu(t^*, u, \vartheta) - \int_t^{t^*} L \mu(t, u, \vartheta) \, dt'
\]
\[
\geq - \int_t^{t^*} L \mu(t, u, \vartheta) \, dt'
\]
\[
\approx (t^* - t) \delta^{-1-\varepsilon_0}.
\]

Together with (3.24), this completes the proof of Proposition 3.3. \( \square \)
4 \( L^\infty \) estimates of higher order derivatives

Since our aim is to close the bootstrap assumptions (3.1), the results obtained in Section 3 are far from enough. For this purpose, we require to derive the uniform \( L^\infty \) estimates of higher order derivatives.

**Proposition 4.1.** Under the assumptions (3.1), for any vector field \( Z \in \{ \rho L, T, R_1, R_2, R_3 \} \), when \( \delta > 0 \) is small, it holds that for \( |\alpha| \leq N - 2 \),

\[
\begin{align*}
\delta^{l+1} \| Z^{\leq \alpha} \chi \|_{L^\infty} & \lesssim M^p \delta^{(1-\varepsilon_0)p}, \\
\delta^{l+1} \| Z^{\leq \alpha+1} \|_{L^\infty} & \lesssim M^p,
\end{align*}
\]

(4.1)

where \( l \) (or \( s \)) is the number of \( T \) (or \( \rho L \)) appearing in the string of \( Z \) with \( s \leq 2 \).

**Proof.** We will prove this proposition by the induction method with respect to the index \( \alpha \). Note that we have proved (4.1) for the case \( |\alpha| \leq 1 \) in Section 3. For any \( 2 \leq |\alpha| \leq N - 2 \), assume that (4.1) holds up to the order \( |\alpha| - 1 \), one needs to show that (4.1) is also true for the order \( |\alpha| \).

We first prove (4.1) in the cases of \( Z \in \{ R_1, R_2, R_3 \} \).

**Part 1. Estimates of \( \mathcal{L}^\alpha_{R_i} \chi \) and \( \mathcal{L}^\alpha_{R_i} (R_i) \chi_L \)**

By (2.31), we commute \( \mathcal{L}^\alpha_{R_i} \) with (3.3) to obtain

\[
\begin{align*}
\mathcal{L}_L \mathcal{L}^\alpha_{R_i} \bar{X} & = \mathcal{L}^\alpha_{R_i} (\mathcal{L}_L X_{AB}) + \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}^\beta_{R_i} \mathcal{L}_{[L,R_i]} \mathcal{L}^{\beta_2}_{R_i} \bar{X} \\
& = \mathcal{L}^\alpha_{R_i} (\mathcal{L}_L X_{AB} + \mathcal{X}_A \mathcal{X}_B + \rho^4 \mathcal{L}_L \bar{X}_{AB} - \bar{H}_{LA} L) \\
& \quad + \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}^\beta_{R_i} \mathcal{L}_{[L,R_i]} (\mathcal{X}_A \mathcal{X}_B + \mathcal{X}_A \bar{X}_{AB} + \mathcal{X}_B \mathcal{X}_C + \mathcal{X}_C \bar{X}_{AB})
\end{align*}
\]

(4.2)

Recalling the expression of \( (R_i) \chi_L \) in (2.30) and utilizing (3.2), together with the induction hypothesis, yield

\[
|\mathcal{L}_L \mathcal{L}^\alpha_{R_i} \bar{X}| \lesssim M^p \delta^{(1-\varepsilon_0)p-|1-(1-\varepsilon_0)p|} \mathcal{L}^\alpha_{R_i} \bar{X} + M^p \delta^{(1-\varepsilon_0)p-|1-(1-\varepsilon_0)p|}.
\]

(4.3)

As in (3.5),

\[
L(\rho^4 |\mathcal{L}^\alpha_{R_i} \bar{X}|^2) = -4 \rho^4 \bar{X}_{AB} \cdot \mathcal{L}^\alpha_{R_i} \bar{X}_{BC} \cdot \mathcal{L}^\alpha_{R_i} \bar{X}_{AC} + 2 \rho^4 \mathcal{L}^\alpha_{R_i} \bar{X}_{AB} \cdot \mathcal{L}_L \mathcal{L}^\alpha_{R_i} \bar{X}_{AB}.
\]

(4.4)

Combining (4.3) and (4.4) derives

\[
|L(\rho^2 |\mathcal{L}^\alpha_{R_i} \bar{X}|)| \lesssim M^p \delta^{(1-\varepsilon_0)p-|1-(1-\varepsilon_0)p|} \rho^2 |\mathcal{L}^\alpha_{R_i} \bar{X}| + M^p \delta^{(1-\varepsilon_0)p-|1-(1-\varepsilon_0)p|}.
\]
It follows from Gronwall inequality that
\[ |\mathcal{L}_{R_i}^{\alpha} \chi| \lesssim M^p \delta^{(1-\varepsilon_0)p}, \]
which also gives the analogous estimate of \( \mathcal{L}_{R_i}^{\alpha} (R_j) \neq L \).

**Part 2. Estimates of** \( R_{i_1}^{\alpha+1} \mu \) **and** \( \mathcal{L}_{R_i}^{\alpha} (R_j) \neq T \)

Similarly to the treatment for \( \mathcal{L}_{R_i}^{\alpha} \chi \), we commute \( R_{i_1}^{\alpha+1} \) with (2.41) to obtain
\[ LR_{i_1}^{\alpha+1} \mu = R_{i_1}^{\alpha+1} L \mu + \sum_{\beta_1+\beta_2=\alpha} R_{i_1}^{\beta_1} [L, R_i] R_{i_1}^{\beta_2} \mu \]
\[ = R_{i_1}^{\alpha+1} (c^{-1} Lc \mu - c Tc) + \sum_{\beta_1+\beta_2=\alpha} \mathcal{L}_{R_i}^{\beta_1} (R_j) \neq L A_{ij} R_{i_1}^{\beta_2} \mu. \]  

(4.5)

By (3.2) and the estimate of \( \mathcal{L}_{R_i}^{\alpha} (R_j) \neq L \) in Part 1, together with the induction hypothesis, one has
\[ |LR_{i_1}^{\alpha+1} \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-\left(1-(1-\varepsilon_0)p\right)} |R_{i_1}^{\alpha+1} \mu| + M^p \delta^{(1-\varepsilon_0)p-1}. \]

Together with Gronwall inequality, this yields
\[ |R_{i_1}^{\alpha+1} \mu| \lesssim M^p, \]
which also gives the corresponding estimate of \( \mathcal{L}_{R_i}^{\alpha} (R_j) \neq T \).

**Part 3. Estimates of** \( R_{i_1}^{\alpha+1} \tilde{L}^j \), \( R_{i_1}^{\alpha+1} \tilde{T}^j \) **and** \( R_{i_1}^{\alpha+1} v_j \)

By (3.16), one has
\[ R_{i_1}^{\alpha+1} \tilde{L}^j = \mathcal{L}_{R_i}^{\alpha} (R_i A_{ij} \tilde{L}^j) \]
\[ = \mathcal{L}_{R_i}^{\alpha} (R_i \chi_{AB} A^B x^j). \]

Thanks to the estimate of \( \mathcal{L}_{R_i}^{\alpha} \chi \) in Part 1, together with the induction hypothesis, we arrive at
\[ |R_{i_1}^{\alpha+1} \tilde{L}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p}. \]

Then the estimates of \( R_{i_1}^{\alpha+1} \tilde{T}^j \) and \( R_{i_1}^{\alpha+1} v_j \) follow immediately from (2.16) and (2.17).

**Part 4. Estimates of the other quantities**

The estimates of \( \mathcal{L}_{R_i}^{\alpha+1} \tilde{x}^j \) and \( \mathcal{L}_{R_i}^{\alpha+1} \tilde{r}_j \) follow directly from (2.18) and Part 3, while the estimates of the other deformation tensors follow from (2.30), (2.29) and the above procedure.

In addition, it follows from (2.43) that the formula of \( \mathcal{L}_T \chi \) can be obtained. After taking the Lie derivatives for this formula with respect to the rotation vector, we may estimate \( \mathcal{L}_{R_i}^{\alpha-1} \mathcal{L}_T \chi \) directly by applying the estimates above and (3.1). Therefore, the bounds of \( \mathcal{L}_{R_i}^{\alpha} \chi \) are obtained when \( Z \in \{ T, R_1, R_2, R_3 \} \) and there is only one \( T \) to appear in \( Z^\alpha \). Similarly, \( Z^{\alpha+1} \tilde{L}^j \) can be estimated by (3.17). By use of (2.41) again, the bounds of \( Z^{\alpha+1} \mu \) are also obtained. On the other hand, the estimates of other left quantities in (4.1) follow immediately. Then by the induction method with respect to the number of \( T \), (4.1) can be derived for \( Z \in \{ T, R_1, R_2, R_3 \} \).

Finally, when the derivatives in \( Z^\alpha \) involve \( \rho L \), the transport equations (2.41), (3.4), (3.13) can be utilized to derive (4.1).

At the end of this section, we close bootstrap assumptions (3.1) for higher order derivatives.
Proposition 4.2. For sufficiently small $\delta > 0$, it holds that for $|\alpha| \leq N - 2$,
\[
\delta^{1+s[1-(1-\varepsilon_0)p]} \| Z^\alpha \varphi \|_{L^\infty(\Sigma^p_t)} \lesssim \delta^{1-\varepsilon_0}, \quad \gamma = 0, 1, 2, 3, \tag{4.6}
\]
where $Z \in \{\rho L, T, R_1, R_2, R_3\}$, $l$ (or $s$) is the number of $T$ (or $\rho L$) included in $Z^\alpha$ with $s \leq 2$.

Proof. We will prove this proposition by the induction method with respect to the index $\alpha$. Note that we have proved (4.6) for the case $|\alpha| \leq 1$ in Section 3. For any $2 \leq |\alpha| \leq N - 2$, assume that (4.6) holds up to the order $|\alpha| - 1$, one requires to show (4.6) for the order $|\alpha|$.

We first prove (4.6) when $Z^\alpha$ only involves the rotation vectors, that is, $Z \in \{R_1, R_2, R_3\}$. To estimate $R^\alpha_i \varphi_\gamma$, by commuting $R^\alpha_i$ with (2.38), one has
\[
\begin{align*}
& L L R^\alpha_i \varphi = R^\alpha_i L L \varphi + [L L, R^\alpha_i] \varphi \\
& = -\frac{1}{2} \text{tr}_g \chi L L R^\alpha_i \varphi + \frac{1}{2} \text{tr}_g \chi L R^\alpha_i \varphi - \frac{1}{2} R^\alpha_i \text{tr}_g \chi L \varphi + R^\alpha_i (\mu \Delta \varphi_\gamma + H_\gamma - F_\gamma) + [L L, R^\alpha_i] \varphi.
\end{align*}
\]
Together with (2.35), Proposition 4.1 and (3.1), this yields
\[
|L L R^\alpha_i \varphi_\gamma| \lesssim |L L R^\alpha_i \varphi_\gamma| + M^{p+1} \delta^{-\varepsilon_0 + (1 - \varepsilon_0)p - [1 - (1-\varepsilon_0)p]}.
\]
Integrating along integral curves of $L$ from 1 to $t$ derives
\[
|L R^\alpha_i \varphi_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0 + (1 - \varepsilon_0)p}
\]
and hence,
\[
|T R^\alpha_i \varphi_\gamma| \lesssim M^{p+1} \delta^{-\varepsilon_0 + (1 - \varepsilon_0)p} \lesssim \delta^{-\varepsilon_0}.
\]
Then we have that by integrating along integral curves of $T$ from 0 to $u$,
\[
|R^\alpha_i \varphi_\gamma| \lesssim \delta^{1-\varepsilon_0}.
\]

When $Z^\alpha$ involves $T$ or $\rho L$, in view of (2.35) and the induction hypothesis, we only need to estimate $L Z^{\alpha-1} \varphi_\gamma$ or $Z^{\alpha-1} L \varphi_\gamma$ by commuting $Z^{\alpha-1}$ with (2.38). This completes the proof of (4.6). \hfill \Box

Remark 4.1. It is pointed out that we have closed the bootstrap assumptions (3.1) for the orders up to $|\alpha| \leq N - 2$. In this case, Proposition 4.1 holds for the orders up to $|\alpha| \leq N - 4$, where the constant $M$ has been removed.

5 Energy estimates for the covariant wave equation

In order to close the estimates of the $(N - 1)^{th}$ and $N^{th}$ order derivatives of $\varphi_\gamma$ in (3.1), we now focus on the energy estimates for (2.36). As in [6] and [16]-[17], the strategy is to deal with linearized equation first and then to act the vector fields $Z^{\alpha+1}$ on (2.36) so that the higher order energies are derived. This procedure is divided into the following six steps.

Step 1. Deriving the divergence form

For the linearized covariant wave equation of (2.36), that is
\[
\mu \Box_g \Psi = \Phi, \tag{5.1}
\]
where $\Psi$ and its derivatives vanish on $C^4_{\alpha \beta}$. By (2.23), one has
\[
\Box_g \Psi \cdot \partial_g \Psi = \mathcal{D}^\alpha Q_{\alpha \beta}.
\]
In addition, for any vector field $V$, it follows from (2.25) that
\[ \square_g \Psi \cdot V \Psi = \mathcal{D}_\alpha (V) J^\alpha - \frac{1}{2} Q^{\alpha \beta} (V) \pi_{\alpha \beta}, \tag{5.2} \]
where $(V) J^\alpha = Q^\alpha_\beta V^\beta$ with $Q^{\alpha \beta} = g^{\alpha \gamma} Q_{\beta \gamma}$ and $Q^{\alpha \beta} = g^{\alpha \kappa} g^{\beta \lambda} Q_{\kappa \lambda}$.

**Step 2. Integrating by parts on domain $D^{t,u}$**

Under the coordinate $\{t, u, \vartheta^1, \vartheta^2\}$, one has
\[ (V) J = (V) J^i \frac{\partial}{\partial t} + (V) J^u \frac{\partial}{\partial u} + (V) J^A \frac{\partial}{\partial \vartheta^A}. \tag{5.3} \]

Let $N = \partial_t = L + c^2 \mu^{-1} T$ be the normal vector. Then taking the inner product with $N$ on both sides of (5.3) yields
\[ (V) J_N = (V) J^i g(L, N) + (V) J^u g(T, N) + (V) J^A g(N_A, N) = -c^2 (V) J^i. \]
Hence, $(V) J^i = -c^{-2} (V) J_N$. Similarly, $(V) J^u = -\mu^{-1} (V) J_L$. Therefore, it follows from $\sqrt{|\det g|} = \mu \sqrt{\det g}$ that
\[ \mathcal{D}_\alpha (V) J^\alpha = \frac{1}{\sqrt{|\det g|}} \left[ \frac{\partial}{\partial t} \left( \sqrt{|\det g|} (V) J^1 \right) + \frac{\partial}{\partial u} \left( \sqrt{|\det g|} (V) J^u \right) + \frac{\partial}{\partial \vartheta^A} \left( \sqrt{|\det g|} (V) J^A \right) \right] = \frac{1}{\sqrt{|\det g|}} \left[ \frac{\partial}{\partial t} (-c^{-2} \mu) (V) J_N \sqrt{\det g} + \frac{\partial}{\partial u} (-c^2 (V) J_L \sqrt{\det g}) + \frac{\partial}{\partial \vartheta^A} (\sqrt{|\det g|} (V) J^A) \right]. \]
Integrating over $D^{t,u}$ to obtain
\[ -\int_{D^{t,u}} \mu \mathcal{D}_\alpha (V) J^\alpha = \int_{\Sigma^u_1} c^{-2} \mu (V) J_N - \int_{\Sigma^t_1} c^{-2} \mu (V) J_N + \int_{C^u_1} (V) J_L, \tag{5.4} \]
where
\[ (V) J_N = (V) J^\alpha N_{\alpha} = Q^\alpha_\beta V^\beta N_{\alpha} = Q_{VN}, \quad (V) J_L = Q_{VL}. \]

**Step 3. Establishing energy identity**

Choosing two vector fields $V_1 = L, V_2 = T$ as multipliers, by $T = \frac{1}{2} (L - c^{-2} \mu L)$ and (2.24), then
\[ Q_{V_1 N} = \frac{1}{2} [(L \Psi)^2 + c^2 |d \Psi|^2], \quad Q_{V_1 L} = (L \Psi)^2, \]
\[ Q_{V_2 N} = \frac{1}{2} [c^2 \mu^{-1} (L \Psi)^2 + \mu |d \Psi|^2], \quad Q_{V_2 L} = \mu |d \Psi|^2. \]
Therefore, by (5.4) and (5.2), we have the following energy identity
\[ E_i[\Psi](t, u) - E_i[\Psi](1, u) + F_i[\Psi](t, u) = -\int_{D^{t,u}} \Phi \cdot V_i \Psi - \int_{D^{t,u}} \frac{1}{2} \mu Q^{\alpha \beta} (V_i) \pi_{\alpha \beta}, \quad i = 1, 2, \tag{5.5} \]
where the energies $E_i[\Psi](t, u)$ and fluxes $F_i[\Psi](t, u)$ are defined as follows
\[ E_1[\Psi](t, u) = \int_{\Sigma^u_1} \frac{1}{2} c^{-2} \mu (L \Psi)^2 + \mu |d \Psi|^2, \quad F_1[\Psi](t, u) = \int_{C^u_1} (L \Psi)^2, \tag{5.6} \]
\[ E_2[\Psi](t, u) = \int_{\Sigma^u_1} \frac{1}{2} [(L \Psi)^2 + c^{-2} \mu^2 |d \Psi|^2], \quad F_2[\Psi](t, u) = \int_{C^u_1} \mu |d \Psi|^2. \]
Step 4. Deriving error estimates and energy inequality

We now deal with the second integral in the right hand side of (5.5). By Lemma 2.2, one has

\[-\frac{1}{2} \mu Q^{\alpha \beta} [\Psi]^{(V)} \pi_{\alpha \beta} = -\frac{1}{2} \mu \left[ -\frac{1}{2} \mu^{-1} (L^\alpha L^\kappa + L^\kappa L^\alpha) + \mathcal{G}^{AB} X_A X_B \right] - \frac{1}{2} \mu^{-1} (L^\beta L^\lambda + L^\lambda L^\beta) + \mathcal{G}^{CD} X_C X_D \right] Q_{\kappa \lambda}^{(V)} \pi_{\alpha \beta} = \frac{1}{8} \mu^{-1} (Q_{LL}^{(V)} \pi_{LL} + Q_{LL}^{(V)} \pi_{LL}) - \frac{1}{4} \mu^{-1} Q_{LL}^{(V)} \pi_{LL} + \frac{1}{2} (Q_L^{(V)} \pi_{LA} + Q_L^{(V)} \pi_{LA}) - \frac{1}{2} \mu Q^{AB} (V) \pi_{AB}.

Then by (2.24), (2.26) and (2.28), we obtain

\[-\frac{1}{2} \mu Q^{\alpha \beta} [\Psi]^{(V_1)} \pi_{\alpha \beta} = -\frac{1}{2} L (c^{-2} \mu) (L \Psi)^2 - \frac{1}{2} \text{tr}_g \chi L \Psi L \Psi + c^2 d_A (c^{-2} \mu) L \Psi d^A \Psi + \frac{1}{2} (L \mu + \mu \text{tr}_g \chi) |d \Psi|^2 - \mu \chi_{AB} d^A \Psi d^B \Psi
\]

and

\[-\frac{1}{2} \mu Q^{\alpha \beta} [\Psi]^{(V_2)} \pi_{\alpha \beta} = \frac{1}{2} c^{-2} \mu \text{tr}_g \chi L \Psi L \Psi - \mu d_A (c^{-2} \mu) L \Psi d^A \Psi - c^2 d_A (c^{-2} \mu) L \Psi d^A \Psi + \frac{1}{2} [L \mu + \mu L (c^{-2} \mu) - c^{-2} \mu^2 \text{tr}_g \chi] |d \Psi|^2 + c^{-2} \mu^2 \chi_{AB} d^A \Psi d^B \Psi.
\]

Applying the results in Sections 3 and 4 to estimate all the terms in (5.7) and (5.8) yields

\[\delta^{-1 - (1-\varepsilon_0)p} \int_{D_{t,u}} -\frac{1}{2} c^{-2} \mu Q^{\alpha \beta} [\Psi]^{(V_1)} \pi_{\alpha \beta} \leq \delta^{-1 - (1-\varepsilon_0)p} \int_0^u \delta^{-1 - (1-\varepsilon_0)p} F_1 (t, u') du' + \delta^{-1} \int_0^u \delta^{-1 - (1-\varepsilon_0)p} F_1 (t, u') du' + \delta^{1 - (1-\varepsilon_0)p} \int_1^t \delta E_2 (t', u) dt' + \delta^{-1} \int_0^u \delta^{-1 - (1-\varepsilon_0)p} F_1 (t, u') du' + \delta^{1 + [(1-\varepsilon_0)p]} K (t, u) + \delta \int_1^t \delta^{1 - (1-\varepsilon_0)p} E_1 (t', u') dt' - K (t, u) + \delta^{1 - (1-\varepsilon_0)p} \int_1^t \delta^{1 - (1-\varepsilon_0)p} E_1 (t', u') dt' + \delta^{(1-\varepsilon_0)p} \int_1^t \delta^{1 - (1-\varepsilon_0)p} E_1 (t', u) dt'.\]
where we have used \( D^{1,u} = (D^{1,u} \cap \{ \mu \geq \frac{1}{10} \}) \cup (D^{1,u} \cap \{ \mu < \frac{1}{10} \}) \), (5.21) and (5.22). In addition, the term \( K(t,u) = \int_{D^{1,u} \cap \{ \mu < \frac{1}{10} \}} |d\Psi|^2 \) in (5.9) plays a key role in controlling \( d\Psi \) in the error terms.

Substituting (5.9) and (5.10) into (5.5) and utilizing the Gronwall inequality, we obtain

\[
\delta^{1-(1-\varepsilon_0)p} E_1(t,u) + \delta^{1-(1-\varepsilon_0)p} F_1(t,u) + \delta E_2(t,u) + \delta F_2(t,u) + K(t,u) \\
\leq \delta^{1-(1-\varepsilon_0)p} E_1(1,u) + \delta E_2(1,u) + \delta^{1-(1-\varepsilon_0)p} \int_{D^{1,u}} |\Phi| \cdot |L\Psi| + \delta \int_{D^{1,u}} |\Phi| \cdot |L\Psi|.
\]

Motivated by (6) and [16]–[17], define the higher order weighted energy and flux as follows

\[
\bar{E}_{i,\leq m+1}(t,u) = \sum_{\gamma=0}^{3} \sum_{|\alpha| \leq m} \delta^{2|\alpha|+2s[1-(1-\varepsilon_0)p]} E_i[Z^\alpha \varphi_\gamma](t,u), \quad i = 1, 2,
\]

\[
\bar{F}_{i,\leq m+1}(t,u) = \sum_{\gamma=0}^{3} \sum_{|\alpha| \leq m} \delta^{2|\alpha|+2s[1-(1-\varepsilon_0)p]} F_i[Z^\alpha \varphi_\gamma](t,u), \quad i = 1, 2,
\]

\[
\bar{K}_{\leq m+1}(t,u) = \sum_{\gamma=0}^{3} \sum_{|\alpha| \leq m} \delta^{2|\alpha|+2s[1-(1-\varepsilon_0)p]} K[Z^\alpha \varphi_\gamma](t,u),
\]

\[
\bar{E}_{i,\leq m+1}(t,u) = \sup_{1 \leq t' \leq t} \left\{ \mu_{\min}^{b_{m+1} t'} (t') \bar{E}_{i,\leq m+1}(t',u) \right\}, \quad i = 1, 2,
\]

\[
\bar{F}_{i,\leq m+1}(t,u) = \sup_{1 \leq t' \leq t} \left\{ \mu_{\min}^{b_{m+1} t'} (t') \bar{F}_{i,\leq m+1}(t',u) \right\}, \quad i = 1, 2,
\]

\[
\bar{K}_{\leq m+1}(t,u) = \sup_{1 \leq t' \leq t} \left\{ \mu_{\min}^{b_{m+1} t'} (t') \bar{K}_{\leq m+1}(t',u) \right\},
\]

where \( l \) (or \( s \)) is the number of \( T \) (or \( \rho L \)) included in \( Z^\alpha \) with \( s \leq 2 \), \( \mu_{\min}(t') = \min_{(u,v)} \mu(t',u,v) \) and the indices \( \{b_k\} \) will be determined in Section [8.1] below.

**Step 5. Obtaining commuted covariant wave equation**
Choosing $\Psi = \Psi_{\gamma}^{[\alpha]+1} = Z_{[\alpha]+1} \cdots Z_1 \varphi_\gamma$ and $\Phi = \Phi_{\gamma}^{[\alpha]+1} = \mu \Box_g \Psi_{\gamma}^{[\alpha]+1}$ in (5.11). By commuting vector fields $Z_k$ ($k = 1, \cdots, [\alpha]+1$) with $\Phi_{\gamma}^{[\alpha]+1}$, the induction argument gives

$$\Phi_{\gamma}^{[\alpha]+1} = \sum_{j=0}^{[\alpha]-1} (Z_{[\alpha]+1} + (Z_{[\alpha]+1}) \lambda) \cdots (Z_{j+2} + (Z_{j+2}) \lambda) (\mu \text{div}_g (Z_{j+1}) C_{\gamma}^j)$$

vanishes when $[\alpha] = 0$

$$+ \mu \text{div}_g (Z_{[\alpha]+1}) C_{\gamma}^0 + (Z_{[\alpha]+1} + (Z_{[\alpha]+1}) \lambda) \cdots (Z_1 + (Z_1) \lambda) \Phi_{\gamma}^0,$$

where

$$\text{div}_g (Z) C_{\gamma}^j = \partial_\beta [((Z) \frac{\partial \psi^j - \frac{1}{2} g^{\beta \nu} \text{tr}_g (Z) \partial_\nu \Psi^j)]_{(Z)} \lambda = \frac{1}{2} \text{tr}_g (Z) \pi - \mu^{-1} Z \mu,$$

$$\Psi_{\gamma}^0 = \varphi_\gamma, \Phi_{\gamma}^0 = \mu \Box_g \varphi_\gamma$$

with $\text{tr}_g (Z) \pi = g^{\alpha \beta (Z)} \pi_{\alpha \beta} = -\frac{1}{2} \mu^{-1} (Z) \pi_{LL} + \frac{1}{2} \text{tr}_g (Z) \pi \neq 0$. Thus,

$$(\rho L) \lambda = \rho \text{tr}_g \chi + 3, (T) \lambda = -c^{-2} \mu \text{tr}_g \chi, (R_i) \lambda = c^{-2} v_i \text{tr}_g \chi.$$

Under the frame $\{L, L, X_1, X_2\}$, $\mu \text{div}_g (Z) C_{\gamma}^j$ can be written as

$$\mu \text{div}_g (Z) C_{\gamma}^j = -\frac{1}{2} [L + L (c^{-2} \mu) - c^{-2} \mu \text{tr}_g \chi] (Z) C_{\gamma,LL} - \frac{1}{2} (L + \text{tr}_g \chi) (Z) C_{\gamma,LL} + \Psi^A (\mu (Z) C_{\gamma,A})$$

where

$$(Z) C_{\gamma,LL} = g (Z) C_{\gamma,LL},$$

$$(Z) C_{\gamma,LL} = -\frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j + (Z) \Psi_{\gamma}^L d^A \Psi^j,$$

$$(Z) C_{\gamma,LL} = g (Z) C_{\gamma,LL},$$

$$(Z) C_{\gamma,LL} = -\frac{1}{2} \mu^{-1} (Z) \pi_{LL} L \Psi^j - \frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j + (Z) \Psi_{\gamma}^L d^A \Psi^j,$$

$$(Z) C_{\gamma,LL} = g (Z) C_{\gamma,LL},$$

$$(Z) C_{\gamma,LL} = -\frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j - \frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j + (Z) \pi_{LL} d^A \Psi^j,$$

$$(Z) C_{\gamma,LL} = g (Z) C_{\gamma,LL},$$

$$(Z) C_{\gamma,LL} = -\frac{1}{2} \mu^{-1} (Z) \pi_{LL} L \Psi^j - \frac{1}{2} \mu^{-1} (Z) \pi_{LL} L \Psi^j + \frac{1}{2} \mu^{-1} (Z) \pi_{LL} d^A \Psi^j,$$

$$(Z) C_{\gamma,LL} = g (Z) C_{\gamma,LL},$$

$$(Z) C_{\gamma,LL} = -\frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j - \frac{1}{2} \text{tr}_g (Z) \pi L \Psi^j + \mu (Z) \Psi_{\gamma}^L d^A \Psi^j.$$

Note that if one substitutes (5.16) into (5.15) directly, then a lengthy and tedious equality for $\mu \text{div}_g (Z) C_{\gamma}^j$ is obtained. To treat these terms more convenient, we will divide $\mu \text{div}_g (Z) C_{\gamma}^j$ into the following three parts as in [10] and [16]:

$$\mu \text{div}_g (Z) C_{\gamma}^j = (Z) \mathcal{A}^1_j + (Z) \mathcal{A}^2_j + (Z) \mathcal{A}^3_j,$$

where

$$(Z) \mathcal{A}^1_j = \frac{1}{4} L (\mu^{-1} (Z) \pi_{LL}) + \frac{1}{4} L (\text{tr}_g (Z) \pi) - \frac{1}{2} \Psi^A (Z) \Psi_{\gamma}^L [L \Psi^j - \frac{1}{2} \mathcal{L} (Z) \Psi_{\gamma}^L] + \frac{1}{2} \mathcal{L} (Z) \Psi_{\gamma}^L$$

$$+ \frac{1}{2} \Psi^L \Psi_{\gamma}^L - \frac{1}{2} \Psi^A (Z) \Psi_{\gamma}^L - \frac{1}{2} \mathcal{L} (Z) \Psi_{\gamma}^L$$

$$(Z) \mathcal{A}^2_j = \frac{1}{2} \text{tr}_g (Z) \pi (L + \frac{1}{2} \text{tr}_g \chi) L \Psi^j + \frac{1}{4} \mu^{-1} (Z) \pi_{LL} L^2 \Psi^j - (Z) \Psi_{\gamma}^L d^A L \Psi^j$$

$$- (Z) \Psi_{\gamma}^L d^A L \Psi^j + \frac{1}{2} \Psi^L \Delta \Psi_{\gamma}^j + (Z) \Psi_{\gamma}^L - \frac{1}{2} \Psi^L (Z) \Psi_{\gamma}^L \Psi_{\gamma}^L.$$
and the first order derivatives of $\delta$ with respect to the first order derivatives of $\Psi^j$. By substituting the components of the deformation tensor, we next derive the expression of $\mathcal{A}$. By (2.26), one has $Z$.

For the term with wavy line in (5.20), it follows from (2.44) that

$$-2\rho \nabla^B (\bar{\chi} \bar{\xi}_{AB}) = -\rho \mu \partial^A \bar{\chi} \partial^B \bar{\chi} - 2\rho \zeta_A \bar{\chi} - 2\zeta_A \bar{\chi}$$

which implies that the troublesome term $\Delta \mu$ in the underline part of (5.20) has been eliminated.

For the term with underline in (5.20), we point out that there exists an important cancelation. In fact, by (2.43), we have

$$T \nabla \bar{\chi} = \bar{\chi} + \mathcal{I}$$

with

$$\mathcal{I} = c^{-2} \mu |\bar{\chi}|^2 - c^{-1} L(c^{-1} \mu) \nabla \bar{\chi} - \nabla^A (c^{-1} \mu \partial^A \bar{\chi}) + c^{-2} \mu |\partial^A \bar{\chi}|^2 - c^{-1} \mu \partial^A \bar{\chi}.$$

It follows from (5.21) that

$$\frac{1}{2} L(\rho \nabla \bar{\chi}) - \rho \nabla^A [c^2 \partial^A (c^{-2} \mu)] = \rho \mathcal{I} - \nabla \bar{\chi} + \frac{1}{2} c^{-2} \mu L(c^{-1} \mu) - \rho \nabla^A [c^2 \partial^A (c^{-2} \mu)],$$

which implies that the troublesome term $\Delta \mu$ in the underline part of (5.20) has been eliminated.

One sees that $(Z) \cdot \mathcal{A}^j_1$ collects the products of the first order derivatives of the deformation tensor and the first order derivatives of $\Psi^j$, and $(Z) \cdot \mathcal{A}^j_2$ contains the terms which are the products of the deformation tensor and the second order derivatives of $\Psi^j$ except the first term which has the good smallness with respect to $\delta$ due to (2.38). In addition, $(Z) \cdot \mathcal{A}^j_3$ is the collections of the products of the deformation tensor and the first order derivatives of $\Psi^j$ which can be treated much easier.

**Step 6. Displaying and analyzing expressions of $(Z) \cdot \mathcal{A}^j_k, \ k = 1, 2, 3$**

By substituting the components of the deformation tensor, we next derive the expression of $(Z) \cdot \mathcal{A}^j_1$.

- **The case of $Z = \rho L$**

By (2.26), one has

$$(\rho L) \cdot \mathcal{A}^j_1 = \left\{ \rho L^2 (c^{-2} \mu) + \frac{1}{2} L(\rho \nabla \bar{\chi}) - \rho \nabla^A [c^2 \partial^A (c^{-2} \mu)] \right\} \nabla \bar{\chi} - \left\{ \mathcal{I} \right\} \nabla \bar{\chi} + \frac{1}{2} L(\rho \nabla \bar{\chi}) \nabla \bar{\chi},$$

(5.20)

For the terms with underline in (5.20), we point out that there exists an important cancelation. In fact, by (2.43), we have

$$T \nabla \bar{\chi} = \Delta \mu + \mathcal{I}$$

with

$$\mathcal{I} = c^{-2} \mu |\bar{\chi}|^2 - c^{-1} L(c^{-1} \mu) \nabla \bar{\chi} - \nabla^A (c^{-1} \mu \partial^A \bar{\chi}) + c^{-2} \mu |\partial^A \bar{\chi}|^2 - c^{-1} \mu \partial^A \bar{\chi}.$$

(5.22)

It follows from (5.21) that

$$\frac{1}{2} L(\rho \nabla \bar{\chi}) - \rho \nabla^A [c^2 \partial^A (c^{-2} \mu)] = \rho \mathcal{I} - \nabla \bar{\chi} + \frac{1}{2} c^{-2} \mu L(c^{-1} \mu) - \rho \nabla^A [c^2 \partial^A (c^{-2} \mu)],$$

(5.23)

which implies that the troublesome term $\Delta \mu$ in the underline part of (5.20) has been eliminated.

For the term with wavy line in (5.20), it follows from (2.44) that

$$-2\rho \nabla^B (\bar{\chi} \bar{\xi}_{AB}) = -\rho \mu \partial^A \bar{\chi} \partial^B \bar{\chi} - 2\rho \zeta_A \bar{\chi} - 2\zeta_A \bar{\chi}$$

$$- 2\rho \partial^B \mu (\bar{\chi} \bar{\xi}_{AB}) - \frac{1}{2} \nabla \bar{\chi} \partial^B \bar{\chi}.$$

Therefore, we arrive at

$$(\rho L) \cdot \mathcal{A}^j_1 = \rho \mu \partial^A \bar{\chi} \cdot \partial^j \bar{\chi} + \text{l.o.t.},$$

(5.24)

where and below “l.o.t.” stands for the lower order derivative term.
• The case of $Z = T$

By (2.29), we have

\[
(\tau) \mathcal{N}^{j}_1 = \left\{ LT(c^{-2} \mu - \frac{1}{2}L(c^{-2} \mu tr g) \chi) + \frac{1}{2} \nabla^A [\mu d_A(c^{-2} \mu)] \right\} L \Psi^j \gamma + \left\{ \frac{1}{2} \mathcal{L}_L [c^2 d_A(c^{-2} \mu)] + \frac{1}{2} \mathcal{L}_L [\mu d_A(c^{-2} \mu)] - d_A T \mu - 2 \nabla^B (c^{-2} \mu^2 (\bar{\chi}_{AB} - \frac{1}{2} tr_g \bar{g}_{AB})) \right\} d^A \Psi^j \gamma
\]

\[
+ \left\{ - \frac{1}{2} L(c^{-2} \mu tr g) \chi + \frac{1}{2} \nabla^A [c^2 d_A(c^{-2} \mu)] \right\} L \Psi^j \gamma.
\]

(5.25)

For the terms with underline in (5.25), one has

\[
\frac{1}{2} \mathcal{L}_L [c^2 d_A(c^{-2} \mu)] - d_A T \mu = \frac{1}{2} c^{-2} \mu d_A L \mu + \frac{1}{2} \mathcal{L}_L [c^2 d_A(c^{-2} \mu)],
\]

which implies that the troublesome term $d_A T \mu$ in the underline part of (5.25) has been eliminated.

For the terms with braces in (5.25), we have

\[
- \frac{1}{2} L(c^{-2} \mu tr g) \chi + \frac{1}{2} \nabla^A [\mu d_A(c^{-2} \mu)]
\]

\[
= - \frac{1}{2} c^{-2} \mu \Delta \mu - c^{-2} \mu L - \frac{1}{2} c^{-2} \mu^2 L \text{tr} g \chi - \frac{1}{2} L(c^{-2} \mu) \text{tr} g \chi + \frac{1}{2} d(c^{-2} \mu) \cdot \text{d} \mu + \frac{1}{2} \nabla^A [\mu^2 d_A(c^{-2} \mu)]
\]

and

\[
\frac{1}{2} \nabla^A [c^2 d_A(c^{-2} \mu)] = \frac{1}{2} \Delta \mu + \frac{1}{2} \nabla^A [c^2 \mu d_A(c^{-2} \mu)].
\]

For the terms with wavy line in (5.25),

\[
- 2 \nabla^B (c^{-2} \mu^2 (\bar{\chi}_{AB} - \frac{1}{2} \text{tr} g \bar{g}_{AB}))
\]

\[
= - c^{-2} \mu^2 d_A \text{tr} g \chi + 2 c^{-2} \mu \zeta^B \bar{\chi}_{AB} - 2 c^{-2} \mu \zeta_A \text{tr} g \chi - 2 c^{-2} \mu^2 \rho^{-1} \zeta_A - 2 d^B (c^{-2} \mu^2)(\bar{\chi}_{AB} - \frac{1}{2} \text{tr} g \bar{g}_{AB}).
\]

Therefore, we arrive at

\[
(\tau) \mathcal{N}^{j}_1 = (\Delta \mu) T \Psi^j - c^{-2} \mu^2 \text{d} \text{tr} g \chi \cdot \text{d} \Psi^j \gamma + \text{l.o.t.}
\]

(5.27)

• The case of $Z = R_i$

By (2.30), we have

\[
(R_i) \mathcal{N}^{j}_1 = \left\{ LR_i(c^{-2} \mu) + \frac{1}{2} L(c^{-1} v_i \text{tr} g \chi) - \frac{1}{2} \nabla^A (c^{-2} \mu R_i^B \bar{\chi}_{AB}) - \nabla^A (c^{-1} v_i d_A \mu) \right\} L \Psi^j \gamma - \frac{1}{2} \nabla^A [c^{-2} \mu \rho^{-1} \zeta^A \delta_{AB} R_i - 3 c^{-2} \mu v_i d_A c + c^{-2} \mu \epsilon_{ijk} \bar{L}^j \delta_{AB} x^k + 2 c^{-1} \mu \epsilon_{ijk} \bar{L}^j \delta_{AB} x^k + \frac{1}{2} \mathcal{L}_L [R_i^B \bar{\chi}_{AB} - \epsilon_{ijk} \bar{L}^j \delta_{AB} x^k + v_i d_A c)]
\]

\[
- \frac{1}{2} \mathcal{L}_L (R_i^B \bar{\chi}_{AB} - \frac{1}{2} \text{tr} g \bar{g}_{AB})) \} d^A \Psi^j \gamma
\]

\[
+ \left\{ \frac{1}{2} L(c^{-1} v_i \text{tr} g \chi) + \frac{1}{2} \nabla^A (R_i^B \bar{\chi}_{AB} - \epsilon_{ijk} \bar{L}^j \delta_{AB} x^k + v_i d_A c) \right\} L \Psi^j \gamma.
\]

(5.28)
For the terms with underline in (5.28), one has that by (2.43),
\[
\frac{1}{2} L(c^{-1} v_i \tr g \chi) - \nabla^A (c^{-1} v_i \slashed{d}_A \mu) = c^{-1} v_i L + \frac{1}{2} c^{-3} \mu v_i L \tr g \chi + \frac{1}{2} L(c^{-1} v_i) \tr g \chi - \slashed{d}(c^{-1} v_i) \cdot \slashed{d} \mu = \text{l.o.t.}
\]
and
\[
\frac{1}{2} \mathcal{L}_L(R^B_i \tilde{X}_{AB}) - \slashed{d}_A R_i \mu = R^B_i \mathcal{L}_T \tilde{X}_{AB} - \nabla_A (R^B_i \slashed{d}_B \mu) + \text{l.o.t.} = \text{l.o.t.},
\]
which implies that the terms $\tilde{\Delta} \mu$ and $\slashed{d}_A R_i \mu$ in the underline part of (5.28) have been eliminated.
For the terms with braces in (5.28), we have
\[
- \frac{1}{2} \nabla^A (c^{-2} \mu R^B_i \tilde{X}_{AB}) = - \frac{1}{2} c^{-2} \mu R_i \tr g \chi + \text{l.o.t.}
\]
and
\[
\frac{1}{2} \nabla^A (R^B_i \tilde{X}_{AB}) = \frac{1}{2} R_i \tr g \chi + \text{l.o.t.}.
\]
For the terms with wavy line in (5.28),
\[
- \nabla^B (2c^{-1} \mu v_i (\tilde{X}_{AB} - \frac{1}{2} \tr g \tilde{g} \tilde{X}_{AB})) = - c^{-1} \mu v_i \slashed{d}_A \tr g \chi + \text{l.o.t.}
\]
Therefore, we arrive at
\[
(R_i) \mathcal{A}^j_1 = R^A_i (\slashed{d}_A \tr g \chi) T \Psi^j_\gamma - c^{-1} \mu v_i \slashed{d}_A \tr g \chi \cdot \slashed{d} \Psi^j_\gamma + \text{l.o.t.} \quad (5.29)
\]
Similarly to the treatment for $(D) \mathcal{A}^j_1$, one has
\[
(pL) \mathcal{A}^j_2 = \rho \tr g \chi (L + \frac{1}{2} \tr g \chi) L \Psi^j_\gamma + \left[ \rho L(c^{-2} \mu) - c^{-2} \mu + 2 \right] L^2 \Psi^j_\gamma - 2 \rho c^2 \frac{\slashed{d}(c^{-2} \mu) \cdot \slashed{d} L \Psi^j_\gamma}{\mu}
\]
\[
- (\rho L \mu + \mu) \tilde{\Delta} \Psi^j_\gamma + 2 \rho \mu (\chi^{AB} - \frac{1}{2} \tr g \tilde{g} \tilde{X}^{AB}) \Psi^2_{AB} \Psi^j_\gamma, 
\]  
(5.30)
\[
(T) \mathcal{A}^j_2 = - c^{-2} \mu \tr g \chi (L + \frac{1}{2} \tr g \chi) L \Psi^j_\gamma + T(c^{-2} \mu) L^2 \Psi^j_\gamma + \mu c^{-2} \frac{\slashed{d}(c^{-2} \mu) \cdot \slashed{d} L \Psi^j_\gamma}{\mu}
\]
\[
+ c^2 \slashed{d}_A(c^{-2} \mu) \slashed{d}_A L \Psi^j_\gamma - T \mu \tilde{\Delta} \Psi^j_\gamma - 2 c^{-2} \frac{\mu (\chi^{AB} - \frac{1}{2} \tr g \tilde{g} \tilde{X}^{AB}) \Psi^2_{AB} \Psi^j_\gamma}{\mu}
\]
(5.31)
\[
(R_i) \mathcal{A}^j_2 = c^{-1} v_i \tr g \chi (L + \frac{1}{2} \tr g \chi) L \Psi^j_\gamma - \frac{1}{2} \slashed{d}_A \tilde{g}_A \Psi^j_\gamma - (c^{-2} \mu (R_i) \tilde{g} \Psi^j_\gamma + 2(R_i) \tilde{g} \Psi^j_\gamma)
\]
\[
+ (R_i) \tilde{g} \Psi^j_\gamma + c^{-1} \mu v_i \tilde{X}^{AB} - \frac{1}{2} \tr g \tilde{g} \tilde{X}^{AB}) \Psi^2_{AB} \Psi^j_\gamma, 
\]
(5.32)
and
\[
(pL) \mathcal{A}^j_3 = \tr g \chi \{ \rho L(c^{-2} \mu) - 2 c^{-2} (\mu - 1) + 2(1 - c^{-2}) - \frac{1}{2} \rho c^{-2} \mu \tr g \chi \} \Psi^j_\gamma
\]
\[
+ 2 \rho c^2 \slashed{d}_A(c^{-2} \mu) \chi^{AB} \slashed{d}_A \Psi^j_\gamma,
\]  
(5.33)
by \[(2.29)\], one has

The elliptic estimates \((6.1)\) and \((6.2)\) hold immediately from \([17, (18.26), (18.15)]\). Moreover, \(G\)

Let \(\Phi\) be such that \(\int_{\mathcal{D}^{1,u}} |\Phi| \cdot |L\Psi|\) and \(\int_{\mathcal{D}^{1,u}} |\Phi| \cdot |L\Psi|\) in \((5.11)\) so that all the energy estimates

\(\text{Lemma 6.1.}\) \(\Psi \in C^1(D^{1,u})\) which vanishes on \(C_0\) and sufficiently small \(\delta > 0\), one has

\[
\int_{\Sigma_u^t} \Psi^2 \lesssim \delta^2 \{ E_1[\Psi](t,u) + E_2[\Psi](t,u) \}.
\]

\(\text{Proof.}\) See the proof in Lemma 7.3 of \([16]\). \(\square\)

\(\text{Lemma 6.2.}\) For any trace-free symmetric \((0,2)\)-type tensor field \(\xi\) on \(S_{t,u}\) or function \(f \in C^2(D^{1,u})\), it holds that

\[
\int_{S_{t,u}} \mu^2 (|\nabla_\xi|^2 + 2G|\xi|^2) \lesssim \int_{S_{t,u}} \left( \mu^2 |\text{div}_g \xi|^2 + |\mathcal{D}\mu|^2 |\xi|^2 \right),
\]

\[
\int_{S_{t,u}} \mu^2 (|\nabla f|^2 + G|\mathcal{D}f|^2) \lesssim \int_{S_{t,u}} \left( \mu^2 |\Delta f|^2 + |\mathcal{D}\mu|^2 |f|^2 \right),
\]

where the Gaussian curvature \(G\) of \(\mathcal{g}\) satisfies

\[
G = 1 + O(\delta^{(1-\varepsilon_0)p}).
\]

\(\text{Proof.}\) The elliptic estimates \([6.1]\) and \([6.2]\) hold immediately from \([17\ (18.26), (18.15)]\). Moreover, by \([16\ (2.29)]\), one has

\[
G = \frac{1}{2} e^{-2} \left[ (\text{tr}_g \chi)^2 - |\chi|^2 \right].
\]

Together with \((2.16)\), it yields

\[
G = \frac{1}{2} e^{-2} \left[ 2 \rho^{-2} + (\text{tr}_g \chi)^2 + 2 \rho^{-1} \text{tr}_g \chi - |\chi|^2 \right] = 1 + O(\delta^{(1-\varepsilon_0)p}).
\]

\(\square\)

\(\text{Lemma 6.3.}\) Let \(\bar{t} = \inf_{t'} \left\{ 1 \leq t' \leq t < t^*: \mu_{\min}(t') < \frac{1}{10} \right\}\). For \(a \geq 1\) and sufficiently small \(\delta > 0\), one has

\[
(T_{\lambda}^2) A^j_i = \left[ T(c^{-2} \mu) \text{tr}_g \chi + \frac{1}{2} c^{-4} \mu^2 (\text{tr}_g \chi)^2 - \frac{1}{2} c^2 |\mathcal{D}(c^{-2} \mu)|^2 \right] L \Psi^j_i + \frac{1}{2} c^2 L(c^{-2} \mu) \mathcal{D}_A(c^{-2} \mu)
\]

\[
- \mu \mathcal{D}_A(c^{-2} \mu) \mathcal{D}^A \Psi^j_i,
\]

\[
(R_{\lambda}^2) A^j_i = \left[ \mathcal{R}_i(c^{-2} \mu) \text{tr}_g \chi - \frac{1}{2} c^{-3} \mu \mathcal{V}_i \text{tr}_g \chi^2 \right] + \frac{1}{2} (R_{\lambda})^j_i \mathcal{D}_A(c^{-2} \mu) \mathcal{L} \Psi^j_i + \frac{1}{2} (R_{\lambda})^j_i \mathcal{D}_A(c^{-2} \mu) \mathcal{D}^A \Psi^j_i,
\]
We now define a time $\tau$ which implies for Eq. (6.1) Non-top order derivative related quantities.

Based on the preparations above, we are ready to derive the non-top order derivative $L^2$ estimates for the related quantities.

**Proposition 6.1.** Under the assumptions \(3.1\), when $\delta > 0$ is small, it holds that for $|\alpha| \leq 2N - 10$,

\[
\begin{align*}
\delta^{l+1}(1-\varepsilon_0) p & \| L^\alpha Z \chi \|_{L^2(\Sigma^u)} \lesssim \delta^{(1-\varepsilon_0) p + \frac{1}{2}} + \Theta_M^1(t, u), \\
\delta^{l+1} & \| Z^{\alpha+1} u \|_{L^2(\Sigma^u)} \lesssim \delta^{1/2} + \Theta_M^2(t, u), \\
\delta^{l+1} & \left( \| Z^{\alpha+1} J \|_{L^2(\Sigma^u)} + \| Z^{\alpha+1} J^2 \|_{L^2(\Sigma^u)} + \| Z^{\alpha+1} v_j \|_{L^2(\Sigma^u)} \right) \lesssim \delta^{(1-\varepsilon_0) p + \frac{1}{2}} + \Theta_M^1(t, u), \\
\delta^{l+1} & \left( \| \mathcal{E}^\alpha Z \mathcal{d} x^j \|_{L^2(\Sigma^u)} + \| \mathcal{E}^\alpha Z R_j \|_{L^2(\Sigma^u)} \right) \lesssim \delta^{1/2} + \Theta_M^1(t, u), \\
\delta^{l+1} & \left( \| \mathcal{E}^\alpha Z (R_i) \mathcal{f} \|_{L^2(\Sigma^u)} + \| \mathcal{E}^\alpha Z (R_j) \mathcal{f} \|_{L^2(\Sigma^u)} + \| \mathcal{E}^\alpha Z (R_i) \mathcal{f} T \|_{L^2(\Sigma^u)} \right) \lesssim \delta^{(1-\varepsilon_0) p + \frac{1}{2}} + \Theta_M^1(t, u), \\
\delta^{l+1} & \left( \| \mathcal{E}^\alpha Z (T) \mathcal{f} \|_{L^2(\Sigma^u)} + \| \mathcal{E}^\alpha Z (T) \mathcal{f} \|_{L^2(\Sigma^u)} \right) \lesssim \delta^{1/2} + \Theta_M^2(t, u),
\end{align*}
\]  

(6.7)
where \( l \) (or \( s \)) is the number of \( T \) (or \( \rho L \)) in the string of \( Z \) with \( s \leq 2 \), and

\[
\Theta_1^1(t, u) = \delta^{(1-\varepsilon_0)(p-1)} \int_1^t \left( \frac{1}{\mu_{\min}(t')} \sqrt{\tilde{E}_{1, \leq |\alpha|+2}(t', u)} + \delta^{(1-\varepsilon_0)p} \sqrt{\tilde{E}_{2, \leq |\alpha|+2}(t', u)} \right) dt', \\
\Theta_2^1(t, u) = \delta^{(1-\varepsilon_0)(p-1)} \int_1^t \left( \frac{1}{\mu_{\min}(t')} \sqrt{\tilde{E}_{1, \leq |\alpha|+2}(t', u)} + \sqrt{\tilde{E}_{2, \leq |\alpha|+2}(t', u)} \right) dt'.
\]

**Proof.** We will prove this proposition by the induction method with respect to the index \( \alpha \). When \( \alpha = 0 \), in view of Proposition 4.1, the corresponding \( L^2 \) estimates can be directly obtained by the fact \( \|1\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{1/2} \) (similarly to the proof of [17], Corollary 11.30.3). For example, \( \|\tilde{\chi}\|_{L^2(\Sigma_u^\omega)} \lesssim \|\tilde{\chi}\|_{L^\infty(\Sigma_u^\omega)} \cdot \|1\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{(1-\varepsilon_0)p+1/2} \). For any \( 1 \leq |\alpha| \leq 2N - 10 \), assume that (6.7) holds up to the order \( |\alpha| - 1 \), one needs to show (6.7) for the order \( |\alpha| \). To this end, we take the \( L^2 \) norms for the factors equipped with the higher order derivatives of the related quantities, meanwhile applying Proposition 4.1 for the corresponding \( L^\infty \) coefficients in these terms.

As in the proof of Proposition 4.1, we first prove (6.7) when \( Z \) only involves the rotation vector, that is, \( Z \in \{R_1, R_2, R_3\} \).

**Part 1. Estimates of \( \mathcal{L}^{\alpha}_{R_i\tilde{\chi}} \) and \( \mathcal{L}^{\alpha}_{R_i}(R_i) \#_L \)**

By (4.2), one has

\[
\mathcal{L}_L \mathcal{L}^{\alpha}_{R_i\tilde{\chi}}(X_{AB}) = \sum_{\beta_1+\beta_2=\alpha} \mathcal{L}^{\beta_1}_{R_i}(c^{-1}L_c) \cdot \mathcal{L}^{\beta_2}_{R_i}(X_{AB}) + \sum_{\beta_1+\beta_2+\beta_3=\alpha} \mathcal{L}^{\beta_1}_{R_i}(\chi^{CD}) \cdot \mathcal{L}^{\beta_2}_{R_i}(X_{AD}) \cdot \mathcal{L}^{\beta_3}_{R_i}(X_{BC}) + \mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB}) \]

\[
- \mathcal{L}^{\alpha}_{R_i}(\tilde{\phi}_{LALB}) + \sum_{\beta_1+\beta_2=\alpha-1} \mathcal{L}^{\beta_1}_{R_i}(\gamma^{DC}) \cdot \mathcal{L}^{\beta_2}_{R_i}(X_{AC}(L_c\phi_{BC}) + \mathcal{L}^{\alpha}_{R_i}(\gamma^{BC}) \cdot \mathcal{L}^{\beta_2}_{R_i}(X_{AC}(L_c\phi_{BC})) + \mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB}) \]

\[
= I_1 + I_2 + I_3 + I_4 + I_5,
\]

where we have neglected the unimportant coefficient constants.

For \( I_1 \), by Lemma 6.1 and (2.35), then

\[
\|I_1\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{(1-\varepsilon_0)p-1-1-(1-\varepsilon_0)p} \|\mathcal{L}^{\alpha}_{R_i}\tilde{\chi}\|_{L^2(\Sigma_u^\omega)} + \delta^{(1-\varepsilon_0)2p} \|\mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB})\|_{L^2(\Sigma_u^\omega)}
\]

\[
+ \delta^{(1-\varepsilon_0)2p} \left( \mu_{\min}(t) \sqrt{\tilde{E}_{1, \leq |\alpha|+1}(t, u)} + \delta^{(1-\varepsilon_0)p} \sqrt{\tilde{E}_{2, \leq |\alpha|+1}(t, u)} \right).
\]

For \( I_2 \), we have

\[
\|I_2\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{(1-\varepsilon_0)p} \|\mathcal{L}^{\alpha}_{R_i}\tilde{\chi}\|_{L^2(\Sigma_u^\omega)} + \delta^{(1-\varepsilon_0)2p} \|\mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB})\|_{L^2(\Sigma_u^\omega)}
\]

In addition,

\[
\|I_3\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{(1-\varepsilon_0)p-1-1-(1-\varepsilon_0)p} \|\mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB})\|_{L^2(\Sigma_u^\omega)}
\]

\[
+ \|R_1^{\leq \alpha} \cdot \left( \frac{1}{2} \frac{p}{p^2 + |\phi_0|^2} L_c\phi_0 \right)\|_{L^2(\Sigma_u^\omega)} \lesssim \delta^{(1-\varepsilon_0)p-1-1-(1-\varepsilon_0)p} \|\mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB})\|_{L^2(\Sigma_u^\omega)}
\]

\[
+ \delta^{(1-\varepsilon_0)p} \|\mathcal{L}^{\alpha}_{R_i}(\rho^{-1}c^{-1}L_c\phi_{AB})\|_{L^2(\Sigma_u^\omega)}
\]

\[
+ \delta^{(1-\varepsilon_0)(p-1)} \left( \mu_{\min}(t) \sqrt{\tilde{E}_{1, \leq |\alpha|+1}(t, u)} + \delta^{(1-\varepsilon_0)p} \sqrt{\tilde{E}_{2, \leq |\alpha|+1}(t, u)} \right),
\]
\[ \|I_4\|_{L^2(\Sigma_t^n)} \lesssim \delta^{(1-\varepsilon_0)(p-1)} \|R^\varphi_0\|_{L^2(\Sigma_t^n)} < \delta^{(1-\varepsilon_0)(p-1)} \mu_{\min}(t) \sqrt{\bar{E}_{1,\leq|a|+2}(t,u)}. \]

For \( I_5 \), by the expression of \( \langle R_i \rangle \# L_A \) in (2.30), we arrive at

\[ \|I_5\|_{L^2(\Sigma_t^n)} \lesssim \delta^{(1-\varepsilon_0)p} \|R^\varphi_{\mathcal{L}_R}\|_{L^2(\Sigma_t^n)} \]
\[ + \delta^{(1-\varepsilon_0)(p+1)} \|R^\varphi_{\mathcal{L}_R} \|_{L^2(\Sigma_t^n)} \]
\[ + \delta^{(1-\varepsilon_0)(p+2)} \delta^{(1-\varepsilon_0)(p+1)} \|R^\varphi_{\mathcal{L}_R} \|_{L^2(\Sigma_t^n)} \]
\[ + \delta^{(1-\varepsilon_0)(p+2)} \|R^\varphi_{\mathcal{L}_R} \|_{L^2(\Sigma_t^n)} \]
\[ + \delta^{(1-\varepsilon_0)(p-1)} \mu_{\min}(t) \sqrt{\bar{E}_{1,\leq|a|+2}(t,u)}. \]

Combining the \( L^2 \) estimates for all \( I_i \) (\( 1 \leq i \leq 5 \)), together with the induction hypothesis, we obtain

\[ \|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha \|_{L^2(\Sigma_t^n)} \lesssim \delta^{(1-\varepsilon_0)p-1} \|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha \|_{L^2(\Sigma_t^n)} \]
\[ + \delta^{(1-\varepsilon_0)p-1} \delta^{(1-\varepsilon_0)(p+1)} \]
\[ + \delta^{(1-\varepsilon_0)(p-1)} \mu_{\min}(t) \sqrt{\bar{E}_{1,\leq|a|+2}(t,u)} + \delta^{(1-\varepsilon_0)p} \sqrt{\bar{E}_{2,\leq|a|+2}(t,u)} \] \hspace{1cm} (6.8)

where the second term in the right hand side of (6.8) comes from the first term in the estimate of \( I_3 \).

For any symmetric \((0, 2)\)-type tensor field \( \xi \), one has

\[ L(\rho^2|\xi|^2) = 2\rho^4\xi^{AB} \mathcal{L}_L \xi_{AB} - 4 \rho^4 \chi_{ABC} \xi^B \mathcal{L}_L \xi^C. \]

Substituting \( \xi = \mathcal{L}_R^\alpha \mathcal{L}_R^\alpha \) into the above identity yields

\[ |L(\rho^2|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha|)| \leq \rho^2 |\mathcal{L}_L \mathcal{L}_R^\alpha \mathcal{L}_R^\alpha| + 2|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha| \cdot \rho^2 |\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha|. \]

According to Newton-Leibniz formula, we have

\[ \rho^2 |\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha|(t) = (1 - u)^2 |\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha|(1) + \int_1^t L[(t' - u)^2 |\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha|](t')dt'. \]

Since \( \rho \sim 1 \), by taking the \( L^2(\Sigma_t^n) \) norm on both sides of the above identity, we obtain

\[ \|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha \|_{L^2(\Sigma_t^n)} \lesssim \delta^{(1-\varepsilon_0)p+1/2} + \int_1^t \|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha\|_{L^2(\Sigma_t^n)} dt'. \] \hspace{1cm} (6.9)

Substituting (6.8) into (6.9), together with Gronwall inequality, yields

\[ \|\mathcal{L}_R^\alpha \mathcal{L}_R^\alpha \|_{L^2(\Sigma_t^n)} \lesssim \delta^{(1-\varepsilon_0)p+1/2} + \delta^{(1-\varepsilon_0)(p-1)} \int_1^t \left( \mu_{\min}(t') \sqrt{\bar{E}_{1,\leq|a|+2}(t', u)} + \delta^{(1-\varepsilon_0)p} \sqrt{\bar{E}_{2,\leq|a|+2}(t', u)} \right) dt', \]

which also derives the analogous estimate of \( L^2 \) norm for \( \mathcal{L}_R^\alpha (R_i) \#_L \).
where we have neglected the unimportant coefficient constants.

For \( I_1' \), we have
\[
\|I_1'\|_{L^2(\Sigma_i^u)} \lesssim \delta^{(1-\varepsilon_0)p-1} \|R_i^{\alpha+1}\|_{L^2(\Sigma_i^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \delta^{(1-\varepsilon_0)p} \left( \frac{\mu_{\min}(t)}{\mu_{\min}(t)} \sqrt{E_{1,\leq|\alpha|+2}(t,u)} + \sqrt{E_{2,\leq|\alpha|+2}(t,u)} \right).
\]

For \( I_2' \), by Lemma 6.1, \( [T, R_i] = (R_i) \#_T \) (due to its expression in (2.30)) and \( T = \frac{1}{2}(L - c^{-2}\mu L) \), we arrive at
\[
\|I_2'\|_{L^2(\Sigma_i^u)} \lesssim \|R_i^{\alpha+1} \left( \frac{1}{2} c^4 \varphi_0^{p-1} T \varphi_0 \right) \|_{L^2(\Sigma_i^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \left( \frac{\mu_{\min}(t)}{\mu_{\min}(t)} \sqrt{E_{1,\leq|\alpha|+1}(t,u)} + \sqrt{E_{2,\leq|\alpha|+2}(t,u)} \right).
\]

For \( I_3' \),
\[
\|I_3'\|_{L^2(\Sigma_i^u)} \lesssim \delta^{(1-\varepsilon_0)p} \|R_i^{\alpha+1}\|_{L^2(\Sigma_i^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_{R_i}^{\alpha-1}(R_j)\|_{L^2(\Sigma_i^u)} + \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \left( \frac{\mu_{\min}(t)}{\mu_{\min}(t)} \sqrt{E_{1,\leq|\alpha|+1}(t,u)} + \sqrt{E_{2,\leq|\alpha|+2}(t,u)} \right).
\]

Combining the \( L^2 \) estimates for all \( I_i' \) (1 \( \leq i \leq 3 \)), we obtain that from the induction hypothesis and the estimates in Part 1
\[
\|LR_i^{\alpha+1}\|_{L^2(\Sigma_i^u)} \lesssim \delta^{(1-\varepsilon_0)p-1} \|R_i^{\alpha+1}\|_{L^2(\Sigma_i^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_{R_i}^{\alpha+1}(R_j)\|_{L^2(\Sigma_i^u)} + \|\mathcal{L}_{R_i}^{\alpha}(R_j)\|_{L^2(\Sigma_i^u)} + \left( \frac{\mu_{\min}(t)}{\mu_{\min}(t)} \sqrt{E_{1,\leq|\alpha|+1}(t,u)} + \sqrt{E_{2,\leq|\alpha|+2}(t,u)} \right),
\]

where the second term in the right hand side of (6.10) comes from the third term in the estimate of \( I_3' \).

According to Newton-Leibniz formula, one has
\[
R_i^{\alpha+1}\mu(t) = R_i^{\alpha+1}\mu(1) + \int_1^t LR_i^{\alpha+1}\mu(t') dt'.
\]

By taking the \( L^2(\Sigma_i^u) \) norm on both sides of the above identity and utilizing the Minkowski inequality, then
\[
\|R_i^{\alpha+1}\mu\|_{L^2(\Sigma_i^u)} \lesssim \delta^{\frac{1}{2}} + \int_1^t \|LR_i^{\alpha+1}\mu\|_{L^2(\Sigma_i^u)} dt'.
\]

Substituting (6.10) into (6.11), by Gronwall inequality, we finally arrive at
\[
\|R_i^{\alpha+1}\mu\|_{L^2(\Sigma_i^u)} \lesssim \delta^{\frac{1}{2}} + \delta^{(1-\varepsilon_0)p-1} \int_1^t \left( \frac{\mu_{\min}(t')}{\mu_{\min}(t')} \sqrt{E_{1,\leq|\alpha|+2}(t',u)} + \sqrt{E_{2,\leq|\alpha|+2}(t',u)} \right) dt',
\]

which also gives the analogous estimate of \( L^2 \) norm for \( \mathcal{L}_{R_i}^{\alpha}(R_j)\) \( \#_T \).
Part 3. Estimates of $R_i^{\alpha+1} \tilde{L}_j$, $R_i^{\alpha+1} \tilde{T}_j$ and $R_i^{\alpha+1} v_j$

By (8.16), one has

$$R_i^{\alpha+1} L_j = \mathcal{L}_R^\alpha (R_i^A \chi_{AB} \tilde{d}B_{x^j}) = \sum_{\beta_1+\beta_2+\beta_3+\beta_4=\alpha} \mathcal{L}_{R_i}^{\beta_1} R_j \cdot \mathcal{L}_{R_i}^{\beta_2} \tilde{\chi} \cdot \mathcal{L}_{R_i}^{\beta_3} \tilde{\gamma}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_4} \tilde{d}x^j.$$ 

Thanks to the estimate of $\mathcal{L}_R^\alpha \tilde{\chi}$ in Part 1, together with the induction hypothesis, we arrive at

$$\|R_i^{\alpha+1} L_j\|_{L^2(\Sigma_t^n)} \lesssim \delta(1-\varepsilon_0)p^2 \delta(1-\varepsilon_0)(p-1) \int_1^t \left( \mu_{\min}^{-\frac{\varepsilon}{2}}(t') \sqrt{\tilde{E}_{1,\leq|\alpha|+2}(t', u)} + \delta(1-\varepsilon_0)p \sqrt{\tilde{E}_{2,\leq|\alpha|+2}(t', u)} \right) dt'.$$

Then the estimates of $R_i^{\alpha+1} \tilde{T}_j$ and $R_i^{\alpha+1} v_j$ follow immediately from (2.16) and (2.17).

Part 4. Estimates of the other quantities

The estimates of $\mathcal{L}_R^\alpha \tilde{d}x^j$ and $\mathcal{L}_{R_i}^{\alpha+1} R_j$ follow directly from (2.18) and Part 3, while the estimates of the other deformation tensors follow from Lemma 2.7 and the estimates in the above parts.

If there are vectorfields $T$ or $\rho L$ in the string of $Z$, we could utilize the structure equations in Lemma 2.11 to get the corresponding $L^2$ bounds as in the end of the proof for Proposition 4.11.

6.2 Top order derivative $L^2$ estimates

When we try to close the energy estimate of the top order derivatives (see Section 7 below) and further complete the proof of bootstrap assumptions in (5.1), it is found that the top orders of derivatives of $\varphi, \chi$ and $\mu$ for the energy estimates are $2N-8$, $2N-9$ and $2N-8$ respectively. However, as shown in Proposition 6.1, the $L^2$ estimates for the $(2N-9)^{th}$ order derivatives of $\chi$ and $(2N-8)^{th}$ order derivatives of $\mu$ can be controlled by the $(2N-7)^{th}$ order derivatives of $\varphi_\gamma$. So there exists one order loss of derivatives in the corresponding energy inequality. To overcome this difficulty, as in [6] and [16 - 17], we need to treat $\chi$ and $\mu$ with the related top order derivatives by introducing some modified quantities and taking the elliptic estimates.

6.2.1 Estimates on $\nabla \chi$

Recalling (2.22), we now define

$$\mu \hat{\mathcal{L}}_{LL} = \mu \hat{g}^{AB} \hat{\mathcal{L}}_{LAB} = -\frac{1}{2} p c^4 \varphi_0^{-p-1} \mu \Delta \varphi_0 + R_0,$$  

where

$$R_0 = -\frac{3}{2} p c^3 \mu \varphi_0^{-p-1} d_A d^A \varphi_0 - \frac{1}{2} p(p-1) c^4 \mu \varphi_0^{-2} |d \varphi_0|^2.$$  

By (2.42) and (2.16), one has

$$L \text{tr}_g \chi = (c^{-1} L c - 2 \rho^{-1} \text{tr}_g \chi + 2 \rho^{-2} - |\chi|^2 - \hat{\mathcal{L}}_{LL},$$  

then

$$L (\mu \text{tr}_g \chi) = (L \mu + c^{-1} L c \mu - 2 \rho^{-1} \mu \text{tr}_g \chi + 2 \rho^{-2} \mu - \mu |\chi|^2 - \mu \hat{\mathcal{L}}_{LL}.$$  

Note that the highest order derivative of $\varphi_0$ in the right hand side of (6.12) is $\Delta \varphi_0$. We will replace $\mu \Delta \varphi_0$ in $\mu \hat{\mathcal{L}}_{LL}$ of (6.15) with $L(\partial \varphi + \text{l.o.t.})$. Indeed, by (2.38), (2.39) and (2.40), one has

$$\mu \Delta \varphi_0 = L \varphi_0 + \text{tr}_g \chi T \varphi_0 - H_0 + F_0.$$  

Hence,

\[
\mu \tilde{\mathcal{H}}_{LL} = -LE_\chi - e_\chi - \frac{1}{2}pc^4 \varphi_0^{p-1} T \varphi_0 \text{tr}_g \chi
\]

\[
= -LE_\chi - e_\chi + cT \alpha \text{tr}_g \chi,
\]

where

\[
E_\chi = \frac{1}{2}pc^4 \varphi_0^{p-1} L \varphi_0,
\]

\[
e_\chi = -L(\frac{1}{2}pc^4 \varphi_0^{p-1})L \varphi_0 - \frac{1}{2}pc^4 \varphi_0^{p-1}(\dot{H}_0 - F_0) - R_0.
\]

Substituting (6.17) into (6.15) yields

\[
L(\mu \text{tr}_g \chi - E_\chi) = 2(L\mu - \rho^{-1}\mu) \text{tr}_g \chi + 2\rho^{-2}\mu - \mu|\chi|^2 + e_\chi.
\]

Define

\[
E_\chi^\alpha = \mu dR_i^\alpha \text{tr}_g \chi - dR_i^\alpha E_\chi.
\]

Then

\[
\mathcal{L}_LE_\chi^0 = 2(\mu^{-1}L\mu - \rho^{-1})E_\chi^0 + 2(\mu^{-1}L\mu - \rho^{-1})dE_\chi - \mu d(|\chi|^2) + e_\chi^0,
\]

where

\[
e_\chi^0 = de_\chi + (dL\mu - 2\rho^{-1}d\mu) \text{tr}_g \chi - d\mu(L \text{tr}_g \chi + |\chi|^2 - 2\rho^{-2}).
\]

By the induction argument on (6.19), we have

\[
\mathcal{L}_LE_\chi^\alpha = 2(\mu^{-1}L\mu - \rho^{-1})E_\chi^\alpha + 2(\mu^{-1}L\mu - \rho^{-1})dR_i^\alpha E_\chi - \mu d(|\chi|^2) + e_\chi^\alpha,
\]

where

\[
e_\chi^\alpha = \mathcal{L}_{R_i}^\alpha [de_\chi + (dL\mu - 2\rho^{-1}d\mu) \text{tr}_g \chi - d\mu(L \text{tr}_g \chi + |\chi|^2 - 2\rho^{-2})]
\]

\[
+ \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_{R_i}^{\beta_1} [(R_i)^\beta_1 \nabla E_{\chi}^{\beta_2} + \nabla (R_i)^\beta_1 E_{\chi}^{\beta_2}]
\]

\[
+ \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_{R_i}^{\beta_1} \{ (R_i L\mu - (R_i)^\beta_1 \nabla A_i \mu - 2\rho^{-1} R_i \mu) dR_i^{\beta_2} \text{tr}_g \chi - R_i \mu [dR_i^{\beta_2}(|\chi|^2) + dLR_i^{\beta_2} \text{tr}_g \chi] \}.
\]

Note that for any one-form \( \xi \), one has

\[
L(|\xi|^2) = \mathcal{L}_L(g^{AB} \xi_A \xi_B)
\]

\[
= -2\chi^{AB} \xi_A \xi_B - 2\rho^{-1}|\xi|^2 + 2\mathcal{L}_L \xi \cdot \xi.
\]

By taking \( \xi = E_\chi^\alpha \) in (6.25) and utilizing (6.23), then

\[
L(|E_\chi|^2) = -2\chi^{AB} (E_\chi^\alpha)_A (E_\chi^\alpha)_B - 2\rho^{-1}|E_\chi|^2
\]

\[
+ [2(\mu^{-1}L\mu - \rho^{-1})E_\chi^\alpha + 2(\mu^{-1}L\mu - \rho^{-1})dR_i^\alpha E_\chi - \mu dR_i^\alpha (|\chi|^2) + e_\chi^\alpha] \cdot E_\chi^\alpha.
\]
For \((\mu^{-1} L \mu - \rho^{-1}) |E_\chi^\alpha|^2\), if \(\mu \geq \frac{1}{16}\), it can be bounded by \(\delta^{-[1-(1-\varepsilon_0)p]} |E_\chi^\alpha|^2\); if \(\mu < \frac{1}{16}\), according to (3.21), the sign of \(\mu^{-1} L \mu\) is negative so that this term can be ignored. Hence,

\[
L(|E_\chi^\alpha|) \lesssim \delta^{-[1-(1-\varepsilon_0)p]} |E_\chi^\alpha| + |\mu^{-1} L \mu - \rho^{-1}||dR_i^\alpha E_\chi| + |\mu dR_i^\alpha (|\chi|^2)| + |\epsilon_\chi|.
\]  

(6.26)

Then by applying Newton-Leibniz formula and taking the \(L^2(\Sigma_t^u)\) norm on both sides of (6.26), we obtain from Gronwall inequality and Minkowski inequality that

\[
\|E_\chi^\alpha\|_{L^2(\Sigma_t^u)} \lesssim \delta^{(1-\varepsilon_0)p-\frac{1}{2}} + \int_1^t \left[\|\mu dR_i^\alpha (|\chi|^2)\|_{L^2(\Sigma_t^u)} + \|\epsilon_\chi\|_{L^2(\Sigma_t^u)} \right] \|dR_i^\alpha E_\chi\|_{L^2(\Sigma_t^u)} dt'.
\]  

(6.27)

Next, we estimate the terms in the integrand of (6.27) one by one.

**1-a) Estimate of \(\mu dR_i^\alpha (|\chi|^2)\)**

According to (2.44), we have

\[
(\text{div}_g \bar{\chi})_A = d_A tr_g \chi + H_A,
\]  

(6.28)

where

\[
H_A = c^{-1} d^B c_{XAB} - c^{-1} d_A c tr_g \chi.
\]  

(6.29)

By commuting \(\mathcal{L}_R^\alpha\) with \(d_g\), it follows from Lemma 2.9 that

\[
\mathcal{L}_R^\alpha (\text{div}_g \bar{\chi})_A = \mathcal{L}_R^\alpha (d_g \bar{\chi})_A = \mathcal{L}_R^\alpha (d_g \bar{\chi})_A + \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_R^\beta_1 \mathcal{L}_R^\beta_2 \mathcal{L}_R^\alpha (d_g \bar{\chi})_A
\]  

(6.30)

where

\[
(\mathcal{H}^\alpha)_A = \mathcal{L}_R^\alpha H_A - \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_R^\beta_1 \mathcal{L}_R^\beta_2 (\mathcal{L}_R^\alpha (d_g \bar{\chi})_A)
\]  

(6.31)

with

\[
\|\mathcal{H}^\alpha\|_{L^2(\Sigma_t^u)} \lesssim \delta^{(1-\varepsilon_0)p-1} \|R_i^{\leq 1+\varphi} \|_{L^2(\Sigma_t^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_R^\alpha \bar{\chi}\|_{L^2(\Sigma_t^u)} + \delta^{(1-\varepsilon_0)p} \|\mathcal{L}_R^\alpha (d_g \bar{\chi})_A\|_{L^2(\Sigma_t^u)}.
\]  

(6.32)

It follows from the elliptic estimate (6.1) and (6.3) with \(\bar{\chi} = (\chi - \frac{1}{2} tr_g \chi) + \frac{1}{2} tr_g \bar{\chi}\) that

\[
\|\mu \nabla \mathcal{L}_R^\alpha - \mu_d \mu \mathcal{L}_R^\alpha \chi\|_{L^2(\Sigma_t^u)} \lesssim \|\mu \text{div}_g \mathcal{L}_R^\alpha \bar{\chi}\|_{L^2(\Sigma_t^u)} + \|\mathcal{L}_R^\alpha \bar{\chi}\|_{L^2(\Sigma_t^u)} + \|\mathcal{L}_R^\alpha (d_g \bar{\chi})_A\|_{L^2(\Sigma_t^u)} + \|\mu dR_i^\alpha tr_g \chi \|_{L^2(\Sigma_t^u)}
\]  

\[
\lesssim \|\mu dR_i^\alpha tr_g \chi\|_{L^2(\Sigma_t^u)} + \|\mathcal{H}^\alpha\|_{L^2(\Sigma_t^u)} + \|\mathcal{L}_R^\alpha \bar{\chi}\|_{L^2(\Sigma_t^u)} + \|\mathcal{L}_R^\alpha (d_g \bar{\chi})_A\|_{L^2(\Sigma_t^u)}.
\]  

(6.33)
Then by (6.32) and (6.33), together with Lemma 6.1 and Proposition 6.1, we obtain

\[
\| \mu \delta R_i^\alpha (|\bar{\chi}|^2) \|_{L^2(\Sigma_i^u)} \\
= \| \mu \delta L_{R_i}^\alpha (\varphi_{AC} \delta^{BD} \bar{\chi} \Lambda_{AB} \bar{\chi} \Lambda_{CD}) \|_{L^2(\Sigma_i^u)} \\
\lesssim \delta (1-\epsilon_0)p \| \mu \delta L_{R_i}^\alpha \chi \|_{L^2(\Sigma_i^u)} + \delta (1-\epsilon_0)p \| \delta L_{R_i}^\alpha \chi \|_{L^2(\Sigma_i^u)} + \delta (1-\epsilon_0)^2p \| \delta L_{R_i}^\alpha \chi \|_{L^2(\Sigma_i^u)} \\
\leq \delta (1-\epsilon_0)p \left( \| \mu \delta R_i^\alpha \varphi_{\bar{g}} \chi \|_{L^2(\Sigma_i^u)} + \| \mu R_i^\alpha \varphi \|_{L^2(\Sigma_i^u)} \right) \\
\lesssim \delta (1-\epsilon_0)p \left( \| \mu \delta R_i^\alpha \chi \|_{L^2(\Sigma_i^u)} + \| \varphi \|_{L^2(\Sigma_i^u)} \right) \\
\lesssim \delta (1-\epsilon_0)p \| \varphi \|_{L^2(\Sigma_i^u)}. 
\]

Hence,

\[
\int_1^t \| \mu \delta R_i^\alpha (|\bar{\chi}|^2) \|_{L^2(\Sigma_i^u)} \, dt' \\
\lesssim \int_1^t \delta (1-\epsilon_0)p \| \varphi \|_{L^2(\Sigma_i^u)} \, dt' \\
\leq \delta (1-\epsilon_0)p \| \varphi \|_{L^2(\Sigma_i^u)}. 
\]

(1-b) Estimate of \( e_X^\alpha \)

Substituting (2.41) and (6.14) into (6.24), one keeps in mind that there are two important eliminations as follows

\[
(\delta L_{\mu} - 2\rho^{-1} \delta \mu) \varphi_{\bar{g}} \chi - \delta \mu (L \varphi_{\bar{g}} \chi + |\bar{\chi}|^2 - 2\rho^{-2}) \\
= \left[ \mu \delta (c^{-1} L c) - \delta (c T c) \right] \varphi_{\bar{g}} \chi + \delta \mu \bar{\varphi}_{LL} \\
\]

and

\[
(R_i \mu - (R_i)^{AB} \delta_{LA} \mu - 2\rho^{-1} R_i \mu ) \delta R_i^{\beta 2} \varphi_{\bar{g}} \chi - R_i \mu [\delta R_i^{\beta 2} (|\bar{\chi}|^2) + \delta L \delta R_i^{\beta 2} \varphi_{\bar{g}} \chi] \\
= \left[ \mu R_i (c^{-1} L c) - R_i (c T c) - (R_i)^{AB} \delta_{LA} \mu \right] \delta R_i^{\beta 2} \varphi_{\bar{g}} \chi + R_i \mu \delta R_i^{\beta 2} \bar{\varphi}_{LL} + 1.o.t.. 
\]

Then by (6.18), (6.20) and Proposition 6.1, one has

\[
\| e_X^\alpha \|_{L^2(\Sigma_i^u)} \\
\lesssim \delta (1-\epsilon_0)p \| \varphi \|_{L^2(\Sigma_i^u)} \\
+ \delta (1-\epsilon_0)(p-1) \| L R_i^{\alpha + 1} \varphi \|_{L^2(\Sigma_i^u)} + \delta (1-\epsilon_0)(p-1) \| L R_i^{\alpha + 1} |\bar{\chi}|^2 \|_{L^2(\Sigma_i^u)} \sqrt{E_{2,|\alpha|+2}(t, u)} \\
+ \delta (1-\epsilon_0)(p-1) \left[ \delta (1-\epsilon_0)p + \delta (1-\epsilon_0)(p-1) \right] \int_1^t \left( \mu_{\min}^{\frac{1}{2}}(t') \sqrt{E_{1,|\alpha|+2}(t', u)} + \delta (1-\epsilon_0)p \sqrt{E_{2,|\alpha|+2}(t', u)} \right) \, dt' \\
+ \delta (1-\epsilon_0)p \left( \delta (1-\epsilon_0)(p-1) \right) \int_1^t \left( \mu_{\min}^{\frac{1}{2}}(t') \sqrt{E_{1,|\alpha|+2}(t', u)} + \delta (1-\epsilon_0)p \sqrt{E_{2,|\alpha|+2}(t', u)} \right) \, dt'.
\]
Hence,

\[
\int_1^t \| e^{\mathcal{A}_t}_{\eta} \|_{L^2(\Sigma^\eta)} \, dt' \\
\lesssim \int_1^t \delta^{(1-\varepsilon_0)p} \| E^n_{\chi} \|_{L^2(\Sigma^\eta)} \, dt' \\
+ \delta^{(1-\varepsilon_0)(\frac{q}{2}-1)-\frac{1}{2}} \int_0^u \mathcal{F}_{1, \leq |\alpha|+2}(t, u') \, du' \\
+ \delta^{(1-\varepsilon_0)(p-1)-1-|\varepsilon_0|p} \int_1^t \left( \frac{\mu_{\min} (t')}{\delta^{1-\varepsilon_0}} \right)^p \sqrt{\mathcal{E}_{1, \leq |\alpha|+2}(t, u)} + \sqrt{\mathcal{E}_{2, \leq |\alpha|+2}(t, u)} \right) \, dt'.
\]

(1-c) Estimate of \((\mu^{-1} L \mu - \rho^{-1}) \mathcal{A}_t \mathcal{E}_\chi\)

Thanks to (6,5), one has

\[
\int_1^t \| \mu^{-1} L \mu - \rho^{-1} \|_{L^\infty(\Sigma^\eta)} \| \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} \, dt' \\
\lesssim \delta^{-1}|-(1-\varepsilon_0)p| \int_1^t \| \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} \, dt' + \delta^{-1}|-(1-\varepsilon_0)p| \int_1^t \mu_{\min}^{-1}(t') \| \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} \, dt' \\
\lesssim \delta^{1-|\varepsilon_0|2p+\frac{1}{2}} + \delta^{1-|\varepsilon_0|3p-\frac{1}{2}} \int_1^t \mu_{\min}^{-1}(t') \, dt' + \delta^{1-\varepsilon_0} \int_1^t \mu_{\min}^{-1}(t') \, dt' \left( \sqrt{\mathcal{E}_{1, \leq |\alpha|+2}(t, u)} + \sqrt{\mathcal{E}_{2, \leq |\alpha|+2}(t, u)} \right).
\]

Finally, substituting the estimates in (1-a), (1-b) and (1-c) into (6.27), we obtain from Gronwall inequality and (6.20) that

\[
\| \mu \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} + \| \mu \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} \\
\lesssim \delta^{1-\varepsilon_0} \int_1^t \mu_{\min}^{-1}(t') \, dt' \\
+ \delta^{1-\varepsilon_0} \frac{\mu_{\min}^{-1}(t)}{\delta^{1-\varepsilon_0}} \int_0^u \mathcal{F}_{1, \leq |\alpha|+2}(t, u') \, du' \\
+ \delta^{1-\varepsilon_0} \mu_{\min}^{-1}(t) \left( \sqrt{\mathcal{E}_{1, \leq |\alpha|+2}(t, u)} + \sqrt{\mathcal{E}_{2, \leq |\alpha|+2}(t, u)} \right).
\]

Moreover, if there is \( T \) or \( \rho L \) in \( Z^\alpha \), by utilizing Lemma 2.11 and (2.35) as in Proposition 6.1, we obtain

\[
\delta^{1+|\varepsilon|s[1-(1-\varepsilon_0)p]} \| \mu \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} + \delta^{1+|\varepsilon|s[1-(1-\varepsilon_0)p]} \| \mu \mathcal{A}_t \mathcal{E}_\chi \|_{L^2(\Sigma^\eta)} \\
\lesssim \delta^{1-\varepsilon_0} \int_1^t \mu_{\min}^{-1}(t') \, dt' \\
+ \delta^{1-\varepsilon_0} \frac{\mu_{\min}^{-1}(t)}{\delta^{1-\varepsilon_0}} \int_0^u \mathcal{F}_{1, \leq |\alpha|+2}(t, u') \, du' \\
+ \delta^{1-\varepsilon_0} \mu_{\min}^{-1}(t) \left( \sqrt{\mathcal{E}_{1, \leq |\alpha|+2}(t, u)} + \sqrt{\mathcal{E}_{2, \leq |\alpha|+2}(t, u)} \right).
\]

6.2.2 Estimates on \( \mathcal{V}^2 \mu \)
Similarly to the treatment for $\text{tr}_g \chi$, we use the transport equation (2.41) to estimate $\Delta \mu$. By Lemma 2.9 one has

\[ L \Delta \mu = (c^{-1}_t L c - 2 \rho^{-1}_t) \Delta \mu - 2 \chi^{AB} \nabla^2_{AB} \mu - d_A \text{tr}_g \chi d^A \mu + \mu L \Delta \ln c - c T \Delta c + J, \]  

where

\[ J = -2c [d_A (c^{-2}_t) \chi^{AB} - 1/2 d_B (c^{-2}_t) \text{tr}_g \chi] d_B \mu - 2 c^{-1}_t \chi^{AB} \nabla^2_{AB} \mu - 2 \rho^{-1}_t \Delta c - 2 d_A d^A T c - \Delta c T c + 2 \mu \chi^{AB} \nabla^2_{AB} \ln c + 2 \rho^{-1}_t \mu \Delta \ln c + 2d_A (c^{-1}_t L c) d^A \mu - 2 \mathcal{H} d^A \mu. \]  

Hence,

\[ L(\mu \Delta c) = L \Delta \mu + \mu L \Delta c \]

\[ = (L \mu + c^{-1}_t L c - 2 \rho^{-1}_t) \mu \Delta c - 2 \mu \chi^{AB} \nabla^2_{AB} \mu - \mu d_A \text{tr}_g \chi d^A \mu \]

\[ + L(\mu^2 \Delta \ln c) - 2 \mu L \mu \Delta \ln c - cp \left[ T \Delta c \right] + \mu J. \]  

The term with underline can be removed to the left hand side of (6.36), while the terms with wavy line will be treated by the elliptic estimate and Gronwall inequality. The strategy to treat the boxed term in (6.36) which contains the third order derivative of the solution is to transfer it into such a form $L(\partial_t \phi^2 \c)$ + l.o.t. Indeed, (6.16) gives that

\[ \mu \Delta c = \mu \nabla^A \left( -1/2 p c^3 \phi_0^{-1} d_A \phi_0 \right) \]

\[ = -1/2 p c^3 \phi_0^{-1} L \mu \phi_0 - 1/2 p c^3 \phi_0^{-1} T \phi_0 \text{tr}_g \chi + 1/2 p c^3 \phi_0^{-1} (\phi_0 - F_0) + \mu d^A (-1/2 p c^3 \phi_0^{-1}) d_A \phi_0. \]

Combining (6.37) and (5.21) yields

\[ -c T \mu \Delta c = -c T (\mu \Delta c) + c T \mu \Delta c \]

\[ = L(1/2 p c^3 \phi_0^{-1} T L \phi_0) - c T \mu \Delta c + C, \quad (6.38) \]

where

\[ C = -L(1/2 p c^3 \phi_0^{-1} T L \phi_0) - 1/2 p c^4 \phi_0^{-1} (T \phi_0) L \phi_0 + 1/2 p c^4 \phi_0^{-1} (T \phi_0) L \phi_0 + c T (1/2 p c^3 \phi_0^{-1} T \phi_0) \text{tr}_g \chi \]

\[ - c T [1/2 p c^3 \phi_0^{-1} (\phi_0 - F_0)] + \mu d^A (-1/2 p c^3 \phi_0^{-1}) d_A \phi_0 + c T \mu \Delta c. \]  

Substituting (6.38) into (6.36), we arrive at

\[ L(\mu \mu - E_\mu) = 2(L \mu - \rho^{-1}_t) \mu \mu - 2 \mu \chi^{AB} \nabla^2_{AB} \mu - d_A \text{tr}_g \chi d^A \mu + e_\mu, \]  

where

\[ E_\mu = 1/2 p c^4 \phi_0^{-1} T L \phi_0 + \mu^2 \Delta \ln c, \]

\[ e_\mu = C + \mu J - 2 \mu L \mu \Delta \ln c. \]
Note that $e_\mu$ is composed of lower order derivative terms such as $\delta^{\leq 2}\varphi$, $\delta^{\leq 1}\mu$ and $\bar{\chi}$.

Define

$$E_\mu^\alpha = \mu \bar{Z}^\alpha \Delta \mu - \bar{Z}^\alpha E_\mu$$

with $\bar{Z} \in \{T, R_1, R_2, R_3\}$. Then

$$LE_\mu^0 = L(\mu \Delta \mu - E_\mu)$$

$$= 2(\mu^{-1} L \mu - \rho^{-1}) E_\mu^0 - 2\mu \bar{\chi}^A \bar{Z}_\alpha^A \mu - \mu \delta \Delta \mu + e_\mu^0,$$  \hspace{1cm} (6.43)

where

$$e_\mu^0 = e_\mu + 2(\mu^{-1} L \mu - \rho^{-1}) E_\mu.$$  \hspace{1cm} (6.44)

By the induction argument on (6.40), we have

$$LE_\mu^0 = 2(\mu^{-1} L \mu - \rho^{-1}) E_\mu^0 - 2\mu \bar{\chi}^A \bar{Z}^\beta_\alpha \bar{Z}^\alpha \mu - \mu \delta \Delta \mu + e_\mu^0,$$  \hspace{1cm} (6.45)

where

$$e_\mu^0 = \bar{Z}^\alpha e_\mu + 2(\mu^{-1} L \mu - \rho^{-1}) E_\mu^0 - \sum_{\beta_1 + \beta_2 = \alpha} \bar{Z}^\beta_1 \left[ \mathcal{L}_Z (\mu \bar{\chi}^A) \mathcal{L}_Z^\beta_2 \bar{Z}_\alpha \right]$$

$$+ \sum_{\beta_1 + \beta_2 = \alpha} \bar{Z}^\beta_1 [L, \bar{Z}] E_\mu^0 + \sum_{\beta_1 + \beta_2 = \alpha} \bar{Z}^\beta_1 \left[ \bar{Z} L \mu - \bar{Z} \mu L \mu - 2 \bar{Z} (\rho^{-1} \mu) \right] \bar{Z}^\beta_2 \Delta \mu$$

$$- \sum_{\beta_1 + \beta_2 = \alpha} \bar{Z}^\beta_1 \left[ (\bar{Z} \mu L \bar{Z} \Delta \mu) + \delta \Delta \mu \bar{Z}_\alpha \right] \left[ \mathcal{L}_Z (\mu d \mu^A) \right].$$  \hspace{1cm} (6.46)

Note that

$$L(|E_\mu^0|^2) = 2 \mu E_\mu^0 \left[ 2(\mu^{-1} L \mu - \rho^{-1}) E_\mu^0 - 2\mu \bar{\chi}^A \bar{Z}^\beta_\alpha \bar{Z}_\alpha \mu - \mu \delta \Delta \mu + e_\mu^0 \right].$$  \hspace{1cm} (6.47)

As in (6.26) and (6.27), we have

$$\delta^l \|E_\mu^0\|_{L^2(\Sigma_u^\alpha)} \leq \delta^{(1-\varepsilon_0)p-\frac{1}{2}} + \delta^l \int_1^t \left( \delta^{(1-\varepsilon_0)p} \|\mu Z_\alpha Z \mu\|_{L^2(\Sigma_u^\alpha)} + \|e_\mu^0\|_{L^2(\Sigma_u^\alpha)} + \|\mu \delta \Delta \mu \|_{L^2(\Sigma_u^\alpha)} \right) dt'.$$  \hspace{1cm} (6.48)

Next, we estimate the terms in the integrand of (6.48) one by one.

(2-a) Estimate of $\mu \mathcal{L}_Z^\alpha \bar{Z}^\alpha \mu$

With the help of Lemma 2.9, we have from (6.3) and the elliptic estimate (6.2) that

$$\delta^l \|\mu \mathcal{L}_Z^\alpha \bar{Z}^\alpha \mu\|_{L^2(\Sigma_u^\alpha)}$$

$$\leq \delta^l \|\mu \bar{Z}^\alpha \mu\|_{L^2(\Sigma_u^\alpha)} + \delta^l \|\mu \mathcal{L}_Z^\alpha \bar{Z}^\alpha \mu\|_{L^2(\Sigma_u^\alpha)}$$

$$\leq \delta^l \|\mu \Delta \mu \|_{L^2(\Sigma_u^\alpha)} + \delta^l \|\mu \delta \Delta \mu \|_{L^2(\Sigma_u^\alpha)} + \delta^l \|\mu \mathcal{L}_Z^\alpha \bar{Z}^\alpha \mu\|_{L^2(\Sigma_u^\alpha)}$$

$$\leq \delta^l \|E_\mu^0\|_{L^2(\Sigma_u^\alpha)} + \delta^{(1-\varepsilon_0)(p-1)-1} \left( \sqrt{\bar{E}_{1,\leq |\alpha|+2}(t, u)} + \sqrt{\bar{E}_{2,\leq |\alpha|+2}(t, u)} \right)$$

$$+ \delta^{(1-\varepsilon_0)p-1} \int_1^t \left( \mu^{-\frac{1}{2}} \sqrt{\bar{E}_{1,\leq |\alpha|+2}(t', u)} + \mu^{-\frac{1}{2}} \sqrt{\bar{E}_{2,\leq |\alpha|+2}(t', u)} \right) dt'.$$
where $\alpha'$ means that the number of $T$ appearing in $L^\alpha_\mathbb{Z}$ is at most $l-1$ and $|\alpha'| \leq |\alpha|$.

Similarly to the treatment for $\chi$, utilizing Lemma [6.3] again, we arrive at

$$
\int_1^t \delta^{(1-\varepsilon_0)p} \mu \mathcal{L}_Z^{\alpha} \mathcal{Y}^2 \| \mu \mathcal{L}_Z^\alpha \mathcal{Y}^2 (\Sigma_\mu^\alpha) \| \mathcal{L}_Z^\alpha \mathcal{Y}^2 (\Sigma_\mu^\alpha) dt' 
\lesssim \int_1^t \delta^{(1-\varepsilon_0)p} \mathcal{L}_Z^{\alpha} \mathcal{Y}^2 \mathcal{L}_Z^\alpha \mathcal{Y}^2 (\Sigma_\mu^\alpha) dt' + \delta^{(1-\varepsilon_0)2p+\frac{1}{2}} + \delta^2 
+ \delta^{(1-\varepsilon_0)(p-1)} \mu_{\min}^{-b_{(\alpha|+2)}} (t) \left( \sqrt{\bar{E}_{1,\leq|\alpha|+2}(t, u)} + \sqrt{\bar{E}_{2,\leq|\alpha|+2}(t, u)} \right).
$$

**(2-b) Estimate of** $e_\mu^{\alpha}$

In view of (6.46), we deal with $2(\mu^{-1} L\mu - \rho^{-1}) \bar{Z}^\alpha E_\mu$, $\bar{Z}^\alpha e_\mu$ and the other left terms one by one. For $2(\mu^{-1} L\mu - \rho^{-1}) \bar{Z}^\alpha E_\mu$, one has from Lemma [6.3] and Proposition [6.1] that

$$
\int_1^t \| \mu^{-1} L\mu - \rho^{-1} \|_{L^\infty(\Sigma_\mu^\alpha)} \cdot \delta^2 \| \bar{Z}^\alpha E_\mu \|_{L^2(\Sigma_\mu^\alpha)} dt' 
\lesssim \delta^{[1-(\varepsilon_0)p]} \int_1^t \mu_{\min}^{-1}(t') \left( \delta^{(1-\varepsilon_0)(p-1)-1} \left( \sqrt{\bar{E}_{1,\leq|\alpha|+2}(t', u)} + \sqrt{\bar{E}_{2,\leq|\alpha|+2}(t', u)} \right) 
+ \delta^{(1-\varepsilon_0)p-1} \mathcal{L}_Z^{\alpha} \mathcal{Y}^2 \mathcal{L}_Z^\alpha \mathcal{Y}^2 (\Sigma_\mu^\alpha) \right) dt' 
\lesssim \delta^{(1-\varepsilon_0)2p+\frac{1}{2}} + \delta^{(1-\varepsilon_0)3p+\frac{1}{2}} \int_1^t \mu_{\min}^{-1}(t') dt' 
+ \delta^{(1-\varepsilon_0)(p-1)-1} \mu_{\min}^{-b_{(\alpha|+2)}} (t) \left( \sqrt{\bar{E}_{1,\leq|\alpha|+2}(t, u)} + \sqrt{\bar{E}_{2,\leq|\alpha|+2}(t, u)} \right).
$$

For $\bar{Z}^\alpha e_\mu$, substituting (6.39) and (6.35) into (6.41) yields

$$
e_\mu = -L \left( \frac{1}{2} pc^4 \varphi_0 - p^{-1} \right) T \mathcal{L}_\varphi + cT \left( \frac{1}{2} pc^4 \varphi_0 - p^{-1} T \mathcal{L}_\varphi \right) tr \gamma + cT \mu \Delta \mathcal{C} - \frac{1}{2} pc^4 \varphi_0 - p^{-1} (T \mathcal{L}_\varphi \varphi_0) + \text{better terms.}
$$

Here and below “better terms” stands for the terms with either lower order derivatives or higher smallness orders of $\delta$, which can be neglected in the related estimates. Then

$$
\int_1^t \delta^\cdot \| \bar{Z}^\alpha e_\mu \|_{L^2(\Sigma_\mu^\alpha)} dt' 
\lesssim \delta^{(1-\varepsilon_0)2p+\frac{1}{2}} + \delta^{(1-\varepsilon_0)(p-1)-1} \mu_{\min}^{-b_{(\alpha|+2)}} (t) \left( \sqrt{\bar{E}_{1,\leq|\alpha|+2}(t, u)} + \sqrt{\bar{E}_{2,\leq|\alpha|+2}(t, u)} \right).
$$

For the other left terms in (6.46), with the help of (2.41) and (6.34), one has

$$
\int_1^t \delta^\cdot \| \text{the other left terms} \|_{L^2(\Sigma_\mu^\alpha)} dt' 
\lesssim \int_1^t \delta^{(1-\varepsilon_0)p} \cdot \delta^\cdot \| E_\mu^\alpha \|_{L^2(\Sigma_\mu^\alpha)} dt' + \text{better terms,}
$$

where the first term on the right hand side of (6.49) comes from the contribution of $\sum_{\beta_1 + \beta_2 = \alpha - 1} \bar{Z}^{\beta_1} [L, \bar{Z}] E_\mu^{\beta_2}$ in (6.46).
Therefore,

\[ \int_{0}^{t} \delta^{\frac{1}{m}} \|e_{\mu}^{\alpha}\|_{L^{2}(\Sigma_{t}^{\mu})} \, dt' \]

\[ \lesssim \int_{0}^{t} \delta^{(1-\varepsilon)a} \cdot \delta^{\frac{1}{m}} \|E_{\mu}^{\alpha}\|_{L^{2}(\Sigma_{t}^{\mu})} \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

(2-c) **Estimate of \( \mu \partial^{\alpha} \frac{\partial}{\partial y} \tilde{\chi} \)**

By (6.2), one has

\[ \int_{0}^{t} \delta \| \mu \partial^{\alpha} \frac{\partial}{\partial y} \tilde{\chi} \|_{L^{2}(\Sigma_{t}^{\mu})} \, dt' \]

\[ \lesssim \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

Finally, substituting the estimates in (2-a), (2-b) and (2-c) into (6.48), we obtain from Gronwall inequality and (6.47) that

\[ \delta \| \mu \partial^{\alpha} \Delta_{\mu} \|_{L^{2}(\Sigma_{t}^{\mu})} + \delta \| \mu \partial^{\alpha} \nabla^{2} \mu \|_{L^{2}(\Sigma_{t}^{\mu})} \]

\[ \lesssim \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{(1-\varepsilon)a} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

Moreover, if there is \( \rho L \) in \( Z^{\alpha} \), by utilizing Lemma 2.11 and 2.35 as in Proposition 6.1, we arrive at

\[ \delta^{[1-(1-\varepsilon)a]p} \| \mu Z^{\alpha} \Delta_{\mu} \|_{L^{2}(\Sigma_{t}^{\mu})} + \delta^{[1-(1-\varepsilon)a]p} \| \mu \partial^{\alpha} \nabla^{2} \mu \|_{L^{2}(\Sigma_{t}^{\mu})} \]

\[ \lesssim \delta^{[1-(1-\varepsilon)a]p} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{[1-(1-\varepsilon)a]p} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

\[ + \delta^{[1-(1-\varepsilon)a]p} \cdot \frac{1}{\mu_{\text{min}}(t')} \int_{0}^{t} \mu_{\text{min}}(t') \, dt' \]

(6.50)
7 Error estimates

In this section, based on the higher order derivative $L^2$ estimates for related quantities in Section 5, we are ready to deal with the last two error terms of (5.11), and hence the energy estimates on (2.36) can be completed.

For convenience, we set

$$\mathcal{M}_k^\alpha = \sum_{j=0}^{|\alpha|-1} (Z_{|\alpha|+1} \cdot (Z_{j+1}^\alpha + (Z_{j+2}^\alpha + \mu_{\min}^{-1}) \cdot \mathcal{M}_k^\alpha, k = 1, 2, 3, (7.1)$$

and use $\mathcal{M}_1^\alpha$ to stand for the summation in $\mathcal{M}_1^\alpha$ excluding the top order derivative terms.

7.1 Error estimates on non-top order derivative terms in (5.13)

\begin{itemize}
  \item Estimate of $\mathcal{M}_1^\alpha$
    
    Utilizing (5.20), (5.25) and (5.28) with Propositions 4.1 and 6.1 one has
    
    $$\delta^{l+s[1-(1-\varepsilon_0)p]} \cdot \mathcal{M}_1^\alpha \cdot L^2(\Sigma_t^\alpha)$$
    
    $$\lesssim \delta^{2-\varepsilon_0} \cdot \left( \delta^2 + \frac{b_{|\alpha|+2}}{2b_{|\alpha|+2} - 1} \cdot \delta^{1-(1-\varepsilon_0)p} \cdot \mathcal{M}_1^\alpha \cdot \tilde{E}_1, \leq |\alpha|+2(t, u) \right)$$
    
    and hence,
    
    $$\delta^{2l+2s[1-(1-\varepsilon_0)p]} \int_{\Omega^\alpha} \mathcal{M}_1^\alpha \cdot (\mathcal{M}_{1}^{\alpha}) \cdot L^2(\Sigma_t^\alpha)$$
    
    $$\lesssim \delta^{2-2\varepsilon_0} \cdot \left( \delta^2 + \frac{b_{|\alpha|+2}}{2b_{|\alpha|+2} - 1} \cdot \delta^{1-(1-\varepsilon_0)p} \cdot \mathcal{M}_1^\alpha \cdot \tilde{E}_1, \leq |\alpha|+2(t, u) \right)$$
    
    $$\left( \delta^2 + \frac{b_{|\alpha|+2}}{2b_{|\alpha|+2} - 1} \cdot \delta^{1-(1-\varepsilon_0)p} \cdot \mathcal{M}_1^\alpha \cdot \tilde{E}_1, \leq |\alpha|+2(t, u) \right)$$
    
    and hence,
    
    $$\delta^{2-2\varepsilon_0} \cdot \left( \delta^2 + \frac{b_{|\alpha|+2}}{2b_{|\alpha|+2} - 1} \cdot \delta^{1-(1-\varepsilon_0)p} \cdot \tilde{E}_2, \leq |\alpha|+2(t, u) \right)$$
    
    $$\left( \delta^2 + \frac{b_{|\alpha|+2}}{2b_{|\alpha|+2} - 1} \cdot \delta^{1-(1-\varepsilon_0)p} \cdot \tilde{E}_2, \leq |\alpha|+2(t, u) \right)$$
    
    \end{itemize}
and hence,

\[
\begin{align*}
\delta^{2l+2s[1-(1-\varepsilon_0)p]} & \int_{D^{1+u}} \mathcal{M}_3^\alpha \left( \delta^{1-(1-\varepsilon_0)p} L \Psi_\gamma^{[\alpha]+1} + \delta L \Psi_\gamma^{[\alpha]+1} \right) \\
\lesssim & \delta^{2-2\varepsilon_0+(1-\varepsilon_0)2p} + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1+(1-\varepsilon_0)p} \tilde{K}_{[\alpha]+2}^b(t, u) + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{F}_{1, [\alpha]+2}^b(t, u) \int_0^u \delta^{1-(1-\varepsilon_0)p} \tilde{F}_{1, [\alpha]+2}(t, u') du' \\
& + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \frac{\delta^{1-(1-\varepsilon_0)p}}{2b_{[\alpha]+2} - 1} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}^b(t, u) + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}(t, u') dt' \\
& + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \frac{1}{2b_{[\alpha]+2} - 1} \delta^{1-(1-\varepsilon_0)p} \tilde{F}_{2, [\alpha]+2}^b(t, u) + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}(t, u') dt'.
\end{align*}
\]

(7.3)

**Estimate of \( \mathcal{M}_3^\alpha \)**

By (5.33), (5.34) and (5.35), Proposition 4.1 and 6.1 we have

\[
\begin{align*}
\delta^{l+s[1-(1-\varepsilon_0)p]} \| \mathcal{M}_3^\alpha \|_{L^2(\Sigma^u)} \\
\lesssim & \delta^{2-2\varepsilon_0-(1-\varepsilon_0)p} + \delta^{1-(1-\varepsilon_0)p} \mu_{\min}^{-b_{[\alpha]+2}}(t) \left( \sqrt{\tilde{E}_{1, [\alpha]+2}^b(t, u)} + \sqrt{\tilde{E}_{2, [\alpha]+2}^b(t, u)} \right)
\end{align*}
\]

and hence,

\[
\begin{align*}
\delta^{2l+2s[1-(1-\varepsilon_0)p]} & \int_{D^{1+u}} \mathcal{M}_3^\alpha \left( \delta^{1-(1-\varepsilon_0)p} L \Psi_\gamma^{[\alpha]+1} + \delta L \Psi_\gamma^{[\alpha]+1} \right) \\
\lesssim & \delta^{2-2\varepsilon_0} + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{F}_{1, [\alpha]+2}^b(t, u') du' \\
& + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \frac{\delta^{1-(1-\varepsilon_0)p}}{2b_{[\alpha]+2} - 1} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}(t, u) + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}(t, u') dt' \\
& + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \frac{1}{2b_{[\alpha]+2} - 1} \delta^{1-(1-\varepsilon_0)p} \tilde{F}_{2, [\alpha]+2}^b(t, u) + \mu_{\min}^{-2b_{[\alpha]+2}}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2, [\alpha]+2}(t, u') dt'.
\end{align*}
\]

(7.4)

**Estimate of \( \mathcal{M}_0^\alpha \)**

Applying (2.36), (2.37), Proposition 4.1 and 6.1 one has

\[
\begin{align*}
\delta^{l+s[1-(1-\varepsilon_0)p]} \| \mathcal{M}_0^\alpha \|_{L^2(\Sigma^u)} \\
\lesssim & \delta^{2-2\varepsilon_0+(1-\varepsilon_0)p-2[1-(1-\varepsilon_0)p]} + \delta^{1-(1-\varepsilon_0)p} \delta^{l+s[1-(1-\varepsilon_0)p]} \| L \Psi_\gamma^{[\alpha]+1} \|_{L^2(\Sigma^u)} \\
& + \delta^{1-(1-\varepsilon_0)p} \mu_{\min}^{-b_{[\alpha]+2}}(t) \left( \sqrt{\tilde{E}_{1, [\alpha]+2}^b(t, u)} + \delta^{1-(1-\varepsilon_0)p} \sqrt{\tilde{E}_{2, [\alpha]+2}^b(t, u)} \right)
\end{align*}
\]
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and hence,

\[
\delta^{2l+2s}[1-(1-\varepsilon_0)p] \int_{D^1,u} \mathcal{M}^0 \left( \delta^{1-(1-\varepsilon_0)p} L \phi_{\gamma}^{\alpha+1} + \delta L \phi_{\gamma}^{\alpha+1} \right) \\
\lesssim \delta^{2-2\varepsilon_0 + (1-\varepsilon_0)2p} + \mu_{\min}^{-2b_{\alpha+2}}(t) \delta^{-1} \int_0^u \delta^{1-(1-\varepsilon_0)p} F_{1,\leq|\alpha|+2}(t, u') du' \\
\quad + \mu_{\min}^{-2b_{\alpha+2}}(t) \frac{1}{2b_{|\alpha|+2}} \delta^{1-(1-\varepsilon_0)p} E_{1,\leq|\alpha|+2}(t, u) + \mu_{\min}^{-2b_{|\alpha|+2}}(t) \delta^{-1-(1-\varepsilon_0)p} \int_1^t \delta^{1-(1-\varepsilon_0)p} E_{2,\leq|\alpha|+2}(t', u) dt'.
\]

(7.5)

\[\delta^l+2s \leq 0 \] and hence,

7.2 Error estimates on top order derivative terms in (5.13)

Note that for the most difficult term \( (Z)_{A_1}^0 \), the number of the top order derivatives is \( |\alpha| \), which means that there will be some terms containing the \( (|\alpha|+1) \)th order derivatives of the deformation tensors. In this case, \( E_{i,\leq|\alpha|+3} \) will appear in the right hand side of (5.11) if one only adopts Proposition 6.1. This leads to that the direct estimate on \( (Z)_{A_1}^0 \) can not be absorbed by the left hand side of (5.11). To overcome this difficulty, we will carefully examine the expression of \( (Z)_{A_1}^j \) and apply the estimates in Section 6 to deal with the top order derivatives of \( \chi \) and \( \mu \). In fact, by substituting \( (Z)_{A_1}^j \) into (5.13), the resulting terms \( R_i^A (d_A Z^\alpha tr_\mu \chi) T \Psi_0^\alpha \) from (5.29) and \( Z^\alpha \Delta \mu T \Psi_0^\alpha \) from (5.27) need to be treated especially.

1) Treatment on \( \int_{D^1,u} R_i^{\alpha+1} tr_\gamma \bar{\chi} T \varphi_\gamma (\delta^{1-(1-\varepsilon_0)p} LR_i^{\alpha+1} \varphi_\gamma + \delta L R_i^{\alpha+1} \varphi_\gamma) \)

Let

\[
\int_{D^1,u} R_i^{\alpha+1} tr_\gamma \bar{\chi} T \varphi_\gamma (\delta^{1-(1-\varepsilon_0)p} LR_i^{\alpha+1} \varphi_\gamma + \delta L R_i^{\alpha+1} \varphi_\gamma) = A_1 + A_2.
\]

At first, we treat \( A_1 \). Integrating by parts yields

\[
A_1 = \delta^{1-(1-\varepsilon_0)p} \int_{D^1,u} R_i^{\alpha+1} tr_\gamma \bar{\chi} T \varphi_\gamma \cdot LR_i^{\alpha+1} \varphi_\gamma \\
= \delta^{1-(1-\varepsilon_0)p} \int_1^t \int_0^u \int_S \left( L + tr_\gamma \chi \right) (R_i^{\alpha+1} tr_\gamma \bar{\chi} T \varphi_\gamma R_i^{\alpha+1} \varphi_\gamma) \\
- \delta^{1-(1-\varepsilon_0)p} \int_1^t \int_0^u \int_S \left( L + tr_\gamma \chi \right) (R_i^{\alpha+1} tr_\gamma \bar{\chi} T \varphi_\gamma R_i^{\alpha+1} \varphi_\gamma) \\
- \delta^{1-(1-\varepsilon_0)p} \int_1^t \int_0^u \int_S R_i^{\alpha+1} tr_\gamma \bar{\chi} LT \varphi_\gamma R_i^{\alpha+1} \varphi_\gamma
\]

\[\chi^1 + \chi^2 + \chi^3.\]

\[\bullet \text{Estimate of } I_\chi^1\]
It follows from the integration by parts that

\[
I_1^1 = \delta^{1-(1-\varepsilon_0)p} \int_1^t \frac{\partial}{\partial t} \int_{\Sigma_i^t} R^{1+1}_i \text{tr}_g \tilde{\chi} T \varphi_\gamma R^{1+1}_i \varphi_\gamma
\]

\[
= -\delta^{1-(1-\varepsilon_0)p} \int_{\Sigma_i^t} R^{1}_i \text{tr}_g \tilde{\chi} T \varphi_\gamma R^{1+2}_i \varphi_\gamma
\]

\[
- \delta^{1-(1-\varepsilon_0)p} \int_{\Sigma_i^t} R^{1}_i \text{tr}_g \tilde{\chi} (R_i T \varphi_\gamma + \frac{1}{2} T \varphi_\gamma \text{tr}_g (R_i) \varphi) R^{1+1}_i \varphi_\gamma
\]

\[
- \delta^{1-(1-\varepsilon_0)p} \int_{\Sigma_i^t} R^{1+1}_i \text{tr}_g \tilde{\chi} T \varphi_\gamma R^{1+1}_i \varphi_\gamma
\]

\[
= I_1^{11} + I_1^{12} + I_1^{13}.
\]

For $I_1^{11}$, by Proposition 6.1 and Lemma 6.3, we arrive at

\[
|I_1^{11}| \leq \mu_{\text{min}}^{2b_{|\alpha|+2}} (t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|\alpha|+2} (t, u) + \mu_{\text{min}}^{-1} (t) \frac{\delta^{1-(1-\varepsilon_0)p}}{\epsilon} \delta^{2-2\varepsilon_0}
\]

\[
+ \mu_{\text{min}}^{-2b_{|\alpha|+2}} (t) \frac{1}{b_{|\alpha|+2}} - \frac{1}{2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|\alpha|+2} (t, u)
\]

\[
+ \mu_{\text{min}}^{-2b_{|\alpha|+2}} (t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|\alpha|+2} (t, u) + \mu_{\text{min}}^{-2b_{|\alpha|+2}} (t) \frac{1}{b_{|\alpha|+2}} - 1 \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2,\leq|\alpha|+2} (t, u)
\]

\[
+ \mu_{\text{min}}^{-2b_{|\alpha|+2}} (t) \epsilon \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|\alpha|+2} (t, u) + \mu_{\text{min}}^{-2b_{|\alpha|+2}} (t) \frac{\delta^{1-(1-\varepsilon_0)p}}{\epsilon} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{2,\leq|\alpha|+2} (t, u)
\]

\[
\text{where } \epsilon > 0 \text{ is a small constant arising from the inequality } ab \lesssim \epsilon a^2 + \frac{1}{\epsilon} b^2.
\]

For $I_1^{12}$, it is a lower order derivative term compared with $I_1^{11}$, which can be estimated similarly.

For $I_1^{13}$, due to $t = 1$, one has

\[
|I_1^{13}| \leq \delta^{1-(1-\varepsilon_0)p} \delta^{\varepsilon_0} ||R^{1+1}_i \text{tr}_g \tilde{\chi}||_{L^2(\Sigma_i^t)} ||R^{1+1}_i \varphi_\gamma||_{L^2(\Sigma_i^t)}
\]

\[
\lesssim \delta^{1-(1-\varepsilon_0)p} \delta^{\varepsilon_0} \cdot \delta^{(1-\varepsilon_0)p} \frac{1}{\delta^{1/2}} \cdot \delta^{1-\varepsilon_0} \frac{1}{\delta^{1/2}}
\]

\[
= \delta^{3-2\varepsilon_0}.
\]

\textbf{• Estimate of } I_1^{2}

By integration by parts, we have that

\[
I_\chi^2 = \delta^{1-(1-\varepsilon_0)p} \int_1^t \left[ \int_0^u (L + \text{tr}_g \chi)(R_i^\alpha \text{tr}_g \tilde{\chi}) T \varphi_\gamma R_i^{\alpha+2} \varphi_\gamma \\
+ \delta^{1-(1-\varepsilon_0)p} \int_1^t \left[ \int_u^T (L + \text{tr}_g \chi)(R_i^\alpha \text{tr}_g \tilde{\chi}) (R_i T \varphi_\gamma + \frac{1}{2} T \varphi_\gamma, \text{tr}_g \tilde{\chi}) R_i^{\alpha+1} \varphi_\gamma \\
- \delta^{1-(1-\varepsilon_0)p} \int_1^t \left[ \int_0^u (R_i) A_i^\delta \text{tr}_g \tilde{\chi} T \varphi_\gamma R_i^{\alpha+1} \varphi_\gamma \\
+ \delta^{1-(1-\varepsilon_0)p} \int_1^t \left[ \int_0^u \left( R_i, \text{tr}_g \tilde{\chi} \right) R_i^{\alpha} T \varphi_\gamma R_i^{\alpha+1} \varphi_\gamma \\
= I_\chi^{21} + I_\chi^{22} + I_\chi^{23} + I_\chi^{24}.
\]

For \( I_\chi^{21} \), by Proposition 6.1, Lemma 6.3 (2.35) and (6.14), one has

\[
|I_\chi^{21}| \lesssim \mu_{\min}^{-2b|\alpha|+2}(t) \frac{1}{b|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t, u) + \frac{\delta^{1+(1-\varepsilon_0)p}}{b|\alpha|+2-\frac{1}{2}} \delta^{2-2\varepsilon_0} \\
+ \mu_{\min}^{-2b|\alpha|+2}(t) \frac{1}{b|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t, u) + \frac{\delta^{1+3\varepsilon_0}}{b|\alpha|+2-\frac{1}{2}} \delta^{2-2\varepsilon_0} \\
+ \delta^{2-2\varepsilon_0} + \mu_{\min}^{-2b|\alpha|+2}(t) \delta \int_1^t \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t', u) dt' \\
+ \delta^{2-2\varepsilon_0} + \mu_{\min}^{-2b|\alpha|+2}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t', u) dt' \\
+ \mu_{\min}^{-2b|\alpha|+2}(t) \frac{1}{2b|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t, u) \\
+ \mu_{\min}^{-2b|\alpha|+2}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1^{b} \lesssim |\alpha|+2(t', u) dt' \\
+ \mu_{\min}^{-2b|\alpha|+2}(t) \delta^{1+(1-\varepsilon_0)}(t') \frac{1}{2b|\alpha|+2-\frac{1}{2}} \delta E_2^{b} \lesssim |\alpha|+2(t, u) \\
+ \mu_{\min}^{-2b|\alpha|+2}(t) \delta^{1+(1-\varepsilon_0)}(t') \frac{1}{2b|\alpha|+2-\frac{1}{2}} \delta E_2^{b} \lesssim |\alpha|+2(t', u) dt' \tag{7.8}
\]

Note that \( I_\chi^{22} \) is a lower order derivative term compared with \( I_\chi^{21} \), \( I_\chi^{23} \) is a better term with higher smallness order of \( \delta \) compared with \( I_\chi^3 \), and \( I_\chi^{24} \) is a lower order derivative term compared with \( I_\chi^{23} \). Then these terms can be treated conveniently.

- **Estimate of \( I_\chi^3 \)**
For $I_\chi^3$, by (5.2) and Lemma 6.3, one has

$$|I_\chi^3| \lesssim \mu_{\min}^{-b_2|\alpha|+2}(t) \frac{1}{b_2|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1, \lesssim \frac{1}{|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1, + \mu_{\min}^{-2\delta}(t) \frac{1}{b_2|\alpha|+2} \delta^{2-2\varepsilon_0}$$

$$+ \mu_{\min}^{-2\delta}(t) \frac{1}{b_2|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1, \lesssim \frac{1}{|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1, + \mu_{\min}^{-2\delta}(t) \frac{1}{b_2|\alpha|+2} \delta^{2-2\varepsilon_0}$$

$$+ \mu_{\min}^{-2\delta}(t) \frac{1}{b_2|\alpha|+2} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_1, \lesssim \frac{1}{|\alpha|+2} \delta^{2-2\varepsilon_0}$$

(7.9)

Secondly, we start to treat $A_2$. Since $\|L_i R_i^{\alpha+1} \varphi_\gamma\|_{L^2(S_i^\alpha)} \lesssim \mu_{\min}^{-b_2|\alpha|+2}(t) \sqrt{\tilde{E}_2, \lesssim |\alpha|+2} \varphi_\gamma$ and $|T \varphi_\gamma| \lesssim (\delta-\varepsilon_0)$, $A_2$ is a better term with higher smallness order of $\delta$ compared with $I_\chi^3$, which can be analogously estimated.

(2) Treatment on $\delta^{2l} \int_{D_i^{\alpha+1}} R_i^l T^{l-1} \Delta \mu T \varphi_\gamma (\delta^{1-(1-\varepsilon_0)p} L R_i^{\alpha+1} \varphi_\gamma + \Delta L R_i^{\alpha+1} \varphi_\gamma), |\beta|+l = |\alpha|+1$

Let

$$\delta^{2l} \int_{D_i^{\alpha+1}} R_i^l T^{l-1} \Delta \mu T \varphi_\gamma (\delta^{1-(1-\varepsilon_0)p} L R_i^{\alpha+1} \varphi_\gamma + \Delta L R_i^{\alpha+1} \varphi_\gamma) = B_1 + B_2.$$

At first, we treat the troublesome term $B_1$. By integration by parts, one has

$$B_1 = \delta^{2l+1-(1-\varepsilon_0)p} \int_{D_i^{\alpha+1}} R_i^l T^{l-1} \Delta \mu T \varphi_\gamma \cdot L R_i^{\alpha+1} \varphi_\gamma$$

$$= \delta^{2l+1-(1-\varepsilon_0)p} \int_{D_i^{\alpha+1}} \int_{S_i^{\alpha+1}} (L + \text{tr}_\chi)(R_i^l T^{l-1} \Delta \mu T \varphi_\gamma R_i^{\alpha+1} T \varphi_\gamma)$$

$$- \delta^{2l+1-(1-\varepsilon_0)p} \int_{D_i^{\alpha+1}} \int_{S_i^{\alpha+1}} (L + \text{tr}_\chi)(R_i^l T^{l-1} \Delta \mu T \varphi_\gamma R_i^{\alpha+1} T \varphi_\gamma)$$

$$- \delta^{2l+1-(1-\varepsilon_0)p} \int_{D_i^{\alpha+1}} \int_{S_i^{\alpha+1}} R_i^l T^{l-1} \Delta \mu T \varphi_\gamma R_i^{\alpha+1} T \varphi_\gamma$$

$$= I_1^l + I_2^l + I_3^l.$$

• Estimate of $I_1^l$
It follows from the integration by parts that

\[ I_\mu^1 = \delta^{2l-1-(1-\varepsilon_0)p} \int_1^t \frac{\partial}{\partial t} \int_{\Sigma_t^u} R_i^\beta T^{l-1} \Delta \mu T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
= -\delta^{2l-1-(1-\varepsilon_0)p} \int_{\Sigma_t^u} R_i^\beta T^{l-1} \Delta \mu T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
- \delta^{2l-1-(1-\varepsilon_0)p} \int_{\Sigma_t^u} R_i^\beta T^{l-1} \Delta \mu (R_i T \varphi_\gamma + \frac{1}{2} T \varphi_\gamma \nabla (R_i) \nabla (R_i) ) R_i^\beta T^l \varphi_\gamma \\
- \delta^{2l-1-(1-\varepsilon_0)p} \int_{\Sigma_t^u} R_i^\beta T^{l-1} \Delta \mu T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
= I_\mu^{11} + I_\mu^{12} + I_\mu^{13}. \]

For \( I_\mu^{11} \), by (6.2) and Lemma 6.3, we have

\[ |I_\mu^{11}| \lesssim \mu^{-1}(t) \delta^{-2-2\varepsilon_0} \left( \mu^{-2b_{|a|+2}}(t) \delta^{1-[(1-\varepsilon_0)p]} \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|a|+2}(t, u) \right) \\
+ \mu^{-2b_{|a|+2}}(t) \left( \frac{\delta}{b_{|a|+2} - \frac{1}{2}} \right) \mu^{-1}(t) \delta^{1-(1-\varepsilon_0)p} \tilde{E}_{1,\leq|a|+2}(t, u) \\
+ \mu^{-2b_{|a|+2}}(t) \left( \frac{\delta}{b_{|a|+2} - \frac{1}{2}} \right) \tilde{E}_{2,\leq|a|+2}(t, u) \\
+ \mu^{-2b_{|a|+2}}(t) \frac{1}{b_{|a|+2} - 1} \delta^{1-[(1-\varepsilon_0)p]} \tilde{E}_{1,\leq|a|+2}(t, u) \\
+ \mu^{-2b_{|a|+2}}(t) \frac{1}{b_{|a|+2} - 1} \delta_{\tilde{E}}^{1-[(1-\varepsilon_0)p]} \tilde{E}_{1,\leq|a|+2}(t, u). \tag{7.10} \]

Note that \( I_\mu^{12} \) is a lower order derivative term compared with \( I_\mu^{11} \), which can be treated similarly.

For \( I_\mu^{13} \), due to \( t = 1 \), one has

\[ |I_\mu^{13}| \lesssim \delta^{1-\varepsilon_0+1-(1-\varepsilon_0)p} \delta^{1-(1-\varepsilon_0)p} \| \delta^{l-1} R_i^\beta T^{l-1} \Delta \mu \| L^2(\Sigma_t^u) \| \delta^{l} R_i^\beta T^l \varphi_\gamma \| L^2(\Sigma_t^u) \]
\[ \lesssim \delta^{-1+\varepsilon_0+1-(1-\varepsilon_0)p} \cdot \frac{1}{\delta} \cdot \delta^{1-\varepsilon_0-\varepsilon_0} \cdot \frac{1}{\delta} \]
\[ = \delta^{3-2\varepsilon_0+1-(1-\varepsilon_0)p}. \tag{7.11} \]

\* Estimate of \( I_\mu^2 \)

It follows from the integration by parts that

\[ I_\mu^2 = \delta^{2l-1-(1-\varepsilon_0)p} \int_1^t \int_0^u \left( L + \nabla \chi \right) (R_i^\beta T^{l-1} \Delta \mu) T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
+ \delta^{2l-1-(1-\varepsilon_0)p} \int_1^t \int_0^u \left( L + \nabla \chi \right) (R_i^\beta T^{l-1} \Delta \mu) (R_i T \varphi_\gamma + \frac{1}{2} T \varphi_\gamma \nabla (R_i) \nabla (R_i) ) R_i^\beta T^l \varphi_\gamma \\
- \delta^{2l-1-(1-\varepsilon_0)p} \int_1^t \int_0^u \left( R_i \nabla \chi \right) \int \Delta A R_i^\beta T^{l-1} \Delta \mu T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
+ \delta^{2l-1-(1-\varepsilon_0)p} \int_1^t \int_0^u \left( R_i \nabla \chi \right) \int \Delta A R_i^\beta T^{l-1} \Delta \mu T \varphi_\gamma R_i^\beta T^l \varphi_\gamma \\
= I_\mu^{21} + I_\mu^{22} + I_\mu^{23} + I_\mu^{24}. \]
For $I_{21}^\mu$, by Proposition 6.1, Lemma 6.3 (2.55) and (6.34), the resulting estimate is the same as $I_{\chi}^2$.

In addition, note that $I_{22}^\mu$ is a lower order derivative term compared with $I_{21}^\mu$, $I_{23}^\mu$ is a better term with higher smallness order of $\delta$ compared with $I_{\delta}^\mu$, and $I_{24}^\mu$ is a lower order derivative term compared with $I_{23}^\mu$. Then these terms can be estimated easily.

- **Estimate of $I_{\delta}^3$**

For $I_{\delta}^3$, by (6.50) and Lemma 6.3, the resulting estimate is the same as $I_{\chi}^3$.

Secondly, we treat $B_2$. Since $\|\mathcal{L} \alpha^{\alpha+1} \varphi_\gamma \|_{L^2(\Sigma^\alpha_t)} \lesssim \mu_{\min}^{-b_\alpha+2}(t) \sqrt{\bar{E}_{2,\leq|\alpha|+2}(t,u)}$ and $|T \varphi_\gamma| \lesssim \delta^{-\varepsilon_0}$, $B_2$ is a better term with higher smallness order of $\delta$ compared with $I_{\delta}^3$, which can be estimated easily.

All in all, combining (7.2)-(7.5) and (7.6)-(7.11), together with Gronwall inequality (choosing $b_k$ sufficiently large), we finally arrive at

$$
\delta^{1-(1-\varepsilon_0)p} \bar{E}_{1,\leq|\alpha|+2}(t,u) + \delta^{1-(1-\varepsilon_0)p} \bar{F}_{1,\leq|\alpha|+2}(t,u) + \delta \bar{E}_{2,\leq|\alpha|+2}(t,u) + \delta \bar{F}_{2,\leq|\alpha|+2}(t,u) + \bar{K}_{\leq|\alpha|+2}(t,u) \lesssim \delta^{2-2\varepsilon_0}.
$$

8 Shock formation

8.1 Descent scheme

Note that the modified energies and fluxes in (5.12) go to zero when $\mu_{\min} \to 0$, which could not be utilized directly to close the bootstrap assumptions (5.1) by Sobolev embedding theorem. As in (6) and (16)-(17), by the descent scheme, when the orders of derivatives decrease, the powers of $\mu_{\min}$ needed in (5.12) also decrease. Therefore, through some finite steps, the related power of $\mu_{\min}$ could become zero. This implies that the standard energy inequality without any $\mu_{\min}$-weights can be eventually obtained.

Indeed, by (6.2), (6.50) and Lemma 6.3 we have

$$
\delta^{2+2s[1-(1-\varepsilon_0)p]} \int_{D^\alpha,u} T \varphi_\gamma Z^\alpha \Delta \varphi_\gamma \lesssim \delta^{4-2\varepsilon_0} + \mu_{\min}^{-b_\alpha+1}(t) \delta^2 - 2\varepsilon_0 + \mu_{\min}^{-b_\alpha+1}(t) \delta \int_0^u \delta^{1-(1-\varepsilon_0)p} \bar{F}_{1,\leq|\alpha|+1}(t,u') du' \\
+ \mu_{\min}^{-b_\alpha+1}(t) \frac{1}{2b_\alpha+1} \delta \bar{E}_{2,\leq|\alpha|+1}(t,u) + \mu_{\min}^{-b_\alpha+1}(t) \delta^{1-(1-\varepsilon_0)p} \int_1^t \delta \bar{E}_{2,\leq|\alpha|+1}(t',u) dt' 
$$

(8.1)

and

$$
\delta^{2+2s[1-(1-\varepsilon_0)p]} \int_{D^\alpha,u} T \varphi_\gamma Z^{\alpha-1} D \varphi_\gamma \lesssim \delta^{2-2\varepsilon_0} + 2[1-(1-\varepsilon_0)p] + \mu_{\min}^{-b_\alpha+1}(t) \delta^2 - 2\varepsilon_0 + \mu_{\min}^{-b_\alpha+1}(t) \delta \int_0^u \delta^{1-(1-\varepsilon_0)p} \bar{F}_{1,\leq|\alpha|+1}(t,u') du' \\
+ \mu_{\min}^{-b_\alpha+1}(t) \frac{1}{2b_\alpha+1} \delta \bar{E}_{2,\leq|\alpha|+1}(t,u) + \mu_{\min}^{-b_\alpha+1}(t) \delta^{1-(1-\varepsilon_0)p} \int_1^t \delta \bar{E}_{2,\leq|\alpha|+1}(t',u) dt',
$$

(8.2)

where the key point is to choose $b_\alpha+1 = b_\alpha+2 - 1$ in (8.1) and (8.2).
Collecting (8.1) and (8.2), together with Gronwall inequality, the next-to-top order energy inequality can be obtained as follows
\[
\delta^{1-(1-\varepsilon_0)p}\hat{E}^{b}_{1,\leq|\alpha|+1}(t, u) + \delta^{1-(1-\varepsilon_0)p}\tilde{F}^{b}_{1,\leq|\alpha|+1}(t, u) \\
+ \delta\hat{E}^{b}_{2,\leq|\alpha|+1}(t, u) + \delta\tilde{F}^{b}_{2,\leq|\alpha|+1}(t, u) + \tilde{K}^{b}_{\leq|\alpha|+1}(t, u) \\
\lesssim \delta^{2-2\varepsilon_0}.
\] (8.3)

Due to \(b_{|\alpha|+2-n} = b_{|\alpha|+2} - n\), then there exists an \(n_0 \in \mathbb{N}\) such that
\[
\hat{E}^{b}_{i,\leq|\alpha|+2-n_0}(t, u) = \sup_{1 \leq t \leq \tau} \hat{E}^{b}_{i,\leq|\alpha|+2-n_0}(t, u), \quad i = 1, 2.
\]
In this case, we have
\[
\delta^{1-(1-\varepsilon_0)p}\hat{E}^{b}_{1,\leq2N-8-n_0}(t, u) + \delta\hat{E}^{b}_{2,\leq2N-8-n_0}(t, u) \lesssim \delta^{2-2\varepsilon_0}.
\]

Together with the following Sobolev embedding formula on \(S_{t,u}\)
\[
\|f\|_{L^\infty(S_{t,u})} \leq \frac{1}{t} \sum_{|\beta| \leq 2} \|R^\beta f\|_{L^2(S_{t,u})},
\] (8.4)
once has that for \(|\alpha| \leq 2N - 11 - n_0,\)
\[
\delta^{4+s[1-(1-\varepsilon_0)p]}|Z^\alpha \varphi_\gamma| \lesssim \sum_{|\beta| \leq 2} \delta^{|\beta|} \|R^\beta Z^\alpha \varphi_\gamma\|_{L^2(S_{t,u})} \\
\lesssim \delta^{\frac{2}{3}} \left( \sqrt{\hat{E}^{b}_{1,\leq2N-8-n_0}} + \sqrt{\hat{E}^{b}_{2,\leq2N-8-n_0}} \right) \\
\lesssim \delta^{\frac{2}{3}} \sqrt{\delta^{1-2\varepsilon_0}} \\
\lesssim \delta^{1-\varepsilon_0},
\] (8.5)
which closes the bootstrap assumptions (3.1) by choosing \(N\) such that \(2N - 11 - n_0 \geq N\).

### 8.2 Shock formation

According to (2.41), (3.19) and Newton-Leibniz formula, we have
\[
L_\mu(t, u, \vartheta) = \left(1 + \frac{1}{t} \right) \delta^{1-\varepsilon_0} \varphi_1 + O(\delta^{1-\varepsilon_0}(1-\varepsilon_0)p) \\
- \frac{1}{t} \delta^{1-\varepsilon_0} \partial_s \varphi_1 + O(\delta^{1-\varepsilon_0}(1-\varepsilon_0)p) \\
+ O(\delta^{1-\varepsilon_0}(1-\varepsilon_0)p - (1-\varepsilon_0)p)]
\]
\[
= -\frac{1}{2} p \varphi_1^{p-1} \partial_s \varphi_1 \frac{1}{p-1} \delta^{1-\varepsilon_0} p = O(\delta^{1-\varepsilon_0}(1-\varepsilon_0)p) - (1-\varepsilon_0)p].
\]

Hence,
\[
\mu(t, u, \vartheta) = \mu(1, u, \vartheta) + \int_1^t \frac{\tau p}{\tau^p} L_\mu(\tau, u, \vartheta) d\tau \\
= 1 - \frac{p}{2} \delta^{1-\varepsilon_0} \varphi_1^{p-1} \partial_s \varphi_1 \frac{1}{p-1} \left(1 - \frac{1}{1} \right) + O(\delta^{1-\varepsilon_0}(1-\varepsilon_0)p).
\]

Therefore, for sufficiently small \(\delta > 0\), \(t\) can not be greater than \(t^*\), otherwise \(\mu\) would be negative in terms of \((1,5)\). In other words, for (1.1) with (1.2), under assumption (1.5), the shock is formed before \(t^*\).
Remark 8.1. Near the blow-up time of (1.1) with (1.2), i.e., \( \mu \to 0 \), then \( \mu < \frac{1}{10} \) holds. By (3.21), one has

\[
-cTc = L\mu - \mu c^{-1}Lc
\]

\[
\lesssim -\delta^{-(1-\varepsilon_0)p} + O(\delta^{(1-\varepsilon_0)p-1})
\]

This derives

\[
c^{-1}\mu |\hat{T}\varphi_0| = |T\varphi_0| = \left| \frac{-cTc}{2pc_0^{4-p-1}} \right| \gtrsim \delta^{-(1-\varepsilon_0)p}.
\]

It follows from \( g(\hat{T}, \hat{T}) = 1 \) that

\[
\delta^{-(1-\varepsilon_0)p} \mu^{-1} \lesssim |\hat{T}^i \partial_i \varphi_0| \lesssim \left( \sum_{i=1}^{3} \left( \hat{T}^i \right)^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{3} (\partial_i \varphi_0)^2 \right)^{\frac{1}{2}} = |\nabla_x \varphi_0|.
\]

Therefore, for the solution \( \phi \) of (1.1) with (1.2),

\[
|\partial^2 \phi| \to +\infty \text{ as } \mu \to 0.
\]

Appendix

A The existence of short pulse initial data

In this section, we give the existence of short pulse initial data in (1.2) which satisfy the outgoing constraint condition (1.3). Indeed, for any fixed \( \phi_0 \in C^\infty_0 ((-1,0) \times \mathbb{S}^2) \), motivated by Appendix in [10], we choose \( \phi_1 \) as

\[
\phi_1 = -\partial_s \phi_0 - \phi_0 \delta - \frac{1}{2(p+1)}(-\partial_s \phi_0)^{p+1} \delta^{(1-\varepsilon_0)p}.
\]  \hspace{1cm} (A.1)

It is pointed out that although the selection of \( \phi_1 \) depends on \( \delta \), the \( C^\infty \)-norm of \( \phi_1 \) is actually independent of \( \delta \).

Due to \( \partial_t^2 \phi = c^2 \Delta \phi = c^2 (\partial_t^2 \phi + \frac{2}{c} \partial_c \phi + \frac{1}{c^2} \Delta \phi) \) from (1.1) and \( r = 1 + s\delta \), one then has

\[
(\partial_t + \partial_c)^2 \phi(1, x)
\]

\[
= 2(\partial_s \phi_1 + \partial_s^2 \phi_0)\delta^{1-\varepsilon_0} + 2\partial_s \phi_0 \delta^{1-\varepsilon_0} - \phi_1^p \partial_s^2 \phi_0 \delta^{(1-\varepsilon_0)p-\varepsilon_0} + O(\delta^{2-\varepsilon_0-2\max\{0,1-(1-\varepsilon_0)p\}})
\]

\[
= 2\partial_s \left[ \phi_1 + \partial_s \phi_0 + \phi_0 \delta + \frac{1}{2(p+1)}(-\partial_s \phi_0)^{p+1} \delta^{(1-\varepsilon_0)p} \right] \delta^{1-\varepsilon_0} + O(\delta^{2-\varepsilon_0-2\max\{0,1-(1-\varepsilon_0)p\}}) \hspace{1cm} (A.2)
\]

and

\[
(\partial_t + \partial_c) \phi(1, x)
\]

\[
= (\phi_1 + \partial_s \phi_0) \delta^{1-\varepsilon_0}
\]

\[
= O(\delta^{\min\{1,1-\varepsilon_0\}}) \delta^{1-\varepsilon_0}
\]

\[
= O(\delta^{2-\varepsilon_0-\max\{0,1-(1-\varepsilon_0)p\}}). \hspace{1cm} (A.3)
\]
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