Numerical and exact analyses of Bures and Hilbert–Schmidt separability and PPT probabilities

Paul B. Slater 1

Received: 23 May 2019 / Accepted: 17 August 2019 / Published online: 22 August 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
We employ a quasirandom methodology, recently developed by Martin Roberts, to estimate the separability probabilities, with respect to the Bures (minimal monotone/statistical distinguishability) measure, of generic two-qubit and two-rebit states. This procedure, based on generalized properties of the golden ratio, yielded, in the course of almost seventeen billion iterations (recorded at intervals of five million), two-qubit estimates repeatedly close to nine decimal places to $\frac{25}{341} = \frac{5^2}{11 \cdot 31} \approx 0.073313783$. However, despite the use of over twenty-three billion iterations, we do not presently perceive an exact value (rational or otherwise) for an estimate of 0.15709623 for the Bures two-rebit separability probability. The Bures qubit–qutrit case—for which Khvedelidze and Rogojin gave an estimate of 0.0014—is analyzed too. The value of $\frac{1}{715} = \frac{1}{5 \cdot 11 \cdot 13} \approx 0.00139860$ is a well-fitting value to an estimate of 0.00139884. Interesting values ($\frac{16}{12375} = \frac{4^2}{3^2 \cdot 5^3 \cdot 11}$ and $\frac{625}{109531136} = \frac{5^4}{2^{12} \cdot 11^2 \cdot 13 \cdot 17}$) are conjectured for the Hilbert–Schmidt (HS) and Bures qubit–qudit ($2 \times 4$) positive-partial-transpose (PPT) probabilities. We re-examine, strongly supporting, conjectures that the HS qubit–qutrit and rebit–retrit separability probabilities are $\frac{27}{1000} = \frac{3^3}{25 \cdot 5^3}$ and $\frac{860}{6561} = \frac{2^{12} \cdot 5 \cdot 43}{3^8}$, respectively. Prior studies have demonstrated that the HS two-rebit separability probability is $\frac{29}{64}$ and strongly pointed to the HS two-qubit counterpart being $\frac{8}{33}$ and a certain operator monotone one (other than the Bures) being $1 - \frac{256}{27 \pi^2}$.

Keywords Separability probabilities · Hilbert–Schmidt measure · Random matrix theory · PPT probabilities · Bures measure · Quasirandom sequences

1 Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, USA
1 Introduction

It has now been formally proven by Lovas and Andai [1, Thm. 2] that the separability probability with respect to Hilbert–Schmidt (flat/Euclidean/Frobenius) measure [2], [3, Sec. 13.3] of the nine-dimensional convex set of two-rebit states [4] is \( \frac{29}{64} = \frac{29}{32} \). (“For quantum mechanics defined over real vector spaces, the simplest composite systems are two-rebits systems” [4].) Additionally, the multifaceted evidence [5–12]—including a recent “master” extension [5,13] of the Lovas–Andai framework to generalized two-qubit states—is strongly compelling that the corresponding value for the 15D convex set of two-qubit states is \( \frac{33}{8} = \frac{21}{3} \) [with that of the 27D convex set of two-quaternionic bits being \( \frac{26}{323} = \frac{23}{19} \) (cf. [14]), among other still higher-dimensional companion random matrix-related results]. A still further extension to the use of induced measures—reducing to the Hilbert–Schmidt case for the case \( k = 0 \)—has been found [13, Sec. XII]—yielding, for example, \( \frac{61}{143} \) for \( k = 1 \). (The parameter \( k \) is the difference \( k = K - N \) between the dimensions \( K, N, \) with \( K \geq N \) of the subsystems of the pure state bipartite system in which the density matrix is regarded as being embedded [15].)

Further, appealing hypotheses parallel to these rational-valued results have been advanced—based on extensive sampling—that the Hilbert–Schmidt separability probabilities for the 35D qubit–qutrit and 20D rebit–retrit states are \( \frac{27}{1000} = \frac{3^3}{2^3} \) and \( \frac{860}{6561} = \frac{2^{2.5} \cdot 43}{3^6} \), respectively [13, Eqs. (15), (20)] [7, Eq. (33)]. (These will be further examined in Sect. 3 below.)

Certainly, one can, however, still aspire to a yet greater “intuitive” understanding of these assertions, particularly in some “geometric/visual” sense [cf. [16–21]], as well as further formalized proofs. It would be of interest, as well, to compare/contrast these finite-dimensional studies with those other quantum-information-theoretic ones, presented in the recent comprehensive volume of Aubrun and Szarek [22], in which the quite different concepts of asymptotic geometric analysis are employed.

By a separability probability, in the above discussion, we mean the ratio of the volume of the separable states to the volume of all (separable and entangled) states, as proposed, apparently first, by Życzkowski, Horodecki, Sanpera and Lewenstein [23] (cf. [24–27]). The present author was, then, led—pursuing an interest in “Bayesian quantum mechanics” [28,29] and the concept of a “quantum Jeffreys prior” [30]—to investigate how such separability probabilities might depend upon the choice of various possible measures on the quantum states [24].

1.1 Partitioning of separability/PPT probabilities

1.1.1 Hilbert–Schmidt and Bures cases

A phenomenon apparently restricted to the Hilbert–Schmidt \( (k = 0) \) case of induced measure is that the positive-partial-transpose (PPT) states are equally divided probability-wise between those for which the determinant \( |\rho^{PT}| \) of the partial transpose of the density matrix \( (\rho) \) exceeds the determinant \( |\rho| \) of the density matrix itself.
and vice versa. (Also, along somewhat similar lines, the Hilbert–Schmidt PPT probability for minimally degenerate [having a single zero eigenvalue] states is half that for nondegenerate states [16]. The PPT property is, of course, equivalent—by the Peres–Horodecki criterion—to separability for $4 \times 4$ and $6 \times 6$ density matrices [3, Sec. 16.6.C].) Quite contrastingly, based on some 122,000,000 two-qubit density matrices randomly generated with respect to Bures measure, of the 8,945,951 separable ones found, 5,894,648 of them (that is, 65.89%) had $|\rho_{PT}| > |\rho|$, clearly distinct from simply 50% (cf. [6, Tabs. 1, 2] [31]).

### 1.1.2 Induced measures, in general

A formula for that part, $Q(k, \alpha)$, of the total separability probability, $P_{sep}(k, \alpha)$, for generalized (real $[\alpha = 1]$, complex $[\alpha = 2]$, quaternionic $[\alpha = 4]$,...) two-qubit states endowed with random induced measure for which the determinantal inequality $|\rho_{PT}| > |\rho|$ holds was given in [32, p. 26]. It took the form $Q(k, \alpha) = G_k^1(\alpha)G_k^2(\alpha)$, for $k = -1, 0, 1, \ldots, 9$. (The factors $G_k^2(\alpha)$ are sums of polynomial-weighted generalized hypergeometric functions $p F_{p-1}$, $p \geq 7$, all with argument $z = \frac{27}{64}$.) Here $\rho$ denotes a $4 \times 4$ density matrix, obtained by tracing over the pure states in $4 \times (4+k)$-dimensions, and $\rho_{PT}$, its partial transpose. Further, $\alpha$ is a Dyson-index-like parameter with $\alpha = 1$ for the standard (15D) convex set of (complex) two-qubit states.

Further, in the indicated ($k = 0$) Hilbert–Schmidt case, we can apparently employ the formula [32, p. 26]

$$P_{sep/PPT}(0, \alpha) = 2 Q(0, \alpha)$$

$$= 1 - \frac{\sqrt{\pi} 2^{-\frac{9\alpha}{2} + \frac{5}{2}} \Gamma \left( \frac{3(\alpha+1)}{2} \right) \Gamma \left( \frac{5\alpha}{4} + \frac{19}{8} \right) \Gamma \left( 2\alpha + 2 \right) \Gamma \left( \frac{5\alpha}{2} + 2 \right)}{\Gamma(\alpha)} \times \tilde{F}_5 \left( \frac{1}{2}, 1, \frac{3}{2}, \frac{5\alpha}{4} + 1, \frac{1}{4}(5\alpha + 6), \frac{5\alpha}{4} + \frac{19}{8}, \frac{3(\alpha+1)}{2}; \frac{\alpha + 4}{2}, \frac{5\alpha}{4}, \frac{11}{8}, \frac{1}{4}(5\alpha + 7), \frac{1}{4}(5\alpha + 9), 2(\alpha + 1); 1 \right).$$

(1)

That is, for $k = 0$, we obtain the previously reported Hilbert–Schmidt formulas, with (the real case) $Q(0, 1) = \frac{29}{128}$, (the standard complex case) $Q(0, 2) = \frac{4}{33}$, and (the quaternionic case) $Q(0, 4) = \frac{13}{323}$—the three simply equaling—by the equipartitioning result noted above—$P_{sep}(0, \alpha)/2$. More generally, $Q(k, \alpha)$ gives that portion, for induced measure, parameterized by $k$, of the total separability/PPT probability for which the determinantal inequality $|\rho_{PT}| > |\rho|$ holds [5, Eq. (84)].

### 2 Use of Bures measure

Of particular initial interest was the Bures/statistical distinguishability (minimal monotone) measure [33–37]. (“The Bures metric plays a distinguished role since it is the
only metric which is also monotone, Fisher-adjusted, Fubini–Study-adjusted and Riemannian” [36]. Bej and Deb have recently “shown that if a qubit gets entangled with another ancillary qubit then negativity, up to a constant factor, is equal to square root of a specific Riemannian metric defined on the metric space corresponding to the state space of the qubit” [38].

In [5, Sec. VII.C], we recently reported, building upon analyses of Lovas and Andai [1, Sec. 4], a two-qubit separability probability equal to $1 - \frac{256}{27\pi^2} = 1 - \frac{8^3}{3^3\pi^2} \approx 0.0393251$. This was based on another (of the infinite family of) operator monotone functions, namely $\sqrt{x}$. (The Bures measure itself is associated with the operator monotone function $\frac{1+x}{2}$.) [Let us note that the complementary “entanglement probability” is simply $\frac{256}{27\pi^2} \approx 0.960675$. There appears to be no intrinsic reason to prefer/privilege one of these two forms (separability and entanglement) of probability to the other (cf. [39]). We observe that the variable denoted $K_s = \frac{(s+1)^{s+1}}{s^s}$, equaling $\frac{256}{27} = \frac{4^4}{3^3}$, for $s = 3$, is frequently employed as an upper limit of integration in the Penson–Życzkowski paper, “Product of Ginibre matrices: Fuss–Catalan and Raney distributions” [40, Eqs. (2), (3)].]

Interestingly, Lovas and Andai “argue that from the separability probability point of view, the main difference between the Hilbert–Schmidt measure and the volume form generated by the operator monotone function $x \rightarrow \sqrt{x}$ is a special distribution on the unit ball in operator norm of $2 \times 2$ matrices, more precisely in the Hilbert–Schmidt case one faces a uniform distribution on the whole unit ball and for monotone volume forms one obtains uniform distribution on the surface of the unit ball” [1, p. 2].

### 2.1 Osipov–Sommers–Żyzckowski interpolation formula

Of central importance in our analyses below will be the construction of Osipov, Sommers and Żyzckowski of an interpolation between the generation of random density matrices with respect to Hilbert–Schmidt and those with respect to Bures measures [41, Eq. (24)] (cf. [42, Eq. (33)]). This formula takes the form

$$\rho_x = \frac{(yI + xU)AA^\dagger(yI + xU^\dagger)}{\text{Tr}(yI + xU)AA^\dagger(yI + xU^\dagger)},$$ (2)

where $y = 1 - x$, with $x = 0$ yielding a Hilbert–Schmidt density matrix $\rho_0$, and $x = \frac{1}{2}$, the Bures counterpart $\rho_{1/2}$. Here, $A$ is an $N \times N$ complex-valued random matrix pertaining to the Ginibre ensemble (with real and imaginary parts of each of the $N^2$ entries being independent standard normal random variates). Further, $U$ is a random unitary matrix distributed according to the Haar measure on $U(N)$. (Of course, $N = 4$ in the basic two-qubit case of first interest here.)

It is an intriguing hypothesis that the Bures two-qubit separability probability also assumes a strikingly elegant form (such as the indicated $\frac{8}{33}, 1 - \frac{256}{27\pi^2}$). (“Observe that the Bures volume of the set of mixed states is equal to the volume of an $(N^2 - 1)$-dimensional hemisphere of radius $R_B = \frac{1}{2}$” [3, p. 415]. It is also noted there that
times the area–volume ratio asymptotically increases with the dimensionality $D = N^2 - 1$, which is typical for hemispheres.

### 2.2 Prior estimations of Bures separability probabilities

In the relatively early (2002) work [43], we had conjectured a Bures two-qubit separability probability equal to $\frac{8}{11 \pi} \approx 0.0736881$. But it was later proposed in 2005 [44], in part motivated by the lower-dimensional exact Bures results reported in [33], that the value might be $\frac{1680 \sigma_{Ag}}{\pi^8} \approx 0.07334$, where $\sigma_{Ag} = \sqrt{2} - 1 \approx 0.414214$ is the “silver mean.” Both of these studies [43,44] were conducted using quasi-Monte Carlo procedures, before the development of the indicated Osipov–Sommers–Życzkowski methodology (2) for generating density matrices, random with respect to Bures measure [41]. More recently, in [13, Sec. X.B.1], we reported, using this Ginibre ensemble-based formula (2) an estimate of 0.0733181043 based on 4,372,000,000 realizations, using simply standard (independent) random normal variate generation. (Khvedelidze and Rogojin gave a value of 0.0733 [6, Table 1] [31].)

Performing a parallel (but much smaller) computation in the two-rebit case, based on forty million random density matrices (6,286,209 of them being separable), we obtained a corresponding (slightly corrected) Bures separability probability estimate of 0.1571469. (In doing so, we took, as required, the now real-entred Ginibre matrix $A$ to be $4 \times 5$ [41, Eqs. (24), (28)] and not $4 \times 4$ as in the two-qubit calculation.)

### 2.3 Application of quasirandom methodology to Bures two-rebit and two-qubit cases

We now importantly examine the question of whether Bures two-qubit and two-rebit separability probability estimation can be accelerated—with superior convergence properties—by, rather than using, as typically done, independently generated normal variates for the Ginibre ensembles at each iteration, making use of normal variates jointly generated by employing low-discrepancy (quasi-Monte Carlo) sequences [45]. In particular, we have employed an “open-ended” sequence (based on extensions of the golden ratio [46]) recently introduced by Martin Roberts in the detailed presentation “The Unreasonable Effectiveness of Quasirandom Sequences” [47].

Roberts notes: “The solution to the $d$-dimensional problem, depends on a special constant $\phi_d$, where $\phi_d$ is the value of the smallest, positive real-value of $x$ such that”

$$x^{d+1} = x + 1,$$

(3)

($d = 1$, yielding the golden ratio, and $d = 2$, the “plastic constant” [48]). The $n$th terms in the quasirandom (Korobov) sequence take the form

$$(\alpha_0 + n\alpha) \mod 1, n = 1, 2, 3, \ldots$$

(4)

where we have the $d$-dimensional vector,
The additive constant $\alpha_0$ is typically taken to be 0. “However, there are some arguments, relating to symmetry, that suggest that $\alpha_0 = \frac{1}{2}$ is a better choice,” Roberts observes.

These points (4), uniformly distributed in the $d$-dimensional hypercube $[0, 1]^d$, can be converted to (quasirandomly distributed) normal variates, required for implementation of the Osipov–Sommers–Życzkowski formula (2), using the inverse of the cumulative distribution function [49, Chap. 2]. Impressively, in this regard, Henrik Schumacher developed for us a specialized algorithm that accelerated the default Mathematica command InverseCDF for the normal distribution approximately ten-fold, as reported in the highly discussed post [50]—allowing us to vastly increase the realization rate.

We take $d = 36$ and 64 in the Roberts methodology, using the Osipov–Sommers–Życzkowski (real and complex) interpolation formulas to estimate the Bures two-rebit and two-qubit separability probabilities, respectively. In the two-qubit case, 32 of the 64 variates are used in generating the Ginibre matrix $A$ and the other 32, for the unitary matrix $U$. (A subsidiary question—which appeared in the discussion with Roberts [48]—is the relative effectiveness of employing—to avoid possible “correlation” effects—the same 32D sequence but at different $n$’s for $A$ and $U$, rather than a single 64D one, as pursued here. A small analysis of ours in this regard did not indicate this to be a meritorious approach.) In the two-rebit case, 20 variates are used to generate the $4 \times 5$ matrix $A$ and the other 16 for an orthogonal $4 \times 4$ matrix $O$.

In Figs. 1 and 2, we show the development of the Bures separability probability estimation procedure in the two cases at hand. (Much earlier versions of these [$\alpha_0 = \frac{1}{2}$] plots—together with [less intensive] estimates using $\alpha_0 = 0$—were displayed as Figs. 5 and 6 in [13].)
Numerical and exact analyses of Bures and Hilbert–Schmidt…

2.3.1 Two-qubit Bures analysis

Using the indicated, possibly superior parameter value \( \alpha_0 = \frac{1}{2} \) in (4), this quasirandom/normal-variate-generation procedure yielded a two-qubit estimate, based on 16,895,000,000 iterations, of 0.073313759. This is closely fitted by the two (themselves very near) values \( \frac{25}{341} \approx 0.07331378299 \) and (as suggested by the WolframAlpha.com site) \( \frac{\sqrt{51}}{\pi^4} \approx 0.07331377752 \). (Informally, Charles Dunkl wrote: “I would hate to think that the answer is \( \frac{\sqrt{51}}{\pi^4} \)—that is just ugly. One hopes for a rational number.”) At 10,850,000,000 iterations, interestingly, the estimate of 0.0733137814 matched \( \frac{25}{341} \) to nearly eight decimal places. The estimate of 0.0733137847 obtained at the considerably smaller number of iterations of 1,445,000,000 was essentially as close too. The Hilbert–Schmidt measure counterpart is (still subject to formal proof) essentially known to be \( \frac{8}{33} = \frac{2^3}{\pi^4} \) [5–12].

2.3.2 Two-rebit Bures analysis

In the two-rebit instance, we obtained a Bures estimate, based on 23,460,000,000 iterations, of 0.157096234. This is presumably, at least as accurate as the considerably, just noted, smaller sample based two-qubit estimate—apparently corresponding to \( \frac{25}{341} \). Nevertheless, we do not presently perceive any possible exact—rational or otherwise—fits to this estimate.

While the Hilbert–Schmidt two-rebit separability probability has been proven by Lovas and Andai to be \( \frac{29}{64} = \frac{29}{29} \) [1, Thm. 2], somewhat similarly to this Bures result, the two-rebit separability probability, 0.2622301318, based on the other monotone (\( \sqrt{x} \)) measure did not seem to have an obvious exact underlying formula.
3 Examination of Hilbert–Schmidt Qubit–Qutrit and Rebit–Retrit separability conjectures

3.1 Prior studies

Based on extensive (standard) random sampling of independent normal variates, in [13, Eqs. (15), (20)], we have conjectured that the Hilbert–Schmidt separability probabilities for the 35D qubit–qutrit and 20D rebit–retrit states are (also interestingly rational-valued) \( \frac{27}{1000} = \frac{3^3}{27.53} = 0.027 \) and \( \frac{860}{6561} = \frac{2^2 \cdot 5 \cdot 43}{38} \approx 0.1310777596 \), respectively. In particular, on the basis of 2,900,000,000 randomly generated qubit–qutrit density matrices, an estimate (with 78,293,301 separable density matrices found) was obtained, yielding an associated separability probability of 0.026997690. (Milz and Strunz had given a confidence interval of 0.02700 \( \pm \) 0.00016 for this probability [7, Eq. (33)], while Khvedelidze and Rogojin reported an estimate of 0.0270 [6, Tab. 1]—but also only 0.0014 for the Bures counterpart [Sec. 7].) Further, on the basis of 3,530,000,000 randomly generated rebit–retrit density matrices, with respect to Hilbert–Schmidt measure, an estimate (with 462,704,503 separable density matrices found) was obtained for an associated separability probability of 0.1310777629. The associated 95% confidence interval is [0.131067, 0.131089].

3.2 New studies

Applying the quasirandom methodology here to further appraise this pair of conjectures, we obtain Figs. 3 and 4. (We take the dimensions \( d \) of the sequences of normal variates generated to be 72 and 42, respectively.)

Interestingly, as shown in Fig. 1, we observe some drift away—with increasing iterations—from early particularly close fits to the two conjectures. But, as shown in Fig. 1—assuming the validity of the conjectures—we might anticipate the esti-
mates re-approaching more closely the conjectured values. It would seem that any presumed eventual convergence is not simply a straightforward monotonic process—perhaps somewhat comprehensible in view of the very high dimensionalities (72, 42) of the sequences involved. (The last recorded separability probabilities—in these ongoing analyses—were 0.0269923 and 0.1310848, based on 1,850,000,000 and 2,415,000,000 iterations, respectively.)

In [5, App. B], we reported an effort to extend the innovative framework of Lovas and Andai [1] to such qubit–qutrit and rebit–retrit settings. (One aspect of interest pertaining to the original $4 \times 4$ density matrix study of Lovas and Andai [1] was that it (surprisingly) appeared possible in [5] to extend the original Lovas–Andai framework by restricting our analyses to $4 \times 4$ density matrices in which the two $2 \times 2$ diagonal blocks were themselves diagonal.)

4 An eight-dimensional (X-states) rebit–retrit scenario

Along similar lines, let us consider an eight-dimensional (X-states) rebit–retrit scenario, in which now the only nonzero entries of $\rho$ are those on the diagonal and anti-diagonal—so that the two $3 \times 3$ diagonal blocks are themselves diagonal. Also, let us employ the “separability function” framework developed in [51, Eq. (5)], where the variable $\eta = \frac{\rho_{11,66} \rho_{33,44}}{\rho_{11,33} \rho_{66,44}}$ was employed.

Then, with the use of the Mathematica command GenericCylindricalDecomposition—employed to enforce the positivity of leading minors of the density matrix and its partial transpose—we are able to formally establish that the associated rebit–retrit Hilbert–Schmidt separability probability is $\frac{16}{3\pi^2} \approx 0.54038 \ [39]$. (This value also holds for the two-rebit and two-retrit X-states, while $\frac{2}{5}$ is the two-qubit X-states probability [13, Sec. VIII].) The value $\frac{16}{3\pi^2}$ is obtained—through integration using the output of this Mathematica command—by taking the ratio of
\[
\int_{\eta=0}^{1} \frac{\pi \eta (-3\eta^2 + (\eta + 4)\eta \log(\eta) + \log(\eta) + 3)}{40320(\eta - 1)^5} \, d\eta = \frac{\pi}{967680} = \frac{\pi}{2^{10} \cdot 3^3 \cdot 5 \cdot 7} \tag{6}
\]

to

\[
\int_{\eta=0}^{1} \frac{\pi \sqrt{\eta} (-3\eta^2 + (\eta + 4)\eta \log(\eta) + \log(\eta) + 3)}{40320(\eta - 1)^5} \, d\eta = \frac{\pi^3}{5160960} = \frac{\pi^3}{2^{14} \cdot 3^2 \cdot 5 \cdot 7}, \tag{7}
\]

where \(\sqrt{\eta}\) plays the role of separability function and is the added factor—that is, \(\eta = (\sqrt{\eta})^2\)—in the first of the two integrands immediately above.

### 5 Application in higher dimensions of master Lovas–Andai generalized two-qubit formulas

We investigated extending this eight-dimensional rebit–retrit analysis to a ten-dimensional one, by replacing two previously zero entries, so that the two off-diagonal \(3 \times 3\) blocks now themselves form \(X\)-patterns. The counterpart of the denominator function (7) is, then,

\[
\int_{\eta=0}^{1} \frac{\pi \eta (3(\eta + 1)(\eta(\eta + 8) + 1) \log(\eta) - (\eta - 1)(\eta(11\eta + 38) + 11))}{1209600(\eta - 1)^7} \, d\eta = \frac{\pi}{29030400} = \frac{\pi}{2^{11} \cdot 3^4 \cdot 5^2 \cdot 7}, \tag{8}
\]

We, then, need to find the appropriate separability function—corresponding to \(\sqrt{\eta}\) in (6)—to insert into this integrand—for the numerator—to complete the calculation of the separability probability ratio. In this regard, we were able to, preliminarily, utilize a sub-optimal separability function (based on the enforcement of the positivity of the determinant of the \(5 \times 5\) leading submatrix of the partial transpose—but not yet the full determinant),

\[
\frac{2 \left(\sqrt{(1 - \eta)\eta} + \sin^{-1}(\sqrt{\eta})\right)}{\pi}, \tag{9}
\]

which yields an upper bound on the separability probability of \(\frac{\sqrt{19}}{5} = 0.809144\).

Then—using the full determinant—we were able to construct the actual separability function [52,53],

\[
\frac{2 \left(\varepsilon^2 (4\text{Li}_2(\varepsilon) - \text{Li}_2 (\varepsilon^2)) + \varepsilon^4 (-\tanh^{-1}(\varepsilon)) + \varepsilon^3 - \varepsilon + \tanh^{-1}(\varepsilon)\right)}{\pi^2 \varepsilon^2}, \tag{10}
\]
where the dilogarithm is indicated and $\epsilon^2 = \eta$. The corresponding separability probability was, then, shown to be [53]

$$\frac{272}{45\pi^2} \approx 0.612430.$$  \hspace{1cm} (11)

(We have also found very strongly convincing numerical evidence that the same separability probability holds, if instead of considering ten-dimensional rebit–retrit scenarios in which the two off-diagonal $3 \times 3$ blocks have $X$-patterns, one considers that the two diagonal $3 \times 3$ blocks do.)

Further, it appears remarkable that the ten-dimensional rebit–retrit separability function (10) turned out to be completely identical with the (polylogarithmic) Lovas–Andai two-rebit function $\tilde{\chi}_1(\epsilon)$ [5, Eq. (2)] [1, Eq. (9)].

Then, in light of this finding, it appears reasonable to entertain an hypothesis that the Lovas–Andai two-qubit function $\tilde{\chi}_2(\epsilon) = \frac{1}{3} \epsilon^2 (4 - \epsilon^2)$ and two-quaterbit function $\tilde{\chi}_4(\epsilon) = \frac{1}{35} \epsilon^4 (15 \epsilon^4 - 64 \epsilon^2 + 84)$ play parallel roles when the associated sets of density matrices share the same zero–nonzero pattern as the two ten-dimensional sets of rebit–retrit density matrices just considered (those with either the two off-diagonal or the two diagonal $3 \times 3$ blocks having $X$-patterns).

Pursuing such an hypothesis, and employing polar and “hyper-polar” coordinates in the very same manner as was done in [5], we can readily perform computations, in these higher-dimensional settings, leading to a presumptive qubit–qutrit separability probability of $\frac{5}{3} \left(\frac{112\pi^2}{3} - 1105\right) = \frac{8962661573}{4725} - 192192\pi^2 \approx 0.67696$ [54]—that is, without simply assuming the applicability of $\tilde{\chi}_2(\epsilon)$, but have only obtained a value of 0.67696 [54].

**6 Enlarged two-retrit $X$-states**

It has been established—as previously noted (Sect. 4)—that the Hilbert–Schmidt separability PPT probabilities are all equal to $\frac{16}{3\pi^2}$ for the two-rebit, rebit–retrit and two-retrit $X$-states. Then, continuing along the lines we have just been investigating, we considered a scenario in which the two-retrit $X$-states gained a nonzero $(1,2)$-entry. Then, we, in fact, were able to determine that the Hilbert–Schmidt PPT probability for this scenario was $\frac{65}{36\pi}$, making use of a separability function $\frac{8}{3\pi u} \left(\sqrt{1-u^2}u^2-\sqrt{1-u^2}+1\right)^2$, where $u = \sqrt{\frac{\rho_{33}}{\rho_{11}\rho_{99}}}$. (We found identical results when the entry chosen to be nonzero was the $(1,4)$—and not the $(1,2)$—one.)
In Table 1 of their recent study, “On the generation of random ensembles of qubits and qutrits: computing separability probabilities for fixed rank states” [6], Khvedelidze and Rogojin report an estimate (no sample size being given) of 0.0014 for the separability probability of the 35D convex set of qubit–qutrit states. We undertook a study of this issue, once again employing the quasirandom methodology advanced by Roberts (with the sequence dimension parameter \(d\) now equal to 144 = 2 \cdot 72), in implementing the Osipov–Sommers–Życzkowski formula (2) given above with \(x = \frac{1}{2}\). (For the companion Bures rebit–retrit estimation, we would have a smaller \(d\), that is, 78—but given our Bures two-rebit analysis above (Sect. 2.3.2), we were not optimistic in being able to advance a possible exact value.) In Fig. 5, we show a (scaled) plot of our corresponding computations. The estimates—recorded at intervals of one million—are in general agreement with the reported value of Khvedelidze and Rogojin. The last value (after 3174 million iterations) was \(\frac{1479997}{1058000000} = \approx 0.001398863\). This can be well fitted by \(\frac{1}{715} = \frac{1}{5 \cdot 11 \cdot 13} \approx 0.00139860\).

7 Bures qubit–qutrit analysis

To estimate the Bures qubit–qudit (“ququart”) bipartite (2 \(\times\) 4) PPT probability, we employed a 256D quasirandom sequence, obtaining 4760 PPT density matrices in 830 million realizations, yielding an estimated probability of 5.7349398 \cdot 10^{-6}. An interesting candidate for a possible corresponding exact value is \(\frac{625}{109531136} = \approx 5.70614003 \cdot 10^{-6}\) (Fig. 6).

For the Bures two-qutrit scenario, employing a 324D sequence, only 43 PPT density matrices were generated in 678 million realizations, yielding an estimate of 6.3421829 \cdot 10^{-8} (Fig. 7). (It would be of interest to relate this last very small PPT probability estimation to the asymptotic analyses of Aubrun and Szarek [22], as well as Ye [55].)
Fig. 6 Qubit–qudit (2 × 4) Bures PPT probability estimates—divided by \( \frac{625}{109531136} = \frac{25^2}{2^{12} \cdot 11^2 \cdot 13 \cdot 17} \)—as a function of the number of iterations of the quasirandom procedure, using \( \alpha_0 = \frac{1}{2} \). Estimates are recorded at intervals of one million iterations.

Fig. 7 Two-qutrit Bures PPT probability estimates as a function of the number of iterations of the quasirandom procedure, using \( \alpha_0 = \frac{1}{2} \). Estimates are recorded at intervals of one million iterations.

8 Higher-dimensional Hilbert–Schmidt analyses

Further, in [13, Sec. 3.5], we had suggested Hilbert–Schmidt PPT probability hypotheses for the 2 × 4 and 2 × 5 qubit–qudit systems of 1612375 = \( \frac{4^2}{3^2 \cdot 5 \cdot 11} \) ≈ 0.001292929 and \( \frac{125}{790016} = \frac{5^3}{2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11} \) ≈ 0.0000260959, and \( \frac{201}{8192} = \frac{3 \cdot 67}{2^3} \) ≈ 0.0245361 and \( \frac{29058}{9765625} = \frac{2 \cdot 3 \cdot 29 \cdot 167}{5^10} \approx 0.00297554 \) for their respective rebit–redit analogues.

For the Hilbert–Schmidt 2 × 4 qubit–qudit and two-qutrit scenarios, using the quasirandom procedure introduced by Martin Roberts [47], we have obtained PPT probability estimates of 0.0012928963 and 0.00010275452 based on 2104 and 1768 million iterations, respectively (Figs. 8 and 9). In further regard to Hilbert–Schmidt two-qutrit probabilities, an estimate of 0.00010218 based on 100 million random realizations was reported in Sec. III.A of “Invariance of Bipartite Separability and PPT-Probabilities over Casimir Invariants of Reduced States” [56]. (An intriguing
possible corresponding exact value is \(\frac{323}{3161088} = \frac{17.19}{2^{10.32.73}} \approx 0.000102180009\)—or \(\frac{11}{107653} = \frac{11}{7^{2.133}} \approx 0.000102180153\.)

8.1 Use of realignment criterion for (bound-)entanglement estimations

Also, in an auxiliary 2 × 4 qubit–qudit analysis, based on 1200 million iterations, use of the realignment criterion [57] yielded an estimate of 0.00023410917 for the bound-entangled probability and 0.94234319 (conjecturally, \(\frac{589}{625} = \frac{17.31}{5^3} \approx 0.9424\)) for the entanglement probability, in general. (The PPT probability was, once again, well fitted—to almost five decimal places—by \(\frac{16}{12375}\).) In that analysis, we were not able to detect any finite probability at all of genuinely tripartite entanglement using the Greenberger–Horne–Zeilinger test set out in Example 3 in [58]. However, in a parallel two-qutrit study, the realignment test for entanglement was not passed by any randomly generated states (cf. [59, Sec. IV]).

8.2 The question of optimality of 64D low-discrepancy sequences

It may be of interest to the reader to here include a response of Martin Roberts to a query as to whether to calculate a 64D integral, it is optimal or not to use a 64D low-discrepancy sequence, as employed above in the two-qubit case. Roberts interestingly replied: “It depends. In theory, the convergence rate of simple random sampling is O(1/n), whereas for low-discrepancy sequences it is O(\(\log(N)^d / N\)). The log(N)^d term implies that in theory for some large D, and very large N, the convergence rate of quasirandom sequences is inferior to simple random sampling. However, the classic Big O notation ignores two things, which in practice are crucial. (1) Big O notation is for \(N \to \infty\). For finite N, the constants of proportionality play a big role in determining which one is more efficient. (2) it has been found that for many high-
dimensional integrals (especially the finance, computer vision, and natural language processing) although they may outwardly look like high-dimensional functions they are in fact really relatively low-dimensional problems embedded in a high-dimensional manifold. Therefore the pragmatic D in the above expression, is really the 'intrinsic' D. This is why finance options-pricing which is based on integrations over a few hundred dimensions are still more efficient with quasirandom sampling."

9 Concluding remarks

We should stress that the problem of formally deriving the Bures two-rebit and two-qubit separability probabilities, and, thus, testing the candidate value ($\frac{25}{341}$) advanced here (Fig. 1), certainly currently seems intractable—even, it would seem, in the pioneering framework of Lovas and Andai [1]. [Perhaps some formal advances can be made, in such regards, with respect to $X$-states (cf. [60]).]

Let us note that the “master Lovas–Andai” formula for generalized two-qubit Hilbert–Schmidt ($k = 0$) separability probabilities reported in [5, Sec. VIII.A]

$$\tilde{\chi}_{d,0}(\varepsilon) \equiv \tilde{\chi}_d(\varepsilon) = \frac{\varepsilon^d \Gamma(d + 1)^3 \, {}_3\tilde{F}_2\left( \frac{-d}{2}, \frac{d}{2}, d; \frac{d}{2} + 1, \frac{3d}{2} + 1; \varepsilon^2 \right)}{\Gamma\left( \frac{d}{2} + 1 \right)^2},$$

(12)

($\varepsilon$ being a singular value ratio and $d$—not the quasirandom dimension parameter—the random matrix Dyson index) has been recently extended to apply to the still more general class of “induced measures” [15], giving expressions for $\chi_{d,k}(\varepsilon)$ [13]. (Also, we have sought to develop an alternative framework to that of Lovas and Andai, in the context of “Slater separability functions,” but not yet fully successfully [61,62].)

As specific illustrations here of (12)—with the assistance of C. Dunkl—are the formulas [13, Sec. B.3.c]—with $z = \varepsilon^2$—for $\chi_{2,k}$, $\chi_{4,k}$ and $\chi_{6,k}$:

$$\chi_{2,k}(z) = 1 + (1 - z)^{k+1} \left(-1 + \frac{1}{k+3} z\right),$$

$$\chi_{4,k}(z) = 1 + (1 - z)^{k+1} \left(-1 + \frac{1}{k+5} z\right),$$

$$\chi_{6,k}(z) = 1 + (1 - z)^{k+1} \left(-1 + \frac{1}{k+7} z\right).$$
\[ \chi_{4,k}(z) = 1 + (1 - z)^{k+1} \left( -1 - (k + 1)z + \frac{2(2k^2 + 14k + 21)}{(k + 5)(k + 6)} z^2 - \frac{6(k + 3)}{(k + 6)(k + 7)} z^3 \right). \]

\[ \chi_{6,k}(z) = 1 + (1 - z)^{k+1} \left\{ -1 - (k + 1)\frac{(k + 2)z}{2} + \frac{3(3k^4 + 60k^3 + 432k^2 + 1230k + 1264)}{2(k + 7)(k + 8)(k + 9)} z^3 - \frac{6(k + 4)(3k^2 + 33k + 80)}{(k + 8)(k + 9)(k + 10)} z^4 \right. \\
+ \left. \frac{30(k + 4)(k + 5)}{(k + 9)(k + 10)(k + 11)} z^5 \right\}. \]

In Sect. 4 of their recent study [1], Lovas and Andai extended their analyses from one involving the (non-monotone [63]) Hilbert–Schmidt measure to one based on the operator monotone function \( \sqrt{x} \). They were able to conclude (for the case \( d = 1 \) [a Dyson-type random matrix index]) that the applicable “separability function” in this case, \( \tilde{\eta}_d(\varepsilon) \), is precisely the same as the Hilbert–Schmidt counterpart \( \tilde{\chi}_d(\varepsilon) \).

Now, quite strikingly, we obtained [5], using this function, for the two-qubit (\( d = 2 \)) analysis, the ratio of \( \frac{\pi^2}{2} - \frac{128}{27} \) to \( \frac{\pi^2}{2} \), that is,

\[ P_{\text{sep}, \sqrt{x}} = 1 - \frac{256}{27\pi^2} = 1 - \frac{4^4}{3^3\pi^2} \approx 0.0393251. \]  

(We observe that such results—as with the Hilbert–Schmidt value of \( \frac{8}{33} \)—interestingly appear to reach their most simple/elegant in the [standard, 15D] two-qubit setting, where the off-diagonal entries of the density matrix are, in general, complex-valued.)

Lovas and Andai have shown that the two-rebit separability probability based on the operator monotone function \( \sqrt{x} \) is approximately 0.26223001318, asserting “that can be evaluated only numerically”. Nevertheless, we investigated—so far, rather not too productively, as with the Bures two-rebit estimate 0.157096234 above (Sect. 2.3.2)—the possibility of finding an exact, underlying value for this statistic. [Our investigation, in this regard, is reported in [64]. It involved first performing a series expansion of the elliptic and hypergeometric functions in their integrand. We were able to then integrate this series expansion, but only over a restricted range—rather than \([0, \infty]\)—of the two indices. Numerical summation over this restricted set yielded a value of only 0.0042727 (reported in [64]) vs. 0.26223001318.]

It would be of substantial interest to compare/contrast the relative merits of our quasirandom estimations above of the two-rebit and two-qubit Bures separability probabilities in the 36- and 64D settings employed with earlier studies (largely involving Euler-angle parameterizations of \( 4 \times 4 \) density matrices [65]), in which nine- and fifteen-dimensional integration problems were addressed [44,66] (cf. [67]). In the higher-dimensional frameworks used here, the integrands are effectively unity, with each randomly generated matrix being effectively assigned equal weight, while not so in the other cases indicated. In [68], we asked the question “Can ‘experimental data from a quantum computer’ be used to test separability probability conjectures?”, following the analyses of Smart et al. in their article [69]. “Experimental data from a quantum computer verifies the generalized Pauli exclusion principle,” in which “quan-
tum many-fermion states are randomly prepared on the quantum computer and tested for constraint violations.”

So, in brief summary, let us state that at this stage of our continuing investigations, it appears that we have a set of three exact-valued measure-dependent two-qubit separability probabilities \((\frac{8}{33} \text{ Hilbert–Schmidt}), 1 - \frac{256}{27\pi^2} \text{ [operator monotone } \sqrt{x} \text{]}, \frac{25}{341} \text{ [Bures—minimal monotone } 1 + x^2 \text{])}, \) but only one two-rebit one \((\frac{29}{64} \text{ Hilbert–Schmidt})). The [apparent lesser than \(\frac{25}{341}\)] separability probabilities for other members—Kubo–Mori, Wigner–Yanase,…—of the monotone family have been estimated in [44]—cf. [26,27]. But since there is, at present, no apparent mechanism available for generating density matrices random with respect to such measures [cf. [70, Sec. V.B] in regard to superfidelity], the quasirandom procedure seems unavailable for them. (Also the use of measures that are non-monotone in nature—in addition to the well-studied Hilbert–Schmidt one—would be of interest, for example, the Monge [71] and Husimi [72,73] measures.) However, separability/PPT probabilities can be so analyzed for the class of induced measures [15].

Let us pose the following problem: Construct a function \(f\) that yields the separability probabilities associated with the monotone metrics. That is, we would have (the Bures case) \(f\left(\frac{1}{2} + t\right) = \frac{25}{341} = 0.0733138, f\left(\sqrt{t}\right) = 1 - \frac{256}{27\pi^2} = 0.0393251\) and \(f\left(\frac{2t}{1+t}\right) = 0\). Additionally, \(f\left(\frac{(t-1)^2}{e}\right) \approx 0.0609965, f\left(\frac{1}{4} \left(\sqrt{t} + 1\right)^2\right) \approx 0.0503391\) and \(f\left(\frac{(t-1)}{\log t}\right) \approx 0.346801 [44, \text{ Tab. II}]\) and also \(f\left(\frac{1+6t+t^2}{4+4t}\right) \approx 0.0475438 [44, \text{ Tab. I}].\)

Acknowledgements This research was supported by the National Science Foundation under Grant No. NSF PHY-1748958.

References

1. Lovas, A., Andai, A.: Invariance of separability probability over reduced states in \(4 \times 4\) bipartite systems. J. Phys. A: Math. Theor. 50, 295303 (2017)
2. Życzkowski, K., Sommers, H.-J.: Hilbert–Schmidt volume of the set of mixed quantum states. J. Phys. A: Math. Gen. 36, 10115 (2003)
3. Bengtsson, I., Życzkowski, K.: Geometry of Quantum States: An Introduction to Quantum Entanglement. Cambridge University Press, Cambridge (2017)
4. Caves, C.M., Fuchs, C.A., Rungta, P.: Entanglement of formation of an arbitrary state of two rebits. Found. Phys. Lett. 14(3), 199–212 (2001). https://doi.org/10.1023/A:1012215309321
5. Slater, P.B.: Master Lovas–Andai and equivalent formulas verifying the \(\frac{8}{33}\) two-qubit Hilbert–Schmidt separability probability and companion rational-valued conjectures. Quantum Inf. Process. 17, 83 (2018)
6. Khvedelidze, A., Rogojin, I.: On the generation of random ensembles of qubits and qutrits: computing separability probabilities for fixed rank states. In: EPJ Web of Conferences (EDP Sciences), vol. 173 (2018)
7. Milz, S., Strunz, W.T.: Volumes of conditioned bipartite state spaces. J. Phys. A: Math. Theor. 48, 035306 (2014)
8. Fei, J., Joynt, R.: Numerical computations of separability probabilities. Rep. Math. Phys. 78, 177 (2016)
9. Shang, J., Seah, Y.-L., Ng, H.K., Nott, D.J., Englert, B.-G.: Monte Carlo sampling from the quantum state space. I. New J. Phys. 17, 043017 (2015)
10. Slater, P.B.: A concise formula for generalized two-qubit Hilbert–Schmidt separability probabilities. J. Phys. A: Math. Theor. 46, 445302 (2013)
11. Slater, P.B., Dunkl, C.F.: Moment-based evidence for simple rational-valued Hilbert–Schmidt generic $2 \times 2$ separability probabilities. J. Phys. A: Math. Theor. 45, 095305 (2012)
12. Slater, P.B.: Dyson indices and Hilbert–Schmidt separability functions and probabilities. J. Phys. A: Math. Theor. 40, 14279 (2007)
13. Slater, P.B.: Extensions of generalized two-qubit separability probability analyses to higher dimensions, additional measures and new methodologies. Quantum Inf. Process. (2018b, to appear). arXiv preprint arXiv:1809.09040
14. Adler, S.L.: Quaternionic Quantum Mechanics and Quantum Fields, vol. 88. Oxford University Press, Oxford (1995)
15. Życzkowski, K., Sommers, H.-J.: Induced measures in the space of mixed quantum states. J. Phys. A: Math. Gen. 34, 7111 (2001)
16. Szarek, S.J., Bengtsson, I., Życzkowski, K.: On the structure of the body of states with positive partial transpose. J. Phys. A: Math. Gen. 39, L119 (2006)
17. Samuel, J., Shivam, K., Sinha, S.: Lorentzian geometry of qubit entanglement (2018). arXiv preprint arXiv:1801.00611
18. Avron, J., Kenneth, O.: Entanglement and the geometry of two qubits. Ann. Phys. 324, 470 (2009)
19. Braga, H., Souza, S., Mizrahi, S.S.: Geometrical meaning of two-qubit entanglement and its symmetries. Phys. Rev. A 81, 042310 (2010)
20. Gamel, O.: Entangled Bloch spheres: Bloch matrix and two-qubit state space. Phys. Rev. A 93, 062320 (2016)
21. Jevtic, S., Pusey, M., Jennings, D., Rudolph, T.: Quantum steering ellipsoids. Phys. Rev. Lett. 113, 020204 (2014)
22. Aubrun, G., Szarek, S.J.: Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory, vol. 223. American Mathematical Soc, Providence (2017)
23. Życzkowski, K., Horodecki, P., Sanpera, A., Lewenstein, M.: Volume of the set of separable states. Phys. A 58, 883 (1998)
24. Petz, D., Sudár, C.: Geometries of quantum states. J. Math. Phys. 37, 2662 (1996)
25. Rexiti, M., Felice, D., Mancini, S.: The volume of two-qubit states by information geometry. Entropy 20, ISSN 1099-4300 (2018). http://www.mdpi.com/1099-4300/20/2/146
26. Singh, R., Kunjwal, R., Simon, R.: Relative volume of separable bipartite states. Phys. Rev. A 89, 022308 (2014)
27. Batle, J., Abdel-Aty, M.: Geometric approach to the distribution of quantum states in bipartite physical systems. JOSA B 31, 2540 (2014)
28. Slater, P.B.: Bayesian quantum mechanics. Nature 367, 328 (1994)
29. Slater, P.B.: Quantum coin-tossing in a Bayesian Jeffreys framework. Phys. Lett. A 206, 66 (1995)
30. Kwek, L., Oh, C., Wang, X.-B.: Quantum Jeffreys prior for displaced squeezed thermal states. J. Phys. A: Math. Gen. 32, 6613 (1999)
31. Reconcile a pair of two-qubit boundary-state separability probability analyses. https://physics.stackexchange.com/questions/422887/reconcile-a-pair-of-two-qubit-boundary-state-separability-probability-analyses. Accessed 2018
32. Slater, P.B.: Formulas for generalized two-qubit separability probabilities. Adv. Math. Phys. 2018, 9365213 (2018)
33. Slater, P.B.: Exact Bures probabilities that two quantum bits are classically correlated. Eur. Phys. J. B–Condens. Matter Complex Syst. 17, 471 (2000)
34. Sarkar, A., Kumar, S.: Bures–Hall ensemble: spectral densities and average entropies (2019). arXiv preprint arXiv:1901.09587
35. Šafránek, D.: Discontinuities of the quantum Fisher information and the Bures metric. Phys. Rev. A 95, 052320 (2017)
36. Forrester, P.J., Kieburg, M.: Relating the Bures measure to the Cauchy two-matrix model. Commun. Math. Phys. 342, 151 (2016)
37. Braunstein, S.L., Caves, C.M.: Statistical distance and the geometry of quantum states. Phys. Rev. Lett. 72, 3439 (1994)
38. Bej, P., Deb, P.: Geometry of quantum state space and entanglement (2018). arXiv preprint arXiv:1805.11292
39. Dunkl, C.F., Slater, P.B.: Separability probability formulas and their proofs for generalized two-qubit X-matrices endowed with Hilbert–Schmidt and induced measures. Random Matrices Theory Appl. 4, 1550018 (2015)
40. Penson, K.A., Žyczkowski, K.: Product of Ginibre matrices: Fus–Catalan and Raney distributions. Phys. Rev. E 83, 061118 (2011)
41. Al Osipov, V., Sommers, H.-J., Žyczkowski, K.: Random Bures mixed states and the distribution of their purity. J. Phys. A: Math. Theor. 43, 055302 (2010)
42. Borot, G., Nadal, C.: Purity distribution for generalized random Bures mixed states. J. Phys. A: Math. Theor. 45, 075209 (2012)
43. Slater, P.B.: A priori probability that two qubits are unentangled. Quantum Inf. Process. 1, 397 (2002)
44. Slater, P.B.: Silver mean conjectures for 15-dimensional volumes and 14-dimensional hyperareas of the separable two-qubit systems. J. Geom. Phys. 53, 74 (2005)
45. Leobacher, G., Pillichshammer, F.: Introduction to Quasi-Monte Carlo Integration and Applications. Springer, Berlin (2014)
46. Livio, M.: The Golden Ratio: The Story of Phi, the World’s Most Astonishing Number. Broadway Books, New York (2008)
47. The unreasonable effectiveness of quasirandom sequences. http://extremelearning.com.au/unreasonable-effectiveness-of-quasirandom-sequences/. Accessed 2018
48. How can one generate an open ended sequence of low discrepancy points in 3D? https://math.stackexchange.com/questions/2231391/how-can-one-generate-an-open-ended-sequence-of-low-discrepancy-points-in-3d. Accessed 2018
49. Devroye, L.: Non-uniform Random Variate Generation. Springer, Berlin (1986)
50. Can I use compile to speed up inverseCDF? https://mathematica.stackexchange.com/questions/181099/can-i-use-compile-to-speed-up-inversecdf. Accessed 2019
51. Slater, P.B.: Extended studies of separability functions and probabilities and the relevance of Dyson indices. J. Geom. Phys. 58, 1101 (2008)
52. Compute a certain ‘separability probability’ via a constrained 4D integration over $[-1, 1]^4$. https://mathematica.stackexchange.com/questions/193379/compute-a-certain-separability-probability-via-a-constrained-4d-integration-ov. Accessed 2019
53. A pair of integrals involving square roots and inverse trigonometric functions over the unit disk. https://mathoverflow.net/questions/325697/a-pair-of-integrals-involving-square-roots-and-inverse-trigonometric-functions-o. Accessed 2019
54. Approximate/estimate the ratio of two multidimensional constrained integrals. https://mathematica.stackexchange.com/questions/193796/approximate-estimate-the-ratio-of-two-multidimensional-constrained-integrals. Accessed 2019
55. Ye, D.: On the Bures volume of separable quantum states. J. Math. Phys. 50, 083502 (2009)
56. Slater, P.B.: Invariance of bipartite separability and PPT-probabilities over Casimir invariants of reduced states. Quantum Inf. Process. 15, 3745 (2016)
57. Chen, K., Wu, L.-A.: A matrix realignment method for recognizing entanglement. Quant. Inform. Comput. 3(3), 193–202 (2003)
58. Bae, J., Chruściński, D., Hiesmayr, B.C.: Entanglement witness 2.0: Compressed entanglement witnesses. (2018). arXiv preprint arXiv:1811.09896
59. Gabdulin, A., Mandilara, A.: Investigating bound entangled two-qudit states via the best separable approximation. (2019). arXiv preprint arXiv:1906.08963
60. Xiong, C., Spehner, D., Wu, J.: Geometric quantum discord for two-qubit X-states (2017). arXiv preprint arXiv:1710.04007
61. Compute the two-fold partial integral, where the three-fold full integral is known. https://mathoverflow.net/questions/322958/compute-the-two-fold-partial-integral-where-the-three-fold-full-integral-is-kno. Accessed 2019
62. Do these polynomials with harmonic number-related coefficients lie in some particular known class? https://math.stackexchange.com/questions/3115582/do-these-polynomials-with-harmonic-number-related-coefficients-lie-in-some-parti. Accessed 2019
63. Ozawa, M.: Entanglement measures and the Hilbert–Schmidt distance. Phys. Lett. A 268, 158 (2000)
64. Sum a certain hypergeometric-function-based expression pertaining to an integration problem. https://mathematica.stackexchange.com/questions/189538/sum-a-certain-hypergeometric-function-based-expression-pertaining-to-an-integrat. Accessed 2019
65. Tilma, T., Byrd, M., Sudarshan, E.: A parametrization of bipartite systems based on SU (4) Euler angles. J. Phys. A: Math. Gen. 35, 10445 (2002)
66. Slater, P.B.: Eigenvalues, separability and absolute separability of two-qubit states. J. Geom. Phys. 59, 17 (2009)
67. Maziero, J.: Random sampling of quantum states: a survey of methods. Braz. J. Phys. **45**, 575 (2015)
68. Can experimental data from a quantum computer be used to test separability probability conjectures? https://quantumcomputing.stackexchange.com/questions/5355/can-experimental-data-from-a-quantum-computer-be-used-to-test-separability-pro. Accessed 2019
69. Smart, S.E., Schuster, D.I., Mazziotti, D.A.: Experimental data from a quantum computer verifies the generalized Pauli exclusion principle. Commun. Phys. **2**, 2 (2019)
70. Puchała, Z., Miszczak, J.A.: Probability measure generated by the superfidelity. J. Phys. A: Math. Theor. **44**, 405301 (2011)
71. Zyczkowski, K., Slomczynski, W.: The Monge metric on the sphere and geometry of quantum states. J. Phys. A: Math. Gen. **34**, 6689 (2001)
72. Slater, P.B.: Quantum and Fisher information from the Husimi and related distributions. J. Math. Phys. **47**, 022104 (2006)
73. Rexiti, M., Felice, D., Mancini, S.: The volume of two-qubit states by information geometry. Entropy **20**, 146 (2018)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.