Mass inequality for the quark propagator

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Abstract

We show that for any gauge-fixing scheme with positive semi-definite functional integral measure, the inverse correlation length of the quark propagator is bounded below by one-half the pion mass.
It is known that quarks develop a dynamically-generated mass in the Nambu Jona-Lasinio model, a self-interacting relativistic quark model without quark confinement [1, 2, 3]. Studies using the Dyson-Schwinger or gap equation for the quark propagator suggest a similar phenomenon occurs in quantum chromodynamics (QCD) [4, 5, 6, 7]. For even light quarks one finds a rather large inverse correlation length or nonperturbative mass function, \( M \), at zero momentum.

This dressed quark mass could be related to the constituent quark mass of the naïve nonrelativistic quark model. Dyson-Schwinger studies suggest a value for \( M(0) \) somewhere between 170 MeV and 500 MeV. Recent quenched lattice results using Landau and Laplacian gauge with the overlap fermions [8] and improved staggered fermions [9] and unquenched lattice results using Landau gauge with improved staggered fermions [10] yield a similar value for \( M(0) \) between 250 and 500 MeV.

In this work we introduce another tool for understanding the infrared behavior of quark propagators in QCD. We prove a general inequality which relates \( M(0) \) and the pion mass, \( m_\pi \). The inequality applies to all gauge-fixing schemes with positive semi-definite functional integral measure.

Weingarten showed that for an even number of degenerate quark flavors, the pion must have the lowest energy of all meson states [11]. The proof required positivity of the QCD action, \( \gamma_5 \)-Hermitivity, and exact quark flavor symmetry. With exactly the same conditions for a gauge-fixed action, we show that

\[
[M(0)]_{\text{gauge-fixed}} \geq \frac{1}{2} [m_\pi]_{\text{gauge-fixed}}.
\]  

(1)

The requirement of positivity is satisfied by all gauge-fixing schemes used in recent lattice simulations. See [12, 13] for reviews of current lattice gauge-fixing methods.

By positivity we mean positivity of the Euclidean functional integral measure and not spectral positivity in a Hamiltonian or transfer matrix formalism. While spectral positivity is violated in certain gauges [14, 15], any gauge-fixing scheme without a positive functional integral measure would have a sign or complex phase problem and be of limited practical use. In the following we say nothing new about the subject of incomplete gauge fixing and Gribov copies for any particular scheme [16, 17]. But if the Gribov problem is in fact handled properly for a given gauge-fixing scheme then we can conclude further that

\[
[M(0)]_{\text{gauge-fixed}} \geq \frac{1}{2} [m_\pi]_{\text{gauge-fixed}} = \frac{1}{2} m_\pi.
\]  

(2)
We begin by establishing our notation for QCD with $N_c$ colors. We let $\lambda^A$ be the matrix generators for $SU(N_c)$, and we use the matrix field notation,

$$A_\mu = A_\mu^B \frac{\lambda^B}{2},$$  

$$F_{\mu\nu} = F_{\mu\nu}^B \frac{\lambda^B}{2},$$

where

$$F_{\mu\nu}^B = \partial_\mu A_\nu^B - \partial_\nu A_\mu^B - g_0 f^{BCD} A_\mu^C A_\nu^D.$$  

Summations are implied by repeated indices. We consider $n_f$ degenerate quark flavors where $n_f$ is even, and label two of the flavors as $u$ and $d$. In Euclidean space, the QCD action prior to gauge fixing is

$$S = S_g + \sum_{i=1}^{n_f} \int d^4x \ \bar{q}_i \left( \gamma_\mu (\partial_\mu + ig_0 A_\mu) + m_0 \right) q_i,$$

where

$$S_g = \frac{1}{2} Tr \left[ \int d^4x \ F_{\mu\nu} F_{\mu\nu} \right].$$

Throughout we assume a gauge-invariant lattice discretization, though we use the more familiar continuum notation.

Let $M$ be the quark matrix due to the background gluon field. $M$ has the block-diagonal structure

$$M = \bigoplus_{i=1}^{n_f} M^{q_i},$$

where all blocks $M^{q_i}$ are the same,

$$M^{q_i} = M^q \equiv \gamma_\mu (\partial_\mu + ig_0 A_\mu) + m_0.$$  

We consider any lattice quark action so long as $\gamma^5$-Hermitivity is maintained and the quark flavor symmetry is exact. For most quark actions this holds true at any lattice spacing. For staggered quarks, however, one must go to the continuum limit in order to regain exact flavor symmetry.

The condition of $\gamma^5$-Hermitivity means

$$\gamma^5 (M^q)^\dagger \gamma^5 = M^q,$$  

and so $\det M^q$ must be real. The block-diagonal structure and flavor symmetry then gives

$$\det M = (\det M^q)^{n_f} \geq 0.$$
MQ has several types of indices corresponding with space-time lattice points, Dirac indices, and color indices in the fundamental representation. When needed we use the notation

\[ M_{ij;ab}^q(x, y) \]  

where \( x, y \) are the space-time lattice points; \( i, j \) are the Dirac indices; and \( a, b \) are the color indices.

We fix the gauge using the constraint functional \( f(A) \) and the gauge-invariant Faddeev-Popov determinant \( \Delta_f(A) \) \cite{12, 13}. \( \Delta_f(A) \) is defined by the condition

\[ \Delta_f^{-1}(A) = \int DG \delta(f(A^G)), \]  

where the integration is over all gauge transformations and \( A^G \) is the gauge orbit of \( A \). We can write \( \Delta_f^{-1}(A) \) as a sum over all solutions \( G_i \) to the constraint equation \( f(A^G) = 0 \),

\[ \Delta_f^{-1}(A) = \sum_i \frac{1}{\det \frac{\delta f(A^G)}{\delta G} \bigg|_{G=G_i}}. \]  

\( \Delta_f(A) \) defined in this way is clearly positive semi-definite. If \( f \) is an ideal gauge-fixing constraint, then there is exactly one simple zero \( f(A^G) = 0 \) for each gauge orbit. Otherwise we must contend with possible complications such as singular configurations where one or more of the determinants in (14) vanish or perhaps even the number of Gribov copies becomes infinite.

We note that

\[ \Delta_f(A^G) = \Delta_f(A), \]  

\[ S_g(A^G) = S_g(A), \]  

\[ \det[M(A^G)] = \det[M(A)], \]  

and for any gauge-invariant observable,

\[ O(A^G) = O(A). \]  

Therefore the expectation value of any gauge-invariant observable is

\[ \langle O \rangle = \frac{\int DA \ O(A)e^{-S_g(A)} \det[M(A)]}{\int DA \ e^{-S_g(A)} \det[M(A)]} \]

\[ = \frac{\int DA \ O(A)e^{-S_g(A)} \det[M(A)]\Delta_f(A)\int DG \delta(f(A^G))}{\int DA \ e^{-S_g(A)} \det[M(A)]\Delta_f(A)\int DG \delta(f(A^G))} \]

\[ = \frac{\int DA \ O(A)e^{-S_g(A)} \det[M(A)]\Delta_f(A)\delta(f(A))}{\int DA \ e^{-S_g(A)} \det[M(A)]\Delta_f(A)\delta(f(A))}. \]
For observables which are not gauge-invariant, we use the last expression in (19) to define the gauge-fixed expectation value,

\[ \langle O \rangle_f \equiv \frac{\int DA \, O(A)e^{-S_g(A)} \det[M(A)]\Delta_f(A) \delta(f(A))}{\int DA \, e^{-S_g(A)} \det[M(A)]\Delta_f(A) \delta(f(A))}. \]  

We note that this gauge-fixed functional integral measure is positive semi-definite.

Unfortunately the action in (20) is too difficult to deploy in actual lattice simulations. In practice, gauge fixing on the lattice is implemented using

\[ \langle O \rangle_f \equiv \frac{\int DA \, O(A^G(A))e^{-S_g(A)} \det[M(A)]}{\int DA \, e^{-S_g(A)} \det[M(A)]} \]  

where \( G(A) \) is a special gauge transformation for which \( f(A^G(A)) = 0 \). This gauge-fixed functional integral measure is also positive semi-definite. If \( f \) is an ideal gauge-fixing constraint, then there is exactly one solution \( f(A^G) = 0 \) for each gauge orbit and the process of selecting \( G(A) \) is unique. Otherwise we must again contend with complications due to multiple Gribov copies.

Following Weingarten [11] we consider the pion correlation function

\[ F_\pi(x) = \langle \bar{d}_i\gamma_5 u_j(x) \bar{u}_k(0)\gamma_5 d_l(0) \rangle. \]  

We can write

\[ F_\pi(x) = \frac{\int DADqD\bar{q} \, \bar{d}_i(x)\gamma_5 \gamma_{ij} u_j(x) \bar{u}_k(0)\gamma_5 d_l(0)e^{-S_g(A)} \Delta_f(A) \delta(f(A))}{\int DADqD\bar{q} \, e^{-S_g(A)} \Delta_f(A) \delta(f(A))}. \]  

Integrating over the quark fields we get

\[ F_\pi(x) = - \sum_{i,l,a,b} \int D\Theta \, |(M^q)^{-1}_{li;ab}(0,x)\gamma_5 (M^q)^{-1}_{jk;ab}(x,0)\gamma_5|_l|^k|. \]  

where \( D\Theta \) is the probability measure,

\[ D\Theta = \frac{DAe^{-S_g(A)} \det[M(A)]\Delta_f(A) \delta(f(A))}{\int DAe^{-S_g(A)} \det[M(A)]\Delta_f(A) \delta(f(A))}. \]  

Since

\[ \gamma^5(M^q)^{-1}\gamma^5 = ((M^q)^{-1})^\dagger, \]  

we have

\[ F_\pi(x) = - \sum_{i,l,a,b} \int D\Theta \, |(M^q)^{-1}_{li;ab}(0,x)|^2. \]
Next we consider the quark correlator,

\[ F_q(x) = \langle \bar{u}_{k\alpha}(x) \Gamma_{kj} u_{j\alpha}(0) \rangle , \]  
(28)

where \( \Gamma \) is an arbitrary matrix in Dirac spinor space. We can write

\[ F_q(x) = \frac{\int \mathcal{D}A \mathcal{D}q \bar{q} u_{k\alpha}(x) \Gamma_{kj} u_{j\alpha}(0) e^{-S_q(A)} \Delta_f(A) \delta(f(A)) \int \mathcal{D}A \mathcal{D}q e^{-S_q(A)} \Delta_f(A) \delta(f(A))}{\int \mathcal{D}A \mathcal{D}q e^{-S_q(A)} \Delta_f(A) \delta(f(A))}. \]  
(29)

Integrating over the quark fields we get

\[ F_q(x) = -N_c \int \mathcal{D} \Theta (M^q)^{-1}_{jk;\alpha\alpha}(0, x) \Gamma_{kj} \]  
(no sum on \( \alpha \)).
(30)

Let us define

\[ \gamma = \sqrt{\sum_{k,j} |\Gamma_{kj}|^2}. \]  
(31)

By the Cauchy-Schwarz inequality we have

\[ |F_q(x)| \leq N_c \gamma \sqrt{\sum_{j,k} \int \mathcal{D} \Theta |(M^u)^{-1}_{jk;\alpha\alpha}(0, x)|^2} \]  
(no sum on \( \alpha \))
\[ \leq N_c \gamma \sqrt{\frac{1}{N_c} \sum_{j,k,\alpha} \int \mathcal{D} \Theta |(M^u)^{-1}_{jk\alpha\alpha}(0, x)|^2}. \]  
(32)

So therefore

\[ |F_q(x)| \leq \gamma \sqrt{N_c |F_\pi(x)|}. \]  
(33)

We now take \( x_1 = x_2 = x_3 = 0 \) and consider the limit as \( x_4 \to \pm \infty \). For any correlation function \( F(x) \) we define an inverse correlation length as the supremum of the set of real numbers \( \alpha \) such that

\[ \lim_{x_4 \to +\infty} |F(x)| e^{\alpha |x_4|} = \lim_{x_4 \to -\infty} |F(x)| e^{\alpha |x_4|} = 0. \]  
(34)

We are implicitly assuming that the set of \( \alpha \)'s is not empty and bounded above. While this seems a safe assumption, it is perhaps not completely trivial given the lack of spectral positivity.

We define \([m_\pi]_f\) as the inverse correlation length of \( F_\pi(x) \) and \([M(0)]_f\) as the inverse correlation length of \( F_q(x) \). We conclude from (33) that

\[ [M(0)]_f \geq \frac{1}{2} [m_\pi]_f. \]  
(35)
If we assume that our gauge-fixing scheme does not affect the calculation of the gauge-invariant pion correlator, then we also have

\[ [M(0)]_f \geq \frac{1}{2} [m_\pi]_f = \frac{1}{2} m_\pi. \quad (36) \]

The mass inequality in (36) should hold for both quenched QCD and unquenched QCD. It holds true at nonzero lattice spacing, any quark mass, any number of colors, any even number of flavors, and any gauge-fixing scheme so long as we have a positive semi-definite functional integral measure, \( \gamma^5 \)-Hermitivity, and exact flavor symmetry. In the chiral limit, the inequality does not require that \([M(0)]_f\) be nonzero. Nor does it imply that \([M(0)]_f\) is the same for different gauge-fixing schemes. However it does require that \([M(0)]_f\) be at least as large as \(\frac{1}{2} m_\pi\), which is proportional to the square root of the current quark mass in unquenched QCD and a slightly modified fractional power in quenched QCD due to quenched chiral logarithms [18, 19].

If QCD did not confine quarks, then our mass inequality would be rather trivial. It would simply say that if the pion exists, then its mass must lie below the quark-antiquark continuum threshold. However in the real world the quark propagator is not directly observable and quarks are confined. Therefore the mass inequality is no longer trivial. If, in some gauge-fixing scheme, \([M(0)]_f\) could be identified with a constituent quark mass, then (36) implies that the pion mass lies below the constituent quark-antiquark threshold.

Recent results from the Dyson-Schwinger equation and lattice simulations at various quark masses suggest that the mass inequality in (36) is indeed satisfied. Further calculations would be useful to understand the dependence of \([M(0)]_f\) on gauge-fixing scheme, lattice spacing, quark masses, different quark actions, number of colors and flavors, and quenching artifacts.

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