A perturbative approach for the study of compatibility between nonminimally coupled gravity and Solar System experiments

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Abstract. We develop a framework for constraining a certain class of theories of nonminimally coupled (NMC) gravity with Solar System observations.

1. Introduction
We consider the possibility of constraining a class of theories of nonminimally coupled gravity [1] by means of Solar System experiments. NMC gravity is an extension of $f(R)$ gravity where the action integral of General Relativity (GR) is modified in such a way to contain two functions $f^1(R)$ and $f^2(R)$ of the space-time curvature $R$. The function $f^1(R)$ has a role analogous to $f(R)$ gravity, and the function $f^2(R)$ yields a nonminimal coupling between curvature and the matter Lagrangian density. For other NMC gravity theories and their potential applications, see, e.g., [2, 3, 4, 5, 6].

NMC gravity has been applied to several astrophysical and cosmological problems such as dark matter [7, 8], cosmological perturbations [9], post-inflationary reheating [10] or the current accelerated expansion of the Universe [11].

In the present communication, by extending the perturbative study of $f(R)$ gravity in [12], we discuss how a general framework for the study of Solar System constraints to NMC gravity can be developed. The approach is based on a suitable linearization of the field equations of NMC gravity around a cosmological background space-time, where the Sun is considered as a perturbation. Solar System observables are computed, then we apply the perturbative approach to the NMC model by Bertolami, Frazão and Páramos [11], which constitutes a natural extension of $1/R^n$ ($n > 0$) gravity [13] to the non-minimally coupled case. Such a NMC gravity model is able to predict the observed accelerated expansion of the Universe. We show that, differently from the pure $1/R^n$ gravity case, the NMC model cannot be constrained by this perturbative method so that it remains, in this respect, a viable theory of gravity. Further details about the subject of the present communication can be found in the manuscript [14].
2. NMC gravity model
We consider a gravity model with an action functional of the type [1],

\[ S = \int \left[ \frac{1}{2} f^1(R) + [1 + f^2(R)] \mathcal{L}_m \right] \sqrt{-g} \, d^4x, \]

where \( f^i(R) \) (\( i = 1, 2 \)) are functions of the Ricci scalar curvature \( R \), \( \mathcal{L}_m \) is the Lagrangian density of matter and \( g \) is the metric determinant. By varying the action with respect to the metric we get the field equations

\[ (f^1_R + 2f^2_R \mathcal{L}_m) R_{\mu\nu} - \frac{1}{2} f^1 g_{\mu\nu} = (1 + f^2) T_{\mu\nu} + \nabla_{\mu\nu} (f^1_R + 2f^2_R \mathcal{L}_m), \tag{1} \]

where \( f^i_R \equiv df^i/dR \) and \( \nabla_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \). We describe matter as a perfect fluid with negligible pressure: the Lagrangian density of matter is \( \mathcal{L}_m = -\rho \) and the trace of the energy-momentum tensor is \( T = -\rho \). We write \( \rho = \rho^{\text{cos}} + \rho^s \), where \( \rho^{\text{cos}} \) is the cosmological mass density and \( \rho^s \) is the Sun mass density.

We assume that the metric which describes the spacetime around the Sun is a perturbation of a flat Friedmann-Robertson-Walker (FRW) metric with scale factor \( a(t) \):

\[ ds^2 = -[1 + 2\Phi(r,t)] dt^2 + a^2(t) \left([1 + 2\Phi(r,t)] dr^2 + r^2 d\Omega^2 \right), \]

where \( |\Phi(r,t)| \ll 1 \) and \( |\Phi(r,t)| \ll 1 \). The Ricci curvature of the perturbed spacetime is expressed as the sum

\[ R(r,t) = R_0(t) + R_1(r,t), \]

where \( R_0 \) denotes the scalar curvature of the background FRW spacetime and \( R_1 \) is the perturbation due to the Sun. Following Ref. [12], we linearize the field equations assuming that

\[ |R_1(r,t)| \ll R_0(t), \tag{2} \]

both around and inside the Sun. This assumption means that the curvature \( R \) of the perturbed spacetime remains close to the cosmological value \( R_0 \) inside the Sun. In GR such a property of the curvature is not satisfied inside the Sun. However, for \( f(R) \) theories which are characterized by a small value of a suitable mass parameter (see next section), condition (2) can be satisfied. For instance, the \( 1/R^n \) (\( n > 0 \)) gravity model [13] satisfies condition (2), as shown in [12, 15].

Eventually, we assume that functions \( f^1(R) \) and \( f^2(R) \) admit a Taylor expansion around \( R = R_0 \) and that terms nonlinear in \( R_1 \) can be neglected in the expansion. We use the notation introduced by [12] (for \( i = 1, 2 \)):

\[ f^i_0 \equiv f^i(R_0), \quad f^i_R \equiv df^i/dR(R_0), \quad f^i_{RR} \equiv d^2f^i/dR^2(R_0). \]

3. Solution of the linearized field equations
The details of the following computations can be found in the paper [14]. First we linearize the trace of the field equations (1). Using condition (2), we neglect \( O(R_1^2) \) contributions but we keep the cross-term \( R_0 R_1 \). Introducing the potential \( U = (f^1_{RR} + 2f^2_{RR} \mathcal{L}_m) R_1 \), we get

\[ \nabla^2 U - m^2 U = -\frac{1}{3} (1 + f^1_0) \rho^s + \frac{2}{3} f^2_{RR} \rho^s R_0 + 2 \rho^s \Box f^2_{RR} + 2 f^2_{RR} \nabla^2 \rho^s, \]

where \( m^2 \) denotes the mass parameter

\[ m^2(r,t) = \frac{1}{3} \left[ \frac{f^1_{RR} - f^2_{RR} \mathcal{L}_m}{f^1_{RR} + 2f^2_{RR} \mathcal{L}_m} - R_0 - \frac{3\Box (f^1_{RR} - 2f^2_{RR} \rho^{\text{cos}}) - 6\rho^s \Box f^2_{RR}}{f^1_{RR} + 2f^2_{RR} \mathcal{L}_m} \right]. \tag{3} \]
When \( f^2(R) = 0 \) we recover the mass formula of \( f(R) \) gravity theory found in [12]. In the following we assume that \(|mr| \ll 1\) at Solar System scale. Under this assumption the solution for \( R_1 \) outside the Sun is given by

\[
R_1(r, t) = \left[ \frac{-\frac{1}{3} \left( 1 + f_0^2 \right) + \frac{2}{5} f_{R0}^2 R_0 + 2 \Box f_{R0}^2}{4 \pi \left( 2 f_{RR0}^2 \rho^{\text{cos}} - f_{R0}^4 \right)} \right] \frac{M_S}{r},
\]

where \( M_S \) is the mass of the Sun. Then we linearize the field equations (1) obtaining

\[
\left( f_{R0}^1 + 2 f_{R0}^2 \mathcal{L}_m \right) \left( \nabla^2 \Psi + \frac{1}{2} R_1 \right) - \nabla^2 \left( \left( f_{R0}^1 + 2 f_{R0}^2 \mathcal{L}_m \right) R_1 \right) = \left( 1 + f_0^2 \right) \rho^S - 2 f_{R0}^2 \nabla^2 \rho^S,
\]

\[
\left( f_{R0}^1 + 2 f_{R0}^2 \mathcal{L}_m \right) \left( -\frac{d^2 \Psi}{dr^2} + \frac{2}{r} \frac{d \Phi}{dr} \right) - \frac{1}{2} f_{R0}^1 R_1 + \frac{2}{r} f_{R0}^1 \frac{d R_1}{dr} + \frac{4}{r} f_{R0}^1 \frac{\partial (\mathcal{L}_m R_1)}{\partial r} = \frac{4}{r} f_{R0}^1 \frac{d \rho^S}{dr}.
\]

Using the divergence theorem and the solution (4) for \( R_1 \), from the first equation we obtain the function \( \Psi \) outside of the Sun:

\[
\Psi(r, t) = -\frac{2}{3r} \left( 1 + f_0^2 + f_{R0}^2 R_0 + 3 \Box f_{R0}^2 \right) \int_0^{R_S} \frac{\rho^S(x)}{f_{R0}^1 + 2 f_{R0}^2 \mathcal{L}_m(x)} r^2 \, dr,
\]

where \( R_S \) is the radius of the Sun. If the following condition is satisfied,

\[
\left| 2 f_{R0}^2 \right| \rho^S(r) \ll \left| f_{R0}^1 - 2 f_{R0}^2 \rho^{\text{cos}}(t) \right|, \quad r \leq R_S,
\]

then the function \( \Psi \) is a Newtonian potential:

\[
\Psi(r, t) = -\frac{G M_S}{r}, \quad G(t) = \frac{1 + f_0^2 + f_{R0}^2 R_0 + 3 \Box f_{R0}^2}{6 \pi \left( f_{R0}^1 - 2 f_{R0}^2 \rho^{\text{cos}} \right)} \quad r \geq R_S,
\]

where \( G(t) \) is an effective gravitational constant. Since \( G \) depends on slowly varying cosmological quantities we have \( G(t) \simeq \text{constant} \), so that \( \Psi(r, t) \simeq \Psi(r) \).

The solution for the function \( \Phi \) is computed from the second of the linearized field equations, and we obtain \( \Phi(r) = -\gamma \Psi(r) \), where the PPN parameter \( \gamma \) depends on cosmological quantities and it is given by

\[
\gamma = \frac{1}{2} \left[ \frac{1 + f_0^2 + 4 f_{R0}^2 R_0 + 12 \Box f_{R0}^2}{1 + f_0^2 + f_{R0}^2 R_0 + 3 \Box f_{R0}^2} \right].
\]

When \( f^2(R) = 0 \) we find the known result \( \gamma = 1/2 \) which holds for \( f(R) \) gravity theories which satisfy the condition \(|mr| \ll 1\) and condition \(|R_1| \ll R_0\), as it has been shown in [12]. The \( 1/R^n \) \((n > 0)\) gravity theory [13], where \( f(R) \) is proportional to \((R + \text{constant}/R^n)\), is one of such theories that, consequently, have to be ruled out by Cassini measurement.

4. Application to a NMC cosmological model

We consider the NMC gravity model proposed in [11] to account for the observed accelerated expansion of the Universe:

\[
f^1(R) = 2\kappa R, \quad f^2(R) = \left( \frac{R}{R_n} \right)^{-n}, \quad n > 0,
\]

where \( \kappa = c^4/16\pi G_N \), \( G_N \) is Newton’s gravitational constant, and \( R_n \) is a constant. This model yields a cosmological solution with a negative deceleration parameter \( q < 0 \), and the scale factor
a(t) of the background metric follows the temporal evolution \( a(t) = a_0 (t/t_0)^{2(1+n)/3} \) where \( t_0 \) is the current age of the Universe. Using the properties of the cosmological solution found in [11] the mass parameter (3) can be computed obtaining (we refer to [14] for details of the computation):

\[
m^2 = \frac{\mu(n)\rho^{\cos} + \nu(n)\rho^8}{\rho^{\cos} + \rho^8} R_0(t), \quad R_0(t) = \frac{4(1 + 4n)(1 + n)}{3t^2},
\]

where \( \mu(n) \) and \( \nu(n) \) are rational functions of the exponent \( n \). In [14] it is shown that the condition \( |mr| \ll 1 \) imposes the extremely mild constraint \( n \gg (1/6)R_s^2R_0 \sim 10^{-25} \). Moreover, from the properties of the cosmological solution [11] we have \( f_{R_0}^2 \rho^{\cos}(t)/\kappa = -2n/(4n + 1) \), from which it follows that condition (5) is incompatible with the previous constraint \( n \gg 10^{-25} \):

\[
\left| \frac{\kappa}{f_{R_0}^2 \rho^{\cos}(t)} - 1 \right| \rho^{\cos} = \left( 3 + \frac{1}{2n} \right) \rho^{\cos}(t) \gg \rho^8(r) \rightarrow n \ll \frac{\rho^{\cos}}{2\rho^8} \sim 10^{-33}.
\]

We now check the assumption \( |R_1| \ll R_0 \). The previous result shows that we can not rely on the validity of Newtonian approximation. Hence we cannot use the effective gravitational constant \( G \) defined in (6) for the estimate of the ratio \( R_1/R_0 \), so that we resort to Newton’s gravitational constant \( G_N = c^4/16\pi\kappa \). The value of this ratio outside the Sun can be computed from the exterior solution (4) for \( R_1 \), while the result for the interior solution requires a more involved computation, based on a polynomial model of the mass density \( \rho^8 \), that can be found in [14]:

\[
\frac{R_1}{R_0} \approx \frac{1 + 4n}{n(1 + n)} \frac{G_N M_S}{r} \quad \text{for } r \geq R_S, \quad \frac{R_1}{R_0} \approx \frac{1}{1 + n} \quad \text{for } r < R_S.
\]

Though \( |R_1| \ll R_0 \) for \( n \gg 1 \), the interior solution shows that non-linear terms in the Taylor expansion of \( f^2(R) \) cannot be neglected, contradicting our assumption at the end of Section 2:

\[
\frac{f^2(R)}{R_0} = f_0^2 \left[ 1 - n \frac{R_1}{R_0} + \frac{n(n + 1)}{2} \left( \frac{R_1}{R_0} \right)^2 - \frac{1}{6} n(n + 1)(n + 2) \left( \frac{R_1}{R_0} \right)^3 \right] + O \left( \left( \frac{R_1}{R_0} \right)^4 \right).
\]

The lack of validity of the perturbative regime leads us to conclude that the model (7) cannot be constrained by this method, so that it remains, in this respect, a viable theory of gravity.

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