COMPACTNESS AND GENERIC FINITENESS FOR FREE BOUNDARY MINIMAL HYPERSURFACES (II)

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Abstract. Given a compact Riemannian manifold with boundary, we prove that the limit of a sequence of embedded, almost properly embedded free boundary minimal hypersurfaces, with uniform area and Morse index upper bound, always inherits a non-trivial Jacobi field. To approach this, we prove a one-sided Harnack inequality for minimal graphs on balls with many holes.

1. Introduction

1.1. Main results. Let \((M^{n+1}, \partial M, g)\) be a compact Riemannian manifold with boundary of dimension \(3 \leq (n + 1) \leq 7\). An \(n\)-submanifold \(\Sigma\) is a critical point of the \(n\)-dimensional area functional if and only if the mean curvature of \(\Sigma\) vanishes everywhere and \(\Sigma\) meets \(\partial M\) orthogonally. Such \(n\)-submanifolds are called free boundary minimal hypersurfaces (see Definition 2.1).

Given a free boundary minimal hypersurface \(\Sigma\), the second variation of area functional produces discrete eigenvalues and eigenfunctions in \(C^\infty(\Sigma)\). Then the dimension of the maximal subspace of \(C^\infty(M)\) that the second variation is negative definite, is called the index of \(\Sigma\), denoted by \(\text{index}(\Sigma)\) (cf. §2). And the eigenfunctions corresponding to the zero eigenvalue are called the Jacobi fields (see Definition 2.4).

Denote by \(\mathcal{M}(\Lambda, I)\) the space of embedded free boundary minimal hypersurfaces with \(\text{Area} \leq \Lambda\) and index \(\leq I\).

The compactness of \(\mathcal{M}(\Lambda, 0)\) was firstly studied by Fraser-Li [4] for 3-manifolds with non-negative Ricci curvature and convex boundary, and Guang-Li-Zhou [6] for higher dimensions without curvature assumptions.

Recently, Ambrozio-Carlotto-Sharp [1] proved the compactness of \(\mathcal{M}(\Lambda, I)\) under additional assumptions. Moreover, they proved that the limit hypersurface has non-trivial Jacobi fields. Later, the compactness result has been proved in [7] for all compact Riemannian manifolds with boundary. The degeneration of the limit hypersurface has also been obtained when the convergence has multiplicity one. In this paper, we enhance this theorem by considering the case of higher multiplicity in convergence.

Theorem 1.1. Let \(\{\Sigma_k\} \subset \mathcal{M}(\Lambda, I)\) locally smoothly converges to \(\Sigma \in \mathcal{M}(\Lambda, I)\) with multiplicity \(m\). Suppose that \(m \geq 2\). Then \(\Sigma\) has a positive Jacobi field.

The first study of such compactness was due to Choi-Schoen [2], who proved compactness for minimal surfaces with bounded topology in closed three-manifolds with positive Ricci curvature. In higher dimensions, Schoen-Simon-Yau [9] and Schoen-Simon [10]
proved interior curvature estimates and compactness for stable closed minimal hypersurfaces with uniform area upper bound.

Their results were later generalized by Sharp [11] to minimal hypersurfaces with uniform Morse index and area upper bound, which also says that the limit hypersurface is degenerate, i.e. has non-trivial Jacobi fields. Combining with the Bumpy Metric Theorem given by White [14], there are only finitely many embedded minimal hypersurfaces with uniform Morse index and area upper bound in manifolds with bumpy metrics. Such results plays an important role in the index estimates of minimal hypersurfaces using min-max construction, proved by Marques-Neves [8].

As a direct consequence of Theorem 1.1 we obtain the following generic finiteness theorem for free boundary minimal hypersurfaces.

**Corollary 1.2.** Let $M^{n+1}$ be a compact manifold with boundary and $3 \leq (n+1) \leq 7$. Fix $I \in \mathbb{N}$ and $\Lambda > 0$. Then for a generic metric on $M$, there are only finitely many almost properly embedded free boundary minimal hypersurfaces in $\mathcal{M}(\Lambda, I)$.

**Remark 1.3.** We can compare the results between minimal hypersurface with free boundary and closed cases. Fraser-Li’s result [14] is a natural free boundary analog of Choi-Schoens result [2]; Guang-Li-Zhou [6] obtained the free boundary version of Schoen-Simon-Yau [9] and Schoen-Simon’s results [10]; [17] and Theorem 1.1 together can be seen as a generalization of [11].

### 1.2. Harnack inequality

We approach Theorem 1.1 by proving a Harnack inequality. Let $N$ be a minimal hypersurface in $M$. Denote by $B(p; r)$ the geodesic ball in $N$. Denote by $A(p; r, s) = B(p; s) \setminus B(p; r)$. Similarly, the geodesic ball in $M$ is denoted by $B(p; r)$.

Let $\Gamma$ and $\Sigma$ be positive minimal graphs with functions $v, u$ on $A(p; r^2, 2\epsilon^2) \setminus \bigcup_{j=1}^I B(q_j; r^2)$ satisfying

$$0 < v(x) - u(x) \leq C_1(|x|^2 + r^2), \quad \text{where } |x| = \text{dist}_{\Sigma}(x, p).$$

Then the key ingredient to approach Theorem 1.1, roughly speaking, is to prove the following:

**Theorem 1.4** (Theorem 3.5). There exists $C = C(M, N, C_1, I)$, $\epsilon_0 = \epsilon_0(M, N, C_1, I)$ so that if $\epsilon < \epsilon_0$, $\{q_j\}_{j=1}^I \subset B(p; cr)$, then

$$\max_{\partial B(p; \epsilon^2)} (v - u)(x) \leq C \min_{\partial B(p; 2r)} (v - u)(x).$$

In high dimensional cases, i.e. $4 \leq (n+1) \leq 7$, we have the following better inequality, which can deduce Theorem 1.4 directly.

**Theorem 1.5** (Theorem 3.5). Let $\Gamma, \Sigma$ be positive minimal graphs with functions $v, u$ on $A(p; r, 2R)$ satisfying

$$0 < v(x) - u(x) \leq C_1|x|^2, \quad \forall x \in A(p; r, 2R), \quad \text{where } |x| = \text{dist}_{\Sigma}(x, p).$$

There exists $C$ and $\epsilon_0$ depending only on $C_1, M, N$ so that if $R \leq \epsilon_0$, then

$$\max_{\partial B(p; R)} (v - u)(x) \leq C \min_{\partial B(p; 2r)} (v - u)(x).$$
We remark that in three-dimensional case, Theorem 1.4 is sharp in some sense, which means that it is impossible get the estimate in Theorem 1.5. For the height of Catenoid in $\mathbb{R}^3$ tends to infinity even if it is small over $A(0, r, 4r)$.

The proofs of Theorem 1.4 and 1.5 are very technical and hence occupy the most pages of this paper (see §5 and §6). However, the idea is quite clear. For $\epsilon$ small enough and the assumption (1.1), the graph function can be seen as an ‘almost harmonic function’ on $\Xi$. By scaling $\Xi$ to a normal size, it can be regarded as a subset of Euclidean space. Then the Harnack inequality looks natural.

The difficulty here is that we can not use such blow-up argument directly because there are no suitable scaling size to make $\epsilon$ to be finite and $r^2$ to be positive simultaneously.

The classical methods from PDE to produce Harnack inequalities does not work since we have many boundaries here. Note that the classical Harnack says the maximum is bounded by the minimum in the interior. However, the Harnack inequality in Theorem 1.4 only states that the value of outside boundary can be bounded by that of inside boundary (so called one-sided Harnack inequality). Namely, the opposite inequality does not holds true by considering the Catenoid in $\mathbb{R}^3$.

Due to the so many boundaries inside, we can not use the minimal foliation argument given by White [13] to obtain the Harnack as in [11].

Therefore, we approach Theorem 1.4 by studying the differential inequality directly.

1.3. Outline of the proof of Theorem 1.1. We first recall the argument in [7]. Given a sequence of $\Sigma_k \in M(\Lambda, I)$, then there exists $\Sigma \in M(\Lambda, I)$ and a finite set $W \subset \Sigma$ with $\#W \leq I$ so that $\Sigma_k$ locally smoothly converges to $\Sigma$ in $M \setminus W$ with multiplicity $m$. Hence $\Sigma_k$ can be regarded as multi-graph on $\Sigma \setminus W$ with graph function $u_1 < u_2 < ... < u_m$.

Then inspired by Simon [12], the difference of top and bottom sheet may converges to a Jacobi field $w$ with possibly singular point on $W$. Then the aim is to prove that $W$ are all removable singular set.

Comparing to [1], the difficulty is that $W$ may have touching set of $\Sigma$, i.e. the set in $\Sigma \cap \partial M \setminus \partial \Sigma$. Let $p \in W$ be a touching point of $\Sigma$. Assume that $\partial M$ is on the non-positive side of $\Sigma$ near $p$. Then [7, Claim D] says that for $\epsilon$ small enough,

$$\max_{\partial B(p; \epsilon)} |u^m| \leq C \max_{\partial B(p; \epsilon)} u^m.$$ 

This gives a removable singularity theorem for the limit of the normalization of $u^m$.

Such a theorem is not enough to prove the existence of entire Jacobi fields because the top sheet near two singularities may not be the same. To overcome this, we need to prove that normalization of $u^1$ also converges to a smooth function.

We argue it by contradiction. Suppose not, then by the Harnack inequality on $\partial B(p; r)$ obtained in [7], the normalization of $u^1$ tends to $-\infty$ at $p$. So we can take $\epsilon \ll 1$ so that $h \geq \kappa u^m$ on $\partial B(p; \epsilon)$ for $\kappa \gg 1$, where $h$ is the minimum of $-u^1$ on $\partial B(p; \epsilon)$. Now let $S_k$ be the subset of $\Sigma_k$ near $B(p; \epsilon)$ such that the $\Sigma_k$ intersects with the level set of $\Sigma$ with large angles. Since index($\Sigma$) $\leq I$, then we need at most $I$ balls...
\( B(q; \rho(q)) \) with \( q \in S_k \) and \( \rho(q) = L(|q|^2 + h/\kappa) \) (see Claim 2). Denote by these balls \( \{B(q_j; \rho(q_j))\} \).

Let \( \Sigma'_k \) be the component of \( \Sigma_k \setminus \bigcup_j B(q_j; \rho(q_j)) \) containing the bottom sheet over \( \partial B(p; \epsilon) \). Then \( \Sigma'_k \) can be seen as a minimal graph over \( B(p; \epsilon) \). After applying Theorem 1.3 at most \( I \) times, we obtain

\[
\max_{\partial B(p; \epsilon)} -u^1 \leq C \min_{\partial B(p; C_0I/h(1/\kappa))} -u^1 \leq Ch/\kappa,
\]

which leads to a contradiction for \( \kappa \) large enough.

To proceed the argument of Theorem 1.1, it suffices to prove Theorem 1.4. Here we give an outline of the proof of it. Denote by

\[
\tau(q; s) = \int_{\partial B(q; s)} \langle \nabla w, \nu \rangle, \quad \text{and} \quad \mathcal{I}(q; s) = s^{n-1} \int_{\partial B(q; s)} w,
\]

where \( w = v - u \). Then by the Harnack inequality obtained in [7] (see also Corollary 3.4), \( \mathcal{I}(q; s) \) can be seen as the value of \( w \) on \( \partial B(q; s) \).

A direct computation (see (6.10)) by divergence theorem gives that

\[
\mathcal{I}(p; \epsilon) \leq 2\mathcal{I}(p; 2\sqrt{\epsilon r}) + 2\tau_0 \log(\sqrt{\epsilon}/r),
\]

where \( \tau_0 = \tau(p; 2\sqrt{\epsilon r}) \). Hence without loss of generality, we assume \( \mathcal{I}(p; 2\sqrt{\epsilon r}) \leq \mathcal{I}(p; 2\sqrt{\epsilon r}) \). Then we can find \( y_1 \in Q := \{q_i\}_{i=1}^I \) in (1.2) and \( \theta_1 \in (1/4^{I+3}, 1/16) \) (see Step A in Proposition 6.1) so that \( \tau(q_1; \theta_1 | q_j |) \geq 1/4^{I+3} \) and

\[
\mathcal{I}(p; r) - c_0 \mathcal{I}(y_1; \theta_1 | y_1 |) \geq c_0 (\log r - \log(\theta_1 | y_1 |)).
\]

Repeating the argument above, we can find a sequence of \( \{y_j\} \subset Q \) so that

\[
\mathcal{I}(p; \theta_j | y_j - y_{j-1}|) - c_0 \mathcal{I}(y_{j+1}; \theta_{j+1} | y_{j+1} - y_j|)
\geq c_0 \left[ \log(\theta_j | y_j - y_{j-1}|) - \log(\theta_{j+1} | y_{j+1} - y_j|) - c_1 \right].
\]

By adding them together with suitable coefficients (see Lemma 6.7), we obtain

\[
\mathcal{I}(p; \sqrt{\epsilon r}) \geq c | \log(\sqrt{\epsilon}/r)|.
\]

Then the desired results follows.

This paper is organized as follows: in Section 2 we will first give some notations; and in Section 3 we state some Harnack inequalities, including the classical one from blowing-up arguments and our new one-sided one; Using these, we construct Jacobi fields in Section 4. The proof of One-sided Harnack inequality is in Section 5 for \( n \geq 3 \) and Section 6 for \( n = 2 \), we give a proof of Harnack inequality in high dimensions; after that, some lemmas and tedious computation will be displayed in Appendix A, B, C and D.

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2. Preliminaries

In this section, we collect some basic definitions and preliminary results for free boundary minimal hypersurfaces. We refer to [7] for detailed notions.

Let $M^{n+1}$ be a smooth compact Riemannian manifold with non-empty boundary $\partial M$. We may assume that $M \hookrightarrow \mathbb{R}^L$ is isometrically embedded in some Euclidean space. By choosing $L$ large, we assume that $M$ is a compact domain of a closed $(n+1)$-dimensional manifold $\tilde{M}$.

Let $\Sigma^n$ be a smooth $n$-dimensional manifold with boundary $\partial \Sigma$ (possibly empty). A smooth embedding $\phi: \Sigma \to M$ is said to be an almost proper embedding of $\Sigma$ into $M$ if $\phi(\Sigma) \subset M$ and $\phi(\partial \Sigma) \subset \partial M$. We write $\Sigma = \phi(\Sigma)$ and $\partial \Sigma = \phi(\partial \Sigma)$.

We use $\text{Touch}(\Sigma)$ to denote the touching set $\text{int}(\Sigma) \cap \partial M$. If the touching set $\text{Touch}(\Sigma)$ is empty, then we say that $\Sigma$ is properly embedded.

**Definition 2.1.** An almost properly embedded hypersurface $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ is called a free boundary minimal hypersurface if and only if the mean curvature of $\Sigma$ vanishes and $\Sigma$ meets $\partial M$ orthogonally along $\partial \Sigma$.

Let $\Sigma^n \subset M^{n+1}$ be an almost properly embedded free boundary minimal hypersurface. The quadratic form of $\Sigma$ associated to the second variation formula is defined as

$$Q(v, v) = \int_{\Sigma} (|\nabla^\perp v|^2 - \text{Ric}_M(v, v) - |A^\Sigma|^2 |v|^2) \, d\mu_{\Sigma} - \int_{\partial \Sigma} h^{\partial M}(v, v) \, d\mu_{\partial \Sigma},$$

where $v$ is a section of the normal bundle of $\Sigma$, $\text{Ric}_M$ is the Ricci curvature of $M$, $A^\Sigma$ and $h$ are the second fundamental forms of the hypersurfaces $\Sigma$ and $\partial M$, respectively.

The Morse index of $\Sigma$ on the proper subset $\Sigma \setminus \text{Touch}(\Sigma)$ is defined to be the maximal dimension of a linear subspace of sections of normal bundle $N\Sigma$ compactly supported in $\Sigma \setminus \text{Touch}(\Sigma)$ such that the quadratic form $Q(v, v)$ is negative definite on this subspace.

**Remark 2.2.** In the following of this paper, the ‘Morse index of $\Sigma$’ always means the ‘Morse index on the proper subset $\Sigma \setminus \partial \Sigma$’, denoted by $\text{index}(\Sigma)$.

**Definition 2.3.** An almost properly embedded free boundary minimal hypersurface $\Sigma^n \subset M$ is said to be stable away from the touching set $\text{Touch}(\Sigma)$ if the Morse index of $\Sigma$ is 0.

**Definition 2.4.** We say that a function $f \in C^\infty(\Sigma)$ is a Jacobi field of $\Sigma$ is $f$ satisfies

$$\begin{align*}
\Delta_{\Sigma} f + (\text{Ric}_M(n, n) + |A^\Sigma|^2) f &= 0 \quad \text{on } \Sigma, \\
\frac{\partial f}{\partial \eta} &= h^{\partial M}(n, n) f \quad \text{on } \partial \Sigma,
\end{align*}$$

where $\eta$ is the co-normal of $\Sigma$.

For simplicity, we will use $\mathcal{M}(\Lambda, I)$ to denote the set of almost properly embedded free boundary minimal hypersurfaces with $\text{Area} \leq \Lambda$ and $\text{index}(\Sigma) \leq I$.

We remark that in the proofs of our results, we often allow a constant $C$ to change from line to line, and the dependence of $C$ should be clear in the context.
3. Harnack inequalities for minimal graphs

In this section, $M^{n+1}$ is always a closed manifold and $\mathcal{N}$ is an embedded compact minimal hypersurface in $M$ so that $\mathcal{B}(p; 1) \cap \partial \mathcal{N} = \emptyset$, where $\mathcal{B}(p; r)$ the intrinsic geodesic ball of $\mathcal{N}$ with radius $r$ and center $p \in \mathcal{N}$.

3.1. The Harnack inequalities on geodesic spheres. In this subsection, we always assume $3 \leq (n + 1) \leq 7$.

We recall Harnack estimate in a disk first. And then we state an second order estimate for minimal graph functions.

Lemma 3.1 (Gradient estimates,[7] Lemma 6.1). Suppose that two sequences of embedded compact minimal graphs (over $\mathcal{B}(p; 1)$) $\{\Sigma_k\}$ and $\{\Gamma_k\}$ with graph functions $\{u_k\}$ and $\{v_k\}$ converge smoothly to $\mathcal{B}(p; 1)$ and $u_k - v_k \geq 0$. Then there exists a constant $C = C(M, \mathcal{N})$ such that for any $r > 0$ and $p' \in \mathcal{B}(p; 1-r)$, we have

$$\limsup_{k \to \infty} \sup_{x \in \mathcal{B}(p'; r)} (r - \text{dist}_{\mathcal{N}}(x, p')) |\nabla \log(u_k - v_k)(x)| \leq C.$$

Definition 3.2. Let $\Omega \subset \mathcal{B}(p; 1)$ be an open set and $u, v$ be two functions on $\Omega$. Given a constant $K > 0$ and a positive function $f \in C^0(\Omega)$, we say that $(v, u)$ is a $(f, K)$-pair if they satisfy

$$(3.1) \quad v(x) - u(x) > 0, \quad |u(x)| + |v(x)| < f, \quad |\nabla u(x)| + |\nabla v(x)| < K.$$  

We say $(v, u)$ is a strong $(f, K)$-pair if it is a $(f, 2)$-pair and

$$|\nabla v(x)| + |\nabla^2 v(x)| \leq K|v(x)| \leq K^2|x|.$$

Furthermore, we have the following estimates:

Lemma 3.3. Let $M^{n+1}$ be a closed manifold with $3 \leq (n + 1) \leq 7$ and $\mathcal{N}$ be an embedded compact minimal hypersurface in $M$ so that $\mathcal{B}(p; 1) \cap \partial \mathcal{N} = \emptyset$. Given a constant $K > 0$, there exist constants $C = C(M, \mathcal{N}, K)$ and $\delta = \delta(M, \mathcal{N}, K)$ so that if $q \in \mathcal{B}(p; 1-r)$ for some $0 < r < 1$, $\Sigma$ and $\Gamma$ are minimal graphs with graph functions $u$ and $v$ over $\mathcal{B}(q; r) \setminus V$ for a compact subset $V \subset \mathcal{N}$ and $(v, u)$ is a $(\delta, K)$-pair, then

$$\text{dist}_{\mathcal{N}}(x, \partial \mathcal{B}(q; r) \cup V) \cdot |\nabla \log(v - u)(x)| < C,$$

$$\text{dist}_{\mathcal{N}}^2(x, \partial \mathcal{B}(q; r) \cup V) \cdot \frac{|\nabla^2(v - u)(x)|}{(v - u)(x)} < C.$$

This lemma can be proved by a standard blow-up process, which is the same with Lemma 3.1. We give the proof in Appendix [3] for the completeness of this paper.

Corollary 3.4. Given a constant $K > 0$, $\theta \in (0, 1/8)$ and $0 < R < 1/2$, there exist constants $C = C(M, \mathcal{N}, K, I, \theta)$, $C_0 = C_0(M, \mathcal{N}, \theta)$ and $\delta = \delta(M, \mathcal{N}, K)$ so that if $\Sigma$ and $\Gamma$ are minimal graphs with functions $u$ and $v$ over $\mathcal{A}(p; \theta R, 2R) \setminus \bigcup_{j=1}^J \mathcal{B}(q_j; r)$, and $(v, u)$ is a $(\delta, K)$-pair (see Definition 3.2) and

- $\mathcal{B}(q_j; r) \subset \subset \mathcal{A}(p; 4\theta R, R/2)$;
- $\theta R \geq C_0 I r$, 

we have

$$|u(x)| + |v(x)| < \theta R,$$

for any $x \in \mathcal{B}(q_j; r) \setminus \bigcup_{j=1}^J \mathcal{B}(q_j; r)$.
then we have
\[
\max_{x \in \partial B(p;R)} (v - u)(x) \leq C \min_{x \in \partial B(p;2\theta R)} (v - u)(x).
\]

**Proof.** For simplicity, denote by \( w(x) = u(x) - v(x) \).

By Lemma 3.1 we can take \( C_0 \) suitable so that there exists a \( C^1 \) curve \( \gamma : [0, 1] \rightarrow \mathcal{A}(p; 2\theta R, R) \) connecting \( \partial B(p; 2\theta R) \) and \( \partial B(p; R) \) so that
\[
\text{Length}(\gamma) \leq C_0 R \quad \text{and} \quad \text{dist}(\gamma, \cup_j B(q_j; r)) \geq \theta R / (C_0 I).
\]
Then Lemma 3.3 gives that there exists \( C_1 = C_1(M, N, K) \) so that for any \( x \in \gamma \),
\[
|\nabla \log w(x)| \leq C_1 / \text{dist}(x, \cup_j B(q_j; r)).
\]
Integrating it over \( \gamma \), together with (3.2) we have
\[
w(\gamma(0)) \leq e^{C_3 C_1 I / 2} w(\gamma(1)).
\]
Moreover, the Lemma 3.3 also implies that
\[
\max_{\partial B(p;R)} w \leq e^{C_0} \min_{\partial B(p;R)} w, \quad \max_{\partial B(p;2\theta R)} w \leq e^{C_1} \min_{\partial B(p;2\theta R)} w.
\]
Hence the desired inequality follows. \( \square \)

### 3.2. One-sided Harnack inequalities.

In this section, \( M^{n+1} \) is always a closed manifold and \( N \) is an embedded compact minimal hypersurface in \( M \) so that \( B(p; 1) \cap \partial N = \emptyset \).

**Theorem 3.5.** Let \( 4 \leq (n + 1) \leq 7 \). Given \( C_1, K > 0, I \in \mathbb{N}, \) there exist \( C = C(M, N, C_1, K, I) \), \( C_0 = C_0(M, N) \) and \( R_0 = R_0(M, N, C_1, K) \) so that if \( \Gamma, \Sigma \) are minimal graphs with functions \( v, u \) on \( \mathcal{A}(p; r, 2R) \setminus \bigcup_{j=1}^{l} B(q_j; r) \) for \( \{q_j\} \subset \mathcal{A}(p; 4r, R/2), \) \( R_0 / 4 \geq R \geq C_0 I r \) and \( (v, u) \) is a strong \( (C_1|x|^2, K) \)-pair (see Definition 3.2), where \( |x| = \text{dist}_{\Sigma}(x, p) \) then there exists \( \tilde{r} \leq C_0 I r \) so that
\[
\max_{x \in \partial B(p;R)} (v - u)(x) \leq C \min_{x \in \partial B(p;\tilde{r})} (v - u)(x).
\]
As a corollary,
\[
\max_{\partial B(p;R)} (v - u)(x) \leq C r^2.
\]

**Theorem 3.6.** Let \( n = 3 \). Given \( C_1, K > 0, I \in \mathbb{N}, \) there exist \( C = C(M, N, C_1, K, I) \), \( \epsilon_0 = \epsilon_0(M, N, C_1, K, I) \) so that if \( \Gamma, \Sigma \) are minimal graphs with functions \( v, u \) on \( \Xi := \mathcal{A}(p; r^2, 2\epsilon) \setminus \bigcup_{j=1}^{l} B(q_j; r^2) \) for some \( \epsilon < \epsilon_0 \) and
- \( (v, u) \) is a strong \( (C_1|x|^2 + r^2, K) \)-pair (see Definition 3.2), where \( |x| = \text{dist}_{\Sigma}(x, p) \);
- \( \{q_j\} \subset \mathcal{B}(p; s) \) for some \( s \in [\sqrt{\epsilon} r / 2, \epsilon / 2] \);
then we have
\[
\max_{\partial B(p;\epsilon)} (v - u)(x) \leq C \min_{\partial B(p;s)} (v - u)(x).
\]

**Remark 3.7.** Theorem 3.6 is equivalent to that statement for \( s = \sqrt{\epsilon} r \). Namely, let \( R = s / \sqrt{\epsilon} \). Then \( R \geq r \) and \( v, u \) are minimal graph functions over \( \mathcal{A}(p; R^2, 2\epsilon) \setminus \bigcup_{j=1}^{l} B(q_j; R^2) \) and \( \{q_j\} \subset \mathcal{B}(p; \sqrt{\epsilon} R) \).
4. Existence of Jacobi fields

Let \((M^{n+1}, \partial M, g)\) be a compact manifold with boundary of dimension \(3 \leq (n+1) \leq 7\). Recall that \(\mathcal{M}(\Lambda, I)\) is the space of almost properly embedded free boundary minimal hypersurfaces with index \(\leq I\) and \(\text{Area} \leq \Lambda\).

We first recall the following compactness theorem:

**Theorem 4.1** ([7, Theorem 4.1]). Let \(\{\Sigma_k\} \subset \mathcal{M}(\Lambda, I)\). Then up to a subsequence, \(\Sigma_k\) converges smoothly and locally uniformly to \(\Sigma\) on \(\Sigma \setminus W\) with finite multiplicity, where \(W \subset \Sigma\) is a finite subset. Moreover, if the convergence has multiplicity one and \(\Sigma_k \neq \Sigma\) eventually, then \(\Sigma\) has a non-trivial Jacobi field.

We now review the convergence. We assume that \(\Sigma\) is two-sided.

Let \(n\) be the unit normal of \(\Sigma\) and \(X \in \mathfrak{X}(M, \Sigma)\) (see [7, §2]) be an extension of \(n\). Suppose that \(\phi_t\) is a one-parameter family of diffeomorphisms of \(\tilde{M}\) generated by \(X\).

For any domain \(U \subset \Sigma\) and small \(\delta > 0\), \(\phi_t\) produces a neighborhood \(U_\delta\) of \(U\) with thickness \(\delta\), i.e., \(U_\delta = \{\phi_t(x) \mid x \in U, |t| \leq \delta\}\). If \(U\) is in the interior of \(\Sigma\), then \(U_\delta\) is the same as \(U \times [-\delta, \delta]\) in the geodesic normal coordinates of \(\Sigma\) for \(\delta\) small. Now fix a domain \(\Omega \subset \subset \Sigma \setminus W\), by the convergence \(\Sigma_k \to \Sigma\), we know that for \(k\) sufficiently large, \(\Sigma_k \cap \Omega_\delta\) can be decomposed as \(m\) graphs over \(\Omega\) which can be ordered by height

\[u_1^k < u_2^k < \cdots < u_m^k.\]

**Theorem 4.1** says that \(\Sigma\) is degenerate when \(m = 1\). In this paper, we improve **Theorem 4.1**:

**Theorem 4.2.** Let \(\{\Sigma_k\} \subset \mathcal{M}(\Lambda, I)\) as in **Theorem 4.1**. Suppose that \(m \geq 2\). Then \(\Sigma\) is degenerate, i.e., \(\Sigma\) has a non-trivial Jacobi field.

**Proof.** If \(\Sigma\) is one-sided, we can then construct a non-trivial Jacobi field over \(\tilde{\Sigma}\) and the construction is similar to the case when \(\Sigma\) is two-sided. Hence, in the following, we will assume that \(\Sigma\) is two-sided.

For any \(p \in W\) and \(\epsilon \ll 1\), set

\[\lambda_k(p, \epsilon) = \max_{\partial B(p, \epsilon)} \{u_k^m, -u_k^1\} \quad \text{and} \quad \Lambda_{k, \epsilon} = \max_{p \in W} \lambda_k(p, \epsilon).\]

Set \(w_k = u_k^m - u_k^1\) and \(\overline{w}_k = w_k / \Lambda_{k, \epsilon}\). Taking an exhaustion \(\{\Omega_i\}\) of \(\Sigma \setminus W\), we obtain a Jacobi field \(w\) on \(\Sigma \setminus W\). Note that \(w\) may be trivial or unbounded.

We pause to give the following claim, which is from [7, Claim D].

**Claim 1.** For each \(p \in W\), there exists a constant \(C = C(M, \Sigma, \overline{\Sigma}, \epsilon)\) such that either \(j = 1\) or \(j = m\) satisfies the following

\[\limsup_{r \to 0} \limsup_{k \to \infty} \frac{\max_{\partial B(p, \epsilon)} |u_k^j|}{\max_{\partial B(p, \epsilon)} u_k^j} \leq C.\]

**Proof of Claim.** First assume that \(p \in \text{Int} \Sigma \cap \partial M\). Then without loss of generality, we assume that \(\partial M\) lies on the negative side of \(\Sigma\) near \(p\) as in [7]. Then [7, Page 19,
Claim C gives that for any \( r \in (0, \epsilon) \),
\[
\max_{x \in \partial B(p; r)} \ u_{m}^{k} > 0 \quad \text{for } k \text{ sufficiently large.}
\]
Then the conclusion of Claim \( \Pi \) for \( j = m \) follows from the argument in \[7\] Page 20-21, Claim D.

It remains to consider \( p \notin \text{Int} \Sigma \cap \partial M \). Then the minimal foliation argument works for both \( u_{1}^{k} \) and \( u_{m}^{k} \). Note that either
\[
\max_{x \in \partial B(p; r)} \ u_{m}^{k} > 0 \quad \text{for } k \text{ sufficiently large},
\]
or
\[
\max_{x \in \partial B(p; r)} -u_{1}^{k} > 0 \quad \text{for } k \text{ sufficiently large},
\]
Then the desired result also follows from the argument in \[7\] Page 20-21, Claim D. \( \square \)

We first consider the case \( w = 0 \) on \( \Sigma \). Then we set \( \overline{\pi}_{k}^{j} = u_{i}^{j}/\Lambda_{k, \epsilon} \) for \( 1 \leq j \leq m \). Then for any \( \Omega \subset \subset \Sigma \setminus \mathcal{W} \), \( \overline{\pi}_{k}^{j} \) is uniformly bounded. Hence \( \overline{\pi}_{k}^{m} \) locally smoothly converges to a Jacobi field \( \overline{\pi} \). It follows that \( \overline{\pi}_{k}^{j} \rightarrow \overline{\pi} \) since \( w = 0 \).

Therefore, \( \overline{\pi} \) is smooth through \( \mathcal{W} \). Then by the definition of \( \Lambda_{k, \epsilon} \), we have either \( \max \overline{\pi}_{k}^{m} = 1 \) or \( \max(-\overline{\pi}_{k}^{1}) = 1 \). This gives that \( \overline{\pi} \) is non-trivial. Thus we also get a nontrivial Jacobi field in this case.

It remains to consider \( w \) is non-trivial. Then Theorem 4.2 follows from this lemma:

**Lemma 4.3.** \( w \) is bounded.

We postpone the proof to the next subsection.

Note that for \( \Omega \subset \subset \Sigma \setminus \mathcal{W} \), \( w \) is uniformly bounded. Denote by \( \mathcal{W}_{0} \) the subset of \( \mathcal{W} \cap \partial M \setminus \partial \Sigma \) so that
\[
\text{given } \epsilon > 0, \ B^{M}(p; \epsilon) \cap \partial \Sigma_{k} \neq \emptyset \quad \text{for } k \text{ sufficiently large.}
\]
It follows from \[11\] Section 6 (see also \[7\] Section 2.3) that \( w \) is smooth through \( W \setminus \mathcal{W}_{0} \). \( \square \)

**Remark 4.4.** Note that in the Proof of Theorem 4.2, for \( p \notin \mathcal{W} \), \( w_{k}/w_{k}(p) \) always converges to a positive Jacobi field \( w' \) with possibly discrete singularities on \( \mathcal{W} \), where \( w' \) may be infinity by Lemma 3.1. Then a classical PDE theory (a cut-off trick) shows that \( \Sigma \) is stable, which implies that the Jacobi field is positive.

The following subsections are devoted to the proof lemma 4.3.

4.1. **Proof of Lemma 4.3.** We prove it by a contradiction argument. Suppose that \( w \) is unbounded.

Without loss of generality, we assume that \( \partial M \) lies on the negative side of \( \Sigma \) around \( p \). Then by the Claim D in \[7\], there exists a constant \( C = C(K, \epsilon) \) such that
\[
\lim_{r \to 0} \limsup_{k \to \infty} \sup_{x \in \partial B(p; r)} \frac{\max_{x \in \partial B(p; x)} |u_{k}^{m}|}{\max_{x \in \partial B(p; x)} u_{k}^{m}} \leq C.
\]
Hence $u_k^m / \Lambda_{k, \varepsilon}$ is uniformly bounded near $p$. Together with the assumption of $w$ is unbounded, then we have $u_k^1 / \Lambda_{k, \varepsilon}$ is unbounded around $p$ as $k \to \infty$. Then for any $\kappa > 0$ (would be fixed later), we can shrink $\varepsilon$ so that for $k$ sufficiently large,

\[(4.1) \max_{\partial B(p; \varepsilon)} -u_k^1 > \kappa \cdot \max_{\partial B(p; \varepsilon)} u_k^m .\]

Recall that $\partial M$ is smooth. Hence there exists a constant $C_1 > 1$ so that the graph function $u_{\partial M}$ of $\partial M$ on $B(p; \varepsilon)$ satisfying

\[(4.2) \quad u_{\partial M} \geq -C_1 |x|^2 \quad \text{for} \quad |x| \leq \varepsilon, \quad \text{where} \quad |x| = \text{dist}_\Sigma(x, p).\]

We can also take $\delta$ small enough so that the minimal foliation near $B(p; \varepsilon)$ containing $B(p; \varepsilon) \times [-\delta, \delta]$. Denote by $\pi$ the projection to $\Sigma$.

Set $h = \min_{\partial B(p; \varepsilon)} -u_k^1$ and

$$S_k = \{ x \in \Sigma_k \cap B(p; \varepsilon) \times [-\delta, \delta] : |\langle n_k, \nabla d \rangle| \leq 1/2 \},$$

where $n_k$ is the unit normal vector field of $\Sigma_k$ and $d$ is the signed distance function to $\Sigma$. Then $S_k$ is a closed set of $\Sigma_k$. Note that $\varepsilon$ can be taken small enough so that $\partial \Sigma_k \cap (B(p; \varepsilon) \times [-\delta, \delta]) \subset S_k$.

Let $\rho(x) = L(|\pi(x)|^2 + h/\kappa)$, where $L$ is a constant (to be specified later).

**Claim 2.** There exist $\{x_j\}_{j=1}^l \subset \Sigma_k$ so that

$$S_k \subset \bigcup_{j=1}^l B^M(x_j; \rho(x_j)).$$

**Proof of Claim 2.** First take any $x_1 \in S_k$ so that

$$\rho(x_1) = \max_{x \in S_k} \rho(x).$$

If we have $x_1, ..., x_j$, then take $x_{j+1} \in S_k \setminus \bigcup_{i=1}^j B^M(x_i; \rho(x_i))$ so that

$$\rho(x_{j+1}) = \max\{ \rho(x) : x \in S_k \setminus \bigcup_{i=1}^j B^M(x_i; \rho(x_i)) \}.$$  

If the process does not stop in $I$ steps, then there exists $\{x_j\}_{j=1}^{l+1} \subset \Sigma_k$ satisfying

$$\text{dist}_M(x_j, x_i) \geq \rho(x_i), \quad \text{for} \quad j > i.$$  

Note that for $j > i$, by the choice of $x_j$, $\rho(x_j) \leq \rho(x_i)$. It follows that

$$B^M(x_j; \rho(x_j)/3) \cap B^M(x_i; \rho(x_i)/3) = \emptyset \quad \text{for} \quad i \neq j.$$  

Then applying [7, Lemma 2.11], there exists $y \in \{x_j\}_{j=1}^{l+1}$ so that $\Sigma_k$ is stable in $B^M(y; \rho(y)/3)$ since the Morse index of $\Sigma_k$ is bounded by $I$. Using the curvature estimate [7, Theorem 3.2], we have

\[(4.3) \quad \sup_{x \in \Sigma_k \cap B^M(y, \rho(y)/4)} |A^{\Sigma_k}|^2(x) \leq C_2/(\rho(y))^2 \]

for some uniform constant $C_2 > 0$. 
Since \( y \in S_k \), then there exists \( \nu \in T_y \Sigma_k \) so that \( \langle \nu, \nabla d \rangle > 1/2 \). let \( \gamma \) be the geodesic starting at \( y \) with direction \( \nu \). By direct computation,

\[
\frac{d}{ds} \langle \gamma'(s), \nabla d \rangle = A_k(\gamma'(s), \gamma'(s)) \langle \nabla d, n_k \rangle + \nabla^2 d(\gamma'(s), \gamma'(s)) = A_k(\gamma'(s), \gamma'(s)) \langle \nabla d, n_k \rangle + \nabla^2 d((\gamma'(s))^\top, (\gamma'(s))^\top),
\]

where \( A_k \) is the second fundamental form of \( \Sigma_k \) and \( (\gamma'(s))^\top \) is the projection to \( \{d^{-1}(d(\gamma_k(s)))) \}. Hence for \( t \in (0, \rho(y)/(10C_2)) \),

\[
\frac{d}{dt} d(\gamma(t)) = \langle \gamma'(t), \nabla d \rangle \geq \frac{1}{2} - \int_0^t (|A_k(\gamma(s))| + 1) ds \geq \frac{1}{2} - 2C_2t/\rho(y) \geq \frac{1}{4}.
\]

Therefore, \( \gamma(t) \notin \partial \Sigma_k \) for \( t \in (0, \rho(y)/(10C_2)) \).

Furthermore,

\[
d(\gamma(\frac{\rho(y)}{10C_2})) = d(\gamma(0)) + \int_0^{\rho(y)/(10C_2)} \langle \nabla d, \gamma'(t) \rangle dt
\geq \frac{1}{4} \cdot \rho(y)/(10C_2)
\geq (\frac{1}{40C_2} - \frac{C_1}{L}) \rho(y).
\]

Then we can take \( L = L(C_1, C_2) \) large enough so that

\[
(\frac{1}{40C_2} - \frac{C_1}{L}) \rho(y) \geq \rho(y)/\sqrt{L} = \sqrt{L} \cdot h/\kappa,
\]

which leads to a contradiction to our assumptions. This completes the proof of Claim \( \Box \)

Now let \( \Sigma'_k \) be the component of \( \Sigma_k \setminus \bigcup_{j=1}^f B(x_j; \rho(x_j)) \) containing the bottom sheet graph on \( \partial B(p; \epsilon) \). By the definition of \( S_k \), we have \(|\langle n_k, \nabla d \rangle| > 1/2 \). Thus we conclude that \( \Sigma'_k \) is a minimal graph on \( \pi(\Sigma'_k) \). Denote by \( u_k \) the minimal graph function. A standard computation (see Appendix A) gives that

\[
|\langle n_k, \nabla d \rangle| = 1/\sqrt{1 + |\nabla u_k|^2}.
\]

It follows that

\[
|\nabla u_k(x)| \leq 1, \text{ for } x \in \pi(\Sigma'_k).
\]

Note that \( u_k \) may not be negative everywhere. To overcome this, we recall the minimal foliation near \( \Sigma \). Let \( t = \max_{\partial B(p; \epsilon)} u_k^m \) and \( \Sigma_t \) be the slice in the minimal foliation, i.e. \( \Sigma_t \) is a minimal graph on \( B(p; \epsilon) \) and \( v_t = t \) on \( \partial B(p; \epsilon) \), where \( v_t \) is the graph function. Then for \( x \in B(p; \epsilon/2) \),

\[
|\nabla^2 v_t| + |\nabla v_t| \leq K|v_t|,
\]

for some universal constant \( K \). Moreover, by the assumption \( (4.1) \),

\[
h \geq \kappa v_t.
\]
Without loss of generality, we can assume that
\[ |\nabla v_t(x)| \leq 1 \quad \text{for all} \ x \in B(p; \epsilon). \tag{4.7} \]

**Claim 3.** For \( x \in \Sigma \) with \( |x| \geq \sqrt{h/\kappa} \),
\[ (v_t - u_k)(x) \leq (C_1 + 1)|x|^2. \]

**Proof of Claim 3.** Note that \( v_t \leq h/\kappa \leq |x|^2 \) for \( |x| \geq \sqrt{k/\kappa} \). Together with (4.2), we have
\[ (v_t - u_k)(x) \leq |x|^2 - u_{\partial \mathcal{M}}(x) \leq (C_1 + 1)|x|^2, \]
which is the desired inequality. \( \square \)

Denote by \( Q = \{ \pi(x_j) \} \) and \( s_1 = \max_{1 \leq j \leq I} \rho(x_j) \). Recall that \( \pi \) is the projection to \( \Sigma \).

**Claim 4.** There exist \( r_1 \in [8s_1, 4M_1 + 3s_1] \) and \( Q_1 \subset Q \) satisfying the following:
(i) for any \( x \in \{ x_j \}_{j=1}^I \), there exists \( x' \in Q_1 \) so that \( \pi(B(x; \rho(x))) \subset B(x'; r_1/4) \) or \( \pi(B(x; \rho(x))) \subset B(p; r_1/4) \);
(ii) \( B(x'; 4r_1) \cap B(x'' ; 4r_1) = \emptyset \) for two different points \( x', x'' \in Q_1 \);
(iii) \( B(x'; 4r_1) \cap B(p; 4r_1) = \emptyset \) for all \( x' \in Q_1 \).

**Proof of the Claim 4.** Note that for \( k \to \infty \), \( \max_{1 \leq j \leq I} \text{dist}_\mathcal{M}(x_j, \Sigma) \to 0 \). So without loss generality, we can assume that for any \( r < 1 \)
\[ \pi(B(x_j; r)) \subset B(\pi(x_j); 2r). \]
Now let \( Q_1 = Q \) and \( r_1 = 8s_1 \). Then the first item follows immediately. If such \( Q_1 \) and \( r_1 \) satisfy all the requirements, then we are done. Otherwise, there exists \( y \in Q_1 \) so that either
\[ B(x'; 4r_1) \cap B(y'; 4r_1) \neq \emptyset, B(x'; 4r_1) \setminus B(y'; 4r_1) \neq \emptyset \quad \text{for some} \ x' \in Q_1, \]
or
\[ B(y; 4r_1) \cap B(p'; 4r_1) \neq \emptyset. \]
In both cases, we replace \( (Q_1, r_1) \) by \( (Q_1 \setminus \{ y \}, 64r_1) \). Then \( B(y; 2s_1) \subset B(x'; r_1/4) \). Hence \( \square \) still holds true for our new \( Q_1 \) and \( r_1 \).

As far, we have proved that if \( Q_1 \) and \( r_1 \) satisfy \( \square \) but not the last two requirements, then we can replace \( (Q_1, r_1) \) by \( (Q_1 \setminus \{ y \}, 64r_1) \) for some \( y \in Q_1 \) so that the new \( Q_1 \) and \( r_1 \) also satisfy \( \square \).

Note that each time we get the new \( Q_1 \) with fewer element. Thus, such a process will stop in \( N(\leq I) \) steps. Then those our desired \( Q_1 \) and \( r_1 \). \( \square \)

Note that \( v_t, u_k \) are positive minimal graph functions on \( \Xi := \mathcal{A}(p; r_1, \epsilon) \setminus \bigcup_{y \in Q_1} B(y; r_1) \). Moreover,

**Claim 5.** If \( r_1 \geq \sqrt{h/\kappa} \), then \( (v_t, u_k) \) is a \( ((C_1 + 1)|x|^2, 2) \)-pair (see Definition 3.3) on \( \Xi \).
Proof of Claim 6. Recall that \( t = \max_{\partial B(p; \epsilon)} u_k^m \) and \( u_k \) is the graph function of \( \Sigma_k \). Then it follows that \( v_t - u_k \geq 0 \) on \( \Xi \). If \( r_1 \geq \sqrt{h/\kappa} \), using Claim 3 then we have
\[
v_t - u_k \leq (1 + C_1)|x|^2, \quad \text{for all } x \in \Xi.
\]
Also, (4.4) and (4.7) gives that
\[
|\nabla u_k(x)| + |\nabla v_t(x)| \leq 2 \quad \text{for all } x \in \Xi.
\]
Therefore, \((v_t, u_k)\) is a \(((C_1 + 1)|x|^2, 2\))

Proof of Claim 5. Recall that \( Q \cap B(\cdot) \) \( \Xi \)

Part 1: In this part, we address the high dimensional case: \( 4 \leq (n + 1) \leq 7 \).

Let \( R_0 = R_0(M, \Sigma) \) and \( C_0 = C_0(M, \Sigma) \) be the constants in Theorem 3.5. Then we can take \( \epsilon \) small enough so that \( \epsilon < R_0/8 \). Then for \( k \) large enough so that \( r_1 \leq 4^{3l+3}s_1 \leq \epsilon/(4^{l+3}C_0) \). Now applying Theorem 3.5 there exists \( \tilde r_1 \leq C_04^{l+1}r_1 \) so that
\[
\max_{\partial B(p; \epsilon)} (v_t - u_k)(x) \leq C \min_{\partial B(p; \tilde r_1)} (v_t - u_k)(x),
\]
and
\[
B(y; r_1) \cap A(p; \tilde r_1/2, 2\tilde r_1) = \emptyset, \quad \text{for all } y \in Q_1.
\]

Now we construct \( Q_j \subset Q \) inductively:

Claim 6. Suppose that, for some \( j \in \mathbb{N} \), \( Q_j \subset Q \) and \( \tilde r_j > C_0 \cdot 4^{10l+10} \sqrt{h/\kappa} \) satisfying
- \( B(y; r_j) \cap A(p; \tilde r_j/2, 2\tilde r_j) = \emptyset \), for all \( y \in Q_j \);
- \( Q_j \neq \emptyset \) and \( Q \cap B(p; \tilde r_j) \neq \emptyset \).

Then
\[
(4.8) \quad \#(Q \cap B(p; \tilde r_j)) \leq I - j;
\]
and there exist a non-empty set \( Q_{j+1} \subset Q \cap B(p; \tilde r_j) \) and \( r_{j+1} < \tilde r_j \) so that \((v_t, u_k)\) is a strong \(((C_1 + 1)|x|^2, K)\)-pair on \( A(p; r_{j+1}, \tilde r_j) \setminus \cup_{y \in Q_{j+1}} B(y; r_{j+1}) \).

Proof. Once we have \( \tilde r_j \) and \( Q_j \), then set
\[
s_{j+1} = \max ( \max_{x \in Q \cap B(p; \tilde r_j)} \rho(x_j), \sqrt{h/\kappa}).
\]
By the same process with Claim 4 we can take \( r_{j+1} \in [8s_{j+1}, 4^{3l+3}s_{j+1}] \) and \( Q_{j+1} \subset Q \cap B(p; \tilde r_j) \) satisfying the following:
- for any \( x \in Q \cap B(p; \tilde r_j) \), there exists \( x' \in Q_{j+1} \) so that \( \pi(B(x; \rho(x))) \subset B(x'; r_{j+1}/4) \) or \( \pi(B(x; \rho(x))) \subset B(p; r_{j+1}/4) \);
- \( B(x'; 4r_{j+1}) \cap B(x''; 4r_{j+1}) = \emptyset \) for two different points \( x', x'' \in Q_{j+1} \);
- \( B(x'; 4r_{j+1}) \cap B(p; 4r_{j+1}) = \emptyset \) for all \( x' \in Q_{j+1} \).

We now check such \( Q_{j+1} \) satisfies our requirements. Recall that \( u_k \) is well-defined on
\[
B(p; \epsilon) \setminus \bigcup_{y \in Q} \pi(B(y; \rho(y))).
\]
Note that for any \( x \in Q \cap B(p; \tilde{r}_j) \), there exists \( x' \in Q_{j+1} \) so that \( \pi(B(x; \rho(x))) \subset B(x'; r_{j+1}/4) \) or \( \pi(B(x; \rho(x))) \subset B(p; r_{j+1}/4) \). Together with \( r_{j+1} \geq s_{j+1} \geq \sqrt{h/\kappa} \), by Claim 5, \((v, u)\) is a \(((C_1 + 1)|x|^2, 2)\)-pair on \( \mathcal{A}(p; r_{j+1}, \tilde{r}_j) \setminus \bigcup_{y \in Q_{j+1}} B(y; r_{j+1}) \). Together with (4.5) and (4.6), we conclude that \((v, u)\) is a strong \(((C_1 + 1)|x|^2, K)\)-pair.

It remains to prove (4.8). By the definition of \( \tilde{r}_j \), there exist \( z \in Q_{j-1} \) (\( Q_0 := Q \)) so that

\[
\tilde{r}_j \leq C_0 \cdot 4^{4I+4} \cdot L(|\pi(z)|^2 + h/\kappa).
\]

Recall that \( \tilde{r}_j > C_0 \cdot 4^{10I+10} \sqrt{h/\kappa} \). This deduces that

\[
L(|\pi(z)|^2 + h/\kappa) \geq \sqrt{h/\kappa}.
\]

Note that \( L = L(C_1, C_2) \) depends only on \( M \) and \( \Sigma \) (see Claim 2). Thus, we can take \( \epsilon \) small enough so that

\[
\sqrt{h/\kappa} \geq 2L \cdot h/\kappa,
\]

which implies that

\[
h/\kappa \leq |\pi(z)|^2.
\]

Then (4.9) becomes

\[
\tilde{r}_j \leq 2C_0L \cdot 4^{4I+4} |\pi(z)|^2 \leq |\pi(z)|.
\]

We conclude that

\[
\#(Q \cap B(p; \tilde{r}_j)) \leq \#(Q \cap B(p; \tilde{r}_{j-1})) - 1, \text{ where } \tilde{r}_0 = \epsilon.
\]

By induction, (4.8) follows. \( \square \)

By Claim 6, \((v, u)\) is a \(((C_1 + 1)|x|^2, 2)\)-pair on \( \mathcal{A}(p; r_{j+1}, \tilde{r}_j) \setminus \bigcup_{y \in Q_{j+1}} B(y; r_{j+1}) \). Then applying Theorem 3.5 again, there exists \( \tilde{r}_{j+1} \leq C_0 4^{I+1} r_{j+1} \) so that

\[
\mathcal{B}(y; r_{j+1}) \cap \mathcal{A}(p; \tilde{r}_{j+1}/2, 2\tilde{r}_{j+1}) = \emptyset, \quad \text{for all } y \in Q_1,
\]

and

\[
\max_{\partial \mathcal{B}(p; \tilde{r}_j)} (v_t - u_k)(x) \leq C \min_{\partial \mathcal{B}(p; \tilde{r}_{j+1})} (v_t - u_k)(x),
\]

which also implies that

\[
\max_{\partial \mathcal{B}(p; \epsilon)} (v_t - u_k)(x) \leq C_{j+1} \min_{\partial \mathcal{B}(p; \tilde{r}_{j+1})} (v_t - u_k)(x).
\]

By (4.8), such processes must stop in \( N(\leq I) \) steps. Using Claim 6 together with (4.10) and \( Q_j \neq \emptyset \) for all \( j \leq N \), we conclude that

- either \( \tilde{r}_N \leq C_0 \cdot 4^{10I+10} \sqrt{h/\kappa}; \)
- or \( Q \cap \mathcal{B}(p; \tilde{r}_N) = \emptyset. \)

In the first case, then for \( x \in \mathcal{B}(p; \tilde{r}_N) \)

\[
(v_t - u_k)(x) \leq 2C_1 \tilde{r}_N^2 \leq Ch/\kappa,
\]

which implies

\[
\max_{\partial \mathcal{B}(p; \epsilon)} (v_t - u_k)(x) \leq C^N \min_{\partial \mathcal{B}(p; \tilde{r}_N)} (v_t - u_k)(x) \leq Ch/\kappa.
\]
In the second case, \((v_t, u_k)\) is a \(((C_1 + 1)|x|^2, 2)\)-pair on \(\mathcal{A}(p; \sqrt{h/\kappa}, 2\tilde{r}_N)\), and hence a strong \(((C_1 + 1)|x|^2, K)\)-pair by (1.5) and (1.6). Using Theorem 3.5 again,

\[
\max_{x \in \partial \mathcal{B}(\mathcal{P}; \tilde{r}_N)} (v_t - u_k)(x) \leq \min_{x \in \partial \mathcal{B}(\mathcal{P}; \sqrt{h/\kappa})} (v_t - u_k)(x) \leq C \cdot C_1 h/\kappa,
\]

which also implies (4.11).

Recall that the constant \(C\) in (4.11) depends only on \(M, \Sigma, I, C_1\). Thus, we can take \(\kappa\) larger than such \(C\), then (4.11) contradicts

\[
h \leq \max_{\partial \mathcal{B}(\mathcal{P}; \tilde{r}_N)} (v_t - u_k)(x).
\]

Therefore, we complete the proof for \(4 \leq (n + 1) \leq 7\).

**Part II:** In this part, we address the three-dimensional case.

Let \(Q_1\) and \(r_1\) be the notion in Claim 4. Assume that \(Q_1 = \{y_j\}\). Recall that \(s_1 = \max_{y \in Q} \rho(y)\).

**Claim 7.** Suppose that \(Q_1 \neq \emptyset\) and \(s_1 \geq 2Lh/\kappa\). There exist \(t_1, \tilde{t}_1\) satisfy

- \(2t_1 \leq \max_{y \in Q} |y| < \tilde{t}_1/2\);
- \(t_1 \leq 2^{10I} + 1\) and \(t_1 \geq \sqrt{\epsilon} \cdot \tilde{r}_1\);
- \(\mathcal{B}(y; r_1) \subset \mathcal{B}(p; \tilde{t}_1)\) for all \(y \in Q_1\).

**Proof of Claim 7** Let \(\alpha_j = \log_2(|y_j|/r_1)\). Then by the Lemma C.1 we can find \(2N(\leq 2I)\) non-negative integers \(\{k_j\}_{j=1}^{2N}\) such that

- \(k_{j+1} - k_j \geq 3\);
- \(k_{2j} - k_{2j-1} \leq 10I + 1\);
- \(\{\alpha_j\}_{j=1}^I \subset \bigcup_j [k_{2j-1} + 1, k_{2j} - 1]\).

The last item is equivalent to say

\[Q_1 \subset \bigcup_{j} \mathcal{A}(p; 2^{k_{2j-1} + 1}r_1, 2^{k_{2j} - 1}r_1).\]

which implies that

\[
\bigcup_{y \in Q_1} \mathcal{B}(y; r_1) \subset \mathcal{B}(p; 2^{k_{2N}} r_1).
\]

Now we set \(t_1 := 2^{k_{2N}} - 1 r_1\) and \(\tilde{t}_1 := 2^{k_{2N}} r_1\). Then it follows that such \(t_1\) and \(\tilde{t}_1\) satisfy the first and the third items.

It remains to prove that \(\tilde{t}_1 \geq \sqrt{\epsilon} \cdot r_1\). Note that \(s_1 \geq 2Lh/\kappa\). Then there exists \(z \in Q\) so that

\[
\rho(z) = \mathcal{L}(|z|^2 + h/\kappa) \geq 2Lh/\kappa,
\]

which deduces that \(|z| \geq \sqrt{h/\kappa}\). Recall that \(r_1 \leq 4^{3I + 3} s_1\). Thus, we have

\[
\sqrt{\epsilon} \cdot r_1 \leq \sqrt{\epsilon} \cdot s_1 \cdot 2^{3I + 3} \leq \sqrt{2\epsilon \mathcal{L} |z|^2} \cdot 2^{3I + 3} \leq |z| \leq 2^{k_{2N}} r_1 = \tilde{t}_1.
\]

Thus, we have proved Claim 7. \(\square\)
Now we can construct $s_j, Q_j, t_j, \tilde{t}_j$ inductively. Suppose we have $t_j$, then set

$$s_{j+1} = \max_{x \in Q \cap B(p; t_j)} \rho(x).$$

By the same process with Claim 4, we can take $r_{j+1} \in [8s_{j+1}, 4^{3I+3}s_{j+1}]$ and $Q_{j+1} \subset Q \cap B(p; t_j)$ so that

- for any $x \in Q \cap B(p; t_j)$, there exists $x' \in Q_{j+1}$ so that $\pi(B(x; \rho(x))) \subset B(x; r_{j+1}/4)$ or $\pi(B(x; \rho(x))) \subset B(p; r_{j+1}/4)$;
- $\mathcal{B}(x'; 4r_{j+1}) \cap \mathcal{B}(x''; 4r_{j+1}) = \emptyset$ for two different $x', x'' \in Q_{j+1}$;
- $\mathcal{B}(x'; 4r_{j+1}) \cap \mathcal{B}(p; 4r_{j+1}) = \emptyset$ for all $x' \in Q_{j+1}$.

**Claim 8.** Suppose that $Q_{j+1} \neq \emptyset$ and $s_{j+1} \geq 2Lh/\kappa$. Then there exist $t_{j+1}$ and $\tilde{t}_{j+1}$ satisfy

- $2t_{j+1} \leq \max_{y \in Q_{j+1}} |y| < \tilde{t}_{j+1}/2$;
- $\tilde{t}_{j+1} \leq 2^{10I+1}t_{j+1}$ and $\tilde{t}_{j+1} \geq \sqrt{\epsilon \cdot r_{j+1}}$;
- $\mathcal{B}(y; r_{j+1}) \subset \mathcal{B}(p; \tilde{t}_{j+1})$ for all $y \in Q_{j+1}$.

**Proof of Claim 8** The proof is almost the same with that of Claim 7.

Let $\beta_j = \log_2(\|y_j\|/r_{j+1})$. Then by the Lemma C.1, we can find $2N(\leq 2I)$ non-negative integers $\{k_m\}_{m=1}^{2N}$ such that

- $k_{m+1} - k_m \geq 3$;
- $k_{2m} - k_{2m-1} \leq 10I + 1$.
- $\{\alpha_m\}_{m=1}^{I} \subset \cup_m [k_{2m-1} + 1, k_{2m} - 1]$.

The last item is equivalent to say

$$Q_{j+1} \subset \bigcup_m \mathcal{B}(p; 2^{k_{2m-1}+1}r_{j+1}, 2^{k_{2m}-1}r_{j+1}).$$

which implies that

$$\bigcup_{y \in Q_{j+1}} \mathcal{B}(y; r_{j+1}) \subset \mathcal{B}(p; 2^{k_{2N}}r_{j+1}).$$

Now we set $t_{j+1} := 2^{k_{2N}+1}r_{j+1}$ and $\tilde{t}_{j+1} := 2^{k_{2N}}r_{j+1}$. Then it follows that such $t_{j+1}$ and $\tilde{t}_{j+1}$ satisfy the first and the third items.

It remains to prove that $\tilde{t}_{j+1} \geq \sqrt{\epsilon \cdot r_{j+1}}$. Note that $s_{j+1} \geq 2Lh/\kappa$. Then there exists $z \in Q \cap \mathcal{B}(p; t_j)$ so that

$$\rho(z) = L(\|z\|^2 + h/\kappa) \geq 2Lh/\kappa,$$

which deduces that $|z| \geq \sqrt{h/\kappa}$. Recall that $r_{j+1} \leq 4^{3I+3}s_{j+1}$. Thus, we have

$$\sqrt{\epsilon \cdot r_{j+1}} \leq \sqrt{\epsilon \cdot s_{j+1}} \cdot 2^{3I+3} \leq \sqrt{2\epsilon L\|z\|^2} \cdot 2^{3I+3} \leq |z| \leq 2^{k_{2N}}r_{j+1} = \tilde{t}_{j+1}.$$

Thus, we have proved Claim 8.

According to the construction, $Q \cap \mathcal{A}(p; t_{j+1}, t_j) \neq \emptyset$. Thus, such inductive process must stop in $N + 1(\leq I)$ steps, that is, there $Q_{N+1} = \emptyset$ or $s_{N+1} \leq 2Lh/\kappa$.\qed
Claim 9. For \( j \leq N \), \((v_t, u_k)\) is a strong \((2C_1(|x|^2 + r_{j+1}), K)\)-pair on \( \mathcal{A}(p; r_{j+1}, 2t_j) \setminus \bigcup_{Q_{j+1}} \mathcal{B}(y; r_{j+1}) \).

Proof of Claim 9. For \( x \in \mathcal{B}(p; \sqrt{h/\kappa}) \), we have
\[
(v_t - u_k)(x) \leq h/\kappa + C_1 h/\kappa \leq 2C_1 r_{j+1};
\]
and if \(|x| \geq \sqrt{h/\kappa}\),
\[
(v_t - u_k)(x) \leq h/\kappa + C_1 |x|^2 \leq 2C_1 |x|^2.
\]
Then together with (4.5) (4.6) and Claim 5, the desired result follows. □

By the definition of \( \tilde{r}_{j+1} \), for \( y \in Q_{j+1} \),
\[
\mathcal{B}(y; r_{j+1}) \subset \mathcal{B}(p; \tilde{r}_{j+1}).
\]
Also, from Claim 8
\[
\sqrt{t_j} \cdot r_{j+1} \leq \epsilon \cdot r_{j+1} \leq \tilde{r}_{j+1}.
\]
Thus, applying Theorem 3.6, we have
\[
\max_{\partial \mathcal{B}(p; t_j)} (v_t - u_k)(x) \leq C \min_{\partial \mathcal{B}(p; t_{j+1})} (v_t - u_k)(x).
\]
Note that \( \tilde{r}_j \leq 2^{10L+1} t_j \). Together with Corollary 3.4 we have
\[
\max_{\partial \mathcal{B}(p; \tilde{r}_j)} (v_t - u_k)(x) \leq C \min_{\partial \mathcal{B}(p; t_j)} (v_t - u_k)(x).
\]
From these two inequalities, we conclude that
\[
(4.12) \quad \max_{\partial \mathcal{B}(p; t_j)} (v_t - u_k)(x) \leq C^{2N} \min_{\partial \mathcal{B}(p; \tilde{r}_j)} (v_t - u_k)(x).
\]
Without loss of generality, we assume that \( t_N \geq \sqrt{h/\kappa} \). Recall that \( Q_{N+1} = \emptyset \) or \( s_{N+1} \leq 2Lh/\kappa \).

If \( Q_{N+1} = \emptyset \), then it follows that \( Q \cap \mathcal{B}(p; t_N) = \emptyset \) (where \( t_0 = \epsilon \)). In this case, \((v_t, u_k)\) is a strong \((2C_1(|x|^2 + h/\kappa), K)\)-pair on \( \mathcal{A}(p; h/\kappa, 2t_N) \). Using Theorem 3.6 we have
\[
\max_{\partial \mathcal{B}(p; t_N)} (v_t - u_k)(x) \leq C \min_{\partial \mathcal{B}(p; \sqrt{h/\kappa})} (v_t - u_k)(x) \leq C h/\kappa.
\]
Together with (4.12), we conclude that
\[
(4.13) \quad \max_{\partial \mathcal{B}(p; \epsilon)} (v_t - u_k)(x) \leq C^{2N+1} h/\kappa.
\]
If \( s_{N+1} \leq 2Lh/\kappa \), then for \( x \in Q \cap \mathcal{B}(p; t_N) \),
\[
\rho(x) = L(|x|^2 + h/\kappa) \leq 2Lh/\kappa,
\]
which implies that \(|x| \leq \sqrt{h/\kappa}\). In this case, for \( x \in \mathcal{A}(p; r_{N+1} h/\kappa, 2t_N) \cup Q_{N+1} \mathcal{B}(y; r_{N+1} h/\kappa), \)
\[
(v_t - u_k)(x) \leq C_1 |x|^2 + h/\kappa \leq 2C_1 (|x|^2 + r_{N+1} h/\kappa).
\]
Therefore, \((v_1, u_k)\) is also a \((2C_1(|x|^2 + r_{N+1} + h/\kappa), 2)\)-pair on \(A(p; r_{N+1} + h/\kappa, 2t_N) \cup_{Q_{N+1}} B(y; r_{N+1} + h/\kappa)\), and hence a strong \((2C_1(|x|^2 + r_{N+1} + h/\kappa), K)\)-pair by (4.5) and (4.6). Moreover,

\[
\sqrt{t_N \cdot (r_{N+1} + h/\kappa)} \leq \sqrt{\epsilon \cdot 4^{3l+3} s_{N+1}} + \sqrt{\epsilon \cdot h/\kappa} \leq 2^{3l+5} L \sqrt{\epsilon \cdot h/\kappa} \leq \frac{1}{2} \sqrt{h/\kappa}.
\]

Then Theorem 3.6 gives that

\[
\max_{\partial B(p; t_N)} (v_t - u_k)(x) \leq C \min_{\partial B(p; 2\sqrt{h/\kappa})} (v_t - u_k)(x) \leq Ch/\kappa,
\]

together with (4.12) which also implies (4.13).

Recall that the constant \(C\) in (4.13) depends only on \(M, \Sigma, I, C_1\). Thus, we can take \(\kappa\) larger than such \(C^{2N+1}\), then (4.13) contradicts

\[
h \leq \max_{\partial B(p; \tau)} (v_t - u_k)(x).
\]

Therefore, we complete the proof for \((n + 1) = 3\).

5. High dimensional case

In this section, we prove the one-sided Harnack inequality for minimal graph functions in high dimensional cases. Such a result is one of key ingredients in the proof of the existence of Jacobi field in Theorem 4.2.

In this section, \((M^{n+1}, g)\) is always a closed manifold and \(N\) is an embedded compact minimal hypersurface in \(M\) so that \(B(p; 1) \cap \partial N = \emptyset\). Recall that \(B(p; r)\) the intrinsic geodesic ball of \(N\) with radius \(r\) and center \(p \in N\) and \(A(p; r, s) = B(p; s) \setminus B(p; r)\).

In order to prove Theorem 3.5, we prove the following lemma first:

**Lemma 5.1.** Given \(C_1 > 0\), there exist \(C = C(M, N, C_1, K)\) and \(R_0 = R_0(M, N, C_1, K)\) so that if \(\Sigma\) and \(\Gamma\) are minimal graphs with functions \(v, u\) over \(A(p; r, 2R)\) for \(8r \leq R \leq R_0\) and \((v, u)\) is a strong \((C_1|x|^2, K)\)-pair (see Definition 3.2), where \(|x| = \text{dist}_N(x, p)\), then we have

\[
(5.1) \quad \max_{\partial B(p; R)} (v - u)(x) \leq C \min_{\partial B(p; 2R)} (v - u)(x).
\]

Now we can use Lemma 5.1 to prove Theorem 3.5 as follows.

**Proof of Theorem 3.5.** Let \(C_0\) be the constant in Lemma 4.4. Then for any \(r_1, r_2 \geq C_0Ir\), there exists a \(C^1\) curve \(\gamma: [0, 1] \to A(p; r_1, r_2)\) satisfying

\begin{itemize}
  \item \(\gamma(0) \in \partial B(p; r_1)\) and \(\gamma(1) \in \partial B(p; r_2)\);
  \item \(\text{Length}(\gamma) \leq C_0(r_2 - r_1)\);
  \item \(\text{dist}(\gamma, \bigcup_{j=1}^l B(q_j, r)) \geq \gamma_1/(C_0I)\).
\end{itemize}

Now take \(\tilde{r} \in (C_0Ir, 3C_0Ir)\) so that

\[
\text{dist}_N(\partial B(p; \tilde{r}), \bigcup_{j=1}^l B(q_j, r)) \geq r.
\]
Set \[ \alpha_j := \log_2(\text{dist}_N(p, q_j)/\bar{r}), \quad \text{for all } j \leq I. \]

Then Lemma [4.1] implies that there exist \(2I(\leq 2I)\) non-negative intergers \(\{k_j\}_{j=1}^{2I}\) such that

- \(k_{j+1} - k_j \geq 3;\)
- \(k_{2j} - k_{2j-1} \leq 10I + 1;\)
- \(\{\alpha_j\}_{j=1}^{2I} \subset \cup_{j}[k_{2j-1} + 1, k_{2j} - 1].\)

The last item deduces that

\[ \bigcup_{j=1}^{I} B(q_j, r) \subset \bigcup_{j=1}^{I} B(p; 2^{k_{2j-1}+1}r, 2^{k_{2j}}r). \]

By the choice of \(C_0\) in the beginning, Corollary [3.4] gives that

\[ \max_{\partial B(p; 2^{2j}r)} (v - u)(x) \leq C \min_{\partial B(p; 2^{2j}r)} (v - u)(x). \]

Notice that there is no \(B(q_j; r)\) in \(B(p; 2^{k_{2j}+1}r, 2^{k_{2j+1}}r)\). Let \(R_0\) satisfy the requirements in Lemma [5.1] then

\[ \max_{\partial B(p; 2^{2j+1}r)} (v - u)(x) \leq C \min_{\partial B(p; 2^{2j}r)} (v - u)(x). \]

Similarly,

\[ \max_{\partial B(p; R)} (v - u)(x) \leq C \min_{\partial B(p; 2^{2j}r)} (v - u)(x). \]

Together with ([5.2]) and ([5.3]) we conclude that

\[ \max_{\partial B(p; R)} (v - u)(x) \leq C^{2I+1} \min_{\partial B(p; 2^{2I}r)} (v - u)(x), \]

which is exactly the desired result. \(\square\)

Then the rest of this section is devoted to prove Lemma [5.1]

5.1. Almost harmonic functions. In this subsection, we prove an one-sided Harnack for a class of ‘almost harmonic functions’ under additional assumptions.

Let \(N^n\) be a compact manifold with boundary of dimension \(n \geq 2\). Denote by \(B(p; r)\) the geodesic ball of \(N\) so that \(\text{inj}_N(p), \text{dist}_N(p, \partial N) > r,\) where \(\text{inj}_N(p)\) is the injective radius. We also denote \(A(p; r, s) = B(p; s) \setminus B(p; r)\).

Given a \(C^2\) function \(w\) on \(\partial B(p; R)\), define

\[ I(s) = s^{1-n} \int_{\partial B(p; s)} w \, d\mu, \]

for \(s \in (0, R]\).

**Lemma 5.2.** Let \(n \geq 3\). Given \(C_2 > 0\), there exists \(R = R(n, C_2) > 0\) so that if \(I\) is a \(C^1\) function on \([r, R]\) with \(R \geq 4r\) and

\[ |(I(s) + cs^{2-n})'| \leq C_2 s I(s), \quad \text{for some constant } c, \]

then \(\max_{s \in [1, R]} I(s) \leq 6I(2r).\)
Hence we have follows.

**Case 1:** $c \leq 0$.
Then $I' \leq C_2 s I$, which implies that $I(s) \leq e^{C_2 s^2} I(2r)$. It follows that $I(s) \leq e I(2r) \leq 6 I(2r)$ whenever $C_2 R^2 \leq 1$.

**Case 2:** $c > 0$.
Set $J(s) = I(s) + c s^{2-n}$. Then we have $|J'| \leq C_2 s J$. The argument in Case 1 gives that $J(s) \leq e^{1/10} J(r) \leq 2 J(r)$ whenever $C_2 R^2 \leq 1/10$.

Integrating (5.5) from $r$ to $2r$, then

$$I(r) + c r^{2-n} - I(2r) - c(2r)^{2-n} \leq 4 C_2 r^2 (I(2r) + c (2r)^{2-n}),$$

which implies that $I(r) + c r^{2-n} \leq 3 I(2r)$. Together with $J(s) \leq 2 J(r)$, we have

$$I(s) \leq J(s) \leq 2 J(r) \leq 6 I(2r).$$

**Corollary 5.3.** Let $n \geq 3$. Given $C_3 > 0$ and $\alpha > 0$, there exists a constant $\epsilon_0 = \epsilon_0(C_3, \alpha, \mathcal{N}) > 0$ so that if $w \geq 0$ is a $C^2$ function on $\mathcal{A}(p; r, \epsilon)(4r \leq \epsilon < \epsilon_0)$ satisfying

1. $I(s) \leq \alpha I(t)$ for $r \leq s \leq t \leq \epsilon$, where $I(s) = s^{1-n} \int_{\partial B(p; s)} w d\mu$;
2. $|\Delta w| \leq C_3 w$,

then $I(s) \leq 6 I(2r)$ for $s \in [r, \epsilon]$.

**Proof.** For $s > t$, set

$$E(s) = s^{1-n} \left( \int_{\partial B(p; s)} \langle \nu, \nabla w \rangle - \int_{\mathcal{A}(p; r, s)} \Delta w \right),$$

where $\nu$ is the unit outward normal vector field of $\partial B(p; s)$. Then by the divergence theorem, there exists a constant $c = c(p)$ such that $E(s) = (cs^{2-n})'$. And a direct computation gives that (cf. [3] Lemma 2.1])

(5.6) $$I'(s) = s^{1-n} \int_{\partial B(p; s)} \left[ \langle \nu, \nabla w \rangle + H(x) - \frac{n-1}{s} \right],$$

where $H(x)$ is the mean curvature of $\partial B(p; s)$. Recall that

$$H(x) = \frac{n-1}{s} + O(s).$$

Hence we have

$$|I' - E| \leq s^{1-n} \int_{B(p; r, s)} |\Delta w| + s^{1-n} \int_{\partial B(p; s)} w(x) \cdot \left| H(x) - \frac{n-1}{s} \right|$$

$$\leq s^{1-n} \int_{r}^{s} \left[ \int_{\partial B(p; t)} C_2 w d\mu \right] dt + C_3 s I(s)$$

$$\leq (C_2 \alpha + C_3) s I(s)$$

Then by Lemma 5.2, there exists $\epsilon_0 = \epsilon_0(n, C_2 \alpha + C_3)$ so that if $\epsilon < \epsilon_0$, then $I(s) \leq 6 I(2r)$ for $s \in [r, \epsilon]$. Note that $C_3$ depends only on $\mathcal{N}$. Then the desired result follows. \qed
5.2. **Minimal graph on annuli in high dimensions.** Recall that \((M^{n+1}, \partial M, g)\) is a closed Riemannian manifold with \(3 \leq (n + 1) \leq 7\) and \(\mathcal{N}\) is an embedded minimal hypersurface.

In this subsection, we give a proof of Lemma 5.1. The main steps are to show that the difference of minimal graph functions on annuli are the ‘almost harmonic function’ in Corollary 5.3.

We now prove a one-sided Harnack inequality for minimal graph functions on annuli.

**Proof of Lemma 5.1.** By Lemma A.2 we can take \(R_0\) small enough so that
\[
|\Delta_N w| \leq C|\nabla^2 w|(|\nabla^2 w|^2 + |\nabla v|^2) + C|\nabla^2 w|(|\nabla v||\nabla^2 v| + |\nabla w|)|v| + C(1 + |\nabla^2 w| + |\nabla^2 v|)w + C|\nabla w| \cdot |v| + C|\nabla w| \cdot |\nabla^2 v|,
\]
where the constant \(C\) depending only on \(M, \mathcal{N}\) and \(K\). Using (4.5) and (4.6), we can rewrite it as
\[
(5.7) \quad |\Delta_N w| \leq C|\nabla^2 w|(|\nabla^2 w|^2 + |v|^2) + C(1 + |\nabla^2 w|)w + C|\nabla w| \cdot |v|.
\]
Recall the gradient and the second order estimates in Lemma 3.3 then we have for \(x \in B(p; \frac{3}{2}r, R)\),
\[
|\nabla w| \leq C(M, \mathcal{N}, K, C_1)|w|/|x|, \quad |\nabla^2 w| \leq C(M, \mathcal{N}, K, C_1)|w|/|x|^2.
\]
Together with \(|v| \leq K|x|\) and \(|w| \leq C_1|x|^2\), we have
\[
|\Delta w| \leq C_4|w|,
\]
for \(x \in A(p; \frac{2}{3}r, R)\).

**Claim 10.** There exists a constant \(\alpha = \alpha(C_4)\) so that either \(I(R) \leq \alpha I(2r)\); or \(I(s) \leq \alpha I(t)\) for all \(2r \leq s \leq t \leq R\), where \(I(s) = s^{1-n} \int_{\partial B(p; s)} w\).

**Proof of Claim 10.** By Lemma 3.1 there exists a constant \(C\) so that
\[
\max_{\partial B(p; 2r)} w \leq C \min_{\partial B(p; 2r)} w, \quad \max_{\partial B(p; R)} w \leq C \min_{\partial B(p; R)} w.
\]
By Lemma A.3 we also have
\[
\min_{\partial A(p; 2r, R)} w \leq C \min_{A(p; 2r, R)} w, \quad \max_{\partial A(p; 2r, R)} w \leq C \max_{A(p; 2r, R)} w.
\]
From this, we have
\[
I(s) \leq \max_{A(2r, R)} w \leq C(\max_{\partial B(p; 2r)} w + \max_{\partial B(p; t)} w) \leq C^2(I(2r) + I(t)).
\]
Now if \(I(R) \geq C^3I(2r)\), then
\[
\min_{\partial B(p; R)} w \geq I(R)/C \geq C^2I(2r) > \min_{\partial B(p; 2r)} w,
\]
which implies that
\[
\min_{\partial B(p; 2r)} w = \min_{\partial A(p; 2r, R)} w \leq C \min_{A(p; 2r, R)} w.
\]
Therefore,
\[ I(s) \leq C^2(I(2r) + I(t)) \leq C^2(C \min_{\partial B(p;2r)} w + I(t)) \]
\[ \leq C^2(C \min_{A(p;2r,R)} w + I(t)) \leq C^4(I(t) + I(t)) \leq 2C^4I(t). \]

Then the desired result follows by setting \( \alpha = 2C^4 \).

Let \( \alpha \) be the constant in Claim 10. Then either
\[ I(R) \leq \alpha I(3r), \]
or
\[ I(s) \leq \alpha I(t) \]
for all \( 4r \leq s \leq t \leq R \). In the latter case, applying Corollary 3.4, we have \( I(R) \leq 6I(3r) \).

Hence we always have \( I(R) \leq \alpha I(3r) \).

Now applying Corollary 3.4,
\[ \max_{\partial B(p;R)} w \leq \min_{\partial B(p;R)} C \min_{\partial B(p;2r)} w \leq C^4 I(R) \leq 6I(3r) \]
\[ \leq \max_{\partial B(p;3r)} w \leq \min_{\partial B(p;2r)} C \min_{\partial B(p;R)} w. \]

Then the desired inequality follows.

\[ \square \]

6. On the three dimensional case

The main purpose of this section is to develop a one-sided Harnack inequality for minimal graph functions over a minimal surface (Theorem 3.6).

In this section, let \( (M^3, g) \) be a three dimensional compact Riemannian manifold and \( \mathcal{N} \) be an embedded minimal surface in \( M \). For \( p \in \mathcal{N} \) and \( s > 0 \), denote by \( B(p; s) \) the geodesic ball in \( \mathcal{N} \) such that \( \text{dist}_{\mathcal{N}}(p, \partial \mathcal{N}) > s \). We also denote \( A(p; r, s) = B(p; s) \setminus B(p; r) \).

6.1. Positive functions satisfying the minimal condition. We also approach Theorem 3.6 by a Harnack inequality for positive functions satisfying additional assumptions.

Denote by \( |x - y| \) the distance between \( x, y \in \mathcal{N} \) and \( d(x, A) \) the distance between \( x \in \mathcal{N} \) and \( A \subset \mathcal{N} \).

Let \( w \) be a \( C^1 \) function on \( B(p; s, r) \). For \( t \in [s, r] \), set
\[ I(p; t) = t^{-1} \int_{\partial B(p; t)} w \, d\mu, \]
\[ \tau(p; t) = \int_{\partial B(p; t)} \langle \nabla w, \nu \rangle \, d\mu, \]
where \( \nu = \nu(p; t) \) is the unit outward normal vector field on \( \partial B(p; t) \).

**Definition 6.1.** We say that a positive \( C^2 \) function defined on subset of \( 0 A(p; r^2, \epsilon) \setminus \bigcup_{j=1}^J B(q_j; r^2) \subset \mathcal{N} \) satisfies minimal condition with constant \( C \) if the following holds true:

(a) \( w \) is positive and \( w(x) \leq C(|x|^2 + r^2) \) for all \( x \in \Xi \), where \( |x| = \text{dist}_{\mathcal{N}}(x, p) \);
(b) \[ \max_{\Xi} |d(x, Q)| \cdot |\nabla \log w(x)| < C, \quad \text{where} \quad Q = \{q_j\}; \]
hence the Harnack inequality in Corollary 3.4 holds true;
(c) for all \( x \in \Xi \),
\[
|\Delta w| \leq Cw + \frac{Cw^3}{d^3(x, Q)}.
\]

In this part, we prove the one-sided Harnack inequality in three dimensional manifolds. Moreover, such a result holds true for more general functions on surfaces.

**Theorem 6.2.** Let \( \mathcal{N} \) be a two-dimensional Riemannian surface and \( \mathcal{B}(p; \epsilon) \) be a geodesic ball. Let \( Q = \{ q_j \}_{j=1}^I \subset \mathcal{B}(p; \epsilon r/2) \). Let \( w \) be a \( C^2 \) function on \( \Xi = \mathcal{B}(p; r^2, \epsilon^2) \setminus \bigcup_{j=1}^I \mathcal{B}(q_j; r^2) \) and \( \epsilon > r^{1/4} \). Given \( C_0 > 0 \), there exists \( \epsilon_0 = \epsilon_0(C_0, \mathcal{N}) \) and \( C = C(C_0, \mathcal{N}) \) so that if \( \epsilon < \epsilon_0 \) and \( w \) satisfies minimal condition with constant \( C_0 \) (see Definition 6.1), then we have
\[
\max_{\partial \mathcal{B}(p; \epsilon r^2/2)} w \leq C \min_{\partial \mathcal{B}(p; \epsilon r)} w.
\]

We postpone the proof in \( \{6.3\} \).

Here we give a proposition which is used in the next subsection.

**Proposition 6.3.** Let \( Q = \{ q_j \}_{j=1}^I \subset \mathcal{B}(p; 4r^2, \epsilon r) \). Suppose that \( r < 4^{-16I-16} \) and \( \epsilon > 4^4 r^{1/4} \). Then there exist a constant \( \theta \in [1/4^{4I+3}, 1/16] \) and subset \( Q' \subset Q \) so that

(i) for any \( q \in Q \), there exists \( x \in Q' \) satisfying \( \mathcal{B}(q; r^2) \subset \mathcal{B}(x; \theta|x|/4) \) or \( \mathcal{B}(q; r^2) \subset \mathcal{B}(p; \theta e^{3/4} r^{5/4}/4) \);

(ii) \( \mathcal{B}(x; 4\theta|x|) \cap \mathcal{B}(y; 4\theta|y|) = \emptyset \) for any \( x, y \in Q' \) with \( x \neq y \);

(iii) \( \mathcal{B}(x; 4\theta|x|) \cap \mathcal{B}(p; 4\theta e^{3/4} r^{5/4}) = \emptyset \) for any \( x \in Q' \),

where \( |x| = \text{dist}(x, p) \).

**Proof.** We say that \( \tilde{Q} \) and \( \alpha > 0 \) satisfy the containing condition if for any \( q \in Q \), there exists \( x \in \tilde{Q} \) satisfying \( \mathcal{B}(q; r^2) \subset \mathcal{B}(x; \alpha|x|/4) \) or \( \mathcal{B}(q; r^2) \subset \mathcal{B}(p; \alpha e^{3/4} r^{5/4}/4) \).

We first note that for any \( \alpha \in [1/4^{4I+3}, 1/16] \), \( Q \) and \( \alpha \) satisfy the containing condition.

Now we proceed to the proof of the proposition. In the first step, take \( \theta_1 = 1/4^{4I+3} \) and a subset \( Q_1 \subset \tilde{Q} \) so that

- \( Q_1 \) and \( \theta_1 \) satisfy the containing condition;
- for any two different point \( x', x'' \in Q_1 \), \( \mathcal{B}(x'; \alpha|x'|/4) \setminus \mathcal{B}(x''; \alpha|x''|/4) \neq \emptyset \), \( \mathcal{B}(x'; \alpha|x'|/4) \setminus \mathcal{B}(p; \alpha e^{3/4} r^{5/4}/4) \neq \emptyset \).

If \( \theta_1 \) satisfies all we need, then we are done. Otherwise, there exist two different points \( x, y \in Q_1 \) so that \( \mathcal{B}(x; 4\theta|x|) \cap \mathcal{B}(y; 4\theta|y|) = \emptyset \) or \( \mathcal{B}(x; 4\theta|x|) \cap \mathcal{B}(p; 4\theta e^{3/4} r^{5/4}) = \emptyset \). In both cases, we can take \( \theta_2 = 4^4 \theta_1 \) and \( Q_2 \subset \tilde{Q} \) so that \( Q_2 \) and \( \theta_2 \) satisfy the above containing condition.

Note that now the number of the elements in \( Q_2 \) is less than that of \( Q_1 \). Hence we can repeat the argument at most \( I \) steps. Suppose that we stop at \( Q_k \) and \( \theta_k \). Then \( \theta = \theta_k \) and \( Q' = Q_k \) are the desired constant and subset. \( \Box \)
6.2. The lower bound. In this subsection, we prove a lower bound of $\mathcal{I}(p; \epsilon r)$ if $w$ satisfying the minimal condition.

**Lemma 6.4.** Suppose that $\epsilon > r^{1/4}$ Then for $\epsilon$ small enough, there exists a constant $C$ so that for any positive function $u$ on $A(q; \rho, R) \subset B(p; \epsilon r)$ satisfying $|\Delta w| \leq C_0 w/(r \epsilon^3)$, we have

$$\min_{\partial A(p; \rho, R)} \ w \leq C \min_{A(p; \rho, R)} \ w, \quad \max_{\partial A(p; \rho, R)} \ w \leq C \max_{A(p; \rho, R)} \ w.$$  

**Proof.** Note that

$$R^2 |\Delta w| \leq 4r^2 \epsilon^2 |\Delta w| \leq C_0 w/r < \epsilon < w/4.$$  

Then our desired result follows from Lemma A.3.

In the following of this subsection, we always assume that

$$\Xi := A(p; r^2, \epsilon r) \setminus \bigcup_{j=1}^I B(q_j; r^2) \subset N.$$  

**Proposition 6.5.** Let $w \in C^2(\Xi)$ be a positive function. Suppose that $w$ satisfies (6.1) in Definition 6.1 with constant $C_0$ and

- $|\Delta w| \leq C_0 w/(r \epsilon^3)$,
- $\mathcal{I}(p; \epsilon r) \leq \tau_0 |\log(\epsilon/r)|$, where $\tau_0 = \tau(p; \epsilon r)$.

Let $Q = \{q_j\}$ and take $Q'$ and $\theta$ by Proposition 6.3.

Then there exists $x_j \in Q'$ such that $\tau(x_j; \theta|x_j|) \geq \tau_0/4^{I+1}$ or $\tau(p; \theta \epsilon^{3/4} r^{5/4}) \geq \tau_0/4^{I+1}$.

**Proof.** We assume that for $y \in Q'$, $\tau(y; \theta|y|) \leq \tau/4^I$. To prove the proposition, it suffices to show that $\tau(p; \theta \epsilon^{3/4} r^{5/4}) \geq \tau_0/4^{I+1}$.

By Lemma C.1, we can find $2N + 1(\leq 2I + 1)$ non-negative integers $\{k_j\}_{j=0}^{2N}$ such that

- $k_{j+1} - k_j \geq 3$;
- $k_{2j} - k_{2j-1} \leq 10I + 1$ for all $j \geq 1$;
- $Q' \subset \bigcup_{j=1}^{2N} A(p; 2^{k_{2j-1}} \theta \epsilon^{3/4} r^{5/4}, 2^{k_{2j}} \theta \epsilon^{3/4} r^{5/4})$.

We now proceed the desired results by several steps:

**Step 1:** We show that $\tau(p; 2^{k_{2N}} \theta \epsilon^{3/4} r^{5/4}) \geq \tau_0/2$ and $\mathcal{I}(p; 2^{k_{2N}} \theta \epsilon^{3/4} r^{5/4}) \leq 2 \tau_0 |\log(\epsilon/r)|$.

Indeed, from (5.6), together with Lemma 6.4, we have

$$|\partial_s \mathcal{I}(p; s) - \tau_0 s^{-1}| \leq s^{-1} \int_{A(p; s, \epsilon r)} |\Delta w| + C s \mathcal{I}(p; s) \leq s^{-1} \int_{A(p; s, \epsilon r)} C_0 w/(r \epsilon^3) + C s \mathcal{I}(p; s) \leq s^{-1} \int_{A(p; s, \epsilon r)} C(\mathcal{I}(p; s) + \mathcal{I}(p; \epsilon r))/(r \epsilon^3) + C s \mathcal{I}(p; s) \leq \frac{C r(\mathcal{I}(p; s) + \mathcal{I}(p; \epsilon r))}{\epsilon s}.$$
Integrating it from $s$ to $\epsilon r$, we get

$$|I(p; \epsilon r) - I(p; s) - \tau_0 \log(\epsilon/r) + \tau_0 \log s| \leq C_0(I(p; s) + I(p; \epsilon r))(r/\epsilon)|\log(\epsilon/r)|.$$ 

Taking $\epsilon$ small enough, such a inequality implies

$$I(p; \epsilon r) - I(p; s) - \tau_0 \log(\epsilon/r) \geq -(I(p; s) + I(p; \epsilon r))/10.$$ 

Hence

$$I(p; s) \leq 2I(p; \epsilon r) \leq 2\tau_0|\log(\epsilon/r)|.$$ 

By the Harnack inequality from Corollary 3.4, we conclude that

$$w(x) \leq C_1(I(p; \epsilon r) + I(p; 2k_2N \theta \epsilon^{3/4}r^{5/4})) \leq C_1\tau_0|\log(\epsilon/r)|,$$

for $x \in A(p; 2k_2N \theta \epsilon^{3/4}r^{5/4}, \epsilon r)$, and $C_1 = C_1(N)$.

Now we show that $\tau(p; s) \geq \tau_0/2$ for $s \in [2k_2N \theta \epsilon^{3/4}r^{5/4}, \epsilon r]$. Indeed, the divergence theorem gives that

$$|\tau(p; s) - \tau_0| \leq \int_{A(p; s, \epsilon r)} |\Delta w| \leq \int_{A(p; s, \epsilon r)} C_0 w/(r\epsilon^3)$$

$$\leq \int_{A(p; s, \epsilon r)} C_0 C_1 \tau_0 |\log(\epsilon/r)|/(r\epsilon^3)$$

$$\leq C_0 C_1 \tau_0 (r/\epsilon)|\log(\epsilon/r)| \leq \tau_0/2.$$

It follows that $\tau(p; s) \geq \tau_0/2$.

**Step 2:**

$$I(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4}) \leq C_1\tau_0|\log(\epsilon/r)|$$

and $\tau(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4}) \geq \tau_0/4$.

Denote by $\Omega_j$ the domain $A(p; 2k_j-1 \theta \epsilon^{3/4}r^{5/4}, 2k_j \theta \epsilon^{3/4}r^{5/4}) \setminus \bigcup_{y \in Q'} B(y; \theta |y|)$.

Since $k_2N - k_2N-1 \leq 10I + 1$, then by the Harnack inequality from Corollary 3.4 there exists $C'_1 = C'_1(N, I)$ satisfying

$$w(x) \leq C'_1\tau_0|\log(\epsilon/r)|,$$ 

for all $x \in \Omega_{2N}$.

Now we use this upper bound of $u$ to estimate $\tau(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4})$. By divergence theorem,

$$|\tau(p; 2k_2N \theta \epsilon^{3/4}r^{5/4}) - \tau(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4})| - \sum_{y \in Q' \cap A(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4}, 2k_2N \theta \epsilon^{3/4}r^{5/4})} \tau(y; \theta |y|)$$

$$\leq \int_{\Omega_{2N}} |\Delta w| \leq \int_{\Omega_{2N}} C_0 C'_1 \cdot \tau_0 |\log(\epsilon/r)|/(r\epsilon^3) \leq \tau_0/4.$$ 

Note that by assumptions, $\tau(y; \theta |y|) \leq \tau_0/4I$ for any $y \in Q'$. Thus we conclude that $\tau(p; 2k_2N-1 \theta \epsilon^{3/4}r^{5/4}) \geq \tau_0/4I$. Thus we finish the proof of Step 2.

Running the argument in Step 1 again, we can prove that $\tau(p; 2k_2N-2 \theta \epsilon^{3/4}r^{5/4}) \geq \tau_0/8$ and $I(p; 2k_2N-2 \theta \epsilon^{3/4}r^{5/4}) \leq C_1\tau_0|\log(\epsilon/r)|$. Then repeating the argument in Step 2, we obtain $\tau(p; 2k_2N-3 \theta \epsilon^{3/4}r^{5/4}) \geq \tau_0/16$ and $I(p; 2k_2N-3 \theta \epsilon^{3/4}r^{5/4}) \leq C_1\tau_0|\log(\epsilon/r)|$.

By induction, we conclude that $\tau(p; \theta \epsilon^{3/4}r^{5/4}) \geq \tau_0/4I$. Thus we complete the proof of Proposition 6.3. \(\square\)
Proposition 6.6. Let \( w \in C^2(\Xi) \). Suppose that \( w \) satisfies minimal condition with constant \( C_0 \) and \( I(p; \epsilon r) \leq \tau_0 |\log(\epsilon/r)| \), where \( \tau_0 = \tau(p; \epsilon r) \). Take \( Q' \subset Q \) and \( \theta \) by Proposition 6.5. Then either
\[
\tau(y, \theta|y|) \leq \tau_0/4^{I+1}, \forall y \in Q',
\]
or there exist \( c_0 = c_0(C_0, N, I), c_1 = c_1(C_0, N, I) \), a sequence \( y_1, y_2, \ldots, y_k \in Q \) and \( \theta_1, \theta_2, \ldots, \theta_{k+1} \in (1/4^{I+3}, 1/16) \) such that
\begin{enumerate}[(i)]
\item \( I(p; \epsilon r) - c_0 I(y_1; \theta_1|y_1|) \geq c_0 \tau_0(\log(\epsilon r) - \log(\theta_1|y_1|) - c_1); \)
\item for \( j \leq k - 1 \), \( I(y_j; \theta_j|y_j - y_{j-1}|) - c_0 I(y_{j+1}; \theta_{j+1}|y_{j+1} - y_j|) \geq c_0 \tau_0(\log(\theta_j|y_j - y_{j-1}|) - \log(\theta_{j+1}|y_{j+1} - y_j|) - c_1), \) where \( y_0 = p; \)
\item \( I(y_k; \theta_k|y_k - y_{k-1}|) - c_0 I(y_k; \theta_{k+1} \epsilon^{3/4} r^{5/4}) \geq c_0 \tau_0(\log(\theta_k|y_k - y_{k-1}|) - \log(\theta_{k+1} \epsilon^{3/4} r^{5/4}) - c_1). \)
\end{enumerate}

Proof. Without loss of generality, we assume that there exist \( y' \in Q' \) so that
\[(6.1) \quad \tau(y'; \theta|y'|) \geq \tau_0/4^{I+1}.
\]
Set \( \theta_1 = \theta \) and take \( y_1 \in Q' \) so that \( \tau(y_1; \theta_1|y_1|) \geq \tau_0/4^{I+1} \) and
\[
\tau(y; \theta_1|y|) < \tau_0/4^{I+1}, \forall y \in Q' \text{ with } |y| > |y_1|.
\]
We remark that such \( y_1 \) is well-defined by (6.1).

**Step A:** We first show that there exists \( c_0 = c_0(C_0, N, I) \) and \( c_1 = c_1(C_0, N, I) \) so that (6.1) is satisfied.

By Lemma 6.1, we can find \( 2N + 1(\leq 2I + 1) \) non-negative integers \( \{k_j\}_{j=0}^{2N} \) such that
\begin{itemize}
\item \( k_{j+1} - k_j \geq 3; \)
\item \( k_{2j} - k_{2j-1} \leq 10I + 1 \) for all \( j \geq 1; \)
\item \( Q' \subset \bigcup_{j=1}^{2N} A(p; 2^{k_{2j} - 1 + 1} \theta_1 \epsilon^{3/4} r^{5/4}, 2^{k_{2j} - 1} \theta_1 \epsilon^{3/4} r^{5/4}). \)
\end{itemize}

Hence there exists \( j' \leq N \) such that \( y_1 \in A(p; 2^{k_{2j'} - 1 + 1} \theta_1 \epsilon^{3/4} r^{5/4}, 2^{k_{2j'} - 1} \theta_1 \epsilon^{3/4} r^{5/4}). \) By the choice of \( y_1, \)
\[
\tau(y; \theta_1|y|) \leq \tau_0/4^{I+1}, \forall y \in Q' \cap A(p; 2^{k_{2j'} - 1} \theta_1 \epsilon^{3/4} r^{5/4}, \epsilon r).
\]

Then by the induction in Step 1 and 2 in Proposition 6.5, we conclude that
\begin{align*}
(6.2) \quad & w(x) \leq C'_1 \tau_0(\log(\epsilon/r)), \quad \text{for all } x \in A(p; 2^{k_{2j'} - 1} \theta_1 \epsilon^{3/4} r^{5/4}, \epsilon r) \setminus \bigcup_{y \in Q'} B(y, \theta_1|y|), \\
(6.3) \quad & \tau(p; 2^{k_{2j}} \theta_1 \epsilon^{3/4} r^{5/4}) \geq \tau_0/4^{I+1}, \quad \text{for all } 2j' \leq j \leq 2N,
\end{align*}
where \( C'_1 = C'_1(N, I). \)
Using the estimates of \( \tau \) for \( 2j' \leq j \leq 2N \), together with divergence theorem implies that for \( s \in [2^{2j_0+1} \theta_1 e^{3/4} r^{5/4}, 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}] \),

\[
|\partial_s \mathcal{I}(p; s) - \tau(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})|s^{-1}| \leq s^{-1} \int_{A(p; s, 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})} |\Delta w| + C_s \mathcal{I}(s) \\
\leq s^{-1} \int_{A(p; s, 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})} C_0 \omega^3/d^4(x, Q) + C_s \mathcal{I}(s).
\]

Note that in this case, \( d(x, Q) \geq e^{3/4} r^{5/4} \). Then the inequality becomes

\[
|\partial_s \mathcal{I}(p; s) - \tau(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})|s^{-1}| \leq s^{-1} \int_{A(p; s, 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})} C_0 \omega^3/(re^3) + C_s \mathcal{I}(s) \\
\leq s^{-1} C_0(r/e)(\mathcal{I}(p; s) + \mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4})) + C_s \mathcal{I}(s) \\
\leq C_0(\mathcal{I}(p; s) + \mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}))/r/(e \epsilon).
\]

Integrating it from \( s \) to \( 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4} \), we get

\[
|\mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}) - \mathcal{I}(p; s) - \tau(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}) \log(2^{2j_1+1} \theta_1 e^{3/4} r^{5/4} / s)| \\
\leq C_0(\mathcal{I}(p; s) + \mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}))(r/e)| \log(e/r)| \\
\leq (\mathcal{I}(p; s) + \mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}))/10.
\]

Together with (6.3), this implies that for \( j' \leq j \leq N - 1 \),

\[
\mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4} - \mathcal{I}(p; 2^{2j_1+1} \theta_1 e^{3/4} r^{5/4} - \frac{70}{4 I^{1/2}} (\log(2^{2j_1+1} \theta_1 e^{3/4} r^{5/4}) - \log(2^{2j_1} \theta_1 e^{3/4} r^{5/4})).
\]

A similar argument for \( [2^{2j_0+1} \theta_1 e^{3/4} r^{5/4}, r] \) gives that

\[
(6.4) \quad \mathcal{I}(p; er) - \frac{1}{2} \mathcal{I}(p; 2^{2j_0+1} \theta_1 e^{3/4} r^{5/4}) \geq \frac{70}{4 I^{1/2}} (\log(er) - \log(2^{2j_0} \theta_1 e^{3/4} r^{5/4})).
\]

Recall the Harnack inequality from Corollary (5.3) gives that

\[
\mathcal{I}(p; 2^{2j_2} \theta_1 e^{3/4} r^{5/4}) \geq \gamma \mathcal{I}(p; 2^{2j_1-1} \theta_1 e^{3/4} r^{5/4}),
\]

for \( 1 \leq j \leq N \) and \( \gamma = \gamma(N, I) < 1/2 \). Particularly,

\[
\mathcal{I}(p; 2^{2j_2} \theta_1 e^{3/4} r^{5/4}) \geq \gamma \mathcal{I}(y_j; \theta_1 |y_1|).
\]

Then putting them together with suitable coefficients (see Appendix E for details), we have

\[
(6.5) \quad \mathcal{I}(p; er) - c_0 \mathcal{I}(y_j; \theta_1 |y_1|) \geq c_0 \gamma_0 (\log er - \log(\theta_1 |y_1|) - c_1)
\]

by setting \( c_0 = \gamma^{2N} / r^{1/2} \) and \( c_1 = \gamma^{-2N} I(10 I + 1) \log 2 \).

Thus we complete Step A.

**Step B:** We now repeat the process in Step A to construct \( \{y_j\} \) and \( \{\theta_j\} \).

Suppose that we have defined \( y_j \) and \( \theta_j \), then by Proposition (6.3), there exist a constant \( \theta_{j+1} \in (1/4 I^3, 1/16) \) and a subset \( Q_{j+1} \subset Q \cap B(y_j; \theta_j |y_j - y_{j-1}|) \) such that

- for any \( y \in Q \cap B(y_j; \theta_j |y_j - y_{j-1}|) \), there exists \( y' \in Q_{j+1} \) satisfying \( B(y'; \theta_{j+1} |y' - y_j|/4) \) or \( B(y'; \theta_{j+1} e^{3/4} r^{5/4} / 4) \).
• $\mathcal{B}(x'; 4\theta_{j+1}|x'-y_j|) \cap \mathcal{B}(x''; 4\theta_{j+1}|x''-y_j|) = \emptyset$ for any $x', x'' \in Q_{j+1}$ with $x' \neq x''$.
• $\mathcal{B}(x'; 4\theta_{j+1}|x'-y_j|) \cap \mathcal{B}(y; 4\theta_{j+1}e^{3/4}r^{5/4}) = \emptyset$ for any $x' \in Q_{j+1}$.

Note that for $x \in \mathcal{A}(y; \theta_{j+1}e^{3/4}r^{5/4}, \theta_j|y_j|) \setminus \bigcup_{y \in Q_{j+1}} \mathcal{B}(y; \theta_{j+1}|y-y_j|)$, we always have some constant $C_I$ depending only on $I$ satisfying
\[
d(x, Q) \geq \epsilon^{3/4}r^{5/4}/C_I,
\]
which implies that
\[
|\Delta w| \leq \frac{C_0C_I}{r^3}.
\]

Claim 11. Either $\tau(y; \theta_{j+1}|y-y_j|) \leq \tau(y_j; \theta_j|y_j-y_{j-1}|)/4^{I+1}$ for all $y \in Q_{j+1}$ ($y_0 = p$), or there exists $x_{j+1} \in Q_{j+1}$ so that
\[
(6.6) \quad \tau(x_{j+1}; \theta_{j+1}|x_{j+1}-y_j|) \geq \tau(y_j; \theta_j|y_j-y_{j-1}|)/4^{I+1}.
\]

Proof of Claim [11]. The proof here is similar to that of Proposition [6.5] with minor modification. Indeed, it follows by replacing $\mathcal{A}(p; r^2, \epsilon r)$ with $\mathcal{A}(y_j; r^2, \theta_j|y_j-y_{j-1}|)$. □

Whenever $Q_{j+1} \neq \emptyset$ and there exists $x_{j+1}$ so that (6.6) is satisfied, then we can define $y_{j+1}$ to be the element in $Q_{j+1}$ so that $\tau(y_{j+1}; \theta_{j+1}|y_{j+1}-y_j|) \geq \tau(y_j; \theta_j|y_j-y_{j-1}|)/4^{I+1}$ and
\[
\tau(y; \theta_{j+1}|y-y_j|) < \tau(y_j; \theta_j|y_j-y_{j-1}|)/4^{I+1}, \forall y \in Q_{j+1} \text{ with } |y-y_j| > |y_{j+1}-y_j|.
\]

Since $y_j \in \mathcal{B}(y_j; \theta_j|y_j-y_{j-1}|) \setminus \mathcal{B}(y_{j+1}; \theta_{j+1}|y_{j+1}-y_j|)$, then such a sequence of $y_j$ is finite, that is, there exists $0 < k < I + 1$ so that $Q_{k+1} = \emptyset$ or $\tau(y; \theta_{k+1}|y-y_k|) \leq \tau(y_k; \theta_k|y_k-y_{k-1}|)/4^{I+1}$ for all $y \in Q_{k+1}$ ($y_0 := p$).

Using the argument in Step A, by replacing $\mathcal{A}(p; r^2, \epsilon r)$ with $\mathcal{A}(y_j; r^2, \theta_j|y_j-y_{j-1}|)$, we can see that (ii) is satisfied. We leave the details to readers.

Step C: We now prove that such a sequence $\{y_j\}$ satisfies (iii).

By the construction in Step B, we know that either $Q_{k+1} = \emptyset$ or $\tau(y; \theta_{k+1}|y-y_k|) \leq \tau(y_k; \theta_k|y_k-y_{k-1}|)/4^{I+1}$ for all $y \in Q_{k+1}$ ($y_0 := p$).

Claim 12. In both case, $\tau(y_k; \theta_{k+1}e^{3/4}r^{5/4}) \geq \tau(y_k; \theta_k|y_k-y_{k-1}|)/4^{I+2}$.

Proof of Claim [12]. Note that for $x \in \mathcal{B}(y_k; \theta_{k+1}e^{3/4}r^{5/4}, \theta_k|y_k-y_{k-1}|) \setminus \bigcup_{y \in Q_{k+1}} \mathcal{B}(y; \theta_{k+1}|y-y_k|)$, we always have
\[
d(x, Q) \geq \epsilon^{3/4}r^{5/4}/C_I,
\]
for $C_I$ depending only on $I$. This implies that
\[
|\Delta w| \leq \frac{CC_I}{r^3}.
\]

Then the desired results follows from Proposition [6.5] by replacing $\mathcal{A}(p; r^2, \epsilon r)$ with $\mathcal{A}(y_k; r^2, \theta_k|y_k-y_{k-1}|)$.

Then the following is similar to that of Step A. For completeness of the proof, we give more details here.

By Lemma [6.1] we can find $2N \leq 2I + 1$ non-negative integers $\{l_j\}_{j=0}^{2N}$ ($l_0 = 0$) such that
\( l_{j+1} - l_j \geq 3; \)
\( l_{2j} - l_{2j-1} \leq 10I + 1 \) for all \( j \geq 1; \)
\( Q_{k+1} \subset \bigcup_{j=1}^{2N} A(p; 2^{l_{2j}+1} \theta_1 \epsilon^{3/4} r^{5/4}, 2^{l_{2j}-1} \theta_1 \epsilon^{3/4} r^{5/4}); \)

Then the argument in Step A implies that for \( j \geq 0, \)
\[
\mathcal{I}(y_k; 2^{l_{2j}+1} \theta_{k+1} \epsilon^{3/4} r^{5/4}) - \frac{1}{2} \mathcal{I}(y_k; 2^{l_{2j}} \theta_{k+1} \epsilon^{3/4} r^{5/4}) \geq \frac{\tau(y_k; \theta_k | y_k - y_{k-1})}{4^{l+2}} \left[ \log(2^{l_{2j}+1} \theta_{k+1} \epsilon^{3/4} r^{5/4}) - \log(2^{l_{2j}} \theta_{k+1} \epsilon^{3/4} r^{5/4}) \right],
\]
and
\[
\mathcal{I}(y_k; \theta_k | y_k - y_{k-1}) - \frac{1}{2} \mathcal{I}(y_k; 2^{l_{2N}} \theta_{k+1} \epsilon^{3/4} r^{5/4}) \geq \frac{\tau(y_k; \theta_k | y_k - y_{k-1})}{4^{l+2}} \left[ \log(\theta_k | y_k - y_{k-1}) - \log(2^{l_{2N}} \theta_{k+1} \epsilon^{3/4} r^{5/4}) \right].
\]

Recall the Harnack inequality from Corollary 3.4 gives that
\[
\mathcal{I}(y_k; 2^{l_{2j}} \theta_{k+1} \epsilon^{3/4} r^{5/4}) \geq \gamma \mathcal{I}(y_k; 2^{l_{2j}-1} \theta_{k+1} \epsilon^{3/4} r^{5/4}),
\]
for \( 1 \leq j \leq N \) and \( \gamma = \gamma(N, I) < 1/2. \) Then putting them together with suitable coefficients (also, see Appendix B for a similar process), we have
\[
\mathcal{I}(y_k; \theta_k | y_k - y_{k-1}) - c_0 \mathcal{I}(y_k; \theta_{k+1} \epsilon^{3/4} r^{5/4}) \geq c_0 \tau_0 \left[ \log(\theta_k | y_k - y_{k-1}) - \log(2 \theta_{k+1} \epsilon^{3/4} r^{5/4}) - c_1 \right].
\]
by setting \( c_0 = \gamma^{-2N}/r^{l+2} \) and \( c_1 = \gamma^{-2N}(10I + 1) \log 2. \)

This is the desired inequality.

**Lemma 6.7.** Let \( w \) be the function in Proposition 6.6. Then there exists a constant \( C_5 = C_5(C_1, N, I) \) so that
\[
\tau_0 \log(\epsilon/r) \leq C_5 \mathcal{I}(p; \epsilon r).
\]

**Proof.** We first assume that
\[
\tau(y, \theta | y) \leq \tau_0 / 4^{l+1},
\]
for all \( y \in Q'. \) Then the argument in Step A and Step C in Proposition 6.6 gives that
\[
\mathcal{I}(p; \epsilon r) - c_0 \mathcal{I}(p; \epsilon r^{3/4} r^{5/4}) \geq c_0 \tau_0 \left( \log(\theta \epsilon^{3/4} r^{5/4}) - c_1 \right) \geq c_0 \tau_0 \left( \frac{1}{4} \log(\epsilon/r) - c_1 \right).
\]
It follows that \( \tau_0 \log(\epsilon/r) \leq C_5 \mathcal{I}(p; \epsilon r) \) for some constant \( C_5 \) depending only on \( c_1, c_0. \)

Note that \( c_0 \) and \( c_1 \) depend only on \( C_1, N, I. \) This is exactly the desired result.

Now we consider that there is \( y \in Q' \) satisfying
\[
\tau(y, \theta | y) > \tau_0 / 4^{l+1}.
\]
Then by Proposition 6.6, there exist \( c_0 \) and \( c_1 \) depending only on the constant \( C_0 \) (in Proposition 6.6) and \( I, \) a sequence \( y_1, y_2, ..., y_k \in Q \) and \( \theta_1, \theta_2, ..., \theta_{k+1} \in (1/4^{4l+3}, 1/16) \) such that
(i) \( \mathcal{I}(p; \epsilon r) - c_0 \mathcal{I}(y_1; \theta_1 | y_1) \geq c_0 \tau_0 \left( \log(\epsilon r) - \log(\theta_1 | y_1) - c_1 \right); \)
(ii) for \( j \leq k - 1 \), \( I(y_j; \theta_j|y_j - y_{j-1}) - c_0 I(y_{j+1}; \theta_{j+1}|y_{j+1} - y_j) \geq c_0 \tau_0 (\log(\theta_j|y_j - y_{j-1}) - \log(\theta_{j+1}|y_{j+1} - y_j)) - c_1 \), where \( y_0 = p \);

(iii)

\[
I(y_k; \theta_k|y_k - y_{k-1}) - c_0 I(y_k; \theta_{k+1}\epsilon^{3/4} r^{5/4}) \\
\geq c_0 \tau_0 [\log(\theta_k|y_k - y_{k-1}) - \log(\theta_{k+1}\epsilon^{3/4} r^{5/4}) - c_1].
\]

By adding them together with suitable coefficients like this:

\[
I(p; \epsilon r) - c_0^{k+1} I(y_k; \theta_{k+1}\epsilon^{3/4} r^{5/4}) \\
= I(p; \epsilon r) - c_0 I(y_1; \theta_1|y_1) + c_0^k \left[ I(y_k; \theta_k|y_k - y_{k-1}) - c_0 I(y_k; \theta_{k+1}\epsilon^{3/4} r^{5/4}) \right] + \\
+ \sum_{j=1}^{k-1} c_0^j \left[ I(y_j; \theta_j|y_j - y_{j-1}) - c_0 I(y_{j+1}; \theta_{j+1}|y_{j+1} - y_j) \right] + \\
\geq c_0 \tau_0 [\log \epsilon r - \log(\theta_1|y_1) - c_1] + c_0^{k+1} \tau_0 (\log(\theta_k|y_k - y_{k-1}) - \log(\theta_{k+1}\epsilon^{3/4} r^{5/4}) - c_1) \\
+ \sum_{j=1}^{k-1} c_0^{j+1} \tau_0 (\log(\theta_j|y_j - y_{j-1}) - \log(\theta_{j+1}|y_{j+1} - y_j)) - c_1 \\
\geq c_0^{k+1} \tau_0 (\log \epsilon r - \log(\theta_{k+1}\epsilon^{3/4} r^{5/4}) - c_1 (I + 1)) \\
\geq c_0^{k+1} \tau_0 \left( \frac{1}{4} \log(\epsilon r) - \log \theta_{k+1} - c_1 (I + 1) \right).
\]

Then the desired result follows from the argument in the first case. \( \square \)

6.3. One-sided Harnack inequality for minimal graph functions on minimal surfaces. In this subsection, we first use the obtained results in previous subsection to prove Theorem 6.2 and then give a proof of Theorem 3.6.

Proof of Theorem 6.2. By the assumptions of \( Q \), it follows that for \( x \in B(p; \epsilon r, \epsilon^2) \),

\[
\text{dist}(x, Q) \geq |x|/2,
\]

which implies that

\[
|\Delta w| \leq C_0 w \left( 1 + \frac{r^4}{|x|^4} \right). \tag{6.7}
\]

Claim 13. There exists a constant \( C_6 \) depending only on \( \mathcal{N}, C_0 \) so that

\[
w(x) \leq C_6 (I(p; \epsilon^2) + I(p; \epsilon r)),
\]

for all \( x \in \mathcal{A}(p; \epsilon r, \epsilon^2) \).

Proof of Claim 13. Note that for \( x \in \mathcal{A}(p; 2C_0 \epsilon r, \epsilon^2) \),

\[
\epsilon^4 |\Delta w| \leq C_0 w \epsilon^4 \left( 1 + \frac{1}{C_0^4 \epsilon^4} \right) < w/4.
\]

By virtue of Lemma A.3, there exists a constant \( C_7 \) so that for \( x \in \mathcal{A}(p; 2C_0 \epsilon r, \epsilon^2) \),

\[
w(x) \leq C_7 \max_{\partial \mathcal{A}(p; 2C_0 \epsilon r, \epsilon^2)} w \leq C_7 (I(p; 2C_0 \epsilon r) + I(p; \epsilon^2)). \tag{6.8}
\]
Recall that for $x \in \mathcal{A}(p; \epsilon r, 2C_0 \epsilon r)$, by Corollary 3.4
\begin{equation}
(6.9) \quad w(x) \leq C' \mathcal{I}(p; \epsilon r),
\end{equation}
for some constant $C' = C'(N, C_0)$. Then Claim 13 follows from (6.8) and (6.9). \qed

Then for $s \in (\epsilon r, \epsilon^2)$,
\[
|\partial_s \mathcal{I}(p; s) - \tau_0 s^{-1}| \leq s^{-1} \int_{\mathcal{A}(p; \epsilon r, s)} |\Delta w| \leq s^{-1} \int_{\mathcal{A}(p; \epsilon r, s)} C_0 w \cdot (1 + \frac{r^4}{|x|^4})
\]
\[
\leq s^{-1} \int_{\mathcal{A}(p; \epsilon r, s)} C_6 (\mathcal{I}(p; \epsilon^2) + \mathcal{I}(p; \epsilon r)) (1 + \frac{r^4}{|x|^4})
\]
\[
\leq C_6 (\mathcal{I}(p; \epsilon^2) + \mathcal{I}(p; \epsilon r)) (s + \frac{r^2}{\epsilon^2 s}).
\]

Integrating it from $\epsilon r$ to $\epsilon^2$, we get
\[
|\mathcal{I}(p; \epsilon^2) - \mathcal{I}(p; \epsilon r) - \tau_0 \log(\epsilon/r)| \leq C_6 (\mathcal{I}(p; \epsilon r) + \mathcal{I}(p; \epsilon^2)) (s^2 + \frac{r^2}{\epsilon^2} |\log(\epsilon/r)|)
\]
\[
\leq (\mathcal{I}(p; \epsilon r) + \mathcal{I}(p; \epsilon^2))/10,
\]
which implies that
\begin{equation}
(6.10) \quad \mathcal{I}(p; \epsilon^2) \leq 2\mathcal{I}(p; \epsilon r) + 2\tau_0 \log(\epsilon/r).
\end{equation}

Now if $\tau_0 \log(\epsilon/r) \leq \mathcal{I}(p; \epsilon r)$, then we have
\[
\mathcal{I}(p; \epsilon^2) \leq 4\mathcal{I}(p; \epsilon r).
\]

If $\mathcal{I}(p; \epsilon r) \leq \tau_0 \log(\epsilon/r)$, then applying Lemma 6.7 together with (6.7), we have
\[
\tau_0 \log(\epsilon/r) \leq C_5 \mathcal{I}(p; \epsilon r),
\]
and it follows that
\begin{equation}
(6.11) \quad \mathcal{I}(p; \epsilon^2) \leq 2C_5 \mathcal{I}(p; \epsilon r).
\end{equation}

In both cases, (6.11) always holds true. Then the desired inequality follows from Corollary 3.4. \qed

Now we are ready to prove our main result in this section:

**Proof of Theorem 3.6** By Lemma A.2 we can take $\epsilon$ small enough so that
\[
|\Delta_N w| \leq C |\nabla^2 w|(|\nabla w|^2 + |\nabla v|^2) + C |\nabla^2 w||\nabla v||\nabla^2 v| + |\nabla w||v| + C(1 + |\nabla^2 w| + |\nabla^2 v||w + C|\nabla w||v| + C|\nabla w||\nabla^2 v|,
\]
where $C = C(M, N, K)$. Recall that $(v, u)$ is a strong $(C_1(|x|^2 + r^2), K)$-pair. Therefore,
\[
|\nabla v(x)| + |\nabla^2 v(x)| \leq K|v(x)| \leq K^2 |x|.
\]
Then the inequality becomes
\[
|\Delta_N w| \leq C |\nabla^2 w|(|\nabla w|^2 + |x|^2) + C(1 + |\nabla^2 w|)w + C|\nabla w||x|,
\]
By virtue of the gradient and the second order estimates in Lemma 3.3, then we have for $x \in \Xi$,
\[
|\nabla w| \leq C'|w|/d(x, Q), \quad |\nabla^2 w| \leq C'|w|/d^2(x, Q),
\]
where $Q = \bigcup_{j=1}^{I}\{q_j\} \cup \{p\}$. Taking it back, we obtain
\[
|\Delta w| \leq C'|w| + \frac{C'|w|^3}{d^4(x, Q)}.
\]

Therefore, $w$ satisfies all the conditions in Theorem 6.2, which implies that there exist $C_6 = C_6(M, \mathcal{N}, K, I, C_0)$ so that
\[
\max_{\partial B(p; \epsilon)} w \leq C_6 \min_{\partial B(p; \sqrt{\epsilon r})} w,
\]
which is exactly the desired inequality. $\square$

**Appendix A. Minimal graph functions**

In this section, we let $\mathcal{N}$ be a two-sided, embedded minimal hypersurface possibly with boundary in $(M^{n+1}, g)$. Denote by $n$ the unit normal vector field on $\mathcal{N}$. Then there exists a local foliation $\{\mathcal{N}_s\}$ around $\mathcal{N}$ by the level set of the distance function to $\Sigma$, where
\[
\Sigma_s = \{\exp_p s n : p \in \Sigma\}.
\]
Here $n$ is the unit normal vector of $\Sigma$.

An embedded hypersurface $\Sigma$ is said to be a graph over $\mathcal{N}$ with function $u$ if the exponential map $\exp(\cdot, u) : \mathcal{N} \to \Sigma$ is a diffeomorphism, where $\exp(p, u) = \exp_p(un(p))$.

Let $\nabla^s$ be the connection on $\mathcal{N}_s$. We will write $\nabla$ with no ambiguity. Denote by $\pi$ the projection to $\mathcal{N}$. Then given a function on $\mathcal{N}$, $f$ can also be regarded as a function on $\mathcal{N}_s$ by defining
\[
\tilde{f}(\exp_x (sn)) := f(x), \quad \forall x \in \mathcal{N}.
\]
Note that $\nabla \tilde{f}|_{\mathcal{N}_s}$ is the extension of $\nabla f$ by parallel moving.

Let $d$ be the oriented distance function to $\mathcal{N}$. Then $\nabla d$ is the unit normal vector field on $\mathcal{N}_s$, which is an extension of $n$. For $p \in \Sigma \cap \mathcal{N}_s$, let $\{e_i\}$ be an orthonormal base of $T_p \mathcal{N}_s$. Then $\{e_i + \langle \nabla u, e_i \rangle \nabla d\}$ is a base of $T_p \Sigma$. It follows that $n_\Sigma = (\nabla d - \nabla u)/\sqrt{1 + |\nabla u|^2}$ is the unit normal vector field of $\Sigma$. Naturally, such $n_\Sigma$ can be extend to $\tilde{n}_\Sigma$ in a neighborhood of $\mathcal{N}$ by parallel moving.

Now let $X$ be a vector field around $\mathcal{N}$ so that $\nabla_{\nabla d} X = 0$. Recall that $\pi$ is the projection to $\mathcal{N}$. Then we have the following:

**Lemma A.1.** There exist $\delta = \delta(M, \mathcal{N})$ and $C = C(M, \mathcal{N})$ so that for $|h|, |s| < \delta$,
\[
|\nabla X|_{\mathcal{N}_h}(\pi^{-1}(x) \cap \mathcal{N}_h) - \nabla X|_{\mathcal{N}_s}(\pi^{-1}(x) \cap \mathcal{N}_s)| \leq C((|X| + |\nabla X|)_{\mathcal{N}} + |h| + |s|)(h - s).
\]
Proof. A standard computation gives that
\[
\frac{\partial}{\partial s} \langle (\nabla_{e_i} X, e_j) g^{ij} \rangle = \langle \frac{\partial}{\partial s} \nabla_{e_i}, e_j \rangle \cdot g^{ij} + \langle \nabla_{e_i}, \nabla_{e_j} \nabla d \rangle \cdot g^{ij} + \langle \nabla_{e_i} X, e_j \rangle \frac{\partial}{\partial s} g^{ij} \\
= -\text{Ric}(\nabla d, X) + \langle \nabla X, \nabla^2 d \rangle - 2\langle \nabla X, \nabla^2 d \rangle \\
= -\text{Ric}(\nabla d, X) - \langle \nabla X, \nabla^2 d \rangle.
\]
Then since \( \frac{\partial}{\partial s} X = 0 \), then
\[
\langle (\frac{\partial}{\partial s} \nabla X)(e_i, e_j) \rangle = \frac{\partial}{\partial s} \langle (\nabla_{e_i}, e_j) \rangle - \langle \nabla X \rangle \langle \frac{\partial}{\partial s} e_i, e_j \rangle - \langle \nabla X \rangle \langle e_i, \frac{\partial}{\partial s} e_j \rangle \\
= R(\nabla d, e_i, X, e_j) + \langle \nabla_{e_i} X, \nabla_{e_j} \nabla d \rangle - \langle \nabla X \rangle \langle \frac{\partial}{\partial s} e_i, e_j \rangle - \langle \nabla_{e_i} X, \frac{\partial}{\partial s} e_j \rangle \\
= R(\nabla d, e_i, X, e_j) - \langle \nabla X, A_s \rangle,
\]
where \( A_s \) is the second fundamental form of \( N_s \). It follows that
\[
\frac{\partial}{\partial s} |\nabla X|_{N_s} \leq a + b|\nabla X|_{N_s},
\]
for some constants \( a, b > 0 \). By the standard ODE inequality, we have
\[
|\nabla X|_{N_s} \leq C(|\nabla X|_{N_s} + |s|).
\]
Combing with the derivative formula, we have
\[
|\text{div}_{N_h} X - \text{div}_{N_s} X| \leq C(|X| + \sup_{s \leq t \leq h} |\nabla X|_{N_t})(h - s) \\
\leq C(|X| + |\nabla X|_{N_s} + |h| + |s|)(h - s).
\]
Then the desired result follows from triangle inequalities.

\[\square\]

Lemma A.2. Given \( K > 0 \), there exist constants \( \delta \) and \( C \) depending only on \( M, \mathcal{N}, K \) so that if \( \Sigma \) and \( \Gamma \) are minimal graphs with function \( v, u \) on a subset of \( \mathcal{B}(p; \epsilon) \) and \((v, u)\) is a \((\delta, K)\)-pair, then
\[
|\Delta_N w| \leq C|\nabla^2 w|(|\nabla v|^2 + |\nabla v|^2) + C|\nabla^2 w|(|\nabla v||\nabla^2 v| + |\nabla w|)|v| \\
+ C(1 + |\nabla^2 v| + |\nabla^2 v|)w + C|\nabla w|v| + C|\nabla w| \cdot |\nabla^2 v|,
\]
where \( w = v - u \).

Proof. For \( p \in \Sigma \cap \mathcal{N}_s \), let \( \{e_i\} \) be an orthonormal base of \( T_p \mathcal{N}_s \). Then \( \{e_i + (\nabla u, e_i) \nabla d\} \) is a base of \( T_p \Sigma \). It follows that \( n_\Sigma = (\nabla d - \nabla u)/\sqrt{1 + |\nabla u|^2} \) is the unit normal vector field of \( \Sigma \). Since \( \Sigma \) is minimal, we have
\[
0 = \text{div}_\Sigma n_\Sigma.
\]
Naturally, such \( n_\Sigma \) can be extend to a neighborhood of \( \mathcal{N} \) by parallel moving. Denote by \( \tilde{n} \) the extended vector filed. Then we have
\[
\text{div}_M \tilde{n}|_\Sigma = \text{div}_\Sigma \tilde{n} = 0.
\]
Denote by $\mathbf{m}$ the unit normal vector of $\Gamma$ and $\tilde{\mathbf{m}}$ the extended vector field around $\mathcal{N}$. Then we also have

$$\text{div}_M \tilde{\mathbf{m}}|_\Gamma = \text{div}_\Gamma \mathbf{m} = 0.$$ 

Take $x \in \Gamma$ and $y \in \Sigma$ so that $\pi(x) = \pi(y) \in \mathcal{N}$. Set $t = d(x)$ and $s = d(y)$. Then by Lemma \[A.1\]

(A.1) \[|\text{div}_{\mathcal{N}'} \tilde{\mathbf{n}} - \text{div}_{\mathcal{N}'} \tilde{\mathbf{m}}| \leq C(1 + |\nabla \tilde{\mathbf{n}}|_\mathcal{N})(t - s).\]

Similarly,

(A.2) \[|\text{div}_{\mathcal{N}'} (\tilde{\mathbf{m}} - \tilde{\mathbf{n}}) - \text{div}_{\mathcal{N}'} (\tilde{\mathbf{m}} - \tilde{\mathbf{n}})| \leq C(|\tilde{\mathbf{m}} - \tilde{\mathbf{n}}|_\mathcal{N} + |\nabla (\tilde{\mathbf{m}} - \tilde{\mathbf{n}})|_\mathcal{N})|t|.

Since $\mathcal{N}$ is also minimal, hence we have

$$\text{div}_{\mathcal{N}'} (\tilde{\mathbf{m}} - \tilde{\mathbf{n}}) = \text{div}_{\mathcal{N}'} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right)$$

$$= \text{div}_{\mathcal{N}'} \left[\frac{-\nabla w}{\sqrt{1 + |\nabla w|^2}} + \frac{\langle w, \nabla u + \nabla v \rangle \cdot \nabla v}{(1 + |\nabla u|^2)\sqrt{1 + |\nabla v|^2} + (1 + |\nabla v|^2)\sqrt{1 + |\nabla u|^2}}\right].$$

Then a direct computation together with (A.1) and (A.2) gives that

$$\left|\text{div}_{\mathcal{N}'} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}}\right| \leq C(1 + |\nabla \tilde{\mathbf{n}}|_\mathcal{N})w + C(|\tilde{\mathbf{m}} - \tilde{\mathbf{n}}|_\mathcal{N} + |\nabla (\tilde{\mathbf{m}} - \tilde{\mathbf{n}})|_\mathcal{N})|t| +$$

$$+ \left|\text{div}_{\mathcal{N}'} \frac{\langle w, \nabla u + \nabla v \rangle \cdot \nabla v}{(1 + |\nabla u|^2)\sqrt{1 + |\nabla v|^2} + (1 + |\nabla v|^2)\sqrt{1 + |\nabla u|^2}}\right|$$

$$\leq C(1 + |\nabla \tilde{\mathbf{n}}|_\mathcal{N})w + C(|\nabla w|_\mathcal{N} + |\nabla (\tilde{\mathbf{m}} - \tilde{\mathbf{n}})|_\mathcal{N})|t| +$$

$$+ |\nabla w| \cdot |\nabla v| \cdot |\nabla^2 (u + v)| + |\nabla^2 w| \cdot |\nabla (u + v)| \cdot |\nabla v| +$$

$$+ |\nabla w| \cdot |\nabla (u + v)| \cdot (|\nabla^2 v| + |\nabla v| \cdot |\nabla^2 u| \cdot |\nabla u| + |\nabla^2 u| \cdot |\nabla v|^2)$$

$$\leq C(1 + |\nabla^2 w| + |\nabla^2 v|)w + C(|\nabla w|_\mathcal{N} + |\nabla (\tilde{\mathbf{m}} - \tilde{\mathbf{n}})|_\mathcal{N})|t| +$$

$$+ C(|\nabla^2 w| \cdot |\nabla v| + |\nabla^2 v| \cdot |\nabla w|)(|\nabla w| + |\nabla v|)$$

By definition,

$$(\tilde{\mathbf{m}} - \tilde{\mathbf{n}})_\mathcal{N} = \frac{\nabla d - \nabla v}{\sqrt{1 + |\nabla v|^2}} - \frac{\nabla d - \nabla u}{\sqrt{1 + |\nabla u|^2}}$$

$$= \frac{\nabla d - \nabla v}{\sqrt{1 + |\nabla v|^2}} - \frac{\nabla d - \nabla v}{\sqrt{1 + |\nabla u|^2}} - \frac{\nabla w}{\sqrt{1 + |\nabla u|^2}}$$

$$= -\langle \nabla w, \nabla u + \nabla v \rangle \cdot (\nabla d - \nabla v)$$

$$\leq \frac{1}{(1 + |\nabla u|^2)\sqrt{1 + |\nabla v|^2} + (1 + |\nabla v|^2)\sqrt{1 + |\nabla u|^2}} \cdot \frac{\nabla w}{\sqrt{1 + |\nabla u|^2}}.$$

Therefore, we have

$$|(\tilde{\mathbf{m}} - \tilde{\mathbf{n}})_\mathcal{N}| \leq |\nabla w|(1 + |\nabla (u + v)|).$$
and
\[ |\nabla (\tilde{m} - \tilde{n})|_{N} \leq |\nabla^{2} w| \cdot |\nabla (u + v)| + |\nabla w| \cdot |\nabla^{2} (u + v)| + |\nabla w| \cdot |\nabla^{2} u| + |\nabla^{2} v| \cdot |\nabla v| \cdot |\nabla (u + v)| + |\nabla w| \cdot |\nabla (u + v)| \cdot (|\nabla^{2} u| \cdot |\nabla u| + |\nabla^{2} v| \cdot |\nabla v|) \]
\[ \leq C (|\nabla w| + |\nabla^{2} w| + |\nabla w|^{3})(|\nabla v| + |\nabla^{2} v|) + |\nabla w| \cdot |\nabla^{2} w| + + (|\nabla w| + |\nabla v| \cdot |\nabla^{2} w|)(|\nabla v| + |\nabla^{2} v|)^{2} + |\nabla^{2} w| \cdot |\nabla w|^{3} + + |\nabla w|^{2} + |\nabla w| \cdot |\nabla v|^{2} \cdot |\nabla^{2} v|) \]
\[ \leq C (|\nabla w| + |\nabla^{2} w|)(|\nabla v| + |\nabla^{2} v| + |\nabla w|). \]

Taking them back, the inequality becomes
\[ |\Delta w| \leq C \left| \text{div}_{N} \frac{\nabla w}{\sqrt{1 + |\nabla u|^{2}}} \right| + C |\nabla w| \cdot |\nabla^{2} u| \cdot |\nabla u| \]
\[ \leq C |\nabla w| \cdot (|\nabla^{2} w| + |\nabla^{2} v|)(|\nabla w| + |\nabla v|) + + C (1 + |\nabla^{2} w| + |\nabla^{2} v|)w + C |\nabla w|t + \]
\[ + C (|\nabla^{2} w| \cdot |\nabla v| + |\nabla^{2} v| \cdot |\nabla w|)(|\nabla w| + |\nabla v|) + + C (|\nabla w| + |\nabla^{2} w|)(|\nabla v| + |\nabla^{2} v| + |\nabla w|)t \]
\[ \leq C |\nabla w|(|\nabla^{2} w| + |\nabla^{2} v|) + C |\nabla^{2} w|(|\nabla v| \cdot |\nabla^{2} v| + |\nabla w|)t + \]
\[ + C (1 + |\nabla^{2} w| + |\nabla^{2} v|)w + C |\nabla w|t + C |\nabla w| \cdot |\nabla^{2} v|, \]

which is exactly the desired result. \(\Box\)

Lemma A.3. Let \(N\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 2\). There exist constants \(C, \epsilon > 0\) so that if \(w > 0\) and \(R^{2} |\Delta w| \leq w/4\) on \(A(p; \rho, R) \subset N\) for \(\rho < R < \epsilon\), then
\[ \min_{\partial A(p; \rho, R)} w \leq C \min_{A(p; \rho, R)} w, \quad \max_{\partial A(p; \rho, R)} w \leq C \max_{A(p; \rho, R)} w. \]

Proof. Denote \(\rho(x) = \text{dist}_{N}(x, p)\). Set \(w_{1} = e^{c^{2}w} w\) and \(w_{2} = e^{br^{2}} w\). Then a direct computation gives that
\[ \Delta w_{1} = w \Delta e^{c^{2}w} + e^{c^{2}w} \Delta w + 2c \langle \nabla r, e^{c^{2}w} \nabla w \rangle \]
\[ = 2c \langle \nabla r, \nabla w_{1} \rangle + (\Delta w + (c \Delta r - e^{2}w) e^{c^{2}w} \]
Take \(c = -1/(2R)\). Then the inequality becomes
\[ \Delta w_{1} + 2 \sqrt{K} \langle \nabla r, \nabla w_{1} \rangle / R \leq 0. \]

By virtue of \([5]\) Theorem 8.1,
\[ \min_{\partial A(p; r, R)} w_{1} = \min_{A(p; r, R)} w_{1}, \]
which implies the desired inequality.
We can also compute the following directly:
\[
\Delta w_2 = w \Delta e^{br^2} + e^{br^2} \Delta w + 4br \langle \nabla r, e^{br^2} \nabla w \rangle \\
= 4br \langle \nabla r, \nabla w_2 \rangle + (\Delta w + (2br \Delta r + 2b - 4b^2 r^2)w)e^{br^2}.
\]

Take \( b = 1/(4R^2) \), then
\[
\Delta w_2 - 4br \langle \nabla r, \nabla w_2 \rangle \geq (\Delta w + w/(4R^2))e^{br^2} \geq 0.
\]

Using [5, Theorem 8.1] again, we have
\[
\max_{\partial A(p; r)} w_2 = \max_{\partial A(p; r)} w_2,
\]
which implies the desired inequality. \( \square \)

**Appendix B. Proof of Lemma 3.3**

**Proof of Lemma 3.3.** Denote by \( A_{\Sigma}(x) \) and \( A_{\Gamma}(x) \) the second fundamental form of \( \Sigma \) and \( \Gamma \) in \( M \). Then a standard blowup argument gives that
\[
(B.1) \quad \sup_{x \in B(q; r) \setminus V} \text{dist}_A(x, \partial B(q; r) \cup V) \cdot (|A_{\Gamma}(x)| + |A_{\Sigma}(x)|) < C.
\]

Now we prove the first inequality. Suppose not, there exist \( K > 0 \) and sequence of \( r_j > 0, p_j \in B(p; 1 - r_j) \) and \( V_j \subset B(p; 1) \) so that \( \{\Sigma_j\} \) and \( \{\Gamma_j\} \) are two sequences of minimal graphs over \( B(p_j; r_j) \setminus V_j \) with positive graph function \( u_j, v_j \) satisfying
\[
u_j(x) - v_j(x) > 0, \quad |u_j(x)| + |v_j(x)| < 1/j, \quad |\nabla u_j(x)| + |\nabla v_j(x)| < K,
\]
and
\[
\sup_{x \in B(p_j; r_j) \setminus V_j} \text{dist}_A(x, \partial B(p_j; r_j) \cup V_j) \cdot |\nabla \log(u_j - v_j)(x)| > j.
\]
Take \( q_j \in B(p_j; r_j) \setminus V_j \) so that
\[
\text{dist}_A(q_j, \partial B(p_j; r_j) \cup V_j) \cdot |\nabla \log(u_j - v_j)(q_j)| = \sup_{x \in B(p_j; r_j) \setminus V_j} \text{dist}_A(x, \partial B(p_j; r_j) \cup V_j) \cdot |\nabla \log(u_j - v_j)(x)|.
\]
Set
\[
\lambda_j = |\nabla \log(u_j - v_j)(q_j)| \quad \text{and} \quad \rho_j = \frac{1}{2} \cdot \text{dist}_A(q_j, \partial B(p_j; r_j) \cup V_j).
\]
Take \( q'_j \in \Sigma_j \) so that its projection to \( N \) is \( q_j \). Then by (B.1), \( (B(q'_j; \rho_j), \lambda_j^2 g, q'_j) \) locally smoothly converges to a minimal graph over a hyperplane. Then it is a hyperplane by Bernstein theorem.

Now denote by \( \tilde{\nabla} \) the Levi-Civita connection under the metric \( \lambda_j^2 g \). Denote by \( \tilde{u}_j, \tilde{v}_j \) the graph functions of \( (\Sigma_j, \lambda_j^2 g) \) and \( (\Gamma_j, \lambda_j^2 g) \) over \( (B(q'_j; \rho_j), \lambda_j^2 g) \). Then we have
\[
(B.2) \quad \frac{\langle \tilde{\nabla}(\tilde{u}_j - \tilde{v}_j)(q_j), \lambda_j^2 g \rangle}{(\tilde{u}_j - \tilde{v}_j)(q_j)} = \frac{\nabla(u_j - v_j)(q_j)}{\lambda_j(u_j - v_j)(q_j)} = 1,
\]
and for any \( x \in \mathcal{B}(q_j; \rho_j) \),
\[
\frac{|
abla (\tilde{u}_j - \tilde{v}_j)(x)\|_{\lambda_j^2 g}(\tilde{u}_j - \tilde{v}_j)(x)}{\lambda_j(u_j - v_j)(x)} = \frac{|\nabla (u_j - v_j)(x)|_{g}}{\lambda_j(u_j - v_j)(x)} < 2.
\]

**Claim 14.** \((\tilde{u}_j - \tilde{v}_j)(q_j) \to 0\).

**Proof of Claim [4]** Note that
\[
|\nabla (\tilde{u}_j - \tilde{v}_j)(q_j)|_{\lambda_j^2 g} = |\nabla (u_j - v_j)(q_j)|_{g}.
\]
Then by our assumptions, it is bounded from above by \( K \). Then using (B.2), we have that \((\tilde{u} - \tilde{v}_j)(q_j)\) is bounded from above by \( K \).

Recall that (B.1) implies that \((\Sigma_j, \lambda_j^2 g, q'_j)\) and \((\Gamma_j, \lambda_j^2 g, q'_j)\) locally smoothly converge to hyperplanes. By the assumption of \( u_j - v_j > 0 \), such two limit hyperplanes are parallelloing to each other. This deduces that \(|\nabla (\tilde{u}_j - \tilde{v}_j)(q_j)| \to 0\). Using (B.2) again, we conclude that \((\tilde{u}_j - \tilde{v}_j)(q_j) \to 0\).

Set \( h_j(x) = (\tilde{u}_j - \tilde{v}_j)(x)/(\tilde{u}_j - \tilde{v}_j)(q_j) \). Then \( h_j \) converges to a positive harmonic function of \( \mathbb{R}^n \). Hence it is a constant. On the other hand,
\[
|\nabla h_j(q_j)| = \frac{|\nabla (\tilde{u}_j - \tilde{v}_j)(q_j)|}{(\tilde{u}_j - \tilde{v}_j)(q_j)} = 1,
\]
which leads to a contradiction.

We now prove the second inequality. Similarly, suppose not, there exist \( K > 0 \) and sequence of \( r_j > 0 \), \( p_j \in \mathcal{B}(p; 1 - r_j) \) and \( V_j \subset \mathcal{B}(p; 1) \) so that \( \{\Sigma_j\} \) and \( \{\Gamma_j\} \) are two sequences of minimal graphs over \( \mathcal{B}(p_j; r_j) \setminus V_j \) with positive graph function \( u_j, v_j \) satisfying
\[
\begin{aligned}
&u_j(x) - v_j(x) > 0, \quad |u_j(x)| + |v_j(x)| < 1/j, \quad |\nabla u_j(x)| + |\nabla v_j(x)| < K, \\
&\sup_{x \in \mathcal{B}(p_j; r_j) \setminus V_j} \text{dist}_{\mathcal{N}}^2(x, \partial \mathcal{B}(p_j; r_j) \cup V_j) \cdot \frac{|\nabla^2 \log(u_j - v_j)(x)|}{(u_j - v_j)(x)} > j.
\end{aligned}
\]
Take \( q_j \in \mathcal{B}(p_j; r_j) \setminus V_j \) so that
\[
\begin{aligned}
&\text{dist}_{\mathcal{N}}^2(q_j, \partial \mathcal{B}(p_j; r_j) \cup V_j) \cdot \frac{|\nabla^2 \log(u_j - v_j)(q_j)|}{(u_j - v_j)(q_j)} \\
&= \sup_{x \in \mathcal{B}(p_j; r_j) \setminus V_j} \text{dist}_{\mathcal{N}}^2(x, \partial \mathcal{B}(p_j; r_j) \cup V_j) \cdot \frac{|\nabla^2 \log(u_j - v_j)(x)|}{(u_j - v_j)(x)}.
\end{aligned}
\]
Set
\[
\lambda_j = \sqrt{\frac{|\nabla^2 \log(u_j - v_j)(q_j)|}{(u_j - v_j)(q_j)}} \quad \text{and} \quad \rho_j = \frac{1}{2} \cdot \text{dist}_{\mathcal{N}}(q_j, \partial \mathcal{B}(p_j; r_j) \cup V_j).
\]
Take \( q'_j \in \Sigma_j \) so that its projection to \( N \) is \( a_j \). Then by (B.1), \((\mathcal{B}(q'_j; \rho_j), \lambda_j^2 g, q'_j)\) locally smoothly converges to a minimal graph over a hyperplane. Then it is a hyperplane by Bernstein theorem.
Now denote by $\nabla$ the Levi-Civita connection under the metric $\lambda_j^2g$. Denote by $\tilde{u}_j, \tilde{v}_j$ the graph functions of $(\Sigma_j, \lambda_j^2g)$ and $(\Gamma_j, \lambda_j^2g)$ over $(B(q_j; \rho_j), \lambda_j^2g)$. Then we have

\begin{equation}
(B.3) \frac{|\nabla^2(\tilde{u}_j - \tilde{v}_j)(q_j)|_{\lambda_j^2g}}{(\tilde{u}_j - \tilde{v}_j)(q_j)} = \frac{|\nabla^2(u_j - v_j)(q_j)|_\alpha}{\lambda_j^2(u_j - v_j)(q_j)} = 1,
\end{equation}

Claim 15. $(\tilde{u}_j - \tilde{v}_j)(q_j) \to 0$.

Proof of Claim \ref{claim:15}. Recall that \ref{eq:b.1} implies that $(\Sigma_j, \lambda_j^2g, q_j')$ and $(\Gamma_j, \lambda_j^2g, q_j')$ locally smoothly converge to hyperplanes. Hence $|\nabla^2(\tilde{u}_j - \tilde{v}_j)| \to 0$. Using \ref{eq:b.3}, we have $(\tilde{u}_j - \tilde{v}_j)(q_j) \to 0$. □

It follows that $(\Sigma_j, \lambda_j^2g, q_j')$ and $(\Gamma_j, \lambda_j^2g, q_j')$ locally smoothly converge to a same hyperplanes. Set $h_j(x) = (\tilde{u}_j - \tilde{v}_j)(x)/(\tilde{u}_j - \tilde{v}_j)(q_j)$. Then $h_j$ converges to a positive harmonic function of $\mathbb{R}^n$. Hence it is a constant. On the other hand,

$$|\nabla^2h_j(q_j)| = \frac{|\nabla^2(\tilde{u}_j - \tilde{v}_j)(q_j)|}{(\tilde{u}_j - \tilde{v}_j)(q_j)} = 1,$$

which leads to a contradiction.

Thus, we have prove Lemma \ref{lem:c.1}. □

APPENDIX C. rearrange lemma

The following lemma is used in this paper frequently.

Lemma C.1. Let $\{\alpha_j\}_{i=1}^I$ be a sequence of positive numbers with $\alpha_j \geq 1$. Then there exist $2N(\leq 2I)$ non-negative integers $\{k_j\}_{j=1}^{2N}$ such that

- $k_{j+1} - k_j \geq 3;
- k_{2j} - k_{2j-1} \leq 10I + 1;
- \{\alpha_j\}_{i=1}^I \subseteq \bigcup_k [k_{2j-1} + 1, k_{2j} - 1].$

Proof. Take $k_1 = \max\{\inf[\alpha_j] - 1, 0\}$. Then define $k_{2s}$ from $k_{2s-1}$ by

$$k_{2s} := \inf\{t \in \mathbb{Z} : t \geq k_{2s-1}, \alpha_j \notin (t, t + 6) \text{ holds for all } j\} + 2.$$

Then define $k_{2s+1}$ from $k_{2s}$ by

$$k_{2s+1} := \sup\{t \in \mathbb{Z} : t \geq k_{2s}, \alpha_j \notin (k_{2s}, t) \text{ holds for all } j\} - 1.$$
Lemma D.1. Then there exists a constant $C_0, \epsilon > 0$ so that for any $r_1, r_2$ satisfy $C_0Ir \leq r_1 \leq r_2 < \epsilon$, there exists a $C^1$ curve $\gamma : [0, 1] \to A(p; r_1, r_2) \setminus \bigcup_{j=1}^{N} B(q_j; r)$ satisfying

- $\gamma(0) \in \partial B(p; r_1)$ and $\gamma(1) \in \partial B(p; r_2)$;
- $\text{Length}(\gamma) \leq C_0(r_2 - r_1)$;
- $\text{dist}(\gamma, \bigcup_{j=1}^{N} B(q_j, r)) \geq r_1/(C_0I)$.

Appendix E. Proof of (6.5)

In this section, we give the proof of (6.5). Such a fundamental process has been used frequently in this paper.

Proof of (6.5). For simplicity, we define the following notions:

$$A := \mathcal{I}(p; \epsilon r) - \gamma \mathcal{I}(p; 2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}),$$

$$B := \sum_{j=j'+1}^{N} \gamma^{2N-2j+1} \left[ \mathcal{I}(p; 2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) - \gamma \mathcal{I}(p; 2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) \right],$$

$$C := \sum_{j=j'}^{N-1} \gamma^{2N-2j} \left[ \mathcal{I}(p; 2^{k_2j} \theta_1 \epsilon^{3/4} r^{5/4}) - \gamma \mathcal{I}(p; 2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) \right],$$

$$D := \gamma^{2N-2j'+1} \left[ \mathcal{I}(p; 2^{k_2j'} \theta_1 \epsilon^{3/4} r^{5/4}) - \gamma \mathcal{I}(y_1; \theta | y_1) \right].$$

Then we have

$$A \geq \frac{\tau_0}{4^{j+2}} \left( \log(\epsilon/r) - \log(2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) \right),$$

$$B \geq \sum_{j=j'+1}^{N} \gamma^{2N-2j+1} \left[ \frac{\tau_0}{4^{j+2}} \left( \log(2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) - \log(2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) \right) \right],$$

$$D \geq \frac{\tau_0}{4^{j'+2}} \left( \log(2^{k_2j'} \theta_1 \epsilon^{3/4} r^{5/4}) - \log(\theta_1 | y_1) \right),$$

and

$$C \geq 0 = \sum_{j=j'}^{N-1} \gamma^{2N-2j} \left[ \frac{\tau_0}{4^{j+2}} \left( \log(2^{k_2j} \theta_1 \epsilon^{3/4} r^{5/4}) - \log(2^{k_2} \theta_1 \epsilon^{3/4} r^{5/4}) \right) \right] +$$

$$+ \sum_{j=j'}^{N-1} \gamma^{2N-2j} \left[ \frac{\tau_0}{4^{j+2}} (k_{2j+1} - k_{2j}) \log 2 \right].$$
Thus,
\[
A + B + C + D \geq \gamma 2^{N} \frac{\tau_0}{4I+2} \left[ \log(\epsilon/r) - \log(\theta_1|y_1|) \right] - \sum_{j=j'}^{N-1} \left[ \frac{\tau_0}{4I+2}(k_{2j+1} - k_{2j}) \log 2 \right]
\]
\[
\geq \gamma 2^{N} \frac{\tau_0}{4I+2} \left[ \log(\epsilon/r) - \log(\theta_1|y_1|) \right] - I(10I + 1)\frac{\tau_0}{4I+2} \log 2.
\]
On the other hand,
\[
A + B + C + D = \mathcal{I}(p; \epsilon r) - 2^{N-2j'}^2 \mathcal{I}(y_1; \theta_1|y_1|) \leq \mathcal{I}(p; \epsilon r) - 2^{N} \mathcal{I}(y_1; \theta_1|y_1|)/4I+2.
\]
Then (6.5) follows by setting \(c_0 = \gamma 2^{N}/r^I+2\) and \(c_1 = \gamma^{-2N}I(10I + 1)\log 2\). □

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