MINIMIMAL HYPERSURFACES AND BORDISM OF POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. Let \((Y, g)\) be a compact Riemannian manifold of positive scalar curvature (psc). It is well-known, due to Schoen-Yau, that any closed stable minimal hypersurface of \(Y\) also admits a psc-metric. We establish an analogous result for stable minimal hypersurfaces with free boundary. Furthermore, we combine this result with tools from geometric measure theory and conformal geometry to study psc-bordism. For instance, assume \((Y_0, g_0)\) and \((Y_1, g_1)\) are closed psc-manifolds equipped with stable minimal hypersurfaces \(X_0 \subset Y_0\) and \(X_1 \subset Y_1\). Under natural topological conditions, we show that a psc-bordism \((Z, g) : (Y_0, g_0) \sim (Y_1, g_1)\) gives rise to a psc-bordism between \(X_0\) and \(X_1\) equipped with the psc-metrics given by the Schoen-Yau construction.

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1. Introduction

1.1. Schoen-Yau minimal hypersurface technique. For a Riemannian metric $g$, we denote by $R_g$ the scalar curvature function, and by $H_g$ the mean curvature of the boundary (if it is not empty). The Schoen-Yau minimal hypersurface technique \cite{23} provides well-known geometric obstructions to the existence of positive scalar curvature. Here is the first fundamental result:

**Theorem 1.** \cite{23} Proof of Theorem 1] Let $(Y, g)$ be a compact Riemannian manifold with $R_g > 0$, and $\dim Y = n \geq 3$. Let $X \subset Y$ be a smoothly embedded stable minimal hypersurface with trivial normal bundle. Then $X$ admits a metric \(\tilde{h}\) with $R_{\tilde{h}} > 0$. Furthermore, the metric \(\tilde{h}\) could be chosen to be conformal to the restriction $g|_X$.

We note that Theorem 1 is proven by analyzing the conformal Laplacian of the hypersurface $X$. It it crucial that $X$ is stable minimal. For arbitrary $(Y, g)$ it is a non-trivial (and possibly obstructed) problem to find a stable minimal hypersurface. However, in low dimensions, geometric measure theory can provide a source of stable minimal hypersurfaces.

**Theorem 2.** (See \cite{17} Chapter 8], \cite{11} Theorem 5.4.15]) Let $(Y, g)$ be a compact orientable Riemannian manifold with $3 \leq \dim Y = n \leq 7$. Assume $\alpha \in H^{n-1}(Y; \mathbb{Z})$ is a nontrivial element. Then there exists a smoothly embedded hypersurface $X \subset Y$ such that

(i) up to multiplicity, $X$ represents the class $\alpha$;
(ii) $X$ minimizes volume among all hypersurfaces which represent $\alpha$ up to multiplicity. In particular, the hypersurface $X$ is stable minimal.

There are several important results based on Theorems 1 and 2. In particular, this gives a geometric proof that the torus $T^n$ does not admit a metric of positive scalar curvature for $n \leq 7$, see \cite{23}. This method was also crucial to provide first counterexample to the Gromov-Lawson-Rosenberg conjecture, see \cite{19}. In this paper we extend these ideas and techniques to the case of manifolds with boundary.

1.2. Stable minimal hypersurfaces with free boundary. Let $(M, \bar{g})$ be a manifold with non-empty boundary $\partial M$ and $W \subset M$ be an embedded hypersurface. We say that a hypersurface $W$ is properly embedded if, in addition, $\partial W = \partial M \cap W$. We say that such a hypersurface $W \subset M$ is stable minimal with free boundary if $W$ is a local minimum of the volume functional among properly embedded hypersurfaces, see Section 2. We establish the following analogue of Theorem 1 for manifolds with boundary in Section 2.3.

**Theorem 3.** Let $(M, \bar{g})$ be a compact Riemannian manifold with non-empty boundary $\partial M$, $R_{\bar{g}} > 0$ and $H_{\bar{g}} \equiv 0$ along $\partial M$, and $\dim M = n + 1 \geq 3$. Let $W \subset M$ be an embedded stable minimal hypersurface with free boundary and trivial normal bundle. Then $W$ admits a metric $\tilde{h}$ with $R_h > 0$ and $H_{\tilde{h}} \equiv 0$. Furthermore, the metric $\tilde{h}$ could be chosen to be conformal to the restriction $\bar{g}|_W$. 
The proof of Theorem 3 is similar to the case of closed manifolds. In particular, we have to analyze the conformal Laplacian on $W$ with minimal boundary conditions. This boundary condition works well with the free boundary stability inequality.

For a compact oriented $(n+1)$-dimensional manifold $M$, we consider the relative integral homology group $H_n(M, \partial M; \mathbb{Z})$. Let $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$ be a non-trivial class which we may assume to be represented by a properly embedded hypersurface $W \subset M$. We notice that the boundary $\partial W$ (which may possibly be empty) represents the class $\partial(\bar{\alpha}) \in H_{n-1}(\partial M; \mathbb{Z})$, where $\partial$ is the connecting homomorphism in the exact sequence

$$\cdots \to H_n(\partial M; \mathbb{Z}) \to H_n(M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(\partial M; \mathbb{Z}) \to \cdots$$

There is an analog of Theorem 2 which relies on a different regularity result, see Section 5.3 for more details.

**Theorem 4.** (See [14, Theorem 5.2]) Let $(M, \bar{g})$ be a compact orientable Riemannian manifold with non-empty boundary $\partial M$ and $3 \leq \dim M = n + 1 \leq 7$. Assume $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$ is a nontrivial element. Then there exists a smooth properly embedded hypersurface $W \subset M$ such that

1. up to multiplicity, $W$ represents the class $\bar{\alpha}$;
2. $W$ minimizes volume among all hypersurfaces which represent $\bar{\alpha}$ up to multiplicity. In particular, $W$ is stable minimal with free boundary.

### 1.3. Positive scalar curvature bordism and minimal hypersurfaces.

The main result of this paper is an application of Theorems 3 and 4 to provide new obstructions for psc-metrics to be psc-bordant.

**Definition 1.** Let $(Y_0, g_0)$ and $(Y_1, g_1)$ be closed oriented $n$-dimensional manifolds with psc-metrics. Then $(Y_0, g_0)$ and $(Y_1, g_1)$ are psc-bordant if there is a compact oriented $(n+1)$-dimensional manifold $(Z, \bar{g})$ such that

- the manifold $Z$ is an oriented bordism between $Y_0$ and $Y_1$, i.e., $\partial Z = Y_0 \cup -Y_1$;
- $\bar{g}$ is psc-metric which restricts to $g_i + dt^2$ near the boundary $Y_i \subset \partial Z$ for $i = 0, 1$.

We write $(Z, \bar{g}) : (Y_0, g_0) \leadsto (Y_1, g_1)$ for a psc-bordism as above.

**Remark.** Sometimes we consider bordisms $(Z, \bar{g}) : (Y_0, g_0) \leadsto (Y_1, g_1)$ as above where the metrics do not necessarily have positive scalar curvature. However we always assume that the metric $\bar{g}$ restricts to a product metric near the boundary.

Now we would like to enrich the psc-bordism relation with an extra structure, namely with a choice of homology classes $\alpha_i \in H_{n-1}(Y_i; \mathbb{Z})$, $i = 0, 1$. Recall the following elementary observation.

Let $\alpha \in H_{n-1}(Y; \mathbb{Z})$, where $Y$ is an oriented closed $n$-dimensional manifold. Then the cohomology class $D\alpha \in H^1(Y; \mathbb{Z})$ Poincare-dual to $\alpha$ can be represented by a smooth map $\gamma : Y \to B\mathbb{Z} = S^1$. Furthermore, we can assume that a given point $s_0 \in S^1$ is a regular value for $\gamma$. It is easy to see that
the inverse image \( X_\gamma := \gamma^{-1}(s_0) \subset Y \) is an embedded hypersurface which represents the homology class \( \alpha \).

If \( M \) is an oriented \((n + 1)\)-dimensional manifold with a map \( \bar{\gamma} : M \to S^1 \), let \( \gamma : \partial M \to S^1 \) be the restriction \( \bar{\gamma}|_{\partial M} \). There is a simple relation between the classes \([\gamma] \in H^1(M;\mathbb{Z})\) and \([\gamma] \in H^1(\partial M;\mathbb{Z})\):  

**Lemma 1.** Let \( \bar{\alpha} \in H_n(M,\partial M;\mathbb{Z}) \) and \( \alpha \in H_{n-1}(\partial M;\mathbb{Z}) \) be Poincare dual to the classes \([\gamma]\) and \([\gamma]\) respectively. Then \( \partial(\bar{\alpha}) = \alpha \), where \( \partial : H_n(M,\partial M;\mathbb{Z}) \to H_{n-1}(\partial M;\mathbb{Z}) \) is the connecting homomorphism. In particular, if \( W = \bar{\gamma}^{-1}(s_0) \subset M \) is a smooth properly embedded hypersurface representing \( \bar{\alpha} \), then the boundary \( \partial W \) represents the class \( \alpha \).

**Definition 2.** Let \((Y_0,g_0)\) and \((Y_1,g_1)\) be closed oriented \(n\)-dimensional Riemannian manifolds with given maps \( \gamma_0 : Y_0 \to S^1 \) and \( \gamma_1 : Y_1 \to S^1 \). We say that the triples \((Y_0,g_0,\gamma_0)\) and \((Y_1,g_1,\gamma_1)\) are **bordant** if there exists a bordism \((Z,\bar{g}) : (Y_0,g_0) \sim (Y_1,g_1)\) and a map \( \bar{\gamma} : Z \to S^1 \) such that \( \bar{\gamma}|_{Y_i} = \gamma_i \) for \( i = 0,1 \).

If the metrics \( g_0, g_1 \) and \( \bar{g} \) are psc-metrics, we say that the triples \((Y_0,g_0,\gamma_0)\) and \((Y_1,g_1,\gamma_1)\) are **psc-bordant**. In both cases we use the notation \((Z,\bar{g},\bar{\gamma}) : (Y_0,g_0,\gamma_0) \sim (Y_1,g_1,\gamma_1)\) for such a bordism.

**Theorem 5.** Let \((Y_0,g_0)\) and \((Y_1,g_1)\) be closed oriented connected \(n\)-dimensional manifolds with psc-metrics, \(3 \leq n \leq 6\), and maps \( \gamma_0 : Y_0 \to S^1 \) and \( \gamma_1 : Y_1 \to S^1 \). Assume that \((Y_0,g_0,\gamma_0)\) and \((Y_1,g_1,\gamma_1)\) are psc-bordant.

Then there exists a psc-bordism \((Z,\bar{g},\bar{\gamma}) : (Y_0,g_0,\gamma_0) \sim (Y_1,g_1,\gamma_1)\) and a properly embedded hypersurface \( W \subset Z \) such that

(i) the hypersurface \( W \) represents the class \( \bar{\alpha} \in H_n(Z,\partial Z;\mathbb{Z}) \) Poincare-dual to \([\gamma]\)  
(ii) the hypersurface \( X_i := \partial W \cap Y_i \subset Y_i \) represents the class \( \alpha_i \in H_{n-1}(Y_i;\mathbb{Z}) \) Poincare-dual to \([\gamma_i]\)  
(iii) there exists a metric \( \bar{h} \) on \( W \) such that \( R_{\bar{h}} > 0 \) and \( H_{\bar{h}} \equiv 0 \) along \( \partial W \), and \( R_{h_i} > 0 \), where \( h_i = \bar{h}|_{X_i} \). In particular, \((W,\bar{h}) : (X_0,h_0) \sim (X_1,h_1)\) is a psc-bordism;  
(iv) the metric \( \bar{h} \) on \( W \) could be chosen to be conformal to the restriction \( \bar{g}|_{W} \).

**Remark.** The psc-bordism \((Z,\bar{g},\bar{\gamma})\) and hypersurface \( W \) may be chosen so that \( \partial W \) is arbitrarily \(C^k\)-close to a desired homologically volume minimizing representative of \( \alpha_0 - \alpha_1 \) for any \( k \) and \( i = 0,1 \).

The first step in the proof of Theorem \([\text{5}]\) is to apply Theorem \([\text{4}]\) to \( \bar{\alpha} \) and obtain a minimal representative \( W \). The main difficulty is that \( \partial W \) is, in general, not a minimal representative of \( \partial \bar{\alpha} \) and so we may not apply Theorem \([\text{4}]\) to conclude that \( \partial W \) even admits a psc-metric. However, in the Main Lemma, we observe that \( \partial W \) becomes closer to minimizing \( \partial \bar{\alpha} \) as longer collars are attached to the psc-bordism \( Z \).

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2. Preliminaries and Theorem 3

2.1. Stable minimal hypersurfaces with free boundary. Let \((M, \bar{g})\) be a compact oriented
\((n+1)\)-dimensional Riemannian manifold with nonempty boundary \(\partial M\). Assume \(W \subset M\) is a
properly embedded hypersurface.

Let \(\bar{h}\) denote the restriction metric \(\bar{h} = \bar{g}|_W\) and fix a unit normal vector field \(\nu^W\) on \(W\)
which is compatible with the orientation. This determines the second fundamental form \(A^W\) on \(W\)
given by the formula \(A^W_g(X, Y) = \bar{g}(\nabla_X Y, \nu^W)\) for vector fields \(X\) and \(Y\) tangential to \(W\). The
trace of \(A^W_g\) with respect to the metric \(\bar{h}\) gives the mean curvature \(H^W_g = \text{tr}_{\bar{h}} A^W_g\). We will often
omit the sub- and super-scripts, writing \(\nu, A\), and \(H\) if there is no risk of ambiguity.

**Definition 3.** Let \(W \subset M\) be hypersurface \(W \subset M\). A **variation** of the hypersurface \(W \subset M\) is a
smooth one-parameter family \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\) of proper embeddings \(F_t : W \to M, t \in (-\epsilon, \epsilon)\) such that
\(F_0\) coincides with the inclusion \(W \subset M\). A variation \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\) is said to be **normal** if the curve
t \(\mapsto F_t(x)\) meets \(W\) orthogonally for each \(x \in W\).

The vector field \(X = \frac{d}{dt} F_t|_{t=0}\) is called the **variational vector field** associated to \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\).
For normal variations, the associated variational vector field takes the form \(\phi \cdot \nu^W\) for some function
\(\phi \in C^\infty(W)\). Clearly, a variation \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\) gives a smooth function \(t \mapsto \text{Vol}(F_t(W))\).

**Definition 4.** A properly embedded hypersurface \(W \subset (M, \bar{g})\) is **minimal with free boundary** if
\[
\frac{d}{dt} \text{Vol}(F_t(W))\Big|_{t=0} = 0
\]
for all variations \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\).

More notation: we denote by \(d\sigma\) and \(d\mu\) the volume forms of \((W, \bar{h})\) and \((\partial W, h)\), where
\(h = \bar{h}|_{\partial W}\) is the induced metric. We denote the outward-pointing unit length normal to \(\partial M\) by \(\nu^\partial\).
Below, Lemmas 2 and 3 contain well-known formulas, see [12].

**Lemma 2.** Let \((M, \bar{g})\) be an oriented Riemannian manifold and let \(W \subset M\) be a properly embedded
hypersurface. If \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\) is a variation of \(W\) with variational vector field \(X\), then
\[
\frac{d}{dt} \text{Vol}(F_t(W)) \bigg|_{t=0} = -\int_W H^W_g(X, \nu^W)d\mu + \int_{\partial W} \bar{g}(X, \nu^\partial) d\sigma.
\]
In particular, a hypersurface \(W\) is minimal with free boundary if and only if \(H^W_g \equiv 0\) and \(W\) meets
the boundary \(\partial M\) orthogonally.

**Definition 5.** A properly embedded minimal hypersurface with free boundary \(W\) is **stable** if
\[
\frac{d^2}{dt^2} \text{Vol}(F_t(W)) \bigg|_{t=0} \geq 0
\]
for all variations \(\{F_t\}_{t \in (-\epsilon, \epsilon)}\).
If a hypersurface $W$ is minimal with free boundary, then any variational vector field must be parallel to $\nu^W$ on $\partial W$ since the variation must go through proper embeddings. Hence, it enough to consider only normal variations to analyze the second variation of the volume functional.

**Lemma 3.** Let $(M, \bar{g})$ be an oriented Riemannian manifold and let $W \subset M$ be a properly embedded minimal hypersurface with free boundary. Let $\{F_t\}_{t \in (-\epsilon, \epsilon)}$ be a normal variation with variational vector field $\phi \cdot \nu^W$. Then

$$
\frac{d^2}{dt^2} \text{Vol}(F_t(W)) \bigg|_{t=0} = \int_W (|\nabla \phi|^2 - \phi^2 (\text{Ric}_\bar{g}(\nu^W, \nu^W) + |A^W|^2)) \, d\mu - \int_{\partial W} \phi^2 A^{BM}(\nu^W, \nu^W) \, d\sigma,
$$

where $\text{Ric}_\bar{g}$ denotes the Ricci tensor of $(M, \bar{g})$.

It will be useful to rewrite equation (2.2). The Gauss-Codazzi equations for any minimal hypersurface $W \subset M$ imply

$$
R_g^M = R_h^W + 2 \text{Ric}_\bar{g}(\nu^W, \nu^W) + |A^W|^2
$$
on $W$. Here $R_g^M$ and $R_h^W$ are the scalar curvatures of $(M, \bar{g})$ and $(W, \bar{h})$, respectively. It follows that the inequality $\frac{d^2}{dt^2} \text{Vol}(F_t(W)) \bigg|_{t=0} \geq 0$ is equivalent to

$$
\int_W |\nabla \phi|^2 \, d\mu \geq \int_W \frac{1}{2} \phi^2 (R_g^M - R_h^W + |A^W|^2) \, d\mu - \int_{\partial W} \phi^2 A^{BM}(\nu^W, \nu^W) \, d\sigma.
$$

2.2. **Conformal Laplacian with minimal boundary conditions.** The proof of Theorem 3 will rely on some basic facts about the conformal Laplacian on manifolds with boundary. Let $(W, \bar{h})$ be an $n$-dimensional manifold with non-empty boundary $(\partial W, h)$ where $h = \bar{h}|_{\partial W}$. We consider the following pair of operators acting on $C^\infty(W)$:

$$
\begin{align*}
L_\bar{h} & = -\Delta_\bar{h} + c_n R_h^W \quad \text{in } W \\
B_\bar{h} & = \partial_\nu + 2c_n H_{\bar{h}}^{BM} \quad \text{on } \partial W,
\end{align*}
$$

where $\nu$ is the outward pointing normal vector to $\partial W$ and $c_n = \frac{n-2}{4(n-1)}$.

Recall that if $\phi \in C^\infty(W)$ is a positive function, then the scalar and boundary mean curvatures of the conformal metric $\bar{h} = \phi^{-\frac{4}{n-2}} \bar{h}$ are given by

$$
\begin{align*}
R_{\bar{h}} & = c_n^{-1} \phi^{-\frac{n+2}{n-2}} \cdot L_\bar{h} \phi \quad \text{in } W \\
H_{\bar{h}} & = \frac{1}{2} c_n^{-1} \phi^{-\frac{n}{n-2}} \cdot B_\bar{h} \phi \quad \text{on } \partial W.
\end{align*}
$$

We consider a relevant Rayleigh quotient and take the infimum:

$$
\lambda_1 = \inf_{\phi \neq 0 \in H^1(W)} \frac{\int_W (|\nabla \phi|^2 + c_n R_{\bar{h}}^W \phi^2) \, d\mu + 2c_n \int_{\partial W} H_{\bar{h}}^{BM} \phi^2 \, d\sigma}{\int_W \phi^2 \, d\mu}.
$$

According to standard elliptic PDE theory, we obtain an elliptic boundary problem, denoted by $(L_\bar{h}, B_\bar{h})$, and the infimum $\lambda_1 = \lambda_1(L_\bar{h}, B_\bar{h})$ is the principal eigenvalue of the minimal boundary
problem \((L_h, B_h)\). The corresponding Euler-Lagrange equations are the following:

\[
\begin{align*}
L_h \phi &= \lambda_1 \phi \quad \text{in } W, \\
B_h \phi &= 0 \quad \text{on } \partial W
\end{align*}
\]

(2.6)

This problem was first studied by Escobar [7] in the context of the Yamabe problem on manifolds with boundary.

Let \(\phi\) be a solution of (2.6). It is well-known that the eigenfunction \(\phi\) is smooth and can be chosen to be positive. A straightforward computation shows that the conformal metric \(\tilde{h} = \phi^{\frac{4}{n-2}} \bar{h}\) has the following scalar and mean curvatures:

\[
\begin{align*}
R_{\tilde{h}} &= \lambda_1 \phi^{-\frac{4}{n-2}} \quad \text{in } W, \\
H_{\tilde{h}} &= 0 \quad \text{on } \partial W.
\end{align*}
\]

(2.7)

In particular, the sign of the eigenvalue \(\lambda_1\) is a conformal invariant, see [7, 10].

2.3. **Proof of Theorem 3** Let \((M, \bar{g})\) and \(W \subset M\) be as in Theorem 3. From the assumption \(H^\partial M \equiv 0\), one can use the Gauss equations to show that \(A^\partial M(\nu, \nu) = -H^\partial W\) where \(H^\partial W\) is the mean curvature of \(\partial W\) as a hypersurface of \(W\). Now, using the condition \(R_{\bar{g}}^M > 0\), the stability inequality (2.3) implies

\[
\int_W (|\nabla \phi|^2 + \frac{1}{2} R_{\bar{h}}^W) \, d\mu + \int_{\partial W} \phi^2 H^\partial W \, d\sigma \geq 0
\]

(2.8)

for all functions \(\phi \in H^1(W)\) with strict inequality if \(\phi \not\equiv 0\). By simple manipulation, the inequality (2.8) may be written as

\[
\int_W (|\nabla \phi|^2 + c_n R_{\bar{h}}^W) \, d\mu + 2c_n \int_{\partial W} \phi^2 H^\partial W \, d\sigma > (1 - 2c_n) \int_W |\nabla \phi|^2 \, d\mu
\]

(2.9)

for all \(\phi \not\equiv 0 \in H^1(W)\).

Clearly, the the right hand side of (2.9) is non-negative since \(1 - 2c_n = \frac{n}{2(n-1)} > 0\). Furthermore, the left hand side of (2.9) coincides with the numerator of the Rayleigh quotient in equation (2.5). We conclude that the principal eigenvalue \(\lambda_1 = \lambda_1(L_{\bar{h}}, B_{\bar{h}})\) is positive. Let \(\phi\) be an eigenfunction corresponding to \(\lambda_1\). Then, according to (2.7), the metric \(\tilde{h} = \phi^{\frac{4}{n-2}} \bar{h}\) has positive scalar curvature and zero mean curvature on the boundary. This completes the proof of Theorem 3.

3. **Cheeger-Gromov convergence of minimizing hypersurfaces**

3.1. **Convergence of hypersurfaces.** Here we introduce the notion of smooth convergence of hypersurfaces we require for the proof of Theorem 3. First, we consider the case when the hypersurfaces are embedded in the same ambient \((n + 1)\)-dimensional manifold \(M\). Below we use coordinate charts \(\Phi_j : U_j \rightarrow M\), where \(U_j\) is an open subset of \(\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}\).
Let $P \subset \mathbb{R}^{n+1}$ be a hyperplane equipped with a normal unit vector $\eta$, and $U \subset \mathbb{R}^{n+1}$ be an open subset. Then for a function $u : P \cap U \to \mathbb{R}$, we denote by $\text{graph}(u)$ its graph, see Fig. 1:

$$\text{graph}(u) = \{x + u(x)\eta \mid x \in P \cap U\}.$$ 

**Definition 6.** Let $k \geq 1$ be an integer. Let $(M, \bar{g})$ be an $(n+1)$-dimensional compact Riemannian manifold and let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of smooth, properly embedded hypersurfaces. Then the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface $\Sigma_\infty$ $C^k$-locally as graphs if there exist

(i) coordinate charts $\Phi_j : U_j \to M$, $j = 1, \ldots, N$;

(ii) a hyperplane $P_j \subset \mathbb{R}^{n+1}$ equipped with a normal unit vector $\eta_j$ for $j = 1, \ldots, N$;

(iii) smooth functions $u_{i,j} : P_j \cap U_j \to \mathbb{R}$ for $j = 1, \ldots, N$ and $i = 1, 2, \ldots$ and $i = \infty$,

which satisfy the following conditions:

(a) $\bigcup_{j=1}^{N} \Phi_j(\text{graph}(u_{i,j}) \cap U_j) = \Sigma_i$ for $i = 1, 2, \ldots$ and $i = \infty$;

(b) for each $j = 1, \ldots, N$, $u_{i,j} \to u_{\infty,j}$ in the $C^k(P_j \cap U_j)$ topology as $i \to \infty$.

We say the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface $\Sigma_\infty$ smoothly locally as graphs if it converges $C^k$-locally as graphs for all $k = 1, 2, \ldots$.

Next, we consider a sequence $\{(M_i, \Sigma_i, \bar{g}_i, S_i)\}$, where $(M_i, \bar{g}_i)$ is a Riemannian manifold, $\Sigma_i \subset M_i$ is a properly embedded smooth hypersurface, and $S_i \subset M_i$ a compact subset, playing a role of a base-point or a finite collection of base points.

**Definition 7.** Let $k \geq 1$ be an integer, and $\{(M_i, \Sigma_i, \bar{g}_i, S_i)\}$ be a sequence as above, where $\dim M_i = n + 1$. We say that the sequence $\{(M_i, \Sigma_i, \bar{g}_i, S_i)\}$ $C^k$-converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, S_\infty)$ if there is an exhaustion of $M_\infty$ by precompact open sets

$$S_\infty \subset U_1 \subset U_2 \subset \cdots \subset M_\infty, \quad M_\infty = \bigcup_{i=1}^{\infty} U_i$$

and maps $\Psi_i : U_i \to M_i$ which are diffeomorphisms onto their images for each $i = 1, 2, \ldots$, such that
(1) $\text{dist}_H^{M_\infty}(S_\infty, \Psi_i^{-1}(S_i)) \to 0$ as $i \to \infty$ where $\text{dist}_H^{M_\infty}$ is the Hausdorff distance between subsets of $M_\infty$;

(2) the sequence $\{\Psi_i^*\bar{g}_i\}$, of metrics converges to $\bar{g}_\infty$ in the $C^k(U_i)$-topology as $i \to \infty$;

(3) the sequence of hypersurfaces $\{\Psi_i^{-1}(\Sigma_i)\}$, $\Psi_i^{-1}(\Sigma_i) \subset M_\infty$, converges $C^k$-locally as graphs in $M_\infty$ to $\Sigma_\infty \cap U_j$ as $i \to \infty$ for each $j$.

Notice that the conditions (1) and (2) imply that the sequence $\{(M_i, \bar{g}_i, S_i)\}$ $C^k$-converges to $(M_\infty, \bar{g}_\infty, S_\infty)$ in the Cheeger-Gromov topology. We say that a sequence $\{(M_i, \Sigma_i, \bar{g}_i, S_i)\}$ smoothly converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, S_\infty)$ if it $C^k$-converges for all $k \geq 1$.

We say that a sequence $\{(M_i, \Sigma_i, \bar{g}_i, S_i)\}$ sub-converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, S_\infty)$ if it has a subsequence which converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, S_\infty)$. In this case we write

$$(M_i, \Sigma_i, \bar{g}_i, S_i) \to (M_\infty, \Sigma_\infty, \bar{g}_\infty, S_\infty).$$

### 3.2. Main convergence result

We are now ready to set the stage for the main result of this section. Let $(Y, g)$ be a closed $n$-dimensional Riemannian manifold with a homology class $\alpha \in H_{n-1}(Y; \mathbb{Z})$. As we discussed in Section 1.3 the class $\alpha$ gives the Poincaré dual class $D\alpha = [\gamma] \in H^1(Y; \mathbb{Z})$ represented by some map $\gamma : Y \to S^1$. Furthermore, we assume that there is a bordism

$$(M, \bar{g}, \bar{\gamma}) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma')$$

for some triple $(Y', g', \gamma')$. In the above, $\bar{\gamma} : M \to S^1$ represents a class $[\bar{\gamma}] \in H^1(M; \mathbb{Z})$ Poincaré dual to a class $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$.

Recall that $Y \subset \partial M$ and $\bar{g} = g + dt^2$ near $Y$. For a real number $L \geq 0$, we consider the following Riemannian manifold

$$(M_L, \bar{g}_L) := (M \cup Y \times [-L, 0], \bar{g}_L),$$

where $\bar{g}_L$ restricts to $\bar{g}$ on $M$ and to the product-metric $g + dt^2$ on $Y \times [-L, 0]$. We obtain a bordism

$$(M_L, \bar{g}_L, \bar{\gamma}_L) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma'),$$

where $[\bar{\gamma}_L]$ is the image of $[\bar{\gamma}]$ under the isomorphism $H^1(M; \mathbb{Z}) \cong H^1(M_L; \mathbb{Z})$. We refer to the bordism $(M_L, \bar{g}_L, \bar{\gamma}_L)$ as the $L$-collaring of $(M, \bar{g}, \bar{\gamma})$. Below we will take $L$ be an integer $i = 1, 2, \ldots$, and write $\bar{\alpha}_L \in H^n(M, \partial M; \mathbb{Z})$ for the class Poincaré dual to $[\bar{\gamma}_L]$.

**Main Lemma.** Let $(M, \bar{g}, \bar{\gamma}) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma')$ be a bordism as in (3.1). Denote by $(M_i, \bar{g}_i, \bar{\gamma}_i)$ the $i$-collaring of $(M, \bar{g}, \bar{\gamma})$ as in (3.2), where $i = 0, 1, 2, \ldots$. Fix a basepoint in each path component of $Y$, denote their union by $S$, and let $S_i$ be the image of $S$ under the inclusion

$$Y \cong Y \times \{0\} \subset Y \times [-i, 0] \subset M_i.$$

Assume $W_i \subset M_i$ is an oriented homologically volume minimizing representative of $\bar{\alpha}_i$ for each $i = 0, 1, 2, \ldots$. If $X \subset Y$ is an embedded hypersurface which is the only volume minimizing representative
of \( \alpha \in H_{n-1}(Y;\mathbb{Z}) \), then there is smooth subconvergence

\[
(M_i, W_i, \bar{g}_i, S_i) \to (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, S_\infty)
\]
as \( i \to \infty \) where \( S_\infty \subset Y \times \{0\} \) is the inclusion of \( S \).

**Remark.** In Main Lemma, we allow the manifold \( Y' \) to be empty.

### 3.3. Proof of the Main Lemma: outline

Consider the limiting space \( Y \times (-\infty, 0] \), with the exhaustive sequence \( U_i = Y \times (-i - 1, 0] \) and maps \( \Psi_i : U_i \to M_i \) taking \( U_i \) identically onto \( Y \times (-i - 1, 0] \subset M_i \). Our choice of \( U_i \) and \( \Psi_i \) satisfy the conditions (1) and (2) from Definition 7 for obvious reasons.

It will be useful to equip \( M \) with a smooth function \( F : M \to [-1, 0] \) satisfying \( Y = F^{-1}(0) \), and \( Y' = F^{-1}(-1) \) and extend this function to \( M_i \) by

\[
F_i(x) = \begin{cases} 
  t & \text{if } x = (y, t) \in Y \times [-i, 0] \\
  F(x) - i & \text{if } x \in M.
\end{cases}
\]

![Figure 2](image)

**Figure 2.** The hypersurface \( W_i^R \hookrightarrow M_i \). In this figure, \( Y' = \emptyset \).

For any positive integer \( i \) and non-negative numbers \( 0 \leq R < R' \leq i \), we write

\[
W_i^R = F_i^{-1}([-R, 0]) \quad \text{and} \quad W_i[-R', -R] = F_i^{-1}([-R', -R]).
\]

Let \( \alpha \in H_{n-1}(Y;\mathbb{Z}) \) be the class from the statement of Main Lemma. For \( L > 0 \) let

\[
\alpha \times [-L, 0] \in H_n(Y \times [-L, 0], Y \times \{-L, 0\}; \mathbb{Z})
\]

be the class given by the product of \( \alpha \) and the fundamental class

\[
[-L, 0] \in H_1([-L, 0], \{-L, 0\}; \mathbb{Z}).
\]
We will break up the proof of Main Lemma into three claims.

**Claim 1.** Let $L > 0$. The hypersurface $X \times [-L, 0] \subset Y \times [-L, 0]$ is the only homologically volume-minimizing representative of $\alpha \times [-L, 0] \in H_n(Y \times [-L, 0], Y \times \{-L, 0\}; \mathbb{Z})$.

**Claim 2.** For each $R > 0$, $\text{Vol}(W^R_i) \to R \cdot \text{Vol}(X)$ as $i \to \infty$.

**Claim 3.** For each $R > 0$, there is a sequence $\{a^R_i\}_{i=1}^\infty$ such that, for each $j = 1, 2, \ldots$, the hypersurfaces $\{\Psi^{-1}(W^R_{a^R_i})\}_{i=1}^\infty$ converge smoothly locally as graphs in $Y \times (-\infty, 0]$.

Now we show how Main Lemma follows from Claims (1), (2), and (3). Indeed, by Claim 3, for each $k = 1, 2, \ldots$, there is a sequence $\{a^k_i\}_{i=1}^\infty$ such that, for each $j = 1, 2, \ldots$, the hypersurfaces $\{\Psi^{-1}(W^k_{a^k_i})\}_{i=1}^\infty$ converges smoothly locally as graphs to some hypersurface $W_{\infty, k} \subset Y \times (-\infty, 0]$. We notice that the hypersurface $W_{\infty, k}$ is contained in $Y \times [-k, 0]$ and represents the class $\alpha \oplus [-k, 0]$. Since the convergence is smooth, we have

$$\text{Vol}(\Psi^{-1}(W_{\infty, k})) = \lim_{i \to \infty} \text{Vol}(\Psi^{-1}(W^k_{a^k_i})) = k \cdot \text{Vol}(X),$$

where the last equality follows from Claim 2. However, according to Claim 1, the only volume minimizing representative of $\alpha \otimes [-k, 0]$ is the hypersurface $X \times [-k, 0]$ which has the volume $k \cdot \text{Vol}(X)$. Thus $W_{\infty, k}$ must be $X \times [-k, 0]$. Evidently, the diagonal sequence $\{\Phi^{-1}(W^k_{a^k_i})\}_{i=1}^\infty$ has the property that, for each $k > 0$, $\Phi^{-1}(W^k_{a^k_i})$ converges smoothly locally as graphs to $X \times [-k, 0]$. This then completes the proof of Main Lemma.

### 3.4. Proof of Claim 1

Let $\Sigma \subset M_L$ be any properly embedded hypersurface representing the class $\alpha \times [-L, 0] \in H_n(Y \times [-L, 0], Y \times \{-L, 0\}; \mathbb{Z})$. Consider the projection function $P : \Sigma \to [-L, 0]$. The coarea formula [15, Theorem 5.3.9] applied to $P$ yields

$$\int_\Sigma |\nabla P| d\mu = \int_{-L}^0 \mathcal{H}^{n-1}(P^{-1}(t)) dt,$$

where $\mathcal{H}^{n-1}$ denotes the $(n - 1)$-dimensional Hausdorff measure associated to the metric $h + dt^2$ on $Y \times [-L, 0]$. Notice that $P$ is weakly contractive in the sense that

$$|P(x) - P(y)| \leq \text{dist}^\Sigma(x, y)$$

for all $x, y \in \Sigma$. Thus we have the pointwise bound $|\nabla P| \leq 1$. Furthermore, since $P^{-1}(t)$ represents the class $\alpha \in H_{n-1}(Y \times \{t\}; \mathbb{Z})$ for each $t \in [-L, 0]$,

$$\mathcal{H}^{n-1}(P^{-1}(t)) \geq \text{Vol}(X)$$

with equality if and only if $P^{-1}(t) = X$. Combining this observation with (3.3), we conclude

$$\text{Vol}(\Sigma) \geq L \cdot \text{Vol}(X)$$

with equality if and only if $\Sigma = X \times [-L, 0]$. This completes the proof of Claim 1.
3.5. **Proof of Claim** Before we begin, we will construct particular (in general, non-minimizing) properly embedded hypersurfaces $N_L \subset M_L$ representing $\alpha_L$ with which to compare $\text{Vol}(W_L)$ against.

Let $X \subset Y$ and $W_0 \subset M_0$ as in Main Lemma. Since the manifolds $\partial W_0 \cap Y$ and $X$ both represent the same homology class, they are bordant via a smooth, properly embedded hypersurface $\iota : U \hookrightarrow Y \times [0,1]$. We identify $[0,1] \cong [-L, -L + 1]$ to obtain the embedding

$$\iota_L : U \hookrightarrow Y \times [0,1] \cong Y \times [-L, -L + 1] \hookrightarrow M_L$$

Clearly the embedding $\iota : U \hookrightarrow Y \times [0,1]$ may chosen so that

$$N_L := W_0 \cup_{\partial W_0} U_L \cup (X \times [-L + 1, 0]),$$

where $U_L = \iota_L(U)$, is a smooth properly embedded hypersurface of $M_L$.

Evidently, $\text{Vol}(N_L) = \text{Vol}(W_0) + \text{Vol}(U_L) + (L - 1)\text{Vol}(X)$ and $N_L$ represents the same homology class as $W_L$. Since $W_L$ is homologically area-minimizing, we have $\text{Vol}(W_L) \leq \text{Vol}(N_L)$.

In other words, we obtain the inequality

$$\text{Vol}(W_R^L) + \text{Vol}(W_L \setminus W_R^L) \leq \text{Vol}(W_0) + \text{Vol}(U_L) + (L - 1)\text{Vol}(X)$$

for any $0 < R < L - 1$.

Now we are ready to prove Claim Assume it fails. Then there exist $\epsilon_0, R_0 > 0$ and an increasing sequence $\{a_i\}_{i=1}^{\infty}$ such that the inequality

$$\text{Vol}(W_{R_0}^{a_i}) > R_0 \cdot \text{Vol}(X) + \epsilon_0$$
holds for all $i$. Combining the inequality (3.4) with the assumption (3.5), we have

$$\text{Vol}(W_0) + \text{Vol}(U_{a_i}) + (a_i - 1)\text{Vol}(X) > \text{Vol}(W_{a_i} \setminus W_{a_i}^{R_0}) + \epsilon_0 + R_0\text{Vol}(X).$$

Now we will inspect the first term in the right hand side of (3.6):

$$\text{Vol}(W_{a_i} \setminus W_{a_i}^{R_0}) = \text{Vol}(W_{a_i}[a_{i-1} - a_i, -R_0]) + \text{Vol}(W_{a_i}[a_i - 1, a_{i-1} - a_i])$$

$$\geq (a_i - a_{i-1} - R_0)\text{Vol}(X) + \text{Vol}(W_{a_{i-1}})$$

$$> (a_i - a_{i-1})\text{Vol}(X) + \epsilon_0 + \text{Vol}(W_{a_{i-1}} \setminus W_{a_{i-1}}^{R_0}).$$

Here we use Claim 1 in the first inequality and the assumption (3.5) in the second.

Combining (3.6) with (3.7), we obtain

$$\text{Vol}(W_0) + \text{Vol}(U_{a_i}) + (a_i - 1)\text{Vol}(X) > (a_i - a_{i-1} + R_0)\text{Vol}(X) + 2\epsilon_0 + \text{Vol}(W_{a_{i-1}} \setminus W_{a_{i-1}}^{R_0}).$$

We iterate the argument to find

$$\text{Vol}(W_0) + \text{Vol}(U_{a_i}) + (a_1 - R_0 - 1)\text{Vol}(X) > i \cdot \epsilon_0 + \text{Vol}(W_{a_1})$$

for every $i = 1, 2, \ldots$. Since the left hand side of (3.8) is independent of $i$, we arrive at a contradiction by taking $i$ to be sufficiently large.

3.6. **Proof of Claim 3** While the proof of Claim 3 is rather technical, it is essentially a consequence of standard tools used in the study of stable minimal hypersurfaces. For instance, see [6] for a similar result in a 3-dimensional context. We divide the proof into three steps, referring to the Appendix when necessary.

To begin, we require the following uniform volume bound.

**Step 1.** For each $R > 0$, there is a constant $V_0 > 0$ such that

$$\text{Vol}(W_{i}[-\lambda - R, -\lambda]) \leq V_0$$

holds for all $i$ and all $\lambda \in [0, i - R]$. In particular, $\text{Vol}(W_{i} \cap B_{R}^{M}(x)) \leq V_0$ for all $i$ and $x \in M_{i}$.

The next key ingredient is the following uniform bound on the second fundamental form $A^{W_L}$.

**Step 2.** There is a constant $C_1 > 0$, depending only on the geometry of $(M, \bar{g})$, such that

$$\sup_{x \in W_{L}} |A^{W_L}(x)|^2 \leq C_1 \quad \text{for } L \geq 0.$$

Step 2 is a consequence of [21, Corollary 1.1]. See Appendix, Section 5.5 for more details.

**Step 3.** For each $R > 0$ and $j = 1, 2, \ldots$, the sequence of hypersurfaces $\Psi_{j}^{-1}(W_{i}^{R})$ sub-converges smoothly locally as graphs as $i \to \infty$.

The following result is a general statement concerning embedded hypersurfaces of Riemannian manifolds with bounded geometry which we state in our present context. It will allow us to effectively apply Schauder theory for the minimal graph equation to the hypersurfaces $W_{L}$. 
Proof of Step 3. We restrict our attention to the tail of the sequence \( \{W_{i}^{R}\}_{i}^{\infty} \), where \( i \geq R+1 \). This allows us to consider each \( W_{i}^{R} \) and \( W_{i}^{R+1} \) as hypersurfaces of \( Y \times (-\infty, 0] \) which is where we will show the convergence. Let \( s > 0 \) satisfy

\[
\frac{1}{\sqrt{sC_{0}}}, \text{inj}(M, \bar{g}), 1
\]

where \( C_{0} \) is the constant from Step 2. By choosing a smaller \( s \), we will assume that the bounds

\[
|\bar{g}_{ij}(x) - \delta_{ij}| \leq \mu_{0}, \quad \left| \frac{\partial \bar{g}_{ij}}{\partial x^{k}}(x) \right| \leq \mu_{0}
\]

hold for all \( i, j, k = 1, \ldots, n+1 \) in geodesic normal coordinates on any ball of radius \( s \) in \( Y \times [-L-1, 0] \) where \( \mu_{0} \) is the constant given in Lemma 6 from the Appendix.

We cover \( Y \times [-R, 0] \) by a finite collection of open balls \( \mathcal{U} = \{B_{s/4}(y_{i})\}_{i=1}^{N} \) of radius \( s/4 \). Notice that each \( B_{s/4}(y_{i}) \subset \subset Y \times [-R-1, 0] \). Consider a ball \( B_{s/4}(y_{i}) \) in \( \mathcal{U} \) with the property that

\[
W_{i}^{R+1} \cap B_{s/4}(y_{i}) \neq \emptyset
\]

for infinitely many \( i \). Unless explicitly stated, we will continue to denote all subsequences by \( W_{i}^{R+1} \).

Our next goal is to show that the sequence of hypersurfaces \( \{W_{i}^{R} \cap B_{s/4}(y_{i})\} \) sub-converges smoothly locally as graphs.

We choose a subsequence of \( W_{i}^{R+1} \) and points \( x_{i} \in W_{i}^{R+1} \cap B_{s/4}(y_{i}) \) which converge to some point \( x_{\infty} \in B_{s/4}(y_{i}) \). Now it will be convenient to work in the tangent space to the point \( x_{\infty} \). We use the short-hand notation \( \phi = \exp_{x_{\infty}}^{Y \times [-L-1, 0]} \) and let

\[
B = \phi^{-1}(B_{s}^{h+dt^{2}}(x_{\infty})) \subset T_{x_{\infty}}(Y \times [-L-1, 0]).
\]

Consider the hypersurfaces \( \Sigma_{i} \subset B \) with basepoints \( p_{i} \in \Sigma_{i} \), given by

\[
\Sigma_{i} = \phi^{-1}(B_{s}(x_{\infty}) \cap W_{i}^{R}), \quad p_{i} = \phi^{-1}(x_{i}).
\]

Since \( W_{i}^{R} \subset M_{i} \) are minimal, \( \Sigma_{i} \) are minimal hypersurfaces in the ball \( B = B_{s}^{n+1}(0) \) with respect to the metric \( \bar{g}_{B} = (\phi^{-1})^{*}(h + dt^{2}) \).

Notice that the choice of \( s \) guarantees

\[
\sup_{\Sigma_{i}} |A^{\Sigma_{i}}|^{2} \leq \sup_{W_{i}^{R}} |A^{W_{i}}|^{2} \leq \frac{1}{20s^{2}}
\]

and we may apply Lemma 6 to each \( \Sigma_{i} \) at \( p_{i} \). Indeed, we consider open subsets \( U_{i} \subset T_{p_{i}} \Sigma_{i} \cap B \), with with normal unit vectors \( \eta_{i} \in S^{n}, \eta_{i} \bot T_{p_{i}} \Sigma_{i} \), and functions \( u_{i} : U_{i} \to \mathbb{R} \) such that \( \text{graph}(u_{i}) = \Sigma_{i}' \) where \( \Sigma_{i}' \) is the component of \( \Sigma \) containing \( p_{i} \). We use compactness of \( S^{n} \) and pass to a subsequence so that the vectors \( \eta_{i} \) converge to some vector \( \eta_{\infty} \in S^{n} \). Let \( P_{\infty} \subset T_{x_{\infty}}(Y \times [-L-1, 0]) \) be the hyperplane perpendicular to \( \eta_{\infty} \). For large enough \( i \), we may translate and rotate the sets \( U_{i} \) to obtain open subsets \( U_{i}' \subset P_{\infty} \) and functions \( u_{i}' : U_{i}' \to \mathbb{R} \) such that

1. \( \Sigma_{i}' = \text{graph}(u_{i}') \);
2. the ball \( B_{s/2}(0) \subset P_{\infty} \) is contained in each of the sets \( U_{i}' \);
(3) the following pointwise bounds hold
\[ |u_i'(x)| \leq s \cdot \frac{3}{2}, \quad |\nabla u_i'| \leq \frac{3}{2}, \quad |\nabla \nabla u_i'| \leq \frac{1}{s^{\sqrt{2}}} \cdot \frac{3}{2}, \]
where the derivatives and lengths are taken using the Euclidian metric. In particular, it follows that the intersection \( U'_\infty = \bigcap_i U'_i \) contains the ball \( B_{s/2}(0) \subset P_\infty \) and the sequence \( \{u_i'|_{U_\infty}\}_i \) is uniformly bounded in \( C^2(U_\infty) \), see Fig. 4.

\[ \text{Figure 4. The functions } u_i' \text{ and hypersurfaces } \Sigma'_i. \]

By Arzela-Ascoli, there exists a subsequence of \( u_i' \) converging in \( C^{1,\alpha}(U'_\infty) \) (for any fixed \( \alpha \in (0, 1) \)) to a function \( u'_\infty : U'_\infty \to \mathbb{R} \). Recall that each \( u_i'|_{U_\infty} \) solves the minimal graph equations with respect to the metric \( \bar{g}_B \). Since \( |\nabla u_i'| \) is uniformly bounded on \( B_{s/2}(0) \), the functions \( u_i' \) solve an elliptic equation with uniform ellipticity bounds. Hence we may apply the Schauder estimate. This allows us to pass to a further subsequence, and upgrade the convergence on the ball \( B_{s/3}(0) \) to \( C^{k,\alpha}(B_{s/3}(0)) \) for any \( k \geq 1 \), see Appendix, Section 5.2 for details. In particular, \( u'_\infty \) is a strong solution to the minimal graph equation on \( B_{s/3}(0) \) with respect to \( \bar{g}_B \). This finishes our work with the hypersurfaces \( \Sigma'_i \).

Now suppose that there is a second sequence of connected components \( W^R_i \cap B_s(y_l) \) which intersect \( B_{s/4}(y_l) \) nontrivially. We can repeat the above process to obtain a second limiting hypersurface. Observe that the number of components of \( W^R_i \cap B_s(y_l) \) intersecting \( B_{s/4}(y_l) \) nontrivially is uniformly bounded in \( i \) and \( l \). Indeed, the lower bound on the diameter of \( U'_\infty \) implies

\[ \text{Vol}_{\bar{g}_B}(\Sigma'_i) \geq \text{Vol}_{\bar{g}_B}(U'_i) \geq \left( \frac{s}{3} \right)^n. \]

However, Step 4 implies \( \text{Vol}(W^R_i \cap B_s(y_l)) \) is uniformly bounded in \( i \) so the number of connected components \( W^R_i \cap B_s(y_l) \) which intersect \( B_{s/4}(y_l) \) nontrivially must be finite and the above process terminates after finitely many iterations. We conclude that the sequence \( \{W^R_i \cap B_{s/4}(y_l)\}_i \) subconverges smoothly locally as graphs to a minimal hypersurface \( \Sigma_{\infty,l} \).
Now, restricting to this subsequence, we turn our attention to another ball $B_{s/4}(y')$ in the cover $U$. We repeat the above argument to obtain a further subsequence and limiting minimal hypersurface $\Sigma_{\infty, l'}$. Repeating this process for each element of $U$ produces a subsequence, denoted by $W_{s'}^R$, converging to a minimal hypersurface $W_{\infty}^R = \bigcup_l \Sigma_{\infty, l}$ smoothly locally as graphs. This completes the proof of Claim 3 and consequently, the proof of Main Lemma. □

4. Proof of Theorem 5

4.1. Positive conformal bordism. In order to prove Theorem 5, we have to use fundamental facts relating conformal geometry and psc-bordism. We briefly recall necessary results, following the conventions in [1]. Let $Y$ be a compact closed manifold with dim $Y = n$ given together with a conformal class $C$ of Riemannian metrics. Then the Yamabe constant $Y$ is defined as

$$Y(Y; C) = \inf_{g \in C} \frac{\int_Y R_g d\mu_g}{\text{Vol}_g(Y)^{\frac{n-2}{n}}}. $$

Then we say that a conformal class $C$ is positive if $Y(Y; C) > 0$. It is well-known that $C$ is positive if and only if there exists a psc-metric $g \in C$.

Now let $Z : Y_0 \sim Y_1$ be a bordism between closed manifolds $Y_0$ and $Y_1$. Assume that we are given a positive conformal classes $C_0$ and $C_1$ on $Y_0$ and $Y_1$ respectively. Let $\bar{C}$ be a conformal class of Riemannian metrics on $Z$, such that $\bar{C}|_{Y_0} = C_0$ and $\bar{C}|_{Y_1} = C_1$, i.e., $\partial \bar{C} = C_0 \cup \partial C_1$. Denote by $\bar{C}^0 = \{\bar{g} \in \bar{C} : h_{\bar{g}} \equiv 0\} \subset \bar{C}$ the subclass of those metrics with vanishing mean curvature of the boundary. Then the relative Yamabe constant $Y_{\bar{C}}((Z; \bar{C}), (Y_0 \sqcup Y_1, C_0 \cup C_1))$ is defined as

$$Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \cup C_1) = \inf_{\bar{g} \in \bar{C}^0} \frac{\int_Z R_{\bar{g}} d\mu_{\bar{g}}}{\text{Vol}_{\bar{g}}(Z)^{\frac{n-2}{n}}}. $$

This gives the relative Yamabe invariant

$$Y(Z, Y_0 \sqcup Y_1; C_0 \cup C_1) = \sup_{\bar{C}, \partial \bar{C} = C_0 \cup C_1} Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \cup C_1). $$

Now we assume that the conformal classes $C_0$ and $C_1$ are positive. Then we say that positive conformal manifolds $(Y_0, C_0)$ and $(Y_1, C_1)$ are positive-conformally bordant if there exists a conformal manifold $(Z, \bar{C})$ and a bordism $Z : Y_0 \sim Y_1$ between $Y_0$ and $Y_1$ such that $\partial \bar{C} = C_0 \cup C_1$, such that the Yamabe constant $Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \cup C_1) > 0$. In this case, we write $(Z, \bar{C}) : (Y_0, C_0) \sim (Y_1, C_1)$.

Remark. We note that existence of such positive-conformal bordism is equivalent to the condition that $Y(Z, Y_0 \sqcup Y_1; C_0 \cup C_1) > 0$. Furthermore, it is shown in [1] that two manifolds $(Y_0, g_0)$ and $(Y_1, g_1)$ with psc-metrics $g_0$ and $g_1$ are psc-bordant if and only if the conformal manifolds $(Y_0, [g_0])$ and $(Y_1, [g_1])$ are positive-conformally bordant.

We need the following result, specialized to our present context.

Theorem 6. [1] Corollary B] Let $Y_0$ and $Y_1$ be closed manifolds of dimension $n \geq 3$, $Z : Y_0 \sim Y_1$ be a bordism between $Y_0$ and $Y_1$, and $g_0$ and $g_1$ be psc-metrics on $Y_0$ and $Y_1$ respectively. Then
Y(Z, Y_0 \sqcup Y_1; [g_0] \sqcup [g_1]) > 0 if and only if the boundary metric \( g_0 \sqcup g_1 \) on \( Y_0 \sqcup Y_1 \) may be extended to a psc-metric \( \bar{g} \) on \( Z \) such that \( \bar{g} = g_j + dt^2 \) near \( Y_j, j = 0, 1 \).

4.2. Long collars.\(^1\) Let \((Y_0, g_0, \gamma_0)\) and \((Y_1, g_1, \gamma_1)\) be the manifolds from Theorem and let \( \alpha_0 \in H_{n-1}(Y_0; \mathbb{Z}) \) and \( \alpha_1 \in H_{n-1}(Y_1; \mathbb{Z}) \) be the classes Poincaré dual to \( \gamma_0 \) and \( \gamma_1 \), respectively. It is convenient to use the notation

\[
Y = Y_0 \sqcup -Y_1
\]

where \( \iota_j : Y_j \hookrightarrow Y \) is the inclusion map for \( j = 0, 1 \). Then we consider hypersurfaces \( X_0 \subset Y_0 \) and \( X_1 \subset Y_0 \) which are homologically volume minimizing representatives of the classes \( \alpha_0 \) and \(-\alpha_1 \). The existence of such smooth \( X_0 \) and \( X_1 \) is guaranteed in this range of dimensions, see \[23\]. Notice that, by a small conformal change which does not effect the assumptions on \((Y_j, g_j, \gamma_j)\), we may assume that \( X_j \) is the only representative of \( \alpha_j \) with minimal volume for \( j = 0, 1 \). We write \((X, h_X)\) for the Riemannian manifold \((X_0 \sqcup X_1, h_X)\), where \( h_X = g_0|\partial Y_0 \sqcup g_1|\partial Y_1 \).

Now we choose a psc-bordism \((Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)\). We will use \((Z, \bar{g}, \bar{\gamma})\) to construct a psc-bordism which satisfies the conclusion of Theorem \[5\]. We denote by \( \bar{\alpha} \in H_n(Z; \mathbb{Z}) \) the homology class Poincaré dual to \( \bar{\gamma} \). Then \( \partial \bar{\alpha} = \alpha \), see Lemma \[4\]. By Theorem \[4\] there exists a properly embedded hypersurfaces \( W \subset Z \) which are homologically volume minimizing and represent \( \bar{\alpha} \).

Now for each \( i = 1, 2, \ldots \), we consider the \( i \)-collaring of the bordism \((Z, \bar{g}, \bar{\gamma})\), denoted by \((Z_i, \bar{g}_i, \bar{\gamma}_i)\), see Section \[3.2\]. Again, by Theorem \[4\] there exists a properly embedded hypersurfaces \( W_i \subset Z_i \) which is homologically volume-minimizing and represents \( \bar{\alpha}_i \). The restrictions of \( \bar{g}_i \) to \( W_i \) and \( \partial W_i \) are denoted by \( \bar{h}_i \) and \( h_i \), respectively.

In preparation to apply Main Lemma, we fix basepoints \( x_j \in X_j \) for each \( j = 0, 1 \) and set \( S = \{x_0, x_1\} \subset X \). Naturally, the set \( S \) is identified with the subsets \( S_i \) in \((X \times \{0\}) \subset \partial Z_i \) for \( i = 1, 2, \ldots \) and with \( S_\infty \) in the boundary of the cylinder \((X \times \{0\}) \subset (Y \times (-\infty, 0]). \) According to Main Lemma we may find a subsequence \( \{a_i\}^\infty_{i=1} \) such that

\[
(Z_{a_i}, W_{a_i}, \bar{g}_{a_i}, S_{a_i}) \rightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, S_\infty)
\]

smoothly as \( i \to \infty \) and the Riemannian manifolds \((\partial W_{a_i}, h_{a_i})\) converge to \((X, h_X)\) in the smooth Cheeger-Gromov topology as \( i \to \infty \).

**Remark.** We note that the manifolds \((\partial W_{a_i}, h_{a_i}), (X, h_X)\) are compact and so there is no need to specify base points.

The following is a special case of a much more general fact on the behavior of elliptic eigenvalue problems under smooth Cheeger-Gromov convergence (see \[4\]).

\(^1\) Here we emphasize a proper orientation on \( Y_0 \) and \( Y_1 \).
Lemma 4. Let \((M_i, g'_i)\) be a sequence of compact Riemannian manifolds smoothly converging to a compact Riemannian manifold \((M_\infty, g'_\infty)\) in the Cheeger-Gromov sense. If \(Y(M_\infty, [g'_\infty]) > 0\), then \(Y(M_i, [g'_i]) > 0\) for all sufficiently large \(i\).

Proof. For each \(i = 1, 2, \ldots\), we denote by \(\lambda_{1,i} = \lambda_1(L_{g'_i})\) the principal eigenvalue of the conformal Laplacian on \((M_i, g'_i)\). Let \(\phi_i \in C^\infty(M_i)\) be the eigenfunction satisfying

\[
L_{g'_i} \phi_i = \lambda_{1,i} \phi_i, \quad \sup_{M_i} \phi_i = 1.
\]

Since \((M_i, g'_i)\) is converging in the Cheeger-Gromov topology to a compact manifold, the coefficients of the operator \(L_{g'_i}\) are bounded in the \(C^1\)-norm uniformly in \(i\). In particular, there is a constant \(C_1 > 0\), independent of \(i\), such that \(|R_{g'_i}| \leq C_1\) on \(M_i\). An obvious estimate on the Rayleigh quotient shows that the sequence \(\{\lambda_{1,i}\}_{i=1}^\infty\) is uniformly bounded above and below.

This allows us to apply standard Schauder estimates to the functions \(\phi_i\) uniformly in \(i\) (see Theorems 3.3 and 3.4 in [4]) with Arzelà-Ascoli to find a subsequence, still denoted by \((M_i, g'_i), \phi_i, \lambda_{1,i}\), a function \(\phi_\infty \in C^\infty(M_\infty)\), and a number \(\lambda_{1,\infty}\) such that \(\phi_i \to \phi_\infty\) \(\lambda_{1,i} \to \lambda_{1,\infty}\).

This allows us to take the limit of equation (4.1) as \(i \to \infty\). Namely, \(\phi_\infty\) is a non-zero solution of the equation

\[
L_{g_\infty} \phi_\infty = \lambda_{1,\infty} \phi_\infty.
\]

By definition, we have \(\lambda_{1,\infty} \geq \lambda_1(L_{g_\infty})\). On the other hand, we have assumed that \(\lambda_1(L_{g_\infty}) > 0\). Hence \(\lambda_{1,i} > 0\) for all sufficiently large \(i\).

We now return to the proof of Theorem 5. Since \(X\) is a stable minimal hypersurface of \(Y\) with trivial normal bundle, Theorem 3 implies that \(Y(X, [g_X]) > 0\). Now we may apply Lemma 4 to find \(Y(\partial W_{a_i}, [h_{a_i}]) > 0\) for sufficiently large \(i\). Since each \(W_{a_i}\) is a stable minimal hypersurface with free boundary and trivial normal bundle, Theorem 3 states that \(Y(W_{a_i}, \partial W_{a_i}; [h_{a_i}]) > 0\) for all \(i \in \mathbb{N}\). By Lemma 4 for large \(i\) we also have \(Y(\partial W_{a_i}; [h_{a_i}]) > 0\). Fix such an \(i\) and let \(h'_{a_i} \in [h_{a_i}]\) be a psc metric on \(\partial W_{a_i}\). Finally, we may use Theorem 6 to find a psc-metric \(\tilde{h}_{a_i}\) on \(W_{a_i}\) which restricts to \(h'_{a_i} + dt^2\) near \(\partial W_{a_i}\). This completes the proof of Theorem 5.

5. Appendix

5.1. Graphical presentations of hypersurfaces. The following lemma is a useful fact concerning hypersurfaces of Euclidian space.

Lemma 5. [5] Let \(\Sigma^n \subset \mathbb{R}^{n+1}\) be an immersed hypersurface with \(x \in \Sigma\). Assume \(s > 0\) is such that

\[
\text{dist}^\Sigma(x, \partial \Sigma) \geq 2s, \quad \sup_{\Sigma} |A^\Sigma|^2 \leq \frac{1}{16s^2}.
\]

Then there is an open subset \(U \subset T_x \Sigma\), a unit vector \(\eta\) normal to \(T_x \Sigma\), and a function \(u : U \to \mathbb{R}\) such that
In the above, the curvature of $\Sigma$ can be expressed by

$$A_x$$

Lemma 6. There is a constant $\mu_0 > 0$ so that if a Riemannian metric $g$ on the unit ball $B = B_1^{n+1}(0) \subset \mathbb{R}^{n+1}$ satisfies the bounds

$$\sup_{x \in B} |g_{ij}(x) - \delta_{ij}| \leq \mu_0, \quad \sup_{x \in B} \left| \frac{\partial g_{ij}}{\partial x^k}(x) \right| \leq \mu_0$$

for all $i, j, k$ in the standard Euclidean coordinates $(x^1, \ldots, x^{n+1})$ on $B$, then the following holds: If $\Sigma \subset B$ is a hypersurface with $x \in \Sigma \cap B_i^{n+1}(0)$ and $s > 0$ satisfying

$$\text{dist}(\Sigma, g(x, \partial \Sigma)) \geq 3s, \quad \sup_{\Sigma} |A_g|^2 \leq \frac{1}{20s^2},$$

then there is a open subset $U \subset T_x \Sigma \subset \mathbb{R}^{n+1}$, a vector $\eta \in S^n$ normal to $T_x \Sigma$, and a function $u : U \rightarrow \mathbb{R}$ such that

1. $\text{graph}(u) = B_s^\Sigma(x)$;
2. the bounds $|\nabla u| \leq 1$ and $|\nabla \nabla u| \leq \frac{1}{s\sqrt{2}}$ hold pointwise, where the derivatives are taken using the Euclidean metric on $U$.

Moreover, the connected component of $B_s^\Sigma(x) \cap \Sigma$ containing $x$ lies in $B_s^\Sigma(x)$.

5.2. The minimal graph equation in a Riemannian manifold. In this section we will work in local coordinates $(x^1, \ldots, x^{n+1})$ on a Riemannian manifold. We adopt the shorthand $x' = (x^1, \ldots, x^n)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $g$ be a Riemannian metric on $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$. For a function $u : \Omega \rightarrow \mathbb{R}$, consider its graph

$$\Sigma = \{(x', u(x')) \in \mathbb{R}^{n+1} : x' \in \Omega\}.$$ 

For $i = 1, \ldots, n$, we have the tangential vector fields $E_i = \frac{\partial}{\partial x^i} + \frac{\partial u}{\partial x^i} \frac{\partial}{\partial u}$ and the upward-pointing unit vector field $\nu$ normal to $\Sigma$. Writing $h_{ij} = g(E_i, E_j)$ for the restriction metric on $\Sigma$, the mean curvature of $\Sigma$ can be expressed by

$$H = h^{ij} g(\nu, \nabla E_i, E_j)$$

$$= h^{ij} \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} + \Gamma^r_{ij} + \Gamma^r_{ji} + \Gamma^r_{ij} \Gamma^n_{r+1} + \Gamma^r_{ji} \Gamma^n_{r+1} + \Gamma^r_{ij} \Gamma^n_{r+1} \Gamma^n_{r+1} \right].$$

In the above, $h^{ij}$ are the components of the inverse of the metric $g(E_i, E_j)$ and can be computed as

$$h^{ij} = g^{ij} - \frac{\nabla_i u \nabla_j u}{\nabla u^2},$$

see [5] Section 7.1 for a detailed exposition in the 3-dimensional case. Supposing there are uniform bounds on the injectivity radius and sectional curvatures of $g$ and $|\nabla u| \in C^{0,\alpha}(\Omega)$, then $(h_{ij})$ is
uniformly positive definite. Inspecting the formula for $H$, this implies that the equation $H = 0$ can be interpreted as a uniformly elliptic linear equation for $u$ with Hölder coefficients. If the function $u$ is a weak solution of the equation $H = 0$, then we may apply the following Schauder estimate.

**Theorem 4.**

We apply these estimates to the case of stable minimal hypersurfaces with free boundary in the Riemannian manifolds $(M_L, \bar{g}_L)$ from the statement of Main Lemma.

**Corollary 1.** Let $(M_L, \bar{g}_L)$ be the manifolds given in the statement of the Main Lemma and let $\Sigma \subset M_L$ be a stable minimal hypersurface with free boundary. There is a radius $r > 0$, independent of $\Sigma$ and $L$, such that the following holds: If $x_0 \in \Sigma$ satisfies $\text{dist}^{\bar{g}_M}(x_0, \partial \Sigma) > r$, there is an open set $\Omega \subset T_{x_0}\Sigma \subset T_{x_0}M_L$ with a normal vector $\eta$ and a function $u : \Omega \rightarrow \mathbb{R}$ such that

(1) $\exp_{x_0}^{\bar{g}_M}(\text{graph}(u)) = \Sigma \cap B_{\bar{g}_M}^{\delta}(x_0)$

(2) $|\nabla u| \leq 1$, $|\nabla \nabla u| \leq \frac{1}{\sqrt{2s}}$ on $U$

(3) for each $k \geq 1$ and $\alpha \in (0, 1)$ there is a constant $C > 0$, depending only on $n, k, \alpha$, and the geometry of $(M, \bar{g})$, such that

(4) $||u||_{C^{k, \alpha}(U)} \leq C$.

**Proof.** Let $C_0$ and $\mu_0$ be the constants given in Claim 2 and Lemma 6. We take $r > 0$ sufficiently small so that $0 < r < \min(\text{inj}_{M_L}, \frac{1}{\sqrt{2s}}, 1)$ and the bounds

(5) $\sup_{x \in B_r(x)} |(\bar{g}_L)_{ij}(x) - \delta_{ij}| \leq \mu_0$, $\sup_{x \in B_r(x)} \left| \frac{\partial (\bar{g}_L)_{ij}}{\partial x^k}(x) \right| \leq \mu_0$

hold for all $i, j, k$ in geodesic normal coordinates on any ball $B_{\bar{g}_M}^{\delta}(y)$.

Lemma 6 gives the graphical representation of $\Sigma \cap B_{\bar{g}_M}^{\delta}(x_0)$. The uniform gradient bound in Lemma 6 and condition (5.5) give a bound on the constant in the estimate (5.3) on smaller balls. The main point is that this constant is independent of both $\Sigma$ and $M_L$. Standard Schauder theory [13, Section 6] gives a similar estimate in the $C^{k, \alpha}$-norm for any $k$. \hfill $\square$

5.3. Details on Theorem 4. Here we review relevant definitions and results to provide details on Theorem 4.

5.3.1. Integral homology using currents. Let us recall some basic notions from theory of integral multiplicity currents. The main reference for this material is [11, Chapter 4].
For an open subset $U \subset \mathbb{R}^{n+k}$, let $\Omega^n(U)$ denote the space of all $n$-forms on $\mathbb{R}^{n+k}$ with compact support in $U$. An $n$-dimensional current on $U$ is a continuous linear functional $T : \Omega^n(U) \to \mathbb{R}$. The vector space of such objects is denoted by $\mathcal{D}_n(U)$. If $n \geq 1$ and $T \in \mathcal{D}_n(U)$ is an $n$-dimensional current, its boundary $\partial T \in D_{n-1}(U)$ is given by

$$(\partial T)(\omega) = T(d\omega), \quad \omega \in \Omega^n(U).$$

For a current $T \in \mathcal{D}_n(U)$, the mass $M(T)$ is given by $M(T) = \sup\{T(\omega) : \omega \in \Omega^n(U), |\omega| \leq 1\}$.

If, for example, $T$ is given by integration along a smooth orientable $n$-dimensional submanifold $M$, then $M(T) = \text{Vol}(M)$.

Let $\mathcal{H}^n$ denote the $n$-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$. A subset $E \subset \mathbb{R}^{n+k}$ is said to be countably $n$-rectifiable if it can be covered by a countable family of Lipschitzian mappings from bounded subsets of $\mathbb{R}^n$. A current $T \in \mathcal{D}_n(U)$ is called integer multiplicity rectifiable if it takes the form

$$(5.6) \quad T(\omega) = \int_M \omega(x) \theta(x) d\mathcal{H}^n(x), \quad \forall \omega \in \Omega^n(U)$$

where

1. $M \subset U$ is $\mathcal{H}^n$-measurable and countably $n$-rectifiable;
2. $\theta : M \to \mathbb{N}$ is locally $\mathcal{H}^n$-integrable;
3. for $\mathcal{H}^n$-almost every $x \in M$, $\xi : M \to \Lambda^n T \mathbb{R}^{n+k}$ takes the form $\xi(x) = e_1 \wedge \cdots \wedge e_n$ where $\{e_i\}_{i=1}^n$ is an orthonormal basis for the approximate tangent space $T_x M$, see [11, Section 3.2].

**Remark.** The above definition of integer multiplicity rectifiable currents can also be extended to Riemannian manifolds $(M, g)$ – one defines the mass of a current using the Hausdorff measure arising from $g$.

The regular set $\text{reg}(T)$ of a rectifiable current $T \in \mathcal{D}_n(U)$ is given by the set of points $x \in \text{spt}(T)$ for which there exists an oriented $n$-dimensional oriented $C^1$-submanifold $M \subset U$, $r > 0$, and $m \in \mathbb{Z}$ satisfying

$$T|_{B_r(x)}(\omega) = m \cdot \int_{M \cap B_r(x)} \omega, \quad \forall \omega \in \Omega^n(U).$$

The singular set $\text{sing}(T)$ is given by $\text{spt}(T) \setminus \text{reg}(T)$.

The abelian group of $n$-dimensional integral flat chains on $U$ is given by

$$\mathcal{F}_n(U) = \{ R + \partial S : R \in \mathcal{D}_n(U) \text{ and } S \in \mathcal{D}_{n+1}(U) \text{ are rectifiable} \}.$$ 

Now we consider subsets $A, B \subset U$ satisfying $B \subset A$. We have the group of integral flat cycles

$$\mathcal{C}_n(A, B) = \{ T \in \mathcal{F}_n(U) : \text{spt}(T) \subset A, \text{spt}(\partial T) \subset B, \text{ or } n = 0 \}$$

and the subgroup of integral flat boundaries

$$\mathcal{B}_n(A, B) = \{ T + \partial S : T \in \mathcal{F}_n(U), \text{spt}(T) \subset B, S \in \mathcal{F}_{n+1}(U), \text{spt}(S) \subset A \}.$$
The quotient groups
\[ H_n(A, B) = C_n(A, B)/B_n(A, B) \]
are the \( n \)-dimensional integral current homology groups.

There is a natural transformation between the integral singular homology functor and the integral current homology functor which induces an isomorphism \( H_n(A, B; \mathbb{Z}) \cong H_n(A, B) \) in the category of local Lipschitz neighborhood retracts \([11\text{ Section 4.4.1}]\). In particular, if \((M, \bar{g})\) is a compact Riemannian manifold, we may embed \( M \) into a Euclidian space and use the above isomorphism to obtain \( H_n(M, \partial M; \mathbb{Z}) \cong H_n(M, \partial M) \) for any \( n = 0, 1, 2, \ldots \).

### 5.3.2. Compactness and regularity

Next, we review some results on integral multiplicity currents which minimize an appropriate functional. A positive parametric integrand degree \( n \) on an open set \( U \subset \mathbb{R}^{n+k} \) is a function
\[
\Phi : U \times \Lambda^n \mathbb{R}^{n+k} \to \mathbb{R}
\]
satisfying, for any \( x \in U \), \( \Phi(x, r\lambda) = r\Phi(x, \lambda) \) for any \( r > 0 \) and \( \Phi(x, \lambda) \neq 0 \) for any \( \lambda \neq 0 \in \Lambda^n \mathbb{R}^{n+k} \). A parametric integrand \( \Phi \) of degree \( n \) may be viewed as a functional on the space of rectifiable currents by setting
\[
\Phi(T) = \int_M \Phi(x, \xi(x))\theta(x)d\mathcal{H}^n(x)
\]
where \( M, \xi, \) and \( \theta \) are the quantities associated to a rectifiable current \( T \in \mathcal{D}_n(U) \) as in equation \([5,6]\). In the context of this paper, it suffices to consider the case in which \( \Phi \) is the parametric area integrand \( \Phi_g(x, \lambda) = |\lambda|_{g(x)} \) arising from a Riemannian metric \( g \) on \( U \). We remark that if \( T \in \mathcal{F}_n(U) \), then \( \Phi_g(T) \) is simply the mass of \( T \) measured with respect to \( g \),
\[
M_g(T) = \text{sup}\{T(\omega) : \omega \in \Omega^n(U), \text{sup}_{x \in U} |\omega(x)|_g = 1\},
\]
see \([11]\) Section 5.1.1. A cycle \( T \in \mathcal{C}_n(A, B) \) is said to be homologically \( \Phi \)-minimizing if
\[
\Phi(T) \leq \Phi(T + S), \quad \forall S \in \mathcal{B}_n(A, B).
\]
In the case that \( \Phi \) is the parametric area integrand \( \Phi_g \), we call \( \Phi \)-minimizing currents volume minimizing. The following lemma guarantees the existence of homologically \( \Phi \)-minimizing currents.

**Lemma 7.** \([11\text{ Section 5.1.6}]\) Let \( B \subset A \) be compact Lipschitz neighborhood retracts in an open set \( U \subset \mathbb{R}^{n+k} \) and let \( \Phi \) be an elliptic parametric integrand of degree \( n \) on \( U \). If \( \alpha \in H_n(A, B) \), then there exists a homologically \( \Phi \)-minimizing current \( T \in \mathcal{C}_n(A, B) \) representing \( \alpha \).

**Corollary 2.** Let \((M, \bar{g})\) be a compact \((n + 1)\)-dimensional Riemannian manifold with boundary and consider an integral homology class \( \alpha \in H_n(M, \partial M; \mathbb{Z}) \). Let \( \bar{\alpha} \in \mathbb{H}_n(M, \partial M) \) be the image of \( \alpha \) under the isomorphism \( H_n(M, \partial M; \mathbb{Z}) \to \mathbb{H}_n(M, \partial M) \). Then there exists a homologically volume minimizing integer multiplicity rectifiable current \( T \in \bar{\alpha} \).

**Proof.** By the Nash embedding theorem there is an isometric embedding \( \iota : M \to \mathbb{R}^{n+1+k} \) for a sufficiently large \( k \). Let \( \hat{M} \) be the image of this embedding and set \( \hat{\alpha} = \iota_* \bar{\alpha} \in H_n(\hat{M}, \partial \hat{M}) \). Applying
Lemma 7, we obtain a homologically volume minimizing current $\hat{T} \in C_n(\hat{M}, \partial \hat{M})$ representing $\hat{\alpha}$. Since $\iota$ is an isometry, $(\iota^{-1})_*\hat{T}$ is the desired current. □

Since Corollary 2 guarantees the existence of homologically volume minimizing representative for the homology class $\alpha$ from the hypothesis of Theorem 4, the final ingredient is the regularity theory of volume-minimizing integer multiplicity rectifiable currents. The following is a regularity theorem due to M. Grünter [14, Theorem 4.7] adapted to the context of an ambient Riemannian metric. See [16, 12, 22] for Riemannian adaptations of similar results.

**Theorem 8.** Let $S \subset \mathbb{R}^{n+1}$ be an $n$-dimensional smooth submanifold, $U \subset \mathbb{R}^{n+1}$ an open set with $\partial S \cap U = \emptyset$, and $g$ a Riemannian metric on $U$ with bounded injectivity radius and sectional curvature. Suppose $T \in F_n(U)$ with spt$(\partial T) \subset S$ satisfies

$$M_g(T) \leq M_g(T + R)$$

for all open $W \subset \subset U$ and all $R \in F_n(U)$ with spt$(R) \subset W$ and spt$(\partial R) \subset S$. Then we have

- $\text{sing}(T) = \emptyset$ if $n \leq 6$
- $\text{sing}(T)$ is discrete for $n = 7$
- $\dim_H(\text{sing}(T)) \leq n - 7$ if $n > 7$

where $\dim_H(A)$ denotes the Hausdorff dimension of a subset $A \subset U$.

We will briefly explain how to show the regularity of a volume minimizing representative $T \in \alpha$ where $\alpha$ is from Theorem 4. For a point $x \in \text{spt}(T)$, set $\phi = \exp^\# x$ and consider

$$U = \phi^{-1}(B_{r'}(x)) \subset T_x M, \quad S = \phi^{-1}(\partial M \cap B_r(x)),$$

$$T' = (\phi^{-1})_*T \in D_n(U), \quad g = (\phi^{-1})_*\bar{g},$$

where $0 < r' < r \leq \text{inj}^\#(\bar{g})$. Theorem 8 implies that there is a neighborhood $V$ of $0 \in U$ such that $T'|_V$ is given by an integer multiple of integration along a $C^1$-submanifold $M \subset V$. Locally, $M$ can be written as the graph of a $C^1$-function which weakly solves the minimal surface equation. Standard elliptic PDE methods imply that $M$ is smooth, see Section 5.4 below for a similar argument.

### 5.4. Doubling minimal hypersurfaces with free boundary.

In this section we consider the reflection of a free boundary stable minimal hypersurface over its boundary. Let $(M, \bar{g})$ be an $(n+1)$-dimensional compact Riemannian manifold with boundary $\partial M$ and restriction metric $g = \bar{g}|_{\partial M}$. We will assume that there is a neighborhood of the boundary on which $\bar{g}$ takes the form $\bar{g} = g_{\partial M} + dt^2$. Now suppose that $\Sigma \subset M$ is a properly embedded minimal hypersurface with free boundary. The double of $(M, \bar{g})$ is the smooth closed manifold $M_D$ given by $M_D = M \cup_{\partial M} (-M)$. Notice that $M_D$ comes equipped with an involution $\iota : M_D \to M_D$ which interchanges the two copies of $M$ and fixes the doubling locus $\partial M \subset M_D$. Since $\bar{g}$ splits as a product near the boundary, one can also form the smooth doubling of $\bar{g}$, denoted by $\bar{g}_D$, by setting $\bar{g}_D = \bar{g}$ on $M$ and $\bar{g}_D = \iota_* \bar{g}$ on $-M$. 
Lemma 8. Let \((M, \bar{g})\) be a compact Riemannian manifold with boundary such that and let there is a neighborhood of the boundary on which \(\bar{g}\) takes the form \(\bar{g} = g_{0M} + dt^2\). If \(\Sigma \subset M\) be a properly embedded minimal hypersurface with free boundary, then double of \(\Sigma\), given by \(\Sigma_D = \Sigma \cup_{\partial \Sigma} \iota(\Sigma)\) is a smooth minimal hypersurface of the double \((M_D, \bar{g}_D)\). Moreover, if \(\Sigma\) is stable, then so is \(\Sigma_D\).

Proof. First, we will show that \(\Sigma_D\) is a smooth hypersurface. Clearly, \(\Sigma_D\) is smooth away from the doubling locus \(\partial \Sigma \subset M_D\). Let \(x_0 \in \partial \Sigma\) and let \(0 < r \leq \text{inj}(M_D, \bar{g}_D)\). Using the exponential map \(\exp_{x_0}^D\), we will work in the tangent space \(T_{x_0}M - \text{set}\)

\[\hat{\Sigma} = (\exp_{x_0}^D)^{-1}(\Sigma \cap B_{r'}^D(x_0)), \quad \hat{\Sigma}_D = (\exp_{x_0}^D)^{-1}(\Sigma_D \cap B_{r'}^D(x_0)), \quad \hat{g}_D = (\exp_{x_0}^D)^*\bar{g}_D\]

and let \(\nu\) denote the unit normal vector field to \(\hat{\Sigma}\) with respect to \(\hat{g}\). Evidently, \(\hat{\Sigma}\) is a minimal hypersurface in \(T_{x_0}M_D\) with free boundary contained in \(T_{x_0}\partial M \subset T_{x_0}M_D\) with respect to the metric \(\hat{g}\). We choose an orthonormal basis \((e^1, \ldots, e^{n+1})\) for \(T_{x_0}M_D\) so that, writing \(x \in T_{x_0}M\) as \((x^1, \ldots, x^{n+1})\) in this basis,

1. \(T_{x_0}\partial \hat{\Sigma} = \{(x^1, \ldots, x^{n-1}, 0, 0)\};\)
2. \(T_{x_0}\Sigma = \{(x^1, \ldots, x^n, 0)\};\)
3. \(T_{x_0}\partial M = \{(x^1, \ldots, x^{n-1}, 0, x^{n+1})\}\).

This can be accomplished using the fact that \(\Sigma\) meets \(\partial M\) orthogonally. In these coordinates, the involution \(\iota\) now takes the form \((x^1, \ldots, x^n, x^{n+1}) \mapsto (x^1, \ldots, -x^n, x^{n+1})\). Notice that, because the second fundamental form of \(\partial M\) vanishes, \((\exp_{x_0}^D)^{-1}(\partial M \cap B_{r'}^D(x_0))\) is contained in the hyperplane \(\{(x^1, \ldots, x^{n+1}) : x^n = 0\}\).

For a radius \(r' > 0\), we consider the \(n\)-dimensional ball \(B_{r'}^n(0) = \{x \in T_{x_0}M : x^{n+1} = 0, ||x|| < r'\}\), the \(n\)-dimensional half-ball \(B_{r'}^n(0) = \{x \in B_{r'}^n(0) : x^n \geq 0\}\), and the cylinder \(C_{r'}(0) = \{x \in T_{x_0}M : (x^1, \ldots, x^n, 0) \in B_{r'}^n(0)\}\). For small enough \(r'\), we may write \(\hat{\Sigma} \cap C_{r'}(0)\) as the graph of a function

\[u : B_{r'}^n(0) \to \mathbb{R}, \quad \text{graph}(u) = \hat{\Sigma} \cap C_{r'}(0)\]

where \(\text{graph}(u) = \{(x^1, \ldots, x^n, u(x^1, \ldots, x^n)) : (x^1, \ldots, x^n, 0) \in B_{r'}^n(0)\}\). Now we may form the doubling of \(u\) to a function \(u_D : B_{r'}^n(0) \to \mathbb{R}\), setting

\[u_D(x^1, \ldots, x^n) = \begin{cases} u(x^1, \ldots, x^n) & \text{if } x^n \geq 0 \\ u(x^1, \ldots, x^{n-1}, -x^n) & \text{if } x^n < 0. \end{cases}\]

Evidently, the graph of \(u_D\) coincides with \(\hat{\Sigma}_D \cap C_{r'}(0)\). Hence, in order to show \(\Sigma_D\) is smooth at \(x_0\), it suffices to show that \(u_D\) is smooth along \(\{x \in B_{r'}^n(0) : x^n = 0\}\).

Since \(\hat{\Sigma}\) meets \(T_{x_0}\partial M\) orthogonally, we have \(\frac{\partial u}{\partial x^n} \equiv 0\) on \(\{x \in B_{r'}^n(0) : x^n = 0\}\) and so \(u_D\) has a continuous derivative on all of \(B_{r'}^n(0)\). Since \(\hat{\Sigma}\) is smooth and minimal, \(u_D\) is smooth and solves the minimal graph equation \([5.2]\) with respect to the metric \(\bar{g}_D\) in the strong sense on \(\{x \in B_{r'}^n(0) : x^n \neq 0\}\). Moreover, the fact that \(\frac{\partial u}{\partial x^n} \equiv 0\) on \(\{x^n = 0\}\) and the \(\iota\)-invariance of \(\bar{g}_D\) can be used to show \(u_D\) solves the minimal graph equation weakly on the entire ball \(B_{r'}^n(0)\).
From this point, the smoothness of \( u_D \) is a standard application of tools from nonlinear elliptic PDE theory, so we will be brief (c.f. [5, Lemma 7.2]). Standard estimates for minimizers implies \( u_D \in H^2(B^n_r(0)) \) (see Evans Section 8.3.1). Writing the equation (5.2) in divergence form

\[
(5.7) \quad \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial u_D}{\partial x^j} + b^i u_D \right) = 0
\]

where \( a^{ij} \) depend on \( u_D \) and are hence only one differentiable. Since \( u_D \) is a weak solution of equation (5.7)

\[
\int_{B^n_r(0)} (a^{ij} \frac{\partial u_D}{\partial x^j} + b^i u_D) \frac{\partial \psi}{\partial x^i} dx = 0.
\]

Taking \( \psi \) to be a test function of the form \( -\frac{\partial \psi}{\partial x^i} \) and integrating by parts, one can show \( \frac{\partial u_D}{\partial x^k} \) is a weak solution of a uniformly elliptic linear equation with \( L^\infty \) coefficients for each \( k = 1, \ldots, n \).

Now we may apply the DeGiorgi-Nash theorem (see [13, Theorem 8.24]) to conclude that, for each \( r'' < r' \) there is an \( \alpha \in (0, 1) \) such that \( \frac{\partial u_D}{\partial x^k} \in C^{0,\alpha}(B^n_r(0)) \) for each \( k = 1, \ldots, n \). Now \( u_D \in C^{1,\alpha}(B^n_r(0)) \) and the functions \( \frac{\partial u_D}{\partial x^k} \) solve a uniformly elliptic linear equation with Hölder coefficients. The Schauder estimates allow us to conclude \( \frac{\partial u_D}{\partial x^k} \in C^{2,\alpha}(B^n_r(0)) \). This argument may be iterated, see [13, Section 8], to conclude \( u_D \in C^{k,\alpha}(B^n_r(0)) \) for any \( k \). This finishes the proof that \( u_D \) is a smooth solution to the mean curvature equation across the doubling locus \( \{ x^n = 0 \} \) and hence \( \Sigma_D \) is a smooth minimal hypersurface.

The last step is to show that \( \Sigma_D \) is stable. Let \( \phi \in C^\infty(\Sigma_D) \) and write \( \phi = \phi_0 + \phi_1 \) where \( \phi_0 \) is invariant under the involution and \( \phi_1 \) is anti-invariant under the involution. Now we will consider the second variation of the volume of \( \Sigma_D \) with respect to \( \phi \).

\[
\delta_\phi^2(\Sigma_D) = \int_{\Sigma_D} |\nabla \phi|^2 - \phi^2(Ric(\nu, \nu) + |A|^2) d\mu
\]

\[
= \int_{\Sigma_D} |\nabla \phi_0|^2 + 2g(\nabla \phi_0, \nabla \phi_1) + |\nabla \phi_1|^2 - (\phi_0^2 + 2\phi_0 \phi_1 + \phi_1^2)(Ric(\nu, \nu) + |A|^2) d\mu
\]

\[
= \delta_{\phi_0}^2(\Sigma_D) + \delta_{\phi_1}^2(\Sigma_D) + \int_{\Sigma_D} 2g(\nabla \phi_0, \nabla \phi_1) - 2\phi_0 \phi_1(Ric(\nu, \nu) + |A|^2) d\mu
\]

\[
= 2\delta_{\phi_01}(\Sigma) + 2\delta_{\phi_11}(\Sigma) \geq 0
\]

where the last equality follows from the fact that both \( g(\nabla \phi_0, \nabla \phi_1) \) and \( \phi_0 \phi_1 \) are both anti-invariant under the involution. This completes the proof of lemma 8. \( \square \)

5.5. Second fundamental form bounds. In this section, we will prove Step 2 in Section 3.6. Let \( (M_i, \bar{g}_i) \) and \( W_i \) be as in Main Lemma. The uniform second fundamental form bounds for the stable minimal hypersurfaces \( W_i \subset M_i \) can be reduced to a classical estimate due to Schoen-Simon [21] for stable minimal hypersurfaces in Riemannian manifolds. In the following, \( (M, \bar{g}) \) is a complete \((n + 1)\)-dimensional Riemannian manifold, \( x_0 \in M \), \( \rho_0 \in (0, \text{inj}(M, \bar{g}; x_0)) \), and \( \mu_1 \) is a constant.
satisfying
\[(5.8) \sup_{B_0(0)} \left| \frac{\partial \bar{g}_{ij}}{\partial x^k} \right| \leq \mu_1, \quad \sup_{B_0(0)} \left| \frac{\partial^2 \bar{g}_{ij}}{\partial x^k \partial x^l} \right| \leq \mu_1^2, \]
on the metric ball $B_{\rho_0}(x_0)$ where $\bar{g} = \bar{g}_{ij} dx^i dx^j$ is written in the geodesic normal coordinates $(x^1, \ldots, x^{n+1})$ centered at $x_0$.

**Theorem 9** (Corollary 1 [21]). Suppose $\Sigma$ is an orientable embedded $C^2$-hypersurface in an $(n+1)$-dimensional Riemannian manifold $(M, \bar{g})$ with $x_0 \in \Sigma$, $\mu_1$ satisfies (5.8), and $\mu$ satisfies $\rho_0^{-n} \mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) \leq \mu$. Assume that $\mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) < \infty$ and $\mathcal{H}^{n-2}(\text{sing}(\Sigma) \cap B_{\rho_0}(x_0)) = 0$. If $n \leq 6$ and $\Sigma$ is stable in $B_{\rho_0}(x_0)$, then
\[
\sup_{B_{\rho_0}(x_0)} |A^\Sigma| \leq C \rho_0,
\]
where $C$ depends only on $n$, $\mu$, and $\mu_1 \rho_0$.

**Proof of Step 2.** By Lemma 8, the doubling $(W_i)_D$ is a smooth stable minimal hypersurface of $(M_i)_D$. In particular, the singular set of $(W_i)_D$ is empty. Moreover, the manifolds $(M_i)_D$ have uniformly bounded geometry so that the injectivity radius is uniformly bounded from below by some $\rho_0 > 0$, and there is a constant $\mu_1$ so that the bounds (5.8) hold in normal coordinates about any $x \in (M_i)_D$, any $\rho \in (0, \rho_0)$, and all $i = 1, 2, \ldots$. According to Step 1, there is a constant $\mu$ such that
\[
\rho_0^{-n} \text{Vol}(W_i \cap B_\rho(x)) \leq \mu
\]
for all $i = 1, 2, \ldots$. Hence, we may uniformly apply Theorem 9 on any ball $B_{\rho_0}(x_0) \subset (M_i)_D$ intersecting $W_i$ to obtain the bound in Step 2. \(\square\)

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