COMPLETE MINORS OF SELF-COMPLEMENTARY GRAPHS
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ABSTRACT. We show that any self-complementary graph with \( n \) vertices contains a \( K_{\frac{n+2}{2}} \) minor. We derive topological properties of self-complementary graphs.

1. Introduction

There are interesting connections between the topological properties of a graph and those of its complement. For example, in \cite{2} it is shown that the complement of a planar graph with nine vertices is non planar. In particular, any self-complementary graph on nine vertices is non planar. Using the results in \cite{4} and \cite{7} on the \( \mu - \) invariant introduced by C. de Verdière in \cite{6}, one can prove that the complement of a planar graph on ten vertices is intrinsically linked.

These topological properties of graphs are connected to the existence of complete minors. For a graph \( G \), a minor of \( G \) is any graph that can be obtained from \( G \) by a sequence of edge deletions and contractions. An edge contraction means identifying its endpoints and deleting any loops and double edges thus created. In this paper, we search for complete minors of self-complementary graphs and we prove that any self-complementary graph \( G \) on \( n \geq 1 \) vertices contains a \( K_{\frac{n+2}{2}} \) minor. We also prove that this bound is the best possible. These results add to the existing body of work on self-complementary graphs started with seminal papers by H. Sachs \cite{10} and G. Ringel \cite{8}.

In this article, all graphs are non-oriented, without loops and without multiple edges. For a graph \( G \) with \( n \) vertices, \( V(G) = \{v_1, v_2, \ldots, v_n\} \) denotes the set of vertices of \( G \), and \( E(G) = \{v_iv_j|v_i \text{ is connected to } v_j\} \) denotes the set of edges of \( G \). The complete graph with \( n \) vertices is denoted by \( K_n \). For \( G \) a graph with \( n \) vertices, \( cG \) denotes the complement of \( G \) in \( K_n \). The graph \( cG \) has the same set of vertices as \( G \) and \( E(cG) = \{v_iv_j|v_iv_j \notin E(G)\} \). A graph \( G \) is called self-complementary (SC) if there exists a graph isomorphism \( \rho: G \to cG \). For a subset of vertices \( \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \), we let \( \langle v_{i_1}, v_{i_2}, \ldots, v_{i_k}\rangle_G \) denote the subgraph of \( G \) induced by this set. For two graphs \( G_1 \) and \( G_2 \), \( G_1 \ast G_2 \) represents the graph defined by \( V(G_1 \ast G_2) = V(G_1) \sqcup V(G_2) \) and \( E(G_1 \ast G_2) = E(G_1) \sqcup E(G_2) \sqcup E \), with \( E = \{ab|a \in V(G_1), b \in V(G_2)\} \). For example, \( K_6 = K_4 \ast K_2 \).

Date: July 28, 2018.
2. Self-Complementary Graphs

The complete graph on \( n \) vertices has \( n(n - 1)/2 \) edges. A SC graph with \( n \) vertices has \( n(n - 1)/4 \) edges. This implies a SC graph has either \( 4k \) or \( 4k + 1 \) vertices. In \([11]\), Sachs describes the cycle structure of \( \rho \) for a general self-complementary graph. With the notations of this paper:

**Theorem 1** (H. Sachs). Let \( G \) be a self-complementary graph on \( n \) vertices and \( \rho : G \to cG \) be a fixed isomorphism. Then

(a) if \( n = 4k \), then \( \rho \) has no fixed points and all the cycles of \( \rho \) have lengths divisible by 4;

(b) if \( n = 4k + 1 \), then \( \rho \) has exactly one fixed point and all the nontrivial cycles of \( \rho \) have lengths divisible by 4.

Let \( a, b \in V(G) \). We call \( a \) and \( b \) similar if \( b = \rho^{2i}(a) \) for some \( i \). Since for any \( i \), \( \rho^{2i} : G \to G \) is a graph isomorphism, any similar vertices have the same incidence structures. Since the restriction of a graph isomorphism is an isomorphism onto its image, we have

**Lemma 2.** For \( S \subset V(G) \) such that \( \rho(S) = S \), \( \langle S \rangle \) is self-complementary.

3. Construction of Complete Minors

In this section, we prove the existence of complete minors of SC graphs. We begin with SC graphs on \( 4n \) vertices and extend the result to SC graphs on \( 4n + 1 \) vertices.

Let \( G \) be a SC graph with \( 4n \) vertices with a fixed isomorphism \( \rho : G \to cG \). In particular, via a labelling of the vertices, \( \rho \) is an element of \( S_{4n} \), the group of permutations on \( 4n \) characters. Since, \( \deg_G(\rho(v)) = 4n - 1 - \deg_{cG}(\rho(v)) = 4n - 1 - \deg_G(v) \), the permutation \( \rho \) takes vertices of \( G \) of degree larger than the average \( 2n - 1/2 \) to vertices of \( G \) of degree smaller than the average, and vice versa. This means that exactly half of the vertices of \( G \) have degree at least \( 2n \).

Let \( N = \langle v_1, v_2, \ldots, v_{2n} \rangle \) denote the subgraph of \( G \) generated by vertices with degree at least \( 2n \) and \( H = \langle v_{2n+1}, v_{2n+2}, \ldots, v_{4n} \rangle \) the subgraph of \( G \) generated by vertices with degree at most \( 2n - 1 \). The isomorphism \( \rho \) takes the vertices of \( N \) to the vertices of \( H \) and the vertices of \( H \) to the vertices of \( N \). Moreover, if \( v_i v_j \) is an edge of \( N \), then \( \rho(v_i) \rho(v_j) \) is an edge of \( cG \) and thus is not an edge of \( H \). It follows that \( N \) and \( H \) are isomorphic to graphs which are complements of each other in \( K_{2n} \).

Let \( L \) denote the subgraph of \( G \) whose edge set is the union of all edges with one endpoint in \( N \) and the other in \( H \). Since \( \rho \) interchanges vertices of \( N \) and \( H \), for an edge \( e \) in \( L \), \( \rho(e) \) is an edge in \( cG \) which has one endpoint in \( H \) and the other in \( N \). This means \( L \) is isomorphic to \( \rho(L) \) and their union is \( K_{2n,2n} \). We say \( L \) is bipartite self-complementary in \( K_{2n,2n} \) (BSC). It follows that \( |E(L)| = 2n^2 \).
Observation 3. Since any cycle of \( \rho \) is set-invariant, Lemma 2 proves that the graph induced by the vertices of this cycle is self–complementary. If \( \rho \) is a cycle of length \( 4k \), then \( 2k \) of its vertices belong to \( N \) and the other \( 2k \) belong to \( H \). Thus the automorphism \( \rho \) induces a BSC graph in \( K_{2k,2k} \), which has \( 2k^2 \) edges. By similarity of vertices within \( N \) and \( H \) respectively, each vertex connects to \( k \) neighbors.

With the notations introduced in this section, we have:

Lemma 4. Let \( G \) be a self-complementary graph on \( 4n \) vertices. Assume \( \rho : G \rightarrow cG \) is a \( 4n \)–cycle. Then \( G \) contains a \( K_{2n} \) minor obtained by contracting \( 2n \) pairwise nonadjacent edges of \( L \).

Proof. Let \( \rho = (a, \rho(a), \ldots, \rho^{4n-1}(a)) \), \( V(N) = \{a, \rho^2(a), \ldots, \rho^{4n-2}(a)\} \) and \( V(H) = \{\rho(a), \rho^3(a), \ldots, \rho^{4n-1}(a)\} \). Let \( 1 \leq t \leq 4n-1 \) odd such that \( a\rho^t(a) \in E(L) \). By Observation 3, we have \( n \) such choices for \( t \). For \( 0 \leq i \leq 2n-1 \), since \( \rho^{2i} \) is a graph isomorphism, \( \rho^{2i}(a)\rho^{2i+t}(a) \in E(L) \).

\[
\begin{align*}
\rho^{2k}(a) & \quad \quad \rho^{2k+t}(a) \\
\quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\rho^{2l}(a) & \quad \quad \rho^{2l+t}(a)
\end{align*}
\]

Figure 1. Green edges are edges of \( L \) which are contracted to obtain the \( K_{2n} \) minor.

For \( 0 \leq k < l \leq 2n-1 \) such that \( \rho^{2k}(a)\rho^{2l}(a) \notin E(N) \), by applying \( \rho^t \), it follows that \( \rho^{2k+t}(a)\rho^{2l+t}(a) \in E(H) \). See Figure 1. This shows that the contraction of the \( 2n \) edges \( \rho^{2i}(a)\rho^{2i+t}(a), 0 \leq i \leq 2n-1 \) produces a \( K_{2n} \) minor of \( G \). \( \square \)

Observation 5. If \( a \) is a vertex of a SC graph \( G \) with \( 4n \) vertices, since \( \rho \) reverses incidence relations, \( a \) neighbors exactly one of \( \{\rho(a), \rho^{-1}(a)\} \). In particular, in any cycle of \( \rho \) one may choose a generator \( a \) such that \( a\rho(a) \in E(G) \).

Lemma 6. Let \( G \) be a self-complementary graph on \( 4n \) vertices. Assume \( \rho : G \rightarrow cG \) factors as the product of two cycles of lengths \( 4k \) and \( 4l = 4(n-k) \), respectively. Then \( G \) contains a \( K_{2n} \) minor obtained by contracting \( 2n \) pairwise nonadjacent edges of \( L \).

Proof. Let \( \rho = \tau\sigma = (a\rho(a), \ldots, \rho^{4k-1}(a))(b\rho(b), \ldots, \rho^{4l-1}(b)) \). By Observation 5, we may assume \( a\rho(a), b\rho(b) \in E(L) \).

Let \( 0 \leq i \leq 2k-1 \) and \( 0 \leq j \leq 2l-1 \). If \( \rho^{2i}(a)\rho^{2j}(b) \notin E(G) \), then, by applying \( \rho \), we obtain \( \rho^{2i+1}(a)\rho^{2j+1}(b) \in E(G) \). See Figure 2. This proves that
Theorem 7. Let $G$ be a self-complementary graph on $4n$ vertices. Then $G$ contains a $K_{2n}$ minor obtained by contracting $2n$ pairwise nonadjacent edges of $L$.

Proof. Let $\rho : G \to cG$ be a fixed isomorphism. Let $\rho = \tau_1\tau_2...\tau_s$ be the cycle decomposition, where the length of each $\tau_i$ is a multiple of 4. We shall prove the conclusion by induction on $s$, the number of disjoint cycles of $\rho$. The base case is done by Lemma 3. For $s > 1$, by Observation 5, for each $1 \leq i \leq s$ there is a vertex $a_i$ such that $\tau_i = (a_i, \rho(a_i), ...)$ and $a_i, \rho(a_i) \in E(L)$. For any cycle $\tau_1 \neq \tau_k$ of $\rho$, the arguments of Lemma 3 show that there is at least one edge between $\{\rho^{2i}(a_1), \rho^{2i+1}(a_1)\}$ and $\{\rho^{2j}(a_k), \rho^{2j+1}(a_k)\}$, for any $0 \leq i, j$. By the induction hypothesis, we obtain a $K_{2n-ord(\tau_1)/2}$ minor of the subgraph of $G$ induced by the vertices not in $\tau_1$. Each of the vertices of this minor connects to at least one vertex of each pair $\{\rho^{2i}(a_1), \rho^{2i+1}(a_1)\}$, $0 \leq i < ord(\tau_1)$. Contracting these $\frac{1}{2}ord(\tau_1)$ edges, we obtain a $K_{2n} = K_{2n-ord(\tau_1)/2} * K_{ord(\tau_1)/2}$ minor of $G$. □

Theorem 8. If $G$ is a self-complementary graph with $4n + 1$ vertices, then $G$ contains a $K_{2n+1}$ minor.

Proof. Let $G$ be a SC graph whose set of vertices is $V(G) = \{v_1, ..., v_{4n+1}\}$. Let $\rho : G \to cG$ be a fixed isomorphism. By Theorem 1, $\rho$ has exactly one fixed point. Assume $\rho(v_{4n+1}) = v_{4n+1}$. Let $\overline{G} = \langle v_1, ..., v_{4n} \rangle_G$. By Lemma 2, $\overline{G}$ is a self complementary graph with $4n$ vertices. Let $N = \langle v_1, ..., v_{2n} \rangle_{\overline{G}}$, $H = \langle v_{2n+1}, ..., v_{4n} \rangle_{\overline{G}}$. We assumed $deg_{\overline{G}}(v_i) \geq 2n$ for $1 \leq i \leq 2n$ and $deg_{\overline{G}}(v_i) \leq 2n-1$ for $2n+1 \leq i \leq 4n$.

For any $a \in V(\overline{G})$, as $v_{4n+1}$ is a fixed point for $\rho$ and $\rho$ reverses incidence relations, $v_{4n+1}$ connects to exactly one of $a$ or $\rho(a)$. This means that for any nontrivial cycle of $\rho$, $v_{4n+1}$ neighbors exactly half of the vertices of this cycle. Moreover, these neighbors are either all vertices of $N$, or all vertices of $H$. By Theorem 7, $\overline{G}$ contains a $K_{2n}$ minor which can be obtained by contracting $2n$
pairwise nonadjacent edges of $L$. Since $v_{4n+1}$ connects to exactly one endpoint of each of these edges, the contraction of the $2n$ edges creates a $K_{2n+1} = K_{2n} \ast K_1$ minor of $G$.

We summarize the results in in theorems 7 and 8 in the following:

**Theorem 9.** Any self-complementary graph $G$ on $n \geq 1$ vertices contains a $K_{\lfloor \frac{n+1}{2} \rfloor}$ minor.

The bound $\lfloor \frac{n+1}{2} \rfloor$ is the best that can be guaranteed as the next theorem proves.

**Theorem 10.**

(a) For $n \geq 1$, there exist self-complementary graphs with $4n$ vertices which do not contain a $K_{2n+1}$ minor.

(b) For $n \geq 0$, there exist self-complementary graphs with $4n + 1$ vertices which do not contain a $K_{2n+2}$ minor.

**Proof.**

(a) Let $N \simeq K_{2n}$ such that $V(N) = \{v_1, \ldots, v_{2n}\}$ and $H$ such that $V(H) = \{w_1, \ldots, w_{2n}\}$ and $E(H) = \emptyset$. Let $L = L_1 \sqcup L_2$, with $L_1$ the complete bipartite graph on $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$, and $L_2$ the complete bipartite graph on $\{v_{n+1}, \ldots, v_{2n}\}$ and $\{w_{n+1}, \ldots, w_{2n}\}$. The graph $G = N \cup L \cup H$ is a self-complementary graph on $4n$ vertices. Assume that $G$ contains a $K_{2n+1}$ minor. Since $|V(N)| = 2n$ and $H$ is a discrete set of points with no edges between its vertices, there exists $w_i \in V(H)$ such that none of the edges adjacent to $w_i$ were contracted in creating this minor. The degree of $w_i$ in the created minor is at most $\deg_G(w_i) = n$, which is a contradiction as the degree of any vertex in $K_{2n+1}$ is $2n$.

(b) Let $G'$ be the self-complementary graph on $4n + 1$ vertices obtained by adding one extra vertex $a$ to $V(G)$ and all edges $av_i$ for $1 \leq i \leq 2n$. The argument of (a) applied verbatim shows that $G'$ cannot contain a $K_{2n+2}$ minor.

\[\square\]

4. **Topological Properties of Self-Complementary Graphs**

In this section we discuss topological properties of self-complementary graphs which derive from Theorem 9, outerplanarity, planarity, intrinsic linkness and intrinsic knottedness. A graph is called *outerplanar* if it can be embedded in the plane such that all its vertices sit on the outer face of the embedding. It is known that a graph is outerplanar if and only if it doesn’t have $K_4$ or $K_{2,3}$ as its minors. A graph is called *planar* if it can be embedded in the plane. The two excluded minors for planarity are $K_5$ and $K_{3,3}$.

A graph $G$ is said to be *intrinsically linked* (IL) if every embedding of $G$ in $\mathbb{R}^3$ contains two disjoint linked cycles. If not IL, then $G$ is *linklessly embeddable* (nIL). By work of Conway and Gordon and Robertson, Seymour, and Thomas, a graph is intrinsically linked if and only if it contains one of the seven graphs...
in the Petersen family as a minor. One of these graphs is $K_6$. A graph $G$ is said to be intrinsically knotted (IK) if every embedding of $G$ in $\mathbb{R}^3$ contains a nontrivial knot. By work of Conway and Gordon [3], we know that $K_7$ is IK. A graph is said to be $n$–apex if deleting $n$ or fewer vertices produces a planar (sub)graph. By [10], a 1–apex graph is nIL. By [1], a 2–apex graph is not IK.

**Corollary 11.** A self-complementary graph on $n \geq 8$ vertices is not outerplanar.

*Proof.* By Theorem 9 any such graph $G$ contains a $K_4$ minor, thus $G$ is not outerplanar. \hfill \Box

On the other hand, all SC graphs on less than 8 vertices are outerplanar. See Figure 3(a).

![Figure 3](image.png)

**Figure 3.** (a) Self-complementary graphs on one, four and five vertices, (b) Planar self-complementary graph on 8 vertices

**Corollary 12.** A self-complementary graph on $n \geq 9$ vertices is not planar.

*Proof.* By Theorem 9 any such graph $G$ contains a $K_5$ minor and thus it is not planar. \hfill \Box

This is also a consequence of [2]. On the other hand, there are SC graphs on 8 vertices which are planar. See Figure 3(b).

**Corollary 13.** A self-complementary graph on $n \geq 12$ vertices is intrinsically linked.

*Proof.* By Theorem 9 any such graph $G$ contains a $K_6$ minor. Since $K_6$ is an excluded minor for linkless embeddability, it follows that $G$ is intrinsically linked. \hfill \Box

On the other hand, there are SC graphs on 9 vertices which are not intrinsically linked. The graph $G$ in Figure 4 is such an example. Deleting the highlighted vertex produces a planar graph. This means $G$ is 1–apex and thus it is not intrinsically linked.

**Corollary 14.** A self-complementary graph on $n \geq 13$ vertices is intrinsically knotted.
Figure 4. (a) The Payley graph on 9 vertices. (b) Deleting the vertex highlighted in red leaves a planar graph. A planar embedding can be obtained by moving the three green edges to the outside of the octagon.

Figure 5. (a) A self-complementary graph on 12 vertices is obtained by pairwise connecting all filled vertices, (b) Removing vertices $a$ and $b$ gives a planar subgraph.

Proof. By Theorem 9 any such graph $G$ contains a $K_7$ minor. Since $K_7$ is an excluded minor for knotless embeddability, it follows that $G$ is intrinsically knotted.

On the other hand, there are SC graphs on 12 vertices which are not intrinsically knotted. We describe the construction of such a graph. Figure 5(a) represents a bipartite self-complementary graph in $K_{6,6}$. Construct a SC graph on 12 vertices $G$ by adding all edges between filled vertices. Deleting vertices $a$ and $b$ of $G$ creates the planar graph in Figure 5(b). This means $G$ is 2–apex and thus it is not intrinsically knotted.

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