A BIRKHOFF-BRUHAT ATLAS FOR PARTIAL FLAG VARIETIES

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Abstract. A partial flag variety \( P_K \) of a Kac-Moody group \( G \) has a natural stratification into projected Richardson varieties. When \( G \) is a connected reductive group, a Bruhat atlas for \( P_K \) was constructed in [7]: \( P_K \) is locally modeled with Schubert varieties in some Kac-Moody flag variety as stratified spaces. The existence of Bruhat atlases implies some nice combinatorial and geometric properties on the partial flag varieties and the decomposition into projected Richardson varieties.

A Bruhat atlas does not exist for partial flag varieties of an arbitrary Kac-Moody group due to combinatorial and geometric reasons. To overcome obstructions, we introduce the notion of Birkhoff-Bruhat atlas. Instead of the Schubert varieties used in a Bruhat atlas, we use the \( J \)-Schubert varieties for a Birkhoff-Bruhat atlas. The notion of the \( J \)-Schubert varieties interpolates Birkhoff decomposition and Bruhat decomposition of the full flag variety (of a larger Kac-Moody group). The main result of this paper is the construction of a Birkhoff-Bruhat atlas for any partial flag variety \( P_K \) of a Kac-Moody group. We also construct a combinatorial atlas for the index set \( Q_K \) of the projected Richardson varieties in \( P_K \). As a consequence, we show that \( Q_K \) has some nice combinatorial properties. This gives a new proof and generalizes the work of Williams [21] in the case where the group \( G \) is a connected reductive group.

1. Introduction

1.1. The flag variety and its decomposition into Richardson varieties. Let \( G \) be a connected reductive group and \( B \) be the full flag variety of \( G \). An open Richardson variety is the intersection of a Bruhat cell with an opposite Bruhat cell. We then have the decomposition of \( B \) into the disjoint union of the open Richardson varieties. This decomposition has many remarkable properties, including:

1. each stratum is smooth;
2. the closure of each stratum is a union of other strata;
3. the closure of each stratum is normal, Cohen-Macaulay, with rational singularities;
4. over positive characteristic, there exists a Frobenius splitting on \( B \) which compatibly splits all the strata;
5. over complex numbers, there exists a Poisson structure on \( B \) for which the \( T \)-leaves are exactly the strata;

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(6) over real numbers, the intersection of the totally nonnegative flag variety $X_{\geq 0}$ with each stratum gives a cellular decomposition of $X_{\geq 0}$;

(7) the poset of the strata is thin and EL-shellable.

Many of these remarkable properties remain valid for the full flag variety of any Kac-Moody group.

1.2. Partial flag varieties. Now we consider the partial flag variety $P_K = G/P_K$ of a Kac-Moody group $G$. The variety $P_K$ has a natural stratification into the projected Richardson varieties. The projected Richardson varieties in $P_K$ are the image of certain Richardson varieties in the full flag $B$ under the projection map $\pi : B \to P_K$. However, the combinatorial and geometric structures of the projected Richardson varieties in $P_K$ are more complicated than the Richardson varieties in $B$.

Let $Q_K$ be the index set of the projected Richardson varieties in $P_K$ and $\preceq$ be the partial order on $Q_K$. Williams [21] showed that if $G$ is a connected reductive group, then the partial order set $Q_K$ has remarkable combinatorial properties: thinness, shellability, etc. Such combinatorial properties are used later by Galashin, Karp and Lam [6] to prove the conjecture of Postnikov and Williams that the totally nonnegative part of $P_K$ is a regular CW complex.

1.3. The Bruhat atlas of [7]. One of the motivations in the unpublished work of Knutson, Lu and the second-named author [7] is to use the Richardson varieties in the full flag variety $\tilde{B}$ of a “much larger” Kac-Moody group $\tilde{G}$ as the model for the decomposition of $P_K$ into projected Richardson varieties, and many other stratified spaces arising in Lie theory. Consequently, many remarkable combinatorial properties and geometric properties on these stratified spaces may be deduced directly from those on the Bruhat order of the Weyl group $\tilde{W}$ of $\tilde{G}$ and the Richardson varieties of $\tilde{B}$.

By definition, a Bruhat atlas for a stratified space $M = \bigsqcup_y M_y$ consists of a large Kac-Moody group $\tilde{G}$ and an open covering $M = \bigsqcup U$, such that

• for each $U$, there exists an isomorphism of stratified spaces from $U$ to a Schubert cell in the flag variety $\tilde{B}$ of $\tilde{G}$;

• for each $y$ and $U$, $U \cap M_y$ is mapped isomorphically to an open Richardson variety of $\tilde{B}$.

The group $\tilde{G}$ is called the atlas group for this Bruhat atlas.

The first example of a Bruhat atlas was constructed by Snider [18] for Grassmannian with the positroid stratification. A Bruhat atlas for the full flag variety of a connected reductive group $G$ was constructed by Knutson, Woo, and Yong in [15]. A Bruhat atlas for any partial flag variety of a connected reductive group and a Bruhat atlas for the wonderful compactification of a semisimple adjoint group was then constructed in [7]. A different “atlas” for the partial flag varieties of a connected reductive group was constructed recently by Galashin, Karp and Lam in [6] and by Huang in [10]. A Bruhat atlas for the wonderful compactification of the symmetric space $PSO(2n)/SO(n-1)$ was recently given by Huang in [11].

1.4. The Birkhoff-Bruhat atlas. In the works discussed above, the group $G$ involved is a connected reductive group, i.e. a Kac-Moody group of finite type.
A Bruhat atlas for $\mathcal{P}_K$ does not exist when $G$ is of infinite type, due to the following reasons. First, the partial flag variety $\mathcal{P}_K$, in general, are infinite-dimensional. Thus one cannot use Schubert cells (which is finite-dimensional) as an atlas for $\mathcal{P}_K$. This gives a geometric obstruction. There is also a combinatorial obstruction arising from comparison of the partial orders. Note that the partial order $\preceq$ on $Q_K$ involved both the Bruhat order and the opposite Bruhat order in the Weyl group $W$ of $G$. By the definition of Bruhat atlas, one needs to embed $Q_K$ into the Weyl group $\tilde{W}$ of an atlas group. It is only possible if the Weyl group $W$ has the longest element, which interchanges the Bruhat order and the opposite Bruhat order on $W$.

The main purpose of this paper is to introduce a suitable “atlas model” for the partial flag varieties of any Kac-Moody group. We use the open $J$-Schubert cells in the full flag variety $\tilde{B}$ of an atlas group $\tilde{G}$ instead of the Schubert cells in $\tilde{B}$ as in the original definition of Bruhat atlas.

The decomposition of $\tilde{B}$ into the $J$-Schubert cell was introduced by Billig and Dyer in [1], which simultaneously generalizes both the Bruhat decomposition of $B$ into the Schubert cells and the Birkhoff decomposition of $B$ into the opposite Schubert cells. A $J$-Schubert cell, in general, is neither finite dimensional nor finite codimensional. A Birkhoff-Bruhat atlas for $\mathcal{P}_K$ consists of an atlas group $\tilde{G}$ and an open covering $\mathcal{P}_K = \bigcup U$ such that

- for each $U$, an embedding of $U$ into the full flag variety $\tilde{B}$ of $\tilde{G}$;
- the intersection of $U$ with any projected Richardson variety is mapped isomorphically to a $J$-Richardson variety in $\tilde{B}$.

The main result of this paper is

**Theorem 1.1.** Any partial flag variety $\mathcal{P}_K$ of a Kac-Moody group admits a Birkhoff-Bruhat atlas.

It is also worth mentioning that even in the finite type case, the atlas groups from the Birkhoff-Bruhat atlas we constructed and those from the Bruhat atlas in [7] are different. Thus our construction provides a new “atlas model” for the partial flag varieties for connected reductive groups.

We also construct a combinatorial “atlas model” for the poset $Q_K$. This “atlas model” identifies the poset $(Q_K, \preceq)$ with a convex subset of the Weyl group $\tilde{W}$ of the atlas group $\tilde{G}$ with respect to a twisted Bruhat order $I^{\flat} \preceq$. This combinatorial “atlas model” is valid for $Q_K$ from an arbitrary Coxeter group.

**Theorem 1.2.** The partial order $\preceq$ on $Q_K$ is thin and EL-shellable.

We refer to section 4.9 for the definition of thinness and EL-Shellability. This result generalizes the previous work of Williams [21].

Galashin, Karp and Lam [6, Conjecture 10.2] conjectured that the totally non-negative part of $\mathcal{P}_K$ is a regular CW complex for a Kac-Moody group $G$, and $Q_K$ is its face poset. Theorem 1.2 confirms the combinatorial aspect of their conjecture.

1.5. **Organization.** This paper is organized as follows. We recall preliminaries of Kac-Moody groups and $J$-Schubert cells in Section 2. We then define a Birkhoff-Bruhat atlas and construct such an atlas for the partial flag variety $\mathcal{P}_K$ of arbitrary type in Section 3. We discuss some combinatorial consequences in Section 4.9. We
We define \( w \) for any \( J \) and \( \alpha \) generated by \( \{ U \} \) associated to the Kac-Moody root datum \( D \) with \( \Delta^\sigma = \Delta^\sigma_+ \cup \Delta^\sigma_- \) is the union of positive real roots and negative real roots.

Let \( k \) be an algebraically closed field. The \textit{minimal Kac-Moody group} \( G \) associated to the Kac-Moody root datum \( D \) is the group generated by the torus \( T = Y \otimes_k k^\times \) and the root subgroup \( U_\alpha \cong k \) for each real root \( \alpha \), subject to the Tits relations [20]. Let \( U^+ \subset G \) (resp. \( U^- \subset G \)) be the subgroup generated by \( U_\alpha \) for \( \alpha \in \Delta^\sigma_+ \) (resp. \( \alpha \in \Delta^\sigma_- \)). Let \( B^\pm \subset G \) be the Borel subgroup generated by \( T \) and \( U^\pm \). We fix a pinning of \( G \) consisting of \( (T, B^+, B^-, x_i, y_i; i \in I) \) with one parameter subgroups \( x_i : k \to U_{\alpha_i} \) and \( y_i : k \to U_{-\alpha_i} \) analogous to [17]. We have an anti-involution \( \Psi \) of \( G \) analogous to [17, §1.2] such that \( \Psi(x_i(a)) = y_i(a) \), \( \Psi(y_i(a)) = x_i(a) \) and \( \Psi(t) = t \) for \( a \in k \), \( t \in T \).

Let \( J \subset I \) (not necessarily of finite type). We denote by \( P^+_J \) the subgroup of \( G \) generated by \( B^+ \) and \( U_{-\alpha_j} \) for \( j \in J \). Let \( W_J \) be the subgroup of \( W \) generated by \( \{ s_j \}_{j \in J} \). Let \( W^J \) be the set of minimal-length coset representatives of \( W/W_J \) and \( ^JW \) be the set of minimal-length coset representatives of \( W \setminus W \). For \( i \in I \), we define

\[ \hat{s}_i = x_i(-1)y_i(1)x_i(-1) \in G. \]

For any \( w \in W \) with reduced expression \( w = s_{i_1} \cdots s_{i_n} \), we define

\[ \hat{w} = \hat{s}_{i_1} \cdots \hat{s}_{i_n} \in G. \]

It is known that \( \hat{w} \) is well-defined and independent of the reduced expression.

Let \( L_J \) be the subgroup of \( P^+_J \) generated by \( T \), \( U_{\pm \alpha_j} \) for \( j \in J \). We denote by \( \Delta^\sigma_j = \{ w(\pm \alpha_j) \in X \mid j \in J, w \in W_J \} \subset \Delta^\sigma \) the set of real roots of \( L_J \). We write \( \Delta^\sigma_\pm = \Delta^\sigma_+ \cap \Delta^\sigma_- \). We denote by \( U_\alpha \) the unipotent radical of \( P^+_J \), which is generated by \( U_{\alpha} \) for \( \alpha \in \Delta^\sigma_+ - \Delta^\sigma_- \). We have following Levi decomposition of \( P^+_J \) [12, Theorem B.39]

\[ P^+_J = L_J \ltimes U_{P^+_J}. \]

(2.1)
We similarly define the subgroup $P_j^+$ of $G^{\text{min}}$ as the subgroup generated by $B^-$ and $U_{\alpha_j}$ for $j \in J$, with the Levi decomposition

$$P_j^+ = L_J \ltimes U_{P_j^-}.$$  \hfill (2.2)

2.2. The full flag variety. In this subsection, we recall several results on the Kac-Moody flag varieties.

We denote by $\mathcal{B}$ the (thin) full flag variety [16], equipped with the ind-variety structure. Let $v, w \in W$. Define, respectively, the Schubert cell, the opposite Schubert cell and the open Richardson variety by

$$X^w = B^+ w B^+ / B^+, \quad \check{X}_v = B^- v B^+ / B^+, \quad \check{R}_{v,w} = \check{X}^w \cap \check{X}_v.$$  

We have the Bruhat decomposition $\mathcal{B} = \bigsqcup_{w \in W} \check{X}^w$ and the Birkhoff decomposition $\mathcal{B} = \bigsqcup_{v \in W} \check{X}_v$. It is known that $\check{R}_{v,w} \neq \emptyset$ if and only if $v \leq w$. In this case, $\check{R}_{v,w}$ is irreducible of dimension $\ell(w) - \ell(v)$. We also have the decomposition

$$\mathcal{B} = \bigsqcup_{v \leq w} \check{R}_{v,w}.$$  

Let $X^w, X_v, R_{v,w}$ be the (Zariski) closure of $\check{X}^w, \check{X}_v, \check{R}_{v,w}$ respectively. By [16, Proposition 7.1.15&7.1.21],

$$X^w = \bigsqcup_{w' \leq w} \check{X}^{w'}, \quad X_v = \bigsqcup_{v' \geq v} \check{X}_{v'}.$$  

As the Schubert varieties and opposite Schubert varieties intersect transversally, we also have (see e.g. [14])

$$R_{v,w} = \bigsqcup_{w' \leq v' \leq w} \check{R}_{v',w'}.$$  

2.3. The $J$-Schubert cells and $J$-Richardson varieties. Let $J \subset I$. Following [11] Closure patterns], we define the partial order $J \leq$ on $W$ as follows. The partial order $J \leq$ on $W$ is generated by the relations $s_{j} w J < w$ for $w \in W$ and $\beta \in \Psi_J$ with $w^{-1}(\beta) \in \Delta^\text{re}-$. Define a (non-standard) length function $J \ell$ on $W$ by

$$J \ell(w) = \ell(w) - 2 \chi(\Delta^\text{re}+ \cap w^{-1}(\Delta^\text{re}-)).$$  

Note that any element in $W$ can be written in a unique way as $x y$ for $x \in W_J$ and $y \in ^J W$. We have $J \ell(xy) = \ell(y) - \ell(x)$, where $\ell(\cdot)$ denotes the usual length function. We define

$$\Psi_J^+ = \Delta^\text{re}- \cup (\Delta^\text{re}+ - \Delta^\text{re}+) \quad \Psi_J^- = \Delta^\text{re}+ \cup (\Delta^\text{re}- - \Delta^\text{re}-).$$

**Remark 2.1.** The relation between our partial order $J \leq$ and the partial order $\leq_{\Psi_J}$ used in [11] Closure pattern] is the following

$$v J \leq w \text{ if and only if } v^{-1} \leq_{\Psi_J} w^{-1}.$$  

Note that $v J \leq w$ is not equivalent to $v^{-1} J \leq w^{-1}$ in general [11] Page 18.

**Remark 2.2.** Let $W_J$ be a finite Weyl group. In this case, we denote by $w_J$ the longest element of $W_J$. Then for $w, w' \in W$, $w' J \leq w$ if and only if $w_J w' \leq w_J w$. Moreover, we have $J \ell(w) = \ell(w_J w) - \ell(w_J).$
Let \( J^+ \) be the subgroup of \( G \) generated by \( T \) and \( U_\alpha \) for \( \alpha \in \Psi^+_J \). Then \( J^+ \) is the opposite Borel subgroup of the standard parabolic subgroup \( P^+_J \). Similarly, let \( J^- \) be the subgroup of \( G \) generated by \( T \) and \( U_\alpha \) for \( \alpha \in \Psi^-_J \). In the case where \( J = \emptyset \), we have \( J^+ = B^+ \) and \( J^- = B^- \). In the case where \( J = I \), we have \( J^+ = B^- \) and \( J^- = B^+ \). Let \( B^+_J = L_J \cap B^+ \) and \( U^+_J = L_J \cap U^+ \).

Thanks to the Levi decompositions \([2,1] & [2,2]\), we have

\[
J^+ = B^- \ltimes U_{P_J^+}, \quad J^- = B^+ \ltimes U_{P_J^-}.
\]

Let \( v, w \in W \). Define, respectively, the \( J \)-Schubert cell, the opposite \( J \)-Schubert cell and the open \( J \)-Richardson variety by

\[
\hat{J}X^w = J^+ \hat{w}B^+/B^+, \quad \hat{J}X_v = J^- \hat{v}B^+/B^+, \quad \hat{J}R_{w,v} = JX^w \cap \hat{J}X_v.
\]

**Lemma 2.3.** We have isomorphisms

\[
(U_J^+ \cap \hat{w}U^- \cdot \hat{w}^{-1}) \times (U_{P_J^+} \cap \hat{w}U^+ \cdot \hat{w}^{-1}) \rightarrow J^+ \hat{X}^w, \quad (x_1, x_2) \mapsto x_1 x_2 \hat{w}B^+/B^+; \]

\[
(U_J^- \cap \hat{v}U^- \cdot \hat{v}^{-1}) \times (U_{P_J^-} \cap \hat{v}U^+ \cdot \hat{v}^{-1}) \rightarrow J^- \hat{X}_v, \quad (x_1, x_2) \mapsto x_1 x_2 \hat{v}B^+/B^+.
\]

**Proof.** We prove the first statement here. The second one is entirely similar. It follows from the Levi decompositions and [16 Theorem 5.2.3] that we have isomorphisms

\[
(U_J^- \cap \hat{w}U^- \cdot \hat{w}^{-1}) \times (U_{P_J^-} \cap \hat{w}U^+ \cdot \hat{w}^{-1}) \rightarrow U_J^-; \]

\[
(U_{P_J^-} \cap \hat{w}U^- \cdot \hat{w}^{-1}) \times (U_{P_J^-} \cap \hat{w}U^+ \cdot \hat{w}^{-1}) \rightarrow U_{P_J^-}.
\]

Therefore we have

\[
J^+ \hat{w}B^+/B^+ = (U_J^- \cap \hat{w}U^- \cdot \hat{w}^{-1}) \cdot (U_{P_J^-} \cap \hat{w}U^+ \cdot \hat{w}^{-1}) \cdot \hat{w}B^+/B^+.
\]

Now the lemma follows from the restriction of the isomorphism

\[
\hat{w}U^- \cdot \hat{w}^{-1} \rightarrow \hat{w}U^- B^+/B^+, \quad g \mapsto g \hat{w}B^+/B^+.
\]

By [1 Theorem 1], we have

\[
\mathcal{B} = \bigsqcup_{w \in W} \hat{J}X^w = \bigsqcup_{v \in W} \hat{J}X_v. \tag{2.3}
\]

Let \( JX^w \) and \( JX_v \) be the (Zariski) closure of \( \hat{J}X^w \) and \( \hat{J}X_v \), respectively. By [1 Theorem 4], we have

\[
\overline{JX^w} = \overline{\hat{J}X^w} = \bigsqcup_{w' \leq w} \hat{X}^{w'}, \quad \overline{JX_v} = \overline{\hat{J}X_v} = \bigsqcup_{v \leq v'} \hat{X}_v. \tag{2.4}
\]

**Proposition 2.4.** Let \( v, w \in W \). Then the following conditions are equivalent:

1. \( JX^w \cap JX_v \neq \emptyset \);
2. \( JX^w \cap JX_v \neq \emptyset \);
3. \( v \leq w \).

Proof. It is obvious that (1) \( \Rightarrow \) (2).

We show that (2) \( \Rightarrow \) (3). If \( JX^w \cap JX_v \neq \emptyset \), then there exists \( z \in W \), such that \( JX^w \cap JX_v \cap X^z \neq \emptyset \). Since \( X^z \) is finite dimensional, \( JX^w \cap JX_v \cap X^z \) is still finite dimensional. It is projective and stable under the left action of \( T \). By [16 Exercise 7.1.E.5], \( JX^w \cap JX_v \cap X^z \) has a \( T \)-fixed point. Hence \( JX^w \cap JX_v \) has a \( T \)-fixed point.

Note that the \( T \)-fixed points in \( X \) are \( \{ \hat{w}B^+/B^+; w \in W \} \). Thus by (2.3), \( \hat{w}B^+/B^+ \) is the only \( T \)-fixed point of \( JX^w \) and of \( JX_v \). By (2.4), the \( T \)-fixed points
in \( J^X_w \) are \( \{ \hat{w}'B^+/B^+; w'^J \leq w \} \) and the \( T \)-fixed points in \( J^X_v \) are \( \{ \hat{v}'B^+/B^+; v'^J \leq v' \} \).

Since \( J^X_w \) and \( J^X_v \) have a common \( T \)-fixed point, we must have \( v'^J \leq w \).

We then show that (3) \( \Rightarrow \) (1). Let \( U_v = \hat{v}U^{-}/B^{+} \subset \mathcal{B} \). Thanks to the Levi decomposition and [16, Theorem 5.2.3], we have the isomorphisms

\[
(U^{-}_J U^+_P \cap \hat{v}U^{-}\hat{v}^{-1}) \times (U^+_J U^{-}_P \cap \hat{v}U^{-}\hat{v}^{-1}) \xrightarrow{\sim} \hat{v}U^{-}\hat{v}^{-1} \xrightarrow{\sim} U_v,
\]

By Lemma 2.3, we have the isomorphism

\[
(U^{-}_J U^+_P \cap \hat{v}U^{-}\hat{v}^{-1}) \times J^X_v \xrightarrow{\sim} U_v.
\]

Since \( J^X_w \) is \( U^{-}_J U^+_P \)-stable, we have, via restriction,

\[
(U^{-}_J U^+_P \cap \hat{v}U^{-}\hat{v}^{-1}) \times (J^X_w \cap U_v) \xrightarrow{\sim} J^X_w \cap U_v.
\]

It remains to prove \( J^X_w \cap U_v \neq \emptyset \) when \( v'^J \leq w \). Since \( U_v \) is open in \( \mathcal{B} \), we have \( J^X_w \cap U_v \neq \emptyset \) if and only if \( J^X_w \cap U_v \neq \emptyset \). Thanks to [2.4], we have \( \hat{v}B^+/B^+ \in J^X_w \) if \( v'^J \leq w \). We clearly have \( \hat{v}B^+/B^+ \in U_v \). Therefore \( \hat{v}B^+/B^+ \in U_v \cap J^X_w \).

The claim follows. \( \square \)

2.4. J-Schubert decompositions. In this subsection, we study a more refined version of Lemma 2.3.

For a group \( K \) and subsets \( K', K_1, K_2, \ldots, K_n \), we write

\[
K' = K_1 \odot K_2 \odot \cdots \odot K_n
\]

if any element \( k' \in K' \) can be written uniquely as \( k' = k_1k_2 \cdots k_n \) with \( k_i \in K_i \).

Lemma 2.5. Let \( v \in W_{J_1} \) and \( w \in W_{J_2} \cap J_1W \). We have

\[
U^{J_1}_{J_1}U^+_P \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = (U^{J_1}_{J_1} \cap \hat{v}U^{J_1}_{J_1}\hat{v}^{-1}) \odot \hat{v}(U^+_P \cap \hat{w}U^{J_1}_{J_1}\hat{w}^{-1}\hat{v}^{-1})\hat{v}^{-1}.
\]

Proof. Note that \( \ell(vw) = \ell(v) + \ell(w) \). We recall the following decompositions from [16, Theorem 5.2.3]:

\[
\begin{align*}
(\hat{v}w)U^{-}(\hat{v}w)^{-1} &= (U^{-} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}) \odot (U^{+} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}); \quad (\odot 1) \\
U^{-} \cap \hat{v}U^{-}\hat{v}^{-1} &= \hat{v}(U^{-} \cap \hat{w}U^{+}\hat{w}^{-1})\hat{v}^{-1} \odot (U^{-} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}); \quad (\odot 2) \\
U^{+} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} &= \hat{v}(U^{+} \cap \hat{w}U^{+}\hat{w}^{-1})\hat{v}^{-1} \odot (U^{+} \cap \hat{v}U^{-}\hat{v}^{-1}). \quad (\odot 3)
\end{align*}
\]

Thanks to the Levi decomposition \( P^{-}_{J_1} = L_{J_1} \times U^+_P \), the decompositions are compatible with the restriction from \( U^{+} \) to \( U^{J_1}_{J_1} \) as well as from \( U^{+} \) to \( U^+_P \).

It follows from (\( \odot 1 \)) that

\[
U^{J_1}_{J_1}U^+_P \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = (U^{J_1}_{J_1} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}) \odot (U^+_P \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}).
\]

Since \( w \in J_1W \), we have \( U^{J_1}_{J_1} \cap \hat{w}U^{+}\hat{w}^{-1} = \{ e \} \). Therefore it follows from (\( \odot 2 \)) that

\[
U^{J_1}_{J_1} \cap \hat{v}U^{J_1}_{J_1}\hat{v}^{-1} = U^{J_1}_{J_1} \cap \hat{v}U^{-}\hat{v}^{-1} = U^{J_1}_{J_1} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}.
\]
Finally, since \( v \in W_{J_1} \), we have \( U_{P_{J_1}^+} \cap \hat{v}U^{-\hat{v}^{-1}} = \{ e \} \). Since \( w \in W_{J_2} \), it follows from (\( \diamondsuit 3 \)) that
\[
U_{P_{J_1}^+} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U^{-\hat{w}^{-1}})\hat{v}^{-1} = \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1}
\]
The lemma follows. \( \square \)

The following result follows easily from Lemma 2.3 and Lemma 2.5

**Corollary 2.6.** Let \( J_1, J_2 \subset I \). Let \( v \in W_{J_1} \) and \( w \in W_{J_2} \cap J^1W \). We have an isomorphism
\[
(U_1^{-} \cap \hat{v}U_{J_2}^{-} \hat{v}^{-1}) \times \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1} \rightarrow J^1X^v, \quad (g_1, g_2) \mapsto g_1g_2(\hat{v}w)B^+.
\]

**Lemma 2.7.** Let \( v \in W_{J_1} \) and \( w \in W_{J_2} \cap J^1W \). We have
\[
U_{P_{J_1}^+}^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = (U_{J_1}^+ \cap \hat{v}U_{J_2}^{-} \hat{v}^{-1}) \odot \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1} \\
\quad \odot \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1}.
\]

**Proof.** Note that \( \ell(vw) = \ell(v)+\ell(w) \). We recall again the following decompositions from [16, Theorem 5.2.3]:
\[
(\hat{v}w)U^{-}(\hat{v}w)^{-1} = (U^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}) \odot (U^- \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}); \quad (\diamondsuit 1)
\]
\[
U^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = \hat{v}(U^+ \cap \hat{w}U^{-\hat{w}^{-1}})\hat{v}^{-1} \odot (U^+ \cap \hat{v}U^{-\hat{v}^{-1}}). \quad (\diamondsuit 3)
\]

Thanks to the Levi decomposition \( P_{J_1}^+ = L_{J_1} \ltimes U_{P_{J_1}^+} \), the decompositions above are compatible with the restriction from \( U^\pm \) to \( U^\pm_{J_1} \) as well as from \( U^+ \) to \( U^+_p \).

It follows from (\( \diamondsuit 1 \)) that
\[
U_{J_1}^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = (U_{J_1}^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}) \odot (U_{P_{J_1}^-} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1}).
\]

Since \( w \in W_{J_2} \cap J^1W \) and \( v \in W_{J_1} \), it follows from (\( \diamondsuit 3 \)) that
\[
U_{J_1}^+ \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = U_{J_1}^+ \cap \hat{v}U^{-\hat{v}^{-1}} = U_{J_1}^+ \cap \hat{v}U_{J_2}^{-\hat{v}^{-1}}\hat{v}^{-1}.
\]

Since \( v \in W_{J_1} \), we have
\[
U_{P_{J_1}^+} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = \hat{v}(U_{P_{J_1}^+} \cap \hat{w}U^{-\hat{w}^{-1}})\hat{v}^{-1}.
\]

Thanks to the Levi decomposition of \( P_{J_2}^- \), we further have
\[
U_{P_{J_1}^-} \cap (\hat{v}w)U^{-}(\hat{v}w)^{-1} = \hat{v}(U_{P_{J_1}^-} \cap \hat{w}U^{-\hat{w}^{-1}})\hat{v}^{-1} \quad = \hat{v}(U_{P_{J_1}^-} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1} \odot \hat{v}(U_{P_{J_1}^-} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1} \quad = \hat{v}(U_{P_{J_1}^-} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1} \odot \hat{v}(U_{P_{J_1}^-} \cap \hat{w}U_{J_2}^{-\hat{w}^{-1}})\hat{v}^{-1}.
\]

The lemma is proved. \( \square \)
2.5. Product of Parabolic subgroups. Let $J_1, J_2 \subset I$. We study the decomposition of $P^+_J P^+_J / B^+$ with respect to the $J_1$-Schubert cells and the opposite $J_1$-Schubert cells.

We first consider the decomposition into the $J_1$-Schubert cells.

**Proposition 2.8.** Let $J_1, J_2 \subset I$. Then we have

$$P^+_J P^+_J = L_{J_1} L_{J_2} B^+ = \bigcup_{\tilde{w} \in W_{J_1} W_{J_2}} J_1^B \cdot \tilde{w} B^+.$$ 

**Proof.** It follows from (2.3) that $\bigcup_{\tilde{w} \in W_{J_1} W_{J_2}} J_1^B \cdot \tilde{w} B^+$ is a disjoint union. We have

$$L_{J_2} B^+ = \bigcup_{w \in W_{J_2} \cap J_1 W} (L_{J_2} \cap P^+_{J_1}) \cdot \tilde{w} B^+$$

Thus

$$L_{J_1} L_{J_2} B^+ = \bigcup_{w \in W_{J_2} \cap J_1 W} L_{J_1} (L_{J_2} \cap U^+_{J_1}) \cdot \tilde{w} B^+$$

$$\subseteq \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1 W} (L_{J_1} \cap U^-) \cdot \tilde{w} (L_{J_1} \cap U^+)(L_{J_2} \cap U^+_{J_1}) \cdot \tilde{w} B^+$$

$$\subseteq \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1 W} (L_{J_1} \cap U^-) \cdot \tilde{w} U^+_{J_1} (L_{J_1} \cap U^+)(\tilde{w}^{-1} (L_{J_1} \cap U^+)) B^+$$

$$\subseteq \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1 W} J_1^B \cdot \tilde{v} \tilde{w} B^+.$$ 

Now we proceed with the reverse inclusion. Note that any element in $W_{J_1} W_{J_2}$ can be written in a unique way as $vw$ for some $v \in W_{J_1}$ and $w \in W_{J_2} \cap J_1 W$. By definition and the Levi decomposition (2.1), $J_1^B = (L_{J_1} \cap U^-) U^+_{J_1} T$. Since $v \in W_{J_1}$, the conjugation action of $\tilde{v}$ stabilizes $U^+_{J_1} T$. Thus

$$J_1^B \cdot \tilde{v} \tilde{w} B^+ = (L_{J_1} \cap U^-) \cdot \tilde{w} (U^+_{J_1} \cdot \tilde{w} B^+) = (L_{J_1} \cap U^-) \cdot \tilde{w} (U^+_{J_1} \cap \tilde{w} B^- \cdot \tilde{w}^{-1}) \cdot \tilde{w} B^+.$$ 

Since $w \in W_{J_2}$, we have $U^+_{J_1} \cap \tilde{w} B^- \cdot \tilde{w}^{-1} \subseteq U^+_{J_1} \cap L_{J_2}$. Thus

$$J_1^B \cdot \tilde{v} \tilde{w} B^+ \subset (L_{J_1} \cap U^-) \cdot \tilde{w} (L_{J_2} \cap U^+ B^+) \subset L_{J_1} L_{J_2} B^+.$$ 

The statement is proved. 

We then consider the decomposition into opposite $J_1$-Schubert cells.

**Proposition 2.9.** We have

$$P^+_J P^+_J = \bigcup_{\tilde{w} \in W_{J_1} W_{J_2}} J_1^B \cdot \tilde{w} B^+ \cap P^+_J P^+_J.$$

Proof. Note that \( P_{J_1}^- \cap L_{J_2} \) is an opposite parabolic subgroup of \( L_{J_2} \). By the Birkhoff decomposition of \( L_{J_2} \) and the Levi decomposition of \( P_{J_2}^+ \), we have

\[
P_{J_2}^+ = \bigcup_{w \in W_{J_2} \cap J_1} (P_{J_1}^- \cap L_{J_2})w B^+ = \bigcup_{w \in W_{J_2} \cap J_1} (L_{J_1} \cap L_{J_2})(U_{J_2}^- \cap U_{P_{J_1}^-})w B^+.
\]

Hence

\[
P_{J_1}^+ P_{J_2}^+ = \bigcup_{w \in W_{J_2} \cap J_1} L_{J_1} (U_{J_2}^- \cap U_{P_{J_1}^-})w B^+
\]

\[
= \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1} U_{J_1}^+ \dot{v} U_{J_1}^- (U_{J_2}^- \cap U_{P_{J_1}^-})w B^+
\]

\[
= \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1} (U_{J_1}^+ \cap \dot{v} U_{J_1}^- \dot{v}^{-1}) \dot{v} U_{J_1}^+ (U_{J_2}^- \cap U_{P_{J_1}^-})w B^+.
\]

We show that

(a) \( U_{J_1}^+ (U_{J_2}^- \cap U_{P_{J_1}^-})w B^+ = (U_{J_2}^- \cap U_{P_{J_1}^-})w B^+ \).

We have the decomposition \( U_{J_1}^+ = (U_{J_1}^+ \cap U_{J_2}^-)(U_{J_1}^+ \cap U_{P_{J_1}^-}) \). Since \( (U_{J_2}^- \cap U_{P_{J_1}^-}) \dot{w} \subset L_{J_2} \), we have \( U_{P_{J_1}^-} (U_{J_2}^- \cap U_{P_{J_1}^-}) \dot{w} = (U_{J_2}^- \cap U_{P_{J_1}^-})w U_{P_{J_1}^-} \) and

\[
(U_{J_1}^+ \cap U_{J_2}^-)(U_{J_1}^+ \cap U_{P_{J_1}^-})(U_{J_2}^- \cap U_{P_{J_1}^-})w B^+ = (U_{J_1}^+ \cap U_{J_2}^-)(U_{J_2}^- \cap U_{P_{J_1}^-})w B^+.
\]

Thanks to the Levi decomposition of \( L_{J_2} \cap P_{J_1}^- \), we further have

\[
(U_{J_1}^+ \cap U_{J_2}^-)(U_{J_1}^+ \cap U_{P_{J_1}^-}) = (U_{J_2}^- \cap U_{P_{J_1}^-})(U_{J_1}^+ \cap U_{J_2}^-).
\]

Since \( w \in J_1 W \), we have \( (U_{J_1}^+ \cap U_{J_2}^-)w B^+ = \dot{w} B^+ \).

Thus (a) is proved.

Now we have

\[
P_{J_1}^+ P_{J_2}^+ = \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1} (U_{J_1}^+ \cap \dot{v} U_{J_1}^- \dot{v}^{-1}) \dot{v} (U_{J_2}^+ \cap U_{P_{J_1}^-})w B^+. \tag{2.5}
\]

Since \( \dot{v} (U_{J_1}^- \cap U_{P_{J_1}^-}) \subset U_{P_{J_1}^-} \dot{v} \), we have

\[
(U_{J_1}^+ \cap \dot{v} U_{J_1}^- \dot{v}^{-1}) \dot{v} (U_{J_2}^+ \cap U_{P_{J_1}^-})w B^+ \subset J_1 B^- \dot{\dot{v}} \dot{\dot{w}} B^+.
\]

Thanks to (2.3), we have

\[
P_{J_1}^+ P_{J_2}^+ \subset \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1} J_1 B^- \dot{\dot{v}} \dot{\dot{w}} B^+.
\]

Then we have that \( (U_{J_1}^+ \cap \dot{v} U_{J_1}^- \dot{v}^{-1}) \dot{v} (U_{J_2}^+ \cap U_{P_{J_1}^-})w B^+ = J_1 B^- \dot{\dot{v}} \dot{\dot{w}} B^+ \cap P_{J_1}^+ P_{J_2}^+ \)

and

\[
P_{J_1}^+ P_{J_2}^+ = \bigcup_{v \in W_{J_1}, w \in W_{J_2} \cap J_1} (J_1 B^- \dot{\dot{v}} \dot{\dot{w}} B^+ \cap P_{J_1}^+ P_{J_2}^+).
\]

Since \( W_{J_1} W_{J_2} = W_{J_1} (W_{J_2} \cap J_1 W) \), we have \( P_{J_1}^+ P_{J_2}^+ = \bigcup_{v \in W_{J_1} W_{J_2}} J_1 B^- \dot{\dot{v}} \dot{\dot{w}} B^+ \cap P_{J_1}^+ P_{J_2}^+ \)

and the proposition is proved. \( \square \)

Combining Proposition 2.9 and Lemma 2.7, we have the following proposition.
Proposition 2.10. Let \( J_1, J_2 \subset I \) and \( v \in W_{J_1}, w \in W_{J_2} \cap J_1W \). We have an isomorphism
\[
(U_{J_1}^+ \cap vU_{J_2}^-) (\hat{v}^{-1}) \rightarrow J_1^*X_{vw} \cap P_{J_1}^+ P_{J_2}^+ / B^+.
\]

Proof. By Proposition 2.9 we have
\[
J_1^*X_{vw} \cap P_{J_1}^+ P_{J_2}^+ / B^+ = (U_{J_1}^+ \cap vU_{J_2}^-)(\hat{v}^{-1})(U_{J_2} \cap U_{J_1}) \hat{w}B^+
\]
\[
= (U_{J_1}^+ \cap vU_{J_2}^-)(\hat{v}^{-1})(U_{J_2} \cap U_{J_1}) \hat{w}U_{J_2}(\hat{v}^{-1})\hat{w}B^+.
\]

By Lemma 2.3 we have the isomorphism
\[
(U_{J_1}^+ \cap (\hat{v}w)U^- (\hat{v}w)^{-1}) \times (U_{J_1}^- \cap (\hat{v}w)U^- (\hat{v}w)^{-1}) \sim J_1^*X_{vw}.
\]

Thanks to Lemma 2.7 the lemma follows from the restriction of the above isomorphism to
\[
(U_{J_1}^+ \cap vU_{J_2}^-)(\hat{v}^{-1})(U_{J_2} \cap U_{J_1}) \hat{w}U_{J_2}(\hat{v}^{-1})\hat{w}B^+.
\]

\[
\blacksquare
\]

3. A Birkhoff-Bruhat atlas

3.1. Definitions. Let \( M \) be an ind-variety over \( k \). A stratification on \( M \) is a family of locally closed, finite dimensional subvarieties \( \{M^y\}_{y \in Y} \) indexed by a poset \( Y \) such that

- \( M = \sqcup_{y \in Y} \hat{M}_y \);
- For any \( y \in Y \), the Zariski closure \( M^y \) of \( \hat{M}_y \) equals \( \sqcup_{y^\prime \leq y} \hat{M}_{y^\prime} \).

Assume furthermore that the minimal strata in the stratification \( Y_{\text{min}} \) of \( M \) are points. A Birkhoff-Bruhat atlas on \((M, Y)\) is the following data:

1. an open covering for \( M \) consisting of open sets \( U_f \) around the minimal strata \( f \in Y \);
2. a (minimal) Kac-Moody group \( \hat{G} \) and a subset \( J \) of the set of simple roots of \( \hat{G} \);
3. for any minimal stratum \( f \in Y \), an embedding \( c_f \) from \( U_f \) into the flag variety \( \hat{B} \) of \( \hat{G} \) such that \( c_f(U_f \cap M_y) \) is an open \( J \)-Richardson variety of \( \hat{B} \) for any \( y \in Y \).

3.2. Partial flag varieties. Let \( K \subset I \) and \( \mathcal{P}_K = G/P_K^+ \) be the partial flag variety. Then we have the decomposition
\[
\mathcal{P}_K = \sqcup_{w \in W_K} B^+ \hat{w}P^+_K / P^+_K = \sqcup_{w \in W_K} B^- \hat{w}P^+_K / P^+_K.
\]

Let \( Q_K = \{(v, w) \in W \times W_K | v \leq w\} \). Define the partial order \( \preceq \) on \( Q_K \) as follows:
\[
(v', w') \preceq (v, w) \text{ if there exists } u \in W_K \text{ such that } v \leq v'u \leq w'u \leq w.
\]

For any \((v, w) \in Q_K\), set
\[
\Pi_{v,w} = \pi_K(\hat{R}_{v,w}) \text{ and } \Pi_{v,w} = \pi_K(R_{v,w}),
\]
where \( \pi_K : \mathcal{B} \rightarrow \mathcal{P}_K \) is the projection map. Then \( \Pi_{v,w} \) is the (Zariski) closure of \( \Pi_{v,w} \) in \( \mathcal{P}_K \). We call \( \Pi_{v,w} \) an open projected Richardson variety and \( \Pi_{v,w} \) a closed projected Richardson variety.
By \cite{[11]} Proposition 3.6, we have
\[ \mathcal{P}_K = \bigsqcup_{(v,w) \in Q_K} \Pi_{v,w} \text{ and } \Pi_{v,w} = \bigsqcup_{(v',w') \in Q_K; (v',w') \leq (v,w)} \Pi_{v',w'}. \quad (3.1) \]

Let \( K \subset I \). The goal of the rest of this section is devoted to construct an Birkhoff-Bruhat atlas for the stratified space \( M = \mathcal{P}_K \) with the stratification \( \{ \Pi_{v,w} \}_{(v,w) \in Q_K} \), considered in \[\text{3.2}\].

### 3.3. The Kac-Moody group \( \tilde{G} \)

We construct the set of simple roots and the associated generalized Cartan matrix of the Kac-Moody \( \tilde{G} \) from the original Kac-Moody group \( G \). We list some examples of such construction in \[\text{3.2}\].

The set \( I \) of simple roots is the union of two copies of \( I \), glued along \( K \). More precisely, let \( \tilde{I} = \{ \breve{i} \mid \breve{i} \in I \} \) and \( I^\circ = \{ i \mid i \in I \} \) be the two copies of \( I \). Then \( \tilde{I} = \tilde{I} \cup I^\circ \) with \( \tilde{I} \cap I^\circ = \{ k^\circ = k^\breve{i} \mid k \in K \} \). For any \( i \in I \), we set \( (i^\circ)^\circ = (i^\breve{i})^\circ = i \).

The generalized Cartan matrix \( \tilde{A} = (\tilde{a}_{i,i'}^\circ)_{i,i' \in \tilde{I}} \) is defined as follows:

- for \( \breve{i}, \breve{i}' \in \tilde{I} \), \( \tilde{a}_{\breve{i},\breve{i}'}^\circ = a_{\breve{i},\breve{i}'}^\circ \);
- for \( \breve{i} \in \tilde{I} - I^\circ \) and \( \breve{i}' \in \tilde{I} - I^\circ \), \( \tilde{a}_{\breve{i},\breve{i}'}^\circ = 0 \).

Since \( I^\circ \cap I^\circ = \{ k^\circ = k^\breve{i} \mid k \in K \} \), we have \( \tilde{a}_{k^\circ,(k')^\circ} = a_{k,k'} = \tilde{a}_{k^\breve{i},(k')^\breve{i}} \) and the generalized Cartan matrix \( \tilde{A} \) is well-defined. The generalized Cartan matrix \( \tilde{A} \) is symmetrizable.

Let \( \tilde{G}^\text{min} \) be the minimal Kac-Moody group of simply connected type associated to \( (I,A) \) and \( \tilde{W} \) is its Weyl group. Let \( \tilde{W}_P \) and \( \tilde{W}_I \) be the parabolic subgroup of \( \tilde{W} \) generated by simple reflections in \( I^\circ \) and \( I^\circ \) respectively. We have natural identifications \( W \to \tilde{W}_P, w \mapsto w^\circ \) and \( W \to \tilde{W}_I, w \mapsto w^\breve{i} \). For \( w \in W_K, w^\circ = w^\breve{i} \).

Similarly, we have natural embedding \( \tilde{G}^\text{min} \to \tilde{L}_P, g \mapsto g^\circ \) and \( \tilde{G}^\text{min} \to \tilde{L}_I, g \mapsto g^\breve{i} \).

We denote by \( \tilde{X} \) the flag variety of \( \tilde{G}^\text{min} \), and denote by \( \breve{P}^\circ \tilde{X}^-, \breve{P}^\circ \tilde{X}^-, \breve{P}^\circ \tilde{R}^-, \) the \( \breve{P}^\circ \)-Schubert cells, the opposite \( \breve{P}^\circ \)-Schubert cells and the open \( \breve{P}^\circ \)-Richardson variety respectively.

### 3.4. The Kac-Moody group in a Birkhoff-Bruhat atlas

Let \( r \in W \). The following multiplication maps are isomorphisms of ind-varieties:
\[
\begin{align*}
(\breve{r}U\breve{r}^{-1} \cap U^+) \times (\breve{r}U\breve{r}^{-1} \cap U^-) & \to \breve{r}U\breve{r}^{-1}, \quad (g_1, g_2) \mapsto g_1 g_2; \quad (3.2) \\
(\breve{r}U\breve{r}^{-1} \cap U^-) \times (\breve{r}U\breve{r}^{-1} \cap U^+) & \to \breve{r}U\breve{r}^{-1}, \quad (h_1, h_2) \mapsto h_1 h_2. \quad (3.3)
\end{align*}
\]

We define morphisms of ind-varieties
\[ \sigma_{r,-} : \breve{r}U\breve{r}^{-1} \to \breve{r}U\breve{r}^{-1} \cap U^-, \quad g_1 g_2 \mapsto g_2; \]
and
\[ \sigma_{r,+} : \breve{r}U\breve{r}^{-1} \to \breve{r}U\breve{r}^{-1} \cap U^+, \quad h_1 h_2 \mapsto h_2. \]

**Lemma 3.1.** \cite{[11]} Proposition 8.2 \ Let \( r \in W \). The map
\[ \sigma_r = (\sigma_{r,+}, \sigma_{r,-}) : \breve{r}U\breve{r}^{-1} \to (U^+ \cap \breve{r}U\breve{r}^{-1}) \times (U^- \cap \breve{r}U\breve{r}^{-1}) \]
is an isomorphism of ind-varieties.
Note that the isomorphism is compatible with Levi decompositions. The restriction of \( \sigma_r \) gives the isomorphism

\[
\tilde{r}U_{P_K}^{-1} \longrightarrow (U^+ \cap \tilde{r}U_{P_K}^{-1}) \times (U^- \cap \tilde{r}U_{P_K}^{-1}).
\]

Let \( r \in W^K \) and \( U_r = \tilde{r}B^- P_K^+ / P_K^+ \subset \mathcal{P}_K \). We have an isomorphism

\[
\tilde{r}U_{P_K}^{-1} \longrightarrow U_r, \quad g \mapsto g \tilde{r}P_K^+/P_K^+.
\]

Finally, for \( g \in \tilde{r}U^{- \tilde{r}} \), we write \( \sigma_{r,\pm}(g)^{\frac{1}{2}} \) for \( \sigma_{r,\pm}(g) \).

**Theorem 3.2.** For \( r \in W^K \), we define the map

\[
\hat{c}_r : U_r \longrightarrow \hat{X},
\]

\[
g \tilde{r}P_K^+/P_K^+ \longmapsto \sigma_{r,+}(g)^{\frac{1}{2}} \cdot \tilde{r}^\flat \cdot \sigma_{r,-}(g)^{-\frac{1}{2}} \cdot \tilde{B}^+ / \hat{B}^+ \text{ for } g \in \tilde{r}U_{P_K}^{-1}.
\]

Then \((\hat{c}_r)_{r \in W^K}\) gives a Birkhoff-Bruhat atlas for \( \mathcal{P}_K \).

**Proof.** Since \( r \in W^K \), we have \( U^+ \cap \tilde{r}U_{P_K}^{-1} = U^+ \cap \tilde{r}U^- \).

Therefore \( \sigma_{r,+}(U_{P_K}) = (U^+)^\flat \cdot \tilde{r}^\flat : (U^-)^\flat \cdot \tilde{B}^+/\hat{B}^+ \). On the other hand, we have

\[
\tilde{r}^\flat (r^{-1})^\flat \cdot \tilde{r}^\flat (r^{-1})^\flat \cdot \tilde{r}^\flat (r^{-1})^\flat = r^\flat \left((r^{-1})^\flat (U^-)^\flat \cap (U_{P_K}^+)^\flat \right) (r^{-1})^\flat
\]

\[
= r^\flat \left((r^{-1})^\flat (U^-)^\flat \cap \tilde{U}_B^- \cap \tilde{U}^{-1}\right) (r^{-1})^\flat.
\]

By Proposition 2.10, \( \hat{c}_r \) is an embedding with image \( \tilde{r} \hat{L}_{P(r)} \cap \tilde{L}_P \hat{B}^+ / \hat{B}^+ \).

We then check the stratifications. Let \((v,w) \in Q_K\). Suppose that \( g \in \tilde{r}U_{P_K}^{-1} \) with \( g \tilde{r}P_K^+/P_K^+ \in \hat{P}_{v,w} \). Then there exists \( l \in L_J \) such that \( g \tilde{r}l \in B^+ \tilde{w}B^+ \cap B^- \tilde{v}B^+ \). By (3.2) and (3.3), we have

\[
\sigma_{r,+}(g) \tilde{r}l \in B^- \tilde{g} \tilde{r}l \subset B^- \tilde{v}B^+ \quad \text{and} \quad \sigma_{r,-}(g) \tilde{r}l \in B^+ \tilde{g} \tilde{r}l \subset B^+ \tilde{w}B^+.
\]

Therefore

\[
\sigma_{r,+}(g)^{\frac{1}{2}} \cdot \tilde{r}^\flat (r^{-1})^\flat \cdot \tilde{r}^\flat (r^{-1})^\flat = (\sigma_{r,+}(g)^{\frac{1}{2}}) \cdot ((\sigma_{r,-}(g)^{\frac{1}{2}})^{-1})^\flat
\]

\[
= (\sigma_{r,+}(g) \tilde{r}l)^{\frac{1}{2}} \cdot ((\sigma_{r,-}(g) \tilde{r}l)^{-1})^\flat
\]

\[
\in (U^-)^\flat \cdot \tilde{v}^\flat \cdot (B^+)^\flat \cdot (\tilde{w}^{-1})^\flat \cdot (B^+)^\flat
\]

\[
\subset (U^-)^\flat \tilde{v}^\flat \cdot \tilde{U}^+_P \cap \tilde{U}^-_{P \cap H} \cdot (\tilde{w}^{-1})^\flat \cdot (B^+)^\flat
\]

\[
= P^- B^+ \tilde{v}^\flat \cdot (\tilde{w}^{-1})^\flat \cdot (B^+)^\flat = P^- \tilde{X}_{(v,w)},
\]

where \((\bigcirc)\) follows from

\[
\tilde{v}^\flat \tilde{U}^+_P \cap \tilde{U}^-_{P \cap H} = \tilde{U}^+_P \cap \tilde{U}^-_{P \cap H} \quad \text{and} \quad \tilde{U}^+_P \cap \tilde{U}^-_{P \cap H} = (\tilde{w}^{-1})^\flat \tilde{U}^+_P \cap \tilde{U}^-_{P \cap H}.
\]

By Proposition 2.9

\[
\hat{c}_r(U_r) = \cup_{(v,w) \in Q_J} \hat{c}_r(U_r \cap \hat{P}_{v,w}) \subset \cup_{(v,w) \in Q_J} P^- \tilde{X}_{(r^{-1})^\flat \cdot \tilde{v}^\flat \cdot (w^{-1})^\flat}
\]

\[
\subset P^- \tilde{X}_{(r,r)} \cap \tilde{L}_P \tilde{L}_H \hat{B}^+ / \hat{B}^+.
\]
Since $\tilde{c}_r(U_r) = p^{\tilde{X}_{\hat{\nu}(r,r)}} \cap \tilde{L}_{L I} \tilde{B}^+ / \tilde{B}^+$, all the inclusion above are actually equalities. In particular, for any $(v, w) \in Q_J$, $\tilde{c}_r(U_r \cap \tilde{N}_{v,w}) = p^{\tilde{R}_{r,\nu(w^{-1})}}$. The theorem is proved. □

4. Combinatorial atlas

In this section, we assume $W$ is an arbitrary Coxeter group and we discuss a combinatorial analog of the geometric “atlas model” in Section 3.

4.1. Posets. Let $Q$ be a poset with partial order $\leq$. For any $x, y \in Q$, let $[x,y] = \{ z \in Q \mid x \leq z \leq y \}$ be the interval from $x$ to $y$. The covering relation is denote by $\rhd$. In other words, $y \rhd x$ if $[x,y] = \{x, y\}$. For any $x, y \in Q$ with $x \leq y$, a maximal chain from $x$ to $y$ is a finite sequence of elements $y = w_0 \rhd w_1 \rhd \cdots \rhd w_n = x$ for some $n \in \mathbb{N}$ and $w_0, w_1, \ldots, w_n \in Q$. The number $n$ is called the length of the chain. Note that the maximal chain may not exists in general.

We say that $Q$ is pure if for any $x, y \in Q$ with $x \leq y$, the maximal chains from $x$ to $y$ always exist and have the same length. Such length is also called the length of the interval $[x,y]$. A pure poset $Q$ is called thin if every interval of length 2 has exactly 4 elements, i.e. has exactly two elements between $x$ and $y$.

A subset $C \subset Q$ is called convex if for any $x, y \in C$, we have $[x,y] \subset C$.

4.2. EL-Shellability. Now we recall the notion of EL-shellability introduced by Bjorner in [2].

Suppose that the poset $Q$ is pure. An edge labeling of $Q$ is a map $\lambda$ from the set of all covering relations in $Q$ to a poset $\Lambda$. The labeling $\lambda$ sends any maximal chain of an interval of $Q$ to a tuple of $\Lambda$. A maximal chain is called increasing if the associated tuple of $\Lambda$ is increasing. The edge labeling $\lambda$ also allows one to order the maximal chains of any interval of $Q$ by ordering the corresponding tuples lexicographically.

An edge labeling of $Q$ is called EL-shellable if for every interval, there exists a unique increasing maximal chain, and all the other maximal chains of this interval are less than this maximal chain (with respect to the lexicographical order).

For a poset $Q$, we define the augmented poset $\hat{Q} = Q \sqcup \{\hat{0}\}$, where $\hat{0}$ is the minimal element of $\hat{Q}$.

Thinness and EL-shellability are important combinatorial properties of posets. For example, Bjorner proved in [3] that if a finite poset $\hat{Q}$ is thin and EL-shellable, then it is the face poset of some regular CW complex homeomorphic to a sphere.

The main result of this section is the following.

**Theorem 4.1.** The poset $\hat{Q}_K$ is thin and EL-shellable.

The strategy of the proof is as follows: we first establish a combinatorial atlas model for the poset $Q_K$; then we prove $Q_K$ is thin in §4.7; we finally prove $Q_K$ is EL-shellable in §4.9.

The case where $W$ is a finite Weyl group was first established by Williams in [21, Theorem 1 & Theorem 2]. Another proof for $Q_K$ when $W$ is a finite Weyl group was given in [8]. Our approach to handle $Q_K$ are different. As to the augmented element $\hat{0}$, we follow a similar idea as in [21].
4.3. Combinatorial atlas. Let $K \subseteq I$. Recall $Q_K = \{(v, w) \in W \times W^K \mid v \leq w\}$ equipped with the partial order $\leq$ defined by

$$(v', w') \leq (v, w)$$

if there exists $u \in W_K$ such that $v \leq v'u \leq w'u \leq w$.

The construction of $\tilde{W}$ in §3.3 remains valid for arbitrary Coxeter group $W$.

**Proposition 4.2.** Define the map

$$\nu : Q_K \to \tilde{W}, (v, w) \mapsto (v)^{\pm}(w^{-1})^x.$$ Then

1. the map $\nu$ induces an isomorphism of the posets $(Q_K, \leq)$ and $(\tilde{W}, {\leq})$;
2. $\nu(Q_K) = \{\tilde{w} \in \tilde{W}_p \mid w^\pm x^r \leq \tilde{w}, \text{for some } r \in W^K\}$;
3. $\nu(Q_K)$ is a convex subset of the poset $(\tilde{W}, {\leq})$.

We shall first give a geometric proof of the proposition in §4.4 when $W$ is a Weyl group of some Kac-Moody group $G$. We then give a combinatorial proof of the proposition in §4.6 for an arbitrary Coxeter group $W$.

4.4. Geometric proof of Proposition 4.2. In this subsection, we assume that $W$ is a Weyl group of some Kac-Moody group $G$. We deduce Proposition 4.2 for such $W$ as a consequence of Theorem 3.2.

Recall 4.2 that the poset $(Q_K, \leq)$ is the index set of the stratification of $P_K$ into the unions of projected Richardson varieties. By (2.3), the poset $(\tilde{W}, {\leq})$ is the index set of the stratification of $\tilde{X}$ into the unions of the $P^\pm$-Schubert varieties.

Proposition 4.2 (1) follows directly from Theorem 3.2.

We show Proposition 4.2 (2). By Proposition 2.4, we have that $c_r(U_r) = \bigcup_{\tilde{w} \in \tilde{W}_p \tilde{W}_r} P^\pm \tilde{X}_{\tilde{w}(r, x)} \cap P^\pm \tilde{X}_{\tilde{w}}$

and for any $\tilde{w} \in \tilde{W}_p \tilde{W}_r$ with $x^r \leq \tilde{w}, P^\pm \tilde{X}_{\tilde{w}(r, x)} \cap P^\pm \tilde{X}_{\tilde{w}} \neq \emptyset$. Hence $\nu(\{(v, w) \in Q_K \mid U_r \cap \tilde{W}_{v, w} \neq \emptyset\}) = \{\tilde{w} \in \tilde{W}_p \tilde{W}_r \mid x^r \leq \tilde{w}\}$.

We then show Proposition 4.2 (3). Let $x, y \in \tilde{W}(Q_K)$ with $x^r \leq \tilde{w}, P^\pm \tilde{X}_{\tilde{w}(r, x)} \cap P^\pm \tilde{X}_{\tilde{w}} \neq \emptyset$. Hence $z \in \tilde{W}(Q_K)$.

Proposition 4.2 then follows for such $W$.

4.5. The partial order $J \leq$. Let $W$ be an arbitrary Coxeter group and $J$ be a subset of the set of simple reflections in $W$. In this subsection, we give an equivalent description of the partial order $J \leq$ defined in §2.3.

**Lemma 4.3.** Let $x \in W_J$ and $y \in W^J$. The partial order $J \leq$ is generated by

(a) $s_\beta^{-1}x^r \leq y^r$ for $\beta \in \Delta_J^r$ with $x < s_\beta x$;
(b) $x^r s_\beta^{-1} \leq y^r$ for $\beta' \in \Delta_J^r$ with $s_\beta^{-1} \leq y^r$.
Proof. Recall $\Psi_J = \Delta_J^r e^- \cup (\Delta_J^r e^+ - \Delta_J^r e^+)$. Let $\beta \in \Psi_J$ with $s_\beta y^{-1} j \leq y x^{-1}$, hence $yx^{-1}(\beta) \in \Delta_J^r e^-$. If $\beta \in \Delta_J^r e^-$, then $x^{-1}(\beta) \in \Delta_J^r e^-$. Since $y \in W^J$, $yx^{-1}(\beta) \in \Delta_J^r e^-$ is equivalent to $x^{-1}(\beta) \in \Delta_J^r e^-$. In this case, we equivalently have $s_\beta x > x$.

If $\beta \in \Delta_J^r e^+ - \Delta_J^r e^+$, then $x^{-1}(\beta) \in \Delta_J^r e^+ - \Delta_J^r e^+$. Set $\beta' = x^{-1}(\beta)$. Then $y(\beta') \in \Delta_J^r e^-$. In this case, $s_\beta y^{-1} < y^{-1}$. On the other hand, if $s_\beta y^{-1} < y^{-1}$ for some $\beta' \in \Delta_J^r e^+$, then we must have $\beta' \notin \Delta_J^r e^+$ since $y \in W^J$. □

**Corollary 4.4.** Let $x \in W_J$ and $y \in W^J$. The partial order $\overset{\lesssim}{J} \leq$ is generated by
(a') $x_1 y_1^{-1} j \lesssim x y^{-1}$ for $x_1 \in W_J$ with $x < x_1$.
(b') $x_1 y_1^{-1} j \lesssim x y^{-1}$ for $y_1 \in W$ with $y_1 < y$.

**Lemma 4.5.** [9, Lemma A.3] Let $x, x', u \in W$ such that $x \leq x'u$. Then there exists $u' \leq u$, such that $x(u')^{-1} \leq x'$.

**Proof.** In the notations of [9, Lemma A.3], Lemma 4.5 is equivalent to

$x \triangleright u^{-1} \leq x'$ if and only if $x \leq x' * u$.

The lemma follows. □

**Proposition 4.6.** Let $x, x' \in W_J$ and $y, y' \in W^J$. The following conditions are equivalent:
(1) $x'(y')^{-1} j \leq xy^{-1}$;
(2) There exists $u \in W_J$ such that $x \leq x'u$ and $y'u \leq y$.

**Proof.** It suffices to replace condition (1) with the generating relations (a') and (b') considered in Corollary 4.4.

Note that Corollary 4.4 (a') $\Rightarrow$ (2) is trivial. For Corollary 4.4 (b'), we write $y_1$ as $y'u$ for $y' \in W^J$ and $u \in W_J$. Then $xy_1^{-1} = (xu^{-1})(y')^{-1}$. And we have $(xu^{-1})u = x$ and $y'u \leq y$. So (1) $\Rightarrow$ (2) is proved.

Now we show that (2) $\Rightarrow$ (1). Suppose that there exists $u \in W_J$ such that $x \leq x'u$ and $y'u \leq y$. By Lemma 4.5, there exists $u' \leq u$ such that $x(u')^{-1} \leq x'$. Since $y' \in W^J$, $y'u' \leq y'u \leq y$. By Corollary 4.4, we have

$x'(y')^{-1} j \leq x(u')^{-1}(y')^{-1} = x(y'u')^{-1} j \leq xy^{-1}$.

This finishes the proof. □

4.6. **Combinatorial proof of Proposition 4.2.** We show (1). For $w \in W^K$, we have $w^{-1} \in K W$ and hence $(w^{-1})^\sharp \in \breve{W}$. It is easy to see that $\breve{v}$ is injective. We prove the compatibility of the partial orders. Let

$$(v, w), (v', w') \in Q_K.$$

By Proposition 4.6 $(v')^\sharp((w')^{-1})^\sharp \leq (v)^\sharp(w^{-1})^\sharp$ if and only if there exists $z \in \breve{W}$, such that $(v)^\sharp \leq (v')^\sharp z$ and $z^{-1}((w')^{-1})^\sharp \leq (w^{-1})^\sharp$.

Note that $((w')^{-1})^\sharp, (w^{-1})^\sharp \in \breve{W}$. Thus $z^{-1}((w')^{-1})^\sharp \leq (w^{-1})^\sharp$ implies that $z \in \breve{W} \cap \breve{W}_p$. In this case, $z^\sharp \in W_K$. Hence

$$v^\sharp \leq (v')^\sharp z \iff v \leq v' z^\sharp,$$

$$z^{-1}((w')^{-1})^\sharp \leq (w^{-1})^\sharp \iff w' z^\sharp \leq w.$$

So $(v')^\sharp((w')^{-1})^\sharp \leq (v)^\sharp(w^{-1})^\sharp$ if and only if $(v', w') \leq (v, w)$. 

We show (2). For any \((v, w) \in Q_K\), we have \((w, w) \preceq (v, w)\). Therefore we have \(w^p(w^{-1})^q \preceq \hat{\nu}(v, w)\) for \(w \in W^K\). Hence
\[
\hat{\nu}(Q_K) \subseteq \{\tilde{w} \in \tilde{W}_p \tilde{W}_t : r^p(r^{-1})^q \preceq \tilde{w} \text{ for some } r \in W^K\}.
\]

Let \(\tilde{w} \in \tilde{W}_p \tilde{W}_t\) with \(r^p(r^{-1})^q \preceq \tilde{w}\) for some \(r \in W^K\). We write \(\tilde{w} = xy^{-1}\) with \(x \in \tilde{W}_p\) and \(y \in \tilde{W}_t\). By Proposition 4.6, there exists \(u \in W_p\) such that \(x \preceq u^p u \preceq y\). Therefore we must have \(u \in \tilde{W}_p \cap \tilde{W}_t\), that is, \(\tilde{u} \in W_K\). So \(x^i \preceq ru^i \preceq y^i\). Hence we have \((x^i, y^i) \in Q_K\) and \(\hat{\nu}((x^i, y^i)) = \tilde{w}\). This shows that
\[
\hat{\nu}(Q_K) \subseteq \{\tilde{w} \in \tilde{W}_p \tilde{W}_t : r^p(r^{-1})^q \preceq \tilde{w} \text{ for some } r \in W^K\}.
\]

We show (3). By (2), it suffices to show the set \(\tilde{W}_p \tilde{W}_t\) is closed in \((\tilde{W}, \preceq)\). Then the argument in the last paragraph of § 4.4 applies. Let \(\tilde{w}^p \preceq xy^{-1}\) with \(x \in \tilde{W}_p\) and \(y \in \tilde{W}_t \cap \tilde{W}_t\). We write \(\tilde{w} = x'(y')^{-1}\) with \(x' \in \tilde{W}_p\) and \(y' \in \tilde{W}_t\). By Proposition 4.6, there exists \(u \in W_p\) such that \(x \preceq x' u\) and \(y' u \preceq y\). Then \(y' \preceq y\) and hence \(y' \in \tilde{W}_t\). So \(\tilde{w} \in \tilde{W}_p \tilde{W}_t\).

4.7. Thinness. Dyer proved in [5, Proposition 2.5] that the poset \((\tilde{W}, \preceq)\) is thin. We have shown in Proposition 4.2 that the poset \((Q_K, \preceq)\) can be identified with a convex subset of the poset \((\tilde{W}, \preceq)\). So \(Q_K\) is thin.

As to the rank 2 intervals involving \(\hat{0}\), one can follow the same proof (the last paragraph) as in [21, Proof of Theorem 1.1]. We shall omit the details here.

4.8. Reflection orders. In order to prove the EL-shellability, we first recall the reflection orders introduced by Dyer in [4, Definition 2.1]. Let \(T\) be the set of reflections in \(\tilde{W}\). A total order \(\preceq\) on \(T\) is called a reflection order if for any \(s, t \in T\), either \(s \prec t\) or \(t \prec s\). For any covering relation \(w > w'\), we label this edge by the reflection \(w(w')^{-1} \in T\).

Dyer proved in [5, Proposition 3.9] (taking \(I = \emptyset\) and \(J = S\) in loc. cit.) that any reflection order on \(T\) gives an EL-labelling on \((\tilde{W}, \preceq)\). In particular, the poset \((\tilde{W}, \preceq)\) is EL-shellable.

Now we take the reflection order \(\preceq\) on \(T\) such that \(\tilde{W}_p \cap T\) is a final section, i.e., for any \(t, t' \in T\) with \(t \in \tilde{W}_p\) and \(t' \notin \tilde{W}_p\), we have \(t' \prec t\). The existence of such reflection order was established by Dyer in [4, Proposition 2.3]. We define the augmented totally ordered set \(\hat{T} = T \cup \{\bot\}\) where \(a < \bot < b\) for any \(a \in T - \tilde{W}_p\) and \(b \in \tilde{W}_p \cap T\).

Recall we can identify \(Q_K\) with a convex subset of \(\tilde{W}\). The edge labelling on \(\tilde{W}\) defined above induces an edge labelling on \(Q_K\). For any edge of \(Q_K\) involving \(\hat{0}\), we label the edge by \(\bot \in \hat{T}\). This gives a labelling on all the edges in \(Q_K\).

4.9. EL-shellability. In this subsection, we show that the edge labelling on \(Q_K\) defined above is an EL-labelling. We have already seen that the edge labelling above on \((\tilde{W}, \preceq)\) is an EL-labelling.

Let \([x, y]\) be an interval in \(Q_K\).

We first consider the case \(x \neq \hat{0}\). By Proposition 4.2, the poset \(Q_K\) can be identified with a convex subset of \((\tilde{W}, \preceq)\). Then \(\hat{\nu}([x, y])\) is an interval in \(\tilde{W}\). The claim follows in this case.
Now we consider the interval $[0, y]$. Similar to the argument in the last two paragraphs of \cite{21}, for any $(v, w) \in Q_K$, any lexicographically minimal chain from $(v, w)$ to $0$ does not involve edges labelled by $\tilde{W}_\mu \cap T$ and must have $(r, r) \succ 0$ as the last two terms, where $r = \min(vW_{\mu})$. The claim for the interval $[0, (v, w)]$ in $\tilde{Q}_K$ now follows from the claim of the interval $[(r, r), (v, w)]$ in $(Q_k, \preceq)$ proved above.

4.10. Comparison with the work \cite{21}. It is worth pointing out that the edges labelled by $T \cap \tilde{W}_\mu$ and $\perp$ corresponds to the edges of type 2 and type 3 in the sense of \cite{21}. Corollary 6.4] respectively. Moreover, we may choose a reflection order on $T$ so that its restriction to $T \cap \tilde{W}_\mu$ matches with the order used for the type 2 edges in \cite{21}. However, it is not clear how to match $T \setminus \tilde{W}_\mu$ with the labeling on the type 1 edges in \cite{21}.

5. A different Birkhoff-Bruhat atlas for $W_K$ finite case

In this section we construct a different Birkhoff-Bruhat atlas for $P_K$ when $K$ is of finite type. Let $w_K$ be the longest element of the finite Weyl group $W_K$. We first construct the atlas group $\tilde{G}$. We list some examples of such construction in \cite{46}.

Let $\tilde{I}$ be the union of two copies of $I$, glued along the $(-w_K)$-graph automorphism of $K$. More precisely, let $I^\parallel$ and $I^\perp$ be the two copies of $I$. Then $\tilde{I} = I^\parallel \cup I^\perp$ with $I^\parallel \cap I^\perp = \{ j^\parallel; j \in K \} = \{ j^\perp; j \in K \}$. For $j, j^\parallel \in K$, $j^\parallel = (j^\parallel)^\perp$ if and only if $\alpha_{j^\parallel} = -w_K(\alpha_j)$. Define $\delta : I^\parallel \to I, (j^\parallel)^\parallel = i$ and $\delta' : I^\perp \to I, (j^\perp)^\parallel = i$. Then for $\tilde{i} \in I^\parallel \cap I^\perp$, $\tilde{j}^\parallel$ and $\tilde{j}^\perp$ are both contained in $K$ and $\alpha_{\tilde{j}^\parallel} = -w_K(\alpha_{\tilde{j}^\perp})$.

The generalized Cartan matrix $\tilde{A} = (\tilde{a}_{\tilde{i}, \tilde{j}}; \tilde{i}, \tilde{j} \in \tilde{I}$ is defined as follows:

- for $\tilde{i}, \tilde{j} \in I^\parallel, \tilde{a}_{\tilde{i}, \tilde{j}} = a_{\tilde{j}, (\tilde{j})^\parallel}$;
- for $\tilde{i}, \tilde{j} \in I^\perp, \tilde{a}_{\tilde{i}, \tilde{j}} = a_{\tilde{j}, (\tilde{j})^\parallel}$;
- for $\tilde{i} \in \tilde{I} - I^\parallel$ and $\tilde{j} \in \tilde{I} - I^\perp, \tilde{a}_{\tilde{i}, \tilde{j}} = \tilde{a}_{\tilde{j}, \tilde{i}} = 0$.

Note that for $j_1, j_1^\parallel, j_2, j_2^\parallel \in K$ with $\alpha_{j_1} = -w_K(\alpha_{j_1})$ and $\alpha_{j_2} = -w_K(\alpha_{j_2})$, we have $a_{j_1, j_2} = a_{j_2, j_1}$. Thus for $\tilde{i}, \tilde{j} \in I^\parallel \cap I^\perp$, we have $a_{\tilde{i}, (\tilde{j})^\parallel} = a_{\tilde{j}, (\tilde{i})^\parallel}$ and the generalized Cartan matrix $\tilde{A}$ is well-defined.

Let $\tilde{G}$ be the minimal Kac-Moody group of simply connected type associated to $(\tilde{I}, \tilde{A})$ and $\tilde{W}$ be its Weyl group. Let $\tilde{W}_\mu$ and $\tilde{W}_\perp$ be the parabolic subgroup of $\tilde{W}$ generated by simple reflections in $I^\parallel$ and $I^\perp$ respectively. We have natural identifications $W \to \tilde{W}_\mu, w \mapsto w^\parallel$ and $W \to \tilde{W}_\perp, w \mapsto w^\perp$. For $w \in \tilde{W}_\mu, w^\parallel = w_{\mu, \mu} w^\perp w_{\mu, \mu}^{-1}$. For any $w \in W$, we set $(w^\parallel)^3 = w$ and $(w^\perp)^3 = w$.

Similarly to Section \cite{46} we have natural embeddings $G \to \tilde{L}_\mu, g \to g^\parallel$ and $G \to \tilde{L}_\perp, g \to g^\perp$.

**Lemma 5.1.** Let $L_{K, \text{der}}$ be the derived group of $L_K$. Then for any $g \in L_{K, \text{der}}$,

\[ (g^\parallel)^{-1} = \Psi(\tilde{w}_K g \tilde{w}_K^{-1})^\parallel. \]

**Proof.** Since $g \in L_{K, \text{der}}$, it suffices to prove that for $i \in K$,

\[ ((x_{\alpha_i})(a)^\parallel)^{-1} = \Psi(\tilde{w}_K x_{\alpha_i}(a) \tilde{w}_K^{-1})^\parallel, \quad (y_{\alpha_i}(a))^\parallel^{-1} = \Psi(\tilde{w}_K y_{\alpha_i}(a) \tilde{w}_K^{-1})^\parallel. \]
We prove the first identity. The second identity is proved in the same way.

Let $j \in K$ with $w_K(\alpha_i) = -\alpha_j$. Then $(s_j^{-1}w_K)(\alpha_i) = \alpha_j$. It follows from \[19, Proposition 9.3.5\] and direct computations that

$$w_K:j \longmapsto s_j^{-1}w_K = s_j\delta_j^{-1}w_K.$$  

Then it follows from Corollary 2.6 that $\hat{\sigma}(\alpha_j) = \alpha_j$.

Proof. We have

$$\sigma_r: U_r \rightarrow \hat{X},$$

and

$$gU_\hat{P}_K \rightarrow \hat{P}_K \rightarrow \hat{P}_K,$$

for $g \in iU_{\hat{P}_K}$. Let $(v, w) \in Q_K$. Suppose that $g \in iU_{\hat{P}_K}$ with $gU_{\hat{P}_K} = U_{\hat{P}_K} \in \mathcal{P}_K$. Then there exists $l \in L_{K,der}$ such that $g \cdot l = b^+ \cdot b^+ \cap B^- \cdot vU_{\hat{P}_K}$. By (3.2) and (3.3), we have

$$\sigma_{\hat{r}^{-1}}(g) \cdot l \in B^- \cdot vU_{\hat{P}_K}, \quad \sigma_{\hat{r}^{-1}}(g) \cdot l \in B^+ \cdot vU_{\hat{P}_K}.\]  

Set $\hat{l} = \hat{w}_K^{-1} \hat{w}_K \in L_{K,der}$. Then $(\hat{l})^{\hat{r}} = \hat{\Psi}(\hat{w}_K \hat{l} \hat{w}_K^{-1}) = \hat{\Psi}(\hat{l})^\hat{r}$ by Lemma 5.1.

So

$$\sigma_{\hat{r}^{-1}}(g) \cdot l \in B^+ \cdot \hat{w}_K \cdot B^- \cdot \hat{w}_K.$$
By Proposition 2.8,

\[ \tilde{c}_r(U_r) = \bigcup_{(v, w) \in Q_K; v \leq w} \tilde{c}_r(U_r \cap \tilde{\Pi}_{v, w}) \subset \bigcup_{(v, w) \in Q_K; v \leq w} I^{\delta \hat{\chi}(\tilde{\nu})^\delta (v - 1)^\delta, (r, w, r)^\delta (r - 1)^\delta} \subset I^{\delta \hat{\chi}(\tilde{\nu})^\delta} . \]

Since \( \tilde{c}_r(U_r) = I^{\delta \hat{\chi}(\tilde{\nu})^\delta} \), all the inclusions above are actually equalities. In particular, we have

\[ \tilde{c}_r(U_r \cap \tilde{\Pi}_{v, w}) = I^{\delta \hat{\chi}(\tilde{\nu})^\delta (v - 1)^\delta, (r, w, r)^\delta (r - 1)^\delta}, \quad \text{for} \quad (v, w) \in Q_K. \]

The theorem is proved. \( \square \)

In the case where \( W \) is finite, the multiplication by \( \dot{w}_p \) interchanges the opposite Schubert cells with the \( I^{\delta} \)-Schubert cells, where \( w_{p}\delta \) is the longest element of \( W_p \).

Thus for \( W \) finite case, the “atlas model” constructed here is essentially the same as the Bruhat atlas constructed in [7]. They differ by the multiplication by \( \dot{w}_p \).

6. Some examples

| \( P_K \) | \( G \) | \( G \) | Bruhat atlas |
|---|---|---|---|
| \( \bigcirc \bigcirc \) | \( \bigtriangleup \bigtriangleup \) | \( \bigtriangleup \bigtriangleup \) | \( \bigtriangleup \bigtriangleup \) |
| | \( \nabla \nabla \) | \( \nabla \nabla \) | \( \nabla \nabla \) |
| | | | |
| \( \bigcirc \bigtimes \bigcirc \) | \( \bigtriangleup \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \) |
| | | | |
| \( \bigcirc \bigtimes \bigtimes \bigcirc \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) |
| | | | |
| \( \bigtimes \bigcirc \bigtimes \bigcirc \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \) |
| | | | |
| \( \bigtimes \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \) |
| | | | |
| \( \infty \bigtimes \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \) | N/A |
| | | | |
| \( \bigtimes \bigtimes \bigtimes \bigtimes \bigcirc \) | \( \bigtriangleup \bigtimes \bigtimes \bigtimes \bigtimes \nabla \) | N/A | N/A |
In this section, we provide some examples of the Dynkin diagrams of the atlas groups. Here $\tilde{G}$ and $\hat{G}$ are the atlas groups constructed in [3] and [5] respectively and “Bruhat atlas” refers to the one constructed in [7]. Note that even when $\tilde{G}$ and $\hat{G}$ are the same, the atlases constructed in [3] and [5] are totally different.

In the first column, we give the Dynkin diagram for the group $G$. The subset $K$ is the set of vertices filled with black color. In the other columns, the Dynkin diagrams consists of three types of vertices: △, the vertices in $(I - J)^2$; ♦, the vertices in $(I - J)^3$; ♣, the vertices in $I^2 \cap P = J^2 = P$.

We also would like to point out that in certain cases, the atlas groups $\tilde{G}$ and $\hat{G}$ constructed in section 3 and section 5 coincide. However, the Birkhoff-Bruhat atlases constructed there are still quite different in these cases, as one may see from the maps $\tilde{c}_r$ and $\hat{c}_r$ in Theorem 3.2 and Theorem 5.2 respectively.

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