Abstract

This paper addresses issues surrounding the concept of fractional quantum mechanics, related to lights propagation in inhomogeneous nonlinear media, specifically restricted to a so-called gravitational optics. Besides Schrödinger Newton equation, we have also concerned with linear and nonlinear Airy beam accelerations in flat and curved spaces and fractal photonics, related to nonlinear Schrödinger equation, where impact of the fractional Laplacian is discussed.

Another important feature of the gravitational optics’ implementation is its geometry with the paraxial approximation, when quantum mechanics, in particular, fractional quantum mechanics, is an effective description of optical effects. In this case, fractional-time differentiation reflexes this geometry effect as well.

Keywords: Parabolic equation approximation, Fractional Schrödinger equation, Fractional Heisenberg equation, Nonlinear Schrödinger equation, Schrödinger Newton equation, Airy beam
1. Introduction

In modern optical experiments of light propagation in nonlinear composite media a suitable description can be developed in the framework of effective quantum equations based on fractional integro-differentiation. This concept of differentiation of non-integer orders arises from works by Leibniz, Liouville, Riemann, Grunwald, and Letnikov, see e.g., Refs. [1, 2]. Its application relates to diffusion-wave processes with power law distributions. This corresponds to the absence of characteristic average values for processes exhibiting many scales in both space and time [3, 4]. This issue by itself is extremely wide and reflected in vast literature. In this concise review we discuss only few examples of specific aspects of fractional quantum mechanics exploring the light propagation in nonlinear inhomogeneous media, related to a so called gravitational optics setups [5]. Gravitational optics effects, considered here in the framework of fractional calculus, concern primary with linear and nonlinear Airy beam accelerations in flat and curved spaces [5, 6] and fractal photonics [7, 8]. Another important feature of the gravitational optics’ implementation is its geometry with the paraxial approximation, when an effective quantum mechanical description of the optical effects is valid [9, 10, 11].

As shown in the seminal papers [12, 13], the appearance of the space fractional derivatives in the Schrödinger equation is natural and relates to the path integral approach, where fractional concept has been introduced by means of the Feynman propagator for non-relativistic quantum mechanics by analogy with fractional Brownian motion [12, 13]. This analogy is natural, since both are Markov processes [14, 15] that indicates equivalence between the Laplace operators for classical diffusion equation and the Schrödinger equation. The same, the appearance of the space fractional derivatives in the Schrödinger equation is natural, since both the standard Schrödinger equation and the space fractional one obey the Markov chain rule. Technically, it is described by a fractional Laplacian in the form of the Riesz fractional derivative (see Appendix A)

\[
(-\Delta)^{\frac{\alpha}{2}} \equiv \frac{1}{2\cos \frac{\alpha \pi}{2}} \left[ \int_{-\infty}^{\alpha} D_{x}^{-\alpha} f(x) + \int_{-\infty}^{\alpha} D_{-x}^{-\alpha} f(-x) \right], \tag{1.1}
\]
where $\alpha \in (0, 2]$. This introduction of Lévy flights due to the power law kernel, as well as the Lévy measure in quantum mechanics is based on the generalization of the self-consistency condition, known as the Bachelor-Smoluchowski-Kolmogorov chain equation (or the Einstein-Smoluchowski-Kolmogorov-Chapman equation, see e.g., [3]), established for the Wiener process for the conditional probability $W(x, t|x_0, t_0)$

$$W(x, t|x_0, t_0) = \int_{-\infty}^{\infty} W(x, t|x', t')W(x', t'|x_0, t_0) \, dx' . \quad (1.2)$$

In the case of the translational symmetry, it reads $W(x, t|x_0, t_0) = W(x - x_0, t|t_0)$. Straightforward generalization of this expression by the Lévy process is expressed through the Fourier transform

$$W(x, t|x_0, t_0) = \int_{-\infty}^{\infty} e^{ik(x-x_0)}e^{-K_\alpha t|k|^\alpha} \, dk , \quad (1.3)$$

where $0 < \alpha \leq 2$ and $K_\alpha$ is a generalized diffusion coefficient [4]. Eventually, Eq. (1.3) in the form of a generalized Feynman-Kac formula [12, 16] results from the fractional space Schrödinger equation (FSSE), which in dimensionless form reads

$$i\hbar_{ef}\partial_t \psi(x, t) = (\hbar_{ef}^2/2)(-\Delta)^\alpha \psi(x, t) + V(x)\psi(x, t) , \quad (1.4)$$

where $V(x)$ is a potential field, while $K_\alpha$ is replaced by $(-i\hbar_{ef})^{\alpha-1}/2$ with $\hbar_{ef}$ being an effective dimensionless Planck constant.

Recently, nonlocal, fractional mechanics has been demonstrated experimentally [17]. This physical implementation of space-fractional Schrödinger equation is based on transverse light dynamics in aspherical optical resonators that realizes the fractional quantum harmonic oscillator [18] (also known as massless relativistic quantum oscillator [19]) in which dual Airy beams can be generated. Further “fractionalization” of optical beams attracts much attention in both theoretical and numerical studies of both linear [20] and nonlinear [21, 22, 23] Schrödinger equations. Another important achievement relates to the fabrication and exploration of optical [24, 25] and quantum fractals [20, 27]. Then the fractional Laplacian (1.1) plays important role in both quantum and wave-diffusion processes [27] (see Sec 2).
Fractional time Schrödinger equations (FTSE) are another large class of quantum problems with long memory effects. This kind of non-Markovian, non-unitary quantum dynamics has been observed in real physical systems. It also corresponds to non-Hermitian extension of quantum mechanics and attracts a growing interest in optics, see, e.g., and references therein. However, an introduction of this fractional time concept in quantum mechanics is not an easy task and needs a special care. This situation is reflected in recent reviews. A FTSE has been introduced by analogy with a fractional Fokker-Planck equation (FFPE) by means of the analytic continuation of time $t \to -it/h$, where $h$ is the Planck constant. This approach attracts much attention and has been extensively studied, see e.g., Refs. (and references therein). Another analytic continuation of time has been suggested as well by $t \to t/(ih)^{1/\beta}$ with $0 < \beta < 1$. At these definitions, a standard partial derivative of the wave function with respect to time is replaced by a so called Caputo fractional derivative, which is a convolution integral of the wave function with a power law kernel:

$$\partial_t^\beta \psi(t) \to \partial_t^\beta \psi(t) \equiv D_C^\beta \psi(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \frac{d}{d\tau} \psi(\tau) d\tau,$$

where $\Gamma(\beta + 1) = \beta \Gamma(\beta)$ is the gamma function, see Appendix A. Contrary to the space fractional concept of Eq. (1.4), the fractional time quantum mechanics is non-unitary. That is, it violates the Stone’s theorem on the one-parameter unitary group, and correspondingly the Feynman path integral approach does not exist.

A general formulation of a nonlinear fractional Schrödinger equation (nonlinear FSE) that envisages possible experimental realizations in optics can be

\[\text{\footnotesize{In each case the Planck constant } h \text{ has a special dimensionality definition. However, in this presentation we use an effective dimensionless Planck constant } h_{\text{eff}}, \text{ introduced above. Note also that this procedure for the dimensionless variables can be always performed, see Refs. 36, 42.}}\]

\[\text{\footnotesize{Stone’s theorem establishes a group property } \hat{U}(t)\hat{U}(s) = \hat{U}(t+s) \text{ for the evolution (unitary) operator } \hat{U}(t).}}\]
formulated as the generalization of Eqs. (1.4) and (1.5) as follows

\[ i\hbar\beta\partial_t\psi(x, t) = (\hbar^2/2)(-\Delta)^{\frac{1}{2}}\psi(x, t) + V(x, t)\psi(x, t) + \mathcal{B}\psi(x, t) \int Q(x - x')|\psi(x', t)|^2 dx'. \]  

(1.6)

Here \( \beta \in (0, 1] \), \( \alpha \in (0, 2] \), while \( \mathcal{B} \) is a nonlinearity parameter and \( Q(x - x') \) is a power law kernel. An effective potential \( V(x, t) \) results from the optical setup. When \( Q(x - x') = \delta(x - x') \), Eq. (1.6) is a fractional generalization of the nonlinear Schrödinger equation (NLSE). The latter is a well known equation with a variety of applications, including nonlinear optics, see recent review [45].

In optics, \( t \equiv z \in [0, \infty) \) in Eq. (1.6) is usually an effective time, which results from the paraxial description of the Helmholtz equation and corresponds to the longitudinal coordinates \( z \in [0, \infty) \), while \( x \in \mathbb{R}^n \) is position coordinates in the \( n \)-dimensional space.

2. FSE in slab geometry: parabolic equation approximation

As admitted above, the Schrödinger equation in optics appears as a formal effective description of diffusive wave transport in complex inhomogeneous media in the parabolic equation approximation that corresponds to the paraxial small angle approximation. Therefore, one can obtain the FSE as an effective way to solve fractional eigenvalue problem in slab geometry, where the paraxial small angle approximation is naturally applied. The method of parabolic equation approximation was first applied by Leontovich in study of radio-waves spreading [46] and later developed in detail by Khokhlov [47] (see also Refs. [48, 49]).

In the section, our main concern is the Helmholtz fractional equation, which relates to the Lévy process in the 2D slab geometry described by dimensionless \((x, z)\) variables, where \( z \in (0, \infty) \) and \( x \in [-L, L] \). Therefore the wave function \( \Psi(z, x, \omega) \) is determined by the fractional Helmholtz equation

\[ \partial_z^\alpha\Psi + D_{||}^\alpha\Psi + \omega\Psi = 0, \]  

(2.1)

where \( \omega \) is the propagating wave/heat frequency. It should be stressed that fractional space derivatives in Eq. (2.1) describe Lévy flights [4]. In particular,
we specify here optical ray dynamics in Lévy glasses, where Lévy flights can be described by Eq. (2.1). Another interesting phenomenon, which is described by Eq. (2.1), is superdiffusion of ultra-cold atoms in optical lattices.

We use the Caputo fractional derivative with respect to the longitudinal coordinate \( z \in [0, \infty) \), namely
\[
\partial_z^\alpha \Psi(z, x, \omega) \equiv D^\alpha_C z
\]
while the Riesz fractional derivative
\[
D^\alpha_{|x|} \Psi(z, x, \omega) = -L \partial^\alpha_x + \frac{RL}{x} \partial^\alpha_L x
\]
is used for the orthogonal direction \( x \in [-L, L] \). When the size \( L \) is less than the Lévy flight lengths in the longitudinal direction, the transport is of a small grazing angle with respect to the longitudinal direction. The solution \( \Psi(z, x, \omega) \) can be presented in the following multiplication form
\[
\Psi(z, x, \omega) = e^{ikz} \psi(z, x, \omega).
\] (2.2)

Substitution Eq. (2.1) in Eq. (2.2) yields the following integration
\[
\partial_z^\alpha \Psi(z, x, \omega) = \frac{1}{\Gamma(2-\alpha)} \int_0^z (z - z')^{2-\alpha-1} \left[ e^{ikz'} \psi(z', x, \omega) \right] dz'.
\] (2.3)

Note that the “initial” conditions at \( z = 0 \) for both \( \Psi \) and \( \psi \) are the same:
\[
\psi(z = 0) = \Psi(z = 0) = \Psi_0(x, \omega).
\]

Now the parabolic equation in the paraxial approximation can be obtained. Taking into account that \( \psi(z, x, \omega) \) is a slowly-varying function of \( z \), such that
\[
\left| \frac{\partial^2 \psi}{\partial z^2} \right| \ll \left| 2k \frac{\partial \psi}{\partial z} \right|,
\] (2.4)

Lévy glasses are specially prepared optical material in which the Lévy flights are controlled by the power law distribution of the step-length of a free ray dynamics, which can be specially chosen in the power law form \( \sim 1/l^{\alpha+1} \).

There are Lévy walks, and the theoretical explanation of this fact, presented within the standard semiclassical treatment of Sisyphus cooling, is based on a study of the microscopic characteristics of the atomic motion in optical lattices and recoil distributions resulting in macroscopic Lévy walks in space, such that the Lévy distribution of the flights depends on the lattice potential depth. The flight times and velocities of atoms are coupled, and these relations, established in asymptotically logarithmic potential, have been studied for different regimes of the atomic dynamics, so the cold atom problem is a variant of the Lévy walks, see also discussion of the Lévy flights in the framework of a superdiffusive comb model.
one obtains

\[
\frac{d^2}{dz^2} [e^{ikz} \psi(z, x, \omega)] \approx 2ike^{ikz} \frac{d}{dz} \psi(z, x, \omega). 
\] (2.5)

After substitution of this approximation in Eq. (2.3), one obtains from Eqs. (2.1), (2.2), and (2.3)

\[
2ik_0 f_2^{2-\alpha} [e^{ikz} \partial_z \psi(z, x, \omega)] + \mathcal{D}_x^{\alpha} \psi(z, x, \omega)e^{ikz} + \omega \psi(z, x, \omega)e^{ikz} = 0. 
\] (2.6)

To remove the exponential \(e^{ikz}\) from Eq. (2.6), we insert it inside the derivative \(e^{ikz} \partial_z \psi(z, x, \omega) = \partial_z [e^{ikz} \psi(z, x, \omega)] + o(k)\). The term \(o(k) = ike^{ikz} \psi(z, x, \omega)\) can be neglected in Eq. (2.6) since it is of the order of \(O(k^2)\), which must be neglected for \(k \ll 1\). Note also that \(\alpha' \neq \alpha\) with \((0 < \alpha' \leq 2)\), then \(\beta = \alpha - 1\) and \(0 < \beta < 1\). In general case, the order of the Caputo fractional derivative in Eq. (2.1) can be \(\alpha' \neq \alpha\) with \((0 < \alpha' \leq 2)\), then \(\beta = \alpha - 1\) does not relate to \(\alpha\).

The Laplace transform can be performed: \(L[e^{ikz} \psi(z)] = \tilde{\psi}(s - ik)\). Therefore, one obtains from Eq. (2.6)

\[
2ik[s^\beta \tilde{\psi}(s - ik) - s^\beta - 1 \psi(z = 0)] + \mathcal{D}_x^{\alpha} \tilde{\psi}(s - ik) + \omega \tilde{\psi}(s - ik). 
\] (2.7)

Performing the shift \(s - ik \rightarrow s\) and neglecting again the terms of the order of \(o(k)\) in Eq. (2.7), and then performing the Laplace inversion, one obtains the Helmholtz equation in the form of the effective fractional Schrödinger equation (FSE), where the \(z\) coordinate plays the role of an effective time

\[
2ik \partial_z^\beta \psi + \mathcal{D}_x^{\alpha} \psi + \omega \psi = 0. 
\] (2.8)

The “initial” condition at \(z = 0\) corresponds to the boundary condition for the initial problem in Eq. (2.1). If one supposes that there is a source of the signal at \(z = 0\), then the initial condition is \(\psi(z = 0, x) = \Psi_0(x, 0)\). The boundary conditions at \(x = \pm L\) are \(\psi(x = \pm L, z) = 0\).

3. Fractional space Schrödinger equation

Taking \(\beta = 1\) in the generalized FSE (1.6), we consider various realizations of the light propagation in nonlinear composite materials describing the phenomenon in the framework of the fractional NLSE, including Airy beams with a
metric determinant different from unite. This so-called gravitational optics can be also realized in the form of a Schrödinger - Newton equation. Another realization of Eq. (1.6) corresponds to the NLSE in random optical potential. An extension method, which is relevant to the quantum field theory consideration is discussed as well.

3.1. Airy-Fox beams

Experimental advances in Airy beam acceleration has been demonstrated in curved optical spaces in both linear and nonlinear optics with predesigned refractive index varying so as to create curved space environment for light, see an extended discussion in Ref. [6]. Therefore, describing this process in the framework of the generalized FSE (1.6), the latter according to Refs. [6, 23] becomes the fractional NLSE as follows

\[ i\hbar \partial_t \psi(x,t) = \left(\frac{\hbar\epsilon}{2g(t)}\right)^\alpha (\Delta)^{\alpha/2} \psi(x,t) - \frac{B}{g(t)} |\psi(x,t)|^2 \psi(x,t). \]  

(3.1)

Here \( g = g(t) \) is a metric determinant (determinant of a metric tensor) that reflects a curved optical space. Potential \( V(t) \) in Eq. (1.6) is a function of time and can be omitted due a gauge transformation \( \psi(x,t) \leftrightarrow e^{-(i/\hbar\epsilon) \int_0^t V(\tau)d\tau} \psi(x,t) \). The nonlinear term is due to the Kerr effect and the effective nonlinear coefficient now is \( B \rightarrow -B/g(t) \) (see Eq. (16) in Ref. [6]). This nonlinear equation stands for numerical investigations [22, 6]. For \( \alpha = 2 \), the solution of the linear part of Eq. (3.1) corresponds to the solution in the form of an accelerating Airy beam, which propagates along the curve \( x = g_1^2(t) \) at the condition \( |\psi(x,0)| = |\psi(x - g_1^2(t), t)| \). This curve is determined by the metric tensor according to \( g_1 = g_1(t) = \frac{1}{x_{at}} \int_0^t g^{-1}(t')dt' \). This longstanding problem attracts much attentions since the seminal result [56]. In particular, the linear Airy beam solution [6] (for \( \alpha = 2 \)) in the form of the Fourier transformation reads

\[ \hat{\psi}_{Ai}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{\pi} \int_0^\infty dke^{i\kappa x/3 + i\kappa X(x,t)}e^{i\epsilon_1(t)X(x,t)} \]

\[ = 2 \exp\left\{ i[k - g_1^2(t)]^3/3 - ikg_1^2(t) \right\}, \]  

(3.2)

where \( X(x,t) \equiv x - g_1^2(t) \).
The linear part of the fractional NLSE (3.1)

\[ i\hbar_{\text{ef}} \partial_t \psi(x, t) = \left( \frac{\hbar_{\text{ef}}}{2g(t)} \right)^\alpha (-\Delta)^\frac{\alpha}{2} \psi(x, t) \]  

(3.3)
does not correspond to the Airy beam. For \( \alpha \neq 2 \) the Green’s function is expressed in the form of the Fox \( H \)-function as follows

\[
G(x, t) = \frac{2}{\pi} \int_0^\infty \cos(kx) \exp\left[-\frac{i}{2} \frac{\hbar_{\text{ef}}^\alpha}{g(t)} |k|^\alpha \right] \text{d}k
\]

\[ = \frac{2}{\pi} \int_0^\infty \cos(kx) H_{0,1}^{1,0} \left[ \frac{\hbar_{\text{ef}}^\alpha}{2} |k|^\alpha g_1(t) \right] \text{d}k
\]

\[ = \frac{2}{x} H_{2,1}^{1,1} \left[ \frac{2}{g_1(t) \hbar_{\text{ef}}^\alpha} |x|^\alpha \right] \right|_{(0, 1)}^{(1, 1), (1, \frac{\alpha}{2}), (1, \alpha), (1, 0), (1, \frac{\alpha}{2})}
\]

(3.4)

where \( \hat{\psi}_{\text{Fox}}(k, t) = 2 \exp[-(i\frac{\hbar_{\text{ef}}^\alpha}{2})g_1(t)|k|^\alpha] \) is the Fourier image of the Fox beam (3.3), see Appendix A.

It is instructive to consider the dynamics (3.3) for the initial Airy function.

\[ \psi(x, 0) = Ai(\alpha x/\hbar_{\text{ef}}^{2/3}) \],

(3.5)

where \( a > 0 \) is arbitrary. For simplicity sake, we take \( a = \hbar_{\text{ef}}^{2/3} \). Therefore, we have that the solution to (3.3) is

\[ \psi_{\text{lin}}(x, t) = \int e^{ikx} e^{-\left(i\frac{\hbar_{\text{ef}}^\alpha}{2}g_1(t)|k|^\alpha\right)} \text{d}k = \int G(x - x', t) Ai(x') \text{d}x'.
\]

(3.6)

Let us consider the asymptotic solution for \( t \to \infty \). If along the time, \( g_1(t) \) increases, then the argument of the Fox \( H \)-function in Eq. (3.4) tends to zero. The Fox \( H \)-function satisfies the asymptotic theorem for small arguments \[ 57 \] that yields \( H_{m,n}^{p,q}(z) = O(|z|) \) for \( |z| \to 0 \). In our case, \( c = 1/\alpha \). Therefore, the wave function is independent of \( x \) and is a function of time only in the large time asymptotics. In particular, in the flat space, when \( g = 1 \), the wave function is

\[ \psi_{\text{lin}}(x, t) \sim 2(1 - i)[g_1(t)]^{-\frac{\alpha}{2}} = 2(1 - i)t^{-\frac{\alpha}{2}}.
\]

(3.7)

Then the nonlinear term in Eq. (3.1) is small enough for a perturbation theory consideration.
3.2. Lévy flights in NLSE

Disregarding the effective potential \( V(x,t) \) in Eq. (1.6) and considering the kernel \( Q = \delta(x-x') \), we arrive at the fractional NLSE

\[
i \hbar \alpha \partial_t \psi(x,t) = \left( \frac{\hbar^2 \alpha}{2} \right) (\Delta)^{\frac{\alpha}{2}} \psi(x,t) + B|\psi(x,t)|^2 \psi(x,t). \tag{3.8}
\]

Fractional NLSE attracts much attention in both physical and mathematical literature. Among the vast literature we only mention that it has been appeared in the form of the continuous limit of the dynamics of nonlinear lattices with long range interactions \([58, 59]\) and it also stems from the fractional generalization of the Ginzburg-Landau model \([60, 61, 62, 63, 64]\). The latter case corresponds to the fractional analogy with the free energy expansion

\[
F = F_0 + \int \left[ |\nabla^\alpha \psi(x)|^2 + |\psi|^2 + \frac{1}{2} |\psi|^4 \right] dx \tag{3.9}
\]

that results from the power law of the order parameter \( x^{-\alpha} \) for the coexisting nonlocal symmetry \([61]\).

3.2.1. Extension method

Another elegant way introducing the fractional Laplacian has been suggested in mathematical literature \([65]\) as a fractional generalization of an extension method for the square root of the Laplacian, \((\Delta)^{\frac{1}{2}} f(x)\), or half Laplacian\(^5\). As admitted in Ref. \([66]\), it is the “revolutionary” result which demonstrates that the fractional Laplacian can be expressed via the Dirichlet to Neumann map associated with a particular extension problem. Namely, it has been shown that the fractional Laplacian \((\Delta)^{\frac{\alpha}{2}} f(x)\) can be related to solutions of an extension problem as follows \([65]\). For a function \( f : \mathbb{R}^n \to \mathbb{R} \), there is the extension \( u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) that satisfies the equation

\[
\Delta_x u + ay^{-1} \partial_y u + \partial_y^2 u = 0, \tag{3.10}
\]

\(^5\)In Ref. \([65]\), the fractional Laplace operator is defined in the following regularized form

\[
(\Delta)^{\alpha/2} f(x) = \frac{1}{\omega_{n-\alpha}} \int \frac{(x-f(\xi))}{|x-\xi|^{n+\alpha}} d\xi = \frac{1}{\omega_{n-1,\alpha}} \int \frac{f'(\xi)}{|x-\xi|^{n-1+\alpha}} d\xi.
\]
with the Dirichlet boundary condition \( u(x, 0) = f(x) \). Here \( a = 1 - \alpha \) and \( x \equiv \mathbf{x} \) is a \( n \) dimensional vector in \( \mathbb{R}^n \). Then the fractional Laplacian is defined as the Neumann boundary condition to the extension problem (3.10), namely it reads

\[
C_{n,\alpha}(-\Delta_x)\Phi f = -\lim_{y\to 0} y^\alpha \partial_y u = \frac{1}{\alpha} \lim_{y\to 0} \frac{u(x, y) - u(x, 0)}{y^\alpha},
\]

where \( C_{n,\alpha} \) is a constant multiplier.

Let us consider half Laplacian. When \( a = 0 \), the stationary NLSE with the half Laplacian reads \[67, 68\]

\[
(-\Delta)^{1/2} u = w(u) + eu \quad \text{with} \quad u(x) > 0 \quad \text{in the domain} \quad \Omega \in \mathbb{R}^m \quad \text{and} \quad u = 0 \quad \text{at the boundary} \quad \partial \Omega, \quad \text{and} \quad w(u) \equiv u^3, \quad \text{in our case.}
\]

The extension problem in a half cylinder \( \Omega \times [0, \infty) \) is \( \Delta v = 0 \) (or \(-\Delta_x v = \partial_y^2 v\)) with \( v(x, y) > 0 \), and the boundary conditions now are \( v = 0 \) at \( \partial \Omega \times [0, \infty) \) and \( \partial_l v(x, \tau)|_{\tau=0} = w(v)|_{\tau=0} \). Here \( l \) is the unit outer normal to \( \Omega \times 0 \).

The fractional, half Laplacian is determined by the extension problem. Applying twice the operator \( T : u \to -\partial_y v(x, 0) \), one obtains for the harmonic function and its derivative: \( T \circ T u = \partial_y^2 v|_{\tau=0} = -\Delta_x v|_{\tau=0} = -\Delta u \). Therefore \( T = (-\Delta)^{1/2} \) is the half Laplacian and in the same time it is the Neumann boundary condition \(-\partial_y v(x, 0)\) for the extension problem.

### 3.2.2. Fractional Laplacian and wave equation in curved space

Let us discuss Eq. (3.10) with respect to the experimental setup in curved optical spaces of Sec. 3.1. To this end we simplify the consideration for the electric field function \( \psi(x, t) \) in Eq. (3.3) and consider it as a harmonic function, which satisfies the wave equation (3.10) as follows

\[
\partial_t^2 \psi(x, t) + \frac{1}{g(t)} \Delta_x \psi(x, t) = 0.
\]

The initial condition, as the Dirichlet boundary condition, is \( \psi(x, 0) = f(x) \). The metric determinant is chosen in the form \( g(t) = t^{1-\alpha} \). Zero boundary conditions are taken at \( x = \pm \infty \). Performing the change of the variables \( \tau = (\tau/\alpha) \) and \( \partial_t \psi = \tau^{1-\alpha} \partial_\tau \psi \), one obtains the Euler-Lagrange equation (3.10)

\[
\Delta_x \psi + \frac{1-\alpha}{\tau} \partial_\tau \psi + \partial_\tau^2 \psi = 0,
\]
which can be produced from the functional (Lagrangian) $\int |\nabla \psi|^2 \tau^{1-\alpha} dx \, dr$, as well [65].

As obtained in Ref. [65], the solution to Eq. (3.12) (or Eq. (3.10)) has the power law form when $n - 1 + a > 1$, which is determined by a Poisson formula

$$\psi(x, \tau) = \int_{\mathbb{R}^n} G(x - x', \tau)f(x')dx'.$$

The Poisson kernel, or the Green’s function reads

$$G(x, \tau) = \frac{C_{n, \alpha}}{|x|^2 + \alpha^2 \tau^{2\alpha}}^{(n+\alpha)},$$

where the constant $C_{n, \alpha} = C_{n+\alpha}$ is $C_k = \pi^{k/2} \Gamma(k/2 - 1)/4$. The Poisson formula (3.14) establishes also the relation between the solution $\psi(x, \tau)$ and the fractional Laplacian in the form of the Neumann boundary condition [65], which reads

$$\partial_{\tau} \psi(x, \tau)|_{\tau=0} = \lim_{\tau \to 0} \frac{\psi(x, \tau) - \psi(x, 0)}{\tau} = -C_{n, \alpha}(-\Delta_x)^{\frac{\alpha}{2}} f(x).$$

### 3.2.3. Fractional NLSE in random potential

Now we account the effective potential $V$ in Eq. (1.6) in the form of a random potential, which describes a space disorder: $V = V(x)$. Then the fractional NLSE (3.8) reads

$$i\hbar_{\text{ef}} \partial_t \psi(x, t) = \left(\hbar_{\text{ef}}^\alpha/2\right)(-\Delta_x)^{\frac{\alpha}{2}} \psi(x, t) + B|\psi(x, t)|^2 \psi(x, t) + V(x) \psi(x, t)$$

$$\equiv \hat{H}(\alpha)\psi + B|\psi|^2 \psi.$$  (3.17)

Here $(-\Delta_x)^{\frac{\alpha}{2}} \equiv \mathcal{D}_{|x|}^{\alpha}$ is defined in $\mathbb{R}$. For the random potential $V = V(x)$, $x \in (-\infty, +\infty)$, Anderson localization takes place for both $\alpha = 2$ [69, 70] and $\alpha < 2$ [66] for the linear case ($B = 0$). For $\alpha = 2$, the situation is well studied and as is well known, the system is described by the exponentially localized Anderson modes (AM)s: $\hat{H}(\alpha = 2)\Psi_{\omega_k}(x) = \omega_k \Psi_{\omega_k}(x)$, where $\Psi_{\omega_k} \equiv \Psi_k(x)$ are real functions and the eigenspectrum $\omega_k$ is discrete and dense [70]. This problem of Eq. (3.17) is relevant to experiments in nonlinear optics, for example disordered photonic lattices [8, 71], where Anderson localization was found in...
the presence of nonlinear effects ($\alpha = 2$), as well as it relates to experiments on Bose-Einstein condensates in disordered optical lattices (see e.g., Refs. 72, 73).

A typical example of fractional dynamics in optics is realized in a competition between localization and nonlinearity that leads to anomalous transport with a transport exponent $1/3$ observed numerically 74 and analytically 75, 76 (see also discussions in Refs. 77, 78 and references therein).

Since the Hamiltonian $\hat{H}(\alpha)$ is a self-adjoint operator for all $\alpha \in (0, 2]$, the AMs $\Psi_\alpha^\alpha(x) \equiv \Psi_k(x)$ are the complete set of eigenfunctions (not obligatory real). Here we also suppose that the random potential is large enough that the spectrum is discrete. Then one projects Eq. (3.17) on the basis of the AMs

$$\psi(x, t) = \sum_{\omega_k} C_{\omega_k}(t) \Psi_{\omega_k}(x) = \sum_k C_k(t) \Psi_k(x)$$

(3.18)

and obtains a system of equations for coefficients of the expansion, $C_k$

$$i\partial_t C_k = \omega_k C_k + B \sum_{k_1, k_2, k_3} A_{k, k_1, k_2, k_3} C_{k_1}^* C_{k_2} C_{k_3},$$

(3.19)

where $A_{k, k_1, k_2, k_3}$ is an overlapping integral of four AMs:

$$A_{k, k_1, k_2, k_3} = \int \Psi_k^*(x) \Psi_{k_1}(x) \Psi_{k_2}(x) \Psi_{k_3}(x) dx.$$  

(3.20)

The initial condition for system of Eqs. (3.19) is $\psi_0(x) = \sum_k a_k \Psi_k(x)$. Equations (3.19) correspond to a system of interacting classical nonlinear oscillators with the Hamiltonian

$$H_{\text{osc}} = \sum_k \omega_k C_k^* C_k + \frac{B}{2} \sum_k A_k C_k^* C_{k_1} C_{k_2} C_{k_3},$$

(3.21)

where $k = (k_1, k_2, k_3, k_4)$.

For $\alpha = 2$, only the nearest neighbor oscillators are effectively interacted 76, 66, and the interaction part of the Hamiltonian (3.21) is

$$H_{\text{int}} = B \sum_k A_{k, k, k, k}^\pm C_k^* C_k C_{k-1}^* C_{k+1}.$$  

(3.22)

$^6$Delocalized states have not been observed yet, and this issue stands for more detailed computational exploration in the framework of a discrete fractional Anderson model 66.
Then the transport in the chain of the nonlinear oscillators is relevant to subdiffusion on a fractal Cayley tree with the transport exponent being $1/3$ \cite{76}.

The situation changes dramatically for Anderson localization of the Lévy flights with $\alpha < 2$. As it follows from section 3.2.1, the Anderson Hamiltonian $\hat{H}(\alpha)$ acts in the two dimensional space that leads to the effective size of the AMs $\Psi_k(x)$ being essentially extended \cite{66}, and the number of effectively interacting oscillators can be large. This large scale interaction resembles a quantum relaxation process \cite{79}, where the transition rate is determined by the Fermi’s golden rule \cite{80}. Therefore, we follow the semiclassical qualitative estimation according to Ref. \cite{81} as follows.

Each nonlinear oscillator with the individual Hamiltonian

$$h_k = \omega_k C_k^* C_k + \frac{B}{2} A_k C_k^* C_k C_k$$ (3.22)

represents one nonlinear eigenstate in the system, identified by its wave number $k$, unperturbed frequency $\omega_k$, and nonlinear frequency shift $\Delta \omega_k = B A_k C_k^* C_k$, where $A_k \equiv A_{k,k,k,k}$. Non-diagonal elements $A_{k,k_1,k_2,k_3}$ characterize couplings between each of the four eigenstates with the wave numbers $k, k_1, k_2,$ and $k_3$. It is understood that the excitation of each eigenstate results from the spreading of the wave field in the wave number space. If the field spreads across a large number of states $\Delta k \gg 1$, then the conservation of the probability

$$\int |\psi(x,t)|^2 dx \sim \sum |C_k|^2 \sim \int |C_k|^2 d\Delta k = 1$$

implies that $|C_k|^2 \sim 1/\Delta k$ and $|\psi(x)|^2 \sim 1/\Delta x$. Note that for the localized states $1/\Delta x \sim 1/\Delta k$ \cite{75}. In the basis of AMs, the evolution of the amplitudes $C_k$ from Eq. (3.19) is controlled by the cubic nonlinearity: $\dot{C}_k \sim BC_k^* C_{k_1} C_{k_2}$ . The rate of excitation of the newly involved modes according to the Fermi’s golden rule is of the order $\frac{d|\psi|^2}{dt}$ and proportional to the cubic power of the probability density, $|\psi|^2$. Taking the conservation of the probability into account, one obtains that the rate of excitation is of the order $\sim (1/\Delta k)^3$. On the other hand, the number of the newly excited modes per unit time is $d\Delta k/dt$, making it possible to assess $d\Delta k/dt \sim 1/(\Delta k)^3$. This eventually yields subdiffusion
with the mean squared displacement \((\Delta x)^2 \sim (\Delta k)^2 \propto t^{\frac{1}{2}}\). This essential increasing of the transport exponent from \(1/3\) (when \(\alpha = 2\)) to \(1/2\) results from the fractional Anderson Hamiltonian \(\tilde{H}(\alpha)\) for \(\alpha < 2\). Note that this result for the transport exponent is corroborated to an experimental observation of the optically induced exciton transport in molecular crystals, which exhibits the intermediate asymptotic subdiffusion \([82]\) with the experimental transport exponent of the order of \(\sim 0.57\).

3.3. Lévy flights in Schrödinger - Newton equation

It has been shown that gravitational effects can be studied in optical experimental setup \([3]\), which can emulate general relativity, gravitational effects with optical wave packets under a long-range nonlocal thermal nonlinearity. The Helmholtz equation, which describes this experiment in paraxial regime, is reduced to the Schrödinger - Newton equation (SNE) \([3]\). The latter has been proposed to describe the gravitational self-interaction of quantum wave packets. In the form of coupling, the Schrödinger equation of quantum mechanics with the Poisson equation from Newtonian mechanics, the SNE reads

\[
\begin{align*}
  i\hbar\partial_t \psi &= -\frac{\hbar^2}{2m} \Delta \psi + U \psi, \\
  \Delta U &= 4\pi G m^2 |\psi|^2.
\end{align*}
\]

Here \(G\) is Newton’s constant, \(m\) is the mass of a quantum particle in the gravitational potential, which is determined by the density of the wave function \(\psi = \psi(r, t)\).

This Schrödinger - Newton equations has been introduced by Penrose in connection with the role of gravity in quantum state reduction (collapse of wave functions) citePenrose1. Another form of the SNE, introduced by Diósi \([84]\) and Penrose \([85]\) reads

\[
\begin{align*}
  i\hbar\partial_t \psi &= \left[ -\frac{\hbar^2}{2m} \Delta - G m^2 \int \frac{|\psi(r')|^2}{|r - r'|} d^3 r' \right] \psi.
\end{align*}
\]

Since the seminal results, these equations have been extensively studied, see Refs. \([86, 87]\) and references therein.
The fractional generalization of the Poisson Eq. (3.23b) for Newtonian gravity has been suggested recently as a novel approach for Galactic dynamics, considered in fractional Newtonian gravity, see discussion in Ref. [88]. Following this consideration of a fractional gravitational potential, we consider the fractional Laplacian for both Eqs. (3.23a) and (3.23b). To give a brief insight into the stability of the dynamics with respect to four Lévy waves processes, we simplify the consideration by reducing it to the one dimensional (1D) case of Eq. (3.24). Therefore, the 1D fractional SNE in the form of Eq. (1.6) reads as follows

\[
i\hbar e_f \partial_t \psi = \left[ \frac{\hbar^\alpha}{2m} (-\Delta_x)^{\alpha/2} - \frac{Gm^2}{\Gamma(-\nu)} \cos \left( \frac{\nu \pi}{2} \right) \int |\psi(x')|^2 dx' \right] \psi,\]

(3.25)

where \( \nu \in (0, 1) \). The last term in Eq. (3.25) is just the Riesz fractional derivative of the density of the wave function. The zero boundary conditions are taken at infinity. The initial conditions will be specified when necessary later. Performing the Fourier transformation of the equation and taking into account the Fourier presentation of the wave function

\[
\psi(x, t) = \frac{1}{2\pi} \int A(k, t) e^{ikx} dk,
\]

(3.26)

we have

\[
i\hbar e_f \dot{A}_k = \hbar^\alpha 2m |k|^\alpha A_k - Gm^2 |k|^{\nu} \int dk_1 dk_2 dk_3 A_{k_1} A_{k_2} A_{k_3} \delta(k_2 + k_3 - k - k_1).\]

(3.27)

where \( A_k(t) \equiv A(k, t) \). The initial conditions are such that at the moment \( t = 0 \) the only mode with fixed \( k = q \) is populated, then

\[
A_k(t = 0) = a_k \delta(k - q).
\]

(3.28)

The solution of Eq. (3.27) with the initial condition (3.28) reads

\[
A(k, t) = \begin{cases} 
\exp \left[ -i \frac{\hbar^\alpha}{2m} |q|^{\alpha} t + \frac{Gm^2}{\hbar e_f} |q|^{\nu} |a_q|^2 t \right] a_q & \text{if } k = q, \\
0 & \text{if } k \neq q.
\end{cases}
\]

(3.29)

In this case, the evolution of the wave function is due to the solution in Eq. (3.29), namely \( \psi(x, t) = A(q, t) e^{iqx} \).
3.3.1. Stability analysis

Let us investigate stability of the solution (3.29) with respect to decay in the nearest modes according to the resonant condition \(2q \to (q + p) + (q - p)\). In this case, at the initial moment all other modes with \(k \neq q\) are populated with amplitudes \(|a_k| \ll |a_q|\). In fact, the \(k\)'s amplitudes can be considered infinitesimally small. Then a perturbation theory can be applied to Eq. (3.27).

Taking into account only linear terms with respect to amplitudes \(A\), the expression

\[
S = \exp\left[\frac{i\hbar}{\epsilon} \sum a_\alpha \mathcal{O}_\alpha \right]
\]

is determined by equation (3.30) by introducing new variables:

\[
i \hbar \epsilon \dot{A}_q = \frac{\hbar^2}{2m} |q|^2 A_q - Gm^2 |q|^\nu A_q^2 A_q
\]
\[
i \hbar \epsilon \dot{A}_{q+p} = \frac{\hbar^2}{2m} |q + p|^\alpha A_{q+p} - 2Gm^2 |q + p|^\nu A_{q+p}^2 A_{q+p} - Gm^2 |q + p|^\nu A_q^2 A_{q+p}
\]
\[
i \hbar \epsilon \dot{A}_{q-p} = \frac{\hbar^2}{2m} |q - p|^\alpha A_{q-p} - 2Gm^2 |q - p|^\nu A_{q-p}^2 A_{q-p} - Gm^2 |q - p|^\nu A_q^2 A_{q+p}
\]

(3.30)

It follows from Eqs. (3.30a) that the dynamics of the “large” wave (3.29) is not changed in the first order of the perturbation theory. In contrast, amplitudes of the “small” waves grow exponentially. To show this, we simplify the form of Eqs. (3.30) by introducing new variables:

\[
F = \frac{\hbar^{\alpha-1}}{2m} |q|^\alpha + \frac{Gm^2}{\hbar \epsilon} |q|^\nu \equiv \omega(q) + \Omega(q) I
\]
\[
F(\pm) = \frac{\hbar^{\alpha-1}}{2m} |q \pm p|^\alpha - 2 \frac{Gm^2}{\hbar \epsilon} |q \pm p|^\nu I \equiv \omega(\pm) - \Omega(\pm) I
\]
\[
\mathcal{F} = F - [F(+) + F(-)]/2, \quad \|A_q(t)\|^2 = |a_q|^2 = I,
\]
\[
A_{q \pm p}(t) \equiv A_\pm(t) = \exp[-i(F + \mathcal{F})t] C_\pm
\]

Then the exponential growth with the increment \(\lambda_p\) is determined by equation

\[
\begin{pmatrix}
\dot{C}_+ \\
\dot{C}_-
\end{pmatrix} =
\begin{pmatrix}
i\mathcal{F} & i\Omega(+) I \\
-i\Omega(-) I & -i\mathcal{F}
\end{pmatrix}
\begin{pmatrix}
C_+ \\
C_-
\end{pmatrix}.
\]

(3.31)

Solving the characteristic equation of the matrix and performing expansion of the expression \(|q + p|^\alpha + |q - p|^\alpha - 2|q|^\alpha \approx \alpha(\alpha - 1)|q|^{\alpha-2} p^2 / 2\), where \(p \ll q\) and \(|q|^\alpha \delta(q) = 0\) since \(q \neq 0\), one obtains the increment of the order of

\[
\lambda_p \sim |q|^\nu Gm^2 \sqrt{T^4 - |q|^{-\alpha-\nu}} + O(p^4)
\]

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Since, according to the normalization of the wave function $|a_q| \sim 1$, while $q > 1$, the real part of increment is always positive value. Therefore, if initially only few modes are populated, these modes are always unstable with respect to the four Lévy waves decay. In opposite case, when the initial condition corresponds to thermalisation and all $k$ modes are equally populated with $|a_k| \to 0$. Then the increment reads

$$\lambda_p \sim \pm ip^2 \frac{\hbar^{\alpha-1}}{4m} |(1 - \alpha)q^{\alpha-2}| \alpha$$

and is imaginary. Then all the modes are stable. In this case, the gravitational, nonlinear term is infinitesimally small and can be neglected. Then the wave function can be estimated in the terms of the Fox $H$-function,

$$\psi(x, t) = \frac{N}{2\pi N} \int_{-\infty}^{\infty} \exp \left[ ikx - i \frac{\hbar^{\alpha-1}}{2m} |k|^\alpha t \right] dk \approx \frac{N}{2\pi N} \int_0^\infty \cos(kx) H_{1,1}^{1,0} \left[ i \frac{\hbar^{\alpha-1}}{2m} k^{\alpha} t \right] \left[ (0, 1) \right] dk$$

$$= \frac{1}{x} \frac{N}{2N} H_{2,2}^{1,1} \left[ -i \frac{2m}{\hbar^{\alpha-1}} |x|^\alpha \left| (1, 1, \alpha, (1, 0), (1, \alpha) \right) \right] \left( (1, 1), (1, \frac{\alpha}{2}) \right) . \quad (3.32)$$

Here, $N \to \infty$ is a number of modes, and eventually it will be absorbed by normalization constant $N = 2\pi N$.

4. Fractional time Heisenberg equation vs FTSE

When parameter $\beta \neq 1$ in Eq. (1.6), the fractional time quantum mechanics is not Markovian \[33\] and nonunitary: it violates the Stone’s theorem on the one-parameter unitary group \[44\]. The Feynman path integral approach does not exist, and as a result, the equivalence between Schrödinger and Heisenberg pictures of the quantum mechanics is broken \[35\]. However, the fractional Heisenberg equation of motion (FHEM) can be obtained by standard quantization of classical equations of motion with memory \[35\] i.e. with fractional time derivative of Eq. (1.5). For better understanding some
arbitrariness in the formulation of the problem, we first, obtain a FTSE

\[ i\hbar \partial_t^\beta \psi(x,t) = \hat{H}\psi(x,t) , \]

where \( \hat{H} \) is the Hamiltonian operator for \( \beta = 1 \) only. Following Refs. \[42, 35\], we consider the FTSE (4.1) suggesting some additional arguments of its inferring.

4.1. Properties of the FTSE: Infinitesimal evolution

In Ref. \[42\], the FTSE (4.1) was obtained as a quantisation of the classical fractional dynamics in the framework of the path integral representation. However, this approach supposes the chain rule in the form of Eq. \[1.2\] together with the unitary dynamics of the Green’s functions. Although this approach looks controversial, it is not so if we avoid a construction of any path integral and consider the fractional dynamics on the infinitely small time scale \( \Delta t \to 0 \) that eventually has been performed in Ref. \[42\] and then in Ref. \[35\]. As is well known, the fractional evolution according to the FTSE is due to the Mittag-Leffler function, which reads from Eqs. (4.1) and \[A.17a\] as follows

\[ \psi(t + \Delta t) = E_\beta \left( -i\hat{H} \Delta t^\beta / \hbar \right) \psi(t) . \]

For the small argument it yields the stretched exponential behavior of the evolution operator on the infinitesimal time scale \( \Delta t \)

\[ \psi(t + \Delta t) = \hat{U} (t + \Delta t, t) \psi(t) = \exp \left[ \frac{-i\hat{H} \Delta t^\beta}{\hbar \Gamma(\beta + 1)} \right] \psi(t) = \]

\[ = \exp \left[ \frac{-i}{\hbar \epsilon \Gamma(\beta)} \int_t^{t+\Delta t} (t + \Delta t - \tau')^{\beta-1} \hat{H} \epsilon' d\tau' \right] \psi(t) . \]

Note, that for \( \Delta t \to 0 \), we perform fractional Taylor expansion, defined in Eqs. \[A.15\] and \[A.16\], for the l.h.s. of Eq. (4.3) and Taylor expansion for the r.h.s.

\[ ^7 \text{When} \beta \neq 1 \text{ the system contains also a specific relaxation and cannot be separated from the environment. Note that independently of} \beta, \text{ operator} \hat{H} \text{ is a well defined Hermitian operator, and in sequel, we consider it as the Hamiltonian operator.} \]
of the equation. This immediately yields the FTSE \[4.1\]. Note that fractional Taylor expansion for the l.h.s. of Eq. \[4.3\] destroys the Markov chain rule.

Now we can obtain the FTSE by means of the Green's function on an infinitesimal time scale. As it has been shown above, the exponential form of the infinitesimal Green's function can be used, and the infinitesimal evolution of the wave function reads

\[
\psi(x, t + \varepsilon) \equiv \langle x | \psi(t + \varepsilon) \rangle = \int_{-\infty}^{\infty} G(x, t + \varepsilon | y, t) \psi(y, t) dy, \quad (4.4)
\]

where the infinitesimal Green's function is determined by conjecture

\[
G(x, t + \varepsilon | y, t) = \frac{1}{A} \exp \left[ i S(\varepsilon, x, y)/\hbar_{\text{ef}} \right], \quad (4.5)
\]

according to the principal Hamilton function, or action \( S(\varepsilon, x, y) \). The latter is defined by the functional

\[
S[f] = \frac{1}{\Gamma(\beta)} \int_{t}^{t+\varepsilon} (t + \varepsilon - \tau)^{\beta-1} L(\partial_{\tau}^0 f(\tau), f(\tau)) d\tau \quad (4.6)
\]

with the boundary conditions \( f(t) = y \) and \( f(t + \varepsilon) = x \). The “fractional” action plays the role of a generating functional of the quantum evolution on the infinitesimal time scale, and as it follows from the fractional evolution operator \[4.3\], it has “fractional time metric”. The Lagrangian \( L = \frac{(\partial^{0} f)^{2}}{2} - V(f) \) corresponds to the classical counterpart of the FTSE with the Hamiltonian \( \hat{H} \). On the infinitesimal time scale \( \varepsilon < \Delta t \to 0 \) the potential \( V(f) \) contributes to the action \( S(\varepsilon, x, y) \) according to the middle point \[14\], which yields \( -V \left( \frac{x+y}{2} \right) \varepsilon^{\beta}/\Gamma(\beta+1) \).

Therefore, we find the fractional classical dynamics due to the kinetic part, which is the fractional integral

\[
S_{\varepsilon}[f] = \frac{1}{\Gamma(\beta)} \int_{t}^{t+\varepsilon} (t + \varepsilon - \tau)^{\beta-1} L(\partial_{\tau}^0 f(\tau)) d\tau,
\]

obtained in Appendix B Taking into account Eq. \[B.13\] for the Green’s function \[4.5\], we obtain the infinitesimal evolution of the wave function in the Feynman’s form (see chapter 4 in Ref. \[14\]), which reads

\[
\psi(x, t + \varepsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \exp \left[ \frac{i}{\hbar \Gamma(\beta+1)} \frac{m_{\beta}(x-y)^{2}}{2\varepsilon^{\beta}} \right] \times \exp \left[ -\frac{i\varepsilon^{\beta}}{\hbar_{\text{ef}} \Gamma(\beta+1)} V \left( \frac{x+y}{2} \right) \right] \psi(y, t) dy. \quad (4.7)
\]
Here
\[ A = \left( \frac{2\pi i \hbar \varepsilon \beta}{m_{\beta}/\Gamma(\beta + 1)} \right)^{\frac{1}{2}} \]
is defined in Eq. (B.13b) and \( m_{\beta} = \Gamma^2(\beta + 1) \) is an effective dimensionless mass due to the fractional time dynamics. Then introducing the infinitesimal shift \( \eta = y - x \) and following Feynman’s instruction, we make the change of the variable \( dy = d\eta \) then perform expansion and integration with respect to \( \eta \). These procedures result in the r.h.s. of Eq. (4.7) as follows
\[ \psi(x, t + \varepsilon) = \psi(x, t) + \varepsilon \frac{\hbar^2}{2m_{\beta}} \partial^2_x \psi(x, t) + O(\varepsilon^2). \]  
(4.8)

To relate the both sides of the equation with the same power \( \varepsilon^\beta \), we perform fractional Taylor expansion, which yields according to Eq. (A.16)
\[ \psi(x, t + \varepsilon) = \psi(x, t) + \frac{\varepsilon^\beta}{\Gamma(\beta + 1)} \partial^\beta_t \psi(x, t) + O(\varepsilon^{\beta+1}). \]  
(4.9)

Eventually, taking the limit \( \varepsilon = 0 \), we obtain the FTSE in Eq. (4.1) with the Hamiltonian operator \( \hat{H} = -\frac{\hbar^2}{2m_{\beta}} \partial^2_x + V(x) \).

4.2. Symmetrical form of FHEM

Since the fractional evolution is not unitary, the equivalence between Schrödinger and Heisenberg representations of fractional time quantum mechanics is broken. In this case, we have some arbitrariness in formulation of the FHEM. A possible convenient way of construction of the FHEM by analogy with the FTSE (4.1), is replacing time derivative \( d/dt \) by fractional time derivative \( \partial^\alpha_t \) in the Hamiltonian structure, known as the Poisson brackets
\[ \{ f, g \} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \]  
(with the classical counterpart of the Hamiltonian \( \hat{H} \)). Here \( f, g \) are arbitrary functions of the momentum \( p \) and the coordinate \( x \). Therefore the fractional evolution of an arbitrary function \( f(x, p) \) is determined by the Poisson brackets
\[ \mathcal{K}f(p, q) \equiv \{ H, f \} = \left( \frac{\partial H}{\partial p} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p} \right) f(p, q), \]  
(4.10)
and the fractional classical dynamics is described by fractional equations of motion
\[ \partial^\beta_t f(x, p) = K f(x, p). \] (4.11)
Here \( x \) and \( p \) are taken at some fixed time, for example at \( t = 0 \).

Now we return to our main consideration of the FHEM by quantization of the Poisson brackets \( \{ x, p \} \rightarrow [\hat{x}, \hat{p}] = -i\hbar ef \{ x, p \} = i\hbar ef \). Therefore, the FHEM for an arbitrary operator \( \hat{f} = \hat{f}(\hat{x}(t), \hat{p}(t)) \) reads
\[ -i\hbar ef \partial^\beta_t \hat{f} = [\hat{H}, \hat{f}], \] (4.12)
where the initial condition of the operator is \( \hat{f}_0 = \hat{f}_0(\hat{x}, \hat{p}) \) and its evolution is the explicit function of time. Note that \((\hat{x}, \hat{p})\) are taken at \( t = 0 \).

4.2.1. Oscillator dynamics in coherent states

An important property of Eq. (4.12) is that the dynamics is not unitary and as a consequence, the commutator \([\hat{f}, \hat{H}] \neq \hat{U}^{-1}(t)[\hat{f}_0, \hat{H}_0] \hat{U}(t)\) is not defined at arbitrary time \( t \). However, the Hamiltonian is the integral of motion since \([\hat{H}, \hat{H}] = 0, \forall t\) and for conservative systems this yields \( \hat{H}(\hat{x}(t), \hat{p}(t), t) = \hat{H}(t = 0) = \hat{H}_0(\hat{x}, \hat{p}) \). In this case, it is possible to consider the commutator at \( t = 0 \).

To explain this situation, we consider a system described in the framework of the Bose creation, \( \hat{a}^\dagger \) and annihilation, \( \hat{a} \) operators, which are taken at \( t = 0 \) with the commutation rule \([\hat{a}, \hat{a}^\dagger] = 1\) and the Hamiltonian \( \hat{H} = \hat{H}(\hat{a}^\dagger, \hat{a}) \). Here \( \hat{a} = \hat{a}(t = 0) \) and \( \hat{a}^\dagger = \hat{a}^\dagger(t = 0) \). We can also rewrite the operator \( \hat{f} \) as a function of the creation and annihilation operators \( \hat{f}(t) = \hat{f}(\hat{a}^\dagger, \hat{a}, t) \).

Now we can introduce a basis of coherent states \(|a\rangle, \langle a|\) at \( t = 0 \) as the eigenfunctions of the creation and annihilation operators \( \hat{a}^\dagger \) and \( \hat{a} \) at \( t = 0 \), respectively:
\[ \hat{a}|a\rangle = a|a\rangle, \quad \langle a|\hat{a}^\dagger = a^* \langle a| . \] (4.13)

---

8It also relates to a specific superposition of the Fock states of a harmonics oscillator. However, for our purpose, Eq. (4.13) is a satisfactory definition.
Using this basis, we introduce the average value of the operator function
\[ f(t) \equiv f(a^*, a, t) = \langle \hat{f}(t) \rangle = \langle a | \hat{f} (\hat{a}^\dagger, \hat{a}, t) | a \rangle. \tag{4.14} \]

This formulation supposes the normal ordering of the operator with respect to the creation and annihilation operators: \( \hat{f} = \sum_{m,n} f_{m,n}(t)\hat{a}^m\hat{a}^n. \) The mapping procedure of FHEM on the basis of the coherent states is based on the following properties \[ \langle a | \hat{f} (\hat{a}^\dagger, \hat{a}, t) \hat{a}^\dagger | a \rangle = e^{-|a|^2} \frac{\partial}{\partial a} e^{|a|^2} f(a^*, a, t), \tag{4.15a} \]
\[ \langle a | \hat{a} \hat{f} (\hat{a}^\dagger, \hat{a}, t) | a \rangle = e^{-|a|^2} \frac{\partial}{\partial a^*} e^{|a|^2} f(a^*, a, t). \tag{4.15b} \]

The averaging procedure results in the c-number (scalar valued) fractional equation of motion as follows
\[ \frac{\partial^2}{\partial t^2} f(t) = i \hbar \mathcal{K} f(t), \tag{4.16} \]
where \( \mathcal{K} \) is a quantum Koopman operator, or \( \mathcal{K} \)-operator, which reads \[ \mathcal{K} = e^{-|a|^2} \left[ H (a^*, \frac{\partial}{\partial a}) - H (\frac{\partial}{\partial a^*}, a) \right] e^{|a|^2}. \tag{4.17} \]

\[ 4.2.2. \quad \text{Hamiltonian form of FHEM} \]

As shown above, the symmetrical form of the FHEM yields a simple and elegant way to find the quantum evolution in the framework of the Koopman operator \[ 4.17. \] However, it is tempting to obtain the FHEM by quantization of the Hamilton equation of motion, obtained from the fraction Lagrangian
\[ L = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} \right)^2 - V(x) \equiv \frac{v^2}{2} - V. \]

Then we obtain the momentum \( p_\beta = (\partial L/\partial v_\beta), \) and the Hamiltonian is
\[ H = p_\beta v_\beta - L = \frac{1}{2} p_\beta^2 + V. \tag{4.18} \]

Therefore, fractional Hamilton equations of motion, defined by the Lagrange-Euler equations are \( (p_\beta \equiv p) \)
\[ \frac{\partial^2}{\partial t^2} x = p \tag{4.19a} \]
\[ R^\beta \frac{\partial}{\partial t} D^\beta p = -\frac{\partial V(x)}{\partial x}. \tag{4.19b} \]
The first equation (4.19a) with the Caputo fractional derivative corresponds to the symmetrical c-number fractional equation of motion (4.16), which eventually yields a decay solution, when \( p = p(t) \) is known. The second equation (4.19b) with right Riemann-Liouville fractional derivative for the averaged momentum \( p(t) = p(a^*, a, t) = \langle a|\hat{p}|a \rangle \) yields

\[
D_{RL}^{\beta} p(t) = \frac{i}{\hbar} \hat{K} p(t),
\]

(4.20)

where we used that \(-\frac{\partial V(x)}{\partial x} \equiv \hat{K} p\). Solving this equation, we first pay attention at the particular solution \( p(t) = C(T - t)^{\beta - 1} \). Taking into account that \( D_{RL}^{\nu} (T - t)^{\nu - 1} = 0 \), and the momentum is invariant with respect to the time shift \( C(T - t)^{\beta - 1} \), where \( C \) is such that \( \hat{K} C = 0 \), then Eq. (4.21) is a particular solution. Therefore, looking for the momentum as a function of \( T - t \), and taking into account the variable separation \( p(t) = \mathcal{U}(t) X(a, a^*) \), then the momentum in Eq. (4.20) is the expansion

\[
p(t) = \sum_{n=0}^{\infty} c_{n, \beta} (T - t)^{n\beta} X(a, a^*). \tag{4.21}
\]

Substituting Eq. (4.21) in Eq. (4.20), one obtains

\[
p(t) = \mathcal{U}(t) X(a, a^*) = E_{\beta} \left[ i(T - t)^{\beta} \hat{K}/\hbar \right] X(a, a^*). \tag{4.22}
\]

Here \( X(a, a^*) \) is a stationary part of the momentum. For example, it can be considered as the “initial condition” at \( t = T \). Then from Eq. (4.22), it follows \( p(t = T) = p(0) = X(a, a^*) \). Another possibility, it can satisfy the eigenvalue equation, \( \hat{K} X(a, a^*) = \hbar \lambda X(a, a^*) \). In this case, the nonunitary dynamics of the momentum is due to the Mittag-Leffler function \( \mathcal{U}(t) = E_{\beta} [i\lambda(T - t)^{\beta}] \).

5. Conclusion

In the review, we have concerned with few examples of fractional Schrödinger equations (FSE) as descriptive models of light propagation in nonlinear com-

\[\text{For example, for the oscillators we have } C = |a|^2. \text{ Another possibility is a “Hamiltonian” } C = H(a^*, a), \text{ since the condition } \hat{K} H(a^*, a) = 0 \text{ is always fulfilled.}\]
posite materials envisaged for real experimental tasks. The general form of the FSE (1.6) describes two large classes of non-local quantum mechanics, namely fractional space Schrödinger equations (FSSEs) and fractional time Schrödinger equations (FTSEs) considered in the review. These two classes belong to completely different quantum phenomena. The former one corresponds to quantum Lévy walks, including Anderson localization of the Lévy flights, which are described by the Hamiltonian quantum mechanics. In contrast, the FTSEs describe non-Markovian quantum dynamics, including quantum decay/friction due to interactions with the environment. However, in optical context, when time is considered as an effective presentation of a longitudinal coordinate in the parabolic equation approximation, the FTSE relates to linear - nonlinear optics in curved spaces and fractional differential geometry. Unfortunately, the latter is a vague issue in fractional calculus. Note also that “fractional time metric” \((T-t)^{\beta-1}\) is reasonable to use only for the infinitesimal evolution for the generating action for the fractional Lagrangian. In this connection, the extension method with fractional metric looks as a possible way to overcome this obstacle, see e.g., Ref. [95]. The similar approach for the FTSE can be suggested in the framework of the non-fractional Lagrangian with fractional metric in the fractional action (4.6), defined on the infinitesimal fractional time. In Sec. 4.1, this situation was discussed in the framework of generating functional for the FTSE, where a fractional time derivative in the classical dynamics invokes fractional time quantum mechanics, see also Appendix B.

In an alternative consideration of this fractional time functional for arbitrary time \(t\) with classical Lagrangian \(L_0 = \frac{d^2}{dt^2} - V(q)\), the fractional action reads

\[
S[q] = \frac{1}{\Gamma(\nu)} \int_0^t L_0(\dot{q}, q, \tau) (t-\tau)^{\nu-1} d\tau, \tag{5.1}
\]

where the variable \(q\) is a coordinate in some \(d\)-dimensional space. This kind of fractional actions, presented in the form of the Lebesgue-Stieltjes integral is considered in quantum gravity [96], where the standard measure in the action is replaced by a nontrivial measure. An example of such fractional action (5.1) has been introduced in the Lagrangian dynamics with friction [97], as well.
The Lagrangian of a non-conservative system

\[ L(\dot{q}, q, \tau) = \frac{1}{\Gamma(\nu)} L_0(\dot{q}, q, \tau)(t - \tau)^{\nu - 1} \]

produces the Lagrangian equations of motion

\[
\begin{align*}
\frac{dL}{dq} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} &= 0, \\
\ddot{q} + \partial_q V(q) + \frac{1 - \nu}{t - \tau} \dot{q} &= 0.
\end{align*}
\]

(5.2a)

(5.2b)

In the new time, \( t - \tau \to t \) with the time derivative \( d/dt \), Eq. (5.2b) reads

\[
\ddot{q} + \partial_q V(q) - \frac{1 - \nu}{t} \dot{q} = 0.
\]

(5.3)

Then the Lagrangian is \( L = |q^2/2 - V(q)|^{\nu - 1} \), and the corresponding action is not anymore fractional \( S[q] = \int_0^t L(\dot{q}, q, \tau) d\tau \) as well. The Hamiltonian dynamics can be constructed as well. Introducing the momentum \( p = \frac{\partial L}{\partial \dot{q}} = \dot{q} t^{\nu - 1} \), we obtain the Hamiltonian

\[
H(p, q, t) = \frac{p^2}{2} t^{1 - \nu} + V(q) t^{\nu - 1},
\]

(5.4)

and the Schrödinger equation reads

\[
i\hbar \partial_t \psi(q, t) = \hat{H}(t) \psi(q, t).
\]

(5.5)

It also follows from Eq. (5.1) that \( H(p, q, t) = -\partial_t S[q] = -D_t^{1-\nu} L_0(\dot{q}, q, \tau) \). Therefore, the Hamiltonian (Markov) quantum mechanics can be constructed to describe the fractional time dynamics. Correspondingly, the Heisenberg equations of motion can be constructed as well\(^{10}\). This issue needs further clarification with respect to local differential geometry in linear - nonlinear optics in curved space\(^9\).

\(^{10}\)Note also that the path integral can be constructed as well, since the quantum Hamiltonian\(^9\) produces the classical action \( S[q] \) rigorously by means of the Trotter product formula, see e.g.\(^9\).

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Appendix A. Elements of fractional integro–differentiation

A basic introduction to fractional calculus can be found, e.g., in Refs. 1, 2. Fractional integration of the order of $\alpha$ is defined by the operator

$$aI^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)} \int^t_a f(\tau)(t-\tau)^{\alpha-1}d\tau, \quad (\alpha > 0),$$  \hspace{1cm} (A.1)

where $\Gamma(\alpha)$ is a gamma function, and there is no constraint on the lower limit $a$.

A fractional derivative is defined as an inverse operator to fractional integration $aI^\alpha_t$ in Eq. (A.1). Its explicit form is the convolution

$$aI^{-\alpha}_t f(t) = aD^\alpha_t f(t) = \frac{1}{\Gamma(-\alpha)} \int^t_0 \frac{f(\tau)d\tau}{(t-\tau)\alpha+1},$$  \hspace{1cm} (A.2)

For arbitrary $\alpha > 0$, this integral is, in general, divergent. As a regularization of the divergent integral, the following two alternative definitions for $aD^\alpha_t$ exist. The first one is the Riemann–Liouville fractional derivative. For $n-1 < \alpha < n, \ n = 1, 2, \ldots$ it reads

$$^{RL}aD^\alpha_t f(t) = D^n_{t} a I^{n-\alpha}_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int^t_0 \frac{f(\tau)d\tau}{(t-\tau)\alpha+1-n}.$$  \hspace{1cm} (A.3)

The second one

$$^{C}aD^\alpha_t f(t) = I^{n-\alpha}_t a D^n_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int^t_0 \frac{f(n)(\tau)d\tau}{(t-\tau)\alpha+1-n},$$  \hspace{1cm} (A.4)

is the fractional derivative in the Caputo form.

For the time, where the limit is $a = 0$, we use $^C_aD^\beta_t f(t) \equiv \partial^\beta_t$ for the Caputo derivative and $^{RL}_0D^\beta_t \equiv D^\beta_t$ for the Riemann–Liouville derivative.

The Laplace transformation can be performed, as well. For example for Eq. (A.4), if $L[f(t)] = \tilde{f}(s)$ is the Laplace transformation, then

$$L \left[ \partial^\beta_t f(t) \right] = s^\beta \tilde{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\beta-1-k}.$$  \hspace{1cm} (A.5)

When $a = -\infty$, the resulting fractional derivative is the Weyl derivative,

$$W^\nu f(x) \equiv -\infty W^\nu_x f(x) \equiv W^\nu a \equiv \frac{1}{\Gamma(-\nu)} \int_{-\infty}^{x} \frac{f(x')dx'}{(x-x')^{1+\nu}}.$$  \hspace{1cm} (A.6)

11Greek letters $\alpha, \beta, \nu$ are arbitrary used for the order of the fractional integro-differentiation, while Latin letters $t$ and $x$ are used as variables.
If we impose the physically reasonable condition \( f(-\infty) = 0 \) together with its \( n \) derivatives, where \( n - 1 < \nu < n \), then

\[
W_{-\infty}^\nu D_x^\nu f(x) = RL_{-\infty}^\nu D_x^\nu f(x) = C_{-\infty}^\nu D_x^\nu f(x) \quad (A.7)
\]

One also has \( W^\nu e^{\lambda x} = \lambda^\nu e^{\lambda x} \). This property is convenient for the Fourier transform \( \mathcal{F}[f(x)](k) = \hat{f}(k) \), which yields

\[
\mathcal{F}[W^\nu f(x)](k) = (-ik)^\nu \hat{f}(k) \quad (A.8)
\]

The fractional derivation with the fixed lower limit is also called the left fractional derivative. One can introduce the right fractional derivative, where the upper limit \( b \) is fixed and \( b > x \). For example, the formal definition of the right fractional derivative is

\[
x^D_b^\nu f(x) = \frac{1}{\Gamma(-\nu)} \int_x^b \frac{f(y)dy}{(y-x)^{1+\nu}} \quad (A.9)
\]

which is regularized in either Riemann-Liouville (A.3) or Caputo (A.4) forms.

The Riesz fractional integral [99] on the finite interval \([a,b]\) is

\[
\frac{1}{2\Gamma(\nu)} \cos \frac{\nu\pi}{2} \int_a^b \int_x^y f(z)dz \frac{dy}{|x-y|^{1-\nu}} \quad (A.10)
\]

where \( a \leq x \leq b \) and \( 0 < \nu < 1 \). It can be represented as the sum of the left and right Riemann-Liouville fractional integrals

\[
\int_a^x \frac{f(y)dy}{(x-y)^{1-\nu}} + \int_x^b \frac{f(y)dy}{(y-x)^{1-\nu}} \quad (A.11)
\]

Consequently, the Riesz fractional derivative \((-\Delta)_{x}^{\nu}\) on the entire \( x\)-axis can also be represented by the Weyl derivatives

\[
(-\Delta)_{x}^{\nu} \equiv \infty D_{|x|}^{\nu} f(x) = \frac{1}{2 \cos \frac{\nu\pi}{2}} \left[ W_{-\infty}^\nu D_x^\nu f(x) + W_{-\infty}^\nu D_{-x}^\nu f(-x) \right] \quad (A.12)
\]

We also use a “group property”

\[
D_t^{-\nu} \left[ D_t^{-\mu} f(t) \right] = D_t^{-(\nu+\mu)} f(t) = D_t^{-\mu} \left[ D_t^{-\nu} f(t) \right] \quad (A.13)
\]

which is based on Dirichlet’s formula \( \int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx \).

Another important property, used in the analysis, is a combination of fractional
integro-differentiation

\[ D^1_\beta \partial^\beta f(t) = D_t I^{1-\beta}_t I^{1-\beta}_t D_t f(t) = D_t[f(t) - f(0)] = \frac{df(t)}{dt} , \]  

(A.14)

where \(0 < \beta < 1\). Using Eq. (A.14), one defines a fractional Taylor expansion as follows

\[ f(t + \varepsilon) - f(t) = I^{\beta}_t t^{\varepsilon} \partial^\beta f(t) = \frac{1}{\Gamma(\alpha)} \int^t_{t+\varepsilon} (t + \varepsilon - t')^\alpha C_t^{\alpha} f(t') dt' . \]  

(A.15)

Applying integration by parts \(N\) times, we have from (A.15)

\[ f(t + \varepsilon) = f(t) + \varepsilon^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + n + 1)} D^n_t \partial^\alpha f(t) + O(\varepsilon^{\alpha+n+1}) . \]  

(A.16)

Appendix A.1. Mittag-Leffler function and Fox \(H\)-function

The fractional derivative of an exponential function can be calculated by virtue of the Mittag-Leffler function. Therefore, applications of the Riemann-Liouville fractional derivative to the exponential results in the Mittag-Leffler function

\[ D^\alpha_t e^{\lambda t} = t^\alpha E^{1,1}_{1,\alpha}(\lambda t) , \]  

where the Mittag–Leffler function is (see e.g., Refs. [2, 101])

\[ E^{\nu,\beta}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\tau^{-\nu - \beta} e^\tau}{\tau - z} \, d\tau = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + \beta)} . \]  

(A.17a)

A detailed description of the Fox \(H\)-function and its application can be found in Refs. [57, 103]. The Fox \(H\)-function is defined in

\[ E^{\nu,1}(z) \equiv E^{\nu}(z) . \]
terms of the Mellin-Barnes integral

\[
H_{m,n}^{p,q}(z) = H_{p,q}^{m,n}
\begin{bmatrix}
(a_p, A_p) \\
(b_q, B_q)
\end{bmatrix}
= \frac{1}{2\pi i} \int_C ds \theta(s) z^{-s}. \quad (A.18)
\]

Here

\[
\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1} \Gamma(1 - b_j - B_j s) \prod_{j=n+1} \Gamma(a_j + A_j s)},
\]

where \( n, m, p, q \) are non-negative integers with \( 0 \leq n \leq p, 1 \leq m \leq q, a_i, b_j \in \mathbb{C}, A_i, B_j \in \mathbb{R}^+, i = 1, \ldots, p, \) and \( j = 1, \ldots, q. \) If the product is empty, then it is set equal to one. The contour \( C \) starts at \( c - i\infty, \) ends at \( c + i\infty, \) and separates the poles \( \xi_{j,k} = -(b_j + k)/B_j \) of the gamma function \( \Gamma(b_j + B_j s) \) with \( j = 1, \ldots, m \)
and \( k = 0, 1, 2, \ldots \) from the poles \( \chi_{i,k} = (1 - a_i + k)/A_i \) of the gamma function \( \Gamma(1 - a_i - A_i s), i = 1, \ldots, n. \)

The Fox \( H \)-function relates to the exponential function and the Mittag-Leffler function \([57, 102, 103]\). Evaluating the Mellin-Barnes integral as a sum of residues in Eq. (A.18), we have

\[
H_{2,1}^{1,1}
\begin{bmatrix}
(0, 1) \\
(0, 1), (1 - \beta, \alpha)
\end{bmatrix}
= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} = E_{\alpha,\beta}(z), \quad (A.20)
\]

\[
H_{0,1}^{1,0}
\begin{bmatrix}
z \\
(0, 1)
\end{bmatrix}
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k = e^{-z}. \quad (A.21)
\]

Here we used the Euler’s reflection formula \( \Gamma(p)\Gamma(1 - p) = \frac{\pi}{\sin(p\pi)}. \)

The Mellin transform is \( M[f(t)](s) = \int_0^\infty f(t) t^{s-1} dt, \) and the contour integral in definition (A.18) is the inverse Mellin transform. Therefore, the Mellin transform of the Fox \( H \)-function yields

\[
\int_0^\infty dt \ t^{s-1} H_{p,q}^{m,n}
\begin{bmatrix}
(a_p, A_p) \\
(b_q, B_q)
\end{bmatrix}
= a^{-s} \theta(s), \quad (A.22)
\]

where \( \theta(s) \) is defined in Eq. (A.18).
The Mellin-cosine transform of the Fox H-function is

\[
\int_{0}^{\infty} d\kappa \kappa^{\rho-1} \cos(\kappa x) H_{p,q}^{m,n} \left[ \begin{array}{c} (a_{p}, A_{p}) \\ (b_{q}, B_{q}) \end{array} \right] \left[ \begin{array}{c} \kappa \\ \beta \end{array} \right] = \pi x^\rho H_{q+1,p+2}^{n+1,m} \left[ \begin{array}{c} \beta \\ (\rho, \delta), (1-a_{p}, A_{p}), (1+\rho, \frac{\delta}{2}) \end{array} \right],
\]

where

\[
\begin{align*}
\text{Re} (\rho + \delta \cdot \min_{1 \leq j \leq m} (b_{j}/B_{j})) > 1, & \quad x^\delta > 0 \\
\text{Re} (\rho + \delta \cdot \min_{1 \leq j \leq n} ((a_{j}-1)/A_{j})) > 3/2, & \quad |\text{arg}(a)| < \pi \tilde{\alpha}/2,
\end{align*}
\]

\[
\tilde{\alpha} = \sum_{j=1}^{n} A_{j} - \sum_{j=n+1}^{p} A_{j} + \sum_{j=1}^{m} B_{j} - \sum_{j=m+1}^{q} B_{j} > 0.
\]

A combination of the Mellin and Laplace transforms also yields a well-known expression

\[
\mathcal{M}[\mathcal{L}[f(t)](s)](1-p) = \Gamma(1-p)\mathcal{M}[f(t)](p).
\]

**Appendix B. Quantization of fractional dynamics: path integral approach**

Let us consider a Lagrangian function that contributes to Green’s function \[^{4.5}\]. The Lagrangian of a fractional “free” particle is

\[
L(x, \partial_{x}^\beta x) = \frac{1}{2} \left( \partial_{x}^\beta x \right)^{2}.
\]

Now, let us introduce a fractional action by means of fractional integration

\[
S(T) = \frac{1}{\Gamma(\alpha)} \int_{0}^{T} L(x, \partial_{x}^\beta x)(T-t)^{\beta-1} dt.
\]

Following Feynman’s heuristic arguments \[^{14}\], the Green’s function of the quantum evolution, or the transition probability amplitude, for the wave function can be constructed by the path integral on the interval \(T = t - t_{0}\). To perform this procedure, it is instructive to introduce some random field \(\xi(t)\), and present the Green’s function by means of the complex Gaussian integral \[^{104, 105, 106}\]

\[
\mathcal{G}(x, T|x_{0}, 0) = \int_{x_{0},0}^{x_{T},T} \delta \left( \xi(t) - \partial_{x}^\beta x \right) \exp \left[ -\frac{i}{2\hbar c \Gamma(\beta)} \int_{0}^{T} [\xi(t)]^{2} (T-t)^{\beta-1} dt \right] [d\xi(t)],
\]

\[
(B.2)
\]
which is an expectation value of the classical propagator. The Dirac delta function ensures that the integration accounts the trajectories, associated with the fractional Langevin equation

\[ \partial_t^\beta x = \xi(t) . \quad (B.3) \]

To proceed, we consider the fractional action in Eq. (B.2) as follows

\[ S[\xi(t)] = \frac{1}{2\Gamma(\beta)} \int_0^T [\xi(t)]^2 (T-t)^{\beta-1} dt . \quad (B.4) \]

Since the integral in Eq. (B.2) is Gaussian, the stationary phase method yields the exact integration. Therefore, we perform the variable change \( \xi(t) = \xi_e(t) + \eta(t) \), where \( \xi_e(t) \) is an extremum solution. This corresponds to Eq. (B.3), which also implies a constraint condition for \( \eta(t) \). Namely, applying the property (A.14) to Eq. (B.3), we have for the extremum path \( \xi_e(t) \)

\[ x_t - x_0 = D_t^{-\beta} \xi_e(t) \equiv J_t^\beta \xi_e(t) , \quad (B.5) \]

which also yields the constraint for the deviation \( D_t^{-\beta} \eta(t) = 0, \) or \( \delta \left( D_t^{-\beta} \eta(t) \right) \).

The path integral (B.2) with respect to \( \xi(t) \) reduces to the path integral with respect to \( \eta(t) \). The latter reads

\[ G(x,T|x_0,0) = e^{\frac{i}{\hbar} \int_0^T S[\eta(t)]} \int_{x_0,0}^{x,T} \delta \left( D_t^{-\beta} \eta(t) \right) e^{\frac{i}{\hbar} \int_0^T S[\eta(t)]} [d\eta(t)] \equiv F(T)e^{\frac{i}{\hbar} \int_0^T S[\xi_e]} . \quad (B.6) \]

The multiplier \( F(T) \equiv A^{-1} \) is the path integral with respect to \( \eta(t) \), which can be obtained from the normalization condition

\[ \int_{-\infty}^{\infty} G(x,T|x_0,0) dx_T = 1 , \quad \forall T . \quad (B.7) \]

The extremum action \( S_e[\xi_e] \) is found from a standard technique of Lagrange multipliers. Following Ref. [106], we add a zero to the extremum action. Taking into account Eqs. (B.3) and (A.13), we have

\[ S[\xi_e(t),\lambda] = \frac{1}{2\Gamma(\beta)} \int_0^T \left[ \xi_e^2(t) + 2\lambda \left( \partial_t^\beta x - \xi_e(t) \right) \right] (T-t)^{\beta-1} dt . \quad (B.8) \]
Variation over $\lambda$ yields Eq. (B.3), and correspondingly Eq. (B.5), while variation over $\xi$ yields

$$\frac{1}{\Gamma(\beta)} \int_0^T (\xi_e(t) - \lambda) (T - t)^{\beta-1} \delta\xi(t) dt = 0. \quad (B.9)$$

Since $\delta\xi(t)$ is arbitrary, the solution to Eq. (B.9) for the extremum path is $\xi_e(t) = \lambda$ and from Eq. (B.5) we obtain

$$x_T - x_0 = 0T^\beta \lambda = \frac{\lambda T^\beta}{\Gamma(\beta + 1)}, \quad (B.10)$$

which yields the Lagrange multiplier and the extremum path

$$\xi_e(t) = \lambda = \Gamma(\beta + 1)(x_T - x_0)/T^\beta. \quad (B.11)$$

Correspondingly, the extremum action reads

$$S_e[\xi_e(t)] = \frac{1}{2\Gamma(\beta)} \int_0^T [\xi_e(t)]^2 (T - t)^{\beta-1} dt = \frac{m_\beta(x_T - x_0)^2}{2T^\beta\Gamma(\beta + 1)}, \quad (B.12)$$

where $m_\beta = [\Gamma(\beta + 1)]^2$ is an effective dimensionless mass due to the fractional time dynamics. For $\beta = 1$, it is $m_\beta = 1$. Inserting it in Eq. (B.6), and accounting the normalization condition (B.7), one obtains the Green’s function as follows

$$G(x_T, T | x_0, 0) = F(T) \exp \left[ \frac{im_\beta(x_T - x_0)^2}{2\hbar \Gamma(\beta + 1)T^\beta} \right], \quad (B.13a)$$

$$F(T) \equiv A^{-1} = \frac{1}{\sqrt{2\hbar \Gamma(\beta + 1)\pi T^\beta/m_\beta}}. \quad (B.13b)$$

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