Oscillatory dynamics in complex recurrent neural networks

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Abstract
Spontaneous oscillations measured by Local field potentials (LFPs), electroencephalograms and magnetoencephalograms exhibits variety of oscillations spanning frequency band (1 – 100 Hz) in animals and humans. Both instantaneous power and phase of these ongoing oscillations have commonly been observed to correlate with pre-stimulus processing in animals and humans. However, despite of numerous attempts it is not fully clear whether the same mechanisms can give rise to a range of oscillations as observed in vivo during resting state spontaneous oscillatory activity of the brain. In the current paper we show how oscillatory activity can arise out of general recurrent on-center off-surround neural network. The current work shows (a) a complex valued input to a class of biologically inspired recurrent neural networks can be shown to be mathematically equivalent to a combination of real-valued recurrent network with real-valued feed forward network, (b) such a network can give rise to oscillatory signatures. We also validate the conjecture with results of simulation of complex valued additive recurrent neural network.

Introduction

It has always been a standard practice to break down the brain signals obtained in electroencephalography (EEG) or magnetoencephalography (MEG) in terms of their frequency components obeying the laws of Fourier decomposition. It is even commonplace to compartmentalize the frequencies into neat bunches like θ (4-8 Hz), α (8-12 Hz), etc. However, the important question is whether these bands are just a matter of convenience or is there an underlying reality to the oscillatory model for the brain. Some recent work has tried to tie the oscillatory framework to underlying spiking model for neurons (Deco et al., 2009; Nakagawa et al., 2014). This question can be solved only by looking at mathematical properties of neural codes.

In recent years the understanding of neural codes has provided us with insights that go beyond the concepts of rate coding and it is increasingly more commonplace to speak
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of temporal codes that use spike timing and phase information in order to transmit and process information reliably (Stanley, 2013; van Rullen & Thorpe, 2001; Masquelier, 2012; Gautrais & Thorpe, 1998). However, most of these attempts are to look at neural codes after the presentation of stimulus. Recently, some studies have looked at pre-stimulus brain states in MEG and EEG based studies and have found that it is possible to predict conscious detection of stimuli based on pre-stimulus oscillatory brain activity (Mathewson et al., 2009; Weisz et al., 2014; May et al., 2012; Keil et al., 2012). For instance, in case of near threshold stimuli some researchers have found the pre-stimulus α frequency band modulation to be important (Weisz et al., 2014). These attempts have drawn a large amount of interest, but have revealed little towards a theoretical or physiological understanding of such phenomena. In the current work we have tried to start from a minimal number of assumptions regarding a neuronal assembly and tried to show how it is possible to theoretically derive such an oscillatory dynamics.

Methods

**Conjecture 1.** If a neural assembly \( S \) of \( N \) neurons consists of both feed-forward and recurrent connections with a finite variable bound refractory period \( \tau \) between the feed-forward and recurrent connections, the equilibrium solution for the assembly can be characterized by a class of oscillatory functions.

**Justification.** If a neural assembly comprises of \( n_1 \) feed-forward neurons and \( n_2 \) recurrent neurons \( (n_1 + n_2 \leq N) \), then the general activations of the pools of neurons will be given by

\[
\dot{x}_1 = \frac{dx_1}{dt} \big|_{n_1} = \text{sgn}(\xi^T \xi x_1 - \theta)
\]

\[
\dot{x}_2 = \frac{dx_2}{dt} \big|_{n_2} = -x_2 + \mathbf{F}.x_2 + \eta I_{x_1}
\]

where \( \xi \) is the weight matrix for the feed-forward connections, \( \theta \) is the threshold, \( \eta \) is a scaling multiplier, \( \mathbf{F} \) is the non-linear transfer function and \( I_{x_1} \) is the input coming from
the feed-forward networks. Now, for the network to be useful there needs to be sustained activity (provided by the recurrent network, see (Sengupta, Surampudi, & Melcher, 2014)) as well as the ability to restart the network dynamics. The latter is provided by assuming that a smoothing regularizer is provided which varies inversely with the network output (Wu & Moody, 1996) (to smooth the network against noisy perturbations), as well as a global decay parameter. In such a network, the overall output function \(<x_1 + x_2>\) (average expectation value of the network output constructed in the line of mean activation in (Sengupta, Surampudi, & Melcher, 2014)), varies between 0 and \(<x_2> + I\), where I is the average normalized input to the network. This alone is sufficient to show the possibility of oscillatory solutions. However, to be more rigorous, let us look at the second order time evolution for the feed-forward and recurrent populations. From Eq. 1 we get.

\[
\ddot{x}_1 = \frac{d^2 x_1}{dt^2}\]  
(6)

Here \(\delta(x)\) is the Dirac delta function. From Eq. 2 we have,

\[
d\dot{x}_2 = -d_{x_2} + d(Fx_2) \]
(7)

\[
= -d_{x_2} + (dF)x_2 + Fdx_2
\]
(8)

If \(F\) is a smooth transfer function then the third term in the right hand side of Eq. 8 becomes negligible. Thus we have

\[
\dot{x}_2 = -\dot{x}_2 + \dot{F}_{x_2}
\]
(9)

\[
\ddot{x}_2 = (1 - F)x_2 + \ddot{F}_{x_2} + O(I_{x_1})
\]
(10)

For the recurrent network, near equilibrium (here we assume absence of input because of the finite refractory period of \(\tau\)), the nonlinear operators \(F\) and \(\dot{F}\) become quasi-linear operators, and thus we have

\[
\ddot{x}_2 = \frac{d^2 x_2}{dt^2}\]  
(11)

\footnote{If we consider a general recurrent shunting networks with dynamics given by

\[
\dot{x}_i = -A_i x_i + (B_i - x_i)(I_i + S(x_i)) - (x_i + C_i) \left( J_i + \sum_{j \neq i} w_{ji} S(x_j) \right)
\]
(3)

where \(I\) and \(J\) are excitatory and inhibitory inputs and \(S\) is a sigmoid function, we can transform it with simple variable change \((y_i = x_i + C_i)\) to the form

\[
y_i = y_i \left( b_i(y_i) - \sum_{j=1}^{n} w_{ji} S(y_j - C_j) \right)
\]
(4)

where,

\[
b_i(y_i) = \frac{1}{y_i} \left\{ A_i C_i - (A_i + J_i) y_i + (B_i + C_i - y_i) [I_i + S(y_i - C_i)] \right\}
\]
(5)

It can be shown that in absence of input and near equilibrium, second order dynamics of this network will be very similar to Eq. 10 with an extra term of the order of \(y_i^2\), which does not change the general conclusions of the paper.}
This has a form of simple eigenvalue problem. The network will have oscillatory solution if the operator $\|F - (1 + \dot{F})\|$ is positive Hermitian. A suitable example is the leaky accumulator network described in (Sengupta, Surampudi, & Melcher, 2014), where the matrix representing the operator becomes real symmetric. Thus near equilibrium, considering $F$ is a quasi-linear approximation $\langle x_2 \rangle$, the mean state of the neural assembly near equilibrium / steady state will have a periodic solution $\xi_n = Ae^{i(\omega_n t - \theta)}$ with $x_1$ contributing to the phase $\theta$. The full neural field dynamics $\psi(x,t)$ can be constructed along the lines of Jirsa and Haken, 1996 with both spatial and temporal components taken into account as the following

$$\psi(x,t) = \sum \xi_n(t) \exp(inkx)$$

Thus the predictions from this analytical framework can be easily ported to neural mass models as well.

**Corollary.** An assembly of recurrent neurons with complex-valued inputs and outputs, is formally equivalent to a neural assembly of independent feed-forward and recurrent neurons.

**Justification.** Let us start with Cohen-Grossberg generalized networks of additive variant with a nonlinear activation function, like the network described by Sengupta, Surampudi, and Melcher, 2014; Usher and Cohen, 1999; Bogacz et al., 2007. If inputs and outputs to the network are given by a vector of complex numbers $\vec{z}_{1 \times m} = \vec{x}_{1 \times m} + i\vec{y}_{1 \times m}$ for a network of $m$ nodes, the network dynamics is governed by

$$\frac{dz_j}{dt} = -z_j + c_1 F(z_j) - c_2 \sum_{k \neq j} F(z_k) + I_j + \text{noise}$$

where $F(z) = z/(1 + z)$. Ignoring noise for the time being, if we decompose the real and imaginary parts of (13) we have

$$\frac{dz_j}{dt} = -z_j + c_1 \frac{z_j + |z_j|^2}{1 + |z_j|^2 + 2\text{Re}(z_j)}$$

$$- c_2 \sum_{k \neq j} \frac{z_k + |z_k|^2}{1 + |z_k|^2 + 2\text{Re}(z_k)} + I_j$$

And thus the separated real and imaginary parts yield two equations given by,

$$\frac{d\text{Re}(z_j)}{dt} = -\text{Re}(z_j) + c_1 \frac{\text{Re}(z_j)}{1 + |z_j|^2 + 2\text{Re}(z_j)}$$

$$- c_1 \sum_{k \neq j} \frac{\text{Re}(z_k)}{1 + |z_k|^2 + 2\text{Re}(z_k)} + \text{Re}(I_j) + O(|z|^2)$$

$$\frac{d\text{Im}(z_j)}{dt} = \sum c_{jk} \text{Im}(z_k) + \text{Im}(I_j)$$

where $c_{jk}$ are constants. Making suitable substitutions ($x_j \leftarrow 2(\text{Re}(z_j) + |z_j|^2/2)$, and $y_j \leftarrow \text{Im}(z_j)$), we have

$$\frac{dx_j}{dt} = -x_j + c'_{1} F(x_j) - c'_{2} \sum_{k \neq j} F(x_k) + I_j + \text{constant terms}$$

$$\frac{dy_j}{dt} = -y_j + c''_{1} F(y_j) - c''_{2} \sum_{k \neq j} F(y_k) + I_j + \text{constant terms}$$
From Eq. 17 and 18 it is clear that a combination of recurrent and feedforward networks can function in a way to handle complex inputs and outputs and thus at least the reverse of Conjecture 1 is true in certain cases.

In the following section we show simulations involving additive neural networks giving rise to oscillatory dynamics under complex inputs.

**Results**

In the above, we have shown the possibility of oscillatory brain states under certain equilibrium conditions for a neural assemble consisting of both feed-forward and recurrent connections. We simulated a recurrent on-center off-surround neural network governed by the dynamical equation (complex version of the neural network used in Sengupta, Surampudi, and Melcher, 2014)

\[
\frac{dz_i}{dt} = -x_i + \alpha F(z_i) - \beta \sum_{j=1,j\neq i}^{N} F(z_j) + I_i + \text{noise}
\]

where \( z_i \) are complex valued activations of the nodes indexed 1 to \( N \) (we used 10 nodes for the simulation). All the nodes are fully connected with self-excitation \( \alpha \) (we used values 0.05, 0.55, 1.05, 1.55, 2.05, 2.55, 3.05, 3.55, 4.05, 4.55, 5.05) and lateral inhibition \( \beta \) (we used 0.03, 0.08, 0.13, 0.18, 0.23, 0.28). Transfer function \( F \) is defined as

\[
F(z) = \frac{z}{1 + z}
\]

\( I_i \) is a transient complex input for 200 ms steps with values clamped at \((0.7 + 0.1i)\).

We ran the numerical simulation for total 2000 ms steps using time steps of 0.01 using Euler method. Noise was sampled from a Gaussian distribution with mean 0 and standard deviation of 0.05. Fig. 1 depicts \( \text{Re}(z) \) for the network for network parameters \( \alpha = 2.0 \) and \( \beta = 0.1 \).

We ran time-frequency analysis on \( \text{Re}(z) \) and the results are shown in Fig. 2. We ran 100 simulations for each values of \( \alpha \) and \( \beta \) and Fig. 3 shows the average dominant frequency values for all \( \alpha \) and \( \beta \).

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2The above analysis also holds for a general sigmoid activation function. If \( \sigma(x) = \frac{1}{1 + e^{-x}} \), then from simple function approximation of a general sigmoidal function \( f(x) = \sum_{i=1}^{n} c_i \sigma(x - a_i) \) can be simply transformed in the complex domain as \( f(x) \rightarrow \sum_{i=1}^{n} \frac{c_i}{e^{a_i} + e^{-a_i}} \), where \( a_i = e^{a_i} \) (see Appendix C of Mandic and Chambers, 2001). Thus the results of conjecture 2 are quite general.
Figure 1. Real values of the activation for the nodes of the network where $\alpha = 2.0$ and $\beta = 0.1$.

Discussion

In the above, we have shown the possibility of oscillatory brain states under certain equilibrium conditions for a neural assemble consisting of both feed-forward and recurrent connections. We also showed with help of simulations (Fig. 1, 2 & 3) that simple fully connected additive recurrent neural network with complex inputs can lead to oscillatory dynamics. We saw in Fig. 3 that dominant frequency of the steady-state dynamics varies linearly with the self-excitation parameter ($\alpha$) of the network, but not with the lateral inhibition ($\beta$). In the future we intend to investigate similar dynamics for other recurrent network variants like multiplicative and self-organized maps, etc.

Appendix

Cognitive dynamics and pre-stimulus oscillations

Conjecture. If a neural assembly $S$ of $N$ neurons consists of both feed-forward and recurrent connections with a finite variable bound refractory period $\tau$ between the feed-forward and recurrent connections, the pre-stimulus brain states are determined by the delay between the regions represented by the neural assemblies.
Figure 2. Frequency power analysis for Re(z) for the nodes of the network where $\alpha = 2.0$ and $\beta = 0.1$.

Justification. The conjecture 1 has some interesting consequences. Firstly, it allows for brain states\textsuperscript{3} to be defined in terms of a frequency or a distribution of frequencies. In a case of a state with a distribution of frequencies we can think of a characteristic frequency range representing the state, the characteristic frequency being the one with maximal power. The conjecture also allows us to think of the brain states as phenomenal superposition of oscillatory dynamics, allowing us to deal with problems such as stimulus related perturbations to brain states more efficiently.

The prevailing additive idea of brain states in the neuroimaging literature needs no introduction. However, we will formally spell out the bare essentials. Before a test condition appears to a subject the family of brain states or the neural assembly $S$ in question ($S$) can be thought of as its resting state. In the test condition, a new perturbation comes from our experimental control ($S_T$). Or we can write, $S \leftarrow \bar{S} + S_T$.

Now if the resting state is imagined as a standing wave, then we have from Eq. 12 $\bar{S} = \sum A \cos(kx) \exp(i\omega_0 t)$. Since resting state can be taken to be not very location

\textsuperscript{3}Here we are using the term brain state to mean the state of the neural assembly involved in a particular task or function, not the entire brain.
specific we can write the rate of change of the state of neural assembly given by $S$, when a perturbative brain state $S_T$ (generally due to oncoming stimulus) interacts with the current state to be given by

$$\frac{\partial S}{\partial t} = i\alpha \exp(\omega_0 t) + \frac{\partial S_T}{\partial t}$$

(21)

From the results of the previous section, if assume the brain states and their perturbations to have solutions of the form $A e^{i(\omega t - \theta)}$ we can write,

$$a_1 i \omega e^{i(\omega t - \theta)} = i\alpha e^{\omega t} + a_2 i \omega' e^{i(\omega' t - \omega_1 t)}$$

(22)

Here $\omega$ is the pre-stimulus brain state frequency that tries to interact with the incoming perturbation characterized by $\omega'$. Separating the real and the imaginary parts and applying the constraint that the imaginary part must go to zero on the left and right hand side of Eq. 22

$$a_1 \omega \cos(\omega t - \theta) = \alpha \cos(\omega_0 t) + a_2 \omega' \cos(\omega' t - \omega_1 t)$$

(23)

Now considering the case $\omega \sim \omega'$, we have from Eq. 23 (if $S$ and $S_T$ are in similar phase) we have for $t \to 0$,

$$(a_1 - a_2) \omega \cos(\omega t - \theta) = \alpha$$

(24)
Thus we have our constraint,

\[-1 \leq \frac{\alpha}{(a_1 - a_2)\omega} \leq 1\] (25)

Thus Eq. 25 shows that the resonant frequency of the pre-stimulus brain states in a region varies with \(\alpha\). Interestingly, \(\alpha\) is a spatial term dependent on the spatial connectivity as seen above. Thus the frequencies generated in the oscillatory pre-stimulus brain will depend upon the delay between the regions.

Thus there are two consequences of the analytical framework. It shows how a general oscillatory activity can be generated in the particular brain region having both feedforward and recurrent connections. Secondly it connects the general oscillatory signals in the brain that arises from connected regions to be dependent upon the delay in the network connectivity arising from spatial factors.

References

Bogacz, Rafal et al. (2007). “Extending a biologically inspired model of choice: multi-alternatives, nonlinearity, and value-based multidimensional choice”. In: Philosophical Transactions of the Royal Society B: Biological Sciences 362, pp. 1655–16709. DOI: 10.1017/cbo9780511731525.009

Deco, G. et al. (2009). “Key role of coupling, delay, and noise in resting brain fluctuations”. In: Proceedings of the National Academy of Sciences USA 106.25, pp. 10302–10307. ISSN: 0027-8424. DOI: 10.1073/pnas.0901831106

Gautrais, Jacques and Simon Thorpe (1998). “Rate coding versus temporal order coding: a theoretical approach”. In: Biosystems 48, pp. 57–65. ISSN: 0303-2647. DOI: 10.1016/s0303-2647(98)00050-1.

Jirsa, V. K. and H. Haken (1996). “Field Theory of Electromagnetic Brain Activity”. In: Physical Review Letters 77, pp. 960–963. ISSN: 0031-9007. DOI: 10.1103/physrevlett.77.960

4Now if we take the imaginary part of the Eq. 22 then we have (for the case \(\omega \sim \omega'\) if \(S\) and \(S_T\) are in similar phase)

\[(a_1 - a_2)\sin(\omega t - \theta) = 0\] (26)

\[\omega t - \theta \sim n\pi\] (27)

\[\mathcal{O}(\omega t) \sim n\pi\] (28)

If \(S\) and \(S_T\) are anti-phase then also we have \(\tan(\omega t - \theta) = \frac{a_2}{a_1} < \infty\) and thus \(\mathcal{O}(\omega t) \sim \pi\). On the other hand for the condition \(\omega \gg \omega'\), we have \(\sin(\omega t - \theta) = 0\), and thus Eq. 28 follows. The condition \(\omega' \gg \omega\) is not very informative as it assumed very high frequency stimulus for low frequency brain states. Thus overall we have that a pre-stimulus oscillatory brain state characterized by the frequency \(\omega\), will be informative about the stimulus if the Eq. 28 holds. Interestingly, most brain dynamics is a time limited phenomena, and the frequency \(\omega\) thus depends upon the delay between the connections as shown in 25. However these delays are related to the biophysical process rates. Thus this formulation gives a very natural way of looking at which frequencies might appear in the dominant. For instance, if the delay \(\tau = 100\) ms, then \(\nu = \frac{1}{2\tau} = \frac{1}{200}\) would be of the order of 10 Hz. This idea is derived here in a non-trivial manner, i.e., just having a refractory period of 100 ms does not guarantee oscillatory solutions. Here we have shown how it is possible to have oscillations in the \(\alpha\) band can be thought to arise from internal brain dynamics.
Keil, J. et al. (2012). “On the Variability of the McGurk Effect: Audiovisual Integration Depends on Prestimulus Brain States”. In: *Cerebral Cortex* 22, pp. 221–231. issn: 1047-3211. doi: [10.1093/cercor/bhr125](https://doi.org/10.1093/cercor/bhr125)

Mandic, Danilo and Jonathon Chambers (2001). *Recurrent neural networks for prediction: learning algorithms, architectures and stability*. Wiley.

Masquelier, Timothée (2012). “Relative spike time coding and STDP-based orientation selectivity in the early visual system in natural continuous and saccadic vision: a computational model”. In: *Journal of Computational Neuroscience* 32, pp. 425–441. issn: 0929-5313. doi: [10.1007/s10827-011-0361-9](https://doi.org/10.1007/s10827-011-0361-9)

Mathewson, K. E. et al. (2009). “To See or Not to See: Prestimulus α Phase Predicts Visual Awareness”. In: *Journal of Neuroscience* 29, pp. 2725–2732. issn: 0270-6474. doi: [10.1523/jneurosci.3963-08.2009](https://doi.org/10.1523/jneurosci.3963-08.2009)

May, Elisabeth S. et al. (2012). “Pre- and post-stimulus alpha activity shows differential modulation with spatial attention during the processing of pain”. In: *NeuroImage* 62, pp. 1965–1974. issn: 1053-8119. doi: [10.1016/j.neuroimage.2012.05.071](https://doi.org/10.1016/j.neuroimage.2012.05.071)

Nakagawa, Tristan T. et al. (2014). “How delays matter in an oscillatory whole-brain spiking-neuron network model for MEG alpha-rhythms at rest”. In: *NeuroImage* 87, pp. 383–394. issn: 1053-8119. doi: [10.1016/j.neuroimage.2013.11.009](https://doi.org/10.1016/j.neuroimage.2013.11.009)

Sengupta, Rakesh, Bapi Raju Surampudi, and David Melcher (2014). “A visual sense of number emerges from the dynamics of a recurrent on-center off-surround neural network”. In: *Brain Research* 1582, pp. 114–224.

Stanley, Garrett B. (2013). “Reading and writing the neural code”. In: *Nature Neuroscience* 16, pp. 259–263. issn: 1097-6256. doi: [10.1038/nn.3330](https://doi.org/10.1038/nn.3330)

Usher, Marius and Jonathan D Cohen (1999). “Short term memory and selection processes in a frontal-lobe model”. In: *Connectionist models in Cognitive Neuroscience*. Springer, pp. 78–91.

van Rullen, R. and Simon J. Thorpe (2001). “Rate Coding Versus Temporal Order Coding: What the Retinal Ganglion Cells Tell the Visual Cortex”. In: *Neural Computation* 13, pp. 1255–1283. issn: 0899-7667. doi: [10.1162/08997660152002852](https://doi.org/10.1162/08997660152002852)

Weisz, N. et al. (2014). “Prestimulus oscillatory power and connectivity patterns predispose conscious somatosensory perception”. In: *Proceedings of the National Academy of Sciences, USA* 111, E417–E425. issn: 0027-8424. doi: [10.1073/pnas.1317267111](https://doi.org/10.1073/pnas.1317267111)

Wu, Lizhong and John Moody (1996). “A Smoothing Regularizer for Feedforward and Recurrent Neural Networks”. In: *Neural Computation* 8, pp. 461–489. issn: 0899-7667. doi: [10.1162/neco.1996.8.3.461](https://doi.org/10.1162/neco.1996.8.3.461)