Applications of the Tarski-Kantorovich Fixed-Point Principle to the study of Infinite Iterated Function Systems

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Abstract. The aim of this paper is to establish some results regarding Infinite Iterated Function Systems with the help of the Tarski-Kantorovich fixed-point principles for maps on partially ordered sets. To this end we introduce two new classes of Infinite Iterated Function Systems which are well suited for applying the aforementioned principle. We also study some properties of the canonical projection from the shift space of an Infinite Iterated Function System belonging to one of the two introduced classes to its attractor.

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1. Introduction

Let \((X, d)\) be a complete metric space and \(\mathcal{S} = (X, (f_i)_{i \in I})\) an Infinite Iterated Function System (IIFS for short) and define the operator \(F_\mathcal{S}\) on the family of nonempty closed and bounded sets of \(X\) as follows: \(F_\mathcal{S}(B) := \bigcup_{i \in I} f_i(B)\) for all \(B \subseteq X\) nonempty, closed and bounded. The fundamental result regarding IIFS’s states that if \(\sup_{i \in I} \text{lip}(f_i) < 1\), where \(\text{lip}(f_i)\) is the Lipschitz constant associated to \(f_i\), then there exists a unique nonempty, closed and bounded subset of \(X\), \(A(\mathcal{S})\), such that \(F_\mathcal{S}(A(\mathcal{S})) = A(\mathcal{S})\), which is called the attractor of the IIFS.

In this paper we follow in the footsteps of the article [8] and study possibilities of applying the Tarski-Kantorovitch fixed-point principle in the theory...
of IIFS’s, which naturally conducts us to consider two new classes of IIFS’s in certain topological spaces. The idea of applying the Tarki-Kantorovitch principle to deduce results about the fixed points of some classes of maps is not necessarily new - for other applications one may consult [1], [3], [7]. Our paper is organized in the following way. In the second section we will recall some definitions and results regarding IIFS’s and introduce two new classes of IIFS’s.

In the third section of the article we shall study possibilities of applying the Tarski-Kantorovitch fixed point principle for the partially ordered set (poset for short) \((\wp(X), \supseteq)\), where \(X\) is an arbitrary set. In this case we give sufficient conditions for the existence of a (greatest) fixed point of the Hutchinson-Barnsley operator associated to an IIFS. These conditions turn out to be also necessary if one is interested in applying the Tarski-Kantorovitch principle.

In the fourth section we turn our attention to the poset \((\mathcal{F}(X), \supseteq)\), where \(\mathcal{F}(X)\) denotes the family of all nonempty closed subsets of a Hausdorff topological space \(X\). As seen in [8], the countable chain condition in this poset forces \(X\) to be countably compact. In the case that \(X\) is also a sequential space, the main result of this section provides sufficient conditions for the existence of a greatest fixed point of the Hutchinson-Barnsley operator associated to an IIFS. As in the previous section, the specified conditions are also necessary for applying the Tarski-Kantorovitch principle.

In the fifth section we are employing similar techniques as in the previous sections in order to apply the Tarski-Kantorovitch principle to the poset \((\mathcal{K}(X), \supseteq)\), where \(\mathcal{K}(X)\) denotes the family of nonempty compact subsets of a topological space \(X\). In particular, we discover that in the same way as in [8], in order to apply the Tarski-Kantorovitch principle in this case, we can actually assume that \(X\) is compact. We also show that if \((X, d)\) is a bounded Heine-Borel metric space, then the Hutchinson-Barnsley operator associated to an IIFS of contractions admits a nonempty compact fixed point (though we cannot posit that this fixed point is unique from the proof provided).

In the final section of the article we turn our attention to the shift space associated to an IIFS and the canonical projection from the shift space to the attractor of said IIFS and investigate what special properties this projection has in the cases of the two new classes of IIFS’s introduced in this article. We also provide the reader with a sufficient condition for the canonical projection to be a homeomorphism and state a few immediate corollaries.

As a final remark in this introductory part of the article, we want to stress the fact that the generality of the setting in which we work, i.e. that of Infinite Iterated Function Systems, has forced us to impose (fairly natural) conditions on the systems we work with in order to apply the Tarski-Kantorovitch Fixed-Point Principle and the two examples we provide tell us the introduction of these classes is in fact necessary.

For more work on Infinite Iterated Function Systems, you can also check [9], [10], [11], [12], [13].
2. Preliminaries

Definition 2.1. Let \((X, d_X), (Y, d_Y)\) be two metric spaces and \((f_i)_{i \in I} \subseteq Y^X\) a family of maps. We say that this family is bounded if the set \(\bigcup_{i \in I} f_i(A)\) is bounded for any \(A \subseteq X\) bounded.

Definition 2.2. Let \((X, d_X), (Y, d_Y)\) be metric spaces and \(f : X \to Y\) a function. The quantity \(\text{lip}(f) := \sup_{x, y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \in [0, \infty]\) is called the Lipschitz constant associated to \(f\). We say that \(f\) is Lipschitz if \(\text{lip}(f) < \infty\) and that \(f\) is a contraction if \(\text{lip}(f) < 1\).

As an immediate consequence of the previous definition, we have the following lemma.

Lemma 2.1. Let \((X, d_X), (Y, d_Y)\) be two metric spaces and \(f : X \to Y\) a map. Then if we denote the diameter of a subset \(A \subseteq X\) with \(\delta(A)\), we have that \(\delta(f(A)) \leq \text{lip}(f) \delta(A)\) for any \(A \subseteq X\). In particular, if \(x, y \in X\), then \(d_Y(f(x), f(y)) \leq \text{lip}(f) d_X(x, y)\).

Remark 2.1. For an arbitrary topological space \(X\), we shall denote by \(\mathcal{P}(X)\) the family of subsets of \(X\), \(\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}\), \(\mathcal{K}(X)\) the family of nonempty compact subsets of \(X\) and by \(\mathcal{F}(X)\) we mean the family of nonempty closed sets of \(X\). If \(X\) is also metrizable, we shall denote by \(\mathcal{B}(X)\) the family of nonempty closed and bounded subsets of \(X\). In the latter case, note that we have the inclusions \(\mathcal{K}(X) \subseteq \mathcal{B}(X) \subseteq \mathcal{P}^*(X)\). However, if \(X\) is not necessarily metrizable, but it is Hausdorff, we have that \(\mathcal{K}(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{P}^*(X)\).

Definition 2.3. Let \(X, Y\) be two arbitrary sets, \(f : X \to Y\) a function and \(y \in Y\). We shall call the set \(f^{-1}(y)\) the fibre of \(f\) over \(y\).

Definition 2.4. We say that a topological space \(X\) is sequential if every sequentially closed subset \(A \subseteq X\) is closed.

We recall the following characterisation of continuity on countably compact sequential spaces from [8].

Theorem 2.1. Let \(X, Y\) be countably compact and sequential spaces and \(f : X \to Y\) a map. The following conditions are equivalent:

a) \(f\) is continuous;
b) if \(A \in \mathcal{F}(X)\), then \(f(A) \in \mathcal{F}(Y)\) and all fibres of \(f\) are closed;
c) if \(A \in \mathcal{F}(X)\), then \(f(A) \in \mathcal{F}(Y)\) and given a decreasing sequence \((A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}(X)\), then \(f(\bigcap_{n \in \mathbb{N}} A_n) = \bigcap_{n \in \mathbb{N}} f(A_n)\);
d) if \(A \in \mathcal{K}(X)\), then \(f(A) \in \mathcal{K}(Y)\) and given a decreasing sequence \((A_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)\), then \(f(\bigcap_{n \in \mathbb{N}} A_n) = \bigcap_{n \in \mathbb{N}} f(A_n)\).

Definition 2.5. Let \((X, d)\) be a metric space. The generalised Hausdorff-Pompeiu semimetric on the family of subsets of \(X\) induced by \(d\) is defined as \(h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, \infty]\), where

\[
h(A, B) := \max\{d(A, B), d(B, A)\}
\]
and
\[ d(A, B) := \sup \inf_{x \in A, y \in B} d(x, y) \] (2)
for all \( A, B \in \mathcal{P} \ast \ast (X) \).

Several properties of the Hausdorff-Pompeiu semimetric can be found in [2], [6], [14].

**Definition 2.6.** Let \( X \) be a topological space. We say that \( S = (X, (f_i)_{i \in I}) \) is an IIFS if \( f_i \) is a selfmap of \( X \) for all \( i \in I \).

**Definition 2.7.** An IIFS \( S = (X, (f_i)_{i \in I}) \) is said to be non-overlapping if \( f_i(B) \cap f_j(B) = \emptyset \) for any \( B \subseteq X \) and \( i, j \in I, i \neq j \).

**Remark 2.2.** Obviously, an IIFS as above is non-overlapping if and only if \( f_i(X) \cap f_j(X) = \emptyset \) for any \( i, j \in I, i \neq j \).

**Definition 2.8.** An IIFS \( S = (X, (f_i)_{i \in I}) \) is said to be locally finite if for any \( y \in X \) there exists a neighbourhood \( V_y \) of \( y \) such that \( \#\{i \in I : V_y \cap f_i(X) \neq \emptyset\} < \infty \).

**Remark 2.3.** Let \( S = (X, (f_i)_{i \in I}) \) be a locally finite IIFS, \( y \in X, A \subseteq X \) and \( V_y \) as in the definition above. Then the set \( \{i \in I : V_y \cap f_i(A) \neq \emptyset\} \) is finite.

**Definition 2.9.** An IIFS \( S = (X, (f_i)_{i \in I}) \) on a metric space \( (X, d) \) is said to be an IIFS of contractions if \( (f_i)_{i \in I} \subseteq \mathcal{X} \) is a bounded family of contractions such that \( \sup_{i \in I} \text{lip}(f_i) =: c < 1 \).

**Definition 2.10.** To an IIFS \( S = (X, (f_i)_{i \in I}) \) we can associate two Hutchinson-Barnsley operators, namely \( F_S, G_S : \mathcal{P} \ast \ast (X) \to \mathcal{P} \ast \ast (X) \) given by
\[ F_S(A) := \bigcup_{i \in I} f_i(A) \] (3)
and
\[ G_S(A) := \bigcup_{i \in I} f_i(A) \] (4)
for all \( \emptyset \neq A \subseteq X \).

**Remark 2.4.** Note that if \( S \) is an IIFS of contractions, then \( G_S \) is a contraction with respect to the Hausdorff-Pompeiu metric on \( \mathcal{B}(X) \) and in fact \( \text{lip}(G_S) \leq c = \sup_{i \in I} \text{lip}(f_i) \).

A straightforward application of the Banach-Caccioppoli-Picard contraction principle yields the following result, which is the fundamental result in the theory of IIFS’s (for the proof, you can check [14]).

**Theorem 2.2.** Let \((X, d)\) be a complete metric space and \( S = (X, (f_i)_{i \in I}) \) an IIFS of contractions. Then we may consider \( G_S : \mathcal{B}(X) \to \mathcal{B}(X) \) and in this case there exists a unique set \( A = A(S) \in \mathcal{B}(X) \) such that \( G_S(A) = A \). We call this set the attractor of \( S \). Moreover, if \( A_0 \in \mathcal{B}(X) \) and \( A_n := G_S(A_{n-1}) \)
for any \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} A_n = A \). As for the speed of convergence, we have the following estimate:

\[
h(A_n, A) \leq \frac{cn}{1-c} h(A_0, A_1)
\]

for all \( n \geq 0 \).

Regarding the shift space associated to an IIFS of contractions on a metric space \((X, d)\), we have the following definitions and main theorem from [10].

**Definition 2.11.** Let \( I \neq \emptyset \). We define:

a) the space \( \Lambda = \Lambda(I) := I^\mathbb{N} \) as the space of infinite words with letters from the alphabet \(I\). An element \( \omega \in \Lambda(I) \) will be written as \( \omega = \omega_1 \omega_2 \ldots \omega_n \omega_{n+1} \ldots \);

b) for \( m \in \mathbb{N} \), the space \( \Lambda_m = \Lambda_m(I) \) of words of length \( m \) with letters from the alphabet \(I\). An element \( \omega \in \Lambda_m(I) \) will be written as \( \omega = \omega_1 \omega_2 \ldots \omega_m \). For \( \omega \in \Lambda_m \) or \( \omega \in \Lambda \) and \( n \in \mathbb{N}, n \leq m \), we denote \([\omega]_n := \omega_1 \ldots \omega_n \);

c) the space \( \Lambda^* = \Lambda^*(I) := \bigcup_{m \in \mathbb{N}} \Lambda_m(I) \cup \{\lambda\} \) of finite words, where \( \lambda \) is the empty word;

d) for \( m, n \in \mathbb{N} \) and \( \alpha \in \Lambda_n(I), \beta \in \Lambda_m(I) \) or \( \beta \in \Lambda(I) \), one may define the concatenated words \( \alpha \beta := \alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_m \) in \( \Lambda_{n+m} \) and \( \alpha \beta = \alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_m \beta_{m+1} \ldots \in \Lambda \), respectively;

e) the metric \( d_{\Lambda} \) on \( \Lambda(I) \) given by \( d_\Lambda(\alpha, \beta) := \frac{1}{\sum_{m \in \mathbb{N}} \frac{1-\delta_{\alpha, \beta}^m}{3^m}} \), where \( \delta_x \) denotes the Kronecker delta of \( x \) and \( y \). Note that \( (\Lambda(I), d_\Lambda) \) is a complete metric space and convergence in \( \Lambda(I) \) coincides with the convergence on components;

f) for \( i \in I \), the right shift function \( F_i : \Lambda(I) \to \Lambda(I) \) given by \( F_i(\omega) := i \omega \) for all \( \omega \in \Lambda(I) \). Note that \( F_i \) is a \( \frac{1}{\lambda} \)-similarity of \( \Lambda(I) \);

g) for \( m \in \mathbb{N} \) and \( \omega \in \Lambda_m(I), F_\omega := F_{\omega_1} \circ \cdots \circ F_{\omega_m} \) and \( \Lambda_\omega := F_\omega(\Lambda) \). By convention \( F_\lambda := id\Lambda(I) \) and \( \Lambda_\lambda = \Lambda \). Note that \( \Lambda(I) = \bigcup_{i \in I} F_i(\Lambda(I)) \), so \( \Lambda(I) \) is the attractor of the IIFS of contractions \((\Lambda(I), (F_i)_{i \in I})\) and for every \( m \in \mathbb{N} \) and \( \omega \in \Lambda^*, \Lambda = \bigcup_{\alpha \in \Lambda_m} \Lambda_\alpha \) and \( \Lambda_\omega = \bigcup_{\alpha \in \Lambda_m} \Lambda_{\omega \alpha} \);

h) if \( (X, d) \) is a metric space, \( S = (X, (f_i)_{i \in I}) \) is an IIFS of contractions on \( X \), \( B \subseteq X \) and \( \omega \in \Lambda_m(I) \), let \( f_\omega := f_{\omega_1} \circ \cdots \circ f_{\omega_m} \) and \( B_\omega := f_\omega(B) \). By convention \( f_\lambda := idX \) and \( B_\lambda = B \);

i) in the setting above, if \( f : X \to X \) is a contraction and \((X, d)\) is complete, we denote its fixed point by \( e_f \). If \( f = f_\omega \) for some \( \omega \in \Lambda^* \), we denote \( e_f = e_{f_\omega} = e_\omega \).

**Theorem 2.3.** Let \( (X, d) \) be a complete metric space, \( S = (X, (f_i)_{i \in I}) \), \( A = A(S) \) its attractor and \( c := \sup_{i \in I} \lip(f_i) < 1 \). Then the assertions below hold:

a) for any \( m \in \mathbb{N} \) and \( \omega \in \Lambda(I) \), we have that \( A_{\omega_{m+1}} \subseteq A_\omega \) and \( \delta(A_{\omega_m}) \to 0 \). More precisely, \( \delta(A_{\omega_m}) = \delta(A_{\omega_m}) \leq c^m \delta(A) \);

b) if \( a_\omega \) is defined by \( \{a_\omega\} := \bigcap_{m \in \mathbb{N}} A_{\omega_m} \), where \( \omega \in \Lambda(I) \), then

\[
\lim_{m \to \infty} d(e_{\omega_m}, a_\omega) = 0;
\]
c) for every $a \in A$ and $\omega \in \Lambda(I)$, we have $\lim_{m \to \infty} f[\omega]_m(a) = a_\omega$;

d) for every $\alpha \in \Lambda^*$, we have $A = A(S) = \bigcup_{\omega \in \Lambda} \{a_\omega\}$ and $A_\alpha = \bigcup_{\omega \in \Lambda} \{a_\omega\}$. If $A = \bigcup_{i \in I} f_i(A)$, then $A = A(S) = \bigcup_{\omega \in \Lambda} \{a_\omega\}$;

e) we have $A = \{e[\omega]_m : \omega \in \Lambda, m \in \mathbb{N}\}$;

f) the function $\pi : \Lambda(I) \to A$, defined by $\pi(\omega) = a_\omega$ for every $\omega \in \Lambda(I)$, has the following properties:

i) $\pi$ is continuous;

ii) $\pi(\Lambda) = A$;

iii) if $A = \bigcup_{i \in I} f_i(A)$, then $\pi$ is surjective;

g) for every $i \in I$, we have that $\pi \circ f_i = f_i \circ \pi$.

Finally we shall state the Tarski-Kantorovitch fixed-point principle.

**Definition 2.12.** Let $(P, \leq)$ be a poset and $F : P \to P$. We say that $F$ is $\leq$-continuous if for every countable chain $C$ admitting a supremum, we have that $F(C)$ has a supremum and $F(\sup C) = \sup F(C)$. Note that in this case $F$ is increasing.

**Theorem 2.4.** (Tarski-Kantorovich) Let $(P, \leq)$ be a poset in which every countable chain admits a supremum and $F : P \to P$ a $\leq$-continuous map such that there exists $a \in P$ with $a \leq F(a)$. Then $F$ has a fixed point. Moreover, $\sup_{n \in \mathbb{N}} F^n(a)$ is the least fixed point of $F$ in the set $\{p \in P : p \geq a\}$.

**Remark 2.5.** Note that we can replace the assumption that every countable chain $C$ admits a supremum with the assumption that each increasing sequence $(p_n)_{n \in \mathbb{N}} \subseteq P$ admits a supremum.

As in [5] we shall assume that every compact or countably compact space is Hausdorff.

3. The Hutchinson-Barnsley operator on $(\wp(X), \supseteq)$

Let $X$ be an arbitrary set and $f : X \to X$ a selfmap of $X$. Proposition 1 from [8] shows that the function $F : \wp(X) \to \wp(X)$ defined by $F(A) := f(A)$ for all $A \subseteq X$ is $\supseteq$-continuous if and only if all fibres of $f$ are finite.

**Remark 3.1.** Note that the poset $(\wp(X), \supseteq)$ satisfies the countable chain condition, as if $(A_n)_{n \in \mathbb{N}} \subseteq \wp(X)$, then $\sup_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} A_n$ (and, of course, $\inf_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A_n$).

Our main result in this section is the following:

**Theorem 3.1.** Let $S = (X, (f_i)_{i \in I})$ be a non-overlapping IIFS such that all the fibres of $f_i$ are finite for each $i \in I$ and let $F_S(A) := \bigcup_{i \in I} f_i(A)$ for any $A \subseteq X$. Then for each $A \subseteq X$ such that $F_S(A) \subseteq A$, the set $\bigcap_{n \in \mathbb{N}} F_S^n(A)$ is a fixed point of $F_S$. In particular, $\bigcap_{n \in \mathbb{N}} F_S^n(X)$ is the greatest fixed point of $F_S$. It follows that $S$ admits a nonempty fixed point if and only if $\bigcap_{n \in \mathbb{N}} F_S^n(X) \neq \emptyset$.
Proof. We have seen that the poset \((\varnothing(X), \supseteq)\) satisfies the countable chain condition and obviously \(F_S(A) \in \varnothing(X)\) for every \(A \subseteq X\). We shall prove that \(F_S\) is \(\supseteq\) -continuous.

Let \((C_n)_{n \in \mathbb{N}} \subseteq \varnothing(X)\) be a \(\supseteq\) -increasing sequence, i.e. decreasing in the usual sense. Obviously, \(F_S\) is increasing, so \((F_S(C_n))_{n \in \mathbb{N}} \subseteq \varnothing(X)\) is a decreasing sequence as well. It remains to prove that \(F_S(\bigcup_{n \in \mathbb{N}} C_n) = \bigcup_{n \in \mathbb{N}} F_S(C_n)\), i.e. \(F_S(\bigcap_{n \in \mathbb{N}} C_n) = \bigcap_{n \in \mathbb{N}} F_S(C_n)\).

Let \(y \in F_S(\bigcap_{n \in \mathbb{N}} C_n) = \bigcup_{i \in I} f_i(\bigcap_{n \in \mathbb{N}} C_n)\). Then there exist \(i \in I\) and \(x \in \bigcap_{n \in \mathbb{N}} C_n\) such that \(y = f_i(x)\). Then \(y \in f_i(C_n)\) for all \(n \in \mathbb{N}\), so \(y \in \bigcap_{n \in \mathbb{N}} \bigcup_{i \in I} f_i(C_n) = \bigcap_{n \in \mathbb{N}} F_S(C_n)\).

Conversely, let \(y \in \bigcap_{n \in \mathbb{N}} \bigcup_{i \in I} f_i(C_n)\). Then for all \(n \in \mathbb{N}\) there exist \(i_n \in I\) and \(x_n \in C_n\) such that \(y = f_{i_n}(x_n)\). Note that since the sequence \((C_n)_{n \in \mathbb{N}}\) is decreasing, we have that \(y \in f_{i_{n+1}}(C_{n+1}) \subseteq f_{i_{n+1}}(C_n)\), so \(y \in f_{i_{n+1}}(C_n) \cap f_{i_n}(C_n)\), and \(i_{n+1} = i_n\) for all \(n \in \mathbb{N}\) since \((f_i)_{i \in I}\) is non-overlapping. Thus, there exists \(i_* \in I\) such that \(y \in f_{i_*}(C_n)\) for all \(n \in \mathbb{N}\) and write \(y = f_{i_*}(x_n)\), where \(x_n \in C_n\) as before. It follows that \((x_n)_{n \in \mathbb{N}} \subseteq f_{i_*}^{-1}(y)\).

Since this fibre is finite, it follows that we may find \((x_{n_k})_{k \in \mathbb{N}} \subseteq (x_n)_{n \in \mathbb{N}}\) a subsequence and \(x_* \in f_{i_*}^{-1}(y)\) such that \(x_{n_k} = x_*\) for all \(k \in \mathbb{N}\). Since \((C_n)_{n \in \mathbb{N}}\) is decreasing and \(x_* = x_{n_k} \in C_{n_k}\) for all \(k \in \mathbb{N}\), it follows that \(x_* \in \bigcap_{n \in \mathbb{N}} C_n\).

Since \(x_* \in f_{i_*}^{-1}(y)\), we deduce that \(y \in \bigcup_{i \in I} f_{i}(\bigcap_{n \in \mathbb{N}} C_n) = F_S(\bigcap_{n \in \mathbb{N}} C_n)\), which concludes the proof that \(F_S\) is \(\supseteq\) -continuous.

Thus, the conditions of the Tarski-Kantorovitch fixed-point principle are satisfied and the first part of the theorem follows from a direct application of this principle.

If \(A\) is a fixed point of \(F_S\), then \(A = F_S(A)\), so \(A = F^n_S(A)\) for all \(n \in \mathbb{N}\). Hence, \(A = \bigcap_{n \in \mathbb{N}} F^n_S(A) \subseteq \bigcap_{n \in \mathbb{N}} F^n_S(X)\), so that \(\bigcap_{n \in \mathbb{N}} F^n_S(X)\) is indeed the greatest fixed point of \(F_S\). The last part of the theorem is trivial. \(\Box\)

Remark 3.2. Consider \(X := [0, 1], (f_m)_{m \geq 2} \subseteq X^X, f_m(x) := \{x + \frac{1}{m}\}, \{0\}\) where \(\{0\}\) symbolises the fractional part of \(x\) and \((C_n)_{n \geq 3}, C_n := [0, \frac{1}{m}] \cup [1 - \frac{1}{m}, 1]\). Obviously, the fibres of each \(f_m\) are finite, but the system is not non-overlapping. Clearly, \(\bigcap_{n \geq 3} C_n = \{0\}\), so \(\bigcup_{m \geq 2} f_m(\bigcap_{n \geq 3} C_n) = \{\frac{1}{2}, \frac{1}{3}, \ldots\}\).

Also note that \(0 \in f_n(C_n)\) for all \(n \geq 3\). It follows that \(0 \in \bigcup_{m \geq 2} f_m(C_n)\) for all \(n \geq 3\), so \(0 \in \bigcap_{n \geq 3} \bigcup_{m \geq 2} f_m(C_n)\). What this simple example tells us is that we actually need to assume that the IIFS considered in the statement of the previous theorem is non-overlapping, otherwise the Hutchinson-Barnsley operator need not be continuous with respect to \(\supseteq\).

4. The Hutchinson-Barnsley operator on \((\mathcal{F}(X), \supseteq)\)

In what follows \(X\) will denote a Hausdorff topological space and \((\mathcal{F}(X), \supseteq)\) is the poset of nonempty closed parts of \(X\) ordered by \(\supseteq\). In order to apply the Tarski-Kantorovitch fixed-point principle, we need to have that each countable chain in \((\mathcal{F}(X), \supseteq)\) admits a supremum. As shown in Proposition 4 from [8], this assumption restricts our attention to countably compact
spaces. In fact we will be looking at countably compact sequential spaces $X$ and establish results regarding the operators $F_S$ and $G_S$ in this setting. Note that in such a space, given a decreasing sequence $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}(X)$, its supremum is simply $\bigcap_{n \in \mathbb{N}} C_n$.

**Remark 4.1.** As $X$ is a countably compact sequential space, Theorem 2.1 assures us that a function $f : X \rightarrow X$ is continuous if and only if $f(A)$ is closed for each $A \in \mathcal{F}(X)$ and all fibres of $f$ are closed.

The main result of this section is the following:

**Theorem 4.1.** Let $X$ be a countably compact sequential space and $S = (X, (f_i)_{i \in I})$ a locally finite non-overlapping IIFS, where each $f_i$ is continuous. Then $F_S(\mathcal{F}(X)) \subseteq \mathcal{F}(X)$, $\bigcap_{n \in \mathbb{N}} F^n_S(X)$ is nonempty and closed and it is the greatest fixed point of $F_S$. Moreover, if $X$ is metrizable, then the sequence $(F^n_S(X))_{n \in \mathbb{N}} \subseteq \mathcal{F}(X)$ converges to $\bigcap_{n \in \mathbb{N}} F^n_S(X)$ with respect to the Hausdorff-Pompeiu metric.

**Proof.** First we prove that we can indeed consider the restriction $F_S : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$. Thus, let $A \in \mathcal{F}(X)$. We shall prove that

$$F_S(A) = \bigcup_{i \in I} f_i(A) = \bigcup_{i \in I} \overline{f_i(A)} = \overline{\bigcup_{i \in I} f_i(A)} = \overline{F_S(A)}.$$  

The direct inclusion is trivial by definition, so we only prove the converse. It suffices to show that $F_S(A)$ is sequentially closed.

Indeed, let $y \in \bigcup_{i \in I} f_i(A)$ and $(y_n)_{n \in \mathbb{N}} \subseteq \bigcup_{i \in I} f_i(A)$ such that $y_n \rightarrow y$. Let $V_y$ be the neighbourhood of $y$ provided by the local-finiteness of the IIFS, i.e. $\#\{i \in I : V_y \cap f_i(X) \neq \emptyset\} < \infty$. Then $V_y$ intersects only finitely many of the sets $(f_i(A))_{i \in I}$. Since $y_n \rightarrow y$, we may assume that $(y_n)_{n \in \mathbb{N}} \subseteq V_y$. Define $i_n \in I$ to be the subscript such that $y_n \in f_{i_n}(A)$ (it is well defined since this IIFS is non-overlapping). It follows that $\#\{i_n : n \in \mathbb{N}\} < \infty$, so we may find a subsequence $(y_{n_k})_{k \in \mathbb{N}} \subseteq (y_n)_{n \in \mathbb{N}}$ such that $i_{n_k} = i_{n_1}$ and $y_{n_k} \in f_{i_{n_1}}(A)$ for all $k \in \mathbb{N}$. Since $X$ is Hausdorff, we deduce that $y_{n_k} \rightarrow y,$ so that $y \in \overline{f_{i_{n_1}}(A)} \subseteq \bigcup_{i \in I} f_i(A) = \bigcup_{i \in I} f_i(A) = F_S(A)$, proving that $F_S(A)$ is indeed closed.

$F_S$ is clearly increasing and the continuity of this operator with respect to $\supseteq$ is shown in the same way as in Theorem 3.1.

It is trivial that $F_S(X) \subseteq X$. Hence, all the conditions stated in the Tarski-Kantorovitch fixed-point principle are satisfied and the first part of the theorem follows directly from this.

For the last part of the theorem, note that if $X$ is metrizable, the sequence $(F^n_S(X))_{n \in \mathbb{N}}$ is $\supseteq$–increasing, so it converges to $\sup_{n \in \mathbb{N}} F^n_S(X) = \bigcap_{n \in \mathbb{N}} F^n_S(X)$ with respect to the Hausdorff-Pompeiu metric (see [4]).

In the proof of the last theorem, we also obtained a result about the other Hutchinson-Barnley operator, $G_S$.

**Corollary 4.1.** Let $X$ be countably compact and sequential space, $S = (X, (f_i)_{i \in I})$ a locally finite non-overlapping IIFS. Then $F_S(A) = \bigcup_{i \in I} f_i(A) = \bigcup_{i \in I} \overline{f_i(A)} = G_S(A)$ for all $A \in \mathcal{F}(X)$.
Remark 4.2. Let $X = [0, 1]$ (which is compact and sequential) and $(f_m)_{m \in \mathbb{N}} \subseteq X^X$ given by $f_m(x) := \frac{x + 1}{2m+1}$ for all $x \in X$ and $m \in \mathbb{N}$. Obviously, the image of $f_m$ is the closed interval $[\frac{1}{2m+1}, \frac{1}{2m-1}]$ and $f_m$ is continuous for any $m \in \mathbb{N}$. It clearly follows that the IIFS $S = (X, (f_m)_{m \in \mathbb{N}})$ is non-overlapping, but not locally finite (because it is not locally finite at 0). It is also clear that $0 \in \bigcup_{m \in \mathbb{N}} f_m(X) \setminus \bigcup_{m \in \mathbb{N}} f_m(X)$, so in this case it is not true that $F_S(S(X)) \subseteq F(X)$. This example shows us that we cannot drop the condition that the IIFS considered in Theorem 4.1 is locally finite, because in that case we could have that $F_S(S(X)) \notin F(X)$ and we wouldn’t be able to apply the Tarski-Kantorovitch fixed-point principle.

Also note that there exist locally finite non-overlapping IIFS’s. Indeed, for a nonempty set $I$ consider the IIFS $S = (\Lambda(I), (F_i)_{i \in I})$. This is clearly non-overlapping. Moreover, if $\omega \in \Lambda(I)$ is arbitrary, consider the open set $U_\omega := \{\eta \in \Lambda(I) : d_\Lambda(\omega, \eta) < \frac{1}{3}\}$. If $\eta \in \Lambda(I)$ and $\eta_1 \neq \omega_1$, then $d_\Lambda(\omega, \eta) \geq \frac{1}{3}$, so $\eta \notin U_\omega$. It follows that $U_\omega$ only intersects $\Lambda_{\omega_1} = F_{\omega_1}(\Lambda(I))$, so this IIFS is also locally finite.

Remark 4.3. Finally, note that we can extend Theorem 4.1 and Corollary 4.1 on the poset $(B(X), \supseteq)$ if $X$ is metrizable, the IIFS considered is also bounded and there exists $B \in B(X)$ such that $F_S(B) \subseteq B$. Indeed, note that in this case $F_S(B(X)) \subseteq B(X)$ since the IIFS is bounded and Theorem 4.1 clearly shows us that $F_S(S(X)) \subseteq F(X)$, so we may apply the Tarski-Kantorovitch fixed-point principle to the poset $(B(X), \supseteq)$ and $F_S$. Note that in the case that such a $B$ exists, then $\bigcap_{n \in \mathbb{N}} F^n_S(B)$ is nonempty, closed and bounded and it is the greatest fixed point of $F_S$ contained in $B$.

5. The Hutchinson-Barnsley operator on $(\mathcal{K}(X), \supseteq)$

Henceforth $X$ will be a Hausdorff topological space and $(\mathcal{K}(X), \supseteq)$ will denote the poset of nonempty compact subsets of $X$ ordered by $\supseteq$. Note that in this case every countable chain admits a supremum and if $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$ is a decreasing sequence, then its supremum in this poset is simply $\bigcap_{n \in \mathbb{N}} C_n$.

Remark 5.1. Note that we may assume that $X$ is compact. Indeed, let $S = (X, (f_i)_{i \in I})$ be an IIFS and $F_S(A) := \bigcup_{i \in I} f_i(A)$ for all $A \subseteq X$ just as before. In order to apply the Tarski-Kantorovitch fixed-point principle to $(\mathcal{K}(X), \supseteq)$ and $F_S$, we would need the existence of a nonempty compact subset $B \subseteq X$ such that $F_S(B) \subseteq B$. Then we may simply consider the restricted IIFS $S|_B := (B, (f_i|_B)_{i \in I})$ and establish the desired result in the poset $(\mathcal{K}(B), \supseteq)$.

The main result of this section is the following:

Theorem 5.1. Let $X$ be a compact space and $S = (X, (f_i)_{i \in I})$ a locally finite non-overlapping IIFS, where $f_i$ is continuous for all $i \in I$. Then $F_S(\mathcal{K}(X)) \subseteq \mathcal{K}(X)$, $\bigcap_{n \in \mathbb{N}} F^n_S(X)$ is nonempty and compact and it is the greatest fixed point of $F_S$. Moreover, if $X$ is metrizable, then the sequence $(F^n_S(X))_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$.
\( \mathcal{K}(X) \) converges to \( \bigcap_{n \in \mathbb{N}} F_S^n(X) \) with respect to the Hausdorff-Pompeiu metric.

**Proof.** Note that all we need to prove is that \( F_S(\mathcal{K}(X)) \subseteq \mathcal{K}(X) \), as the continuity of \( F_S \) with respect to \( \sup \) follows in the same way as in Theorem 3.1 and we clearly have that \( F_S(X) \subseteq X \) (and \( X \) is compact), so we may apply the Tarski-Kantorovitch fixed point principle to establish the first part of the theorem. Also the last part of the theorem can be proven in the same way as in Theorem 4.1.

To show that \( F_S(\mathcal{K}(X)) \subseteq \mathcal{K}(X) \), note that all compact sets are closed in Hausdorff topological spaces, so \( F_S(\mathcal{K}(X)) \subseteq F_S(\mathcal{F}(X)) \subseteq \mathcal{F}(X) \). But since closed subsets of compact spaces are compact, we deduce that indeed \( F_S(\mathcal{K}(X)) \subseteq \mathcal{K}(X) \).

**Corollary 5.1.** Let \( X \) be an arbitrary topological space and \( S = (X, (f_i)_{i \in I}) \) a locally finite non-overlapping IIFS, where \( f_i \) is continuous for all \( i \in I \). The following assertions are equivalent:

a) there exists \( A \in \mathcal{K}(X) \) such that \( F_S(A) = A \);

b) there exists \( A \in \mathcal{K}(X) \) such that \( F_S(A) \subseteq A \).

**Proof.** Obviously, \( a) \implies b) \). The converse follows from applying Theorem 5.1 to the restricted IIFS \( S \upharpoonright A \) described in Remark 5.1.

A direct application of Corollary 5.1 (thus a direct application of Theorem 5.1) is the next result, which establishes the existence of a fixed point of the Hutchinson-Barnsley operator associated to a locally finite non-overlapping IIFS of contractions on a bounded Heine-Borel metric space.

**Corollary 5.2.** Let \( (X, d) \) be a bounded Heine-Borel metric space and \( S = (X, (f_i)_{i \in I}) \) a locally finite non-overlapping IIFS of contractions, where the contractive constant of \( f_i \) is \( h_i \in (0, 1) \). Then there exists a nonempty compact subset \( A \subseteq X \) such that \( F_S(A) = A \).

**Proof.** The proof is the same as the proof of Corollary 2 from [8] with the remark that in this case, we have that \( M := \sup_{i \in I} d(e_1, e_i) < \infty \), where \( e_i \) is the fixed point of \( f_i \) for each \( i \in I \) and \( h := \sup_{i \in I} h_i < 1 \) by the definition of an IIFS of contractions.

### 6. Remarks regarding the canonical projection \( \pi : \Lambda(I) \to A(S) \) for an IIFS of contractions

Throughout this section, \((X, d)\) is a complete metric space and \( S = (X, (f_i)_{i \in I}) \) is an IIFS of contractions on \( X \). As in Definition 2.9, we will denote \( c := \sup_{i \in I} \text{lip}(f_i) < 1 \). The attractor of \( S \) will be denoted by \( A = A(S) \).

By \( \Lambda(I) \) we mean the shift space associated to this IIFS (as in Definition 2.11) and \( \pi : \Lambda(I) \to A(S) \) is the canonical projection from the shift space to the attractor of \( S \). Note that each metric space is sequential.
Proposition 6.1. With the notations above, if $c \leq \frac{1}{3}$, then $\pi$ is a contraction and $\text{lip}(\pi) \leq 3\delta(A)$.

Proof. Indeed, let $\alpha, \beta \in \Lambda(I)$, $\alpha \neq \beta$ and write $\alpha = \alpha_1\alpha_2\ldots \alpha_n\alpha_{n+1}\ldots$, $\beta = \beta_1\beta_2\ldots \beta_n\beta_{n+1}\ldots$. Define $m := \max\{i \geq 0 : \alpha_i = \beta_i\}$, where we define $\alpha_0 = \beta_0 := \lambda$. Then $\alpha_j = \beta_j$ for all $0 \leq j \leq m$ and $\alpha_{m+1} \neq \beta_{m+1}$. It follows from the definition of $d_\Lambda$ that $\frac{1}{3^{m+1}} \leq d_\Lambda(\alpha, \beta) \leq \sum_{j \geq m+1} \frac{1}{3^j} = \frac{1}{2} \frac{1}{3^m}$.

Moreover, note that $a_\alpha, a_\beta \in \overline{A[A[m]_\alpha]} = \overline{A[A[m]_\beta]}$, so $d(\pi(\alpha), \pi(\beta)) = d(a_\alpha, a_\beta) \leq \delta(A[A[m]_\alpha]) \leq c^m \delta(A)$ (by part a) of Theorem 2.3). Thus, $d(\pi(\alpha), \pi(\beta)) \leq c^m \delta(A) \leq \frac{1}{3^m} \delta(A) = \frac{1}{3^m} 3\delta(A) \leq 3\delta(A) d_\Lambda(\alpha, \beta)$. Since the inequality is also valid when $\alpha = \beta$, the conclusion follows. \hfill \Box

Proposition 6.2. If $X$ is countably compact and $S$ is also locally finite and non-overlapping, then the canonical projection $\pi : \Lambda(I) \to A$ is surjective.

Proof. It follows directly from Remark 4.3, Theorem 4.1, Corollary 4.1, the definition of the attractor of this IIFS and Proposition 5.1 from [10] (stating that $\pi$ is onto if and only if $A = \bigcup_{i \in I} f_i(A)$). \hfill \Box

Remark 6.1. Note that if the IIFS considered has the property that $f_\omega(B) \cap f_\gamma(B) = \emptyset$ for all $B \in B(X)$ and $\omega, \gamma \in \Lambda^*(I)$, $\omega \neq \gamma$ (we shall say in this case that the IIFS is strongly non-overlapping) and $X$ and $S$ satisfy the conditions in the last proposition, then $\pi$ is also injective. Indeed, let $\omega, \gamma \in \Lambda(I)$, $\omega \neq \gamma$ and let $m \in \mathbb{N}$ be such that $\omega_m \neq \gamma_m$. Since $\pi$ is surjective, Proposition 5.1 from [10] tells us that $A = A_\alpha = \bigcup_{\omega \in \Lambda(I)} \{a_\omega\}$ for any $\alpha \in \Lambda^*(I)$. Then $[\omega]_m \neq [\gamma]_m$, so $f_{\omega}[\omega]_m(A) \cap f_{\gamma}[\gamma]_m(A) = A_{[\omega]_m} \cap A_{[\gamma]_m} = \emptyset$. But $a_\omega \in A_{[\omega]_m}$ and $a_\gamma \in A_{[\gamma]_m}$, so it follows that $\pi(\omega) = a_\omega \neq a_\gamma = \pi(\gamma)$, i.e. $\pi$ is injective.

The last remark proves the following:

Proposition 6.3. If $X$ is countably compact and $S$ is also locally finite and strongly non-overlapping, then the canonical projection $\pi : \Lambda(I) \to A$ is bijective.

Finally, we will give sufficient conditions for the canonical projection to be a homeomorphism and give a few corollaries.

Theorem 6.1. Let $(X, d)$ be a complete metric space and $S = (X, (f_i)_{i \in I})$ an IIFS of bi-Lipschitz contractions with attractor $A = A_S \in B(X)$ admitting a seed space such that:

a) the coding map $\pi : I^\omega \to A_S$ is continuous and bijective (in particular, this implies that $S$ satisfies (SSC)- the strong separation condition);

b) if $c_{ij} := \inf_{x,y \in A} d(f_i(x), f_j(y)) > 0$ (from (SSC)), we ask that $c := \inf_{i,j \in I} c_{ij} > 0$.

Then $\pi : I^\omega \to A$ is a homeomorphism.

Proof. All we need to show is that the inverse of $\pi$ is continuous, i.e. $\pi^{-1} : A \to I^\omega$ is continuous. Let $l, L \in (0, 1)$ such that $ld(x,y) \leq d(f_i(x), f_i(y)) \leq Ld(x,y)$ for all $i \in I$ and $x, y \in X$. 

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Let us fix $\varepsilon > 0$ and define $\delta_\varepsilon := c \cdot l^{-1 \log_3 \varepsilon}$. For $\alpha \in I^\omega$, we want to show that $\pi^{-1}(A \cap B_X(a_{\alpha}, \delta_\varepsilon)) \subseteq B_f(\alpha, \varepsilon)$. We define $n(\cdot, \cdot)$ in the following way: $n(\alpha, \beta) := \sup\{n \geq 0 : [\alpha]_n = [\beta]_n\}$ for all $\alpha, \beta \in I^\omega$. Let us fix $\alpha, \beta \in I^\omega$, $\alpha \neq \beta$ and $n > n(\alpha, \beta) =: p$. Then we have that $d(f_{[\alpha]}(x), f_{[\beta]}(y)) = d(f_{[\alpha]}(f_{[\alpha]}(x)), f_{[\beta]}(f_{[\beta]}(y))) \geq l^p d(f_{\alpha+1}(u), f_{\beta+1}(v))$, where $u$ and $v$ are some elements in the attractor of $S$. Then by hypothesis we get $d(f_{[\alpha]}(x), f_{[\beta]}(y)) \geq l^p c_{\alpha+1}\beta+1 \geq l^p \cdot c > 0$. Keeping in mind the statement of $c)$ of Theorem 2.3, we deduce that if $d(a_{\alpha}, a_{\beta}) < \delta_\varepsilon$, then we must have that $l^p \cdot c < \delta_\varepsilon = c \cdot l^{-1 \log_3 \varepsilon}$. Therefore, we infer that $p > -\log_3 \varepsilon$. Consequently, we have that $d_I(\alpha, \beta) < \frac{1}{\log_3 \varepsilon} = \frac{1}{3^\gg \varepsilon} < 3^{\log_3 \varepsilon} = \varepsilon$. Hence the desired inclusion: $\pi^{-1}(A \cap B_X(a_{\alpha}, \delta_\varepsilon)) \subseteq B_f(\alpha, \varepsilon)$. 

**Remark 6.2.**

i) Note that the $\delta_\varepsilon$ we defined in the proof of the previous theorem does not depend on $\alpha \in I^\omega$, so $\pi^{-1}$ is actually uniformly continuous;

ii) Recall that in a metric space, a set is compact if and only if it is complete and totally bounded;

iii) Note that if $\#I < \infty$, then the second assumption is superfluous since it is always true. We want to explain why the last theorem is not necessarily very restrictive. One of the main points of the theorem is that the system considered consists in bi-Lipschitz function, which we have seen that is pivotal point of the proof. A large class of interest in the theory of iterated function systems is that of self-similar systems, i.e. systems of similarities. Obviously, every similarity is in particular a bi-Lipschitz function, so the class of systems considered is larger than that of self-similar systems. The second important assumption is that the canonical projection is continuous and bijective. If $I$ is finite, then the only condition here is that the system satisfies (SSC), which is not a big ask. If $I$ is infinite, then we also ask that this projection is surjective. Once again, a sufficiently large class of systems satisfy this condition. Finally, we asked that $\inf_{i,j \in I} c_{ij} > 0$. We are not entirely sure how much this reduces the class of functions considered when $I$ is infinite. However, having gained some insight from the proof of the last theorem, we actually deduce a lesser condition which allows us to conclude that the inverse of the coding map is continuous. More exactly, we want the following condition to hold: given $\alpha \in I^\omega$, the number $c_\alpha := \inf_{n \in \mathbb{N}, j \in I} c_{\alpha n j}$ is strictly positive. In this case we lose the uniform continuity of $\pi^{-1}$, but what matters is that $\pi^{-1}$ is still continuous;

iv) Note that $I^\omega$ is compact if and only if $I$ is finite. Indeed, since $I^\omega$ is a metric space, it is easier to prove that $I^\omega$ is sequentially compact if and only if $I$ is finite, which is fairly easy to see.

**Corollary 6.1.** If $\pi : I^\omega \to A_S$ is a homeomorphism, then the attractor of $S$ is totally disconnected.

**Proof.** This follows immediately from the fact that $I^\omega$ is totally disconnected in the topology induced by $d_I$. 

\[ \Box \]
Corollary 6.2. All finite iterated function systems of bi-Lipschitz functions satisfying (SSC) have the property that their attractor is homeomorphic to their associated shift space.

Proof. This follows immediately from the comments made in the last remark and the last theorem.

Corollary 6.3. Let $S$ be an IIFS of bi-Lipschitz functions satisfying (SSC) and whose coding map is surjective. Assume that the attractor of $S$ is compact. Then $\pi$ cannot be a homeomorphism. In particular, neither condition $b)$, nor the condition stated in the previous remark holds.

Corollary 6.4. If $S$ is an IIFS of bi-Lipschitz functions satisfying the conditions of the last theorem, then the attractor of $S$ is not totally bounded.

Proof. Obviously, it would be true that $A$ is homeomorphic to the shift space of $S$. But $I^\omega$ is not compact. Therefore, neither is the attractor of $S$. However, $A$ is closed in a complete metric space, so it is also complete. Since it is not compact, we deduce that it is not totally bounded.

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