PV criticality of Achucarro-Ortiz black hole in the presence of higher order quantum and GUP corrections

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In this paper, we study quantum-corrected and GUP-(generalized uncertainty principle) corrected thermodynamics of the (2+1)-dimensional charged-rotating Achucarro-Ortiz (AO) black hole. The corrected parameters include temperature, entropy, and heat capacity which help in investigating the instability phases of the AO black hole. We show that AO black hole with small mass possesses unstable regions. However, we reveal that those instabilities can be removed by the GUP corrections. Finally, we also compute the maximum temperature that can be reached by the AO black hole.

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**I. INTRODUCTION**

Banados-Teitelboim- Zanelli (BTZ) black hole [1] is one of the important kinds of black holes in three dimensional space-time. The rotating charged BTZ black hole solution was studied by Achucarro and Ortiz [2], theoretical physics and cosmology [3, 4], [8, 9]. For example, the effects of quantum fluctuations (first order) on the properties of a charged BTZ black hole in massive gravity were studied in the Ref. [8]. Recently, rotating BTZ black hole with higher order corrections of the entropy has been investigated and shown that it elicits some instabilities [10]. Besides, it was seen that when a logarithmic correction is considered for the uncharged BTZ black hole, the leading-order corrections yield an instability while the higher order corrections remove them [11]. It is worth noting that thermodynamics of higher dimensional black holes with higher order thermal fluctuations have been studied in [12].

In the present study, our main goal is to study the thermodynamics of the AO black hole with higher order quantum corrections. In particular, we study how the correction terms affect the stability of the AO black hole. Correction terms include logarithmic one [13, 14] together with the higher order term, which is proportional to the inverse of the entropy [15]. These correction terms indeed arise from the thermal fluctuations of the statistical physics, which may be interpreted as quantum corrections when the size of the black hole becomes infinitesimally small [16]. Thermal fluctuations are important in several backgrounds like a hyperscaling violation background [17]. When we have infinitesimal black hole with quantum effects, then one can consider the quantum gravity effects [18, 19] (see, for example, [5–7, 20–23]). Moreover, the logarithmic and higher order corrections are considered to study the critical thermodynamics behaviors of some black object like a dyonic charged AdS black hole [24], charged dilatonic black Saturn [25, 26], and AdS black holes in massive gravity [27]. In that case, it is possible to have the holographic dual of a Van der Waals fluid. Hence, we shall investigate the $P-V$ diagram of the AO black hole to find such a Van der Waals behavior in the presence of higher order corrections [28–42]. On the other hand, the emergence of a minimal observable distance yields to the generalized uncertainty principle (GUP) [43–45], which may be affect the black hole temperature and entropy. One of the key frontiers in modern theoretical physics is to construct a renormalizable, UV complete and non-perturbative theory of quantum gravity which could explain the features near the singularity of black hole and the Big Bang. Although numerous candidates of such theories are proposed, however, most of them offer no testable predictions or even untestable experimentally. Any quantum gravity (QG) theory should offer additional terms or correction terms to the results of semi-classical physics. In this connection, one of the aims of 2+1 dimensional QG is construct toy models of black holes and analyze their thermal properties. Since GUP and log corrections are motivated by numerous QG theories, it is imperative to apply these corrections to lower space dimensional black holes. These corrections play prominent role at the smaller scales closer to the Planck scale. Hence, we would like to consider such quantum effect and study modified thermodynamics.

The plan of the paper is organized as follows: In Sect. 2, we briefly review the AO black hole space-time and its thermodynamics. In Sect. 3, we compute the higher order quantum corrected temperature of the AO black hole. Section 4 is devoted to the GUP corrected entropy and temperature of AO black hole. In Sect. 5, we summarize our results.

**II. ACHUCARRO-ORTIZ BLACK HOLE**

The Einstein-Maxwell action coupled with a charged scalar field in 2 + 1 dimensions is given by

$$I = \int d^3x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F_{ab} F^{ab} \right).$$

The Einstein field equations for (2+1)-dimensional space-time with negative cosmological constant take the following form

$$G_{ab} + \Lambda g_{ab} = \pi T_{ab} \quad (a, b = 0, 1, 2),$$

which results in the BTZ black hole solution with electric charge and spin:

$$ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\phi^2 - \frac{J}{2r^2} dt \right)^2.$$  \hspace{1cm}  \text{(3)}$$

The above line-element is also called AO black hole [2]. In Eq. (3), the metric function $f(r)$ reads

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{l^4 r^2} - \frac{\pi}{2} Q^2 \ln r, \hspace{1cm} \text{(4)}$$

where $M$, $Q$, $J$ denote the mass, electric charge, and angular momentum of the black hole, respectively. $\Lambda = -1/l^2$ is the negative cosmological constant. The event horizon (or the stationary limit surface) which is a null hypersurface of this
black hole occurs when $g^{rr} = 0$:

$$- M + \frac{r^2}{l^2} + \frac{J^2}{4r^4_+} - \frac{\pi}{2} Q^2 \ln r_+ = 0,$$

which yields the mass of the black hole as

$$M = \frac{4r^4_+ - 2\pi Q^2 l^2 r^2 + J^2 l^2}{4l^2 r^2_+}.$$  \hspace{1cm} (6)

The Hawking temperature of the black hole is given by

$$T_H = \frac{1}{4\pi} \left. \frac{\partial f(r)}{\partial r} \right|_{r=r_+} = \frac{1}{4\pi} \left( \frac{2r_+ - J^2}{2r^3_+} - \frac{\pi Q^2}{2r^2_+} \right).$$  \hspace{1cm} (7)

The entropy is associated with the event horizon as $S_0 = 4\pi r_+$. The thermodynamic volume is $V = \pi r^2_+$, which means that event horizon can be expressed as $r_+ = \sqrt{\frac{V}{\pi}}$.

The logarithmic corrected entropy expression (or the first order correction) is given by [14],

$$S = S_0 - \frac{\alpha}{2} \ln(S_0^2 T_H),$$

where the constant $\alpha$ is added to track the correction term [47, 48]. In that case if we choose $\alpha = 0$, the expression for the entropy without any corrections recovered. Moreover, in the case of $\alpha = 1$, we obtain the corrections due to thermal fluctuations. Hence, for the large black holes in low temperature, we can take the limit $\alpha \to 0$, and for the small black holes in high temperature, we can take the limit $\alpha \to 1$. In general, one can consider arbitrary $\alpha$ and fix it by observational data. One can find out the explicit form of Eq. (8) as follows

$$S = 4\pi r_+ - \frac{1}{2} \alpha \left[ - 4 \ln 2 - \ln \pi + \ln \left( \frac{(\pi l^2 Q^2 r^2_+ + J^2 l^2 - 4r^4_+)^2}{l^4 r^2_+} \right) \right].$$  \hspace{1cm} (9)

The Helmholtz function is given by

$$F = - \int SdT_H = - \int S(r_+) \frac{dT_H}{dr_+} dr_+,$$

which yields

$$F = \frac{1}{\pi r^3_+ l^2} \left[ - \frac{1}{2} \pi^2 Q^2 r^2_+ l^2 \ln r_+ + \frac{3}{4} \pi^4 r^2_+ l^2 - \pi r^5_+ \right] + F_1(\alpha),$$

where $F_1(\alpha)$ is a long expression of the first correction term. We now proceed to the second order correction to the entropy as follows:

$$S_c = S_0 - \frac{\alpha}{2} \ln(S_0^2 T_H) + \frac{\beta}{S_0},$$  \hspace{1cm} (11)

where the constant $\beta$ shows the higher order correction. In general, it is possible to state that all the different approaches to quantum gravity in leading order generate logarithmic corrections to the area-entropy law of a black hole while inverse of entropy in higher order. It should be noted that even though the leading order corrections to this area-entropy law are logarithmic, the coefficient of such a term depends on the approach to the quantum gravity, which can be used as a free parameter of the model. Since the values of the coefficients depend on the chosen approach to quantum gravity, it can be argued that such terms are generated from quantum fluctuations of the space–time geometry rather than from matter fields on that space–time. Hence we consider general $\alpha$ and $\beta$ to obtain modified thermodynamics. The above corrected entropy yields the following relevant correction term to the Helmholtz function:

$$F_c = F + \beta \left( \frac{Q^2}{64\pi^2 r^2_+} + \frac{3J^2}{128\pi^4 r^4_+} - \frac{\ln r_+}{8\pi^2 l^2} \right).$$  \hspace{1cm} (12)

In Fig. 1 one can see the typical behavior of the Helmholtz free energy for the corrected and uncorrected cases. We see some infinitesimal variation in the Helmholtz function as being observed in [49, 50]. It can be also seen that there is a
FIG. 1: Helmholtz free energy in terms of $V$ (left plot) and in terms of $r_+$ (middle plot) with $Q = J = l = 1$. Dashed red lines represent the ordinary case with $\alpha = \beta = 0$. Solid blue lines represent corrected case with $\alpha = \beta = 1$. The right figure shows Helmholtz free energy against volume of black hole horizon with chosen values $\alpha = 1, \beta = 0.2, J = 3, l = 2$ and $Q = 0$.

maximum value for the Helmholtz free energy. The correction terms reduce the maximum value.

To find the pressure, we employ the first derivative of $F_c$ with respect to volume, more specifically:

$$P = -\frac{\partial F_c}{\partial V},$$

(13)

which gives

$$P = \frac{16\pi^{7/2}Q^2V^{5/2}l^2 + 48J^2\pi^{7/2}V^{3/2}l^2 + 64\pi^{3/2}V^{7/2}}{64\pi^{5/2}l^2V^{7/2}} + P_1(\alpha) + P_2(\beta).$$

To avoid cumbersome mathematical expressions like above, from now on we shall write only the leading order term, however all terms will be taken into account for the numerical purposes.

In Fig. 2 one can deduce from the $P$-$V$ diagram that there is no Van der Waals behavior as well as no critical points. When the size of the black hole is at microscopic scale, the pressure becomes negative and black hole exhibits unstable phases. On the other hand, logarithmic and higher order corrections yield instability for the small black hole. In other words, when the black hole size is reduced by the Hawking radiation, thermal fluctuations of the quantum corrections become important and their effects cause to the instability.

To obtain internal energy, we use the following thermodynamic relation:

$$E = F_c + S_c T_H,$$

(14)

which gives us the following expression

$$E = F + \beta \left( \frac{Q^2}{64\pi r_+^2} + \frac{3J^2}{128\pi^2 r_+^4} - \frac{\ln r_+}{8\pi^2 l^2} \right) + S_0 T_H - \frac{\alpha}{2} T \ln (S_0^2 T_H) + \frac{\beta}{S_0} T_H.$$  

(15)

Further, we can calculate enthalpy

$$H = E + PV,$$

(16)

and the Gibbs free energy:

$$G = H - T_H S_c.$$  

(17)

Our numerical analysis show that the changes of entropy and Gibbs energy with internal energy show similar behaviors. To investigate the stability of the physical system, we consider the specific heat definition:

$$C = T_H \frac{\partial S_c}{\partial T_H}$$  

(18)
FIG. 2: Pressure in terms of $V$. Left plot: Dashed red lines represent the ordinary case with $\alpha = \beta = 0$, while solid blue lines represent corrected case with $\alpha = \beta = 1$. Right plot: $\alpha = 1$, $\beta = 0.2$, $J = 3$, $l = 2$.

which gives

$$C = 4\pi \frac{2r_+ - J^2}{2r_+^2} - \frac{\pi Q^2}{2r_+} + C_1(\alpha) + C_2(\beta), \quad (19)$$

where $C_1(\alpha)$ and $C_2(\beta)$ are complicated $\alpha$ and $\beta$ dependent terms. We shall make their physical interpretation, graphically. Sign of the specific heat and its asymptotic behavior can give us information about the stability and phase transition [51].

In Fig. 3 we draw the specific heat versus event horizon graph. In the case of chargeless static AO black hole (dotted green line), we see completely stable black hole. However, with the inclusion of rotation, some instabilities appear for small radii. In the presence of correction terms, unstable regions increase. Although there are some unstable phases, however there is no critical point and phase transition corresponding to asymptotic behavior.

III. HIGHER ORDER QUANTUM CORRECTED TEMPERATURE

In this section, we shall attempt to derive higher order quantum corrected temperature of BTZ black hole, which is dual of one dimensional holographic superconductors [52]. To this end, we shall study the Parikh-Wilczek’s quantum tunneling method [53] together with the entropy (11) to add higher order quantum corrections to the tunneling probability by considering the back reaction effects. However, the effects of Heisenberg uncertainty principle i.e., GUP corrections will be discussed in the next section. Finally, the modified $T_H$ due to the back reaction effect will be computed. However, as can be seen from the operations that are detailed below, it is necessary to express the entropy in terms of mass to be able to do all of these. For this, it is necessary to write the horizon in terms of mass. But, the transcendental structure of Eq. (6) does not allow this. To overcome this difficulty, we simply consider the chargeless case: $Q = 0$. Thus, from Eq. (6), one can get event and inner horizons as follows

$$r_+ = \frac{1}{\sqrt{2}} \sqrt{Ml^2 + l\sqrt{M^2l^2 - J^2}}, \quad (20)$$

and

$$r_- = \frac{1}{\sqrt{2}} \sqrt{Ml^2 - l\sqrt{M^2l^2 - J^2}}, \quad (21)$$

such that we have
FIG. 3: Specific heat in terms of $r_+$. Left plot: Dotted green line represent the uncharged static case ($Q = J = 0$), dashed red lines represent the ordinary case with $\alpha = \beta = 0$ and $Q = J = 1$, while solid blue lines represent corrected case with $\alpha = \beta = 1$ and $Q = J = 1$. Right plot: $\alpha = 1$, $\beta = -0.2$, $J = 3$, $Q = -1$, and $l = 2$.

\begin{align}
M &= \frac{r_+^2 - r_-^2}{l}, \\
J &= \frac{2r_+r_-}{l},
\end{align}

and also the Hawking temperature (7) becomes

\begin{equation}
T_H = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{J^2}{2r_+^2} \right).
\end{equation}

The coordinate system in Eq. (3) is described for the observer located at spatial infinity. After transforming the metric (3) to the dragging coordinate system (25):

\begin{equation}
\psi = \phi - \frac{J}{2r^2}t, \quad T = t,
\end{equation}

we see that the physics near the horizon can be effectively ($\psi = \text{const.}$) described by the following two-dimensional metric

\begin{equation}
ds^2 = f(r)dT^2 - \frac{dr^2}{f(r)},
\end{equation}

On the other hand, the near horizon metric (26) can be expressed in the regular Painleve-Gulstrand (PG) coordinates (55-56) by applying the following transformation

\begin{equation}
dT_{PG} = dT + \sqrt{1 - f(r)}dr,
\end{equation}

where $T_{PG}$ is called the PG time, which is nothing but the proper time. Thus, the metric (26) recasts in

\begin{equation}
ds^2 = -f(r)dT_{PG}^2 + 2\sqrt{1 - f(r)}dT_{PG}dr + dr^2,
\end{equation}
and it admits the following radial null geodesics of a test particle:

\[ \dot{r} = \frac{dr}{d\tau_{PG}} = -\sqrt{1 - f(r)} \pm 1, \]  

(29)

where plus (minus) sign corresponds to outgoing (ingoing) geodesics. After expanding Eq. (29) around the event horizon, one can find that the radial outgoing null geodesics, \( \dot{r} \), is approximated to the following

\[ \dot{r} \cong \kappa(r - r_+). \]  

(30)

in which the surface gravity \([57]\) (in the PG coordinates) reads

\[ \kappa = \frac{1}{2} \left. \frac{\partial f(r)}{\partial r} \right|_{r=r_+}. \]  

(31)

On the other hand, the imaginary part of the action \( (I) \) for an outgoing particle with positive energy that crosses the event horizon from inside \( (r_\odot) \) to outside \( (r_\oplus) \) is given by \([58]\)

\[ \text{Im}I = \text{Im} \int_{r_\odot}^{r_\oplus} p_r dr = \text{Im} \int_{r_\odot}^{r_\oplus} \int_0^{p_r} d\tilde{p}_r dr. \]  

(32)

Let us recall that the Hamilton’s equation for the classical trajectory is given by

\[ dp_r = \frac{dH}{\dot{r}}, \]  

(33)

where \( p_r \) and \( H \) are the radial canonical momentum and Hamiltonian, respectively. Thus, we have

\[ \text{Im}I = \text{Im} \int_{r_\odot}^{r_\oplus} \int_0^H \frac{dH}{\dot{r}} dr. \]  

(34)

Now, let us assume that we have a circularly symmetric space-time having constant total mass \( M \). By making one more assumption, we consider the system as if containing a radiating BTZ black hole with varying mass \( M - \omega \) that emits a circular shell of energy \( \omega \): \( \omega \ll M \). This scenario describes the self-gravitational effect \([59, 60]\). In this framework, Eq. (34) becomes

\[ \text{Im}I = \text{Im} \int_{r_\odot}^{r_\oplus} \int_0^{M-\omega} \frac{dH}{\dot{r}} dr, \]  

\[ = -\text{Im} \int_{r_\odot}^{r_\oplus} \int_0^{\omega} \frac{d\tilde{\omega}}{\dot{r}} dr, \]  

(35)

in which the Hamiltonian \( H = M - \omega \) and \( dH = -d\tilde{\omega} \) are used. Following Eq. (30), the radial outgoing null geodesics, \( \dot{r} \), of the radiating black hole is defined as follows \([58]\)

\[ \dot{r} \cong \kappa_{QGC}(r - r_+), \]  

(36)

where \( \kappa_{QGC} = \kappa(M - \omega) \) is the quantum gravity corrected (QGC) surface gravity \([61, 62]\). Therefore, after \( r \) integration (the integration over \( r \) is done by deforming the contour), Eq. (35) becomes

\[ \text{Im}I = -\pi \int_0^\omega \frac{d\tilde{\omega}}{\kappa_{QGC}}. \]  

(37)

After defining the QGC Hawking temperature as \( T_{QGC} = \frac{\kappa_{QGC}}{2\pi} \), we get
\[ ImI = -\frac{1}{2} \int_0^\omega \frac{d\omega}{T_{QGC}} = -\frac{1}{2} \int_{S_{QGC}(M-\omega)} dS = -\frac{1}{2} \Delta S_{QGC}. \tag{38} \]

The above expression yields the modified tunneling rate:
\[ \Gamma_{QGC} \sim e^{-2ImI} = e^{\Delta S_{QGC}}. \tag{39} \]

Taking cognizance of Eq. (11), one can compute \( \Delta S_{QGC} \) as follows
\[ \Delta S_{QG} = S_{QGC}(M-\omega) - S_{QGC}(M) = 4\pi (r_+(\omega) - r_+) + \frac{\alpha}{2} \left[ \ln \left( \frac{r_+}{4\pi} \left( 2l^{-2} + \frac{6J^2}{r_+^2} \right)^2 \right) \right] - \left[ \ln \left( \frac{r_+}{4\pi} \left( 2l^{-2} + \frac{6J^2}{r_+^2} \right)^2 \right) \right] - \frac{\beta}{2\pi} \left( \frac{1}{2r_+} - \frac{1}{2r_+(\omega)} \right), \tag{40} \]

where \( r_+ \) is given by the equation (20) and we defined
\[ r_+(\omega) = \frac{1}{\sqrt{2}} \sqrt{(M-\omega)l^2 + l\sqrt{(M-\omega)^2 l^2 - J^2}}. \tag{41} \]

When we expand \( \Delta S_{QG} \) and recast terms up to leading order in \( \omega \), we find
\[ \Delta S_{QGC} \approx \Psi \omega + O(\omega^2), \tag{42} \]

where
\[ \Psi = -\frac{l (12 J^2 Ml + 11 J^2 \sqrt{M^2 l^2 - J^2} - 2 \sqrt{M^2 l^2 - J^2} M^3 l^3 - 2 M^2 l^2 \sqrt{M^2 l^2 - J^2})}{4 \sqrt{M^2 l^2 - J^2} (Ml + \sqrt{M^2 l^2 - J^2}) (M^2 l^2 + Ml \sqrt{M^2 l^2 - J^2} - J^2)} \alpha + \frac{\sqrt{2} \beta}{8 \pi \sqrt{Ml + \sqrt{M^2 l^2 - J^2}} \sqrt{M^2 l^2 - J^2}} \frac{\sqrt{2} l^3}{\sqrt{Ml + \sqrt{M^2 l^2 - J^2}}} \frac{1}{\sqrt{M^2 l^2 - J^2}}. \tag{43} \]

Based on Eqs. (42) and (43) and recalling the Boltzmann factor [63],
\[ \Gamma_{QGC} \sim e^{\Delta S_{QGC}} = e^{-\frac{\Psi}{\omega}}, \tag{44} \]
we find out the \( QGC \) temperature as follows
\[ T_{QGC} = -\frac{1}{\Psi}. \tag{45} \]

It is easy to verify that when suppressing the \( QGC \) effects (i.e., \( \alpha = \beta = 0 \)), \( T_{QGC} \) reduces to
\[ T_{QGC}|_{\alpha=\beta=0} = \frac{\sqrt{M^2 l^2 - J^2}}{\sqrt{2} l^3 \sqrt{Ml + \sqrt{M^2 l^2 - J^2}}} = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{J^2}{2r_+^2} \right), \tag{46} \]
which is nothing but the standard Hawking temperature (24) of the rotating BTZ black hole.
IV. GUP CORRECTIONS TO ENTROPY OF AO BLACK HOLE

In this section, the entropy of the AO black hole will be calculated using the GUP corrections [64]. At the size of Planck length, semiclassical methods do not work properly, so that one needs to use the theory of the quantum gravity which is not complete yet. For this purpose, some corrections to the classical theory is used to approach quantum gravity regime [64, 65]. Now, we study the GUP corrections, which were first applied in string theory and in loop quantum gravity, on the Hawking temperature and Bekenstein entropy [66-70]. GUP can directly be applied to the modified thermodynamical quantities by counting the number of states with the help of quantum corrections [71-72]. Furthermore, GUP provides the modification of the Heisenberg principle at Planck scales [64-71]:

\[
\Delta x \Delta p \geq \hbar \left( 1 - \frac{\gamma l_p^2}{\hbar^2} \Delta x + \frac{\gamma^2 l_p^4}{\hbar^4} (\Delta p)^2 \right),
\]

(47)

where \( \gamma \) is a dimensionless positive parameter, \( l_p = \sqrt{\hbar G/c^3} = M_p G/c^2 \approx 10^{-45} m \) is for the Planck length, \( M_p = \sqrt{\hbar c/G} \) is the Plank mass, and \( c \) is the velocity of light. Here \( \gamma \) is the Newtonian coupling constant, the correction terms in the uncertainty relation (47) are due to the effects of gravity. Equation (47) can be rewritten as follows

\[
\Delta p \geq \frac{\hbar (\Delta x + \gamma l_p)}{2\gamma l_p^2} \left( 1 - \frac{1}{1 - \frac{4\gamma^2 l_p^4}{(\Delta x + \gamma l_p)^2}} \right).
\]

(48)

Equation (48) implies that the minimum measurable length is \( \Delta x \geq (\Delta x)_{\text{min}} \approx \gamma l_p \), moreover the maximum measurable momentum is \( \Delta p \leq (\Delta p)_{\text{max}} \approx \hbar/\gamma l_p \). Then we compare \( l_p/\Delta x \) with unity which is small, and the above equation can be written as follows: [61]

\[
\Delta p \geq \frac{1}{2\Delta x} \left[ 1 - \frac{\gamma}{2\Delta x} + \frac{\gamma^2}{2(\Delta x)^2} + \cdots \right].
\]

(49)

Here we take \( G = c = k_B = 1 \), so that \( \hbar = 1 \) and \( l_p = 1 \). For \( \gamma = 0 \) in Eq. (49), the uncertainty principle is \( \Delta x \Delta p \geq 1 \),

(50)

Note that the factor 2 is in \( \Delta x \). Then use the Eq. (50), we obtain a bound on the energy as follows: [75]

\[
E \Delta x \geq 1,
\]

(51)

where we use \( E^2 = p^2 + m^2 \) [76]. Afterwards, we take \( p \sim \Delta p \geq 1/\Delta x \), and we find that \( E \geq \Delta p \geq 1/\Delta x \) [77], \( \Delta x \Delta p \geq 1 \), where \( E \) is the particle’s energy. Hence, the relation becomes (49) as follows:

\[
E_{\text{GUP}} \geq E \left[ 1 - \frac{\gamma}{2(\Delta x)} + \frac{\gamma^2}{2(\Delta x)^2} + \cdots \right].
\]

(52)

Now, one can also obtain the corrected probability as follows: [80]

\[
\Gamma \approx \exp[-2\text{Im}(I)] = \exp \left[ \frac{-4\pi E_{\text{GUP}}}{\kappa} \right].
\]

Then it is straightforward to find temperature of the AO black hole

\[
T \leq T_H \left[ 1 - \frac{\gamma}{2(\Delta x)} + \frac{\gamma^2}{2(\Delta x)^2} + \cdots \right]^{-1},
\]

with (7):

\[
T_H = \frac{1}{4\pi} \left( \frac{2r_+}{l_+^2} - \frac{j^2}{2r_+^3} - \frac{\pi Q^2}{2r_+} \right).
\]

(53)

For the case of near horizon, we use that \( \Delta x = 2r_+ \), and temperature with GUP correction is found as follows

\[
T_{\text{GUP}} \leq \frac{1}{4\pi} \left( \frac{2r_+}{l_+^2} - \frac{j^2}{2r_+^3} - \frac{\pi Q^2}{2r_+} \right) \left( 1 - \frac{\gamma}{4r_+} + \frac{\gamma^2}{8r_+^2} + \cdots \right)^{-1}
\]

\[
\approx \frac{1}{4\pi} \left( \frac{2r_+}{l_+^2} - \frac{j^2}{2r_+^3} - \frac{\pi Q^2}{2r_+} \right) \left( 1 + \frac{\gamma}{4r_+} - \frac{\gamma^2}{8r_+^2} + \cdots \right). \quad (54)
\]
Inserting event horizon radius in terms of the black hole hairs and fix coefficients one can re produce corrected temperature given by the equation (45). Hence we have,

\[ S_{GUP} = \int \frac{1}{T_{GUP}} \frac{\partial M}{\partial r_+} dr_+, \]  
(55)

and in terms of the entropy \( S_0 = 4\pi r_+ \), we find

\[ S_{GUP} \leq S_0 - \gamma \pi \ln \left( \frac{S_0}{S_0} \right) + \cdots. \]  
(56)

The temperature is calculated as follows:

\[ T_{GUP} = 2T_H \left( 1 + \frac{\gamma l_p}{\Delta x} \right)^{-1} \left[ 1 + \sqrt{1 - \frac{4}{(1 + \frac{\Delta x}{l_p})^2}} \right]^{-1}, \]  
(57)

From above, we have deduced that the maximum temperature satisfies the following relation: \( T_{GUP} \leq T_{max} = T_H \).

V. CONCLUSION

In this paper, we have considered the AO black hole and study its thermodynamics by taking into account of quantum corrections (considering the effects of back reaction and GUP separately. We have computed the modified Helmholtz free energy and used it to investigate the \( P - V \) criticality for the AO black hole. From the results obtained, we have revealed that there are no Van der Waals behavior and critical points. Performing the numerical analysis for the specific heat, we have shown that the quantum corrected terms of the entropy signal the instability. We have also derived the QGC (with back reaction effect) and GUP corrected expressions of thermodynamic parameters: temperature, heat capacity, and entropy of the AO black hole. In particular, for the microscopic AO black holes, those corrections remove the thermal instability (see Fig. 3). Furthermore, we have also calculated the upper limit of the Hawking temperature and prove that \( T_{GUP} \leq T_{max} = T_H \). To this end, we have focused on the second order corrections that were ignored in the earlier studies about this subject.

The present study motivates us for doing further research in this direction. We plan to extend our analytical analysis to the higher dimensional black holes and explore the effects of dimension on the quantum corrected temperature and entropy. Finally it may be interesting to consider the corrections arising from the classical geometry which is called extended uncertainty principle (EUP) on the thermodynamics of black hole [78]. Following Ref. [79] in which the Hawking-Page transition of the BTZ black hole where discussed in the framework of the EUP, and its GUP corrections (GEUP), we also aim to study the EUP and GEUP corrections on the AO black hole thermodynamics. This is the next stage of study that interests us.

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