Nonlinear deformation of a solid body on the basis of flow theory and realization of FEM in mixed formulation

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Abstract. In the mixed formulation, an algorithm for the formation of a hexahedral finite element deformation matrix at the loading step to determine the stress-strain state of a solid body beyond the elastic limit based on the theory of plastic flow is developed in a curvilinear coordinate system. Stresses and displacements are taken as nodal unknowns. The approximation of the required values of the inner point of the finite element through the nodal unknowns was carried out by trilinear functions. To obtain the deformation matrix of the hexahedral finite element, a mixed functional on the equality of the real and possible works of external and internal forces with the replacement of the actual work of internal forces by the difference of the total and additional work of internal forces at the loading step is used.

1 Introduction

The stress-strain state (SSS) theory of solids developed in the present time enough [1-5]. The complexity of the differential equations obtained in determining SSS of solids does not allow to obtain analytical solutions. In order to find the main deformation parameters, numerical methods are actively used, the main of which is the FEM. The most common finite element method in the formulation of the displacement method [6-10], rarely used in a mixed formulation [6, 11]. The unknown functions determined by the method of displacements are obtained continuous only at the nodal points of the finite element, and at its boundaries are not continuous. In the case of a mixed method, the found unknown functions have continuity both at the nodes and at the boundaries of the elements. Therefore, the use of the FEM in a mixed formulation is preferable.

The problems of reducing the material consumption of the designed engineering structures for various purposes dictate the need to consider the processes of deformation of solids beyond the elastic limit. For this purpose, in this paper, an algorithm for obtaining the defining relations at the loading step is developed for use in the formation of the deformation matrix of a bulk hexahedral finite element in the mixed formulation of the finite element method (FEM). As a research tool, an algorithm for obtaining relations between stress increments and strain increments is developed in a curvilinear coordinate system based on the hypothesis of the theory of plastic flow on the proportionality of the components of the plastic strain increment tensor to the components of the stress deviator. The obtained relations are used in the algorithm for the development of the deformation matrix of a three-dimensional hexagonal finite element in a mixed formulation.

2 Materials and Methods

The geometry of a rigid body. The position of an arbitrary point of the \( M \) body in the coordinate system \( xOz \) is described by the radius vector

\[
\vec{R} = x_k \left( \theta^m \right)_k; \quad k, m = 1, 2, 3,
\]
where \( x^k \) - cartesian coordinates of the point \( M \); \( \theta^m \) - curvilinear coordinates of the point \( M \); \( \vec{r}_k \) - orts of the cartesian coordinate system.

The vectors of the local basis of a point \( M \) are defined by the expression

\[
\vec{g}_i = \vec{R}_i.
\]

The basis vectors (2) and their derivatives can be represented by a matrix expression

\[
\left\{ \vec{g} \right\}_{3x1} = [s] \left[ \vec{g} \right]_{3x3} \left[ g \right]_{3x1}; \quad \left\{ \vec{g}_i \right\}_{3x1} = [m_i] \left\{ \vec{g} \right\}_{3x1}; \quad i = 1, 2, 3,
\]

where

\[
\left( \vec{g} \right)_{1x3} = \left( \vec{g}_1 \vec{g}_2 \vec{g}_3 \right); \quad \left( \vec{g}_i \right)_{1x3} = \left( \vec{g}_{i1} \vec{g}_{i2} \vec{g}_{i3} \right); \quad \left( \vec{g}_j \right)_{1x3} = \left( \vec{g}_{j1} \vec{g}_{j2} \vec{g}_{j3} \right).
\]

Based on the displacements from the deformation at loading step. When a rigid body is deformed, its arbitrary point is considered in three states: initial \( M \), deformed after \( j \) loading steps \( M^j \) (displacement vector \( V \)) and deformed after \((j + 1)\) loading step \( M^{j+1} \) (displacement vector \( \vec{w} \)).

Displacement vectors \( V \) are \( \vec{w} \) expressed by the components in the basis of the \( M^i \) initial state point in the form

\[
\vec{V} = v^i \vec{g}_i, \quad \vec{w} = w^i \vec{g}_i
\]

Their derivatives are defined by expressions

\[
\vec{V}_r = v^i g^k g_{kr}, \quad \vec{w}_r = w^i g^k g_{kr} = \alpha^i_k g_k,
\]

where \( f^i_k \) - functions of components \( v^i \) and their derivatives; \( \alpha^i_k \) - functions component \( w^i \) and their derivatives.

Deformations and their increments at the points \( M^j \), \( M^{j+1} \) and after \( j \) the \((j + 1)\) loading steps are determined as the differences between the metric tensors of the initial and deformed states of the body and are written in the form [4], which, taking into account (4) and (5), are written in matrix form

\[
\left\{ \Delta e \right\}_{6x1} = \left[ L \right] \left\{ v \right\}_{3x1}; \quad \left\{ \Delta e \right\}_{6x1} = \left[ L \right] \left\{ w \right\}_{3x1},
\]

where \( \left\{ e \right\}_{1x6} = \left\{ e_{11} e_{22} e_{33} 2 e_{12} 2 e_{13} 2 e_{23} \right\} \) - matrix - line deformation; \( \left[ L \right] \) - the matrix of differential operators;

\[
\left\{ \Delta e \right\}_{1x6} = \left\{ \Delta e_{11} \Delta e_{22} \Delta e_{33} 2 \Delta e_{12} 2 \Delta e_{13} 2 \Delta e_{23} \right\} - matrix - string deformations;
\]

\[
\left\{ \Delta w \right\}_{1x3} = \left\{ \Delta w^1 \Delta w^2 \Delta w^3 \right\} - the matrix-line displacements of the point \( M^{j+1} \).
\]

Relations of the theory of plastic flow. The total strain increments are determined by the sum of the elastic and plastic strain increments

\[
\Delta e = \Delta e^e + \Delta e^p,
\]

where \( \Delta e^e \) - the total increment of deformation; \( \Delta e^i \) - the increments of elastic deformation; \( \Delta e^p \) - the increment of plastic deformation.
The increment of elastic strain using the stress increment is expressed by the Hooke’s law in matrix form

\[
\begin{bmatrix} \Delta \varepsilon' \end{bmatrix}_{\text{6x1}} = [C]\begin{bmatrix} \Delta \sigma \end{bmatrix}_{\text{6x1}},
\]

where \( \begin{bmatrix} \Delta \varepsilon' \end{bmatrix}_{\text{6x1}} = \begin{bmatrix} \Delta \varepsilon'_{11} & \Delta \varepsilon'_{12} & \Delta \varepsilon'_{13} & 2\Delta \varepsilon'_{12} & 2\Delta \varepsilon'_{13} & 2\Delta \varepsilon'_{23} \end{bmatrix} \) - matrix-row increments of elastic deformation; 
\( \begin{bmatrix} \Delta \sigma \end{bmatrix}_{\text{6x1}} = \begin{bmatrix} \Delta \sigma_{11} & \Delta \sigma_{12} & \Delta \sigma_{13} & \Delta \sigma_{22} & \Delta \sigma_{23} & \Delta \sigma_{33} \end{bmatrix} \) - the matrix-the row of increments of stress.

The increments of plastic deformations at the loading step are determined on the basis of the hypothesis of the theory of plastic flow about the proportional dependence of increments of plastic deformations on the components of the stress deviator, which is written in the form

\[
\Delta \varepsilon_p^i = k \left( \sigma_{ij} - \frac{1}{3} P(\sigma) g_{ij} \right),
\]

where \( k \) is the coefficient of proportionality; \( P(\sigma) = \sigma_{mm} g_{mm} \) - is the first invariant of the stress tensor.

The coefficient of proportionality in the expression (9) is the formula [3]

\[
k = \frac{3 \varepsilon_i^{np}}{2 \sigma_i},
\]

where \( \varepsilon_i^{np} \) is the intensity of increments of plastic deformations; \( \sigma_i \) – the intensity of stresses.

At small steps of loading the intensity of increments of plastic deformations and increments of their intensity can be taken equal to

\[
\varepsilon_i^{np} = \Delta \varepsilon_p^i,
\]

where \( \Delta \varepsilon_p^i \) – the increment of the intensity of plastic deformations.

And we can also assume that the increment of the intensity of total deformations \( \Delta \varepsilon_p^i \) is equal to the sum of the increments of the intensities of elastic \( \Delta \varepsilon_e^i \) and plastic \( \Delta \varepsilon_p^i \) deformations

\[
\Delta \varepsilon_p^i = \Delta \varepsilon_e^i + \Delta \varepsilon_p^i,
\]

The increment of the intensity of elastic deformations is according to the formula

\[
\Delta \varepsilon_e^i = \frac{\Delta \sigma_{ij}}{E_u},
\]

where \( E_u \) is the deformation module of the initial section of the deformation diagram.

At elastic-plastic deformation at a loading step the relation takes place

\[
\Delta \sigma_{ij} = E_i \Delta \varepsilon_e^i,
\]

where \( E_i \) is the tangent module of the deformation diagram.

The increment of the intensity of plastic deformations is determined from the expression (13)

\[
\Delta \varepsilon_p^i = \Delta \varepsilon_p^i - \Delta \varepsilon_e^i.
\]
\[ \sigma_i = \sqrt{3S^i S_j} \, , \]  
(16)

where \( S_{ij} = \sigma_{ij} - \frac{1}{3} P(\sigma) g_{ij} \) is the stress deviator.

At the loading step, the stress intensity increment is determined using the derived expression (16).

The step of loading the dependencies between the total increments of strain increments of stress taking into account (8) and (9) can be written in matrix form

\[ \{\Delta \varepsilon\}_i = [C_n] \{\Delta \sigma_i\} \, . \]  
(17)

3 Results and discussion

Finite element deformation matrix at the loading step. The finite element is selected in the form of a hexagon, with nodes \( i, j, k, l, m, n, p, h \). The nodal unknowns are taken as increments of displacements and increments of stresses. To perform numerical integration, an arbitrary hexagon is mapped to a rectangular cube with local coordinates \( \xi, \eta, \zeta \), varying from -1 to 1. The global coordinates of an \( x^1, x^2, x^3 \) arbitrary point of the hexahedronal finite element, depending on their nodal values, are expressed by relations

\[ x' = \{\phi(\xi, \eta, \zeta)\}^T \{x'\}_i ; \quad i = 1, 2, 3 \, , \]  
(18)

where \( \{\phi(\xi, \eta, \zeta)\}^T = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\} \) - matrix - a string of form functions;

\[ \{x'\}_i = \{x'^i, x'^j, x'^k, x'^m, x'^n, x'^p\} \} - matrix - a string of nodal values of the coordinates of the finite element.

The displacements of an arbitrary point of \( M^i \) a finite element are approximated by the displacements of its nodal points by functions (18) and are represented in matrix form

\[ \{v_i\} = [A]\{v_r\}_i \, , \]  
(19)

where \( \{v_r\}_i = \left( \begin{array}{c} v_{r, }_{1} \\ v_{r, }_{2} \\ v_{r, }_{3} \\ v_{r, }_{4} \\ v_{r, }_{5} \\ v_{r, }_{6} \end{array} \right) \),

The relations between deformations and displacements, increments of deformations and displacements at the loading step (6) taking into account (19) are written in the form

\[ \{\varepsilon_i\} = [L][A]\{v_r\} = [B]\{v_r\}_i \; ; \; \{\Delta \varepsilon_i\} = [L][A]\{w_r\} = [B]\{w_r\}_i \, . \]  
(20)

where \( \{w_r\}_i = \left( \begin{array}{c} w_{r, }_{1} \\ w_{r, }_{2} \\ w_{r, }_{3} \\ w_{r, }_{4} \\ w_{r, }_{5} \\ w_{r, }_{6} \end{array} \right) \)

The stresses at an arbitrary point of a finite element are approximated through the stress components in its nodal stresses by expressions

\[ \sigma_{ij} = \{\phi(\xi, \eta, \zeta)\}^T \{\sigma_{ij} \}_n ; \quad i = 1, 2, 3, \quad j = 1, 2, 3 \, . \]  
(21)
where \( \sigma_{ij} \) - matrix - stress lines in the nodes of the finite element.

The approximation of stress increments at an arbitrary point of a finite element through its nodal stresses and their increments is performed in the same way as the expressions (21) and is written in matrix form

\[
\{ \sigma \} = \{ G \} \{ \sigma_i \}^T, \quad \{ \Delta \sigma \} = \{ G \} \{ \Delta \sigma_i \}^T, \tag{22}
\]

where \( \{ \sigma \}^T = \{ \sigma_{11} \, \sigma_{22} \, \sigma_{33} \, \sigma_{12} \, \sigma_{13} \, \sigma_{23} \} \);

\[
\{ \sigma_i \}^T = \{ \sigma_{11,i} \, \sigma_{22,i} \, \sigma_{33,i} \, \sigma_{12,i} \, \sigma_{13,i} \, \sigma_{23,i} \};
\]

\[
\{ \Delta \sigma \}^T = \{ \Delta \sigma_{11} \, \Delta \sigma_{22} \, \Delta \sigma_{33} \, \Delta \sigma_{12} \, \Delta \sigma_{13} \, \Delta \sigma_{23} \};
\]

\[
\{ \Delta \sigma_i \}^T = \{ \Delta \sigma_{11,i} \, \Delta \sigma_{22,i} \, ... \, \Delta \sigma_{13,i} \, \Delta \sigma_{23,i} \}.
\]

The functional expressing the equality of possible and actual works of internal and external forces at the loading step for an arbitrary stress state of the shell of rotation is written in the form

\[
II = \int \{ \Delta \sigma \}^T \{ L \} \{ w \} dV - \frac{1}{2} \{ \Delta \sigma \}^T \{ \Delta \sigma \}^T \{ \Delta \sigma \} \{ \sigma \} dV - \frac{1}{2} \{ \Delta \sigma \}^T \int \{ w \}^T \{ \Delta \sigma \} dS - \int \{ w \}^T \{ \sigma \} dS + \int \{ \sigma \}^T \{ \sigma \} dV,
\]

where \( \{ q \}^T = \{ q_1 \, q_2 \, q_3 \} \); \( \{ \Delta \sigma \}^T = \{ \Delta q_1 \, \Delta q_2 \, \Delta q_3 \} \) - matrix - row components of the total external loads and their increments, respectively.

Taking into account (17), (21) and (22) the functional (23) for the finite element will take the form

\[
II = \int \{ \Delta \sigma_i \}^T \{ G \} \{ [ A ] \} \{ [ B ] \} dV \{ w_i \} - \frac{1}{2} \{ \Delta \sigma_i \}^T \{ [ A ] \} \{ [ C ] \} \{ [ A ] \} \{ \Delta \sigma_i \} \{ w_i \} - \frac{1}{2} \{ w_i \}^T \{ [ A ] \} \{ \Delta \sigma_i \} dS - \{ w_i \}^T \{ [ A ] \} \{ q \} dS - \{ [ A ] \}^T \{ \sigma \} dV - \int \{ [ A ] \}^T \{ q \} dS - \int \{ [ B ] \}^T \{ \sigma \} dV,
\]

where \( \{ w_i \}^T \) are \( \{ \Delta \sigma_i \}^T \) obtained

\[
\frac{\partial II}{\partial \{ \Delta \sigma \}^T} = \{ G \} \{ w_i \} - \{ H \} \{ \Delta \sigma_i \} = 0; \tag{25}
\]

\[
\frac{\partial II}{\partial \{ w \}^T} = \{ G \} \{ \Delta \sigma_i \} - \{ \Delta f \} - \{ R \} = 0.
\]

where \( \{ H \} = \{ G \}^T \{ [ C ] \} \{ G \} dV; \quad \{ Q \} = \{ G \}^T \{ [ B ] \} dV; \)
\[ \left\{ \Delta f_y \right\} = \int A^T \left\{ \Delta q \right\} ds; \quad \left\{ R \right\} = \int A^T \left\{ q \right\} ds - \int B^T \left\{ \sigma \right\} dv \] - Raphson’s residual.

Systems of equations (25) are presented in the traditional form for the finite element method

\[
[K] \left\{ z_y \right\} = \left\{ f_y \right\} \tag{26}
\]

where \([K]_{72 \times 72}\) is the deformation matrix of the finite element;

\[
\left\{ z_y \right\} = \left\{ \Delta \sigma_y \right\}_{1 \times 72} + \left\{ w_y \right\}_{1 \times 72} \] - vector of nodal unknowns of the finite element;

\[
\left\{ f_y \right\} = \left\{ 0 \right\}_{1 \times 48} + \left\{ \Delta q \right\}_{1 \times 24} + \left\{ R \right\}_{1 \times 24} \] - vector of nodal loads.

The formation of the deformation matrix of the entire structure is performed using the traditional FEM procedure.

The resulting deformation matrix of the hexahedral finite element in the mixed formulation has a size of \((72 \times 72)\), which is smaller than the size of the stiffness matrix of the same finite element in the formulation of the displacement method \((96 \times 96)\). In addition, the advantage of the algorithm for obtaining the finite element stiffness matrix in a mixed formulation is to provide stress compatibility between elements not only at nodal points, but also on the faces of adjacent elements.

**Conclusion**

The mixed functional used to obtain the deformation matrix of a bulk finite element allows us to take into account the residual at the loading step as a difference in the numerical values of the external and internal forces for the previous loading steps, as well as to take into account the boundary stress conditions.

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**References**

[1] Novozhilov V V 1962 *Theory of thin shells* (Leningrad: Sudpromgiz) p 432

[2] Sedov L I 1976 *Continuum Mechanics* Vol 1 (Moscow: Nauka) p 536

[3] Grigolyuk E I and Lopanitzin E A 2004 *Finite deflections, stability and supercritical behavior of thin shallow shells* (Moscow: Moscow State Technological University MAMI) p 162

[4] Petrov V V 2017 *Nonlinear incremental construction mechanics* (Moscow: Infra-Engineering) p 479

[5] Petrov V V 2013 Calculation of flexible inhomogeneous shallow shells made of a physically nonlinear material *Collection of scientific works of VolGAS* pp 373 - 377

[6] Badriev I B, Makarov M V and Paymushin V N 2012 Solvability of physically and geometrically nonlinear problem of three-layer plates with transversally soft filler *Izv. Higher Educational. Mathematics* 2 pp 171-183

[7] Malinin N N 1975 *Applied theory of plasticity and creep* (Moscow: Mechanical Engineering) p 399
[8] Bathe K J 2010 *Finite element Methods* (Moscow: Fizmatlit) p 1022
[9] Golovanov A I, Tuleneva O N and Shigabutdinov A F 2006 *Finite element Method in statics and dynamics of thin-walled structures* (Moscow: Fizmatlit) p 391
[10] Klochkov Yu V, Nikolaev A P, Vakhnina O V and Kiseleva T A 2016 Analysis of the stress-strain state of the fuselage fragment in the form of a thin shell using a triangular finite element with Lagrange multipliers *Izv. Higher Educational. Aviation* 3 pp 20–26.
[11] Gureeva N A 2007 Eight-node volumetric finite element of the shell of rotation with unknown stresses and displacements in the nodes *Izv. Higher Educational. Construction* 4 pp 33–39
[12] Ignatiev V A and Ignatiev A V 2006 Mixed form of FEM in problems of construction mechanics *Construction mechanics and calculation of structures* 1 pp 59-64.
[13] Myoung-Gyu Lee and Chung-Souk Han 2012 An explicit final element approach with paten projection technique for strain gradient plasticity formulas *Computational Mechanics* 2 pp 171-183
[14] Beirao L, Veiga Da, Brezzi F, and Marini L D 2013 Virtual Elements for linear elasticity problems *SIAM Journal on Numerical Analysis* 2 pp 794-812.
[15] Gain A L, Talischi C and Paulino G H 2014 On the Virtual Element Method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes *Computer Methods in Applied Mechanics and Engineering* pp 132-160.
[16] Beirao L, Lovadina C and Mora D 2015 A Virtual Element Method for elastic and inelastic problems on polytope meshes *Computer Methods in Applied Mechanics and Engineering* pp 327-346.
[17] Aldakheel F, Hudobivnik B and Wriggers P 2018 A Virtual element formulation for phase-field modeling of ductile fracture *Computational Engineering*.
[18] Artioli E, Veiga L B D, Lovadina C and Sacco E. 2017 Arbitrary order 2d virtual elements for polygonal meshes: Part ii, inelastic problem *Computational Mechanics* pp 643-657.