New results on integrability of the Kahan-Hirota-Kimura discretizations

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Abstract
R. Hirota and K. Kimura discovered integrable discretizations of the Euler and the Lagrange tops, given by birational maps. Their method is a specialization to the integrable context of a general discretization scheme introduced by W. Kahan and applicable to any vector field with a quadratic dependence on phase variables.

We report several novel observations regarding integrability of the Kahan-Hirota-Kimura discretization. For several of the most complicated cases for which integrability is known (Clebsch system, Kirchhoff system, and Lagrange top),

\begin{itemize}
  \item we give nice compact formulas for some of the more complicated integrals of motion and for the density of the invariant measure, and
  \item we establish the existence of higher order Wronskian Hirota-Kimura bases, generating the full set of integrals of motion.
\end{itemize}

While the first set of results admits nice algebraic proofs, the second one relies on computer algebra.

1 Introduction
The Kahan-Hirota-Kimura discretization method was introduced in the geometric integration literature by Kahan in the unpublished notes \cite{Kahan} as a method applicable to any system of ordinary differential equations with a quadratic vector field:

\begin{equation}
\dot{x} = f(x) = Q(x) + Bx + c,
\end{equation}

where each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form, while $B \in \text{Mat}_{n \times n}(\mathbb{R})$ and $c \in \mathbb{R}^n$. Kahan’s discretization (with stepsize $2\epsilon$) reads as

\begin{equation}
\frac{\bar{x} - x}{2\epsilon} = Q(x, \bar{x}) + \frac{1}{2}B(x + \bar{x}) + c,
\end{equation}

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where
\[ Q(x, \tilde{x}) = \frac{1}{2} (Q(x + \tilde{x}) - Q(x) - Q(\tilde{x})) \]
is the symmetric bilinear form corresponding to the quadratic form \( Q \). We say that the expression on the right-hand side of (2) is the *polarization* of the expression on the right-hand side of (1). Equation (2) is linear with respect to \( \tilde{x} \) and therefore defines a rational map \( \tilde{x} = \Phi_f(x, \epsilon) \). Clearly, this map approximates the time \( 2\epsilon \) shift along the solutions of the original differential system. Since equation (2) remains invariant under the interchange \( x \leftrightarrow \tilde{x} \) with the simultaneous sign inversion \( \epsilon \mapsto -\epsilon \), one has the *reversibility* property
\[ \Phi_f^{-1}(x, \epsilon) = \Phi_f(x, -\epsilon). \]  
In particular, the map \( \Phi_f \) is *birational*.

Kahan applied this discretization scheme to the famous Lotka-Volterra system and showed that in this case it possesses a very remarkable non-spiralling property. This property was explained by Sanz-Serna [18] by demonstrating that in this case the numerical method preserves an invariant Poisson structure of the original system.

The next intriguing appearance of this discretization was in two papers by Hirota and Kimura who (being apparently unaware of the work by Kahan) applied it to two famous *integrable* system of classical mechanics, the Euler top and the Lagrange top [10, 13]. Surprisingly, the discretization scheme produced in both cases *integrable* maps.

In [16, 14, 15] the authors undertook an extensive study of the properties of the Kahan’s method when applied to integrable systems (we proposed to use in the integrable context the term “Hirota-Kimura method”). It was demonstrated that, in an amazing number of cases, the method preserves integrability in the sense that the map \( \Phi_f(x, \epsilon) \) possesses as many independent integrals of motion as the original system \( \dot{x} = f(x) \).

Further remarkable geometric properties of the Kahan’s method were discovered by Celledoni, McLachlan, Owren and Quispel in [5], see also [6, 7]. These properties are unrelated to integrability. They demonstrated that for an arbitrary Hamiltonian vector field with a constant Poisson tensor and a cubic Hamilton function, the map \( \Phi_f(x, \epsilon) \) possesses a rational integral of motion and an invariant measure with a polynomial density.

The goal of the present paper is to communicate several novel observations regarding integrability of the Kahan’s method. These observations hold for several of the most complicated cases for which integrability of the Kahan-Hirota-Kimura discretization is established (Clebsch system, \( so(4) \) Euler top, Kirchhoff system, and Lagrange top). However, some of our new findings here are verifiable by hands and do not require heavy computer algebra computations. This refers to nice compact formulas for some of the more complicated integrals of motion and for the density of the invariant measure. See Theorem 1 and Observation 2 in Section 3. We give these results for all of the above mentioned systems, but provide detailed proofs for the first Clebsch flow only (see Section
Another set of results still relies on the computer algebra. This refers to the so-called higher order Wronskian Hirota-Kimura bases. See Observation 3 in Section 3. We expect that understanding of the latter phenomenon could be crucial for the whole integrability picture of the Kahan-Hirota-Kimura discretizations, but to this moment the origin of this phenomenon remains obscure.

2 General properties of Kahan-Hirota-Kimura discretization

The explicit form of the map $\Phi_f$ defined by (2) is

$$\tilde{x} = \Phi_f(x, \epsilon) = x + 2\epsilon(I - \epsilon f'(x))^{-1} f(x),$$

(4)

where $f'(x)$ denotes the Jacobi matrix of $f(x)$. As a consequence, each component $x_i$ of $x$ is a rational function,

$$x_i = \frac{p_i(x; \epsilon)}{\Delta(x; \epsilon)},$$

(5)

with a common denominator

$$\Delta(x; \epsilon) = \det(I - \epsilon f'(x)).$$

(6)

Clearly, the degrees of all polynomials $p_i$ and $\Delta$ are equal to $n$.

One has the following expression for the Jacobi matrix of the map $\Phi_f$:

$$d\Phi_f(x) = \frac{\partial \tilde{x}}{\partial x} = (I - \epsilon f'(x))^{-1} (I + \epsilon f'(\tilde{x})),$$

(7)

so that

$$\det(d\Phi_f(x)) = \frac{\Delta(\tilde{x}; -\epsilon)}{\Delta(x; \epsilon)}.$$

(8)

For our investigations here, the notion of a Hirota-Kimura basis will be relevant. It was introduced and studied in some detail in [14]. We recall here the main facts.

For a given birational map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a set of functions $(\varphi_1, \ldots, \varphi_m)$, linearly independent over $\mathbb{R}$, is called a Hirota-Kimura basis (HK basis), if for every $x \in \mathbb{R}^n$ there exists a non-vanishing vector of coefficients $c = (c_1, \ldots, c_m)$ such that

$$c_1\varphi_1(\Phi^i(x)) + \ldots + c_m\varphi_m(\Phi^i(x)) = 0 \quad \text{for all} \quad i \in \mathbb{Z}.$$

For a given $x \in \mathbb{R}^n$, the vector space consisting of all $c \in \mathbb{R}^m$ with this property, say $K(x)$, is called the null-space of the basis $(\varphi_1, \ldots, \varphi_m)$ at the point $x$.

The notion of HK-bases is closely related to the notion of integrals, even if they cannot be immediately translated into one another. For instance, if $\dim K(x) = 1$, and $K(x)$ is spanned by the vector $(c_1(x), \ldots, c_m(x))$, then the quotients $c_i(x) : c_j(x)$ are integrals of the map $\Phi$.  

3
3 Novel observations and results

The most complicated cases where integrability of the Kahan-Hirota-Kimura discretization was established in [14, 15] are 6-dimensional systems including the Clebsch and the Kirchhoff cases of the motion of a rigid body in an ideal fluid (the first one being linearly isomorphic to the \(so(4)\)-Euler top), and the Lagrange top. A common feature of these systems is the they are Hamiltonian with respect to linear Lie-Poisson brackets, and completely integrable in the Liouville-Arnold sense (possess four independent integrals in involution, two of them being the Casimir functions of the Lie-Poisson bracket). The discretizations turn out to possess four integrals of motion. These integrals are very complicated. However, for each of these discretizations, one simple quadratic-fractional integral was found. Integrals which are not simple tend to be extremely complex. Nevertheless, we find a compact representation for some of those complex integrals.

**Theorem 1.** a) Suppose that there exists a symmetric bilinear expression \(\hat{P}(x, \bar{x}; \epsilon^2)\) such that for \(\bar{x} = \Phi_f(x; \epsilon)\) we have

\[
\hat{P}(x, \bar{x}; \epsilon^2) = \frac{p(x; \epsilon^2)}{\Delta(x; \epsilon)}.
\]  

Then the map \(\Phi_f(x; \epsilon)\) has an invariant measure

\[
\frac{dx_1 \wedge \ldots \wedge dx_n}{p(x; \epsilon^2)}.
\]  

b) Suppose that there exists another symmetric bilinear expression \(\hat{Q}(x, \bar{x}; \epsilon^2)\) such that for \(\bar{x} = \Phi_f(x; \epsilon)\) we have

\[
\hat{Q}(x, \bar{x}; \epsilon^2) = \frac{q(x; \epsilon^2)}{\Delta(x; \epsilon)}.
\]  

Then the map \(\Phi_f(x; \epsilon)\) has an integral of motion

\[
J(x; \epsilon^2) = \frac{\hat{P}(x, \bar{x}; \epsilon^2)}{\hat{Q}(x, \bar{x}; \epsilon^2)}.
\]

**Proof.** Changing in (9) \(\epsilon\) to \(-\epsilon\), we see that

\[
\hat{P}(x, \bar{x}; \epsilon^2) = \frac{p(x; \epsilon^2)}{\Delta(x; -\epsilon)}.
\]  

Applying the map \(x \mapsto \bar{x}\) and taking into account symmetry of \(\hat{P}\), we find:

\[
\hat{P}(x, \bar{x}; \epsilon^2) = \frac{p(\bar{x}; \epsilon^2)}{\Delta(\bar{x}; -\epsilon)}.
\]
Comparing this with (9), we arrive at

\[
\frac{\Delta(\bar{x}; -\epsilon)}{\Delta(x; \epsilon)} = \frac{p(\bar{x}; \epsilon^2)}{p(x; \epsilon^2)}.
\]

According to (8), this is equivalent to the first statement of the theorem. The second one follows from (13). Indeed, we see that

\[
\frac{\hat{P}(x, x; \epsilon^2)}{\hat{Q}(x, x; \epsilon^2)} = \frac{\hat{P}(x, \bar{x}; \epsilon^2)}{\hat{Q}(x, \bar{x}; \epsilon^2)}.
\]

This finishes the proof.

In general, it is not clear how to find bilinear expressions with the property described in Theorem 1. However, the following observation mysteriously holds true in a big number of cases.

**Observation 2.** Suppose that the Kahan-Hirota-Kimura discretization of \( \dot{x} = f(x) \) possesses a quadratic-fractional integral of the form

\[
I(x, \epsilon) = \frac{P(x; \epsilon^2)}{Q(x; \epsilon^2)},
\]

where \( P \) and \( Q \) as functions of \( x \) are polynomials of degree \( \leq 2 \), while as functions of \( \epsilon \) they are polynomials of \( \epsilon^2 \). In many cases, the polarizations \( \hat{P}, \hat{Q} \) of the polynomials \( P, Q \), with \( \epsilon^2 \) replaced by \(-\epsilon^2\), satisfy conditions of Theorem 1. In particular, to the quadratic-fractional integral (14), there corresponds a further bilinear-fractional integral

\[
J(x, \epsilon) = \frac{\hat{P}(x, \bar{x}; -\epsilon^2)}{\hat{Q}(x, \bar{x}; -\epsilon^2)},
\]

while either of the numerators of \( \hat{P}(x, \bar{x}; -\epsilon^2), \hat{Q}(x, \bar{x}; -\epsilon^2) \) serves as a density of an invariant measure.

The second observation is related to linear Wronskian relations with constant coefficients, which turn out to exist for all systems we consider in the present paper. These are relations of the type

\[
\sum_{(i, j) \in J} \gamma_{ij} w_{ij}(x) = 0, \quad \text{where} \quad w_{ij}(x) = \dot{x}_i x_j - x_i \dot{x}_j,
\]

satisfied on solutions of \( \dot{x} = f(x) \). Here \( J \subset \{1, 2, \ldots, n\}^2 \) is some set, and \( \gamma_{ij} \) for \((i, j) \in J\) are certain constants. Note that the existence of such a relation can be formulated in a different way, by saying that the Wronskians \( w_{ij}(x) = \dot{x}_i x_j - x_i \dot{x}_j = x_j f_i(x) - x_i f_j(x) \) with \((i, j) \in J\) build a HK basis for the continuous time system \( \dot{x} = f(x) \).
**Observation 3.** In many cases, to a Wronskian relation \( (16) \) satisfied on solutions of \( \dot{x} = f(x) \), the discrete Wronskians

\[
W_{ij}^{(\ell)}(x) = x_i^{(\ell)} x_j - x_i x_j^{(\ell)}, \quad (i, j) \in J,
\]

of all orders \( \ell \geq 1 \), form a HK basis for the map \( \Phi_f \). Here we use the notation \( x_i^{(\ell)} = x_i \circ \Phi^\ell(x; \epsilon) \) for the components of the iterates of \( x \) under the map \( \Phi_f \).

### 4 General Clebsch flow

The motion of a heavy top and the motion of a rigid body in an ideal fluid can be described by the so-called Kirchhoff equations

\[
\begin{cases}
\dot{m} = m \times \frac{\partial H}{\partial m} + p \times \frac{\partial H}{\partial p}, \\
\dot{p} = p \times \frac{\partial H}{\partial m},
\end{cases}
\]

where \( H = H(m, p) \) is a quadratic polynomial in \( m = (m_1, m_2, m_3) \in \mathbb{R}^3 \) and \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \); here \( \times \) denotes vector product in \( \mathbb{R}^3 \). The physical meaning of \( m \) is the total angular momentum, whereas \( p \) represents the total linear momentum of the system. System \( (18) \) is Hamiltonian with the Hamilton function \( H(m, p) \), with respect to the Lie-Poisson bracket of \( e(3)^* \):

\[
\{m_i, m_j\} = m_k, \quad \{m_i, p_j\} = p_k,
\]

where \((i, j, k)\) is a cyclic permutation of \((1,2,3)\) (all other pairwise Poisson brackets of the coordinate functions are obtained from these by the skew-symmetry, or otherwise vanish). The functions

\[ K_1 = p_1^2 + p_2^2 + p_3^2, \quad K_2 = m_1 p_1 + m_2 p_2 + m_3 p_3 \]

are Casimirs of the bracket \( (19) \), thus integrals of motion of an arbitrary Hamiltonian system \( (18) \). The complete integrability of Kirchhoff equations is guaranteed by the existence of a fourth integral of motion, functionally independent of \( H, K_1, K_2 \) and in Poisson involution with the Hamiltonian \( H \).

Consider a homogeneous quadratic Hamiltonian

\[
H = \frac{1}{2} \langle m, Am \rangle + \frac{1}{2} \langle p, Bp \rangle,
\]

where \( A = \text{diag}(a_1, a_2, a_3) \) and \( B = \text{diag}(b_1, b_2, b_3) \). The corresponding Kirchhoff equations read

\[
\begin{cases}
\dot{m} = m \times Am + p \times Bp, \\
\dot{p} = p \times Am,
\end{cases}
\]
or, in components,

\[
\begin{aligned}
\dot{m}_1 &= (a_3 - a_2)m_2m_3 + (b_3 - b_2)p_2p_3, \\
\dot{m}_2 &= (a_1 - a_3)m_1m_3 + (b_1 - b_3)p_3p_1, \\
\dot{m}_3 &= (a_2 - a_1)m_1m_2 + (b_2 - b_1)p_1p_2, \\
\dot{p}_1 &= a_3m_3p_2 - a_2m_2p_3, \\
\dot{p}_2 &= a_1m_1p_3 - a_3m_3p_1, \\
\dot{p}_3 &= a_2m_2p_1 - a_1m_1p_2.
\end{aligned}
\]  

(22)

An additional (fourth) integral of motion exists, if the following Clebsch condition is satisfied:

\[
\frac{b_1 - b_2}{a_3} + \frac{b_2 - b_3}{a_1} + \frac{b_3 - b_1}{a_2} = 0.
\]  

(23)

Note that condition (23) is equivalent to the existence of \(\omega_1, \omega_2, \omega_3\) such that

\[
\omega_1 - \omega_2 = \frac{b_1 - b_2}{a_3}, \quad \omega_2 - \omega_3 = \frac{b_2 - b_3}{a_1}, \quad \omega_3 - \omega_1 = \frac{b_3 - b_1}{a_2}.
\]  

(24)

Another way to express condition (23) is to say that the following three expressions have the same value:

\[
\frac{b_1 - b_2}{a_3(a_1 - a_2)} = \frac{b_2 - b_3}{a_1(a_2 - a_3)} = \frac{b_3 - b_1}{a_2(a_3 - a_1)} = \frac{1}{\beta}.
\]  

(25)

With this notation, and taking into account (24), we easily derive that there exists a constant \(\alpha\) such that

\[
a_i = \alpha + \beta \omega_i, \quad b_i = \alpha \omega_i - \beta \omega_j \omega_k.
\]  

(26)

Thus, under condition (23), the flow (22) is a linear combination of two flows:

- the first Clebsch flow corresponding to \(\alpha = 1\) and \(\beta = 0\), so that
  \[
  a_1 = a_2 = a_3 = 1, \quad b_1 = \omega_1, \quad b_2 = \omega_2, \quad b_3 = \omega_3,
  \]  

with the Hamilton function \(\frac{1}{2}H_1\), where

\[
H_1 = m_1^2 + m_2^2 + m_3^2 + \omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2.
\]  

(28)

- and the second Clebsch flow corresponding to \(\alpha = 0\) and \(\beta = 1\), so that
  \[
  \begin{aligned}
  a_1 &= \omega_1, & a_2 &= \omega_2, & a_3 &= \omega_3, \\
  b_1 &= -\omega_2 \omega_3, & b_2 &= -\omega_3 \omega_1, & b_3 &= -\omega_1 \omega_2.
  \end{aligned}
  \]  

(29)

with the Hamilton function \(\frac{1}{2}H_2\), where

\[
H_2 = \omega_1 m_1^2 + \omega_2 m_2^2 + \omega_3 m_3^2 - \omega_2 \omega_3 p_1^2 - \omega_3 \omega_1 p_2^2 - \omega_1 \omega_2 p_3^2.
\]  

(30)
The key statement is:

**Theorem 4.** Hamilton functions (28) and (30) are in involution, so that the first and the second Clebsch flows commute. Thus, under condition (23), the flow (22) is completely integrable.

As a matter of fact, Clebsch condition (23) admits a different characterization.

**Theorem 5.** Condition (23) is equivalent to the existence of a Wronskian relation

\[
A_1(\dot{m}_1p_1 - m_1\dot{p}_1) + A_2(\dot{m}_2p_2 - m_2\dot{p}_2) + A_3(\dot{m}_3p_3 - m_3\dot{p}_3) = 0
\]  

(31)

with constant coefficients \(A_i\), satisfied on solutions of (22). In this case, coefficients \(A_i\) are given, up to a common factor, by

\[
A_1 = \frac{1}{a_2} + \frac{1}{a_3} - \frac{1}{a_1}, \quad A_2 = \frac{1}{a_3} + \frac{1}{a_1} - \frac{1}{a_2}, \quad A_3 = \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3}.
\]  

(32)

**Proof.** Wronskian relation (31) is satisfied by virtue of equations of motion (22), if and only if coefficients \(A_i\) satisfy

\[
\begin{cases}
A_1(a_3 - a_2) + A_2a_3 - A_3a_2 = 0, \\
-A_1a_3 + A_2(a_1 - a_3) + A_3a_1 = 0, \\
A_1a_2 - A_2a_1 + A_3(a_2 - a_1) = 0,
\end{cases}
\]  

(33)

and

\[
A_1(b_3 - b_2) + A_2(b_1 - b_3) + A_3(b_2 - b_1) = 0.
\]  

(34)

System (33) is equivalent to

\[(A_1 + A_2)a_3 = (A_3 + A_1)a_2 = (A_2 + A_3)a_1,
\]

so that, up to an inessential common factor, we find:

\[
A_1 + A_2 = \frac{2}{a_3}, \quad A_2 + A_3 = \frac{2}{a_1}, \quad A_3 + A_1 = \frac{2}{a_2},
\]  

(35)

leading to (32). With these values of \(A_i\), equation (34) is equivalent to Clebsch condition (23):

\[
A_1(b_3 - b_2) + A_2(b_1 - b_3) + A_3(b_2 - b_1)
= b_1(A_2 - A_3) + b_2(A_3 - A_1) + b_3(A_1 - A_2)
= b_1 \left( \frac{1}{a_3} - \frac{1}{a_2} \right) + b_2 \left( \frac{1}{a_1} - \frac{1}{a_3} \right) + b_3 \left( \frac{1}{a_2} - \frac{1}{a_1} \right)
= \frac{b_1 - b_2}{a_3} + \frac{b_2 - b_1}{a_1} + \frac{b_3 - b_1}{a_2} = 0,
\]

which concludes the proof. \(\square\)
In what follows, we assume that Clebsch condition (25) is satisfied with \( \beta \neq 0 \), and we define the constants \( A_i \) as in (32).

Applying the Kahan-Hirota-Kimura scheme to the general flow (22) of Clebsch system, we arrive at the following discretization:

\[
\begin{align*}
\tilde{m}_1 - m_1 &= \epsilon(a_3 - a_2)(\tilde{m}_2m_3 + m_2\tilde{m}_3) + \epsilon(b_3 - b_2)(\tilde{p}_2p_3 + p_2\tilde{p}_3), \\
\tilde{m}_2 - m_2 &= \epsilon(a_1 - a_3)(\tilde{m}_3m_1 + m_3\tilde{m}_1) + \epsilon(b_1 - b_3)(\tilde{p}_3p_1 + p_3\tilde{p}_1), \\
\tilde{m}_3 - m_3 &= \epsilon(a_2 - a_1)(\tilde{m}_1m_2 + m_1\tilde{m}_2) + \epsilon(b_2 - b_1)(\tilde{p}_1p_2 + p_1\tilde{p}_2), \\
\tilde{p}_1 - p_1 &= ea_3(\tilde{m}_3p_2 + m_3\tilde{p}_2) - ea_2(\tilde{m}_2p_3 + m_2\tilde{p}_3), \\
\tilde{p}_2 - p_2 &= ea_1(\tilde{m}_1p_3 + m_1\tilde{p}_3) - ea_3(\tilde{m}_3p_1 + m_3\tilde{p}_1), \\
\tilde{p}_3 - p_3 &= ea_2(\tilde{m}_2p_1 + m_2\tilde{p}_1) - ea_1(\tilde{m}_1p_2 + m_1\tilde{p}_2).
\end{align*}
\]
(36)

Linear system (36) defines a birational map \( \Phi_f : \mathbb{R}^6 \to \mathbb{R}^6 \), \((m, p) \to (\tilde{m}, \tilde{p})\).

A simple (quadratic-fractional) integral of \( \Phi_f \) and a first order Wronskian HK basis for this map were found in our previous work.

**Theorem 6.** (Quadratic-fractional integral, [14]) The function

\[
I_0(m, p, \epsilon) = \frac{A_1a_2a_3g_1 + A_2a_3a_1g_2 + A_3a_1a_2g_3}{1 + \epsilon^2\frac{a_1a_2a_3}{\beta}(g_1 + g_2 + g_3)},
\]
(37)

where

\[
g_i(m, p) = p_i^2 + \frac{\beta a_i}{a_ja_k}m_i^2,
\]
(38)

is an integral of motion of the map \( \Phi_f \). The set

\[
\Psi_0 = (g_1, g_2, g_3, 1)
\]
(39)

is a HK basis for the map \( \Phi_f \), with a one-dimensional null space

\[
K_{\Psi_0}(m, p) = [c_1a_2a_3 : c_2a_3a_1 : c_3a_1a_2 : -c_0],
\]
(40)

where

\[
\begin{align*}
c_1 &= A_1 + \epsilon^2A_3a_1(b_1 - b_2)g_2 + \epsilon^2A_2a_1(b_1 - b_3)g_3, \\
c_2 &= A_2 + \epsilon^2A_1a_2(b_2 - b_3)g_3 + \epsilon^2A_3a_2(b_2 - b_1)g_1, \\
c_3 &= A_3 + \epsilon^2A_2a_3(b_1 - b_3)g_1 + \epsilon^2A_1a_3(b_1 - b_2)g_2, \\
c_0 &= A_1a_2a_3g_1 + A_2a_3a_1g_2 + A_3a_1a_2g_3.
\end{align*}
\]
(41)-(44)

**Theorem 7.** (Discrete Wronskians HK basis, [15]) Functions \( W_i^{(1)}(m, p) = \tilde{m}_i p_i - m_i \tilde{p}_i, \ i = 1, 2, 3 \), form a HK basis for the map \( \Phi_f \), with a one-dimensional null space spanned by \([c_1 : c_2 : c_3]\), with the functions \( c_i \) given in (11) - (13). In other words, on orbits of the map \( \Phi_f \) there holds:

\[
\sum_{i=1}^{3} c_i(\tilde{m}_i p_i - m_i \tilde{p}_i) = 0.
\]
(45)
Novel results, exemplifying Observations 2 and 3, are as follows.

**Theorem 8. (Bilinear-fractional integral)** The function

\[ J_0(m, p, \epsilon) = \frac{A_1a_2a_3G_1 + A_2a_3a_1G_2 + A_3a_1a_2G_3}{1 - \epsilon^2 \frac{a_1a_2a_3}{\beta} (G_1 + G_2 + G_3)}, \quad (46) \]

where

\[ G_i(m, p) = \frac{\beta a_i}{a_ja_k} m_i \tilde{m}_i, \quad (47) \]

is an integral of motion of the map \( \Phi_f \). The set

\[ \Psi_1 = (G_1, G_2, G_3, 1) \]

is a HK basis for the map \( \Phi_f \), with a one-dimensional null space

\[ K_{\Psi_1}(m, p) = [C_1a_2a_3 : C_2a_3a_1 : C_3a_1a_2 : -C_0], \]

where

\[ C_1 = A_1 - \epsilon^2 A_3a_1(b_1 - b_2)G_2 - \epsilon^2 A_2a_1(b_1 - b_3)G_3, \quad (50) \]
\[ C_2 = A_2 - \epsilon^2 A_1a_2(b_2 - b_3)G_3 - \epsilon^2 A_3a_2(b_2 - b_1)G_1, \quad (51) \]
\[ C_3 = A_3 - \epsilon^2 A_2a_3(b_3 - b_1)G_1 - \epsilon^2 A_1a_3(b_3 - b_2)G_2, \quad (52) \]
\[ C_0 = A_1a_2a_3G_1 + A_2a_3a_1G_2 + A_3a_1a_2G_3. \quad (53) \]

Actually, the numerator and the denominator of the fraction \(46\) satisfy conditions of Theorem 1.

**Corollary 9. (Density of an invariant measure)** The map \( \Phi_f(x; \epsilon) \) has an invariant measure

\[ \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(m, p; \epsilon)}, \]

where \( \phi(m, p; \epsilon) \) can be taken as the numerator of either of the functions \( C_i \), \( i = 0, 1, 2, 3 \).

**Theorem 10. (Second order Wronskians HK basis)** The functions

\[ W_i^{(2)}(m, p) = \tilde{m}_i p_i - m_i \tilde{p}_i, \quad i = 1, 2, 3, \]

form a HK basis for the map \( \Phi_f \), with a one-dimensional null space spanned by \([C_1 : C_2 : C_3]\), with the functions \( C_i \) given in \(50\)–\(52\). In other words, on orbits of the map \( \Phi_f \) there holds:

\[ \sum_{i=1}^{3} C_i(\tilde{m}_i p_i - m_i \tilde{p}_i) = 0. \quad (54) \]
Theorem 11. (Higher order Wronskians HK bases)

1. The functions $W^{(3)}_i(m, p) = \tilde{m}_i p_i - \tilde{m}_i \tilde{p}_i$, $i = 1, 2, 3$, form a HK basis for the map $\Phi_f$, with a one-dimensional null space. On orbits of the map $\Phi_f$ there holds

$$3 \sum_{i=1}^3 D_i (\tilde{m}_i p_i - \tilde{m}_i \tilde{p}_i) = 0.$$ 

The two integrals of motion $J_1 = D_1/D_3$ and $J_2 = D_2/D_3$ are functionally independent.

2. The functions $W^{(4)}_i(m, p) = \tilde{m}_i p_i - \tilde{m}_i \tilde{p}_i$, $i = 1, 2, 3$, form a HK basis for the map $\Phi_f$, with a one-dimensional null space. On orbits of the map $\Phi_f$ there holds

$$3 \sum_{i=1}^3 E_i (\tilde{m}_i p_i - \tilde{m}_i \tilde{p}_i) = 0.$$ 

The two integrals of motion $J_3 = E_1/E_3$ and $J_4 = E_2/E_3$ are functionally independent.

3. Among the integrals of motion $I_0, J_0, \ldots, J_4$ four are functionally independent. In particular, each of the sets $\{I_0, J_0, J_1, J_2\}$, $\{I_0, J_0, J_3, J_4\}$, $\{J_1, J_2, J_3, J_4\}$ consists of four independent integrals of motion.

5 The first Clebsch flow

We collect here the formulas for the first Clebsch flow, since this will be our main example. Equations of motion:

$$\begin{align*}
\dot{m}_1 &= (\omega_3 - \omega_2)p_2 p_3, \\
\dot{m}_2 &= (\omega_1 - \omega_3)p_3 p_1, \\
\dot{m}_3 &= (\omega_2 - \omega_1)p_1 p_2, \\
\dot{p}_1 &= m_3 p_2 - m_2 p_3, \\
\dot{p}_2 &= m_1 p_3 - m_3 p_1, \\
\dot{p}_3 &= m_2 p_1 - m_1 p_2.
\end{align*}$$

(55)

Wronskian relation satisfied on solutions of (55) is:

$$(\dot{m}_1 p_1 - m_1 \dot{p}_1) + (\dot{m}_2 p_2 - m_2 \dot{p}_2) + (\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0.$$ 

(56)

Applying the Kahan-Hirota-Kimura scheme to the first Clebsch flow (55),

11
we arrive at the following discretization:

\[
\begin{align*}
\tilde{m}_1 - m_1 &= \epsilon(\omega_3 - \omega_2)(\tilde{p}_2p_3 + p_2\tilde{p}_1), \\
\tilde{m}_2 - m_2 &= \epsilon(\omega_1 - \omega_3)(\tilde{p}_3p_1 + p_3\tilde{p}_1), \\
\tilde{m}_3 - m_3 &= \epsilon(\omega_2 - \omega_1)(\tilde{p}_1p_2 + p_1\tilde{p}_2), \\
\tilde{p}_1 - p_1 &= \epsilon(\tilde{m}_3p_2 + m_3p_2) - \epsilon(\tilde{m}_2p_3 + m_2\tilde{p}_3), \\
\tilde{p}_2 - p_2 &= \epsilon(\tilde{m}_1p_3 + m_1\tilde{p}_3) - \epsilon(\tilde{m}_3p_1 + m_3\tilde{p}_1), \\
\tilde{p}_3 - p_3 &= \epsilon(\tilde{m}_2p_1 + m_2\tilde{p}_1) - \epsilon(\tilde{m}_1p_2 + m_1\tilde{p}_2).
\end{align*}
\] (57)

Linear system (57) defines a birational map \( \Phi_f : \mathbb{R}^6 \to \mathbb{R}^6, (m, p) \mapsto (\tilde{m}, \tilde{p}) \).

**Theorem 12.** (Quadratic-fractional integral, [14]) The function

\[
I_0(m, p, \epsilon) = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}
\] (58)

is an integral of motion of the map \( \Phi_f : (m, p) \mapsto (\tilde{m}, \tilde{p}) \). The set

\[
\Psi_0 = (p_1^2, p_2^2, p_3^2, 1)
\] (59)

is a HK-basis for the map \( \Phi_f \), with a one-dimensional null-space

\[
K_{\Psi_0}(m, p) = [c_1 : c_2 : c_3 : -c_0],
\] (60)

where

\[
\begin{align*}
c_1 &= 1 + \epsilon^2(\omega_1 - \omega_2)p_2^2 + \epsilon^2(\omega_1 - \omega_3)p_3^2, \\
c_2 &= 1 + \epsilon^2(\omega_2 - \omega_1)p_1^2 + \epsilon^2(\omega_2 - \omega_3)p_3^2, \\
c_3 &= 1 + \epsilon^2(\omega_3 - \omega_1)p_1^2 + \epsilon^2(\omega_3 - \omega_2)p_2^2, \\
c_0 &= p_1^2 + p_2^2 + p_3^2.
\end{align*}
\] (61-63)

Equivalently:

\[
K_{\Psi_0}(m, p) = \left[ 1 + \epsilon^2\omega_1 I_0 : 1 + \epsilon^2\omega_2 I_0 : 1 + \epsilon^2\omega_3 I_0 : -I_0 \right] = \left[ \frac{1}{I_0} + \epsilon^2\omega_1 : \frac{1}{I_0} + \epsilon^2\omega_2 : \frac{1}{I_0} + \epsilon^2\omega_3 : -1 \right].
\] (65-66)

**Theorem 13.** (Discrete Wronskians HK basis, [15]) The functions \( W_1^{(1)}(m, p) = \tilde{m}_i p_i - m_i \tilde{p}_i, i = 1, 2, 3 \), form a HK basis for the map \( \Phi_f \), with a one-dimensional null space spanned by \([c_1 : c_2 : c_3]\), with the functions \( c_i \) given in (61)-(63). In other words, on orbits of the map \( \Phi_f \) there holds:

\[
\sum_{i=1}^{3} c_i(\tilde{m}_i p_i - m_i \tilde{p}_i) = 0.
\] (67)

Novel results, illustrating Observations 2 and 3, are as follows.
Theorem 14. (Bilinear-fractional integral) The function
\[ J_0(m, p, \epsilon) = \frac{p_1\tilde{p}_1 + p_2\tilde{p}_2 + p_3\tilde{p}_3}{1 + \epsilon^2(\omega_1p_1\tilde{p}_1 + \omega_2p_2\tilde{p}_2 + \omega_3p_3\tilde{p}_3)} \]  
(68)
is an integral of motion of the map \( \Phi_f \). The set
\[ \Psi_1 = (p_1\tilde{p}_1, p_2\tilde{p}_2, p_3\tilde{p}_3, 1) \]  
(69)
is a HK-basis for the map \( \Phi_f \), with a one-dimensional null space
\[ K_{\Psi_1}(m, p) = [C_1 : C_2 : C_3 : -C_0], \]  
(70)
where
\[
\begin{align*}
C_1 &= 1 + \epsilon^2(\omega_2 - \omega_1)p_2\tilde{p}_2 + \epsilon^2(\omega_3 - \omega_1)p_3\tilde{p}_3, \\
C_2 &= 1 + \epsilon^2(\omega_1 - \omega_2)p_1\tilde{p}_1 + \epsilon^2(\omega_3 - \omega_2)p_3\tilde{p}_3, \\
C_3 &= 1 + \epsilon^2(\omega_1 - \omega_3)p_1\tilde{p}_1 + \epsilon^2(\omega_2 - \omega_3)p_2\tilde{p}_2, \\
C_0 &= p_1\tilde{p}_1 + p_2\tilde{p}_2 + p_3\tilde{p}_3. 
\end{align*}
\]
Equivalentally:
\[
K_{\Psi_1}(m, p) = \left[ 1 - \epsilon^2\omega_1J_0 : 1 - \epsilon^2\omega_2J_0 : 1 - \epsilon^2\omega_3J_0 : -J_0 \right] 
= \left[ \frac{1}{J_0} - \epsilon^2\omega_1 : \frac{1}{J_0} - \epsilon^2\omega_2 : \frac{1}{J_0} - \epsilon^2\omega_3 : -1 \right].
\]
(75)
(76)
Remark 1. One can show that the numerators of the expressions \( C_i, i = 0, \ldots, 3 \), are irreducible polynomials of \( m, p \) depending on \( \epsilon^2 \) rather than on \( \epsilon \), thus they satisfy Observation 2.

Corollary 15. (Density of an invariant measure) The map \( \Phi_f(x; \epsilon) \) has an invariant measure
\[
\frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(m, p, \epsilon)},
\]
where for \( \phi(m, p; \epsilon) \) one can take the numerator of the function \( p_1\tilde{p}_1 + p_2\tilde{p}_2 + p_3\tilde{p}_3 \),
or the numerator of the function \( 1 + \epsilon^2(\omega_1p_1\tilde{p}_1 + \omega_2p_2\tilde{p}_2 + \omega_3p_3\tilde{p}_3) \) (the quotient of both densities is an integral of motion \( J_0 \)).

Theorem 16. (Second order Wronskians HK basis) The functions
\[ W_i^{(2)}(m, p) = \tilde{m}_i p_i - m_i\tilde{p}_i, \quad i = 1, 2, 3, \]
form a HK basis for the map \( \Phi_f \), with a one-dimensional null space spanned by \([C_1 : C_2 : C_3] \), with the functions \( C_i \) given in (71)–(73). In other words, on orbits of the map \( \Phi_f \) there holds:
\[
\sum_{i=1}^{3} C_i(\tilde{m}_i p_i - m_i\tilde{p}_i) = 0.
\]  
(77)
Finally, there holds a theorem which reads literally as Theorem 11 on higher order Wronskians HK bases.

We now turn to the proofs of the above results.

Proof of Theorem 12. We show that the function $I_0$ in (68) is an integral of motion, i.e., that

$$\frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2 (\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)} = \frac{\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2}{1 - \epsilon^2 (\omega_1 \tilde{p}_1^2 + \omega_2 \tilde{p}_2^2 + \omega_3 \tilde{p}_3^2)}.$$ 

This is equivalent to

$$\tilde{p}_1^2 - p_1^2 + \tilde{p}_2^2 - p_2^2 + \tilde{p}_3^2 - p_3^2 = \epsilon^2 \left[ (\omega_2 - \omega_1)(\tilde{p}_1 p_2^2 - \tilde{p}_2 p_1^2) + (\omega_3 - \omega_2)(\tilde{p}_2 p_3^2 - \tilde{p}_3 p_2^2) + (\omega_1 - \omega_3)(\tilde{p}_3 p_1^2 - \tilde{p}_1 p_3^2) \right].$$

On the left-hand side of this equation we replace $\tilde{p}_i - p_i$ through the expressions from the last three equations of motion (57), on the right-hand side we replace $\epsilon(\omega_i - \omega_j)(\tilde{p}_i p_k + p_j p_k)$ by $\tilde{m}_i - m_i$, according to the first three equations of motion (57). This brings the equation we want to prove into the form

$$(\tilde{p}_1 + p_1)(\tilde{m}_3 p_2 + m_3 \tilde{p}_2 - m_2 \tilde{p}_3 - m_2 p_3) + \epsilon(\omega_2 - \omega_1)(\tilde{p}_2 p_3 - p_2 \tilde{p}_3)(\tilde{m}_1 - m_1) + (\tilde{p}_3 p_1 - p_3 \tilde{p}_1)(\tilde{m}_2 - m_2).$$

But this is an algebraic identity in twelve variables $m_k, p_k, \tilde{m}_k, \tilde{p}_k$. ■

Proof of Theorem 14. We show that the function $J_0$ in (68) is an integral of motion, i.e., that

$$\frac{p_1 p_1 + p_2 p_2 + p_3 p_3}{1 + \epsilon^2 (\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)} = \frac{\tilde{p}_1 p_1 + \tilde{p}_2 p_2 + \tilde{p}_3 p_3}{1 + \epsilon^2 (\omega_1 \tilde{p}_1^2 + \omega_2 \tilde{p}_2^2 + \omega_3 \tilde{p}_3^2)}.$$ 

This is equivalent to

$$p_1(\tilde{p}_1 - p_1) + p_2(\tilde{p}_2 - p_2) + p_3(\tilde{p}_3 - p_3) = \epsilon^2 \left[ (\omega_1 - \omega_2)(\tilde{p}_1 p_2 p_3 - \tilde{p}_2 p_1 p_3) + (\omega_2 - \omega_3)(\tilde{p}_2 p_3 p_1 - \tilde{p}_3 p_2 p_1) + (\omega_3 - \omega_1)(\tilde{p}_3 p_1 p_2 - \tilde{p}_1 p_3 p_2) \right].$$

On the left-hand side of this equation we use

$$2p_i(\tilde{p}_i - p_i) = (\tilde{p}_i + p_i)(p_i - p_i) + (\tilde{p}_i - p_i)(p_i + p_i),$$

and then replace $\tilde{p}_i - p_i$ and $p_i - p_i$ through the expressions from the last three equations of motion (64). On the right-hand side we use

$$2(\tilde{p}_i p_j p_k - \tilde{p}_k p_i p_j) = (\tilde{p}_j p_k + \tilde{p}_k p_j)(p_i p_k - p_k p_i) + (\tilde{p}_j p_k - \tilde{p}_k p_j)(p_i p_k + p_k p_i),$$

14
and then replace $\epsilon(\omega_k - \omega_j)(\bar{p}_j p_k + p_j \bar{p}_k)$ by $\bar{m}_i - m_i$, and $\epsilon(\omega_k - \omega_j)(\bar{p}_j p_k + p_k \bar{p}_j)$ by $m_i - \bar{m}_i$, according to the first three equations of motion (57). This brings the equation we want to prove into the form

\[
(p_1 + p_1)(\bar{m}_3 p_2 + m_3 \bar{p}_2 - m_2 p_3 - m_2 \bar{p}_3) + (\bar{p}_1 + p_1)(m_3 p_2 + m_3 p_2 - m_2 \bar{p}_3 - m_2 \bar{p}_3) + \\
(p_2 + p_2)(\bar{m}_1 p_3 + m_1 \bar{p}_3 - m_3 \bar{p}_1 - m_3 p_1) + (\bar{p}_2 + p_2)(m_1 p_3 + m_1 p_3 - m_3 \bar{p}_1 - m_3 p_1) + \\
(p_3 + p_3)(\bar{m}_2 p_1 + m_2 \bar{p}_1 - m_1 \bar{p}_2 - m_1 p_2) + (\bar{p}_3 + p_3)(m_2 p_1 + m_2 p_1 - m_1 \bar{p}_2 - m_1 p_2)
\]

\[
= (p_2 p_1 - p_2 p_1)(\bar{m}_3 - m_3) + (\bar{p}_2 p_1 - p_2 \bar{p}_1)(m_3 - m_3) \\
+ (p_3 p_2 - p_3 p_2)\bar{m}_1 - m_1 + (\bar{p}_3 p_2 - p_3 \bar{p}_2)(m_3 - m_3) \\
+ (p_1 p_3 - p_1 \bar{p}_3)(\bar{m}_2 - m_2) + (\bar{p}_1 p_3 - p_1 \bar{p}_3)(m_2 - m_2).
\]

On the left-hand side there are many cancellations, so that we get:

\[
p_1(m_3 p_2 - m_2 p_3) + p_1(m_3 p_2 - m_2 p_3) + p_1(m_3 p_2 - m_2 p_3) + \\
p_2(m_1 p_3 - m_3 p_1) + p_2(m_1 p_3 - m_3 p_1) + p_2(m_1 p_3 - m_3 p_1) + \\
p_3(m_2 p_1 - m_1 p_2) + p_3(m_2 p_1 - m_1 p_2) + p_3(m_2 p_1 - m_1 p_2)
\]

\[
= (p_2 p_1 - p_2 p_1)(\bar{m}_3 - m_3) + (\bar{p}_2 p_1 - p_2 \bar{p}_1)(m_3 - m_3) \\
+ (p_3 p_2 - p_3 p_2)\bar{m}_1 - m_1 + (\bar{p}_3 p_2 - p_3 \bar{p}_2)(m_3 - m_3) \\
+ (p_1 p_3 - p_1 \bar{p}_3)(\bar{m}_2 - m_2) + (\bar{p}_1 p_3 - p_1 \bar{p}_3)(m_2 - m_2).
\]

But this is an algebraic identity in the variables $m_k, p_k, \bar{m}_k, \bar{p}_k, m_3, p_3$.

**Proof of Theorem 13** is based on the following four identities which hold on orbits of the map $\Phi_J$:

\[
\sum_{i=1}^{3} c_i \bar{m}_i p_i = \sum_{i=1}^{3} C_i m_i p_i, \quad (78)
\]

\[
\sum_{i=1}^{3} c_i m_i \bar{p}_i = \sum_{i=1}^{3} C_i m_i p_i, \quad (79)
\]

\[
\sum_{i=1}^{3} \bar{c}_i \bar{m}_i \bar{p}_i = \sum_{i=1}^{3} C_i \bar{m}_i \bar{p}_i, \quad (80)
\]

\[
\sum_{i=1}^{3} \bar{c}_i \bar{m}_i p_i = \sum_{i=1}^{3} C_i \bar{m}_i p_i. \quad (81)
\]

Indeed, from these relations there follows immediately:

\[
\sum_{i=1}^{3} c_i (\bar{m}_i p_i - m_i \bar{p}_i) = \sum_{i=1}^{3} \bar{c}_i (\bar{m}_i p_i - m_i \bar{p}_i) = 0,
\]

which proves the theorem.
Proof of formula (78). Using the first three equations of motion (57), we compute:

\[ \sum_{i=1}^{3} (\ddot{m}_i - m_i) p_i \]

\[ = \epsilon(\omega_3 - \omega_2)(p_1 p_2 \ddot{p}_3 + p_1 p_3 \ddot{p}_2) + \epsilon(\omega_1 - \omega_3)(p_2 p_3 \ddot{p}_1 + p_2 p_1 \ddot{p}_3) + \epsilon(\omega_2 - \omega_1)(p_3 p_1 \ddot{p}_2 + p_3 p_2 \ddot{p}_1) \]

\[ = \epsilon(\omega_1 - \omega_2)p_1 p_2 \ddot{p}_3 + \epsilon(\omega_2 - \omega_3)p_2 p_3 \ddot{p}_1 + \epsilon(\omega_3 - \omega_1)p_3 p_1 \ddot{p}_2 \]

\[ = \epsilon(\omega_1 - \omega_2)p_1 p_2 (\ddot{p}_3 - p_3) + \epsilon(\omega_2 - \omega_3)p_2 p_3 (\ddot{p}_1 - p_1) + \epsilon(\omega_3 - \omega_1)p_3 p_1 (\ddot{p}_2 - p_2). \]

Using the last three equations of motion (57), we continue the computation:

\[ = \epsilon^2(\omega_1 - \omega_2)p_1 p_2 ((\ddot{m}_2 p_1 + m_2 \ddot{p}_1) - (\ddot{m}_1 p_2 + m_1 \ddot{p}_2)) + \epsilon^2(\omega_2 - \omega_3)p_2 p_3 ((\ddot{m}_3 p_2 + m_3 \ddot{p}_2) - (\ddot{m}_2 p_3 + m_2 \ddot{p}_3)) + \epsilon^2(\omega_3 - \omega_1)p_3 p_1 ((\ddot{m}_1 p_3 + m_1 \ddot{p}_3) - (\ddot{m}_3 p_1 + m_3 \ddot{p}_1)) \]

\[ = (\epsilon^2(\omega_2 - \omega_1)p_2^2 + \epsilon^2(\omega_3 - \omega_1)p_3^2) \ddot{m}_1 p_1 + (\epsilon^2(\omega_2 - \omega_1)p_2 p_2 + \epsilon^2(\omega_3 - \omega_1)p_3 p_3) m_1 p_1 + (\epsilon^2(\omega_3 - \omega_2)p_3^2 + \epsilon^2(\omega_1 - \omega_2)p_1^2) \ddot{m}_2 p_2 + (\epsilon^2(\omega_3 - \omega_2)p_3 p_3 + \epsilon^2(\omega_1 - \omega_2)p_1 p_1) m_2 p_2 + (\epsilon^2(\omega_1 - \omega_3)p_1^2 + \epsilon^2(\omega_2 - \omega_3)p_2^2) \ddot{m}_3 p_3 + (\epsilon^2(\omega_1 - \omega_3)p_1 p_1 + \epsilon^2(\omega_2 - \omega_3)p_2 p_2) m_3 p_3. \]

This proves (78). We remark that this relation is the discrete time analog of \( \sum_{i=1}^{3} m_i p_i = 0 \); the crucial point in this analogy is that \( \sum_{i=1}^{3} (\ddot{m}_i - m_i) p_i \) turns out to be of order \( \epsilon^2 \).

Proof of formula (79). Using the first three equations of motion (57), we compute:

\[ \sum_{i=1}^{3} m_i (\ddot{p}_i - p_i) \]

\[ = \epsilon m_1 ((\ddot{m}_3 p_2 + m_3 \ddot{p}_2) - (\ddot{m}_2 p_3 + m_2 \ddot{p}_3)) + \epsilon m_2 ((\ddot{m}_1 p_3 + m_1 \ddot{p}_3) - (\ddot{m}_3 p_1 + m_3 \ddot{p}_1)) + \epsilon m_3 ((\ddot{m}_2 p_1 + m_2 \ddot{p}_1) - (\ddot{m}_1 p_2 + m_1 \ddot{p}_2)) \]

\[ = \epsilon (m_1 p_2 - m_2 p_1) \ddot{m}_3 + \epsilon (m_2 p_3 - m_3 p_2) \ddot{m}_1 + \epsilon (m_3 p_1 - m_1 p_3) \ddot{m}_2 \]

\[ = \epsilon (m_1 p_2 - m_2 p_1) (\ddot{m}_3 - m_3) + \epsilon (m_2 p_3 - m_3 p_2) (\ddot{m}_1 - m_1) + \epsilon (m_3 p_1 - m_1 p_3) (\ddot{m}_2 - m_2). \]
Using the last three equations of motion (57), we continue:

\[ e^2(\omega_2 - \omega_1)(m_1p_2 - m_2p_1)(p_1\ddot{p}_2 + p_2\ddot{p}_1) + e^2(\omega_3 - \omega_2)(m_2p_3 - m_3p_2)(p_2\ddot{p}_3 + p_3\ddot{p}_2) + e^2(\omega_1 - \omega_3)(m_3p_1 - m_1p_3)(p_3\ddot{p}_1 + p_1\ddot{p}_3) \]

\[ = (e^2(\omega_2 - \omega_1)p_2^2 + e^2(\omega_3 - \omega_1)p_2^2)m_1\ddot{p}_1 + (e^2(\omega_2 - \omega_1)p_2^2 + e^2(\omega_3 - \omega_1)p_2^2)m_1p_1 \
+ (e^2(\omega_3 - \omega_2)p_3^2 + e^2(\omega_1 - \omega_2)p_3^2)m_2\ddot{p}_2 + (e^2(\omega_3 - \omega_2)p_3^2 + e^2(\omega_1 - \omega_2)p_3^2)m_2p_2 \
+ (e^2(\omega_1 - \omega_3)p_1^2 + e^2(\omega_2 - \omega_3)p_1^2)m_2\ddot{p}_3 + (e^2(\omega_1 - \omega_3)p_1^2 + e^2(\omega_2 - \omega_3)p_1^2)m_2p_2 \]

This proves (80). Again, this relation is the discrete time analog of \( \sum_{i=1}^{3} m_i\ddot{p}_i = 0 \), since \( \sum_{i=1}^{3} m_i(\ddot{p}_i - p_i) \) turns out to be of order \( e^2 \).

**Proof of formula (80).** We start by using the first three equations of motion (57):

\[ \sum_{i=1}^{3}(\ddot{m}_i - m_i)p_i \]

\[ = e(\omega_1 - \omega_2)(\ddot{p}_1p_3p_2 + \ddot{p}_1p_2p_3) + e(\omega_1 - \omega_3)(\ddot{p}_2p_1p_3 + \ddot{p}_2p_3p_1) + e(\omega_2 - \omega_1)(\ddot{p}_3p_2p_1 + \ddot{p}_3p_1p_2) \]

\[ = e(\omega_1 - \omega_2)p_1\ddot{p}_3p_2 + e(\omega_2 - \omega_3)p_3\ddot{p}_2p_1 + e(\omega_3 - \omega_1)p_3\ddot{p}_1p_2 \]

and continue by using the last three equations of motion (57):

\[ = e^2(\omega_2 - \omega_1)p_1\ddot{p}_3 + (\ddot{m}_3p_1 - m_3\ddot{p}_1 - (\ddot{m}_1p_2 + m_1\ddot{p}_2) \]

\[ + e^2(\omega_3 - \omega_2)p_2\ddot{p}_3 + (\ddot{m}_3p_2 - m_3\ddot{p}_2 - (\ddot{m}_2p_3 + m_2\ddot{p}_3) \]

\[ + e^2(\omega_1 - \omega_3)p_3\ddot{p}_1 + (\ddot{m}_3p_1 - m_3\ddot{p}_1 - (\ddot{m}_1p_3 + m_1\ddot{p}_3) \]

\[ = (e^2(\omega_2 - \omega_1)p_2^2 + e^2(\omega_3 - \omega_1)p_2^2)m_2p_1 + (e^2(\omega_3 - \omega_2)p_3^2 + e^2(\omega_1 - \omega_2)p_3^2)m_2p_2 + (e^2(\omega_1 - \omega_3)p_1^2 + e^2(\omega_2 - \omega_3)p_1^2)m_2p_2 \]

This proves (80). Again, this relation is the discrete time analog of \( \sum_{i=1}^{3} m_i\ddot{p}_i = 0 \).
Proof of formula (81). We compute:

\[
\sum_{i=1}^{3} \tilde{m}_i (\tilde{p}_i - p_i) = \epsilon \tilde{m}_1 ((\tilde{m}_3 p_2 + m_3 \tilde{p}_2) - (\tilde{m}_2 p_3 + m_2 \tilde{p}_3)) + \epsilon \tilde{m}_2 ((\tilde{m}_1 p_3 + m_1 \tilde{p}_3) - (\tilde{m}_3 p_1 + m_3 \tilde{p}_1)) \\
+ \epsilon \tilde{m}_3 ((\tilde{m}_2 p_1 + m_2 \tilde{p}_1) - (\tilde{m}_1 p_2 + m_1 \tilde{p}_2))
\]

\[
= \epsilon (\tilde{m}_1 \tilde{p}_2 - \tilde{m}_2 \tilde{p}_1) m_3 + \epsilon (\tilde{m}_2 \tilde{p}_3 - \tilde{m}_3 \tilde{p}_2) m_1 + \epsilon (\tilde{m}_3 \tilde{p}_1 - \tilde{m}_1 \tilde{p}_3) m_2
\]

\[
= \epsilon (m_2 \tilde{p}_1 - \tilde{m}_1 \tilde{p}_2) (m_3 - m_3) + \epsilon (\tilde{m}_3 \tilde{p}_2 - m_2 \tilde{p}_3) (m_1 - m_1) + \epsilon (\tilde{m}_1 \tilde{p}_3 - m_3 \tilde{p}_1) (m_2 - m_2)
\]

\[
= \epsilon^2 (\omega_2 - \omega_1) (m_2 \tilde{p}_1 - \tilde{m}_1 \tilde{p}_2) (p_1 \tilde{p}_2 + p_2 \tilde{p}_1) + \epsilon^2 (\omega_3 - \omega_2) (m_3 \tilde{p}_2 - m_2 \tilde{p}_3) (p_2 \tilde{p}_3 + p_3 \tilde{p}_2)
+ \epsilon^2 (\omega_1 - \omega_3) (m_1 \tilde{p}_3 - \tilde{m}_3 \tilde{p}_1) (p_3 \tilde{p}_1 + p_1 \tilde{p}_3)
\]

This proves (81) and provides us with another discrete time analog of \(\sum_{i=1}^{3} m_i \tilde{p}_i = 0\). 

Corollary 17. The function

\[
K(m, p) = \sum_{i=1}^{3} \frac{C_i m_i \tilde{p}_i}{C_0 c_0}
\]

is an integral of motion of the map \(\Phi_f\).

Proof. Compare (78) with (81), taking into account that \(c_i/c_0\) are integrals of motion, that is, \(c_i/c_0 = \tilde{c}_i/\tilde{c}_0\). We arrive at

\[
\sum_{i=1}^{3} C_i \frac{m_i \tilde{p}_i}{c_0} = \sum_{i=1}^{3} C_i \frac{\tilde{m}_i \tilde{p}_i}{\tilde{c}_0}.
\]

Since \(C_i/C_0\) are integrals of motion, that is \(C_i/C_0 = \tilde{C}_i/\tilde{C}_0\), we see that

\[
\sum_{i=1}^{3} C_i \frac{m_i \tilde{p}_i}{c_0} = \sum_{i=1}^{3} \tilde{C}_i \frac{\tilde{m}_i \tilde{p}_i}{\tilde{c}_0}.
\]

This proves the statement. 

\[\Box\]
Proof of Theorem 16. We start with
\[\sum_{i=1}^{3} (\tilde{m}_i - m_i)p_i + \sum_{i=1}^{3} (\tilde{m}_i - m_i)\tilde{p}_i\]
\[= \epsilon(\omega_3 - \omega_2)(\tilde{p}_2\tilde{p}_3 + \tilde{p}_2\tilde{p}_3)p_1 + \epsilon(\omega_1 - \omega_3)(\tilde{p}_3\tilde{p}_1 + \tilde{p}_3\tilde{p}_1)p_2 + \epsilon(\omega_2 - \omega_1)(\tilde{p}_1\tilde{p}_2 + \tilde{p}_1\tilde{p}_2)p_3\]
\[+ \epsilon(\omega_3 - \omega_2)(\tilde{p}_2p_3 + p_2\tilde{p}_3)\tilde{p}_1 + \epsilon(\omega_1 - \omega_3)(\tilde{p}_3p_1 + p_3\tilde{p}_1)\tilde{p}_2 + \epsilon(\omega_2 - \omega_1)(\tilde{p}_1p_2 + p_1\tilde{p}_2)\tilde{p}_3\]
\[= \epsilon(\omega_3 - \omega_1)\tilde{p}_1\tilde{p}_2p_3 + \epsilon(\omega_1 - \omega_2)\tilde{p}_1\tilde{p}_3p_2 + \epsilon(\omega_1 - \omega_2)\tilde{p}_2\tilde{p}_3p_1 + \epsilon(\omega_2 - \omega_1)\tilde{p}_2\tilde{p}_1p_3\]
\[+ \epsilon(\omega_3 - \omega_1)\tilde{p}_3\tilde{p}_1p_2 + \epsilon(\omega_3 - \omega_1)\tilde{p}_3\tilde{p}_2p_1\]
\[= \epsilon(\omega_3 - \omega_2)\tilde{p}_1\tilde{p}_2p_3 + \epsilon(\omega_1 - \omega_3)\tilde{p}_2\tilde{p}_3p_1 + \epsilon(\omega_2 - \omega_1)\tilde{p}_3p_1p_2\]
\[+ \epsilon(\omega_3 - \omega_1)\tilde{p}_1(\tilde{p}_2 - p_2)p_3 + \epsilon(\omega_1 - \omega_2)\tilde{p}_1(\tilde{p}_3 - p_3)p_2 + \epsilon(\omega_1 - \omega_2)\tilde{p}_2(\tilde{p}_3 - p_3)p_1\]
\[+ \epsilon(\omega_3 - \omega_1)\tilde{p}_2(\tilde{p}_1 - p_1)p_3 + \epsilon(\omega_2 - \omega_3)\tilde{p}_3(\tilde{p}_1 - p_1)p_2 + \epsilon(\omega_3 - \omega_1)\tilde{p}_3(\tilde{p}_2 - p_2)p_1\]

This is equal to

\[= \epsilon(\omega_3 - \omega_2)(\tilde{p}_1 - \tilde{p}_1)p_2p_3 + \epsilon(\omega_1 - \omega_3)(\tilde{p}_2 - \tilde{p}_2)p_3p_1 + \epsilon(\omega_2 - \omega_1)(\tilde{p}_3 - \tilde{p}_3)p_1p_2\]
\[+ \epsilon(\omega_3 - \omega_2)\tilde{p}_1\tilde{p}_2p_3 + \epsilon(\omega_1 - \omega_3)\tilde{p}_2\tilde{p}_3p_1 + \epsilon(\omega_2 - \omega_1)\tilde{p}_3p_1p_2\]
\[+ \epsilon(\omega_3 - \omega_1)\tilde{p}_1(\tilde{p}_2 - p_2)p_3 + \epsilon(\omega_1 - \omega_2)\tilde{p}_1(\tilde{p}_3 - p_3)p_2 + \epsilon(\omega_1 - \omega_2)\tilde{p}_2(\tilde{p}_3 - p_3)p_1\]
\[+ \epsilon(\omega_3 - \omega_1)\tilde{p}_2(\tilde{p}_1 - p_1)p_3 + \epsilon(\omega_2 - \omega_3)\tilde{p}_3(\tilde{p}_1 - p_1)p_2 + \epsilon(\omega_3 - \omega_1)\tilde{p}_3(\tilde{p}_2 - p_2)p_1\]

Here the second line is equal to

\[-\sum_{i=1}^{3} (\tilde{m}_i - m_i)p_i,

so that, upon use of equations of motion, we find:

\[\sum_{i=1}^{3} (\tilde{m}_i - m_i)p_i - \sum_{i=1}^{3} m_i p_i + \sum_{i=1}^{3} \tilde{m}_i \tilde{p}_i\]
\[= \epsilon^2(\omega_3 - \omega_2)(\tilde{p}_2p_3 + \tilde{p}_2p_3) - (\tilde{p}_2p_3 + \tilde{p}_2p_3)p_2p_3\]
\[+ \epsilon^2(\omega_1 - \omega_3)(\tilde{p}_3p_1 + \tilde{m}_1\tilde{p}_1) - (\tilde{m}_3\tilde{p}_1 + \tilde{m}_3\tilde{p}_1)p_3p_1\]
\[+ \epsilon^2(\omega_2 - \omega_1)(\tilde{p}_3p_1 + \tilde{m}_1\tilde{p}_1) - (\tilde{m}_3\tilde{p}_1 + \tilde{m}_3\tilde{p}_1)p_1p_2\]
\[+ \epsilon^2(\omega_3 - \omega_1)(\tilde{p}_1(\tilde{m}_3p_1 + m_3\tilde{p}_1) - (\tilde{m}_3p_1 + m_3\tilde{p}_1))p_3\]
\[+ \epsilon^2(\omega_1 - \omega_2)(\tilde{p}_1(\tilde{m}_3p_1 + m_3\tilde{p}_1) - (\tilde{m}_3p_1 + m_3\tilde{p}_1)p_2\]
\[+ \epsilon^2(\omega_1 - \omega_2)(\tilde{p}_2(\tilde{m}_3p_1 + m_3\tilde{p}_1) - (\tilde{m}_3p_1 + m_3\tilde{p}_1)p_1\]
\[+ \epsilon^2(\omega_2 - \omega_3)(\tilde{p}_2(\tilde{m}_3p_1 + m_3\tilde{p}_1) - (\tilde{m}_3p_1 + m_3\tilde{p}_1)p_2\]
\[+ \epsilon^2(\omega_3 - \omega_1)(\tilde{p}_3(\tilde{m}_3p_1 + m_3\tilde{p}_1) - (\tilde{m}_3p_1 + m_3\tilde{p}_1)p_1).\]
(So, the strategy of transformations is: leave one variable with double tilde, one with single tilde and two without tilde.) Collecting terms, we have:

\[
\sum_{i=1}^{3} (\tilde{m}_i p_i - m_i \tilde{p}_i) - \sum_{i=1}^{3} m_i p_i + \sum_{i=1}^{3} \tilde{m}_i \tilde{p}_i
\]

\[
= \tilde{m}_1 p_1 \left( c^2(\omega_1 - \omega_3) \tilde{p}_3 p_3 + c^2(\omega_1 - \omega_2) \tilde{p}_2 p_2 \right) + \tilde{m}_2 p_2 \left( c^2(\omega_2 - \omega_3) \tilde{p}_3 p_3 + c^2(\omega_2 - \omega_1) \tilde{p}_1 p_1 \right) + \tilde{m}_3 p_3 \left( c^2(\omega_3 - \omega_2) \tilde{p}_2 p_2 + c^2(\omega_3 - \omega_1) \tilde{p}_1 p_1 \right) + m_1 \tilde{p}_1 \left( c^2(\omega_3 - \omega_1) \tilde{p}_3 p_3 + c^2(\omega_3 - \omega_2) \tilde{p}_3 \tilde{p}_3 \right) + m_2 \tilde{p}_2 \left( c^2(\omega_2 - \omega_1) \tilde{p}_2 p_2 + c^2(\omega_2 - \omega_3) \tilde{p}_2 \tilde{p}_2 \right) + m_3 \tilde{p}_3 \left( c^2(\omega_1 - \omega_2) \tilde{p}_1 \tilde{p}_1 + c^2(\omega_1 - \omega_3) \tilde{p}_1 \tilde{p}_3 \right).
\]

This can be put as

\[
\sum_{i=1}^{3} C_i (\tilde{m}_i p_i - m_i \tilde{p}_i) = \sum_{i=1}^{3} \tilde{C}_i m_i p_i - \sum_{i=1}^{3} c_i \tilde{m}_i \tilde{p}_i.
\]

From \(82\), \(80\), and from relations \(\tilde{C}_i / \tilde{C}_0 = C_i / C_0 \) and \(c_i / c_0 = \tilde{c}_i / \tilde{c}_0 \), we derive:

\[
\sum_{i=1}^{3} \tilde{C}_i m_i p_i = \tilde{C}_0 \cdot c_0 K(m, p), \quad \sum_{i=1}^{3} c_i \tilde{m}_i \tilde{p}_i = c_0 \cdot \tilde{C}_0 K(\tilde{m}, \tilde{p})
\]

By Corollary \(17\) \(K(m, p) = K(\tilde{m}, \tilde{p})\). This finishes the proof. \(\blacksquare\)

As for Theorem \(11\) at present we only have a proof based on symbolic computations by Maple, even for the first Clebsch flow.
6 The Kirchhoff case

The Kirchhoff case of the motion of the rigid body in an ideal fluid corresponds to the following values of the parameters in (20), (22):

\[ a_1 = a_2, \quad b_1 = b_2. \] (83)

Equations of motion read:

\[
\begin{align*}
\dot{m}_1 &= (a_3 - a_1)m_2m_3 + (b_3 - b_1)p_2p_3, \\
\dot{m}_2 &= (a_1 - a_3)m_1m_3 + (b_1 - b_3)p_1p_3, \\
\dot{m}_3 &= 0, \\
\dot{p}_1 &= a_3p_2m_3 - a_1p_3m_2, \\
\dot{p}_2 &= a_1p_3m_1 - a_3p_1m_3, \\
\dot{p}_3 &= a_1(p_1m_2 - p_3m_1).
\end{align*}
\] (84)

Thus, \( m_3 \) is an obvious fourth integral, due to the rotational symmetry of the system. It is easy to see that for (83) the Clebsch condition (23) is satisfied, as well. Thus, formally the Kirchhoff case is the particular case of the Clebsch case. One can choose the parameters \( \omega_1 \) in (24) as

\[ \omega_1 = \omega_2 = \frac{b_1}{a_1}, \quad \omega_3 = \frac{b_3}{a_1}. \] (85)

Correspondingly, integral \( H_1 \) becomes proportional to

\[ a_1H_1 = a_1(m_1^2 + m_2^2 + m_3^2) + b_1(p_1^2 + p_2^2) + b_3p_3^2. \]

Taking into account that the Hamilton function \( H \) is given by

\[ 2H = a_1(m_1^2 + m_2^2) + a_3m_3^2 + b_1(p_1^2 + p_2^2) + b_3p_3^2, \]

we see that the fourth integral \( H_1 \) can be replaced just by \( m_3^2 \).

Wronskian relation satisfied on solutions of (84):

\[
(m_1p_1 - m_1\dot{p}_1) + (m_2p_2 - m_2\dot{p}_2) + \left(\frac{2a_3}{a_1} - 1\right)(m_3p_3 - m_3\dot{p}_3) = 0.
\] (86)

Applying the Kahan-Hirota-Kimura scheme to the Kirchhoff system (84), we arrive at the following discretization:

\[
\begin{align*}
\tilde{m}_1 - m_1 &= \epsilon(a_3 - a_1)(\tilde{m}_2m_3 + m_2\tilde{m}_3) + \epsilon(b_3 - b_1)(\tilde{p}_2p_3 + p_2\tilde{p}_3), \\
\tilde{m}_2 - m_2 &= \epsilon(a_1 - a_3)(\tilde{m}_3m_1 + m_3\tilde{m}_1) + \epsilon(b_1 - b_3)(\tilde{p}_3p_1 + p_3\tilde{p}_1), \\
\tilde{m}_3 - m_3 &= 0, \\
\tilde{p}_1 - p_1 &= \epsilon a_3(\tilde{m}_3p_2 + m_3\tilde{p}_2) - \epsilon a_1(\tilde{m}_2p_3 + m_2\tilde{p}_3), \\
\tilde{p}_2 - p_2 &= \epsilon a_1(\tilde{m}_1p_3 + m_1\tilde{p}_3) - \epsilon a_3(\tilde{m}_3p_1 + m_3\tilde{p}_1), \\
\tilde{p}_3 - p_3 &= \epsilon a_1(\tilde{m}_2p_1 + m_2\tilde{p}_1 - \tilde{m}_1p_2 - m_1\tilde{p}_2).
\end{align*}
\] (87)

As usual, linear system (87) defines a birational map \( \Phi_f : \mathbb{R}^6 \to \mathbb{R}^6, (m, p) \mapsto (\tilde{m}, \tilde{p}) \).
Theorem 18. (Quadratic-fractional integral, [15]) The function
\[ I_0(m, p; \epsilon) = \frac{c_3(m, p; \epsilon)}{c_1(m, p; \epsilon)}, \tag{88} \]
where
\begin{align*}
  c_1 &= 1 + \epsilon^2 a_3 (a_1 - a_3) m_3^2 + \epsilon^2 a_1 (b_1 - b_3) p_3^2, \\
  c_3 &= \frac{2a_3}{a_1} - 1 + \epsilon^2 a_1 (a_3 - a_1)(m_1^2 + m_2^2) + \epsilon^2 a_3 (b_3 - b_1)(p_1^2 + p_2^2), \tag{90}
\end{align*}
is an integral of motion of the map \( \Phi_f \).

Theorem 19. (Discrete Wronskians HK basis, [15]) Functions \( W^{(1)}(m, p) = \bar{m}_i p_i - m_i \bar{p}_i, \ i = 1, 2, 3, \) form a HK basis for the map \( \Phi_f \) with a one-dimensional null space spanned by \([c_1 : c_1 : c_3] = [1 : 1 : I_0]\), where \( I_0 \) is the integral of \( \Phi_f \) given by (88).

Novel results, illustrating Observations 2 and 3, are as follows.

Theorem 20. (Bilinear-fractional integral) The function
\[ J_0(m, p; \epsilon) = \frac{C_3(m, p; \epsilon)}{C_1(m, p; \epsilon)}, \tag{91} \]
where
\begin{align*}
  C_1 &= 1 - \epsilon^2 a_3 (a_1 - a_3) m_3^2 - \epsilon^2 a_1 (b_1 - b_3) p_3^2, \\
  C_3 &= \frac{2a_3}{a_1} - 1 - \epsilon^2 a_1 (a_3 - a_1)(m_1 \bar{m}_1 + m_2 \bar{m}_2) - \epsilon^2 a_3 (b_3 - b_1)(p_1 \bar{p}_1 + p_2 \bar{p}_2), \tag{93}
\end{align*}
is an integral of motion of the map \( \Phi_f \).

Corollary 21. (Density of an invariant measure) The map \( \Phi_f(x; \epsilon) \) has an invariant measure
\[ \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(m, p; \epsilon)}, \]
where for \( \phi(m, p; \epsilon) \) one can take the numerator of either of the functions \( C_1, C_3 \).

Theorem 22. (Second order Wronskians HK basis) The functions
\[ W^{(2)}_i(m, p) = \bar{m}_i p_i - m_i \bar{p}_i, \ i = 1, 2, 3, \]
form a HK basis for the map \( \Phi_f \), with a one-dimensional null space spanned by \([C_1 : C_1 : C_3] = [1 : 1 : J_0]\), with the function \( J_0 \) given in (91).

Theorem 23. (Third order Wronskians HK basis) Functions \( W^{(3)}_i(m, p) = \bar{m}_i p_i - m_i \bar{p}_i, \ i = 1, 2, 3, \) form a HK basis for the map \( \Phi_f \), with a one-dimensional null space. On orbits of the map \( \Phi_f \) there holds
\[ J_1(\bar{m}_1 p_1 - m_1 \bar{p}_1) + J_2(\bar{m}_2 p_2 - m_2 \bar{p}_2) + J_3(\bar{m}_3 p_3 - m_3 \bar{p}_3) = 0, \]
where the function \( J_1(m, p; \epsilon) \) is an integral of motion. The four integrals of motion \( \{I_0, J_0, J_1, m_3\} \) are functionally independent.
7 Lagrange top

The Hamilton function of the Lagrange top is \( H = \frac{1}{2} H_1 \), where
\[
H_1 = m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3.
\]

Unlike the Clebsch and the Kirchhoff cases, this function is not homogeneous. Equations of motion of Lagrange top read
\[
\begin{align*}
\dot{m}_1 &= (\alpha - 1)m_2 m_3 + \gamma p_2, \\
\dot{m}_2 &= (1 - \alpha)m_1 m_3 - \gamma p_1, \\
\dot{m}_3 &= 0, \\
\dot{p}_1 &= \alpha p_2 m_3 - p_3 m_2, \\
\dot{p}_2 &= p_3 m_1 - \alpha p_1 m_3, \\
\dot{p}_3 &= p_1 m_2 - p_2 m_1.
\end{align*}
\]

So, like in the Kirchhoff case, \( m_3 \) is an obvious fourth integral, due to the rotational symmetry of the system.

Wronskian relation satisfied on solutions of (95):
\[
(m_1 \dot{p}_1 - m_1 \dot{p}_1) + (m_2 \dot{p}_2 - m_2 \dot{p}_2) + (2\alpha - 1)(\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0.
\]

Applying the Kahan-Hirota-Kimura scheme to the vector field \( f \) from (95), we arrive at a birational map \( \Phi_f : \mathbb{R}^6 \to \mathbb{R}^6, (m, p) \mapsto (\tilde{m}, \tilde{p}) \), defined by the following linear system:
\[
\begin{align*}
\tilde{m}_1 - m_1 &= \epsilon(\alpha - 1)(\tilde{m}_2 m_3 + m_2 \tilde{m}_3) + \epsilon \gamma (p_2 + \tilde{p}_2), \\
\tilde{m}_2 - m_2 &= \epsilon(1 - \alpha)(\tilde{m}_3 m_1 + m_3 \tilde{m}_1) - \epsilon \gamma (p_1 + \tilde{p}_1), \\
\tilde{m}_3 - m_3 &= 0, \\
\tilde{p}_1 - p_1 &= \epsilon \alpha (p_2 \tilde{m}_3 + \tilde{p}_2 m_3) - \epsilon (p_3 \tilde{m}_2 + \tilde{p}_3 m_2), \\
\tilde{p}_2 - p_2 &= \epsilon (p_3 \tilde{m}_1 + \tilde{p}_3 m_1) - \epsilon \alpha (p_1 \tilde{m}_3 + \tilde{p}_1 m_3), \\
\tilde{p}_3 - p_3 &= \epsilon (p_1 \tilde{m}_2 + \tilde{p}_1 m_2 - p_2 \tilde{m}_1 - \tilde{p}_2 m_1).
\end{align*}
\]

Theorem 24. (Quadratic-fractional integral, [15])
\[
I_0(m, p; \epsilon) = \frac{r(m, p; \epsilon)}{s(m, p; \epsilon)},
\]
where
\[
\begin{align*}
r &= (2\alpha - 1) + \epsilon^2(\alpha - 1)(m_1^2 + m_2^2) + \frac{\epsilon^2 \gamma}{m_3}(m_1 p_1 + m_2 p_2), \\
s &= 1 + \epsilon^2 \alpha(1 - \alpha)m_3^2 - \epsilon^2 \gamma p_3,
\end{align*}
\]
is an integral of motion of the map \( \Phi_f \).
Theorem 25. (Discrete Wronskians HK basis, [13]) Functions $W_i^{(1)}(m, p) = \tilde{m}_i p_i - m_i \tilde{p}_i, i = 1, 2, 3$, form a HK basis for the map $\Phi_f$ with a one-dimensional null space spanned by $[1 : 1 : I_0]$, where $I_0$ is the integral of $\Phi_f$ given by (98).

Novel results, supporting Observations 2 and 3, are as follows.

Theorem 26. (Bilinear-fractional integral) The function

$$J_0(m, p; \epsilon) = R(m, p; \epsilon) S(m, p; \epsilon),$$

where

$$R = (2\alpha - 1) - \epsilon^2(\alpha - 1)(m_1 \tilde{m}_1 + m_2 \tilde{m}_2) - \frac{\epsilon^2 \gamma}{2m_3}(\tilde{m}_1 p_1 + m_1 \tilde{p}_1 + m_2 \tilde{p}_2 + m_2 \tilde{p}_2),$$

$$S = 1 - \epsilon^2 \alpha(1 - \alpha)m_3^2 + \frac{1}{2} \epsilon^2 \gamma(p_3 + \tilde{p}_3),$$

is an integral of motion of the map $\Phi_f$.

Corollary 27. (Density of an invariant measure) The map $\Phi_f(x; \epsilon)$ has an invariant measure

$$\frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(m, p; \epsilon)},$$

where $\phi(m, p; \epsilon)$ can be taken as the numerator of either of the functions $R, S$.

Theorem 28. (Second order Wronskians HK basis) The functions

$$W_i^{(2)}(m, p) = \tilde{m}_i p_i - m_i \tilde{p}_i, \quad i = 1, 2, 3,$$

form a HK basis for the map $\Phi_f$, with a one-dimensional null space spanned by $[1 : 1 : J_0]$, with the function $J_0$ given in (101).

There holds a theorem which reads literally as Theorem 23 on the third order Wronskians HK basis.

8 Concluding remarks

We would like to remark that the existence of quadratic-fractional integrals of the Kahan discretizations is a rather common phenomenon which even is not related to integrability. In this connection, we refer to the recent paper [8], where the following result is established.

Theorem 29. Let two components a quadratic vector field be of the form

$$\begin{cases}
\dot{x}_1 = \ell(x) \frac{\partial H}{\partial x_2} = \ell(x)(bx_1 + cx_2), \\
\dot{x}_2 = -\ell(x) \frac{\partial H}{\partial x_1} = -\ell(x)(ax_1 + bx_2),
\end{cases}$$

(104)
admits an quadratic-fractional integral of motion \( \epsilon \)

\[ H(x_1, x_2) = \frac{1}{2} (ax_1^2 + 2bx_1x_2 + cx_2^2) \]

is a quadratic form of \( x_1, x_2 \). The Kahan discretization of this vector field, with the first two equations of motion

\[
\begin{align*}
\dot{x}_1 - x_1 &= \epsilon \ell(x)(b\bar{x}_1 + c\bar{x}_2) + \epsilon \ell(x)(bx_1 + cx_2), \\
\dot{x}_2 - x_2 &= -\epsilon \ell(x)(a\bar{x}_1 + b\bar{x}_2) - \epsilon \ell(x)(ax_1 + bx_2),
\end{align*}
\]

admits an quadratic-fractional integral of motion

\[ F(x, \epsilon) = \frac{ax_1^2 + 2bx_1x_2 + cx_2^2}{1 + \epsilon^2(ac - b^2)\ell^2(x)} \]

Actually, their result holds true for any (not necessarily homogeneous) quadratic polynomial \( H(x_1, x_2) \), but this generalization easily follows by shift of variables. This result does not depend on equations of motion for \( x_k \) with \( 3 \leq k \leq n \), and therefore it is unrelated to integrability. It turns out that the procedure described in Observation 2 (polarization of the quadratic polynomials in the numerator and in the denominator, accompanied by the change \( \epsilon^2 \rightarrow -\epsilon^2 \)) works for the whole class of vector fields described in Theorem 29, but, amazingly, it does not lead to new integrals. We have the following result.

**Theorem 30.** On orbits of the map \( \text{[106]} \), we have: \( \hat{F}(x, \epsilon) = F(x, \epsilon) \), where

\[ \hat{F}(x, \epsilon) = \frac{ax_1\bar{x}_1 + b(x_1\bar{x}_2 + x_2\bar{x}_1) + cx_2\bar{x}_2}{1 - \epsilon^2(ac - b^2)\ell(x)\ell(\bar{x})}. \]

**Proof.** Relation

\[ \frac{ax_1\bar{x}_1 + b(x_1\bar{x}_2 + x_2\bar{x}_1) + cx_2\bar{x}_2}{1 - \epsilon^2(ac - b^2)\ell(x)\ell(\bar{x})} = \frac{ax_1^2 + 2bx_1x_2 + cx_2^2}{1 + \epsilon^2(ac - b^2)\ell^2(x)} \]

is equivalent to

\[
ax_1(\bar{x}_1 - x_1) + bx_1(\bar{x}_2 - x_2) + bx_2(\bar{x}_1 - x_1) + cx_2(\bar{x}_2 - x_2)
\]

\[ = -\epsilon^2(ac - b^2)\ell(x)(ax_1(\bar{x}_1\ell(x) + x_1\ell(\bar{x}))) + bx_1(\bar{x}_2\ell(x) + x_2\ell(\bar{x})) + bx_2(\bar{x}_1\ell(x) + x_1\ell(\bar{x})), \]

We transform the left-hand side, using equations of motion \( \text{[106]} \):

\[
(ax_1 + bx_2)(\bar{x}_1 - x_1) + (bx_1 + cx_2)(\bar{x}_2 - x_2)
\]

\[ = \epsilon(ax_1 + bx_2)((b\bar{x}_1 + c\bar{x}_2)\ell(x) + (bx_1 + cx_2)\ell(\bar{x})) - \epsilon(bx_1 + cx_2)((a\bar{x}_1 + b\bar{x}_2)\ell(x) + (ax_1 + bx_2)\ell(\bar{x})) = \epsilon(ac - b^2)(x_1\bar{x}_2 - x_2\bar{x}_1)\ell(x). \]
A similar transformation of the right-hand side leads to:

\[-\epsilon^2(ac - b^2)\ell(x)\left(x_1((a\bar{x}_1 + b\bar{x}_2)\ell(x) + (ax_1 + bx_2)\ell(\bar{x}))
+ x_2((b\bar{x}_1 + c\bar{x}_2)\ell(x) + (bx_1 + cx_2)\ell(\bar{x}))\right)
= \epsilon(ac - b^2)\ell(x)\left(x_1(\bar{x}_2 - x_2) - x_2(\bar{x}_1 - x_1)\right)
= \epsilon(ac - b^2)\ell(x)(x_1\bar{x}_2 - x_2\bar{x}_1)\]

This proves the theorem.

The latter result makes the applicability of Observation 2 all the more intriguing. In the majority of cases when the Kahan discretization possesses a quadratic-fractional integral of motion $F(x, \epsilon)$, the polarization of the latter (i.e., the function $\hat{F}(x, \epsilon)$ obtained by the polarization of the numerator and of the denominator of $F(x, \epsilon)$, accompanied by the change $\epsilon^2 \rightarrow -\epsilon^2$) turns out to be an integral, as well. Further examples of this observation are delivered by integrable systems with the Lax representation $\dot{L} = [L^2, A]$ in the following situations:

- $L$ a $3 \times 3$ or a $4 \times 4$ skew-symmetric matrix and $A$ a constant diagonal matrix (Euler top on the algebras $so(3)$ and $so(4)$);
- $L$ a symmetric $3 \times 3$ matrix and $A$ is a constant skew-symmetric $3 \times 3$ matrix [3, 4];
- $L$ a $3 \times 3$ matrix with vanishing diagonal and $A$ a constant diagonal matrix (3-wave system, see [15]).

The only counterexample we are aware of, is given by the system $\dot{L} = [L^2, A]$ with a general $3 \times 3$ matrix $L$ and a constant diagonal matrix $A$ [2]. For this system, polarization applied to quadratic-fractional integrals of the Kahan-Hirota-Kimura discretization (there are two independent such integrals) does not lead to integrals of motion.

Clarifying all the mysterious observations related to the Kahan-Hirota-Kimura discretization remains an intriguing and a rewarding task.

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