The Kosterlitz-Thouless-Berezinskii transition of homogeneous and trapped Bose gases in two dimensions

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We derive the scaling structure of the Kosterlitz-Thouless-Berezinskii (KTB) transition temperature of a homogeneous Bose gas in two dimensions within diagrammatic perturbation theory. Approaching the system from above the transition, we calculate the critical temperature, $T_{KT}$, and show how the superfluid mass density emerges from Josephson’s relation as an interplay between the condensate density in a finite size system, and the infrared structure of the single particle Green’s function. We then discuss the trapped two-dimensional Bose gas, where the interaction changes the transition qualitatively from Bose-Einstein in an ideal gas to a KTB transition in the thermodynamic limit. We show that the transition temperature lies below the ideal Bose-Einstein transition temperature, and calculate the first correction in terms of the interparticle interactions. The jump of the total superfluid mass at the transition is suppressed in a trapped system.

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I. INTRODUCTION

The ability to produce a two-dimensional atomic gas trapped in an optical potential has stimulated interest in the Kosterlitz-Thouless-Berezinskii (KTB) transition in such a system 1, 2, 3, 4. Recently, interesting phase patterns have been measured in a quasi two-dimensional situation 5. A homogeneous Bose gas in two dimensions undergoes Bose-Einstein condensation only at zero temperature, since long wavelength phase fluctuations destroy long range order 6; nonetheless interparticle interactions drive a phase transition to a superfluid state at finite temperature, as first pointed out by Berezinskii 7 and by Kosterlitz and Thouless 8. The phase transition is characterized by an algebraic decay of the off-diagonal one-body density matrix below the transition temperature, $T_{KT}$. Furthermore, the superfluid mass density, $\rho_s$, jumps with falling temperature, $T$, from 0 just above $T_{KT}$ to a universal value $\rho_s = 2m^2T_{KT}/\pi$ just below 9, where $m$ is the atomic mass. We use units $\hbar = k_B = 1$.

In a non-interacting homogeneous Bose gas in two dimensions, the density is given in terms of the chemical potential $\mu$ by

$$n\lambda^2 = -\log[1 - e^{\beta\mu}], \quad (1)$$

where $\beta = 1/T$, and $\lambda = (2\pi/mT)^{1/2}$ is the thermal wavelength. As $\mu$ approaches zero, the phase-space density grows arbitrarily, implying that the ideal Bose gas does not undergo Bose condensation at finite temperature. On the other hand, a trapped non-interacting gas undergoes a Bose-Einstein condensation at a finite temperature 10. In the semi-classical limit, the total number of particles is given by $N(\beta\mu) = g_2(-\beta\mu)(T/\omega)^2$, where $g_2(x) = \sum_{n=1}^{\infty} e^{-nx^2}/n^2 \simeq \pi^2/6 - x(1 - \log x)$, for $x \to 0$, and $\omega$ is the trapping frequency; thus the condensation temperature is $T_{BEC} = \sqrt{6N\omega/\pi}$.

The question of whether an interacting trapped Bose gas undergoes an ordinary Bose-Einstein or a KTB has not completely been settled 11. In this paper we show that in the thermodynamic limit – in which the total particle number $N$ goes to $\infty$, and the trap frequency, $\omega$, goes to 0, with $N\omega^2$ constant – interactions at the mean field level destroy ordinary Bose condensation; instead, the system undergoes a KTB transition at a temperature slightly below the ideal condensation temperature, $T_{BEC}$.

We approach the phase transition of the homogeneous two-dimensional Bose gas by carrying out a scaling analysis (Sec. II and III) similar to the one we used earlier to discuss the phase transition in a dilute three-dimensional homogeneous Bose gas 12, 13, 14. The phase below $T_{KT}$ is characterized by an algebraic decay in space of the single particle Green’s function, $G$. As in three dimensions, the transition point occurs when the single particle spectrum becomes gapless, $G^{-1}(0,0) = 0$, where $G(k, z_t)$ is the Fourier transform of $G$ in space and time. We first discuss, in Sec. II, the scaling structure of $G$ as $T_{KT}$ is approached from above, and rederive the relation between the temperature and density at the transition:

$$\frac{T_{KT}}{n} = \frac{2\pi}{m \log(C/\alpha)},$$

where $C$ is a constant 15, $\approx (380 \pm 3)/2\pi$ according to numerical simulation 16, and we take the interparticle coupling constant in two dimensions to have the form $g = 2\pi\alpha/m$, where $\alpha$ is a dimensionless parameter 21.

We then turn, in Sec. III, to the structure below $T_{KT}$, working in terms of the Green’s function in momentum space. Below $T_{KT}$ we expect $G(r - r') \sim 1/|r - r'|^q$ for
an infinite system and \(|r - r'| \to \infty\), where \(\eta\) depends on \(T\) (and equals 1/4 at \(T_{KT}\) ); however, \(G(k)\) is not well-defined in an infinite system. We therefore analyze the structure starting with a finite size system, of characteristic dimension \(L\); such a system has a non-zero condensate density, \(n_0 \sim G(r \sim L) \sim 1/L^\nu\), which vanishes in the thermodynamic limit, \(L \to \infty\). Nonetheless, we show using Josephson’s relation between the condensate density, the superfluid mass density, \(\rho_s\), and the infrared behavior of \(G(k)\) \cite{17,18,19},

\[
\rho_s = -\lim_{k \to 0} \frac{n_0 m^2}{k^2 G(k, 0)}.
\]

that \(\rho_s\) is finite in the thermodynamic limit. Our analysis of the scaling structure is valid for all coupling strengths, and thus, going beyond perturbation theory, is an extension of the work of Popov \cite{20}.

Then, in Sec. IV, we apply our results to a trapped gas within the local density approximation. We show that the KTB transition temperature, for a weakly interacting system, lies below the Bose-Einstein condensation temperature of the ideal trapped gas by terms of order \(\alpha \log^2 \alpha\). Furthermore, the jump of the total superfluid mass at the transition is \(\sim \alpha\), and is thus highly suppressed compared with that in a homogeneous system.

II. SCALING STRUCTURE ABOVE THE TRANSITION

In this section we derive the KTB transition of a two-dimensional weakly interacting homogeneous Bose gas by studying, as in \cite{12}, the scaling structure of \(G\) just above the transition. For wavevector \(k\) and complex frequency \(z\), \(G(k, z)\) is given in terms of the self-energy, \(\Sigma(k, z)\), by

\[
G^{-1}(k, z) = z + \mu - \frac{k^2}{2m} - \Sigma(k, z),
\]

where \(\mu\) is the chemical potential. The transition is defined, for fixed density, by the temperature at which the single particle spectrum becomes gapless, and consequently off-diagonal single particle density matrix decays algebraically; at the transition point, \(G^{-1}(0, 0) = 0\), or \(\mu = \Sigma(0, 0)\). In terms of \(G\) the density is

\[
n = -T \sum_\nu \int \frac{d^2 k}{(2\pi)^2} G(k, z_\nu).
\]

where the \(z_\nu = 2\pi i \nu T\) are the Matsubara frequencies (\(\nu = 0, \pm 1, \pm 2, \ldots\)).

The mean-field contribution, \(\Sigma_{mf} = 2gn\), to the self-energy, is independent of frequency and momentum, and can be absorbed in a shift of the chemical potential. We introduce the mean-field coherence length, \(\zeta\), by

\[
\frac{1}{2m\zeta^2} = \Sigma_{mf} - \mu;
\]

at the transition,

\[
1/2m\zeta^2 = \Sigma_{mf} - \Sigma(0, 0).
\]

In terms of \(\zeta\), the mean-field Green’s function is,

\[
G_{mf}(k, z_\nu) = -\frac{2m\zeta^2}{(k\zeta)^2 + 1 - 8\pi^2(\zeta/\lambda)^2 \nu}.
\]

In the zero Matsubara frequency sector, the self-energy has the scaling structure:

\[
\Sigma(k, 0) - \Sigma_{mf} = \frac{1}{2m} \sigma_0 \left( k\zeta, \frac{\alpha \zeta^2}{\lambda^2} \right),
\]

where \(\sigma_0\) is dimensionless and \(\sim \alpha^2\) as \(\alpha \to 0\). In contrast to the case of three-dimensions, self-energy diagrams beyond mean field are ultraviolet convergent even in the zero Matsubara frequency sector.

Using Eqs. (9) and (10) we may write the transition condition, (10), as

\[
\sigma_0 \left( 0, \frac{\alpha \zeta^2}{\lambda^2} \right) + 1 = 0;
\]

at the phase transition the parameter \(\alpha \zeta^2/\lambda^2 \equiv J\) approaches a finite value (fixed point), \(J^*\), determined by this equation. The contribution of non-zero Matsubara frequencies in the denominator of Eq. (8) is \(\sim (\alpha \zeta^2/\lambda^2)/\alpha\), and thus in the limit of small \(\alpha\), the contributions of non-zero Matsubara frequencies to the self-energy are of relative order \(\alpha\) and higher, and can be neglected.

To calculate the critical density at given temperature we use the mean-field density as reference:

\[
n_c(\alpha, T) = n_{mf}(\mu = -1/2m\zeta^2)
\]

\[
- T \sum_\nu \int \frac{d^2 k}{(2\pi)^2} [G(k, z_\nu) - G_{mf}(k, z_\nu)].
\]

In leading order, we neglect non-zero Matsubara frequencies in the summation on the right, and derive

\[
\lambda^2 n_c = \log \frac{4\pi \zeta^2}{\lambda^2} + 2 \int k dk \left( \frac{1}{k^2 + \zeta^{-2}(1 + \sigma_0)} - \frac{1}{k^2 + \zeta^{-2}} \right) + O(\alpha),
\]

Using Eq. (12) we see that the integral in (12) is a constant of order unity, independent of \(\alpha\). Thus we arrive at the critical density,

\[
\lambda^2 n_c = \log \frac{C}{\alpha} + O(\alpha),
\]

as in Eq. (2).

The true correlation length, \(\xi\), above \(T_{KT}\) is given by

\[
\frac{1}{2m\xi^2} \equiv \Sigma(0, 0) - \mu = \Sigma(0, 0) - \Sigma_{mf} + \frac{1}{2m\zeta^2}.
\]
To make contact with the theory of critical phenomena, we rewrite this equation as

$$M(J) = \frac{\lambda^2}{\alpha \xi^2} = 1 + \frac{2m^2}{\alpha} (\Sigma(0,0) - \Sigma_{mf}),$$  \hspace{1cm} (15)$$

where from the previous discussion we know that the right side is a function only of $J = \alpha \xi^2 / \lambda^2$. At the transition, $J$ goes to the fixed point, $J^*$, given by $M(J^*) = 0$. Furthermore, approaching the transition from above, we see from Eq. (12) that $\Delta J = J - J^* \propto (T - T_{KT}) / \alpha$. The dependence of $M(J)$ on $\Delta J$ near the fixed point determines the critical index $\nu$, of the correlation length, $\xi \sim |T - T_{KT}|^{-\nu}$, in particular that $M(J) \sim |\Delta J|^{2\nu} \sim |T - T_{KT}|^{2\nu}$, in the neighborhood of the fixed point.

### III. SCALING STRUCTURE BELOW THE TRANSITION

At the transition temperature, $T_{KT}$, the single particle Green’s function decays algebraically. Below, the scaling structure is most readily analyzed in momentum space, as above; however this approach is made difficult by the fact that below $T_{KT}$ the single particle Green’s functions continue to decay algebraically in real space sufficiently slowly that its Fourier transform is not absolutely convergent. In order to avoid this problem, and to use the same approach as above, we adopt the strategy of working in a finite size system, of characteristic dimension $L$, in which the condensate density, $n_0$, is non-zero. At the very end, we take the thermodynamic limit, $L \rightarrow \infty$, at fixed density, in which case $n_0$ also goes to zero, since long-range order is prohibited in two dimensions.

In a finite size system the condensate density is given by $n_0 = n - \tilde{n}$, where

$$\tilde{n} = -T \sum_n \int_0^\infty \frac{d^2k}{(2\pi)^2} G(k,n),$$  \hspace{1cm} (16)$$

and $k_0 \sim \pi/L$; in leading order, $G$ here can be taken to be the infinite size Green’s function. Since for $T \leq T_{KT},$

$$G(k \rightarrow 0,0) = -2mK \xi^2 / k^{2-\eta},$$  \hspace{1cm} (17)$$

where the constants $K$ and $\eta$ ($0 \leq \eta \leq 2$) in general depend on $T$, we find to leading order,

$$\tilde{n}_0 = \frac{2K}{\eta \lambda^2} (k_0 \xi)^\eta,$$  \hspace{1cm} (18)$$

The result (18) is given for a circular box: in general it is modified by a numerical factor close to unity, weakly dependent on the geometry.

Below $T_{KT}$ the system is superfluid even though the condensate density is not extensive and vanishes in the thermodynamic limit. However, in two dimensions, an algebraically decaying correlation function is sufficient to yield a non-vanishing superfluid density, $\rho_s$, in the thermodynamic limit, as can be seen from the Josephson relation between $\rho_s$ and $n_0$ in a finite system:

$$\rho_s = -\lim_{k \rightarrow k_0} \frac{n_0 m^2}{k^2 G(k,0)}.$$  \hspace{1cm} (19)$$

Thus from Eqs. (17) and (18) and find

$$\rho_s = \frac{m^2 T}{2\pi \eta}.$$  \hspace{1cm} (20)$$

Inverting the relation we obtain $\eta = m^2 T / 2\pi \rho_s$; thus this prediction of the spin-wave approximation for the critical exponent is prediction below $T_{KT}$. Since $\rho_s \leq mn$, we see that

$$\eta \geq \frac{1}{n \lambda^2}.$$  \hspace{1cm} (21)$$

Furthermore, since $\eta \leq 2$, the superfluid mass density can never vanish at finite temperature below $T_{KT}$, and thus it must be discontinuous across the phase transition. At $T_{KT}$, the value of the universal jump of the superfluid density predicted by Nelson and Kosterlitz [3], $\rho_s = 4T_{KT}m^2 / 2\pi$, leads to $\eta = 1/4$.

Let us now discuss the detailed structure of the Green’s function below the transition. We basically follow the scaling approach used in [23] in three dimensions. Our strategy is to expand the self-energies formally in powers of $\alpha$, $n_0$, and $k_0$. In the infinite size system, $k_0 \rightarrow 0$, the self-energies diverge as $n_0 \rightarrow 0$; the point, $k_0 \rightarrow 0$, $n_0 \rightarrow 0$ is singular. However Josephson’s relation constrains the limit $n_0 \rightarrow 0$ and $k_0 \rightarrow 0$ in terms of $\rho_s$.

The particle density, $n$, in the condensed phase is a function of $\alpha$, $n_0$, and $T$, and has the form, $n(\alpha, n_0, k_0, T) = n_0 + \tilde{n}(\alpha, n_0, k_0, T)$, where $\tilde{n}(\alpha, n_0, T)$ is the density of non-condensed particles (with momentum $k > k_0$). At the transition temperature, $\tilde{n}(\alpha, 0, T_c) = n_c$. We calculate $\tilde{n}(\alpha, n_0, T)$ in terms of the matrix Green’s function,

$$G(rt, r't') = -i \left( \langle \Psi(r) \Psi^\dagger(r') \rangle \right)$$

$$-\langle \Psi^\dagger(r') \Psi(r) \rangle,$$  \hspace{1cm} (22)$$

where the two component field operator is $\Psi(rt) = (\psi(rt), \psi^\dagger(rt))$. The Fourier components of $G^{-1}$ have the form,

$$G^{-1} (k, z_n) = \begin{pmatrix}
z_n + \mu - \varepsilon_k - \Sigma_{11} & -\Sigma_{12} \\
-\Sigma_{21} & -z_n + \mu - \varepsilon_k - \Sigma_{22}
\end{pmatrix},$$  \hspace{1cm} (23)$$

where $\varepsilon_k = k^2 / 2m$, and the $\Sigma_{ij}(k, z_n)$ are the corresponding self-energies. The chemical potential, $\mu$, depends on $n_0$ and $k_0$, and is specified by the Hugenholtz-Pines relation [24],

$$\mu = \Sigma_{11}(0,0) - \Sigma_{12}(0,0).$$  \hspace{1cm} (24)$$
The lowest order mean field self-energies, $\Sigma_{11} = \Sigma_{11}^{mf} = 2g(n_0 + \bar{n})$, $\Sigma_{12} = \Sigma_{12}^{mf} = gn_0$ are independent of momenta and Matsubara frequency, and, as above $T_{KT}$, we absorb them in a mean field coherence length, $\zeta$.

$$\mu - (\Sigma_{11}^{mf} - \Sigma_{12}^{mf})$$
$$= (\Sigma_{11}(0, 0) - \Sigma_{12}(0, 0)) - (\Sigma_{11}^{mf} - \Sigma_{12}^{mf})$$
$$\equiv -1/2m\zeta^2.$$  \tag{25}

Since the propagators remain formally infrared convergent we can derive the scaling structure of the self-energies by power-counting. As above the transition, we may neglect non-zero Matsubara contributions to leading order. As opposed to the three-dimensional case, there are no formal ultraviolet divergencies in the expansion beyond those in mean field, and therefore need for renormalization. The expansion of the self-energies beyond mean field starts at order $\alpha^2\zeta^2/\lambda^2$; furthermore $\Sigma_{12}$ is formally at least of order $n_0$. Diagrams of order $g^\kappa$ with $\kappa \geq 3$ in the formal expansion contain vertices with two Green’s functions entering; similar to the structure at $T_c$, they involve the dimensionless combinations $\alpha^2\zeta^2/\lambda^2 \equiv P$ and $n_0\lambda^2$. The latter part originates from the dependence of $G^{mf}$ on $2m\Sigma_{12}^{mf} \sim \alpha n_0$. Any diagram with an explicit power, $p$, of $n_0$ can be generated from a corresponding diagram of power $p - 1$ in which a line is replaced by $\sqrt{n_0}$ at each of its ends. Thus each power of $n_0$ involves one fewer two-momentum loop to be integrated over. The explicit $n_0$ dependences enter in two ways. Terms involving $G_{11} - G_{12}$ lead to the combination $P^2n_0\lambda^2$, which vanishes as $n_0 \to 0$. On the other hand terms involving the combination $G_{11} + G_{12}$, which in mean field diverges as $n_0 \to 0$ in the infrared limit, lead to divergences which are cutoff by $k_0$ and thus produce an additional $(k_0\zeta)^2/(n_0\lambda^2 P)$ dependence. In the limit $n_0 \to 0$, $k_0 \to 0$, only the dependence on $(k_0\zeta)^2/n_0\lambda^2 \equiv Q$ survives.

Then with all momenta $k$ scaled by $1/\zeta$, we find the following scaling structure for the self-energies in this limit, 

$$2m\zeta^2 \left( \Sigma_{ij}(k) - \Sigma_{ij}^{mf}(0) \right) = \sigma_{ij}(k\zeta, P, Q),$$  \tag{26}

where the $\sigma_{ij}$ are dimensionless functions of dimensionless variables, and we neglect terms proportional to positive powers of $n_0$ and $k_0$. In particular, for $k \to 0$,

$$2m\zeta^2 \left( (\Sigma_{11}(0, 0) - \Sigma_{12}(0, 0)) - (\Sigma_{11}^{mf} - \Sigma_{12}^{mf}) \right)$$
$$= s(P, Q),$$  \tag{27}

where $s$ is a dimensionless function. Comparing with Eq. (26) we see that $s + 1 = 0$, an equation that determines $P$ as a function of $Q$, so that

$$\zeta = \frac{\lambda}{\sqrt{\alpha}} h(Q),$$  \tag{28}

where $h$ is dimensionless.

We now take the limit $n_0 \to 0$ and $k_0 \to 0$. Equations (26)-(27) imply that $G_{11}(k_0, 0) = -m\zeta^2/\ell(Q)$, and thus the Josephson relation implies that the superfluid mass density has the structure,

$$\rho_s = \frac{m}{\lambda^2 Q\ell(Q)}.$$  \tag{29}

where $\ell$ is a dimensionless function; this equation relates $n_0$ and $k_0$ to $\rho_s$ in the limiting process. In this limit $P$ as well as $Q$ are constants dependent only on $\rho_s/T$. Similarly the other dimensionless functions depend in this limit only on $\rho_s$. These forms are valid over the temperature range from $T_{KT}$ down to $T = 0$ (with non-zero Matsubara frequency contributions neglected); this entire region is therefore critical.

We can gain further insight into the structure below $T_{KT}$ by reformulating it in terms of the correlation lengths, $\zeta_T$ and $\zeta_L$, that control the infrared behavior of the transverse and longitudinal Green’s functions, $G_T = G_{11} - G_{12}$, and $G_L = G_{11} + G_{12}$. We regularize the infrared divergent structure below $T_{KT}$ by assuming a finite condensate density, $n_0$, and finite correlation lengths. In this description, the low temperature phase of the KTB transition is characterized by the ratio of amplitude (L) and phase (T) fluctuations of the order parameter, even in the absence of long-range order in the thermodynamic limit in which $n_0 \to 0$. In the end we take the limit, $n_0 \to 0$, $\zeta_T \to \infty$, and $\zeta_L \to \infty$. In this way, we do not have to introduce an explicit infrared cutoff, $k_0$, as we had to above. We define

$$\frac{1}{2m\zeta_T^2} = \Sigma_{11}(0, 0) - \Sigma_{12}(0, 0) - \mu,$$  \tag{30}

and

$$\frac{1}{2m\zeta_L^2} = \Sigma_{11}(0, 0) + \Sigma_{12}(0, 0) - \mu,$$  \tag{31}

in terms of which $G_T$ and $G_L$ are given by,

$$G_T^{-1}(k, 0) = -\left( \frac{k^2}{2m} + [\Sigma_{11}(k, 0) - \Sigma_{12}(k, 0)]$$
$$- [\Sigma_{11}(k, 0) - \Sigma_{12}(0, 0)] + \frac{1}{2m\zeta_T^2} \right).$$  \tag{32}

and

$$G_L^{-1}(k, 0) = -\left( \frac{k^2}{2m} + [\Sigma_{11}(k, 0) + \Sigma_{12}(k, 0)]$$
$$- [\Sigma_{11}(k, 0) + \Sigma_{12}(0, 0)] + \frac{1}{2m\zeta_L^2} \right).$$  \tag{33}

The equilibrium state of the system is specified below the transition by $\zeta_T \to \infty$, as seen from the Hugenholtz-Pines relation (24), as well as $n_0 \to 0$. From Eqs. (30) and (31), we obtain the ratio

$$\left( \frac{\zeta_T}{\zeta_L} \right)^2 = 1 + 4m\zeta_T^2\Sigma_{12}(0, 0).$$  \tag{34}
The second term is of order \( n_0 \zeta^2 \). From the previous discussion, we deduce that in the limit \( n_0 \lambda^2 \to 0 \), \( 4m\zeta^2 \Sigma_{12}(0,0) \to 8\pi a_0 \zeta^2 f(\zeta_r/\zeta) \), where \( f \) is a dimensionless function. Equation (34) thus implies that \( n_0 \zeta^2 \) remains finite. The value of this parameter is determined in terms of \( \rho_s \) by Josephson’s relation, which we write in the form (19)

\[
\rho_s = \lim_{\zeta_r \to \infty} mn_0 \left( 1 + \frac{2m}{\partial k^2} [\Sigma_{11}(k,0) - \Sigma_{12}(k,0)] \right). (35)
\]

For finite \( \zeta_r \), the derivative of the self energies with respect to \( k^2 \) are finite in the limit \( k \to 0 \), but must diverge as \( \zeta^2 \). The right side is a function only of \( n_0 \zeta^2 \) in the limit \( \zeta^2 \to \infty \), and \( n_0 \to 0 \), thus defining \( n_0 \zeta^2 \) in terms of \( \rho_s \).

**IV. THE TRANSITION IN A TRAPPED GAS**

We turn now to the behavior of a two-dimensional system trapped in an oscillator potential, of frequency \( \omega \). We consider for simplicity only the thermodynamic limit \( N \to \infty \), \( \omega \to 0 \), with \( N\omega^2 \) constant, where \( N \) is the total particle number.

In the absence of interactions the system undergoes a Bose-Einstein condensation at the critical temperature \( T_{BEC} = \sqrt{6N\omega/\pi} \). However, at the mean field level, interactions destroy simple Bose-Einstein condensation. To see how the critical temperature is reduced to zero, we begin with the density profile calculated in the local density approximation,

\[
n_{mf}(r) = \int \frac{d^2k}{(2\pi)^2} e^{\beta(k^2/2m + V_{\text{eff}}(r) - \mu)} - 1
= -\frac{1}{\lambda^2} \log \left( 1 - e^{\beta(\mu - V_{\text{eff}}(r))} \right), \quad (36)
\]

where \( V_{\text{eff}}(r) = m\omega^2 r^2/2 + 2gn(r) \). Since \( n(r) \) has negative curvature at the origin, interactions tend to reduce the effective trapping frequency. Explicitly, expanding (36) to order \( r^2 \) about the origin, we find that the self-consistent trap frequency at the origin is given by,

\[
m_{\text{eff}}^2 = m\omega^2 + 2g \left( \frac{\partial n_{\text{mf}}(r)}{\partial r^2} \right)_{r=0}
= m\omega^2 - \frac{1 - e^{\beta(\mu - 2gn(0))}}{4\beta g \lambda^2 (1 + 1 - e^{\beta(\mu - 2gn(0))})}. \quad (37)
\]

As \( \mu \to 2gn(0) \), the limit in which Bose-Einstein condensation occurs in mean field, we have

\[
\omega_{\text{eff}}^2 \approx \omega^2 \frac{\beta}{2\alpha} (2gn(0) - \mu); \quad (38)
\]

the potential becomes arbitrarily flat as \( \mu \to 2gn(0) \). In the thermodynamic limit, \( N\omega^2 \) constant, the effective oscillator length \( d_{\text{eff}} \equiv 1/\sqrt{m\omega_{\text{eff}}} \) is \( (NT/(2gn(0) - \mu))^{1/4} \). At the point of mean-field Bose condensation, \( 2gn(0) - \mu = 1/2m\zeta(0)/\lambda^2 \) is therefore \( T/N \); using the estimate (1), we see that the number of particles in the center, \( \sim n_0(0)\pi d_{\text{eff}}^2 \), grows as \( (T/\omega)\sqrt{\alpha(\zeta(0)/\lambda)} \log(\zeta(0)/\lambda) \). Now, an extensive population of the condensate mode implies \( \zeta(0)/\lambda \sim N^{1/2} \) within mean field. From our estimate of the number of (excited) particles in the center of the trap we obtain an upper bound for the critical temperature of Bose condensation, \( \sim N^{1/2}/(\alpha^{1/2} \log N) \). In the thermodynamic limit of the trapped system, the critical temperature of Bose condensation goes to zero.

Mean fields destroy Bose-Einstein condensation of an ideal gas. However, the system does undergo a KT transition in the thermodynamic limit, at a temperature calculable, to leading order in \( \alpha \), in terms of the mean field density profile, Eq. (36). The KT transition occurs when the chemical potential reaches the critical chemical potential, \( \Sigma(0,0) \), calculated for a homogeneous system of the same density and temperature as in the center of the trap, or equivalently, Eq. (19) is satisfied. As we see from Eqs. (11) and (12) in the homogeneous case, the critical density is given to logarithmic accuracy by the mean field density evaluated at the critical \( \zeta \) given by Eq. (7); the corrections arise from critical fluctuations, which are, however, important only at small distances where \( m\omega^2 r^2/2 \gtrsim |gn(0) - \mu| \), or

\[
r < r_c \approx \frac{1}{m\omega\zeta}. \quad (39)
\]

In the thermodynamic limit the critical region produces corrections to the total number, \( \Delta N/N \sim \alpha \log \alpha \), which can be neglected, and thus, to leading order in \( \alpha \), we can calculate the transition from density profile, Eq. (36). Expanding \( n_{\text{mf}}(r) \) to first order in \( \alpha \) we have,

\[
n_{\text{mf}}(r) = n_0(r) + \frac{2\beta gn_0(r)}{\lambda^2} \frac{1}{e^{\beta(\mu - 2gn(0))}/2 - \beta \mu - 1}; \quad (40)
\]

where

\[
n_0(r) = -\frac{1}{\lambda^2} \log \left( 1 - e^{\beta(\mu - 2gn(0))}/2 \right), \quad (41)
\]

is the ideal gas density. Integrating over space, we find the first correction to the total number of particles,

\[
N = \int d^2r n_{\text{mf}}(r) = \int d^2r n_0(r) - \alpha \left( \frac{2\pi n_0(0)}{m\omega} \right)^2
= g_2(-\beta \mu) \frac{T^2}{\omega^2} - \alpha \left( \frac{2\pi \log(-\beta \mu)}{m\omega^2} \right)^2, \quad (42)
\]

where \( \mu = 2gn(0) - 1/2m\zeta^2 \). At the Kosterlitz-Thouless transition, \( \mu \sim \alpha \). To within logarithmic corrections we
may replace \( n_0(0) \) by \(-\lambda^{-2} \log(\alpha)\), and \( g_2(-\beta \mu) \) by \( g_2(0)\). Thus at given \( T \),

\[
N_{KT}(T, \alpha) = N_{BEC}(T) - \alpha \left( \frac{2 \pi \log \alpha}{m \omega / \lambda^2} \right)^2, \tag{43}
\]

where \( N_{BEC}(T) = (\pi^2/6)(T/\omega)^2 \) is the relation between particle number and temperature of the ideal Bose gas transition in the trap. Equivalently

\[
\frac{T_{KT} - T_{BEC}}{T_{BEC}} = -\frac{3}{\pi^2} \alpha \log^2 \alpha + O(\alpha \log \alpha). \tag{44}
\]

The existence of the Kosterlitz-Thouless transition just below \( T_{BEC} \) in a trap involves physics beyond mean-field.

In a KTB transition in a homogeneous system, the superfluid mass density jumps with falling temperature discontinuously at \( T_{KT} \) from zero to \( 2 m^2 T_{KT}/\pi \). In a trap, however, the transition first occurs in the center, and extends over a region of size \( r_c \), Eq. (39). The total superfluid mass, \( M_s \), is therefore

\[
M_s = \int \rho_s(r) d^2r \sim \pi \rho_s(r = 0) r_c^2,
\]

so that

\[
\frac{M_s}{M} \sim \frac{1}{N} \left( \frac{T_c}{\omega} \right)^2 \frac{\lambda^2}{\zeta r_c^2} \sim \frac{\alpha}{P}, \tag{46}
\]

where \( M = mN \), and \( P = \alpha \zeta^2/\lambda^2 \) is a constant (fixed point) at the central transition point. The jump in \( M_s \) is thus highly suppressed, and goes to zero in the limit of an ideal gas.

A key indication of superfluid behavior below the transition would be creation of a vortex at the center of the trap, where the system first becomes superfluid, e.g., by cooling a rotating system through \( T_{KT} \). In order to create a vortex at \( T_{KT} \) it is necessary that the vortex core, of radius \( \zeta \), fit within the critical region, of size \( \sim r_c \), at the transition. From Eq. (39), \( \zeta/r_c \sim m \omega \zeta^2 \), so that in the thermodynamic limit, \( \zeta/r_c \sim P/\alpha \sqrt{N} \). Thus for large \( N \), creation of a vortex within the critical region becomes possible. The critical rotation frequency, \( \Omega_c \), for creation of a vortex is of order \( (1/m^2 \zeta^2) \log(r_c/\zeta) \), and therefore \( \Omega_c/\omega \sim (\zeta/r_c) \log(r_c/\zeta) \sim (1/\alpha \sqrt{N}) \log(\alpha \sqrt{N}) \).

A further probe of the state of the system below \( T_{KT} \) would be determination of the density correlations, as have been recently measured in Bose gases trapped in optical lattices [19]. These correlations will depend strongly on the amplitude fluctuations, described by \( \zeta_T \) [20].

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