SHARP WEIGHTED LOG-SOBOLEV INEQUALITIES: CHARACTERIZATION OF EQUALITY CASES AND APPLICATIONS

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Abstract. By using optimal mass transport theory, we provide a direct proof to the sharp \(L^p\)-log-Sobolev inequality \((p \geq 1)\) involving a log-concave homogeneous weight on an open convex cone \(E \subseteq \mathbb{R}^n\). The perk of this proof is that it allows to characterize the extremal functions realizing the equality cases in the \(L^p\)-log-Sobolev inequality. The characterization of the equality cases is new for \(p \geq n\) even in the unweighted setting and \(E = \mathbb{R}^n\). As an application, we provide a sharp weighted hypercontractivity estimate for the Hopf-Lax semigroup related to the Hamilton-Jacobi equation, characterizing also the equality cases.

1. Introduction

The first version of the optimal Euclidean \(L^p\)-log-Sobolev inequality for \(1 < p < n\) goes back to Del Pino and Dolbeault [16], whose proof is based on the Gagliardo-Nirenberg inequality. It states that for Sobolev functions \(u \in W^{1,p}(\mathbb{R}^n)\) with \(\int_{\mathbb{R}^n} |u|^p dx = 1\) one has the inequality

\[
\int_{\mathbb{R}^n} |u|^p \log |u|^p dx \leq \frac{n}{p} \log \left( \mathcal{L}_p \int_{\mathbb{R}^n} |\nabla u|^p dx \right); \tag{1.1}
\]

here, the sharp constant has the form

\[
\mathcal{L}_p = \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \left( \Gamma \left( \frac{n}{p'} + 1 \right) \omega_n \right)^{-\frac{p}{n}},
\]

where \(p' = p/(p-1)\) and \(\omega_n\) is the volume of the unit ball in \(\mathbb{R}^n\). Furthermore, in [16] it was shown by a symmetrization technique that equality in (1.1) holds if and only if the extremal functions \(u \in W^{1,p}(\mathbb{R}^n)\) belong to the family of Gaussians, i.e., they are given by

\[
u_{\lambda,x_0}(x) = \lambda^{\frac{n}{p'}} \left( \Gamma \left( \frac{n}{p'} + 1 \right) \omega_n \right)^{-\frac{p}{n}} e^{-\lambda \frac{|x-x_0|^p}{p}}, \quad x \in \mathbb{R}^n, \tag{1.2}
\]

where \(\lambda > 0\) and \(x_0 \in \mathbb{R}^n\) are arbitrarily fixed. When \(p = 2\), inequality (1.1) appears in the paper of Weissler [35], which turns out to be equivalent to Gross’ log-Sobolev inequality [23] stated for the Gaussian measure. Very soon after the paper [16] has been published, by using the Prékopa-Leindler inequality, Gentil [22] (see also [26]) showed that in fact (1.1) is valid for all \(p > 1\) and the equality in (1.1) holds true for functions of the type (1.2). However, an ’if and only if’ characterization of the equality in (1.1) remained still open for \(p \geq n\).

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The purpose of the present paper is twofold. First, we prove a more general, weighted version of (1.1) for all values of $1 \leq p < \infty$ on open convex cones of $\mathbb{R}^n$. Second, we provide an 'if and only if' characterization of the equality cases in our version of the $L^p$-log-Sobolev inequality. In particular, this result provides a concluding answer to the open problem concerning the characterization of the equality case for $p \geq n$ in $\mathbb{R}^n$, see e.g. Gentil [22, p. 599].

To formulate our results we need to introduce some notation. Let us consider $E \subseteq \mathbb{R}^n$, an open convex cone and $\omega: E \to (0, \infty)$ be a log-concave homogeneous weight of class $C^1$ with degree $\tau \geq 0$; that means that the function $x \to \log \omega(x)$ is concave on $E$ and it satisfies $\omega(\lambda x) = \lambda^\tau \omega(x)$ for all $x \in E$ and $\lambda \geq 0$. For $p \geq 1$ we denote by $L^p(\omega; E)$ the space that contains all measurable functions $u : E \to \mathbb{R}$ such that $\int_E \|u\|^p \omega dx < \infty$ and we also introduce the weighted Sobolev space

$$W^{1,p}(\omega; E) = \{ u \in L^p(\omega; E) : \nabla u \in L^p(\omega; E) \}. \quad (1.3)$$

We denote by $B(x, r) \subseteq \mathbb{R}^n$ the ball with center in $x \in \mathbb{R}$ and radius $r > 0$; in particular, we let $B := B(0, 1)$.

As expected, the cases $p > 1$ and $p = 1$ should be treated separately. We start with the main result for the case $p > 1$:

**Theorem 1.1.** (Case $p > 1$) Let $E \subseteq \mathbb{R}^n$ be an open convex cone and $\omega: E \to (0, \infty)$ be a log-concave homogeneous weight of class $C^1$ with degree $\tau \geq 0$, and $p > 1$. Then for every function $u \in W^{1,p}(\omega; E)$ with $\int_E \|u\|^p \omega dx = 1$ we have

$$\mathcal{E}_\omega,E(|u|^p) := \int_E \|u\|^p \log \|u\|^p \omega dx \leq \frac{n + \tau}{p} \log \left( \mathcal{L}_{\omega,p} \int_E \|\nabla u\|^p \omega dx \right), \quad (1.4)$$

where

$$\mathcal{L}_{\omega,p} = \frac{p}{n + \tau} \left( \frac{p - 1}{e} \right)^{p-1} \left( \Gamma \left( \frac{n + \tau}{p'} + 1 \right) \int_{B \cap E} \omega \right)^{-\frac{1}{p' + \tau}}. \quad \text{Equality holds in (1.4) if and only if the extremal function belongs to the family of Gaussians}$$

$$u_{\lambda, x_0}(x) = \lambda^{\frac{n + \tau}{p'}} \left( \Gamma \left( \frac{n + \tau}{p'} + 1 \right) \int_{B \cap E} \omega \right)^{-\frac{1}{p' + \tau}} e^{-\lambda \frac{\|x - x_0\|^p}{p}}, \quad x \in E, \lambda > 0, \quad (1.5)$$

where

- $x_0 \in -\partial E \cap \partial E$ and $\omega(x + x_0) = \omega(x)$ for every $x \in E$ whenever $\tau > 0$, and
- $x_0 \in -\overline{E} \cap \overline{E}$ and $\omega$ is constant in $E$ whenever $\tau = 0$.

Although the log-concavity gives a regularity of $\omega$ (i.e., an a.e. differentiability on $E$), we assume for simplicity that it is of class $C^1$; a suitable modification of our proofs allows to relax this regularity assumption, see e.g. Balogh, Gutiérrez and Kristály [3].

As we noticed, Theorem 1.1 provides an extension of (1.1) to weighted Sobolev spaces as well as a full characterization of the extremal functions, answering in particular also the question from Gentil [22]. Furthermore, Theorem 1.1 gives the full class of extremal functions in the $L^p$-log-Sobolev inequality ($p > 1$) for monomial weights, see Nguyen [33], where only the 'if' part is stated.
In the case \( p = 1 \), the framework for stating log-Sobolev inequality is the space of functions with weighted bounded variation \( BV(\omega; E) \), rather than the Sobolev space \( W^{1,1}(\omega; E) \subset BV(\omega; E) \). In fact, \( BV(\omega; E) \) contains all functions \( u \in L^1(\omega; E) \) such that

\[
\sup \left\{ \int_E u \text{div}(\omega X) dx : X \in C_0^1(E; \mathbb{R}^n), |X| \leq 1 \right\} < \infty.
\]

By the Riesz representation theorem we can associate to any function \( u \in BV(\omega; E) \) its weighted variation measure \( \|Du\|_{\omega} \) similarly as in the usual non-weighted case, see e.g. Evans and Gariepy [20]. Using this notation and \( \mathds{1}_S \) for the characteristic function of the nonempty set \( S \subset \mathbb{R}^n \), we can formulate the appropriate version of the weighted \( L^1 \)-log-Sobolev inequality:

**Theorem 1.2.** (Case \( p = 1 \)) Let \( E \subseteq \mathbb{R}^n \) be an open convex cone and \( \omega: E \to (0, \infty) \) be a log-concave homogeneous weight of class \( C^1 \) with degree \( \tau \geq 0 \). Then for every function \( u \in BV(\omega; E) \) with \( \int_u |u| \omega dx = 1 \) we have

\[
\mathcal{E}_{\omega,E}(\|u\|) := \int_E |u| \log |u| \omega dx \leq (n + \tau) \log \left( \frac{\int_{B \cap E} \omega}{n + \tau} \|D(|u|)\|_{\omega}(E) \right).
\]  

(1.6)

Moreover, equality holds in (1.6) if and only if

\[
u_{\lambda,x_0}(x) = \lambda^{-n-\tau} \mathds{1}_{B(-x_0,\lambda) \cap E}(x), \quad x \in E, \ \lambda > 0,
\]  

(1.7)

where

- \( x_0 \in -\partial E \cap \partial E \) and \( \omega(x + x_0) = \omega(x) \) for every \( x \in B(-x_0, \lambda) \cap E \) whenever \( \tau > 0 \), and

- \( x_0 \in -\partial E \cap \partial E \) and \( \omega \) is constant in \( E \) whenever \( \tau = 0 \).

Theorem 1.2 extends the result of Beckner [6] and Ledoux [28] to the weighted setting, providing also the full characterization of extremals. When \( \omega \equiv 1 \) and \( E = \mathbb{R}^n \), Theorem 1.2 follows by the quantitative log-Sobolev inequality established by Figalli, Maggi and Pratelli [25]. Furthermore, the results concerning monomial weights, treated by Nguyen [33], directly follows by Theorem 1.2 together with the explicit form of the extremals.

Log-Sobolev inequalities are typically deduced by using a \emph{limiting argument} in a Gagliardo-Nirenberg-type inequality, see e.g. Beckner and Pearson [7], Del Pino and Dolbeault [16], Lam [27], Nguyen [33] and references therein. In fact there are two different limiting arguments involved in this method: with respect to the \emph{parameter} in the Gagliardo-Nirenberg inequality and also with respect to \emph{approximation} by compactly supported smooth functions. Because of the usage of this double approximation, an \"if and only if\" characterization of the extremals is no longer possible by this method.

Having as main goal the characterization of the extremals, we follow in this paper a different approach: by using optimal transport theory inspired by Cordero-Erausquin, Nazaret and Villani [15], [34] – where sharp Sobolev and Gagliardo-Nirenberg inequalities are established.
– we give a direct proof of the log-Sobolev inequalities (1.4) and (1.6). In this way we avoid the use of the limiting argument with respect to the parameter in the Gagliardo-Nirenberg inequality. Next, we shall prove the inequality for a general $W^{1,p}(\omega;E)$ resp. $BV(\omega;E)$ function to avoid the necessity of approximation with compactly supported smooth functions. However, an important technical difficulty arises: the proofs of inequalities (1.4) and (1.6) are much simpler on $C_c^\infty(\mathbb{R}^n)$ because the usual tools of calculus are readily available for this class of functions. In order to handle this situation, we have to develop these tools of calculus for non-smooth functions. In this sense we shall prove two important technical propositions containing ‘integration by part’ formulas/inequalities for functions in the general space $W^{1,p}(\omega;E)$ and $BV(\omega;E)$, which will play crucial roles in the proof of (1.4) and (1.6), see Propositions 3.1 and 4.1, respectively. These formulas together with a scaling argument, a weighted entropy-type control (see Proposition 2.2) and properties of optimal mass transport maps give not only an elegant proof for (1.4) and (1.6), but also the full characterization of the equality cases by tracking back the sharp estimates in the proofs.

The optimal mass transport theory has been successfully applied in proving weighted Sobolev and Gagliardo-Nirenberg inequalities, see e.g. Agueh, Ghoussoub and Kang [1], Ciraolo, Figalli and Roncoroni [13], Lam [27], Nguyen [33], and Balogh, Gutiérrez and Kristály [3]. Moreover, Figalli, Maggi and Pratelli [24, 25] applied this method to prove quantitative isoperimetric and Sobolev inequalities in the unweighted setting. Cordero-Erausquin [14], Barthe and Kolesnikov [5] and Fathi, Indrei and Ledoux [21] gave a direct proof of the Gaussian log-Sobolev inequality (even in quantitative form) for $p = 2$ based on optimal mass transport theory. However, we are not aware of the existence of a direct proof of the sharp Euclidean $L^p$-log-Sobolev inequality based on the optimal transport method, $p > 1$.

We notice that – by using the ABP-method – further sharp Sobolev inequalities with weights have been also stated by Cabré, Ros-Oton and Serra [10, 11]. Similarly to the aforementioned works (see [13], [27], [33] and [3]), in the papers [10, 11] the initial assumption is the concavity of $\omega^{\frac{1}{n}}$ on $E \subseteq \mathbb{R}^n$ (with the convention that $\omega$ is constant whenever $\tau = 0$). Even though the concavity of $\omega^{\frac{1}{n}}$ implies generically the log-concavity of $\omega$, – the latter being crucial in the proof of the log-Sobolev inequalities – one can observe that the converse also holds due to the homogeneity of $\omega$, see Proposition 2.1. Consequently, the assumptions of Theorems 1.1 and 1.2 are in a full concordance with the ones used in the usual literature, and they essentially require the weight $\omega$ to verify the curvature-dimension condition $CD(0, n + \tau)$ on $E$, see [11].

In several of these works the authors established Sobolev-type inequalities with general norms instead of the Euclidean one, see e.g. Cordero-Erausquin, Nazaret and Villani [15], Cabré, Ros-Oton and Serra [11], Figalli, Maggi and Pratelli [24], Gentil [22], Lam [27], Nguyen [33]. Although Theorems 1.1 and 1.2 can be stated in the same general context, we avoid to do this for the sake of simplicity. Such an anisotropic extension relies on the definition of the dual norm $\| \cdot \|_*$ of an arbitrary norm $\| \cdot \|$ given by $\|X\|_* = \sup_{\|Y\| \leq 1} X \cdot Y$, where $X \cdot Y$ stands for the usual Euclidean inner product of $X,Y \in \mathbb{R}^n$, see Remark 3.2. It is worth to notice that this extension does not require any modification of the optimal transport theory of Brenier-McCann (see [9] and [31]), the cost function still remaining the Euclidean one.

As an application of the sharp $L^p$-log-Sobolev inequality (Theorem 1.1), we establish a sharp weighted hypercontractivity estimate for the Hopf-Lax semigroup related to Hamilton-Jacobi equation and characterize the equality cases. To be more precise, for $p > 1$ and for a
function $g : E \to \mathbb{R}$ we recall the Hopf-Lax formula

$$Q_t g(x) := Q^E_{t} g(x) = \inf_{y \in E} \left\{ g(y) + \frac{|y - x|^{p'}}{p't^{p'-1}} \right\}, \quad t \in (0,t_0), x \in E. \quad (1.8)$$

Here $t_0 > 0$ is chosen such that $Q_t g(x) \in \mathbb{R}$ for $t \in (0,t_0)$ and $x \in E$ (by convention, we consider $Q_t g = g$). Clearly, $t_0 = \infty$ whenever $g$ is a bounded Lipschitz function on $E$. Moreover, $(x,t) \mapsto Q_t g(x)$ solves the classical Hamilton-Jacobi equation; see e.g. Gentil [22] or the book of Evans [19]. However, in order to characterize the equality cases in the hypercontractivity estimate (see Theorem 1.3 below), we need a larger class of functions $g$ than bounded and Lipschitz, that eventually contains the candidates for extremals in the hypercontractivity estimate. To achieve this, for a fixed $t_0 > 0$, we consider the family of functions

$$\mathcal{F}_{t_0}(E) := \left\{ g : E \to \mathbb{R} : g \text{ is measurable, bounded from above and there exists } x_0 \in E \text{ such that } Q_{t_0} g(x_0) > -\infty \right\}.$$

In the sequel we show that if $g \in \mathcal{F}_{t_0}(E)$ then $Q_t g(x) \in \mathbb{R}$ for every $t \in (0,t_0)$ and $x \in E$, and the following weighted hypercontractivity estimate holds.

**Theorem 1.3.** Let $E \subseteq \mathbb{R}^n$ be an open convex cone and $\omega : E \to (0,\infty)$ be a log-concave homogeneous weight of class $C^1$ with degree $\tau \geq 0$, and $p > 1$. Given the numbers $t_0 > 0$ and $0 < \alpha \leq \beta$, for every function $g \in \mathcal{F}_{t_0}(E)$ with $e^g \in L^\alpha(\omega;E)$ and every $t \in (0,t_0)$ we have

$$\|e^{Q^E_{t} g}\|_{L^\beta(\omega;E)} \leq \|e^g\|_{L^\alpha(\omega;E)} \left( \frac{\beta - \alpha}{t} \right)^{\frac{n+\tau}{p} - \frac{\beta - \alpha}{p}} \frac{\alpha^{\frac{n+\tau}{p}}}{\beta^{\frac{n+\tau}{p}}} \left( \frac{n + \tau}{p'} + 1 \right) \int_{B^\omega E} \omega \right)^{\frac{\alpha - \beta}{\alpha p}}. \quad (1.9)$$

In addition, equality holds in (1.9) for some $\tilde{t} \in (0,t_0)$ and $g \in \mathcal{F}_{\tilde{t}}(E)$ with $e^g \in L^\alpha(\omega;E)$ if and only if $\alpha < \beta$ and

$$g(x) = C - \frac{1}{p'} \left( \frac{\beta - \alpha}{\beta t} \right)^{\frac{1}{p}} |x + x_0|^{p'}, \quad x \in E, \quad (1.10)$$

for some $C \in \mathbb{R}$ and

- $x_0 \in -\partial E \cap \partial E$ and $\omega(x + x_0) = \omega(x)$ for every $x \in E$ whenever $\tau > 0$, and
- $x_0 \in -\overline{E} \cap \overline{E}$ and $\omega$ is constant in $E$ whenever $\tau = 0$.

The proof of Theorem 1.3 is based on the a.e. validity of the Hamilton-Jacobi equation associated with the Hopf-Lax solution $(x,t) \mapsto Q_t g(x)$ and its integrability properties (see Proposition 5.1) combined with the weighted $L^p$-log-Sobolev inequality. The equality cases in (1.9) are obtained via the characterization of the extremal functions in the $L^p$-log-Sobolev inequality formulated in Theorem 1.1 and by using properties of the map $(x,t) \mapsto Q_t g(x)$.

The paper is organized as follows. In Section 2 we present some basic properties of the weights, including a weighted entropy-type integrability, based on a two-weighted Sobolev inequality established in [3]. In Section 3 we prove Theorem 1.1, where one of the crucial steps is an integration by parts formula (see Propositions 3.1) for functions belonging to $W^{1,p}(\omega;E)$, $p > 1$. Section 4 is a kind of counterpart of Section 3, where we prove Theorem 1.2, and one of the main ingredients is again an integration by parts formula (see Proposition 4.1) for functions in $BV(\omega;E)$. In Section 5 we present the proof of Theorem 1.3.
2. Preparatory results

We start with an elementary observation related to (log-)concave homogeneous weights.

**Proposition 2.1.** Let \( E \subseteq \mathbb{R}^n \) be an open convex cone and \( \omega: E \rightarrow (0, \infty) \) be a homogeneous weight of class \( C^1 \) with degree \( \tau \geq 0 \).

(i) If \( \tau > 0 \), the following statements are equivalent:

(a) \( \omega \) is log-concave in \( E \);

(b) for every \( x, y \in E \) one has

\[
\log \left( \frac{\omega(y)}{\omega(x)} \right) \leq -\tau + \frac{\nabla \omega(x) \cdot y}{\omega(x)}; \tag{2.1}
\]

(c) \( \omega^{\frac{1}{\tau}} \) is concave in \( E \).

Moreover, any of the above properties imply that \( 0 \leq \nabla \omega(x) \cdot y \) for every \( x, y \in E \).

(ii) If \( \omega \) is log-concave in \( E \), then \( \tau = 0 \) if and only if \( \omega \) is constant in \( E \); in particular, if \( E = \mathbb{R}^n \) then \( \omega \) is constant in \( \mathbb{R}^n \).

**Proof.** (i) We assume that \( \tau > 0 \). The equivalence between (a) and (b) is a consequence of the definition and the Euler’s relation: \( \nabla \omega(x) \cdot x = \tau \omega(x) \) for every \( x \in E \) for the \( \tau \)-homogeneous function \( \omega \).

The implication (c) \( \implies \) (a) is generically valid even in the absence of homogeneity.

We now prove the implication (a) \( \implies \) (c). For simplicity, let \( v := \omega^{\frac{1}{\tau}} \), which is log-concave in \( E \) by assumption. Let \( x_1, x_2 \in E \) be two fixed elements and \( v(x_1) = a_1 > 0 \) and \( v(x_2) = a_2 > 0 \). By the 1-homogeneity of \( v \), one has that \( v(x_1/a_1) = v(x_2/a_2) = 1 \), thus the log-concavity of \( v \) implies that for every \( t \in [0, 1] \) we have

\[
v \left( \frac{x_1}{a_1} + (1 - t) \frac{x_2}{a_2} \right) \geq v \left( \frac{x_1}{a_1} \right)^t v^{1-t} \left( \frac{x_2}{a_2} \right) = 1.
\]

If \( \lambda \in [0, 1] \) is arbitrarily fixed, let \( t := \frac{\lambda a_1}{\lambda a_1 + (1 - \lambda) a_2} \in [0, 1] \) be in the above estimate; then, we obtain

\[
v \left( \frac{\lambda x_1 + (1 - \lambda) x_2}{\lambda a_1 + (1 - \lambda) a_2} \right) \geq 1.
\]

Again by the 1-homogeneity of \( v \), the latter relation becomes equivalent to

\[
v(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda a_1 + (1 - \lambda)a_2 = \lambda v(x_1) + (1 - \lambda)v(x_2),
\]

which is precisely the concavity of \( v = \omega^{\frac{1}{\tau}} \).

By contradiction, let us assume that there exist \( x_0, y_0 \in E \) such that \( \nabla \omega(x_0) \cdot y_0 < 0 \). We choose \( x := x_0 \) and \( y := \lambda y_0 \) with \( \lambda > 0 \) in (2.1); by the homogeneity of \( \omega \), it turns out that

\[
\tau \log \lambda + \log \left( \frac{\omega(y_0)}{\omega(x_0)} \right) \leq -\tau + \lambda \frac{\nabla \omega(x_0) \cdot y_0}{\omega(x_0)}, \quad \lambda > 0.
\]

Letting \( \lambda \rightarrow \infty \), we get a contradiction.

(ii) If \( \omega \) is constant, then trivially \( \tau = 0 \). Conversely, if \( \tau = 0 \), we have \( \omega(\lambda y) = \omega(y) \) for every \( \lambda > 0 \) and \( y \in E \). Clearly, (2.1) is also valid (with \( \tau = 0 \)); in particular, by putting...
\( y := \lambda y \), we obtain
\[
\log \left( \frac{\omega(y)}{\omega(x)} \right) \leq \frac{\lambda \nabla \omega(x) \cdot y}{\omega(x)} \quad \text{for all } \lambda > 0, \ x, y \in E.
\]

Letting \( \lambda \to 0 \), the latter inequality implies that \( \omega(y) \leq \omega(x) \) for every \( x, y \in E \), which shows that \( \omega \) is constant in \( E \).

Let \( E = \mathbb{R}^n \). Since \( 0 \leq \nabla \omega(x) \cdot y \) for every \( x, y \in \mathbb{R}^n \) (similarly to the case (i)), we necessarily have \( \nabla \omega(x) = 0 \) for every \( x \in \mathbb{R}^n \), i.e., \( \omega \) is constant in \( \mathbb{R}^n \). \( \square \)

**Remark 2.1.** Large classes of weights verifying the properties of Proposition 2.1 can be found in Cabré, Ros-Oton and Serra \([11]\), Section 2 and Balogh, Gutiérrez and Kristály \([3]\), Section 4, including – among others – the monomial weight \( \omega \): \( E \to (0, \infty) \) defined as \( \omega(x) = x^{a_1} \ldots x^{a_n} \) with \( a_i \geq 0 \) for every \( i = 1, \ldots, n \) and
\[
E = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ whenever } a_i > 0 \}.
\]

The following auxiliary result concerns a weighted entropy-type estimate which is crucial in the proof of our main theorems. To prove it, we need a two-weighted Sobolev inequality by Balogh, Gutiérrez and Kristály \([3]\) that we recall in the sequel.

Let \( E \subseteq \mathbb{R}^n \) be an open convex cone. First, we assume that the two weights \( \omega_1, \omega_2 : E \to (0, \infty) \) satisfy the homogeneity condition
\[
\omega_i(\lambda x) = \lambda^{\tau_i} \omega_i(x) \quad \text{for all } \lambda > 0, x \in E,
\]
where the parameters \( \tau_1, \tau_2 \in \mathbb{R} \) verify
\[
1 \leq s < \tau_2 + n \leq \tau_1 + s + n,
\]
and
\[
\tau_2 \geq \left( 1 - \frac{s}{n} \right) \tau_1,
\]
for some \( s \in \mathbb{R} \). Furthermore, for the given values \( s, \tau_1, \tau_2 \) as above choose \( t \in \mathbb{R} \) such that the following balance condition holds:
\[
\frac{\tau_1 + n}{t} = \frac{\tau_2 + n - s}{s} - 1.
\]
By relations (2.5) and (2.3), we obtain that \( t = \frac{s(\tau_1 + n)}{\tau_2 + n - s} \geq s \). The fractional dimension \( n_a \) is given by
\[
\frac{1}{n_a} = \frac{1}{s} - \frac{1}{t}.
\]
By (2.5), relation (2.4) turns out to be equivalent to \( n_a \geq n \). The cases \( n_a = n \) and \( n_a > n \) should be discussed separately, both of them being crucial in our further investigations. It turns out that the weighted Sobolev inequality
\[
\left( \int_E |v(x)|^t \omega_1(x) \, dx \right)^{1/t} \leq K_0 \left( \int_E |\nabla v(x)|^s \omega_2(x) \, dx \right)^{1/s} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n),
\]
holds, see \([3, \text{Theorem 1.1}]\), where \( K_0 > 0 \) is independent on \( v \in C_c^\infty(\mathbb{R}^n) \), whenever
In addition, for \( x \in E \) and for all \( y \in E \),

\[
0 \leq \left( \frac{1}{s'} \frac{\nabla \omega_1(x)}{\omega_1(x)} + \frac{1}{s} \frac{\nabla \omega_2(x)}{\omega_2(x)} \right) \cdot y
\]  

for a.e. \( x \in E \) and for all \( y \in E \),

(II) or \( n_a > n \) and there exists a constant \( C_0 > 0 \) such that

\[
\left( \frac{\omega_2(y)}{\omega_2(x)} \right)^{\frac{s}{2}} \left( \frac{\omega_1(x)}{\omega_1(y)} \right)^{\frac{t}{2}} \leq C_0 \left( \frac{1}{s'} \frac{\nabla \omega_1(x)}{\omega_1(x)} + \frac{1}{s} \frac{\nabla \omega_2(x)}{\omega_2(x)} \right) \cdot y,
\]

for a.e. \( x \in E \) and for all \( y \in E \).

Here, as before, \( s' = s/(s-1) \) denotes the conjugate of \( s \geq 1 \) (with \( s' = +\infty \) if \( s = 1 \)). Moreover, by density arguments, the inequality (WSI) is valid for functions \( v : \mathbb{R}^n \to \mathbb{R} \) belonging to \( W^{1,s}(\omega_2; E) \). Note that by scaling arguments, (WSI) necessarily requires the dimensional balance condition (2.5).

After this preparatory part, we state the following result.

**Proposition 2.2.** Let \( p \geq 1 \), \( E \subseteq \mathbb{R}^n \) be an open convex cone and \( \omega : E \to (0, \infty) \) be a log-concave homogeneous weight of class \( C^1 \) with degree \( \tau \geq 0 \). Then for every \( u \in W^{1,p}(\omega; E) \) such that \( \int_E |u|^p \omega dx = 1 \) the following inequalities hold:

\[
\int_E |u|^p \omega \log(|u|^p \omega) < +\infty,
\]

(2.9)

\[
\int_E |u|^p \omega \log |u|^p < +\infty,
\]

(2.10)

\[-\infty < \int_E |u|^p \omega \log \omega. \]

(2.11)

In addition, for \( p = 1 \), the same statements hold for functions \( u \in BV(\omega; E) \).

**Proof.** We prove relation (2.9) first. We consider only the case when \( \tau > 0 \); otherwise, if \( \tau = 0 \), the function \( \omega \) is constant and the claim holds by the Euclidean log-Sobolev inequality (1.1). Since \( n \geq 2 \), we may choose the parameters \( s = 1, t = \frac{n}{n-1} \) and the weights \( \omega_1 = \omega^{\frac{n}{n-1}}, \omega_2 = \omega \). In particular, we have \( \tau_1 = \frac{n}{n-1} \tau \) and \( \tau_2 = \tau \). Moreover, simple computations confirm that the assumptions (2.2)–(2.5) are verified. Note that for the fractional dimension we have \( \frac{1}{n_a} = \frac{1}{s} - \frac{1}{t} = \frac{1}{n} \), thus \( n_a = n \). In addition,

\[
\frac{\omega_1(x)^{1/t}}{\omega_2(x)^{1/s}} = \frac{\omega(x)}{\omega(x)} = 1, \quad x \in E.
\]

Finally, it remains to verify (2.7), which directly follows by Proposition 2.1. According to the above reasons, – keeping the above notations, – it follows by (I) that the weighted Sobolev inequality (WSI) holds, i.e.,

\[
\left( \int_E |v(x)|^{\frac{n}{n-1}} \omega(x)^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq K_0 \int_E |\nabla v(x)| \omega(x) dx \quad \text{for all } v \in W^{1,1}(\omega; E).
\]  

(2.12)
Fix $u \in W^{1,p}(\omega; E)$ such that $\int_E |u|^p \omega dx = 1$ and let $v := |u|^p$. We prove that $v \in W^{1,1}(\omega; E)$. Let us assume first that $p > 1$. We observe that
\[
\int_E |v(x)| \omega(x) \, dx = \int_E |u(x)|^p \omega(x) \, dx = 1.
\]
Moreover, since $\nabla v = p|u|^{p-1}\nabla |u|$, by Hölder’s inequality and $u \in W^{1,p}(\omega; E)$ we have that
\[
\int_E |\nabla v(x)| \omega(x) \, dx = p \int_E |\nabla u(x)||u(x)|^{p-1} \omega(x) \, dx \\
\leq p \left( \int_E |\nabla u(x)|^p \omega(x) \, dx \right)^{\frac{1}{p}} \left( \int_E |u(x)|^p \omega(x) \, dx \right)^{\frac{1}{p}} < +\infty.
\]
Therefore, $v \in W^{1,1}(\omega; E)$. Now, by (2.12) we obtain
\[
\int_E |u(x)|^p \frac{\tau}{n} \omega(x)^{\frac{n}{n-\tau}} \, dx = \int_E |v(x)|^{\frac{n}{n-\tau}} \omega(x)^{\frac{n}{n-\tau}} \, dx < +\infty.
\]
By Jensen’s inequality and the above estimate we have that
\[
\frac{1}{n-1} \int_E |u|^p \omega \log(|u|^p \omega) = \int_E |u|^p \omega \log\left(|u|^p \omega^{\frac{1}{n}}\right) \leq \log \left( \int_E |u(x)|^{p \frac{n}{n-\tau}} \omega(x)^{\frac{n}{n-\tau}} \right) < +\infty,
\]
which concludes the proof of (2.9).

The proof of (2.10) conceptually is similar to the above argument. In this case we choose the parameters $s = 1$, $t = \frac{\tau+n}{\tau+n-1}$ and the weights $\omega_1 = \omega_2 = \omega$; therefore, $\tau_1 = \tau_2 = \tau$ and the fractional dimension is $n_\omega = n + \tau > n$. Moreover, due to Proposition 2.1, inequality (2.8) becomes equivalent to the concavity of $\omega^{\frac{1}{\tau}}$ once we choose $C_0 = \frac{1}{\tau}$. Therefore, by (II) the Sobolev inequality (WSI) reads as
\[
\left( \int_E |v(x)|^{\frac{\tau+n}{\tau+n-1}} \omega(x) \, dx \right)^{\frac{\tau+n-1}{\tau+n}} \leq K_0 \int_E |\nabla v(x)| \omega(x) \, dx \quad \text{for all } v \in W^{1,1}(\omega; E).
\] (2.13)

In the same way as before we can combine Jensen’s inequality and the estimate (2.13) to deduce that
\[
\frac{1}{\tau+n-1} \int_E |u|^p \omega \log |u|^p = \int_E |u|^p \omega \log\left(|u|^p \omega^{\frac{1}{\tau+n-1}}\right) \leq \log \left( \int_E |u(x)|^{p \frac{\tau+n}{\tau+n-1}} \omega(x) \right) < +\infty,
\]
which concludes the proof of (2.10).

When $p = 1$, by standard approximation arguments the inequality (2.12) extends to functions $v \in BV(\omega; E)$ by replacing the right hand side to the $\omega$-bounded variation of $v \in BV(\omega; E)$, see [8, Theorem 5.1]. The rest of the proof of inequalities (2.9) and (2.10) is similar to the case $p > 1$.

Finally we prove inequality (2.11). Let us fix a point $x_0 \in E$. By the log-concavity of $\omega$ (see (2.1)) we can write
\[
\log \omega(x_0) + \tau \leq \log \omega(x) + \frac{\nabla \omega(x)}{\omega(x)} \cdot x_0 \quad \text{for all } x \in E.
\]
Multiplying by $u^p \omega$ the latter inequality and integrating over $E$ we obtain

$$(\log \omega(x_0) + \tau) \int_E |u|^p \omega - \int_E |u|^p \nabla \omega(x) \cdot x_0 \leq \int_E |u|^p \omega \log \omega.$$  

Since $\int_E |u|^p \omega dx = 1$, we only need to check that

$$\int_E |u|^p \nabla \omega(x) \cdot x_0 < \infty.$$  

To verify this, we integrate by parts to obtain

$$\int_E |u|^p \nabla \omega(x) \cdot x_0 dx = \int_{\partial E} |u|^p \omega x_0 \cdot n dH^{n-1} - p \int_E |u|^{p-1} |\nabla| \cdot x_0 \omega dx,$$

where $n(x)$ is the unit outer normal vector to $E$ at $x \in \partial E$. Since $x_0 \in E$, by the convexity of the cone $E$ we have that the vectors $x_0$ and $n(x)$ form an obtuse angle, i.e., $x_0 \cdot n(x) \leq 0$ for all points $x \in \partial E$ where the normal vector $n(x)$ exists; thus the first integral on the right hand side is non-positive. Therefore, it is enough to estimate the second integral from above, which is obtained by Hölder’s inequality, i.e.,

$$-\int_E |u|^{p-1} (|\nabla| \cdot x_0) \omega dx \leq |x_0| \int_E |u|^{p-1} |\nabla| \omega dx \leq |x_0| \left( \int_E |u|^p \omega \right)^{\frac{1}{p'}} \left( \int_E |\nabla|^p \omega \right)^{\frac{1}{p'}} < \infty,$$

finishing the proof in the case $p > 1$. If $p = 1$, we do not need to apply Hölder’s inequality but we need to perform an approximation argument and work with the $\omega$-bounded variation of measure as indicated before. □

### 3. Proof of Theorem 1.1: the case $p > 1$

The main idea of the application of the optimal mass transport method in proving geometric inequalities lies in the fact that the optimal transport map is given as the gradient of a convex function $\phi : E \to \mathbb{R}$. We recall that by the theorem of Alexandrov [20] a convex function is twice differentiable almost everywhere. We denote by $\Delta_{D'} \phi$ the distributional Laplacian of $\phi$; this is a positive Radon measure whose absolutely continuous part is denoted by $\Delta_A \phi$ and is called the Alexandrov Laplacian of $\phi$. Clearly the inequality

$$\Delta_A \phi \leq \Delta_{D'} \phi$$

holds in the sense of distributions.

Before presenting the proof of Theorem 1.1, we need an integration by part formula for functions $u \in W^{1,p}(\omega; E)$ whose proof is inspired by Cordero-Erausquin, Nazaret and Villani [15] and Nguyen [33].

**Proposition 3.1.** Let $p > 1$, $u \in W^{1,p}(\omega; E)$ be non-negative, $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $u^{p-1} \nabla \phi \in L^p(\omega; E)$ and $\nabla \phi(x) \in \overline{E}$ for a.e. $x \in E$. Then we have

$$\int_E u^p \omega \Delta_A \phi dx \leq -p \int_E u^{p-1} \omega \nabla \phi \cdot \nabla u dx - \int_E u^p \nabla \omega \cdot \nabla \phi dx. \quad (3.1)$$
Proof. Let $S := \text{int}\{x \in \mathbb{R}^n : \phi(x) < +\infty\}$; by assumption, $E \cap S$ is open, convex and contains the support of $u$ except, possibly, for a Lebesgue null set.

For every $k \geq 1$, let $\theta_k^0 : [0, \infty) \to [0, 1]$ be a non-decreasing $C^\infty$ function such that $\theta_k^0(s) = 0$ for $s \leq \frac{1}{2k}$ and $\theta_k^0(s) = 1$ for $s \geq \frac{1}{k}$; in addition, let $\theta_k : E \to [0, 1]$ be the locally Lipschitz function $\theta_k(x) = \theta_k^0(d(x, \partial E))$, $x \in E$. Let $u_k = u \theta_k$, $k \geq 1$. A simple argument shows that $u_k \in W^{1,p}(\omega; E)$ for every $k \geq 1$ and supp$u_k \subseteq S_k$ where $S_k = \{x \in E : d(x, \partial E) \geq \frac{1}{2k}\}$. Furthermore, for every fixed $k \geq 1$, by a trivial extension, the function $u_k$ belongs to the usual Sobolev space $W^{1,p}(\mathbb{R}^n)$ as well.

Consider now a cut-off function $\chi : \mathbb{R}^n \to [0, 1]$ of class $C^\infty$ such that $\chi \equiv 1$ on $B = B(0, 1)$, $\chi \equiv 0$ on $\mathbb{R}^n \setminus B(0, 2)$. Fix $x_0 \in E$ and $k \geq 1$. For $\varepsilon \in (0, 1)$ small enough, we define

$$ u_{k,\varepsilon}(x) = \min \left\{ u_k \left( x_0 + \frac{x - x_0}{1 - \varepsilon} \right), \chi(\varepsilon x) u_k(x) \right\}, \quad x \in E. $$

Then $u_{k,\varepsilon} \geq 0$ has compact support in $E \cap S$ and since $|\nabla \chi| \leq C_0$ for some $C_0 > 0$, one has that $u_{k,\varepsilon} \in W^{1,p}(\omega; E) \cap W^{1,p}(\mathbb{R}^n)$.

Given $\kappa \in C_c^\infty(B)$ be a non-negative function such that $\int_B \kappa \, dx = 1$, we consider the convolution kernel $\kappa_\delta(x) = \frac{1}{\delta^n} \kappa(x/\delta)$, $\delta > 0$, and define the convolution function

$$ u_{k,\varepsilon}^\delta(x) = (u_{k,\varepsilon} * \kappa_\delta)(x) := \int_E u_{k,\varepsilon}(y) \kappa_\delta(x - y) \, dy \geq 0. $$

It is standard that the functions $u_{k,\varepsilon}^\delta$ are smooth on $E$ for every $\varepsilon \in (0, 1)$ belonging to the above range, and $\delta > 0$, and $u_{k,\varepsilon}^\delta$ converges to $u_{k,\varepsilon}$ in $W^{1,p}(\omega; E)$ as $\delta \to 0$. Moreover, since $\phi$ is convex, and supp$u_{k,\varepsilon}^\delta$ is compact in $E$, then $\nabla \phi$ is essentially bounded on supp$(u_{k,\varepsilon}^\delta)$ for every sufficiently small $\delta > 0$.

The idea is to apply for the smooth function $u_{k,\varepsilon}^\delta$ the usual divergence theorem and pass to the limit as $\delta \to 0$, $\varepsilon \to 0$ and $k \to \infty$, in this order, obtaining the expected inequality (3.1).

Taking into account that $u_{k,\varepsilon}^\delta = 0$ on $\partial E$, and $\Delta_A \phi \leq \Delta_{\mathcal{D}} \phi$ in the distributional sense, the divergence theorem implies

$$ \int_E (u_{k,\varepsilon}^\delta)^p \omega \Delta_A \phi \, dx \leq \int_E (u_{k,\varepsilon}^\delta)^p \omega \Delta_{\mathcal{D}} \phi \, dx $$

$$ \quad = -p \int_E (u_{k,\varepsilon}^\delta)^p \nabla u_{k,\varepsilon}^\delta \cdot \nabla \phi \omega \, dx - \int_E (u_{k,\varepsilon}^\delta)^p \nabla \omega \cdot \nabla \phi \, dx. \quad (3.2) $$

Thus, by inequality (3.2) we have

$$ \int_E (u_{k,\varepsilon}^\delta)^p (\omega \Delta_A \phi + \nabla \omega \cdot \nabla \phi) \, dx \leq -p \int_E (u_{k,\varepsilon}^\delta)^p \nabla u_{k,\varepsilon}^\delta \cdot \nabla \phi \omega \, dx. \quad (3.3) $$

Since $u_{k,\varepsilon}^\delta$ converges to $u_{k,\varepsilon}$ in $W^{1,p}(\omega; E)$, one has that $\nabla u_{k,\varepsilon}^\delta$ converges to $\nabla u_{k,\varepsilon}$ in $L^p(\omega; E)$ and $(u_{k,\varepsilon}^\delta)^p$ converges to $(u_{k,\varepsilon})^p$ in $L^p(\omega; E)$ whenever $\delta \to 0$, the latter property coming from the property of superposition operators, see e.g. Willem [36, Appendix A]. In addition, since $\nabla \phi$ is essentially bounded on supp$(u_{k,\varepsilon}^\delta)$, we have that

$$ \int_E (u_{k,\varepsilon}^\delta)^p \nabla u_{k,\varepsilon}^\delta \cdot \nabla \phi \omega \, dx \to \int_E (u_{k,\varepsilon})^p \nabla u_{k,\varepsilon} \cdot \nabla \phi \omega \, dx \quad \text{as} \quad \delta \to 0. $$
Since $\phi$ is convex, then $\Delta_A \phi \geq 0$; moreover, by Proposition 2.1 we also have that $\nabla \phi \cdot \nabla \omega \geq 0$ a.e. on $E$, since $\nabla \phi(x) \in \overline{E}$ for a.e. $x \in \overline{E}$. Therefore, as $\delta \to 0$, the latter properties together with (3.3) and Fatou’s lemma imply that

$$
\int_E u_{k,\varepsilon}^p (\omega \Delta_A \phi + \nabla \omega \cdot \nabla \phi) \, dx \leq -p \int_E u_{k,\varepsilon}^{p-1} \nabla u_{k,\varepsilon} \cdot \nabla \phi \omega \, dx. \quad (3.4)
$$

In the sequel we pass to the limit in (3.4) as $\varepsilon \to 0$. By the definition of $u_{k,\varepsilon}$, we first observe that – extracting eventually a subsequence of $\varepsilon = (\varepsilon_{k,l})_{l \in \mathbb{N}}$ – the sequence $u_{k,\varepsilon}$ converges to $u_k$ a.e. as $\varepsilon \to 0$. By definition one has $u_{k,\varepsilon} \leq u_k \leq u$ and recall that by assumption $u^{p-1} \nabla \phi \in L^p(\omega; E)$; thus, by the dominated convergence theorem we have that

$$
u_{k,\varepsilon}^{p-1} \nabla \phi \to u_k^{p-1} \nabla \phi \quad \text{in } L^p(\omega; E) \text{ as } \varepsilon \to 0.
$$

One can also prove that $\nabla u_{k,\varepsilon}$ converges to $\nabla u_k$ in the sense of distributions and $\nabla u_{k,\varepsilon}$ is uniformly bounded in $L^p(E)$ w.r.t. $\varepsilon > 0$; hence, $\nabla u_{k,\varepsilon}$ converges weakly in $L^p(E)$ to $\nabla u_k$ as $\varepsilon \to 0$. Combining these facts, we obtain that

$$
\int_E u_{k,\varepsilon}^{p-1} \nabla u_{k,\varepsilon} \cdot \nabla \phi \omega \, dx \to \int_E u_k^{p-1} \nabla u_k \cdot \nabla \phi \omega \, dx \quad \text{as } \varepsilon \to 0.
$$

Taking into account Fatou’s lemma and letting $\varepsilon \to 0$ in (3.4) we get

$$
\int_E u_k^p (\omega \Delta_A \phi + \nabla \omega \cdot \nabla \phi) \, dx \leq -p \int_E u_k^{p-1} \nabla u_k \cdot \nabla \phi \omega \, dx. \quad (3.5)
$$

Now, it remains to take the limit $k \to \infty$ in (3.5). To do this, we first observe that since $x \mapsto d(x, \partial E)$ is locally Lipschitz, it is differentiable a.e. in $E$; furthermore, for a.e. $x \in E$ one has that $\nabla d(\cdot, \partial E)(x) = -n(x^*)$, where $x^* \in \partial E$ is the unique point with $|x - x^*| = d(x, \partial E)$ and $n(x^*)$ is the unit outward normal vector at $x^* \in \partial E$. Since $E$ is convex and $\nabla \phi(x) \in \overline{E}$ for a.e. $x \in \overline{E}$, it turns out that $n(x^*) \cdot \nabla \phi(x) \leq 0$ for a.e. $x \in E$. In particular, the monotonicity of $\theta_k^p$ implies that for a.e. $x \in E$ one has

$$
\nabla u_k(x) \cdot \nabla \phi(x) = \theta_k(x) \nabla u(x) \cdot \nabla \phi(x) - u(x) (\theta_k^p)'(d(x, \partial E)) n(x^*) \cdot \nabla \phi(x)

\geq \theta_k(x) \nabla u(x) \cdot \nabla \phi(x).
$$

The above arguments together with (3.5) imply

$$
\int_E \theta_k^p u^p (\omega \Delta_A \phi + \nabla \omega \cdot \nabla \phi) \, dx \leq -p \int_E \theta_k^p u^{p-1} \nabla u \cdot \nabla \phi \omega \, dx.
$$

Passing to the limit in the latter inequality as $k \to \infty$ and taking into account that $\theta_k \to 1$ for a.e. $x \in E$ as $k \to \infty$, Fatou’s lemma and the dominated convergence theorem imply the desired inequality (3.1).

We are ready to prove Theorem 1.1; we distinguish the cases when $\tau > 0$ and $\tau = 0$, respectively.
3.1. Proof of the inequality (1.4): case $p > 1$, $\tau > 0$. Fix $u \in W^{1,p}(\omega; E)$ such that $\int_E |u|^p \omega dx = 1$. On account of relation (2.10) by Proposition 2.2, we have $\mathcal{E}_{\omega,E}(|u|^p) < +\infty$.

We first notice that when $\mathcal{E}_{\omega,E}(|u|^p) = \int_E |u|^p \log |u|^p \omega dx = -\infty$, we have nothing to prove in (1.4). Accordingly, we may assume in the sequel that

$$-\infty < \mathcal{E}_{\omega,E}(|u|^p) < +\infty.$$  

(3.6)

Using this relation, combined with relations (2.9)-(2.11) from Proposition 2.2, we can conclude that

$$-\infty < \int_E |u|^p \omega \log \omega dx < +\infty.$$  

(3.7)

We begin now by introducing some constants that will be used throughout the proof. Let $\omega_{SE} := \int_{\mathbb{R}^{n-1} \cap E} \omega$ and for further convenience, we introduce the real numbers $C_i = C_i(\omega, E, n, p, \tau)$, $i = 1, \ldots, 4$, as follows:

$$C_1 := p' \left( \omega_{SE} \Gamma \left( \frac{n + \tau}{p'} \right) \right)^{-1},$$

$$C_2 := \log C_1 - C_1 \int_E e^{-y|\nu'|} |y|^{p'} \omega(y) dy = \log \left( p' \left( \omega_{SE} \Gamma \left( \frac{n + \tau}{p'} \right) \right)^{-1} \right) - \frac{n + \tau}{p'},$$

$$C_3 := C_2 + C_1 \int_E e^{-|y|^{p'}} \omega(y) \log \omega(y) dy,$$

and

$$C_4 := pC_1^\frac{1}{p'} \left( \int_E e^{-|y|^{p'}} |y|^{p'} \omega(y) dy \right)^{-\frac{1}{p'}} = p \left( \frac{n + \tau}{p'} \right)^{\frac{1}{p'}}.$$  

We notice that the numbers $C_i \in \mathbb{R}$ are well-defined, $i = 1, \ldots, 4$.

We consider the scaled function $u_t(x) = t^\alpha u(tx)$ for $t > 0$, $x \in E$; by the homogeneity of $\omega$ we observe that $\int_E |u_t|^p \omega dx = 1$ for every $t > 0$ if and only if $\alpha p = n + \tau$. By standard scaling arguments we also have that

$$\mathcal{E}_{\omega,E}(|u_t|^p) = \mathcal{E}_{\omega,E}(|u|^p) + \alpha p \log t, \quad t > 0,$$  

(3.8)

and

$$\int_E |\nabla u_t|^p \omega = t^p \int_E |\nabla u|^p \omega,$$

therefore, $u_t \in W^{1,p}(\omega; E)$ and inequality (1.4) holds for $u \in W^{1,p}(\omega; E)$ if and only if it holds for $u_t$. In particular, due to (3.7), we may choose $t > 0$ such that

$$\int_E |u_t(x)|^p \omega(x) \log \omega(x) dx = \int_E |u(x)|^p \omega(x) \log \omega(x) dx - \tau \log t = C_3 - C_2.$$

From now on we will simply write $u$ instead of $u_t$ and assume without loss of generality that

$$\int_E |u(x)|^p \omega(x) \log \omega(x) dx = C_3 - C_2.$$  

(3.9)
Now, we shall focus to the proof of the log-Sobolev inequality (1.4). Let us consider the probability measures $d\mu(x) = |u(x)|^p \omega(x) dx$ and $d\nu(x) = C_1 e^{-|y|^{p'}} \omega(y) dy$. By the theory of optimal mass transport developed by Brenier [9] and further extended by McCann [31, Main Theorem] for general probability measures, there exists a convex function $\phi: \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \phi$ pushes $\mu$ forward to $\nu$, i.e. $\nabla \phi \# \mu = \nu$, and $\nabla \phi(\mathcal{E}) \subseteq \mathcal{E}$. In addition, according to Alexandrov’s classical result, $\nabla \phi$ is a.e. differentiable, and the push-forward relation $\nabla \phi \# \mu = \nu$ implies the validity of the Monge-Ampère equation

$$|u(x)|^p \omega(x) = C_1 e^{-|\nabla \phi(x)|^{p'}} \omega(\nabla \phi(x)) \det(D^2_A \phi(x)) \quad \text{a.e. } x \in E \cap U, \quad (3.10)$$

where $E = \text{supp } \mu$ and $D^2_A \phi$ denotes the Hessian of $\phi$ in the sense of Alexandrov (namely, the absolutely continuous part of the distributional Hessian of $\phi$), see McCann [32, Remark 4.5]; for further discussion, one may consult the monographs by Ambrosio,Brué and Semola [2, Lecture 5], Maggi [29, Chapter 8] and Villani [34, Section 3.3].

In the sequel, we will use several times the change of variables via the Monge-Ampère equation (3.10) which is understood in the sense that $\nabla \phi \# \mu = \nu$, equivalent to

$$\int_E b(\nabla \phi(x)) |u(x)|^p \omega(x) dx = C_1 \int_E b(y) e^{-|y|^{p'}} \omega(y) dy \quad (3.11)$$

for every Borel function $b : E \to \mathbb{R}$.

By taking the logarithm of relation (3.10), and integrating with respect to the measure $|u(x)|^p \omega(x) dx$, we obtain that

$$I := \mathcal{E}_{\omega,E}(|u|^p) = \int_E |u(x)|^p \log(|u(x)|^p) \omega(x) dx = \log C_1 - \int_E |\nabla \phi(x)|^{p'} |u(x)|^p \omega(x) dx + \int_E |u(x)|^p \omega(x) \log \omega(\nabla \phi(x)) dx$$

$$- \int_E |u(x)|^p \omega(x) \log \omega(x) dx + \int_E |u(x)|^p \omega(x) \log \det(D^2_A \phi(x)) dx.$$

The change of variables formula (3.11) applied to the second and third terms (for $b(y) := |y|^{p'}$ and $b(y) := \log \omega(y)$, respectively) gives

$$I = C_3 - \int_E |u(x)|^p \omega(x) \log \omega(x) dx + \int_E |u(x)|^p \omega(x) \log \det(D^2_A \phi(x)) dx.$$

On the other hand, by the weighted AM-GM inequality we have that

$$\det(D^2_A \phi(x))^{\frac{1}{n+\tau}} \leq \frac{\tau}{n+\tau} + \frac{\text{tr} D^2_A \phi(x)}{n+\tau} = \frac{\tau}{n+\tau} + \frac{\Delta A \phi(x)}{n+\tau} \quad \text{for a.e. } x \in E \cap U. \quad (3.12)$$
In order to do this, we first observe that by a change of variables we have that

\[ III := \int_E |u(x)|^p \omega(x) \log(\Delta A \phi(x)) \, dx \]

\[ \leq (n + \tau) \int_E |u(x)|^p \omega(x) \log \left( \frac{\tau}{n + \tau} + \frac{\Delta A \phi(x)}{n + \tau} \right) \, dx \]

\[ \leq (n + \tau) \log \left( \int_E |u(x)|^p \omega(x) \left( \tau + \Delta A \phi(x) \right) \, dx \right) - (n + \tau) \log(n + \tau) \]

\[ = (n + \tau) \log \left( \tau + \int_E |u(x)|^p \omega(x) \Delta A \phi(x) \, dx \right) - (n + \tau) \log(n + \tau). \tag{3.13} \]

We are going to estimate the integral in the first term on the right side of the above expression. In order to do this, we first observe that by a change of variables we have that \( u^{p-1} \nabla \phi \in L^p(\omega; E) \). Accordingly, the integration by parts formula from Proposition 3.1 and Hölder’s inequality imply that

\[ III := \int_E |u(x)|^p \omega(x) \Delta A \phi(x) \, dx \]

\[ \leq -p \int_E |u(x)|^{p-1} \omega(x) \nabla \phi(x) \cdot \nabla |u(x)| \, dx - \int_E |u(x)|^p \omega(x) \cdot \nabla \phi(x) \, dx \]

\[ \leq p \left( \int_E |\nabla u|^p \omega \right)^{\frac{1}{p}} \left( \int_E |u(x)|^p \omega(x) |\nabla \phi(x)|^p \right)^{\frac{1}{p}} - \int_E |u(x)|^p \omega(x) \cdot \nabla \phi(x) \, dx \]

\[ = C_4 \left( \int_E |\nabla u|^p \omega \right)^{\frac{1}{p}} - \int_E |u(x)|^p \omega(x) \cdot \nabla \phi(x) \, dx, \tag{3.14} \]

where we used again the change of variables formula (3.11) and the exact form of \( C_4 > 0 \), respectively.

Since \( \omega \) is log-concave on \( E \), by the fact that \( \nabla \phi(E) \subseteq \overline{E} \) and Proposition 2.1, we have that

\[ \log \left( \frac{\omega(\nabla \phi(x))}{\omega(x)} \right) \leq -\tau + \frac{\nabla \omega(x) \cdot \nabla \phi(x)}{\omega(x)} \text{ for a.e. } x \in E. \tag{3.15} \]

By relations (3.15), (3.10) and (3.9), it follows that

\[ III \leq C_4 \left( \int_E |\nabla u|^p \omega \right)^{\frac{1}{p}} - \int_E |u(x)|^p \omega(x) \cdot \nabla \phi(x) \, dx \]

\[ \leq C_4 \left( \int_E |\nabla u|^p \omega \right)^{\frac{1}{p}} - \tau. \tag{3.16} \]

Summing up relations (3.13) and (3.16) we conclude that

\[ \int_E |u(x)|^p \log(|u(x)|^p) \omega(x) \, dx \leq C_3 - \int_E |u(x)|^p \omega(x) \log \omega(x) \, dx - (n + \tau) \log(n + \tau) \]

\[ + (n + \tau) \log \left( C_4 \left( \int_E |\nabla u|^p \omega \right)^{\frac{1}{p}} \right). \tag{3.17} \]
Taking relation (3.9) again into account, the inequality (3.17) reduces to

\[ \int_E |u(x)|^p \log(|u|^p) \omega(x) dx \leq C_2 - (n + \tau) \log(n + \tau) + \frac{n + \tau}{p} \log \left( C_4^p \int_E |\nabla u|^p \omega \right) \]

\[ = \frac{n + \tau}{p} \log \left( \mathcal{L}_{\omega,p} \int_E |\nabla u|^p \omega dx \right), \]

where by using the values of \( C_2 \) and \( C_4 \) and the fact that \( \omega_{SE} = \int_{S^{n-1}} \omega = (n + \tau) \int_{B \cap E} \omega \), we obtain that

\[ \mathcal{L}_{\omega,p} = \frac{e^{\frac{\mu_c^2}{n + \tau}}}{(n + \tau)^2} C_4^p = \frac{p}{n + \tau} \left( \frac{p - 1}{e} \right)^{p-1} \left( \Gamma \left( \frac{n + \tau}{p} + 1 \right) \int_{B \cap E} \omega \right)^\frac{1}{p}, \]

which is precisely the expected constant, see (1.4).

\[ \square \]

**Remark 3.1.** The proof of (1.4) is substantially simpler for functions belonging to \( C_c^\infty(\mathbb{R}^n) \) (restricted to \( E \)) than for functions in \( W^{1,p}(\omega; E) \); indeed, both Propositions 2.2 and 3.1 as well as relation (3.7) – which are crucial in the proof – trivially hold for functions in \( C_c^\infty(\mathbb{R}^n) \). However, the \( C_c^\infty(\mathbb{R}^n) \)-version is not sufficient to characterize the equality cases in (1.4) as the candidates for the extremal functions do not belong to \( C_c^\infty(\mathbb{R}^n) \); see §3.2.

**Remark 3.2.** As we mentioned in the Introduction, it is possible to give an anisotropic version of Theorem 1.1 w.r.t. a general norm \( \| \cdot \| \). Indeed, if \( \| \cdot \|_* \) is its dual, then in (1.4) the expression of the gradient norm is replaced by \( \int_E \| \nabla u \|_*^p \omega dx \), while the ball \( B \) in the best constant \( \mathcal{L}_{\omega,p} \) is understood w.r.t. the norm \( \| \cdot \| \). In the same way, in the expression of the extremal function (1.5) the Euclidean norm is replaced by the norm \( \| \cdot \| \). The only modification of the proof is in the estimate (3.14), where we use the inequality \( X \cdot Y \leq \| X \|_* \| Y \| \) for every \( X, Y \in \mathbb{R}^n \), combined with the corresponding Hölder inequality; namely

\[ \int_E |u(x)|^{p-1} \nabla u(x) \cdot \nabla \phi(x) \omega(x) dx \leq \int_E |u(x)|^{p-1} \| \nabla u(x) \|_* \| \nabla \phi(x) \| \omega(x) dx \]

\[ \leq \left( \int_E \| \nabla u \|_*^p \omega \right)^\frac{1}{p} \left( \int_E |u|^p \| \nabla \phi \|^{p'} \omega \right)^\frac{1}{p'} . \]

### 3.2. Equality in (1.4): case \( p > 1, \tau > 0 \)

We are going to characterize the equality in (1.4). We assume that there exists a function \( u \in W^{1,p}(\omega; E) \) with equality in (1.4) and \( \int_E u^p \omega dx = 1; \) without loss of generality, we may assume that \( u \geq 0 \). Let \( S \) be the interior of the effective domain of \( \phi \), i.e., \( \{ x \in \mathbb{R}^n : \phi(x) < +\infty \} \). We observe that \( U = \text{supp} u \) is contained in \( E \cap S \).

As a first step we claim that for every compact subset \( K \subset E \cap S \), there exist \( c_K, C_K > 0 \) such that \( c_K \leq u(x) \leq C_K \) for a.e. \( x \in K \). The proof of this property is inspired by [15, Proposition 6]). By tracking back the equality cases in the proof of (1.4), we have – among others – that we should have equality in Hölder’s inequality, see (3.14); in particular, there exists \( M > 0 \) such that

\[ |\nabla u(x)|^p \omega(x) = Mu(x)^p |\nabla \phi(x)|^{p'} \omega(x) \quad \text{for a.e. } x \in E \cap S. \]
For every \( k \geq 1 \), we consider the function \( u_k(x) = \max\{\frac{1}{k}, u(x)\} > 0 \), \( x \in E \cap S \). Note that \( \nabla u_k \in L^p(\omega; E \cap S) \). In fact, we have for every \( k \geq 1 \) and a.e. \( x \in E \cap S \) that
\[
|\nabla u_k(x)|^p \leq |\nabla u(x)|^p = M u(x)^p |\nabla \phi(x)|^p' \leq M u_k(x)^p |\nabla \phi(x)|^p' .
\]
This estimate implies that for every \( k \geq 1 \) one has
\[
|\nabla \log u_k(x)|^p \leq M |\nabla \phi(x)|^p' \text{ for a.e. } x \in E \cap S .
\]
Taking into account that \( \phi \) is convex, \( |\nabla \phi| \) is locally bounded on \( E \cap S \). Thus, the functions \( \log u_k \) are uniformly locally Lipschitz on \( E \cap S \) with respect to \( k \); in particular, \( \log u_k \) are also locally bounded on \( E \cap S \), i.e., there are \( c_0^k, C_0^k \in \mathbb{R} \) such that
\[
c_0^k \leq \log u_k(x) \leq C_0^k, \; \forall k \geq 1 \text{ and for a.e. } x \in K .
\]
Letting \( k \to \infty \), it turns out that
\[
c_K \leq u(x) \leq C_K \text{ for a.e. } x \in K , \tag{3.18}
\]
where \( c_K = e^{c_0^0} > 0 \) and \( C_K = e^{C_0^0} > 0 \) which concludes the claim. In particular, we have that \( U = \text{supp } = E \cap S \).

Similarly to Cordero-Erausquin, Nazaret and Villani [15] (see also Nguyen [33]), we prove that \( \Delta_s \phi \) vanishes, where \( \Delta_s \phi \) stands for the singular part of the distributional Laplacian \( \Delta_{D'} \phi \). Note that \( \Delta_s \phi \) is a non-negative measure and \( \Delta_{D'} \phi = \Delta_A \phi + \Delta_s \phi \). Since we should have equality in (3.2), repeating the approximation argument of Proposition 3.1 for the function \( u \), we necessarily have that
\[
\lim_{k \to \infty} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \langle (u_{k, \varepsilon}^\delta)^p \omega, \Delta_s \phi \rangle_{D'} = 0 . \tag{3.19}
\]

Fix a convex and compact set \( K \subset E \cap S \) containing \( \tilde{x} \) in its interior, and define \( d_K := d(K, (E \cap S)^c) > 0 \). The set \( K' := \{ x \in E \cap S : d(x, K) \leq d_K/2 \} \) is a convex compact subset of \( E \cap S \) which contains \( K \). Let \( k \geq \frac{1}{d(K', \partial (E \cap S))} \) and \( \varepsilon > 0 \) be small enough such that \( \frac{K}{1 - \varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \tilde{x} \subset K' \) and \( \varepsilon < \frac{1}{\max_{x \in K} |x|} \). By using (3.18) and the fact that \( \theta_k(x) = 1 \) for \( d(x, \partial E) \geq \frac{1}{k} \), it turns out that \( u_{k, \varepsilon} \geq c_{K'} \) a.e. on \( K' \). In particular, if \( 0 < \delta < d_K/2 \), then by the definition of \( K' \) we have \( K + \delta B(0, 1) \subset K' \) and consequently, a change of variables implies that for every \( x \in K \), we have
\[
u_{k, \varepsilon}(x) = \int_E u_{k, \varepsilon}(y) \kappa_\delta(x - y) dy = \frac{1}{\delta^n} \int_{|x - y| < \delta} u_{k, \varepsilon}(y) \kappa \left( \frac{x - y}{\delta} \right) dy
\]
\[
= \int_{B(0, 1)} u_{k, \varepsilon}(x - \delta z) \kappa(z) d z
\]
\[
\geq c_{K'} \int_{B(0, 1)} \kappa(z) d z
\]
\[
= c_{K'} .
\]
Therefore, for such values of \( \delta \) and \( \varepsilon \) we have
\[
\langle (u_{k, \varepsilon}^\delta)^p \omega, \Delta_s \phi \rangle_{D'} \geq c_{K'}^k \omega_K \Delta_s \phi[K] \geq 0 ,
\]
where \( \omega_K = \min_K \omega > 0 \). The latter estimate together with (3.19) imply that \( \Delta_s \phi[K] = 0 \).
By the arbitrariness of \( K \), \( \Delta_s \phi \) should vanish. In particular, \( \nabla \phi \in W^{1,1}(E) \).
The equality in (3.12) implies that $D^2_0\phi(x) = I_n$ for a.e. $x \in E$. Since $\Delta_\omega \phi = 0$ we can apply Figalli, Maggi and Pratelli [24, Lemma A.2] and find $x_0 \in \overline{E}$ such that

$$\nabla \phi(x) = x + x_0 \quad \text{for a.e. } x \in E. \tag{3.20}$$

In particular, $\overline{E} + x_0 \subseteq \overline{E}$. Moreover, the equality in (3.15) with (3.20) and the smoothness of $\omega$ implies that

$$\log \left( \frac{\omega(x + x_0)}{\omega(x)} \right) = \frac{\nabla \omega(x) \cdot x_0}{\omega(x)}, \quad x \in E. \tag{3.21}$$

If we assume by contradiction that $x_0 \in E$, then the latter relation with $x := x_0$ gives that $\tau \log 2 = \tau$, a contradiction to $\tau > 0$. Thus $x_0 \in \partial E$.

Moreover, if we consider for every $x \in E$ the function $r_x : E \to \mathbb{R}$ defined by

$$r_x(y) = \log \left( \frac{\omega(y)}{\omega(x)} \right) + \tau - \frac{\nabla \omega(x) \cdot y}{\omega(x)}, \quad y \in E,$$

then the log-convavity of $\omega$ on $E$ is equivalent to $r_x(y) \leq 0$ for every $y \in E$. Moreover, (3.21) implies that $r_x(x + x_0) = 0$, thus $y := x + x_0 \in E$ is a maximum point of $r_x$; in particular, $\nabla r_x|_{y=x+x_0} = 0$, i.e.,

$$\frac{\nabla \omega(x + x_0)}{\omega(x + x_0)} - \frac{\nabla \omega(x)}{\omega(x)} = 0, \quad x \in E.$$

Therefore, there exists $c > 0$ such that $\omega(x + x_0) = c \omega(x)$ for every $x \in E$.

We are going to prove that $c = 1$. First, by (3.21) we have that $\omega(x) \log c = \nabla \omega(x) \cdot x_0$ for every $x \in E$. Differentiating $\omega(x + x_0) = c \omega(x)$, the homogeneity of $\omega$ implies that for every $x \in E$ we have

$$c \tau \omega(x) = \tau \omega(x + x_0) = \nabla \omega(x + x_0) \cdot (x + x_0) = c \nabla \omega(x) \cdot (x + x_0)$$

$$= c \tau \omega(x) + c \omega(x) \log c, \tag{3.22}$$

i.e., we necessarily have that $c = 1$. Thus, $\omega(x + x_0) = \omega(x)$ for every $x \in E$. In particular, by the Monge-Ampère equation (3.10) and (3.20) one has that

$$u(x)^p = C_1 e^{-\frac{|x + x_0|^p}{p}}, \quad x \in E,$$

i.e., the extremal function $u$ in (1.4) is necessarily a Gaussian of the type (1.5) with the usual normalization property.

We now prove that $x_0 \in (-\partial E) \cap (\partial E)$. Notice that since $\nabla \phi(x) = x + x_0$ for a.e. $x \in E$, we can assume without loss of generality that $\nabla \phi(x) = x + x_0$ for every $x \in E$. Since $\nabla \phi$ is bijective (see [34, Theorem 2.12(iv)]), we have $E + x_0 = E$. In particular, $x_0 \in (-\overline{E}) \cap \overline{E}$.

Conversely, if $u = u_{\lambda, x_0}$ from (1.5), by a change of variables and using the fact that $\omega(x + x_0) = \omega(x)$ for every $x \in E$, a simple computation shows that equality holds in (1.4). \qed

3.3. Proof of the inequality (1.4): case $p > 1$, $\tau = 0$. Since $\tau = 0$, Proposition 2.1 implies that $\omega$ is constant in $E$, i.e. $\omega(x) = \omega_0 > 0$ for every $x \in E$; moreover, without loss of generality, a simple computation shows that $\omega$ by choosing $v := \omega_0^{1/p}$ in the original inequality (1.4) – we may consider $\omega_0 = 1$. Accordingly, the proof performed in §3.1 becomes even simpler, e.g. the validity of (3.7) is trivial. Furthermore, it turns out that $C_2 = C_3$ (as $\omega_0 = 1$) which simplifies the scaling argument, see (3.9), obtaining directly the inequality (1.4). \qed
3.4. Equality in (1.4): case $p > 1$, $\tau = 0$. Analogously to §3.2, the equality in (1.4) implies that $\Delta_c \phi$ vanishes, thus $\nabla \phi \in W^{1,1}(E)$. The equality in (3.12) (with $\tau = 0$) implies that for some $c_0 > 0$ we have $D^2 \phi(x) = c_0 I_n$ for a.e. $x \in E$. In particular, there exists $x_0 \in \overline{E}$ such that

$$\nabla \phi(x) = c_0 x + x_0 \quad \text{for a.e. } x \in E.$$  

(3.23)

Thus, by the Monge-Ampère equation (3.10) and (3.23), since the weight is constant, we have that

$$u(x)^p = C_1 c_0^n e^{-|c_0 x + x_0|^p}, \quad x \in E,$$

i.e., the only class of extremal functions (up to scaling) is provided by the Gaussians (1.5). Arguing as in the case $\tau > 0$, we can assume without loss of generality that $\nabla \phi(x) = c_0 x + x_0$ for every $x \in E$. Since $\nabla \phi$ is bijective, $c_0 E + x_0 = E$. Since $0 \in \overline{E}$, we have that $-x_0/c_0 \in \overline{E}$, whence $x_0 \in (\overline{E}) \cap \overline{E}$, as required. \hfill $\square$

4. Proof of Theorem 1.2: the case $p = 1$

The proof of Theorem 1.2 requires the counterpart of Proposition 3.1 in the case $p = 1$, i.e., an integration by parts formula/inequality for functions belonging to $BV(\omega; E)$.

**Proposition 4.1.** Let $E \subseteq \mathbb{R}^n$ be an open convex cone, $u \in BV(\omega; E)$ be non-negative and let $\phi: \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $|\nabla \phi(x)| \leq 1$ and $\nabla \phi(x) \in E$ for a.e. $x \in E$. Then we have

$$\int_E u \omega \Delta_A \phi \, dx \leq \|Du\|_{\omega}(E) - \int_E u \nabla \omega \cdot \nabla \phi \, dx.$$  

(4.1)

**Proof.** If $u \in L^1(\omega; E) \cap C^\infty_c(\mathbb{R}^n)$, then the divergence theorem yields

$$\int_E u \omega \Delta_A \phi \, dx \leq \int_E u \omega \Delta_{\partial E} \phi \, dx \quad (4.2)$$

and

$$= -\int_E \nabla \phi \cdot \nabla u \omega \, dx - \int_E u \nabla \omega \cdot \nabla \phi \, dx + \int_{\partial E} u \omega \tilde{\nabla} \phi \cdot d\mathcal{H}^{n-1}, \quad (4.3)$$

where $\tilde{\nabla} \phi$ denotes the trace of $\nabla \phi$ on $\partial E$, which is well-defined up to $\mathcal{H}^{n-1}$-null sets. One can verify that since $\nabla \phi(x) \in E$ for a.e. $x \in \overline{E}$, then $\tilde{\nabla} \phi \cdot n \leq 0$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial E$, whence

$$\int_{\partial E} u \omega \tilde{\nabla} \phi \cdot d\mathcal{H}^{n-1} \leq 0.$$  

(4.4)

On the other hand, since $|\nabla \phi| \leq 1$ a.e. on $E$, one has

$$-\int_E \nabla \phi \cdot \nabla u \omega \, dx \leq \|Du\|_{\omega}(E).$$  

(4.5)

Combining (4.4), (4.5) and (4.2) we obtain (4.1) for $u \in L^1(\omega; E) \cap C^\infty_c(\mathbb{R}^n)$.

Assume now $u \in BV(\omega; E)$. If we have a family of $u_\varepsilon \in C^\infty(\overline{E}) \cap BV(\omega; E)$ such that

$$\lim_{\varepsilon \to 0} \int_E |u_\varepsilon - u| \omega \, dx = 0,$$

$$\lim_{\varepsilon \to 0} \|Du_\varepsilon\|_{\omega}(E) = \|Du\|_{\omega}(E),$$  

(4.6)
the proof is concluded. Indeed, by the previous part we have that
\[
\int_E u_\varepsilon \omega \Delta_A \phi \, dx + \int_E u_\varepsilon \nabla \omega \cdot \nabla \phi \, dx \leq \|Du_\varepsilon\|_{\omega}(E),
\]
(4.7)
and taking the limit of (4.7) as \(\varepsilon \to 0\), the approximations from (4.6) together with Fatou’s lemma imply the required inequality (4.1).

In the sequel we shall prove (4.6) on a generic open set \(\Omega \subseteq E\), i.e., we provide the existence of \(u_\varepsilon \in C^\infty(\Omega) \cap BV(\omega; \Omega), \varepsilon \in (0, 1)\), such that the approximation properties
\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\Omega} |u_\varepsilon - u| \omega \, dx &= 0, \\
\lim_{\varepsilon \to 0} \|Du_\varepsilon\|_{\omega}(\Omega) &= \|Du\|_{\omega}(\Omega)
\end{align*}
\]
(4.8)
hold. In fact, this property is well-known by Bellettini, Bouchitté and Fragalà [8, Theorem 5.1] for general finite Radon measures on \(\mathbb{R}^n\) and one can extend by a standard diagonal argument to \(\sigma\)-finite measures. We however need to have an explicit expression for the approximating sequence, hence we give a direct proof of the result by adapting essentially the argument of Evans and Gariepy [20, Theorem 5.3].

To do this, fix \(\varepsilon > 0\). For any \(m, k \in \mathbb{N}\), we define
\[
U_k = \left\{ x \in \Omega : d(x, \partial \Omega) > \frac{1}{m+k} \right\} \cap B(0, m+k),
\]
and fix \(m \in \mathbb{N}\) so large that \(\|Du\|_{\omega}(\Omega \setminus U_1) < \varepsilon\). Observe that \(U_k\) has compact closure and \(d(U_k, \partial \Omega) > 0\). By setting \(U_0 := \emptyset\), we introduce the open sets \(V_k\) by letting
\[
V_k := U_{k+1} \setminus \overline{U}_{k-1}, \quad k \geq 1.
\]
Clearly, \(\bigcup_{k \geq 1} V_k = \Omega\). According to the partition of unity we can find a family \(\zeta_k \in C^\infty_c(V_k)\) with \(0 \leq \zeta_k \leq 1\) and \(\sum_{k=1}^{\infty} \zeta_k \equiv 1\). By the construction of \(V_k\)’s, the sum above evaluated at any point has only at most three non-zero elements. Consider a symmetric mollifier \(\eta \in C^\infty_c(B(0, 1))\) such that \(\int \eta \, dx = 1\) and define \(\eta_\delta(x) := \delta^{-n} \eta(x/\delta)\). Denote the convolution between the kernel \(\eta_\delta\) and a function \(f \in L^1_{\text{loc}}(\Omega)\) with
\[
\eta_\delta \ast f(x) = \int \eta_\delta(x-y)f(y) \, dy.
\]
It is classical that if \(f \in L^p(\Omega)\), then \(\eta_\delta \ast f \in L^p(\Omega)\) and
\[
\lim_{\delta \to 0} \int_{\Omega} |(\eta_\delta \ast f) - f|^p \, dx = 0.
\]
(4.9)
Moreover \(\eta_\delta \ast f \in C^\infty\) and if \(f \in C^1\), one has
\[
\partial_j (\eta_\delta \ast f) = \eta_\delta \ast (\partial_j f).
\]
(4.10)
For any \(k \in \mathbb{N}\), the function \(u\zeta_k\) is supported in \(V_k\), whose closure is compact, hence \(u\zeta_k \in L^1(\Omega)\). This follows by the estimate
\[
\int_{\Omega} |u| \zeta_k \, dx = \int_{V_k} |u| \zeta_k \, dx \leq \frac{1}{\min_{V_k} \omega} \int_{V_k} |u| \omega \, dx < +\infty.
\]
Analogously, $u \nabla \zeta_k$ is compactly supported in $V_k$ and $u \nabla \zeta_k \in L^1(\Omega)$. Fix $X \in C^1_c(\Omega; \mathbb{R}^n)$ such that $|X| \leq 1$. By (4.9) and the fact that $V_k$ is compact and $\omega$ of class $C^1$, we can choose $\varepsilon_k > 0$ such that the following conditions hold:

\[
\begin{cases}
\text{supp}(\eta_{\varepsilon_k} * (u \zeta_k)) \subset V_k \\
\int_{\Omega} |(\eta_{\varepsilon_k} * (u \zeta_k)) - u \zeta_k| \omega dx < \frac{\varepsilon}{2^k} \\
\int_{\Omega} |(\eta_{\varepsilon_k} * (u \nabla \zeta_k)) - u \nabla \zeta_k| \omega dx < \frac{\varepsilon}{2^k}.
\end{cases}
\] (4.11)

Since $\omega X$ is uniformly continuous on $V_k$, we can moreover assume that $\varepsilon_k$ is small enough to ensure

\[
|\zeta_k(\eta_{\varepsilon_k} * \omega X)|(x) \leq \omega(x)|x| + \varepsilon \min_{V_k} \omega \leq \omega(x)(1 + \varepsilon) \quad \text{for all } x \in \Omega. \tag{4.12}
\]

We then define

\[
u_{\varepsilon} := \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (u \zeta_k).
\]

Then $u_{\varepsilon} \in C^\infty(\Omega)$ and by (4.11) and the fact that $u = \sum_{k=1}^{\infty} u \zeta_k$ it holds that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} |u_{\varepsilon} - u| \omega dx = 0.
\]

By the lower semicontinuity of the total variation with respect to $L^1(\omega; \Omega)$-convergence we also have

\[
||D u||(\omega(\Omega)) \leq \liminf_{\varepsilon \to 0} ||D u_{\varepsilon}||(\omega(\Omega)).
\]

To prove the opposite inequality we proceed as follows:

\[
\int_{\Omega} u_{\varepsilon} \text{div}(\omega X) dx = \sum_{k=1}^{\infty} \int_{\Omega} \eta_{\varepsilon_k} * (u \zeta_k) \text{div}(\omega X) dx = \sum_{k=1}^{\infty} \int_{\Omega} (u \zeta_k)(\eta_{\varepsilon_k} * \text{div}(\omega X)) dx
\]

\[
= \sum_{k=1}^{\infty} \int_{\Omega} (\zeta_k) \text{div}(\eta_{\varepsilon_k} * \omega X) dx
\]

\[
= \sum_{k=1}^{\infty} \int_{\Omega} u \text{div}(\zeta_k(\eta_{\varepsilon_k} * \omega X)) dx - \sum_{k=1}^{\infty} \int_{\Omega} u \nabla \zeta_k \cdot (\eta_{\varepsilon_k} * \omega X) dx
\]

\[
= \sum_{k=1}^{\infty} \int_{\Omega} u \text{div}(\zeta_k(\eta_{\varepsilon_k} * \omega X)) dx - \sum_{k=1}^{\infty} \int_{\Omega} X \cdot (\eta_{\varepsilon_k} * u \nabla \zeta_k) \omega dx.
\] (4.13)

Here, we have used (4.10) and the fact that, if $f \in L^1_{\text{loc}}(\Omega)$ and $g \in L^1_{\text{loc}}(\Omega)$ with $(\text{supp}(g))_\delta \subset \Omega$, then

\[
\int_{\Omega} (\eta_\delta * f) g dx = \int_{\Omega} f (\eta_\delta * g) dx.
\]

Taking into account that $\sum_{k=1}^{\infty} \nabla \zeta_k \equiv 0$, the second sum in (4.13) is

\[
I_{\varepsilon} := - \sum_{k=1}^{\infty} \int_{\Omega} X \cdot (\eta_{\varepsilon_k} * u \nabla \zeta_k) \omega dx = - \sum_{k=1}^{\infty} \int_{\Omega} X \cdot [(\eta_{\varepsilon_k} * u \nabla \zeta_k) - u \nabla \zeta_k] \omega dx.
\]
By (4.11), one has $|I_1^ε| < ε$. The first sum of (4.13) can be written as
\[
I_1^ε := \sum_{k=1}^{∞} \int_{Ω} u \text{div}(ζ_k(η_{εk} \ast ωX))dx = \int_{Ω} u \text{div}(ζ_1(η_{ε1} \ast ωX))dx + \sum_{k=2}^{∞} \int_{Ω} u \text{div}(ζ_k(η_{εk} \ast ωX))dx.
\]
Taking into account (4.12) and that $ζ_k(η_{εk} \ast ωX)$ is compactly supported on $V_k$ we can write
\[
|I_1^ε| \leq (1 + ε)∥Du∥_ω(Ω) + (1 + ε)\sum_{k=2}^{∞} ∥Du∥_ω(V_k)
\]
\[
\leq (1 + ε)∥Du∥_ω(Ω) + 3(1 + ε)∥Du∥_ω(Ω \setminus U_1)
\]
\[
\leq (1 + ε)∥Du∥_ω(Ω) + 3ε(1 + ε).
\]
Therefore, we proved that
\[
\int_{Ω} u \text{div}(ωX)dx \leq (1 + ε)∥Du∥_ω(Ω) + 3ε(1 + ε) + ε.
\]
The proof is concluded by the arbitrariness of $X ∈ C^1_c(Ω; ℝ^n)$ with $|X| ≤ 1$, by taking the limit $ε → 0$. □

Remark 4.1. In the proof of Proposition 4.1 is not restrictive to choose $ε_k$ also such that
\[
|(η_{εk} \ast ζ_k)(x) − ζ_k(x)| < \frac{ε}{2k} \quad \text{for all } x ∈ Ω.
\]
This is a consequence of the fact that, if $f$ is uniformly continuous, then the convolution $η_δ \ast f$ uniformly converges to $f$ as $δ → 0$.

4.1. Proof of the inequality (1.6): case $p = 1$, $τ > 0$. The proof is similar to the one presented in §3.1; we only focus on the differences. Let $u ∈ BV(ω; E)$ be such that $\int_{E} |u|ωdx = 1$.

If $u_t(x) = t^αu(λx)$ for $t > 0$, $x ∈ E$, the homogeneity of $ω$ implies that $\int_{E} |u_t|ωdx = 1$ for every $t > 0$ whenever $α = n + τ$. Moreover, since
\[\mathcal{E}_{ω,E}(|u_t|) = \mathcal{E}_{ω,E}(|u|) + α \log t \quad \text{and} \quad ∥D(|u_t|)∥_ω(E) = t∥D(|u|)∥_ω(E), \quad t > 0,\]
one has that $u_t ∈ BV(ω; E)$ and inequality (1.6) holds for some $u ∈ BV(ω; E)$ if and only if it holds for $u_t$. Therefore may choose $t > 0$ such that
\[
\int_{E} |u_t(x)|ω(x) log(ω(x))dx = \int_{E} |u(x)|ω(x) log(ω(x))dx − τ log t = \tilde{C}_1 \int_{E \cap E} ω(x) log(ω(x))dx,
\]
where $\tilde{C}_1 := \left( \int_{E \cap E} ω \right)^{-1}$. Without loss of generality, we will write $u$ in place of $u_t$ and assume
\[
\int_{E} |u(x)|ω(x) log(ω(x))dx = \tilde{C}_1 \int_{E \cap E} ω(x) log(ω(x))dx. \quad (4.14)
\]

Let us consider the probability measures $|u(x)|ω(x)dx$ and $\tilde{C}_1 1_{B \cap E}(y) ω(y)dy$. By the theory of optimal mass transport, we find a convex function $φ: ℝ^n → ℝ$ such that $∇φ(E) ⊆ E$ and the Monge-Ampère equation holds, i.e.,
\[
|u(x)|ω(x) = \tilde{C}_1 1_{B \cap E}(∇φ(x))ω(∇φ(x)) det(D^2φ(x)) \quad \text{for a.e. } x ∈ E \cap U, \quad (4.15)
\]
where $U$ is the support of $u$. In particular, by (4.15) one has that

$$|\nabla \phi(x)| \leq 1 \quad \text{for a.e. } x \in E \cap U.$$ 

The latter bound for $\nabla \phi$ and the integration by parts formula from Proposition 4.1 imply that

$$\tilde{I} := \int_E |u(x)| \omega(x) \Delta_A \phi(x) dx \leq \|D(|u|)\|_\omega(E) - \int_E |u(x)| \nabla \omega(x) \cdot \nabla \phi(x) dx. \quad (4.16)$$

By the log-concavity of $\omega$, see (3.15), similarly to (3.16), the latter inequality implies that

$$\tilde{I} \leq \|D(|u|)\|_\omega(E) - \tau + \int_E |u(x)| \omega(x) \log \omega(x) dx - \tilde{C}_1 \int_{B \cap E} \omega(x) \log \omega(x) dx. \quad (4.17)$$

A similar argument as above, combined with the equation (4.15) shows that

$$\tilde{II} := \mathcal{E}_{\omega,E}(|u|)$$

$$= \int_E |u(x)| \log(|u(x)|) \omega(x) dx$$

$$= \log \tilde{C}_1 - \int_E |u(x)| \omega(x) \log \omega(x) dx + \tilde{C}_1 \int_{B \cap E} \omega(x) \log \omega(x) dx$$

$$+ \int_E |u(x)| \omega(x) \log \det(D^2 \phi(x)) dx.$$ 

This relation together with the AM-GM inequality (3.12), the Jensen inequality and estimate (4.17) imply that

$$\tilde{II} \leq \log \tilde{C}_1 - \int_E |u(x)| \omega(x) \log \omega(x) dx + \tilde{C}_1 \int_{B \cap E} \omega(x) \log \omega(x) dx - (n + \tau) \log(n + \tau)$$

$$+ (n + \tau) \log \left(\|D(|u|)\|_\omega(E) + \int_E |u(x)| \omega(x) \log \omega(x) dx - \tilde{C}_1 \int_{B \cap E} \omega(x) \log \omega(x) dx\right).$$

By (4.14), the latter estimate reduces to

$$\mathcal{E}_{\omega,E}(|u|) \leq \log \tilde{C}_1 - (n + \tau) \log(n + \tau) + (n + \tau) \log \left(\|D(|u|)\|_\omega(E)\right).$$

We observe that this inequality is equivalent to

$$\mathcal{E}_{\omega,E}(|u|) \leq (n + \tau) \log \left(\frac{\tilde{C}_1^{n+\tau}}{n + \tau} \|D(|u|)\|_\omega(E)\right),$$

which is exactly the required relation (1.6). \qed
4.2. Equality in (1.6): case \( p = 1, \tau > 0 \). By the Monge-Ampère equation (4.15) and \( \nabla \phi(\overline{E}) \subseteq \overline{E} \) we obtain that, up to a null-measure set, \( \Omega := \text{int}(E \cap U) \) coincides with the set \( \{ x \in E : |\nabla \phi(x)| < 1 \} \). In particular, \( u(x) = 0 \) for a.e. \( x \in E \setminus \Omega \). On the other hand, we are going to prove that

\[
|u(x)| \geq \tilde{C}_1 e^{-\tau} \text{ for a.e. } x \in \Omega. \tag{4.18}
\]

To do this, the equality in (1.6) implies that equalities should occur both in the AM-GM inequality (3.12) and (3.15); namely, \( D_A^2 \phi(x) = I_n \) for a.e. \( x \in E \) and

\[
\log \left( \frac{\omega(\nabla \phi(x))}{\omega(x)} \right) = -\tau + \frac{\nabla \omega(x) \cdot \nabla \phi(x)}{\omega(x)} \text{ for a.e. } x \in E. \tag{4.19}
\]

In particular, the former relation implies that \( \det(D_A^2 \phi(x)) = 1 \) for a.e. \( x \in E \), thus the Monge-Ampère equation (4.15) reduces to

\[
|u(x)|\omega(x) = \tilde{C}_1 \omega(\nabla \phi(x)) \text{ for a.e. } x \in \Omega. \tag{4.20}
\]

Moreover, since \( \nabla \phi(\overline{E}) \subseteq \overline{E} \), Proposition 2.1 implies that \( \nabla \omega(x) \cdot \nabla \phi(x) \geq 0 \) for a.e. \( x \in E \); thus, combining this inequality with (4.19) and (4.20), the requested inequality (4.18) immediately follows.

In the sequel, we prove that \( \Delta_x \phi[K] = 0 \) for every compact set \( K \subseteq \Omega \). The equality in (1.6) also implies that

\[
\lim_{\varepsilon \to 0} \langle |u|_{\varepsilon} \omega, \Delta_x \phi \rangle_{D'} = 0, \tag{4.21}
\]

see inequality (4.1), where \((|u|)_{\varepsilon} \in BV(\omega; E) \cap C^\infty_c(\Omega)\) is the sequence approximating \(|u|\) described in the proof of Proposition 4.1, namely

\[
|u|_{\varepsilon} = \sum_{k=1}^\infty \eta_{k,\varepsilon} * (|u|\zeta_k),
\]

where \( \zeta_k \in C^\infty_c(V_k) \) are relative to a partition of unity associated to the family of open sets \( V_k \) and, according to Remark 4.1, \( \varepsilon_k \) is such that

\[
|(\eta_{k,\varepsilon} \ast \zeta_k)(x) - \zeta_k(x)| < \frac{\varepsilon}{2k}, \quad \forall x \in \Omega. \tag{4.22}
\]

Let \( K \subseteq \Omega \) be a compact set; we claim that

\[
|u|_{\varepsilon}(x) \geq \frac{\tilde{C}_1}{2} e^{-\tau}, \quad \forall x \in K, \forall \varepsilon \in (0, 1/2). \tag{4.23}
\]

In fact, by (4.22), for every \( x \in K \) we have

\[
|u|_{\varepsilon}(x) = \sum_{k=1}^\infty \eta_{k,\varepsilon} \ast (|u|\zeta_k) = \sum_{k=1}^\infty \int_{\Omega} \eta_{k,\varepsilon}(x-y)|u|(y)\zeta_k(y)dy \geq \tilde{C}_1 e^{-\tau} \sum_{k=1}^\infty (\eta_{k,\varepsilon} \ast \zeta_k)(x) \geq \tilde{C}_1 e^{-\tau} \sum_{k=1}^\infty \left( \zeta_k(x) - \frac{\varepsilon}{2k} \right) = \tilde{C}_1 e^{-\tau} \left( 1 - \varepsilon \right).
\]
Finally, since \( \Omega \subseteq E \), we have that \( K \cap \partial E = \emptyset \) and \( \omega_K := \min K \omega > 0 \). Therefore, by (4.21) and (4.23), it yields that

\[
0 = \lim_{\epsilon \to 0} \langle |u| \omega, \Delta_s \phi \rangle_D \geq \frac{\tilde{C}_1}{2} e^{-\tau \omega_K \Delta_s \phi[K]}.
\]

Note that \( \Delta_s \phi[K] \geq 0 \); thus, we necessarily have that \( \Delta_s \phi[K] = 0 \), which ends the proof.

In particular, we have proved that \( \phi \in W^{2,1}(\Omega) \). We aim to prove that \( u \) is constant a.e. on \( \Omega \). Since \( D^2 \phi = I_n \) a.e. on \( E \), there exist a family of connected open sets \( \Omega_i \subset E \) and points \( x_0^i \in \mathbb{R}^n, i \in I \), such that \( \Omega = \bigcup_{i \in I} \Omega_i \) and

\[
\nabla \phi(x) = x + x_0^i \quad \text{for a.e. } x \in \Omega_i.
\]

Notice that by the Monge-Ampère equation, up to a null-measure set, \( \Omega_i \subseteq \{ x \in E : x + x_0^i \in E, |x + x_0^i| < 1 \} \), so that

\[
\Omega_i \subseteq (B(0, 1) \cap E - x_0^i) \cap E.
\]

In particular \( \mathcal{L}^n(\partial \Omega) = 0 \). We can now localize (4.19) on \( \Omega_i \) and write

\[
\log \left( \frac{\omega(x + x_0^i)}{\omega(x)} \right) = -\tau + \frac{\nabla \omega(x) \cdot (x + x_0^i)}{\omega(x)} \quad \text{for a.e. } x \in \Omega_i.
\]

Arguing as in the case \( p > 1 \), we can find \( c_i > 0 \) such that

\[
\omega(x + x_0^i) = c_i \omega(x), \quad \forall x \in \Omega_i;
\]

in fact, it turns out that \( c_i = 1 \) for every \( i \in I \). Thus, the Monge-Ampère equation (4.15) implies that for every \( i \in I \),

\[
|u|(x) = \tilde{C}_1 \mathbb{1}_{B \cap E}(x + x_0^i) \quad \text{for a.e. } x \in \Omega_i.
\]

We now exploit the equality in (4.1). In particular, we have

\[
n \int_{\Omega_i} \tilde{C}_1 \omega(x) dx + \sum_{i \in I} \int_{\Omega_i} \tilde{C}_1 \nabla \omega(x) \cdot (x + x_0^i) dx = \tilde{C}_1 \sum_{i \in I} P_\omega(\Omega_i; E),
\]

where \( P_\omega(\Omega; E) \) stands for the weighted perimeter of \( \Omega \) relative to the cone \( E \); see e.g. Cabré, Ros-Oton and Serra [11, p. 2977]; indeed, one has that \( \|D \mathbb{1}_\Omega\|_{\omega}(E) = P_\omega(\Omega; E) \). Taking into account that \( \nabla \omega(x) \cdot x_0^i = 0 \) for every \( x \in E \) and \( i \in I \) (see (4.24)), the latter relation implies that

\[
(n + \tau) \int_{\Omega} \omega(x) dx = \sum_{i \in I} P_\omega(\Omega_i; E).
\]

We now prove that \( \Omega \) has just one connected component. Since \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \), then

\[
\sum_{i \in I} P_\omega(\Omega_i; E) = P_\omega(\Omega; E)
\]

and – since the optimal mass transport map \( \nabla \phi \) is essentially one-to-one (see [34, Theorem 2.12(iv)]) – we have that

\[
\mathcal{L}^n((\Omega_i + x_0^i) \cap (\Omega_j + x_0^j)) = 0, \quad i \neq j.
\]
In particular, up to null-measure sets, we have that $\bigcup_{i \in I} (\Omega_i + x_i^0) = B \cap E$. By (4.25) with $c_i = 1$ we obtain that
\[
\int_{\Omega_i} \omega(x)dx = \int_{\Omega_i} \omega(x + x_i^0)dx = \int_{\Omega_i + x_i^0} \omega(x)dx, \quad i \in I.
\]
Combining the previous fact with (4.28), it follows that
\[
\int_{\Omega} \omega(x)dx = \int_{B \cap E} \omega(x)dx.
\]
Consequently, by using (4.29), the equation (4.27) can be written into the equivalent form
\[
(n + \tau) \left( \int_{B \cap E} \omega(x)dx \right)^{1 - \frac{1}{n+\tau}} \left( \int_{\Omega} \omega(x)dx \right)^{\frac{1}{n+\tau}} = P_{\omega}(\Omega; E).
\]
The latter relation means that we have equality in the weighted isoperimetric inequality, and since $\omega^{\frac{1}{n+\tau}}$ is concave (see Proposition 2.1), we are within the setting of Cinti, Glaudo, Pratelli, Ros-Oton and Serra [12]; namely, $\Omega$ has the form $\Omega = (B \cap E) - x_0$ for some $x_0 \in \partial E \cap (-\partial E)$. In particular, $\Omega$ has only one connected component.

An alternative, self-contained proof of this fact can be also obtained by observing – similarly as above by $c_i = 1$ and relation (4.25) – that
\[
P_{\omega}(\Omega_i; E) = P_{\omega}(\Omega_i + x_i^0; E), \quad i \in I.
\]
By a radial change of coordinates we also have
\[
(n + \tau) \int_{B(0,1) \cap E} \omega(x)dx = \int_{\mathbb{R}^{n-1}} \omega d\mathcal{H}^{n-1} = P_{\omega}(B \cap E; E).
\]
Combining (4.31), (4.30) and (4.29) with (4.27) we get
\[
\sum_{i \in I} P_{\omega}(\Omega_i + x_i^0; E) = \sum_{i \in I} P_{\omega}(\Omega_i; E) = P_{\omega}(\Omega; E) = (n + \tau) \int_{\Omega} \omega dx
\]
\[
= (n + \tau) \int_{B \cap E} \omega dx = P_{\omega}(B \cap E; E).
\]
If, by contradiction we would have $\# I \geq 2$, then the perimeter of $B \cap E$ would equal the sum of the perimeters of some of its open subsets, contradicting the connectedness of $B \cap E$. Therefore, there is $x_0 \in \mathbb{R}^n$ such that $\Omega = ((B \cap E) - x_0) \cap E$, as before.

Notice that we can replace $\nabla \phi$ with a representative such that $\nabla \phi(x) = x + x_0$ for every $x \in \Omega$. Moreover, since $\nabla \phi(\Omega) = B \cap E$, we also have that $\Omega = B \cap E - x_0$. Thus, by the fact that $B \cap E - x_0 = \Omega \subset E$, we have $-x_0 \in \overline{E}$ and relation (4.25) implies
\[
\omega(x + x_0) = \omega(x), \quad \forall x \in B \cap E - x_0.
\]
Assume by contradiction that $-x_0 \in E$. Then, for every $t \in (0,1/|x_0|)$, one has that $-tx_0 \in B \cap E$. Applying (4.33) to $x = -(t+1)x_0 \in B \cap E - x_0$ yields $t^{\tau} \omega(-x_0) = (t+1)^{\tau} \omega(-x_0)$, a contradiction. Hence $-x_0 \in \partial E$. Since $\nabla \phi(B \cap E - x_0) = B \cap E$, we also have $x_0 \in \overline{E}$. Since $-x_0 \notin E$ and $E$ is convex, it cannot be $x_0 \in E$. In particular $x_0 \in \partial E \cap (-\partial E)$, as required.

Relation (4.26) implies that
\[
|u|(x) = \tilde{C}_1 1_{B \cap E}(x + x_0) \quad \text{for a.e. } x \in B \cap E - x_0,
\]
thus a change of variables and a scaling argument provide the general form of the extremal functions (1.7). In fact, it is enough to observe that the equality in inequality (1.2) holds for \( u \) if and only if it holds for \( x \mapsto \lambda^{n+\tau} u(\lambda x) \), \( \lambda > 0 \).

4.3. Proof of the inequality (1.6): case \( p = 1, \tau = 0 \). Since \( \omega \) is constant in \( E \), see Proposition 2.1, the proof from §4.1 can be easily adapted to the present setting.

4.4. Equality in (1.6): case \( p = 1, \tau = 0 \). The proof works exactly as in §4.2 with some minor changes. In this case, the bijectivity of \( \nabla \phi \) only gives \( x_0 \in (-\overline{E}) \cap \overline{E} \). Alternatively, we may obtain the same result by Figalli, Maggi and Pratelli [25].

5. Application: sharp weighted hypercontractivity

We recall that for a given function \( g : E \to \mathbb{R} \), the Hopf-Lax formula has the expression

\[
Q_t g(x) := Q^p_t g(x) = \inf_{y \in E} \left\{ g(y) + \frac{|y - x|^{p'}}{p' p^{p'-1}} \right\}, \tag{5.1}
\]

where \( E \subseteq \mathbb{R}^n \) is an open convex cone and \( p > 1 \); we assume that \( t > 0 \) is fixed such that \( Q_t g(x) > -\infty \) for all \( x \in E \).

We recall that in the classical literature (see e.g. Evans [19]) it is assumed that \( g \) is bounded Lipschitz and as a consequence one obtains that the function \( (x,t) \mapsto Q_t(g)(x) \) is well-defined for all \( (x,t) \in E \times (0,\infty) \). Moreover, this function is locally Lipschitz and for a.e. \( (x,t) \in E \times (0,\infty) \) the Hamilton-Jacobi equation holds:

\[
\frac{\partial}{\partial t} Q_t g(x) + \frac{|\nabla Q_t g(x)|^p}{p} = 0. \tag{5.2}
\]

For our purposes the space of bounded Lipschitz functions does not work well. Indeed, observe that if \( g \) is bounded and Lipschitz, then \( \|e^g\|_{L^p(\omega;E)} = +\infty \) and thus relation (1.9) becomes trivial.

In order to handle this issue we shall consider the family of functions already considered in the Introduction of the paper: for some \( t_0 > 0 \), let

\[
\mathcal{F}_{t_0}(E) := \left\{ g : E \to \mathbb{R} : g \text{ is measurable, bounded from above and there exists } x_0 \in E \text{ such that } Q_{t_0} g(x_0) > -\infty \right\}.
\]

Our first result shows that if \( t \in (0,t_0) \) then \( (x,t) \mapsto Q_t g(x) \) is well-defined, locally Lipschitz, satisfies the Hamilton-Jacobi equation almost everywhere and also has strong global integrability properties:

**Proposition 5.1.** Let \( p > 1, t_0 > 0, E \subseteq \mathbb{R}^n \) be an open convex cone and \( g \in \mathcal{F}_{t_0}(E) \). Then the following statements hold.

(i) For every \( (x,t) \in E \times (0,t_0) \) one has \( Q_t g(x) > -\infty \) and for a.e. \( (x,t) \in E \times (0,t_0) \) the Hamilton-Jacobi equation (5.2) holds.

(ii) If \( \omega : E \to (0,\infty) \) is a homogeneous function and \( 0 < \alpha_1 \leq \alpha_2 \) with \( e^g \in L^{\alpha_1 p}(\omega;E) \cap L^{\alpha_2 p}(\omega;E) \) then for a.e. \( t \in (0,t_0) \) the function \( x \mapsto e^{q(t)} Q_t g(x) \) belongs to \( W^{1,p}(\omega;E) \), where \( q : [0,t_0] \to [\alpha_1,\alpha_2] \) is any function.
Proof. (i) Let $K \subset E$ be any compact set and $0 < t_1 < t_2 < t_0$. We are going to prove that the function $\mathcal{Q}_{t_0}g(x)$, introduced in (5.1), is well-defined and Lipschitz continuous on $K \times [t_1, t_2]$. Since $t_2 < t_0$ then for every $t \in [t_1, t_2]$, we have the coercivity property

$$
\lim_{y \in E, |y| \to \infty} \left\{ \frac{|y - x|^p}{p't_0^{p'}} - \frac{|y - x_0|^p}{p't_0^{p'}} \right\} = +\infty,
$$

uniformly in $x \in K$. Consequently, the latter coercivity property and the assumption $\mathcal{Q}_{t_0}g(x_0) > -\infty$ imply that for every $(x, t) \in K \times [t_1, t_2]$ one has

$$
\mathcal{Q}_{t_0}g(x) = \inf_{y \in E} \left\{ g(y) + \frac{|y - x|^p}{p't_0^{p'}} \right\} 
\geq \inf_{y \in E} \left\{ g(y) + \frac{|y - x_0|^p}{p't_0^{p'}} \right\} + \inf_{y \in E} \left\{ \frac{|y - x|^p}{p't_0^{p'}} - \frac{|y - x_0|^p}{p't_0^{p'}} \right\} 
= \mathcal{Q}_{t_0}g(x_0) + \inf_{y \in E} \left\{ \frac{|y - x|^p}{p't_0^{p'}} - \frac{|y - x_0|^p}{p't_0^{p'}} \right\} > -\infty.
$$

In addition, we can find $R > 0$ such that for every $t \in [t_1, t_2]$, we have

$$
\mathcal{Q}_{t_0}g(x) = \inf_{y \in E \cap B(0, R)} \left\{ g(y) + \frac{|y - x|^p}{p't_0^{p'}} \right\}.
$$

Note that the function $(x, t) \mapsto g(y) + \frac{|y - x|^p}{p't_0^{p'}}$, $(x, t) \in K \times [t_1, t_2]$, is uniformly Lipschitz in $y$, thus $(x, t) \mapsto \mathcal{Q}_{t_0}g(x)$ is also Lipschitz, being the infimum of a family of Lipschitz functions. Thus, by Rademacher’s theorem it follows that $(x, t) \mapsto \mathcal{Q}_{t_0}g(x)$ is differentiable a.e. in $K \times [t_1, t_2]$. In particular, by the arbitrariness of $K$ and $t_1, t_2 \in (0, t_0)$, it turns out that $(x, t) \mapsto \mathcal{Q}_{t_0}g(x)$ is differentiable a.e. on $\mathbb{R}^n \times (0, t_0)$. Now, we can apply the proof of [19, Theorem 5, p. 128] to check that in the differentiable points of $(x, t) \mapsto \mathcal{Q}_{t_0}g(x)$, the Hamilton-Jacobi equation (5.2) holds. In fact, the quoted result is proved on the whole $\mathbb{R}^n$, for bounded Lipschitz initial datum $g$, but it can be easily adapted to $E \subset \mathbb{R}^n$ and $g \in \mathcal{F}_{t_0}(E)$.

(ii) Let $0 < \alpha_1 \leq \alpha_2$ be two numbers with $e^g \in L^{\alpha_1p}(\omega; E) \cap L^{\alpha_2p}(\omega; E)$ and any function $q : [0, t_0] \to [\alpha_1, \alpha_2]$. We first claim that $e^q(t)\mathcal{Q}_{t_0}g \in L^p(\omega; E)$ for every $t \in (0, t_0)$. To do this, we observe by definition that $\mathcal{Q}_{t_0}g \leq g$ for every $t \in (0, t_0)$. Consequently, since $q(t) \in [\alpha_1, \alpha_2]$ for every $t \in [0, t_0]$, we have for every $t \in (0, t_0)$ and $x \in E$ that

$$
0 \leq e^{pq(t)}\mathcal{Q}_{t_0}g(x) \leq e^{p\max\{\alpha_1, \alpha_2\} \mathcal{Q}_{t_0}g(x)} = \max\{e^{p\alpha_1 \mathcal{Q}_{t_0}g(x)}, e^{p\alpha_2 \mathcal{Q}_{t_0}g(x)}\} \leq \max\{e^{p\alpha_1g(x)}, e^{p\alpha_2g(x)}\}.
$$

Since by assumption one has $e^g \in L^{\alpha_1p}(\omega; E) \cap L^{\alpha_2p}(\omega; E)$, the claim directly follows by the latter estimate.

Now, for every $t \in (0, t_0)$ we consider the functions

$$
H_i(t) = \int_E e^{p\alpha_i \mathcal{Q}_{t_0}g(x)} \omega(x) dx, \quad i = 1, 2.
$$

By the previous step, $H_i$ is well-defined on $(0, t_0)$; moreover, since $t \mapsto \mathcal{Q}_{t_0}g(x)$ is non-increasing for every $x \in E$, the same holds for $H_i$ as well, $i = 1, 2$. In particular, $H_i$ is differentiable a.e.
on $(0,t_0)$ and $-\infty < \frac{d}{dt} H_i(t) < \infty$ for a.e. $t \in (0,t_0)$. In addition, by the Hamilton-Jacobi equation (5.2) it follows that for $i = 1,2$ and for a.e. $t \in (0,t_0)$ we have

$$-\infty < \frac{d}{dt} H_i(t) = p \alpha_i \int_E \frac{\partial}{\partial t} Q_i t g(x) e^{p \alpha_i Q_i t g(x)} \omega(x) dx = -\alpha_i \int_E \nabla Q_i t g(x) e^{p \alpha_i Q_i t g(x)} \omega(x) dx,$$

i.e.,

$$\int_E |\nabla Q_i t g(x)|^p e^{p \alpha_i Q_i t g(x)} \omega(x) dx < +\infty, \quad i = 1,2, \text{ for a.e. } t \in (0,t_0).$$

By trivial interpolation, since $q(t) \in [\alpha_1,\alpha_2]$ for every $t \in [0,t_0]$, we obtain that

$$\int_E |\nabla Q_i t g(x)|^p e^{p q(t) Q_i t g(x)} \omega(x) dx < +\infty \quad \text{for a.e. } t \in (0,t_0),$$

which is equivalent to

$$\int_E |\nabla e^{q(t) Q_i t g(x)}|^p \omega(x) dx < +\infty \quad \text{for a.e. } t \in (0,t_0).$$

In particular, the fact that $e^{q(t) Q_i t g} \in L^p(\omega; E)$ combined with the last estimate implies that $e^{q(t) Q_i t g} \in W^{1,p}(\omega; E)$ for a.e. $t \in (0,t_0)$. \hfill $\square$

Let us observe that if $g : E \to \mathbb{R}$ is bounded from above, then we can consider

$$T = T(g) := \sup \{t_0 : g \in F_{t_0}(E)\}.$$

Clearly, the statement of the above proposition holds for the number $T$ replacing $t_0$. On the other hand, if $t > T$ then it also follows from the statement of the proposition that $Q_i(g)(x) = -\infty$ for all $x \in E$, and thus the statement of Theorem 1.3 becomes trivial for the values $\beta > T$.

After this preparation, we are ready to prove Theorem 1.3.

5.1. **Proof of the inequality** (1.9). We first assume that $\alpha < \beta$. We claim that $g \in F_{t_0}(E)$ with $e^g \in L^\alpha(\omega; E)$ implies that $e^g \in L^\gamma(\omega; E)$ for every $\gamma \geq \alpha$. Indeed, let $M > 0$ be such that $g(x) \leq M$ for every $x \in E$. In addition, let $S = \{x \in E : g(x) \geq 0\}$. In particular, we have that

$$\mu_\omega(S) := \int_S \omega(x) dx \leq \int_S e^{\alpha g(x)} \omega(x) dx \leq \|e^g\|_{L^\alpha(\omega; E)}^\alpha < +\infty.$$

Moreover, since $\gamma \geq \alpha$, one has that

$$\|e^g\|_{L^\gamma(\omega; E)} = \int_S e^{\gamma g(x)} \omega(x) dx + \int_{E \setminus S} e^{\gamma g(x)} \omega(x) dx \leq e^{\gamma M} \mu_\omega(S) + \int_{E \setminus S} e^{\alpha g(x)} \omega(x) dx \leq (e^{\gamma M} + 1)\|e^g\|_{L^\alpha(\omega; E)}^\gamma < +\infty,$$

which ends the proof of the claim.

Consequently, since $p > 1$ and $\beta > \alpha$, we have that $e^g \in L^{\alpha p}(\omega; E) \cap L^{\beta p}(\omega; E)$; moreover, by Proposition 5.1/(ii), for every function $q : [0,t_0] \to [\alpha,\beta]$ one has that

$$e^{q(t) Q_i t g} \in W^{1,p}(\omega; E) \quad \text{for a.e. } t \in (0,t_0). \quad (5.3)$$

Fix arbitrarily $\tilde{t} \in (0,t_0)$ and let $q : [0,\tilde{t}] \to [\alpha,\beta]$ be the function

$$q(t) = \frac{\alpha \beta}{(\alpha - \beta) t/\tilde{t} + \beta}, \quad t \in [0,\tilde{t}]. \quad (5.4)$$
Note that $q$ is increasing on $[0, \tilde{t}]$ and $q(0) = \alpha$ and $q(\tilde{t}) = \beta$. Since relation (5.3) is valid a.e. on $(0, \tilde{t})$, the function

$$F(t) = \|e^{Q_{q}}g\|_{L^{q}(\omega; E)} > 0, \quad t \in [0, \tilde{t}],$$

is well-defined for a.e. $t \in [0, \tilde{t}]$ and absolutely continuous on $[0, \tilde{t}]$, see the proof of Proposition 5.1/(i). A direct computation and the Hamilton-Jacobi equation (5.2) imply that for a.e. $t \in [0, \tilde{t}]$ we have

$$F'(t) = F(t)^{1-q(t)} q'(t) \frac{q'(t)}{q^{2}(t)} \left( E_{\omega, E}(e^{q(t)Q_{q}}g) - F(t)^{q(t)} \log(F(t))^{q(t)} \right. \left. - \frac{q^{2}-p(t)}{p q^{2}(t)} \int_{E} e^{q(t)Q_{q}g(x)} |\nabla(q(t)Q_{q}g(x))|^{p} \omega(x) dx \right),$$

where we used the notation for the entropy from relation (5.4). In fact, due to (5.3), we may apply the log-Sobolev inequality (1.4) for the normalized function $u := \frac{\omega_{E}}{F(t)^{\frac{q(t)}{p}}} \in W^{1,p}(\omega; E)$ for a.e. $t \in (0, \tilde{t})$, obtaining that

$$\frac{E_{\omega, E}(e^{q(t)Q_{q}}g)}{F(t)^{q(t)}} - \log(F(t))^{q(t)} \leq \frac{n + \tau}{p} \log \left( \frac{\mathcal{L}_{\omega, p}}{e_{p}^{\beta}} \int_{E} e^{q(t)Q_{q}g(x)} |\nabla(q(t)Q_{q}g(x))|^{p} \omega(x) dx \right),$$

(5.6)

By using the elementary inequality $\log(ey) \leq y$ for every $y > 0$ (with equality only for $y = 1$), the right hand side RHS of the above inequality can be estimated for every $s > 0$ by

$$RHS \leq \frac{n + \tau}{p} \log \left( \frac{\mathcal{L}_{\omega, p}}{e_{p}^{\beta}} \right) + \frac{n + \tau}{ps} \int_{E} e^{q(t)Q_{q}g(x)} |\nabla(q(t)Q_{q}g(x))|^{p} \omega(x) dx.$$

(5.7)

Combining relations (5.5)-(5.7) with the choice

$$s := \frac{(n + \tau)q'(t)}{q^{2}(t)} > 0,$$

it turns out that

$$\frac{F'(t)}{F(t)} \leq \frac{n + \tau}{p} \frac{q'(t)}{q^{2}(t)} \log \left( \frac{\mathcal{L}_{\omega, p}}{e_{p}^{\beta}} \frac{(n + \tau)q'(t)}{q^{2}(t)} \right) \quad \text{for a.e. } t \in [0, \tilde{t}].$$

(5.8)

After an integration of the above inequality on $[0, \tilde{t}]$, we obtain that

$$\log \frac{F(t)}{F(0)} \leq \frac{n + \tau}{p} \int_{0}^{\tilde{t}} \frac{q'(t)}{q^{2}(t)} \log \left( \frac{\mathcal{L}_{\omega, p}}{e_{p}^{\beta}} \frac{(n + \tau)q'(t)}{q^{2}(t)} \right) dt.$$
Since $F(t) = \|e^{Q_t g}\|_{L^\beta(\omega;E)}$ and $F(0) = \|e^g\|_{L^\alpha(\omega;E)}$, a direct computation of the above integral yields

$$\|e^{Q_t g}\|_{L^\beta(\omega;E)} \leq \|e^g\|_{L^\alpha(\omega;E)} \left( \frac{\beta - \alpha}{t} \right)^{\frac{\alpha + \beta}{\alpha p} \left( \frac{\beta - \alpha}{\beta - 1} \right) - 1} \left( p' \right)^{\frac{\alpha + \beta}{\beta - 1}} \left( \frac{n + \tau}{p' + 1} + 1 \right) \int_{B \cap E} \omega \right)^{\frac{\alpha - \beta}{\beta - 1}},$$

which is precisely the inequality (1.9) whenever $\alpha < \beta$.

When $\alpha = \beta$, the inequality (1.9) reduces to $\|e^{Q_t g}\|_{L^\alpha(\omega;E)} \leq \|e^g\|_{L^\alpha(\omega;E)}$, which directly follows by $Q_t g \leq g$.

5.2. Equality in (1.9). We first assume again that $\alpha < \beta$. If the function $g : E \to \mathbb{R}$ has the form from (1.10), after a direct computation we observe that equality holds in (1.9); a similar proof can be found in Gentil [22] for the unweighted case and $E = \mathbb{R}^n$.

Conversely, assume that $\alpha < \beta$ and there exists equality in (1.9) for some $\tilde{t} \in (0, t_0)$ and $g \in F_t(E)$ with $e^g \in L^\alpha(\omega;E)$. In particular, by tracking back the inequalities in the proof, it turns out that we should have equalities both in (5.6) and (5.7); namely, we have that

$$\frac{e^{Q_{\tilde{t}} g(x)}}{F(\tilde{t})^{Q_{\tilde{t}}}} = u_{\lambda_t, x_0}(x) \text{ for every } x \in E \text{ and a.e. } t \in [0, \tilde{t}],$$

for some $\lambda_t > 0$ and $x_0$ such that $x_0 \in -\partial E \cap \partial E$ and $\omega(x + x_0) = \omega(x)$ for every $x \in E$ whenever $\tau > 0$, and $x_0 \in -\overline{E} \cap \overline{E}$ and $\omega$ is constant in $E$ whenever $\tau = 0$, see (1.5), and

$$\int_E e^{Q_{\tilde{t}} g(x)} |\nabla (g(t)Q_{\tilde{t}} g(x))|^p \omega(x)dx = \frac{(n + \tau)q'(t)}{q^2 - p(t)} \text{ for a.e. } t \in [0, \tilde{t}].$$

By replacing (5.9) into (5.10), we obtain that

$$p^p \int_E |\nabla u_{\lambda_t, x_0}(x)|^p \omega(x)dx = \frac{(n + \tau)q'(t)}{q^2 - p(t)} \text{ for a.e. } t \in [0, \tilde{t}].$$

Relation (1.5), the fact that $\omega(x + x_0) = \omega(x)$ for every $x \in E$ whenever $\tau > 0$ (and $\omega$ is constant in $E$ whenever $\tau = 0$) and a change of variables imply that

$$\int_E |\nabla u_{\lambda_t, x_0}(x)|^p \omega(x)dx = \int_E |\nabla u_{\lambda_t, 0}(x)|^p \omega(x)dx = \lambda_t^{p - \frac{n + \tau}{p'}} \frac{(p')^p}{p^p}. $$

Therefore, one has that

$$\lambda_t = \frac{1}{p} \left( \frac{\beta - \alpha}{t} \right) \frac{1}{p - 1} \frac{\alpha \beta}{((\alpha - \beta)t/\beta + \beta)^p}, \text{ t } \in [0, \tilde{t}].$$
Note that relation (5.9) can be equivalently transformed into

\[ Q_t g(x) = \log F(t) + \frac{p}{q(t)} \log u_{\lambda_t x_0^t}(x) \]

\[ = \log F(t) + \frac{p}{q(t)} \log \left( \frac{n+\tau}{p'} \left( \frac{n}{p'} + 1 \right) \int_{B \cap E} n \omega \right)^{-\frac{1}{p'}} - \frac{\lambda_t}{q(t)} |x + x_0^t|^{p'} \]

\[ =: h(t) - \frac{b_t}{p'} |x + x_0^t|^{p'}, \]

which is valid for every \( t \in (0, \tilde{t}] \).

On one hand, we notice that we have equality also in (5.8). Moreover, a direct computation yields that the equality in (5.8) is equivalent to the vanishing of the derivative of \( t \mapsto h(t) \) on \((0, \tilde{t}]\); in particular, for every \( t \in (0, \tilde{t}] \) we have

\[ h(t) = \lim_{t \to 0} h(t) =: C \in \mathbb{R}. \]

On the other hand, by the definition of \( Q_t g \), one has that

\[ Q_t g(x) = \inf_{y \in E} \left\{ \frac{|y - x|^{p'}}{p't^{p'-1}} - (-g)(y) \right\}, \quad x \in E, \ t \in (0, \tilde{t}]. \]

The latter relation gives the hint to use properties of \( c \)-concave functions associated with the cost function \( c(x, y) := c_t(x, y) = \frac{|x-y|^{p'}}{p't^{p'-1}} \).

Let us recall from Villani [34] that a function \( \phi : E \to \mathbb{R} \) is \( c \)-concave if it can be written as

\[ \phi(x) = \inf_{y \in E} \{ c(x, y) - \psi(y) \}, \quad x \in E, \]

for some function \( \psi : E \to \mathbb{R} \) that is not identically \(-\infty\). We can denote the right side of the above expression by \( \psi^c \) and call it the \( c \)-transform of \( \psi \). In this way, a function is \( c \)-concave if it can be written as \( \phi = \psi^c \) for some other function \( \psi \). It can be easily verified just by the definition that \( \phi \leq (\phi^c)^c \) and \( \phi \) is \( c \)-concave if and only if \( \phi = (\phi^c)^c \).

Let us assume momentarily that \( \phi = -g \) is \( c_t \)-concave. It follows by [34, Proposition 5.47] that \(-g\) will be \( c_t \)-concave for all \( 0 < t \leq \tilde{t} \). Using that

\[ -g = ((-g)^c)^c \]

and writing the \( c_t \)-transform in terms of \( Q_t \) we obtain the relation

\[ -g(y) = \inf_{x \in E} \left\{ \frac{|y - x|^{p'}}{p't^{p'-1}} - Q_t g(x) \right\} = Q_t (-Q_t g)(y), \quad y \in E. \]

Therefore, by (5.11) and (5.12), we have that

\[ -g(y) = -C + Q_t \left( \frac{b_t}{p'} |y + x_0^t|^{p'} \right) (y), \quad y \in E. \]

A direct computation gives that for every \( b > 0 \) and \( x_0 \in E \), one has

\[ Q_t \left( \frac{b}{p'} |y + x_0^t|^{p'} \right) (y) = \frac{b}{p'(1 + t b^{p-1})^{p'-1}} |y + x_0^t|^{p'}, \quad y \in E; \]
applying the latter formula for \((5.13)\) and recalling that \(b_t = \frac{\mathcal{L}t}{q(t)} > 0\), it turns out that

\[
g(y) = C - \frac{1}{p^q} \left( \frac{\beta - \alpha}{\beta t} \right)^{\frac{1}{p-1}} |y + x_0|^{p'}, \ y \in E. \tag{5.15}
\]

Since \(g\) is independent on \(t > 0\), we necessarily have that \(x_0 := x_0_t\) for every \(t \in (0, \tilde{t})\); this concludes the proof of the relation \((1.10)\) whenever \(-g\) is \(c_t\)-concave. A straightforward computation shows that \(e^g \in L^\alpha(\omega; E)\).

We also observe that \(g \in \mathcal{F}_t(E)\). To see this, by a direct computation we have that

\[
\mathcal{Q}_t \left( -\frac{b}{p^q} \cdot + x_0|^{p'} \right)(x) = -\frac{b}{p^q (1 - t|b|^{p-1})} |x + x_0|^{p'}, \ x \in E, \tag{5.16}
\]

for every \(x_0 \in E\) and \(0 < t|b|^{p-1} < 1\). Clearly, \(g\) is bounded from above, and as for

\[
b_0 := \left( \frac{\beta - \alpha}{\beta t} \right)^{\frac{1}{p-1}},
\]

we have that \(t|b|^{p-1} < 1\), it yields that \(g \in \mathcal{F}_t(E)\), see \((5.16)\).

Finally, we shall remove the assumption that \(-g\) is \(c_t\)-concave. Assume by contradiction that there exists another extremal function \(\tilde{g}\) in \((1.9)\) such that \(-\tilde{g}\) is not \(c_t\)-concave. By tracking back again the equality cases like in the first part of the proof, see \((5.11)\), we obtain also for this function \(\tilde{g}\) that

\[
\mathcal{Q}_t \tilde{g}(x) = C - \frac{b_t}{p^q} |x + x_0|^{p'}, \ x \in E, \ t \in (0, \tilde{t}). \tag{5.17}
\]

Moreover, if \(g\) is the \(c_t\)-concave function from \((5.15)\), it turns out by relations \((5.17)\) and \((5.14)\) that

\[-\tilde{g}(y) \leq \mathcal{Q}_t(-\tilde{g})(y) = -g(y), \ y \in E, \ t \in (0, \tilde{t}),
\]

thus \(\tilde{g} \geq g\) on \(E\). On the other hand, both functions \(g\) and \(\tilde{g}\) are extremals in the hypercontractivity inequality \((1.9)\), i.e., in particular, we have

\[
\frac{\|e^{\mathcal{Q}_t \tilde{g}}\|_{L^\beta(\omega; E)}}{\|e^{\tilde{g}}\|_{L^\alpha(\omega; E)}} = \frac{\|e^{\mathcal{Q}_t g}\|_{L^\beta(\omega; E)}}{\|e^g\|_{L^\alpha(\omega; E)}}.
\]

Furthermore, since \(\mathcal{Q}_t g = \mathcal{Q}_t \tilde{g}\), up to a translation (see \((5.11)\) and \((5.17)\)), it follows that

\[
\|e^{\mathcal{Q}_t \tilde{g}}\|_{L^\beta(\omega; E)} = \|e^{\mathcal{Q}_t g}\|_{L^\beta(\omega; E)},
\]

which implies

\[
\|e^g\|_{L^\alpha(\omega; E)} = \|e^{\tilde{g}}\|_{L^\alpha(\omega; E)}.
\]

This relation together with \(\tilde{g} \geq g\) on \(E\) implies that \(\tilde{g} = g\) a.e. on \(E\), which is a contradiction.

It remains to analyze the equality in \((1.9)\) whenever \(\alpha = \beta\); namely, we assume that equality holds in \((1.9)\) for some \(\tilde{t} \in (0, t_0)\) and \(g \in \mathcal{F}_t(E)\) with \(e^g \in L^\alpha(\omega; E)\), i.e., \(\|e^{\mathcal{Q}_t \tilde{g}}\|_{L^\alpha(\omega; E)} = \|e^g\|_{L^\alpha(\omega; E)}\). Since we have that \(\mathcal{Q}_t g \leq g\) on \(E\), the latter equality implies that \(\mathcal{Q}_t g = g\). Moreover, the monotonicity of \(t \mapsto \mathcal{Q}_t g\) implies that for every \(t \in [0, \tilde{t}]\),

\[
\mathcal{Q}_t g \leq \mathcal{Q}_{\tilde{t}} g \leq g = \mathcal{Q}_t g,
\]

thus \(\mathcal{Q}_t g = g\) for every \(t \in [0, \tilde{t}]\). In particular, \(\frac{\partial}{\partial t} \mathcal{Q}_t g = 0\) and by the Hamilton-Jacobi equation \((5.2)\) it follows that \(|\nabla \mathcal{Q}_t g(x)|^p = 0\) for every \(t \in (0, \tilde{t})\) and a.e. \(x \in E\). Since \(\mathcal{Q}_t g\) is
locally Lipschitz, it turns out that \( g = Q_t g \equiv C \) for some \( C \in \mathbb{R} \), which contradicts the fact that \( e^g \in L^\alpha(\omega; E) \). This concludes the proof. \( \square \)

**Remark 5.1.** As a final remark we should mention that there is a well-known equivalence between the log-Sobolev inequality, hypercontractivity of the Hopf-Lax semigroup and Prékopa-Leindler inequalities, see [22], [17]. It would be therefore feasible that this equivalence is also reflected in the characterization of the equality cases in these inequalities. Consequently, one possible route to obtain a characterization of the extremals in the weighted log-Sobolev inequality could have been done as follows: Start from weighted Prékopa-Leindler inequalities as shown by Milman and Rotem [30]. Next, obtain a characterization of the equality cases here using the techniques of Balogh and Kristály [4] or Dubuc [18]. Using the characterization of the equality cases in the weighted Prékopa-Leindler inequality, the characterization of the equality case in the Hopf-Lax hypercontractivity estimate could be deduced and could be used for a characterization of the extremals in the weighted log-Sobolev inequality. We think that this strategy could be worked out, however the technical complexity involved in this process would be at least comparable if not higher to the one of the present paper.

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