Variational inequalities for bilinear averages

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Abstract

We obtain variational inequalities for some classes of bilinear averages of one variable, generalizing the variational inequalities for averages of R. Jones et al. As an application we get almost everywhere convergence for the ergodic averages along cubes on a dynamical system.

1 Introduction

The variational inequalities have been the subject of many recent articles in probability, ergodic theory and harmonic analysis. For linear version, the first variational inequality was proved by Lépingle [17] for martingales (see [23] for a simple proof). Bourgain [2] used Lépingle’s result to obtain corresponding variational estimates for the Birkhoff ergodic averages and then directly deduce pointwise convergence results without previous knowledge that pointwise convergence holds for a dense subclass of functions, which is quite difficult in some ergodic models. A few years later, Jones and his collaborators systematically studied variational inequalities for ergodic averages in [12], [13], [3] and [4], see also [11, 19, 18]. Recently, several results on variational inequalities for discrete averaging operators of Radon type have also been established (cf. e.g. [15], [20], [21], [22], [28]).

In this paper we concern with variational inequalities for some classes of bilinear averages, and their application to ergodic theory. In fact, the problem of almost everywhere convergence of multilinear ergodic averages plays an important role in ergodic theory. For instance, Demeter et al [7] considered the following multilinear averages and related ergodic averages:

\begin{equation}
T_{A,R,r}(f_1, \cdots, f_{n-1}) = \frac{1}{(2^r)^m} \int_{|t_1|, \cdots, |t_m| \leq r} \prod_{i=1}^{n-1} f_i(x + \sum_{j=1}^{m} a_{i,j} t_j) \, dt',
\end{equation}

and

\begin{equation}
T_{A,X,L}(f_1, \cdots, f_{n-1}) = \frac{1}{(2L + 1)^m} \sum_{|l_1|, \cdots, |l_m| \leq L} \prod_{i=1}^{n-1} f_i(S^{\sum_{j=1}^{m} a_{i,j} l_j} x),
\end{equation}

where \( n > 1, \ m \geq 1, \ A = (a_{i,j}) \) is a \((n-1) \times m\) integer-valued matrix and \((X, \Sigma, m, S)\) is a dynamical system. This kind of averages are related to the Furstenberg recurrence theorem [8] and to Szemerédi’s theorem [26] on arithmetic progressions, and are also connected to the result in [9] that primes contain arbitrarily long progressions. To get the convergence, authors established the almost everywhere convergence for \( T_{A,X,L} \) for \( f_1, \cdots, f_{n-1} \in L^\infty(X) \), proved \( \sup_{L>0} |T_{A,X,L}| \) maps \( L^{p_1}(X) \times \cdots \times L^{p_{n-1}}(X) \) to \( L^p(X) \) and extended the convergence result to the case when \( f_i \in L^{p_i}(X) \). The boundedness of \( \sup_{L>0} |T_{A,X,L}| \) is a consequence of an analogous boundedness
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for $\sup_{r > 0} |T_{A,r,r^*}|$, because of transference arguments. But the problem of almost everywhere convergence of $T_{A,X,L}$ for $f_1, \cdots, f_{n-1} \in L^\infty(X)$ is quite difficult except some special cases. An alternate method would be to prove variational inequalities for $T_{A,X,L}$ in $L$ without consider the almost everywhere convergence of $T_{A,X,L}$ for $f_1, \cdots, f_{n-1} \in L^\infty(X)$. In 2008, Demeter et al [6] established an oscillation result (a weak variational inequality) which is used to prove the convergence for the signed average analog of Bourgain’s return times theorem, and to provide a separate proof of Bourgain’s theorem.

Precisely, we primarily consider the almost everywhere convergence of the following bilinear averages:

$$Q_t(f,g)(x) = \frac{1}{t^2} \int_{|y| \leq \frac{1}{2}} \int_{|z| \leq \frac{1}{2}} f(x-y)g(x-z)dydz,$$

where $t > 0$, and $f,g$ are arbitrary measurable functions on $\mathbb{R}$. Note that averages $Q_t(f,g)$ are special cases of multilinear averages defined in (1.1) when $n = 3, m = 2$ and $A = I_{2 \times 2}$. We denote the family $\{Q_t(f,g)\}_{t > 0}$ by $Q(f,g)$. Before we can get into more details we need some definitions.

For sequence $\{a_n\}$ and $\rho \geq 1$ define the variational norm $V_\rho$ by

$$\|\{a_n\}\|_{V_\rho} = \sup_{\{n_i\}} \left( \sum_{i} |a_{n_i} - a_{n_{i+1}}|^\rho \right)^{1/\rho},$$

where the supremum is taken over all systems of indices $n_1 < n_2 < \cdots$. Given an interval $I \in (0, \infty)$ and a family of complex numbers $a = \{a_t\}_{t \in I}$, the variational norm of the family $a$ is defined as

$$\|a\|_{V_\rho(I)} = \sup_{t \geq 1} \left( \sum_{i \geq 1} |a_{t_i} - a_{t_{i+1}}|^\rho \right)^{1/\rho},$$

where the supremum runs over all increasing sequences $\{t_i \in I : i \geq 1\}$. It is trivial that

$$\|a\|_{L^\infty(I)} := \sup_{t \in I} |a_t| \leq |a_{t_0}| + \|a\|_{V_\rho(I)} \quad \text{for any } t_0 \in I \text{ and } \rho \geq 1. \quad (1.3)$$

If $I = (0, \infty)$, we denote the variational norm $V_\rho(I)$ by $V_\rho$ for short.

Given a family of Lebesgue measurable functions $F = \{F_t(x)\}_{t > 0}$ defined on $\mathbb{R}$, for fixed $x$ in $\mathbb{R}$ the value of the strong $\rho$-variation operator $V_\rho(F)$ of the family $F$ at $x$ is defined by

$$V_\rho(F)(x) = \|\{F_t(x)\}_{t > 0}\|_{V_\rho}, \quad \rho \geq 1. \quad (1.4)$$

It is easy to observe from the definition of $\rho$-variation norm that for fixed $x$ if $V_\rho(F)(x) < \infty$, then $\{F_t(x)\}_{t > 0}$ converges when $t \to 0$ and $t \to \infty$. In particular, if $V_\rho(F)$ belongs to some function spaces such as $L^p$ or $L^{p,\infty}$, then the family $\{F_t(x)\}_{t > 0}$ converges almost everywhere
without any additional condition. This is why mapping property of strong $\rho$-variation operator is so interesting in ergodic theory and harmonic analysis.

The following theorem is a variational inequality for bilinear averages over cubes.

**Theorem 1.1.** For $\rho > 2$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have
\[
\|V_\rho (Q(f, g))\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.
\]

In addition to averages $\{Q_t(f, g)\}_{t>0}$, we introduce averages $\{Q_L(\phi, \psi)\}_{L \in \mathbb{N}}$ defined on $\phi, \psi : \mathbb{Z} \to \mathbb{R}$ of compact support:
\[
Q_L(\phi, \psi)(i) = \frac{1}{(2L+1)^2} \sum_{|l|, |k| \leq L} \phi(i-l)\psi(i-k).
\]

The family of discrete averages $\{Q_L(\phi, \psi)\}_{L \in \mathbb{N}}$ is denoted by $Q(\phi, \psi)$. Moreover, we obtain the discrete version of Theorem 1.1 as follows.

**Corollary 1.2.** For $\rho > 2$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have
\[
\|V_\rho (Q(\phi, \psi))\|_{L^p(\mathbb{Z}^n)} \leq C\|\phi\|_{L^{p_1}(\mathbb{Z}^n)}\|\psi\|_{L^{p_2}(\mathbb{Z}^n)}.
\]

Let $(X, \Sigma, m, S)$ denote a dynamical system with $(X, \Sigma, m)$ a complete probability space and $S$ an invertible bimeasurable transformation such that $mS^{-1} = m$. The closely related ergodic averages are given by
\[
Q_L(f, g)(x) = \frac{1}{(2L+1)^2} \sum_{|l_1|, |l_2| \leq L} f(S^{l_1}x)g(S^{l_2}x).
\]

The sequence $\{Q_L(f, g)\}_L$ is denoted by $\mathcal{Q}(f, g)$. Appealing to Corollary 1.2 and standard transfer methods like in [7, 5], we get

**Corollary 1.3.** For $\rho > 2$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have
\[
\|V_\rho (\mathcal{Q}(f, g))\|_{L^p(X)} \leq C\|f\|_{L^{p_1}(X)}\|g\|_{L^{p_2}(X)}.
\]

Moreover, for every dynamical system $(X, \Sigma, m, S)$, the averages over squares
\[
\frac{1}{(2N+1)^2} \sum_{i=-N}^{N} \sum_{j=-N}^{N} f(S^i x)g(S^j x)
\]
converge a.e. for $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X)$.
For $j, m \in \mathbb{Z}$, the dyadic interval in $\mathbb{R}$ is an interval of the form $[m2^j, (m+1)2^j)$. The set of all dyadic intervals with side-length $2^j$ is denoted by $D_j$. The conditional expectation of a local integrable $f$ with respect to the increasing family of $\sigma$–algebras $\sigma(D_j)$ generated by $D_j$ is given by

$$E_j f(x) = \sum_{I \in D_j} \frac{1}{|I|} \int_I f(y) dy \cdot \chi_I(x)$$

for all $j \in \mathbb{Z}$. In view of the Lebesgue differentiation theorem, we have that

$$\lim_{j \to \infty} E_j f \to f, \text{ a.e.}$$

for $f \in L^2(\mathbb{R})$. \{E_j f\}_j can be looked as a family of averages which are constructed from $f$ by certain averaging process. Moreover, there is a close connection between the martingale sequence \{E_j f\}_j and averages over cubes \[12, 13, 14\]. Therefore, we consider the bilinear conditional expectation of two local integrable $f$ and $g$, which is given by

$$E_j(f,g)(x) = \sum_{I,J \in D_j} \frac{1}{|I \times J|} \int_{I \times J} f(y) g(z) dydz \cdot \chi_{I \times J}(x,x).$$

For the bilinear conditional expectation, we obtain the following variational inequality.

**Theorem 1.4.** For $\rho > 2$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_\rho(\{E_j(f,g)\}_j)\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}.$$

Other family of bilinear averages are carried out by a suitable ”approximation of the identity” as follows. Fix $\phi \in \mathcal{S}(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \phi(x) dx = 1$. For $t > 0$, set $\phi_t(x,y) = t^{-2} \phi(x/t, y/t)$. The bilinear convolution operators are given by

$$\phi_t(f,g)(x) = \int_{\mathbb{R}^2} \phi_t(x-y, x-z) f(y) g(z) dydz.$$

We denote \{\phi_t(f,g)\}_{t>0} by $\Phi(f,g)$. In this setting we obtain the variational estimate as follows.

**Theorem 1.5.** For $\rho > 2$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_\rho(\Phi(f,g))\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}.$$

In the next section we give the proof of the variational inequality for averages over cubes, which is a consequence of an vector-valued bilinear interpolation and an endpoint estimate for certain vector-valued operator. The discrete analogue is proved at the end of this section. The variational inequality for conditional expectations is treated in the same way in section 3. In final section we prove the variational estimate for approximations of the identity in the similar way. But, the $L^{p_1} \times L^{p_2} \to L^p$ bounds for all $1 < p, p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and endpoint estimate can not be established directly, since those kernels are not multiplicatively separable. We apply bilinear vector-valued Calderón-Zygmund theory to deal with those problems.
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In order to prove Theorem 1.1, we present a $\mathcal{B}$-valued bilinear interpolation, where $\mathcal{B}$ is a Banach space, see [10] and [24].

**Lemma 2.1.** Suppose and $T$ is a bilinear $\mathcal{B}$-valued operator. If $T$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p,\infty}(\mathcal{B})$ for all $1 < p, p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and from $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ into $L^{1/2,\infty}(\mathcal{B})$, then $T$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p,\infty}(\mathcal{B})$ for all $1 < p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{p}$. 

We take the Banach space $\mathcal{B} = \{a(t) : \|a\|_\mathcal{B} = \|a\|_{V^p} < \infty\}$. Then, $V_\rho(Q(f,g))(x) = \|\{Q_t(f,g)(x)\}_{t>0}\|_\mathcal{B}$. Lemma 2.1 implies Theorem 1.4 is a consequence of the following two propositions.

**Proposition 2.2.** For $\rho > 2$, $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_\rho(Q(f,g))\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}.$$ 

**Proof.** Similarly, we get

$$V_\rho(Q(f,g))(x) = V_\rho(M(f) \cdot M(g))(x) \leq M(f)(x) \cdot V_\rho(M(g))(x) + M(g)(x) \cdot V_\rho(M(f))(x).$$

By using Hölder’s inequality and the variational inequalities for averages [12, 3], we get the desired result. \[\square\]

**Lemma 2.3.** For $\rho > 2$, we have

$$\lambda |\{x \in \mathbb{R} : V_\rho(Q(f_1,f_2))(x) > \lambda\}|^2 \leq C \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}$$

uniformly in $\lambda > 0$.

**Proof.** By scaling, we can assume that $\lambda = 1$. Suppose that $f_1, f_2$ are step functions given by a finite linear combination of characteristic functions of disjoint dyadic intervals. In proving above weak endpoint type estimate, we may assume that

$$\|f_1\|_{L^1} = \|f_2\|_{L^1} = 1.$$

The general case follows immediately by scaling. It suffices to prove

$$|\{x \in \mathbb{R} : V_\rho(Q(f_1,f_2))(x) > 1\}| \leq C.$$
We apply the Calderón-Zygmund decomposition to functions $f_i$ at height 1 to obtain functions $g_i$, $b_i$ and finite families dyadic intervals $\{I_{i,k}\}_k$ with disjoint interiors such that

$$f_i = g_i + b_i \quad \text{and} \quad b_i = \sum_k b_{i,k}.$$

For $i = 1, 2$, we have

$$\text{support}(b_{i,k}) \subseteq I_{i,k}$$

$$\int_{I_{i,k}} b_{i,k}(x) \, dx = 0$$

$$\int_{I_{i,k}} |b_{i,k}(x)| \, dx \leq C |I_{i,k}|$$

$$|\cup_k I_{i,k}| \leq C$$

$$\|g_i\|_{L^1} \leq \|f_i\|_{L^1} = 1$$

$$\|g_i\|_{L^\infty} \leq 2.$$

For interval $I$, $\tilde{I}$ denotes the interval that is concentric with $I$ and has length $3|I|$. For convenience, we denote $\cup_k \tilde{I}_{1,k}$ and $\cup_j \tilde{I}_{2,j}$ by $\Omega_1$ and $\Omega_2$, respectively. Since

$$|\{x \in \mathbb{R} : V_{\rho}(Q(f_1, f_2))(x) > 1/4\}| \leq |\{x \in \mathbb{R} : V_{\rho}(Q(f_1, f_2))(x) > 1/4\}| + |\Omega_1|$$

$$+ |\Omega_2| + |\{x \notin \Omega_1 : V_{\rho}(Q(b_1, g_2))(x) > 1/4\}|$$

$$+ |\{x \notin \Omega_2 : V_{\rho}(Q(g_1, b_2))(x) > 1/4\}|$$

$$+ |\{x \notin \Omega_1 \cup \Omega_2 : V_{\rho}(Q(b_1, b_2))(x) > 1/4\}|,$$

it suffices to estimate each of above six sets. Let us start with the first one. Applying Proposition 2.2, we observe

$$|\{x \in \mathbb{R} : V_{\rho}(Q(g_1, g_2))(x) > 1/4\}| \leq C \|V_{\rho}(Q(g_1, g_2))\|_{L^1} \leq C \|g_1\|_{L^2} \|g_2\|_{L^2} \leq C.$$

Obviously, $|\Omega_1| + |\Omega_2| \leq C$. Now we turn to the fourth term. For $x \notin \Omega_1$ and $t \in (0, \infty)$, there are at most two $k$’s for which

$$\frac{1}{t} \int_{x - \frac{t}{2}}^{x + \frac{t}{2}} b_{1,k}(y) \, dy \neq 0.$$
Indeed, it happens only if $I_{1,k}$ contains the starting point or endpoint of $(x - \frac{t_j}{2}, x + \frac{t_j}{2})$. Hence,

$$V_{\rho}(Q(b_1, g_2))(x) = \sup_{\{t_j\} \searrow 0} \left( \sum_j \left| \sum_k [M_{t_j}(b_{1,k}, g_2)(x) - M_{t_{j+1}}(b_{1,k}, g_2)(x)] \right|^\rho \right)^{1/\rho}$$

$$\leq C \sup_{\{t_j\} \searrow 0} \left( \sum_j \sum_k \left| M_{t_j}(b_{1,k}, g_2)(x) - M_{t_{j+1}}(b_{1,k}, g_2)(x) \right|^\rho \right)^{1/\rho}$$

$$\leq C \left( \sum_k V_{\rho}(Q(b_{1,k}, g_2))^\rho(x) \right)^{1/\rho}.$$ 

For $x \notin \tilde{I}_{1,k}$, we assume $x$ is on the right of $I_{1,k}$, the other case can be treated in the same way. We can choose a monotone decreasing sequence $\{t_j(x)\}_j$ approaching 0 such that

$$V_{\rho}(Q(b_{1,k}, g_2))(x) \leq C \sum_j |Q_{t_j(x)}(b_{1,k}, g_2)(x) - Q_{t_{j+1}(x)}(b_{1,k}, g_2)(x)|$$

$$\lesssim |Q_{t_{j_0}}(b_{1,k}, g_2)(x)| + \sum_{j=j_0}^{j_1-1} |Q_{t_j(x)}(b_{1,k}, g_2)(x) - Q_{t_{j+1}(x)}(b_{1,k}, g_2)(x)|$$

$$+ |Q_{t_{j_1}}(b_{1,k}, g_2)(x)|$$

$$\lesssim \frac{1}{t_{j_1}(x)} \|b_{1,k}\|_{L^1} + \sum_{j=j_0}^{j_1-1} \left| M_{t_j(x)}(b_{1,k})(x) - M_{t_{j+1}(x)}(b_{1,k})(x) \right|$$

$$+ \sum_{j=j_0}^{j_1-1} \left| M_{t_j(x)}(g_2)(x) - M_{t_{j+1}(x)}(g_2)(x) \right| \|M_{t_{j+1}(x)}(b_{1,k})(x)\|,$$

where $x - t_{j_0}(x) \in I_{1,k}$ and $x - t_{j_0-1}(x) \notin I_{1,k}$, $x - t_{j_1}(x) \in I_{1,k}$ and $x - t_{j_1+1}(x) + x \notin I_{1,k}$, and we have used the fact that $\|M(g_2)\|_{L^\infty} \leq 2$. Clearly, $t_{j_1}(x) \sim d(x, I_{1,k})$ for $x \notin I_{1,k}$. Then, the second summand is dominated by

$$\sum_{j=j_0}^{j_1-1} \frac{1}{t_{j_1}(x)} \|b_{1,k}\|_{L^1} \leq \frac{C\|b_{1,k}\|_{L^1}}{d(x, I_{1,k})},$$

For the third summand, it is controlled by

$$\sum_{j=j_0}^{j_1-1} \frac{1}{t_{j_1}(x)} \|b_{1,k}\|_{L^1} \lesssim \frac{d(x, I_{1,k}) + |I_{1,k}|}{d(x, I_{1,k})^2} \|b_{1,k}\|_{L^1},$$

$$\sum_{j=j_0}^{j_1-1} \frac{1}{t_{j_1}(x)} \|b_{1,k}\|_{L^1} \lesssim \frac{d(x, I_{1,k}) + |I_{1,k}|}{d(x, I_{1,k})^2} \|b_{1,k}\|_{L^1}.$$
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where we used the fact \( \|g_2\|_{L^\infty} \leq 2 \) and \( \|g_2\|_{L^1} \leq 1 \).

As a result, we get

\[
\left| \{ x \notin \Omega_1 : V_\rho(\mathcal{Q}(b_1, g_2))(x) > 1/4 \} \right| \leq C \sum_k \int_{(I_{1,k})^c} V_\rho(\mathcal{Q}(b_{1,k}, g_2))^\rho(x) dx
\]

\[
\leq C \sum_k \| b_{1,k} \|_{L^1}^\rho \int_{(I_{1,k})^c} \frac{(d(x, I_{1,k}) + |I_{1,k}|)^\rho}{d(x, I_{1,k})^{2\rho}} dx
\]

\[
\leq C \sum_k \| b_{1,k} \|_{L^1}^\rho |I_{1,k}|^{1-\rho} \leq C \sum_k |I_{1,k}| \leq C.
\]

The fifth term can be treated in the similar way, we obtain

\[
\left| \{ x \notin \Omega_2 : V_\rho(\mathcal{Q}(g_1, b_2))(x) > 1/4 \} \right| \leq C.
\]

For the last one, we write

\[
b_1(y) b_2(z) = \sum_k b_{1,k}(y) \sum_{i: |I_{2,i}| \leq |I_{1,k}|} b_{2,i}(z) + \sum_i b_{2,i}(z) \sum_{k: |I_{1,k}| \leq |I_{2,i}|} b_{1,k}(y)
\]

\[
:= \sum_k b_{1,k}(y) b_k^{(2)}(z) + \sum_i b_{2,i}(z) b_1^{(i)}(y).
\]

Then, for \( x \notin \Omega_1 \cup \Omega_2 \), we observe that

\[
V_\rho(\mathcal{Q}(b_1, b_2))(x) \leq \left( \sum_k V_\rho(\mathcal{Q}(b_{1,k}, b_2^{(k)}))^\rho(x) \right)^{1/\rho} + \left( \sum_i V_\rho(\mathcal{Q}(b_1^{(i)}, b_2,i))^\rho(x) \right)^{1/\rho},
\]

where we use the fact that for \( x \notin \Omega_1 \cup \Omega_2 \) and \( t \in (0, \infty) \), there are at most two \( k \)'s and two \( i \)'s for which

\[
\frac{1}{t} \int_{x-t}^{x+t} b_{1,k}(y) dy \neq 0 \quad \text{and} \quad \frac{1}{t} \int_{x-t}^{x+t} b_{2,i}(z) dz \neq 0.
\]

Hence, we see that

\[
\left| \{ x \notin \Omega_1 \cup \Omega_2 : V_\rho(\mathcal{Q}(b_1, b_2))(x) > 1/4 \} \right| \leq \left| \{ x \notin \Omega_1 \cup \Omega_2 : \left( \sum_k V_\rho(\mathcal{Q}(b_{1,k}, b_2^{(k)}))^\rho(x) \right)^{1/\rho} > 1/8 \} \right|
\]

\[
+ \left| \{ x \notin \Omega_1 \cup \Omega_2 : \left( \sum_i V_\rho(\mathcal{Q}(b_1^{(i)}, b_2,i))^\rho(x) \right)^{1/\rho} > 1/8 \} \right|.
\]

It suffices to consider the first term, the other one can be treated in the same way. For \( x \notin \Omega_1 \cup \Omega_2 \) and \( t > d(x, I_{1,k}) \) such that \( M_t(b_{1,k})(x) \neq 0 \), there are at most two summands \( b_{2,i} \) in \( b_2^{(k)} \) for which

\[
\int_{x-t}^{x+t} b_{2,i}(z) dz \neq 0 \quad \text{and} \quad \left| \int_{x-t}^{x+t} b_{2,i}(z) dz \right| \leq |I_{2,i}| \leq |I_{1,k}|.
\]
Notice that dyadic intervals \( \{I_{2,i}\}_i \) are with disjoint interiors. Moreover, for above \( x \) and \( t \), we obtain
\[
|M_{t}(b_2^{(k)})| \leq \frac{2|I_{1,k}|}{d(x, I_{1,k})} \leq 2 \quad \text{and} \quad |Q_{t}(b_1,k, b_2^{(k)})| \leq \frac{1}{\ell} \|b_1,k\|_{L^1} M_{t}(b_2^{(k)}) \leq \frac{2}{\ell} \|b_1,k\|_{L^1}.
\]
For \( x \notin I_{1,k} \cup \Omega_2 \), we assume \( x \) is on the right of \( I_{1,k} \). We can choose a monotone decreasing sequence \( \{t_j(x)\}_j \) approaching 0 such that
\[
V_\rho(\Omega(b_1,k, b_2^{(k)}))(x) \leq C \sum_j |Q_{t_j}(x)(b_1,k, b_2^{(k)})(x) - Q_{t_{j+1}}(x)(b_1,k, b_2^{(k)})(x)|
\]
\[
\leq |Q_{t_{j_0}}(x)(b_1,k, b_2^{(k)})(x)| + \sum_{j=j_0}^{j_1-1} |Q_{t_j}(x)(b_1,k, b_2^{(k)})(x) - Q_{t_{j+1}}(x)(b_1,k, b_2^{(k)})(x)|
\]
\[
+ |Q_{t_{j_1}}(x)(b_1,k, b_2^{(k)})(x)| + \sum_{j=j_0}^{j_1-1} |M_{t_j}(x)(b_1,k)(x) - M_{t_{j+1}}(x)(b_1,k)(x)||M_{t_j}(x)(b_2^{(k)})(x)|
\]
\[
+ \sum_{j=j_0}^{j_1-1} |M_{t_j}(x)(b_2^{(k)})(x) - M_{t_{j+1}}(x)(b_2^{(k)})(x)||M_{t_j}(x)(b_1,k)(x)|,
\]
where \( x - t_{j_0}(x) \in I_{1,k} \) and \( x - t_{j_0-1}(x) \notin I_{1,k}, x - t_{j_1}(x) \in I_{1,k} \) and \( x - t_{j_1+1}(x) \notin I_{1,k} \). The second summand is dominated by
\[
\sum_{j=j_0}^{j_1-1} \frac{1}{t_j(x)} - \frac{1}{t_{j+1}(x)} \big\|b_1,k\|_{L^1} + \sum_{j=j_0}^{j_1-1} \frac{1}{t_j(x)} \int_{x-t_j(x)}^{x-t_{j+1}(x)} b_1,k(y)dy - \int_{x-t_{j+1}(x)}^{x-t_{j}(x)} b_1,k(y)dy
\]
\[
\leq \frac{1}{t_{j_1}(x)} \big\|b_1,k\|_{L^1} \leq \frac{C \|b_1,k\|_{L^1}}{d(x, I_{1,k})}.
\]
We estimate the third summand as
\[
\sum_{j=j_0}^{j_1-1} \frac{1}{t_j(x)} - \frac{1}{t_{j+1}(x)} \big\|I_{1,k}\|_{L^1} + \sum_{j=j_0}^{j_1-1} \frac{|I_{1,k}|}{t_j(x)} \big\|b_1,k\|_{L^1} + \sum_{j=j_0}^{j_1-1} \frac{|I_{1,k}|}{t_j(x)} \big\|b_2^{(k)}\|_{L^1}
\]
\[
\leq \frac{|I_{1,k}|}{t_{j_1}^2(x)} \big\|b_1,k\|_{L^1} \leq \frac{C \|b_1,k\|_{L^1}}{d(x, I_{1,k})},
\]
where we use the fact \( |I_{1,k}| \leq t_{j_1}(x) \). Finally, using Chebyshev’s inequality,
\[
\left| \left\{ x \notin \Omega_1 \cup \Omega_2 \mid \left( \sum_k V_\rho(\Omega(b_1,k, b_2^{(k)}))^\rho(x) > 1/8 \right) \right\} \right| \leq C \sum_k \int_{I_k} V_\rho(\Omega(b_1,k, b_2^{(k)}))^\rho(x)dx
\]
\[
\leq C \sum_k \|b_1,k\|_{L^1}^\rho \int_{I_k} \frac{1}{d(x, I_{1,k})^\rho}dx
\]
\[
\leq C \sum_k \|b_1,k\|_{L^1}^\rho |I_{1,k}|^{1-\rho} \leq C.
\]
This completes the proof of Proposition 2.3.

Now let turn to the proof of Corollary 1.2.

Proof. For each $\phi, \psi : \mathbb{Z} \to \mathbb{Z}$ we consider functions like $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} \phi([x]), & [x] + \frac{1}{4} \leq x \leq [x] + \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$ with

$$g(x) = \begin{cases} \psi([x]), & [x] + \frac{1}{4} \leq x \leq [x] + \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

For $L \in \mathbb{N}$ and $i \in \mathbb{Z}$, we observe that

$$Q_L(\phi, \psi)(i) = 4Q_{L+\frac{1}{4}}(f, g)(x), \quad x \in [i, i + \frac{3}{4}].$$

Further, we get that

$$V_\rho(Q(\phi, \psi))(i) \leq 4V_\rho(Q(f, g))(x), \quad x \in [i, i + \frac{3}{4}].$$

For the variational inequality for averages over cubes in Theorem 1.1 we deduce that

$$\|V_\rho(Q(\phi, \psi))\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$

\[3\] Variational inequality for conditional expectations

In the same way, we apply Lemma 2.1 and take the Banach space $B = \{a(j) : \|a\|_B = \|a\|_{V_\rho} < \infty\}$. Then, $V_\rho([E_j(f, g)]_j) = \|[E_j(f, g)]_j\|_B$. Lemma 2.1 implies Theorem 1.4 is a consequence of the following two propositions.

**Proposition 3.1.** For $\rho > 2$, $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_\rho([E_j(f, g)]_j)\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$
This completes the proof of Proposition 3.1.

Analogously, we apply the Calderón-Zygmund decomposition to functions $g$ by applying Hölder’s inequality and Lépingle’s inequality [17], we obtain immediately by scaling. It suffices to prove Proposition 3.3.

Proof. Obviously, we have

$$
\mathbb{E}_j(f, g)(x) = \sum_{I, J \in \mathcal{D}_j} \frac{1}{|I|} \int_I f(y) dy \chi_I(x) \frac{1}{|J|} \int_J g(z) dz \chi_J(x) = \mathbb{E}_j(f)(x) \mathbb{E}_j(g)(x).
$$

Then, we get

$$
|\mathbb{E}_{j+1}(f, g) - \mathbb{E}_j(f, g)| = |\mathbb{E}_{j+1}(f) \mathbb{E}_{j+1}(g) - \mathbb{E}_j(f) \mathbb{E}_j(g)|
\leq |\mathbb{E}_{j+1}(f) - \mathbb{E}_j(f)| \cdot |\mathbb{E}_{j+1}(g) - \mathbb{E}_j(g)| + |\mathbb{E}_{j+1}(g) - \mathbb{E}_j(g)| \cdot |\mathbb{E}_j(f)|.
$$

By applying Hölder’s inequality and Lépingle’s inequality [17], we obtain

$$
\|V_p(\{\mathbb{E}_j(f, g)\})\|_{L^p(\mathbb{R})} \leq \|M(f) \cdot V_p(\{\mathbb{E}_j(f)\})\|_{L^p} + \|M(f) \cdot V_p(\{\mathbb{E}_j(g)\})\|_{L^p}
\leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.
$$

This completes the proof of Proposition 3.3.

Remark 3.2. In fact, above bilinear variational inequality holds for $p = 1, 1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

The second proposition is the variational weak endpoint type estimate for conditional expectation sequence.

Proposition 3.3. For $\rho > 2$, we have

$$
\lambda |\{x \in \mathbb{R} : V_\rho(\{\mathbb{E}_j(f_1, f_2)\})_j(x) > \lambda\}|^2 \leq C\|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}
$$

uniformly in $\lambda > 0$.

Proof. By scaling, we assume that $\lambda = 1$ and $\|f_1\|_{L^1} = \|f_2\|_{L^1} = 1$, the general case follows immediately by scaling. It suffices to prove

$$
|\{x \in \mathbb{R} : V_\rho(\{\mathbb{E}_j(f_1, f_2)\})_j(x) > 1\}| \leq C.
$$

Analogously, we apply the Calderón-Zygmund decomposition to functions $f_i$ at height 1 to obtain functions $g_i, b_i$ and dyadic intervals $\{I_{i,k}\}_k$ such that $f_i = g_i + b_i$ and $b_i = \sum_k b_i,k$. Since

$$
|\{x \in \mathbb{R} : V_\rho(\{\mathbb{E}_j(f_1, f_2)\})_j(x) > 1\}| \leq |\{x \in \mathbb{R} : V_\rho(\{\mathbb{E}_j(g_1, g_2)\})_j(x) > 1/4\}| + |\Omega_1|
+ |\Omega_2| + |\{x \notin \Omega_1 : V_\rho(\{\mathbb{E}_j(b_1, g_2)\})_j(x) > 1/4\}|
+ |\{x \notin \Omega_2 : V_\rho(\{\mathbb{E}_j(g_1, b_2)\})_j(x) > 1/4\}|
+ |\{x \notin \Omega_1 \cup \Omega_2 : V_\rho(\{\mathbb{E}_j(b_1, b_2)\})_j(x) > 1/4\}|,
$$

we have...
it suffices to estimate each of above six sets. Applying Proposition 3.1, we observe
\[ |\{ x \in \mathbb{R} : V_\rho(\{E_j(g_1, g_2)\}_j(x) > 1/4)\} | \leq C\| V_\rho(\{E_j(g_1, g_2)\}_j)\|_{L^1} \leq C\| g_1 \|_{L^2} \| g_2 \|_{L^2} \leq C. \]

Clearly, \(|\Omega_1| + |\Omega_2| \leq C\). Note that \(E_j(b_{1,k})(x) = 0\) for \(x \not\in \tilde{I}_{1,k}\). Hence \(E_j(b_1, g_2)(x) = E_j(b_1)(x) \cdot E_j(g_2)(x) = 0\) for \(x \not\in \Omega_1\). Consequently,
\[ |\{ x \not\in \Omega_1 : V_\rho(\{E_j(b_1, g_2)\}_j(x) > 1/4)\} | = |\{ x \not\in \Omega_1 \cup \Omega_2 : V_\rho(\{E_j(b_1, g_2)\}_j(x) > 1/4)\} | = 0. \]

Similarly,
\[ |\{ x \not\in \Omega_2 : V_\rho(\{E_j(g_1, b_2)\}_j(x) > 1/4)\} | = 0. \]

This proves Proposition 3.3.

\[ \square \]

4 Variational inequality for approximations of the identity

In order to prove Theorem 1.5, we view the kernel \(\{\phi_t(y, z)\}_{t>0}\) as having values in the Banach space
\[ (4.1) \quad B = \{a(t) : \|a\|_B = \|a\|_{V_\rho} < \infty \}. \]

Then, \(V_\rho(\Phi(f, g))(x) = \{\phi_t(f, g)(x)\}_{t>0}\|_B\). Lemma 2.1 implies Theorem 1.5 is a consequence of the following two propositions:

**Proposition 4.1.** For \(\rho > 2\), \(1 < p, p_1, p_2 < \infty\) and \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\), we have
\[ \|V_\rho(\Phi(f, g))\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}. \]

**Proposition 4.2.** For \(\rho > 2\), then
\[ \lambda^2 |\{ x \in \mathbb{R} : V_\rho(\Phi(f, g))(x) > \lambda \} | \leq C\|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})} \]
for any \(\lambda > 0\).

4.1 Variational inequality with \(1 < p, p_1, p_2 < \infty\).

The goal of this subsection is to prove Proposition 4.1. Let \(\varphi \in \mathcal{S}(\mathbb{R})\) and \(\int_\mathbb{R} \varphi(x)dx = 1\). Then, we have the following pointwise estimate:
\[ V_\rho(\Phi(f, g)) \leq V_\rho(\{\varphi_t(f) \cdot \varphi_t(g)\}_{t>0}) + V_\rho(\{\phi_t(f, g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0}). \]

Hence, it suffices to estimate the \(L^p\) norms of \(V_\rho(\{\varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\) and \(V_\rho(\{\phi_t(f, g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\).
Lemma 4.3. For $\rho > 2$, $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_{\rho}(\{\varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$ 

Proof. Note that

$$|\varphi_{t_i}(f) \cdot \varphi_{t_i}(g) - \varphi_{t_i+1}(f) \cdot \varphi_{t_i+1}(g)|$$

$$\leq |\varphi_{t_i}(f) - \varphi_{t_{i+1}}(f)| \cdot |\varphi_{t_i}(g)| + |\varphi_{t_i}(g) - \varphi_{t_i+1}(g)| \cdot |\varphi_{t_{i+1}}(f)|.$$ 

Then, by using Hölder’s inequality and Theorem 2.6 in [18], we get

$$\|V_{\rho}(\{\varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq \|M(g) \cdot V_{\rho}(\{\varphi_t(f)\}_{t>0})\|_{L^p} + \|M(f) \cdot V_{\rho}(\{\varphi_t(g)\}_{t>0})\|_{L^p}$$

$$\leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$ 

\[\square\]

The long variation operator $V_{\rho}^L(\mathcal{F})$ of the family $\mathcal{F}$ at $x$ is defined by

$$(4.2) \quad V_{\rho}^L(\mathcal{F})(x) = \|\{F_{2^n}(x)\}_{n\in\mathbb{Z}}\|_{V_{\rho}}, \quad \rho \geq 1.$$ 

Moreover, the short variation operator

$$S_2(\mathcal{F})(x) = \left( \sum_{j \in \mathbb{Z}} \|\{F_{1}(x)\}_{i>0}\|_{V_{2^j,2^{j+1}}^2}^2 \right)^{1/2}.$$ 

Then the following pointwise comparison holds.

Lemma 4.4. (see [14] Lemma 1.3)

$$(4.3) \quad V_{\rho}(\mathcal{F})(x) \lesssim V_{\rho}^L(\mathcal{F})(x) + S_2(\mathcal{F})(x).$$ 

Lemma 4.5. For $\rho > 2, 1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\|V_{\rho}(\{\varphi_t(f,g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$ 

Proof. To estimate the $L^p$ norm of $V_{\rho}(\{\varphi_t(f,g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0})$, we denote the function $\phi(y,z) - \varphi(y)\varphi(z)$ by $\psi(y,z)$ for convenience. [14,15] reduces above desired estimate to

$$(4.4) \quad \|V_{\rho}^L(\{\psi_t(f,g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}$$

and

$$(4.5) \quad \|S_2(\{\psi_t(f,g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}.$$
We show (4.4) first. Clearly, for $\rho > 2$ we have

$$V^L_\rho(\{\phi_t(f,g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0}) = V^L_\rho(\{\psi_t(f,g)\}_{t>0}) \leq \left( \sum_j |\psi_{2j}(f,g)|^2 \right)^{1/2}.$$ 

Hence, it suffices to prove

$$\|\left( \sum_j |\psi_{2j}(f,g)|^2 \right)^{1/2}\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})},$$

for $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

To obtain (4.6), we apply [10, Theorem 1.1] and verify $\psi$ satisfying related conditions. Note that $\phi \in \mathcal{S}(\mathbb{R}^2)$ and $\varphi \in \mathcal{S}(\mathbb{R})$, then $\psi \in \mathcal{S}(\mathbb{R}^2)$. Hence, for any $N \in \mathbb{N}$ and multi-indices $\alpha$ we have

$$|\partial^\alpha \psi(y,z)| \leq C_N \frac{1}{(1 + |y| + |z|)^2N} \leq C_N \frac{1}{(1 + |y|)^N(1 + |z|)^N}.$$ 

Moreover, it satisfies the cancellation condition

$$\int_{\mathbb{R}^2} \psi(y,z)dydz = \int_{\mathbb{R}^2} \phi(y,z)dydz - \int_{\mathbb{R}} \varphi(y)dy \cdot \int_{\mathbb{R}} \varphi(z)dz = 0.$$ 

As a result, we obtain

$$\|V^L_\rho(\{\phi_t(f,g) - \varphi_t(f) \cdot \varphi_t(g)\}_{t>0})\|_{L^p(\mathbb{R})} \leq \|\left( \sum_j |\psi_{2j}(f,g)|^2 \right)^{1/2}\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})},$$

and complete the proof of (4.4).

Next we turn to proof of (4.5). By using Bergh and Peetre's [11] estimate

$$\|a\|_{V_\rho} \leq \|a\|_{L^{p'}}^{1/p} \|a'\|_{L^{p'}}^{1/p},$$

we observe that

$$S^2_2(\{\psi_t(f,g)\}_{t>0})(x) = \sum_k \|\psi_t(f,g)\|^2_{L^2[2^k,2^{k+1}]} \leq \sum_k \|\psi_t(f,g)\|_{L^2[2^k,2^{k+1}]} \|\frac{d}{dt} \psi_t(f,g)\|_{L^2_t[2^k,2^{k+1}]} \leq C \left( \int_0^\infty |\psi_t(f,g)(x)|^2 \frac{dt}{t} \right)^{1/2} \left( \int_0^\infty |\tilde{\psi}_t(f,g)(x)|^2 \frac{dt}{t} \right)^{1/2} := CG(f,g)(x)\tilde{G}(f,g)(x),$$
where \( \hat{\psi}(y, z) = 2\psi(y, z) + y\partial_y \psi(y, z) + z\partial_z \psi(y, z) \). Note that \( \psi, \hat{\psi} \in \mathcal{S}(\mathbb{R}^2) \), for any \( N \in \mathbb{N} \) we have

\[
|\psi(\xi, \eta)| + |\hat{\psi}(\xi, \eta)| \leq \frac{C}{(1 + |(\xi, \eta)|)^N} \quad \text{and} \quad \hat{\psi}(0, 0) = \hat{\psi}(0, 0) = 0.
\]

Using [25, Example 2.1] and [27, Theorem 1.2], we get

\[
\|G(f, g)\|_{L^p(\mathbb{R})} + \|\hat{G}(f, g)\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}
\]

for \( 1 < p, p_1, p_2 < \infty \). Furthermore, by Hölder’s inequality

\[
\|S_2(\{\psi_t(f, g)\}_{t > 0})\|_{L^p(\mathbb{R})}^p = \int_\mathbb{R} S_2(\{\psi_t(f, g)\}_{t > 0})^2 \frac{1}{t} (x) dx \leq \int_\mathbb{R} G(f, g)^{\frac{p}{2}}(x) \hat{G}(f, g)^{\frac{p}{2}}(x) dx
\]

\[
\leq C\|G(f, g)\|_{L^p(\mathbb{R})}^p \|\hat{G}(f, g)\|_{L^p(\mathbb{R})}^p \leq C\|f\|_{L^{p_1}(\mathbb{R})}^p \|g\|_{L^{p_2}(\mathbb{R})}^p.
\]

This completes the proof of (4.5). \( \square \)

### 4.2 Variational weak endpoint type estimate

To prove Proposition 4.2, we use bilinear vector-valued Calderón-Zygmund theory. Let \( \mathcal{B} \) be the Banach space given by (4.1) and \( F \) be a bilinear function defined on \( \mathbb{C} \times \mathbb{C} \) to \( \mathcal{B} \), we define

\[
\|F\|_{BL(\mathbb{C} \times \mathbb{C} \rightarrow \mathcal{B})} = \sup_{|\xi_1|, |\xi_2| \leq 1} \|F(\xi_1, \xi_2)\|_{\mathcal{B}}.
\]

Let \( T \) be a bilinear operator defined on \( \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \) and taking values in \( \mathcal{S}'(\mathbb{R}; \mathcal{B}) \). Assume that the restriction of its distributional kernel away from the diagonal \( x = y = z \) in \( \mathbb{R}^3 \) coincides with a \( \mathcal{B} \)-valued function \( K \), satisfying the size condition

\[
\|K(x, y, z)\|_{BL(\mathbb{C} \times \mathbb{C} \rightarrow \mathcal{B})} \leq \frac{C}{(|x - y| + |x - z|)^2} \quad \text{for} \quad |x - y| + |x - z| \neq 0,
\]

the regularity conditions

\[
\|K(x, y, z) - K(x + h, y, z)\|_{BL(\mathbb{C} \times \mathbb{C} \rightarrow \mathcal{B})} + \|K(x, y, z) - K(x, y + h, z)\|_{BL(\mathbb{C} \times \mathbb{C} \rightarrow \mathcal{B})}
\]

\[
+ \|K(x, y, z) - K(x, y, z + h)\|_{BL(\mathbb{C} \times \mathbb{C} \rightarrow \mathcal{B})} \leq \frac{C|h|}{(|x - y| + |x - z|)^3}
\]

for \( |h| \leq \max(|x - y|, |x - z|)/2 \), and such that

\[
T(f, g)(x) = \int_{\mathbb{R}^2} K(x, y, z) f(y) g(z) dy dx
\]

whenever \( f, g \in \mathcal{D}(\mathbb{R}) \) and \( x \notin \text{supp } f \cap \text{supp } g \). Under above assumptions and \( T \) is bounded \( L^{p_1} \times L^{p_2} \rightarrow L^p(\mathcal{B}) \) for some \( 1 < p, p_1, p_2 < \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), we will say \( T \) is a bilinear \( \mathcal{B} \)-valued Calderón-Zygmund operator. We state a weak endpoint result in [10] for bilinear vector-valued Calderón-Zygmund operators as follows.
Lemma 4.6. If $T$ is a bilinear $B$-valued Calderón-Zygmund operator, then $T$ is bounded $L^1 \times L^1 \rightarrow L^{1/2, \infty}(B)$.

Proof. Lemma 4.6 implies that it suffices to verify $\{\phi_t(f, g)\}_{t > 0}$ be a bilinear $B$-valued Calderón-Zygmund operator. We have proved that $\{\phi_t(f, g)\}_{t > 0}$ is bounded $L^{p_1} \times L^{p_2} \rightarrow L^p(B)$ for $1 < p, p_1, p_2 < \infty$ in Proposition 4.1, it suffices to verify the kernel $\{\phi_t(y, z)\}_{t > 0}$ satisfying related size condition and regularity conditions.

We consider the size condition first. Note that $\|a\|_B = \|a\|_{V_\rho} \leq \|a\|_{V_1} \leq \int_0^\infty |a'(t)|dt$. Then,

$$\|\{\phi_t(y, z)\}_{t > 0}\|_B \leq \int_0^\infty \left| \frac{d}{dt} \phi_t(y, z) \right| dt \leq C \int_0^\infty \left[ \frac{1}{t^3} \left| \phi\left( \frac{y}{t}, \frac{z}{t} \right) \right| + \frac{1}{t^4} \left| \frac{y}{t} \phi_1\left( \frac{y}{t}, \frac{z}{t} \right) + \frac{z}{t} \phi_2\left( \frac{y}{t}, \frac{z}{t} \right) \right| \right] dt \leq C \int_0^\infty \left| \frac{(y, z)}{(1 + \frac{|y, z|}{t})} \right| dt \leq C \frac{|(y, z)|^2}{(|y| + |z|)^2},$$

where $\phi_1(y, z) = \partial_y \phi(y, z)$ and $\phi_2(y, z) = \partial_z \phi(y, z)$.

For the regularity condition, we have

$$\|\{\phi_t(y, z) - \phi_t(y', z)\}_{t > 0}\|_B \leq C \int_0^\infty \frac{|y - y'|}{t^4 \left( 1 + \frac{|(y, z)|}{t} \right)} dt \leq C \frac{|y - y'|}{|y| + |z|},$$

In the same way, we have

$$\|\{\phi_t(y, z) - \phi_t(y, z')\}_{t > 0}\|_B \leq C \int_0^\infty \frac{|z - z'|}{t^4 \left( 1 + \frac{|(y, z)|}{t} \right)} dt \leq C \frac{|z - z'|}{|y| + |z|}.$$

This completes the proof of Proposition 4.2.

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