SOMMERFELD PARTICLE IN STATIC MAGNETIC FIELD: TUNNELING AND DELAYED UNTWISTING IN CYCLOTRON

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Motion of a charged particle with finite size, described by Sommerfeld model, in static magnetic field has two peculiar features: 1.) there is the effect of tunneling - Sommerfeld particle overcomes the barrier and finds itself in the forbidden, from classical point of view, area; 2.) the untwisting of trajectory in cyclotron for Sommerfeld particle is strongly delayed compared to that of a classical particle.

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Here we continue our investigation of peculiar features of motion of Sommerfeld particle [1]. Let us remind that long time ago [2] Sommerfeld proposed a model of a charged particle of finite size - sphere with uniform surface charge $Q$ and mechanical mass $m$. In nonrelativistic approximation such sphere obeys the equation (see also [3]):

$$m\ddot{\vec{v}} = \vec{F}_{ext} + \eta [\vec{v}(t-2a/c) - \vec{v}(t)]$$ (1)

here $a$ - radius of the sphere, $\eta = \frac{Q^2}{3ca^3}$, $\vec{v} = d\vec{R}/dt$, $\vec{R}$ - coordinate of the center of the shell, $\vec{F}_{ext}$ - some external force.

This model is a good tool to consider effects of radiation reaction of a charged particle of finite size, free of problems of classical point-like Lorentz-Dirac description.

A.

If Sommerfeld particle moves in the external static magnetic field $\vec{H}$, the
force $\vec{F}_{\text{ext}} = \int d\vec{r} \rho \cdot [\frac{\vec{r}}{||\vec{r}||} - \vec{R}]$ for $\rho = Q\delta(||\vec{r} - \vec{R}|| - a)/4\pi a^2$ has the form

$$F_{\text{ext}} = \frac{Q}{c} [\vec{R}, \vec{H}]$$

If magnetic field has non-zero values only in the shell of finite size $S$ ($0 < Y < S$, $\vec{H}$ is parallel to z-axis, $\vec{R} = (X, Y, 0)$), then, as the particle has finite size $2a$, force $\vec{F}_{\text{ext}}$ must be multiplied by the factor $f$:

$$f = \begin{cases} 
0, & Y < -a; \\
\frac{Y}{2a} + \frac{1}{2}, & -a < Y < a; \\
1, & a < Y < S - a; \\
\frac{S - Y}{2a} + \frac{1}{2}, & S - a < Y < S + a; \\
0, & S + a < y;
\end{cases} \quad (2)$$

For dimensionless variables $x = X/M, \ y = Y/M, \ \tau = ct/M$ (M -scale factor) equation (1) takes the form

$$\ddot{y} = K \cdot [\dot{y}(\tau - d) - \dot{y}(\tau)] - \lambda \cdot \dot{x} \cdot f,$$
$$\ddot{x} = K \cdot [\dot{x}(\tau - d) - \dot{x}(\tau)] + \lambda \cdot \dot{y} \cdot f, \quad (3)$$

here

$$f = \begin{cases} 
0, & y < -\frac{d}{2}; \\
\frac{y}{d} + \frac{1}{2}, & -\frac{d}{2} < y < \frac{d}{2}; \\
1, & \frac{d}{2} < y < L - \frac{d}{2}; \\
\frac{L - y}{d} + \frac{1}{2}, & L - \frac{d}{2} < y < L + \frac{d}{2}; \\
0, & L + \frac{d}{2} < y;\end{cases} \quad (4)$$

and

$$K = \frac{Q^2 M}{3a^2 mc^2}, \ \lambda = \frac{QHM}{mc^2}, \ d = \frac{2a}{M}, \ L = \frac{S}{M}.$$  

Classical analog of equation (3) for point-like particle without radiation reaction reads

$$\ddot{y} = -\lambda \cdot \dot{x} \cdot g,$$
$$\ddot{x} = \lambda \cdot \dot{y} \cdot g, \quad (5)$$

here

$$g = \begin{cases} 
0, & y < 0; \\
1, & 0 < y < L; \\
0, & L < y;\end{cases} \quad (6)$$
For initial conditions \( x(0) = 0, \ y(0) = 0, \ \dot{x}(0) = 0, \ \dot{y}(0) = v \) solution of (5) is

\[
\begin{align*}
x &= -\frac{v}{\lambda} + \frac{v}{\lambda} \cos(\lambda \tau), \\
y &= \frac{v}{\lambda} \sin(\lambda \tau) \quad (0 < y < L)
\end{align*}
\]

We see that for initial velocities \( v \) smaller, then the critical velocity \( v_{cr} = \lambda L \), particle trajectory (half-circle) lies inside the shell, i.e. particle cannot overcome the barrier. If \( L = 10^4, \ \lambda = 10^{-4} \) then \( v_{cr} = 1 \).

We numerically investigated the particle motion governed by equation (3) for the following values of initial velocity:

\[
v = 0.43, \quad v = 0.44
\]

and for

\[
L = 10^4, \ \lambda = 10^{-4}, \ d = 1.0, \ K = 4/(3d^2),
\]

i.e. particle is of electron size and mass, magnetic field approximately equals \( 10^{12} \) gauss and \( S \approx 5.6 \cdot 10^{-9} \) sm.

The result is shown on Fig. A, compared with classical trajectory, governed by (7) with \( v = 0.44 \). Horisontal axis is \( x \) and vertical axis is \( y \).

The effect of tunneling for Sommerfeld particle is vividly seen: velocity \( v = 0.44 \) is smaller then the critical \( v_{cr} = 1 \), but the particle overcomes the barrier and finds itself in the forbidden from classical point of view area \( y > L = 10^4 \).

**B.**

If magnetic field is parallel to \( z \)-axis for \( y < 0 \) and \( y > L \) and equals to zero for \( 0 < y < L \), and for \( 0 < y < L \) there is static electric field \( E \), parallel to \( y \)-axis in such a way, that it is always collinear to \( y \)-component of particle velocity (i.e. particle is always accelerates in the clearance \( 0 < y < L \)), then there is a model of cyclotron.

Equation of motion for Sommerfeld particle in cyclotron reads

\[
\begin{align*}
\ddot{y} &= K \cdot [\dot{y}(\tau - d) - \dot{y}(\tau)] - \lambda \cdot \dot{x} \cdot f + \epsilon \cdot Sgn(\dot{y}) \cdot (1 - f), \\
\ddot{x} &= K \cdot [\dot{x}(\tau - d) - \dot{x}(\tau)] + \lambda \cdot \dot{y} \cdot f,
\end{align*}
\]

\[
(8)
\]
Here
\[ \epsilon = \frac{QEM}{mc^2} \]

Classical analog of (8) one can construct replacing in (8) \( K \) by zero and \( f \) by \( g \) (6):
\[ \dot{y} = -\lambda \cdot \dot{x} \cdot g + \epsilon \cdot Sgn(\dot{y}) \cdot (1 - g), \]
\[ \ddot{x} = \lambda \cdot \dot{y} \cdot g, \]  
(9)

Initial conditions are:
\[ x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0 \]

Due to classical equation of motion without radiation reaction (9) particle moves along untwisting trajectory. Total increase of kinetic energy \( W_c = (\dot{x})^2/2 + (\dot{y})^2/2 \) of particle is \( N \cdot \epsilon \cdot L \):
\[ W_c = N \cdot \epsilon \cdot L \]

where \( N \) - is the total number of passing of particle through the accelerating field \( E \).

If \( N = 10, \ \epsilon = \lambda = 10^{-7}, \ \ L = 10^5 \), then
\[ W_c = 10^{-1}. \]

We numerically calculated the particle motion governed by equation (8) with zero initial conditions for the following values of parameters:
\[ L = 10^5, \ \lambda = 10^{-7} = \epsilon, \ \ d = 0.3, \ \ K = 2.0, \]
i.e. particle is of electron size and mass, magnetic field approximately equals to \( 8.1 \cdot 10^7 \) gauss and electric field produces in the clearance potential difference equal to \( 10^4 \) eV.

The results of calculations are shown on Fig. B.1 - classical case and on Fig. B.2 - case of Sommerfeld particle. Horizontal axis is \( x \cdot \lambda \) and vertical axis is \( y \cdot \lambda \).

We see that for the same ”time” \( \tau \approx 10^8 \) (i.e. \( t \approx 10^{-4} \) sec) classical particle (without radiation reaction) made \( N = 10 \) passings through the accelerating field \( E \) with total energy increase \( W_c = 10^{-1}, \) while Sommerfeld particle made only \( N = 6 \) passings with total energy increase \( W_s = 0.0375 \) ( \( W_c \)
for $N = 6$ is equal to 0.06). Thus untwisting of trajectory for Sommerfeld particle is strongly delayed compared to that of a classical one.

Delay in energy increase falls mainly on the moments of passing through the clearance. It can be explained by difference in accelerations in electric field (proportional to $\epsilon \approx 10^{-7}$) and in magnetic field (proportional to $v \cdot \lambda \approx 10^{-8}$) as flux of radiating energy is proportional to square of acceleration.

REFERENCES

1. Alexander A. Vlasov, physics/9905050, physics/9911059.
2. A. Sommerfeld, Gottingen Nachrichten, 29 (1904), 363 (1904), 201 (1905).
3. L. Page, Phys. Rev., 11, 377 (1918). T. Erber, Fortschri. Phys., 9, 343 (1961). P. Pearle in "Electromagnetism", ed. D. Tepliz, (Plenum, N.Y., 1982), p. 211. A. Yaghjian, "Relativistic Dynamics of a Charged Sphere". Lecture Notes in Physics, 11 (Springer-Verlag, Berlin, 1992).
Fig. A
Fig. B.1.
