Compositeness Condition

Keiichi Akama
Department of Physics, Saitama Medical College
Kawakado, Moroyama, Saitama, 350-04, Japan

By solving the compositeness condition, under which the Yukawa-type model coincides the NJL type model, we obtain the expressions for the effective coupling constants in terms of the compositeness scale at the next-to-leading order in $1/N$. In the NJL model with a scalar composite, the next to leading contributions are too large for $N_c = 3$. In the induced gauge theory with abelian gauge symmetry, the correction term is reasonably suppressed, while in the non-abelian gauge theory the corrections are suppressed only when it is asymptotically non-free.

1. Introduction

This talk is based on the recent works done in collaboration with Takashi Hattori from Kanagawa Dental College. The origin of the generations is one of the most challenging problem in physics today and future. Among various ideas, compositeness may be a natural strong candidate for the solution of the problem. Repeated appearance of the color triplets in the quark lepton spectrum seems to suggest existence of the common subconstituent $c_i (i = 1, 2, 3)$ which carries color [1], and repeated appearance of the weak doublets suggests existence of the common subconstituent $w_i (i = 1, 2)$ which carries weak isospin [2]. Then the quarks $q$ and leptons $l$ are composed of them as

$$ q \sim whc, \quad l \sim wh', \quad (1) $$

where $h$ and $h'$ are subconstituents or sets of subconstituents (including empty sets) depending on details of models. The weak boson $W^i_\mu$ and the Higgs scalar $\phi$ can also be composites like

$$ W^i_\mu \sim \overline{w}_L \tau^i \gamma_\mu w_L \text{ or } \overline{q}_L \tau^i \gamma_\mu q_L \quad \phi \sim \overline{w}_R w_L \text{ or } \overline{q}_R q_L \quad (2) $$

About twenty years ago, we considered a composite model of the Nambu-Jona-Lasinio type [3], where the gauge bosons and the Higgs scalar appear as composites of the quarks and leptons, or subquarks [4]. In this type of model, the compositeness condition [3] plays an important role. This is the condition that

$$ Z_3 = 0 \quad (3) $$

where $Z_3$ is the wave-function renormalization constant of a boson in a Yukawa-type model. Under this condition, the Yukawa type model with the elementary boson becomes equivalent to the Nambu-Jona-Lasinio type model [3], and the elementary boson becomes a composite. The compositeness condition imposes relations among the coupling constants, masses and the compositeness scale. When it is applied to the standard model where gauge bosons and Higgs scalar are taken as composites of the quarks and leptons, the relations indicate that at least one of the quarks should have a mass of the order of the weak interaction scale [3]. It looked puzzling because the known quarks at that time were much lighter than the weak scale. Today, however, we know that the top quark has the mass of the order of the weak scale [7], and the relation derived from the compositeness condition becomes rather natural. This fact called the revived attentions to the NJL-type model of the spontaneously broken electroweak symmetry [8]. Numerically, however, it does not precisely hold. We need to consider how to make it more precise beyond the leading approximation in $1/N$ [9].

2. Nambu-Jona-Lasinio Model

2.1 Compositeness Condition

We consider the NJL model for the fermion $\psi = \{ \psi_1, \psi_2, \cdots, \psi_N \}$ with $N$ colors given by the Lagrangian

---

* Invited talk given at Shizuoka Workshop on Masses and Mixing of Quarks and Leptons, Shizuoka, Japan, March 1997. To be published in Proceedings.
† E-mail: akama@tanashi.kek.jp, akama@saitama-med.ac.jp
\[ \mathcal{L}_{\text{NJL}} = \overline{\psi} i \gamma \psi + f |\overline{\psi}_L \psi_R|^2 \]  

(4)

with \( U(1) \times U(1) \) chiral symmetry. In 3+1 dimensions, it is not renormalizable, and we assume a very large but finite momentum cutoff. This Lagrangian \( \mathcal{L}_{\text{NJL}} \) is known to be equivalent to the linearized Lagrangian \[ \mathcal{L}_{\text{NJL}}' = \overline{\psi} i \gamma \psi + (\overline{\psi}_L \phi \psi_R + \text{h.c.}) - \frac{1}{f} |\phi|^2 \]  

(5)

which is written in terms of the auxiliary field \( \phi \). Now compare it with this Lagrangian of the renormalized Yukawa model,

\[ \mathcal{L}_{\text{Yukawa}} = Z \overline{\psi} \psi = Z g_i \overline{\psi}_L \phi \psi_R + \text{h.c.} - \frac{1}{f} |\phi|^2 \]  

(6)

where the quantities indicated by suffices \( r \) are renormalized ones, and \( Z \)'s are the renormalization constants. We can see that if \( Z_\phi = Z_\lambda = 0 \),

(7)

the Lagrangians \( \mathcal{L}_{\text{NJL}}' \) and \( \mathcal{L}_{\text{Yukawa}} \) coincide, where we identify \( \psi, \phi \) and \( f \) with

\[ \psi = \sqrt{Z_\psi} \psi_r, \quad \phi = \frac{Z_g}{Z_\psi} g_r \phi_r, \quad f = \frac{Z_g^2 g_r^2}{Z_\psi^2 Z_\mu^2} \]  

(8)

in the Yukawa model. This condition \( \mathcal{L}_{\text{NJL}}' = \mathcal{L}_{\text{Yukawa}} \) is called the compositeness condition \( \mathcal{L}_{\text{NJL}}' = \mathcal{L}_{\text{Yukawa}} \). Thus this Lagrangian for NJL model is the special case of the renormalized Yukawa model specified by the compositeness condition. The compositeness condition gives rise to relations among coupling constants \( g_r, \lambda_r \), and the cut off \( \Lambda \). If the chiral symmetry is spontaneously broken, they imply relations among the fermion mass \( m_f \), the Higgs-scalar mass \( M_H \), and the cutoff \( \Lambda \). Thus we can analyze everything in the NJL model by investigating the well-understood Yukawa model, and by imposing the compositeness condition on the coupling constants and masses. Then what is urgent is to work out the compositeness condition, and solve it for the coupling constants.

### 2.2 Lowest order

For an illustration, we begin with the lowest-order contributions in \( 1/N \) expansion. In the Yukawa model, the boson self-energy part and the four-boson vertex part are given by the diagrams

\[ \sim g_r^2 NI \rho^2, \quad \sim (Z_\phi - 1) \rho^2, \]  

(9)

where \( \circ \) and \( \bullet \bullet \bullet \bullet \) are the fermion and boson propagator, respectively, and \( I \) is the divergent integral

\[ I = \left\{ \begin{array}{ll} \frac{1}{16\pi^2} \frac{1}{\epsilon} & \text{(dimensional regularization)} \quad (\epsilon = 4 - \frac{d}{2}) \\ \frac{1}{16\pi^2} \log \Lambda^2 & \text{(Pauli Villars regularization)} \end{array} \right. \]  

(10)

The renormalization constants \( Z_\phi \) and \( Z_\lambda \) should be chosen as

\[ Z_\phi = 1 - g_r^2 NI, \quad Z_\lambda \lambda_r = \lambda_r - g_r^4 NI \]  

(11)

so as to cancel out all the divergences in \( I \). Then the compositeness condition is obtained by putting \( Z_\phi \) and \( Z_\lambda \) vanishing,

\[ 0 = 1 - g_r^2 NI, \quad 0 = \lambda_r - g_r^4 NI, \]  

(12)

and it is easily solved to give the expressions for the coupling constants

\[ g_r^2 = \frac{1}{NI}, \quad \lambda_r = \frac{1}{NI}. \]  

(13)
If the chiral symmetry is spontaneously broken the masses of the physical fermion and physical Higgs scalar are given in terms of cutoff:

\[ m_f = g_r \langle \phi \rangle / \sqrt{N} I, \quad M_H = 2 \sqrt{\lambda_r} \langle \phi \rangle / \sqrt{N} I \]  

(14)

The Higgs mass is twice the fermion mass.

\[ 2m_f = M_H \]  

(15)

These reproduce the well known results of the lowest order Nambu-Jona-Lasinio model [3].

### 2.3 Next-to-leading order

Now we turn to the next-to-leading order in \( \frac{1}{\sqrt{N}} \) [12]. In the Yukawa model, the boson self-energy part is given by the diagram

\[ + \text{the counter terms for all the sub-diagram divergences}, \]  

(16)

where \( \ldots \ldots = \ldots \ldots + \ldots \ldots + \ldots \ldots + \ldots \ldots \). The renormalization constant \( Z_\phi \) is calculated to be

\[ Z_\phi = 1 - g_r^2 N I - g_r^2 I - \frac{1}{N} (1 - g_r^2 N I) \log(1 - g_r^2 N I) \]  

(17)

so as to cancel out all the divergences in (16). The logarithm arises from the infinite sum over the fermion loop insertions into the internal boson lines. The four boson vertex part is given by the diagrams

\[ + \text{counter terms for the sub-diagram divergences}. \]  

(18)

The renormalization constant \( Z_\lambda \) is calculated to be

\[ Z_\lambda = \lambda_r - g_r^4 N I + 8 g_r^4 I + \frac{20(\lambda_r - g_r^2)^2 I}{1 - g_r^2 N I} \]  

\[ - \frac{1}{N} \left[ 2g_r^2 (1 - g_r^2 N I) + 20(\lambda_r - g_r^2) \right] \log(1 - g_r^2 N I) \]  

(19)

so as to cancel out all the divergences in (18). The compositeness condition is given by putting these expressions vanishing. Though it looks somewhat complex at first sight, it can be solved by iteration to give the very simple solution:

\[ g_r^2 = \frac{1}{NI} \left[ 1 - \frac{1}{N} + O\left( \frac{1}{N^2} \right) \right], \quad \lambda_r = \frac{1}{NI} \left[ 1 - \frac{10}{N} + O\left( \frac{1}{N^2} \right) \right]. \]  

(20)

The next-to-leading correction to the ratio of \( M_H \) and \( m_f \), which was 2 in the lowest order, is calculated to be:

\[ \frac{M_H}{m_f} = \frac{g_r}{\sqrt{\lambda_r}} = 2 \left[ 1 - \frac{9}{2N} + O\left( \frac{1}{N^2} \right) \right] \]  

(21)

For the case of \( N = 3 \) of the practical interest, the corrections turn out to be too large, and the coupling constant \( \lambda \) is negative, which implies that the Higgs potential is unstable.

### 3. Induced Gauge Theory — Abelian —

We can apply [13] this method to the induced gauge theory [14], namely, the gauge theory with a composite gauge field. It is given by the strong coupling limit \( f \to \infty \) [13] of the vector-type four Fermi interaction model for the fermion \( \psi \) with the mass \( m \):

\[ \mathcal{L}_{4F} = \overline{\psi_j} (i \not\!\!D - m) \psi - f \left( \overline{\psi} \gamma_\mu \psi \right)^2, \]  

(22)

where \( f \) is the coupling constant. The Lagrangian \( \mathcal{L}_{4F} \) is equivalent to

\[ \mathcal{L}'_{4F} = \overline{\psi} \left( i \not\!\!D - m - \Delta \right) \psi \]  

(23)
written in terms of the vectorial auxiliary field $A_\mu$. Then we can see that this is the special case of the renormalized gauge theory

$$L_G = \bar{\psi}_i (iZ_2 \partial - Z_m m - Z_1 e_i A_\mu) \psi_i - \frac{1}{4} Z_3 (F_\mu^\nu)^2,$$

specified by the compositeness condition

$$Z_3 = 0,$$

where the quantities indicated by suffixes $r$ are renormalized ones, $e$ is the effective coupling constant, $Z$’s are the renormalization constants, and $F_\mu^\nu$ is the field strength of $A_\mu$.

The self-energy part of the gauge boson is given by the diagrams

![Diagram](image)

(26)

at the leading and the next-to-leading order. The renormalization constant $Z_3$ is chosen so as to cancel out the divergences in these diagrams. After a lengthy calculation we obtain the following expression for $Z_3$:

$$Z_3 = 1 - \frac{e_i^2 N}{12\pi^2 \epsilon} - \frac{3e_i^2}{16\pi^2} \left[ 1 + \frac{12}{16}\epsilon \ln \left( 1 - \frac{e_i^2 N}{12\pi^2 \epsilon} \right) \right],$$

(27)

where $\epsilon = (4 - d)/2$ with the dimension $d$. Then the compositeness condition $Z_3 = 0$ is solved to give the simple solution:

$$e_i^2 = \frac{12\pi^2 \epsilon}{N} \left[ 1 - \frac{9\epsilon}{4N} \right].$$

(28)

The correction term $9\epsilon/4N$ is naturally suppressed by the small factor $\epsilon$. It justifies the lowest order approximation of this model unlike in the case of the aforementioned NJL model of the scalar composite. The origin of the suppression factor is traced back to the gauge cancellation of the leading divergence in the next-to-leading (in $1/N$) diagrams in (26).

So far we assumed that all the fermions have the same charges for simplicity. If the charges $Q_i$ are different, the expression is modified as

$$e_i^2 = \frac{12\pi^2 \epsilon}{\sum_j Q_j^2} \left[ 1 - \frac{9\epsilon \sum_j Q_j^2}{4(\sum_j Q_j^2)^2} \right].$$

(29)

If we apply this to the quantum electrodynamics with 3 generations of quarks and leptons, $\epsilon$ is estimated to be $6 \times 10^{-3}$, which implies the next-to-leading order correction amounts only to 0.1% of the lowest order term.

4. Induced Gauge Theory — Nonabelian —

Now we apply it to the non-abelian induced gauge theory. This is the new result obtained after my talk at SCGT96, International Workshop on Perspectives of Strong Coupling Gauge Theories, held in Nagoya, Japan, in November 1996. We start with the four Fermi Lagrangian

$$L_{4F} = \bar{\psi} (i\partial - m) \psi - f (\psi \lambda^a \gamma_\mu \nu \psi)^2$$

(30)

for the fermion $\psi$ with $N_c$ gauged color and $N_f$ ungauged flavor, where $\lambda_a (a = 1, \cdots, N_c^2 - 1)$ is the $N_c \times N_c$ Gell-Mann matrix, $m$ is the mass of the fermion and $f$ is the coupling constant. In the strong coupling limit $f \to \infty$, the Lagrangian $L_{4F}$ is equivalent to the linearized one

$$L'_{4F} = \bar{\psi} (i\partial - m - \lambda^a A^a_\mu) \psi$$

(31)

written in terms of the auxiliary field $A^a_\mu$. Then $L'_{4F}$ is the special case of the renormalized gauge theory

$$L_G = \bar{\psi}_i (iZ_2 \partial - Z_m m - Z_1 e_i A^a_\mu) \psi_i - \frac{1}{4} Z_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z_3^{1/2} Z_9 g_f f^{abc} A^{b}_\mu A^{c}_\nu)^2,$$

(32)
specified by the compositeness condition

\[ Z_3 = 0, \]  

where the quantities indicated by suffices \( r \) are renormalized ones, \( g \) is the effective coupling constant, \( Z \)'s are the renormalization constants, and \( F_{\mu \nu} \) is the field strength of \( A^a_{\mu} \). The self-energy part of the gauge boson, at the leading order and the next-to-leading order in \( 1/N_f \), is given by the diagrams

\[ \text{(34)} \]

where \( \cdots \cdots \) is the Fadeev Popov ghost propagator. The renormalization constant \( Z_3 \) is chosen so as to cancel out the divergences in the diagrams in \( \text{(34)} \). After a lengthy calculation we obtain the expression:

\[
\begin{align*}
Z_3 &= 1 - \frac{2}{3}Nfg_r^2I + \frac{11}{3}Nc g_r^2 I - \frac{\alpha}{2}Nc g_r^2 I (1 - \frac{2}{3}Nfg_r^2I) \\
&\quad + \frac{3}{2}Nc (\frac{3}{2N_f} - g_r^2 I) \ln(1 - \frac{2}{3}Nfg_r^2I) + O(\frac{1}{N_f^2}).
\end{align*}
\]

(35)

Then the compositeness condition \( Z_3 = 0 \) is solved to give the simple solution:

\[
g_r^2 = \frac{3}{2N_fI} \left[ 1 + \frac{11N_c}{2N_f} + O(\frac{1}{N_f^2}) \right].
\]

(36)

This results exhibit several interesting features.

1) The gauge boson self-energy part is purely transversal (i.e. proportional to \( (p^2 g_{\mu\nu} - p_{\mu}p_{\nu}) \)) at all order in \( g_r \) at the next-to-leading order. In the practical calculation, this is realized, at the lowest order in \( g_r \), by adding the Fadeev Popov loop, and in the higher order, by cancellations of the non-transversal parts.

2) Though the \( Z_3 \) itself does depend on gauge parameter \( \alpha \), the solution \( g_r \) to the compositeness condition \( Z_3 = 0 \) does not. This should be so because the coupling constants and the compositeness scale are observable object.

3) The condition that the next-to-leading contribution should not exceed the leading contribution implies that

\[ N_f > \frac{11N_c}{2}. \]  

(37)

Note that this critical value for \( N_f \) and \( N_c \) coincides with that for asymptotic freedom. When the gauge theory is asymptotically free, the next-to-leading contributions are too large, so that the gauge bosons cannot be a composite of the above type. And when it is asymptotically non-free, the next-to-leading order contributions are reasonably suppressed, and the gauge boson can be interpreted as a composite.

5. Summary

In summary, by solving the compositeness condition, under which the Yukawa-type model coincides the NJL type model, we obtained the expressions for the effective coupling constants in terms of the compositeness scale at the next-to-leading order in \( 1/N \). In the NJL model with a scalar composite, the next to leading contribution to \( g^2_f \) is \(-1/N\), and that to \( \lambda_r \) is \(-10/N\), which are too large for \( N_c = 3 \) of our practical interest, and imply unstable Higgs potential. On the other hand, in the induced gauge theory with abelian gauge symmetry, the correction term to \( e_r^2 \) is \(-9\epsilon/4N\), which is reasonably suppressed by the small factor \( \epsilon \), and amounts to only 0.1% when it is applied to the QED. In the non-abelian gauge theory, the correction term to \( g^2_r \) is \( 11N_c/2N_f \). This implies that, when the gauge theory is asymptotically free, the next to leading contribution is too large, and we can not take the gauge boson as a composite of this type. On the other hand, when it is asymptotically non-free, the corrections are suppressed, and the gauge boson is safely taken as a composite. We expect that these methods and results will be useful in disclosing the nature of composite objects in particle and nuclear physics in the future.

Acknowledgment

The author would like to thank Professors H. Terazawa and T. Matsuki for stimulating discussions, and Professor T. Hattori for collaborations. He wishes also to thank Professor Y. Koide and the members of the organizing committee of this workshop for their kind hospitalities extended on him during his stay in Shizuoka and for arranging the opportunity for him to present this work in the workshop.
[1] J.C. Pati and A. Salam, Phys. Rev. D10, 275 (1974).
[2] K. Akama and H. Terazawa, INS-Rep-257 (1976); H. Terazawa, Y. Chikashige and K. Akama, Phys. Rev. D15, 480 (1977).
[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
[4] T. Saito and K. Shigemoto, Prog. Theor. Phys. 57, 242 (1977).
[5] B. Jouvet, Nuovo Cim. 5, 1133 (1956); M. T. Vaughn, R. Aaron and R. D. Amado, Phys. Rev. 124, 1258 (1961); A. Salam, Nuovo Cim. 25, 224 (1962); S. Weinberg, Phys. Rev. 130, 776 (1963).
[6] D. Lurié and A. J. Macfarlane, Phys. Rev. 136, B816 (1964).
[7] Particle Data Group, Phys. Rev. D54, 1 (1996).
[8] V.A. Miransky, M. Tanabashi and K. Yamawaki, Phys. Lett. B221, 177 (1989); Mod. Phys. Lett. A4, 1043 (1989); W.A. Bardeen, C.T. Hill and M. Lindner, Phys. Rev. D41, 1647 (1990).
[9] S. Hands, A. Kocić, and J. B. Kogut, Phys. Lett. B273, 111 (1991); Ann. Phys. B224, 29 (1993); J. Zinn-Justin, Nucl. Phys. B367, 105 (1991); H. Yamamoto and I. Ichinose, Nucl. Phys. B370, 695 (1992); P. Fishbane, R.E. Norton and T.N. Truong, Phys. Rev. D46, 1768 (1992); P. Fishbane and R.E. Norton, Phys. Rev. D48, 4924 (1993); D. Lurié and G.B. Tupper, Phys. Rev. D47 (1993) 3580; R.S. Willey, Phys. Rev. D48, 2877 (1993); J. A. Gracey, Phys. Lett. B308, 65 (1993); B342, 297 (1995). E. Pallante, NORDITA preprint, NORDITA 96/31 N, P (1996); G. Cvetić and N. D. Vlachos, Phys. Lett. B377, 102 (1996); R. de Mello Koch and J. P. Rodorigues, Phys. Rev. D54, 7794 (1996); G. Cvetić, Dortmund preprint, DO-TH-96-07 (1996), hep-ph/9605433.
[10] D. J. Gross and A. Neveu, Phys. Rev. D10, 3235 (1974); T. Kugo, Prog. Theor. Phys. 55, 2032 (1976); T. Kikkawa, Prog. Theor. Phys. 56, 947 (1976).
[11] T. Eguchi, Phys. Rev. D14, 2755 (1976); D17, 611 (1978); D. Campbell, F. Cooper, G. S. Guralnik and N. Snyderman, Phys. Rev. D19, 549 (1979); K. Shizuya, Phys. Rev. D21, 2327 (1980).
[12] K. Akama, Phys. Rev. Lett. 76, 184 (1996).
[13] K. Akama and T. Hattori, Phys. Lett. B392, 383 (1997).
[14] W. Heisenberg, Rev. Mod. Phys. 29, 269 (1957); J. D. Bjorken, Ann. Phys. 24, 174 (1963). For further references, see Akama and Hattori [13].
[15] I. Białynicki-Birula, Phys. Rev. 130, 465 (1963).
[16] K. Akama and T. Hattori, in preparation.
[17] K. Akama, talk given at International Workshop on Perspectives of Strong Coupling Gauge Theories, Nagoya, Japan, November 1996. To be published in Proceedings.