Repulsive Electromagnetic Stresses in the Casimir Piston

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Abstract: We present explicit expressions for the electromagnetic Casimir energy and the pressures acting at the interface of a perfectly conducting rectangular piston. We show that the attractive or repulsive character of the net pressure at the interface is determined both by its relative position and the piston aspect ratio. In particular, for pistons with very narrow aspect ratios, this force may be repulsive with respect to both piston ends. In that case, the interface could perform a vacuum-induced oscillatory motion about the piston middle point.

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The development of experimental techniques [1] with the necessary accuracy to test in detail theoretical predictions on the Casimir effect between parallel conducting plates [2] has opened the way to study vacuum forces in more complicated geometric configurations such as spheres [3], rectangular cavities [4, 5, 6, 7, 8], or cylinders [9]. Some of these studies predict that in closed cavities repulsive Casimir stresses should be exerted at the cavity walls. However, although at least in rectangular cavities the finite contributions to the Casimir forces may be neatly isolated, some doubts have been raised over the physical significance of these results, since the regularization process involves discarding contributions not present in the parallel plate configuration. In addition, there exist intrinsic experimental difficulties in testing those predictions.

A related setup in which some of these ambiguities may be cured is the rectangular piston model. It consists of two joint perfectly-conducting rectangular cavities with sides \((a_1, a_2, a_3)\) and \((a_1, a_2, L-a_3)\), with a freely moving interface (see Fig.1). This model, introduced by Boyer [10], and Cavalcanti [11] for one and two dimensions, respectively, has the advantage of being cutoff independent because the infinite contributions to the Casimir energy on both sides of the interface cancel each other. By using a formalism based on summations over optical paths [12, 13], Hertzberg et al. [12] extended the theory to three-dimensional electromagnetic fields and found the exact solution for pistons with rectangular cross sections. In their model the net pressure at the interface has a finite value and it is always attracted towards the closer end of the piston. This kind of results motivated extensive research on the Casimir piston model [13, 14, 15, 16, 17, 18, 19, 20]. In some cases, repulsive Casimir forces may be attained by introducing non-electromagnetic interactions such as scalar fields subject to mixed boundary conditions [14], or quantum star graphs [19]. Interestingly, a perturbative analysis by Barton [15] based on electromagnetic fluctuations, also yields repulsive Casimir forces in a weakly reflecting semi-infinite piston, although the attractive character of the forces is recovered for thick enough materials.

In this work we show that even standard vacuum electromagnetic fluctuations may induce repulsive Casimir stresses at the interface of perfectly conducting rectangular piston in a cutoff-independent way. With that purpose, regularized expressions for the configuration energy and the pressures at the piston interface are directly deduced from the corresponding quantities already derived for single rectangular cavities [4, 5, 6]. The components of the energy-momentum tensor \(T_{\mu\nu}\) have been expressed in [6] in terms of two-point correlation functions of the vacuum electromagnetic field, calculated at equal space coordinates and a time separation, \(t-t' = \sigma\). The energy per unit volume \(T_{00} \equiv \mathcal{E}\) is given by the limit \(\sigma \to 0\) of

\[
\mathcal{E}(\sigma) = -\frac{1}{\pi^2} \sum_n \frac{3\sigma^2 + u_n^2}{[u_n^2 - \sigma^2]^3} + \sum_{i=1}^{3} \frac{a_i}{4\pi V} \sum_l \frac{\sigma^2 + (2a_i l)^2}{[(2a_i l)^2 - \sigma^2]^2},
\]

with \(n = \{n_1, n_2, n_3\}\), \(V = a_1 a_2 a_3\), \(u_n^2 = \sum_i (2a_i n_i)^2\), and we have set \(h = 1\), \(c = 1\). The terms in \((1)\) with

![FIG. 1: Sketch of perfectly conducting rectangular piston, with lateral sizes \(a_1, a_2\), total length \(L\), and interface located at \(a_3\).](image)
all \( n_i = 0 \) lead to an energy density contribution that diverges as \( \sigma \to 0 \) irrespective of the box size, namely:

\[
E(\sigma) = \frac{3}{\pi^2 \sigma^4} + \frac{(a_1 + a_2 + a_3)}{4\pi V \sigma^2} + E^I(\sigma),
\]

where \( E^I \) is finite as \( \sigma \to 0 \), and tends to zero as \( a_i \to \infty \). The divergent terms here have a natural physical interpretation. They arise from the Fourier transform of the leading contribution in Weyl’s asymptotic mode distribution for very large (but finite) cavities, valid in the \( kV^{1/3} \gg 1 \) regime, \( k \) being the magnitude of the wave vector \[21\]. As for the pressure acting at the wall with a normal directed along \( n_i \), it is

\[
T_{ii}(\sigma) = -\frac{1}{\pi^2} \sum_n \frac{4(2a_i n_i)^2 - u_n^2 + \sigma^2}{[u_n^2 - \sigma^2]^3} + \frac{a_i}{4\pi V} \sum_n \frac{\sigma^2 + (2a_i n_i)^2}{[(2a_i n_i)^2 - \sigma^2]^2}.
\]

As before, the terms with all \( n_i = 0 \) yield divergent contributions in the limit \( \sigma \to 0 \), which may be explicitly isolated:

\[
T_{ii}(\sigma) = \frac{1}{\pi^2 \sigma^4} + \frac{a_i}{4\pi V \sigma^2} + T^I_{ii}(\sigma),
\]

with \( T^I_{ii}(\sigma) \) finite. In general, the predictions arising from the finite contributions in Eqs. \[11\] and \[3\] coincide with those obtained by means of other regularization schemes, such as the introduction of an exponential convergence factor \[4\], or the use of properties of Riemann \( \zeta \) functions \[5\]. The structure of these equations implies that the pressures exerted at the cavity walls may be either attractive or repulsive, in accordance with the traceless nature of the electromagnetic stress-tensor \( E = T_{11} + T_{22} + T_{33} \). If we consider, for example, a cavity with a Casimir-like configuration, \( i.e. \ a_3 \ll a_1, a_2 \), then \( E \approx -1/720a^3_3 \), \( T_{33} = \mathcal{E} \), \( T_{11} = T_{22} = -\mathcal{E} \); for an elongated cavity with \( a_3 \gg a_1 = a_2 \), then \( E \approx -G/24\pi a^4_3 \), \( T_{33} = -\mathcal{E} \), \( T_{11} = T_{22} = \mathcal{E} \), where \( G \) is Catalan’s constant.

Thus, repulsive stresses arise even if the energy density is a negative monotonous decreasing function of the distance. This reflects the fact that the energy density is a global quantity, whereas the stress distribution is a local one. Notice that, if it is assumed that the cavity is built by joining two separated shells in vacuum, no contradiction exists with theorems on the concavity of the Casimir energy of mirror-reflected probes \[22\] [23], as these theorems describe the behavior of the inter-shell separation potential. Furthermore, as pointed out by Bachas \[22\], the process of building a cavity from two shells is mathematically singular, as it introduces divergent edge contributions to the energy.

We now employ the former elements to determine the Casimir stresses within the perfectly conducting rectangular piston. For that sake, we first calculate the total Casimir energy \( E \equiv V \mathcal{E} \) as the sum of the zero-point energies of the single cavities. Following Boyer \[10\], we fix a fiducial level of the energy by subtracting out that associated to the equilibrium configuration, with the interface placed just in the middle of the piston. This procedure cancels out exactly the divergent contributions to the energy:

\[
E^\infty(\sigma) = E^\infty(\sigma, a_3) + E^\infty(\sigma, L - a_3) - 2E^\infty(\sigma, L/2) \equiv 0,
\]

so that \( E(\sigma) = E^I(\sigma, a_3) + E^I(\sigma, L - a_3) - 2E^I(\sigma, L/2) \). The pressure difference between the left-hand and the right-hand side of the interface, \( \Delta P_3 \), may be obtained from the work performed when this is displaced from \( L/2 \) up to \( a_3 \). Energy conservation demands that the is work equals the change in the zero-point energy and, taking into account that \( E^I(\sigma, L/2) \) does not contribute to the force, we get:

\[
\Delta P_3(\sigma, a_3; L) = -\frac{1}{a_1 a_2 a_3} \left[ E^I(\sigma, a_3) + E^I(\sigma, L - a_3) \right] = T^I_{33}(\sigma, a_3) - T^I_{33}(\sigma, L - a_3),
\]

where the last equality follows from direct application...
of the derivative operator and the chain rule, and $T_{33}^f$ is identical to the expression given by (3) and (4).

Consistency of (6) with formulas presented in previous works on the piston model is shown by employing (6) to evaluate the Casimir pressure at the interface of a semi-infinite piston aligned in the $e_3$ direction. This is given by $\Delta P_3 = T_{33}^f(\sigma, a_3) - T_{33}^f(\sigma, a_3)$. For a piston with a square cross section ($a_1 = a_2$), the resulting expression is further simplified by means of the formula $\sum_{n=1}^{\infty} [x^2 + n^2]^{-1} = \coth \pi x - (2x/\pi)$ which allows to evaluate the summation over $a_3$. We are finally led to

$$\Delta P_3(\sigma, a_3; L) = \frac{\pi}{8a_3^3} \sum_{n_1,n_2} \frac{\coth[\pi u_{n_1,n_2,0}/a_3]}{u_{n_1,n_2,0} \sinh^2[\pi u_{n_1,n_2,0}/a_3]}$$

$$-\frac{\pi^2}{240a_3^3} + \frac{\pi}{24a_1^2a_2^2} - T_{33}^f(\sigma, a_3),$$

which coincides with the expression for the pressure within a semi-infinite piston derived in [12].

In order to study the behavior of the total Casimir energy and local pressures at the piston interface for a manifold of geometric configurations, we introduce the parameters $z \equiv a_3/a_1$, and $y \equiv a_2/a_1$. They define the relative position of the interface, and the piston aspect ratio, respectively. In Fig.(2) we present the energy and pressure surfaces arising from the variation of $y$ and $z$ for a very long piston ($L = 100a_1$). We observe that, depending on the aspect ratio, two qualitatively different behaviors appear. In the case $y > y_{\text{crit}}$, the energy and pressure show the intuitively expected behavior, already discussed in previous works: the energy is a monotonous increasing function of the interface separation, and similarly for the pressure difference, so that the interface is always attracted towards the closer end of the piston. On the other hand, for $y \leq y_{\text{crit}}$, the energy develops a bi-modal structure with a minimum located at the center of the piston. Consequently, the pressure difference at the interface becomes repulsive at an intermediate position between one closing end and the middle point. After reaching a maximum positive value, it decreases and vanishes at $a_3 = L/2$. For $a_3 > L/2$, the interface is now attracted towards the left-hand piston end.

This behavior may be appreciated in greater detail in Figs. (3) and (4), where we present cross sections of the energy and pressure surfaces for a piston of length $L = 5a_1$, as a function of $z$, and three particular values $y = 0.01 < y_{\text{crit}}$, $y = 0.02 \approx y_{\text{crit}}$, and $y = 1 > y_{\text{crit}}$, while in Fig.(5) we plot the ratio of $\Delta P_3$ with respect to the square piston configuration ($y = 1$). In contrast, for a narrow piston it develops an unexpectedly large value $\Delta P_3 \gg P_{\text{Cas}}$, which could be subject, in principle, of experimental verification in micrometric cavities. For example, in the configuration considered in Figs. (3)-(5), for a piston of length $L \sim 5 \mu$m, the other quantities would be $a_1 \sim 1 \mu$m, $a_2 \sim 10$ nm, and $\Delta P_3 \approx 50P_{\text{Cas}}$ for an interface separation $a_3 = 1\mu$m. These values seem accessible to current experimental techniques. The former results may be affected by finite conductivity, temperature fluctuations, rugosity, etc. In particular, we have analyzed the effect of the cutoff $\sigma$ in the finite terms of (4) and (3). If we assume that $\sigma/a_1 \approx 10^{-2} - 10^{-4}$, the energy and pressure

![FIG. 3: Total energy as a function of the relative interface position $z = a_3/a_1$ for fixed values of the piston aspect ratio $y = a_2/a_1$, for a piston with total length $L/a_1 = 5$.](image1)

![FIG. 4: Net pressure at the piston interface as a function of the relative interface position $z = a_3/a_1$ for fixed values of the piston aspect ratio $y = a_2/a_1$, for a piston with total length $L/a_1 = 5$. Here, the interface is assumed to be displaced from the left-hand side of the piston to the right.](image2)
The possible existence of repulsive Casimir forces (see also the last reference in [3]).

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![Figure 5](image)

FIG. 5: Ratio of the net pressure to the magnitude of the Casimir pressure for parallel plates as a function of $z = a_3/a_1$ for fixed values of the piston aspect ratio $y = a_2/a_1$, for a piston with total length $L/a_1 = 5$. The inset is an amplification of this ratio for a piston with squared aspect ratio ($y = 1$).

The curves develop a behavior (not shown in the figures) indistinguishable from that observed in Figs. (3) and (4), except for a strong repulsive pressure appearing at extremely small interface separations, rendering finite the Casimir force even at zero distance. This had been observed in several works, where the cutoff had been related with electron-hole pair excitations [24], finite plasma frequency [8], or finite interatomic distance [20] (see also the last reference in [3]).

The extension of this work to consider the role of finite temperature or realistic models of finite conductivity in pistons is in progress [25].