Solution of spherical equation in 3 dimensions for hydrogen atom with quantum numbers $4 \leq n \leq 5$

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**Abstract.** The lightest and simplest arrangement of protons and electrons is a hydrogen atom. One of the wave characteristics of an atom can be known using the Schrodinger equation. Schrodinger's equations of spherical coordinates consist of radial equations and angular equations. Quantum numbers $4 \leq n \leq 5$ from the Schrodinger equation sphere coordinates produce different angular wave functions. To find out the angular wave equation using normalization conditions. Based on the normalization conditions for $n = 4$ a wave function is produced, while for $n = 5$ a wave function is.

1. Introduction

One of the most studied in quantum mechanic is Hydrogen atom. Hydrogen atom consists of a proton and an electron that around in neutron. Wave equation as representation of electron or microscopic particle in non-relativistic condition and also second order equation is called Schrodinger equation that found by Erwin Schrodinger[1,2]. Schrodinger equation in three-dimensional space contains a Laplacian operator that depends on that's coordinate.

Position or move from a particle in three-dimensional can be represented by using a coordinate system. Coordinate system in three-dimensional can be stated in cartesian coordinate, tube coordinate and spherical coordinate which interrelated each other. If particle position in cartesian coordinate is $M(x, y, z)$, so particle position in tube coordinate is $M(r, \theta, z)$ and particle position in spherical coordinate is $M(r, \theta, \phi)$[3].

Schrodinger equation that used for Hydrogen atom is stated in spherical coordinate because atom is considered symmetry with sphere[4]. Schrodinger equation in spherical coordinate consist of two equation namely radial equation and spherical equation. Spherical equation is wave equation that propagates angular ($\theta$) or rotate based on polar angle ($\phi$). Physically, polar equation is equation that describes orbital shape that intersects the xy axis ($0 - \pi$), whereas azimuth equation is equation that describes a rotate in direction of z axis ($0 - 2\pi$).

Solution from angular equation called wave function that consists of polar wave function and azimuth wave function. Polar wave function can be trigonometry or polar exponential that one of important studied in physics quantum[5]. Angular wave function that has been normalization can use to describes characteristic from a wave.
Angular wave function contains orbital quantum number \((l)\) and magnetic quantum number \((m)\). Orbital quantum number \((l)\) express angular velocity and orbital shape from electron. Then, magnetic quantum number\((m)\) express orbital space orientation[6]. The value of orbital quantum number \((l)\) and magnetic quantum number \((m)\) depends on energy level of the electron orbitals expressed by principal quantum number \((n)\). In this research will discuss solution of angular equation with principal quantum number \(4 \leq n \leq 5\).

2. Methods

Time-independent Schrodinger equation for Hydrogen atom can be given by:

\[
-\left(\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}\right) \psi = E\psi
\]

(1)

Where \(\nabla^2\) is Laplacian that depends on coordinate that used to solve the Schrodinger equation[7]. Mathematically, Laplacian operator spherical coordinate can be written:

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]

(2)

Then by substitution equation (2) to equation (1) will get Schrodinger equation in spherical coordinate which is two-dimensional differential equation that contains two tribes namely first tribe depends on radius\((r)\) and second tribe namely depend on angle \((\theta, \phi)\).

Second tribe depends on angle\((\theta, \phi)\) called angular equation and given by:

\[
\frac{1}{Y(\theta, \phi) \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2} = -l(l + 1)
\]

(3)

Where \(-l(l + 1)\) is separate constant between first tribe that depends on radius \((r)\) and second tribe that depends on angle \((\theta, \phi)\).

Equation solution (3) called angular wave function \(Y(\theta, \phi)\) which is linear combination from polar wave function \(\Theta(\theta)\) and azimuth wave function \(\Phi(\phi)\) with using separation variation method [9]:

\[
\frac{\sin \theta}{\Theta(\theta)} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta(\theta)}{\partial \theta} \right) + l(l + 1)\sin^2 \theta = m^2
\]

\[
-\frac{1}{\Phi(\phi)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = m^2
\]

(4) (5)

Where \(m^2\) is separating constant chosen so that the solution obtained later has physical meaning.

Now we will solve equation (5) and (6) to getting angular wave function \(Y(\theta, \phi)\). First equation that will be solved is equation (5) because need special function is the following Legendre polynomial:

\[
P_l^m(v) = (1 - v^2)^{\frac{|m|}{2}} \left( \frac{d}{dv} \right)^{|m|} P_l(v)
\]

(6)

Where \(P_l(v)\) is Legendre polynomial which is defined by the following Rodrigues formula:

\[
P_l(v) = \frac{1}{2^l l!} \left( \frac{d}{dv} \right)^l (v^2 - 1)^l
\]

(7)

Equation (7) and (8) is solution from differential equation:

\[
(1 - v^2) \frac{d^2 P_l^m(v)}{dv^2} + 2v \frac{d P_l^m(v)}{dv} + [l(l + 1) - \frac{m^2}{1 - v^2}] P_l^m(v) = 0
\]

(8)

The step taken is to re-enter (5) in the form:

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta(\theta)}{\partial \theta} \right) + \left( l(l + 1) - \frac{m^2}{\sin^2 \theta} \right) \Theta(\theta) = 0
\]

(9)

Than with replace variable \(\cos \theta = v\) will get that equation (11) identical to the equation (10), so we can get a conclusion that polar equation function given by:

\[
\Theta_{lm}(\theta) = F_{lm} P_l^m(\cos \theta)
\]

(11)

\[
F_{lm} = (-1)^{(m+|m|)/2} \sqrt{\frac{2l+1}{2} \frac{(l-|m|)!}{(l+|m|)!}}
\]

(12)
Where $F_{lm}$ is normalization constant of polar equation function which its value determined by normalization requirements:

$$\int_0^{\pi} [\Theta_{lm}(\theta)]^* [\Theta_{lm}(\theta)] \sin \theta \, d\theta = 1$$  \hspace{1cm} (13)

And using Ortogonalitas characteristic $P_{lm}^m(\cos \theta)$:

$$\int_0^{\pi} P_{lm}^m(\cos \theta) P_{lm}^m(\cos \theta) \sin \theta \, d\theta = \frac{2}{2l+1} \delta_{l\ell} \delta_{mm'}$$  \hspace{1cm} (14)

Next is to solve equation (6) by rewriting it in the form

$$\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + m^2 \Phi(\phi) = 0$$  \hspace{1cm} (15)

Solution for equation (13) is called Azimuth wave function that given by [10]:

$$\Phi_m(\phi) = C_m e^{im\phi}$$  \hspace{1cm} (16)

$$C_m = \frac{1}{\sqrt{2\pi}}$$  \hspace{1cm} (17)

Where $(m)$ here can be valuable $m = 0, \pm 1, \pm 2, \ldots$ and $C_m$ is the normalization constant of the azimuth wave function whose value can be determined with the condition of normalizing the following azimuth wave function

$$\int_0^{2\pi} [\Phi_m(\phi)]^* [\Phi_m(\phi)] d\phi = 1$$  \hspace{1cm} (18)

The angular wave function as a linear combination of the polar wave function and the azimuth wave function is given by

$$Y_{lm}(\theta, \phi) = B_{lm} P_{lm}^m(\cos \theta) e^{im\phi}$$  \hspace{1cm} (19)

$$B_{lm} = F_{lm} C_m = (-1)^{(m+|m|)/2} \frac{2l+1}{2} \frac{(l+|m|)!}{(l-|m|)!} \frac{1}{\sqrt{2\pi}}$$  \hspace{1cm} (20)

Where $B_{lm}$ is a normalizing constant of the angular wave function, $(l)$ is called orbital quantum number and $(m)$ is called magnetic quantum number. The price $(l)$ allowed is $(l = 0, 1, 2, \ldots n - 1)$ and the price $(m)$ allowed is $(m = -l, \ldots, 0, \ldots, +l)$, while the price of the main quantum number $(n)$ which states the electron orbitals’ energy levels in this study are $(n) = 4$ and $5$.

3. Result and Discussion

The following are the results of an angular equation with the main quantum number $4 \leq n \leq 5$.

**Table 1.** The angular equation of Hydrogen Atom $4 \leq n \leq 5$

| $n$ | $l$ | $m$ | $\Theta_{lm}(\theta)$ | $\Phi_m(\phi)$ | $Y_{lm}(\theta, \phi)$ |
|-----|-----|-----|------------------------|----------------|------------------------|
| 0   | 0   | 0   | $\frac{1}{\sqrt{2}}$   | $\frac{1}{\sqrt{2\pi}}$ | $\frac{1}{\sqrt{4\pi}}$ |
|     |     | ±1  | $\frac{3}{\sqrt{4}} \sin \theta$ | $\frac{1}{\sqrt{2\pi}} e^{\pm il\phi}$ | $\frac{3}{\sqrt{8\pi}} \sin \theta e^{\pm il\phi}$ |
| 4   | 1   | 0   | $\frac{3}{\sqrt{2}} \cos \theta$ | $\frac{1}{\sqrt{2\pi}}$ | $\frac{3}{\sqrt{4\pi}} \cos \theta$ |
|     |     | ±2  | $\frac{15}{16} \sin^2 \theta$ | $\frac{1}{\sqrt{2\pi}} e^{\pm 2il\phi}$ | $\frac{15}{\sqrt{32\pi}} \sin^2 \theta e^{\pm 2il\phi}$ |
| n  | l  | m  | $\Theta_{lm}(\theta)$                              | $\Phi_m(\phi)$                              | $Y_{lm}(\theta, \phi)$                          |
|----|----|----|---------------------------------|---------------------------------|-----------------------------------|
| ±1 | 0  | 0  | $\pm \frac{15}{4 \sqrt{2}} \sin \theta \cos \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm i \phi}$ | $\pm \frac{15}{8 \pi \sqrt{2}} \sin \theta \cos \theta e^{\pm i \phi}$ |
|    | ±1 | 0  | $\pm \frac{5}{8} (3 \cos^2 \theta - 1)$ | $\pm \frac{1}{2 \sqrt{2 \pi}}$ | $\pm \frac{5}{16 \pi} (3 \cos^2 \theta - 1)$ |
| ±3 | 0  | ±2 | $\pm \frac{1575}{1440} (1 - \cos^2 \theta)^{3/2}$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm 3 i \phi}$ | $\pm \frac{1575}{2880 \pi} (1 - \cos^2 \theta)^{3/2} e^{\pm 3 i \phi}$ |
| ±2 | 0  | ±1 | $\pm \frac{1575}{240} \sin^2 \theta \cos \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm 2 i \phi}$ | $\pm \frac{1575}{480 \pi} \sin^2 \theta \cos \theta e^{\pm 2 i \phi}$ |
| 3  | ±1 | 0  | $\pm \frac{126}{192} \sin \theta (5 \cos^2 \theta - 1)$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm i \phi}$ | $\pm \frac{126}{384 \pi} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i \phi}$ |
|    | ±1 | ±2 | $\pm \frac{7}{8} (5 \cos^3 \theta - 3 \cos \theta)$ | $\pm \frac{1}{2 \sqrt{2 \pi}}$ | $\pm \frac{7}{16 \pi} (5 \cos^3 \theta - 3 \cos \theta)$ |
|    | ±1 | ±1 | $\pm \frac{1}{2}$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm i \phi}$ | $\pm \frac{1}{4 \pi}$ |
| ±2 | 0  | ±1 | $\pm \frac{3}{4} \sin \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm i \phi}$ | $\pm \frac{3}{8 \pi} \sin \theta e^{\pm i \phi}$ |
| ±1 | 0  | ±1 | $\pm \frac{3}{2} \cos \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}}$ | $\pm \frac{3}{4 \pi} \cos \theta$ |
| 5  | ±2 | 0  | $\pm \frac{15}{16} \sin^2 \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm 2 i \phi}$ | $\pm \frac{15}{32 \pi} \sin^2 \theta e^{\pm 2 i \phi}$ |
| 2  | ±1 | 0  | $\pm \frac{15}{4} \sin \theta \cos \theta$ | $\pm \frac{1}{2 \sqrt{2 \pi}} e^{\pm i \phi}$ | $\pm \frac{15}{8 \pi} \sin \theta \cos \theta e^{\pm i \phi}$ |
| ±1 | 0  | ±1 | $\pm \frac{5}{8} (3 \cos^2 \theta - 1)$ | $\pm \frac{1}{2 \sqrt{2 \pi}}$ | $\pm \frac{5}{16 \pi} (3 \cos^2 \theta - 1)$ |
The angular wave function consists of 2 wave functions, namely the polar wave function and the azimuth wave function. The polar wave function shows the rotational motion that intersects the XY axis. While the azimuth wave function shows rotational motion on the z-axis or at points 0 to 2π.

Based on table 1, it is known that the quantum numbers used to find angular wave functions are the main quantum numbers, azimuth quantum numbers and magnetic quantum numbers. The quantum number is an integer 1, 2, 3, ... which shows the energy level of an electron. The quantum number is 0 to which shows the atomic sub-cape and the quantum number is ..., for each quantum number l which indicates the direction of orbit [11]. The quantum number n = 4 has 16 angular equation solutions and n = 5 has 25 different angular equation solutions.

| n | l | m | θ_{lm}(θ) | Φ_{m}(φ) | Y_{lm}(θ,φ) |
|---|---|---|---|---|---|
| ±3 | ±3 | 1575 \(\sqrt{1440} \) \((1 - \cos^2θ)^{3/2}\) | \(\frac{1}{2\pi} e^{±3iφ}\) | 1575 \(\frac{1}{2880\pi} (1 - \cos^2θ)^{3/2} e^{±3iφ}\) |
| ±2 | ±2 | 1575 \(\sqrt{240} \) \(\sin^2θ \cosθ\) | \(\frac{1}{2\pi} e^{±2iφ}\) | 1575 \(\sqrt{480\pi} \) \(\sin^2θ \cosθ e^{±2iφ}\) |
| ±1 | ±1 | 126 \(\sqrt{192} \) \(\sinθ (5 \cos^2θ - 1)\) | \(\frac{1}{2\pi} e^{±iφ}\) | 126 \(\sqrt{384\pi} \) \(\sinθ (5 \cos^2θ - 1) e^{±iφ}\) |
| 0 | 0 | 7 \(\sqrt{8} \) \(5\cos^3θ - 3\cosθ\) | | 7 \(\sqrt{16\pi} \) \(5\cos^3θ - 3\cosθ\) |
| ±4 | ±4 | 99225 \(\sqrt{80640} \) \(\sin^4θ\) | \(\frac{1}{2\pi} e^{±4iφ}\) | 99225 \(\sqrt{161280\pi} \) \(\sin^4θ e^{±4iφ}\) |
| ±3 | ±3 | 99225 \(\sqrt{10080} \) \(\sin^3θ \cosθ\) | \(\frac{1}{2\pi} e^{±3iφ}\) | 99225 \(\sqrt{20160\pi} \) \(\sin^3θ \cosθ e^{±3iφ}\) |
| 4 | 2 | 1800 \(\sqrt{25920} \) \(\sin^2θ (42 \cos^2θ - 6)\) | \(\frac{1}{2\pi} e^{±2iφ}\) | 1800 \(\sqrt{51840\pi} \) \(\sin^2θ (42 \cos^2θ - 6) e^{±2iφ}\) |
| ±2 | ±2 | 5400 \(\sqrt{15360} \) \(\sinθ (14 \cos^3θ - 6 \cosθ)\) | \(\frac{1}{2\pi} e^{±iφ}\) | 5400 \(\sqrt{30720\pi} \) \(\sinθ (14 \cos^3θ - 6 \cosθ) e^{±iφ}\) |
| ±1 | ±1 | 9 \(\sqrt{128} \) \(35 \cos^4θ - 30 \cos^2θ + 3\) | | 9 \(\sqrt{256\pi} \) \(35 \cos^4θ - 30 \cos^2θ + 3\) |

The angular wave function consists of 2 wave functions, namely the polar wave function and the azimuth wave function. The polar wave function shows the rotational motion that intersects the XY axis. While the azimuth wave function shows rotational motion on the z-axis or at points 0 to 2π. Based on table 1, it is known that the quantum numbers used to find angular wave functions are the main quantum numbers (n), azimuth quantum numbers (l) and magnetic quantum numbers (m). The quantum number n is an integer 1, 2, 3, ... which shows the energy level of an electron. The quantum number l is 0 to n – l which shows the atomic sub-cape and the quantum number m is 0, ± 1, ± 2, .... for each quantum number l which indicates the direction of orbit [11]. The quantum number n = 4 has 16 angular equation solutions and n = 5 has 25 different angular equation solutions.

In the main quantum number n = 4 the set of permitted quantum numbers is (4,0,0), (4,1, -1), (4,1,0), (4,1,1), (4,2, -2), (4,2, -1), (4,2,0), (4,2,1), (4,2,2), (4,3, -3), (4, 3, -2), (4,3, -1), (4,3,0), (4,3,1), (4,3,2), (4,3,3). Whereas the quantum number n = 5 has the following set of permissible quantum
numbers \((5,0,0), (5,1,1), (5,1,0), (5,2,0), (5,2, -2), (5,2, -1), (5,2,0), (5,2,1), (5,2,2), (5 , 3, -3), (5,3, -2), (5,3, -1), (5,3,0), (5,3,1), (5,3,2), (5,3,3), (5,4, -4), (5,4, -3), (5,4, -2), (5,4, -1), (5,4,0) , (5,4,1), (5,4,2), (5,4,3), (5,4,4)\). Each set number has a different wave function.

The following is a three-dimensional angular graph with principal quantum number \(4 \leq n \leq 5\) using informatics program:

![Three-dimensional angular graph](image)

**Figure 1.** Three-dimensional angular graph with set of quantum number \((4,0,0),(4,1,0),(4,2,0)\) and \((4,3,0)\).

Figure 1 above show that in set of quantum numbers \((4,0,0),(4,1,0),(4,2,0)\) and \((4,3,0)\). Produce only real-wave functions. Therefore, in the set of numbers only one graph is obtained. In orbital quantum number \(|l| = 1\) and magnetic quantum number \(m = 0\) the possibility of finding an electron is found on the z axis. Where as in the orbital quantum number \(|l| = 0\) and magnetic quantum number \(m = 0\) the possibility of finding an electron is on the xyz axis. For magnetic quantum number \(m = 0\) and orbital quantum number \(l = 2,3\) the possibility of and electron being found is on the xyz axis.

![Three-dimensional angular graph](image)

**Figure 2.** Three-dimensional angular graph with set of quantum number \((4,1,\pm 1)\).

Three graphs in set of quantum numbers \((4,1,\pm 1)\) produces 3 graphs. The graphs consist of complex, imaginary and real graphics. Complex graphs mean that graphs are generated from the value
of a complex wave function which consists of imaginary and real values. In imaginary and real graphs produce 2 orbital spaces. Possibility of finding an electron is found on my axis when the magnetic quantum number \((m = \pm 1)\) and orbital quantum number \((l = 1)\). That all shown in figure 2.

Figure 3. Three-dimensional angular graph with set of quantum number \((4,1,\pm 2)\).

In the set of quantum numbers \((4,2,\pm 2)\) and \((4,2,\mp 2)\) produces 3 graphs. Imaginary and real graph each produces 4 orbital spaces. When orbital quantum number \((l = 2)\) and magnetic quantum number \((m = \pm 2)\) the possibility of finding an electron is found on the xy axis. All of that shown in Figure 3.

Figure 4. Three-dimensional angular graph with set of quantum number \((4,2,\pm 1)\).

Graph of Figure 4 produces 2 orbital spaces. In the imaginary and real graph produces 4 orbital spaces, as well as imaginary and real graphs in the set of quantum number \((4,2,\pm 2)\) and \((4,2,\mp 2)\). When magnetic quantum number \((m = \pm 1)\) and orbital quantum number \((l = 2)\) the possibility of finding an electron is found on the yz axis.

Figure 5. Three-dimensional angular graph with set of quantum number \((4,3,\pm 1)\).

Imaginary and real graphs in the set of quantum numbers \((4,3,\pm 1)\) respectively produces 6 orbital spaces. Then on complex graph produces 3 orbital spaces. The possibility of finding an electron is
found on the XYZ axis when the magnetic quantum number \((m = \pm 1)\) and orbital quantum number \((l = 3)\). All shown in figure 5.

\[
\begin{align*}
&4.3,\pm 2 \\
&\text{Complex} \\
&4.3,\pm 2 \\
&\text{Imaginary} \\
&4.3,\pm 2 \\
&\text{Real} \\
\end{align*}
\]

**Figure 6.** Three-dimensional angular graph with set of quantum number \((4,3,\pm 2)\).

The set of quantum numbers in figure 6 consist of 6 orbital spaces on imaginary and real graphs. The possibility of finding an electron is found on the XYZ axis when the magnetic quantum number \((m = \pm 2)\) and orbital quantum number \((l = 3)\).

\[
\begin{align*}
&5,0,0 \\
&\text{Real} \\
&5,1,0 \\
&\text{Real} \\
&5,2,0 \\
&\text{Real} \\
&5,3,0 \\
&\text{Real} \\
&5,4,0 \\
&\text{Real} \\
\end{align*}
\]

**Figure 7.** Three-dimensional angular graph with set of quantum number \((5,0,0),(5,1,0),(5,2,0),(5,3,0)\) and \((5,4,0)\)

Five graphs in Figure 7 show in the set of quantum numbers \((5,0,0),(5,1,0),(5,2,0),(5,3,0)\) and \((5,4,0)\) only produce 1 graph. This is because the set of quantum numbers only produces real value. When magnetic quantum number \((m = 0)\) and orbital quantum number \((l = 1)\) the possibility of finding an electron is found on the z axis. When magnetic quantum number \((m = 0)\) and orbital quantum number \((l = 0)\) the possibility of finding an electron is on the xyz axis. And when magnetic quantum number \((m = 0)\) and orbital quantum number \((l = 2,3,4)\) an electron may be found in the xyz axis.
The set of quantum numbers $(5,1, + 1)$ and $(5,1, -1)$ produces 2 orbitals in imaginary and real graphs. Figure 8 shows that when magnetic quantum number $(m = \pm 1)$ and orbital quantum number $(l = 1)$ the possibility of finding an electron is on the xy axis.

Figure 8. Three-dimensional angular graph with set of quantum number $(5,1, \pm 1)$

Three graphs in figure 9 show that the set of quantum numbers $(5,2, \pm 2)$ produces 4 orbitals in imaginary and real graphs. For orbital quantum number $(l = 2)$ and magnetic quantum number $(m = \pm 2)$ the possibility of an electron being found is on the xy axis.

Figure 9. Three-dimensional angular graph with set of quantum number $(5,1, \pm 2)$

The set of quantum numbers $(5,2, + 1)$ and $(5,2, -1)$ produces 4 orbital spaces on imaginary and real graphs, and 2 orbital spaces on a complex graph which is combination of imaginary graph and real graph.

Figure 10. Three-dimensional angular graph with set of quantum number $(5,2, \pm 1)$
Three graphs in figure 11 show that the set of quantum numbers $(5,3, \pm 3)$ produces 6 orbital spaces on imaginary and real charts. The possibility of an electron being found is on the $XY$ axis when the orbital quantum number ($l = 3$) and magnetic quantum number ($m = \pm 3$).

Three graphs in figure 12 show that the set of quantum numbers $(5,3, \pm 2)$ produces 8 orbital spaces and 2 orbital spaces on a complex graph. The possibility of an electron being found is on the $XYZ$ axis when the magnetic quantum number ($m = \pm 2$) and orbital quantum number ($l = 3$).

The set of quantum numbers $(5,4, +1)$ and $(5,4, -1)$ produces 8 orbitalspaces in imaginary and real graphs. And 4 orbital spaces in complex graph which is a combination of imaginary and real graphs. The possibility of an electron being found is on the $XYZ$ axis when the magnetic quantum number ($m = \pm 1$) and orbital quantum number ($l = 4$).
In the set of quantum numbers $(5, 4, \pm 2)$ and $(5, 4, \pm 2)$ produces 4 orbital spaces in imaginary and real graphs. When orbital quantum number $(l = 4)$ and magnetic quantum number $(m = \pm 2)$ the possibility of finding an electron is on the $xyz$ axis.

The set of quantum numbers in figure 15 produces 11 orbitals in imaginary and real graphs. Then in the complex graph there are two orbital spaces. When orbital quantum number $(l = 4)$ and magnetic quantum number $(m = \pm 3)$ the possibility of finding an electron is on the $xyz$ axis.

Figure 16 shows that imaginary and real graphs in the set of quantum numbers $(5, 3, \pm 2)$ produces 8 orbital spaces and 2 orbital spaces on a complex graph. The possibility of an electron being found is on the $XY$ axis is when the orbital quantum number $(l = 4)$ and magnetic quantum number $(m = \pm 4)$. 

Figure 14. Three-dimensional angular graph with set of quantum number $(5,4,\pm 2)$

Figure 15. Three-dimensional angular graph with set of quantum number $(5,4,\pm 3)$

Figure 16. Three-dimensional angular graph with set of quantum number $(5,4,\pm 2)$
4. Conclusion
Schrodinger's Equations Hydrogen atoms use the Schrodinger equation for spherical coordinates because Hydrogen atoms are in the form of spherical symmetry. Spherical coordinate consists of radial equations and angular equations. The polar and azimuth equations are angular equations. The polar equation depends on the angle $\theta$. While the azimuth equation depends on the angle $\phi$. Angular wave functions of hydrogen atoms can be described with 3 graphs consisting of complex graphs, imaginary graphs and real graphs. Imaginary graphs and real graphs show that the resulting value is a complex value. At 0 orbital and magnetic quantum only 1 graph is produced. This is because the quantum number only contains real numbers.

Acknowledgement
We gratefully acknowledge the support and prayers of 3rd Research group Physics Education from FKIP – University of Jember of year 2019.

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