The diameter and radius of radially maximal graphs

Pu Qiao\textsuperscript{a}, Xingzhi Zhan\textsuperscript{b}\textdagger

\textsuperscript{a}Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China

\textsuperscript{b}Department of Mathematics, East China Normal University, Shanghai 200241, China

Abstract

A graph is called radially maximal if it is not complete and the addition of any new edge decreases its radius. In 1976 Harary and Thomassen proved that the radius \( r \) and diameter \( d \) of any radially maximal graph satisfy \( r \leq d \leq 2r - 2 \). Dutton, Medidi and Brigham rediscovered this result with a different proof in 1995 and they posed the conjecture that the converse is true, that is, if \( r \) and \( d \) are positive integers satisfying \( r \leq d \leq 2r - 2 \), then there exists a radially maximal graph with radius \( r \) and diameter \( d \). We prove this conjecture and a little more.

Key words. Radially maximal; diameter; radius; eccentricity

1 Introduction

We consider finite simple graphs. Denote by \( V(G) \) and \( E(G) \) the vertex set and edge set of a graph \( G \) respectively. The complement of \( G \) is denoted by \( \bar{G} \). The radius and diameter of \( G \) are denoted by \( \text{rad}(G) \) and \( \text{diam}(G) \) respectively.

Definition. A graph \( G \) is said to be radially maximal if it is not complete and

\[
\text{rad}(G + e) < \text{rad}(G) \quad \text{for any } e \in E(\bar{G}).
\]

Thus a radially maximal graph is a non-complete graph in which the addition of any new edge decreases its radius. Since adding edges in a graph cannot increase its radius,
every graph is a spanning subgraph of some radially maximal graph with the same radius. It is well-known that the radius \( r \) and diameter \( d \) of a general graph satisfy \( r \leq d \leq 2r \) [4, p.78]. In 1976 Harary and Thomassen [3, p.15] proved that the radius \( r \) and diameter \( d \) of any radially maximal graph satisfy
\[
    r \leq d \leq 2r - 2. \tag{1}
\]

Dutton, Medidi and Brigham [1, p.75] rediscovered this result with a different proof in 1995 and they [1, p.76] posed the conjecture that the converse is true, that is, if \( r \) and \( d \) are positive integers satisfying (1) then there exists a radially maximal graph with radius \( r \) and diameter \( d \).

We prove this conjecture and a little more.

We denote by \( d_G(u, v) \) the distance between two vertices \( u \) and \( v \) in a graph \( G \). The eccentricity, denoted by \( e_G(v) \), of a vertex \( v \) in \( G \) is the distance to a vertex farthest from \( v \). The subscript \( G \) might be omitted if the graph is clear from the context. Thus \( e(v) = \max\{d(v, u) | u \in V(G)\} \). If \( e(v) = d(v, x) \), then the vertex \( x \) is called an eccentric vertex of \( v \). By definition the radius of a graph \( G \) is the minimum eccentricity of all the vertices in \( V(G) \), whereas the diameter of \( G \) is the maximum eccentricity. A vertex \( v \) is a central vertex of \( G \) if \( e(v) = \rad(G) \). A graph \( G \) is said to be self-centered if \( \rad(G) = \diam(G) \). Thus self-centered graphs are those graphs in which every vertex is a central vertex. \( N_G(v) \) will denote the neighborhood of a vertex \( v \) in \( G \). The order of a graph is the number of its vertices. The symbol \( C_k \) denotes a cycle of order \( k \).

## 2 Main Results

We will need the following operation on a graph. The extension of a graph \( G \) at a vertex \( v \), denoted by \( G\{v\} \), is the graph with \( V(G\{v\}) = V(G) \cup \{v'\} \) and \( E(G\{v\}) = E(G) \cup \{vv'\} \cup \{v'x | vx \in E(G)\} \) where \( v' \notin V(G) \). Clearly, if \( G \) is a connected graph of order at least 2, then \( e_{G\{v\}}(u) = e_G(u) \) for every \( u \in V(G) \) and \( e_{G\{v\}}(v') = e_{G\{v\}}(v) = e_G(v) \). In particular, \( \rad(G\{v\}) = \rad(G) \) and \( \diam(G\{v\}) = \diam(G) \).

Gliviak, Knor and Šoltész [2, Lemma 5] proved the following result.

**Lemma 1.** Let \( G \) be a radially maximal graph. If \( v \in V(G) \) is not an eccentric vertex of any central vertex of \( G \), then the extension of \( G \) at \( v \) is radially maximal.

Now we are ready to state and prove the main result.

**Theorem 2.** Let \( r, d \) and \( n \) be positive integers. If \( r \geq 2 \) and \( n \geq 2r \), then there exists
a self-centered radially maximal graph of radius $r$ and order $n$. If $r < d \leq 2r - 2$ and $n \geq 3r - 1$, then there exists a radially maximal graph of radius $r$, diameter $d$ and order $n$.

Proof. We first treat the easier case of self-centered graphs. Suppose $r \geq 2$ and $n \geq 2r$. The even cycle $C_{2r}$ is a self-centered radially maximal graph of radius $r$ and order $2r$. Choose any but fixed vertex $v$ of $C_{2r}$. For $n > 2r$, successively performing extensions at vertex $v$ starting from $C_{2r}$ we obtain a graph $G(r, n)$ of order $n$. $G(4, 11)$ is depicted in Figure 1.

Denote $G(r, 2r) = C_{2r}$. Since $G(r, n)$ has the same diameter and radius as $C_{2r}$, it is self-centered with radius $r$. Let $xy$ be an edge of the complement of $G(r, n)$. Denote by $S$ the set consisting of $v$ and the vertices outside $C_{2r}$. Then $S$ is a clique. If one end of $xy$, say, $x$ lies in $S$, then $y \not\in N[v]$, the closed neighborhood of $v$ in $G(r, n)$. We have $e(x) < r$. Otherwise $x, y \in V(C_{2r}) \setminus S$. We then have $e(x) < r$ and $e(y) < r$. In both cases, $\text{rad}(G(r, n) + xy) < \text{rad}(G(r, n))$. Hence $G(r, n)$ is radially maximal.

Next suppose $r < d \leq 2r - 2$ and $n \geq 3r - 1$. We define a graph $H = H(r, d, 3r - 1)$ of order $3r - 1$ as follows. $V(H) = \{x_1, x_2, \ldots, x_{2r-1}\} \cup \{y_1, y_2, \ldots, y_r\}$ and

$$E(H) = \{x_i x_{i+1}|i = 1, 2, \ldots, 2r - 1\} \cup \{x_{2r-1} y_1\} \cup \{x_{2r-2j+2} y_j|j = 1, 2, \ldots, 2r - d\}$$

$$\cup \{x_{d-r+1} y_{2r-d+1}\} \cup \{y_t y_{t+1}|t = 2r - d + 1, \ldots, r - 1 \text{ if } d \geq r + 2\}$$

where $x_{2r} = x_1$. $H$ is obtained from the odd cycle $C_{2r-1}$ by attaching edges and one path. A sketch of $H$ is depicted in Figure 2, and $H(6, d, 17)$ with $d = 7, 8, 9, 10$ are depicted in Figure 3.
Clearly, $H$ has radius $r$, diameter $d$ and order $3r - 1$. To see this, verify that $x_{d-r+1}$ is a central vertex and $e_H(y_r) = d$.

Now we show that $H$ is radially maximal. Let $C$ be the cycle of length $2r - 1$; i.e., $C = x_1x_2\ldots x_{2r-1}x_1$. We specify two orientations of $C$. Call the orientation $x_1, x_2, \ldots, x_{2r-1}, x_1$ clockwise and call the orientation $x_{2r-1}, x_{2r-2}, \ldots, x_1, x_{2r-1}$ counterclockwise. For two vertices $a, b \in V(C)$, we denote by $\rightarrow C(a, b)$ the clockwise $(a, b)$-path on $C$ and by $\leftarrow C(a, b)$ the counterclockwise $(a, b)$-path on $C$.

For $uv \in E(H)$, denote $T = H + uv$. To show $\text{rad}(T) < r$, it suffices to find a vertex $z$ such that $e_T(z) < r$. Denote

$$A = V(C) = \{x_1, x_2, \ldots, x_{2r-1}\} \quad \text{and} \quad B = V(H) \setminus V(C) = \{y_1, y_2, \ldots, y_r\}.$$  

We distinguish three cases.

**Case 1.** $u, v \in A$. Let $u = x_i$ and $v = x_j$ with $i > j$. 

**Fig. 2. A sketch of $H(r, d, 3r-1)$**

**Fig. 3. $H(6, d, 17)$ with $d = 7, 8, 9, 10$**
Since $d - r + 1 \leq 2r - 3$, the vertex $y_2$ is a leaf whose only neighbor is $x_{2r-2}$. Note that in $H$, the three vertices $x_{r}$, $x_{r-1}$ and $x_{r-2}$ are central vertices, $y_1$ is the unique eccentric vertex of $x_r$, and $y_2$ is the unique eccentric vertex of $x_{r-1}$ and $x_{r-2}$. If $j \geq r$ or $i \leq r$, then $e_T(x_r) < r$. Indeed, in the former case $\overrightarrow{C}(x_r, v) \cup vu \cup \overrightarrow{C}(u, x_{2r-1}) \cup x_{2r-1}y_1$ is an $(x_r, y_1)$-path of length less than $r$ and in the latter case, $\overrightarrow{C}(x_r, u) \cup uv \cup \overrightarrow{C}(v, x_1) \cup x_1y_1$ is an $(x_r, y_1)$-path of length less than $r$.

Next suppose $i > r > j$. If $| (i - r) - (r - j) | \geq 2$, then in $T$ there is an $(x_r, y_1)$-path of length less than $r$, which implies that $e_T(x_r) < r$. It remains to consider the case $| (i - r) - (r - j) | \leq 1$. If $(i - r) - (r - j) = 0$ or 1, then in $T$, there is an $(x_{r-1}, y_2)$-path of length less than $r$ and hence $e_T(x_{r-1}) < r$. If $(r - j) - (i - r) = 1$, then in $H$, there is an $(x_{r-2}, y_2)$-path of length $r - 1$ and hence $e_T(x_{r-2}) < r$.

Case 2. $u, v \in B$. Let $u = y_i$ and $v = y_j$ with $1 \leq i < j \leq r$.

Subcase 2.1. $i = 1$ and $j \leq 2r - d$. In the sequel the subscript arithmetic for $x_k$ is taken modulo $2r - 1$. $x_{r-2j+2}$ is a central vertex of $H$ whose unique eccentric vertex is $y_j$. To see this, note that if $r - 2j + 2 \leq d - r + 1$ then $d_H(x_{r-2j+2}, y_r) \leq d - r + 1 - (r - 2j + 2) + r - (2r - d) = 2d - 3r + 2j - 1 \leq r - 1$ since $j \leq 2r - d$, and if $r - 2j + 2 > d - r + 1$ then $d_H(x_{r-2j+2}, y_r) \leq r - 2j + 2 - (d - r + 1) + r - (2r - d) = r - 2j + 1 \leq r - 3$ since $j \geq 2$.

If $r - 2j + 2 \geq 1$, in $T$ there is the $(x_{r-2j+2}, y_j)$-path $\overrightarrow{C}(x_{r-2j+2}, x_1) \cup x_1y_1 \cup y_1y_j$. Hence $d_T(x_{r-2j+2}, y_j) \leq r - 2j + 2 - 1 + 2 = r - 2j + 3 \leq r - 1$ since $j \geq 2$, implying $e_T(x_{r-2j+2}) < r$. If $r - 2j + 2 \leq 0$, in $T$ there is the path $\overrightarrow{C}(x_{r-2j+2}, x_{2r-1}) \cup x_{2r-1}y_1 \cup y_1y_j$. Hence $d_T(x_{r-2j+2}, y_j) \leq 0 - (r - 2j + 2) + 2 = 2j - r \leq r - 2$ since $j \leq 2r - d$ and $d \geq r + 1$, implying $e_T(x_{r-2j+2}) < r$.

Subcase 2.2. $i = 1$ and $2r - d + 1 \leq j \leq r$. First suppose $j = r$. Observe that $x_{2d-3r+1}$ is a central vertex of $H$ whose unique eccentric vertex is $y_r$. Also the condition $d \leq 2r - 2$ implies $2d - 3r + 1 < d - r + 1$. If $2d - 3r + 1 \geq 1$, then $d_T(x_{2d-3r+1}, y_r) \leq 2d - 3r + 1 - 1 + 2 \leq r - 2$. If $2d - 3r + 1 \leq 0$, then $d_T(x_{2d-3r+1}, y_r) \leq 0 - (2d - 3r + 1) + 2 \leq r - 1$, where we have used the fact that $d \geq r + 1$. Hence $e_T(x_{2d-3r+1}) < r$.

Next suppose $2r - d + 1 \leq j \leq r - 1$. Observe that $x_{r}$ is a central vertex of $H$ whose unique eccentric vertex is $y_1$. Note also that $r > d - r + 1$. Now in $T$, there is the $(x_r, y_1)$-path $\overrightarrow{C}(x_r, x_{d-r+1}) \cup x_{d-r+1}y_{2r-d+1} \cdots y_j \cup y_jy_1$. Hence $d_T(x_r, y_1) \leq r - (d - r + 1) + j - (2r - d) + 1 = j \leq r - 1$, implying $e_T(x_r) < r$.  

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Subcase 2.3. \( i \geq 2 \) and \( j \leq 2r - d \). First suppose \( 2(j - i) \leq r - 1 \). Then \( 2r - 2j + 2 \geq r - 2i + 3 \). Clearly \( x_{2r-2j+2} \) is the unique neighbor of \( y_j \) in \( H \). By considering the two possible cases \( r - 2i + 3 \leq d - r + 1 \) and \( r - 2i + 3 > d - r + 1 \), it is easy to verify that \( x_{r-2i+3} \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_i \). In \( T \) there is the \( (x_{r-2i+3}, y_i) \)-path \( \overrightarrow{C}(x_{r-2i+3}, x_{2r-2j+2}) \cup x_{2r-2j+2}y_j \cup y_jy_i \). Hence \( d_T(x_{r-2i+3}, y_i) \leq 2r - 2j + 2 - (r - 2i + 3) + 1 + 1 = r - 2(j - i) + 1 \leq r - 1 \), implying \( e_T(x_{r-2i+3}) < r \).

Next suppose \( 2(j - i) \geq r \). Then \( r - 2i + 2 \geq 2r - 2j + 2 \). Observe that \( x_{r-2i+2} \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_i \). Also \( j - i \leq 2r - d - 2 \). Similarly we have

\[
d_T(x_{r-2i+2}, y_i) \leq r - 2i + 2 - (2r - 2j + 2) + 1 + 1
= 2 - r + 2(j - i)
\leq 2 - r + 2(2r - d - 2)
\leq r - 2,
\]

implying \( e_T(x_{r-2i+2}) < r \).

Subcase 2.4. \( 2 \leq i \leq 2r - d \) and \( 2r - d + 1 \leq j \leq r \). First suppose \( 2r + 2 \leq 2i + d \). Then \( d - r + 1 \geq r - 2i + 3 \). Note that \( x_{r-2i+3} \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_i \). In \( T \) we have the \( (x_{r-2i+3}, y_i) \)-path \( \overrightarrow{C}(x_{r-2i+3}, x_{d-r+1}) \cup x_{d-r+1}y_{2r-d+1} \ldots y_j \cup y_jy_i \). Thus

\[
d_T(x_{r-2i+3}, y_i) \leq d - r + 1 - (r - 2i + 3) + j - (2r - d) + 1
\leq d - r + 1 - (r - 2i + 3) + r - (2r - d) + 1
= 2d - 3r + 2i - 1
\leq r - 1,
\]

implying \( e_T(x_{r-2i+3}) < r \).

Next suppose \( 2r + 2 \geq 2i + d + 1 \). Then \( r - 2i + 2 \geq d - r + 1 \). Observe that \( x_{r-2i+2} \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_i \). Similarly we have

\[
d_T(x_{r-2i+2}, y_i) \leq r - 2i + 2 - (d - r + 1) + j - (2r - d) + 1
\leq r - 2i + 2 - (d - r + 1) + r - (2r - d) + 1
= r - 2i + 2
\leq r - 2,
\]
implying \( e_T(x_{r-2i+2}) < r \).

Subcase 2.5. \( 2r - d + 1 \leq i < j \leq r \). Observe that \( x_{r+1} \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_r \). Clearly \( e_T(x_{r+1}) < r \).

**Case 3.** \( u \in A \) and \( v \in B \). Let \( u = x_i \) and \( v = y_j \).

Observe that \( x_r \) is a central vertex of \( H \) whose unique eccentric vertex is \( y_1 \). If \( j = 1 \), then \( e_T(x_r) < r \). Now suppose \( 2 \leq j \leq 2r - d \). Then both \( x_{r-2j+2} \) and \( x_{r-2j+3} \) are central vertices of \( H \) whose unique eccentric vertex is \( y_j \). If \( u \) lies on the path \( \overrightarrow{C}(x_{r-2j+2}, x_{r-2j+2}) \), then \( e_T(x_{r-2j+2}) < r \); if \( u \) lies on the path \( \overrightarrow{C}(x_{r-2j+2}, x_{r-2j+3}) \), then \( e_T(x_{r-2j+3}) < r \).

Finally suppose \( 2r - d + 1 \leq j \leq r \). We have \( 2d - 3r + 1 < d - r + 1 < r + 1 \). Observe that both \( x_{r+1} \) and \( x_{2d-3r+1} \) are central vertices of \( H \) whose unique eccentric vertex is \( y_r \). If \( 2d - 3r + 1 \leq i \leq d - r + 1 \), then \( d_T(x_{2d-3r+1}, y_r) \leq r - 1 \) and hence \( e_T(x_{2d-3r+1}) < r \). Similarly, if \( d - r + 2 \leq i \leq r + 1 \) then \( e_T(x_{r+1}) < r \).

It remains to consider the case when \( u = x_i \) lies on the path \( \overrightarrow{C}(x_{r+2}, x_{2d-3r}) \). We assert that \( e_T(u) < r \). First note that if \( w \in \{y_{2r-d+1}, y_{2r-d+2}, \ldots, y_r\} \) then \( d_T(x_i, w) \leq d - r \leq r - 2 \). Also if \( w \in V(C) \) we have \( d_T(x_i, w) \leq r - 1 \) since \( \text{diam}(C) = r - 1 \). Next suppose \( w = y_s \) with \( 1 \leq s \leq 2r - d \). Let \( x_k \) and \( x_{k+1} \) be the two vertices on \( C \) with \( d_C(x_i, x_k) = d_C(x_i, x_{k+1}) = r - 1 \). Since \( x_i \) lies on the path \( \overrightarrow{C}(x_{r+2}, x_{2d-3r}) \), we have \( k \geq 2 \) and \( k + 1 \leq 2d - 2r < 2(d - r + 1) \). It follows that \( d_H(x_i, w) \leq r - 1 \), since \( N_H(y_i) = \{x_{2r-1}, x_1\} \) and \( N_H(y_{2r-d}) = \{x_{2(d-r+1)}\} \). This completes the proof that \( H \) is radially maximal.

Note that by the two inequalities in (1), any non-self-centered radially maximal graph has radius at least 3. Obviously, the vertex \( x_{2r-2} \) is not an eccentric vertex of any vertex in \( H \). Hence by Lemma 1, the extension of \( H \) at \( x_{2r-2} \), denoted \( H_{3r} \), is radially maximal. Also, \( H_{3r} \) has the same diameter and radius as \( H \), and has order \( 3r \). Again, the vertex \( x_{2r-2} \) is not an eccentric vertex of any vertex in \( H_{3r} \). For any \( n > 3r - 1 \), performing extensions at the vertex \( x_{2r-2} \) successively, starting from \( H \), we can obtain a radially maximal graph of radius \( r \), diameter \( d \) and order \( n \). This completes the proof. \( \square \)

Combining the restriction (1) on the diameter and radius of a radially maximal graph and Theorem 2 we obtain the following corollary.

**Corollary 3.** There exists a radially maximal graph of radius \( r \) and diameter \( d \) if and only if \( r \leq d \leq 2r - 2 \).
3 Final Remarks

Since any graph with radius $r$ has order at least $2r$, Theorem 2 covers all the possible orders of self-centered radially maximal graphs.

Gliviak, Knor and Šoltés [2, p.283] conjectured that the minimum order of a non-self-centered radially maximal graph of radius $r$ is $3r − 1$. This conjecture is known to be true for the first three values of $r$; i.e., $r = 3, 4, 5$ [2, p.283], but it is still open in general. If this conjecture is true, then Theorem 2 covers all the possible orders of radially maximal graphs with a given radius.

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