Initial-amplitude dependence in weakly damped oscillators

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A pedagogically instructive experimental procedure is suggested for distinguishing between different damping terms in a weakly damped oscillator, which highlights the connection between non-linear damping and initial-amplitude dependence. The most common damping terms such as contact friction, air resistance, viscous drag, and electromagnetic damping have velocity dependences of the form constant, \(v\), or \(v^2\). The corresponding energy dependences of the form \(\sqrt{E}\), \(E\), or \(E\sqrt{E}\) in the energy loss equation give rise to characteristic dependence of the amplitude decay slope on the initial amplitude.

I. INTRODUCTION

The most commonly studied source of damping in oscillating systems is a resistive force \(F_{\text{res}} = -nv/\tau\) proportional to velocity, such as due to fluid drag at very low relative speed, when the Reynolds number \(R\) is of the order 1 or less. The equation of motion involves a linear differential equation of the type

\[ \ddot{x} + \frac{\dot{x}}{\tau} + \omega_0^2 x = 0 \]  

which yields an exponential decay \(x_0(t)/x_0(0) = e^{-t/2\tau}\) for the normalized amplitude of oscillations. The decay is independent of the initial amplitude, a characteristic of the linear nature of the system. Here \(x(t)\) refers to the instantaneous displacement of a block in a spring-mass oscillator or the angular displacement of a physical pendulum.

For weakly damped oscillations, it is convenient to consider the oscillator energy \(E\) averaged over one cycle, during which period the amplitude \(x_0(t)\) is nearly constant. With \(x(t) = x_0 \sin \omega t\), we have \(E = m\omega_0^2 x_0^2/2\), and the change in the oscillator energy over one cycle is \(\Delta E = \int F_{\text{res}} v dt = -E T/\tau\). This yields the linear energy loss equation

\[ \frac{dE}{dt} = -E \frac{E}{\tau} \]  

over long time scales compared to the oscillation time period. The normalized energy \(E(t)/E(0) = e^{-t/\tau}\) also decays exponentially, independently of the initial energy.

In most laboratory oscillators, however, non-linear terms actually dominate in the energy loss equation. A \(\sqrt{E}\) term results from the nearly velocity-independent dry friction, for instance at the pivot of a physical pendulum or due to a brake pad, whereas air resistance leads to a \(E\sqrt{E}\) term as the Reynolds number is typically of the order of 1000, in which regime the air resistance is proportional to \(v^2\). Another potential contribution to the energy loss equation arises from a centripetal correction to the normal reaction at the pivot, and therefore to the friction. As the average centripetal force \(F_c \propto \omega_{\text{max}}^2 \sim E\), this leads to a \(E\sqrt{E}\) correction to the contact friction contribution in the energy loss equation.

In certain situations, even electromagnetic damping in an oscillating system leads to a \(\sqrt{E}\) damping term, as recently reported for a simple experimental setup involving a magnet oscillating through a coil. \[1\] Similar damping behavior is expected in an oscillating pendulum consisting of a copper or aluminum disk, periodically passing between the pole pieces of two strong magnets placed near the mean position, provided the conducting disk passes through the magnetic field in a very short time compared to the oscillation time period. On the other hand, a continuously acting electromagnetic damping force, as on a conducting spinning wheel placed between two strong magnets, leads to a \(E\) damping term.

In practice, the oscillator energy loss is therefore typically described by the following non-linear equation

\[ \frac{dE}{dt} = -k\sqrt{E} - k'E - k''E \sqrt{E} \]  

In this article we describe a quantitative study of the expected initial-condition dependence of the amplitude decay due to the non-linear damping, and discuss a practical application of this initial-condition dependence. Even though the different damping terms have characteristic decay signatures—a linear decay of amplitude with time for \(\sqrt{E}\) damping, an exponential decay for \(E\) damping, and an inverse power decay for \(E\sqrt{E}\) damping—a practical difficulty often encountered is how to distinguish between several weak damping terms present simultaneously. Towards this end, we suggest a sensitive experimental procedure for quantitatively identifying the separate damping contributions in Eq. (3). Furthermore, as this procedure requires only the initial amplitude decay and not the full decay over long time, it is especially useful in situations where very small oscillation amplitudes cannot be reliably obtained.

Either separately, or in combination, the different damping terms have been considered in several earlier studies. These include a harmonic oscillator with sliding friction \[2,3\] and viscous force, \[4\] an oscillating sphere with fluid drag, \[5\] and a physical pendulum with air resistance, \[6,7\] dry friction, \[8,9\] and electromagnetic damping. \[10,11\]
II. EXPERIMENTAL SETUP

We have used a common laboratory setup for studying electromagnetic induction to monitor the oscillation amplitude. An oscillating magnet passes through a coil periodically, generating a series of emf pulses. In a practical realization of this concept, a rigid semi-circular frame of aluminum, pivoted at the center (O) of the circle (see Fig. 1) and free to oscillate in its own plane about a horizontal axis through O, has a rectangular bar magnet mounted at the center of the arc passing through a coil C of suitable area of cross section. A convenient way of monitoring the induced emf pulses, and therefore the oscillation amplitude, is through a PC interface, which can be readily realized by low-cost, convenient data-acquisition modules available in the market. The amplitude can also be directly monitored by connecting the oscillator shaft to a precision potentiometer, and recording the instantaneous voltages through a PC interface. [10]

The underlying electromagnetic induction phenomenon in this oscillating system has been discussed earlier. [1] The induced emf is significant only in a very narrow angular range about the mean position, and if the angular amplitude is not too small ($\theta_0 \gg 5^\circ$), the angular velocity of the bar magnet is very nearly constant in this narrow angular range. The peak emf $E_{\text{max}}$ is then approximately given by

$$E_{\text{max}} \approx \left( \frac{d\Phi}{d\theta} \right)_{\text{max}} \omega_{\text{max}} \approx \left( \frac{d\Phi}{d\theta} \right)_{\text{max}} \frac{2\pi}{T} \theta_0 ,$$

(4)

where $T$ is the time period of (small) oscillations, and the maximum angular velocity $\omega_{\text{max}} = (4\pi/T) \sin(\theta_0/2) \approx (2\pi/T)\theta_0$, if $\theta_0/2$ (in radians) is small compared to 1. Thus, the peak emf $E_{\text{max}}$ provides a measure of the angular amplitude $\theta_0$, and the oscillator energy $E = \frac{1}{2} I \omega_{\text{max}}^2$ is directly proportional to $\theta_0^2$.

III. DECAY OF OSCILLATOR ENERGY AND AMPLITUDE

While a solution of Eq. (3) is easily obtained, it is more instructive to proceed in steps, and we consider three separate cases.

A. $k' = k'' = 0$

When only a $\sqrt{E}$ damping term is present, as due to contact friction and intermittent electromagnetic damping, the solution of Eq. (3) yields a linear decay of the normalized oscillator amplitude with time

$$\frac{\theta_0(t)}{\theta_0(0)} = \frac{\sqrt{E(t)}}{\sqrt{E(0)}} = 1 - at ,$$

(5)

where the decay slope $a = k/2\sqrt{E(0)}$ is inversely proportional to the initial amplitude. A phase-space plot between the normalized angular momentum $L/L_0$ and angular displacement $\theta/\theta_0$ is a linear spiral, and the number of cycles executed before the oscillations come to a stop is proportional to the initial amplitude.

B. $k'' = 0$

When a resistive force proportional to velocity is included, which would be appropriate for low relative velocities when the Reynolds number $R \sim 1$, the solution is modified to

$$\frac{k + k'\sqrt{E(t)}}{k + k'\sqrt{E(0)}} = e^{-k't/2} ,$$

(6)

which leads to an exponential decay $E(t)/E(0) = e^{-k't}$ in the limit $k \to 0$. It is instructive to consider the $k' \to 0$ limit. Expanding the exponential term, we obtain

$$\frac{\theta_0(t)}{\theta_0(0)} = \frac{\sqrt{E(t)}}{\sqrt{E(0)}} = 1 - at + bt^2 + \cdots$$

(7)

where $a = k/2\sqrt{E(0)} + k'/4$ and $b/a = k'/4$. The $E$ damping term contributes to the deviation from linearity, and as $k' \to 0$, all second- and higher-order terms vanish, leaving only the linear term, as in Eq. (5).

C. General case

We again consider the small-time behaviour of the oscillator energy $E(t)$ or amplitude $\theta_0(t)$ given by Eq. (7), and substituting in Eq. (3) we obtain
damping terms which modify the quadratic coefficient $b$ depend on the initial amplitude. More importantly, the time $t$ the non-linear terms $k$, $k'$, and $k''$, which implies that the linear part of the amplitude decay itself contains information about all the damping terms.

A typical amplitude decay with time is shown in Fig. 2 for both the open- and short-circuit configurations. A dominantly linear decay is seen in both cases, with a deviation from linearity becoming pronounced at large times. In the open-circuit configuration, the damping is due to friction and air resistance. When the coil is short-circuited (through a low resistance (220 Ω) so that the emf pulses can be monitored by tapping the voltage across the resistor), intermittent electromagnetic damping is activated due to the induced eddy current when the magnet passes through the coil.

Least-square fits with the quadratic form given in Eq. (7) yield $a = 49 \times 10^{-4}$ sec$^{-1}$ and $b = 51 \times 10^{-7}$ sec$^{-2}$ for the open-circuit case, whereas for the short-circuit case $a = 117 \times 10^{-4}$ sec$^{-1}$ and $b = 214 \times 10^{-7}$ sec$^{-2}$. The linear decay rate $a$ is significantly larger in the short-circuit case due to the additional contribution to $k$ from electromagnetic damping. From Eq. (9), we note that the ratio $b/a$ depends only on the air-resistance damping coefficients $k'$ and $k''$. If the electromagnetic damping only modifies the $k$ term, then this ratio should be identical for both cases, provided the initial amplitude is identical. The near doubling of this ratio in the short-circuit case is therefore a clear indication that electromagnetic damping also introduces small $E$ and $E\sqrt{E}$ damping terms which modify the quadratic coefficient $b$.

\[ 2a = \frac{k}{\sqrt{E(0)}} + k' + k'' \sqrt{E(0)}, \]  
\[ \frac{b}{a} = \frac{k'}{4} + \frac{k'' \sqrt{E(0)}}{2}. \]

As expected, the initial energy $E(0)$ appears with both the non-linear terms $k$ and $k''$, and hence both $a$ and $b$ depend on the initial amplitude. The slope remains constant at 5 times the initial angular amplitudes shows no change in the decay slope, hence indicating no observable effect of air resistance.

**IV. INITIAL-AMPLITUDE DEPENDENCE OF THE DECAY SLOPE**

Equation (9) shows that both $k'$ and $k''$ contribute to the coefficient $b$ of the quadratic term in Eq. (7). Therefore, it is not possible to distinguish between the $E$ and $E\sqrt{E}$ damping terms from the initial amplitude decay, unless the dependence on the initial energy (amplitude) is taken into account. The form of Eq. (8) suggests a pedagogically instructive experimental procedure for identifying the signature of the weak damping terms $k$, $k'$, and $k''$ by studying the initial amplitude dependence.

If the amplitude decay is written as $\theta_0(t) = \theta_0(0) - \dot{\alpha}t + \beta t^2$, and the peak emf $E_{\text{max}}(t) = K\theta_0(t)$ as

\[ E_{\text{max}}(t) = E_{\text{max}}(0) - \alpha t + \beta t^2, \]

then from Eqs. (7) and (8), the slope $\alpha = K\theta_0(0)a$ of the linear part of the decay can be written as

\[ \alpha = \frac{K}{2C} [k + k' C\theta_0(0) + k'' C^2 \theta_0''(0)] , \]

where $\sqrt{E(0)} = C\theta_0(0)$ defines the constant $C$. Therefore, a constant, linear, and quadratic dependence of the slope $\alpha$ on the initial amplitude $\theta_0(0)$ will be signatures of the weak damping terms $k$, $k'$, and $k''$, respectively.

Figure 3 shows the amplitude decay for different initial amplitudes $\theta_0(0)$ in the open-circuit configuration. There is no noticeable change in the slope, as confirmed from the plot of the decay slope $\alpha$ vs. $\theta_0(0)$ shown in Fig. 4. The slope remains constant at $5.15 \times 10^{-2}$ V/sec, and this rules out both the damping terms $k'$ and $k''$. It thus appears that air resistance and centripetal correction to friction do not have any observable contribution to the amplitude decay, and the friction contribution is given by $(K/2C)k = 5.15 \times 10^{-2}$ V/sec.
FIG. 4. The decay slope $\alpha$ shows no change with the initial angular amplitude $\theta_0(0)$, showing no observable contribution to the amplitude decay from air resistance.

FIG. 5. A variation in the decay slope with the initial angular amplitudes is clearly observed when a $8\pi'' \times 11''$ cardboard sheet is attached to the oscillator to provide air resistance.

When a $8\pi'' \times 11''$ cardboard sheet was attached to the oscillator frame to provide air resistance, the amplitude decay showed a clear dependence on the initial amplitude $\theta_0(0)$, as seen in Fig. 5. Least-square fit of the peak-emf decay with the quadratic form of Eq. (10) yields the decay slope $\alpha$, which shows a significant increase with $\theta_0(0)$ (see Fig. 6). Least-square fit of the decay slope $\alpha$ with the quadratic form of Eq. (11) yields $(K/2C)k'C = 3.1 \times 10^{-2}$ V/sec/deg and $(K/2C)k''C^2 = 3.2 \times 10^{-3}$ V/sec/deg$^2$ for the linear and quadratic coefficients, respectively. It is instructive to note that the air resistance contribution to the decay slope $\alpha$ due to the cardboard sheet is comparable in magnitude to that due to friction at the pivot. The presence of a small $k'$ term, although ideally this term should vanish if the air resistance were purely proportional to $v^2$, has also been noted earlier.\[11\]

V. SUMMARY

Qualitatively different damping terms can be distinguished from each other by studying the variation of the amplitude decay slope with the initial amplitude. A constant, linear, and quadratic dependence of the decay slope on the initial amplitude are signatures of the $E^{1/2}$, $E$, and $E^{3/2}$ damping terms, respectively. A quantitative determination of the contributions of contact friction and air resistance due to an attached vane is demonstrated.

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