A DISCRETE STOCHASTIC INTERPRETATION OF THE DOMINATIVE \( p \)-LAPLACIAN

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Contents

1. Introduction 1
2. Statements of Results 2
3. Proofs 7
References 24

1. Introduction

The interplay between Stochastic Game Theory and nonlinear Partial Differential Equations has been of increasing importance, beginning with the pioneering work of Kohn and Serfaty [KS09, KS10] and Peres, Schramm, Sheffield and Wilson [PS08, PSSW09], involving discrete processes. We shall develop this connection for the so-called Dominative \( p \)-Laplace Equation, which is akin to the well-known normalized \( p \)-Laplace Equation. Thus, we shall present a discrete stochastic interpretation and prove uniform convergence of the discretizations.

The Dominative \( p \)-Laplacian is the operator defined for \( 2 \leq p < \infty \) as follows:

\[
\mathcal{L}_p u(x) = \frac{1}{p} (\lambda_1 + \ldots + \lambda_{N-1}) + \frac{(p-1)}{p} \lambda_N,
\]

where we have ordered the eigenvalues of \( D^2 u(x) \) as \( \lambda_1 \leq \lambda_2 \ldots \leq \lambda_N \). The operator \( \mathcal{L}_p u(x) \) has been introduced by Brustad in [Bru18b], where it was used to give a natural explanation of the superposition principle for the \( p \)-Laplace equation (see [CZ03] and [LM08]). This operator is interesting in its own right. The case \( p = 2 \) reduces to a constant multiple of the Laplace operator \( \Delta u(x)/2 \).

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The operator $L_p$ is sublinear, therefore convex and uniformly elliptic. Thus, the viscosity solutions of the equation $L_p u(x) = 0$ are locally in the class $C^{2,\alpha}$. See Chapter 6 in [CC95] for the regularity result and [Bru18a] for the general theory of sublinear operators.

In this paper, we present a discrete stochastic approximation to the unique viscosity solution of the Dirichlet problem for the Dominative $p$-Laplace Equation

$$
\begin{cases}
L_p u(x) = 0 & \text{for } x \in \Omega \\
u(x) = F(x) & \text{for } x \in \partial\Omega
\end{cases}
$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. The prescribed boundary values $F: \partial\Omega \mapsto \mathbb{R}$ are assumed to be Lipschitz continuous.

For $\epsilon > 0$ we construct an approximation $u_\epsilon$ that satisfies a non-linear mean value property, or Dynamic Programming Principle,

$$u_\epsilon(x) = \frac{1}{q-1} \int_{B_\epsilon(x)} u_\epsilon(y) \, dy + \left(\frac{q-2}{q-1}\right) \sup_{\sigma} \left\{ \frac{u_\epsilon(x + \epsilon \sigma(x)) + u_\epsilon(x - \epsilon \sigma(x))}{2} \right\},$$

where

$$q = \frac{p + 4N + 6}{2N + 4},$$

and $\sigma$ is a Borel function $\sigma: \Omega \mapsto S^{N-1}$, which we call a control.

We give a game-theoretic interpretation of $u_\epsilon$ and show that $u_\epsilon \to u$ uniformly in $\overline{\Omega}$, where the limit function $u$ is the unique solution of the Dirichlet problem (1.2).

It is also of interest to consider the case $p = \infty$ with the following interpretation

$$L_\infty u(x) = \lambda_N.$$

This is the largest eigenvalue equation, or the equation for the concave envelope, which has been studied in [Obe07] and [OS11]. For $p = \infty$ viscosity solutions with $C^{1,\alpha}$ boundary values are in the class $C^{1,\alpha}[OS11]$.

2. Statements of Results

$\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$. For a fixed $0 < \epsilon \ll 1$ we set

$$\Gamma_\epsilon = \{ x \in \mathbb{R}^N \setminus \Omega : d(x, \partial\Omega) \leq \epsilon \},$$

the outer boundary strip of width $\epsilon$. We also set $X = \Omega \cup \Gamma_\epsilon$. Note that for $x \in \Omega$, we always have $B_\epsilon(x) \subset X$. 
We extend the given bounded Lipschitz function $F: \partial \Omega \mapsto \mathbb{R}$ to $\Gamma_{\epsilon}$, preserving the same Lipschitz constant. Let $\mathcal{A}$ denote the class of bounded and Borel measurable functions $v: X \mapsto \mathbb{R}$ such that $v = F$ on $\Gamma_{\epsilon}$.

When $q \in (2, \infty)$ we define a non-linear Mean Value Operator acting on $v \in \mathcal{A}$ as follows

$$
\mathcal{M}_q(v, B_{\epsilon}(x)) = \frac{1}{q-1} \int_{\partial \Omega} v(y) dy + \left(\frac{q-2}{q-1}\right) \sup_{\sigma} \left\{ \frac{v(x+\epsilon \sigma(x)) + v(x-\epsilon \sigma(x))}{2} \right\},
$$

and the corresponding averaging operator $T_q: \mathcal{A} \mapsto \mathcal{A}$ as follows:

$$
(2.1) \quad \begin{cases}
\text{for } x \in \Omega, & T_q v(x) = \mathcal{M}_q(v, B_{\epsilon}(x)) \\
\text{for } x \in \Gamma_{\epsilon}, & T_q v(x) = v(x).
\end{cases}
$$

Note that we can also write

$$
\mathcal{M}_q(v, B_{\epsilon}(x)) = \frac{1}{q-1} \int_{B_{\epsilon}(x)} v(y) dy + \left(\frac{q-2}{q-1}\right) \sup_{|h|=1} \left\{ \frac{v(x + \epsilon h) + v(x - \epsilon h)}{2} \right\}.
$$

The mean value operator $\mathcal{M}_q$ has the following properties:

1. $\mathcal{M}_q(c, B_{\epsilon}(x)) = c$ for any constant $c$ and $x \in \Omega$

2. For $v \in \mathcal{A}$, $v \geq 0$ and $\lambda > 0$, we have

$$
\mathcal{M}_q(\lambda v, B_{\epsilon}(x)) = \lambda \mathcal{M}_q(v, B_{\epsilon}(x))
$$

for any $x \in \Omega$

3. For $v_1, v_2 \in \mathcal{A}$ such that $v_1 \leq v_2$ in $B_{\epsilon}(x)$ we have

$$
\mathcal{M}_q(v_1, B_{\epsilon}(x)) \leq \mathcal{M}_q(v_2, B_{\epsilon}(x))
$$

4. For smooth functions we have

$$
(2.2) \quad \lim_{\epsilon \to 0} \frac{\mathcal{M}_q(v, B_{\epsilon}(x)) - v(x)}{\epsilon^2} = D_q v(x),
$$

where

$$
D_q u(x) = \frac{1}{(q-1)} \frac{1}{2(N+2)} \Delta u + \left(\frac{q-2}{q-1}\right) \lambda_N(D^2 u(x)).
$$

Note that we have

$$
D_q = \frac{p}{2(N+2) + p - 2} \mathcal{L}_p,
$$

where $p = 2 + 2(N+2)(q - 2)$ as in (1.3) and $q \in [2, \infty)$. 

Lemma 2.1. There exists a unique function \( v_\epsilon \in A \) such that \( T_q v_\epsilon(x) = v_\epsilon(x) \) for all \( x \in X \).

We keep the subindex \( \epsilon \) to emphasize the dependence on the step-size. We call \( v_\epsilon \) the \( \epsilon \)-mean value solution.

Given a fixed control \( \sigma: \Omega \mapsto S^{N-1} \) and a stepsize \( 0 < \epsilon << 1 \) we define a discrete random process as follows. Start at a point \( x_0 \in X \). If \( x_0 \in \Gamma_\epsilon \) we set \( x_1 = x_0 \) and stop; otherwise \( B_\epsilon(x_0) \subset X \). In the latter case, we move one step according to

- with probability \( \frac{1}{q-1} \) select \( x_1 \in B_\epsilon(x_0) \) at random,
- with probability \( \frac{q-2}{2(q-1)} \) select \( x_1 = x_0 + \epsilon \sigma(x_0) \), and
- with probability \( \frac{q-2}{2(q-1)} \) select \( x_1 = x_0 - \epsilon \sigma(x_0) \).

(Observe that the probabilities sum up to 1, as they should.) We continue this process so that we always have \(|x_i - x_{i-1}| \leq \epsilon\), and stop when we first reach \( \Gamma_\epsilon \), say at \( x_{\tau_\sigma} \), when \( k = \tau_\sigma \). More formally, fix \( x_0 \in X \) and define the space

\[
X^{\infty,x_0} = \{ \omega = (x_0, x_1, x_2, \ldots): x_n \in X \}.
\]

For \( n \geq 1 \) let \( \mathcal{F}^{x_0}_n \) be the \( \sigma \)-algebra generated by the cylinders

\[
A_1 \times A_2 \times \cdots \times A_n \times X \times X \cdots
= \{ \omega \in X^{\infty,x_0}: x_i \in A_i, i = 1, \ldots, n \}
= A_1 \times A_2 \times \cdots \times A_n \quad \text{(abuse of notation),}
\]

where \( A_i \subset X \) are Borel sets.

Clearly we have \( \mathcal{F}^{x_0}_n \subset \mathcal{F}^{x_0}_{n+1} \) so that \( \{ \mathcal{F}^{x_0}_n \}_{n \geq 1} \) is a filtration of the \( \sigma \)-algebra

\[
\mathcal{F}^{x_0} = \sigma \left( \bigcup_{n \geq 1} \mathcal{F}^{x_0}_n \right).
\]

The coordinate functions \( x_n(\omega) = x_n \) are \( \mathcal{F}^{x_0}_n \) and \( \mathcal{F}^{x_0} \) measurable.

Let \( \tau_\sigma: X^{\infty,x_0} \to \mathbb{N} \cup \{ \infty \} \) be the random variable

\[
\tau_\sigma(\omega) = \min \{ n \in \mathbb{N}: x_n \in \Gamma_\epsilon \},
\]

where we follow the convention \( \min \emptyset = \infty \). We say that \( \tau_\sigma \) is a stopping time with respect to the filtration \( \{ \mathcal{F}^{x_0}_n \}_{n \geq 1} \).

For \( x \in X \) define the transition probability measures \( \gamma[x] \) as follows:

If \( x \in \Gamma_\epsilon \) we set \( \gamma[x] = \delta_x \).
If \( x \in \Omega \) we set
\[
\gamma(x) = \frac{1}{q-1} \frac{\mathcal{L}^N[B_\epsilon(x)]}{\mathcal{L}^N(B_\epsilon(x))} + \frac{q-2}{2(q-1)} (\delta_{x+\epsilon \sigma(x)} + \delta_{x-\epsilon \sigma(x)}).
\]
Here \( \mathcal{L}^N[B_\epsilon(x)] \) denotes the \( N \)-dimensional Lebesgue measure restricted to the ball \( B_\epsilon(x) \) so that \( \gamma(x) \) is always a probability.

For \( n \geq 1 \) define the probability measures \( \mathbb{P}^{n,x_0}_\sigma \) on the measurable space \((X^{\infty,x_0}, \mathcal{F}^{x_0}_n)\) as follows:

\[
\mathbb{P}^{1,x_0}_\sigma(A_1) = \int_{A_1} 1 \, d\gamma(x_0)(y_1),
\]
(Note that \( x_0 \) is fixed and the integration variable \( y_1 \in A_1 \).)

\[
\mathbb{P}^{2,x_0}_\sigma(A_1 \times A_2) = \int_{A_1} \left( \int_{A_2} 1 \, d\gamma(y_2) \right) d\gamma(x_0)(y_1)
\]

In the general case we get

\[
\mathbb{P}^{n,x_0}_\sigma(A_1 \times \cdots \times A_n)
= \int_{A_1} \left( \int_{A_2} \left( \cdots \int_{A_n} 1 \, d\gamma_{n-1}(y_{n-1}) \right) \cdots d\gamma_2(y_2) \right) d\gamma_1(x_0)(y_1).
\]

Note that we have used the following:

**Claim 2.1.** The mapping \( x \rightarrow \gamma(x)(A) \) is Borel measurable for a fixed Borel set \( A \subset X \).

We write
\[
\gamma(x)(A) = \frac{1}{q-1} \frac{|B_\epsilon(x) \cap A|}{|B_\epsilon(x)|} + \frac{q-2}{2(q-1)} (\delta_{x+\epsilon \sigma(x)}(A) + \delta_{x-\epsilon \sigma(x)}(A)).
\]

The first term is, in fact, continuous and the second one is easily seen to be Borel measurable, since \( x \rightarrow \sigma(x) \) is so. The family of probabilities \( \{\mathbb{P}^{n,x_0}_\sigma\}_{n \geq 1} \) is consistent in the sense of Kolmogorov. Thus the limit probability
\[
\mathbb{P}^{x_0}_\sigma = \lim_{n \to \infty} \mathbb{P}^{n,x_0}_\sigma
\]
exists and we have
\[
\mathbb{P}^{n,x_0}_\sigma(A_1 \times \cdots \times A_n) = \mathbb{P}^{x_0}_\sigma(A_1 \times \cdots \times A_n)
\]
for all cylinders \( A_1 \times \cdots \times A_n \).

The following lemma tells us that the conditional expectation of the process at step \( n \) relative to its past history, reflected in the sigma-algebra \( \mathcal{F}^{x_0}_{n-1} \), is precisely the integral of \( \nu \) with respect to the transition probability from step \( n-1 \) to \( n \).
Lemma 2.2. Let \( v: X \mapsto \mathbb{R} \) be a bounded Borel measurable function. Then we have

\[
\mathbb{E}^{x_0} \left[ v(x_n) \mid \mathcal{F}_{n-1}^x \right](x_{n-1}) = \frac{1}{q - 1} \int_{B(x_{n-1})} v(y) \, dy + \frac{q - 2}{q - 1} \left( \frac{v(x_{n-1} + \epsilon \sigma(x_{n-1})) + v(x_{n-1} - \epsilon \sigma(x_{n-1}))}{2} \right).
\]

Lemma 2.3. For any fixed \( y_0 \in \mathbb{R}^N \) the sequence of random variables

\[
\{ |x_{n \wedge \tau_{\sigma}} - y_0|^2 - c_N(n \wedge \tau_{\sigma}) \epsilon^2 \}_{n \geq 1}
\]

is a martingale with respect to the natural filtration \( \{ \mathcal{F}_n \}_{n \geq 1} \). Here \( c_N \) is some constant depending only on \( N \).

Applying Doob’s optional stopping to the finite stopping times \( \tau_{\sigma} \wedge n \) and letting \( n \to \infty \), we have

\[
\epsilon^2 \mathbb{E}^{x_0}_{\sigma}[\tau_{\sigma}] \leq C(N, \Omega)
\]

and the process ends almost surely:

\[
\mathbb{P}^{x_0}_{\sigma}(\{ \tau_{\sigma} < \infty \}) = 1.
\]

Therefore, when we run the process we will hit \( \Gamma_\epsilon \) almost surely. Thus, the random variable \( F(x_{\tau_{\sigma}}) \) is well defined. Averaging over all possible runs we get the expected value

\[
u^{\sigma}_\epsilon(x_0) = \mathbb{E}^{x_0}_{\sigma}[F(x_{\tau_{\sigma}})].\]

Optimizing over all strategies we get

\[
u_\epsilon(x_0) = \sup_{\sigma} (\nu^{\sigma}_\epsilon(x_0)) = \sup_{\sigma} (\mathbb{E}^{x_0}_{\sigma}[F(x_{\tau_{\sigma}})]),\]

which we call the \( \epsilon \)-stochastic solution.

Recall that the \( \epsilon \)-mean value solution was defined in Lemma 2.1.

Theorem 2.1. The following hold:

i) \( \nu_{\epsilon}(x) = F(x) \) for \( x \in \Gamma_{\epsilon} \).

ii) \( \nu_{\epsilon}(x_0) = v_{\epsilon}(x_0) \), where \( v_{\epsilon} \) is the \( \epsilon \)-mean value solution. That is, \( u_{\epsilon} \)

also satisfies the dynamic programming principle \( u_{\epsilon}(x_0) = T_{\epsilon} u_{\epsilon}(x_0) \).

We will need later the following comparison principle for \( \epsilon \)-mean value solutions, which follows at once from formula (2.6) and Theorem 2.1.

Lemma 2.4. Let \( v_{\epsilon} \) be the \( \epsilon \)-mean value solution with boundary values \( F \) and let \( w_{\epsilon} \) be the \( \epsilon \)-mean value solution with boundary values \( G \). Suppose that \( F \leq G \) on \( \partial \Omega \), then we have \( v_{\epsilon} \leq w_{\epsilon} \) in \( \Omega \).
We next adapt the Barle-Souganidis technique [BS91] of semi-continuous regularizations. We remark that in [BS91] the domain \( \Omega \) must be of class \( C^2 \) and the equation must satisfy a strong uniqueness property involving the viscosity interpretation of the boundary Dirichlet data. We replace the strong uniqueness property with uniform boundary estimates for the discretizations \( u_\epsilon \) to reach the same uniform convergence conclusion as in [BS91].

**Lemma 2.5.** Given \( \eta > 0 \) we can find \( \delta_1 > 0 \) and \( \epsilon_1 > 0 \) such that whenever \( x_0 \in \Omega, \ y_0 \in \partial \Omega, \ |x_0 - y_0| < \delta_1 \) and \( \epsilon < \epsilon_1 \) we have

\[
|u_\epsilon(x_0) - F(y_0)| \leq \eta.
\]

For \( x \in \Omega \) define the upper-semicontinuous envelope

\[
\overline{u}(x) = \limsup_{\epsilon \to 0} \limsup_{y \to x} u_\epsilon(y)
\]

and the lower-semicontinuous envelope

\[
\underline{u}(x) = \liminf_{\epsilon \to 0} \liminf_{y \to x} u_\epsilon(y)
\]

Lemma 2.5 implies

\[
\limsup_{x \in \Omega, y \in \partial \Omega} \overline{u}(x) \leq F(y)
\]

(2.7)

\[
\liminf_{x \in \Omega, y \in \partial \Omega} \underline{u}(x) \geq F(y)
\]

(2.8)

**Lemma 2.6.** \( \overline{u} \) is a viscosity subsolution and \( \underline{u} \) is a viscosity supersolution of \( L_p u = 0 \).

**Theorem 2.2.** We have \( \overline{u} = \underline{u} \), denoted by \( u \). It is the unique solution to the Dirichlet problem (1.2). Moreover \( u_\epsilon \to u \) uniformly in \( \overline{\Omega} \).

3. Proofs

3.1. The linear case. We shall keep the control \( \sigma \) fixed for now. For \( v \in A \) we define the mean value operator

\[
\mathcal{M}_\sigma(v, B_\epsilon(x)) = \frac{1}{q-1} \int_{B_\epsilon(x)} v(y) \, dy + \frac{q-2}{q-1} \left\{ \frac{v(x + \epsilon \sigma(x)) + v(x - \epsilon \sigma(x))}{2} \right\}
\]

and the corresponding averaging operator \( T_\sigma v \in A \) as follows:

\[
\begin{cases}
\text{for } x \in \Omega, & T_\sigma v(x) = \mathcal{M}_\sigma(v, B_\epsilon(x)) \\
\text{for } x \in \Gamma_\epsilon, & T_\sigma v(x) = v(x).
\end{cases}
\]
For smooth functions we have
\[
\lim_{\epsilon \to 0} \frac{\mathcal{M}_\sigma(v, B_\epsilon(x)) - v(x)}{\epsilon^2} = \mathcal{L}_\sigma v(x),
\]
where \(\mathcal{L}_\sigma\) is the differential operator
\[
\mathcal{L}_\sigma u(x) = \frac{1}{(q - 1) 2(N + 2)} \Delta u + \frac{q - 2}{q - 1} (D^2 u(x) \sigma(x), \sigma(x)).
\]
Note that
\[
\mathcal{L}_\sigma u(x) = \text{trace} \{A(x) D^2 u(x)\},
\]
where
\[
A(x) = \frac{1}{(q - 1) 2(N + 2)} I_N + \frac{q - 2}{q - 1} \sigma(x) \otimes \sigma(x).
\]
We see that
\[
\langle A(x) \xi, \xi \rangle = \frac{1}{(q - 1) 2(N + 2)} |\xi|^2 + \frac{q - 2}{q - 1} \langle \sigma(x), \xi \rangle^2.
\]
The operator \(\mathcal{L}_\sigma\) is uniformly elliptic:
\[
\frac{1}{(q - 1) 2(N + 2)} |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq \left( \frac{1}{(q - 1) 2(N + 2)} + \frac{q - 2}{q - 1} \right) |\xi|^2.
\]
We now follow the scheme described in \[LPS14\]. Set \(v_0 = \chi_{\Gamma_\epsilon} F + \chi_{\Omega} \inf F\) and set \(v_1 = T_\sigma v_0\). We see that \(v_1 \geq v_0\). We now set \(v_{n+1} = T_\sigma v_n\) and observe that the sequence \(v_n\) is non-decreasing and the boundary condition \(v_n = F\) is satisfied on \(\Gamma_\epsilon\).

**Claim 3.1.** \(v_n \leq \sup_{\Gamma_\epsilon} F\) (Clear by induction on \(n\)).

Hence \(v = \lim_{n \to \infty} v_n\) exists for some function \(v \in \mathcal{A}\).

**Claim 3.2.** \(v_n \to v\) uniformly in \(X\).

Let \(M = \lim_{n \to \infty} \sup_X (v - v_n)\) and suppose that \(M > 0\). Fix \(\delta > 0\) and select \(n > 1\) so that
\[
\sup_X (v - v_n) < M + \delta
\]
and
\[
\frac{1}{q - 1} \int_{B_\epsilon(x)} (v - v_n) \leq \frac{1}{(q - 1)|B_\epsilon(x)|} \int_X (v - v_n) < \delta
\]
uniformly in \(x\) (recall that \(\epsilon\) is fixed) by the monotone convergence theorem.

Choose \(x_0 \in X\) such that \(v(x_0) - v_{n+1}(x_0) > M - \delta\) and note that \(x_0 \in \Omega\).
Choose $m > n$ so that $v_{m+1}(x_0) - v_{n+1}(x_0) > M - 2\delta$.

\[
M - 2\delta < v_{m+1}(x_0) - v_{n+1}(x_0) = T_\sigma(v_m - v_n)(x_0)
\]

\[
= \frac{1}{q-1} \int_{B_\epsilon(x_0)} (v_m - v_n) + \frac{q-2}{q-1} (v_m - v_n)(x_0 + \epsilon \sigma(x_0)) + (v_m - v_n)(x_0 - \epsilon \sigma(x_0))
\]

\[
\leq \frac{1}{q-1} \int_{B_\epsilon(x_0)} (v - v_n) + \frac{q-2}{q-1} \sup_X (v_m - v_n)
\]

\[
\leq \delta + \frac{q-2}{q-1} \sup_X (v - v_n)
\]

\[
\leq \delta + \frac{q-2}{q-1} (M + \delta)
\]

Thus we see that for $\delta$ small we must have

\[
M - 2\delta \leq \delta + \frac{q-2}{q-1} (M + \delta),
\]

which is clearly not possible since $0 < \frac{q-2}{q-1} < 1$. Thus $M = 0$, as desired.

**Lemma 3.1.** There exists a unique Borel function $v_\epsilon \in A$ such that

\[
T_\sigma v_\epsilon(x) = v_\epsilon(x)
\]

for all $x \in X$.

**Proof.** Existence follows by Claim 3.2. Suppose that we have two solutions $u$ and $w$. Let

\[
M = \sup_X (u - w) > 0.
\]

Choose $x_n \in X$ such that $\lim_{n \to \infty} (u(x_n) - w(x_n)) = M$. Note that indeed $x_n \in \Omega$. We have

\[
(u - w)(x_n) \leq \frac{1}{q-1} \int_{B_\epsilon(x_n)} (u - w)
\]

\[
+ \frac{q-2}{q-1} \frac{1}{q-1} (u - w)(x_n + \epsilon \sigma(x_n)) + (u - w)(x_n - \epsilon \sigma(x_n))
\]

Let $n \to \infty$. A subsequence of $x_n$ will approach some point $x_0 \in X$, so that we have

\[
M \leq \frac{1}{q-1} \int_{B_\epsilon(x_0)} (u - w) + \frac{q-2}{q-1} M.
\]

Simplifying, it becomes

\[
M \leq \int_{B_\epsilon(x_0)} (u - w).
\]

We conclude that $(u - w)(x) = M$ for a.e. $x \in B_\epsilon(x_0)$. Note that this implies that $x_0 \in \Omega$ and $B_\epsilon(x_0) \subset \Omega$. Define the set

\[
G = \{x \in \Omega: (u - w)(x) = M \text{ a.e. in a neighborhood of } x\}.
\]

We have shown that $G \neq \emptyset$. The same proof shows that $G$ is closed, and since it is clearly open, we have $G = \Omega$ so that $(u - w)(x) = M$ a.e. in $\Omega$. 
To reach a contradiction, take \( y \in \partial \Omega \) and choose \( x_n \in \Omega \) such that \( x_n \to y \) and \( (u - w)(x_n) = M \).

\[ \Box \]

3.1.1. **Proof of Lemma 2.2.**

**Proof.** The conditional expectation \( \mathbb{E}^{x_0}_\sigma [v \circ x_n | \mathcal{F}^{x_0}_{n-1}] \) is \( \mathcal{F}^{x_0}_{n-1} \) measurable, and thus a function of \( (x_1 \ldots, x_{n-1}) \) such that

\[
\mathbb{E}^{x_0}_\sigma \left[ \chi_A \mathbb{E}^{x_0}_\sigma [v \circ x_n | \mathcal{F}^{x_0}_{n-1}] \right] = \mathbb{E}^{x_0}_\sigma [\chi_A (v \circ x_n)]
\]

for every cylinder \( A = A_1 \times \cdots \times A_{n-1} \). Do this for \( v = \chi_B \), then for simple functions, and then use the monotone convergence theorem.

Temporarily, set \( G(x_1, \ldots, x_{n-1}) = \mathbb{E}^{x_0}_\sigma [\chi_B \circ x_n | \mathcal{F}^{x_0}_{n-1}] \). This function must satisfy

\[
\mathbb{E}^{x_0}_\sigma [\chi_A G(x_1, \ldots, x_{n-1})] = \mathbb{E}^{x_0}_\sigma [\chi_A (\chi_B \circ x_n)]
\]

for every cylinder \( A = A_1 \times \cdots \times A_{n-1} \). We have

\[
\mathbb{E}^{x_0}_\sigma [\chi_A (\chi_B \circ x_n)] = \mathbb{P}^{x_0}_\sigma (A_1 \times \cdots \times A_{n-1} \times B)
\]

and

\[
\mathbb{P}^{x_0}_\sigma (A_1 \times \cdots \times A_{n-1} \times B) = \int_{A_1} \left( \int_{A_2} \left( \cdots \int_B 1 \, d\gamma[y_{n-1}] (y_n) \right) \cdots d\gamma[y_1] (y_2) \right) d\gamma[x_0] (y_1)
\]

\[
= \int_{A_1} \int_{A_2} \left( \cdots \int_B \frac{1}{q-1} |B_\epsilon(y_{n-1}) \cap B| \right) d\gamma[y_1] (y_2) \right) d\gamma[x_0] (y_1)
\]

\[
= \int_{A_1} \left( \int_{A_2} \left( \cdots G(y_1, \ldots, y_{n-1}) \right) \right) d\gamma[y_1] (y_2) \right) d\gamma[x_0] (y_1)
\]

for all cylinders \( A = A_1 \times \cdots \times A_{n-1} \). Thus, we must have

\[
G(y_1, \ldots, y_{n-1}) = \frac{1}{q-1} \frac{|B_\epsilon(y_{n-1}) \cap B|}{|B_\epsilon(y_{n-1})|} + \frac{q-2}{2(q-1)} (\delta_{y_{n-1}+\sigma(y_{n-1})}(B) + \delta_{y_{n-1}-\sigma(y_{n-1})}(B))
\]

\[ \Box \]
3.1.2. Proof of Lemma 2.3.

Proof. Let us compute

\[
\mathbb{E}_{x_0}^{x_0} \left[ |x_n|^2 \right| \mathcal{F}_{n-1}^{x_0} ] (x_{n-1}) = \frac{1}{q-1} \int_{B_\epsilon(x_{n-1})} |y|^2 \, dy + \frac{q-2}{q-1} |x_{n-1} + \epsilon \sigma(x_{n-1})|^2 + \frac{q-2}{q-1} |x_{n-1} - \epsilon \sigma(x_{n-1})|^2
\]

\[
= |x_{n-1}|^2 + \left( \frac{1}{q-1} N + \frac{q-2}{q-1} \right) \epsilon^2,
\]

where we have used the fact that

\[
\int_{B_\epsilon(0)} |z|^2 \, dz = \epsilon^2 \frac{N}{N+2}.
\]

Writing

\[
C_{N,q} = \frac{1}{q-1} \frac{N}{N+2} + \frac{q-2}{q-1} \epsilon^2,
\]

we get

\[
\mathbb{E}_{x_0}^{x_0} \left[ |x_n|^2 - n C_{N,q} \epsilon^2 \right| \mathcal{F}_{n-1}^{x_0} ] (x_{n-1}) = |x_{n-1}|^2 + C_{N,q} \epsilon^2 - n C_{N,q} \epsilon^2 = |x_{n-1}|^2 - (n-1) C_{N,q} \epsilon^2.
\]

\[
\square
\]

Next, we present the linear version of Theorem 2.1.

Lemma 3.2. The following hold:

i) \( u_\epsilon(x_n) \) is a martingale with respect to the filtration \( \{ \mathcal{F}_n \}_{n \geq 1} \)

ii) \( u_\epsilon(x) = F(x) \) for \( x \in \Gamma_\epsilon \).

iii) \( u_\epsilon(x_0) = v_\epsilon(x_0) \), where \( v_\epsilon \) is the solution given in Lemma 3.1. That is, we have \( u_\epsilon(x_0) = T_\sigma u_\epsilon(x_0) \).

Proof. Let us first note that \( v_\epsilon(x_n) \) is a martingale with respect to the filtration \( \{ \mathcal{F}_n \}_{n \geq 1} \). Let us start by using Lemma 2.2:
\[ E_{\sigma}^{x_0} \left[ v_\epsilon \circ x_n \mid F_{x_0}^{x_n} \right](x_{n-1}) \]
\[ = \int_X v_\epsilon(y) \, d\gamma_\sigma[x_{n-1}](y) \]
\[ = \frac{1}{q-1} \int_{B_\epsilon(x_{n-1})} v_\epsilon(y) \, dy + \frac{q-2}{q-1} \left\{ \frac{v_\epsilon(x_{n-1} + \epsilon \sigma(x_{n-1})) + v_\epsilon(x_{n-1} - \epsilon \sigma(x_{n-1}))}{2} \right\} \]
\[ = T_\sigma v_\epsilon(x_{n-1}) = v_\epsilon(x_{n-1}) \]

We have used the definition of \( T_\sigma \) and the fact that \( v_\epsilon \) is the mean value solution given in Lemma 3.1.

We use now Doob’s theorem to move from the boundary back to \( x_0 \):
\[
 u_\epsilon^\sigma(x_0) = E_{\sigma}^{x_0} [F(x_\tau)] = E_{\sigma}^{x_0} [v_\epsilon(x_\tau)] = E_{\sigma}^{x_0} [v_\epsilon(x_0)] = v_\epsilon(x_0)
\]

\[
 {\Box}
\]

3.2. The General Case. In this section we consider the set of all possible Borel controls \( \sigma : \Omega \mapsto \mathbb{S}^{N-1} \).

Recall that for \( v \in \mathcal{A} \) we define the non-linear mean value operator
\[
 M(v, B_\epsilon(x)) = \frac{1}{q-1} \int_{B_\epsilon(x)} v(y) \, dy + \left( \frac{q-2}{q-1} \right) \sup_{\sigma} \left( \frac{v(x + \epsilon \sigma(x)) + v(x - \epsilon \sigma(x))}{2} \right)
\]
and the corresponding averaging operator \( T v \in \mathcal{A} \) as follows:
\[
 (3.2) \begin{cases}
  & \text{for } x \in \Omega, \quad T v(x) = M(v, B_\epsilon(x)) \\
  & \text{for } x \in \Gamma_\epsilon, \quad T v(x) = v(x).
\end{cases}
\]

Note that
\[
 M_\sigma(v, B_\epsilon(x)) \leq M(v, B_\epsilon(x))
\]
and
\[
 \sup_{\sigma} M_\sigma(v, B_\epsilon(x)) = M(v, B_\epsilon(x)).
\]

Lemma 3.3. There exists a unique Borel function \( v_\epsilon \in \mathcal{A} \) such that
\[ T v_\epsilon(x) = v_\epsilon(x) \]
for all \( x \in X \).
Proof. Set $v_0 = \chi_{\Gamma_\epsilon} F + \chi_{\Omega} \inf F$ and set $v_1 = T v_0$. Since $F \geq 0$, we see that $v_1 \geq v_0$. We now set $v_{n+1} = T v_n$ and observe that the sequence $v_n$ is non-decreasing and the functions satisfy the boundary condition $v_n = F$ on $\Gamma_\epsilon$.

Claim 3.3. $v_n \leq \sup_{\Gamma_\epsilon} F$ (Clear by induction on $n$).

Hence $v = \lim_{n \to \infty} v_n$ exists for some function $v \in \mathcal{A}$.

Claim 3.4. $v_n \to v$ uniformly in $X$.

Let $M = \lim_{n \to \infty} \sup_X (v - v_n)$ and suppose that $M > 0$. Fix $\delta > 0$ and select $n > 1$ so that

$$\sup_X (v - v_n) < M + \delta$$

and

$$\frac{1}{q - 1} \int_{B_\epsilon(x)} (v - v_n) \leq \frac{1}{(q - 1)|B_\epsilon(x)|} \int_X (v - v_n) < \delta$$

uniformly in $x$ (recall that $\epsilon$ is fixed) by the monotone convergence theorem. Choose $x_0 \in X$ such that $v(x_0) - v_{n+1}(x_0) > M - \delta$ and note that $x_0 \in \Omega$. 
Choose $m > n$ so that $v_{m+1}(x_0) - v_{n+1}(x_0) > M - 2\delta$.

\[
M - 2\delta < v_{m+1}(x_0) - v_{n+1}(x_0) = T(v_m)(x_0) - T(v_n)(x_0)
\]

\[
= \frac{1}{q - 1} \int_{B_r(x_0)} (v_m - v_n) + \frac{q-2}{q-1} \sup_{|h|=1} \left\{ \frac{v_m(x_0 + \epsilon h) + v_m(x_0 - \epsilon h)}{2} - \frac{v_n(x_0 + \epsilon h) + v_n(x_0 - \epsilon h)}{2} \right\}
\]

\[
\leq \frac{1}{q - 1} \int_{B_r(x_0)} (v_m - v_n)
\]

\[
+ \frac{q-2}{q-1} \sup_{|h|=1} \left\{ \frac{(v_m - v_n)(x_0 + \epsilon h) + (v_m - v_n)(x_0 - \epsilon h)}{2} \right\}
\]

\[
< \delta + \frac{q-2}{q-1} \sup_x (v_m - v_n) \leq \delta + \frac{q-2}{q-1} \sup_x (v - v_n)
\]

\[
\leq \delta + \frac{q-2}{q-1} (M + \delta)
\]

Thus we see that for $\delta$ small we must have

\[
M - 2\delta \leq \delta + \frac{q-2}{q-1} (M + \delta),
\]

which is clearly not possible since $0 < \frac{q-2}{q-1} < 1$.

Existence follows by Claim 3.4. To obtain uniqueness, suppose that we have two solutions $u$ and $w$. Let

\[
M = \sup_x (u - w) > 0.
\]
Choose \( x_n \in X \) such that \( \lim_{n \to \infty} (u(x_n) - w(x_n)) = M \). Note that indeed \( x_n \in \Omega \). We have

\[
(u - w)(x_n) = \frac{1}{q - 1} \int_{B_r(x_n)} (u - w)
+ \left( \frac{q - 2}{q - 1} \right) \sup_{|h| = 1} \left\{ \frac{u(x_n + \epsilon h) + u(x_n - \epsilon h)}{2} \right\}
- \frac{q - 2}{q - 1} \sup_{|h| = 1} \left\{ \frac{w(x_n + \epsilon h) + w(x_n - \epsilon h)}{2} \right\}
\leq \frac{1}{q - 1} \int_{B_r(x_n)} (u - w)
+ \left( \frac{q - 2}{q - 1} \right) \sup_{|h| = 1} \left\{ \frac{(u - w)(x_n + \epsilon h) + (u - w)(x_n - \epsilon h)}{2} \right\}
\]

Let \( x_n \to x_0 \in X \), we get

\[
M \leq \frac{1}{q - 1} \int_{B_r(x_0)} (u - w) + \frac{q - 2}{q - 1} M.
\]

Simplifying, it becomes

\[
M \leq \int_{B_r(x_0)} (u - w).
\]

We conclude that \( (u - w)(x) = M \) for a.e. \( x \in B_r(x_0) \). Note that this implies \( x_0 \in \Omega \) and also that \( B_r(x_0) \subset \Omega \). Define the set

\[
G = \{ x \in \Omega: (u - w) = M \text{ a.e. in a neighborhood of } x \}.
\]

We have shown that \( G \neq \emptyset \). The same proof shows that \( G \) is closed, and since it is clearly open, we have \( G = \Omega \) so that \( (u - w)(x) = M \) a.e. in \( \Omega \).

To reach a contradiction, take \( y \in \partial \Omega \) and choose \( x_n \in \Omega \) such that \( x_n \to y \) and \( (u - w)(x_n) = M \).

\[\square\]

### 3.2.1. Proof of Theorem 2.1.

Let us start by using Lemma 2.2:

\[
\mathbb{E}_{x_0}^x \left[ v_\epsilon \circ x_n | \mathcal{F}_{x_0}^{x_{n-1}} \right] (x_{n-1}) = \int_X v_\epsilon(y) d\gamma_n[x_{n-1}](y)
= \frac{1}{q - 1} \int_{B_r(x_{n-1})} v_\epsilon(y) dy + \frac{q - 2}{q - 1} \left( v_\epsilon(x_{n-1} + \epsilon \sigma(x_{n-1})) + v_\epsilon(x_{n-1} - \epsilon \sigma(x_{n-1})) \right)
\leq \frac{1}{q - 1} \int_{B_r(x_{n-1})} v_\epsilon(y) dy + \frac{q - 2}{q - 1} \sup_{\sigma} \left( v_\epsilon(x_{n-1} + \epsilon \sigma(x_{n-1})) + v_\epsilon(x_{n-1} - \epsilon \sigma(x_{n-1})) \right)
= T v_\epsilon(x_{n-1}) = v_\epsilon(x_{n-1})
\]
We have shown that \( \{v_\epsilon \circ x_n\}_{n \geq 1} \) is a supermartingale with respect to the filtration \( \{F_n^{x_0}\}_{n \geq 1} \) for all controls \( \sigma \).

We use now Doob’s theorem for supermartingales to move from the boundary back to \( x_0 \):

\[
u_\epsilon(x_0) = \sup_\sigma (E^{x_0}_\sigma [F(x_\tau)])
= \sup_\sigma (E^{x_0}_\sigma [v_\epsilon(x_\tau)])
\leq \sup_\sigma (E^{x_0}_\sigma [v_\epsilon(x_0)])
= v_\epsilon(x_0).
\]

Suppose now that we can find a Borel quasi-optimal control \( \tilde{\sigma} \) such that

\[
v_\epsilon(x) \leq \frac{1}{q-1} \int_{B_r(x)} v_\epsilon(y) dy + \frac{q-2}{q-1} \left( \frac{v_\epsilon(x + \epsilon \tilde{\sigma}(x)) + v_\epsilon(x - \epsilon \tilde{\sigma}(x))}{2} \right) + \delta.
\]

We claim that \( \{v_\epsilon \circ x_n + n \delta\}_{n \geq 1} \) is a submartingale with respect to the filtration \( \{F_n^{x_0}\}_{n \geq 1} \):

\[
E^{x_0}_\sigma [v_\epsilon \circ x_n + n \delta | F_n^{x_0}] (x_{n-1}) = \int_X v_\epsilon(y) d\gamma_n,_{\tilde{\sigma}} [x_{n-1}](y) + n \delta
= \frac{1}{q-1} \int_{B_r(x_{n-1})} v_\epsilon(y) dy
+ \frac{q-2}{q-1} \left( \frac{v_\epsilon(x_{n-1} + \epsilon \tilde{\sigma}(x_{n-1})) + v_\epsilon(x_{n-1} - \epsilon \tilde{\sigma}(x_{n-1}))}{2} \right) + n \delta
\geq v_\epsilon(x_{n-1}) - \delta + n \delta = v_\epsilon(x_{n-1}) + (n-1) \delta.
\]

We use Doob’s theorem for submartingales to move from the boundary back to \( x_0 \):

\[
u_\epsilon(x_0) = \sup_\sigma (E^{x_0}_\sigma [F(x_\tau)])
\geq E^{x_0}_\sigma [F(x_{\tau_\delta})]
= E^{x_0}_\sigma [v_\epsilon(x_{\tau_\delta})]
= E^{x_0}_\sigma [v_\epsilon(x_{\tau_\delta}) + \tau_\delta \delta - \tau_\delta \delta]
= E^{x_0}_\sigma [v_\epsilon(x_{\tau_\delta}) + \tau_\delta \delta] - E^{x_0}_\sigma [\tau_\delta \delta]
\geq v_\epsilon(x_0) - \delta E^{x_0}_\sigma [\tau_\delta]
\]

Since \( E^{x_0}_\sigma [\tau_\delta] < \infty \) independently on the strategy used (2.3) and \( \delta \) is arbitrary, we conclude that \( u_\epsilon(x_0) \geq v_\epsilon(x_0) \).

We need to find a quasi-optimal Borel control \( \tilde{\sigma} \). We do know that for all \( x \in X \) we have \( v_\epsilon(x) = T v_\epsilon(x) \). Thus, given \( \delta > 0 \) and any given point \( x^* \in \Omega \) we can find a Borel control \( \sigma_{x^*} \) such that

\[
\frac{1}{q-1} \int_{B_r(x^*)} v_\epsilon(y) dy + \frac{q-2}{q-1} \left( \frac{v_\epsilon(x^* + \epsilon \sigma_{x^*}(x^*)) + v_\epsilon(x^* - \epsilon \sigma_{x^*}(x^*))}{2} \right) \geq v_\epsilon(x^*) - \delta.
\]
Start at a point \( x_0 \in \Omega \), get the control \( \sigma_{x_0} \). Note that the knowledge of the unit vector \( \sigma_{x_0}(x_0) \) is enough to determine \( x_1 \). Get the control \( \sigma_{x_1} \) so that (3.3) holds. Note that the knowledge of the unit vector \( \sigma_{x_1}(x_1) \) is enough to determine \( x_2 \). We continue in this way to produce a sequence of controls \( \sigma_{x_n} \) so that

\[
\frac{1}{q - 1} \int_{B_\epsilon(x_n)} v_\epsilon(y) \, dy + \frac{q - 2}{q - 1} \left( \frac{v_\epsilon(x_n + \epsilon \sigma_{x_n}(x_n)) + v_\epsilon(x_n \epsilon \sigma_{x_n}(x_n))}{2} \right) \geq v_\epsilon(x_n) - \delta.
\]  

(3.4)

as long as \( x_n \in \Omega \). We then select a quasi-optimal controls by setting \( \tilde{\sigma}(x_n) = \sigma_{x_n}(x_n) \) and otherwise \( \tilde{\sigma}(y) = v_0 \), some fixed unit vector not in the set \( \{ \sigma_{x_0}(x_0), \sigma_{x_1}(x_1), \ldots \} \). It is easily seen that \( \tilde{\sigma} \) is Borel.

3.2.2. Proof of Lemma 2.5. The strategy to prove this lemma is as follows. First, we prove the convergence for smooth functions as done in [PS08] for the \( p \)-Laplacian for functions with non-vanishing gradient. We apply this result to the radial barriers which are translations and scaling of the fundamental solution, and then iterate following the argument of [MPR12] for \( p \)-harmonic functions.

Consider the case of smooth functions \( v \in C^3(\overline{\Omega}) \) satisfying \( \mathcal{L}_p v = 0 \) in \( \Omega \). Consider the control \( \nu(x) \) defined as a unit eigenvector corresponding to the largest eigenvalue \( \lambda_N(D^2v(x)) \). Then from the expansion (2.2) we have, uniformly in \( \Omega \) that

\[
v(x) = \mathcal{M}_q(v, B_\epsilon(x)) + O(\epsilon^3).
\]  

(3.5)

Fix a control \( \sigma \) and run the corresponding process \( x_0, x_1, \ldots \). From estimate (3.5) we get

**Lemma 3.4.** There exists a constant \( C_1 > 0 \) that depends on \( v \) and \( \Omega \) but it is independent of \( \epsilon > 0 \), such that:

(i) For an arbitrary control \( \sigma \) the sequence of random variables

\[
M_k = v(x_k) - C_1 k \epsilon^3
\]

is a supermartingale.

(ii) For the control \( \sigma_0(x) = \nu(x) \) the sequence of random variables

\[
N_k = v(x_k) + C_1 k \epsilon^3
\]

is a submartingale.
Proof. We choose $C_1$ given by (3.5) and calculate:

$$
\mathbb{E}_{\sigma}^{x_0} [M_k \mid \mathcal{F}_{n-1}^{x_0}] = \mathbb{E}_{\sigma}^{x_0} [v(x_k) \mid \mathcal{F}_{n-1}^{x_0}] - C_1 ke^3
= MV_{\sigma}(v, B_e(x_{k-1})) - C_1 ke^3
\leq MV_{\sigma}(v, B_e(x_{k-1})) - C_1 ke^3
\leq v(x_{k-1}) + C_1 e^3 - C_1 ke^3
= v(x_{k-1}) - C_1 (k - 1)e^3
= M_{k-1}
$$

$$
\mathbb{E}_{\sigma}^{x_0} [N_k \mid \mathcal{F}_{n-1}^{x_0}] = \mathbb{E}_{\sigma}^{x_0} [v(x_k) \mid \mathcal{F}_{n-1}^{x_0}] + C_1 ke^3
= MV_{\sigma}(v, B_e(x_{k-1})) + C_1 ke^3
= MV_{\sigma}(v, B_e(x_{k-1})) + C_1 ke^3
\geq v(x_{k-1}) - C_1 e^3 + C_1 ke^3
= v(x_{k-1}) + C_1 (k - 1)e^3
= N_{k-1}
$$

Corollary 3.1. There exists a constant $C_2 > 0$ depending on $v$ and $\Omega$ but independent of $\epsilon$ such that for all $x \in \Omega$ we have

$$
|v(x) - v^\epsilon(x)| \leq C_2 \epsilon
$$

Proof. From Lemma 3.4 (i) we have

$$
v^\epsilon(x_0) = \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[v(x_{\tau_\sigma})]) = \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[v(x_{\tau_\sigma}) - C_1 \tau_\sigma e^3 + C_1 \tau_\sigma e^3])
\leq \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[v(x_{\tau_\sigma}) - C_1 \tau_\sigma e^3]) + \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[C_1 \tau_\sigma e^3])
\leq v(x_0) + C_1 e^3 \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[\tau_\sigma]),
$$

and from Lemma 3.4 (ii) we have

$$
v^\epsilon(x_0) = \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[v(x_{\tau_\sigma})]) \geq (\mathbb{E}_{\sigma_0}^{x_0}[v(x_{\tau_\sigma})] + C_1 \tau_{\sigma_0} e^3 - C_1 \tau_{\sigma_0} e^3])
= \mathbb{E}_{\sigma_0}^{x_0}[v(x_{\tau_{\sigma_0}}) + C_1 \tau_{\sigma_0} e^3] - \mathbb{E}_{\sigma_0}^{x_0}[C_1 \tau_{\sigma_0} e^3]
\geq v(x_0) - C_1 e^3 \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[\tau_\sigma]).
$$

Therefore we have that

$$
|v(x) - v^\epsilon(x)| \leq C_1 e^3 \sup_\sigma (\mathbb{E}_{\sigma}^{x_0}[\tau_\sigma]) \leq C_1 C(\Omega, N) \epsilon
$$

by the stopping time bound (2.3). □

Next, we follow the argument used in [MPR12] $p$-harmonious functions.

First, we construct the barriers. Consider the ring domain $B_R(x_0) \setminus B_r(x_0)$ and assign boundary values $m$ on the inner boundary $|x - x_0| = r$ and $M$ on the outer boundary $|x - x_0| = R$ satisfying $m \leq M$. Set $b = -(N + p - 4)$. 

If $b = 0$, then we must have $N = p = 2$ since $N \geq 2$ and $p \geq 2$. In this case, we define

$$U(x) = \frac{M - m}{\log(R/r)} \log \left( \frac{|x - x_0|}{r} \right) + m.$$ (3.6)

When $b < 0$ we set instead

$$U(x) = \frac{M - m}{R^b - r^b} \left( |x - x_0|^b - r^b \right) + m.$$ (3.7)

In each case we have $\mathcal{L}_p U = 0$ in $B_R(x_0) \setminus B_r(x_0)$ with boundary values $m$ on the inner boundary $|x - x_0| = r$ and $M$ on the outer boundary $|x - x_0| = R$.

Since $\Omega$ is Lipschitz, it is clear that $\Omega$ satisfies the following regularity condition:

There exists $\tilde{\delta} > 0$ and $\mu \in (0, 1)$ such that for every $\delta \in (0, \tilde{\delta})$ and $y \in \partial \Omega$ there exists a ball $B_{\mu \delta}(z)$ strictly contained in $B_{\delta}(y) \setminus \Omega$.

Let $u_\varepsilon$ be as in Lemma 2.5. Fix $\delta \in (0, \tilde{\delta})$. For $y \in \partial \Omega$ consider:

$$m^\varepsilon(y) := \sup_{B_{\mu \delta}(y) \cap \Gamma^\varepsilon} F \text{ and } M^\varepsilon := \sup_{\Gamma^\varepsilon} F.$$ (3.8)

Let $\theta \in (0, 1)$ depending only on $\mu$, $N$ and $p$ to be determined later. Set $\delta_k = \delta/4^{k-1}$ for $k \geq 0$ and define

$$M_k^\varepsilon(y) = m^\varepsilon(y) + \theta^k (M^\varepsilon - m^\varepsilon(y)).$$ (3.9)

By the regularity assumption on $\Omega$, there exist balls $B_{\mu \delta_{k+1}}(z_k)$ contained in $B_{\delta_{k+1}}(y) \setminus \Omega$ for all $k \in \mathbb{N}$. Note that $\mu$ is independent of $k$ and $\delta$. The iteration lemma is the following:

**Lemma 3.5.** There exists $\theta \in (0, 1)$ depending only on $\mu$, $N$ and $p$ such that the following holds: Fix $\eta > 0$ and let $y \in \partial \Omega$ and $\varepsilon_k > 0$. Under the above notations, suppose that for all $\varepsilon < \varepsilon_k$ we have:

$$u_\varepsilon \leq M_k^\varepsilon(y) \text{ in } B_{\delta_k}(y) \cap \Omega.$$ (3.10)

Then, either $M_k^\varepsilon(y) - m^\varepsilon(y) \leq \frac{\eta}{4} \varepsilon_k$ or there exists $\varepsilon_{k+1} = \varepsilon_{k+1}(\eta, \mu, \delta, N, p, G) \in (0, \varepsilon_k)$ such that:

$$u_\varepsilon \leq M_{k+1}^\varepsilon(y) \text{ in } B_{\delta_{k+1}}(y) \cap \Omega$$ (3.11)

for all $\varepsilon \leq \varepsilon_{k+1}$.

**Proof.** We will present the case $b < 0$. Suppose that we are in the case $M_k^\varepsilon(y) - m^\varepsilon(y) > \frac{\eta}{4} \varepsilon_k$. For notational convenience set $m = m^\varepsilon(y)$ and $M_k = M_k^\varepsilon(y)$. Consider the barrier $U_k$ defined on the ring $R_k = B_{\delta_k}(z_k) \setminus B_{\mu \delta_{k+1}}(z_k)$

$$U_k(x) = \frac{M_k - m}{\delta_k - (\mu \delta_{k+1})} \left( |x - z_k|^b - (\mu \delta_{k+1})^b \right) + m.$$
Note that $U_k$ is increasing in $|x - z_k|$ is smooth and solves the problem:

\[
\begin{cases}
    \mathcal{L}_p(U_k) = 0 & \text{in } B_{\delta_k}(z_k) \setminus \overline{B_{\mu\delta_{k+1}}(z_k)} \\
    U_k = m & \text{on } \partial B_{\mu\delta_{k+1}}(z_k) \\
    U_k = M_k & \text{on } \partial B_{\delta_k}(z_k).
\end{cases}
\]

We will establish several upper bounds for $\varepsilon_{k+1}$, and take $\varepsilon_{k+1}$ to be the minimum of such bounds.

First, let $\varepsilon_{k+1} = \frac{\mu\delta_{k+1}}{2}$. For $\varepsilon \leq \varepsilon_{k+1}$, extend the barrier $U_k$ to the ring

$R_{k,\varepsilon} = B_{\delta_k+2\varepsilon}(z_k) \setminus \overline{B_{\mu\delta_{k+1}}(z_k)}$.

Let $U^\varepsilon_k$ be $\varepsilon$-mean value solution in $R_k = B_{\delta_k}(z_k) \setminus \overline{B_{\mu\delta_{k+1}}(z_k)}$ with boundary values $U_k$ on $R_{k,\varepsilon} \setminus R_k$, the outer $\varepsilon$-neighborhood of $R_k$. Since $R_k$ is a smooth domain, by Corollary 3.1 we have that $U^\varepsilon_k$ converges to $U_k$ uniformly in $\tilde{X}$ as $\varepsilon \to 0$. Hence, given

$\gamma = \frac{(1/2)^b - ((2 - \mu)/4)^b}{8} \eta$,

there exists $\varepsilon_{k+1} = \varepsilon_{k+1}(\gamma) > 0$ such that:

$|U^\varepsilon_k - U_k| \leq \gamma$

for $\varepsilon \leq \varepsilon_{k+1}$ and for every $p \in \tilde{X}$.

Next, define

$\alpha = 1 - \frac{(1/2)^b}{1 - (\mu/4)^b}$ and $\beta = \frac{(1/2)^b - (\mu/4)^b}{1 - (\mu/4)^b}$

and note that $\alpha$ and $\beta$ are non-negative and that $\alpha + \beta = 1$.

We now prove the following claim:

Claim 3.5.

$\alpha u^\varepsilon + \beta m \leq U_k + \gamma$ in $B_{\delta_k/2}(z_k) \cap \Omega$,

for $\varepsilon \leq \varepsilon_{k+1}$.

From the comparison principle (Lemma 2.4) we get

$\partial \varepsilon(B_{\delta_k/2}(z_k) \cap \Omega) \subseteq \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$,

where $\Gamma_1^\varepsilon = B_{\delta_k/2+\varepsilon}(z_k) \cap \Gamma^\varepsilon$ and $\Gamma_2^\varepsilon = (B_{\delta_k/2+\varepsilon}(z_k) \setminus \overline{B_{\delta_k/2}(z_k)}) \cap \Omega$.

On $\Gamma_1^\varepsilon$, we have $u^\varepsilon = F \leq m$, hence: $\alpha u^\varepsilon + \beta m \leq m = \inf_{R_k} U_k \leq U_k \leq U_k^\varepsilon + \gamma$, since $\Gamma_1^\varepsilon \subset R_{k,\varepsilon}$. 
On $\Gamma_2^\varepsilon$, we have $v^\varepsilon \leq M_k$ by assumption, because $B_{\delta_k/2+\varepsilon}(z_k) \subset B_{\delta_k}(y)$. For $x \in \partial B_{\delta_k/2}(z_k)$, we have $|x - z_k| = \delta_k/2$, hence:

$$U_k(x) = \frac{M_k - m}{\delta_k^+ - (\mu \delta_{k+1})^+} \left((\delta_k/2)^b - (\mu \delta_{k+1})^b\right) + m$$

(3.10)

$$= \frac{M_k - m}{1 - (\mu/4)^b} \left((1/2)^b - (\mu/4)^b\right) + m$$

$$= \frac{1 - (1/2)^b}{1 - (\mu/4)^b} m + \frac{(1/2)^b - (\mu/4)^b}{1 - (\mu/4)^b} M_k$$

$$= \alpha m + \beta M_k$$

and by monotonicity $U_k \geq \alpha m + \beta M_k$ in $\Gamma_2^\varepsilon$. Hence:

$$\alpha m + \beta v^\varepsilon \leq \alpha m + \beta M_k \leq U_k \leq U_k^{\varepsilon} + \gamma$$

in $\Gamma_2^\varepsilon$. In conclusion, we have: $\alpha m + \beta v^\varepsilon \leq U_k^{\varepsilon} + \gamma$ in $\partial \varepsilon(B_{\delta_k/2}(z_k) \cap \Omega)$, and the claim follows again by the comparison principle Lemma 2.4.

Consider next the intersection $B_{\delta_{k+1}}(y) \cap \Omega$. We have $B_{\delta_{k+1}}(y) \subset B_{(2-\mu)\delta_{k+1}}(z_k)$ and for $x \in B_{(2-\mu)\delta_{k+1}}(z_k)$ we have:

$$U_k(x) \leq \frac{M_k - m}{\delta_k^+ - (\mu \delta_{k+1})^+} \left(((2 - \mu) \delta_{k+1})^b - (\mu \delta_{k+1})^b\right) + m$$

(3.11)

$$= \frac{M_k - m}{1 - (\mu/4)^b} \left(((2 - \mu)/4)^b - (\mu/4)^b\right) + m$$

$$= \alpha' m + \beta' M_k,$$

where we have set

$$\alpha' = \frac{1 - ((2 - \mu)/4)^b}{1 - (\mu/4)^b} \quad \text{and} \quad \beta' = \frac{(2 - \mu)/4)^b - (\mu/4)^b}{1 - (\mu/4)^b}.$$

Also, note that $B_{\delta_{k+1}}(y) \subset B_{\delta_k/2}(z_k)$, hence by (3.5) we get:

(3.12) $\alpha m + \beta v^\varepsilon \leq U_k + \gamma$ in $B_{\delta_{k+1}}(y) \cap \Omega$.

Combining (3.11) and (3.12), for $p \in B_{\delta_{k+1}}(y) \cap \Omega$ and $\varepsilon < \varepsilon_{k+1}$, we get:

$$v^\varepsilon(p) \leq \frac{\alpha' - \alpha}{\beta} m + \frac{\beta'}{\beta} M_k + \frac{\gamma}{\beta}$$

$$= m + \frac{\beta'}{\beta} (M_k - m) + \frac{\gamma}{\beta}.$$

Observe that $\beta'/\beta \in (0,1)$ and that $\beta' < \beta$. Recall that we have chosen

$$\gamma = \frac{(1/2)^b - ((2 - \mu)/4)^b}{8} \eta \leq \frac{(1/2)^b - ((2 - \mu)/4)^b}{2} (M_k - m).$$
Thus, we get
\[ v^\varepsilon(p) \leq m + \left( \frac{\beta'}{\beta} + \frac{(1/2)^b - ((2 - \mu)/4)^b}{2\beta} \right) (M_k - m), \]
and setting
\[ \theta = \frac{\beta'}{\beta} + \frac{(1/2)^b - ((2 - \mu)/4)^b}{2\beta} \]
we get
\[ v^\varepsilon(p) \leq m + \theta(M_k - m) \leq m + \theta^{k+1}(M^\varepsilon - m). \]

The next Corollary, whose proof follows in a standard from Lemma 3.5, implies Lemma 2.5.

**Corollary 3.2.** Given \( \eta > 0 \), there exist \( \delta = \delta(\eta, F, \delta) \), \( k_0 = k_0(\eta, \mu, p, F) \), \( \varepsilon_0 = \varepsilon_0(\eta, \delta, \mu, k_0) \) such that:
\[ |v^\varepsilon(p) - F(y)| \leq \frac{\eta}{2}, \]
for all \( y \in \partial\Omega, p \in B_{\delta/4\varepsilon_0}(y) \cap \Omega \) and \( \varepsilon \leq \varepsilon_0 \).

**3.2.3. Proof of Lemma 2.6.** Let us prove that \( \bar{u} \) is a viscosity subsolution; that is, it satisfies \( \mathcal{L}_p \bar{u} \geq 0 \) in the viscosity sense. Let \( x_0 \in \Omega \) and choose \( \phi \in C^2(\Omega) \) such that \( \phi \) touches \( \bar{u} \) from above at \( x_0 \); i.e. we have \( \bar{u}(x_0) = \phi(x_0) \) and \( \bar{u}(x) < \phi(x) \) for \( x \in \Omega \setminus \{x_0\} \). The following claim is standard ([BS91]):

**Claim 3.6.** There exist a sequence \( \varepsilon_n \to 0 \) and a sequence of points \( x_n \to x_0 \) such that \( u_{\varepsilon_n}(x_n) \to \bar{u}(x_0) \) and \( \phi - u_{\varepsilon_n} \) has an interior minimum at \( x_n \).

Starting with \( \phi(x_n) - u_{\varepsilon_n}(x_n) \leq \phi(x) - u_{\varepsilon_n}(x) \) and integrating over \( B_{\varepsilon_n}(x_n) \) we get
\[ \phi(x_n) - u_{\varepsilon_n}(x_n) \leq \mathcal{M}_\sigma(\phi - u_{\varepsilon_n}, B_{\varepsilon_n}(x_n)) \]
\[ = \mathcal{M}_\sigma(\phi, B_{\varepsilon_n}(x_n)) - \mathcal{M}_\sigma(u_{\varepsilon_n}, B_{\varepsilon_n}(x_n)) \]
Therefore, we have
\[ \phi(x_n) - u_{\varepsilon_n}(x_n) + \mathcal{M}_q(u_{\varepsilon_n}, B_{\varepsilon_n}(x_n)) \leq \mathcal{M}_\sigma(\phi, B_{\varepsilon_n}(x_n)), \]
and taking supremum among all strategies we get
\[ \phi(x_n) - u_{\varepsilon_n}(x_n) + \mathcal{M}_q(u_{\varepsilon_n}, B_{\varepsilon_n}(x_n)) \leq \mathcal{M}_q(\phi, B_{\varepsilon_n}(x_n)) \]
from which we, using the fact that \( u_{\varepsilon_n}(x_n) = \mathcal{M}_q(u_{\varepsilon_n}, B_{\varepsilon_n}(x_n)) \), conclude that
\[ \phi(x_n) \leq \mathcal{M}_q(\phi, B_{\varepsilon_n}(x_n)) \]
\[ = \phi(x_n) + \mathcal{D}_q\phi(x_n) + o(\varepsilon_n^2) \]
\[ = \phi(x_n) + \frac{p}{2(N+2)+p-2} L_p\phi(x_n) + o(\varepsilon_n^2), \]
Therefore we have $\mathcal{L}_p\phi(x_0) \geq 0$ and thus $\mathcal{L}_\sigma \pi(x_0) \geq 0$ in the viscosity sense. A similar proof shows that $u$ is a viscosity supersolution.

3.2.4. Proof of Theorem 2.2. Given the boundary estimates (2.7) and (2.8), we use the comparison principle for viscosity solution of $\mathcal{L}_p = 0$ to conclude that

$$\overline{u} = u$$

and

$$\lim_{\epsilon \to 0} u_\epsilon = \underline{u} = \underline{u} = u$$

uniformly in $\overline{\Omega}$, where $u$ is only solution to the Dirichlet problem (1.2).

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