On Dirichlet’s Derivation of the Ellipsoid Potential

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Abstract

Newton’s potential of a massive homogeneous ellipsoid is derived via Dirichlet’s discontinuous factor. At first we review part of Dirichlet’s work in an English translation of the original German, and then continue with an extension of his method into the complex plane. With this trick it becomes possible to first calculate the potential and thereafter the force components exerted on a test mass by the ellipsoid. This is remarkable in so far as all other famous researchers prior to Dirichlet merely calculated the attraction components. Unfortunately, Dirichlet’s derivation is to a large extent mathematically unacceptable which, however, can be corrected by treating the problem in the complex plane.

1 Introduction: The Homogeneous Ellipsoid

The calculation of the attraction and the potential of the homogeneous ellipsoid is one of the most-discussed problems in mathematical physics. Only at the beginning of the 19th century was a satisfactory solution found. It was at this time and the time thereafter
that the work of Laplace (1782), Ivory (1809), Gauß (1813), Chasles (1838) and Dirichlet (1839) took place [1].

This article is not so much about C.F. Gauss and his seminal contribution to the calculation of the attractive force components of a homogeneous massive ellipsoid in inner and outer space, but a homage to his successor in Göttingen, P.G. Lejeune-Dirichlet, who, for the first time, tried to compute the potential prior to the force components that an ellipsoid exerts in inner and outer space.

Gauß’ contribution is even more important for another reason. In his article of 1813 on the ellipsoid, he uses for the first time in history the divergence theorem which carries his name. The world of mathematics and physics would be unthinkable without this integral theorem. True, the attraction components of the ellipsoid are calculated, but not the potential, a concept that Gauss introduced later in 1840. However, from a handwritten remark which is re-printed in Vol. V, pp. 285-286 of Gauß’ works, he lets us know that one can compute the potential as well using a method similar to that employed in his ellipsoid paper.

The emphasis in the current article is on Dirichlet’s so-called discontinuous factor which he uses to handle multiple integrals with function-like variables at the lower and upper limit of the integral. One should consider this idea as a “predecessor” to δ-like functions (distributions) which were introduced by P.M.A. Dirac and are at the center of J. Schwinger’s and others’ treatment of any type of field theory with Green’s functions in modern physics. It is precisely the δ function which can be represented by a limiting process of “reasonable functions” that can be used under an integral to pick out special values and forget about the complicated upper and lower limits of the integral - exactly what Dirichlet had in mind when he discovered how easy it can be when computing multiple integrals as in the case of the potential of a homogeneous massive ellipsoid. This is reason enough to take a closer look at Dirichlet’s trick to simplify complicated integrals introduced in 1839.

2 On a New Method for Calculating Multiple Integrals by P.G. Lejeune-Dirichlet

Dirichlet’s article, “Über eine neue Methode zur Bestimmung vielfacher Integrale”, was published in extenso in Treatises of the Berlin Academy from the Year 1839, Berlin 1841
The following represents an English translation of the introductory parts of Dirichlet’s contribution, originally published in German.

It is well known that the calculation of a multiple integral or its reduction to a lower order is generally one of the more difficult problems encountered when the limits of integration for the individual variables are not constant but are mutually dependent, so that the domain of integration is expressed by one or more inequalities containing more than one variable. When dealing with various physical problems which lead back to the calculation of a class of multiple integrals of an undetermined order, the author came across the method that is the subject of this article and which not only yields the value of the integral on which the present investigation relies, but also on the many other different kinds of integrals that it can be applied to. Nevertheless this method is so simple that one wonders why it hasn’t been applied to similar studies well before now. The principle behind this approach to multiple integrals which are to be taken between variable limits is based on the well-known property of certain integrals which represent discontinuous functions of those constants that are contained in the integrals and are dependent in different intervals in various ways. For example, we know that the simple expression

$$\left( \frac{2}{\pi} \right) \int_{0}^{\infty} \cos(g\varphi) \frac{\sin \varphi}{\varphi} d\varphi$$

(1)

is equal to unity as long as $g$ lies between $-1$ and $+1$, but disappears if $g$ lies outside this interval. If one has a three-fold integral - we are not considering one of higher order because with three variables the procedure takes on a geometric dimension, which allows us to describe the process - which is to extend over a defined space, e.g., over an ellipsoid surface, so that one can say that if $\alpha, \beta, \gamma$ describe the semi-axes of this surface in which direction the coordinate axes coincide, the expression

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2}$$

lies below or above unity, depending on whether the point $(x, y, z)$ lies within or outside the specified space, to see immediately that the integral

$$\left( \frac{2}{\pi} \right) \int d\varphi \frac{\sin \varphi}{\varphi} \cos \left[ \left( \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} \right) \varphi \right]$$

(2)

has the value of unity inside of the ellipsoid, but disappears outside of it. So if one multiplies the given differential expression $Pdxdydz$, where $P$ indicates some function
of \(x, y, z\), under the above integral in question, one no longer needs to take the original
limits into account when integrating, i.e., one can perform the integrations with respect
the variables \(x, y, z\) between the constant limits \(-\infty\) and \(\infty\) in which, due to the added
discontinuous factors, the elements on which the integration should not be extended drop
out by themselves. This method can be described in two words in such a way that an
integral extending in all directions over a limited mass distribution can be immediately
transformed into one that stretches over an infinite space and in most cases will be much
easier to handle because one permits the density outside of a given volume to become equal
to zero; this can easily be done with a discontinuous factor. It is surprising to what extent
the most difficult integrations can be easily performed with these transformations, which
initially seem to hardly promise success, and how these problems, which often demand
intricate and time-consuming calculations, can be solved without difficulty, simply with
the help of a few well-known integrals.

In this paper we can only give a short description of a few of the applications of this
method. One example is the attraction of the ellipsoid, a problem that mathematicians
have studied more than any other involving integral calculus.

Normally one reduces the case of a point external to the ellipsoid to an internal one, which
is easier to calculate, or, if both are to be solved independently of each other, then quite
different means are used.

Using the above-described method, both cases can be treated in a similar manner and
independently. First one must distinguish between the two in order to express the result
in a final and simple form. Furthermore, the procedure should not be limited to the
requirement that the attraction be inversely proportional to the distance squared, but
rather remains applicable for any other integer or fractional power of the distance. Nor
need the density of the attracting mass be assumed constant but can be expressed by any
rational, integer function of the coordinates \(x, y, z\). For simplicity’s sake, however, the
density will be assumed to be constant and equal to unity.

Let \(\alpha, \beta, \gamma\) be the semi-axes of the ellipsoids; \(a, b, c\) the coordinates of the attracted point;
and \(x, y, z\) of any point of the attracting mass. Furthermore, let

\[ \rho^2 = (x - a)^2 + (y - b)^2 + (z - c)^2 \]

and \(\frac{1}{\rho^p}\) be the law of attraction (where \(p\) is assumed to lie between 2 and 3; beyond these
limits the procedure requires a few minor modifications), then the force component \(A\) of
the attraction parallel to the \(x\) axis (and considered to be positive from the side where \(x\)’s
decrease), is obtained by taking the derivative with respect to $a$ of the integral covering the entire ellipsoid:

$$
-\frac{1}{(p-1)} \int \frac{dxdydz}{\rho^{p-1}}.
$$

(3)

According to the above, the integral is transformed into

$$
-\frac{2}{\pi(p-1)} \int_0^\infty d\varphi \frac{\sin \varphi}{\varphi} \int \cos \left[ \left( \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} \right) \varphi \right] \frac{dxdydz}{\rho^{p-1}}
$$

(4)

where the integrations for $x, y, z$ can be extended from $-\infty$ to $\infty$. The calculation is much easier if, instead of this integral, we observe the following one, whose real part coincides with that which we are looking for:

$$
-\frac{2}{\pi(p-1)} \int_0^\infty d\varphi \frac{\sin \varphi}{\varphi} \int \exp \left[ i\varphi \left( \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} \right) \right] \frac{dxdydz}{\rho^{p-1}}.
$$

(5)

Integrating for $x, y, z$ cannot be performed in this form, but can easily be done if one expresses the factor $\frac{1}{\rho^{p-1}}$ with the help of a certain integral in such a way that the coordinates $y, y, z$, as in the other factor, appear only in the exponents.

So much for the introduction to Dirichlet’s original paper. Later in the text we will return to the entire calculation, then treating it in the complex plane. Before continuing, it would be appropriate to derive formula (1) because this equation lies at the heart of Dirichlet’s computational trick.

### 3 Dirichlet’s Discontinuous Integral -“Light”

Let us start with the function $h(t)$, defined as

$$
h(t) = \int_0^\infty e^{-xt} \sin(gx) \frac{x}{x} dx, \quad t > 0, g = \text{const.}
$$

(6)

Differentiating with respect to the parameter $t$,

$$
\frac{dh}{dt} = -\int_0^\infty e^{-xt} \sin(gx) dx,
$$

and integrating by parts twice, gives

$$
\frac{dh}{dt} = -\frac{g}{(g^2 + t^2)}
$$
Figure 1: Dirichlet’s discontinuous integral

which, when integrated, yields

\[ h(t) = A - \tan^{-1}\left(\frac{t}{g}\right) \]

with \( A \) a constant of integration. From the integral definition of \( h(t) \) (6) we observe that \( h(\infty) = 0 \). Thus, \( 0 = C - \tan^{-1}(\pm \infty) \), where we use the + sign if \( g > 0 \) and the - sign if \( g < 0 \). Hence, \( A = \pm \frac{\pi}{2} \) and we have

\[ h(t) = \pm \frac{\pi}{2} - \tan^{-1}\left(\frac{t}{g}\right). \tag{7} \]

At this point we set \( t = 0 \) and with it, \( \tan^{-1}\left(\frac{0}{g}\right) = 0 \), so that we obtain Dirichlet’s discontinuous integral

\[ \int_{0}^{\infty} \frac{\sin(gx)}{x} \, dx = \begin{cases} \frac{\pi}{2} & \text{if } g > 0 \\ 0 & \text{if } g = 0 \\ -\frac{\pi}{2} & \text{if } g < 0 \end{cases} \tag{8} \]

The plot in Fig. 1 shows Dirichlet’s discontinuous integral with the sudden jump as \( g \) goes from negative to positive values. By the way, Euler derived the special case \( g = 1 \) at the end of his life in 1783:

\[ \frac{2}{\pi} \int_{0}^{\infty} dx \frac{\sin x}{x} = 1. \tag{9} \]

This famous formula is sufficient to derive Dirichlet’s expression of (1). Here are two ways to prove this statement:
(a) Let us put \( x = tu \) in \([9]\) where \( t \) is a positive number. This gives
\[
\frac{2}{\pi} \int_0^\infty du \frac{\sin(tu)}{u} = 1.
\]  
(10)

Now let \( h > g \) be two positive numbers. Then \( h + g \) and \( h - g \) are also positive. If we substitute these two numbers for \( t \) into the former integral, we obtain \((u = \varphi)\)
\[
\frac{2}{\pi} \int_0^\infty \varphi \sin(h + g) \varphi = 1, \quad \frac{2}{\pi} \int_0^\infty \varphi \sin(h - g) \varphi = 1.
\]
Adding and subtracting these two integrals gives
\[
\frac{2}{\pi} \int_0^\infty \varphi \sin(h \varphi) \cos(g \varphi) = 1, \quad \frac{2}{\pi} \int_0^\infty \varphi \sin(g \varphi) \cos(h \varphi) = 0,
\]
or
\[
\frac{2}{\pi} \int_0^\infty \varphi \sin(h \varphi) \cos(g \varphi) = 1 \text{ or } 0,
\]
depending on \( h > g \) or \( h < g \). Finally we put \( h = 1 \) and so obtain
\[
\frac{2}{\pi} \int_0^\infty \varphi \frac{\sin \varphi}{\varphi} \cos(g \varphi) = \begin{cases} 1 & \text{for } g < 1 \\ 0 & \text{for } g > 1 \\ \frac{1}{2} & \text{for } g = 1 \end{cases}.
\]  
(11)

Since the value of the integral in unchanged when we replace \(+g\) by \(-g\), we can also write
\[
\frac{2}{\pi} \int_0^\infty \varphi \frac{\sin \varphi}{\varphi} \cos(g \varphi) = \begin{cases} 1 & \text{for } -1 < g < 1 \\ 0 & \text{for } g < -1 \text{ or } g > 1 \\ \frac{1}{2} & \text{for } g = \pm 1 \end{cases}.
\]  
(12)

Equation (11) is precisely Dirichlet’s discontinuity factor which is everywhere inside of an ellipsoid defined by
\[
g(x, y, z) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} < 1,
\]  
(13)
equal to 1 and equals zero for every point \((x, y, z)\) outside, i.e.,
\[
g(x, y, z) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} > 1.
\]
(b) Here is another derivation of Dirichlet's discontinuity factor. It starts with Fourier's integral formula

\[
f(g) = \frac{1}{\pi} \int_{0}^{\infty} d\varphi \int_{-\infty}^{+\infty} dt f(t) \cos(\varphi(t - g))
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} d\varphi \cos(\varphi g) \int_{-\infty}^{+\infty} dt f(t) \cos(\varphi t)
\]

\[
+ \frac{1}{\pi} \int_{0}^{\infty} d\varphi \sin(\varphi g) \int_{-\infty}^{+\infty} dt f(t) \sin(\varphi t)
\].

If \( f(g) \) is an even function, we have

\[
f(g) = 2 \frac{1}{\pi} \int_{0}^{\infty} d\varphi \cos(\varphi g) \int_{0}^{\infty} dt f(t) \cos(\varphi t) \tag{14}
\]

and for odd function \( f(g) \) we obtain

\[
f(g) = 2 \frac{1}{\pi} \int_{0}^{\infty} d\varphi \sin(\varphi g) \int_{0}^{\infty} dt f(t) \sin(\varphi t)
\].

Now, according to \([12]\), Dirichlet’s discontinuity function, \( f(g) \) is even and

\[
f(g) = 1 \text{ for } g < 1,
\]

\[
f(g) = \frac{1}{2} \text{ for } g = 1,
\]

\[
f(g) = 0 \text{ for } g > 1.
\]

Therefore we obtain from \([14]\)

\[
f(g) = 2 \frac{1}{\pi} \int_{0}^{\infty} d\varphi \cos(\varphi g) \int_{0}^{1} dt f(t) \cos(\varphi t) = 2 \frac{1}{\pi} \int_{0}^{\infty} d\varphi \cos(\varphi g) \left( \frac{\sin \varphi}{\varphi} \right),
\]

which is the desired result.

4 Dirichlet’s Discontinuity Factor in the Complex Plane

We could continue with section \([2]\) and follow for the rest of the paper Dirichlet’s original work where he generalizes Newton’s law to \( r^{-p} \), in which \( p \) is not necessarily an integer
number. In the course of the calculation, the exponent must be subjected to a two-fold limiting condition, so that for finite \( p \) the interval \( 2 < p < 3 \) is left over and the case of interest to us, \( p = 2 \) even, would not be permitted. This was shown in detail in Dirichlet’s lectures, published by G. ARENDT [3], where a further extension of the range of validity to \( 1 < p < 3 \) is claimed, whose grounds are however not satisfactory. This is probably the “unimportant modification” that Dirichlet ([4] p. 404) mentions without further explanation. The limitation to the attraction components also follows due to convergence difficulties when integrating; the potential itself is in fact not derived correctly by DIRCHLET; rather, a formula without proof or limits of validity for \( p \) ([4] p. 408) is given. This ambiguity is probably also the reason that these elegant methods have not found entry into the textbook literature (including TISSERAND, Méc. cél. volume II). Wherever this was attempted ([5]), the derivation of the potential was dispensed with due to the above-mentioned difficulties, and only the force components were determined. This failing can be remedied if one assumes a complex formulation of the discontinuous factor rather than Dirichlet’s version using e.g., the real Fourier integral. This is, by the way, also desirable, since DIRICHLET has to change over to complex integrals in the course of the calculation. We define with real \( g > 0 \) as discontinuous factor

\[
\frac{1}{\pi} \int_C d\varphi e^{ig\varphi} \frac{(\sin \varphi)}{\varphi} = \begin{cases} 
1, & g < 1 \\
0, & g > 1
\end{cases}
\quad (15)
\]

and take the whole real axis from \(-\infty\) to \(+\infty\) as the integration path \( C \), bypassing the zero point of the complex \( \varphi \) plane by a small half-circle in the upper half-plane (see Fig. 2). Here is a proof of formula (15).

The path \( C \) is given by \( \overrightarrow{c} \) and \( \sin \varphi = \frac{1}{(2i)}(e^{i\varphi} - e^{-i\varphi}) \).
So we have to study the integral
\[
\frac{1}{\pi} \int_C d\varphi e^{i\varphi} \sin \varphi = \frac{1}{\pi} \int_C \frac{d\varphi}{\varphi} \left( \frac{1}{2i} \right) \left[ e^{i(g+1)\varphi} - e^{i(g-1)\varphi} \right]. \tag{16}
\]

For \( g > 1 \), we integrate along \( C + C_1 \), where \( C_1 \) denotes the upper semicircle with radius \( R_1 \to \infty \). Then, on the closed path we apply Cauchy’s integral theorem. Since inside this path we have no singularity in the \( \varphi \) plane, therefore the integral together with the limit \( R_1 \to \infty \) is equal to zero.

For \( g < 1 \) and since \( g > 0 \) so that \( (g + 1) > 1 \), the first part of the integral \textbf{[16]} vanishes and we are left with the clockwise path integral \( C + C_2 \):
\[
-\frac{1}{\pi} \int_{C+C_2} \frac{d\varphi}{\varphi} \left( \frac{1}{2i} \right) e^{-i(1-g)\varphi}.
\]

For \( R_2 \to \infty \) we again obtain no contribution. Expanding the exponential in this expression we obtain
\[
-\frac{1}{\pi} \left( \frac{1}{2i} \right) \int_{C+C_2} \frac{d\varphi}{\varphi} \left[ 1 - i(1-g)\varphi + \ldots \right].
\]

Here we apply Cauchy’s residue theorem, which gives
\[
-\frac{1}{\pi} \left( \frac{1}{2i} \right) (-)2\pi i \cdot 1 = 1 \text{ for } g < 1
\]
and finishes our proof for \textbf{(15)}.

Furthermore, DIRICHLET uses EULER’S formula (which actually was later proved by Poisson)
\[
\int_0^\infty d\nu e^{i\nu} \nu^{s-1} = \frac{\Gamma(s)}{(\pm q)^s} e^{\pm i\pi s} \tag{17}
\]
with \( 0 < s < 1 \) and where the upper (lower) sign holds for positive (negative) \( q \) values.

For points \((x, y, z)\) on the surface of the ellipsoid with semiaxes \( a, b, c \) we have
\[
g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,
\]
and for points \((x, y, z)\) inside (outside) of the ellipsoid we have \( g(x, y, z) < 1(> 1) \). The potential of the homogeneous ellipsoid at a point \((\xi, \eta, \zeta)\) is then given by the volume integral over the entire ellipsoid:
\[
V(\xi, \eta, \zeta) = Gp \int_V dxdydz \frac{1}{r}, \quad r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2, \tag{18}
\]
where $\rho$ is the constant density and $G$ denotes Newton’s gravitational constant. Now with the aid of (15) (not Dirichlet’s variant!), we can extend the volume integral (18) over the entire space:

$$V(\xi, \eta, \zeta) = G \rho \int_0^\infty \frac{d\varphi}{\nu} \frac{(\sin \varphi)}{\varphi} \int_0^{2\pi} d\psi \int_{-\infty}^\infty dx dy dz \frac{e^{i\varphi g(x,y,z)}}{r}.$$  \hfill (19)

For the representation of the inverse distance $r^{-1}$ we make use of Euler’s formula (17), with $s = \frac{1}{2}, q = r^2 > 0$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ so that

$$\frac{1}{r} = e^{-i\frac{\pi}{4}} \int_0^{\infty} \frac{dv}{\sqrt{v}} e^{iv^2}.$$  

The explicit expression for the potential is then given by

$$V(\xi, \eta, \zeta) = G \rho e^{-i\frac{\pi}{4}} \int_0^{\infty} \frac{dv}{\sqrt{v}} \int_{-\infty}^{\infty} dx u(x,\xi) \int_{-\infty}^{\infty} dy u(y,\eta) \int_{-\infty}^{\infty} dz u(z,\zeta),$$  \hfill (20)

where

$$\int_{-\infty}^{\infty} dx u(x,\xi) = \int_{-\infty}^{\infty} dx \exp[i(\frac{x^2}{a^2})\varphi + i(x-\xi)^2 v] = a\sqrt{\pi} \frac{e^{i\frac{\pi}{4}}}{\sqrt{\varphi + a^2 v}} \exp[e^{iv^2}]$$

with similar expressions for the integrals over $y$ and $z$. After substituting these expressions into (20) we obtain

$$V(\xi, \eta, \zeta) = iG \rho abc \int_0^{\infty} \frac{dv}{\sqrt{v}} \int \frac{d\varphi}{\nu} (\sin \varphi) \times \exp[i\varphi \frac{\xi^2}{(\varphi + a^2 v)} + \frac{\eta^2}{(\varphi + a^2 v)} + \frac{\zeta^2}{(\varphi + c^2 v)}][(\varphi + a^2 v)(\varphi + b^2 v)(\varphi + c^2 v)]^{-\frac{1}{2}}$$

Replacing the integration variable $v$ by $v = \frac{\varphi}{\lambda}$ with fixed $\varphi$, we finally obtain

$$V(\xi, \eta, \zeta) = iG \rho abc \int_0^{\infty} \frac{d\lambda}{\sqrt{\Psi(\lambda)}} \int \frac{d\varphi}{\varphi^2} (\sin \varphi) e^{iS\varphi}$$  \hfill (21)

with

$$\Psi(\lambda) = [(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)]$$  \hfill (22)

and

$$S(\xi, \eta, \zeta, \lambda) = \frac{\xi^2}{(a^2 + \lambda)} + \frac{\eta^2}{(b^2 + \lambda)} + \frac{\zeta^2}{(c^2 + \lambda)}.$$
The integral over \( C \) now replaces DIRICHLET’s integration over the positive real semi-axis where the use of EULER’s formula \((17)\) fails. If one performs the partial derivatives of \((21)\) with respect to \( \xi, \eta, \zeta \), only the first power of \( \varphi \) remains in the denominator of the integral, and formula \((15)\) becomes applicable (compare F. HOPFNER \([6]\)), whereas with DIRICHLET’s integration the implementation of \((17)\) is not permitted, according to the above-mentioned exclusion of the NEWTONian case of \( p = 2 \). According to DIRICHLET’s method, the potential could only be treated for \( p > 4 \) and the attraction components only for \( p > 2 \). Since in the first use of EULER’s formula in \((17)\) the limit \( 1 < p < 3 \) has already been included, DIRICHLET has to forgo the derivation of the potential and limit himself to the attraction components for \( 2 < p < 3 \). The complex formulation removes this difficulty and leads to

\[
i \int_{C} \frac{d\varphi}{\varphi^2} \frac{(\sin \varphi)}{e^{iS\varphi}} = \frac{1}{2} \int_{C} \frac{d\varphi}{\varphi^2} (e^{i(S+1)\varphi} - e^{i(S-1)\varphi}). \tag{23}\]

Similarly to the computation of \((16)\), we now have to discuss the cases \( S > 1 \) and \( S < 1 \). Again, we employ Cauchy’s integral theorem and close the path \( C_2 \) clockwise in the lower complex half plane and so obtain for \((23)\)

\[
\frac{1}{2} \int_{C+C_2} \frac{d\varphi}{\varphi^2} e^{-i(1-S)\varphi} = \frac{1}{2} \int_{C+C_2} \frac{d\varphi}{\varphi^2} (1 - i(1 - S)\varphi - \ldots) = \frac{1}{2} 2\pi i(-i)(1 - S) = \pi(1 - S). \tag{24}\]

We conclude that if the test point \((\xi, \eta, \zeta)\) lies inside the ellipsoid, \( S(\xi, \eta, \zeta; \lambda) < 1 \), we find for the potential

\[
V_i(\xi, \eta, \zeta; \lambda) = G\rho abc \pi \int_{0}^{\infty} \frac{d\lambda}{\sqrt{\Psi(\lambda)}} (1 - S(\xi, \eta, \zeta; \lambda)), \quad S(\xi, \eta, \zeta; \lambda) < 1. \tag{25}\]

If the test point lies outside, there exists exactly one value \( \lambda = u \) that denotes the positive real root of \( S(\xi, \eta, \zeta; \lambda) = 1 \); for all other values \( \lambda > u, S(\xi, \eta, \zeta; \lambda) \) is smaller than 1. The lower value \( u \) is a function of \( \xi, \eta, \zeta \) of the test point. Therefore the potential of the ellipsoid in external space is given by

\[
V_e = G\rho abc \pi \int_{u}^{\infty} \frac{d\lambda}{\sqrt{\Psi(\lambda)}} (1 - S(\xi, \eta, \zeta; \lambda)). \tag{26}\]

Finally let us apply our formula to the simple case, namely the potential of a homogeneous sphere with radius \( R = a = b = c \). Admittedly, this case belongs to a first-semester course
in mechanics. However, it illustrates very nicely Dirichlet’s path to reproducing Newton’s result. We start with

\[ S(\xi, \eta, \zeta; \lambda) = (r^2 + \lambda)^{-1}(\xi^2 + \eta^2 + \zeta^2) < 1, \]

and the inner point is given by \( r^2 = \xi^2 + \eta^2 + \zeta^2 \). So we are given \( S(\xi, \eta, \zeta; \lambda) = \frac{1}{(R^2 + \lambda)}r^2 \) and \( [(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)]^{\frac{1}{2}} = (R^2 + \lambda)^{3/2}. \)

According to formula (25), the potential for a test point \( r \) inside the massive homogeneous sphere is given by

\[
V_i(r) = G\rho R^3 \pi \int_0^\infty d\lambda \left[ \frac{1}{(R^2 + \lambda)^{3/2}} - \frac{r^2}{(R^2 + \lambda)^{5/2}} \right].
\] (27)

The two integrals can easily be calculated so that we get

\[
V_i(r) = G\rho R^3 \pi \left[ \frac{2}{R} - \frac{2}{3} \frac{r^2}{R^2} \right] = G\rho \pi \frac{2}{3} \left[ 3R^2 - r^2 \right]. (4.14)
\] (28)

For the potential of the external point we first have to determine the lower limit \( u \), which follows from \( S(\xi, \eta, \zeta; u) = 1 = \frac{r^2}{(R^2 + u)} \), i.e., \( u = r^2 - R^2 \). This requires the value of the integral, as in (26):

\[
V_e(r) = G\rho R^3 \pi \left[ \int_{r^2 - R^2}^\infty d\lambda \frac{1}{(R^2 + \lambda)^{3/2}} - r^2 \int_{r^2 - R^2}^\infty d\lambda \frac{1}{(R^2 + \lambda)^{5/2}} \right]
\]

\[
= G\rho R^3 \pi \left[ \frac{2}{r} - \frac{2}{3} \frac{(R)}{r} \right] = G\rho \left( \frac{4}{3}\pi R^3 \right) \frac{1}{r} = G\rho V\left( \frac{1}{r} \right) = G\frac{M}{r}. (Newton). (29)
\]

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References

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