Equilibration and order in quantum Floquet matter

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Equilibrium thermodynamics is characterized by two fundamental ideas: thermalization—that systems approach a late time thermal state; and phase structure—that thermal states exhibit singular changes as various parameters characterizing the system are changed. We summarize recent progress that has established generalizations of these ideas to periodically driven, or Floquet, closed quantum systems. This has resulted in the discovery of entirely new phases which exist only out of equilibrium, such as the \( S \)-spin glass/Floquet time crystal.

Remarkable progress in the physics of closed quantum systems away from equilibrium has occurred over the past decade. This has been experimental—most strikingly in cold atomic systems\(^1\), computational—often involving quantum information ideas\(^2\), and intellectual—ranging from a systematic use of entanglement ideas to the long-sought demonstration that localization exists in many-body systems\(^3\). Here, we report very recent progress, building particularly on the latter, in our understanding of periodically driven or Floquet many-body systems.

Closed Floquet systems comprise a vast family of systems generally defined by ‘drives’ or time-dependent Hamiltonians with \( \mathcal{H}(t+T) = \mathcal{H}(t) \) for a fixed period \( T \). The promise of Floquet systems is that the periodic drive can lead to new physical phenomena, but their peril is the risk of heating up to a ‘fully scrambled’ or ‘infinite temperature’ state, supporting no non-trivial correlations as all configurations occur with the same probability. The progress reviewed here has established that the peril can be avoided; that interesting long-time steady states can be obtained; and that sharply different behaviours can be distinguished and classified, providing generalizations of the foundational thermodynamic notions of thermalization and phase structure into the non-equilibrium regime. These are generalizations in that they reduce to the familiar ideas in the setting of ergodic, time-dependent Hamiltonian systems. Indeed, Floquet systems arguably represent the maximum known extension of equilibrium phase structure in that generic driven systems lacking temporal periodicity are believed to heat to infinite temperature. Pioneering experiments\(^1\)-\(^6\) have very recently started exploring this universe of many-body Floquet drives.

Our viewpoint is statistical mechanical and restricted to closed/isolated systems. There is also a large and older literature on single-particle Floquet systems\(^7\) and much recent work on using Floquet physics to engineer non-trivial Hamiltonians as well as on open system physics to use such engineering to interesting ends. We make contact with this larger Floquet universe only where it intersects with our main theme and direct the reader to the literature for this complementary work\(^8\)-\(^14\).

Floquet basics

Most broadly, the quantum mechanics of closed systems is concerned with their unitary time evolution governed by the Schrödinger equation (\( \hbar = 1 \))

\[
\frac{d}{dt} U(t, t_0) = \mathcal{H}(t) U(t, t_0)
\]

where \( U(t, t_0) \) is the unitary time evolution operator that relates states at time \( t_0 \) to states at time \( t \). For completely general \( \mathcal{H}(t) \) there is not much else to do than to buckle down and solve (1). For static systems, \( \mathcal{H}(t) \equiv \mathcal{H}_0 \) life is much simpler as \( U(t, t_0) = e^{-i(\mathcal{H}_0 T)/\hbar} \), and so we learn vast amounts by solving the eigensystem problem for \( \mathcal{H}_0 \). Specifically, the eigenstates give rise to special, stationary, solutions of the Schrödinger equation that form a basis for general time evolution.

The fundamental difference between the Hamiltonians of Floquet and static systems is that the latter are fully independent of time, while the former are only invariant under discrete time translations by a period \( T \). This difference is analogous to the difference between translation invariance of the continuum and of a lattice. There, the former allows us to study the spectrum of the generator of translations (the momentum) while the latter requires that we study the spectrum of the discrete translation operator itself, with states in different bands corresponding to the same quasi-momentum. Correspondingly, for Floquet systems one needs to study the properties of the family of single-period time evolution operators

\[
U(t_0 + T, t_0) = T e^{-i(\mathcal{H}_0 T)/\hbar} U(t, t_0)
\]

where \( 0 \leq t_0 < T \).

Let us define \( U(T) = U(T, 0) \), whose eigenstates

\[
U(T)|\psi_\gamma(\tau)\rangle = e^{-i\epsilon_\gamma T}|\psi_\gamma(\tau)\rangle
\]

define special solutions of (1), the Floquet eigenstates

\[
|\psi_\gamma(t)\rangle = U(t, 0)|\phi_\gamma\rangle
\]

which satisfy \( |\psi_\gamma(t + T)\rangle = e^{-i\epsilon_\gamma T}|\psi_\gamma(t)\rangle \). Like the stationary solutions of the static problem, they explicitly exhibit the temporal periodicity of the Hamiltonian and form a basis for general time evolution. The choice of quasienergy \( \epsilon_\gamma \) is not unique as \( \epsilon_\gamma \equiv \epsilon_\gamma + n_\gamma (2\pi/\hbar T) \). This is the freedom in choosing the operator logarithm in \( U(T) = e^{-i\mathcal{H}_0 T} \), to obtain what is called the Floquet Hamiltonian \( \mathcal{H}_F \). In the following, we consider only systems where the local Hilbert space is finite-dimensional, and the coefficients of all terms appearing in \( \mathcal{H}(t) \) are bounded. A final piece of jargon: one refers to a time series spaced \( T \) apart as being stroboscopic.

To heat or not to heat

We begin with the textbook thermodynamic viewpoint, which notes that systems without continuous time-translation symmetry do

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not conserve energy; in particular in periodically driven systems, energy is conserved only modulo $2\pi/T$. For generic systems lacking any other local conserved quantities, thermodynamics predicts an entropy maximizing state at late times that is just the infinite temperature state\cite{Berry1984,Deutsch1991,Polkovnikov2011}, with all local operator expectation values time-independent at long times irrespective of the starting state. We can reach the same conclusion by noting that linear response theory implies absorption at non-zero frequencies and thus a heating cascade that can only terminate at $T = \infty$. In this unique ergodic phase, all Floquet eigenstates must individually yield $T = \infty$ correlations and exhibit volume law entanglement with the maximum thermodynamic entropy. This requirement is an incarnation of the eigenstate thermalization hypothesis (ETH), originally formulated for static ergodic systems\cite{Srednicki1994,GellMann1990,Balistrieri1996} which states that the value of any local observable in an eigenstate is a smooth function of eigenenergy. The second class is Floquet systems exhibiting many-body localization (MBL). Their discovery\cite{Basko2006,Altshuler2007,Imbrie2014} came as a by-product of the explosion of interest in MBL\cite{Rammer1979}, which generalizes the venerable Anderson localization of non-interacting particles to the interacting setting. For these systems, it was established that there exists a set of $O(N)$ spatially localized, mutually commuting, ‘$l$-bit’ operators $\tau_i^z$ (which depend on details of the drive) such that

$$[H(t), \tau_i^z] = 0$$

Here, spatial localization means that the $\tau_i^z$ operators have admixtures of operators whose amplitude decays exponentially in the distance away from site $i$. Floquet-MBL is most intuitive when adding a weak drive to a static MBL system (although not restricted to this case). The reference MBL system is itself transformation in $d = 1$. Such systems are described by a quadratic Floquet Hamiltonian (for $N$ sites)

$$H_i = \sum_{\alpha=1}^{N} \epsilon_{\alpha} |\alpha\rangle \langle \alpha|$$

where each eigenstate $\alpha$ comes with a conserved quantity

$$I_\alpha = |\alpha\rangle \langle \alpha|$$

For a local $H(t)$, linear combinations of these constants probably always yield quasi-local conserved quantities. We will return to the implications of this below. We also note very recent work that generalizes free fermion Floquet integrability to classes of interacting systems\cite{Gu2014}.

Existence of the Floquet-ergodic phase and applicability of ETH to its Floquet eigenstates has been confirmed computationally. There is considerable evidence that clean, interacting drives generically give rise to this behaviour, as assumed in the following. However, exceptions\cite{Michailides2017} and apparent exceptions\cite{Luitz2017,Chattopadhyay2017} are known and deserve more investigation, although there is presently no good reason to assume that they represent stable behaviour. However, there is at the very least the challenge of getting a better understanding of the various timescales involved in the heating process.

Leaving such worries aside, the suggestion is that to avoid heating we need $O(N)$ integrals of the motion, that is, quantities that commute with $U(T)$, and which can be written as sums of quasi-local terms. Here we use avoidance of heating in a strong sense. (There are Floquet systems with a much smaller number of conserved quantities that do not have the character of the energy. We expect that their Floquet evolution would have the character of heating to infinite temperature in sectors labelled by the conserved quantities.) There are two well-studied classes of systems where this is the case.

The first class is driven free fermion systems\cite{Khemani2014} and equivalent interacting spin systems obtained via Jordan–Wigner transformation in $d = 1$. Such systems are described by a quadratic Floquet Hamiltonian (for $N$ sites)

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Note that Floquet-MBL systems avoid heating generically—weak perturbations of Floquet-MBL drives that leave the period unchanged are also Floquet-MBL. By contrast, free fermion systems are stable to interactions only when Anderson localized by disorder.

We next discuss how these systems host generalizations of the two central ideas of thermodynamics—of equilibrium and phase structure. We take these in reverse order.

**Eigenstate order and phase structure**

As the Floquet-ETH phase is the only ergodic phase, all other phases must be localized. To define such phases it is fruitful to generalize the notions of eigenstate order and eigenstate phase transitions from the study of undriven MBL\cite{1,2} to Floquet systems. Eigenstate order exists when individual many-body eigenstates exhibit ordering, of which the spectrum exhibits a characteristic signature; at eigenstate phase transitions the eigenstates and eigenvalues can exhibit singular changes as a parameter is varied. For static/Floquet ergodic systems, this reduces to the conventional notion of order in the standard ensembles of statistical mechanics, as nearby/all eigenstates by ETH all yield the same answer. For Floquet systems eigenstate order can involve non-trivial variations of the eigenstates inside the Floquet period.

To get a sense of how more, and fundamentally new, phases arise\cite{3}, we discuss the simplest setting—that of Floquet-MBL chains with an Ising ($\mathbb{Z}_2$) symmetry. Consider the binary drive protocol

$$H(t) = \begin{cases} -\sum_i J_i \sigma_i^x \sigma_i^z + H_{\text{int}} & \text{for } 0 \leq t < T_1 \\ -\sum_i J_i \sigma_i^x \sigma_{i+1}^z + H_{\text{int}} & \text{for } T_1 \leq t < T \end{cases}$$

(4)

where $\sigma_i^x, \sigma_i^y$ are Pauli-matrix operators at site $s$, and the $h_i, J_i$ are weakly random about mean values $h$ and $J$ to obtain localization; the additional interaction terms, weaker still to preserve localization, prevent a possible reduction to free fermions. All terms commute with a global Ising symmetry $P = \prod_i \sigma_i^z$.

This family of drives exhibits four localized phases\cite{33}. These are shown in the phase diagram (Fig. 2a) for the free fermion limit; with interactions the Floquet-ergodic phase will also appear. These phases are characterized as follows in terms of the spectrum of $U(T)$ and the correlations $C_{ij} = \langle \sigma_i^+ \sigma_j^- \rangle$ at long distance $|i - j| \to \infty$ of the local Ising-odd operators $\sigma_i^z$ (Fig. 2).

**Paramagnet PM (no symmetry-breaking):** in all eigenstates $C_{ij} \to 0$.

**Spin glass SG:** in all eigenstates $C_{ij} \neq 0$. The spectrum contains quasi-degenerate (to exponential accuracy) pairs of cat states (Fig. 2a) that are superpositions of states with spin glass order and their Ising-reversed counterparts. Equivalently, in the thermodynamic limit it consists entirely of states with broken Ising symmetry and spin glass long-range order. Over each period, the order parameter (the local magnetization) returns to itself as detected by the dependence of $C_{ij}$ within the period\cite{33}.

**$\pi$-spin glass $\pi$SG:** in all eigenstates $C_{ij} \neq 0$. The spectrum contains pairs of cat states, with splitting exponentially close to $\pi/T$ (Fig. 2b). These are superpositions of states with spin glass order and their Ising-reversed counterparts. Even in the thermodynamic limit these cannot be rearranged into eigenstates with explicitly broken Ising symmetry. Thus, while the symmetry is broken as indicated by the two-point function, the catness is intrinsic. Over each period, the order parameter changes sign.

**0$\pi$-paramagnet 0$\pi$PM:** in all eigenstates $C_{ij} \to 0$ in the bulk. However, in open chains the spectrum comes in multiplets of four with splittings exponentially close to 0 and $\pi$; in closed chains the states are unique. Such phases are known as symmetry-protected topological phases (SPTs)\cite{34}: trivial in the bulk, but with edge states on open chains. There is also interesting dynamics at the edge.

We emphasize that all these phases exhibit a breakdown of ETH in that the correlators fluctuate strongly between neighbouring eigenstates. Thus, while an average over all states yields $T = \infty$ correlators, individual eigenstates do not (see Fig. 1). Also, the eigenstates exhibit area law entanglement, which then also serves
as an additional eigenstate diagnostic of the passage between any one of these phases and the ergodic phase. Interestingly, the two new phases, πSG and 0πPM, can also be classified by means of local order parameters for time-translational symmetry which is generated by $U(T)$ itself\(^3\). Of these, the πSG breaks time-translational symmetry in its bulk\(^3\),\(^3\),\(^3\), while the 0πPM breaks it only at its boundaries: these provide examples of the ‘time crystals’ first hypothesized for undriven systems\(^4\), although the term ‘spatio-temporally ordered’ is more accurate\(^5\). We describe the dynamical consequences of this identification below.

Finally, we note that the πSG is an exceptionally interesting phase. It is not merely stable to Ising-invariant perturbations, instead it is absolutely stable\(^6\)—that is, it is stable to all weak perturbations that do not alter the drive period. The enlarged phase breaks an emergent Ising symmetry as well as time-translation symmetry.

**Late time states**

Thus far we have made sharp statements about many-body eigenstates. As these are in general not easy to prepare, it is important to ask what degree of universality is present in late time states reached by time evolution from more easily prepared initial states; and whether the above phases and transitions can be detected in such late time states. For the ergodic phase, but not for our case, ETH ensures that eigenstate and late time averages agree. Nevertheless, the late time states are sufficiently robust that the phase structure can indeed be detected. To see this, consider a general state

$$|\chi(t+nT)\rangle = \sum_{\alpha} \epsilon_{\alpha} e^{-i\epsilon_{\alpha}nT} |\psi_{\alpha}(t)\rangle$$

which gives rise to the stroboscopic expectation value

$$\langle \chi(t+nT)|O|\chi(t+nT)\rangle = \sum_{\alpha,\beta} c^*_\alpha c_\beta e^{-i\epsilon_{\alpha}nT-i\epsilon_{\beta}nT} \langle \psi_{\alpha}(t)|O|\psi_{\beta}(t)\rangle$$

(5)

For MBL-Floquet systems $\epsilon_{\alpha} - \epsilon_{\beta}$ is essentially continuously distributed in the thermodynamic limit, except for the splittings internal to the spectral multiplets of the kind discussed above. Thus, at late times, $n \gg 1$, the expectation value reduces to its value in the quasi-diagonal ensemble

$$\langle \chi(t+nT)|O|\chi(t+nT)\rangle \sim \sum_{\alpha,\beta,\mu,\nu} e^{i\epsilon_{\mu}nT-i\epsilon_{\nu}nT} c^*_\alpha c_{\beta\mu} \langle \psi_{\alpha}(t)|O|\psi_{\beta\nu}(t)\rangle$$

(6)

so the late time density matrix is effectively

$$\rho \sim \sum_{\beta,\mu} e^{i\epsilon_{\mu}nT-i\epsilon_{\nu}nT} c^*_\alpha c_{\beta\mu} \langle \psi_{\alpha}(t)|O|\psi_{\beta\nu}(t)\rangle$$

with $\beta(\mu)$ the member of the multiplet that contains $\alpha$. Thus, at late times, roughly half the parameters present in the specification of the initial state (the phases) can no longer be recovered by local measurements.

For the phases of our model Ising drive we obtain the following characteristic behaviour of late time states. For the paramagnet the states are synchronized and paramagnetic, and expectation values are strictly periodic with $T$, with those of Ising-odd operators vanishing for all starting states. For the spin glass the states are synchronized and break Ising symmetry. For an initial state that breaks Ising symmetry, the one-point functions of Ising-odd operators are non-zero while for Ising-symmetric initial states we need to examine the two-point functions at large distances to detect the broken symmetry. For the 0π paramagnet the states are synchronized and paramagnetic, except at the boundary, where they exhibit period doubling. Finally, for the π spin glass the states break Ising symmetry with period doubling. For an initial state that breaks Ising symmetry, one-point functions of Ising-odd operators are non-zero while for Ising-symmetric initial states we need to examine the two-point functions at large distances. Stroboscopic snapshots look like Fig. 2c. In regions of the πSG phase lacking a microscopic Ising symmetry, generic local operators will exhibit period doubling; this has been seen in an experiment\(^7\).

Finally we turn to free fermion systems, which turn out to behave differently. Of these, Anderson localized systems share much with their MBL cousins, but they do not exhibit dephasing and so exhibit late time states with no particular periodicity. For free fermion Floquet systems without Anderson localization, stroboscopic evolution with $H_F$ is believed to lead to late time states which are well captured by a generalized Gibbs ensemble (GGE)

$$\rho \sim e^{-\sum k \epsilon_k a_k^\dagger a_k}$$

With the non-trivial but periodic intra-period evolution included, this has been called the periodic Gibbs ensemble (PGE; ref. 25) or the Floquet-GGE\(^8\). It is worth noting that the PGE density matrix leads to a volume law entanglement entropy that is less than the infinite temperature value, thus confirming a lack of heating\(^9\). The moral of this part of the story is that much less information survives in the free fermion late time states than does in the diagonal ensembles that describe Floquet-MBL systems, but more than survives for the Floquet-ETH case.

**Recent developments and outlook**

In a flurry of work, the programme of identifying stable interacting Floquet phases has been pushed quite far already\(^10\\)\(^-\)\(^45\). This builds on an essentially complete classification analogous to that of topological insulators and superconductors for free fermion systems\(^4\)\(^-\)\(^6\). The free fermion classification classifies single-particle unitaries and does not always lead to stable many-body phases upon the addition of weak interactions, as is the case for the analogous question for undriven free fermion systems\(^4\). Among the examples which are stable is the anomalous Floquet Anderson insulator\(^8\), which exhibits chiral edge modes without delocalized bulk states and is readily realized via a binary drive that appears to be experimentally feasible. The free fermion classification is, of course, relevant to experiments that probe few-particle physics.

Cold atomic systems, combining long coherence times and tunability of geometry, disorder and interactions, provide an ideal platform for testing those ideas. An important development is the demonstration of (static) MBL in a disordered two-dimensional optical lattice, finding a transition into a regime at which memory of the initial state with an asymmetric boson occupancy became long-lived. Very recently, an analogous study\(^4\) of a Floquet system with a (quasi-)disorder potential oscillating in time around a non-zero mean. Here, the memory indicative of MBL disappears as the driving frequency is lowered, in keeping with the above-mentioned predictions\(^27\)\(^-\)\(^28\). Finally, a first experiment claiming the observation of a discrete time crystal in the time domain has also appeared\(^5\). An experimental tour de force, it involves a mesoscopic system, with the experimental verification of the full spatiotemporal order in the πSG remaining an outstanding challenge.

An important line of work continues to be the search for conditions under which heating is not observed, and—highly relevant to experiments—the study of pre-thermal regimes for Floquet systems wherein they can exhibit plateaux characterized by equilibration with an effective $H_F$ over a long period before finally heating up to the ergodic steady state\(^30\)\(^-\)\(^32\). In principle this makes it possible to observe non-trivial effective phases, such as time crystals, even in systems that are not localized. Excitingly, a very recent experiment sees such behaviour in a three-dimensional system of nitrogen vacancy centres\(^6\), also in the time domain, although the precise connection to pre-thermalization theory is not settled. There
clearly remains much scope for further theoretical and experimental studies of the increasingly rich and complex phenomena in many-body Floquet systems.

Received 12 December 2016; accepted 22 March 2017; published online 24 April 2017

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Acknowledgements

We would like to thank A. Das, D. Huse, C. von Keyserlingk, V. Khemani, A. Lazarides and A. Polkovnikov for many useful discussions and for comments on the manuscript. This work was supported by the NSF-DMR via Grant No. 131781 and the Alexander von Humboldt foundation via a Humboldt award (S.L.S.) as well as the Deutsche Forschungsgemeinschaft via SFB 1143 (R.M.).

Additional information

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Competing financial interests

The authors declare no competing financial interests.