A Möbius scalar curvature rigidity on compact conformally flat hypersurfaces in $S^{n+1}$

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Abstract

In this paper, we study conformally flat hypersurfaces of dimension $n(\geq 4)$ in $S^{n+1}$ using the framework of Möbius geometry. First, we classify and explicitly express the conformally flat hypersurfaces of dimension $n(\geq 4)$ with constant Möbius scalar curvature under the Möbius transformation group of $S^{n+1}$. Second, we prove that if the conformally flat hypersurface with constant Möbius scalar curvature $R$ is compact, then

$$R = (n - 1)(n - 2)r^2, \quad 0 < r < 1,$$

and the compact conformally flat hypersurface is Möbius equivalent to the torus

$$S^1(\sqrt{1 - r^2}) \times S^{n-1}(r) \hookrightarrow S^{n+1}.$$

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1 Introduction

A Riemannian manifold \((M^n, g)\) is conformally flat, if every point has a neighborhood which is conformal to an open set in the Euclidean space \(\mathbb{R}^n\). A hypersurface of the sphere \(S^{n+1}\) is said to be conformally flat if so it is with respect to the induced metric. Due to conformal invariant objects, the theory of conformally flat hypersurfaces is essentially the same whether it is considered in the space forms \(\mathbb{R}^{n+1}, S^{n+1}\) or \(\mathbb{H}^{n+1}\). In fact, there exists conformal diffeomorphism between the space forms. The \((n+1)\)-dimensional hyperbolic space \(\mathbb{H}^{n+1}\) defined by

\[\mathbb{H}^{n+1} = \{ (y_0, y_1, \cdots, y_{n+1}) | -y_0^2 + y_1^2 + \cdots + y_{n+1}^2 = -1, y_0 > 0 \}.\]

The conformal diffeomorphisms \(\sigma, \tau\) are defined by

\[\sigma : \mathbb{R}^{n+1} \to S^{n+1}\{(-1, \vec{0})\}, \quad \sigma(u) = \left( \frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2} \right),\]

\[\tau : \mathbb{H}^{n+1} \to S^{n+1}_+ \subset S^{n+1}, \quad \tau(y) = \left( \frac{1}{y_0}, \frac{\vec{y}}{y_0} \right), \quad y = (y_0, \vec{y}) \in \mathbb{H}^{n+1},\]

where \(S^{n+1}_+\) is the hemisphere in \(S^{n+1}\) whose the first coordinate is positive. By conformal diffeomorphisms \(\sigma, \tau\), the conformally flat hypersurfaces in space forms are equivalent to each other.

The dimension of the hypersurface seems to play an important role in the study of conformally flat hypersurfaces. For \(n \geq 4\), the immersed hypersurface \(f : M^n \to S^{n+1}\) is conformally flat if and only if at least \(n-1\) of the principal curvatures coincide at each point by the result of Cartan-Schouten ([1],[10]). Cartan-Schouten’s result is no longer true in dimension 3. Lancaster ([6]) gave some examples of conformally flat hypersurfaces in \(\mathbb{R}^4\) having three different principal curvatures. For \(n = 2\), the existence of isothermal coordinates means that any Riemannian surface is conformally flat. Do Carmo, Dajczer and Mercuri in [2] have studied Diffeomorphism types of the compact conformally flat hypersurfaces in \(\mathbb{R}^{n+1}\). Pinkall in [9] was studied the intrinsic conformal geometry of compact conformally flat hypersurfaces. Suyama in [11] explicitly constructs compact conformally flat hypersurfaces in space forms using codimension one foliation by \((n-1)\)-spheres. Standard examples of the conformally flat hypersurfaces come from cones, cylinders, or rotational hypersurfaces over a curve in...
Euclidean 2-space \( \mathbb{R}^2 \), 2-sphere \( S^2 \), or hyperbolic 2-space \( \mathbb{R}^2_+ \), respectively (see section 3). In [4], Lin and Guo showed that if the conformally flat hypersurface has closed Möbius form, then it is Möbius equivalent to one of the standard examples.

It is known that the conformal transformations group of a sphere is isomorphic to its Möbius transformation group. As conformal invariant objects, conformally flat hypersurfaces are investigated in this paper using the framework of Möbius geometry. If the conformally flat hypersurface is no umbilical point everywhere, then there exists a global Möbius metric (see section 2), which is invariant under the Möbius transformation group of \( S^{n+1} \). The scalar curvature with respect to the Möbius metric is called Möbius scalar curvature. First, we classify locally the conformally flat hypersurfaces of dimension \( n(\geq 4) \) with constant Möbius scalar curvature under the Möbius transformation group of \( S^{n+1} \).

**Theorem 1.1.** Let \( f : M^n \to S^{n+1}, n \geq 4 \), be a conformally flat hypersurface without umbilical points. If the Möbius scalar curvature is constant, then the Möbius form is closed and \( f \) is Möbius equivalent to one of the following hypersurfaces in \( S^{n+1} \),

(i) the image of \( \sigma \) of a cylinder over a curvature-spiral in \( \mathbb{R}^2 \subset \mathbb{R}^{n+1} \);
(ii) the image of \( \sigma \) of a cone over a curvature-spiral in \( S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^{n+1} \);
(iii) the image of \( \sigma \) of a rotational hypersurface over a curvature-spiral in \( \mathbb{R}^2_+ \subset \mathbb{R}^{n+1} \).

Here the so-called curvature-spiral in a 2-dimensional space form \( \mathbb{N}^2(\epsilon) = S^2, \mathbb{R}^2, \mathbb{R}^2_+ \) (of Gaussian curvature \( \epsilon = 1, 0, -1 \) respectively) is determined by the intrinsic equation

\[
- \frac{\kappa_{ss}}{\kappa^3} + \frac{(n + 2)\kappa^2}{2\kappa^4} + \epsilon \frac{n - 2}{2\kappa^2} = R, \quad \kappa_s = \frac{d}{ds}\kappa.
\]

Here \( s \) is the arc-length parameter, \( \kappa \) denotes the geodesic curvature of the curve \( \gamma \), and \( R \) is a real constant. In [3], authors classified locally the hypersurfaces with constant Möbius sectional curvature, which is some special conformally flat hypersurfaces with Möbius scalar curvature by the equation (1.1).

For compact conformally flat hypersurfaces, we obtain the following Möbius scalar curvature rigidity theorem, which means that the closed curve in \( \mathbb{R}^2_+ \) satisfying the intrinsic equation (1.1) with geodesic curvature \( \kappa > 0 \) is circle \( S^1 \).
Theorem 1.2. Let $f : M^n \to S^{n+1}$, $n \geq 4$, be a compact conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature $R$ is constant, then

$$R = (n-1)(n-2)r^2, \quad 0 < r < 1,$$

and the compact conformally flat hypersurface is Möbius equivalent to the torus

$$f : S^1(\sqrt{1-r^2}) \times S^{n-1}(r) \to S^{n+1}.$$

Remark 1.1. Theorem 1.1 and Theorem 1.2 is true for $n = 3$ provided that the 3-dimensional conformally flat hypersurface has only two distinct principal curvatures.

The paper is organized as follows. In section 2, we review the elementary facts about Möbius geometry of hypersurfaces in $S^{n+1}$. In section 3, we prove the theorem 1.1. In section 4, we prove the theorem 1.2.

2 Möbius invariants of hypersurfaces in $S^{n+1}$

In this section, we recall some facts about the Möbius invariants of hypersurfaces in $S^{n+1}$. For details we refer to [12].

Let $f : M^n \to S^{n+1}$ be a hypersurface without umbilical points. In this section we use the range of indices: $1 \leq i, j, k, l \leq n$. We assume that $\{e_i\}$ is an orthonormal basis with respect to the induced metric with $\{\theta_i\}$ the dual basis. Let $II = \sum_{ij} h_{ij} \theta_i \theta_j$ and $H = \sum_i \frac{h_{ii}}{n}$ be the second fundamental form and the mean curvature of $x$, respectively. We define the Möbius metric $g$, the Möbius second fundamental form $B$, the Blaschke tensor $A$ and the Möbius form $C$ as follows, respectively,

$$g = \rho^2 dx \cdot dx, \quad \rho^2 = \frac{n}{n-1}(|h|^2 - nH^2),$$

$$B = \rho \sum_{ij} (h_{ij} - H \delta_{ij}) \theta_i \otimes \theta_j,$$

$$C = -\rho^{-1} \sum_i [e_i(H) + \sum_j (h_{ij} - H \delta_{ij}) e_j] \theta_i,$$

$$A = \sum_{ij} \left\{ e_i(\log \rho) e_j(\log \rho) - \nabla_{e_i} \nabla_{e_j} \log \rho + H h_{ij} + \frac{1}{2} [1 - H^2 - |\nabla \log \rho|^2] \delta_{ij} \right\} \theta_i \otimes \theta_j.$$

(2.2)
Note that the conformal compactification space $S^{n+1}$ unifies the space forms $S^{n+1}, \mathbb{R}^{n+1}, \mathbb{H}^{n+1}$ and the formula above defining the Möbius metric $g$ and the Möbius second fundamental form $B$ are the same for any of them.

**Theorem 2.1.**\cite{12} Two hypersurfaces $f: M^n \to S^{n+1}$ and $\bar{f}: M^n \to S^{n+1}(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\varphi: M^n \to M^n$ which preserves the Möbius metric and the Möbius second fundamental form.

Let $E_i = \rho^{-1} e_i, \omega_i = \rho \theta_i$, then $\{E_1, \cdots, E_n\}$ is an orthonormal basis with respect to the Möbius metric $g$ with the dual basis $\{\omega_1, \cdots, \omega_n\}$. Let $\{\omega_{ij}\}$ be the connection 1-form of the Möbius metric under the orthonormal basis $\{\omega_i\}$, and

$$A = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad B = \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \quad C = \sum_i C_i \omega_i.$$  

The covariant derivative of $C_i, A_{ij}, B_{ij}$ are defined by

$$\sum_j C_{i,j} \omega_j = dC_i + \sum_j C_{j,ji},$$  
$$\sum_k A_{ij,k} \omega_k = dA_{ij} + \sum_k A_{ikj} \omega_k + \sum_k A_{kj} \omega_{ki},$$  
$$\sum_k B_{ij,k} \omega_k = dB_{ij} + \sum_k B_{ikj} \omega_k + \sum_k B_{kj} \omega_{ki}.$$  

The integrability conditions of the Möbius invariants are given by

\begin{align}
(2.3) \quad & A_{ij,k} - A_{ik,j} = B_{ik} C_j - B_{ij} C_k, \\
(2.4) \quad & C_{i,j} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{jk} A_{ki}), \\
(2.5) \quad & B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \\
(2.6) \quad & R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk} + \delta_{ik} A_{jl} + \delta_{jk} A_{il} - \delta_{il} A_{jk} - \delta_{jl} A_{ik},
\end{align}  

where $R_{ijkl}$ denote the curvature tensor of $g$. Moreover,

\begin{align}
(2.7) \quad & \sum_i B_{ii} = 0, \quad \sum_{ij} (B_{ij})^2 = \frac{n-1}{n}, \\
& \sum_i A_{ii} = \frac{1}{2n} + \frac{R}{2(n-1)}, \quad \sum_j B_{ij,j} = -(n-1) C_i,
\end{align}  

where $R = \sum_{i>j} R_{ijij}$ is the Möbius scalar curvature.
By equation (2.4), we have

\[(2.8) \quad dC = 0 \iff \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}) = 0,\]

which implies that the matrix \((B_{ij})\) and \((A_{ij})\) can be diagonalizable simultaneously.

### 3 Local geometry of conformally flat hypersurfaces

In this section, we will give the Möbius invariants of the standard examples of conformally flat hypersurfaces in \(\mathbb{R}^{n+1}\). Then we prove that the conformally flat hypersurfaces with constant Möbius scalar curvature come from these examples.

A key observation is that the Möbius metric of those standard examples are of the form

\[g = \kappa^2(s) \left( ds^2 + I_{n-1}^{n-1} - \epsilon \right), \]

where \(I_{n-1}^{n-1}\) is the metric of \(n - 1\) dimensional space form of constant curvature \(-\epsilon\). For such metric forms we have

**Lemma 3.1.** The metric \(g = \kappa^2(s)(ds^2 + I_{n-1}^{n-1})\) given above is of constant scalar curvature \(R\) if and only if the function \(\kappa(s)\) satisfies

\[-\frac{\kappa_{ss}}{\kappa^3} + \frac{(n+2)\kappa_s^2}{2\kappa^4} + \frac{\epsilon n - 2}{2\kappa^2} = R, \quad \kappa_s = \frac{d}{ds}\kappa.\]

This lemma is easy to prove using exterior differential forms and we omit the proof at here. Below we give the explicit construction of the standard examples of conformally flat hypersurfaces as well as their Möbius metric.

**Example 3.1.** Let \(\gamma : I \rightarrow \mathbb{R}^2\) be a regular curve, and \(s\) denote the arclength of \(\gamma(s)\). we define cylinder in \(\mathbb{R}^{n+1}\) over \(\gamma\),

\[f(s, y) = (\gamma(s), y) : I \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n+1},\]

where \(y : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}\) is identical mapping.

The first fundamental form \(I\) and the second fundamental form \(II\) of the cylinder \(f\) are, respectively,

\[I = ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = \kappa ds^2,\]
where $\kappa(s)$ is the geodesic curvature of $\gamma$, $I_{\mathbb{R}^{n-1}}$ is the standard Euclidean metric of $\mathbb{R}^{n-1}$. So we have $(h_{ij}) = \text{diag}(\kappa, 0, \cdots, 0)$, $H = \frac{\kappa}{n}$, $\rho = \kappa$. Thus the Möbius metric $g$ of the cylinder $f$ is

$$g = \rho^2 I = \kappa^2 (ds^2 + I_{\mathbb{R}^{n-1}}).$$

where $I_{\mathbb{R}^{n-1}}$ is the standard hyperbolic metric of $\mathbb{H}^{n-1}(-1)$. Because at least $n-1$ of the principal curvatures coincide at each point, the cylinder $f$ is a conformally flat hypersurface. When $\gamma = S^1$, the cylinder $f$ is the isoparametric hypersurface $S^1 \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$.

**Example 3.2.** Let $\gamma : I \to \mathbb{S}^2(1) \subset \mathbb{R}^3$ be a regular curve, and $s$ denote the arclength of $\gamma(s)$. we define cone in $\mathbb{R}^{n+1}$ over $\gamma$,

$$f(s, t, y) = (t\gamma(s), y) : I \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1},$$

where $y : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is identical mapping and $\mathbb{R}^+ = \{t \mid t > 0\}$.

The first and second fundamental forms of the cone $f$ are, respectively,

$$I = t^2 ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = \kappa t ds^2.$$

So we have $(h_{ij}) = \text{diag}(\frac{t}{\kappa}, 0, \cdots, 0)$, $H = \frac{\kappa}{nt}$, $\rho = \frac{\kappa}{t}$. Thus the Möbius metric $g$ of the cone $f$ is

$$g = \rho^2 I = \frac{\kappa^2}{t^2} (t^2 ds^2 + I_{\mathbb{R}^{n-1}}) = \kappa^2 (ds^2 + I_{\mathbb{H}^{n-1}}),$$

where $I_{\mathbb{H}^{n-1}}$ is the standard hyperbolic metric of $\mathbb{H}^{n-1}(-1)$. Clearly the cone $f$ is a conformally flat hypersurface. When $\gamma = S^1$, the cone $f$ is the image of $\tau^{-1} \circ \sigma$ of the isoparametric hypersurface $S^1(r) \times \mathbb{H}^{n-1}(\sqrt{1 + r^2}) \to \mathbb{H}^{n+1}$.

**Example 3.3.** Let $\mathbb{R}^+_2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half-space endowed with the standard hyperbolic metric

$$ds^2 = \frac{1}{y^2} [dx^2 + dy^2].$$

Let $\gamma = (x, y) : I \to \mathbb{R}^+_2$ be a regular curve, and $s$ denote the arclength of $\gamma(s)$. we define rotational hypersurface in $\mathbb{R}^{n+1}$ over $\gamma$,

$$f : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}, \quad f(x, y, \theta) = (x, y\theta),$$

where $\theta : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is a standard immersion of a round sphere.
In the Poincare half plane $\mathbb{R}^2_+$ we denote the covariant differential of the hyperbolic metric as $D$. Choose orthonormal frames $e_1 = y \frac{\partial}{\partial x}, e_2 = y \frac{\partial}{\partial y}$. It is easy to find

$$D_{e_1}e_1 = e_2, \quad D_{e_1}e_2 = -e_1, \quad D_{e_2}e_1 = D_{e_2}e_2 = 0.$$ 

For $\gamma(s) = ((x(s), y(s))) \subset \mathbb{R}^2_+$ let $x'$ denote derivative $\partial x/\partial s$ and so on. Choose the unit tangent vector $\alpha = \frac{1}{y}(x'(s)e_1 + y'(s)e_2)$ and the unit normal vector $\beta = \frac{1}{y}(-y'(s)e_1 + x'(s)e_2)$. The geodesic curvature is computed via

$$\kappa = \langle D_{\alpha} \alpha, \beta \rangle = \frac{x'y'' - x''y'}{y^2} + \frac{x'}{y}.$$ 

After these preparation, we see that the rotational hypersurface $f(x, y, \theta) = (x, y\theta)$ has differential $df = (x' ds, y' ds + y d\theta)$ and unit normal vector $\eta = \frac{1}{y}(-y', x' \theta)$. Thus the first and second fundamental forms of hypersurface $f$ are, respectively,

$$I = df \cdot df = y^2(ds^2 + I_{S^n-1}), \quad II = -df \cdot d\eta = (y\kappa - x')ds^2 - x'I_{S^n-1},$$

where $I_{S^n-1}$ is the standard metric of $S^n-1(1)$. Thus principal curvatures are

$$\kappa = \frac{x'y'' - x''y'}{y^2}, \quad \frac{x'}{y}, \cdots, \frac{x'}{y}.$$ 

So $\rho = \frac{2}{y}$, and the M"{o}bius metric of $f$ is

$$g = \rho^2 I = \kappa^2(ds^2 + I_{S^n-1}).$$

Clearly the hypersurface $f$ is a conformally flat hypersurface. When $\gamma = S^1$, the cone $f$ is the image of $\sigma$ of the isoparametric hypersurface $S^1(\sqrt{1-r^2}) \times S^{n-1}(r) \to S^{n+1}$.

Next, we compute the M"{o}bius invariant of the conformally flat hypersurfaces. From (2.7), We can choose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the M"{o}bius metric $g$ such that

$$(B_{ij}) = \text{diag}(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}).$$ 

In the following section we make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq n; \quad 2 \leq \alpha, \beta, \gamma \leq n.$$ 

Since $B_{\alpha\beta} = \frac{1}{n} \delta_{\alpha\beta}$, we can rechoose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect
to the Möbius metric $g$ such that

$$(3.9) \quad (B_{ij}) = \text{diag} \left( \frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n} \right), \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & a_2 & 0 & \cdots & 0 \\ A_{31} & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & 0 & 0 & \cdots & a_n \end{pmatrix}$$

Let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis, and $\{\omega_{ij}\}$ the connection forms.

**Lemma 3.2.** Let $f : M^n \to S^{n+1}$ ($n \geq 4$) be a conformally flat hypersurface without umbilical points. If the Möbius scalar curvature is constant, then we can choose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the Möbius metric $g$ such that

$$(3.10) \quad (B_{ij}) = \text{diag} \left\{ \frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n} \right\}, \quad (A_{ij}) = \text{diag} \{a_1, a_2, \cdots, a_2\}.$$ 

Moreover, the distribution $\mathbb{D} = \text{span}\{E_2, \cdots, E_n\}$ is integrable.

**Proof.** Using $dB_{ij} + \sum_k B_{kj} \omega_{ki} + \sum_k B_{ik} \omega_{kj} = \sum_k B_{ij,k} \omega_k$, the equation (2.5), we get

$$(3.11) \quad B_{1\alpha,\alpha} = -C_1, \text{ otherwise, } \quad B_{ij,k} = 0;$$

$$\omega_{1\alpha} = -C_1 \omega_\alpha, \quad C_\alpha = 0.$$ 

Thus $d\omega_1 = 0$ and the distribution $\mathbb{D} = \text{span}\{E_2, \cdots, E_n\}$ is integrable.

Using $dC_i + \sum_k C_{k\omega_{ki}} = \sum_k C_{i,k} \omega_k$ and (3.11), we can obtain

$$(3.12) \quad C_{\alpha,\alpha} = -C_1^2, \quad C_{\alpha,k} = 0, \alpha \neq k.$$ 

From (3.11),

$$d\omega_{1\alpha} = -dC_1 \wedge \omega_\alpha - C_1 d\omega_\alpha = -dC_1 \wedge \omega_\alpha + C_1^2 \omega_1 \wedge \omega_\alpha - C_1 \sum_\gamma \omega_\gamma \wedge \omega_{\gamma\alpha},$$

and $d\omega_{1\alpha} - \sum_j \omega_{1j} \wedge \omega_{j\alpha} = -\frac{1}{2} \sum_{kl} R_{1\alpha kl} \omega_k \wedge \omega_l$, we get that

$$(3.13) \quad R_{1\alpha \alpha 1} = C_{1,1} - C_1^2, \quad R_{1\alpha \beta \alpha} - C_{1,\beta} = 0.$$
Since \( R_{1\alpha 1\alpha} = -\frac{a-1}{n^2} + a_1 + a_\alpha = C_{1,1} - C_1^2 \) and \( R_{1\alpha \beta \alpha} = A_{1\beta}, \alpha \neq \beta \), thus we have

\[
(3.14) \quad a_2 = a_3 = \cdots = a_n, \quad A_{1\beta} = C_{1,\beta}.
\]

Thus \( A|_\mathbb{D} = aI \), \( a = a_2 \). Since \( E_1 \) is principal vector field, then vector \( E = A_{12}E_2 + \cdots + A_{1n}E_n \) is well defined. If \( E = 0 \), then \( A_{12} = \cdots = A_{1n} = 0 \). If \( E \neq 0 \), we can rechoose a local orthonormal basis \( \{ \tilde{E}_2 = \frac{E}{|E|}, \tilde{E}_3, \cdots, \tilde{E}_n \} \) of \( \mathbb{D} \) with respect to the Möbius metric \( g \) such that

\[
(3.15) \quad (B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2 \end{pmatrix},
\]

To finish the proof of the Lemma, we need to prove that \( A_{12} = 0 \). Using \( dA_{ij} + \sum_k A_{kj}\omega_{ki} + \sum_k A_{ik}\omega_{kj} = \sum_k A_{ij,k}\omega_k \), the equation (2.3) and (3.15), we get

\[
(3.16) \quad \begin{aligned}
\sum_m A_{12,m}\omega_m &= dA_{12} + (A_{11} - A_{22})\omega_{12}, \\
\sum_m A_{1\alpha,m}\omega_m &= (A_{11} - a_2)\omega_{1\alpha} + A_{12}\omega_{2\alpha}, \quad \sum_m A_{2\alpha,m}\omega_m = A_{12}\omega_{1\alpha}, \quad \alpha \geq 3, \\
\sum_k A_{11,k}\omega_k &= dA_{11} + 2A_{12}\omega_{21}, \quad \sum_k A_{22,k}\omega_k = dA_{22} + 2A_{12}\omega_{12}, \\
\sum_k A_{\alpha\alpha,k}\omega_k &= dA_{\alpha\alpha}, \quad A_{\alpha\beta,k} = 0, \quad \alpha \neq \beta, \quad \alpha, \beta \geq 3.
\end{aligned}
\]

From the fourth and seventh equation in (3.16), we get

\[
(3.17) \quad E_\alpha(a_2) = A_{\beta\beta,\alpha} = A_{\beta\alpha,\beta} = 0, \quad \alpha \geq 3.
\]

Since the Möbius scalar curvature is constant, \( tr(A) = A_{11} + (n - 1)a_2 \) is constant. Thus

\[
(3.18) \quad A_{1\alpha,1} = A_{11,\alpha} = E_\alpha(A_{11}) = 0, \quad \alpha \geq 3.
\]

From the first, second and third equation in (3.16), we get

\[
(3.19) \quad A_{12,2} = E_2(A_{12}) - (A_{11} - a_2)C_1, \quad A_{1\beta,\beta} = -(A_{11} - a_2)C_1 + A_{12}\omega_{2\beta}(E_\beta).
\]
On the other hand, From (2.3), we have
\[
E_1(A_{22}) = A_{22,1} = A_{12,2} + \frac{1}{n} C_1 = E_2(A_{12}) - (A_{11} - a_2)C_1 + \frac{1}{n} C_1, \\
E_1(A_{\alpha\alpha}) = A_{\alpha,1} = A_{1\alpha,\alpha} + \frac{1}{n} C_1 = -(A_{11} - a_2)C_1 + A_{12}A_{2\beta}(E_\beta) + \frac{1}{n} C_1,
\]
which implies that
\[
E_1(A_{\alpha\alpha}) = A_{\alpha,1} = A_{1\alpha,\alpha} + \frac{1}{n} C_1 = -(A_{11} - a_2)C_1 + A_{12}A_{2\beta}(E_\beta) + \frac{1}{n} C_1,
\]
which implies that
\[
E_2(A_{12}) = A_{12}A_{2\beta}(E_\beta).
\]
Since \(A_{1\alpha,\beta} = A_{\alpha\beta}, \gamma \neq \beta\), from the second equation in (3.16) we can obtain
\[
E_2(A_{12}) = A_{12}A_{2\beta}(E_\beta), \quad \beta \geq 3, \quad k \neq \beta.
\]
From the first and third equation in (3.16), we get
\[
E_1(A_{12}) = A_{12,1} = A_{11,2} = E_2(A_{12}) + 2A_{12}C_1 = E_2(-(n - 1)a_2) + 2A_{12}C_1 = (n + 1)A_{12}C_1.
\]
Now we assume that \(A_{12} \neq 0\), From (3.20) and (3.21), we have
\[
\omega_{2\alpha} = \frac{E_2(A_{12})}{A_{12}} \omega_{\alpha} := \mu \omega_{\alpha}, \quad \alpha \geq 3.
\]
Thus
\[
d\omega_{2\alpha} = d\mu \wedge \omega_{\alpha} + d\mu \omega_{\alpha} \\
= d\mu \wedge \omega_{\alpha} - \mu C_1^2 \omega_1 \wedge \omega_{\alpha} + \mu^2 \omega_2 \wedge \omega_{\alpha} + \mu \sum_{\gamma \geq 3} \omega_{\gamma} \wedge \omega_{\gamma\alpha}.
\]
Using \(d\omega_{2\alpha} - \sum_{j} \omega_{2j} \wedge \omega_{j\alpha} = -\frac{1}{2} \sum_{kl} R_{2\alpha kl} \omega_k \wedge \omega_l\), we get that
\[
E_1(\mu) - \mu C_1 = -A_{12}.
\]
On the other hand, using (3.11) and (3.22), we have
\[
E_1(\mu) = E_1\left[ \frac{E_2(A_{12})}{A_{12}} \right] = \frac{E_1 E_2(A_{12})}{A_{12}} - \frac{E_2(A_{12})E_1(A_{12})}{A_{12}^2} \\
= \frac{E_1 E_2(A_{12})}{A_{12}} - (n + 1) \frac{E_2(A_{12})C_1}{A_{12}} \\
= \frac{(E_2 E_1 + C_1 E_2)(A_{12})}{A_{12}} - (n + 1) \frac{E_2(A_{12})C_1}{A_{12}} \\
= \frac{E_2[(n + 1)A_{12}C_1]}{A_{12}} - n \frac{E_2(A_{12})C_1}{A_{12}} \\
= (n + 1)C_{1,2} + \frac{E_2(A_{12})C_1}{A_{12}},
\]
which implies that 
\[(n + 1)C_{1,2} = -A_{12}.
\]
This is a contradiction by \(A_{12} = C_{1,2}\). Therefore \(A_{12} = 0\) and we finish the proof. \(\square\)

By Lemma 3.2 and equation (2.3), we can derive that \(dC = 0\). Combining the results in [4] and Lemma 3.1 we finish the proof of Theorem 1.1.

4 Global rigidity of Möbius scalar curvature

A hypersurface in \(S^{n+1}\) is called a Möbius isoparametric hypersurface if its Möbius form vanishes and all the eigenvalues of the Möbius second fundamental form \(B\) with respect to \(g\) are constants. In [5], authors gave the following classification theorem.

**Theorem 4.1.** [5] Let \(f : M^n \to S^{n+1}\) be a Möbius isoparametric hypersurface with two distinct principal curvatures. Then \(f\) is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurfaces in \(S^{n+1}\):

(i) the standard torus \(S^k(r) \times S^{n-k}(\sqrt{1 - r^2})\);

(ii) the image of \(\sigma\) of the standard cylinder \(S^k(1) \times R^{n-k} \subset R^{n+1}\);

(iii) the image of \(\tau\) of the standard \(S^k(r) \times R^{n-k}(\sqrt{1 + r^2})\) in \(H^{n+1}\).

To prove Theorem 4.1, we only need to prove \(C = 0\). The way of the proof is to use divergence theorem. First, we need some local computation.

**Lemma 4.1.** Let \(f : M^n \to S^{n+1}\) be a conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature \(R\) is constant, then under the local orthonormal basis \(\{E_1, \cdots, E_n\}\) in Lemma 3.2, we have

\[
\begin{align*}
    a_1 &= \frac{2n - 1}{2n^2} - \frac{R}{2(n - 1)(n - 2)} - \frac{n - 1}{n - 2}(C_{1,1} - C_1^2), \\
    a_2 &= \frac{R}{2(n - 1)(n - 2)} - \frac{1}{2n^2} - \frac{1}{n - 2}(C_{1,1} - C_1^2), \\
    A_{\alpha\alpha,1} &= \frac{R}{(n - 1)(n - 2)} C_1 - \frac{n - 1}{n - 2}(C_1 C_{1,1} - C_1^3).
\end{align*}
\]

(4.23)

Except these coefficients \(A_{11,1}, A_{1\alpha,\alpha}\), and \(A_{\alpha\alpha,1}\) the coefficients of \(\nabla A\) are equal to zero.
Proof. The first and second equation in (4.23) can derive directly from the equation

\[ \text{tr}(A) = a_1 + (n - 1)a_2 = \frac{1}{2n} + \frac{R}{2(n-1)} \quad \text{and} \quad R_{1\alpha_1\alpha} = -\frac{n-1}{n^2} + a_1 + a_2 = C_{1,1} - C_1^2 \text{ in (3.13)}. \]

From (3.16), we can get

\[ (4.24) \quad A_{1\alpha,\alpha} = (a_2 - a_1)C_1 = \left[ \frac{R}{(n-1)(n-2)} - \frac{1}{n} \right] C_1 - \frac{n}{n-2} (C_{1,1} - C_1^3). \]

By (2.3), we have

\[ A_{\alpha\alpha,1} = A_{1\alpha,\alpha} + \frac{1}{n} C_1. \]

Combining above equation we get the third equation in (4.23).

Since \( \text{tr}(A) = a_1 + (n - 1)a_2 \) is constant, we have \( A_{11,1} = -(n - 1)A_{\alpha\alpha,1} \). Thus, by lemma 3.2 we know that except these coefficients \( A_{11,1}, A_{1\alpha,\alpha} \) and \( A_{\alpha\alpha,1} \) the coefficients of \( \nabla A \) are equal to zero.

Lemma 4.2. Let \( f : M^n \to S^{n+1} \) \( (n \geq 4) \) be a conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature \( R \) is constant. then under the local orthonormal basis \( \{E_1, \cdots, E_n\} \) in Lemma 3.2 we have

\[ (4.25) \quad C_{1,11} = E_1(C_{1,1}) = (n + 2)C_1C_{1,1} - nC_1^3 - \frac{R}{n-1} C_1, \]
\[ C_{\alpha,\alpha} = E_1(C_{\alpha,\alpha}) = -2C_1C_{1,1}, \quad C_{1,\alpha\alpha} = C_{1,1\alpha} = -(C_{1,1} + C_1^2)C_1. \]

Except these coefficients \( C_{1,11}, C_{1,\alpha\alpha} \) and \( C_{\alpha,\alpha,1} \) the coefficients of \( \nabla^2 C \) are equal to zero.

Proof. Since \( (A_{ij}) = \text{diag}\{a_1, a_2, \cdots, a_2\} \) under the local orthonormal basis, we have

\[ A_{\alpha\alpha,1} = E_1(A_{\alpha\alpha}) = E_1(a_2) = -\frac{1}{n-2}(C_{1,11} - 2C_1C_{1,1}) \]

by Lemma 4.1 combining the first equation in (4.23), we get the first equation in (4.25).

By the equation (3.12) and the equation (3.14), \( (C_{ij}) = \text{diag}(C_{1,1}, -C_1^2, \cdots, -C_1^2) \) under the local orthonormal basis, thus we have

\[ (4.26) \quad C_{\alpha,\alpha} = E_1(C_{\alpha,\alpha}) = -2C_1C_{1,1}, \quad C_{1,\alpha\alpha} = C_{1,1\alpha} = -(C_{1,1} + C_1^2)C_1. \]

And the rest coefficients of \( \nabla^2 C \) are zero.

Since the hypersurface is conformally flat, the Schouten tensor \( S = \sum_{ij} S_{ij} \omega_i \otimes \omega_j \) is a Codazzi tensor (i.e., \( S_{ij,k} = S_{ik,j} \)), which defined by

\[ S_{ij} = R_{ij} - \frac{R}{2(n-1)} \delta_{ij}. \]
Noting that the scaler curvature $R$ is constant, $tr(A)$ and $tr(S)$ are constant by the equation (2.7). Furthermore, we have

\begin{equation}
\sum_j A_{ij,j} = -\sum_j B_{ij}C_j, \quad \sum_j S_{ij,j} = 0.
\end{equation}

Under the local orthonormal basis $\{E_1, \cdots, E_n\}$ in Lemma 3.2, we have

\begin{equation}
(S_{ij}) = \text{diag}(S_1, S_2, \cdots, S_2),
\end{equation}

\begin{equation}
S_1 = -\frac{(2n-1)(n-2)}{2n^2} + (n-2)a_1, \quad S_2 = \frac{n-2}{2n} + (n-2)a_2.
\end{equation}

Thus we have

\begin{equation}
S_{\alpha\alpha,1} = S_{1\alpha,\alpha} = (n-2)A_{\alpha\alpha,1} = \frac{R}{n-1}C_1 - n(C_1C_{1,1} - C_1^3),
\end{equation}

\begin{equation}
S_{11,1} = -(n-1)(n-2)A_{\alpha\alpha,1} = -RC_1 + n(n-1)(C_1C_{1,1} - C_1^3).
\end{equation}

**Lemma 4.3.** Let $f : M^n \to S^{n+1}$ ($n \geq 4$) be a conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature $R$ is constant, then under the local orthonormal basis $\{E_1, \cdots, E_n\}$ in Lemma 3.2, the coefficients of $\nabla^2 S$ satisfy

\begin{equation}
S_{11,11} = -RC_1 + n(n-1)C_{1,1}^2 + n(n-1)^2C_1^2C_{1,1} - n^2(n-1)C_1^4 - nRC_1^2,
\end{equation}

\begin{equation}
S_{11,\alpha} = \frac{(n+1)R}{n-1}C_1^2 - n(n+1)[C_1^2C_{1,1} - C_1^4], \quad S_{\alpha,11} = -(n-1)S_{11,11},
\end{equation}

\begin{equation}
S_{\alpha\alpha,\alpha\alpha} = 3S_{\alpha\alpha,\beta\beta} = 3\left\{-\frac{R}{n-1}C_1^2 + n[C_1^2C_{1,1} - C_1^4]\right\}, \quad \alpha \neq \beta.
\end{equation}

**Proof.** Since $(S_{ij}) = \text{diag}(S_1, S_2, \cdots, S_2)$, we know that except these coefficients $S_{11,1}$, $S_{1\alpha,\alpha}$ and $S_{\alpha\alpha,1}$ the coefficients of $\nabla S$ are equal to zero. Using the definition of the second covariant derivative of $S$, we can compute these equations in (4.30). \qed

Since $E_1$ is principal vector corresponding the eigenvalue $\frac{n-1}{n}$ of the Möbius second fundamental form $B$, the $C_1 = C(E_1)$, $C_{1,1} = \nabla C(E_1, E_1)$ are well-defined functions on $M^n$ up to a sign.

**Lemma 4.4.** Let $f : M^n \to S^{n+1}$ ($n \geq 4$) be a compact conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature $R$ is constant, then

\begin{equation}
\int_{M^n} C_1^2C_{1,1}dV_g = \frac{n-1}{3} \int_{M^n} |C|^4dV_g, \quad \int_{M^n} C_{1,1}^2dV_g = \int_{M^n} |C|^4dV_g + \frac{R}{n-1} \int_{M^n} |C|^2dV_g.
\end{equation}
Proof. Using the coefficients of the tensor $C$ and $S$, we define two smooth vector fields

$$X_S = \sum_{ij} C_i S_{ij} E_j, \quad X_C = \sum_{ij} C_i E_i.$$ 

From Lemma 3.2 and the equation (4.23), we can get the divergence of $X_S, X_C$,

$$\text{div}X_C = \sum_i C_{i,i} = C_{1,1} - (n-1)C_1^2,$$

$$\text{div}X_S = (n-1)(C_{2,1,1}^2 - C_1^4 - \frac{R}{n-1} C_1^2) + \frac{R}{2(n-1)} \text{div}X_C.$$ 

Since the hypersurface is compact, we have

$$\int_{M^n} C_{1,1} dV_g = (n-1) \int_{M^n} |C|^2 dV_g,$$

$$\int_{M^n} C_{2,1,1} dV_g = \int_{M^n} |C|^4 dV_g + \frac{R}{n-1} \int_{M^n} |C|^2 dV_g, \quad (4.32)$$

On the other hand, we compute $\Delta |C|^2$,

$$\Delta |C|^2 = \sum_i (E_i E_i - \nabla E_i E_i)|C|^2 = \sum_i (E_i E_i - \nabla E_i E_i) C_1^2$$

$$= E_1 E_1 (C_1^2) - \sum_i \nabla E_i E_i (C_1^2) = 2C_{1,1}^2 + 2C_1 C_{1,11} - 2(n-1) C_1^2 C_{1,1}$$

$$= 2C_{1,1}^2 + 6C_1^2 C_{1,1} - 2nC_1^4 - \frac{2R}{n-1} C_1^2.$$ 

Since the hypersurface is compact, we have

$$\int_{M^n} C_{2,1,1}^2 dV_g + 3 \int_{M^n} C_1^2 C_{1,1}^2 dV_g - n \int_{M^n} C_1^4 dV_g - \frac{R}{n-1} \int_{M^n} |C|^2 dV_g = 0.$$ 

Combining the equation (4.32), we can derive the first equation in (4.31). \qed

Lemma 4.5. Let $f : M^n \to S^{n+1}$ $(n \geq 4)$ be a compact conformally flat hypersurface without umbilical points everywhere. If the M"obius scalar curvature $R$ is constant, then

$$\int_{M^n} \{ C_{1,1}^3 + (n+5)C_1^2 C_{1,1}^2 - (2n+5)C_1^4 C_{1,1} + (n-1)C_1^6 - \frac{2R}{3} C_1^4 \} dV_g = 0. \quad (4.33)$$

Proof. Using the coefficients of the tensor $C$ and $S$, we define a smooth vector field

$$Y_S = \sum_{ijk} C_{i,j} S_{ij,k} E_k.$$
Using Lemma 4.2 and the equation (4.30), we compute the divergence of $Y_s$,

$$\frac{1}{n(n-1)}\text{div}Y_s = \sum_{ijk} C_{i,jk}S_{ij,k} + \sum_{ijk} C_{i,j}S_{ij,kk},$$

$$= C^3_{1,1} + (n + 5)C^2_{1,1} - (2n + 5)C^4_{1,1,1} + (n - 1)C^6_1 + \frac{(2n - 1)R}{n(n-1)}C^4_1 - \frac{R}{n(n-1)}C^2_{1,1,1} - \frac{2(n + 3)R}{n(n-1)}C^2_{1,1} + \frac{R^2}{n(n-1)^2}C^2_2.$$

Integrating this equation and using (4.31), we can derive the second equation in (4.33).

Lemma 4.6. Let $f : M^n \rightarrow S^{n+1}$ ($n \geq 4$) be a compact conformally flat hypersurface without umbilical points everywhere. If the M"obius scalar curvature $R$ is constant, then

(4.34)

$$\int_{M^n} \{ C^2_{1,1} + C^4_{1,1} - \frac{n}{3}C^6_1 - \frac{R}{3(n-1)}C^4_1 \} dV_g = 0,$$

$$\int_{M^n} \{ C^2_{1,1} + (n - 1)C^2_{1,1} - (2n + 1)C^4_{1,1} + (n + 1)C^6_1 + \frac{2(n + 1)R}{3(n-2)}C^4_1 \} dV_g = 0.$$

Proof. Using (4.25),

$$\frac{1}{4} \Delta |C|^4 = \frac{1}{4} \sum_i (E_iE_i - \nabla E_i E_i)|C|^4 = 3C^2_{1,1} + C^2_{1,1,1} - (n - 1)C^4_{1,1}$$

$$= 3C^2_{1,1} + 3C^4_{1,1,1} - nC^6_1 - \frac{R}{n-1}C^4_1.$$

Since the hypersurface is compact, then

$$\int_{M^n} \{ 3C^2_{1,1} + 3C^4_{1,1,1} - nC^6_1 - \frac{R}{n-1}C^4_1 \} dV_g = 0.$$

Since $S_{ij}$ is a Codazzi tensor and $tr(S)$ is constant, we can compute $\Delta |S|^2$ by (4.29),

$$\frac{1}{n^2(n-1)}\Delta |S|^2 = \frac{1}{2n^2(n-1)}\sum_{ijk} |S_{ij,k}|^2 + \frac{1}{2} \sum_{ij} (S_i - S_j)^2 R_{ijij}$$

$$= C^3_{1,1} - (2n + 1)C^4_{1,1} + (n - 1)C^2_{1,1,1} - \frac{2R}{n-1}C^4_{1,1} - \frac{2R}{n(n-1)}C_{1,1}$$

$$+ (n + 1)C^6_1 + \frac{2(n + 1)R}{n(n-1)}C^4_1 + \frac{(n + 1)R^2}{n^2(n-1)^2}C^2_2 + \frac{R^2}{n^2(n-1)^2}C^2_1.$$
Lemma 4.7. Let \( f : M^n \to \mathbb{S}^{n+1} \) \((n \geq 4)\) be a compact conformally flat hypersurface without umbilical points everywhere. If the Möbius scalar curvature \( R \) is constant. then

\[
\int_{M^n} \left\{ C_{1,1}^3 + \frac{5n+4}{2} C_{1,1}^4 C^2_{1,1} - 3(n+1) C_{1,1}^4 - \frac{n}{2} C_1^6 - \frac{(7n-10)R}{3(n-2)} C_1^4 \right\} dV_g = 0
\]

\[
\int_{M^n} \left\{ C_{1,1}^3 + \frac{5n+16}{2} C_{1,1}^4 C^2_{1,1} - 3(n-1) C_{1,1}^4 - \frac{3n}{2} C_1^6 - \frac{(7n+2)R}{6(n-1)} C_1^4 \right\} dV_g = 0.
\]

Proof. Using the coefficients of the tensor \( C \) and \( A \), we have two following smooth functions,

\[|C|_A^2 = \sum_{ij} C_i A_{ij} C_j = C_1^2 a_1, \quad |C|_C^2 = \sum_{ij} C_i C_{ij} C_j = C_1^2 C_{1,1}.\]

Next we compute \( \triangle(|C|_A^2) \) and \( \triangle(|C|_C^2). \)

\[
\frac{n-2}{2(n-1)} \triangle(|C|_A^2) = \frac{n-2}{2(n-1)} (E_1 E_1 (C_1^2 a_1) - (n-1) C_1 E_1 (C_2^1 a_1))
\]

\[= C_{1,1}^3 - 3(n+1) C_{1,1}^4 C_{1,1} + \frac{5n+4}{2} C_{1,1}^2 C_{1,1}^2 + \frac{n}{2} C_1^6 + \frac{R}{(n-1)(n-2)} C_{1,1}^4 - \frac{2n-1}{n} - \frac{(2n-1)R}{(n-1)(n-2)} C_1^4 \]

\[\frac{n-2}{2(n-1)} \triangle(|C|_C^2) = \frac{3(2n-1)}{n^2} \frac{R}{(n-1)(n-2)} C_1^2 C_{1,1} - \frac{R}{(n-1)(n-2)} C_1^2 C_{1,1} \]

\[- \frac{2n-1}{n^2} \frac{R}{(n-1)(n-2)} C_1^2 C_{1,1} \]

Integrating this equation and using (4.31), we can derive the first equation in (4.35).

\[
\frac{1}{2} \triangle(|C|_A^2) = \frac{1}{2} (E_1 E_1 (C_1^2 C_{1,1}) - (n-1) C_1 E_1 (C_1^2 C_{1,1}))
\]

\[= C_{1,1}^3 - 3(n-1) C_{1,1}^4 C_{1,1} + \frac{5n+16}{2} C_{1,1}^2 C_{1,1}^2 - \frac{3n}{2} C_1^6 - \frac{7R}{2(n-1)} C_{1,1}^4 C_{1,1} - \frac{3R}{2(n-1)} C_1^4.
\]

Integrating this equation and using (4.31), we can derive the second equation in (4.35).

Now we combine these equation system in (4.33), (4.34) and (4.35), we can derive that \( \int_{M^n} C_1^4 dV_g = 0 \), which implies that \( C_1 = 0 \) and the Möbius form vanishes. Thus we finish the proof of Theorem 1.2.
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