Constructing the CKVs of Bianchi III and V spacetimes

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Abstract

We determine the conformal algebra of Bianchi III and Bianchi V spacetimes or, equivalently, we determine all Bianchi III and Bianchi V spacetimes which admit a proper conformal Killing vector. The algorithm that we use has been developed in Class. Quantum. Grav. 15, 29 09 (1998) and concerns the computation of the CKVs of decomposable spacetimes. The main point of this method is that a decomposable space admits a CKV if the reduced space admits a gradient homothetic vector the latter being possible only if the reduced space is flat or a space of constant curvature. We apply this method in a stepwise manner starting from the two dimensional spacetime which admits an infinite number of CKVs and we construct step by step the Bianchi III and V spacetimes by assuming that CKVs survive as we increase the dimension of the space. We find that there is only one Bianchi III and one Bianchi V spacetime which admit at maximum one proper CKV. In each case we determine the conformal Killing vector and the corresponding conformal factor. As an application in the spacetimes we found we study the kinematics of the comoving observers and the dynamics of the corresponding cosmological fluid. As a second application we determine in these spacetimes generators of the Lie symmetries of the wave equation.

Keywords: Bianchi spacetimes; Conformal vector fields; Collineations; Symmetries; Lie symmetries of wave equation;

1 Introduction

A conformal Killing vector (CKV) \( X \) of a metric \( g_{ab} \) is a vector field that satisfies the condition \( L_X g_{ab} = 2\psi g_{ab} \) where \( \psi (x^r) \) is the conformal factor. The CKVs are classified as Killing Vectors (KVs) for \( \psi = 0 \); Homothetic Vectors (HVs) for \( \psi = \text{const} \); Special Conformal Killing Vectors (SCKVs) for \( \psi_{,ab} = 0 \); and proper CKVs \( \psi_{,ab} \neq 0 \).

The knowledge of the proper CKVs of a given spacetime is important because they act as geometric constraints which can be used in the study of the kinematics and the dynamics of a given spacetime. For example a CKV can be used to reduce the number of unknowns of a gravitational (or cosmological) model and also to increase the possibility of finding new solutions of Einstein’s field equations (see for example \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}, \cite{15}, \cite{16}). Furthermore the conformal algebra can be used in order to classify spaces (e.g. Finsler manifolds, pseudo-Euclidean manifolds) (see \cite{17}, \cite{18}). For example one may use the CKVs of a space in order to determine the classes of manifolds which are conformally related to the given space; or to use them in order to study the locally conformal flatness of a space around a singularity (i.e. a point \( x_0 \) where the CKV vanishes).
Apart of the above applications, another important area where the CKVs (along with the other types of collineations) are used is the geometric study of Lie symmetries of differential equations. Early studies of the geodesic equations [19], [20], [21], [22] have shown a unique connection of the Lie point symmetries for the geodesic equations in a space with the elements of the projective algebra of the space where motion occurs. In [23], [24] it has been shown that the Lie point symmetries of the dynamic equations are given by the special projective algebra of the kinetic metric and the Noether point symmetries by the Homothetic algebra of the kinetic metric. The generic symmetry vector has been presented in terms of the collineations. Similar results have been found for some partial differential equations of special interest in curved spacetimes, as the wave and the heat equation (see [25], [26], [27] and references therein).

In the present work we apply the propositions and the methodology developed in [4], [7] and [5] in order to determine all Bianchi III and V spacetimes that admit proper Conformal Killing Vectors (CKVs). The Bianchi I spacetimes which admit proper CKVs have been determined in [7]. Bianchi spacetimes are spatially homogeneous spacetimes of the general form

\[ ds^2 = -dt^2 + A^2(t)(\omega_1)^2 + B^2(t)(\omega_2)^2 + C^2(t)(\omega_3)^2 \]

where \( \omega_i, i = 1, 2, 3, \) are basis 1-forms and \( A(t), B(t), C(t) \) are functions of the time coordinate (see [1], [2], [3]). For instance,

Bianchi I: \( \omega_1 = dx, \omega_2 = dy, \omega_3 = dz \)
Bianchi III: \( \omega_1 = dx, \omega_2 = dy, \omega_3 = e^{-\xi}dz \)
Bianchi V: \( \omega_1 = dx, \omega_2 = e^\xi dy, \omega_3 = e^\xi dz. \)

In case \( B^2(t) = C^2(t) \) the Bianchi spacetimes contain a fourth isometry which is the rotation of the \( yz \) plane and reduce to the important subclass of Locally Rotational Symmetric (LRS) spacetimes (see for example [3] and citations therein).

The structure of the paper is as follows. In section 2 we briefly discuss the method we apply for the determination of the proper CKVs of a decomposable spacetime. Sections 3 and 4 contain the main calculations and citations therein. In section 5 we apply our results in the cases of Bianchi III and Bianchi V cosmological models. Section 6 contains an application of the CKVs on the theory of symmetries of differential equations. More specifically, we compute the generic Lie point symmetries of the wave equation for the Bianchi spacetimes constructed in sections 3 and 4. Finally, in section 7 we draw our conclusions.

## 2 Preliminaries

In the following we briefly discuss the algorithm developed in [4] which determines the proper CKVs of an \( n \)-dimensional decomposable Riemannian manifold with \( n \geq 3 \) in terms of the (gradient) proper CKVs of the \( (n-1) \) non-decomposable space.

In particular, it has been shown that an \( n \)-dimensional decomposable space \( M^n \) admits proper CKVs if and only if the \( (n-1) \) non-decomposable space \( M^{n-1} \) admits a gradient proper CKV whose conformal factor is the gradient factor which constructs the (gradient) CKV. In addition, any gradient proper CKV of the \( M^{n-1} \) provides two proper CKVs for the \( M^n \). Specifically the following result is shown in [4].

If \( M^n, n = 4 \) is a decomposable Riemannian manifold with line element

\[ ds^2 = \varepsilon dt^2 + h_{\mu\nu}(x^\sigma)\, dx^\mu dx^\nu, \]

where \( \varepsilon = \pm 1 \), the vector field

\[ X^\alpha \partial_\alpha = -\frac{\varepsilon}{p} \dot{\lambda}(t) \, \psi(x^\sigma)\, \partial_t + \frac{1}{p} \lambda(t) \, \xi^\mu(x^\sigma) \, \partial_\mu + L^\mu \partial_\mu, \]

is a proper CKV of [2], where

a. \( L^\mu \) is a non-gradient KV or HV of \( M^{n-1} \)

b. \( \xi^\mu(x^\sigma) \) is a gradient proper CKV of \( M^{n-1} \) with conformal factor \( \psi(x^\sigma) \) i.e. \( L_\xi h_{\mu\nu}(x^\sigma) = 2\psi(x^\sigma) \, h_{\mu\nu}(x^\sigma) \)
c. function $\lambda(t)$ is given by
\[ \lambda(t) = \lambda_1 e^{t \sqrt{\varepsilon p}} + \lambda_2 e^{-t \sqrt{\varepsilon p}}, \text{ for } \varepsilon p > 0 \]
(4)
or
\[ \lambda(t) = \lambda_3 e^{t \sqrt{\varepsilon p}} + \lambda_4 e^{-t \sqrt{\varepsilon p}}, \text{ for } \varepsilon p < 0 \]
(5)
where $p$ is a non-vanishing constant and $\lambda_1, \lambda_2$ are independent constants

provided the function $\psi(x^\sigma)$ satisfies the condition
\[ \psi_{\mu \nu} = p \psi h_{\mu \nu}. \]
(6)

Concerning the homothetic vector it has been shown in [3] that when the $M^{n-1}$ space admits a HV $H^{\mu}(x^\sigma)$ with conformal factor $C$, the $M^n$ admits the HV
\[ H^{\alpha} \partial_\alpha = C t \partial_t + H^{\mu} \partial_\mu. \]
(7)

Finally concerning the Killing vector fields it has been shown that the Killing vector fields of $M^n$ are
\[ K^a = k_0 \partial_t + k_1 h^{\mu \nu}(x^\sigma) K^I_\nu(x^\sigma) + k_2 h^{\mu \nu}(x^\sigma) S^I_\nu(x^\sigma) + k_3 (\varepsilon S^I(x^\sigma) \partial_t + h^{\mu \nu}(x^\sigma) S^I_\mu(x^\sigma)), \]
(8)
where $K^I_\nu(x^\sigma)$ are the non-gradient KVs of $M^{n-1}$ and $S^I_\nu(x^\sigma)$ are the gradient KVs of $M^{n-1}$. Finally, $k_0, k_1, k_2, k_3$ are independent constants.

However another possibility that the $M^n$ space [2] admits proper CKVs is when it is conformally flat. That case was found to be important in the classification of Bianchi I spacetimes in [5] according to the admitted CKVs, but it does not provide any result in the case of Bianchi III and Bianchi V spacetimes, thus we omit it from the present discussion.

The concept of conformally related metrics plays a crucial role in the computation of the CKVs in the following sections. Two metrics $\hat{g}_{ab}, g_{ab}$ are said to be conformally related iff there is a function $N^2(x^\nu)$ such that $\hat{g}_{ab} = N^2(x^\nu)g_{ab}$. The conformally related metrics share the same conformal algebra but with different conformal factors. For a given vector field $X$ we have the decompositions/identities
\[ L_X \hat{g}_{ab} = 2\hat{\psi}(X) \hat{g}_{ab} + 2\hat{H}_{ab}(X) \]
and
\[ L_X g_{ab} = 2\psi(X) g_{ab} + 2H_{ab}(X). \]

where $H_{ab}(X), \hat{H}_{ab}(X)$ are symmetric traceless tensors. Then it can be shown that
\[ \hat{\psi}(X) = \psi(X) \ln N + \psi(X), \quad \hat{H}_{ab}(X) = N^2 H_{ab}(X) \]
and
\[ \hat{F}_{ab}(X) = N^2 F_{ab}(X) - 2NN_{[a}X_{b]} \]
where $\hat{F}_{ab}(X) = \hat{X}_{[a;b]} = \hat{X}_{[a;b]}$ and $F_{ab}(X) = X_{[a;b]}$. Moreover
\[ \hat{X}_{a;b} = \frac{1}{2} L_X \hat{g}_{ab} + \hat{F}_{ab}(X) \]
\[ X_{a;b} = \frac{1}{2} L_X g_{ab} + F_{ab}(X). \]

A metric $g_{ab}$ is called conformally flat iff it is conformal to the flat metric $\eta_{ab}$. A metric conformally related to a conformally flat metric is also conformally flat. It is well known that all the 2d-spacetimes are conformally flat and admit an infinity number of CKVs while only the flat 2d-metrics admit special CKVs.

The flat $n$-dimensional metric $\eta_{ab}$ admits an algebra of $n(n+1)/2$ CKVs which consists of $n + n(n-1)/2 = n(n+1)/2$ KVs, one HV and $n$ proper SCVs. The generic (not proper!) SCKV of a flat metric is given by the formula
\[ \chi^a \partial_a = \alpha^a P_a + \alpha^{BA} r_{AB} + \beta H + 2\beta^a K_a \]
(9)
with conformal factor $\psi = \beta + 2\beta_2 x^a$ and integration constants $\alpha, \beta, \alpha^a, \beta^a$, $\alpha_{ab} = -\alpha_{ba}$. In this expression $P_a$ are the $n$ KVs (translations) $r_{AB}$ are $n(n-1)/2$ KVs (rotations), $H$ is the HV (dilatation) and $K_a$ are the $n$ proper SCVs. The summation over $A, B$ satisfies the condition $1 \leq A < B \leq n$.

In a coordinate system in which the metric has its reduced form $\eta_{ab} = \text{diag}(-1,...,-1, +1,..., +1)$ the above vectors are given by the following expressions $P_a = \delta^a_{[b} \partial_{c]}$, $r_{ab} = 2\delta^a_{[a} \delta^b_{c]} x^c$, $\partial_a = x^a \partial_a$ and $K_a = \left(x_a x^b - \frac{1}{2} \delta_a^b x^c x^c\right) \partial_b = x_a H - \frac{1}{2} (x_b x^b) P_a$. These vector fields span the conformal algebra of the flat space $\eta_{ij}$.
3 CKVs of Bianchi III spacetime

Consider the three-dimensional decomposable spacetime of Lorentzian signature

\[ ds^2_{(1+2)} = \Gamma^2 (\tau) (-d\tau^2 + dx^2) + dy^2. \] (10)

The line element \((10)\) for arbitrary \(\Gamma (\tau)\) admits a two-dimensional conformal Killing algebra consisting by the KVs \(\partial_y\) and \(\partial_x\).

For the conformal spacetime

\[ d\tilde{s}^2_{(1+2)} = B^2 (\tau) e^{2x} ds^2_{(1+2)}, \] (11)

the vector field \(\partial_y\) remains a KV but \(\partial_x\) now becomes a proper HV.

Consider now the four-dimensional decomposable spacetime

\[ ds^2_{(1+3)} = d\tilde{s}^2_{(1+2)} + dz^2, \] (12)

which admits a three-dimensional conformal algebra consisting of the KVs \(\partial_y\), \(\partial_z\) and the proper HV \(\partial_x + z\partial_z\). Then, the conformally related spacetime \(ds^2_{(III)} = A^2 (\tau) e^{-2x} ds^2_{(1+3)}\) which can be written equivalently

\[ ds^2_{(III)} = -dt^2 + \alpha^2 (t) dx^2 + \beta^2 (t) dy^2 + \gamma^2 (t) e^{-2x} dz^2 \] (13)

is a Bianchi III spacetime and the vector fields \(\partial_y\), \(\partial_z\), \(\partial_x + z\partial_z\) form the Killing algebra of \((13)\). Therefore, in order the Bianchi III \((13)\) to admit greater conformal algebra the functions \(\alpha (t)\), \(\beta (t)\) and \(\gamma (t)\) must be specified. Recall that when \(\alpha (t) = \gamma (t)\) spacetime \((13)\) is locally rotational and admits as extra KV the rotation in the two dimensional space \(ds^2 = dx^2 + e^{-2x} dz^2\).

The three-dimensional space \((10)\) admits a greater conformal algebra for specific functions \(\Gamma (\tau)\). From the discussion of Section 2 it follows that \(\Gamma (\tau)\) must be such that the two-dimensional space

\[ ds^2_{(2)} = \Gamma^2 (\tau) (-d\tau^2 + dx^2), \] (14)

admits proper gradient CKVs or a greater Killing algebra. For two-dimensional spaces it is well-known that the admitted KVs can be zero, one or three and in the latter case the space is maximally symmetric. Since \((13)\) admits always the KV \(\partial_x\), the \(\Gamma (\tau)\) must be specified so that \((13)\) is maximally symmetric. Without loss of generality we can select \(\Gamma^2 (\tau) = e^{mt}\) in which case \((13)\) is the flat space with Ricci Scalar \(R_{(2)} = 0\), or \(\Gamma^2 (\tau) = \kappa^{-2} \cos^{-2} (\tau)\) in which case \(R_{(2)} = 2\kappa^2\).

Furthermore, all the two-dimensional spaces admit infinity CKVs, however, the requirement that at least one of the proper CKVs is to be gradient, specifies the spacetime to be of nonzero constant curvature, which is a maximally symmetric space and it admits five gradient proper CKVs.

3.1 Case \(\Gamma^2 (\tau) = e^{mt}\)

In case \(\Gamma^2 (\tau) = e^{mt}\) the three-dimensional space

\[ ds^2_{(1+2)} = e^{mt} (-d\tau^2 + dx^2) + dy^2. \] (15)

is flat and admits a ten-dimensional conformal algebra. This algebra consists of the six KVs

\[
\begin{align*}
Y_1 &= \frac{2}{m} e^{-\frac{m}{2} (\tau-x)} \partial_x - \frac{2}{m} e^{-\frac{m}{2} (\tau-x)} \partial_x \\
Y_2 &= -\frac{2}{m} e^{-\frac{m}{2} (\tau+x)} \partial_x - \frac{2}{m} e^{-\frac{m}{2} (\tau+x)} \partial_x \\
Y_3 &= \partial_x, \quad Y_4 = \partial_y \\
Y_5 &= ye^{-\frac{m}{2} (\tau+x)} \partial_x + ye^{-\frac{m}{2} (\tau+x)} \partial_x + \frac{2}{m} e^{\frac{m}{2} (\tau-x)} \partial_y
\end{align*}
\]

\footnote{Where \(\alpha^2 (t) = A^2 (t) B^2 (t) \Gamma^2 (t), \beta^2 (t) = A^2 (t) B^2 (t)\) and \(\gamma^2 (t) = A^2 (t)\) while \(t = \int a (\tau) d\tau\).}
\[ \mathbf{Y}_6 = -y e^{-\frac{Y}{2}(r-x)} \partial_r + y e^{-\frac{Y}{2}(r-x)} \partial_x - \frac{2}{m} e^{\frac{Y}{2}(r+x)} \partial_y \]

the HV

\[ \mathbf{Y}_7 = \frac{2}{m} \partial_r + y \partial_y, \quad \psi_{(1+2)}(\mathbf{Y}_7) = 1 \]

and the three special CKVs

\[ \mathbf{Y}_8 = \left[ \frac{2}{m^2} e^{\frac{Y}{2}(r-x)} + \frac{y^2}{2} e^{\frac{Y}{2}(r+x)} \right] \partial_r + \left[ -\frac{2}{m^2} e^{\frac{Y}{2}(r-x)} + \frac{y^2}{2} e^{\frac{Y}{2}(r+x)} \right] \partial_x + \frac{2y}{m} e^{\frac{Y}{2}(r-x)} \partial_y \]

\[ \mathbf{Y}_9 = -\left[ \frac{2}{m^2} e^{\frac{Y}{2}(r+x)} + \frac{y^2}{2} e^{-\frac{Y}{2}(r-x)} \right] \partial_r + \left[ -\frac{2}{m^2} e^{\frac{Y}{2}(r+x)} + \frac{y^2}{2} e^{-\frac{Y}{2}(r-x)} \right] \partial_x - \frac{2y}{m} e^{\frac{Y}{2}(r+x)} \partial_y \]

\[ \mathbf{Y}_{10} = my \partial_r + \left[ \frac{m^2 y^2}{4} + e^{mt} \right] \partial_y \]

with conformal factors \( \psi_{(1+2)}(\mathbf{Y}_8) = \frac{2}{m} e^{\frac{Y}{2}(r-x)}, \quad \psi_{(1+2)}(\mathbf{Y}_9) = -\frac{2}{m} e^{\frac{Y}{2}(r+x)}, \) and \( \psi_{(1+2)}(\mathbf{Y}_{10}) = \frac{m^2 y^2}{2} \) respectively.

The conformally flat space

\[ ds^2_{(1+2)} = B^2(\tau) e^{2x} \left[ e^{mt} \left( -dr^2 + dx^2 \right) + dy^2 \right] \]

(16)

admits the same elements of the conformal algebra with (15) but with different conformal factors \( \tilde{\psi}_{(1+2)} \). More specifically it follows that

\[ \tilde{\psi}_{(1+2)}(\mathbf{Y}_A) = \mathbf{Y}_A \left[ \ln(B(\tau)) \right] + \psi_{(1+2)}(\mathbf{Y}_A). \]

(17)

When we impose condition (6) we find that there does not exist function \( B(\tau) \) such that \( \tilde{\psi}_{(1+2)}(\mathbf{Y}_A) \) to satisfy (6). On the other hand, we observe that for

\[ B(\tau) = e^{\mu \tau}, \quad \mu = \frac{m(\lambda - 1)}{2} \]

(18)

it follows \( \tilde{\psi}_{(1+2)}(\mathbf{Y}_7) = \lambda \), which means that \( \mathbf{Y}_7 \) is reduced to a HV for (16). At this point it is important to mention that \( \tilde{\psi}_{1+2}(\mathbf{Y}_3) = 1 \), however there is only one proper HV and not two, as expected. We assume \( \mathbf{Y}_7 \) to be the proper HV and \( \mathbf{Y}_3 - \frac{1}{\lambda} \mathbf{Y}_7 \) to be a KV.

For the four-dimensional decomposable spacetime

\[ ds^2_{(1+3)} = e^{2x} e^{2\mu \tau} \left[ e^{mt} \left( -dr^2 + dx^2 \right) + dy^2 \right] + dz^2, \]

(19)

from \( \mathbf{Y}_7 \) we find the proper HV

\[ \mathbf{L}_1 \equiv \mathbf{Y}_7 + \lambda z \partial_z = \frac{2}{m} \partial_r + y \partial_y + \lambda z \partial_z. \]

(20)

We conclude that the Bianchi III spacetime

\[ ds^2_{(111)} = e^{m \lambda \tau} A^2(\tau) \left( -dr^2 + dx^2 + e^{-m \tau} dy^2 + e^{-m \lambda \tau} e^{-2x} dz^2 \right), \]

(21)

admits the proper CKV \( \mathbf{L}_1 \) with conformal factor \( \psi_{(111)}(\mathbf{L}_1) = \frac{2}{m \lambda} + \lambda \) which reduces to a HV when \( A(\tau) \) is an exponential in which case the line element is:

\[ ds^2_{(111)} = -e^{m \kappa \tau} dr^2 + e^{m \kappa \tau} dx^2 + e^{m(\kappa - 1) \tau} dy^2 + e^{m(\kappa - \lambda) \tau} e^{-2x} dz^2, \]

(22)

or in equivalently form

\[ ds^2_{(111)} = -dt^2 + \frac{m^2 \kappa^2 l^2}{4} dx^2 + \left( \frac{m^2 \kappa^2 l^2}{4} \right) e^{2x} dy^2 + \left( \frac{m^2 \kappa^2 l^2}{4} \right) e^{-2x} dz^2. \]

(23)

where now we write \( \mathbf{L}_1 = \kappa \partial_t + y \partial_y + \lambda z \partial_z \) with \( \psi_{(111)}(\mathbf{L}_1) = \text{const} \equiv \kappa \neq 0 \), recall that \( dt = e^{\frac{Y}{2}(r-x)} d\tau \).

Performing the same analysis for the second case of \( \Gamma^2(\tau) = \kappa^{-2} \cos^{-2}(\tau) \) we find that the resulting Bianchi III spacetime does not admit any proper CKV or a proper HV, hence we omit the presentation of this analysis.

We summarize our results in the following proposition
In this section we study some of the physical properties of spacetime for the comoving observers.

5.1 Bianchi III cosmological fluid

Proposition 1 The only Bianchi III spacetime which admits a proper CKV is

\[ ds^2 = A^2(\tau) \left[ e^{m\lambda^+}(\tau) (-dt^2 + dx^2) + e^{m(\lambda-1)^+} (dy^2 + e^{-2x}dz^2) \right] \]  

The CKV is \( L_1 = \frac{\omega}{m} \partial_r + y \partial_y + \lambda z \partial_z \) and has conformal factor \( \psi_{(III)}(L_1) = \frac{\omega}{m} + \lambda \), where \( A(\tau) \) is an arbitrary function.

4 Bianchi V spacetimes which admit a CKV

For the computation of the CKVs for the Bianchi V spacetime we apply the same procedure with Section 3, but for this case we start from the two-dimensional spacetime

\[ ds^2_{(2)} = \Gamma^2(\tau) e^{-2x} (-dt^2 + dx^2) . \]  

The latter space is maximally symmetric only for \( \Gamma^2(\tau) = e^{\tau} \) where the Ricci Scalar is calculated to be \( R_{(2)} = 0. \) It is important to mention that there is not any function \( \Gamma(\tau) \) where space \( (25) \) is of constant curvature.

We omit the intermediary calculations and we summarize the results in the following proposition

Proposition 2 The Bianchi V spacetime

\[ ds^2 = A^2(\tau) [\Gamma^2(\tau) (-dt^2 + dx^2) + e^{2x} (B^2(\tau) dy^2 + dz^2)] \]  

admits the unique proper CKV \( L_1 = \frac{\omega}{m} \partial_r + y \partial_y + \lambda z \partial_z \) with \( \psi_{(V)}(L_1) = \frac{\omega}{m} + \lambda \) only when \( \Gamma^2(\tau) = e^{m\lambda^+}, B^2(\tau) = e^{m(\lambda-1)^+} \). For \( A^2(\tau) = e^{m(\kappa-\lambda)^+} \) the CKV reduces to a HV with homothetic \( \psi_{(V)}(L_1) = \) const = \( \kappa \neq 0 \).

5 Applications

5.1 Bianchi III cosmological fluid

In this section we study some of the physical properties of spacetime \( (20) \). For the comoving observers \( u^a = \frac{-m \theta}{A(\tau)^2} \delta^a_\tau, u^a u_a = -1 \). As it is well known (see e.g. \( (28) \)) the four velocity of a class of observers introduces the 1+3 decomposition of tensor fields in spacetime. The decomposition of \( u_{a:b} \) gives the kinematic quantities \( \theta, \sigma^2, \omega^2 \) and \( \alpha \) defined by the identity

\[ u_{a:b} = -\alpha_a u_b + \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} \]  

(27)

where \( \alpha^a = \dot{u}^a = u^a_{,\tau} u^b, \omega_{ab} = h^c_{\ a} h^d_{\ b} u_{(c:d)}, \sigma_{ab} = (h^c_{\ a} h^d_{\ b} - \frac{1}{3} h_{cd} h_{ab}) u_{(c:d)}, \theta = h_{ab} u_{a:b} = u^a_{,b}, \sigma^2 = \frac{1}{3} \sigma_{ab} \sigma_{ab}, \omega^2 = \frac{1}{2} \omega_{ab} \omega_{ab} \). Similarly the 1+3 decomposition of the Einstein tensor \( G_{ab} \) defines the dynamic quantities by the identity

\[ G_{ab} = \rho u_a u_b + 2q(a u_b) + \pi_{ab} \]  

(28)

where \( \rho = G_{ab} u^a u^b, p = \frac{1}{2} h_{ab} G_{ab}, q^a = -h^a c G_{cd} u^d \) and \( \pi_{ab} = (h^c_{\ a} h^d_{\ b} - \frac{1}{3} h_{cd} h_{ab}) G_{cd} \).

Applying the above for the comoving observers in Bianchi III spacetime \( (21) \) we compute that the kinematic quantities \( \omega^2 = 0, \alpha^a = 0 \) while

\[ \theta = e^{-\frac{m \lambda^+}{A}} \left[ 3 \frac{d(ln A)}{d\tau} + \frac{m(2\lambda - 1)}{2} \right] \]  

(29)

and

\[ \sigma^2 = \frac{m^2(\lambda^2 - \lambda + 1)}{12} e^{-m\lambda^+} \]  

(30)

Similarly for the dynamic qualities we find that the (non-zero) components for the cosmological fluid defined by the Bianchi III spacetime \( (20) \) are:

\[ \rho = \frac{e^{-m\lambda^+}}{4A^2} \left[ \frac{d(ln A)}{d\tau} \left( 3 \frac{d(ln A)}{d\tau} + m(2\lambda - 1) \right) + m^2 \lambda(\lambda - 1) - 4 \right] \]  

(31)
\[ p = \frac{e^{-m\lambda\tau}}{A^2} \left[ -\frac{2}{A} \frac{d^2 A}{d\tau^2} + \frac{d (\ln A)}{A d\tau} \left( \frac{d (\ln A)}{d\tau} + \frac{m}{3} (2 - \lambda) \right) - \frac{m^2}{12} (\lambda - 1) (\lambda - 2) + \frac{1}{3} \right], \quad (32) \]

\[ q^a = \left( 0, \frac{m\lambda e^{-\frac{3m\lambda\tau}{2}}}{2A^3}, 0, 0 \right) \]

\[ \pi_{xx} = \frac{m(\lambda + 1)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2(\lambda^2 - 1)}{12} - \frac{1}{3}, \quad (34) \]

\[ \pi_{yy} = e^{-m\tau} \left[ \frac{m(\lambda - 2)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2}{12} (\lambda - 1)(\lambda - 2) + \frac{2}{3} \right], \quad (35) \]

and

\[ \pi_{zz} = -e^{-m\lambda\tau - 2x} \left[ \frac{m(2\lambda - 1)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2}{12} (\lambda - 1)(2\lambda - 1) + \frac{1}{3} \right]. \quad (36) \]

In the case \( A^2(\tau) = e^{m(\kappa - \lambda)\tau} \), where the CKV \( L_1 \) becomes a HV, the above non-zero quantities are simplified as follows

\[ \theta = \frac{m(3\kappa - \lambda - 1)}{2} e^{-\frac{m\lambda\tau}{2}}, \quad (37) \]

\[ \sigma^2 = \frac{m^2(\lambda^2 - 1)}{12} e^{-m\kappa\tau}, \quad (38) \]

\[ \rho = \rho_0 (m, \kappa, \lambda) e^{-m\kappa\tau}, \quad p = p_0 (m, \kappa, \lambda) e^{-m\kappa\tau}, \quad (39) \]

\[ q^a = \left( 0, \frac{m\lambda e^{-\frac{3m\lambda\tau}{2}}}{2A^3}, 0, 0 \right), \quad (40) \]

\[ \pi_{xx} = \pi_{xx0} (m, \kappa, \lambda), \quad \pi_{yy} = \pi_{yy0} (m, \kappa, \lambda) e^{-m\kappa\tau}, \quad \pi_{zz} = \pi_{zz0} (m, \kappa, \lambda) e^{-m\lambda\tau - 2x}. \quad (41) \]

From the latter expressions we infer that for large \( \tau \) and \( m\kappa > 0 \) all the kinematical quantities, the mass density, the isotropic pressure and the heat flux vector vanish. If, in addition, \( \pi_{xx0} (m, \kappa, \lambda) = 0, m > 0 \) and \( \lambda > 0 \), then for large \( \tau \) the fluid source vanishes and the solution describes an isotropic empty spacetime.

### 5.2 Bianchi V cosmological fluid

We consider the extended Bianchi V spacetime of proposition\[2\]

\[ ds^2_{(V)} = -A^2(\tau) e^{m\lambda\tau} d\tau^2 + A^2(\tau) e^{m\lambda\tau} dx^2 + A^2(\tau) e^{m(\lambda-1)\tau} e^{2\tau} dy^2 + A^2(\tau) e^{2\tau} dz^2 \quad (42) \]

and repeat the calculations for the comoving observers. We find that the kinematic quantities are exactly the same with those of the Bianchi III spacetime while the dynamic (non-zero) dynamic variables of the cosmological fluid are

\[ \rho = \frac{e^{-m\lambda\tau}}{4A^2} \left[ 3 \frac{d (\ln A)}{d\tau} \left( \frac{d (\ln A)}{d\tau} + m(2\lambda - 1) \right) + m^2\lambda(\lambda - 1) - 12 \right], \quad (43) \]

\[ p = \frac{e^{-m\lambda\tau}}{A^2} \left[ -\frac{2}{A} \frac{d^2 A}{d\tau^2} + \frac{d (\ln A)}{A d\tau} \left( \frac{d (\ln A)}{d\tau} + \frac{m}{3} (2 - \lambda) \right) - \frac{m^2}{12} (\lambda - 1)(\lambda - 2) + 1 \right], \quad (44) \]

\[ q^a = \left( 0, -\frac{m(\lambda + 1) e^{-\frac{3m\lambda\tau}{2}}}{2A^3}, 0, 0 \right) \]

\[ \pi_{xx} = \frac{m(\lambda + 1)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2(\lambda^2 - 1)}{12}, \quad (46) \]

\[ \pi_{yy} = e^{-m\tau + 2x} \left[ \frac{m(\lambda - 2)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2}{12} (\lambda - 1)(\lambda - 2) \right], \quad (47) \]

and

\[ \pi_{zz} = -e^{-m\lambda\tau + 2x} \left[ \frac{m(2\lambda - 1)}{3} \frac{d (\ln A)}{d\tau} + \frac{m^2}{12} (\lambda - 1)(2\lambda - 1) \right]. \quad (48) \]

In the case \( L_1 \) is a HV we deduce the same conclusions with the Bianchi III case of section\[5.3\]
6 Lie point symmetries of the wave equation

Collineations of spacetimes can be used to construct symmetries and conservation laws for some differential equations defined in curved spacetimes. In [29] it has been shown that there exists a unique connection between the Noether symmetries for the geodesic Lagrangian of a given Riemannian space and the elements of the admitted homothetic algebra. Similar results have been shown for other partial differential equations of special interest [30, 31].

In this work we consider the wave equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} u (x^\lambda) \right) = 0,$$

(49)

in the Bianchi III spacetime (21) and in the Bianchi V spacetime (26) and determine its Lie symmetries. By following the generic results of [31], we find that the admitted Lie point symmetries for the line element (1), are 3 + 1 + infinity, where 3 are the admitted KVs, and 1 + infinity are the vector fields $Y_u = u \partial_u$, $Y_\infty = b (x^\nu) \partial_u$ where $b (x^\nu)$ is a solution of the original equation (49). The latter symmetry vector fields exists because equation (49) is a linear partial differential equation.

Thus, for a greater dimensional conformal algebra, equation (49) admits extra Lie point symmetries. Indeed, from our analysis and for the case where the Bianchi III and Bianchi V spacetimes admit a proper HV the wave equation becomes

$$(-u_{tt} + u_{xx} + u_{yy} + e^{m\lambda t + 2x} u_{zz} \right) + \frac{m}{2} (\lambda - 2\kappa + 1) u_t - u_x = 0,$$

or

$$(-u_{tt} + u_{xx} + e^{m\lambda t - 2x} u_{yy} + e^{m\lambda t - 2x} u_{zz} \right) + \frac{m}{2} (\lambda + 2\kappa - 1) u_t + 4u_x = 0.$$

(50)

Then we find that equation (50) admits the generic Lie point symmetry vector

$$Y_{III} = \left( \frac{a_1}{m} \right) \partial_t + a_2 \partial_x + (a_1 y + a_3) \partial_y + (a_1 \lambda z + a_2 z + a_4) \partial_z + (a_u u + a_\infty b (t, x, y, z)) \partial_u,$$

(52)

while equation (51) is invariant under the one parameter point transformation with generator

$$Y_V = \left( \frac{a_1}{m} \right) \partial_t + a_2 \partial_x + (a_1 y - a_1 y + a_3) \partial_y + (a_1 \lambda z - a_2 z + a_4) \partial_z + (a_u u + a_\infty b (t, x, y, z)) \partial_u.$$

(53)

The latter symmetry vectors can be applied to construct conservation laws or similarity solutions for the wave equation. Such an analysis is beyond the scope of this work and we omit it.

7 Conclusion

In this paper we have shown that there is only one type of Bianchi III and Bianchi V spacetime given respectively in (21) and (26) which admit a single proper CKV. In order to prove that we apply an algorithm which relates the CKVs of decomposable spacetimes with the collineations of the non-decomposable subspace. The kinematics of the comoving fluid of observers in all these four spacetimes is not accelerating and rotating and has only expansion and shear a result compatible with the anisotropy of the Bianchi spacetimes. Concerning the dynamics it has been shown that the fluid of these observers is heat conducting and anisotropic, that is it is a general fluid. Finally we have used the conformal vectors we found in each case in order to determine the generators of the Lie symmetries of the wave equation in the Bianchi III spacetime (21) and in the Bianchi V spacetime (26).

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