COBOUNDARY DYNAMICAL POISSON GROUPOIDS AND INTEGRABLE SYSTEMS

LUEN-CHAU LI

Abstract. In this paper, we present a general scheme to construct integrable systems based on realization in the coboundary dynamical Poisson groupoids of Etingof and Varchenko. We also present a factorization method for solving the Hamiltonian flows. To illustrate our scheme and factorization theory, we consider a family of hyperbolic spin Ruijsenaars-Schneider models related to affine Toda field theories and solve the equations of motion in a simple case.

1. Introduction.

Many important examples of integrable systems are related to Lie groups and Lie algebras and can be studied using a well-known group-theoretic scheme (see, for example, [A], [K], [S], [RSTS1], [RSTS2], [AvM], [STS1],[STS2], [RSTS3], [FT], [LP1] and the references therein). However, as the author has come to realize in the last few years, Lie algebroids and Lie groupoids are also of relevance [LX1], [LX2], [L1], [HM]. More precisely, the requisite objects are connected with the so-called classical dynamical r-matrices (or generalizations), which first appeared in the context of Wess-Zumino-Witten (WZW) conformal field theory [BDF], [F]. While classical r-matrices play a role in Poisson Lie group theory [D], the authors in [EV] showed that an appropriate geometric setting for the classical dynamical r-matrices is that of a special class of Poisson groupoids (a notion due to Weinstein [W]), the so-called coboundary dynamical Poisson groupoids. If $R$ is an $H$-equivariant classical dynamical r-matrix, and $(\Gamma, \{\cdot, \cdot\}_R)$ is the associated coboundary dynamical Poisson groupoid, then it follows from Weinstein’s coisotropic calculus [W] or otherwise that the Lie algebroid dual $A^*\Gamma$ also has a natural Lie algebroid structure [LP2], [BKS]. We shall call $A^*\Gamma$ the coboundary dynamical Lie algebroid associated to $R$ and it is this class of Lie algebroids which provides the natural setting in [LX1], [LX2].

At this point, it is convenient to summarize a few characteristics of the class of invariant Hamiltonian systems which admit either a realization in the dual vector...
bundle $A^*\Gamma$ of $A^*\Gamma$ \cite{LX1}, \cite{LX2}, or in the coboundary dynamical Poisson groupoid $\Gamma$ itself \cite{L1}: (a) the systems are defined on a Hamiltonian $H$-space $X$ with equivariant momentum map $J$ and the Hamiltonians are the pull-back of natural invariant functions by an $H$-equivariant realization map, (b) the pullback of natural invariant functions do not Poisson commute everywhere on $X$, but they do so on a fiber $J^{-1}(\mu)$ of the momentum map, and (c) the reduced Hamiltonian systems on $X_\mu = J^{-1}(\mu)/H_\mu$ ($H_\mu$ is the isotropy subgroup at $\mu$) admit a natural collection of Poisson commuting integrals.

We should emphasize that while the reduced systems are the goal, the unreduced ones are the key players in the analysis. Clearly, the geometric objects $\Gamma$ and $A^*\Gamma$ encode the hidden symmetries of the Hamiltonian systems and this is one of their virtues. Now, while this is true for any $\Gamma$ or $A^*\Gamma$ regardless of the underlying $r$-matrix, we shall focus on those which correspond to the solutions of the modified dynamical Yang-Baxter equation (mDYBE). As the reader will see in the present work, the (mDYBE) is connected with a factorization problem on the trivial Lie groupoid $\Gamma$. Moreover, there is a prescription to integrate the Hamiltonian flows on $J^{-1}(\mu)$ explicitly based on this factorization. Hence, we can obtain the induced flows on $X_\mu$ by reduction. Surely, this is the most important aspect of our theory, and one in which the criticality of the Lie groupoid $\Gamma$ is clearly demonstrated.

Our objective in this paper is to present some of the essentials of the factorization theory which we mentioned above. Since several constructions are more transparent for the coboundary dynamical Poisson groupoids, we have decided in this first publication to present our theory for this particular case (see \cite{L2} and \cite{L3} for the algebroid case). However, it is clear that several basic results (Prop.4.4, Thm.4.6 and Corollary 4.8) are common to both frameworks.

The paper is organized as follows. In Section 2 and 3, we present a general scheme to construct integrable systems based on realization in coboundary dynamical Poisson groupoids. In the case of a constant $r$-matrix, our theory here is a slight extension of the standard $r$-matrix framework for Lax equations on Lie groups \cite{STS1}, \cite{STS2} (see Theorem 2.5). In Section 4, we discuss the algebraic and geometric structures associated with the (mDYBE), leading up to a factorization method for solving the Hamiltonian flows. As an illustration of our theory in these sections, we introduce a family of hyperbolic spin Ruijsenaars-Schneider (RS) models in Section 5 and solve the equations of motion in a simple case. Remarkably, one of the models in the $SL(N+1,\mathbb{C})$ case has been shown to govern the soliton dynamics of the so-called $A_N^{(1)}$ affine Toda field theory \cite{BH}. We plan to study this
family of spin RS models as well as others in subsequent publications.

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### 2. Realization of Hamiltonian systems in coboundary dynamical Poisson groupoids.

Let $G$ be a connected Lie group and $H \subset G$ a connected Lie subgroup. We shall denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to $G$ and $H$ respectively and let $\iota : \mathfrak{h} \rightarrow \mathfrak{g}$ be the Lie inclusion.

We begin by recalling a fundamental construction of [EV] using the formulation in [LP2]. Let $U \subset \mathfrak{h}^*$ be a connected $Ad^*_H$-invariant open subset, we say that a smooth map $R : U \rightarrow L(\mathfrak{g}^*, \mathfrak{g})$ (here and henceforth we denote by $L(\mathfrak{g}^*, \mathfrak{g})$ the set of linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$) is a classical dynamical $r$-matrix if and only if it is pointwise skew-symmetric:

$$< R(q)(A), B > = - < A, R(q)B >$$  \tag{2.1}

and satisfies the classical dynamical Yang-Baxter condition

$$[R(q)A, R(q)B] + R(q)(ad^*_{R(q)}A B - ad^*_{R(q)}B A) + dR(q)\iota ^*A(B) - dR(q)\iota ^*B(A) + d < R(A), B > (q) = \chi (A, B),$$  \tag{2.2}

where $\chi : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}$ is $G$-equivariant, that is,

$$\chi (Ad^*_{g^{-1}}A, Ad^*_{g^{-1}}B) = Ad_g \chi (A, B)$$  \tag{2.3}

for all $A, B \in \mathfrak{g}^*$, $g \in G$, and all $q \in U$.

The dynamical $r$-matrix is said to be $H$-equivariant if and only if

$$R(Ad^{*}_{h^{-1}}q) = Ad_h \circ R(q) \circ Ad^{*}_{h}$$  \tag{2.4}

for all $h \in H, q \in U$. We shall equip $\Gamma = U \times G \times U$ with the trivial Lie groupoid structure over $U$ [M] with target and source maps

$$\alpha (u, g, v) = u, \quad \beta (u, g, v) = v$$  \tag{2.5}
and multiplication map
\[ m((u, g, v), (v, g', w)) = (u, gg', w). \] (2.6)

For \( \varphi \in C^\infty(\Gamma) \), we define its partial derivatives and its left and right gradients (with respect to \( G \)) by
\[
<\delta_1 \varphi, u'> = \frac{d}{dt} \bigg|_{t=0} \varphi(u + tu', g, v), \quad <\delta_2 \varphi, v'> = \frac{d}{dt} \bigg|_{t=0} \varphi(u, g, v + tv'), \quad u', v' \in \mathfrak{h}^*
\]
\[
<D \varphi, X> = \frac{d}{dt} \bigg|_{t=0} \varphi(u, e^{tx} g, v), \quad <D' \varphi, X> = \frac{d}{dt} \bigg|_{t=0} \varphi(u, ge^{tx}, v), \quad X \in \mathfrak{g}.
\]

**Theorem 2.1 [EV].**

(a) The bracket
\[
\{\varphi, \psi\}_R(u, g, v) = <u, [\delta_1 \varphi, \delta_1 \psi]> - <v, [\delta_2 \varphi, \delta_2 \psi]>
\]
\[
- <i\delta_1 \varphi, D\psi> - <i\delta_2 \varphi, D'\psi>
\]
\[
+ <i\delta_1 \psi, D\varphi> + <i\delta_2 \psi, D'\varphi>
\]
\[
+ <R(v)D'\varphi, D'\psi> - <R(u)D\varphi, D\psi>
\]

defines a Poisson structure on \( \Gamma \) if and only if \( R: U \rightarrow L(\mathfrak{g}^*, \mathfrak{g}) \) is an \( H \)-equivariant dynamical \( r \)-matrix.

(b) The trivial Lie groupoid \( \Gamma \) equipped with the Poisson bracket \( \{\cdot, \cdot\}_R \) is a Poisson groupoid.

We shall call the pair \((\Gamma, \{\cdot, \cdot\}_R)\) the coboundary dynamical Poisson groupoid associated to \( R \).

**Definition 2.2.** A Poisson manifold \((X, \{\cdot, \cdot\}_X)\) is said to admit a realization in the coboundary dynamical Poisson groupoid \((\Gamma, \{\cdot, \cdot\}_R)\) if there is a Poisson map \( \rho: (X, \{\cdot, \cdot\}_X) \rightarrow (\Gamma, \{\cdot, \cdot\}_R) \).

**Definition 2.3.** Suppose a Poisson manifold \((X, \{\cdot, \cdot\}_X)\) admits a realization \( \rho: (X, \{\cdot, \cdot\}_X) \rightarrow (\Gamma, \{\cdot, \cdot\}_R) \) and \( \mathcal{H} \in C^\infty(X) \). The map \( \rho \) is said to give a realization of the Hamiltonian system \( \dot{x} = X_{\mathcal{H}}(x) \) in \( \Gamma \) if there exists \( \varphi \in C^\infty(\Gamma) \) such that \( \mathcal{H} = \rho^* \varphi \).

In what follows, we shall work with a Poisson manifold \((X, \{\cdot, \cdot\}_X)\) together with a realization \( \rho: X \rightarrow \Gamma \). Let \( Pr_2: \Gamma \rightarrow G \) be the projection map onto the second factor of \( \Gamma \) and set
\[ L = Pr_2 \circ \rho : X \to G, \quad (2.8) \]

\[ m_1 = \alpha \circ \rho : X \to U, \quad (2.9) \]

and

\[ m_2 = \beta \circ \rho : X \to U, \quad (2.10) \]

i.e. \( \rho = (m_1, L, m_2) \).

**Proposition 2.4.** For all \( f_1, f_2 \in C^\infty(G) \),

\[
\{ L^* f_1, L^* f_2 \}_X (x) = \langle R(m_2(x)) D' f_1(L(x)), D' f_2(L(x)) \rangle \\
- \langle R(m_1(x)) D f_1(L(x)), D f_2(L(x)) \rangle, \quad \forall x \in X,
\]

**Proof.** Since \( \rho \) is a Poisson map, we have

\[
\{ L^* f_1, L^* f_2 \}_X (x) = \langle Pr^*_2 f_1, Pr^*_2 f_2 \rangle_R (\rho(x)) \\
= \langle R(m_2(x)) D' f_1(L(x)), D' f_2(L(x)) \rangle \\
- \langle R(m_1(x)) D f_1(L(x)), D f_2(L(x)) \rangle,
\]

as \( \delta_i(Pr^*_2 f_1) = \delta_i(Pr^*_2 f_2) = 0, i = 1, 2. \)

Let \( I(G) \) be the space of central functions on \( G \). We now examine the class of Hamiltonian systems \( \dot{x} = X_H(x) \) on \( X \) which can be realized in \( (\Gamma, \{ \cdot, \cdot \}_R) \) by means of \( \rho \) with \( H \in \rho^*(Pr^*_2 I(G)) = L^* I(G) \).

**Theorem 2.5.**

(a) If \( H = L^* f, \) where \( f \in I(G) \), then under the flow \( \phi_t \) generated by the Hamiltonian \( H \), the following equation hold:

\[
\frac{d}{dt} m_1(\phi_t) = -\iota^* D f(L(\phi_t)),
\]

\[
\frac{d}{dt} L(\phi_t) = Te^r_{L(\phi_t)} R(m_1(\phi_t))(D f(L(\phi_t))) \\
- Te^l_{L(\phi_t)} R(m_2(\phi_t))(D f(L(\phi_t))),
\]

\[
\frac{d}{dt} m_2(\phi_t) = -\iota^* D f(L(\phi_t)).
\]
(b) For all \( f_1, f_2 \in I(G) \),
\[
\{ L^* f_1, L^* f_2 \}_X (x) = \langle (R(m_2(x)) - R(m_1(x)))Df_1(L(x)), Df_2(L(x)) \rangle, \quad \forall x \in X.
\]

Proof. (a) Since \( \rho \) is a Poisson map, we have \( \frac{d}{dt} \rho(\phi_t) = X_{Pr^2 f}(\rho(\phi_t)) \). The equations then follow from a direct computation using (2.7).
(b) This is obvious from Proposition 2.4. \( \square \)

Consider the important special case where \( X = \Gamma \) and \( \rho = id_\Gamma \). If \( R \) is a constant r-matrix, it is immediate from the above results that the family of functions \( Pr^2 I(G) \) Poisson commutes on \( \Gamma \). So in this case, what we have here is a slight extension of the standard r-matrix framework for Lax equations on Lie groups [STS1],[STS2]. On the other hand, if \( R \) is genuinely dynamical, the functions in \( Pr^2 I(G) \) no longer Poisson commute on all of \( \Gamma \), but they do so on the gauge group bundle (see (3.3)), where \( u = v \). In this case, a reduction is required to obtain the associated integrable systems. In the rest of the paper, we shall deal exclusively with the genuinely dynamical case.

3. Reduction to integrable systems.

Theorem 2.5 (b) shows that the functions in \( L^* I(G) \) do not necessarily Poisson commute in \( X \). In the following, we shall describe a situation where we can obtain integrable flows on a reduced Poisson space. In particular, this applies to the important case where \( X \) is \( \Gamma \) itself and where \( \rho \) is the identity map. We shall make the following assumptions:

A1. \( X \) is a Hamiltonian \( H \)-space with an equivariant momentum map \( J : X \to h^* \),
A2. the realization map \( \rho : X \to \Gamma \) is \( H \)-equivariant, where \( H \) acts on \( \Gamma \) via the formula
\[
h \cdot (u, g, v) = (Ad_{h^{-1}}^* u, hgh^{-1}, Ad_{h^{-1}}^* v). \tag{3.1}
\]
A3. for some regular value \( \mu \in h^* \) of \( J \),
\[
\rho(J^{-1}(\mu)) \subset I\Gamma, \tag{3.2}
\]
where
\[
I\Gamma = \{(u, g, u) \mid u \in U, g \in G\} \tag{3.3}
\]
is the gauge group bundle of $\Gamma$.

Now, recall that $\Gamma$ has a pair of commuting Hamiltonian $H$-actions [EV]: a left action given by $h \cdot (u, g, v) = (Ad_h^* u, hg, v)$, and a right action given by $(u, g, v) \cdot h = (u, gh, Ad_h^* v)$. By combining these actions, we obtain the $H$-action in assumption A2. As can be easily verified, this is a Hamiltonian action with equivariant momentum map

$$\gamma = \alpha - \beta : \Gamma \to \mathfrak{h}^*$$

and we have $\gamma^{-1}(0) = \mathcal{I}\Gamma$. Consequently, $X = \Gamma$ and $\rho = \text{id}_{\Gamma}$ satisfy assumptions A1-A3 with $\mu = 0$.

We shall denote by $H_\mu$ the isotropy subgroup of $\mu$ for the $H$-action on $X$. Then it follows by Poisson reduction [MR], [OR] (see [OR] for the singular case) that the variety $X_\mu = J^{-1}(\mu)/H_\mu$ inherits a unique Poisson structure $\{\cdot, \cdot\}_{X_\mu}$ satisfying

$$\pi^*_\mu \{f_1, f_2\}_{X_\mu} = i^*_\mu \{\tilde{f}_1, \tilde{f}_2\}_X.$$  \hfill (3.5)

Here, $i_\mu : J^{-1}(\mu) \to X$ is the inclusion map, $\pi_\mu : J^{-1}(\mu) \to X_\mu$ is the canonical projection, $f_1, f_2 \in C^\infty(X_\mu)$, and $\tilde{f}_1, \tilde{f}_2$ are (locally defined) smooth extensions of $\pi^*_\mu f_1$, $\pi^*_\mu f_2$ with differentials vanishing on the tangent spaces of the $H$-orbits. For the case where $X = \Gamma$, $\rho = \text{id}_{\Gamma}$, it is clear that the isotropy subgroup $H_0$ of $\mu = 0$ is $H$ and so we have the reduced Poisson variety

$$(\Gamma_0 = \gamma^{-1}(0)/H, \{\cdot, \cdot\}_{\Gamma_0}),$$  \hfill (3.6)

with the inclusion map $i_H : \gamma^{-1}(0) \to \Gamma$ and the canonical projection $\pi_H : \gamma^{-1}(0) \to \Gamma_0$.

Clearly, functions in $i_H^* \text{Pr}_2^* I(G) \subset C^\infty(\gamma^{-1}(0))$ are $H$-invariant, hence they descend to functions in $C^\infty(\Gamma_0)$. On the other hand, it follows from assumption A2 that the functions in $i_\mu^* L^* I(G) \subset C^\infty(J^{-1}(\mu))$ drop down to functions in $C^\infty(X_\mu)$. Now, by assumptions A2-A3, and the fact that $\rho$ is Poisson, it follows from [OR] that $\rho$ induces a unique Poisson map

$$\hat{\rho} : X_\mu \to \Gamma_0 = \mathcal{I}\Gamma/H$$  \hfill (3.7)

characterized by $\pi_H \circ \rho \circ i_\mu = \hat{\rho} \circ \pi_\mu$. Hence $X_\mu$ admits a realization in the Poisson variety $\Gamma_0$.
We shall use the following notation. For \( f \in I(G) \), the unique function in \( C^\infty(\Gamma_0) \) determined by \( i_H^* Pr_2^* f \) will be denoted by \( \tilde{f} \); while the unique function in \( C^\infty(X_\mu) \) determined by \( i_\mu^* L^* f \) will be denoted by \( F_\mu \). From the definitions, we have

\[
F_\mu \circ \pi_\mu = (\tilde{\rho}^* \tilde{f}) \circ \pi_\mu = i_\mu^* L^* f.
\]

(3.8)

**Theorem 3.1.** Let \( (X, \{ \cdot, \cdot \}_X) \) be a Poisson manifold which admits a realization \( \rho : X \to \Gamma \) and assume that A1-A3 are satisfied. Then there exist a unique Poisson structure \( \{ \cdot, \cdot \}_{X_\mu} \) on the reduced space \( X_\mu = J^{-1}(\mu)/H_\mu \) and a unique Poisson map \( \rho \) such that

(a) for all \( f_1, f_2 \in I(G) \) and \( x \in J^{-1}(\mu) \),

\[
\{ \tilde{\rho}^* f_1, \tilde{\rho}^* f_2 \}_{X_\mu} \circ \pi_\mu(x) = < R(m_1(x)) Df_1(L(x)), Df_2(L(x)) > < R(m_1(x)) Df_1(L(x)), Df_2(L(x)) >,
\]

(b) functions \( \tilde{\rho}^* \tilde{f} \), \( f \in I(G) \), Poisson commute in \( (X_\mu, \{ \cdot, \cdot \}_{X_\mu}) \),

(c) if \( \psi_t \) is the induced flow on \( \mathcal{I}\Gamma \) generated by the Hamiltonian \( Pr_2^* f \), \( f \in I(G) \), and \( \phi_t \) is the Hamiltonian flow of \( F = L^* f \) on \( X \), then the reduction \( \phi_t^{red} \) of \( \phi_t \circ i_\mu \) on \( X_\mu \) defined by \( \phi_t^{red} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu \) is a Hamiltonian flow of \( F_\mu = \tilde{\rho}^* \tilde{f} \) and \( \tilde{\rho} \circ \phi_t^{red}(\pi_\mu(x)) = \pi_H \circ \psi_t(\rho(x)), \quad x \in J^{-1}(\mu) \).

**Proof.** (a) Since \( \rho(J^{-1}(\mu)) \subset \mathcal{I}\Gamma \), we have \( m_1(x) = m_2(x) \) for \( x \in J^{-1}(\mu) \). Therefore,

\[
\{ \tilde{\rho}^* f_1, \tilde{\rho}^* f_2 \}_{X_\mu} \circ \pi_\mu(x) = \{ f_1, f_2 \}_{\Gamma_0} \circ \pi_H(\rho(x)) = \{ Pr_2^* f_1, Pr_2^* f_2 \}_{R(\rho(x))}
\]

and so the assertion now follows from the proof of Proposition 2.4.

(b) This assertion is clear from part (a).

(c) The first part of the assertion follows from Theorem 2.16 of [OR] while the second part is a consequence of the relation \( \rho \circ \phi_t \circ i_\mu = \psi_t \circ \rho \circ i_\mu \) and the definition of \( \tilde{\rho} \) and \( \phi_t^{red} \).

**Remark 3.2** (a) If we assume the existence of an \( H \)-equivariant map \( g : X \to H \), then we can define the gauge transformation \( \tilde{\rho} \) of the realization map \( \rho \) in the obvious way. Therefore, if the reduced Poisson space \( X_\mu \) is smooth (or has a smooth component), then \( \tilde{\rho}|_{J^{-1}(\mu)} \) descends to a uniquely determined map \( \rho_\mu \) on
and we can write down the (generalized) Lax equations for the the reduced Hamiltonian systems on $X_{\mu}$ (or a smooth component of $X_{\mu}$). Details of this has been work out in [L1] but we do not need it in this work.

(b) Clearly, our results here can be generalized in several directions. For example, we can consider coboundary Poisson groupoids $(\mathcal{M} \times G \times \mathcal{M}, \{\cdot,\cdot\})$ (where $\mathcal{M}$ is a manifold) corresponding to the Lie bialgebroids considered in [LiX2]. Another possibility is to consider realization in $\Gamma^N = \Gamma \times \cdots \times \Gamma$ ($N$ times), equipped with the product structure, and use the class of twisted invariant functions on $\mathcal{G}^N$. We shall report on these results elsewhere.

(c) An example of integrable systems which fits into our framework is given by the elliptic Sklyanin systems in [HM]. Therefore, the Hamilton's equations of such systems can be solved via our factorization theory in the next section (see [L3]). In Section 5 below, we shall present an example of physical origin.

4. Exact solvability and factorization problems on Lie groupoids.

It is well-known that an important class of Poisson Lie groups is associated with the Sklyanin bracket $[D]_{[u]}$[STS2]. For Poisson groupoids, an analog of the Sklyanin construction was considered in [LiX1]. We shall begin by relating the bracket $\{\cdot,\cdot\}_R$ in Theorem 2.1 to the consideration in [LiX1]. Let $A\Gamma := \bigcup_{u \in U} T_{\epsilon(u)}^{-1}(u) = \bigcup_{u \in U} \{0\} \times g \times \mathfrak{h}^*$ be the Lie algebroid of $\Gamma$. By Weinstein’s coisotropic calculus [W] or otherwise, the Lie algebroid dual $A^*\Gamma = \bigcup_{u \in U} \{0\} \times g^* \times \mathfrak{h}$ also has a natural Lie algebroid structure [BKS],[LP2] such that the pair $(A\Gamma, A^*\Gamma)$ is a Lie bialgebroid in the sense of Mackenzie and Xu [MX]. We shall denote the Lie brackets on $\text{Sect}(U,A\Gamma)$ and $\text{Sect}(U,A^*\Gamma)$, respectively, by $[\cdot,\cdot]_A\Gamma$ and $[\cdot,\cdot]_{A^*\Gamma}$.

For our purpose, we introduce the bundle map

$$\mathcal{R} : A^*\Gamma \longrightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, -iZ + R(q)A, \iota^*A - ad^*_Z(q)).$$ (4.1)

For $\varphi \in C^\infty(\Gamma)$, we also define the left and right gradients $\mathcal{D}'\varphi(u,g,v)$ and $\mathcal{D}\varphi(u,g,v)$ as follows:

$$\langle \mathcal{D}'\varphi(u,g,v), X(v) \rangle = \frac{d}{dt}|_{t=0} \varphi(\epsilon t, x(u,g,v)), \quad (4.2a)$$

$$\langle \mathcal{D}\varphi(u,g,v), X(u) \rangle = \frac{d}{dt}|_{t=0} \varphi(l_\epsilon x(u,g,v)), \quad (4.2b)$$
where \( X \in \text{Sect}(U, A\Gamma) \), and \( r_{e^tX} \), \( l_{e^tX} \) are the right and left translations corresponding to the local bisection \( e^{tX} \), defined by \( r_{e^tX}(u, g, v) = (u, g, v)e^{tX}(v) \) and \( l_{e^tX}(u, g, v) = e^{tX}((\beta \circ e^{tX})^{-1}(u))(u, g, v) \), respectively, and where \( e^{tX}(u) = f_t(u, 1, u) \), \( f_t \) being the (local) flow generated by the left invariant vector field \( \overrightarrow{X} \) (see [M] for more details on the exponential map, bisections and the corr. left and right translations).

**Proposition 4.1.** For all \( \varphi, \psi \in C^\infty(\Gamma) \),

\[
\{ \varphi, \psi \}_R(u, g, v) = \langle R_u(\mathcal{D}'\varphi(u, g, v)), \mathcal{D}'\psi(u, g, v) \rangle \quad - \quad \langle R_u(\mathcal{D}\varphi(u, g, v)), \mathcal{D}\psi(u, g, v) \rangle.
\]

**Proof.** From the definition of the left and right gradients, we can check that \( \mathcal{D}'\varphi(u, g, v) = (0_v, D'\varphi, \delta_2\varphi) \) and \( \mathcal{D}\varphi(u, g, v) = (0_u, D\varphi, -\delta_1\varphi) \). The rest of the proof is plain. \( \square \)

If \( (u, g, v) \in \Gamma \), we shall denote by \( l(u, g, v) \) and \( r(u, g, v) \) the left translation and right translation in \( \Gamma \) defined by \( l(u, g, v)(v, g', w) = (u, gg', w) \) and \( r(u, g, v)(w, g', u) = (w, g'g, v) \) respectively.

**Corollary 4.2.** The Hamiltonian vector field generated by \( \psi \in C^\infty(\Gamma) \) is given by

\[
X_\psi(u, g, v) = -T_{\epsilon(v)}(u, g, v)R_u(\mathcal{D}'\psi(u, g, v)) \quad - \quad \langle R_u(\mathcal{D}\psi(u, g, v)), \mathcal{D}\psi(u, g, v) \rangle,
\]

where \( \epsilon \) is the identity section and \( i \) is the inversion map of \( \Gamma \).

**Proof.** This proof follows from Proposition 4.1 and the relations

\[
\langle \mathcal{D}'\varphi(u, g, v), X(v) \rangle = \overrightarrow{X} \varphi(u, g, v), \quad \langle \mathcal{D}\varphi(u, g, v), X(u) \rangle = \overleftarrow{X} \varphi(u, g, v),
\]

where \( \overrightarrow{X} \) (resp., \( \overleftarrow{X} \)) is the left (resp., right) invariant vector field on \( \Gamma \) generated by \( X \in \text{Sect}(U, A\Gamma) \). \( \square \)

**Definition 4.3.** The bundle map \( R : A^*\Gamma \longrightarrow A\Gamma \) is called an r-matrix of the Lie algebroid \( A^*\Gamma \).

An important sufficient condition for \( \{ \cdot, \cdot \}_R \) to be a Poisson bracket is the modified dynamical Yang-Baxter equation (mDYBE):

\[
[R(q)A, R(q)B] + R(q)(ad^*_{R(q)A}B - ad^*_{R(q)B}A) \\
+ dR(q)\epsilon^* A(B) - dR(q)\epsilon^* B(A) + d < R(A), B > (q) \\
= - [K(A), K(B)]
\]
where $K \in L(g^*, g)$ is a nonzero symmetric map which satisfies $ad_X \circ K + K \circ ad_X = 0$ for all $X \in g$.

Using $K$, we define

$$K : A^\ast \Gamma \rightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, K(A), 0),$$

and set $R^\pm = R \pm K, R^\pm(q) = R(q) \pm K.$

We shall give a sketch of the proof of our next result, details can be found in [L2].

**Proposition 4.4.** If $R$ satisfies the (mDYBE), then

(a) $R^\pm$ are morphisms of transitive Lie algebroids. In particular,

$$[R^\pm(0, A, Z), R^\pm(0, A', Z')]_{A\Gamma} = R^\pm([0, A, Z), (0, A', Z')]_{A^\ast \Gamma}$$

for all smooth maps $A, A' : U \rightarrow g^*, Z, Z' : U \rightarrow h$.

(b) $\text{Im} R^\pm$ are transitive Lie subalgebroids of $A\Gamma$.

**Proof.** (a) First of all, we can show that the r-matrix $R$ satisfies the equation

$$[R(0, A, Z), R(0, A', Z')]_{A\Gamma} - R([0, A, Z), (0, A', Z')]_{A^\ast \Gamma} = (0, -[K(A), K(A')], 0)$$

for all smooth maps $A, A' : U \rightarrow g^*, Z, Z' : U \rightarrow h$. If $a$ and $a_\ast$ are the anchor maps of the Lie algebroids $A\Gamma$ and $A^\ast \Gamma$, it is easy to check that $a \circ R^\pm = a_\ast$. On the other hand, it follows from (4.6) that (4.5) holds if and only if

$$K([0, A, Z), (0, A', Z')]_{A^\ast \Gamma} = [R(0, A, Z), K(0, A', Z')_{A\Gamma} + [K(0, A, Z), R(0, A', Z')]_{A\Gamma}$$

This latter relation can then be verified in a direct manner by using the property that $ad_X \circ K + K \circ ad_X = 0$ for $X \in g$.

(b) This property is a consequence of (a). □

**Remark 4.5.** An upshot of Proposition 4.4 (a) is that the dual maps $(R^\pm)^* = -R^\mp$ are Poisson maps, when the domain and target are equipped with the corresponding Lie-Poisson structures. This fact has been used in [L2] to construct a family of hyperbolic spin Calogero-Moser systems. In particular, the associated integrable models contain as a special case the system considered in [R].

In the rest of the section, we shall assume that there is an ad-invariant nondegenerate pairing $(\cdot, \cdot)$ on $g$ and without loss of generality, we shall take the map
$K : \mathfrak{g}^* \to \mathfrak{g}$ in the above discussion to be the identification map induced by $(\cdot, \cdot)$. Indeed, with the identification $\mathfrak{g}^* \simeq \mathfrak{g}$, we shall regard $R$ as taking values in $\text{End}(\mathfrak{g})$, and the left and right gradients as well as the dual maps are computed using $(\cdot, \cdot)$.

Also, we have $ad^* \simeq -ad$ and $\iota^* \simeq \Pi_h$, where $\Pi_h$ is the projection map to $\mathfrak{h}$ relative to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. We shall keep, however, the notation $A^*\Gamma$ although as a set it can be identified with $A\Gamma$.

The connection between (mDYBE) and our factorization theory is the following decomposition

$$(0_q, X, 0) = \frac{1}{2} R^+_q(0_q, X, 0) - \frac{1}{2} R^-_q(0_q, X, 0) \tag{4.7}$$

where the element $(0_q, X, 0)$ on the left hand side is in the adjoint bundle $Ker a$ of $A\Gamma$. The reader should note that the vector bundles $\{R^+_q(0_q, X, 0) \mid q \in U, X \in \mathfrak{g}\}$ are not Lie subalgebroids of $A\Gamma$ unless $R$ is a constant $r$-matrix (see Remark 4.11 (c)). This fact has repercussion when we try to formulate a global version of the decomposition in (4.7).

In order to state our next result, introduce the Lie algebroid direct sum $A\Gamma \oplus TU A\Gamma$. Clearly, this is the Lie algebroid of the product groupoid $P = \Gamma \times U \times \Gamma \simeq U \times (G \times G) \times U$. For later usage, we shall denote the structure maps (target, source etc.) of $P$ by $\alpha_P$, $\beta_P$, and so forth.

**Theorem 4.6.** The map $(R^+, R^-) : A^*\Gamma \to A\Gamma \oplus TU A\Gamma$ is a monomorphism of transitive Lie algebroids. In particular, the coboundary dynamical Lie algebroid $(A^*\Gamma, \cdot, \cdot, A^*\Gamma)$ is integrable.

**Proof.** Since $R^+_q(0_q, X, Z) - R^-_q(0_q, X, Z) = (0_q, 2X, 0)$, it is clear that $(R^+, R^-)$ is 1:1. On the other hand, it is consequence of Proposition 4.4 (a) and the definition of $A\Gamma \oplus TU A\Gamma$ that $(R^+, R^-)$ is a morphism of Lie algebroids. Finally, as $A^*\Gamma$ is isomorphic to a Lie subalgebroid of $A\Gamma \oplus TU A\Gamma$ and the latter is integrable, it follows that $A^*\Gamma$ is also integrable [MM]. \qed

**Remark 4.7** (a) Note that in contrast to the case of finite dimensional Lie algebras, not all Lie algebroids are integrable even in finite dimensions [AM]. In Theorem 4.6, the integrability of the Lie algebroid $A\Gamma \oplus TU A\Gamma$ has essentially allowed us to bypass the necessity of checking the integrability conditions in [CF].

(b) In addition to its application below in our factorization theory, this result also implies the existence of the dual Poisson groupoids for the the class of coboundary
dynamical Poisson groupoids $(\Gamma, \{\cdot, \cdot\}_R)$, where $R$ is a solution of $(m\text{DYBE})$.

(c) From the proof of Theorem 4.6, it is clear that $(\mathcal{R}^+, \mathcal{R}^-)(0_q, X, Z)$ is an element of the diagonal of $\mathcal{A} \oplus \mathcal{A}$ if and only if $(0_q, X, Z) = (0_q, 0, Z)$. We shall use the global version of this fact in Corollary 4.8 below.

In the rest of the section, we shall assume both $G$ and $U$ are simply-connected. Let $\Gamma^*$ be the unique source-simply connected Lie groupoid which integrates $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma})$. Then $(\mathcal{R}^+, \mathcal{R}^-)$ can be lifted up to a unique monomorphism of Lie groupoids $\Gamma^* : \longrightarrow \Gamma \times \Gamma$ which we shall denote by the same symbol. Now, let $j : \Gamma \times \Gamma \longrightarrow \mathcal{T} \Gamma$ be the map defined by $j(a, b) = ab^{-1}$ and let $\bar{m} = j \circ (\mathcal{R}^+, \mathcal{R}^-)$.

Our next corollary is a global version of the decomposition in (4.7). For its formulation, note that the Lie groupoid of $(\{0_q, 0, Z\} | q \in U, Z \in \mathfrak{h}) \subset A^*\Gamma$ is $H \times U$, with target and source maps $\alpha'(h, u) = u, \beta'(h, u) = Ad_h u$ and multiplication map $m'((h, u), (k, Ad_k u)) = (kh, u)$ (this is isomorphic to the Hamiltonian unit in [LP2]). On the other hand, the Lie groupoid of $\mathcal{R}^\pm \{0_q, 0, Z\} | q \in U, Z \in \mathfrak{h}$ is given by $E = \{(u, h, Ad_{h^{-1}} u) | u \in U, h \in H\}$ and $\mathcal{R}^\pm$ embeds $H \times U$ in $E, \mathcal{R}^\pm | H \times U : (h, u) \mapsto (u, h^{-1}, Ad_h u)$. Clearly, the diagonal $\Delta(E)$ of $E \times E$ acts on $\text{Im}(\mathcal{R}^+, \mathcal{R}^-)$ from the right via the simple formula

$$((u, k_+, v), (u, k_-, v)) \cdot ((v, h, Ad_{h^{-1}} v), (v, h, Ad_{h^{-1}} v))$$

$$=((u, k_+ h, Ad_{h^{-1}} v), (u, k_- h, Ad_{h^{-1}} v))$$

and the map $j | \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$ is constant on the orbits of this action.

**Corollary 4.8.** Suppose $U$ is simply-connected, then $j | \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$ induces a one-to-one map $\tilde{j} : \text{Im}(\mathcal{R}^+, \mathcal{R}^-) / \Delta(E) \longrightarrow \mathcal{T} \Gamma$. Therefore, for each $\gamma \in \text{Im} \bar{m}$, there exists a unique [$(\gamma_+, \gamma_-)$] in the homogeneous space $\text{Im}(\mathcal{R}^+, \mathcal{R}^-) / \Delta(E)$ such that $\tilde{j}([(\gamma_+, \gamma_-)]) = \gamma$.

**Proof.** Suppose $[((u, k_+, v), (u, k_-, v))]$ and $[((u, k'_+, v'), (u, k'_-, v'))]$ are two elements in $\text{Im}(\mathcal{R}^+, \mathcal{R}^-) / \Delta(E)$ with the same image under the map $\tilde{j}$, then $k_+ k_-^{-1} = k'_+ k'_-^{-1}$ and therefore, $k_+^{-1} k'_+ = k_-^{-1} k'_-$. Now,

$$((v, k_+^{-1} k'_+, v'), (v, k_-^{-1} k'_-, v'))$$

$$= ((u, k_+, v), (u, k_-, v))^{-1} ((u, k'_+, v'), (u, k'_-, v'))$$

$$= (\mathcal{R}^+ (\gamma^*), \mathcal{R}^- (\gamma^*))$$

for unique $\gamma^* \in \Gamma^*$. Hence from the fact that $\mathcal{R}^+ (\gamma^*) = \mathcal{R}^- (\gamma^*)$, we must have $\gamma^* \in H \times U$ (from Remark 4.7 (c) and the above discussion) and so $k_+^{-1} k'_+$ =
\[ k_+^{-1}k'_- = h \in H \text{ and } v' = Ad_{h^{-1}}v. \] Consequently, \[ \{((u, k_+, v), (u, k_-, v))\} \]

In what follows, we shall rescale the Poisson bracket \( \{\cdot, \cdot\}_R \) by the factor 1/2. Let \( f \in I(G) \) and let \( F = Pr_2^*f \). Then \( \mathcal{D}'F(u, g, v) = (0_v, Df(g), 0) \) and \( \mathcal{D}F(u, g, v) = (0_u, Df(g), 0) \). Therefore, according to Corollary 4.2, the Hamilton’s equation generated by \( F \) when restricted to \( \mathcal{I} \Gamma \) takes the form

\[
\begin{align*}
\frac{d}{dt}(u, g, u) & = -\frac{1}{2} T_{(u)} l(u, g, u) R_u(0_u, Df(g), 0) \\
& \quad - \frac{1}{2} T_{(u)} (r(u, g, u) \circ i) R_u(0_u, Df(g), 0).
\end{align*}
\]

**Theorem 4.9.** Suppose that \( f \in I(G) \), \( F = Pr_2^*f \) and \( u_0 \in U \), where \( U \) is simply-connected. Then for some \( 0 < T \leq \infty \), there exists a unique element \( (\gamma_+(t), \gamma_-(t)) = ((u_0, k_+(t), u(t)), (u_0, k_-(t), u(t))) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-) \) for \( 0 \leq t < T \) which is smooth in \( t \), solves the factorization problem

\[
\exp\{-t(0, Df(g_0), 0)\}(u_0) = \gamma_+(t) \gamma_-(t)^{-1}
\]

and satisfies

\[
(T_{\gamma_+(t)} l_{\gamma_+(t)^{-1}}, \gamma_+(t), T_{\gamma_-(t)} l_{\gamma_-(t)^{-1}} \gamma_-(t)) \in (\mathcal{R}^+, \mathcal{R}^-)_{u(t)}(\{0_u(t)\} \times \mathfrak{g} \times \{0\})
\]

with

\[
\gamma_\pm(0) = (u_0, 1, u_0).
\]

Moreover, the solution of Eqn.(4.8) with initial data \((u, g, u)(0) = (u_0, g_0, u_0)\) is given by the formula

\[
(u, g, u)(t) = (u_0, k_\pm(t), u(t))^{-1}(u_0, g_0, u_0)(u_0, k_\pm(t), u(t))
\]

**Proof.** We first prove the uniqueness of the element \((\gamma_+(t), \gamma_-(t))\). Suppose \((\gamma'_+(t), \gamma'_-(t)) = ((u_0, k'_+(t), u'(t)), (u_0, k'_-(t), u'(t))) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-) \) is a second element with the properties in (4.9) and (4.10). Then by Corollary 4.8, we have \( k_+^{-1}(t)k'_+(t) = k_+^{-1}(t)k'_+(t) = h(t) \in H \) and \( u'(t) = Ad_{h(t)}^{-1}u(t) \). Consider \( h(t) = k_+^{-1}(t)k'_+(t) \). By differentiation, and using (4.10), we have
\[ T_{h(t)} l_{h(t)^{-1}} \dot{h}(t) \]
\[ = T_{k_+^l(t)} l_{k_+^l(t)^{-1}} \dot{k}_+^l(t) - Ad_{h(t)^{-1}} T_{k_+^l(t)} l_{k_+^l(t)^{-1}} \dot{k}_+^l(t) \]
\[ = R^+(u'(t)) X'(t) - Ad_{h(t)^{-1}} R^+(u(t)) X(t) \]
(4.12)
for some \( X(t), X'(t) \in g \). Similarly, by taking \( h(t) = k_-^l(t) k_-^r(t) \), we obtain
\[ T_{h(t)} l_{h(t)^{-1}} \dot{h}(t) \]
\[ = R^- (u'(t)) X'(t) - Ad_{h(t)^{-1}} R^- (u(t)) X(t). \]
(4.13)

Therefore, upon equating the two expressions, we find \( X'(t) = Ad_{h(t)^{-1}} X(t) \). Substituting this back in (4.12), and using (2.4) and the relation \( u'(t) = Ad_{h(t)^{-1}} u(t) \), we learn that \( T_{h(t)} l_{h(t)^{-1}} \dot{h}(t) = 0 \). Therefore, \( h(t) = 1 \) and so \( (\gamma_+(t), \gamma_-(t)) = (\gamma_+(t), \gamma_-(t)) \).

Assuming the existence of the factors for the moment, we claim that \((u(t), g(t), u(t))\) as given by (4.11) solves (4.8). First of all, we have
\[
\gamma_+(t)^{-1}(u_0, g_0, u_0) \gamma_+(t)
\]
\[
= (u(t), k_+(t)^{-1} g_0 k_+(t), u(t))
\]
\[
= (u(t), k_-(t)^{-1} g_0 k_-(t), u(t))
\]
\[
= \gamma_-(t)^{-1}(u_0, g_0, u_0) \gamma_-(t)
\]
since, from the fact that \( f \in I(G) \), we have \( Df(g_0) = Ad_{g_0} Df(g_0) \). Take
\[
(u(t), g(t), u(t)) = \gamma_+(t)^{-1}(u_0, g_0, u_0) \gamma_+(t).
\]

By differentiating the expression, we obtain
\[
\frac{d}{dt}(u(t), g(t), u(t))
\]
\[
= T_{e(u(t))} l_{(u(t), g(t), u(t))} T_{\gamma_+(t)} l_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t)
\]
\[
+ T_{e(u(t))} (r_{u(t), g(t), u(t)} \circ i) T_{\gamma_+(t)} l_{\gamma_+(t)^{-1}} \dot{\gamma}_-(t).
\]
(*

On the other hand, by rewriting (4.9) in the form
\[
\exp\{-t(0, Df(g_0), 0)\}(u_0) \gamma_-(t) = \gamma_+(t),
\]
we have
\[
T_{\gamma_+(t)} l_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t)
\]
\[
= T_{\gamma_-(t)} l_{\gamma_-(t)^{-1}} \dot{\gamma}_-(t)
\]
\[
- T_{\gamma_-(t)^{-1}} r_{\gamma_-(t)} T_{e(u_0)} l_{\gamma_-(t)^{-1}} (0, u_0, Df(g_0), 0).
\]
But
\[ T_{\gamma_-(t)}^{-1} r_{\gamma_-(t)} T_{\epsilon(u_0)} I_{\gamma_-(t)}^{-1} (0_{u_0}, Df(g_0), 0) \]
\[ = (0_{u(t)}, Df(g(t)), 0), \]
as \( f \in I(G) \). Hence it follows that
\[ T_{\gamma_+(t)} I_{\gamma_+(t)}^{-1} \dot{\gamma}_+(t) - T_{\gamma_-(t)} I_{\gamma_-(t)}^{-1} \dot{\gamma}_-(t) \]
\[ = - (0_{u(t)}, Df(g(t)), 0). \]
From the property of \( \gamma_\pm \) in (4.10), we can now conclude that
\[ T_{\gamma_\pm(t)} I_{\gamma_\pm(t)}^{-1} \dot{\gamma}_\pm(t) \]
\[ = - \frac{1}{2} R_{u(t)}^\pm (0_{u(t)}, Df(g(t)), 0). \]
Therefore, on substituting into (*), we find
\[
\frac{d}{dt}(u(t), g(t), u(t)) \\
= - \frac{1}{2} T_{\epsilon(u(t))} I_{(u(t), g(t), u(t))} \mathcal{R}_{u(t)}^+ (0_{u(t)}, Df(g(t)), 0) \\
- \frac{1}{2} T_{\epsilon(u(t))} (r_{(u(t), g(t), u(t))} \circ i) \mathcal{R}_{u(t)}^+ (0_{u(t)}, Df(g(t)), 0).
\]
Hence our claim follows from the fact that
\[ T_{\epsilon(u(t))} I_{(u(t), g(t), u(t))} (0_{u(t)}, Df(g(t)), 0) \]
\[ = - T_{\epsilon(u(t))} (r_{(u(t), g(t), u(t))} \circ i)(0_{u(t)}, Df(g(t)), 0) \]
\[ = (0_{u(t)}, T_{\epsilon} l_{g(t)} Df(g(t)), 0). \]
To prove the existence of the factors \( \gamma_\pm(t) \), we solve the initial value problems
\[ \dot{k}_\pm(t) = - \frac{1}{2} T_{\epsilon} k_\pm(t) R_\pm(u(t)) Df(g(t)), \quad k_\pm(0) = 1, \quad (**\*)\]
where \( u(t) \) and \( g(t) \) are the solutions of (4.8) with initial data \( u(0) = u_0, g(0) = g_0 \)
(which are known to exist by ODE theory). Set \( \gamma_\pm(t) = (u_0, k_\pm(t), u(t)) \). As can be
easily verified, we can combine the equations for \( u(t), k_\pm(t) \) into one single equation
for \( (\gamma_+(t), \gamma_-(t)) \):
\[
\frac{d}{dt}(\gamma_+(t), \gamma_-(t)) \\
= (T_{\epsilon(u(t))} I_{\gamma_+(t)} \mathcal{R}_+ (0_{u(t)}, - \frac{1}{2} Df(g(t)), 0), T_{\epsilon(u(t))} I_{\gamma_-(t)} \mathcal{R}_- (0_{u(t)}, - \frac{1}{2} Df(g(t)), 0)) \\
= T_{\epsilon p(\beta_p(\gamma_+(t), \gamma_-(t)))) I_{(\gamma_+(t), \gamma_-(t))} (\mathcal{R}_+, \mathcal{R}_-)(0_{u(t)}, - \frac{1}{2} Df(g(t)), 0) \quad (****) \]
where $l_P^{\gamma_+(t),\gamma_-(t)}$ represents left translation by $(\gamma_+(t), \gamma_-(t))$ in the product groupoid $P = \Gamma \times \Gamma \rightrightarrows U$. Clearly, what we have just written down is a well-defined equation for $(\gamma_+(t), \gamma_-(t)) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$. Moreover, from the initial conditions for $k_\pm(t)$ and $u(t)$, we have $(\gamma_+(0), \gamma_-(0)) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$.

Now, from the equations for $k_\pm$ in (**), we find

$$
\frac{d}{dt} \gamma_+(t) \gamma_-(t)^{-1} = (0_{u_0}, -T_{\ell_{k_+(t)} r_{k_-(t)^{-1}} \ell_{k_+(t)}}) Df(g(t), 0) = (0_{u_0}, -T_{\ell_{k_+(t) k_-(t)^{-1}}} Df(k_-(t) g(t) k_-(t)^{-1}), 0).
$$

But it follows on using the equations for $k_\pm$ and $g$ that

$$
\frac{d}{dt} k_-(t) g(t) k_-(t)^{-1} = 0.
$$

Therefore, $k_-(t) g(t) k_-(t)^{-1} = g_0$ and so

$$
\frac{d}{dt} \gamma_+(t) \gamma_-(t)^{-1} = (0_{u_0}, -T_{\ell_{k_+(t) k_-(t)^{-1}}} Df(g_0), 0).
$$

As $\gamma_+(t) \gamma_-(t)^{-1} = (u_0, k_+(t) k_-(t)^{-1}, u_0)$, this shows that $k_+(t) k_-(t)^{-1} = e^{-t Df(g_0)}$ and consequently,

$$
\exp\{-t(0, Df(g_0), 0)\}(u_0) = \gamma_+(t) \gamma_-(t)^{-1}.
$$

Thus it remains to show that condition (4.10a) is satisfied. But this is immediate from (**). This completes the proof.

\begin{corollar}
Let $\psi_t$ be the induced flow on $\mathcal{IG}$ as defined in (4.11) and let $\phi_t$ be the Hamiltonian flow of $\mathcal{F} = L^* f$ on $X$, where $L = P \rho_2 \circ \rho$ for a realization map $\rho : X \rightarrow \Gamma$ satisfying assumptions A1-A3. If we can solve for $\phi_t(x)$, $x \in J^{-1}(\mu)$ explicitly from the relation $\rho(\phi_t(x)) = \psi_t(\rho(x))$, then the formula $\phi_t^{\text{red}} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu$ gives an explicit expression for the flow of the reduced Hamiltonian $\mathcal{F}_\mu = \hat{\rho}^\ast \mathcal{F}$.
\end{corollar}

\begin{remark}
(a) The reader should not feel uneasy about the use of equations (**) above (which involve the solutions $u(t)$ and $g(t)$) to show the existence of the factors $k_\pm(t)$, and which are then used in turn to construct $u(t)$ and $g(t)$. As the reader will see in Section 5, knowledge of the existence of the factorization facilitates its construction.
\end{remark}
(b) For Hamiltonian systems which admit realization in the dual vector bundles of coboundary dynamical Lie algebroids (where \( R \) solves \((\text{mDYBE})\)) and satisfy assumptions A1, A2 and A4 in [LX2], there is also a method for solving the flows similar to what we discussed above. We shall refer the reader to [L2], [L3] for details.

(c) Although we can apply Theorem 4.9 even when \( R \) is a constant \( r \)-matrix, it would be simpler to use the fact that the vector bundles \( \{ R_q^\pm (0_q, X, 0) \mid q \in U, X \in \mathfrak{g} \} \) are Lie subalgebroids of \( \mathcal{A} \Gamma \) in this case. An analog of Theorem 4.9 using these objects can be formulated for the constant \( r \)-matrix case, but we provide no details here.

5. A family of hyperbolic spin Ruijsenaars-Schneider models.

In this section, we shall construct a family of hyperbolic spin Ruijsenaars-Schneider models using coboundary dynamical Poisson groupoids. To illustrate the factorization method of Section 4, we shall solve the Hamilton’s equations in a simple case. To do so, we shall make use of the solutions of \((\text{CDYBE})\) for pairs \((\mathfrak{g}, \mathfrak{h})\) of Lie algebras, as classified in [EV]. Here, \( \mathfrak{g} \) is simple, and \( \mathfrak{h} \subset \mathfrak{g} \) is a Cartan subalgebra.

We begin with some notation. Let \( \mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \) be the root space decomposition of the simple Lie algebra \( \mathfrak{g} \) and let \((\cdot, \cdot)\) denote its Killing form. We fix a simple system of roots \( \pi = \{ \alpha_1, \cdots, \alpha_N \} \) and denote by \( \Delta^\pm \) the corresponding positive/negative system. For any positive root \( \alpha \in \Delta^+ \), we choose root vectors \( e_\alpha \in \mathfrak{g}_\alpha \) and \( e_{-\alpha} \in \mathfrak{g}_{-\alpha} \) which are dual with respect to \((\cdot, \cdot)\) so that \([e_\alpha, e_{-\alpha}] = h_\alpha\). We also fix an orthonormal basis \((x_i)_{1 \leq i \leq N}\) of \( \mathfrak{h} \). Lastly, for a subset of simple roots \( \pi' \subset \pi \), we shall denote the root span of \( \pi' \) by \( <\pi' > \subset \Delta \) and set \( \pi'^\pm = \Delta^\pm \setminus <\pi' >^\pm \).

For any subset \( \pi' \subset \pi \), we consider the following \( H \)-equivariant solution of the \((\text{mDYBE})\) (with \( K = \frac{1}{2} id_\mathfrak{h} \)):

\[
R(q)X = - \sum_{\alpha \in \Delta} \phi_\alpha(q) X_\alpha e_\alpha
\]  

(5.1)

where

\[
\phi_\alpha(q) = \begin{cases} 
\frac{1}{2} & \text{for } \alpha \in \pi'^+, \\
-\frac{1}{2} & \text{for } \alpha \in \pi'^- \\
\frac{1}{2} \coth \left( \frac{1}{2} (\alpha(q)) \right) & \text{for } \alpha \in <\pi'>,
\end{cases}
\]

and \( X_\alpha = (X, e_{-\alpha}) \), \( \alpha \in \Delta \).
Throughout this section, the coboundary dynamical Poisson groupoid \((\Gamma, \{\cdot, \cdot\}_R)\)
will refer to the one which corresponds to this choice of \(R\). Also, we assume the Lie groups \(G\) and \(H\) are simply-connected.

Let \(\omega_1, \ldots, \omega_N\) be fundamental weights of \(\mathfrak{g}\), and let \(\chi_1, \ldots, \chi_N\) denote the characters of the irreducible representations corresponding to these fundamental weights [St].

**Definition 5.1.** The spin Ruijsenaars-Schneider models associated to \(R\) are the Hamiltonian systems on \(\Gamma\) generated by nonzero multiples of \(H_i = Pr_2^\ast \chi_i, \ i = 1, \ldots, N\).

**Example 5.2.** Take \(G = SL(N + 1, \mathbb{C})\) and let \(H\) be the Cartan subgroup consisting of diagonal matrices. We shall denote the corresponding Lie algebra by \(\mathfrak{g}\) and \(\mathfrak{h}\) respectively and we shall use the pairing \((A, B) = tr(AB)\) on \(\mathfrak{g}\).

Consider the spin Ruijsenaars-Schneider model generated by \(H_1 = (R(q)g, g, q')\) for the case where \(\pi' = \pi\). Since on \(\Gamma\), the Hamilton’s equation is of the form
\[
\dot{q}_i = -\frac{1}{2}\Pi_h (g - \frac{1}{N+1}tr(g)I), \quad \dot{g}_{ij} = \frac{1}{2}(R(q)g)g - \frac{1}{2}g(R(q)g),
\]
it follows that in terms of the components of \(q\) and \(g\), we have
\[
\ddot{q}_i = -\frac{1}{2}\dot{g}_{ii} = \frac{1}{4} \sum_{k \neq i} \coth((q_i - q_k)/2)g_{ik}g_{ki},
\]
\[
\dot{g}_{ij} = \frac{1}{4} \left[ \coth((q_i - q_j)/2)g_{ij}(g_{ii} - g_{jj})
\right.
\]
\[
\left. + \frac{1}{4} \sum_{k \neq i, j} \left( \coth((q_k - q_j)/2) - \coth((q_i - q_k)/2) \right)g_{ik}g_{kj}, \quad i \neq j \right)
\]
Thus up to constants, these are exactly Eqns.(14)-(15) in the paper [BH] (compare also Eqns.(1.21)-(1.23) of [KZ]) if we take \(q\) to be real diagonal and \(g\) to be Hermitian. It is a remarkable fact that these are the equations which govern the soliton dynamics of the so-called \(A^{(1)}_N\) affine Toda field theory [BH].

Rather than spelling out the equations of the other systems explicitly in terms of components, our next goal is to solve the equations via the factorization method in Section 4. For simplicity, we shall do it for the special case where \(\pi' = \pi\), the lengthy analysis of the case where \(\pi' \neq \pi\) is given in [L2].

From the definition of \(R\) in (5.1), it is straightforward to check that in this case, the bundle maps \(\mathcal{R}^\pm\) are isomorphisms of Lie algebroids so that \(Im \mathcal{R}^\pm = A\Gamma\). To set up the factorization problem, it is important to give a precise description of \(Im(\mathcal{R}^+, \mathcal{R}^-) \subset A\Gamma \oplus A\Gamma\). In this connection, note that the bundle map
\[
\theta : Im \mathcal{R}^+ \rightarrow Im \mathcal{R}^- : \mathcal{R}^+(0_q, X, Z) \mapsto \mathcal{R}^-(0_q, X, Z)
\]
is well-defined and moreover is a Lie algebroid isomorphism (note that this is not so if \( \pi' \neq \pi \)). Indeed, we have \( \theta(0_q, X, Z) = (0_q, -tZ + Ad_{e^q}X, Z) \). Therefore,

\[
\text{Im}(R^+, R^-) = \{(0_q, X, Z, \theta(0_q, X, Z)) \mid q \in U, X \in \mathfrak{g}, Z \in \mathfrak{h}\}. \tag{5.4}
\]

Now, we can check that \( \theta \) integrates to an isomorphism of the Lie groupoid \( \Gamma \), given by \((u, g, v) \mapsto e^{-tDf}(g_0) e^{u(0)} \). Hence the factorization problem in (4.9) reduces to

\[
e^{-tDf(g_0)} e^{u_0} = k_+(t) e^{u(t)} k_+(t)^{-1}. \tag{5.5}
\]

As the union of the conjugates of \( H \) forms an open dense subset of \( G \), we can find (for at least small values of \( t \)) \( x_+(t) \in G \) (\( x_+(t) \) is unique up to transformations \( x_+(t) \rightarrow x_+(t) \delta(t) \), where \( \delta(t) \in H \)) and unique \( d(t) \in H \) such that

\[
e^{-tDf(g_0)} e^{u_0} = x_+(t)d(t)x_+(t)^{-1} \tag{5.6}
\]

with \( x_+(0) = 1, d(0) = e^{u_0} \). This uniquely determines \( u(t) \) via the formula

\[
u(t) = \log d(t) \tag{5.7}
\]

On the other hand, let us fix one such \( x_+(t) \). We shall seek \( k_+(t) \) in the form

\[
k_+(t) = x_+(t)h(t), \quad h(t) \in H. \tag{5.8}
\]

To determine \( h(t) \), we shall impose the condition in Eqn. (4.10). When we write out this condition, we find it natural to introduce \( g_+(t) = k_+(t)e^{-\frac{1}{2}(u(t)-u_0)} \), in terms of which the condition becomes

\[
\Pi_\mathfrak{h}(T_{g_+(t)} l_{g_+(t)^{-1}} \dot{g_+}(t)) = 0. \tag{5.9}
\]

From this, we find that \( h(t) \) satisfies the equation

\[
\dot{h}(t) = T_\nu l_{h(t)} \left( \frac{1}{2} \dot{u}(t) - \Pi_\mathfrak{h}(T_{x_+(t)} l_{x_+(t)^{-1}} \dot{x_+}(t)) \right) \tag{5.10}
\]

with \( h(0) = 1 \). Solving this equation explicitly, we find that

\[
k_+(t) = x_+(t) \exp \left\{ \frac{1}{2}(u(t) - u_0) - \int_0^t \Pi_\mathfrak{h}(T_{x_+(\tau)} l_{x_+(\tau)^{-1}} \dot{x_+}(\tau)) d\tau \right\}. \tag{5.11}
\]

Hence we can write down the solution using (4.11).
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L.-C. Li, Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

E-mail address: luenli@math.psu.edu