Abstract

We study multi-type housing markets, where there are \( p \geq 2 \) types of items, each agent is initially endowed one item of each type, and the goal is to design mechanisms without monetary transfer to (re)allocate items to the agents based on their preferences over bundles of items, such that each agent gets one item of each type. In sharp contrast to classical housing markets, previous studies in multi-type housing markets have been hindered by the lack of natural solution concepts, because the strict core might be empty.

We break the barrier in the literature by leveraging AI techniques and making natural assumptions on agents' preferences. We show that when agents’ preferences are lexicographic, even with different importance orders, the classical top-trading-cycles mechanism can be extended while preserving most of its nice properties. We also investigate computational complexity of checking whether an allocation is in the strict core and checking whether the strict core is empty. Our results convey an encouragingly positive message: it is possible to design good mechanisms for multi-type housing markets under natural assumptions on preferences.

Introduction

In this paper, we ask the following question: is it possible at all to design good mechanisms for multi-type housing markets? In multi-type housing markets (Moulin 1995; Konishi, Quint, and Wako 2001; Wako 2005; Klaus 2008), there are multiple types of items, each agent is initially endowed one item of each type. The goal is to design mechanisms without monetary transfer to (re)allocate items to the agents based on their preferences over bundles of items, such that each agent gets one item of each type.

Multi-type housing markets are often described using examples of houses and cars as metaphors for indivisible items. However, the allocation problem is applicable to many other types of items and scarce resources. For example, students may want to exchange papers and dates for exam preparation (Mackin and Xia 2016); in cloud computing, agents may want to allocate multiple types of resources, including CPU, memory, and storage (Ghodsi et al. 2011; Ghodsi et al. 2012); patients may want to allocate multiple types of medical resources, including surgeons, nurses, rooms, and equipments (Huh, Liu, and Truong 2013).

Mechanism design for single-type housing markets is a well-established field in economics, often referred to as housing markets (Shapley and Scarf 1974). In housing markets, the most sensible solution concept is the strict core, which is the set of allocations where no group of agents have incentive to deviate by exchanging their initial endowments within the group. Strict core is desirable because it is an intuitive stable solution, and when agents' preferences are linear orders, the strict core allocation always exists and is unique, which can be computed in polynomial time by Gale’s celebrated Top-Trading-Cycles (TTC) algorithm (Shapley and Scarf 1974; Roth and Postlewaite 1977; Abdulkadiroğlu and Sönmez 1999). TTC enjoys many desirable axiomatic properties including individual rationality, Pareto optimality, and group strategy-proofness. Many extensions of TTC to other single-type housing markets have been proposed and studied. See more details in Related Work.

In sharp contrast to the popularity of housing markets, there is little research on multi-type housing markets, despite their importance and generality. A potential reason for the absence of positive results is that the strict core can be empty or multi-valued in multi-type housing markets (Konishi, Quint, and Wako 2001). Therefore, as Sönmez and Unver (2011) noted: “Positive results of this section on housing markets no longer hold in an economy in which one agent can consume multiple houses or multiple types of houses”. This is the problem we address in this paper and provide a number of positive results, a first in this field.

Our contributions

In this paper, we present novel algorithms building on AI techniques in preference representation and reasoning for allocation in multi-type housing markets. We assume that agents’ preferences are represented by arbitrary acyclic CP-nets (Boutilier et al. 2004). Different agents may have arbitrarily different CP-net structure. We also assume that agents’ preferences are lexicographic, meaning that agents have arbitrary importance orders over item types.

We propose the following natural extension of TTC, which we call Multi-type TTC (MTTC) for multi-type housing markets. MTTC builds a directed bipartite graph in each.
round, where agents and items are two separated groups of vertices. There is an edge \((a, b_i)\), where \(a\) is an agent and \(b_i\) is agent \(b\)'s initial type-\(i\) endowment, if (1) the most important type for agent \(a\), from which she has not obtained an item, is type \(i\), and (2) \(b_i\) is agent \(a\)'s top-ranked type-\(i\) item among remaining type-\(i\) items. For any type \(i\) and any agent \(b\), there is an edge \((b_i, b)\). Then, MTTC finds and implements all cycles as TTC does, but only removes the items in the cycle. The agents always remain in the graph.

We note that in MTTC, a cycle may involve items of multiple types. Our main theorem is the following:

**Theorem [1]** For lexicographic preferences, MTTC runs in polynomial-time and satisfies strict-core-selection (which implies Pareto optimality and individual rationality), non-bossiness, and strong group strategy-proofness when agents cannot lie about importance orders over types.

We note that Theorem [1]'s assumption on lexicographic preferences are not only very mild: given agents can have arbitrarily different CP-net structure and importance orders, but also an extension of naturalistic decision making structures studied in cognitive science literature for ordering items based on multiple criteria (Luan, Schoolder, and Gigerenzer 2014). This positive result is especially surprising when it is put in comparison with similar research in combinatorial voting, where negative results emerge as soon as agents do not share similar CP-net structures (Lang and Xia 2016).

We also prove an impossibility theorem to show that Theorem [1] cannot be improved by allowing agents to lie about the importance order. As for computing other strict cores, we prove that, it is coNP-complete to check whether a given allocation is in the strict core, even for two types of items and agents with separable preferences w.r.t. the same importance order. This hardness result is unexpected due to relatively small number of bundles \((n^2)\), and the simplicity of the underlying preference structures. Furthermore, the same problem for single-type housing market can be easily computed in polynomial time. We also prove that it is NP-hard to check whether the strict core is non-empty.

We expect that the availability of MTTC will allow development of many applications for multi-type housing markets, especially those discussed in the beginning of the Introduction.

**Related Work and Discussions**

Housing markets are closely related to *house allocation*, where agents do not have initial endowments, and *matching*, where houses also have preferences over agents. See (Sonmez and Unver 2011) for a recent survey. In the past decade, housing markets has been a popular topic under multi-agent resource allocation (Chevaleyre et al. 2006).

Many subsequent works in Economics and AI extend the standard, single-type housing markets. For example, agents may be indifferent between houses (Quint and Wako 2004; Yilmaz 2009; Alcalde-Unzu and Molis 2011; Jaramillo and Manjunath 2012; Aziz and de Keijzer 2012; Plaxton 2013; Saban and Sethuraman 2013). Agents may desire multiple houses (Papai 2007; Todo, Sun, and Yokoo 2014; Sonoda et al. 2014; Fujita et al. 2015; Sun et al. 2015). Some agents may not have initial endowments (Abdulkadiroglu and Sonmez 1999; Chen and Sonmez 2002; Sonmez and Unver 2010).

In our setting, agent’s preferences are represented by the celebrated CP-nets (Boutilier et al. 2004). A CP-net allows agents to specify preferences over items within each type given other items allocated to her. In general, a CP-net represents a partial order over bundles of items. CP-nets have been heavily used in combinatorial voting, see for example (Rossi, Venel and Walsh 2004; Lang 2007; Lang and Xia 2016), but negative results often emerge as soon as agents’ CP-nets do not line up. Lexicographic orders (Booth et al. 2010) are special linear orders that extend some CP-nets, where agents can specify importance orders over types.

Previously, Bouvieret, Endriss, and Lang (2009) proposed a CP-net-like language and Fujita et al. (2015) used lexicographic orders to model agents’ preferences over bundles of items of a single type. Monte and Tumennasan (2015) characterized strategy-proof mechanisms for multi-type house allocation, when agents’ preferences include lexicographic preferences. To the best of our knowledge, our paper is the first time that CP-nets and lexicographic preferences are investigated in multi-type housing markets.

Konishi, Quint, and Wako (2001) assumed that agents have separable and additive preferences. This assumption is in general incomparable to our assumptions on agents’ preferences, because preferences represented by CP-nets are generally not separable, except when the CP-net has no edges. Therefore, our positive results are not obtained by putting further restrictions on agents’ preferences.

There has been very little work on multi-type housing markets after Konishi, Quint, and Wako’s negative results (Wako 2005; Klaus 2008). These papers focus on a different solution concept called coordinate-wise core rule, which are composed of type-wise strict-core allocations. We show that this naturally corresponds to the output of MTTC when agents’ have separable and lexicographic preferences with a common importance order.

**Preliminaries**

We consider a market consisting of a set \(N = \{1, \ldots, n\}\) of agents with \(p \geq 2\) types of indivisible items. For any \(i \leq p\), there are \(n\) items of type \(i\), denoted by \(T_i = \{i_1, \ldots, n_i\}\). For each item \(o\), Type\((o)\) is the type of \(o\), that is, \(o \in T_{\text{Type}(o)}\). Each agent \(j \in N\) initially owns exactly one item of each type, and her endowment is denoted by a \(p\)-vector \(O(j)\). W.l.o.g. in this paper we let \(O(j) = (j_1, \ldots, j_p)\). Let \(T = T_1 \times \cdots \times T_p\) be the set of all bundles, each of which is represented by a \(p\)-vector. We will often use vectors such as \(\vec{d}\) and \(\vec{c}\) to represent bundles, and for any \(i \leq p\), let \(\vec{d}_i\) denote the type-\(i\) item in \(\vec{d}\). A multi-type housing market \(M\) is given by the tuple \((N, \{T_1, \ldots, T_p\}, O)\).

Each agent desires to consume exactly one item of each type, and her preferences are represented by a linear order over \(T\). A preference profile \(P = (R_1, \ldots, R_n)\) is a collection of agents’ preferences. In any multi-type housing mar-
ket $M$, an allocation $A$ is a mapping from $N$ to $T$ such that for any $j \leq n$, $A(j)$ is the bundle allocated to $j$. Since no item is allocated twice, we have that for any $j \neq j'$ and any $i \leq p$, $A(j) \neq A(j')$. Given a market $M$, a mechanism $f$ is a function that maps agents' profile $P$ to an allocation in $M$.

**Axiomatic Properties**

A mechanism $f$ satisfies *individually rationality* if for any profile $P$, no agent prefers her initial endowment to her allocation by $f$. $f$ satisfies *Pareto optimality* if for any profile $P$, there does not exist an allocation $A$ such that (1) every agent weakly prefers her allocation in $A$ to her allocation in $f(P)$, and (2) some agent strictly prefers her allocation in $A$ to that in $f(P)$. $f$ is *non-bossy* if for any profile $P$, no agent can change any other agent’s allocation without changing her own by reporting differently. $f$ is *strategy-proof* if for each agent, falsely reporting her preferences is not beneficial. A mechanism satisfies *strong group strategy-proofness* if there is no group of agents $S$ who can falsely report their preferences so that (1) every agent in $S$ gets a weakly preferred bundle, and (2) at least one agent in $S$ gets a strictly preferred bundle.

An allocation $A$ is said to be *weakly blocked* by a coalition $S \subseteq N$, if the agents in $S$ can find an allocation $B$ of their initial endowments so that each agent weakly prefers allocation in $B$ to that in $A$, and some agent is strictly better off in $B$ than in $A$. The *strict core* of a market is the set of all allocations that are not weakly blocked by any coalition. A mechanism $f$ is *strict-core-selecting*, if for any profile $P$, $f(P)$ is always in the strict core.

**CP-nets and Lexicographic Preferences**

A (directed) CP-net $\mathcal{N}$ over $T$ is defined by (i) a directed graph $G = (\{T_1, \ldots, T_p\}, E)$, called the dependency graph, and (ii) for each $T_i$, there is a conditional preference table $\text{CPT}_i$, which contains a linear order $\succ_i$ over $T_i$ for each valuation $\vec{u}$ of the parents of $T_i$ in $G$, denoted $\text{Pa}(T_i)$. Each CPT-entry $\succ_i \vec{u}$ carries the following meaning: my preferences over type $i$ is $\succ_i \vec{u}$ given that I get items $\vec{u}$, and these preferences are independent of other items I get. An agents’ preferences are *separable* if there are no edges in the dependency graph.

Each CP-net $\mathcal{N}$ represents a partial order $\succ_{\mathcal{N}}$, which is the transitive closure of preference relations represented by all CPT entries, which are $\{(a_i, \vec{u}, \vec{z}) \succ_{\mathcal{N}} (b_i, \vec{u}, \vec{z}) : i \leq p; a_i, b_i \in T_i; \vec{u} \in \text{Pa}(T_i); \vec{z} \in \mathcal{T}_{-(\text{Pa}(T_i) \cup \{T_i\})}\}$.

For example, Figure 1 illustrates a separable CP-net. There are two types: houses (H) and cars (C), with two items per type. Since the preferences are separable, there is no edge in the dependency graph (in the left of the figure). The CPTs are shown in the middle of the Figure 1 and the partial order represented by the CP-net is shown in the right.

Let $\mathcal{O} = \{T_1 \succ \cdots \succ T_p\}$ be a linear order over the types. A CP-net is $\mathcal{O}$-legal, if there is no edge $(T_k, T_l)$ with $k > l$ in its dependency graph. A *lexicographic extension* of an $\mathcal{O}$-legal CP-net $\mathcal{N}$ is a linear order $\mathcal{V}$ over $T$, where for any $i \leq p$, any $\vec{x} \in T_1 \times \cdots \times T_{i-1}$, any $a_i, b_i \in T_i$, and any $\vec{y}, \vec{z} \in T_{i+1} \times \cdots \times T_p$, if $a_i \succ_x b_i$ in $\mathcal{N}$, then $(\vec{x}, a_i, \vec{y}) \succ_{\mathcal{V}} (\vec{x}, b_i, \vec{z})$. In other words, the agent believes that type $T_1$ is most important type to her, $T_2$ is the second most important type, etc. In a lexicographic extension, $\mathcal{O}$ is called the *importance order*.

In this paper an agent’s preferences are *lexicographic*, which means that each agent’s ranking is a lexicographic extension of a CP-net. We note that the CP-net does not need to be separable and the importance order can be different.

**Example 1.** Suppose the agent’s preferences is lexicographic w.r.t. the separable CP-net in Figure 1 and the importance order $H \succ C$, then her preferences are $(1_H, 1_C) \succ (1_H, 2_C) \succ (2_H, 1_C) \succ (2_H, 2_C)$. If her importance order is $C \succ H$, then her preferences are $(1_H, 1_C) \succ (2_H, 1_C) \succ (1_H, 2_C) \succ (2_H, 2_C)$. \hfill $\Box$

**The Multi-Type TTC Mechanism**

We propose the multi-type TTC (MTTC) mechanism as Algorithm 1. MTTC assumes that agents’ preferences are lexicographic (w.r.t. possibly different importance orders and possibly different CP-net structures).

**Algorithm 1 MTTC**

1: **Input:** A multi-type housing market $M$ and a profile $P$ of lexicographic preferences.
2: $t \leftarrow 1$. Let $L \leftarrow \cup_{i \in T} T_i$ be the set of unallocated items. Let $A$ be the empty assignment. For each $j \leq n$, let $i_j^*$ is agent $j$’s most desirable type.
3: while $L \neq \emptyset$ do
4: Build a directed graph $G_i = (N \cup L, E)$. For every $j \in N, i_j \in L$, add edge $(j, i_j)$ to $E$. For every $j \in N$, add edge $(j, i_j^*)$ to her most preferred item in $L$ of type $i_j^*$, to $E$.
5: **Implement cycles in** $G_i$. For each cycle $C$, for every $(j, i_j^*) \in C$, assign $[A(j)]_{i_j^*} = [O(j)]_{i_j^*}$.
6: Remove assigned items from $L$.
7: For any agent $j$ who is assigned an item, set $i_j^*$ to be the next type according to $j$’s importance order.
8: $t \leftarrow t + 1$.
9: end while
10: **Output:** The allocation $A$.

**Example 2.** Consider the market with 3 agents and 2 types: Houses (H) and Cars (C) with items $\{1_H, 2_H, 3_H\}$ and $\{1_C, 2_C, 3_C\}$, respectively. Let the initial endowments of each agent $j$ be $(j_H, j_C)$. Figure 2 shows agents’ lexicographic preferences.

\[ H \text{ Pref. over } H \]

\[ C \text{ Pref. over } C \]

| $H$ | $P$ | $C$ |
|-----|-----|-----|
| $1_H > 2_H$ | $(1_H, 1_C)$ | $(1_H, 2_C)$ |
| $(2_H, 1_C)$ | $(2_H, 2_C)$ |

Figure 1: A CP-net.

Figure 2: Showing agents’ lexicographic preferences.
It is not hard to see that when all agents have the same rationality, the output order of MTTC is the same as the desired type. It takes $O(n)$ time to find the most preferred remaining item. The algorithm runs in $O(n^2p)$ time.

**Step 2:** MTTC satisfies strict-core-selecting. For the sake of contradiction, let the allocation $A$ produced by the algorithm does not belong to the strict core. Then, there exists a coalition $S$, and an allocation $B$ on $S$ that blocks $A$. Let $t^*$ be the first round in which the algorithm’s assignments restricted to $S$ at the end of the round differs from the assignments in $B$. Then, there is some $j \in S$, a type $i$ such that at least one of the following cases hold:

1. $j$ is assigned a different item of type $i$ in $B$ than in $A$, or
2. the item $j_i$ initially endowed to an agent in $S$ is assigned to a different agent in $B$ than in $A$.

Suppose Case (1) holds. There exists some agent $j \in S$, a type $i$ such that $j_i^* \neq [B(j)]_i = [A(j)]_i = j_i$. Note that $t^*$ is the first iteration where the allocations differ, and the algorithm assigns items of agents’ most desired type. Then, we must have that $j_i^* \succ j_i$ since for every type that takes precedence over $i$, the allocation in $B$ is the same as the allocation in $A$. Now, if $j_i^*$ was available at round $t^*$, then we must have that the only outgoing edge from $j_i$ points at the owner $j_i^*$ and not at $j_i$ at round $t$. This is a contradiction to the assumption that $A$ is an output of the algorithm. If $j_i^*$ was unavailable at round $t^*$, then it must already have been assigned to another agent in a strictly earlier round $t' < t^*$ which is a contradiction to our assumption that $t^*$ is the first round where assignments restricted to $S$ differ.

Now, consider Case (2). We will show that it reduces to Case (1). Suppose different agents $j$ and $j^*$ receive item $j_i$ in $A$ and $B$ respectively i.e. $[A(j)]_i = j_i = [B(j^*)]_i \neq j_i$. Let $C$ be the cycle that is implemented at round $t^*$ of the algorithm. W.L.O.G. let $C$ consist of edges $(1, 2_{i_1}), \ldots, (k, 1_{i_k}), (1_{i_1}, 3)$, involving agents $1, \ldots, k$ whose most desired types are $i_1, \ldots, i_k$ at round $t^*$. 

![Figure 2: A lexicographic and separable profile $P$ and an execution of MTTC on $P$.](image-url)
Let agent $1 \in S$, $1_{ik}$ is an item owned by an agent in $S$, and the agents who receive $1_{ik}$ differ in $A,B$. Now, if $k \in S$, we must have that $[B(k)]_{ik} \neq [A(k)]_{ik} = 1_{ik}$, since $1_{ik}$ is being assigned to a different agent in $B$ and this reduces to Case (1). If $k \notin S$, there must be some consecutive pair of nodes $j, j + 1_{ik}$ in $C$ such that $2 \leq j \leq k, j + 1 \notin S$. Then, we must have that $[B(j)]_{ik} \neq [A(j)]_{ik}$, since $j$ can only be assigned items initially endowed to agents in $S$ in the allocation $B$. Again, this reduces to Case (1).

**Step 3: Defining MTTC* the single-cycle-elimination variant of MTTC.** To prove the non-bossiness, monotonicity, and strong group-strategy-proofness, we will consider a class of algorithms called MTTC*, which is similar to MTTC except that in each round a single trading cycle is implemented. There are multiple MTTC* algorithms, each corresponds to a different order of implementing cycles. Just by definition we do not know yet whether different MTTC* algorithms correspond to different allocations. Later we will show that, indeed different MTTC* algorithms on the same preferences must output the same allocation. In fact, we will prove a stronger lemma (Lemma [1]) that characterizes the executions of all MTTC* algorithms.

**Definition 1.** Given a housing market $M$ and any profile $P$, let $\text{MTTC}^*(P)$ denote the set of algorithms, each of which is a modification of MTTC (Algorithm 1), where instead of implementing all cycles in each round, the algorithm implements exactly one available cycle in each round.

We note that MTTC*(P) depends on $P$ because for different profiles the cycles in each round might be different. For each $\mathcal{A} \in \text{MTTC}^*(P)$, we let $\text{Order}(\mathcal{A})$ denote the linear order over the cycles that $\mathcal{A}$ implements. That is, if $\text{Order}(\mathcal{A}) = C_1 \succ \cdots \succ C_k$, then it means that for any $t \leq k$, $C_t$ is the cycle implemented by $\mathcal{A}$ in round $t$.

**Example 3.** Let $M,P$ be the same as in Figure 2. Let $C_1,C_2,C_3,C_4$ be the same cycles in Figure 2. Let $\mathcal{A}$ be the MTTC* algorithm such that for any $t \leq 4$, in the $t$-th round $C_t$ is implemented. Then, $\text{Order}(\mathcal{A}) = C_1 \succ C_2 \succ C_3 \succ C_4$.

**Definition 2.** For any multi-type housing market $M$, let $\text{Cycles}(P)$ denote the set of cycles implemented in the execution of MTTC on $P$. We define a partial order $\text{PO}(P)$ over $\text{Cycles}(P)$ as follows. For every pair of cycles $C_k,C_l$, $C_k \succ C_l$ in $\text{PO}(P)$ if one of the following two conditions hold:

1. There is an agent who gets an item of a more important type in $C_k$ than in $C_l$. That is, $\exists j \in C_k \cap C_l$ such that $\text{Type}(C_k(j)) \succ j \text{ Type}(C_l(j))$.

2. There is an agent in $C_l$ who prefers an item in $C_k$ over the item she is pointing to in $C_l$ of the same type, conditioned on the item she got in previous rounds. That is, there exists $j \in C_l$ and $j' \in C_k$, such that, $\text{Type}(C_k(j')) = \text{Type}(C_l(j))$ and $C_k(j') \succ j C_l(j)$ conditioned on the items $j$ got in previous rounds.

Then, $\text{PO}(P)$ is the transitive closure of the two classes of binary relations mentioned above. Let $\text{Ext}(P)$ denote the set of linear orders that extend $\text{PO}(P)$.

**Example 4.** Continuing Example 3, $\text{PO}(P)$ is illustrated in Figure 3. $\text{PO}(P) = \{C_1 \succ C_2, C_1 \succ C_3, C_1 \succ C_4, C_3 \succ C_4\}$. For all $2 \leq i \leq 4$, $C_i \succ C_l$ because houses are more important than cars to all agents. $C_3 \succ C_4$ since agent 1 has a more preferred car in $C_3$ than in $C_4$.

![Figure 3: PO(P) in Example 4](image)

We are now ready to present the key lemma that establishes the equivalence between $\text{MTTC}^*(P)$ and $\text{Ext}(P)$.

**Lemma 1.** For any multi-type housing market $M$ and any profile $P$, we have $\text{Order}(\text{MTTC}^*(P)) = \text{Ext}(P)$.

**Proof.** We first prove the $\supseteq$ direction. For any $W \in \text{Ext}(P)$, we will prove that there exists an MTTC* algorithm $\mathcal{A}$ that implements cycles exactly as in $W$. W.l.o.g. let $W = C_1 \succ \cdots \succ C_k$. Suppose for the sake of contradiction, $W$ is not in $\text{Order}(\text{MTTC}^*(P))$. Let $1 \leq h \leq k$ denote the smallest number such that there exists $\mathcal{A} \in \text{MTTC}^*(P)$ whose top $h-1$ cycles in $\text{Order}(\mathcal{A})$ are exactly $C_1 \succ \cdots \succ C_{h-1}$ but whose $h$-th cycle in $\text{Order}(\mathcal{A})$ is not $C_h$; let $\mathcal{A}$ denote this MTTC* algorithm. Let $G_h$ denote the graph at the beginning of round $h$ of algorithm $\mathcal{A}$.

We now prove that $C_h$ must be in $G_h$ by showing that each edge $(j, o_1)$ in $G_h$ must be in $G_h$. Suppose for the sake of contradiction that this is not true and $(j, o_1)$ is in $G_h$ but not in $G_h$. There are three cases: (1) agent $j$ has not obtained an item in a more important type than $i$ yet; (2) agent $j$ is pointing to a more preferable item in type $i$ than $o_1$; and (3) agent $j$ has obtained an item in type $i$ in a previous round. Case (1) and (2) correspond to the two cases in the definition of $\text{PO}(P)$ (Definition 2), respectively. Therefore, neither can hold because $W$ is an extension of $\text{PO}(P)$, which means that the more preferable items must have been allocated in the first $h-1$ rounds of $\mathcal{A}$. The third case cannot be true either, because otherwise it means that there exists $C_l$ with $l \leq h-1$ where $j$ points to a different item in type $i$ than $o_1$. This means that agent $j$ gets two items in type $i$ in MTTC*, which is a contradiction. Therefore, $C_h$ must be in $G_h$.

However, this contradicts the minimality of $h$. Therefore, $W \in \text{Order}(\text{MTTC}^*(P))$. This proves the $\supseteq$ direction.

We now prove the $\subseteq$ direction. Suppose for the sake of contradiction there exists $\mathcal{A} \in \text{MTTC}^*(P)$ such that $\text{Order}(\mathcal{A}) \notin \text{Ext}(P)$. W.l.o.g. let $\text{Order}(\mathcal{A}) = [C_1 \succ \cdots \succ C_{h-1} \succ C_{h} \succ \cdots]$, such that there exists no $L \in \text{Ext}(P)$ that agrees with $\text{Order}(\mathcal{A})$ with the order of the top $h-1$ elements (cycles), but there does not exist $L^* \in \text{Ext}(P)$ that agrees with $\text{Order}(\mathcal{A})$ with the order of the top $h$ cycles. W.l.o.g. let $L = [C_1 \succ \cdots \succ C_k]$, where $C_{h} \neq C_k$. By...
the ⊆ direction, there exists \( A_L \in \text{MTTC}^*(P) \) such that \( \text{Order}(A_L) = L \).

We claim that \( C_h^* \in \{C_{h+1}, \ldots, C_h\} \). Let \( G_h \) denote the graph at the beginning of round \( h \) in \( A \). \( G_h \) must also be the graph at the beginning of round \( h \) in \( A_L \), because the first \( h - 1 \) cycles in \( \text{Order}(A) \) and those in \( \text{Order}(A_L) \) are exactly the same. It follows that \( C_h^* \) is in \( G_h \). Therefore, in one of the following rounds in \( A_L \), \( C_h^* \) must be implemented, otherwise the items involved in it will never be allocated.

Let \( C_h^* = C_i \) for some \( h \leq l \leq k \). There must exist \( h + 1 \leq l' \leq k \) such that \( C_{l'} \succ C_i \) in \( \text{PO}(P) \), otherwise there is an extension \( \text{PO}(P) \) where the top \( h \) cycles are \( C_1 \succ \cdots \succ C_h \succ C_i \), which contradicts the assumption that no order in \( \text{Order}(A) \) agrees with \( \text{PO}(P) \) on the top \( h \) cycles.

However, by the definition of \( \text{PO}(P) \) and by the fact that \( C_i \) has not been implemented in the first \( h - 1 \) round of \( A \) (as well as \( A_L \)), it is impossible that \( C_i \) is in \( G_h \). This contradicts the assumption that \( C_h^* = C_i \) is implemented in round \( h \) by \( A \). This proves the ⊆ direction. \( \square \)

It follows directly from Lemma 1 that the outcomes of all \( \text{MTTC}^*(P) \) are the same, which is \( \text{MTTC}(P) \).

**Lemma 2.** For any multi-type housing market \( M \), any profile \( P \), and any agent \( j \), let \( P' \) be any market where agent \( j \) changes her top bundle to be \( \text{MTTC}(P)(j) \) and other agents’ preferences are the same, then \( \text{MTTC}(P') = \text{MTTC}(P) \).

**Proof.** W.l.o.g. let \( j = 1 \) and \( C_1 = \{C_{k_1}, C_{k_2}, \ldots, C_{k_h}\} \) be the set of cycles that involve agent \( j \) in \( \text{Cycles}(P) \). Let \( L \in \text{Ext}(P) \). By Lemma 1 there exists \( A \in \text{MTTC}^*(P) \) such that \( \text{Order}(A) = L \).

We next prove that \( A \) can be applied to \( P' \) and the output is the same as \( A \). To see this, we will examine two parallel runs of the execution of \( A \) with input \( P \) and with input \( P' \), respectively. For any \( t \geq 0 \), let \( G_t \) and \( G_t' \) denote the graphs of \( \text{MTTC}^* \) in the beginning of round \( t \) w.r.t. \( P \) and \( P' \), respectively. We claim that for all \( t \leq k \), \( G_t = G_t' \) by induction. The base case of \( t = 0 \) is straightforward. Suppose the claim is true for all \( t \leq h - 1 \). Then if the \( h \)-th cycle in \( L \) is not in \( C_1 \), then \( G_h = G_h' \) because all agents except agent 1 have the same preferences in \( P \) and in \( P' \). If the \( h \)-th cycle in \( L \) is in \( C_1 \), then suppose agent 1 points to an item \( o_1 \in T_i \) in \( G_t \). This means that \( |\text{MTTC}(P)(1)|_i = o_1 \). Therefore, agent 1 also points to \( o_1 \) in \( G_t' \). This proves that \( G_t = G_t' \) for all \( t \leq k \).

Therefore, \( A \) can be successfully applied to \( P' \) and the allocation is the same as \( \text{MTTC}(P) \). By Lemma 1 \( \text{MTTC}(P') \) is the same as the output of \( A \) on \( P' \), which proves the lemma. \( \square \)

**Step 4: MTTC satisfies non-bossiness.** Suppose for the sake of contradiction \( \text{MTTC} \) does not satisfy non-bossiness. W.l.o.g. let \( M \) and \( M' \) denote two markets where only agent 1’s preferences are different, \( \text{MTTC}(P)(1) = \text{MTTC}(P')(1) \), yet \( \text{MTTC}(P) \neq \text{MTTC}(P') \). Let \( M \) denote the market obtained from \( M \) by letting agent 1’s top-ranked bundle to be \( \text{MTTC}(P)(1) \). By Lemma 2 \( \text{MTTC}(M) = \text{MTTC}(M) \) and \( \text{MTTC}(M) = \text{MTTC}(P') \), which is a contradiction.

**Step 5: MTTC is strong group-strategyproof when the agents cannot lie about the importance orders.** Suppose for the sake of contradiction that the proposition is not true in a multi-type housing market \( M \). Let \( P \) denote the truthful profile, \( S \subseteq N \) denote the group of strategic agents, and \( P' \) denote the untruthful profile where preferences of all truthful agents are the same as in \( P \). Let \( P \) denote the profile obtained from \( P' \) by letting the top-ranked bundle of all agents \( j \in S \) be \( \text{MTTC}(P')(j) \). By sequentially applying Lemma 1 to all agents in \( S \), we have that \( \text{MTTC}(P) = \text{MTTC}(P') \). In particular, all agents in \( S \) get the same bundles in \( \text{MTTC}(P) \) as those in \( \text{MTTC}(P') \).

We now compare side by side two parallel runs of two \( \text{MTTC}^* \) algorithms: \( A \in \text{MTTC}^*(P) \) and \( \bar{A} \in \text{MTTC}^*(P) \). We will define \( A \) and \( \bar{A} \) dynamically. Starting with \( t = 0 \), let \( G_t \) and \( \bar{G}_t \) denote the graphs of \( \text{MTTC}^* \) at the beginning of round \( t \) for input \( P \) and input \( P' \), respectively. If there is a common cycle \( C \) in \( G_t \) and \( \bar{G}_t \), then we let both \( A \) and \( \bar{A} \) implement \( C \) and move on to the next round.

Because \( \text{MTTC}(P) \neq \text{MTTC}(P') \), there exists a round \( t \) such that \( G_t \) and \( \bar{G}_t \) does not have a common cycle. Let \( t^* \) be the earliest such round. We note that for any \( j \notin S \), the outgoing edge of \( j \) in \( G_t \) and that in \( \bar{G}_t \) must be same, because \( j \) is truthful and the remaining items in \( G_t \) and in \( \bar{G}_t \) are the same. Let \( C \) denote an arbitrary cycle in \( G_t \). It follows that there exists \( j \in S \) such that \( (j, o_t) \in C \) and \( (j, s_t) \in \bar{G}_t \), where \( o_t \neq s_t \). Because no agent is allowed to lie about the importance order, and the allocation of all previous rounds are the same in \( A \) and in \( \bar{A} \), we must have \( i = i^*, s_t \) is in \( G_t \), and the items agent \( j \) gets in all previous rounds in \( A \) is the same as that in \( \bar{A} \). Therefore, agent \( j \) strictly prefers \( o_t \) to \( s_t \), give her allocation of more important types in previous rounds in \( A \). We note that because agent \( j \)’s top-ranked bundle is her final allocation in \( \text{MTTC}(P) \), agent \( j \) must get \( s_t \) in \( \text{MTTC}(P) \). Because agent \( j \) gets \( o_t \) in \( \text{MTTC}(P') \), we have that \( \text{MTTC}(P)(j) \succ_j \text{MTTC}(P) \). This contradicts the assumption that none of agents in \( S \) is strictly worse off in \( \text{MTTC}(P) \). Therefore, \( \text{MTTC} \) is strong group-strategyproof when agents cannot lie about the importance orders. \( \square \)

**Proposition 1.** \( \text{MTTC} \) is not strategy-proof w.r.t. only misreporting the importance order (i.e. without misreporting local preferences over types).

**Proof.** Consider the preferences in Figure 2. We recall that when agents are truthful, the output of \( \text{MTTC} \) is \((2_H, 1_C), (3_H, 2_C), (1_H, 3_C)\) to agents 1, 2, 3, respectively (Example 2). If agent 1 misreport the importance order as \( C \succ H \) without misreporting any preferences over types,
then the output of MTTC is \((2_H, 3_C), (3_H, 2_C), (1_H, 1_C)\) to agents 1, 2, 3, respectively. We note that agent 1 prefers \((2_H, 3_C)\) to \((2_H, 1_C)\). This proves the proposition.

**Proposition 2.** The strict core of a multi-type housing market can be multi-valued, even when agents’ preferences are separable and lexicographic w.r.t. the same order.

**Proof.** Consider the preferences in Figure 2. Let \(B\) denote the allocation such that \(B(1) = (2_H, 3_C), B(2) = (3_H, 2_C), B(3) = (1_H, 1_C)\). For the sake of contradiction, suppose \(S\) be a blocking coalition to \(B\). We can observe that agents 1, 2 each receive their top bundles in \(B\) and have no incentive to participate in a coalition. Therefore \(S = N\). However, it can be verified that \(B\) is Pareto optimal. Further, agent 3 cannot benefit by not participating since \((1_H, 1_C) \succ (3_H, 3_C)\). This means that there is no coalition that blocks \(B\), which is a contradiction.

The next Theorem states that, not only is MTTC susceptible to misreporting importance order, but also, all mechanisms that satisfies individual rationality and Pareto optimality do. We note the MTTC satisfies strict-core-selecting, which implies individual rationality and Pareto optimality.

**Theorem 2.** For any multi-type housing market \(M\) with \(n \geq 3\) and \(p \geq 2\), there is no mechanism that satisfies individually rationality, Pareto optimality, and strategy-proofness, even when agents’ preferences are lexicographic and separable.

**Proof.** For the sake of contradiction, let \(f\) be any mechanism that is individually rational, Pareto-optimal and strategy-proof. Consider the agents’ lexicographic and separable preferences \(P\) as in Figure 4. Explicitly, \(P\) is the following:

1. \((2_H, 1_C) \succ (2_H, 3_C) \succ (2_H, 2_C) \succ (1_H, 1_C) \succ \text{others}\).
2. \((3_H, 2_C) \succ (2_H, 2_C) \succ \text{others}\).
3. \((1_H, 1_C) \succ (1_H, 3_C) \succ (1_H, 2_C) \succ (3_H, 1_C) \succ (3_H, 3_C) \succ \text{others}\).

The only individually rational allocations are:

1. \(((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))\).
2. \(((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))\).
3. \(((1_H, 1_C), (2_H, 2_C), (3_H, 3_C))\).

Only (i) and (ii) are also Pareto-optimal.

Since \(f\) is individually rational and Pareto optimal, \(f(P)\) must be either \(((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))\), or \(((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))\). We will show that in either case there is some agent who has an incentive to misreport her preferences.

Suppose \(f(P) = ((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))\). Then, consider the case where agent 1 misreports her lexicographic order as \(2 \succ 1\). Agent 1’s preferences over bundles is \((2_H, 1_C) \succ (1_H, 1_C) \succ \text{others}\). The allocation \(((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))\) is not individually rational w.r.t. the profile with agent 1’s misreported preferences.

The only allocation that is both individually rational and Pareto optimal w.r.t. the misreported preferences is \(((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))\). Therefore, \(f\) must select this allocation. Note that agent 1 does strictly better, \((2_H, 1_C) \succ (2_H, 3_C)\), when she misreports. This contradicts the assumption that \(f\) is strategy-proof.

Suppose \(f(P) = ((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))\). Then consider the case where agent 3 misreports her lexicographic order as \(2 \succ 1\), and her local preferences over type 1 as \(3_H \succ 1_H \succ 2_H\). Agent 3’s preference over bundles is \((3_H, 1_C) \succ (1_H, 1_C) \succ (2_H, 1_C) \succ (3_H, 3_C) \succ \text{others}\). The only allocation that is both individually rational and Pareto optimal w.r.t. the profile with agent 3’s misreported preferences is \(((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))\). Therefore, \(f\) must select this allocation. Note that agent 3 gets a strictly better bundle, \((1_H, 1_C) \succ (1_H, 3_C)\), when she misreports her preferences. This contradicts the assumption that \(f\) is strategy-proof.

Therefore, such a mechanism \(f\) does not exist. This proves the theorem.

**Computing the Membership of the Strict Core**

**Definition 3 (InStrictCore).** Given a multi-type housing market \(M\), agents’ preferences \(P\), and an allocation \(A\), we are asked whether \(A\) is a strict core allocation w.r.t. \(M\).

**Theorem 3.** InStrictCore is co-NPC even when agents have separable lexicographic preferences over \(p \geq 2\) types.

**Proof.** We start by noting that given a blocking coalition \(S\) and an allocation \(B\) it is easy to check whether \(S\) blocks \(A\) when agents have lexicographic preferences.

We show a reduction from 3-SAT. An instance of 3-SAT is given by a formula \(F\) in 3-CNF, consisting of clauses \(c_1, \ldots, c_n\) involving Boolean variables \(x_1, \ldots, x_m\) that can take on values in \(\{0,1\}\), and we are asked whether there is a valuation of the variables that satisfies \(F\).

**Example 5.** \(F = (x_1 \lor x_2 \lor x_3) \land (\bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_3)\) is a formula in 3-CNF involving variables \(x_1, x_2, x_3\) and clauses \(c_1 = x_1 \lor x_2 \lor x_3, c_2 = \bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_3\). We will use this example to illustrate the proof.

Given an arbitrary instance \(I\) of 3-SAT, we construct an instance \(J\) of InStrictCore where \(P\) is a profile of lexicographic preferences over \(p = 2\) types where all agents have the same importance order \(1 \succ 2\), and an allocation \(A\) as follows:

**Agents:**

- For every clause \(c_j\), we have an agent \(c_j\).
• For every Boolean variable $x_i$, we have agents $1_i^j, 0_i^j$, one for every clause $c_j$ that involves literals of the variable $x_i$.

• For every variable $x_i$, we have additional agents $\bar{x}_i$, and $\bar{x}_i$.

• Additionally, for every variable $x_i$, we have an agent $d_i$.

• Lastly, we add agents $c_1, e_2$.

**Initial Endowments:** Every agent $a$ is initially endowed with the bundle $([a]_1, [a]_2)$.

**Preferences and Allocations:** We will represent agents’ lexicographic preferences in $P$ with importance order $1 \gg 2$ for every agent, and their allocated items in $A$ by 2 directed graphs $G_1, G_2$, one for each type. $G_1, G_2$ have a node for each agent.

A *dashed* edge $(a, b)$ in $G_k$ represents (i) the preference $[b]_k \succ [a]_k$, and (ii) the allocation $[A(a)]_k = [b]_k$. A *solid* edge $(a, b)$ indicates a strict preference $[b]_k \succ [A(a)]_k$. The absence of an edge $(a, b)$ indicates $[b]_k \prec [a]_k$.

– The graph $G_1$ corresponding to Example 5 is in Figure 5.

![Figure 5: The graph $G_1$ represents agents’ preferences and their allocations in $A$ over type 1 corresponding to Example 5. Here, dashed edges represent allocations in $A$, and preference over initial endowment, and solid edges represent strict preference over allocation in $A$. For example, for agent $1_i^j$, $[A(1_i^j)]_1 = [1_i^j]_1 \succ [1_i^j]_2$ and $[0_i^j]_1 \prec [0_i^j]_2$.](image)

Formally, we construct $G_1$ as follows:

• For each variable $x_i$, for each $j \leq k_{i-1}$, we add dashed edges $(1_i^j, 1_i^{j+1})$ and $(0_i^j, x_i)$.

• For each variable $x_i$, for each $j \leq k_{i-1}$, we add dashed edges $(0_i^j, 0_i^{j+1})$ and $(0_i^j, x_i)$.

• For $i \leq m$, we add a dashed edge $(x_i, 1_i^{i+1})$ and solid edges $(x_i, 0_i^{i+1}), (\bar{x}_i, 1_i^{i+1}), (\bar{x}_i, 0_i^{i+1})$.

• For $i \leq m$, we add dashed edges $(\bar{x}_i, d_i), (d_i, 0_i^j)$.

• For each of $j \leq n - 1$, we add dashed edges $(c_j, c_{j+1})$.

• We add dashed edges $(c_1, e_1), (c_n, e_2)$.

• Finally, we add dashed edges $(e_2, 1_i^j), (x_m, e_1)$, solid edges $(e_2, 0_i^j), (\bar{x}_m, e_1)$.

– We illustrate the construction of $G_2$ in Figure 6.

![Figure 6: Graph $G_2$ representing agents’ preferences and allocations in $A$ over type 2 corresponding Example 5](image)

Formally, $G_2$ is constructed as:

• For $i \leq m$, for $j \leq k_i$, we add dashed edges $(1_i^j, 0_i^j), (0_i^j, 0_i^j)$.

• For $i \leq m$, we add dashed edges $(x_i, x_i), (\bar{x}_i, \bar{x}_i)$.

• We add dashed edges $(c_1, c_1), (c_n, c_n)$.

• For every clause $c_j$, if $x_i$ is a literal in $c_j$, we add solid edges $(0_i^j, 0_i^j)$ in $G_2$.

$P$ is the profile of lexicographic preferences where agent has the importance order $1 \gg 2$ over types and every agents’ local preferences over items of type 1 and type 2 are represented by the graphs $G_1$ and $G_2$ respectively. This completes the construction.

We briefly illustrate the idea behind the rest of the proof using Example 5. Consider the satisfying valuation $\phi = (1, 1, 0)$. It is easy to verify that there is a coalition of agents $S = \{1_i^1, 1_i^2, x_1, x_2, 0_i^2, 1_i^3\}$, which agents in $S$ all receive an item of type 1 that is weakly preferred over their item in $A$ according to their local preferences given by $G_1$. Agents $c_1$ and $c_2$ strictly improve over their allocations in $A$ according to their preferences in $G_2$. By exchanging items with agents $1_i^1$ and $0_i^2$, respectively, we can verify that $S$ is a blocking coalition of agents in $G_1$. And agents $c_1$ and $c_2$ strictly improve over their allocations in $A$ by exchanging items with agents $1_i^1$ and $0_i^2$, respectively. Now, consider the negative example of the valuation $\psi = (1, 1, 1)$ and the coalition $\{1_i^1, 1_i^2, x_1, x_2, 1_i^3\}$, which agents in $S$ all receive an item of type 2. However, agent $c_2$ can only receive a strictly worse item of type 2 compared to her allocation in $A$ from the agents in $S$.

At a high level, we use the construction in graph $G_1$ to ensure that the valuation of variables is consistent by ensuring that for any variable $x_i$, we may only exclusively have either agents $1_i^j$ or $0_i^j$ in any blocking coalition. We use $G_2$ to ensure that if there is a coalition that includes every agent $c_j$, then there is a satisfying valuation of the Boolean variables that satisfies all clauses. We now proceed with the formal proof.

**Claim 1.** $(\Leftarrow)$ If $I$ is a Yes instance, then $J$ is a No instance.
Proof. Let \( \phi \) be a valuation that satisfies the formula \( F \) in instance \( I \). We will show the existence of a blocking coalition in \( J \).

Consider the coalition \( S \):
- For every \( i \leq m \), if \( \phi_i = 1 \) (similarly 0), then \( \forall j \leq k_i, 1_i^j \in S \) and \( x_i \in S \) (similarly \( 0_i^j, x_i \)).
- \( \forall j \leq n, c_j \in S \).
- \( e_1, e_2 \in S \).

We will now construct an allocation \( B \) on \( S \) that weakly improves every agents’ allocation over \( A \) and is strictly improving for some agent. We will construct \( B \) by identifying cycles in the graph and assigning every agent in a cycle the item of the agent she is pointing at.

Consider the graph \( G_1 \) and the allocations corresponding to the cycle:
- Edges \( (e_1, c_1), (c_1, c_2), \ldots, (c_{m+1}, c_n), (c_n, e_2) \).
- If \( \phi_1 = 1 \), the edge \( (e_2, 1_i^1) \). (Similarly, \( \phi_1 = 0, 0_i^1 \))
- For every \( i \leq m \), if \( \phi_i = 1 \), the edges \( j \leq k_i - 1, (1_i^j, 1_i^{j+1}) \) and the edge \( (1_i^m, x_i) \).
- For every \( i \leq m - 1 \), if \( \phi_i = 1 \) and \( \phi_{i+1} = 1 \), the edge \( (x_i, 1_i^{i+1}) \).
- If \( \phi_m = 1 \), the edge \( (x_m, e_1) \).

In \( G_2 \), consider the allocations corresponding to the cycles:
- For \( j \leq n \), let \( i \) be the lowest number such that \( x_i = \phi_i \) satisfies clause \( c_j \), and let \( \phi_i = 1 \), then a cycle involving solid edges \( (c_j, 1_i^j), (1_i^j, c_j) \). (similarly if \( \phi_i = 0 \), a cycle involving \( c_j, 0_i^j \)).
- For the remaining agents \( 1_i^j \in S \) (or \( 0_i^j \in S \)), agents \( x_i \in S \) (or \( x_i \in S \)) and agents \( e_1, e_2 \), a self loop.

In the allocation \( B \) constructed by implementing the cycles we identified, every agent receives a weakly improving bundle over their allocated bundle in \( A \) since they receive items of both types that are weakly preferred according to their local preferences. The agents \( c_j \) and \( 1_i^j \) (or \( 0_i^j \)) corresponding to a satisfying valuation of the variable \( x_i \) involved in clause \( c_j \), receive strictly improving bundles by receiving strictly preferred items of type 2 in \( B \) compared to \( A \) according to their local preferences. Lastly, note that \( B \) is an allocation on \( S \) since none of the items allocated in \( B \) were initially endowed to any agent outside \( S \). This shows that \( S \) is a blocking coalition and \( B \) is a weakly improving allocation on \( S \), and strictly improving for some agents in \( S \). Therefore, \( J \) is a No instance.

Claim 2. \( (\Rightarrow) \text{If } J \text{ is a No instance, } I \text{ is a Yes instance.} \)

Proof. We are given that \( J \) is a No instance. Let \( S \) be a coalition that blocks \( A \) w.r.t. preferences in \( P \) and let \( B \) be an allocation on \( S \) such that (i) \( \forall a \in S, B(a) \succeq A(a) \), and (ii) \( \exists a \in S, B(a) \succ A(a) \).

We start by proving some simple properties about the membership of agents in \( S \) that are specific to our construction. Throughout, we will use the fact that \( B \) must be an allocation where (a) \( \exists a \in S \) such that \( [B(a)]_1 < [A(a)]_1 \), and (b) \( \forall a \in S, [B(a)]_1 = [A(a)]_1 \), then we must have that \( [B(a)]_2 \geq [A(a)]_2 \). Otherwise, there is some agent \( a^* \) for which the allocation \( B \) fails to satisfy either of these conditions, then \( B(a^*) \prec A(a^*) \).

Lemma 3. For any \( i \leq m \), any \( j \leq k_i - 1 \), (i) if \( 1_i^j \in S \), then \( [B(1_i^j)]_1 = 1_i^{j+1} + 1 \in S \), and (ii) if \( 0_i^j \in S \), then \( [B(0_i^j)]_1 = 0_i^{j+1} + 1 \in S \).

Proof. Let there be some agent \( 1_i^j \in S \), \( i \leq j \leq k_i - 1 \), such that \( [B(1_i^j)]_1 \neq 1_i^{j+1} + 1 \). Then, according to her local preferences in \( G_1 \), \( [B(1_i^j)]_1 \prec [A(1_i^j)]_1 \), and by her importance order \( 1 \succ J, B(1_i^j) \prec A(1_i^j) \). This is a contradiction to our assumption that \( B \) is an improving allocation for every agent in \( S \).

Lemma 4. For any \( i \leq m \), any \( 2 \leq j \leq k_i \), (i) if \( 1_i^j \in S \), then \( [B(1_i^{j-1})]_1 = 1_i^j + 1 \), and (ii) if \( 0_i^j \in S \), then \( [B(0_i^{j-1})]_1 = 0_i^j \).

Proof. By Lemma 3 \( 1_i^j \) must be assigned \( 1_i^{j+1} + 1 \) (or \( j = k_i \), one of the items \( [1_i^{j+1}]_1, [0_i^{j+1}]_1 \), or \( [e_2]_1 \)). Then, the item \( [1_i^j]_1 \) must be assigned to another agent in \( S \). Suppose an agent \( a \in S, x \neq 1_i^j \) is assigned \( [1_i^j]_1 \) in \( B \). However, according to the preferences in \( G_1 \), for every agent \( a \) other than \( 1_i^j \), the item \( [1_i^j]_1 \) is strictly worse than their allocated type-1 item in \( A \). Then, \( B(a) \prec A(a) \), a contradiction to our assumption that \( B \) is weakly improving for every agent in \( S \).

Lemma 5. For any \( i \leq m \), any \( j^* \leq k_i \), (i) if \( 1_i^j \in S \), then for every \( j \leq k_i, 1_i^j \in S \), \([B(1_i^j)]_1 = [A(1_i^j)]_1 \), and \( x_i \in S \), and (ii) if \( 0_i^j \in S \), then for every \( j \leq k_i, 0_i^j \in S \), \([B(0_i^j)]_1 = [A(0_i^j)]_1 \), and \( x_i \in S \).

Proof. If \( 1_i^j \in S \), by Lemma 3 for every \( j = j^* + 1, \ldots \), \( k_i \), agent \( 1_i^j \in S \) and \([B(1_i^j)]_1 = [A(1_i^j)]_1 \), and by Lemma 4 for every \( j^* = 1, \ldots, j^* - 1 \), agent \( 1_i^j \in S \) and \([B(1_i^j)]_1 = [A(1_i^j)]_1 \). Then, for \( 1_i^{k_i} \in S, [B(1_i^{k_i})]_1 = [A(1_i^{k_i})]_1 \), and \( x_i \in S \).

Lemma 6. For any \( i \leq m \), any \( j \leq k_i \), if \( 1_i^j \in S \) and \([B(1_i^j)]_1 \leq [A(1_i^j)]_1 \), then either \([B(1_i^j)]_2 = [1_i^{j+1}]_2 \) or \([B(1_i^j)]_2 = [c_i^j]_2 \).

Proof. Every other type-2 item is strictly worse than \([A(1_i^j)]_2 \) according to agent \( 1_i^j \)'s local preferences in \( G_2 \).

Claim 3. Let any agent \( c_j \in S \), then for every \( j \leq n \), (i) \( [B(c_j)]_1 = [A(c_j)]_1 \), (ii) \( c_{j+1} \in S \), and (iii) \( c_1 \leq e_2 \in S, B(e_1) = A(e_1), B(e_2) = A(e_2) \).
Proof. If \( c_j \in S \), by the preferences in \( G_1 \), we must have that \( B(c_j) = A(c_j) \) and \( c_j + 1 \in S \) (and if \( j = m \), \( B(c_m) = A(c_m) \) and \( c_2 \in S \)) since every other item of type 1 is strictly worse. Now, the item \( [c_j]_1 \) must be assigned to some agent in \( S \). However, the only agent for whom the assignment of \( [c_j]_1 \) is weakly beneficial is the agent \( c_j - 1 \) (or \( j = 1 \), the agent \( c_1 \)). By induction we must have that if \( c_j \in S \), every agent \( c_j \) and the agents \( e_1, e_2 \) are in \( S \) and their allocation of items of type 1 is the same in \( B \) and in \( A \). This proves part (i).

To prove part (ii), let us examine \( G_2 \). Every other item of type 2 is strictly worse than \( \bar{c}_j \) according to \( c_j \)'s local preferences in \( G_2 \). If she receives any other item of type 2, then \( B(c_j) < A(c_j) \), a contradiction to our assumption that \( B \) is improving for every agent in \( S \).

Finally, agents \( e_1 \) and \( e_2 \) must be assigned \( [c_1]_2 \) and \( [e_2]_2 \) respectively since they do not strictly improve on their item of type 1 in \( B \). This completes the proof of part (iii).

\[ \square \]

Lemma 7. For every \( i \leq m, d_i \neq S \).

Proof. We provide a proof by contradiction that if \( d_i \in S \), then there is no allocation \( B \) that improves over \( A \).

Claim 4. If \( d_i \in S \), then \( \{d_i\}_1 = \{0\}_1 = \{A(d_i)\}_1 \) and \( \{d_i\}_1 \in S \).

Proof. Let us examine \( G_1 \). If \( d_i \in S \), we must have that \( B(d_i) = \{0\}_1 = \{A(d_i)\}_1 \), \( \{d_i\}_1 \in S \), since every other item is strictly worse according to her local preferences over type 1.

Claim 5. If \( d_i \in S \), then for every \( 0^j_i \), \( 0^i_j \in S \), \( B(0^i_j) = \{0\}_1 = \{A(0^i_j)\}_1 \), and \( \bar{x}_i \in S \), \( B(0^i_j) = \{A(0^i_j)\}_1 \) if \( d_i \).

Proof. If \( d_i \in S \), the claim follows from Claim 4 and the fact that \( \{x_i\} \) must be assigned to \( d_i \) if \( x_i \in S \) according to the preferences of every other agent in \( G_1 \).

Claim 6. If \( d_{m+1} = \{d_{m+1}\} + 1 \), \( \{d_{m+1}\} + 1 \in S \), for every \( i \leq m, d_i \in S \) and \( B(d_i) = \{d_i\}_1 \) and \( \{d_i\}_1 \in S \).

Proof. Let us examine \( G_2 \). For every \( i = 1, \ldots, m \), if \( d_i \in S \), we must have that \( \{B(d_i)\}_2 = \{d_i\}_2 + 1 \), \( \{d_i\}_2 + 1 \in S \). Suppose for some \( i \leq m - 1 \), \( B(d_i) = \{d_i\}_2 \), then \( B(d_i) < \{A(d_i)\}_2 \). For \( i = m \), we must have that \( \{B(d_m)\}_2 = \{c_n\}_2 = \{A(d_m)\}_2 \). Together with Claim 4, this proves part (ii) of the claim.

Part (ii) of the claim is proved by Claim 3 and the following argument. First we show that \( c_n \) cannot enter into an exchange with an agent \( 1^j_i \) (or \( 0^j_i \)) corresponding to a valuation of \( x_i \) that satisfies \( c_n \). If \( c_n \) receives \( \{1^j_i\}_2 \) (or \( \{0^j_i\}_2 \)), then by Lemma 5, Claim 3, and Lemma 6, the agent \( 1^j_i \) (or \( 0^j_i \)) must be assigned \( [c_n]_2 \) since she cannot be assigned a strictly improving item of type 1 in 1.

Let us consider agents \( c_j \) in the order \( j = n - 1, \ldots, 1 \). We know that \( c_j \in S \) and the item \( [c_j]_2 \) must already be assigned to another agent. Then, \( c_j \) must exchange items with an agent \( 1^j_i \) (or \( 0^j_i \)) corresponding to a valuation of the variable \( x_i \) that satisfies \( c_j \).

– If \( d_i \in S \), then by Claim 3, Claim 4, Claim 5, Lemma 6, and Lemma 7, \( S \) must be the set of all agents, and that the weakly improving allocation \( A \) is exactly the allocation \( A \). This is a contradiction to our assumption that \( B \) is weakly improving for every agent but also strictly improving for some agent. 

\[ \square \]

Lemma 8. Every weakly blocking coalition \( S \) consists of all the agents \( c_j, j \leq n \), and \( \forall i \leq n \), either (i) all agents \( x_i \), or (ii) all agents \( x_i \).

Proof. We start by showing that every blocking coalition \( S \) must include at least one of the agents \( 1^j_i, 0^j_i, x_i, \bar{x}_i, c_i \), where these are the only agents with solid incoming or outgoing edges and every allocation \( B \) that strictly improves over \( A \) must involve an assignment corresponding to one of these edges.

Claim 7. If \( c_j \in S \) or any \( 0^j_i \in S \) or any \( 1^j_i \in S \), then \( e_1, e_2 \in S \), every \( c_j \in S \) and for every \( i \leq m \), either every agent \( 1^j_i \in S \) or every \( 0^j_i \in S \)

– Suppose \( c_j \in S \). Let us examine her preferences in \( G_1 \). By Claim 3, for \( j \leq n - 1 \), we must have that \( B(c_j) = \{c_j + 1\}_1 \), and \( c_j + 1 \in S \). By induction, \( c_n \in S \).

– If \( c_n \in S \), we must have that \( B(c_n) = \{c_2\}_1 = \{A(c_n)\}_1 \), \( e_2 \in S \) since every other item of type 1 is worse according to \( G_1 \).

– If \( e_2 \in S \) then either \( B(e_2) = \{1\}_1 \), \( 1^j_i \in S \) or \( B(e_2) = \{0\}_1 \), \( 0^j_i \in S \) since these are the only two items that are weakly preferred over \( A(e_2) \).

– For any \( i \leq m \), if \( 1^j_i \in S \), then by Lemma 5, for every \( j \leq k \), \( 1^j_i \in S \) and \( x_i \in S \). (Similarly, if for any \( i \leq m \), any \( 0^j_i \in S \), then for every \( j \leq k \), agent \( 0^j_i \in S \).

– For \( i = m - 1 \), if \( x_i \in S \), then either \( 1^j_i + 1 \in S \) or \( 0^j_i + 1 \in S \) since in the case that \( x_i \) is not assigned one of the items \( 1^j_i + 1 \), \( 0^j_i + 1 \), \( A(e_2) \), \( B(e_2) \) in \( B \), any other item of type 1 assigned to her must be worse than \( A(1^j_i) \), a contradiction to the preferences in \( G_1 \), a contradiction to our assumption that \( B \) is weakly improving for every agent in \( S \). Similarly, if \( \bar{x}_i \in S \), then either \( 1^j_i + 1 \in S \) or \( 0^j_i + 1 \in S \).

– Finally, if \( x_m \in S \) (or \( x_m \)), then we must have that \( e_1 \in S \), since \( [e_1]_1 \) is the only item that is weakly better than her allocation \( A \) according to \( G_1 \).

– If \( e_1 \in S \), we must have that \( c_1 \in S \) since \( c_1 \) is the only item that is weakly preferred over \( A(c_1) \), according to \( G_1 \).
Thus, if any agent $1^j_i, 0^j_i, x_i$ or $\overline{x}_i$ is in $S$, then we must have $c_1 \in S$. We have already established that the existence of any $c_j \in S$ implies that for every $i \leq m$, either all $1^j_i \in S$ or all $0^j_i \in S$.

Finally, we will show that there is no blocking coalition $S$ such that for some $i \leq m$, any $j \leq k_i, k'_i \leq k_i$ the agents $1^j_i \in S$ and $0^j_i \in S$. Suppose for the sake of contradiction that such a coalition exists. We must have by Lemma 8, Lemma 4 and Lemma 7 that both $x_m, \overline{x}_m \in S$. However, by Lemma 7 we know that $d_m \notin S$ which implies that both $x_m$ and $\overline{x}_m$ must get $[c_1]_1$ in $B$ since every other item is strictly worse than their allocations in $A$ according to preferences in $G_1$. This is impossible since items are unique and indivisible.

**Lemma 9.** If $1^j_i \in S$, $[B(1^j_i)]_1 = [A(1^j_i)]_1 = [1^j_i + 1]_1$ (if $j = k_i, [x_i]_1$).

**Proof.** Every other item of type 1 is strictly worse for agent $1^j_i$ according to $G_1$. Similarly, if $0^j_i \in S$, $[B(0^j_i)]_1 = [A(0^j_i)]_1 = [0^j_i + 1]_1$ (if $j = k_i, [\overline{x}_i]_1$).

We will now prove that: If there is a coalition $S$ that blocks $A$, there is a satisfying assignment to $F$.

We now describe the construction of a valuation $\phi$ that satisfies the formula $F$ in the instance $I$ by examining $G_2$ and argue that such a valuation must exist. We start by noting that since agents $c_j$ and agents $1^j_i, 0^j_i$ cannot strictly improve over their allocations of type 1 according to preferences in $G_1$, they must weakly improve their allocation of items of type 2.

Let us examine $G_2$. We know by Lemma 7 that none of the agents $d_i$ are in $S$. We have also shown that $c_1 \in S$. Agent $c_1$ must receive an item $[1^1_i]_2$ (or $[0^1_i]_2$) where the corresponding valuation of the variable $x_i$ satisfies $c_1$ and the agent $1^1_i$ (or $0^1_i$) is in $S$. Otherwise, every other item of type 2 is strictly worse than her allocated item of type 2 in $A$ and together with the fact that $c_1$ does not receive a strictly better item of type 1 in $B$, we have that $B(c_1) < A(c_1)$, a contradiction to our assumption that $B$ is weakly improving for every agent in $S$. Further, agent $1^1_i$ (or $0^1_i$) must receive the item $[c_1]_2$ since every other item is strictly worse. Set $\phi_i$ to be 1 if $c_1$ gets $[1^1_i]_2$ and to 0 if $c_1$ gets $[0^1_i]_2$ in $B$.

Similarly, agent $c_2$ cannot assigned $[c_1]_2$ since it must be assigned to some agent $1^1_i$ (or $0^1_i$) and agent $c_2$ must exchange items with agent an agent $1^2_i$ (or $0^2_i$) that corresponds to a satisfying valuation of a variable $x_i$ that satisfies clause $c_2$. By an inductive argument, we must have that for every $j \leq m$, agent $c_j$ receives the item of an agent $1^j_i$ or $0^j_i$ that is in $S$ which corresponds to a satisfying valuation of variables $x_i$ that satisfies clause $c_j$. For each clause, set $\phi_i$ to be 1 if $c_j$ gets $[1^j_i]_2$ and to 0 if $c_j$ gets $[0^j_i]_2$ in $B$. Note that each clause $c_j$ is satisfied by a variable whose value is consistent with the final value according to $\phi_i$ since the value of $\phi_i$ can never change once set by Lemma 8. The agents involved correspond to a consistent valuation of the variables and all of the clauses must be satisfied simultaneously.

This completes the proof. Given an instance $I$ of 3-SAT, we can construct in polynomial time, a corresponding instance $J$ of InStrictCore, such that $I$ is a Yes instance iff $J$ is a No instance.

**Definition 4** (StrictCoreNonEmpty). Given a multi-type housing market $M$, agents’ preferences $P$, and an allocation $A$, we are asked whether the strict core of $M$ is non-empty.

**Theorem 4.** StrictCoreNonEmpty is NP-hard.

**Proof.** We will show a reduction from 3-SAT. An instance $I$ of 3-SAT is given by a formula $F$ in $3$-CNF consisting of clauses $c_1, \ldots, c_n$ involving Boolean variables $x_1, \ldots, x_m$, and we are asked whether there is a valuation of the variables that satisfies $F$. Each clause $c_j$ is a disjunction of exactly 3 literals $c_j = l_{j1} \lor l_{j2} \lor l_{j3}$. Each literal $l_i$ is either the variable (positive literal) $x_i$ or its negation (negative literal) $\overline{x}_i$. Each of $l_{j1}, l_{j2}, l_{j3}$ corresponds to the positive or negative literal of one of the variables $x_1, \ldots, x_m$.

- We define the following ordering $W$ on positive and negative literals of the Boolean variables where positive literals are ranked over negative literals and literals of variables with lower index are ranked over literals of higher index. Let $l_i, l_{i'}$ be any two literals. Then, $l_i \succ_W l_{i'}$ if either: (1) $i \leq i'$ and if either (i) $l_i = x_i$ or (ii) $l_i = \overline{x}_i$ and $l_{i'} = \overline{x}_{i'}$ (2) $i \geq i'$, $l_i = x_i$ and $l_{i'} = \overline{x}_{i'}$. For every clause $c_j$, add the agents $c_j^1, c_j^2, c_j^3$.

- For every variable $x_i$, add agents $x_i$.

- For every variable $x_i$, add agents $1^j_i, 0^j_i$ for every clause $c_j$ that involves variable $x_i$.

**Preferences:**

- Agents $c_j^1: (1^j_{j1}, c_j^1 \lor 3^j_{j1}, c_j^1 \lor 1^j_{j2}, c_j^1 \lor 3^j_{j2}, c_j^1 \lor 1^j_{j3}, c_j^1 \lor 3^j_{j3}, c_j^1 \lor (c_j^1, c_j^2) \lor (c_j^2, c_j^1) \lor (c_j^1, c_j^3) \lor (c_j^3, c_j^1) \lor (c_j^2, c_j^3) \lor (c_j^3, c_j^2) \lor (c_j^1, c_j^4) \lor (c_j^4, c_j^1) \lor (c_j^2, c_j^4) \lor (c_j^4, c_j^2) \lor (c_j^3, c_j^4) \lor (c_j^4, c_j^3))$ others.

- Agents $c_j^2: (c_j^1, c_j^2 \lor c_j^2, c_j^1 \lor (c_j^2, c_j^3) \lor (c_j^3, c_j^2) \lor (c_j^1, c_j^3) \lor (c_j^3, c_j^1) \lor (c_j^2, c_j^1) \lor (c_j^1, c_j^2) \lor (c_j^2, c_j^3) \lor (c_j^3, c_j^2) \lor (c_j^1, c_j^4) \lor (c_j^4, c_j^1) \lor (c_j^2, c_j^4) \lor (c_j^4, c_j^2) \lor (c_j^3, c_j^4) \lor (c_j^4, c_j^3))$ others.

- Agents $c_j^3: (c_j^1, c_j^3 \lor c_j^1, c_j^3 \lor (c_j^3, c_j^2) \lor (c_j^2, c_j^3) \lor (c_j^1, c_j^2) \lor (c_j^2, c_j^1) \lor (c_j^1, c_j^4) \lor (c_j^4, c_j^1) \lor (c_j^2, c_j^4) \lor (c_j^4, c_j^2) \lor (c_j^3, c_j^4) \lor (c_j^4, c_j^3))$ others.

- Agents $1^j_i$:...
If \( l_i = x_i \in c_j, j \leq k_i, \ (c_j^1, 1^j_{i+1}) \succ (1^j_i, 1^j_i) \succ \text{others} \).
If \( j = k_i \), replace \( 1^j_{i+1} \) with \( x(i+1) \mod m \).
- If \( l_i = x_i \notin c_j, j \leq k_i, \ (1^j_i, 1^j_{i+1}) \succ (1^j_i, 1^j_i) \succ \text{others} \).
If \( j = k_i \), replace \( 1^j_{i+1} \) with \( x(i+1) \mod m \).

\[ \text{Agents } 0^j_i: \]
- If \( l_i = \bar{x}_i \in c_j, j \leq k_i, \ (c_j^3, 0^j_{i+1}) \succ (0^j_i, 0^j_i) \succ \text{others} \).
If \( j = k_i \), replace \( 0^j_{i+1} \) with \( 0(i+1) \mod m \).
- If \( l_i = \bar{x}_i \notin c_j, j \leq k_i, \ (0^j_i, 0^j_{i+1}) \succ (0^j_i, 0^j_i) \succ \text{others} \).
If \( j = k_i \), replace \( 0^j_{i+1} \) with \( 0(i+1) \mod m \).

\[ \text{Agents } x^j_i: \ (x_i, 1^j_i) \succ (x_i, 0^j_i) \succ (x_i, x_i) \succ \text{others}. \]

**Initial endowments:** Each agent \( x \) is initially endowed with a bundle \( O(x) = (x, x) \).

**High level idea:**

- The preferences are structured so that type-1 items are used to track satisfaction of individual clauses, while type-2 are used to ensure a consistent valuation of the Boolean variables and satisfaction of all clauses.

- Borrowing from the Example 2.2 in [Konishi, Quint, and Wako 2001], if \( c_j^1, c_j^2, c_j^3 \) only receive each others’ items in an allocation, there is a blocking coalition. For an allocation to be individually rational and stable, agent \( c_j^3 \) must receive an item of type 1 from an agent \( 1^j_i \) or \( 0^j_i \).

**Example 7.** [Example 2.2 in Konishi, Quint, and Wako 2001] Consider the multi-type housing market with agents \( N = \{1, 2, 3\}, p = 2 \) types and the following preferences.

Agent 1: \((1, 3) \succ (3, 3) \succ (1, 2) \succ \text{others} \).
Agent 2: \((2, 3) \succ (2, 1) \succ (3, 3) \succ (3, 1) \succ (2, 2) \succ \text{others} \).
Agent 3: \((2, 1) \succ (3, 2) \succ (3, 1) \succ (1, 1) \succ (3, 2) \succ (1, 2) \succ (2, 3) \succ (3, 3) \succ (1, 3) \succ \text{others} \).

The strict core is empty for the market above. \( \square \)

**Claim 8.** \((\Rightarrow) \) If \( I \) is a Yes instance, \( J \) is a Yes instance.

**Proof.** We start by define a lexicographic order \( Q \) on valuations. Given two valuations \( \psi, \omega \in \Phi \), let \( i \) be the lowest value such that \( \psi_i \) and \( \omega_i \) differ. We rank \( \psi \) over \( \omega \) if \( \psi_i = 1 \) \((\omega_i \mod m = 0)\).

**Example 8.** In the Example 5 the valuation \((1, 1, 0)\) is ranked over \((1, 0, 1)\).

- Let \( \Phi \) be the set of satisfying valuations to \( F \). Let \( \phi \) be the top ranked satisfying valuation according to \( Q \). We will construct an allocation \( A \) w.r.t. the valuations in \( \phi \) that is a core allocation.

**Allocation \( A \):**

- \( A(c_j^1) = (1^j_i, c_j^1_{i+1}) \) if \( l_i = x_i \) is the highest ranked literal according to \( W \) that satisfies \( c_j \) when \( x_i = \phi_i \). If \( l_i = \bar{x}_i \), replace \( 1^j_i \) with \( 0^j_i \).
- \( A(c_j^2) = (c_j^2, c_j^3) \).
- \( A(c_j^3) = (c_j^1, c_j^3) \).
- \( A(1^j_i) = (c_j^1, 1^j_{i+1}) \), if \( l_i = x_i \) is the highest ranked literal according to \( W \) that satisfies \( c_j \) when \( x_i = \phi_i \).
- \((1^j_i, 1^j_{i+1})\), otherwise.
If \( j = k_i \), replace \( 1^j_{i+1} \) with \( x(i+1) \mod m \).
- \( A(0^j_i) = (c_j^1, 0^j_{i+1}) \), if \( l_i = \bar{x}_i \) is the highest ranked literal according to \( W \) that satisfies \( c_j \) when \( x_i = \phi_i \).
- \((0^j_i, 0^j_{i+1})\), otherwise.
If \( j = k_i \), replace \( 0^j_{i+1} \) with \( x(i+1) \mod m \).
- \( A(x_i) = (x_i, x_i^1) \).

- Suppose, for the sake of contradiction that \( S \) blocks \( A \). Let \( B \) be an improving allocation on \( S \). We will show that \( B \) corresponds to a satisfying assignment to \( F \) that is ranked lower than \( \phi \), and \( B \) must be strictly worse for some agent in \( S \).

- We start by showing that \( S \) must involve at least one of the agents \( c_j^1, 1^j_i, 0^j_i \).

If either of \( c_j^2 \) or \( c_j^3 \) are in \( S \) for some \( j \leq n \), then we must have that all the agents \( c_j^1, c_j^2, c_j^3 \) are in \( S \). If \( c_j^3 \in S \), we must have that \( c_j^3 \in S \). Otherwise, \( c_j^3 \) gets a strictly worse allocation in \( B \). If \( c_j^3 \in S \), we must have at least one of the agents \( c_j^3 \) or \( c_j^1 \) in \( S \), otherwise she gets a strictly worse bundle in \( B \) than in \( A \). If only \( c_j^1, c_j^3 \in S \), and \( c_j^1 \notin S \), neither agent can strictly improve over their allocation in \( A \). Therefore, we must have \( c_j^3 \in S \).

- Let \( [A(c_j^3)]_1 = [O(1^j_i)]_1 \) (or \( 0^j_i \)) and \( [B(c_j^3)]_1 = [O(1^j_i)]_1 \) (or \( 0^j_i \)). Then agent \( 1^j_i \) (or \( 0^j_i \)) corresponds to a literal \( l_i \) in clause \( c_j \) that is ranked weakly on top of \( l_i \) according to \( W \). Otherwise, \( c_j^1 \) is strictly worse off in \( B \) than in \( A \) according to the preferences in the construction, a contradiction to our assumption that \( B \) is improving for every agent in \( S \).

- If any one of \( c_j^1 \in S \), then every agent \( c_j^1 \in S \). Every agent \( c_j^1 \) must be assigned \( [B(c_j^1)]_2 = [O(1^j_i)]_2 \), otherwise she is strictly worse off in \( B \). This also implies that \( c_j^1 \in S \) (if \( j = k_i \), replace \( j + 1 \) with 1).

- If \( 1^j_i \in S \) (or \( 0^j_i \)), then \( 1^j_{i+1} \in S \) (or \( 0^j_{i+1} \)) and \( 1^j_{i-1} \in S \) (or \( 0^j_{i-1} \)). For agent \( 1^j_i \) to weakly improve her allocation in \( B \) over her allocation in \( A \), we must have \([B(1^j_i)]_2 = [O(1^j_i)]_2 \) (if \( j = k_i \), replace \( 1^j_{i+1} \) with \( 1^j_{i+1} \)). Otherwise, \( 1^j_i \) is strictly worse off in \( B \).

Some agent in \( S \) must receive the item \([O(1^j_i)]_1 \). Let agent \( a \in S \), \( a \neq 1^j_i \) be assigned item \([O(1^j_i)]_1 \) in \( B \), then by her reported preferences, agent \( a \) is strictly worse off with
her allocation in $B$ compared to her allocation in $A$, a contradiction to our assumption that $B$ is improving for every agent in $S$.

For some pair $j, j'$, we cannot have both $1_j, 0_{j'} \in S$. Suppose we have both $1_j$ and $0_{j'}$ in $S$, then we must have both $1_k$ and $0_k$ in $S$. However both of the agents $1_k$ and $0_k$ must be assigned the single item $[O(x_{i+1})]_{2}$ in $B$ to weakly improve over their allocation in $A$. Therefore one of the two agents must be strictly worse off in $B$, a contradiction to our assumption that allocations in $B$ are weakly preferred over $A$ by every agent in $S$.

We have shown that $S$ must include all the agents $c_j^i$ corresponding to clauses $c_j$, for every $i$ either all of the agents $1_j^i$ or all of the agents $0_j^i$ and that for every $c_j$, an agent $1_j^i$ or $0_j^i$ corresponding to a valuation of a variable $x_i$ that satisfies $c_j$.

Let $\psi$ be the valuation that is constructed as follows: for every clause $c_j$, if $c_j^i$ receives a type-1 item endowed to agent $1_j^i$ we set $\psi = 1$. Note that every $c_j$ receives such an item and this corresponds to a valuation of the variable $x_i$ that satisfies $c_j$ and that once a variable takes a value, it is never changed again i.e. the value of $\psi$ is always consistent with a value that satisfies all clauses $c_j$ considered previously. Therefore, the values of the variables in $\psi$ constitutes a satisfying valuation.

Now, consider the allocation $B$ where the pairs of agents $c_j^1, 1_j^1$ (or $0_j^1$) exchanging items of type 1 is the same as in the allocation $A$. Then $B$ is not a strictly improving allocation over $A$ for any agent in $S$, a contradiction to our assumption on the allocation $B$.

Let $i^*$ be the smallest value such that $\phi_{i^*} \neq \psi_{i^*}$. If $\psi_{i^*} = 1$, then $\psi$ is ranked over $\phi$ by $Q$, a contradiction to our assumption that $\phi$ is the highest ranked satisfying valuation.

If $\psi_{i^*} = 0$, then $\phi_{i^*} = 1$, then consider the agent $c_j$ who was assigned the type-1 item of $1_j^i$ in $A$. By the construction of $A$, $c_j$ must either receive the type-1 item of an agent $1_j^i$ in $B$ where $i > i^*$, or of an agent $0_j^i$ in $B$. In either case, $c_j$ is strictly worse off in $B$ compared to $A$ since it corresponds to a lower ranked literal according to $W$. This is a contradiction to our assumption that every agent is weakly better off with their allocations in $B$ compared to $A$.

Claim 9. \((\Leftarrow)\) If $J$ is a Yes instance, $I$ is a Yes instance.

Proof. Suppose $A$ is a core allocation.

Every agent $c_j^1$ must receive an item of type 1 from an agent $1_j^1$ (or $0_j^1$) where the corresponding literal of $x_i$ satisfies clause $c_j$ i.e. we must have that $[A(c_j^1)]_1 = [O(1_j^i)]_1$ (similarly $[O(0_j^i)]_1$).

For the sake of contradiction let $c_j^1$ not receive an item from an agent $1_j^1$ (or $0_j^1$). Then there are two cases: either $c_j^1, c_j^2, c_j^3$ are assigned each others’ items or one of them is assigned the initial endowment of some other agent.

If $c_j^1, c_j^2, c_j^3$ are only assigned each others’ initial endowments, then by our construction of the preferences of agents $c_j^1, c_j^2, c_j^3$ by adapting Example 7 there is always a blocking coalition to every such allocation, a contradiction to our assumption that $A$ is in the core. If $c_j^1$ is assigned the initial endowment of an agent other than $c_j^1, c_j^2$ or $c_j^3$, then the resulting allocation is not individually rational by the construction of agents’ preferences, a contradiction. If $c_j^1$ is assigned the initial endowment of an agent other than $c_j^1$, then we must have a type-1 or $0_j^i$ corresponding to a satisfying assignment for the clause $c_j$, then again the resulting allocation is not individually rational.

If $c_j^1$ exchanges her item type-1 item with an agent $c_j^1$, then none of the agents $0_j^i$ (similarly, $1_j^i$) can exchange items with some other agent $c_j^1$.

For the sake of contradiction, let $1_j^1$ and $0_j^1$ exchange their items of type 1 with agents $c_j^1$ and $c_j^2$ i.e they get $[A(1_j^1)]_1 = [O(1_j^i)]_1$ and $[A(0_j^1)]_1 = [O(1_j^i)]_1$, respectively. Then agents $1_j^1$ and $0_j^1$ must get the assignments $[A(1_j^i)]_2 = [O(1_j^i)]_2$ and $[A(0_j^1)]_2 = [O(0_j^i)]_2$, respectively.

If $1_j^1 \in S$ (or similarly, $0_j^1$), and $[O(1_j^i)]_2$ is assigned to another agent, then we must have that $[A(1_j^i)]_2 = [O(1_j^i)]_2$, otherwise the allocation $A$ is not individually rational.

By an inductive argument, we must have that the agents $1_j^i$ and $0_j^i$ must both be assigned the single indivisible item $[O(x_{i+1})]_2$ for $A$ to be individually rational, a contradiction to our assumption that $A$ is in the core since every core allocation must also be individually rational.

Each agent $x_i$ receives either one of $[O(1_j^i)]_2$ or $[O(0_j^i)]_2$ since an agent can only receive a single item of a given type. By our argument immediately above, if $x_i$ receives $[O(1_j^i)]_2$ in $A$, then only agents $1_j^i$ can exchange items with agents $c_j^1$ and none of the agents $0_j^i$ can exchange items with an agent $c_j^1$.

Now, consider the construction of a satisfying valuation $\phi$ as follows: For every clause $c_j$, if $[A(c_j^1)]_1 = [O(1_j^i)]_1$ (similarly, $0_j^i$), set $\phi_i = 1$ which must be an assignment to the variable $x_i$ that satisfies clause $c_j$. We have already shown that such an assignment exists for every clause $c_j$. Further, once $\phi_i$ is set to a value, it can never change as we have already shown that if there is some $c_j$ such that $[A(c_j^1)]_1 = [O(1_j^i)]_1$, there is no $c_j$ such that $[A(c_j^1)]_1 = [O(0_j^i)]_1$. Therefore at every point where we verified the satisfaction of a clause $c_j$, the value of the variable $x_i$ in $\phi_i$ that satisfied $c_j$ is the same as the final value of $\phi_i$. Therefore, $\phi$ satisfies $F$.

This completes the proof.

Summary and Future Work

We propose MTTC for multi-type housing markets with lexicographic preferences, and prove that it satisfies many de-
sirable axiomatic properties. There are many future directions in mechanism design for multi-type housing markets. Are there good mechanisms when agents demand more than one item of some type? Can we design strategy-proof mechanisms under other assumptions about agents’ preferences, such as LP-trees [Booth et al. 2010]? What is the computational complexity of manipulation under MTTC? What if agents’ preferences are partial orders such as CP-nets only?

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