Lorentz gauge theory of gravity in electron–positron colliders

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Abstract

Lorentz gauge theory (LGT) is a feasible candidate for a theory of quantum gravity in which routine field theory calculations can be carried out perturbatively without encountering too many divergences. In LGT, the spin of matter also gravitates; spin-generated gravity is expected to be very much stronger than that generated by mass, and could be explored in current colliders. In this article, the signals of the theory observable in an electron–positron collider are investigated. We specifically study pair annihilation into two gravitons, and LGT corrections to processes like $e^- + e^+ \rightarrow \mu^- + \mu^+$ and $e^- + e^+ \rightarrow e^- + e^+$.

Keywords: quantum gravity, Lorentz gauge theory, linear colliders

(Some figures may appear in colour only in the online journal)

1. Introduction

Gravity is expected to be quantized at small scales [1, 2]. However, there is as yet no theory of quantum gravity that meets all the expectations. Several directions have been tried so far. Canonical quantization of general relativity [3, 4] is perhaps the oldest approach. String theory [5] and loop quantum gravity [6, 7] are other familiar endeavors. Lorentz gauge theory (LGT) [8] should be added to the list of candidates for a quantum theory of gravity. It is a Yang–Mills theory, based on internal Lorentz symmetry of fermions, in which the metric is not dynamic, and the energy-momentum tensor is not the source of gravity. In LGT, interactions of gravitons with fermionic spins are expected to become significant at energies much lower than the Planck scale. Moreover, several electron–positron accelerators, including the Large Electron Positron Collider (LEP) [9] and the SLAC Linear Collider (SLC) [10], have...
already collected lots of data. Furthermore, there are ongoing studies for high luminosity colliders with center-of-mass energies in the TeV range. Examples are the International Linear Collider (ILC) [11] and the Compact Linear Collider (CLIC) [12]. Therefore, searches for LGT signals in current or near future experiments are well motivated, and in the present paper we will study these interactions and their signatures in electron–positron colliders.

In this paper, the following conventions are adopted. For the sake of simplicity, the study is restricted to the important case of relativistic collisions where masses can be neglected altogether. Moreover, calculations are all restricted to the center of mass frame, in which all the incoming and outgoing particles have the same energy. To reduce the effects of the weak interactions, the center of mass energy is restricted to be below the mass of the Z boson. We further assume that the incoming electrons and positrons are moving in the positive and negative $z$ directions respectively. Due to the cylindrical symmetry in such interactions, calculations are restricted in the $x$–$z$ plane. Therefore, the scatterings can be characterized with the outgoing fermion’s angle $\theta$.

In this article, we first summarize Lorentz gauge theory in section 2. Next, in sections 3 and 4 quantum corrections from LGT into $e^- + e^+ \rightarrow \mu^- + \mu^+$ and $e^- + e^+ \rightarrow e^- + e^+$ are investigated respectively. In section 5, after studying the gravitational plane waves and deriving their physical modes, the process of pair annihilation into gravitons is investigated. A conclusion is given in the end in section 6.

2. Lorentz gauge theory of gravity

General relativity was first proposed to remain invariant only under general coordinate transformations

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x),$$  \hspace{1cm} (1)

where $\xi^\mu(x)$ is an arbitrary vector and can be written as

$$\xi^\mu(x) = \Lambda^\nu_{\mu}(x) x^\nu + (\xi^\mu(x) - \Lambda^\nu_{\mu}(x) x^\nu).$$  \hspace{1cm} (2)

Here, $\Lambda^\mu_{\nu}(x)$ indicates a Lorentz transformation, and the last term is a translation—i.e. the whole change is a Poincaré transformation. This original version of general relativity is good as far as fermions can be neglected. The reason is that the group of general linear $4 \times 4$ matrices has no representation that behaves like spinors under the Lorentz group. In order to incorporate fermionic fields in general theory of relativity, people have utilized the concept of Minkowskian tangent spaces defined at each point of a semi-Riemannian space-time. This opens the possibility of defining the spinor fields in these tangent spaces, and requiring that their Lagrangian remains invariant under Lorentz transformations that are solely defined in the new spaces. Matter Lagrangians have to remain invariant under any general coordinate transformation in the space-time, and also under any Lorentz transformation in the tangent spaces. Hence, GR can be regarded as a theory that is invariant under [13]$^3$

General Covariance $\otimes$ Lorentz.

$^3$Look for the paragraph: ‘There are now two invariance principles which must be met in constructing a suitable matter action $I_M(\Lambda)$ The action must be generally covariant, ... with respect to Lorentz transformation $\Lambda^\nu_{\mu}(x)$ that can depend on position in space-time ... These two invariance principles lead to a dual classification of physical quantities. A coordinate scalar or coordinate tensor transforms as a scalar or a tensor under changes in the coordinate system. A Lorentz scalar or Lorentz tensor or Lorentz spinor transforms according to a rule like ... under changes in the choice of the locally inertial coordinate frame’.  

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Consequently, physical objects belong to one of the two groups. The metric and vector bosons belong to space-time, and are coordinate tensors, while fermions belong to the Lorentz space and objects like \( \bar{\psi} (\text{combination of } \gamma^i) \psi \) are Lorentz tensors. The Lorentz spaces are connected with space-time through a set of four vector fields called the tetrad. The tetrad can be decomposed to its space-time and Lorentz components at each point of space-time:

\[
e_{i\mu} = \eta^{\bar{l}\bar{k}} e_{i\bar{k}} e_{\bar{l}\mu},
\]

where components of the objects in the free falling frame are indicated with a bar and

\[
e_{i\mu} \equiv \hat{e}_i \cdot \hat{e}_\mu
\]

\[
e_{\bar{l}\mu} \equiv \hat{e}_{\bar{l}} \cdot \hat{e}_\mu.
\]

while \( \hat{e}_i \) are the unit vectors in the Lorentz space at that point, \( \hat{e}_{\bar{l}} \) are the unit vectors carried by the free falling observer at that point, and \( \hat{e}_\mu \) are the unit vectors tangent to coordinates at that point. The general covariance leads to conservation of both energy momentum and angular momentum. The Lorentz invariance, on the other hand, leads to a conserved current with angular momentum dimension, to which Lagrangians of Bosons or vacuum energy do not contribute. The current is derived in appendix A. Having two independent sets of conserved currents, we have to decide which one is the source of gravity. In GR the energy-momentum tensor is chosen as the source of gravity. That corresponds to assuming that the space-time component of the tetrad is dynamic, i.e. \( \delta e_{i\mu} = \eta^{\bar{l}\bar{k}} e_{i\bar{k}} \delta e_{\bar{l}\mu} \). This, however, leads to several unsolved problems. Examples are the cosmological constant problem [14, 15], the problem of time [16, 17], and nonrenormalizability [18, 19]. In Lorentz gauge theory of gravity (LGT), it is assumed that the conserved current of the Lorentz space is the source of gravity, which means the Lorentz component of the tetrad is dynamic \( \delta e_{\bar{l}\mu} = \eta^{\bar{l}\bar{k}} \delta e_{\bar{k}\mu} \). This enables us to define a Yang–Mills theory whose equations determine the spin connections; the latter is sufficient to determine the tetrad, and thereby the metric and the metric-compatible Christoffel symbols. A simple dimensional analysis shows that the coupling constant of LGT is dimensionless compared to that of GR, which has negative two dimension. Therefore, LGT is expected to have a much better high energy behavior than GR. LGT is formally defined with the following action [8]

\[
S = \int e d^4x \left[ \mathcal{L}_A + \mathcal{L}_M + \mathcal{L}_C \right],
\]

where \( \mathcal{L}_A \) is the gauge field’s Lagrangian, and is given by

\[
\mathcal{L}_A = -\frac{1}{4g^2} F^{\mu\nu ij} F_{\mu\nu ij},
\]

where the strength tensor is defined in just the same way as in any Yang–Mills theory:

\[
F_{\mu\nu ij} = g \partial_\mu A_{ij\nu} - g \partial_\nu A_{ij\mu} + g^2 A_{i\mu}^m A_{mj\nu} - g^2 A_{i\nu}^m A_{mj\mu}.
\]

It should be mentioned that the symmetries of the theory allow other terms in the Lagrangian as well, which can be found in [20, 21]. However, they lead to interactions that are not familiar from the standard model, and our preliminary evaluation suggests that they are not renormalizable; they are abandoned for this reason. \( \mathcal{L}_M \) is the Lagrangian of the standard model, while \( \mathcal{L}_C \) is just the tetrad postulate times a Lagrange multiplier.
\[
\mathcal{L}_C = S^{\mu
u}D_{\mu}e_{\nu} = 0. \tag{8}
\]

The reason for the latter Lagrangian is that the tetrad in the action should be expressed in terms of the spin connections, which itself requires solving an integral equation. To avoid this cumbersome task, the dependence of the tetrad on the spin connections is entered into the action as a constraint, and the two fields are treated independently afterward. The field equations are found by varying the action in the Lorentz spaces, i.e. under which coordinate tensors like the metric remain unchanged. A variation with respect to the tetrad gives a constraint equation:

\[
D_{\mu}S^{\mu
u} = \frac{\delta \mathcal{L}_M}{\delta e_{\nu}}, \tag{9}
\]

in which \(\frac{\delta \mathcal{L}_M}{\delta e_{\nu}}\) is just the energy-momentum tensor. A variation with respect to the spin connections, on the other hand, gives the dynamical field equations

\[
D_{\nu}F^{\mu
u} = -\frac{\delta \mathcal{L}_M}{\delta A_{\mu
u}} + S^{\mu\nu[i}e_{j]}, \tag{10}
\]

where the first term on the right-hand side is just the spin angular momentum of fermions. Therefore, it is the second term that gives rise to the Newtonian gravity.

Comparing the two source terms, it is seen that the coupling constant of LGT appears alone in the first term, but is multiplied by a vector—with dimension of length—in the second. To arrive at Newtonian gravity in the classical regimes, we have to assume that the latter length belongs to the Planck scale. By absorbing the small length, labeled \(\delta\), into the coupling constant of LGT, a classically effective version is developed [22], in which the effective coupling constant is Newton’s gravitational constant

\[
G = \frac{g\delta^2}{40\pi}. \tag{11}
\]

Since this equation is derived after assuming that \(\delta \sim \sqrt{G}\), we can conclude that the strength of \(g\) is comparable to the coupling constants in the standard model, i.e. \(g \sim 1\). On the other hand, the coupling constant of the first source term in equation (10) is \(g\), and is not multiplied by \(\delta\). This suggests that interactions contained in this term become significant at energies much lower than the Planck mass. Study of these interactions is the subject of the present paper, in which the second term on the right-hand side of equation (10) is totally neglected. After dropping the irrelevant interactions, and fixing the gauge by \(\partial_{\nu}A_{\mu\nu} = 0\), Feynman rules for LGT in the momentum space are as follows [8]. The propagator is just the inverse of \(\partial^2\) in the linear field equations, given in equation (30),

\[
ij\mu \quad \quad \quad \quad mm\nu = -\frac{i}{2} \frac{\eta_{\mu\nu}(\eta_{mi}\eta_{nj} - \eta_{mj}\eta_{ni})}{q^2 + i\epsilon}. \tag{12}
\]

The only important self interaction at the tree level, which is the subject of the present study, is obtained by dropping \(g^2\) terms, and varying the Lagrangian three times with respect to \(A_{ij\mu}\).
Also, the only relevant matter interaction is obtained after varying the matter Lagrangian, given in equation (A.2), with respect to $\bar{\psi}, \psi,$ and $A_{ij\mu}$ and putting $\delta = 0$, equivalent to assuming $\delta e_i^{\mu} \delta A_{mn}^{\nu} = 0$.

\[ mm\nu = g\delta^\nu_k \{ \gamma^k, S^{mn} \}, \]  

where the tetrad in flat background is shown with a delta and $\{a, b\} \equiv \frac{1}{4} (a \cdot b + b \cdot a)$.

So far, we have found two exact vacuum solutions for LGT, namely the Schwarzschild and the de Sitter spaces. The former is crucial to passing the Solar System tests of gravity, while the latter is needed to explain inflationary expansions in the very early times and in the late times in the Universe. It should be noted that the solution holds for when there is no matter, nor dark energy, in the Universe. Therefore, LGT does not need dark energy to explain such expansions. Moreover, in LGT transition from decelerating to accelerating expansion in the late times is spontaneous, and is driven by geometrical terms. Also, unlike in GR, in LGT the vacuum energy does not gravitate [23]. See appendix for more details.

Therefore, there will be no contradiction with quantum field theory—which predicts a very large value for the vacuum energy. All the supplementary materials that are needed to reproduce our results in this and preceding papers are gathered in a repository that can be found at [24].

3. LGT corrections to $e^- + e^+ \rightarrow \mu^- + \mu^+$

The effects of LGT in processes like $e^- + e^+ \rightarrow \mu^- + \mu^+$ are investigated in this section. Because of the similarity between the LGT diagram in equation (14) and the only interaction in QED, it is already clear that there are corrections from LGT into such processes. The Feynman diagrams for this interaction from both QED and LGT are given in the following
where the internal line is a photon $\gamma$ in QED and a graviton $A_{ij\mu}$ in LGT. The amplitude corresponding to the two diagrams is given by

$$iM_{ss's'} = -e^2 \bar{\nu}_r(k) \delta^{ij} \gamma \nu_l(p) \frac{-\eta^{ij\mu\nu}}{(p + p')^2} \bar{\nu}'(p') \delta^{ij} \gamma \nu_l'(k')$$

$$+ g^2 \bar{\nu}_r(k) \delta^{ij} \{\gamma_k, S^l\} \nu_l(p) \frac{-\eta^{ij\mu\nu}(\eta^{ij\rho\sigma} - \eta^{ij\sigma\rho})}{(p + p')^2} \bar{\nu}'(p') \delta^{ij} \{\gamma_l, S'^m\} \nu_l'(k').$$

(16)

After a long but straightforward calculation, the non-zero components of the amplitude turn out to be

$$M_{1212} = \left(\frac{3}{2} g^2 - 2e^2\right) \cos^2(\theta/2)$$

$$M_{2112} = \left(\frac{3}{2} g^2 + 2e^2\right) \sin^2(\theta/2),$$

$$M_{1221} = \left(\frac{3}{2} g^2 + 2e^2\right) \sin^2(\theta/2)$$

$$M_{2121} = \left(\frac{3}{2} g^2 - 2e^2\right) \cos^2(\theta/2).$$

(17)

Since particle detectors are usually blind to polarization, we are also interested in unpolarized amplitude. Hence, a sum on the outgoing spins and an average over the incoming ones is in order:

$$\frac{1}{4} \sum_{\text{spin}} |M|^2 = \left(e^4 + \frac{9}{16} g^4\right) (1 + \cos^2(\theta)) - 3e^2 g^2 \cos(\theta).$$

(18)

Therefore the differential cross section in the center of mass frame takes the following form

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{cm}}^2} \left(1 + \frac{9}{16} g^4\right) (1 + \cos^2(\theta)) - 3e^2 g^2 \cos(\theta).$$

(19)

where $\alpha = \frac{e^2}{\pi}$ is the QED coupling constant, while $r = \frac{g}{\alpha}$ and is zero if LGT is switched off. While the QED term of the cross section is symmetric between the forward and backward hemispheres of particle detectors, separated by the $\theta = \frac{\pi}{2}$ plane, the LGT term is not. Fermions that undergo gravitational interactions tend to scatter more into the backward region. Therefore, one viable search for LGT is to subtract the number of events in the backward hemisphere from those in the forward one, and see if the number is different from zero. The difference in the total number of observed events is given by

$$N(\theta : \frac{\pi}{2} - \pi) - N(\theta : 0 - \frac{\pi}{2}) = \text{luminosity} \times 2\pi \left(\int_{\frac{\pi}{2}}^\pi \sin(\theta) \frac{d\sigma}{d\Omega} d\theta - \int_0^{\frac{\pi}{2}} \sin(\theta) \frac{d\sigma}{d\Omega} d\theta\right),$$

(20)

which is plotted in figure 1 as a function of the coupling constant of LGT over that of QED at a center of mass energy of 50 GeV and luminosity of 2 $\text{fb}^{-1}$. If the value of the LGT coupling constant is measured one day, we can use equation (11) to find the value of the scale $\delta$, beyond which
the neglected interactions are also important. The top x-axis in the same figure shows this scale divided by the Planck length. Finally we use a simple $\chi^2(r)$ fit to set an upper limit on $r = \frac{\delta}{\sqrt{3}}$.

$$\chi^2(r) = \text{luminosity} \times \frac{\sigma_{\text{LGT}}}{\sigma_{\text{QED}}},$$

where $\sigma_{\text{QED}}$ refers to the first term in equation (19), and $\sigma_{\text{LGT}}$ refers to the second one, both integrated over the whole phase space. A 95% CL upper limit on $r$ will be obtained by requiring that $\chi^2(r) < 0.004$. Solving the inequality for a center of mass energy of 50 GeV and luminosity of $2 \text{ fb}^{-1}$

$$r < 0.014.$$  

### 4. LGT corrections to $e^- + e^+ \rightarrow e^- + e^+$

This section is devoted to Bhabha scattering to which not only the S-channel amplitude of the previous section, but also a T-channel amplitude, given by the following diagram, contribute:

$$i\mathcal{M}^{s's'} =$$

$$= -i\epsilon^2 \left( \bar{\nu}^{s'}(p')\delta^{\mu k}u^l(p)\frac{-i\eta_{\mu k}}{(p-p')^2} \bar{\nu}^{l'}(k') \right)$$

$$-g^2 \left( \bar{\nu}^{s'}(p')\delta^{\mu l}(\gamma_k, S^{\mu}) \bar{\nu}^l(p)\frac{-i\eta_{\mu k}}{(p-p')^2} \bar{\nu}^{l'}(k') \right).$$

Figure 1. There exists an asymmetry in the observed number of events in the left and right sides of the detector. The asymmetry is given as a function of the ratio of LGT to QED coupling constants. The top x-axis shows the ratio of the scale $\delta$, beyond which other gravity–matter interactions become significant, to the Planck length.
Therefore, the non-zero amplitudes for polarized beams read
\[ \mathcal{M}^{1111} = (2e^2 + 3g^2) \csc^2 \left( \frac{\theta}{2} \right), \quad \mathcal{M}^{2121} = \frac{2e^2 - 3g^2}{csc^2 \left( \frac{\theta}{2} \right)}, \]
\[ \mathcal{M}^{2112} = \frac{2e^2 - 3g^2}{\cot^2 \left( \frac{\theta}{2} \right)}, \quad \mathcal{M}^{2222} = \frac{2e^2 + 3g^2}{\csc^2 \left( \frac{\theta}{2} \right)}. \] (24)

After adding these T-channel amplitudes to those from the S-channel, averaging over initial polarizations, and summing the final ones, we get
\[ \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = 2e^4 \left( \frac{1 + \cos^4 \left( \frac{\theta}{2} \right)}{\sin^4 \left( \frac{\theta}{2} \right)} + \frac{1 + \cos^2 (\theta)}{2} - \frac{2 \cos^4 \left( \frac{\theta}{2} \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \right) \]
\[ + 3e^2 g^2 \left( 5 + 2 \cos (\theta) + \cos (2\theta) \right) \csc^2 \left( \frac{\theta}{2} \right) + \frac{9}{64} g^4 \left( 7 + \cos (2\theta) \right)^2 \csc^4 \left( \frac{\theta}{2} \right). \] (25)

The differential cross section of the Bhabha scattering for three different LGT couplings' strengths is shown in figure 2 for a center-of-mass energy of 50 GeV. To set an upper limit on the coupling constant of LGT, equation (21) can be used again. It turns out that, with the same center-of-mass energy and luminosity, Bhabha scattering will set a tighter limit of
\[ r < 0.006. \] (26)

5. Pair annihilation into gravitons

So far, we have studied the LGT corrections to the interactions that are dominated by QED. In this section, we would like to focus on purely LGT dominated events of pair annihilation into gravitons. This work is not possible until the Feynman rules for gravitons are worked out. In the following, first a plane gravitational wave is studied, and physical and non-physical modes are investigated. Next, the amplitudes of interest are calculated by drawing the relevant Feynman diagrams.

5.1. Gravitational plane wave and Feynman rules for gravitons

A plane gravitational wave that is freely propagating in space should satisfy the linearized LGT field equations
\[ \partial^2 \Lambda_{ij\mu} - \partial^\mu \partial_\nu \Lambda_{ij\nu} = 0. \] (27)

LGT is invariant under both local Lorentz transformations in the Lorentz spaces and arbitrary change of coordinates in the space-time, given by
\[ \Lambda^m_i = \delta^m_i + \omega^m_i, \]
\[ \frac{\partial x^\nu}{\partial \tilde{x}^\mu} = \delta^\mu_j + \partial_j \xi^\mu, \] (28)
respectively, where \( \omega^m_i \) and \( \xi^\mu \) are arbitrary but small parameters. Under these two transformations, the propagating field in LGT changes as
\[ \tilde{\Lambda}_{ij\mu} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \left( \Lambda_{ij\nu} \Lambda_{mm\nu} + \Lambda_{ij\nu} \partial_\nu A_{mm} \right) + A_{ij\mu} + \Lambda_{ij\mu} \partial_\nu \omega_{ij} + \mathcal{O}(2). \] (29)
Due to this gauge freedom, we can choose to work with the class of divergence-free spin connections. Therefore, the field equations reduce to

\[ \partial^2 A_{ij\mu} = 0, \]
\[ \partial^\nu A_{ij\nu} = 0. \]  

(30)

These equations are satisfied if a free plane wave is described by

\[ A_{ij\mu} = e_{ij\mu} e^{ik\cdot x} + e_{ij\mu}^* e^{-ik\cdot x}, \]
\[ e_{ij\mu} k^\mu = 0. \]  

(31)

Since \( A_{ij\mu} \) is antisymmetric in the two Lorentz indices, it has at most 24 independent components. However, the condition above kills six of these. To see which of the remaining components are physical, the field should be transformed according to equation (29), after using

\[ \omega_{ij} = i e_{ij\mu} e^{ik\cdot x} - i e_{ij\mu}^* e^{-ik\cdot x}, \]
\[ e_{ij} = \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j \end{cases}. \]  

(32)

The transformation is therefore equivalently represented by the following

\[ \tilde{e}_{ij\mu} = e_{ij\mu} - k^\mu e_{ij}. \]  

(33)

Without loss of generality, the wave is assumed to propagate in the \( z \) direction, i.e. \( k^\mu = (k, 0, 0, k) \). Hence, the latter equation implies

\[ \tilde{e}_{ij3} = e_{ij3} - ke_{ij}, \]
\[ \tilde{e}_{ij2} = e_{ij2}, \]
\[ \tilde{e}_{ij1} = e_{ij1}, \]  

(34)

while equation (31) leads to

\[ \tilde{e}_{ij0} = -\tilde{e}_{ij3}. \]  

(35)

It is now easy to see that \( e_{ij1} \) and \( e_{ij2} \) cannot be eliminated under any transformation. This means the gravitational waves are transverse just like the electromagnetic waves are. The
transverse plane can be spanned with two orthogonal unit vectors \( \frac{1}{\sqrt{2}} (0, 1, \pm i, 0) \). The twelve remaining components can thereby be written as

\[
e^1_{ij} (k \hat{z}) = \frac{e_{ij}}{2\sqrt{2}} (0, 1, +i, 0),
\]

\[
e^2_{ij} (k \hat{z}) = \frac{e_{ij}}{2\sqrt{2}} (0, 1, -i, 0).
\] (36)

The study presented above suggests two Feynman rules for gravitons

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array} \quad \begin{array}{c} \leftarrow \\ k \\
\end{array} \quad \begin{array}{c} \rightarrow \\ r \\
\end{array} &= e^{r}_{ij \mu} (p) \\
\begin{array}{c} k \end{array} \quad \begin{array}{c} \rightarrow \\ r \\
\end{array} &= e^{* r}_{ij \mu} (p).
\end{align*}
\] (37)

According to equation (29), \( A_{ij\mu} \) does not transform like a tensor under Lorentz transformations, and as a result it is not possible to define spin angular momentum for gravitons. This may look odd at first, because every other known particle carries spin angular momentum. However, the spin of those particles is to leave the quantum amplitudes invariant under global Lorentz transformations, and the sole reason for the existence of the spin connections is to locally preserve the same symmetry. In the following, when Feynman diagrams with external gravitons are considered, we conservatively will make sure that the amplitudes remain invariant under Lorentz transformations, by checking that the following (which is similar to the Ward identity) holds:

\[
\mathcal{M} = \mathcal{M}^\mu_{ij} e^{\mu}_{ij} = \mathcal{M}^\mu_{ij} \tilde{e}^{\mu}_{ij},
\] (38)

where \( \tilde{e}_{ij\mu} \) is given by equation (33).

5.2. Computation of cross section

Now that Feynman rules for gravitons are known, we can draw the diagrams for pair annihilation into two gravitons in all S-, T-, and U-channels, and calculate the amplitude as usual. The diagrams are shown below

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array} \quad \begin{array}{c} \leftarrow \\ s' \\
\end{array} \quad \begin{array}{c} A \\
\end{array} \quad \begin{array}{c} A \\
\end{array} \quad \begin{array}{c} \rightarrow \\ r' \\
\end{array} \quad \begin{array}{c} \rightarrow \\ r \\
\end{array} \quad \begin{array}{c} \rightarrow \\ s \\
\end{array} \\
\end{align*}
\] (39)

Their amplitude reads

\[
i\mathcal{M}^\mu_{ij} \gamma^\nu = \bar{v} \left( k \right) \left( g \{ \gamma^\nu, S^m \} \right) u(p) \left( \frac{\eta_{\mu \nu} \eta_{\alpha \beta} - \eta_{\mu \beta} \eta_{\alpha \nu}}{(p + k)^2} \right) \chi_{ij \alpha \beta \mu \nu} \psi^{*}_{ij \mu \nu} (p') \psi^{*}_{ij \mu \nu} (k')
\]

\[
+ \bar{v} \left( k \right) \psi^{*}_{ij \mu \nu} (k') \left( g \{ \gamma^\nu, S^m \} \right) \left( \frac{\eta_{\mu \nu} \eta_{\alpha \beta} - \eta_{\mu \beta} \eta_{\alpha \nu}}{(p - p')^2} \right) \psi^{*}_{ij \mu \nu} (p') \psi(p)
\]

\[
+ \bar{v} \left( k \right) \psi^{*}_{ij \mu \nu} (p') \left( g \{ \gamma^\nu, S^m \} \right) \left( \frac{\eta_{\mu \nu} \eta_{\alpha \beta} - \eta_{\mu \beta} \eta_{\alpha \nu}}{(p - k')^2} \right) \psi^{*}_{ij \mu \nu} (k') \psi(p).
\] (40)
The polarized amplitudes are found to be

\begin{align*}
M^{1211} &= -\frac{1}{2} g^2 \left( -2 + (2 + i) \cos (\theta) - \cos (2\theta) + i \sin (\theta) + \tan \left( \frac{\theta}{2} \right) \right), \\
M^{2112} &= \frac{1}{2} g^2 \left( 2 - (2 - i) \cos (\theta) + \cos (2\theta) + i \sin (\theta) - \tan \left( \frac{\theta}{2} \right) \right), \\
M^{1212} &= \frac{1}{2} g^2 \left( 2 + (2 + i) \cos (\theta) + \cos (2\theta) + i \sin (\theta) + \cot \left( \frac{\theta}{2} \right) \right), \\
M^{2121} &= \frac{1}{2} g^2 \left( 2 + (2 - i) \cos (\theta) + \cos (2\theta) - i \sin (\theta) + \cot \left( \frac{\theta}{2} \right) \right), \\
M^{1211} &= \frac{1}{2} g^2 (i + \cos (2\theta)), \\
M^{2122} &= \frac{1}{2} g^2 (-i + \cos (2\theta)), \\
M^{2111} &= \frac{1}{2} g^2 (i + \cos (2\theta)), \\
M^{2212} &= \frac{1}{2} g^2 (-i + \cos (2\theta)).
\end{align*}

The unpolarized amplitude is

\begin{equation}
\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{g^4}{16} \left( \frac{18 (1 + \sin (2\theta)) - 5 \cos (2\theta) - 4 \cos (4\theta) - \cos (6\theta) + \sin (4\theta)}{\sin^2 (\theta)} \right),
\end{equation}

which is singular at \( \theta = 0 \), due to neglecting electron mass. The singular point is, however, out of pseudorapidity coverage of most particle detectors. Therefore, we can leave the singularity out, and integrate the rest to get the total number of such events. Assuming that a given detector covers only the pseudorapidities \( |\eta| \leq 2.44 \), the total number of graviton pairs created, assessed after collecting 2 fb\(^{-1}\) data at center-of-mass energy 50 GeV, is shown in figure 3 as a function of the coupling constant of LGT over that of QED, \( r \).
6. Conclusions

The number of divergences in a renormalizable theory of gravity remains limited, no matter what order of perturbation is desired. This is met only if the coupling constant of the theory does not have a negative dimension. Fortunately, LGT has a dimensionless coupling constant, and is expected to have good high energy behavior, which makes it a viable candidate for a quantum theory of gravity. In LGT, not only the mass of fermions but also their spin gravitate. The mass-generated gravity is expected to become significant only at distance scales as small as the Planck length. Nevertheless, studies suggest that this is not true for the spin-generated gravity that is important at much lower scales. In this paper, we have investigated the observable signals of the latter type of interaction between matter and gravity. It has been shown that their Feynman vertex diagram can be converted to the QED vertex if the graviton line is replaced by that of a photon. Due to this similarity, there are corrections from LGT to QED dominated processes. Since electron–positron colliders are very popular, only those effects of LGT that can be observed in the data from such accelerators have been studied. We have specifically studied $e^- + e^+ \rightarrow \mu^- + \mu^+$, and shown that—unlike in QED, where the differential cross section is symmetric between the backward and forward detector hemispheres—in LGT, fermions tend to scatter more into the backward region. Therefore, an asymmetry between the number of events in the two hemispheres can be interpreted as a signal of LGT. Bhabha scattering has been studied as well. The two studies suggest that the coupling constant of LGT is at least three orders of magnitude smaller than that of QED. However, it is still far from the Planck scale.

Gravitational plane waves have been investigated as well. We have shown that these waves are transverse, just like their electromagnetic counterparts, and have derived the corresponding Feynman diagrams. Pair annihilation into gravitons is the last process studied. We have shown that the cross section of such processes is much smaller than those of the other two QED dominated ones. Moreover, since current or near future particle detectors are blind to gravitons, the two outgoing particles will just disappear without further track.

Appendix. A conserved current made of fermions

Normally, any invariance comes with a conserved current, which can be found by infinitesimally transforming the action of matter and setting that to zero

$$\delta \int \sqrt{-g} d^4x L_M = 0.$$  \hspace{1cm} (A.1)

For the sake of simplicity, we assume that matter is described by a Dirac Lagrangian plus the vacuum energy, instead of using the standard model Lagrangian:

$$L_M = \frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi - \frac{i}{2} D_\mu \bar{\psi} \gamma^\mu \psi + \text{constant},$$  \hspace{1cm} (A.2)

where

$$D_\mu \psi = \partial_\mu \psi - g S^{\mu\nu} A_{\nu\mu} \psi,$$

$$D_\mu \bar{\psi} = \overline{\partial}_\mu \psi + \bar{\psi} S^{\mu\nu} A_{\nu\mu},$$  \hspace{1cm} (A.3)

and $S^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu]$ refers to the generators of the Lorentz group. Now we need to see how different objects transform under infinitesimal Lorentz transformations in the tangent spaces.
\[ \delta \psi = S^{mn} \omega_{mn} \psi, \]
\[ \delta A_{
+\mu} = D_{\mu} \omega_{\mu}, \]
\[ \delta e_{mn} = \omega_{mn} e^\mu_{\mu}, \]
\[ \delta g_{\mu\nu} = 0, \]
\[ \text{(A.4)} \]

where \( \omega_{mn} \) is an antisymmetric arbitrary parameter. The metric remains unchanged under Lorentz transformations, because it purely belongs to space-time. To see the consistency of the equations
\[ \delta g_{\mu\nu} = \eta^{mn} (\delta e_{mn} e_{\mu} + e_{mn} \delta e_{\mu}) = e_{mn} (\omega_{mn} + \omega_{mn}), \]
\[ \text{(A.5)} \]

which is zero because \( \omega_{mn} \) is antisymmetric. Since the metric compatible Christoffel symbols and the determinant of the metric are made of the metric, they also remain unchanged under such transformations
\[ \delta \sqrt{-g} = 0, \]
\[ \delta \Gamma^\mu_{\mu\nu} = 0. \]
\[ \text{(A.6)} \]

An important consequence of the latter equation is that the vacuum energy will have no contribution to the conserved source. Inserting everything into equation (A.1)
\[ \int \sqrt{-g} d^4x \left[ \frac{i}{2} \bar{\psi} \gamma^\mu e_\mu \left( \partial_\mu - S^{nm} A_{nm} \right) \psi - \frac{i}{2} \bar{\psi} \left( \partial_\mu + S^{nm} A_{nm} \right) \gamma^\mu e_\mu \psi \right] = 0. \]
\[ \text{(A.7)} \]

Also, \( \frac{\partial L_M}{\partial e_{\mu}} \delta \psi = \delta \bar{\psi} \frac{\partial L_M}{\partial e_{\mu}} = 0 \) are just the Dirac field equations, and can be removed. The latter equation can thereby be written as
\[ \int \sqrt{-g} d^4x \left[ \frac{i}{2} \bar{\psi} \gamma^\mu e_\mu \left( \partial_\mu - S^{nm} A_{nm} \right) \psi - \frac{i}{2} \bar{\psi} \left( \partial_\mu + S^{nm} A_{nm} \right) \gamma^\mu e_\mu \psi \right] = 0. \]
\[ \text{(A.8)} \]

It should be noted that the tetrad postulate can be solved such that the tetrad is expressed in terms of the spin connection, at least perturbatively. Therefore, we could in principle write \( \delta e_{\mu} = \frac{\lambda_{\mu}}{\lambda_{\nu}} \delta A_{mn}. \) Nevertheless, a better approach is to define \( S^{\mu\nu}_{mn}, \) such that \( D_{\mu} S^{\mu\nu}_{mn} = \frac{\delta L_M}{\delta e_{\mu}}. \)
This is in fact the Lagrange multiplier in equations (8) and (9). Inserting this back to the equation, and using the tetrad postulate \( D_{\mu} e_{\nu} = 0, \) and consequently \( D_{\mu} g = 0, \)
\[ \int \sqrt{-g} d^4x D_{\mu} \left( S^{\mu \nu}_{mn} + i \bar{\psi} e_{\mu} \left\{ \gamma^\mu, S^{mn} \right\} \psi \right) \omega_{mn} = 0. \]
\[ \text{(A.9)} \]

Since \( \omega_{mn} \) is an arbitrary antisymmetric parameter, the term by which it is multiplied has to be zero:
\[ D_{\mu} \left( S^{\mu \nu}_{mn} + i \bar{\psi} e_{\mu} \left\{ \gamma^\mu, S^{mn} \right\} \psi \right) = 0. \]
\[ \text{(A.10)} \]

Finally, the conserved current is
\[ J^{\mu \nu}_{mn} = - S^{\mu \nu}_{mn} + i \bar{\psi} e_{\mu} \left\{ \gamma^\mu, S^{mn} \right\} \psi. \]
\[ \text{(A.11)} \]
References

[1] Albers M, Kiefer C and Reginatto M 2008 Phys. Rev. D 78 064051
[2] Kiefer C 2007 Quantum Gravity 2nd edn (Oxford: Oxford University Press)
[3] DeWitt B S 1967 Phys. Rev. 160 1113
[4] Arnowitt R, Deser S and Misner C W 2008 Gen. Relativ. Gravit. 40 1997
[5] Polchinski J 1998 String Theory (Cambridge: Cambridge University Press)
[6] Ashtekar A and Lewandowski J 2004 Class. Quantum Grav. 21 R53152
[7] Rovelli C 2011 Class. Quantum Grav. 28 114005
[8] Borzou A 2016 Class. Quantum Grav. 33 025008
[9] Bachy G, Hofmann A, Myers S, Picasso E and Plass G 1990 Part. Accel. 26 19
[10] Loew G A 1984 The SLAC linear collider and a few ideas on future linear colliders, SLAC-PUB-3327 (http://cds.cern.ch/record/153469)
[11] Brau J, Barklow T, Fujii K, Gao J, List J, Walker N and Yokoya K 2016 PoS ICHEP2016 062
[12] The CLIC Collaboration 2016 PoS ICHEP2016 064
[13] Weinberg S 1972 Gravitation And Cosmology: Principles and Applications Of The General Theory Of Relativity (New York: Wiley) p 365
[14] Weinberg S 1989 Rev. Mod. Phys. 61 1
[15] Martin J 2012 C. R. Phys. 13 566
[16] Kiefer C 2013 ISRN Math. Phys. 2013 509316
[17] Isham C J 1993 Canonical Quantum Gravity and the Problem of Time Integrable Systems, Quantum Groups and Quantum Field Theories ed L A Ibort and M A Rodriguez (Dordrecht: Kluwer)
[18] ’t Hooft G and Veltman M 1974 Ann. Inst. Henri Poincaré 20 69
[19] Deser S 2000 Ann. Phys. 9 299
[20] Hayashi K and Shirafuji T 1980 Prog. Theor. Phys. 64 866
[21] Nair V P, Randjbar-Daemi S and Rubakov V 2009 Phys. Rev. D 80 104031
[22] Borzou A 2016 Class. Quantum Grav. 33 235006
[23] Borzou A and Mirza B 2017 Class. Quantum Grav. 34 145005
[24] LGT in Linear Colliders (https://github.com/ahmadborzou/Lorentz-Gauge-Theory-Of-Gravity-LGT/wiki/LGT-in-Linear-Colliders)