Reduction of Lie–Jordan Banach algebras and quantum states

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Abstract
In this paper, it is shown that the concept of dynamical correspondence for Jordan Banach algebras is equivalent to a Lie structure compatible with the Jordan one. Then a theory of reduction of Lie–Jordan Banach algebras in the presence of quantum constraints is presented and compared to the standard reduction of $C^\ast$-algebras of observables of a quantum system. The space of states of the reduced Lie–Jordan Banach algebra is characterized in terms of Dirac states on the physical algebra of observables and its GNS representations described in terms of states on the unreduced algebra.

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1. Introduction

In this paper, we present a theory of reduction of Lie–Jordan algebras that can be used as an alternative to deal with symmetries and local constraints in quantum physics and quantum field theories. Haag’s algebraic approach to quantum systems [1] has had a profound influence on both the foundations and applications of quantum physics. The background for that approach is to consider a quantum system as described by a $C^\ast$-algebra $\mathcal{A}$ whose real parts are the observables of the system, and its quantum states $\omega$ are normalized positive complex functionals on it. The state space $\mathcal{S}$ becomes a convex $\ast$-compact topological space and pure states are its extremal points. Moreover, given a state $\omega$, the GNS construction allows us to represent the $C^\ast$-algebra $\mathcal{A}$ on a cyclic complex Hilbert space $\mathcal{H}_\omega$.

A geometrical approach to quantum mechanics [4, 5] has been developed in the last 20 years (see also Cirelli and Lanzavecchia [2] and Ashtekar and Schilling [3]). The geometrical
picture of quantum mechanics could be useful to gain an insight into such properties of quantum systems as integrability [6], the intrinsic nature of different measures of entanglement [7, 8], etc (see for instance [9] and references therein). Often, a geometrical description of dynamical systems provides a natural setting to describe symmetries and/or constraints. For instance, if the system carries a symplectic or Poisson structure, several procedures were introduced along the years to cope with them, like Marsden–Weinstein reduction, symplectic reduction, Poisson reduction, reduction of contact structures, etc. However, it was soon realized that the algebraic approach to reduction provided a convenient setting to deal with the reduction of classical systems [10, 11].

Whenever constraints are imposed on a quantum system or symmetries are present, both dynamical or gauge, some reduction on the state space must be considered either because not all states are physical and/or because families of states are equivalent. In the standard approach to quantum mechanics, constraints are imposed on the system by selecting subspaces determined by the quantum operators corresponding to the constraints of the theory. Dirac states and equivalence of quantum states were dealt with by using the representation theory of the corresponding group of symmetries. However, many difficulties emerge when implementing this analysis for arbitrary singular Lagrangian systems or other singularities arise (like singular level sets of momentum maps for instance).

Taking as a departure point the algebraic approach to quantum mechanics and quantum field theory the problem of reduction of the quantum system becomes the problem of reducing the C*-algebra of the system. Such a programme was successfully developed for some gauge theories and was called T-reduction (see [12] and references therein). The analysis for quantum local constraints can be found in [13]. However, the state space S of the quantum system does not determine univocally the C*- algebra structure of the system but only its Jordan–Banach real algebra part [14, 15]. In fact the real (or self-adjoint) part of a C*-algebra A, because of a theorem by Kadison [16], is isometrically isomorphic to the space of all w*-continuous affine functions on the state space. Connes on one hand [17] and Alfsen and Shultz on the other [18] solved the problem of when a given Jordan–Banach algebra is the real part of a C*-algebra. The characterization obtained by Alfsen and Schultz in terms of the existence of a dynamical correspondence on a Jordan– Banach algebra amounts to stating that the relevant structure to discuss the properties of the state space of a quantum system is that of a Lie–Jordan–Banach algebra [19, 20]. In fact, the topological properties of the state space are completely captured by the Jordan–Banach algebra structure and the Lie algebra structure allows us to construct the C*-algebra setting for them, their GNS representations, etc.

Thus, as stated above, we will address the problem of the reduction of quantum systems as the reduction of Lie–Jordan–Banach algebras. Inspired by the reduction theory of C*-algebra we will obtain in section 4 a reduction theorem for Lie–Jordan Banach algebras with respect to quantum constraints. These results will be compared in section 5 to the T-procedure reduction of C*-algebra and shown to be equivalent to the theory discussed in this paper. An explicit description of the states of the reduced Lie–Jordan–Banach algebras will be obtained and the corresponding GNS representations will be discussed in sections 6 and 7.

2. States and Lie–Jordan Banach algebras

States are positive linear functionals ω on a C*-algebra A of observables, thus ω(a*a) ≥ 0, ∀ a ∈ A. We will assume in what follows that the C*-algebra is unital and states are
normalized, i.e. \( \omega(1) = 1 \), where 1 denotes the unit element of \( \mathcal{A} \). We will denote by \( \mathcal{A}_{ss} \) the real (or self-adjoint) part of \( \mathcal{A} \), i.e.

\[
\mathcal{A}_{ss} = \{ a \in \mathcal{A} \mid a^* = a \},
\]
which inherits the structure of a real Lie–Jordan algebra. It is a Jordan algebra with the non-associative commutative Jordan product

\[
a \circ b = \frac{1}{2}(ab + ba), \quad \forall a, b \in \mathcal{A}_{ss},
\]
and by setting \( a \circ a = a^2 \), the Jordan product \( \circ \) satisfies

\[
(a^2 \circ b) \circ a = a^2 \circ (b \circ a)
\]
which is the usual replacement for associativity of Jordan algebras.

The Lie product is obtained by taking a scaled commutator

\[
[a, b] = i\lambda(ab - ba), \quad \forall a, b \in \mathcal{A}_{ss},
\]
with \( \lambda \in \mathbb{R} \). Note that \( [a, b] \in \mathcal{A}_{ss} \forall a, b \in \mathcal{A}_{ss} \). The skew-symmetric bilinear form \( [\cdot, \cdot] \) equips \( \mathcal{A}_{ss} \) with a Lie algebra structure since it verifies the Jacobi identity

\[
[[a, b], c] + [[c, a], b] + [[b, c], a] = 0.
\]
These two operations are compatible in the sense that Leibniz identity is satisfied:

\[
[a, b \circ c] = [a, b] \circ c + b \circ [a, c],
\]
or, in other words, the linear map \( D_a(\cdot) = [a, \cdot] \) is a derivation of the Jordan product \( \circ \). By abstracting the previous properties, one says in general that a vector space with a symmetric operation \( \circ \) and an antisymmetric one \( [\cdot, \cdot] \) satisfying the properties (2), (4) and (5), is called a pre-Lie–Jordan algebra. The complete definition of a Lie–Jordan algebra requires that the associator of the structure product is related to the Lie bracket by

\[
(a \circ b) \circ c - a \circ (b \circ c) = \kappa [b, [c, a]],
\]
\( \kappa \) being a positive real number. When using the Lie bracket (3), we obtain that (6) requires

\[
\kappa \lambda^2 = 1/4,
\]
if \( \kappa \neq 0 \). Notice that if \( \kappa = 0 \) the Jordan product is associative.

If we consider a classical carrier space, for instance a Poisson manifold, the algebra of smooth functions on the manifold becomes a Lie–Jordan algebra when equipped with the (associative) pointwise product \( f \circ g(x) = f(x)g(x) \), and Lie bracket \( [f, g] = \{f, g\} \), with \( \{\cdot, \cdot\} \) being the Poisson bracket defined on the manifold. Thus, it follows that from an algebraic point of view it is quite appropriate to consider a Poisson algebra as a Lie–Jordan algebra with \( \kappa = 0 \). From this perspective, we may consider the parameter \( \kappa \) a sort of deformation parameter connecting the classical and the quantum picture of a system. With this intuition in mind, we may call Lie–Jordan algebras with \( \kappa = 0 \) classical.

**Definition 1.** A pre-Lie–Jordan–Banach algebra (or pre-LJB algebra for short) is a pre-Lie-Jordan algebra \((\mathcal{L}, \circ, [\cdot, \cdot])\) such that it carries a complete norm \( \| \cdot \| \) verifying

\[
\begin{align*}
(i) \quad & \|a \circ b\| \leq \|a\| \|b\| \\
(ii) \quad & \|a^2\| = \|a\|^2 \\
(iii) \quad & \|a^2\| \leq \|a^2 + b^2\|,
\end{align*}
\]

\( \forall a, b \in \mathcal{L} \) and \( [\cdot, \cdot] \) is continuous. A Lie–Jordan–Banach algebra (LJB algebra for short) is a pre-LJB algebra that satisfies the associator compatibility condition equation (6).
In particular, an LJB algebra is a Jordan–Banach algebra (JB algebra) when considered with the Jordan product alone. On the other hand, if we are given an LJB-algebra $L$, by taking combinations of the two products we can define an associative product on the complexification $L^C = L \oplus iL$. Specifically, we define
\[ ab = a \circ b - i\sqrt{\kappa} [a, b] . \]
Such associative algebra equipped with the norm $\|x\| = \|x^*x\|^{1/2}$ where $x = a + ib$, is the unique $C^*$-algebra whose real part is precisely $L$ (see section 3).

Note that if the LJB algebra $L$ is classical, $\kappa = 0$, its associated $C^*$-algebra is isomorphic to the space of continuous functions on a compact topological space with the supremum norm, hence if such space carries a differentiable structure the Lie bracket will define a family of unbounded derivations on the dense subspace of smooth functions, otherwise trivial. In other words, we will need weaker topologies to accommodate classical LJB algebra in the same picture. Then, from now on, we will just consider non-classical LJB algebras, i.e. $\kappa \neq 0$.

The space of states $\mathcal{S}(L)$ of a Jordan–Banach algebra consists of all real normalized positive linear functionals on $L$, i.e.
\[ \rho : L \rightarrow \mathbb{R}, \]
such that $\rho(1) = 1$ and $\rho(a^2) \geq 0$, $\forall a \in L$. The state space is convex and compact with respect to the $w^*$-topology. Moreover, it can be shown that the states satisfy the following Cauchy–Schwarz like inequalities:
\[ \rho(a \circ b)^2 \leq \rho(a^2)\rho(b^2), \quad (7) \]
\[ \rho([a, b])^2 \leq \frac{1}{\kappa} \rho(a^2)\rho(b^2). \quad (8) \]

It was an important problem to determine which state spaces of Jordan algebras are the state spaces of a $C^*$-algebra. In fact, it follows from the results of Kadison [16] that if $\mathcal{S}(A)$ denotes the state space of a $C^*$-algebra $A$, then $A_{sa}$ is naturally isomorphic to the space of affine $w^*$-continuous real functions on $\mathcal{S}(A)$; hence the state space $\mathcal{S}(A)$ as a topological space is determined just by the self-adjoint part of $A$. The previous problem was solved by Alfsen and Shultz [18] and Connes [17] independently. We will describe the reconstruction of the $C^*$-algebra in the following section.

3. Dynamical correspondence and LJB algebras

Following [18] we will define a derivation of a JB algebra $L$ by focusing first only on the order structure, ignoring the algebraic multiplicative aspect. All the proofs contained in [18] will be omitted. First note that a unital JB algebra $L$ is a complete order unit space with respect to the positive cone
\[ L^+ = \{ a^2 \mid a \in L \}. \quad (9) \]

**Definition 2.** A bounded linear operator $\delta$ on a JB algebra $L$ is called an order derivation if $e^{itL}(L^+) \subset L^+$, $\forall t \in \mathbb{R}$.

We denote the Jordan multiplier determined by an element $b \in L$ by $\delta_b$. Thus, for all $a \in L$
\[ \delta_b(a) = b \circ a. \]
It can be shown that Jordan multipliers $\delta_b$ are order derivations $\forall b \in L$. 

\[ 4 \]
Definition 3. An order derivation $\delta$ on a unital JB algebra $L$ is self-adjoint if there exists $a \in L$ such that $\delta = a$ and is skew-adjoint if $\delta(1) = 0$.

Again, it can be shown that if $\delta$ is an order derivation, then $\delta$ is skew if and only if $\delta$ is a Jordan derivation, i.e. is a derivation with respect to the Jordan product, that is

$$\delta(a \circ b) = \delta a \circ b + a \circ \delta b, \quad \forall a, b \in L.$$  \hspace{1cm} (10)

Definition 4. A dynamical correspondence on a unital JB algebra $L$ is a linear map

$$\psi : a \rightarrow \psi a$$  \hspace{1cm} (11)

from $L$ into the set of skew order derivations on $L$ which satisfies

(i) there exists $\kappa \in \mathbb{R}$ such that $\kappa [\psi a, \psi b] = -[\delta a, \delta b], \quad \forall a, b \in L$, and

(ii) $\psi a a = 0, \quad \forall a \in L$.

It follows immediately from the definitions that

$$\psi b a = -\psi a b, \quad \forall a, b \in L.$$  \hspace{1cm} (12)

Definition 5. Let $L$ be a unital JB algebra. A $C^*$-product compatible with $L$ is an associative product on the complex linear space $L \oplus iL$ which induces the given Jordan product on $L$ and makes $L \oplus iL$ into a $C^*$-algebra with involution $(a + ib)^* = a - ib$ and norm $\|x\| = \|x^*x\|^{1/2}$ where $x = a + ib$.

Note that if a JB algebra $L$ is the self-adjoint part of a $C^*$-algebra $A$, then there are a natural product and a norm induced in $L \oplus iL$ by using the representation $A = a + ib$ with $A \in A$ and $a, b \in L$. Such product and norm organize $L \oplus iL$ into a $C^*$-algebra. It follows that a JB algebra is the self-adjoint part of a $C^*$-algebra if and only if there exists a $C^*$-product compatible with $L$ on $L \oplus iL$. We can now state the main result in [18], relating the existence of a dynamical correspondence on a JB algebra to the existence of a compatible $C^*$ product.

Theorem 1. A unital JB algebra $L$ is Jordan isomorphic to the self-adjoint part of a $C^*$-algebra if and only if there exists a dynamical correspondence on $L$. Each dynamical correspondence $\psi$ on $L$ determines a unique associative $C^*$ product compatible with $L$ defined as

$$ab = a \circ b - i \sqrt{\kappa} \psi a b$$  \hspace{1cm} (13)

and each $C^*$ product compatible with $L$ arises in this way from a unique dynamical correspondence $\psi$ on $L$.

We will now show that the existence of a dynamical correspondence on $L$ is equivalent to the existence of a Lie product organizing $L$ into an LJB algebra. First we need the following lemmas.

Lemma 1. Let $(L, [\cdot, \cdot]_L, \circ)$ be an LJB algebra. Then there exists an associative bilinear product on $L \times L$ defined as

$$a \cdot b = a \circ b - i \sqrt{\kappa} [a, b]_L, \quad \forall a, b \in L.$$  \hspace{1cm} (14)
Proof. Bilinearity of the product follows directly from the bilinearity of the Jordan and Lie products. We have to prove the associativity, i.e.

\[ a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall \ a, b, c \in \mathcal{L}. \]  

(15)

The lhs of the previous equation leads to

\[ a \cdot (b \cdot c) = a \circ (b \circ c) - i\sqrt{\kappa} a \circ [b, c]_{\mathcal{L}} - i\sqrt{\kappa} [a, b]_{\mathcal{L}} \circ c - i\sqrt{\kappa} b \circ [a, c]_{\mathcal{L}}, \]

and the rhs

\[ (a \cdot b) \cdot c = (a \circ b) \circ c - i\sqrt{\kappa} b \circ [a, c]_{\mathcal{L}} - i\sqrt{\kappa} a \circ [b, c]_{\mathcal{L}} - i\sqrt{\kappa} [a, b]_{\mathcal{L}} \circ c - \kappa [a, [b, c]_{\mathcal{L}}]. \]

Then

\[ a \cdot (b \cdot c) - (a \cdot b) \cdot c = a \circ (b \circ c) - (a \circ b) \circ c - \kappa [a, [b, c]_{\mathcal{L}}] - \kappa [c, [a, b]_{\mathcal{L}}] \]
\[ = \kappa ([b, c, a]_{\mathcal{L}} + [a, [b, c]_{\mathcal{L}}] + [c, [a, b]_{\mathcal{L}}]) = 0, \]

where we have used (4)–(6).

Note that the Jordan and Lie products can be obviously expressed in terms of the associative product as

\[ a \circ b = \frac{1}{2} (a \cdot b + b \cdot a), \]  

(16)

\[ [a, b]_{\mathcal{L}} = \frac{i}{2\sqrt{\kappa}} (a \cdot b - b \cdot a). \]  

(17)

Lemma 2. Let \((\mathcal{L}, [, , ]_{\mathcal{L}}, \circ)\) be an LJB algebra. Then \(e^{[\cdot , ]_{\mathcal{L}}^a}\) is a Jordan automorphism \(\forall a \in \mathcal{L}\).

Proof. We have to prove that

\[ e^{[\cdot , ]_{\mathcal{L}}^a} (b \circ c) = (e^{[\cdot , ]_{\mathcal{L}}^a} b) \circ (e^{[\cdot , ]_{\mathcal{L}}^a} c). \]  

(18)

By Hadamard’s formula, the lhs of the previous equation is

\[ e^{[\cdot , ]_{\mathcal{L}}^a} (b \circ c) = e^a \cdot (b \circ c) \cdot e^{-a}. \]

By using formula (16), the rhs of (18) becomes

\[ (e^{[\cdot , ]_{\mathcal{L}}^a} b) \circ (e^{[\cdot , ]_{\mathcal{L}}^a} c) = (e^a \cdot b \cdot e^{-a}) \circ (e^a \cdot c \cdot e^{-a}) \]
\[ = \frac{1}{2} e^a \cdot (b \circ c) \cdot e^{-a} + \frac{1}{2} e^a \cdot (c \cdot b) \cdot e^{-a} \]
\[ = e^a \cdot (b \circ c) \cdot e^{-a}. \]

\[ \square \]

Lemma 3. Let \((\mathcal{L}, [, , ]_{\mathcal{L}}, \circ)\) be an LJB algebra. Then \([a, , ]_{\mathcal{L}}\) is an order derivation on \(\mathcal{L}\) \(\forall a \in \mathcal{L}\).

Proof. From definition 2, we have to prove that \(e^{[\cdot , ]_{\mathcal{L}}^a} (\mathcal{L}^+) \subset \mathcal{L}^+, \forall a \in \mathcal{L}\) and \(\forall t \in \mathbb{R}\). Since \(e^{[\cdot , ]_{\mathcal{L}}^a}\) is a Jordan automorphism (lemma 2), we have

\[ e^{[\cdot , ]_{\mathcal{L}}^a} (b \circ b) = (e^{[\cdot , ]_{\mathcal{L}}^a} b) \circ (e^{[\cdot , ]_{\mathcal{L}}^a} b), \]

\(\forall a, b \in \mathcal{L}\) and \(\forall t \in \mathbb{R}\), i.e. \(e^{[\cdot , ]_{\mathcal{L}}^a}\) preserves the positive cone (9) \(\mathcal{L}^+\).

\[ \square \]
**Theorem 2.** Let \( \mathcal{L} \) be a unital JB algebra. There exists a dynamical correspondence \( \psi \) on \( \mathcal{L} \) if and only if \( \mathcal{L} \) is an LJB algebra with a Lie product \( [\cdot, \cdot]_L \) such that
\[
[a, b]_L = \psi_{ab}.
\]

**Proof.** First assume that \( \mathcal{L} \) is an LJB algebra. From definition 4, we have to check that
\[
\kappa [\psi_a, \psi_b] = -[\delta_a, \delta_b]
\]
that is
\[
\kappa ([a, [b, c]]_L - [b, [a, c]]_L) = b \circ (a \circ c) - a \circ (b \circ c),
\]
which is an easy computation once the Jordan and Lie products are expressed as in (17) and (16). From the antisymmetry of the Lie product, it is also true that \( \psi_{aa} = [a, a]_L = 0 \) \( \forall a \in \mathcal{L} \). Hence, the linear map \( a \rightarrow [a, \cdot]_L \) from the LJB algebra \( \mathcal{L} \) to the skew-order derivations on \( \mathcal{L} \) is a dynamical correspondence.

Conversely, assume \( \mathcal{L} \) is a JB algebra with a dynamical correspondence \( \psi \). Then from (12) \( \psi_{ab} = [a, b]_L \) is antisymmetric. The Jacobi property (4) follows from the defining property (i) of the dynamical correspondence (definition 4), the Leibniz identity (5) follows from (10) and also the compatibility condition (6) is easy to check with a simple computation using the properties of the dynamical correspondence (definition 4). Hence a JB algebra with a dynamical correspondence is an LJB algebra. \( \square \)

**Corollary 1.** A unital JB algebra \( \mathcal{L} \) is Jordan isomorphic to the self-adjoint part of a C*-algebra if and only if it is an LJB algebra.

**Proof.** This is an obvious consequence of theorems 1 and 2. \( \square \)

**Corollary 2.** Let \( (\mathcal{L}, \circ, [\cdot, \cdot]_L) \) be an LJB algebra and \( \mathcal{A} = \mathcal{L}^\mathbb{C} \) the natural C*-algebra defined by the complexification of \( \mathcal{L} \). Then there is a natural identification between the states \( \mathcal{S}(\mathcal{L}) \) of \( \mathcal{L} \) and the states \( \mathcal{S}(\mathcal{A}) \) of the C*-algebra \( \mathcal{A} \).

**Proof.** Given a state \( \omega \) of \( \mathcal{L} \), we define a linear functional \( \tilde{\omega} \) of \( \mathcal{A} \) by extending it linearly. The linear functional \( \tilde{\omega} \) is positive and normalized because \( \omega \) is positive and normalized. The converse is trivial. \( \square \)

### 4. Reduction of Lie–Jordan–Banach algebras

In this section, we show how to deal with quantum constraints in the LJB algebra setting. A quantum system with constraints is a pair \( (\mathcal{L}, \mathcal{C}) \) where the field algebra \( (\mathcal{L}, \circ, [\cdot, \cdot]) \) is a unital LJB algebra containing the constraint set \( \mathcal{C} \). The constraints select the physical state space, also called Dirac states
\[
\mathcal{S}_D = \{\omega \in \mathcal{S}(\mathcal{L}) \mid \omega(c^2) = 0, \ \forall c \in \mathcal{C}\},
\]
where \( \mathcal{S}(\mathcal{L}) \) is the state space of \( \mathcal{L} \). We define the vanishing subalgebra \( \mathcal{V} \) as
\[
\mathcal{V} = \{a \in \mathcal{L} \mid \omega(a^2) = 0, \ \forall \omega \in \mathcal{S}_D\}.
\]

**Proposition 1.** \( \mathcal{V} \) is a non-unital LJB subalgebra.
Proof. Let \( a, b \in \mathcal{V} \). From (6) it follows:

\[(a \circ b)^2 = \kappa [b, [a \circ b, a]] + a \circ (b \circ (a \circ b)).\]  

(20)

If we introduce \( c = [a \circ b, a] \) and \( d = b \circ (a \circ b) \), equation (20) becomes

\[(a \circ b)^2 = \kappa [b, c] + (a \circ d).\]  

(21)

From the inequalities (7) and (8) it is easy to show that if \( \omega(a^2) = 0 \) then

\[\omega(a \circ b) = 0 = \omega([a, b]) \quad \forall b \in \mathcal{L}.\]  

(22)

Then if we apply the state \( \omega \) to the expression (21), from (22) it follows:

\[\omega((a \circ b)^2) = \kappa \omega([b, c]) + \omega(a \circ d) = 0.\]  

(23)

By definition of \( \mathcal{V} \), this means that \( \forall a, b \in \mathcal{V}, a \circ b \in \mathcal{V} \).

By applying the state \( \omega \) to the relation

\[(a \circ b)^2 = \kappa [a, b]^2 + a \circ (b \circ (a \circ b)) - \kappa a \circ [b, [a, b]],\]

we obtain \( \omega((a \circ b)^2) = \omega((a \circ b)^2) = 0 \), that is \( \forall a, b \in \mathcal{V}, [a, b] \in \mathcal{V} \). Hence, \( \mathcal{V} \) is a Lie–Jordan subalgebra.

\( \mathcal{V} \) also inherits the Banach structure since it is defined as the intersection of closed subspaces. \( \Box \)

We can use the vanishing subalgebra to give an alternative description of the Dirac states that will be useful in the following.

**Proposition 2.** With the previous definitions we have

\[\mathfrak{S}_D = \{\omega \in \mathfrak{S}(\mathcal{L}) \mid \omega(a) = 0, \forall a \in \mathcal{V}\}.\]

**Proof.** As \( \mathcal{V} \) is a subalgebra and it contains \( \mathcal{C} \) it is clear that the right-hand side is included into \( \mathfrak{S}_D \).

To see the other inclusion, it is enough to consider that for any state \( \omega(a)^2 \leq \omega(a^2) \), therefore any Dirac state should vanish on \( \mathcal{V} \).

Define now the Lie normalizer as

\[\mathcal{N}_\mathcal{V} = \{a \in \mathcal{L} \mid [a, \mathcal{V}] \subset \mathcal{V}\},\]  

(24)

which corresponds roughly to Dirac’s concept of ‘first class variables’ [22].

**Proposition 3.** \( \mathcal{N}_\mathcal{V} \) is a unital LJB algebra and \( \mathcal{V} \) is a Lie–Jordan ideal of \( \mathcal{N}_\mathcal{V} \).

**Proof.** Let \( a, b \in \mathcal{N}_\mathcal{V} \) and \( v \in \mathcal{V} \). Then by definition of the normalizer it immediately follows:

\[[a, b], v] = [[a, v], b] + [[v, b], a] \in \mathcal{V}.\]

Let us now prove that \( \forall v \in \mathcal{V}, v \circ a \in \mathcal{V} \), this is \( \mathcal{V} \) is a Jordan ideal of \( \mathcal{N}_\mathcal{V} \):

\[\omega((v \circ a)^2) = \kappa \omega([a, [v \circ a, v]]) + \omega(v \circ (a \circ (a \circ v))),\]

which gives zero by repeated use of properties (7) and (8).

Then it becomes easy to prove that \( \mathcal{N}_\mathcal{V} \) is a Jordan subalgebra:

\[[a \circ b, v] = [a, v] \circ b + a \circ [b, v] \in \mathcal{V}.\]
Finally, since the Lie bracket is continuous with respect to the Banach structure, it also follows that $N_V$ inherits the Banach structure by completeness. □

In the spirit of Dirac, the physical algebra of observables in the presence of the constraint set $C$ is represented by the LJB algebra $N_V$ which can be reduced by the closed Lie–Jordan ideal $V$ which induces a canonical Lie–Jordan algebra structure in the quotient:

$$\tilde{L} = N_V/V.$$  \hfill (25)

We will denote in the following the elements of $\tilde{L}$ by $\tilde{a}$.

The quotient Lie–Jordan algebra $\tilde{L}$ carries the quotient norm,

$$\|\tilde{a}\| = \| [a] \| = \inf_{b \in V} \|a + b\|,$$

where $a \in N_V$ is an element of the equivalence class $[a]$ of $N_V$ with respect to the ideal $V$.

Hence, the reduction of the Lie–Jordan algebra $L$ with respect to the constraint set $C$ is given by the short exact sequence

$$0 \rightarrow V \rightarrow N_V \rightarrow \tilde{L} \rightarrow 0.$$  \hfill (26)

In section 6, we will prove that the states on the reduced LJB algebra $\tilde{L}$ are exactly the Dirac states restricted to the physical algebra of observables $N_V$.

5. Reduction of Lie–Jordan algebras and constraints in $C^*$-algebras

Following [12, 13], we briefly recall how to deal with quantum constraints in a $C^*$-algebra setting. The aim of this section is to prove that the reduction procedure of $C^*$-algebras used to analyze quantum constraints, also called T-reduction, can be equivalently described by using the theory of reduction of LJB algebras discussed above.

A quantum system with constraints is a pair $(F, C)$ where now the field algebra $F$ is a unital $C^*$-algebra containing the self-adjoint constraint set $C$, i.e. $C = C^* \forall C \in C$. The constraints select the Dirac states $S_D \equiv \{\omega \in S(F) \mid \omega(C^2) = 0, \ \forall \ C \in C\}$, where $S(F)$ is the state space of $F$.

Define $D = [FC] \cap [CF]$ where the notation $[\cdot]$ denotes the closed linear space generated by its argument. It satisfies the following.

**Theorem 3.** $D$ is the largest non-unital $C^*$-algebra in $\bigcap_{\omega \in S_D} \ker \omega$.

For any set $\Omega \subset F$, define as before its normalizer or ‘weak commutant’ as

$$\Omega_W = \{F \in F \mid [F, H] \in \Omega, \ \forall H \in \Omega\}. \hfill (27)$$

Consider now the multiplier algebra of $\Omega$ as

$$\mathcal{M}(\Omega) = \{F \in F \mid FH \in \Omega \text{ and } HF \in \Omega, \ \forall H \in \Omega\}, \hfill (28)$$

i.e. the largest set for which $\Omega$ is a bilateral ideal. $\mathcal{M}(\Omega)$ is clearly an unital $C^*$-algebra and we have the following.

**Theorem 4.** $\mathcal{O} = D_W = \mathcal{M}(D)$.  

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That is, the weak commutant of $D$ is also the largest set for which $D$ is a bilateral ideal and it will be denoted by $\mathcal{O}$. It follows that the maximal (and unital) $C^*$-algebra of physical observables determined by the constraints $C$ is given by

$$\tilde{\mathcal{F}} = \mathcal{O} / D.$$  \hfill (29)

To show that this procedure is equivalent to the reduction of the corresponding LJB algebra (as discussed in section 4), we need to prove some simple statements.

**Lemma 4.** Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie–Jordan subalgebras of an LJB algebra $\mathcal{L}$. Then $\mathcal{Z}^C = \mathcal{Z} \oplus i\mathcal{Z}$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^C = \mathcal{I} \oplus i\mathcal{I}$ if and only if $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$, i.e. $\mathcal{Z} = N_{\mathcal{I}}$.

**Proof.** Assume first $\mathcal{Z}^C$ is the weak commutant of $\mathcal{I}^C$ and let $a + ib \in \mathcal{Z}^C$ with $a, b \in \mathcal{Z}$. By definition

$$[a + ib, \mathcal{I} \oplus i\mathcal{I}] \subset \mathcal{I} \oplus i\mathcal{I},$$

that is

$$[a + b, \mathcal{I}] \subset \mathcal{I} \hspace{1em} \text{and} \hspace{1em} [a - b, \mathcal{I}] \subset \mathcal{I}.$$

Since the normalizer is a vector space, this implies

$$[a, \mathcal{Z}] \subset \mathcal{I} \hspace{1em} \text{and} \hspace{1em} [b, \mathcal{Z}] \subset \mathcal{I}, \hspace{1em} \forall a, b \in \mathcal{Z},$$

that is $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$. Conversely assume $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$

$$[a, \mathcal{I}] \subset \mathcal{I}, \hspace{1em} \forall a \in \mathcal{Z},$$

then it follows:

$$[a + ib, x + iy] \in \mathcal{I}, \hspace{1em} \forall a, b \in \mathcal{Z} \hspace{1em} \text{and} \hspace{1em} \forall x, y \in \mathcal{I},$$

that is $\mathcal{Z}^C$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^C$. \hfill \Box

**Lemma 5.** Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie–Jordan subalgebras of $\mathcal{L}$. Then $\mathcal{I}$ is a Lie–Jordan ideal of $\mathcal{Z}$ if and only if $\mathcal{I}^C = \mathcal{I} \oplus i\mathcal{I}$ is an associative bilateral ideal of $\mathcal{Z}^C = \mathcal{Z} \oplus i\mathcal{Z}$.

**Proof.** Using the expressions provided by equations (16) and (17), the statement becomes an easy computation. \hfill \Box

Let us define $\mathcal{L}$ and $\tilde{\mathcal{L}}$ such that $\mathcal{F} = \mathcal{L} \oplus i\mathcal{L}$ and $\tilde{\mathcal{F}} = \tilde{\mathcal{L}} \oplus i\tilde{\mathcal{L}}$, i.e. they are the self-adjoint parts of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively. From corollary 1, it follows that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are unital LJB algebras. Similarly, define the LJB algebras $\mathcal{N}_{\mathcal{F}}$ and $\mathcal{N}_{\tilde{\mathcal{F}}}$ as the self-adjoint parts of $\mathcal{O}$ and $\mathcal{D}$, respectively, i.e. $\mathcal{O} = \mathcal{N}_{\mathcal{F}} \oplus i\mathcal{N}_{\mathcal{F}}$, $\mathcal{D} = \mathcal{N}_{\mathcal{F}} \oplus i\mathcal{N}_{\mathcal{F}}$.

**Theorem 5.** With the notations above, let $\mathcal{F} = \mathcal{L} \oplus i\mathcal{L}$ be the field algebra of the quantum system and $\mathcal{C}$ a real constraint set. Let $\mathcal{D} = [\mathcal{F} \mathcal{C}] \cap [\mathcal{C} \mathcal{F}]$, $\tilde{\mathcal{O}} = \mathcal{D}_{\mathcal{V}}$ be as in theorem 4, and $\tilde{\mathcal{F}} = \tilde{\mathcal{O}} / \tilde{\mathcal{D}} = \tilde{\mathcal{L}} \oplus i\tilde{\mathcal{L}}$ be the reduced field algebra. Then

$$\tilde{\mathcal{L}} = \mathcal{N}_{\mathcal{F}} / \mathcal{V},$$

with $\mathcal{V}$ and $\mathcal{N}_{\mathcal{F}}$ being the vanishing subalgebra of $\mathcal{L}$ and its Lie normalizer, respectively.
Proof. Observe that the space of states on $F$ is the space of states on $L$ extended linearly by complexification and conversely $\mathcal{S}(L) = \mathcal{S}(F)|_L$. Then from theorem 3, it follows that $D$ is exactly the vanishing subalgebra for $\mathcal{S}_D$; that is, $D = V \oplus iV$. Then from lemmas 4 and 5, everything is straightforward and the two procedures are clearly equivalent. □

The equivalence of the two approaches can be illustrated pictorially by the following ‘functorial’ diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{\mathcal{F} = L \oplus iL} & \tilde{L} \\
\downarrow & & \downarrow \\
V, \mathcal{N}_V & \xrightarrow{D, \mathcal{O}} & \tilde{V}, \mathcal{N}_V \\
\end{array}
\]

6. The space of states of the reduced LJB algebra

The purpose of the remaining two sections is to discuss the structure of the space of states and the GNS construction of reduced states for reduced LJB algebras with respect to the space of states of the unreduced one.

As discussed in the previous section, let $A$ be a \(C^*\)-algebra, $L = A_{sa}$ its real part and $V$ the vanishing subalgebra of $L$ with respect to a constraint set $C$ and let $\mathcal{N}_V$ be the Lie normalizer of $V$. Then we will denote as before by $\tilde{L}$ the reduced Lie–Jordan Banach algebra $\mathcal{N}_V / V$ and its elements by $\tilde{a}$.

Let $\tilde{\mathcal{S}} = \mathcal{S}(\tilde{L})$ be the state space of the reduced LJB algebra $\tilde{L}$, i.e. $\tilde{\omega} \in \tilde{\mathcal{S}}$ means that $\tilde{\omega}(\tilde{a}^2) \geq 0 \ \forall \ \tilde{a} \in \tilde{L}$, and $\tilde{\omega}$ is normalized. Note that if $L$ is unital, then $1 \in \mathcal{N}_V$ and $1 + V$ is the unit element of $\tilde{L}$. We will denote it by $\tilde{1}$.

We have the following.

Lemma 6. There is a one-to-one correspondence between normalized positive linear functionals on $\tilde{L}$ and normalized positive linear functionals on $\mathcal{N}_V$ vanishing on $V$.

Proof. Let $\omega' : \mathcal{N}_V \to \mathbb{R}$ be positive. The positive cone on $\tilde{L}$ consists of elements of the form $\tilde{a}^2 = (a + V)^2 = a^2 + V$, i.e.

\[
K^+_{\tilde{L}} = \{a^2 + V \mid a \in \mathcal{N}_V\} = K^+_{\mathcal{N}_V} + V.
\]

Thus, if $\omega'$ is positive on $\mathcal{N}_V$, $\omega'(a^2) \geq 0$, hence

\[
\omega'(a^2 + V) = \omega'(a^2) + \omega'(V),
\]

and if $\omega'$ vanishes on the closed ideal $V$, then $\omega'$ induces a positive linear functional on $\tilde{L}$. Clearly, if $\omega'$ is normalized then the induced functional is normalized too because $\tilde{1} = 1 + V$.

Conversely, if $\tilde{\omega} : \tilde{L} \to \mathbb{R}$ is positive and we define

\[
\omega'(a) = \tilde{\omega}(a + V),
\]

then $\omega'$ is well-defined, positive, normalized and $\omega'|_V = 0$. □

Note also that given a positive linear functional on $\mathcal{N}_V$, there exists an extension of it to $L$ which is positive too.
Lemma 7. Given a closed Jordan subalgebra $Z$ of an LJB algebra $L$ such that $1 \in Z$ and $\omega'$ is a normalized positive linear functional on $Z$, then there exists $\omega : L \to \mathbb{R}$ such that $\omega(a) = \omega'(a)$, $\forall a \in Z$ and $\omega \geq 0$.

Proof. Since $L$ is a JB algebra, it is also a Banach space. Due to the Hahn–Banach extension theorem, there exists a continuous extension $\omega$ of $\omega'$, i.e. $\omega(a) = \omega'(a)$, $\forall a \in Z$, and moreover $\|\omega\| = \|\omega'\|$.

From the equality of norms and the fact that $\omega'$ is positive, we have $\|\omega\| = \omega'(1)$, but $\omega$ is an extension of $\omega'$ then $\|\omega\| = \omega(1)$, which implies that $\omega$ is a positive functional and satisfies all the requirements stated in the lemma. □

We can now prove the following.

Theorem 6. The set $\mathcal{S}_D(N_V)$ of Dirac states on $L$ restricted to $N_V$ is in one-to-one correspondence with the space of states of the reduced LJB algebra $\tilde{L}$.

Proof. In proposition 2, we characterized the Dirac states as those that vanish on $V$. Combining this result with that of lemma 6 the proof follows. □

7. The GNS representation of reduced states

Finally, we will describe the GNS representation of a reduced state in terms of data from the unreduced LJB algebra. Let $\tilde{L}$ be, as before, the reduced LJB algebra of $L$ with respect to the constraint set $\mathcal{C}$. Denote by $\tilde{A} = \tilde{L} \oplus i\tilde{L}$ the corresponding $C^*$-algebra and by $\tilde{\mathfrak{S}}$ its state space. Let $\tilde{\omega} \in \tilde{\mathfrak{S}}$ be a normalized state on $\tilde{A}$. The GNS representation of $\tilde{A}$ associated with the state $\tilde{\omega}$, denoted by

$$\pi_{\tilde{\omega}} : \tilde{A} \to B(H_{\tilde{\omega}}),$$

is defined as

$$\pi_{\tilde{\omega}}(\tilde{A})(\tilde{B} + J_{\tilde{\omega}}) = \tilde{A}\tilde{B} + J_{\tilde{\omega}}, \quad \forall\tilde{A}, \tilde{B} \in \tilde{A},$$

where the Hilbert space $H_{\tilde{\omega}}$ is the completion of the pre-Hilbert space defined on $\tilde{A}/J_{\tilde{\omega}}$ by the inner product

$$\langle \tilde{A} + J_{\tilde{\omega}}, \tilde{B} + J_{\tilde{\omega}} \rangle \equiv \tilde{\omega}(\tilde{A}^*\tilde{B}),$$

and $J_{\tilde{\omega}} = \{\tilde{A} \in \tilde{A} | \tilde{\omega}(\tilde{A}^*\tilde{A}) = 0\}$ is the Gelfand left-ideal of $\tilde{\omega}$. Let $\omega$ be a state on $A = L \oplus iL$ that extends the state $\omega'$ on $N_{\mathcal{V}}^C$ induced by $\tilde{\omega}$ according to lemmas 6 and 7. Note that $\omega$ vanishes on $V$; thus, the Gelfand ideal $J_{\omega}$ of $\omega$ contains $V$. We will have then

Theorem 7. There is a unitary equivalence between $H_{\omega}$ and the completion of the pre-Hilbert space:

$$H' = N_{\mathcal{V}}^C/(N_{\mathcal{V}}^C \cap J_{\omega})$$

with the inner product defined by

$$\langle A + N_{\mathcal{V}}^C \cap J_{\omega}, B + N_{\mathcal{V}}^C \cap J_{\omega} \rangle' \equiv \omega(A^*B), \quad \forall A, B \in N_{\mathcal{V}}^C.$$
Proof. Note first that $\langle \cdot, \cdot \rangle'$ is well defined because of the properties of the Gelfand ideal $\mathcal{J}_\omega$. Moreover, we have that

$$H_\widetilde{\omega} = \widetilde{A} / \mathcal{J}_\widetilde{\omega}$$

and from theorem 5, $\widetilde{A} = N_C^C / V^C$ and $\mathcal{J}_\omega = \mathcal{J}_\omega' \cap V^C)$. Hence, because $\mathcal{J}_\omega = (N_C^C \cap \mathcal{J}_\omega)$ and $V^C \subset \mathcal{J}_\omega'$, we have

$$H_\widetilde{\omega} = \widetilde{A} / \mathcal{J}_\widetilde{\omega} = (N_C^C / V^C) / (N_C^C \cap \mathcal{J}_\omega / V^C) \cong N_C^C / (N_C^C \cap \mathcal{J}_\omega).$$

Note that

$$H' = N_C^C / (N_C^C \cap \mathcal{J}_\omega) \cong (N_C^C + \mathcal{J}_\omega) / \mathcal{J}_\omega.$$ 

Thus, the reduced GNS construction corresponding to the state $\widetilde{\omega}$ is the GNS construction of any extension $\omega$ of $\widetilde{\omega}$ restricted to $N_C^C + \mathcal{J}_\omega$. Note that $\widetilde{\omega}$ will be a pure state if and only if $\pi_{\widetilde{\omega}}$ is irreducible, i.e. if the representation of $\pi_\omega$ of $\omega$ restricted to $N_C^C + \mathcal{J}_\omega$ is irreducible. Then if $N_C^C + \mathcal{J}_\omega = A$, $\pi_{\omega}$ will be irreducible if $\omega$ is a pure state. If $N_C^C + \mathcal{J}_\omega \subset A$, then the state $\omega$ extending $\widetilde{\omega}$ might be non-pure.

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