RATIONAL INTERPOLATION AND MIXED INVERSE SPECTRAL PROBLEM FOR FINITE CMV MATRICES

LEONID GOLINSKII AND MIKHAIL KUDRYAVTSEV

Abstract. For finite dimensional CMV matrices the mixed inverse spectral problem of reconstruction the matrix by its submatrix and a part of its spectrum is considered. A general rational interpolation problem which arises in solving the mixed inverse spectral problem is studied, and the description of the space of its solutions is given. We apply the developed technique to give sufficient conditions for the uniqueness of the solution of the mixed inverse spectral problem.

1. Introduction

The theory of CMV matrices, rapidly developing in the recent years ([1], [2], [3], see also the expositions in [4], [5] and the references therein), has a strong background in the theory of Jacobi matrices. The similarity between the Jacobi and CMV matrices not only provides a general concern of investigation, but also permits sometimes to predict the most probable answers. Some of the facts known for the Jacobi matrices can be easily extended to the CMV case, some require considerable adjustment. In turn, the spectral theory of the Jacobi matrices is paralleled by the spectral theory for the Sturm-Liouville differential operators, which forms another important front of study in this area of research.

Mixed inverse spectral problems (MISP) are of special interest in the inverse spectral theory. For this kind of problems one reconstructs a differential or difference operator by a part of its potential and some additional spectral data. Started by Hochstadt and Lieberman [7] for the Sturm-Liouville operators, these problems were extended and refined for many other cases (see [4] for the references). Compared to the “ordinary” inverse spectral problems where the whole potential is to be reconstructed, for MISP one needs to know “less” spectral data.

In what follows, we refer to several key studies of the development of MISP. Let $J$ be an $N \times N$ three-diagonal matrix of the form

$$J = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots & \cdots & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & b_{N-1} & a_N \end{pmatrix}, \quad a_n, b_n \in \mathbb{R}, \ b_n > 0.$$
Consider the $a$'s and the $b$'s as a single sequence $a_1, b_1, a_2, b_2, \ldots$, that is,
\[ c_{2n-1} = a_n, \quad c_{2n} = b_n, \quad n \in \mathbb{N}. \]
In [8] Hochstadt proved the discrete version of the Hochstadt-Lieberman theorem:

**Theorem.** Let $n \in \mathbb{N}$. Suppose that $c_{N+1}, \ldots, c_{2N-1}$ are known, as well as the eigenvalues $\lambda_1, \ldots, \lambda_N$ of $J$. Then $c_1, \ldots, c_N$ are uniquely determined.

It is important to remark that in certain complicated physical systems it is not always possible to know the entire spectrum. A natural question arises if it is possible to reconstruct the Jacobi matrix when we know more than a half of the potential, but less than the whole spectrum. The positive answer is given by Gesztesy and Simon in [5].

**Theorem.** Suppose that $1 \leq j \leq N$ and $c_{j+1}, \ldots, c_{2N-1}$ are known, as well as (any) $j$ of the eigenvalues. Then $c_1, \ldots, c_j$ are uniquely determined.

The goal of this paper is to study the MISP for CMV matrices. Although the algorithm of solving the problem is similar to that of the Jacobi case, essential difference arises when the uniqueness of the reconstruction is concerned. In the Jacobi case the MISP is reduced to the interpolation of a rational function (specifically, the Weyl function of the unknown submatrix) by its values in the known eigenvalues of the whole matrix. The degree of the rational function (i.e., the sum of the degrees of the numerator and denominator) corresponds to the number of the interpolation points, and both numerator and denominator are monic. So, the uniqueness of the interpolating rational function drops out immediately from the simple fact that a polynomial of degree $k$ with $k+1$ zeros is identically zero, and thus we prove the uniqueness of the reconstructed Jacobi matrix. However, in the CMV case the degree of the interpolating rational function (the Weyl function of the “reduced CMV matrix”) is greater by 1 than the number of the interpolation points, and the lacking “piece of information” is given by a restricting condition on the free term of the numerator to be 1. Here the trivial consideration of the Jacobi case fails, and a theory is required of how to find the interpolating rational function with such restriction.

So, the work consists of two parts. In Section 2 we give an approach to the rational interpolation theory adapted for solution of the interpolation problem related to the MISP. The MISP itself is studied in Section 3. Note that, although in the Jacobi case we always have uniqueness in the MISP (provided the number of lacking entries agrees with the number of the given parameters), in the CMV case a degenerated case is possible where the MISP has infinitely many solutions. The main results of this paper are Theorems 2.12 with a description of the solutions of the rational interpolation problem, and 3.3 with a sufficient condition for the MISP to have a unique solution.

2. TWO-DIMENSIONAL VECTOR-POLYNOMIALS AND RATIONAL INTERPOLATION

We start with the following interpolation problem: given points $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and numbers $\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{C}$ find a “nice” description of all rational functions $P^{(1)}/P^{(2)}$, where $P^{(1)}$ and $P^{(2)}$ are polynomials with complex coefficients, for which
\[
\frac{P^{(1)}(z_j)}{P^{(2)}(z_j)} = \omega_j, \quad j = 1, \ldots, n,
\]
(\omega_j = \infty \text{ means that the rational function must have a pole at the point } z_j). By a “nice” description we understand a description whose form allow us to “control” the degrees of \(P^{(j)}(z), j = 1, 2\), and some more parameters such as their leading coefficients or the free term, required in the concrete application of the rational interpolation problem (we will return later on to this topic).

It is advisable to reformulate this problem as a linear problem in the space of two-dimensional polynomial vector-functions (vector-polynomials). Define numbers \(\alpha_j^{(1)}\) and \(\alpha_j^{(2)}\) by the following rule

\[
\begin{cases}
\alpha_{j,1} := 1, & \alpha_{j,2} := -\omega_j, \quad \text{if } \omega_j \neq \infty, \\
\alpha_{j,1} := 0, & \alpha_{j,2} := 1, \quad \text{if } \omega_j = \infty.
\end{cases}
\]

Then (2.1) implies

\[
\alpha_{j,1}P^{(1)}(z_j) + \alpha_{j,2}P^{(2)}(z_j) = 0, \quad j = 1, \ldots, n,
\]

where \(|\alpha_{j,1}| + |\alpha_{j,2}| > 0\).

Conversely, if \(|P^{(1)}(z_j)| + |P^{(2)}(z_j)| > 0, j = 1, \ldots, n\), then (2.3) implies (2.1) with

\[
\omega_j := -\frac{\alpha_{j,1}}{\alpha_{j,2}} \in \mathbb{C}, \quad j = 1, \ldots, n.
\]

However, if \(|P^{(1)}(z_j)| + |P^{(2)}(z_j)| = 0\) for some \(j\), then (2.1) does not make sense at the point \(z_j\).

Thus, we reformulated the initial interpolation problem in the following way: find all the pairs of polynomials \(P^{(1)}\) and \(P^{(2)}\), for which (2.3) holds. The second problem is “almost equivalent” to the first one: if the rational function \(P^{(1)}/P^{(2)}\) is a solution of (2.1), then the pair of polynomials \(P^{(1)}\) and \(P^{(2)}\) is a solution of (2.3) with \(\alpha_{j,1}\) and \(\alpha_{j,2}\) defined in (2.2). If \(P^{(1)}\) and \(P^{(2)}\) are polynomials such that \(|P^{(1)}(z_j)| + |P^{(2)}(z_j)| > 0\) solving (2.3), then the rational function \(P^{(1)}/P^{(2)}\) is a solution of (2.1) with \(\omega_j\) defined in (2.4).

In what follows we refer to interpolation problem (2.3) as the problem \((I_n)\), and denote by \(P_\infty(I_n)\) the set of all its solutions. Some of the results described below for the polynomial vector-functions are taken from [10], [11] where the interpolation problems also appears as well as its main objects like generators, etc.

2.1. The space of vector-polynomials. To describe the class of solutions of (2.3), we introduce the space

\[
P_\infty := \left\{ p(z) = \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix}, \quad P^{(i)}, i = 1, 2, \text{ are complex polynomials} \right\}.
\]

This is a linear space with the standard operations. The zero element on this space is \(0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\). We point out that \(P_\infty\) is also a module over the ring of polynomials:

\[
Sp = \begin{pmatrix} SP^{(1)} \\ SP^{(2)} \end{pmatrix} \in P_\infty
\]

for any polynomial \(S\).
Definition 2.1. The height of the vector-polynomial \( p = \left( \begin{array}{c} P^{(1)} \\ P^{(2)} \end{array} \right) \neq 0 \) is the number
\[
(2.5) \quad h(p) := \begin{cases} 
2 \deg P^{(1)}, & \deg P^{(1)} > \deg P^{(2)}, \\
2 \deg P^{(2)} + 1, & \deg P^{(1)} \leq \deg P^{(2)},
\end{cases}
\]
As usual, \( \deg 0 = -\infty \), so we put \( h(0) := -\infty \).

It is obvious that
\[
(2.6) \quad h(p) = \max\{2 \deg P^{(1)}, 2 \deg P^{(2)} + 1\}
\]
and
\[
(2.7) \quad h(Sp) = h(p) + 2 \deg S.
\]
The degrees of the components of the vector-polynomials can be written down in the following table:

| height \( p \) | 0 | 1 | 2 | 3 | 4 | ... | 2k | 2k+1 | ...
|----------------|---|---|---|---|---|-----|----|------|-----|
| \( \deg P^{(1)} \) | 0 | ≤ 0 | 1 | ≤ 1 | ≤ 2 | ... | = k | ≤ k | ... |
| \( \deg P^{(2)} \) | −∞ | ≤ 0 | 1 | ≤ 1 | ... | ≤ k−1 | = k | ... |    |

The following proposition demonstrates that the notion of the height is a natural extension of the degree of polynomials.

**Proposition 2.2.** If \( h(p) \neq h(q) \), then \( \forall a, b \in \mathbb{C} \)
\[
h(ap + bq) = \max(h(p), h(q))
\]
If \( h(p) = h(q) = n \), then
\[
(1) \quad \forall a, b \in \mathbb{C} \quad h(ap + bq) \leq n
(2) \quad \exists c \in \mathbb{C} : \quad h(p + cq) \leq n - 1.
\]

**Proof.** We prove (2), the rest is plain. If \( n = 2k \), then \( k = \deg P^{(1)} > \deg P^{(2)}, k = \deg Q^{(1)} > \deg Q^{(2)} \). So, there exists \( c \in \mathbb{C} \) such that \( \deg(P^{(1)} + cQ^{(1)}) \leq k - 1 \). Also, since \( \deg P^{(2)} \leq k - 1 \) and \( \deg Q^{(2)} \leq k - 1 \), we see that \( \deg(P^{(2)} + cQ^{(2)}) \leq k - 1 \). Then, by (2.6)
\[
h(p + cq) \leq 2k - 1 = n - 1.
\]
If \( n = 2k + 1 \), then \( k = \deg P^{(2)} \geq \deg P^{(1)}, k = \deg Q^{(2)} \geq \deg Q^{(1)} \). So, \( \exists c \in \mathbb{C} : \deg(P^{(2)} + cQ^{(2)}) \leq k - 1, \deg(P^{(1)} + cQ^{(1)}) \leq k \). Then, by (2.6)
\[
h(p + cq) \leq 2k = n - 1.
\]

Consider the following basic system of vectors in \( \mathbb{P}_\infty \):
\[
(2.8) \quad e_{2k}(z) := \begin{pmatrix} z^k \\ 0 \end{pmatrix}, \quad e_{2k+1}(z) := \begin{pmatrix} 0 \\ z^k \end{pmatrix}, \quad k \in \mathbb{Z}_+ = \{0, 1, \ldots\}
\]
It is clear that \( h(e_n) = n \) for all \( n \).

**Proposition 2.3.** \( \{e_n\}_{n \geq 0} \) is a basis in \( \mathbb{P}_\infty \), i.e.,
\[
p \in \mathbb{P}_\infty, \quad h(p) = m \quad \implies \quad p(z) = \sum_{k=0}^m c_k e_k(z), \quad c_m \neq 0,
\]
and this representation is unique.
Proof. We use induction on $m$:

1) The cases $m = 0, 1$ are checked immediately.

2) Let $m = 2k + 1$, then $k = \deg P^{(2)} \geq \deg P^{(1)}$, i.e., $P^{(2)}(z) = az^k + Q^{(2)}(z)$, $a \neq 0$, $\deg Q^{(2)}(z) \leq k - 1$. Then

$$q(z) := p(z) - ae^{2k+1}(z) = \left(\begin{array}{c} Q^{(1)}(z) \\ Q^{(2)}(z) \end{array}\right),$$

where $Q^{(1)}(z) = P^{(1)}(z)$ and $\deg Q^{(1)} \leq k$.

So, $h(q) \leq 2k = m - 1$, and we can apply the inductive hypothesis. The representation $q(z) = \sum_{k=0}^{m-1} c_k e_k(z)$ is unique, so the representation

$$p(z) = ae_m(z) + \sum_{k=0}^{m-1} c_k e_k(z)$$

is also unique.

3) Let $m = 2k$. Then $P^{(1)}(z) = bz^k + Q^{(1)}(z)$, $b \neq 0$ and $\deg Q^{(1)} \leq k - 1$. Define $q(z) := p(z) - be^{2k} = \left(\begin{array}{c} Q^{(1)}(z) \\ Q^{(2)}(z) \end{array}\right)$. Here $\deg Q^{(2)} \leq k - 1$, so $h(q) \leq 2k - 1 = m - 1$ and we can again apply the inductive hypothesis.

The latter proposition can be extended in a natural way.

**Proposition 2.4.** Let $\{g_n(z)\}_{n \geq 0}$ be an arbitrary sequence of vector-polynomials, such that $h(g_n) = n$, $n \in \mathbb{Z}_+$. Then $\{g_n\}_{n \geq 0}$ is a basis in $P_\infty$.

**Proof.** By Proposition 2.3

$$g_m(z) = \sum_{k=0}^{m} c_{m,k} e_k(z), \quad c_{m,m} \neq 0,$$

or in a vector-matrix form

$$\begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix} = \begin{pmatrix} c_{00} & c_{10} & c_{11} & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ c_{m0} & \cdots & \cdots & c_{mm} \end{pmatrix} \begin{pmatrix} e_0 \\ \vdots \\ e_m \end{pmatrix},$$

where the matrix of $(c_{m,k})$ is triangular. So,

$$\begin{pmatrix} e_0 \\ \vdots \\ e_m \end{pmatrix} = \begin{pmatrix} \tilde{c}_{00} & \tilde{c}_{10} & \tilde{c}_{11} & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{c}_{m0} & \cdots & \cdots & \tilde{c}_{mm} \end{pmatrix} \begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix}, \quad \tilde{c}_{m,m} \neq 0.$$

Since $\{e_n\}_{n \geq 0}$ is a basis, then so is $\{g_n\}_{n \geq 0}$. □
2.2. Transforms in $\mathbb{P}_\infty$. A $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a transform of the vector-polynomials:

$$Ap(z) = A \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix} = \begin{pmatrix} aP^{(1)}(z) + bP^{(2)}(z) \\ cP^{(1)}(z) + dP^{(2)}(z) \end{pmatrix}$$

**Proposition 2.5.**

1. For arbitrary matrix $A$ and $p \in \mathbb{P}_\infty$

   $$h(Ap) \leq h(p) + 1.$$

2. If $A$ is upper-triangular, then for arbitrary $p \in \mathbb{P}_\infty$

   $$h(Ap) \leq h(p).$$

3. If $A$ is lower-triangular, then

   $$h(p) \leq 2k + 1 \implies h(Ap) \leq 2k + 1.$$

**Proof.**

1. By (2.6)

   $$h(Ap) = \max \left( 2 \deg(aP^{(1)} + bP^{(2)}), \ 2 \deg(cP^{(1)} + dP^{(2)}) + 1 \right),$$

   $$2 \deg(aP^{(1)} + bP^{(2)}) \leq 2 \max(\deg P^{(1)}, \ \deg P^{(2)}) = \max(2 \deg P^{(1)}, \ 2 \deg P^{(2)}) \leq h(p),$$
   $$2 \deg(cP^{(1)} + dP^{(2)}) \leq \max(2 \deg P^{(1)}, \ 2 \deg P^{(2)}) \leq h(p),$$

   $$2 \deg(cP^{(1)} + dP^{(2)}) + 1 \leq h(p) + 1.$$

2. If $c = 0$, then $2 \deg(dP^{(2)}) + 1 \leq 2 \deg P^{(2)} + 1 \leq h(P)$.

3. Let now $h(p) \leq 2k + 1$ and $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Then

   $$Ap(z) = \begin{pmatrix} aP^{(1)}(z) \\ cP^{(1)}(z) + dP^{(2)}(z) \end{pmatrix},$$

   and we have by assumption $2 \deg P^{(1)} \leq 2k + 1$, $2 \deg P^{(2)} \leq 2k$. So,

   $$\deg(cP^{(1)} + dP^{(2)}) \leq k \implies h(Ap) \leq 2k + 1,$$

   as claimed. \[\square\]

Later on we will use the following property of the height, which is a simple consequence of (2.6): for $p = \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix} \in \mathbb{P}_\infty$ we have

$$h(p) \leq 2k + 1 \implies h\left( \begin{pmatrix} (z-a)P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix} \right) \leq 2k + 2,$$

$$h(p) \leq 2k \implies h\left( \begin{pmatrix} P^{(1)}(z) \\ (z-a)P^{(2)}(z) \end{pmatrix} \right) \leq 2k + 1.$$
2.3. The generators of interpolation problem. It is clear that solutions of (2.3) form a module over the ring of polynomials in \( \mathbb{P}_\infty \), i.e., if \( r \) and \( q \) are solutions of (2.3), then so is \( Sr + Tq \) for arbitrary polynomials \( S \) and \( T \). The goal of this subsection is to show that this module has exactly two generators, and to study their properties.

Recall that \( \mathbb{P}_\infty(I_n) \) is the set of all solutions of (2.3). Set
\[
\begin{align*}
    h(I_n) := & \min \{ h(q) : q \in \mathbb{P}_\infty(I_n), \quad q \neq 0 \},
\end{align*}
\]
which we call the height of the interpolation problem.

**Definition 2.6.** We say that \( r \in \mathbb{P}_\infty(I_n) \) is a minimal generator of (2.3), if \( h(r) = h(I_n) \).

**Proposition 2.7.** The minimal generator of \( I_n \) is unique up to a constant factor.

**Proof.** Let \( r_1 \) and \( r_2 \) be two minimal generators. By Proposition 2.2, \( \exists a \in \mathbb{C} : h(ar_1 + r_2) = h(I_n) - 1 \). But \( ar_1 + br_2 \) is a solution of (2.3). Since \( r_1 \) and \( r_2 \) are minimal non-trivial solutions of (2.3), we conclude \( ar_1 + br_2 = 0 \).

A trivial (nonzero) solution of (2.3) \( P^{(1)} = P^{(2)} = \prod_j (z - z_j) \) provides the bound \( h(I_n) \leq 2n + 1 \). It turns out that this bound can be improved immensely.

**Theorem 2.8.** \( h(I_n) \leq n \).

**Proof.** The following non-negative matrices of rank 1 play a key role in our consideration:
\[
\sigma_k = \begin{pmatrix}
|\alpha_{k,1}|^2 & \bar{\alpha}_{k,1}\alpha_{k,2} \\
\bar{\alpha}_{k,2}\alpha_{k,1} & |\alpha_{k,2}|^2
\end{pmatrix} = \begin{pmatrix}
\bar{\alpha}_{k,1} \alpha_{k,2} \\
\bar{\alpha}_{k,2} \alpha_{k,1}
\end{pmatrix} (\alpha_{k,1}, \alpha_{k,2}), \quad k = 1, \ldots, n.
\]
It is clear that
\[
|\alpha_{k,1} P^{(1)}(z_k) + \alpha_{k,2} P^{(2)}(z_k)|^2 = p^*(z_k) \sigma_k p(z_k),
\]
\[
p^*(z) := \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix}, \quad p(z) := \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix},
\]
so the problem (2.3) is equivalent to
\[
(2.10) \quad p^*(z_k) \sigma_k p(z_k) = 0, \quad k = 1, \ldots, n.
\]

We proceed by induction on \( n \).
1. For \( n = 1 \) we have a non-trivial solution
\[
p := \begin{pmatrix} -\alpha_{1,2} \\ \alpha_{1,1} \end{pmatrix}, \quad h(p) \leq 1.
\]
2. Suppose that we have already proved the result for \( n \), and we want to prove it for \( n + 1 \). The forthcoming construction depends on whether \( n \) is odd or even.

Let \( n = 2k + 1 \), and consider the problem \( (I_{n+1}) \) with \( n + 1 \) data. If \( \alpha_{j,1} = 0 \) for all \( j = 1, \ldots, n + 1 \), then the vector-polynomial \( p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( h(p) = 0 \), is a solution of \( (I_n) \), and we are done. So, suppose without loss of generality that \( \alpha_{n+1,1} \neq 0 \) (otherwise enumerate the points \( z_1, \ldots, z_{n+1} \)). The upper-triangular matrix
\[
\Omega_0 := \begin{pmatrix} (\alpha_{n+1,1})^{-1} & -\alpha_{n+1,2}(\alpha_{n+1,1})^{-1} \\ 0 & 1 \end{pmatrix},
\]
By the induction hypothesis, there exists a solution \( q \) \((2.14)\) satisfying \( (2.13) \)

\[
\begin{pmatrix}
\beta_{j,1}^2 \\
\beta_{j,2}^2
\end{pmatrix} = \begin{pmatrix}
|\beta_{j,1}|^2 \\
|\beta_{j,2}|^2
\end{pmatrix} \\
\alpha_{j,1} \beta_{j,2} \\
\beta_{j,1} \beta_{j,2}
\end{pmatrix}; \quad j = 1, \ldots, n,
\]

with some numbers \( \beta_{j,1}, \beta_{j,2}, j = 1, \ldots, n \). Put

\[
\gamma_{j,1} := (z_{n+1} - z_j) \beta_{j,1}; \quad \gamma_{j,2} := \beta_{j,2}, \quad j = 1, \ldots, n,
\]

and consider an auxiliary interpolation problem \((I_n)\):

\[
\gamma_{j,1} P^{(1)}(z_j) + \gamma_{j,2} P^{(2)}(z_j) = 0, \quad j = 1, \ldots, n.
\]

By the induction hypothesis, there exists a solution \( q \in \mathbb{P}_\infty \), \( h(q) \leq n = 2k + 1 \), of this problem:

\[
q^*(z_j) \sigma_j q(z_j) = 0, \quad j = 1, \ldots, n,
\]

where

\[
\bar{\sigma}_j = \begin{pmatrix}
|\gamma_{j,1}|^2 \\
|\gamma_{j,2}|^2
\end{pmatrix} = \begin{pmatrix}
(z_{n+1} - z_j) \\
0
\end{pmatrix} \Omega^* \sigma_j \Omega \begin{pmatrix}
z_{n+1} - z_j \\
0
\end{pmatrix}.
\]

Define a vector-polynomial

\[
r(z) = \begin{pmatrix}
R^{(1)}(z) \\
R^{(2)}(z)
\end{pmatrix} := \Omega^* \begin{pmatrix}
z_{n+1} - z \\
0
\end{pmatrix} q(z).
\]

For \( r \) we have

\[
r^*(z_j) \sigma_j r(z_j) = q^*(z_j) \bar{\sigma}_j q(z_j) = 0, \quad j = 1, \ldots, n,
\]

and for \( j = n + 1 \)

\[
r^*(z_{n+1}) \sigma_{n+1} r(z_{n+1}) = q^*(z_{n+1}) \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} q(z_{n+1}) = 0,
\]

by \((2.11)\). So, \( r \) is a solution of \((I_{n+1})\).

Since \( h(q) \leq 2k + 1 \), then by the upper inequality in \((2.9)\) we have

\[
h \begin{pmatrix}
(z_{n+1} - z) Q^{(1)}(z) \\
Q^{(2)}(z)
\end{pmatrix} \leq 2k + 2 = n + 1,
\]

so, by Proposition \((2.5)\) \( h(r) \leq n + 1 \), as needed.

Let \( n = 2k \), and \((I_{n+1})\) \((2.10)\) be the interpolation problem with \( n + 1 \) data. If \( \alpha_{j,2} = 0 \) for all \( j = 1, \ldots, n + 1 \), then the vector-polynomial \( p = \begin{pmatrix}
0 \\
1
\end{pmatrix} \), \( h(p) = 1 \), is a solution of \((I_n)\), and we are done. So, suppose as above, that \( \alpha_{n+1,2} \neq 0 \). The lower-triangular matrix

\[
\Omega_1 := \begin{pmatrix}
1 \\
-\alpha_{n+1,1} \alpha_{n+1,2}^{-1} \\
\alpha_{n+1,2}^{-1}
\end{pmatrix},
\]

satisfies

\[
(2.14) \quad \Omega^*_1 \sigma_{n+1} \Omega_1 = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix},
\]
and
\begin{equation}
\Omega_1^* \sigma_j \Omega_1 = \left( \frac{|\delta_{j,1}|^2}{\delta_{j,2} \delta_{j,1}}, \frac{ \bar{\delta}_{j,1} \delta_{j,2}}{|\delta_{j,2}|^2} \right), \quad j = 1, \ldots, n.
\end{equation}

Put
\[ \lambda_{j,2} := (z_{n+1} - z_j) \delta_{j,2}; \quad \lambda_{j,1} := \delta_{j,1}, \quad j = 1, \ldots, n, \]
and consider an auxiliary interpolation problem \((I_n)\):
\[ \lambda_{j,1} P^{(1)}(z_j) + \lambda_{j,2} P^{(2)}(z_j) = 0, \quad j = 1, \ldots, n, \]
By the induction hypothesis, there exists a solution \( q \in \mathbb{P}_n \), \( h(q) \leq n = 2k \), of this problem:
\[ q^*(z_j) \bar{\sigma}_j q(z_j) = 0, \quad j = 1, \ldots, n, \]
where
\[ \bar{\sigma}_j = \left( \frac{|\lambda_{j,1}|^2}{\lambda_{j,2} \lambda_{j,1}}, \frac{\bar{\lambda}_{j,1} \lambda_{j,2}}{|\lambda_{j,2}|^2} \right) = \left( 1, 0 \right) \Omega_1^* \sigma_j \Omega_1 \left( 0, 0 \right) \left( 0, z_{n+1} - z_j \right). \]
Define a vector-polynomial
\begin{equation}
r(z) = \left( \frac{P^{(1)}(z)}{P^{(2)}(z)} \right) := \Omega_1 \left( 1, 0 \right) \frac{1}{z_{n+1} - z} q(z).
\end{equation}
For \( r \) we have
\[ r^*(z_j) \sigma_j r(z_j) = q^*(z_j) \bar{\sigma}_j q(z_j) = 0, \quad j = 1, \ldots, n, \]
and for \( j = n + 1 \)
\[ r^*(z_{n+1}) \sigma_{n+1} r(z_{n+1}) = q^*(z_{n+1}) \left( 1, 0 \right) \left( 0, 0 \right) \left( 0, 0 \right) q(z_{n+1}) = 0, \]
by (2.14). So, \( r \) is a solution of \((I_{n+1})\).
Since \( h(q) \leq 2k \), then by the lower inequality in (2.9) we have
\[ h \left( \frac{Q^{(1)}(z)}{(z_{n+1} - z) Q^{(2)}(z)} \right) \leq 2k + 1 = n + 1, \]
so, by (3), Proposition 2.5 (\( \Omega_1 \) is lower-triangular), \( h(r) \leq 2k + 1 = n + 1 \). The proof is complete. \( \square \)

If \( r \) is a minimal generator of \((I_n)\), then \( S r \in \mathbb{P}_\infty(I_n) \) for any polynomial \( S \). So, the question arises naturally whether \( \mathbb{P}_\infty(I_n) = \{S r\} \), \( S \) a polynomial. The answer is negative: it turns out that the module of solutions of \((I_n)\) has exactly one more generator. Denote \( \mathbb{P}'_\infty(I_n) = \mathbb{P}_\infty(I_n) \setminus \{S r\} \). It is shown in Theorem 2.10 below that this set is nonempty, so the following definition makes sense.

**Definition 2.9.** We say that \( q \in \mathbb{P}_\infty(I_n) \) is a second generator of \((I_n)\), if
\[ h(q) = \min \{h(p), \ p \in \mathbb{P}'_\infty(I_n)\}. \]

**Theorem 2.10.** The set \( \mathbb{P}_\infty(I_n) \) is nonempty. Furthermore, the height of any second generator is \( h(q) = 2n + 1 - h(I_n) \).
Proof. We show that there is a solution of $(I_n)$ of the height $\leq 2n+1-h(I_n)$, which is not of the form $Sr$.

Let $h(I_n) = k \leq n$. Pick arbitrary different numbers $z_{n+1}, z_{n+2}, \ldots, z_{2n+1-k} \in \mathbb{C}$ distinct from $z_1, \ldots, z_n$. Take numbers $\alpha_{j,1}, \alpha_{j,2}, j = n+1, n+2, \ldots, 2n+1-k$ in such a way that

$$\alpha_{j,1}R^{(1)}(z_j) + \alpha_{j,2}R^{(2)}(z_j) \neq 0, \quad j = n+1, n+2, \ldots, 2n+1-k,$$

where $r = \begin{pmatrix} R^{(1)} \\ R^{(2)} \end{pmatrix}$ is the minimal generator of $(I_n)$. Consider the interpolation problem

$$\alpha_{j,1}p^{(1)}(z_j) + \alpha_{j,2}p^{(2)}(z_j) = 0, \quad j = 1, 2, \ldots, 2n+1-k.$$

By Theorem 2.8, there exists a nonzero vector-polynomial $p = \begin{pmatrix} P^{(1)} \\ P^{(2)} \end{pmatrix} \neq 0$, which solves this problem, and $h(p) \leq 2n+1-k$.

Suppose that $p = Sr$ for a polynomial $S$. Then, by (2.17),

$$S(z_1)\left[\alpha_{j,1}R^{(1)}(z_j) + \alpha_{j,2}R^{(2)}(z_j)\right] = 0, \quad j = n+1, n+2, \ldots, 2n+1-k,$$

so by (2.17),

$$S(z_{n+1}) = S(z_{n+2}) = \ldots = S(z_{2n+1-k}) = 0.$$

Since $S(z) \neq 0$, we see that $\deg S(z) \geq n+1-k$, and by (2.18),

$$h(p) = h(Sr) = h(r) + 2 \deg S \geq h(r) + 2(n+1-k) = 2n+2-k,$$

which leads to contradiction with $h(p) \leq 2n+1-k$. So $p \in P'_\infty(I_n)$ and $h(p) \leq 2n+1-k$, as claimed.

Let now $q$ be any second generator, so $h(q) \leq 2n+1-k$. We prove next $h(q) \geq 2n+1-k$.

We have

$$\begin{cases} \alpha_{j,1}R^{(1)}(z_j) + \alpha_{j,2}R^{(2)}(z_j) = 0, \\ \alpha_{j,1}Q^{(1)}(z_j) + \alpha_{j,2}Q^{(2)}(z_j) = 0, \end{cases} \quad j = 1, \ldots, n,$$

so

$$\det \begin{pmatrix} R^{(1)}(z_j) & R^{(2)}(z_j) \\ Q^{(1)}(z_j) & Q^{(2)}(z_j) \end{pmatrix} = R^{(1)}(z_j)Q^{(2)}(z_j) - R^{(2)}(z_j)Q^{(1)}(z_j) = 0,$$

$j = 1, \ldots, n$. Suppose that $h(q) \leq 2n-k$, which implies by (2.9)

$$\deg Q^{(1)} \leq n - \frac{k}{2}, \quad \deg Q^{(2)} \leq n - \frac{k}{2} - 1.$$

If $h(r) = k$ is even, then

$$\deg R^{(1)} = \frac{k}{2}, \quad \deg R^{(2)} = \frac{k}{2} - 2,$$

and so

$$\deg(R^{(1)}Q^{(2)} - R^{(2)}Q^{(1)}) \leq n - 1.$$

If $k$ is odd, then

$$\deg R^{(2)} = \frac{k-1}{2}, \quad \deg R^{(1)} = \frac{k-1}{2},$$

and again

$$\deg(R^{(1)}Q^{(2)} - R^{(2)}Q^{(1)}) \leq n - 1.$$
It follows now from (2.19) that

\[ R^{(1)}(z)Q^{(2)}(z) - R^{(2)}(z)Q^{(1)}(z) = 0. \]  

(2.20)

Let \( T \) be the greatest common divisor of \( R^{(1)} \) and \( R^{(2)} \), so (2.20) turns into

\[ X^{(1)}TQ^{(2)} = X^{(2)}TQ^{(1)}, \]

with relatively prime \( X^{(1)} := R^{(1)}/T \) and \( X^{(2)} := R^{(2)}/T \). Hence \( S^{(1)} := Q^{(1)}/X^{(1)} \) is a polynomial, and so is \( S^{(2)} := Q^{(2)}/X^{(2)} \). Now \( X^{(1)}TQ^{(2)} = X^{(2)}TQ^{(1)} \) implies \( S^{(1)} = S^{(2)} = S \) and

\[ Q^{(i)} = \frac{R^{(i)}}{T}S, \quad i = 1, 2. \]  

(2.21)

Note that all roots of \( T \) are among the nodes of interpolation. Indeed, if \( T(w) = 0 \) and \( w \not\in \{z_1, \ldots, z_n\} \), then

\[ \hat{r}(z) = \left( \frac{\hat{R}^{(1)}}{\hat{R}^{(2)}} \right) = \frac{1}{z - w} \left( \frac{R^{(1)}}{R^{(2)}} \right) \in \mathbb{P}_\infty(I_n) \]

and \( h(\hat{r}) < h(r) \), which is impossible since \( r \) is a minimal generator of \( (I_n) \). Hence, \( w = z_i \).

It remains only to show that \( S(z_i) = 0 \). Assume that \( S(z_i) \neq 0 \). Then

\[ \alpha_{1}Q^{(1)}(z_i) + \alpha_{2}Q^{(2)}(z_i) = 0 \]

and (2.21) imply

\[ \lim_{z \to z_i} \left\{ \alpha_{1} \frac{R^{(1)}(z)}{T(z)} + \alpha_{2} \frac{R^{(2)}(z)}{T(z)} \right\} = 0. \]

So

\[ \lim_{z \to z_i} \left\{ \alpha_{1} \frac{R^{(1)}(z)}{(z - z_i)^{n_1}} + \alpha_{2} \frac{R^{(2)}(z)}{(z - z_i)^{n_2}} \right\} = 0, \]

where \( T = (z - z_i)^{n_1}T, \) \( T(z_i) \neq 0 \). Since \( R^{(i)} = (z - z_i)^{n_i}R^{(i)}, \) \( i = 1, 2 \), we see that

\[ \hat{r}(z) = \frac{1}{(z - z_i)^{n_i}} \hat{r}(z) \in \mathbb{P}_\infty(I_n) \]

and clearly \( h(\hat{r}) < h(r) \), which again leads to contradiction with \( r \) being the minimal generator.

Finally, since all the roots of \( T \) are among \( \{z_1, \ldots, z_k\} \) and \( S(z_1) = \ldots = S(z_k) = 0 \), \( P = S/T \) is a polynomial, and by (2.21) \( Q^{(i)} = PR^{(i)} \), which contradicts to \( q \in \mathbb{P}_\infty(I_n) \). The proof is complete.

Remark. In fact we have proven that each solution \( q \in \mathbb{P}_\infty(I_n) \) with \( h(q) \leq 2n + 1 - h(I_n) \) is a second generator. It is also not hard to see that each solution \( q \in \mathbb{P}(I_n) \) with \( h(q) = 2n + 1 - h(I_n) \) is a second generator.

Theorem 2.11. Each solution of the problem \((I_n)\) has the form

\[ p(z) = S(z)r(z) + T(z)q(z), \]

(2.22)

where \( r \) and \( q \) are the minimal and second generators of \( (I_n) \), respectively, and \( S \) and \( T \) are polynomials. Conversely, each vector-polynomial of the form (2.22) with arbitrary polynomials \( S \) and \( T \) belongs to \( \mathbb{P}_\infty(I_n) \).
Proof. We only prove the first statement. Let $h(I_n) = k$. Consider a system of vector-polynomials $\{f_j\}_{j \geq 0}$ defined as follows:

\[
\begin{align*}
f_k &= r, & f_{2n+1-k} &= q, \\
f_j &= e_j, & j &= 0, 1, \ldots, k-1; & f_{k+2j-1} &= e_{k+2j-1}, & j &= 1, 2, \ldots, n-k, \\
f_{k+2j} &= z^j r, & f_{2n+1-k+2j} &= z^j q; & j &\in \mathbb{N},
\end{align*}
\]

$e_i$ are in (2.8). It is easy to check that $h(f_j) = j$ for all $j$, so by Proposition 2.14 this system is a basis in $\mathbb{P}_n^\infty$, and in particular each $p \in \mathbb{P}_n^\infty(I_n)$ admits a unique representation in the form

\[
p(z) = \sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i} + S_1(z)r(z) + T_1(z)q(z).
\]

Since $p, r, q \in \mathbb{P}_n^\infty(I_n)$ then

\[
\sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i}
\]

is also solution of $(I_n)$. But its height is less then $2n+1-k$, so, according to the definition of a second generator,

\[
\sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i} = \tilde{S}(z)r(z).
\]

Hence $p$ is of the form (2.22) with $S = S_1 + \tilde{S}$, $T = T_1$. \hfill \Box

Thus, the following theorem holds for interpolation problem 2.1:

**Theorem 2.12.** Each solution of problem (2.1) has the form

\[
\frac{S(z)R^{(1)}(z) + T(z)Q^{(1)}(z)}{S(z)R^{(2)}(z) + T(z)Q^{(2)}(z)},
\]

where $r = \begin{pmatrix} R^{(1)} \\ R^{(2)} \end{pmatrix}$ and $q = \begin{pmatrix} Q^{(1)} \\ Q^{(2)} \end{pmatrix}$ are minimal and second generators of $(I_n)$, and $S$ and $T$ are polynomials. Conversely, if $r$ and $q$ are the minimal and second generators of $(I_n)$, and $S$ and $T$ are such polynomials that the numerator and the denominator in (2.23) have no common roots, then (2.23) is a solution of (2.1).

**Remark.** Roughly speaking, the task of giving a description for the set of the solutions of rational interpolation problem (2.1) is obvious. Let $\frac{R^{(1)}}{R^{(2)}}$ be any rational function solving problem 2.1. Then all the functions of the type $\frac{R^{(1)}}{R^{(2)}} + \frac{Q^{(1)}}{Q^{(2)}}$, where $\frac{Q^{(1)}}{Q^{(2)}}$ is an arbitrary rational function vanishing in the nodes of interpolation, will be all the solutions of the rational interpolation problem. However, such a description does not permit us to predict the degrees of the numerator and denominator as well as other properties needed in applications. So, we cannot obtain in this way a rational function which solves the interpolation problem and has the prescribed properties. For example, in the interpolation problem appearing in the next section we will need a rational function with monic numerator and denominator of concrete degrees, such that the free term of the numerator equals 1. Thus,
more elaborated results are required. Certainly, we do not think that the two descriptions for the solutions of the interpolation problem, mentioned above, are the only possible.

3. Reduction of MISP to Rational Interpolation

For the definitions, notations and basic properties of finite CMV matrices see, for example, [12, 6]. We will add some more to the list.

Let \( C = C(\alpha_0, \ldots, \alpha_{n-2}; \beta) \) be a finite CMV matrix with Verblunsky’s parameters \((\alpha_0, \ldots, \alpha_{n-2}; \beta)\) and the system of the Szegő polynomials \( \{\Phi_0, \ldots, \Phi_{n-1}; \Phi_n\} \). They satisfy the famous Szegő recurrence relations

\[
\Phi_k(z) = z\Phi_{k-1}(z) - \alpha_{k-1}\Phi_{k-1}(z), \quad k = 1, 2, \ldots, n - 1, \quad \Phi_0 \equiv 1,
\]

(3.1)

\[
\Phi_n(z) = z\Phi_{n-1}(z) - \beta\Phi_{n-1}(z).
\]

As is known,

\[
\Phi_n(z) = \prod_{j=1}^{n} (z - \zeta_j), \quad \Sigma(C) = \{\zeta_j\}_{1}^{n}
\]

a spectrum of \( C \), \( \zeta_j \neq \zeta_i, j \neq i \). Put

\[
\kappa_m = \prod_{j=0}^{m-1} (1 - |\alpha_j|^2)^{-1/2}, \quad m = 0, 1, \ldots, n - 1; \quad \kappa_0 = 1,
\]

and for appropriate values of the indices define

\[
x_{2k}(z) := z^{-k}\kappa_{2k}\Phi_{2k}(z), \quad x_{2k+1}(z) := z^{-k-1}\kappa_{2k+1}\Phi_{2k+1}(z),
\]

where by definition \( \varphi_k(z) = z^k\varphi_k(1/z) \).

For each eigenvalue \( \zeta_j \) the following equality

\[
CX_j = \zeta_jX_j, \quad X_j = [x_0(\zeta_j), \ldots, x_{n-1}(\zeta_j)]^t,
\]

(3.4)

gives (along with (3.3)) an explicit expression for the eigenvectors of \( C \) in terms of the Szegő polynomials and Verblunsky parameters. (3.4) is proved in [12, Lemma 4.3.14], for infinite CMV matrices. For finite matrices the argument is similar. As a matter of fact, the following more precise result holds.

**Proposition 3.1.** For \( z \in \mathbb{C}\setminus\{0\} \) and \( X(z) = [x_0(z), \ldots, x_{n-1}(z)]^t \) the equality holds

\[
(z - C)X = z^{-[n/2]n-1}\Phi_n(z)v_n, \quad v_n \in \mathbb{C}^n, \quad \|v_n\| = 1.
\]

Due to the sieving procedure we will assume without loss of generality that \( n \) is an even number: \( n = 2l \). Throughout the rest of the paper we assume that the last Verblunsky coefficient is known, and for simplicity put \( \beta = 1 \). The \( LM \) factorization now takes the form \( C(\alpha_0, \ldots, \alpha_{2l-2}; 1) = LM \) with

\[
L = \begin{pmatrix} \theta(\alpha_0) & \vdots & \theta(\alpha_{2l-2}) \\ \vdots & \ddots & \vdots \\ \theta(\alpha_{2l-2}) & \vdots & \theta(\alpha_0) \end{pmatrix}, \quad M = \begin{pmatrix} 1 & \theta(\alpha_1) & \vdots & \theta(\alpha_{2l-3}) \\ \theta(\alpha_1) & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \theta(\alpha_{2l-3}) & \vdots & \ddots & 1 \end{pmatrix},
\]

(3.5)

\[
\Theta(\alpha_j) = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad |\alpha_j| < 1, \quad \rho_j = \sqrt{1 - |\alpha_j|^2} > 0.
\]
Put

\begin{equation}
U = \begin{pmatrix}
\bigcirc & \ldots & \bigcirc & J \\
\bigcirc & \ldots & J & \bigcirc \\
\ldots & \ldots & \ldots & \ldots \\
J & \bigcirc & \ldots & \bigcirc
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\end{equation}

the orthogonal $2l \times 2l$ matrix, and consider the reflection of $C$

\begin{equation}
C_r := UCU = ULU : UMU = \mathcal{L}_r : \mathcal{M}_r.
\end{equation}

It is clear from (3.5)–(3.7), that

\begin{equation}
\mathcal{L}_r = \begin{pmatrix}
\theta(-\bar{\alpha}_{2l-2}) & & \\
& \ddots & \\
& & \theta(-\bar{\alpha}_0)
\end{pmatrix},
\end{equation}

\begin{equation}
\mathcal{M}_r = \begin{pmatrix}
1 & \theta(-\bar{\alpha}_{2l-3}) & & \\
& \ddots & \ddots & \\
& & \ddots & \theta(-\bar{\alpha}_1) \\
& & & 1
\end{pmatrix},
\end{equation}

so

\begin{equation}
C_r = C(\lambda_0, \ldots, \lambda_{2l-2}; 1), \quad \lambda_k := -\bar{\alpha}_{2l-2-k}, \quad k = 0, 1, \ldots, 2l - 2,
\end{equation}

is also a CMV matrix corresponding to the “reversed” Verblunsky parameters. We denote the Szegö polynomials for $C_r$ by $\{\Lambda_0, \ldots, \Lambda_{n-1}; \Lambda_n\}$.

Obviously, $\Sigma(C_r) = \Sigma(C) = \{\zeta_j\}^n_{j=1}$ and

\begin{equation}
C_r Y_j = \zeta_j Y_j, \quad Y_j = [y_0(\zeta_j), \ldots, y_{n-1}(\zeta_j)]^t,
\end{equation}

where $y_k$ are in (3.3) for the matrix $C_r$. On the other hand, by (3.4) and (3.7)

\begin{equation}
C_r \hat{X}_j = \zeta_j \hat{X}_j, \quad \hat{X}_j = [x_{n-1}(\zeta_j), \ldots, x_0(\zeta_j)]^t.
\end{equation}

Since the spectrum is simple, the vectors $Y_j$ and $\hat{X}_j$ are proportional:

\begin{equation}
Y_j = c_j \hat{X}_j, \quad y_{k-1}(\zeta_j) = c_j x_{n-k}(\zeta_j); \quad k, j = 1, 2, \ldots, n,
\end{equation}

or

\begin{equation}
y_{k-1}(\zeta_j) = \frac{x_{n-k}(\zeta_j)}{x_{n-k-1}(\zeta_j)}, \quad k = 1, 2, \ldots, n - 1.
\end{equation}

Under the mixed inverse spectral problem (MISP) we mean the reconstruction of a CMV matrix $C = C(\alpha_0, \ldots, \alpha_{n-2}; 1)$, or equivalently, of a set of Verblunsky parameters $\alpha_0, \ldots, \alpha_{n-2} \in \mathbb{D}$, when a part $\{\zeta_j\}_{j=1}^m$ of its spectrum and a part of the system $(\alpha_0, \ldots, \alpha_{n-2})$ are known.

Here is the simplest problem of this type. Assume that we know $(\alpha_0, \ldots, \alpha_{n-3})$ as well as two eigenvalues $\zeta_1 \neq \zeta_2$, and $\alpha_{n-2}$ is to be found so that $\zeta_{1,2} \in \Sigma(C)$. Once $\Phi_{n-2}$ is known, we apply the Szegö recurrences to obtain

\begin{equation}
\Phi_n(z) = z(z + \alpha_{n-2})\Phi_{n-2}(z) - (z\bar{\alpha}_{n-2} + 1)\Phi_{n-2}^*,
\end{equation}

so

\begin{equation}
b(\zeta_j) = \tau_j, \quad b(\lambda) = \frac{\lambda + \alpha_{n-2}}{1 + \lambda\bar{\alpha}_{n-2}}, \quad j = 1, 2,
\end{equation}

where $\tau_j, \alpha_{n-2}$ are to be chosen so that $\zeta_{1,2} \in \Sigma(C)$. The trouble here is that once the first part of the spectrum is known, it is possible to choose the corresponding parameters so that the second part of the spectrum is also known.
\[ \tau_j = \frac{\Phi^*_n(-\zeta_j)}{\zeta_j \Phi_n(-\zeta_j)} \quad j = 1, 2. \]

The question is whether \( \alpha_{n-2} \) is uniquely determined from the interpolation problem (3.11). An elementary analysis of (3.11) shows that it has a unique solution as long as \( \tau_1 \zeta_1 \neq \tau_2 \zeta_2 \), that is,

\[ \frac{\Phi^*_n(-\zeta_1)}{\Phi_n(-\zeta_1)} \neq \frac{\Phi^*_n(-\zeta_2)}{\Phi_n(-\zeta_2)}, \]

it has infinitely many solutions if \( \tau_2 = -\zeta_1 \) and \( \tau_1 = -\zeta_2 \), or

\[ \frac{\Phi^*_n(-\zeta_1)}{\Phi_n(-\zeta_1)} = \frac{\Phi^*_n(-\zeta_2)}{\Phi_n(-\zeta_2)} = -\zeta_1 \zeta_2, \]

and it has no solutions at all, if \( \tau_1 \zeta_1 = \tau_2 \zeta_2 \), but \( \tau_1 \neq -\zeta_2, \tau_2 \neq -\zeta_1 \). It is not hard to check that each situation may occur for interpolation problem (3.11). However, if the existence of CMV matrix \( \mathcal{C} \) with \( \zeta_1, \zeta_2 \in \Sigma(\mathcal{C}) \) is supposed, the existence of the solution of problem (3.11) is guaranteed and the problem of finding \( \alpha_{n-2} \) may have either unique or infinitely many solutions (see example 1 below).

Since the Blaschke product \( \Phi_n/\Phi^*_n \) of order \( n-2 \) cannot take the same value on the \( n \)-point set \( \Sigma(\mathcal{C}) \), there always exists such a pair \( \zeta_1 \neq \zeta_2 \) in \( \Sigma(\mathcal{C}) \), that (3.12) holds, so \( \alpha_{n-2} \) is uniquely determined.

The general MISP for CMV matrices we study here looks as follows. Let \( n = 2l \) be even. Given first \( n-m-1 \) Verblunsky parameters \( \alpha_0, \ldots, \alpha_{n-m-2} \), and \( 2m \) eigenvalues \( \zeta_1, \ldots, \zeta_{2m}, 1 \leq m \leq n/2 = l \), find the rest \( m \) parameters \( \alpha_{n-m-1}, \ldots, \alpha_{n-2} \) and thereby restore the whole matrix \( \mathcal{C} \). Our main result provides the conditions for this problem to have a unique solution.

Consider a pair of CMV matrices with the “known” parameters \( \mathcal{C}(\alpha_0, \ldots, \alpha_{n-m-3}; 1) \) and \( \mathcal{C}(\alpha_0, \ldots, \alpha_{n-m-2}; 1) \) and the systems of the Szegő polynomials

\[ \{ \Phi_0, \ldots, \Phi_{n-2}; \Phi_{n-1} \}, \quad \{ \Phi_0, \ldots, \Phi_{n-m-1}; \Phi_{n-m} \}, \]

respectively. By the Szegő recurrences (3.14)

\[ \Phi_{n-1}(z) - \Phi_{n-1}(z) = (1 - \Phi_{n-2}) \Phi_{n-2}^*, \]

so

\[ \Phi_{n-1}(z) - \Phi_{n-1}(z) = (1 - \Phi_{n-2}) \Phi_{n-2}^*, \]

Similarly, for the pair \( \mathcal{C}(\lambda_0, \ldots, \lambda_{m-1}; 1) \) and \( \mathcal{C}(\lambda_0, \ldots, \lambda_m; 1) \) of “unknown” CMV matrices, \( \lambda_j \) from (3.29) with the Szegő polynomials \( \{ \Lambda_0, \ldots, \Lambda_m; \Lambda_{m+2} \} \), respectively, one has

\[ \Lambda_{m+1}(z) - \Lambda_{m+1}(z) = (1 - \Lambda_m) \Lambda^*_m. \]

Now write (3.10) with \( k = m + 1 \):

\[ \frac{y_n(z)}{y_{n+1}(z)} = \frac{x_{n-1}(z)}{x_{n-2}(z)}, \quad j = 1, \ldots, n, \]

\[ 2m \] “real” parameters are given to find \( m \) “complex” ones.
and observe that the right hand side of (3.15) is known for \( j = 1, 2, \ldots, 2m \). Indeed, let, e.g., \( m \) be odd (for even \( m \) the calculation is the same). Then by (3.3)

\[
x_{n-m-2}(z) = z^{-\frac{n-m-1}{2}} \kappa_{n-m-2} \Phi^*_{n-m-2}(z); \quad x_{n-m-1}(z) = z^{-\frac{n-m}{2}} \kappa_{n-m-1} \Phi_{n-m-1}(z),
\]

so in view of (3.13)

\[
\frac{x_{n-m-1}(\zeta)}{x_{n-m-2}(\zeta)} = \frac{\kappa_{n-m-1} \Phi_{n-m-1}(\zeta)}{\kappa_{n-m-2} \Phi_{n-m-2}(\zeta)} = (1 - |\alpha_{n-m-2}|^2)^{-1/2} \frac{\Phi_{n-m-1}(\zeta) + (1 - \alpha_{n-m-2}) \Phi^*_{n-m-2}(\zeta)}{\Phi^*_{n-m-2}(\zeta)} = \rho_{n-m-2}^{-1} \left\{ \Phi_{n-m-1}(\zeta) + 1 - \alpha_{n-m-2} \right\}, \quad \rho_i = (1 - |\alpha_i|^2)^{1/2}.
\]

In the same way

\[
y_m = z^{-\frac{m+1}{2}} \kappa_{m,r} \Lambda^*_m(z), \quad y_{m+1} = z^{-\frac{m+1}{2}} \kappa_{m+1,r} \Lambda_{m+1}(z),
\]

and with

\[
\kappa_{m,r} = \prod_{j=0}^{m-1} (1 - |\lambda_j|^2)^{-1/2} = \prod_{j=n-m-1}^{n-2} (1 - |\alpha_j|^2)^{-1/2}
\]

we have

\[
\frac{y_m(\zeta)}{y_{m+1}(\zeta)} = \frac{\kappa_{m,r} \Lambda^*_m(\zeta)}{\kappa_{m+1,r} \Lambda_{m+1}(\zeta)} = \rho_{m,r}^{-1} \left\{ \frac{\Phi_{n-m-1}(\zeta) + (1 - \Lambda_m^*(\zeta)) \Lambda_{m+1}(\zeta)}{\Lambda_{m+1}(\zeta)} \right\}^{-1} \left\{ \frac{\Phi_{n-m-1}(\zeta) + 1 + \alpha_{n-m-2}}{\Phi_{n-m-2}(\zeta)} \right\}, \quad \rho_{m,r} = (1 - |\Lambda_m|^2)^{-1/2} = \rho_{n-m-2}.
\]

Using (3.15), we end up with the following equalities for \( j = 1, 2, \ldots, 2m \)

\[
(3.16) \quad \frac{\Lambda_{m+1}(\zeta)}{\Lambda_m(\zeta)} = -1 - \alpha_{n-m-2} + \frac{1 - |\alpha_{n-m-2}|^2}{\Phi_{n-m-1}(\zeta) \Phi_{n-m-2}(\zeta)}.
\]

As the last step, we express the ratios in terms of the Weyl functions (more precisely, their reciprocals)

\[
W(z) = \frac{1}{w(z)} = \frac{\Phi_{n-m-1}(z)}{\Phi_{n-m-2}(z)}, \quad W_r(z) = \frac{1}{w_r(z)} = \frac{\Lambda_{m+1}(z)}{\Lambda_m(z)}
\]

of the known \( \mathcal{C}(\alpha_0, \ldots, \alpha_{n-m-3}; 1) \) and unknown \( \mathcal{C}(\lambda, \ldots, \lambda_{m-1}; 1) \), respectively. Indeed, for \(|z| = 1\)

\[
\Phi_{n-m-1}(z) = \prod_{i=1}^{n-m-1} (z - z_i), \quad |z_i| = 1, \quad \Phi_{n-m-1}(0) = (-1)^{n-m-1} \prod_{i=1}^{n-m} z_i = -\beta = -1,
\]

so

\[
\Phi_{n-m-1}(\zeta) = -\zeta_j^{-n+m+1} \Phi_{n-m-1}(\zeta), \quad \frac{\Phi_{n-m-1}(\zeta)}{\Phi_{n-m-2}(\zeta)} = \frac{\zeta_j^{n-m+1} \Phi_{n-m-1}(\zeta)}{\zeta_j^{n-m} \Phi_{n-m-2}(\zeta)} = -\zeta_j W(\zeta),
\]

Indeed, for \( |z| = 1 \)
and, similarly,\[ \frac{\tilde{\Lambda}_{m+1}(\zeta_j)}{\Lambda_m^*(\zeta_j)} = -\zeta_j W_r(\zeta_j). \]

Finally, we come to the following interpolation problem for the Weyl function of the “unknown” CMV matrix \( C(\lambda_0, \ldots, \lambda_{m-1}; 1) \)
\[ (3.17) \]
\[ W_r(\zeta_j) = \frac{(1 + \alpha_{n-m-2}) \{ 1 - \alpha_{n-m-2} - \zeta_j W(\zeta_j) \} - (1 - |\alpha_{n-m-2}|^2)}{1 - \alpha_{n-m-2} - \zeta_j W(\zeta_j)} =: \omega_j, \]
\[ j = 1, 2, \ldots, 2m, \text{ or} \]
\[ (3.18) \]
\[ P^{(1)}(\zeta_j) - \omega_j P^{(2)}(\zeta_j) = 0, \quad j = 1, \ldots, 2m, \]
with \( \omega_j \) defined in \((3.17)\), which we have denoted by \((I_{2m})\) in the previous section.
Now \( \omega_j \neq \infty \) since all zeros of \( \Lambda_m \) are in the open unit disk \( \mathbb{D} \). The above argument shows that \((3.18)\) has a nontrivial solution \( h(\lambda) = 2m + 2 \)

**Proposition 3.2.** For the problem \((3.18)\) \( h(I_{2m}) \geq 2m - 1. \)

**Proof.** Let \( r \) be the minimal generator of \((3.18)\), and suppose that \( h(r) \leq 2m - 2. \) By Theorem \( 2.11 \) for the second generator \( q \) one has \( h(q) \geq 2m + 3. \) It follows now from Theorem \( 2.11 \) and \((3.19)\) that \( \lambda = Sr \) with \( \deg S \geq 2, \) so
\[ \tilde{\Lambda}_{m+1}(z) = S(z)R^{(1)}(z), \quad \Lambda_m(z) = S(z)R^{(2)}(z), \]
which is impossible, for \( \tilde{\Lambda}_{m+1} \) and \( \Lambda_m \) have no common zeros. \( \square \)

There is some more information available about the solution \( \lambda \). Specifically,
\[ \deg \tilde{\Lambda}_{m+1} = m + 1, \quad \deg \Lambda_m = m \]
and
\[ (3.21) \]
\[ \tilde{\Lambda}_{m+1}(0) = -1. \]

In view of Theorem \( 2.8 \) it is easy to conclude from the Proposition \( 3.2 \) that if the data of interpolation problem \((3.18)\) correspond to a CMV matrix, then either \( h(I_{2m}) = 2m \) or \( h(I_{2m}) = 2m - 1. \)

**Theorem 3.3.** Let for the minimal generator \( r \) of problem \((3.18)\) \( R^{(1)}(0) \neq 0 \) holds. Then \((3.18)\) has a unique solution and, hence, the solution of the MISP is unique.

**Proof.** By Theorem \( 2.11 \)
\[ \Lambda^{(1)}(z) = S(z)R^{(1)}(z) + T(z)Q^{(1)}(z), \]
\[ \Lambda^{(2)}(z) = S(z)R^{(2)}(z) + T(z)Q^{(2)}(z), \]
\[ \Lambda^{(1)}(0) \neq 0. \]
where \( r \) and \( q \) are the minimal and second generators for the problem \((3.18)\), respectively. Proposition \((3.2)\) reads that either \( h(r) = 2m \) or \( h(r) = 2m - 1 \), so by Theorem \((2.10)\) either \( h(q) = 2m + 1 \) or \( h(q) = 2m + 2 \). In the first case

\[
\deg R^{(1)} = m, \quad \deg R^{(2)} \leq m - 1, \quad \deg Q^{(1)} \leq m, \quad \deg Q^{(2)} = m,
\]

and in the second one

\[
\deg R^{(1)} \leq m - 1, \quad \deg R^{(2)} = m - 1, \quad \deg Q^{(1)} = m + 1, \quad \deg Q^{(2)} \leq m.
\]

In view of the degrees of \( \Lambda^{(1)} \) and \( \Lambda^{(2)} \), in both cases

\[
\deg S(z) = 1, \quad \deg T(z) = 0,
\]

i.e., for \( \Lambda^{(j)} \) we have

\[
\Lambda^{(1)}(z) = (az + b)R^{(1)}(z) + cQ^{(1)}(z),
\]

\[
\Lambda^{(2)}(z) = (az + b)R^{(2)}(z) + cQ^{(2)}(z),
\]

For fixed \( r \) and \( q \), it is easy to see that \( a \) and \( c \) are uniquely determined by condition \((3.20)\). If \( R^{(1)}(0) \neq 0 \), then \( b \) is also uniquely determined by \((3.21)\), but if \( Q^{(1)}(0) = 0 \), then the problem has either no solutions, or infinitely many solutions. But since the data of the problem is taken from the CMV matrix, the solution does exist, so if \( R^{(1)}(0) = 0 \), the interpolation problem has infinitely many solutions and, hence, the MISP may have infinitely many solutions.

In view of this theorem, two natural questions arise.

1) Is it possible for the minimal generator to have \( R^{(1)}(0) = 0 \) when \((3.17)\) is related to MISP for a certain CMV matrix?

2) Is it possible for the MISP to have more than one solution if \( R^{(1)}(0) = 0 \)?

Both answers are positive, so in some special cases MISP with non-unique solutions does exist, although the number of the “pieces of information” in the inverse data is equal to the number of parameters to reconstruct. We provide examples for both possible cases \( h(I_{2m}) = 2m - 1 \) (example 1) and \( h(I_{2m}) = 2m \) (example 2).

**Example 1.** Let \(-1 < b < 1 \) and \( C = C(0, 0, b; 1) \). By the Szegő recurrences

\[
\Phi_1(z) = z, \quad \Phi_2(z) = z^2, \quad \Phi_3(z) = z^3 - b, \quad \Phi_4(z) = -bz^3 + 1,
\]

and

\[
\bar{\Phi}_4(z) = z\Phi_3(z) - \Phi_3^*(z) = (z^2 - 1)(z^2 + bz + 1),
\]

so the eigenvalues are

\[
\zeta_{1,2} = \pm 1, \quad \zeta_{3,4} = \frac{-b}{2} \pm i\sqrt{1 - \frac{b^2}{4}}.
\]

We see that the pair \( \zeta_1, \zeta_2 \) does not determine \( b \) uniquely, although any other pair does.

However, the MISP of (non-unique) reconstruction of \( b \) by the two eigenvalues \( \zeta_{1,2} = \pm 1 \) is still possible. Find the right-hand side of \((3.17)\) for this case. First, consider the Weyl function of the “known” left matrix \( C(0; 1) \). Its Szegő polynomials are \( \Phi_1(z) = z \) and \( \bar{\Phi}_2(z) = z\Phi_1 - 1 \cdot \Phi_1^* = z^2 - 1 \). So, the reciprocal of its Weyl function is \( W(z) = \frac{z^2 - 1}{z} \) and the right-hand side of \((3.17)\) is

\[
\omega_j = \zeta_j \cdot \frac{(1 + 0)\{1 - 0 - \zeta_j W(\zeta_j)\} - (1 - 0)}{1 - \zeta_j W(\zeta_j)} = \zeta_j \cdot \frac{-\zeta_j W(\zeta_j)}{1 - \zeta_j W(\zeta_j)}.
\]
For \( \zeta_j = \pm 1 \) we have \( \omega_j = 0 \), so, according to (3.17), the reciprocal of the Weyl function of the matrix \( C_r = C(-b; 1) \) satisfies

\[
W_r(\pm 1) = 0.
\]

Remind that it also must satisfy additional conditions (3.20) and (3.21) with \( m = 2 \).

So, following the procedure of solving the MISP, described above, to reconstruct the inverse of the Weyl function of \( C(-b; 1) \), we need to reconstruct a rational function \( \frac{P^{(1)}}{P^{(2)}} \) such that

\[
(3.23) \quad \frac{P^{(1)}(z)}{P^{(2)}(z)}|_{z=\pm 1} = 0, \quad \deg P^{(1)} = 2, \quad \deg P^{(2)}(z) = 1,
\]

\( P^{(1)}, P^{(2)} \) are monic and \( P^{(1)}(0) = 1 \).

The corresponding interpolation problem for the vector-functions is

\[
(3.24) \quad P^{(1)}(\pm 1) = 0.
\]

According to Proposition 3.2, the minimal generator of the problem (3.24) must have the height \( \geq 1 \), and according to Theorem 2.8, it must have the height \( \leq 2 \). In fact, it can be immediately checked that the non-trivial vector-function of minimal height, corresponding to this problem, is \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), whose height is 1. Further, since the height of the minimal generator is 1 and there is 2 points of interpolation, according to Theorem 2.10, the second generator must have the height 4 and there is no solutions of height 2. In fact, it is evident that the vector-polynomial \( \left( \frac{(z + 1)(z - 1)}{0} \right) \), whose height is 4, solves (3.24), and there is no solutions of height 2. Finally, the general solution of (3.23), is

\[
\frac{P^{(1)}(z)}{P^{(2)}(z)} = \frac{1 \cdot (z + 1)(z - 1) + (z + a) \cdot 0}{1 \cdot 0 + (z + b) \cdot 1} = \frac{(z + 1)(z - 1)}{z + b},
\]

with arbitrary number \( b \) (cf. (3.22)). However, only those solutions with additional condition \( |b| < 1 \) give us not only a solution of (3.23), but also the Weyl function of a CMV matrix of the type \( C(-b; 1) \). In fact, let us find directly the Weyl function of \( C(-b; 1) \). Its Szegö polynomials are:

\[
\begin{align*}
A_1(z) &= z + b; \\
A_2(z) &= zA_1(z) - A_1^*(z) = z(z + b) - (bz + 1) = (z + 1)(z - 1).
\end{align*}
\]

So,

\[
W_r(z) = \frac{A_2}{A_1} = \frac{(z + 1)(z - 1)}{z + b},
\]

as was to be checked.

**Example 2.** Consider a family of CMV matrices of order 4: \( C(0, -y, -x; 1) \); \( -1 < x, y < 1 \), and analyze the MISP of reconstruction of the unknown \( x, y \) by the four eigenvalues. Calculate for them the Szegö polynomials:

\[
\begin{align*}
\Phi_1(z) &= z, \quad \Phi_1^*(z) = 1; \\
\Phi_2 &= z\Phi_1 + y\Phi_1^* = z^2 + y, \quad \Phi_2^* = 1 + z^2 y; \\
\Phi_3 &= z\Phi_2 + x\Phi_2^* = (z^2 + y) + x(1 + z^2 y) = z^3 + xy^2 + yz + x, \quad \Phi_3^* = 1 + xy + yz^2 + xz^3;
\end{align*}
\]
\[
\Phi_4(z) = z\Phi_3 - 1 \cdot \Phi_0 = z(z^3 + x y z^2 + y z + x) - 1 - x y z - y z^2 - x^2 z^3 = (z^4 - 1) + (x y - x)z(z^3 - 1) = (z^2 - 1)(z^2 + (x y - x)z + 1).
\]

Introducing the notation
\[
(3.25) \quad k := xy - x,
\]
we express the eigenvalues of \(C\) as
\[
(3.26) \quad \Sigma: \quad \zeta_{1,2} = \pm 1; \quad \zeta_{3,4} = -\frac{k}{2} \pm i \sqrt{1 - \frac{k^2}{4}}; \quad \zeta_3 \neq \zeta_4 \text{ and } \zeta_4 = \bar{\zeta}_3.
\]

Hence, if \(x\) and \(y\) are related by \((3.26)\) and \(k\) is fixed, we have an infinite family of CMV matrices \(C(x, y, 0; 1)\) with the same spectrum \(\Sigma\) \((3.26)\). According to the general theory the auxiliary Weyl functions \(W_r(z)\) and \(W(z)\) of the matrices \(C_r = C(x, y; 1)\) and \(C(1)\), resp., take the same values on \(\Sigma\) for different \(x\) and \(y\), related by \((3.26)\). Find the right-hand side of \((3.17)\) for this case. First, consider the Weyl function of the “known” left matrix \(C(1)\). Its Szegő polynomials are \(\Phi_0(z) = 1\) and \(\Phi_1(z) = z\Phi_0 - 1 - \Phi_0 = z - 1\). So, the inverse of its Weyl function is \(W(z) = z - 1\) and the right-hand side of \((3.17)\) is
\[
\omega_j = \zeta_j \cdot \frac{(1 + 0)(1 - 0 - \bar{\zeta}_j W(\zeta_j)) - (1 - 0)}{1 - 0 - \bar{\zeta}_j W(\zeta_j)} = \zeta_j \cdot \frac{-\bar{\zeta}_j(\zeta_j - 1)}{1 - \zeta_j(\zeta_j - 1)} = \zeta_j(1 - \zeta_j).
\]

Let us directly check that the left-hand side of \((3.17)\) coincide with the obtained numbers. The Szegő polynomials of \(C_r = C(x, y; 1)\) are
\[
\Lambda_1(z) = z - x, \quad \Lambda_1^*(z) = 1 - x z;
\]
\[
\Lambda_2(z) = z\Lambda_1 - y\Lambda_1^* = z^2 + (x y - x)z - y, \quad \Lambda_2^*(z) = -y z^2 + (x y - x)z + 1;
\]
\[
\bar{\Lambda}_3(z) = z\Lambda_2 - \Lambda_2^* = z^3 + (x y - x)z^2 - y z + y z^2 - (x y - x)z - 1 = (z - 1)(z^2 + (k + y + 1)z + 1);
\]
\[
W_r(z) = \frac{\bar{\Lambda}_3(z)}{\Lambda_2(z)} = \frac{(z - 1)(z^2 + (k + y + 1)z + 1)}{z^2 + k z - y};
\]
\[
W_r(1) = 0 = \omega_1;
\]
\[
W_r(-1) = \frac{-2(2 - (k + y + 1))}{1 - k - y} = -2 = \omega_2;
\]
and, since \(\zeta_j^2 + k\zeta_j + 1 = 0; \ j = 3, 4\), we have
\[
W_r(\zeta_j) = \frac{(\zeta_j - 1)(y + 1)\zeta_j}{-(y + 1)} = \zeta_j(1 - \zeta_j) = \omega_j; \quad j = 3, 4.
\]

If we solved the MISP following the procedure described above, we have to find (non-uniquely) the inverse Weyl function \(W_r(z)\) from its value in the four eigenvalues and the additional conditions for the numerator and denominator. In this example we will restrict ourselves by illustrating that the minimal generator of the corresponding interpolation problem does not satisfy the conditions of theorem \((3.33)\), which actually cause the existence of infinitely many solutions.

Consider the interpolation problem, corresponding to this case:
\[
\left\{ \begin{array}{l}
R^{(1)}(\zeta_j) - \omega_j R^{(2)}(\zeta_j); \ j = 1, 2, 3, 4; \quad \zeta_j \in \Sigma; \\
\omega_1 = 0, \quad \omega_2 = -2, \quad \omega_3, 4 = \zeta_3, 4(1 - \zeta_3, 4).
\end{array} \right.
\]
(since $\zeta_j \neq 1$, we have $\omega_j \neq 0$). We are looking for the solution of height 4:

$$R^{(1)} = z^2 + \alpha z + \beta, \quad R^{(2)} = \gamma z + \delta,$$

so

$$\begin{cases}
1 + \alpha + \beta = 0; \\
1 - \alpha + \beta + 2(-\gamma + \delta) = 0; \\
\zeta_j^2 + \alpha \zeta_j + \beta - \omega_j (\gamma \zeta_j + \delta) = 0, \quad j = 3, 4.
\end{cases}$$

Since $\zeta_j^2 + k \zeta_j + 1 = 0$, the last 2 equations can be rewritten as

$$(\alpha - k) \zeta_j + \beta - 1 - \omega_j (\gamma \zeta_j + \delta) = 0, \quad (\alpha - k) \zeta_j - \alpha - 2 - \omega_j (\gamma \zeta_j + \delta) = 0; \quad j = 3, 4.$$

Exclude from these equations first $\gamma$, then $\delta$:

1) $\begin{cases}
(\alpha - k) \zeta_3 \omega_4 - (\alpha + 2) \omega_4 - \gamma \zeta_3 \omega_3 \omega_4 - \delta \omega_3 \omega_4 = 0; \\
(\alpha - k) \zeta_4 \omega_3 - (\alpha + 2) \omega_3 - \gamma \zeta_4 \omega_4 \omega_3 - \delta \omega_3 \omega_4 = 0.
\end{cases}$

$$\gamma = \frac{\alpha - k}{1 - \zeta_3^2} + (\alpha + 2) \frac{\omega_3 - \omega_4}{(\zeta_3 - \zeta_4)(1 - \zeta_3^2)} = \frac{\alpha - l}{1 - \zeta_3^2} + (\alpha + 2) \frac{1 + k}{1 - \zeta_3^2} = \frac{(\alpha + 1)(k + 2)}{1 - \zeta_3^2}.$$

But $|1 - \zeta_3^2| = 2 - \zeta_3 - \zeta_3^2 = 2 + k \Rightarrow \gamma = \alpha + 1$.

2) $\begin{cases}
(\alpha - k) \zeta_3 \omega_4 - (\alpha + 2) \omega_4 - \gamma \zeta_3 \omega_3 \omega_4 - \delta \omega_3 \omega_4 = 0; \\
(\alpha - k) \zeta_4 \omega_3 - (\alpha + 2) \omega_3 - \gamma \zeta_4 \omega_4 \omega_3 - \delta \omega_4 \omega_3 = 0.
\end{cases}$

$$\zeta_4 \omega_4 - \zeta_3 \omega_3 = \zeta_4^2(1 - \zeta_4) - \zeta_3^2(1 - \zeta_3) = (\zeta_4 - \zeta_3)(\zeta_4 + \zeta_3 - \zeta_4^2 - \zeta_3^2 - 1) = (\zeta_4 - \zeta_3)(-k^2 - k + 1),$$

$\omega_4 - \omega_3 = (\zeta_4 - \zeta_3)(1 - \zeta_3 - \zeta_4) = (\zeta_4 - \zeta_3)(1 + k)$.

Hence,

$$(\alpha - k)(1 + k) + (\alpha + 2)(k^2 + k - 1) = \delta (2 + k);$$

$$ak(k + 2) + (k - 1)(k + 2) = \delta (k + 2) \Rightarrow \delta = ak + k - 1.$$

3) We have from the second equation $\alpha = \delta - \gamma \Rightarrow \alpha = ak + k - 2 \Rightarrow \alpha = -1; \quad \beta = 0; \quad \gamma = 0; \quad \delta = -1,$

$$R^{(1)}(z) = z^2 - z; \quad R^{(2)}(z) = -1.$$ It can be immediately checked that the solution of height 3 (such that $R^{(1)} = \alpha z + \beta$, $R^{(2)} = z + \gamma$ does not exist, so $r = \left( \frac{R^{(1)}}{R^{(2)}} \right)$ is the minimal solution. As we see, $R^{(1)}(0) = 0$. Finally, rewriting

$$W_r(z) = \frac{(z - 1)(z^2 + (k + 1)z + 1) + (z^2 - z) \cdot y}{z^2 + k\ z + (-1) \cdot y},$$

we see in the non-determined (arbitrary) up to a constant factor terms of the numerator $(z^2 - z) \cdot y$ and denominator $-1 \cdot y$ the components of the minimal generator (cf. (3.22) with $b = y$).

**Remark.** Assume that $2m + 1$ eigenvalues $\zeta_1, \ldots, \zeta_m, \zeta_{m+1}$ are known. Since

$$w_2(z) = \frac{\Psi_m(z)}{\Psi_{m+1}(z)} = \frac{z^m + \ldots}{z^{m+1} + \ldots}.$$
the interpolation problem $w_2(\zeta_j) = \Omega_j; \ j = 1, 2, \ldots, 2m + 1$ has obviously the unique solution.

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Institute for Low Temperature Physics and Engineering, 47, Lenin Ave., Kharkov, Ukraine
E-mail address: golinsky@ilt.kharkov.ua

Institute for Low Temperature Physics and Engineering, 47, Lenin Ave., Kharkov, Ukraine
E-mail address: kudryavtsev@ilt.kharkov.ua; kudryavstev@onet.com.ua