ELLIPITC PSEUDO-DIFFERENTIAL EQUATIONS AND SOBOLEV SPACES OVER $p$-ADIC FIELDS

J. J. RODRÍGUEZ-VEGA AND W. A. ZÚÑIGA-GALINDO

Abstract. We study the solutions of equations of type $f(D, \alpha)u = v$, where $f(D, \alpha)$ is a $p$-adic pseudo-differential operator. If $v$ is a Bruhat-Schwartz function, then there exists a distribution $E_\alpha$, a fundamental solution, such that $u = E_\alpha \ast v$ is a solution. However, it is unknown to which function space $E_\alpha \ast v$ belongs. In this paper, we show that if $f(D, \alpha)$ is an elliptic operator, then $u = E_\alpha \ast v$ belongs to a certain Sobolev space. Furthermore, we give conditions for the continuity and uniqueness of $u$. By modifying the Sobolev norm, we can establish that $f(D, \alpha)$ gives an isomorphism between certain Sobolev spaces.

1. Introduction

In recent years $p$–adic analysis has received a lot of attention due to its applications in mathematical physics, see e.g. [1], [2], [3], [9], [10], [12], [16], [19], [20] and references therein. As a consequence new mathematical problems have emerged, among them, the study of $p$-adic pseudo-differential equations, see e.g. [4], [6], [11], [12], [13], [14], [15], [17], [20], [21], [23] and references therein. In this paper, we study the solutions of $p$-adic elliptic pseudo-differential equations on Sobolev spaces.

A pseudo-differential operator $f(D, \beta)$ is an operator of the form

$$(f(D, \alpha)\varphi)(x) = \mathcal{F}_{\xi}^{-1} (|f(\xi)|^p \mathcal{F}_{\xi} \varphi(x)), \quad \varphi \in S,$$

where $\mathcal{F}$ denotes the Fourier transform, $\alpha$ is a positive real number, $S$ denotes the $\mathbb{C}$-vector space of Bruhat-Schwartz functions over $\mathbb{Q}_p^n$, and $f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n]$. If $f(\xi)$ is a homogeneous polynomial of degree $d$ satisfying

$$f(\xi) = 0 \text{ if and only if } \xi = 0,$$

then the corresponding operator is called an elliptic pseudo-differential operator. At any case, the operator $f(D, \beta)$ is continuous and has a self-adjoint extension with dense domain in $L^2(\mathbb{Q}_p^n)$. This operator is considered to be a $p$-adic analogue of a linear partial elliptic differential operator with constant coefficients. A $p$-adic pseudo-differential equation is an equation of type

$$f(D, \alpha)u = v.$$

If $v \in S$, then there exists a distribution $E_\alpha$, a fundamental solution, such that $u = E_\alpha \ast v$ is a solution. The existence of a fundamental solution for general pseudo-differential operators was established by the second author in [21] by adapting

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the proof given by Atiyah for the Archimedean case [5]. However, it is unknown
to which function space $E_\alpha \ast v$ belongs. In this paper, we show that if $f(D, \alpha)$
is an elliptic operator, then $u = E_\alpha \ast v$ belongs to a certain Sobolev space (see
Theorem 3). Furthermore, we give conditions for the continuity and uniqueness
of $u$. By modifying the Sobolev norm, we can establish that $f(D, \alpha)$ gives an
isomorphism between certain Sobolev spaces, (see Propositions 1, 2 and Theorem
4). Our approach is based on the explicit calculation of fundamental solutions of
pseudo-differential operators on certain function spaces and the fact that elliptic
pseudo-differential operators behave like the Taibleson operator when acting on
certain function spaces (see Theorems 1, 2).

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2. Preliminary Results

We summarize some basic facts about $p$-adic analysis that will be used in this
paper. For a complete exposition, we refer the reader to [18], [20].

Let $\mathbb{Q}_p$ be the field of the $p$-adic numbers, and let $\mathbb{Z}_p$ be the ring of $p$-adic
integers. For $x \in \mathbb{Q}_p$, let $v(x) \in \mathbb{Z} \cup \{\infty\}$ denote the valuation of $x$ normalized
by the condition $v(p) = 1$. By definition $v(x) = \infty$ if and only if $x = 0$. Let
$|x|_p = p^{-v(x)}$ be the normalized absolute value. Here, by definition $|x|_p = 0$ if and
only if $x = 0$. We extend the $p$-adic absolute value to $\mathbb{Q}_p^n$ as follows:

$$||x||_p := \max\{|x_1|_p, \ldots, |x_n|_p\}, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define the exponent of local constancy of $\varphi(x) \in S(\mathbb{Q}_p^n)$ as the smallest
integer, $l \geq 0$, with the property that, for any $x \in \mathbb{Q}_p^n$,

$$\varphi(x + x') = \varphi(x) \text{ if } ||x'||_p \leq p^{-l}.$$

For $x, y \in \mathbb{Q}_p^n$, we put $x \cdot y = \sum_{i=1}^n x_i y_i$.

Let $\Psi$ denote an additive character of $\mathbb{Q}_p$, trivial on $\mathbb{Z}_p$, but not on $p^{-1}\mathbb{Z}_p$. For
$\varphi \in S(\mathbb{Q}_p^n)$, we define its Fourier transform as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-x \cdot \xi) \varphi(x) \, dx,$$

where $dx$ denotes the Haar measure of $\mathbb{Q}_p^n$ normalized in such a way that $\mathbb{Z}_p^n$ has
measure one.

We denote by $\chi_r$, $r \in \mathbb{Z}$, the characteristic function of the polydisc $B_r(0) :=
(p^{r}\mathbb{Z}_p)^n$. For any $\varphi \in S$, we set

$$r_\varphi := \min\{r \in \mathbb{N} \mid \varphi|_{B_r(0)} = \varphi(0)\}.$$

Definition 1. We set $\mathcal{L} := \mathcal{L}(\mathbb{Q}_p^n) = \{\varphi \in S \mid \int_{\mathbb{Q}_p^n} \varphi(x) \, dx = 0\}$, and $\mathcal{W} := \mathcal{W}(\mathbb{Q}_p^n)$
to be the $\mathbb{C}$-vector space generated by the functions $\chi_r$, $r \in \mathbb{Z}$.

We note that any $\varphi \in S$ can be written uniquely as $\varphi = \varphi_\mathcal{L} + \varphi_\mathcal{W}$, where $\varphi_\mathcal{W} = \int_{\mathbb{Q}_p^n} \varphi(x) \, dx \chi_{r_\varphi} \in \mathcal{W}$, and $\varphi_\mathcal{L} = \varphi - \varphi_\mathcal{W} \in \mathcal{L}$. However, $S$ is not the direct
sum of $\mathcal{L}$ and $\mathcal{W}$. The space $\mathcal{W}$ was introduced in [22], and $\{\mathcal{F}(\varphi) \mid \varphi \in \mathcal{L}\}$ is a
Lizorkin space of second class [4].
2.1. Elliptic Pseudo-differential Operators. Let \( f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n] \) be a nonconstant polynomial. A pseudo-differential operator \( f(D, \alpha), \alpha > 0, \) with symbol \(|f(\xi)|_p^\alpha\), is an operator of the form
\[
(f(D, \alpha)\varphi) = \mathcal{F}^{-1}\left(|f|_p^\alpha \mathcal{F}\varphi\right),
\]
where \( \varphi \in S \).

**Lemma 2.** Let \( f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n] \) be a nonconstant polynomial. We say that \( f(\xi) \) is an elliptic polynomial of degree \( d \), if it satisfies: (i) \( f(\xi) \) is a homogeneous polynomial of degree \( d \), and (ii) \( f(\xi) = 0 \Leftrightarrow \xi = 0 \).

**Lemma 1.** [23, Lemma 1] Let \( f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n] \) be an elliptic polynomial of degree \( d \). There exist positive constants, \( C_0(f) \) and \( C_1(f) \), such that
\[
C_0(f)||\xi||^d_{p} \leq |f(\xi)|_p \leq C_1(f)||\xi||^d_{p}, \quad \text{for every } \xi \in \mathbb{Q}_p^n.
\]

We note that if \( f(\xi) \) is elliptic, then \( cf(\xi) \) is elliptic for any \( c \in \mathbb{Q}_p^\times \). For this reason, we will assume from now on that the elliptic polynomials have coefficients in \( \mathbb{Z}_p \).

**Lemma 2.** [23, Lemma 3] Let \( f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n] \) be an elliptic polynomial of degree \( d \). Let \( \mathcal{A} \subset \mathbb{Q}_p^n \) be a compact subset such that \( 0 \not\in \mathcal{A} \). Then there exists a positive integer \( m = m(A, f) \) such that \( |f(\xi)|_p \geq p^{-m} \), for any \( \xi \in \mathcal{A} \). Furthermore, for any covering of \( \mathcal{A} \) by the form \( \cup_{i=1}^k B_i \), with \( B_i = z_i + (p^n\mathbb{Z}_p)^n \), we have \( |f(\xi)|_p = |f(z_i)|_p \) for any \( \xi \in B_i \).

**Definition 3.** Let \( f(\xi) \in \mathbb{Z}_p[\xi_1, \ldots, \xi_n] \) be an elliptic polynomial of degree \( d \). We will say that \( f|_p^\alpha \) is an elliptic symbol, and that \( f(D, \beta) \) is an elliptic pseudo-differential operator of order \( d \).

2.2. Igusa’s local zeta functions. Let \( g(x) \in \mathbb{Q}_p[x], \ x = (x_1, \ldots, x_n) \), be a non-constant polynomial. Igusa’s local zeta function associated to \( g(x) \) is the distribution
\[
\langle |g|^s_p, \varphi \rangle = \int_{\mathbb{Q}_p^n \sim g^{-1}(0)} |g(x)|^s_p \varphi(x) \, dx,
\]
for \( s \in \mathbb{C}, \ \text{Re}(s) > 0, \) where \( \varphi \in S \), and \( dx \) denotes the normalized Haar measure of \( \mathbb{Q}_p^n \). The local zeta functions were introduced by Weil and their basic properties for general \( g(x) \) were first studied by Igusa. A central result in the theory of local zeta functions established that \( |g|^s_p \) admits a meromorphic continuation to the complex plane such that \( \langle |g|^s_p, \varphi \rangle \) is rational function of \( p^{-s} \) for each \( \varphi \in S \). Furthermore, there exists a finite set \( \cup_{E \in \mathcal{E}} \{(N_E, n_E)\} \) of pairs of positive integers such that
\[
\prod_{E \in \mathcal{E}} (1 - p^{-n_E-N_E s}) |g|^s_p
\]
is a holomorphic distribution on \( S \). In particular, the real parts of the poles of \( |g|^s_p \) are negative rational numbers see [24, Chap. 8]. The existence of a meromorphic continuation for the distribution \( |g|^s_p \) implies the existence of a fundamental solution for the pseudo-differential operator with symbol \( |g|^s_p \).

For a fixed \( \varphi \in S \), we denote the integral \( \langle |g|^s_p, \varphi \rangle \) by \( Z_\varphi(s, g) \). In particular, \( Z(s, g) = Z_{\chi_0}(s, g) \).
Lemma 3. Let \( f(x) \in \mathbb{Z}_p[x] \), \( x = (x_1, \ldots, x_n) \), be an elliptic polynomial of degree \( d \). Then

\[
Z(s, f) = \frac{L(p^{-s})}{1 - p^{-ds-n}},
\]

where \( L(p^{-s}) \) is a polynomial in \( p^{-s} \) with rational coefficients. Furthermore, \( s = -n/d \) is a pole of \( Z(s, f) \).

Proof. Let \( A = \{ x \in \mathbb{Z}_p^n \mid \text{ord}(x_i) \geq d, \ i = 1, \ldots, n \} \), and \( A' = \{ x \in \mathbb{Z}_p^n \mid \text{ord}(x_i) < d, \ \text{for some} \ i \} \). Then \( \mathbb{Z}_p^n \) is the disjoint union of \( A \) and \( A' \) and

\[
Z(s, f) = \int_A |f(x)|_p^s dx + \int_{A'} |f(x)|_p^s dx = p^{-ds-n}Z(s, f) + \int_{A'} |f(x)|_p^s dx,
\]

i.e., \( Z(s, f) = \frac{1}{1 - p^{-ds-n}} \int_{A'} |f(x)|_p^s dx \). Since \( A' \) is compact, by applying Lemma 2, we find a covering of \( A' = \bigcup_{i=1}^L B_i \), where \( |f|_p \) is constant on each \( B_i \). Hence,

\[
\int_{A'} |f(x)|_p^s dx = p^{-nm} \sum_{i=1}^L |f(z_i)|_p^s,
\]

and

\[
Z(s, f) = \frac{p^{-nm} \sum_{i=1}^L |f(z_i)|_p^s}{1 - p^{-ds-n}}.
\]

2.3. The Riesz Kernel. We collect some well-know results about the Riesz kernel that will be used in the next sections, we refer the reader to [18] or [20] for further details.

The \( p \)-adic Gamma function \( \Gamma_p^{(n)}(s) \) is defined as follows:

\[
\Gamma_p^{(n)}(s) = \frac{1 - p^{s-n}}{1 - p^{-s}}, \ s \in \mathbb{C}, \ s \neq 0.
\]

The Gamma function is meromorphic with simple zeros at \( n + \frac{2\pi i}{\ln p} \mathbb{Z} \) and unique simple pole at \( s = 0 \). In addition, it satisfies

\[
\Gamma_p^{(n)}(s)\Gamma_p^{(n)}(n-s) = 1, \ \text{for} \ s \neq \{0\} \cup \{n + \frac{2\pi i}{\ln p} \mathbb{Z}\}.
\]

The Riesz kernel \( R_s \) is the distribution determined by the function

\[
R_s(x) = \frac{||x||_p^{s-n}}{\Gamma_p^{(n)}(s)}, \ \text{Re}(s) > 0, \ s \notin \{n + \frac{2\pi i}{\ln p} \mathbb{Z}\}, \ x \in \mathbb{Q}_p^n.
\]

The Riesz kernel has, as a distribution, a meromorphic continuation to \( \mathbb{C} \) given by

\[
\langle R_s(x), \varphi(x) \rangle = \frac{1 - p^{-n}}{1 - p^{s-n}} \varphi(0) + \frac{1 - p^{-s}}{1 - p^{s-n}} \int_{||x||_p > 1} ||x||_p^{s-n} \varphi(x) \ dx
\]

\[
+ \frac{1 - p^{-s}}{1 - p^{s-n}} \int_{||x||_p \leq 1} ||x||_p^{s-n} (\varphi(x) - \varphi(0)) \ dx,
\]
with poles at $n + \frac{2\pi i}{\ln p}$. In particular, for $\text{Re}(s) > 0$,

$$
\langle R_s(x), \varphi(x) \rangle = \frac{1 - p^{-s}}{1 - p^{s-n}} \int_{Q_p} \varphi(x)||x||_p^{s-n} \, dx, \quad s \notin n + \frac{2\pi i}{\ln p},
$$

(2.1) \hspace{1cm} \langle R_{-s}(x), \varphi(x) \rangle = \frac{1 - p^s}{1 - p^{-s-n}} \int_{Q_p^1} (\varphi(x) - \varphi(0))||x||_p^{-s-n} \, dx.

In the case $s = 0$, by passing to the limit, we obtain

$$
\langle R_0(x), \varphi(x) \rangle = \lim_{s \to 0} \langle R_s(x), \varphi(x) \rangle = \varphi(0),
$$

i.e., $R_0(x) = \delta(x)$, the Dirac delta function. Therefore, $R_s \in S'(Q_p^0)$, for $s \in \mathbb{C} \setminus \left\{ n + \frac{2\pi i}{\ln p} \right\}$.

**Remark 1.** The distribution $||x||_p^s$, $\text{Re}(s) > 0$, admits the following meromorphic continuation,

$$
\langle ||x||_p^s, \varphi(x) \rangle = \frac{1 - p^{-n}}{1 - p^{-s-n}} \varphi(0) + \int_{||x||_p > 1} ||x||_p^s \varphi(x) \, dx
$$

$$
+ \int_{||x||_p \leq 1} ||x||_p^s (\varphi(x) - \varphi(0)) \, dx, \quad \varphi \in S.
$$

In particular, all the poles of $||x||_p^s$ have real part equal to $-n$.

**Lemma 4 ([15] Chap. III, Theorem 4.5).** As element of $S'(Q_p^0)$, $(\mathcal{F} R_s)(x)$ equals $||x||_p^{-s}$, for $s \notin n + \frac{2\pi i}{\ln p}$.

The following explicit formula will be used in the next sections.

**Lemma 5.** Let $f(x) \in Q_p[x]$, $x = (x_1, \ldots, x_n)$, be an elliptic polynomial of degree $d$. Then

$$
||f||_p^s = \frac{(1 - p^{ds})L(p^{-s})}{(1 - p^{-s})(1 - p^{-ds-n})} R_{ds+n}, \quad s \in \mathbb{C}
$$

as distributions on $W$. Here $L(p^{-s})$ is the numerator of $Z(s, f)$ which is a polynomial in $p^{-s}$ with rational coefficients.

**Proof.** Let $\varphi \in W$, then

$$
\varphi(x) = \sum_i c_i \chi_{r_i}(x),
$$

where $c_i \in \mathbb{C}$, $r_i \in \mathbb{Z}$ (recall that $\mathcal{F}(\chi_r) = p^{-nr} \chi_{-r}$). The action of $||f||_p^s$ on $\mathcal{F}\varphi$ can be explicitly described as follows:

$$
\langle ||f||_p^s, \mathcal{F}\varphi \rangle = \sum_i c_i \langle ||f||_p^s, p^{-nr_i} \chi_{-r_i} \rangle,
$$

but

$$
\langle ||f||_p^s, p^{-nr_i} \chi_{-r_i} \rangle = p^{-nr_i} \int_{Q_p^1} |f(x)||x|_p^{s-nr_i} \chi_{-r_i}(x) \, dx = p^{dr_i}Z(s, f),
$$

for $\text{Re}(s) > 0$, thus

$$
\langle ||f||_p^s, \mathcal{F}\varphi \rangle = Z(s, f) \sum_i c_i p^{dr_i}, \quad \text{Re}(s) > 0.
$$
On the other hand,
\[
\langle 1 - p^{d_s} 1 - p^{-n} R_{ds+n} , p^{-n_r} \chi_{-r_i} \rangle = \langle 1 - p^{d_s} 1 - p^{-n} \|x\|_p^{d_s} , p^{-n_r} \chi_{-r_i} \rangle = p^{d_r},
\]
for every \( r_i \in \mathbb{Z} \) and \( \text{Re}(s) > 0 \). Then we have
\[
\langle |f|_{p'} \mathcal{F} \varphi \rangle = 1 - p^{d_s} 1 - p^{-n} Z(s, f) \langle R_{ds+n} , \mathcal{F} \varphi \rangle,
\]
for \( \text{Re}(s) > 0 \). Now \( Z(s, f) \) and \( R_{ds+n} \) have a meromorphic continuation to the complex plane, therefore this formula extends to \( \mathbb{C} \). Finally, since the Fourier transform establishes a \( \mathcal{C} \)-isomorphism on \( \mathcal{W} \), it is possible remove the Fourier transform symbol. \( \Box \)

2.4. The Taibleson Operator.

**Definition 4.** The Taibleson pseudo-differential operator \( D_T^\alpha \), \( \alpha > 0 \), is defined as
\[
(D_T^\alpha \varphi)(x) = \mathcal{F}_{\xi^{-1}}^{-1}(\|\xi\|_p^\alpha \mathcal{F}_{x-\xi} \varphi), \text{ for } \varphi \in S.
\]

As a consequence of the Lemma 4 and (2.1), one gets
\[
(D_T^\alpha \varphi)(x) = (k_{-\alpha} \ast \varphi)(x) = \frac{1 - p^{d_s}}{1 - p^{-\alpha-n}} \int_{\mathbb{Q}_p} \|y\|_p^{-\alpha-n} (\varphi(x-y) - \varphi(x)) \, dy.
\]

The right-hand side of previous formula makes sense for a wider class of functions than \( S(\mathbb{Q}_p) \), for example, for the class \( \mathcal{E}_\alpha(\mathbb{Q}_p) \) of locally constant functions \( \varphi(x) \) satisfying
\[
\int_{||x||_p \geq 1} ||x||_p^{-\alpha-n} |\varphi(x)| \, dx < \infty.
\]

**Remark 2.** As a consequence of the previous observations we may assume that the constant functions are contained in the domain of \( D_T^\alpha \), and that \( D_T^\alpha \varphi = 0 \), for any constant function.

3. Fundamental Solutions for the Taibleson Operator

We now consider the following pseudo-differential equation:

\[
(3.1) \quad D_T^\alpha u = v, \text{ with } v \in \mathcal{S}, \text{ and } \alpha > 0.
\]

We say that \( E_\alpha \in \mathcal{S}' \) is a fundamental solution of (3.1) if \( E_\alpha \ast v \) is a solution.

**Lemma 6.** If \( E_\alpha \) is a fundamental solution of (3.1), then for any constant \( c \), \( E_\alpha + c \) is also a fundamental solution.

**Proof.** Let \( E_\alpha \) a fundamental solution for (3.1), then
\[
D_T^\alpha ((E_\alpha + c) \ast v) = D_T^\alpha ((E_\alpha \ast v) + (c \ast v)) = u + D_T^\alpha (c \ast v) = u,
\]
because \( u \) and the constant function, \( c \ast v \), are in the domain of \( D_T^\alpha \). \( \Box \)
Theorem 1. A fundamental solution of (3.1) is

\[ E_\alpha(x) = \begin{cases} 
    \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} |x|^{\alpha-n} & \text{if } \alpha \neq n \\
    \frac{1 - p^{\alpha-n}}{p^n \ln p} & \text{if } \alpha = n.
\end{cases} \]

Proof. The proof is based on the ideas introduced in [21]. The existence of a fundamental solution \( E_\alpha \) is equivalent to the existence of a distribution \( F E_\alpha \) satisfying

\[ ||x||^\alpha_p F E_\alpha = 1, \]

as distributions. Let \( ||x||^s_p = \sum_{m \in \mathbb{Z}} c_m (s + \alpha)^m \) be the Laurent expansion at \(-\alpha\) with \( c_m \in S'\) for all \( m \). The existence of this expansion is a consequence of the completeness of \( S' \) (see e.g. [7, pp. 65-66]). Since the real parts of the poles of the meromorphic continuation of \( ||x||^s_p \) are negative rational numbers (cf. Remark [1]), \( ||x||^{s+\alpha}_p = ||x||^s_p ||x||^\alpha_p \) is holomorphic at \( s = -\alpha \). Therefore, \( ||x||^\alpha_p c_m = 0 \) for all \( m < 0 \) and \( ||x||^{s+\alpha}_p = ||x||^\alpha_p c_0 + \sum_{m=1}^\infty ||x||^s_p c_m (s + \alpha)^m \).

By using the Lebesgue dominated convergence theorem, one verifies that

\[ \lim_{s \to -\alpha} \langle ||x||^{s+\alpha}_p, \phi \rangle = \int_{Q_p^n} \phi(x) dx = \langle 1, \phi \rangle, \]

and then we can take \( FE_\alpha = c_0 \). Furthermore, if \(-\alpha\) is not a pole of \( ||x||^s_p \),

\[ FE_\alpha = \lim_{s \to -\alpha} ||x||^s_p. \]

To calculate \( c_0 \), consider the following two cases.

Case \( \alpha \neq n \).

We use (3.3) and the Lemma [4] i.e.,

\[ \int_{(Q_p^n)^n} ||x||^s_p \mathcal{F}(\varphi)(x) dx = \frac{1 - p^s}{1 - p^{s-n}} \int_{(Q_p^n)^n} ||x||^{s-n} \varphi(x) dx, \]

for \( s \neq n + (2\pi i / \ln p) \mathbb{Z} \). If \( \alpha \neq n \), by (3.3),

\[ \langle E_\alpha, \mathcal{F}(\varphi) \rangle = \lim_{s \to -\alpha} \int_{(Q_p^n)^n} ||x||^s_p \mathcal{F}(\varphi)(x) dx. \]

If \( \alpha > n \), by the Lebesgue dominated convergence theorem, we can interchange the limit and the integral. If \( 0 < \alpha < n \), by taking into account that

\[ \int_{||x||_p \leq 1} ||x||^{\alpha-n}_p dx < +\infty, \quad 0 < \alpha < n, \]

and by using Lebesgue dominated convergence theorem, we can exchange the limit and the integral. Therefore,

\[ \langle E_\alpha, \varphi \rangle = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{(Q_p^n)^n} ||x||^{\alpha-n}_p \varphi(x) dx = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \int_{Q_p^n} ||x||^{\alpha-n}_p \varphi(x) dx. \]
Set \( \tilde{\varphi}(x) = \varphi(-x) \), with \( \varphi \in S \). The results follow by replacing \( \varphi \) by \( \mathcal{F}(\tilde{\varphi}) \) because \( \mathcal{F}(\mathcal{F}(\tilde{\varphi})) = \varphi \).

**Case** \( \alpha = n \).

We compute the constant term, \( c_0 \), in the expansion
\[
\langle \langle |x|^s \rangle_p, \mathcal{F}(\varphi) \rangle = \sum_{m \in \mathbb{Z}} \langle c_m, \mathcal{F}(\varphi) \rangle (s + n)^m.
\]
Since
\[
\langle \langle |x|^s \rangle_p, \mathcal{F}(\varphi) \rangle = \frac{1 - p^s}{1 - p^{-s-n}} \int_{\mathbb{R}^n} \langle |x|^{-s-n} \varphi(x) \rangle \, dx
\]
\[
= (1 - p^s) \int_{\mathbb{R}^n} \frac{p^s(x)(s+n)}{1 - p^{-s-n}} \varphi(x) \, dx,
\]
where \( x = (x_1, \ldots, x_n) \), \( v(x) := \min_{1 \leq i \leq n} v(x_i) \), and \( \langle |x|_p \rangle = p^{-v(x)} \), by expanding
\[
\frac{(1 - p^s)p^v(x)(s+n)}{1 - p^{-s-n}} = \frac{1 - p^{-n}}{\ln p} (s + n)^{-1} \frac{1 - p^{-v(x)} \ln p - \frac{\ln p}{2} \ln p}{\ln p} + O((s + n)),
\]
one gets
\[
\langle E_n, \varphi \rangle = \langle c_0, \varphi \rangle = \int_{\mathbb{R}^n} \left( \frac{1 - p^0}{p^0 \ln(p)} \ln(|x|_p) + \frac{p^n - 3}{2p^n} \right) \varphi(x) \, dx.
\]

The announced results follow by replacing \( \varphi \) by \( \mathcal{F}(\tilde{\varphi}) \), \( \varphi \in S \), and using the fact that the fundamental solution is determined up to the addition of a constant (cf. Lemma 5).

In the case \( n = 1 \), the previous result is already known, see e.g. [14, Theorem 2.1].

### 4. Fundamental Solutions for Elliptic Operators

**Theorem 2.** Let \( f(D, \alpha) \) be an elliptic operator of order \( d \). Then, a fundamental solution \( E_\alpha \) of \( f(D, \alpha)u = v, \alpha > 0 \), and \( v \in \mathcal{W} \), is given by
\[
E_\alpha(x) = \begin{cases} 
\frac{L(p^\alpha)(1 - p^{-\alpha})}{(1 - p^{-n})(1 - p^\alpha - n)} |x|^{(1 - d \alpha - n)} & \text{as a distribution on } \mathcal{W}, \text{ with } \alpha \neq n/d \\
\frac{L(p^\alpha/(d \alpha))}{(1 - p^{-n})(p^\alpha \ln p)} \ln |x|_p & \text{as a distribution on } \mathcal{W}, \text{ with } \alpha = n/d,
\end{cases}
\]
where \( L(p^{-s}) \) is the numerator of \( Z(s, f) \).

**Proof.** As we mention before, the problem of the existence of a fundamental solution, \( E_\alpha \), is equivalent to the existence of a distribution \( \mathcal{FE}_\alpha \) satisfying
\[
|f|^\alpha \mathcal{FE}_\alpha = 1 \text{ in } S'.
\]

By Lemma 5
\[
\langle |f|^\alpha, \varphi \rangle = \left( \frac{(1 - p^\alpha)L(p^{-\alpha})}{(1 - p^{-n})(1 - p^{-\alpha} - n)} \right) R_{d\alpha + n}, \varphi
\]
\[ \varphi \in \mathcal{W}, \ s \in \mathbb{C}. \] The result follows by reasoning as in the proof of Theorem 1 and by the fact that the space \( \mathcal{W} \) is invariant under the Fourier transform. \( \square \)

**Corollary 1.** With the hypotheses of the previous theorem, and assuming that \( \alpha \neq n/d \), we have

\[ |\mathcal{F}(E_{\alpha} \ast \varphi)(x)| \leq C(\alpha)||x||_{p}^{-d_{\alpha}}|\mathcal{F}(\varphi)(x)|, \]

for all \( x \in \mathbb{Q}_{p}^{n} \), and \( \varphi \in \mathcal{W} \).

5. **Solutions of Elliptic Pseudo-Differential Equations in Sobolev Spaces**

Given \( \phi \in \mathcal{S} \) and \( l \) a non-negative number, we define

\[ ||\phi||_{H^{l}}^{2} = \int_{\mathbb{Q}_{p}^{n}} [\max(1, ||\xi||_{p})]^{2l} |\mathcal{F}(\phi)(\xi)|^{2} d\xi. \]

We call the completion of \( \mathcal{S} \) with respect to \( || \cdot ||_{H^{l}} \) the \( l \)-Sobolev space \( H^{l} := H^{l}(\mathbb{Q}_{p}^{n}) \).

We note that \( H^{l} \) contains properly the space of test functions, \( \mathcal{S} \). Indeed, consider the function

\[ f(x) = \begin{cases} 0 & \text{if } ||x||_{p} \leq 1 \\ ||x||_{p}^{-\beta} & \text{if } ||x||_{p} > 1 \end{cases} \]

with \( \beta > n \). A direct calculation shows that

\[ ||f||_{H^{l}}^{2} = \int_{||\xi||_{p} \leq 1} \left[ \frac{(1-p^{-n})(1-||\xi||_{p}^{\beta-n})}{(1-p^{n-\beta})} - p^{-\beta}||\xi||_{p}^{\beta-n} \right]^{2} d\xi. \]

Thus, \( ||f||_{H^{l}}^{2} < \infty \), but \( f \) does not have compact support.

**Lemma 7.** If \( l > n/2 \), then there exists an embedding of \( H^{l} \) into the space of uniformly continuous functions.

**Proof.** Let \( \phi \in H^{l} \). Since the Fourier transform of a function in \( L^{1} \) is uniformly continuous, it is sufficient to show that \( \mathcal{F}(\phi) \in L^{1} \). By using the Hölder inequality and the fact that

\[ \int_{\mathbb{Q}_{p}^{n}} (\max(1, ||\xi||_{p}))^{-2l} d\xi < +\infty, \text{ for } l > n/2, \]

we have

\[ \int_{\mathbb{Q}_{p}^{n}} |\mathcal{F}(\phi)(\xi)| d\xi = \int_{\mathbb{Q}_{p}^{n}} \frac{(\max(1, ||\xi||_{p}))^{l}}{(\max(1, ||\xi||_{p}))^{l}} |\mathcal{F}(\phi)(\xi)| d\xi \leq C||\phi||_{H^{l}}. \]

**Lemma 8.** For any \( \alpha > 0 \) and \( l \geq 0 \), the mapping \( f(D, \alpha) : H^{l+d_{\alpha}} \to H^{l} \) is a well-defined continuous mapping between Banach spaces.

**Proof.** Let \( \phi \in \mathcal{S} \). Since \( f(D, \alpha) \) is an elliptic operator, by Lemma \( \square \) we have that

\[ ||f(D, \alpha)\phi||_{H^{l}}^{2} = \int_{\mathbb{Q}_{p}^{n}} [\max(1, ||\xi||_{p})]^{2l} |f(\xi)|^{2\alpha} |\mathcal{F}(\phi)(\xi)|^{2} d\xi \]

\[ \leq C_{1} \int_{\mathbb{Q}_{p}^{n}} [\max(1, ||\xi||_{p})]^{2(l+\alpha)} |\mathcal{F}(\phi)(\xi)|^{2} d\xi = C_{1}||\phi||_{H^{l+d_{\alpha}}}. \]
The result follows from the fact that $S$ is dense in $H^{l+d\alpha}$. \qed

**Remark 3.** Let $\beta$ be a positive real number, and let

$$I(\beta) := \int_{||\varepsilon||_p \leq 1} ||\varepsilon||_p^\beta d\varepsilon.$$

Then

$$I(\beta) = \frac{1 - p^{-n}}{1 - p^{-n-n\beta}}, \text{ for } \beta > -n.$$

Indeed,

$$I(\beta) = \int_{||\varepsilon||_p < 1} ||\varepsilon||_p^\beta d\varepsilon + \int_{||\varepsilon||_p = 1} d\varepsilon$$

$$= \int_{||\varepsilon||_p < 1} ||\varepsilon||_p^\beta d\varepsilon + 1 - p^{-n}.$$

By making the change of variables $\varepsilon_i = px_i$, $i = 1, \ldots, n$, we have

$$I(\beta) = p^{-n-n\beta} I(\beta) + 1 - p^{-n}.$$

**Theorem 3.** Let $f(D, \alpha)$, $0 < \alpha < n/2d$ be an elliptic pseudo-differential operator of order $d$. Let $l$ be a positive real number satisfying $l > n/2$. Then, the equation

$$f(D, \alpha)u = v \quad (v \in S),$$

has a unique uniformly continuous solution $u \in H^{l+d\alpha}$.

**Proof.** Let $v \in S$, then $v = v_W + v_L$, where $v_W \in W$ and $v_L \in L$. Thus, in order to prove the existence of a solution $u$, it is sufficient to show that the two following equations have solutions:

(5.1) \quad $f(D, \alpha)u_W = v_W,$

(5.2) \quad $f(D, \alpha)u_L = v_L.$

We first consider equation (5.1). By Theorem 2, $u_W = E_\alpha * v_W$ is a solution of (5.1), and by Corollary 1, we have

$$||u_W||_{H^{l+d\alpha}}^2 = \int_{Q^n_p} [\max(1, ||\xi||_p)]^{2(l+d\alpha)} |\mathcal{F}(u_W)(\xi)|^2 d\xi$$

$$= C(\alpha, d, n) \int_{Q^n_p} [\max(1, ||\xi||_p)]^{2(l+d\alpha)} ||\xi||_p^{-2d\alpha} |\mathcal{F}(v_W)(\xi)|^2 d\xi$$

$$= C(\alpha, d, n) \left\{ \int_{||\xi||_p \leq 1} ||\xi||_p^{-2d\alpha} |\mathcal{F}(v_W)(\xi)|^2 d\xi \right\}$$

$$+ \int_{||\xi||_p > 1} ||\xi||_p^{2l} |\mathcal{F}(v_W)(\xi)|^2 d\xi.$$
We now recall that \(v_\mathcal{W}(\xi) = p^r n C \chi_r(\xi)\), with \(r > 0\). Then, \(\mathcal{F}(v_\mathcal{W})(\xi) = C \chi_{-r}(\xi)\) and
\[
||u_\mathcal{W}||_{H_l^{r+\alpha}}^2 \leq C(\alpha, d, n) \left\{ C^2 p^2r n \int_{||\xi||_p \leq 1} ||\xi||_{p}^{-2d\alpha} \, \, d\xi 
+ ||v_\mathcal{W}||_{H_l^r}^2 \right\}
\leq C(\alpha, d, n) \left\{ C_1(\alpha, d, n) + ||v_\mathcal{W}||_{H_l^r}^2 \right\},
\]
since \(-2d\alpha > -n\), cf. Remark 3. Therefore \(u_\mathcal{W} \in H_l^{r+\alpha}\).

We now consider equation (5.2). Since \(f(\alpha)u = v\), for any \(v \in \mathcal{S}\)
which is uniformly continuous, by Lemma 7, such that \(f\) is elliptic, indeed, if \(f(D, \alpha)u' = v\), then
\[
\mathcal{F}(D, \alpha)u = v, \quad \text{for any } v \in \mathcal{S}. 
\]
Finally, we show that \(u\) is unique. Indeed, if \(f(D, \alpha)u' = v\), then
\[
f(D, \alpha)(u - u') = 0, \quad \text{i.e., } |f|_p^\alpha \mathcal{F}(u - u') = 0,
\]
and thus \(\mathcal{F}(u - u')(\xi) = 0\) if \(\xi \neq 0\), since \(f\) is elliptic. Then \(\Psi(x \cdot \xi)(u - u')(\xi) = 0\) almost everywhere, and a fortiori \((u - u')(\xi) = 0\) almost everywhere, and by the continuity of \(u - u'\), \(u(\xi) = u'(\xi)\) for any \(\xi \in \mathbb{Q}_p^n\).

6. Solutions of Elliptic Pseudo-Differential Equations in Singular Sobolev Spaces

In this section, we modify the Sobolev norm to obtain spaces of functions on which \(f(D, \alpha)\) gives a surjective mapping.

**Definition 5.** Given \(\varphi \in \mathcal{S}\) and \(l\) a non-negative number, we set
\[
||\varphi||_{H_l^l}^2 := \int_{\mathbb{Q}_p^n} ||\xi||_p^2 |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi.
\]
We call the completion of \(\mathcal{S}\) with respect to \(|| \cdot ||_{H_l^l}\) the \(l\)-singular Sobolev Space \(\mathcal{H}_l^l := \mathcal{H}(\mathbb{Q}_p^n)\). Note that \(H^l \subset H_l^l\), \(l \geq 0\), since \(||\varphi||_{H_l^l} \leq ||\varphi||_{H_l^l}\).
Lemma 9. For any, $\alpha > 0$, $l \geq 0$, the mapping $f(D, \alpha) : \mathcal{H}^{l+da} \to \mathcal{H}^l$ is a well-defined continuous mapping between Banach Spaces.

Proof. Similar to the proof of Lemma 8. \hfill \Box

We denote by $\mathcal{L}^l$ and $\mathcal{W}^l$, the respective completions of $\mathcal{L}$ and $\mathcal{W}$ with respect to $\| \cdot \|_{\mathcal{H}^l}$; furthermore, we set

$$\mathcal{H}^l_0 := \mathcal{L}^l + \mathcal{W}^l \subseteq \mathcal{H}^l.$$ 

Proposition 1. Let $f(D, \alpha)$, $\alpha > 0$, be an elliptic pseudo-differential operator of order $d$, and let $l$ be a non-negative real number. Then $f(D, \alpha) : \mathcal{H}^{l+da} \to \mathcal{W}^l$, is a surjective mapping between Banach spaces.

Proof. By Lemma 9, the mapping is well-defined. Let $v \in \mathcal{W}^l$, and let $\{v_n\}$ a Cauchy sequence in $\mathcal{W}$ converging to $v$. By Theorem 3 there exits a sequence $\{u_n\}$ in $H^{l+da}$ such that $f(D, \alpha)u_n = v_n$. We now show that $\{u_n\}$ is a Cauchy sequence in $H^{l+da}$ as follows:

$$\|u_n - u_m\|_{H^{l+da}} \leq C \int_{Q_p^d} \|\xi\|_p^{2(l+da)} \|\xi\|_p^{-2da} |\mathcal{F}(v_n - v_m)(\xi)|^2 d\xi$$

$$\leq C \|v_n - v_m\|_{H^l}^2.$$ 

Thus, there exists $u \in \mathcal{H}^{l+da}$ such that $u_n \to u$, and by the continuity of $f(D, \alpha)$, $f(D, \alpha)u = v$. \hfill \Box

Proposition 2. Let $f(D, \alpha)$, $\alpha > 0$, be an elliptic pseudo-differential operator of order $d$, and let $l$ be a non-negative real number. Then, $f(D, \alpha) : \mathcal{H}^{l+da} \to \mathcal{L}^l$ is a surjective mapping between Banach spaces.

Proof. By Lemma 9, the mapping is well-defined. Let $v \in \mathcal{L}^l$, and let $\{v_n\}$ a Cauchy sequence in $\mathcal{L}$ converging to $v$. By the same reasoning given in proof Theorem 3 for establishing the existence of a solution for equation (5.2), we obtain a sequence $\{u_n\}$ in $H^{l+da}$ such that $f(D, \alpha)u_n = v_n$. We now show that $\{u_n\}$ is a Cauchy sequence in $H^{l+da}$.

By using

$$|\mathcal{F}(u_n)(\xi)| \leq C \|\xi\|^{-da} |\mathcal{F}(v_n)(\xi)|,$$

one gets

$$\|u_n - u_m\|_{H^{l+da}} \leq C \int_{Q_p^d} \|\xi\|_p^{2(l+da)} \|\xi\|_p^{-2da} |\mathcal{F}(v_n - v_m)(\xi)|^2 d\xi$$

$$\leq C \|v_n - v_m\|_{H^l}^2.$$ 

Thus, there exists $u \in \mathcal{H}^{l+da}$ such that $u_n \to u$, and by the continuity of $f(D, \alpha)$, $f(D, \alpha)u = v$. \hfill \Box

From the previous two lemmas we obtain the following result.

Theorem 4. Let $f(D, \alpha)$ be an elliptic pseudo-differential operator of order $d$. Let $l$ be a positive real number. Then the equation

$$f(D, \alpha)u = v, \quad v \in \mathcal{H}^l_0$$

has a unique solution $u \in \mathcal{H}^{l+da}$. 

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E-mail address: jjrodriguezv@unal.edu.co
E-mail address: wzuniga@math.cinvestav.mx

Departamento de Matemáticas, Universidad Nacional de Colombia, Ciudad Universitaria, Bogotá D.C., Colombia.

Centro de Investigación y de Estudios Avanzados del I.P.N., Departamento de Matemáticas, Av. Instituto Politécnico Nacional 2508, Col. San Pedro Zacatenco, México D.F., C.P. 07360, México.