Asymptotics of Solutions to the Modified Nonlinear Schrödinger Equation: Solitons on a Non-Vanishing Continuous Background

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Abstract

Using the matrix Riemann-Hilbert factorization approach for nonlinear evolution systems which take the form of Lax-pair isospectral deformations and whose corresponding Lax operators contain both discrete and continuous spectra, the leading-order asymptotics as $t \to \pm \infty$ of the solution to the Cauchy problem for the modified nonlinear Schrödinger equation, $i \partial_t u + \frac{1}{2} \partial_x^2 u + |u|^2 u + is \partial_x(|u|^2 u) = 0$, $s \in \mathbb{R}_{>0}$, which is a model for nonlinear pulse propagation in optical fibers in the subpicosecond time scale, are obtained: also derived are analogous results for two gauge-equivalent nonlinear evolution equations; in particular, the derivative nonlinear Schrödinger equation, $i \partial_t q + \partial_x^2 q - i \partial_x(|q|^2 q) = 0$. As an application of these asymptotic results, explicit expressions for position and phase shifts of solitons in the presence of the continuous spectrum are calculated.

Key words. asymptotics, Riemann-Hilbert problem, solitons, optical fibers

AMS subject classifications. 35Q15, 35Q55, 58F07, 78A60

Abbreviated title. Modified Nonlinear Schrödinger Equation
1 Introduction

With the current emphasis on the utilization of optical fibers, capable of supporting solitons, as the communication channel in the practical realization and implementation of all-optical (lightwave), ultrahigh-bit-rate, long-distance communication systems using the return-to-zero (RZ) format for generating the optical bit stream, design issues requiring the consideration of several factors, e.g., soliton widths and inter-soliton spacings, are intimately related to the study of the fundamental dynamical processes associated with the propagation of high-power ultrashort pulses in optical fibers (at the present stage of technology, these systems can at best still only be called near-soliton(ic)-based) [1]. The standard, classical mathematical model for nonlinear pulse propagation in the picosecond time scale in the anomalous dispersion regime in an isotropic, homogeneous, lossless, non-amplifying, polarization-preserving single-mode optical fiber is the nonlinear Schrödinger equation (NLSE) [1, 2]. However, experiments and theories on the propagation of high-power ultrashort pulses in the subpicosecond-femtosecond time scale in monomode optical fibers have shown that the NLSE is no longer a valid model, and that additional nonlinear terms (dispersive and dissipative) and higher-order linear dispersion must be taken into account: in this case, pulse-like propagation is described (in dimensionless and normalized form) by the following nonlinear evolution equation (NLEE) [1],

$$i\partial_z u + \frac{1}{2} \partial^2_{\tau} u + |u|^2 u + is\partial_{\tau}(|u|^2 u) = -i\bar{\Gamma} u + i\bar{\delta}\partial^3_{\tau} u + \frac{\bar{\tau}_n}{\bar{\tau}_0} u\partial_{\tau}(|u|^2),$$  

(1)

where $u$ is the slowly-varying amplitude of the complex field envelope, $z$ is the propagation distance along the fiber length, $\tau$ is the time measured in a frame of reference moving with the pulse at the group velocity, $s \in \mathbb{R}_{>0}$ governs the effects due to the intensity dependence of the group velocity (self-steepening), $\bar{\Gamma}$ is the intrinsic fiber loss, $\bar{\delta}$ governs the effects of the third-order linear dispersion, and $\bar{\tau}_n/\bar{\tau}_0$ governs the soliton self-frequency shift effect.

Since, under typical operating conditions, $\bar{\Gamma}$, $\bar{\delta}$, and $\bar{\tau}_n/\bar{\tau}_0$ are small parameters [1], a strategy to study the solutions of Eq. (1), for which the nonlinear effects dominate the higher-order linear dispersive one, is to set the right-hand side equal to zero, thus obtaining the following NLEE (integrable in the sense of the inverse scattering method (ISM) [3]),

$$i\partial_t u + \frac{1}{2} \partial^2_x u + |u|^2 u + is\partial_x(|u|^2 u) = 0,$$

(2)

which, hereafter, is called the modified nonlinear Schrödinger equation (MNLSE) (the physical variables, $z$ and $\tau$, have been mapped isomorphically onto the mathematical $t$ and $x$ variables, which are standard in the ISM context), and to treat Eq. (1) as a non-integrable perturbation of the MNLSE. From the above discussion, it is clear that perturbations of multi-soliton solutions of the MNLSE can be very important in the physical context related to optical fibers [1]. Since practical lasers excite not only the soliton(ic) mode(s), but also an entire continuum of linear-like dispersive (radiative) waves, to have physically meaningful and practically representative results, it is necessary to investigate solutions of the MNLSE under general initial (launching, in the optical fiber literature [1]) conditions, without any artificial restrictions and/or constraints, which have both soliton(ic) and non-soliton(ic) (continuum) components: it is towards such a solution that the initial pulse launched into an optical fiber is evolving asymptotically [1]. In physical terms, the pulse adjusts its width as it propagates along the optical fiber to evolve into a (multi-) soliton, and a part of the pulse energy is shed away in the form of dispersive waves in the process: normally, these dispersive waves form a low-level broadband background radiation that accompanies the (multi-) soliton [1, 3]. From the physical point of view, therefore, it is seminal to understand how the continuum and the (near-) soliton(s) interact, and to be able to derive an explicit functional form for this process. Since Eq. (2) is integrable via the ISM, one can use one of the techniques developed in the framework of this approach to solve the aforementioned problem; in particular, in this paper, the Riemann-Hilbert (RH) factorization method is applied.
For several soliton-bearing equations, e.g., KdV, Landau-Lifshitz, NLS, sine-Gordon and MKdV, it is known that the dominant ($O(1)$) asymptotic ($t \rightarrow \pm \infty$) effect of the continuous spectra on the multi-soliton solutions is a shift in phase and position of their constituent solitons [3]: as will be shown in this paper, an analogous, though analytically more complicated, situation takes place for the MNLSE (the additional complexification occurs due to the non-standard normalization of the associated RH problem). While the above-mentioned works deal only with the leading-order ($O(1)$) asymptotic term, in this paper, for the MNLSE, not only the leading-order, but the next-to-leading-order ($O(t^{-1/2})$) term as well is derived; in particular, besides inducing an $O(1)$ position and phase shift on the multi-soliton solution, this $O(t^{-1/2})$ term represents the evolution of the continuum component (dispersive wavetrain [1]) as well as the non-trivial interaction (overlap) of the soliton and continuum components of the solution. It is worth mentioning that, even though there have been several papers [6] devoted to studying the soliton solutions of the MNLSE, to the best of our knowledge, very little, if anything, was known about its solution(s) for the class of non-reflectionless initial data until very recently [7].

This paper is organized as follows. In Sec. 2, a matrix RH problem for the solution of a NLEE gauge-equivalent to Eq. (2) is stated, and the results of this paper are summarized as Theorems 2.1–2.3. In Sec. 3, an extended RH problem is formulated and shown to be equivalent to the original one stated in Sec. 2, and as $t \rightarrow +\infty$, it is shown that the solution of the extended RH problem converges, modulo exponentially decreasing terms, to the solution of a model RH problem. In Sec. 4, the Beals-Coifman [13] formulation for the solution of a RH problem on an oriented contour is succinctly recapitulated, and the model RH problem is solved asymptotically as $t \rightarrow +\infty$ for the Schwartz class of non-reflectionless generic potentials. In Sec. 5, a phase integral which is associated with the non-standard normalization of the above-mentioned RH problem is evaluated asymptotically as $t \rightarrow +\infty$. Finally, in Sec. 6, the asymptotic analysis as $t \rightarrow -\infty$ is presented.

2 The Riemann-Hilbert (RH) Problem and Summary of Results

In this section, the matrix RH problem is stated, and the main results of the paper are formulated as Theorems 2.1–2.3. Before doing so, however, it is necessary to introduce some notation and definitions which are used throughout the paper:

Notational Conventions

(1) $e_{\alpha\beta}, \alpha, \beta \in \{1, 2\}$, denote $2 \times 2$ matrices with entry 1 in ($\alpha \beta$), $(e_{\alpha\beta})_{ij} := \delta_{\alpha i} \delta_{\beta j}$, $i, j \in \{1, 2\}$, where $\delta_{ij}$ is the Kronecker delta;

(2) $I:=e_{11}+e_{22}:=\text{diag}(1,1)$ denotes the $2 \times 2$ identity matrix;

(3) $\sigma_3:=e_{11}-e_{22}:=\text{diag}(1,-1)$, $\sigma_-:=e_{21}$, $\sigma_+:=e_{12}$, and $\sigma_1:=\sigma_-+\sigma_+$;

(4) for a scalar $\varpi$ and a $2 \times 2$ matrix $\Upsilon$, $\varpi^{\text{ad}()} \Upsilon:=\varpi^{\sigma_3} \Upsilon \varpi^{-\sigma_3}$;

(5) $\overline{()}$ denotes complex conjugation of (\);\n
(6) $M_2(\mathbb{C})$ denotes the $2 \times 2$ complex matrix algebra with the following inner product $(\cdot|\cdot):M_2(\mathbb{C}) \times M_2(\mathbb{C}) \rightarrow \mathbb{C}$, $\forall a, b \in M_2(\mathbb{C})$, $(a, b):=\text{tr}(\overline{b}a)$, and (for $a \in M_2(\mathbb{C})$) the norm on $M_2(\mathbb{C})$ is defined as $|a|:=\sqrt{(a,a)}$.
(7) \( \mathcal{L}^p(D; M_2(\mathbb{C})) := \{ f; f: D \to M_2(\mathbb{C}), \|f\|_{\mathcal{L}^p(D; M_2(\mathbb{C}))} := (\int_D |f(q)|^p dq)^{1/p} < \infty, p \in \{1, 2\} \}; \)

(8) \( \mathcal{L}^\infty(D; M_2(\mathbb{C})) := \{ g; g: D \to M_2(\mathbb{C}), \|g\|_{\mathcal{L}^\infty(D; M_2(\mathbb{C}))} := \max_{1 \leq i, j \leq 2} \sup_{q \in D} |g_{ij}(q)| < \infty \}; \)

(9) for \( D \) an unbounded domain of \( \mathbb{R} \cup i\mathbb{R} \), let \( \mathcal{S}(D; \mathbb{C}) \) (resp. \( \mathcal{S}(D; M_2(\mathbb{C})) \)) denote the Schwartz class on \( D \), i.e., the class of smooth \( \mathbb{C} \)-valued (resp. \( M_2(\mathbb{C}) \)-valued) functions \( f(x): D \to \mathbb{C} \) (resp. \( F(x): D \to M_2(\mathbb{C}) \)) which together with all derivatives tend to zero faster than any positive power of \( |x|^{-1} \) as \( |x| \to \infty \).

In this paper, as in [7], along with the MNLSE, the following NLEEs are studied:

\[
i\partial_t Q + \partial_x^2 Q + iQ^2 \partial_x Q + \frac{1}{2} Q|Q|^4 = 0,
\]

with initial condition \( Q(x,0) \in \mathcal{S}(\mathbb{R}; \mathbb{C}) \), and the derivative nonlinear Schrödinger equation (DNLSE),

\[
i\partial_t q + \partial_x^2 q - i\partial_x (|q|^2 q) = 0,
\]

with initial condition \( q(x,0) \in \mathcal{S}(\mathbb{R}; \mathbb{C}) \). To recall the relations between the solutions of these NLEEs, the following propositions are formulated:

\textbf{Proposition 2.1 ([14])} The necessary and sufficient condition for the compatibility of the following system of linear ODEs (the Lax pair) for arbitrary \( \lambda \in \mathbb{C} \),

\[
\partial_x \Psi(x,t;\lambda) = U(x,t;\lambda) \Psi(x,t;\lambda), \quad \partial_t \Psi(x,t;\lambda) = V(x,t;\lambda) \Psi(x,t;\lambda),
\]

where

\[
U(x,t;\lambda) = -i\lambda^2 \sigma_3 + \lambda(\overline{\sigma}_- + \sigma_+) - \frac{i}{2} |Q|^2 \sigma_3,
\]

\[
V(x,t;\lambda) = 2\lambda^2 U(x,t;\lambda) - i\lambda((\partial_x \overline{Q}) \sigma_- - (\partial_x Q) \sigma_+) + \left( \frac{i}{4} |Q|^4 + \frac{1}{2} (\partial_x \overline{Q} - \sigma_3 \partial_x \overline{Q}) \sigma_3 \right),
\]

is that \( Q(x,t) \) satisfies Eq. (3).

\textit{Proof.} Eq. (3) is the Frobenius compatibility condition for system (5). \( \blacksquare \)

\textbf{Proposition 2.2} Let \( Q(x,t) \) be a solution of Eq. (3). Then there exists a corresponding solution of system (5) such that \( \Psi(x,t;0) \) is a diagonal matrix.

\textit{Proof.} For given \( Q(x,t) \), let \( \tilde{\Psi}(x,t;\lambda) \) be a solution of system (5) which exists in accordance with Proposition 2.1. Setting \( \lambda = 0 \) in system (5), one gets that \( \tilde{\Psi}(x,t;\lambda) = \exp\left\{-\frac{it}{2} \int x_0 |Q(q,t)|^2 dq\right\} \tilde{K}(\lambda) \), for some \( x_0 \in \mathbb{R} \) and non-degenerate matrix \( \tilde{K}(\lambda) \) which is independent of \( x \) and \( t \). The function \( \Psi(x,t;\lambda) := \tilde{\Psi}(x,t;\lambda)(\tilde{K}(\lambda))^{-1} \) is the solution of system (5) which is diagonal at \( \lambda = 0 \). \( \blacksquare \)

\textbf{Proposition 2.3 ([15])} Let \( Q(x,t) \) be a solution of Eq. (3) and \( \Psi(x,t;\lambda) \) the corresponding solution of system (5) given in Proposition 2.2. Set \( \Psi_q(x,t;\lambda) := \Psi^{-1}(x,t;0) \Psi(x,t;\lambda) \). Then

\[
\partial_x \Psi_q(x,t;\lambda) = U_q(x,t;\lambda) \Psi_q(x,t;\lambda), \quad \partial_t \Psi_q(x,t;\lambda) = V_q(x,t;\lambda) \Psi_q(x,t;\lambda),
\]

where

\[
U_q(x,t;\lambda) = -i\lambda^2 \sigma_3 + \lambda(\overline{\sigma}_- + q \sigma_+),
\]

\[
V_q(x,t;\lambda) = \left( \begin{array}{cc}
-2i\lambda^4 - i\lambda^2 |q|^2 & 2\lambda^3 q + i\lambda \partial_x q + \lambda |q|^2 q \\
2\lambda^3 q - i\lambda \partial_x q + \lambda |q|^2 q & 2i\lambda^4 + i\lambda^2 |q|^2 q
\end{array} \right),
\]

with \( q(x,t) \) defined by

\[
q(x,t) := Q(x,t)((\Psi^{-1}(x,t;0))_{11})^2,
\]

is the “Kaup-Newell” \([16]\) Lax pair for the DNLSE.
Proof. Differentiating $\Psi_q(x,t;\lambda) := \Psi^{-1}(x,t;0)\Psi(x,t;\lambda)$ with respect to $x$ and $t$ and using the fact that $\Psi(x,t;0) = \exp\{-\frac{it}{2}\int_0^x |Q(p,t)|^2 dp\}$, for some $x_0 \in \mathbb{R}$, and $\Psi(x,t;\lambda)$ satisfy system (5) for $\lambda = 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$, respectively, defining $q(x,t)$ as in Eq. (9), one gets that $\Psi_q(x,t;\lambda)$ satisfies system (6), where $\mathcal{U}_q(x,t;\lambda)$ and $\mathcal{V}_q(x,t;\lambda)$ are given by Eqs. (7) and (8): Eq. (4) is the Frobenius compatibility condition for system (6).

Proposition 2.4 If $q(x,t)$ is a solution of the DNLSE such that $q(x,0) \in \mathcal{S}(\mathbb{R};\mathbb{C})$, then

$$u(x,t) := \frac{1}{\sqrt{2s}} \exp\{\frac{s}{2}(x - \frac{t}{2})\} q(\frac{t}{2} - x, \frac{t}{2})$$

(10)

satisfies the MNLSE with initial condition $u(x,0) \in \mathcal{S}(\mathbb{R};\mathbb{C})$.

Proof. Direct substitution.

Remark 2.1 A convention is now adopted which is adhered to sensus strictu throughout the paper: for each segment of an oriented contour, according to the given orientation, the “+” side is to the left and the “−” side is to the right as one traverses the contour in the direction of the orientation; hence, ($\bullet$) and ($\bullet$) denote, respectively, the non-tangential limits (boundary values) of ($\bullet$) on an oriented contour from the “+” (left) and “−” (right) sides.

Before stating the matrix RH problem which is investigated asymptotically (as $t \to \pm \infty$) in this paper (see Lemma 2.1), it will be convenient to introduce the following notation: let $\mathcal{Z}_d := \bigcup_{i=1}^N (\{\pm \lambda_i\} \cup \{\pm \overline{\lambda_i}\})$ and $\hat{\Gamma} := \{\lambda; \Im(\lambda^2) = 0\}$ (oriented as in Fig. 1) denote, respectively, the discrete and continuous spectra of the operator $\partial_x - U(x,t;\lambda)$, and $\sigma_L := \text{Spec}(\partial_x - U) = \mathcal{Z}_d \cup \hat{\Gamma}$ ($\mathcal{Z}_d \cap \hat{\Gamma} = \emptyset$).

![Figure 1: Continuous spectrum $\hat{\Gamma}$.](image)

Lemma 2.1 Let $Q(x,t)$, as a function of $x, \in \mathcal{S}(\mathbb{R};\mathbb{C})$. Set $m(x,t;\lambda) := \Psi(x,t;\lambda) \exp\{i(\lambda^2 x + 2\lambda^4 t)\sigma_3\}$. Then: (1) the bounded discrete set $\mathcal{Z}_d$ is finite (card($\mathcal{Z}_d$) < $\infty$); (2) the poles of $m(x,t;\lambda)$ are simple; (3) the first (resp. second) column of $m(x,t;\lambda)$ has poles at $\{\pm \lambda_i\}_{i=1}^N$ (resp. $\{\pm \overline{\lambda_i}\}_{i=1}^N$); and (4) $\forall t \in \mathbb{R}$ the function $m(x,t;\lambda): \mathbb{C} \setminus (\mathcal{Z}_d \cup \hat{\Gamma}) \to \text{SL}(2,\mathbb{C})$ solves the following RH problem:

a. $m(x,t;\lambda)$ is meromorphic $\forall \lambda \in \mathbb{C} \setminus \hat{\Gamma}$;

b. $m(x,t;\lambda)$ satisfies the following jump conditions,

$$m_+(x,t;\lambda) = m_-(x,t;\lambda)v(x,t;\lambda), \quad \lambda \in \hat{\Gamma},$$

where

$$v(x,t;\lambda) = \exp\{-i(\lambda^2 x + 2\lambda^4 t)\sigma_3\}\left(\begin{array}{cc}
1 - r(\lambda) & \frac{r(\lambda)}{r(\overline{\lambda})} \\
-r(\overline{\lambda}) & 1
\end{array}\right),$$

$r(\lambda)$, the reflection coefficient associated with the direct scattering problem for the operator $\partial_x - U(x,t;\lambda)$, satisfies $r(-\lambda) = -r(\lambda)$, and $r(\lambda) \in \mathcal{S}(\hat{\Gamma};\mathbb{C})$;
c. for the simple poles of $m(x,t;\lambda)$ at $\{\pm \lambda_j\}_{j=1}^N$ and $\{\pm \lambda_j\}_{j=1}^N$, there exist nilpotent matrices $\{v_j(x,t)\sigma_-\}_{j=1}^N$ and $\{\overline{v_j(x,t)}\sigma_+\}_{j=1}^N$, respectively, such that the residues, for $1 \leq j \leq N$, satisfy the (Beals-Coffman) polar conditions,

$$\text{res}(m(x,t;\lambda_j)) = \lim_{\lambda \to \lambda_j} m(x,t;\lambda)\sigma_j,$$

$$\text{res}(m(x,t;\lambda_j)) = -\sigma_3 \text{res}(m(x,t;\lambda_j))$$

$$\text{res}(m(x,t;\lambda_j)) = \lim_{\lambda \to \lambda_j} m(x,t;\lambda)\sigma_j,$$

$$\text{res}(m(x,t;\lambda_j)) = -\sigma_3 \text{res}(m(x,t;\lambda_j))$$

where $v_j(x,t) := C_j \exp\{2i(\lambda_j^2 x + 2\lambda_j^4 t)\}$, and $C_j$ are complex constants associated with the direct scattering problem for the operator $\partial_x - U(x,t;\lambda)$.

d. as $\lambda \to \infty$, $\lambda \in \mathbb{C} \setminus (\mathbb{Z}_d \cup \hat{\Gamma})$,

$$m(x,t;\lambda) = 1 + \mathcal{O}(\lambda^{-1}).$$

**Proof.** Conditions (1)–(4) follow from the results given in \cite{1, 13, 17}. □

**Lemma 2.2** Let $||r||_{L^\infty(\mathbb{R},\mathbb{C})} := \sup_{\lambda \in \mathbb{R}} |r(\lambda)| < 1$. Then: (1) the RH problem formulated in Lemma 2.1 is uniquely solvable; (2) $\Psi(x,t;\lambda) = m(x,t;\lambda)\exp\{-i(\lambda^2 x + 2\lambda^4 t)\sigma_3\}$ is the solution of system (7) with

$$Q(x,t) := 2i \lim_{\lambda \to \infty} (\lambda m(x,t;\lambda))_{12};$$

(11)

(3) the function $Q(x,t)$ defined by Eq. (11) satisfies Eq. (3), and

$$q(x,t) := Q(x,t)((m^{-1}(x,t;0))_{11})^2$$

(12)

satisfies the DNLSE; and (4) $m(x,t;\lambda)$ possesses the following symmetry reductions, $m(x,t;\lambda) = \sigma_3 m(x,t;\lambda)$ and $m(x,t;\lambda) = \sigma_1 m(x,t;\lambda)\sigma_1$.

If $r(\lambda) \in S(\hat{\Gamma};\mathbb{C})$, then, for any $t \in \mathbb{R}$, $Q(x,t)$ (resp. $q(x,t)$), as a function of $x, x \in S(\mathbb{R};\mathbb{C})$.

**Proof.** The solvability of the RH problem (formulated in Lemma 2.1) is a consequence of Theorem 9.3 in \cite{18} and the vanishing winding number of $1 - r(\lambda)r(\lambda)$, \text{card}(1 - r(\lambda)r(\lambda)) = \sum_{t \in \{0, 1\}} s(t)n(t) = 0$, where $s(>) = -s(<) = 1$, and $n(\leq) := \text{card}((\lambda_j; \exists (\lambda_j^2 \geq 0))$, items (2) and (4) can be verified through straightforward calculations, and the fact that $q(x,t)$ (Eq. (12)) satisfies the DNLSE follows from Proposition 2.3 and the definition of $m(x,t;\lambda)$. □

**Remark 2.2** In fact, in this paper, the solvability of the RH problem for $||r||_{L^\infty(\mathbb{R},\mathbb{C})} < 1$ is proved for all sufficiently large $|t|$: the solvability of the RH problem for $||r||_{L^\infty(\hat{\Gamma})} < 1$ in the solitonless sector, $\mathbb{Z}_d \equiv \emptyset$, for all sufficiently large $|t|$ was proved in \cite{11}. Note: the condition $||r||_{L^\infty(\hat{\Gamma})} < 1$ which appears in \cite{11} is restrictive, and can be replaced by the weaker condition $||r||_{L^\infty(\mathbb{R},\mathbb{C})} < 1$.

Before summarizing the main results of this paper, namely, Theorems 2.1–2.3, some further preamble is required: (1) the Kaup-Newell parametrization \cite{16} is adopted for the discrete eigenvalues, $\lambda_j := \Delta_j \exp\{\eta_j \pm \gamma_j\}$, $\Delta_j > 0$, $\gamma_j \in (0, \pi)$, $\pm \gamma_j \in (0, \pi)$, and $\lambda_j^2 := \xi_j + i \eta_j$, where $\xi_j = -\Delta_j^2 \cos \gamma_j + i \eta_j = \Delta_j^2 \sin \gamma_j$ (note that, with this parametrization, $\{\pm \lambda_j\}_{j=1}^N$, (resp. $\{\pm \lambda_{N+1} \}_{i=1}^N$) lie in the 1st and 3rd quadrants (resp. 2nd and 4th quadrants) of the complex plane of the auxiliary spectral parameter, $\lambda$); and (2) it is supposed throughout that $\xi_i \neq \xi_j, 1 \leq i \neq j \leq N$, so that it is convenient to choose the following enumeration for the points of the discrete spectrum (ordering of the solitons), $\xi_1 > \cdots > \xi_N > \cdots > \xi_N$.

**Remark 2.3** Even though the “symbol” (“notation”) $C(z)$ appearing in the various final error estimations is not the same and should properly be denoted as $C_1(z), C_2(z)$, etc., the
simplified “notation” $C(z)$ is retained throughout since the principal concern here is not its explicit functional $z$-dependence, but rather, the functional class(es) to which it belongs. Throughout the paper, $M \in \mathbb{R}_{>0}$ is a fixed constant.

**Remark 2.4** In Theorems 2.1–2.3 (see below), one should keep the upper signs as $t \to +\infty$ and the lower signs as $t \to -\infty$ everywhere.

**Theorem 2.1** Let $m(x,t;\lambda)$ be the solution of the RH problem formulated in Lemma 2.1 with the condition $|r(r)|_{L^\infty(\mathbb{R}\cup \mathbb{C})} < 1$ and $Q(x,t)$ be defined by Eq. (11). Then as $t \to \pm \infty$ and $x \to \mp \infty$ such that $\lambda_0 := \frac{1}{2} \sqrt{-\frac{i}{\pi}} > M$ and $(x,t) \in \Omega_n := \{(x,t); x-4t\Delta_n^2 \cos \gamma_n := l_n(t) = O(1)\}$, for those $\gamma_n \in \left(\frac{\pi}{2}, \pi\right)$,

$$Q(x,t) = Q_{\text{as}}^+(x,t) + O\left(\frac{C(\lambda_0) \ln |t|}{t}\right),$$

where

$$Q_{\text{as}}^+(x,t) := Q_+^S(x,t) + Q_+^C(x,t) + Q_+^SC(x,t),$$

with

$$Q_+^S(x,t) = \frac{2i\Delta_n \sin(\gamma_n) \exp\left(\frac{i\lambda_n}{\lambda_0}\right) \exp\left(2i\Delta_n^2(2t\Delta_n^2 + l_n(t) \cos \gamma_n + \phi_n^\pm)\right)}{\cosh(\frac{x+x_n}{\Delta_n}) \sin(\gamma_n) l_n(t) - x_n},$$

$$\phi_n^\pm = -\frac{1}{2} \arg C_n + \arg \delta^\pm(\lambda_0; \lambda_0) + \sum_{l \in L^\pm} \arg \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right),$$

$$\hat{\phi}_n^\pm = -\ln(\Delta_n \sin \gamma_n) + \ln|C_n| - 2 \ln|\delta^\pm(\lambda_0; \lambda_0)| + 2 \sum_{l \in L^\pm} \ln \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right),$$

$$\delta^+(\lambda; z) = \exp\left\{ \int_0^z \frac{2 \ln(1 - r(\phi)^2)}{(\phi - \lambda)^2} \, d\phi \right\} - \exp\left\{ \int_0^z \frac{\ln(1 + r(\phi)^2)}{(\phi - \lambda)^2} \, d\phi \right\},$$

$$\delta^-(\lambda; z) = \exp\left\{ \int_z^0 \frac{2 \ln(1 - r(\phi)^2)}{(\phi - \lambda)^2} \, d\phi \right\} - \exp\left\{ \int_z^0 \frac{\ln(1 + r(\phi)^2)}{(\phi - \lambda)^2} \, d\phi \right\},$$

$$Q_+^C(x,t) = \sqrt{\frac{\nu(\lambda_0)}{2\lambda_0^3}} \exp\left\{ i\phi^\pm(\lambda_0) + \hat{\Phi}^\pm(\lambda_0; t) + \frac{\pi}{2}\right\},$$

$$\nu(\lambda) = -\frac{1}{\lambda} \ln(1 - |\lambda|^2),$$

$$\phi^+(\lambda) = \frac{1}{\pi} \int_0^\infty \ln|\phi'^2 + 2|d\ln(1 - |\phi|)^2 - \frac{1}{\pi} \int_0^\infty \ln|\phi'^2 + 2|d\ln(1 + |\phi|)^2),$$

$$\phi^- (\lambda_0; t) = 4\lambda_0^2 t \mp \nu(\lambda_0) \ln |t| \mp \arg \Gamma(i\nu(\lambda_0)) + \arg r(\lambda_0) \mp 3\nu(\lambda_0) \ln 2$$

$$+ (2 \pm 1)\frac{\pi}{2} + 2 \sum_{l \in L^\pm} \arg \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right),$$

$$Q_{\text{as}}^SC(x,t) = -\frac{4(\mp)^2 \pi^2 |\phi_n^\pm|}{\eta_n} \sqrt{\pm \nu(\lambda_0)} \exp\left\{ \exp(2i\Delta_n^2 \sin(\gamma_n) l_n(t)) \right\} \exp\left\{ 2 \sum_{l \in L^\pm} \ln \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right)\right\},$$

$$|g_n^\pm| = |C_n| |\delta^\pm(\lambda_0; \lambda_0)|^{-2} \exp\left\{ -2\Delta_n^2 \sin(\gamma_n) l_n(t)\right\} \exp\left\{ 2 \sum_{l \in L^\pm} \ln \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right)\right\},$$

$$g_n^\pm = \arg G_n^\pm = \arg C_n - 2 \arg \delta^\pm(\lambda_0; \lambda_0) + 2 \sum_{l \in L^\pm} \arg \left(\frac{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}{\lambda_0 - \lambda_0 ||\lambda_n + \lambda_0||}\right),$$

$$\pm^\pm = \exp\left(\frac{i\lambda_n}{\lambda_0}\right) \exp(2i\Delta_n^2 \sin(\gamma_n) l_n(t) - x_n^\pm)\frac{2 \cosh(\frac{x+x_n}{\Delta_n}) \sin(\gamma_n) l_n(t) - x_n^\pm)}{2 \cosh(\frac{x+x_n}{\Delta_n}) \sin(\gamma_n) l_n(t) - x_n^\pm)},$$

$$\varphi_n^\pm(\lambda_0; t) := \arg g_n^\pm + \phi^\pm(\lambda_0) + \hat{\Phi}^\pm(\lambda_0; t),$$

$$\sum_{l \in L^+} := \sum_{l=n+1}^{N}, \sum_{l \in L^-} := \sum_{l=0}^{n-1}, \Gamma(\cdot) \text{ is the gamma function \cite{15}, and } C(\lambda_0) \in S(\mathbb{R}_{>M}; \mathbb{C}),$$

and, as $t \to \pm \infty$ and $x \to \pm \infty$ such that $\mu_0 := \frac{1}{2} \sqrt{-\frac{i}{\pi}} > M$ and $(x,t) \in U_n := \{(x,t); x+ \frac{\pi}{2} \sqrt{-\frac{i}{\pi}} > M \}.$
4t\Delta_n^2 \cos \gamma_n := -l_n(t) = \mathcal{O}(1) \}, \text{ for those } \gamma_n \in (0, \frac{\pi}{2}),

Q(x, t) = Q_{as}^{\pm}(x, t) + \mathcal{O}\left(\frac{C(\mu_0) |\ln |t|\|^2}{t}\right),

\text{where}

Q_{as}^{\pm}(x, t) := Q_{S}^{\pm}(x, t) + Q_{S}^{\mp}(x, t) + Q_{S}^{\mp}(x, t),

\text{with}

Q_{S}^{\pm}(x, t) = \frac{2\Delta_n \sin(\gamma_n) \exp\left\{-\frac{\mu_0}{2\mu_0 t}\right\} \exp\left\{2i(\Delta_n^2 (2t\Delta_n^2 + l_n(t) \cos \gamma_n) + \widehat{\phi}_{s}^{\pm})\right\}}{\sinh(\Delta_n^2 + 2\Delta_n^2 \sin(\gamma_n) l_n(t) + \widehat{\phi}_{s}^{\pm})},

\widehat{\phi}_{s}^{\pm} = -\frac{1}{\pi} \arg C_n + \arg \delta_n^{\pm}(\overline{\gamma_n}; \mu_0) - \arg \sum_{l \in L_{\pm}} \left(\frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} (\lambda_n + \lambda_l)\right),

\widehat{x}_{n}^{\pm} = -\ln(\Delta_n \sin \gamma_n) + \ln|C_n| + 2 \ln|\delta_n^{\pm}(\overline{\gamma_n}; \mu_0)| + 2 \sum_{l \in L_{\pm}} \ln\left(\frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} (\lambda_n + \lambda_l)\right),

\delta_n^{\pm}(\lambda; z) = \exp\left\{\int_{z}^{\gamma_n} \frac{\phi_{n}^{\pm}(\mu_0)}{1 + r(|z|)^2} \rho \, d\rho \right\} - \exp\left\{\int_{z}^{\gamma_n} \frac{\phi_{n}^{\pm}(\mu_0)}{1 + r(|z|)^2} \rho \, d\rho \right\},

\phi_{n}^{\pm}(\mu_0; t) = 4\mu_0 t + \nu(i\mu_0) \ln|t| + \pi \arg \Gamma(i\nu(i\mu_0)) + \arg r(i\mu_0) + 3\nu(i\mu_0) \ln 2

- (2 + 1) \frac{\pi}{4} - 2 \sum_{l \in L_{\pm}} \arg\left(\frac{\mu_0 - \lambda_l}{\mu_0 - \lambda_l} (\lambda_n + \lambda_l)\right),

Q_{S}^{\mp}(x, t) = -\frac{4i(\pm)\gamma_n \exp\left\{i |\ln |t|\| \right\}}{\eta_n} \exp\left\{i \varphi_{n}^{\mp}(\mu_0; t)\right\} - 2i \cot(\gamma_n) \cos(\varphi_{n}^{\mp}(\mu_0; t)),

|g_{n}^{\pm}| = \left|C_n \delta_n^{\pm}(\overline{\gamma_n}; \mu_0)\right|^{-2} \exp\left\{2 \Delta_n^2 \sin(\gamma_n) l_n(t)\right\} \exp\left\{2 \sum_{l \in L_{\pm}} \ln\left(\frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} (\lambda_n + \lambda_l)\right)\right\},

\arg g_{n}^{\pm} = \arg C_n - 2 \arg \delta_n^{\pm}(\overline{\gamma_n}; \mu_0) - \arg \sum_{l \in L_{\pm}} \left(\frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} (\lambda_n + \lambda_l)\right)

- 2\Delta_n^2 (2t\Delta_n^2 + l_n(t) \cos \gamma_n),

\Xi^{\pm} = -\frac{\exp\left\{-\frac{\mu_0}{2\mu_0 t}\right\} \exp\left\{-2\Delta_n^2 \sin(\gamma_n) l_n(t) - \widehat{x}_{n}^{\pm}\right\}}{2 \sinh(\Delta_n^2 + 2\Delta_n^2 \sin(\gamma_n) l_n(t) + \widehat{x}_{n}^{\pm})},

\varphi_{n}^{\mp}(\mu_0; t) := \arg g_{n}^{\pm} + \phi_{n}^{\mp}(\mu_0; t) + \widehat{\phi}_{s}^{\pm}(\mu_0; t),

\text{and } C(\mu_0) \in S(\mathbb{R}_{> M}; C).

\textbf{Theorem 2.2} Let \(m(x, t; \lambda)\) be the solution of the RH problem formulated in Lemma 2.1 with the condition \(|r|\|_{C_{∞}(\mathbb{R}; \mathbb{C})} < 1\) and \(q(x, t)\), the solution of the DNLSE (Eq. (4)), be defined by Eq. (12) in terms of the function \(Q(x, t)\) given in Theorem 2.1. Then as \(t \to \pm\infty\) and \(x \to \pm\infty\) such that \(\lambda_0 := \frac{1}{2} \sqrt{-\frac{1}{\epsilon^2}} > M\) and \((x, t) \in \Omega_n := \{(x, t); x - 4t\Delta_n^2 \cos \gamma_n := l_n(t) = \mathcal{O}(1)\}, \text{ for those } \gamma_n \in (\frac{\pi}{2}, \pi),

q(x, t) = Q_{as}^{\mp}(x, t) \exp\{i \arg g_{n}^{\pm}(x, t)\} + \mathcal{O}\left(\frac{C(\lambda_0) |\ln |t|\|^2}{t}\right),

\text{where } Q_{as}^{\pm}(x, t) \text{ are given in Theorem 2.1, Eqs. (14)–(29),}

\arg g_{n}^{\pm}(x, t) = -4 \sum_{l \in L_{\pm}} \gamma_l + 4 \arctan(\eta_n |g_{n}^{\pm}|^{-2} + \cot \gamma_n) + \mathcal{V}_{\pm}(\lambda_0)
Theorem 2.3

Let \( m(x,t;\lambda) \) be the solution of the RH problem formulated in Lemma 2.1 with the condition \( ||r||_{L^\infty(\mathbb{R};\mathbb{C})} < 1 \) and \( u(x,t) \), the solution of the MNLSE (Eq. (2)), be defined by Eq. (10) in terms of the function \( q(x,t) \) given in Theorem 2.2. Then as \( t \to \pm\infty \) and \( x \to \pm\infty \) such that \( \tilde{x}_0 := \sqrt{2/3} > M, \tilde{\lambda}_0 > \tilde{x}_0^2, s \in \mathbb{R}_{>0}, \) and \( (x,t) \in \Omega := \{(x,t); -x + t(\frac{e^{-\Delta^2_n \cos \gamma_n}}{2}) \tilde{n}(t) = \mathcal{O}(1)\} \), for those \( \gamma_n \in \{\frac{\pi}{2}, \pi\}, \)

\[
\begin{align*}
u(x,t) &= v^{\pm}(x,t)w^{\pm}(x,t) + \mathcal{O}\left(\frac{C(\nu_n)(\ln|t|)^2}{t}\right),
\end{align*}
\]
\[ v_S^C(x, t) = \left\{ \begin{array}{ll} \sqrt{\Delta_n} \sin(\gamma_n) & \text{for } \gamma_n \neq 0, \\
 & \text{for } \gamma_n = 0, \\
 & \end{array} \right. \]

\[ \bar{\phi}_n = -\frac{1}{2} \arg C_n + \arg \delta^\pm(\overline{\lambda}_n; \overline{\mu}_n) - \sum_{l \in L_\pm} \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l}, \]

\[ \tilde{x}_n^\pm = -\ln(\Delta_n \sin(\gamma_n)) + \ln|C_n| + 2 \ln|\delta^\pm(\overline{\lambda}_n; \overline{\mu}_n)| + \sum_{l \in L_\pm} \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l}, \]

\[ u(x, t) = v_s^\pm(x, t)w_s^\pm(x, t) + \mathcal{O}\left( \left( \frac{C}{\mu^0} \right)^2 \right), \]

\[ v_s^\pm(x, t) = \frac{\sqrt{\Delta_n} \sin(\gamma_n) \exp\left\{ \frac{-i \phi^\pm(\overline{\mu}_n)}{2m} \right\} \exp\left\{ 2i(\Delta_n^2 + \lambda_n(t) \cos(\gamma_n)) + \bar{\phi}^\pm(\overline{\mu}_n, t) \right\}}{\sqrt{\sinh(\Delta_n^2 + 2 \Delta_n \sin(\gamma_n) \lambda_n(t) + \bar{x}_n^\pm)}}, \]

\[ \bar{\phi}^\pm(\overline{\mu}_n; t) = 2\mu_n t + \nu(i\overline{\mu}_n) \ln|t| \pm \arg \Gamma(\nu(i\overline{\mu}_n)) + \arg r(i\overline{\mu}_n) = 2\nu(i\overline{\mu}_n) \ln 2 \]

\[ - (2 + 1) \mathcal{P} - 2 \sum_{l \in L_\pm} \frac{(\overline{\mu}_0 - \overline{\lambda}_n)(\overline{\mu}_0 + \overline{\lambda}_n)}{(\overline{\mu}_0 - \overline{\lambda}_n)(\overline{\mu}_0 + \overline{\lambda}_n)}, \]

\[ |\bar{g}_n^\pm| = |C_n|\delta^\pm(\overline{\lambda}_n; \overline{\mu}_0)|^{-2} \exp\left\{ 2 \Delta_n^2 \sin(\gamma_n) \overline{\lambda}_n(t) \right\} \exp\left\{ 2 \sum_{l \in L_\pm} \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} \right\}. \]
arg \bar{g}_n^{\pm} = \arg C_n - 2 \arg \delta_0^+ (|\lambda_n|; \bar{\mu}_0) - 2 \sum_{l \in L_\pm} \arg \left( \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} \right)
- 2 \Delta_n^+ (t \Delta_n^+ + \tilde{\eta}_n(t) \cos \gamma_n),
\bar{\Theta}^{\pm}(\mu; \tilde{\mu}_0, t) := \arg \bar{g}^{\pm}_n + \delta^{\pm}(\tilde{\mu}_0) + \bar{\Theta}^{\pm}(\tilde{\mu}_0, t),
\lambda_n = 4 \arg \left( \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} \right) - 2 \sum_{l \in L_\pm} \arg \left( \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} \right) \delta_0^+ (|\lambda_n|; \bar{\mu}_0)
- 2 \Delta_n^+ (t \Delta_n^+ + \tilde{\eta}_n(t) \cos \gamma_n),
\bar{\Theta}^{\pm}(\mu; \tilde{\mu}_0, t) := \arg \bar{g}^{\pm}_n + \delta^{\pm}(\tilde{\mu}_0) + \bar{\Theta}^{\pm}(\tilde{\mu}_0, t),
\bar{\Theta}^{\pm}(\mu; \tilde{\mu}_0, t) := \bar{\Theta}^{\pm}(\tilde{\mu}_0, t) + \varphi^{\pm}(\tilde{\mu}_0) + \bar{\Theta}^{\pm}(\tilde{\mu}_0),
and \bar{C}(\mu) \in \mathcal{S}(\mathbb{R} \times \mathcal{M}; \mathbb{C}).

One possible application of the asymptotic results obtained in Theorems 2.1–2.3 is associated with the so-called “soliton scattering”, namely, the calculation of the position and phase shifts of the \(n\)th soliton (\(1 \leq n \leq N\)) for \(Q(x, t), q(x, t),\) and \(u(x, t)\) in the presence of the continuous spectrum: other physical applications of these asymptotic results include, for example, the calculation of the temporal and spectral intensities for the solutions of the DNLSE and MNLSE.

**Corollary 2.1**

(A) \(Q(x, t)\):

\[ \Delta x_n^{Q^S} := (2 \eta_n)^{-1} (\tilde{x}_n^+ - \tilde{x}_n^-) \]
\[ = \eta_n^{-1} \left\{ \sum_{l = 1}^{N} \text{sgn}(l - n) \ln \left( \frac{|\lambda_n - \lambda_l|}{|\lambda_n - \lambda_l|} \right) - \ln \left( \frac{|\delta^+ (\lambda_n; \lambda_l)|}{|\delta^- (\lambda_n; \lambda_l)|} \right) \right\}, \]
\[ \Delta \phi_n^{Q^S} := 2 (\tilde{\phi}_n^+ - \tilde{\phi}_n^-) \]
\[ = 2 \left\{ \sum_{l = 1}^{N} \text{sgn}(l - n) \arg \left( \frac{\lambda_n - \lambda_l}{\lambda_n - \lambda_l} \right) + \arg \left( \frac{\delta^+ (\lambda_n; \lambda_l)}{\delta^- (\lambda_n; \lambda_l)} \right) \right\}, \]

(B) \(q(x, t)\) (DNLSE):

\[ \Delta x_n^{q^S} = \Delta x_n^{Q^S}, \]
\[ \Delta \phi_n^{q^S} = \Delta \phi_n^{Q^S} - 4 \sum_{l = 1}^{N} \text{sgn}(l - n) \gamma_l + \gamma_+(\lambda_0) + \gamma_-(\lambda_0), \]
Theorems 2.1–2.3, Eqs. (16), (17), (33), (34), (48), (53), (60), (61), (69), (74), (75) and Remark 2.6

Proposition 3.1 (the transformations from the original RH problem to the model one is elucidated in the
Lemma 2.1, a simpler, model RH problem (see Lemma 3.3) is derived in this section. As an
In order to simplify the asymptotic analysis of the original RH problem formulated in
asymptotic expansions, while the remaining terms are exponentially small and negligible
sides of the asymptotic expansions become the leading-order terms of the corresponding
x, t
the remaining domains of the (x, t)-plane are obtained analogously. If the conditions on
Remark 2.5 The expressions for the soliton phase shifts given in Corollary 2.1, namely,

\[ \delta^+(\lambda;\omega) = \left(\frac{\lambda - \omega}{\lambda + \omega}\right)^\nu \exp\left\{ \sum_{\ell \in \{\pm\}} (\rho(\ell) + \tilde{\rho}(\ell)) \right\} \]

\[ \rho(\lambda) = \frac{1}{2\pi} \int_0^{\pm\lambda_0} \ln(1 - |r(t)|^2)\left(\frac{d\omega}{\omega - \lambda}\right) \]

\[ \tilde{\rho}(\lambda) = \int_{-\infty}^{0} \left\{ \ln(1 - |r(\omega)|^2) \right\} \left(\frac{d\omega}{\omega - \lambda}\right) \]

\[ \nu := \nu(\lambda_0) \text{ is given by Eq. } (21), \quad |(\delta^+(\cdot;\lambda_0))^{\pm 1}|_{L^\infty(\mathbb{C};\mathbb{C})} := \sup_{\lambda \in \mathbb{C}} |(\delta^+(\lambda;\lambda_0))^{\pm 1}| < \infty, \]

\[ \left(\delta^+(\pm\lambda_0)\right)^{-1} = \delta^+(\lambda;\omega), \text{ the principal branch of the logarithmic function is taken, } \ln(\mu - \lambda) := \ln|\mu - \lambda| + i \arg(\mu - \lambda), \right. \]

and \( C(\lambda_0) \in S(\mathbb{R} \setminus \{0\}; M_2(\mathbb{C})) \).

Proof. Follows from the definition of soliton position and phase shifts given in [2] and
Theorems 2.1–2.3, Eqs. (16), (17), (33), (34), (48), (53), (60), (61), (69), (74), (75) and

Remark 2.6 For the asymptotics of the \( \mathbb{C} \)-valued functions \( Q(x, t), q(x, t) \), and \( u(x, t) \),
one must actually consider four different cases, depending, respectively, on the quadrant of
the (x, t)-plane. In this paper, the proof of the asymptotic expansions for \( Q(x, t) \) and \( q(x, t) \)
(resp. \( u(x, t) \)) is presented for the cases \( (x, t) \to (\pm \infty, \pm \infty) \) (resp. \( (x, t) \to (\pm \infty, \pm \infty) \)) such
that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \) (resp. \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \)) for those \( \gamma_n \in \left(\frac{\pi}{2}, \pi\right) \); the results for
the remaining domains of the (x, t)-plane are obtained analogously. If the conditions on \( \gamma_n \)
stated in Theorems 2.1–2.3 are violated, then \( (x, t) \in \{\mathbb{R}^2 \setminus \Omega_n, \mathbb{R}^2 \setminus \Omega_n, \mathbb{R}^2 \setminus \Omega_n, \mathbb{R}^2 \setminus \Omega_n\} \), but
the asymptotic results stated still remain valid although the second terms on the right-hand
sides of the asymptotic expansions become the leading-order terms of the corresponding
asymptotic expansions, while the remaining terms are exponentially small and negligible
with respect to the given error estimations.

3 The Model RH Problem

In order to simplify the asymptotic analysis of the original RH problem formulated in
Lemma 2.1, a simpler, model RH problem (see Lemma 3.3) is derived in this section. As an
intermediate step towards the formulation of the model RH problem, it will be convenient
to derive an “extended” RH problem (see Lemma 3.2): the general idea pertaining to
the transformations from the original RH problem to the model one is elucidated in the
paragraph following Lemma 3.1 (see below).

Proposition 3.1 ([7]) In the solitonless sector \( (Z_d \equiv \emptyset) \), as \( t \to +\infty \) and \( x \to -\infty \) such
that \( \lambda_0 := \frac{1}{2\pi} \sqrt{-\frac{x}{t}} > M \),

\[ m(x, t; \lambda) = \Delta(\lambda) + O\left(\frac{C(\lambda_0)}{\sqrt{t}}\right) \]

where \( \Delta(\lambda) := (\delta^+(\lambda; \lambda_0))^\nu \)

\[ \delta^+(\lambda; \lambda_0) = \left(\frac{\lambda - \lambda_0}{\lambda + \lambda_0}\right)^\nu \exp\left\{ \sum_{\ell \in \{\pm\}} (\rho(\ell) + \tilde{\rho}(\ell)) \right\} \]

\[ \rho(\lambda) = \frac{1}{2\pi} \int_0^{\pm\lambda_0} \ln(1 - |r(\omega)|^2)\left(\frac{d\omega}{\omega - \lambda}\right) \]

\[ \tilde{\rho}(\lambda) = \int_{-\infty}^{0} \left\{ \ln(1 - |r(\omega)|^2) \right\} \left(\frac{d\omega}{\omega - \lambda}\right) \]

\[ \nu := \nu(\lambda_0) \text{ is given by Eq. } (21), \quad ||(\delta^+(\cdot;\lambda_0))^{\pm 1}||_{L^\infty(\mathbb{C};\mathbb{C})} := \sup_{\lambda \in \mathbb{C}} ||(\delta^+(\lambda;\lambda_0))^{\pm 1}|| < \infty, \]

\[ \left(\delta^+(\pm\lambda)\right)^{-1} = \delta^+(\lambda;\omega), \text{ the principal branch of the logarithmic function is taken, } \ln(\mu - \lambda) := \ln|\mu - \lambda| + i \arg(\mu - \lambda), \right. \]

and \( C(\lambda_0) \in S(\mathbb{R} \setminus \{0\}; M_2(\mathbb{C})) \).
Remark 3.1 For notational convenience, until the end of Sec. 5, all explicit $x, t$ dependencies are suppressed, except where absolutely necessary, and $\delta^+(\lambda; \lambda_0) := \delta(\lambda)$.

Lemma 3.1 There exists a unique solution $m^\Delta(\lambda) : \mathbb{C} \setminus (\mathbb{Z}_d \cup \hat{\Gamma}) \to SL(2, \mathbb{C})$ of the following RH problem,

1. $m^\Delta(\lambda)$ is meromorphic $\forall \lambda \in \mathbb{C} \setminus \hat{\Gamma}$,

where

$$m^\Delta_+(\lambda) = m^\Delta_-(\lambda) v^\Delta(\lambda), \quad \lambda \in \hat{\Gamma},$$

and $\theta(\lambda) := \lambda^2 x + 2 \lambda^4 t$.

2. $m^\Delta(\lambda)$ has simple poles at $\{\pm \lambda_i, \pm \lambda_i^1\}_{i=1}^N$ with $(1 \leq i \leq N)$

$$\text{res}(m^\Delta(\lambda); \lambda_i) = \lim_{\lambda \to \lambda_i} m^\Delta(\lambda) v_i(\delta(\lambda_i))^{-2} \sigma_i,$$

$$\text{res}(m^\Delta(\lambda); -\lambda_i) = -\sigma_i \text{res}(m^\Delta(\lambda); \lambda_i) \sigma_3,$$

$$\text{res}(m^\Delta(\lambda); \lambda_i^1) = \lim_{\lambda \to \lambda_i^1} m^\Delta(\lambda) \overline{v_i(\delta(\lambda_i^1))}^2 \sigma_3,$$

$$\text{res}(m^\Delta(\lambda); -\lambda_i^1) = -\sigma_i \text{res}(m^\Delta(\lambda); \lambda_i^1) \sigma_3,$$

3. as $\lambda \to \infty$, $\lambda \in \mathbb{C} \setminus (\mathbb{Z}_d \cup \hat{\Gamma})$

$$m^\Delta(\lambda) = I + \mathcal{O}(\lambda^{-1});$$

moreover, $Q(x, t) = 2i \lim_{\lambda \to \infty} (\lambda m^\Delta(x, t; \lambda))_{12}$ is equal to $Q(x, t)$ in Lemma 2.2, Eq. (11).

Proof. Let $m(\lambda)$ be the solution of the RH problem formulated in Lemma 2.1. Define $m^\Delta(\lambda) := m(\lambda) (\Delta(\lambda))^{-1}$. $\blacksquare$

In order to motivate Proposition 3.2 and Lemma 3.2 (see below), consider the trajectory of the $n$th soliton with $\gamma_n \in \left(\frac{\pi}{2}, \pi\right)$ in the $(x, t)$-plane which belongs to the set $\Omega_n := \{(x, t); x = 4t_0^2 \lambda_0 \cos \gamma_n = \mathcal{O}(1)\}$, and note from Lemma 2.1 and the soliton ordering in Sec. 2 that, as $t \to +\infty$ and $x \to -\infty$ such that $\lambda_0 > M$ and $(x, t) \in \Omega_n$: (1) $\mathcal{R}(v_i|_{\Omega_n}) \sim \mathcal{O}(\exp\{-8t\eta_i(\xi_i - \xi_n)\}) \to 0 \forall i \leq n$ ($i \in \{1, 2, \ldots, n-1\}$); (2) $\mathcal{R}(v_n|_{\Omega_n}) \to +\infty \forall i > n$ ($i \in \{n+1, n+2, \ldots, N\}$); and (3) $\mathcal{R}(v_i|_{\Omega_n}) \sim \mathcal{O}(1)$ for $i = n$. Thus, along the trajectory of the arbitrarily fixed $n$th soliton, there are exponentially growing polar conditions for solitons $i$ with $n+1 \leq i \leq N$.

One must effectively deal with such growing polar conditions in a self-consistent manner. In a recent paper [20] devoted to the asymptotics of the Toda rarefaction problem, Deift et al. showed how this could be done: they noticed that it is possible to replace the poles with the exponentially growing polar conditions by jump matrices on small, mutually disjoint (and disjoint with respect to $\hat{\Gamma}$) circles such that these jump matrices behave like $I +$ exponentially decreasing terms as $t \to +\infty$. Thus, instead of the original RH problem, one gets a new, (“extended”) RH problem with $4(N-n)$ fewer poles, and $4(N-n)$ additional circles with jump conditions stated on them. Finally, by removing the added circles from the specification of the extended RH problem, one arrives at the model RH problem: the estimation of the “difference” between the extended and model RH problems shows that the solution of the model RH problem approximates the solution of the original one modulo terms which are decaying exponentially as $t \to +\infty$.

Proposition 3.2 Introduce arbitrarily small, clockwise- and counter-clockwise-oriented, mutually disjoint (and disjoint with respect to $\hat{\Gamma}$) circles $K^+_j$ and $L^+_j$, $n+1 \leq j \leq N$, around
the eigenvalues \( \{\pm \lambda_j\}_{j=n+1}^N \) and \( \{\pm \lambda_j^\prime\}_{j=n+1}^N \), respectively, and define

\[
m^b(\lambda) := \begin{cases} 
m^A(\lambda), & \lambda \in \mathbb{C} \setminus (\hat{I} \cup (\bigcup_{i=n+1}^N (K_i^\pm \cup L_i^\pm))), \\
m^A(\lambda) \left( I - \frac{v_1(\delta(\pm \lambda_i)) - 2}{(\lambda \mp \lambda_i)} \sigma_- \right), & \lambda \in \text{int} K_i^\pm, \quad n + 1 \leq i \leq N, \\
m^A(\lambda) \left( I + \frac{v_1(\delta(\pm \lambda_i)) - 2}{(\lambda \mp \lambda_i)} \sigma_+ \right), & \lambda \in \text{int} L_i^\pm, \quad n + 1 \leq i \leq N.
\end{cases}
\] (85)

Then \( m^b(\lambda) \) solves a RH problem on \( (\sigma^- \cup \cup_{i=n+1}^N (\{\pm \lambda_i\} \cup \{\pm \lambda_i^\prime\}) \cup (\bigcup_{i=n+1}^N (K_i^\pm \cup L_i^\pm))) \) with the same jumps as \( m^A(\lambda) \) on \( \hat{I} \), \( m^b_+(\lambda) = m^b_-(\lambda) \nu(\lambda) \), and

\[
m^b_+(\lambda) := \begin{cases} 
m^b_-(\lambda) \left( I - \frac{v_1(\delta(\pm \lambda_i)) - 2}{(\lambda \mp \lambda_i)} \sigma_- \right), & \lambda \in K_i^+, \quad n + 1 \leq i \leq N, \\
m^b_-(\lambda) \left( I + \frac{v_1(\delta(\pm \lambda_i)) - 2}{(\lambda \mp \lambda_i)} \sigma_+ \right), & \lambda \in L_i^+, \quad n + 1 \leq i \leq N.
\end{cases}
\]

**Proof.** Follows from Lemma 3.1 and the definition of \( m^b(\lambda) \). \( \blacksquare \)

**Remark 3.2** The superscripts \( \pm \) on \( \{K_i^\pm\}_{i=n+1}^N \) and \( \{L_i^\pm\}_{i=n+1}^N \), which are related with \( \{\pm \lambda_i\}_{i=n+1}^N \) and \( \{\pm \lambda_i^\prime\}_{i=n+1}^N \), respectively, should not be confused with the subscripts \( \pm \) appearing in the various RH problems in Secs. 3–5, namely, \( m_{\pm}(\lambda), m_{\pm}^A(\lambda), m_{\pm}^b(\lambda), m_{\pm}^b(\lambda), \chi_{\pm}(\lambda), E_{\pm}(\lambda), \) and \( \chi_{\pm}^\prime(\lambda) \).

**Remark 3.3** Even though the exponentially growing polar (residue) conditions have been replaced by jump matrices, it should be noted that, along the trajectory of soliton \( n \), these jump matrices are also exponentially growing as \( t \to +\infty \). These lower/upper diagonal, exponentially growing jump matrices are now replaced, through a sequence of \( N-n \) similar transformations, by upper/lower diagonal jump matrices which converge, along the trajectory of soliton \( n \), to \( I \) as \( t \to +\infty \).

**Lemma 3.2** Set

\[
m^b(\lambda) := \prod_{l=i+1}^N (d_{i+l}(\lambda))^{-\sigma_3}, \quad \lambda \in \mathbb{C} \setminus (\hat{I} \cup (\bigcup_{i=n+1}^N (K_i^\pm \cup L_i^\pm))),
\]

\[
m^b_+(\lambda) := \prod_{l=i+1}^N (d_{i-l}(\lambda))^{-\sigma_3}, \quad \lambda \in \text{int} K_i^\pm, \quad n + 1 \leq i \leq N,
\]

\[
m^b_-(\lambda) := \prod_{l=i+1}^N (d_{i+l}(\lambda))^{-\sigma_3}, \quad \lambda \in \text{int} L_i^\pm, \quad n + 1 \leq i \leq N,
\]

where

\[
d_{i+}(\lambda) := \frac{(\lambda - \lambda_i)(\lambda + \lambda_i)}{(\lambda - \lambda_i)(\lambda + \lambda_i)}, \quad \lambda \in \mathbb{C} \setminus (\bigcup_{i=n+1}^N (K_i^\pm \cup L_i^\pm)), \quad n + 1 \leq l \leq N,
\]

\[
d_{i-}(\lambda) := \left\{ \begin{array}{l} \frac{(\lambda - \lambda_i)(\lambda + \lambda_i)}{(\lambda - \lambda_i)(\lambda + \lambda_i)}, \quad \lambda \in \bigcup_{i=n+1}^N \text{int} K_i^\pm, \quad n + 1 \leq l \leq N, \\
(\lambda - \lambda_i)(\lambda + \lambda_i), \quad \lambda \in \bigcup_{i=n+1}^N \text{int} L_i^\pm, \quad n + 1 \leq l \leq N,
\end{array} \right.
\]

and the \( SL(2,\mathbb{C}) \)-valued, holomorphic in \( \text{int} K_i^\pm \) and \( \text{int} L_i^\pm \), respectively, functions \( J_{K_i^\pm}(\lambda) \) and \( J_{L_i^\pm}(\lambda), n+1 \leq i \leq N \), are given by

\[
J_{K_i^\pm}(\lambda) = \left( \begin{array}{cc} \prod_{l=i+1}^N \frac{d_{l-1}(\lambda)}{d_{l+1}(\lambda)} & -v_1(\delta(\pm \lambda_i))^{-2} \prod_{l=i+1}^N \frac{d_{l-1}(\lambda)}{d_{l+1}(\lambda)} \\
-v_1(\delta(\pm \lambda_i))^{-2} \prod_{l=i+1}^N \frac{d_{l-1}(\lambda)}{d_{l+1}(\lambda)} & \prod_{l=i+1}^N \frac{d_{l-1}(\lambda)}{d_{l+1}(\lambda)} \end{array} \right).
\]
\[
J_{l_+^i}^\pm(\lambda) = \left( (\lambda \mp \lambda_i) \prod_{l \neq i}^{N} \frac{d_{l^-}^{-1}(\lambda)}{d_{l^+}^{-1}(\lambda)} \right) \frac{v_i(\delta(\pm \lambda_i))^{2}}{\prod_{l \neq i}^{N} \frac{d_{l^-}^{-1}(\lambda)}{d_{l^+}^{-1}(\lambda)}} \left( \prod_{l=n+1}^{N} \frac{d_{l^+}^{-1}(\lambda)}{d_{l^-}^{-1}(\lambda)} - \frac{v_i(\delta(\pm \lambda_i))^{2}}{\prod_{l \neq i}^{N} \frac{d_{l^+}^{-1}(\lambda)}{d_{l^-}^{-1}(\lambda)}} \right),
\]

with

\[
C_i^\pm = (v_i)^{-1}(\delta(\pm \lambda_i))^{2} \prod_{l \neq i}^{N} (d_{l^+}(\pm \lambda_i)) \prod_{l=n+1}^{N} (d_{l^+}(\pm \lambda_i)), \quad n+1 \leq i \leq N.
\]

Then \(m_\pm^x(\lambda) : \mathbb{C} \setminus \left( \bigcup_{i=n+1}^{N} (\{\pm \lambda_i\} \cup \{\pm \lambda_i^\pm\}) \right) \cup (\bigcup_{i=n+1}^{N} (K_i^\pm \cup L_i^\pm)) \rightarrow \text{SL}(2, \mathbb{C})\) solves the following, extended RH problem on \((\sigma_x \setminus \bigcup_{i=n+1}^{N} (\{\pm \lambda_i\} \cup \{\pm \lambda_i^\pm\}) \cup (\bigcup_{i=n+1}^{N} (K_i^\pm \cup L_i^\pm))),\]

\[
m_\pm^x(\lambda) = m_\pm(\lambda)e^{-i\theta(\lambda)\text{ad}(\lambda)}v_\pm(\lambda),
\]

where

\[
v_\pm(\lambda) = \begin{cases}
1 + \frac{(v_i)^{-1}(\delta(\pm \lambda_i))^{2}}{(\lambda \mp \lambda_i)} \cdot \frac{(\lambda - \lambda_i)}{\lambda_i^\pm} \cdot \frac{2}{\lambda_i^\pm - \lambda_i} \cdot \prod_{l \neq i}^{N} \frac{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)}{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)} \cdot \sigma_+ & , \quad \lambda \in \bigcup_{i=n+1}^{N} K_i^\pm,
1 + \frac{(v_i)^{-1}(\delta(\pm \lambda_i))^{2}}{(\lambda \mp \lambda_i)} \cdot \frac{(\lambda - \lambda_i)}{\lambda_i^\mp} \cdot \frac{2}{\lambda_i^\mp - \lambda_i} \cdot \prod_{l \neq i}^{N} \frac{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)}{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)} \cdot \sigma_- & , \quad \lambda \in \bigcup_{i=n+1}^{N} L_i^\pm,
\end{cases}
\]

with polar (residue) conditions,

\[
\begin{align*}
\text{res}(m_\pm^x(\lambda); \lambda_i) &= \lim_{\lambda \to \lambda_i} m_\pm(\lambda)v_i(\delta(\lambda_i))^{-2} \prod_{l \neq i}^{N} \frac{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)}{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)} \cdot \sigma_+, \quad 1 \leq i \leq n, \\
\text{res}(m_\pm(\lambda); -\lambda_i) &= -\sigma_3 \text{res}(m_\pm(\lambda); \lambda_i)\sigma_3, \quad 1 \leq i \leq n, \\
\text{res}(m_\pm^x(\lambda); \lambda_i) &= \lim_{\lambda \to \lambda_i^\pm} m_\pm^x(\lambda)v_i(\delta(\lambda_i))^{-2} \prod_{l \neq i}^{N} \frac{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)}{(\lambda - \lambda_i)(\lambda_i^\pm + \lambda_i^+)} \cdot \sigma_+, \quad 1 \leq i \leq n, \\
\text{res}(m_\pm(\lambda); -\lambda_i) &= -\sigma_3 \text{res}(m_\pm(\lambda); \lambda_i^\pm)\sigma_3, \quad 1 \leq i \leq n,
\end{align*}
\]

and, as \(\lambda \to \infty, \lambda \in \mathbb{C} \setminus \left( \bigcup_{i=n+1}^{N} (\{\pm \lambda_i\} \cup \{\pm \lambda_i^\pm\}) \cup (\bigcup_{i=n+1}^{N} (K_i^\pm \cup L_i^\pm)) \right),\]

\[
m_\pm^x(\lambda) = \mathcal{I} + \mathcal{O}(\lambda^{-1});
\]

moreover, \(Q(x, t) = 2i \lim_{\lambda \to \infty} (\lambda m_\pm^x(x, t; \lambda))_{12} \) is equal to \(Q(x, t) \) in Lemma 2.2, Eq. (11).

**Proof.** The proof is presented for the eigenvalues \(\lambda_i \}_{i=n+1}^{N}\), around which are defined the small, clockwise-oriented, mutually disjoint circles \(K_i^\pm \}_{i=n+1}^{N}\) the proof for the eigenvalues \(-\lambda_i \}_{i=n+1}^{N}\) follows in an analogous manner. From the definition of \(m_\pm^x(\lambda)\) and Proposition 3.2, one sees that, on \(\{K_i^\pm \}_{i=n+1}^{N}\), \(m_\pm^x(\lambda)\) solves the following RH problem \((\lambda \in \bigcup_{i=n+1}^{N} K_i^\pm),\)

\[
m_\pm^x(\lambda) = m_\pm^x(\lambda) \prod_{l=n+1}^{N} (d_{l^+}(\lambda))^{\sigma_3} J_{K_i^\pm}(\lambda) \left( 1 + \frac{v_i(\delta(\lambda_i))^{2}}{(\lambda - \lambda_i)} \right) \prod_{l=n+1}^{N} (d_{l^+}(\lambda))^{-\sigma_3}.
\]
Demanding that the above “jump matrix” be equal to the following upper triangular form,
\[
J_{K_i}^+(\lambda) = \begin{pmatrix}
  \frac{d_{-i}(\lambda)}{d_{+i}(\lambda)} & \frac{N}{(\lambda - \lambda_i)^2} \sum_{l=n+1}^N \frac{d_{-i}(\lambda)}{d_{+i}(\lambda)} d_{-i}(\lambda) d_{+i}(\lambda) & \frac{N}{(\lambda - \lambda_i)^2} \sum_{l=n+1}^N \frac{d_{-i}(\lambda)}{d_{+i}(\lambda)} d_{+i}(\lambda) \\
  \frac{N}{(\lambda - \lambda_i)^2} \sum_{l=n+1}^N \frac{d_{+i}(\lambda)}{d_{-i}(\lambda)} d_{-i}(\lambda) d_{+i}(\lambda) & 0 & \frac{N}{(\lambda - \lambda_i)^2} \sum_{l=n+1}^N \frac{d_{+i}(\lambda)}{d_{-i}(\lambda)} d_{+i}(\lambda) \\
  0 & 0 & \frac{N}{(\lambda - \lambda_i)^2} \sum_{l=n+1}^N \frac{d_{+i}(\lambda)}{d_{-i}(\lambda)} d_{+i}(\lambda)
\end{pmatrix}
\]

Note that \(\det(J_{K_i}^+(\lambda)) = 0\) \((n+1 \leq i \leq N)\). Defining, for \(n+1 \leq l \leq N\), \(d_{+i}(\lambda)\) and \(d_{-i}(\lambda)\) as in Eqs. (87), and choosing \(C_i^\pm\), \(n+1 \leq i \leq N\), as in Eq. (88) (with \(+\lambda_i\)), one gets the expression for \(J_{K_i}^+(\lambda)\) (which is holomorphic \(\forall \lambda \in \bigcup_{i=n+1}^N \text{int}\{K_i^+\}\) given in the Lemma; also, because of the symmetry properties of \(\delta(\lambda)\) (Proposition 3.1), \(C_i^\pm = (v_i)^{-1}(\delta(\pm \lambda_i))^{-2} \cdot (d_{-i}(\pm \lambda_i))^{-2} \prod_{l=n+1}^N (d_{+i}(\pm \lambda_i))^{-2}\). The remainder of the proof is a consequence of Lemma 3.1, Proposition 3.2, and the definition of \(m^2(\lambda)\).

Remark 3.4 Even though, along the trajectory of soliton \(n\), all the initial, exponentially growing nilpotent residue matrices have been replaced by jump matrices which tend to \(I\) as \(t \to \pm \infty\), i.e., \(\exists \varepsilon \in \mathbb{R}_{\geq 0}\) such that \(\forall i \in \{n+1, n+2, \ldots, N\}\), \(|v_i|_{\Omega_n}^{-1} \sim O(\exp\{-ct\})\), it does not necessarily follow that elements in the solution of the extended RH problem for \(m^2(\lambda)\) cannot grow exponentially; for example, note that the \((21)\)-elements of \(J_{K_i}^+(\lambda)\) and the \((12)\)-elements of \(J_{L_i}^+(\lambda)\), \(n+1 \leq i \leq N\), grow exponentially.

By estimating the error, along the trajectory of soliton \(n\) \((1 \leq n \leq N)\), when the jump matrices on \(\{K_i^+, L_i^\pm\}_{i=n+1}^N\) are removed from the specification of the RH problem for \(m^2(\lambda)\), one gets the following—asymptotically solvable—model RH problem.

Lemma 3.3 Let \(\chi(\lambda)\) solve the following RH problem on \(\tilde{\Omega} \setminus \bigcup_{i=n+1}^N \{\{\pm \lambda_i\} \cup \{\pm \lambda_i\}\}\),
\[
\chi_+(\lambda) = \chi_-(\lambda) e^{-i\theta(\lambda) \lambda d(\sigma_3)\lambda^N} \left|_{\tilde{\Omega}}\right.; \quad \lambda \in \tilde{\Omega};
\]
with polar (residue) conditions,
\[
\text{res}(\chi(\lambda); \lambda_i) = \lim_{\lambda \to \lambda_i} \chi(\lambda) v_i(\delta(\lambda))^{-2} \sum_{l=n+1}^N \left(\frac{(\lambda_i - \lambda)(\lambda_i + \lambda_l)}{(\lambda_i - \lambda_l)(\lambda_i + \lambda_l)}\right)^2 \sigma_-, \quad 1 \leq i \leq n,
\]
\[
\text{res}(\chi(\lambda); -\lambda_i) = -\sigma_3 \text{res}(\chi(\lambda); \lambda_i) \sigma_3, \quad 1 \leq i \leq n,
\]
\[
\text{res}(\chi(\lambda); -\lambda_i) = \lim_{\lambda \to -\lambda_i} \chi(\lambda) \text{res}(\delta(\lambda))^{-2} \sum_{l=n+1}^N \left(\frac{(\lambda_i - \lambda)(\lambda_i + \lambda_l)}{(\lambda_i - \lambda_l)(\lambda_i + \lambda_l)}\right)^2 \sigma_+, \quad 1 \leq i \leq n,
\]
\[
\text{res}(\chi(\lambda); -\lambda_i) = -\sigma_3 \text{res}(\chi(\lambda); -\lambda_i) \sigma_3, \quad 1 \leq i \leq n,
\]
and, as \(\lambda \to \infty\), \(\lambda \in \mathbb{C} \setminus \bigcup_{i=1}^N \{\{\pm \lambda_i\} \cup \{\pm \lambda_i\}\}\),
\[
\chi(\lambda) = I + O(\lambda^{-1}).
\]

Then as \(t \to +\infty\) and \(x \to -\infty\) such that \(\lambda_0 > M\) and \((x, t) \in \Omega_n\), the function \(E(\lambda) := m^2(\lambda)(\chi(\lambda))^{-1}\) has the following asymptotics,
\[
E(\lambda) = I + O(F(\lambda; \lambda_0) \exp\{-bt\}),
\]
where \(\|F(\cdot; \lambda_0)\|_{L^\infty(C; M^2(\mathbb{C}))} < \infty\), \(\|F(\cdot)\|_{L^\infty(R > M; M^2(\mathbb{C}))} < \infty\), \(F(\lambda; \lambda_0) \sim O\left(C(\lambda_0)\frac{\lambda}{\lambda}ight)\) as \(\lambda \to \infty\) with \(C(\lambda_0) \in L^\infty(R > M; M^2(\mathbb{C}))\), \(a := \min\{\{\eta_i\}_{i=n+1}^N\} > 0\), and \(b := \min\{|\xi_n - \xi_i|\}_{i=n+1}^N\).
Proof. Writing, for \( n+1 \leq i \leq N \), Eqs. (89) in the following form,
\[
v^\sharp(\lambda) := \begin{cases} 
I + (v_i)^{-1}\hat{W}_{K_i}^\pm(\lambda)\sigma_+ & \lambda \in \bigcup_{i=n+1}^N K_i^\pm, \\
I + (\overline{v_i})^{-1}\hat{W}_{L_i}^\pm(\lambda)\sigma_- & \lambda \in \bigcup_{i=n+1}^N L_i^\pm,
\end{cases}
\]
consider the “error function” \( E(\lambda) \) defined in the Lemma. One notes that: (1) \( \det(E(\lambda)) = 1 \); (2) \( E(\lambda) \) has no poles; and (3) \( E(\lambda) \) solves the following RH problem on the oriented contour \( \Sigma_E := \bigcup_{i=n+1}^N (K_i^\pm \cup K_i^- \cup L_i^+ \cup L_i^-) \),
\[
E_+(\lambda) = E_-(\lambda) \begin{pmatrix} 1 + (v_i)^{-1}\hat{W}_{K_i}^\pm(\lambda) & \left( -\chi_{11}(\lambda)\chi_{21}(\lambda) - (\chi_{11}(\lambda))^2 \chi_{21}(\lambda) \right) \\
I + (\overline{v_i})^{-1}\hat{W}_{L_i}^\pm(\lambda) & \left( \chi_{12}(\lambda)\chi_{22}(\lambda) - (\chi_{12}(\lambda))^2 \chi_{22}(\lambda) \right) \end{pmatrix}, \quad \lambda \in K_i^-, \\
E_+(\lambda) = E_-(\lambda) \begin{pmatrix} 1 + (v_i)^{-1}\hat{W}_{K_i}^\pm(\lambda) & \left( \chi_{11}(\lambda)\chi_{21}(\lambda) - (\chi_{11}(\lambda))^2 \chi_{21}(\lambda) \right) \\
I + (\overline{v_i})^{-1}\hat{W}_{L_i}^\pm(\lambda) & \left( -\chi_{12}(\lambda)\chi_{22}(\lambda) - (\chi_{12}(\lambda))^2 \chi_{22}(\lambda) \right) \end{pmatrix}, \quad \lambda \in L_i^+,
\]
\[n+1 \leq i \leq N, \text{ and, as } \lambda \to \infty, \lambda \in \mathbb{C} \setminus \Sigma_E, E(\lambda) = I + O(\lambda^{-1}).\]
Now, writing the RH problem for \( E(\lambda) \) on the oriented contour \( \Sigma_E \) in terms of an equivalent system of linear singular integral equations, using the explicit asymptotic solution of the model RH problem for \( \chi(\lambda) \) given in Sec. 4, recalling that, as \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \), \((v_i)_{\Omega_n})^{-1} \sim \mathcal{O}(\exp\{-8\pi n|\xi_n - \xi_i|\}) \), \( n+1 \leq i \leq N \), and proceeding as in the proof of Lemma 3.3 in [14], one deduces the estimate in Eq. (90).

\section{4 Asymptotic Solution of the Model RH Problem}

In this section, the asymptotic (as \( t \to +\infty \) and \( x/t \sim \mathcal{O}(1) \)) solution of the model RH problem (Lemma 3.3) for the Schwartz class of non-reflectionless generic potentials \((r(\lambda) \in \mathcal{S}(\hat{\Gamma}; \mathbb{C})\)) is presented. Before doing so, however, recall the following well-known fact from matrix RH theory [11, 13].

\textbf{Proposition 4.1} The solution of the model RH problem (Lemma 3.3), \( \chi(\lambda): \mathbb{C} \setminus (\hat{\Gamma} \cup (\cup_{i=1}^n (\{\pm \lambda_i\} \cup \{\pm \overline{\lambda_i}\})) \to \text{SL}(2, \mathbb{C}) \), has the following representation,
\[
\chi(\lambda) = \chi_d(\lambda) + \int_{\hat{\Gamma}} \frac{\chi_{-}(\lambda)(v^\sharp(\lambda)|_{\hat{\Gamma}} - I)}{(\lambda - \lambda)} d\lambda, \tag{91}
\]
where
\[
\chi_d(\lambda) = 1 + \sum_{i=1}^n \left( \frac{\text{res}(\chi(\lambda); \lambda_i)}{\lambda - \lambda_i} - \frac{\sigma_3\text{res}(\chi(\lambda); \lambda_i)\sigma_3}{\lambda + \lambda_i} + \frac{\text{res}(\chi(\lambda); \overline{\lambda_i})}{\lambda - \overline{\lambda_i}} - \frac{\sigma_3\text{res}(\chi(\lambda); \overline{\lambda_i})\sigma_3}{\lambda + \overline{\lambda_i}} \right). \tag{92}
\]
The solution of Eq. (91) can be written as the following ordered product,
\[
\chi(\lambda) = \chi_d(\lambda)\chi^c(\lambda), \tag{93}
\]
where \( \chi_d(\lambda) \) is given by Eq. (92), and \( \chi^c(\lambda) \) solves the following RH problem: (1) \( \chi^c(\lambda) \) is piecewise holomorphic \( \forall \lambda \in \mathbb{C} \setminus \hat{\Gamma} \); (2) \( \chi^c_+(\lambda) = \chi^c_-(\lambda) \exp\{-i\theta(\lambda)\text{ad}(\sigma_3)\}(v^\sharp(\lambda)|_{\hat{\Gamma}}) \), \( \lambda \in \hat{\Gamma} \); and (3) as \( \lambda \to \infty, \lambda \in \mathbb{C} \setminus \hat{\Gamma}, \chi^c(\lambda) = I + \mathcal{O}(\lambda^{-1}).\)

\textbf{Remark 4.1} From Proposition 4.1, Eq. (93), it is seen that, in order to solve the model RH problem, explicit knowledge of \( \chi_d(\lambda) \) and \( \chi^c(\lambda) \) is necessary. The determination of \( \chi^c(\lambda) \) is technically more complicated than that of \( \chi_d(\lambda) \); actually, the determination of \( \chi_d(\lambda) \) depends on the explicit knowledge of \( \chi^c(\lambda) \) (see Proposition 4.2); hence, the asymptotic solution of \( \chi^c(\lambda) \) is presented first (see Lemma 4.1).

In order to more fully comprehend certain elements of the proof of Lemma 4.1 given below, the Beals-Coifman [13] formulation for the solution of a (matrix) RH problem on an oriented contour is requisite: a self-contained synopsis of this formulation as it applies to the solution of the RH problem for \( \chi^c(\lambda) \) stated in Proposition 4.1 now follows.
Writing the jump matrix in the following factorized form, 
\[ v^t(\lambda)_{I_\Gamma} := (I - w^{-}_{x,t}(\lambda))^{-1}(I + w^{+}_{x,t}(\lambda)), \lambda \in \bar{\Gamma}, \]
where \( w^{\pm}_{x,t}(\lambda) \in \cap_{k \in \{2, \infty\}} L^k(\bar{\Gamma}; M_2(\mathbb{C})) \) (with \( ||w^{\pm}_{x,t}(\lambda)||_{k \in \{2, \infty\}} L^k(\bar{\Gamma}; M_2(\mathbb{C})) := \sum_{k \in \{2, \infty\}} ||w^{\pm}_{x,t}(\lambda)||_{L^k(\bar{\Gamma}; M_2(\mathbb{C}))} \)), respectively, are nilpotent off-diagonal upper/lower triangular matrices, define \( w_{x,t}(\lambda) := w^{-}_{x,t}(\lambda) + w^{+}_{x,t}(\lambda) \), and introduce the operator \( C_{w_{x,t}} \) on \( L^2(\bar{\Gamma}; M_2(\mathbb{C})) \) as \( C_{w_{x,t}} f := C_{+}(f w^{-}_{x,t}) + C_{-}(f w^{+}_{x,t}) \), where \( f \in L^2(\bar{\Gamma}; M_2(\mathbb{C})) \), and \( C_{\pm} : L^2(\bar{\Gamma}; M_2(\mathbb{C})) \rightarrow L^2(\bar{\Gamma}; M_2(\mathbb{C})) \) denote the Cauchy operators, \( (C_{\pm} f)(\lambda) := \lim_{\lambda^+ \in \text{side of} \ \Gamma} \int_{\Gamma} f(\lambda) \frac{d\rho}{(\rho - \lambda)^2}. \)

**Theorem 4.1** ([3]) If \( \mu^c(\lambda) \in I \oplus L^2(\bar{\Gamma}; M_2(\mathbb{C})) \) solves the following linear singular integral equation,

\[
\text{Id} - C_{w_{x,t}} \mu^c = I,
\]
where \( \text{Id} \) is the identity operator on \( I \oplus L^2(\bar{\Gamma}; M_2(\mathbb{C})) \), then the solution of the RH problem for \( \chi^c(\lambda) \) is

\[
\chi^c(\lambda) = I + \int_{\Gamma} \mu^c(\theta) w_{x,t}(\theta) \frac{d\theta}{(\theta - \lambda)} \frac{2\pi i}{\lambda - \lambda_{0}} \left( \exp\{ -i(\phi^+(\lambda_0) + \hat{\Phi}^+(\lambda_0; t)) \} \right) \sigma_+ + O\left( \frac{G(\lambda; \lambda_0) \ln t}{t} \right),
\]

where \( \nu(\lambda_0), \phi^+(\lambda_0), \) and \( \hat{\Phi}^+(\lambda_0; t) \) are given in Theorem 2.1, Eqs. (21), (22), and (24), \( \|G(\cdot; \lambda_0)\|_{L^\infty(C \cup \{0, \pm \lambda_0\} N(\mathbb{R}; \epsilon_0); M_2(\mathbb{C}))} < \infty, G(\lambda; \cdot) \in S(\mathbb{R} \setminus M; M_2(\mathbb{C})), G(\lambda; \lambda_0) \sim O\left( \frac{C(\lambda_0)}{\lambda} \right) \) as \( \lambda \rightarrow \infty \) with \( C(\lambda_0) \in S(\mathbb{R} \setminus M; M_2(\mathbb{C})) \), and satisfies the following involutions, \( \chi^c(-\lambda) = -\sigma_3 \chi^c(\lambda) \sigma_3 \) and \( \chi^c(\lambda) = \sigma_1 \chi^c(\lambda) \sigma_1 \).

**Proof.** In Secs. 5 and 6 of [3], it was shown that, for \( \lambda \in C \cup \{0, \pm \lambda_0\} N(\mathbb{R}; \epsilon_0) \), as \( t \rightarrow +\infty \) and \( x \rightarrow -\infty \) such that \( \lambda_0 > M \), for arbitrary \( t' \in \mathbb{Z}_{\geq 1} \),

\[
\chi^c(\lambda) = I + \sum_{m_3 \in M_{\mathbb{Z}_2}} \frac{b(m_3)}{m_3} \int_{\mathbb{R}} \frac{m_2 \mu^c(\delta(\varphi))_{\text{sgn}(m_2)}(\delta(\varphi))^{2m_2} e^{-2im_2\theta(\varphi)} R(m_2)_{\text{sgn}(m_2)} \sigma_{\text{sgn}(m_2)} d\varphi}{(\lambda - \lambda_{0})} + O\left( \frac{a(\lambda_0)}{(\lambda_0 t')^3} \right),
\]

where: (1) \( m_k \in M_{\mathbb{Z}_2} \) denotes the set of vectors with \( k \) components each of which take the values \( \pm 1 \), \( \text{card}(M_{\mathbb{Z}_2}) = 2^k \); (2) \( a(m_3) = m_1 \lambda_0 + m_2 \exp\left( -\frac{i\pi m_3}{4} \right) \), \( a(m_3) = m_1 \lambda_0 \), and \( b(m_2) = m_1 \exp\left( -\frac{i\pi m_2}{4} \right) \), where \( \varepsilon \) is an arbitrarily fixed, sufficiently small positive real number; (3) \( \mu^c(s) := \mu^c(s)_{L'}, \mu^c(s)_{-} := \mu^c(s)_{L''} \), with \( L' = \{ \lambda; \lambda = \hat{\lambda}(\varphi) \} \), \( \hat{u} \in (\varepsilon, \bar{u}) \} \cup (u_{\in} \{\pm 1\} \{ \lambda = \lambda_0' + \hat{u} \exp\left( -\frac{i\pi}{4} \right), \hat{u} \in (\varepsilon, \bar{u}) \}) \), and \( \mu^c(\cdot) \) is the solution of the Beals-Coifman [3] linear singular integral equation (Theorem 4.1); (4) \( \theta(\varphi) := 2\pi^2 (\lambda - 2\lambda_0); \) (5) \( \delta(\varphi) = ((\lambda - \lambda_0)(\lambda - \lambda_0'))^{\theta(\varphi)} \exp\left( \sum_{l \in \{\pm 1\}}(\nu_l(\varphi) + \hat{\nu}_l(\varphi)) \right), \nu := \nu(\lambda_0), \rho_{\pm}(\varphi) = \frac{1}{2\pi i} \int_{\mathbb{R}} \psi_{\lambda_0}(\lambda) \ln(1 - (x_0 - \lambda)_{\lambda_0} d\lambda) \rho_{\pm}(\lambda) \left( \frac{1}{(\lambda - \lambda_0)_{\lambda_0}} \right), \) and \( \hat{\rho}_{\pm}(\lambda) = \int_{0}^{1} \int_{0}^{1} \frac{1}{(x_0 - \lambda)_{\lambda_0}} d\lambda \rho_{\pm}(\lambda) \left( \frac{1}{(\lambda - \lambda_0)_{\lambda_0}} \right), \) \( \mathcal{R}^{-1,-1}(\varphi) = \mathcal{R}^{-1,1}(\varphi) = \mathcal{R}^{1,-1}(\varphi) = \mathcal{R}^{1,1}(\varphi) = (\mathcal{R}^{1,1}(\varphi)^*)^\dagger. \)
and $\mathcal{R}^{-1,1,-1}(\zeta) = \mathcal{R}_{-1,1,-1}(\zeta) = (\mathcal{R}^{1,1,1}(\zeta))^*$, where $\mathcal{P}(z) := \prod_{l=0}^{N} \frac{\lambda(z+i\lambda_0)}{(z+i\lambda_0)(z+i\lambda_d)}$, and $\alpha(\cdot) = (\beta(\cdot))^*$ means that $\alpha(\cdot)$ is the same piecewise-rational function as $\beta(\cdot)$ except with the complex conjugated coefficients; (7) $\mathcal{R}_{1,1,1}(\zeta) = \mathcal{R}^{-1,1,1}(\zeta) = \frac{r(\zeta)P(\zeta)}{(1-|\zeta|)P} - \frac{r(\zeta)P(\zeta)}{(1+|\zeta|)^2}$ and $\mathcal{R}_{-1,1,-1}(\zeta) = \mathcal{R}^{1,1,1}(\zeta) = (\mathcal{R}^{1,1,1}(\zeta))^*$; and (8) $\mathcal{A}(\lambda) \in L_\infty(\mathbb{R}_{>0}; M_2(\mathbb{C}))$. Since, in the above expression for $\chi(\lambda)$, the estimation of all the integrals is analogous, without loss of generality, the following integral is considered,

$$I_0 := \int_{\lambda_0 + \epsilon e^{-\frac{t}{4}}}^{\lambda_0} A_0(\zeta) B_0(\zeta) \frac{d\zeta}{2\pi i},$$

where $A_0(\zeta) := \mu^\epsilon(\zeta) |L|^t$, and $B_0(\zeta) := -\alpha(\zeta) \exp\left\{-2i\theta(\zeta)r(\zeta)\right\} \sigma_+$. Begin by estimating $B_0(\zeta)$ \cdot \frac{d\zeta}{2\pi i}$: (1) expand $B_0(\zeta)$ in a Taylor series about $\lambda_0$; (2) make the following change of variable $\tilde{\zeta} := \zeta(\tilde{w}) = \lambda_0 + \bar{w}(16\lambda_0^4 t)^{-1/2}$, and express the expansion obtained in (1) above in terms of $\tilde{w}$; and (3) use the following identity, $ab = (a-b)(b-1) + (a-1)(b-1) + 1$. Carrying out steps (1)–(3), one gets that,

$$B_0(\zeta) \frac{d\zeta}{2\pi i}(\zeta(\tilde{w})) = -\frac{3i\varphi(\lambda)(\zeta)}{\lambda_0} \exp\left\{-6i\varphi(\lambda)(\zeta)\right\} \frac{1}{\sqrt{16\lambda_0^4 t}} \sum_{k=1}^{n} \left(p_k(\tilde{w})\right) \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \exp\left\{\frac{i\lambda_0^4}{2} \cdot 2i\varphi(\lambda)(\zeta) \right\} \exp\left\{-i\lambda_0^2 \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-\frac{i\lambda_0^2}{2} \cdot 2i\varphi(\lambda)(\zeta) \right\} \exp\left\{-i\lambda_0^2 \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\},$$

where $\lambda_0 \in \mathbb{C}$. Now, proceed to estimate $\mu^\epsilon(\zeta) |L|^t$ for $\zeta \in (\lambda_0, \lambda_0 + \epsilon \exp\{-1/t\})$; for this, the Beals-Coifman $[13]$ formulation for the solution of a RH problem on an oriented contour is necessary (Theorem 4.1 and the paragraph preceding it for discussion and notation); in particular, one has to estimate the functions $\tilde{w}_x(\zeta)$ on $L(\mathbb{T})$. In Sec. 5 of $[13]$, it was shown that, on $(\lambda_0, \lambda_0 + \epsilon \exp\{-1/t\})$, $\tilde{w}_x(\zeta) = -(\delta(\zeta) \exp\{-i\theta(\zeta)\}) \cdot r(\zeta) \mathcal{P}(\zeta) \sigma_+ \tilde{w}_x(\zeta) + 0$; hence, from the Beals-Coifman $[13]$ formulation, for any $f \in \mathcal{L}^2(\mathbb{T}; M_2(\mathbb{C}))$, $C_{w_x,f} := C_0(f \tilde{w}_x)$. To estimate $\tilde{w}_x(\zeta)$, one proceeds as follows: (1) recalling that $\mathcal{P}(\zeta) \in \mathcal{S}(\mathbb{T}; \mathbb{C})$ and $|\mathcal{P}(\zeta)| \in L_\infty(\mathbb{T}; \mathbb{C}) < 1$, expand $\mathcal{P}(\zeta)$ by parts, for $\zeta \in (\lambda_0, \lambda_0 + \epsilon \exp\{-1/t\})$, via an integration by parts argument; (2) expand $\exp\{-i\theta(\zeta)\} \cdot r(\zeta) \mathcal{P}(\zeta)$ in a Taylor series about $\lambda_0$; and (3) change variables $\tilde{w}_x(\zeta) := \zeta(\tilde{w})$. Carrying out steps (1)–(3), one shows that,

$$w_{\text{ad}}^+ \left(\zeta(\tilde{w}) \right) = e^{\frac{i}{4} \lambda_0^4 t - 2i \varphi(\lambda)(\tilde{w})} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\} \exp\left\{-2i \ln \left(\frac{C^1(\lambda_0)}{|\lambda - \lambda_0|} + C^2(\lambda_0) \lambda_0 \right) \right\}$$
\[
\phi(\lambda_0) = 2\nu \ln \lambda_0 + \frac{1}{2} \int_0^1 \mu_0^2 \ln|z^2 - \lambda_0^2|d\ln(1 - |r(z)|^2) - \frac{1}{2} \int_0^\infty \ln|z^2 + \lambda_0^2|d\ln(1 + |r(i\lambda)|^2),
\]
\[
v_{00}(\bar{\omega};\lambda_0) := R(\lambda_0), \quad v_{10}(\bar{\omega};\lambda_0), \quad v_{11}(\bar{\omega};\lambda_0), \quad \text{and} \quad v_{22}(\bar{\omega};\lambda_0) \text{ are nilpotent matrix polynomials whose elements are sums of products of terms of the type} \ \bar{\omega}^j \ \text{and} \ \ln(\bar{\omega})^k, \ j \in \mathbb{Z}_{\geq 1}, \ k \in \mathbb{Z}_{\geq 0}, \ \text{with} \ \lambda_0\text{-dependent coefficients, and for} \ 0 \leq j \leq 2, \ 0 \leq k \leq j, \ \text{and} \ ||(\cdot)||_{L^2(L^2(\lambda_0);\mathbb{C})} < \infty, \ \text{with} \ ||(\cdot)||_{\cap L^2(L^2(\lambda_0);\mathbb{C})} := \sum_{l \in \{1, 2, \infty\} ||(\cdot)||_{L^2(L^2(\lambda_0);\mathbb{C})}}. \ \text{Hence, for any} \ f \in L^2(L^2(\lambda_0);\mathbb{C}), \ \text{let} \ \lambda_0 \text{-dependent coefficients, and for} \ 0 \leq j \leq 2, \ 0 \leq k \leq j, \ \text{and} \ ||C_{\lambda_0}^k(\cdot;\lambda_0)||_{M(L^2(\lambda_0);\mathbb{C})} \leq K^2(\lambda_0) < \infty, \ \text{with} \ M(\bullet;\mathbb{C}) \ \text{denoting the space of bounded linear operators acting from} \ L^2(\bullet;\mathbb{C}) \ \text{into} \ L^2(\bullet;\mathbb{C}), \ \text{for} \ \lambda_0 > M, \ \ker(\mathbf{I} - C_{\lambda_0}) = 0 \ \text{and} \ ||(\mathbf{I} - C_{\lambda_0}^k)(\cdot;\lambda_0)||_{M(L^2(\lambda_0);\mathbb{C})} < K^2(\lambda_0) < \infty. \ \text{Using the method of successive approximations, one shows that, as} \ t \to +\infty \ \text{and} \ x \to -\infty \ \text{such that} \ \lambda_0 > M, \ \mu^C(\cdot;\lambda_0) \ \text{can be expanded in the following Neumann-type series (see, also, Part II of [12], and [21]),}
\]
\[
A_0(\bar{\omega})_{\mid \lambda_0} = \mu^C(\bar{\omega})_{\mid \lambda_0} = \mu_0^C(\bar{\omega};\lambda_0) + \mu_0^C(\bar{\omega};\lambda_0)\ln t + O\left(\frac{\mu_0^C(\bar{\omega};\lambda_0)(\ln t)^2}{t}\right),
\]
\[
\mu_0^C(\bar{\omega};\lambda_0) := (\mathbf{I} - C_{00}(\bar{\omega};\lambda_0))^{-1}I, \ ||(\mathbf{I} - C_{00}(\cdot;\lambda_0))^{-1}||_{M(L^2(\lambda_0);\mathbb{C})} < \infty, \ 1 \leq j \leq 2, \ 0 \leq k \leq j: \ \text{an explicit expression for} \ \mu_0^C(\bar{\omega};\lambda_0) = \mu_0^C(\bar{\omega};\lambda_0) = \mu_0^C(\bar{\omega};\lambda_0) = \mu_0^C(\bar{\omega};\lambda_0)\ln t \ \text{in terms of parabolic-cylinder functions was given in Sec. 7 of [7] (see below). Making one more change of variable,} \ \vartheta = \sqrt{2\bar{\omega}}\exp\left(\frac{\sigma}{\vartheta}\right), \ \text{and recalling the definition of} \ I_0, \ \text{one shows that}
\]
\[
I_0 - I_{1/2} = I_{0,a} + I_{0,b} + I_{0,c} + I_{0,d} + \varepsilon_r,
\]
where
\[
I_{1/2} := Y_a(\lambda,\lambda_0; t) \int_0^\infty \mu_0^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta, \\
I_{0,a} := Y_a(\lambda,\lambda_0; t) \sum_{l_4 \in \mathbb{L}_4} \int_0^\alpha \mu_0^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) \prod_{k=1}^4 (p_k(\frac{\vartheta - \lambda_0}{\sqrt{2}}))^{l_k} e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta, \\
I_{0,b} := Y_b(\lambda,\lambda_0; t) \sum_{l_4 \in \mathbb{L}_4} \int_0^\alpha \mu_0^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) \prod_{k=1}^4 (p_k(\frac{\vartheta - \lambda_0}{\sqrt{2}}))^{l_k} e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta, \\
I_{0,c} := Y_c(\lambda,\lambda_0; t) \sum_{l_4 \in \mathbb{L}_4} \int_0^\alpha \mu_0^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) \prod_{k=1}^4 (p_k(\frac{\vartheta - \lambda_0}{\sqrt{2}}))^{l_k} e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta, \\
I_{0,d} := Y_d(\lambda,\lambda_0; t) \sum_{l_4 \in \mathbb{L}_4} \int_0^\alpha \mu_0^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) \prod_{k=1}^4 (p_k(\frac{\vartheta - \lambda_0}{\sqrt{2}}))^{l_k} e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta, \\
\varepsilon_r := O\left(\frac{\lambda_0^2}{\mathcal{A}_0(\lambda_0;\lambda_0)} \sum_{k=1}^2 \mu_k^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta\right) + O\left(\frac{\lambda_0^2}{\mathcal{A}_0(\lambda_0;\lambda_0)} \sum_{k=1}^2 \mu_k^C(\frac{\vartheta - \lambda_0}{\sqrt{2}};\lambda_0) e^{2i\vartheta^2/2}e^{-\vartheta^2/2}\sigma_d\vartheta\right),
\]
\(\hat{\alpha} := (32\varepsilon^2\lambda_0^2 t)^{1/2}\), the prime on the summation in the expression for \(I_{0,a}\) means that the term corresponding to \((l_1, l_2, l_3, l_4) = (0, 0, 0, 0)\) is omitted from the sum,

\[
Y_a(\lambda, \lambda_0; t) = \frac{y(\lambda_0; t) e^{-2\Phi(t) R(\lambda_0)}}{(\lambda - \lambda_0)^{3/2} \lambda_0^2 t}, \quad Y_0(\lambda, \lambda_0; t) = \frac{i y(\lambda_0; t) (3i\mu R(\lambda_0) - \lambda_0 R' (\lambda_0))}{(\lambda - \lambda_0)^{3/2} \lambda_0^2 t},
\]

\[
Y_c(\lambda, \lambda_0; t) = \frac{y(\lambda_0; t) e^{-2\Phi(t) R(\lambda_0)}}{(\lambda - \lambda_0)^{3/2} \lambda_0^2 t}, \quad Y_d(\lambda, \lambda_0; t) = \frac{-i y(\lambda_0; t) R(\lambda_0)}{(\lambda - \lambda_0)^{5/2} \lambda_0^2 t},
\]

\[y(\lambda_0; t) := \frac{e^{2\Phi(t)} \mathcal{C}^2_1(\lambda_0)(\lambda_0)^{4i\lambda_0^2 t}}{(2\pi i)^4 \lambda_0^{4i\lambda_0^2 t}}, \quad \mathcal{C}^2_1(\lambda_0) \in \mathcal{S}(\mathbb{R} > M; \mathbb{C}), \quad 1 \leq i \leq 4.\]

As will be shown below, \(I_{1/2}\) gives rise to the leading-order (\(\mathcal{O}(t^{-1/2})\)) term: towards the proof of this statement, one proceeds by estimating the difference, \(I_0 - I_{1/2}\). Recall, first, the following inequality, \(|\exp\{\cdot\} - 1| \leq |\cdot| \sup_{s \in [0,1]}|\exp\{s \cdot\}|\); hence, \(|\exp\{\Delta_i^2\} - 1| \leq |\Delta_i^2| \sup_{s \in [0,1]}|\exp\{s \Delta_i^2\}|\), where \(\Delta_i^2 \equiv 2\Delta_i(\lambda_0 + \frac{e^{2\Phi(t)} \mathcal{C}^2_1(\lambda_0)(\lambda_0)^{4i\lambda_0^2 t}}{(2\pi i)^4 \lambda_0^{4i\lambda_0^2 t}}), \ i \in \{1, 2\}\). Since, as shown in [7], \(||(\delta(\cdot))^{1/2}||_{\mathcal{L}_\infty(\mathbb{C} \mathbb{C})} < \infty\), from the definitions of \(\rho_\pm(\lambda_0), \ \tilde{\rho}_\pm(\lambda_0), \) and \(\Delta_i^2, \ i \in \{1, 2\}\), it follows that \(\sup_{s \in [0,1]}|\exp\{s \Delta_i^2\}| < \infty\); furthermore, using the Lipschitz property of \(\ln\left(\frac{1 - i(\rho(\lambda))^2}{1 - (\rho(\lambda))^2}\right), \ |\lambda| < \lambda_0, \) and the fact that \(\rho(\lambda) \in \mathcal{S}(\mathbb{R}^2; \mathbb{C})\) and \(||r||_{\mathcal{L}_\infty(\mathbb{C} \mathbb{C})} < 1\), via an integration by parts argument, one deduces that

\[
|\Delta_i^2| \leq \frac{K^3_1(\lambda_0) \Theta + K^3_2(\lambda_0) \Theta \ln \Theta + K^3_2(\lambda_0) \Theta \ln \Theta}{\lambda^2_0 t}, \quad |\Delta_2^2| \leq \frac{K^3_1(\lambda_0) \Theta}{\lambda^2_0 t},
\]

with \(K^3_1(\lambda_0) \in \mathcal{L}_\infty(\mathbb{R} > M; \mathbb{R} > 0), \ i \in \{1, 4\}\). Similarly, one gets that,

\[
|\exp\{-i^2 \frac{\Phi(t)}{3\lambda^2_0 t} \} - 1| \leq \frac{e^{3\lambda^2_0 t}}{8\sqrt{2} \lambda^2_0 t} \sup_{s \in [0,1]}|\exp\{-i^2 \frac{\lambda^2_0 t}{16\lambda^2_0 t}\}| := \frac{\bar{K}_1(\lambda_0) \Theta}{\lambda^2_0 t},
\]

\[
|\exp\{i^2 \frac{\lambda^2_0 t}{3\lambda^2_0 t}\} - 1| \leq \frac{e^{3\lambda^2_0 t}}{8\sqrt{2} \lambda^2_0 t} \sup_{s \in [0,1]}|\exp\{i^2 \frac{\lambda^2_0 t}{4\lambda^2_0 t}\}| := \frac{\bar{K}_2(\lambda_0) \Theta}{\lambda^2_0 t},
\]

with \(\bar{K}_1(\lambda_0) \in \mathcal{L}_\infty(\mathbb{R} > M; \mathbb{R} > 0), \ i \in \{1, 2\}\). Although the expression for the difference, \(I_0 - I_{1/2}\), contains many terms, estimations for the respective terms are analogous. Consider, say, the bound for the term corresponding to \((l_1, l_2, l_3, l_4) = (1, 0, 0, 0)\) in \(I_{0,a}\), which is denoted by \(I^1_{0,a}\):

\[
I^1_{0,a} := Y_a(\lambda, \lambda_0; t) \int_0^\infty \mu_0^c(\rho e^{-2\Phi(t)/\sqrt{2}}; \lambda_0) p_1\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}\right) g^{2iu} e^{-\varepsilon t/2} \sigma d\Theta.
\]

Using the fact that \(0 < \nu \leq \nu_{\text{max}} := \frac{1}{2\pi} \ln(1 - \sup_{\lambda \in \mathbb{R}^2} |\rho(\lambda)|^2) < \infty\), and recalling the definitions of \(s(\lambda_0)\) and \(\mathcal{R}(\lambda_0)\), one gets that, for \(\lambda \in \mathbb{C} \setminus \mathcal{N}(\lambda_0; \varepsilon_0), \ |Y_0(\lambda, \lambda_0; t)| \leq \frac{\exp\{\epsilon_{\text{max}}\} |r(\lambda)|}{2\pi |\lambda - \lambda_0| \sqrt{2}\lambda^2_0 t};\)

hence, letting the upper limit of integration tend to \(+\infty\) (for brevity, the following notation is used: for matrices \(A\) and \(B\), the inequality \(|A| \leq |B|\) means that \(|A_{ij}| \leq |B_{ij}|\) \(\forall\ i, j)\),

\[
|I^1_{0,a}| \leq \frac{\exp\{\epsilon_{\text{max}}\} \epsilon_{\text{max}} |r(\lambda)|}{2\pi |\lambda - \lambda_0| \sqrt{2}\lambda^2_0 t} \int_0^\infty \mu_0^c\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}; \lambda_0\right) p_1\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}\right) |e^{2\varepsilon / 2} / 2\sigma d\Theta.
\]

In [4], it was shown that \(\mathcal{U}_{00} := ||\mu_0^c\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}; \lambda_0\right)||_{\mathcal{L}^2(\mathbb{R} > M; \mathbb{C})} < \infty\); hence, from this estimate and the Cauchy-Schwarz inequality for integrals,

\[
|I^1_{0,a}| \leq \frac{\exp\{\epsilon_{\text{max}}\} \epsilon_{\text{max}} |r(\lambda)|}{2\pi |\lambda - \lambda_0| \sqrt{2}\lambda^2_0 t} ||p_1\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}\right) \exp\{-\varepsilon t/2\}||_{\mathcal{L}^2(\mathbb{R} > M; \mathbb{C})}.
\]

Recalling the estimate for \(|\exp\{\hat{\Delta}^2_i\} - 1|\), one shows that,

\[
||p_1\left(\frac{e^{-2\Phi(t)/\sqrt{2}}}{\sqrt{2}}\right) \exp\{-\varepsilon t/2\}||_{\mathcal{L}^2(\mathbb{R} > M; \mathbb{C})} \leq \frac{\bar{K}^3(\lambda_0) \ln t}{\sqrt{\lambda^2_0 t}}.
\]
where $\widetilde{K}^3(\lambda_0) \in L^\infty(\mathbb{R}_M; \mathbb{R}_0)$; hence, uniformly for $\lambda \in \mathbb{C} \setminus \mathcal{N}(\lambda_0; \varepsilon_0)$,

$$|I_{\lambda,a}| \leq \frac{\widetilde{K}_1^3(\lambda_0) \ln t}{|\lambda - \lambda_0| \lambda_0^t},$$

with $\widetilde{K}_1^3(\lambda_0) \in S(\mathbb{R}_M; M_2(\mathbb{R}_0))$. Similarly, recalling the estimates for $|\exp\{-i\theta^3 \exp(-\frac{m^2}{8\pi^2})\} - 1|$, $|\exp\{\frac{i\theta^3}{4\pi \sigma_0}\} - 1|$, and $|\exp\{\frac{\Delta_2^0}{16\pi^2}\} - 1|$, and using the triangle inequality for $L^2$-norms, one shows that the remaining terms for $I_{\lambda,a}$ are of the type $O(t^{-\frac{m}{2}})$ and $O(t^{-\frac{m}{2}} \ln t)$, $2 \leq m \leq 5$, $3 \leq n \leq 6$. Estimating the remaining terms of $I_0 - I_{1/2}$ analogously, one shows that, as $t \to +\infty$ and $x \to -\infty$ such that $\lambda_0 > M$, uniformly for $\lambda \in \mathbb{C} \setminus \mathcal{N}(\lambda_0; \varepsilon_0)$,

$$|I_0 - I_{1/2}| \leq \frac{\widetilde{K}_2^3(\lambda; \lambda_0) \ln t}{\lambda_0^t},$$

where $|\widetilde{K}_2^3(\lambda; \lambda_0)| |L^\infty(\mathbb{C} \setminus \mathcal{N}(\lambda_0; \varepsilon_0); M_2(\mathbb{R}_0)) < \infty$, $\widetilde{K}_2^3(\lambda; \lambda_0) \in S(\mathbb{R}_M; M_2(\mathbb{R}_0))$, and, as $\lambda \to \infty$, $\widetilde{K}_2^3(\lambda; \lambda_0) \sim O(\frac{\lambda_0}{|\lambda - \lambda_0| \lambda_0^t})$, with $\widetilde{K}_2^3(\lambda_0) \in S(\mathbb{R}_M; M_2(\mathbb{R}_0))$.

Repeating the whole of the above analysis mutatis mutandis for each term on the right-hand side of the original integral expression for $\chi^c(\lambda)$ which appears at the very beginning of the proof, one shows that, as $t \to +\infty$ and $x \to -\infty$ such that $\lambda_0 > M$, uniformly for $\lambda \in \mathbb{C} \setminus \cup_{t \in \{\pm \}} \mathcal{N}(\lambda_0; \varepsilon_0)$,

$$|\chi^c(\lambda) - \chi^c_{1/2}(\lambda)| \leq \frac{(h_1^+ + h_2^+)(\lambda; \lambda_0) \ln t}{\lambda_0^t},$$

where $h_1^+(\lambda; \lambda_0) := \sum_{t' \in \{0, \pm t\}} e^{i t_0} e^{it_0} \rho(t'\lambda) \rho(t'\lambda_0) |r\Sigma_0^c|$, $h_2^+(\lambda; \lambda_0) := \sum_{t' \in \{0, \pm t\}} e^{i t_0} e^{it_0} \rho(t'\lambda) \rho(t'\lambda_0) \rho(t'\lambda_0) |r\Sigma_0^c|$, and the functions $e_0^\pm(\lambda; \lambda_0)$ and $e_0^\pm(\lambda; \lambda_0)$ have the following properties as $\lambda \to \infty$, $h_1^+(\lambda; \lambda_0) + h_2^+(\lambda; \lambda_0) \sim O(\sum_{t' \in \{0, \pm t\}} e^{i t_0} e^{it_0} \rho(t'\lambda) \rho(t'\lambda_0) |r\Sigma_0^c| \lambda^- \lambda_0^0)$, $e_0^\pm(\lambda; \lambda_0) \in L^\infty(\mathbb{R}_M; M_2(\mathbb{R}_0))$, $t' \in \{0, \pm t\}$, and $e_0^\pm(\lambda) \in S(\mathbb{R}_M; M_2(\mathbb{R}_0))$. Moreover, $h_1^+(\lambda; \lambda_0) \sim O(\mathcal{N}(\lambda_0; \varepsilon_0); M_2(\mathbb{R}_0)) < \infty$, $h_1^+(\lambda; \lambda_0) \in L^\infty(\mathbb{R}_M; M_2(\mathbb{R}_0))$, $h_1^+(\lambda; \lambda_0) \in S(\mathbb{R}_M; M_2(\mathbb{R}_0))$, and $\chi^c_{1/2}(\lambda)$ represents the sum over all $I_{1/2}$-like terms in which the upper limits of integration tend to $+\infty$. One can write $\chi_{1/2}^c(\lambda)$ in the following form,

$$\chi_{1/2}^c(\lambda) = I + \frac{\widetilde{A}^0}{\sqrt{16\lambda_0^t}} \chi^c_0(\lambda) + \frac{\widetilde{A}^0_{\mathrm{ad}(\sigma)}}{\sqrt{8\lambda_0^t}} \chi^c_0(\lambda),$$

where

$$\widetilde{A}^0 = \frac{\exp(2\lambda_0^t \rho^t)}{(16\lambda_0^t)^{1/2}} \exp\{\sum_{l \in \{\pm \}} (\rho_l(0) + \rho_l(0))\},$$

$$\widetilde{A}^0_{\mathrm{ad}(\sigma)} = \frac{\exp(2\lambda_0^t \rho^t)}{(16\lambda_0^t)^{1/2}} \exp\{\sum_{l \in \{\pm \}} (\rho_l(0) + \rho_l(0))\},$$

$$\rho_{\pm}(\lambda_0) = \frac{1}{2\pi t} \int_{0}^{\pm t} \ln\left(\frac{1}{|1 - r(\lambda_0)|^t}\right) \frac{dk}{(\lambda - \lambda_0)^t}, \quad \rho_{\pm}(\lambda) = \frac{1}{2\pi t} \int_{0}^{\pm t} \ln\left(\frac{1}{|1 - r(\lambda_0)|^t}\right) \frac{dk}{(\lambda - \lambda_0)^t},$$

$$\chi^c_{11}(\lambda) = \chi^c_0(\lambda), \quad \chi^c_{12}(\lambda) = \chi^c_{11}(\lambda),$$

$$\chi^c_{21}(\lambda) = \chi^c_{21}(\lambda), \quad \chi^c_{22}(\lambda) = \chi^c_{21}(\lambda),$$

$$\chi^c_{22}(\lambda) = \chi^c_{21}(\lambda).$$
\[
X^{\Sigma_{B, r}}_{+, 12} (s) = \frac{-i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{+, 21} (s) = \frac{-i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{+, 22} (s) = \frac{i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{-, 22} (s) = \frac{-i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{-, 21} (s) = \frac{-i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{-, 11} (s) = \frac{i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}), \\
X^{\Sigma^2}_{+, 11} (s) = \frac{-i \nu e^{-i \frac{\pi}{4} \rho}}{2 \pi^2 e^{-i \frac{\pi}{4} r \rho}} D_{-i \nu} (\sqrt{2} e^{-i \frac{\pi}{4} r \rho}).
\]

The integrals are evaluated along the rays \((0, \varepsilon_k)\) (and their complex conjugates), \(\varepsilon_1 := \infty \exp\{\frac{\pi}{4}\}, \varepsilon_2 := \infty \exp\{-\frac{3 \pi}{4}\}\), \(D_{\pm i \nu}(\cdot)\) is the parabolic-cylinder function [19], and

\[
\hat{\alpha}^0_C = \exp\left\{ \sum \left( \rho^\pm (0) + \hat{\beta}^\pm (0) \right) \right\},
\]

\[
\rho^\pm (0) = -\frac{1}{2 \pi i} \int_0^{\pm \lambda_0} \ln|\ell| d |\ln(1 - |r|)^2|, \quad \hat{\beta}^\pm (0) = \int_0^{\infty} \frac{\ln(1 + |r|) \ell^2}{|\ell|} d \ell,
\]

where \(r(0) := (r(\lambda) |_{\lambda \in \mathbb{R}}) |_{\lambda = 0}\), and \(r(\ell) := (r(\lambda) |_{\lambda \in \mathbb{R}^+}) |_{\lambda = \ell}\): the explicit expressions for \(X^{\Sigma^2}_{\pm, ij}(s), i \in \{1, 2, 3, 4\}, j, k \in \{1, 2\}\), are not written down here since they will not actually be needed. Since \(\int (r(\lambda) |_{\lambda \in \mathbb{R}}) |_{\lambda = 0} = (r(\lambda) |_{\lambda \in \mathbb{R}^+}) |_{\lambda = 0} = 0\), \(\hat{R}^+ (0) = \hat{R}^- (0) = h^+ (\lambda; \lambda_0) = 0\); hence, \(X^{\Sigma^2}_{ij}(\lambda) = 0, i, j \in \{1, 2\}\). To obtain the expression for \(X_{ij}^{\Sigma^2}(\ell), i, j \in \{1, 2\}\), given above, use was made of the explicit representation for \(\mu^\ell (\cdot; \lambda_0) := (\mathbf{I}_d - C^\ell_0 (\cdot; \lambda_0))^{-1} \mathbf{I}\) on \(L' \cup \overline{L'}\) (recall the definition of \(I_{1/2}\)) in terms of parabolic-cylinder functions given in Sec. 7 of [4]. Now, substituting the expressions given above for \(X^{\Sigma^2}_{\pm, ij}(s)\) and \(X^{\Sigma^2}_{\pm, ij}(s), i, j \in \{1, 2\}\), into the corresponding integrals for \(X^{\Sigma^2}_{ij}(\lambda), i, j \in \{1, 2\}\), and using the following identities [13],

\[
\partial_{\ell} D_A (s) = \frac{1}{2} (a D_{A-1} (s) - D_{A+1} (s)), \quad \zeta D_A (s) = D_{A+1} (s) + a D_{A-1} (s), \quad |\Gamma (i \nu)|^2 = \pi / (\nu \sinh \pi \nu),
\]
where as well as the following integral \[19\],
\[
f_0^\infty \exp(-\frac{x^2}{4})x^{a-1}D_{-b}(x)dx = \frac{\sqrt{\pi} \exp(-\frac{1}{2}(a+b) \ln 2) \Gamma(a)}{\Gamma(\frac{1}{2}(a+b) + \frac{1}{2})}, \quad \Re(a) > 0,
\]
one obtains the result stated in the Lemma.

**Proposition 4.2** As \(\lambda \to \infty\), \(\lambda \in \mathbb{C} \setminus (\Gamma \cup (\bigcup_{j=1}^{p} (\{\pm \lambda_j\} \cup \{\pm \overline{\lambda}_j\})))\), \(\chi(\lambda)\) has the following asymptotic expansion,
\[
\chi(\lambda) = I + \frac{1}{2\lambda} \left( \left( Q^x(x, t) + 4 \sum_{i=1}^{n} \left( \beta_i - \frac{\chi_{i1}(\lambda)}{\chi_{i2}(\lambda)} \delta_i \right) \right) \sigma_+ + \left( Q^x(x, t) + 4 \sum_{i=1}^{n} \left( \omega_i - \frac{\chi_{i2}(\lambda)}{\chi_{i2}(\lambda)} \alpha_i \right) \right) \sigma_- \right) + O(\lambda^{-2}), \quad (94)
\]
where \(\lim (\chi^c(x,t;\lambda))_{12} := Q^x(x, t)/2\lambda\), \(\{\alpha_i, \omega_i\}_{i=1}^{n}\) satisfy the following non-degenerate system of \(2n\) linear inhomogeneous algebraic equations,
\[
\begin{bmatrix}
\hat{A}^+ \\
\hat{C}^+
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n \\
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_n
\end{bmatrix} = \begin{bmatrix}
g_1^+ \chi_{i2}(\lambda_1) \\
g_2^+ \chi_{i2}(\lambda_2) \\
\vdots \\
g_n^+ \chi_{i2}(\lambda_n) \\
g_1^+ \chi_{11}(\lambda_1) \\
g_2^+ \chi_{11}(\lambda_2) \\
\vdots \\
g_n^+ \chi_{11}(\lambda_n)
\end{bmatrix}, \quad (95)
\]
where, for \(i, j \in \{1, 2, \ldots, n\}\), the \(n \times n\) matrix blocks, \(\hat{A}^+, \hat{B}^+, \hat{C}^+, \text{ and } \hat{D}^+\), are defined as follows,
The set \( \{ \beta_i, \delta_i \}_{i=1}^n \) satisfies the following non-degenerate system of \( 2n \) linear inhomogeneous algebraic equations,

\[
\begin{bmatrix}
\bar{\mathcal{E}}^+ & \bar{\mathcal{F}}^+ \\
\bar{\mathcal{G}}^+ & \bar{\mathcal{H}}^+
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n \\
\delta_1 \\
\delta_2 \\
\vdots \\
\delta_n
\end{bmatrix}
= \begin{bmatrix}
g_1^+ \chi_{22}^\delta(\lambda_1) \\
g_2^+ \chi_{22}^\delta(\lambda_2) \\
\vdots \\
g_n^+ \chi_{22}^\delta(\lambda_n) \\
g_1^+ \chi_{21}^\delta(\lambda_1) \\
g_2^+ \chi_{21}^\delta(\lambda_2) \\
\vdots \\
g_n^+ \chi_{21}^\delta(\lambda_n)
\end{bmatrix},
\]  

(96)

where, for \( i, j \in \{1, 2, \ldots, n\} \), the \( n \times n \) matrix blocks, \( \bar{\mathcal{E}}^+ \), \( \bar{\mathcal{F}}^+ \), \( \bar{\mathcal{G}}^+ \), and \( \bar{\mathcal{H}}^+ \), are defined as follows,

\[
\bar{\mathcal{E}}^+_{ij} := \begin{cases} \lambda_i g_i^+ \chi_{ij}^\delta(\lambda_i), & i = j, \\ 2g_i^+ \chi_{22}^\delta(\lambda_i) \chi_{22}^\delta(\lambda_i)/(\lambda_i - \lambda_j^2), & i \neq j, \end{cases}
\]

\[
\bar{\mathcal{F}}^+_{ij} := \begin{cases} 2g_i^+ \chi_{22}^\delta(\lambda_i) \chi_{22}^\delta(\lambda_i)/(\lambda_i - \lambda_i^2), & i = j, \\ 2g_i^+ \chi_{22}^\delta(\lambda_i) \chi_{22}^\delta(\lambda_i)/(\lambda_i - \lambda_j^2), & i \neq j, \end{cases}
\]

\[
\bar{\mathcal{G}}^+_{ij} := \begin{cases} \lambda_i g_i^+ \chi_{ij}^\delta(\lambda_i)/(\lambda_i - \lambda_i^2), & i = j, \\ 2g_i^+ \chi_{22}^\delta(\lambda_i) \chi_{22}^\delta(\lambda_i)/(\lambda_i - \lambda_j^2), & i \neq j, \end{cases}
\]

\[
\bar{\mathcal{H}}^+_{ij} := \begin{cases} \lambda_i g_i^+ \chi_{ij}^\delta(\lambda_i)/(\lambda_i - \lambda_i^2), & i = j, \\ 2g_i^+ \chi_{22}^\delta(\lambda_i) \chi_{22}^\delta(\lambda_i)/(\lambda_i - \lambda_j^2), & i \neq j, \end{cases}
\]

with

\[
g_j^+ := C_j e^{2i\beta_j^0 x + 4i\lambda_j^0 t} (\delta^+(\alpha_j; \lambda_0))^{-2} \prod_{i=n+1}^N \left( (\lambda_i - \lambda_0)/(\lambda_i + \lambda_0) \right)^2, \quad 1 \leq j \leq n,
\]

\( \delta^+(\lambda_k; \lambda_0), k \in \{1, 2, \ldots, n\} \), given in Theorem 2.1, Eq. (18), and \( W(\chi_{ij}^\delta(z), \chi_{i'j'}^\delta(z)) \) is the Wronskian of \( \chi_{ij}^\delta(\lambda) \) and \( \chi_{i'j'}^\delta(\lambda) \) evaluated at \( z \) \( (i, j, i', j' \in \{1, 2\}) \): \( W(\chi_{ij}^\delta(z), \chi_{i'j'}^\delta(z)) := (\chi_{ij}^\delta(\lambda) \partial_\lambda \chi_{i'j'}^\delta(\lambda) - \chi_{i'j'}^\delta(\lambda) \partial_\lambda \chi_{ij}^\delta(\lambda))|_{\lambda=z} \).

\textbf{Proof.} For \( 1 \leq i \leq n \), set

\[
\text{res}(\chi(\lambda); \alpha_i) = \begin{pmatrix} \alpha_i \\ \beta_i \\ a_i \\ b_i \end{pmatrix}, \quad \text{res}(\chi(\lambda); \omega_i) = \begin{pmatrix} \omega_i \\ d_i \\ \delta_i \end{pmatrix}.
\]

(97)

From Eqs. (92), (93), and (97), and the polar (residue) conditions in Lemma 3.3, one gets a system of linear algebraic equations for \( \{\alpha_i, \beta_i, a_i, b_i, c_i, d_i, \omega_i, \delta_i \}_{i=1}^n \): from this system, one shows that, for \( 1 \leq i \leq n \),

\[
\begin{align*}
(\chi_{12}^\delta(\lambda_i) \alpha_i + \chi_{22}^\delta(\lambda_i) b_i) g_i^+ &= 0 \Rightarrow a_i = -\frac{\chi_{12}^\delta(\lambda_i)}{\chi_{22}^\delta(\lambda_i)} \alpha_i, \\
(\chi_{12}^\delta(\lambda_i) \beta_i + \chi_{22}^\delta(\lambda_i) b_i) g_i^+ &= 0 \Rightarrow b_i = -\frac{\chi_{12}^\delta(\lambda_i)}{\chi_{22}^\delta(\lambda_i)} \beta_i, \\
(\chi_{11}^\delta(\lambda_i) c_i + \chi_{21}^\delta(\lambda_i) \omega_i) g_i^+ &= 0 \Rightarrow c_i = -\frac{\chi_{11}^\delta(\lambda_i)}{\chi_{21}^\delta(\lambda_i)} \omega_i, \\
(\chi_{11}^\delta(\lambda_i) d_i + \chi_{21}^\delta(\lambda_i) \delta_i) g_i^+ &= 0 \Rightarrow d_i = -\frac{\chi_{11}^\delta(\lambda_i)}{\chi_{21}^\delta(\lambda_i)} \delta_i.
\end{align*}
\]

(98)
using Eqs. (98), which show that the matrices \( \{ \text{res}(\chi(\lambda); \lambda_i) \}_{i=1}^n \) and \( \{ \text{res}(\chi(\lambda); \lambda_i) \}_{i=1}^n \), respectively, are degenerate, one simplifies the resulting system of linear algebraic equations for \( \{ \alpha_i, \beta_i, a_i, b_i, c_i, d_i, \sigma_i, \delta_i \}_{i=1}^n \) and obtains Eqs. (95) and (96): the non-degeneracy of systems (95) and (96) is a consequence of the unique solvability of the original RH problem (Lemma 2.2). Substituting Eqs. (97) into Eq. (92) and defining \( Q^\chi(x, t) \) as in the Proposition, one obtains, from Eq. (93), Lemma 4.1, and Eqs. (98), Eq. (94).

**Corollary 4.1** As \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \),

\[
Q(x, t) = Q^C_+(x, t) + 4i \sum_{j=1}^n \left( \omega_j - \frac{\chi_{j2}(\lambda)}{\chi_{j2}(\lambda_n)} \alpha_j \right) + O(C(\lambda_0) \exp(-abt) ),
\]

where \( Q^C_+(x, t) := iQ^\chi(x, t), a \) and \( b \) are given in Lemma 3.3, and \( C(\lambda_0) \in L^\infty(\mathbb{R}_{>M}; \mathbb{C}) \).

**Proof.** Since, from Lemma 3.2, \( Q(x, t) = 2i \lim_{\lambda \to +\infty} \lambda \text{m}^t(x, t; \lambda) \nu \)_{12} \), the result follows from Lemma 3.3 and Proposition 4.2.

**Proposition 4.3** As \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \),

\[
Q(x, t) = Q^C_+(x, t) + 4i \left( \omega_n - \frac{\chi_{12}(\lambda_0)}{\chi_{12}(\lambda_n)} \alpha_n \right) + O(C(\lambda_0) \ln t) + O(C(\lambda_0) e^{-a_0 b_0 t}),
\]

where

\[
\alpha_n = \frac{\mu_{12} x_1^2(\lambda_0) + \mu_{12} x_1^2(\lambda_n)}{(a_{11} \mu_{22} - a_{12} \mu_{21})},
\]

\[
\omega_n = \frac{\mu_{12} x_1^2(\lambda_0) + \mu_{12} x_1^2(\lambda_n)}{(a_{11} \mu_{22} - a_{12} \mu_{21})},
\]

\[
\hat{a}_{11} := \frac{\lambda_0 + \mu_{12} x_1^2(\lambda_0) \chi_{12}(\lambda_n) + \lambda_0 \mu_{12} W(x_1^2(\lambda_0), \chi_{12}(\lambda_n))}{\lambda_n \chi_{12}(\lambda_n)},
\]

\[
\hat{a}_{12} := \frac{2 \mu_{12} x_1^2(\lambda_0) \chi_{12}(\lambda_n) - \lambda_0 \mu_{12} x_1^2(\lambda_n) \chi_{12}(\lambda_n)}{\chi_{11}(\lambda_n) (\lambda_n^2 - \lambda_0^2)},
\]

\[
\hat{a}_{21} := \frac{2 \mu_{12} x_1^2(\lambda_0) \chi_{12}(\lambda_n) - \lambda_0 \mu_{12} x_1^2(\lambda_n) \chi_{12}(\lambda_n)}{\chi_{22}(\lambda_n) (\lambda_n^2 - \lambda_0^2)},
\]

\[
\hat{a}_{22} := \frac{-\lambda_0 - \mu_{12} x_1^2(\lambda_0) \chi_{11}(\lambda_n) + \lambda_0 \mu_{12} W(x_1^2(\lambda_0), \chi_{11}(\lambda_n))}{\lambda_n \chi_{11}(\lambda_n)},
\]

\( Q^C_+(x, t) \) is given in Theorem 2.1, Eqs. (20)–(22) and (24), \( a_0 := \min(a, 8 \min_{1 \leq i \leq n-1} \{ \kappa_i - \xi_i \} \nu_{i-1} ) > 0 \), \( b_0 := \min(b, \min_{i \leq n-1} \{ \kappa_i - \xi_i \} \nu_{i-1} ) \), \( C_1(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M}; \mathbb{C}) \), and \( C_2(\lambda_0) \in L^\infty(\mathbb{R}_{>M}; \mathbb{C}) \).

**Proof.** Solving Eqs. (95) as \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \) for \( \{\alpha_i\}_{i=1}^n \) and \( \{\omega_i\}_{i=1}^n \) via Cramer’s rule, one shows that

\[
\alpha_i, \ \omega_i \sim O(\exp\{-a^b \min_{1 \leq i \leq n-1} \{ \kappa_i - \xi_i \} \nu_{i-1} \}, \ \ 1 \leq i \leq n-1,
\]

where \( a^b := \min_{1 \leq i \leq n-1} \{ \kappa_i - \xi_i \} \nu_{i-1} > 0 \), and \( \alpha_n \) and \( \omega_n \) are given by Eqs. (100) and (101): the result now follows from Corollary 4.1 and the estimates in Eq. (106).

**Proposition 4.4** As \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \),

\[
Q(x, t) = Q^\infty_{as}(x, t) + O\left( \frac{C(\lambda_0) \ln t}{t} \right),
\]

where \( Q^\infty_{as}(x, t) \) is given in Theorem 2.1, Eqs. (14)–(29), and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M}; \mathbb{C}) \).

**Proof.** Substitute \( \chi_{ij}^\infty() \), \( i, j \in \{ 1, 2 \} \), from Lemma 4.1 into Eqs. (99)–(105) and neglect exponentially small terms.

5 Asymptotic Evaluation of \( ((\Psi^{-1}(x, t; 0))_{11})^2 \)

In this section, the phase integral, \( ((\Psi^{-1}(x, t; 0))_{11})^2 \), which appears in the gauge transformation (Proposition 2.3, Eq. (9)) is evaluated asymptotically as \( t \to +\infty \) \( (x/t \sim O(1)) \).
Lemma 5.1 As \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \) and \((x,t) \in \Omega_n\),

\[
((\Psi^{-1}(x,t;0))_{11})^2 = \exp\{2\ln(\chi_{22}(0))\} \exp\left\{ \frac{2i}{\pi} \left( \int_0^{\lambda_0} \frac{\ln(1-|r(\phi)|^2)}{e} d\phi - \int_0^\infty \frac{\ln(1+|r(\phi)|^2)}{e} d\phi \right) \right\} 
\times \exp\left\{ -4i \sum_{l=n+1}^N \gamma_l \right\} \exp\{2\ln(1 - \sum_{i=1}^n \left( \frac{2b_i}{\lambda_i} + \frac{2\beta_i}{\gamma_i} \right))\} + O(C(\lambda_0) e^{-\alpha t b})
\]

where \( b_i = -\chi_i(\lambda)(\lambda) \beta_i, 1 \leq i \leq n, \{\beta_i, \tilde{\beta}_i\}_{i=1}^n \) satisfy Eqs. (96), \( a \) and \( b \) are given in Lemma 3.3, and \( C(\lambda_0) \in \mathcal{L}^\infty(\mathbb{R}_M; \mathbb{C}) \).

Proof. From Lemma 2.1, the proof of Lemma 3.1, Proposition 3.2 (Eqs. (85)), Lemma 3.2 (Eqs. (86)), and Lemma 3.3, one gets that

\[
\Psi(x,t;0) = \chi(0)(\delta(0))^{\sigma_3} \prod_{l=n+1}^N (d_{l+}(0))^{\sigma_3} + O(C_1(\lambda_0) \exp\{-\alpha t b\})
\]

where \( C_1(\lambda_0) \in \mathcal{L}^\infty(\mathbb{R}_M; \mathbb{M}_2(\mathbb{C})) \). From Propositions 2.1–2.3, the parametrization for the discrete eigenvalues (Sec. 2), Lemma 3.2 (Eqs. (87)), Proposition 4.1 (Eqs. (92) and (93)), the proof of Proposition 4.2 (Eqs. (97) and (98)), and the \( \sigma_1 \) and \( \sigma_3 \) symmetry reductions for \( \chi(\lambda) \), one shows that,

\[
\Psi^{-1}(x,t;0) = (\tilde{h}(0))^{\sigma_3} \left( \begin{array}{cc} 1 - \frac{n}{\lambda_i} + \frac{2\tilde{\gamma}_i}{\gamma_i} & 0 \\ 0 & 1 - \frac{n}{\lambda_i} + \frac{2\tilde{\gamma}_i}{\gamma_i} \end{array} \right) + O(C_2(\lambda_0) e^{-\alpha t b})
\]

where \( \tilde{h}(0) = \chi_{22}(0)(\delta(0))^{-1} \exp\{-2i\frac{\sum_{i=n+1}^N \gamma_i}{\lambda_i} \}, b_i, 1 \leq i \leq n, \{\beta_i, \tilde{\beta}_i\}_{i=1}^n \) are as given in the Lemma, \( c_i = -\frac{\chi_i(\lambda)(\lambda) \omega_i}{\chi_{11}(\lambda)} \omega_i, 1 \leq i \leq n, \{\alpha_i, \omega_i\}_{i=1}^n \) are defined by system (95), and \( C_2(\lambda_0) \in \mathcal{L}^\infty(\mathbb{R}_M; \mathbb{M}_2(\mathbb{C})) \); using the expression for \( \delta^+(\lambda; \lambda_0) \) given in Proposition 3.1 (and Remark 3.1), one obtains the result stated in the Lemma.

In order to estimate \( (\chi_{22}(0))^2 \), the following proposition and lemma are necessary:

Proposition 5.1 Define \( Q^x(t,x):=2i \lim_{\lambda \to \infty} (\lambda \chi^c(x,t;\lambda))_{12} \). Then

\[
(||Q^x(\cdot,t)||_{L^2(\mathbb{R}_M; \mathbb{C})})^2 = \frac{2}{\pi} \left( \int_0^\infty \frac{\ln(1+|r(\phi)|^2)}{e} d\phi - \int_0^\infty \frac{\ln(1-|r(\phi)|^2)}{e} d\phi \right),
\]

\[
((\chi_{22}(0))^2) = (\delta^+(0; \lambda_0))^2 \exp\{i \int_{-\infty}^x |Q^x(\phi,t)|^2 d\phi\}.
\]

Proof. Follows from the definition of \( \chi^c(\lambda) \) given in Proposition 4.1, Proposition 2.2, and Proposition 8.1 in [2]}

Lemma 5.2 \( \left(22\right) \) As \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 > M \),

\[
Q^x(x,t) = \frac{u^+_{1,1,0}(\lambda_0)e^{(4\lambda_{0}^t-x(\lambda_0)^{1/2})}}{\sqrt{t}} + \frac{u^+_{1,2,0}(\lambda_0)}{t^{1/2}} + O\left(\frac{C(\lambda_0)(\ln t)^2}{t^{1/2}}\right),
\]

where

\[
u_{1,1,0}(\lambda_0) = \sqrt{\frac{\nu(\lambda_0)}{2\lambda_0}} \exp\{i\theta^+(\lambda_0)\}.
\]

\[
\theta^+(\lambda_0) = \phi^+(\lambda_0) - \frac{2\pi}{8\nu^2} \arg\Gamma(i\nu(\lambda_0)) + \arg r(\lambda_0) - 3\nu(\lambda_0) \ln 2 + 2\sum_{l=n+1}^N \arg\left(\frac{\lambda_0 - \lambda_l}{\lambda_0 + \lambda_l}\right)\left(\frac{\lambda_0 - \lambda_l}{\lambda_0 - \lambda_l}\right) + \left(\gamma_l + 2\theta^+(\lambda_0)\right).
\]

\[
u_{1,2,0}(\lambda_0) = \frac{i}{8\nu^2} \left(\frac{d(r(\phi))}{d\phi}\right|_{\phi=0} - \frac{d(r(\phi))}{d\phi}\right|_{\phi=0}) \exp\left\{ i \left( \frac{4}{\pi} \sum_{l=n+1}^N \gamma_l + 2\theta^+(\lambda_0) \right) \right\},
\]

\[
\phi^+(\cdot) \text{ is given in Theorem 2.1, Eq. (22), and } C(\lambda_0) \in \mathcal{S}(\mathbb{R}_M; \mathbb{C}).
\]
Comment to Proof. Up to the leading \((O(t^{-1}))\) term, the asymptotic expansion was proved in [7]. The \(O(t^{-1})\) term constitutes the leading-order contribution from the first-order stationary phase point at \(\lambda = 0\): the complete proof of this asymptotic expansion can be found in [23].

**Proposition 5.2** As \(t \to +\infty\) and \(x \to -\infty\) such that \(\lambda_0 > M\),

\[
(\chi_{22}(0))^2 = \exp \left\{ i \left( \frac{1}{t} \int_{\lambda_0}^{\infty} \sqrt{\frac{n!}{\mu}} \left( R_+^*(0) \cos(\kappa^+(\mu; t)) - R_+^*(0) \sin(\kappa^+(\mu; t)) \right) \frac{d\mu}{\pi} \right) \right\} + O \left( \frac{C(\lambda_0)(\ln t)^2}{t} \right),
\]

where \(R_+^*(0) = \Re \{ R_+^*(0) \}, R_+^*(0) = \Re \{ R_+^*(0) \}, R_+^*(0) = \left( \frac{d(r(\varphi)|_{\varphi=\Re})}{d\varphi} \right)_{\varphi=0} - \frac{d(r(\varphi)|_{\varphi=\Re})}{d\varphi} \big|_{\varphi=0} \).

\[
\cdot \exp \{ 4i \sum_{i=1}^{N} E \{ \gamma_i \}, \kappa^+(\lambda_0; t) := 4\lambda_0 t - \nu(\lambda_0) \ln t + \theta^+(\lambda_0) - 2\theta^+(\lambda_0), \text{ and } C(\lambda_0) \in (\Re \{ R_+^*(0) \}, \Re \{ R_+^*(0) \}).
\]

\[
\text{Proof.} \text{ Writing } \int_{-\infty}^{+\infty} \left| Q^2(q, t) \right|^2 d\varphi = |1| + \int_{-\infty}^{+\infty} |Q^2(q, t)|^2 d\varphi, \text{ using the expressions for } |1| + \int_{-\infty}^{+\infty} |Q^2(q, t)|^2 d\varphi \text{ and } (\chi_{22}(0))^2 \text{ given in Proposition 5.1, the asymptotic expansion for } Q^2(x, t) \text{ given in Lemma 5.2, the following inequalities, } |\exp \{ (\cdot) \} - 1| \leq |(\cdot)| \sup_{s \in [0, 1]} |\exp \{ s(\cdot) \}| \text{ and } 0 < \nu(\lambda_0) \leq \nu_{\max} = -\frac{\pi}{4} \ln(1 - \sup_{\lambda \in \Re \{ R_+^*(0) \}} |r(\lambda)|^2) \text{, and the fact that } r(\lambda) \in \mathcal{S}(\Re; \C) \text{, one obtains the result stated in the Proposition.}
\]

**Lemma 5.3** As \(t \to +\infty\) and \(x \to -\infty\) such that \(\lambda_0 > M\) and \((x, t) \in \Omega_n\),

\[
((\Psi^{-1}(x, t; 0)))_1^2 = \exp \{ i \arg q_{\text{as}}(x, t) \} + O \left( \frac{C(\lambda_0)(\ln t)^2}{t} \right),
\]

where \(\arg q_{\text{as}}(x, t) \) is given in Theorem 2.2, Eqs. (48)–(51), and \(C(\lambda_0) \in (\Re \{ R_+^*(0) \}, \Re \{ R_+^*(0) \}).
\]

**Proof.** According to Lemma 5.1, in order to evaluate \(((\Psi^{-1}(x, t; 0)))_1^2\), estimates for \(2 \ln(\chi_{22}(0))\) and \(\{ b_i, \delta_i \}_{i=1}^{n+1} \) are required: the estimation for \(2 \ln(\chi_{22}(0))\) is given in Proposition 5.2; hence, it remains to estimate \(\{ b_i, \delta_i \}_{i=1}^{n+1} \). Solving system (96) as \(t \to +\infty\) and \(x \to -\infty\) such that \(\lambda_0 > M\) and \((x, t) \in \Omega_n\) for \(\{ b_i \}_{i=1}^{n+1} \text{ and } \{ \delta_i \}_{i=1}^{n+1} \text{ via Cramer’s rule, one shows that, } \beta_i \sim O(\exp(-\alpha^y \min_{1 \leq i \leq n-1} |x_n - x_i|)), \text{ } 1 \leq i \leq n-1, \text{ and }
\]

\[
\beta_n = \frac{\tilde{\beta}_n}{(\tilde{E}_n + \tilde{H}_n - \tilde{F}_n + \tilde{G}_n)}, \quad \delta_n = \frac{\tilde{\delta}_n}{(\tilde{E}_n + \tilde{H}_n - \tilde{F}_n + \tilde{G}_n)},
\]

where

\[
\beta_n := \frac{\alpha_1}{\lambda_1} \chi_{11}(\lambda_n) + \frac{\alpha_2}{\lambda_2} \chi_{22}(\lambda_n), \quad \alpha_1 = \chi_{11}^2(\lambda_n), \quad \alpha_2 = \chi_{22}^2(\lambda_n), \quad \alpha_1 \sim \alpha_2 \sim 2M \frac{\chi_{11}^2(\lambda_n) \chi_{22}^2(\lambda_n)}{(\lambda_1^2 - \lambda_n^2)}.
\]

\[
\tilde{E}_n + \tilde{H}_n - \tilde{F}_n + \tilde{G}_n := \frac{1}{\chi_{11}(\lambda_n) \chi_{11}(\lambda_n)} + \frac{\alpha_1}{\lambda_1} \chi_{11}(\lambda_n) \chi_{22}(\lambda_n) + \frac{\alpha_2}{\lambda_2} \chi_{22}(\lambda_n) \chi_{11}(\lambda_n),
\]

Substituting the expressions for \(\chi_{ij}^2(\cdot)\), \(i, j \in \{1, 2\}\), given in Lemma 4.1 into the above equations for \(\beta_n, \delta_n, \text{ and } \tilde{E}_n + \tilde{H}_n - \tilde{F}_n + \tilde{G}_n\), and recalling that \((Eqs. (98)) b_i = \frac{\chi_{11}(\lambda_1) - \chi_{22}(\lambda_n)}{\chi_{22}(\lambda_n)} \beta_i, 1 \leq i \leq n, \text{ one obtains, as a result of Lemma 5.1, keeping only } O(1) \text{ and } O(t^{-1/2}) \text{ terms, the result stated in the Lemma.}
\]

**Corollary 1** As \(t \to +\infty\) and \(x \to -\infty\) such that \(\lambda_0 > M\) and \((x, t) \in \Omega_n\),

\[
q(x, t) = Q^+(x, t) \exp \{ i \arg q^+(x, t) \} + O \left( \frac{C(\lambda_0)(\ln t)^2}{t} \right),
\]

\[
\text{\textit{Proof.}} \text{ The proof follows from the result stated in the \textit{Lemma 5.3}.}
\]
where \( Q_{\alpha}^+(x,t) \) is given in Theorem 2.1, Eqs. (14)–(29), \( \arg q_{\alpha}^+(x,t) \) is given in Theorem 2.2, Eqs. (48)–(51), and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{C}) \).

**Proof.** Consequence of Proposition 2.3 and Lemma 5.3. ■

**Corollary 5.2** As \( t \to +\infty \) and \( x \to +\infty \) such that \( \hat{\lambda}_0 := \sqrt{\frac{1 - \frac{x}{t} - \frac{1}{s}}{\frac{x}{t} + \frac{1}{s}}} > M, \frac{x}{t} > \frac{1}{s}, s \in \mathbb{R}_{>0}, \) and \( (x,t) \in \bar{\Omega}_n, \)

\[
u(x,t) = v_{\alpha}^+(x,t)w_{\alpha}^+(x,t) + O\left(\frac{C(\lambda_0)\ln t}{t}\right),
\]

where \( v_{\alpha}^+(x,t) \) and \( w_{\alpha}^+(x,t) \) are given in Theorem 2.3, Eqs. (58)–(70), and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{C}) \).

**Proof.** Consequence of Proposition 2.4 and Corollary 5.1. ■

## 6 Asymptotics as \( t \to -\infty \)

In this section, the asymptotic paradigm presented in Secs. 3–5 is reworked for the case when \( t \to -\infty \); since the proofs of all obtained results are analogous, they will be omitted. This section is divided into three parts: (1) in Subsection 6.1, extended and model RH problems are formulated as \( t \to -\infty \); (2) in Subsection 6.2, the model RH problem formulated in (1) above is solved asymptotically as \( t \to -\infty \) for the Schwartz class of non-reflectionless generic potentials; and (3) in Subsection 6.3, the phase integral, \( (\Psi^{-1}(x,t;0))_{11}^2 \), is evaluated asymptotically as \( t \to -\infty \).

### 6.1 Extended and Model RH Problems

**Proposition 6.1.1** In the solitonless sector (\( \mathcal{Z}_{\alpha} = \emptyset \)), as \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M, \)

\[
m(x,t;\lambda) = \tilde{\Delta}(\lambda) + O\left(\frac{C(\lambda_0)}{\sqrt{-t}}\right),
\]

where \( \tilde{\Delta}(\lambda) := (\delta^-(\lambda;\lambda_0))^{\rho_+} \)

\[
\delta^-(\lambda;\lambda_0) = ((\lambda - \lambda_0)(\lambda + \lambda_0))^{-i\nu}\exp\left\{ \sum_{\rho \in \{\pm\}} \tilde{\rho}(\lambda) \right\},
\]

\[
\tilde{\rho}(\lambda) = -\frac{1}{2\pi i} \int_{\pm\lambda_0} \ln(\zeta - \lambda) d\ln(1 - |r(\zeta)|^2),
\]

\( \nu := \nu(\lambda_0) \) is given by Eq. (21), \( ||(\delta^-(:,:,\lambda_0))^{\pm1}||_{\mathcal{L}_{\infty}(\mathbb{C};\mathbb{C})} := \sup_{\lambda \in \mathbb{C}} |(\delta^-(:,:,\lambda_0))^{\pm1}| < \infty, \)

\( (\delta^-(:,:,\lambda_0))^{-1} = \delta^-(\lambda;\lambda_0) \), the principal branch of the logarithmic function is taken, \( \ln(\mu - \lambda) := \ln(|\mu - \lambda|) + i \arg(\mu - \lambda), \) \( \arg(\mu - \lambda) \in (-\pi, \pi), \) and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{M}_2(\mathbb{C})) \).

**Remark 6.1.1** Hereafter, all explicit \( x,t \) dependences are suppressed, except where absolutely necessary, and \( \delta^-(:,:,\lambda_0) := \tilde{\delta}(\lambda) \).

**Lemma 6.1.1** There exists a unique solution \( \tilde{m}\tilde{\Delta}(\lambda) := m(\lambda)(\tilde{\Delta}(\lambda))^{-1} : \mathbb{C} \setminus (\mathcal{Z}_{\alpha} \cup \hat{\mathcal{P}}) \to \text{SL}(2,\mathbb{C}) \) of the following RH problem,

1. \( \tilde{m}\tilde{\Delta}(\lambda) \) is meromorphic \( \forall \lambda \in \mathbb{C} \setminus \hat{\mathcal{P}} \),
2. \( \tilde{m}\tilde{\Delta}(\lambda) = \tilde{m}\tilde{\Delta}(\lambda)\tilde{v}\tilde{\Delta}(\lambda), \lambda \in \hat{\mathcal{P}}, \)

where \( \tilde{v}\tilde{\Delta}(\lambda) = e^{-i\theta(\lambda)\text{ad}(\sigma_3)} \begin{pmatrix} (1 - r(\lambda)r(\lambda))\tilde{\delta}^-(\lambda)(\tilde{\delta}^+(\lambda))^{-1} & r(\lambda)\tilde{\delta}^-(\lambda)\tilde{\delta}^+(\lambda) \\ -r(\lambda)(\tilde{\delta}^-(\lambda))^{-1}(\tilde{\delta}^+(\lambda))^{-1} & (\tilde{\delta}^-(\lambda))^{-1}\tilde{\delta}^+(\lambda) \end{pmatrix} \).
3. \( m^b(\lambda) \) has simple poles at \( \{\pm \lambda_i, \pm \lambda_j\}_{i=1}^N \) with \( 1 \leq i \leq N \)

\[
\text{res}(m^b(\lambda); \lambda_i) = \lim_{\lambda \to \lambda_i} m^b(\lambda) \frac{1}{n_i(\hat{\delta}(\lambda_i))^{2\sigma_-}},
\]

\[
\text{res}(m^b(\lambda); -\lambda_i) = -\sigma_3 \text{res}(m^b(\lambda); \lambda_i) \sigma_3,
\]

\[
\text{res}(m^b(\lambda); \lambda_j) = \lim_{\lambda \to \lambda_j} m^b(\lambda) \frac{1}{n_j(\hat{\delta}(\lambda_j))^{2\sigma_+}},
\]

\[
\text{res}(m^b(\lambda); -\lambda_j) = -\sigma_3 \text{res}(m^b(\lambda); \lambda_j) \sigma_3,
\]

4. as \( \lambda \to \infty \), \( \lambda \in \mathbb{C} \setminus (\mathbb{Z}_d \cup \hat{\Gamma}) \),

\[
m^b(\lambda) = I + \mathcal{O}(\lambda^{-1});
\]

moreover, \( Q(x, t) = 2i \lim_{\lambda \to \infty} (\lambda m^b(x, t; \lambda))_{12} \) is equal to \( Q(x, t) \) in Lemma 2.2, Eq. (11).

**Proposition 6.1.2** Introduce arbitrarily small, clockwise- and counter-clockwise-oriented, mutually disjoint (and disjoint with respect to \( \hat{\Gamma} \)) circles \( K^\pm_j \) and \( \bar{L}^\pm_j \), \( 1 \leq j \leq n-1 \), around the eigenvalues \( \{\pm \lambda_j\}_{j=1}^{n-1} \) and \( \{\pm \lambda_i\}_{i=1}^{n-1} \), respectively, and define

\[
m^b(\lambda) := \begin{cases}
m^\pm(\lambda), & \lambda \in \mathbb{C} \setminus (\hat{\Gamma} \cup (\bigcup_{i=1}^{n-1}(K^\pm_i \cup \bar{L}^\pm_i))), \\
n^b(\lambda) \left( 1 - \frac{n_i(\hat{\delta}(\lambda_j))^{2\sigma_-}}{(\lambda - \lambda_j)^2} \right), & \lambda \in \text{int} K^\pm_i, \quad 1 \leq i \leq n-1, \\
^b(\lambda) \left( 1 + \frac{n_i(\hat{\delta}(\lambda_j))^{2\sigma_+}}{(\lambda - \lambda_j)^2} \right), & \lambda \in \text{int} \bar{L}^\pm_i, \quad 1 \leq i \leq n-1.
\end{cases}
\]

Then \( m^b(\lambda) \) solves a RH problem on \( (\sigma_x \cup \bigcup_{i=1}^{n-1}(\{\pm \lambda_j\} \cup \{\pm \lambda_i\}) \cup (\bigcup_{i=1}^{n-1}(K^\pm_i \cup \bar{L}^\pm_i))) \) with the same jumps as \( m^\pm(\lambda) \) on \( \hat{\Gamma} \), \( m^b_+(\lambda) = m^b_-(\lambda) \sigma^\pm(\lambda) \), and

\[
m^b_+(\lambda) := \begin{cases}
m^\pm_+(\lambda), & \lambda \in K^\pm_i, \quad 1 \leq i \leq n-1, \\
m^\pm_-(\lambda), & \lambda \in L^\pm_i, \quad 1 \leq i \leq n-1.
\end{cases}
\]

**Remark 6.1.2** The superscripts \( \pm \) on \( \{K^\pm_i\}_{i=1}^{n-1} \) and \( \{\bar{L}^\pm_i\}_{i=1}^{n-1} \), which are related with \( \{\pm \lambda_j\}_{j=1}^{n-1} \) and \( \{\pm \lambda_i\}_{i=1}^{n-1} \), respectively, should not be confused with the subscripts \( \pm \) appearing in the various RH problems in this and the next subsection, namely, \( m_\pm(\lambda), m^\pm_\pm(\lambda), \)

\( m^\pm_\mp(\lambda), m^\pm_\perp(\lambda), \bar{\chi}_\pm(\lambda), \bar{E}_\pm(\lambda), \) and \( \bar{\chi}_\perp(\lambda). \)

**Lemma 6.1.2** Set

\[
m^b(\lambda) = \prod_{l=1}^{n-1} (d_{l+}(\lambda))^{-\sigma_3}, \quad \lambda \in \mathbb{C} \setminus (\hat{\Gamma} \cup (\bigcup_{i=1}^{n-1}(K^\pm_i \cup \bar{L}^\pm_i))),
\]

where

\[
d_{l+}(\lambda) := \frac{(\lambda - \lambda_i)(\lambda + \lambda_j)}{(\lambda - \lambda_i)(\lambda + \lambda_j)}, \quad \lambda \in \mathbb{C} \setminus (\bigcup_{i=1}^{n-1}(K^\pm_i \cup \bar{L}^\pm_i)), \quad 1 \leq l \leq n-1,
\]

\[
d_{l-}(\lambda) := \begin{cases}
\frac{(\lambda - \lambda_i)(\lambda + \lambda_j)}{(\lambda - \lambda_i)(\lambda + \lambda_j)}, & \lambda \in \bigcup_{i=1}^{n-1} \text{int} K^\pm_i, \quad 1 \leq l \leq n-1, \\
\frac{(\lambda - \lambda_i)(\lambda + \lambda_j)}{(\lambda - \lambda_i)(\lambda + \lambda_j)}, & \lambda \in \bigcup_{i=1}^{n-1} \text{int} \bar{L}^\pm_i, \quad 1 \leq l \leq n-1.
\end{cases}
\]
and the SL(2,ℂ)-valued, holomorphic in int\(\tilde{K}_i^\pm\) and int\(\tilde{L}_i^\pm\), respectively, functions \(\tilde{J}_{K_i}^\pm(\lambda)\) and \(\tilde{J}_{L_i}^\pm(\lambda)\), 1 ≤ i ≤ n−1, are given by

\[
\tilde{J}_{K_i}^\pm(\lambda) = \left( \begin{array}{cc}
\prod_{l=1}^{n-i} \frac{d_{i-1}^{-1}(\lambda)}{d_{i+1}^{-1}(\lambda)} & \frac{\bar{c}_i^2}{(d_{i+1}^{-1}(\lambda))^2} \prod_{l=1}^{n-i} \frac{d_{i-1}^{-1}(\lambda)}{d_{i+1}^{-1}(\lambda)} \\
-\bar{v}_i(\tilde{\delta}(\pm\lambda_i)) - 2^{n-1} \prod_{l=1}^{n-i} \frac{d_{i-1}(\lambda)}{d_{i+1}(\lambda)} & \frac{\bar{c}_i}{(d_{i+1}^{-1}(\lambda))^2} \prod_{l=1}^{n-i} \frac{d_{i-1}(\lambda)}{d_{i+1}(\lambda)}
\end{array} \right),
\]

\[
\tilde{J}_{L_i}^\pm(\lambda) = \left( \begin{array}{cc}
\prod_{l=1}^{n-i} \frac{d_{i+1}^{-1}(\lambda)}{d_{i-1}^{-1}(\lambda)} & \frac{\bar{c}_i^2}{(d_{i-1}^{-1}(\lambda))^2} \prod_{l=1}^{n-i} \frac{d_{i+1}(\lambda)}{d_{i-1}(\lambda)} \\
-\bar{v}_i(\tilde{\delta}(\pm\lambda_i)) - 2^{n-1} \prod_{l=1}^{n-i} \frac{d_{i+1}(\lambda)}{d_{i-1}(\lambda)} & \frac{\bar{c}_i}{(d_{i-1}^{-1}(\lambda))^2} \prod_{l=1}^{n-i} \frac{d_{i+1}(\lambda)}{d_{i-1}(\lambda)}
\end{array} \right),
\]

with

\[
\bar{c}_i = (\bar{v}_i)^{-1}(\tilde{\delta}(\pm\lambda_i))^2(\bar{d}_{i-1}(\pm\lambda_i))^2 \prod_{l=1}^{n-i} (\bar{d}_{i+1}(\pm\lambda_i))^2, \quad 1 ≤ i ≤ n−1.
\]

Then \(\tilde{m}_\lambda^\sharp(\lambda) : \mathbb{C} \setminus ((\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\}) \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\}) \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\})) \to \text{SL}(2,\mathbb{C})\) solves the following, extended RH problem on \((\sigma_L \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\} \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\} \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\})))\),

\[
\tilde{m}_\lambda^\sharp(\lambda) = \tilde{m}_\lambda^\sharp(\lambda)e^{-i\theta(\lambda)\text{ad}(\sigma_3)}\bar{v}(\lambda),
\]

where

\[
\bar{v}(\lambda) = \left( \begin{array}{cc}
(1 - r(\lambda)r(\overline{\lambda})) - 2(\overline{\lambda} - \lambda) & r(\lambda)(\overline{\lambda} - \lambda) - 2(\overline{\lambda} - \lambda) \\
-(\overline{\lambda} - \lambda) & -(\overline{\lambda} - \lambda)
\end{array} \right),
\]

\[
\bar{v}(\lambda) = \left( \begin{array}{cc}
\frac{1}{(\lambda - \lambda_i)(\lambda + \lambda) \delta_i(-\lambda)} & \frac{1}{(\lambda - \lambda_i)(\lambda - \lambda) \delta_i(-\lambda)} \\
\frac{1}{(\lambda - \lambda_i)(\lambda + \lambda) \delta_i(-\lambda)} & \frac{1}{(\lambda - \lambda_i)(\lambda - \lambda) \delta_i(-\lambda)}
\end{array} \right),
\]

with polar (residue) conditions,

\[
\text{res}(\tilde{m}_\lambda^\sharp(\lambda); \lambda_i) = \lim_{\lambda \to \lambda_i} \tilde{m}_\lambda^\sharp(\lambda)v_i(\tilde{\delta}(\lambda_i)) - 2^{n-1} \prod_{l=1}^{n-i} \frac{(\lambda_i - \lambda_i)(\lambda_i + \lambda)}{(\lambda_i - \lambda_i)(\lambda_i + \lambda)} \sigma_-,
\]

\[
\text{res}(\tilde{m}_\lambda^\sharp(\lambda); -\lambda_i) = -\sigma_3 \text{res}(\tilde{m}_\lambda^\sharp(\lambda); \lambda_i) \sigma_3,
\]

\[
\text{res}(\tilde{m}_\lambda^\sharp(\lambda); \overline{\lambda}_i) = \lim_{\lambda \to \overline{\lambda}_i} \tilde{m}_\lambda^\sharp(\lambda)v_i(\tilde{\delta}(\overline{\lambda}_i)) - 2^{n-1} \prod_{l=1}^{n-i} \frac{(\overline{\lambda}_i - \lambda_i)(\overline{\lambda}_i + \lambda)}{(\overline{\lambda}_i - \lambda_i)(\overline{\lambda}_i + \lambda)} \sigma_+,
\]

\[
\text{res}(\tilde{m}_\lambda^\sharp(\lambda); -\overline{\lambda}_i) = -\sigma_3 \text{res}(\tilde{m}_\lambda^\sharp(\lambda); \overline{\lambda}_i) \sigma_3,
\]

and, as \(\lambda \to \infty\), \(\lambda \in \mathbb{C} \setminus ((\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\}) \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\}) \cup (\mathbb{Z}_d \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\}))\),

\[
\tilde{m}_\lambda^\sharp(\lambda) = 1 + O(\lambda^{-1});
\]

moreover, \(Q(x,t) = 2i \lim_{\lambda \to \infty} (\lambda \tilde{m}_\lambda^\sharp(x,t;\lambda))_{12} = \text{equal to} \ Q(x,t) \text{ in Lemma 2.2, Eq. (11)}.
\]

**Lemma 6.1.3** Let \(\tilde{\chi}(\lambda)\) solve the following RH problem on \((\sigma_L \cup \mathbb{Z}_{i=1}^{n-1}\{\pm\lambda_i\} \cup \{\pm\lambda_i\})\),

\[
\tilde{\chi}_+ (\lambda) = \tilde{\chi}_-(\lambda)e^{-i\theta(\lambda)\text{ad}(\sigma_3)}\bar{v}(\lambda)|_{\Gamma}, \quad \lambda \in \Gamma,
\]
with polar (residue) conditions,
\[
\text{res}(\tilde{\chi}(\lambda);\lambda_i) = \lim_{\lambda \to \lambda_i} \tilde{\chi}(\lambda)v_i(\tilde{\delta}(\lambda_i))^{-2} \prod_{l=1}^{n-1} \left( \frac{(\lambda_i-\lambda_l)(\lambda_i+\lambda_l)}{(\lambda_i-\lambda_l)(\lambda_i+\lambda_l)} \right)^2 \sigma_-, \quad n \leq i \leq N,
\]
\[
\text{res}(\tilde{\chi}(\lambda);-\lambda_i) = -\sigma_3\text{res}(\tilde{\chi}(\lambda);\lambda_i)\sigma_3, \quad n \leq i \leq N,
\]
\[
\text{res}(\tilde{\chi}(\lambda);\lambda_i) = \lim_{\lambda \to \lambda_i} \tilde{\chi}(\lambda)v_i(\tilde{\delta}(\lambda_i))^{-2} \prod_{l=1}^{n-1} \left( \frac{(\lambda_i-\lambda_l)(\lambda_i+\lambda_l)}{(\lambda_i-\lambda_l)(\lambda_i+\lambda_l)} \right)^2 \sigma_+, \quad n \leq i \leq N,
\]
\[
\text{res}(\tilde{\chi}(\lambda);-\lambda_i) = -\sigma_3\text{res}(\tilde{\chi}(\lambda);\lambda_i)\sigma_3, \quad n \leq i \leq N,
\]
and, as \(\lambda \to \infty\), \(\lambda \in \mathbb{C}\setminus(\mathbb{U}_i)\),
\[
\tilde{\chi}(\lambda) = I + \mathcal{O}(\lambda^{-1}).
\]
Then as \(t \to -\infty\) and \(x \to +\infty\) such that \(\lambda_0 > M\) and \((x,t) \in \Omega_n\), the function \(\tilde{E}(\lambda) := \tilde{m}^\circ(\tilde{\chi}(\lambda))^{-1}\) has the following asymptotics,
\[
\tilde{E}(\lambda) = I + \mathcal{O}(\tilde{E}(\lambda;\lambda_0)\exp\{\tilde{a}t\}),
\]
where \(\|\tilde{E}(\cdot;\lambda_0)\|_{L^\infty(\mathbb{C};M_2(\mathbb{C}))} < \infty\) and \(\|\tilde{E}(\lambda;\cdot)\|_{L^\infty(\mathbb{R}_x;M_2(\mathbb{C}))} < \infty\), \(\tilde{E}(\lambda;\lambda_0) \sim \mathcal{O}\left(\frac{C(\lambda_0)}{\lambda}\right)\) as \(\lambda \to \infty\) with \(C(\lambda_0) \in L^\infty(\mathbb{R}_x;M_2(\mathbb{C}))\), \(\tilde{a} := 8\min\{\eta_i\}_{i=1}^{n-1}(>0)\), and \(\tilde{b} := \min\{\xi_n-\xi_i\}_{i=1}^{n-1}\).

### 6.2 Asymptotic Solution for \(\tilde{\chi}(\lambda)\)

#### Proposition 6.2.1

The solution of the model RH problem formulated in Lemma 6.1.3, \(\tilde{\chi}(\lambda) : \mathbb{C}\setminus(\mathbb{U}_i) \to \tilde{\text{SL}}(2;\mathbb{C}),\) has the following representation,
\[
\tilde{\chi}(\lambda) = \tilde{\chi}_d(\lambda) + \int_{\tilde{\Gamma}} \frac{\tilde{\chi}(\tilde{\theta})|\tilde{\theta}(\tilde{\theta})|^{\tilde{\theta}} - I}{(\tilde{\theta} - \lambda)} \frac{d\tilde{\theta}}{2\pi i}, \tag{107}
\]
where
\[
\tilde{\chi}_d(\lambda) = I + \sum_{i=1}^N \left( \frac{\text{res}(\tilde{\chi}(\lambda);\lambda_i)}{\lambda - \lambda_i} - \frac{\sigma_3\text{res}(\tilde{\chi}(\lambda);\lambda_i)\sigma_3}{\lambda + \lambda_i} + \frac{\text{res}(\tilde{\chi}(\lambda);\lambda_i)}{\lambda - \lambda_i} - \frac{\sigma_3\text{res}(\tilde{\chi}(\lambda);\lambda_i)\sigma_3}{\lambda + \lambda_i} \right). \tag{108}
\]
The solution of Eq. (107) can be written as the following ordered product,
\[
\tilde{\chi}(\lambda) = \tilde{\chi}_d(\lambda)\tilde{\chi}^c(\lambda),
\]
where \(\tilde{\chi}_d(\lambda)\) is given by Eq. (108), and \(\tilde{\chi}^c(\lambda)\) solves the following RH problem:
1. \(\tilde{\chi}^c(\lambda)\) is piecewise holomorphic \(\forall \lambda \in \mathbb{C}\setminus\tilde{\Gamma}\);
2. \(\tilde{\chi}^c(\lambda) = \tilde{\chi}^c(\lambda)\exp\{-i\theta(\lambda)\text{ad}(\sigma_3)\}(\tilde{\theta}(\lambda)|\tilde{\theta}|^{\tilde{\theta}}), \lambda \in \tilde{\Gamma};\) and
3. as \(\lambda \to \infty, \lambda \in \mathbb{C}\setminus\tilde{\Gamma}, \tilde{\chi}^c(\lambda) = I + \mathcal{O}(\lambda^{-1})\).

#### Lemma 6.2.1

Let \(\tilde{\epsilon}_0\) denote an arbitrarily fixed, sufficiently small positive real number. For \(\tilde{\epsilon} \in \{0, \pm \lambda_0\}\), set \(\tilde{\mathcal{N}}(\tilde{\epsilon};\tilde{\epsilon}_0) := \{\lambda; |\lambda - \tilde{\epsilon}_0| \leq \tilde{\epsilon}_0\}\). Then as \(t \to -\infty\) and \(x \to +\infty\) such that \(\lambda_0 > M\) and \(\lambda \in \mathbb{C}\setminus\cup_{\tilde{\epsilon}_0 \in \{0, \pm \lambda_0\}} \tilde{\mathcal{N}}(\tilde{\epsilon};\tilde{\epsilon}_0)\), \(\tilde{\chi}(\lambda)\) has the following asymptotic expansion,
\[
\tilde{\chi}(\lambda) = 1 + \frac{1}{4} \sqrt{-\frac{\nu(\lambda_0)}{2\lambda_0^2}} \left( \frac{1}{\lambda - \lambda_0} + \frac{1}{\lambda + \lambda_0} \right) \left( \exp\{-i\phi^c(\lambda_0) + \tilde{\phi}^c(\lambda_0; t)\}\right) \sigma_-
\]
\[
+ \exp\{i\phi^c(\lambda_0) + \tilde{\phi}^c(\lambda_0; t)\}\sigma_+ + \mathcal{O}\left(\frac{G(\lambda;\lambda_0)\ln|t|}{t}\right),
\]
where \(\nu(\lambda_0), \phi^c(\lambda_0),\) and \(\tilde{\phi}^c(\lambda_0; t)\) are given in Theorem 2.1, Eqs. (21), (23), and (24), \(\|\tilde{G}(\cdot;\lambda_0)\|_{L^\infty(\mathbb{C}\setminus\cup_{\tilde{\epsilon}_0 \in \{0, \pm \lambda_0\}} \tilde{\mathcal{N}}(\tilde{\epsilon};\tilde{\epsilon}_0))} < \infty\), \(\tilde{G}(\cdot;\lambda) \in \tilde{\mathcal{S}}(\mathbb{R}_x;M_2(\mathbb{C}))\), \(\tilde{G}(\lambda;\lambda_0) \sim \mathcal{O}\left(\frac{C(\lambda_0)}{\lambda}\right)\) as \(\lambda \to \infty\) with \(C(\lambda_0) \in \tilde{\mathcal{S}}(\mathbb{R}_x;M_2(\mathbb{C}))\), and satisfies the following involutions, \(\tilde{\chi}^c(-\lambda) = \sigma_3\tilde{\chi}^c(\lambda)\sigma_3\) and \(\tilde{\chi}^c(\lambda) = \sigma_1\tilde{\chi}^c(\lambda)\sigma_1\).
Proposition 6.2.2 For \( n \leq i \leq N \), set

\[
\text{res}(\tilde{x}(\lambda); \lambda_i) = \begin{pmatrix} a_i^- & a_i^- \cr \beta_i^- & b_i^- \end{pmatrix}, \quad \text{res}(\tilde{x}(\lambda); \lambda_\infty) = \begin{pmatrix} e_i^- & \omega_i^- \cr d_i^- & \delta_i^- \end{pmatrix}.
\]

Then as \( \lambda \to \infty \), \( \lambda \in \mathbb{C} \setminus (\Gamma \cup (\cup_{i=1}^N \{ \pm \lambda_i \} \cup \{ \pm \lambda_\infty \})) \), \( \tilde{x}(\lambda) \) has the following asymptotic expansion,

\[
\tilde{x}(\lambda) = I + \frac{1}{2\lambda} \left( \begin{pmatrix} Q(X, t) + 4 \sum_{i=n}^{N} \left( \beta_i^- - \frac{\chi_{21}^c(\lambda_i)}{\chi_{11}^c(\lambda_i)} \delta_i^- \right) \end{pmatrix} \sigma_- \right. \\
+ \left. \begin{pmatrix} Q(X, t) + 4 \sum_{i=n}^{N} \left( \omega_i^- - \frac{\chi_{12}^c(\lambda_i)}{\chi_{22}^c(\lambda_i)} \alpha_i^- \right) \end{pmatrix} \sigma_+ \right) + O(\lambda^{-2}),
\]

where \( \lim_{\lambda \to \infty} \chi^c(x, t; \lambda)_{12} := \frac{Q(X, t)}{2\lambda} \), \( \{ \alpha_i^-, \omega_i^- \}_{i=n}^{N} \) satisfy the following non-degenerate system of \( 2(N-n+1) \) linear inhomogeneous algebraic equations,

\[
\begin{pmatrix}
\hat{A}^- & \hat{B}^- \\
\hat{C}^- & \hat{D}^-
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix}
\alpha_n^- \\
\alpha_{n+1}^-
\end{pmatrix} \\
\vdots \\
\begin{pmatrix}
\alpha_N^- \\
\omega_n^- \\
\omega_{n+1}^- \\
\vdots \\
\omega_N^-
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix}
g_n^- \chi_{12}^c(\lambda_n^-) \\
g_{n+1}^- \chi_{12}^c(\lambda_{n+1}^-)
\end{pmatrix} \\
\vdots \\
\begin{pmatrix}
g_N^- \chi_{11}^c(\lambda_N^-) \\
g_{n+1}^- \chi_{11}^c(\lambda_{n+1}^-)
\end{pmatrix}
\end{pmatrix},
\]

where, for \( i, j \in \{n, n+1, \ldots, N\} \), the \((N-n+1) \times (N-n+1)\) matrix blocks, \( \hat{A}^-, \hat{B}^-, \hat{C}^-, \) and \( \hat{D}^- \), are defined as follows,

\[
\hat{A}_{ij} := \begin{cases}
\frac{\lambda_i^+ \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-) + \lambda_i^- \chi_{21}^c(\lambda_i^-) \chi_{22}^c(\lambda_i^-)}{\chi_{22}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^+)} + \lambda_i^- \chi_{21}^c(\lambda_i^-) \chi_{22}^c(\lambda_i^-), & i = j, \\
-2\lambda_i^+ (-\lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) + \lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-)) \chi_{22}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^+), & i \neq j,
\end{cases}
\]

\[
\hat{B}_{ij} := \begin{cases}
-\frac{2\gamma_i^- (\lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) - \lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-)) \chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}, & i = j, \\
-\frac{2\gamma_i^- (\lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) - \lambda_i \chi_{22}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-)) \chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}, & i \neq j,
\end{cases}
\]

\[
\hat{C}_{ij} := \begin{cases}
\frac{2\chi_i^- \chi_{11}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)} + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-), & i = j, \\
-\frac{2\chi_i^- \chi_{11}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}, & i \neq j,
\end{cases}
\]

\[
\hat{D}_{ij} := \begin{cases}
\frac{\lambda_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-) + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)} + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-), & i = j, \\
-\frac{2\chi_i^- \chi_{11}^c(\lambda_i^-) \chi_{12}^c(\lambda_i^-) + \chi_i^- \chi_{12}^c(\lambda_i^-) \chi_{11}^c(\lambda_i^-)}{\chi_{11}^c(\lambda_i^-) (\lambda_i^- - \lambda_i^-)}, & i \neq j,
\end{cases}
\]
\{ \beta_i^-, \delta_i^- \}_{i=1}^N \) satisfy the following non-degenerate system of \( 2(N-n+1) \) linear inhomogeneous algebraic equations,

\[
\left[ \begin{array}{cc}
\check{\mathcal{E}}^- & \check{\mathcal{F}}^- \\
\hat{\mathcal{G}}^- & \hat{\mathcal{H}}^-
\end{array} \right]
\left[ \begin{array}{c}
\beta_n^- \\
\beta_{n+1}^- \\
\vdots \\
\beta_N^- \\
\delta_n^- \\
\delta_{n+1}^- \\
\vdots \\
\delta_N^-
\end{array} \right] =
\left[ \begin{array}{c}
g_n^- \tilde{\chi}_{22}^\dagger(\lambda_n) \\
g_{n+1}^- \tilde{\chi}_{22}^\dagger(\lambda_{n+1}) \\
\vdots \\
g_N^- \tilde{\chi}_{22}^\dagger(\lambda_N) \\
g_{n+1}^- \tilde{\chi}_{21}^\dagger(\lambda_{n+1}) \\
\vdots \\
g_N^- \tilde{\chi}_{21}^\dagger(\lambda_N)
\end{array} \right],
\]

where, for \( i, j \in \{n, n+1, \ldots, N\} \), the \((N-n+1) \times (N-n+1)\) matrix blocks, \( \check{\mathcal{E}}^- \), \( \check{\mathcal{F}}^- \), \( \hat{\mathcal{G}}^- \), and \( \hat{\mathcal{H}}^- \), are defined as follows,

\[
\check{\mathcal{E}}_{ij}^- := \begin{cases}
\lambda_i^- - \gamma_i^- \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) + \lambda_i g_i^- W(\tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i = j, \\
2 \gamma_i^- (\lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i \neq j,
\end{cases}
\]

\[
\check{\mathcal{F}}_{ij}^- := \begin{cases}
2 \gamma_i^- (\lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i = j, \\
2 \gamma_i^- (\lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i \neq j,
\end{cases}
\]

\[
\hat{\mathcal{G}}_{ij}^- := \begin{cases}
\lambda_i^- + \gamma_i^- \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i g_i^- W(\tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i = j, \\
2 \gamma_i^- (\lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i \neq j,
\end{cases}
\]

\[
\hat{\mathcal{H}}_{ij}^- := \begin{cases}
\lambda_i^- + \gamma_i^- \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i g_i^- W(\tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i = j, \\
2 \gamma_i^- (\lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i) - \lambda_i \tilde{\chi}_{12}^\dagger(\lambda_i) \tilde{\chi}_{22}^\dagger(\lambda_i)), & i \neq j,
\end{cases}
\]

with \( a_i^- = -\frac{\gamma_{12}}{\lambda_{22}^\dagger(\lambda_i)} \alpha_i^- \), \( b_i^- = -\frac{\gamma_{12}}{\lambda_{22}^\dagger(\lambda_i)} \beta_i^- \), \( c_i^- = -\frac{\gamma_{21}}{\lambda_{11}^\dagger(\lambda_i)} \omega_i^- \), \( d_i^- = -\frac{\gamma_{21}}{\lambda_{11}^\dagger(\lambda_i)} \delta_i^- \), \( n \leq i \leq N \),

\[
g_j^- := C_j e^{2 \lambda_j^2 x + 4 i \lambda_j^2 t} (\delta^- (\lambda_j; \lambda_0))^{-2} \prod_{l=1}^{n-1} \left( \frac{\lambda_l^2 - \lambda_j^2}{\lambda_l^2 - \lambda_0^2} \right), \quad n \leq j \leq N,
\]

\( \delta^- (\lambda_k; \lambda_0), k \in \{n, n+1, \ldots, N\} \), given in Theorem 2.1, Eq. (19), and \( W(\tilde{\chi}_{ij}^\dagger(\lambda), \tilde{\chi}_{ij}^\dagger(\lambda)) \) is the Wronskian of \( \tilde{\chi}_{ij}^\dagger(\lambda) \) and \( \tilde{\chi}_{ij}^\dagger(\lambda) \) evaluated at \( z \) \((i, j, i', j' \in \{1, 2\}): W(\tilde{\chi}_{ij}^\dagger(\lambda), \tilde{\chi}_{ij}^\dagger(\lambda)) := (\tilde{\chi}_{ij}^\dagger(\lambda) \partial_i \tilde{\chi}_{ij}^\dagger(\lambda) - \tilde{\chi}_{ij}^\dagger(\lambda) \partial_i \tilde{\chi}_{ij}^\dagger(\lambda))_{|z=z}.
\]

**Corollary 6.2.1** As \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \),

\[
Q(x, t) = Q_C^\mathcal{E}(x, t) + 4 i \sum_{j=n}^N \left( \omega_j^- - \frac{\gamma_{12}}{\lambda_{22}^\dagger(\lambda_j)} \alpha_j^- \right) + O(C(\lambda_0)^{\exp\{\tilde{a} bt\}}),
\]

where \( Q_C^\mathcal{E}(x, t) := i Q_N(x, t) \), \( \tilde{a} \) and \( \tilde{b} \) are given in Lemma 6.1.3, and \( C(\lambda_0) \in L^\infty(\mathbb{R}>M; \mathbb{C}) \).

**Proposition 6.2.3** As \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M \) and \( (x, t) \in \Omega_n \),

\[
Q(x, t) = Q_C^\mathcal{E}(x, t) + 4 i \left( \omega_n^- - \frac{\gamma_{12}}{\lambda_{22}^\dagger(\lambda_n)} \alpha_n^- \right) + O\left( \frac{C_1(\lambda_0) \ln t}{t} \right) + O(C_2(\lambda_0)e^{\tilde{a} \tilde{b} t}),
\]

\[
\frac{\gamma_{12}}{\lambda_{22}^\dagger(\lambda_n)} \alpha_n^-.
\]
where \( \alpha^-_n, \omega^-_n \sim \mathcal{O}(\exp\{\bar{a}^0 \min_{n+1 \leq i \leq N} |\xi_n - \xi_i|\}) \), \( n+1 \leq i \leq N \), \( \bar{a}^0 := 8 \min \{\eta_i\}_{i=n+1}^N (>0) \),
\[
\begin{align*}
\alpha^-_n &= \overline{a}_{11} g_n \chi^c_{11}(\lambda_n) + \overline{a}_{12} g_n \chi^c_{12}(\lambda_n), \\
\omega^-_n &= \overline{a}_{11} g_n \chi^c_{11}(\lambda_n) + \overline{a}_{12} g_n \chi^c_{12}(\lambda_n), \\
\overline{a}_{11} &= g_n \chi^c_{11}(\lambda_n) + g_n \chi^c_{11}(\lambda_n), \\
\overline{a}_{12} &= g_n \chi^c_{11}(\lambda_n) + g_n \chi^c_{12}(\lambda_n), \\
\overline{a}_{21} &= g_n \chi^c_{11}(\lambda_n) + g_n \chi^c_{11}(\lambda_n), \\
\overline{a}_{22} &= g_n \chi^c_{12}(\lambda_n) + g_n \chi^c_{12}(\lambda_n),
\end{align*}
\]
\( Q^c(x,t) \) is given in Theorem 2.1, Eqs. (20), (21), (23) and (24), \( \bar{a}_0 := \min(\bar{a}, \bar{a}^2) (>0) \), \( \bar{b}_0 := \min(b, \min\{|\xi_n - \xi_i|\}_{i=n+1}^N) \), \( C_1(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{C}) \), and \( C_2(\lambda_0) \in \mathcal{L}^\infty(\mathbb{R}_{>M};\mathbb{C}) \).

**Proposition 6.2.4** As \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M \) and \( (x,t) \in \Omega_n \),
\[
Q(x,t) = Q^-_x(x,t) + \mathcal{O}\left(\frac{C(\lambda_0) \ln |t|}{t}\right),
\]
where \( Q^-_x(x,t) \) is given in Theorem 2.1, Eqs. (14)–(29), and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{C}) \).

**6.3 Asymptotics of \( (\Psi^{-1}(x,t;0))_1 \) as \( t \to -\infty \)**

**Proposition 6.3.1** Define \( Q^c(x,t) := 2i \lim_{\lambda \to -\infty} (\lambda \chi^c(x,t;\lambda))_{12} \). Then
\[
(\chi^c_{22}(0))^2 = (\delta^-(0;\lambda_0))^2 \exp\{\int_{-\infty}^t |Q^c(\rho, t)|^2 d\rho\}.
\]

**Lemma 6.3.1** \( (\overline{\chi}^c_{22}(0))^2 \)

As \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M \),
\[
Q^c(x,t) = u_{-1,1,0}(\lambda_0) e^{i(4\lambda^2 + v(\lambda_0) \ln |t|)} + u_{-1,2,0}(\lambda_0) + \mathcal{O}\left(\frac{C(\lambda_0)(\ln |t|)^2}{(-t)^{1/2}}\right),
\]
where
\[
u(\lambda_0) = \sqrt{-t} \exp\{i\theta^-(\lambda_0)\},
\]
\[
\theta^-(\lambda_0) = \phi^-(\lambda_0) + \frac{3\pi}{4} - \arg \Gamma(i\nu(\lambda_0)) + \arg r(\lambda_0) + 3\nu(\lambda_0) \ln 2 + 2 \sum_{l=1}^{n-1} \arg\left(\frac{(\lambda_0 - \lambda_l)(\lambda_0 + \lambda_l)}{(\lambda_0 - \lambda_l)(\lambda_0 + \lambda_l)}\right),
\]
\[
\begin{align*}
&u_{-1,1,0}(\lambda_0) = \sqrt{\frac{\nu(\lambda_0)}{2\lambda_0^2}} \exp\{i\theta^-(\lambda_0)\}, \\
&\frac{d\nu(\rho)}{d\rho}\bigg|_{\rho=0} - \frac{d\nu(\rho)}{d\rho}\bigg|_{\rho=\infty} \exp\{i\left(4 \sum_{l=1}^{n-1} \gamma_l + 2\theta^-(\lambda_0)\right)\}, \\
&\theta^-(\lambda_0) = -\int_0^{\infty} \ln(1-r(\rho)^2) d\rho.
\end{align*}
\]
\( \phi^-(-) \) is given in Theorem 2.1, Eq. (23), and \( C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M};\mathbb{C}) \).

**Proposition 6.3.2** As \( t \to -\infty \) and \( x \to +\infty \) such that \( \lambda_0 > M \),
\[
(\chi^c_{22}(0))^2 = \exp\left\{i\left(\frac{2}{-t} \int_{-\lambda_0}^{\infty} \frac{\sqrt{\nu(\mu)}}{\mu^2} \left(R_i(0) \cos(\kappa^- - \mu; t) - R_r(0) \sin(\kappa^- - \mu; t)\right) \frac{d\mu}{\pi}\right)\right\}
\]
\[
+ \mathcal{O}\left(\frac{C(\lambda_0)(\ln |t|)^2}{\lambda_0^2 t}\right),
\]
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where $R_i^-(0) = \mathbb{3}\{R^-(0)\}$, $R_r^-(0) = \mathbb{R}\{R^-(0)\}$, $R^-(0) := \left( \left. \frac{d(\rho(\theta))}{d\theta} \right|_{\theta=0} - \left. \frac{d(\rho(\theta))}{d\theta} \right|_{\theta=0} \right) 
abla \exp \left\{ 4i \sum_{r}^{n-1} \gamma_r \right\}$, $\kappa^-(\lambda_0; t) := 4\lambda_0^{\frac{1}{2}} + \lambda_0^{\frac{1}{2}} \ln|t| + \theta^- (\lambda_0) - 2\theta^- (\lambda_0)$, and $C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M}; \mathbb{C})$.

Lemma 6.3.2 As $t \to -\infty$ and $x \to +\infty$ such that $\lambda_0 > M$ and $(x, t) \in \Omega_n$,

$$((\Psi^{-1}(x; t; 0))_{11})^2 = \exp \left\{ 2\ln(\tilde{x}_2(0)) \right\} \exp \left\{ \frac{2i}{\pi} \int_{\lambda_0}^{\infty} \frac{\ln(1-|r(\phi)|^2)}{\phi} d\phi \right\} \exp \left\{ -4i \sum_{r}^{n-1} \gamma_r \right\}$$

$$\times \exp \left\{ 2\ln(1 - \sum_{i=1}^{N} \frac{2\rho_i^-}{\lambda_i} + \frac{2\tilde{\gamma}_i^-}{\lambda_i}) \right\} + O(C(\lambda_0), \exp(\tilde{a} t),)$$

where $(\tilde{x}_2(0))^2$ is given in Proposition 6.3.2, $b_i^- = \frac{\tilde{W}_i(x, \lambda_0)}{\tilde{x}_2(x, \lambda_0)} \beta_i^-$, $n \leq i \leq N$, $b_j^-, \hat{\gamma}_j^- \sim O(\exp{\tilde{a} \min{n+1 \leq j \leq N} |\xi_n - \xi_j|})$, $n+1 \leq j \leq N$,

$$\beta_n^- = \frac{\rho_n^- N}{(E_{nn} - \hat{x}_n \tilde{\gamma}_n)}, \quad \hat{\gamma}_n^- = \frac{\tilde{\gamma}_n^- N}{(E_{nn} - \hat{x}_n \tilde{\gamma}_n)},$$

with

$$\beta_n^- := \frac{g_n \tilde{W}_n(x, \lambda_n)}{\lambda_n} + \frac{1}{\lambda_n} \frac{g_n}{\lambda_n} \tilde{W}_n(x, \lambda_n),$$

$$\hat{\gamma}_n^- := \frac{g_n \tilde{W}_n(x, \lambda_n)}{\lambda_n} \tilde{W}_n(x, \lambda_n) - \frac{g_n \tilde{W}_n(x, \lambda_n)}{\lambda_n} \tilde{W}_n(x, \lambda_n),$$

$$\hat{\gamma}_n^- := \frac{g_n \tilde{W}_n(x, \lambda_n)}{\lambda_n} \tilde{W}_n(x, \lambda_n) - \frac{g_n \tilde{W}_n(x, \lambda_n)}{\lambda_n} \tilde{W}_n(x, \lambda_n),$$

and $C(\lambda_0) \in \mathcal{L}^\infty(\mathbb{R}_{>M}; \mathbb{C})$.

Corollary 6.3.1 As $t \to -\infty$ and $x \to +\infty$ such that $\lambda_0 > M$ and $(x, t) \in \Omega_n$,

$$q(x, t) = Q_{as}(x, t) \exp \left\{ i \arg Q^{-}_{as}(x, t) \right\} + O \left( \frac{C(\lambda_0) (|\ln|t||)^2}{t} \right),$$

where $Q_{as}(x, t)$ is given in Theorem 2.1, Eqs. (14)–(29), $\arg Q^{-}_{as}(x, t)$ is given in Theorem 2.2, Eqs. (48)–(51), and $C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M}; \mathbb{C})$.

Corollary 6.3.2 As $t \to -\infty$ and $x \to -\infty$ such that $\lambda_0 := \sqrt{\frac{1}{2} (\frac{2}{t} - \frac{1}{s})} > M$, $\frac{2}{t} > \frac{1}{s}$, $s \in \mathbb{R}_{>0}$, and $(x, t) \in \Omega_n$,

$$u(x, t) = v_{as}^-(x, t) w_{as}^-(x, t) + O \left( \frac{C(\lambda_0) (|\ln|t||)^2}{t} \right),$$

where $v_{as}^-(x, t)$ and $w_{as}^-(x, t)$ are given in Theorem 2.3, Eqs. (58)–(70), and $C(\lambda_0) \in \mathcal{S}(\mathbb{R}_{>M}; \mathbb{C})$.

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