FREE PRODUCT OF OPERADS AND FREE BASIS OF
LIE-ADMISSIBLE OPERAD

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Abstract. In this paper, we introduce the definition of a free product of operads following the definition of a free product of algebras. We present a method for finding the basis and dimension of the free product of operads. We prove that the Lie-admissible operad is isomorphic to the free product of Lie and commutative (but non-associative) operads.

1. Introduction

A wide range of algebraic systems in nonassociative algebra is presented by linear spaces with two operations satisfying certain axioms. So are Poisson algebras, Novikov–Poisson algebras [10], Gelfand–Dorfman (bi)algebras [5], [11]. Each operad governing either of the corresponding varieties of algebras is a quotient of the operad $O$ generated by two different operations without identities that contain both these products. A natural construction that describes such an operad from a more general point of view is the free product of operads.

Obviously, if operads $O_i$, $i = 1, 2$, have the graded spaces of generators $V_i$ and defining relations $\Sigma_i \subset \bigcup_{n \geq 1} F(V_i)(n)$ then $O_1 * O_2$ (* symbol of free product of operads) is the operad generated by $V_1 \oplus V_2$ with defining relations $\Sigma_1 \cup \Sigma_2$. For example, the operad governing the variety of 2-As algebras [7] is a free product of two operads $A$ governing the class of associative algebras. We may also consider the class of 2-Com-As algebras as an analogue of 2-As. There is a natural connection between $O_1 * O_2$ and series-parallel networks.

The notion of a Lie-admissible algebra was introduced by A. Albert. By the definition, an algebra $L$ with a single product $(x, y) \mapsto xy$ is Lie-admissible if and only if the commutator algebra $L^{(-)}$ with the product $[x, y] = xy - yx$ is a Lie algebra. Lie admissible algebra and operad was studied in [9] and [11]. It would be interesting to find other free algebras $A$ as a Lie admissible that is isomorphic to a free product of $A^{(-)}$ and $A^{(+)}$.

The theory of Gröbner bases is a useful tool for solving the word problem and finding normal forms in commutative algebras in the most efficient algorithmic way. For noncommutative and nonassociative algebras the Gröbner–Shirshov bases theory addresses the same problems (see [1]). The theory of Gröbner bases for operads was

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established in [2] and [3]. To prove the isomorphism of operads, we use the theory developed by them.

2. Free product of operads

Let \( S_n \) stand for the symmetric group of the set \( \{1, \ldots, n\} \). A symmetric operad \( \mathcal{P} \) is a symmetric collection of a \( S_n \)-modules \( \mathcal{P}(n) \), \( n \geq 1 \), equipped with linear composition maps

\[
\gamma_{m_1, \ldots, m_n}^m : \mathcal{P}(m) \otimes \mathcal{P}(n_1) \otimes \ldots \otimes \mathcal{P}(n_m) \to \mathcal{P}(n_1 + \ldots + n_m)
\]

which satisfy the associativity condition. The space \( \mathcal{P}(1) \) contains an element \( \gamma \) that acts as an identity relative to the compositions. Finally, the compositions are equivariant with respect to the symmetric group actions.

An operad ideal in an operad \( \mathcal{P} \) is a collection of \( \mathcal{P} \)-invariant subspaces \( I(n) \subseteq \mathcal{P}(n) \), \( n \geq 1 \), such that

\[
\gamma_{m_1, \ldots, m_n}^m(I(m), \mathcal{P}(n_1), \ldots, \mathcal{P}(n_m)) \subseteq I(n_1 + \ldots + n_m),
\]

\[
\gamma_{m_1, \ldots, m_n}(\mathcal{P}(m), \mathcal{P}(n_1), \ldots, I(n_k), \ldots, \mathcal{P}(n_m)) \subseteq I(n_1 + \ldots + n_m).
\]

A morphism of operads \( \psi : \mathcal{O} \to \mathcal{P} \) is a collection of \( \mathcal{O} \)-linear maps \( \psi(n) : \mathcal{O}(n) \to \mathcal{P}(n) \), \( n \geq 1 \), preserving the compositions and identity.

**Definition 1.** Let \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be two operads. Then the free product of two operads \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) is an operad \( \mathcal{O} = \mathcal{O}_1 * \mathcal{O}_2 \) satisfying the following conditions.

1. There exist morphisms \( \pi_i : \mathcal{O}_i \to \mathcal{O} \) for \( i = 1, 2 \);
2. For every operad \( \mathcal{P} \) and for every pair of morphisms \( \psi_i : \mathcal{O}_i \to \mathcal{P} \) there exists a unique morphism \( \psi : \mathcal{O} \to \mathcal{P} \) such that \( \psi_i = \psi \pi_i \) for \( i = 1, 2 \).

For every graded space \( V = \bigoplus_{n \geq 1} V(n) \) there exists a uniquely defined free operad \( \mathcal{F}(V) \) generated by \( V \). An operad ideal of \( \mathcal{F}(V) \) may be presented as a minimal one that contains a given series of elements from \( \bigcup_{n \geq 1} \mathcal{F}(V)(n) \). Therefore, an operad may be defined by generators and relations.

For example, the operad Com-As defining the variety of commutative-associative algebras is generated by \( V_1 = V_1(2) \), where \( \dim V_1(2) = 1 \), this is a symmetric \( S_2 \)-module. The free operad \( \mathcal{F}(V_1) \) is exactly the operad of commutative algebras. The defining relations of Com-As consist of associative identity:

\[
\gamma_{2,1}^2(\mu, \mu, 1) + \gamma_{1,2}^2(\mu, 1, \mu).
\]

It is well-known that \( \dim \text{Com-As}(n) = 1 \).

The operad Lie of the variety of Lie algebras is generated by \( V_2 = V_2(2) \), where \( \dim V_2(2) = 1 \), this is a skew-symmetric \( S_2 \)-module. The free operad \( \mathcal{F}(V_2) \) is exactly the operad of anti-commutative algebras. The set of defining relations of Lie consists of the Jacobi identity:

\[
\gamma_{2,1}^2(\mu, \mu, 1) + \gamma_{2,1}^2(\mu, \mu, 1)^{(123)} + \gamma_{2,1}^2(\mu, \mu, 1)^{(132)}.\]
It is well-known that \( \dim \text{Lie}(n) = (n - 1)! \).

The graphical presentation of the free product of two operads naturally comes from the theory of Gröbner bases for operads. Let \( X(n) \) and \( Y(n) \), \( n \geq 1 \), be linear bases of \( \mathcal{O}_1(n) \) and \( \mathcal{O}_2(n) \), respectively. Then each of the operads \( \mathcal{O}_1 \) or \( \mathcal{O}_2 \) may be considered as a shuffle operad (see [2] for the definition) generated by \( X = \bigcup_{n \geq 1} X(n) \) or \( Y = \bigcup_{n \geq 1} Y(n) \), respectively. The sets of defining relations \( \Sigma_1 \) and \( \Sigma_2 \) then consist of all possible compositions of elements in \( X \) and \( Y \), respectively. (It is similar to the observation that in every reasonable class of algebras the multiplication table of an algebra is a Gröbner–Shirshov basis.) It remains to describe those elements of the free operad generated by the linear span of \( X \cup Y \) reduced with respect to \( \Sigma_1 \cup \Sigma_2 \).

**Example 1.** For every \( n \geq 1 \), consider the set \( B(n) \) of planar rooted trees with \( n \) leaves enumerated by integers \( 1, \ldots, n \), with internal vertices labelled by \( \circ \) and \( \bullet \) satisfying the following conditions:

1. Each internal vertex has at least two leaves;
2. There are no edges connecting internal vertices with similar labels.

Let us define a composition of such trees. In order to compose a tree \( t \in B(n) \) with a sequence of trees \( t_i \in B(m_i), \ i = 1, \ldots, n \), we first perform the ordinary grafting with re-numeration of leaves and then suppress those edges (if any) that connect vertices with similar edges. The resulting tree \( \gamma_{m_1, \ldots, m_n}^n(t, t_1, \ldots, t_n) \) belongs to \( B(m_1 + \cdots + m_n) \). For example,

\[
\gamma_{1,2,2}^3(\begin{array}{c}
\bullet \\
1 3 2
\end{array}, \text{id}, \begin{array}{c}
\bigcirc \\
2 1 1
\end{array}, \begin{array}{c}
\bullet \\
2 2
\end{array}) = \begin{array}{c}
\bigcirc \\
3 2 1 4 5
\end{array}
\]

The family of spaces \( B(n) \) together with the composition defined above forms an operad \( B \).

This is an easy exercise in the Gröbner bases technique for operads to show \( B \) is isomorphic to \( \text{As} \ast \text{As} \) (or two-associative algebra in [7]).

Suppose \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are two binary symmetric operads. Let us construct an operad \( \mathcal{T} = \mathcal{T}(\mathcal{O}_1, \mathcal{O}_2) \) as follows. First, construct the family of linear spaces \( \mathcal{T}(n), n \geq 1 \), by induction. Set \( \dim \mathcal{T}(1) = 1 \), so \( \mathcal{T}(1) \) is the linear span of the identity \( \text{id} = \text{id}_T \). For \( n = 2 \), set \( \mathcal{T}(2) = \mathcal{T}(2)^\bullet \oplus \mathcal{T}(2)^\circ \), where \( \mathcal{T}(2)^\bullet = \mathcal{O}_1(2), \mathcal{T}(2)^\circ = \mathcal{O}_2(2) \).

Suppose \( n > 2 \). Consider all partitions \( \lambda = (n_1, \ldots, n_m), n_1 + \cdots + n_m = n \), where \( n_1 \geq \cdots \geq n_m \geq 1 \). Assume each \( \mathcal{T}(n_i) \) is already defined, and for \( n_i \geq 2 \) it is presented in a form

\[
\mathcal{T}(n_i) = \mathcal{T}(n_i)^\bullet \oplus \mathcal{T}(n_i)^\circ.
\]
Then define
\[ T_\lambda(n)^\bullet = O_1(m) \otimes T(n_1)^\circ \otimes \cdots \otimes T(n_m)^\circ, \]
\[ T_\lambda(n)^\circ = O_2(m) \otimes T(n_1)^\bullet \otimes \cdots \otimes T(n_m)^\bullet. \]

Denote by \( S(m, \lambda) \) the set of all \( \sigma \in S_m \) such that \( n_{\sigma(i)} = n_i \) for all \( i = 1, \ldots, m \).

There is a natural embedding
\[ S(\lambda) := S(m, \lambda) \times S_{n_1} \times \cdots \times S_{n_m} \to S_n \]
given by the composition of permutations. The subgroup \( S(\lambda) \) acts on \( T_\lambda(n)^\bullet \) and \( T_\lambda(n)^\circ \) in the obvious way. Finally,
\[ T(n)^\bullet = \bigoplus_{\lambda \in P(n)} \text{Ind}^S_{S(\lambda)} T_\lambda(n)^\bullet, \quad T(n)^\circ = \bigoplus_{\lambda \in P(n)} \text{Ind}^S_{S(\lambda)} T_\lambda(n)^\circ, \]
\[ T(n) = T(n)^\bullet \oplus T(n)^\circ. \]

In order to define a composition rule on \( T \), note that the elements of \( T(n) \) are in one-to-one correspondence with formal linear combinations of planar rooted trees from Example 1 whose internal vertices have additional labels: \( \bullet \)-vertices (resp., \( \circ \)-vertices) with \( m \) leaves are marked by elements from \( O_1(m) \) (resp., \( O_2(m) \)). The composition of such trees is a straightforward generalization of that from Example 1 assuming that the labels of gluing vertices are composed accordingly. For example,
\[ \gamma^3_{1,2,3}(1, 2, 3, \text{id}, \mu, \nu, 1, 2) = (1, 2, 3, 2, 4, 5, \gamma^2_{2,1}(\mu, \nu, \text{id}) \]

**Theorem 1.** The operad \( T = T(O_1, O_2) \) constructed above is isomorphic to \( O_1 \ast O_2 \).

**Proof.** The trees representing basic elements of \( T(n) \) are exactly the reduced trees of the free operad generated by
\[ \bigoplus_{n \geq 1} (O_1(n) \oplus O_2(n)) \]
relative to the defining relations representing all possible compositions on \( O_1 \) and on \( O_2 \). Neither of the compositions contains vertices from both \( O_1 \) and \( O_2 \). \( \square \)

Let us calculate \( \dim T(n) \) according to the construction from Theorem 1. Denote \( \dim O_1(m) = x_m, \dim O_2(k) = y_k, \dim T^\bullet(n) = d_n^\bullet, \dim T^\circ(n) = d_n^\circ \). Then\( \dim T(n) = d_n^\bullet + d_n^\circ \) for \( n \geq 2 \), and
\[ d_n^\bullet = \sum_{\lambda \in P(n)} \frac{n!}{n_1! \cdots n_m! |S(m, \lambda)|} x_m d_{n_1}^\circ \cdots d_{n_m}^\circ, \]
\[ d_n^\circ = \sum_{\lambda \in P(n)} \frac{n!}{n_1! \cdots n_m! |S(m, \lambda)|} y_m d_{n_1}^\bullet \cdots d_{n_m}^\bullet. \]
Assuming \( d^*_k = x_2, \) \( d^*_2 = y_2, \) we may calculate all dimensions in terms of \( x_i, y_i \) by induction.

**Corollary 1.** The dimensions \( d^*_n \) for \( n = 3, 4, 5 \) are given by

\[
\begin{align*}
d^*_3 &= x_3 + 3x_2y_2, \\
d^*_4 &= x_4 + 6x_3y_2 + 3x_2y_2^2 + 4x_2y_3 + 12x_2y_2, \\
d^*_5 &= x_5 + 10x_4y_2 + 5x_3y_2 + 15x_3y_2^2 + 30x_2y_3 + 50x_2x_3y_2 + 10x_2y_2y_3 \\
&+ 90x_2y_2^2 + 15x_3y_2 + 10x_3y_3.
\end{align*}
\]

In order to get \( d^*_n \) one needs to interchange \( x \) and \( y. \)

**Proof.** Let us show how to derive \( d^*_5 \) from \( d^*_k, \) \( 2 < k < 5. \) There are six partitions of \( n = 5: \)

\[
(4, 1); \quad (3, 2); \\
(3, 1, 1); \quad (2, 2, 1); \\
(2, 1, 1, 1); \quad (1, 1, 1, 1).
\]

For \( \lambda = (4, 1), S(2, \lambda) = \{ e \}, \) so \( |S(\lambda)| = 1 \times 4! \times 1 = 24. \) The dimension of the corresponding subspace is \( 5x_2d^*_1. \)

For \( \lambda = (3, 2), S(2, \lambda) = \{ e \}, \) so \( |S(\lambda)| = 1 \times 3! \times 2! = 12. \) The dimension of the corresponding subspace is \( 10x_2y_2d^*_1. \)

For \( \lambda = (3, 1, 1), S(3, \lambda) \cong S_2, \) so \( |S(\lambda)| = 2! \times 3! \times 1 \times 1 = 12. \) The dimension of the corresponding subspace is \( 10x_3d^*_1. \)

For \( \lambda = (2, 2, 1), S(3, \lambda) \cong S_2, \) so \( |S(\lambda)| = 2! \times 2! \times 2! \times 1 = 8. \) The dimension of the corresponding subspace is \( 15x_3d^*_2. \)

For \( \lambda = (2, 1, 1, 1), S(4, \lambda) \cong S_3, \) so \( |S(\lambda)| = 3! \times 2! \times 1 \times 1 \times 1 = 12. \) The dimension of the corresponding subspace is \( 10x_4d^*_2. \)

For \( \lambda = (1, 1, 1, 1, 1), S(5, \lambda) \cong S_5, \) so \( |S(\lambda)| = 5! \times 1 = 120. \) The dimension of the corresponding subspace is \( x_5. \)

Hence,

\[
d^*_5 = 5x_2d^*_1 + 10x_2d^*_2 + 10x_3d^*_2 + 15x_3d^*_2^2 + 10x_4d^*_2 + x_5 \\
= 5x_2(y_4 + 6y_3x_2 + 3y_2^2 + 4y_2x_3 + 12y_2^2x_2) + 10x_2(y_3 + 3x_2y_2)y_2 \\
+ 10x_3(y_3 + 3x_2y_2) + 15x_3y_2 + 10x_4y_2 + x_5 = x_5 + 10x_4y_2 \\
+ 5x_2y_4 + 15x_3y_2 + 30x_2y_3 + 50x_2x_3y_2 + 10x_2y_2y_3 + 90x_2y_2^2 + 15x_2^2y_2 + 10x_3y_3.
\]

In particular,

\[
d_5 = d^*_5 + d^*_5 = x_5 + y_5 + 15(x_4y_2 + x_2y_4) + 20x_3y_3 \\
+ 60(x_2x_3y_2 + x_2y_2y_3) + 45(x_3y_2^2 + x_2y_3) + 180x_2y_2^2 + 15(x_2y_2 + y_2^2x_2).
\]

\[\square\]
Given the construction of the basis for free product of operads is a useful tool to find the Gröbner basis of algebras equipped with two binary operations. Let us show some examples.

**Example 2.** Let us apply Corollary 1 to calculate the dimension of the two-associative algebra up to degree 5:

$$
\begin{array}{c|cccc}
 n & 1 & 2 & 3 & 4 & 5 \\
\hline
\dim(\text{As} \ast \text{As}(n)) & 1 & 4 & 36 & 528 & 10800 \\
\end{array}
$$

The obtained result is the same as the dimension of two-associative operad given in [2].

**Example 3.** Suppose that $\mathcal{O}_1$ is the operad Com-As and $\mathcal{O}_2$ is the operad Lie generated by operations $\nu = (\cdot \cdot \cdot)$ and $\mu = [, ,]$, respectively. By Corollary 1, \( \dim(\text{Lie} \ast \text{Com-As}(4)) = 67 \). Note that the operad Pois (governing Poisson algebras) can be defined as a quotient

$$\text{Lie} \ast \text{Com-As} / (\gamma_{2,1}^2(\mu, 1, \nu) - \gamma_{2,1}^2(\nu, \mu, 1) - \gamma_{2,1}^2(\nu, \mu, 1))^{(23)}.$$

It is well-known that \( \dim\text{Pois}(n) = n! \). The Leibniz identity written in terms of Lie brackets and associative-commutative multiplication can be converted into shuffle polynomials as follows:

- \([a, bc] \rightarrow [a, b]c + [a, c]b \iff [a, bc] \rightarrow [a, b]c + [a, c]b, \]
- \([b, ac] \rightarrow [b, a]c + [b, c]a \iff [ac, b] \rightarrow [a, b]c - a[b, c], \]
- \([c, ab] \rightarrow [c, a]b + [c, b]a \iff [ab, c] \rightarrow [a, c]b + a[b, c]. \]

It is easy to show that the given rewriting system forms Gröbner basis generated by Leibniz identity. For that, we have to check 3 compositions:

1. \([a, bc] \text{ and } [ac, b]. \) Those are \([ac, bd] \text{ and } [ad, bc]. \)
2. \([a, bc] \text{ and } [ab, c]. \) That is \([ab, cd]. \)

Let us check only one of them (the remaining two can be checked in the same way). On the one hand,

\([ac, bd] \rightarrow [ac, d]b + [ac, b]d \rightarrow ([a, d]c)b + ([a, c]d)b + ([a, b]c)d - (a[b, c])d \rightarrow ([a, d]b)c + (ab)c + d + ([a, b]c)d - (a[b, c])d, \]

on the other hand,

\([ac, bd] \rightarrow [a, bd]c - a[bd, c] \rightarrow ([a, b]d)c + ([a, d]b)c - a([b, c]d) + a[b, c, d] \rightarrow ([a, b]c)d + ([a, d]b)c - (a[b, c])d + (ab)c. \]

In other words, it means that we should reduce basic trees of Lie $\ast$ Com-As that contain compositions of the form $\mathcal{O}_2(m) \otimes \mathcal{T}(n_1)^{\circ} \otimes \cdots \otimes \mathcal{T}(n_m)^{\circ}$, where at least one $n_i > 1$. There are exactly 43 terms like that. So \( \dim\text{Pois}(4) = 67 - 43 = 24 \) as expected.

**Example 4.** Let $\mu$ and $\nu$ be the generators of the operads Lie and Nov, respectively, where Nov corresponds to the class of Novikov algebras. Let us apply Corollary 1 to calculate the dimensions of the free product of operads Lie and Nov up to degree 5:
\[
\begin{array}{c|ccccc}
 n & 1 & 2 & 3 & 4 & 5 \\
 \text{dim}(\text{Lie} \ast \text{Nov}(n)) & 1 & 3 & 20 & 216 & 3274 \\
\end{array}
\]

In a similar way to the previous example, the quotient

\[
\text{Lie} \ast \text{Nov}/([a, b \circ c] - [c, b \circ a] + [b, a] \circ c - [b, c] \circ a - b \circ [a, c])
\]

represents the operad GD governing the class of Gelfand–Dorfman algebras \[^5\].

Let us find the dimensions of GD\((n)\) up to \(n = 4\) as in the previous example omitting all the notions of a shuffle operad. For degrees \(n = 1, 2\) there is nothing to do. Denote the identity from (2) by \(gd(1)\), and start with the set of identities \(S := \{gd(1)\}\). If

\[
b \circ [a, c] \to [a, b \circ c] - [c, b \circ a] + [b, a] \circ c - [b, c] \circ a,
\]

then, for degree 3, the rewriting system \(S\) reduces elements of the form \(a \circ [b, c]\). There are exactly three different relations like that, so

\[
dim(\text{GD}(3)) = 20 - 3 = 17.
\]

Generally, by using \(gd(1)\) we reduce all elements of the form \(a \circ [b, c]\), where \(a, b\) and \(c\) some monomials of the GD-operad.

For degree 4, consider the composition of \((a \circ b) \circ c - (a \circ c) \circ b = 0\) and \(b = [u, v]\). Consider

\[
(a \circ [u, v]) \circ c - (a \circ c) \circ [u, v] \pmod{S},
\]

where \(u < c < v\) and \(\pmod{S}\) reduces all monomials by the rule \(gd(1)\). In this way, we obtain the following relation in GD:

\[
gd(2) = [u, (a \circ c) \circ v] \to [v, (a \circ c) \circ u] + [u, a \circ c] \circ v - [v, a \circ c] \circ u \\
+ [u, a \circ v] \circ c - [v, a \circ u] \circ c + ([u, u] \circ v) \circ c - ([u, v] \circ u) \circ c - ([a, u] \circ v) \circ c - ([a, v] \circ u) \circ c.
\]

Let us add \(gd(2)\) to \(S\), so now \(S = \{gd(1), gd(2)\}\).

Computing the composition of \((a \circ b) \circ c - (b \circ a) \circ c - a \circ (b \circ c) + b \circ (a \circ c) = 0\) and \(b = [u, v]\) modulo \(S\) we get

\[
gd(3) := (a \circ [u, v]) \circ c - ([u, v] \circ a) \circ c - a \circ ([u, v] \circ c) + [u, v] \circ (a \circ c) = [u, a \circ v] \circ c - [v, a \circ u] \circ c + ([a, u] \circ v) \circ c - ([u, v] \circ u) \circ c - ([a, v] \circ u) \circ c - [u, v] \circ (a \circ c).
\]

Add \(gd(3)\) to the set \(S\). The rewriting rule corresponding to \(gd(3)\) is

\[
a \circ ([u, v] \circ c) \to [u, a \circ v] \circ c - [v, a \circ u] \circ c + ([a, u] \circ v) \circ c - ([a, v] \circ u) \circ c - ([u, v] \circ a) \circ c + [u, v] \circ (a \circ c).
\]

Reducing elements of the form \(x_1 \circ [x_2, x_3]\) by \(gd(1)\) and \([u, (a \circ c) \circ v]\) by \(gd(2)\) (for \(u < c < v\)), and \(a \circ ([u, v] \circ c)\) by \(gd(3)\), we get

\[
dim(\text{GD}(4)) = \text{Lie} \ast \text{Nov}(4)/\{gd(1), gd(2), gd(3)\} = \\
216 - 56(by \ \text{gd}(1)) - 8(by \ \text{gd}(2)) - 12(by \ \text{gd}(3)) = 140.
\]
The result agrees with the computer computation in [6] given dimension of GD-operad up to degree 5, which coincides with the obtained result.

Let $\text{2-Com-As}(X)$ be the free algebra equipped with two binary operations generated by the set $X = \langle x_1, x_2, \ldots \rangle$, where both multiplications are associative and commutative.

**Theorem 2.** The dimension of the degree $n$ component of $\text{2-Com-As}(x_1)$ is equal to the number of series-parallel networks (or MacMahon numbers [8]) with $n$ unlabeled edges.

**Proof.** By Theorem 1 the operad $\text{Com-As} \ast \text{Com-As}$ is isomorphic to the multilinear part of algebra $\text{2-Com}(X)$. To find the desired dimensions we should consider the basis trees of $\text{Com-As} \ast \text{Com-As}$ with unlabeled leaves. There is a natural bijection between these basis trees and series-parallel networks. To construct a bijection between them, we identify the first associative-commutative multiplication $\bullet$ with the parallel network connection, and second associative-commutative multiplication $\circ$—with the series connection. $\square$

Let us state the first seven terms of MacMahon numbers [8]:

| $n$ unlabeled edges | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|----------------------|---|---|---|---|---|---|---|-----|
| series-parallel networks | 1 | 2 | 4 | 10 | 24 | 66 | 180 | ... |

**Example 5.** If $n = 2, 3$, then

\[
\begin{align*}
\bullet & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ; \\
\begin{array}{c}
\bullet
\end{array} & \rightarrow \begin{array}{c}
\circ
\end{array} ;
\end{align*}
\]

For every two operads $\mathcal{O}_1$ and $\mathcal{O}_2$ there is a natural mapping from monomials in $(\mathcal{O}_1 \ast \mathcal{O}_2)(n)$ to the set of series-parallel networks with $n$ edges as given above. For every particular series-parallel network $B_i$, its pre-image span a subspace $\mathcal{N}_i$ in $(\mathcal{O}_1 \ast \mathcal{O}_2)(n)$ The sum of all such subspaces form the entire space $(\mathcal{O}_1 \ast \mathcal{O}_2)(n)$. Then:

\[(\mathcal{O}_1 \ast \mathcal{O}_2)(n) = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_{k_n},\]

where $k_n$ are the MacMahon numbers.

**3. Basis of Lie-admissible operad**

**Definition 2.** A Lie-admissible algebra over a field $\mathbb{K}$ of characteristic not equal to 2 is a vector space equipped with a binary operation satisfying the following identity:

\[(ab)c = (ba)c + c(ab) - c(ba) - (bc)a + (cb)a + a(bc) - a(cb) - (ca)b + (ac)b + b(ca) - b(ac). \quad (3)\]
If $\mathcal{L}$ is a Lie-admissible algebra generated by a set $X = \{x_1, x_2, \ldots\}$ then $\mathcal{L}^{(-)}$ is a subalgebra of $\mathcal{L}$ generated by $X$ relative to the multiplication $[\cdot, \cdot]$, such that

$$[x_i, x_j] = x_i x_j - x_j x_i.$$  

It is well-known that $\mathcal{L}^{(-)}$ is a Lie algebra and $\mathcal{L}^{(+)}$ is a commutative (non-associative) algebra, where $\mathcal{L}^{(+)}(X)$ is a subalgebra of $\mathcal{L}$ defined by multiplication $\{\cdot, \cdot\}$, such that:

$$\{x_i, x_j\} = x_i x_j + x_j x_i.$$  

The operad Lie-adm is a Lie-admissible algebra generated by $W_2 = W_2(2)$, where dim $W_2(2) = 2$. The free operad $\mathcal{F}(W_2)$ is exactly the operad of magma algebras. The set of defining relations of Lie-adm consists of the identity

$$\gamma_{2,1}^2(\rho, \rho, 1) - \gamma_{2,1}^2(\rho, \rho, 1)^{(12)} - \gamma_{1,2}^2(\rho, 1, \rho)^{(13)} + \gamma_{1,2}^2(\rho, 1, \rho)^{(13)}$$

$$+ \gamma_{2,1}^2(\rho, \rho, 1)^{(123)} - \gamma_{2,1}^2(\rho, \rho, 1)^{(13)} - \gamma_{1,2}^2(\rho, 1, \rho)^{(23)} + \gamma_{1,2}^2(\rho, 1, \rho)^{(23)}$$

$$+ \gamma_{2,1}^2(\rho, \rho, 1)^{(132)} - \gamma_{2,1}^2(\rho, \rho, 1)^{(23)} - \gamma_{1,2}^2(\rho, 1, \rho)^{(132)} + \gamma_{1,2}^2(\rho, 1, \rho)^{(132)} (4)$$

**Definition 3.** A shuffle operad is a monoid in the category of nonsymmetric collections of vector spaces with respect to the shuffle composition product defined as follows:

$$\mathcal{V} \circ_{\Pi} \mathcal{W}(n) = \bigoplus_{r \geq 1} \mathcal{V}(r) \otimes \bigoplus_{\pi} \mathcal{W}(|I^{(1)}|) \otimes \mathcal{W}(|I^{(2)}|) \otimes \ldots \otimes \mathcal{W}(|I^{(r)}|),$$

where $\mathcal{W}$ and $\mathcal{V}$ are nonsymmetric collections, $\pi$ ranges in all set partitions $\{1, \ldots, n\}$ $= \bigsqcup_{j=1}^{\ell} I^{(j)}$ for which all parts $I^{(j)}$ are nonempty and $\min(I_1) < \ldots < \min(I_\ell)$.

**Lemma 1.** The operads $\text{Com}$ and $\text{Com} \ast \text{Anti-Com}$ may be presented as free shuffle operads with operation alphabet $\mathcal{X}(2) = \{x\}$ and $\mathcal{X}(2) = \{x, y\}$, respectively.

**Proof.** Firstly, let us prove this statement for operad $\text{Com}$ by induction of a monomial length. Suppose that a generator of operad $\text{Com}$ is $\mu$. For $n = 2$, there are two monomials: $\mu(1, 2) \leftrightarrow x(12)$, $\mu(2, 1) = \mu(1, 2) \leftrightarrow x(12)$. Every monomial of length $n$ can be presented as: $\mu(k, \mu(A_1)), \mu(\mu(A_2), l) \mu(\mu(A_3), \mu(A_4))$, where $1 \leq k, l \leq n$ and $A_i$ are monomials of $\text{Com}$ that length less that $n$. To prove this statement, we have to consider the following cases:

1. For $\mu(k, \mu(A_1))$, if $k = 1$, then $\mu(1, \mu(A_1)) \in T_{\Pi}(\mathcal{X})$ by the inductive hypothesis.
2. For $\mu(k, \mu(A_1))$, if $k \neq 1$, then $\mu(k, \mu(A_1)) = \mu(\mu(A_1), k) \in T_{\Pi}(\mathcal{X})$ by the inductive hypothesis.
3. For $\mu(\mu(A_2), l)$, if $l = 1$, then $\mu(\mu(A_2), 1) = \mu(1, \mu(A_2)) \in T_{\Pi}(\mathcal{X})$ by the inductive hypothesis.
4. For $\mu(\mu(A_2), l)$, if $l \neq 1$, then $\mu(\mu(A_2), l) \in T_{\Pi}(\mathcal{X})$ by the inductive hypothesis.
(5) For $\mu(\mu(A_3), \mu(A_4))$, if monomial $A_3$ contains index 1, then $\mu(\mu(A_3), \mu(A_4)) \in T_{\text{III}}(X)$ by the inductive hypothesis.

(6) For $\mu(\mu(A_3), \mu(A_4))$, if monomial $A_4$ contains index 1, then $\mu(\mu(A_3), \mu(A_4)) = \mu(\mu(A_4), \mu(A_3)) \in T_{\text{III}}(X)$ by the inductive hypothesis.

In the same way, this statement can be proved for the case $\text{Com} \ast \text{Anti-Com}$. □

Example 6. Let us consider the operad $\text{Lie}$ with operation $\mu$. Using a forgetful functor $f$, we can convert this operad to the shuffle operad $T_{\text{III}}(X)$ as follows [2]:

$$\mu(1 2) \leftrightarrow x(1 2), \quad \mu(2 1) \leftrightarrow y(1 2),$$

for the operation alphabet $X(2) = \{x, y\}$. By anti-commutativity, $x(1 2) = -y(1 2)$. For $T_{\text{III}}(X)$, the Jacobi identity can be written in the following form:

$$x(x(1 2) 3) - x(1 x(2 3)) - x(1 3) 2).$$

This element generates an ideal of relations defining the operad $\text{Lie}^f$ as a quotient of $T_{\text{III}}(X)$.

To find the Gröbner basis of $\text{Lie}^f$ we have to check only one composition of the Jacobi identity: $x(x(1 2) 3) 4)$, where the leading monomial is $x(x(1 2) 3)$. On the one hand,

$$x(x(x(1 2) 3) 4) \rightarrow x(x(1 2) 4) 3) + x(x(1 2) x(3 4)) \rightarrow x(x(1 4) 2) 3) + x(x(1 x(2 4)) 3) + x(x(1 x(3 4)) 3) 2) + x(1 x(3) x(2 4)) + x(x(1 4) x(2 3)) + x(1 x(2 4) 3) + x(1 x(2 x(3 4))).$$

On the other hand,

$$x(x(x(1 2) 3) 4) \rightarrow x(x(1 3) 2) 4) + x(x(1 x(2 3)) 4) \rightarrow x(x(1 3) 4) 2) + x(x(1 3) x(2 4)) + x(x(1 4) x(2 3)) + x(1 x(2 4) 3) + x(x(1 x(3) x(2 4))) + x(x(1 4) x(2 3)) + x(1 x(2 4) 3) + x(1 x(2 x(3 4))).$$

The final expressions coincide, so the single defining relation of the operad $\text{Lie}^f$ forms a Gröbner basis.

Let us apply the forgetful functor $f$ to the operad $\text{Lie}$-adm and find it’s Gröbner basis. Choose the operation alphabet $X$ with $X(2) = \{x, y\}$ to convert the operad $\text{Lie}$-adm to shuffle operad $T_{\text{III}}(X)$ as in Example [6]

$$x(1 2) \leftrightarrow \rho(1 2), \quad y(1 2) \leftrightarrow \rho(2 1).$$

Converting the identity [4] of the operad $\text{Lie}$-adm into an element of $T_{\text{III}}(X)$, we obtain the following:

$$x(x(1 2) 3) - x(y(1 2) 3) - y(x(1 2) 3) + y(1 x(2 3)) - y(1 y(2 3)) - x(1 x(2 3)) + x(1 y(2 3)) + x(y(1 3) 2) - x(1 3) 2) - y(y(1 3) 2) + y(x(1 3) 2)$$

This element generates the ideal of relations defining the quotient of $T_{\text{III}}(X)$ isomorphic to the operad $\text{Lie}$-adm$^f$. 

Theorem 3. \( X \) corresponds to all monomials in the free shuffle operad with operation alphabet \( \mathcal{X} \).

On the other hand, \( X \) corresponds to all monomials in the free shuffle operad with operation alphabet \( \mathcal{X} \).

### Lemma 2.

The defining relation \( [5] \) of the operad Lie-adm is a Gröbner basis.

**Proof.** The leading monomial of \( [5] \) is \( x(x(1 2) 3) \). There is only one composition that we have to check: \( x(x(1 2) 3) \). On the one hand, \( [5] \) implies

\[
\begin{align*}
& x(x(x(1 2) 3) 4) \rightarrow x(x(1 2) x(3 4)) - x(x(1 2) y(3 4)) + x(x(x(1 2) 4) 3) \\
& - x(y(x(1 2) 4) 3) + x(y(x(1 2) 3) 4) - y(x(1 2) x(3 4)) + y(x(1 2) y(3 4)) \\
& - y(x(x(1 2) 4) 3) + y(x(x(1 2) 3) 4) + y(y(x(1 2) 4) 3) - y(y(x(1 2) 3) 4) \\
& \rightarrow x(1 x(2 x(3 4))) - x(1 x(2 y(3 4))) + x(1 x(x(2 4) 3)) - x(1 x(2 4) 3)) \\
& - x(1 y(2 x(3 4))) + x(1 y(2 y(3 4))) - x(1 y(x(2 4) 3)) + x(1 y(2 y(4) 3)) \\
& + x(x(x(1 4) x(2 3)) - x(x(1 4) y(2 3)) + x(1 y(2 3) x(2 4)) - x(x(1 3) y(2 4)) \\
& + x(x(y(1 2) 4) 3) - x(y(1 y(2 3) 4)) + x(x(1 y(2 4) 3)) - x(y(x(2 3) 3) 4) \\
& - x(y(y(1 3) 2) 4) - x(y(1 x(2 3) 4)) + y(x(1 x(2 4) 3)) - y(x(y(2 3) 4)) + y(x(1 y(2 3) 4)) \\
& + y(1 x(y(2 3) 3)) + y(1 y(2 y(3 4))) - y(1 y(y(2 3) 4)) + y(1 y(x(2 3) 4)) \\
& - y(1 y(2 4) 3)) - y(1 y(x(2 3) 4)) + y(1 x(x(2 4) 3)) - y(x(1 x(2 3) 4) \\
& + y(1 x(2 3) 4) 2) - y(x(1 x(1 4) 3) 2) + x(y(x(1 h 3) 2) 4) + y(x(1 x(4) 3) 2) \\
& - y(x(y(1 2) 4) 3) + y(y(x(1 2) 3) 4) - y(x(y(1 3) 2) 4) + y(x(1 4) y(2 3)) \\
& - y(y(1 2) x(3 4)) - y(y(1 2) y(3 4)) + y(y(1 2) x(3 4)) + y(x(1 y(2 3) 4)) \\
& - y(y(1 y(3 4) 2) - y(y(x(1) 4) 3) 2) + y(x(1 x(1 4) 3) 2) + y(x(y(1 x(2) 3)) 4) \\
& + y(y(1 y(3) 3) 2) - y(y(y(1 2) 3) 4) + y(y(1 y(1 2) 3) 4) = S.
\end{align*}
\]

On the other hand,

\[
\begin{align*}
& x(x(x(1 2) 3) 4) \rightarrow x(x(x(1 2) 3) 4) - x(x(1 y(2 3)) 4) + x(x(x(1 3) 2) 4) \\
& + x(x(y(1 2) 3) 4) - x(x(y(1 3) 2) 4) - x(y(x(1 2) 3) 4) + x(y(x(1 y(2) 3)) 4) \\
& + x(y(x(1 2) 3) 4) - x(y(x(1 3) 2) 4) - x(y(x(1 2) 3) 4) + x(y(x(1 y(2) 3)) 4) \rightarrow S.
\end{align*}
\]

\( \square \)

**Theorem 3.** The operad Lie-adm is isomorphic to the free product Lie * Com.

**Proof.** If the operations of the anti-commutative (Anti-Com) and commutative (Com) operads are denoted by \( x \) and \( y \), respectively, then by Lemma 1, Com * Anti-Com corresponds to all monomials in the free shuffle operad with operation alphabet \( \mathcal{X}(2) = \{ x, y \} \). So
\[ \dim(\text{Com} \ast \text{Anti-Com}) = \dim(\mathcal{T}_{\text{III}}(\mathcal{X})). \]

By Lemma 2, we can reduce all trees in \( \mathcal{T}_{\text{III}}(\mathcal{X}) \) that contain a shuffle subtree of the form \( x(x(1 2) 3) \).

By Example 6, the basis elements of the shuffle operad \( \text{Lie}^\mathcal{F} \) with operation alphabet \( \mathcal{X}(2) = \{x\} \) are those shuffle trees that contain no shuffle subtrees of the form \( x(x(1 2) 3) \).

Hence, the reduced form of a tree in \( \text{Com} \ast \text{Lie} \) is the same as in \( \text{Lie}-\text{adm} \), and \( \dim(\text{Com} \ast \text{Lie}) = \dim(\text{Lie}-\text{adm}) \).

Finally, by the definition of the free product there is a morphism of operads \( \text{Lie} \ast \text{Com} \to \text{Lie}-\text{adm} \) sending the generators of \( \text{Lie} \) and \( \text{Com} \) to \( x(1 2) - y(1 2) \) and \( x(1 2) + y(1 2) \), respectively. The morphism is surjective since every monomial in the free \( \text{Lie} \)-admissible algebra can be written as a sum of monomials obtained from an associative word by applying \([\cdot, \cdot]\) and \(\{\cdot, \cdot\}\) via

\[ ab = \frac{\{a, b\} + [a, b]}{2} \]

and

\[ ba = \frac{\{b, a\} + [b, a]}{2}. \]

The morphism is injective since the dimension of \( \text{Com} \ast \text{Lie} \) is the same as the dimension of \( \text{Lie}-\text{adm} \).

Graphically it can be presented as follows:

\[
\begin{array}{c}
\text{Com} \ast \text{Anti-Com} & \xrightarrow{f} & \text{Com} \ast \text{Lie} \\
\text{Shuffle}(x, y) & \xrightarrow{g} & \text{Lie}-\text{adm}
\end{array}
\]

where kernel of functions \( f \) and \( g \) are ideals generated by Jacoby identity and \( (5) \), respectively.

\[ \square \]

Using the calculation method of dimension \( \text{Lie} \ast \text{Com} \), there can be obtained the dimension of \( \text{Lie} \)-admissible operad:

| \( n \) | \( \text{dim}(\text{Lie} \ast \text{Com}(n)) \) |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 11 |
| 4 | 101 |
| 5 | 1299 |
| 6 | 21484 |
| 7 | 434314 |
| \ldots | \ldots |

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