Hylomorphic solitons for the Benjamin-Ono and the fractional KdV equations

Vieri Benci, Donato Fortunato
February 2, 2018

Abstract
This paper concerns with the existence of solitons, namely stable solitary waves, for the Benjamin-Ono and the fractional KdV equations.

AMS subject classification: 74J35, 35C08, 35A15, 35Q74, 35B35

Key words: Benjamin-Ono equation, fractional KdV equation, fractional Schrödinger equation, travelling solitary waves, hylomorphic solitons, variational methods.

Dedicated to the memory of Enrico Magenes.

Contents
1 Introduction 2
1.1 Notations 4
2 Abstract theory 5
2.1 Orbitally stable states and solitons 5
2.2 An abstract theorem 6
3 The nonlinear fractional Schrödinger equation 9
3.1 Main results 9
3.2 Proof of Th. 14 12
4 Hylomorphic solitons for the generalized BO equation 17

* Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, Pisa, ITALY and Centro Interdisciplinare "Beniamino Segre", Accademia dei Lincei. e-mail: vieri.benci@unipi.it
† Dipartimento di Matematica, Università di Bari "Aldo Moro" Via Orobona 4, 70125 Bari and INFN, Sezione 4 - email: donato.fortunato@uniba.it
1 Introduction

The Benjamin-Ono equation (BO) is a model of one dimensional waves in deep water
\[ \partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0 \]  
(BO)
where \( u = u(t,x) \), and \( \mathcal{H} \) denotes the spatial Hilbert transform:
\[ \mathcal{H} w(x) = \frac{1}{\pi} \text{p.v.} \int \frac{w(y)}{x-y} dy = -i \sqrt{\frac{2}{\pi}} \int \hat{w}(\xi) e^{ix\xi} d\xi \]
where \( \hat{w}(\xi) \) denotes the Fourier transform of \( w \), namely
\[ \hat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int w(y) e^{-iy\xi} dy. \]

It is well known that (BO) admits soliton solutions (see e.g. [12]). In this paper, we shall use the method developed in [4] to prove that a large class of equations including equation (BO) admits hylomorphic solitons. Following [1] and [4], a soliton is called hylomorphic if its stability is due to a particular interplay between the energy \( E \) and the hylenic charge
\[ C := \int u^2 dx \]
which is another integral of motion. More precisely, a soliton \( u_0 \) is hylomorphic if
\[ E(u_0) = \min \left\{ E(u) \mid \int u^2 dx = C(u_0) \right\}. \]

In this paper we give a general theorem which, if it is applied to (BO), gives the following theorem:

**Theorem 1** Equation (BO) has a one parameter family \( u_\delta, \delta \in (0, \delta_\infty) \), of hylomorphic solitons (see Def. [17]). Moreover, \( u_\delta \) is a (weak) solution of the equation
\[ \mathcal{H} \partial_x^2 u + u \partial_x u = \lambda_\delta \partial_x u \]
(1)
and
\[ U_\delta(t,x) = u_\delta(x - \lambda_\delta t) \]
(2)
solves (BO) for suitable \( \lambda_\delta \).

It is well known that, in this case (see e.g. [12]), eq. (BO) has explicit solutions, namely
\[ u_\delta(x - \lambda_\delta t) = \frac{4\lambda_\delta}{1 + \lambda_\delta^2 (x - x_0 - \lambda_\delta t)^2}. \]

We get the above theorem as a particular case of the study of the following two families of equations (see Th. [15] and Th. [25]):
\[ \partial_t u + \partial_x \left[ D_x^{2s} u + W'(u) \right] = 0, \quad u(t, x) \in \mathbb{R}, \; s \in \mathbb{R}, \; s \geq \frac{1}{2} \]  

(FKdV)

and

\[ \frac{i}{2} \frac{\partial \psi}{\partial t} = \frac{1}{2} D^{2s} \psi + \frac{1}{2} W'(|\psi|) \frac{\psi}{|\psi|}, \quad \psi(t, x) \in \mathbb{C}, \; s \in \mathbb{R}, \; s \geq \frac{1}{2} \]  

(FNS)

where

\[ D^s w(x) = \frac{1}{\sqrt{2\pi}} \int |\xi|^s \hat{w}(\xi) e^{ix\xi} d\xi; \]  

and \( W \in C^2(\mathbb{R}) \). We will refer to these equations as to the Fractional Korteweg–de Vries equation (FKdV) and the Fractional Nonlinear Schroedinger (FNS) equation respectively. Moreover, it is immediate to see that

\[ D = \mathcal{H} \partial_x. \]

The above equations, for particular choices of \( s \) and \( W \), reduce to well known PDE’s of physics. In particular if \( W(r) = \frac{1}{6} r^3 \) and \( s = 1/2 \), (FKdV) reduces to (BO).

If \( s = 1 \), and \( W(r) = -\frac{1}{6} r^3 \), then \( D^{2s} = D^2 = -\partial_x^2 \) and hence (FKdV) reduces to the KdV equation:

\[ \partial_t u - \partial_x^3 u - u \partial_x u = 0 \]  

(4)

When \( s = m \in \mathbb{N} \), and \( W(r) = -\frac{|r|^p}{p(p-1)} \), we have the following result:

**Theorem 2** Consider the equation

\[ \partial_t u + (-1)^m \partial_x^{2m+1} u - |u|^{p-2} \partial_x u = 0 \]  

(5)

and assume that the Cauchy problem is globally well posed for (5) in \( H^{2m} \). Then (5) admits hylomorphic solitons \( u_\delta \), \( \delta \in (0, \delta_\infty) \) provided that

\[ 2 < p < 4m + 2 \]  

(6)

Moreover, \( u_\delta \) is a (weak) solution of the equation

\[ (-1)^m \partial_x^{2m+1} u - |u|^{p-2} \partial_x u = \lambda_\delta \partial_x u \]  

(7)

and

\[ U_\delta(t, x) = u_\delta(x - \lambda_\delta t) \]  

(8)

solves (5) for suitable \( \lambda_\delta \).

Observe that the Cauchy problem for (5) with \( m = 1 \) is globally well posed in \( H^2 \) (see[17], [9]).

**Theorem 2** generalizes a result of ([11]).
We consider also the generalized fractional nonlinear Schrödinger equation (FNS), since it is crucial in the study of (FKdV).

For $s = 1$ and $W(u) = -\frac{1}{4}u^4$, (FNS) reduces to the Gross-Pitaevskii equation:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\partial_x^2 \psi - |\psi|^2 \psi, \quad \psi(t, x) \in \mathbb{C}. \quad (9)$$

Taking in (FNS) $s = \frac{m}{p}$ and $W(r) = -\frac{|r|^p}{p}$, we have the following theorem which generalizes some results relative to (9):

**Theorem 3** Consider the equation

$$i\frac{\partial \psi}{\partial t} = \frac{(-1)^m}{2}\partial_x^{2m} \psi - |\psi|^{p-2} \psi, \quad \psi(t, x) \in \mathbb{C}, \quad (10)$$

and assume that the Cauchy problem is globally well posed in $H^m$. Then (10) admits hylomorphic solitons $u_\delta$, $\delta \in (0, \delta_\infty)$ provided that (6) holds. Moreover, for suitable $\omega_\delta$, $u_\delta$ is a (weak) solution of the equation

$$\frac{(-1)^m}{2}\partial_x^{2m} u - |u|^{p-2} u = \omega_\delta u \quad (11)$$

and

$$U_\delta(t, x) = u_\delta(x)e^{-i\omega_\delta t} \quad (12)$$

solves (10).

Observe that the Cauchy problem for (10) with $m = 1$ is globally well posed in $H^1$ (see [14] and its references)

### 1.1 Notations

Let $\Omega$ be a subset of $\mathbb{R}^N$: then

- $C^k(\mathbb{R})$ denotes the set of real functions which have continuous derivatives up to the order $k$;
- $D(\mathbb{R})$ denotes the set of the infinitely differentiable functions with compact support; $D'$ denotes the topological dual of $D(\mathbb{R})$, namely the set of distributions;
- $H^k(a, b)$ is the closure in $L^1_{loc}$ of $D(\mathbb{R})$ with respect to the norm

$$||u||^2_{H^k(a, b)} = \int_a^b (|D^k u(x)|^2 + |u(x)|^2) \, dx$$

- $\dot{H}^s$ is the closure in $L^1_{loc}$ of $D(\mathbb{R})$ with respect to the norm

$$||u||^2_{\dot{H}^s} = \int |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi$$

4
• $H^s$ is the closure in $L^1_{loc}$ of $\mathcal{D}(\mathbb{R})$ with respect to the norm

\[
\|u\|^2_{H^s} = \int (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi = \|u\|^2_{H^s} + \|u\|^2_{L^2}
\]

• $W^{s,2}(a,b)$, $s \in (0,1)$, is the closure in $L^1_{loc}$ of $\mathcal{D}(\mathbb{R})$ with respect to the norm

\[
\|u\|^2_{W^{s,2}(a,b)} = \int_a^b \int_a^b \frac{|u(x) - u(y)|^2}{|x-y|^{2s+1}} dx dy + \|u\|^2_{L^2(a,b)}
\]

• $W^{s,2}(a,b)$, $s = k+\theta$, $k \in \mathbb{N}$, $\theta \in (0,1)$, is the closure in $L^1_{loc}$ of $\mathcal{D}(\mathbb{R})$ with respect to the norm

\[
\|u\|^2_{W^{k,2}(a,b)} = \|u\|^2_{H^s(a,b)} + \|D^k u\|^2_{W^{\theta,2}(a,b)}
\]

For the sake of the reader we will recall the properties of the Sobolev spaces which we will use:

The norms of $W^{s,2}(\mathbb{R})$ and $H^s$ are equivalent \hspace{1cm} (13)

For $a < b < c$, we have that

\[
\|u\|^2_{W^{s,2}(a,b)} + \|u\|^2_{W^{s,2}(b,c)} \leq \|u\|^2_{W^{s,2}(a,c)} \hspace{1cm} (14)
\]

2 Abstract theory

In this section we construct an abstract functional framework which allows to define solitary waves, solitons and hylomorphic solitons.

2.1 Orbitally stable states and solitons

Let us consider the following field equation

\[
\frac{\partial u}{\partial t} = \mathcal{A}(u) \hspace{1cm} (15)
\]

where $u(t, \cdot) \in X$, $X$ is a functional Hilbert space and $\mathcal{A} : X \to Y$ is a differential operator.

We make the following assumptions:

• we have that

\[
X \subseteq L^1_{loc}(\mathbb{R}^N, V) \hspace{1cm} (16)
\]

where $V$ is a vector space with norm $\cdot |_{V}$ and which is called the internal parameters space.

For the sake of the reader we will recall the properties of the Sobolev spaces which we will use:

\[\|u\|^2_{H^s} = \int (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi = \|u\|^2_{H^s} + \|u\|^2_{L^2} \]
• the dynamics has two constants of motion, the energy $E$ and the hylenic charge $C$; more precisely there are two continuous functionals on $X$, which are constant along the smooth solutions of (15); of course at this level of abstraction the names energy and hylenic charge are conventional.

For every $u_0 \in X$, the number $T(u_0) \geq 0$ is defined as the supremum of the $\tau \in [0, +\infty)$ such that $\forall t \in [0, \tau)$, the solution $u(t,x)$ of eq. (15) with initial value $u_0 \in X$ satisfies the following requests:

• $u(t, \cdot)$ is the unique solution in $X$;
• $u(\cdot, \cdot)$ is continuous in every point $(t, u) \in Z$ where $Z := \{(t, u) \in \mathbb{R} \times X \mid t \in [0, T(u))\}$
• the functions $t \mapsto E(u(t, \cdot))$ and $t \mapsto C(u(t, \cdot))$ are constant.

Also, we shall use the following notation: for every $u_0 \in X$, we set
$$\gamma_t u_0 := u(t,x), \ t \in [0, T(u_0)).$$
where $u(t,x)$ is the solution of eq. (15) with initial value $u_0$. So that $(X, \gamma)$ defines a dynamical system. Finally, we set
$$X_0 = \{u \in X \mid T(u) = +\infty\}$$
In the case of KdV equation we have that $X = H^1$ (see [6]) and $X_0 = H^2$ (see [9], [17]).

Roughly speaking a soliton is a localized state whose evolution preserves this localization and which exhibits some form of stability so that it has a particle-like behavior. To give a precise definition of soliton at this level of abstractness, we need to recall some well known notions in the theory of dynamical systems.

We start with the definition of dynamical system suitable to our purposes:

**Definition 4** A dynamical system is a couple $(X, \gamma)$ where $X$ is a metric space and
$$\gamma : Z \to X$$
is a continuous map such that
$$\gamma_0(u) = u$$
$$\gamma_t \circ \gamma_s(u) = \gamma_{t+s}(u)$$
provided that $t + s \in [0, T(u))$.

**Definition 5** A set $\Gamma \subset X$ is called invariant if $\forall u \in \Gamma, \forall t \in [0, T(u)), \ \gamma_t u \in \Gamma$. 

Definition 6 An invariant set $\Gamma \subset X$ is called (orbitally) stable, if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall u \in X$, $d(u, \Gamma) \leq \delta$, implies that $\forall t \in [0, T(u))$, $d(\gamma_t u, \Gamma) \leq \varepsilon$.

Remark 7 Notice that, under reasonable assumptions, if $\Gamma$ is stable, then $\exists \delta > 0$, $d(u, \Gamma) \leq \delta \Rightarrow T(u) = +\infty$

Now we can give our definition of soliton:

Definition 8 A soliton is a state $u \in \Gamma \subset X_0$, where $\Gamma$ satisfies the following properties:

- (i) $\Gamma$ is an invariant, stable set,
- (ii) $\Gamma$ is "compact up translations", namely for any sequence $u_n(x) \in \Gamma$ there is a subsequence $u_{n_k}$ and a sequence $\tau_k \in \mathbb{R}^n$ such that $u_{n_k}(x - \tau_k)$ is convergent.

Remark 9 The above definition needs some explanation. For simplicity, we assume that $\Gamma$ is a manifold (actually, it is possible to prove that this is the generic case if the problem is formulated in a suitable function space). Then (ii) implies that $\Gamma$ is finite dimensional. Since $\Gamma$ is invariant, $u_0 \in \Gamma \Rightarrow \gamma_\tau u_0 \in \Gamma$ for every time. Thus, since $\Gamma$ is finite dimensional, the evolution of $u_0$ is described by a finite number of parameters. By the stability of $\Gamma$, a small perturbation of $u_0$ remains close to $\Gamma$. However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite dimensional system with a small perturbation.

2.2 An abstract theorem

We recall that our dynamical system $(X, \gamma)$ has two constants of motion: the energy $E$ and the hylenic charge $C$.

Definition 10 A soliton $u_0 \in X$ is called hylomorphic if the set $\Gamma$ (given by Def. 8) has the following structure

$$\Gamma = \Gamma(e_0, c_0) = \{ u \in X \mid E(u) = e_0, |C(u)| = c_0 \} \quad (18)$$

where $e_0 = \min \{ E(u) \mid |C(u)| = c_0 \}$. \quad (19)

Notice that, by (19), we have that a hylomorphic soliton $u_0$ minimizes the energy on

$$\mathcal{M}_{c_0} = \{ u \in X \mid |C(u)| = c_0 \} \quad (20)$$
If \( \mathfrak{M}_{c_0} \) is a manifold and \( E \) and \( C \) are differentiable, then \( u_0 \) satisfies the following nonlinear eigenvalue problem:

\[
E'(u_0) = \lambda C'(u_0).
\]

We assume that \( E \) and \( C \) satisfy the following assumptions:

- \((EC-1)\) \textbf{(Value at 0)} \( E, C \) are \( C^1 \), bounded on bounded sets functionals and such that
  \[
  E(0) = 0, \ C(0) = 0; \ E'(0) = 0; \ C'(0) = 0.
  \]

- \((EC-2)\) \textbf{(Invariance)} \( E \) and \( C \) are \( \tau \)–invariant i.e.
  invariant under space translations.

- \((EC-3)\) \textbf{(Coercivity)} we assume that \( C(u) > 0 \Leftrightarrow u \neq 0 \) and that
  there exists \( a \geq 0 \) and \( \beta > 1 \) such that
  \[
  - (i) \ E(u) + aC(u)\beta \geq 0; \\
  - (ii) \text{ if } \|u\| \to \infty, \text{ then } E(u) + aC(u)\beta \to \infty; \\
  - (iii) \text{ for any bounded sequence } u_n \text{ in } X \text{ such that } E(u_n) + aC(u_n)\beta \to 0, \text{ we have that } u_n \to 0.
  \]

- \((EC-4)\) \textbf{(Splitting property)} \( E \) and \( C \) satisfy the splitting property.
  We say that a functional \( F \) on \( X \) has the splitting property if given a sequence
  \( u_n = u + w_n \) in \( X \) such that \( w_n \) converges weakly to 0, we have that
  \[
  F(u_n) = F(u) + F(w_n) + o(1).
  \]

We need also the following definition:

\textbf{Definition 11} A sequence \( \psi_n \) is called vanishing if

- for any sequence \( \{x_n\} \subset \mathbb{R}^N \), the translated sequence \( \{\psi_n(-x_n)\} \) converges weakly in \( X \) to 0.

\textbf{Example.} If \( \psi_n \) converges strongly to zero then it is a vanishing sequence, but the converse is not true; for example the sequence

\[
\psi_n = e^{-x^2} \sin nx
\]

in \( L^2 \); it does not converge strongly to 0 and it is vanishing.

We set

\[
\Lambda(u) := \frac{E(u)}{C(u)}.
\]

Since \( E \) and \( C \) are constants of motion, also \( \Lambda \) is a constant of motion; it will be called \textbf{hylenic ratio} and, as we will see it will play a central role in this theory. Finally we set

\[
\Lambda_0 := \inf \{\liminf \Lambda(u_n) \mid u_n \text{ is a vanishing sequence}\}
\]
**Theorem 12** Assume that $E$ and $C$ satisfy (EC-1),(EC-2),(EC-3). Moreover assume that the following condition
\[ \inf_u \Lambda(u) < \Lambda_0. \] (24)
is satisfied. Then for every $\delta \in (0,\delta_\infty)$, $\delta_\infty > 0$, there exist $c_\delta > 0$ and $\Gamma_\delta \subset X_0$ satisfying i),ii) of definition 8 and such that any soliton $u_\delta \in \Gamma_\delta$ is hylomorphic, i.e. it minimizes the energy on the manifold
\[ \mathcal{M}_{c_\delta} = \{ u \in X \mid C(u) = c_\delta \}. \]

Moreover if $\delta_1 < \delta_2$ we have that $c_{\delta_1} > c_{\delta_2}$).

The proof of this theorem is in [4] Th. 34 pag. 39.

The inequality (24) plays a crucial role in this theory; we will refer to it as to the hylomorphy condition.

3 The nonlinear fractional Schrödinger equation

3.1 Main results

The solitons for eq. (FKdV), as we will see, are related to the solitons of the Fractional Nonlinear Schrödinger equation (FNS).

Here we shall use a method to prove the existence of hylomorphic solitons for (FKdV) similar to the one presented in [4] (see also [3] and [5]). In this section we will resume this method.

The Fractional Nonlinear Schrödinger equation is given by
\[ i \frac{\partial \psi}{\partial t} = \frac{1}{2} D^s_\psi + \frac{1}{2} W'(\psi); \quad s > 0, \] (25)
where $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $D$ is defined by (3), $W : \mathbb{C} \to \mathbb{R}$ and
\[ W'(\psi) = \frac{\partial W}{\partial \psi_1} + i \frac{\partial W}{\partial \psi_2}. \] (26)

We assume that $W$ depends only on $|\psi|$, namely
\[ W(\psi) = F(|\psi|) \] and so $W'(\psi) = F'(|\psi|) \frac{\psi}{|\psi|}$.

for some smooth function $F : [0,\infty) \to \mathbb{R}$.

**Proposition 13** Let us consider the dynamical system $(H^s,\gamma)$ related to eq. (25). Then the energy
\[ E = \int \left( \frac{1}{2} |D^s \psi|^2 + W(\psi) \right) dx \] (27)
and the charge
\[ C = \int |\psi|^2 \, dx \quad \text{(28)} \]
are constant along a smooth solution \( \psi \) which decay in space sufficiently fast.

**Proof:** Let \( \psi \) be a smooth solution of (25) with initial condition \( \psi_0 \). Then, we have

\[
\frac{d}{dt} E(\psi(t)) = \text{Re} \int \left( D^s \psi \partial_t D^s \overline{\psi} + W'(\psi) \partial_t \overline{\psi} \right) dx
\]

\[
= \text{Re} \int \left( D^{2s} \psi + W'(\psi) \right) \overline{\partial_t \psi} \, dx
\]

\[
= \text{Re} \int \left( D^{2s} \psi + W'(\psi) \right) \left( \frac{1}{2} i D^{2s} \psi + \frac{1}{2} i W'(\psi) \right)
\]

\[
= - \text{Re} \left[ \frac{1}{2} i \int \left( D^{2s} \psi + W'(\psi) \right) \left( D^{2s} \psi + W'(\psi) \right) \right]
\]

\[
= - \text{Re} \left[ \frac{1}{2} i \int \left| D^{2s} \psi + W'(\psi) \right|^2 \, dx \right] = 0.
\]

Moreover

\[
\frac{d}{dt} \int |\psi|^2 \, dx = 2 \text{Re} \int \left( \psi \partial_t |\psi| \right) dx
\]

\[
= -2 \text{Re} i \int \left( \psi \overline{D^{2s} \psi + W'(\psi)} \right) dx = 0
\]

\[ \square \]

We make the following assumptions on the function \( W \):

\[ W(0) = W'(0) = 0. \quad \text{(W-0)} \]

Set

\[ W(r) = E_0 r^2 + N(r), \quad E_0 = \frac{1}{2} W''(0) \quad \text{(29)} \]

and assume that

\[ \exists r_0 \in \mathbb{R}^+ \text{ such that } N(r_0) < 0. \quad \text{(W-1)} \]

There exist \( q_1 \leq q_2 \) in \((2, +\infty)\), s. t.

\[ |N'(r)| \leq c_1 \left( r^{q_1 - 1} + r^{q_2 - 1} \right) \quad \text{(W-2)} \]

Moreover assume that, \( \exists p \in (2, 4s + 2) \), with \( s \geq \frac{1}{2} \), s. t.

\[ N(r) \geq -cr^p, \quad c \geq 0, \text{ for } r \text{ large} \quad \text{(W-3)} \]

We can apply the abstract theory of section \[ \text{[2]} \] taking \( X = H^s \):
Theorem 14 Let $W$ satisfy (W-0),..., (W-3), and
\[ \frac{1}{2} W''(0) = E_0 > 0 \] (30)
Then equation (24) admits a family of hylomorphic solitons $u_\delta$, $\delta \in (0, \delta_\infty)$, $u_\delta \in H^{2s}$.

Proof: The proof of this theorem will be given in the next section.

The following theorem gives more information on the structure of the solitons and of their dynamics; moreover assumption (30) is avoided.

Theorem 15 Assume that all the hypotheses of Theorem 14 hold with exception of assumption (30). Then equation (24) admits hylomorphic solitons $u_\delta \in H^{2s}$, $\delta \in (0, \delta_\infty)$. Moreover, $u_\delta$ is a (weak) solution of the equation
\[ \frac{1}{2} D^{2s} u + \frac{1}{2} W'(u) = \omega u \] (31)
and
\[ \psi_\delta(t, x) := u_\delta(x) e^{-i\omega t} \] (32)
solves (25).

Proof. First let us assume (30). By Theorem 14 (24) admits a family of hylomorphic solitons $u_\delta$, $\delta \in (0, \delta_\infty)$, $u_\delta \in H^{2s}$.

Let $u_\delta$ be a hylomorphic soliton, then it is a minimizer of the energy $E$ defined in (27) on $\mathcal{M}_c$. Then we get
\[ E'(u_\delta) = \omega C'(u_\delta) \] (33)
where $\omega$ is a Lagrange multiplier. Clearly (33) gives
\[ \frac{1}{2} D^{2s} u_\delta + \frac{1}{2} W'(u_\delta) = \omega u_\delta \]
which implies that $\psi = u_\delta e^{-i\omega t}$ solves (25). Observe that, since $u_\delta \in H^s$ solves (33), by elliptic regularization we have $u_\delta \in H^{2s}$. It remains to show that the same result holds even when (30) is violated. So assume that
\[ \frac{1}{2} W''(0) = E_0 \leq 0; \]
we can reduce the problem to the case (30). To do this, we replace $W(r)$ with
\[ W_1(r) = W(r) + \frac{1}{2} (1 - E_0) \]
So
\[ \frac{1}{2} W''_1(0) = 1 > 0. \]
and (30) is satisfied by \( W_1 \). Then we can apply the previous considerations. So there exists a hylomorphic soliton \( u_1 \) and \( \omega_1 \) s.t.

\[
\psi_1 = u_1 e^{-i\omega_1 t}.
\]

solves the equation

\[
\frac{\partial \psi_1}{\partial t} = \frac{1}{2} D_2^2 \psi_1 + \frac{1}{2} W_1'(\psi_1),
\]

(34)

It can be easily seen that \( \psi = \psi_1(t,x) e^{\frac{(E_0-1)}{2} t} \) is a solution of (26)

\[ \square \]

Remark 16 Th. 3 is a particular case of the above theorem when \( s \) in (25) is a positive integer.

3.2 Proof of Th. 14

In this section, we will prove Theorem 14 by using Th.12 with \( X = H^s \), \( E \) and \( C \) as in (27) and (28) and

\[
\Lambda(\psi) := \frac{E(\psi)}{|C(\psi)|} = \frac{\int \left( \frac{1}{2} |D^s \psi|^2 + W(\psi) \right) dx}{\int |\psi|^2 dx}.
\]

(35)

Let us first prove the Coercivity assumption (EC-3);

Lemma 17 If \( W \) satisfies (W-0),...(W-3) then assumption (EC-3) holds

Proof: by (29) and (W-3), we have

\[
E(u) + aC(u)^\beta = \int \left( \frac{1}{2} |D^s \psi|^2 + W(\psi) \right) dx + a \left( \int |\psi|^2 dx \right)^\beta 
\]

\[
\geq \frac{1}{2} \|u\|_{H^s}^2 + E_0 \|u\|^2 - b \|u\|_{L^p}^p + a \|u\|_{L^2}^{2\beta}.
\]

(36)

where \( b \) is a suitable constant and \( a, \beta \) will be choosen later.

The Gagliardo Nirenberg inequalities in our case take the following form:

\[
\|u\|_{L^p} \leq c \|u\|_{H^s}^\theta \|u\|_{L^2}^{1-\theta}
\]

provided that

\[
\frac{1}{p} = \frac{1-\theta}{2} + \theta \left( \frac{1}{2} - s \right) \quad \text{and} \quad \theta \in (0,1)
\]

namely

\[
\theta = \frac{1}{s} \left( \frac{1}{2} - \frac{1}{p} \right) \quad \text{and} \quad \theta \in (0,1)
\]

(37)
Notice that for $s \geq \frac{1}{2}$ and $p > 2$, the above conditions are satisfied. Then,

$$
\|u\|_{L^p}^p \leq c^p \|u\|_{H^s}^{p\theta} \|u\|_{L^2}^{p-p\theta}
$$

$$
= \frac{1}{2bp^\theta} \left( \|u\|_{H^s}^{2(\theta/2)} \right)^{(p\theta)/2} \left( c_1 \|u\|_{L^2}^{p-p\theta} \right)
$$

By (29) $p < 4s + 2$, and hence $\frac{2}{p\theta} = \frac{4s}{p-2} > 1$. We now use Young’s inequality:

$$
\|u\|_{L^p}^p \leq \frac{1}{2bp^\theta} \left( \frac{p\theta}{2} \left[ \left( \|u\|_{H^s}^{2(\theta/2)} \right)^{2/(p\theta)} + c_2 \left( \|u\|_{L^2}^{p-p\theta} \right)^{(2/(p\theta))} \right] \right)
$$

(38)

where $\beta = \left( \frac{2-p\theta}{2} \right) \left( \frac{2}{p\theta} \right)$. We want to show that $\beta > 1$. We have that

$$
\beta = (1-\theta) \frac{p}{2-p\theta}
$$

and using (37), since $\beta > 0$ and $p > 2$, we get

$$
\beta = \frac{(2ps + 2) - p}{(4s + 2) - p} > 1
$$

So, by the above inequalities (38) and (39),

$$
E(u) + aC(u)^\beta \geq \frac{1}{2} \|u\|_{H^s}^2 - b \left[ \frac{1}{4b} \|u\|_{H^s}^2 + \frac{c_2}{b} \|u\|_{L^2}^{2\beta} \right] + a \|u\|_{L^2}^{2\beta}.
$$

(39)

$$
= \frac{1}{4} \|u\|_{H^s}^2 + (a - c_3) \|u\|_{L^2}^{2\beta}
$$

(40)

If $a \geq c_3$, then all the assumptions (EC-3) are easily verified.

Lemma 18  If $W$ satisfies (W-0), ..., (W-3) then the splitting property (EC-4) holds.

Proof:  See the proof Lemma 5.3 at pg. 74 in [4].

Next we will verify that the hylomorphy condition (24) is satisfied. The following lemma, which is in the same spirit of some compactness results in [13], [2] and [7], plays a fundamental role in proving (24):

Lemma 19  Let $\psi_n$ be a vanishing sequence in $H^s$, $s \geq \frac{1}{2}$ (see Def. 11); then for any $p > 2$ we have $\|\psi_n\|_{L^p} \to 0$. 

13
Proof. Now let \( \{ \psi_n \} \subset H^s \) be a vanishing sequence and prove that \( \| \psi_n \|_{L^p} \to 0 \). Arguing by contradiction, assume that, up to a subsequence, \( \| \psi_n \|_{L^p} \geq a > 0 \). Since \( \psi_n \) is vanishing, there exists \( M > 0 \) such that \( \| \psi_n \|_{H^s}^2 \leq M \). Then, if \( L \) is the constant for the Sobolev embedding \( H^s \subset L^p (j, j + 1) \), we have

\[
0 < a^p \leq \int |\psi_n|^p = \sum_j \int_{j}^{j+1} |\psi_n|^p = \sum_j \| \psi_n \|_{L^p(j,j+1)}^p \cdot \| \psi_n \|^2_{L^p(j,j+1)} \leq L \left( \sup_j \| \psi_n \|_{L^p(j,j+1)}^{p-2} \right) \cdot \sum_j \| \psi_n \|^2_{L^p(j,j+1)}
\]

\[
\leq L \left( \sup_j \| \psi_n \|_{L^p(j,j+1)}^{p-2} \right) \cdot \sum_j \| \psi_n \|^2_{W^{2,s}(j,j+1)} \quad \text{(by (13))}
\]

\[
\leq L \left( \sup_j \| \psi_n \|_{L^p(j,j+1)}^{p-2} \right) \cdot \| \psi_n \|^2_{W^{2,s}(\mathbb{R})} \quad \text{(by (13))}
\]

\[
\leq LM_1 \left( \sup_j \| \psi_n \|_{L^p(j,j+1)}^{p-2} \right) \cdot \| \psi_n \|^2_{H^s} \leq LM_1 \left( \sup_j \| \psi_n \|_{L^p(j,j+1)}^{p-2} \right).
\]

Then

\[
\left( \sup_j \| \psi_n \|_{L^p(j,j+1)} \right) \geq \left( \frac{a^p}{LM_1} \right)^{1/(p-2)}
\]

Then, for any \( n \), there exists \( j_n \in \mathbb{Z} \) such that

\[
\| \psi_n \|_{L^p(j_n, j_n+1)} \geq \alpha > 0. \tag{41}
\]

Then, we easily have

\[
\| \psi_n (\cdot - j_n) \|_{L^p(0,1)} = \| \psi_n \|_{L^p(j_n, j_n+1)} \geq \alpha > 0. \tag{42}
\]

Since \( \psi_n \) is bounded, also \( \psi_n (\cdot - j_n) \) is bounded (in \( H^s \)). Then we have, up to a subsequence, that \( \psi_n (\cdot - j_n) \rightharpoonup \psi_0 \) weakly in \( H^s \) and hence strongly in \( L^p(0,1) \). By (42), \( \psi_0 \neq 0 \) and this contradicts the fact that \( \psi_n \) is vanishing.

Lemma 20 If the assumptions of Theorem 14 are satisfied, we have

\[
\liminf_{\psi \in H^s, \| \psi \|_{L^p} \to 0} \Lambda(\psi) \geq E_0
\]

Proof. Clearly

\[
\liminf_{\psi \in H^s, \| \psi \|_{L^p} \to 0} \Lambda(\psi) = \liminf_{\psi \in H^s, \| \psi \|_{L^p} = 1, \varepsilon \to 0} \frac{E(\varepsilon \psi)}{C(\varepsilon \psi)}
\]

\[
= \inf_{\psi \in H^s, \| \psi \|_{L^p} = 1} \left( \frac{\int \left( \frac{1}{2} |D^s \psi|^2 + E_0 |\psi|^2 \right) dx}{\int |\psi|^2} \right) \quad + \quad \liminf_{\psi \in H^s, \| \psi \|_{L^p} = 1, \varepsilon \to 0} \frac{\int N(\varepsilon \psi) dx}{\varepsilon^2 \int |\psi|^2}
\]

\[
\geq E_0 + \liminf_{\psi \in H^s, \| \psi \|_{L^p} = 1, \varepsilon \to 0} \frac{\int N(\varepsilon \psi) dx}{\varepsilon^2 \int |\psi|^2}
\]

14
So the proof of Lemma will be achieved if we show that
\[ \liminf_{\psi \in H^s, \|\psi\|_{L^p}=1, \varepsilon \to 0} \frac{\int N(\varepsilon \psi)}{\varepsilon^2 \int |\psi|^2} = 0. \]  
(43)

By assumptions (W-2) and (W-3) we have
\[ -c r^p \leq N(r) \leq c_1 (r^{q_1-1} + r^{q_2-1}). \]  
(44)

Then by (44) we have
\[-cA \varepsilon^{p-2} \leq \inf_{\|\psi\|_{L^p}=1} \int \frac{N(\varepsilon \psi)}{\varepsilon^2 \int |\psi|^2} \leq c_1 B (\varepsilon^{q_1-1} + \varepsilon^{q_2-1}) \]  
(45)

where
\[ A = \inf_{\psi \in H^s, \|\psi\|_{L^p}=1} \frac{\int |\psi|^p}{\int |\psi|^2}, \quad B = \inf_{\psi \in H^s, \|\psi\|_{L^p}=1} \frac{\int \left( |\psi|^{q_1-1} + |\psi|^{q_2-1} \right)}{\int |\psi|^2}. \]

By (45) we easily get (43).

\[ \square \]

**Corollary 21** If the assumptions of Theorem [14] are satisfied, then
\[ E_0 \leq \Lambda_0 \]

**Proof.** By Lemma [19] and Lemma [20]
\[ \Lambda_0 = \inf \{ \liminf \Lambda(u_n) \mid u_n \text{ is a vanishing sequence} \} \geq \liminf_{\|\psi\|_{L^p} \to 0} \Lambda(\psi) \geq E_0 \]
\[ \square \]

Finally we can prove that the hylomorphy condition is satisfied.

**Lemma 22** If the assumptions of Theorem [14] are satisfied, then the hylomorphy condition [24] holds, namely we have
\[ \inf_{\psi \in H^s} \Lambda(\psi) < \Lambda_0 \]
Proof. We need to construct a function \( u \in H^s \) such that \( \Lambda(u) < \Lambda_0 \). Such a function can be constructed as follows. Let \( u_R \geq 0 \) be a \( C^\infty \) function such that
\[
u_R = \begin{cases} s_0 & \text{if } |x| < R \\ 0 & \text{if } |x| > R + 1 \end{cases}.
\]
and there exists a constant \( C \) such that, for any integer \( m \geq s \) we have
\[|D^m u_R(x)| \leq C\]
There is a constant \( c \) such that
\[
\|u\|_{H^m}^2 \leq c \left[ \int |D^m u(x)|^2 \, dx + \int |u(x)|^2 \, dx \right]
\]
then
\[\int |D^s u_R(x)|^2 \, dx \leq c_1 \|u\|_{H^m}^2 \leq cc_1 \left[ \int |D^m u_R(x)|^2 \, dx + \int |u_R(x)|^2 \, dx \right]
\leq 2C^2cc_1 + 2s_0^2(R + 1) \leq c_2\]
Moreover,
\[2s_0^2R \leq \int |u_R|^2 \, dx \leq 2s_0^2(R + 1)\]
then
\[
\frac{\int \left[ \frac{1}{2} |D^s u_R|^2 + E_0u_R^2 \right] \, dx}{\int u_R^2 \, dx} \leq E_0 + O \left( \frac{1}{R} \right). \quad (46)
\]
Moreover
\[
\int N(u_R) \, dx = 2RN(s_0) + \int_R^{R+1} N(u_R) \, dx + \int_{-R-1}^{-R} N(u_R) \, dx.
\]
So
\[
\frac{\int N(u_R) \, dx}{\int u_R^2 \, dx} \leq \frac{2RN(s_0) + c_3}{\int u_R^2 \, dx} \leq (\text{since } N(s_0) < 0) \quad (47)
\leq \frac{2RN(s_0)}{2s_0^2(R + 1)} + \frac{c_3}{2s_0^2R} = \frac{N(s_0)}{s_0^2} + O \left( \frac{1}{R} \right).
\]
Then, by (46) e (47) we get
\[
\Lambda(u_R) = \frac{\int \left( \frac{1}{2} |D^s u_R|^2 + W(u_R) \right) \, dx}{\int u_R^2 \, dx} \quad (48)
\]
\[
= \frac{\int \left( \frac{1}{2} |D^s u_R|^2 + E_0u_R^2 \right) \, dx}{\int u_R^2 \, dx} + \frac{\int N(u_R) \, dx}{\int u_R^2 \, dx} \leq E_0 + \frac{N(s_0)}{s_0^2} + O \left( \frac{1}{R} \right) \quad (49)
\]

Then by (W-1) we can easily deduce that for $R$ large enough we have
\[ \Lambda(u_R) < E_0. \] (50)

Finally by (50) and Corollary 21 we get
\[ \Lambda(u_R) < \Lambda_0 \]

Now we are ready to prove the theorem.

**Proof of Theorem 14** We just need to check that all the assumptions of Th. 12 are satisfied. (EC-1),(EC-2) hold trivially. (EC-3) and (EC-4) hold by Lemma 17 and 18 respectively and (24) is verified in Lemma 22.

\[ \square \]

4  **Hylomorphic solitons for the generalized BO equation**

In this section we will study equation (FKdV).

**Proposition 23.** Let $W$ be a $C^1$ function and $u(t, \cdot) \in H^{2s}$ be a smooth solution of equation (FKdV) decayng sufficiently fast in space. Then $u$ has the following constants of motion: the energy
\[ E = \int \left( \frac{1}{2} \left[ D^s u \right]^2 + W(u) \right) \ dx \] (51)
and the charge
\[ C = \frac{1}{2} \int u^2 \ dx \] (52)

**Proof.** We have
\[ dE(u) [v] = \int (D^s u \ D^s v + W'(u)v) \ dx \]
\[ = \int [D^{2s} u + W'(u)] \ v dx \]

hence, using the equation (FKdV)
\[ \frac{d}{dt} E(u(t)) = \int [D^{2s} u + W'(u)] \ \partial_t u \]
\[ = \int [D^{2s} u + W'(u)] \ \partial_x [D^{2s} u + W'(u)] \]
\[ = \frac{1}{2} \int \partial_x (D^{2s} u + W'(u))^2 \ dx = 0 \]
Then $E$ is constant along the solution $u$.
Let us now show that also $C$ is constant along $u$. By \text{FKdV} we have
\begin{equation}
\frac{d}{dt} C(u) = \int u \partial_t u dx = \int u \partial_x \left[ D_x^{2s} u + W'(u) \right] dx \tag{53}
\end{equation}
\begin{equation}
= \int u \partial_x D_x^{2s} u dx + \int u \partial_x W'(u) dx \tag{54}
\end{equation}
Let us compute each piece separately:
\begin{equation}
\int u \partial_x D_x^{2s} u dx = \int u \partial_x D_x^s D_x^s u dx = \int u D_x^s \partial_x D_x^s u dx \tag{55}
\end{equation}
\begin{equation}
= \int D_x^s u \partial_x D_x^s u dx = \frac{1}{2} \int \partial_x \left( D_x^s u \right)^2 = 0 \tag{56}
\end{equation}
Moreover
\begin{equation}
\int \partial_x W'(u) u dx = - \int W'(u) \partial_x u dx \tag{57}
\end{equation}
\begin{equation}
= - \int \partial_x W(u) dx = 0 \tag{58}
\end{equation}
Substituting \eqref{57} and \eqref{55} in \eqref{53} we get
\begin{equation}
\frac{d}{dt} C(u) = 0 \tag{59}
\end{equation}
\hfill $\square$

We will apply the abstract theory of section 2 taking $X = H^s(\mathbb{R})$; in this case a function $u(t, \cdot) \in H^s$ is a weak solution of \text{FKdV} if $\forall \varphi \in D(\mathbb{R})$
\begin{equation}
\int \left[ \partial_t u \varphi + u \partial_x D_x^{2s} \varphi + W'(u) \partial_x \varphi \right] dx = 0 \tag{59}
\end{equation}

\textbf{Theorem 24} Let all the assumptions of Theorem (14) hold, $\gamma$ being here the evolution operator for eq. \text{FKdV}. Then the equation \text{FKdV} admits a family of hylomorphic solitons $u_\delta$, $\delta \in (0, \delta_\infty)$, $u_\delta \in H^{2s}$.

\textbf{Proof}: The proof of this theorem is essentially the same as the proof of Th.14. The reason for this relies on the fact that the energy and the charge for eq. \eqref{25} given by \eqref{27} and \eqref{28} are formally the same as the energy and the charge of equation \text{FKdV} given by \eqref{51} and \eqref{52}. The fact that in the first case $\psi$ is complex while in the second case $u$ is real-valued does not affect the estimates.
\hfill $\square$

The next theorem is the analogous of Th. 15 and it gives more information on the structure of the solitons and of its dynamics. Moreover it permits to eliminate assumption \eqref{30}.  

18
Theorem 25 Let all the assumptions of Theorem 15 hold, \( \gamma \) being here the evolution operator for eq. (FKdV). Then the equation (FKdV) admits a family of hylomorphic solitons \( u_\delta, \delta \in (0, \delta_\infty), u_\delta \in H^{2s} \). Moreover, \( u_\delta \) is a (weak) solution of the equation

\[
\partial_x D^{2s} u + \partial_x W'(u) = \lambda_\delta \partial_x u
\]

and

\[
U_\delta(t, x) = u_\delta(x - \lambda_\delta t)
\]

solves (FKdV).

Proof. First let us assume (30). Then, by Theorem 24, the equation (FKdV) admits a family of hylomorphic solitons \( u_\delta, \delta \in (0, \delta_\infty), u_\delta \in H^{2s} \).

Since \( u_\delta \) is a minimizer of the energy \( E \) on the manifold \( \mathcal{M}_\varepsilon \), there exists a Lagrange multiplier \( \lambda_\delta \) s.t.

\[
E'(u_\delta) = \lambda_\delta C'(u_\delta).
\]

The above equality can be written as follows

\[
D^{2s} u + W'(u_\delta) = \lambda_\delta u_\delta
\]

So, if we take the derivative \( \frac{\partial}{\partial x} \) on both side, we get (60). Finally (60) implies that the travelling wave \( U_\delta(t, x) = u_\delta(x - \lambda_\delta t) \) solves (FKdV) and consequently \( u_\delta \) is a soliton.

It remains to show that the same result holds even when (30) is violated.

Consider the following equation

\[
\partial_t u + \partial_x \left[ D^{2s}_x u + W'_0(u) \right] = 0,
\]

where \( W'_0(0) = -2E_0 < 0 \). In this case is convenient to consider the equation

\[
\partial_t v + \partial_x \left[ D^{2s}_x v + W'(v) \right] = 0,
\]

where \( W(r) = W_0(r) + (1 + E_0) r^2 \). We have that

\[
W''(0) = 2 > 0
\]

and to every solution \( v \) of eq. (63) corresponds a solution

\[
u(t, x) = v(t, x - \lambda t) \quad \text{with} \quad \lambda = 2(1 + E_0)
\]

of eq. (62). In fact

\[
\partial_t u + \partial_x \left[ D^{2s}_x u + W'_0(u) \right] = \partial_t v - \lambda \partial_x v + \partial_x \left[ D^{2s}_x v + W'(v) \right]
\]

\[
= \partial_t v - \lambda \partial_x v + \partial_x \left[ D^{2s}_x v + W'(v) + 2(1 + E_0) v \right]
\]

\[
= \partial_t v - \lambda \partial_x v + \partial_x \left[ D^{2s}_x v + W'(v) + \lambda v \right]
\]

\[
= \partial_t v - \partial_x \left[ D^{2s}_x v + W'(v) \right] = 0
\]
Remark 26 Th. 2 is a particular case of the above theorem when $D^{2s}$ reduces to a differential operator. Th. 1 is obtained by Th. 25 taking $s = 1/2$ in (FKdV).

Remark 27 The Cauchy problem for (BO) is globally well posed in $H^1$ [15], whereas the Cauchy problem for KdV equation (4) is globally well posed in $H^2$ (see [9], [17]).

References

[1] V. Benci, Hylomorphic solitons, Milan J. Math., 77 (2009), 271-332.
[2] V. Benci, G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, Arch. Rational Mech. Anal. 99 (1987), 283-300.
[3] V. Benci, D. Fortunato, A minimization method and applications to the study of solitons, Nonlinear Anal. T. M. A., 75, (2012), 4398-4421.
[4] V. Benci, D. Fortunato, Variational methods in nonlinear field equations, Springer Monographs in Mathematics, Springer Cham Heidelberg, (2014), ISBN: 978-3-319-06913-5, DOI 10-1007/978-3-319-06914-2
[5] V. Benci, D. Fortunato, Solitons in Schrödinger-Maxwell equations, J. Fixed Point Theory Appl. 15 (2014), 101-132.
[6] V. Benci, D. Fortunato, Hylomorphic solitons for the generalized KdV equation, (2015), 61-86
[7] H. Brezis, E. H. Lieb, Minimum action solutions of some vector field equations, Comm. Math. Phys. 96 (1984), 97-113.
[8] T. Cazenave, P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982), 549-561.
[9] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR 759907 (86f:35160).
[10] C. E. Kenig, C. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries Equation via the Contraction Principle, Comm. Pure and Appl. Math. 46, (1993), 527-620.
[11] C. E. Kenig, H. Takaoka, Global wellposedness of the modified Benjamin-Ono equation with initial data in $H^{1/2}$, arxiv.org/pdf/math/0509573
[12] C. E. Kenig, Y. Martel, Asymptotic stability of solitons for the Benjamin-Ono equation, Revista Matematica Iberoamericana, 2009
[13] E. H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, Invent. Math. 74 (1983), 441-448.
[14] C. Sulem, P. L. Sulem, *The Nonlinear Schrödinger Equation*, Springer New York (1999)

[15] T. Tao, *Global well-posedness of the Benjamin–Ono equation in $H^1(\mathbb{R})$*, Journal of Hyperbolic Differential Equations, 01, 27 (2004). DOI: 10.1142/S0219891604000032

[16] M. Tsutsumi, T. Mukasa, Iino, Richi, *On the generalized Korteweg–de Vries equation*, Proc. Japan Acad. Volume 46, Number 9 (1970), 921-925.

[17] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. 39 (1986), no. 1, 51–67.

[18] G. B. Whitham, *Linear and nonlinear waves*, John Wiley & Sons New York (1974).