Significance of the spinorial basis in relativistic quantum mechanics

Valeri V. Dvoeglazov†

Escuela de Física, Universidad Autónoma de Zacatecas
Antonio Dovalí Jaime s/n, Zacatecas 98000, ZAC., México
Internet address: VALERI@CANTERA.REDUAZ.MX
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Abstract

The problems connected with a choice of the spinorial basis in the $(j,0) \oplus (0,j)$ representation space are discussed. It is shown to have profound significance in relativistic quantum theory. From the methodological viewpoint this fact is related with the important dynamical role played by space-time symmetries for all kind of interactions.

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†On leave of absence from Dept. Theor. & Nucl. Phys., Saratov State University, Astrakhanskaya ul., 83, Saratov RUSSIA. Internet address: dvoeglazov@main1.jinr.dubna.su
We have become accustomed to thinking of the particle world from a viewpoint of the principle of gauge invariance. Profound significance of this principle seems to be clear for everybody and it deserves to be in the place that it occupies now. Remarkable experimental confirmations of both quantum electrodynamics and its non-Abelian extensions (Weinberg-Salam-Glashow model, quantum chromodynamics), ref. [2,3], proved its applicability. Nevertheless, let us still not forget that firstly the principle has been deduced from the interaction of charge particles with electromagnetic potential. At the same time, it has been long recognized that for other kind of particles (namely, for truly neutral particles that are supposed to be described by self/anti-self charge conjugate states) a change of phase leads to destroying self/anti-self conjugacy [4]. It is in this field of modern science (neutrino physics, gluon contributions in QCD etc.) that we have now most consistent indications for new physics. Without any intention to shadow great achievements of the theories based on the use of 4-vector potentials I am going to look into the subject from a little bit different point of view. I would like to discuss here the constructs based on the use of the \((j,0) \oplus (0,j)\) Lorentz group representations for description of the particle world and interactions in it. I hope that presented thoughts could be useful for deeper understanding surprising symmetries of the Dirac equation and an unexpected rich structure of the \((j,0) \oplus (0,j)\) representation space by students and, perhaps, by higher school professors.

The spinorial basis in the standard representation of the Dirac equation:

\[
\begin{align*}
    u^{(1)}(\hat{p}^\mu) &= \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & u^{(2)}(\hat{p}^\mu) &= \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & v^{(1)}(\hat{p}^\mu) &= \sqrt{m} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & v^{(2)}(\hat{p}^\mu) &= \sqrt{m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
\end{align*}
\]

(1)

is well understood and acceptable by everybody for description of a Dirac particle. However, let ask ourselves, what forced us to choose it?.. Let me attack the problem of the choice of a spinorial basis from the most general position.

I am going to consider theories based on the following four postulates:

- For arbitrary \(j\) the right \((j,0)\) and the left \((0,j)\) handed spinors transform in the following ways (according to the Wigner’s ideas [13,7]):

\[
\begin{align*}
    \phi_R(p^\mu) &= \Lambda_R(p^\mu \leftarrow \hat{p}^\mu) \phi_R(\hat{p}^\mu) = \exp(+J \cdot \varphi) \phi_R(\hat{p}^\mu), \\
    \phi_L(p^\mu) &= \Lambda_L(p^\mu \leftarrow \hat{p}^\mu) \phi_L(\hat{p}^\mu) = \exp(-J \cdot \varphi) \phi_L(\hat{p}^\mu).
\end{align*}
\]

(2a)

(2b)

\(\Lambda_{R,L}\) are the matrices for Lorentz boosts; \(J\) are the spin matrices for spin \(j\); \(\varphi\) are parameters of the given boost. If restrict ourselves by the case of bradyons they are defined, e. g., refs. [10,11], by means of:

---

1Of course, spin-1/2 fermions, that transform on the \((1/2,0) \oplus (0,1/2)\) representation of the Lorentz group, could be considered as particular cases. The discussion of recent achievements in development of the Weinberg \(2(2j + 1)\) component theory could be found in ref. [9].

2I use here and below a notation of refs. [10] [12]. For the 4-momentum of a particle in the rest one uses \(\hat{p}^\mu\).

---
\[
\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m}, \quad \sinh(\varphi) = v\gamma = \frac{|p|}{m}, \quad \hat{\varphi} = \hat{n} = \frac{p}{|p|}.
\] (3)

• \(\phi_L\) and \(\phi_R\) are the eigenspinors of the helicity operator \((\mathbf{J} \cdot \mathbf{n})\):

\[
(\mathbf{J} \cdot \mathbf{n}) \phi_{R,L} = h \phi_{R,L}
\] (4)

\((h = -j, -j + 1, \ldots j\) is the helicity quantum number).

• The relativistic dispersion relation \(E^2 - p^2 = m^2\) is hold for free particles.

• Physical results do not depend on rotations of spatial coordinate axes (in other words: the 3-space is uniform).

Since spin-1/2 particles are most important in physical applications and, moreover, the Maxwell’s spin-1 equations can be written in the similar 4-component form, e.g., ref. [14], let me concentrate in the analysis of the \((1/2, 0) \oplus (0, 1/2)\) representation space. For the sake of compact description let denote 2-spinors (left- or right-handed) as \(\xi\). From the condition (see the second item):

\[
\frac{1}{2} (\sigma \cdot \mathbf{n}) \xi = \pm \frac{1}{2} \xi
\] (5)

and by using the expressions for \(\mathbf{n}\) in spherical coordinates:

\[
\begin{align*}
n_x &= \sin \theta \cos \phi, \\
n_y &= \sin \theta \sin \phi, \\
n_z &= \cos \theta,
\end{align*}
\] (6a)

we find that the Pauli spinor \(\xi = column(\xi_1 \quad \xi_2)\) answering for the eigenvalue \(h = 1/2\) of the helicity operator can be parametrized as

\[
\xi_{+1/2} = \begin{pmatrix} \xi_1 \\ \tan \left(\theta/2\right) e^{i\phi} \xi_1 \end{pmatrix} \quad \text{or} \quad \xi_{+1/2} = \begin{pmatrix} \cot \left(\theta/2\right) e^{-i\phi} \xi_2 \\ \xi_2 \end{pmatrix}
\] (7)

in terms of the azimuthal \(\theta\) and the polar \(\phi\) angles associated with the vector \(\mathbf{p} \to 0\), refs. [15,12]. The one, answering for the \(h = -1/2\) eigenvalue, as

\[
\xi_{-1/2} = \begin{pmatrix} \xi_1 \\ -\cot \left(\theta/2\right) e^{i\phi} \xi_1 \end{pmatrix} \quad \text{or} \quad \xi_{-1/2} = \begin{pmatrix} \cot \left(\theta/2\right) e^{-i\phi} \xi_2 \\ \xi_2 \end{pmatrix}
\] (8)

From the normalization condition \(\xi_{\pm1/2}^\dagger \xi_{\pm1/2} = N^2\) (with \(N^2\) being a normalization factor) we have that the form of spinors can be chosen[^3]

[^3]: The second parametrization differs from the first one by an overall phase factor, what does not have influence on physical results.
\[ \xi_{+1/2} = N e^{i \phi_1^+} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \quad \text{or} \quad \xi_{+1/2} = N e^{i \phi_2^+} \begin{pmatrix} \cos(\theta/2) e^{-i \phi} \\ \sin(\theta/2) \end{pmatrix}, \quad (9a) \]

\[ \xi_{-1/2} = N e^{i \phi_1^+} \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) e^{i \phi} \end{pmatrix} \quad \text{or} \quad \xi_{-1/2} = N e^{i \phi_2^+} \begin{pmatrix} -\sin(\theta/2) e^{-i \phi} \\ \cos(\theta/2) \end{pmatrix}. \quad (9b) \]

This parametrization coincides with Eqs. (22a,22b) of ref [12b] and with the formulas of ref. [15, p.87] within definitions of overall phase factors \( \vartheta^\pm \). Let me note useful identities:

\[ \xi_{+1/2}(\hat{p}' \mu) = e^{i(\vartheta^+ - \vartheta^-)} \xi_{-1/2}(\hat{p} \mu), \quad \xi_{-1/2}(\hat{p}' \mu) = e^{i(\vartheta^- - \vartheta^+)} \xi_{+1/2}(\hat{p} \mu), \quad (10) \]

where \( \hat{p}' \mu \) is the parity conjugated 4-momentum in the rest \( (\theta' = \pi - \theta, \phi' = \phi + \pi) \).

If we know spin matrices for arbitrary \( j \) one could find similar parametrizations for spinors of higher dimensions by resolving the set of equations of the \( (2j+1) \)-order for each value of the helicity\(^4\). Let me note that one has a certain freedom in a choice of the spinorial basis in the \( (1/2, 0) \) (or \( 0, 1/2 \)) space since, according to the fourth postulate, physical results do not depend on rotations of the spatial axes and, furthermore, one has arbitrary phase factors \( e^{i \vartheta^\pm} \). Therefore, the common-used choice \( (p \mid OZ) \)

\[ \xi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (11) \]

is only a convenience.

In the Dirac equation one has two kind of spinors \( (\phi_R \text{ and } \phi_L) \). In refs. [10–12] the relation between them in the rest frame

\[ \phi_R(\hat{p} \mu) = \pm \phi_L(\hat{p} \mu) \quad (12) \]

has been named as the Ryder-Burgard relation (see also [14]). It was shown (see footnote \#1 in [12b]) that the relation (12) can be used to derive the Dirac equation, the equation that describes eigenstates of the charge operator. Moreover, if accept this form of the relation for \( (1,0) \oplus (0,1) \) bispinors one can construct an example of the Foldy-Nigam-Bargmann-Wightman-Wigner (FNBWW) type quantum field theory \( [17,13,11] \). The remarkable feature of this Dirac-like modification of the Weinberg theory \( [3] \) is the fact that boson and its antiboson have opposite relative intrinsic parities (like the Dirac fermion).

However, nobody forbids us to take more general form of Eq. (12). Let assume that \( \phi_R(\hat{p} \mu) \) and \( \phi_L(\hat{p} \mu) \) are connected by an arbitrary linear transformation with the complex matrix \( \mathcal{A} \), namely, \( \phi_R(\hat{p} \mu) = \mathcal{A} \phi_L(\hat{p} \mu) \). The unit matrix and Pauli three \( \sigma \)-matrices form a complete set. Therefore, the matrix corresponding to the linear transformation \( \mathcal{A} \) can be expanded in this complete set with the complex coefficients \( c_1 \)

\[ \phi_R(\hat{p} \mu) = \mathcal{A} \phi_L(\hat{p} \mu) = \left[ \begin{pmatrix} \Re c_1 & -\Im c_1 \\ \Im c_1 & \Re c_1 \end{pmatrix} \begin{pmatrix} \phi_L^+(\hat{p} \mu) \\ \phi_L^-(\hat{p} \mu) \end{pmatrix} = \\ = \left[ \begin{pmatrix} c_1^0 & -c_1^1 \\ -c_1^1 & c_1^0 \end{pmatrix} \begin{pmatrix} \phi_L^+(\hat{p} \mu) \\ \phi_L^-(\hat{p} \mu) \end{pmatrix} = e^{i \alpha} \phi_L^+(\hat{p} \mu), \quad (13) \right. \]

\(^4\text{See, e. g., the formulas (23a-c) in ref. [12b] and below.}\)

\(^5\text{The signs } \pm \text{ should be referred to the helicity of the spinors.}\)
Above we have used that $\phi_L$ and $\phi_R$ are the eigenspinors of the helicity operator and have chosen the parametrization of the coefficients $[c_1^+ \pm (|\Re c_1| + i|\Im c_1|)] = e^{i\alpha \pm}$. The modulus of the bracketed quantity (the determinant of the $\mathcal{A}$ matrix) should be equal to the unit from the condition of invariance of the norm of spinors. The equation (12) answers for the particular choices of $\alpha = 0, \pm \pi$. By using the generalized Ryder-Burgard relation and the fact that

$$\left[\Lambda_{L,R}(p^\mu \leftarrow \tilde{p}^\mu)\right]^{-1} = \left[\Lambda_{R,L}(p^\mu \leftarrow \tilde{p}^\mu)\right]^\dagger$$ \hspace{1cm} (14)

we immediately obtain the “generalized” Dirac equation:

$$\phi_R^\pm(p^\mu) = \Lambda_R(p^\mu \leftarrow \tilde{p}^\mu) \phi_R^\pm(\tilde{p}^\mu) = e^{i\alpha \pm} \Lambda_R(p^\mu \leftarrow \tilde{p}^\mu) \Lambda_R^{-1}(p^\mu \leftarrow \tilde{p}^\mu) \phi_R^\pm(p^\mu) = e^{i\alpha \pm} \Lambda_R(p^\mu \leftarrow \tilde{p}^\mu)$$ \hspace{1cm} (15a)

$$\phi_L^\pm(p^\mu) = \Lambda_L(p^\mu \leftarrow \tilde{p}^\mu) \phi_R^\pm(\tilde{p}^\mu) = e^{-i\alpha \pm} \Lambda_L(p^\mu \leftarrow \tilde{p}^\mu) \Lambda_R^{-1}(p^\mu \leftarrow \tilde{p}^\mu) \phi_R^\pm(p^\mu) = e^{-i\alpha \pm} \Lambda_L(p^\mu \leftarrow \tilde{p}^\mu)$$ \hspace{1cm} (15b)

By using definitions of the Lorentz boost (2,3) one can re-write the equations (15a,15b) in the matrix form (provided that $m \neq 0$):

$$\begin{pmatrix} -me^{-i\alpha \pm} & p_0 + (\sigma \cdot \mathbf{p}) \\ p_0 - (\sigma \cdot \mathbf{p}) & -me^{i\alpha \pm} \end{pmatrix} \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} = 0$$ \hspace{1cm} (16)

or

$$(\hat{p} - m\mathcal{T}) \Psi(p^\mu) = 0$$ \hspace{1cm} (17)

with

$$\mathcal{T} = \begin{pmatrix} e^{-i\alpha \pm} & 0 \\ 0 & e^{i\alpha \pm} \end{pmatrix}$$ \hspace{1cm} (18)

Let us note the particular cases:

$$\alpha \pm = 0, 2\pi \hspace{1cm} (\hat{p} - m)\Psi = 0 \hspace{1cm} (19a)$$

$$\alpha \pm = \pm \pi \hspace{1cm} (\hat{p} + m)\Psi = 0 \hspace{1cm} (19b)$$

$$\alpha \pm = +\pi/2 \hspace{1cm} (\hat{p} + im\gamma_5)\Psi = 0 \hspace{1cm} (19c)$$

$$\alpha \pm = -\pi/2 \hspace{1cm} (\hat{p} - im\gamma_5)\Psi = 0 \hspace{1cm} (19d)$$

The equations (19a,19b) are the well-known Dirac equations for positive- and negative-energy bispinors in the momentum space. The equations of the type (19c,19d) had also been discussed in the old literature, e.g., [18]. They have been named as the Dirac equations for 4-spinors of the second kind [19,20]. Their possible relevance to description of neutrino

\[\text{6Please, do not forget that the Lorentz boost matrices are Hermitian for a finite representation of the group.}\]
By using the mentioned property of the Wigner operator we transform Eqs. (23a,23b) to:

\[
\phi^+_{R}(p^\mu) = -i e^{i\beta} \Theta_{[1/2]} \phi^+_{L}(p^\mu) \quad \text{and} \quad \phi^+_{L}(p^\mu) = i e^{i\beta} \Theta_{[1/2]} \phi^+_{R}(p^\mu) \, .
\]

We have used above that \( \sigma_2 \) matrix is connected with the Wigner operator \( \Theta_{[1/2]} = -i \sigma_2 \) and the property of the Wigner operator for any spin \( \Theta_{[j]} J \Theta_{[j]}^{-1} = -J^* \). So, if \( \phi_{L,R} \) is an eigenstate of the helicity operator, then \( \Theta_{[j]} \phi^*_{L,R} \) is the eigenstate with the opposite helicity quantum number:

\[
(J \cdot n) \Theta_{[j]} \left[ \phi^h_{L,R}(p^\mu) \right]^* = -\hbar \Theta_{[j]} \left[ \phi^h_{L,R} \right]^* \, .
\]

Therefore, from Eqs. (20,21) we have:

\[
\phi^+_{R}(p^\mu) = +i e^{i\beta} \Lambda_R (p^\mu \leftarrow \hat{p}^\mu) \Theta_{[1/2]} \left[ \Lambda_L^{-1} (p^\mu \leftarrow \hat{p}^\mu) \right]^* \left[ \phi^+_{L}(p^\mu) \right]^* \, .
\]

(23a)

\[
\phi^+_{L}(p^\mu) = -i e^{i\beta} \Lambda_L (p^\mu \leftarrow \hat{p}^\mu) \Theta_{[1/2]} \left[ \Lambda_R^{-1} (p^\mu \leftarrow \hat{p}^\mu) \right]^* \left[ \phi^+_{R}(p^\mu) \right]^* \, .
\]

(23b)

By using the mentioned property of the Wigner operator we transform Eqs. (23a,23b) to:

\[
\phi^+_{R}(p^\mu) = +i e^{i\beta} \Theta_{[1/2]} \phi^+_{L}(p^\mu) \quad \text{and} \quad \phi^+_{L}(p^\mu) = -i e^{i\beta} \Theta_{[1/2]} \phi^+_{R}(p^\mu) \, .
\]

(24a)

(24b)

In the matrix form one has:

\[
\begin{pmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{pmatrix} = e^{i\beta} \begin{pmatrix}
0 & i \Theta_{[1/2]} \\
i \Theta_{[1/2]} & 0
\end{pmatrix}
\begin{pmatrix}
\phi^*_R(p^\mu) \\
\phi^*_L(p^\mu)
\end{pmatrix} = S^c_{[1/2]} \begin{pmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{pmatrix}
\]

(25)

with \( S^c_{[1/2]} \) being the operator of charge conjugation in the \((1/2,0) \oplus (0,1/2)\) representation space, e. g. [21]. In fact, we obtain conditions of self/anti-self charge conjugacy:

\[
\Psi(p^\mu) = \pm \Psi^c(p^\mu) \, .
\]

(26)

\[\text{We know that after a multiplication by a non-singular matrix the property of being a complete set is hold.}\]
Thus, depending on relations between left- and right-handed spinors (in fact, depending on the choice of the spinorial basis) we obtain physical excitations of the different physical nature. In the first version of the Ryder-Burgard relation we have the Dirac equations; in the framework of the second version, neutral fermions.

At last, the most general form of the Ryder-Burgard relation is

$$\phi_R(p^\mu) = A \phi_L(p^\mu) + B \phi_L^*(p^\mu) \quad ,$$

what results in

$$\begin{align*}
\phi_R(p^\mu) &= A e^{i\alpha} \phi_L(p^\mu) + i B e^{i\beta} \Theta_{[1/2]} \phi_L^*(p^\mu) \quad , \\
\phi_L(p^\mu) &= A e^{-i\alpha} \phi_R(p^\mu) - i B e^{i\beta} \Theta_{[1/2]} \phi_R^*(p^\mu) \quad .
\end{align*}$$

The equation, that could be considered as a mathematical generalization of the Dirac equation, is then

$$\begin{pmatrix}
A e^{i\alpha} \Lambda_R \Lambda_L^{-1} - i B e^{i\beta} \Theta_{[1/2]} K \\
-1
\end{pmatrix}

\begin{pmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{pmatrix} = 0 \quad ,$$

where $A^2 + B^2 = 1$ and $K$ is the operation of complex conjugacy. In a symbolic form it is re-written to

$$\left[ A \frac{\hat{p}}{m} + B T S_{[1/2]}^c - T \right] \Psi(p^\mu) = 0 \quad .$$

By using the computer algebra system MATEMATICA 2.2 it is easy to check that the equation has the correct relativistic dispersion (see the third item of the set of postulates).

What physical sense could be attached to this equation?

Let me now regard the problem of a choice of spinorial basis in a $j = 1$ case. In the presented consideration I use the Weinberg $2(2j + 1)$ component formalism. It is easy to show, by using the same procedure, that $j = 1$ spinors $\xi$ can be parametrized, e. g., in the following form

$$\begin{align*}
\xi_+ = N e^{i\vartheta} \begin{pmatrix}
\frac{1}{2} (1 + \cos \theta) e^{-i\phi} \\
\sqrt{\frac{1}{2}} \sin \theta
\end{pmatrix} \quad , & \quad \xi_- = N e^{i\vartheta} \begin{pmatrix}
-\frac{1}{2} (1 - \cos \theta) e^{-i\phi} \\
\sqrt{\frac{1}{2}} \sin \theta
\end{pmatrix} \quad , \\
\xi_0 = N e^{i\vartheta} \begin{pmatrix}
-\sqrt{\frac{1}{2}} \sin \theta e^{-i\phi} \\
\cos \theta
\end{pmatrix} \quad ,
\end{align*}$$

In fact, the second definition of the Ryder-Burgard relation leads to the Majorana-McLennan-Case spinors. Recent discussions of this construct could be found in refs. 24, 25.

We deal above with the spinors of same helicities. The student can reveal without any troubles, what happens if we were connect the spinors of different helicities, $\phi_R^\pm = A \phi_L^\mp$ or $\phi_R^\mp = B [\phi_L^\mp]^*$. 

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provided that they are eigenspinors of the helicity operator. In the isotropic-basis representation the \( j = 1 \) spin operators are expressed, ref. \[26\], in the following way:

\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\] (33)

The eigenvalues of the operator \( J \cdot n \) could be \( h = \pm 1, 0 \). As opposed to a spin-1/2 case one has \( 9 = 3^2 \) linear independent matrices forming the complete set. They can be chosen from the following set of the ten symmetric matrices

\[
J_{00} = 1, \quad J_{0i} = J_{i0} = J_i, \quad J_{ij} = J_i J_j + J_j J_i - \delta_{ij}.
\] (34)

\[
J_{\mu\nu} = 0 \quad \text{eliminates one of } J_{\mu\nu} \text{ matrix (e. g., } J_{00}).\]

Following to main points of the preceding discussion let me consider relations between left- and right- spinors. The following form of the Ryder-Burgard relation:

\[
\phi_R^{\pm,0}(\vec{p}^\mu) = e^{i\alpha_{\pm,0}} \phi_L^{\pm,0}(\vec{p}^\mu), \quad \phi_L^{\pm,0}(\vec{p}^\mu) = e^{-i\alpha_{\pm,0}} \phi_R^{\pm,0}(\vec{p}^\mu)
\] (36)

is very similar to the first form of the relation in a spin-1/2 case. In the process of deriving this relation we used that any tensor can be expanded in a direct product of two vectors. The equation obtained by using the Wigner postulate (item 1, \( m^2 \neq 0 \))

\[
[\gamma_{\mu\nu} p^\mu p^\nu - m^2 T] \Psi(p^\mu) = 0
\] (37)

in the case \( \alpha_{\pm,0} = 0 \) coincides with the Weinberg equation and, after taking into account \( \alpha_{\pm,0} = \pm \pi \), with the modified equation obtained by Ahluwalia \[11\] in the framework of the FNBWW-type quantum field theory \[17,13\].

As for the second form (connecting \( \phi_{L,R}^* \) and \( \phi_{L,R}^* \)) one has an essential difference from the spin-1/2 consideration. Expanding the \( B \) matrix, \( \phi_R(\vec{p}^\mu) = B \phi_L(\vec{p}^\mu) \), in the other complete set \[1\] namely, \( J_{\mu\nu} \Theta_{[1]} \) we come to

\[
\Theta_{[1]} = \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\] (38)

---

10 Of course, it is possible to choose the so-called ‘orthogonal’ basis \((J_i)_{jk} = -i \epsilon_{ijk}\) because they are connected by the unitary matrix \( J_{\text{isotr}} = U J_{\text{orth}} U^{-1} \):

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -i & 0 \\
0 & 0 & -\sqrt{2} \\
-1 & -i & 0
\end{pmatrix}
\] (32)

11 The explicit form of the Wigner operator \( \Theta_{[1]} \) for spin 1 has been given in refs. \[12,27\] in the isotropic basis:

\[
\Theta_{[1]} = \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]
\[\phi_{R}^{\pm,0}(\hat{p}^\mu) = e^{i\beta \mp \theta} \Omega \left[ \phi_{R}(\hat{p}^\mu) \right]^* , \quad \phi_{L}^{\pm,0}(\hat{p}^\mu) = e^{i\beta \pm \theta} \Omega \left[ \phi_{L}(\hat{p}^\mu) \right]^* . \] (39)

This fact is connected with another property of the Wigner operator: \( \Theta_{[j]} = (-1)^j. \) As a result, we obtain

\[\Psi(p^\mu) = \left( \begin{array}{c} \phi_{R}(p^\mu) \\
\phi_{L}(p^\mu) \end{array} \right) = e^{i\beta \pm \theta} \left( \begin{array}{cc} 0 & \Theta \mathbb{K} \\
\Theta \mathbb{K} & 0 \end{array} \right) \left( \begin{array}{c} \phi_{R}(p^\mu) \\
\phi_{L}(p^\mu) \end{array} \right) = \Gamma_5 S_{[1]} \Psi(p^\mu) , \] (40)

provided that the charge conjugation operator \( S_{[1]} \) is chosen like ref. [11,12] in accordance with the FNBWW construct.

Next, let me draw your kindness attention to other possibilities of description of arbitrary spin particles. In general, it is possible to choose other representation of the Lorentz group for describing higher spin particles (see ref. [6c]). It is interesting to note that the well-known the Dirac-Fierz-Pauli equation for any spin has been re-written in ref. [27] (see also my recent work [28]) to the form very similar to the spin-1/2 case:

\[\alpha^\mu \partial_\mu \Phi = + m \Upsilon , \] (41a)
\[\bar{\alpha}^\mu \partial_\mu \Upsilon = - m \Phi , \] (41b)

where \( \bar{\alpha}^\mu = \alpha_\mu \) are the matrices that satisfy all the algebraic relations that the Pauli \( 2 \times 2 \) matrices \( \sigma_\mu \) do, except for completeness. The object \( \Phi \) belongs to the \((j, 0) \oplus (j - 1, 0)\) representation of the Lorentz group and the \( \Upsilon \), to the \((j - 1/2, 1/2)\) representation. Is there exist the analog of the Ryder-Burgard relation which could be proposed in the framework of the Dirac-Fierz-Dowker construct for any spin?

Finally, in my presentation the attempt was undertaken to understand, how all possible relations between basis vectors of the different representation space (e.g., between right- \( \phi_{R}(\hat{p}^\mu) \) and left-handed spinors \( \phi_{L}(\hat{p}^\mu) \) that are known to become interchanged under parity conjugation [14]) define dynamical equations. It was found that from a mathematical viewpoint the well-known equations are the particular cases only. The analysis reveals that the choice of spinorial basis in \((j, 0) \oplus (0, j)\) representation space has profound significance for dynamical evolution of the physical systems.

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REFERENCES

[1] V. V. Dvoeglazov, Yu. N. Tyukhtyaev and R. N. Faustov, Mod. Phys. Lett. A8 (1993) 3263; Fiz. Elem. Chast. At. Yadra 25 (1994) 144 [English translation: Phys. Part. Nucl. 25 (1994) 58]

[2] J. L. Rosner, Overview of the Standard Model. Preprint EFI 94-59 [hep-ph/9411396], Nov. 1994, to be published in Proc. of the Joint US-Polish Workshop on Physics from Planck Scale to Electroweak Scale. Warsaw, Poland, 21-24 Sep., 1994

[3] R. K. Ellis, Status of QCD. Preprint FERMILAB-CONF-93/011-T, Jan. 1993, presented at the 7th Meeting of the APS Division of Part. and Fields, Fermilab, Nov. 10-14, 1992

[4] W. H. Furry, Phys. Rev. 54 (1938) 56; ibid 56 (1939) 1184

[5] Not all physicists are so enthusiastic in evaluations of the status of modern physics theories. For instance, let me remind what claimed P. A. M. Dirac in his last works. E. g., in [Mathematical Foundations of Quantum Theory. (Academic Press, Inc., 1978), p. 1] he doubts mathematical grounds of modern physics: “Any physical or phylosophical ideas that one has must be adjusted to fit the mathematics. Not the other way around. Too many physicists are inclined to start from some preconceived physical ideas and then to try to develop them and find a mathematical scheme that incorporate them. Such a line of attack is unlikely to lead to success...

The appearance of this [Dirac] equation did not solve the general problem of making quantum mechanics relativistic. It applied only to the problem of a single electron, not several particles in interaction... When one tried to solve it, one always obtained divergent integrals... Rules for discarding the infinities [(renormalization) have been developed]. Most physicists are very satisfied with this situation. They argue that if one has rules for doing calculations and the results agree with observation, that is all that one requires. But it is not all that one requires. One requires a single comprehensive theory applying to all physical phenomena. Not one theory for dealing with non-relativistic effects and a separate disjoint theory for dealing with certain relativistic effects. Furthermore, the theory has to be based on sound mathematics, in which one neglects only quantities that are small. One is not allowed to neglect infinitely large quantities. The renormalization idea would be sensible only if it was applied with finite renormalization factors, not infinite ones. For these reasons I find the present quantum electrodynamics quite unsatisfactory. One ought not to be complacent about its faults. The agreement with observation is presumably a coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits. Such coincidences are no reason for turning a blind eye to the faults of a theory. Quantum electrodynamics ... was built up from physical ideas that were not correctly incorporated into the theory and it has no sound mathematical foundation. One must seek a new relativistic quantum mechanics and one’s prime concern must be to base it on sound mathematics.”

[6] S. Weinberg, Phys. Rev. B133 (1964), 1318; ibid 134 (1964) 882; ibid 181 (1969) 1893. In these papers the pioneer study of the $(j,0) \oplus (0,j)$ representation space has been undertaken. In fact, this way of description of particles of arbitrary spin originates from the classical work of E. Wigner, ref. [7]. Unfortunately, one of remarkable statements of the second paper of S. Weinberg was not realised before an appearance of the paper [8]. It deals with the massless first-order equations (4.19,4.20) for arbitrary spin (as a particular case, with the first-order equations (4.21,4.22) for spin $j = 1$). “The fact that these field equations are of first order for any spin seems to me to be of no great significance...” (II, p. B888). “A field with $2j + 1$ components, which are constructed as a linear combination of $2j + 1$ independent creation and/or annihilation operators, can satisfy only the trivial wave equations $({\Box}^2 - m^2)^N \psi = 0$” (III, p. 1896).

Cf. with the thoughts of P. A. M. Dirac [9].
[7] E. P. Wigner, Ann. Math. **40** (1939) 149
[8] D. V. Ahluwalia and D. J. Ernst, Mod. Phys. Lett. **A7** (1992) 1967
[9] V. V. Dvoeglazov, Rev. Mex. Fis. Suppl. **40** (1994) 352
[10] L. H. Ryder, “Quantum Field Theory” (Cambridge University Press, Cambridge, UK, 1987), §2.3
[11] D. V. Ahluwalia, M. B. Johnson and T. Goldman, Phys. Lett. **B316** (1993) 102; D. V. Ahluwalia and T. Goldman, Mod. Phys. Lett. **A8** (1993) 2623
[12] D. V. Ahluwalia, “Incompatibility of Self-Charge Conjugation with Helicity Eigenstates and Gauge Interactions.” Preprint LA-UR-94-1252, Los Alamos, Apr. 1994; “McLennan-Case Construct for Neutrino, its Generalization, and a Fundamentally New Wave Equation.” Preprint LA-UR-94-3118, Los Alamos, Sept. 1994
[13] E. P. Wigner, in “Group theoretical concepts and methods in elementary particle physics – Lectures of the Istanbul Summer School of Theoretical Physics, 1962”. Ed. F. Gürsey
[14] T. Ohmura (Kikuta), Progr. Theor. Phys. **16** (1956) 684; M. Sachs, Ann. Phys. **6** (1959) 244; M. Sachs and S. L. Schwebel, J. Math. Phys. **3** (1962) 843
[15] C. Itzykson and J.-B. Zuber, “Quantum field theory.” (McGraw-Hill Book Co., 1980)
[16] Yu. V. Novozhilov, “Vvedenie v teoriyu elementarnyh chastits.” (Nauka, Moscow, 1971) [English translation: “Introduction to Elementary Particle Theory.” (Pergamon Press, Oxford, UK, 1975), p. 96]
[17] B. P. Nigam and L. L. Foldy, Phys. Rev. **102** (1956) 1410
[18] G. A. Sokolik, ZhETF **33** (1957) 1515 [English translation: Sov. Phys. JETP **6** (1958) 1170]
[19] É. Cartan, “Lecons sur la Théorie des spineurs.” (Hermann, Paris, 1938)
[20] I. M. Gelfand and M. L. Tsetlin, ZhETF **31** (1956) 1107 [English translation: Sov. Phys. JETP **4** (1957) 947]
[21] P. Ramond, Field Theory: A Modern Primer. (Addison-Wesley Pub. Co., USA, 1989)
[22] E. Majorana, Nuovo Cim. **14** (1937) 171 [English translation: Tech. Trans. TT-542, Nat. Res. Council of Canada]
[23] K. M. Case, Phys. Rev. **107** (1957) 307
[24] D. V. Ahluwalia, M. B. Johnson and T. Goldman, Mod. Phys. Lett. **A9** (1994) 439
[25] V. V. Dvoeglazov, “A note on the Majorana theory for $j = 1/2$ and $j = 1$ particle states.” Preprint EFUAZ FT-94-10, Dec. 1994, reported at the XVIII Oaxtepec Nuclear Physics Symp., Jan. 4-7, 1995
[26] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, Kvantovaya teoriya uglovogo momenta. (Nauka, Leningrad, 1975) [English translation: Quantum Theory of Angular Momentum. (World Sci. Pub., Singapore, 1988)]
[27] J. S. Dowker and Y. P. Dowker, Proc. Roy. Soc. **A294** (1966) 175; J. S. Dowker, ibid **297** (1967) 351
[28] V. V. Dvoeglazov, Nuovo Cim. **107A** (1994) 1758