Frequency Limited $\mathcal{H}_2$ Optimal Model Reduction
of Large-Scale Sparse Dynamical Systems

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Abstract
We mainly consider the frequency limited $\mathcal{H}_2$ optimal model order reduction of large-scale sparse generalized systems. For this purpose we need to solve two Sylvester equations. This paper proposes efficient algorithm to solve them efficiently. The ideas are also generalized to index-1 descriptor systems. Numerical experiments are carried out using Python Programming Language and the results are presented to demonstrate the approximation accuracy and computational efficiency of the proposed techniques.

keywords Frequency limited model reduction, $\mathcal{H}_2$ optimal condition, frequency limited Gramenas, Sylvester and Lyapunov equations

1 Introduction
Model order reduction (MOR) is a process to approximate a high-order dynamical system by a substantially low-order system with a maximum accuracy. This tool is now widely used in different disciplines of science, engineering and technology to reduce the complexity of the model. In general, the reduced order models are used in controller design, simulation and optimization. For motivation, applications and techniques of MOR see, e.g., [1, 2].

The commonly useful methods for the model reduction of large-scale linear time invariant dynamical systems are the balanced truncation and $\mathcal{H}_2$ optimal

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model reduction \[1\]. Both the methods are well established and successfully investigated to find the model reduction of large-scale sparse dynamical systems. In the recent times frequency and time limited model reduction methods have taken a lot of attentions due to its demand in real-life applications. In many applications, a specific frequency interval is more important, i.e., the ROM should maintain a superior accuracy within that desired frequency interval. Balanced truncation based frequency limited model reduction was discussed by Gawronski and Juang in \[3\]. The computational techniques for the time and frequency limited balanced truncation were discussed in \[4\].

The optimal $\mathcal{H}_2$ model reduction methods have been studied and investigated in \[5, 6, 7, 8, 9\]. See, also the references cited therein. In all these papers the proposed technique is based on either Gramian assistance first-order optimality conditions \[5\] or the tangential interpolation \[6\] of the transfer function. In fact both the conditions are coincided which is shown in \[9\]. These papers only discuss the model reduction on the infinite frequency interval. For the time limited case the we refer the readers to \[10\]. Although, in \[11\] authors briefly introduced optimal $\mathcal{H}_2$ model reduction problem of standard state space systems considering a restricted frequency interval, there the implementation details were not given.

This paper focuses on the computational techniques of the frequency limited optimal $\mathcal{H}_2$ model reduction method of large-scale sparse systems. We mainly generalized the idea as in \[12, 9\] in which the proposed algorithm is called two sided iteration algorithm (TSIA). Moreover, to implement the frequency limited TSIA we need to solve two frequency limited Sylvester equations. This paper also discusses how to solve the Sylvester equations efficiently by preserving the sparsity of the system. Besides the generalized systems the idea is also extended for index-1 descriptor systems. The benefits of the algorithmic improvements presented in this paper are illustrated by several numerical examples. We have generated the the results by using Python Programming Language. Rest of this paper is organized as follows.

Section 2 overview the TSIA and the optimal $\mathcal{H}_2$ model reduction of generalized system. Then the ideas of this section are discussed in the next sections for the frequency-limited model order reduction. Section 5 presents the algorithm for solving frequency limited Sylvester equations which provides projectors to carry out the FLMOR. The results of the numerical experiments are depicted in Section 6 which show the efficiency and capability of the proposed methods.

2 The TSIA and $\mathcal{H}_2$ optimal model order reduction of generalized Systems

The goal of this section is to review the basic idea of $\mathcal{H}_2$ optimal model order reduction of generalized Systems. Let us consider a linear time invariant continuous-time system of the form

$$
E \dot{x}(t) = Ax(t) + Bu(t),
$$

$$
y(t) = Cx(t) + Du(t),
$$

where $E \in \mathbb{R}^{n \times n}$ is non-singular, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times p}$. The transfer-function matrix of this system is defined by $G(s) = \ldots$
\[ C(sE - A)^{-1}B + D, \] where \( s \in \mathbb{C}. \) The controllability and the observability Gramians of the system on the infinite frequency range can be defined as

\[
P = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega E - A)^{-1}BB^T(\omega E^T - A^T) d\omega \quad \text{and} \quad Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega E^T - A^T)^{-1}C^TC(\omega E - A) d\omega,
\]

and they are the solutions of the continuous-time algebraic Lyapunov equations

\[
AP^T + EP^T + BB^T = 0 \quad \text{and} \quad A^TQE + E^TQA + C^TC = 0,
\]

respectively. We want to construct a substantially reduced-order model

\[
\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\
\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t),
\]

where \( \hat{A} \in \mathbb{R}^{r \times r}, \hat{B} \in \mathbb{R}^{r \times p}, \hat{C} \in \mathbb{R}^{m \times r} \) and \( \hat{D} \in \mathbb{R}^{m \times p}. \) The goal is to minimize the error

\[
\Xi = \|G - \hat{G}\|_{\mathcal{H}_2},
\]

where \( \|\cdot\|_{\mathcal{H}_2} \) denotes the system’s \( \mathcal{H}_2 \)-norm [1] and \( \hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \) is the transfer-function matrix of the reduced system. The \( \mathcal{H}_2 \)-norm of the error system as defined in (1) can be measured by

\[
\Xi =: \sqrt{\text{Tr}(C_\Xi P_\Xi C_\Xi^T)} \quad \text{or} \quad \sqrt{\text{Tr}(B_\Xi^T Q_\Xi B_\Xi)},
\]

where \( P_\Xi \) and \( Q_\Xi \) are the solutions of the Lyapunov equations

\[
A_\Xi P_\Xi E_\Xi^T + E_\Xi P_\Xi A_\Xi^T + B_\Xi B_\Xi^T = 0 \quad \text{and} \quad A_\Xi^T Q_\Xi E_\Xi + E_\Xi^T Q_\Xi A_\Xi + C_\Xi^TC_\Xi = 0,
\]

in which

\[
E_\Xi = \begin{bmatrix} E \\ I \end{bmatrix}, \quad A_\Xi = \begin{bmatrix} A \\ \hat{A} \end{bmatrix}, \quad B_\Xi = \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \quad \text{and} \quad C_\Xi = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.
\]

Now, partitioning \( P_\Xi \) and \( Q_\Xi \) as

\[
P_\Xi = \begin{bmatrix} P & M \\ MT & \hat{P} \end{bmatrix} \quad \text{and} \quad Q_\Xi = \begin{bmatrix} Q & N \\ NT & \hat{Q} \end{bmatrix},
\]

and plugging into (6) we obtain the following algebraic matrix equations

\[
APE^T + EPA^T + BB^T = 0, \\
A\hat{P} + \hat{P}\hat{A}^T + B\hat{B}^T = 0, \\
AM + EM\hat{A}^T + B\hat{B}^T = 0, \\
A^TQE + E^TQA + C^TC = 0, \\
\hat{A}^T\hat{Q} + \hat{Q}\hat{A} + \hat{C}^T\hat{C} = 0, \\
A^TN + E^TN\hat{A} - C^T\hat{C} = 0,
\]

respectively.
where $\hat{P}$ and $\hat{Q}$ are respectively known as the controllability and observability Gramians of the reduced systems. Therefore, the $H_2$ norm of the error system in (5) can be measured by

$$\Xi = \begin{cases} 
\sqrt{\text{Tr}(CPC^T) + \text{Tr}(\hat{C}\hat{P}\hat{C}^T) + 2\text{Tr}(CM\hat{C}^T)} \\
\text{or} \\
\sqrt{\text{Tr}(B^TQB) + \text{Tr}(\hat{B}\hat{Q}\hat{B}) + 2\text{Tr}(B^T\hat{N}\hat{B})}. 
\end{cases}$$

(8)

The first-order optimality conditions for the optimal $H_2$ model reduction was given in [14], which is known as Wilson conditions. Wilson conditions are based on the first derivatives of (4) with respect to $\hat{A}$, $\hat{B}$ and $\hat{C}$ as follows:

$$\nabla \Xi_{\hat{A}} = 2(\hat{Q}\hat{P} + WT EV), \quad \nabla \Xi_{\hat{B}} = 2(\hat{Q}\hat{B} + W^T E^{-1} B), \quad \nabla \Xi_{\hat{C}} = 2(\hat{C}\hat{P} - CV).$$

Setting these three derivatives to zero leads to the Wilson conditions,

$$\hat{Q}\hat{P} + N^T EM = 0,$$  
(9)

$$\hat{Q}\hat{B} + N^T E^{-1} B = 0,$$  
(10)

$$\hat{C}\hat{P} - CM = 0.$$  
(11)

These three conditions in fact yield the left and right projection matrices to compute an optimal reduced order system (3) and in the optimal reduced order system the reduced matrices are formed as

$$\hat{A} = W^T E^{-1} AV, \quad \hat{B} = W^T B, \quad \text{and} \quad \hat{C} = CV,$$  
(12)

where $V = M\hat{P}^{-1}$ and $W^T = -\hat{Q}^{-1} N^T$ and hence it can be proved that $W^T EV = I$. However, we can not guarantee that $\hat{P}$ and $\hat{Q}$ are invertible, since to assure this the reduced model should be completely controllable and observable [1]. In the case that they are invertible, the multiplication from the right is only a transformation of bases and does not change the subspace. The idea by Xu and Zeng [9] was to satisfy the Wilson conditions by setting

$$W = N \quad \text{and} \quad V = M.$$  
(13)

Note that $V$ and $W$ can be computed by solving the matrix equations (1c) and (7f), respectively which are known as Sylvester’s equations. Another important observation is that if we want to compute the optimal projection subspace we already need the optimal solution $\hat{A}$, $\hat{B}$ and $\hat{C}$. However, this is not known prior. A possible solution is to start with a reduced model, which emerged from an arbitrary projection of the original model, solve matrix equations (7c) and (7f), compute the projectors, and restart the process with the newly obtained reduced model until we are satisfied. In this way we get a kind of a fixed point iteration. This procedure is called two sided iteration algorithm (TSIA) by Xu and Zeng in [9].

The Wilson conditions are Gramian-based conditions since they are related to Gramians of the system. The Hyland-Bernstein conditions [7] are another gramian-based first-order optimal conditions, which were shown to be equivalent to the Wilson conditions [15]. Van Dooren et al., were characterized the
Algorithm 1: Two-sided iteration algorithm (TSIA).

**Input**: \( E, A, B, C, D \).

**Output**: \( \hat{A}, \hat{B}, \hat{C}, \hat{D} := D \).

1. Choose matrices \( W_0 \in \mathbb{R}^{n \times r} \) and \( V_0 \in \mathbb{R}^{n \times r} \) such that \( W_0^T V_0 = I \).

2. Construct the reduced-order matrices:
   \[
   \hat{A} = W_0^T E^{-1} A V_0, \quad \hat{B} = W_0^T E^{-1} B \quad \text{and} \quad \hat{C} = C V_0.
   \]

3. **while** \( i \leq N - 1 \) **do**

4. Compute \( V_i \) and \( W_i \) by solving Sylvester’s equations:
   \[
   AV + EV \hat{A}^T + BB^T = 0 \quad (14a)
   
   A^T W + E^T W \hat{A} = C^T \hat{C} = 0, \quad (14b)
   \]

5. Compute \( W_{i+1} = W_i (V_i^T W_i)^{-1} \) and \( V_{i+1} = V_i \)

6. Construct the reduced-order matrices
   \[
   \hat{A} = W_{i+1}^T E^{-1} A V_{i+1}, \quad \hat{B} = W_{i+1}^T E^{-1} B \quad \text{and} \quad \hat{C} = C V_{i+1}.
   \]

7. \( i = i + 1 \).

8. **end while**

Tangential interpolation based \( H_2 \) optimal conditions in [12]. One drawback of interpolation based model reduction is the selection of interpolation points. However in [15] authors proposed the Iterative Rational Krylov Algorithms (IRKA) to resolve this problem. On the other hand Xu and Zeng showed in [9] that both the Gramian and interpolation-based optimality conditions are the same. In [9] authors also presented the two-sided iterative algorithm (TSIA) for a standard system which is slightly modified as in [12]. For our convenience the TSIA for the \( H_2 \) optimal model reduction for the generalized system [11] is summarized in Algorithm 1.

## 3 FL \( H_2 \) optimal MOR of generalized systems

This section moves to the frequency limited \( H_2 \) optimal model reduction of the system [11]. For this purpose we first define frequency limited Gramians. In the previous section the system Gramians have been defined on the infinite frequency interval. If we replace the infinite interval \((-\infty, \infty)\) into a finite interval \( \omega = [\omega_1, \omega_2] \) then the controlability and the observability Gramians can be defined as

\[
P_\omega = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} (\nu E - A)^{-1} B B^T (\nu E^T - A^T) d\nu \quad (15)
\]

\[
Q_\omega = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} (\nu E^T - A^T)^{-1} C^T C (\nu E - A) d\nu, \quad (16)
\]

which satisfies the frequency limited Lyapunov equations.
\[ \begin{align*}
    AP_\omega E^T + EP_\omega A^T + B_\omega BB^T + BB^T B_\omega^* &= 0, \\
    A^T Q_\omega E + E^T Q_\omega A + C_\omega^* C^T C + C^T CC_\omega &= 0,
\end{align*} \]

with

\[ \begin{align*}
    B_\omega &= \frac{i}{2\pi} \ln(A + i\omega_2 E)(A + i\omega_1 E)^{-1} \\
    C_\omega &= \frac{i}{2\pi} \ln(A + i\omega_1 E)(A + i\omega_2 E).
\end{align*} \]

The goal in this paper is to construct a reduced model \( \hat{\mathcal{G}} = (\hat{A}, \hat{B}, \hat{C}) \) from a given model \( G = (E, A, B, C) \) that minimizes the error

\[ \Xi_\omega = \|G - \hat{\mathcal{G}}\|_{\mathcal{H}_2, \omega}, \]

where \( \|\cdot\|_{\mathcal{H}_2, \omega} \) denotes the \( \mathcal{H}_2 \)-norm on the prescribed frequency range \( \omega \). The transfer-function matrix of the reduced system is the \( \mathcal{H}_2 \)-norm of the error system as defined in [1] can be measured efficiently by

\[ \Xi_\omega = \sqrt{\text{Tr} \left( C \Xi_\omega C^T \right)} \quad \text{or} \quad \sqrt{\text{Tr} \left( B^T Q \Xi_\omega B \right)}, \]

where \( P_{\Xi, \omega} \) and \( Q_{\Xi, \omega} \) are the solutions of the Lyapunov equations

\[ \begin{align*}
    A_{\Xi} P_{\Xi, \omega} E_{\Xi}^T + E_{\Xi} P_{\Xi, \omega} A_{\Xi}^T + B_{\Xi} B_{\Xi} E_{\Xi}^T + B_{\Xi} E_{\Xi}^T B_{\Xi}^* &= 0, \\
    A^T_{\Xi} Q_{\Xi, \omega} E_{\Xi} + E^T_{\Xi} Q_{\Xi, \omega} A_{\Xi} + C_{\Xi}^* C^T_{\Xi} C_{\Xi} + C^T_{\Xi} C_{\Xi} E_{\Xi}^T &= 0,
\end{align*} \]

where

\[ \begin{align*}
    B_{\Xi, \omega} &= \frac{i}{2\pi} \ln(A_{\Xi} + i\omega_2 E_{\Xi})(A_{\Xi} + i\omega_1 E_{\Xi})^{-1} \\
    C_{\Xi, \omega} &= \frac{i}{2\pi} \ln(A_{\Xi} + i\omega_1 E_{\Xi})(A_{\Xi} + i\omega_2 E_{\Xi}).
\end{align*} \]

Due to the structure of \( E_{\Xi}, A_{\Xi}, B_{\Xi} \) and \( C_{\Xi} \) we can partition \( P_{\Xi, \omega}, Q_{\Xi, \omega}, B_{\Xi, \omega} \) and \( C_{\Xi, \omega} \) as follows

\[ \begin{align*}
    P_{\Xi, \omega} &= \begin{bmatrix} P_{\omega} & M_{\omega} \\ M^T_{\omega} & P_{\omega} \end{bmatrix}, & Q_{\Xi, \omega} &= \begin{bmatrix} Q_{\omega} & N_{\omega} \\ N^T_{\omega} & Q_{\omega} \end{bmatrix}, \\
    B_{\Xi, \omega} &= \begin{bmatrix} B_{\omega} & 0 \\ 0 & B_{\omega} \end{bmatrix}, & C_{\Xi, \omega} &= \begin{bmatrix} C_{\omega} & 0 \\ 0 & C_{\omega} \end{bmatrix}.
\end{align*} \]

Therefore (20) yields

\[ \begin{align*}
    AP_\omega E^T + EP_\omega A^T + B_\omega BB^T + BB^T B_\omega^* &= 0, \\
    \dot{A}P_\omega + \dot{P}_\omega A^T + B_\omega \dot{B}B^T + \dot{B}B^T B_\omega^* &= 0, \\
    AM_\omega + EM_\omega A^T + B_\omega BB^T + BB^T B_\omega^* &= 0, \\
    AQ_\omega E^T + EQ_\omega A^T + C_\omega^* C^T C + C^T CC_\omega &= 0, \\
    \dot{A}Q_\omega + \dot{Q}_\omega A^T + C_\omega^* \dot{C}^T C + \dot{C}^T \dot{C} C_\omega &= 0, \\
    AN_\omega + EN_\omega A^T + C_\omega^* \dot{C}^T C - \dot{C}^T \dot{C} C_\omega &= 0,
\end{align*} \]
Algorithm 2: Two-sided iteration algorithm (TSIA).

| Step | Description |
|------|-------------|
| 1    | Choose matrices $W_0 \in \mathbb{R}^{n \times r}$ and $V_0 \in \mathbb{R}^{n \times r}$ such that $W_0^T V_0 = I$. |
| 2    | Construct the reduced-order matrices $\hat{A} = W_0^T E^{-1} A V_0^{-1}$, $\hat{B} = W_0^T E^{-1} B$ and $\hat{C} = CV_0$. |
| 3    | while $(i \leq N - 1)$ do |
| 4    | Compute $V_i = M_\omega$ and $W_i = N_\omega$ by solving the Sylvester equations: |
|      | $A M_\omega + E M_\omega \hat{A}^T + B_\omega B \hat{B}^T + B \hat{B}^T \hat{B}^* = 0$, \hspace{1cm} (22a) |
|      | $A N_\omega + E N_\omega \hat{A}^T + C_\omega^* C^T \hat{C} - C^T \hat{C} \hat{C}_\omega = 0$, \hspace{1cm} (22b) |
|      | Compute $W_{i+1} = W_i (V_i^T W_i)^{-1}$ and $V_{i+1} = V_i$. |
| 5    | Construct the reduced-order matrices $\hat{A} = W_{i+1}^T E^{-1} A V_{i+1}$, $\hat{B} = W_{i+1}^T E^{-1} B$ and $\hat{C} = CV_{i+1}$. |
| 6    | $i = i + 1$. |
| 7    | end while |

with

$$\hat{B}_\omega = \frac{i}{2\pi} \ln(\hat{A} + i \omega_1 I)(\hat{A} + i \omega_1 I)^{-1}$$

and

$$\hat{C}_\omega = \frac{i}{2\pi} \ln(\hat{A} + i \omega_1 I)^{-1}(\hat{A} + i \omega_2 I).$$

Following the discussion in the above section here we also construct reduced order model by constructing the reduced matrices as in [12]. We solve the Sylvester equations (22a) and (22b) to construct $V = M_\omega$ and $W = N_\omega$. The constructed reduced order model is $H_2$ optimal on the limited frequency range and satisfies Wilson’s first-order optimality conditions. The whole procedure is summarized in Algorithm 2. The main computation tasks in this algorithm is to solve sparse-dense Sylvester equations (22a) and (22b). Following section will presents how to solve them efficiently.

### 4 Solution of semi-generalized Sylvester equations

Above section shows that to perform the frequency limited model reduction of system (1) we need to solve two frequency limited matrix equations namely Sylvester equations (22a) and (22b). This section discusses how to solve them efficiently. Since the Sylvester equations (22a) and (22b) are duel of each other we only interested to elaborate the solution of (22a). Another one can be solved by applying the same procedure. For our convenient we rewrite the equation (22a) as

$$AX + EX \hat{A}^T + F = 0,$$

where $F = B_\omega B B^T + B B^T B^* \omega_2$ and $X = M_\omega$. The technique that we have followed here was presented in [16] where $E = I$ is an identity matrix. In [17]
Algorithm 3: Solution of semi-generalized Sylvester equations.

Input : $E, A, \hat{A}, F$ from (23).
Output: $X \in \mathbb{R}^{n \times r}$ solution of (23).

1. Compute the Schur decomposition $\hat{A} = QSQ^*$ and Define $\tilde{F} = FQ$
2. for $i = 1, \ldots, r$
3. Compute $\hat{F} = -\tilde{F} - E \sum_{i=1}^{j-1} S_{i,j} \tilde{X}_i$
4. Solve $(A + S_{jj}E) \tilde{X}_j = \hat{F}$
5. end for
6. $X = \tilde{X}Z^*$.

authors generalized the idea of [16] for the equation like (23) where $F = B\hat{B}$.

Considering the Schur decomposition of $\hat{A}$ as $QSQ^*$ such that $QQ^* = Q^*Q = I$ and inserting this into (23) we get

$$AX + EXQS^* + F = 0. \quad (24)$$

By multiplying this from the right with $Q$ we obtain

$$AXQ + EXQ S + FQ = 0, \quad (25)$$

Observing that $S$ is a upper triangular matrices leads to a formula for the first column of $\tilde{X}$:

$$A\tilde{X}_1 + E\tilde{X}_1 S_{1,1} + \tilde{F}_1 = 0, \quad (26)$$

$$\Leftrightarrow (A + S_{11}E)\tilde{X}_1 = -\tilde{F}_1. \quad (27)$$

For all other columns we have to take care of the linear combination of $E$ matrix. If we consider the second column of the solution

$$(A + S_{22}E)\tilde{X}_2 = -\tilde{F}_2 - S_{12}E\tilde{X}_1. \quad (28)$$

In this way the arbitrary column $j$ of $\tilde{X}$ we find

$$(A + S_{jj}E)\tilde{X}_j = -\tilde{F}_j - E \sum_{i=1}^{j-1} S_{ij} \tilde{X}_i. \quad (29)$$

To obtain the solution of the original system we multiply $\tilde{X}$ by $U^*$ from the right.

5 FLMOR of structured index-1 systems

The index 1 descriptor system that we consider in the section has the following form.

$$E_1 \dot{x}(t) = J_1 x(t) + J_2 z(t) + B_1 u(t)$$
$$0 = J_3 x(t) + J_4 z(t) + B_2 u(t)$$
$$y(t) = C_1 x(t) + C_2 z(t) + D_3 u(t), \quad (30)$$
Where \( x(t) \in \mathbb{R}^{n_1} \) is the vector with differential variables and \( z(t) \in \mathbb{R}^{n_2} \) is the vector with algebraic variables. Model reduction of such descriptor system has been discussed in a couple of previous research articles, e.g., [18, 19, 20, 21, 22, 20, 23, 24] on an unrestricted frequency limit. In [18] author uses spectral projector to split the system into finite and infinite parts and balancing based method was applied to the finite part. Other papers implemented MOR without computing the spectral projector, rather eliminating the algebraic part the system was converted into an ODE system. However, practical implementation was carried out without computing the ODE system explicitly. This paper generalizes this idea for of the FLMOR discussed in the above section.

By eliminating the algebraic variables i.e., \( z(t) \in \mathbb{R}^{n_2} \) of the system we obtain a generalized system (1) where the coefficients matrices are defined as

\[
E := E_1, \quad A := J_1 - J_2 J_4^{-1} J_3, \quad B := B_1 - J_2 J_4^{-1} B_2, \\
C := C_1 - C_2 J_4^{-1} J_3, \quad D := D_a - C_2 J_4^{-1} B_2.
\]  

(31)

The index-1 and generalized systems are equivalent since the responses of the systems are same and their finite eigenvalues are coincided. Such structured system are arising in power system model [25]. For the FLMOR of index-1 system we define \( V \) and \( W \) by solving the corresponding Sylvester’s equations of the generalized system as discussed in Section 3. Now, applying these transformations the reduced system matrices can be constructed as:

\[
\tilde{A} := \hat{J}_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3, \quad \tilde{B} := \hat{B}_1 - \hat{J}_2 \hat{J}_4^{-1} B_2, \\
\tilde{C} := \hat{C}_1 - C_2 \hat{J}_4^{-1} \hat{J}_3, \quad \tilde{D} := D_a - C_2 \hat{J}_4^{-1} B_2,
\]  

(32)

where \( \hat{J}_1 = W^{T} E_1^{-1} J_1 V, \hat{J}_2 = W^{T} J_2, \hat{J}_3 = J_3 V, \hat{B}_1 = W^{T} E_1^{-1} B_1 \) and \( \hat{C}_1 = C_1 V \). To compute the the projectors \( V \) and \( W \) by solving the corresponding Sylvester’s equations is a challenging task since the input matrices in (32) are highly dense. In the following text we discuss how to solve the Sylvester’s equations related to the index-1 system (30) efficiently.

To solve the Sylvester’s equations we can use Algorithm 3. In this algorithm the main expensive task is to solve a linear system at each iteration step. At Step 4 of the algorithm we need to solve the linear system

\[
(A + S_t E) \tilde{X}_j = \hat{F}.
\]

Plugging \( A \) and \( E \) from (31) we obtain

\[
(J_1 - J_2 J_4^{-1} J_3 + S_t E_1) \tilde{X}_j = \hat{F},
\]

which can be rewritten as

\[
(J_1 + S_t E_1 - J_2 J_4^{-1} J_3) \tilde{X}_j = \hat{F}.
\]  

(33)

A close observation reveal that instead of this we can solve the following linear system

\[
\begin{bmatrix}
J_1 + S_t E_1 & J_2 \\
J_3 & J_4
\end{bmatrix}
\begin{bmatrix}
\tilde{X}_j \\
1
\end{bmatrix} = \begin{bmatrix}
\hat{F} \\
0
\end{bmatrix}
\]

(34)

for \( \tilde{X}_j \). Although, linear system in (34) is larger than the system in (33) it is sparse and hence can be solved by any sparse solvers (e.g., direct [26, 27] or iterative e.g., [28, 29]) efficiently.
Algorithm 4: Sylvester equation for index-1 system.

Input : $E_1, J_1, J_2, J_3, J_4, B_1, B_2, \hat{A}$.
Output: $X \in \mathbb{R}^{n \times r}$.

1 Form $F = B_1B\hat{B}^T + B\hat{B}^T \hat{B}_2^*$, where

\[
B_\omega = \frac{1}{2\pi} \ln(J_1 + \omega_2 E_1 - J_2 J_3^{-1} J_3)(J_1 + \omega_1 E_1 - J_2 J_4^{-1} J_4)^{-1},
\]

\[
\hat{B}_\omega = \frac{1}{2\pi} \ln(\hat{A} + \omega_1 I)^{-1}(\hat{A} + \omega_2 I), \quad \text{and} \quad B = B_1 - J_2 J_4^{-1} B_2
\]

Compute the Schur decomposition $\hat{A} = QSQ^*$ and Define $\bar{F} = FQ$

2 for $i = 1, \cdots, r$ do

3 Compute $\hat{F} = -\bar{F}_j - E_1 \sum_{i=1}^{i-1} S_{i,j} \bar{X}_i$

4 Solve

\[
\begin{bmatrix}
J_1 + S_{ii} E_1 & J_2 \\
J_3 & J_4
\end{bmatrix}
\begin{bmatrix}
\hat{X}_j \\
\Gamma
\end{bmatrix} = \left[\hat{F}\right]
\]

for $\hat{X}_j$

5 end for

6 $X = \bar{X} Z^*$.

6 Numerical results

To assess the efficiency of our proposed techniques in this section we discuss the numerical results. For our convenience we have split the section into several subsections.

6.1 Model examples

We consider the following model examples for the numerical experiments.

Example 6.1 (International Space Station (ISS)). This is a model of stage 1R (Russian Service Module) of the ISS. It has $n=270$ states, $p=3$ inputs and $m=3$ outputs. The details of the model can be obtained in [30].

Example 6.2 (Clamped beam model (CBM)). This structural model was obtained by spatial discretization of an appropriate partial differential equation (see [31]). The dimension of the model is $n=348$ and it is a single input single output i.e., SISO system. The input represents the force applied to the structure at the free end, and the output is the resulting displacement.

Example 6.3 (Triple chain oscillator (TCO) model). Although this example was originated in [32], the setup was described in [33] which resulted in second-order system. We convert into first-order form in which the dimension of the system is $n=10000$. It is also an SISO system and input, output matrices are transpose of each other.

Example 6.4 (Power system model). We consider several Brazilian Power System (BPS) models from [21] which are in index-1 descriptor form. Table I shows
### Table 1: Dimension of differential and algebraic variables of different Brazilian power system models.

| Model     | \(n_1\) | \(n_2\) | \(m/p\) |
|-----------|---------|---------|---------|
| BPS-606   | 606     | 1142    |         |
| BPS-1142  | 1142    | 8593    | 4/4     |
| BPS-1450  | 1450    | 9855    |         |
| BPS-1693  | 1693    | b11582  |         |

### Table 2: Dimension of reduced order models and \(H_2\) norms of the error systems with and without limited frequency intervals.

| Model     | \(r\) | \(\Xi_{\omega}\) | \(\Xi\) |
|-----------|-------|------------------|--------|
| ISS       | 30    | 7.3244 \times 10^{-8} | 2.8000 \times 10^{-3} |
| TCO       | 30    | 3.4850 \times 10^{-3}  | 9.6000 \times 10^{-3} |
| BPS-606   | 30    | 7.8992 \times 10^{-8} | 1.0000 \times 10^{-2} |
| BPS-1142  | 35    | 3.4421 \times 10^{-5} | 5.3000 \times 10^{-3} |
| BPS-1450  | 35    | 9.3818 \times 10^{-8} | 1.5063 \times 10^{-5} |
| BPS-1693  | 45    | 4.2995 \times 10^{-6} | 1.3000 \times 10^{-3} |

6.2 Setup Hardware and Software

The experiments are carried out with Python 3.7.9 on a board with AMD Ryzen™ Threadripper™ 1920X 12-Core Processor with a 3.5 GHz clock speed and 128 GB RAM.

6.3 Error analysis of reduce-order model

The proposed techniques was applied to the all model examples mentioned above. For the ISS, CBM, TCO we apply Algorithm 2 to obtain frequency limited reduced order model. On the other for the BPS models we have applied the techniques discussed in Section 5. We have computed different dimensional reduced order models for different model examples which are mentioned in Table 2. The table also shows \(H_2\) norm of the error systems in both frequency restricted and unrestricted cases. For all the models frequency restricted reduced order model show much more better accuracy than the frequency frequency underused ones within the assigned frequency intervals. We also have investigated the frequency domain analysis including the errors of the original and reduced order models by using sigma plots. Exemplary, we have depicted the frequency responses of TCO and BPS models only. Figures 1 and 2 show comparisons of the frequency responses of the of the original and the reduced order models of TCO and BPS models, respectively. In both the figures from absolute and the relative errors we observe that frequency restricted reduced order models approximate with the original models with higher accuracy within on the prescribed frequency ranges.
Figure 1: Comparison of original and 30 dimensional reduced systems on the frequency range [1,2] for TCO.
Figure 2: Comparison of original and 45 dimensional reduced systems on the frequency range $[6,10]$ for BPS-1693.
6.4 Comparison of sparse and dense system

We already mentioned that in the TSIA the main computational task is solving two Sylvester equations. This paper presents Algorithm 4 to solve the Sylvester equations efficiently for index-1 system. We know that by converting index-1 system into a generalized system we can solve the Sylvester equation using Algorithm 3. Figure 3 shows the time comparisons of Algorithms 3 and 4 for solving Sylvester equations of index-1 system. We see that if the dimension of the system is increased then the computational times for Algorithms 3 is increased rapidly. On the other hand the computational time with Algorithm 4 is nominal in compare to Algorithms 3.

7 Conclusions

In this paper we have discussed the frequency limited $\mathcal{H}_2$ optimal model order reduction of large-scale sparse dynamical systems. For this purpose we have solved two mixed (Sparse and dense) Sylvester equations which are dual of each other. We have shown that how to solve the Sylvester equations efficiently without losing the sparsity of large sparse system matrices. The ideas are also generalized to index-1 descriptor systems. Index-1 system can be converted into a generalized system by eliminating the algebraic equations, which however convert the system from sparse to dense. We have discussed how to perform the model reduction without converting the dense system explicitly. Numerical experiments has been carried out to demonstrate the approximation accuracy and computational efficiency of the proposed algorithm using Python Programming Language.
Acknowledgments. This research work was funded by NSU-CTRG research grant under the project No.: CTRG-19/SEPS/05. It was also supported by National Natural Science Foundation of China under Grant No. (61873336, 61873335), the Fundamental Research Funds for the Central Universities under Grant (FRF-BD-19-002A), and the High-end foreign expert program of Shanghai University.

References

[1] A. Antoulas, *Approximation of Large-Scale Dynamical Systems*, ser. Advances in Design and Control. Philadelphia, PA: SIAM Publications, 2005, vol. 6.

[2] M. M. Uddin, *Computational Methods for Approximation of Large-Scale Dynamical Systems*. New York, USA: Chapman and Hall/CRC, 2019.

[3] W. Gawronski and J. Juang, “Model reduction in limited time and frequency intervals,” *Int. J. Syst. Sci.*, vol. 21, no. 2, pp. 349–376, 1990.

[4] P. Benner, P. Kürschner, and J. Saak, “Frequency-limited balanced truncation with low-rank approximations,” *SIAM J. Sci. Comput.*, vol. 38, no. 1, pp. A471–A499, Feb. 2016.

[5] P. Van Dooren, K. Gallivan, and P.-A. Absil, “$\mathcal{H}_2$-optimal model reduction of MIMO systems,” *Appl. Math. Lett.*, vol. 21, pp. 1267–1273, 2008.

[6] S. Gugercin, A. C. Antoulas, and C. A. Beattie, “$\mathcal{H}_2$ model reduction for large-scale dynamical systems,” *SIAM J. Matrix Anal. Appl.*, vol. 30, no. 2, pp. 609–638, 2008.

[7] D. Hyland and D. Bernstein, “The optimal projection equations for model reduction and the relationships among the methods of wilson, skelton, and moore.”

[8] W. Y. Yan and J. Lam, “An approximate approach to $\mathcal{H}_2$ optimal model reduction,” *IEEE Trans. Autom. Control*, vol. 44, no. 7, pp. 1341–1358, 1999.

[9] Y. Xu and T. Zeng, “Optimal $\mathcal{H}_2$ model reduction for large scale MIMO systems via tangential interpolation,” *International Journal of Numerical Analysis and Modeling*, vol. 8, no. 1, pp. 174–188, 2011.

[10] P. Goyal and M. Redmann, “Time-limited $\mathcal{H}_2$-optimal model order reduction,” *Applied Mathematics and Computation*, vol. 355, pp. 184–197, 2019.

[11] D. Petersson and J. Löfberg, “Model reduction using a frequency-limited $\mathcal{H}_2$-cost,” *Systems & Control Letters*, vol. 67, pp. 32–39, 2014.

[12] P. Van Dooren, K. A. Gallivan, and P. A. Absil, “$\mathcal{H}_2$-optimal model reduction of mimo systems,” *Appl. Math. Lett.*, vol. 21, no. 12, pp. 1267–1273, 2008.
[13] K. Zhou, G. Salomon, and E. Wu, “Balanced realization and model reduction for unstable systems,” *Internat. J. Robust and Nonlinear Cont.*, vol. 9, no. 3, pp. 183–198, 1999.

[14] D. A. Wilson, “Optimum solution of model-reduction problem,” *Proceedings of the Institution of Electrical Engineers*, vol. 117, no. 6, pp. 1161–1165, 1970.

[15] S. Gugercin, A. C. Antoulas, and C. Beattie, “H2 model reduction for large-scale linear dynamical systems,” *SIAM journal on matrix analysis and applications*, vol. 30, no. 2, pp. 609–638, 2008.

[16] D. C. Sorensen and A. C. Antoulas, “The Sylvester equation and approximate balanced reduction,” *Numer. Lin. Alg. Appl.*, vol. 351–352, pp. 671–700, 2002.

[17] P. Benner, M. Köhler, and J. Saak, “Sparse-dense Sylvester equations in H2-model order reduction,” 2011.

[18] T. Stykel, “Analysis and numerical solution of generalized Lyapunov equations,” Ph.D. Thesis, Technische Universität Berlin, Berlin, 2002.

[19] S. Gugercin, T. Stykel, and S. Wyatt, “Model reduction of descriptor systems by interpolatory projection methods,” *SIAM J. Sci. Comput.*, vol. 35, no. 5, pp. B1010–B1033, 2013.

[20] M. M. Uddin, J. Saak, B. Kranz, and P. Benner, “Computation of a compact state space model for an adaptive spindle head configuration with piezo actuators using balanced truncation,” *Production Engineering*, vol. 6, pp. 577–586, 2012.

[21] F. Freitas, J. Rommes, and N. Martins, “Gramian-based reduction method applied to large sparse power system descriptor models,” *IEEE Trans. Power Syst.*, vol. 23, no. 3, pp. 1258–1270, Aug. 2008.

[22] M. M. Uddin, “Model reduction for piezo-mechanical systems using Balanced Truncation,” Master’s thesis, Stockholm University, Stockholm, Sweden, 2011. [Online]. Available: http://www.qucosa.de/fileadmin/data/qucosa/documents/7822/MasterThesis_Uddin.pdf

[23] M. S. Hossain and M. M. Uddin, “Iterative methods for solving large sparse Lyapunov equations and application to model reduction of index 1 differential-algebraic-equations,” *Numerical Algebra, Control & Optimization*, vol. 9, no. 2, p. 173, 2019.

[24] M. M. Uddin, M. S. Hossain, and M. F. Uddin, “Rational krylov subspace method (rksm) for solving the lyapunov equations of index-1 descriptor systems and application to balancing based model reduction,” in *9th International Conference on Electrical & Computer Engineering (ICECE) 2016*. IEEE, 2016, pp. 451–454.

[25] R. W. Freund, “Structure-preserving model order reduction of RCL circuit equations,” in *Model Order Reduction: Theory, Research Aspects and Applications*, W. H. A. Schilders, H. A. van der Vorst, and J. Rommes, Eds. Springer-Verlag, 2008, pp. 49–73.

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[26] T. A. Davis, *Direct Methods for Sparse Linear Systems*, ser. Fundamentals of Algorithms. Philadelphia, PA, USA: SIAM, 2006, no. 2.

[27] I. S. Duff, A. M. Erisman, and J. K. Reid, *Direct methods for sparse matrices*. Oxford, UK: Clarendon Press, 1989.

[28] H. A. Van der Vorst, *Iterative Krylov Methods for Large Linear Systems*. Cambridge: Cambridge University Press, 2003.

[29] Y. Saad, *Iterative Methods for Sparse Linear Systems*. Philadelphia, PA, USA: SIAM, 2003.

[30] S. Gugercin, A. C. Antoulas, and N. Bedrossian, “Approximation of the international space station 1r and 12a models,” in *Proc. of the 40th IEEE Conferences on Decision and Control*, 2001, pp. 1515–1516.

[31] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, “A survey of model reduction methods for large-scale systems,” *Contemp. Math.*, vol. 280, pp. 193–219, 2001.

[32] N. Truhar and K. Veselić, “An efficient method for estimating the optimal dampers’ viscosity for linear vibrating systems using Lyapunov equation,” *SIAM J. Matrix Anal. Appl.*, vol. 31, no. 1, pp. 18–39, 2009.

[33] J. Saak, “Efficient numerical solution of large scale algebraic matrix equations in pde control and model order reduction,” Dissertation, TU Chemnitz, July 2009, available from [http://nbn-resolving.de/urn:nbn:de:bsz:ch1-200901642](http://nbn-resolving.de/urn:nbn:de:bsz:ch1-200901642)