Abstract. We estimate the density of tubes around the algebraic variety of decomposable univariate polynomials over the real and the complex numbers.

Keywords. Polynomial composition, real polynomials, complex polynomials, volume, tubes.

Subject classification. 12D05, 26B15, 51M25, 68W30

1. Introduction

For two univariate polynomials $g, h \in F[x]$ of degrees $d, e$, respectively, over a field $F$, their composition

$$f = g(h) = g \circ h \in F[x]$$

(1.1)

is a polynomial of degree $n = de$. If such $g$ and $h$ exist with degree at least 2, then $f$ is called decomposable (or composed, composite, a composition).

Since the foundational work of Ritt, Fatou, and Julia in the 1920s on compositions over $\mathbb{C}$, a substantial body of work has been concerned with structural properties (e.g., Friedman & MacRae 1969, Dorey & Whaples 1974, Schinzel 1982, 2000, Zannier 1993), with algorithmic questions (e.g., Barton & Zippel 1985, Kozen & Landau 1989, Blankertz 2014), and with enumeration over finite fields, exact and approximate (e.g., Giesbrecht 1988, von zur Gathen et al. 2010, Blankertz et al. 2013, von zur Gathen 2014a, Ziegler 2014).
This paper presents analogs for the case of the real or complex numbers of the latter counting results. What does counting mean here? The dimensions and degrees of its irreducible components as algebraic varieties? These quantities turn up in our argument, but we bound here the density of these components. Any proper algebraic subvariety $X$ of $\mathbb{R}^n$ has volume and density 0. However, we can bump up the dimension of $X$ to $n$ by taking an $\epsilon$-tube $U_\epsilon$ around $X$, replacing each point in $X$ by a hypercube $(-\epsilon, \epsilon)^{\text{codim} X}$ for “small” positive $\epsilon$. If this is done properly, $U_\epsilon$ has dimension $n$. Its volume may be infinite, and we make it finite by intersecting with a hypercube $(-B, B)^n$ for some “large” positive $B$. Then the density of $X$ in $(-B, B)^n$ is this finite volume divided by the volume $(2B)^n$ of the large hypercube. A similar approach works in the complex case.

Let $X$ be an equidimensional real or complex algebraic variety embedded in a $k$-dimensional affine space with codimension $m$, and consider the $\epsilon$-neighborhood $\{y: |x - y| < \epsilon\}$ of some point $x$ in the space. This is a real hypercube or a complex polycylinder, respectively. We also use the corresponding notion for a projective space. There are several notions for forming an $\epsilon$-tube around $X$, namely, as the union of all

- $k$-dimensional $\epsilon$-neighborhoods of $x \in X$,
- $m$-dimensional $\epsilon$-neighborhoods in the direction normal to $x \in X$,
- $m$-dimensional $\epsilon$-neighborhoods in a fixed direction around $x \in X$.

Singular points may be disregarded. The first two tubes comprise the points in the affine space whose distance to $X$ in a normal direction is less than $\epsilon$. Thus the two notions coincide, at least for smooth varieties. The ratio to the volume in the third notion is locally $\cos \alpha$, where $\alpha$ is the angle between the two $m$-dimensional linear spaces, namely the normal space (at a nonsingular point) and the space in the chosen fixed direction.

The present paper uses exclusively the third notion, the others are included only for perspective. Almost tight bounds such as our results look unattainable in general settings.
Weyl (1939), answering a question posed by Hotelling (1939), proved fundamental results on tubes around manifolds. Since then, the topic has been studied in topology and differential geometry and is the subject of the textbook of Gray (1990), which includes many further references. The first notion and generalizations of it are commonly used.

For algebraic varieties and the first notion, Demmel (1988) and Beltrán & Pardo (2007) show upper bounds of the form $c \cdot \deg X \cdot (\epsilon/B)^{2m}$ on the density of complex $\epsilon$-tubes inside the $B$-neighborhood of 0, where $c$ does not depend on $\epsilon$ or $B > \epsilon$. In the real case, $2m$ is replaced by $m$. Often it is sufficient to consider $B = 1$. Lower bounds, with various values for $c$, are also available. Smale (1981) uses the third notion and shows in his Theorem 4A an upper bound of $k(\epsilon/B)^{2}$ for the hypersurface of monic squareful univariate polynomials of degree $k$ in $\mathbb{C}^k$. These papers investigate the condition number which is large for inputs at which (iterative) numerical methods behave badly, such as the matrices close to singular ones for matrix inversion or the (univariate) polynomials close to squareful ones for Newton’s root finding method. These hypersurfaces are also the topic of Shub & Smale (1993).

Yet another notion is the $2d$-dimensional volume of a $d$-dimensional variety in a complex affine space. Demmel (1988), Section 7, provides bounds on this volume.

We study the density of tubes around the (affine closed, usually reducible) variety of decomposable univariate polynomials. An isomorphism with an affine space (Theorem 2.4) suggests a preferred (constant) direction in which to attach $\epsilon$-neighborhoods, thus following the third one of the recipes sketched above.

Cheung et al. (2013) also consider decomposable polynomials. They bound the density of a hypersurface containing them, using the third notion. This provides, by necessity, a weaker bound than ours.

This paper is organized as follows. Section 2 presents a decomposition algorithm which is central for our approach, and results on the dimensions and degrees of various varieties of decomposable polynomials. Section 3 defines our tubes over $\mathbb{R}$ and presents asymptotically matching upper and lower bounds for the resulting
density (Theorem 3.11). Section 4 considers the analogous problem over \( \mathbb{C} \). Section 5 takes up the above discussion of other notions of tubes and of related work, with some more detail.

2. The Newton-Taylor decomposition algorithm

It is well known that we may assume all three polynomials in (1.1) to be monic (leading coefficient 1) and original (constant coefficient 0, so that the graph contains the origin). All other compositions can be obtained from this special case by composing (on the left and on the right) with linear polynomials (polynomials of degree 1); see, e.g., von zur Gathen (2013).

Thus we consider for a proper divisor \( d \) of \( n \) and \( e = n/d \)

\[
P_n(F) = \{ f \in F[x] : \deg f = n, f \text{ monic original} \},
\]

\[
\gamma_{n,d} : P_d(F) \times P_e(F) \to P_n(F) \text{ with } \gamma_{n,d}(g, h) = g \circ h,
\]

\[
C_{n,d}(F) = \{ f \in P_n(F) : \exists g, h \in P_d(F) \times P_e(F) \quad f = g \circ h \}
\]

\[
= \text{im } \gamma_{n,d},
\]

\[
(2.1) \quad C_n(F) = \bigcup_{d \mid n} C_{n,d}(F).
\]

\( P_n(F) \) is an \((n - 1)\)-dimensional vector space over \( F \) and \( C_n(F) \) is the algebraic variety of decomposable polynomials. We drop the argument \( F \) when it is clear from the context. When \( n \) is prime, then \( C_n(F) \) is empty, and in the following we always assume \( n \) to be composite.

We recall the decomposition algorithm from von zur Gathen (1990). It computes \( \gamma_{n,d}^{-1}(f) \) for \( f \in C_{n,d} \) by taking the reverse \( \tilde{f} = x^n \cdot f(x^{-1}) \) of \( f \) and computing its \( d \)th root \( \tilde{h} \) modulo \( x^e \), via Newton iteration with initial value 1. Thus \( \tilde{h}^d \equiv \tilde{f} \mod x^e \), \( \deg \tilde{h} < e \), and \( \tilde{h}(0) = 1 \). Then the reverse \( h = x^e \cdot \tilde{h}(x^{-1}) \) of \( \tilde{h} \) has degree \( e \) and is the unique candidate for the right component. The Newton iteration is well-defined unless \( \text{char}(F) \) divides \( d \). Like any polynomial of degree at most \( n = de \), \( f \) has a (unique) generalized Taylor expansion \( f = \sum_{0 \leq i \leq d} G_i h^i \) around \( h \), with all \( G_i \in F[x] \) of degree less than \( e \). Then \( \tilde{f} \in C_{n,d} \) if and only if all \( G_i \) are constants, and if so, indeed \( f = g \circ h \) with \( g = \sum_{0 \leq i \leq d} G_i x^i \). We
call this the Newton-Taylor (NT) method for decomposing. This computation expresses each coefficient of $g$ and $h$ as a polynomial in the coefficients of $f$, as illustrated in Example 2.3. It can be executed with $O(n \log^2 n \log \log n)$ operations in $F$. For more details on the computer algebra machinery, see the cited article and von zur Gathen & Gerhard (2013), Sections 9.2 and 9.4.

When $f$ is known to be in $C_{n,d}$ and $h$ has been calculated from its $e$ highest coefficients, then only the coefficients of $f$ at powers $x^i$ with $d$ dividing $i$ are needed to compute $g$. We let $N = \{1, 2, \ldots, n - 1\}$ be the support of a general $f \in P_n$. The NT method only uses the coefficients $f_i$ of $f$ at $x^i$ for those $i \in N$ which are in the Newton-Taylor set

$$NT_d = \{n - 1, \ldots, n - e + 1\} \cup \{i \in N : e \mid i\}.$$ 

We also take the complement $cNT_d = N \setminus NT_d$. Then

$$#NT_d = n/d - 1 + d - 1 = d + n/d - 2,$$

$$m_d = #cNT_d = n - d - n/d + 1.$$ (2.2)

![Figure 2.1: The Newton-Taylor set for $n = 20$ and $d = 5$.](image)

**Example 2.3.** For $n = 20$, $d = 5$, and $e = 4$, we have $NT_5 = \{19, 18, 17, 16, 12, 8, 4\}$ and $#NT_5 = d + n/d - 2 = 7$. The leading and trailing coefficients of any $f \in P_{20}$ are fixed as 1 and 0, respectively. The bullets and open circles in Figure 2.1 are positions of coefficients used in the Newton and Taylor algorithms, respectively. Using the binomial expansion (also known to Newton) instead of
the Newton iteration and \( u = f_{19}x + f_{18}x^2 + f_{17}x^3 \), we find
\[
\tilde{h} = 1 + h_3x + h_2x^2 + h_1x^3 \equiv (1 + u)^{1/5}
\]
\[
= \sum_{i \geq 0} \left( \frac{1}{5} \right) u^i \equiv 1 + \frac{1}{5} u - \frac{2}{25} u^2 + \frac{6}{125} u^3 \mod x^4,
\]
\[h = x^4 + h_3x^3 + h_2x^2 + h_1x\]
\[= x^4 + \frac{f_{19}}{5} \cdot x^3 + \frac{-2f_{19}^2 + 5f_{18}}{25} \cdot x^2 + \frac{6f_{19}^3 - 20f_{18}f_{19} + 25f_{17}}{125} \cdot x.
\]
Now
\[g_4 = f_{16} - \frac{1}{125} (21f_{19}^4 - 90f_{18}f_{19}^2 + 50f_{18}^2 + 100f_{17}f_{19})
\]
is the coefficient of \( x^{16} \) in \( f - h^5 \). Similarly, \( g_i \) for \( i = 3, 2, 1 \) is determined as the coefficient of \( x^{4i} \) in \( f - \sum_{i<j} g_jh^j \). These four coefficients of \( g \) have degrees 4, 8, 12, and 16, respectively, in the \( \text{NT}_5 \)-coefficients of \( f \).

To complete the picture, one can express the twelve \( c\text{NT}_5 \)-coefficients of \( f \in C_{20,5} \) as polynomials in the seven \( \text{NT} \)-coefficients. The polynomials have the following degrees at \( x^{15}, x^{14}, x^{13}, x^{11}, \ldots \): 5, 5, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19. The Bézout number, that is, the product of these degrees, is much larger than the bound in Theorem 2.4 below.

Using these facts, we give a geometric description of \( C_{n,d} \).

**Theorem 2.4.** Let \( d \) be a proper divisor of \( n \) and assume that \( \text{char}(F) \) does not divide \( d \). Then \( C_{n,d}(F) = \text{im} \gamma_{n,d} \) is a closed irreducible algebraic subvariety of \( P_n(F) \) of dimension \( d + n/d - 2 \) and codimension \( m_d \). The Newton-Taylor method provides a polynomial section \( \nu_{n,d}: C_{n,d} \to P_d(F) \times P_{n/d}(F) \) of \( \gamma_{n,d} \). Furthermore, \( \gamma_{n,d} \) and \( \nu_{n,d} \) are defined over \( \mathbb{Z} \) and \( \mathbb{Z}[d^{-1}] \), respectively. For \( f = \sum_{1 \leq i \leq n} f_i x^i \in C_{n,d}, \nu_{n,d}(f) \) depends only on the coefficients \( f_i \) with \( i \in \text{NT}_d \). The degree of \( C_{n,d} \) is at most \( d^{d+n/d-2} \).

**Proof.** Let \( e = n/d \). To show that \( C_{n,d} \) is closed and irreducible, we embed \( P_n(F) = \{(f_{n-1}, \ldots, f_1) \in A^{n-1}(F)\} \) in \( P^{n-1}(F) \) via
\[
(f_{n-1}, \ldots, f_1) \mapsto \bar{f} = (1 : f_{n-1} : \ldots : f_1).
\]
The projective version $\tilde{\gamma}_{n,d}$ of $\gamma_{n,d}$ is then
\[
\tilde{\gamma}_{n,d}((g_d : g_{d-1} : \ldots : g_1), (h_e : h_{e-1} : \ldots : h_1)) = \sum_{1 \leq i \leq d} g_i h_e^{d-i} \tilde{h}^i \in \mathbb{P}^{n-1}(F),
\]
where $\tilde{h} = \sum_{1 \leq j \leq e} h_j x^i$ and, with a slight abuse of notation, $\tilde{h}^i$ stands for the (projective) vector of $n$ coefficients of the polynomial $\tilde{h}^i$. This vector is homogeneous in the variables $h_j$ of degree $i$.

The scalar extension of $\tilde{\gamma}_{n,d}$ to $\mathbb{P}^{d-1}(F) \times \mathbb{P}^{e-1}(F) \to \mathbb{P}^{n-1}(F)$ is well-defined, where $\overline{F}$ is an algebraic closure of $F$. As this extension is a closed mapping which is defined over $F$, we conclude that $\tilde{\gamma}_{n,d}$ is also a closed mapping. In particular, $\tilde{C}_{n,d} = \text{im} \tilde{\gamma}_{n,d}$ is closed in $\mathbb{P}^{n-1}(F)$.

We now show that $\tilde{C}_{n,d}(F) \cap \mathbb{A}^{n-1}(F) = C_{n,d}(F)$. The inclusion $C_{n,d}(F) \subseteq \tilde{C}_{n,d}(F) \cap \mathbb{A}^{n-1}(F)$ is clear. For the other inclusion, we take some $f \in \tilde{C}_{n,d}(F) \cap \mathbb{A}^{n-1}(F)$ and $\bar{g} \in \mathbb{P}^{d-1}(F)$ and $\bar{h} \in \mathbb{P}^{e-1}(F)$ with $f = \bar{g} \circ \bar{h}$. Since $f \in \mathbb{A}^{n-1}(F) = P_n(F)$, the leading coefficient of $\bar{f}$ (at $x^n$) is nonzero. We normalize $\bar{f}$ so that this coefficient equals 1. The coefficient of $\bar{g} \circ \bar{h} = \sum_{1 \leq i \leq d} g_i h_e^{d-i} \tilde{h}^i$ at $x^n$ equals $g_d \cdot \text{lc}(\tilde{h}^d) = g_d \cdot h_e^d$. It follows that $g_d h_e \neq 0$. After normalizing $\bar{g}$ and $\bar{h}$ by dividing by their leading coefficients, we obtain polynomials $g \in P_d(F)$ and $h \in P_e(F)$ with $f = g \circ h$. This shows the desired inclusion and the claim that $C_{n,d}$ is closed in $P_n$. Furthermore, as $P_d(F) \times P_e(F)$ is irreducible, it follows that $C_{n,d} = \text{im} \gamma_{n,d}$ is also irreducible.

We prove the degree estimate. The existence of the section $\nu_{n,d}$ implies that $\dim C_{n,d} = d + e - 2$. Let $H_1, \ldots, H_{d+e-2}$ be hyperplanes of $P_n(F)$ with $\#(C_{n,d} \cap H_1 \cap \cdots \cap H_{d+e-2}) = \deg C_{n,d}$. Let $S = C_{n,d} \cap H_1 \cap \cdots \cap H_{d+e-2}$. Then $\#S = \deg C_{n,d}$ and
\[
\gamma^{-1}_{n,d}(S) = \gamma^{-1}_{n,d}(H_1) \cap \cdots \cap \gamma^{-1}_{n,d}(H_{d+e-2}).
\]

The polynomial map $\gamma_{n,d}$ consists of $n - 1$ integer polynomials in the coefficients of $g$ and $h$, all of total degree at most $d$. Furthermore, for each $i \leq d + e - 2$ there exists a linear combination $m_i$ of the polynomials which define the coordinates of $\gamma_{n,d}$ so that $\gamma^{-1}_{n,d}(H_i) = \{m_i = 0\}$. Therefore, $\deg \gamma^{-1}_{n,d}(H_i) \leq d$.
and, by the Bézout inequality (see, e.g., [Heintz (1983), Fulton (1984), Vogel (1984)]), it follows that $\deg \gamma_{n,d}^{-1}(S) \leq d^{d+e-2}$. Let $\gamma_{n,d}^{-1}(S) = \bigcup_{1 \leq j \leq k} X_j$ be the decomposition of $\gamma_{n,d}^{-1}(S)$ into irreducible components. Since $\gamma_{n,d}(\gamma_{n,d}^{-1}(S)) = S$ and each irreducible component $X_j$ of $\gamma_{n,d}^{-1}(S)$ is mapped by $\gamma_{n,d}$ to a point of $S$, we deduce that

$$\deg C_{n,d} = \# S \leq k \leq \sum_{1 \leq j \leq k} \deg X_j = \deg \gamma_{n,d}^{-1}(S) \leq d^{d+e-2}.$$ 

The Newton and Taylor algorithms are integral algorithms, except that divisions by $d$ occur in Newton iteration. □

This precise description of $C_{n,d}$, with the section $\nu_{n,d}$, is the basis for our tight bounds on the real and complex densities that we consider.

The convex function $d + n/d$ of $d$ assumes its maximum among the proper divisors of $n$ at $d = \ell$ and $d = n/\ell$, where $\ell$ is the least prime number dividing $n$; see von zur Gathen (2013). Thus the two “large” components of $C_n$ are $C_{n,\ell}$ and $C_{n,n/\ell}$, unless $n = \ell^2$, when they coincide. We will deal with the other components of smaller dimension at the end of Section 3.

We also want to show that the sum of the two “large” densities bounds the density of $C_n$ from below. To this end, it suffices to prove that the intersection of the two components has small dimension. Since both are irreducible, it suffices to show that they are distinct. This follows easily from Ritt’s Second Theorem. In fact, the following geometric variant of the normal form for Ritt’s Theorem in von zur Gathen (2014b) provides precise bounds.

**Theorem 2.5.** Let $n$, $d$, and $e = n/d$ be as above, with $e > d \geq 2$ in addition, $i = \gcd(d,e)$, $s = \lfloor e/d \rfloor$ and $F$ a field of characteristic either 0 or coprime to $n$. Then $X = C_{n,d}(F) \cap C_{n,e}(F)$ is a closed algebraic subvariety of $P_n(F)$.

When $d \geq 3i$, then $X$ has exactly two irreducible components, one of dimension $2i + s - 1$ and another one of dimension $2i$, and the intersection of the two is irreducible of dimension $2i - 1$. When $d \leq 2i$, then $X$ is irreducible. It has dimension $d + e/d - 3/2$ if $d = 2i$, and dimension $2d + e/d - 3$ if $d = i$. 


Decomposable polynomials over $\mathbb{R}$ and $\mathbb{C}$

In all cases, $\dim X < \dim C_{n,d}(F) = \dim C_{n,e}(F)$.

**Proof.** We first assume that $d \geq 2i$. Theorem 6.3 from von zur Gathen (2014b) provides, expressed in geometric language, two polynomial functions

$$
\alpha_{\text{exp}} : P_i \times F^s \times F \times P_i \to P_n,
\alpha_{\text{trig}} : P_i \times F \times F \times P_i \to P_n,
$$

so that $X = \text{im} \alpha_{\text{exp}} \cup \text{im} \alpha_{\text{trig}}$. Here $\text{exp}$ stands for exponential and $\text{trig}$ for trigonometric collisions. More precisely,

$$
\alpha_{\text{exp}}(u, w, a, v) = u \circ (x^{d(e-sd)/i^2} w^{d/i} (x^{d/i}))^a \circ v,
\alpha_{\text{trig}}(u, z, a, v) = u \circ T_{n/i^2}(x, z)^a \circ v.
$$

Here $T_n$ is the Dickson polynomial of degree $n$, closely related to the Chebyshev polynomial and satisfying $T_n(x, 0) = x^n$. The monic (but possibly not original) polynomial $w = \sum_{0 \leq i \leq s} w_i x^i$ of degree $s$ corresponds to the vector $(w_{s-1}, \ldots, w_0) \in F^s$, with $w_s = 1$. The original shift $p[a]$ of a polynomial $p \in F[x]$ by $a \in F$ is $(x - p(a)) \circ p \circ (x + a)$. Furthermore, $\alpha_{\text{exp}}$ and $\alpha_{\text{trig}}$ are injective, $\dim \text{im} \alpha_{\text{exp}} = 2i + s - 1$, and $\dim \text{im} \alpha_{\text{trig}} = 2i$. If $d = 2i$ then $\text{im} \alpha_{\text{trig}} \subseteq \text{im} \alpha_{\text{exp}}$. Otherwise we have $d \geq 3i$ and

$$
\text{im} \alpha_{\text{exp}} \cap \text{im} \alpha_{\text{trig}} = P_i \circ (x^{n/i^2})^{[F]} \circ P_i
$$

$$
= \{u \circ ((x + a)^{n/i^2} - a^{n/i^2}) \circ v : u, v \in P_i, a \in F\}
$$

has dimension $2i - 1$. For the dimension inequality in this case, we have $2i + s - 1 \leq d + e/2 - 1 \leq d + e - 3$.

When $d = i$, the same Theorem 6.3 shows that $C_{n,d} \cap C_{n,e} = P_i \circ P_{e/d} \circ P_i$ is irreducible of dimension $2i + s - 3 = 2d + e/d - 3 < d + e - 2 = \dim C_{n,d}$. For more details, see the cited paper. \(\square\)

The main point here is that the dimension of this intersection is less than the dimension of its two arguments.

**3. Density estimates for $C_{n,d}(\mathbb{R})$ and $C_n(\mathbb{R})$**

In this section we consider the set $P_n(\mathbb{R})$ of monic original polynomials of composite degree $n$ with real coefficients and the subsets
$C_n(\mathbb{R})$ and $C_{n,d}(\mathbb{R})$ of $P_n(\mathbb{R})$ for a proper divisor $d$ of $n$. Our aim is to obtain density estimates on tubes around $C_{n,d}(\mathbb{R})$ and $C_n(\mathbb{R})$. We drop the field $F = \mathbb{R}$ from our notation in this section.

We identify $P_n$ with $\mathbb{R}^{n-1}$ by mapping $x^n + a_{n-1}x^{n-1} + \cdots + a_1x \in P_n$ to $(a_{n-1}, \ldots, a_1) \in \mathbb{R}^{n-1}$. As shown above, $C_{n,d}$ is a real variety of dimension $d + n/d - 2$. In particular, $C_{n,d}$ has codimension at least $n/2$, and thus its (standard Lebesgue) volume is 0. For a meaningful concept, we take a specific $\epsilon$-tube around $C_{n,d}$. Namely, for each $f = \sum_{1 \leq i \leq n} f_i x^i \in C_{n,d}$ and $\epsilon > 0$, we define the $\epsilon$-neighborhood of $f$ as

$$U_\epsilon(f) = \left\{ u = \sum_{1 \leq i \leq n} u_i x^i \in P_n(\mathbb{R}) : u_i = f_i \text{ for } i \in \text{NT}_d, \quad |u_i - f_i| < \epsilon \text{ for } i \in \text{cNT}_d \right\}.$$ 

Thus $U_\epsilon(f)$ is an open $m_d$-dimensional hypercube $(-\epsilon, \epsilon)^{m_d}$ in $P_n$. Around each coefficient $f_i$ with $i \in \text{cNT}_d$ we have a real interval of length $2\epsilon$. We also set

$$U_\epsilon(C_{n,d}) = \bigcup_{f \in C_{n,d}} U_\epsilon(f).$$

In order to have finite volumes, we take a bound $B > 0$ on the coefficients and consider the $(n-1)$-dimensional hypercube

$$P_{n,B} = \left\{ f = \sum_{1 \leq i \leq n} f_i x^i \in P_n : |f_i| < B \text{ for } 1 \leq i < n \right\}$$

around $P_n$ and its intersection with the $\epsilon$-tube

$$U_{\epsilon,B}(C_{n,d}) = U_\epsilon(C_{n,d}) \cap P_{n,B}.$$ 

Our main purpose is to obtain estimates on the density $\text{den}_{\epsilon,B}(C_{n,d})$ of the $\epsilon$-tube in $P_{n,B}$, namely

$$\text{den}_{\epsilon,B}(C_{n,d}) = \frac{\text{vol} \left( U_{\epsilon,B}(C_{n,d}) \right)}{\text{vol}(P_{n,B})} = \frac{\text{vol} \left( U_{\epsilon,B}(C_{n,d}) \right)}{(2B)^{n-1}}.$$ 

In a slightly different model of our situation, we might allow arbitrary leading coefficients in our polynomials, rather than just 1. It would then be sufficient to just consider the unit hypercube with $B = 1$ and scale the resulting density. However, our approach is overall more convenient and allows an easier comparison with previous work; see Section 5.
Example 3.3. For perspective, we calculate the density of the linear subspace $L = \mathbb{R}^k \times \{0\}^{n-k} \subseteq \mathbb{R}^n$. For $x \in L$, we take $U_\epsilon(x) = \{(x_1, \ldots, x_k)\} \times (-\epsilon, \epsilon)^{n-k}$ and have, for $B > \epsilon$,

\[
U_\epsilon(L) = \mathbb{R}^k \times (-\epsilon, \epsilon)^{n-k},
\]

\[
\text{den}_{\epsilon,B}(L) = \frac{\text{vol} \left((-B, B)^k \times (-\epsilon, \epsilon)^{n-k}\right)}{(2B)^n} = \left(\frac{\epsilon}{B}\right)^{n-k}.
\]

\[\Box\]

Let $\chi: P_{n,B} \to \{0,1\}$ be the characteristic function of $U_{\epsilon,B}(C_{n,d}) \subseteq P_{n,B}$. Then the density is

\[
\text{den}_{\epsilon,B}(C_{n,d}) = \frac{1}{(2B)^{n-1}} \int_{P_{n,B}} \chi(a) \, da.
\]

For a subset $S \subseteq N$ of cardinality $s$, we consider the projection $\pi^S: \mathbb{R}^{n-1} \to \mathbb{R}^s$ onto the coordinates in $S$: $\pi^S(a_{n-1}, \ldots, a_1) = (a_i: i \in S)$. Furthermore, for a subset $C \subseteq P_n$, we write $C^S$ for $\pi^S(C)$. By reordering the coordinates, we can express the hypercube $P_{n,B}$ as the Cartesian product $P_{n,B} = P_{n,B}^{NT} \times P_{n,B}^{cNT}$. According to Fubini’s theorem we have

\[
\text{vol}(U_{\epsilon,B}(C_{n,d})) = \int_{P_{n,B}} \chi(a) \, da = \int_{P_{n,B}^{NT}} \left(\int_{P_{n,B}^{cNT}} \chi(a^{NT}, a^{cNT}) \, da^{cNT}\right) \, da^{NT}.
\]

Here $(a^{NT}, a^{cNT})$ are the coordinates of $a \in P_{n,B}$ in the product representation, and $a^{NT}$ refers to a point in $P_{n,B}^{NT}$.

Lemma 3.6. Let $0 < \epsilon < B$ and $b \in P_{n,B}^{NT}$.

(i) We have

\[
\int_{P_{n,B}^{cNT}} \chi(b, c) \, dc \leq (2\epsilon)^{m_d},
\]

where $c$ ranges over $P_{n,B}^{cNT}$. 
(ii) Let $f \in C_{n,d}$ be the unique element with $\pi^{NT_d}(f) = b$. If $U_\epsilon(f) \subseteq P_{n,B}$, then equality holds in (i).

**Proof.** The existence and uniqueness of $f$ follow from the section $\nu_{n,d}: C_{n,d} \to P_d \times P_{n/d}$ (Theorem 2.4). For any $c \in P_{n,B}^{cNT_d}$, we have

$$\chi(b, c) = 1 \iff (b, c) \in U_\epsilon, B(C_{n,d}) \iff (b, c) \in U_\epsilon(f) \cap P_{n,B}.$$ 

Therefore

$$\int_{P_{n,B}^{cNT_d}} \chi(b, c) \, dc \leq \int_{U_\epsilon(f)} dc = (2\epsilon)^{md}.$$ 

If $U_\epsilon(f) \subseteq P_{n,B}$, then equality holds. This shows both claims. \[\square\]

We derive the following upper bound on the density of $C_{n,d}$.

**Proposition 3.7.** With notation and assumptions as above, we have

$$\text{den}_{\epsilon, B}(C_{n,d}) \leq \left( \frac{\epsilon}{B} \right)^{md}.$$ 

**Proof.** Combining (3.5) and Lemma 3.6 we obtain

$$\text{vol}(U_\epsilon, B(C_{n,d})) = \int_{P_{n,B}} \chi(a) \, da$$

$$= \int_{P_{n,B}^{NT_d}} \left( \int_{P_{n,B}^{cNT_d}} \chi(a_{NT_d}, a^{cNT_d})da^{cNT_d} \right) \, da^{NT_d}$$

$$\leq \int_{P_{n,B}^{NT_d}} (2\epsilon)^{md} \, da^{NT_d} = (2\epsilon)^{md}(2B)^{n-1-md}.$$ 

Using (3.2) it follows that

$$\text{den}_{\epsilon, B}(C_{n,d}) \leq \frac{(2\epsilon)^{md}(2B)^{n-1-md}}{(2B)^{n-1}} = \left( \frac{\epsilon}{B} \right)^{md}.$$ \[\square\]

We complement this upper bound with an asymptotically matching lower bound.
Proposition 3.8. With notation and assumptions as above, we have
\[ \text{den}_{\epsilon, B}(C_{n,d}) \geq \left( \frac{\epsilon}{B} \right)^{m_d} \left( 1 - \frac{\epsilon}{B} \right)^{d+n/d-2}. \]

Proof. Let
\[ V = \bigcup_{f \in C_{n,d} \cap P_{n,B-\epsilon}} U_\epsilon(f) \subseteq U_{\epsilon,B}(C_{n,d}) \]
and \( \chi_\epsilon : P_{n,B} \to \{0, 1\} \) be its characteristic function. Since \( V \subseteq U_{\epsilon,B}(C_{n,d}) \), it follows that \( \chi_\epsilon(a) \leq \chi(a) \) for every \( a \in P_{n,B} \). Using Lemma 3.6(ii), we find
\[
\int_{P_{n,B}} \chi(a) \, da \geq \int_{P_{n,B}} \chi_\epsilon(a) \, da \\
= \int_{P_{n,NT_d}} \left( \int_{P_{n,B}} \chi_\epsilon(a^{NT_d}, a^{cNT_d}) \, da^{cNT_d} \right) da^{NT_d} \\
= \int_{P_{n,NT_d}} (2\epsilon)^{m_d} da^{NT_d} = (2\epsilon)^{m_d} (2(B - \epsilon))^{n-1-m_d}.
\]
Dividing by \( \text{vol}(P_{n,B}) = (2B)^{n-1} \) and using (2.2) yields the claimed bound. \( \square \)

We summarize the results of Propositions Proposition 3.7 and Proposition 3.8 as follows.

Theorem 3.9. Let \( 0 < \epsilon < B \) and let \( d \) be a proper divisor of \( n \). Then we have the following bounds on the density of the \( \epsilon \)-tube around \( C_{n,d}(\mathbb{R}) \):
\[
\left( \frac{\epsilon}{B} \right)^{n-d-n/d+1} \left( 1 - \frac{\epsilon}{B} \right)^{d+n/d-2} \leq \text{den}_{\epsilon,B}(C_{n,d}(\mathbb{R})) \leq \left( \frac{\epsilon}{B} \right)^{n-d-n/d+1}.
\]

We thus have good bounds on the irreducible components of \( C_n \) in (2.1). How to get such bounds for \( C_n \) itself? In Theorem 3.9 we consider \( \epsilon \)-tubes around \( C_{n,d} \) of a direction and a dimension that varies with \( d \). For \( C_n \), it seems appropriate to consider \( \epsilon \)-tubes of the same dimension for all \( d \), as follows.
We let $\ell$ be the least prime number dividing the composite integer $n$. If $n = \ell^2$, then $C_n = C_{n,\ell}$ has just one component. Otherwise, [Theorem 2.4] shows that $C_n(\mathbb{R}) \subseteq \mathbb{R}^{n-1}$ has two “large” components, namely $C_{n,\ell}$ and $C_{n,n/\ell}$ of dimension $\ell + n/\ell - 2 = \dim C_n$. As a consequence, for the density of $C_n$ we consider $\epsilon$-tubes of the same dimension $m_\ell = n - 1 - \dim C_n$ around each component of $C_n$. For a proper divisor $d \not\in \{\ell, n/\ell\}$ of $n$, we have $\dim C_{n,d} < \dim C_n$ and thus $\dim C_{n,d} + m_\ell < n - 1$. Then any $m_\ell$-dimensional $\epsilon$-tube around $C_{n,d}$ has volume and density equal to zero. Furthermore, [Theorem 2.5] implies that the sum of the two “large” densities bounds the density of $C_n$ from below. In other words, we define $\text{vol}_{\epsilon,B}(C_n) = \text{vol}(U_{\epsilon,B}(C_{n,\ell}) \cup U_{\epsilon,B}(C_{n,n/\ell}))$ and $\text{den}_{\epsilon,B}(C_n) = \text{vol}_{\epsilon,B}(C_n)/\text{vol}(P_{n,B})$. Setting

$$\delta_n = \begin{cases} 1 & \text{if } n = \ell^2, \\ 2 & \text{otherwise}, \end{cases}$$

we obtain the following result.

**Theorem 3.11.** Let $0 < \epsilon < B$ and let $\ell$ be the least prime number dividing the composite integer $n$. Then

$$\delta_n \left( \frac{\epsilon}{B} \right)^{n - \ell - n/\ell + 1} \left( 1 - \frac{\epsilon}{B} \right)^{\ell + n/\ell - 2} \leq \text{den}_{\epsilon,B}(C_n(\mathbb{R})) \leq \delta_n \left( \frac{\epsilon}{B} \right)^{n - \ell - n/\ell + 1}.$$

The approximation factor $(1 - \epsilon/B)^{\ell + n/\ell - 2}$ in the lower bound tends exponentially to 1 (from below) when $\epsilon/B$ gets small compared to $n$.

### 4. Density estimates for $C_{n,d}(\mathbb{C})$ and $C_n(\mathbb{C})$

In this section, we take $F = \mathbb{C}$ and consider the real volume on $P_n(\mathbb{C})$ as a $(2n - 2)$-dimensional real vector space. We discuss briefly the density estimates we obtain for $C_{n,d}(\mathbb{C})$. The approach is similar to that of [Section 3], therefore, we merely sketch the proofs and summarize the results we obtain. We drop the field $F = \mathbb{C}$ from our notation.

As in (3.1), we take an $\epsilon$-tube around $C_{n,d}$, namely given $f = \sum_{1 \leq i \leq n} f_i x^i \in C_{n,d}$ and $\epsilon > 0$, we define the (complex) $\epsilon$-neighborhood
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of $f$ as

$$U_\varepsilon(f) = \left\{ u = \sum_{1 \leq i \leq n} u_i x^i \in P_n : u_i = f_i \text{ for } i \in \text{NT}_d, \right. \left. |u_i - f_i| < \varepsilon \text{ for } i \in \text{cNT}_d \right\}. $$

Thus $U_\varepsilon(f)$ is an open $m_d$-dimensional complex polycylinder in $P_n$, of real dimension $2m_d$. Around each coefficient $f_i$ with $i \in \text{cNT}_d$, we have a real circle of radius $\varepsilon$ and area $\pi \varepsilon^2$. For $B > 0$, we set

$$U_\varepsilon(C_{n,d}) = \bigcup_{f \in C_{n,d}} U_\varepsilon(f), $$

$$P_{n,B} = \left\{ f = \sum_{1 \leq i \leq n} f_i x^i \in P_n : |f_i| < B \text{ for } 1 \leq i < n \right\}, $$

$$U_{\varepsilon,B}(C_{n,d}) = U_\varepsilon(C_{n,d}) \cap P_{n,B}. $$

Then $\text{vol}(P_{n,B}) = (\pi B^2)^{n-1}$. Let $\chi : P_{n,B} \to \{0, 1\}$ be the characteristic function of $U_{\varepsilon,B}(C_{n,d})$. As before, we express the polycylinder $P_{n,B}$ as the Cartesian product $P_{n,B} = P_{n_B}^{\text{NT}_d} \times P_{n_B}^{\text{cNT}_d}$ and apply Fubini’s theorem to obtain

$$\text{(4.1)}$$

$$\text{vol} \left( U_{\varepsilon,B}(C_{n,d}) \right) = \int_{P_{n_B}^{\text{NT}_d}} \left( \int_{P_{n_B}^{\text{cNT}_d}} \chi(a^{\text{NT}_d}, a^{\text{cNT}_d}) \, da^{\text{cNT}_d} \right) \, da^{\text{NT}_d}. $$

For an arbitrary element $b \in P_{n,B}^{\text{NT}_d}$, there exists a unique $f \in C_{n,d}$ with $\pi^{\text{NT}_d}(f) = b$ by Theorem 2.4. Then the function $\chi(b, f^{\text{cNT}_d})$ takes the value 1 on an $m_d$-dimensional complex polycylinder of radius $\varepsilon$ whose center is the vector of coefficients of $f$ corresponding to indices in $\text{cNT}_d$. As a consequence, we have

$$\text{vol} \left( U_{\varepsilon,B}(C_{n,d}) \right) = \int_{P_{n_B}^{\text{NT}_d}} (\pi \varepsilon^2)^{m_d} \, da^{\text{NT}_d} \leq (\pi \varepsilon^2)^{m_d} (\pi B^2)^{n-1-m_d}. $$

This yields the complex analog of the upper bound of Proposition 3.7:

$$\text{den}_{\varepsilon,B}(C_{n,d}) = \frac{\text{vol}(U_{\varepsilon,B}(C_{n,d}))}{(\pi B^2)^{n-1}} \leq \left( \frac{\varepsilon}{B} \right)^{2m_d}. $$
On the other hand, for a lower bound we consider as in Proposition 3.8 the characteristic function $\chi_\epsilon : P_{n,B} \to \{0, 1\}$ of the set

$$V = \bigcup_{f \in C_{n,d} \cap P_{n,B-\epsilon}} U_\epsilon(f) \subseteq U_{\epsilon,B}(C_{n,d})$$

for $\epsilon < B$ and argue as before to obtain

$$\text{vol}(U_{\epsilon,B}(C_{n,d})) \geq \text{vol}(V) \geq (\pi \epsilon^2)^{m_d}(\pi \cdot (B - \epsilon)^2)^{d + n/d - 2}.$$

Finally, in order to obtain a meaningful notion of density of $C_n$, we consider, as in Section 3, the $\epsilon$–tube $U_{\epsilon,B}(C_n) = U_{\epsilon,B}(C_{n,\ell}) \cup U_{\epsilon,B}(C_{n,n/\ell})$ around $C_n$, where $\ell$ is the smallest prime divisor of $n$. Summarizing, we have the following results on the density of these tubes.

**Theorem 4.2.** Let $0 < \epsilon < B$, $n$ be a composite integer, and $\delta_n$ as in (3.10).

(i) Let $d$ be a proper divisor of $n$. Then

$$\left(\frac{\epsilon}{B}\right)^{2(n-d-n/d+1)} \cdot \left(1 - \frac{\epsilon}{B}\right)^{2(d+n/d-2)} \leq \text{den}_{\epsilon,B}(C_{n,d}(\mathbb{C})) \leq \left(\frac{\epsilon}{B}\right)^{2(n-d-n/d+1)}.$$

(ii) Let $\ell$ be the smallest prime divisor of $n$. Then

$$\delta_n \left(\frac{\epsilon}{B}\right)^{2(n-\ell-n/\ell+1)} \cdot \left(1 - \frac{\epsilon}{B}\right)^{2(\ell+n/\ell-2)} \leq \text{den}_{\epsilon,B}(C_n(\mathbb{C})) \leq \delta_n \left(\frac{\epsilon}{B}\right)^{2(n-\ell-n/\ell+1)}.$$

**5. Discussion**

For an arbitrary irreducible algebraic subvariety $X$ of $\mathbb{R}^n$ with codimension $m$, we might attach a hypercube $(-\epsilon, \epsilon)^m$ to each smooth point $x$ of $X$ in the normal direction to $X$, thus following the second recipe listed in the introduction. The singular points do not contribute to the volume. Then this tube around $X$ has real dimension...
n. In an analog of Lemma 3.6, the coordinates in \( \text{NT}_d \) are replaced by local coordinates at the point and those of \( \text{cNT}_d \) by coordinate functions normal to them. Instead of having a unique \( f \) as in the proof of that lemma, we only know that the normal linear space, of complementary dimension, intersects generically in at most \( \deg X \) points. The resulting upper bound then is \( \deg X \cdot (2\epsilon)^m \).

Our construction in (3.1) of the \( \epsilon \)-tube around \( C_{n,d} \) does not follow this general recipe, since the \( \epsilon \)-hypercube in the direction of the coordinates from \( \text{cNT}_d \) is, in general, not normal to \( C_{n,d} \). It is not clear whether one can obtain tight upper and lower bounds as in Theorems Theorem 3.11 and Theorem 4.2 for other choices of the \( \epsilon \)-tubes.

Cheung et al. (2013) also provide bounds on the density of \( C_n(\mathbb{C}) \). Instead of the precise information provided by the Newton-Taylor method of Section 2, they use the fact that \( C_n(\mathbb{C}) \subseteq X \) for a certain hypersurface \( X \) and then a specific one-dimensional \( \epsilon \)-tube around \( X \) chosen to suit their argument, following the third of the options listed in the introduction. They show that \( \text{den}_{\epsilon,B}(C_n(\mathbb{C})) \leq \left( n^2 - 2n \right) \cdot (\epsilon/B)^2 \), which is to be compared with our result in Theorem 4.2.

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