ORLICZ-BESOV IMBEDDING AND GLOBALLY $n$-REGULAR DOMAINS

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Abstract Denote by $\dot{B}^{\alpha,\phi}(\Omega)$ the Orlicz-Besov space, where $\alpha \in \mathbb{R}$, $\phi$ is a Young function and $\Omega \subset \mathbb{R}^n$ is a domain. For $\alpha \in (-n, 0)$ and optimal $\phi$, in this paper we characterize domains supporting the imbedding $\dot{B}^{\alpha,\phi}(\Omega)$ into $L^{n/|\alpha|}(\Omega)$ via globally $n$-regular domains. This extends the known characterizations for domains supporting the Besov imbedding $\dot{B}^s(\Omega)$ into $L^{n/|s|}(\Omega)$ with $s \in (0,1)$ and $1 \leq p < n/s$. The proof of the imbedding $\dot{B}^{\alpha,\phi}(\Omega) \to L^{n/|\alpha|}(\Omega)$ in globally $n$-regular domains $\Omega$ relies on a geometric inequality involving $\phi$ and $\Omega$, which extends a known geometric inequality of Caffarelli et al.

1. Introduction

Suppose that $\phi$ is a Young function in $[0, \infty)$, that is,
\begin{equation}
\phi \in C([0, \infty)) \text{ is convex and satisfies } \phi(0) = 0, \phi(t) > 0 \text{ for } t > 0 \text{ and } \lim_{t \to \infty} \phi(t) = \infty.
\end{equation}

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a domain. For $\alpha \in \mathbb{R}$, the homogenous Orlicz-Besov space $\dot{B}^{\alpha,\phi}(\Omega)$ is defined as the space of all measurable functions $u$ in $\Omega$ whose (semi-)norms
\[ ||u||_{\dot{B}^{\alpha,\phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\lambda |x-y|^\alpha} \right) \frac{dxdy}{|x-y|^{2\alpha}} \leq 1 \right\} \]
are finite. Modulo constant functions, $\dot{B}^{\alpha,\phi}(\Omega)$ is a Banach space. Recall that the space $\dot{B}^0(\Omega)$ in metric spaces was introduced by Piaggio [15] in the study of geometric group theory, and the extension and imbedding properties of $\dot{B}^{0,\phi}(\Omega)$ were considered by Liang-Zhou [13].

The Orlicz-Besov spaces extend the Besov (or fractional Sobolev) spaces. For $s > 0$ and $p \geq 1$, denote by $\dot{B}_p^s(\Omega)$ the Besov spaces (also called fractional Sobolev space) equipped with the (semi-)norms
\begin{equation}
||u||_{\dot{B}_p^s(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{s+p}} \frac{dxdy}{|x-y|^{2\alpha}} \right)^{1/p}.
\end{equation}

Letting $\alpha = s - n/p$ and $\phi(t) = t^p$, we always have $||u||_{\dot{B}^{\alpha,\phi}(\Omega)} = ||u||_{\dot{B}_p^s(\Omega)}$ for all possible $u$, and hence $\dot{B}^{\alpha,\phi}(\Omega) = \dot{B}_p^s(\Omega)$. Thus, Orlicz-Besov spaces include the Besov spaces as special examples.

Note that when $s \in (0,1)$ and $p \geq 1$, $\dot{B}_p^s(\Omega)$ is non-trivial, indeed, $C^1_c(\Omega) \subset \dot{B}_p^s(\Omega)$. But when $s \geq 1$ and $p \geq 1$, thanks to the Poincaré inequality, $\dot{B}_p^s(\Omega)$ is trivial, that is, contain at most constant functions; see for example [5, Theorems 4.1 & 4.2]. In general, to guarantee $C^1_c(\Omega) \subset \dot{B}^{\alpha,\phi}(\Omega)$ and hence the non-triviality of $\dot{B}^{\alpha,\phi}(\Omega)$, we assume that $\phi$ be a Young function satisfying
\begin{equation}
\Delta_\phi(\alpha) := \sup_{\alpha > 0} \int_0^1 \frac{\phi(t^{-\alpha} x)}{\phi(x)} \frac{dt}{t^{n+1}} < \infty
\end{equation}
and
\begin{equation}
\overline{\Delta}_\phi(\alpha) := \sup_{\alpha > 0} \int_0^\infty \frac{\phi(t^{-\alpha} x)}{\phi(x)} \frac{dt}{t^{n+1}} < \infty;
\end{equation}
see Lemma 2.3. For the optimality of (1.3) and (1.4) we refer to Remark 1.4 and Remark 2.4. Under (1.3) and (1.4), the interesting range of $\alpha$ is $(-n,1)$. Indeed, by the convexity of $\phi$, when $\alpha \geq 1$, we always have
\[ \Delta_\phi(\alpha) \geq \int_0^1 t^{1-\alpha} \frac{dt}{t^{n+1}} = \infty, \]
that is, (1.3) fails. Moreover, (1.4) always holds when $\alpha \geq 0$, but when $\alpha \leq -n$, by the convexity of $\phi$ and letting $0 < c \in \partial \phi(1)$, we always have
\[
\overline{\mathcal{A}}_\phi(\alpha) \geq \sup_{x > 0} \int_1^{\infty} \frac{\phi(1) + c(r^{-\alpha} x - 1)}{\phi(x)} \frac{dt}{t^{\alpha+1}} = \sup_{x > 0} \left[ \frac{\phi(1) - c}{n \phi(x)} + \frac{c x}{\phi(x)} \int_1^{\infty} \frac{dt}{t^{\alpha+\alpha+1}} \right] = \infty,
\]
that is, (1.4) fails. Young functions satisfies (1.3) and (1.4) include $t^{p/(1-\alpha)}[\ln(1 + t)]^\gamma$ with $\gamma > 1$ and $\alpha \in (-n + 1, 0)$, $t^p[\ln(1 + t)]^\gamma$ with $p \in (n/(1-\alpha), n/|\alpha|) \cap [1, n/|\alpha|)$ and $\gamma \geq 1$ or $\gamma = 0$, and also their convex combinations.

In this paper, we are interesting in $\alpha \in (-n, 0)$. In this case, the homogeneity of $\dot{B}^{\alpha,\phi}(\mathbb{R}^n)$ is the same as that of the Lebesgue space $L^{n/|\alpha|}(\mathbb{R}^n)$, that is,
\[
\|u(r)\|_{\dot{B}^{\alpha,\phi}(\mathbb{R}^n)} = r^{-\alpha}\|u\|_{\dot{B}^{\alpha,\phi}(\mathbb{R}^n)} \quad \text{and}\quad \|u(r)\|_{L^{n/|\alpha|}(\mathbb{R}^n)} = r^{-\alpha}\|u\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \quad \text{for any } r > 0 \text{ and function } u.
\]
Viewing the theory for Sobolev spaces and Besov spaces, this enable us to establish the following imbeddings of $\dot{B}^{\alpha,\phi}(\mathbb{R}^n)$ into $L^{n/|\alpha|}(\mathbb{R}^n)$, and $\dot{B}^{\alpha,\phi}(B)$ into $L^{n/|\alpha|}(B)$ for all balls $B \subset \mathbb{R}^n$.

**Theorem 1.1.** Let $\alpha \in (-n, 0)$ and $\phi$ be a Young function satisfying (1.3) and (1.4). Then there exists a constant $C \geq 1$ depending only on $n$, $\alpha$ and $\phi$ such that
\[
\|u - u_B\|_{L^{n/|\alpha|}(B)} \leq C\|u\|_{\dot{B}^{\alpha,\phi}(B)} \quad \forall \text{ balls } B \text{ and } u \in \dot{B}^{\alpha,\phi}(B)
\]
and
\[
\inf_{c \in \mathbb{R}}\|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \leq C\|u\|_{\dot{B}^{\alpha,\phi}(\mathbb{R}^n)} \quad \forall u \in \dot{B}^{\alpha,\phi}(\mathbb{R}^n).
\]

For any $s \in (0, 1)$ and $1 < p < n/s$, let $\alpha = s - n/p \in (-n, 0)$. By $\dot{B}^{s,p}_p(\Omega) = \dot{B}^{\alpha,\phi}(\Omega)$ for all $\Omega \subset \mathbb{R}^n$ and $n/|\alpha| = np/(n - ps)$, Theorem 1.1 is exactly the following well-known imbedding of Besov spaces $\dot{B}^{s,p}_p$: there exists a constant $C > 0$ depending only on $n, s, p$ such that
\[
\|u - u_B\|_{L^{np/(n-s)p}(B)} \leq C\|u\|_{\dot{B}^{s,p}_p(B)} \quad \forall \text{ balls } B \text{ and } u \in \dot{B}^{s,p}_p(B)
\]
and
\[
\inf_{c \in \mathbb{R}}\|u - c\|_{L^{np/(n-s)p}(\mathbb{R}^n)} \leq C\|u\|_{\dot{B}^{s,p}_p(\mathbb{R}^n)} \quad \forall u \in \dot{B}^{s,p}_p(\mathbb{R}^n)
\]
see for example [1, 4, 12, 14].

Moreover, we characterize all domains $\Omega \subset \mathbb{R}^n$ supporting the imbedding of $\dot{B}^{\alpha,\phi}(\Omega)$ into $L^{n/|\alpha|}(\Omega)$ via globally $n$-regular domains. Here, a domain $\Omega \subset \mathbb{R}^n$ is globally $n$-regular if there exists a constant $\theta \in (0, 1)$ such that
\[
|B(x, r) \cap \Omega| \geq \theta^r^n \quad \forall x \in \Omega \text{ and } 0 < r < 2 \text{ diam } \Omega.
\]
Recall that in the literature a domain $\Omega \subset \mathbb{R}^n$ is $n$-regular (or satisfies the measure density property) if there exists a constant $\theta \in (0, 1)$ such that
\[
|B(x, r) \cap \Omega| \geq \theta r^n \quad \forall x \in \Omega \text{ and } 0 < r < 1,
\]
see [11, 12, 18, 8, 22] and references therein. Obviously, a bounded domain is $n$-regular if and only if it is globally $n$-regular. A unbounded globally $n$-regular domain is always $n$-regular but the converse is not true in general.

**Theorem 1.2.** Let $\alpha \in (-n, 0)$ and $\phi$ be a Young function satisfying (1.3) and (1.4). Then, a domain $\Omega \subset \mathbb{R}^n$ is globally $n$-regular if and only if there exists a constant $C \geq 1$ such that
\[
\|u - u_\Omega\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{\dot{B}^{\alpha,\phi}(\Omega)} \quad \forall u \in \dot{B}^{\alpha,\phi}(\Omega) \text{ when diam } \Omega < \infty,
\]
and
\[
\|u\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{\dot{B}^{\alpha,\phi}(\Omega)} \quad \forall u \in \dot{B}^{\alpha,\phi}(\Omega) \text{ having bounded supports when diam } \Omega = \infty.
\]
As a consequence of Theorem 1.2, we have the following results for inhomogeneous Orlicz-Besov spaces \( \mathcal{B}^{\alpha,\phi}(\Omega) := L^\phi(\Omega) \cap \dot{\mathcal{B}}^{\alpha,\phi}(\Omega) \) which are equipped with the norms \( \|u\|_{\mathcal{B}^{\alpha,\phi}(\Omega)} = \|u\|_{L^\phi(\Omega)} + \|u\|_{\dot{\mathcal{B}}^{\alpha,\phi}(\Omega)} \). Recall that \( L^\phi(\Omega) \) is the Orlicz space, that is, the collection of measurable functions \( u \) in \( \Omega \) with

\[
\|u\|_{L^\phi(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \frac{|u(x)|}{\lambda} \, dx \leq 1 \right\} < \infty.
\]

**Corollary 1.3.** Let \( \alpha \in (-n, 0) \) and \( \phi \) be a Young function satisfying (1.3) and (1.4). A bounded domain \( \Omega \subset \mathbb{R}^n \) is \( n \)-regular if and only if there exists a constant \( C \geq 1 \) such that

\[
\|u\|_{L^{\phi(n)}(\Omega)} \leq C\|u\|_{\mathcal{B}^{\alpha,\phi}(\Omega)} \quad \forall u \in \mathcal{B}^{\alpha,\phi}(\Omega).
\]

For any \( s \in (0, 1) \) and \( 1 < p < n/s \), by considering extension of \( \mathcal{B}^{s}_{p,p}(\Omega) \) functions as in [11, 12, 18, 19, 22], we know that a domain \( \Omega \subset \mathbb{R}^n \) is \( n \)-regular if and only if there exists a constant \( C > 0 \) such that

\[
\|u\|_{L^{\phi(n/p)}(\Omega)} \leq C\|u\|_{\mathcal{B}^{s}_{p,p}(\Omega)} \quad \forall u \in \mathcal{B}^{s}_{p,p}(\Omega).
\]

By considering the extension of \( \mathcal{B}^{s}_{p,p}(\Omega) \) functions similarly, one also would prove that a bounded domain \( \Omega \subset \mathbb{R}^n \) is globally \( n \)-regular if and only if there exists a constant \( C > 0 \) such that

\[
\|u - u_0\|_{L^{\phi(n/p)}(\Omega)} \leq C\|u\|_{\mathcal{B}^{s}_{p,p}(\Omega)} \quad \forall u \in \mathcal{B}^{s}_{p,p}(\Omega).
\]

Letting \( \alpha = s - n/p \in (-n, 0) \) and \( \phi(t) = t^p \), since \( \mathcal{B}^{s}_{p,p}(\Omega) = \mathcal{B}^{\alpha,\phi}(\Omega) \) and \( \dot{\mathcal{B}}^{s}_{p,p}(\Omega) = \dot{\mathcal{B}}^{\alpha,\phi}(\Omega) \), we know that Corollary 1.3 and Theorem 1.2 generalize both of (1.10) and (1.11) when \( \Omega \) is bounded domain. We refer to [6, 7, 20, 8, 9, 10] for the characterization of domains supporting the Hajłasz Sobolev (Besov) imbedding into Lebesgue spaces via \( n \)-regular domains.

**Remark 1.4.** We remark that the assumption (1.4) in Theorem 1.2 and Corollary 1.3 is optimal for bounded domains in the following sense. Given any \( \alpha \in (-n, 0) \), the Young function \( t^p \) with \( p \geq 1 \) satisfies the assumption (1.4) if and only if \( p > -n/\alpha \). In the critical case \( \phi_0(x) = x^{p/|\alpha|} \) with \( \alpha \in (-n, 0) \), (1.7) holds for any bounded domain \( \Omega \subset \mathbb{R}^n \). Indeed, for any \( u \in \mathcal{B}^{\alpha,\phi}(\Omega) \), by the Hölder inequality we have

\[
\|u - u_B\|_{L^{|\alpha|/|\alpha|}(\Omega)} \leq |B|^{1/|\alpha|} \left\{ \int_\Omega \int_\Omega |u(x) - u(y)|^{p/|\alpha|} \, dx \, dy \right\}^{1/|\alpha|/n} \leq \frac{|B|^{1/|\alpha|}}{(\text{diam } \Omega)^{n/|\alpha|}} \left( \int_\Omega \int_\Omega \phi_0 \left( \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right) \, dx \, dy \right)^{1/|\alpha|/n},
\]

where \( B \) is a fixed ball so that \( 2B \subset \Omega \). Therefore,

\[
\int_\Omega \int_\Omega \phi_0 \left( \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right) \, dx \, dy \geq 1,
\]

which yields that

\[
\|u - u_B\|_{L^{|\alpha|/|\alpha|}(\Omega)} \leq \frac{(\text{diam } \Omega)^p}{|B|^{n/|\alpha|}} \|u\|_{\mathcal{B}^{\alpha,\phi}(\Omega)}
\]
as desired.

To prove Theorems 1.1&1.2, we establish the following geometric inequality, which extend a geometric inequality of Caffarelli-Valdinoci [2] and Savin-Valdinoci [17] not only to general Young functions but also to globally \( n \)-regular domains; see Remark 3.3 (i).

**Lemma 1.5.** Suppose that \( \Omega \subset \mathbb{R}^n \) is a globally \( n \)-regular domain. Let \( \alpha \in (-n, 0) \) and \( \phi : [0, \infty) \rightarrow [0, \infty) \) be a Young function satisfying (1.3) and (1.4). There exists constants \( C_1, C_2 > 0 \) depending on \( n, \alpha, \phi, \Omega \) such that

\[
\int_{\Omega \setminus E} \frac{\phi(t|x-y|^{-\alpha})}{|x-y|^{2n}} \, dy \geq C_1 \frac{1}{|E|} \frac{\Omega \setminus E}{\Omega} \phi(C_2 t|E|^{1/|\alpha|/n})
\]

whenever \( t > 0 \), \( x \in \Omega \) and \( E \subset \Omega \) with \( 0 < |E| < \infty \). Here, if \( |\Omega| = \infty \), we let \( \frac{|\Omega \setminus E|}{|\Omega|} = 1 \).
In Section 3, applying Lemma 1.5, using median values and improving some argument of Di Nezza et al [3] we prove in a direct way that globally $n$-regular domains supporting $\dot{B}^{0,\phi}$-imbedding (1.7) or (1.8). By approximating $\mathbb{R}^n$ by balls $B(0,R)$, which have uniform globally $n$-regular constants, we also derive Theorem 1.1. Conversely, by precise estimates of $\dot{B}^{0,\phi}$-norms of certain cut-off functions as in Section 2, borrowing the ideas from Hajłasz et al [8] and Zhou [22] we prove that domains supporting $\dot{B}^{0,\phi}$-imbeddings as in (1.7) or (1.8) are globally $n$-regular. Thus Theorem 1.2 holds. We remark that this gives a direct and new proof for (1.5) and (1.6), and also for (1.11) and (1.10) in bounded $n$-regular domains; see Remark 3.3 (ii).

**Remark 1.6.** Under the assumption of Theorem 1.2, when $\Omega$ is an unbound domain, we also have the following characterization: $\Omega$ is globally $n$-regular if and only if there exists constant $C > 0$ such that

\[
\inf_{c \in \mathbb{R}} \|u - c\|_{L^\infty(\Omega)} \leq C\|u\|_{\dot{B}^{0,\phi}(\Omega)} \quad \forall u \in \dot{B}^{0,\phi}(\Omega),
\]

which is better than (1.8). Moreover, Corollary 1.3 also holds when $\Omega$ is unbounded. However, the direct approach above fails to prove them; see Remark 3.3 for a reason. Instead, in a forth-coming paper [21], for full range $\alpha \in (-n, 1)$ and Young functions $\phi$ satisfying (1.3) and (1.4), we consider the extension of Orlicz-Besov spaces, then together with Theorems 1.1 & 1.2 and Corollary 1.3 above, fully characterize Orlicz-Besov imbedding domains.

Finally, we make some convention on the notations or notion used in this paper. Throughout the paper, $C$ denotes a positive constant, which depend only on $n, \alpha, \phi, \Omega$ but whose value might be change from line to line. We write $A \lesssim \langle \rangle B$ if there exists a constant $C > 0$ such that $A \leq \langle \rangle CB$. We use $\int_B f(x)dx$ to denote the average $\frac{1}{|B|} \int_B f(x)dx$ of a function $f$ in a set $B$ with positive measure. For any $x \in \Omega$ and $A \subset \mathbb{R}$, $\text{dist}(x, A)$ denote the distance from $x$ to $A$, $\text{diam}(A)$ denote the diameter of the set $A$.

## 2. Some basic properties

We list several basic properties of Orlicz-Besov spaces in this section. Let us begin with two simple properties of Young functions. Note that Young functions are always strictly increasing.

**Lemma 2.1.** Let $\alpha \in (-n, 0)$ and $\phi$ be a Young function.

(i) If $\phi$ satisfies (1.3), then

\[
\phi(xs) \leq 2^{2n} \Lambda_\alpha(\alpha) \phi(2^{1-\alpha}x)^{s/(1-\alpha)} \quad \forall 0 < s \leq 1, x > 0.
\]

(ii) If $\phi$ satisfies (1.4), then $\phi(xs^{-\alpha})s^{-\alpha} \to 0$ as $s \to \infty$ for any $x > 0$ and

\[
\phi(xs) \leq 2^{3n} \Lambda_\alpha(\alpha) \phi(x)s^{-n/\alpha} \quad \forall s \geq 1, x > 0.
\]

**Proof.** (i) Since $\phi$ is increasing, by (1.3) we have

\[
\sup_{s \geq 0} \frac{\phi(xs^{1-\alpha})}{\phi(x)^{s^{1-\alpha}}} \leq \sup_{j \geq 0} \sup_{s \in [2^{-j-1}, 2^{-j}]} \frac{\phi(xs^{1-\alpha})}{\phi(x^{2^{1-j}})^{s^{1-\alpha}}} \leq 2^{3n} \sup_{j \geq 0} \int_{2^{-j}}^{2^{-j+1}} \frac{\phi(xs^{1-\alpha})}{\phi(x^{2^{1-j}})^{s^{1-\alpha}}} ds \leq 2^{2n} \sup_{j \geq 0} \int_{2^{-j-1}}^{2^{-j}} \frac{\phi(xs^{1-\alpha})}{\phi(x^{2^{1-j}})^{s^{1-\alpha}}} ds \leq 2^{2n} \Lambda_\alpha(\alpha) \quad \forall x > 0,
\]

which gives (2.1). (ii) Similarly, for all $x > 0$ and $j \geq 0$ we have

\[
\sup_{s \geq 1} \frac{\phi(xs^{-\alpha})}{s^{\alpha}\phi(x)} \leq 2^{2n+1} \int_{2^{j+1}}^{2^{j+2}} \frac{\phi(xs^{-\alpha})}{\phi(x)^{s^{1-\alpha}}} ds \leq \int_{2^{j+1}}^{\infty} \frac{\phi(xs^{-\alpha})}{\phi(x)^{s^{1-\alpha}}} ds \leq 2^{3n} \Lambda_\alpha(\alpha) \quad \forall x > 0,
\]

which implies that $\phi(xs^{-\alpha})s^{-\alpha} \to 0$ as $s \to 0$. Moreover, this also implies that

\[
\sup_{s \geq 1} \frac{\phi(xs^{-\alpha})}{s^{\alpha}\phi(x)} \leq \sup_{j \geq 0} \sup_{s \in [2^{j}, 2^{j+1}]} \frac{\phi(xs^{-\alpha})}{s^{\alpha}\phi(x)} \leq 2^{3n} \int_{2}^{\infty} \frac{\phi(xs^{-\alpha})}{\phi(x)^{s^{1-\alpha}}} ds \leq 2^{3n} \Lambda_\alpha(\alpha) \quad \forall x > 0,
\]

which gives (2.2). \qed
Lemma 2.2. Let $\alpha \in (-n, 0)$ and $\phi$ be a Young function. For any domain $\Omega \subset \mathbb{R}^n$, we have $B^{\alpha, \phi}(\Omega) \subset \dot{B}^{\alpha, \phi}(\Omega) \subset L^1_{loc}(\Omega)$ as sets.

Proof. Let $u \in \dot{B}^{\alpha, \phi}(\Omega)$ and $\lambda > ||u||_{B^{\alpha, \phi}(\Omega)}$, we have

$$\int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{\alpha}} \right) \frac{dydx}{|x - y|^{2n}} \leq 1.$$  

By Fubini’s theorem, for almost all $x \in \Omega$ we have

$$\int_{\phi} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{\alpha}} \right) \frac{dy}{|x - y|^{2n}} < \infty,$$

For any $B = B(z, r)$ with $z \in \Omega$ and $r < \frac{1}{2} \text{diam} \Omega$, choose an $x \in 3B \setminus 2B$ satisfying above inequality. Then $r \leq |x - y| \leq 4r$ for all $y \in B$, and hence

$$\int_{\phi} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{\alpha}} \right) \frac{dy}{(4r)^{2n}} \leq \int_{\phi} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{\alpha}} \right) \frac{dydx}{|x - y|^{2n}} < \infty.$$  

By Jessen’s inequality, we have

$$\phi \left( \int_{\phi} \phi \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^{\alpha}} \right) \frac{dy}{|x - y|^{2n}} \right) < \infty$$

which implies that $\int_{\phi} |u(x) - u(y)| / dy < \infty$ that is, $u \in L^1(B)$. This completes the proof of Lemma 2.2. \hfill \Box

Lemma 2.3. Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $\alpha \in (-n, 0)$ and $\phi$ is a Young function satisfying (1.3) and, when $\text{diam} \Omega = \infty$, also satisfying (1.4). Then $C_1^1(\Omega) \subset B^{\alpha, \phi}(\Omega) \subset \dot{B}^{\alpha, \phi}(\Omega)$ as sets.

Proof. Let $u \in C_1^1(\Omega)$. Obviously, $u \in L^\phi(\Omega)$. To see $u \in \dot{B}^{\alpha, \phi}(\Omega)$, let $V = \text{supp} \ u$, and $W \subset \Omega$ be a bounded open set so that $V \subset W$. Then

$$H = \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(z) - u(w)|}{\lambda|z - w|^{\alpha}} \right) \frac{dzdw}{|z - w|^{2n}}$$  

$$= \int_{W} \int_{W} \phi \left( \frac{|u(z) - u(w)|}{\lambda|z - w|^{\alpha}} \right) \frac{dzdw}{|z - w|^{2n}} + 2 \int_{\Omega \setminus W} \int_{V} \phi \left( \frac{u(z)}{\lambda|z - w|^{\alpha}} \right) \frac{dzdw}{|z - w|^{2n}}$$  

$$=: H_1 + H_2.$$  

It then suffices to show that $H_1 \leq 1/2$ and $H_2 \leq 1/2$ when $\lambda$ is sufficiently large. Write $L = ||Du||_{L^\phi(\Omega)}$. By (1.3) one has

$$H_1 \leq \int_{W} \int_{B(w, \text{diam} W)} \phi \left( \frac{|z - w|^{1-\alpha}}{\lambda/L} \right) \frac{dz}{|z - w|^{2n}} \frac{dw}{|z - w|^{2n}}$$  

$$= \omega_n \int_{W} \int_{0}^{\text{diam} W} \phi \left( \frac{t^{1-\alpha}}{\lambda/L} \right) \frac{dt}{t^{n+1}} \frac{dw}{|z - w|^{2n}}$$  

$$= \omega_n |W| (\text{diam} W)^{-n} \int_{0}^{1} \phi \left( \frac{(\text{diam} W)^{1-\alpha}t^{1-\alpha}}{\lambda/L} \right) \frac{dt}{t^{n+1}}$$  

$$\leq \Delta_\phi(\alpha) \omega_n |W| (\text{diam} W)^{-n} \phi \left( \frac{(\text{diam} W)^{1-\alpha}L}{\lambda} \right),$$

and hence $H_1 \leq 1/2$ when $\lambda > 0$ is large enough. If $\Omega$ is bounded, then

$$H_2 \leq 2 \text{dist} (V, \partial W)^{-2n} |\Omega \setminus W||V| \phi \left( \frac{||u||_{L^\phi(\Omega)} (\text{diam} \Omega)^{\phi}}{\lambda} \right),$$

and hence $H_2 \leq 1/2$ when $\lambda$ is large enough. If $\Omega$ is unbounded, by (1.4) we have

$$H_2 \leq 2 \int_{V} \int_{\mathbb{R}^n \setminus W} \phi \left( \frac{||u||_{L^\phi(\Omega)}}{\lambda|z - w|^{\alpha}} \right) \frac{dw}{|z - w|^{2n}} dz$$  

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\begin{align*}
&\leq 2\omega_n \int_V \int_0^\infty \phi \left( \frac{||t||_{L^\infty(\Omega_r)}}{\lambda^{p+1}} \right) dt dz \\
&\leq 2\omega_n \int_1^\infty \phi \left( \frac{||t||_{L^\infty(\Omega)}}{\lambda(\text{dist} (V, \partial W))^p t^p} \right) dt \\
&\leq 2\omega_n \int_1^\infty \phi \left( \frac{||t||_{L^\infty(\Omega)}}{\lambda \text{dist} (V, \partial W)^p} \right),
\end{align*}
\]

hence \( H_2 \leq 1/2 \) when \( \lambda \) is large enough. This completes the proof of Lemma 2.3. \( \square \)

**Remark 2.4.** (i) The assumption (1.3) is optimal to guarantee \( C^1(\Omega) \subset \hat{B}^{\alpha,\phi}(\Omega) \) and hence the non-triviality of \( \hat{B}^{\alpha,\phi}(\Omega) \) in the following sense. For \( \alpha \geq 1 - n \) and \( p \geq 1 \), by a direct calculation, the Young function \( \phi(t) = t^p \) satisfies (1.3) if and only if \( p > n/(1-\alpha) \). Note that \( \hat{B}^{\alpha,\phi}(\Omega) = \hat{B}^{\alpha/p+\alpha}(\Omega) \). By [5, Theorems 4.1&4.2], \( \hat{B}^{\alpha/p+\alpha}(\Omega) \), and hence \( \hat{B}^{\alpha,\phi}(\Omega) \), is nontrivial (or contains \( C^1(\Omega) \)) if and only if \( s = n/p + \alpha < 1 \), that is, \( p > n/(1-\alpha) \).

(ii) In the case \( \text{diam} \Omega = \infty \), (1.4) is optimal to guarantee \( C^1(\Omega) \subset \hat{B}^{\alpha,\phi}(\Omega) \) or the nontriviality of \( \hat{B}^{\alpha,\phi}(\Omega) \) in the following sense. Indeed, the Young function \( t^p \) with \( p \geq 1 \) satisfies (1.4) if and only if \( p < n/\alpha \). Let \( \phi(t) = t^{p/\alpha} \) and \( \Omega \) be any unbounded globally \( n \)-regular domain. We see that \( ||t||_{\hat{B}^{\alpha,\phi}(\Omega)} = \infty \) for any \( u \in C^1(\Omega) \) and \( u \neq 0 \). Indeed, for any \( \lambda > 0 \), let \( H_2 \) be as in the proof of Lemma 2.3. Moreover, let \( V_0 = \{ x \in \Omega : |u(x)| > ||u||_{L^\infty(\Omega)} / 2 \} \) and \( W = B(z_0, 4 \text{diam} V) \) with \( z_0 \in V \). For all \( w \in \Omega \setminus W \), \( z \in V \), \( |z-z_0| < 4 \text{diam} V \), \( |z-w| \geq 4 \text{diam} V \), it deduces that \( |z-z_0| \leq 3/4 |z_0-w| \). Then \( |z-w| \leq |z_0-w| + |z-z_0| \leq 5/4 |z_0-w| \).

Thus
\[
H_2 \geq \frac{||u||_{L^\infty(\Omega)}}{(2\lambda)^{n/\alpha}} \int_V \int_{\Omega,|w| < |z-w|^n} dw \geq \frac{||u||_{L^\infty(\Omega)}}{(2\lambda)^{n/\alpha}(\frac{4}{3})^n} |V_0| \int_{\Omega,|w| < |z-w|^n} dw.
\]

But
\[
\int_{\Omega \setminus \Omega,|w| < |z-w|^n} dw \geq \sum_{j \geq 2} 2^{-j k_0 n} (\text{diam} V)^{-n} |\Omega \cap [B(z_0, 2^{-j+k_0} \text{diam} V) \setminus B(z_0, 2^j \text{diam} V)]| \geq \sum_{j \geq 2} 2^{-j k_0 n} = \infty,
\]

where \( k_0 \) is a postiche integer satisfying \( \theta_0 k_0 n \geq 2\omega_n \) so that
\[
|\Omega \cap [B(z_0, 2^{-j+k_0} \text{diam} V) \setminus B(z_0, 2^j \text{diam} V)]| = |\Omega \cap B(z_0, 2^{-j+k_0} \text{diam} V)| - |\Omega \cap B(z_0, 2^j \text{diam} V)| \geq \theta_0 k_0 n (\text{diam} V)^n - \omega_n 2^{in} (\text{diam} V)^n \geq 2^{in} (\text{diam} V)^n.
\]

Therefore, we always \( H \geq H_2 = \infty \), which means that \( u \notin \hat{B}^{\alpha,\phi}(\Omega) \).

To end this section, we calculate \( B^{\alpha,\phi}(\Omega) \) and \( \hat{B}^{\alpha,\phi}(\Omega) \)-norms of some special functions, which will be used in Sections 3 and 4.

For \( x \in \Omega \) and \( 0 < r < t < \frac{1}{2} \text{diam} \Omega \), let \( B_\Omega(x,t) := \Omega \cap B(x,t) \) and \( B_{\Omega}(x,r) := \Omega \cap B(x,r) \), set the function
\[
u_{r} \in \hat{B}^{\alpha,\phi}(\Omega) \text{ and } \nu_{r} \in \hat{B}^{\alpha,\phi}(\Omega) \text{ and }
\]

\[
\begin{align*}
\|u_{x,r,t}\|_{B^{\alpha,\phi}(\Omega)} &\leq C(t-r)^{-\alpha} \left[ \phi^{-1} \left( \frac{(t-r)^n}{|B_\Omega(x,t)|} \right) \right]^{-\alpha}.
\end{align*}
\]
Proof. Write $u = u_{x,r}$ for simple. Note that

$$H := \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(z) - u(w)|}{\lambda|z-w|^\alpha} \right) \frac{dzdw}{|z-w|^{2n}}$$

$$= \int_{B_1(x,t)} \int_{B_1(x,t)} \phi \left( \frac{|u(z) - u(w)|}{\lambda|z-w|^\alpha} \right) \frac{dzdw}{|z-w|^{2n}} + 2 \int_{\Omega} \int_{B_2(x,t)} \phi \left( \frac{|u(z)|}{\lambda|z-w|^\alpha} \right) \frac{dzdw}{|z-w|^{2n}}$$

$$\leq \int_{B_1(x,t)} \int_{B_1(x,t)} \phi \left( \frac{|z-w|^{1-\alpha}}{\lambda(t-r)} \right) \frac{dz}{|z-w|^{2n}} dw$$

$$+ \int_{B_1(x,t)} \int_{B_1(x,t)} \phi \left( \frac{|z-w|^{-\alpha}}{\lambda} \right) \frac{dz}{|z-w|^{2n}} dw$$

$$+ 2 \int_{B_1(x,t)} \int_{B_2(x,t)} \phi \left( \frac{(t - |z-x|)|z-w|^{-\alpha}}{\lambda(t-r)} \right) \frac{dw}{|z-w|^{2n}} dz$$

$$+ 2 \int_{B_1(x,t)} \int_{\Omega\setminus B_2(x,t)} \phi \left( \frac{|z-w|^{-\alpha}}{\lambda} \right) \frac{dw}{|z-w|^{2n}} dz$$

$$=: H_1 + H_2 + 2H_3 + 2H_4.$$
Choosing $8$ HONGYAN SUN

Note that this is equivalent to

$$
\lambda > M
$$

Combining all above estimates together we conclude

$$
H \leq M \left[ \frac{B_\Omega(x, t)}{(t-r)^n} \right] M \left[ \frac{1}{M} \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right] \leq 1.
$$

Thus $\|u_{x,r,t}\|_{B^{\alpha,\Phi}(\Omega)} \leq \lambda_0$ as desired.

\[\square\]

**Lemma 2.6.** Let $\alpha \in (-n, 0)$ and $\Phi$ be a Young function satisfying (1.3) and (1.4). Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain. There exists a constant $C > 0$ depending on $n, \alpha, \Omega$ and $\Phi$ such that $x \in \Omega$ and $0 < r < t < \frac{1}{2} \text{diam} \Omega, u_{x,r,t} \in B^{\alpha,\Phi}(\Omega)$ and

$$
\|u_{x,r,t}\|_{B^{\alpha,\Phi}(\Omega)} \leq C(t-r)^{-\alpha} \left[ \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right]^{-1}.
$$

**Proof.** Note that

$$
\|u_{x,r,t}\|_{L^{\Phi}(\Omega)} \leq \left[ \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right]^{-1}.
$$

Indeed, for $\lambda > [\phi^{-1} \left( \frac{1}{|B_\Omega(x,t)|} \right)]^{-1}$, since $u_{x,r,t}$ is supported in $B_\Omega(x,t)$ and $0 \leq u_{x,r,t} \leq 1$, we have

$$
\int_\Omega \phi \left( \frac{u_{x,r,t}}{\lambda} \right) \leq \phi(1/\lambda)|B_\Omega(x,t)| < 1
$$

as desired.

It then suffices to prove that there exists a constant $C > 0$ such that

$$
\left[ \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right]^{-1} \leq C(t-r)^{-\alpha} \left[ \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right]^{-1}.
$$

Note that this is equivalent to

$$
\frac{(t-r)^n}{|B_\Omega(x, t)|} \leq \phi \left( \frac{1}{|B_\Omega(x, t)|} \right) C(t-r)^{-\alpha}.
$$

Since $\phi(x(\text{diam} \Omega)^{-\alpha}) \leq 2^{3n} \overline{\Lambda}_\Phi(\alpha) (\frac{\text{diam} \Omega}{t-r})^n \phi(x(t-r)^{-\alpha})$ as in (2.2), it suffices to prove

$$
\frac{2^{3n} \overline{\Lambda}_\Phi(\alpha) (\text{diam} \Omega)^n}{|B_\Omega(x, t)|} \leq \phi \left( \frac{1}{|B_\Omega(x, t)|} \right) C \left( (\text{diam} \Omega)^{-\alpha} \right),
$$

Choosing $C = (\text{diam} \Omega)^{\alpha} (2^{3n} \overline{\Lambda}_\Phi(\alpha) (\text{diam} \Omega)^n + 1)$, by the convexity of $\phi$ we have

$$
\phi \left( \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) C \left( (\text{diam} \Omega)^{-\alpha} \right) \right) \geq \phi \left( \phi^{-1} \left( \frac{1}{|B_\Omega(x, t)|} \right) \right) \left[ C \left( (\text{diam} \Omega)^{-\alpha} \right) \right].
\[
\frac{C(\text{diam } \Omega)^{-\alpha}}{|B_{\Omega}(x, t)|} \geq \frac{2^{3n} \Lambda_\phi(\alpha)}{|B_{\Omega}(x, t)|} \]

as desired. \qed

3. Proofs of Theorems 1.1 & 1.2 and Corollary 1.3

We begin with the proof of Lemma 1.5, which is motivated by [2, 17] and also [3].

**Proof of Lemma 1.5.** Let \( \kappa = [2\omega_n / \theta]^{1/n} + 2 \). Then

\[
\Omega \cap (B(z, \kappa s) \setminus B(z, s)) \neq \emptyset \quad \forall z \in \Omega \quad \text{and} \quad 0 < s < \frac{2}{\kappa} \text{ diam } \Omega.
\]

Indeed, we have \(|\Omega \cap B(z, \kappa s)| \geq \theta \kappa^s s^a\) and \(|\Omega \cap B(z, s)| \leq \omega_n s^a\) for all \( z \in \Omega \) and \( 0 < s < \frac{2}{\kappa} \text{ diam } \Omega \). Since \( \theta \kappa^n > 2\omega_n \), we know that \( \Omega \cap (B(z, \kappa s) \setminus B(z, s)) \) has positive measure.

Let \( r \in (0, 2 \text{ diam } \Omega) \) such that \(|E| = |\Omega \cap B(x, r)|\), and moreover, \( \theta r^a \leq |E| \leq \omega_n r^a\). If \( r \geq \frac{1}{8 \kappa} \text{ diam } \Omega \), then \(|E| \geq C|\Omega|\) and \( \text{diam } \Omega < \infty \). By (2.2), for all \( y \in \Omega \), we have

\[
\phi(t|x - y|^{-\alpha}) \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \frac{\phi(t(\text{diam } \Omega)^{-\alpha})}{\text{diam } \Omega} \left( \frac{|x - y|}{\text{diam } \Omega} \right)^n \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \phi(t(\text{diam } \Omega)^{-\alpha}) \left( \frac{|x - y|}{\text{diam } \Omega} \right)^{2n},
\]

and hence there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\int_{\Omega \setminus E} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy \geq C_1 \left( \frac{|\Omega \setminus E|}{|\Omega|^2} \right) \phi(|\Omega|^{-\alpha/n}) \geq C_1 C \left( \frac{|\Omega \setminus E|}{|\Omega|^2} \right) \phi(|\Omega|^{-\alpha/n}) \left( \frac{|x - y|}{\text{diam } \Omega} \right)^{2n}
\]

as desired.

If \( r < \frac{1}{8 \kappa} \text{ diam } \Omega \), write

\[
\int_{\Omega \setminus E} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy = \int_{(\Omega \setminus E) \cap B(x, r)} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy + \int_{(\Omega \setminus E) \setminus B(x, r)} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy.
\]

By (2.2), for \( y \in B(x, r) \) we have

\[
\phi(t|x - y|^{-\alpha}) \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \phi(tr^{-\alpha}) \left( \frac{|x - y|}{r} \right)^n \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \phi(tr^{-\alpha}) \left( \frac{|x - y|}{r} \right)^{2n}.
\]

Thus

\[
\int_{(\Omega \setminus E) \cap B(x, r)} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \frac{\phi(tr^{-\alpha})}{r^{2n}} |(\Omega \setminus E) \cap B(x, r)|.
\]

Note that

\[
|(\Omega \setminus E) \cap B(x, r)| = |\Omega \cap B(x, r)| - |E \cap B(x, r)| = |E| - |E \cap B(x, r)| = |\Omega \setminus B(x, r)|.
\]

By (2.2), for \( y \in E \setminus B(x, r) \) we have

\[
\phi(tr^{-\alpha}) \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \phi(t|x - y|^{-\alpha}) \left( \frac{r}{|x - y|} \right)^n \geq \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \phi(t|x - y|^{-\alpha}) \left( \frac{r}{|x - y|} \right)^{2n}.
\]

Therefore,

\[
\int_{(\Omega \setminus E) \setminus B(x, r)} \frac{\phi(t|x - y|^{-\alpha})}{|x - y|^{2n}} dy \geq \left( \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \right)^2 \int_{E \setminus B(x, r)} \phi(t|x - y|^{-\alpha}) \frac{1}{|x - y|^{2n}} dy.
\]

Since \((\Omega \setminus E) \setminus B(x, r) \cup E \setminus B(x, r) = \Omega \setminus B(x, r)\), we obtain

\[
\int_{\Omega \setminus E} \phi(t|x - y|^{-\alpha}) \frac{1}{|x - y|^{2n}} dy \geq \min \left\{ \left( \frac{1}{2^{3n} \Lambda_\phi(\alpha)} \right)^2, 1 \right\} \int_{\Omega \setminus B(x, r)} \phi(t|x - y|^{-\alpha}) \frac{1}{|x - y|^{2n}} dy.
\]
By (3.1), \( \Omega \cap (B(x, 2kr) \setminus B(x, 2r)) \) is not empty set, and hence containing some point, say \( z \). Then
\[
\Omega \cap B(z, r) \subset \Omega \cap [(B(x, 3kr) \setminus B(x, r)) \subset \Omega \setminus B(x, r)
\]
and
\[
|\Omega \cap [(B(x, 3kr) \setminus B(x, r))] \geq |B(z, r) \cap \Omega| \geq \theta r^n.
\]
Thus, we have
\[
\int_{\Omega \setminus B(x, r)} \phi(t|x - y|^{-\alpha}) \frac{1}{|x - y|^{2n}} dy \geq \int_{\Omega \setminus [(B(x, 3kr) \setminus B(x, r))] \setminus B(x, r)} \phi(t|x - y|^{-\alpha}) \frac{1}{|x - y|^{2n}} dy \geq C_3 r^{-\alpha} \phi(t r^{-\alpha}),
\]
where \( C_3 \) is positive constant. Note that \( \theta r^n \leq |E| \leq \omega_n r^n \) and \( \frac{|\Omega| |E|}{|\Omega|} \leq 1 \). The proof of Lemma 1.5 is completed. \( \square \)

**Lemma 3.1.** Let \( \alpha \in (-n, 0) \) and \( \phi \) be a Young function satisfying (1.3) and (1.4). Suppose that \( \Omega \) is a globally \( n \)-regular domain. Then there exists a constant \( C > 0 \) depending only on \( n, \alpha, \phi \) and \( \theta \) such that
\[
||u||_{L^{\theta,\phi}(\Omega)} \leq C||u||_{\dot{B}^{\alpha,\phi}(\Omega)}
\]
whenever \( u \in \dot{B}^{\alpha,\phi}(\Omega) \) satisfies
\[
(3.2) \quad u \geq 0 \text{ in } \Omega \text{ and } ||x \in \Omega, u = 0|| \geq \frac{1}{2} |\Omega| \text{ if } |\Omega| < \infty
\]
or
\[
(3.3) \quad u \geq 0 \text{ in } \Omega \text{ and } ||x \in \Omega, u > a|| < \infty \forall a > 0 \text{ if } |\Omega| = \infty.
\]

**Proof.** Let \( u \in \dot{B}^{\alpha,\phi}(\Omega) \) satisfy (3.2) or (3.3). Obviously, we may assume that \( u \neq 0 \). Without loss of generality, we may also assume that \( u \) is bounded. Indeed, for \( N \geq 1 \) let
\[
u^N = u|_{\{x \in \Omega : u < 2^N\}} + 2^N |_{\{x \in \Omega : u \geq 2^N\}}.
\]
Note that \( ||u||_{L^{\theta,\phi}(\Omega)} \leq \lim_{N \to \infty} ||u^N||_{L^{\theta,\phi}(\Omega)} \) and \( \sup_{N \geq 1} ||u^N||_{\dot{B}^{\alpha,\phi}(\Omega)} \leq ||u||_{\dot{B}^{\alpha,\phi}(\Omega)} \). If \( ||u^N||_{L^{\theta,\phi}(\Omega)} \leq C||u^N||_{\dot{B}^{\alpha,\phi}(\Omega)} \) hold for all \( N \geq 1 \), by sending \( N \to \infty \), we have \( ||u||_{L^{\theta,\phi}(\Omega)} \leq C||u||_{\dot{B}^{\alpha,\phi}(\Omega)} \) as desired. Moreover, when \( |\Omega| = \infty \), we may further assume \( ||x \in \Omega, u > 0|| < \infty \). Indeed, for \( N \leq 0 \) let
\[
u_N = (u - 2^N)|_{\{x \in \Omega : u \geq 2^N\}}.
\]
By (3.3), we have
\[
||x \in \Omega, u_N(x) > 0|| = ||x \in \Omega, u(x) > 2^N|| < \infty.
\]
Note that \( ||u||_{L^{\theta,\phi}(\Omega)} = \lim_{N \to \infty} ||u_N||_{L^{\theta,\phi}(\Omega)} \) and \( \sup_{N \geq 0} ||u_N||_{\dot{B}^{\alpha,\phi}(\Omega)} \leq ||u||_{\dot{B}^{\alpha,\phi}(\Omega)} \). If \( ||u_N||_{L^{\theta,\phi}(\Omega)} \leq C||u_N||_{\dot{B}^{\alpha,\phi}(\Omega)} \) hold for all \( N \leq 0 \), by sending \( N \to -\infty \), we have \( u \in L^{\theta,\phi}(\Omega) \) and \( ||u||_{L^{\theta,\phi}(\Omega)} \leq C||u||_{\dot{B}^{\alpha,\phi}(\Omega)} \) as desired.

Under above assumptions on \( u \), we have \( u \in L^{\theta,\phi}(\Omega) \). Indeed, in the case \( |\Omega| < \infty \), the boundedness of \( u \) implies that \( u \in L^{\theta,\phi}(\Omega) \). In the case \( |\Omega| = \infty \), the assumption \( ||x \in \Omega : u(x) > 0|| < \infty \) and the boundedness of \( u \) also gives \( u \in L^{\theta,\phi}(\Omega) \). Write
\[
A_k := \{ z \in B : u(z) > 2^k \} \quad \text{and} \quad D_k := A_k \setminus A_k+1 = \{ z \in B : 2^k < u(z) \leq 2^{k+1}\},
\]
and \( a_k := |A_k| \) and \( d_k := |D_k| \) for \( k \in \mathbb{Z} \). Then
\[
||u||_{L^{\theta,\phi}(\Omega)} \leq \sum_{i \in \mathbb{Z}} d_i 2^{(i+1)n/\alpha} \leq 2^{n/\alpha} \sum_{i \in \mathbb{Z}} a_i 2^{in/\alpha}
\]
and
\[
(3.4) \quad \sum_{i \in \mathbb{Z}} a_i 2^{in/\alpha} \leq \sum_{j = 1} \sum_{i \in \mathbb{Z}} d_j 2^{jn/\alpha} \leq \sum_{j} d_j \sum_{i \in \mathbb{Z}} 2^{jn/\alpha} \leq \frac{1}{1 - 2^{n/\alpha}} \sum_{j} d_j 2^{jn/\alpha} \leq \frac{1}{1 - 2^{n/\alpha}} ||u||_{L^{\theta,\phi}(\Omega)}.
\]
On the other hand, observe that \( \{D_l\}_{l \in \mathbb{Z}} \), and hence \( \{D_l \times (\Omega \setminus A_{l-1})\}_{l \in \mathbb{Z}} \), are disjoint for each other, and that for any \((x, y) \in D_l \times (\Omega \setminus A_{l-1})\), we have \( u(x) \geq 2^l \) and \( u(y) \leq 2^{l-1} \), and hence \( |u(x) - u(y)| \geq 2^{l-1} \). Therefore,

\[
H := \int_{\Omega \setminus \Omega_{A_{l-1}}} \int_{\Omega_{A_{l-1}}} \phi \left( \frac{|u(x) - u(y)|}{d(x, y)} \right) \frac{dx}{x-y} \geq \sum_{l \in \mathbb{Z}} \sum_{l \neq l-1} \int_{\Omega \setminus \Omega_{A_{l-1}}} \int_{\Omega_{A_{l-1}}} \phi \left( \frac{2^{l-1}}{d(x, y)} \right) \frac{dy}{x-y} \int_{\Omega_{A_{l-1}}} \int_{\Omega_{A_{l-1}}} \phi \left( \frac{2^{l-1}}{d(x, y)} \right) \frac{dy}{x-y} \int_{\Omega_{A_{l-1}}} \int_{\Omega_{A_{l-1}}} \phi \left( \frac{2^{l-1}}{d(x, y)} \right) \frac{dy}{x-y} .
\]

If \( |\Omega| < \infty \), by (3.2), we know that \( a_k \leq \frac{1}{2}|\Omega| \) for all \( k \in \mathbb{Z} \). If \( |\Omega| = \infty \), then, by (3.3) we have \( a_l < \infty \) for all \( l \in \mathbb{Z} \). Thus, applying Lemma 1.5, we obtain

\[
H \geq C_1 \sum_{l \in \mathbb{Z} \setminus A_{l-1} \neq 0} \frac{d_l}{a_l-1} \frac{|\Omega \setminus A_{l-1}|}{|\Omega|} \phi \left( \frac{C_2 2^{l-1} a_l^{|\alpha|/n}}{\lambda} \right) \geq \frac{1}{2} C_1 \sum_{l \in \mathbb{Z} \setminus A_{l-1} \neq 0} \frac{d_l}{a_l-1} \phi \left( \frac{C_2 2^{l-1} a_l^{|\alpha|/n}}{\lambda} \right) .
\]

Let \( \lambda = M(\sum_{i \in \mathbb{Z}} a_i 2^{in/|\alpha|})^{1/n} \) with

\[
M = C_2 \left[ \frac{1}{\phi^{-1} \left( \frac{2^{3n+2n/|\alpha|} \Lambda_\phi(\alpha)}{C_1 (1 - 2^{n/|\alpha|})} \right)} \right]^{-1} .
\]

Noting \( \phi(sC_2/M) \geq 2^{-3n/|\alpha|} \Lambda_\phi(\alpha)^{1-1} \phi(C_2/M) \) for any \( s \in (0, 1) \) as given in (2.2), by (3.5) we obtain

\[
H \geq \frac{C_1}{2} \sum_{l \in \mathbb{Z} \setminus A_{l-1} \neq 0} \frac{d_l}{a_l-1} \phi \left( \frac{C_2 2^{l-1} a_l^{|\alpha|/n}}{M \left( \sum_{l \in \mathbb{Z}} 2^{n|\alpha|/|\alpha|} a_l \right)^{1/n}} \right) \geq \frac{C_1}{2^{3n+1} \Lambda_\phi(\alpha)} \sum_{l \in \mathbb{Z} \setminus A_{l-1} \neq 0} \frac{d_l}{a_l-1} \phi \left( \frac{C_2 \left( \sum_{i \in \mathbb{Z}} 2^{n|\alpha|/|\alpha|} a_i \right)^{1/n}}{M} \right) .
\]

Since \( a_{l-1} = 0 \) implies that \( d_l = 0 \), we have \( \sum_{l \in \mathbb{Z} \setminus A_{l-1} \neq 0} \frac{d_l / 2^{(l-1)n/|\alpha|}}{a_l-1} \leq \sum_{l \in \mathbb{Z}} d_l / 2^{(l-1)n/|\alpha|} \). Thus by (3.4),

\[
H \geq \frac{C_1}{2^{3n+1} \Lambda_\phi(\alpha)} \phi \left( \frac{C_2}{M} \right) = 2,
\]

which implies that \( M \left( \sum_{i \in \mathbb{Z}} a_i 2^{in/|\alpha|} \right)^{1/n} \leq \|u\|_{B^{\alpha, \phi}(\Omega)} \), that is, \( \|u\|_{L^{n/|\alpha|}(\Omega)} \leq 2M^{-1}\|u\|_{B^{\alpha, \phi}(\Omega)} \) as desired. The proof of Lemma 3.1 is completed. \( \square \)

From Lemma 3.1 and the media value we conclude the following Lemma 3.2.

**Lemma 3.2.** Let \( \alpha \in (-n, 0) \) and \( \phi \) be a Young function satisfying (1.3) and (1.4).

(i) If \( \Omega \subset \mathbb{R}^n \) is a bounded globally \( n \)-regular domain, then there exists a constant \( C > 0 \) depending only on \( n, \alpha, \phi \) and \( \phi \) such that

\[
\|u - u_\Omega\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{B^{\alpha, \phi}(\Omega)}, \quad \forall u \in \dot{B}^{\alpha, \phi}(\Omega).
\]

(ii) If \( \Omega \subset \mathbb{R}^n \) is an unbounded globally \( n \)-regular domain, then there exists a constant \( C > 0 \) depending only on \( n, \alpha, \phi \) and \( \phi \) such that

\[
\|u\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{B^{\alpha, \phi}(\Omega)}, \quad \forall u \in \dot{B}^{\alpha, \phi}(\Omega) \text{ with } \{x \in \Omega : |u| > a\} < \infty \text{ for all } a > 0.
\]

**Proof.** (i) Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded globally \( n \)-regular domain. For any \( u \in \dot{B}^{\alpha, \phi}(\Omega) \), set the median value

\[
m_u(\Omega) := \inf \left\{ c \in \mathbb{R} : \left| \{ x \in B : u > c \} \right| \leq \frac{1}{2}|\Omega| \right\}.
\]

Then

\[
|x \in \Omega : u > m_u(\Omega)| \leq \frac{1}{2}|\Omega| \quad \text{and} \quad |x \in \Omega : u < m_u(\Omega)| \leq \frac{1}{2}|\Omega|.
\]
Write $u_+ = [u - m_n(\Omega)]\chi_{\Omega \setminus m_n(\Omega)}$ and $u_- = -[u - m_n(\Omega)]\chi_{\Omega \setminus m_n(\Omega)}$. Then $u_\pm$ satisfies (3.2), and hence by Lemma 3.1, we obtain $\|u_\pm\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u_\pm\|_{B^{\alpha, \phi}(\Omega)}$.

On the other hand, note that $u - m_n(\Omega) = u_+ - u_-$ and

$$\|u - m_n(\Omega)\|_{L^{n/|\alpha|}(\Omega)} = \|u_+\|_{L^{n/|\alpha|}(\Omega)} + \|u_-\|_{L^{n/|\alpha|}(\Omega)}.$$  

Moreover, by Proof of Theorem 1.1.

Note that $\|u\|_{L^{n/|\alpha|}(\Omega)}$.

Then for any $z$ and $\rho$, we have $\|u\|_{L^{n/|\alpha|}(\Omega)}$. Combining with $\|u\|_{L^{n/|\alpha|}(\Omega)} = \|u_+\|_{L^{n/|\alpha|}(\Omega)} + \|u_-\|_{L^{n/|\alpha|}(\Omega)}$, we conclude $\|u\|_{L^{n/|\alpha|}(\Omega)} \leq C\|u\|_{B^{\alpha, \phi}(\Omega)}$ as desired. This completes the proof of Lemma 3.2.

Therefore Theorem 1.1 then follows from Lemma 3.2.

**Proof of Theorem 1.1.** Note that $\{B(z, R)\}_{z \in \mathbb{R}^n, R > 0}$ are globally n-regular domains with the same constant $\theta$. Indeed, let $\theta > 0$ such that

$$\|B(0, 1) \cap B(x, r)\| \geq \theta r^n$$

for all $x \in B(0, 1)$ and $r < 2$.

Then for any $z \in \mathbb{R}^n$ and $R > 0$, we have

$$\|B(z, R) \cap B(x, r)\| = \|B(0, R) \cap B(x - z, r/R)\| = R^n \|B(0, 1) \cap B((x - z)/R, r/R)\| \geq \theta r^n$$

whenever $0 < r < 2R$ and $x \in B(z, R)$. Thus by Lemma 3.2 (i) we know that there exists a constant $C > 0$ such that

$$\|u - u_B\|_{L^{n/|\alpha|}(B)} \leq C\|u\|_{B^{\alpha, \phi}(\Omega)}$$

as desired. Especially, given any $u \in \tilde{B}^{\alpha, \phi}(\mathbb{R}^n)$ we have

$$\|u - u_B\|_{L^{n/|\alpha|}(B)} \leq C\|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)} \forall k \in \mathbb{N}.$$  

Therefore

$$\|u_{B(0, 2^k)} - u_B\|_{L^{n/|\alpha|}(B)} \leq \omega_n^{n/|\alpha|} 2^{-k|\alpha|} \|u - u_{B(0, 2^{k+1})}\|_{L^{n/|\alpha|}(B(0, 2^{k+1}))} \leq C 2^{-k|\alpha|} \|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)}$$

for all $k \in \mathbb{N}$.

Thus $u_{B(0, 2^k)}$ converges to some $c \in \mathbb{R}$ as $k \to \infty$, and

$$\|u_{B(0, 2^k)} - c\|_{L^{n/|\alpha|}(B(0, 2^k))} \leq \sum_{l \leq k} |u_{B(0, 2^l)} - u_{B(0, 2^{l+1})}| \leq \sum_{l \leq k} C 2^{-l|\alpha|} \|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)} \leq C \frac{2^{-k|\alpha|}}{1 - 2^{-n}} \|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)}.$$  

Since

$$\|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \leq \|u - u_B\|_{L^{n/|\alpha|}(B(0, 2^k))} + |B(0, 2^k)|^{n/|\alpha|} \|u_{B(0, 2^k)} - c\| \leq C\|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)},$$

letting $k \to \infty$, we obtain $\|u - c\|_{L^{n/|\alpha|}(\mathbb{R}^n)} \leq C\|u\|_{B^{\alpha, \phi}(\mathbb{R}^n)}$ as desired. This completes the proof of Theorem 1.1. \qed
Remark 3.3. (i) Let \( 0 < s < 1 \) and \( 1 \leq p < n/s \). It is proved by [2, Corollary 25] and [17, Lemma A.1] that \( n \in \mathbb{R} \), \( \alpha = s - n/p \) and \( \phi(t) = t^p \). Using (3.6), Di Nezza et al [3] proved that

\[
\int_{\mathbb{R}^n \setminus E} |x - y|^{\alpha + sp} \, dy \geq \frac{C}{|E|^p} \quad \forall E \subset \mathbb{R}^n \text{ with } |E| < \infty,
\]

that is, Lemma 1.5 with \( \Omega = \mathbb{R}^n \), \( \alpha = s - n/p \) and \( \phi(t) = t^p \). Below assume that (3.3), and by borrowing some ideas from [8, 22], we will show that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy \geq \sum_{j \in \mathbb{Z}, a_{j-1} \neq 0} a_{j|a_{j-1}|^{sp}} \quad \forall u \in \dot{B}^s_{pp}(\mathbb{R}^n) \text{ having bounded supports},
\]

which is Lemma 3.1 with \( \Omega = \mathbb{R}^n \), \( \alpha = s - n/p \) and \( \phi(t) = t^p \) essentially. After several technical arguments, this allows them to obtain

\[
(3.7) \quad \|u\|_{L^{sp/(n-sp)}(\mathbb{R}^n)} \leq \|u\|_{\dot{B}^s_{pp}(\mathbb{R}^n)} \quad \forall u \in \dot{B}^s_{pp}(\mathbb{R}^n) \text{ having bounded supports},
\]

See [3, Section 6] for details. To get (3.7), our proof in Lemma 3.1 via Orlicz norm simplify the argument in [3, Section 6] by dropping several technical arguments therein.

(ii) Lemma 1.5 extends (3.6) not only to general \( \phi \) but also to globally \( n \)-regular domains \( \Omega \). Applying Lemma 1.5 and an argument simpler than [3, Section 6], we extend (3.7) to not only general \( \phi \) but also to globally \( n \)-regular domains \( \Omega \) as in Lemma 3.1. Moreover, with the aid of the median value, we further obtain the desired imbedding of \( B^{s, \phi}(\Omega) \) in Lemma 3.2. In particular, we give a new and direct proof to the well-known facts (1.5) and (1.6), and also (1.11) for bounded domains.

(iii) When \( \Omega \subset \mathbb{R}^n \) is a globally \( n \)-regular domain and \( |\Omega| \rightleftharpoons \infty \), the direct argument above fails to prove (1.12); the difficulty is to find a sequence domains \( \Omega_r \) which are globally \( n \)-regular with the same \( \theta \) so that \( \Omega_r \) is increasing and converges to \( \Omega \).

Now we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. If \( \Omega \subset \mathbb{R}^n \) is a globally \( n \)-regular domain, then (1.7) and (1.8) follows from Lemma 3.2 directly. Below assume that \( \Omega \subset \mathbb{R}^n \) is a domain satisfying (1.7) or (1.8). With the aid of Lemma 2.5, (3.2) and (3.3), and by borrowing some ideas from [8, 22], we will show that \( \Omega \) is globally \( n \)-regular. To this end, take arbitrary \( z \in \Omega \) and \( 0 < r < \frac{1}{2} \text{diam} \Omega \), and for \( j \geq 0 \), let \( 0 < b_j \leq 1 \) such that

\[
B(z, b_jr) \cap \Omega = \frac{1}{2^j} |B(z, r) \cap \Omega|.
\]

Obviously, \( 1 = b_0 > b_j > b_{j+1} > 0 \) for all \( j \geq 1 \) and

\[
|B(z, b_jr) \cap \Omega| = \frac{1}{2} |B(z, b_{j-1}r) \cap \Omega| \quad \forall j \geq 0.
\]

Case \( \text{diam} \Omega = \infty \). It suffices to prove that there exists a constant \( C > 0 \) independent of \( z, r \) such that

\[
(3.10) \quad \phi(C \max(B(z, b_jr)^{n/j}, (b_jr - b_{j+1}r)^n)^{-1}) \geq 1 \quad \forall j \geq 0.
\]

Indeed, since \( \phi(Cs^{-n})s^{-n} \rightarrow 0 \) as \( s \rightarrow \infty \) as given in Lemma 2.1 (ii), we know that \( \phi(Cs^{-n})s^{-n} \geq 1 \) implies that \( s \leq \Lambda_{C} \) for some constant \( \Lambda_{C} > 0 \). By this and (3.10), we obtain

\[
\frac{(b_jr - b_{j+1}r)}{|B_{\Omega}(z, b_jr)|^{1/n}} \leq \Lambda_{C}.
\]

This together with (3.8) yields that

\[
b_jr - b_{j+1}r \leq \Lambda_{C}|B_{\Omega}(z, b_jr)|^{1/n} = \Lambda_{C} 2^{-j/n}|B_{\Omega}(z, r)|^{1/n},
\]

which gives

\[
r = \sum_{j \geq 0} (b_jr - b_{j+1}r) \leq \sum_{j \geq 0} \Lambda_{C} 2^{-j/n}|B_{\Omega}(z, r)|^{1/n} = \Lambda_{C}|B_{\Omega}(z, r)|^{1/n}
\]

as desired.
To prove (3.10), for \( j \geq 0 \) let \( u_{z,b_{j+1}r} \) be the function defined by (2.3). By Lemma 2.5, we have \( u_{z,b_{j+1}r} \in \dot{B}^{\alpha,\phi}(\Omega) \) and
\[
\|u_{z,b_{j+1}r}\|_{\dot{B}^{\alpha,\phi}(\Omega)} \leq C(b_{j+1}r)^{-\alpha} \left[ \phi^{-1}\left( \frac{(b_{j+1}r)^n}{|B_\Omega(z,b_{j+1}r)|} \right) \right]^{-1}.
\]
Since \( \|x \in \Omega : u_{z,b_{j+1}r}(x) \neq 0\| < \infty \), by (1.8) we have
\[
\|u_{z,b_{j+1}r}\|_{L^p(\Omega)} \leq C\|u_{z,b_{j+1}r}\|_{\dot{B}^{\alpha,\phi}(\Omega)}.
\]
Note that
\[
\|u_{z,b_{j+1}r}\|_{L^p(\Omega)} \geq \|\Omega \cap B(x,b_{j+1}r)\|^{-\alpha/n} = 2^{\alpha/n}\|\Omega \cap B(x,b_{j+1}r)\|^{-\alpha/n}.
\]
We conclude that
\[
2^{\alpha/n}|B_\Omega(z,b_{j+1}r)|^{-\alpha/n} \leq C(b_{j+1}r)^{-\alpha} \left[ \phi^{-1}\left( \frac{(b_{j+1}r)^n}{|B_\Omega(z,b_{j+1}r)|} \right) \right]^{-1},
\]
which implies that
\[
\phi^{-1}\left( \frac{(b_{j+1}r)^n}{|B_\Omega(z,b_{j+1}r)|} \right) \leq C \frac{|B_\Omega(z,b_{j+1}r)|^{\alpha/n}}{(b_{j+1}r)^{\alpha}}.
\]

Case \( \text{diam}\ \Omega < \infty \). Note that by a similar argument as in the case \( |\Omega| = \infty \), we have \( b_{j+1}r < \Lambda C|B_\Omega(z,b_{j+1}r)|^{1/n} \). Indeed, for \( j \geq 1 \),
\[
(3.11)
\|u_{z,b_{j+1}r} - (u_{z,b_{j+1}r})_{\Omega}\|_{L^p(\Omega)} \geq \frac{1}{2}\|\Omega \cap B(x,b_{j+1}r)\|^{-\alpha/n} = \left( \frac{1}{2} \right)^{1-\alpha/n}\|\Omega \cap B(x,b_{j+1}r)\|^{-\alpha/n}.
\]
Note that \( u_{z,b_{j+1}r} - (u_{z,b_{j+1}r})_{\Omega} \geq \frac{1}{2} \) either in \( B_\Omega(x,b_{j+1}r) \) or in \( \Omega \setminus B_\Omega(x,b_{j+1}r) \). Since \( j \geq 1 \) implies that \( B_\Omega(x,b_{j+1}r) \setminus B_\Omega(x,b_{j+1}r) \subset \Omega \setminus B_\Omega(x,b_{j+1}r) \) and hence
\[
|\Omega \setminus B_\Omega(x,b_{j+1}r)| \geq |B_\Omega(x,b_{j+1}r) \setminus B_\Omega(x,b_{j+1}r)| = |B_\Omega(x,b_{j+1}r)| > |B_\Omega(x,b_{j+1}r)|,
\]
with \( |B_\Omega(x,b_{j+1}r)| = \frac{1}{2}|B_\Omega(x,b_{j+1}r)| \) we obtain (3.11). Then by the same argument as in the case \( \text{diam}(\Omega) = \infty \), we are able to prove that \( b_{j+1}r < \Lambda C|B_\Omega(z,b_{j+1}r)|^{1/n} \), where the value \( \Lambda C \) needed to be adjusted.

If \( b_{j+1} \geq 1/10 \), by \( b_{j+1} < \Lambda C|B_\Omega(z,b_{j+1}r)|^{1/n} \), we have \( r < 10\Lambda C|B_\Omega(z,r)|^{1/n} \) as desired. Assume that \( b_{j+1} < 1/10 \). Let \( R = \frac{2}{5}r \) and \( y \in B_\Omega(z,r) \) with \( |y - z| = b_{j+1}r + R/2 \). Note that since \( \Omega \) is path-connected and \( b_{j+1}r + R/2 < 3 \text{diam}\ \Omega/20 \), such \( y \) exists. Then \( B(z,b_{j+1}r) \subseteq B(y,R) \subseteq B(z,r) \), and \( B(z,b_{j+1}r) \cap B(y,R/2) = \emptyset \). Thus if \( |B_\Omega(y,b_{j+1}r)| = \frac{1}{2}|B_\Omega(z,y,R)|, \) by \( |B_\Omega(z,b_{j+1}r)| = \frac{1}{2}|B_\Omega(z,r)| > \frac{1}{2}|B_\Omega(y,R)| \), we have \( B_\Omega(y,b_{j+1}r) \cap B_\Omega(z,b_{j+1}r) = \emptyset \), so \( b_{j+1} \geq 1/2 \). Note that by the above argument, we already have \( b_{j+1}r < \Lambda C|B_\Omega(y,b_{j+1}r)|^{1/n} \). Hence
\[
\frac{2}{5}r = R \leq 2\Lambda C|B_\Omega(y,R)|^{1/n} \leq 2\Lambda C|B_\Omega(z,r)|^{1/n},
\]
which gives \( 5\Lambda C|B_\Omega(z,r)|^{1/n} \geq r \) as desired.

\(\square\)

\textbf{Proof of Corollary 1.3.} (i) Assume that \( \Omega \subset \mathbb{R}^n \) is a bounded \( n \)-regular domain. Then it is also bounded globally \( n \)-regular domain. Let \( u \in \dot{B}^{\alpha,\phi}(\Omega) \). Since Jessen’s inequality implies that
\[
\phi\left( \frac{|u|}{\lambda} \right) \leq \phi\left( \int_{\Omega} \frac{|u|}{\lambda} \, dx \right) \leq \int_{\Omega} \phi\left( \frac{|u|}{\lambda} \right) \, dx \leq \frac{1}{|\Omega|} \int_{\Omega} \phi\left( \frac{|u|}{\lambda} \right) \, dx \quad \forall \lambda > ||u||_{L^p(\Omega)},
\]
we have \( |u_{\Omega}| = ||u||_{L^p(\Omega)}\phi^{-1}(\Omega^{-1}) \). By Lemma 3.2 (i), we then have
\[
||u||_{L^p(\Omega)} \leq ||u - u_{\Omega}||_{L^p(\Omega)} + ||u_{\Omega}||_{\Omega^{1/n}} \leq C||u||_{\dot{B}^{\alpha,\phi}(\Omega)}
\]
as desired.

Conversely, assume that \( \Omega \subset \mathbb{R}^n \) is a bounded domain satisfying that
\[
||u||_{L^p(\Omega)} \leq C||u||_{\dot{B}^{0,\phi}(\Omega)} \quad \forall u \in \dot{B}^{\alpha,\phi}(\Omega)
\]
for some constant \( C > 0 \). Considering Lemma 2.6, by the same argument as in the case \( |\Omega| = \infty \) in Theorem 1.1, we show that \( \Omega \) is \( n \)-regular; the details are omitted. The proof of Corollary 1.3 is completed. \(\square\)
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References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Elsevier/Academic Press, Amsterdam, 2003.
[2] L. Caffarelli and E. Valdinoci, Uniform estimates and limiting arguments for nonlocal minimal surfaces, Calc. Var. Partial Differential Equations 41 (2011) 203–240.
[3] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136 (2012), 521–573.
[4] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001.
[5] A. Gogatishvili, P. Koskela and Y. Zhou, Characterizations of Besov and Triebel-Lizorkin spaces on metric measure spaces, Forum Math. 25 (2013), 787-819.
[6] P. Hajłasz, Sobolev spaces on an arbitrary metric spaces, Potential Anal. 5 (1996), 403-415.
[7] P. Hajłasz, Sobolev spaces on metric-measure spaces, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173-218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2004.
[8] P. Hajłasz, P. Koskela and H. Tuominen, Sobolev imbeddings, extensions and measure density condition, J. Funct. Anal. 254 (2008), 1217-1234.
[9] P. Hajłasz, P. Koskela and H. Tuominen, Measure density and extendability of Sobolev functions, Rev. Mat. Iberoam. 24 (2008), 645-669.
[10] T. Heikkinen, L. Ihnatsyeva and H. Tuominen, Measure density and extension of Besov and Triebel-Lizorkin functions J. Fourier Anal. Appl. 22 (2016), 334-382.
[11] A. Jonsson and H. Wallin, A Whitney extension theorem in $L^p$ and Besov spaces, Ann. Inst. Fourier (Grenoble) 28 (1978), 139-192.
[12] A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbb{R}^n$, Math. Rep. 2 (1984), no. 1, xiv+221 pp.
[13] T. Liang and Y. Zhou, Orlicz-Sobolev extension and Ahlfors $n$-regular domains, 2018. to appear.
[14] J. Peetre, New thoughts on Besov spaces, Duke University Mathematics Series, No. 1. Mathematics Department, Duke University, Durham, N.C., 1976. vi+305 pp.
[15] M. C. Piaggio, Orlicz spaces and the large scale geometry of Heintze groups, Math. Ann. 368 (2017), 433-481.
[16] O. Savin, E. Valdinoci, Density estimates for a nonlocal variational model via the Sobolev inequality, SIAM J. Math. Anal. 43 (2011), 2675-2687.
[17] O. Savin, E. Valdinoci, Density estimates for a variational model driven by the Gagliardo norm, J. Math. Pures Appl. (101) 2014, 1-26.
[18] P. Shvartsman, Local approximations and intrinsic characterizations of spaces of smooth functions on regular subsets of $\mathbb{R}^n$, Math. Nachr. 279 (2006), 1212-1241.
[19] P. Shvartsman, On extensions of Sobolev functions defined on regular subsets of metric measure spaces, Journal of Approximation Theory, 214 (2007), 139-161.
[20] D. Yang, New characterizations of Hajłasz-Sobolev spaces on metric spaces, Sci. China Ser. A 46 (2003), 675-689.
[21] H. Sun, Orlicz-Besov extension and imbedding, preprint.
[22] Y. Zhou, Fractional Sobolev extension and imbedding, Trans. Amer. Math. Soc. 367 (2015), 959-979.

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