Metrics that realize all Lorentzian holonomy algebras

Anton S. Galaev *

March 29, 2022

Dedicated to Dmitri Vladimirovich Alekseevsky on his 65th birthday

Abstract

All candidates to the weakly-irreducible not irreducible holonomy algebras of Lorentzian manifolds are known. In the present paper metrics that realize all these candidates as holonomy algebras are given. This completes the classification of the Lorentzian holonomy algebras. Also new examples of metrics with the holonomy algebras $g_2 \ltimes \mathbb{R}^7 \subset \mathfrak{so}(1, 8)$ and $\text{spin}(7) \ltimes \mathbb{R}^8 \subset \mathfrak{so}(1, 9)$ are constructed.

Keywords: Lorentzian manifold, holonomy algebra, local metric

Mathematical subject codes: 53C29, 53C50, 53B30

Introduction

The classification of the holonomy algebras for Riemannian manifolds is a well-known classical result. By the Borel-Lichnerowicz theorem a Riemannian manifold is locally a product of Riemannian manifolds with irreducible holonomy algebras ([9]). In 1955 M. Berger gave a list of possible irreducible holonomy algebras of Riemannian manifolds, see [8]. Later, in 1987 R. Bryant constructed metrics for the exceptional algebras of this list, see [13]. For more details see [1 7 21].

The classification problem for the holonomy algebras of pseudo-Riemannian manifolds is still open. The main difficulty is that the holonomy algebra can preserve an isotropic subspace of the tangent space. A subalgebra $g \subset \mathfrak{so}(r, s)$ is called weakly-irreducible if it does not preserve any nondegenerate proper subspace of $\mathbb{R}^{r,s}$. The Wu theorem states that a pseudo-Riemannian manifold is locally a product of pseudo-Riemannian manifolds with weakly-irreducible holonomy algebras, see [28]. So, it is enough to consider only weakly-irreducible holonomy algebras. If

*EMail: galaev@mathematik.hu-berlin.de
a holonomy algebra is irreducible, then it is weakly-irreducible. In [8] M. Berger gave also a classification of possible irreducible holonomy algebras for pseudo-Riemannian manifolds, but there is no classification for weakly-irreducible not irreducible holonomy algebras for general pseudo-Riemannian manifolds. About the Lorentzian case see below. There are some partial results for holonomy algebras of pseudo-Riemannian manifolds of signature \((2,N)\), see [20, 17, 18], and \((N,N)\), see [6].

We consider the holonomy algebras of Lorentzian manifolds. From Be rger’s list it follows that the only irreducible holonomy algebra of Lorentzian manifolds is \(\mathfrak{so}(1, n + 1)\), see [14] and [12] for direct proofs of this fact. In 1993 L. Berard Bergery and A. Ikemakhen classified weakly-irreducible not irreducible subalgebras of \(\mathfrak{so}(1, n + 1)\), see [5]. More precisely, they divided these subalgebras into 4 types, and associated to each such subalgebra \(\mathfrak{g} \subset \mathfrak{so}(1, n + 1)\) a subalgebra \(\mathfrak{h} \subset \mathfrak{so}(n)\), which is called the orthogonal part of \(\mathfrak{g}\). In [16] more geometrical proof of this result was given. The Lie algebras of type 1 and 2 have the forms \((\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n\) and \(\mathfrak{h} \ltimes \mathbb{R}^n\) respectively. The Lie algebras of type 3 and 4 can be obtained from the first two by some twistings. Recently T. Leistner proved that if \(\mathfrak{g} \subset \mathfrak{so}(1, n + 1)\) is the holonomy algebra of a Lorentzian manifold, then its orthogonal part \(\mathfrak{h} \subset \mathfrak{so}(n)\) is the holonomy algebra of a Riemannian manifold, see [22, 23, 24, 25]. This gives a list of possible holonomy algebras for Lorentzian manifolds (Berger algebras). To complete the classification of holonomy algebras one must prove that all Berger algebras can be realized as holonomy algebras. In [5] were given metrics that realize all Berger algebras of type 1 and 2. In [10, 11] Ch. Boubel studied possible shapes of local metrics for Lorentzian manifolds with weakly-irreducible not irreducible holonomy algebras. In particular, he gave equivalent conditions for such manifolds to have the holonomy of type 1, 2, 3 or 4 and parameterized the set of germs of metrics giving a holonomy algebra of each type. In [26] K. Sfetsos and D. Zoakos constructed metrics with the holonomy algebras \(\mathfrak{su}(2) \ltimes \mathbb{R}^4 \subset \mathfrak{so}(1,5)\), \(\mathfrak{su}(3) \ltimes \mathbb{R}^6 \subset \mathfrak{so}(1,7)\) and \(g_2 \ltimes \mathbb{R}^7 \subset \mathfrak{so}(1,8)\).

In the present paper we construct metrics that realize all Berger algebras as holonomy algebras. The method of the construction generalizes an example of A. Ikemakhen given in [19]. The coefficients of the constructed metrics are polynomial functions, hence the holonomy algebra at a point is generated by the values of the curvature tensor and of its derivatives at this point, and it can be computed. This completes the classification of the holonomy algebras of Lorentzian manifolds. As application we construct new examples of metrics with the holonomy algebras \(g_2 \ltimes \mathbb{R}^7 \subset \mathfrak{so}(1,8)\) and \(\mathfrak{spin}(7) \ltimes \mathbb{R}^8 \subset \mathfrak{so}(1,9)\).

Acknowledgements. I would like to thank Dmitri Vladimirovich Alekseevsky for introducing me to the Lorentzian holonomy algebras and for his careful attention to my work during the last four years. I am grateful to Charles Boubel, who took my attention to the problem of construction of metrics. I thank the Erwin Schrödinger Institute, where the work on this paper
1 Preliminaries

Let \((\mathbb{R}^{1,n+1}, \eta)\) be a Minkowski space of dimension \(n+2\), where \(\eta\) is a metric on \(\mathbb{R}^{n+2}\) of signature \((1, n+1)\). We fix a basis \(p, e_1, ..., e_n, q\) of \(\mathbb{R}^{1,n+1}\) such that the Gram matrix of \(\eta\) has the form
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & E_n & 0 \\
1 & 0 & 0
\end{pmatrix},
\]
where \(E_n\) is the \(n\)-dimensional identity matrix. We will denote by \(\mathbb{R}^n \subset \mathbb{R}^{1,n+1}\) the Euclidean subspace spanned by the vectors \(e_1, ..., e_n\).

**Definition 1** A subalgebra \(g \subset \mathfrak{so}(1,n+1)\) is called irreducible if it does not preserve any proper subspace of \(\mathbb{R}^{1,n+1}\); \(g\) is called weakly-irreducible if it does not preserve any non-degenerate proper subspace of \(\mathbb{R}^{1,n+1}\).

Obviously, if \(g \subset \mathfrak{so}(1,n+1)\) is irreducible, then it is weakly-irreducible.

Denote by \(\mathfrak{so}(1,n+1)_{\mathbb{R}p}\) the subalgebra of \(\mathfrak{so}(1,n+1)\) that preserves the isotropic line \(\mathbb{R}p\). The Lie algebra \(\mathfrak{so}(1,n+1)_{\mathbb{R}p}\) can be identified with the following matrix algebra
\[
\mathfrak{so}(1,n+1)_{\mathbb{R}p} = \left\{ \begin{pmatrix}
a & X & 0 \\
0 & A & -X^t \\
0 & 0 & -a
\end{pmatrix} \right\},
\]
where \(a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n)\).

The above matrix can be identified with the triple \((a,A,X)\). Define the following subalgebras of \(\mathfrak{so}(1,n+1)_{\mathbb{R}p}\), \(\mathcal{A} = \{(a,0,0)|a \in \mathbb{R}\}, \mathcal{K} = \{(0,A,0)|A \in \mathfrak{so}(n)\}\) and \(\mathcal{N} = \{(0,0,X)|X \in \mathbb{R}^n\}\).

We see that \(\mathcal{A}\) commutes with \(\mathcal{K}\), and \(\mathcal{N}\) is a commutative ideal. We also see that
\[
[(a,A,0), (0,0,X)] = (0,0,aX + AX).
\]

We have the decomposition
\[
\mathfrak{so}(1,n+1)_{\mathbb{R}p} = (\mathcal{A} \oplus \mathcal{K}) \ltimes \mathcal{N} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.
\]

If a weakly-irreducible subalgebra \(g \subset \mathfrak{so}(1,n+1)\) preserves a degenerate proper subspace \(U \subset \mathbb{R}^{1,n+1}\), then it preserves the isotropic line \(U \cap U^\perp\), and \(g\) is conjugated to a weakly-irreducible subalgebra of \(\mathfrak{so}(1,n+1)_{\mathbb{R}p}\).

Let \(\mathfrak{h} \subset \mathfrak{so}(n)\) be a subalgebra. Recall that \(\mathfrak{h}\) is a compact Lie algebra and we have the decomposition \(\mathfrak{h} = \mathfrak{h'} \oplus \mathfrak{z}(\mathfrak{h})\), where \(\mathfrak{h'}\) is the commutant of \(\mathfrak{h}\) and \(\mathfrak{z}(\mathfrak{h})\) is the center of \(\mathfrak{h}\) \((\mathfrak{27})\).

The following result is due to L. Berard Bergery and A. Ikemakhen.

**Theorem** \([5]\) A subalgebra \(g \subset \mathfrak{so}(1,n+1)_{\mathbb{R}p}\) is weakly-irreducible if and only if \(g\) belongs to one of the following types
type 1. $g^{1,h} = (\mathbb{R} \oplus h) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} a & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & -a \end{pmatrix} \middle| a \in \mathbb{R}, A \in h, X \in \mathbb{R}^n \right\}, \text{where } h \subset \mathfrak{so}(n)$

is a subalgebra;

type 2. $g^{2,h} = h \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} 0 & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & 0 \end{pmatrix} \middle| A \in h, X \in \mathbb{R}^n \right\};$

type 3. $g^{3,h,\varphi} = \{(\varphi(A), A, 0)|A \in h\} \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} \varphi(A) & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & -\varphi(A) \end{pmatrix} \middle| A \in h, X \in \mathbb{R}^n \right\},$

where $h \subset \mathfrak{so}(n)$ is a subalgebra with $\mathfrak{z}(h) \neq \{0\}$, and $\varphi : h \to \mathbb{R}$ is a non-zero linear map with $\varphi|_{h'} = 0$;

type 4. $g^{4,h,m,\psi} = \{(0, A, X + \psi(A))|A \in h, X \in \mathbb{R}^m\}$

$= \left\{ \begin{pmatrix} 0 & X & \psi(A) & 0 \\ 0 & A & 0 & -X^t \\ 0 & 0 & 0 & -\psi(A)^t \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| A \in h, X \in \mathbb{R}^m \right\}, \text{where } 0 < m < n \text{ is an integer},$

$h \subset \mathfrak{so}(m)$ is a subalgebra with $\dim \mathfrak{z}(h) \geq n - m$, and $\psi : h \to \mathbb{R}^{n-m}$ is a surjective linear map with $\psi|_{h'} = 0$.

Definition 2 The subalgebra $h \subset \mathfrak{so}(n)$ associated to a weakly-irreducible subalgebra $g \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}p}$ in the above theorem is called the orthogonal part of $g$.

Let $(M, g)$ be a Lorentzian manifold of dimension $n + 2$ and $g$ the holonomy algebra (that is the Lie algebra of the holonomy group) at a point $x \in M$. By Wu’s theorem (see [28]) $(M, g)$ is locally indecomposable, i.e. is not locally a product of two pseudo-Riemannian manifolds if and only if the holonomy algebra $g$ is weakly-irreducible. If the holonomy algebra $g$ is irreducible, then $g = \mathfrak{so}(T_xM, g_x)$ ([3]). So we may assume that it is weakly-irreducible and not irreducible.

Then $g$ preserves an isotropic line $\ell \subset T_xM$. We can identify the tangent space $T_xM$ with $\mathbb{R}^{1,n+1}$ such that $g_x$ corresponds to $\eta$ and $\ell$ corresponds to the line $\mathbb{R}p$. Then $g$ is identified with a weakly-irreducible subalgebra of $\mathfrak{so}(1, n + 1)_{\mathbb{R}p}$.

Let $W$ be a vector space and $\mathfrak{g} \subset \mathfrak{gl}(W)$ a subalgebra.

Definition 3 The vector space

$\mathcal{R}(\mathfrak{g}) = \{ R \in \text{Hom}(W \wedge W, \mathfrak{g}) | R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \text{ for all } u, v, w \in W \}$
is called the space of curvature tensors of type \( \mathfrak{f} \). Denote by \( L(\mathcal{R}(\mathfrak{f})) \) the vector subspace of \( \mathfrak{f} \) spanned by \( R(u \wedge v) \) for all \( R \in \mathcal{R}(\mathfrak{f}) \), \( u, v \in W \),

\[
L(\mathcal{R}(\mathfrak{f})) = \text{span}\{R(u \wedge v)|R \in \mathcal{R}(\mathfrak{f}), u, v \in W\}.
\]

A subalgebra \( \mathfrak{f} \subset \mathfrak{gl}(W) \) is called a Berger algebra if \( L(\mathcal{R}(\mathfrak{f})) = \mathfrak{f} \).

From the Ambrose-Singer theorem ([2]) it follows that if \( \mathfrak{g} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}^p} \) is the holonomy algebra of a Lorentzian manifold, then \( \mathfrak{g} \) is a Berger algebra.

Let \( \mathfrak{h} \subset \mathfrak{so}(n) \) be a subalgebra.

**Definition 4** The vector space

\[
\mathcal{P}(\mathfrak{h}) = \{P \in \text{Hom}(\mathbb{R}^n, \mathfrak{h})|\eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \text{ for all } u, v, w \in \mathbb{R}^n\}
\]

is called the space of weak-curvature tensors of type \( \mathfrak{h} \). A subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) is called a weak-Berger algebra if \( L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h} \), where

\[
L(\mathcal{P}(\mathfrak{h})) = \text{span}\{P(u)|P \in \mathcal{P}(\mathfrak{h}), u \in \mathbb{R}^n\}
\]

is the vector subspace of \( \mathfrak{h} \) spanned by \( P(u) \) for all \( P \in \mathcal{P}(\mathfrak{h}) \) and \( u \in \mathbb{R}^n \).

The following theorem was proved in [15].

**Theorem [15]** A weakly-irreducible subalgebra \( \mathfrak{g} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}^p} \) is a Berger algebra if and only if its orthogonal part \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a weak-Berger algebra.

Recently T. Leistner proved the following theorem.

**Theorem [22, 23, 24, 25]** A subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a weak-Berger algebra if and only if \( \mathfrak{h} \) is a Berger algebra.

Recall that from the classification of Riemannian holonomy algebras it follows that a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a Berger algebra if and only if \( \mathfrak{h} \) is the holonomy algebra of a Riemannian manifold.

Thus a subalgebra \( \mathfrak{g} \subset \mathfrak{so}(1, n + 1) \) is a weakly-irreducible not irreducible Berger algebra if and only if \( \mathfrak{g} \) is conjugated to one of the subalgebras \( \mathfrak{g}^{1, \mathfrak{h}}, \mathfrak{g}^{2, \mathfrak{h}}, \mathfrak{g}^{3, \mathfrak{h}, \psi}, \mathfrak{g}^{4, \mathfrak{h}, m, \psi} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}^p} \), where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is the holonomy algebra of a Riemannian manifold.

To complete the classification of Lorentzian holonomy algebras we must realize all weakly-irreducible Berger subalgebra of \( \mathfrak{so}(1, n + 1)_{\mathbb{R}^p} \) as the holonomy algebras. There are some examples.

**Example [5].** In 1993 L. Berard Bergery and A. Ikemakhen realized the weakly-irreducible Berger subalgebra of \( \mathfrak{so}(1, n + 1)_{\mathbb{R}^p} \) of type 1 and 2 as the holonomy algebras. They constructed
the following metrics. Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be the holonomy algebra of a Riemannian manifold. Let $x^0, x^1, ..., x^n, x^{n+1}$ be the standard coordinates on $\mathbb{R}^{n+2}$, $h$ be a metric on $\mathbb{R}^n$ with the holonomy algebra $\mathfrak{h}$, and $f(x^0, ..., x^{n+1})$ be a function with $\frac{\partial f}{\partial x^1} \neq 0, ..., \frac{\partial f}{\partial x^n} \neq 0$. If $\frac{\partial f}{\partial x^1} = 0$, then the holonomy algebra of the metric

$$g = 2dx^0dx^{n+1} + h + f \cdot (dx^{n+1})^2$$

is $\mathfrak{g}^{1, \mathfrak{h}}$. If $\frac{\partial f}{\partial x^1} = 0$, then the holonomy algebra of the metric $g$ is $\mathfrak{g}^{2, \mathfrak{h}}$.

In the next section we will construct metrics that realize all weakly-irreducible Berger algebras. We will use the space $\mathcal{P}(\mathfrak{h})$ and the fact that $\mathfrak{h} = L(\mathcal{P}(\mathfrak{h}))$. The idea of the constructions is given by the following example of A. Ikemakhen.

**Example [19].** Let $x^0, x^1, ..., x^5, x^6$ be the standard coordinates on $\mathbb{R}^7$. Consider the following metric

$$g = 2dx^0dx^6 + \sum_{i=1}^{5}(dx^i)^2 + 2\sum_{i=1}^{5}u^i dx^i dx^6,$$

where

$$u^1 = -(x^3)^2 - 4(x^4)^2 - (x^5)^2, \quad u^2 = u^4 = 0,$$

$$u^3 = -2\sqrt{3}x^2x^3 - 2x^4x^5, \quad u^5 = 2\sqrt{3}x^2x^5 + 2x^3x^4.$$  

The holonomy algebra of this metric at the point 0 is $\mathfrak{g}^{2, \rho(\mathfrak{so}(3))} \subset \mathfrak{so}(1, 6)$, where $\rho: \mathfrak{so}(3) \rightarrow \mathfrak{so}(5)$ is the representation given by the highest irreducible component of the representation $\otimes^2 \text{id} : \mathfrak{so}(3) \rightarrow \otimes^2 \mathfrak{so}(3)$. The image $\rho(\mathfrak{so}(3)) \subset \mathfrak{so}(5)$ is spanned by the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 1 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \sqrt{3} \\ 0 & 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \sqrt{3} & 1 & 0 & 0 \end{pmatrix}.$$

We have $\text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}))_0 = A_1$, $\text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^4}))_0 = A_2$, $\text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^5}))_0 = A_3$ and $\text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^6}))_0 = \text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^6}))_0 = 0$.

Note the following. Let $P \in \text{Hom}(\mathbb{R}^n, \mathfrak{h})$ be a linear map defined as follows $P(e_1) = P(e_2) = 0$, $P(e_3) = A_1$, $P(e_4) = A_2$ and $P(e_5) = A_3$. We have $P \in \mathcal{P}(\mathfrak{h})$, $P(\mathbb{R}^n) = \mathfrak{h}$ and $\text{pr}_{\mathfrak{so}(n)}(R(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^5}))_0 = P(e_i)$ for all $1 \leq i \leq 5$.

## 2 Main results

In this section we will construct metrics that for any Riemannian holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ realize the Lie algebras $\mathfrak{g}^{1, \mathfrak{h}}$, $\mathfrak{g}^{2, \mathfrak{h}}$, $\mathfrak{g}^{3, \mathfrak{h}, \varphi}$ and $\mathfrak{g}^{4, \mathfrak{h}, m, \psi}$ (if $\mathfrak{g}^{3, \mathfrak{h}, \varphi}$ and $\mathfrak{g}^{4, \mathfrak{h}, m, \psi}$ exist) as holonomy algebras.
Recall that the Lie algebra \( g^{3,\mathbb{h},\mathbb{p}} \) exists only for \( \mathbb{h} \subset \mathfrak{so}(n) \) with \( \mathfrak{z}(\mathbb{h}) \neq \{0\} \) and the Lie algebra \( g^{4,\mathbb{h},\mathbb{m},\mathbb{p}} \) exists only for \( \mathbb{h} \subset \mathfrak{so}(m) \) with \( \dim \mathfrak{z}(\mathbb{h}) \geq n - m \).

**Constructions of the metrics.** Let \( \mathbb{h} \subset \mathfrak{so}(n) \) be the holonomy algebra of a Riemannian manifold. The Borel-Lichnerowicz theorem \([9]\) states that we have an orthogonal decomposition

\[
\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_s} \oplus \mathbb{R}^{n_{s+1}}
\]

and the corresponding decomposition into the direct sum of ideals

\[
\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s \oplus \{0\}
\]

such that \( \mathfrak{h} \) annihilates \( \mathbb{R}^{n_{s+1}} \), \( \mathfrak{h}_i(\mathbb{R}^{n_i}) = 0 \) for \( i \neq j \), and \( \mathfrak{h}_i \subset \mathfrak{so}(n_i) \) is an irreducible subalgebra for \( 1 \leq i \leq s \). Moreover, the Lie algebras \( \mathfrak{h}_i \) are the holonomy algebras of Riemannian manifolds. Note that we have \([22, 15]\)

\[
\mathcal{P}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h}_1) \oplus \cdots \oplus \mathcal{P}(\mathfrak{h}_s).
\]

We will assume that the basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) is compatible with the above decomposition of \( \mathbb{R}^n \).

Let \( n_0 = n_1 + \cdots + n_s = n - n_{s+1} \). We see that \( \mathfrak{h} \subset \mathfrak{so}(n_0) \) and \( \mathfrak{h} \) does not annihilate any proper subspace of \( \mathbb{R}^{n_0} \). Note that in the case of the Lie algebra \( g^{4,\mathbb{h},\mathbb{m},\mathbb{p}} \) we have \( 0 < n_0 \leq m \).

We will always assume that the indices \( b, c, d, f \) run from 0 to \( n + 1 \), the indices \( i, j, k, l \) run from 1 to \( n \), the indices \( \hat{i}, \hat{j}, \hat{k}, \hat{l} \) run from 1 to \( n_0 \), the indices \( \hat{i}, \hat{j}, \hat{k}, \hat{l} \) run from \( n_0 + 1 \) to \( n \), and the indices \( \alpha, \beta, \gamma \) run from 1 to \( N \). In case of the Lie algebra \( g^{4,\mathbb{h},\mathbb{m},\mathbb{p}} \) we will also assume that the indices \( \hat{i}, \hat{j}, \hat{k}, \hat{l} \) run from \( n_0 + 1 \) to \( m \) and the indices \( \hat{i}, \hat{j}, \hat{k}, \hat{l} \) run from \( m + 1 \) to \( n \). We will use the Einstein rule for sums.

Let \( (P_\alpha)_{\alpha=1}^N \) be linearly independent elements of \( \mathcal{P}(\mathfrak{h}) \) such that the subset \( \{P_\alpha(u)|1 \leq \alpha \leq N, u \in \mathbb{R}^{n_0}\} \subset \mathfrak{h} \) generates the Lie algebra \( \mathfrak{h} \). For example, it can be any basis of the vector space \( \mathcal{P}(\mathfrak{h}) \). We have \( P_\alpha|_{\mathbb{R}^{n_{s+1}}} = 0 \) and \( P_\alpha \) can be considered as linear maps \( P_\alpha : \mathbb{R}^{n_0} \to \mathfrak{h} \subset \mathfrak{so}(n_0) \). For each \( P_\alpha \) define the numbers \( P^k_{\alpha ji} \) such that \( P_\alpha(e_i)e_j = P^k_{\alpha ji}e_k \). Since \( P_\alpha \in \mathcal{P}(\mathfrak{h}) \), we have

\[
P^\hat{j}_{\alpha k\hat{i}} = -P^k_{\alpha j\hat{i}} \quad \text{and} \quad P^k_{\alpha ji} + P^i_{\alpha jk} + P^\hat{i}_{\alpha k\hat{i}} = 0.
\]

Define the following numbers

\[
a^k_{\alpha ji} = \frac{1}{3 \cdot (\alpha - 1)!} \left( P^k_{\alpha ji} + P^k_{\alpha j\hat{i}} \right).
\]

We have

\[
a^k_{\alpha ji} = a^\hat{k}_{\alpha j\hat{i}}.
\]

From (4) it follows that

\[
P^k_{\alpha ji} = (\alpha - 1)! \left( a^k_{\alpha ji} - a^\hat{k}_{\alpha k\hat{i}} \right) \quad \text{and} \quad a^k_{\alpha ji} + a^\hat{k}_{\alpha k\hat{i}} + a^\hat{i}_{\alpha k\hat{i}} = 0.
\]
Let $x^0, ..., x^{n+1}$ be the standard coordinates on $\mathbb{R}^{n+2}$. Consider the following metric

\[ g = 2dx^0dx^{n+1} + \sum_{i=1}^{n}(dx^i)^2 + 2\sum_{i=1}^{n_0}u^i dx^i dx^{n+1} + f \cdot (dx^{n+1})^2, \quad (8) \]

where

\[ u^i = a^i_{\alpha j k} x^j x^k (x^{n+1})^{\alpha - 1} \quad (9) \]

and $f$ is a function that will depend on the type of the holonomy algebra that we wish to obtain.

For the Lie algebra $\mathfrak{g}^{3, h, \varphi}$ (if it exists) define the numbers

\[ \varphi_{\alpha i} = \frac{1}{(\alpha - 1)!} \varphi(P_\alpha(e_i)) \quad (10) \]

For the Lie algebra $\mathfrak{g}^{4, h, m, \psi}$ (if it exists) define the numbers $\psi_{\alpha i}$ such that

\[ \frac{1}{(\alpha - 1)!} \psi(P_\alpha(e_i)) = \sum_{\hat{i} = m+1}^{n} \psi_{\alpha i}(e_{\hat{i}}) \quad (11) \]

Suppose that $f(0) = 0$, then $g_0 = \eta$ and we can identify the tangent space to $\mathbb{R}^{n+2}$ at 0 with the vector space $\mathbb{R}^{1, n+1}$.

**Theorem 1** The holonomy algebra $\mathfrak{hol}_0$ of the metric $g$ at the point $0 \in \mathbb{R}^{n+2}$ depends on the function $f$ in the following way:

| $f$ | $\mathfrak{hol}_0$ |
| --- | --- |
| $(x^0)^2 + \sum_{i=n_0+1}^{n}(x^i)^2$ | $\mathfrak{g}^1, h$ |
| $\sum_{i=n+1}^{n_0}(x^i)^2$ | $\mathfrak{g}^2, h$ |
| $2x^0\varphi_{\alpha i} x^i (x^{n+1})^{\alpha - 1} + \sum_{i=n_0+1}^{n}(x^i)^2$ | $\mathfrak{g}^{3, h, \varphi}$ (if $3(h) \neq \{0\}$) |
| $2\psi_{\alpha i} x^i (x^{n+1})^{\alpha - 1} + \sum_{i=n_0+1}^{n}(x^i)^2$ | $\mathfrak{g}^{4, h, m, \psi}$ (if $\dim 3(h) \geq n - m$) |

As the corollary we get the classification of the weakly-irreducible not irreducible Lorentzian holonomy algebras.

**Theorem 2** A weakly-irreducible not irreducible subalgebra $\mathfrak{g} \subset \mathfrak{so}(1, n + 1)$ is the holonomy algebra of a Lorentzian manifold if and only if $\mathfrak{g}$ is conjugated to one of the subalgebras $\mathfrak{g}^{1, h}, \mathfrak{g}^{2, h}, \mathfrak{g}^{3, h, \varphi}, \mathfrak{g}^{4, h, m, \psi} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}^p}$, where $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.

From theorem [2] Wu’s theorem and Berger’s list it follows that the holonomy algebra $\mathfrak{hol} \subset \mathfrak{so}(1, N + 1)$ of any Lorentzian manifold of dimension $N + 2$ has the form $\mathfrak{hol} = \mathfrak{g} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$,
where either \( g = \mathfrak{so}(1, n + 1) \) or \( g \) is a Lie algebra from theorem 2 and \( h_i \subset \mathfrak{so}(n_i) \) are the irreducible holonomy algebras of Riemannian manifolds (\( N = n + n_1 + \cdots + n_r \)).

**Explanation of the idea of the constructions.** Now we compare our method of constructions with the example of A. Ikemakhen. Let us construct the metric for \( g^2, \rho(\mathfrak{so}(3)) \subset \mathfrak{so}(1, 6) \) by our method. Take \( P \in \mathcal{P}(\rho(\mathfrak{so}(3))) \) defined as \( P(e_1) = P(e_2) = 0, P(e_3) = A_1, P(e_4) = A_2 \) and \( P(e_5) = A_3 \). By our constructions, we have

\[
g = 2dx^0dx^6 + \sum_{i=1}^{5}(dx^i)^2 + 2\sum_{i=1}^{5}u^i dx^i dx^6,
\]

where

\[
u^1 = -\frac{2}{3}(x^3)^2 + 4(x^4)^2 + (x^5)^2), \quad u^2 = \frac{2\sqrt{2}}{3}(x^3)^2 - (x^5)^2,\
u^3 = \frac{2}{3}(x^1x^3 - \sqrt{3}x^2x^3 - 3x^4x^5 - (x^5)^2), \quad u^4 = \frac{2}{3}x^1x^4,\
u^5 = \frac{2}{3}(x^1x^5 + \sqrt{3}x^2x^5 + 3x^3x^4 + x^3x^5).
\]

We still have \( \text{pr}_{\mathfrak{so}(n)}(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_0) = A_1, \text{pr}_{\mathfrak{so}(n)}(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_0) = A_2, \text{pr}_{\mathfrak{so}(n)}(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_0) = A_3 \) and \( \text{pr}_{\mathfrak{so}(n)}(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_0) = 0 \).

The reason why we obtain another metric is the following. The idea of our constructions is to find the constants \( a^k_{ij} \) such that

\[
\text{pr}_{\mathfrak{so}(n)}\left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_0\right) = P_1(e_i),
\]

\[\cdots,\]

\[
\text{pr}_{\mathfrak{so}(n)}\left(\nabla^{N-1}R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \cdots, \frac{\partial}{\partial x^{n+1}}\right)_0\right) = P_N(e_i).
\]

These conditions give us the system of equations

\[
\left\{
\begin{array}{c}
(\alpha - 1)! \left( a^k_{ij} - a^k_{ji} \right) = \frac{\partial^2 f}{\partial x^0} \\
\alpha a^k_{ij} = 0.
\end{array}
\right.
\]

One of the solutions of this system is given by (5), but this system can have other solutions. In the example we use the solution given by (5), taking another solution of the above system, we can obtain the metric constructed by A. Ikemakhen.

Thus the choice of the functions \( u^i \) given by (5) guarantees us that the orthogonal part of the holonomy algebra \( \mathfrak{hol}_0 \) coincides with the given Riemannian holonomy algebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) (the other values of \( \text{pr}_{\mathfrak{so}(n)}(\nabla^\alpha R) \) does not give us anything new). This also guarantees us the inclusion \( \mathbb{R}^{n_0} \subset \mathfrak{hol}_0 \). The reason why we choose the function \( f \) as in theorem 1 can be easily understood from the following formulas

\[
\text{pr}_{\mathbb{R}}\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^{n+1}}\right)_0 = \frac{1}{2}(\frac{\partial^2 f}{\partial x^0})^2 \quad \text{(we use this for } \mathfrak{g}^{1,h}),
\]

\[
\text{pr}_{\mathbb{R}}\left(\nabla^{\alpha-1}R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{n+1}}, \cdots, \frac{\partial}{\partial x^{n+1}}\right)_0 = \frac{1}{2}(\frac{\partial^{\alpha+1} f}{\partial x^0})^2 \quad \text{(we use this for } \mathfrak{g}^{3,h,p}).
\]
where $\text{Lie algebra } g$

The holonomy algebra of the metric

Consider the linear map $P$ (7) 

It can be checked that $\text{dim } 32 \subset \text{so}(7)$ 

Example (Metric with the holonomy algebra $g^{2, g_2} \subset \text{so}(1, 8)$). Consider the Lie subalgebra $g_2 \subset \text{so}(7)$. The vector subspace $g_2 \subset \text{so}(7)$ is spanned by the following matrices (8) 

$A_1 = E_{12} - E_{34}, \quad A_2 = E_{12} - E_{56}, \quad A_3 = E_{13} + E_{24}, \quad A_4 = E_{13} - E_{67}, \quad A_5 = E_{14} - E_{23},$

$A_6 = E_{14} - E_{57}, \quad A_7 = E_{15} + E_{26}, \quad A_8 = E_{15} + E_{47}, \quad A_9 = E_{16} - E_{25}, \quad A_{10} = E_{16} + E_{37},$

$A_{11} = E_{17} - E_{36}, \quad A_{12} = E_{17} - E_{45}, \quad A_{13} = E_{27} - E_{35}, \quad A_{14} = E_{27} + E_{46},$

where $E_{ij} \in \text{so}(7)$ ($i < j$) is the skew-symmetric matrix such that $(E_{ij})_{ii} = 1$, $(E_{ij})_{ji} = -1$ and $(E_{ij})_{kl} = 0$ for other $k$ and $l$.

Consider the linear map $P \in \text{Hom}(\mathbb{R}^7, g_2)$ defined as

$P(e_1) = A_6, \quad P(e_2) = A_4 + A_5, \quad P(e_3) = A_1 + A_7, \quad P(e_4) = A_1,$

$P(e_5) = A_4, \quad P(e_6) = -A_5 + A_6, \quad P(e_7) = A_7.$

It can be checked that $P \in \mathcal{P}(g_2)$. Moreover, the elements $A_1, A_4, A_5, A_6, A_7 \in g_2$ generate the Lie algebra $g_2$.

The holonomy algebra of the metric

$g = 2dx^0 dx^8 + \sum_{i=1}^{7} (dx^i)^2 + 2 \sum_{i=1}^{7} u^i dx^i dx^8,$

where

$u^1 = \frac{2}{3}(2x^2 x^3 + x^1 x^4 + 2x^2 x^4 + 2x^3 x^5 + x^5 x^7),$  

$u^2 = \frac{2}{3}(-x^1 x^3 - x^2 x^3 - x^1 x^4 + 2x^3 x^6 + x^6 x^7),$  

$u^3 = \frac{2}{3}(-x^1 x^2 + x^2 x^3 - x^3 x^4 - x^4 x^5 - x^5 x^6),$  

$u^4 = \frac{2}{3}(-x^1 x^2 - x^1 x^2 + (x^3)^2 + x^3 x^4),$  

$u^5 = \frac{2}{3}(-x^1 x^3 - 2x^1 x^7 - x^6 x^7),$  

$u^6 = \frac{2}{3}(-x^2 x^3 - 2x^2 x^7 - x^5 x^7),$  

at the point $0 \in \mathbb{R}^9$ is $g^{2, g_2} \subset \text{so}(1, 8)$.

Using computer it can be checked that $\text{dim } \mathcal{P}(g_2) = 64$. This means that we can construct quite a big number of metrics with the holonomy algebra $g^{2, g_2} \subset \text{so}(1, 8)$.

Example (Metric with the holonomy algebra $g^{2, \text{spin}(7)} \subset \text{so}(1, 9)$). Consider the Lie subalgebra $\text{spin}(7) \subset \text{so}(8)$. The vector subspace $\text{spin}(7) \subset \text{so}(8)$ is spanned by the following
matrices \(^{(4)}\).

\[
A_1 = E_{12} + E_{34}, \quad A_2 = E_{13} - E_{24}, \quad A_3 = E_{14} + E_{23}, \quad A_4 = E_{56} + E_{78}, \quad A_5 = -E_{57} + E_{68}, \\
A_6 = E_{58} + E_{67}, \quad A_7 = -E_{15} + E_{26}, \quad A_8 = E_{12} + E_{56}, \quad A_9 = E_{16} + E_{25}, \quad A_{10} = E_{37} - E_{48}, \\
A_{11} = E_{38} + E_{47}, \quad A_{12} = E_{17} + E_{28}, \quad A_{13} = E_{18} - E_{27}, \quad A_{14} = E_{35} + E_{46}, \quad A_{15} = E_{36} - E_{45}, \\
A_{16} = E_{18} + E_{36}, \quad A_{17} = E_{17} + E_{35}, \quad A_{18} = E_{26} - E_{48}, \quad A_{19} = E_{25} + E_{38}, \quad A_{20} = E_{23} + E_{67}, \\
A_{21} = E_{24} + E_{57}.
\]

Consider the linear map \(P \in \text{Hom}(\mathbb{R}^8, \text{spin}(7))\) defined as

\[
P(e_1) = 0, \quad P(e_2) = -A_{14}, \quad P(e_3) = 0, \quad P(e_4) = A_{21}, \\
P(e_5) = A_{20}, \quad P(e_6) = A_{21} - A_{18}, \quad P(e_7) = A_{15} - A_{16}, \quad P(e_7) = A_{14} - A_{17}.
\]

It can be checked that \(P \in \mathcal{P}(\text{spin}(7))\). Moreover, the elements \(A_{14}, A_{15} - A_{16}, A_{17}, A_{18}, A_{20}, A_{21} \in \text{spin}(7)\) generate the Lie algebra \(\text{spin}(7)\).

The holonomy algebra of the metric

\[
g = 2dx^0 dx^9 + \sum_{i=1}^{8} (dx^i)^2 + 2 \sum_{i=1}^{8} u^i dx^i dx^9,
\]

where

\[
u^1 = -\frac{4}{3} x^7 x^8, \quad \nu^2 = \frac{2}{3} (x^4)^2 + x^3 x^5 + x^4 x^6 - (x^6)^2, \\
u^3 = -\frac{4}{3} x^2 x^5, \quad \nu^4 = \frac{2}{3} (-x^2 x^4 - 2 x^2 x^6 - x^5 x^7 + 2 x^6 x^8), \\
u^5 = \frac{2}{3} (x^2 x^3 + 2 x^4 x^7 + x^6 x^7), \quad \nu^6 = \frac{2}{3} (x^2 x^4 + x^2 x^6 + x^5 x^7 - x^4 x^8), \\
u^7 = \frac{2}{3} (-x^4 x^5 - 2 x^5 x^6 + x^1 x^8), \quad \nu^8 = \frac{2}{3} (-x^4 x^6 + x^1 x^7),
\]

at the point \(0 \in \mathbb{R}^9\) is \(g^{2,\text{spin}(7)} \subset \mathfrak{so}(1,9)\).

Note that \(\text{dim} \mathcal{P}(\text{spin}(7)) = 112\).

**Proof of theorem** \(^{(4)}\). We will prove the theorem for the case of algebras of type 4. For other types the proof is analogous.

Since the coefficients of the metric \(g\) are polynomial functions, the Levi-Civita connection given by \(g\) is analytic and the Lie algebra \(\mathfrak{hol}_0\) is generated by the operators

\[
R(X, Y)_0, \nabla R(X, Y; Z_1)_0, \nabla^2 R(X, Y; Z_1; Z_2)_0, ... \in \mathfrak{so}(1, n + 1),
\]

where \(\nabla^r R(X, Y; Z_1; ...; Z_r) = (\nabla_{Z_r} \cdots \nabla_{Z_1} R)(X, Y)\) and \(X, Y, Z_1, Z_2, ...\) are vectors at the point 0.

To prove the theorem we will compute the components of the curvature tensor and of its derivatives.
The non-zero Christoffel symbols for the metric $g$ given by (8) are the following

\[
\Gamma^0_{0n+1} = \frac{1}{2} \frac{\partial f}{\partial x^0}, \\
\Gamma^0_{n+1n+1} = \frac{1}{2} \left( -\sum_{i=1}^{n_0} u^i \left( 2 \frac{\partial u^i}{\partial x^{n+1}} - \frac{\partial f}{\partial x^{n+1}} \right) + \frac{\partial f}{\partial x^0} \left( f - \sum_{i=1}^{n_0} (u^i)^2 \right) \right), \\
\Gamma^0_{ij} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right), \\
\Gamma^0_{in+1} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^{n+1}} - \frac{\partial u^{n+1}}{\partial x^i} \right) + \frac{1}{2} \frac{\partial f}{\partial x^0}, \\
\Gamma^0_{i,n+1} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i} \right), \\
\Gamma^i_{jn+1} = \frac{1}{2} \left( \frac{\partial u^i}{\partial x^{n+1}} - \frac{\partial f}{\partial x^i} + u^i \frac{\partial f}{\partial x^0} \right), \\
\Gamma^i_{n+1n+1} = -\frac{1}{2} \frac{\partial f}{\partial x^i}, \\
\Gamma^{n+1}_{n+1} = -\frac{1}{2} \frac{\partial f}{\partial x^0}.
\]

Suppose that dim $\mathfrak{z}(\mathfrak{h}) \geq n - m$. Let $f = 2\psi^i x^i x^j (x^{n+1})^\alpha - 1 + \sum_{i=n_0+1}^{m} (x^i)^2$. We must prove that $\mathfrak{o}_0 = g^{4,h,m,\psi}$. We have the following non-zero Christoffel symbols

\[
\Gamma^0_{n+1n+1} = -\sum_{i=1}^{n_0} u^i \left( (\alpha - 1) a^{i}_{\alpha jk} x^j x^k (x^{n+1})^{\alpha - 2} - \psi_{\alpha ii} x^i x^j (x^{n+1})^{\alpha - 1} \right) + (\alpha - 1) \psi_{\alpha ii} x^j x^i (x^{n+1})^{\alpha - 2}, (12)
\]
\[
\Gamma^0_{ij} = \frac{1}{2} \left( a^i_{\alpha jk} + a^i_{\alpha ik} \right) x^k (x^{n+1})^{\alpha - 1}, (13)
\]
\[
\Gamma^0_{i,n+1} = \frac{1}{(\alpha - 1)!} u^j \frac{\partial^i}{\alpha jk} x^k (x^{n+1})^{\alpha - 1} + \psi_{\alpha ii} x^j (x^{n+1})^{\alpha - 1}, (14)
\]
\[
\Gamma^i_{jn+1} = \frac{1}{(\alpha - 1)!} \frac{\partial^i}{\alpha jk} x^k (x^{n+1})^{\alpha - 1}, (15)
\]
\[
\Gamma^0_{i,n+1} = x^i, (16)
\]
\[
\Gamma^0_{i,n+1} = \psi_{\alpha ii} x^i (x^{n+1})^{\alpha - 1}, (17)
\]
\[
\Gamma^i_{n+1n+1} = -x^i, (18)
\]
\[
\Gamma^{n+1}_{n+1} = -\psi_{\alpha ii} x^i (x^{n+1})^{\alpha - 1}. (19)
\]

In particular, note the following

\[
\Gamma^k_{ij} = \Gamma^n_{n+1} = \Gamma^b_{bc} = \Gamma^i_{n+1} = 0. (20)
\]

For $r \geq 0$ let $R^b_{c,d,f;f_1,\ldots,f_r}$ be the functions such that $\nabla^r R(\frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d}, \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}}) \frac{\partial}{\partial x^0} = R^b_{c,d,f;f_1,\ldots,f_r}$.
One can compute the following components of the curvature tensor

\[
R_{ji}^{k}(n+1)\alpha = \frac{1}{(\alpha - 1)!} P_{\alpha j}^{i}(x^{n+1})^{\alpha - 1}, \quad R_{ji}^{k}(n+1) = 0, \tag{21}
\]

\[
R_{bc}^{k} = 0 \text{ if } (b, c) \notin \{1, \ldots, n\} \times \{n + 1\} \cup \{1, \ldots, n\}, \tag{22}
\]

\[
R_{in+1}^{0} = \psi_{aiz}(x^{n+1})^{\alpha - 1}, \tag{23}
\]

\[
R_{bc}^{0} = 0 \text{ if } (b, c) \notin \{1, \ldots, n\} \times \{n + 1\} \cup \{1, \ldots, n\}, \tag{24}
\]

\[
R_{kij}^{0} = \frac{1}{(\alpha - 1)!} P_{\alpha i}^{j}(x^{n+1})^{\alpha - 1}, \tag{25}
\]

\[
R_{i+1}^{0} = 1, \quad R_{0+1}^{n+1} = -1, \quad R_{c+1}^{0} = 0 \text{ if } (b, c) \neq (0, \tilde{i}) \text{ and } (b, c) \neq (\tilde{i}, n+1), \tag{26}
\]

\[
R_{0bc}^{0} = 0. \tag{27}
\]

Using (21), we get

\[
\Gamma_{bi}^{j} R_{cdg}^{f} = \Gamma_{bi}^{j} R_{cdg}^{f}, \quad \Gamma_{bh}^{f} R_{dfg}^{c} = \Gamma_{bh}^{f} R_{dfg}^{c}, \quad \Gamma_{bh}^{f} R_{cdg}^{e} = \Gamma_{bh}^{f} R_{cdg}^{e}, \quad \Gamma_{bh}^{f} R_{deg}^{c} = \Gamma_{bh}^{f} R_{deg}^{c}. \tag{28}
\]

From equalities (12) – (19) it follows that

\[
\Gamma_{bc}^{d} R_{ief}^{d} = \Gamma_{bc}^{d} R_{ief}^{d}, \quad \Gamma_{bc}^{d} R_{eif}^{d} = \Gamma_{bc}^{d} R_{eif}^{d}, \quad \Gamma_{bc}^{d} R_{efi}^{d} = \Gamma_{bc}^{d} R_{efi}^{d} \text{ if } b \neq n+1 \text{ or } c \neq n+1, \tag{29}
\]

\[
\Gamma_{bc}^{d} R_{def}^{d} = \Gamma_{bc}^{d} R_{def}^{d} \text{ if } b \neq 0 \text{ or } c \neq n+1. \tag{30}
\]

Proof of the inclusion \(g^{4,b,m,\psi} \subset \mathfrak{h} \Omega_{0}\).

**Lemma 1** For any \(1 \leq r \leq N\) we have

1. \[
R_{j+1}^{k}(n+1; \ldots; n+1) = \sum_{\alpha=r}^{n} (\frac{1}{(\alpha - 1)!} P_{\alpha j}^{i}(x^{n+1})^{\alpha - r} + y_{\alpha j}^{i}), \text{ where } y_{\alpha j}^{i} \text{ are functions such that } y_{\alpha j}^{i}(0) = \cdots = 0; \tag{31}
\]

2. \[
R_{i+1}^{0}(n+1; \ldots; n+1) = \sum_{\alpha=r}^{n} (\frac{1}{(\alpha - 1)!} \psi_{aiz}(x^{n+1})^{\alpha - r} + z_{\alpha j}^{i}), \text{ where } z_{\alpha j}^{i} \text{ are functions such that } z_{\alpha j}^{i}(0) = \cdots = 0. \tag{32}
\]

**Proof.** We will prove this lemma by induction over \(r\). For \(r = 1\) the lemma follows from (21) and (23). Fix \(r_0 > 1\) and assume that the lemma is true for all \(r < r_0\). We must prove that the lemma is true for \(r = r_0\). We have
From (11), (31), (32) and (33) it follows that

$$R_{j in+1;n + 1; \ldots ; n + 1}^k =$$

$$\left. \frac{\partial R_{j in+1;n + 1; \ldots ; n + 1}^k}{\partial x^{n+1}} \right|_{r \text{ times}} + \Gamma_{j in+1}^k R_{i j n+1}^l \left. R_{i j n+1}^k \right|_{r-1 \text{ times}} + \Gamma_{j in+1}^l R_{i j n+1}^k \left. R_{i j n+1}^l \right|_{r-1 \text{ times}}$$

$$- \Gamma_{j in+1}^l R_{i j n+1}^k \left. R_{i j n+1}^l \right|_{r-1 \text{ times}} - \Gamma_{j in+1}^k R_{i j n+1}^l \left. R_{i j n+1}^k \right|_{r-1 \text{ times}}$$

$$- \Gamma_{j in+1}^l R_{i j n+1}^k \left. R_{i j n+1}^l \right|_{r-1 \text{ times}} - \cdots - \Gamma_{j in+1}^{l+n+1} R_{i j n+1}^{k+n+1} \left. R_{i j n+1}^{l+n+1} \right|_{r-2 \text{ times}}$$

$$= \sum_{\alpha=r+1}^{N} \frac{1}{(\alpha-\alpha)!} \cdot \left( x^{n+1} \right)^{\alpha-1} + \frac{\partial y^{\alpha}}{\partial x^{n+1}} + y^{\alpha}$$

Claim 1) follows from the fact that all Christoffel symbols and all their derivatives with respect to $x^{n+1}$ vanish at the point 0. The proof of claim 2) is analogous. The lemma is proved. \( \square \)

From lemma [1] it follows that for any $1 \leq r \leq N$ we have

$$R_{j in+1;n + 1; \ldots ; n + 1}^k \left. (0) = \right|_{r-1 \text{ times}}$$

$$P_{r j i}^k \left. (0) = \right|_{r-1 \text{ times}}$$

Similarly we can prove that from (32) it follows that

$$R_{i j n+1;n + 1; \ldots ; n + 1}^k \left. (0) = \right|_{r-1 \text{ times}}$$

$$= 0.$$ (33)

From (11), (31), (32) and (33) it follows that

$$R(e_1, e_{n+1}, \ldots ; e_{n+1})_0 = (0, P_r(e_i), X_{r \cdot i} + \psi(P_r(e_i)))$$

where $X_{r \cdot i} \in \text{span} \{ e_1, \ldots, e_{n+1} \}$. Since the elements $P_r(e_i)$ generate the Lie algebra $\mathfrak{h}$ and $\text{pr}_{so(n)} \mathfrak{h} \mathfrak{o} \mathfrak{l}_0$ is a Lie algebra, from (34) we get

$$\mathfrak{h} \subset \text{pr}_{so(n)} \mathfrak{h} \mathfrak{o} \mathfrak{l}_0.$$ (35)

Using (35) we can prove that for all $1 \leq r \leq N$ we have

$$P_{r j i}^k \left. (0) = \right|_{r-1 \text{ times}}$$

This means that

$$(0, 0, P_{r j i}^k e_i) \in \mathfrak{h} \mathfrak{o} \mathfrak{l}_0.$$ (36)
Recall that we have decompositions (1), (2) and (3). Since \( h \) is generated by the images of the elements \( P_\alpha \), for any \( 2 \leq i \leq s \) there exist \( \alpha, \hat{i}, \hat{j}, \hat{k} \) such that \( n_1 + \cdots + n_{i-1} + 1 \leq \hat{j} \leq n_1 + \cdots + n_i \) and \( P_{\alpha \hat{i} \hat{k}} \neq 0 \). Combining this with (36), we get

\[
\{(0, 0, X) | X \in \mathbb{R}^{n_i}\} \cap \mathfrak{hol}_0 \neq \{0\}.
\]

Since \( \mathfrak{h}_i \subset \mathfrak{so}(n_i) \) is an irreducible subalgebra, \( \mathfrak{h}_i \subset \mathfrak{pr}_{\mathfrak{so}(n)} \mathfrak{hol}_0 \), and for any \( A \in \mathfrak{h}, Z \in \mathbb{R}^n, Y \in \mathbb{R}^{n_i} \) holds

\[
[(0, A, Z), (0, 0, Y)] = (0, 0, AY) \in \{(0, 0, X) | X \in \mathbb{R}^{n_i}\} \cap \mathfrak{hol}_0,
\]

we see that

\[
\{(0, 0, X) | X \in \mathbb{R}^{n_i}\} \subset \mathfrak{hol}_0,
\]

hence,

\[
\{(0, 0, X) | X \in \text{span}\{e_1, \ldots, e_{n_0}\}\} \subset \mathfrak{hol}_0. \tag{37}
\]

From (26) it follows that \( R(e_{\hat{i}}, q) = (0, 0, e_{\hat{i}}) \), hence,

\[
\{(0, 0, X) | X \in \text{span}\{e_{n_0+1}, \ldots, e_m\}\} \subset \mathfrak{hol}_0. \tag{38}
\]

From (34), (37), (38) and the fact that \( \mathfrak{h} \) is generated by the elements \( P_\alpha(e_{\hat{i}}) \) it follows that \( g^{4, h, m, \psi} \subset \mathfrak{hol}_0 \).

**Proof of the inclusion \( \mathfrak{hol}_0 \subset g^{4, h, m, \psi} \).**

**Lemma 2** For any \( r \geq 0 \) we have

1) \( R_{j i d : f_1 \cdots : f_r}^k = 0 \);
2) \( R_{j i m + 1 : f_1 \cdots : f_r}^k = 0 \);
3) \( R_{i i d : f_1 \cdots : f_r}^0 = 0 \);
4) \( R_{j i m + 1 : f_1 \cdots : f_r}^0 = 0 \);
5) \( R_{i i m + 1 : f_1 \cdots : f_r}^0 = 0 \);
6) \( R_{i 0 c : f_1 \cdots : f_r}^0 = 0 \);
7) \( R_{j i m + 1 : f_1 \cdots : f_r}^k = \sum_{t \in T_{i f_1 \cdots f_r}} \sum_{A_t} A_t^k_{i j} \), where \( T_{i f_1 \cdots f_r} \) is a finite set of indeces, \( z_t \) are functions and \( A_t \in \mathfrak{h} \).
8) \( R_{i i m + 1 : f_1 \cdots : f_r}^0 = \sum_{t \in T_{i f_1 \cdots f_r}} z_t \psi_{\hat{i} \hat{i}} \), where \( \psi_{\hat{i} \hat{i}} \) are numbers such that \( \psi(A_t) = \sum_{i=m+1}^{n} \psi_{\hat{i} \hat{i}} e_{\hat{i}} \).
Proof. We will prove the claims of the lemma by induction over \( r \). For \( r = 0 \) the claims follow from (21) – (27). Fix \( r > 0 \) and assume that the lemma is true for \( r \). We must prove that the lemma is true for \( r + 1 \).

1) We have
\[
R_{jil; f_1; \ldots; f_r; i_1}^k = \frac{\partial R_{jil; f_1; \ldots; f_r}^k}{\partial x_{i_1}} + \Gamma_{l_{i_2}}^k R_{jil; f_1; \ldots; f_r}^k - \Gamma_{l_{i_3}}^k R_{l_{i_2}il; f_1; \ldots; f_r}^k - \cdots - \Gamma_{l_{i_r}}^k R_{l_{i_{r-1}}l_{i_r}; f_1; \ldots; f_r}^k.
\]
From (20) and the inductive hypothesis it follows that
\[
R_{jil; f_1; \ldots; f_r; i_1}^k = 0.
\]

Claim 1) is proved. The proofs of claims 2) – 6) are analogous.

7) We have
\[
R_{jil; f_1; \ldots; f_r; i_1}^k = \frac{\partial R_{jil; f_1; \ldots; f_r}^k}{\partial x_{i_1}} + \Gamma_{l_{i_2}}^k R_{jil; f_1; \ldots; f_r}^k - \Gamma_{l_{i_3}}^k R_{l_{i_2}il; f_1; \ldots; f_r}^k - \cdots - \Gamma_{l_{i_r}}^k R_{l_{i_{r-1}}l_{i_r}; f_1; \ldots; f_r}^k = 0.
\]
From (20), claim 1) of the lemma and the inductive hypothesis it follows that
\[
R_{jil; f_1; \ldots; f_r; i_1}^k = 0.
\]
Using this, claim 1) and inductive hypothesis, we get

\[ R^k_{jn+1;f_1;\ldots;f_r;n+1} = \sum_{t \in T_{f_1, \ldots, f_r}} \frac{\partial z_t}{\partial x^{n+1}} A^k_{ij} + \frac{1}{(x-1)!} x^2 (x^{n+1})^{\alpha-1} \left[ P_\alpha(e_{i_j}), R(e_{i_1}, e_{n+1}; e_{f_1}; \ldots; e_{f_r}) \right]_{ij} \]

\[ - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1}; \ldots;f_r}} z_t \Gamma_{l_1}^{d_1} A^k_{ij} \]

\[ - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1}; \ldots;f_r-1}} z_t \Gamma_{l_1}^{d_1} A^k_{ij} \]

\[ = \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1}; \ldots;f_r-1}} z_t \Gamma_{l_1}^{d_1} A^k_{ij} \]

This proves claim 7).

8) We have

\[ R^0_{jn+1;f_1;\ldots;f_r,n+1} = \frac{\partial R^0_{jn+1;f_1;\ldots;f_r}}{\partial x^n} + \Gamma_{l_1}^{d_1} R^0_{jn+1;f_{l_1};\ldots;f_r} - \Gamma_{l_1}^{d_1} R^0_{jn+1;f_{l_1};\ldots;f_r} - \Gamma_{l_1}^{d_1} \Gamma_{l_1}^{d_1} R^0_{jn+1;f_{l_1};\ldots;f_r-1} \]

Using this, (20), claims 2) and 3), and the inductive hypothesis, we get

\[ R^0_{jn+1;f_1;\ldots;f_r,n+1} = \sum_{t \in T_{f_1, \ldots, f_r}} \frac{\partial z_t}{\partial x^n} \psi_{i_1} - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r-1}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} \]

Finally,

\[ R^0_{jn+1;f_1;\ldots;f_r,n+1} = \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r-1}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} \]

Using this, (20), claims 2) and 3), and the inductive hypothesis, we get

\[ R^0_{jn+1;f_1;\ldots;f_r,n+1} = \sum_{t \in T_{f_1, \ldots, f_r}} \frac{\partial z_t}{\partial x^n} \psi_{i_1} - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} - \sum_{l_1=1}^{n} \sum_{t \in T_{l_1;f_{l_1};\ldots;f_r-1}} z_t \Gamma_{l_1}^{d_1} \psi_{i_1} \]

Combining (39) with (41) and (40) with (42) and using the fact that \( \psi|_{y'} = 0 \), we see that claim 8) is true. The lemma is proved. \( \square \)

From lemma 2 it follows that

\[ \text{hol}_0 \subseteq g^4_{\mathbb{h,m,\psi}} \]

Thus,

\[ \text{hol}_0 = g^4_{\mathbb{h,m,\psi}} \]

The theorem is proved. \( \square \)
References

[1] D.V. Alekseevsky, *Riemannian manifolds with exceptional holonomy groups*, Funksional Anal. i Prilozhen. 2 (2), 1-10, 1968.

[2] W. Ambrose, I.M. Singer *A theorem on holonomy*, Trans. Amer. Math. Soc. 79(1953), 428-443.

[3] V.V.Astrahantsev, *On the holonomy groups of 4-dimensional pseudo-Riemannian manifolds*, Matematicheskie zametki, 9, N1(1971), 59-66.

[4] H. Baum, I. Kath, *Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds*, SFB 288 Preprint No 276, 1997.

[5] L. Berard Bergery, A. Ikemakhen, *On the Holonomy of Lorentzian Manifolds*, Proceeding of symposia in pure math., volume 54, 27 – 40, 1993.

[6] L. Berard Bergery, A. Ikemakhen, *Sur l’holonomie des variétés pseudo-riemanniennes de signature (n,n)*, Bull. Soc. Math. France. t 125, f1 (1997), 93-114.

[7] A.L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[8] M.Berger, *Sur les groupes d’holonomie des variétés a connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955), 279-330.

[9] A. Borel, A. Lichnerowicz, *Groupes d’holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris 234 (1952), 279-300.

[10] Ch. Boubel, *Sur l’holonomie des variétés pseudo-riemanniennes*, PhD thesis, Université Henri Poincaré, Nancy, 2000.

[11] Ch. Boubel, *On the holonomy of Lorentzian metrics*, Prépublication de l’ENS Lyon no. 323 (2004).

[12] Ch. Boubel, A. Zeghib, *Dynamics of some Lie subgroups of O(n,1), applications*, Prépublication de l’ENS Lyon no. 315 (2003).

[13] R. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. (2) 126 (1987), 525-576.

[14] A.J. Di Scala, C.Olmos, *The geometry of homogeneous submanifolds of hyperbolic space*, Math. Z. 237, 199 – 209 (2001).

[15] A.S. Galaev, *The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds*, Diff. Geom. and its Applications 22, 1-18 (2005).
[16] A.S. Galaev, *Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidean spaces and Lorentzian holonomy groups*, arXiv:math.DG/0404426, 2004.

[17] A.S. Galaev, *Classification of connected holonomy groups of pseudo-Kählerian manifolds of index 2*, arXiv:math.DG/0405098 v2, 2005.

[18] A.S. Galaev, *Remark on holonomy groups of pseudo-Riemannian manifolds of signature (2,n+2)*, Arxiv:math.DG/0406397, 2004.

[19] A. Ikemakhen, *Examples of indecomposable non-irreducible Lorentzian manifolds*, Ann. Sci. Math. Québec 20(1996), no. 1, 53-66.

[20] A. Ikemakhen, *Sur l’holonomie des variétés pseudo-riemanniennes de signature (2,2+n)*, Publ. Mat. 43 (1999), no. 1, 55–84.

[21] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, 2000.

[22] T. Leistner, *Berger algebras, weak-Berger algebras and Lorentzian holonomy*, sfb 288-preprint no. 567, 2002.

[23] T. Leistner, *Towards a classification of Lorentzian holonomy groups*, arXiv:math.DG/0305139 2003.

[24] T. Leistner, *Towards a classification of Lorentzian holonomy groups. Part II: Semisimple, non-simple weak-Berger algebras*, arXiv:math.DG/0309274 2003.

[25] T. Leistner, *Holonomy and parallel spinors in Lorentzian geometry*, PhD thesis, Humboldt-Universität zu Berlin, 2003.

[26] K. Sfetsos, D. Zoakos, *Supersymmetry and Lorentzian holonomy in various dimensions*, J. High Energy Phys. JHEP09(2004)010.

[27] E.B. Vinberg, A.L. Onishchik, *Seminar on Lie groups and algebraic groups*, Moscow, 1995.

[28] H. Wu, *Holonomy groups of indefinite metrics*, Pacific J. of Math., 20, 351 – 382, 1967.