A generalization of the Gołąb-Schinzel functional equation

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Dedicated to Professor Janos Aczel on the occasion of his 85th birthday

Abstract. For every fixed real \( p \), the continuous real valued functions \( f \) defined on a linear topological space and satisfying the functional equation

\[
f (p[f(y)x + y] + (1 - p)[f(x)y + x]) = f(x)f(y)
\]

are determined. For \( p = 0 \) or \( p = 1 \) this equation coincides with the classical Gołąb-Schinzel equation.

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1. Introduction

The functional equation

\[
f (x + yf(x)) = f(x)f(y), \quad x, y \in X,
\]

where \( X \) is a real linear space and \( f : X \to \mathbb{R} \) an unknown function is the well-known Gołąb-Schinzel equation (cf. [8] also Aczél [1], Aczél [2, pp 132–135], Aczél and Dhombres [3, Chapter 19], Brillouet and Dhombres [6], Javor [9], Wołodźko [12] and Baron [5] where the complex-valued solutions are considered). This equation arises in the problem to determine all subsemigroups of a semigroup which have a faithful continuous parametrization ([4], also [3, Chapter 19]) and, as it contains the superpositions of the unknown functions, it is of the composite type. There are several papers devoted to some generalizations of the (G-S) equation, cf. a survey paper Brzdek [7] and a recent paper by Mureńko [11].

Let \( X = \mathbb{R} \). Suppose that \( f : X \to \mathbb{R} \) is an injective solution of the (G-S) equation, and \( p : X \times X \to \mathbb{R} \) is an arbitrary function. Then, obviously,
\[ x + yf(x) = f^{-1}(f(x)f(y)), \quad y + xf(y) = f^{-1}(f(x)f(y)), \quad (x, y \in X). \]

Multiplying these equations by \(1 - p(x, y)\) and \(p(x, y)\), respectively, and then adding the resulting equations by sides we get

\[ p[y + xf(y)] + (1 - p)[x + yf(x)] = f^{-1}(f(x)f(y)), \quad (x, y \in X), \]

which implies that \(f\) satisfies the functional equation

\[ f(p(x, y)[f(y)x + y] + (1 - p(x, y))[f(x)y + x]) = f(x)f(y), \quad x \in X. \quad (P) \]

In this paper we consider the following special case of equation (P):

\[ f(p[f(y)x + y] + (1 - p)[f(x)y + x]) = f(x)f(y), \quad x \in X, \quad (1) \]

where \(p \in \mathbb{R}\) is fixed, \(X\) is a real linear space and \(f : X \to \mathbb{R}\) an unknown function. For \(p = 0\) or \(p = 1\) it becomes the Golab-Schinzel equation. For \(p = \frac{1}{2}\) this equation is symmetric with respect to \(x\) and \(y\), contrary to the (G-S) equation. According to my best knowledge, Eq. 1 is a new generalization of Eq. G-S.

Theorem 1 gives all the continuous solutions \(f : \mathbb{R} \to \mathbb{R}\) of Eq. 1. Since the method of periodic functions (applied in the case of the Golab-Schinzel equation) does not work in the case when \(0 \neq p \neq 1\), an essential part of the proof is based on an iterative functional equation (Lemma 2). Applying Theorem 1 we determine the continuous solutions of the form \(f : X \to \mathbb{R}\) where \(X\) is a linear topological space (Theorem 2).

At the end of this paper we propose an open problem.

2. A remark and two lemmas

The following is easy to verify.

**Remark 1.** Let \(X\) be a real linear space and \(p \in \mathbb{R}\) fixed. If a function \(f : X \to \mathbb{R}\) satisfies Eq. 1, then, for every \(a \in \mathbb{R}\), the function \(X \ni x \mapsto f(ax)\) also satisfies Eq. 1.

**Lemma 1.** Let \(X\) be a real linear space and \(p \in \mathbb{R}\) fixed. If a function \(f : X \to \mathbb{R}\) satisfies Eq. 1 then, for every \(k \in \mathbb{N}\),

\[ f \left( \prod_{j=0}^{k-1} \left( 1 + [f(x)]^{2^j} \right) \right) x = [f(x)]^{2^k}, \quad x \in X. \quad (2) \]

**Proof.** Taking \(y = x\) in (1) we see that \(f\) satisfies the iterative functional equation

\[ f((1 + f(x))x) = [f(x)]^2, \quad x \in X, \]

so the lemma holds true for \(k = 1\). Suppose (2) holds true for some \(k \in \mathbb{N}\). Replacing \(x\) by \((1 + f(x))x\) in (2) and using this functional equation we obtain equality (2) for \(k + 1\) and the induction completes the proof. \(\square\)
Lemma 2. Let $X$ be a real linear topological space and $p \in \mathbb{R}$ fixed. If a continuous function $f : X \to \mathbb{R}$ satisfies (1) and $f(X) \cap (0,1) \neq \emptyset$, then $0 \in f(X)$.

Proof. By assumption there exists an $x \in X$ such that $0 < f(x) < 1$. Since the infinite product $\prod_{j=0}^{\infty} \left( 1 + [f(x)]^{2^j} \right)$ converges, the point

$$z := \left( \prod_{j=0}^{\infty} \left( 1 + [f(x)]^{2^j} \right) \right) x$$

is an element of $X$. By the continuity of $f$ and Lemma 1, we hence get

$$f(z) = f \left( \lim_{k \to \infty} \left( \prod_{j=0}^{k-1} \left( 1 + [f(x)]^{2^j} \right) \right) x \right)$$

$$= \lim_{k \to \infty} f \left( \prod_{j=0}^{k-1} \left( 1 + [f(x)]^{2^j} \right) \right) x$$

$$= \lim_{k \to \infty} [f(x)]^{2^k} = 0,$$

which completes the proof. \(\square\)

3. Continuous solutions of real variable

Theorem 1. Let $p \in \mathbb{R}$ be fixed. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f \left( p[xf(y) + y] + (1-p)[yf(x) + x] \right) = f(x)f(y), \quad x, y \in \mathbb{R}, \quad (3)$$

if, and only if, either

$$f(x) = 0, \quad x \in \mathbb{R},$$

or there is $c \in \mathbb{R}$ such that

$$f(x) = 1 + cx, \quad x \in \mathbb{R},$$

or $p \in [0; 1]$ and

$$f(x) = \max (1 + cx, 0), \quad x \in \mathbb{R},$$

for some $c \in \mathbb{R}$, $c \neq 0$.

Proof. Suppose that a continuous $f : \mathbb{R} \to \mathbb{R}$ satisfies (3). Setting $x = y = 0$ in (3) we get $f(0) = [f(0)]^2$ and, consequently, either $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 0$, setting $y = 0$ in (3) we obtain $f((1-p)x) = 0$ for all $x \in \mathbb{R}$, and setting $x = 0$ in (3) we obtain $f(py) = 0$ for all $y \in \mathbb{R}$. It follows that, independently of the value of $p$, $f(x) = 0$ for all $x \in \mathbb{R}$. 

Now assume that $f(0) = 1$. Put

$$Z := \{x \in \mathbb{R} : f(x) = 0\}. \quad (4)$$

By the continuity of $f$, the set $Z$ is closed.

Part I. We assume that $Z \neq \emptyset$. If $z_1, z_2 \in Z$ then, by (3),

$$f((1 - p)z_1 + pz_2) = 0,$$

which proves that $((1 - p)z_1 + pz_2) \in Z$. 

Case $p \notin [0; 1]$. If $p < 0$ then $1 - p > 1$. Therefore, Eq. 3 with $p < 0$ and Eq. 3 with $p > 1$ coincide. Thus, without any loss of generality we can assume that $p > 1$.

We shall show that $Z$ defined by (4) is a singleton. For an indirect argument suppose that there are $z_1, z_2 \in Z$ such that $z_1 < z_2$. Then

$$(1 - p)z_2 + pz_1 < z_1 < z_2 < (1 - p)z_1 + pz_2$$

and

$$(1 - p)z_2 + pz_1 \in Z, \quad (1 - p)z_1 + pz_2 \in Z.$$

Since $Z$ is closed, it follows that $Z$ is unbounded from both sides.

Hence, taking into account the continuity of $f$ and $f(0) = 1$, we conclude that there are $z_1, z_2 \in Z$ such that

$$z_1 < 0 < z_2 \quad \text{and} \quad f(x) > 0 \quad \text{for all} \quad x \in (z_1, z_2).$$

Setting $x := z_k$ and $y := x$ in (3) we get

$$f(p[z_k f(x) + x] + (1 - p)z_k) = 0, \quad k = 1, 2, \quad x \in [z_1, z_2].$$

Since $f(u) \neq 0$ for $u \in (z_1, z_2)$, by the continuity of $f$, we infer that, for each $k \in \{1, 2\}$, either

$$p[z_k f(x) + x] + (1 - p)z_k \leq z_1, \quad x \in [z_1, z_2]. \quad (5)$$

or

$$p[z_k f(x) + x] + (1 - p)z_k \geq z_2 \quad x \in [z_1, z_2]. \quad (6)$$

Setting $x = z_2$ in inequality (5) we get

$$pz_2 + (1 - p)z_k \leq z_1$$

which is false for $k = 1$ and $k = 2$. Setting $x = z_1$ in (6) we get

$$pz_1 + (1 - p)z_k \geq z_2$$

which is also false for $k = 1$ and $k = 2$. This contradiction proves that if $p \notin [0, 1]$ then $Z$ must be a singleton, that is $Z = \{z_0\}$ for some $z_0 \neq 0$.

Case $p = 1$ or $p = 0$. 

Since in each of these two cases Eq. 3 with the classical Gołąb-Schinzel functional equation, it is well known that either $Z = \{z_0\}$ or $Z = (z_0, \infty)$.

Case $p \in (0; 1)$.

Now $Z$ is $p$-convex. We shall prove that $Z$ is convex. Assume, on the contrary, that it is not true. Then we would find $x, y \in Z, x < y$ such that $[x, y] \setminus Z$ is nonempty. Since $[x, y] \cap Z$ is closed, the set $[x, y] \setminus Z$ is open in $[x, y]$. Consequently, there exist $u, v \in [x, y] \cap Z, u \neq v$, such that $(u, v) \cap Z = \emptyset$. On the other hand, as $u, v \in Z$, from the $p$-convexity of the set $Z$ we have $pu + (1 - p)v \in Z \cap A$. This contradiction proves that $Z$ is convex. (For a more general fact see [10]).

Taking into account that $0 \notin Z$, one of the following situations must occur:

(i): for some $z_0 > 0$,

$$Z = [z_0, \infty);$$

(ii): for some $z_0 < 0$,

$$Z = (-\infty, z_0];$$

(iii): for some $z_0 \in \mathbb{R}, z_0 \neq 0$,

$$Z = \{z_0\};$$

(iv): for some $a, b \in \mathbb{R}, b < a < 0$,

$$Z = [b, a];$$

(v): for some $a, b \in \mathbb{R}, 0 < a < b$,

$$Z = [a, b].$$

Note that the (iv) and (v) cannot happen. Indeed, if for instance (iv) occurs then, setting $y = a$ in Eq. 3, we infer that, for all $x \in \mathbb{R}$,

$$b \leq pa + (1 - p)[af(x) + x] \leq a,$$

whence, for all $x \in \mathbb{R}$,

$$1 - \frac{x}{a} \leq f(x),$$

and, setting $y = b$ in Eq. 3 we infer that, for all $x \in \mathbb{R}$,

$$b \leq pb + (1 - p)[bf(x) + x] \leq a,$$

whence, for all $x \in \mathbb{R}$,

$$f(x) \leq 1 - \frac{x}{b}.$$

Consequently we would have

$$1 - \frac{x}{a} \leq 1 - \frac{x}{b}, \quad x \in \mathbb{R},$$

which is impossible. To exclude (v) we can argue in the same way.
Thus, we have shown that either $Z = \{z_0\}$ or $Z = (-\infty, z_0]$ or $Z = [z_0, \infty)$. To conclude Part I of the proof assume that $Z = \{z_0\}$.

Setting $y := z_0$ in (3) we obtain

$$f(pz_0 + (1 - p)[z_0f(x) + x]) = 0, \quad x \in \mathbb{R}.$$  

It follows that

$$pz_0 + (1 - p)[z_0f(x) + x] = z_0, \quad x \in \mathbb{R},$$  

whence

$$f(x) = 1 - \frac{x}{z_0}, \quad x \in \mathbb{R}.$$

Assume $Z = (-\infty, z_0]$ for some $z_0 < 0$. In this case, as $f$ satisfies Eq. 3, we infer that:

for all $x, y \in \mathbb{R}$, $y \leq z_0$ (as $f(y) = 0$),

$$py + (1 - p)[yf(x) + x] \leq z_0,$$

and, for all $x, y \in (z_0, +\infty)$,

$$p[xf(y) + y] + (1 - p)[yf(x) + x] > z_0.$$  

Hence, taking $y = z_0$ in the first of these inequalities and, letting $y \to z_0$ in the second one and taking into account that $f(z_0) = 0$, we obtain, respectively,

$$\begin{align*}
(1 - p)[z_0f(x) + x] & \leq (1 - p)z_0, \quad x \in \mathbb{R}, \\
(1 - p)[z_0f(x) + x] & \geq (1 - p)z_0, \quad x > z_0.
\end{align*}$$

Hence

$$z_0f(x) + x = z_0, \quad x > z_0,$$

whence

$$f(x) = 1 - \frac{x}{z_0}, \quad x > z_0.$$

Thus, we have shown that in the case when $Z = (-\infty, z_0]$ the function $f$ must be of the form

$$f(x) = \max\left(1 - \frac{x}{z_0}, 0\right), \quad x \in \mathbb{R}.$$  

By Remark 1, the case $Z = [z_0, +\infty)$ reduces to the previous one.

Part 2. Assume that $Z = \emptyset$.

By Lemma 2 we have $f(x) \geq 1$ for all $x \in \mathbb{R}$. Put

$$S := \{x \in \mathbb{R} : f(x) = 1\}.$$
The set $S$ is nonempty as $0 \in S$. If $x, y \in S$ then, by (3),
\[
f(x + y) = f(p[xf(y) + y]) + (1 - p)[yf(x) + x] = 1,
\]
whence $x + y \in S$ and, consequently, $S$ is an additive semigroup of $\mathbb{R}$.

First consider the case when there exists an $x \in S$ such that $x \neq 0$. Without any loss of generality we can assume that $x > 0$ (cf. Remark 1). Put
\[
S_+ := \{ x \in S : x > 0 \}.
\]
We shall show that $a := \inf S_+ = 0$. Assume for an indirect argument that $a > 0$. Since $na \in S_+$ for every $n \in \mathbb{N}$, setting $y = na$ in (3) we obtain
\[
f(x + npa + n(1 - p)a f(x)) = f(x), \quad x \in \mathbb{R}, \ n \in \mathbb{N}.
\]
This equation implies that $f([na, (n+1)a]) = f([0, a])$ for every $n \in \mathbb{N}$. Putting
\[
M = \sup f([0; \infty))
\]
we infer that there exists an $x_0 \in [0; a]$ such that $f(x_0) = M$. Setting $x = y = x_0$ in (3) we get
\[
f(x_0f(x_0) + x_0)) = [f(x_0)]^2 = M^2.
\]

Since $M \geq 1$, from the definition of $M$ we conclude that $M^2 \leq M$ and, consequently, $M = 1$. Thus $S_+ = (0, \infty)$ and, obviously, $a = 0$ which is a desired contradiction. This fact easily implies that $S$ is dense in $[0, \infty)$, and, by the continuity of the function $f$,
\[
f(x) = 1, \quad x \geq 0.
\]
Now, setting $y \geq 0$ in (3) and taking into account that $f(y) = 1$, we obtain
\[
f(x + py + (1 - p)yf(x)) = f(x), \quad y \in [0, \infty), \ x \in \mathbb{R}.
\]
We can assume without any loss of generality that $1 - p > 0$. Let $x < 0$ be arbitrarily fixed. Since $f(x) \geq 1$, taking $y := -x$ we have
\[
x + py + (1 - p)yf(x) \geq x + py + (1 - p)y = 0,
\]
whence
\[
f(x) = f(x + py + (1 - p)yf(x)) = 1, \quad x < 0.
\]
Assume that $S = \{0\}$. By Lemma 2 we would have
\[
f(x) > 1 \quad \text{for all} \quad x \in \mathbb{R}, x \neq 0.
\]
To show that this case cannot occur put
\[
F(x, y) := p[xf(y) + y] + (1 - p)[yf(x) + x], \quad x, y \in \mathbb{R}.
\]
and observe that, by (3),
\[
F(x, y) = 0 \iff x = y = 0.
\]
Since the function $F$ is continuous in $\mathbb{R}^2$ and $f(0) = 1$ we infer that $F$ would be positive in $\mathbb{R}^2 \setminus \{(0, 0)\}$. This is impossible as $F(0, y) = y$ for all $y \in \mathbb{R}$. So the case $S = \{0\}$ cannot occur.

This concludes the proof. □

**Remark 2.** It is easy to verify, that every continuous solution $f$ of Eq. 3 satisfies the following conditional functional equation

$$f(x) \neq 0 \implies f \left( \frac{-x}{f(x)} \right) f(x) = 1, \quad x \in \mathbb{R}.$$  

Replacing $x$ by $y$ and $y$ by $-\frac{x}{f(x)}$ in (3) and using this relation we obtain

$$f(x) \neq 0 \implies f \left( \frac{p(y - x) + (1 - p)(yf(x) - xf(y))}{f(x)} \right) = \frac{f(y)}{f(x)}, \quad x, y \in \mathbb{R}.$$  

**Remark 3.** Let $p \in \mathbb{R}$ be fixed. If a function $f : \mathbb{R} \to \mathbb{R}$ is one to one and satisfies Eq. 3 then, for some $c \in \mathbb{R}$, $c \neq 0$,

$$f(x) = cx + 1, \quad x \in \mathbb{R}.$$  

4. **Main result**

**Theorem 2.** Let $X$ be a real linear topological space and $p \in \mathbb{R}$ be fixed. A continuous function $f : X \to \mathbb{R}$ satisfies Eq. 1:

$$f (p[f(y)x + y] + (1 - p)[f(x)y + x]) = f(x)f(y), \quad x \in X,$$

if, and only if, either

$$f(x) = 0, \quad x \in X,$$

or there is an $x^* \in X^*$ such that

$$f(x) = 1 + <x, x^*>, \quad x \in \mathbb{R},$$

or $p \in [0; 1]$ and there exists an $x^* \in X^* \setminus \{0\}$ such that

$$f(x) = \sup (1 + <x, x^*>, 0), \quad x \in \mathbb{R}.$$  

**Proof.** Setting $y = 0$ in (1) we deduce that either $f(0) = 1$ or $f \equiv 0$. The latter is (7). From now on suppose that $f(0) = 1$.

Take arbitrarily $x_0 \in X$, $x_0 \neq 0$, and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) := f(tx_0), \quad t \in \mathbb{R}.$$  

From (1) we have, for all $s, t \in \mathbb{R}$,

$$g \left( p[sg(t) + t] + (1 - p)[tg(s) + s] \right) = f \left( (p[sg(tx_0) + t] + (1 - p)[tg(sx_0) + s])x_0 \right) = f \left( p[f(tx_0)sx_0 + tx_0] + (1 - p)[f(sx_0)tx_0 + sx_0] \right) = f(sx_0)f(tx_0) = g(s)g(t).$$
Since $g$ is continuous, in view of Theorem 1, either, for some $c \in \mathbb{R}$, we have
\[ g(t) = 1 + ct, \quad t \in \mathbb{R}, \quad (11) \]
or $p \in [0, 1]$ and, for some $c \in \mathbb{R}$, $c \neq 0$,
\[ g(t) = \sup(1 + ct, 0), \quad t \in \mathbb{R}. \quad (12) \]

Define $Y := \{ x \in X : f(x) = 1 \}$. Of course $0 \in Y$ and, by $(1)$, if $x, y \in Y$ then $x + y \in Y$. If $x_0 \in Y$, using $g$ defined by $(10)$, we infer that $g(0) = g(1) = 1$. This fact excludes form $(12)$ for $g$ as, in the opposite case, we would have $1 = g(1) = \sup(1 + c, 0)$ whence $c = 0$. Thus $g$ must be of the form $(11)$ with $c = 0$, and, consequently, $g(t) = 1$ for all $t \in \mathbb{R}$. It follows that $tx_0 \in Y$ for all $t \in \mathbb{R}$. We thus have proved that $Y$ is a linear subspace of $X$. The continuity of $f$ implies that $Y$ is closed. Suppose that there are two linearly independent vectors $x_1, x_2 \in X$ such that
\[ \{ sx_1 + tx_2 : s, t \in \mathbb{R} \} \cap Y = \{ 0 \}. \quad (13) \]

Using $(10)$ with $x_0$ replaced by $x_1$ and $x_2$, respectively, we define $g_1, g_2 : \mathbb{R} \to \mathbb{R}$. Since $x_1, x_2 \notin Y$, according to what we have already shown, these functions are of forms $(11)$ or $(12)$ with $c \neq 0$. It follows that there exist $t_1, t_2 \in \mathbb{R}$ such that $g_1(t_1) < 1$ and $g_2(t_2) > 1$. Consider the segment with endpoints $t_1x_1$ and $t_2x_2$:
\[ [t_1x_1, t_2x_2] := \{ t(t_1x_1) + (1 - t)t_2x_2 : t \in [0, 1] \}. \]

Since each segment is a connected set and
\[ f(t_1x_1) = g_1(t_1) < 1 < g_2(t_2) = f(t_2x_2), \]
the Darboux property of $f$ restricted to $[t_1x_1, t_2x_2]$ implies the existence of a number $t \in (0; 1)$ such that $f(t(t_1x_1) + (1 - t)t_2x_2) = 1$ and, moreover, $t(t_1x_1) + (1 - t)t_2x_2 \neq 0$. This contradicts $(13)$. Therefore, either $Y = X$, yielding $(8)$ with $x^* = 0$ or $Y$ is a proper closed hyperplane of $X$. In this last case there exists $y^* \in X^* \setminus \{ 0 \}$ such that $Y = \{ x \in X : < x, y^* >= 0 \}$. Take $x_0 \in X \setminus Y$ and put
\[ < x_0, y^* >= c. \quad (14) \]

Note that for any $x \in X$ there are unique $t \in \mathbb{R}$ and $u \in Y$ such that $x = u + tx_0$. For $x := tx_0$ and $y := u$ in $(1)$, taking into account that $f(u) = 1$, we hence get
\[ f \left( [(1 - p)f(tx_0) + p]u + tx_0 \right) = f(tx_0), \quad u \in Y, \ t \in \mathbb{R}. \quad (15) \]

If the function $g$ given by $(10)$ is of form $(12)$ then $p \in [0, 1]$ and, as $g$ is non-negative, we infer that $(1 - p)f(tx_0) + p \neq 0$ for all $t \in \mathbb{R}$ when $p \neq 0$. Replacing $u$ by $\frac{u}{(1 - p)f(tx_0) + p}$ in $(15)$ we get
\[ f(u + tx_0) = f(tx_0), \quad u \in Y, \ t \in \mathbb{R}. \quad (16) \]
If \( p = 0 \) we obtain (16) by putting \( x = u, y = tx_0 \) in (1). Thus, for every \( x \in X \) we have \( x = u + tx_0 \) and
\[
f(x) = g(t) = \sup(1 + ct, 0).
\]
Since, by (14),
\[
ct = \langle x_0, x^* \rangle - t = \langle tx_0, x^* \rangle = \langle u + tx_0, x^* \rangle = \langle x, x^* \rangle,
\]
we hence get
\[
f(x) = \sup(1 + \langle x, x^* \rangle, 0), \quad x \in X,
\]
that is \( f \) is of form (9).

Suppose that the function \( g \) is of form (11). Since Eq. 1 with \( p = 0 \) and Eq. 1 with \( p = 1 \) coincide, we can assume without any loss of generality, that \( p \neq 1 \). Simple calculation shows that
\[
(1 - p)f(tx_0) + p = (1 - p)g(t) + p = 0 \quad \text{iff} \quad t = \frac{1}{(p - 1)c}.
\]
Therefore, replacing \( u \) by \( \frac{u}{(1-p)f(tx_0)+p} \) in (15) we get
\[
f(u + tx_0) = f(tx_0), \quad u \in Y, \quad t \in \mathbb{R} \setminus \left\{ \frac{1}{(p - 1)c} \right\}.
\]
The continuity of \( f \) implies that in this case (16) also holds true. Now, similarly as in the previous case, for every \( x \in X \), we have
\[
f(x) = g(t) = 1 + ct = 1 + \langle x, x^* \rangle,
\]
that is \( f \) is of form (8).

To show the “if” part of the theorem take arbitrary \( x, y \in X \). Since \( x = u + sx_0 \) and \( y = v + tx_0 \) for some (uniquely determined) \( u, v \in Y \) and \( s, t \in \mathbb{R} \), it is easy to see that functions (7), (8) and (9) satisfy Eq. 1 iff the real variable function \( g \) defined by (10) satisfies Eq. 3. An application of Theorem 1 completes the proof. \( \square \)

**Remark 4.** In the proof of the above theorem we apply a modified method which is due to N. Brilouet and J. Dhombres [6] (cf. also [3]).

**Remark 5.** Let \( X \) be a real linear space, \( h : X \to \mathbb{Q} \) an arbitrary additive function, and \( p \in \mathbb{Q} \) where \( \mathbb{Q} \) denotes the set of rational numbers. Then the function \( f : X \to \mathbb{Q} \) given by
\[
f(x) = 1 + h(x), \quad x \in \mathbb{R},
\]
is a solution of Eq. 3 ([2, pp. 134–135]).
Indeed, making use of the additivity of $h$ and, repeatedly, the rational homogeneity of additive functions, we have, for all $x, y \in X$,

$$f \left( p[xf(y) + y] + (1 - p)[yf(x) + x] \right)$$

$$= h \left( p \left( x[h(y) + 1] + y \right) + (1 - p)(y[h(x) + 1] + x) \right) + 1$$

$$= ph \left( x[h(y) + 1] \right) + ph(y) + (1 - p)h(y[h(x) + 1]) + (1 - p)h(x) + 1$$

$$= ph \left( x[h(y) + 1] \right) + ph(y) + (1 - p)[h(x) + 1]h(y) + (1 - p)h(x) + 1$$

$$= (h(x)h(y) + h(x) + h(y) + 1) = [h(x) + 1][h(y) + 1] = f(x)f(y).$$

Of course, if $h$ is not identically zero, the function $f$ in this remark is discontinuous everywhere.

We end the paper with the following open

**Problem 1.** Determine all (or all continuous) functions $f$ and $p$ satisfying Eq. P.

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