QUANTUM SPIN CHAINS FROM ONSAGER ALGEBRAS
AND REFLECTION K-MATRICES

ATSUO KUNIBA AND VINCENT PASQUIER

Abstract

We present a representation of the generalized $p$-Onsager algebras $O_p(A_{n-1}^{(1)})$, $O_p(D_n^{(2)})$, $O_p(D_n^{(1)})$, $O_p(D_n^{(1)})$ and $O_p(D_n^{(1)})$ in which the generators are expressed as local Hamiltonians of XXZ type spin chains with various boundary terms reflecting the Dynkin diagrams. Their symmetry is described by the reflection $K$ matrices which are obtained recently by a $q$-boson matrix product construction related to the 3D integrability and characterized by Onsager coideals of quantum affine algebras. The spectral decomposition of the $K$ matrices with respect to the classical part of the Onsager algebra is described conjecturally. We also include a proof of a certain invariance property of boundary vectors in the $q$-boson Fock space playing a key role in the matrix product construction.

1. Introduction

The generalized $p$-Onsager algebra $O_p(A_{n-1}^{(1)})$ is generated by $b_1, \ldots, b_n$ with the relations

\begin{align}
 b_i b_j - b_j b_i &= 0 \quad (a_{ij} = 0), \\
 b_i^2 &= (p^2 + p^{-2}) b_i - (p + p^{-1}) b_i b_i = b_j \quad (a_{ij} = -1),
\end{align}

where we assume $n \geq 3$ and the parameter $p$ is generic. The data $(a_{ij})_{i,j \in \mathbb{Z}_n}$ is the Cartan matrix of the affine Lie algebra $A_{n-1}^{(1)}$ [14]. The above relation with $p = 1$ goes back to [28, eqs.(11),(12)]. In what follows, the algebra $O_p(\mathfrak{g})$ introduced for any affine Lie algebra $\mathfrak{g}$ [8] will simply be called an Onsager algebra for short. We refer to [27, Rem. 9.1] for the early history of the Onsager algebra starting from [23 and 15, Sec.1(1)] for an account on more recent studies and the references therein.

Let $q$ be a parameter such that $q^2 = -p^{-2}$. Then $O_p(A_{n-1}^{(1)})$ has a representation $b_i \mapsto b_i$ defined by

\begin{align}
 b_i = z^{-\delta_{i,0}} \sigma^+_i \sigma^-_{i+1} + z^{\delta_{i,0}} \sigma^-_i \sigma^+_{i+1} + \frac{q + q^{-1}}{4} \sigma^+_i \sigma^-_{i+1} + \frac{q - q^{-1}}{4} (\sigma^+_i - \sigma^-_{i+1}) - \frac{(q - q^{-1})^2}{4(q + q^{-1})} \quad (i \in \mathbb{Z}_n),
\end{align}

where $z$ is a spectral parameter and $\sigma^\pm_i = \frac{1}{2}(\sigma^+_i \pm i \sigma^-_i)$, $\sigma^+_i$ are the Pauli matrices acting on the $i$ th component of $(\mathbb{C}^2)^{\otimes n}$. Thus the generators of $O_p(A_{n-1}^{(1)})$ are expressed as local Hamiltonians of XXZ type spin chain on a length $n$ periodic lattice.

In this paper we explain the origin of the representation [22] and extend it to the Onsager algebra $O_p(\mathfrak{g})$ associated with the non-exceptional affine Lie algebra $\mathfrak{g} = D_n^{(2)}$, $B_n^{(1)}$, $B_n^{(1)}$ and $D_n^{(1)}$. It is based on the two recent works; the $q$-boson matrix product construction of the reflection $K$ matrices connected to the three dimensional (3D) integrability [20] and the characterization of those $K$ matrices by Onsager coideals [19]. The resulting Hamiltonians contain various boundary terms reflecting the shape of the relevant Dynkin diagrams. They yield a systematic realization of the Onsager algebras by spin chain Hamiltonians, providing examples beyond free-fermions which were sought eagerly in the very end of [28].

Let us sketch some more detail of our approach and the results. Our representation of $O_p(\mathfrak{g})$ is an elementary consequence of the composition

\begin{align}
 O_p(\mathfrak{g}) &\to U_p(\mathfrak{g}) \to \text{End} V \\
 b_i &\mapsto g_i \mapsto b_i,
\end{align}

1 $B_n^{(1)}$ is isomorphic to $B_n^{(1)}$ but with a different numeration of the vertices of the Dynkin diagram. See Section 5. It is included for uniformity of the description.

2 The coexistence of $p$ and $q$ related by $p = \pm iq^{-1}$ originates in the $q$-boson for $U_p$ in [20].
where \( U_p(g) \) denotes the Drinfeld-Jimbo quantum affine algebra \( \mathfrak{g} \). The space \( V \) is taken as \((\mathbb{C}^2)^\otimes n\) and the latter arrow stands for the fundamental representations for \( U_p(A_{n-1}^{(1)}) \) and the spin representations for \( U_p(g) \) for the other \( g \) mentioned in the above. They carry a spectral parameter \( z \) whose dependence is incorporated into \( b_0 \) only. (See Remark 3.3 however.) A natural basis of \( V \) is parametrized as \( |\alpha_1, \ldots, \alpha_n\rangle \) with \( \alpha_i \in \{0, 1\} \), which may be viewed as a state of a spin \( \frac{1}{2} \) chain with \( n \) sites. In this interpretation, the generators \( e_i, f_i, k_i^{\pm 1} \) of \( U_p(g) \) are expressed as exchange type interactions among local spins around site \( i \) of the lattice. The first arrow in (3) stands for an algebra homomorphism defined by

\[
\begin{align*}
\mathfrak{b}_i &\mapsto g_i = f_i + p^2k_i^{-1}e_i + \frac{1}{q+q^{-1}}k_i^{-1} (i \in \mathbb{Z}_n)
\end{align*}
\]

for \( g = A_{n-1}^{(1)} \). A similar embedding is known for all \( \mathfrak{g} \) [3]. In this context, (1) can be viewed as a modified \( p \)-Serre relation. Observe in general that the elements of the form \( g_i' = f_i + c_i k_i^{-1} e_i + d_i k_i^{-1} \in U_p(g) \) with arbitrary coefficients \( c_i, d_i \) behave under the coproduct \( \Delta \) (defined in (8)) as

\[
\Delta g_i' = f_i \otimes 1 + k_i^{-1} \otimes f_i + c_i (k_i^{-1} \otimes k_i^{-1})(1 \otimes e_i + e_i \otimes k_i) + d_i k_i^{-1} \otimes k_i^{-1}
\]

(4)

It implies that the subalgebra \( \mathfrak{B}_p \subset U_p \) generated by \( g_i' \)’s becomes a left coideal subalgebra \( \Delta \mathfrak{B}_p \subset U_p \otimes \mathfrak{B}_p \). In this vein, the coideal subalgebra of \( U_p(A_{n-1}^{(1)}) \) generated by \( g_i \) [3] whose coefficients are deliberately chosen to further fit (1) was called an Onsager coideal in [19]. Its natural analogue for \( g \) other than \( A_{n-1}^{(1)} \) can also be formulated, albeit that a couple of variants are allowed for the coefficients \( c_i, d_i \). See [37–90] and the remarks following them.

Having the Onsager coideals of \( U_p(g) \) of a decent origin, it is tempting to seek the associated reflection \( K \) matrices governed by them via the boundary intertwining relation [3]. In our setting it is represented as the symmetry of local Hamiltonians

\[
(b_i|_{z \to z^{-1}})K(z) = K(z)b_i \quad (0 \leq i \leq n'),
\]

(5)

where the replacement \( z \to z^{-1} \) is relevant only for \( i = 0 \). The integer \( n' \) denotes the rank of \( \mathfrak{g} \), i.e.,

\[
n' = n - 1 \quad \text{for} \quad g = A_{n-1}^{(1)} \quad \text{and for the other type under consideration. It was shown in [19] that [3] admits a unique (up to normalization) solution} \quad K(z) : V \to V \quad \text{satisfying the reflection equation [5, 10, 25].}
\]

Moreover it reproduces the reflection \( K \) matrix constructed by the matrix product method connected to the 3D integrability [29]. In other words, these \( K \) matrices are characterized by the commutativity with the local Hamiltonians.

Introduce the Hamiltonian \( H(z) = \kappa_0 b_0 + \kappa_1 b_1 + \cdots + \kappa_{n'} b_{n'} \) with constant coefficients \( \kappa_0, \ldots, \kappa_{n'} \). It depends on \( z \) via \( b_0 \) only. Then (3) implies the quasi-commutativity \( H(z^{-1})K(z) = K(z)H(z) \) for arbitrary \( \kappa_0, \ldots, \kappa_{n'} \). In the special case \( \kappa_0 = 0 \), \( H(z) \) reduces to a \( z \)-independent operator \( H \) enjoying the symmetry \( [K(z), H] = 0 \). On the other hand, for each \( g \) under consideration, we will show that there is one special choice of \( \kappa_0, \ldots, \kappa_{n'} \) (up to overall normalization) such that all of them are non-vanishing and

\[
[K^\vee(z), H(z)] = 0.
\]

Here \( K^\vee(z) = \sigma^z K(z) \) and \( \sigma^z \) is the global spin reversal operator [13]. In this way, the \( K \) matrices in [29] are shown to serve as various versions of symmetry operators of the Hamiltonians consisting of Onsager algebra generators. See also the ending remarks in Section 12.

Our second main result is the spectral decomposition of the \( K \) matrices with respect to the classical part \( O_p(\mathfrak{g}) \) of the Onsager algebra \( O_p(g) \). The former is defined as the subalgebra of the latter by dropping the generator \( b_0 \). The relation (3) tells that \( K(z) \) commutes with \( O_p(\mathfrak{g}) \) in the representation \( V \) under consideration. Therefore it is a scalar on each irreducible \( O_p(\mathfrak{g}) \) component within \( V \). We present detailed conjectures on the eigenspectra and the decompositions. A typical formula of such kind is [19, 29]. They are boundary analogues of the celebrated spectral decomposition of quantum \( R \) matrices

\[\text{\footnotesize 3 \, \mathfrak{g} \text{ is obtained from } \mathfrak{g} \text{ by removing the } 0 \text{ th vertex in its Dynkin diagram.} \]
with respect to $U_p(\mathcal{F})$, and deserve further studies from the viewpoint of representation theory of Onsager algebras.

Our third main result is a proof of Theorem 3.4 in Appendix 3. It states certain vectors in the $q$-boson Fock space remain invariant under the action of the intertwiner of the quantized coordinate ring $A_q(\text{Sp}_d)$ [18]. The content is apparently independent from the other parts of the paper. However the claim is essential and has been used as a key conjecture in [20] to perform the $q$-boson matrix product construction of the $K$ matrices for $g = D_{n+2}^{(2)}, B_n^{(1)}, B_n^{(1)}$ and $D_n^{(1)}$ treated in this paper. So the proof included here really completes the 3D approach by the authors [20] and establishes the reflection equation independently from the representation theoretical method using Onsager coideals [19].

The layout of the paper is as follows. In Section 2, quantum affine algebras $U_p(g)$ and the $q$-boson matrix product construction of the reflection $K$ matrices [20] are recalled. In Section 3, fundamental representations of $U_p(A_{n-1}^{(1)})$ are recalled and the Hamiltonian associated with $O_p(A_{n-1}^{(1)})$ is given. A simple connection to the Temperley-Lieb algebra [20] is pointed out in Remark 3.1. In Section 4, spectral decomposition of the type $A_{n-1}^{(1)}$ is described. Section 3 is a guide to the subsequent sections devoted to presenting parallel results for $O_p(g)$ with $g$ other than $A_{n-1}$. It summarizes common and general features in these cases. Concrete formulas for the spin representations of $U_p(g)$, Hamiltonians associated with $O_p(g)$ and their $K$ matrix symmetry are given in Section 3 for $D^{(2)}$. Section 7 for $B^{(1)}$, Section 8 for $B^{(1)}$ and Section 9 for $D^{(1)}$. Section 10 and Section 11 describe the spectral decompositions of the $K$ matrices when the classical part of $O_p(g)$ is $O_p(B_n)$ and $O_p(D_n)$, respectively. Section 12 is a summary. Appendix A is a proof of commutativity of the $K$ matrix for type $A_{n-1}^{(1)}$. Appendix B contains a proof of the important Theorem B.1. Throughout the paper the parameters $q, p$ are related by (1) and assumed to be generic. We use the notation

$$(z;q)_m = \prod_{j=1}^{m}(1 - zq^j).$$

2. General remarks and definitions

In this section we introduce the definitions that will be commonly used in the paper.

2.1. Quantum affine algebra $U_p$. Let $U_p = U_p(A_{n-1}^{(1)})$ ($n \geq 3$), $U_p(D_{n+1}^{(2)})$ ($n \geq 2$), $U_p(B_n^{(1)})$ ($n \geq 3$), $U_p(B_n^{(1)})$ ($n \geq 3$), $U_p(B_n^{(1)})$ ($n \geq 3$) be quantum affine algebras without derivation operator [10, 13]. The affine Lie algebra $B_n^{(1)}$ is just $B_n^{(1)}$ but with different enumeration of the nodes as shown in Section 8.1. Note that $U_p(A_{n}^{(1)})$ has been excluded. We assume that $p$ is generic throughout. For convenience set $n' = n-1$ for $A_{n-1}^{(1)}$ and $n' = n$ for the other cases. $U_p$ is a Hopf algebra generated by $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq n'$) satisfying

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$

$$k_i e_j k_i^{-1} = p_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = p_i^{-a_{ij}} f_j, \quad e_i f_j - f_j e_i = \delta_{ij} \left( \frac{k_i - k_i^{-1}}{p_i - p_i^*} \right),$$

(6)

and the Serre relations which will be described later. The Cartan matrix $(a_{ij})_{0 \leq i, j \leq n'}$ [14] will also be given later for each case. The constants $p_i$ ($0 \leq i \leq n'$) in (6) are $p_i = p^2$ except for $p_0 = p_n = p$ for $D_{n+1}^{(2)}, p_n = p$ for $B_n^{(1)}$ and $p_0 = p$ for $B_n^{(1)}$. In addition to $p$, we allow the coexistence of the parameters $q, t$ and the sign factors $\epsilon, \mu$ related as

$$q^{\frac{1}{2}} = i t, \quad p = - i e q^{-1} = i e t^{-2}, \quad \epsilon = \pm 1, \quad \mu = \pm 1.$$  

(7)

The second relation is the same with [19 eq.(96)]. The coproduct $\Delta$ is taken as

$$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i.$$  

(8)

2.2. $U_p$ module $V$ and local spins. We will be concerned with the $U_p$ module $V$ with dim $V = 2^n$ presented as

$$V = \bigoplus_{\alpha \in \{0,1\}^n} \mathbb{C}[\alpha] \simeq (\mathbb{C}^2)^{\otimes n},$$

(9)

$$|\alpha\rangle = |\alpha_1, \ldots, \alpha_n\rangle, \quad |\alpha\rangle = \alpha_1 + \cdots + \alpha_n \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in \{0,1\}. \quad \text{(10)}$$
Vectors $|\beta\rangle$ with $\beta \not\in \{0,1\}^n$ should be understood as 0. The space $V$ will be an irreducible $U_p$ module for $U_p(D_{n+1})$, $U_p(B_n^{(1)})$ and $U_p(B_n^{(1)})$. For $U_p(A_n^{(1)})$ and $U_p(D_n^{(1)})$, one needs to introduce the finer subspaces $V_l$ and $V_{\pm}$ as

$$V_l = \bigoplus_{\alpha \in \{0,1\}^n, |\alpha| = l} \mathbb{C}|\alpha\rangle,$$

$$V_{\pm} = \bigoplus_{\alpha \in \{0,1\}^n, (-1)^{|\alpha|} = \pm 1} \mathbb{C}|\alpha\rangle,$$

which leads to the decompositions

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n, \quad V = V_+ \oplus V_-.$$

Let $\sigma^x_i, \sigma^y_i, \sigma^z_i$ and $\sigma^{\pm}_i = \frac{1}{2}(\sigma^x_i \pm \imath \sigma^y_i)$ ($1 \leq i \leq n$), denote the Pauli matrices acting on the $i$ th component of $V$ regarding $\alpha_i = 1$ as an up-spin and $\alpha_i = 0$ as a down-spin. Namely,

$$\sigma^x_i |\ldots, 1, \ldots, 0\rangle = |\ldots, 0, \ldots, 1\rangle, \quad \sigma^x_i |\ldots, 0, \ldots, 1\rangle = |\ldots, 1, \ldots, 0\rangle,$$

$$\sigma^y_i |\ldots, 1, \ldots, 0\rangle = \imath |\ldots, 0, \ldots, 1\rangle, \quad \sigma^y_i |\ldots, 0, \ldots, 1\rangle = -\imath |\ldots, 1, \ldots, 0\rangle,$$

$$\sigma^z_i |\ldots, 1, \ldots, 0\rangle = 0, \quad \sigma^z_i |\ldots, 0, \ldots, 1\rangle = 0,$$

$$\sigma^+_i |\ldots, 1, \ldots, 0\rangle = |\ldots, 0, \ldots, 1\rangle, \quad \sigma^+_i |\ldots, 0, \ldots, 1\rangle = 0,$$

$$\sigma^-_i |\ldots, 1, \ldots, 0\rangle = |\ldots, 0, \ldots, 1\rangle, \quad \sigma^-_i |\ldots, 0, \ldots, 1\rangle = 0.$$

The global spin reversal operator will be denoted by

$$\sigma^x = \sigma^x_1 \sigma^x_2 \cdots \sigma^x_n.$$

It acts on a base vector as $\sigma^x|\alpha\rangle = |1 - \alpha\rangle$ where $1 = e_1 + \cdots + e_n$.

2.3. $K$ matrices. Let us recall the matrix product construction of the $K$ matrices related to the 3D integrability [20]. We will not use the reflection equations satisfied by them in this paper. They have been described in detail in [20][19].

Let $F_q = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ and $F_q^* = \bigoplus_{m \geq 0} \mathbb{C}\langle m|$ be the Fock space and its dual equipped with the inner product $\langle m|m'\rangle = (q^2;q^2)_m \delta_{m,m'}$. We define the $q$-boson operators $a^+, a^-, k$ on them by

$$a^+|m\rangle = |m + 1\rangle, \quad a^-|m\rangle = (1 - q^{2m})|m - 1\rangle, \quad k|m\rangle = q^m|m\rangle,$$

$$\langle m|a^- = \langle m + 1|, \quad \langle m|a^+ = \langle m - 1| (1 - q^{2m}), \quad \langle m|k = \langle m|q^m. $$

They satisfy $(\langle m|X|m\rangle) = (m|X|m\rangle)$ and the relations

$$k a^+ = q^{-1} a^+ k, \quad a^+ a^- = 1 - k^2, \quad a^- a^+ = 1 - q^2 k^2.$$

We also use the number operator $h$ acting as $h|m\rangle = m|m\rangle$ and $\langle m|h = \langle m|m$ so that $k$ may be identified with $q^h$. Set

$$\begin{pmatrix} K^0_0 & K^0_1 \\ K^1_0 & K^1_1 \end{pmatrix} = \begin{pmatrix} a^+ & -q k \\ k & a^- \end{pmatrix}. $$

The $K$ matrix $K_{tr}(z)$ related to $U_p(A_n^{(1)})$ is given by the matrix product formula [20]:

$$K_{tr}(z) : V \rightarrow V, \quad K_{tr}(z)|\alpha\rangle = \sum_{\beta \in \{0,1\}^n} K_{tr}(z)^{\beta}_{\alpha}|\beta\rangle, $$

$$K_{tr}(z)^{\beta}_{\alpha} = \kappa_{tr,|\alpha}\langle z|K^{\alpha_1}_{\alpha_1} \cdots K_{\alpha_n}, $$

$$\kappa_{tr,l}(z) = (-1)^l q^{\min(0,2l-n)} (1 - q^{-2l+n}) z. $$

The trace here is evaluated by means of [15] and $\text{Tr}(z^h k^m) = \frac{1}{1 - z q^m}$. All the elements $K_{tr}(z)^{\beta}_{\alpha}$ is a rational function of $z$ and $q$. Moreover it is easily seen that $K_{tr}(z)^{\beta}_{\alpha} = 0$ unless $|\alpha| + |\beta| = n$. Thus (19) is actually refined as

$$K_{tr}(z) = K_{tr,0}(z) \oplus \cdots \oplus K_{tr,n}(z), \quad K_{tr,l}(z) : V_l \rightarrow V_{n-l},$$

$$K_{tr}(z)|\alpha\rangle = \sum_{\beta \in \{0,1\}^n, |\beta| = n-|\alpha|} K_{tr}(z)^{\beta}_{\alpha}|\beta\rangle. $$


Some examples from \( n = 3 \) read
\[
\begin{align*}
K_{tt}(z)|011\rangle &= -\frac{q(1-q^2)z}{1-q^2z}|001\rangle + \frac{q^2(1-q^2)z^2}{1-q^2z^2}|010\rangle + \frac{q^2(1-qz)}{1-q^2z}|100\rangle, \\
K_{tt}(z)|101\rangle &= \frac{q^2(1-q^2)z}{1-q^2z}|001\rangle - \frac{q(1-q^2)}{1-q^2z}|100\rangle, \\
K_{tt}(z)|110\rangle &= -\frac{q^2(1-qz)}{1-q^2z}|001\rangle - \frac{q(1-q^2)}{1-q^2z}|100\rangle - \frac{q^2(1-q^2)}{1-q^2z}|110\rangle,
\end{align*}
\]
which are actually the action of \( K_{tt,2}(z) \). We have slightly changed the gauge in \([136]\) from \([20]\) eq.(6) and the normalization factor from \([20]\) eq.(77) to \([19]\) so that
\[
K_{tt}(z)K_{tt}(z^{-1}) = \text{id}_V
\]
is satisfied. Another notable property is the commutativity
\[
[K_{tt}(z), K_{tt}(w)] = 0,
\]
where \([\cdot, \cdot]\) denotes the commutator defined after \([22]\). A proof of \([24]\) is given in Appendix A.

To present the \( K \) matrices \( K_{kk'}(z) \) related to \( U_p(D^{(2)}_{n+1}), U_p(B^{(1)}_n), U_p(B^{(1)}_n), U_p(D^{(1)}_{n+1}) \), we prepare the boundary vectors
\[
\langle \eta_k \rangle = \sum_{m=0}^{\langle km \rangle} \frac{\langle km \rangle}{(q^k; q^k)_m} \in F_q^*, \quad \| \eta_k \| = \sum_{m=0}^{\langle km \rangle} \frac{|km|}{(q^{k}; q^k)_m} \in F_q \quad (k, k' = 1, 2).
\]
Then \( K_{kk'}(z) \) are given by the matrix product construction \([20]\):
\[
\begin{align*}
K_{kk'}(z): V &\to V, \quad K_{kk'}(z)|\alpha\rangle = \sum_{\beta \in \{0,1\}^n} K_{kk'}(z)_{\alpha}^{\beta} |\beta\rangle \quad (k, k' = 1, 2), \\
K_{kk'}(z)_{\alpha}^{\beta} &= \kappa_{kk'}(z) \langle \eta_k | z^n H K_{kk'}^{(1)} \cdots K_{kk'}^{(n)} | \eta_k \rangle \quad (k, k' \neq (2, 2)), \\
K_{22}(z)_{\alpha}^{\beta} &= \kappa_{22}(z)^{-1} \langle \eta_2 | z^n H K_{22}^{(1)} \cdots K_{22}^{(n)} | \eta_2 \rangle,
\end{align*}
\]
where the normalization factors are specified as
\[
\kappa_{kk'}(z) = \langle \eta_k | z^{|h|} \eta_k \rangle^{-1} = \frac{(z^{\max(k,k')}; q^{kk'})_{\infty}}{((-q)^{\min(k,k')} z^{\max(k,k')}; q^{kk'})_{\infty}}.
\]
The quantity \( \langle \eta_k | z^n X | \eta_k \rangle \) for any polynomial \( X \) in \( a^\pm, k \) can be calculated by using \([136]\) and the explicit formula
\[
\begin{align*}
\langle \eta_k | z^n (a^\pm)^j k^n w^n | \eta_k \rangle &= \langle \eta_k | w^n h^n k^n (a^\pm)^j z^n | \eta_k \rangle \quad (k, k' = 1, 2), \\
\langle \eta_1 | z^n (a^\pm)^j k^n w^n | \eta_2 \rangle &= \frac{z^{-l} q^j (q^{l+2m+1} z^2 w^2 - q^{2l+2m} z^4 w^2; q^2)_{\infty}}{(q^m z^2 w; q^2)_{\infty}}, \\
\langle \eta_2 | z^n (a^\pm)^j k^n w^n | \eta_2 \rangle &= \frac{q^j (q^{l+2m+1} z^2 w^2 - q^{2l+2m} z^4 w^2; q^2)_{\infty}}{(q^m z^2 w^2; q^2)_{\infty}},
\end{align*}
\]
where \( \theta(true) = 1, \theta(false) = 0 \). Obviously \( K_{22}(z)_{\alpha}^{\beta} = 0 \) unless \( |\alpha| + |\beta| - n \in 2 \mathbb{Z} \). Therefore \([24]\) for \( (k, k') = (2, 2) \) is refined as
\[
\begin{align*}
K_{22}(z) &= K_{22, +}(z) \oplus K_{22, -}(z), \quad K_{22, \pm}(z): V_0 \to V_0 (-1)^n, \\
K_{22}(z)|\alpha\rangle &= \sum_{\beta \in \{0,1\}^n, |\beta| \in 0 - |\alpha|+2 \mathbb{Z}} K_{22}(z)_{\alpha}^{\beta} |\beta\rangle.
\end{align*}
\]
The normalization factors have been chosen so that all the elements of $K_{k,k'}(z)$ are rational function of $z$, $q$ and

$$K_{k,k'}(z)^{1,1,...,1}_{0,0,...,0} = \frac{(z^{\text{max}(k,k')}; q^{k,k'})_n}{(-q^{z\text{max}(k,k')}; q^{k,k'})_n} \quad ((k,k') \neq (2,2)), \quad (32)$$

$$K_{2,2}(z)^{1,1,...,1}_{0,0,...,0} = -q^{-1} K_{2,2}(z)^{0,1,...,1}_{1,0,...,0} = \theta(n \in 2\mathbb{Z}) \frac{(z^{2}; q^{4})_{n}^{2}}{(q^{z^{2}}; q^{4})_{n}^{2}} + \theta(n \in 2\mathbb{Z} + 1) \frac{(q^{z^{2}}; q^{4})_{n+1}^{2}}{(z^{2}; q^{4})_{n+1}^{2}}. \quad (33)$$

For instance one has

$$K_{1,1}(z)|00\rangle = \frac{(-q; q)_2 z^{2}|00\rangle}{(-qz; q)_2} + \frac{(1 + q)(1 - z)z|01\rangle}{(q - qz; q)_2} + \frac{q(1 + q)(1 - z)z|10\rangle}{(-qz; q)_2} + \frac{(z; q)_2|11\rangle}{(-qz; q)_2},$$

$$K_{1,2}(z)|00\rangle = \frac{(1 + q)z^{2}(1 + q - q^{2}z^{2} + q^{3}z^{2})|00\rangle}{(q - qz; q^2)_2} + \frac{(1 + q)(1 - z^{2})z|01\rangle}{(-qz; q^2)_2} + \frac{q(1 + q)(1 - z^{2})z|10\rangle}{(-qz; q^2)_2} + \frac{(z; q^2)_2|11\rangle}{(-qz; q^2)_2},$$

$$K_{2,1}(z)|00\rangle = \frac{(1 + q)z^{2}(1 - q + q^{2}z^{2} + q^{3}z^{2})|00\rangle}{(-qz; q^2)_2} + \frac{q(1 + q)(1 - z^{2})z|01\rangle}{(q - qz; q^2)_2} + \frac{q^{2}(1 - q^{2}z^{2})|10\rangle}{(z; q^2)_2} + \frac{(1 - q^{2}z^{2})|11\rangle}{(-qz; q^2)_2},$$

$$K_{2,2}(z)|00\rangle = \frac{(1 - q^{2})z^{2}|00\rangle}{(z; q^2)_2} + \frac{q(1 - q^{2}z^{2})|01\rangle}{(q - qz; q^2)_2} + \frac{q^{2}(1 - q^{2}z^{2})|10\rangle}{(z; q^2)_2} + \frac{(1 - q^{2})|11\rangle}{(-qz; q^2)_2}.$$
3.1. $U_p(A^{(1)}_{n-1})$ and fundamental representations. We assume $n \geq 3$. The Dynkin diagram and the Cartan matrix are given by

$$A^{(1)}_{n-1}$$

$$(a_{ij})_{0 \leq i, j \leq n-1} = \begin{pmatrix} 2 & -1 & \cdots & 0 & -1 \\ -1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 2 & \cdots & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

The Serre relations have the form

$$e_i e_j - e_j e_i = 0 \quad (a_{ij} = 0),$$
$$e_i^2 e_j - (p^2 + p^{-2})e_i e_j e_i + e_j e_i^2 = 0 \quad (a_{ij} = -1),$$
and the same ones for $f_j$’s. The fundamental representations are defined on the subspaces $V_0, V_1, \ldots, V_n$ of $V$ in (11) as

$$e_j |\alpha\rangle = z^\delta_{j,a} |\alpha - e_j + e_{j+1}\rangle, \quad f_j |\alpha\rangle = z^{-\delta_{j,a}} |\alpha + e_j - e_{j+1}\rangle, \quad k_j |\alpha\rangle = p^{2(a_{j+1} - a_j)} |\alpha\rangle \quad (j \in \mathbb{Z}_n),$$

where $z$ is a spectral parameter. The symbol $e_j$ denotes the $j$th elementary vector

$$e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n \quad (1 \leq j \leq n).$$

This should not be confused with the generator $e_j$ of $U_p$.

3.2. Onsager algebra $O_p(A^{(1)}_{n-1})$ and the classical part $O_p(A_{n-1})$. Again we assume $n \geq 3$. The algebra $O_p(A^{(1)}_{n-1})$ is generated by $b_0, \ldots, b_{n-1}$ obeying the relations [3]:

$$b_i b_j - b_j b_i = 0 \quad (a_{ij} = 0),$$
$$b_i^2 b_j - (p^2 + p^{-2})b_i b_j b_i + b_j b_i^2 = b_j \quad (a_{ij} = -1).$$

The classical part of $A^{(1)}_{n-1}$ without the vertex 0 is $A_{n-1}$. Thus the subalgebra of $O_p(A^{(1)}_{n-1})$ generated by $b_1, \ldots, b_{n-1}$ is the Onsager algebra for $A_{n-1}$. We denote it by $O_p(A_{n-1})$. The reason to employ $p^{\pm 2}$ here instead of $p^{\pm 1}$ is to avoid $p^{\pm 1}$ in the forthcoming formulas like [13] and [50]–[52] via $q^{\mp 1}$ by [7].

**Remark 3.1.** Let $T_{q,n}$ denote the Temperley-Lieb algebra [20] generated by $t_1, \ldots, t_{n-1}$ obeying the relations

$$t_i t_j - t_j t_i = 0 \quad (|i - j| \geq 2),$$
$$t_i^2 = (q + q^{-1}) t_i,$n
$$t_i t_j t_i = t_i \quad (|i - j| = 1).$$

Under the relation $p^2 = -q^{-2}$ according to [7], it is easy to see that

$$b_i \mapsto t_i - \frac{1}{q + q^{-1}}$$

yields an algebra homomorphism $O_p(A_{n-1}) \to T_{q,n}$. The case $p^2 = 1$ studied in [25] corresponds to the singular situation $q + q^{-1} = 0$.

3.3. Representation $\pi^{lr}_i$. The representation $\pi^{lr}_i$ of $O_p(A^{(1)}_{n-1})$ on $V_l$ is obtained by the composition

$$\pi^{lr}_i : O_p(A^{(1)}_{n-1}) \to U_p(A^{(1)}_{n-1}) \to \text{End} V_l,$n

where the latter is the $l$-th fundamental representation [43] and the former embedding is given by

$$b_i = f_i + p^2 k_i^{-1} e_i + \frac{1}{q + q^{-1}} k_i^{-1} \quad (i \in \mathbb{Z}_n).$$

This corresponds to [19] eq.(34) with $p = -i\epsilon q^{-1}$ according to [7].
The summands in (51) are expressed by the local spins (13) as follows:

$$f_i = z^{-\delta_i} \sigma_i^+ \sigma_i^- + p^2 \epsilon_i = z^{\delta_i} \sigma_i^- \sigma_i^+,$$

$$\frac{1}{q + q^{-1}} k_i^{-1} = \frac{q + q^{-1}}{4} \sigma_i^+ \sigma_i^- + \frac{q - q^{-1}}{4} (\sigma_i^z - \sigma_i^{-z} - (q - q^{-1})^2 (4(q + q^{-1})).$$

The sum of two terms in (51) with $i = 0$ is also written as

$$z^{-1} \sigma_n^+ \sigma_i^- + z \sigma_n^- \sigma_i^+ = \frac{z + z^{-1}}{4}(\sigma_n^+ \sigma_i^+ + \sigma_n^- \sigma_i^-) - \frac{z - z^{-1}}{4}(\sigma_n^z \sigma_i^z - \sigma_n^- \sigma_i^-),$$

where the second summand is a Dzyaloshinskii-Moriya (DM) interaction term. The constant term appearing in (52) is redundant for $\sigma_n^z$. We denote the image of $\sigma_n^z$ by the composition $b_i$, i.e., $b_i = \pi_i^r(b_i)$. Thus it is identified with a local Hamiltonian of XXZ type:

$$b_i = z^{-\delta_i} \sigma_i^+ \sigma_i^- + z^{\delta_i} \sigma_i^- \sigma_i^+ + \frac{q + q^{-1}}{4} \sigma_i^+ \sigma_i^- + \frac{q - q^{-1}}{4} (\sigma_i^z - \sigma_i^{-z} + \Gamma (i \in \mathbb{Z}).$$

Remark 3.2. According to [3 Prop.2.1], setting

$$b_i = f_i + p^2 k_i^{-1} \epsilon_i + d_i k_i^{-1} (i \in \mathbb{Z}),$$

provides an embedding $O_p(A_n^{(1)}) \hookrightarrow U_p(A_n^{(1)})$ if and only if the coefficients $d_i$ satisfy

$$d_i \left( d_j - \frac{1}{q + q^{-1}} \right) = 0 \quad \forall (i,j) \text{ such that } (a_{ij}, a_{ji}) = (-1, -1).$$

The formula (50) corresponds to $\forall d_i = \frac{1}{q + q^{-1}}$. Another choice $\forall d_i = \frac{1}{q + q^{-1}}$ followed by a similarity transformation $b_i \mapsto \sigma^+ \sigma_i^0 \sigma_i^x$ by (14) leads to another representation of $O_p(A_n^{(1)})$:

$$b_i' = \sigma_i^+ \sigma_i^- + \sigma_i^- \sigma_i^+ + \frac{q + q^{-1}}{4} \sigma_i^+ \sigma_i^- + \frac{q - q^{-1}}{4} (\sigma_i^z - \sigma_i^{-z}) - \Gamma (1 \leq i \leq n).$$

Its constant shift according to (48), i.e.,

$$b_i' + \frac{1}{q + q^{-1}} = \sigma_i^+ \sigma_i^- + \sigma_i^- \sigma_i^+ + \frac{q + q^{-1}}{4} \sigma_i^+ \sigma_i^- + \frac{q - q^{-1}}{4} (\sigma_i^z - \sigma_i^{-z}) + \frac{q + q^{-1}}{4}$$

reproduces the well-known realization of the Temperley-Lieb generators by an $n$ site spin $\frac{1}{2}$ chain [21, 22].

It has been shown [19] that the $K$ matrix $K_{tr}(z)$ (17–18) is characterized, up to normalization, by the commutativity with the Onsager algebra:

$$K_{tr}(z) b_i = \left. b_i \right|_{z \rightarrow z^{-1}} K_{tr}(z) \quad (0 \leq i \leq n),$$

where the replacement $z \rightarrow z^{-1}$ matters only for $i = 0$.

Set

$$H_{tr}(z) = b_0 + b_1 + \cdots + b_{n-1} = \sum_{i \in \mathbb{Z}} \left( z^{-\delta_i} \sigma_i^+ \sigma_i^- + z^{\delta_i} \sigma_i^- \sigma_i^+ + \frac{q + q^{-1}}{4} \sigma_i^+ \sigma_i^- \right) + n \Gamma,$$

where the $z$-dependence comes only from $b_0$. We have taken the coefficients of $b_i$’s so that the sum eliminates the $\sigma_i^z$-linear terms in (55), and therefore $\sigma^z H_{tr}(z) \sigma^z = H_{tr}(z^{-1})$ holds with $\sigma^z$ defined by (14). Then (60) and (55) lead to the commutativity

$$[K_{tr}(z), H_{tr}(z)] = 0.$$

To construct higher order commuting Hamiltonians within the Onsager algebra $O_p(A_n^{(1)})$ is an outstanding problem whose solution has been known only at $p = 1$ [28, 31]. See also the ending remarks in Section 12. As far as $O_p(A_n^{(1)})$ is concerned, it may be useful to combine Remark 5.3 and 12.

Let us comment on the hermiticity of the Hamiltonians. The local ones $b_0, \ldots, b_{n-1}$ (55) are all hermite if and only if $|z| = 1$ and $q \in \mathbb{R}$. When $|z| = 1$ and $q \in i\mathbb{R}$, they are hermite except for the summand

$\lambda$. The condition $a_{ij} = a_{ji}$ is redundant for $A_n^{(1)}$, but it is included for the later use [91] in non simply-laced algebras.
Systematizing such investigations leads to the conjecture that there are irreducible even, we also define

\[ e^z \]

Remark 3.3. It is possible to formulate an \( n \)-parameter version of the above result. This is due to the algebra automorphism \( e_i \mapsto z_i e_i, f_i \mapsto z_i^{-1} f_i, k_i \mapsto k_i \) involving the nonzero parameters \( z_i \) \((i \in \mathbb{Z}_n)\). Alternatively, one may keep \( \mathbb{Z}_n \) and modify the representation \( \Pi \) into

\[ c_j |\alpha\rangle = z_j (|\alpha\rangle - e_j + e_{j+1}), \quad f_j |\alpha\rangle = z_j^{-1} (|\alpha\rangle + e_j - e_{j+1}), \quad k_j |\alpha\rangle = p^{2(\alpha_{j+1} - \alpha_j)} |\alpha\rangle \quad (j \in \mathbb{Z}_n). \quad (63) \]

Then \( \Pi \) is changed into

\[ H(\alpha, \cdots, \beta, \gamma) = \sum_{i<j} (z_i^{-1} \sigma_i^+ \sigma_i^- + z_j \sigma_j^+ \sigma_j^- + \frac{q+q^{-1}}{4} \sigma_i^+ \sigma_j^-) + n \Gamma. \quad (64) \]

The choice \( z_i = z \) for all \( i \in \mathbb{Z}_n \) is the model involving uniform DM terms studied in \([1]\). Introduce \( K_r(z_0, \cdots, z_{n-1}) = \sigma^r K_{\text{tr}}(z_0, \cdots, z_{n-1}) \) similarly to \( \Pi \), where elements of the latter is defined by generalizing \( \Pi \) to

\[ K_{\text{tr}}(z_0, z_1, \cdots, z_{n-1}) = \text{Tr}(z_0^{h_1^{\sigma_i^+} z_1^{h_2^{\sigma_j^+}} \cdots z_{n-1}^{h_{n-1}^{\sigma_i^+}}) \quad (65) \]

up to overall normalization. Then the following commutativity is valid:

\[ [K_{\text{tr}}(z_0, \cdots, z_{n-1}), H(z_0, \cdots, z_{n-1})] = 0. \quad (66) \]

This kind of multi-parameter generalizations are possible also for \( U_p \) treated in later sections, although they will be omitted for simplicity.

4. Spectral Decomposition of \( K_{\text{tr}} \) by \( O_p(A_n) \)

The classical part \( O_p(A_n) \) of the Onsager algebra \( O_p(A_n) \) introduced in Section 3.2 has the representation

\[ O_p(A_n) \to O_p(A_n) - \pi_{1r} - \pi_{2r} \to \text{End} V_l \quad (0 \leq l \leq n). \quad (67) \]

We denote this restriction also by \( \pi_{1r} \). The relation \( \Pi \) with \( i \neq 0 \) tells the commutativity

\[ [K_{\text{tr}}(z), \pi_{1r}(O_p(A_n)))] = 0. \quad (68) \]

The representation \( \pi_{1r} \) of \( O_p(A_n) \) on \( V_l \) is irreducible \([19]\). On the other hand it is not so with respect to the classical subalgebra \( O_p(A_n) \). The \( K \) matrix \( K_{\text{tr}}(z) \) should be a scalar on each irreducible component. For instance when \( (n, l) = (4, 2) \), it acts on the \( \binom{4}{2} = 6 \) dimensional space \( V_2 \), and its eigenvalues read

\[ 1, 1, \quad \frac{q^2 - z}{-1 + q^2 z}, \quad \frac{q^2 - z}{-1 + q^2 z}, \quad \frac{q^2 - z}{-1 + q^2 z}, \quad \frac{(q^2 - z)(q^4 - z)}{(1 - q^2 z)(1 - q^4 z)}. \quad (69) \]

The multiplicities 2, 3 and 1 here are equal to the Kostka numbers \( \binom{4}{2} - \binom{1}{2}, \binom{4}{2} - \binom{2}{2} \) and \( \binom{4}{2} \), respectively. Systematizing such investigations leads to the conjecture that there are irreducible \( O_p(A_n) \) modules \( W_{l,j} \) with \( 0 \leq l \leq \frac{n}{2} \) or \( \frac{n}{2} < l \leq n \) having the properties (i), (ii) and (iii) described below:

(i) \( V_0, V_1, \ldots, V_n \) are decomposed as

\[ V_l = \begin{cases} \oplus V_l, & (0 \leq l \leq \frac{n}{2}), \\ \oplus V_l, & (\frac{n}{2} < l \leq n). \end{cases} \quad (70) \]

\[ \dim W_{l,j} = \begin{cases} \binom{n}{j} - \binom{n}{j-1}, & (0 \leq j \leq l \leq \frac{n}{2}), \\ \binom{n}{j} - \binom{n}{j+1}, & (\frac{n}{2} < l \leq j \leq n). \end{cases} \quad (71) \]

This is consistent with \( V_l = \binom{n}{l} \) and satisfies \( \dim W_{l,j} \) \( = \dim W_{n-l,n-j} \). For convenience when \( n \) is even, we also define \( W_{\frac{n}{2}, j} \) with \( j > \frac{n}{2} \) by setting \( W_{\frac{n}{2}, j} = W_{\frac{n}{2}, n-j} \) for all \( 0 \leq j \leq n \).

(ii) The decomposition \( K_{\text{tr}}(z) \in \bigoplus_{l=0}^{n} \text{Hom}(V_l, V_{l-1}) \) in \( (20) \) is refined into

\[ K_{\text{tr}}(z) = \bigoplus_{0 \leq j \leq \frac{n}{2}} \text{Hom}(W_{l,j}, W_{n-l,n-j}) \oplus \bigoplus_{\frac{n}{2} < l \leq n} \text{Hom}(W_{l,j}, W_{n-l,n-j}), \quad (72) \]

where each component is an isomorphism of \( O_p(A_n) \) modules.
(iii) There exists a basis \( \{ \xi_{i,j} \mid 1 \leq i \leq \dim W_{i,j} \} \) of \( W_{i,j} \) in terms of which the isomorphism in (ii) is explicitly described as the spectral decomposition:

\[
K_{ij}(z) = \bigoplus_{0 \leq j \leq l < \frac{p}{2}} p_{i,j}(z)P_{i,j} \oplus \bigoplus_{\frac{p}{2} < j \leq n} p_{i,j}(z)P_{i,j},
\]

\[
P_{i,j}(z) = \beta_{ij}^\alpha(z^{1-l-n-j}),
\]

We have used infinite products in order to make the formula uniform. However all the eigenvalues of the \( K \) matrices appearing here and in what follows are rational functions of \( z \). It is easy to see \( p_{ij}(z)\rho_{n-l-n-j}(z^{-1}) = 1 \), which is consistent with (22).

In view of Remark 5.1, we expect that \( W_{i,j} (0 \leq j \leq \frac{p}{2}) \) is the irreducible representation \( V_{[n-j,i]} \) of the Temperley-Lieb algebra \( T_{q,n} \) labeled with the two row Young diagram \( [n-j,i] \) in [13] p126. The decomposition \( (70) \) corresponds to \( (2) \) eq.(57).

5. TYPES OTHER THAN \( U_p(A_{n-1}^{(1)}) \): GENERAL FEATURES

This brief section is a guide to Sections \([8,11]\) where contents analogous to \( U_p(A_{n-1}^{(1)}) \) case in Section \([3,4]\) will be presented individually for the other \( U_p \) under consideration. They consist of so many cases that one may wonder if it is possible to grasp them in a unified manner. Our aim here is to indicate how to do so at least partially. We note that these variety of cases have originated in the solutions of the reflection equation listed in [20] Sec.6 and the corresponding coideals in [19] App.B.

For convenience we set

\[
g^{1,1} = D_{n+1}^{(2)}, \quad g^{2,1} = B_n^{(1)}, \quad g^{1,2} = \tilde{B}_n^{(1)}, \quad g^{2,2} = D_n^{(1)}.
\]

The superscript \( r \) in \( g^{r, r'} \) indicates that the Dynkin diagram around the 0 th node is an outward double arrow for \( r = 1 \) and trivalent for \( r = 2 \). The shape around the \( n \) th node is specified by \( r' \) similarly. The quantity \( p_i \) defined after (4) is written as \( p_0 = p^\rho, \quad p_i = p^2 (0 < i < n) \) and \( p_n = p^{r'} \).

For each \( g^{r, r'} \), we will consider the quantum affine algebra \( U_p(g^{r, r'}) \) and the Onsager algebra \( O_p(g^{r, r'}) \) [3]. The Serre relations in \( U_p(g^{r, r'}) \) read

\[
e_i e_j - e_j e_i = 0 \quad (a_{ij} = 0),
\]

\[
e_i e_j - (p^2 + p^{-2}) e_i e_j = 0 \quad (a_{ij} = -1),
\]

\[
e_i^2 - e_j^2 - (p^2 + 1 + p^{-2}) e_i e_j - (p^2 + 1 + p^{-2}) e_j e_i + (p^2 + 1 + p^{-2}) e_i e_j e_i = 0 \quad (a_{ij} = -2),
\]

and the same ones for \( f \)'s. The other relations have been already given in [3].

The Onsager algebra \( O_p(g^{r, r'}) \) is generated by \( b_0, \ldots, b_n \) obeying modified \( p \)-Serre relations [3]:

\[
b_i b_j - b_j b_i = 0 \quad (a_{ij} = 0),
\]

\[
b_i^p b_j - (p^2 + 1 + 1) b_i b_j = b_j \quad (a_{ij} = -1),
\]

\[
b_i^p b_j - (p^2 + 1 + p^{-2}) b_i b_j = (p + 1 + p^{-2}) b_i b_j - b_j b_i^p = (p + 1 + p^{-2}) b_i b_j - b_j b_i^p \quad (a_{ij} = -2).
\]

Except for (70) and (92), which are void for the simply-laced \( U_p(g^{2, 2}) \) and \( O_p(g^{2, 2}) \), these relations are formally the same with those in type \( A_{n-1}^{(1)} \). In terms of commutators \( [X, Y] = [X, Y]_1, [X, Y]_r = X Y - r Y X \), the relations (70) - (92) are written more compactly as

\[
[a_i, a_j] = 0 \quad (a_{ij} = 0),
\]

\[
[a_i, \{a_i, a_j\}_r]_{r=2} = a_j \quad (a_{ij} = -1),
\]

\[
[a_i, \{a_i, \{a_i, a_j\}_r\}_{r=2}] = (p + 1 + p^{-2}) [a_i, a_j] \quad (a_{ij} = -2).
\]

The quartic relation of the form (85) with \( p^2 = 1 \) is often referred to as the Dolan-Grady condition [9]. It is typical for the situation \( a_{ij} = -2 \), which was indeed utilized to reformulate the original Onsager algebra for \( A_1^{(1)} \) in only two generators. The Onsager algebra \( O_p(D_n^{(1)}) \) with \( p = 1 \) was introduced in [6]. It is an interesting open question if there is an analogue of Remark 5.1 for \( g \neq A_{n-1}^{(1)} \) related to a boundary extension of the Temperley-Lieb algebra like [4].
We will deal with the representations of $O_p(\mathfrak{g}^{r,r'})$ constructed as
\[
\pi^{r,r'}_{k,k'} : O_p(\mathfrak{g}^{r,r'}) \hookrightarrow U_p(\mathfrak{g}^{r,r'}) \to \text{End} V \quad ((r,k),(r',k') = (1,1),(1,2),(2,2))
\] (86)
with $V = \mathbb{C}^n$. Thus there are nine cases to consider. We remark that the strange condition $r \leq k, r' \leq k'$ originates in (227) to validate Theorem B.1 which was a key in the 3D approach [20]. The latter arrow in (86) is the spin representations of $U_p(\mathfrak{g}^{r,r'})$ which will be specified in later sections. They carry a spectral parameter $z$. The former embedding depends on $k, k'$ and is given by
\[
b_0 = f_0 + p' k_0 e_0 + d_k k_0^{-1},
\] (87)
\[
b_i = f_i + p^2 k_i e_i + \frac{1}{q + q^{-1}} k_i^{-1} (0 < i < n),
\] (88)
\[
b_n = f_n + p' k_n e_n + d_k' k_n^{-1},
\] (89)
\[
d_1^2 = \frac{q^2 + q^{-2}}{q + q^{-1}}, \quad d_2^2 = \frac{1}{q + q^{-1}}.
\] (90)
See [7] for the relations among the parameters $p, q, \epsilon$ etc. Recall also that $p_0, \ldots, p_n$ were specified after [5]. In general, according to [3] Prop.2.1, setting $b_i = f_i + p_i k_i^{-1} e_i + d_i k_i^{-1} (0 \leq i \leq n)$ provides an embedding $O_p(\mathfrak{g}^{r,r'}) \to U_p(\mathfrak{g}^{r,r'})$ if and only if the following condition is satisfied:
\[
\text{det} \left( d_i^2 - \frac{1}{(q + q^{-1})^2} \right) = 0 \quad \forall (i,j) \text{ such that } (a_{ij}, a_{ji}) = (-2,-1).
\] (91)

One can check that (87)–(90) fulfills this and the relations (80)–(82) directly.

As in $U_p(A^{(1)}_{n-1})$, we shall write $b_i = \pi^{r,r'}_{k,k'}(b_i)$ to mean the representation (86) of $b_i \in O_p(\mathfrak{g}^{r,r'})$. Its dependence on $(k, k')$ should not be forgotten although it is suppressed in the notation for simplicity. Among $b_0, b_1, \ldots, b_n$, there is a special (affine) one $b_0$ which includes the spectral parameter $z$ built in the spin representations.

It has been shown [19] that the $K$ matrix $\bar{K}_{k,k'}(z)$ is characterized, up to normalization, by the commutativity with the Onsager algebra:
\[
\bar{K}_{k,k'}(z)b_i = (b_i|_{z \to z^{-1}})\bar{K}_{k,k'}(z) \quad (0 \leq i \leq n),
\] (92)
where $\bar{K}_{k,k'}(z)$ has been defined in (33).

It turns out that the analogue of $H_{k_1}(z)$ in (61) can be constructed for the representation $\pi^{r,r'}_{k,k'}$ if and only if $(r,r') = (k,k')$. In fact, for the generators $b_0, \ldots, b_n$ in $\pi^{k,k'}_{k,k'}(O_p(\mathfrak{g}^{k,k'}))$, it is possible to choose the constant ($z$-independent) coefficients $\kappa_0, \ldots, \kappa_n$ so that
\[
H_{k,k'}(z) = \kappa_0 b_0 + \kappa_1 b_1 + \cdots + \kappa_n b_n
\] (93)
becomes free from $\sigma_i^z$-linear terms and fulfills $\sigma^z H_{k,k'}(z) \sigma^z = H_{k,k'}(z^{-1})$. As a result, (92) leads to
\[
[K^\vee_{k,k'}(z), H_{k,k'}(z)] = 0,
\] (94)
where $K^\vee_{k,k'}(z)$ has been introduced in (38). The concrete forms of $H_{k,k'}(z)$ will be presented in (106), (142), (161) and (179).

The local Hamiltonians $b_0, \ldots, b_n$ are all hermite if and only if $|z| = 1$ and $t_{\text{min}(k,k')} \in \mathbb{R}$, where $t$ is related to $p$ and $q$ as in (7). When $|z| = 1$ and either $t_{\text{min}(k,k')} \in \mathbb{R}$ or $t_{\text{min}(k,k')} \in i\mathbb{R}$, some of them acquire a pure imaginary magnetic field term. The Hamiltonian $H_{k,k'}(z)$ is hermite if and only if $|z| = 1$ and either $t_{\text{min}(k,k')} \in \mathbb{R}$ or $t_{\text{min}(k,k')} \in i\mathbb{R}$.

Next let us motivate Section 10 and 11 In view of the Dynkin diagrams, it is natural to denote the subalgebra of $O_p(\mathfrak{g}^{r,r'})$ generated by $b_0, \ldots, b_n$ by $O_p(B_n)$ for $r' = 1$ and $O_p(D_n)$ for $r' = 2$. By inspection it is easy to see that $\pi^{r,1}_{k,1}$ of $O_p(\mathfrak{g}^{r,1})$ defines the same representation of $O_p(B_n)$ for any choice $(r,k) = (1,1),(1,2),(2,2)$. This common representation will naturally be denoted by $\pi^1$. The relation (92) implies that $K_{1,1}(z)$ commutes with $O_p(B_n)$ in the representation $\pi^1$. The spectral decomposition of $K_{1,1}(z)$ with respect to $\pi^1(O_p(B_n))$ will be given in Section 10. Similarly, $\pi^{r,2}_{k,2}$ of $O_p(\mathfrak{g}^{r,2})$ yields the same representation of $O_p(D_n)$ for $(r,k) = (1,1),(1,2),(2,2)$. It will be denoted by $\pi^2$. The relation (92) also implies that $K_{k,2}(z)$ commutes with $O_p(D_n)$ in the representation $\pi^2$. The spectral decomposition of $K_{k,2}(z)$ with respect to $\pi^2(O_p(D_n))$ will be given in Section 11 (We do not treat $r' \neq k'$ case to avoid technical complexity.)
6.2. Onsager algebra \( O_p(D_{n+2}^{(2)}) \) and the classical part \( O_p(B_n) \). The algebra \( O_p(D_{n+2}^{(2)}) \) is generated by \( b_0, \ldots, b_n \) obeying (S0–S2). The classical part of \( D_{n+2}^{(1)} \) without the vertex 0 is \( B_n \). Thus the subalgebra of \( O_p(D_{n+2}^{(2)}) \) generated by \( b_1, \ldots, b_n \) is the Onsager algebra for \( B_n \). We denote it by \( O_p(B_n) \).

6.3. Representations \( \pi_{1,1}^{1,1} \). The representation \( \pi_{1,1}^{1,1} \) of \( O_p(D_{n+1}^{(2)}) \) on \( V \) in (E) is obtained by the composition

\[
\pi_{1,1}^{1,1} : O_p(D_{n+2}^{(2)}) \rightarrow U_p(D_{n+1}^{(2)}) \rightarrow \text{End}V,
\]
where the latter is the spin representation (65)–(67) and the former embedding (68)–(70) reads as

\[
b_0 = f_0 + pk_0^{-1}e_0 - ie\mu \frac{t-t^{-1}}{t^2 + t^{-2}}k_0^{-1}, \quad b_i = f_i + pk_i^{-1}e_i - \frac{1}{t^2 + t^{-2}}k_i^{-1}. \quad (0 < i < n),
\]
\[
b_n = f_n + pk_n^{-1}e_n - ie\mu \frac{t-t^{-1}}{t^2 + t^{-2}}k_n^{-1}.
\]

This corresponds to (139) eqs.(167)-(169)] with \( s = s' = q^2 \) according to (19) eq.(96)]. These generators are represented as local Hamiltonians:

\[
b_0 = z\sigma_1^+ + z^{-1}\sigma_1^- - \mu \frac{t-t^{-1}}{2}\sigma_1^z - \mu \frac{(t-t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})},
\]
\[
b_i = \sigma_i^z\sigma_{i-1}^+ + \sigma_{i-1}^-\sigma_i^+ - \frac{t^2 + t^{-2}}{4}\sigma_i^z\sigma_{i+1}^z - \frac{t^2 - t^{-2}}{4}(\sigma_i^z - \sigma_{i+1}^z) + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})}, \quad (0 < i < n),
\]
\[
b_n = \sigma_n^z + \mu \frac{t-t^{-1}}{2}\sigma_n^z - \mu \frac{(t-t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})}.
\]

They commute with the \( K \) matrix (64) up to \( z \) as (19):

\[
\tilde{K}_{1,1}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\tilde{K}_{1,1}(z) \quad (0 \leq i \leq n).
\]

Set

\[
H_{1,1}(z) = -\frac{\mu(t+t^{-1})}{2}b_0 + b_1 + \cdots + b_{n-1} - \frac{\mu(t+t^{-1})}{2}b_n
\]
\[
= -\frac{\mu(t+t^{-1})}{2}(z\sigma_1^+ + z^{-1}\sigma_1^- + \sigma_1^z) + \sum_{i=1}^{n-1} \left( \sigma_i^+\sigma_{i+1}^+ + \sigma_i^-\sigma_{i+1}^- - \frac{t^2 + t^{-2}}{4}\sigma_i^z\sigma_{i+1}^z \right) + (n+1)\Gamma.
\]

It satisfies \( \sigma^x H_{1,1}(z) \sigma^x = H_{1,1}(z^{-1}) \), therefore (103) and (38) lead to the commutativity

\[
[K_{1,1}^{
u}(z), H_{1,1}(z)] = 0. \tag{107}
\]

The Hamiltonian \( H_{1,1}(z) \) has appeared for example in [21 eq.(1.3)] with \( \beta_\pm = \theta_\pm = 0, e^{\theta_\pm} = z, e^{\varphi} = -t^2, \quad \sinh\alpha_\pm = \frac{\mu(t-t^{-1})}{2} \).
6.4. Representation $\pi_{2,1}^{1,1}$. The representation $\pi_{2,1}^{1,1}$ of $O_p(D_{n+1}^{(2)})$ on $V$ in (104) is obtained by the composition

$$\pi_{2,1}^{1,1} : O_p(D_{n+1}^{(2)}) \rightarrow U_p(D_{n+1}^{(2)}) \rightarrow EndV,$$

where the latter is the spin representation [85]-[87] and the former embedding [87]-[90] reads as

$$b_0 = f_0 + pk_0^{-1}e_0, \quad b_i = f_i + p^2 k_i^{-1}e_i - \frac{1}{t_i^2 + t^{-2}}k_i^{-1} (0 < i < n), \quad b_n = f_n + pk_n^{-1}e_n - i\epsilon \frac{t - t^{-1}}{t^2 + t^{-2}}k_n^{-1}.$$

This corresponds to [19] eqs.(167), (172)-(173) with $s = s' = q^\frac{\pi}{2}$. These generators are represented as local Hamiltonians:

$$b_0 = z\sigma_1^+ + z^{-1}\sigma_1^-,$$

$$b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ - \frac{t_i^2 + t^{-2}}{4}\sigma_i^+\sigma_{i+1}^- = \frac{t_i^2 - t^{-2}}{4}(\sigma_i^+ - \sigma_{i+1}^-) + \frac{(t_i^2 - t^{-2})^2}{4(t_i^2 + t^{-2})} \quad (0 < i < n),$$

$$b_n = \sigma_n^+ + \mu \frac{t - t^{-1}}{2}\sigma_n^- - \mu \frac{(t - t^{-1})(t^2 - t^{-2})}{2(t_1^2 + t^{-2})}.$$

They commute with the $K$ matrix [34] up to $z$ as [19]:

$$K_{2,1}(z)b_i = (b_i|_{z \rightarrow z^{-1}})K_{2,1}(z) \quad (0 \leq i \leq n).$$

6.5. Representation $\pi_{1,2}^{1,1}$. The representation $\pi_{1,2}^{1,1}$ of $O_p(D_{n+1}^{(2)})$ on $V$ in (105) is obtained by the composition

$$\pi_{1,2}^{1,1} : O_p(D_{n+1}^{(2)}) \rightarrow U_p(D_{n+1}^{(2)}) \rightarrow EndV,$$

where the latter is the spin representation [85]-[87] and the former embedding [87]-[90] reads as

$$b_0 = f_0 + pk_0^{-1}e_0 - i\epsilon \frac{t - t^{-1}}{t^2 + t^{-2}}k_0^{-1}, \quad b_i = f_i + p^2 k_i^{-1}e_i - \frac{1}{t_i^2 + t^{-2}}k_i^{-1} (0 < i < n), \quad b_n = f_n + pk_n^{-1}e_n.$$

This corresponds to [19] eqs.(167), (176)-(177) with $s = s' = q^\frac{\pi}{2}$ according to [19] eq.(96)]. These generators are represented as local Hamiltonians:

$$b_0 = z\sigma_1^+ + z^{-1}\sigma_1^- - \mu \frac{t - t^{-1}}{2}\sigma_1^2 - \mu \frac{(t - t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})},$$

$$b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ - \frac{t_i^2 + t^{-2}}{4}\sigma_i^+\sigma_{i+1}^- = \frac{t_i^2 - t^{-2}}{4}(\sigma_i^+ - \sigma_{i+1}^-) + \frac{(t_i^2 - t^{-2})^2}{4(t_i^2 + t^{-2})} \quad (0 < i < n),$$

$$b_n = \sigma_n^+,$$

They commute with the $K$ matrix [34] up to $z$ as [19]:

$$K_{1,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})K_{1,2}(z) \quad (0 \leq i \leq n).$$

6.6. Representation $\pi_{2,2}^{1,1}$. The representation $\pi_{2,2}^{1,1}$ of $O_p(D_{n+1}^{(2)})$ on $V$ in (106) is obtained by the composition

$$\pi_{2,2}^{1,1} : O_p(D_{n+1}^{(2)}) \rightarrow U_p(D_{n+1}^{(2)}) \rightarrow EndV,$$

where the latter is the spin representation [85]-[90] and the former embedding [87]-[90] reads as

$$b_0 = f_0 + pk_0^{-1}e_0, \quad b_i = f_i + p^2 k_i^{-1}e_i + \frac{1}{q + q^{-1}}k_i^{-1} (0 < i < n), \quad b_n = f_n + pk_n^{-1}e_n.$$

$$\sigma_i^+ = \frac{1}{2}(\sigma_i^+ + \sigma_{i+1}^-), \quad \sigma_i^- = \frac{1}{2}(\sigma_i^- + \sigma_{i+1}^+),$$

$$\sigma_i^0 = \frac{1}{2}(\sigma_i^+ - \sigma_{i+1}^-).$$
This corresponds to [19] eqs.(167), (184)-(185) with $s = s' = q^\frac{1}{2}$. These generators are represented as local Hamiltonians:

$$b_0 = z\sigma_1^+ + z^{-1}\sigma_1^-,$$  \(128\)

$$b_i = \sigma_i^+\sigma_{i+1}^+ + \sigma_i^-\sigma_{i+1}^- + \frac{q + q^{-1}}{4}\sigma_i^2\sigma_{i+1}^2 + \frac{q - q^{-1}}{4}(\sigma_i^- - \sigma_{i+1}^-) - \frac{(q - q^{-1})^2}{4(q + q^{-1})} (0 < i < n),$$  \(129\)

$$b_n = \sigma_n^x.$$  \(130\)

They commute with the $K$ matrix [34] up to $z$ as [19]:

$$\tilde{K}_{2,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\tilde{K}_{2,2}(z) \quad (0 \leq i \leq n).$$  \(131\)

7. $O_P(B_n^{(1)})$ Hamiltonians

7.1. $U_p(B_n^{(1)})$ and spin representations. The Dynkin diagram and the Cartan matrix are given by

$$g^{2,1} = B_n^{(1)}$$

$\begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array}$

\[\left(\begin{array}{cccc}
2 & 0 & -1 & \cdots & 0 \\
0 & 2 & -1 & \cdots & 0 \\
-1 & -1 & 2 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -2 & 1 \\
0 & 0 & \cdots & -2 & 2
\end{array}\right)\]  \(132\)

The spin representation on $V$ is given by

$$e_0|\alpha\rangle = z^2|\alpha + e_1 + e_2\rangle, \quad f_0|\alpha\rangle = z^{-2}|\alpha - e_1 - e_2\rangle, \quad k_0|\alpha\rangle = p^{2(a_1 + a_2 - 1)}|\alpha\rangle,$$  \(133\)

$$e_j|\alpha\rangle = |\alpha - e_j + e_{j+1}\rangle, \quad f_j|\alpha\rangle = |\alpha + e_j - e_{j+1}\rangle, \quad k_j|\alpha\rangle = p^{2(a_{j+1} - a_j)}|\alpha\rangle \quad (0 < j < n),$$  \(134\)

$$e_n|\alpha\rangle = |\alpha - e_n\rangle, \quad f_n|\alpha\rangle = |\alpha + e_n\rangle, \quad k_n|\alpha\rangle = p^{2a_n}|\alpha\rangle.$$  \(135\)

7.2. Onsager algebra $O_P(B_n^{(1)})$ and the classical part $O_P(B_n)$. The algebra $O_P(B_n^{(1)})$ is generated by $b_0, \ldots, b_n$ obeying the relations (50)–(52). The classical part of $B_n^{(1)}$ without the vertex 0 is $B_n$. Thus the subalgebra of $O_P(B_n^{(1)})$ generated by $b_1, \ldots, b_n$ agrees with the Onsager algebra $O_P(B_n)$ introduced in Section 6.2.

7.3. Representation $\pi_{2,1}^{2,1}$. The representation $\pi_{2,1}^{2,1}$ of $O_P(B_n^{(1)})$ on $V$ in [9] is obtained by the composition

$$\pi_{2,1}^{2,1} : O_P(B_n^{(1)}) \hookrightarrow U_p(B_n^{(1)}) \rightarrow \mathrm{End}V,$$  \(136\)

where the latter is the spin representation [132], [134] and the former embedding [87], [90] reads as

$$b_i = f_i + p2k_i^{-1}e_i \frac{1}{t^2 + t^{-2}}k_i^{-1} (0 \leq i < n),$$  \(137\)

$$b_n = f_n + pk_n^{-1}e_n - i\mu \frac{t - t^{-1}}{t^2 + t^{-2}}k_n^{-1}.$$  \(138\)

This corresponds to [19] eqs.(167),(170)-(171) with $s = s' = q^\frac{1}{2}$ according to [19] eq.(96)]. These generators are represented as local Hamiltonians:

$$b_0 = z^2\sigma_1^+\sigma_2^- + z^{-2}\sigma_1^-\sigma_2^+ - \frac{t^2 + t^{-2}}{4}\sigma_1^2\sigma_2^2 + \frac{t^2 - t^{-2}}{4}(\sigma_1^2 + \sigma_2^2) + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})}.$$  \(139\)

$$b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ - \frac{t^2 + t^{-2}}{4}\sigma_i^2\sigma_{i+1}^2 + \frac{t^2 - t^{-2}}{4}(\sigma_i^2 - \sigma_{i+1}^2) + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})} (0 < i < n),$$  \(140\)

$$b_n = \sigma_n^x + \mu \frac{t - t^{-1}}{2} \sigma_n^z - \mu \frac{(t - t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})}.$$  \(141\)

They commute with the $K$ matrix [34] up to $z$ as [19]:

$$\tilde{K}_{2,1}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\tilde{K}_{2,1}(z) \quad (0 \leq i \leq n).$$  \(142\)
Set
\[ H_{2,1}(z) = b_0 + b_1 + 2(b_2 + \cdots + b_{n-1}) - \mu(t + t^{-1})b_n, \]
\[ = z^2\sigma_1^+\sigma_2^- + z^{-2}\sigma_1^-\sigma_2^+ + 2\sum_{i=1}^{n-1}\left((1 - \frac{1}{2}\delta_{i1})(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) - \frac{t^2 + t^{-2}}{4}\sigma_i^+\sigma_{i+1}^-\right) \]
\[ - \mu(t + t^{-1})\sigma_n^x + 2n\Gamma. \]
It satisfies \( \sigma^x H_{2,1}(z)\sigma^x = H_{2,1}(z^{-1}) \). Therefore \( (141) \) and \( (38) \) lead to the commutativity
\[ [K_{2,1}^\uparrow(z), H_{2,1}(z)] = 0. \]

7.4. **Representation** \( \pi_{2,2}^{2,1} \). The representation \( \pi_{2,2}^{2,1} \) of \( O_p(B_n^{(1)}) \) on \( V \) in \( (9) \) is obtained by the composition
\[ \pi_{2,2}^{2,1} : O_p(B_n^{(1)}) \rightarrow U_p(B_n^{(1)}) \rightarrow \text{End}V, \]
where the latter is the spin representation \( (132)-(134) \) and the former embedding \( (87)-(90) \) reads as
\[ b_i = f_i + p^2k_i^{-1}e_i + \frac{1}{q + q^{-1}}k_i^{-1} (0 \leq i < n), \]
\[ b_n = f_n + pk_n^{-1}e_n. \]
This corresponds to \( (19) \) eqs.(167), (180)-(181) \( s = s' = q^4 \) according to \( (19) \) eq.(96)]. These generators are represented as local Hamiltonians:
\[ b_0 = z^2\sigma_1^+\sigma_2^- + z^{-2}\sigma_1^-\sigma_2^+ + \frac{q + q^{-1}}{4}\sigma_1^+\sigma_2^- - \frac{q - q^{-1}}{4}(\sigma_1^+ - \sigma_2^-) - \frac{(q - q^{-1})^2}{4(q + q^{-1})}, \]
\[ b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{q + q^{-1}}{4}\sigma_i^+\sigma_{i+1}^- - \frac{q - q^{-1}}{4}(\sigma_i^+ - \sigma_{i+1}^-) - \frac{(q - q^{-1})^2}{4(q + q^{-1})} (0 < i < n), \]
\[ b_n = \sigma_n^x. \]
They commute with the \( K \) matrix \( (34) \) up to \( z \) as \( (19) \):
\[ \tilde{K}_{2,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\tilde{K}_{2,2}(z) \quad (0 \leq i \leq n). \]

8. **\( O_p(B_n^{(1)}) \) Hamiltonians**

8.1. **\( U_p(\tilde{B}_n^{(1)}) \) and spin representation**. The Dynkin diagram and the Cartan matrix are given by
\[ g^{1,2} = \tilde{B}_n^{(1)} \]

\[ (a_{ij})_{0 \leq i, j \leq n} = \begin{pmatrix}
2 & -2 & \cdots & \cdots & 0 & 0 \\
-1 & 2 & \cdots & & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots \\
0 & \cdots & -1 & 2 & 0 \\
0 & 0 & \cdots & -1 & 2 & 0
\end{pmatrix}. \]

The spin representation on \( V \) is given by
\[ e_0|\alpha\rangle = z|\alpha + e_1\rangle, \quad f_0|\alpha\rangle = z^{-1}|\alpha - e_1\rangle, \quad k_0|\alpha\rangle = p^{2\alpha_1 - 1}|\alpha\rangle, \]
\[ e_j|\alpha\rangle = |\alpha - e_j + e_{j+1}\rangle, \quad f_j|\alpha\rangle = |\alpha + e_j - e_{j+1}\rangle, \quad k_j|\alpha\rangle = p^{2(\alpha_{j+1} - \alpha_j)}|\alpha\rangle \quad (0 < j < n), \]
\[ e_n|\alpha\rangle = |\alpha - e_{n-1} - e_n\rangle, \quad f_n|\alpha\rangle = |\alpha + e_{n-1} + e_n\rangle, \quad k_n|\alpha\rangle = p^{2(1 - \alpha_{n-1} - \alpha_n)}|\alpha\rangle. \]

8.2. **Onsager algebra** \( O_p(B_n^{(1)}) \) and the classical part \( O_p(D_n) \). The algebra \( O_p(B_n^{(1)}) \) is generated by \( b_0, \ldots, b_n \) obeying the relations \( (50) \) \( (52) \). The classical part of \( B_n^{(1)} \) without the vertex 0 is \( D_n \). Thus the subalgebra of \( O_p(B_n^{(1)}) \) generated by \( b_1, \ldots, b_n \) is the Onsager algebra for \( D_n \). We denote it by \( O_p(D_n) \).
8.3. **Representation $\pi_{1,2}^{1,2}$**. The representation $\pi_{1,2}^{1,2}$ of $O_p(\hat{B}_n^{(1)})$ on $V$ is obtained by the composition

$$\pi_{1,2}^{1,2} : O_p(\hat{B}_n^{(1)}) \to U_p(\hat{B}_n^{(1)}) \to \text{End}V,$$

(154)

where the latter is the spin representation \([151], [153]\) and the former embedding \([87]-[90]\) reads as

$$b_0 = f_0 + pk_0^{-1}e_0 - i\mu \frac{t - t^{-1}}{t^2 + t^{-2}} k_0^{-1},$$

(155)

$$b_i = f_i + p^2k_i^{-1}e_i - \frac{1}{t^2 + t^{-2}} k_i^{-1} \quad (0 < i \leq n).$$

(156)

This corresponds to \([19\) eqs.\((167), (174)-(175)]\) $s = s' = q^{\frac{1}{2}}$ according to \([19\) eq.(96)]. These generators are represented as local Hamiltonians:

$$b_0 = z\sigma_1^+ + z^{-1}\sigma_1^- - \mu \frac{t - t^{-1}}{2} \sigma_1^z - \mu \frac{(t - t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})},$$

(157)

$$b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ - \frac{t^2 + t^{-2}}{4} \sigma_i^z\sigma_{i+1}^z - \frac{t^2 - t^{-2}}{4}(\sigma_i^z - \sigma_{i+1}^z) + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})} \quad (0 < i < n),$$

(158)

$$b_n = \sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ + \frac{t^2 + t^{-2}}{4} \sigma_{n-1}\sigma_n + \frac{t^2 - t^{-2}}{4}(\sigma_{n-1} + \sigma_n)^z + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})}.$$

(159)

They commute with the $K$ matrix \([34]\) up to $z$ as \([19]\):

$$\hat{K}_{1,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\hat{K}_{1,2}(z) \quad (0 \leq i \leq n).$$

(160)

Set

$$H_{1,2}(z) = -\mu(t + t^{-1})b_0 + 2(b_1 + \cdots + b_{n-2}) + b_{n-1} + b_n$$

$$= -\mu(t + t^{-1})(z\sigma_1^+ + z^{-1}\sigma_1^-) + 2 \sum_{i=1}^{n-1} (1 - \frac{1}{4}k_{i+1})(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) - \frac{t^2 + t^{-2}}{4} \sigma_i^z\sigma_{i+1}^z)$$

$$+ \sigma_{n-1}\sigma_n^+ + \sigma_{n-1}\sigma_n^- + 2n\Gamma.$$  

(161)

It satisfies $\sigma^z H_{1,2}(z)\sigma^z = H_{1,2}(z^{-1}).$ Therefore \([100\) and \([83\] lead to the commutativity

$$[K_{1,2}^\dagger(z), H_{1,2}(z)] = 0.$$

(162)

8.4. **Representation $\pi_{2,2}^{1,2}$**. The representation $\pi_{2,2}^{1,2}$ of $O_p(\hat{B}_n^{(1)})$ on $V$ is obtained by the composition

$$\pi_{2,2}^{1,2} : O_p(\hat{B}_n^{(1)}) \to U_p(\hat{B}_n^{(1)}) \to \text{End}V,$$

(163)

where the latter is the spin representation \([151], [153]\) an and the former embedding \([87]-[90]\) reads as

$$b_0 = f_0 + pk_0^{-1}e_0,$$

(164)

$$b_i = f_i + p^2k_i^{-1}e_i - \frac{1}{q + q^{-1}} k_i^{-1} \quad (0 < i \leq n).$$

(165)

This corresponds to \([19\) eqs.\((167), (182)-(183)]\) $s = s' = q^{\frac{1}{2}}$ according to \([19\) eq.(96)]. These generators are represented as local Hamiltonians:

$$b_0 = z\sigma_1^+ + z^{-1}\sigma_1^-,$$

(166)

$$b_i = \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{q + q^{-1}}{4} \sigma_i^z\sigma_{i+1}^z - \frac{q - q^{-1}}{4}(\sigma_i^z - \sigma_{i+1}^z) - \frac{(q - q^{-1})^2}{4(q + q^{-1})} \quad (0 < i < n),$$

(167)

$$b_n = \sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ + \frac{q + q^{-1}}{4} \sigma_{n-1}\sigma_n - \frac{q - q^{-1}}{4}(\sigma_{n-1} + \sigma_n)^z - \frac{(q - q^{-1})^2}{4(q + q^{-1})}.$$

(168)

They commute with the $K$ matrix \([34]\) up to $z$ as \([19]\):

$$\hat{K}_{2,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})\hat{K}_{2,2}(z) \quad (0 \leq i \leq n).$$

(169)
9. $O_p(D_n^{(1)})$ Hamiltonians

9.1. $U_p(D_n^{(1)})$ and spin representations. The Dynkin diagram and the Cartan matrix are given by

\[
\begin{align*}
\begin{array}{cccccccccccc}
2 & 0 & -1 & \cdots & \cdots & 0 & 0 \\
0 & 2 & -1 & \cdots & \cdots & 0 \\
-1 & -1 & 2 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots \cdots & \ddots & \vdots \\
0 & \cdots & 2 & -1 & -1 \\
0 & \cdots & \cdots & -1 & 0 & 2
\end{array}
\end{align*}
\]

There are two spin representations $V_+, V_- \subset V$ in (11) with dimension $\dim V_\pm = 2^{n-1}$. They are given by

\[
\begin{align*}
\begin{array}{cccccccc}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{array}
\end{align*}
\]

9.2. Onsager algebra $O_p(D_n^{(1)})$ and the classical part $O_p(D_n)$. The algebra $O_p(D_n^{(1)})$ is generated by $b_0, \ldots, b_n$ obeying the relations (80)-(82). The classical part of $D_n^{(1)}$ without the vertex 0 is $D_n$. Thus the subalgebra of $O_p(D_n^{(1)})$ generated by $b_1, \ldots, b_n$ agrees with the Onsager algebra $O_p(D_n)$ introduced in Section 8.2.

9.3. Representation $\pi^{(2,2)}_{2,2}$. The representation $\pi^{(2,2)}_{2,2}$ of $O_p(D_n^{(1)})$ on $V_\pm$ is obtained by the composition

\[
\begin{align*}
\pi^{(2,2)}_{2,2} : O_p(D_n^{(1)}) \rightarrow U_p(D_n^{(1)}) \rightarrow \text{End} V_+ \oplus \text{End} V_-,
\end{align*}
\]

where the latter is the spin representation (170)-(172) and the former embedding (87) reads as

\[
b_i = f_i + p^2k_i^{-1}e_i + \frac{1}{q + q^{-1}}k_i^{-1} (0 \leq i \leq n).
\]

This corresponds to [19] eqs.(167), (178)-(179)] with $s = s' = q^2$ according to [19] eq.(96)]. These generators are represented as local Hamiltonians:

\[
\begin{align*}
b_0 &= z^2\sigma_1^+\sigma_2^+ + z^{-2}\sigma_1^-\sigma_2^- + \frac{q + q^{-1}}{4}\sigma_1^2\sigma_2^2 - \frac{q - q^{-1}}{4}(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) - \frac{(q - q^{-1})}{2}(q + q^{-1}), \\
b_i &= \sigma_i^+\sigma_{i-1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{q + q^{-1}}{4}\sigma_i^2\sigma_{i+1}^2 - \frac{q - q^{-1}}{4}(\sigma_i^+\sigma_{i-1}^- + \sigma_i^-\sigma_{i+1}^+) - \frac{(q - q^{-1})}{2}(q + q^{-1}) (0 < i < n), \\
b_n &= \sigma_{n-1}^+\sigma_n^- + \sigma_n^+\sigma_{n-1}^- + \frac{q + q^{-1}}{4}\sigma_n^2\sigma_{n-1}^2 - \frac{q - q^{-1}}{4}(\sigma_n^+\sigma_{n-1}^- + \sigma_n^-\sigma_{n-1}^+) - \frac{(q - q^{-1})}{2}(q + q^{-1}).
\end{align*}
\]

They commute with the $K$ matrix [34] up to $z$ as [19]:

\[
\begin{align*}
K_{2,2}(z)b_i = (b_i|_{z \rightarrow z^{-1}})K_{2,2}(z) \quad (0 \leq i \leq n).
\end{align*}
\]

Set

\[
\begin{align*}
H_{2,2}(z) &= b_0 + b_1 + 2(b_2 + \cdots + b_{n-2}) + b_{n-1} + b_n \\
&= z^2\sigma_1^+\sigma_2^+ + z^{-2}\sigma_1^-\sigma_2^- + 2\sum_{i=1}^{n-1}(1 - \frac{1}{2}\delta_i, n-1)(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + \frac{q + q^{-1}}{4}\sigma_1^2\sigma_{n+1}^2 \\
&+ \sigma_{n-1}^+\sigma_n^+ + \sigma_{n-1}^-\sigma_n^- + (2n - 2)\Gamma.
\end{align*}
\]

It satisfies $\sigma^\mp H_{2,2}(z)\sigma^\mp = H_{2,2}(z^{-1})$. Therefore [178] and [38] lead to the commutativity

\[
[K_{2,2}(z), H_{2,2}(z)] = 0.
\]
10. Spectral decomposition of $K_{1,1}$ and $K_{2,1}$ by $O_p(B_n)$

The Onsager algebra $O_p(B_n)$ defined in Section 6.3 shows up either as the classical part of $O_p(D_{n+1}^{(2)})$ or $O_p(B_n^{(1)})$. Consider the resulting representations of $O_p(B_n)$ constructed as

$$O_p(B_n) \hookrightarrow O_p(D_{n+1}^{(2)}) \xrightarrow{\pi_{1,1}^1} \text{End} V,$$

$$O_p(B_n) \hookrightarrow O_p(D_{n+1}^{(2)}) \xrightarrow{\pi_{2,1}^1} \text{End} V,$$

$$O_p(B_n) \hookrightarrow O_p(B_n^{(1)}) \xrightarrow{\pi_{2,1}^2} \text{End} V.$$  

(181) 

(182) 

(183)

From the definitions (185) and (135), they actually yield the same representation

$$b_i = \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ - \frac{t^2 - t^{-2}}{4} \sigma_i^z \sigma_{i+1}^z - \frac{t^2 - t^{-2}}{4} (\sigma_i^z - \sigma_{i+1}^z) + \frac{(t^2 - t^{-2})^2}{4(t^2 + t^{-2})} \quad (1 \leq i < n),$$

$$b_n = \sigma_n^+ + \mu \frac{t - t^{-1}}{2} \sigma_n^z - \mu \frac{(t - t^{-1})(t^2 - t^{-2})}{2(t^2 + t^{-2})}.$$  

(184) 

(185)

We denote this by $\pi_1^1 : O_p(B_n) \rightarrow \text{End} V$.

The relation (105) with $i \neq 0$ tells that $\pi_1^1(O_p(B_n))$ commutes with $\hat{K}_{1,1}(z)$. Similarly, the both (115) and (111) imply that it also commutes with $\hat{K}_{2,1}(z)$. We summarize these facts as

$$[\hat{K}_{1,1}(z), \pi_1^1(O_p(B_n))] = 0, \quad [\hat{K}_{2,1}(z), \pi_1^1(O_p(B_n))] = 0.$$  

(186)

In other words, there are at least two affinizations that are compatible with the classical Onsager algebra symmetry $O_p(B_n)$ in the representation $\pi_1^1$ under consideration.

The representations $\pi_{1,1}^{1,1}, \pi_{2,1}^{1,1}$ of $O_p(D_{n+1}^{(2)})$ and $\pi_{2,1}^{2,1}$ of $O_p(B_n^{(1)})$ on $V$ are irreducible [19]. On the other hand $V$ is no longer irreducible with respect to their common classical subalgebra $O_p(B_n)$. The $K$ matrices should be scalar on each irreducible component. We conjecture that there are irreducible $O_p(B_n)$ modules $X_0, X_1, \ldots, X_n$ allowing the joint spectral decomposition of $K_{1,1}(z)$ and $K_{2,1}(z)$ as follows:

$$V = X_0 \oplus X_1 \oplus \cdots \oplus X_n, \quad \dim X_i = \binom{n}{l},$$

$$K_{1,1}(z) = \bigoplus_{0 \leq l \leq n} z^{n-2l} \frac{(-q^{n+1+2l} z; q)_{\infty} (-q^{-1} z; q)_{\infty}}{(-q^l z; q)_{\infty} (-q^{n+1-2l} z^{-1}; q)_{\infty}} \text{id}_{X_i},$$

$$K_{2,1}(z) = \bigoplus_{0 \leq l \leq n} z^{n-2l} \frac{(-q^{2n+3-4l} z; q^4)_{\infty} (-q^3 z^{-2}; q^4)_{\infty}}{(-q^l z^2; q^4)_{\infty} (-q^{2n+3-4l} z^{-2}; q^4)_{\infty}} \text{id}_{X_i} \quad (n\text{ even}),$$

$$= \bigoplus_{0 \leq l \leq n} z^{n+1-2l} \frac{(-q^{2n+3-4l} z; q^4)_{\infty} (-q z^{-2}; q^4)_{\infty}}{(-q^{l+2} z^2; q^4)_{\infty} (-q^{2n+3-4l} z^{-2}; q^4)_{\infty}} \text{id}_{X_i} \quad (n\text{ odd}).$$  

(187) 

(188) 

(189) 

(190)

Here $\text{id}_{X_i}$ denotes the orthonormal projector $\text{id}_{X_i} \text{id}_{X_j} = \delta_{l,l'} \text{id}_{X_i}$. Similar notations will also be used in the sequel. (Note that $\hat{K}_{1,k'}(z)$ and $\hat{K}_{k,k'}(z)$ possess the same spectrum due to (184).)

As an example (188) means

$$K_{1,1}(z) = \bigoplus_{l \geq \left\lceil \frac{n}{4} \right\rceil} \bigoplus_{j=1}^{2l-n} \frac{q^j + z}{1 + q^j z} \text{id}_{X_i} \oplus \bigoplus_{l \leq \left\lfloor \frac{n}{4} \right\rfloor} \bigoplus_{j=1}^{n-2l} \frac{q^j + z}{1 + q^j z} \text{id}_{X_i},$$

$$K_{2,1}(z) = \bigoplus_{l \geq \left\lceil \frac{n}{4} \right\rceil} \bigoplus_{j=1}^{2l-n} \frac{q^j + z}{1 + q^j z} \text{id}_{X_i} \oplus \bigoplus_{l \leq \left\lfloor \frac{n}{4} \right\rfloor} \bigoplus_{j=1}^{n-2l} \frac{q^j + z}{1 + q^j z} \text{id}_{X_i},$$  

(191)

where $[x]$ stands for the largest integer not exceeding $x$. 
11. Spectral decomposition of $K_{1,2}$ and $K_{2,2}$ by $O_{p}(D_{n})$

The Onsager algebra $O_{p}(D_{n})$ defined in Section 8.2 shows up either as the classical part of $O_{p}(\tilde{B}_{n}^{(1)})$ or $O_{p}(D_{n}^{(1)})$. Consider the resulting representations of $O_{p}(D_{n})$ constructed as

\[
O_{p}(D_{n}) \hookrightarrow O_{p}(\tilde{B}_{n}^{(1)}) \xrightarrow{\sigma_{1,2}} \text{End}V, \\
O_{p}(D_{n}) \hookrightarrow O_{p}(\tilde{B}_{n}^{(1)}) \xrightarrow{\sigma_{2,2}} \text{End}V, \\
O_{p}(D_{n}) \hookrightarrow O_{p}(D_{n}^{(1)}) \xrightarrow{\sigma_{2,2}} \text{End}V_{+} \oplus \text{End}V_{-}.
\]

For the definition of $V_{\pm}$, see [11]. From (154), (163) and (173), they actually yield the same representation

\[
\text{det}(\tilde{K}_{1,2}(z)) = \frac{z^{2} - q^{2}}{z^{2} - q^{2}}, \\
\text{det}(\tilde{K}_{2,2}(z)) = \frac{z^{2} q^{2} - 1}{z^{2} q^{2} - 1}.
\]

Obviously this defines the representation either on $V_{+}$ or $V_{-}$ separately. We denote them by $\sigma_{2,\pm}^{2} : O_{p}(D_{n}) \rightarrow \text{End}V_{\pm}$ and their direct sum by $\sigma_{2}^{2} : O_{p}(D_{n}) \rightarrow \text{End}V$.

The $K$ matrix $\tilde{K}_{1,2}(z)$ does not preserve $V_{+}$ and $V_{-}$ individually. However the relation (160) tells that it commutes with $\sigma_{2}^{2}(O_{p}(D_{n}))$. On the other hand, $\tilde{K}_{2,2}(z)$ maps $V_{\pm}$ to $V_{\pm(-1)^{n}}$ as seen in (39). Then (160) and (178) imply

\[
[\tilde{K}_{1,2}(z), \sigma_{2}^{2}(O_{p}(D_{n}))] = 0, \quad [\sigma_{2}^{2}, (O_{p}(D_{n}))] = 0.
\]

The representations $\sigma_{2}^{1,2}, \sigma_{2}^{1,2}$ of $O_{p}(\tilde{B}_{n}^{(1)})$ on $V$ and $\sigma_{2}^{2,2}$ of $O_{p}(D_{n}^{(1)})$ on $V_{\pm}$ are irreducible [19]. On the other hand they are no longer irreducible with respect to their common classical subalgebra $O_{p}(D_{n})$. The $K$ matrices should be a scalar on each irreducible component.

For $n$ even, we conjecture that there are irreducible $O_{p}(D_{n})$ modules $Z_{0}^{\pm}, \ldots, Z_{n}^{\pm}$ having the properties (i) and (ii) described below:

(i) $V_{\pm}$ are decomposed as

\[
V_{\pm} = Z_{0}^{\pm} \oplus \cdots \oplus Z_{n}^{\pm}, \quad \dim Z_{l}^{\pm} = \begin{cases} \binom{n}{l} & (0 \leq l < \frac{n}{2}), \\ \binom{n}{\frac{n}{2}} & (l = \frac{n}{2}), \end{cases}
\]

which is consistent with $\dim V_{\pm} = 2^{n-1}$.

(ii) There exists a basis $\{\zeta_{l,i}^{\pm} \mid 1 \leq i \leq \dim Z_{l}^{\pm}\}$ of $Z_{l}^{\pm}$ in terms of which the spectral decomposition of the $K$ matrices is described as

\[
K_{1,2}(z) = \bigoplus_{0 \leq i \leq n} (-t^{2} z^{-1} - t^{4} \zeta_{l,i}^{+} + \zeta_{l,i}^{-}) \leq \begin{cases} \frac{n}{2} \end{cases}, \\
K_{2,2}(z) = \bigoplus_{0 \leq i \leq \frac{n}{2}} (-t^{2} z^{-1} - t^{4} \zeta_{l,i}^{+} + \zeta_{l,i}^{-}) \leq \begin{cases} \frac{n}{2} \end{cases},
\]

where the spaces $Y_{0}, \ldots, Y_{n}$ are given by

\[
Y_{l} = \bigoplus_{1 \leq i \leq \binom{n}{l}} \mathbb{C} \zeta_{l,i}^{+} + \zeta_{l,i}^{-} (0 \leq l < \frac{n}{2}), \\
Y_{l} = \bigoplus_{1 \leq i \leq \binom{n}{\frac{n}{2}}} \mathbb{C} \zeta_{l,i}^{+} + \zeta_{l,i}^{-} (l = \frac{n}{2}), \\
Y_{l} = \bigoplus_{1 \leq i \leq \binom{n}{\frac{n}{2}}} \mathbb{C} \zeta_{l-i,i}^{+} - \zeta_{l-i,i}^{-} (\frac{n}{2} < l \leq n).
\]

Thus the following relations hold:

\[
Y_{l} \cap V_{\pm} = Z_{l}^{\pm}, \quad (Y_{l} + Y_{n-l}) \cap V_{\pm} = Z_{l}^{\pm} (0 \leq l < \frac{n}{2}).
\]

\[
[\tilde{K}_{1,2}(z), \sigma_{2}(O_{p}(D_{n}))] = 0, \quad [\sigma_{2}, (O_{p}(D_{n}))] = 0.
\]
For \( n \) odd, we conjecture that there are irreducible \( O_p(D_n) \) modules \( Z^+_0, \ldots, Z^+_\frac{n-1}{2} \) having the properties (iii) and (iv) described below:

(iii) \( V_\pm \) are decomposed as

\[
V_\pm = Z^+_0 \oplus \cdots \oplus Z^+_\frac{n-1}{2}, \quad \dim Z^+_i = \binom{n}{i},
\]

which is consistent with \( \dim V_\pm = 2^{n-1} \).

(iv) There exists a basis \( \{ Z^+_i \mid 1 \leq i \leq \dim Z^+_i \} \) of \( Z^+_i \) in term of which the spectral decomposition of the \( K \) matrices is described as

\[
K_{1,2}(z) = \bigoplus_{0 \leq l \leq n} z^{-t(z^{-1}t^4)}(t-2n^2+34z^{2+l})^l(t-2n^2+4z-1)^l t^4(t-4z^{2+l})^l v_t, \quad K_{2,2}(z) = \bigoplus_{0 \leq l \leq n} -z^{-t(-t^4z^{-1}+4z^2)}(t-2n^2+4z^{2+l})^l(t-4z^{2+l})^l t^4(t-4z^{2+l})^l v_t (P^+_t \oplus P^-_t).
\]

Here the spaces \( Y_0, \ldots, Y_n \) are given again by (201) and (203), hence the latter relation in (204) is valid. The operators \( P^+_t \) are defined by

\[
P^+_t \zeta^+_{i,i} = \delta_{t,t'} \zeta^+_{i'\bar{i}'}, \quad P^+_t \zeta^+_{\bar{i}i} = 0,
\]

giving isomorphism \( Z^+_i \to Z^+_{i'} \). We note that the eigenvalues appearing in (201) and (207) are actually even functions of \( z \) as with \( K_{2,2}(z) \).

12. Summary

In this paper we have pointed out that the generators of the Onsager algebras \( O_p(A^{(1)}_{n-1}) \) in the fundamental representations and \( O_p(B^{(2)}_{n+1}), O_p(B^{(1)}_n), O_p(D^{(1)}_n) \) in the spin representations are naturally regarded as local Hamiltonians of XXZ type spin chains involving various boundary terms reflecting the relevant Dynkin diagrams. The reflection \( K \) matrices due to the matrix product construction are shown to serve as symmetry operators of these Hamiltonians. The spectra of the latters are yet to be analyzed in general. We have given the spectral decomposition of the \( K \) matrices with respect to the classical part of the Onsager algebras conjecturally. They exhibit an intriguing structure which deserves further investigations from the viewpoint of the representation theory of Onsager algebras. We have also included a proof of Theorem 15.1, which was formulated as a conjecture in 20 and played a key role in the matrix product construction there.

Let us remark a related result from 28, where a family of mutually commuting Hamiltonians of the form

\[
I_m = \kappa_0 S^{(m)}_0 + \kappa_1 S^{(m)}_1 + \cdots + \kappa_n S^{(m)}_n, \quad S^{(m)}_i = O_p(g), \quad S^{(0)}_i = \mathfrak{g}, \quad (m \geq 0)
\]

were constructed for \( g = A^{(1)}_{n-1} \) (hence \( n' = n - 1 \)) with \( n \geq 3 \) and \( p = 1 \). Here \( \kappa_0, \ldots, \kappa_n \) are free parameters. (A slightly more general one is given in 24 eq.(2.44).) Our \( H_{tr}(z) = b_0 + \cdots + b_{n-1} \) in (200) and (204) formally correspond to a \( p \)-analogue of the representation of \( I_0 \) on \( V \) with \( \forall \kappa_i = 1 \). It is an interesting problem to construct a \( p \)-analogue of \( I_m \) within \( O_p(g) \) for general \( g \).

APPENDIX A. PROOF OF [28]

Define

\[
K(z|\beta) = \text{Tr}(z^bK_{\alpha_1}^\beta \cdots K_{\alpha_n}^\beta), \quad K(z|\alpha) = \sum_{\beta \in \{0,1\}^n} K(z|\beta)\alpha^\beta,
\]

which are just (18) and (17) without an overall scalar for simplicity. In order to describe the elements of \( K(z)K(w) \), we prepare two copies of (16) and their product:

\[
L_1 = \begin{pmatrix} \mathbf{a}_1^+ & -q\mathbf{k} \end{pmatrix}, \quad M_1 = \begin{pmatrix} M_0^0 & \mathbf{M}_0^1 \\ \mathbf{M}_1^0 & \mathbf{M}_1^1 \end{pmatrix} = L_1 \cdot L_2 = \begin{pmatrix} \mathbf{a}_1^+ \mathbf{a}_2^+ - q\mathbf{k}_1 \mathbf{k}_2 \mid -q(\mathbf{a}_1^+ \mathbf{k}_2 + \mathbf{k}_1 \mathbf{a}_2^+) \end{pmatrix},
\]

where \( i = 1, 2 \) and \( \cdot \) signifies the usual product as 2 by 2 matrices. We will also use the copies \( \mathbf{h}_1, \mathbf{h}_2 \) of the number operator \( \mathbf{h} \) defined after (15). Operators with different indices are commutative as they act on different \( q \)-boson Fock spaces.
By the definition the matrix element of $K(z)K(w)$ is expressed as
\begin{equation}
(K(z)K(w))^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} = \text{Tr}_{12} \left( z^{h_1} w^{h_2} M^{\beta_1}_{\alpha_1} \ldots M^{\beta_n}_{\alpha_n} \right),
\end{equation}
where the trace extends over the two $q$-boson Fock spaces 1 and 2.

Let $r$ be the exchange operator of the two $q$-bosons:
\begin{equation}
r^2 = 1, \quad r a^\pm_i = a^\mp_{3-i} r, \quad r k_i = k_{3-i} r, \quad r h_i = h_{3-i} r \quad (i = 1, 2).
\end{equation}
One can easily check the following relations for any $\alpha, \beta \in \{0, 1\}$:
\begin{equation}
r M^\beta_\alpha = (-q)^{\alpha-\beta} M^\beta_\alpha r, \quad M^0_1 M^1_1 = M^1_0 M^0_0, \quad M^0_1 M^0_1 = q M^0_0 M^0_0, \quad M^{-1}_1 M^1_1 = q^{-1} M^1_1 M^{-1}_1.
\end{equation}

Note that the relation (215) is also satisfied by the elements of $L_i$ (211) for each $i = 1, 2$. The product $L_1 \cdot L_2$ preserves the relation because it coincides with the coproduct of the $q$-oscillator representation $\pi_i$ of the quantized coordinate ring $A_q(SL_n)$ in [18] eqs.(2.3)–(2.6)].

Insert 1 = $r^2$ anywhere in the trace of (212) and let one the $r$‘s encircle the whole array once using (213) and (214). The result gives
\begin{equation}
\text{Tr}_{12} \left( z^{h_1} w^{h_2} M^{\beta_1}_{\alpha_1} \ldots M^{\beta_n}_{\alpha_n} \right) = \text{Tr}_{12} \left( w^{h_1} z^{h_2} M^{\alpha_1}_{\beta_1} \ldots M^{\alpha_n}_{\beta_n} \right) (-q)^{\alpha-\beta},
\end{equation}
where the symbol $|\alpha|$ is defined in (10). From (20) we know $(K(z)K(w))^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} = 0$ unless $|\alpha| = |\beta|$. Thus the factor $(-q)^{|\alpha|-|\beta|}$ in the above can be removed, leading to
\begin{equation}
(K(z)K(w))^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} = (K(w)K(z))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}.
\end{equation}

Next consider the expression (212) again. Under the assumption $|\alpha| = |\beta|$, the number of $M^0_1$ and $M^1_1$ in the trace is equal, which we denote by $m$. Then by means of (214) one can send $M^0_1$ and $M^1_1$ to the left to rewrite (212) uniquely in the form
\begin{equation}
(K(z)K(w))^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n} = q^m \text{Tr}_{12} \left( z^{h_1} w^{h_2} (M^0_1 M^1_1)^m N_1 \ldots N_{n-2m} \right),
\end{equation}
where $N_i = M^0_1$ or $M^1_1$ are in the original order and $\Phi$ is some integer. Starting from $(K(z)K(w))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}$, the same rewriting procedure leads to the identical expression to (214). Thus we find
\begin{equation}
(K(z)K(w))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = (K(w)K(z))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}.
\end{equation}

Combining (219) with (217) we conclude
\begin{equation}
(K(w)K(z))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = (K(z)K(w))^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n},
\end{equation}
which completes a proof of (20).

Appendix B. Proof of the invariance of boundary vectors under 3D $K$ matrix

The matrix product construction of the reflection $K$ matrices $K_{k,k'}(z)$ in [20] was based on the fact that certain boundary vectors remain invariant under the action of the $3D$ $K$ matrix $K$ which is the intertwiner of quantized coordinate ring $A_q(Sp_4)$ [18]. See [20] eq.(78)]. In this appendix we prove this crucial property which had been left as a conjecture in [20], thereby completing the 3D approach there. For simplicity we shall concentrate on the latter relation in [20] eq.(78)] on the ket-vectors. The former relation corresponding to the bra-vector version follows from it by an argument similar to the proof of [18] Prop.2.4]. We leave an detailed description of the 3D $K$ matrix to the original work [18]. A quick exposition is available in [20] Sec.3.2].

Let $F_q^2$ be the Fock space obtained by formally replacing $q$ by $q^2$ in $F_q$ in Section 2.3. The $q^2$-boson operators are denoted by $A^±, K$, i.e.,
\begin{equation}
A^± |m\rangle = (1 - q^{4m}) |m\rangle - 1, \quad A^\pm |m\rangle = |m\rangle + 1, \quad K |m\rangle = q^{2m} |m\rangle.
\end{equation}
We introduce the boundary vectors by
\begin{equation}
|\chi_k\rangle = \sum_{m \geq 0} \frac{|km\rangle}{(q^{2k^2}; q^{2k^2})_m} \in F_q^2 \quad (k = 1, 2),
\end{equation}
which is equal to $|\eta_k\rangle$ in \[24\] with $q$ replaced by $q^2$. Up to normalization, the vector $|\eta_1\rangle (|\chi_1\rangle)$ is characterized by any one of the following three conditions in the left (right) column:

\[ (a^+ - 1 + k)|\eta_1\rangle = 0, \quad (A^+ - 1 + K)|\chi_1\rangle = 0, \]  
\[ (a^- - 1 - qk)|\eta_1\rangle = 0, \quad (A^- - 1 - q^2K)|\chi_1\rangle = 0, \]  
\[ (a^+ - a^- + (1 + q^2)k)|\eta_1\rangle = 0, \quad (A^+ - A^- + (1 + q^2)K)|\chi_1\rangle = 0. \]  

(223)

(224)

(225)

Up to normalization, the vectors $|\eta_2\rangle$ and $|\chi_2\rangle$ are characterized by

\[ (a^+ - a^-)|\eta_2\rangle = 0, \quad (A^+ - A^-)|\chi_2\rangle = 0. \]  

(226)

Define the three boundary vectors by

\[ |\Xi_{r,k}\rangle = |\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle \quad ((r, k) = (1, 1), (1, 2), (2, 2)). \]  

(227)

Let $K \in \text{End}(F_q \otimes F_q \otimes F_q \otimes F_q)$ be the 3D $K$ matrix in [18] Th.3.4] which only depends on the parameter $q$. It satisfies the intertwining relation

\[ \Delta(t_{ij})K = K\Delta^{op}(t_{ij}) \quad (i, j \in \{1, 2, 3, 4\}), \]  

(228)

where $\Delta$ and $\Delta^{op}$ are shorthand for the tensor product representations $(\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1) \circ \Delta$ and $(\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1) \circ \Delta^{op}$ of $A_q(\text{Sp}_4)$ defined by

\[ \Delta(t_{ij}) = \sum_{1 \leq k, l, m \leq 4} \pi_2(t_{ik}) \cdot \pi_1(t_{kl}) \cdot \pi_2(t_{lm}) \cdot \pi_1(t_{mk}), \]  

(229)

\[ \Delta^{op}(t_{ij}) = \sum_{1 \leq k, l, m \leq 4} \pi_2(t_{km}) \cdot \pi_1(t_{lm}) \cdot \pi_2(t_{tk}) \cdot \pi_1(t_{ik}), \]  

(230)

where the symbol $\cdot$ is the abbreviation of $\otimes$, and each component is given by

\[ \pi_1(t_{ij}) = \begin{pmatrix} a^- & \nu_1 k & 0 & 0 \\ -q\nu_1 k & a^+ & 0 & 0 \\ 0 & 0 & a^- & -\nu_1 k \\ 0 & 0 & q^2\nu_1 k & a^+ \end{pmatrix}, \quad \pi_2(t_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A^- & \nu_k K & 0 \\ 0 & -q^2\nu_2 K & A^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  

(231)

By this we mean that the LHS is given by the element in the RHS at the $i$th row and the $j$th column from the top left. The parameters $\nu_1, \nu_2$ are free and do not influence the subsequent argument, so we set $\nu_1 = \nu_2 = 1$ below. The following was conjectured in [20 eq.(78)].

**Theorem B.1.**

\[ K|\Xi_{r,k}\rangle = |\Xi_{r,k}\rangle \quad ((r, k) = (1, 1), (1, 2), (2, 2)). \]  

(232)

By a direct calculation one can show

**Lemma B.2.**

\[ 1 \cdot k^2 \cdot K^2 \cdot 1 = \Delta(t_{14} - q^{-3}t_{24}t_{13}), \]  

(233)

\[ 1 \cdot k \cdot K \cdot k = \Delta(-t_{14}) = \Delta(q^{-4}t_{41}), \]  

(234)

\[ 1 \cdot k \cdot K \cdot a^+ = \Delta(-q^{-3}t_{42}), \]  

(235)

\[ A^+ \cdot k^2 \cdot K \cdot 1 = \Delta(-q^{-5}t_{34}t_{41} - t_{34}t_{13}), \]  

(236)

\[ 1 \cdot k \cdot a^+ \cdot K \cdot 1 = \Delta(q^2t_{44}t_{13} + q^{-4}t_{43}t_{41}), \]  

(237)

\[ 1 \cdot k^2 \cdot KA^+ \cdot 1 = \Delta(-q^{-1}t_{44}t_{41} - q^{-2}t_{43}t_{42}), \]  

(238)
Proof of Theorem B.1} In view of the definition \((224)\) and the characterization \((225)\)–\((226)\), it suffices to show
\[
((A^+ - A^-) \cdot k^2 \cdot K \cdot 1)\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(239)}
\]
\[
(1 \cdot k (a^+ - a^-) \cdot K \cdot 1)\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(240)}
\]
\[
(1 \cdot k^2 \cdot K (A^+ - A^-) \cdot 1)\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(241)}
\]
\[
(1 \cdot k \cdot K (a^+ - a^-))\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(242)}
\]
\[
((A^+ - A^- + (1 + q^2)K) \cdot k^2 \cdot K \cdot 1)\mathcal{X}(\Xi_{1,2}) = 0, \quad \text{(243)}
\]
\[
(1 \cdot k (a^+ - a^-) \cdot K \cdot 1)\mathcal{X}(\Xi_{1,2}) = 0, \quad \text{(244)}
\]
\[
((A^+ - A^- + (1 + q^2)K) \cdot k^2 \cdot K \cdot 1)\mathcal{X}(\Xi_{1,1}) = 0, \quad \text{(245)}
\]
\[
(1 \cdot k \cdot K (a^+ - a^-))\mathcal{X}(\Xi_{1,1}) = 0, \quad \text{(246)}
\]
\[
((A^+ - A^- + (1 + q^2)K) \cdot k^2 \cdot K \cdot 1)\mathcal{X}(\Xi_{1,1}) = 0, \quad \text{(247)}
\]
\[
(1 \cdot k (a^+ - a^-) \cdot K \cdot 1)\mathcal{X}(\Xi_{1,1}) = 0, \quad \text{(248)}
\]
\[
(1 \cdot k^2 \cdot K (A^+ - A^- + (1 + q^2)K) \cdot 1)\mathcal{X}(\Xi_{1,1}) = 0, \quad \text{(249)}
\]
\[
(1 \cdot k \cdot K (a^+ - a^- + (1 + q)k))\mathcal{X}(\Xi_{1,1}) = 0. \quad \text{(250)}
\]
As an illustration, consider \((238)\). It is verified as
\[
\Delta(-q^n t_{33} t_{41} - t_{33} t_{13} - q t_{22} t_{14} - q^{-4} t_{21} t_{42})\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(251)}
\]
where the last equality is checked directly, although tedious, by using \((231)\) and applying \((220)\) to \((227)\). The other relations can be shown similarly. Namely one can always find a polynomial \(T(t_{ij})\) which is a linear combination of those appearing in \((231)\)–\((234)\) such that the relation in question is expressed and shown as
\[
\Delta(T(t_{ij}))\mathcal{X}(\Xi_{r,k}) = \mathcal{X}\Delta^p(-q^n t_{33} t_{41} - t_{33} t_{13} - q t_{22} t_{14} - q^{-4} t_{21} t_{42})\mathcal{X}(\Xi_{2,2}) = 0, \quad \text{(252)}
\]
by applying \((225)\)–\((220)\) in the last step.

The only exception is \((218)\) involving \(X = 1 \cdot k^2 \cdot K \cdot 1\) which is not contained in the list \((231)\)–\((234)\). In fact, from \((217)\), LHS of \((218)\) is written as
\[
(\Delta(q t_{44} t_{13} + q^{-4} t_{43} t_{41} + q^{-1} t_{42} t_{11} - q^{-4} t_{41} t_{12}) + (1 + q)X)\mathcal{X}(\Xi_{1,1}), \quad \text{(253)}
\]
To treat this, we rely on \((231)\) which can be proved independently as explained in the above. It then tells that the third component of \(\Xi_{1,1}\) is proportional to \(|\chi_1\rangle\). Therefore from \((229)\) we may claim that it also satisfies
\[
(1 \cdot k^2 \cdot K (A^+ + K) \cdot 1)\mathcal{X}(\Xi_{1,1}) = 0. \quad \text{(254)}
\]
This leads to
\[
X\mathcal{X}(\Xi_{1,1}) = (1 \cdot k^2 \cdot K (A^+ + K) \cdot 1)\mathcal{X}(\Xi_{1,1}) = \Delta(-q^{-1} t_{44} t_{41} - q^{-2} t_{43} t_{42} + t_{14}^2 - q^{-3} t_{42} t_{13})\mathcal{X}(\Xi_{1,1}) \quad \text{(255)}
\]
due to \((233)\) and \((238)\). Substituting this into \((254)\) and using \((228)\) one can check that it indeed vanishes.

**Acknowledgments**

The authors thank Pascal Baseilhac for comments. A.K. thanks Masato Okado and Akihito Yoneyama for collaboration in their previous works. He is supported by Grants-in-Aid for Scientific Research No. 16H03922, 18H01141 and 18K03452 from JSPS.

**References**

[1] F. C. Alcaraz and W. F. Wesselski, The Heisenberg XXZ Hamiltonian with Dzyaloshinsky-Moriya interactions, J. Stat. Phys. 58 45-56 (1990).

[2] B. Aufegbauer and A. Klümper, Quantum spin chains of Temperley-Lieb type: periodic boundary conditions, spectral multiplicities and finite temperature, J. Stat. Mech. P05018 (2010)

[3] P. Baseilhac and S. Belliard, Generalized q-Onsager algebras and boundary affine Toda field theories, Lett. Math. Phys. 93 213–228 (2010).
[4] P. Baseilhac, N. Crampe and R. A. Pimenta, Higher rank classical analogs of the Askey-Wilson algebra from the $sl_N$ Onsager algebra, arXiv:1811.02763.

[5] I. V. Cherednik, Factorizing particles on a half-line and root systems, Theor. Math. Phys. 61 35–44 (1984).

[6] E. Date and K. Usami, On an analogue of the Onsager algebra of type $D^{(1)}_n$, Contemp. Math. 343 43–51 (2004).

[7] J. de Gier and A. Nichols The two-boundary Temperley-Lieb algebra, J. Alg. 321 1132–1167 (2009).

[8] G. Delius and N. J. MacKay, Quantum group symmetry in sine-Gordon and affine Toda field theories on the half-line, Commun. Math. Phys. 233 173–190 (2003).

[9] L. Dolan and M. Grady, Conserved charges from self-duality, Phys. Rev. D25 1587–1604 (1982).

[10] V. G. Drinfeld, Quantum groups, in Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 798–820 (1987).

[11] F. M. Goodman, P. de la Harpe and V. F. R. Jones, Coxeter graphs and towers of algebras, Springer Verlag (1989).

[12] T. Halverson, M. Mazzocco and A. Ram, Commuting families in Hecke and Temperley-Lieb algebras, Nagoya Math. J. 195 125–152 (2009).

[13] M. Jimbo, A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. 10 63–69 (1985).

[14] V. G. Kac, Infinite dimensional Lie algebras, third ed., Cambridge University Press (1990).

[15] S. Kolb, Quantum symmetric Kac-Moody pairs, Adv. Math. 267 395–469 (2014).

[16] P. P. Kulish, Yang-Baxter equation and reflection equations in integrable models, in Low-dimensional models in statistical physics and quantum field theory (Schladming, 1995), Lect. Note. Phys. 469 125–144.

[17] A. Kuniba, S. Maruyama and M. Okado, Multispecies TASEP and the tetrahedron equation, J. Phys. A: Math. Theor. 49 114001 (22pp) (2016).

[18] A. Kuniba and M. Okado, Tetrahedron and 3D reflection equations from quantized algebra of functions, J. Phys. A: Math.Theor. 45 465206 (27pp) (2012).

[19] A. Kuniba, M. Okado and A. Yoneyama, Reflection $K$ matrices associated with an Onsager coideal of $U_p(A_{n-1}^{(1)})$, $U_p(B_n^{(1)})$, $U_p(D_n^{(1)})$ and $U_p(D_{n+1}^{(2)})$, arXiv:1904.05653v2.

[20] A. Kuniba and V. Pasquier, Matrix product solutions to the reflection equation from three dimensional integrability, J. Phys. A: Math. Theor. 51 255204 (26pp) (2018).

[21] R. I. Nepomechie, Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms, J. Phys. A: Math. Gen. 37 433–440 (2004).

[22] A. Nichols, V. Rittenberg and J. de Gier, One-boundary Temperley-Lieb algebras in the XXZ and loop models, J. Stat. Mech. P03003 (2005).

[23] L. Onsager, Crystal statistics. I. A two-dimensional model with an order-disorder transition, Phys. Rev. 65 117–149 (1944).

[24] V. Pasquier and H. Saleur, Common structures between finite systems and conformal field theories through quantum groups, Nucl. Phys. B330 523–556 (1990).

[25] E. K. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A: Math. Gen. 21 2375–2389 (1988).

[26] H. N. V. Temperley and E. H. Lieb, Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem, Proc. R. Soc. Lond. A322 251–280 (1971).

[27] P. Terwilliger, An algebraic approach to the Askey scheme of orthogonal polynomials, in Orthogonal Polynomials and Special Functions, Lect. Note. in Math. 1883 Springer 225–330 (2006).

[28] D. B. Uglov and I. T. Ivanov, $sl(N)$ Onsager’s algebra and integrability, J. Stat. Phys. 82 87–113 (1996).