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INDECOMPOSABLE GENERALIZED WEIGHT MODULES OVER THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS

V. V. BAVULA, V. BEKKERT AND V. FUTORNY

Abstract. For the algebra \( I_1 = K\langle x, \frac{d}{dx}, \int \rangle \) of polynomial integro-differential operators over a field \( K \) of characteristic zero, a classification of indecomposable, generalized weight \( I_1 \)-modules of finite length is given. Each such module is an infinite dimensional uniserial module. Ext-groups are found between indecomposable generalized weight modules, it is proven that they are finite dimensional vector spaces.

Key Words: the algebra of polynomial integro-differential operators, generalized weight module, indecomposable module, simple module.

Mathematics subject classification 2000: 16D60, 16D70, 16P50, 16U20.

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; \( \mathbb{N} := \{0, 1, \ldots\} \) is the set of natural numbers; \( \mathbb{N}_+ := \{1, 2, \ldots\} \) and \( \mathbb{Z}_{\leq 0} := -\mathbb{N} \); \( K \) is a field of characteristic zero and \( K^* \) is its group of units; \( P_1 := K[x] \) is a polynomial algebra in one variable \( x \) over \( K \); \( \partial := \frac{d}{dx} \); \( \text{End}_K(P_1) \) is the algebra of all \( K \)-linear maps from \( P_1 \) to \( P_1 \), and \( \text{Aut}_K(P_1) \) is its group of units (i.e. the group of all the invertible linear maps from \( P_1 \) to \( P_1 \)); the subalgebras \( A_1 := K\langle x, \partial \rangle \) and \( I_1 := K\langle x, \partial, \int \rangle \) of \( \text{End}_K(P_1) \) are called the (first) Weyl algebra and the algebra of polynomial integro-differential operators respectively where \( \int : P_1 \to P_1, p \mapsto \int p \, dx \), is the integration, i.e. \( \int : x^n \mapsto \frac{x^{n+1}}{n+1} \) for all \( n \in \mathbb{N} \). The algebra \( I_1 \) is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals. \[2\].

In Section 2 a classification of indecomposable, generalized weight \( I_1 \)-modules of finite length is given (Theorem 2.5). A similar classification is given in \[1\] for the generalized Weyl algebras where a completely different approach was taken. Properties of the algebras \( I_n := I_1 \otimes_k \cdots \otimes_k I_1 \) of polynomial integro-differential operators in arbitrary many variables are studied in \[2\] and \[5\]. The groups \( \text{Aut}_{K-\text{alg}}(I_n) \) are found in \[3\]. The simple \( I_1 \)-modules are classified in \[4\].

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2. Classification of indecomposable, generalized weight \( I_1 \)-modules of finite length

In this section, a classification of indecomposable, generalized weight \( I_1 \)-modules of finite length is given (Theorem 2.5).

As an abstract algebra, the algebra \( I_1 \) is generated by the elements \( \partial, H := \partial x \) and \( \int \) (since \( x = \int H \) that satisfy the defining relations, \[2\] Proposition 2.2) (where \( [a, b] := ab - ba \)):

\[ \partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial. \]

The elements of the algebra \( I_1 \),

\[ e_{ij} := \int_i^{i+1} \partial^j - \int_{i+1}^{i+1} \partial^j, \quad i, j \in \mathbb{N}, \quad (1) \]
satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where $\delta_{jk}$ is the Kronecker delta function. Notice that $e_{ij} = \int e_{ii} \partial^i$. The matrices of the linear maps $e_{ij} \in \text{End}_K(K[x])$ with respect to the basis $\{x^s := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^s = \begin{cases} x^i & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \text{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js}x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij},$$

and $K e_{ij} = K E_{ij}$, and $F := \bigoplus_{i,j \geq 0} K e_{ij} = \bigoplus_{i,j \geq 0} K E_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices.

**Z-grading on the algebra $I_1$ and the canonical form of an integro-differential operator, [2].** The algebra $I_1 = \bigoplus_{i \in \mathbb{Z}} I_{1,i}$ is a Z-graded algebra ($I_{1,i}I_{1,j} \subseteq I_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$I_{1,i} = \begin{cases} D_1 \int = \int D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{\mid i \mid} D_1 = D_1 \partial^{\mid i \mid} & \text{if } i < 0, \end{cases}$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i \in \mathbb{N}} K e_{ii}$ is a commutative non-Noetherian subalgebra of $I_1$, $He_{ii} = e_{ii}H = (i + 1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i \in \mathbb{N}} K e_{ii}$ is the direct sum of non-zero ideals of $D_1$): $(\int D_1)D_1 \simeq D_1$, $\int d \mapsto d$, $D_1(\partial^i) \simeq D_1$, $\partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int = 1$. Notice that the maps $\cdot \int : D_1 \to D_1 \int$, $d \mapsto d \int$, and $\partial^i : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{i \geq 1} K e_{ij}$.

Each element $a$ of the algebra $I_1$ is the unique finite sum

$$a = \sum_{i > 0} a_i \partial^i + a_0 + \sum_{i > 0} \int a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the canonical form of the polynomial integro-differential operator [2].

Let $v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{\mid i \mid} & \text{if } i < 0. \end{cases}$

Then $I_{1,i} = D_1v_i = v_i D_1$ and an element $a \in I_1$ is the unique finite sum

$$a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij}$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^i \partial^i, H^i \partial^j, e_{st} | i \geq 1; j, s, t \geq 0\}$ is a $K$-basis for the algebra $I_1$. The multiplication in the algebra $I_1$ is given by the rule:

$$\int H = (H - 1) \int, \quad H \partial = \partial (H - 1), \quad \int e_{ij} = e_{i+1,j},$$

$$e_{ij} \int = e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij} \partial = \partial e_{i,j+1},$$

$$He_{ii} = e_{ii}H = (i + 1)e_{ii}, \quad i \in \mathbb{N},$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The algebra $I_1$ has the only proper ideal $F = \bigoplus_{i,j \in \mathbb{N}} K e_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra $I_1/F$ is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H + 1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{-1} \alpha = \tau^{-1}(\alpha) \partial^{-1}$ for all elements $\alpha \in K[H]$). The algebra $B_1$ is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra $A_1$ at the powers of the element $\partial$ (notice that $x = \partial^{-1}H$).
An $I_1$-module $M$ is called a weight module if $M = \oplus_{\lambda \in K} M_{\lambda}$ where $M_{\lambda} := \{ m \in M \mid Hm = \lambda m \}$. An $I_1$-module $M$ is called a generalized weight module if $M = \oplus_{\lambda \in K} M_{\lambda}$ where $M_{\lambda} := \{ m \in M \mid (H - \lambda)^n m = 0 \text{ for some } n = n(m) \}$. The set $\text{Supp}(M) := \{ \lambda \in K \mid M_{\lambda} \neq 0 \}$ is called the support of the generalized weight module $M$. For all $\lambda \in K$ and $n \geq 1$,
\[
\partial^n M_{\lambda} \subseteq M_{\lambda-n} \quad \text{and} \quad \int^n M_{\lambda} \subseteq M_{\lambda+n}.
\]
Let $0 \to N \to M \to L \to 0$ be a short exact sequence of $I_1$-modules. Then $M$ is a generalized weight module iff so are the modules $N$ and $L$, and in this case
\[
\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(L).
\]
For each $I_1$-module $M$, there is a short exact sequence of $I_1$-modules
\[
0 \to FM \to M \to \underline{M} := M/FM \to 0
\]
where
(i) $F \cdot FM = FM$, and
(ii) $F \cdot \underline{M} = 0$,
and the properties (i) and (ii) determine the short exact sequence uniquely, i.e. if $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of $I_1$-modules such that $FM_1 = M_1$ and $FM_2 = 0$ then $M_1 \cong FM$ and $M_2 \cong \underline{M}$.

Notice that
\[
FM \cong K[x]^I,
\]
i.e. the $I_1$-module $FM$ is isomorphic to the direct sum of $I$ copies of the simple weight $I_1$-module $K[x]$. Clearly, $\underline{M}$ is a $B_1$-module.

The indecomposable $I_1$-modules $M(n, \lambda)$. For $\lambda \in K$ and a natural number $n \geq 1$, consider the $B_1$-module
\[
M(n, \lambda) := B_1 \otimes_{K[H]} K[H]/(H - \lambda)^n.
\]
Clearly,
\[
M(n, \lambda) \cong B_1/B_1(H - \lambda)^n \cong I_1/(F + I_1(H - \lambda)^n).
\]
The $I_1$-module/$B_1$-module $M(n, \lambda)$ is a generalized weight module with $\text{Supp}(M(n, \lambda)) = \lambda + \mathbb{Z}$,
\[
M(n, \lambda) = \bigoplus_{i \in \mathbb{Z}} M(n, \lambda)^{\lambda+i} \quad \text{and} \quad \dim M(n, \lambda)^{\lambda+i} = n \quad \text{for all } i \in \mathbb{Z}.
\]
For an algebra $A$, we denote by $A - \text{Mod}$ its module category. The next proposition describes the set of indecomposable, generalized weight $I_1$-modules of finite length $M$ with $FM = 0$.

**Proposition 2.1.**

1. $M(n, \lambda)$ is an indecomposable, generalized weight $I_1$-module of finite length $n$.
2. $M(n, \lambda) \cong M(m, \mu)$ if and only if $n = m$ and $\lambda - \mu \in \mathbb{Z}$.
3. Let $M$ be a generalized weight $B_1$-module of length $n$ (i.e. let $M$ be a generalized weight $I_1$-module such that $FM = 0$, by [6]). Then $M$ is indecomposable if and only if $M \cong M(n, \lambda)$ for some $\lambda \in K$.

**Proof.**

1. Since $(B_1)_{K[H]} = \oplus_{i \in \mathbb{Z}} \partial^i K[H]$ is a free right $K[H]$-module, the functor
\[
B_1 \otimes_{K[H]} - : K[H] - \text{Mod} \to B_1 - \text{Mod}, \quad N \mapsto B_1 \otimes_{K[H]} N,
\]
is an exact functor. The $K[H]$-module $K[H]/(H - \lambda)^n$ is an indecomposable, hence the $B_1$-module $M(n, \lambda)$ is indecomposable and generalized weight of length $n$.

2. Suppose that $I_1$-modules $M(n, \lambda)$ and $M(m, \mu)$ are isomorphic. Then $\text{Supp}(M(n, \lambda)) = \text{Supp}(M(m, \mu))$, i.e. $\lambda + \mathbb{Z} = \mu + \mathbb{Z}$, i.e. $\lambda - \mu \in \mathbb{Z}$. Then $n = m$, by [10].

3. Suppose that $\lambda := \lambda - \mu \in \mathbb{Z}$ and $n = m$. We may assume that $k \geq 1$. Using the equality $(H - \lambda)^n \partial^k = \partial^k (H - \lambda - k)^n$, we see that the $B_1$-homomorphism
\[
M(n, \lambda) = B_1/B_1(H - \lambda)^n \to M(n, \mu) = B_1/B_1(H - \mu)^n, \quad 1 + B_1(H - \lambda)^n \mapsto \partial^k + B_1(H - \mu)^n,
\]
is an isomorphism with the inverse given by the rule $1 + B_1(H - \mu)^n \mapsto \partial^{-k} + B_1(H - \lambda)^n$.

3. This implication follows from statement 2.
Each indecomposable, generalized weight $B_1$-module $M$ is of the type $B_1 \otimes_{K[H]} N$ for an indecomposable $K[H]$-module $N$ of length $n$. Notice that $N \simeq K[H]/(H - \lambda)^n$ for some $\lambda \in K$. Therefore, $M \simeq M(n, \lambda)$. □

Lemma 2.2. Let $M$ be an indecomposable, generalized weight $I_1$-module. Then $\text{Supp}(M) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in K$.

Proof. Let $M = \bigoplus_{\mu \in \text{Supp}(M)} M^\mu$ be a generalized weight $I_1$-module. Then

\[ M = \bigoplus_{\mu + z \in \text{Supp}(M)/\mathbb{Z}} M_{\mu + z} \]

is a direct sum of $I_1$-submodules $M_{\mu + z} := \bigoplus_{i \in \mathbb{Z}} M_{\mu + i}$ where $\text{Supp}(M)/\mathbb{Z}$ is the image of the support $\text{Supp}(M)$ under the abelian group epimorphism $K \to K/\mathbb{Z}, \gamma \mapsto \gamma + \mathbb{Z}$. The $I_1$-module $M$ is indecomposable, hence $M = M_{\lambda + Z}$ for some $\lambda \in K$, i.e. $\text{Supp}(M) \subseteq \lambda + \mathbb{Z}$. □

The next lemma describes the set of indecomposable, generalized weight $I_1$-modules $M$ with $FM = M$.

Lemma 2.3. Let $M$ be an indecomposable, generalized weight $I_1$-modules $M$. Then the following statements are equivalent.

1. $FM = M$.
2. $M \simeq K[x]$.
3. $\text{Supp}(M) \subseteq \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) : If $FM = M$ then $M \simeq K[x]^{(I)}$ for some set $I$ necessarily with $|I| = 1$ since $M$ is indecomposable, i.e. $M \simeq K[x]$.

(2) $\Rightarrow$ (3) : $\text{Supp}(K[x]) = \{1, 2, \ldots\} \subseteq \mathbb{N}$.

(3) $\Rightarrow$ (1) : Suppose that $\text{Supp}(M) \subseteq \mathbb{N}$. Using the short exact sequence of $I_1$-modules $0 \to FM \to M \to \overline{M} := M/FM \to 0$ we see that $\text{Supp}(M) = \text{Supp}(FM) \cup \text{Supp}(\overline{M})$. Since $\text{Supp}(FM) = \text{Supp}(K[x]) = \{1, 2, \ldots\}$ and $\text{Supp}(\overline{M})$ is an abelian group, we must have $\overline{M} = 0$ (since $\text{Supp}(M) \subseteq \mathbb{N}$), i.e. $M = FM$. □

The following result is a key step in obtaining a classification of indecomposable, generalized weight $I_1$-modules of finite length.

Theorem 2.4. Let $M$ be a generalized weight $I_1$-module of finite length. Then the short exact sequence (5) splits.

Proof. We can assume that $FM \neq 0$ and $\overline{M} \neq 0$. It is obvious that $FM \simeq K[x]^s$ for some $s \geq 1$ and the $B_1$-module $\overline{M} \simeq \bigoplus_{i=1}^t M(n_i, \lambda_i)$ for some $n_i \geq 1, \lambda_i \in K$ and $t \geq 1$. It suffices to show that

\[ \text{Ext}^1_{I_1}(M(n, \lambda), K[x]) = 0 \] (11)

for all $n \geq 1$ and $\lambda \in K$. If $\lambda \in \mathbb{Z}$ we can assume that $\lambda = 0$, by Proposition 2.1.(2).

(i) $F(H - \lambda)^n = F$: The equality follows from the equalities $e_{ij}(H - \lambda)^n = e_{ij}(j + 1 - \lambda)$ and the choice of $\lambda$.

(ii) $M(n, \lambda) = \mathbb{I}_1/I_1(H - \lambda)^n$: By (i), $\mathbb{I}_1(H - \lambda)^n \supseteq F(H - \lambda)^n = F$. Hence,

\[ M(n, \lambda) = \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^n) = \mathbb{I}_1/I_1(H - \lambda)^n. \]

(iii) The equality (6) holds: Let $M = M(n, \lambda)$. By (ii), the short exact sequence of $I_1$-modules

\[ 0 \to \mathbb{I}_1(H - \lambda)^n \to \mathbb{I}_1 \to M \to 0 \] (12)

is a projective resolution of the $I_1$-module $M$ since the map

\[ (H - \lambda)^n : \mathbb{I}_1 \to \mathbb{I}_1(H - \lambda)^n, \ a \mapsto a(H - \lambda)^n, \]

is an isomorphism of $I_1$-modules, by the choice of $\lambda$. Then

\[ \text{Ext}^1_{I_1}(M, K[x]) \simeq Z^1/B^1 \]

where $Z^1 = \text{Hom}_{I_1}(\mathbb{I}_1(H - \lambda)^n, K[x]) \simeq K[x]$ and $B^1 \simeq (H - \lambda)^n K[x] = K[x]$, by the choice of $\lambda$. Hence, the equality (6) holds. The proof of the theorem is complete. □
The next theorem is a classification of the set of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length.

**Theorem 2.5.** Each indecomposable, generalized weight $\mathbb{I}_1$-module of finite length is isomorphic to one of the modules below:

1. $K[x]$,
2. $M(n, \lambda)$ where $n \geq 1$ and $\lambda \in \Lambda$ where $\Lambda$ is any fixed subset of $K$ such that the map $\Lambda \to (K/\mathbb{Z})$, $\lambda \mapsto \lambda + \mathbb{Z}$, is a bijection.

The $\mathbb{I}_1$-modules above are pairwise non-isomorphic, indecomposable, generalized weight and of finite length.

**Proof.** The theorem follows from Theorem 2.4, Proposition 2.7 and Lemma 2.3.

**Corollary 2.6.** Every indecomposable, generalized weight $\mathbb{I}_1$-module is an uniserial module.

**Proof.** The statement follows from Theorem 2.5.

**Homomorphisms and Ext-groups between indecomposables.**

**Proposition 2.7.**

1. Let $M$ and $N$ be generalized weight $\mathbb{I}_1$-modules such that $\text{Supp}(M) \cap \text{Supp}(N) = \emptyset$. Then $\text{Hom}_{\mathbb{I}_1}(M, N) = 0$.
2. $\text{Hom}_{\mathbb{I}_1}(M(n, \lambda), K[x]) = 0$.
3. $\text{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda)) = 0$.
4. $\text{Hom}_{\mathbb{I}_1}(M(n, \lambda), M(m, \mu)) \simeq \text{Hom}_{K[H]}((K[H]/((H - \lambda)^n), (K[H]/((H - \lambda)^m)) \simeq K[H]/((H - \lambda)^{\min(n, m)})$.

**Proof.**

1. Statement 1 is obvious.
2. Statement 2 follows from the fact that $FM(n, \lambda) = 0$ and $FP = K[x]$ for all nonzero elements $p \in K[x]$ (since $K[x]$ is a simple $\mathbb{I}_1$-module, $F$ is an ideal of the algebra $\mathbb{I}_1$ such that $FK[x] = K[x]$).
3. Statement 3 follows from the fact that $FK[x] = K[x]$ and $FM(n, \lambda) = 0$: $f(K[x]) = f(FK[x]) = f(f(K[x]) = 0$ for any $f \in \text{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda))$.
4. The first isomorphism is obvious. Then the second isomorphism follows.

**Proposition 2.8.**

1. $\text{Ext}^1_{\mathbb{I}_1}(K[x], K[x]) = 0$.
2. $\text{Ext}^1_{\mathbb{I}_1}(M(n, \lambda), K[x]) = 0$.
3. $\text{Ext}^1_{\mathbb{I}_1}(K[x], M(n, \lambda)) = 0$.
4. $\text{Ext}^1_{\mathbb{I}_1}(M(n, \lambda), M(m, \mu)) = \begin{cases} K & \text{if } \lambda - \mu \in \mathbb{Z}, \\ 0 & \text{if } \lambda - \mu \notin \mathbb{Z}. \end{cases}$

**Proof.**

1. Let $0 \to K[x] \to N \to K[x] \to 0$ be a s.e.s. of $\mathbb{I}_1$-modules. Then $FN = N$ (since $FK[x] = K[x]$), and so $N$ is an epimorphic image of the semisimple $\mathbb{I}_1$-module $F \oplus F$. Hence, $N \simeq K[x] \oplus K[x]$ (since $I_1 F \simeq K[x]$).
2. See 1.
3. Let $0 \to M = M(n, \lambda) \to L \to K[x] \to 0$ be a s.e.s. of $\mathbb{I}_1$-modules. Since $FM = 0$, we have $FL = FK[x] \simeq K[x]$ is a submodule of $L$ such that $FL \cap M = 0$ (since otherwise $FL \subseteq M$ by simplicity of the $\mathbb{I}_1$-module $FL \simeq K[x]$, and so $0 \neq K[x] \simeq FL = F^2L \subseteq FM = 0$, a contradiction). Then $FL \oplus M \subseteq L$. Furthermore, $FL \oplus M = L$ since $l_i(FL \oplus M) = l_i(L)$. This means that the s.e.s. splits.
4. Let $0 \to M_1 \to M \to M_1 \to 0$ be a s.e.s. of generalized weight $\mathbb{I}_1$-modules. If $\text{Supp}(M_1) \cap \text{Supp}(M_2) = \emptyset$, it splits. In particular, $\text{Ext}^1_{\mathbb{I}_1}(M(n, \lambda), M(m, \mu)) = 0$ if $\lambda - \mu \notin \mathbb{Z}$. If $\lambda - \mu \in \mathbb{Z}$ we can assume that $\lambda = \mu$ (since $M(m, \lambda) \simeq M(m, \mu)$). Using 12, where we assume that $\lambda = 0$ if $\lambda \in \mathbb{Z}$, we see that $\text{Ext}^1_{\mathbb{I}_1}(M(n, \lambda), M(m, \mu)) \simeq M(n, \lambda)/(H - \lambda)M(m, \mu) \simeq K$. □

Since the left global dimension of the algebra $\mathbb{I}_1$ is 1, Proposition 2.7 and Proposition 2.8 describe all the Ext-groups between indecomposable, generalized weight $\mathbb{I}_1$-modules. This is also obvious from the proofs of the propositions.
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