Compositional Construction of Abstractions for Infinite Networks of Discrete-Time Switched Systems

Maryam Sharifi\textsuperscript{a}, Abdalla Swikir\textsuperscript{b}, Navid Noroozi\textsuperscript{c}, Majid Zamani\textsuperscript{c,d}

\textsuperscript{a}School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden  
\textsuperscript{b}Department of Electrical and Computer Engineering, Technical University of Munich, Germany  
\textsuperscript{c}Institute for Informatics, LMU Munich, Germany  
\textsuperscript{d}Computer Science Department, University of Colorado Boulder, USA

Abstract
In this paper, we develop a compositional scheme for the construction of continuous approximations for interconnections of infinitely many discrete-time switched systems. An approximation (also known as abstraction) is itself a continuous-space system, which can be used as a replacement of the original (also known as concrete) system in a controller design process. Having designed a controller for the abstract system, it is refined to a more detailed one for the concrete system. We use the notion of so-called simulation functions to quantify the mismatch between the original system and its approximation. In particular, each subsystem in the concrete network and its corresponding one in the abstract network are related through a notion of local simulation functions. We show that if the local simulation functions satisfy certain small-gain type conditions developed for a network containing infinitely many subsystems, then the aggregation of the individual simulation functions provides an overall simulation function quantifying the error between the overall abstraction network and the concrete one. In addition, we show that our methodology results in a scale-free compositional approach for any finite-but-arbitrarily large networks obtained from truncation of an infinite network. We provide a systematic approach to construct local abstractions and simulation functions for networks of linear switched systems. The required conditions are expressed in terms of linear matrix inequalities that can be efficiently computed. We illustrate the effectiveness of our approach through an application to AC islanded microgrids.

Keywords: Compositionality, continuous abstractions, infinite networks, small-gain theorem, switched systems.

1. Introduction
Recent technological advances in sensing, computation, and data management have enabled us to develop smart networked systems providing more autonomy and flexibility. Smart grids, swarm robotics, connected automated vehicles and smart manufacturing are just a few examples of such emerging smart networked systems, in which a large numbers of dispersed agents interact and communicate with each other to achieve a common objective. The size and the structure of such networks can be arbitrarily large, time-varying or even unknown, and agents can be constantly plugged into and out from the network. Emerging control networks necessitate also sophisticated control objectives, which go beyond standard goals pursued in classical control theory. For instance, a sophisticated objective is to control connected autonomous vehicles merging at a traffic intersection while ensuring safety and fuel economy constraints.

The complexity of control objectives, the large number of participating agents, as well as safety concerns call for automated and provably correct techniques to verify or synthesize controllers for the emerging applications of control systems. A promising methodology to address the above issues is achieved by a careful integration of concepts from control theory (e.g. Lyapunov methods and small-gain theory) and those of computer science (e.g. formal methods and assume-guarantee rules)\cite{1,2}. Discrete abstractions (a.k.a. symbolic models) is one particular technique to provide automated synthesis of correct-by-design controllers for concrete systems. In this approach, controller synthesis problems can be algorithmically solved over finite abstractions of concrete systems by resorting to automata-theoretic approaches\cite{3}. Then, the constructed controllers can be refined back to the original systems based on some behavioral relations between original systems and their finite abstractions such as approximate alternating simulation relations\cite{4} or feedback refinement relations\cite{5}.

The computational complexity of constructing finite ab-
stractions of the concrete systems makes the practical applicability of these methods considerably challenging. Hence, applying such approaches to large-scale systems is not feasible at all. An appropriate technique to overcome this challenge is to introduce a pre-processing step by constructing so-called continuous abstractions. In that way, a continuous-space system, but possibly with a lower dimension, is obtained as a substitute of the concrete system [3, 7, 8, 9]. We note that the applicability of continuous abstractions is not limited to the context of symbolic controllers. In fact, they can be used in other hierarchical control approaches in the lower layers, where a simplified model of the system is used for controller design purposes.

For large-scale networks, it is often more useful to maintain the structure (i.e. topology) of the network while abstractions are constructed. In that way, corresponding to each participating subsystem of the network, a continuous abstraction is constructed individually. Therefore, the complexity of synthesizing continuous abstractions of infinite-dimensional systems is managed in an efficient way. The methodology by which an abstraction for the overall network is achieved via the interconnection of the individual abstractions is called a compositional approach [10, 11, 12]. In order to guarantee that the aggregation of the individual abstractions provides an abstraction for the overall network, the interaction between subsystems should be weak enough which can be technically described by a small-gain condition [10, 11, 12].

Small-gain type conditions are intrinsically dependent on the size of the network. Hence, one can readily find that the satisfaction of compositionality conditions dramatically degrades as the the number of subsystems increases and may not be valid anymore, see [11, Remark 6.1]. The works in the literature regarding stability analysis of large-scale systems, e.g. [16, 17, 18, 19, 20, 21], inspired us to address the scalability issue using an over-approximation of a finite-but-large network with a network composed of infinitely many subsystems. We call such aggregated system an infinite network. It is widely accepted that an infinite network captures the essence of its corresponding finite network; see e.g. a vehicle platooning application in [22]. This treatment leads to an infinite-dimensional system and calls for a more rigorous and detailed setting. In particular, we adapt the notion of simulation functions [6] to the case of infinite-dimensional switched systems. The existence of a simulation function ensures that the error between the output trajectories of the abstract and concrete system is quantitatively bounded in a certain sense (cf. Definition 5). By exploiting the compositionality approach, we assign an individual simulation function to each subsystem and construct the corresponding local abstraction accordingly. Then we aggregate them to construct an abstraction for the overall network. We show that if a certain small-gain condition recently developed in [17] is satisfied, then the aggregation yields a continuous abstraction for the overall concrete network. Particularly, for linear networks, our conditions are expressed in terms of linear matrix inequalities, where we explicitly construct the individual abstractions as well as the controller refinement formulation.

Motivated by the scale-dependency issue in the classic compositionality methods, in this paper, a scale-free compositional approach for the construction of continuous abstractions for arbitrarily large-scale networks of discrete-time switched systems is provided. We elucidate the scale-free property of our approach by truncating the infinite network to a finite-but-arbitrary large network and show that the compositional abstraction results are preserved under any truncation. To the best of our knowledge, our work is the first to provide a scale-free compositional approach for construction of continuous abstractions. In addition to the scalability issue, in a large number of applications, the structure of the network is time-varying in the sense that the communication links between subsystems change over time. In power networks, for instance, there exist line switches and the agents are constantly plugged into and out. This calls for considering switched dynamics describing this time dependency of the network structure. Our setting, therefore, considers an infinite network of switched systems. To validate the effectiveness of our approach, we apply our results to AC microgrids operating in an islanded mode. In particular, we show through simulations that the behavior of the network remains independent of the size of the network, while the network size dramatically increases.

This paper expands on the conference paper [23], where uniformity conditions with respect to the switching modes were made. The present work provides a completely non-uniform structure for the simulation functions with respect to the switching signals in the network. Moreover, the scale-free property of the result is established, which leads to constructing compositional abstractions for any finite-but-arbitrarily large network. Therefore, the current setting allows us to consider more general and realistic scenarios, including the new AC microgrid case study.

The rest of the paper is organized as follows. Section 2 provides the systems description. In Section 3 we first introduce the notion of simulation functions for switched systems, and then show the importance of the existence of such functions in the construction of abstractions. Section 4 contains the main result of the paper, that is the construction of continuous abstractions compositionally using small-gain theory. In Section 5 we focus on linear subsystems and provide easier-to-check conditions for the construction of continuous abstractions. In Section 6 we apply our results to a network of AC islanded microgrids.

2. Preliminaries and System Description

2.1. Notation

We write $\mathbb{N}_0(\mathbb{N})$ for the set of nonnegative (positive) integers. For vector norms on finite- and infinite-dimensional vector spaces, we write $\| \cdot \|$. By $\ell^p$, $p \in [1, \infty)$, we denote
the Banach space of all real sequences $x = (x_i)_{i \in \mathbb{N}}$ with finite $\ell^p$-norm $|x|_p < \infty$, where $|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $p < \infty$. If $X$ is a Banach space, we write $r(T)$ for the spectral radius of a bounded linear operator $T : X \to X$.

The identity function is denoted by $id$. Throughout this work, we will consider $\mathcal{K}$ and $\mathcal{K}_\infty$ comparison functions; see [24 Chapter 4.4] for definitions.

We consider discrete-time switched subsystems $\Sigma_i$, defined later. The arbitrary switching signals are defined as $\sigma_i : N_0 \to S_i$ for each subsystem $\Sigma_i$, $i \in \mathbb{N}$, and $S_i = \{1, 2, \ldots, r_i\}$ is a finite index set with $r_i \in \mathbb{N}$. The set of such switching signals are denoted by $\mathcal{S}_i$.

2.2. Infinite networks

First, we define discrete-time switched subsystems which are interconnected to form an infinite network consisting of countably infinite number of control subsystems.

**Definition 1.** A discrete-time switched system $\Sigma_i$, $i \in \mathbb{N}$, is defined by the tuple

$$\Sigma_i = (X_i, W_i, U_i, Y_i, h_{i,s,i}, f_{i,s,i}, S_i),$$

where $X_i \subseteq \mathbb{R}^n$, $W_i \subseteq \mathbb{R}^m$, $U_i \subseteq \mathbb{R}^p$, and $Y_i \subseteq \mathbb{R}^q$, are the state set, internal input set, external input set, and output set, respectively. We use symbol $U_i$ to denote the set of functions $u_i : N_0 \to U_i$. Functions $f_{i,s,i} : X_i \times W_i \times U_i \to X_i$ are the transition functions for $s_i \in S_i$. Moreover, $h_{i,s,i} : X_i \to Y_i$ are the output maps.

The discrete-time switched subsystems $\Sigma_i$, $i \in \mathbb{N}$, are represented by the difference equation of the form

$$\begin{align*}
\Sigma_i : \quad & \begin{cases}
x_i(k+1) = f_{i,s,i}(x_i(k), w_i(k), u_i(k)), \\
y_i(k) = h_{i,s,i}(x_i(k)),
\end{cases}
\end{align*}$$

where $x_i : N_0 \to X_i$, $w_i : N_0 \to W_i$, $u_i : N_0 \to U_i$, and $y_i : N_0 \to Y_i$ are the state signal, internal input signal, external input signal, and output signal, respectively.

The finite set $I_{i,s,i}^m \subseteq N \setminus \{i\}$ collects mode-dependent in-neighbors of $\Sigma_i$, i.e., systems $\Sigma_j, j \in I_{i,s,i}^m$, directly influencing $\Sigma_i$. On the other hand, the finite set $I_{i,s,i}^o \subseteq N \setminus \{i\}$ collects mode-dependent out-neighbors of $\Sigma_i$, i.e., $\Sigma_j, j \in I_{i,s,i}^o$, influenced by $\Sigma_i$. Note that we assume $i \not\in I_{i,s,i}^m \cup I_{i,s,i}^o$, $\forall i \in \mathbb{N}$. The input-output structure of each subsystem $\Sigma_i$, $i \in \mathbb{N}$, is given by

$$\begin{align*}
w_i(k) &= (w_{ij}(k))_{j \in I_{i,s,i}^m} \in W_i := \prod_{j \in I_{i,s,i}^m} W_{ij}, \\
y_i(k) &= (y_{ij}(k))_{j \in I_{i,s,i}^o} \in Y_i := \prod_{j \in I_{i,s,i}^o} Y_{ij}, \\
h_{i,s,i}(x_i(k)) &= (h_{ij,s,i}(x_i(k)))_{j \in I_{i,s,i}^o}.
\end{align*}$$

We denote $w_i(k)$ for $N_i := \sum_{j \in I_{i,s,i}^m} n_j$, as the internal inputs describing the interconnections among subsystems.

The outputs $y_i(k), j \in I_{i,s,i}^o$, are considered as internal outputs which are used to construct interconnections between subsystems, whereas $y_i(k) \in Y_i$ are denoted as external outputs. Note that $w_i(k)$ and $y_i(k)$ are partitioned into sub-vectors and we aggregate all the subsystems $\Sigma$ through the interconnection constraints given by $w_{ij}(k) = y_{ji}(k)$ for all $i \in \mathbb{N}$ and for all $j \in I_{i,s,i}^o$.

To model the state (resp. input) space of the overall network, we introduce a Banach space of sequences $x = (x_i)_{i \in \mathbb{N}}$ (resp. $u = (u_i)_{i \in \mathbb{N}}$). The most natural choice is the $\ell^p$-space, precisely, defined as follows: we first fix a norm on each $X_i$; then, for every $p \in [1, \infty)$, we put

$$\ell^p(N, (n_i)) := \left\{ x = (x_i)_{i \in \mathbb{N}} : x_i \in X_i, \sum_{i \in \mathbb{N}} |x_i|^p < \infty \right\},$$

and equip this space with the norm $|x|_p := (\sum_{i \in \mathbb{N}} |x_i|^p)^{1/p}$.

Now, we provide a formal definition of the infinite network.

**Definition 2.** Consider subsystems $\Sigma_i = (X_i, W_i, U_i, Y_i, h_{i,s,i}, f_{i,s,i}, S_i)$, $i \in \mathbb{N}$, with the input-output structure as in (3). A discrete-time infinite network $\Sigma$ is defined by the tuple $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, \mathcal{h}, \mathcal{f}, \mathcal{S})$, where $\mathcal{X} = \ell^p(N, (n_i)) \subset \prod_{i \in \mathbb{N}} X_i$ with a fixed $p \in [0, \infty)$ and $\mathcal{U} = \ell^q(N, (n_i)) \subset \prod_{i \in \mathbb{N}} U_i$ with a fixed $q \in [0, \infty)$. The space of admissible external input functions $u$ is defined by $\mathcal{U} := \{ u : N_0 \to \mathcal{U} \}$. Moreover, $h_{i}(x) = (h_{i,s,i}(x_i))_{i \in \mathbb{N}}, s \in \mathcal{S}, \mathcal{S} := \prod_{i \in \mathbb{N}} S_i$ denotes the output function, where $h_i : \mathcal{X} \to \mathcal{Y}, \mathcal{Y} \subset \prod_{i \in \mathbb{N}} \mathcal{Y}_i$. In addition, we restrict $f_s(x, u) = (f_{i,s,i}(x_i, w_i, u_i))_{i \in \mathbb{N}}$ to $f_s : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$.

In that way, the interconnection of subsystems $\Sigma_i$, $i \in \mathbb{N}$, is described by

$$\begin{align*}
\Sigma : \quad & \begin{cases}
x(k+1) = f_{\sigma}(x(k), u(k)), \\
y(k) = h_{\sigma}(x(k)),
\end{cases}
\end{align*}$$

where $x(k) = (x_i(k))_{i \in \mathbb{N}}, u(k) = (u_i(k))_{i \in \mathbb{N}}, y(k) = (y_i(k))_{i \in \mathbb{N}}, \sigma(k) = (\sigma_i(k))_{i \in \mathbb{N}}, f_{\sigma}(x(k), u(k)) = (f_{i,\sigma_i}(x_i(k), w_i(k), u_i(k)))_{i \in \mathbb{N}}$, and $h_{\sigma}(x(k)) := (h_{i,\sigma_i}(x_i(k)))_{i \in \mathbb{N}}$. We call the overall system an infinite network and denote the corresponding solutions by $\mathbf{x}(k, x, \sigma, u)$ for any $k \in N_0$, any initial value $x \in \mathcal{X}$, any switching signal $\sigma : N_0 \to \mathcal{S}, \mathcal{S} := \{ \sigma : N_0 \to \mathcal{S} \}$, and any control input $u \in \mathcal{U}$.

We refer to system (4) as the concrete system, which is often hard to control or analyze. To simplify the controller design process, we, instead, use a simpler and less precise system called an abstract system.

3. Abstractions for Discrete-Time Switched Systems

In this section, we introduce a notion of simulation functions for discrete-time switched systems. A simulation function quantifies a relation between the concrete system and its abstraction in the sense that the mismatch between
their output trajectories remains bounded (cf. Proposition 4). A simulation function is formally defined as follows.

Definition 3. Consider two systems \( \Sigma = (X, U, \tilde{U}, Y, h_s, f_s, S) \) and \( \tilde{\Sigma} = (\tilde{X}, \tilde{U}, \tilde{Y}, \tilde{h}_s, \tilde{f}_s, \tilde{S}) \) with the same output space dimensions. Let \( p, q \in [1, \infty) \) be given. Let \( V_s : X \times \tilde{X} \to \mathbb{R}_+ \), \( s \in S \), be a family of functions. Assume that there exist constants \( a, b > 0 \) such that for all \( s \in S \) and all \( x, \tilde{x} \in X \),

\[
\alpha \left| h_s(x) - \tilde{h}_s(\tilde{x}) \right|^b \leq V_s(x, \tilde{x}),
\]

and there exist a function \( p_{ext} \in K \) and a constant \( 0 < \lambda < 1 \), such that for all \( s' \), \( s \in S \) and all \( x, \tilde{x} \in X \) and \( \tilde{u} \in \tilde{U} \), there exists \( u \in U \) so that we have

\[
V_s(f_s(x, u), \tilde{f}_s(\tilde{x}, \tilde{u})) - V_s(x, \tilde{x}) \leq -\lambda V_s(x, \tilde{x}) + p_{ext}(|\tilde{u}|_q).
\]

Functions \( V_s \) satisfying (5) and (6) are called simulation functions from \( \Sigma \) to \( \tilde{\Sigma} \) and \( \tilde{\Sigma} \) is called an abstraction of \( \Sigma \).

Now we show that the existence of a simulation function ensures that the output trajectories of the abstract and concrete systems remain within a bounded distance from each other.

Proposition 4. Consider systems \( \Sigma = (X, U, \tilde{U}, Y, h_s, f_s, S) \) and \( \tilde{\Sigma} = (\tilde{X}, \tilde{U}, \tilde{Y}, \tilde{h}_s, \tilde{f}_s, \tilde{S}) \) with the same output space dimensions. Let a set of simulation functions \( V_s, s \in S \), from \( \Sigma \) to \( \tilde{\Sigma} \) and \( \tilde{\Sigma} \) is given. Then there exists a function \( \gamma_{ext} \in K \) and positive constants \( \vartheta \) and \( \beta < 1 \), such that for any \( s \in S \), \( x \in X \), \( \tilde{x} \in \tilde{X} \), and \( \tilde{u} \in \tilde{U} \) there exists \( u \in U \) so that we have

\[
|y(k, x, \sigma, u) - \tilde{y}(k, \tilde{x}, \sigma, \tilde{u})|_p \leq \vartheta \beta^k \gamma_{ext}(|\tilde{u}|_{q, \infty}) + \gamma_{ext}(|\tilde{u}|_{q, \infty}),
\]

where \( |\tilde{u}|_{q, \infty} := sup_{\tilde{u} \in \tilde{U}} |\tilde{u}|_q \) and \( b \) as in (3).

Proof. The proof follows similar arguments as those in the proof of \cite[Lemma 3.5]{8}. Take any \( \varepsilon > 0 \) and define \( D := \{(x, \tilde{x}) \in X \times \tilde{X} : V_s(x, \tilde{x}) \leq \frac{1}{1 - \lambda} p_{ext}(|\tilde{u}|_q) \} \) for all \( s \in S \). It follows from (8) that

\[
V_s(f_s(x, u), \tilde{f}_s(\tilde{x}, \tilde{u})) - V_s(x, \tilde{x}) \leq -\varepsilon V_s(x, \tilde{x}) + p_{ext}(|\tilde{u}|_q) - (\lambda - \varepsilon) V_s(x, \tilde{x}).
\]

(8)

For all \((x, \tilde{x}) \in (X \times \tilde{X}) \setminus D\), we have \( V_s(x, \tilde{x}) > \frac{1}{\lambda - \varepsilon} p_{ext}(|\tilde{u}|_q) \). Thus, we have

\[
V_s(f_s(x, u), \tilde{f}_s(\tilde{x}, \tilde{u})) - V_s(x, \tilde{x}) \leq -\varepsilon V_s(x, \tilde{x}),
\]

for all \((x, \tilde{x}) \in (X \times \tilde{X}) \setminus D\), that can be written as

\[
V_{s(k)}(x(k, x, \sigma, u), \tilde{x}(k, \tilde{x}, \sigma, \tilde{u})) \leq (1 - \varepsilon)V_{s(k-1)}(x(k-1, x, \sigma, u), \tilde{x}(k-1, \tilde{x}, \sigma, \tilde{u})),
\]

for all \( k \in \mathbb{N}_0 \). Therefore, we obtain

\[
V_{s(k)}(x(k, x, \sigma, u), \tilde{x}(k, \tilde{x}, \sigma, \tilde{u})) \leq (1 - \varepsilon)^k V_{s(0)}(x, \tilde{x}).
\]

(9)

Now consider \((x, \tilde{x}) \in D\). It follows from (8) that

\[
V_s(f_s(x, u), \tilde{f}_s(\tilde{x}, \tilde{u})) \leq (1 - \lambda) V_s(x, \tilde{x}) + p_{ext}(|\tilde{u}|_q) \leq \frac{1 - \lambda}{\lambda - \varepsilon} p_{ext}(|\tilde{u}|_q) + p_{ext}(|\tilde{u}|_q) = \frac{1 - \varepsilon}{\lambda - \varepsilon} p_{ext}(|\tilde{u}|_q).
\]

(10)

Inequalities (9) and (10) imply that

\[
V_{s(k)}(x(k, x, \sigma, u), \tilde{x}(k, \tilde{x}, \sigma, \tilde{u})) \leq (1 - \varepsilon)^k V_{s(0)}(x, \tilde{x}) + \frac{(1 - \varepsilon)}{\lambda - \varepsilon} p_{ext}(|\tilde{u}|_{q, \infty}).
\]

(11)

It follows from (5) and (11) that

\[
\alpha |y(k, x, \sigma, u) - \tilde{y}(k, \tilde{x}, \sigma, \tilde{u})|_p \leq (1 - \varepsilon)^k V_{s(0)}(x, \tilde{x}) + \frac{(1 - \varepsilon)}{\lambda - \varepsilon} p_{ext}(|\tilde{u}|_{q, \infty}),
\]

which implies that

\[
\gamma_{ext}(|\tilde{u}|_{q, \infty}) \leq \vartheta \beta^k \gamma_{ext}(|\tilde{u}|_{q, \infty}) + \gamma_{ext}(|\tilde{u}|_{q, \infty}),
\]

where \( \vartheta = (2\frac{\lambda}{\lambda - \varepsilon})^k \), \( \beta = (1 - \varepsilon)^k \), \( \gamma_{ext}(\cdot) \). This completes the proof.

Remark 5. Suppose that we are given an interface function \( \nu \), which maps every \( x, \tilde{x} \), \( u \), and \( s \) to an input \( u = \nu(x, \tilde{x}, u, s) \) so that (9) is satisfied. Then, the input \( u \) that realizes (9) is readily given by \( u(k) = \nu(x(k), \tilde{x}(k), u(k), s(k)) \); see [29, Theorem 1].

Due to the size of the systems, a simulation function from \( \Sigma \) to \( \tilde{\Sigma} \) is quite hard to be directly computed. To address this complexity, we follows a compositional approach and define local simulation functions for each finite-dimensional subsystem (cf. Definition 6). This enables us to verify (5) and (6) in a bottom-up way. The next section develops this strategy with the use of small-gain theory for infinite networks.

4. Compositional Construction of Abstractions and Simulation Functions

In the following, we provide a method for compositional construction of simulation functions between the infinite networks \( \Sigma \) and \( \tilde{\Sigma} \). We assume that each subsystem \( \Sigma_i = (X_i, \tilde{X}_i, U_i, \tilde{U}_i, Y_i, h_{i,s}, f_{i,s}, S_i) \) and \( \tilde{\Sigma}_i = (\tilde{X}_i, \tilde{U}_i, \tilde{Y}_i, \tilde{h}_{i,s}, \tilde{f}_{i,s}, \tilde{S}_i) \) admits a local simulation function as defined below.

Definition 6. Consider subsystems \( \Sigma_i = (X_i, \tilde{X}_i, U_i, \tilde{U}_i, Y_i, h_{i,s}, f_{i,s}, S_i) \) and \( \tilde{\Sigma}_i = (\tilde{X}_i, \tilde{U}_i, \tilde{Y}_i, \tilde{h}_{i,s}, \tilde{f}_{i,s}, \tilde{S}_i) \), \( i \in \mathbb{N} \). Let \( p, q \in [1, \infty) \) be given. Assume that there exist functions \( V_{s_i} : X_i \times \tilde{X}_i \to \mathbb{R}_+ \), \( s_i \in S_i \), satisfying the following properties:

1. \( V_{s_i}(x, \tilde{x}) \geq 0 \) for all \( x, \tilde{x} \in X_i \times \tilde{X}_i \). Then, \( V_{s_i} \) is called a simulation function from \( \Sigma_i \) to \( \tilde{\Sigma}_i \).
• There are constants $\alpha_i > 0$ so that for all $x_i \in X_i$ and all $\hat{x}_i \in \hat{X}_i$

$$\alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p \leq V_{i,s_i}(x_i, \hat{x}_i).$$ (12)

• There are positive constants $\lambda_i < 1, \rho_{i,\text{int}}, \rho_{i,\text{ext}}$ such that for all $s_i', s_i \in S_i$, all $x_i \in X_i$, all $\hat{x}_i \in \hat{X}_i$, all $\hat{u}_i \in \hat{U}_i$, there exists $u_i \in U_i$ so that the following holds for all $v_i \in V_i$ and all $\hat{w}_i \in \hat{V}_i$

$$V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) - V_{i,s_i}(x_i, \hat{x}_i) + \rho_{i,\text{int}}|\hat{u}_i|^q + \rho_{i,\text{ext}}|w_i - \hat{w}_i|^p.$$ (13)

Then functions $V_{i,s_i}$ are called local simulation functions from $\hat{\Sigma}_i$ to $\Sigma_i$ and $\hat{\Sigma}_i$ are called abstractions of $\Sigma_i$ for each $i \in N$.

Assume that each $\Sigma_i$ admits an abstraction $\hat{\Sigma}_i, \forall i \in N$, given as in Definition 6. We establish a compositional approach for the construction of continuous abstractions of infinite networks by aggregating individual continuous abstractions $\hat{\Sigma}_i$. To do so, we need interaction between subsystems to be sufficiently weak, which is quantitatively described by a small-gain condition, see Assumption 9 below.

To employ the small-gain theorem, the following conditions are required. The first one makes uniformity conditions on the constants given by Definition 6.

**Assumption 7.** There are constants $\underline{\alpha} \leq \alpha_i \leq \overline{\alpha}$ and $\underline{\rho} \leq \rho_{i,\text{int}}, \rho_{i,\text{ext}} \leq \overline{\rho}$.

We collect the coefficients from (12) and (13) to define

$$\gamma_{ij} := \left\{ \begin{array}{ll} \rho_{i,\text{int}}N_i & \text{if } j \in I_{i,s_i}' \setminus I_{i,s_i}, \\ 0 & \text{if } j \notin I_{i,s_i}', \end{array} \right. \quad \forall j \in N_i,$$ (14)

where $N_i$ denotes the cardinality of the set $I_{i,s_i}'$. We additionally introduce the following matrices.

$$\Lambda := \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \Gamma := (\gamma_{ij})_{i,j \in N}.$$ (15)

Now, we define the following matrix by which we express our small-gain condition

$$\Psi := \Lambda^{-1/2}(\psi_{ij})_{i,j \in N}, \quad \psi_{ij} = \gamma_{ij}/\lambda_i.$$ (16)

We also make an assumption on the boundedness of the operator $\Gamma$.

**Assumption 8.** The operator $\Gamma = (\gamma_{ij})_{i,j \in N}$ satisfies

$$\sup_{j \in N} \sum_{i=1}^{\infty} \gamma_{ij} < \infty.$$ (17)

Note that Assumption 8 always holds if each subsystem is interconnected to finitely many subsystems and no global communication is used.

The following spectral radius condition provides a quantitative bound on the strength of couplings between the subsystems. This is, in fact, the small-gain condition that is required to guarantee that the aggregation of $\hat{\Sigma}_i$ gives a continuous abstraction for network $\Sigma$.

**Assumption 9.** The spectral radius $r(\Psi) < 1$.

The following theorem gives the main result of the paper, which is a compositional approach to construct the abstractions of infinite interconnected switched control systems and their corresponding simulation functions.

**Theorem 10.** Consider infinite networks $\Sigma = (X, U, Y, h_s, f_s, S)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{Y}, \hat{h}_s, \hat{f}_s, S)$. Let $p, q \in [1, \infty)$ be given. Let local simulation functions $V_{i,s_i} : X_i \times \hat{X}_i \to \mathbb{R}_+, s \in S_i$, satisfy Assumptions 7 and 8 and 9. Then there exists a vector $\mu = (\mu_i)_{i \in N} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ with constants $\underline{\mu}, \overline{\mu} > 0$ such that the following is satisfied

$$\left[ \mu^\top(-\Lambda + \Gamma)\right]_i \leq -\lambda_i, \quad \forall i \in N,$$ (18)

for a constant $\lambda_i \in (0, 1)$. Moreover, the following family of functions $V_s : X \times \hat{X} \to \mathbb{R}_+$, with $S = \bigcup_{i \in N} S_i$,

$$V_s(x, \hat{x}) = \sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i, \hat{x}_i), \quad V_s : X \times \hat{X} \to \mathbb{R}_+,$$

are simulation functions from $\hat{\Sigma}$ to $\Sigma$ with $b = p, \alpha = \underline{\mu}$, and $\lambda = \lambda_\infty$ and $\rho_{\text{ext}} : t \mapsto \overline{\mu} p_{\text{ext}} t^q$ as in (16).

**Proof.** From [17, Lemma V.10], Assumption 9 (i.e. $r(\Psi) < 1$) implies that there exists a vector $\mu = (\mu_i)_{i \in N} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ such that (17) holds.

Now we show that $V$ in (14) satisfies (5) with $\alpha = \underline{\mu}$. For any $s \in S, s_i \in S_i, x \in X$ and $\hat{x} \in \hat{X}$ and taking $b = p$, it follows from (12) and Assumption 7 that

$$\sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i, \hat{x}_i) \geq \sum_{i=1}^{\infty} \mu_i \alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p$$

$$\geq \underline{\mu} \alpha \sum_{i=1}^{\infty} |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p$$

$$\geq \underline{\mu} \alpha \lambda_i |x_i - \hat{x}_i|^p.$$ (19)

Next we show the inequality (18) holds as well. Considering (13) and (14), we obtain the chain of inequality in (18) for all $s_i', s_i \in S_i, s_j \in S_j, s' \in S \in N$.

Letting $V_{\text{ext}}(x, \hat{x}) := (V_{i,s_i}(x, \hat{x}_i))_{i \in N}$ and using (18) and (17), we have that

$$V_{\text{ext}}(f_s(x, u), \hat{f}_s(\hat{x}, \hat{u})) - V_s(x, \hat{x})$$

$$\leq \left[ \mu^\top(-\Lambda + \Gamma)V_{\text{ext}}(x, \hat{x}) + \sum_{i=1}^{\infty} \mu_i \rho_{i,\text{ext}}|\hat{u}_i|^q \right]$$

$$\leq -\lambda_\infty V_s(x, \hat{x}) + \rho_{\text{ext}}(|\hat{u}_i|^q),$$

where $\rho_{\text{ext}}(t) = \overline{\mu} p_{\text{ext}} t^q$ for all $t \geq 0$. \qed

**Remark 11.** The significance of Assumptions 7 and 8 in Theorem 10 has been discussed in [17]. Specifically, the
small-gain condition \( r(\Psi) < 1 \) is tight and cannot be relaxed. In view of Gelfand’s formula, the spectral small-gain condition is equivalent to the existence of \( k \in \mathbb{N} \) such that \( \|k\Psi\|^k < 1 \). For networks with some special structure, e.g., (quasi) spatially invariant systems, one can easily check Assumption 4 with the use of Gelfand’s formula, see Section 7 for more details.

4.1. From infinite to finite networks

The main purpose of dealing with infinite networks is to develop scale-free tools for the analysis and design of finite, but arbitrarily large networks. In this section we truncate the infinite network \( \Sigma \) and keep only the first \( n \) subsystems of the network. Roughly speaking, we show that if conditions required by Theorem 10 hold, then for any truncation of infinite network \( \Sigma \) and accordingly that of \( \Sigma \), the same conclusion as in Theorem 10 is obtained for truncated networks.

Consider the first \( n \in \mathbb{N} \) subsystems of \( \Sigma \) and denote the truncated system by \( \Sigma^{<n} \). The truncated system is a network \( \Sigma^{<n} \) whose dynamics are described by

\[
\begin{aligned}
\Sigma^{<n}_i(x^{<n}_i(k+1)) &= f^{<n}_{\sigma^{<n}}(k)(x^{<n}_i(k), \tilde{x}(k), u^{<n}_i(k)), \\
\Sigma^{<n}_i(x^{<n}_i(k)) &= h^{<n}_{\sigma^{<n}}(k)(x^{<n}_i(k)),
\end{aligned}
\]

(19)

where \( x^{<n}_i(k) = (x_i(k), i \leq n) \) are elements of \( X^{<n} \subseteq \mathbb{R}^N \), \( N := \sum_{i=1}^n n_i \), \( u^{<n}_i(k) = (u_i(k))_{1 \leq i \leq n} \) are elements of \( U^{<n} \subseteq \mathbb{R}^M \), and \( M := \sum_{i=1}^n m_i \). Moreover, we denote by \( I^{<n}_{\sigma^{<n}}(k) = \bigcup_{i=1}^n I^{<n}_{\sigma^{<n}}(k) \subseteq \{1, \ldots, n\} \), the finite set of neighbors of the first \( n \) subsystems. Then, \( \tilde{x}(k) = (\tilde{x}_i(k))_{j \in I^{<n}_{\sigma^{<n}}(k)} \in \mathbb{R}^L \), \( L := \sum_{j \in I^{<n}_{\sigma^{<n}}(k)} n_j \), is considered as the additional input vector. Note that we do not neglect subsystems \( \Sigma_i, i > n \), instead we consider them as additional external inputs \( \tilde{x}(k) \) to the network \( \Sigma^{<n} \). Clearly, the case in which subsystems \( \Sigma_i, i > n \), are entirely removed from the network is covered by our setting by taking \( \tilde{x} \equiv 0 \). We denote the set of input functions of the truncated network as \( U^{<n} \) and the output maps are viewed as \( h^{<n}_{\sigma^{<n}} : X^{<n} \to \mathbb{R}^{<n} \) with \( S^{<n} = \prod_{1 \leq i \leq n} S_i \). Moreover, functions \( f^{<n}_{\sigma^{<n}} : X^{<n} \times \mathbb{R}^L \to \mathbb{R}^{<n} \) are defined accordingly.

In the following, we construct the compositional construction of abstractions for the network \( \Sigma^{<n} \) under the assumption of Theorem 10.

Theorem 12. Consider the truncated networks \( \Sigma^{<n}_i = (X^{<n}_i, X^{<n}_i, U^{<n}_i, Y^{<n}_i, h^{<n}_{\sigma^{<n}}_i, f^{<n}_{\sigma^{<n}}_i, S^{<n}_i) \) and \( \Sigma^{<n} = (X^{<n}, X^{<n}, U^{<n}, Y^{<n}, h^{<n}_{\sigma^{<n}}, f^{<n}_{\sigma^{<n}}, S^{<n}) \). Let \( p, q \in [1, \infty) \) be given. Consider local simulation functions \( V_{i,s_i} : X^{<n}_i \times \tilde{x}_i \to \mathbb{R}_+, s_i \in S_i \) and suppose that Assumptions 7 and 8 hold. Assume that there exists a vector \( \mu = (\mu_i)_{i \in \mathbb{N}} \in \mathbb{R}^\infty \), \( \mu_i \leq \mu \), with some constants \( \mu_i \), \( \mu \) satisfying 14. Then, the family of functions \( V_{<n} : X^{<n} \times \mathbb{R}^L \to \mathbb{R}_+, S^{<n} \in S^{<n} \), where

\[
V_{<n}(x^{<n}, \tilde{x}^{<n}) = \sum_{i=1}^n \mu_i V_{i,s_i}(x_i, \tilde{x}_i),
\]

are simulation functions from \( \Sigma^{<n} \) to \( \Sigma^{<n} \) with \( b = p, \alpha = \mu \), as in 5 and satisfy the following

\[
\begin{aligned}
V_{<n}(f^{<n}_{\sigma^{<n}}(x^{<n}, \tilde{x}, u^{<n}), f^{<n}_{\sigma^{<n}}(x^{<n}, \tilde{x}, u^{<n})) &
\leq -\lambda V_{<n}(x^{<n}, \tilde{x}^{<n}) + \rho_{\text{ext}}(|u^{<n}|_q) + \rho_{\text{ext}}(|\tilde{x}^{<n}|_q),
\end{aligned}
\]

(20)

for all \( s^{<n}, s^{<n} \in S^{<n} \), where \( \lambda = \lambda_{\infty} \) and \( \rho_{\text{ext}} : t \to \mathbb{R}^p \).

Proof. By following similar arguments as in Theorem 10, one can obtain

\[
\sum_{i=1}^n \mu_i V_{i,s_i}(x_i, \tilde{x}_i) \geq \sum_{i=1}^n \mu_i \alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\tilde{x}_i)|^p
\]
Moreover, by letting \( V_{s<n}^<(x^{<n}, \tilde{x}^{<n}) := (V_{s,x}(x_i, \tilde{x}_i))_{1 \leq i \leq n} \) using the chain of inequalities in (15) for all \( s_i, s_{i'} \in S_i, s_j \in S_j, s^{<n'}, s^{<n} \in S^{<n}, 1 \leq i \leq n, \) and (17), we have
\[
V_{s<n}^<(f_{s<n}^<(x^{<n}, \tilde{x}, u^{<n})), f_{s<n}^<(x^{<n}, \tilde{x}, u^{<n})) - V_{s<n}^<(x^{<n}, \tilde{x}^{<n}) \leq \sum_{i=1}^{n} \mu_i \rho_i, \|\hat{u}_i\|^2 \|
\]
where \( \mu_\infty = \sup_{i \leq n} V_{s<n}^<(x, \tilde{x}) + \sum_{i=1}^{n} \mu_i \rho_i, \|\hat{u}_i\|^2 \|
\]
and
\[
\sum_{i=1}^{n} \mu_i \rho_i, \|\hat{u}_i\|^2 \|
\]

As can be seen from Theorem 12, the decay rate \( \lambda_\infty \) as well as the gain function due to external input \( u \) are preserved under truncation. Thus, the indices of the proposed compositional method are independent of the network size.

5. Construction of Abstractions for Linear Systems

In this section, we explicitly construct local abstractions and corresponding simulation functions for linear switched subsystems.

We make the following assumption on the simulation functions, which is an incremental version of a similar assumption used to achieve the input-to-state stability of switched systems under constrained switching conditions.

**Assumption 13.** There exist uniformly bounded constants \( \tau_i \geq 1, i \in \mathbb{N} \), such that for all \( x_i \in \mathcal{X}_i \), all \( \tilde{x}_i \in \hat{\mathcal{X}}_i \), and every \( s_i, s'_i \in S_i \)
\[
V_{s,i}(x_i, \tilde{x}_i) \leq \tau_i V_{s',i}(x_i, \tilde{x}_i).
\]

Consider the following class of linear switched subsystems
\[
\Sigma_i : \begin{cases}
\dot{x}_i(k+1) &= A_{i,s_i(k)}x_i(k) + D_{i,s_i(k)}w_i(k) + B_{i,s_i(k)}u_i(k), \\
\dot{y}_i(k) &= C_{i,s_i(k)}x_i(k),
\end{cases}
\]
where \( s_i \in S_i, A_{i,s_i(k)} \in \mathbb{R}^{n_i \times n_i}, B_{i,s_i(k)} \in \mathbb{R}^{n_i \times m_i}, C_{i,s_i(k)} \in \mathbb{R}^{1 \times n_i} \) and \( D_{i,s_i(k)} \in \mathbb{R}^{1 \times p_i} \), for \( i \in \mathbb{N} \).

Choose \( \mathcal{X} = \ell^2(\mathbb{N}, (n_i)) \) and \( U = \ell^2(\mathbb{N}, (m_i)) \) for the overall infinite network. By slight abuse of notation, we use the tuples \( \Sigma_i = (A_{i,s_i}, B_{i,s_i}, C_{i,s_i}, D_{i,s_i}) \) to refer to switched subsystem with transition and output functions of the form (21) with the specified matrices dimensions.

Assume that there exist a family of matrices \( K_{i,s_i} \), positive definite matrices \( M_{i,s_i} \), real numbers \( \epsilon_i > 0 \), and \( 0 < \kappa_i < 1 \) such that the following matrix inequalities hold for all \( s_i \in S_i, i \in \mathbb{N} \)
\[
C_{i,s_i}^\top C_{i,s_i} \preceq M_{i,s_i},
\]

and
\[
\sum_{i=1}^{n} \mu_i \rho_i, \|\hat{u}_i\|^2 \| (1 + \frac{\epsilon_i}{\kappa_i}) (A_{i,s_i} + B_{i,s_i}, K_{i,s_i})^\top M_{i,s_i} \left(A_{i,s_i} + B_{i,s_i}, K_{i,s_i}\right) \preceq \kappa_i M_{i,s_i}.
\]

**Remark 14.** Given \( \kappa_i \) and \( \epsilon_i \), inequality (22) is not jointly convex on the decision variables \( M_{i,s_i} \) and \( K_{i,s_i} \). Then, this inequality is not amenable to existing semidefinite tools for linear matrix inequalities (LMI). By using the Schur complement lemma, (22b) could be transformed to the following LMI over decision variables \( Q_{i,s_i} \) and \( Z_{i,s_i} \):
\[
\begin{bmatrix}
-\kappa_i Q_{i,s_i} & Q_{i,s_i} A_{i,s_i}^\top + Z_{i,s_i} B_{i,s_i}^\top \\
A_{i,s_i} Q_{i,s_i} + B_{i,s_i} Z_{i,s_i} & (1 + \frac{1}{\epsilon_i}) B_{i,s_i}^\top B_{i,s_i}
\end{bmatrix} \preceq 0, \quad Q_{i,s_i} > 0,
\]
where \( Q_{i,s_i} = M_{i,s_i}^{-1} \) and \( Z_{i,s_i} = K_{i,s_i} Q_{i,s_i} \).

Consider the simulation function candidates \( V_{s,i}(x_i, \tilde{x}_i) = (x_i - P_{i,s_i} \hat{x}_i)^\top M_{i,s_i} (x_i - P_{i,s_i} \hat{x}_i) \).

The control inputs of the concrete subsystems are given by
\[
u_i = \nu_i(x_i, \tilde{x}_i, \hat{u}_i, \tilde{w}_i, s_i)
\]
\[
= K_{i,s_i}(x_i - P_{i,s_i} \hat{x}_i) + Q_{i,s_i} \hat{x}_i + R_{i,s_i} \hat{u}_i + T_{i,s_i} \tilde{w}_i,
\]
where \( P_{i,s_i}, i \in \mathbb{N} \), are some matrices of appropriate dimensions. Assume that the following inequalities hold for some matrices of appropriate dimensions \( Q_{i,s_i}, T_{i,s_i} \):
\[
A_{i,s_i} P_{i,s_i} = P_{i,s_i} A_{i,s_i} - B_{i,s_i} Q_{i,s_i},
\]
\[
D_{i,s_i} = P_{i,s_i} D_{i,s_i} - B_{i,s_i} T_{i,s_i},
\]
\[
C_{i,s_i} P_{i,s_i} = \hat{C}_{i,s_i},
\]
Next theorem shows that functions \( V_{s,i}, s_i \in S_i \), defined in (23), are simulation functions from \( \hat{\Sigma}_i \) to \( \Sigma_i \).

**Theorem 15.** Consider systems \( \Sigma_i = (A_{i,s_i}, B_{i,s_i}, C_{i,s_i}, D_{i,s_i}) \) and \( \hat{\Sigma}_i = (\hat{A}_{i,s_i}, \hat{B}_{i,s_i}, \hat{C}_{i,s_i}, \hat{D}_{i,s_i}) \) for \( i \in \mathbb{N} \). Suppose that for all \( s_i \in S_i \), there exist appropriate matrices \( M_{i,s_i}, P_{i,s_i}, K_{i,s_i}, Q_{i,s_i} \) and \( T_{i,s_i} \) satisfying (22) and (23). Moreover, assume that \( \tau_i \kappa_i < 1 \). Then, functions in (23) are simulation functions from \( \Sigma_i \) to \( \hat{\Sigma}_i \) with concrete inputs given by (24).

Proof. According to (21), we have
\[
|C_{i,s_i} x_i - \hat{C}_{i,s_i} \hat{x}_i| =
\]

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\[(x_i - P_{i,s_i} \hat{C}_{i,s_i})^\top C_{i,s_i} C_{i,s_i} (x_i - P_{i,s_i} \hat{C}_{i,s_i}) \geq 0.\]

Using (24), it is clear that \(|C_{i,s_i} x_i - \hat{C}_{i,s_i} \hat{x}_i|^2 \leq V_{i,s_i}(x_i, \hat{x}_i)\) holds for all \(x_i \in \mathbb{X}_i, \hat{x}_i \in \mathbb{X}_i\). Then, (12) is satisfied with \(\alpha_i = 1, i \in \mathbb{N}, p = 2\).

Now, we proceed to show that (13) is satisfied, too.

Using Assumption (13), one gets the following inequality for all switchings \(s_i, s_i \in S_i\)
\[V_{i,s_i} (f_{i,s_i}(x_i, u_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)) \leq \tau_i V_{i,s_i} (f_{i,s_i}(x_i, u_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)).\] (26)

Using the system dynamics (24) and the candidate simulation function in (23), the inequality (26) can be written as
\[V_{i,s_i} (f_{i,s_i}(x_i, u_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)) \leq \tau_i [A_{i,s_i} x_i + B_{i,s_i} u_i + D_{i,s_i} w_i - P_{i,s_i} (\hat{A}_{i,s_i} \hat{x}_i + \hat{B}_{i,s_i} \hat{u}_i + \hat{D}_{i,s_i} \hat{w}_i)]^\top M_{i,s_i} \times [A_{i,s_i} x_i + B_{i,s_i} u_i + D_{i,s_i} w_i - P_{i,s_i} (\hat{A}_{i,s_i} \hat{x}_i + \hat{B}_{i,s_i} \hat{u}_i + \hat{D}_{i,s_i} \hat{w}_i)].\] (27)

Substituting \(u_i\) from (24) and employing (25a) to (25b) yield
\[V_{i,s_i} (f_{i,s_i}(x_i, u_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)) \leq \tau_i \left[\kappa_i V_{i,s_i} (x_i, \hat{x}_i) + (1 + \frac{1}{\epsilon_i} + \epsilon_i) \sqrt{M_{i,s_i} (B_{i,s_i} R_{i,s_i} - P_{i,s_i} \hat{B}_{i,s_i})} \right] [u_i]^2.\]

Thus, (13) holds with \(p = q = 2, \lambda_i = \kappa_i, \rho_{\text{ext}} = \tau_i (1 + \frac{1}{\epsilon_i} + \epsilon_i) \max \{ \sqrt{M_{i,s_i}(B_{i,s_i} R_{i,s_i} - P_{i,s_i} \hat{B}_{i,s_i})} \}^2 \) and \(\rho_{\text{int}} = \tau_i (1 + \frac{1}{\epsilon_i} + \epsilon_i) \max \{ \sqrt{M_{i,s_i} \hat{D}_{i,s_i}} \}^2 \).

Therefore, the candidate functions in (23) are simulation functions from \(\Sigma_i\) to \(\Sigma_i\), for all \(i \in \mathbb{N}\).

\[\square\]

6. Example

To verify the effectiveness of our results, we apply them to a voltage regulation problem in AC islanded microgrids.

Islanded microgrids are self-sufficient small-scale power grids composed of several Distributed Generation Units (DGUs). They are designed to operate safely and reliably in the absence of connection to the main grid (28). When the microgrids are working in connected mode, voltage and frequency are set by the main grid. However, in the islanded mode, they must be controlled by DGUs. Therefore, their connection should be robust against line faults or variations in the topology of DGUs’ connections. Treating time-varying communication topologies is beneficial to evaluate the system performance in the presence of the line switches or plug-and-play operations.

We consider a switched AC islanded microgrid network modeled by an interconnection of fourth-order DGUs as underlying subsystems. In particular, we consider two circular topologies as shown in Figures 1 and 2, and assume that the network topology switches between these two configurations at certain times. Let \(\sigma_i(k)\) be the switching signal which takes values in the set \{1, 2\}, where \(\sigma_i(k) = 1\) corresponds to the topology shown in Figure 1 and \(\sigma_i(k) = 2\) pertains to that in Figure 2.

The discrete-time dynamics of each DGU in the microgrid with sampling time \(t_s\) is described by (29), adapted from (28). In (29), \(V_{i,d}\) (resp. \(V_{i,q}\)) are the \(d\) (resp. \(q\)) components of the load voltage. Similarly, \(I_{i,d}\) (resp. \(I_{i,q}\)) denote the \(d\) (resp. \(q\)) components of the current of DGU \(\Sigma_i\). In addition, the integrators \(\nu_{i,d}, \nu_{i,q}\) are added for disturbance rejection reasons (28). The control inputs (the voltage of corresponding voltage source converter (VSC)) and outputs are denoted by \(u_i(k) = [V_{i,d}(k), V_{i,q}(k)]^\top\) and \(y_i(k) = [V_{i,d}(k), V_{i,q}(k)]^\top\), respectively. Furthermore, \(D_{i,\sigma_i(k)} w_i(k)\) models the coupling of DGU \(\Sigma_i\) with its neighbors \(\Sigma_j, j \in I_{i,\sigma_i(k)}\), corresponding to each switching mode. In addition, \(H_{i,d}\) represents the collection of load currents \(I_{i,d}\) and \(I_{i,q}\) which are considered as constant exogenous inputs acting as a disturbance and tracking references \(y_{i,\sigma_i}(k) = [y_{i,d}(k), y_{i,q}(k)]^\top\).

The parameters \(R_{ij}, L_{ij}\) are the resistance and inductance, respectively, corresponding to DGU \(\Sigma_i\) and \(\Sigma_j\). \(L_{ij}\) is chosen as the line between DGU \(\Sigma_i\) and DGU \(\Sigma_j\) which are connected through a three-phase line. In addition, \(X_{ij} = \omega_0 L_{ij}\) and \(Z_{ij} = |R_{ij} + jX_{ij}|\) with the rotation...
speed $\omega_0$. Moreover, $k_i$ is the transformer ratio which connects DGU $\Sigma_i$ to the remainder of the network. The other transformer parameters are included in $R_i$ and $L_i$. A shunt capacitance $C_i$ is used for attenuating the impact of high-frequency harmonics of the load voltage.

The interconnection structure switches between two circular topologies shown in Figures 1 and 2. In these topologies, each subsystem $\Sigma_i$ is fed by subsystems $\Sigma_{i-1}$ for $\sigma_i(k) = 1$ ($I_{i,n}^a = \{i-1\}$) and $\Sigma_{i+1}$ for $\sigma_i(k) = 2$ ($I_{n,i}^a = \{i+1\}$), respectively.

We denote $\Sigma_i$ as the augmented infinite network consisting of infinite subsystems $\Sigma_i$. To construct an overall abstraction for $\Sigma_i$, we construct abstractions of subsystems $\Sigma_i$, $i \in \mathbb{N}$, with dimensions $\hat{n}_i$ for both $s_i = 1, 2$. Necessary and sufficient conditions on the geometrical properties of the involved matrices $P_i, D_i, A_i, B_i, L_i$ are provided in [10, Sec. 4.3], which determine the largest possible state dimension for $\hat{\Sigma}_i$, $i \in \mathbb{N}$, as $\hat{n}_i = 3$.

Now we compute the abstraction matrices satisfying (25). Considering (25a) and (25b) and taking $\hat{D}_{i,s}$ as 

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and $T_{i,s} = 0$, we get

\[
P_{i,s} = \frac{1}{2C_i} \sum_{j \in I_{i,s}} \begin{bmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}^T, \]

\[
Q_{i,s} = \frac{t_i k_i}{2C_i} \sum_{j \in I_{i,s}} \begin{bmatrix}
0 & -\frac{X_j}{Z_{ij}} & \frac{X_j}{Z_{ij}} & \frac{Z_{ij}}{X_j} & \frac{Z_{ij}}{X_j} \\
0 & -\frac{X_j}{Z_{ij}} & \frac{X_j}{Z_{ij}} & \frac{Z_{ij}}{X_j} & \frac{Z_{ij}}{X_j}
\end{bmatrix}, s_i = 1, 2.
\]

Furthermore, $\hat{A}_{i,s}$ is obtained by solving $\hat{n}_i \times \hat{n}_i$ equations provided that matrix $\sum_{j \in I_{i,s}} \begin{bmatrix}
\frac{R_j}{Z_{ij}} & -\frac{X_j}{Z_{ij}} \\
\frac{X_j}{Z_{ij}} & \frac{Z_{ij}}{X_j}
\end{bmatrix}$ is invertible.

In addition, $\hat{C}_{i,s} = \frac{1}{2C_i} \sum_{j \in I_{i,s}} \begin{bmatrix}
0 & \frac{X_j}{Z_{ij}} & -\frac{R_j}{Z_{ij}} \\
0 & \frac{X_j}{Z_{ij}} & -\frac{R_j}{Z_{ij}}
\end{bmatrix}$.

By considering the computed matrices $\hat{A}_{i,s}$ and taking $\hat{B}_{i,s} = I_{\hat{n}_i}$, we choose appropriate matrices $\hat{K}_{i,s}$ for local controllers $\hat{u}_i = -\hat{K}_{i,s} \hat{x}_i$, which stabilize abstract subsystems $\hat{\Sigma}_i$ at the origin.

We also choose $R_{i,s} = (B_{i,s}^T L_i R_{i,s})^{-1} B_{i,s}^T L_i P_i B_{i,s}$ to minimize $\rho_{i,ext}$ as suggested in [4].

We illustrate the scale-free property of our approach with respect to the size of network via simulations. Following Theorem [12] we consider three truncated networks of microgrids shown in Figures 4 and 5 respectively, consisting of $10^2$, $10^3$ and $10^4$ subsystems. The parameters are set as $R_{i,s} = 1.5\, \Omega$, $L_{i,s} = 300\, \mu\text{H}$, $C_{i,s} = 460\, \mu\text{F}$, $k_i = 1$ for all subsystems $\Sigma_i$. Additionally, we choose $R_{ij} = 1\, \text{m}\Omega$, $L_{ij} = 10\, \text{m}\Omega$ for all subsystems $\Sigma_i$, $\Sigma_j$ with $s_i = 1$ and $R_{ij} = 1.2\, \text{m}\Omega$, $L_{ij} = 8\, \text{mH}$ for all subsystems $\Sigma_i$, $\Sigma_j$ with $s_i = 2$. The microgrids frequency and the sampling time are set as $f_0 = 60\, \text{Hz}$ and $t_s = 10^{-4}\, \text{s}$, respectively. The switchings between $s_i = 1$ and $s_i = 2$ occur at time steps $k = 4n$, $n \in \mathbb{N}$. We choose $\kappa_i = 0.01$ and take matrices $K_{i,s}$ such that the eigenvalues of pairs $(A_{i,s}, B_{i,s})$ in closed loop are $[0.3; 0.15; 0.6; 0.2; 0.4; 0.5]$.
\[
\begin{bmatrix}
V_{i,d}(k + 1) \\
V_{i,q}(k + 1) \\
I_{i,d}(k + 1) \\
I_{i,q}(k + 1) \\
\nu_{i,d}(k + 1) \\
\nu_{i,q}(k + 1)
\end{bmatrix}
= \mathbf{x}_i(k + 1)
\]
\[
\begin{bmatrix}
\dot{V}_{i,d}(k) \\
\dot{V}_{i,q}(k) \\
\dot{I}_{i,d}(k) \\
\dot{I}_{i,q}(k) \\
\dot{\nu}_{i,d}(k) \\
\dot{\nu}_{i,q}(k)
\end{bmatrix}
= \mathbf{A}_{i,s}(k)
\]
\[
\begin{bmatrix}
\dot{V}_{i,d}(k) \\
\dot{V}_{i,q}(k) \\
\dot{I}_{i,d}(k) \\
\dot{I}_{i,q}(k) \\
\dot{\nu}_{i,d}(k) \\
\dot{\nu}_{i,q}(k)
\end{bmatrix}
= \begin{bmatrix}
-\frac{t_k}{C_{ii}} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{t_k}{L_{ii}} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{t_k}{L_{ii}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{t_k}{L_{ii}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{t_k}{L_{ii}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{t_k}{L_{ii}}
\end{bmatrix}
\begin{bmatrix}
V_{i,d}(k) \\
V_{i,q}(k) \\
I_{i,d}(k) \\
I_{i,q}(k) \\
\nu_{i,d}(k) \\
\nu_{i,q}(k)
\end{bmatrix}
\]
\[
\mathbf{u}_i(k)
\]

Figure 1: The interconnected system \( \Sigma \) for \( s_i = 1 \).

Figure 2: The interconnected system \( \Sigma \) for \( s_i = 2 \).

For both \( s_i = 1, 2 \). Then, we compute \( M_{i,s_i} = \).

With the choice of (28) for \( V_{i,s_i} \), we get \( \tau_i \leq \max\{\lambda_{\text{max}}(M_{i,s_i})\} \) for \( s_i = 1, 2 \). Thus, \( 1 \leq \tau_i \leq 67.61 \).

Therefore, the parameters in Definition 2 satisfying Assumption 2 are as \( \alpha_i = 1, \lambda_i = \kappa_i \in [0.3239, 0.99], \epsilon_i = 1, \rho_i, \text{int} \leq 0.321, \text{ and } \rho_i, \text{ext} \leq 512.312 \). Recalling the circular interconnection topologies, each subsystem is directly fed
by one other subsystem at each time instant. Thus, (13) gives $\gamma_{ij} = \tau_i(1 + \epsilon_i + \frac{1}{\xi_i}) \max\left\{ \sqrt{M_{ij}} \right\}$ for $j \in I_{s,i}$, and $\gamma_{ij} = 0$ for $j \notin I_{s,i}$. Then, we get

$$r(\Psi) < \sup_{j \in \mathbb{C}} \sum_{i=1}^{\infty} \psi_{ij} < (1 + \epsilon_i + \frac{1}{\xi_i}) \max\left\{ \sqrt{M_{ij}} \right\} \frac{\tau_i}{\xi_i} \leq 0.991,$$

which implies the satisfaction of Assumption 10 on the spectral radius condition. Therefore, all the hypotheses of Theorem 10 are satisfied.

The norm of the overall error between the output trajectories of the abstract and concrete systems for three different sizes of networks are shown in Figures 5. From the choice of $\hat{u}$ and stabilizability of $\Sigma$ at the origin, $\lim_{k \to \infty} |\hat{u}(k)|_2 \to 0$. This together with (10) implies that the mismatch between output trajectories converges to zero, illustrated by Figure 3. The reference signals of DGUs are set as $y_{i,ref} = [0, 0.2]^T$, $i \in \mathbb{N}$. The closed-loop output trajectories of the concrete subsystems in a set-point tracking scenario are depicted by Figure 4 in per unit system. From Figure 3 one can see that the overall behavior of the network remains almost identical, though the network size grows dramatically. This admits that performances indices are independent of the network size.

7. Conclusions

We proposed a compositional approach on the construction of continuous abstractions for infinite networks of switched discrete-time systems with arbitrary switching signals. To do this, we extended the notion of simulation functions to infinite-dimensional systems (networks of infinitely many finite-dimensional switched systems). Following the compositionality approach, we assigned to each subsystem an individual simulation function and constructed its local abstraction accordingly. Finally, we composed local abstractions to provide an abstraction of the overall network. We showed that the aggregation yields a continuous abstraction of the overall concrete network if a small-gain condition, expressed in terms of a spectral radius criterion, is satisfied. We also established that our result leads to scale-free compositional method for any finite-but-arbitrarily large networks. For linear systems, our conditions for constructing local abstractions boil down to some linear matrix inequality conditions which can be computed efficiently. We applied our results to AC islanded microgrids under switched topologies and showed the scale-freeness of our proposed approach.

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Figure 3: The error norm between the output trajectories of \( \Sigma \) and \( \hat{\Sigma} \) in per unit system.

Figure 4: The external outputs \( V_{i,d}, V_{i,q} \) in per unit system.

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