MOVING FRAMES ON THE TWISTOR SPACE OF 
SELF-DUAL POSITIVE EINSTEIN 4-MANIFOLDS

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ABSTRACT. The twistor space $Z$ of self-dual positive Einstein manifolds naturally
admits two 1-parameter families of Riemannian metrics, one is the family of canonical
defformation metrics and the other is the family introduced by B. Chow and D. Yang
in [C-Y]. The purpose of this paper is to compare these two families. In particular we
compare the Ricci tensor and the behavior under the Ricci flow of these families. As
an application, we propose a new proof to the fact that a locally irreducible self-dual
positive Einstein 4-manifold is isometric to either $S^4$ with a standard metric or $\mathbb{P}^2(\mathbb{C})$
with a Fubini-Study metric.

0. Introduction

Let $\mathbb{R}^4$ be the oriented Euclidean 4-space which is identified with $\mathbb{H}$ and we consider
the right multiplication of the group $\text{Sp}(1)$ of unit quaternions. This defines a
subgroup $\text{Sp}(1)_-$ in $\text{SO}(4)$ which is isomorphic to $\text{Sp}(1)$. The centralizer of this
subgroup in $\text{SO}(4)$ is again isomorphic to $\text{Sp}(1)$ identified with the left multiplication
of unit quaternions, which we denote by $\text{Sp}(1)_+$. We thus have the group
homomorphism $\text{Sp}(1) \times \text{Sp}(1) \to \text{SO}(4)$ defined by the left/right multiplication of
unit quaternions on $\mathbb{H}$. Its kernel is $\pm(\text{id}, \text{id}) \cong \mathbb{Z}_2$ and therefore we have the de-
composition of the Lie algebra $\text{so}(4) = \text{sp}(1)_+ \oplus \text{sp}(1)_-$. Here, $\text{sp}(1)_\pm$ are copies of
$\text{sp}(1)$ and $\text{sp}(1)_+$ (resp. $\text{sp}(1)_-$) corresponds to the left (resp. right) multiplication
of unit quaternions. In terms of the identification $\text{so}(4) = \Lambda^2 \mathbb{R}^4$, the component
$\text{sp}(1)_+$ (resp. $\text{sp}(1)_-$) corresponds to the self-dual (resp. anti-self-dual) 2-forms.

A Riemannian 4-manifold $(M, g_M)$ is said to be self-dual positive Einstein if its
self-dual part $W_-$ of the Weyl tensor vanishes and Einstein condition is satisfied
with positive scalar curvature$^1$. The twistor space of a self-dual Riemannian 4-
manifold $(M, g)$ is defined as follows. Let $\mathcal{P}$ be the holonomy reduction of the
bundle of oriented orthonormal frames of $(M, g)$. For instance, if $M = (S^4, g_{\text{std}})$, $\mathcal{P}$
is a principal $\text{SO}(4)$-bundle over $S^4$ and if $M = (\mathbb{P}^2, g_{\text{FS}})$, $\mathcal{P}$ is a principal $\text{SU}(1) \times \text{U}(2)$-bundle. The holonomy group of a self-dual positive Einstein manifold $(M, g)$
is a subgroup of $\text{SO}(4)$ and we write it as $H$. The fiber of $\mathcal{P}$ over $m \in M$ consists

$^1$ The twistor space of a self-dual positive Einstein 4-manifold is a positive Kähler-Einstein
manifold and hence simply connected. This implies that any self-dual positive Einstein 4-manifold
is simply connected.
of the $H$-rotation of an oriented orthonormal frame of the tangent space $M_m$. The twistor space $\mathcal{Z}$ of $(M, g)$ is defined as

$$\mathcal{Z} = \mathcal{P}/H \cap U(2)$$

and the twistor fibration is the $\mathbb{P}^1$-bundle

$$\pi : \mathcal{Z} \to M.$$ 

Note that the subgroup of $H$ which induces a holomorphic transformation w.r.t. to the orthogonal complex structure represented by a point of $\mathbb{P}^1_m$ is identified with $H \cap U(2)$ and therefore $\mathbb{P}^1_m$ is identified with $H/H \cap U(2) = SO(4)/U(2) = U(2)$.

As $H \cap U(2)$ is precisely the subgroup of $H$ consisting of all rotations in $H$ which are holomorphic w.r.t. an orthogonal complex structure of $M_m$ (say, that induced from the right $j$-multiplication), the $\mathbb{P}^1$-fiber of the twistor fibration is identified with the set of all orthogonal complex structures of $M_m$. The meaning $U(2)$ is explained as follows. Take $j$ as an orthogonal complex structure. The subgroup of $SO(4)$ which centralizes the right multiplication of $j$ is the product (in $SO(4)$) of $Sp(1)_+$ and the subgroup of $Sp(1)_-$ which centralizes $j$, i.e., the $S^1$-subgroup of $Sp(1)_-$ consisting of $x + yj$, $x^2 + y^2 = 1$. The product group in $SO(4)$ is isomorphic to $U(2)$. The twistor space $\mathcal{Z}$ of a self-dual positive Einstein 4-manifold $(M, g_M)$ is naturally equipped with two kinds of 1-parameter families of Riemannian metrics in the following way:

1. The canonical deformation metrics (see, for instance, [Be]). This is defined as follows. The Levi-Civita connection of $(M, g_M)$ canonically defines a horizontal distribution on the twistor fibration $\pi : \mathcal{Z} \to M$. This allows one to define

$$g_{\lambda}^{\text{can}} := \lambda^2 g_{FS} + g_M$$

where $g_{FS}$ is Fubini-Study metric of constant curvature 4 along the $\mathbb{P}^1$-fiber of the twistor fibration, $g_M$ is the base self-dual positive Einstein metric on $M$ normalized so that its scalar curvature is 48 and $\lambda > 0$ is a partial scaling parameter. If $T_z\mathcal{Z} = V \oplus H$ is the decomposition into the vertical and horizontal subspaces, then we have $g_{\lambda}(V, H) = 0$, $g_{\lambda}^{\text{can}}|_V = \lambda^2 g_1|_V$ and $g_{\lambda}^{\text{can}}|_H = g_1^{\text{can}}|_H$.

2. The Chow-Yang metrics ([C-Y]). This is very much involved compared to the canonical deformation metrics. This family was introduced by B. Chow and D. Yang in [C-Y] for the twistor space of positive quaternion Kähler manifolds. Roughly speaking the Chow-Yang metrics are defined as follows. Let $\mathbb{P}^1_m$ be the $\mathbb{P}^1$-fiber of the twistor fibration $\pi : \mathcal{Z} \to M$ over $m \in M$. Along $\mathbb{P}^1_m$ we locally associate a smooth family of “rotating” oriented orthonormal frames where the rotation is chosen from $H$ modulo $H \cap U(2)$. These local associations define local sections

\[2\] In this paper we normalize the Fubini-Study metric on $\mathbb{P}^1$ so that its curvature is 4 and the self-dual positive Einstein metric on $M^4$ so that its scalar curvature is 48.
s : \mathcal{Z} \rightarrow \mathcal{P} of the principal \( H \cap U(2) \)-bundle \( p : \mathcal{P} \rightarrow \mathcal{Z}. \) Now the Chow-Yang metric is defined by requiring the dual orthonormal coframes of \( M \) together with the image in the twistor space of the \( \text{sp}(1) \)-part of the connection form (of the Levi-Civita connection of \( (M, g) \)) determined by the “rotation” of the oriented orthonormal frames along the fiber of \( \mathcal{P} \) to be the system of orthonormal coframe on \( \mathcal{Z} \). Here, the contribution from the \( \text{sp}(1) \)-part of the connection form corresponds to the tangent space of the point \( z \) of the fiber \( \mathbb{P}^1_m \) in \( \mathcal{Z} \) which represents the infinitesimal deformation of the orthogonal complex structure at \( z \in \mathbb{P}^1_m \). For instance, as an orthogonal complex structure is represented by \( a_i + b_j + c_k \) with \( a^2 + b^2 + c^2 = 1 \), if the orthogonal complex structure represented by \( z \) is the right \( j \) multiplication (\( j \) is defined by \( a = c = 0 \) and \( b = 1 \)), then the infinitesimal deformation of the orthogonal complex structure is represented by \( \alpha_1 i + \alpha_3 k \). Therefore the contribution from the \( \text{sp}(1) \)-part of the connection form is \( \alpha_1 \) and \( \alpha_3 \) in this case. If we introduce a partial scaling parameter \( \lambda > 0 \) in the connection form part, Chow-Yang metric looks like

\[
g^\text{CY}_\lambda = \lambda^2 (\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^{3} X_i^2
\]

where the \( X_i \)-part represents the “rotating” (here, “rotation” is considered modulo \( H \cap U(2) \)) oriented orthonormal frame along the \( \mathbb{P}^1 \)-fiber, the pair \( \{\alpha_1, \alpha_3\} \) (say) represents the connection form part representing the “rotation modulo (orthogonal to) \( H \cap U(2) \)-rotation” of the \( X_i \)-part in the \( \mathbb{P}^1 \)-fiber and \( \lambda > 0 \) is a partial scaling parameter.

In this paper, we are concerned with the comparison of the above introduced two 1-parameter families of Riemannian metrics on the twistor space \( \mathcal{Z} \) of self-dual positive Einstein 4-manifolds \( (M, g) \). In Section 1, we compare these two families from topological viewpoint and produce a semi-global Riemannian invariant which distinguishes these two families. Then we compute the Ricci tensor of metrics in these two families. The computation shows that the Ricci tensor of the Chow-Yang (resp. the canonical deformation) metrics is again the Chow-Yang (resp. the canonical deformation) type. In Section 2, we compare these two families from the Ricci flow viewpoint.

A convention. We always normalize the scaling of the self-dual positive Einstein 4-manifold under consideration so that its scalar curvature is 48 (if \( M = S^4 \) with the standard metric \( g_{\text{std}} \), then its sectional curvature is identically 4, and if \( M = \mathbb{P}^2(\mathbb{C}) \) with the Fubini-Study metric, then its holomorphic sectional curvature is identically 8) and normalize the Fubini-Study metric of \( \mathbb{P}^1 \) so that its Gaussian curvature is 4.

Then our main results are concerned with compact self-dual Einstein 4-manifolds with positive scalar curvature:

**Theorem 0.1.** A canonical deformation metric and a Chow-Yang metric on the twistor space \( \mathcal{Z} \) coincide if and only if the partial scaling parameter \( \lambda \) satisfies the condition \( \lambda = 1 \). In this case \( g^\text{can}_1 = g^\text{CY}_1 \) is a Kähler-Einstein metric on \( \mathcal{Z} \).

**Theorem 0.2.** (1) If \( \lambda^2 < 3 \), the Ricci tensor of the canonical deformation metric \( g^\text{can}_\lambda \) is positive and given by the formula

\[
\text{Ric}(g^\text{can}_\lambda) = 4(1 + \lambda^4)g_{FS} + 4(3 - \lambda^2)g_M = 4(3 - \lambda^2)g^\text{can}_\lambda \sqrt{\frac{1 + \lambda^4}{3 - \lambda^2}}.
\]
In particular $g^\text{can}_\lambda$ is Kähler-Einstein if and only if $\lambda^2 = 1$ and Einstein non-Kähler if and only if $\lambda^2 = \frac{1}{2}$.

(2) The Ricci tensor of the Chow-Yang metric $g^\text{CY}_\lambda$ is given by
\[
\text{Ric}(g^\text{CY}_\lambda) = 2\lambda^{-2}(1 + 3\lambda^2) \frac{g^\text{CY}_\lambda}{\sqrt{1 + 3\lambda^2}}.
\]
In particular, $g^\text{CY}_\lambda$ is Einstein if and only if $\lambda^2 = 1$ and in this case $g^\text{CY}_1$ is Kähler-Einstein.

**Theorem 0.3.** (1) Consider a 2-parameter family
\[
\mathcal{F}^\text{can} = \{\rho g^\text{can}_\lambda\}_{\rho > 0, \sqrt{3} > \lambda > 0}
\]
of canonical deformation metrics. The explicit form of the Ricci flow with initial metric in $\mathcal{F}^\text{can}$ is given by
\[
\begin{cases}
\frac{d\lambda^2}{dt} = -\frac{8}{\rho}(\lambda^2 - 1)(2\lambda^2 - 1) \\
\frac{d\rho}{dt} = -8(3 - \lambda^2).
\end{cases}
\]
The trajectory is given by the equation
\[
\rho = (\text{const.}) \frac{|\lambda^2 - 1|^2}{|2\lambda^2 - 1|^2}.
\]

(2) Consider a 2-parameter family
\[
\mathcal{F}^\text{CY} = \{\rho g^\text{CY}_\lambda\}_{\rho > 0, \lambda > 0}
\]
of the Chow-Yang metrics. The explicit form of the Ricci flow with initial metric in $\mathcal{F}^\text{CY}$ is given by
\[
\begin{cases}
\frac{d}{dt}(\rho(t)\lambda(t)^2) = -8(1 + \lambda^2(t)) \\
\frac{d}{dt}\rho(t) = -4(\lambda(t)^{-2} + 3).
\end{cases}
\]
The trajectory is given by the equation
\[
\rho = (\text{const.}) \frac{\lambda^2}{|1 - \lambda^2|^4}.
\]
The family $\mathcal{F}^\text{CY}$ is interpreted as a “Ricci flow unstable cell” centered at a Kähler-Einstein metric in the sense that each solution of this family is an ancient solution with the Kähler-Einstein metric as its asymptotic soliton.

The motivation of Theorem 0.3 is the gradient flow interpretation of the Ricci flow proposed by Perelman [P]. Here, a Ricci flow solution is called an ancient solution if the existence time is $(-\infty, T)$ ($T$ being finite) (see [Ha] and [P]). In the trajectory of the Ricci flow on the family $\mathcal{F}^\text{CY}$ (identified with the $(\lambda^2, \rho)$-plane), if $\lambda^2 > 1$ increases, then $\rho$ decreases. Moreover, in the Ricci flow in the family $\mathcal{F}^\text{CY}$ the scaling parameter $\rho(t)$ decreases along the flow. Therefore the partial scaling parameter $\lambda^2(t)$ (if $\lambda^2 > 1$) in $\mathcal{F}^\text{CY}$ increases along the flow. On the other hand, if $\lambda^2 > 1$, $\lambda^2$ decreases along the Ricci flow in the family $\mathcal{F}^\text{can}$.
**Theorem 0.4.** A locally irreducible self-dual positive Einstein 4-manifold is isometric to either \((S^4, g_{\text{std}})\) or \((\mathbb{P}^2(\mathbb{C}), g_{\text{FS}})\).

Here \((S^4, g_{\text{std}})\) is \(S^4\) with the standard metric of constant sectional curvature 4 (scalar curvature 48) and \((\mathbb{P}^2(\mathbb{C}), g_{\text{FS}})\) is \(\mathbb{P}^2(\mathbb{C})\) with the Fubini-Study metric of constant holomorphic sectional curvature 8 (scalar curvature 48).

Theorem 0.4 is a classical result in 4-dimensional geometry (see Hitchin [Hi], Friedrich-Kurke [F-K] and [S2]). A new proof proposed in this paper is based on the fact that the set of the family of the Chow-Yang metrics for \(\lambda > 1\) is foliated by the trajectories of the Ricci flow ancient solutions whose asymptotic soliton is the trajectory of the shrinking homothetical family of the Kähler-Einstein metrics on \(\mathcal{Z}\). The advantage of our proof lies in the fact that we can apply the same strategy to the positive quaternion Kähler case of all dimensions \(\geq 8\) in a uniform way. In [K-O], we will extend our methods to positive quaternion Kähler case and construct a “Ricci flow unstable cell” centered at the Kähler-Einstein metric on the twistor space of positive quaternion Kähler manifolds and apply this Ricci flow unstable cell to the study of the structure of positive quaternion Kähler manifolds. In particular we will show that any locally irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces (this is an affirmative answer to the LeBrun-Salamon conjecture ([Leb], [L-S])).

1. **Moving Frames on the Twistor Space**

Let \((M, g)\) be a compact self-dual positive Einstein 4-manifold. As introduced in the introduction, let \(\mathcal{P}\) be the holonomy reduction of the bundle of oriented orthonormal frames of \((M, g)\). Then the holonomy group \(H(\subset \text{SO}(4))\) acts from the right by rotation of a given oriented orthonormal frame. We define the twistor space \(\mathcal{Z}\) of \((M, g)\) by

\[
\mathcal{Z} = \mathcal{P}/H \cap \text{U}(2)
\]

and write \(\pi : \mathcal{Z} \to M\) for the twistor fibration. The fibration \(\mathcal{P} \to \mathcal{Z}\) is a principal \(H \cap \text{U}(2)\)-bundle and each fiber over a point of \(M\) is a copy of \(H/H \cap \text{U}(2) = \text{SO}(4)/\text{U}(2) = \text{Sp}(1)_-/\text{Sp}(1)_- \cap \text{U}(2) = \mathbb{P}^1\), i.e., the fibration \(\mathcal{P} \to \mathcal{Z}\) restricted on the \(\mathbb{P}^1\)-fiber of the twistor fibration is identified with the upper or the lower row of the diagram

\[
\begin{align*}
U(2) & \longrightarrow \text{SO}(4) \longrightarrow \mathbb{P}^1 \\
\downarrow & \downarrow \downarrow \text{id} \\
\text{Sp}(1)_- \cap H & \longrightarrow \text{Sp}(1)_- \longrightarrow \mathbb{P}^1
\end{align*}
\]

where the lower row is just the Hopf fibration \(S^1 \to S^3 \to \mathbb{P}^1\). The twistor space \(\mathcal{Z}\) fibers over \(M\) with fiber \(\mathbb{P}^1\) and the fiber \(\mathbb{P}^1_m\) over \(m \in M\) is identified with the set of all orthogonal complex structures on the tangent spaces \(M_m\). The Levi-Civita connection of \((M, g)\) canonically defines a horizontal distribution on the twistor fibration. This allows us to define a 1-parameter family of Riemannian metrics \(g_{\text{can}}^\lambda\) which is the so called family of canonical deformation metrics (the sum \(g_{\text{can}}^\lambda = \lambda^2 g_{\text{FS}} + g_M\) of the base metric \(g_M\) on \(M\) and the scaled fiber Fubini-Study metric \(\lambda^2 g_{\text{FS}}\)).
In [C-Y], B. Chow and D. Yang introduced another 1-parameter family of metrics (called the family of Chow-Yang metrics) $g^{CY}_\lambda$ on the twistor space $\mathcal{Z}$ of positive quaternion Kähler manifolds by using the moving frame technique.

We proceed to the description of the Chow-Yang metrics. First we consider topologically. Let $(M, g)$ be a self-dual positive Einstein 4-manifold, $\mathcal{Z}$ its twistor space and $\pi : \mathcal{Z} \to M$ the twistor fibration. Let $m \in M$, $U$ an open neighborhood of $m \in M$ and consider a local oriented orthonormal frame on $U$. We consider any local frame field on $U$ which defines a section $U \to \mathcal{P}|_U$ of the $H$-principal bundle $\mathcal{P} \to M$ (the holonomy reduction of the bundle of oriented orthonormal frames). We extend the section in the direction of the fiber $\mathcal{P}_m$ over $m$ of $\mathcal{P} \to M$ by rotating the given oriented orthonormal frame at $m$ by elements of $H$. This defines a “local” section $s : \mathcal{Z}|_U \to \mathcal{P}|_U$ of the principal $H \cap U(2)$-bundle $\mathcal{P} \to \mathcal{Z}$, where “local” means that the section is defined locally along the $\mathbb{P}^1$-fiber of the twistor fibration. This section corresponds to the “rotation” of the given oriented orthonormal frame by elements of $H$ modulo $U(2)$. Because there exists no global section of the $S^1$-bundle $H \to H/H\cap U(2) \cong \mathbb{P}^1$, the section is defined only locally along the $\mathbb{P}^1$-fiber of the twistor fibration. Thus, given local oriented orthonormal frame field on $U$, we defines a “local” section $s : \mathcal{Z}|_U \to \mathcal{P}|_U$ and therefore we get the system of 1-forms on $\mathcal{Z}$

$$\{X^0, X^1, X^2, X^3\}$$

which is the pull-back of the unitary coframes defined globally on $\mathcal{P}$ and the system of 1-forms on $\mathcal{Z}$

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

which is considered modulo $H \cap U(2)$-action and is the pull-back of the Levi-Civita connection form of $(M, g)$. For instance, if $z \in \mathbb{P}_m^1$ (the $\mathbb{P}^1$-fiber of the twistor fibration over $m \in M$) corresponds to the orthogonal complex structure $j$ represented by $(0, 1, 0)$, the infinitesimal deformation of the orthogonal complex structure at $j$ is given by $\alpha_1 i + \alpha_3 k$. The Chow-Yang metrics are defined by

$$g^{CY}_\lambda = \lambda^2 (\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^{3} X_i^2$$

by declaring that the system

$$\{\lambda \alpha_1, \lambda \alpha_3, X_0, X_1, X_2, X_3\}$$

being an orthonormal coframe ($\lambda > 0$ being a partial scaling parameter). We are now ready to compare the family of Chow-Yang metrics and canonical deformation metrics. To state the result, we normalize the base space $(M, g)$ so that the canonical deformation metric for $\lambda = 1$, i.e., $g^{\text{can}}_1 = g_{FS} + g_M$ is Kähler-Einstein.

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3 This is the unique way to extend the local orthonormal frame field on $U(\subset M)$ to the inverse image of $U$ under the twistor fibration $\pi : \mathcal{Z} \to M$ in the way compatible with the geometric meaning (i.e., the rotation by the left $H$-action) of $\mathcal{P}$.
Yang proved that mentioned “rotation” along the orthogonal complex structure of the twistor space and therefore the above manifolds. We will prove that means that a canonical deformation metric a self-dual positive Einstein manifold below by moving frame computation. This Einstein. In this case the parallel translation along curves in the the canonical deformation metric object for the Chow-Yang metric and this set has different topological structures for two metrics coincide if and only if \( \lambda = \lambda' = 1 \). In particular, the Chow-Yang metric \( g_{\lambda'} \) for \( \lambda' \neq 1 \) do not belong to the family of canonical deformation metrics.

Proof. It is well-known that the canonical deformation metric is Kähler-Einstein for a unique suitable partial scaling. Our normalization is that \( g_1^{\text{can}} \) is Kähler-Einstein. In this case the parallel translation along curves in the \( \mathbb{P}^1 \)-fiber preserves the orthogonal complex structure of the twistor space and therefore the above mentioned “rotation” along the \( \mathbb{P}^1 \)-fiber must belong to \( U(2) \). In [C-Y], Chow and Yang proved that \( g_1^{\text{CY}} \) is Kähler-Einstein in the case of positive quaternion Kähler manifolds. We will prove that \( g_1^{\text{CY}} \) is Kähler-Einstein in the case that \( (M,g) \) is a self-dual positive Einstein manifold below by moving frame computation. This means that a canonical deformation metric \( g_{\lambda}^{\text{can}} \) and a Chow-Yang metric \( g_{\lambda'}^{\text{CY}} \) coincide if \( \lambda = \lambda' = 1 \).

Suppose next that \( \lambda, \lambda' \neq 1 \). For \( g_{\lambda}^{\text{can}} \), there exists an oriented orthonormal frame field for the horizontal subspaces defined globally along a \( \mathbb{P}^1 \)-fiber of the twistor fibration (namely, the constant horizontal frame along the \( \mathbb{P}^1 \)-fiber). We show that such a global object does not exist for the Chow-Yang metric \( g_{\lambda'}^{\text{CY}} \) for \( \lambda' \neq 1 \). To see this we fix a \( \mathbb{P}^1 \)-fiber of the twistor fibration. We note that for any value of \( \lambda' > 0 \), any \( \mathbb{P}^1 \)-fiber is totally geodesic w.r.t. the metric \( g_{\lambda'}^{\text{CY}} \). Therefore the parallel translation along curves in the \( \mathbb{P}^1 \)-fiber preserves tangent spaces of the \( \mathbb{P}^1 \)-fiber and therefore preserves the horizontal subspaces. On the other hand, as was shown in [C-Y] by Chow and Yang for positive quaternion Kähler manifolds and will be shown below by moving frame computation for self-dual positive Einstein 4-manifolds, the Chow-Yang metric \( g_{\lambda'}^{\text{CY}} \) is never Kähler if \( \lambda' \neq 1 \). Therefore the holonomy along closed curves in the \( \mathbb{P}^1 \)-fiber is not contained in \( U(2) \). Therefore the set of all parallel translations along curves in the \( \mathbb{P}^1 \)-fiber of a given oriented horizontal orthonormal frame must be identical to \( H/H \cap U(2) \cong \mathbb{P}^1 \). This implies that there exists no smooth horizontal oriented orthonormal frame field defined globally along the \( \mathbb{P}^1 \)-fiber. Indeed, the existence of such a global object would correspond to a global section of the principal \( H \cap U(2) \)-bundle \( H \rightarrow H/H \cap U(2) \cong \mathbb{P}^1 \) which is either a copy of the Hopf fibration \( S^1 \rightarrow S^3 \rightarrow S^2 \) or \( U(2) \rightarrow SO(4) \rightarrow S^2 \). However, such a global section does not exist. We have thus proved that \( g_{\lambda'}^{\text{CY}} \) is never a canonical deformation metric, because the set of all parallel translations along curves in the \( \mathbb{P}^1 \)-fiber of a given horizontal oriented orthonormal frame is a Riemannian invariant and this set has different topological structures for two metrics \( g_{\lambda}^{\text{can}} \) and \( g_{\lambda'}^{\text{CY}} \) (here, \( \lambda, \lambda' \neq 1 \)). Indeed, the set has a global horizontal section along the \( \mathbb{P}^1 \)-fiber for the canonical deformation metric \( g_{\lambda}^{\text{can}} \), while this set does not admit such a global object for the Chow-Yang metric \( g_{\lambda'}^{\text{CY}} \).

Next we proceed to the description of the Chow-Yang metrics and compute its Ricci tensor. The right multiplication of \( i \) and \( j \) on \( \mathbb{H} \) is given by the matrices:

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]
Therefore the Lie algebra of $\text{Sp}(1)_+$ (the $\text{Sp}(1)$ subgroup of $\text{SO}(4)$ stemming from the right multiplication of unit quaternions) is given by

\[
\begin{pmatrix}
0 & -a_1 & -a_2 & -a_3 \\
-a_1 & 0 & a_3 & -a_2 \\
-a_2 & a_3 & 0 & a_1 \\
-a_3 & a_2 & -a_1 & 0
\end{pmatrix}.
\]

Similarly, the Lie algebra of $\text{Sp}(1)_+$ (the centralizer of the right multiplication of unit quaternions) is computed as

\[
\begin{pmatrix}
0 & -A_1 & -A_2 & -A_3 \\
A_1 & 0 & -A_3 & A_2 \\
A_2 & A_3 & 0 & -A_1 \\
A_3 & -A_2 & A_1 & 0
\end{pmatrix}.
\]

Therefore the Lie algebra of $\text{SO}(4)$ is expressed as

\[
\begin{pmatrix}
0 & -A_1 - a_1 & -A_2 - a_2 & -A_3 - a_3 \\
A_1 + a_1 & 0 & -A_3 + a_3 & A_2 - a_2 \\
A_2 + a_2 & A_3 - a_3 & 0 & -A_1 + a_1 \\
A_3 + a_3 & -A_2 + a_2 & A_1 - a_1 & 0
\end{pmatrix}.
\]

Let $(M, g)$ be a self-dual 4-manifold and $P$ the principal $H$-bundle over $M$ obtained by the holonomy reduction of the bundle of oriented orthonormal frames of $(M, g)$. Any local oriented orthonormal frame field $(e_A)_{A=0}^3$ defined on an open set $U \subset M$ defines a section $e : U \to P$ and therefore the system $\{X^A\}_{A=0}^3$ of the dual coframes is globally defined on $P$. The Levi-Civita connection of $(M, g)$ defines a unique Lie($H$)-valued 1-form which is called the connection form. The connection form is characterized by the following properties (1) and (2):

1. Along the fiber over $m \in M$ of the fibration $p : P \to M$, the connection form is just the infinitesimal version of the “rotation” given by the right action of $H$ on an oriented orthonormal frame of $M_m$ (this consists of the contribution from the left action of elements of $\text{Sp}(1)_+$ and the right action of elements of $\text{Sp}(1)_- \cap H$ on $M_m \cong \mathbb{R}^4$).

2. If $X$ is a tangent vector and $(e_A)_{A=0}^3$ is a local oriented orthonormal frame field on an open set $U \subset M$, then

\[
\nabla_X e_B = e_A \Gamma^A_B(X)
\]

where $\Gamma = (\Gamma^A_B)_{A,B=0,1,2,3}$ represents the $\text{sp}(1)_+$-valued connection 1-form corresponding to the Levi-Civita connection of $(M, g)$.

It follows that the connection form $\Gamma$ is written as

\[
\Gamma = (\Gamma^A_B) = \begin{pmatrix}
0 & -\Gamma_1 - \alpha_1 & -\Gamma_2 - \alpha_2 & -\Gamma_3 - \alpha_3 \\
\Gamma_1 + \alpha_1 & 0 & -\Gamma_3 + \alpha_3 & \Gamma_2 - \alpha_2 \\
\Gamma_2 + \alpha_2 & \Gamma_3 - \alpha_3 & 0 & -\Gamma_1 + \alpha_3 \\
\Gamma_3 + \alpha_3 & -\Gamma_2 + \alpha_2 & \Gamma_1 - \alpha_1 & 0
\end{pmatrix}.
\]
The 1-forms $X^A (A = 0, 1, 2, 3)$ and $\Gamma$ satisfy the following first and second structure equations:

$$dX^A + \Gamma^A_B X^B = 0.$$  
$$d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B = \Omega^A_B$$

where $\Omega = (\Omega^A_B)$ is the skew symmetric matrix of 2-forms on $P$ which is identified with the curvature tensor of $(M, g)$ as follows. For each point $e = (e_A) \in P$ over $m \in M$ we have

$$R(X, Y)e_B = e_A \Omega^A_B(X, Y)$$

for all $X, Y \in M_m$. The sectional curvature for the 2-plane spanned by $\{e_A, e_B\}$ is given by

$$K(e_A, e_B) = g(R(e_A, e_B)e_B, e_A) = \Omega^A_B(e_A, e_B).$$

Let $(e_A) \in P$ be an oriented orthonormal frame of $M_m$. This canonically defines an identification

$$\mathbb{R}^4 = H \ni (x^0 + ix^1 + jx^2 + kx^3) \leftrightarrow (x^0 + jx^2, x^1 + jx^3) \in \mathbb{C}^2.$$

The right multiplication of $j$ on $M_m \cong \mathbb{C}^2$ induces the canonical almost complex structure on $\mathbb{C}^2$. The infinitesimal deformation of the orthogonal complex structures at $j$ is given by $\alpha_1 i + \alpha_3 k$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the connection form of the Levi-Civita connection. In this situation, we pick a point $z$ in the $\mathbb{P}^1$-fiber over $m \in M_m$ corresponding to the right $j$ multiplication. The canonical almost complex structure of $Z$ at $z$ is defined by determining the basis of the space of $(1, 0)$-forms in the following way:

$$\zeta^0 = \alpha_1 + i\alpha_3,$$

$$Z^1 = X^0 + iX^2,$$

$$Z^2 = X^1 + iX^3.$$

Recall that the Chow-Yang metric on the twistor space $Z$ of $(M, g)$ is defined by declaring that $\{\alpha_1, \alpha_3\}$ (representing the “rotation modulo $H \cap U(2)$” of oriented orthonormal frames along the $\mathbb{P}^1$-fiber of the twistor fibration) together with

$$\{X_0, X_1, X_2, X_3\}$$

(the rotating oriented orthonormal frame of the horizontal subspace, where the rotation means the $H$-rotation modulo $H \cap U(2)$-action) form an orthonormal frame. From here on we lower the indices and write $X_0, X_1, X_2, X_3$ instead of $X^0, X^1, X^2, X^3$ (and same for $Z$’s). To get a moving frame expression of the first structure equation of the Chow-Yang metric $g^{CY}_1$, we begin with the first structure
equation for the metric $g_M$. It follows from the above expression of the connection form $\Gamma = (\Gamma^3_1) = \cdots$ that the first structure equation

$$dX + \Gamma \wedge X = 0$$
on the $\mathcal{P}$ is written as

$$\begin{cases}
\{ \displaystyle dZ_1 + \bar{Z}_2 \wedge \zeta_0 + (\Gamma_0 + i(\Gamma_2 + \alpha_2)) \wedge Z_1 + (-\Gamma_1 + i\Gamma_3) \wedge Z_2 = 0 \\
\{ \displaystyle dZ_2 - \bar{Z}_1 \wedge \zeta_0 + (\Gamma_1 + i\Gamma_3) \wedge Z_1 + (\Gamma_0 - i(\Gamma_2 - \alpha_2)) \wedge Z_2 = 0
\end{cases}$$

which in matrix form is expressed as

$$(1) \quad d\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = -\begin{pmatrix} \Gamma_0 + i\Gamma_2 & -\Gamma_1 + i\Gamma_3 \\ \Gamma_1 + i\Gamma_3 & \Gamma_0 - i\Gamma_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} - \begin{pmatrix} i\alpha_2 & 0 \\ 0 & i\alpha_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha_1 + i\alpha_3 \\ \alpha_1 + i\alpha_3 & 0 \end{pmatrix} \begin{pmatrix} \bar{Z}_1 \\ \bar{Z}_2 \end{pmatrix} .$$

To get the first structure equation of the Chow-Yang metric on $\mathcal{Z}$, we need to compute $d\zeta_0$. For the computation of $d\zeta_0$, we use the second structure equation

$$d\Gamma + \Gamma \wedge \Gamma = \Omega .$$

The curvature form can be expressed as

$$\Omega = \begin{pmatrix} 0 & \Omega^0_1 & \Omega^0_2 & \Omega^0_3 \\ \Omega^1_0 & 0 & \Omega^1_2 & \Omega^1_3 \\ \Omega^2_0 & \Omega^2_1 & 0 & \Omega^2_3 \\ \Omega^3_0 & \Omega^3_1 & \Omega^3_2 & 0 \end{pmatrix}$$

where $\Omega^\mu_{\nu} = -\Omega^\nu_{\mu}$. Let $(\mu, \eta, \nu)$ be any cyclic permutation of $(1, 2, 3)$. Then the above expression of the connection form implies

$$\Omega^\mu_{\nu} = d(\Gamma_{\mu} + \alpha_{\mu}) + (\Gamma_{\mu} + \alpha_{\mu}) \wedge \Gamma_0 + (\Gamma_{\eta} - \alpha_{\eta}) \wedge (\Gamma_{\nu} - \alpha_{\nu}) + \Gamma_0 \wedge (\Gamma_{\mu} + \alpha_{\mu}) + (\Gamma_{\mu} + \alpha_{\mu}) \wedge (\eta_{\alpha} + \alpha_{\eta}) +$$
$$+ d\Gamma_{\mu} + \Gamma_{\mu} \wedge \Gamma_0 + \Gamma_{\eta} \wedge \Gamma_{\nu} + \Gamma_0 \wedge \Gamma_{\mu} - \Gamma_{\nu} \wedge \Gamma_0 +$$
$$+ d\alpha_{\mu} - 2\alpha_{\eta} \wedge \alpha_{\nu} +$$
$$= d\Gamma_{\mu} + 2\Gamma_{\eta} \wedge \Gamma_{\nu} + d\alpha_{\mu} - 2\alpha_{\eta} \wedge \alpha_{\nu}$$

$$\Omega^\mu_{\nu} = d(-\Gamma_{\mu} + \alpha_{\mu}) + (\Gamma_{\eta} + \alpha_{\eta}) \wedge (\Gamma_{\nu} - \alpha_{\nu}) + (\Gamma_{\mu} + \alpha_{\mu}) \wedge \Gamma_0 +$$
$$+ (\Gamma_{\nu} - \alpha_{\nu}) \wedge (\Gamma_{\eta} - \alpha_{\eta}) + \Gamma_0 \wedge (\Gamma_{\mu} + \alpha_{\mu}) +$$
$$- d\Gamma_{\mu} - \Gamma_{\eta} \wedge \Gamma_{\nu} - \Gamma_{\mu} \wedge \Gamma_0 + \Gamma_{\nu} \wedge \Gamma_{\eta} - \Gamma_0 \wedge \Gamma_{\mu} +$$
$$+ d\alpha_{\mu} - 2\alpha_{\eta} \wedge \alpha_{\nu} +$$
$$= -d\Gamma_{\mu} - 2\Gamma_{\eta} \wedge \Gamma_{\nu} + d\alpha_{\mu} - 2\alpha_{\eta} \wedge \alpha_{\nu} .$$

Therefore we have

$$(2) \quad \Omega^\mu_{0} + \Omega^\eta_{\mu} = 2d\alpha_{\mu} - 4\alpha_{\eta} \wedge \alpha_{\nu} .$$
To compute further, we need some representation theory of the curvature tensor of Riemannian 4-manifolds ([A-H-S]). We first normalize that the standard metric on the 4-dimensional sphere \((S^4, g_{std})\) has constant sectional curvature 4 (and therefore the scalar curvature 48). We consider the curvature operator of a Riemannian 4-manifold \((M, g)\) as an element of an endomorphism of the bundle of 2-forms which decomposes according to the Lie algebra decomposition \(so(4) = so(3) + so(3)\) into self-dual and anti-self-dual 2-forms:

\[
\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_- (M) .
\]

According to the above decomposition of \(\Lambda^2(M)\), the curvature operator of \((M, g)\) decomposes as

\[
(3) \quad \mathcal{R} = \frac{s}{48} \mathcal{R}_0 + \mathcal{R}'
\]

where \(s\) is the scalar curvature of \((M, g)\), \(\mathcal{R}_0 = \begin{pmatrix} 4\text{id} & 0 \\ 0 & 4\text{id} \end{pmatrix}\) is the curvature operator of \((S^4, g_{std})\) of scalar curvature 48 and the remaining part \(\mathcal{R}'\) is expressed in block matrix form as

\[
\mathcal{R}' = \begin{pmatrix} W_+ & B \\ B & W_- \end{pmatrix}
\]

where \(W_+\) (resp. \(W_-\)) is the self-dual (resp. anti-self-dual) part of the Weyl tensor and \(B\) is the traceless Ricci tensor. The self-dual (resp. anti-self-dual) part of the curvature operator consists of \((W_+ + \frac{s}{12}\text{id} \quad B)\) (resp. \((B \quad W_- + \frac{s}{12}\text{id})\)). In particular, If \((M, g)\) is a self-dual positive Einstein 4-manifold, the \(\mathcal{R}'\) part has a matrix form

\[
\mathcal{R}' = \begin{pmatrix} W_+ & 0 \\ 0 & 0 \end{pmatrix}
\]

with respect to the above decomposition of \(\Lambda^2(M)\). Therefore \(\mathcal{R}'\) has the same form as the curvature operator of a self-dual Einstein manifold of zero scalar curvature (i.e., a complex K3 surface with a Ricci-flat Kähler metric orientation reversed). This part may be called the “hyper-Kähler part” of the curvature operator. Note that the hyper-Kähler part has no contribution to the computation of the Ricci curvature of a self-dual positive Einstein manifold (for a quaternion Kähler generalization of \(\dim \geq 8\) of the above story, see [S1]).

The curvature form \(\tilde{\Omega}\) of \((S^4, g_{std})\) of sectional curvature 4 is expressed as

\[
(4) \quad \tilde{\Omega}_\kappa^\lambda = 4X_\kappa \wedge X_\lambda \quad (\kappa, \lambda = 0, 1, 2, 3).
\]

Let \(\tilde{\alpha}_\mu\) \((\mu = 1, 2, 3)\) denote the \(\alpha_\mu\)'s for \((S^4, g_{std})\) of curvature 4. From (2) and (4) we have for any cyclic permutation \((\mu, \eta, \nu)\) of \((1, 2, 3)\) :

\[
(5) \quad d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu = 2(X_\mu \wedge X_0 + X_\eta \wedge X_\nu).
\]
We write $\Omega'$ for the part of the curvature form corresponding to $\mathcal{R}'$. Then it follows from (3) and (5) (or directly from (4)) that

$$
\begin{align*}
d\alpha_\mu - 2\alpha_\eta \wedge \alpha_\nu &= \frac{1}{2}(\Omega^\mu_0 + \Omega^\eta_\nu) \\
&= \frac{1}{248}(\tilde{\Omega}^\mu_0 + \tilde{\Omega}^\eta_\nu) + \frac{1}{2}(\Omega'^\mu_0 + \Omega'^\eta_\nu) \\
&= \frac{s}{48}(d\tilde{\alpha}_\mu - 2\tilde{\alpha}_\eta \wedge \tilde{\alpha}_\nu) \\
&= \frac{s}{48}(X_\mu \wedge X_0 + X_\eta \wedge X_\nu)
\end{align*}
$$

where the third equality is a direct consequence from the fact that the $\Omega'$-part does not involve the $\alpha$-part. Indeed, the $\Omega'$-part (in the positive Einstein case) consists of $W_+$ in the above block form which is contained in the $\text{sp}(1)_+$-part of the curvature form. On the other hand, the $\alpha$-part stems from the “rotation modulo $H \cap \text{U}(2)$” of oriented orthonormal frames along the $\mathbb{P}^1$-fiber of the twistor fibration. Since the $\text{sp}(1)_+$-part (stemming from the left multiplication of unit quaternions) acts holomorphically w.r.t. the orthogonal complex structures (introduced by the right multiplication of unit imaginary quaternions), this does not contribute to the above mentioned “rotation” because the “rotation” induced by the $W_+$-part is contained in the action of $H \cap \text{U}(2)$. This means that the $W_+$ part has no contribution to the $\alpha$-part.

From the above identity we have

$$
(6) \quad d\zeta_0 = d(\alpha_1 + i\alpha_3) = 2\alpha_2 \wedge \alpha_3 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) + 2i\alpha_1 \wedge \alpha_2 + i(d\alpha_3 - 2\alpha_1 \wedge \alpha_2) = -2i\alpha_2 \wedge \zeta_0 + \frac{s}{48}(Z_2 \wedge Z_1 - Z_1 \wedge Z_2).
$$

Combining (1) and (6) we have the first structure equation for the Hermitian metric $g^{\text{CY}}_1$ on $\mathcal{Z}$ with respect to the basis of $(1,0)$-forms (we take the normalization $s = 48$ into account):

$$
(7) \quad d\begin{pmatrix} \zeta_0 \\ Z_1 \\ Z_2 \end{pmatrix} = -\begin{pmatrix} 2i\alpha_2 & -Z_2 & Z_1 \\ Z_2 & i\Gamma_2 + i\alpha_2 & -\Gamma_1 + i\Gamma_3 \\ -\overline{Z}_2 & \Gamma_1 + i\Gamma_3 & -i\Gamma_2 + i\alpha_2 \end{pmatrix} \wedge \begin{pmatrix} \zeta_0 \\ Z_1 \\ Z_2 \end{pmatrix}.
$$

First, the right hand side contains no $(0,2)$-forms and this implies that the almost complex structure defined by specifying by the space $\{\zeta_0, Z_1, Z_2\}$ of $(1,0)$-forms (i.e., the orthogonal complex structure) is integrable. Second, the connection matrix is certainly skew-Hermitian and this means that the Hermitian metric $g^{\text{CY}}_1$ on $\mathcal{Z}$ is Kähler with respect to the orthogonal complex structure.

Moreover, the direct computation shows that the curvature form of $g^{\text{CY}}_1$ is ex-
pressed as

$$
\begin{pmatrix}
2\zeta_0 \wedge \bar{\zeta}_0 + Z_1 \wedge \bar{Z}_1 & \zeta_0 \wedge \bar{Z}_1 & \zeta_0 \wedge \bar{Z}_2 \\
+ Z_2 \wedge \bar{Z}_2 & i\Omega_0^2 \ - \frac{1}{2}\{\Omega_0^1 + \Omega_2^3 - i(\Omega_0^3 + \Omega_1^3)\} & -\frac{1}{2}\{\Omega_0^1 + \Omega_2^3 - i(\Omega_0^3 + \Omega_1^3)\} \\
Z_1 \wedge \bar{\zeta}_0 & -\bar{Z}_2 \wedge Z_2 + \zeta_0 \wedge \bar{\zeta}_0 & +\bar{Z}_2 \wedge Z_1 \\
Z_2 \wedge \bar{\zeta}_0 & \frac{1}{2}\{\Omega_0^1 + \Omega_2^3 + i(\Omega_0^3 + \Omega_1^3)\} & -\bar{Z}_1 \wedge Z_1 + \zeta_0 \wedge \bar{\zeta}_0
\end{pmatrix}.
$$

Therefore, the Ricci form in the complex notation is

$$\text{Ric}(\Omega) = \text{tr}(\Omega) = 4(\zeta_0 \wedge \overline{\zeta}_0 + Z_1 \wedge \overline{Z}_1 + Z_2 \wedge \overline{Z}_2).$$

This implies that $g_1^{\text{CY}}$ is a Kähler-Einstein metric on $\mathcal{Z}$ ([C-Y]).

On the other hand, it is well-known that the canonical deformation metric $g_1^{\text{can}}$ is Kähler-Einstein (see, for instance, [Be]). Therefore we have

$$g_1^{\text{CY}} = g_1^{\text{can}} = \text{Kähler-Einstein metric on } \mathcal{Z}.$$

Next we proceed to the study of the moving frame of the metric

$$g_\lambda^{\text{CY}} = \lambda^2(\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^{3} X_i \cdot X_i$$

for $\lambda \neq 1$. From the real version of (7) we have

$$d \begin{pmatrix}
\lambda \alpha_1 \\
\lambda \alpha_3 \\
X_0 \\
X_1 \\
X_2 \\
X_3
\end{pmatrix} = -\begin{pmatrix}
2\alpha_2 & \lambda X_1 & \lambda X_0 & \lambda X_3 & -\lambda X_2 \\
0 & -\lambda X_3 & -\lambda X_2 & -\lambda X_1 & \lambda X_0 \\
\lambda^{-1}X_1 & \lambda^{-1}X_3 & 0 & -\Gamma_1 & -\Gamma_2 - \alpha_2 & -\Gamma_3 \\
-\lambda^{-1}X_0 & -\lambda^{-1}X_2 & \Gamma_1 & 0 & -\Gamma_3 & \Gamma_2 - \alpha_2 \\
-\lambda^{-1}X_3 & \lambda^{-1}X_1 & \Gamma_2 + \alpha_2 & \Gamma_3 & 0 & -\Gamma_1 \\
\lambda^{-1}X_2 & -\lambda^{-1}X_0 & \Gamma_3 & -\Gamma_2 + \alpha_2 & \Gamma_1 & 0
\end{pmatrix} \begin{pmatrix}
\lambda \alpha_1 \\
\lambda \alpha_3 \\
X_0 \\
X_1 \\
X_2 \\
X_3
\end{pmatrix}.$$

The big matrix in (8) is the Levi-Civita connection form of the metric $g_\lambda^{\text{CY}}$ on $\mathcal{Z}$. WE write this connection form as $\Gamma_\lambda$. Write

$$\Omega_\lambda = \begin{pmatrix}
\Omega_{\lambda -2} & \Omega_{\lambda -2} & \Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda -2} & \Omega_{\lambda -2} \\
\Omega_{\lambda -1} & \Omega_{\lambda -1} & \Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda -1} & \Omega_{\lambda -1} \\
\Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda 0} & \Omega_{\lambda 0} \\
\Omega_{\lambda 1} & \Omega_{\lambda 1} & \Omega_{\lambda 1} & \Omega_{\lambda 1} & \Omega_{\lambda 1} & \Omega_{\lambda 1} \\
\Omega_{\lambda 2} & \Omega_{\lambda 2} & \Omega_{\lambda 2} & \Omega_{\lambda 2} & \Omega_{\lambda 2} & \Omega_{\lambda 2} \\
\Omega_{\lambda 3} & \Omega_{\lambda 3} & \Omega_{\lambda 3} & \Omega_{\lambda 3} & \Omega_{\lambda 3} & \Omega_{\lambda 3}
\end{pmatrix}.$$
for the curvature form of the metric $g^\text{CY}_\lambda$ on $\mathcal{Z}$. The second structure equation

$$d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda = \Omega_\lambda$$

computes the curvature form $\Omega_\lambda$. From (8) and the second structure equation we have by direct computation the following expressions:

(9)

\[
\begin{align*}
\Omega^{-1}_\lambda &= 0, \\
\Omega^{-2}_\lambda &= 4\alpha_3 \wedge \alpha_1 + 2(X_2 \wedge X_0 + X_3 \wedge X_1), \\
\Omega^0_\lambda &= \lambda^{-1}(X_0 \wedge \alpha_1 + X_2 \wedge \alpha_3), \\
\Omega^1_\lambda &= \lambda^{-1}(X_1 \wedge \alpha_1 + X_3 \wedge \alpha_3), \\
\Omega^2_\lambda &= \lambda^{-1}(-X_0 \wedge \alpha_3 + X_2 \wedge \alpha_1), \\
\Omega^3_\lambda &= \lambda^{-1}(X_0 \wedge \alpha_1 + X_2 \wedge \alpha_3), \\
\Omega_0^{-2} &= \lambda(\alpha_1 \wedge X_0 + \alpha_3 \wedge X_2), \\
\Omega_0^{-1} &= \lambda(\alpha_3 \wedge X_0 - \alpha_1 \wedge X_2), \\
\Omega_1^{-2} &= \lambda(\alpha_1 \wedge X_1 + \alpha_3 \wedge X_3), \\
\Omega_1^{-1} &= \lambda(\alpha_3 \wedge X_1 - \alpha_1 \wedge X_3), \\
\Omega_2^{-2} &= \lambda(-\alpha_3 \wedge X_0 + \alpha_1 \wedge X_2), \\
\Omega_2^{-1} &= \lambda(\alpha_1 \wedge X_0 + \alpha_3 \wedge X_2), \\
\Omega_3^{-2} &= \lambda(-\alpha_3 \wedge X_1 + \alpha_1 \wedge X_3), \\
\Omega_3^{-1} &= \lambda(\alpha_1 \wedge X_1 + \alpha_3 \wedge X_3), \\
\Omega_0^0 &= \Omega_1^1 = \Omega_2^2 = \Omega_3^3 = 0, \\
\Omega_0^1 &= \Omega_0^1 + X_0 \wedge X_1 + X_2 \wedge X_3 - (d\alpha_1 - 2\alpha_2 \wedge \alpha_3), \\
\Omega_0^2 &= \Omega_0^2 + 2X_3 \wedge X_1 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2, \\
\Omega_0^3 &= \Omega_0^3 - X_2 \wedge X_1 + X_0 \wedge X_3 - (d\alpha_3 - 2\alpha_1 \wedge \alpha_2), \\
\Omega_1^2 &= \Omega_1^2 - X_3 \wedge X_0 + X_1 \wedge X_2 + (d\alpha_3 - 2\alpha_1 \wedge \alpha_2), \\
\Omega_1^3 &= \Omega_1^3 + 2X_2 \wedge X_0 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2, \\
\Omega_2^3 &= \Omega_2^3 + X_2 \wedge X_3 + X_0 \wedge X_1 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3). 
\end{align*}
\]

We recall the fact that the curvature form of self-dual Einstein 4-manifolds decomposes as

(10)

\[
\begin{align*}
\Omega^\mu_0 &= \tilde{\Omega}_0^\mu + \Omega^\mu_0 = 4X_\mu \wedge X_0 + \Omega^\mu_0, \\
\Omega^\mu_\nu &= \tilde{\Omega}_\nu^\mu + \Omega^\mu_\nu = 4X_\eta \wedge X_\nu + \Omega^\mu_\nu,
\end{align*}
\]

where $(\mu, \eta, \nu)$ is any cyclic permutation of $(1, 2, 3)$, $\tilde{\Omega}$ is the curvature form of $S^4$ (with the standard metric with sectional curvature identically 4) and $\Omega'$ is the $(\begin{smallmatrix}
W_+ & 0 \\
0 & 0
\end{smallmatrix})$-part (which has no contribution to the Ricci tensor).

From (9), (10) and the fact that $\Omega'$-part does not contribute to the Ricci tensor, we can compute the Ricci tensor by direct calculation using the definition of the Ricci tensor

\[
R(e_i, e_j) = \sum_{k=1}^n g(R(e_i, e_k)e_k, e_j) = \sum_{k=1}^n g(\Omega_k^j(e_i, e_k)e_j, e_j)
\]
where \( \{e_1, \ldots, e_n\} \) is a system of orthonormal frame on an \( n \)-dimensional Riemannian manifold and \( \Omega \) is the curvature form with respect to this orthonormal frame.

In the following computation of the Ricci tensor, we write \( \{\lambda^{-1}\xi_2, \lambda^{-1}\xi_1, \xi_0, \xi_2, \xi_3\} \) for the oriented orthonormal frame dual to the orthonormal coframe \( \{\lambda\alpha_1, \lambda\alpha_3, X_0, X_1, X_2, X_3\} \).

The result of the computation of the Ricci tensor is as follows:

\[
\begin{align*}
\text{Ric}_\lambda(\lambda^{-1}\xi_2, \lambda^{-1}\xi_2) &= \text{Ric}_\lambda(\lambda^{-1}\xi_1, \lambda^{-1}\xi_1) = \frac{4}{\lambda^2} + 4, \\
\text{Ric}_\lambda(\lambda^{-1}\xi_2, \lambda^{-1}\xi_1) &= 0, \\
\text{Ric}_\lambda(\lambda^{-1}\xi_2, \xi_0) &= \text{Ric}_\lambda(\lambda^{-1}\xi_2, \xi_1) = \text{Ric}_\lambda(\lambda^{-1}\xi_2, \xi_2) = \text{Ric}_\lambda(\lambda^{-1}\xi_2, \xi_3) \\
&= \text{Ric}_\lambda(\lambda^{-1}\xi_1, \xi_0) = \text{Ric}_\lambda(\lambda^{-1}\xi_1, \xi_1) \text{Ric}_\lambda(\lambda^{-1}\xi_1, \xi_2) = \text{Ric}_\lambda(\lambda^{-1}\xi_1, \xi_3) \\
&= 0, \\
\text{Ric}_\lambda(\xi_0, \xi_0) &= \text{Ric}_\lambda(\xi_1, \xi_1) = \text{Ric}_\lambda(\xi_2, \xi_2) = \text{Ric}_\lambda(\xi_3, \xi_3) = \frac{2}{\lambda^2} + 6, \\
\text{Ric}_\lambda(\xi_0, \xi_1) &= \text{Ric}_\lambda(\xi_0, \xi_2) = \text{Ric}_\lambda(\xi_0, \xi_3) \\
&= \text{Ric}_\lambda(\xi_1, \xi_2) = \text{Ric}_\lambda(\xi_1, \xi_3) = \text{Ric}_\lambda(\xi_2, \xi_3) \\
&= 0.
\end{align*}
\]

Summing up the above computation, we have:

**Theorem 1.2 (Theorem 0.2 (2)).** The Ricci tensor of the Chow-Yang metric

\[
g^{\text{CY}}_\lambda = \lambda^2(\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^{3} X_i^2
\]

is given by the formula

\[
(11) \quad \text{Ric}(g^{\text{CY}}_\lambda) = 4(1 + \lambda^2)(\alpha_1^2 + \alpha_3^2) + 2(\lambda^{-2} + 3)(X_0^2 + X_1^2 + X_2^2 + X_3^2)
\]

\[
= 2\lambda^{-2}(1 + 3\lambda^2) \frac{g^{\text{CY}}_\lambda}{\sqrt{\frac{2\lambda^2(1 + \lambda^2)}{1 + 3\lambda^2}}}
\]

Therefore, the Chow-Yang metric \( g^{\text{CY}}_\lambda \) has positive Ricci curvature and the Ricci tensor defines a Chow-Yang metric with different parameter, i.e., the Ricci map (the map which associate to each metric its Ricci tensor) is identified with a map on the \((\lambda^2, \rho)\) plane defined by

\[
\text{Ric} : (\lambda^2, \rho) \mapsto \left( \frac{2\lambda^2(1 + \lambda^2)}{1 + 3\lambda^2}, 2\lambda^{-2}(1 + 3\lambda^2) \right).
\]
2. Ricci Flow

In this section we study the Ricci flow on $\mathcal{Z}$ with initial metric taken from the family of the Chow-Yang metrics. We consider the two parameter family $\mathcal{F}^{\text{CY}}$ of scaled Chow-Yang metrics $\{\rho g^{\text{CY}}_\lambda\}_{\rho>0,\lambda>0}$ where $\rho g_\lambda$ is expressed as, say,

$$
\rho g^{\text{CY}}_\lambda = \rho\left\{\lambda^2(\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^{3} X_i^2\right\}.
$$

Then the family $\mathcal{F}^{\text{CY}}$ is invariant under the sum with nonnegative coefficients. Moreover, Theorem 1.2 implies that the Ricci tensor of a Chow-Yang metric is again contained in the family $\mathcal{F}^{\text{CY}}$. Therefore, the Ricci flow with initial metric in $\mathcal{F}^{\text{CY}}$ stays in $\mathcal{F}^{\text{CY}}$ as long as the solution exists as Riemannian metrics on $\mathcal{Z}$.

From Theorem 1.2 we have the explicit form of the Ricci flow with initial metric in the family $\mathcal{F}^{\text{CY}}$:

**Theorem 2.1 (Theorem 0.3 (2)).** (1) The Ricci flow equation $\partial_t g = -2\text{Ric}_g$ on the twistor space with initial metric in $\mathcal{F}^{\text{CY}}$ is given by the system of ordinary differential equations

$$
\begin{align*}
\frac{d}{dt} \rho(t)\lambda(t)^2 &= -8(1 + \lambda^2(t)) , \\
\frac{d}{dt} \rho(t) &= -4(\lambda(t)^{-2} + 3) .
\end{align*}
$$

The trajectory of the equation (12) in the $(\lambda^2, \rho)$-plane (this is identified with the Ricci flow trajectory) is given by

$$
\rho = (\text{const.}) \frac{\lambda^2}{|1 - \lambda^2|^4} .
$$

(2) For any initial metric at time $t = 0$ in the family $\mathcal{F}^{\text{CY}}$, the system of ordinary differential equations (12) has a solution defined on $(-\infty, T)$, i.e. the solution is extended for all negative reals and extinct at some finite time $T$ (i.e., as $t \to T$ the solution shrinks the space and become extinct at time $T$). The extinction time $T$ depends on the choice of the initial metric.

(3) Suppose that $\rho(0) = 1$. If $\lambda(0) = 1$, then the metric remains Kähler-Einstein ($\lambda(t) \equiv 1$) and the solution evolves just by homothety $\rho(t) = 1 - 8t$ (in this case $T = \frac{1}{8}$).

If $\lambda(0) < 1$, then $\lim_{t \to -\infty} \lambda(t) = 1$, $\lim_{t \to -\infty} \rho(t) = \infty$, $\lim_{t \to -T} \lambda(t) = 0$ and $\lim_{t \to -T} \rho(t) = 0$.

If $\lambda(0) > 1$, then $\lim_{t \to -\infty} \lambda(t) = 1$, $\lim_{t \to -\infty} \rho(t) = \infty$, $\lim_{t \to -T} \lambda(t) = \infty$, $\lim_{t \to -T} \rho(t) = 0$ and $\lim_{t \to -T} \rho(t)\lambda^2(t) = 0$.

Suppose that $\lambda(0) \neq 1$. Then, as $t$ becomes larger in the future direction, the deviation $|1 - \lambda(t)|$ of the solution from being Kähler-Einstein becomes larger as well. As $t$ becomes larger in the past direction, then the solution becomes backward asymptotic to the solution in the case of $\lambda(0) = 1$, i.e., the Kähler-Einstein metric is the asymptotic soliton of the Ricci flow under consideration.
(4) Suppose that \( \lambda(0) < 1 \). Then the Gromov-Hausdorff limit of the Ricci flow solution as \( t \to T \), scaled with the factor \( \rho(t)^{-1} \), is the original self-dual positive Einstein metric on \( M \).

(5) Suppose that \( \lambda(0) > 1 \). Then the Gromov-Hausdorff limit of the Ricci flow solution as \( t \to T \), scaled with the factor \( \rho(t)^{-1} \), is the sub-Riemannian metric defined on the horizontal distribution of the twistor space \( Z \) which projects isometrically to the original self-dual positive Einstein metric on \( M \).

Proof. Putting \( \mu(t) = \lambda^2(t) \) and assuming \( \mu \neq 1 \), we get from (12) the system of ordinary differential equations
\[
\begin{align*}
\frac{d\rho}{dt} &= -4\mu^{-1} - 12, \\
\frac{d(\rho\mu)}{dt} &= -8 - 8\mu.
\end{align*}
\]
Eliminating \( dt \) from the above two equations, we have
\[
d\log \rho = d\left\{ \log \mu - 4 \log(1 - \mu) \right\}.
\]
The solution of this equation represents a family of curves in the \((\mu, \rho)\)-plane. Explicitly, we have (13):
\[
\rho = \text{(const.)} \frac{\mu}{|1 - \mu|^4}.
\]
The meaning of (13) is this: Each Ricci flow solution of (12) is identified with the oriented curve in the \((\mu, \rho)\) \((\mu = \lambda^2)\) plane, each of whose defining equation is given by (13) with a specified positive constant determined by the initial metric. The proof of Theorem 2.1 is now direct from (13).

Remark 2.2 (comparison with canonical deformation metrics, cf. [Be] and [D-M]). Let \( g^\text{can}_\lambda = \lambda^2 g_{FS} + g_M \) \((\lambda > 0)\) be the canonical deformation metrics on \( Z \), where \( g_{FS} \) is the Fubini-Study metric of the fiber \((\text{curvature} \equiv 4)\), \( g_M \) is a quaternion Kähler metric with scalar curvature \( S = 48 \) and the sum is defined with respect to the horizontal distribution corresponding to \( g_M \). Then O’Neill’s formula implies
\[
\text{Ric}(g^\text{can}_\lambda) = 4(1 + \lambda^4)g_{FS} + 4(3 - \lambda^2)g_M \left[= 4(3 - \lambda^2) g^\text{can}_\lambda \sqrt{\frac{1 + \lambda^4}{3 - \lambda^2}} \quad \text{if} \ \lambda^2 < 3 \right].
\]
We have two consequences from this formula. First, the Ricci tensor of \( g^\text{can}_\lambda \) is positive if and only if \( \lambda^2 < 3 \). Second, \( g^\text{can}_\lambda \) is Einstein if and only if \( \lambda^2 = 1 \) or \( \lambda^2 = \frac{1}{2} \). The case \( \lambda^2 = 1 \) corresponds to the submersion metric coming from \( P \to Z \) which turns out to be Kähler-Einstein. Another Einstein metric (corresponding to the case \( \lambda^2 = \frac{1}{2} \)) is non-Kähler. The Ricci flow equation for the canonical deformation metrics becomes
\[
\begin{align*}
\frac{d\lambda^2}{dt} &= -\frac{8}{\rho}(\lambda^2 - 1)(2\lambda^2 - 1), \\
\frac{d\rho}{dt} &= -8(3 - \lambda^2).
\end{align*}
\]
For instance, if \( \lambda > 1 \), then \( \lambda \) decreases as \( t \) increases (i.e., \( \lambda \) decreases along the Ricci flow). Therefore the behavior of the Ricci flow in the canonical deformation is very different from Theorem 2.1.
3. Uniformization of Self-Dual Positive Einstein 4-Manifolds

Analyzing the ancient solutions in Theorem 2.1 we prove the following:

**Theorem 3.1.** Let \((M, g)\) be a self-dual positive Einstein 4-manifold and \(g^\text{CY}_\lambda\) be the Chow-Yang metric. Then we have the limit formula

\[
\lim_{\lambda \to 0} |\nabla g^\text{CY}_\lambda \cdot \text{Rm}(g^\text{CY}_\lambda)|_{g^\text{CY}_\lambda} = 0.
\]

**Proof.** We fix a small positive number \(\delta\) and pick a sequence of points \(\{(1 + \delta, \rho_k)\}_{k=1}^\infty\) in the \((\mu, \rho)\) plane \((\mu = \lambda^2)\) satisfying the property \(\lim_{k \to \infty} \rho_k = \infty\). Correspondingly, we consider the Ricci flow solutions \(\{g_k(t)\}_{k=1}^\infty\) in Theorem 2.1 with initial metric \(g_k(0) = \rho_k g^\text{CY}_{1+\delta}\). Let \(T_k\) be the time when the solution \(g_k(t)\) passes the line \(\rho = 1\), i.e., \(g_k(T_k) = g^\text{CY}_\lambda\) where \(\{\lambda_k\}_{k=1}^\infty\) is a sequence of positive numbers satisfying the property \(\lim_{k \to \infty} \lambda_k = \infty\). We note that \(T_k \to \infty\) as \(k \to \infty\). To see this, we note that \(\rho_k \to \infty\) as \(k \to \infty\). We have from (12) that \(\frac{dt}{d\rho} = -\frac{1}{4(1+3\lambda^2)}\) and therefore

\[
T_k = \frac{1}{4} \int_1^{\rho_k} \frac{\lambda^2}{1 + 3\lambda^2} d\rho.
\]

As the trajectory of \(g_k\) is given by the equation \(\rho = c_k \frac{\lambda^2}{|1 - \lambda^2|}\) in the \((\lambda, \rho)\) plane where \(c_k\) is a positive constant satisfying the condition \(\lim_{k \to \infty} c_k = \infty\), we see that as \(\rho_k\) becomes large, the contribution from large \(\lambda\) (which is \(\approx \frac{1}{3}\)) in the above integral becomes more dominant. Therefore we have \(T_k \to \infty\) as \(k \to \infty\).

Write \(\text{Rm}_k(x, t)\) for the Riemann curvature tensor of the solution metric \(g_k(x, t)\). It follows from the computation in §2 we have

\[
|Rm_k(x, t)| \leq \frac{c_1}{T_k - t + 1}
\]

whenever \(0 \leq t \leq T_k\) where the constant \(c_1\) is chosen uniformly in \(k\). We would like to estimate the asymptotic behavior of \(\max_{x \in Z} |\nabla \text{Rm}_k(x, t)|\) when \(t \to \infty\). For this purpose, we introduce

\[
F(x, t) = t |\nabla \text{Rm}(x, t)| + \beta(t) |\text{Rm}(x, t)|^2
\]

where the function \(\beta(t)\) is chosen in the form

\[
\beta(t) = 1 + \frac{c t}{T_k - t + 1}
\]

with constant \(c > 0\) satisfying the condition

\[
1 + c_1 t |\text{Rm}| - 2\beta(t) < 0.
\]

The strategy is to apply the parabolic maximum principle to the evolution inequality satisfied by the function \(F(x, t)\) (see [Ba], [Sh1,2] and also [C-K, Chapt 7]). Following the notation in [C-K, p.227], we have

\[
\frac{\partial}{\partial t} |\text{Rm}|^2 = \Delta |\text{Rm}|^2 - 2|\nabla \text{Rm}|^2 + (\text{Rm})^3
\]
and 
\[ \frac{\partial}{\partial t}|\nabla Rm|^2 = \Delta |Rm|^2 - 2|\nabla^2 Rm|^2 + Rm \ast (\nabla Rm)^2. \]

Combining the above two parabolic equations and (15), we conclude that the function \( F(x, t) \) satisfies the parabolic inequality

\[
\frac{\partial F}{\partial t} = |\nabla Rm|^2 + t \frac{\partial}{\partial t}|\nabla Rm|^2 + \beta'(t)|Rm|^2 + \beta(t) \frac{\partial}{\partial t}|Rm|^2
\]

\[
= \Delta F + |\nabla Rm|^2 + t Rm \ast (Rm)^2 - 2\beta(t)|\nabla Rm|^2
\]

\[
+ \beta(t)|\nabla^3 Rm|^3 + \beta'(t)|Rm|^2 - 2t|\nabla^2 Rm|^2.
\]

\[
\leq \Delta F + c_2 \beta(t)|Rm|^3 + \beta'(t)|Rm|^2
\]

where \( c_2 \) is a positive constant which can be chosen uniformly in \( k \). Applying the parabolic maximum principle (see, for instance, [C-K, pp.95-96]) to (16) with the help of (14) and (15), we have

\[
\sup_{x \in Z} F(x, T_k) \leq C \left\{ \beta(0) \left( \frac{c_1}{T_k + 1} \right)^2 + \int_0^{T_k} \left( \frac{T_k}{(T_k - t)^3} + \frac{T_k}{(T_k - t + 1)^4} \right) dt \right\}
\]

\[
\leq \frac{C}{T_k}.
\]

Therefore we have

\[
|\nabla Rm(T_k)|^2 \leq \frac{1}{T_k} \sup_{x \in Z} F(x, T_k) \leq \frac{C}{T_k^2} \to 0
\]

as \( k \to \infty \). This implies

\[
\lim_{\lambda \to \infty} |\nabla g^{CY}_{\lambda} Rm_{\lambda}^{CY}|_{g^{CY}_{\lambda}} = 0
\]

because \( g_k = g^{CY}_{\lambda} \) can be chosen arbitrary as far as \( \lim_{k \to \infty} \lambda_k = \infty \). \( \square \)

We use the limit formula in Theorem 3.1 to prove Theorem 0.4 (Theorem 3.2). Intuitively, the behavior of \( \nabla g^{CY}_{\lambda} Rm_{\lambda}^{CY} \) reduces that of \( \nabla Rm \) for the original \((M, g)\) because the fiber \( \mathbb{P}^1 \) of the twistor fibration \( Z \to M \) blows up in the limit \( \lambda \to \infty \).

**Theorem 3.2 (Theorem 0.4).** Any locally irreducible self-dual positive Einstein 4-manifold is isometric to either \( S^4 \) with standard metric or \( \mathbb{P}^2(\mathbb{C}) \) with Fubini-Study metric.

**Proof.** We choose an orthonormal basis \((e_A)\) of the tangent space \((T_m M, g_m) \ (m \in M)\) and extend it to an orthonormal frame on a neighborhood of \( m \) by parallel transportation along geodesics emanating from \( m \). This defines a 4-dimensional surface \( S \) centered at \((e_A) \in \mathcal{P} \) (\( \mathcal{P} \) being the holonomy reduction of the principal
bundle of orthonormal frames of $M$) which is transversal to the vertical foliation. This determines a 4-dimensional surface $S'$ in the twistor space $\mathcal{Z}$ centered at a point $\tilde{m}$ on a $\mathbb{P}^1$-fiber over $m$, which is transversal to the $\mathbb{P}^1$-fibration. The covariant derivative of the curvature tensor at $m$ is computed by differentiating the components of the curvature tensor w.r.t. the orthonormal frames represented by points of $S$ (identified with $S'$ in $\mathcal{Z}$) in the direction of a horizontal tangent vector at $\tilde{m}$ (identified with a tangent vector of $M$ at $m$). In §3, we computed the curvature form of the metric $g_\lambda = \lambda^2(\alpha_1^2 + \alpha_2^2) + \sum_{i=0}^3 \pi_i X_i \cdot X_i$ on $\mathcal{Z}$. For our purpose, we need:

\[
\begin{align*}
\Omega_{\lambda_0}^0 &= \Omega_0^0 - X_1 \wedge X_1 - X_3 \wedge X_3, \\
\Omega_{\lambda_0}^1 &= \Omega_0^0 - X_0 \wedge X_0 - X_2 \wedge X_2 \\
\Omega_{\lambda_0}^2 &= \Omega_0^0 - X_1 \wedge X_1 - X_3 \wedge X_3, \\
\Omega_{\lambda_0}^3 &= \Omega_0^0 - X_0 \wedge X_0 - X_2 \wedge X_2 \\
\end{align*}
\]

and

\[
\begin{align*}
\Omega_{\lambda_1}^0 &= \Omega_1^0 + X_0 \wedge X_1 + X_2 \wedge X_3 - (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) \\
\Omega_{\lambda_1}^2 &= \Omega_0^2 + X_3 \wedge X_1 - X_1 \wedge X_3 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2 \\
\Omega_{\lambda_1}^3 &= \Omega_0^3 - X_2 \wedge X_1 + X_0 \wedge X_3 - (d\alpha_3 - 2\alpha_1 \wedge \alpha_2) \\
\Omega_{\lambda_2}^1 &= \Omega_1^2 - X_3 \wedge X_0 + X_1 \wedge X_2 + (d\alpha_3 - 2\alpha_1 \wedge \alpha_2) \\
\Omega_{\lambda_2}^3 &= \Omega_1^3 + X_2 \wedge X_0 - X_0 \wedge X_2 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2 \\
\Omega_{\lambda_3}^3 &= \Omega_2^3 + X_2 \wedge X_3 + X_0 \wedge X_1 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3). \\
\end{align*}
\]

Taking the component in the $X_i$ $(i = 0, 1, 2, 3)$ direction of the curvature tensor and taking the covariant derivative in the $X_i$ $(i = 0, 1, 2, 3)$ direction, we immediately conclude that the covariant derivatives of the $X_i$ $(i = 0, 1, 2, 3)$ part of the curvature tensor of the metric $g_\lambda$ of the twistor space $\mathcal{Z}$ at $\tilde{m}$ in the horizontal direction is equal to the covariant derivative in the corresponding direction of the curvature tensor of the quaternion Kähler manifold $(M, g)$ under question. On the other hand, we have from Theorem 4.2 the limit formula

\[
\lim_{\lambda \to \infty} |\nabla^{g_\lambda^{\text{CV}}} \text{Rm}(g_\lambda^{\text{CV}})|_{g_\lambda^{\text{CV}}} = 0.
\]

This implies that the curvature tensor of the positive quaternion Kähler manifold $(M, g)$ must satisfy the condition $\nabla R \equiv 0$ from the beginning. This implies that $(M, g)$ is a symmetric space. Since we assumed that $(M, g)$ is irreducible, $(M, g)$ must be isometric to one of the locally irreducible 4-dimensional compact Riemannian symmetric spaces. $\square$

**Remark 3.3.** Here we explain the “intuition” behind Theorem 3.1 and 3.2. Let us recall that Perelman’s No Local Collapsing Theorem [P] asserts the following. Suppose that a finite time singularity occurs in the space-time of the Ricci flow on a closed manifold. Then, as a limit of suitable parabolic scalings, we get a Ricci flow ancient solution which encodes all information of the singularity. From this viewpoint, Theorem 2.1 means the following. Suppose that $\lambda > 1$. Then the collapse of the twistor space where the base $M$ shrinks faster is realized as a finite time
singularity arising in the Ricci flow starting at the metric $\rho g^\text{CY}_\lambda$ ($\lambda > 1$). Moreover, this Ricci flow coincides with the above mentioned ancient solution, which encodes all information of the collapse of $Z$ where the base $M$ shrinks faster. Moreover, the shrinking family of Kähler-Einstein metrics on $Z$ is its asymptotic soliton. Therefore, we can view the Kähler-Einstein metric of $Z$ not only as a $t \to \infty$ limit (thermo-dynamical equilibrium) of the “heat”-type equation (“stability” under the Kähler-Ricci flow of the positive Kähler-Einstein metric of Fano manifolds proved by Perelman and Tian-Zhu [T-Z]), but as a $t \to -\infty$ limit of the (non-Kähler) Ricci flow ancient solution. The $t \to -\infty$ limit should correspond to the “minimum entropy”. If we recall the “intuitive” meaning of the thermo-dynamical entropy, we can expect that we can extract the local information of the Kähler-Einstein metric on $Z$ (for instance, the “local symmetry” of the original self-dual positive Einstein metric on the base $M$).

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