A Tutorial on Asymptotic Properties of Regularized Least Squares Estimator for Finite Impulse Response Model

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Abstract—In this paper, we give a tutorial on asymptotic properties of the Least Square (LS) and Regularized Least Squares (RLS) estimators for the finite impulse response model with filtered white noise inputs. We provide three perspectives: the almost sure convergence, the convergence in distribution and the boundedness in probability. On one hand, these properties deepen our understanding of the LS and RLS estimators. On the other hand, we can use them as tools to investigate asymptotic properties of other estimators, such as various hyper-parameter estimators.

Index Terms—Least squares estimator, Regularized least squares estimator, Asymptotic properties

I. LEAST SQUARES ESTIMATOR FOR FINITE IMPULSE RESPONSE MODEL

We focus on the $n$th-order finite impulse response (FIR) model as follows,

$$y(t) = \sum_{i=1}^{n} g_i u(t-i) + v(t), \quad t = 1, \ldots, N$$

where $n$ is the order of the FIR model, $N$ denotes the sample size, $t$ denotes the time index, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $v(t) \in \mathbb{R}$ are the input, output and disturbance at time $t$, respectively, and $g_1, \ldots, g_n \in \mathbb{R}$ are FIR model parameters to be estimated.

Assumption 1: The input $u(t)$ with $t = 1, \ldots, N - 1$ is the filtered white noise with the stable filter $H(q)$, i.e.

$$H(q) = \sum_{k=0}^{\infty} h(k)q^{-k} \quad \text{with} \quad \sum_{k=0}^{\infty} |h(k)| < \infty,$$  

where $q^{-1}$ represents the backward shift operator: $q^{-1}u(t) = u(t-1)$, and $v(t)$ is independent and identically distributed (i.i.d.) with zero mean, variance $\sigma_v^2 > 0$, bounded moments of order $4 + \delta$ for some $\delta > 0$, and $\mathbb{E}[v(t)^4] = c\sigma_v^4$ with $c \in \mathbb{R}$ being a constant. Moreover, let $\Sigma \in \mathbb{R}^{n \times n}$ be

$$\Sigma \triangleq \text{COV} \left( \begin{bmatrix} u(0) & u(1) & \cdots & u(n-1) \end{bmatrix}^T \right),$$

where $\text{COV}()$ denotes the covariance matrix of a random vector. In addition, $\Sigma$ is assumed to be positive definite, i.e. $\Sigma > 0$.

Assumption 2: The measurement noise $v(t)$ with is i.i.d. with zero mean, variance $\sigma_v^2 > 0$ and bounded moments of order $4 + \delta$ for some $\delta > 0$.

Assumption 3: $\{u(t)\}_{t=1}^{N-1}$ and $\{v(t)\}_{t=1}^{N}$ are mutually independent, which means that for $i = 1, \ldots, N-1$ and $j = 1, \ldots, N$, $u(i)$ and $v(j)$ are independent.

By Assumption 1, $u(t)$ is a stationary stochastic process with

$$\mathbb{E}[u(t)] = 0,$$  

$$\mathbb{E}[u(t)u(t+\tau)] \triangleq R_u(\tau) = \sigma_u^2 \sum_{k=0}^{\infty} h(k)h(k+\tau),$$

where $\mathbb{E}(\cdot)$ denotes the mathematical expectation, $\tau \geq 0$, and $R_u(\tau) = R_u(-\tau)$, and moreover, the $(i,j)$th element of $\Sigma$ is $R_u(|i-j|)$.

The model (1) can also be rewritten in matrix form as

$$Y = \Phi \theta + V,$$  

where

$$Y = \begin{bmatrix} y(1) & y(2) & \cdots & y(N) \end{bmatrix}^T,$$  

$$\Phi = \begin{bmatrix} \phi(1) & \phi(2) & \cdots & \phi(N) \end{bmatrix}^T,$$  

$$\theta = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix}^T,$$  

$$V = \begin{bmatrix} v(1) & v(2) & \cdots & v(N) \end{bmatrix}^T.$$
with \( \phi(t) = [u(t - 1), u(t - 2), \ldots, u(t - n)]^T \) and \( u(t) = 0 \) for \( t < 0 \).

Assume that \( \Phi \in \mathbb{R}^{n \times n} \) with \( n > n \) is full column rank, i.e. \( \text{rank}(\Phi) = n \). To estimate the unknown \( \theta \) based on data \( \{y(t), \phi(t)\}_{t=1}^N \), one classic estimation method is the Least Squares (LS):

\[
\hat{\theta}^{LS} = \arg \min_{\theta \in \mathbb{R}^n} \| Y - \Phi \theta \|_2^2 \tag{7a}
\]

\[
= (\Phi^T \Phi)^{-1} \Phi^T Y, \tag{7b}
\]

where \( \| \cdot \|_2 \) denotes the Frobenius norm. The LS estimator has some interesting asymptotic properties, which will be discussed in the following context.

II. ASYMPTOTIC PROPERTIES OF LS ESTIMATOR FOR FIR MODEL

Let the true FIR parameter be \( \theta_0 \in \mathbb{R}^n \). Based on (7b), the LS estimator can be rewritten as

\[
\hat{\theta}^{LS} = (\Phi^T \Phi)^{-1} \Phi^T (\theta_0 + V) = \theta_0 + N(\Phi^T \Phi)^{-1} \Phi^T V / N, \tag{8}
\]

which consists of three building blocks: \( \theta_0 \), \( N(\Phi^T \Phi)^{-1} \) and \( \Phi^T V / N \). Firstly, apart from the fixed \( \theta_0 \), we consider asymptotic properties of \( N(\Phi^T \Phi)^{-1} \) and \( \Phi^T V / N \) in Theorem 1.

In the following part, we adopt the concepts of almost sure convergence and convergence in distribution. We define that the random sequence \( \{\xi_N\} \in \mathbb{R}^d \) converges almost surely to a random variable \( \xi \in \mathbb{R}^d \) if \( \text{Pr}(\lim_{N \to \infty} \|\xi_N - \xi\|_2 = 0) = 1 \), which can be written as \( \xi_N \xrightarrow{a.s.} \xi \) as \( N \to \infty \). Define that the random sequence \( \{\xi_N\} \) converges in distribution to a random variable \( \xi \), if \( \text{lim}_{N \to \infty} \text{Pr}(\xi_N \leq x) = \text{Pr}(\xi \leq x) \) for every \( x \) at which the limit distribution function \( \text{Pr}(\xi \leq x) \) is continuous, where the map \( x \mapsto \text{Pr}(\xi \leq x) \) denotes the distribution function of \( \xi \). It can be written as \( \xi_N \xrightarrow{d} \xi \).

**Theorem 1:** Under Assumption 1, 2 and 3, as \( N \to \infty \), we have

\[
\frac{\Phi^T \Phi}{N} \xrightarrow{a.s.} \Sigma, \tag{9}
\]

\[
\frac{\Phi^T V}{N} \xrightarrow{a.s.} 0, \tag{10}
\]

\[
\frac{V^T V}{N} \xrightarrow{a.s.} \sigma^2, \tag{11}
\]

\[
\left( \sqrt{N} \left( \frac{\Phi^T \Phi}{N} - \Sigma \right), \sqrt{N} \left( \frac{\Phi^T V}{N} \right), \sqrt{N} \left( \frac{V^T V}{N} - \sigma^2 \right) \right) \xrightarrow{d} \left( \Gamma, \nu, \rho \right), \tag{12}
\]

where \( \Gamma \in \mathbb{R}^{n \times n}, \nu \in \mathbb{R}^{n \times 1} \) and \( \rho \in \mathbb{R} \) are jointly Gaussian distributed with zero mean and satisfy

\[
C_{\Gamma} \triangleq \text{E}(\Gamma \otimes \Gamma) = \lim_{N \to \infty} N \text{E} \left[ \left( \frac{\Phi^T \Phi}{N} - \Sigma \right) \otimes \left( \frac{\Phi^T \Phi}{N} - \Sigma \right) \right] , \tag{13}
\]

\[
\text{E}(\nu^T \nu) = \sigma^2, \tag{14}
\]

\[
\text{E}(\nu^T) = \text{E}[\nu^T(t)] - \sigma^T, \tag{15}
\]

\[
\text{E}(\nu \otimes \Gamma) = 0 \in \mathbb{R}^{n \times n}, \tag{16}
\]

\[
\text{E}(\nu \nu^T) = 0 \in \mathbb{R}^{n \times 1}, \tag{17}
\]

\[
\text{E}(\rho \Gamma) = 0 \in \mathbb{R}^{n \times n}. \tag{18}
\]

Here \( \otimes \) denotes the Kronecker product.

Moreover, for \( i, j = 1, \ldots, n \), the \((i, j)\)th element of \( C_{\Gamma} \) can be represented as

\[
[C_{\Gamma}]_{i,j} = R_u((i-j)); \tag{19}
\]

for \( i, j = 1, \ldots, n^2 \), the \((i, j)\)th element of \( C_{\Gamma} \) can be represented as

\[
[C_{\Gamma}]_{i,j} = \{ \text{E}[\nu^T(t)/\sigma^2] - 3 \} R_u(k)R_u(l) + \sum_{\tau=-\infty}^{\infty} [R_u(\tau)R_u(\tau+k-l) + R_u(\tau+k)R_u(\tau-l)], \tag{20}
\]

where \( R_u(\tau) \) is defined in (4b), and

\[
k = |[(i-1)/n] - [(j-1)/n]|, \tag{21a}
\]

\[
l = |i-j| - [(i-1)/n]n + [(j-1)/n]n. \tag{21b}
\]

Here \( \lfloor \cdot \rfloor \) denotes the floor operation, i.e. \( \lfloor x \rfloor = \max \{ \hat{x} \in \mathbb{Z} | \hat{x} \leq x \} \).

Then, combining (8) with Theorem 1, we can derive the almost sure convergence and the convergence in distribution of the LS estimator.

**Theorem 2:** Under Assumption 1, 2 and 3, as \( N \to \infty \), we have

\[
N(\Phi^T \Phi)^{-1} \xrightarrow{a.s.} \Sigma^{-1}, \tag{22}
\]

\[
\hat{\theta}^{LS} \xrightarrow{a.s.} \theta_0, \tag{23}
\]

\[
\hat{\sigma}^2 \xrightarrow{a.s.} \sigma^2 \tag{24}
\]

\[
\left( \sqrt{N} \left( \frac{\Phi^T \Phi}{N} - \Sigma^{-1} \right), \sqrt{N} \left( \frac{\Phi^T V}{N} \right), \sqrt{N} \left( \frac{V^T V}{N} - \sigma^2 \right) \right) \xrightarrow{d} \left( -\Sigma^{-1} \Gamma \Sigma^{-1}, \Sigma^{-1} \Gamma \nu, \rho \right), \tag{25}
\]

where \( \Gamma, \nu \) and \( \rho \) are defined as Theorem 1, and

\[
\hat{\sigma}^2 = \frac{\| Y - \Phi \hat{\theta}^{LS} \|^2_2}{N - n}. \tag{26}
\]

Moreover, the boundedness in probability of building blocks and the LS estimator can also be derived according to Theorem 1 and 2, from which we can observe their convergence rates. For the nonzero sequence \( \{a_N\} \), we let \( \xi_N = O_p(a_N) \) denote that \( \{\xi_N/a_N\} \) is bounded in probability, which means that \( \forall \epsilon > 0, \exists L > 0 \) such that \( \sup_N \text{Pr}(\|\xi_N/a_N\|_2 > L) < \epsilon. \)
Theorem 3: Under Assumption 1, 2 and 3, we have
\[ \Phi^T \Phi = O_p(N), \]  
\[ V^TV = O_p(N), \]  
\[ \frac{\Phi^T \Phi}{N} - \Sigma = O_p(1/\sqrt{N}), \]  
\[ \frac{\Phi^T V}{N} = O_p(1/\sqrt{N}), \]  
\[ \frac{V^TV}{N} - \sigma^2 = O_p(1/\sqrt{N}), \]  
where \( \sigma^2 \) is defined as (26).

III. BOUNDEDNESS OF MOMENTS OF SEVERAL TERMS

In this section, we show the boundedness of moments of several terms.

Assumption 4: The 8th moments of \( e(t) \) and \( v(t) \) are both bounded.

Assumption 5: The 16th moments of \( e(t) \) and \( v(t) \) are both bounded.

Theorem 4: There exists \( \tilde{M} > 0 \), which is irrepsasive of \( N \), such that under Assumptions 1-3, we have
\[ E \left( \left\| \frac{\Phi^T \Phi}{N} \right\|_2^4 \right) \leq \frac{1}{N^2} \tilde{M}. \]  

Theorem 5: There exists \( \tilde{M} > 0 \), which is irrepsasive of \( N \), such that under Assumptions 1-4, we have
\[ E \left( \left\| \frac{\Phi^T V}{N} \right\|_2^4 \right) \leq \frac{1}{N^4} \tilde{M}, \]  
\[ E \left( \left\| \frac{\Phi^T \Phi}{N} - \Sigma \right\|_F^4 \right) \leq \frac{1}{N^4} \tilde{M}, \]  
\[ E \left( \left\| V^TV - \sigma^2 \right\|_F^4 \right) \leq \frac{1}{N^4} \tilde{M}. \]

Theorem 6: There exists \( \tilde{M} > 0 \), which is irrepsasive of \( N \), such that under Assumptions 1-5, we have
\[ E \left( \left\| \frac{\Phi^T \Phi}{N} - \Sigma \right\|_F^8 \right) \leq \frac{1}{N^4} \tilde{M}, \]  
\[ E \left( \left\| V^TV - \sigma^2 \right\|_F^8 \right) \leq \frac{1}{N^4} \tilde{M}. \]

Remark 1: On the one hand, if Assumption 5 is true, then Assumption 4 must be true, because the boundedness of higher order moments always implies the boundedness of lower order moments. On the other hand if \( e(t) \) and \( v(t) \) are both Gaussian distributed, as long as their second moments are bounded, both Assumption 4 and 5 are true.

IV. PRELIMINARY RESULTS ABOUT REGULARIZED LEAST SQUARE ESTIMATOR FOR FIR MODEL

To handle the ill-conditioned problem, one can introduce a regularization term in (7a) to obtain the regularized least squares (RLS) estimator:
\[ \hat{\theta}_R = \arg \min_{\theta \in \mathbb{R}^n} \| Y - \Phi \theta \|^2 + \sigma^2 \theta^T P^{-1} \theta \]  
\[ = (\Phi^T \Phi + \sigma^2 P^{-1})^{-1} \Phi^T Y \]  
\[ = P \Phi^T Q^{-1} Y, \]  
where \( P \in \mathbb{R}^{n \times n} \) is positive semidefinite, its \( (i, j) \)th element \( P_{ij} \) can be designed through a positive semidefinite kernel \( \kappa(i,j; \eta) \colon \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R} \) with \( \eta \in \Omega \subset \mathbb{R}^p \) being the hyperparameter and thus \( P \) is often called the kernel matrix, and
\[ Q = P \Phi \Phi^T + \sigma^2 I_N, \]  
and \( I_N \) denotes the \( N \)-dimensional identity matrix. Moreover, we define that
\[ \hat{S}(\eta) = P(\eta) + \sigma^2 (\Phi^T \Phi)^{-1}. \]

Theorem 7: For the FIR model (5), under Assumption 1-3, we have the following results.
1) For any given \( \eta \in \mathbb{R}^p \), we have
\[ \sqrt{N} (\Phi^T \Phi)^{-1} \overset{a.s.}{\rightarrow} 0, \]  
\[ \hat{S}(\eta) \overset{a.s.}{\rightarrow} P(\eta)^{-1}, \]  
\[ \sqrt{N} (\hat{S}(\eta)^{-1} - P(\eta)^{-1}) \overset{a.s.}{\rightarrow} 0, \]  
\[ \frac{\partial \hat{S}(\eta)^{-1}}{\partial \eta_k} - \frac{\partial P(\eta)^{-1}}{\partial \eta_k} \overset{a.s.}{\rightarrow} 0, \]  
where \( \eta_k \) denotes the \( k \)th element of \( \eta \) and \( k = 1, \ldots, p \).
2) Suppose that as \( N \rightarrow \infty \), \( \eta \overset{a.s.}{\rightarrow} \eta^* \). If \( P(\eta) \) is continuous for every \( \eta \in \Omega \) and there exists a compact set \( \Omega \) containing \( \eta^* \) such that \( \| \hat{S}^{-1} \|_F < \| P^{-1} \|_F \) is bounded, we have as \( N \rightarrow \infty \),
\[ \hat{S}(\eta_N) \overset{a.s.}{\rightarrow} P(\eta^*)^{-1}. \]
3) Suppose that as \( N \rightarrow \infty \), \( \eta \overset{a.s.}{\rightarrow} \eta^* \). If \( P(\eta) \) is differentiable for every \( \eta \in \Omega \), then we have
\[ \hat{S}(\eta_N)^{-1} - P(\eta^*)^{-1}, \]  
\[ = \hat{S}(\eta_N)^{-1} \left[ \sum_{k=1}^p \frac{\partial P(\eta)}{\partial \eta_k} \right] \left( \eta_N - \eta^* \right) \]  
\[ = -\sigma^2 \hat{S}(\eta_N)^{-1} (\Phi^T \Phi)^{-1} P(\eta^*)^{-1}, \]  
where \( \epsilon_k \in \mathbb{R}^p \) denotes a column vector with \( k \)th element being one and others zero, and \( \eta_N \) belongs to a neighborhood of \( \eta^* \) with radius \( \| \eta_N - \eta^* \|_2 \). In particular, if we consider \( P = \eta I_\eta \), with \( \eta > 0 \), we have
\[ \hat{S}(\eta_N)^{-1} - P(\eta^*)^{-1}, \]  
\[ = (\eta_N - \eta^*) \hat{S}(\eta_N)^{-1} P(\eta^*)^{-1} \]  
\[ = \sigma^2 \hat{S}(\eta_N)^{-1} (\Phi^T \Phi)^{-1} P(\eta^*)^{-1}. \]
4) We have
\[ \hat{S}(\eta)^{-1} - P(\eta)^{-1} = -\frac{1}{N} \hat{S}(\eta)^{-1} N (\Phi^T \Phi)^{-1} P(\eta)^{-1}. \]
For $k, l = 1, \cdots, p$, we have
\[
\frac{\partial P(\eta)^{-1}}{\partial \eta_k} = - P^{-1} \frac{\partial P(\eta)}{\partial \eta_k} P^{-1},
\]
\[
\frac{\partial \hat{S}(\eta)^{-1}}{\partial \eta_k} = - \hat{S}(\eta)^{-1} \frac{\partial P(\eta)}{\partial \eta_k} \hat{S}(\eta)^{-1},
\]
\[
\frac{\partial^2 P(\eta)^{-1}}{\partial \eta_k \partial \eta_l} = \frac{\partial P(\eta)^{-1}}{\partial \eta_k} \frac{\partial P(\eta)^{-1}}{\partial \eta_l} P(\eta)^{-1} - P(\eta)^{-1} \frac{\partial^2 P(\eta)^{-1}}{\partial \eta_k \partial \eta_l} P(\eta)^{-1} + \frac{\partial P(\eta)^{-1}}{\partial \eta_l} \frac{\partial P(\eta)^{-1}}{\partial \eta_k} P(\eta)^{-1},
\]
\[
\frac{\partial^2 \hat{S}(\eta)^{-1}}{\partial \eta_k \partial \eta_l} = \hat{S}(\eta)^{-1} \frac{\partial^2 P(\eta)^{-1}}{\partial \eta_k \partial \eta_l} \hat{S}(\eta)^{-1} - \hat{S}(\eta)^{-1} \frac{\partial^2 P(\eta)^{-1}}{\partial \eta_k \partial \eta_l} \hat{S}(\eta)^{-1} + \hat{S}(\eta)^{-1} \frac{\partial P(\eta)^{-1}}{\partial \eta_l} \frac{\partial P(\eta)^{-1}}{\partial \eta_k} \hat{S}(\eta)^{-1}.
\]

V. CONCLUSION

For the FIR model with filtered white noise inputs, we mainly consider asymptotic properties of the LS estimator in terms of: the almost sure convergence, the convergence in distribution and the boundedness in probability. Moreover, some preliminary results of the RLS estimator are also contained. These properties help us to have a better understanding of the LS and RLS estimators. In addition, we can use these tools to investigate asymptotic properties of other estimators, such as hyper-parameter estimators.

APPENDIX A

All proofs of theorems and corollaries are included in Appendix A, and all required lemmas and their corresponding proofs are shown in Appendix B.

A.1. Proof of Theorem 1

1) Proof of (9)
It can be seen that for $i, j = 1, \cdots, n$, the $(i, j)$th element of $\Phi^T \Phi / N$ is
\[
\left[ \frac{\Phi^T \Phi}{N} \right]_{i,j} = \sum_{t=1-i}^{N-i} u(t)u(t+i-j).
\]
Using Lemma B.2 with
\[
x(t) = u(t), \ m(t) = 0, \ R_s(\tau) = R_u(\tau),
\]
we obtain that as $N \to \infty$,
\[
\frac{1}{N} \sum_{t=1}^{N} u(t)u(t-\tau) \overset{a.s.}{\to} R_u(\tau).
\]
Applying (A.3) to each element of $\Phi^T \Phi / N$, since the almost sure convergence of a matrix is equivalent to that of its all elements, it completes the proof of (9).

2) Proof of (10)
Since
\[
\frac{\Phi^T V}{N} = \left[ \begin{array}{c} \frac{1}{N} \sum_{t=0}^{N-1} u(t)v(t+1) \\ \frac{1}{N} \sum_{t=-1}^{N-2} u(t)u(t+2) \\ \vdots \\ \frac{1}{N} \sum_{t=-n+1}^{N-n} u(t)v(t+n) \end{array} \right],
\]
we can consider the almost sure convergence of the general form of elements. It means that as long as
\[
\frac{1}{N} \sum_{t=1}^{N} u(t)v(t+i) \overset{a.s.}{\to} 0
\]
with $i = 1, \cdots, n$ when $N \to \infty$, $\Phi^T V / N \overset{a.s.}{\to} 0$ holds.

Recall that
\[
\frac{1}{N} \sum_{t=1}^{N} u(t)v(t+i) = \frac{1}{N} \sum_{t=1}^{N} \sum_{k=0}^{\infty} h(k)e(t-k)v(t+i),
\]
Firstly, using Lemma B.25 with
\[
w_1(t) = u(t), \quad w_3(t) = v(t), \quad r = 1,
\]
we obtain that
\[
\mathbb{E} \left[ \sum_{t=1}^{N} u(t)v(t+i) \right]^2 \leq C_1 N,
\]
where $C_1 = [\sum_{k=0}^{\infty}|h(k)|^2 \sigma_x^2 \sigma_v^2$.
Then we apply Lemma B.27 with
\[
s(t) = u(t)v(t+i), \quad C_s = C_1,
\]
hence as $N \to \infty$, it yields that
\[
\frac{1}{N} \sum_{t=1}^{N} u(t)v(t+i) \overset{a.s.}{\to} 0,
\]
which can be extended to each element of $(\Phi^T V)/N$. It means that as $N \to \infty$, we have (10).

3) Proof of (11)
For
\[
\frac{V^T V}{N} = \frac{1}{N} \sum_{t=1}^{N} v^2(t),
\]
since for $t = 1, 2, \cdots, N$,
\[
\mathbb{E}[v^2(t)] = \mathbb{E}[v^2(t)] = \sigma_v^2 < \infty,
\]
we can derive (11) from Lemma B.14.

4) Proof of (12), (13), (14), (15), (16), (17) and (18)
It is equivalent to derive the convergence in distribution of the following vector
\[
S_N = \left[ \begin{array}{c} \text{vec}(\sqrt{N}(\Phi^T \Phi / N - \Sigma)) \\ \sqrt{N}(\Phi^T V / N - \sigma_v^2) \end{array} \right],
\]
The $k$th element of $S_N \in \mathbb{R}^{n^2+n+1}$ is
\[
\left\{ \begin{array}{ll}
\frac{1}{\sqrt{N}} \sum_{t=1}^{N} [u(t-i)u(t-j)] & \text{if } 1 \leq k \leq n^2, \\
-\frac{1}{\sqrt{N}} \sum_{t=1}^{N} [R_u([i-j])], & \\
\frac{1}{\sqrt{N}} \sum_{t=1}^{N} [u(t-k+n^2)v(t)], & \text{if } n^2 + 1 \leq k \leq n^2 + n, \\
\frac{1}{\sqrt{N}} \sum_{t=1}^{N} v^2(t), & \text{if } k = n^2 + n + 1.
\end{array} \right.
\]
where \( k = 1, 2, \ldots, n^2 + n + 1 \) and
\[
\begin{align*}
i &= \lfloor (k-1)/n \rfloor + 1 \quad \text{(A.14a)} \\
j &= k - \lfloor (k-1)/n \rfloor n. \quad \text{(A.14b)}
\end{align*}
\]

Here \( \text{vec}(\cdot) \) denotes the vectorization of a matrix and \( \lfloor \cdot \rfloor \) denotes the floor operation, i.e., \( \lfloor x \rfloor = \max\{\hat{x} \in \mathbb{Z} | \hat{x} \leq x \} \).

For a positive integer \( M \), let \( u(t) = u^M(t) + \bar{u}^M(t) \) with
\[
\begin{align*}
u^M(t) &= \sum_{k=0}^{M} h(k)e(t-k) \tag{A.15} \\
\bar{u}^M(t) &= \sum_{k=M+1}^{\infty} h(k)e(t-k). \tag{A.16}
\end{align*}
\]

Thus, we have
\[
S_N = Z_M(N) + X_M(N), \tag{A.17}
\]
where
\[
\begin{align*}
Z_M(N) &= \sum_{t=1}^{N} \kappa_{M,N}(t) \tag{A.18} \\
X_M(N) &= \frac{1}{\sqrt{N}} \tilde{\kappa}_M(N) \tag{A.19}
\end{align*}
\]
with their structure satisfying
\[
\begin{align*}
[k_{M,N}(t)]_k &= \begin{cases} \frac{1}{\sqrt{N}} \{ u^M(t-i)u^M(t-j) \}, & \text{if } 1 \leq k \leq n^2, \\
-\mathbb{E}\left[ u^M(t-i)u^M(t-j) \right], & \text{if } n^2 + 1 \leq k \leq n^2 + n, \\
\frac{1}{\sqrt{N}} \left[ u^2(t) - \sigma^2 \right], & \text{if } k = n^2 + n + 1,
\end{cases} \\
[\tilde{\kappa}_M(N)]_k &= \begin{cases} \sum_{t=1}^{N} \left\{ \bar{u}^M(t-i)u(t-j) \right\}, & \text{if } 1 \leq k \leq n^2, \\
+u^M(t-i)\bar{u}^M(t-j), & \text{if } n^2 + 1 \leq k \leq n^2 + n, \\
-\mathbb{E}\left[ \bar{u}^M(t-i)u(t-j) \right], & \text{if } k = n^2 + n + 1,
\end{cases}
\end{align*}
\]
where \( i \) and \( j \) are defined as (A.14). In the following part, we shall apply Lemma B.9 for \( S_N \).

– Firstly, we will show that as \( N \to \infty \),
\[
\begin{align*}
Z_M(N) &\xrightarrow{d} N(0, Q_M), \tag{A.22} \\
Q_M &= \lim_{N \to \infty} \mathbb{E}[Z_M(N)Z_M(N)^T]. \tag{A.23}
\end{align*}
\]
It can be seen that \( \mathbb{E}(\kappa_{M,N}(t)) = 0 \). And \( \kappa_{M,N}(t) \) is \( (M+n-1) \)-dependent, which means that
\[
\{\kappa_{M,N}(1), \ldots, \kappa_{M,N}(s)\} \text{ and } \{\kappa_{M,N}(t), \kappa_{M,N}(t+1), \ldots\}
\]
are independent if \( t-s > M+n-1 \), which can be derived from \( \max_{i,j} |i-j| = n-1 \).

Then for
\[
\begin{align*}
\mathbb{E}[u^M(t)|^{4+\delta} &\leq \mathbb{E} \left| \sum_{k=0}^{M} h(k)e(t-k) \right|^{4+\delta} \\
&\leq 2^{(3+\delta)M} \sum_{k=0}^{M} \mathbb{E}|h(k)e(t-k)|^{4+\delta}
\end{align*}
\]
[using Lemma B.6]
\[
\begin{align*}
&\leq 2^{(3+\delta)M} \sup_{t} \mathbb{E}|e(t)|^{4+\delta} \left[ \sum_{k=0}^{M} |h(k)| \right]^{4+\delta} \\
&\leq 2^{(3+\delta)M} \sup_{t} \mathbb{E}|e(t)|^{4+\delta} \left[ \sum_{k=0}^{\infty} |h(k)| \right]^{4+\delta}, \tag{A.25}
\end{align*}
\]
since \( \sum_{k=0}^{\infty} |h(k)| < \infty \) and \( \mathbb{E}|e(t)|^{4+\delta} < \infty \) for some \( \delta > 0 \), we know that \( u^M(t) \) also has bounded moments of order \( 4+\delta \) for some \( \delta > 0 \). Then we apply Lemma B.28 with
\[
\begin{align*}
* &\text{ when } 1 \leq k \leq n^2, \beta_k = u^M(t-i) \text{ and } \gamma_k = u^M(t-j); \\
* &\text{ when } n^2 + 1 \leq k \leq n^2 + n, \beta_k = u^M(t-k+n^2) \text{ and } \gamma_k = v(t); \\
* &\text{ when } k = n^2 + n + 1, \beta_k = v(t) \text{ and } \gamma_k = v(t)
\end{align*}
\]
to obtain that there exists a constant \( C_1 > 0 \) such that
\[
\mathbb{E}(\|\kappa_{M,N}(t)\|_2^{4+\delta}) \leq \frac{1}{N} N^{-\delta/2} C_1. \tag{A.26}
\]
It follows that
\[
\lim_{N \to \infty} \sup_{t} \sum_{i=1}^{N} \mathbb{E}(\|\kappa_{M,N}(t)\|_2) < \infty \tag{A.27}
\]
\[
\lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E}(\|\kappa_{M,N}(t)\|_2^{4+\delta}) = 0 \text{ for some } \delta > 0. \tag{A.28}
\]

Thus, we can apply Lemma B.8 with \( x_N(t) = \kappa_{M,N}(t) \) to yield as \( N \to \infty \),
\[
Z_M(N) \xrightarrow{d} N(0, Q_M), \tag{A.29} \\
Q_M = \lim_{N \to \infty} \mathbb{E}[Z_M(N)Z_M(N)^T]. \tag{A.30}
\]

– The second step is to show that there exists a constant \( C_M > 0 \) such that
\[
\mathbb{E}(\|X_M(N)\|_2^2) \leq C_M, \tag{A.31} \\
\lim_{M \to \infty} C_M = 0. \tag{A.32}
\]
Note that when \( k = n^2 + n + 1 \), the \( k \)th element of
\[ \tilde{\kappa}_M(N) \text{ is zero. Then for} \]
\[ E\|\tilde{\kappa}_M(N)\|^2 = \sum_{k=1}^{n^2} E \left\{ \sum_{t=1}^{N} \left[ \tilde{u}^M(t-i)u(t-j) + u^M(t-i)\tilde{u}^M(t-j) \right] + E[\tilde{u}^M(t-i)u(t-j) + u^M(t-i)\tilde{u}^M(t-j)] \right\}^2 \]
\[ + \sum_{k=n^2+1}^{n^2+n} E \left\{ \sum_{t=1}^{N} \tilde{u}^M(t-k+n^2)u(t) \right\}^2, \]
(A.33)
if we apply the adjusted version of Lemma B.25, there must exist a constant \( C_3 \) such that
\[ E\|\tilde{\kappa}_M(N)\|^2 \leq C_3 \left[ \sum_{k=M+1}^{\infty} |h(k)| \right]^2 N, \quad (A.34) \]
which gives that
\[ E\|X_M(N)\|^2 \leq C_M = C_3 \left[ \sum_{k=M+1}^{\infty} |h(k)| \right]^2. \quad (A.35) \]
In addition, since \( \sum_{k=0}^{\infty} |h(k)| \) is always finite, it can be seen that \( \sum_{k=0}^{N} |h(k)| \) is convergent, leading to
\[ \lim_{M \to \infty} C_M = \lim_{M \to \infty} C_3 \left[ \sum_{k=M+1}^{\infty} |h(k)| \right]^2 = 0. \quad (A.36) \]

Combining two steps above with (A.17), we can use Lemma B.9 to know that as \( N \to \infty \),
\[ S_N \xrightarrow{d} (0, Q_s), \quad (A.37) \]
\[ Q_s = \lim_{M \to \infty} Q_M = \lim_{M \to \infty} E[S_N S_N^T], \quad (A.38) \]
where the last step comes from \( \lim_{M \to \infty} \lim_{N \to \infty} E\|X_M(N)\|^2 = 0 \).
Hence, from (A.37), we can equivalently derive (12) as \( N \to \infty \). Moreover, from (A.38), we have (13) and the following results.

For (14), we have
\[ E(\nu \nu^T) = \lim_{N \to \infty} \frac{1}{N} E(\Phi^T V V^T \Phi) \]
\[ = \lim_{N \to \infty} \frac{\sigma^2 \Phi^T \Phi}{N} \]
\[ = \sigma^2 \Sigma. \quad (A.39) \]

For (16), we have
\[ E(\nu \otimes \Gamma) = \lim_{N \to \infty} N E \left\{ [\Phi^T V/N] \otimes [(\Phi^T \Phi)/N - \Sigma] \right\}. \quad (A.40) \]

Then for \( k_1, k_2, k_2 = 1, \ldots, n \), each element of \( (A.40) \) can be represented as follows,
\[ E \left\{ \left[ \frac{\phi_T V}{N} \right]_{k_1} \left[ \frac{\Phi^T \Phi}{N} - \Sigma \right]_{k_2, k_2} \right\} \]
\[ = E \left\{ \frac{1}{N^2} \sum_{t_1=1}^{N} u(t_1 - k_1) v(t_1) \right\} \]
\[ \sum_{t_2=1}^{N} [u(t_2 - k_2)u(t_2 - l_2) - R_u(|k_2 - l_2|)] \]
\[ = \frac{1}{N^2} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} E[v(t_1)] \]
\[ E[u(t_1 - k_1)] (u(t_2 - k_2)u(t_2 - l_2) - R_u(|k_2 - l_2|)] \]
(A.41)
where the last step comes from the mutual independence of \( \{v(t)\}_{t=1}^{N} \) and \( \{u(t)\}_{t=1}^{N-1} \). Since
\[ E(v(t_1)) = 0, \quad (A.42) \]
\[ E[u(t_1 - k_1)] (u(t_2 - k_2)u(t_2 - l_2) - R_u(|k_2 - l_2|)] \]
\[ = E[u(t_1 - k_1)]u(t_2 - k_2)u(t_2 - l_2)] \]
\[ - E[u(t_1 - k_1)]R_u(|k_2 - l_2|) \]
\[ = E \left[ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} h(m_1) h(m_2) h(m_3) \right] \]
\[ e(t_1 - k_1 - m_1)e(t_2 - k_2 - m_2)e(t_2 - l_2 - m_3) \]
\[ = \sup_e \left[ e^{\lambda}(t_1 - m_1)e(t_2 - k_2 - m_2)e(t_2 - l_2 - m_3) \right] \]
\[ \leq \sup_e \left[ e^{\lambda}(t_1 - m_1)e(t_2 - k_2 - m_2)e(t_2 - l_2 - m_3) \right] < \infty, \quad (A.43) \]
we obtain (16).

For (15), we have
\[ E(\nu \nu) \]
\[ = \lim_{N \to \infty} \frac{1}{N} E \left[ \sum_{t=1}^{N} (v^2(t) - \sigma^2) \right] \]
\[ = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \sum_{j=1}^{N} E \left[ (v^2(t) - \sigma^2)(v^2(j) - \sigma^2) \right] \]
\[ = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \{ E[v^4(t)] - 2\sigma^2 E[v^2(t)] + \sigma^4 \} \]
\[ \text{[since } v(t) \text{ and } v(j) \text{ are independent when } t \neq j \} \]
\[ = E[v^4(t)] - \sigma^4. \quad (A.44) \]

For (17), we have
\[ E(\nu \nu) \]
\[ = \lim_{N \to \infty} E \left[ \Phi^T V \left( V^T V/N - \sigma^2 \right) \right]. \quad (A.45) \]
The \( k \)th element of \( E(\nu \nu) \) with \( k = 1, 2, \ldots, n \) can
be represented as

\[
[E(\rho v)]_k = \lim_{N \to \infty} E \left\{ \sum_{i=1}^{N} u(t-k)v(t) \left[ \frac{1}{N} \sum_{j=1}^{N} v^2(j) - \sigma^2 \right] \right\}
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left[ \sum_{i=1}^{N} u(t-k)v(t)v^2(j) \right]
\]

\[
= \lim_{N \to \infty} \sum_{i=1}^{N} E \left[ u(t-k)v^3(t) \right]
= 0.
\]

(A.46)

○ For (18), we have

\[
E(\rho v) = \lim_{N \to \infty} \mathbb{E} \left[ \left( \frac{v^T v}{N} - \sigma^2 \right) \left( \frac{\Phi^T \Phi}{N} - \Sigma \right) \right]
= \lim_{N \to \infty} \mathbb{E} \left[ \frac{v^T v}{N} - \sigma^2 \right] \mathbb{E} \left( \frac{\Phi^T \Phi}{N} - \Sigma \right)
= 0,
\]

(A.47)

where the last second step derives from the independence between $\Phi$ and $V$.

5) Proof of (19)
Combining (4a), (4b) and (3), for $i, j = 1, \cdots, n$, it can be derived that

\[
[\Sigma]_{i,j} = \mathbb{E} [u(i-1)u(j-1)] = R_u(|i-j|).
\]

(A.48)

6) Proof of (20)
For $i, j = 1, \cdots, n^2$, the $(i, j)\text{th}$ element of $C_T$ is

\[
[C_T]_{i,j} = \lim_{N \to \infty} \mathbb{E} \left[ \left( \frac{\Phi^T \Phi}{N} - \Sigma \right)_{k_1,l_1} \left( \frac{\Phi^T \Phi}{N} - \Sigma \right)_{k_2,l_2} \right]
= \lim_{N \to \infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{t_1=1}^{N} u(t_1-k_1)u(t_1-l_1) - R_u(|k_1-l_1|) \right\}
\]

\[
\left\{ \frac{1}{N} \sum_{t_2=1}^{N} u(t_2-k_2)u(t_2-l_2) - R_u(|k_2-l_2|) \right\}
= \lim_{N \to \infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{t_1=1}^{N} u(t_1)u(t_1+|k_1-l_1|) - R_u(|k_1-l_1|) \right\}
\]

\[
\left\{ \frac{1}{N} \sum_{t_2=1}^{N} u(t_2)u(t_2+|k_2-l_2|) - R_u(|k_2-l_2|) \right\},
\]

(A.49)

where $k_1$, $k_2$, $l_1$ and $l_2$ satisfy

\[
k_1 = [(i-1)/n] + 1
\]

(A.50)

\[
l_1 = [(j-1)/n] + 1
\]

(A.51)

\[
k_2 = i - [(i-1)/n] n
\]

(A.52)

\[
l_2 = j - [(j-1)/n] n.
\]

(A.53)

For convenience, define

\[
k = |k_1 - l_1|,
\]

(A.54a)

\[
l = |k_2 - l_2|.
\]

(A.54b)

Then applying Lemma B.18, we can derive (20).

A.2. Proof of Theorem 2

1) Proof of (22)
Applying Lemma B.11 to (9), we can obtain (22).

2) Proof of (23)
Combining (9) with (10), we can apply Lemma B.12 and B.13 to (8) to obtain (23).

3) Proof of (24)
We have

\[
\left\| Y - \hat{\Phi}^L S \right\|^2 \quad \frac{N-n}{N-n} = \left\| \Phi(\theta_0 - \hat{\theta}^L S) + V \right\|^2
\]

\[
= - \frac{N}{N-n} (\hat{\theta}^L S - \theta_0)^T \Phi^T V + \frac{N}{N-n} \frac{N}{N-n} V^T V.
\]

(A.55)

Applying (9), (10), (11), (23) and Lemma B.11, it leads to (24).

4) Proof of (25)
Firstly, based on (8), (9) and Lemma B.12 and B.11, we can obtain as $N \to \infty$,

\[
\sqrt{N}(\hat{\theta}^L S - \theta_0) \xrightarrow{d} \Sigma^{-1} v.
\]

(A.56)

Then, we consider

\[
\sqrt{N} \left( \hat{\sigma}^2 - \sigma^2 \right)
\]

\[
= \sqrt{N} \left( \left\| Y - \hat{\Phi}^L S \right\|^2 \frac{N-n}{N-n} - \sigma^2 \right)
\]

\[
= - \frac{N}{N-n} \sqrt{N}(\hat{\theta}^L S - \theta_0)^T \Phi^T V
\]

\[
+ \frac{N}{N-n} \sqrt{N} \left( V^T V - \sigma^2 \right) + \frac{n\sigma^2}{N-n}.
\]

(A.57)

Due to (A.56), (9), (10) and (12) and Lemma B.12 and B.11, it follows that as $N \to \infty$,

\[
- \frac{N}{N-n} \sqrt{N}(\hat{\theta}^L S - \theta_0)^T \Phi^T V + \sqrt{N} n \sigma^2 / (N-n) \xrightarrow{d} 0.
\]

(A.58)

Finally, combining (A.56) and (A.55), we can apply (A.58) and Lemma B.11 to derive (25).

A.3. Proof of Theorem 3

Based on Lemma B.10 and B.12, we can derive Theorem 3 from Theorem 1 and 2.
A.4. Proof of Theorems 4, 5 and 6

For \( \mathbb{E}(\|\Phi^T V/N\|^2_2) \), there exists \( \hat{M}_1 > 0 \), irrespective of \( N \), such that

\[
\mathbb{E}\left( \frac{\|\Phi^T \Phi/N - \Sigma\|^2}{N} \right) = \frac{1}{N^4} \mathbb{E} \left\{ \sum_{i=1}^{n} \left( \sum_{t=1}^{N} u(t) v(t+i) \right)^2 \right\} \\
\leq \frac{1}{N^2} 2^{n-1} \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ \sum_{t=1}^{N} u(t) v(t+i) \right]^2 \right\}
\]

[using Lemma B.6]

\[
\leq \frac{1}{N^2} \hat{M}_1,
\]

(A.59)

where the last step is derived from Assumptions 1-3 and Lemma B.26 with \( m = 4 \). Similarly, under Assumption 1-4, we can obtain (35) using Lemma B.26 with \( m = 8 \).

For \( \mathbb{E}(\|\Phi^T \Phi/N - \Sigma\|^2_2) \), there exists \( \hat{M}_2 > 0 \), irrespective of \( N \), such that

\[
\mathbb{E}\left( \frac{\|\Phi^T \Phi/N - \Sigma\|^2}{N} \right) \leq \mathbb{E} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{1}{N} \sum_{t=1}^{N} u(t-i) u(t-j) \right]^2 \right\}
\]

\[
- \mathbb{E}(u(t-i) u(t-j)) \right\}^2 \right\} \\
\leq \frac{1}{N^2} 2^{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left\{ \sum_{t=1}^{N} u(t-i) u(t-j) \right\}
\]

\[
- \mathbb{E}(u(t-i) u(t-j)) \right\}^4 \right\}
\]

\[
\leq \frac{1}{N^2} \hat{M}_2,
\]

(A.60)

where the second step is derived using Lemma B.6 and the last step we apply Assumptions 1, 4 and Lemma B.26 with \( m = 4 \). Similarly, we can obtain (38) using Assumptions 1, 5 and Lemma B.26 with \( m = 8 \).

At the same time, there exists \( \hat{M}_4 > 0 \), irrespective of \( N \), such that

\[
\mathbb{E}\left( \frac{V^T V}{N} - \sigma_i^2 \right)^4 = \frac{1}{N^4} \mathbb{E} \left\{ \sum_{t=1}^{N} \left[ v^2(t) - \sigma^2 \right]^4 \right\}
\]

\[
\leq \frac{1}{N^2} \hat{M}_4,
\]

(A.61)

where we apply Assumptions 2, 4 and Lemma B.26 with \( m = 4 \). Similarly, we can obtain (39) using Assumption 2, 5 and Lemma B.26 with \( m = 8 \).

A.5. Proof of Theorem 7

1) For (43), we can apply (9) and Lemma B.11.
2) Proof of (44), (45) and (51)

For

\[
\hat{S}^{-1} = [P + \widehat{\sigma}^2(\Phi^T \Phi)^{-1}]^{-1}.
\]

(A.62)

using (9), (24) and Lemma B.11, we can derive (44) as \( N \rightarrow \infty \). The derivation of (51) is straightforward. Then for (51), we can apply (9), (24), (44), Lemma B.11 and B.13 to derive (45).

3) Proof of (52), (53), (46) and (47)

Using the basic identity of the inverse matrix derivative [3, Page 9, Section 2.2, (59)], we can derive (52) and (53).

Then from (52), we have

\[
\frac{\partial \hat{S}^{-1}}{\partial \eta_k} - \frac{\partial P^{-1}}{\partial \eta_k} = (P^{-1} - \hat{S}^{-1}) \frac{\partial P}{\partial \eta_k} P^{-1} + \hat{S}^{-1} \frac{\partial P}{\partial \eta_k} (P^{-1} - \hat{S}^{-1}),
\]

(A.63)

which derives (46) as \( N \rightarrow \infty \) based on Lemma B.12, B.13 and (44).

Furthermore, combining (51) with (A.63), we can derive (47) using (9), (24), (44), Lemma B.11 and B.13.

4) Proof of (48)

It can be known that there exists a constant \( M > 0 \) such that \( \|\hat{S}^{-1}\|_F < \|P^{-1}\|_F \leq M \) for any \( \eta \in \Omega \). Then we can obtain

\[
\sup_{\eta \in \Omega} |\hat{S}^{-1} - P^{-1}| = \sup_{\eta \in \Omega} |\hat{S}^{-1}(\hat{S} - P)P^{-1}| \leq \sigma^2 M^2 \frac{1}{N} \|N(\Phi^T \Phi)^{-1}\|_F \rightarrow \frac{\sigma^2}{M^2} 0,
\]

(A.64)

as \( N \rightarrow \infty \), where the last step uses (9), (24), Lemma B.11 and B.13. It means that \( \hat{S}^{-1} \) converges to \( P^{-1} \) almost surely and uniformly in \( \Omega \). Further applying Lemma B.17 with \( \eta \rightarrow \eta^* \) as \( N \rightarrow \infty \), we have (48).

5) Proof of (49) and (50)

According to (42), we have

\[
\hat{S}^{-1}(\hat{\eta}_N) - P^{-1}(\eta^*) = - \hat{S}^{-1}(\hat{\eta}_N) \left[ \hat{S}(\hat{\eta}_N) - P(\eta^*) \right] P^{-1}(\eta^*)
\]

\[
= - \hat{S}^{-1}(\hat{\eta}_N) [P(\hat{\eta}_N) - P(\eta^*)] P^{-1}(\eta^*)
\]

\[
- \sigma^2 \hat{S}^{-1}(\hat{\eta}_N)(\Phi^T \Phi)^{-1} P^{-1}(\eta^*).
\]

(A.65)

Then we apply the first-order Taylor expansion of \( P(\hat{\eta}_N) \)

\[
P(\hat{\eta}_N) = P(\eta^*) + \sum_{k=1}^{p} \frac{\partial P(\eta)}{\partial \eta_k} \bigg|_{\eta = \eta_N, k} (\hat{\eta}_N - \eta_k^*)
\]

\[
= P(\eta^*) + \sum_{k=1}^{p} \frac{\partial P(\eta)}{\partial \eta_k} \bigg|_{\eta = \eta_N} v_k^T (\hat{\eta}_N - \eta_k^*),
\]

(A.66)

where we use the Lagrange’s form of the remainder term, \( \hat{\eta}_N, k \) and \( \eta_k^* \) denote the \( k \)th element of \( \hat{\eta}_N \) and \( \eta^* \), respectively. Inserting (A.66) into (A.65), we can obtain (49).

In particular, for \( P = \eta I_n \), the derivation of (50) is straightforward.
APPENDIX B

Fundamental lemmas and some preliminary results with their proofs are shown in Appendix B.

B.1. Matrix Norm Inequalities

Lemma B.1: ([3] Chapter 10.3 Page 61-62, [4] Page 68-72) For $a \in \mathbb{R}^{m \times 1}$, $b \in \mathbb{R}^{m \times 1}$, $B \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{m \times m}$, we have

$$
\|a + b\|_2 \leq \|a\|_2 + \|b\|_2, \quad (B.1)
$$

$$
|a^T b| \leq \|a\|_2 \|b\|_2, \quad (B.2)
$$

$$
\|BB^T\|_2 \leq \|B\|_F \|B\|_F, \quad (B.3)
$$

$$
\|BC\|_F \leq \|B\|_F \|C\|_F, \quad (B.4)
$$

$$
\|B + C\|_F \leq \|B\|_F + \|C\|_F, \quad (B.5)
$$

$$
\|\text{Tr}(B)\| \leq \sqrt{m} \|B\|_F, \quad (B.6)
$$

Proof. Suppose that $a_i$ and $b_i$ are $i$th elements of $a$ and $b$, respectively. And let $b_{ij}$ denote the $(i, j)$th element of $B$. For (B.2), it can be derived from

$$
|a^T b| = \sum_{i=1}^{m} a_i b_i \leq \sqrt{\sum_{i=1}^{m} a_i^2 \sum_{j=1}^{m} b_j^2} = \|a\|_2 \|b\|_2, \quad (B.7)
$$

which uses Cauchy-Schwarz inequality.

For (B.3), we have

$$
\|BB^T\|_2 \leq \|B\|_F \|B\|_F \quad [\text{using properties of induced norm}],
$$

$$
\leq \|B\|_F \|B\|_F, \quad (B.8)
$$

where the last step comes from $\|B\|_2 \leq \|B\|_F$ shown in Section 10.4.4 in [3].

For (B.6), it can be seen that

$$
|\text{Tr}(B)| = \sum_{i=1}^{m} |B_{i,i}| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} |B_{i,j}|^2} \quad [\text{using Cauchy-Schwarz inequality}],
$$

$$
\leq \sqrt{m} \|B\|_F, \quad (B.9)
$$

B.2. Ergodic Theory

Lemma B.2: ([5] Page 43 Theorem 2.3) Let $\{s(t)\}$ be a stationary stochastic process with

$$
\mathbb{E}[s(t)] = m(t), \quad (B.10)
$$

$$
R_s(\tau) = \mathbb{E}[s(t)s(t - \tau)], \quad (B.11)
$$

Assume that

$$
s(t) - m(t) = x(t) = \sum_{k=0}^{\infty} h(k)e(t - k) = H(q)e(t), \quad (B.12)
$$

where $\{e(t)\}$ is a sequence of independent random variables with zero mean values, $\mathbb{E}[e^2(t)] = \lambda_t$ and bounded fourth moments, and $H(q)$ is a stable filter, i.e. $\sum_{k=0}^{\infty} |h(k)| < \infty$.

Then, as $N \to \infty$,

$$
\frac{1}{N} \sum_{t=1}^{N} s(t)s(t - \tau) \xrightarrow{a.s.} R_s(\tau), \quad (B.13)
$$

$$
\frac{1}{N} \sum_{t=1}^{N} \{s(t)m(t - \tau) - \mathbb{E}[s(t)m(t - \tau)]\} \xrightarrow{a.s.} 0. \quad (B.14)
$$

This lemma is a special case of Theorem 2.3 in [5].

B.3. Markov’s Inequality

Lemma B.3: ([6] Page 120 Theorem 1.1) For a random variable $X \in \mathbb{R}$, suppose that $\mathbb{E}|X|^r < \infty$ for some $r > 0$. Then for any $\epsilon > 0$,

$$
\Pr(|X| > \epsilon) \leq \frac{\mathbb{E}|X|^r}{\epsilon^r}. \quad (B.15)
$$

In particular, when $r = 2$, it is also known as the Chebyshev’s inequality.

B.4. Borel-Cantelli’s Lemma

Lemma B.4: ([5] Page 542 (I.18)) For the random variable $X_N \in \mathbb{R}$, $\forall \epsilon > 0$, if

$$
\sum_{N=1}^{\infty} \Pr(|X_N| > \epsilon) < \infty, \quad (B.16)
$$

then as $N \to \infty$, we have

$$
X_N \xrightarrow{a.s.} 0. \quad (B.17)
$$

B.5. Cauchy-Schwarz Inequality in Probability Theory

Lemma B.5: ([6] Page 130 Theorem 3.1) Suppose that $X_1 \in \mathbb{R}$ and $X_2 \in \mathbb{R}$ are random variables with finite variances. Then we have

$$
|\mathbb{E}(X_1 X_2)| \leq \mathbb{E}|X_1 X_2| \leq \sqrt{\mathbb{E}(X_1^2)\mathbb{E}(X_2^2)}. \quad (B.18)
$$

B.6. The C_r Inequality

Lemma B.6: ([6] Page 127 Theorem 2.2) For random variables $X_1 \in \mathbb{R}$ and $X_2 \in \mathbb{R}$, suppose that $\mathbb{E}|X_1|^r < \infty$ and $\mathbb{E}|X_2|^r < \infty$ for $r > 0$. Then we have

$$
\mathbb{E}|X_1 + X_2|^r \leq c_r (\mathbb{E}|X_1|^r + \mathbb{E}|X_2|^r), \quad (B.19)
$$

where $c_r = 1$ when $r \leq 1$ and $c_r = 2^{r-1}$ when $r \geq 1$.

B.7. Lyapunov Inequality

Lemma B.7: ([6] Page 127 Theorem 2.5) For the random variable $X \in \mathbb{R}$ and $0 < r \leq p$, we have

$$
(\mathbb{E}|X|^r)^{1/r} \leq (\mathbb{E}|X|^p)^{1/p}. \quad (B.20)
$$
B.8. Center Limit Theorem (CLT) for M-dependent Sequence

**Lemma B.8:** ([5] Lemma 9.A1, [7]) Consider the sum of doubly indexed random vectors \( \{x_N(t)\}_{t=1}^N \):

\[
Z_N = \sum_{t=1}^N x_N(t),
\]

(B.21)

where \( \mathbb{E}[x_N(t)] = 0 \). Suppose that \( \{x_N(t)\} \) is \( M \)-dependent for an integer \( M \), i.e., \( \{x_N(1), x_N(2), \ldots, x_N(s)\} \) and \( \{x_N(t), x_N(t+1), \ldots, x_N(n)\} \) are independent if \( t-s > M \). In addition, assume that

\[
\limsup_{N \to \infty} \sum_{k=1}^N \mathbb{E}[x_N(k)]^2 < \infty,
\]

(B.22)

\[
\lim_{N \to \infty} \sum_{k=1}^N \mathbb{E}[x_N(k)]^2 + \delta = 0 \text{ for some } \delta > 0.
\]

(B.23)

Let

\[
Q_z = \lim_{N \to \infty} \mathbb{E}(Z_NZ_N^T).
\]

(B.24)

Thus we have as \( N \to \infty \),

\[
Z_N(t) \overset{d}{\to} N(0,Q_z).
\]

(B.25)

B.9. Limiting Theorem of Sum

**Lemma B.9:** ([5] Lemma 9.A2, [8]) Let

\[
S_N = Z_M(N) + X_M(N), \quad M, N = 1,2, \ldots
\]

such that

\[
\mathbb{E}||X_M(N)||_2^2 \leq C_M, \quad \lim_{M \to \infty} C_M = 0
\]

(B.27a)

\[
\mathbb{P}(Z_M(N) \leq z) = F_{M,N}(z)
\]

(B.27b)

\[
\lim_{M \to \infty} F_{M,N}(z) = F_M(z)
\]

(B.27c)

\[
\lim_{M \to \infty} F_M(z) = F(z).
\]

(B.27d)

Then we have

\[
\lim_{N \to \infty} \mathbb{P}(S_N \leq z) = F(z).
\]

(B.28)

B.10. Convergence in Distribution and Boundedness in Probability

**Lemma B.10:** ([9] Lemma 3.8, [10] Page 8 Theorem 2.4) Let \( X_N \in \mathbb{R} \) be a random variable satisfying as \( N \to \infty \)

\[
X_N \overset{d}{\to} X.
\]

(B.29)

Then \( X_N \) is uniformly tight, which is also known to be bounded in probability, i.e., \( X_N = O_p(1) \).

B.11. Continuous Mapping Theorem

**Lemma B.11:** ([10] Theorem 2.3) Let \( g: \mathbb{R}^k \to \mathbb{R}^m \) be continuous at every point of a set \( D \) such that \( \mathbb{P}(X \in D) = 1 \).

1) If \( X_N \overset{d}{\to} X \) as \( N \to \infty \), then \( g(X_N) \overset{d}{\to} g(X) \);
2) If \( X_N \overset{p}{\to} X \) as \( N \to \infty \), then \( g(X_N) \overset{p}{\to} g(X) \);
3) If \( X_N \overset{a.s.}{\to} X \) as \( N \to \infty \), then \( g(X_N) \overset{a.s.}{\to} g(X) \).

B.12. Relationships between modes of convergence

**Lemma B.12:** ([10] Theorem 2.7) Let \( X_N, X \) and \( Z_N \) be random vectors. Then as \( N \to \infty \),

1) \( X_N \overset{a.s.}{\to} X \) implies \( X_N \overset{d}{\to} X \);
2) \( X_N \overset{p}{\to} X \) implies \( X_N \overset{d}{\to} X \);
3) \( X_N \overset{p}{\to} c \) for a constant \( c \) if and only if \( X_N \overset{d}{\to} c \).

B.13. Slutsky Theorem

**Lemma B.13:** ([10] Theorem 2.8) Let \( X_N, X \) and \( Y_N \) be random vectors or variables. If as \( N \to \infty \), \( X_N \overset{d}{\to} X \) and \( Y_N \overset{d}{\to} c \) for a constant \( c \), then

1) \( X_N + Y_N \overset{d}{\to} X + c \);
2) \( X_N Y_N \overset{d}{\to} cX \);
3) \( Y_N^{-1}X_N \overset{d}{\to} c^{-1}X \) provided \( c \neq 0 \).

B.14. The Kolmogorov Strong Law

**Lemma B.14:** ([6], Page 295, Theorem 6.1(a)) For the random sequence \( \{X_i\}_{i=1}^N \) with \( X_i \in \mathbb{R} \) for \( i = 1,2, \ldots, N \), if \( \mathbb{E}|X_i| < \infty \) and \( \mathbb{E}(X_i) = \mu \), as \( N \to \infty \), we have

\[
\frac{1}{N} \sum_{i=1}^N X_i \overset{a.s.}{\to} \mu.
\]

(B.30)

B.15. Squeeze Theorem for Convergence in Distribution

**Lemma B.15:** Let \( X_N \in \mathbb{R}, Y_{N,1} \in \mathbb{R}, Y_{N,2} \in \mathbb{R} \) and \( X \in \mathbb{R} \) be random. If as \( N \to \infty \), \( Y_{N,1} \overset{d}{\to} X, Y_{N,2} \overset{d}{\to} X \) and \( X_N \) satisfies \( Y_{N,1} \leq X_N \leq Y_{N,2} \), we have

\[
X_N \overset{d}{\to} X,
\]

(B.31)

as \( N \to \infty \).

**Proof.** According to the definition of the convergence in distribution, we can know that

\[
\lim_{N \to \infty} \mathbb{P}(Y_{N,1} \leq x) = \mathbb{P}(X \leq x)
\]

(B.32)

\[
\lim_{N \to \infty} \mathbb{P}(Y_{N,2} \leq x) = \mathbb{P}(X \leq x),
\]

(B.33)

for every \( x \) at which the distribution function \( F_X : x \mapsto \mathbb{P}(X \leq x) \) is continuous. Meanwhile, \( Y_{N,1} \leq X_N \leq Y_{N,2} \) derives

\[
\mathbb{P}(Y_{N,1} \leq x) \leq \mathbb{P}(X_N \leq x) \leq \mathbb{P}(Y_{N,2} \leq x).
\]

(B.34)

Then based on the standard Squeeze theorem ([11, Page 104, Theorem 3.3.6]), we have \( X_N \overset{d}{\to} X \) as \( N \to \infty \).

B.16. Inequalities of Determinant and Trace

**Lemma B.16:** For a positive definite matrix \( A \in \mathbb{R}^{m \times m} \), we have

\[
\text{Tr}(I_n - A^{-1}) \leq \log \det(A) \leq \text{Tr}(A - I_n).
\]

(B.35)

**Proof.** Our first step is to show that for any \( x > 0 \),

\[
f_1(x) = 1 - \frac{1}{x} - \log(x) \leq 0
\]

(B.36)

\[
f_2(x) = \log(x) - x + 1 \leq 0.
\]

(B.37)
The first order derivatives of $f_1(x)$ and $f_2(x)$ with respect to $x$, respectively are
\[
\frac{df_1(x)}{dx} = \frac{1}{x^2}(1 - x) = 0 \Rightarrow x = 1, \quad (B.38)
\]
\[
\frac{df_2(x)}{dx} = \frac{1}{x}(1 - x) = 0 \Rightarrow x = 1, \quad (B.39)
\]
which derives that $\forall x > 0$, $f_1(x) \leq f_1(1) = 0$ and $f_2(x) \leq f_2(1) = 0$.

Then we rewrite (B.35) using eigenvalues of $A$:
\[
\text{Tr}(I_n - A^{-1}) - \log \det(A) = \sum_{i=1}^{m} \left[ 1 - \frac{1}{\lambda_i(A)} - \log(\lambda_i(A)) \right],
\]
\[
\log \det(A) - \text{Tr}(A - I_n) = \sum_{i=1}^{m} [\log(\lambda_i(A)) - \lambda_i(A) + 1].
\]
Since $A$ is positive definite, we have $\lambda_i(A) > 0$. Then applying $f_1(\lambda_i(A)) \leq 0$ and $f_2(\lambda_i(A)) \leq 0$ can lead to (B.35). \(\blacksquare\)

**B.17. Almost Sure Convergence of Convergent Function at Convergent Estimate**

**Lemma B.17**: Suppose that as $N \to \infty$, $M_N(\eta)$ converges almost surely to a non-stochastic function $M(\eta)$ uniformly in a compact set $D$ containing $\eta^*$ and $\hat{\eta}$. If $\hat{\eta} \xrightarrow{a.s.} \eta^*$ as $N \to \infty$, and $M(\eta)$ is continuous in $D$, we have as $N \to \infty$,
\[
M_N(\hat{\eta}) \xrightarrow{a.s.} M(\eta^*). \quad (B.40)
\]

**Proof**: It can be known that
\[
|M_N(\hat{\eta}) - M(\eta^*)| = |M_N(\hat{\eta}) - M(\hat{\eta}) + M(\hat{\eta}) - M(\eta^*)| \\
\leq |M_N(\hat{\eta}) - M(\hat{\eta})| + |M(\hat{\eta}) - M(\eta^*)|,
\]
where we need to consider two terms $|M_N(\hat{\eta}) - M(\hat{\eta})|$ and $|M(\hat{\eta}) - M(\eta^*)|$, respectively.
For the first term, since $M_N(\eta)$ converges almost surely to a non-stochastic function $M_\eta$ uniformly in a compact set $D$, we have
\[
|M_N(\hat{\eta}) - M(\hat{\eta})| \leq \sup_{\eta \in D} |M_N(\eta) - M(\eta)| \xrightarrow{a.s.} 0, \quad (B.41)
\]
which comes from the almost surely uniform convergence of $M_N(\eta)$ to $M_\eta$ in $D$.

For the second term, since $M(\eta)$ is continuous and $\hat{\eta} \xrightarrow{a.s.} \eta^*$, we can apply Lemma B.11 to obtain as $N \to \infty$,
\[
|M(\hat{\eta}) - M(\eta^*)| \xrightarrow{a.s.} 0. \quad (B.42)
\]
Consequently, as $N \to \infty$, $M_N(\hat{\eta}) \xrightarrow{a.s.} M(\eta^*)$. \(\blacksquare\)

**B.18. Moments of Filtered White Noise**

**Lemma B.18**: ([12, (A.48)-(A.52)]) For a stationary process with $t = 1, 2, \cdots$,
\[
x(t) = \sum_{k=0}^{\infty} h(k)w(t-k), \quad (B.43)
\]
where $w(t)$ is independent and identically distributed (i.i.d.) with zero mean, variance $\sigma_w^2$, and fourth moment
\[
E[w^4(t)] = \varepsilon \sigma_w^4 < \infty, \quad \varepsilon \in \mathbb{R} \text{ is some constant}, \quad (B.44)
\]
and $\sum_{k=0}^{\infty} |h(k)| < \infty$, for fixed $\tau, \tau' \in \mathbb{Z}$, if we define the covariance function
\[
\gamma(\tau) = E[x(t)x(t + \tau)] \text{ with } \gamma(-\tau) = \gamma(\tau), \quad (B.45)
\]
and its sample estimator
\[
\hat{\gamma}(\tau) = \frac{1}{N} \sum_{t=1}^{N} x(t)x(t + \tau), \quad (B.46)
\]
then we have
\[
\gamma(\tau) = \sigma^2_n \sum_{k=0}^{\infty} h(k)h(k + |\tau|), \quad \quad \lim_{N \to \infty} N[\hat{\gamma}(\tau) - \gamma(\tau)] = \gamma(\tau) - \gamma(\tau'), \quad (B.47)
\]
(proof is similar to Section 2.2). If $\gamma(\tau) = 0$, we can obtain (B.51).

**B.19. Relationship of Convergence in Distribution and Convergence of Moments**

**Lemma B.19**: (13, Page 100 Theorem 4.5.2) For random $x_N \in \mathbb{R}$ and $x \in \mathbb{R}$, if $x_N \xrightarrow{d} x$ as $N \to \infty$, and for some $p > 0$, $\exists M > 0$ such that
\[
\sup_{N} E[|x_N|^p] \leq M, \quad (B.48)
\]
then for each $0 < r < p$ and $r \in \mathbb{N}$, we have
\[
\lim_{N \to \infty} E(x_N^r) = E(x^r). \quad (B.49)
\]
Moreover, Lemma B.19 can be expended to the multidimensional case to obtain the following results.

**Lemma B.20**: For random $X_N \in \mathbb{R}^{m \times 1}$ and $X \in \mathbb{R}^{m \times 1}$, if $X_N \xrightarrow{d} X$ as $N \to \infty$, and $\exists M > 0$, where $M$ is irrespective of $N$, such that
\[
\sup_{N} E\|X_N\|_2^2 \leq M, \quad (B.50)
\]
then we have
\[
\lim_{N \to \infty} E\|X_N\|_2^2 = E\|X\|_2^2. \quad (B.51)
\]

**Proof**: Define that the $i$th elements of $X_N$ and $X$ with $i = 1, \cdots, m$ are $X_{N,i}$ and $X_i$, respectively. Since $\exists M_1 > 0$ such that
\[
\sup_{N} E\|X_N\|_2^2 = \sup_{N} E\left(\sum_{i=1}^{m} X_{N,i}^2\right) \leq M_1, \quad (B.52)
\]
we can see that for each $i = 1, \cdots, m$,
\[
\sup_{N} E(X_{N,i}^2) \leq \sup_{N} E\|X_N\|_2^2 \leq M_1. \quad (B.53)
\]
At the same time, since as $N \to \infty$, $X_N \xrightarrow{d} X$, we have as $N \to \infty$, for $i = 1, \cdots, m$,
\[
X_{N,i} \xrightarrow{d} X_i. \quad (B.54)
\]
Then applying Lemma B.19 with $x_N = X_{N,i}$, $x = X_i$, $p = 2$, $r = 1$, $i = 1, \cdots, m$, we can obtain (B.51). \(\blacksquare\)
Lemma B.21: For random \( X_N \in \mathbb{R}^{m \times 1} \) and \( X \in \mathbb{R}^{m \times 1} \), if \( X_N \overset{d}{\to} X \) as \( N \to \infty \), and \( \exists M > 0 \), where \( M \) is irrespective of \( N \), such that
\[
\sup_N \mathbb{E}\|X_N\|_2^2 \leq M, \tag{B.56}
\]
then we have
\[
\lim_{N \to \infty} \mathbb{E}(X_N X_N^T) = \mathbb{E}(X X^T). \tag{B.57}
\]

Proof. Define that the \( i \)th elements of \( X_N \) and \( X \) with \( i = 1, \cdots, m \) are \( X_{N,i} \) and \( X_i \), respectively. It can be seen that the derivation of (B.57) is equivalent to showing that for every \( i, j = 1, \cdots, m \),
\[
\lim_{N \to \infty} \mathbb{E}(X_{N,i}X_{N,j}) = \mathbb{E}(X_iX_j). \tag{B.58}
\]
Since
\[
\sup_N \mathbb{E}\|X_N\|_2^2 = \sup_N \mathbb{E} \left[ \left( \sum_{i=1}^{m} X_{N,i}^2 \right)^2 \right] \tag{B.59}
\]
is bounded, it can be derived that for each \( i = 1, \cdots, m \),
\[
\sup_N \mathbb{E} \left( X_{N,i}^4 \right) \leq \sup_N \mathbb{E}\|X_N\|_2^2 \leq M_2, \tag{B.60}
\]
which leads to that for each \( i, j = 1, \cdots, m \),
\[
\sup_N \mathbb{E}(|X_{N,i}X_{N,j}|^2) \leq \sup_N \sqrt{\mathbb{E}(X_{N,i}^4)\mathbb{E}(X_{N,j}^4)} \\leq M_2. \tag{B.61}
\]

Then applying (B.61) and Lemma B.19 with
\[
x_N = X_{N,i}X_{N,j}, \quad x = X_iX_j, \quad p = 2, \quad r = 1, \quad i, j = 1, \cdots, m, \tag{B.62}
\]
we can obtain (B.58), which leads to (B.57).

B.20. Relationship of Essentially Boundedness and Boundedness of Moments

Lemma B.22: For the random variable \( X_N > 0 \), if \( X_N \) is essentially bounded, i.e. there exists \( M > 0 \) such that for all \( N = 1, 2, \cdots \), the subset of event space \( \Omega_1 = \{ \omega | X_N > M \} \) satisfies
\[
\Pr(\Omega_1) = 0, \quad \Pr(\overline{\Omega_1}) = 1, \tag{B.63}
\]
where \( \overline{\Omega_1} \) is the complement of \( \Omega_1 \), then we have
\[
\mathbb{E}(X_N^k) \leq M^k, \quad \forall k \geq 1, \quad k \in \mathbb{N}. \tag{B.64}
\]

Proof. We consider that \( X_N \) is a continuous random variable (the discrete case is similar). It follows that
\[
\mathbb{E}(X_N^k) = \int_{\Omega_1} X_N^k dP(\omega) + \int_{\overline{\Omega_1}} X_N^k dP(\omega)
= \int_{\Omega_1} X_N^k dP(\omega)
\leq \int_{\Omega_1} M^k dP(\omega)
= M^k \Pr(\overline{\Omega_1})
= M^k. \tag{B.65}
\]

B.21. Upper and Lower Bounds of a Trace

Lemma B.23: For \( A \in \mathbb{R}^{m_1 \times m_2}, B \in \mathbb{R}^{m_1 \times m_1} \) and \( k \in \mathbb{Z}^+ \), if \( B \) is positive definite, define that the largest and smallest eigenvalue of \( B \) are \( \lambda_1(B) \) and \( \lambda_{m_1}(B) \), respectively. Let \( u_{B,m_1} \in \mathbb{R}^{m_1} \) denote the eigenvector associated with \( \lambda_{m_1}(B) \) and \( \text{cond}(B) \) denote the condition number of \( B \), defined as \( \text{cond}(B) = \lambda_1(B)/\lambda_{m_1}(B) \).

- If \( A \) is irrespective of \( B \) and \( u_{B,m_1}^T A \neq 0 \), then there exists \( B_L, B_U > 0 \), irrespective of \( \text{cond}(B) \), such that
\[
\frac{B_L}{\lambda_1^k(B)} \text{cond}^k(B) \leq \text{Tr}(A^T B^{-k} A) \leq \frac{B_U}{\lambda_{m_1}^k(B)} \text{cond}^k(B), \tag{B.66}
\]
where
\[
B_L = u_{B,m_1}^T A A^T u_{B,m_1}, \quad B_U = \text{Tr}(A A^T). \tag{B.67}
\]
- If \( m_1 = m_2, A \) is irrespective of \( B \) and \( u_{B,m_1}^T A u_{B,m_1} \neq 0 \), there exists \( B_L, B_U > 0 \), irrespective of \( \text{cond}(B) \), such that
\[
\frac{B_L}{\lambda_1^3(B)} \text{cond}^k(B) \leq \text{Tr}(B^{-1} A^T A B^{-1} A^T) \leq \frac{B_U}{\lambda_{m_1}^3(B)} \text{cond}^k(B), \tag{B.68}
\]
where
\[
B_L = (u_{B,m_1} A u_{B,m_1})^2, \quad B_U = \text{Tr}(A A^T). \tag{B.69}
\]

Proof. Define the eigenvalue decomposition (EVD) of \( B \) as follows,
\[
B = U_B \Lambda_B U_B^T = \sum_{i=1}^{m_1} \lambda_i(B) u_{B,i} u_{B,i}^T, \tag{B.70}
\]
where \( \Lambda_B \in \mathbb{R}^{m_1 \times m_1} \) is diagonal with \( \lambda_1(B) \geq \cdots \geq \lambda_{m_1}(B) > 0 \), eigenvalues of \( B \), and \( U_B \in \mathbb{R}^{m_1 \times m_1} \) is an orthogonal matrix with \( u_{B,i} \in \mathbb{R}^{m_1} \), being its \( i \)th column vector. Let \( \text{cond}(B) = \lambda_1(B)/\lambda_{m_1}(B) \) denote the condition number of \( B \).

For (B.66), inserting (B.70), we have
\[
\text{Tr}(A^T B^{-k} A)
= \text{Tr} \left[ A^T \sum_{i=1}^{m_1} \frac{1}{\lambda_i^k(B)} u_{B,i} u_{B,i}^T A \right]
= \frac{1}{\lambda_1^k(B)} \sum_{i=1}^{m_1} \lambda_i^k(B) u_{B,i} A A^T u_{B,i}
= \frac{1}{\lambda_1^k(B)} \left[ u_{B,1} A A^T u_{B,1} + \sum_{i=2}^{m_1} \frac{\lambda_i^k(B)}{\lambda_1^k(B)} u_{B,i} A A^T u_{B,i} \right]. \tag{B.71}
\]

Then if we let
\[
B_L = u_{B,m_1} A A^T u_{B,m_1}, \tag{B.72}
B_U = \sum_{i=1}^{m_1} u_{B,i} A A^T u_{B,i}
= \text{Tr} \left( A A^T \sum_{i=1}^{m_1} u_{B,i} u_{B,i}^T \right)
= \text{Tr}(A A^T), \tag{B.73}
\]
where
\[
\sum_{i=1}^{m_1} u_{B,i} u_{B,i}^T = U_B U_B^T = I_{m_1}, \quad \text{(B.74)}
\]
we can obtain (B.66).

For \( m_1 = m_2 \), the derivation of (B.68) is analogous.

Inserting (B.70), we have
\[
\text{Tr}(B^{-1} A^T B^{-1} A B^{-1}) = \text{Tr}
\left[
\frac{1}{\lambda_i(B)} \sum_{j=1}^{m_1} \lambda_j^2(B) (u_{B,j}^T u_{B,j})^2
\right]
\]
\[
= \frac{1}{\lambda_i^2(B)} \sum_{i=1}^{m_1} \sum_{j=1}^{m_1} \lambda_i^2(B) (u_{B,i}^T u_{B,j})^2.
\quad \text{(B.75)}
\]

Then if let
\[
B_L = \left(u_{B,m_1}^T A u_{B,m_1}\right)^2,
\quad B_U = \sum_{i=1}^{m_1} \sum_{j=1}^{m_1} (u_{B,i}^T A u_{B,j})^2
\]
\[
= \sum_{i=1}^{m_1} u_{B,i}^T A \sum_{j=1}^{m_1} u_{B,j} u_{B,j}^T A^T u_{B,i}
\]
\[
= \text{Tr} \left(A A^T \sum_{i=1}^{m_1} u_{B,i} u_{B,i}^T\right)
\]
\[
= \text{Tr}(A A^T),
\quad \text{(B.77)}
\]
we can obtain (B.68).

Since as \( N \to \infty \),
\[
\frac{1}{N} \sum_{i=1}^{N} X_i = \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{m} u(i-j) a_s \to 0,
\quad \text{(B.81)}
\]
\[
\frac{1}{N} \sum_{i=1}^{N} X_i^2 = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} u(i-j) u(i-k)
\]
\[
\to a_s \sum_{j=1}^{n} \sum_{k=1}^{n} g_j^0 g_k^0 B_u (|j-k|) = \theta_0^T \Sigma \theta_0,
\quad \text{(B.82)}
\]
which can be derived using Lemma B.14, then we can obtain (B.78).

**B.23. Some Preliminary Results**

**Lemma B.25:** Let
\[
w_1(t) = \sum_{k=0}^{\infty} \alpha_1(k) x(t-k),
\quad \text{(B.83)}
\]
where \( \sum_{k=0}^{\infty} |\alpha_1(k)| < \infty \) and \( \{x(t)\} \) is independent with zero mean.

1) If \( \{x(t)\} \) is equipped with bounded fourth moments, we have
\[
E \left\{ \sum_{t=1}^{N} [w_1(t-\tau_1) w_1(t-\tau_2)
- E(w_1(t-\tau_1) w_1(t-\tau_2))] \right\}^2 \leq 4 C_{\alpha}^2 C_{x^4} N,
\quad \text{(B.84)}
\]
where
\[
C_{\alpha} = \sum_{k=0}^{\infty} |\alpha_1(k)|, \quad C_{x^4} = \sup_t E[x^4(t)].
\quad \text{(B.85)}
\]

2) If \( \{x(t)\} \) is equipped with bounded second moments and there exists a random sequence \( \{w_2(t)\} \) independent of \( \{x(t)\} \), which is also independent with zero mean and bounded second moments, we have
\[
E \left[ \sum_{t=1}^{N} w_1(t) w_2(t) \right]^2 \leq C_{\alpha}^2 C_{x^2} C_{z^2} N,
\quad \text{(B.86)}
\]
where
\[
C_{x^2} = \sup_t E[x^2(t)], \quad C_{z^2} = \sup_t E[z^2(t)].
\quad \text{(B.87)}
\]

**Proof.** The inequality (B.84) can be derived from Lemma 2B.1 in [5].
For (B.86), it can be known that
\[
E \left[ \sum_{t=1}^{N} w_1(t)w_2(t) \right] = E \left[ \sum_{t=1}^{N} \sum_{k=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_1(k_1) x(t-k_1) w_2(t) \alpha_1(k_2) x(t-k_2)w_2(l) \right]
\]
\[
= \sum_{t=1}^{N} \sum_{k=0}^{\infty} \alpha_1^2(k_1) E[x^2(t-k_1)] E[w_2(t)]
\]
\[
\leq \sum_{t=1}^{N} \left[ \sum_{k=0}^{\infty} |\alpha_1(k_1)| \right]^2 \sup_{t} E[x^2(t-k_1)] \sup_{t} E[w_2^2(t)]
\]
\[
\leq C_0^2 C_{xy}^2 C_{\alpha} N. \quad (B.88)
\]
where the last third step applies the respective independence of \( \{x(t)\} \) and \( \{w_2(t)\} \) and their mutual independence. \( \blacksquare \)

The following lemma is a strengthened version of Lemma B.25.

**Lemma B.26:** For \( m = 4 \) or \( m = 8 \), let
\[
w_1(t) = \sum_{k=0}^{\infty} \alpha_1(k)x(t-k), \quad (B.89)
\]
where \( \sum_{k=0}^{\infty} |\alpha_1(k)| < \infty \) and \( \{x(t)\} \) is independent with zero mean.

1) If \( \{x(t)\} \) is equipped with bounded \( 2m \) moments, there exists \( C_1 > 0 \), which is irrespective of \( N \), such that
\[
E \left\{ \sum_{t=1}^{N} \left[ w_1(t-\tau_1)w_1(t-\tau_2) - E(w_1(t-\tau_1)w_1(t-\tau_2)) \right] \right\} \leq C_1 N^{m/2}. \quad (B.90)
\]

2) If \( \{x(t)\} \) is equipped with bounded \( m \) moments and there exists a random sequence \( \{w_2(t)\} \) independent of \( \{x(t)\} \), which is also independent with zero mean and bounded \( m \) moments, there exists \( C_2 > 0 \), which is irrespective of \( N \), such that
\[
E \left[ \sum_{t=1}^{N} w_1(t)w_2(t) \right] \leq C_2 N^{m/2}. \quad (B.91)
\]

**Proof:** Here we only derive the case of \( m = 4 \), and the case of \( m = 8 \) is similar and thus omitted. The main thought is analogous to the proof of Lemma B.25.

For (B.90) with \( m = 4 \), without loss of generality (W.L.O.G.), we take \( \tau_1 = \tau_2 = 0 \). Define that
\[
(S_{w_1}^N)^4 = \sum_{t=1}^{N} \sum_{t_4=1}^{N} \sum_{k_1=0}^{\infty} \sum_{k_4=0}^{\infty} \alpha_1(k_1)\alpha_1(k_4)\alpha_1(l_1)\alpha_1(l_4)
\]
\[
\gamma(t_1, \ldots, t_4, k_1, l_1, \ldots, k_4, l_4), \quad (B.95)
\]
where
\[
\gamma(t_1, \ldots, t_4, k_1, l_1, \ldots, k_4, l_4) = \beta(t_1, k_1, l_1) \cdots \beta(t_4, k_4, l_4). \quad (B.96)
\]
Since \( \{x(t)\} \) is independent, we can see that \( E(\gamma) \) is equal to zero unless at least some indices in \( \beta(t_i, k_i, l_i) \) with \( i = 1, \ldots, 4 \) coincide. Thus, we have
\[
E \left\{ \sum_{t=1}^{N} \left[ w_1(t-\tau_1)w_1(t-\tau_2) - E(w_1(t-\tau_1)w_1(t-\tau_2)) \right] \right\} \leq C_1 N^{m/2}. \quad (B.90)
\]
\[
E \left[ \sum_{t=1}^{N} w_1(t)w_2(t) \right] \leq C_2 N^{m/2}. \quad (B.91)
\]
where
\[
C_1 \triangleq 16 \sum_{k=0}^{\infty} |\alpha_1(k)|^4 \sup_{t} E[x^8(t)]. \quad (B.98)
\]

For (B.91), we have
\[
E \left[ \sum_{t=1}^{N} w_1(t)w_2(t) \right] \leq C_2 N^{m/2}. \quad (B.99)
\]
where the first step is based on the independence of \( \{x(t)\} \) and \( \{w_2(t)\} \) and their mutual independence, and
\[
C_2 \triangleq \left[ \sum_{k=0}^{\infty} |\alpha_1(k)| \right]^4 \sup_{t} E[x^4(t)] \sup_{t} E[w_2^4(t)]. \quad (B.100)
\]

**Lemma B.27:** If there exists a constant \( C_\alpha \) such that
\[
E \left[ \sum_{t=1}^{N} \bar{s}(t) \right]^2 \leq C_\alpha N, \quad (B.101)
\]
as \( N \to \infty \), we have
\[
\frac{1}{N} \sum_{t=1}^{N} \tilde{s}(t) \xrightarrow{a.s.} 0. \tag{B.102}
\]

**Proof.** Similar proof is included in the proof of Theorem 2B.2 in [5]. Our proof consists of two steps.

- Firstly, we show that \( \frac{1}{N^2} \sum_{t=1}^{N^2} (\tilde{s}(t))^2 \to 0 \) as \( N \to \infty \).
  
  Since \( \mathbb{E}[\sum_{t=1}^{N^2} \tilde{s}(t)^2] \leq C_s \cdot N \), it gives that
  \[
  \mathbb{E} \left[ \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t)^2 \right] \leq \frac{CN^2}{N^4} = \frac{C_s}{N^2}. \tag{B.103}
  \]

  Based on the Chebyshev’s inequality as shown in Lemma B.3 with \( r = 2 \), for any \( \epsilon > 0 \), we have
  \[
  \Pr \left( \left| \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t) \right| > \epsilon \right) \leq \frac{1}{\epsilon^2} \mathbb{E} \left[ \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t)^2 \right] \leq \frac{C_s}{\epsilon^2 N^2},
  \tag{B.104}
  \]
  which yields that
  \[
  \sum_{N=1}^{\infty} \Pr \left( \left| \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t) \right| > \epsilon \right) \leq \frac{C_s}{\epsilon} \sum_{N=1}^{\infty} \frac{1}{N^2} < \infty.
  \tag{B.105}
  \]

  Using Lemma B.4 (Borel-Cantelli’s lemma), it implies that as \( N \to \infty \),
  \[
  \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t) \xrightarrow{a.s.} 0. \tag{B.106}
  \]

- Secondly, we show that \( \sup_{N^2 \leq k \leq (N + 1)^2} \left| \frac{1}{k} \sum_{t=1}^{k} \tilde{s}(t) \right| \xrightarrow{a.s.} 0 \) as \( N \to \infty \), which proves Lemma B.27.

  Suppose that \( \sup_{N^2 \leq k \leq (N + 1)^2} \frac{1}{k} \sum_{t=1}^{k} \tilde{s}(t) \) is achieved at \( k = k_N \). Then we can obtain that
  \[
  \sup_{N^2 \leq k \leq (N + 1)^2} \frac{1}{k} \left| \frac{1}{k_N} \sum_{t=1}^{k_N} \tilde{s}(t) \right| \leq \frac{1}{k_N} \sum_{t=1}^{N^2} \tilde{s}(t) + \frac{1}{k_N} \sum_{t=k_N+1}^{k} \tilde{s}(t). \tag{B.107}
  \]

  For the first term of (B.107), since \( k_N \geq N^2 \), it is known that
  \[
  \frac{1}{k_N} \sum_{t=1}^{N^2} \tilde{s}(t) \leq \frac{1}{N^2} \sum_{t=1}^{N^2} \tilde{s}(t). \tag{B.108}
  \]

  Combining with (B.106), we know that as \( N \to \infty \),
  \[
  \frac{1}{k_N} \sum_{t=1}^{N^2} \tilde{s}(t) \xrightarrow{a.s.} 0. \tag{B.109}
  \]

  The second term of (B.107), we have
  \[
  \mathbb{E} \left[ \left( \frac{1}{k_N} \sum_{t=N^2+1}^{k_N} \tilde{s}(t) \right)^2 \right] \leq \frac{1}{N^2} \mathbb{E} \left[ \sum_{t=N^2+1}^{k_N} \tilde{s}(t)^2 \right] \leq \frac{1}{N^2} C_s [(N + 1)^2 - N^2 - 1] (k_N - N^2 - 1) \tag{B.101}
  \]
  \[
  \leq \frac{1}{N^4} C_s [(N + 1)^2 - N^2 - 1]^2 \tag{since \( k_N \leq (N + 1)^2 \)}
  \]

  Similarly, according to the Chebyshev’s inequality (Lemma B.3 with \( r = 2 \)) and Borel-Cantelli’s lemma (Lemma B.4), we can derive that as \( N \to \infty \),
  \[
  \frac{1}{k_N} \left| \frac{1}{k} \sum_{t=1}^{k} \tilde{s}(t) \right| \xrightarrow{a.s.} 0.
  \tag{B.111}
  \]

  Consequently, inserting (B.109) and (B.111) into (B.107), we show that as \( N \to \infty \),
  \[
  \sup_{N^2 \leq k \leq (N + 1)^2} \frac{1}{k} \sum_{t=1}^{k} \tilde{s}(t) \xrightarrow{a.s.} 0,
  \tag{B.112}
  \]
  which implies (B.102).

Then we can derive the following result using Lemma B.5 and B.6.

**Lemma B.28:** Let \( \beta_i \in \mathbb{R} \) and \( \gamma_i \in \mathbb{R} \) for \( i = 1, \ldots, m \) be random variables and have bounded moments with order \( 4 + \delta \) for some \( \delta > 0 \) for \( \beta_i \gamma_i \).

\[
X = \begin{bmatrix}
\beta_1 \gamma_1 - \mathbb{E}(\beta_1 \gamma_1) \\
\vdots \\
\beta_m \gamma_m - \mathbb{E}(\beta_m \gamma_m)
\end{bmatrix},
\tag{B.113}
\]

there exists a constant \( C_{\beta \gamma} \) satisfying
\[
\mathbb{E}\|X\|^{2+\delta} \leq C_{\beta \gamma}.
\tag{B.114}
\]

In particular, if \( \beta_i \) and \( \gamma_i \) are mutually independent, as long as they have bounded moments of order \( 2+\delta \) for some \( \delta > 0 \), \( \mathbb{E}\|X\|^{2+\delta} \) is still bounded.

**Proof.**
\[
\mathbb{E}\|X\|^{2+\delta} = \mathbb{E}\left\{ \sum_{i=1}^{m} |\beta_i \gamma_i - \mathbb{E}(\beta_i \gamma_i)|^{2+\delta} \right\}^{(2+\delta)/2}
\leq 2^{\delta m/2} \sum_{i=1}^{m} |\beta_i \gamma_i - \mathbb{E}(\beta_i \gamma_i)|^{2+\delta}
\leq 2^{\delta m/2 + 1+\delta} \sum_{i=1}^{m} (\mathbb{E}(\beta_i \gamma_i)^{2+\delta} + |\mathbb{E}(\beta_i \gamma_i)|^{2+\delta})
\]

[using Lemma B.6 with \( r = (2 + \delta)/2 \)]
\[
\leq 2^{\delta m/2 + 1+\delta} \sum_{i=1}^{m} (\mathbb{E}(\beta_i \gamma_i)^{2+\delta} + |\mathbb{E}(\beta_i \gamma_i)|^{2+\delta})
\]

[using Lemma B.6 with \( r = 2 + \delta \).]
Apply the Cauchy-Schwarz inequality as shown in Lemma B.5 to obtain

\[ E(\beta_i^2 + \delta) \leq E(\beta_i^4 + 2\delta) E(\gamma_i^4 + 2\delta)^{1/2}, \]  
(B.116a)

\[ |E(\beta_i^2 + \delta) \leq E(\beta_i^2 E(\gamma_i^2)^{(2+\delta)/2}. \]  
(B.116b)

Note that for a random variable \(X\), if \(E|X|^r < \infty\), for any \(0 < r \leq s\), we have \(E|X|^s < \infty\). Combining (B.115) and (B.116), since \(\beta_i\) and \(\gamma_i\) have bounded moments of order \(4 + \delta\) (or lower order) and (B.115) is a finite sum of bounded moments, hence \(E\|X\|^{2+\delta}\) is also bounded for some fixed \(\delta\).

Define that

\[ C_{\beta} = \sup_i E(\beta_i)^2, \quad \tilde{C}_{\beta} = \sup_i E(\beta_i)^{4+\delta}, \]

(B.117)

\[ C_{\gamma} = \sup_i E(\gamma_i)^2, \quad \tilde{C}_{\gamma} = \sup_i E(\gamma_i)^{4+\delta}. \]

(B.118)

Thus we have

\[ E\|X\|^{2+\delta} \leq C_{\beta\gamma}, \]  
(B.119)

where

\[ C_{\beta\gamma} = 2^{m/2+1+\delta} m[(\tilde{C}_{\beta}\tilde{C}_{\gamma})^{1/2} + (C_{\beta}C_{\gamma})^{(2+\delta)/2}]. \]  
(B.120)

In particular, if \(\beta_i\) and \(\gamma_i\) are mutually independent, (B.116a) can be adjusted as

\[ E|\beta_i\gamma_i|^{2+\delta} = E|\beta_i|^{2+\delta} E|\gamma_i|^{2+\delta}, \]  
(B.121)

which will still be bounded if \(\beta_i\) and \(\gamma_i\) have bounded moments of order \(2 + \delta\) for some \(\delta > 0\).

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