Fourier Transforms and Bent Functions on Finite Abelian Group-Acted Sets

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June 18, 2014

Abstract

Let $G$ be a finite abelian group acting faithfully on a finite set $X$. As a natural generalization of the perfect nonlinearity of Boolean functions, the $G$-bentness and $G$-perfect nonlinearity of functions on $X$ are studied by Poinset et al. \cite{6,7} via Fourier transforms of functions on $G$. In this paper we introduce the so-called $G$-dual set $\hat{X}$ of $X$, which plays the role similar to the dual group $\hat{G}$ of $G$, and the Fourier transforms of functions on $X$, a generalization of the Fourier transforms of functions on finite abelian groups. Then we characterize the bent functions on $X$ in terms of their own Fourier transforms on $X$. Bent (perfect nonlinear) functions on finite abelian groups and $G$-bent ($G$-perfect nonlinear) functions on $X$ are treated in a uniform way in this paper, and many known results in \cite{4,2,6,7} are obtained as direct consequences. Furthermore, we will prove that the bentness of a function on $X$ can be determined by its distance from the set of $G$-linear functions. In order to explain the main results clearly, examples are also presented.

Keywords: group actions; $G$-linear functions; $G$-dual sets; Fourier transforms on $G$-sets; bent functions; $G$-perfect nonlinear functions

1 Introduction

Bent functions, perfect nonlinear functions, and their generalizations have been studied in many papers. The notion of a Boolean bent function was introduced by Rothaus \cite{10}. More than a decade ago, Logachev, Salnikov, and Yashchenko \cite{4} generalized this concept to bent functions on finite abelian groups. As a further generalization, Poinset \cite{5} studied bent functions on finite nonabelian groups. Recently, a closely related notion, perfect nonlinear functions between finite abelian groups as well as between arbitrary finite groups, has been studied in quite a few papers; for example, see \cite{2,8,9,12,13,14}. These functions have numerous applications in cryptography, coding theory, and other fields. A critical tool in these studies is the Fourier transforms of functions on finite groups.

The perfect nonlinearity of a function $f : G \rightarrow H$ between finite abelian groups $G$ and $H$ is characterized by its derivatives $f'_\alpha : G \rightarrow H$, $x \mapsto f(\alpha x)f(x)^{-1}$, for all non-identity $\alpha \in G$. 

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By observing that, for any $\alpha \in G$, the mapping $G \to G, x \mapsto \alpha x$ is just the regular action of $G$ on its base set $G$, Poinsot et al. [6, 7] generalized the concept of the perfect nonlinearity to a function $g : X \to H$, where $X$ is a finite set with an action of $G$ on it. The derivatives of $g$ are $g'_\alpha : X \to H, x \mapsto f(\alpha x)f(x)^{-1}$, for any $\alpha \in G$. In order to characterize the perfect nonlinearity of functions from the set $X$ to $H$, the Fourier transforms of functions on the group $G$ are used in [5, 7]. For any $x \in X$, let $g_x : G \to H$ be the function defined by $g_x(\alpha) = g(\alpha x)$, for any $\alpha \in G$. Let $\hat{G}, \hat{H}$ be the dual groups of $G$ and $H$, respectively, and let $\zeta_0$ be the principal (irreducible) character of $H$. Then $g$ is $G$-perfect nonlinear (cf. [7, Theorems 2 and 3]) if and only if for any $\zeta \in \hat{H}\setminus\{\zeta_0\}$,

$$\frac{1}{|X|} \sum_{x \in X} \left| \zeta \widehat{g_x}(\xi) \right|^2 = |G|, \quad \text{for any } \xi \in \hat{G},$$

where $|X|$ and $|G|$ are the cardinalities of $X$ and $G$, respectively, and $\zeta \widehat{g_x}$ is the Fourier transform of $\zeta g_x$ on $\hat{G}$. Poinsot [6] also defined the bentness of functions on $X$ in a similar way. Let $\mathbb{C}$ be the complex field, $T$ the unit circle in $\mathbb{C}$, and $f : X \to T$ a function. For any $x \in X$, let $f_x$ be a function on $G$ defined by $f_x(\alpha) := f(\alpha x)$, for any $\alpha \in G$. Then using the Fourier transforms $\hat{f}_x$ of the functions $f_x$ on $G$, the $G$-bentness of the function $f$ on $X$ is defined as follows:

$$f \text{ is } G\text{-bent if and only if for all non-identity element } \alpha \in G, \text{ the derivatives } f'_\alpha \text{ are balanced.}$$

As the Fourier transform on finite groups is the key tool in the study of bent and perfect nonlinear functions on finite groups, in this paper for a finite abelian group $G$ acting on a finite set $X$ ($X$ is called a $G$-set), we will develop the Fourier analysis on $X$, and use it as our tool to study the bentness and perfect nonlinearity of functions on $X$. By characterizing the bent functions on $X$ in terms of their own Fourier transforms, we are able to treat bent and perfect nonlinear functions on finite abelian groups as well as $G$-bent and $G$-perfect nonlinear functions on $X$ in a uniform way.

The set of functions from a $G$-set $X$ to $\mathbb{C}$, denoted by $\mathbb{C}^X$, is a $\mathbb{C}G$-module, where $\mathbb{C}G$ is the group algebra of $G$ over $\mathbb{C}$. $\mathbb{C}^X$ is also a unitary space with a natural $G$-invariant inner product. For each irreducible character $\psi \in \hat{G}$, the $G$-linear component of $\mathbb{C}^X$ with respect to $\psi$ is the $\mathbb{C}G$-submodule of $\mathbb{C}^X$ consisting of $\psi$-linear functions (see Definition 2.2 below). $\mathbb{C}^X$ can be decomposed into the orthogonal direct sum of its $G$-linear components (see Proposition 2.7 below). A key step is that by using this decomposition we obtain an orthogonal basis $\hat{X}$ of $\mathbb{C}^X$ consisting of $G$-linear functions such that $\hat{X}$ is closed under complex conjugation (see Theorem 2.10 below). Such a basis $\hat{X}$, called a $G$-dual set of $X$, plays a role in $\mathbb{C}^X$ similar to $\hat{G}$ in $\mathbb{C}G$. For any function $f \in \mathbb{C}^X$, we define the Fourier transform $\hat{f}$ of $f$ as a function on $\hat{X}$ (see Definition 3.4 below), and define the bentness of $f$ in terms of $\hat{f}(\lambda)$ for all $\lambda \in \hat{X}$ (see Definition 4.1 below). Our definitions of the Fourier transforms and bentness of functions on $G$-sets are natural generalizations of the Fourier transforms and bentness of functions on finite abelian groups, respectively.

Then using the Fourier analysis on a $G$-set $X$, we study the characterizations of bent functions on $X$. We will prove that (Theorem 4.6) a function $f : X \to T$ is bent if and only if the derivatives of $f$ in all nontrivial directions are balanced. Since the bentness and perfect nonlinearity of functions on finite abelian groups and $G$-sets are treated in a uniform way, many known results in [4, 2, 6, 7] are obtained as immediate consequences. Furthermore, we
will prove that (Theorem 5.2) $f \in T^X$ is a bent function if and only if the distance from $f$ to the set $(\mathbb{C}^X)_G$ of $G$-linear functions reaches the best possible upper bound of the distance between $(\mathbb{C}^X)_G$ and any function in $T^X$. This result gives another geometric interpretation of the importance of bent functions in cryptography. The perfect nonlinearity of functions from $X$ to a finite abelian group $H$ is also characterized in terms of Fourier transforms of functions on $X$ (see Theorem 5.2). To explain the theory established in this paper, several examples are also included.

The rest of the paper is organized as follows. In Section 2 we present the classical decomposition of the $\mathbb{C}G$-module $\mathbb{C}^X$, and prove the existence of the $G$-dual set $\hat{X}$ of $X$. Then in Section 3 we introduce the Fourier transforms of functions in $\mathbb{C}^X$, and investigate their basic properties. Section 4 is devoted to the study of the characterizations of bent functions on $X$. Finally, $G$-perfect nonlinear functions are discussed in Section 5 and explanatory examples are presented in Section 6.

2 Complex functions and $G$-dual sets

Throughout the paper, $G$ is always a finite abelian group of order $|G| = m$ with multiplicative operation, $X$ a finite $G$-set with cardinality $|X| = n$, $\mathbb{C}$ the complex field, and $T$ the unit circle in $\mathbb{C}$. For any sets $R$ and $S$, by $R^S$ we denote the set of functions from $S$ to $R$. Note that $\mathbb{C}^S$ is equipped with scalar multiplication, function addition and function multiplication such that it is a complex algebra. Also for $f \in \mathbb{C}^S$ by $\overline{f}$ we denote the complex conjugation function, i.e. $\overline{f}(s) = \overline{f(s)}$ for $s \in S$, where $\overline{f(s)}$ is the complex conjugate of $f(s) \in \mathbb{C}$. Furthermore, $\mathbb{C}^X$ is a $\mathbb{C}G$-module (see below). In this section we discuss the structure of $\mathbb{C}^X$, and prove that it has a special basis $\hat{X}$, called the $G$-dual set, which plays a similar role of $\hat{G}$ for the Fourier transforms. Some of the results in this section are known for general $\mathbb{C}G$-modules, but our treatment is different.

2.1 Dual groups and Fourier transforms on groups

An irreducible character of a finite abelian group $G$ is a homomorphism from $G$ to the multiplicative group of non-zero complex numbers. By $\hat{G}$ we denote the dual group of $G$, i.e. the group consisting of irreducible characters of $G$. For the fundamentals of representation theory of finite groups, the reader is referred to [11]. We include some needed known facts here.

For any $\sigma \in \mathbb{C}^G$ we have a function $\widehat{\sigma} \in \mathbb{C}\hat{G}$, called the Fourier transform of $\sigma$, defined by $\widehat{\sigma}(\psi) = \sum_{\alpha \in G} \sigma(\alpha) \psi(\alpha)^{-1}$, for all $\psi \in \hat{G}$. On the other hand, for any $\tau \in \mathbb{C}\hat{G}$ we have a function $\widetilde{\tau} \in \mathbb{C}^G$, called the Fourier inverse transform of $\tau$, defined as follows: $\widetilde{\tau}(\alpha) = \frac{1}{m} \sum_{\psi \in \hat{G}} \overline{\psi(\alpha)} \psi(\alpha)^{-1}$, for all $\alpha \in G$. Note that $\overline{\psi(\alpha)}$ runs over $\hat{G}$ as $\psi$ runs over $\hat{G}$. Also $\overline{\psi(\alpha)} = \overline{\psi(\alpha)^{-1}} = \overline{\psi(\alpha^{-1})}$, for all $\alpha \in G$.

It is well-known that $\widehat{\sigma} = \sigma$ for all $\sigma \in \mathbb{C}^G$, and $\widetilde{\tau} = \tau$ for all $\tau \in \mathbb{C}\hat{G}$.

Remark 2.1. By $\rho$ we denote the regular character of $G$, i.e.

$$\rho(\alpha) = \begin{cases} m & \alpha = 1, \\ 0 & \alpha \neq 1; \end{cases} \quad \forall \alpha \in G,$$

where 1 denotes the identity element of $G$. It is known that $\rho = \sum_{\psi \in \hat{G}} \psi$. Since $\hat{G}$ is a basis of the $m$-dimensional space $\mathbb{C}^G$, any $\sigma \in \mathbb{C}^G$ is a linear combination of $\hat{G}$, and the coefficients of the linear combination are uniquely determined by $\sigma$. As $\sigma = \widehat{\sigma}$, by the Fourier inverse
transform the linear combination of $\sigma$ is $\sigma = \frac{1}{m} \sum_{x \in G} \bar{\sigma}(x) \psi$. Note that $\sigma(\alpha) = 0$ for all $\alpha \in G \setminus \{1\}$ if and only if $\sigma = \frac{\sigma(1)}{m} \rho$, where $G \setminus \{1\}$ denotes the difference set of $G$ removing the identity element $1$. Thus,

(i) $\sigma$ takes zero on $G \setminus \{1\}$ if and only if $\hat{\sigma}$ is constant on $\hat{G}$.

Since $G$ and $\hat{G}$ are dual to each other through the Fourier transforms, by interchanging the roles of $\sigma$ and $\hat{\sigma}$ in (i), we further get

(ii) $\sigma$ is constant on $G$ if and only if $\hat{\sigma}$ takes zero on $\hat{G} \setminus \{1\}$,

where by abuse of notation, 1 denotes the unity (principal) character: $1(\alpha) = 1$ for all $\alpha \in G$. No ambiguity of this notation can arise from the context.

### 2.2 $G$-linear functions and the classical decomposition

As mentioned before, $X$ is a $G$-set with cardinality $|X| = n$. That is, there is a map $G \times X \to X$, $(\alpha, x) \mapsto \alpha x$, such that for all $x \in X$ we have $(\alpha \beta) x = \alpha (\beta x)$ for all $\alpha, \beta \in G$, and $1x = x$. For the fundamentals of group actions, we refer the reader to [1].

If all the values of a function $f \in \mathbb{C}^X$ have length 1, i.e. $f \in T^X$, then we say that $f$ is a unitary function.

The complex space $\mathbb{C}^X$ is a $\mathbb{C}G$-module with the following $G$-action:

$$(\alpha f)(x) = f(\alpha^{-1} x), \quad \forall f \in \mathbb{C}^X \forall \alpha \in G \forall x \in X.$$  \hfil (2.1)

In the literature, a $\mathbb{C}G$-module is also called a complex $G$-space, see [1, Ch 1]. In this paper we will use both terms. Furthermore, $\mathbb{C}^X$ is a unitary space with the following inner product:

$$\langle f, g \rangle = \sum_{x \in X} f(x) \overline{g(x)}, \quad \forall f, g \in \mathbb{C}^X.$$  \hfil (2.2)

This inner product is $G$-invariant in the following sense: $\langle \alpha f, \alpha g \rangle = \langle f, g \rangle$, or equivalently $\langle \alpha f, g \rangle = \langle f, \alpha^{-1} g \rangle$, for all $f, g \in \mathbb{C}^X$ and $\alpha \in G$. The length (or norm) $|f|$ of any $f \in \mathbb{C}^X$ is then defined as

$$|f| = \sqrt{\langle f, f \rangle} = \sqrt{\sum_{x \in X} f(x) \overline{f(x)}},$$

and the distance between $f, g \in \mathbb{C}^X$ is defined as $d(f, g) = |f - g|$. Further, for any subsets $S_1, S_2 \subseteq \mathbb{C}^X$ we can define the distance between $S_1$ and $S_2$ as follows:

$$d(S_1, S_2) = \min \{d(f_1, f_2) \mid f_1 \in S_1, f_2 \in S_2\}. \hfil (2.3)$$

**Definition 2.2.** (i) A function $f \in \mathbb{C}^X$ is said to be $G$-linear if there is a $\psi \in \hat{G}$ such that

$$f(\alpha x) = \psi(\alpha) f(x), \quad \forall \alpha \in G \forall x \in X,$

i.e. $\alpha f = \overline{\psi(\alpha)} f$ for all $\alpha \in G$. More precisely, in that case we say that the function $f$ is $\psi$-linear.

(ii) For any $\psi \in \hat{G}$, the $\psi$-component $(\mathbb{C}^X)_\psi$ of $\mathbb{C}^X$ is defined by

$$(\mathbb{C}^X)_\psi = \{f \mid f \in \mathbb{C}^X, f \text{ is } \psi \text{-linear}\}.$$

The $\psi$-components for all $\psi \in \hat{G}$ are also simply called $G$-linear components of $\mathbb{C}^X$, without mentioning the irreducible characters.

(iii) By $(\mathbb{C}^X)_G$ we denote the set of $G$-linear functions on $X$, i.e.

$$(\mathbb{C}^X)_G = \{f \mid f \in \mathbb{C}^X, f \text{ is } G \text{-linear}\} = \bigcup_{\psi \in \hat{G}} (\mathbb{C}^X)_\psi.$$
Lemma 2.4. (i) For any $\psi \in \widehat{G}$, $(\mathbb{C}X)_\psi$ is a $G$-invariant subspace (i.e. $\mathbb{C}G$-submodule) of $\mathbb{C}X$.

(ii) For any $\psi, \varphi \in \widehat{G}$ such that $\psi \neq \varphi$, we have the orthogonality:
$$\langle (\mathbb{C}X)_\psi, (\mathbb{C}X)_\varphi \rangle = 0.$$

In particular, $(\mathbb{C}X)_\psi \cap (\mathbb{C}X)_\varphi = \{0\}$ if $\psi \neq \varphi$.

Proof. (i) is straightforward. Now we prove (ii). Since $\psi \neq \varphi$, there is an $\alpha \in G$ such that $\psi(\alpha) \neq \varphi(\alpha)$. So for any $f \in (\mathbb{C}X)_\psi$ and $g \in (\mathbb{C}X)_\varphi$,
$$\psi(\alpha)\langle f, g \rangle = \langle \psi(\alpha)f, g \rangle = \langle \alpha^{-1}f, g \rangle = \langle f, \alpha g \rangle = \langle f, \varphi(\alpha)g \rangle = \varphi(\alpha)\langle f, g \rangle.$$

Hence $\psi(\alpha) \neq \varphi(\alpha)$ implies that $\langle f, g \rangle = 0$. This proves that $\langle (\mathbb{C}X)_\psi, (\mathbb{C}X)_\varphi \rangle = 0$. The rest of (ii) is clear. \hfill $\square$

Remark 2.5. The zero function 0 is clearly $\psi$-linear for all $\psi \in \widehat{G}$. However, for a non-zero $G$-linear function $f$, Lemma 2.4 implies that there is a unique $\psi \in \widehat{G}$ such that $f$ is $\psi$-linear. In other words, the union $(\mathbb{C}X)_G = \bigcup_{\psi \in \widehat{G}} (\mathbb{C}X)_\psi$ is a disjoint union, in the sense that only the zero function is in the intersections.

For $\sigma \in \mathbb{C}G$ and $f \in \mathbb{C}X$, we define a function $\sigma*f \in \mathbb{C}X$, called the convolution product of $\sigma$ and $f$, as follows:
$$(\sigma*f)(x) = \sum_{\alpha \in G} \sigma(\alpha)f(\alpha^{-1}x), \quad \forall \; x \in X.$$

Specifically, taking $X = G$ to be the regular $G$-set, then for $\sigma, \tau \in \mathbb{C}G$ the above formula gives the usual convolution product $\tau*\sigma \in \mathbb{C}G$ of functions on the group:
$$(\tau*\sigma)(\alpha) = \sum_{\beta \in G} \tau(\beta)\sigma(\beta^{-1}\alpha), \quad \forall \; \alpha \in G.$$
Proposition 2.7. (i) For any $\psi \in \hat{G}$ and $f \in \mathbb{C}^X$, we have $\psi*f \in (\mathbb{C}^X)_\psi$.
(ii) For any $\psi \in \hat{G}$ and $f \in \mathbb{C}^X$, $f \in (\mathbb{C}^X)_\psi$ if and only if $f = \frac{1}{m}\psi*f$.
(iii) For any $f \in \mathbb{C}^X$, we have $f = \frac{1}{m}\psi*f$.
(iv) We have the orthogonal direct sum: $\mathbb{C}^X = \bigoplus_{\psi \in \hat{G}}(\mathbb{C}^X)_\psi$.

Proof. (i). For any $\alpha \in G$ and $x \in X$, since $\alpha\beta$ runs over $G$ as $\beta$ runs over $G$, we have

$$(\psi*f)(ax) = \sum_{\beta \in G} \psi(\alpha\beta)f((\alpha\beta)^{-1}ax) = \sum_{\beta \in G} \psi(\alpha)\psi(\beta)f(\beta^{-1}x) = \psi(\alpha) \cdot \sum_{\beta \in G} \psi(\beta)f(\beta^{-1}x) = \psi(\alpha) \cdot (\psi*f)(x).$$

(ii). If $f = \frac{1}{m}\psi*f$, then $f \in (\mathbb{C}^X)_\psi$ by (i). If $f \in (\mathbb{C}^X)_\psi$, then for any $x \in X$,

$$\frac{1}{m}(\psi*f)(x) = \frac{1}{m} \sum_{\alpha \in G} \psi(\alpha)f(\alpha^{-1}x) = \frac{1}{m} \sum_{\alpha \in G} \psi(\alpha)\psi(\alpha)f(x) = f(x).$$

(iii). Since the regular character $\rho$ satisfies that $\rho(\alpha) = \begin{cases} 0, & \alpha \neq 1; \\ m, & \alpha = 1; \end{cases}$, we have $\frac{1}{m} \rho*f = f$, for all $f \in \mathbb{C}^X$. Furthermore, since $\rho = \sum_{\psi \in \hat{G}} \psi$, we get that

$$f = \frac{1}{m} \rho*f = \frac{1}{m} \left( \sum_{\psi \in \hat{G}} \psi \right)*f = \frac{1}{m} \sum_{\psi \in \hat{G}} \psi*f.$$

(iv) follows directly from (i), (iii), and Lemma 2.3.

Definition 2.8. (i) For any $f \in \mathbb{C}^X$ and $\psi \in \hat{G}$, the $\psi$-component of $f$ is $f_\psi = \frac{1}{m}\psi*f$, and the classical decomposition of $f$ is $f = \sum_{\psi \in \hat{G}} f_\psi$.
(ii) The orthogonal direct sum $\mathbb{C}^X = \bigoplus_{\psi \in \hat{G}}(\mathbb{C}^X)_\psi$ is called the classical decomposition of the $G$-space $\mathbb{C}^X$; cf. [III] §2.6.

2.3 The $G$-dual set $\hat{X}$

For an unitary space $V$ and a basis $u_1, \ldots, u_n$ of $V$, if there is a non-zero $\pi \in \mathbb{C}$ such that

$$\langle u_i, u_j \rangle = \begin{cases} 0, & i \neq j; \\ \pi, & i = j; \end{cases},$$

then we say that $u_1, \ldots, u_n$ is an $\bar{n}$-normal orthogonal basis of the unitary space.

Definition 2.9. A basis $\bar{X}$ of the unitary $G$-space $\mathbb{C}^X$ is called a $G$-dual set of $X$ if the following three conditions are satisfied:

(i) $\bar{X}$ is an $n$-normal orthogonal basis;
(ii) any $\lambda \in \bar{X}$ is $G$-linear;
(iii) $\bar{X}$ is closed under complex conjugation, i.e. $\overline{\lambda} \in \bar{X}$ for all $\lambda \in \bar{X}$.
Theorem 2.10. For any $G$-set $X$, there exists a $G$-dual set $\hat{X}$.

Proof. Since $\bar{\psi} \in \hat{G}$ for any $\psi \in \hat{G}$, it follows from Lemma 2.9(iii) and Proposition 2.7(ii) that for any $f \in (\mathbb{C}^X)_\psi$, $\bar{f} \in (\mathbb{C}^X)_{\bar{\psi}}$. That is,

$$(\mathbb{C}^X)_\psi = (\mathbb{C}^X)_{\bar{\psi}}, \quad \forall \ \psi \in \hat{G},$$

where $(\mathbb{C}^X)_{\psi} = \{ \bar{f} \mid f \in (\mathbb{C}^X)_\psi \}$. For any $\psi \in \hat{G}$, it is known that there is an $n$-normal orthogonal basis $(\hat{X})_\psi$ for the $\psi$-component $(\mathbb{C}^X)_\psi$ of $\mathbb{C}^X$. Hence, $(\hat{X})_\psi = \{ \lambda \mid \lambda \in (\hat{X})_\psi \}$ is also an $n$-normal orthogonal basis of the $\bar{\psi}$-component $(\mathbb{C}^X)_{\bar{\psi}}$. Thus, if $\psi \neq \bar{\psi}$, then $(\hat{X})_\psi \cup (\hat{X})_{\bar{\psi}}$ is an $n$-normal orthogonal basis of $(\mathbb{C}^X)_\psi \oplus (\mathbb{C}^X)_{\bar{\psi}}$.

In the following we prove that if $\psi = \bar{\psi}$, then there is an $n$-normal orthogonal basis $(\hat{X})_\psi$ of $(\mathbb{C}^X)_\psi$ such that for any $\lambda \in (\hat{X})_\psi$, $\lambda = \bar{\lambda}$. Let $f \in (\mathbb{C}^X)_\psi$ such that $f \neq 0$. Then at least one of $f + \bar{f}$ and $\sqrt{-1}(f - \bar{f})$ is not zero. Thus, $(\mathbb{C}^X)_\psi = (\mathbb{C}^X)_{\bar{\psi}}$ implies that there is a $\lambda_1 \in (\mathbb{C}^X)_\psi$ such that $\lambda_1 \neq 0$, and $\lambda_1 = \bar{\lambda}_1$. We may also assume that $\langle \lambda_1, \lambda_1 \rangle = n$. Note that $(\mathbb{C}^X)_\psi = \mathbb{C}\lambda_1 \oplus (\mathbb{C}\lambda_1)_{\perp}$. Also for any $f \in (\mathbb{C}\lambda_1)$, it follows from $\lambda_1 = \bar{\lambda}_1$ that $f \in (\mathbb{C}\lambda_1)_{\perp}$. Hence, if $(\mathbb{C}\lambda_1)_{\perp} \neq \{0\}$, then as above, there is $\lambda_2 \in (\mathbb{C}\lambda_1)_{\perp}$ such that $\lambda_2 = \bar{\lambda}_2$, $\langle \lambda_2, \lambda_2 \rangle = n$, and $(\mathbb{C}\lambda_1)_{\perp} = \mathbb{C}\lambda_2 \oplus (\mathbb{C}\lambda_1 \oplus \mathbb{C}\lambda_2)_{\perp}$. Continuing this process, we see that $\lambda_1, \lambda_2, \cdots$ form an $n$-normal orthogonal basis of $(\mathbb{C}^X)_\psi$.

Therefore, the orthogonal direct sum $\mathbb{C}^X = \bigoplus_{\psi \in \hat{G}} (\mathbb{C}^X)_\psi$ implies that the union $\hat{X}$ of the $n$-normal orthogonal bases of the $G$-linear components of $\mathbb{C}^X$ chosen in the above two paragraphs is a basis of $\mathbb{C}^X$ that satisfies the conditions (i), (ii) and (iii) of Definition 2.9.

Remark 2.11. (i) If $\hat{X}$ is a $G$-dual set of $X$, then $\hat{Y} = \{ \varepsilon \lambda \mid \lambda \in \hat{X}, \varepsilon \in T \}$ is also a $G$-dual set of $X$. We call $\hat{Y}$ a rescaling of $\hat{X}$ by $T$.

(ii) If $X$ is a transitive $G$-set, then every $G$-linear component $(\mathbb{C}^X)_\psi$ of $\mathbb{C}^X$ is 1-dimensional, and hence $(\hat{X})_\psi$ consists of exactly one function of length $\sqrt{n}$. Thus, $X$ has a unique $G$-dual set $\hat{X}$ up to rescaling by $T$.

(iii) In particular, if $X = G$ is the regular $G$-set, then $X$ has a unique $G$-dual set up to rescaling by $T$. Usually, the dual group $\hat{G}$ is chosen as $\hat{X}$.

(iv) However, if the number of the $G$-orbits of $X$ is greater than 1, then the $G$-dual set $\hat{X}$ is not unique up to rescaling by $T$. The proof of Theorem 2.10 provides a way to chose a $G$-dual set. Later we will show another way to get a $G$-dual set (see Examples 6.2 and 6.3 below).

From now on, for the $G$-set $X$ we fix a $G$-dual set $\hat{X}$. Then we have the disjoint union

$$\hat{X} = \bigcup_{\psi \in \hat{G}} (\hat{X})_\psi,$$

where $(\hat{X})_\psi$ is an $\psi$-component of $\mathbb{C}^X$ is $\mathbb{C}^X$, and $\mathbb{C}^X = (\hat{X})_{\bar{\psi}}$. Thus, the $\psi$-component of $\mathbb{C}^X$ is

$$(\mathbb{C}^X)_\psi = \bigoplus_{\lambda \in (\hat{X})_\psi} \mathbb{C}\lambda, \quad \forall \ \psi \in \hat{G}. \quad (2.4)$$

Note that some subsets $(\hat{X})_\psi$ may be empty (correspondingly, some component $(\mathbb{C}^X)_\psi$ may be zero).

Let $\hat{X} = \{ \lambda_1, \cdots, \lambda_n \}$ and $X = \{ x_1, \cdots, x_n \}$. Then we have an $n \times n$ matrix $\Lambda = (\lambda_i(x_j))_{1 \leq i,j \leq n}$. The $n$-normal orthogonality of $\hat{X}$ implies that $\Lambda \cdot \overline{\Lambda}^T = nI$, where $I$ is the
identity matrix and $Λ^T$ is the conjugate transpose of $Λ$. Hence we also have $Λ^T \cdot Λ = nI$. Thus, we have the following

**Lemma 2.12.** (Orthogonality Relations) The following hold:
\[
\sum_{x \in X} \lambda(x) \overline{p(x)} = \begin{cases} n, & \lambda = \mu; \\ 0, & \lambda \neq \mu; \end{cases} \quad \forall \lambda, \mu \in \hat{X}. \tag{2.5}
\]
\[
\sum_{\lambda \in \hat{X}} \lambda(x) \overline{\lambda(y)} = \begin{cases} n, & x = y; \\ 0, & x \neq y; \end{cases} \quad \forall x, y \in X. \tag{2.6}
\]

3 **Fourier transforms of functions on $G$-sets**

Given a $G$-dual set $\hat{X}$ of the $G$-set $X$, in this section we define the Fourier transform of $f \in \mathbb{C}^X$ on $\hat{X}$, and discuss its basic properties. We will need to consider the space $\mathbb{C}^{\hat{X}}$ of complex functions on $\hat{X}$, which is also a unitary space with the inner product
\[
\langle g, h \rangle = \sum_{\lambda \in \hat{X}} g(\lambda) \overline{h(\lambda)}, \quad \forall g, h \in \mathbb{C}^{\hat{X}}.
\]

For any $\sigma \in \mathbb{C}^G$, the Fourier transform $\hat{\sigma}$ of $\sigma$ at any $\psi \in \hat{G}$ is $\hat{\sigma}(\psi) = \sum_{\alpha \in G} \sigma(\alpha) \psi(\alpha)$. The next definition generalizes this notion to the functions on $G$-sets.

**Definition 3.1.** For any $f \in \mathbb{C}^X$, the **Fourier transform** of $f$, $\hat{f} \in \mathbb{C}^{\hat{X}}$, is defined as
\[
\hat{f}(\lambda) = \sum_{x \in X} f(x) \lambda(x), \quad \forall \lambda \in \hat{X}.
\]

For any $g \in \mathbb{C}^{\hat{X}}$, the **Fourier inversion** of $g$, $\hat{g} \in \mathbb{C}^X$, is defined as
\[
\hat{g}(x) = \frac{1}{n} \sum_{\lambda \in \hat{X}} g(\lambda) \overline{\lambda(x)}, \quad \forall x \in X.
\]

It is clear that if $X = G$ is the regular $G$-set, then the Fourier transform of $f \in \mathbb{C}^X$ in the above definition is exactly the Fourier transform of functions on $G$.

**Remark 3.2.** For $x \in X$ we have the characteristic function $1_x$ (i.e. $1_x(y) = 0$, if $y \neq x$, and $1_x(x) = 1$), whose Fourier transform is $\hat{1}_x(\lambda) = \lambda(x)$, for any $\lambda \in \hat{X}$. Thus, we can rewrite the definitions of $\hat{f}$ and $\hat{g}$ in Definition 3.1 as follows:
\[
\hat{f}(\lambda) = \langle f, \hat{1}_x \rangle, \quad \forall f \in \mathbb{C}^X, \quad \forall \lambda \in \hat{X};
\]
\[
\hat{g}(x) = \frac{1}{n} \langle g, \hat{1}_x \rangle, \quad \forall g \in \mathbb{C}^{\hat{X}}, \quad \forall x \in X.
\]

**Lemma 3.3.** $\hat{f} = f$, $\forall f \in \mathbb{C}^X$, and $\hat{g} = g$, $\forall g \in \mathbb{C}^{\hat{X}}$.

**Proof.** For any $x \in X$ we have
\[
\hat{f}(x) = \frac{1}{n} \sum_{\lambda \in \hat{X}} \left( \sum_{y \in X} f(y) \lambda(y) \right) \overline{\lambda(x)} = \sum_{y \in X} f(y) \cdot \frac{1}{n} \sum_{\lambda \in \hat{X}} \lambda(y) \overline{\lambda(x)}. \tag{8}
\]
By the second orthogonality relation \(2.6\), we have \(\hat{f}(x) = f(x)\) for all \(x \in X\). Similarly, by the first orthogonality relation \(2.5\), we have \(\hat{g}(\lambda) = g(\lambda)\) for all \(\lambda \in \hat{X}\). \(\square\)

For any \(\lambda \in \hat{X}\), there is a unique irreducible character \(\psi_{\lambda}\) of \(G\) such that \(\lambda \in (\mathbb{C}^X)_{\psi_{\lambda}}\). Also for any \(g \in \mathbb{C}^X\), the length of \(g\) is \(|g| = \sqrt{\langle g, g \rangle}\).

**Lemma 3.4.** Let \(\sigma \in \mathbb{C}^G\) and \(f \in \mathbb{C}^X\). Then the following hold.

(i) \(\hat{\sigma \ast f}(\lambda) = \hat{\sigma}(\psi_{\lambda}) \hat{f}(\lambda)\) for all \(\lambda \in \hat{X}\).

(ii) If \(\sigma \in \mathcal{G}\), then \(\frac{1}{m} \hat{\sigma \ast f}(\lambda) = \begin{cases} \hat{f}(\lambda), & \lambda \in (\hat{X})_{\sigma}, \\ 0, & \lambda \not\in (\hat{X})_{\sigma}. \end{cases}\)

(iii) \(|\hat{f}_\psi|^2 = |\frac{1}{m} \hat{\psi \ast f}|^2 = \sum_{\lambda \in (X)_{\psi}} |\hat{f}(\lambda)|^2\), for all \(\psi \in \mathcal{G}\).

**Proof.** (i) Since \(\lambda(\alpha^{-1}x) = \psi_{\lambda}(\alpha^{-1})\lambda(x)\) for \(\alpha \in G\) and \(x \in X\), we have
\[
\hat{\sigma \ast f}(\lambda) = \sum_{x \in X} (\sigma \ast f)(x)\lambda(x) = \sum_{x \in X} \sum_{\alpha \in G} \sigma(\alpha) f(\alpha^{-1}x)\lambda(x)
= \sum_{\alpha \in G} \sigma(\alpha) \psi_{\lambda}(\alpha) \sum_{x \in X} f(\alpha^{-1}x)\lambda(\alpha^{-1}x)
= \hat{\sigma}(\psi_{\lambda}) \hat{f}(\lambda).
\]

(ii) If \(\sigma \in \mathcal{G}\), then by Remark 3.2 \(\hat{\sigma}(\psi_{\lambda}) = \langle \sigma, \psi_{\lambda} \rangle = \begin{cases} m, & \sigma = \overline{\psi_{\lambda}}; \\ 0, & \sigma \neq \psi_{\lambda}. \end{cases}\) So (ii) follows from (i).

(iii) \(|\hat{f}_\psi|^2 = |\frac{1}{m} \hat{\psi \ast f}|^2 = \sum_{\lambda \in \hat{X}} |\frac{1}{m} \hat{\psi \ast f}(\lambda)|^2\). Since \(\hat{X} = \bigcup_{\psi \in \mathcal{G}} (\hat{X})_{\psi}\), by (ii) we get
\[
|\hat{f}_\psi|^2 = \sum_{\lambda \in (X)_{\psi}} |\hat{f}(\lambda)|^2 = \sum_{\lambda \in (\hat{X})_{\psi}} |\hat{f}(\lambda)|^2.
\]

The following is an easy but useful fact.

**Lemma 3.5.** Any function \(f \in \mathbb{C}^X\) is a unique linear combination of \(\hat{X}\) as follows:
\[
f = \frac{1}{n} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda)\lambda.
\]

Hence, for the classical decomposition \(f = \sum_{\psi \in \mathcal{G}} f_\psi\), the \(\psi\)-component
\[
f_\psi = \frac{1}{m} \psi \ast f = \frac{1}{n} \sum_{\lambda \in (X)_{\psi}} \hat{f}(\lambda)\lambda.
\]

**Proof.** For all \(x \in X\), Lemma 3.3 implies that
\[
f(x) = \hat{f}(x) = \frac{1}{n} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda)\lambda(x) = \frac{1}{n} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda)\lambda(x).
\]

Since \(\frac{1}{n} \sum_{\lambda \in (X)_{\psi}} \hat{f}(\lambda)\lambda \in (\mathbb{C}^X)_{\psi}\), Proposition 2.4 and 2.1 imply that the \(\psi\)-component of \(f\) is
\[
f_\psi = \frac{1}{n} \sum_{\lambda \in (\hat{X})_{\psi}} \hat{f}(\lambda)\lambda. \quad \square
\]
Lemma 3.6. Let $f, g \in \mathbb{C}^X$ and $\alpha \in G$. Then

$$\langle \alpha^{-1} f, g \rangle = \frac{1}{n} \sum_{\psi \in \hat{G}} \sum_{\lambda \in (\hat{X})_\psi} \hat{f}(\lambda) \overline{g(\lambda)}.$$ 

In particular $(f, g) = \frac{1}{n} (\hat{f}, \hat{g})$.

**Proof.** Recall that $(\alpha^{-1} f)(x) = f(\alpha x)$ (cf. Eqn (2.1)). So by Lemma 3.5 we have

$$\langle \alpha^{-1} f, g \rangle = \sum_{x \in X} f(\alpha x) \overline{g(x)} = \sum_{x \in X} \frac{1}{n} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda) \psi(\lambda) \overline{\lambda(x)} \overline{g(\lambda)} \mu(x).$$

Note that $\hat{X}$ is the disjoint union $\hat{X} = \bigcup_{\psi \in \hat{G}} (\hat{X})_\psi$, and for $\lambda \in (\hat{X})_\psi$ we have $\lambda(\alpha x) = \psi(\lambda) \lambda(x)$. So

$$\langle \alpha^{-1} f, g \rangle = \frac{1}{n} \sum_{x \in X} \sum_{\psi \in \hat{G}} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda) \psi(\lambda) \overline{\lambda(x)} \overline{g(\lambda)} \mu(x) \mu(x).$$

By the first orthogonality relation (2.5), we get that

$$\langle \alpha^{-1} f, g \rangle = \frac{1}{n} \sum_{\psi \in \hat{G}} \sum_{\lambda \in \hat{X}} \hat{f}(\lambda) \overline{g(\lambda)}.$$

Taking $\alpha = 1$ in the above formula, we have $(f, g) = \frac{1}{n} (\hat{f}, \hat{g})$. □

The next corollary is immediate from Lemma 3.6

**Corollary 3.7.** If $f \in T^X$, then $(f, f) = n$ and $(\hat{f}, \hat{f}) = n^2$.

For any $\lambda \in \hat{X}$, $\hat{\lambda} \in \mathbb{C}^{\hat{X}}$, and $\hat{\lambda}(\mu) = \begin{cases} 0, & \mu \neq \overline{\lambda} \\ n, & \mu = \overline{\lambda} \end{cases}$. So $\{\hat{\lambda} \mid \lambda \in \hat{X}\}$ is an $n^2$-normal orthogonal basis of $\mathbb{C}^{\hat{X}}$.

4 Bent functions on $G$-sets

In this section we define the bentness of functions on the $G$-set $X$, and study its characterizations. For a finite abelian group $G$, a unitary function $f : G \to T$ is called a bent function (cf. [4]) if for any $\psi \in \hat{G}$, $|\hat{f}(\psi)|^2 = |G|$. The next definition generalizes this notion to unitary functions on $G$-sets.

**Definition 4.1.** A unitary function $f : X \to T$ on the $G$-set $X$ is called a bent function if

$$\sum_{\lambda \in (\hat{X})_\psi} |\hat{f}(\lambda)|^2 = |X|^2 |G|^{-1}, \quad \text{for all } \psi \in \hat{G}.$$

If $X = G$ is the regular $G$-set, then for any $\psi \in \hat{G}$, $(\hat{X})_\psi = \{\psi\}$, and the above definition of a bent function on the $G$-set $X$ is exactly the same as the definition of a bent function on
G. The bentness of functions on G-sets are also defined in [3] Definition 6, and called G-bent functions. But the definition in [3] is different; it uses the Fourier transforms of functions on G. However, we will show that the definition in [3] is equivalent to Definition 4.11 (see Corollary 4.12 below).

Although the bent function is defined by the use of \( \lambda \in \hat{X} \), the next lemma says that the bentness of a function on X is independent of the choice of \( \hat{X} \). Recall that the length of a function is defined by Eqn (2.2).

**Lemma 4.2.** For a unitary function \( f : X \to T \), the following are equivalent.

(i) \( f \) is a bent function.

(ii) For any \( \psi, \varphi \in \hat{G} \), \( |\hat{f}_\psi| = |\hat{f}_\varphi| \).

(iii) For any \( \psi, \varphi \in \hat{G} \), \(|f_\psi| = |f_\varphi| \).

**Proof.** By Lemma 3.4 (i) implies (ii). Assume (ii). From Lemma 3.4 and Corollary 3.7 we see that

\[
\sum_{\psi \in \hat{G}} |\hat{f}_\psi|^2 = \sum_{\psi \in \hat{G}} \sum_{\lambda \in (X)} |\hat{f}(\lambda)|^2 = (\hat{f}, \hat{f}) = n^2.
\]

Hence, for any \( \psi \in \hat{G} \), \( \sum_{\lambda \in (X)} |\hat{f}(\lambda)|^2 = |\hat{f}_\psi|^2 = n^2/m \), and (i) holds.

(ii) and (iii) are equivalent by Lemma 3.6. \( \square \)

The zero-point set \( \text{Ann}(f) \) of \( f \in C^X \) and its complement \( \overline{\text{Ann}(f)} \) in X are introduced in Remark 2.3. Note that \( f \) is non-zero if and only if \( \overline{\text{Ann}(f)} \neq \emptyset \).

**Definition 4.3.** If \( f \in C^X \) is a non-zero function and \( \text{Ann}(f) \) is G-invariant, then \( f \) is said to be differentiable. For any differentiable function \( f \in C^X \) we define a function \( f'_\alpha \) on \( \overline{\text{Ann}(f)} \) as follows:

\[
f'_\alpha(x) = f(\alpha x)f(x)^{-1}, \quad \forall \ x \in \overline{\text{Ann}(f)}.
\]

\( f'_\alpha \) is called the derivative of \( f \) in direction \( \alpha \).

Any unitary function \( f \in T^X \) is differentiable and \( f'_\alpha \in T^X \). By Remark 2.3(ii), any non-zero G-linear function is differentiable. The following lemma is a geometric explanation of the G-linearity of a function by its derivative.

**Lemma 4.4.** Let \( f \in C^X \) be differentiable. Then \( f'_\alpha \) is a constant function on \( \overline{\text{Ann}(f)} \) for any \( \alpha \in G \) if and only if \( f \) is G-linear, i.e. there is a unique character \( \psi \in \hat{G} \) such that \( f \in (C^X)_\psi \).

**Proof.** It is clear that if \( f \) is \( \psi \)-linear for \( \psi \in \hat{G} \), then for any \( \alpha \in G \), \( f'_\alpha(x) = \psi(\alpha) \) for \( x \in \overline{\text{Ann}(f)} \) is a constant function. Now assume that for any \( \alpha \in G \), \( f'_\alpha(x) = \psi_f(\alpha) \), for all \( x \in \overline{\text{Ann}(f)} \). Then for any \( \alpha, \beta \in G \) we have

\[
\psi_f(\alpha \beta) = f'_\alpha(\beta x) = f(\alpha \beta x)f(x)^{-1} = f(\alpha \beta x)f(\beta x)^{-1} \cdot f(\beta x)f(x)^{-1} = f'_\alpha(\beta x) \cdot f'_\beta(x) = \psi_f(\alpha) \psi_f(\beta).
\]

So \( \psi_f \) is an irreducible character of \( G \), and

\[
f(\alpha x) = \psi_f(\alpha)f(x), \quad \forall \ x \in X. \quad \square
\]

Unitary functions far away from G-linear functions on X are more useful and interesting in cryptography. So by Lemma 4.4 we want to investigate those unitary functions whose derivatives in all nontrivial directions are far away from constant functions. As for unitary functions on finite groups, a unitary function \( h : X \to T \) is said to be balanced if \( \sum_{x \in X} h(x) = 0 \).
Definition 4.5. A unitary function $f : X \to T$ is said to have totally balanced derivatives if

$$\sum_{x \in X} f'_\alpha(x) = 0, \quad \forall \ \alpha \in G \setminus \{1\}.$$ 

Now we are ready to present the characterizations of bent functions on $G$-sets.

Theorem 4.6. A unitary function $f \in T^X$ is bent if and only if $f$ has totally balanced derivatives.

Proof. Note that

$$\sum_{x \in X} f'_\alpha(x) = \sum_{x \in X} f(\alpha x) \overline{f}(x) = (\alpha^{-1}f, f).$$

By Lemma 3.4 we have

$$\sum_{x \in X} f'_\alpha(x) = \frac{1}{n} \sum_{\psi \in \hat{G}} \psi(\alpha) \sum_{\lambda \in \hat{X}_\psi} \overline{\hat{f}(\lambda)} \hat{f}(\lambda)$$

$$= \frac{1}{n} \sum_{\psi \in \hat{G}} \left( \sum_{\lambda \in \hat{X}_\psi} |\hat{f}(\lambda)|^2 \right) \psi(\alpha).$$

Thus, Lemma 3.4(ii) implies that

$$\sum_{x \in X} f'_\alpha(x) = \frac{1}{n} \sum_{\psi \in \hat{G}} |\hat{f}_\psi|^2 \psi(\alpha). \quad (4.1)$$

If $f$ has totally balanced derivatives, i.e. $\sum_{x \in X} f'_\alpha(x) = 0$ for all $\alpha \in G \setminus \{1\}$, then Eqn (4.1) implies that the function $\sum_{\psi \in \hat{G}} |\hat{f}_\psi|^2 \psi$ on $G$ takes zero on $G \setminus \{1\}$, and hence it must be a multiple of the regular character $\rho = \sum_{\psi \in \hat{G}} \psi$ of $G$, cf. Remark 2.1. Thus, for any $\psi, \phi \in \hat{G}$ we have $|\hat{f}_\psi|^2 = |\hat{f}_\phi|^2$, and $f$ is bent by Lemma 4.2.

Conversely, if $f$ is bent, i.e. $|\hat{f}_\psi|^2 = \frac{n^2}{m}$ for all $\psi \in \hat{G}$, then by Eqn 4.1 we have

$$\sum_{x \in X} f'_\alpha(x) = \frac{1}{n} \sum_{\psi \in \hat{G}} \frac{n^2}{m} \psi(\alpha) = \frac{n}{m} \sum_{\psi \in \hat{G}} \psi(\alpha) = \frac{n}{m} \rho(\alpha).$$

So for all $\alpha \in G \setminus \{1\}$, $\sum_{x \in X} f'_\alpha(x) = 0$ by Remark 2.1 and $f$ has totally balanced derivatives.

Corollary 4.7. If there is a $\psi \in \hat{G}$ such that $(\mathbb{C}^X)_\psi = 0$ (i.e. $(\overline{X})_\psi = \emptyset$), then there exists no bent function $f \in T^X$.

Proof. In that case $|\hat{f}_\psi| = 0$. 

Remark 4.8. The above corollary says that the condition “$(\mathbb{C}^X)_\psi \neq 0$ for all $\psi \in \hat{G}$” is a necessary condition for the existence of bent functions.

If the $G$-action on $X$ is not faithful, i.e. the kernel $K$ of the action is nontrivial, then there must be an irreducible character $\psi$ of $G$ which takes nontrivial values on $K$, and hence $(\mathbb{C}^X)_\psi = 0$, cf. Remark 2.3(ii). So by the above corollary, there exists no bent functions on $X$.

However, even if the $G$-action on $X$ is faithful, there may still exist some $\psi \in \hat{G}$ such that $(\mathbb{C}^X)_\psi = 0$, and hence the bent functions on $X$ do not exist. See Example 6.2 below for such an example.
Our next characterization of a bent function is given by its distance from the set \((\mathbb{C}^X)_G\) of \(G\)-linear functions (cf. Definition 2.2). Recall that the distance between two subsets of \(\mathbb{C}^X\) is defined by Eqn (2.3). The next theorem says that the distance from a bent function to \((\mathbb{C}^X)_G\) is greater than the distance from any non-bent unitary function to \((\mathbb{C}^X)_G\). It also says that \(\sqrt{(m-1)n/m}\) is the best possible upper bound of the distance between any unitary function and \((\mathbb{C}^X)_G\).

**Theorem 4.9.** Let \(f \in T^X\). Then the following hold.

(i) \(d(f, (\mathbb{C}^X)_G) \leq \sqrt{(m-1)n/m}\).

(ii) \(f\) is bent if and only if \(d(f, (\mathbb{C}^X)_G) = \sqrt{(m-1)n/m}\).

**Proof.** We have seen that

\[(\mathbb{C}^X)_G = \bigcup_{\psi \in \hat{G}} (\mathbb{C}^X)_\psi.
\]

For any \(G\)-linear function \(g\) there is a \(\varphi \in \hat{G}\) such that \(g\) is \(\varphi\)-linear, i.e. \(g = g_\varphi \in (\mathbb{C}^X)_\varphi\), and \(g_\psi = 0\) for any \(\psi \in \hat{G}\) such that \(\psi \neq \varphi\). Since any two different \(G\)-linear components are orthogonal to each other, we can compute the distance between \(f\) and \(g\) as follows:

\[
d(f, g)^2 = |f - g|^2 = \left|\sum_{\psi \in \hat{G}} (f_\psi - g_\psi)\right|^2 = |f_\varphi - g_\varphi|^2 + \sum_{\psi \neq \varphi} |f_\psi|\}
\]

and the equality holds when \(g = f_\varphi\). By Corollary 3.7,

\[
\sum_{\psi \in \hat{G}} |f_\psi|^2 = \sum_{\psi \in \hat{G}} \langle f_\psi, f_\psi \rangle = \left\langle \sum_{\psi \in \hat{G}} f_\psi, \sum_{\psi \in \hat{G}} f_\psi \right\rangle = \langle f, f \rangle = |f|^2 = n.
\]

So according to the definition of the distance in Eqn (2.3), we have

\[
d(f, (\mathbb{C}^X)_\varphi)^2 = n - |f_\varphi|^2.
\]

Hence the square of the distance between \(f\) and \((\mathbb{C}^X)_G\) is

\[
d(f, (\mathbb{C}^X)_G)^2 = \min_{\varphi \in \hat{G}} \{n - |f_\varphi|^2\} = n - \max_{\varphi \in \hat{G}} \{|f_\varphi|^2\}.
\]

By the equality \(\sum_{\psi \in \hat{G}} |f_\psi|^2 = n\) again, \(|\hat{G}| = m\) implies that

\[
\max_{\varphi \in \hat{G}} \{|f_\varphi|^2\} \geq \frac{n}{m},
\]

where the equality holds if and only if \(|f_\psi|^2 = |f_\varphi|^2\) for all \(\psi, \varphi \in \hat{G}\). In conclusion,

\[
d(f, (\mathbb{C}^X)_G)^2 \leq n - \frac{n}{m} = \frac{(m-1)n}{m}, \tag{4.2}
\]

and the equality in (4.2) holds if and only if \(|f_\psi|^2 = |f_\varphi|^2\) for all \(\psi, \varphi \in \hat{G}\). By Lemma 4.2, the equality in (4.2) holds if and only if \(f\) is bent. \(\Box\)

By taking \(X = G\) as the regular \(G\)-set, we have the next corollary from Theorem 4.6, Theorem 4.9, and Lemma 4.2. Note that the equivalence of (i) and (ii) in Corollary 4.10 below was proved in [4].
Corollary 4.10. Let $f : G \rightarrow T$ be a unitary function. Then the following are equivalent.

(i) $f$ is a bent function.
(ii) $f$ has totally balanced derivatives.
(iii) Among all functions in $T^G$, $f$ has the greatest distance $\sqrt{|G| - 1}$ from the set $(\mathbb{C}^G)_G$ of $G$-linear functions.
(iv) $|\langle f, \psi \rangle|$ are equal for all $\psi \in \hat{G}$.

Proof. The equivalence of (i), (ii), and (iii) is immediate from Theorems 4.6 and 4.9. Since $\hat{G}$ is a basis of $\mathbb{C}^G$, we may assume that $f = \sum_{\psi \in \hat{G}} c_{\psi} \psi$, where $c_{\psi} \in \mathbb{C}$. Hence, the $\psi$-component of $f$ is $f_{\psi} = c_{\psi} \psi$, for any $\psi \in \hat{G}$. Thus,

$$|\langle f, \psi \rangle| = |\langle c_{\psi} \psi, \psi \rangle| = |c_{\psi}| = \sqrt{|\langle f_{\psi}, f_{\psi} \rangle|}, \quad \text{for any } \psi \in \hat{G}.$$  

So the equivalence of (i) and (iv) holds by Lemma 4.2.

Lemma 4.11. For any $f \in \mathbb{C}^X$ and $x \in X$, let $f_x \in \mathbb{C}^G$ be defined by $f_x(\alpha) = f(\alpha x)$ for all $\alpha \in G$. Then $\hat{f}_x(\psi) = m f_{\psi}(x)$ for all $\psi \in \hat{G}$.

Proof. It is a straightforward computation to see that

$$\hat{f}_x(\psi) = \sum_{\alpha \in G} f_x(\alpha) \psi(\alpha) = \sum_{\alpha \in G} f(\alpha x) \overline{\psi}(\alpha^{-1}) = \overline{(\psi \ast f)}(x) = m f_{\psi}(x).$$

The next corollary is one of the main results of [6,7], where the $G$-bentness of $f \in T^X$ is defined by the condition (ii) of Corollary 4.12. So Corollary 4.12 implies that the $G$-bentness defined in [6,7] is equivalent to the bentness defined by Definition 6.

Corollary 4.12. (Cf. [6,7]) Let $f \in T^X$. Then the following two statements are equivalent to each other:

(i) $f$ has totally balanced derivatives. That is, $f$ is a bent function by Definition 4.1.
(ii) $\frac{1}{n} \sum_{x \in X} |\hat{f}_x(\psi)|^2 = m$ for all $\psi \in \hat{G}$. That is, $f$ is a $G$-bent function by [6, Definition 6].

Proof. By Lemmas 4.11 and 5.6, we have

$$\frac{1}{n} \sum_{x \in X} |\hat{f}_x(\psi)|^2 = \frac{m^2}{n} \sum_{x \in X} |f_{\psi}(x)|^2 = \frac{m^2}{n} \langle f_{\psi}, f_{\psi} \rangle = \frac{m^2}{n^2} \langle \hat{f}_{\psi}, \hat{f}_{\psi} \rangle = \frac{m^2}{n^2} |\hat{f}_{\psi}|^2.$$

Thus, (ii) holds if and only if $|\hat{f}_{\psi}|^2 = \frac{m^2}{n}$ for all $\psi \in \hat{G}$ if and only if $f$ has totally balanced derivatives by Theorem 4.6 and Lemma 4.2.

Remark 4.13. For any $f, g \in \mathbb{C}^X$, the pseudo-convolution $f \boxtimes g$ of $f$ and $g$ is defined as (cf. [7])

$$f \boxtimes g : G \rightarrow \mathbb{C}, \quad \alpha \mapsto \sum_{x \in X} f(x) g(\alpha x).$$

By Lemmas 2.12 and 3.3, it is straightforward to show that

$$\hat{(f \boxtimes g)}(\psi) = \frac{m}{n} \sum_{\lambda \in (X)_\psi} \hat{f}(\lambda) \hat{g}(\lambda), \quad \text{for any } \psi \in \hat{G}.$$

The equivalence of (i) and (ii) of Corollary 4.12 can also be proved by the above equality.
5 Perfect nonlinear functions on $G$-sets

As an application of the characterizations of bent functions on $G$-sets, in this section we discuss the characterizations of perfect nonlinear functions from a $G$-set to an abelian group. Our approach here is different from that of [6, 7]. Let $X$ be a $G$-set as before, and let $H$ be an abelian group with multiplicative operation. Let $H^X$ denote the set of all functions from $X$ to $H$. An $f \in H^X$ is said to be evenly-balanced (cf. [13, 14]) if $|H|$ divides $|X|$ and

$$\{|x \in X | f(x) = h| = \frac{|X|}{|H|}$$

for any $h \in H$.

An evenly-balanced function is also called a balanced or uniformly distributed function in some literature. The derivative of $f \in H^X$ in direction $\alpha \in G$ is

$$f'_\alpha : X \rightarrow H, \quad x \mapsto f(\alpha x) f(x)^{-1}.$$

**Definition 5.1.** (cf. [7, Definition 1]) A function $f : X \rightarrow H$ is said to be $G$-perfect nonlinear if for any $\alpha \in G \backslash \{1\}$, the function $f'_\alpha$ is evenly-balanced.

Any $g \in H^X$ induces a non-negative integral function $g^#$ on $H$ as follows:

$$g^# : H \rightarrow \mathbb{N} \cup \{0\}, \quad h \mapsto |\{x \in X | f(x) = h\}|.$$

Hence, $g^#$ is constant on $H$ if and only if $g$ is evenly-balanced. Thus, a function $f : X \rightarrow H$ is $G$-perfect nonlinear if and only if for any $\alpha \in G$, $f^#_\alpha$ is constant on $H$.

**Theorem 5.2.** Let $f \in H^X$. Then following are equivalent.

(i) For any $\xi \in \hat{H} \backslash \{1\}$ the composition function $\xi \circ f : X \rightarrow \mathbb{C}$ has totally balanced derivatives.

(ii) For any $\xi \in \hat{H} \backslash \{1\}$ the composition function $\xi \circ f : X \rightarrow \mathbb{C}$ is bent.

(iii) The function $f : X \rightarrow H$ is $G$-perfect nonlinear.

**Proof.** It is enough to show that (i) $\Leftrightarrow$ (iii). Since $(\xi \circ f)(x) \in T$, we have $(\xi \circ f)(x)^{-1} = (\xi \circ f)(x)$, for any $x \in X$. So

$$\sum_{x \in X} (\xi \circ f)'_\alpha (x) = \sum_{x \in X} (\xi \circ f)(\alpha x)(\xi \circ f)(x)$$

$$= \sum_{x \in X} \xi(f(\alpha x))\xi(f(x)) = \sum_{x \in X} \xi(f(\alpha x))\xi(f(x)^{-1})$$

$$= \sum_{x \in X} \xi(f(\alpha x)f(x)^{-1}) = \sum_{x \in X} \xi(f'_\alpha(x)).$$

For any $h \in H$, let $X(f'_\alpha, h) = \{x \in X | f'_\alpha(x) = h\}$. Then $X$ is the disjoint union $X = \bigcup_{h \in H} X(f'_\alpha, h)$, and the cardinality $|X(f'_\alpha, h)| = f^#_\alpha(h)$. So

$$\sum_{x \in X} (\xi \circ f)'_\alpha (x) = \sum_{h \in H} \sum_{x \in X(f'_\alpha, h)} \xi(h) = \sum_{h \in H} f^#_\alpha(h)\xi(h) = f^#_\alpha(\xi). \quad (5.1)$$

Thus, $(\xi \circ f)'_\alpha$ is balanced if and only if $f^#_\alpha(\xi) = 0$. Hence for any $\xi \in \hat{H} \backslash \{1\}$, the function $(\xi \circ f)'_\alpha$ is balanced if and only if $f^#_\alpha$ is zero on $\hat{H} \backslash \{1\}$ and if and only if $f^#_\alpha$ is constant on $H$ by Remark 2.1 (ii). That is, for any $\xi \in \hat{H} \backslash \{1\}$, the function $\xi \circ f$ has totally balanced derivatives if and only if $f$ is $G$-perfect nonlinear.

Taking $X = G$ to be the regular $G$-set, we have the next
Corollary 5.3. (Cf. [2]) Let $G, H$ be abelian groups, and $f : G \to H$ a function. Then the following are equivalent.

(i) $f$ is perfect nonlinear.

(ii) For any $\xi \in \hat{H}\{1\}$ the composition function $\xi \circ f : G \to \mathbb{C}$ is bent.

Let $f \in H^X$. Then for any $x \in X$, there is a function (cf. [3, 7])

$$f_x : G \to H, \quad \alpha \mapsto f(\alpha x).$$

Also for any $\xi \in \hat{H}$, there is a function $(\xi \circ f)_x : G \to \mathbb{T}, \alpha \mapsto (\xi \circ f)(\alpha x)$. Note that $(\xi \circ f)_x = \xi \circ f_x$, for any $x \in X$. The next corollary is immediate from Theorem 5.2 and Corollary 4.12.

Corollary 5.4. (cf. [6, Theorems 5 and 7]) Let $f \in H^X$. Then the following are equivalent.

(i) $f$ is $G$-perfect nonlinear.

(ii) For any $\xi \in \hat{H}\{1\}$ and $\alpha \in G$,

$$\frac{1}{|X|} \sum_{x \in X} \left| (\xi \circ f_x)(\alpha) \right|^2 = |G|. $$

6 Examples

In this section we present a few examples that explain the theory developed in the previous sections.

Example 6.1. Assume that $X = G$ is the regular $G$-set. As mentioned in Remark 2.11, the $G$-dual set $\hat{X}$ is unique up to rescaling by $T$, and the typical choice of $\hat{X}$ is just the dual group $\hat{G}$. So the theory developed in previous sections includes the corresponding theory for finite abelian groups as a special case. For example, some well-known results in [2, 4, 12] as well as other properties of bent functions on finite abelian groups are given in Corollary 4.10 and Corollary 5.3 as immediate consequences.

The next example gives a $G$-set on which there exists no bent function.

Example 6.2. Let $G = \{1, \alpha, \beta, \gamma\}$ be the Klein four group. That is, $G$ is an abelian group such that

$$\alpha^2 = \beta^2 = \gamma^2 = 1, \quad \alpha \beta = \gamma, \quad \beta \gamma = \alpha, \quad \gamma \alpha = \beta.$$ 

Then $\hat{G} = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ is given by Table 6.1.

$$
\begin{array}{|c|c|c|c|}
\hline
 & 1 & \alpha & \beta \\
\hline
\psi_1 & 1 & 1 & 1 \\
\psi_2 & 1 & 1 & -1 \\
\psi_3 & 1 & -1 & 1 \\
\psi_4 & 1 & -1 & -1 \\
\hline
\end{array}
$$

Table 6.1: Character Table of the Klein Four Group

Let $X = \{x_1, x_2, x_3, x_4\}$ be a faithful $G$-set with two orbits $X_1$ and $X_2$ as follows:

- $X_1 = \{x_1, x_2\}$, 1 and $\alpha$ fix both points $x_1$ and $x_2$, while $\beta$ and $\gamma$ interchange the two points;
• $X_2 = \{x_3, x_4\}$, 1 and $\beta$ fix both points $x_3$ and $x_4$, while $\gamma$ and $\alpha$ interchange the two points.

We can take $\tilde{X} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ as in Table 6.2 (to simplify the table, we list $\sqrt{2}\lambda_i$ instead of $\lambda_i$).

We can take $\hat{X} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ as in Table 6.2 (to simplify the table, we list $\sqrt{2}\lambda_i$ instead of $\lambda_i$).

Table 6.2: The $G$-dual set in Example 6.2

| $\lambda_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|------------|-------|-------|-------|-------|
| $\sqrt{2}$ | 1     | 1     | 0     | 0     |
| $\sqrt{2}$ | 1     | -1    | 0     | 0     |
| $\sqrt{2}$ | 0     | 0     | 1     | 1     |
| $\sqrt{2}$ | 0     | 0     | 1     | -1    |

Table 6.3: The $G$-dual set in Example 6.3

| $\lambda_i$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|------------|-------|-------|-------|-------|-------|-------|
| $\sqrt{3}$ | 1     | 1     | 0     | 0     | 0     | 0     |
| $\sqrt{3}$ | 1     | -1    | 0     | 0     | 0     | 0     |
| $\sqrt{3}$ | 0     | 0     | 1     | 1     | 0     | 0     |
| $\sqrt{3}$ | 0     | 0     | 1     | -1    | 0     | 0     |
| $\sqrt{3}$ | 0     | 0     | 0     | 0     | 1     | 1     |
| $\sqrt{3}$ | 0     | 0     | 0     | 1     | -1    | 1     |

Since one of the $G$-linear components is empty, there exists no bent function $f \in T^X$ by Corollary 4.7.

The next example gives a $G$-set $X$ and a bent function on $X$.

**Example 6.3.** As above in Example 6.2 let $G = \{1, \alpha, \beta, \gamma\}$ be the Klein four group and $\hat{G} = \{\psi_1, \psi_2, \psi_3, \psi_4\}$. But this time we consider the $G$-set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with three orbits:

• $X_1 = \{x_1, x_2\}$, 1 and $\alpha$ fix both points $x_1$ and $x_2$, while $\beta$ and $\gamma$ interchange the two points;
• $X_2 = \{x_3, x_4\}$, 1 and $\beta$ fix both points $x_3$ and $x_4$, while $\gamma$ and $\alpha$ interchange the two points;
• $X_3 = \{x_5, x_6\}$, 1 and $\gamma$ fix both points $x_5$ and $x_6$, while $\alpha$ and $\beta$ interchange the two points.

We can take $\tilde{X} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ as in Table 6.3 (to simplify the table, we list $\sqrt{3}\lambda_i$ instead of $\lambda_i$).

We can check that the $G$-linear components of $\mathbb{C}^X$ are

$(\tilde{X})_{\psi_1} = \{\lambda_1, \lambda_3, \lambda_5\}$, $(\tilde{X})_{\psi_2} = \{\lambda_2\}$, $(\tilde{X})_{\psi_3} = \{\lambda_4\}$, $(\tilde{X})_{\psi_4} = \emptyset$. 

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Let \( \omega = \frac{1 + \sqrt{-3}}{2} \) be a primitive third root of unity. Take \( f \in T^X \) as follows:

\[
f(x_j) = \omega^{(1+(-1)^j)/2} = \begin{cases} 1, & j = 1, 3, 5; \\ \omega, & j = 2, 4, 6. \end{cases}
\]

Then

\[
\sum_{x \in X_j} f'_\alpha(x) = \sum_{x \in X_j} f(\alpha x)f(x)^{-1} = \begin{cases} 1+1 = 2, & j = 1; \\ 1 \cdot \omega^{-1} + \omega \cdot 1 = -1, & j = 2, 3. \end{cases}
\]

So \( \sum_{x \in X} f'_\alpha(x) = 0 \). Similarly, \( \sum_{x \in X} f'_\beta(x) = \sum_{x \in X} f'_\gamma(x) = 0 \). That is, \( f \) has totally balanced derivatives.

On the other hand,

\[
\langle \hat{f}_{\psi_1}, \hat{f}_{\psi_1} \rangle = \sum_{\lambda \in (X)_{\psi_1}} |\hat{f}(\lambda)|^2 = \sum_{j=1,3,5} |\sum_{x \in X} f(x)\lambda_j(x)|^2
\]

\[
= \sum_{j=1,3,5} |1 + \omega|^2 = 3|1 + \omega|^2 = 3,
\]

\[
\langle \hat{f}_{\psi_2}, \hat{f}_{\psi_2} \rangle = \sum_{\lambda \in (X)_{\psi_2}} |\hat{f}(\lambda)|^2 = |\sum_{x \in X} f(x)\lambda_2(x)|^2 = |1 - \omega|^2 = 3.
\]

Similarly, \( \langle \hat{f}_{\psi_3}, \hat{f}_{\psi_3} \rangle = \langle \hat{f}_{\psi_4}, \hat{f}_{\psi_4} \rangle = |1 - \omega|^2 = 3 \). In conclusion, we have \( \langle \hat{f}_{\psi}, \hat{f}_{\psi} \rangle = 3, \forall \psi \in \hat{G} \), and \( f \) is a bent function.

Finally, we give an example of a \( G \)-perfect nonlinear function.

**Example 6.4.** We continue Example 6.3 and further take \( H = \{1, h, h^2\} \) with \( h^3 = 1 \) to be a cyclic group of order 3. Let \( g : X \to H \) be as follows:

\[
g(x_j) = h^{(1+(-1)^j)/2} = \begin{cases} 1, & j = 1, 3, 5; \\ h, & j = 2, 4, 6. \end{cases}
\]

It is known that \( \hat{H} = \{1, \xi, \xi^2\} \), where \( \xi(h^i) = \omega^i, i = 0, 1, 2 \). Then the composition function \( \xi \circ g : X \to \mathbb{C} \) is just the function \( f \) in Example 6.3, and hence \( \xi \circ g \) is a bent function on \( X \). Similarly we can check that \( \xi^2 \circ g \) is also a bent function on \( X \). So \( g : X \to H \) is a \( G \)-perfect nonlinear function from the \( G \)-set \( X \) to the abelian group \( H \). In fact, one can check directly that

\[
g'_\alpha(x_j) = g(\alpha x_j)g(x_j)^{-1} = \begin{cases} 1, & j = 1, 2; \\ h, & j = 3, 5; \\ h^2, & i = 4, 6. \end{cases}
\]

Hence, \( g'_\alpha(h^i) = 2 \) for \( i = 0, 1, 2 \). Similarly, \( g'_\beta = g'_\gamma = 2 \) are constant functions on \( H \), too.

**Acknowledgments**

This work was done while the first author was visiting the second author at Eastern Kentucky University in Spring 2014; he is grateful for the hospitality. The work of the first author is supported by NSFC with grant numbers 11171194 and 11271005.
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