Semiclassical and quantum Liouville theory on the sphere

Pietro Menotti
Dipartimento di Fisica dell’Università, Pisa 56100, Italy
and INFN, Sezione di Pisa
e-mail: menotti@df.unipi.it

Gabriele Vajente
Scuola Normale Superiore, Pisa 56100, Italy
and INFN, Sezione di Pisa
e-mail: g.vajente@sns.it

Abstract

We solve the Riemann-Hilbert problem on the sphere topology for three singularities of finite strength and a fourth one infinitesimal, by determining perturbatively the Poincaré accessory parameters. In this way we compute the semiclassical four point vertex function with three finite charges and a fourth infinitesimal. Some of the results are extended to the case of $n$ finite charges and $m$ infinitesimal. With the same technique we compute the exact Green function on the sphere with three finite singularities. Turning to the full quantum problem we address the calculation of the quantum determinant on the background of three finite charges and the further perturbative corrections. The zeta function technique provides a theory which is not invariant under local conformal transformations. Instead by employing a regularization suggested in the case of the pseudosphere by Zamolodchikov and Zamolodchikov we obtain the correct quantum conformal dimensions from the one loop calculation and we show explicitly that the two loop corrections do not change such dimensions. We expect such a result to hold to all order perturbation theory.

\footnote{This work is supported in part by M.I.U.R.}
1 Introduction

The quantization of Liouville field theory has been subject to intensive study. The hamiltonian quantization on a compact space, i.e. the circle, has been carried through in [1, 2]; the main result is that provided one properly tunes the anomalous contribution to the energy momentum tensor, the regularized theory satisfies the Virasoro algebra.

From the functional point of view in the euclidean formulation a major difficulty is that in absence of sources there is no stable background with sphere topology. The situation is different in the case of the pseudosphere topology [3] where a stable background solution exists around which a perturbative expansion can be developed [3, 4, 5]. Divergences occur in the calculation of the graphs. In [3, 4, 5] it was proven that provided one chooses a proper regularization of the Green function at coincident points one reproduces the dimensions of the vertex operators and in particular of the cosmological term as derived in the hamiltonian approach in [1, 2].

On the sphere, on the other hand, it is well known that in order to have a stable background, sources have to be present, and such sources cannot be arbitrarily small as they are subject to the Picard inequalities. Such inequalities in particular tell us that the sources must be at least three. On the other hand the classical solution in presence of three singularities is known in terms of hypergeometric functions [6].

The purpose of the present paper is to develop a perturbation theory starting from the three point background such as to render the functional approach to Liouville theory self consistent and completely independent of the hamiltonian approach. To such end it is necessary to compute the Green function on the classical background of three generic charges.

This is done in Sect.3 where the problem of computing the semiclassical four point function with three finite charges plus a fourth infinitesimal is addressed. The result is obtained by solving perturbatively the Riemann-Hilbert problem, i.e. providing a perturbative calculation of the Poincaré accessory parameters which appears in the four singularity problem. The result is obtained in terms of quadratures and as a by product the exact Green function on the sphere with three arbitrary singularities is obtained, provided such singularities satisfy the necessary Picard bounds.

Several of the obtained results can be extended to the semiclassical $n + m$ point functions where $n$ charges are finite and the remaining $m$ small and this is done in Sect.4.

In order to produce the quantum correlation functions one has to proceed with the perturbative expansion where as in [3, 4, 5] the expansion parameter is the Liouville coupling constant $b$. 
The first problem is to compute the functional determinant of the linearized problem on the classical background. The problem is addressed in Sect. 6. The most natural tool in defining the functional determinant is the well known zeta function regularization. However such regularization gives the cosmological term weights which are in contrast with those required by local conformal invariance with the result that only the global $SL(2, C)$ invariance survives.

Thus a different regularization is developed for computing such a determinant. The procedure, which is similar in spirit to the procedure of [7] is to compute the variation of the determinant under small variation of the sources; the advantage of such a procedure is to expose the role of the value of the Green function at coincident points. If the Green function at coincident points is regularized according the proposal of Zamolodchikov and Zamolodchikov in the context of the pseudosphere [3], one obtains dimensions for the vertex operators which agree with those found in the hamiltonian approach and in particular the cosmological term assumes weight $(1, 1)$. The next problem is to examine the higher order correction. In Sect. 7 we perform a detailed computation of the two loop contributions and we find that using the same regularization such contributions do not alter the quantum anomalous dimensions found in the hamiltonian approach. In providing the necessary cancellations the boundary terms appearing in the action play a fundamental role.

In principle such computation can be carried on to arbitrary order perturbation theory and one could use it to furnish a perturbative analysis of the three-point vertex function conjectured in [8, 9] and derived in [10] and of higher point functions. Such investigation will be the subject of a future paper.

2 Classical Liouville theory

The regularized classical action for the Liouville theory on the sphere in presence of $N$ sources is given by [8, 17]

$$S_L[\phi] = \lim_{\varepsilon_n \to 0} \left\{ \int_{\Gamma_{\varepsilon,R}} \left[ \frac{1}{\pi} \partial_z \phi \partial_{\bar{z}} \phi + \mu e^{2b\phi} \right] \frac{dz \wedge d\bar{z}}{2} + \frac{Q}{2\pi i} \oint_{\partial \Gamma_R} \phi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + Q^2 \log R^2 - \frac{1}{2\pi i} \sum_{n=1}^N \alpha_n \oint_{\partial \Gamma_n} \phi \left( \frac{dz}{z - z_n} - \frac{d\bar{z}}{\bar{z} - \bar{z}_n} \right) - \sum_{n=1}^N \alpha_n^2 \log \varepsilon_n^2 \right\}$$

where $z_n$ and $\alpha_n$ are position and charge of the $n$-th source. The domain of integration is the region $\Gamma_{\varepsilon,R} = \{ |z| < R \} \cup \bigcup_n \{ |z - z_n| < \varepsilon_n \}$, $\partial \Gamma_R$ is the border around infinity
while $\partial \Gamma_n$ is the border around the $n$-th source. Here $Q$ is a parameter linked to the transformation law of the Liouville field. Classically its value is $Q = \frac{1}{b}$.

In order to examine the semiclassical limit of the $N$-point function it is useful [8] to go over to the field $\varphi = 2b\psi$. The corresponding charges are $\eta_n = \alpha_n b$ and the action takes the form

$$S[\varphi] = b^2 S_L[\varphi] = \int_{\Gamma_{l,r}} \left[ \frac{1}{4\pi} \partial_z \varphi \partial_{\bar{z}} \varphi + b^2 \mu e^\varphi \right] \frac{dz \wedge d\bar{z}}{2}$$

$$+ \frac{bQ}{4\pi i} \oint_{\partial \Gamma_R} \varphi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + (bQ)^2 \log R^2$$

$$- \frac{1}{4\pi i} \sum_{n=1}^N \eta_n \oint_{\partial \Gamma_n} \varphi \left( \frac{dz}{z-z_n} - \frac{d\bar{z}}{\bar{z}-\bar{z}_n} \right) - \sum_{n=1}^N \eta_n^2 \log \varepsilon_n^2. \tag{2}$$

The field $\varphi$ behaves like

$$\begin{cases} 
\varphi(z) = -2\eta_n \log |z-z_n|^2 + O(1) & \text{for } z \to z_n \\
\varphi(z) = -2bQ \log |z|^2 + O(1) & \text{for } z \to \infty. 
\end{cases} \tag{3}$$

We decompose the field $\varphi$ into the sum of a classical background $\varphi_B$ and a quantum fluctuation $\psi = 2b\chi$, $\varphi = \varphi_B + 2b\chi$. The action $S_L$ becomes

$$S_L[\varphi_B, \chi] = S_{cl}[\varphi_B] + S_Q[\varphi_B, \chi] \tag{4}$$

where

$$S_{cl}[\varphi_B] = \frac{1}{b^2} \left[ \frac{1}{8\pi} \int_{\Gamma} \left( \frac{1}{2} (\partial_z \varphi_B)^2 + 8\pi \mu b^2 e^{\varphi_B} \right) d^2 z \\
- \sum_{n=1}^N \left( \eta_n \frac{1}{4\pi i} \oint_{\partial \Gamma_n} \varphi_B \left( \frac{dz}{z-z_n} - \frac{d\bar{z}}{\bar{z}-\bar{z}_n} \right) + \eta_n^2 \log \varepsilon_n^2 \right) \\
+ \frac{1}{4\pi i} \oint_{\partial \Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \log R^2 \right] \tag{5}$$

and

$$S_Q[\varphi_B, \chi] = \frac{1}{4\pi} \int_{\Gamma} \left( (\partial_z \chi)^2 + 4\pi \mu e^{\varphi_B} (e^{2b\chi} - 1 - 2b\chi) \right) d^2 z$$

$$+ (2 + b^2) \ln R^2 + \frac{1}{4\pi i} \oint_{\partial \Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \frac{b}{2\pi i} \oint_{\partial \Gamma_R} \chi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right). \tag{6}$$

The terms in the second row arise from having chosen $Q = 1/b + b$. With regard to the classical action (5) we recall the Picard inequalities

$$\begin{cases} 
2\eta_n < 1 \\
\sum_{n=1}^N \eta_n > 1 
\end{cases} \tag{7}$$
which are implied the first by the finiteness of the area and the second by the Gauss-Bonnet theorem.

From the classical action (5) one derives the equation of motion

$$-\Delta \varphi + 8\pi \mu b^2 e^\varphi = 8\pi \sum_{n=1}^N \eta_n \delta^2(z - z_n)$$

whose solution is obtained [6] in terms of two independent solutions of the fuchsian equation

$$y''(z) + \left(\sum_{n=1}^N \frac{1 - \lambda_n^2}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)}\right) y(z) = 0.$$

Here in addition to the parameters $\lambda_n = 1 - 2\eta_n$ related to the charges, also the Poincaré accessory parameters $\beta_n$ appear. These are in principle fixed by the monodromy requirements on the field $\varphi$. It can be shown that these accessory parameters must obey three constraints

$$\begin{cases}
\sum_{n=1}^N \beta_n = 0 \\
\sum_{n=1}^N [2\beta_n z_n + (1 - \lambda_n^2)] = 0 \\
\sum_{n=1}^N [\beta_n z_n^2 + z_n(1 - \lambda_n^2)] = 0.
\end{cases}$$

In the case of only three singularities these constraints fix completely the form of the $\beta_n$, which we report here as we shall need them later

$$\begin{cases}
\beta_1 = \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - 1}{2(z_1 - z_2)} + \frac{\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - 1}{2(z_1 - z_3)} \\
\beta_2 = \frac{-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 1}{2(z_2 - z_3)} - \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - 1}{2(z_1 - z_2)} \\
\beta_3 = \frac{-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 1}{2(z_2 - z_3)} - \frac{\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - 1}{2(z_1 - z_3)}.
\end{cases}$$

The solution of equation (8) is given by

$$e^{\varphi_c} = \frac{1}{\pi \mu b^2} \frac{|w_{12}|^2}{(y_2 y_2' - y_1 y_1')^2}$$

where $w_{12} = y_1 y_2' - y_1' y_2$ is the constant wronskian and the two solutions $y_1$ and $y_2$ of (9) must be chosen in such a way that their monodromy group is $SU(1, 1)$ in order to ensure that the Liouville field $\varphi(z)$ is one-valued on the whole complex plane. In the case of only three singularities the conformal factor is given in terms of hypergeometric functions [6].

We recall two important relations to which the classical action, i.e. the action (5) computed on the solution of the equation of motion (8), is subject. The first is easily derived from the form of the action and reads

$$\frac{\partial S_{cl}}{\partial \eta_i} = -X_i$$
where \( X_i \) is the finite part of the field \( \varphi \) at \( z_i \)
\[
\varphi(z) = -2\eta_i \log |z - z_i|^2 + X_i + o(|z - z_i|).
\]

The second relation is the so called Polyakov relation [11, 12, 13, 14]
\[
\frac{\partial S_{cl}}{\partial z_i} = -\frac{\beta_i}{2}
\]
which directly relate the accessory parameters to the classical Liouville action.

Using these two relations it is possible to compute the semiclassical limit of the three-point function, which is related to the value of the classical action. We first consider the case of singularities placed in \( z_1 = 0, z_2 = 1, z_3 = \infty \). The finite part of the Liouville field \( \varphi \) can be computed starting from equation (12) and using the explicit form of the solutions \( y_1, y_2 \) [6]. The result is
\[
X_1 = -\log(\pi \mu b^2) - \log \frac{\gamma(\eta_1 + \eta_2 + \eta_3 - 1)\gamma(\eta_1 - \eta_2 + \eta_3)\gamma(\eta_1 + \eta_2 - \eta_3)}{\gamma^2(2\eta_1)\gamma(\eta_2 + \eta_3 - \eta_1)}
\]
and cyclic permutations. Here we have defined as usual \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \). Integrating the differential system (13) one obtains [8]
\[
S_{cl}[0, 1, \infty; \eta_1, \eta_2, \eta_3] = S_0 + \left( \eta_1 + \eta_2 + \eta_3 - \frac{3}{2} \right) \log(\pi \mu b^2) + 3F(1)
-F(2\eta_1) - F(2\eta_2) - F(2\eta_3) + F(\eta_1 + \eta_2 + \eta_3 - 1)
+F(\eta_3 + \eta_2 - \eta_1) + F(\eta_2 + \eta_1 - \eta_3) + F(\eta_3 + \eta_1 - \eta_2)
\]
where the new function \( F \) is given by
\[
F(x) = \int_{1/2}^{x} \log \gamma(s) ds.
\]
The next step is to compute the \( z_n \) dependence of the classical action, that is the semiclassical conformal dimensions of the Liouville vertex operators. To this purpose we use the Polyakov relation (15) and the explicit form of the accessory parameters for three singularities (11). Equation (15) can be easily integrated to give
\[
S_{cl}[z_1, z_2, z_3; \eta_1, \eta_2, \eta_3] = (\delta_1 + \delta_2 - \delta_3) \log |z_1 - z_2|^2 + (\delta_2 + \delta_3 - \delta_1) \log |z_2 - z_3|^2
+ (\delta_3 + \delta_1 - \delta_2) \log |z_3 - z_1|^2 + S_{cl}[0, 1, \infty; \eta_1, \eta_2, \eta_3]
\]
where \( \delta_i = \eta_i(1 - \eta_i) \).

In conclusion we have the following expression for the semiclassical limit of the three-point function
\[
\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle_{sc} = C_{sc}(\eta_1, \eta_2, \eta_3) |z_1 - z_2|^{-2(\Delta_1^{sc} + \Delta_2^{sc} - \Delta_3^{sc})} |z_2 - z_3|^{-2(\Delta_2^{sc} + \Delta_3^{sc} - \Delta_1^{sc})} |z_3 - z_1|^{-2(\Delta_3^{sc} + \Delta_1^{sc} - \Delta_2^{sc})}
\]
where
\[ C_{sc}(\eta_1, \eta_2, \eta_3) = \exp \left( -\frac{1}{b^2} S_{cl}[0, 1, \infty, \eta_1, \eta_2, \eta_3] \right) \]
\[ \Delta_{i}^{sc} = \alpha_{i} \left( \frac{1}{b} - \alpha_{i} \right) . \]  
(21)

From eq.(8) we can obtain the area of the surface
\[ A = \int e^{\phi_{c}} d^2 z = \frac{1}{\mu b^2} \left( \sum_{j} \eta_{j} - 1 \right) . \]  
(22)

### 3 The semiclassical four point function

In this section we shall determine the classical action in presence of three finite singularities and a fourth infinitesimal; such a calculation gives the semiclassical four point function for vertices with three finite charges and the fourth small. As a byproduct we shall derive in Sect.5 the exact Green function on the sphere with three arbitrary singularities satisfying the Picard bounds.

The conformal factor with three arbitrary conical defects can be computed in terms of hypergeometric functions and is well known [6]. The possibility of performing such a calculation is related to the fact that for three singularities the Fuchs relations (10) are sufficient for determining the accessory parameters \( \beta_j \). The procedure we shall use in presence of a fourth weak singularity is to solve perturbatively the fuchsian equation associated to the Liouville equation leaving the fourth small accessory parameter \( \beta_4 \) free, and then determine it by imposing the monodromy condition on the conformal factor.

Given four singularities, by means of an \( SL(2, C) \) transformation we can take three of them in \( 0, 1, \infty \). The position of the fourth will be called \( t \) and the coefficient \( Q \) in the fuchsian equation becomes
\[ Q(z) = \frac{1 - \lambda_1^2}{4 z^2} + \frac{1 - \lambda_2^2}{4(z - 1)^2} + \frac{1 - \lambda_3^2}{4(z - t)^2} + \frac{\beta_1}{2z} + \frac{\beta_2}{2(z - 1)} + \frac{\beta_4}{2(z - t)} . \]  
(23)

We notice that \( 1 - \lambda_i = 2 \eta_i \) and in the case of three singularities the Picard inequalities (7) impose \( 0 < \eta_i < 1/2 \). In presence of the fourth singularity the Fuchs relations
\[ \beta_1 = \frac{2 + 2(t - 1) \beta_4 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2} \]
\[ \beta_2 = -\frac{2 + 2t \beta_4 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2} \]  
(24)
are not sufficient to determine the \( \beta \)'s. For the source in \( t \) of infinitesimal strength we shall write \( \lambda_4 = 1 - 2 \varepsilon \) and \( \beta_4 = \varepsilon \beta \) and our aim will be to determine \( \beta \). We have
\[ \beta_1 = \frac{1 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2}{2} + \varepsilon [(t - 1) \beta + 2] + O(\varepsilon^2) \]
\[ \beta_2 = -\frac{1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2}{2} - \varepsilon [2 + t\beta] + O(\varepsilon^2) \]  

(25)

and we write

\[ Q(z) = Q_0(z) + \varepsilon q(z) \]  

(26)

where \( Q_0(z) \) stays for the coefficient of the three singularity problem, while \( q(z) \) is the perturbation

\[ q(z) = \frac{1}{2} \left[ \frac{(t-1)\beta + 2}{z} - \frac{2 + t\beta}{z-1} + \frac{\beta}{z-t} + \frac{2}{(z-t)^2} \right] \]  

(27)

After writing \( y = y_0 + \delta y \), being \( y_0 \) a solution of the unperturbed equation, we have to first order in \( \varepsilon \) the inhomogeneous equation

\[ (\delta y)'' + Q \delta y = -q y_0 . \]  

(28)

Such an equation can be solved by a well known method [15] and in our case we have

\[ \delta y_i = -\frac{1}{w_{12}} \int_{z_0}^z dx \ [y_1(x)y_2(z) - y_1(z)y_2(x)] \ q(x)y_i(x). \]  

(29)

being \( w_{12} = y_1 y_2' - y_1' y_2 \) the constant wronskian and \( z_0 \) an arbitrary base point in the complex plane. It will be useful to define the following integrals

\[ I_{ij}(z) \equiv \int_{z_0}^z dx \ y_i(x)y_j(x) \ q(x) \]  

(30)

and in term of them we have the following two independent solutions of the perturbed problem

\[ Y_1(z) = \left[ 1 + \varepsilon \frac{I_{12}(z)}{w_{12}} \right] y_1(z) - \varepsilon \frac{I_{11}(z)}{w_{12}} y_2(z) \]

\[ Y_2(z) = \varepsilon \frac{I_{22}(z)}{w_{12}} y_1(z) + \left[ 1 - \varepsilon \frac{I_{12}(z)}{w_{12}} \right] y_2(z). \]  

(31)

We must now compute the monodromy matrices around 0, 1, t and impose on them the \( SU(1,1) \) nature. This will determine uniquely the parameter \( \beta \) and thus the perturbed conformal factor. The calculation is given in Appendix A. We find

\[ \beta = -\frac{4\kappa}{\bar{y}_1 y_1' - \bar{y}_2 y_2'} \]  

(32)

being \( \kappa = |k_0|^4 \) with \( k_0 \) the parameter which appears in the three-singularity conformal factor

\[ e^{2\phi_0} = \frac{1}{\pi \mu b^2} \frac{w_{12}^2}{(|k_0|^2 y_1 - |k_0|^{-2} y_2)^2} \]  

(33)
and
\[ e^{2\Delta_e \phi_c} = e^{\varphi_c} = \frac{1}{\pi \mu b^2} \frac{w_{12}^2}{(Z_1 \bar{Z}_1 - Z_2 \bar{Z}_2)^2} \]  

with
\[ Z_1(z) = k_0 \left[ \left( 1 + \varepsilon \frac{I_{12}(z) + h}{w_{12}} \right) y_1(z) - \varepsilon \frac{I_{11}(z)}{w_{12}} y_2(z) \right] \]
\[ Z_2(z) = \frac{1}{k_0} \left[ \varepsilon \frac{I_{22}(z)}{w_{12}} y_1(z) + \left( 1 - \varepsilon \frac{I_{12}(z) + h}{w_{12}} \right) y_2(z) \right] \]  

where the $h$ is also given in Appendix A. The functions $Z_1, Z_2$ have $SU(1, 1)$ monodromies around all singularities and as such determine a globally monodromic conformal factor satisfying the Liouville equation. We can now compute the conformal factor in presence of our four sources to first order in $\varepsilon$

\[ e^{\varphi_c} = e^{\varphi_c^0} \left\{ 1 - \varepsilon \frac{2}{w_{12}(\kappa y_1 \bar{y}_1 - y_2 \bar{y}_2)} \right\} \left[ \left( \kappa y_1 \bar{y}_1 + y_2 \bar{y}_2 \right) \left( I_{12} + \bar{I}_{12} + h + \bar{h} \right) \right. \\
- y_1 \bar{y}_2 \left( I_{22} + \kappa \bar{I}_{11} \right) - \bar{y}_1 y_2 \left( \bar{I}_{22} + \kappa I_{11} \right) \right] + O(\varepsilon^2) \}
\[ \equiv e^{\varphi_c^0} (1 + \varepsilon \chi + O(\varepsilon^2)). \]  

Thus
\[ \varphi_c(z, \bar{z}) = \varphi_c^0(z, \bar{z}) + \varepsilon \chi(z, \bar{z}) + O(\varepsilon^2). \]  

The perturbation $\chi$ has a singularity in $t$
\[ \chi(z, \bar{z}) \sim -4 \log |z - t| + o(t) + o(z - t) \quad \text{for } z \to t \]  

and in Sect.5 it is proven that $\chi$ is regular in $0, 1, \infty$. Eq.(32) gives the value of $\beta_4$ to first order $\beta_4 = \varepsilon \beta$. Recalling the expression of the unperturbed conformal factor $e^{\varphi_c^0}$ with only three sources we have
\[ \beta_4 = -4\varepsilon \frac{e^{\varphi_c^0/2} \partial_z e^{-\varphi_c^0/2}}{\partial t} \bigg|_{z=t} = 2\varepsilon \frac{\partial_z \varphi_c^0(z)}{\partial t} \bigg|_{z=t}. \]  

We can exploit such a result and Polyakov relation to compute to order $\varepsilon$ the classical action for the new solution
\[ \frac{\partial S_{cl}[\eta_1, \eta_2, \eta_3, \varepsilon]}{\partial t} = -\frac{\beta_4}{2} = -\varepsilon \frac{\partial \varphi_c^0}{\partial t} \]  

and thus
\[ S_{cl}[0, 1, \infty; t; \eta_1, \eta_2, \eta_3, \eta_4] = S_{cl}[0, 1, \infty; \eta_1, \eta_2, \eta_3] - \eta_4 \varphi_c^0(t) + \eta_4 f(\eta_1, \eta_2, \eta_3) + O(\eta_4^3). \]  


We exploit now eq.(13)

\[
- \frac{\partial S_d(0,1,\infty,t,\eta_1,\eta_2,\eta_3,\eta_4)}{\partial \eta_4} = \varphi^0_c(t) - f(\eta_1,\eta_2,\eta_3) + O(\eta_4) = \\
\text{Finite}_{z \to t} \left( \varphi^0_c(z) + \eta_1(-4 \log |z - t| + c(t) + o(z - t)) + O(\eta_4^2) = \\
\varphi^0_c(t) + \eta_4 c(t) + O(\eta_4^2) \right) 
\]

from which \( f(\eta_1,\eta_2,\eta_3) = 0. \)

Thus for the semiclassical four point function with small \( \alpha_4 \) we have

\[
\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)V_{\alpha_4}(t) \rangle_{sc} = \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle_{sc} e^{2\alpha_4 \varphi^0_c(t)}. \tag{43} \]

It is easily checked that the four point function (43) has the correct transformation properties with dimensions \( \alpha_4/b \) for the vertex field \( V_{\alpha_4}(z_4) \) in agreement with the semiclassical dimensions \( \alpha_4(1/b - \alpha_4) \) keeping in mind that we have been working to first order in \( \alpha_4 \), and thus we can write to first order in \( \alpha_4 \)

\[
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4) \rangle_{sc} = \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3) \rangle_{sc} e^{2\alpha_4 \varphi^0_c(z_4)}. \tag{44} \]

4 Generalization to \( n \)-point functions

We can generalize some of the results obtained above to \( n \) arbitrary sources and \( m \) infinitesimal sources. Let us start from the case in which we have \( n \) finite sources plus one infinitesimal. Let us suppose to know a pair of solutions \( y_1, y_2 \) which produce the monodromic conformal factor with \( n \) finite sources and which we know to exist. The discussion we have been performing on the case of three sources which leads to the inhomogeneous equation (28) remains valid also in this case; the only difference is that now we do not know the explicit form of the unperturbed solutions \( y_1, y_2 \). Let us suppose the first three finite sources to be in \( 0,1,\infty \). Imposition of monodromy in 1 fixes the parameter \( k \) it happens in the case of three finite singularities plus one infinitesimal. Then the accessory parameter \( \beta_t = \varepsilon \beta \) is again given by eq.(39)

\[
\beta = 2 \partial_z \varphi^0_c(z) \bigg|_{z=t} \tag{45} \]

where now \( \varphi^0_c \) is the conformal field which solves the problem in presence of the \( n \) finite sources. Thus we have a general relation between the value of the accessory parameter relative to the infinitesimal source in \( t \) and the conformal factor for the unperturbed background and thus we can extend the result (44) to \( n \) finite sources plus an infinitesimal
one. Finally due to the additive nature of the perturbation with \( m \) infinitesimal sources we have in this case for the classical action

\[
S_{cl}[z_1, \ldots, z_n, t_1, \ldots, t_m; \eta_1, \ldots, \eta_n, \varepsilon_1, \ldots, \varepsilon_m] = S[z_1, \ldots, z_n; \eta_1, \ldots, \eta_n] - \sum_{j=1}^{m} \varepsilon_j \varphi^0_c(t_j)
\]  

(i.e. the \( n + m \) semiclassical correlation function has the value

\[
\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) V_{\gamma_1}(t_1) \ldots V_{\gamma_m}(t_m) \rangle_{sc} = \langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) \rangle_{sc} \prod_{j=0}^{m} e^{2\gamma_j \phi^0_c(t_j)}. \]

We stress again that the difference between the case of \( n \) finite sources plus \( m \) infinitesimal ones and the case of three finite sources and one infinitesimal is that in the latter case we have an explicit form, in terms of quadratures, of the four point function.

5 The Green function on the sphere with three singularities

From the above derived results we can extract the exact Green function on the sphere in presence of three finite singularities. The equation for the Green function is

\[
-\Delta g(z, t) + 8\pi \mu b^2 e^{\varphi_B(z)} g(z, t) = 2\pi \delta^2(z - t)
\]

where \( \varphi_B \) is the classical solution in presence of three finite singularities. Such a Green function can be computed from the result obtained in Sect.3. In fact we have found a solution to

\[
-\Delta \varphi + 8\pi \mu b^2 e^{\varphi} = 8\pi \sum_{i=1}^{3} \eta_i \delta^2(z - z_i) + 8\pi \varepsilon \delta^2(z - t)
\]

for infinitesimal \( \varepsilon \) i.e. \( \varphi = \varphi_B + \varepsilon \chi \). Substituting we obtain

\[
-\Delta \chi + 8\pi \mu b^2 e^{\varphi_B} \chi = 8\pi \delta^2(z - t)
\]

i.e. we have \( g(z, t) = \frac{\chi}{4} \). From eq.(36) we have

\[
g(z, t) = -\frac{1}{2w_{12} \left[ \kappa y_1(z) \bar{y}_1(\bar{z}) - y_2(z) \bar{y}_2(\bar{z}) \right]} \left[ \kappa y_1(z) y_1(z) + y_2(z) \bar{y}_2(\bar{z}) \right] \cdot \left[ I_{12}(z, t) + \bar{I}_{12}(\bar{z}, \bar{t}) + h(t) + \bar{h}(\bar{t}) \right] - y_1(z) \bar{y}_2(\bar{z}) \left[ I_{22}(z, t) + \kappa \bar{I}_{11}(\bar{z}, \bar{t}) \right] - \bar{y}_1(\bar{z}) y_2(z) \left[ \bar{I}_{22}(\bar{z}, \bar{t}) + \kappa I_{11}(z, t) \right].
\]
It is possible to verify directly that (51) satisfies eq.(48) by using
\[
\frac{\partial I_{ij}(z,t)}{\partial z} = y_i(z)y_j(z)q(z)
\] (52)
and the fact that the wronskian \(w_{12} = y_1(z)y'_2(z) - y'_1(z)y_2(z)\) is constant and real. From this it follows that expression (51) is completely general, i.e. it applies also for the case of a background given by \(n\) finite sources with \(y_i\) solutions of the related fuchsian equation. Again the difference with the \(n = 3\) case is that in the latter case we know the explicit form of \(y_i\) which are not known for \(n > 3\). Equation (48) for the Green function is obviously invariant under \(SL(2,C)\) transformations. Thus we can write the Green function for a general location of the singularities in terms of the standard one with singularities in \(0, 1, \infty\)
\[
g(z_1, z_2, z_3; z, t) = g\left(0, 1, \infty; \frac{(z - z_1)(z_3 - z_2)}{(z - z_1)(z_2 - z_1)}, \frac{(t - z_1)(z_3 - z_2)}{(z_3 - t)(z_2 - z_1)}\right).
\] (53)
It is possible to check that the explicit from (51) has the correct logarithmic singularity at \(t\)
\[
g(z, t) \sim -\frac{1}{2} \log|z - t|^2 + \ldots \quad \text{for } z \to t.
\] (54)
In addition \(g(z, t)\) is regular on the three finite sources. By symmetry it is sufficient to check this e.g. at \(z = 0\). The integrals \(I_{ij}(z)\) vanish for \(z = 0\) and thus for \(z\) near to 0 we have
\[
g(z, t) \sim -\frac{1}{2w_{12}|z|^{1 - \Delta}} \left(h + \bar{h}\right) = -\frac{h + \bar{h}}{2w_{12}}.
\] (55)
Such regularity result allows us to derive a simple relation which will be useful in the following; integrating eq.(48) on the whole complex plane excluding a disk of radius \(\varepsilon\) around \(t\), and then letting \(\varepsilon\) going to zero we obtain
\[
\int e^{\phi_B(z)} g(z, t) \, d^2z = 2\pi.
\] (56)
One would expect the Green function \(g(z, t)\) to be symmetric in the arguments. This is far from evident from the expression (51). The differential operator \(D = -\Delta_{LB} + 1\) is hermitean in the background metric \(e^{\phi_B} d^2z\). As a result also its inverse \(G = D^{-1}\) is hermitean \(G = G^+\). \(G\) is represented by \(g(x, t)\) which is also real and thus we have \(g(z, t) = g(t, z)\).

6 The quantum determinant

The complete action is given by eqs.(5) and (6) and the quantum \(n\)-point function by
\[
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \ldots V_{\alpha_n}(z_n) \rangle = e^{-S_{\alpha}[\phi_B]} \int D[\chi] \, e^{-S_{\phi}}.
\] (57)
We recall that $S_d$ is $O(1/b^2)$ while the quantum action is

$$S_Q[\varphi_B, \chi] = \frac{1}{2\pi} \int_\Gamma \left( \frac{1}{2} (\partial_a \chi)^2 + 2\pi \mu e^{\varphi_B} (e^{2b\chi} - 1 - 2b\chi) \right) d^2 z$$

$$+ \frac{1}{4\pi i} \oint_{\partial\Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + 2 \log R^2$$

$$+ \frac{b}{2\pi i} \oint_{\partial\Gamma_R} \chi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + b^2 \ln R^2$$

(58)

where the first integral can be expanded as

$$\frac{1}{4\pi} \int_\Gamma \left( (\partial_a \chi)^2 + 4\pi \mu e^{\varphi_B} (e^{2b\chi} - 1 - 2b\chi) \right) d^2 z =$$

$$\frac{1}{4\pi} \int_\Gamma \left( (\partial_a \chi)^2 + 8\pi \mu b^2 e^{\varphi_B} \chi^2 + 8\pi \mu b^2 e^{\varphi_B} \left( \frac{4b^3 \chi^3}{3!} + \frac{8b^2 \chi^4}{4!} + \ldots \right) \right) d^2 z.$$  (59)

From now on we shall denote by $\varphi_B$ the classical solution with three singularities at $z_1, z_2, z_3$ and with charges $\eta_1, \eta_2, \eta_3$. In performing the perturbative expansion in $b$ we have to keep the $\eta_1, \eta_2, \eta_3$ constant [8]. The $O(b^0)$ contribution to the three point function is given by

$$(\text{Det}D)^{-\frac{1}{2}} = \int D[\chi] e^{-\int f(z) D\chi(z) f(z) d^2 z}$$

(60)

where $f(z) = 8\pi \mu b^2 e^{\varphi_B(z)}$ and

$$D = \frac{1}{4\pi} (-\Delta_{LB} + 1)$$

(61)

being $\Delta_{LB} = f^{-1}\Delta$ the Laplace-Beltrami operator on the background $f(z)$. It provides the one loop quantum correction to the semiclassical result we have been discussing above.

The usual technique for defining the functional determinant is provided by the $Z$-function procedure

$$Z_D(s) = \text{Tr}(D^{-s}) = \sum_n \lambda_n^{-s}$$

(62)

and

$$Z_D'(0) = -\log(\text{Det}D) = \gamma_E Z_D(0) + \text{Finite}_{\epsilon \to 0} \int_0^\infty \frac{dt}{t} \text{Tr}(e^{-tD}).$$

(63)

We notice that the operator $D$ is covariant under $SL(2, \mathbb{C})$ transformations, thus all the eigenvalues are invariant, and this gives rise to a definition of $\text{Det}D$ invariant under $SL(2, \mathbb{C})$. This means that the semiclassical result for the dimensions of the vertex fields $e^{2\alpha\phi}$

$$\Delta^{sc}(\alpha) = \alpha \left( \frac{1}{b} - \alpha \right) = \frac{1}{b^2} \eta(1 - \eta)$$

(64)

does not receive any $O(b^0)$ correction, as further quantum corrections which start from order $O(b^2)$ cannot alter such a result. The situation is similar to the one discussed by
D’Hoker, Freedman and Jackiw [16] and the one considered by Takhtajan [17] with the use of an invariant regularization of the Green function (Hadamard regularization [18, 19]) and $Q = 1/b$. As already mentioned in the introduction such a regularization scheme gives the cosmological term the weights $(1 - b^2, 1 - b^2)$. In the following we shall pursue a different regularization scheme.

To compute the determinant (60) we shall use a variational procedure similar to the one employed in the standard heat kernel approach [7], with the simplifying feature that we shall compute simply the derivative with respect to the three parameters $\eta_j$. We have

$$\frac{\partial}{\partial \eta_j} \left( \log(\text{Det}D)^{-\frac{1}{2}} \right) = -2\pi b^2 \int \frac{\partial \varphi_B}{\partial \eta_1}(z) g(z, z) e^{\varphi_B(z)} d^2 z. \quad (65)$$

In the above equation the Green function at coincident points appears and such a quantity has to be regularized. We have already seen that the invariant Hadamard regularization gives rise to a theory in which the cosmological term $e^{2b\phi(z)}$ does not have weights $(1, 1)$ and as such does not give rise to a theory invariant under the whole (infinite dimensional) conformal group.

We shall adopt here the regularization proposed by Zamolodchikov and Zamolodchikov [3] (ZZ regulator) for perturbative calculations on the pseudosphere i.e.

$$g(z, z) = \lim_{z' \to z} (g(z, z') + \log |z - z'|). \quad (66)$$

As under an $SL(2, C)$ transformation

$$w = \frac{az + b}{cz + d} \quad (67)$$

the Green function is invariant in value

$$g^w(w, w') = g(z, z') \quad (68)$$

we have

$$g^w(w, w) = g(z, z) + \log \left| \frac{\partial w}{\partial z} \right| = g(z, z) + \log \left| \frac{1}{(cz + d)^2} \right|. \quad (69)$$

The above relation will play a major role in all subsequent developments.

Let us consider first the dilatation $w = \lambda z$; we have $g^w(w, w) = g(z, z) + \log |\lambda|$ and thus using eq.(65) keeping in mind that from eq.(22)

$$2\pi b^2 \int \frac{\partial \varphi_B}{\partial \eta_1}(z) e^{\varphi_B(z)} d^2 z = 2 \quad (70)$$

we have

$$\frac{\partial}{\partial \eta_j} \log \left( (\text{Det}D)^{-\frac{1}{2}}(\lambda z_1, \lambda z_2, \lambda z_3) \right) = \frac{\partial}{\partial \eta_j} \log \left( (\text{Det}D)^{-\frac{1}{2}}(z_1, z_2, z_3) \right) - 2 \log |\lambda| \sum_j \eta_j \quad (71)$$
where we evidenced the dependence of the determinant on the position of the singularities. We remark that such a result is consistent with the known exact structure of the three-point function given by eq.(20) with $\Delta_j$ replaced by the quantum anomalous dimensions

$$\Delta_j = \alpha_j (Q - \alpha_j) = \Delta_j^{\text{sc}} + b \alpha_j = \frac{1}{b^2} \eta_j (1 - \eta_j) + \eta_j .$$

(72)

Thus what is left is to extend the above argument to the general $SL(2, C)$ transformation and to prove that the subsequent perturbative corrections $O(b^{2n})$ do not alter the $z_j$ dependence of the three-point function. We want to express

$$\frac{\partial}{\partial \eta_j} \log \left( (\text{Det} D)^{-\frac{1}{2}} (z_1, z_2, z_3) \right)$$

in terms of

$$\frac{\partial}{\partial \eta_j} \log \left( (\text{Det} D)^{-\frac{1}{2}} (0, 1, \infty) \right).$$

(74)

The transformation which takes from $z$ to $w$ is given by

$$w = \frac{(z - z_1)(z_3 - z_2)}{(z_3 - z)(z_2 - z_1)}$$

(75)

and consequently

$$\varphi_B(z) = \varphi_B^w(w) + 2 \log \left| \frac{(z_3 - z_1)(z_3 - z_2)}{(z_3 - z)^2(z_2 - z_1)} \right|$$

(76)

and for the Green function at coincident points

$$g^w(w, w) = g(z, z) + \log \left| \frac{(z_3 - z_1)(z_3 - z_2)}{(z_3 - z)^2(z_2 - z_1)} \right| .$$

(77)

We obtain

$$\frac{\partial}{\partial \eta_1} \log \left( (\text{Det} D)^{-\frac{1}{2}} (z_1, z_2, z_3) \right) = -2 \mu b^2 \int \frac{\partial \varphi_B}{\partial \eta_1} (z) g(z, z)e^{\varphi_B(z)}d^2 z =$$

$$-2 \mu b^2 \log \left| \frac{z_2 - z_1}{(z_3 - z_1)(z_3 - z_2)} \right| \frac{\partial A}{\partial \eta_1} - 2 \mu b^2 \int \frac{\partial \varphi_B}{\partial \eta_1} (w) g^w(w, w)e^{\varphi_B^w(w)}d^2 w$$

$$-2 \mu b^2 \int \frac{\partial \varphi_B}{\partial \eta_1} (z) \log |z - z_3|^2 e^{\varphi_B(z)}d^2 z =$$

$$-2 \log \left| \frac{z_2 - z_1}{(z_3 - z_1)(z_3 - z_2)} \right| - 2 \mu b^2 \int \frac{\partial \varphi_B}{\partial \eta_1} (w) g^w(w, w)e^{\varphi_B^w(w)}d^2 w - \frac{\partial X_3}{\partial \eta_1} + \frac{\partial X_\infty}{\partial \eta_1}$$

(78)

being $X_3$ and $X_\infty$ the finite parts of the field $\varphi_B(z)$ at $z_3$ and $\infty$. The last three terms are the result of performing the integral containing $\log |z - z_3|^2$ which can be computed
by using the equation for the field $\varphi_B(z)$. Thus

$$
\frac{\partial}{\partial \eta_1} \log \left( (\text{Det} D)^{-\frac{1}{2}}(z_1, z_2, z_3) \right) = 
-2\mu b^2 \int \frac{\partial \varphi^w_B(z)}{\partial \eta_1} g^w(w, w) e^{\varphi^w_B(z)} d^2w - 2 \log \left| \frac{(z_1 - z_2)(z_1 - z_3)}{(z_2 - z_3)} \right| 
- \frac{\partial X_3}{\partial \eta_1} + \frac{\partial X_\infty}{\partial \eta_1}
$$

(79)

and similarly

$$
\frac{\partial}{\partial \eta_2} \log \left( (\text{Det} D)^{-\frac{1}{2}}(z_1, z_2, z_3) \right) = 
-2\mu b^2 \int \frac{\partial \varphi^w_B(z)}{\partial \eta_2} g^w(w, w) e^{\varphi^w_B(z)} d^2w - 2 \log \left| \frac{(z_2 - z_3)(z_2 - z_1)}{(z_3 - z_1)} \right| 
- \frac{\partial X_3}{\partial \eta_2} + \frac{\partial X_\infty}{\partial \eta_2}.
$$

(80)

On the other hand for the derivative with respect to $\eta_3$ we find

$$
\frac{\partial}{\partial \eta_3} \log \left( (\text{Det} D)^{-\frac{1}{2}}(z_1, z_2, z_3) \right) = 
-2\mu b^2 \int \frac{\partial \varphi^w_B(z)}{\partial \eta_3} g^w(w, w) e^{\varphi^w_B(z)} d^2w - 2 \log \left| \frac{(z_2 - z_1)}{(z_3 - z_1)(z_3 - z_2)} \right| 
- \frac{\partial X_3}{\partial \eta_3} + \frac{\partial X_\infty}{\partial \eta_3}.
$$

(81)

Recalling now eqs.(13,19) we see that

$$
\frac{\partial X_3}{\partial \eta_1} = -\frac{\partial S_{cl}[z_1, z_2, z_3, \eta_1, \eta_2, \eta_3]}{\partial \eta_1 \partial \eta_3}
$$

(82)

is independent of $z_i$, and similarly for $\frac{\partial X_3}{\partial \eta_2}$ while we have

$$
\frac{\partial X_3}{\partial \eta_3} = -\frac{\partial S_{cl}[z_1, z_2, z_3, \eta_1, \eta_2, \eta_3]}{\partial \eta_3^2} - 4 \log \left| \frac{z_1 - z_2}{(z_2 - z_3)(z_1 - z_3)} \right|.
$$

(83)

We must now take into account the additional boundary term contribution in the action (58)

$$
\frac{1}{4\pi i} \oint_{\Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + 2 \log R^2 = X_\infty
$$

(84)

which has its origin in the fact that $Q$ is not 1/b but 1/b + b. Summing up we have for $j = 1, 2, 3$, $k = 2, 3, 1$, $l = 3, 1, 2$

$$
\frac{\partial}{\partial \eta_j} \log((\text{Det} D)^{-\frac{1}{2}}) - \frac{\partial X_\infty}{\partial \eta_j} = f_j(\eta_1, \eta_2, \eta_3) - 2 \log \left| \frac{(z_j - z_k)(z_j - z_l)}{(z_j - z_l)} \right|.
$$

(85)
Integration of the above equation gives
\[ c(\eta_1, \eta_2\eta_3) - 2(\eta_1 + \eta_2 - \eta_3) \log |z_1 - z_2| - 2(\eta_2 + \eta_3 - \eta_1) \log |z_2 - z_3| - 2(\eta_3 + \eta_1 - \eta_2) \log |z_3 - z_1| \]
(86)
as \(O(b^0)\) correction i.e. the one loop correction, and we have obtained the three-point function with the correct quantum dimensions \(\Delta_j = \eta_j (1 - \eta_j) / b^2 + \eta_j\). Thus the situation is very similar to what happens on the pseudosphere, where the one loop corrections with the ZZ regulator provide the exact quantum dimensions. On the pseudosphere is not too difficult to prove that higher loop corrections do not change the quantum anomalous dimensions of the vertex fields \(e^{2\alpha \phi}\). On the sphere due to the appearance of the boundary terms the analysis is more complicated. We shall give here below the explicit calculation proving that at two loop no change is induced in the dimensions of the vertex fields.

7 Two loop calculation

In this section we shall compute explicitly the two loop corrections to the three point function. The interaction vertices relevant to the two loop calculation are
\[ \frac{b}{6\pi} f(z) \chi^3(z), \quad \frac{b^2}{12\pi} f(z) \chi^4(z). \]
(87)
with \(f(z) = 8\pi \mu b^2 e^{\varphi_B(z)}\). One should not forget the interaction originating from the boundary term in eq.(58) with action given by
\[ S_B = \frac{b}{2\pi i} \oint_{\partial R} \chi(z) \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right). \]
(88)
The relevant graphs are shown in fig.1. The explicit computation gives
\[ (a) = \frac{b^2}{8\pi^2} \int g(z, z) f(z) d^2 z g(z, z') d^2 z' f(z') g(z', z') \]
(89)
\[ (b) = -\frac{b^2}{4\pi} \int g(z, z) f(z) g(z, z) d^2 z \]
(90)
\[ (c) = \frac{b^2}{4\pi^2 i} \oint_{\partial R} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) g(z, z') f(z') d^2 z' g(z', z') \]
(91)
and
\[ (d) = \frac{b^2}{2(2\pi i)^2} \oint_{\partial R} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) g(z, z') \oint_{\partial R} \left( \frac{dz'}{z'} - \frac{d\bar{z}'}{\bar{z}'} \right). \]
(92)
With regard to contribution \((d)\) we notice it can be rewritten by performing the change of variable \(w = -1/z\) as follows
\[ (d) = b^2 \log R^2 - \frac{b^2}{2\pi} \int_0^{2\pi} \log(4 \sin^2 \phi) \, d\phi + 2b^2 g^w(0, 0). \]
(93)
Fig. 1
We notice that the divergent part in log $R^2$ is canceled by the term $-2b^2\log R$ appearing to order $b^2$ in the expansion of $\exp(-S)$ and this assures the finiteness of the $O(b^2)$ computation for $R \to \infty$.

We have now to compute the change of the sum $(a) + (b) + (c) + (d)$ under an $SL(2,C)$ transformation. For doing that it is simpler to prove separately the invariance under translations, dilatations and inversions.

The invariance under translations is trivial; with regard to dilatations $w = \lambda z$, $(a)$ goes over to

$$
(a') = \frac{b^2}{8\pi^2} \int g^w(w, w)f^w(w)d^2w g^w(w, w')d^2w' f^w(w')g^w(w', w') =
$$

$$
(a) + \frac{b^2}{4\pi^2} \log |\lambda| \int f(z)d^2z g(z, z')d^2z' f(z')g(z', z')
$$

$$
+ \frac{b^2}{8\pi^2} (\log |\lambda|)^2 \int f(z)d^2z g(z, z')d^2z' f(z')
$$

(94)

while $(b)$ becomes

$$
(b') = (b) - \frac{b^2}{2\pi} \log |\lambda| \int f(z)g(z, z)d^2z - \frac{b^2}{4\pi}(\log |\lambda|)^2 \int f(z)d^2z.
$$

(95)

Using the relation

$$
\int f(z) g(z, z') d^2z = 2\pi
$$

(96)

we see that the variation of $(a) + (b)$ vanishes. Similarly one finds using eq.(93) that the variation of $(c) + (d)$ vanishes.

We are left to prove the invariance under the inversion $w = -1/z$. The variation of $(a)$ is computed as usual by performing integrations by parts by

$$
(a') = (a) - \frac{ib^2}{2\pi^2} \oint_{\partial \Gamma_R} d\tilde{z} \log z\tilde{z} \partial_{\tilde{z}} g(z, z')d^2z' f(z') g(z', z')
$$

$$
- \frac{b^2}{2\pi} \int d^2z' \log z\tilde{z} g(z', z') - \frac{ib^2}{2\pi^2} \oint_{\partial \Gamma_R} \frac{dz}{z} g(z, z') f(z')d^2z' g(z', z')
$$

$$
- \frac{b^2}{\pi} \int g(0, z')d^2z' f(z')g(z', z') + \frac{ib^2}{4\pi^2} \oint_{\partial \Gamma_R} d\tilde{z}\partial_{\tilde{z}} g(z, z') \log z\tilde{z} f(z')d^2z' \log z\tilde{z}'
$$

$$
+ \frac{b^2}{4\pi} \int \log z\tilde{z}'d^2z' f(z') \log z\tilde{z}' + \frac{ib^2}{4\pi^2} \oint_{\partial \Gamma_R} g(z, z') \frac{dz}{z} f(z')d^2z' \log z\tilde{z}'
$$

$$
+ \frac{b^2}{2\pi} \int g(0, z')f(z')d^2z' \log z\tilde{z}'
$$

(97)

where the first and the fifth integral vanish in the $R \to \infty$ limit due to the appearance of $\partial_{\tilde{z}} g(z, z')$. On the other hand the variation of $(b)$ is given by

$$
(b') = (b) + \frac{b^2}{2\pi} \int \log z\tilde{z} f(z)d^2z g(z, z) - \frac{b^2}{4\pi} \int (\log z\tilde{z})^2 f(z)d^2z.
$$

(98)
Similarly one computes the variation of \((c)\)

\[
(c') = (c) + \frac{b^2}{\pi} \int g(0,z) d^2 z f(z) g(z,z) + \frac{b^2}{4\pi^2 i} \oint_{\partial \Gamma_R} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) g(z,z') d^2 z' f(z') g(z',z)
\]

\[- \frac{b^2}{\pi} \int g(0,z) d^2 z f(z) \log zz \]

and the variation of \((d)\)

\[
(d') = (d) - \frac{b^2}{8\pi^2} \left( \oint_{\partial \Gamma_{1/R}} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) g(z,z') \oint_{\partial \Gamma_{1/R}} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \right) - \oint_{\partial \Gamma_R} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) g(z,z') \oint_{\partial \Gamma_R} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right). \]

Performing integrations by parts in the third and fourth integrals appearing in eq.(97) we find that the variation of \((a) + (b) + (c) + (d)\) under inversion vanishes and thus we have invariance under the whole \(SL(2,C)\) group. In Appendix B we give the details of the calculation.

8 Outlook and conclusions

In this paper we have provided a functional approach to the quantum Liouville theory on the sphere, which does not make any appeal to the canonical quantization. We start from the stable classical background solution in presence of three finite sources satisfying the Picard bounds. We compute the semiclassical four point function with three finite sources and the fourth weak which as a by product furnishes in terms of quadratures the exact Green function on the background generated by three finite sources. Several of the results obtained for the four point functions extend to the case of the semiclassical vertex functions with \(n\) finite charges and \(m\) infinitesimal charges. The lowest order quantum correction is provided by the determinant of the linearized problem on the Liouville three-source background. The regularization suggested by Zamolodchikov and Zamolodchikov in the context of the pseudosphere [3] gives rise to the quantum dimensions found in the hamiltonian approach while the invariant (Hadamard) regularization of the Green function fails to give the cosmological term the weight \((1,1)\). An explicit calculation shows that the two loop correction do not alter such dimensions. We expect such a result to hold to all order perturbation theory even if the presence of the contour terms, which are essential to obtain the two loop result, makes the calculation not so straightforward as in the case of the pseudosphere. The obtained results can be useful to perform a perturbative analysis of the three-point vertex function conjectured in [8, 9] and derived in [10] and of higher point functions.
Acknowledgments

One of us (P.M.) is grateful to Domenico Seminara for a useful discussion.

Appendix A

We give here the calculations of the monodromy matrices around 0, 1, $\infty$, $t$ and the procedure to impose on all of them the $SU(1, 1)$ nature.

The behavior of the canonical unperturbed solutions around 0 is

$$y_1(z) \simeq z^{\frac{1-\lambda_1}{2}}, \quad y_2(z) \simeq z^{\frac{1+\lambda_1}{2}} \quad \text{and} \quad q(z) \simeq \frac{(t - 1)\beta + 2}{2z} + O(1) \quad (101)$$

and one can obtain the behavior of the integrals $I_{ij}(z)$ as follows

$$I_{11}(z) \simeq I_{11}(0) + \int_0^z dx \; x^{1-\lambda_1}\frac{(t - 1)\beta + 2}{2} x^{-1} = I_{11}(0) + \frac{(t - 1)\beta + 2}{2(1 - \lambda_1)} z^{1-\lambda_1} \quad (102)$$

$$I_{12}(z) \simeq I_{12}(0) + \frac{(t - 1)\beta + 2}{2} z \quad (103)$$

$$I_{22}(z) \simeq I_{22}(0) + \frac{(t - 1)\beta + 2}{2(1 + \lambda_1)} z^{1+\lambda_1} \quad (104)$$

and of the perturbed solutions

$$Y_1(z) \simeq \left[ 1 + \varepsilon \frac{I_{12}(0)}{w_{12}} \right] z^{\frac{1-\lambda_1}{2}} - \varepsilon \frac{I_{11}(0)}{w_{12}} z^{\frac{1+\lambda_1}{2}} \quad (105)$$

$$Y_2(z) \simeq \varepsilon \frac{I_{22}(0)}{w_{12}} z^{\frac{1-\lambda_1}{2}} + \left[ 1 - \varepsilon \frac{I_{12}(0)}{w_{12}} \right] z^{\frac{1+\lambda_1}{2}} \quad (106)$$

Thus a pair of solution canonical around 0, i.e. providing around 0 a monodromic conformal factor is

$$U_1 = \left[ 1 - \varepsilon \frac{I_{12}(0)}{w_{12}} \right] Y_1 + \varepsilon \frac{I_{11}(0)}{w_{12}} Y_2 \simeq z^{\frac{1-\lambda_1}{2}} \quad (107)$$

$$U_2 = -\varepsilon \frac{I_{22}(0)}{w_{12}} Y_1 + \left[ 1 + \varepsilon \frac{I_{12}(0)}{w_{12}} \right] Y_2 \simeq z^{\frac{1+\lambda_1}{2}} \quad (107)$$

Obviously we can apply to the couple of eq.(107) a linear transformation which leaves unchanged the diagonal form of the monodromy around 0

$$M_0 = \begin{pmatrix} -e^{-i\pi\lambda_1} & 0 \\ 0 & -e^{i\pi\lambda_1} \end{pmatrix} \quad (108)$$
and look for the solution which is monodromic everywhere in the class of linear combinations. It will be useful to introduce a matrix notation to deal with such pair of solutions

\[ Y = \left[ 1 + \varepsilon N(z) \right] y \]
\[ U = (1 - \varepsilon P) Y = \Lambda Y \]
\[ N(z) = \frac{1}{w_{12}} \begin{pmatrix} I_{12}(z) & -I_{11}(z) \\ I_{22}(z) & -I_{12}(z) \end{pmatrix} \]
\[ P = \frac{1}{w_{12}} \begin{pmatrix} I_{12}(0) & -I_{11}(0) \\ I_{22}(0) & -I_{12}(0) \end{pmatrix}. \] (109)

At this stage we recall the well known result about the monodromy of the canonical unperturbed solution \( y_1(z) \) and \( y_2(z) \) around the point 1; setting \( \zeta = 1 - z \) we have

\[ y_1(z) \simeq A_{11} \zeta^{1-\lambda_2} + A_{12} \zeta^{1+\lambda_2} \]
\[ y_2(z) \simeq A_{21} \zeta^{1-\lambda_2} + A_{22} \zeta^{1+\lambda_2} \] (110)

where the matrix \( A_{ij} \) is given by

\[ A = \begin{pmatrix} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \end{pmatrix}. \] (111)

Similarly to what done around \( z = 0 \) we can develop the integrals \( I_{ij}(z) \) around 1 obtaining

\[ I_{11}(z) \simeq I_{11}(1) - \frac{2 + \beta t}{2} \left[ \frac{A_{11}^2}{1-\lambda_2} \zeta^{1-\lambda_2} + 2A_{11}A_{12} \zeta + \frac{A_{12}^2}{1+\lambda_2} \zeta^{1+\lambda_2} \right] \] (112)
\[ I_{12}(z) \simeq I_{12}(1) - \frac{2 + \beta t}{2} \left[ \frac{A_{11}A_{21}}{1-\lambda_2} \zeta^{1-\lambda_2} + (A_{11}A_{22} + A_{12}A_{21}) \zeta + \frac{A_{21}A_{22}}{1+\lambda_2} \zeta^{1+\lambda_2} \right] \] (113)
\[ I_{22}(z) \simeq I_{22}(1) - \frac{2 + \beta t}{2} \left[ \frac{A_{21}^2}{1-\lambda_2} \zeta^{1-\lambda_2} + 2A_{21}A_{22} \zeta + \frac{A_{22}^2}{1+\lambda_2} \zeta^{1+\lambda_2} \right]. \] (114)

Again from eq.(31) we have around \( z = 1 \), \( Y = y + \varepsilon N y \), where

\[ N = N(1) = \frac{1}{w_{12}} \begin{pmatrix} I_{12}(1) & -I_{11}(1) \\ I_{22}(1) & -I_{12}(1) \end{pmatrix}. \] (115)

Thus, if we denote by \( M_1^{(0)} \) the monodromy matrix of the unperturbed problem around 1, we have after encircling 1

\[ Y' = M_1^{(0)} y + \varepsilon N M_1^{(0)} y = M_1^{(0)}(1 - \varepsilon N) Y + \varepsilon N M_1^{(0)} Y \]
\[ = \left( M_1^{(0)} + \varepsilon \left[ N, M_1^{(0)} \right] \right) Y. \] (116)

21
Thus the monodromy matrix around 1 for the canonical solutions $U$ is

\[
M_1 = (1 - \varepsilon P) \left( M_1^{(0)} + \varepsilon \left[ N, M_1^{(0)} \right] \right) (1 + \varepsilon P) = M_1^{(0)} + \varepsilon \left[ N - P, M_1^{(0)} \right].
\]

(117)

We come to examine the monodromy matrix around the weak singularity located in $t$. At $t$ the unperturbed solutions are analytic and thus the only contribution to the monodromy matrix comes from the integrals $I_{ij}(z)$

\[
I_{ij}(z) \to I_{ij}(z) + \oint_I y_i(x)y_j(x)q(x) \, dx
\]

(118)

to obtain

\[
\delta I_{ij} = \frac{1}{2} \oint_I y_i(x)y_j(x) \left[ \frac{\beta}{z - t} + \frac{2}{(z - t)^2} \right] \, dx = i\pi \left[ \beta y_i(x)y_j(x) + 2 (y_i(x)y_j(x))' \right]_{x=t}
\]

(119)

i.e. we obtain for $Y$ the transformation

\[
Y_1 \to Y_1 + \frac{\varepsilon}{w_{12}} [\delta I_{12} y_1 - \delta I_{11} y_2],
Y_2 \to Y_2 + \frac{\varepsilon}{w_{12}} [\delta I_{22} y_1 - \delta I_{12} y_2]
\]

(120)

and thus to order $O(\varepsilon)$ we have for the monodromy matrix of the canonical solutions $U$

\[
M_t = \Lambda M_t(Y) \Lambda^{-1} = \begin{pmatrix}
1 + \varepsilon \frac{\delta I_{12}}{w_{12}} & -\varepsilon \frac{\delta I_{11}}{w_{12}} \\
\varepsilon \frac{\delta I_{12}}{w_{12}} & 1 - \varepsilon \frac{\delta I_{11}}{w_{12}}
\end{pmatrix} = M_t(Y).
\]

(121)

By construction the canonical solutions $U$ have a monodromy matrix around 0 which is an element of $SU(1, 1)$. The only freedom left is the conjugation by a matrix of the form

\[
K = \begin{pmatrix}
k & 0 \\
0 & 1/k
\end{pmatrix}.
\]

(122)

We shall start by imposing the monodromy around $t$. We must impose

\[
\tilde{M}_t \equiv K M_t K^{-1} = \begin{pmatrix}
1 + \frac{\varepsilon \delta I_{12}}{w_{12}} & -\frac{k^2 \varepsilon \delta I_{11}}{w_{12}} \\
\frac{k^2 \varepsilon \delta I_{12}}{w_{12}} & 1 - \frac{\varepsilon \delta I_{11}}{w_{12}}
\end{pmatrix} \in SU(1, 1)
\]

(123)

i.e. as such a matrix already belong to $SL(2, C)$ it sufficient to impose

\[
(\tilde{M}_t)_{11} = \overline{(M_t)_{22}}, 
(\tilde{M}_t)_{12} = \overline{(M_t)_{21}}.
\]

(124)
The first condition gives

\[ \beta y_1 y_2 + 2(y_1 y_2)' = \bar{\beta} \bar{y}_1 \bar{y}_2 + 2(\bar{y}_1 \bar{y}_2)' \]  \hspace{1cm} (125)

where all \( y_i \) are computed at \( t \). The second condition gives \( k \) as a function of \( \beta \)

\[ |k|^4 = -\frac{\delta I_{22}}{\delta I_{11}} = -\frac{\bar{\beta} \bar{y}_2 \bar{y}_2 + 4\bar{y}_2 \bar{y}_2'}{\beta y_1 + 4y_1 y_1'} \]  \hspace{1cm} (126)

which by the way implies that \( \delta I_{11} \delta I_{22} \) is real and negative which can also be written as

\[ \left[ \bar{\beta} \bar{y}_1 \bar{y}_2 + 2(\bar{y}_1 \bar{y}_2)' \right]^2 - 4w_{12}^2 > 0. \]  \hspace{1cm} (127)

We shall check after determining \( \beta \) such inequality. We come now to the imposition of the monodromy at 1. The zero order value of \( K \) which we shall call \( K_0 \) can be computed from the three singularity monodromy matrix (111). Thus we shall pose

\[ K = \begin{pmatrix} 1 + \varepsilon \frac{h}{w_{12}} & 0 \\ 0 & 1 - \varepsilon \frac{h}{w_{12}} \end{pmatrix} \begin{pmatrix} k_0 & 0 \\ 0 & 1/k_0 \end{pmatrix} = (1 + \varepsilon H) K_0. \]  \hspace{1cm} (128)

Notice that as we are free to multiply \( K \) by an element \( \text{diag}(e^{i\alpha}, e^{-i\alpha}) \) of \( SU(1,1) \), only the real part of the parameter \( h \) is meaningful. The monodromy matrix of the new solutions is given by

\[ K M_1 K^{-1} = (1 + \varepsilon H) K_0 \left( M_1^{(0)} + \varepsilon [N - P, M_1^{(0)}] \right) K_0^{-1} (1 - \varepsilon H) \]

\[ = K_0 M_1^{(0)} K_0^{-1} + \varepsilon \left[ H, K_0 M_1^{(0)} K_0^{-1} \right] + \varepsilon K_0 [N - P, M_1^{(0)}] K_0^{-1} \]

\[ = D + \varepsilon \left[ H + K_0 (N - P) K_0^{-1}, K_0 M_1^{(0)} K_0^{-1} \right] \]  \hspace{1cm} (129)

where we set \( D \equiv K_0 M_1^{(0)} K_0^{-1} \), and we must impose

\[ \tilde{M}_1 \equiv K M_1 K^{-1} = D + \varepsilon \left[ H + K_0 (N - P) K_0^{-1}, D \right] \equiv D + \varepsilon [B, D] \]  \hspace{1cm} (130)

to be an element of \( SU(1,1) \). The request \( D \in SU(1,1) \) is equivalent to the three source problem which has already been solved by fixing

\[ |k_0|^4 = \frac{M_{21}^{(0)}}{M_{12}^{(0)}} \equiv \kappa. \]  \hspace{1cm} (131)

We notice that such a value of \( |k_0|^4 \) is already sufficient to determine the value of \( \beta \). In fact Eqs.(125,126) furnish the system

\[ \begin{cases} \kappa \bar{y}_1 (\bar{\beta} \bar{y}_1 + 4\bar{y}_1') = y_2 (\beta y_2 + 4y_2') \\
\beta y_1 y_2 + 2(y_1 y_2)' = \bar{\beta} \bar{y}_1 \bar{y}_2 + 2(\bar{y}_1 \bar{y}_2)' \end{cases} \]  \hspace{1cm} (132)
which gives
\[ \beta = -2 \frac{2|y_2|^2 y_2' + \kappa \bar{y}_1 (\bar{y}_2 y'_1 - \bar{y}_1 y'_2 + y_2 y'_1 + y_1 y'_2)}{y_2 (\kappa \bar{y}_1 y_1 - \bar{y}_2 y_2)} . \] (133)

Exploiting the fact that the wronskian of the two solution is real such expression can be simplified to
\[ \beta = -4 \frac{\kappa \bar{y}_1 y'_1 - \bar{y}_2 y'_2}{\kappa \bar{y}_1 y_1 - \bar{y}_2 y_2} . \] (134)

We are left to determine the parameter \( h \). We have \( \det D = 1 \) and as a consequence also \( \det \tilde{M}_1 = 1 + O(\varepsilon^2) \). The explicit form of the matrix \( B \) is
\[
B = \begin{pmatrix}
\frac{h}{w_{12}} + \frac{I_{12}(1) - I_{12}(0)}{w_{12}} & -k_0^2 \frac{I_{11}(1) - I_{11}(0)}{w_{12}} \\
\frac{I_{22}(1) - I_{22}(0)}{w_{12}} & -\frac{h}{w_{12}} - \frac{I_{12}(1) - I_{12}(0)}{w_{12}}
\end{pmatrix}
\] (135)

which as expected is independent of the base point \( z_0 \). From now on we shall fix \( z_0 = 0 \) so that \( I_{ij}(0) = 0 \). Defined \( C = [B, D] \) we must impose \( C_{12} = \bar{C}_{21} \) and \( C_{11} = \bar{C}_{22} \) keeping in mind that already \( D \in SU(1, 1) \). The first relation gives the condition
\[
B_{11} + B_{11} - B_{22} - B_{22} = \frac{D_{11} - \bar{D}_{11}}{D_{12}} (B_{12} - B_{21})
\] (136)

which can be rewritten as
\[
h + \bar{h} = -I_{12}(1) - \bar{I}_{12}(1) + \frac{A_{12} A_{21} + A_{11} A_{22}}{2} \left( \frac{I_{11}(1)}{A_{11} A_{12}} + \frac{\bar{I}_{22}(1)}{A_{21} A_{22}} \right) .
\] (137)

The second relation can be rewritten as
\[
A_{21} A_{22} I_{11}(1) - A_{11} A_{12} I_{22}(1) \in R.
\] (138)

Due to the solubility of the four source problem as assured by Picard theorem such a relation has to be satisfied and furnishes a non trivial relation among the integrals \( I_{11} \) and \( I_{22} \) containing hypergeometric functions. We did not find such a relation in the standard tables but as a check we verified it numerically to \( 10^{-12} \) precision. Thus we have reached the following pair of solutions
\[
Z_1(z) = k_0 \left[ \left( 1 + \varepsilon \frac{I_{12}(z) + h}{w_{12}} \right) y_1(z) - \varepsilon \frac{I_{11}(z)}{w_{12}} y_2(z) \right]
\]
\[
Z_2(z) = \frac{1}{k_0} \left[ \varepsilon \frac{I_{22}(z)}{w_{12}} y_1(z) + \left( 1 - \varepsilon \frac{I_{12}(z) + h}{w_{12}} \right) y_2(z) \right]
\] (139)

which have \( SU(1, 1) \) monodromies around all singularities and as such determine a globally monodromic conformal factor satisfying the Liouville equation
\[
e^{2b_\phi c} = e^{\phi c} = \frac{1}{\pi \mu b^2} \frac{w_{12}^2}{(Z_1 \bar{Z}_1 - Z_2 \bar{Z}_2)^2}
\] (140)
as $Z_1Z'_2 - Z'_1Z_2 = w_{12} = \text{const.}$ At this stage we can verify the inequality (127). Substituting eq.(134) into (127) we have

$$
\left[\bar{\beta}y_1\bar{y}_2 + 2(\bar{y}_1\bar{y}_2)\right]^2 - 4w_{12}^2 = 16\frac{\kappa|y_1|^2|y_2|^2w_{12}^2}{(\kappa|y_1|^2 - |y_2|^2)^2} > 0. \tag{141}
$$

**Appendix B**

We give here the details of the calculation which proves the invariance under inversion of the two-loop contribution.

The first integral in eq.(97) vanishes for $R \to \infty$ as the infinity is a regular point for the Green function $g(z, z')$ and as such $\partial_z g(z, z') = O(1/|z|^2)$ and the same holds for the fifth integral. The $(d') - (d)$ term performing the change of variable $w = -1/z$ can be rewritten as

$$(d') - (d) = 2b^2 (g(0, 0) - g^w(0, 0)). \tag{142}$$

Then summing all contributions we find

$$(a') + (b') + (c') + (d') - (a) - (b) - (c) - (d) = \frac{b^2 i}{4\pi^2} \int_{\partial_{R'}R} \frac{dz}{z} g(z, z')d^2z' \log z'z'
- \frac{b^2}{2\pi} \int g(0, z')d^2z'f(z') \log z'z' + 2b^2 (g(0, 0) - g^w(0, 0))
= \frac{b^2}{2\pi} \int g^w(0, w')f^w(w')d^2w' \log w'w' - \frac{b^2}{2\pi} \int g(0, z')f(z')d^2z' \log z'z'
+ 2b^2 (g(0, 0) - g^w(0, 0)). \tag{143}$$

Using the differential equation for $g(0, z')$ and integrating by parts we find

$$\frac{b^2}{2\pi} \int g(0, z)f(z)d^2z \log z\bar{z} = 2b^2g(0, 0) - 2b^2g(0, \infty) \tag{144}$$

and thus we find for eq.(143) the value

$$2b^2g(0, \infty) - 2b^2g^w(0, \infty) = 0 \tag{145}$$

due to the invariance in value of the Green function $g(z, z') = g^w(-1/z, -1/z')$ and the symmetry of the Green function.

**References**

[1] T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309; E. Brateen, T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 51 (1983) 19; Ann. Phys. 147 (1983) 365.
[2] G. Jorjadze and G. Weigt, Phys.Lett. B581 (2004) 133; Theor.Math.Phys. 139 (2004) 654.

[3] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Liouville Field Theory on a Pseudosphere*, arXiv: hep-th/0101152.

[4] P. Menotti and E. Tonni, Phys. Lett. B586 (2004) 425.

[5] P. Menotti and E. Tonni, *Standard and geometric approaches to quantum Liouville theory on the pseudosphere* arXiv: hep-th/0406014, to appear in Nuclear Physics B (FS).

[6] See e.g.: A. Bilal, J.-L. Gervais, Nucl. Phys. B305 (1988) 33; A. Bilal, J.-L. Gervais, J. Geom. Phys. 5 (1988) 277.

[7] O. Alvarez, Nucl. Phys. B216 (1982) 125.

[8] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B477 (1996) 577.

[9] H. Dorn and H.J. Otto, Nucl. Phys. B429 (1994) 375.

[10] J. Teschner, Phys. Lett. B363 (1995) 65; Class. Quant. Grav. 18 (2001) R153; *A lecture on the Liouville vertex operators*, Contribution to the proceedings of the 6th International Conference on CFTs and Integrable Models, Chernogolovka, Russia, 2002, arXiv: hep-th/0303150; *From Liouville theory to the quantum geometry of Riemann surfaces*, Contribution to 14th International Congress on Mathematical Physics (ICMP 2003), Lisbon, Portugal, 28 Jul - 2 Aug 2003, arXiv: hep-th/0308031.

[11] P. G. Zograf, L. A. Takhtajan, Math. USSR Sbornik 60 (1988) 143; Math. USSR Sbornik 60 (1988) 297.

[12] L. Cantini, P. Menotti and D. Seminara, Phys.Lett. B517 (2001) 203.

[13] L. Cantini, P. Menotti and D. Seminara, Nucl. Phys. B638 (2002) 351.

[14] L. Takhtajan and P. Zograf, *Hyperbolic 2 spheres with conical singularities, accessory parameters and Kaehler metric on M(0,N)*, arXiv: math.cv/0112170.

[15] E. L. Ince, *Ordinary Differential Equations*, Dover Publ. (1944).

[16] E. D’Hoker and R. Jackiw, Phys. Rev. D26 (1982) 3517; E. D’Hoker, D.Z. Freedman and R. Jackiw, Phys. Rev. D28 (1983) 2583.
[17] L.A. Takhtajan, *Topics in Quantum Geometry of Riemann Surfaces: Two Dimensional Quantum Gravity*, Proc. Internat. School Phys. Enrico Fermi, 127, IOS, Amsterdam, 1996, arXiv: hep-th/9409088; L.A. Takhtajan, Mod. Phys. Lett. A11 (1996) 93.

[18] J. Hadamard, *Lectures on Cauchy’s Problem in Linear Partial Differential Equations*, Yale Univ. Press, New Haven, Conn., 1923.

[19] P.R. Garabedian, *Partial Differential Equations*, John Wiley & Sons, New York, 1964.