A SALAD OF COCYCLES

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ABSTRACT. We study the centraliser of locally compact group extensions of ergodic probability preserving transformations. New methods establishing ergodicity of group extensions are introduced, and new examples of squashable and non-coalescent group extensions are constructed. Smooth versions of some of the constructions are also given.

§0 Introduction

Let $T$ be an ergodic probability preserving transformation of the probability space $(X, B, m)$.

Let $(G, T)$ be a locally compact, second countable, topological group ($T = T(G)$ denotes the family of open sets in the topological space $G$), and let $\varphi : X \to G$ be a measurable function.

The (left) skew product or $G$-extension $T \varphi : X \times G \to X \times G$, is defined by

$$T \varphi(x, g) = (Tx, \varphi(x)g).$$

The skew product preserves the measure $\mu = m \times m_G$, where $m_G$ is left Haar measure on $G$. There is an ergodic skew product $T \varphi : X \times G \to X \times G$ iff the group $G$ is amenable (see [G-S], references therein, and [Zim]). In this paper, we are mainly concerned with Abelian $G$. Recall that on any locally compact, Abelian, second countable, topological group $G$, there is defined a norm $\| \cdot \|_G$ (satisfying $\|x\| = \|-x\| \geq 0$ with equality iff $x = 0$, and $\|x + y\| \leq \|x\| + \|y\|$) which generates the topology of $G$.

The centraliser.

Recall that the centraliser of a non-singular transformation $R : X \to X$ is the collection of commutors of $R$, that is, non-singular transformations of $X$ which commute with $R$. The collection of invertible commutors (the invertible centraliser) is denoted by $C(R)$.

We study those commutors $Q$ of $T \varphi$, of form

$$Q(x, y) = (Sx, f(x)w(y))$$

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where \( w : G \to G \) is a surjective, continuous group endomorphism, \( S \) is a commutator of \( T \), and \( f : X \to G \) is measurable. It is shown in proposition 1.1 of [A-L-M-N] that if \( T \) is a Kronecker transformation, and \( T_\varphi \) is ergodic, then every commutator of \( T_\varphi \) is of form \((*)\).

Let \( \text{End}(G) \) denote the collection of surjective, continuous group endomorphisms of \( G \) (a semigroup under composition) and let

\[
\mathcal{E}_\varphi = \{ w \in \text{End}(G) : \exists \text{ a commutor } Q \text{ of } T_\varphi \text{ of form } (*) \text{ with } w = wQ \},
\]

a sub-semigroup of \( \text{End}(G) \).

The study of \( \mathcal{E}_\varphi \) yields counterexamples:

- if \( \mathcal{E}_\varphi \) contains non-invertible endomorphisms, then \( T_\varphi \) is not coalescent, i.e. its centraliser contains some non-invertible transformation (see [H-P]); and
- if \( \mathcal{E}_\varphi \) contains endomorphisms which do not preserve \( m_G \) (a possibility only for non-compact \( G \)), then \( T_\varphi \) is squashable, i.e. i.e. its centraliser contains some non-singular transformation which is not measure preserving (see [Aa1] and below).

Counterexamples like these (and others) will be discussed below.

In case \( T \) is an odometer, for any Abelian, locally compact, second countable \( G \),

\[
\{ \varphi : X \to G : T_\varphi \text{ ergodic} , \mathcal{E}_\varphi = \{ \text{Id} \} \}
\]

is residual in the collection of measurable functions \( \varphi : X \to G \) considered in the topology of convergence in measure (see below). Analogous results hold, when \( T \) is a rotation of the circle, for smooth \( \varphi : \Theta \to \mathbb{R} \).

**Semigroup homomorphisms.**

Let \( \mathcal{L}_\varphi \) denote the collection of those commutors \( S \) of \( T \), for which \( \exists \) a commutor \( Q \) of \( T_\varphi \) of form \((*)\) with \( S = SQ \).

When \( G \) is Abelian and \( T_\varphi \) is ergodic, there is a surjective semigroup homomorphism \( \pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi \) such that if \( S \in \mathcal{L}_\varphi \), and \( Q \) is a commutor of \( T_\varphi \) of form \((*)\) with \( S = SQ \), then \( wQ = \pi_\varphi(S) \). This result (called the semigroup embedding lemma) is proved at the end of this introduction.

It implies that \( \mathcal{E}_\varphi \) is Abelian whenever the commutors of \( T \) form an Abelian semigroup, for instance when \( T \) is a Kronecker transformation.

The restriction of \( \pi_\varphi \) to \( L_\varphi(T) = \{ SQ : Q \in C(T_\varphi) \text{ of form } (*) \} \) is continuous with respect to the relevant Polish topologies by the continuous embedding lemma established in §1 (c.f. [G-L-S] for the case where \( G \) is compact).

The question arises as to when a homomorphism \( \pi \) from a sub-semigroup \( S \) of commutors of \( T \) into \( \text{End}(G) \) occurs in this manner. That is, when does there exist a measurable function \( \varphi : X \to G \) such that \( T_\varphi \) is ergodic, \( S \subset \mathcal{L}_\varphi \), and \( \pi = \pi_\varphi|_S \).

In [L-L-T] it is shown that for \( T \) an invertible, ergodic probability preserving transformation with some invertible commutor \( S \) so that \( \{ S^mT^n : m, n \in \mathbb{Z} \} \) acts freely, and \( G = \Theta \), \( \exists \varphi : X \to \Theta \) such that \( S \in \mathcal{L}_\varphi \), \( \mathcal{E}_\varphi \ni [x \mapsto 2x \mod 1] \), and indeed, \( \pi_\varphi(S) = [x \mapsto 2x \mod 1] \). This includes the first example of a non-coalescent Anzai skew product (i.e. \( \Theta \)-extension of a rotation of \( \Theta \)).
The main results. We generalise this to all Abelian, locally compact, second countable $G$:

**Theorem 1.** Suppose that $T$ is an ergodic probability preserving transformation, $d \leq \infty$, and $S_1, \ldots, S_d \in C(T)$ ($d \leq \infty$) are such that $(T, S_1, \ldots, S_d)$ generate a free $\mathbb{Z}^{d+1}$ action of probability preserving transformations of $X$.

If $w_1, \ldots, w_d \in \text{End}(G)$ commute (i.e. $w_i \circ w_j = w_j \circ w_i \forall 1 \leq i, j \leq d$), then there is a measurable function $\varphi : X \to G$ such that

$$T_\varphi \text{ is ergodic},$$

$$S_1, \ldots, S_d \in \mathcal{L}_\varphi, \ w_1, \ldots, w_d \in \mathcal{E}_\varphi;$$

and

$$\pi_\varphi(S_i) = w_i \quad (1 \leq i \leq d).$$

Any Kronecker transformation of an uncountable compact group satisfies the preconditions of theorem 1.

**Theorem 2.** Suppose that $T$ is an ergodic probability preserving transformation, and that $\{S_t : t \in \mathbb{R}\} \subset C(T)$ is such that $T$ and $\{S_t : t \in \mathbb{R}\}$ generate a free $\mathbb{Z} \times \mathbb{R}$ action of probability preserving transformations of $X$.

There is a measurable function $\varphi : X \to \mathbb{R}$ such that

$$T_\varphi \text{ is ergodic; \quad and there is a flow } \{Q_t : t \in \mathbb{R}\} \subset C(T_\varphi) \text{ of form }$$

$$Q_t(x, y) = (S_t x, e^t y + \psi_t(x)).$$

In particular,

$$S_t \in \mathcal{L}_\varphi, \ w_t \in \mathcal{E}_\varphi \quad \forall \ t \in \mathbb{R}$$

where $w_t(y) = e^t y$; and

$$\pi_\varphi(S_t) = w_t \quad \forall \ t \in \mathbb{R}.$$

Remarks.  

1) Theorem 1 can be extended (with analogous proof) to enable ”realisation” of a semigroup homomorphism defined on a discrete subgroup of the centraliser which is amenable, and which has Følner sets which tile (see [O-W]).

2) In view of the reliance here on Rokhlin lemmas (see the proof of lemmas 4.1 and 4.2), we ask if there is an ergodic probability preserving transformation $(X, \mathcal{B}, m, T)$, and an ergodic $\varphi : X \to \mathbb{T}^2$ such that $SL(2, \mathbb{Z}) \subset \mathcal{E}_\varphi$.

Note that $SL(2, \mathbb{Z}) \subset \text{End}(\mathbb{T}^2)$, and that if $T$ is the 4-shift with symmetric product measure, then $SL(2, \mathbb{Z}) \subset C(T)$.

Squashability and laws of large numbers.

Let $T = (X_T, \mathcal{B}_T, m_T)$ be a conservative, ergodic measure preserving transformation of the $\sigma$-finite measure space $(X_T, \mathcal{B}_T, m_T)$. Each commutor of $T$ is a measure multiplying transformation. This is because for $Q$ a commutor of $T$, the measure $m_T \circ Q^{-1}$ is $m_T$-absolutely continuous, $T$-invariant, and hence

$$\frac{dm_T \circ Q^{-1}}{dm_T} \circ T = \frac{dm_T \circ Q^{-1}}{dm_T}.$$
which is constant by ergodicity.

The dilation of a measure multiplying transformation \( Q \) is defined by
\[
D(Q) = \frac{dm \circ Q}{dm} \in (0, \infty).
\]

Recall from [Aa1] that the transformation \( T \) is called squashable if it has a commutor with non-unit dilation.

Let \( C \subset B_T \), \( C = \) either \( B_T \) or \( F_T = \{ B \in B_T : m_T(B) < \infty \} \). A law of large numbers for \( T \) with respect to \( C \) is a function \( L : \{0, 1\}^\mathbb{N} \rightarrow [0, \infty] \) such that
\[
L(1_A, 1_A \circ T, \ldots) = m_T(A) \text{ a.e.}
\]
\( \forall \ A \in C \). If \( L \) is a law of large numbers for \( T \) with respect to \( C \), and \( Q \) is a commutor of \( T \) such that \( Q^{-1}C \subseteq C \), then \( D(Q) = 1 \), as \( \forall \ A \in C \), \( Q^{-1}A \in C \), and for a.e. \( x \in X \),
\[
m_T(A) = L(1_A(Qx), 1_A(TQx), \ldots) = L(1_{Q^{-1}A}(x), 1_{Q^{-1}A}(Tx), \ldots) = m_T(Q^{-1}A).
\]

Consequently, if \( T \) has a law of large numbers with respect to \( B_T \), then \( T \) is non-squashable, and if \( T \) has a law of large numbers with respect to \( F_T \), then no commutor of \( T \) has non-unit, finite dilation.

It was shown in [Aa2, corollary 2.3, & theorem 3.4] that if \( G \) is a countable group without arbitrarily large finite normal subgroups (e.g. \( G = \mathbb{Z}^k \times \mathbb{Q}^l \) or \( G = \mathbb{Z}^\infty = \{(n_1, n_2, \ldots) \in \mathbb{Z}^\mathbb{N} : n_k \rightarrow 0\} \)), then any ergodic \( G \)-extension of a Kronecker transformation has a law of large numbers with respect to \( F \).

**Example 1.**

Let \( T \) be a Kronecker transformation, then \( \exists \ S \in C(T) \) so that \( \{S, T\} \) generate a free \( \mathbb{Z}^2 \) action. Let \( G = \mathbb{Z}^\infty \), and let \( w = w_1 \in \text{End}(G) \) be the shift \( w((n_1, n_2, \ldots)) = (n_2, n_3, \ldots) \). By theorem 1, \( \exists \ \varphi : X \rightarrow G \) such that \( T_\varphi \) is ergodic, and \( w \in \mathcal{E}_\varphi \).

It follows that:
\( T_\varphi \) has a law of large numbers with respect to \( F_{T_\varphi} \).

but also a commutor \( Q(x, y) = (Sx, w(y) + g(x)) \) which has infinite dilation since (as shown in the proof of proposition 1.1 of [A-L-M-N]) \( D(Q) = D(w) = \infty \), whence \( T_\varphi \) has no law of large numbers with respect to \( B_{T_\varphi} \).

**Complete squashability and Maharam transformations.**

Evidently \( D : C(T) \rightarrow \mathbb{R}_+ \) is a multiplicative homomorphism. Set \( \Delta_0(T) = D(C(T)) \). The group \( \Delta_0(T) \) was first considered in [H-I-K] (see also [Aa2]). If \( T \) has a law of large numbers with respect to \( F_T \), then \( \Delta_0(T) = \{1\} \). In particular, \( \{[Aa2], \text{ or } [A-L-M-N]\} \) if \( T \) is a \( \mathbb{Z} \)-extension of a Kronecker transformation, then \( \Delta_0(T) = \{1\} \). Our results on \( R \)-extensions show that this result fails dramatically for other transformations \( T \), an \( R \)-extension of \( T \) being a \( \mathbb{Z} \)-extension of a \( \mathbb{Z} \)-extension of \( T \). Moreover, (see proposition 2.5) for Bernoulli \( T \), any ergodic \( R \)-extension of \( T \) is isomorphic to a \( \mathbb{Z} \)-extension of \( T \).
It is standard (see §1 where we recall some well known facts about Polish groups of measure multiplying transformations) that \( \Delta_0(T) \) is a Borel subgroup of \( \mathbb{IR}_+ \). The class

\[
\{ \Delta_0(T) : T \text{ a conservative, ergodic measure preserving transformation} \}
\]

includes

\( \mathbb{IR}_+ \), all countable subgroups of \( \mathbb{IR}_+ \), and subgroups of \( \mathbb{IR}_+ \) with any Hausdorff dimension (see [Aa2]).

Call a conservative, ergodic measure preserving transformation \( T \) with the property that \( \Delta_0(T) = \mathbb{IR}_+ \) completely squashable. Any ergodic Maharam transformation (defined below) is completely squashable.

For a non-singular conservative, ergodic transformation \( R \) of \( (\Omega, \mathcal{A}, p) \), the transformation \( T : X = \Omega \times \mathbb{IR} \rightarrow X \) defined by

\[
T(x, y) = (Rx, y - \log \frac{dp \circ R}{dp})
\]

preserves the measure \( dm_T(x, y) = dp(x)e^ydy \), and is called the Maharam transformation of \( R \), as it was shown to be conservative in [Mah]. If \( Q_t(x, y) = (x, y + t) \), then \( Q_t \in C(T) \) and \( D(Q_t) = e^t \).

Conservative, ergodic Maharam transformations were constructed in [Kr], and smooth Maharam transformations of \( T \times \mathbb{IR} \) are constructed in [H-S].

We show in §5 that the transformations \( T_\varphi \) constructed in theorem 2 are isomorphic to Maharam transformations (proposition 5.1), and we obtain \( \mathbb{Z} \)-extensions of Bernoulli transformations which are Maharam transformations (see the remarks after proposition 5.1).

In §§6 and 7, we present completely squashable \( \mathbb{IR} \)-extensions \( T_\varphi \) which are not isomorphic to any Maharam transformation, for \( T \) an odometer, and \( T \) a rotation of \( \mathbb{IR} \). For \( T \) a rotation of \( \mathbb{IR} \), our examples are as smooth as possible given the Diophantine properties of the rotation number of \( T \) (including \( C^\infty \), ergodic, completely squashable \( \mathbb{IR} \)-extensions for suitable rotation numbers).

It is not hard to construct real analytic, ergodic, completely squashable \( \mathbb{IR} \)-extensions of a suitable irrational rotation using §5 and [Kw-Le-Ru1] and [Kw-Le-Ru2].

**Conditions for ergodicity, and non-squashability of skew products.**

Let \( T \) be an ergodic probability preserving transformation of the probability space \( (X, \mathcal{B}, m) \), assume that \( G \) is Abelian, and let \( \varphi : X \rightarrow G \) be measurable.

Recall from [Sch], that the **essential values** of \( \varphi \) are defined by

\[
E(\varphi) = \{ a \in G : \forall A \in \mathcal{B}_+, \ a \in U \in T, \ \exists \ n \geq 1 \ \exists \ m(A \cap T^{-n}A \cap [\varphi_n \in U]) > 0 \},
\]

which is a closed subgroup of \( G \). It is shown in [Sch] that \( T_\varphi \) is ergodic iff \( E(\varphi) = G \).

Set

\[
\bar{D}(\varphi) = \{ a \in G : \exists q_n \ni T^{q_n} \stackrel{\text{M}(X)}{\longrightarrow} \text{Id, } & \varphi_{q_n} \rightarrow a \ a.e. \}
\]
where $\mathcal{M}(X)$ denotes convergence in the topology of measure preserving transformations on $X$ (see §1 below), then (see [A-L-M-N]) $E(\varphi) \supset \text{Gp}(\tilde{D}(\varphi))$ (the group generated by $\tilde{D}(\varphi)$).

If $\text{Gp}(\tilde{D}(\varphi))$ is dense in $G$, then $T_\varphi$ is not only ergodic, but also non-squashable.

If $T$ is an odometer, then (see [A-L-M-N]) for any Abelian, locally compact, second countable $G$, there is a measurable function $\varphi : X \to G$ such that $\text{Gp}(\tilde{D}(\varphi))$ is dense in $G$. Because of the density of coboundaries, such functions are residual in the collection of measurable functions $\varphi : X \to G$ considered in the topology of convergence in measure. It is also shown in [A-L-M-N] that there are rotations of $TT$ for which there is an ergodic, real analytic $\varphi : TT \to IR$ with $\text{Gp}\tilde{D}(\varphi)$ dense in $IR$, whence such functions are residual in any space (containing real analytic functions) where the coboundaries are dense.

It is shown in [L-V], that for a rotation $T$ of $TT$, there exist dense $G_\delta$ sets in the spaces of absolutely continuous, Lipschitz, $k$ times continuously or infinitely differentiable functions $f$ with zero mean on $T$ for which the distributions of $f_{n_k}$ converge to a continuous distribution along a rigid sequence $n_k \to \infty$ whenever theses spaces contain non-trivial cocycles (i.e. not $T$-cohomologous to a constants).

For irrational rotations with bounded partial quotients, nontrivial cocycles exist in the space of absolutely continuous functions (while every zero mean Lipschitz function is a coboundary). For rotations with unbounded partial quotients, non-trivial cocycles exist in the space $C^p$ of $p$ times continuously differentiable functions if and only if

$$\limsup_{n \to \infty} q_{n+1}/q_n^p = \infty$$

where $\{q_n : n \in \mathbb{N}\}$ are the principal denominators of the rotation (cf. [Ba-Me]).

For irrational rotations satisfying this condition $\forall p \in \mathbb{N}$, there are non-trivial infinitely differentiable cocycles.

Every such cocycle (the distributions of which converge to a continuous one along a rigid sequence) is ergodic (see [L-V]). But also $E_f \subset \{\pm 1\}$, since, if $f \circ S = cf + g \circ T - g$, where $S$ is another rotation of $TT$;

on the one hand, the distributions $f_{n_k} \circ S$ converge to the same limit as the distributions of $f_{n_k}$,

while on the other hand, $(g - g \circ T)_{n_k} \to 0$ in measure, hence the limit distribution is invariant under multiplication by $c$ which implies $c = \pm 1$ and $T_f$ non-squashable.

**New conditions for ergodicity allowing squashability.** The conditions for ergodicity of skew products discussed in [A-L-M-N] and [L-V] are unsuitable for our constructions of squashable skew products as they eliminate squashability.

We need new conditions for the ergodicity of a measurable function $\varphi : X \to G$ which are flexible enough to allow squashability.

Such conditions, called *essential value conditions* are introduced in §3.

Cocycles are constructed as infinite sums of coboundaries. Each coboundary "contributes" a particular essential value condition, which the subsequent coboundaries are "too small" to destroy. The essential value conditions remaining for the infinite sum gives its ergodicity.

The simplest version of our essential value conditions is the *rigid* one (see proposition 6.2) used in the constructions of §6 and §7 which could form an introduction
To conclude this introduction, we prove the

**Semigroup embedding lemma.** Suppose that $G$ is Abelian, and that $\varphi : X \to G$ is such that $T_\varphi$ is ergodic. There is a surjective semigroup homomorphism

$$\pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi$$

such that if $Q(x, y) = (Sx, f(x)w(y))$ defines a commutor of $T_\varphi$, then $w = \pi_\varphi(S)$.

**Proof.** We must show that if $S \in \mathcal{L}_\varphi$, $w_1, w_2 \in \mathcal{E}(G)$, $f_i : X \to G$, $(i = 1, 2)$ are measurable, and $Q_i(x, y) = (Sx, f_i(x)w_i(y))$ are such that $Q_i \circ T_\varphi = T_\varphi \circ Q_i$, $(i = 1, 2)$, then $w_1 = w_2$.

To this end, let $U = w_1 - w_2$, then $T_\varphi \circ U$ is an ergodic transformation of $X \times U(G)$ (being a factor of $T_\varphi$ via $(x, y) \mapsto (x, U(y))$). The condition $Q_i \circ T_\varphi = T_\varphi \circ Q_i$ means that

$$\varphi \circ S = w_1 \circ \varphi + f_1 \circ T - f_1 \quad (i = 1, 2),$$

whence

$$U \circ \varphi = g \circ T - g$$

where $g = f_1 - f_2$. Define $\tilde{g} : X \to G/U(G)$ by $\tilde{g}(x) = g(x) + U(G)$. It follows that $\tilde{g} \circ T = \tilde{g}$, whence by ergodicity of $T$, $\exists \gamma \in G$ such that $\tilde{g} = \gamma + U(G)$ a.e. Therefore $h := g - \gamma : X \to U(G)$ is measurable and satisfies

$$U \circ \varphi = h \circ T - h.$$ 

The ergodicity of $T_\varphi \circ U$ on $X \times U(G)$ now implies $U(G) = \{0\}$, i.e. $U \equiv 0$, or $w_1 = w_2$.

We’ve shown that $\forall S \in \mathcal{L}_\varphi$, $\exists w =: \pi_\varphi(S) \in \mathcal{E}_\varphi$ such that $\exists f_S : X \to G$ measurable so that $Q(x, y) = (Sx, f_S(x)\pi_\varphi(S)(y))$ defines a commutor of $T_\varphi$. The rest of the lemma follows easily from this. □
§1 POLISH GROUPS OF MEASURE MULTIPLYING TRANSFORMATIONS

Given a $\sigma$-finite measure space $(Y,\mathcal{C},\nu)$ let $\mathcal{M}(Y,\mathcal{C},\nu)$ denote the group of invertible measure multiplying transformations of $(Y,\mathcal{C},\nu)$, i.e. non-singular transformations $Q : Y \to Y$ such that $\nu \circ Q^{-1} = c\nu$ for some constant $c \in \mathbb{R}_+$. This is a Polish group when equipped with the weak topology inherited from that of the invertible, bounded linear operators on $L^2(Y,\mathcal{C},\nu)$. A metric for this topology is defined by

$$\rho(Q, R) = \sum_{n=1}^{\infty} \left( \| f_n \circ Q - f_n \circ R \|_2 + \| f_n \circ Q^{-1} - f_n \circ R^{-1} \|_2 \right)$$

where $\{f_n : n \in \mathbb{N}\}$ is a C.O.N.S. in $L^2(mG)$. The dilation function $D : \mathcal{M} \to \mathbb{R}_+$ defined (as above) by $Q \mapsto \frac{\text{dvol}(Q^{-1})}{\text{dvol}(Q)} := D(Q)$ is a continuous homomorphism.

Let $T$ be a conservative, ergodic measure preserving transformation of the standard $\sigma$-finite measure space $(X,\mathcal{B},m)$, then $C(T)$ is a closed subgroup of $\mathcal{M}(X,\mathcal{B},m)$ and hence Polish. The multiplicative homomorphism $D : C(T) \to \mathbb{R}$ is continuous, and so $\ker D$ is a closed, normal subgroup of $C(T)$. The natural topology on $C(T)/\ker D$ is Polish, and $D : C(T)/\ker D \to \mathbb{R}$ is continuous and injective. By Souslin’s theorem (see [Kur, §36, IV]),

$$\Delta_0(T) = D(C(T)/\ker D)$$

is a Borel set in $\mathbb{R}$.

Let $G$ be a locally compact, second countable topological group, then $G$ is $\sigma$-compact, and Polish. Let $mG$ be left Haar measure on $G$.

The action of $G$ on $(G,\mathcal{B}(G),m_G)$ by left translation is ergodic. To see this, suppose $A \in \mathcal{B}(G)_+$ and $m_G(gA\Delta A) = 0 \forall g \in G$. The measure $m'$ defined by $dm' = 1_A dm_G$ is a left Haar measure on $G$, and by unicity of such, $m' = m_G$ whence $A = G \mod m_G$. The ergodicity of the action of $G$ by right translation is obtained in a similar manner, as right Haar measure is equivalent to $m_G$.

The maps $L, R : G \to \mathcal{M}(G)$ given by $L_g f(x) := f(gx)$, $R_g f(x) := f(xg)$ are continuous, and their ranges are closed in $\mathcal{M}(G)$. This follows from the ergodicity of the actions of $G$ by translation; indeed, if $R = \lim_{n \to \infty} R_{g_n}$, set $f(x) = x^{-1}R(x)$, which is $L_g$-invariant $\forall g \in G$, hence constant, and $R = R_h$ for some $h \in G$. Let the ranges of these maps in $\mathcal{M}(G)$ be $\hat{G}_L$ and $\hat{G}_R$, considered with their inherited (Polish) topologies. By Souslin’s theorem ([Kur, §36, IV]), the inverse maps $L^{-1} : \hat{G}_L \to G$ and $R^{-1} : \hat{G}_R \to G$ are both measurable, and (being group isomorphisms) are continuous by Banach’s theorem ([Ban, p.20]). In particular, the metric $d$ on $G$, defined by $d(x,y) = \rho(L_x, L_y)$, generates the topology of $G$.

In case $G$ is Abelian, $\|x\|_G := d(L_x, \text{Id})$ defines a topology generating norm on $G$.

Let $\text{Aut}(G)$ denote the group of continuous group automorphisms of $G$. For example, $\tau_g(x) := g^{-1}xg$ is a continuous group automorphisms of $G$ (called an inner automorphism). Also, consider $\text{Aff}(G)$, the group of invertible, affine transformations of $G$ of form $L_g \circ w \equiv R_g \circ w'$ where $g \in G$ and $w, w' \in \text{Aut}(G)$ (and $w' = \tau_g \circ w$).

Both $\text{Aff}(G)$ and $\text{Aut}(G)$ are closed subgroups of $\mathcal{M}(G)$. To show that $\text{Aff}(G)$ is closed in $\mathcal{M}(G)$, we note first that if $Q \in \mathcal{M}(G)$, then $Q \in \text{Aff}(G)$ iff $\exists w \in \text{Aut}(G)$ such that

$$Q \circ L_g = L_{w(g)} \circ Q \quad \forall g \in G.$$
Indeed, supposing this condition, the function \( x \mapsto w(x)^{-1}Q(x) \) is \( L_g \)-invariant for all \( g \in G \), hence constant, and \( Q = R_h \circ w \in \mathcal{M}(G) \).

Also, the topology on \( \text{Aff}(G) \) inherited from \( \mathcal{M}(G) \) coincides with the compact-open topology. This follows because \( \text{Aff}(G) \), equipped with the compact-open topology, is a Polish space continuously embedded (by the identity) onto \( \text{Aff}(G) \) equipped with the topology inherited from \( \mathcal{M}(G) \); this identity necessarily being a homeomorphism (as shown above by the theorems of Banach and Souslin).

It is not hard to show that the Polish topology on \( \text{Aff}(G) \) also coincides with the topology of pointwise convergence.

It follows from the above that \( \text{Aut}(G) \) is closed in \( \text{Aff}(G) \). We’ll assume throughout that \( \text{Im}(G) := \{ \tau_g : g \in G \} \) is also closed in \( \text{Aff}(G) \). As pointed out to us by Danilenko and Glasner, this is the case e.g. when \( G \) is Abelian, compact or a connected Lie group; but not in general. Let \( T \) be an ergodic probability preserving transformation of the standard non-atomic probability space \((X, \mathcal{B}, m)\), and let \( G \) be a locally compact, second countable, topological group with left Haar measure \( m_G \), and consider the \( \sigma \)-finite measure space \((X \times G, \mathcal{B} \times \mathcal{B}(G), \mu)\) where \( \mu = m \times m_G \).

The main point of this section is to establish the continuous embedding lemma (see below) which is a topological version of the semigroup embedding lemma. Let \( \mathcal{M} = \mathcal{M}(X \times G, \mathcal{B} \times \mathcal{B}(G), \mu) \), and let \( \tilde{\mathcal{M}} \) denote those \( Q \in \mathcal{M} \) of form

\[
Q(x, y) = (Sx, h(x)w(y))
\]

where \( w \in \text{Aut}(G) \) is a continuous group automorphism, \( S \in \mathcal{M}(X) \) and \( h : X \to G \) is measurable.

Write

\[
S_f(x, y) = (Sx, f(x)y)
\]

for \( S \in \mathcal{M}(X) \) and \( f : X \to G \) measurable. Also, for \( w \in \text{Aut}(G) \), write

\[
W_w(x, y) = (x, w(y)).
\]

If \( Q \in \tilde{\mathcal{M}} \) is as in (\( * \)), then

\[
Q = S_f \circ W_w.
\]

It is clear that the representation \( S_f(x, y) = (Sx, f(x)y) \) is unique in the sense that \( S_f = S'_f \) implies \( S = S' \) and \( f = f' \). The unicity of the representation of \( Q \in \tilde{\mathcal{M}} \) by (\( * \)) follows from this, and \( \sigma_{w(g)} = Q \circ Q^{-1} \) where \( \sigma_g(x, y) = (x, yg) \).

Now let \( T : X \to X \) be an ergodic probability preserving transformation, and let \( \varphi : X \to G \) be measurable.

We study \( \tilde{C}(T_\varphi) := C(T_\varphi) \cap \tilde{\mathcal{M}} \) - a closed subgroup of \( \mathcal{M} \). If \( Q \in \tilde{\mathcal{M}} \), then \( Q \in \tilde{C}(T_\varphi) \) if and only if \( S_Q \in C(T) \), and

\[
\varphi \circ S_Q = h_Q \circ T \cdot w_Q \circ \varphi \circ h_Q^{-1}.
\]

Set

\[
A_\varphi(T) = \{ w_Q : Q \in \tilde{C}(T_\varphi) \}, \quad L_\varphi(T) = \{ S_Q : Q \in \tilde{C}(T_\varphi) \} = L_\varphi \cap C(T).
\]
Continuous embedding lemma. Suppose that $T$ is an ergodic probability preserving transformation of the standard probability space $(X, \mathcal{B}, m)$, that $G$ is a locally compact, second countable topological group, and that $\varphi : X \to G$ is measurable such that $T \varphi$ is ergodic.

There is a Polish topology on $L_\varphi$, stronger than the topology inherited from $\mathcal{M}(X)$, and a continuous homomorphism

$$\pi_\varphi : L_\varphi \to Aut(G)/Inn(G)$$

such that if $Q \in C(T \varphi)$ is of form $(\ast)$, then

$$w_Q Inn(G) = \pi_\varphi(S_Q).$$

In case $G$ is Abelian, then $\pi_\varphi$ is the restriction to $L_\varphi$ of the homomorphism in the semigroup embedding lemma, and there is a Polish topology on $A_\varphi(T)$ stronger than that inherited from $Aut(G)$ such that

$$\pi_\varphi : L_\varphi(T) \to A_\varphi(T)$$

is continuous.

The proof of the continuous embedding lemma uses four lemmas, two of which concern the structure of $\widetilde{\mathcal{M}}$.

Let $\overline{G} := \{\sigma_g : g \in G\}$, then $\overline{G} = \{Id\} \times \hat{G}_R$ is a closed subgroup of $\mathcal{M}$, and the embedding $g \mapsto \sigma_g$ is a homeomorphism ($G \leftrightarrow \overline{G}$). Also, $\overline{G} \subset \widetilde{\mathcal{M}}$ because $\sigma_g(x, y) = (x, g(g^{-1}yg))$. We’ll need

**Lemma 1.1.** If $Z$ is a separable metric space, and $f : X \times G \to Z$ is measurable and $f \circ \sigma_g = f$ a.e. $\forall g \in G$, then $\exists g : X \to Z$ such that $f(x, y) = g(x)$ a.e.

**Proof.** Choose $h : Z \to [0, 1]$ injective, and (Borel) measurable. For $A \in \mathcal{B}$, the function $f_A : G \to IR$ defined by

$$f_A(y) = \int_A h(f(x, y)) dm(x)$$

is $R_g$-invariant $\forall g \in G$, and hence, by ergodicity of $\hat{G}_R$ on $G$, $\exists c(A) \in IR$ such that $f_A = c(A)$ a.e. Since $c : \mathcal{B} \to IR$ is a $m$-absolutely continuous signed measure, $\exists k : X \to IR$ such that $h(f(x, y)) = k(x)$ a.e., and the required function is $g(x) = h^{-1}(k(x))$. □

Suppose that $Q \in \widetilde{\mathcal{M}}$, then, as mentioned above

$$\forall g \in G, \exists g' \in G \ni Q \circ \sigma_g = \sigma_{g'} \circ Q.$$  

We obtain the converse statement as an immediate consequence of lemma 1.1.

Supposing that $Q \in \mathcal{M}$ and that

$$\forall g \in G, \exists g' \in G \ni Q \circ \sigma_g \circ Q^{-1} = \sigma_{g'},$$

we obtain a continuous group endomorphism $w : G \to G$ such that

$$Q \circ \sigma_g \circ Q^{-1} = \sigma_{w(g)}.$$
Writing
\[ Q(x, y) = (S(x, y), F(x, y)), \]
we have that
\[ S(x, yg) = S(x, y), \quad F(x, yg) = F(x, y)w(g). \]
The functions \( (x, y) \mapsto S(x, y) \) and \( (x, y) \mapsto F(x, y)w(y)^{-1} \) are \( \sigma_g \)-invariant \( \forall g \in G \), and hence, by lemma 1.1, for a.e. \((x, y) \in X \times G\)
\[ S(x, y) = S(x) \text{ and } F(x, y) = f(x)w(y). \]
The assumption that \( Q \in \mathcal{M} \) now gives that \( Q \in \widetilde{\mathcal{M}} \).

We now discuss the topology of \( \widetilde{\mathcal{M}} \). It can easily be shown that the topology inherited by
\[ \{ \text{Id}_f : f : X \to G \text{ measurable} \} \]
from \( \mathcal{M} \) is the topology of convergence in measure.

**Lemma 1.2.** \( \widetilde{\mathcal{M}} \) is closed in \( \mathcal{M} \), and the projections
\[ Q \mapsto S_Q, \; Q \mapsto f_Q \text{ and } Q \mapsto w_Q \]
are continuous.

**Proof.** Firstly, suppose that \( Q_n \in \widetilde{\mathcal{M}} \), and \( Q_n \to Q \) in \( \mathcal{M} \). We have that \( \sigma_{w_n(g)} = Q_n \circ \sigma_g \circ Q_n^{-1} \) converges, necessarily to \( Q \circ \sigma_g \circ Q^{-1} = \sigma_{w(g)} \) since \( \overline{G} \) is closed in \( \mathcal{M} \), and by the above application of lemma 1.1, \( Q \in \widetilde{\mathcal{M}} \). This also proves that
\[ w_{Q_n} \to w_Q \text{ pointwise,} \]
and hence in \( \text{Aut}(G) \).

To see that \( Q \mapsto S_Q \) is continuous, let \( Q, R \in \widetilde{\mathcal{M}} \). Let \( A \in \mathcal{B}(X) \), then \( \exists C \subset A, \; C \in \mathcal{B}(X) \) such that \( m(C) > \frac{m(S_Q^{-1}A \Delta S_R^{-1}A)}{2} \), and either (a) \( S_Q^{-1}C \cap S_R^{-1}A = \emptyset \), or (b) \( S_Q^{-1}A \cap S_R^{-1}C = \emptyset \).

Now let \( F \in \mathcal{B}(G) \), \( m_G(F) < \infty \).

In case (a), \( Q^{-1}(C \times F) \cap R^{-1}(A \times F) = \emptyset \), and
\[ \mu(Q^{-1}(A \times F) \Delta R^{-1}(A \times F)) \geq \mu(Q^{-1}(C \times F)) = D(Q)m(C)m_G(F), \]
and similarly, in case (b), \( Q^{-1}(A \times F) \cap R^{-1}(C \times F) = \emptyset \), and
\[ \mu(Q^{-1}(A \times F) \Delta R^{-1}(A \times F)) \geq \mu(R^{-1}(C \times F)) = D(R)m(C)m_G(F). \]
This shows that
\[ m(S_Q^{-1}A \Delta S_R^{-1}A) \leq \frac{2\mu(Q^{-1}(A \times F) \Delta R^{-1}(A \times F))}{\min\{D(Q), D(R)\}m_G(F)}. \]

The continuity of \( Q \mapsto D(Q) = D(w_Q) \) now shows the continuity of \( Q \mapsto S_Q \).

Finally, the continuity \( Q \mapsto f_Q \) follows from
\[ (\text{Id})_{f_Q} = (S_Q)^{-1} \circ Q \circ W^{-1}_{w_Q}. \]
\( \square \)
Lemma 1.3.

There is a topology on $L_\varphi(T)$ with respect to which it is a Polish group, continuously embedded in $C(T)$, and

$$L_\varphi(T) \cong \tilde{C}(T_\varphi)/\mathbb{G}.$$ 

Proof. Write $p(Q) = S_Q$, then $p : \tilde{M} \to M(X)$ is continuous, and $p(\tilde{C}(T_\varphi)) = L_\varphi(T)$. Clearly, $p \circ \sigma_g = p$ for all $g \in G$, whence $p : \tilde{C}(T_\varphi)/\mathbb{G} \to L_\varphi(T)$ is well-defined, onto, and continuous when $\tilde{C}(T_\varphi)/\mathbb{G}$ is equipped with the quotient (Polish) topology. We claim that $p|_{\tilde{C}(T_\varphi)/\mathbb{G}}$ is actually injective.

To see this, suppose that $Q \in \tilde{C}(T_\varphi)$ and $Q(x,y) = (x,h(x)w(y))$, then

$$h \circ T \cdot w \circ \varphi = \varphi \cdot h.$$ 

Now set $F(x,y) = y^{-1}h(x)w(y)$, then

$$F \circ T_\varphi(x,y) = y^{-1}\varphi(x)^{-1}[h(Tx)w(\varphi(x))]w(y)$$

$$= y^{-1}\varphi(x)^{-1}[\varphi(x)h(x)]w(y)$$

$$= F(x,y)$$

and $F$ is constant by ergodicity of $T_\varphi$, whence $Q \in \mathbb{G}$.

The group isomorphism $p : \tilde{C}(T_\varphi)/\mathbb{G} \to L_\varphi(T)$ can be used to transport the Polish structure to $L_\varphi(T)$ which is a Polish group, continuously embedded in $C(T)$. □

Remark. By Souslin’s theorem (see [Kur, §36, IV]: $L_\varphi(T)$ is a Borel subset of $C(T)$, and

$$\mathcal{B}(L_\varphi(T)) = \mathcal{B}(C(T)) \cap L_\varphi(T).$$

Let $\tilde{C}_0(T_\varphi) = \{Q \in \tilde{C}(T_\varphi) : w_Q = \text{Id} \}$, a normal, closed subgroup of $\tilde{C}(T_\varphi)$.

Lemma 1.4.

There is a topology on $\mathcal{A}_\varphi(T) := \mathcal{E}_\varphi \cap \text{Aut}(G)$ with respect to which it is a Polish group, continuously embedded in $\text{Aut}(G)$, and, as Polish groups,

$$\mathcal{A}_\varphi(T) \cong \tilde{C}(T_\varphi)/\tilde{C}_0(T_\varphi).$$

Proof. Write $q(Q) = w_Q$ for $Q \in \tilde{C}(T_\varphi)$. By lemma 1.2, $q : \tilde{C}(T_\varphi) \to \text{Aut}(G)$ is continuous, and $q(\tilde{C}(T_\varphi)) = \mathcal{A}_\varphi(T)$. Clearly,

$$\text{Ker} \ q = \tilde{C}_0(T_\varphi),$$

whence $q : \tilde{C}(T_\varphi)/\tilde{C}_0(T_\varphi) \to \mathcal{A}_\varphi(T)$ is well-defined and bijective.

As before, the group isomorphism can be used to transport the quotient Polish topology on $\tilde{C}(T_\varphi)/\tilde{C}_0(T_\varphi)$ to $\mathcal{A}_\varphi(T)$ which thus becomes a Polish group, continuously embedded in $\text{Aut}(G)$. □
Proof of the continuous embedding lemma. Let
\[ \tilde{C}_I(T_\varphi) = \{ Q \in \tilde{C}(T_\varphi) : w_Q \in \text{Inn}(G) \}. \]

Note that \( \tilde{C}_I(T_\varphi) \) is a closed normal subgroup of \( \mathcal{M} \), and is generated by \( \tilde{C}_0(T_\varphi) \), and \( \overline{G} \).

It now follows that
\[ \tilde{C}(T_\varphi)/\tilde{C}_I(T_\varphi) \]

is a Polish group, continuously embedded in \( \text{Aut}(G)/\text{Inn}(G) \) by
\[ Q\tilde{C}_I(T_\varphi) \mapsto w_Q\text{Inn}(G). \]

A natural map \( L_\varphi(T) \rightarrow \text{Aut}(G)/\text{Inn}(G) \) is now generated by
\[ L_\varphi(T) \cong \tilde{C}(T_\varphi)/\overline{G} \rightarrow \tilde{C}(T_\varphi)/\tilde{C}_I(T_\varphi) \rightarrow \text{Aut}(G)/\text{Inn}(G). \]

It follows from the above that this map is continuous.

In case \( G \) is Abelian, \( \text{Inn}(G) = \{ \text{Id} \} \), and the above becomes a statement of the continuity of:
\[ L_\varphi(T) \cong \tilde{C}(T_\varphi)/\overline{G} \rightarrow \tilde{C}(T_\varphi)/\tilde{C}_0(T_\varphi) \cong \mathcal{A}_\varphi(T). \]

\[ \square \]
§2 Eigenvalues of skew products

Let $T$ be an ergodic probability preserving transformation of the probability space $(X, B, m)$, let $G$ be an locally compact, second countable, topological group, and let $\varphi : X \to G$ be a cocycle with $T \varphi$ ergodic on $X \times G$.

Recall that if $R$ is non-singular, conservative, ergodic and $f : X \to \mathbb{C}$ is measurable such that $f \circ R = \lambda f$, where $\lambda \in \mathbb{C}$, then $|f|$ is constant (w.l.o.g. = 1), $|\lambda| = 1$.

We consider the situation where the measurable function $\varphi : X \to G$ is aperiodic in the sense that all eigenvalues for the skew product $T \varphi$ are eigenvalues for $T$ (that is, if $f : X \times G \to \mathbb{T}$ is measurable and $f \circ T \varphi = \lambda f$ where $\lambda \in \mathbb{T}$, then $\exists g : X \to \mathbb{T}$ measurable such that $f(x,y) = g(x)$ a.e.). The main result of this section is

**Proposition 2.1.**

If $G = \mathbb{R}$ or $\mathbb{T}$, $T \varphi$ is ergodic, and $E_{\varphi} \neq \{Id\}$, then $\varphi$ is aperiodic.

**Lemma 2.2.**

Suppose that $T \varphi$ is ergodic and $f : X \times G \to \mathbb{T}$ is measurable such that $f \circ T \varphi = \lambda_0 f$ where $\lambda_0 \in \mathbb{T}$, then

$$f(x,y) = f_0(x)\gamma(y)$$

where $\gamma \in \hat{G}$ and $f_0 : X \to \mathbb{T}$ is measurable.

**Remark.** Note that we do not assume that $G$ is Abelian here.

**Proof.** For $Q \in C(T \varphi)$, we have that

$$(f \circ Q) \circ T \varphi = f \circ T \varphi \circ Q = \lambda_0 f \circ Q,$$

whence, by ergodicity of $T \varphi$, $\exists \lambda(Q) \in \mathbb{T}$ such that $f \circ Q = \lambda(Q)f$ (note that $\lambda(T \varphi) = \lambda_0$). The mapping $\lambda(Q) : C(T \varphi) \to \mathbb{T}$ is a continuous homomorphism.

Since $G \subset C(T \varphi)$, we obtain $\gamma \in \hat{G}$ by taking $\gamma(g) := \lambda(\sigma_g)$. Thus

$$f \circ \sigma_g = \gamma(g)f \ \forall \ g \in G.$$ 

Set $F(x,y) = \gamma(y)^{-1}f(x,y)$, then $F \circ \sigma_g = F \ \forall \ g \in G$, whence by lemma 1.1, for a.e. fixed $x \in X$, $F(x,\cdot)$ is constant. This proves the lemma. \hfill \Box

**Lemma 2.3.**

Suppose that $T \varphi$ is ergodic and $f = f_0 \otimes \mu$ where $f_0 : X \to \mathbb{T}$ is measurable, $\gamma \in \hat{G}$, and $f \circ T \varphi = \lambda_0 f$ for some $\lambda_0 \in \mathbb{T}$, then

$$\gamma \circ w = \gamma \ \forall \ w \in E_{\varphi}.$$ 

**Proof.** Let $\lambda : C(T \varphi) \to \mathbb{T}$ be such that

$$f \circ Q = \lambda(Q)f \ \forall \ Q \in C(T \varphi).$$
Suppose that \( w \in \mathcal{E}_\varphi \), and \( Q \in \tilde{\mathcal{C}}(T_\varphi) \) with \( Q(x, y) = (Sx, h(x)w(y)) \), then
\[
\lambda(Q)f_0 \otimes \gamma(x, y) = \lambda(Q)f(x, y) \\
= f \circ Q(x, y) \\
= f_0(Sx)\gamma(h(x))\gamma(w(y)) \\
= [(f_0 \circ S) \cdot (\gamma \circ h)] \otimes \gamma \circ w(x, y),
\]
and since the character \( \gamma \in \hat{G} \) appearing in the eigenfunction \( f_0 \otimes \gamma \) is unique,
\[
\gamma \circ w = \gamma.
\]

\[\square\]

**Proof of Proposition 2.1.** This now follows from lemma 2.3, as if \( G = \mathcal{I}; \mathbb{R}, \gamma \in \hat{G}, w \in \text{End}(G) \), then \( \gamma \circ w = \gamma \) iff either \( \gamma \equiv 1 \) or \( w = \text{Id} \). \[\square\]

**Corollary 2.4** (cf [Rob]).
If \( T_\varphi \) is ergodic, then the only eigenvalues of \( T_\varphi \) are the eigenvalues of \( T_{\varphi[G,G]} : X \times G/[G,G] \rightarrow X \times G/[G,G] \).

**Proof.** By lemma 2.2, an eigenfunction of \( T_\varphi \) must be of form \( f_0 \otimes \mu \) where \( \mu \in \hat{G} \). But
\[
\hat{G} = G/[G,G],
\]
so that any eigenfunction of \( T_\varphi \) is actually an eigenfunction of \( T_{\varphi[G,G]} \). \[\square\]

**Proposition 2.5.**
If \( T \) is Bernoulli, then any ergodic \( \mathbb{R} \)-extension of \( T \) is (isomorphic to) a \( \mathbb{Z} \)-extension of \( T \).

**Proof.** Let \( \varphi : X \rightarrow \mathbb{R} \) be an measurable such that \( T_\varphi \) is ergodic. For \( c > 0 \) let \( \varphi^{(c)} : X \rightarrow \mathbb{R} \cong [0, c) \) be defined by \( \varphi^{(c)} = \varphi \mod c \). For each \( c > 0 \) there is a measurable function \( \psi^{(c)} : X \times \mathbb{R} \rightarrow \mathbb{Z} \) such that
\[
T_\varphi \cong (T_{\varphi^{(c)}})_{\psi^{(c)}}.
\]
It is known that for some \( c > 0 \), \( \varphi^{(c)} \) is not cohomologous to a constant, else (see [M-S] and [H-O-O]) \( \varphi \) would be cohomologous to a constant and not ergodic. For this \( c > 0 \), \( T_{\varphi^{(c)}} \) is weakly mixing, whence by theorem 1 of [Rud] \( T_{\varphi^{(c)}} \) is Bernoulli, and since \( h(T_{\varphi^{(c)}}) = h(T) \), we have by [Or] that \( T_{\varphi^{(c)}} \cong T \). The conclusion is that \( (T_{\varphi^{(c)}})_{\psi^{(c)}} \) is a \( \mathbb{Z} \)-extension of \( T \). \[\square\]

§3 ESSENTIAL VALUE CONDITIONS

Let \( T \) be an invertible, ergodic probability preserving transformation of the standard probability space \( (X, \mathcal{B}, m) \), let \( G \) be a locally compact, second countable Abelian group, and let \( \varphi : X \rightarrow G \) be measurable. We develop here a countable condition for ergodicity of \( T_\varphi \). The EVC’s to be defined are best understood in terms of orbit cocycles, and the groupoid of \( T \) (see [Fe-Mo]).

A partial probability preserving transformation of \( X \) is a pair \((R, A)\) where \( A \in \mathcal{B} \) and \( R : A \rightarrow RA \) is invertible and \( m|_{RA} \circ R^{-1} = m|_A \). The set \( A \) is called the
The domain of \((R, A)\). We’ll sometimes abuse this notation by writing \(R = (R, A)\) and \(A = \mathcal{D}(R)\). Similarly, the image of \((R, A)\) is the set \(\Im(R) = RA\).

The equivalence relation generated by \(T\) is
\[
\mathcal{R} = \{(x, T^n x) : x \in X, \ n \in \mathbb{Z}\}.
\]

For \(A \in \mathcal{B}(X)\) and \(\phi : A \to \mathbb{Z}\), define \(T^\phi : A \to X\) by \(T^\phi(x) := T^{\phi(x)}x\).

The groupoid of \(T\) is
\[
[T] = \{T^\phi : T^\phi\text{ is a partial probability preserving transformation}\}.
\]

It’s not hard to see that
\[
[T] = \{R : R \text{ a partial probability preserving transformation, } \& \ (x, Rx) \in \mathcal{R} \text{ a.e.}\}.
\]

For \(A \in \mathcal{B}(X)\) and \(T^\phi \in [T]\), write \(\phi_{(R)} := \phi\). Let
\[
[T]^+ = \{R \in [T] : \phi_{(R)} \geq 1 \text{ a.e.}\}.
\]

Recall from [Halm1]:

E. Hopf’s Equivalence lemma. If \(T\) is an ergodic measure preserving transformation of \((X, \mathcal{B}, m)\) and \(A, B \in \mathcal{B}\) with \(m(A) = m(B)\), then
\[
\exists R \in [T]^+ \text{ such that } \mathcal{D}(R) = A, \ \Im(R) = B.
\]

We’ll also need a quantitative version of this lemma when \(A = B\).

Lemma 3.1.

Suppose that \(T\) is an ergodic probability preserving transformation of \((X, \mathcal{B}, m)\), \(A \in \mathcal{B}_+, \text{ and } c, \varepsilon > 0, \text{ then } \forall p, q \in \mathbb{N} \text{ large enough, } \exists R \in [T]^+ \text{ such that}
\[
\mathcal{D}(R), \ \Im(R) \subset A, \ m(A \setminus \mathcal{D}(R)) < \varepsilon, \text{ and } \phi_{(R)} = cpq(1 \pm \varepsilon).
\]

The proof of lemma 3.1 will be given at the end of this section.

Let \(\mathcal{R}\) be the equivalence relation generated by \(T\). An orbit cocycle is a measurable function \(\tilde{\varphi} : \mathcal{R} \to G\) such that if \((x, y), (y, z) \in \mathcal{R}\), then
\[
\tilde{\varphi}(x, z) = \tilde{\varphi}(x, y) + \tilde{\varphi}(y, z).
\]

Let \(\varphi : X \to G\) be measurable, and let \(\varphi_n (n \in \mathbb{Z})\) denote the cocycle generated by \(\varphi\) under \(T\). The orbit cocycle \(\tilde{\varphi} : \mathcal{R} \to G\) corresponding to \(\varphi\) is defined by
\[
\tilde{\varphi}(x, T^n x) = \varphi_n(x).
\]

For \(R \in [T]\), the function \(\varphi_R : \mathcal{D}(R) \to G\) is defined by
\[
\varphi_R(x) = \tilde{\varphi}(x, Rx).
\]

Clearly \(\varphi(R \circ S, x) = \varphi(S, x) + \varphi(R, Sx)\) on \(\mathcal{D}(R \circ S) = \mathcal{D}(S) \cap S^{-1}\mathcal{D}(R)\).
Definition. Let $\alpha$ be a measurable partition of $X$, $U$ a subset of $G$, and $\varepsilon > 0$. We say that the measurable cocycle $\varphi : X \rightarrow \Gamma$ satisfies EVC$_T(U, \varepsilon, \alpha)$ if for $\varepsilon$-almost every $a \in \alpha$, $\exists R = R_a \in [T]_+$ such that

$$\mathcal{D}(R), \mathcal{Z}(R) \subset a, \varphi_R \in U \text{ on } \mathcal{D}(R_a), \ m(\mathcal{D}(R))) > (1 - \varepsilon)m(a).$$

Definition. We say that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$ if

$$\forall B \in \mathcal{B}(X), \varepsilon > 0 \exists k_0 \geq 1 \exists k \geq k_0, \exists A_k \in \mathcal{A}(\alpha_k) \ni m(B \Delta A_k) < \varepsilon.$$ 

Here $\mathcal{A}(\alpha)$ denotes the algebra generated by $\alpha$. It is not hard to see that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$, if and only if $E(1_B|\mathcal{A}(\alpha_k)) \rightarrow 1_B$ in probability $\forall B \in \mathcal{B}$, and in this case,

$$\forall \varepsilon > 0, \quad B \in \mathcal{B}, \quad k_0 \text{ such that } \sum_{a \in \alpha_k, \ 1 - m(B(a)) \leq \varepsilon} m(a) \geq (1 - \varepsilon)m(B) \quad \forall k \geq k_0.$$ 

**Proposition 3.1.** Suppose that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$, and let $\varepsilon_k \downarrow 0$, $\gamma \in \Gamma$, and $U_k \subset G$ satisfy $U_n \downarrow \{\gamma\}$, and $\text{diam} U_n \downarrow 0$.

If $\varphi$ satisfies EVC$_T(U_k, \varepsilon_k, \alpha_k) \forall k \geq 1$, then

$$\gamma \in E(\varphi).$$

**Proof.** Suppose that $B \in \mathcal{B}_+$, and that $V \subset G$ is an open neighbourhood of $\gamma$. We’ll show that

$$\exists n \geq 1 \ni m(B \cap T^{-n}B \cap \{\varphi_n \in V\}) > 0.$$ 

Evidently, $V \supset U_k$ for all $k$ sufficiently large. It follows from the definitions, that $\forall k$ sufficiently large, $\exists a \in \alpha_k$ such that

$$m(a \setminus B) < 0.1m(a),$$

and $\exists R = R_a \in [T]_+$ such that

$$\mathcal{D}(R), \mathcal{Z}(R) \subset a, \varphi_R \in U_k \text{ on } \mathcal{D}(R), \text{ and } m(a \setminus \mathcal{D}(R)) < 0.1m(a).$$

It follows that

$$m(B \setminus \mathcal{D}(R)) < 0.2m(a).$$

Let $R = T^\phi$, where $\phi : \mathcal{D}(R) \rightarrow \mathbb{Z}$. We have that

$$\sum_{n \in \mathbb{Z}} m(B \cap [\phi = n] \cap T^{-n}B \cap \{\varphi_n \in U_k\})$$

$$\geq m(B \cap \mathcal{D}(R) \cap R^{-1}(B \cap \mathcal{Z}(R)) \cap \{\varphi_R \in U_k\})$$

$$\geq 0.6m(a),$$

whence $\exists n \in \mathbb{Z}$ such that

$$m(B \cap T^{-n}B \cap \{\varphi_n \in V\}) \geq m(B \cap [\phi = n] \cap T^{-n}B \cap \{\varphi_n \in U_k\}) > 0.$$ 

$\square$
Corollary 3.2. Suppose that the partitions \{\alpha_k : k \geq 1\} approximately generate \mathcal{B}, let \{U_k : k \geq 1\} be a basis of neighbourhoods for the topology of \Gamma, and let \varepsilon_k \downarrow 0.

If \varphi satisfies \text{EVC}_T(U_k, \varepsilon_k, \alpha_k) \forall k \geq 1, then \mathcal{T}_\varphi is ergodic.

This sufficient condition for ergodicity is actually necessary.

Proposition 3.3. If \mathcal{T}_\varphi is ergodic, then \forall A \in \mathcal{B}_+ \cup \emptyset open in \Gamma, \exists R \in [T]_+ such that

\[ D(R) = \exists(R) = A, \; \& \; \varphi_R \in U \; \text{a.e. on} \; A, \]

and hence, \varphi satisfies \text{EVC}_T(U, \varepsilon, \alpha) for any measurable partition \alpha of \Gamma, \cup \emptyset open in \Gamma, \varepsilon > 0.

Proof. This follows from the ergodicity of \mathcal{T}_\varphi. Let \Gamma be open in \Gamma. Choose \varphi \in \Gamma, then \Gamma := \Gamma - \varphi is a neighbourhood of 0 \in \Gamma. Choose \Gamma open in \Gamma such that \Gamma + \Gamma \subset \Gamma. By ergodicity of \mathcal{T}_\varphi, for every \varphi, \phi \in \mathcal{B}_+, \exists n \in \mathbb{N} such that \mu((A \times \Gamma) \cap T^{-n}(B \times (\Gamma + \Gamma))) > 0, whence \mu((A \cap T^{-n}B) \cap \{\varphi \in \Gamma\}) > 0. The proposition follows from this via a standard exhaustion argument. \qed

We'll need a finite version of \text{EVC} more suited to sequential constructions.

Definition. Let \alpha be a measurable partition of \Gamma, \cup \emptyset open in \Gamma, \varepsilon > 0, and \mathbb{N} \geq 1. We say that the measurable cocycle \varphi : \Gamma \to \Gamma satisfies \text{EVC}^T(U, \varepsilon, \alpha, \mathbb{N}) if:

for \varepsilon\text{-almost every } a \in \alpha, \exists R = R_a \in [T]_+ with \phi(R) \leq \mathbb{N} such that

\[ D(R), \; \exists(R) \subset a, \; \varphi_R \in U \text{ on } D(R), \text{ and } m(a \setminus D(R)) < \varepsilon m(a). \]

Proposition 3.4. Let \alpha be a measurable partition of \Gamma, \cup \emptyset open in \Gamma, \varepsilon > 0. The measurable cocycle \varphi : \Gamma \to \Gamma satisfies \text{EVC}^T(U, \varepsilon, \alpha, \mathbb{N}) iff it satisfies \text{EVC}^T(U, \varepsilon, \alpha, \mathbb{N}) for some \mathbb{N} \geq 1.

The next lemma shows that addition of a sufficiently small cocycle does not affect \text{EVC}^T conditions too much.

Lemma 3.5. Let \alpha be a partition, \varepsilon, \delta, \varepsilon > 0, \mathbb{N} \in \mathbb{N}, U \subset \Gamma, and \phi : \Gamma \to \Gamma be a cocycle satisfying \text{EVC}^T(U, \varepsilon, \alpha, \mathbb{N}) where \Gamma \subset \Gamma.

If \varphi : \Gamma \to \Gamma is measurable, and

\[ m(\{\varphi \notin U\}) < \frac{\delta^2}{\mathbb{N}}, \]

then \phi + \varphi satisfies \text{EVC}(U + V, \varepsilon + \delta, \alpha, \mathbb{N}).

Proof. Let \Gamma = \{\varphi \circ T^j \in V \forall j \leq N - 1\}, then since

\[ \varphi_n \in V \text{ on } B \forall 1 \leq n \leq N, \]

\[ \varphi_R \in V \text{ on } B \cap D(R) \forall R \in [T]_+ \text{ with } \phi(R) \leq \mathbb{N}. \]

Let \alpha consist of those \alpha \in \alpha such that \exists R = R_a \in [T]_+ with \phi(R) \leq \mathbb{N} such that

\[ D(R), \; \exists(R) \subset a, \; \varphi_R \in U \text{ on } D(R), \text{ and } m(a \setminus D(R)) < \varepsilon m(a). \]

We have that

\[ m(\bigcup_{a \in \alpha} a) > 1 - \varepsilon. \]
Let $\alpha_2$ consist of those $a \in \alpha$ for which

$$m(B \cap a) > (1 - \delta)m(a).$$

It follows from Chebyshev’s inequality that

$$m\left( \bigcup_{a \in \alpha_2} a \right) > 1 - \frac{m(B)}{\delta} > 1 - \delta.$$

Therefore

$$m\left( \bigcup_{a \in \alpha_1 \cap \alpha_2} a \right) > 1 - \varepsilon - \delta.$$

If $a \in \alpha_1 \cap \alpha_2$, and $R' = R'_a := (R_a, D(R_a) \cap B) \in [T]_+$, then:

$$D(R'), 3(R') \subset a, (\phi + \varphi)_{R'} \in U + V \text{ on } D(R'), \text{ and } m(a \setminus D(R')) < (\varepsilon + \delta)m(a).$$

□

Our main result in this section is a sufficient condition for a group element to be an essential value of a sum of coboundaries.

**Theorem 3.6.**

Suppose that $g \in G$, the partitions $\{\alpha_j\}$ approximately generate $B$; $N_k \in \mathbb{N}$, $N_k \uparrow \infty$, and $\varepsilon_k > 0$, $\sum_{k=1}^{\infty} \varepsilon_k < \infty$.

If for $k \in \mathbb{N}$, $f_k : X \to G$ is measurable,

$$\sum_{j=1}^{k} (f_j \circ T - f_j) \text{ satisfies } EVC^T(N(g, \varepsilon_k), \varepsilon_k, \alpha_k, N_k),$$

and

$$m([|f_k \circ T - f_k| \geq \frac{\varepsilon_{k-1}}{N_{k-1}}]) \leq \frac{\varepsilon_{k-1}^2}{N_{k-1}},$$

then

$$\sum_{k=1}^{\infty} |f_k \circ T - f_k| < \infty \text{ a.e., and } g \in E\left( \sum_{k=1}^{\infty} (f_k \circ T - f_k) \right).$$

**Proof.** By the Borel Cantelli lemma, $\sum_{k=1}^{\infty} |f_k \circ T - f_k| < \infty$ a.e.. Write

$$\phi := \sum_{k=1}^{\infty} (f_k \circ T - f_k), \quad \tilde{\phi}_k = \sum_{j=1}^{k} (f_j \circ T - f_j), \quad \hat{\phi}_k = \sum_{j=k+1}^{\infty} (f_j \circ T - f_j).$$

Since

$$\phi = \tilde{\phi}_k + \hat{\phi}_k \ \forall \ k \geq 1,$$
\( \phi_k \) satisfies \( \text{EVC}^T(N(g, \varepsilon_k), \varepsilon_k, N_k) \), and
\[
m((|\hat{\phi}_k| \geq \frac{1}{N_k} \sum_{j=k+1}^{\infty} \varepsilon_j)) \leq \sum_{j=k+1}^{\infty} m((|f_j \circ T - f_j| \geq \frac{\varepsilon_j}{N_k})) \\
\leq \sum_{j=k+1}^{\infty} m((|f_j \circ T - f_j| \geq \frac{\varepsilon_j}{N_{j-1}})) \\
\leq \frac{1}{N_k} \sum_{j=k}^{\infty} \varepsilon^2_j,
\]
it follows from lemma 3.5 that \( \phi \) satisfies
\( \text{EVC}^T(N(g, \sum_{j=k}^{\infty} \varepsilon_j), 2\sqrt{\sum_{j=k}^{\infty} \varepsilon^2_j}, \alpha_k, N_k) \).
\( \square \)

As promised above, we conclude this section with the

**Proof of lemma 3.1.** Let
\[
A_n = \left[ \left| \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T^k - m(A) \right| < \varepsilon m(A) \right].
\]

By Birkhoff’s ergodic theorem, \( \exists \ p_0 \in \mathbb{N} \) such that \( m(A_{p_0}^c) < \frac{\varepsilon^2}{2} \ \forall \ p \geq p_0 \). Fix \( p \geq p_0 \). Now fix \( q \geq \frac{p}{\varepsilon} := q_0 \). Set
\[
B = A_p \cap T^{-[cq]} A_p.
\]
Evidently \( m(B) > 1 - \varepsilon^2 \).

By Birkhoff’s ergodic theorem \( \exists N_0 \in \mathbb{N} \) such that
\[
m(C_n^c) < \frac{\varepsilon^2}{2p} \ \forall \ n \geq N_0
\]
where
\[
C_n = \left[ \left| \frac{1}{n} \sum_{k=0}^{n-1} 1_B \circ T^k \geq E(1_B|I_Tp) - \varepsilon^2 \right| \right].
\]

Let \( N > \frac{p}{\varepsilon} \ \forall p N_0 \). By Rokhlin’s theorem, \( \exists F \in \mathcal{B} \) such that
\[
\{T^jF : 0 \leq j \leq N - 1\}
\]
are disjoint, and \( m\left( X \setminus \bigcup_{j=0}^{N-1} T^jF \right) < \frac{\varepsilon}{p} \).

Note that since \( E(1_B|I_Tp) \) is \( T^p \)-invariant, we have
\[
\frac{N}{p} \sum_{k=0}^{p-1} \int_{T^kF} E(1_B^c|I_Tp) dm \leq \int_X E(1_B^c|I_Tp) dm
\]
\[
= m(B^c) < \varepsilon^2,
\]
whence \( 0 \leq k \leq p - 1 \) such that

\[
\int_{T^kF} E(1_{B^c}|I_{T^p})dm < \varepsilon^2 m(F).
\]

There is no loss of generality in assuming \( k = 0 \) as this merely involves taking \( T^kF \) as the base for a slightly shorter Rokhlin tower, and adding \( \bigcup_{j=0}^{k-1} T^jF \) to the "error set".

Set

\[ X_0 = \bigcup_{j=0}^{N-pq} T^jF, \quad \text{and} \quad J = X_0 \cap \bigcup_{j\geq 0, jp\leq N} T^{jp}F, \]

then \( m(J) > \frac{1}{2p} \) so

\[ m(C_n^c \cap J) \leq \varepsilon^2 m(J) \quad \forall \ n \geq N_0. \]

For \( y \in J \), set \( \kappa(y) = \#\{0 \leq j \leq p - 1 : T^jy \in A\} \) and write

\[ \{T^jy : 0 \leq j \leq p - 1, T^jy \in A\} = \{T^{j_i(y)}y : 1 \leq i \leq \kappa(y)\} \]

in case \( \kappa(y) \geq 1 \), where \( j_i(y) < j_{i+1}(y) \). Note that

\[ \kappa = pm(A)(1 \pm \varepsilon) \quad \text{on} \ J \cap A_p. \]

To estimate \( m(J \cap B) \):

\[
\sum_{0 \leq j \leq \frac{N}{p} : m(C_n^c \cap T^{jp}F) \geq \varepsilon} m(T^{jp}F) \leq \sum_{0 \leq j \leq \frac{N}{p}} \frac{m(C_n^c \cap T^{jp}F)}{\varepsilon} \leq \frac{m(C_n^c \cap J)}{\varepsilon} \leq \frac{m(C_n^c)}{\varepsilon} \leq \frac{\varepsilon}{2p} \leq \varepsilon m(J),
\]

whence, \( \exists \ i \leq \frac{\varepsilon N}{p} \) such that

\[ m(C_n^c \cap T^{ip}F) = m(T^{-ip}C_n^c \cap F) \geq (1 - \varepsilon)m(F). \]

For \( y \in T^{-ip}C_n^c \cap F \),

\[
\#\{0 \leq j \leq \frac{N}{p} : T^{jp}y \in B\} \geq \#\{0 \leq j \leq \frac{N}{p} : T^{(i+j)p}y \in B\} - \varepsilon \frac{N}{p} \geq \frac{N}{p}(E(1_B|I_{T^p}) - 2\varepsilon).
\]
Therefore,
\[
m(J \cap B) = \sum_{k=0}^{p-1} m(T^j F \cap B)
\]
\[
= \int_F \left( \sum_{k=0}^{p-1} 1_B \circ T^j \right) dm
\]
\[
\geq \frac{N}{p} \int_{T^{-1p} C \cap F} (E(1_B | I_T) - 2\varepsilon) dm
\]
\[
\geq \frac{N}{p} \int_F (E(1_B | I_T) - 3\varepsilon) dm
\]
\[
\geq (1 - 4\varepsilon)m(F)\frac{N}{p} = (1 - 4\varepsilon)m(J).
\]

For \(x \in \bigcup_{j=0}^{p-1} T^j J\), let \(j(x)\) be such that \(T^{-j(x)}x \in J\), and let \(y(x) = T^{-j(x)}x\).
Define \(\psi : A \cap \bigcup_{j=0}^{p-1} T^j J \to \{1, 2, \ldots p\}\) by
\[
\psi(x) = \sum_{k=0}^{j(x)} 1_A(T^{-k}x) = \sum_{k=0}^{j(x)} 1_A(T^k y(x)).
\]
Note that
\[
x = T^{j(\psi(x))}y(x).
\]
Now define \(D \subset A \cap X_0\) by
\[
D \cap \bigcup_{j=0}^{p-1} T^j J_0 = \{ x \in A \cap J_0 : \psi(x) \leq \kappa(y(T^{[cq]p}x)) \},
\]
and define \(\phi : D \to \mathbb{N}\) by
\[
\phi(x) = [cq]p + j_{\psi(x)}(y(T^{[cq]p}x)) \quad x \in D \cap \bigcup_{j=0}^{p-1} T^j J.
\]
We claim that if \(R \in [T]_+\) is defined by \(\mathcal{D}(R) = D\) and \(\phi(R) = \phi\), then \(\phi\) is as advertised. To see this, check that \(\kappa \geq (1 - \varepsilon)m(A)p\) on \(J \cap B\), whence
\[
m(D) \geq m(J_0 \cap B)(1 - \varepsilon)m(A)p \geq (1 - 6\varepsilon)m(J)pm(A) \geq (1 - 7\varepsilon)m(A).
\]

§4 PROOF OF THEOREMS 1 AND 2

In this section, we prove theorems 1 and 2. The proofs are sequential using theorem 3.6. The inductive steps are lemmas 4.1 and 4.2. Their proofs use the Rohlin lemmas for Abelian group actions of Katznelson and Weiss [K-W], and Lind [Lin] respectively (see also [O-W] for a general Rohlin lemma for amenable group actions implying these).

Let \(G\) be a locally compact, second countable Abelian group with invariant metric \(d\), and let \(T\) be an ergodic probability preserving transformation of the standard probability space \((X, \mathcal{B}, m)\).
Definition.

A measurable function \( f : X \to G \) is called a \( T \)-coboundary if \( f = h - h \circ T \) for some measurable function \( h : X \to G \).

Measurable functions \( f, g : X \to G \) are said to be \( T \)-cohomologous, written \( f \sim_T g \), if \( f - g \) is a \( T \)-coboundary.

Let \( \varphi : X \to G \) be measurable. Suppose \( S \in \mathcal{L}_\varphi(T) \), and \( w \in \text{End}(G) \), then

\[ w = \pi_\varphi(S) \iff \varphi \circ S \sim_T w \circ \varphi. \]

We prove the following version of theorem 1:

**Theorem 1'**.
Suppose that \( T \) is an ergodic probability preserving transformation, \( S_1, \ldots, S_d \in C(T) (d \leq \infty) \) are such that \((T, S_1, \ldots, S_d)\) generate a free \( \mathbb{Z}^{d+1} \) action of probability preserving transformations of \( X \).

If \( w_1, \ldots, w_d \in \text{End}(G) \) commute (i.e. \( w_i \circ w_j = w_j \circ w_i \forall 1 \leq i, j \leq d \)), then there is a measurable function \( \varphi : X \to G \) such that \( T \varphi \) is ergodic, and

\[ \varphi \circ S_i \sim_T w_i \circ \varphi \quad (1 \leq i \leq d). \]

**Lemma 4.1**.  Let \( \phi : X \to G \) be a \( T \)-coboundary, let \( S_1, \ldots, S_d \) be probability preserving transformations generating a free \( \mathbb{Z}^{d+1} \) action together with \( T \), and let \( w_1, \ldots, w_d \in \text{End}(G) \), \( w_i \circ w_j = w_j \circ w_i \). If \( \alpha \) is a finite, measurable partition of \( X \), and \( \varepsilon > 0 \), then there is a measurable function \( f : X \to G \) such that

\begin{align*}
(1) & \quad m(\{f \circ T - f \neq 0\}) < \varepsilon, \\
(2) & \quad m(\{f \circ S_j \neq w_j \circ f\}) < \varepsilon, \quad (1 \leq j \leq d) \\
\end{align*}

and such that

\[ \phi + f - f \circ T \text{ satisfies } \text{EVC}_T(N(\gamma, \varepsilon), \varepsilon, \alpha). \]

**Proof**.  Write \( \phi = H - H \circ T \). Possibly refining \( \alpha \), we may assume that for \( \frac{\varepsilon}{2} \)-a.e. \( a \in \alpha \), the oscillation of \( H \) on \( a \) is less than \( \frac{\varepsilon}{2} \).

For \( \underline{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d_+ \), we’ll write

\[ S_{\underline{i}} := S_{i_1} \circ \cdots \circ S_{i_d} \quad \& \quad w_{\underline{i}} := w_{i_1} \circ \cdots \circ w_{i_d}. \]

Then

\[ S_{\underline{i} + \underline{j}} = S_{\underline{i}} \circ S_{\underline{j}} \quad \& \quad w_{\underline{i} + \underline{j}} = w_{\underline{i}} \circ w_{\underline{j}} \]

since \( S_i \circ S_j = S_j \circ S_i \) and \( w_i \circ w_j = w_j \circ w_i \).

Fix \( k > \frac{10}{\varepsilon} \). There is an ergodic cocycle \( \varphi : X \to G \) such that

\[ m(\{\varphi \neq 0\}) < \frac{\varepsilon}{3kd}. \]
It follows that \( w_i \circ \varphi \circ S_{-i} \) is ergodic for \( i \geq 0 \) (as \( w_i \) is surjective, and \( S_{-i} \) commutes with \( T \) for \( i \geq 0 \)), whence \( \phi + w_i \circ \varphi \circ S_{-i} \) is ergodic for \( i \geq 0 \) (as \( \phi \) is a coboundary), and so satisfies \( \text{EVC}_T(N(\gamma, \frac{\epsilon}{4k^d}), \frac{\epsilon}{4k^d}, \alpha) \). Therefore (by propositions 3.3 and 3.4),

\[
\exists M \in \mathbb{N} \text{ such that } \phi + w_i \circ \varphi \circ S_{-i} \text{ satisfies } \text{EVC}_T(N(\gamma, \frac{\epsilon}{4}), \frac{\epsilon}{4k^d}, \alpha, M)
\]

for \( 0 \leq i \leq k \) where \( k := (k, \ldots, k) \), and \( 0 \leq i < k \) means \( 0 \leq i_j < k_j \forall 1 \leq j \leq d \).

Now choose \( N \geq 1 \) such that

\[
\frac{M}{N} < \frac{\epsilon \eta_\alpha}{5}
\]

where \( \eta_\alpha := \min \{m(a) : a \in \alpha\} \). By the Katzenelson-Weiss Rohlin lemma ([K-W]), \( \exists F \in \mathcal{B}(X) \) such that

\[
\{T^jS_jF : 0 \leq j \leq N-1, \ 0 \leq i < k\} \text{ are disjoint,}
\]

and

\[
m\left( X \setminus \bigcup_{0 \leq j \leq N-1, \ 0 \leq i < k} T^jS_jF \right) < \frac{\epsilon \eta_\alpha}{6}.
\]

Let

\[
C = \bigcup_{j=0}^{N-1} T^jF, \quad \tilde{C} = \bigcup_{j=0}^{N-M} T^jF, \quad \mathcal{T} = \bigcup_{0 \leq i < k} S_iC, \quad \tilde{T} = \bigcup_{0 \leq i < k} S_i\tilde{C}.
\]

There is a measurable function \( f_0 : X \to G \) such that

\[
\varphi = f_0 - f_0 \circ T \text{ on } \mathcal{T}.
\]

Set \( \varphi' = f_0 - f_0 \circ T \), then \( m([\varphi \neq \varphi']) < \frac{\epsilon \eta_\alpha}{6} \).

Now define \( f : \mathcal{T} \to G \) by

\[
f = \begin{cases} 
    w_i \circ f_0 \circ S_{-i} & \text{on } S_iC \quad (0 \leq i < k) \\
    0 & \text{else,}
\end{cases}
\]

and define

\[
\psi = f - f \circ T.
\]

To establish (1),

\[
m([\psi \neq 0]) < m([\psi \neq 0] \cap \tilde{T}) + m(X \setminus \tilde{T}) \leq k^d m([\varphi \neq 0] \cap \tilde{C}) + m(X \setminus \tilde{T}) \leq \frac{\epsilon}{3} + \frac{M}{N} < \epsilon.
\]
Next, to prove (2), suppose that \( 0 \leq i < k \), \( 1 \leq j \leq d \) and \( i_j < k - 1 \). If \( x \in S_j C \), then
\[
\begin{align*}
f(S_j x) &= w_{i+e_j} \circ f_0 \circ S_{-i}(S_j x) \\
&= w_j \circ w_{i+e_j} \circ f_0 \circ S_{-i}(x) \\
&= w_j \circ f(x)
\end{align*}
\]
whence
\[
m([f \circ S_j \neq w_j \circ f]) < m\left( \bigcup_{0 \leq i < k, i_j = k-1} S_j C \right) + m(X \setminus T) \\
< \frac{1}{k} + \frac{\varepsilon \eta \alpha}{6} < \varepsilon.
\]

To complete the proof, we show (3).

We know that \( \phi + w_i \circ \varphi \circ S_{-i} \) satisfies EVC
\( T(N(\gamma, \varepsilon), \varepsilon, k, \alpha, M) \) \( \forall i \), whence
\( \phi + w_i \circ \varphi' \circ S_{-i} \) satisfies EVC
\( T(N(\gamma, \varepsilon), \varepsilon, k, \alpha, M) \) \( \forall i \).

It follows that for \( \frac{\varepsilon}{3} \) a.e. \( a \in \alpha \), and for each \( 0 \leq i < k \),
\( \exists R_i = R_{a,i} \subseteq [R]_+ \) such that \( D(R_i) \subseteq a \),
\( m(a \setminus D(R_i)) < \frac{\varepsilon}{3} m(a) \), and
\( (\phi + w_i \circ \varphi \circ S_{-i}) R_i \subseteq N(\gamma, \varepsilon) \) on \( D(R_i) \).

Define \( R = R_a \subseteq [T]_+ \) by
\[
D(R) = \bigcup_{0 \leq i < k} D(R_i) \cap S_i \tilde{C},
\]
and
\[
R = R_i \text{ on } S_i \tilde{C}, \quad (0 \leq i < k).
\]

For \( x \in D(R) \), \( \exists i = i(x) \) such that \( x \in D(R_i) \cap S_i \tilde{C} \), and we have that
\[
(\phi + \psi) R(x) = (\phi + w_i \circ \varphi \circ S_{-i}) R_i(x) \\
\in N(\gamma, \varepsilon).
\]

Lastly,
\[
m(a \setminus D(R)) = \sum_{0 \leq i < k} m(a \setminus D(R)) \cap S_i \tilde{C} + m(T \setminus \tilde{T}) + m(X \setminus T) \\
< \sum_{0 \leq i < k} m(a \cap S_i \tilde{C} \setminus D(R_i)) + \frac{M}{N} + m(X \setminus T) \\
\leq \sum_{0 \leq i < k} m(a \setminus D(R_i)) + \frac{\varepsilon}{5} \eta \alpha + \frac{\varepsilon}{6} \eta \alpha \\
\leq \varepsilon m(a).
\]

□
Proof of theorem 1' in case $d$ finite.
Choose a countable, dense subset $\Gamma$ of $G$. Let $(\gamma_1, \gamma_2, \ldots) \in \Gamma^\mathbb{N}$ satisfy

$$\{\gamma_k : k \geq 1\} = \Gamma, \quad \text{and} \quad \forall \gamma \in \Gamma, \quad \gamma_k = \gamma \text{ for infinitely many } k,$$

let the partitions $\{\alpha_j\}$ approximately generate $B$, and let $\varepsilon_k = 2^{-k^2}$.

Construct (sequentially) using lemma 4.1, a sequence of coboundaries

$$\phi_k = f_k - f_k \circ T$$

such that

$$m([f_k \circ S_j \neq E \circ f_k]) \leq \varepsilon_k \quad (1 \leq j \leq d),$$

$$\check{\phi}_k := \sum_{j=1}^k \phi_j \text{ satisfies EVCT}^{\Gamma}(N(\gamma_k, \varepsilon_k), \varepsilon_k, \alpha_k, N_k) \text{ where } N_k \in \mathbb{N}, \ N_k \uparrow,$$

$$m([\check{\phi}_k \neq 0]) \leq \frac{\varepsilon_k}{N_{k-1}}.$$

Clearly $\phi := \sum_{k=1}^\infty \phi_k$ converges a.e. Also

$$\psi_j := \sum_{k=1}^\infty (f_k \circ S_j - w_j \circ f_k) \quad (1 \leq j \leq d)$$

converges a.e., whence

$$\phi \circ S_j - w_j \circ \phi = \psi_j - \psi_j \circ T \quad (1 \leq j \leq d).$$

Theorem 3.6 now shows that $\Gamma \subset E(\phi)$, and the ergodicity of $\phi$ is established. $\square$

We prove the following version of theorem 2.

**Theorem 2'**. Suppose that $T$ is an ergodic probability preserving transformation, $\{S_t : t \in \mathbb{R}\} \subset C(T)$ are such that $T$ and $\{S_t : t \in \mathbb{R}\}$ generate a free $\mathbb{Z} \times \mathbb{R}$ action of probability preserving transformations of $X$.

There is a measurable function $\varphi : X \rightarrow \mathbb{R}$ such that $T_\varphi$ is ergodic; and

$\exists \ g : \mathbb{R} \times X \rightarrow \mathbb{R}$ measurable (with respect to $m_\mathbb{R} \times m$) such that

(1) $$\varphi \circ S_t(x) - e^t \varphi(x) = g(t, Tx) - g(t, x),$$

and

(2) $$g(t + u, x) = g(t, S_u x) + e^u g(u, x).$$

If $Q_t(x, y) := (S_t x, e^t y + g(t, x))$, then (1) implies that $Q_t \in C(T_{\varphi}) \ \forall \ t \in \mathbb{R}$, and by (2), $\{Q_t : t \in \mathbb{R}\}$ is a flow, whence $T_{\varphi}$ is a Maharam transformation.
Lemma 4.2. Let $\phi : X \to \mathbb{R}$ be a $T$-coboundary, let $\{S_t : t \in \mathbb{R}\}$ be probability preserving transformations generating a free $\mathbb{Z} \times \mathbb{R}$ action together with $T$.

If $\alpha$ is a finite, measurable partition of $X$, $\varepsilon > 0$, and $J \subset \mathbb{R}^+$ is an open interval, then there is a measurable function $f : X \to \mathbb{R}$ such that

1. $m(\{|f \circ T - f| \geq \varepsilon\}) < \varepsilon$,
2. $m(\{|f \circ S_t \neq e^t f\}) < \varepsilon$, $(0 \leq t \leq 1)$

and such that

3. $\phi + f - f \circ T$ satisfies $EVC_T(J, \varepsilon, \alpha)$.

Proof of lemma 4.2. Write $J = ((1 - \delta)b, (1 + \delta)b)$ where $b$, $\delta > 0$. We’ll sometimes use the notation $x = (1 \pm \delta)b$ which means $x \in J$.

Write $\phi = \psi \circ T - \psi$ where $\psi : X \to \mathbb{R}$ is measurable.

Choose a refinement $\alpha_1$ of $\alpha$ with the property that

$$\forall a \in \alpha_1, \exists y_a \in \mathbb{R} \ni |\psi - y_a| < \frac{b\delta}{2} \text{ a.e. on } a,$$

and set $\eta_\alpha := \min \{m(a) : a \in \alpha\}$.

Fix $K = \frac{10}{\varepsilon}$, and $0 = t_0 < t_1 < \cdots < t_M = K$ such that $e^{t_{i+1}} < (1 + \frac{\delta}{2})e^{t_i}$.

By lemma 3.1, $\exists p, q \in \mathbb{N}$ such that $\frac{\delta e^K pq}{pq} < \varepsilon$, and

$$\forall a \in \alpha_1, \ 0 \leq k \leq M - 1, \exists R_{a,k} \in [T]^+$$

such that

$$\mathcal{D}(R_{a,k}), \exists(R_{a,k}) \subset a, \ m(a \setminus \mathcal{D}(R_{a,k})) < \frac{\varepsilon}{tM} m(a), \ \text{and} \ \phi^{(R_{a,k})} = e^{-tkpq(1 \pm \frac{\delta}{9})}.$$

Now choose $N \geq 1$ such that

$$\frac{\delta e^K pq}{N} < \frac{\varepsilon \eta_\alpha}{5}.$$

By the Rokhlin theorem for continuous groups ([Lin], [O-W])

$$\exists F \in \mathcal{B}(X) \text{ such that } T^k S_t F \text{ are disjoint for } 0 \leq k \leq N, \ 0 \leq t \leq K,$$

and

$$m\left(X \setminus \bigcup_{0 \leq k \leq N-1, \ 0 \leq t \leq K} T^k S_t F\right) < \frac{\varepsilon \eta_\alpha}{6}.$$

Let

$$C = \bigcup_{j=0}^{N-1} T^j F, \ \bar{C} = \bigcup_{j=0}^{N-2} T^j F, \ \mathcal{T} = \bigcup_{0 \leq t \leq K} S_t C, \ \bar{\mathcal{T}} = \bigcup_{0 \leq t \leq K} S_t \bar{C}.$$
There is a measurable function \( f : T \to \mathbb{R} \) such that
\[
f \circ T - f = \frac{b}{pq} e^t \text{ on } S_t \tilde{C}.
\]

Complete the definition of \( f : X \to \mathbb{R} \) by setting \( f = 0 \) on \( T^c \).

It is immediate from this construction that \( f \) satisfies (1) and (2). We establish (3) by showing that \( f \circ T - f \) satisfies \( \text{EVC}_T(J, \varepsilon, \alpha_1) \). Let
\[
\tilde{C} = \bigcup_{j=0}^{N-pq} T^j F, \quad \tilde{T} = \bigcup_{0 \leq t \leq K} S_t \tilde{C}.
\]

Let, for \( 0 \leq k \leq M - 1 \),
\[
\tilde{T}_k = \bigcup_{t_k \leq t \leq t_{k+1}} S_t \tilde{C}.
\]

Fix \( a \in \alpha_1 \), and define \( R'_a \in [T]_+ \) by
\[
R'_a = R_{a,k} \text{ on } D(R_{a,k}) \cap \tilde{T}_k.
\]

It follows that \( D(R'_a) \), \( \mathcal{S}(R'_a) \subseteq a \),
\[
m(a \setminus D(R'_a)) = \sum_{k=0}^{M-1} m(\tilde{T}_k \cap [a \setminus D(R_{a,k})])
\leq \sum_{k=0}^{M-1} m(a \setminus D(R_{a,k}))
\leq \varepsilon m(a);
\]
and, on \( D(R'_a) \cap \tilde{T}_k \):
\[
|\psi \circ R'_a - \psi| < \frac{b\delta}{2},
\]
whence, on \( S_t \tilde{C}, \ t \in [t_k, t_{k+1}] \),
\[
\varphi_{R'_a} = \frac{e^t b}{pq} \phi_{R'_a} = \frac{b\delta}{2}
= e^{t-t_k} b(1 \pm \delta) \pm \frac{b\delta}{2}
= b(1 \pm \frac{\delta}{9})(1 \pm \frac{\delta}{3})(1 \pm \frac{\delta}{2}) \in J.
\]

\( \square \)

Proof of theorem 2'. Fix \((g_1, g_2, \ldots) = (1, \sqrt{2}, 1, \sqrt{2}, \ldots)\).
Construct using lemma 4.2, a sequence of coboundaries
\[
f_k \circ T - f_k
\]
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such that

\[ m([f_k \circ S_t \neq e^t f_k]) \leq \frac{1}{2^k} \quad (0 \leq t \leq 1), \]

\[ \phi_k := \sum_{j=1}^{k} (f_j \circ T - f_j) \text{ satisfies } \text{EVC}^T((\gamma_k - \frac{1}{2^k}, \gamma_k + \frac{1}{2^k}), \varepsilon_k, \alpha_k, N_k) \]

where \( N_k \in \mathbb{N} \), \( N_k \uparrow \), and

\[ m([|f_k \circ T - f_k| \geq \frac{1}{2^k N_{k-1}}]) \leq \frac{1}{2^k N_{k-1}}. \]

The ergodicity of

\[ \sum_{k=1}^{\infty} (f_k \circ T - f_k) \]

follows from

\[ 1, \sqrt{2} \in E\left(\sum_{k=1}^{\infty} (f_k \circ T - f_k)\right) \]

which follows from theorem 3.6 \( \Box \)

§5 Maharam transformations

In this section, we give conditions for a conservative, ergodic, measure preserving transformation to be isomorphic to a Maharam transformation. The first proposition shows that the transformations constructed in theorem 2 are Maharam transformations, and the second (a converse of theorem 2 for Kronecker transformations) will be used to construct completely squashable, ergodic \( \mathbb{R} \)-extensions which are not isomorphic to any Maharam transformation.

**Proposition 5.1.** A conservative, ergodic, measure preserving transformation \( T \) of the standard, non atomic, \( \sigma \)-finite measure space \((X, \mathcal{B}, m)\) is isomorphic to a Maharam transformation if, and only if there is a flow \( \{ Q_t : t \in \mathbb{R} \} \subset C(T) \) such that \( D(Q_t) = e^t \forall t \in \mathbb{R} \).

**Proof.**

Suppose first that \( T \) is a Maharam transformation, i.e. \( T : X = \Omega \times \mathbb{R} \to X \) is defined by

\[ T(x, y) = (Rx, y - \log \frac{dp \circ R}{dp}) \]

and preserves the measure \( dm(x, y) := dp(x)e^y dy \), where \( R \) is a non-singular conservative, ergodic transformation of the standard probability space \((\Omega, \mathcal{A}, p)\). Set \( Q_t(x, y) = (x, y + t) \), then \( \{ Q_t : t \in \mathbb{R} \} \subset C(T) \) is a flow, and \( D(Q_t) = e^t \).

Conversely, suppose that there is a flow \( \{ Q_t : t \in \mathbb{R} \} \subset C(T) \) such that \( D(Q_t) = e^t \forall t \in \mathbb{R} \). The flow \( \{ Q_t : t \in \mathbb{R} \} \) is dissipative on \( X \). It is well known that up to measure theoretic isomorphism, \( X = \Omega \times \mathbb{R} \) where \( \Omega \) is some probability space, \( Q_t(x, y) = (x, y + t) \), and \( dm(x, y) = e^y dp(x) dy \) where \( p \) is the probability on \( \Omega \).

Since \( \{ Q_t : t \in \mathbb{R} \} \subset C(T) \), \( \exists \) a non-singular transformation \( R : \Omega \to \Omega \) such that

\[ T(x, y) = (Rx, Y(x, y)). \]
A calculation shows that indeed $Y(x, y) = y - \log R'(x)$ where $R' = \frac{d\lambda \circ R}{d\lambda}$, i.e. $T$ is the Maharam transformation of $R$. The ergodicity of $T$ implies that $\Omega$ is non-atomic, and hence standard. □

Remarks.
1) By proposition 5.1, the skew products constructed in theorem 2 are isomorphic to Maharam transformations.
2) Let $T$ be a Bernoulli transformation. We claim that there is a $\mathbb{Z}$-extension of $T$ which is isomorphic to a Maharam transformation.

Indeed, by theorem 2 and the above remark, there is such an $\mathbb{R}$-extension of $T$. By proposition 2.5, this $\mathbb{R}$-extension of $T$ is isomorphic to a $\mathbb{Z}$-extension of $T$.

Proposition 5.2.
Let $T$ be a Kronecker transformation of the compact, metric, Abelian group $X$.

If there is an ergodic $\mathbb{R}$-extension of $T$ which is isomorphic to some Maharam transformation, then there is a continuous, injective group homomorphism $\mathbb{R} \to X$.

Proof. Let $T_\varphi$ be an ergodic $\mathbb{R}$-extension of $T$ which is isomorphic to some Maharam transformation. By proposition 5.1, there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T_\varphi)$ such that $D(Q_t) = e^t \forall t \in \mathbb{R}$. It follows from [A-L-M-N] that $Q_t$ has form $(\ast) \forall t \in \mathbb{R}$, i.e.

$$Q_t(x, y) = (S_t x, e^t y + g_t(x)).$$

Clearly, the map $t \mapsto S_t$ is a measurable homomorphism from $\mathbb{R} \to C(T) \cong X$, whence by Banach’s theorem, continuous. To see that $t \mapsto S_t$ is injective, suppose otherwise, that $S_a = \text{Id}$ for some $a \neq 0$. Then $Q_a(x, y) = (x, e^a y + g_a(x))$, whence

$$\varphi = G - G \circ T$$

where $G = \frac{g_a}{e^a - 1}$

contradicting ergodicity of $T_\varphi$. □

§6 COMPLETELY SQUASHABLE $\mathbb{R}$-EXTENSIONS OF ODOMETERS

For $a_n \in \mathbb{N}$, $(n \in \mathbb{N})$, set

$$X := \prod_{n=1}^{\infty} \{0, \ldots, a_n - 1\}$$

equipped with the addition

$$(x + x')_n = x_n + x'_n + \varepsilon_n \mod a_n$$

where

$$\varepsilon_1 = 0, \quad \varepsilon_{n+1} = \begin{cases} 0 & x_n + x'_n + \varepsilon_n < a_n \\ 1 & x_n + x'_n + \varepsilon_n \geq a_n. \end{cases}$$

Clearly, $X$ equipped with the product discrete topology, is a compact Abelian topological group, with Haar measure

$$m = \prod_{n=1}^{\infty} \left( \frac{1}{a_n}, \ldots, \frac{1}{a_n} \right).$$
Also if \( \tau = (1, 0, \ldots) \) then \( X = \overline{\{n\tau\}}_{n \in \mathbb{Z}} \) whence \( x \mapsto Tx := x + \tau \) is ergodic.

Set \( q_1 = 1, q_{n+1} = \prod_{k=1}^{n} a_k \), then

\[
(q_n \tau)_k = \begin{cases} 
1 & k = n \\
0 & k \neq n,
\end{cases}
\]

whence

\[
T^{q_n \tau} = (x_1, \ldots, x_{n-1}, \tilde{T}_n(x_n, \ldots))
\]

where \( \tilde{T}_n : \prod_{k=n}^{\infty} \{0, \ldots, a_k - 1\} \to \prod_{k=n}^{\infty} \{0, \ldots, a_k - 1\} \) is defined by \( \tilde{T}_n(x) = x + \tilde{\tau}_n \) where \( \tilde{\tau}_n = (1, 0, \ldots) \).

The transformation \( T \cong (X, T) \) is called the \textit{odometer with digits} \( \{a_n : n \in \mathbb{N}\} \).

Let \( G \) be a second countable LCA group, and let \( (X, T) \) be an odometer.

We consider cocycles \( \varphi : X \to G \) of form

\[
\varphi(x) := \sum_{n=1}^{\infty} [\beta_n((Tx)_k) - \beta_n(x_k)],
\]

the sum being a finite sum. Cocycles of this form are called \textit{of product type}. We’ll call the functions \( \{\beta_k : k \in \mathbb{N}\} \) the partial transfer functions of \( \varphi \). Clearly if the sum of the partial transfer functions converges, then indeed the limit is a transfer function for the cocycle, which is a coboundary.

We prove

**Theorem 6.1.** There is an odometer \( (X, T) \), and an ergodic cocycle \( \varphi : X \to \mathbb{R} \) of product type such that \( T_\varphi \) is completely squashable, indeed

\[
\forall c > 0, \exists a \text{ measurable function } \psi_c : X \to G, \text{ and a translation } S_c : X \to X\text{ satisfying}
\]

\[
\varphi \circ S_c = c\varphi + \psi_c \circ T - \psi_c.
\]

We claim that \( T_\varphi \) is not isomorphic to a Maharam transformation.

Otherwise, by proposition 5.2, there would be a continuous, injective group homomorphism \( \mathbb{R} \to X \), whose existence is prevented by the disconnectedness of \( X \).

We prove ergodicity of \( \varphi \) using rigid essential value conditions.

**Proposition 6.2.** Let \( \varphi : X \to G \) be a cocycle, and let \( \gamma \in G \). If \( \forall \varepsilon > 0, \exists \delta_k \to 0 \), and a sequence of partitions \( \alpha_k \) which approximately generate \( \mathcal{B} \), such that

\[
m\bigg( \bigcup_{a \in \alpha_k} a \bigg) > 1 - \delta_k,
\]

and for every \( k \geq 1 \), for \( \delta_k \)-a.e. \( a \in \alpha_k \), \( \exists n = n(a) \) such that

\[
m(a \Delta T^{-n} a) < \delta_k m(a), \text{ and } m(a \cap [\varphi_n \in N(\gamma, \varepsilon)]) > \frac{m(a)}{25},
\]

respectively.
\[ \gamma \in E(\varphi). \]

**Proof.** This is a special case of lemma 3.1. \( \square \)

The functions \( \beta_k \) are defined by means of blocks. To \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m \), associate a *canonical difference block* \( B_\gamma = (b(0), b(1), \ldots, b(2^m-1)) \in \mathbb{R}^{2^m} \) defined by
\[
b \left( \sum_{k=1}^{m} \varepsilon_k 2^{k-1} \right) = \sum_{k=1}^{m} \varepsilon_k \gamma_k \quad (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m.
\]
It is evident that for \( 0 \leq \nu \leq 2^m - 1, \ 1 \leq j \leq m \) with \( \varepsilon_j(\nu) = 0 \), we have \( \nu + 2^{j-1} \leq 2^m - 1 \), and \( b(\nu + 2^{j-1}) - b(\nu) = \gamma_j \). It follows that \( \forall \ 1 \leq j \leq m, \ \exists \ 1 \leq n = n(j) \leq 2^m \) such that
\[
\# \left\{ 1 \leq \nu \leq 2^m - 1 : \nu + n \leq 2^m - 1, \ b(\nu + n) - b(\nu) = \gamma_j \right\} \geq \frac{2^m}{2}.
\]
We'll need some control over the size of \( |b(j)|, \ (0 \leq j \leq 2^m - 1) \) and to obtain this, we need the *balanced canonical difference block* associated to \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m \), defined by
\[
B = (b(0), b(1), \ldots, b(4^m - 1)) \in \mathbb{R}^{4^m}
\]
where
\[
b \left( \sum_{k=1}^{m} \varepsilon_k 2^{k-1} + \sum_{\ell=1}^{m} \delta_\ell 2^{m+\ell-1} \right) = \sum_{k=1}^{m} (\varepsilon_k - \delta_k) \gamma_k, \quad (\varepsilon, \delta) \in \{0, 1\}^m.
\]
Let \( B \) be the balanced canonical difference block associated to \( (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m \). Since \( B \) is also the canonical difference block associated to \( (\gamma_1, \ldots, \gamma_m, -\gamma_1, \ldots, -\gamma_m) \in \mathbb{R}^{2^m} \), we have by (1) that
\[
\# \left\{ 1 \leq \nu \leq 4^m - 1 : \nu + n \leq 4^m - 1, \ b(\nu + n) - b(\nu) = \gamma_j \right\} \geq \frac{4^m}{2}.
\]
Also, we claim that
\[
\# \left\{ 0 \leq \nu \leq 4^m - 1 : |b(\nu)| \geq m^2 \right\} \leq \max_{1 \leq j \leq m} |\gamma_j|^2 \frac{4^m}{\sqrt{m}}.
\]
To see this
\[
|\left\{ 0 \leq \nu \leq 4^m - 1 : |b(\nu)| \geq m^2 \right\}| \leq \frac{1}{m^2} \sum_{\varepsilon, \delta \in \{0, 1\}^m} \left( \sum_{k=1}^{m} (\varepsilon_k - \delta_k) \gamma_k \right)^2 = \frac{4^m}{m^2} \sum_{k=1}^{m} \gamma_k^2 = \frac{4^m}{\sqrt{m}} \max_{1 \leq j \leq m} |\gamma_j|^2.
\]
We now construct the odometer and cocycle. We construct our cocycle $\varphi$ to have $1, \frac{1}{\sqrt{2}} \in E(\varphi)$ thus ensuring ergodicity. Let $g_{2n} = 1$, and $g_{2n+1} = \frac{1}{\sqrt{2}}$.

For $k \geq 1$, choose natural numbers $\nu_k$ and $\mu_k$ satisfying

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty, \quad (3)$$

$$\sum_{k=1}^{\infty} \frac{e^{2\mu_k}}{\sqrt{\mu_k \nu_k}} < \infty, \quad (4)$$

and

$$\sum_{k=1}^{\infty} \frac{\mu_k}{\nu_k^4} < \infty. \quad (5)$$

For example:

$$\mu_k = k^2, \quad \nu_k = k^2 3^{4k^2}.$$  

Set $m_k = \mu_k \nu_k$, and let

$$B_k = (b_k(0), b_k(1), \ldots, b_k(4^{m_k} - 1))$$

be the balanced canonical difference block associated to $(\gamma_k(1), \ldots, \gamma_k(m_k))$ where

$$\gamma_k(j) = g_k e^{-\frac{j}{\nu_k}}.$$  

Now let

$$a_k = m_k 4^{m_k}$$

and let $(X, T)$ be the odometer with digits $\{a_n : n \in \mathbb{N}\}$.

We specify a cocycle $\varphi : X \to \mathbb{R}$ of product type, defining it’s partial transfer functions $\beta_k : \{0, \ldots, a_k - 1\} \to \mathbb{R}$ by

$$\beta_k(j 4^{m_k} + \nu) = e^{\nu_k} b_k(\nu) \quad (0 \leq j \leq m_k - 1, \quad 0 \leq \nu \leq 4^{m_k} - 1).$$

Note that for $0 \leq j \leq m_k - 1$,

$$|\{j 4^{m_k} \leq \nu < (j + 1) 4^{m_k} : \beta_k(\nu + n(j, k)) - \beta_k(\nu) = g_k\}|$$

$$\geq |\{0 \leq \nu < 4^{m_k} : b_k(\nu + n(j, k)) - \beta_k(\nu) = \gamma_k(j)\}|$$

$$\geq \frac{4^{m_k}}{2}. \quad (6)$$
To check ergodicity of $\varphi$, we show, using the essential value condition that $1, \frac{1}{\sqrt{2}} \in E(\varphi)$. For $k \geq 1$, we let

$$\alpha_k = \{ A((u_1, \ldots, u_{k-1}), j) : 0 \leq u_\nu < a_\nu, \ 1 \leq \nu \leq k - 1, \ 0 \leq j \leq m_k - 2 \}$$

where

$$A((u_1, \ldots, u_{k-1}), j) = \{ x \in X : x_\nu = u_\nu, \ 1 \leq \nu \leq k - 1, \ \&\ jN_k \leq u_k < (j+1)N_k \}.$$ 

It follows that

$$m \left( \bigcup_{a \in \alpha_k} a \right) = 1 - \frac{1}{m_k}.$$ 

Also, if $n = n(j, k)q_k$ where $n(j, k)$ is as in (6), then

$$m(A(u, j) \Delta T^{-n} A(u, j)) < \frac{1}{m_k} m(A(u, j)),$$

and, by (6)

$$m(A(u, j) \cap [\varphi_{n(j,k)} = \gamma_k]) \geq \frac{m(A(u, j))}{2}.$$ 

The essential value condition now shows that

$$1, \frac{1}{\sqrt{2}} \in E(\varphi).$$

To conclude, we show that $\forall \ c \in (1, e), \ \exists S : X \to X$ such that $\varphi \circ S - c\varphi$ is a coboundary. Fix $c \in (1, e)$ and let

$$r_k = [\nu_k \log c] \leq u_k.$$ 

Let

$$S = (r_1 4^{m_1}, r_2 4^{m_2}, \ldots).$$

We claim that $\forall \ k \geq 1,$

$$m(\{ x \in X : |\beta_k(x_k)| \geq m_k^2 \}) \leq \frac{e^{2\mu_k}}{\sqrt{m_k}}.$$ 

To see this

$$m(\{ x \in X : |\beta_k(x_k)| \geq m_k^2 \})$$

$$= \frac{1}{a_k} \# \{ 0 \leq \nu \leq a_k - 1 : |\beta_k(\nu)| \geq m_k^2 \}$$

$$= \frac{1}{a_k} \sum_{j=0}^{m_k-1} \# \{ 0 \leq \nu \leq 4^{m_k} - 1 : |\beta_k(j4^{m_k} + \nu)| \geq m_k^2 \}$$

$$= \frac{1}{a_k} \sum_{j=0}^{m_k-1} \# \{ 0 \leq \nu \leq 4^{m_k} - 1 : e^{\mu_k} |b_k(\nu)| \geq m_k^2 \}$$

$$\leq \frac{1}{4^{m_k}} \# \{ 0 \leq \nu \leq 4^{m_k} - 1 : e^{\mu_k} |b_k(\nu)| \geq m_k^2 \}$$

$$\leq \frac{e^{2\mu_k}}{\sqrt{m_k}}.$$
It now follows from (4), and the Borel-Cantelli lemma that for a.e. \( x \in X \), \( \exists k_x \geq 1 \) such that \( \beta_k(x_k) \leq m_k^{\frac{3}{4}} \forall k \geq k_x \).

It follows from (3) that
\[
\sum_{k=1}^{\infty} m(\{ x \in X : x_k \geq 2r_k4^{m_k} \}) < \infty,
\]
whence by the Borel-Cantelli lemma, for a.e. \( x \in X \), \( \exists K_x \geq 1 \) such that \( (Sx)_k = x_k + r_kN_k \forall k \geq K_x \).

It follows that a.e. \( x \in X \), for \( k \geq k_x, K_x \),
\[
|\beta_k((Sx)_k) - c\beta_k(x_k)| = (c - e\nu_k)|\beta_k(x_k)| \leq \frac{cm_k^{\frac{3}{4}}}{\nu_k} = \frac{cm_k^{\frac{3}{4}}}{\nu_k}
\]
whence by (5),
\[
\sum_{k=1}^{\infty} |\beta_k((Sx)_k) - c\beta_k(x_k)| < \infty \text{ for a.e. } x \in X
\]
and \( \varphi \circ S - c\varphi \) is a coboundary, being a product type cocycle, the sum of whose partial transfer functions converges.

§7 Smooth completely squashable \( R \)-extensions

In this section, we construct smooth completely squashable \( R \)-extensions of rotations of the circle. Ergodicity is established again using proposition 6.2.

**Theorem 7.1.** Let \( T : x \mapsto x + \alpha \mod 1 \) be an irrational circle rotation (the circle is represented as the unit interval \([0,1)\)). Let \( \alpha \) have unbounded partial quotients. Then there exists a real smooth ergodic cocycle \( F \) such that for every \( c \neq 0 \) there exists a rotation \( S \) for which

\[
f \circ S - c \cdot F
\]
is a \((T-)\)coboundary:

1. If \( 1 \leq p < \infty \) is an integer and \( \lim \sup_{n \to \infty} q_{n+1}/q_n^p = \infty \), the cocycle \( F \) can be found in \( C^p \). (For \( p = 1 \) the condition means unbounded partial quotients.)

2. If \( \lim \sup_{n \to \infty} q_{n+1}/q_n^p = \infty \) for all positive integers \( p \), \( F \) can be found in \( C^\infty \).

It is known (cf. [Ba-Me]) that if \( \lim \sup_{n \to \infty} q_{n+1}/q_n^p < \infty \), every \( F \in C^0 \) is a coboundary.

**Proof.** We shall prove Statement 1; then, the other one can be derived rather easily.

Let \( a_n \) be the partial quotients, \( q_n \) the convergents. We define
\[
\alpha_k = k^3e^k,\quad r_k = k^4e^k, \quad k = 1, 2, \ldots.
\]
Since \( \lim \sup_{n \to \infty} q_{n+1}/q_n^p = \infty \), there exist sequences of positive integers
\[
\ell_k = 2[(a_{n_k} - 1)/2r_k] \quad \text{where} \quad [x] \text{ denotes the integer part of } x,
\]
\[ \ell'_k = \ell_k/2, \]
\[ d_k = e^{k_kk}/q_{nk}, \]
\[ d_k = d_k(r_kq_{nk-1})^p = e^{k(p+1)k_kk+6q_{nk-1}}/q_{nk}, \]
\[ \text{where} \]
\[ \sum_{k=1}^{\infty} d_k e^k < \infty. \]

Let us suppose that \( n = n_k \) is odd (the case with \( n_k \) even is similar). From the continued fraction expansion we get two Rohlin towers:
\[ ([j\alpha], \{(q_n-1+j)\alpha\}), j = 0, \ldots, q_n - 1 \text{ and} \]
\[ ([q_n\alpha], 1), \{((j+q_n)\alpha),(j+1)\alpha\}, j = 1, \ldots, q_n - 1 \]
(for \( 0 \leq x < 1 - \|q_n-1\alpha\| \) we thus have \( T^{q_n-1}x = x + \|q_n-1\alpha\| \)). Let us denote \( I_0 = [0, \|q_n-1\alpha\|) \)
\[ I_i = T^i I_0, \quad i = 1, \ldots, q_n - 1. \]
For \( j = 0, \ldots, q_n - 1 \), the intervals \( I_j + q_n = 1, \ldots, I_j + q_n \) are adjacent.

For \( j = 0, \ldots, r_k - 1 \) and \( u = 0, \ldots, q_{nk-1} - 1 \) we define
\[ J'_{0,0} = \bigcup_{i=0}^{\ell'_k-1} I_{i:q_{nk-1}}, \quad J'_{0,j} = T^{\ell'_k} q_{nk-1} J'_{0,0}, \]
\[ J''_{0,j} = T^{\ell'_k} q_{nk-1} J'_{0,j}, \]
\[ J''_{u,j} = T^u J''_{0,j}, \]
\[ J_{u,j} = J_{u,j} \cup J''_{u,j}, \]
\[ J_u = \bigcup_{j=0}^{r_k-1} J_{u,j}. \]

Notice that for every \( u \) the sets \( J_{u,j} \) are adjacent intervals composing the interval \( J_u = \{u\alpha, r_k \cdot \ell_k \cdot \|q_{nk-1}\alpha\|\} \); each \( J_{u,j} \) is cut in the middle into \( J''_{u,j} \) and \( J''_{u,j} \).

Let \( F_k \) be a \( C^\infty \) function on \([0, 1)\) which is
- zero out of \( J''_{0,0} \),
- \( d_k \) on the middle half of \( J''_{0,0} \)
  (i.e. on the interval \([\ell'_k \|q_{nk-1}\alpha\|/4, 3\ell'_k \|q_{nk-1}\alpha\|/4]\)),
- has values between 0 and \( d_k \) on the rest of \( J''_{0,0} \),
\[ F^{(i)}(0) = 0 = F^{(i)}(\ell'_k \|q_{nk-1}\alpha\|) \] for \( i = 1, \ldots, p \).
Moreover, the functions \( F_k \) can be found such that there exists a positive constant \( C \),
\[ \|F_k\|_{C^p} < C \bar{d}_k \]
for all \( k \). Let us show this for \( p = 1 \):
Let \( \ell \) be a function which is zero on
\[ [0, 1) \setminus \{(0, \ell'_k \|q_{nk-1}\alpha\|/4) \cup (3\ell'_k \|q_{nk-1}\alpha\|/4, \ell'_k \|q_{nk-1}\alpha\|)\}, \]
on the interval
\[ (0, \ell'_k \|q_{nk-1}\alpha\|/4) \]
it is a tent-like function, and on the interval
it is a reversed tent-like function, both of height \(8d_k/(k\alpha\|q_k\alpha\|)\). The indefinite integral \(F_k(t) = \int_0^t \tilde{f}(x)\,dx\) will thus be zero on \([0,1)\setminus J'_{0,0}\), \(d_k\) on the middle half of \(J'_{0,0}\) and monotone on the remaining part of \(J'_{0,0}\); there exists a constant \(C\) such that \(\|\tilde{F}_k\|_C < C\) for every \(k\).

The case of larger exponents \(p\) is left to the reader (it can be done in a recursive way; the constant \(C\) depends on \(p\)).

We define \(\tilde{F}_k = \tilde{F}_k - \tilde{F}_k \circ T^{-\ell_k q_{p_k-1}}\) (i.e. \(\tilde{F}_k = \tilde{F}_k\) on \(J'_{0,0}\); on \(J''_{0,0}\), \(\tilde{F}_k\) is got by shifting \(\tilde{F}_k\) by \(\ell_k\|q_{p_k-1}\alpha\|\) and changing the sign);

\[
F_k = (-1)^j(1 + \frac{1}{c_k})J^{(n+j\ell_k q_{p_k-1})} 0 \leq x \leq r_k - 1, u = 0, \ldots, q_{p_k-1} - 1
\]

\[
F_k = 0 \quad \text{otherwise.}
\]

We have

\[
|F_k| \leq d_k \cdot (1 + \frac{1}{c_k})^{r_k} \leq d_k \cdot e^k,
\]

\[
\|F_k\|_{C^p} \leq C \cdot d_k \cdot e^k.
\]

By (2) we have \(\|F_k\|_{C^p} < \infty\), hence for each subset \(K\) of \(\mathbb{N}\) there exists a \(C^p_0\) function

\[
F(K) = \sum_{k \in K} F_k.
\]

Let \(0 \leq j \leq r_k - 1\) and \(x \in I_j q_{p_k-1}\).

For \(i = u + p \cdot q_{p_k-1}\) where \(0 \leq u \leq q_{p_k-1} - 1, 0 \leq p \leq \ell_k - 1\) we have \(T^i x \in J'_{u,j}\), and for \(i = u + p \cdot q_{p_k-1} + \ell_k q_{p_k-1} (0 \leq u \leq q_{p_k-1} - 1, 0 \leq p \leq \ell_k - 1)\) we have \(T^i x \in J''_{u,j}\).

From the definition of \(F_k\) we thus get

\[
\sum_{i=0}^{\ell_k q_{p_k-1}-1} F_k(T^i x) = -\sum_{i=\ell_k q_{p_k-1}}^{\ell_k q_{p_k-1}-1} F_k(T^i x) \quad (x \in I_j q_{p_k-1})(3)
\]

hence for every \(0 \leq j \leq r_k - 2\) and \(x \in I_0\),

\[
\sum_{i=j}^{(j+1)\ell_k q_{p_k-1}-1} F_k(T^i x) = 0
\]

and

\[
F_k = G_k - G_k \circ T
\]

where

\[
G_k(T^u x) = -\sum_{i=0}^{u-1} F_k(T^i x) \quad \text{for} \ x \in I_0, \ u = 0, \ldots, q_{p_k-1}
\]

\[
G_k(x) = 0 \quad \text{for} \ x \in [0,1) \setminus \bigcup_{i=0}^{q_{p_k-1}-1} I_i.
\]
Therefore, \( F_k \) is a coboundary with the transfer function \( G_k \in C^p \).
Let us compute \( \sup |G_k| \). We have
\[
\sup |G_k| = \sup_{x \in J_0} \max \left\{ \left| \sum_{i=0}^{n-1} F_k(T^i x) \right| : 0 \leq u \leq q_{nk} - 1 \right\}
\]
by (3'), the partial sums are zero for every \( u = j \cdot \ell_k \cdot q_{nk-1}, 1 \leq j \leq r_k - 1 \). From this and from (3) we get
\[
|G_k| \leq \sup \left\{ \left| \sum_{i=0}^{u-1} F_k \circ T^i (x) \right| : x \in X, 1 \leq u \leq \ell_k \cdot q_{nk-1} \right\}
\]
(5)
\[
\ell_k \cdot q_{nk-1} \sup |F_k| = \ell_k \cdot q_{nk-1} \cdot C \cdot e_k \cdot (1 + \frac{1}{c_k})^k \leq C \cdot e^k \cdot k^2.
\]
As \( T^{q_{nk-1}} \) is the shift (mod 1) by \( ||q_{nk-1} \alpha|| \), \( T^{j \cdot q_{nk-1}}, j = 1, \ldots, [\ell_k'/k], k = 1, 2, \ldots \), is a rigid time. For any fixed positive integer \( p \) we thus have
\[
\lim_{k \to \infty} \max_{j = 1, \ldots, [\ell_k'/k]} |S_{j \cdot q_{nk-1}}(\sum_{i=0}^{p} F_i)| = 0.
\]
(6)
From this and from \( |F_k| \leq d_k \cdot e^k \to 0 \) (cf. (2)) follows that there exists an infinite subset \( K \subset \mathbb{N} \) s.t.
\[
\lim_{k \to \infty} \max_{j = 1, \ldots, [\ell_k'/k]} |S_{j \cdot q_{nk-1}}(F_{(K)} - F_k)| = 0
\]
where \( F_{(K)} = \sum_{k \in K} F_k \). The set \( K \) can be chosen such that
\[
\sum_{k \in K} \frac{1}{k} < \infty.
\]
(7)
Let \( A_k \) be the partition of \([0,1]\) into the sets \( J_{u,j} \) and the complement of their union; as \( a_n \to \infty \), \( A_k \not\to A \) (for a subsequence of the numbers \( k \)).

Let \( 0 \leq u \leq q_{nk-1} - 1 \), \( 0 \leq j \leq r_k - 1 \) be fixed, \( E \) be the middle third of the interval \( J_{u,j}' \). If \( 1 \leq i \leq \ell_k'/k \) and \( k \geq 12 \),
\[
S_{i \cdot q_{nk-1}}(F_k) = i \cdot q_{nk-1} \cdot (-1)^i (1 + \frac{1}{c_k})^i d_k.
\]
on \( E \).
Let \( a \) be a fixed number of the same sign as \( (-1)^i \) and let \( \epsilon > 0 \). Without loss of generality we can suppose that \( j \) is even, \( a \geq 0 \). By (2) we have
\[
q_{nk-1} d_k (1 + \frac{1}{c_k})^k \leq q_{nk-1} d_k e^k \to 0
\]
and for \( k \) sufficiently big,
\[
d_k q_{nk-1} \ell_k'/k \geq e^k q_{nk}^6 \cdot q_{nk-1} (a_n k \cdot 2k^4 e^k - 1) \frac{1}{k} \geq k/3,
\]
for every $1 \leq S \leq \ell_k$ exists a rotation $\hat{T}$. From this and from

The rotation $T^{\eta_{k-1}}$ is the shift by $\|q_{n-1}\|$ (mod 1), hence, if $k \geq 30$, we have

$$\lambda(E \cap T^{-i \cdot q_{n-1}} E) > 0, \quad \lambda(E)$$

for every $1 \leq i \leq \ell'_k/k$.

From this and from $\lambda(E) \geq \lambda(J_{u,j})/6$ we get the rigid EVC for $F_k$. From the rigid EVC for $F_k$ and (6) follows that for any infinite subset $K'$ of $K$, $F(K') = \sum_{k \in K'} F_k$ satisfies the essential value condition. This way we have found an uncountable set of ergodic cocycles $F = F(K') \in C^0$.

It remains to prove that the set $K$ can be chosen so that for every $c \neq 0$ there exists a rotation $S$ for which

(1) $$F \circ S - c \cdot F$$

is a $T$-coboundary.

If $F \circ S' - c' \cdot F$ and $F \circ S'' - c'' \cdot F$ are coboundaries, then $F \circ S' \circ S'' - c' \cdot c'' \cdot F = F \circ S' \circ S'' - (c' \cdot F \circ S') + c'' \cdot (F \circ S' - c')$ is a coboundary, too, hence the set of numbers $c$ for which (1) holds true is a group. It thus suffices to find $S$ for $|c| > 1$. First we shall show the proof for $c > 1$. Let us suppose that the $c > 1$ is fixed.

Let $j(k)$ be the greatest positive even integer for which

$$(1 + \frac{1}{c_k})^{j(k)} < c.$$ 

For nonnegative integers $k$, $v$ let us define a number

$$\sigma(k, v) = \{(v(k) + j(k) \cdot \ell_k q_{n-1})\alpha\}$$

and a rotation

$$\hat{\sigma}(k, v) = \{(v(k) + j(k) \cdot \ell_k q_{n-1})\alpha\}$$

(we denote both by the same symbol).

We'll recursively define $K' = \{k_0 < k_1 < \ldots \} \subset K$, nonnegative integers $\{v(k) : k \in K')$, $k = k_0, k_1, \ldots$, numbers and rotations $\{\sigma(k) : k \in K')$ (denoted by the same symbol):

For $k_0$ we choose the smallest element of $K$ and define

$v(k_0) = 0, \sigma(k_0) = \sigma(k_0, v(k_0)).$

If $k_i, v(k_i)$ have been defined for $i = 0, \ldots, m$, we define $k_{m+1}$ as the smallest $k \in K$ such that:

1. $k > k_m$.
2. There exists an integer $0 \leq v = v(k) < q_{n-1}/k$ such that

$$|\sigma(k_m) - \sigma(k, v)| < 1/2^m,$$

$$\sup_{j} |G_j \circ \sigma(k_m) - G_j \circ \sigma(k, v)| < 1/2^m \quad \text{for } j = k_0, \ldots, k_m.$$ 

Set $\sigma(k_{m+1}) = \sigma(k_{m+1}, v(k_{m+1}))$.

The numbers $\sigma(k_m)$ then for $m \to \infty$ converge to a limit $\sigma$.

By $S$ we denote the rotation $x \mapsto x + \sigma$ mod 1, $K' = \{k_0, k_1, \ldots\}$.

For $k = k_m \in K'$

$$\|G_k \circ S - G_k \circ \sigma(k)\|_\infty \leq$$

$$\sum_{i=0}^{\infty} \|G_k \circ \sigma(k_{m+i}) - G_k \circ \sigma(k_{m+i+1})\|_\infty < \sum_{i=0}^{\infty} 1/2^{m+i} = 1/2^{m-1},$$
\[ \sum_{k \in K'} \| G_k \circ S - G_k \circ \sigma(k) \|_{\infty} \]

converges.

We have

\[ F \circ S - c \cdot F = \sum_{k \in K'} (F_k \circ S - c \cdot F_k) = \]

\[ \sum_{k \in K'} ((F_k \circ S - F_k \circ \sigma(k)) + (F_k \circ \sigma(k) - (1 + \frac{1}{c_k})j(k)F_k) + ((1 + \frac{1}{c_k})j(k) - c)F_k). \]

By (4), each of the functions \( F_k \) is a coboundary, hence all summands in the last sum are coboundaries, too. For proving (1) it suffices to show that the sum of the corresponding transfer functions converges:

1. \( \sum_{k \in K'} (F_k \circ S - F_k \circ \sigma(k)) \).

Every \( F_k \) is a coboundary with a transfer function \( G_k \). We have shown that \( \sum_{k \in K'} (G_k \circ S - G_k \circ \sigma(k)) \) converges; it is a transfer function of \( \sum_{k \in K'} (F_k \circ S - F_k \circ \sigma(k)) \).

2. \( \sum_{k \in K'} (F_k \circ \sigma(k) - (1 + \frac{1}{c_k})j(k)F_k) \).

The function \( F_k \circ \sigma(k) - (1 + \frac{1}{c_k})j(k)F_k \) is a coboundary with a transfer function \( \tilde{G}_k = G_k \circ \sigma(k) - (1 + \frac{1}{c_k})j(k)G_k \).

From the definitions of \( F_k, G_k \), and \( \sigma(k) \) follows that for \( x \in I_i, i = u + t \cdot \ell_k \cdot q_{n_k - 1} + p \cdot q_{n_k - 1}, 0 \leq u, u + v(k) \leq q_{n_k - 1}, 0 \leq t, t + j(k) \leq r_k - 1 \), and \( 0 \leq p \leq \ell_k - 1 \), we have

\[ \tilde{G}_k = G_k \circ \sigma(k) - (1 + \frac{1}{c_k})j(k)G_k = 0. \]

From \( v(k) \leq q_{n_k - 1}/k \) and \( j(k) \leq c_k \cdot (\log c + 1) = (\log c + 1) \cdot r_k/k \) (for \( k \) sufficiently big) we get that

\[ \lambda(\tilde{G}_k \neq 0) < (3 + \log c)/k \]

for \( k \) big enough. From this and (7) follows that the sum \( \sum_{k \in K'} \tilde{G}_k \) converges almost surely.

3. \( \sum_{k \in K'} ((1 + \frac{1}{c_k})j(k) - c)F_k \).

By (4), \( F_k \) is a coboundary with a transfer function \( G_k \) and by (5), \( |G_k| \) is bounded by \( (1/2)e^{k^2} \). We can easily see that \( c - (1 + \frac{1}{c_k})j(k) \leq 2c/c_k \), hence

\[ \sum_{k \in K'} |((1 + \frac{1}{c_k})j(k) - c)G_k| \leq \sum_{k \in K'} c \cdot e^{k^2}/c_k \leq \sum_{k \in K'} 1/k \]

where the last sum is finite by (7).

If we define \( j(k) \) as the biggest odd number for which

\[ (1 + \frac{1}{c_k})j(k) < c, \]

we get the rotation \( S \) for \( c < -1 \), hence (1) holds true for all \( c \neq 0 \).

All the cocycles \( F_k \) which we defined are from \( C^\infty \). If \( \limsup_{n_k \to \infty} q_{n_k+1}/q_n = \infty \), we can find the cocycles \( F_k \) such that \( \sum_{k=1}^\infty F_k \) converges in every \( C^p, 1 \leq p < \infty \), hence \( F = \sum_{k=1}^\infty F_k \in C^\infty \). This proves the second statement of the theorem. \( \square \)
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