Equivalences of PDE systems associated to degenerate para-CR structures: foundational aspects

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Abstract
Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We study basic invariants of submanifolds of solutions $\mathcal{M} = \{y = Q(x, a, b)\}$ in coordinates $x \in \mathbb{K}^{n \geq 1}, y \in \mathbb{K}, a \in \mathbb{K}^{m \geq 1}, b \in \mathbb{K}$ under split-diffeomorphisms $(x, y, a, b) \mapsto (f(x, y), g(x, y), \varphi(a, b), \psi(a, b))$. Two Levi forms exist, and have the same rank $r \leq \min(n, m)$. If $\mathcal{M}$ is $k$-nondegenerate with respect to parameters and $l$-nondegenerate with respect to variables, $\text{Aut}(\mathcal{M})$ is a local Lie group of dimension:

$$\dim \text{Aut}(\mathcal{M}) \leq (n + 1) \left(\frac{n+1+2k+2l}{n+1}\right) + (m + 1) \left(\frac{m+1+2k+2l}{m+1}\right).$$

Mainly, our goal is to set up foundational material addressed to CR geometers. We focus on $n = m = 2$, assuming $r = 1$. In coordinates $(x, y, z, a, b, c)$, a local equation is:

$$z = c + xa + \beta xx b + \beta y aa + c O_{x,y,a,b}(2) + O_{x,y,a,b,c}(4),$$

with $\beta$ and $\beta$ representing the two 2-nondegeneracy invariants at 0. The associated para-CR PDE system:

$$zy = F(x, y, z, z_x, z_{xx}) \quad \text{and} \quad z_{xxx} = H(x, y, z, z_x, z_{xx}),$$

satisfies $F_{zzx} \equiv 0$ from Levi degeneracy. We show in details that the hypothesis of 2-nondegeneracy with respect to variables is equivalent to $F_{z_xz} \neq 0$. This gives a CR-geometric meaning to the first two para-CR relative differential invariants encountered independently in another paper, joint with Paweł Nurowski.

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1 Introduction

The first goal of this article is to develop some foundational aspects of para-CR structures, neither touched in the first two memoirs on the subject [12,14], nor published somewhere, since then. The second goal is to provide a detailed entrance to an advanced paper [16], joint with Paweł Nurowski, devoted to the classification of Levi degenerate 5-dimensional para-CR structures. Thus, the focus is mainly on accessible aspects of para-CR structures, at the threshold of more advanced equivalence results about equivalences of Partial Differential Equations, which are highly demanding, computationally speaking.

This introductory Sect. 1 presents, in a self-contained and condensed way, some of the concepts and results of the paper, emphasizing three new theorems taken from its core.

We work over \( \mathbb{K} := \mathbb{R} \) or \( \mathbb{K} := \mathbb{C} \). We study codimension 1 para-CR structures [12], called submanifolds of solutions in [14]. Let therefore \( x \in \mathbb{K}^n \), with \( n \geq 1 \), let \( y \in \mathbb{K} \), let \( a \in \mathbb{K}^m \), with \( m \geq 1 \), and let \( b \in \mathbb{K} \), be coordinates. In the CR setting \( x := z \in \mathbb{C}^n \), \( y := w \in \mathbb{C} \), \( m = n \) (due to complex conjugation), \( a := \bar{z}, b := \bar{w} \). In fact, CR objects will be only seldom mentioned in this paper, since they are less general than the para-CR ones.

A (local) submanifold of solutions \( \mathcal{M} \subset \mathbb{K}^{n+1} \times \mathbb{K}^{m+1} \) is an analytic submanifold (hypersurface) passing through the origin:

\[
\mathcal{M} = \{(x, y, a, b): \, R(x, y, a, b) = 0\}
\]

with \( R_x \neq 0 \neq R_b \). By solving for \( y \) or for \( b \), we may represent \( \mathcal{M} = \{y = Q(x, a, b)\} \) with \( Q_b \neq 0 \) or \( \mathcal{M} = \{b = P(a, x, y)\} \) with \( P_y \neq 0 \). Such two graphing functions satisfy identically:

\[
y = Q(x, a, P(a, x, y)) \quad \text{and} \quad b = P(a, x, Q(x, a, b)). \tag{1.1}
\]

The goal is to understand in a precise manner para-CR invariants under local maps:

\[
(F, \Phi): (x, y, a, b) \mapsto (f(x, y), g(x, y), \varphi(a, b), \psi(a, b)) =: (x', y', a', b'), \tag{1.2}
\]

which send \( \mathcal{M} \) diffeomorphicallo to another submanifold of solutions \( \mathcal{M}' \) represented by similar equivalent equations:

\[
y' = Q'(x', a', b') \quad \text{and} \quad b' = P'(a', x', y').
\]

In the CR setting, \( m = n \), and equivalences are (local) biholomorphisms \( \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \). Equivalently, the following power series identities hold:

\[
g(x, Q(x, a, b)) = Q'\left( f(x, Q(x, a, b)), \varphi(a, b), \psi(a, b) \right) \quad \text{in} \ \mathbb{C}\{x, a, b\},
\]

\[
\psi(a, P(a, x, y)) = P'\left( \varphi(a, P(a, x, y)), f(x, y), g(x, y) \right) \quad \text{in} \ \mathbb{C}\{a, x, y\}).
\]

The link with completely integrable PDE systems whose general solution depends on a finite number of constants is explained in [12,14]. Accordingly, \((x, y)\) are called variables, "var" (of the concerned PDE system), while \((a, b)\) are called parameters, "par" (namely "constants of integration"). Thus \( y := Q(x, a, b) \) can be thought of as being the general solution to some PDE system, and this explains the terming "submanifold of solutions".

Fixing either \((a, b)\) or \((x, y)\), we introduce two kinds of leaves which foliate \( \mathcal{M} \):

\[
\mathcal{L}_{a,b} := \{(x, y): \ y = Q(x, a, b)\} \quad \text{and} \quad \mathcal{P}_{x,y} := \{(a, b): \ b = P(a, x, y)\}.
\]
They are preserved through any para-CR equivalence:

\[(F, \Phi)(\mathcal{D}_{a,b}) \subset \mathcal{D}'_{\Phi(a,b), \Psi(a,b)} \quad \text{and} \quad (F, \Phi)(\mathcal{P}_{x,y}) \subset \mathcal{P}'_{f(x,y), g(x,y)}.\]

In Sect. 8, we introduce two kinds of Levi forms, whose two matrices in some natural local frames are the \(n \times m\) matrix and the \(m \times n\) matrix:

\[
\text{Levi}_{\text{par}}(Q) := \begin{pmatrix}
-Q_b Q_{x_1 a_1} + Q_{a_1 Q_{x_1 b}} & \ldots & -Q_b Q_{x_m a_m} + Q_{a_m Q_{x_1 b}} \\
Q_b & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
-Q_b Q_{x_m a_m} + Q_{a_m Q_{x_1 b}} & \ldots & -Q_b Q_{x_1 a_1} + Q_{a_1 Q_{x_1 b}}
\end{pmatrix},
\]

\[
\text{Levi}_{\text{var}}(P) := \begin{pmatrix}
-P_y P_{x_1 a_1} + P_{a_1 P_{x_1 y}} & \ldots & -P_y P_{x_m a_m} + P_{a_m P_{x_1 y}} \\
-\frac{P_y}{P_x} & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
-\frac{P_y P_{a_m Q_{x_1 b} + P_{x_m P_{a_1 y}}} & \ldots & -\frac{P_y P_{x_1 a_1} + P_{a_1 P_{x_1 y}}}{P_x P_y}
\end{pmatrix}.
\]

By differentiating (1.1), we see that \(P_y = \frac{1}{Q_b}\) and we show in Lemma 8.1 that for all indices \(1 \leq i \leq n\) and \(1 \leq j \leq m\), it holds identically on \(\mathcal{M}\):

\[-\frac{Q_b Q_{x_i a_j} + Q_{a_j Q_{x_i b}}}{Q_b Q_b} \equiv -P_y \left(-\frac{P_y P_{x_i a_j} + P_{a_j P_{x_i y}}}{P_y P_y}\right).
\]

From this, it follows that the two Levi forms have the same rank, since:

\[\text{Levi}_{\text{par}}(Q) = -P_y^T \text{Levi}_{\text{var}}(P) \iff -Q_b^T \text{Levi}_{\text{par}}(Q) = \text{Levi}_{\text{var}}(P).\]

Denoting this common rank by \(r \leq \min(n, m)\), we deduce that \(\mathcal{M}\) can always be represented, in suitably normalized coordinates \((x, y, a, b)\), as:

\[y = b + x_1 a_1 + \cdots + x_r a_r + O_{x,a}(3) + b O_{x,a,b}(2),\]

\[\iff b = y - x_1 a_1 - \cdots - x_r a_r + O_{a,x}(3) + y O_{a,x,y}(2).\]

In Sect. 11, we show that the two Levi forms are related to jets of order 1 of \(\mathcal{D}_{a,b}\) and of \(\mathcal{P}_{x,y}\). More generally, we study jet-maps of any orders \(k \geq 1\) and \(l \geq 1\):

\[J^k_{x} \mathcal{D}_{a,b} := \left(x, (\partial^\beta_x Q(x,a,b))_{|\beta| \leq k}\right) \quad \text{and} \quad J^l_{a} \mathcal{P}_{x,y} := \left(a, (\partial^\gamma_a P(a,x,y))_{|\gamma| \leq l}\right).\]

Assume a para-CR equivalence (1.2) is given. In the less general CR context, the following theorem already appeared in [13,19,20], and in Sect. 12, Theorem 12.3 will provide more details.

**Theorem 1.3** For any two integers \(k \geq 1\) and \(l \geq 1\), there are two maps \(R^k_{f,g}\) and \(S^l_{\varphi, \psi}\) of the form:

\[R^k_{f,g} \left(x, y, \left(y_{x^\beta}\right)_{1 \leq |\beta| \leq k}\right) \quad \text{and} \quad S^l_{\varphi, \psi} \left(a, b, \left(b_{a^\gamma}\right)_{1 \leq |\gamma| \leq l}\right),\]

which make commutative the two diagrams:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{(f,g,\varphi, \psi)} & \mathcal{M}' \\
J^k_x \mathcal{D}_{a,b} \quad (n+k)! \mapsto \quad R^k_{f,g} \mathcal{D}'_{a,b'} \quad (n+k)! \mapsto \\
\mathbb{K}^{n+(m+k)!} \xrightarrow{r_{f,g}} \mathbb{K}^{m+(n+k)!}. & \quad & \quad \mathcal{M} & \xrightarrow{(\varphi, f, g)} & \mathcal{M}' \\
J^l_a \mathcal{P}_{x,y} \quad (m+l)! \mapsto \quad S^l_{\varphi, \psi} \mathcal{P}'_{x', y'} \quad (m+l)! \mapsto \\
\mathbb{K}^{m+(n+l)!} \xrightarrow{r_{f,g}} \mathbb{K}^{n+(m+l)!}. & \quad & \quad & \quad & \quad
\end{array}
\]
A submanifold of solutions $\mathcal{M}$ is called:

- $k$-nondegenerate with respect to parameters at a point $(x_0, a_0, b_0)$ if there exists an integer $k \geq 1$ (smallest possible) such that the invariant jet map $J^k_x Q_{a,b}$:

$$(x, a, b) \mapsto (x, (\partial^{\beta}_{a} Q(x, a, b))_{|\beta| \leq k})$$

is of maximal possible rank $n + 1 + m$ at $(x_0, a_0, b_0)$;

- $l$-nondegenerate with respect to variables at a point $(a_0, x_0, y_0)$ if there exists an integer $l \geq 1$ (smallest possible) such that the invariant jet map $J^l_x P_{x,y}$:

$$(a, x, y) \mapsto (a, (\partial^{\gamma}_{y} P(a, x, y))_{|\gamma| \leq l})$$

is of maximal possible rank $m + 1 + n$ at $(a_0, x_0, y_0)$.

A general result not contained in [14] can be formally written by taking inspiration from [9, 10]. Its dimension bounds are general, but not sharp.

**Theorem 1.4** If $\mathcal{M}$ is $k$-nondegenerate with respect to parameters and $l$-nondegenerate with respect to variables, then:

$$\text{Aut}(\mathcal{M}) := \{(F, \Phi) \in \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} : (F, \Phi)(\mathcal{M}) \subset \mathcal{M}\},$$

is a local Lie group of dimension:

$$\dim \text{Aut}(\mathcal{M}) \leq (n + 1) \left(\frac{n + 1 + 2k + 2l}{n + 1}\right) + (m + 1) \left(\frac{l + 1 + 2k + 2l}{m + 1}\right).$$

As is usual with any general result, there is a price to pay: its hidden coarseness. For instance, with $n = m = 2$, and with $k = l = 2$, the bound is:

$$\dim \text{Aut}(\mathcal{M}) \leq (2 + 1) \left(\frac{2 + 1 + 2 + 2 + 2 + 2}{2 + 1}\right) + (2 + 1) \left(\frac{2 + 1 + 2 + 2 + 2 + 2}{2 + 1}\right) = 990,$$

but the right bound, attained, is [16]:

$$\dim \text{Aut}(\mathcal{M}) \leq 10.$$

The present paper, preliminary to the articles [16, 17], examines mainly the dimensions $n = m = 2$. More suitable notations for the two graphed equations of $\mathcal{M}$ then are:

$$z = Q(x, y, a, b, c) \iff c = P(a, b, x, y, z).$$

Also, we assume throughout that the Levi rank $r$ is constant $r = 1 < 2$, so that both Levi forms are degenerate. Thus, we assume that:

$$Q_b \neq 0 \neq P_y, \quad -Q_c Q_{xa} + Q_a Q_{xc} \neq 0 \neq -P_z P_{ax} + P_x P_{az},$$

and we assume the two (equivalent) identical vanishings:

$$0 \equiv \text{det Levi}_{\text{par}}(Q) = \begin{pmatrix}
- Q_c Q_{xa} + Q_a Q_{xc} & - Q_c Q_{xb} + Q_b Q_{xc} \\
- Q_c Q_{ya} + Q_a Q_{yc} & - Q_c Q_{yb} + Q_b Q_{yc} \\
- Q_c Q_{za} + Q_a Q_{zc} & - Q_c Q_{zb} + Q_b Q_{zc}
\end{pmatrix},$$

$$0 \equiv \text{det Levi}_{\text{var}}(P) = \begin{pmatrix}
P_z P_{ax} + P_x P_{az} & - P_y P_{az} + P_z P_{ay} \\
-P_z P_{ax} + P_x P_{az} & P_y P_{az} - P_z P_{ay}
\end{pmatrix}.$$
In Sect. 19, we show that there exist normalized coordinates \((x, y, z, a, b, c)\) in which these two equations read:

\[
\begin{align*}
    z &= c + xa + \beta xxb + \beta yaa + c O_{x,y,a,b}(2) + O_{x,y,a,b,c}(4), \\
    c &= z - ax - \beta aay - \beta bxx + z O_{a,b,x,y}(2) + O_{a,b,x,y,z}(4).
\end{align*}
\]

Also, we verify that, at the origin \((x_0, y_0, z_0, a_0, b_0, c_0) = (0, 0, 0, 0, 0, 0)\), one has:

\[
\mathcal{M} \text{ is 2-nondegenerate with respect to parameters } \iff \beta \neq 0 \iff \Delta_{123}(Q) \neq 0,
\]

\[
\mathcal{M} \text{ is 2-nondegenerate with respect to variables } \iff \beta \neq 0 \iff \square(P) \neq 0,
\]

where:

\[
\Delta(Q) := \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{xxa} & Q_{xxb} & Q_{xxc} \end{vmatrix}, \\
\square(P) := \begin{vmatrix} P_x & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{axx} & P_{ayy} & P_{azz} \end{vmatrix}.
\]

In the CR setting, \(\beta = \overline{\beta}\), hence both 2-nondegeneracy conditions are equivalent. From now on, we shall assume that \(\mathcal{M}\) is 2-nondegenerate with respect to parameters.

Next, in terms of the two equivalent choices of fundamental vector fields:

\[
\begin{align*}
    \mathcal{K}_x &:= \partial_x, \quad \mathcal{K}_y := \partial_y, \quad \mathcal{L}_a := \partial_a - \frac{Q_a}{Q_c} \partial_c, \quad \mathcal{L}_b := \partial_b - \frac{Q_b}{Q_c} \partial_c, \\
    \mathcal{K}_x &:= \partial_x - \frac{P_x}{P_z} \partial_z, \quad \mathcal{K}_y := \partial_y - \frac{P_y}{P_z} \partial_z, \quad \mathcal{L}_a := \partial_a, \quad \mathcal{L}_b := \partial_b.
\end{align*}
\]

we may introduce the two kernels of the two Levi forms:

\[
\begin{align*}
    \text{Ker Levi}_{\text{var}} &:= \left\{ \mathcal{K} \in \Gamma(T^{\text{var}} \mathcal{M}) : 0 = \text{Levi}_{\text{var}}(\mathcal{K}, \mathcal{L}), \ \forall \mathcal{L} \in \Gamma(T^{\text{par}} \mathcal{M}) \right\}, \\
    \text{Ker Levi}_{\text{par}} &:= \left\{ \mathcal{L} \in \Gamma(T^{\text{par}} \mathcal{M}) : 0 = \text{Levi}_{\text{par}}(\mathcal{L}, \mathcal{K}), \ \forall \mathcal{K} \in \Gamma(T^{\text{var}} \mathcal{M}) \right\}.
\end{align*}
\]

Abbreviating:

\[
\begin{align*}
    k &:= -\frac{P_z P_{ay} + P_y P_{az}}{-P_z P_{ax} + P_x P_{az}} \quad \text{and} \quad l := -\frac{-Q_c Q_{xb} + Q_b Q_{xc}}{-Q_c Q_{xa} + Q_a Q_{xc}},
\end{align*}
\]

we see that these two kernels are line bundles generated by:

\[
\mathcal{K}_{\text{ker}}^{\text{var}} = k \mathcal{K}_x + \mathcal{K}_y \quad \text{and} \quad \mathcal{L}_{\text{ker}}^{\text{par}} = l \mathcal{L}_a + \mathcal{L}_b.
\]

We will come back to these two Levi kernel bundles in a while.

For the moment, by eliminating the three parameters \((a, b, c)\) from the three equations

\[
y = Q, \ y_x = Q_x, \ y_{xx} = Q_{xx},
\]

and by substituting the solved \((a, b, c)\) in \(z_y = Q_y, \ z_{xxx} = Q_{xxx}\), we reconstitute the associated PDE system which is the main object of study in [16,17]:

\[
\begin{align*}
    z_y &= F(x, y, z, z_x, z_{xx}) \quad \text{and} \quad z_{xxx} = H(x, y, z, z_x, z_{xx}). \quad (1.5)
\end{align*}
\]

From \((Q_y)_{xxx} = (Q_{xxx})_y\), we deduce a compatibility constraint on \(F\) and \(H\):

\[
D_x(D_x(D_x(F))) = D_y(H),
\]

in terms of the two total differentiation operators:

\[
D_x := \partial_x + z_x \partial_z + z_{xx} \partial_{z_x} + H \partial_{z_{xx}}, \quad D_y := \partial_y + F \partial_z + D_x(F) \partial_{z_x} + D_x(D_x(F)) \partial_{z_{xx}}.
\]

The converse holds true, and is an elementary consequence of the Frobenius theorem [14].
As a preliminary to the articles [16,17] (see also [1,2,7]), we can now explain how the geometric (invariant) hypotheses on submanifolds of solutions do transfer to the para-CR PDE systems. At first, we realize in Proposition 22.1 the following nice fact.

**Proposition 1.6** If $\mathcal{M}$ has (degenerate) Levi form of constant rank 1 and if it is 2-nondegenerate with respect to parameters, then $F$ is independent of $z_{xx}$:

$$0 \equiv F_{z_{xx}}.$$ 

Next, due to the independency between $\beta \neq 0$ and $\beta \neq 0$, there comes a

**Question 1** How can one view the hypothesis of 2-nondegeneracy with respect to variables in the PDE system (1.5)?

By some (non-straightforward) differential-algebraic computations, we show in Lemmas 24.1 and in Proposition 24.2 that:

$$F_{z_x} = -Q_c Q_{ya} + Q_a Q_{yc} = -P_z P_{ay} + P_y P_{az} = -k.$$ 

Further, in Lemma 24.3, we show the nowhere vanishing invariant expression:

$$\frac{\partial}{\partial a} (F_{z_x}) = P_z \left[ \begin{array}{ccc} P_t & P_y & P_z \\ P_{ax} & P_{ay} & P_{az} \\ P_{aax} & P_{aay} & P_{aaz} \end{array} \right] \left( -P_z P_{ax} + P_x P_{az} \right)^2 \neq 0,$$

from which we deduce:

**Theorem 1.7** Let (1.5) be a PDE system coming from a submanifold of solutions $\mathcal{M}$ which is 2-nondegenerate with respect to parameters. Then $\mathcal{M}$ is also 2-nondegenerate with respect to variables if and only if:

$$F_{z_x z_x} \neq 0.$$ 

This theorem therefore explains, in standard CR terms or para-CR language, why in [16,17], we encountered the natural differential invariant $F_{z_x z_x}$, which we assumed $\neq 0$.

In order to present a bit more of foundational material preparatory to [16,17], let us describe how to launch Cartan’s equivalence method, both from the point of view of $y = Q(x, a, b)$ and from the point of view of $z_y = F, z_{xxx} = H$.

Coming back to our two Levi kernel bundles $T^\text{var}_{\ker \mathcal{M}}$ and $T^\text{par}_{\ker \mathcal{M}}$, we observe that, through any equivalence $(F, \Phi): \mathcal{M} \rightarrow \mathcal{M}'$:

$$(F, \Phi)_* (T^\text{var}_{\ker \mathcal{M}}) = T^\text{var}_{\ker \mathcal{M}'} \quad \text{and} \quad (F, \Phi)_* (T^\text{par}_{\ker \mathcal{M}}) = T^\text{par}_{\ker \mathcal{M}'}.$$ 

It follows that there exist 11 functions on $\mathcal{M}$ such that:

$$\begin{pmatrix} T_c' \\ T_a' \\ L' L_a' + L_b' \\ \mathcal{H}_x' + \mathcal{H}_y' \end{pmatrix} = \begin{pmatrix} a & b_1 & b_2 & c_1 & c_2 \\ 0 & f_1 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & 0 & 0 \\ 0 & 0 & 0 & h_1 & h_2 \end{pmatrix} \begin{pmatrix} T_c \\ T_a \\ L' L_a + L_b \\ \mathcal{H}_x + \mathcal{H}_y \end{pmatrix}.$$
Hence passing to dual 1-forms, a plain transposition yields (with new functions):

\[
\begin{pmatrix}
Q'_a da' + Q'_b db' + Q'_c dc' \\
da' - l'db'
\end{pmatrix}
= \begin{pmatrix}
a & 0 & 0 & 0 \\
b & f_1 & 0 & 0 \\
b & f_2 & f_3 & 0 \\
c & 0 & h_1 & 0 \\
c & 0 & h_2 & h_3
\end{pmatrix}
\begin{pmatrix}
Q_a da + Q_b db + dc \\
da - ldb \\
db \\
dx - kdy \\
dy
\end{pmatrix}.
\]

From the point of view of the PDE system (1.5), the natural coframe to start with when running Cartan’s equivalence method is:

\[
\lambda := dz - z_x dx - F dy, \\
\mu_1 := dz_x - z_{xx} dx - D(F) dy, \\
\mu_2 := dz_{xx} - H dx - D_x(D_x(F)) dy, \\
v_1 := dx - F_{z \bar{z}} dy, \\
v_2 := dy.
\]

This is under the assumption that \( F_{z \bar{z}} \neq 0 \) only! In conclusion, this explains why the lifted coframe we start with in [16] is:

\[
\begin{pmatrix}
\lambda \\
\mu_1 \\
\mu_2 \\
v_1 \\
v_2
\end{pmatrix} := \begin{pmatrix}
a & 0 & 0 & 0 \\
b & f_1 & 0 & 0 \\
b & f_2 & f_3 & 0 \\
c & 0 & h_1 & 0 \\
c & 0 & h_2 & h_3
\end{pmatrix} \begin{pmatrix}
\lambda \\
\mu_1 \\
\mu_2 \\
v_1 \\
v_2
\end{pmatrix}.
\]

Beyond these elementary foundational considerations, readers interested in advanced Cartan-type results are referred to [16,17].

### 2 Motivation from CR geometry

Already in 1907, Poincaré observed that there are many more real-analytic ($C^\infty$) hypersurfaces \( M^3 \subset \mathbb{C}^2 \), represented in coordinates \( (z, w) = (x + \sqrt{-1} y, u + \sqrt{-1} v) \) as graphs by means of converging real power series:

\[
u = \varphi(x, y, v) = \sum_{i,j,k} \varphi_{i,j,k} x^i y^j v^k \quad (\varphi_{i,j,k} \in \mathbb{R}),
\]

than there are local biholomorphic transformations:

\[
(z, w) \mapsto (f(z, w), g(z, w)) = \left( \sum_{i,j} f_{i,j} z^i w^j \right) \left( \sum_{i,j} g_{i,j} z^i w^j \right) \quad (f_{i,j}, g_{i,j} \in \mathbb{C}),
\]

since for instance if all objects are polynomials of a certain large degree \( d \gg 1 \):

\[
\text{dim}_{\mathbb{R}} \{ \varphi_{i,j,k} \in \mathbb{R} : i + j + k \leq d \} = \frac{(d+3)}{2} \gg 2 \cdot \frac{(d+2)}{2} = \text{dim}_{\mathbb{R}} \{ f_{i,j}, g_{i,j} \in \mathbb{C} : i + j \leq d \}.
\]

Hence there is a problem of classification, which can be formulated generally in any dimension \( N \geq 1 \), and in a local setting: given two real submanifolds \( M \subset \mathbb{C}^N \) and \( M' \subset \mathbb{C}^N \), determine whether there exist local biholomorphic transformations \( h: \mathbb{C}^N \rightarrow \mathbb{C}^N \) with \( h(M) = M' \) (up to restriction to open subsets), and classify such real submanifolds of \( \mathbb{C}^N \) up to local biholomorphisms.
Here, to avoid dwelling into the analysis of embedability problems and to benefit of Zariski-generic properties, only real-analytic ($C^\omega$) objects will be dealt with. Even in this context, the problem is too ample to be solved completely in any dimension.

To start with, two basic facts must be reminded, see e.g. [19, Section 1] for details. Let $N \geq 1$ and let:

$$(z_1, \ldots, z_N) \in \mathbb{C}^N.$$

Let $J : T\mathbb{C}^N \rightarrow T\mathbb{C}^N$ be the standard complex structure which expresses the multiplication by $\sqrt{-1}$ of real vector fields:

$$J\left(\frac{\partial}{\partial x_k}\right) := \frac{\partial}{\partial y_k} \quad \text{and} \quad J\left(\frac{\partial}{\partial y_k}\right) := -\frac{\partial}{\partial x_k} \quad (1 \leq k \leq n).$$

A $C^\omega$ real submanifold $M \subset \mathbb{C}^N$ is called Cauchy-Riemann (CR) when the smallest $J$-invariant subspaces of the tangent spaces $T_p M$:

$$\dim_{\mathbb{R}} \left( T_p M \cap J(T_p M) \right) =: 2n,$$

are of constant (necessarily even) dimension when $p \in M$ varies. It is called CR-generic when its tangent spaces $T_p M$ generate the whole ambient tangent spaces:

$$T_p M + J(T_p M) = T_p \mathbb{C}^N \quad (\forall p \in M).$$

Elementary linear algebra shows that CR-generic implies CR. The complex tangent spaces are then:

$$T^c_p M := T_p M \cap J(T_p M) \quad (p \in M).$$

The first fact is that any $C^\omega$ real-analytic submanifold $M \subset \mathbb{C}^N$ (even every real-analytic subset) is CR at a Zariski-generic point, that is, outside some proper real-analytic subset. Agreeing to study problems only at generic points like Lie and Cartan did, one disregards such exceptional sets.

The second basic fact is that every $C^\omega$ CR submanifold $M \subset \mathbb{C}^N$ is locally straightenable, by means of some appropriate biholomorphism, to a CR-generic one times zero:

$$M = M_1 \times \{0\}^{N-N_1} \subset \mathbb{C}^{N_1} \times \mathbb{C}^{N-N_1},$$

with $M_1 \subset \mathbb{C}^{N_1}$ being CR-generic. Hence the problem reduces to equivalences of CR-generic submanifolds, and in this context, a deep link exists with equivalences of certain systems of partial differential equations, as we will see.

Before coming to PDE’s, consider a CR-generic $M \subset \mathbb{C}^N$ of real codimension:

$$c := \text{codim}_{\mathbb{R}} M,$$

represented locally in some neighborhood of some ‘central’ point $p_0 \in M$ as the zero-set:

$$M = \{ z \in \mathbb{C}^N : \rho_1(z, \bar{z}) = \cdots = \rho_c(z, \bar{z}) = 0 \} \quad (\rho(p_0) = 0),$$

of $c$ real-analytic functions, which, to guarantee smoothness, have independent differentials:

$$0 \neq d\rho_1 \wedge \cdots \wedge d\rho_c(p) \quad (\forall \ p \in M).$$

The reality $\bar{\rho} = \rho$ of this $\mathbb{R}^c$-valued function $\rho = (\rho_1, \ldots, \rho_c)$ means, when expanding it in converging power series:

$$\sum_{\alpha, \beta} \rho_{\alpha, \beta} z^\alpha \bar{z}^\beta = \rho(z, \bar{z}) = \bar{\rho}(z, \bar{z}) = \sum_{\alpha, \beta} \bar{\rho}_{\alpha, \beta} z^\alpha z^\beta,$$

Springer
that its coefficients satisfy by identification:

$$\rho_{\beta,\alpha} = \rho_{\alpha,\beta} \quad (\forall \alpha \in \mathbb{N}^n, \forall \beta \in \mathbb{N}^n),$$

so that if one defines, by conjugating only coefficients:

$$\tilde{\rho}(z, \bar{z}) := \sum_{\alpha,\beta} \rho_{\alpha,\beta} z^\alpha \bar{z}^\beta,$$

one indeed has an identity which expresses $$\rho = \tilde{\rho}$$ with arguments:

$$\rho(z, \bar{z}) \equiv \tilde{\rho}(z, \bar{z}) \quad (\text{in } \mathbb{C}[z, \bar{z}]^c).$$

As is known, the CR-genericity of $$M$$ is then equivalent to:

$$0 \neq \partial \rho_1 \wedge \cdots \wedge \partial \rho_c(p) \quad (\forall p \in M),$$

where, for any $$\mathcal{C}^\omega$$ function $$\chi$$, the holomorphic $$\partial$$ and antiholomorphic $$\overline{\partial}$$ differential operators are defined by:

$$\partial \chi := \sum_k \frac{\partial \chi}{\partial z_k} dz_k, \quad \overline{\partial} \chi := \sum_k \frac{\partial \chi}{\partial \bar{z}_k} d\bar{z}_k, \quad d\chi = \partial \chi + \overline{\partial} \chi,$$

and have sum equal to the standard real differential. Then the integer $$n - c =: 2n$$ is even [19], and its half:

$$n := CRdim M$$

is called the CR dimension of the CR-generic submanifold $$M \subset \mathbb{C}^{n+c}$$. Always, to avoid purely real and purely complex geometries, we will assume:

$$c \geq 1 \text{ and } n \geq 1,$$

hence:

$$N = n + c \geq 2.$$

With any $$c \times c$$ invertible matrix $$(\alpha_{j,k})$$ of $$\mathcal{C}^\omega$$ functions, the functions $$\rho'_j := \sum_k \alpha_{j,k} \rho_k$$ are still defining $$M = \{\rho'_1 = \cdots = \rho'_c = 0\}$$, which shows a (known) lack of adequate correspondence $$M \longleftrightarrow \rho$$.

Actually, most of the time, it is more appropriate to work with graphed representations of submanifolds, since the link with intrinsic geometric properties becomes one-to-one.

**Lemma 2.1** [19] Given a $$\mathcal{C}^\omega$$ CR-generic $$M \subset \mathbb{C}^{n+c}$$ with codim_{\mathbb{R}} M = c \geq 1 and CRdim M = n \geq 1, at each point $$p_0 \in M$$, there exist centered holomorphic coordinates:

$$(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_c) \in \mathbb{C}^n \times \mathbb{C}^c,$$

and a complex-analytic $$\mathbb{C}^c$$-valued function $$\Theta(z, \bar{z}, \bar{w})$$ such that, near $$p$$:

$$M = \{(z, w) \in \mathbb{C}^N: w_1 = \Theta_1(z, \bar{z}, \bar{w}), \ldots, w_c = \Theta_c(z, \bar{z}, \bar{w})\}. \qed$$

This vector-valued graphing function $$\Theta = (\Theta_1, \ldots, \Theta_c)$$ is constructed by first performing a complex-linear transformation in order that:

$$0 \neq det \left( \frac{\partial \rho'_j}{\partial w_j} (p_0) \right).$$
and then by applying the analytic implicit function theorem to solve for \( w \) in the \( c \) equations:

\[
0 = \rho_1(z, w, \bar{z}, \bar{w}) = \cdots = \rho_c(z, w, \bar{z}, \bar{w}).
\]

With the convention, seen above, that the conjugation of power series distributes bars over coefficients and over variables as well:

\[
\Theta(z, \bar{z}, \bar{w}) = \sum_{\alpha, \beta, \gamma} \Theta_{\alpha, \beta, \gamma} z^\alpha \bar{z}^\beta \bar{w}^\gamma = \sum_{\alpha, \beta, \gamma} \Theta_{\alpha, \beta, \gamma} z^\alpha \bar{z}^\beta \bar{w}^\gamma =: \Theta(\bar{z}, z, w),
\]

this graphed \( \mathbb{C}^c \)-valued function representing:

\[ M = \{ w = \Theta(z, \bar{z}, \bar{w}) \}, \]

then satisfies identically by construction:

\[
0 \equiv \rho(z, \Theta(z, \bar{z}, \bar{w}), \bar{z}, \bar{w}).
\]

Conjugating this and remembering \( \bar{\rho}(\bar{z}, z) = \rho(z, \bar{z}) \) yields:

\[
0 \equiv \rho(z, w, \bar{z}, \Theta(\bar{z}, z, w)),
\]

which means that the equations \( \rho = 0 \) can also be solved with respect to \( \bar{w} \) as:

\[
\bar{w} = \Theta(\bar{z}, z, w).
\]

A more accurate and complete analysis conducts to the basic

**Lemma 2.2** The complex \( \mathbb{C}^c \)-valued graphing function \( \Theta(z, \bar{z}, \bar{w}) \) for a CR-generic submanifold \( M \subset \mathbb{C}^{n+c} \) satisfies the two (equivalent by conjugation) functional equations:

\[
w \equiv \Theta(z, \bar{z}, \Theta(\bar{z}, z, w)), \]

\[
\bar{w} \equiv \Theta(\bar{z}, z, \Theta(z, \bar{z}, \bar{w})).
\]

Conversely, given a \( \mathbb{C}^c \)-valued power series \( \Theta \in \mathbb{C}[z, \bar{z}, \bar{w}]^c \) vanishing at the origin which satisfies these identities, the two zero-sets:

\[
\{ w = \Theta(z, \bar{z}, \bar{w}) \} = \{ \bar{w} = \Theta(\bar{z}, z, w) \},
\]

coincide and define a local real-analytic CR-generic submanifold \( M \subset \mathbb{C}^{n+c} \).

Next, assume that two CR-generic submanifolds passing through the origin \( 0 \in M \subset \mathbb{C}^{n+c} \) and \( 0' \in M' \subset \mathbb{C}^{n'+c} \) having the same codimension \( c \) and the same CR dimension \( n \) represented as:

\[
M = \{ w = \Theta(z, \bar{z}, \bar{w}) \}\quad \text{and} \quad M' = \{ w' = \Theta'(z', \bar{z}', \bar{w}') \},
\]

are equivalent through a local biholomorphism:

\[
h: (z, w) \mapsto (f(z, w), g(z, w)) =: (z', w'),
\]

that is to say, assume that \( h(M) \subset M' \). This inclusion can be expressed as two (equivalent) relations holding for \( (z, w) \in M \):

\[
g(z, w) = \Theta'(f(z, w), \bar{f}(\bar{z}, \bar{w}), g(\bar{z}, \bar{w})),
\]

\[
\bar{g}(\bar{z}, \bar{w}) = \Theta(\bar{f}(\bar{z}, \bar{w}), f(z, w), g(z, w)).
\]
As is visible in these relations, simultaneously with the biholomorphic equivalence \( h \), there comes its antiholomorphic conjugation:

\[
\tilde{h} : (z, w) \mapsto (\bar{f}(\bar{z}, \bar{w}), \bar{g}(\bar{z}, \bar{w})),
\]

and consequently, it is natural to consider simultaneously both maps:

\[
(z, w, \bar{z}, \bar{w}) = (z, \bar{z}) \mapsto (h(z), \bar{h}(\bar{z})) = (f(z, w), g(z, w), \bar{f}(\bar{z}, \bar{w}), \bar{g}(\bar{z}, \bar{w})).
\]

These aspects of thought will now become more transparent when passing to equivalences of PDE systems.

### 3 Equivalences of submanifolds of solutions for PDE systems

As explained in [12,14], point equivalences of completely integrable systems of partial differentials equations whose solutions depend on a finite number of parameters can be understood as point equivalences of their respective submanifolds of solutions, which share remarkable common properties with CR-generic submanifolds. The theory works equally well over the fields \( K = \mathbb{R} \) or \( K = \mathbb{C} \).

The general set up is a local product space of variables \((x, y)\) times parameters \((a, b)\), in any dimension:

\[
x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_c), \quad a = (a_1, \ldots, a_m), \quad b = (b_1, \ldots, b_c),
\]

with \( n \geq 1 \), with \( m \geq 1 \) and with \( c \geq 1 \) being arbitrary integers, the number of parameters \( b \)'s being the same as that of variables \( y \)'s, because we will understand \( y(x) = Q(x, a, b) \) as a function of the variables \( x \) depending on the parameters \( (a, b) \), with \( b = Q(0, a, b) = y(0) \) being its value at the origin. Abbreviate:

\[
N := n + c \quad \text{and} \quad MM := m + c,
\]

For CR-generic \( M \subset \mathbb{C}^{n+c} \), one has the correspondence:

\[
x \leftrightarrow z, \quad y \leftrightarrow w, \quad a \leftrightarrow \bar{z}, \quad b \leftrightarrow \bar{w},
\]

which imposes \( m = n \), a restriction that will hence be removed from now on. Equivalences of CR manifolds will nevertheless remain inspirational for these more general structures, termed para-CR in [12].

By definition, point equivalences of PDE systems transform graphs of solutions into graphs of solutions, and as these graphs are marked out by parameters, such equivalences naturally induce equivalences in the parameter space. Consequently, the natural infinite pseudogroup of equivalences is the product:

\[
Diff_{\text{var}} \times Diff_{\text{par}}
\]

of diffeomorphisms in the space of variables times diffeomorphisms in the space of parameters, namely invertible transformations of the form:

\[
(x, y, a, b) \mapsto (f(x, y), g(x, y), \varphi(a, b), \psi(a, b)).
\]

In the CR case, we have:

\[
\varphi \leftrightarrow \bar{f}, \quad \psi \leftrightarrow \bar{g}.
\]
Denoting symbolically:
\[ \text{var} := (x, y) \in \mathbb{K}^N \quad \text{and} \quad \text{par} := (a, b) \in \mathbb{K}^M, \]
the infinite pseudogroup \( \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} \) can be thought of as the group of split diffeomorphisms in the total space \( \mathbb{K}^N \times \mathbb{K}^M \ni (x, y, a, b) \).

**Definition 3.1** A (local) submanifold of solutions \( \mathcal{M} \subset \mathbb{K}^{n+c} \times \mathbb{K}^{m+c} \) is a \( c \)-codimensional \( \mathbb{K} \)-analytic submanifold passing through the origin:
\[ \mathcal{M} = \{ (x, y, a, b) : R(x, y, a, b) = 0 \} \quad (0 \in \mathcal{M}), \]
with its defining \( \mathbb{K}^c \)-valued function \( R \in \mathbb{K}\{x, y, a, b\}^c \) satisfying:
\[
det \left( \frac{\partial R_{j_1}}{\partial y_{j_2}} (0) \right) \neq 0 \neq det \left( \frac{\partial R_{j_1}}{\partial b_{j_2}} (0) \right) \quad (1 \leq j_1, j_2 \leq c). \]

The analytic implicit function theorem then yields two graphing functions \( Q = Q(x, a, b) \) and \( P = P(a, x, y) \) so that
\[ R(x, y, a, b) = 0 \iff y = Q(x, a, b) \]
\[ \iff b = P(a, x, y), \]
and this can be expressed as two functional equations holding identically:
\[ 0 \equiv R(x, Q(x, a, b), a, b), \quad (3.2) \]
\[ 0 \equiv R(x, y, a, P(a, x, y)), \quad (3.3) \]
in \( \mathbb{C}\{x, a, b\}^c \) and in \( \mathbb{C}\{a, x, y\}^c \).

The problem of equivalence considered here is the one of submanifolds of solutions considered up to split diffeomorphisms belonging to the infinite pseudogroup \( \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} \).

### 4 Normal coordinates for submanifolds of solutions

A first normalization, inspired from CR geometry, provides (for free) useful normalizing coordinates in which, later, some invariant concepts of the theory will be more transparently seen.

**Proposition 4.1** After a \( \mathbb{K} \)-analytic change of coordinates belonging to \( \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} \) having the shape:
\[ x = x', \quad y = y'(x', y'), \quad a = a', \quad b = b'(a', b'), \]
\( \mathcal{M} \) is transformed into a new \( \mathcal{M}' \) whose new \( \mathbb{K}^c \)-valued implicit defining function:
\[ R'(x', y', a', b') := R(x', y'(x', y'), a', b'(a', b')), \]
satisfies the normalization:
\[ R'(0, b', a', b') \equiv 0 \quad \text{and} \quad 0 \equiv R'(x', y', 0, y'). \]

Formally in the proof below and in some later ones, thinking of \( \mathbb{K}^c \)-valued functions or assuming for simplicity that \( c = 1 \) works equally well.
Proof After a linear change of coordinates belonging to $\text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}}$ which stabilizes $x' = x$ and $a' = a$, we can assume that:

$$R = y - b + O_2(x, y, a, b).$$

(4.2)

Set $x := 0$ and solve for $y$ in the equation:

$$0 = R(0, y, a, b) \iff y = U(a, b) = b + O_2(a,b),$$

with a certain unique analytic map $U$ hence satisfying:

$$0 \equiv R(0, U(a, b), a, b).$$

(4.3)

Next, solve:

$$U(a, b) = y \iff b = B(a, y),$$

and then take as defining equation for the transformed $\mathcal{M}'$:

$$R'(x, y, a, b') := R(x, y, a, B(a, b')).$$

Using $U(a, B(a, b')) = b'$ when replacing $b := B(a, b')$ in (4.3) conducts to the first claimed normalization:

$$0 \equiv R(0, U(a, B(a, b')), a, B(a, b'))$$

$$= R(0, b', a, B(a, b'))$$

$$= R'(0, b', a, b').$$

which we rewrite, dropping primes, simply as:

$$0 \equiv R(0, b, a, b).$$

(4.4)

To reach the second normalization, observe at first that setting $a := 0$ gives:

$$0 \equiv R(0, b, 0, b).$$

(4.5)

Now, solve for $b$ in the equation:

$$0 = R(x, y, 0, b) \iff b = V(x, y) = y + O_2(x, y),$$

to get a certain unique analytic map $V$ with $V(0, y) \equiv y$ then satisfying:

$$0 \equiv R(x, y, 0, V(x, y)).$$

(4.6)

Next, solve:

$$V(x, y) = b \iff y = \Upsilon(x, b),$$

and observe that $\Upsilon(0, b) \equiv b$ by uniqueness in the implicit function theorem, since $V(0, y) \equiv y$.

Change defining equation as:

$$R'(x, y', a, b) := R(x, \Upsilon(x, y'), a, b),$$

and, importantly, observe that the first normalization is preserved:

$$R'(0, b, a, b) = R(0, \Upsilon(0, b), a, b)$$

$$= R(0, b, a, b)$$

$$\equiv 0.$$
Lastly, verify that the second normalization is also attained, by coming back to (4.6), in which \( y \) is replaced by \( y := \mathcal{Y}(x, y') \):

\[
0 \equiv R(x, \mathcal{Y}(x, y'), 0, \mathcal{V}(x, \mathcal{Y}(x, y'))) \\
\equiv R(x, \mathcal{Y}(x, y'), 0, y') \\
\equiv R'(x, y', 0, y').
\]

In conclusion, the composition of the two changes of coordinates employed above is indeed of the announced form stabilizing \( x' = x \) and \( a' = a \).

Any system of coordinates \( (x, y, a, b) \) in which the considered submanifold of solutions \( \mathcal{M} \subset \mathbb{K}^N_{\text{var}} \times \mathbb{K}^M_{\text{par}} \) has a defining function \( R \) enjoying:

\[
0 \equiv R(0, b, a, b) \equiv R(x, y, 0, y), \tag{4.7}
\]

will be called normalized coordinates. What precedes shows that they always exist, for free.

**Corollary 4.8** In normalized coordinates \( (x, y, a, b) \) at the origin, the two graphing functions \( Q \) and \( P \) for:

\[
\mathcal{M} = \{ y = Q(x, a, b) \} = \{ b = P(a, x, y) \},
\]

enjoy the normalization conditions:

\[
Q(0, a, b) \equiv b \equiv Q(x, 0, b) \quad \text{and} \quad P(0, x, y) \equiv y \equiv P(a, 0, y).
\]

**Proof.** Treat only the first pair of normalizations for \( Q \), the one for \( P \) being similar. In (3.2), setting \( x := 0 \) and then \( a := 0 \), we have:

\[
0 \equiv R(0, Q(0, a, b), a, b) \quad \text{to be compared with} \quad 0 \overset{(4.7)}{=}= R(0, b, a, b),
\]

\[
0 \equiv R(x, Q(x, 0, b), 0, b) \quad \text{to be compared with} \quad 0 \overset{(4.7)}{=}= R(x, b, 0, b),
\]

and the uniqueness property in the implicit function theorem when solving \( 0 = R(0, y, a, b) \) for \( y \) and when solving \( 0 = R(x, y, 0, b) \) also for \( y \) yields as wanted:

\[
Q(0, a, b) \equiv b \quad \text{and} \quad Q(x, 0, b) \equiv b. \quad \square
\]

## 5 Two fundamental foliations \( \mathcal{F}_{\text{var}} \) and \( \mathcal{F}_{\text{par}} \) on \( \mathcal{M} \)

As before, working locally, we adopt the convention of almost never mentioning open (sub)sets, in order to lighten formalism, preserve clarity, and save symbols.

Equivalences belonging to \( \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} \) are of the shape:

\[
(x, y, a, b) \mapsto (f(x, y), g(x, y), \varphi(a, b), \psi(a, b)) =: (x', y', a', b'),
\]

hence they stabilize vertical sets \( \{ \text{var} = \text{constant} \} \) and horizontal sets \( \{ \text{par} = \text{constant} \} \).

The tangent bundle \( T(\mathbb{K}^N_{\text{var}} \times \mathbb{K}^M_{\text{par}}) \) to the ambient space \( \mathbb{K}^N_{\text{var}} \times \mathbb{K}^M_{\text{par}} \) is therefore equipped with two invariant subbundles:

\[
\mathcal{F}_{\text{var}} := \text{Span} \frac{\partial}{\partial x} + \text{Span} \frac{\partial}{\partial y} \quad \text{and} \quad \mathcal{F}_{\text{par}} := \text{Span} \frac{\partial}{\partial a} + \text{Span} \frac{\partial}{\partial b},
\]
which are trivially (Frobenius) integrable:
\[
\left[ \Gamma(\mathcal{F}^{\text{var}}), \Gamma(\mathcal{F}^{\text{var}}) \right] \subset \Gamma(\mathcal{F}^{\text{var}}) \quad \text{and} \quad \left[ \Gamma(\mathcal{F}^{\text{par}}), \Gamma(\mathcal{F}^{\text{par}}) \right] \subset \Gamma(\mathcal{F}^{\text{par}}),
\]
hence define two foliations:
\[\mathcal{F}_{\text{var}} := \bigcup_{a,b} \{(x, y, a, b)\} \quad \text{and} \quad \mathcal{F}_{\text{par}} := \bigcup_{x,y} \{(x, y, a, b)\}.\]

Thanks to the assumption that \(\mathcal{M} = \{y = Q(x, a, b)\} = \{b = P(a, x, y)\}\), these two foliations intersect \(\mathcal{M}\) transversely and define two foliations on \(\mathcal{M}\), still denoted:
\[\mathcal{F}_{\text{var}} := \bigcup_{a,b} \mathcal{Q}_{a,b} \quad \text{and} \quad \mathcal{F}_{\text{par}} := \bigcup_{x,y} \mathcal{P}_{x,y},\]
whose leaves are defined as:
\[\mathcal{Q}_{a,b} := \{(x, y) : y = Q(x, a, b)\}, \quad \mathcal{P}_{x,y} := \{(a, b) : b = P(a, x, y)\}.\]

At the level of vector bundles, these \(n\)-dimensional leaves \(\mathcal{Q}_{a,b}\) and these \(m\)-dimensional leaves \(\mathcal{P}_{x,y}\) are just integral manifolds of the integrable subbundles:
\[T^{\text{var}}.\mathcal{M} := T.\mathcal{M} \cap \mathcal{F}^{\text{var}} \quad \text{and} \quad T^{\text{par}}.\mathcal{M} := T.\mathcal{M} \cap \mathcal{F}^{\text{par}}.\]

Next, assume that two submanifolds of solutions \(0 \in \mathcal{M} \subset \mathbb{K}^N \times \mathbb{K}^M\) and \(0' \in \mathcal{M}' \subset \mathbb{K}'^N \times \mathbb{K}'^M\) having the same type \((n, c, m, c)\) and passing through the origin, represented as:
\[\mathcal{M} = \{y = Q(x, a, b)\} \quad \text{and} \quad \mathcal{M}' = \{y' = Q'(x', a', b')\}\]
\[= \{b = P(a, x, y)\} \quad \text{and} \quad = \{b' = P'(a', x', y')\}\]
are equivalent under an allowed transformation \((f, g, \varphi, \psi)\). Such an equivalence expresses as two (equivalent) relations holding for \((x, y, a, b) \in \mathcal{M}\):
\[g(x, y) = Q'(f(x, y), \varphi(a, b), \psi(a, b)), \quad \psi(a, b) = P'(\varphi(a, b), f(x, y), g(x, y)),\]
that is as power series identities:
\[g(x, Q(x, a, b)) = Q'(f(x, Q(x, a, b)), \varphi(a, b), \psi(a, b)) \quad \text{(in } \mathbb{C}[x, a, b]^c),\]
\[ \psi(a, P(a, x, y)) = P'(\psi(a, P(a, x, y)), f(x, y), g(x, y)) \quad \text{(in } \mathbb{C}[a, x, y]^{e}). \]

Abbreviating:

\[ F := (f, g) \quad \text{and} \quad \Phi := (\psi, \psi), \]

these identities can be read off as expressing that:

\[ (F, \Phi)(\sigma_{a,b}) = \phi'_{\Phi(a,b)}, \]
\[ (F, \Phi)(\sigma_{x,y}) = \phi'_{F(x,y)}, \]

which means that the allowed transformations \((F, \Phi) \in \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}}\) stabilize the pairs of foliations:

\[ (F, \Phi)(\mathcal{F}_{\text{var}}) \subset \mathcal{F}'_{\text{var}}, \]
\[ (F, \Phi)(\mathcal{F}_{\text{par}}) \subset \mathcal{F}'_{\text{par}}. \]

### 6 Two systems of coordinates and functional relations

A submanifold of solutions \(\mathcal{M}\) of codimension \(c\) in \(\mathbb{K}^n \times \mathbb{K}^c \times \mathbb{K}^m \times \mathbb{K}^c\) naturally comes equipped with two charts represented by the following two projections:

\[ \tau_{\text{par}}: \mathcal{M} \longrightarrow \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K}^c, \quad (x, y, a, b) \longmapsto (x, a, b) \]
\[ \tau_{\text{var}}: \mathcal{M} \longrightarrow \mathbb{K}^m \times \mathbb{K}^n \times \mathbb{K}^c, \quad (x, y, a, b) \longmapsto (a, x, y), \]

which correspond to the arguments \((x, a, b)\) in the graphing function \(Q(x, a, b)\) and to the arguments \((a, x, y)\) in the graphing function \(P(a, x, y)\). The theory will be systematic by playing simultaneously with these two systems of coordinates.

Once the implicit function theorem has been applied, it is natural to set, in the first system of coordinates \((x, a, b)\):

\[ R(x, y, a, b) := -y + Q(x, a, b), \]

and in the second system of coordinates \((a, x, y)\):

\[ R(x, y, a, b) := -b + P(a, x, y). \]
Functional Relations 6.1  The two graphing functions $Q(x, a, b)$ and $P(a, x, y)$ satisfy identically:

\[
\begin{align*}
    y &\equiv Q(x, a, P(a, x, y)), \\
    b &\equiv P(a, x, Q(x, a, b)).
\end{align*}
\]

\[\square\]

7 Two fundamental collections of vector fields $\mathcal{K}_x_i$ and $\mathcal{L}_a_j$

From now on, assume $c = 1$. In the ambient, extrinsic coordinates $(x, y, a, b)$, some natural basic vector fields generating the two vector bundles $T_{\text{par}}\mathcal{M}$ of rank $m$ and $T_{\text{var}}\mathcal{M}$ of rank $n$ are:

\[
\begin{align*}
    \mathcal{L}_{a_j} &:= \frac{\partial}{\partial a_j} - \frac{Q_{a_j}(x, a, b)}{Q_b} \frac{\partial}{\partial b} \\
    \mathcal{K}_x_i &:= \frac{\partial}{\partial x_i} - \frac{P_{x_i}(a, x, y)}{P_y} \frac{\partial}{\partial y} \\
    (1 &\leq j \leq m), \\
    (1 &\leq i \leq n).
\end{align*}
\]

One easily convinces oneself that, in the two systems of coordinates $(x, a, b)$ and $(a, x, y)$, the pushes-forward of these vector fields consist in just dropping the extrinsic coordinate fields $\frac{\partial}{\partial b}$ and $\frac{\partial}{\partial y}$:

\[
\begin{align*}
    \tau_{\text{var}} &\left( \frac{\partial}{\partial a_j} - \frac{Q_{a_j}}{Q_b} \frac{\partial}{\partial b} \right) = \frac{\partial}{\partial a_j}, \\
    \tau_{\text{par}} &\left( \frac{\partial}{\partial x_i} - \frac{P_{x_i}}{P_y} \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial x_i}.
\end{align*}
\]

So intrinsically on $\mathcal{M}$, generators of $T_{\text{var}}\mathcal{M}$ and of $T_{\text{par}}\mathcal{M}$ write out in two ways as:

\[
\begin{align*}
    \mathcal{L}_{a_j} &= \frac{\partial}{\partial a_j} - \frac{Q_{a_j}}{Q_b} \frac{\partial}{\partial b}, & \text{or} & & \mathcal{L}_{a_j} &= \frac{\partial}{\partial a_j}, \\
    \mathcal{K}_x_i &= \frac{\partial}{\partial x_i}, & \mathcal{K}_x_i &= \frac{\partial}{\partial x_i} - \frac{P_{x_i}}{P_y} \frac{\partial}{\partial y},
\end{align*}
\]

depending on the choice of coordinates, $(x, a, b)$ or $(a, x, y)$.

As seen above, the two integrable distributions generated by the $\mathcal{L}_{a_j}$ and by the $\mathcal{K}_x_i$ are invariant. It is useful to also introduce two transversal vector fields:

\[\mathcal{T}_b := \frac{\partial}{\partial b} \quad \text{and} \quad \mathcal{U}_y := \frac{\partial}{\partial y},\]

whose directions are not invariant under allowed transformations $(F, \Phi) \in \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}}$.

In the coordinates $(x, a, b)$ on $\mathcal{M}$, a frame for the tangent bundle $T\mathcal{M}$ is:

\[
\{ \mathcal{K}_x_1, \ldots, \mathcal{K}_x_n, \mathcal{L}_{a_1}, \ldots, \mathcal{L}_{a_m}, \mathcal{T}_b \}.
\]

In the other coordinates $(a, x, y)$ on $\mathcal{M}$, a frame for $T\mathcal{M}$ is:

\[
\{ \mathcal{L}_{a_1}, \ldots, \mathcal{L}_{a_m}, \mathcal{K}_x_1, \ldots, \mathcal{K}_x_n, \mathcal{U}_y \}.
\]
Notice that, by construction:
\[ \{ dR = 0 \} = \text{Span} \{ \mathcal{X}_i, \mathcal{L}_a \}. \]
a property that is independent of the choice of a defining function \( R \) for the submanifold of solutions, cf. [19]; this is clear with either \( R := -y + Q(x, a, b) \) or \( R := -b + P(a, x, y) \), since in this case:
\[ 0 = dR(\mathcal{X}_i) = dR(\mathcal{L}_a). \]
It is also clear that:
\[ dR(\mathcal{T}_b) \neq 0 \quad \text{and} \quad dR(\mathcal{U}_y) \neq 0, \]
which can also be seen while making in advance the linear normalization (4.2) which implies:
\[ dR = dy + db + O(1). \]
Consequently, the analytic differential 1-forms on \( \mathcal{M} \):
\[ \rho := d \left( \frac{R}{dR(\mathcal{T}_b)} \right) \big|_{\mathcal{M}} \quad \text{and} \quad \sigma := d \left( \frac{R}{dR(\mathcal{U}_y)} \right) \big|_{\mathcal{M}} \]
satisfy:
\[ \rho(\mathcal{T}_b) \equiv 1 \quad \text{and} \quad \sigma(\mathcal{U}_y) \equiv 1. \]

8 Nonholonomy levi bilinear forms \( \text{Levi}_{\text{var}} \) and \( \text{Levi}_{\text{par}} \)

On any CR manifold, the so-called Levi form is a CR-invariant Hermitian form on the CR bundle. Locally, it is represented by a square matrix, equal to its transposed conjugate, which means that the other Hermitian form on the conjugate CR bundle coincides with it.

On a submanifold of solutions, by analogy, there is not just one (generalized) Levi form, but two. Similarly, their matrices, not necessarily square, are transposed of one another (up to a nowhere vanishing factor), so that their ranks and kernels are anyway essentially the same.

However, we will see later that some higher order jet invariants of submanifolds of solutions can differ strongly, when seen either from the variables space, or from the parameters space.

General sections of \( T^{\text{var}} \mathcal{M} \) and of \( T^{\text{par}} \mathcal{M} \) decompose along the chosen frames as:
\[ \mathcal{X} = \mu_1 \mathcal{X}_{x_1} + \cdots + \mu_n \mathcal{X}_{x_n}, \]
\[ \mathcal{L} = \nu_1 \mathcal{L}_{a_1} + \cdots + \nu_m \mathcal{L}_{a_m}, \]
in terms of certain arbitrary analytic functions \( \mu_i \) and \( \nu_j \).

The Levi form of \( \mathcal{M} \) can be intrinsically defined as taking Lie brackets modulo the sum of the two invariant distributions:
\[ \Gamma(\mathcal{T}^{\text{var}} \mathcal{M}) \times \Gamma(\mathcal{T}^{\text{par}} \mathcal{M}) \rightarrow \Gamma(T \mathcal{M}) / \text{mod} \left( \Gamma(\mathcal{T}^{\text{var}} \mathcal{M}) \oplus \Gamma(\mathcal{T}^{\text{par}} \mathcal{M}) \right) \]
\[ (\mathcal{X}, \mathcal{L}) \mapsto [\mathcal{X}, \mathcal{L}] \text{mod} \left( \mathcal{T}^{\text{var}} \mathcal{M} \oplus \mathcal{T}^{\text{par}} \mathcal{M} \right), \]
but it is preferable, in order to fix ideas about such a moding out, to employ some differential 1-forms like \( \rho \) and \( \sigma \) above whose kernel distributions represent exactly this direct sum.
\( T \var M \oplus T \par M \), the result being independent, up to a nowhere vanishing factor, of such a choice.

Then two versions \( \text{Levi}_{\var} \) and \( \text{Levi}_{\par} \) of this Levi form exist, both of which will be useful later.

Firstly, work in coordinates \((x, a, b)\). For \(1 \leq i \leq n\) and for \(1 \leq j \leq m\), compute the basic Lie brackets:
\[
\begin{bmatrix} \mathcal{X}_{x_i} \mid \mathcal{L}_{a_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial a_j} - \frac{Q_{a_j}}{Q_b} \frac{\partial}{\partial b} \end{bmatrix} = \begin{bmatrix} -\frac{Q_b Q_{x_i a_j} + Q_{a_j} Q_{x_i b}}{Q_b Q_b} \frac{\partial}{\partial b} \end{bmatrix}.
\]
With the 1-form \( \rho \) introduced above, compute then:
\[
\rho \left( \begin{bmatrix} \mathcal{X}_{x_1}, \ldots, \mathcal{X}_{x_n} \mid \mathcal{L}_{a_1}, \ldots, \mathcal{L}_{a_m} \end{bmatrix} \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_i \nu_j \rho \left( \begin{bmatrix} \mathcal{X}_{x_i} \mid \mathcal{L}_{a_j} \end{bmatrix} \right)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_i \nu_j - \frac{Q_b Q_{x_i a_j} + Q_{a_j} Q_{x_i b}}{Q_b Q_b},
\]
so that by introducing the \( n \times m \) matrix:
\[
\text{Levi}_{\par}(Q) := \begin{bmatrix} -\frac{Q_b Q_{x_1 a_1} + Q_{a_1} Q_{x_1 b}}{Q_b Q_b} & \cdots & -\frac{Q_b Q_{x_1 a_m} + Q_{a_m} Q_{x_1 b}}{Q_b Q_b} \\
\vdots & \ddots & \vdots \\
-\frac{Q_b Q_{x_n a_1} + Q_{a_1} Q_{x_n b}}{Q_b Q_b} & \cdots & -\frac{Q_b Q_{x_n a_m} + Q_{a_m} Q_{x_n b}}{Q_b Q_b} \end{bmatrix},
\]
it comes in terms of the row vectors \( \tau \nu = (\nu_1, \ldots, \nu_m) \) and \( \tau \mu = (\mu_1, \ldots, \mu_n) \):
\[
\rho \left( \begin{bmatrix} \mathcal{X} \mid \mathcal{L} \end{bmatrix} \right) = \tau \nu \cdot \text{Levi}_{\par}(Q) \cdot \tau \mu.
\]

Secondly, work in coordinates \((a, x, y)\). The basic Lie brackets become:
\[
\begin{bmatrix} \mathcal{L}_{a_j} \mid \mathcal{X}_{x_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial a_j}, \frac{\partial}{\partial x_j} - \frac{P_{x_j}}{P_y} \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{P_y P_{a_j x_j} + P_{x_j} P_{a_j y}}{P_y P_y} \frac{\partial}{\partial y} \end{bmatrix},
\]
hence:
\[
\sigma \left( \begin{bmatrix} \mathcal{L}_{a_1} + \cdots + \mathcal{L}_{a_m}, \mu_1 \mathcal{X}_{x_1} + \cdots + \mu_n \mathcal{X}_{x_n} \end{bmatrix} \right) = \sum_{j=1}^{m} \sum_{i=1}^{n} \nu_j \mu_i - \frac{P_y P_{a_j x_j} + P_{x_j} P_{a_j y}}{P_y P_y},
\]
so that with the \( m \times n \) matrix:
\[
\text{Levi}_{\var}(P) := \begin{bmatrix} -\frac{P_y P_{a_1 x_1} + P_{x_1} P_{a_1 y}}{P_y P_y} & \cdots & -\frac{P_y P_{a_1 x_n} + P_{x_n} P_{a_1 y}}{P_y P_y} \\
\vdots & \ddots & \vdots \\
-\frac{P_y P_{a_m x_1} + P_{x_1} P_{a_m y}}{P_y P_y} & \cdots & -\frac{P_y P_{a_m x_n} + P_{x_n} P_{a_m y}}{P_y P_y} \end{bmatrix},
\]
it comes analogously:
\[
\sigma \left( \begin{bmatrix} \mathcal{L} \mid \mathcal{X} \end{bmatrix} \right) = \tau \nu \cdot \text{Levi}_{\var}(P) \cdot \tau \mu.
\]
We now need to compare these two matrices of the two Levi forms. The result is that they are transposed one to another, up to a nowhere vanishing factor.
Lemma 8.1 For all indices $1 \leq i \leq n$ and $1 \leq j \leq m$, it holds identically on $\mathcal{M}$:

$$
- \frac{Q_b Q_{x_i a_j} + Q_{a_j} Q_{x_i b}}{Q_b Q_b} \equiv - P_y \left( \frac{-P_y P_{x_i a_j} + P_{x_i} P_{a_j y}}{P_y P_y} \right).
$$

**Proof** Start from the Functional Relations 6.1, rewritten in length as:

- $y \equiv Q(x_1, \ldots, x_n, a_1, \ldots, a_m, P(a_1, \ldots, a_m, x_1, \ldots, x_n, y))$,
- $b \equiv P(a_1, \ldots, a_m, x_1, \ldots, x_n, Q(x_1, \ldots, x_n, a_1, \ldots, a_m, b))$,

differentiate the first line with respect to $x_i$, to $a_j$, to $y$:

$$
0 \equiv Q_{x_i} + P_{x_i} Q_{b},
0 \equiv Q_{a_j} + P_{a_j} Q_{b},
1 \equiv P_y Q_{b},
$$

and solve:

$$
P_{a_j} = - \frac{Q_{a_j}}{Q_{b}},
Q_{b} = \frac{1}{P_y}.
$$

Next, differentiate up to order 2 with respect to $x_i a_j$, to $y a_j$:

$$
0 \equiv Q_{x_i a_j} + \hat{P}_{a_j} Q_{x_i b} + P_{x_i a_j} \hat{Q}_{b} + P_{x_i} (Q_{a_j b} + P_{a_j} Q_{bb}),
0 \equiv P_y Q_{a_j b} + P_{a_j y} \hat{Q}_{b} + P_{a_j} P_y Q_{bb},
$$

solve the underlined terms in the second equation, replace the result in the first equation, and replace the wide hats to conclude by multiplying at the end by $\frac{1}{Q_{b}} = P_y$:

$$
0 \equiv Q_{x_i a_j} - \frac{Q_{a_j}}{Q_{b}} Q_{x_i b} + P_{x_i a_j} \frac{1}{P_y} + P_{x_i} \left( - \frac{P_{a_j y}}{P_y} \frac{1}{P_y} \right) \\
\equiv \frac{Q_b Q_{x_i a_j} - Q_{a_j} Q_{x_i b}}{Q_b} + \frac{P_{x_i a_j} P_y - P_{x_i} P_{a_j y}}{P_y P_y}.
$$

With this uniform relation between the entries of the two matrices of the two Levi forms, equal to each other up to the nowhere vanishing factors:

$$
- P_y \neq 0 \quad \text{and} \quad - Q_b \neq 0,
$$

we obtain:

$$
\text{Levi}_{\text{par}}(Q) = - P_y^\top \text{Levi}_{\text{var}}(P) \iff - Q_b^\top \text{Levi}_{\text{par}}(Q) = \text{Levi}_{\text{var}}(P).
$$

9 Invariance of the nonholonomy bilinear forms $\text{Levi}_{\text{var}}$ and $\text{Levi}_{\text{par}}$

Consider an equivalence $(F, \Phi): \mathcal{M} \rightarrow \mathcal{M}'$ between two submanifolds of solutions, as in Sect. 5, hence satisfying:

$$
(F, \Phi)_*(T_{\text{var}} \mathcal{M}) = T_{\text{var}} \mathcal{M}',
(F, \Phi)_*(T_{\text{par}} \mathcal{M}) = T_{\text{par}} \mathcal{M}'.
$$
As in Sect. 6, consider two frames for $T \mathcal{M}$ and $T \mathcal{M}'$, expressed in terms of graphing functions $P$, $Q$ in coordinates $(x, y, a, b)$ and in terms of graphing functions $P'$, $Q'$ in coordinates $(x', y', a', b')$:

\[
\begin{align*}
\mathcal{K}_{x_1}, \ldots, \mathcal{K}_{x_1}, & \quad \mathcal{K}'_{x'_1}, \ldots, \mathcal{K}'_{x'_n}, \\
\mathcal{L}_{a_1}, \ldots, \mathcal{L}_{a_n}, & \quad \mathcal{L}'_{a'_1}, \ldots, \mathcal{L}'_{a'_m}, \\
\mathcal{T}_b, & \quad \mathcal{T}'_{b'}, \\
\mathcal{U}_y, & \quad \mathcal{U}'_{y'}.
\end{align*}
\]

Dropping the symbol $(F, \Phi)^{-1}$ for the push-forward of vector fields under the inverse map $\mathcal{M} \leftarrow \mathcal{M}' : (F, \Phi)^{-1}$, we can therefore write:

\[
\begin{align*}
\mathcal{K}_{x_i} &= \sum_{1 \leq i' \leq n} X'_{i,i'} \mathcal{K}'_{x'_{i'}}, \\
\mathcal{L}_{a_j} &= \sum_{1 \leq j' \leq m} A'_{j,j'} \mathcal{L}'_{a'_{j'}},
\end{align*}
\]

in terms of certain two invertible $n \times n$ and $m \times m$ matrices of functions on $\mathcal{M}'$. Furthermore, there exists a nowhere vanishing function $c' : \mathcal{M}' \to \mathbb{K}$ such that, after (still unwritten) pull-back:

\[
\rho = c' \rho'.
\]

Then:

\[
\frac{-Q_b Q_{x_ia_j} + Q_{a_j} Q_{x_ib}}{Q_b Q_b} = \rho\left(\mathcal{K}_{x_i}, \mathcal{L}_{a_j}\right) = c' \rho' \left(\sum_{i'} X'_{i,i'} \mathcal{K}'_{x'_{i'}}, \sum_{j'} A'_{j,j'} \mathcal{L}'_{a'_{j'}}\right) = c' \sum_{i'} \sum_{j'} X'_{i,i'} \rho' \left(\mathcal{K}'_{x'_{i'}}, \mathcal{L}'_{a'_{j'}}\right) A'_{j,j'} = c' \sum_{i'} \sum_{j'} X'_{i,i'} \left(\frac{-Q_b' Q'_{x'_{i'}a'_{j'}} + Q_{a'_j} Q'_{x'_{i'}b'}}{Q_{b'} Q_{b'}}\right) A'_{j,j'},
\]

and the result writes in matrix form as:

\[
\begin{pmatrix}
\frac{-Q_b Q_{x_1a_1} + Q_{a_1} Q_{x_1b}}{Q_b Q_b} & \cdots & \frac{-Q_b Q_{x_1a_m} + Q_{a_m} Q_{x_1b}}{Q_b Q_b} \\
\vdots & \ddots & \vdots \\
\frac{-Q_b Q_{x_na_1} + Q_{a_1} Q_{x_nb}}{Q_b Q_b} & \cdots & \frac{-Q_b Q_{x_na_m} + Q_{a_m} Q_{x_nb}}{Q_b Q_b}
\end{pmatrix}
\begin{pmatrix}
X'_{1,1} & \cdots & X'_{1,n} \\
\vdots & \ddots & \vdots \\
X'_{n,1} & \cdots & X'_{n,n}
\end{pmatrix}
= c' \begin{pmatrix}
\frac{-Q'_b Q'_{x'_1a'_1} + Q'_{a'_1} Q'_{x'_1b'}}{Q_{b'} Q_{b'}} & \cdots & \frac{-Q'_b Q'_{x'_1a'_m} + Q'_{a'_m} Q'_{x'_1b'}}{Q_{b'} Q_{b'}} \\
\vdots & \ddots & \vdots \\
\frac{-Q'_b Q'_{x'_n a'_1} + Q'_{a'_1} Q'_{x'_nb'}}{Q_{b'} Q_{b'}} & \cdots & \frac{-Q'_b Q'_{x'_n a'_m} + Q'_{a'_m} Q'_{x'_nb'}}{Q_{b'} Q_{b'}}
\end{pmatrix}
\begin{pmatrix}
A'_{1,1} & \cdots & A'_{m,1} \\
\vdots & \ddots & \vdots \\
A'_{1,m} & \cdots & A'_{m,m}
\end{pmatrix},
\]

that is to say:

\[
\text{Levi}_{par}(Q) = c' \cdot X' \cdot \text{Levi}_{par}(Q') \cdot A'.
\]
Similarly, with $\sigma = d' \sigma'$:

$$Levi_{\text{var}}(P) = d' \cdot A' \cdot Levi_{\text{var}}(P') \cdot {^TX'}.$$ 

**Corollary 9.1** The common rank $r$ with $0 \leq r \leq \min(n, m)$ of the two matrices $Levi_{\text{par}}(Q)$ and $Levi_{\text{var}}(P)$ of the two Levi forms of $\mathcal{M}$ is independent of coordinates. 

## 10 Normalizations of $Levi_{\text{par}}$ and of $Levi_{\text{var}}$ in local coordinates

Once this basic invariant is at hand:

$$r := rank \ Levi_{\text{par}}(Q) = rank \ Levi_{\text{var}}(P),$$

it is interesting to perform normalizing changes of coordinates in order to view it in the equations of $\mathcal{M}$.

Assuming from the beginning that coordinates $(x, y, a, b)$ are normal in the sense of Proposition 4.1, when expanding in power series:

$$y = b + \sum_{k=0}^{\infty} Q_k(x, a) \ b^k = b + O_{x,a,b}(2),$$

it then comes:

$$0 = Q_k(0, a) = Q_k(x, 0) \quad (\forall k \geq 0),$$

whence:

$$Q_k(x, a) = O_{x,a}(2) \quad (\forall k \geq 0).$$

By specifying homogeneous terms of degree 2 in $Q_0(x, a)$, observing that $b^k \ Q_k(x, a) = b \ O_{x,a}(2)$ for all $k \geq 1$, we can write the equation of $\mathcal{M}$ as:

$$y = b + \Lambda(x, a) + O_{x,a}(3) + b \ O_{x,a,b}(2),$$

with a certain bilinear form:

$$\Lambda(x, a) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{i,j} x_i a_j,$$

which, as is easily devised, represents $Levi_{\text{par}}(Q)$ at the origin.

Indeed, a computation of the entries of this matrix:

$$\frac{-Q_b Q_{x,a} + Q_{a} Q_{x,b}}{Q_b Q_b} = \frac{-(1 + O_2) (\lambda_{i,j} + O_1) + O_1 O_1}{(1 + O_1)^2} = -\lambda_{i,j} + O_{x,a,b}(1),$$

confirms that at the origin, up to an innocuous overall minus sign:

$$Levi_{\text{par}}(Q) = \left( - \lambda_{i,j} \right)_{1 \leq i \leq n, 1 \leq j \leq m}.$$
Lemma 10.1 Any $\mathbb{K}$-bilinear form on $\mathbb{K}^n \times \mathbb{K}^m$ of rank $0 \leq r \leq \min(n, m)$:

$$\Lambda(x, a) = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \lambda_{i,j} x_i a_j,$$

can be brought, by means of some linear transformation in $\text{Diff}_x \times \text{Diff}_a$, to the form:

$$x_1 a_1 + \cdots + x_r a_r.$$

Proof Rewrite it as:

$$\Lambda(x, a) = \sum_{i=1}^n x_i \Lambda_i(a_1, \ldots, a_r) \quad \text{with} \quad \Lambda_i(a) := \sum_{j} \lambda_{i,j} a_j.$$

Among these $n$ linear forms $\Lambda_1, \ldots, \Lambda_n$, a certain maximal number $r$ are linearly independent, with of course $r$ being the rank in question. After a renumeration, $\Lambda_1, \ldots, \Lambda_r$ are independent, hence can be taken as new coordinates $a_1 := \Lambda_1, \ldots, a_r := \Lambda_r$, in terms of which the remaining linear forms therefore express:

$$\Lambda_{r+1}(a_1, \ldots, a_r), \ldots, \Lambda_n(a_1, \ldots, a_r),$$

whence a clever (easy) reorganization:

$$\Lambda(x, a) = x_1 a_1 + \cdots + x_r a_r + x_{r+1}(\lambda_{r+1,1} a_1 + \cdots + \lambda_{r+1,r} a_r) +$$

$$+ \cdots + x_n(\lambda_{n,1} a_1 + \cdots + \lambda_{n,r} a_r)$$

$$= (x_1 + \lambda_{r+1,1} x_{r+1} + \cdots + \lambda_{n,1} x_n) a_1 +$$

$$\cdots + (x_r + \lambda_{r+1,r} x_{r+1} + \cdots + \lambda_{n,r} x_n) a_r$$

$$=: x_1 a_1 + \cdots + x_r a_r,$$

yielding an obvious linear change of coordinates in the $x$-space, concludes.

Corollary 10.2 A submanifold of solutions $\mathcal{M} \subset \mathbb{K}^{n+1} \times \mathbb{K}^{m+1}$ whose Levi forms at the origin have (always equal) rank $0 \leq r \leq \min(n, m)$ can always be represented, in suitable normalized coordinates $(x, y, a, b)$, as:

$$y = b + x_1 a_1 + \cdots + x_r a_r + O_{x,a}(3) + b O_{x,a,b}(2),$$

and equivalently as:

$$b = y - x_1 a_1 - \cdots - x_r a_r + O_{a,x}(3) + y O_{a,x,y}(2).$$

Proof An application of the implicit function theorem shows that $P(a, x, y)$ is indeed of this form once $Q(x, a, b)$ has been normalized.

11 Levi forms and order 1 jets of invariant leaves

In CR geometry, and in integrable PDE geometry as well, the Levi form is a too elementary invariant, insufficient to even characterize some initial data before launching the search for deeper invariants by means of Cartan’s method of equivalence.

A wealth of results in CR geometry, based on finer invariants that enjoy natural transfer to PDE geometry, have not yet been set up in the literature (just a few of them appear in [14]).
Let us present a family of fundamental higher order invariants, and link them with the Levi form(s) before going further.

Knowing, from Sect. 5, that the two foliations:

\[ F_{\text{var}} := \bigcup_{a,b} \{(x, y) : y = Q(x, a, b)\} , \]
\[ F_{\text{par}} = \bigcup_{x,y} \{(a,b) : b = P(a, x, y)\} , \]

are invariant under equivalences in Diff_{\text{var}} \times Diff_{\text{par}}, the main idea is to look at their leaves:

\[ D_{a,b} = \{(x, y) : y = Q(x, a, b)\} , \]
\[ D_{x,y} = \{(a,b) : b = P(a, x, y)\} , \]

considered as parametrized by \( x \) and by \( a \), and then, for every integer \( k \geq 0 \) and every integer \( l \geq 0 \), to introduce the two fundamental collections of jets-of-leaves maps:

\[ J^k_x D_{a,b} := (x, (\partial_\beta^x Q(x, a, b))_{|\beta| \leq k}) , \]
\[ J^l_{a} D_{x,y} := (a, (\partial_\gamma^a P(a, x, y))_{|\gamma| \leq l}) , \]

using multiindex notation:

\[ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n , \quad |\beta| := \beta_1 + \cdots + \beta_n , \]
\[ \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m , \quad |\gamma| := \gamma_1 + \cdots + \gamma_m . \]

In the next Sect. 12, we will show rigorously that these two maps are invariants of submanifolds of solutions considered modulo transformations in the pseudogroup Diff_{\text{var}} \times Diff_{\text{par}}, but before doing this, to make a natural transition with what precedes, let us look at the special case of order 1 jets:

\[ k = 1 = l . \]

Since the first entry \( x \) of the order 1 jet map \( J^1_x D_{a,b} \) does not depend on \( (a, b) \), its rank amounts to the rank of the ‘submap’:

\[ J^1_{\text{var}} Q : (a, b) \mapsto (Q, Q_{x1}, \ldots, Q_{xn})(x, a, b) \]

in which \( x \) is fixed—and whose Jacobian \((n + 1) \times (m + 1)\) matrix is just:

\[ \text{Jac} (J^1_{\text{var}} Q) = \begin{pmatrix} Q_{a1} & \cdots & Q_{am} & Q_{b} \\ Q_{x1a1} & \cdots & Q_{x1am} & Q_{x1b} \\ \vdots & \vdots & \vdots & \vdots \\ Q_{xn a1} & \cdots & Q_{xna m} & Q_{xnb} \end{pmatrix} . \]

**Question 2** Can this \( \text{Jac} (J^1_{\text{var}} Q) \) be compared with the matrix:

\[ \text{Levi}_{\text{par}}(Q) = \begin{pmatrix} -Q_b Q_{x1a1} + Q_{a1} Q_{x1b} & \cdots & -Q_b Q_{x1a m} + Q_{a m} Q_{x1b} \\ Q_b Q_{b} & \cdots & Q_b Q_{b} \\ \vdots & \vdots & \vdots \\ -Q_b Q_{xn a1} + Q_{a1} Q_{xnb} & \cdots & -Q_b Q_{xn a m} + Q_{a m} Q_{xnb} \end{pmatrix} ? \]
Yes! It suffices, since $Q_b \neq 0$ vanishes nowhere by assumption, to divide all entries of $\text{Jac}(J^1_{\text{var}} Q)$ by $Q_b$ and then to perform obvious row operations which leave the rank unchanged:

\[
\text{Jac}(J^1_{\text{var}} Q) \to Q_b \begin{pmatrix}
\frac{Q_{x_1}}{Q_b} & \cdots & \frac{Q_{x_m}}{Q_b} & 1 \\
\frac{Q_{x_1} a_1}{Q_b} & \cdots & \frac{Q_{x_1} a_m}{Q_b} & \frac{Q_{x_1} b}{Q_b} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{Q_{x_m} a_1}{Q_b} & \cdots & \frac{Q_{x_m} a_m}{Q_b} & \frac{Q_{x_m} b}{Q_b}
\end{pmatrix} \\
\to Q_b \begin{pmatrix}
\frac{Q_{x_1}}{Q_b} & \cdots & \frac{Q_{x_m}}{Q_b} & 1 \\
\frac{Q_{x_1} a_1}{Q_b} & \cdots & \frac{Q_{x_1} a_m}{Q_b} & \frac{Q_{x_1} b}{Q_b} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{Q_{x_m} a_1}{Q_b} & \cdots & \frac{Q_{x_m} a_m}{Q_b} & \frac{Q_{x_m} b}{Q_b}
\end{pmatrix}
\]

\[
, = Q_b \begin{pmatrix}
* & 1 \\
-\text{Levi}_{\text{var}}(Q) & 0
\end{pmatrix},
\]

to realize that the rank of this order 1 jet map coincides with the rank of the Levi form!

Of course, a similar procedure can also be applied to the map:

\[
J^1_{\text{par}} P : (x, y) \mapsto (P, P_{a_1}, \ldots, P_{a_m})(a, x, y)
\]

having $(m + 1) \times (n + 1)$ Jacobian matrix:

\[
\text{Jac}(J^1_{\text{par}} P) = \begin{pmatrix}
P_{x_1} & \cdots & P_{x_n} & P_y \\
P_{a_1 x_1} & \cdots & P_{a_1 x_n} & P_{a_1 y} \\
\vdots & \vdots & \vdots & \vdots \\
P_{a_m x_1} & \cdots & P_{a_m x_n} & P_{a_m y}
\end{pmatrix} \\
\to P_y \begin{pmatrix}
* & 1 \\
-\text{Levi}_{\text{par}}(P) & 0
\end{pmatrix}.
\]

**Corollary 11.1** At every point $(x, y, a, b) \in \mathcal{M}$:

\[
\text{rank } (J^1_{\text{var}} Q) = 1 + \text{rank } (\text{Levi}_{\text{var}}(Q)) = 1 + \text{rank } (\text{Levi}_{\text{par}}(P)) = \text{rank } (J^1_{\text{par}} P),
\]
or equivalently:

\[
\text{rank } (J^1_{\text{par}} \mathcal{D}_{a,b}) = n + 1 + \text{rank } (\text{Levi}_{\text{var}}(Q)),
\]
\[
\text{rank } (J^1_{\text{par}} \mathcal{P}_{x,y}) = m + 1 + \text{rank } (\text{Levi}_{\text{par}}(P)).
\]

\[
\square
\]

12 Higher order jets of the pair of invariant leaves

Having justified the interest of jets by linking them with the (too) classical concept(s) of Levi form(s), let us now study our two favorite jet maps\textsuperscript{1} of arbitrary orders:

\textsuperscript{1} In a CR context, partial aspects of the topics presented in this Sect. 12 and in the next Sects. 13, 14, 15, 16, 17, have already been studied in [13,19,20].
\[ J^k_x \mathcal{P}_{a,b} := \left( x, \left( \partial^k_x Q(x,a,b) \right)_{|\beta| \leq k} \right), \]
\[ J^l_a \mathcal{P}_{x,y} := \left( a, \left( \partial^l_y P(a,x,y) \right)_{|y| \leq l} \right), \]

which, for \( k = 0 = l \), reduce plainly to:
\[ (x, a, b) \mapsto (x, Q(x,a,b)), \]
\[ (a, x, y) \mapsto (a, P(a,x,y)). \]

The goal is to understand in a precise manner their invariancy under the pseudogroup of allowed equivalences:
\[ (x, y, a, b) \mapsto (f(x,y), g(x,y), \varphi(a,b), \psi(a,b)) \]
\[ =: (x', y', a', b'), \]

which send \( \mathcal{M} \) into another \( \mathcal{M}' \) represented by similar equivalent equations:
\[ y' = Q'(x', a', b'), \]
\[ b' = P'(a', x', y'). \]

For jets of order \( k = 0 = l \), the task is easy.

**Observation 12.1** The two maps:
\[ (x, y) \xrightarrow{R^0_{f,g}} (f(x,y), g(x,y)), \]
\[ (a, b) \xrightarrow{S^0_{\varphi,\psi}} (\varphi(a,b), \psi(a,b)), \]

make commutative the two diagrams:
\[ \xymatrix{ \mathcal{M} \ar[r]^{(f,g,\varphi,\psi)} & \mathcal{M}' \ar[d]^{J^0_x \mathcal{P}_{a',b'}} \ar[r]^{J^0_y \mathcal{P}_{x',y'}} & \mathcal{M}' \ar[d]^{S^0_{\varphi,\psi}} \ar[r] & \mathcal{M}' \ar[d]^{S^0_{\varphi,\psi}} \ar[r] & \mathcal{M}' } \]

**Proof.** By symmetry, we can treat only the first diagram. The up \( \circ \) right composition is:
\[ (x, a, b) \xrightarrow{R^0_{f,g}} (f(x, Q(x,a,b)), \varphi(a,b), \psi(a,b)) \]
\[ \downarrow \]
\[ (f(x, Q(x,a,b)), \varphi(a,b), \psi(a,b)) \]
\[ \downarrow \]
\[ \left( f(x, Q(x,a,b)), Q'(f(x, Q(x,a,b)), \varphi(a,b), \psi(a,b)) \right), \]

while the down \( \circ \) right one is:
\[ (x, a, b) \]
\[ \downarrow \]
\[ (x, Q(x,a,b)) \xrightarrow{R^0_{f,g}} (f(x, Q(x,a,b)), g(x, Q(x,a,b))), \]

\[ \text{ Springer} \]
and the coincidence of the two bottom-right results comes instantly from the assumption $(f, \varphi, \psi)(\mathcal{M}) \subseteq \mathcal{M}'$ which writes out as the following identity in $\mathbb{C}\{x, a, b\}$:

$$g(x, Q(x, a, b)) \equiv Q'(f(x, Q(x, a, b)), \varphi(a, b), \psi(a, b)).$$

\[\square\]

Similar pairs of commutative diagrams exist for jets $k$ and $l$ of arbitrary orders. To present them, some preliminaries are needed.

**Lemma 12.2** For every equivalence $(f, g, \varphi, \psi) \in \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}}$ between two submanifolds of solutions $\mathcal{M} \rightarrow \mathcal{M}'$, one has the two everywhere nonvanishings:

$$0 \neq \Delta(f, Q) := \det \left( \left( f_{i', x_i} + Q_{x_i} f_{i', y} \right)_{1 \leq i \leq n} \right),$$

$$0 \neq \Delta(\varphi, P) := \det \left( \left( \varphi_{j', a_j} + P_{a_j} \varphi_{j', b} \right)_{1 \leq j \leq n} \right).$$

**Proof.** By symmetry, we can focus on just $\Delta(f, Q)$.

We know that the restriction of $(F, \Phi)$ to the $n$-dimensional leaf in $\mathcal{M}$:

$$\mathcal{L}_{a, b} = \{(x, y) : y = Q(x, a, b)\},$$

maps it into the corresponding $n$-dimensional leaf in $\mathcal{M}'$:

$$(F, \Phi)(\mathcal{L}_{a, b}) \subseteq \mathcal{L}_{\varphi(a, b), \psi(a, b)}.$$ It is well known that the restriction of any (local) diffeomorphism to a submanifold becomes a diffeomorphism onto its (submanifold) image, hence we receive an analytic diffeomorphism:

$$(F, \Phi)|_{\mathcal{L}_{a, b}} : \mathcal{L}_{a, b} \sim \mathcal{L}_{\varphi(a, b), \psi(a, b)}.$$

But since $x'$ is a natural coordinate on all the $\mathcal{L}'_{a', b'}$, which are graphed as $y' = Q'(x', a', b')$, and since $x$ parametrizes all the $\mathcal{L}_{a, b}$, the restricted map in question identifies with the bottom map below:

$$(x, Q(x, a, b), a, b) \rightarrow (f(x, Q(x, a, b)), g(x, Q(x, a, b)), \varphi(a, b), \psi(a, b))$$

$$x \rightarrow f(x, Q(x, a, b)),

\frac{\partial}{\partial x_i} \left[f_{i'}(x, Q(x, a, b))\right] = f_{i', x_i} + Q_{x_i} f_{i', y}.$$

\[\square\]

The fact that $(f, g, \varphi, \psi) : \mathcal{M} \sim \mathcal{M}'$ is an equivalence expresses as two pairs of equivalent identities:

$$g(x, Q(x, a, b)) \equiv Q'(f(x, Q(x, a, b)), \varphi(a, b), \psi(a, b)) \quad \text{(in } \mathbb{C}\{x, a, b\}),$$

$$g(x, y) \equiv Q'(f(x, y), \varphi(a, P(a, x, y)), \psi(a, P(a, x, y))) \quad \text{(in } \mathbb{C}\{a, x, y\}),$$

$$\psi(a, P(a, x, y)) \equiv P'(\varphi(a, P(a, x, y)), f(x, y), g(x, y)) \quad \text{(in } \mathbb{C}\{a, x, y\}),$$

$$\psi(a, b) = P'(\varphi(a, b), f(x, Q(x, a, b)), g(x, Q(x, a, b))) \quad \text{(in } \mathbb{C}\{x, a, b\}).$$

\[\square\]
By repeatedly differentiating these identities and then reorganizing the outcomes, we will soon be in a position to realize the invariancy of the two jet maps under the form of an appropriate statement. For that purpose, let us introduce some independent jet-coordinates:

\[(y_{x}^\beta)_{|\beta| \leq k} \quad \text{and} \quad (b_{a^r})_{|\gamma| \leq l},\]

corresponding to \(y\) and \(b\) being considered as functions of \(x\) and \(a\), a point of view which will also be useful later on when we will introduce partial differential equations associated to submanifolds of solutions. The two jet maps take therefore their values into the two jet spaces:

\[J^k_x \mathcal{Q}_{a,b} \in \mathbb{K}^{n+\frac{(n+k)l}{m+k}} = \{(x, (y_{x}^\beta)_{|\beta| \leq k})\},\]

\[J^l_a \mathcal{P}_{x,y} \in \mathbb{K}^{m+\frac{(m+l)l}{m+l}} = \{(a, (b_{a^r})_{|\gamma| \leq l})\}.

Lastly, let us employ abbreviated notations for jets:

\[f^k_s Q := (\partial^\beta_s Q)_{|\beta| \leq k}, \quad j^l_a P := (\partial^\alpha_a P)_{|\gamma| \leq l},\]

\[j^l_a \phi := (\partial^\gamma a^l \phi)_{|\gamma|+l \leq k}, \quad j^l_a \phi := (\partial^\alpha a^l \phi)_{|\beta| \leq k} \quad \text{and} \quad j^l_a \phi := (\partial^\beta a^l \phi)_{|\beta| \leq k}.

**Theorem 12.3** For each multiindex \(\beta' \in \mathbb{N}^n\) with \(1 \leq |\beta'|\), and for each multiindex \(\gamma' \in \mathbb{N}^m\) with \(1 \leq |\gamma'|\), there exist two rational expressions satisfying identically in \(\mathbb{C}\{x, a, b\}\) and in \(\mathbb{C}\{a, x, y\}\):

\[
\begin{align*}
\text{Polynomial}_{\beta'}\left((\partial^\beta_s Q)_{1 \leq |\beta| \leq |\beta'|}, (\partial^\alpha a^l \phi)_{1 \leq |\gamma| \leq |\gamma'|}\right) & \equiv Q^\beta_{x,y}(f, \phi, \psi), \\
\text{Polynomial}_{\gamma'}\left((\partial^\alpha a^l \phi)_{1 \leq |\beta| \leq |\beta'|}, (\partial^\gamma a^l \phi)_{1 \leq |\gamma| \leq |\gamma'|}\right) & \equiv P^\gamma_{a^r}(\phi, \psi, f, g).
\end{align*}
\]

Moreover, for any two integers \(k \geq 0\) and \(l \geq 0\), the two maps:

\[R^k_{f,g} : \quad \mathbb{K}^{n+\frac{(n+k)l}{m+k}} \rightarrow \mathbb{K}^{n+\frac{(n+k)l}{m+k}},\]

\[S^l_{\phi,\psi} : \quad \mathbb{K}^{m+\frac{(m+l)l}{m+l}} \rightarrow \mathbb{K}^{m+\frac{(m+l)l}{m+l}},\]

defined by:

\[R^k_{f,g}(x, y, (y_{x}^\beta)_{1 \leq |\beta| \leq k}) := \left(f(x, y), g(x, y), \text{Polynomial}_{\beta'}\left((x, y, (y_{x}^\beta)_{1 \leq |\beta| \leq |\beta'|}, (\partial^\alpha a^l \phi)_{1 \leq |\gamma| \leq |\gamma'|}\right)_{1 \leq |\beta| \leq k} \right) \right),\]

\[S^l_{\phi,\psi}(a, b, (b_{a^r})_{1 \leq |\gamma| \leq l}) := \left(\phi(a, b), \psi(a, b), \text{Polynomial}_{\gamma'}\left((a, b, (b_{a^r})_{1 \leq |\gamma| \leq |\gamma'|}\right)_{1 \leq |\gamma| \leq l} \right).\]
make commutative the two diagrams:

\[
\begin{align*}
\mathcal{M} & \xrightarrow{(f, g, \varphi, \psi)} \mathcal{M}' \\
\mathbb{K}^n & \xrightarrow{J^k_{x,y}} \mathbb{K}^{n+m} \\
\mathbb{K}^{n+m} & \xrightarrow{R^k_{f,g}} \mathbb{K}^{n+m} \\
\mathcal{M}' & \xleftarrow{(\varphi, \psi, f, g)} \mathcal{M}'
\end{align*}
\]

**Proof** By symmetry, we can focus on the first statement, which has already been explained above for \( k = 0 \).

For simplicity, let us treat the case \( k = 1 \). We start from the identity:

\[
g(x, Q(x, a, b)) = Q'(f(x, Q(x, a, b)), \varphi(a, b), \psi(a, b)),
\]

which we differentiate with respect to the \( x_i \) for all \( 1 \leq i \leq n \):

\[
g_{x_i} + Q_{x_i} g_y = \sum_{1 \leq i' \leq n} \left( f_{i',x_i} + Q_{x_i} f_{i',y} \right) Q'_{x_i}.
\]

(12.4)

This amounts to apply the already seen first-order differentiation operators:

\[
\mathcal{H}_{x_i} := \frac{\partial}{\partial x_i} + Q_{x_i} \frac{\partial}{\partial y},
\]

namely in matrix form:

\[
\begin{pmatrix}
\mathcal{H}_{x_1}(g) \\
\vdots \\
\mathcal{H}_{x_n}(g)
\end{pmatrix} =
\begin{pmatrix}
\mathcal{H}_{x_1}(f_1) & \cdots & \mathcal{H}_{x_1}(f_n) \\
\vdots & \ddots & \vdots \\
\mathcal{H}_{x_n}(f_1) & \cdots & \mathcal{H}_{x_n}(f_n)
\end{pmatrix}
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]

Since Lemma 12.2 guarantees that this \( n \times n \) matrix is invertible, we deduce:

\[
\begin{pmatrix}
\mathcal{H}_{x_1}(f_1) & \cdots & \mathcal{H}_{x_1}(f_n)
\end{pmatrix}^{-1}
\begin{pmatrix}
\mathcal{H}_{x_1}(g) \\
\vdots \\
\mathcal{H}_{x_n}(g)
\end{pmatrix} =
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]

For instance, when \( n = 1 \), this writes simply as:

\[
\frac{g_x + Q_x g_y}{f_x + Q_x f_y} = Q'_{x}.
\]

If we therefore define:

\[
R^1_{f,g}(x, y, y_{x_1}, \ldots, y_{x_n}) =: (x', y', y'_{x_1}', \ldots, y'_{x_n}'),
\]

with:

\[
x' := f(x, y),
\]

\[
y' := g(x, y),
\]

and with:

\[
\begin{pmatrix}
y'_{x_1} \\
\vdots \\
y'_{x_n}
\end{pmatrix} :=
\begin{pmatrix}
f_{1,x_1} + y_{x_1} f_y & \cdots & f_{n,x_1} + y_{x_1} f_{n,y} \\
\vdots & \ddots & \vdots \\
f_{1,x_n} + y_{x_n} f_y & \cdots & f_{n,x_n} + y_{x_n} f_{n,y}
\end{pmatrix}^{-1}
\begin{pmatrix}
g_{x_1} + y_{x_1} g_y \\
\vdots \\
g_{x_n} + y_{x_n} g_y
\end{pmatrix},
\]

\[
\square
g_{x_1} + Q_{x_1} g_y
\]

\[
\frac{f_x + Q_x f_y}{f_x + Q_x f_y}
\]

\[
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]

\[
\begin{pmatrix}
\mathcal{H}_{x_1}(f_1) & \cdots & \mathcal{H}_{x_1}(f_n) \\
\vdots & \ddots & \vdots \\
\mathcal{H}_{x_n}(f_1) & \cdots & \mathcal{H}_{x_n}(f_n)
\end{pmatrix}
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]

\[
\begin{pmatrix}
\mathcal{H}_{x_1}(f_1) & \cdots & \mathcal{H}_{x_1}(f_n)
\end{pmatrix}^{-1}
\begin{pmatrix}
\mathcal{H}_{x_1}(g) \\
\vdots \\
\mathcal{H}_{x_n}(g)
\end{pmatrix} =
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]

\[
\begin{pmatrix}
\mathcal{H}_{x_1}(f_1) & \cdots & \mathcal{H}_{x_1}(f_n)
\end{pmatrix}^{-1}
\begin{pmatrix}
\mathcal{H}_{x_1}(g) \\
\vdots \\
\mathcal{H}_{x_n}(g)
\end{pmatrix} =
\begin{pmatrix}
Q'_{x_1} \\
\vdots \\
Q'_{x_n}
\end{pmatrix}.
\]
a check left shows that commutativity of the 1-jet diagram:

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^1_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f, g, \varphi, \psi) \\
J^1_x \mathcal{D}_{a',b'} \\
\downarrow J^1_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}
\]

holds true, by virtue of the identity (12.4).

The construction of the maps \( R^k_{f,g} \) for higher jet orders \( k \) proceeds by induction.

### 13 Invariances of jet mappings ranks

Now, suppose that we have a third submanifold of solutions \( \mathcal{M}'' \) represented by:

\[
y'' = Q''(x'', a'', b'') \quad \text{or} \quad b'' = P''(a'', x'', y''),
\]

and that we have a sequence of 2 composable equivalences:

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f', g', \varphi', \psi') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

and let us write the second map as:

\[
f''(x', y') =: x'', \quad g''(x', y') =: y'', \quad \varphi''(a', b') =: a'', \quad \psi''(a', b') =: b''.
\]

Denote the composition of the two maps as:

\[
(f'', g'', \varphi'', \psi'') := (f', g', \varphi', \psi') \circ (f, g, \varphi, \psi),
\]

that is to say:

\[
\begin{align*}
f''(x, y) &= f'(f(x, y), g(x, y)), \\
g''(x, y) &= g'(f(x, y), g(x, y)), \\
\varphi''(a, b) &= \varphi'(\varphi(a, b), \psi(a, b)), \\
\psi''(a, b) &= \psi'(\varphi(a, b), \psi(a, b)).
\end{align*}
\]

The naturality of the above construction yields:

**Lemma 13.1** *In the composition diagram:*

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f, g, \varphi, \psi) \\
J^1_x \mathcal{D}_{a',b'} \\
\downarrow J^1_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^1_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^1_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f'', g'', \varphi'', \psi'') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f', g', \varphi', \psi') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f'', g'', \varphi'', \psi'') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f', g', \varphi', \psi') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow J^k_x \mathcal{D}_{a,b} \\
\mathbb{R}^{n+1+n}
\end{array}
\quad \begin{array}{c}
(f'', g'', \varphi'', \psi'') \\
J^k_x \mathcal{D}_{a',b'} \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\end{array} \\
\mathcal{M}' \\
\downarrow J^k_x \mathcal{D}_{a',b'} \\
\mathbb{R}^{n+1+n}
\]

\[
\mathcal{M}'' \\
\downarrow J^k_x \mathcal{D}_{a,b'} \\
\mathbb{R}^{n+1+n}
\]
one has:

\[ R^k_{f',g'} = R^k_{f,g'} \circ R^k_{f,g}.\]

As an application, take the inverses:

\[ (f', g') := (f, g)^{-1} \quad \text{and} \quad (\varphi', \psi') := (\varphi, \psi)^{-1}. \]

Since an elementary check convinces that:

\[ \text{Id} = R^k_{\text{Id}} = R^k_{(f,g)}^{-1} \circ R^k_{f,g}, \]

we deduce that \( R^k_{f,g} \) is also an equivalence (an invertible map):

\[ \mathbb{K}^{n + \frac{(n+k)!}{m!k!}} \rightarrow \mathbb{K}^{n + \frac{(n+k)!}{m!k!}}. \]

**Corollary 13.2** For every equivalence \( \mathcal{M} \sim \mathcal{M}' \), and for every integers \( k \geq 0 \) and \( l \geq 0 \), the ranks at all points of the two pairs of jet maps:

\[
\begin{align*}
&\mathcal{M} \xrightarrow{J^k_{a,b}} \mathcal{M}' \\
&\mathcal{M} \xrightarrow{J^l_{a,b}} \mathcal{M}'
\end{align*}
\]

are the same:

\[
\begin{align*}
\text{rank} \left( J^k_{x} Q_{a,b} \right) &= \text{rank} \left( J^k_{f(x,y)} Q'_{\varphi(a,b), \psi(a,b)} \right), \\
\text{rank} \left( J^l_{a} P_{x,y} \right) &= \text{rank} \left( J^l_{\varphi(a,b)} P'_{f(x,y),g(x,y)} \right),
\end{align*}
\]

and their generic ranks:

\[
\text{genrank} \left( J^k_{x} Q_{a,b} \right) = \text{genrank} \left( J^k_{x} Q_{a,b} \right) \quad \text{and} \quad \text{genrank} \left( J^l_{a} P_{x,y} \right) = \text{genrank} \left( J^l_{a} P_{x,y} \right),
\]

are also identical.

These ranks are therefore invariants of the para-CR structures under study.

Another basic fact is a stabilization property.

**Proposition 13.3** If, for some integer \( k \geq 1 \):

\[ \text{genrank} \left( J^k_{x} Q_{a,b} \right) = \text{genrank} \left( J^{k+1}_{x} Q_{a,b} \right), \]

then for all integers \( l \geq 1 \) as well:

\[ \text{genrank} \left( J^k_{x} Q_{a,b} \right) = \text{genrank} \left( J^{k+l}_{x} Q_{a,b} \right). \]
Proposition 13.4 If:
\[
\text{genrank } (J_x^1 \mathcal{Q}_{a,b}) = n + 1,
\]
then $\mathcal{M}$ is equivalent to the flat hyperplane:
\[
\{ y = b + 0 \}.
\]

This statement uses the fact that already:
\[
\text{genrank } (J_0^0 \mathcal{Q}_{a,b}) = n + 1 \text{ (at every point)}.\]

14 Finite nondegeneracy with respect to parameters and to variables

Contrary to what happens for CR manifolds, in the context of submanifolds of solutions, there are two distinct and non-equivalent conditions of finite nondegeneracy.

Definition 14.1 A submanifold of solutions $\mathcal{M} \subset \mathbb{K}_{\text{var}}^{n+1} \times \mathbb{K}_{\text{par}}^{m+1}$ is called finitely nondegenerate with respect to parameters at a point $(x_0, a_0, b_0)$ if there exists an integer $k \geq 0$ such that the invariant jet map $J_x^k \mathcal{Q}_{a,b}$:
\[
(x, a, b) \mapsto (x, (\partial_\xi^\beta Q(x, a, b))_{|\beta| \leq k})
\]
is of maximal possible rank $n + 1 + m$ at $(x_0, a_0, b_0)$.

Equivalently, the map:
\[
(a, b) \mapsto (\partial_\xi^\beta Q(x_0, a, b))_{|\beta| \leq k}
\]
is of maximal possible rank $1 + m$ at $(a_0, b_0)$.

Terminology 14.2 Say that $\mathcal{M}$ is $k$-nondegenerate with respect to parameters when $k < \infty$ is the smallest such integer.

Thanks to the fundamental Corollary 13.2, this first condition is independent of coordinates, as is the next, second, symmetric condition.

Definition 14.3 A submanifold of solutions $\mathcal{M} \subset \mathbb{K}_{\text{var}}^{n+1} \times \mathbb{K}_{\text{par}}^{m+1}$ is called finitely nondegenerate with respect to variables at a point $(a_0, x_0, y_0)$ if there exists an integer $l \geq 0$ such that the invariant jet map $J_a^l \mathcal{P}_{x,y}$:
\[
(a, x, y) \mapsto (a, (\partial_a^\gamma P(a, x, y))_{|\gamma| \leq l})
\]
is of maximal possible rank $m + 1 + n$ at $(a_0, x_0, y_0)$.

Equivalently, the map:
\[
(x, y) \mapsto (\partial_a^\gamma P(a_0, x, y))_{|\gamma| \leq l}
\]
is of maximal possible rank $1 + n$ at $(x_0, y_0)$.
Terminology 14.4 Say that $\mathcal{M}$ is $l$-nondegenerate with respect to variables when $l < \infty$ is the smallest such integer.

Example 14.5 For $n = m = 1$, the submanifold of solutions $\{y = b + xa\}$ is simultaneously finitely nondegenerate with respect to parameters and to variables, but with $n = m = 2$, the submanifold:

$$\{y = b + xa + xxb\},$$

is 2-nondegenerate with respect to parameters, and not $l$-nondegenerate with respect to variables for any $l \in \mathbb{N}$.

Our goal in the subsequent paragraphs is to explore more what happens when $m = n = 2$. But before studying specifically 5-dimensional submanifolds $\mathcal{M} \subset \mathbb{K}^{2+1} \times \mathbb{K}^{2+1}$, we need to yet expose some further aspects of the general theory.

15 Generic ranks, degeneracies, and local product structures

Another view on the two invariant jet maps, in which the jet order is now pushed to infinity:

$$J^\infty_x \mathcal{D}_{a,b}: \ (x,a,b) \mapsto (x, (\partial^\beta_x Q(x,a,b))_{\beta \in \mathbb{N}^n}) \in \mathbb{K}^\infty,$$

$$J^\infty_a \mathcal{P}_{x,y}: \ (a,x,y) \mapsto (a, (\partial^\gamma_a P(a,x,y))_{\gamma \in \mathbb{N}^m}) \in \mathbb{K}^\infty,$$

consists in partly expanding the two equations of $\mathcal{M}$ as:

$$y = Q(x,a,b) = \sum_{\beta \in \mathbb{N}^n} x^\beta Q_\beta(a,b),$$

$$b = P(a,x,y) = \sum_{\gamma \in \mathbb{N}^m} a^\gamma P_\gamma(x,y).$$

Then by introducing some two infinite coefficients maps:

$$\mathcal{D}^\infty_{par} := (a,b) \mapsto (Q_\beta(a,b))_{\beta \in \mathbb{N}^n},$$

$$\mathcal{P}^\infty_{var} := (x,y) \mapsto (P_\gamma(x,y))_{\gamma \in \mathbb{N}^m},$$

a detailed analysis of the power series expansions:

$$\partial^\beta_x Q(x,a,b) = \sum_{\beta_1 \in \mathbb{N}^n} x^{\beta_1} (\frac{\beta + \beta_1)!}{\beta! \beta_1!} Q_{\beta+\beta_1}(a,b),$$

$$\partial^\gamma_a P(a,x,y) = \sum_{\gamma_1 \in \mathbb{N}^m} a^{\gamma_1} (\frac{\gamma + \gamma_1)!}{\gamma! \gamma_1!} P_{\gamma+\gamma_1}(x,y),$$

conducts to a proof of

**Proposition 15.1** The generic ranks of these 4 infinite maps are related by:

$$\text{genrank} (J^\infty_\cdot \mathcal{D}_\cdot) = n + \text{genrank} (\mathcal{D}^\infty_{par}),$$

$$\text{genrank} (J^\infty_\cdot \mathcal{P}_\cdot) = m + \text{genrank} (\mathcal{P}^\infty_{var}).$$
Recall that from the beginning, we assume that the implicit defining function of \( \mathcal{M} = \{ R(x, y, a, b) = 0 \} \) satisfies:

\[
R_y(0, 0, 0, 0) \neq 0 \neq R_b(0, 0, 0, 0),
\]

an assumption equivalent to:

\[
Q_b(0, 0, 0, 0) \neq 0 \neq P_y(0, 0, 0),
\]

so that it comes that the ranks of the 0-jet maps:

\[
(x, a, b) \mapsto Q(x, a, b), \quad (a, b) \mapsto Q_0(a, b),
\]

\[
(a, x, y) \mapsto P(a, x, y), \quad (x, y) \mapsto P_0(x, y),
\]

are all already equal to 1. Consequently, the above generic ranks are all \( \geq 1 \).

Let us therefore introduce a notation for them:

\[
1 \leq \text{genrank}(Q_\infty^{\text{par}}) = 1 + m_M \leq 1 + m,
\]

\[
1 \leq \text{genrank}(P_\infty^{\text{var}}) = 1 + n_M \leq 1 + n.
\]

These two invariant integers \( m_M \) and \( n_M \) are intrinsic to \( \mathcal{M} \), and they represent the true-dimensional space \( \mathbb{K}^{n_M + 1} \times \mathbb{K}^{m_M + 1} \) in which \( \mathcal{M} \) really lives (at least generically), as expresses the next

**Theorem 15.2** (1) Locally in a neighborhood of a generic point of \( \mathcal{M} \), there exists a change of coordinates:

\[
(x, y, a, b) \mapsto (f(x, y), g(x, y), \varphi(a, b), \psi(a, b)),
\]

which transforms \( \mathcal{M} \) into a new \( \mathcal{M}' \subset \mathbb{K}^{n+1} \times \mathbb{K}^{m+1} \) having equations:

\[
y' = Q'(x', a', b') \iff b' = P'(a', x', y'),
\]

in which both \( Q' \) and \( P' \) are independent of the coordinates:

\[
(x_{n_M+1}, \ldots, x_n) \text{ and } (a_{m_M+1}, \ldots, a_m).
\]

(2) Furthermore, the reduced submanifold:

\[
\mathcal{M}'_{\text{reduced}} \subset \mathbb{K}^{n_M} \times \mathbb{K} \times \mathbb{K}^{m_M} \times \mathbb{K}
\]

having these equations is finitely nondegenerate at every point both with respect to parameters and to variables.

(3) Lastly, \( \mathcal{M}'_{\text{reduced}} \) is not locally equivalent to any product with either \( \mathbb{K}^1_{\text{var}} \) or \( \mathbb{K}^1_{\text{par}} \).

A particular case is when \( n_M = m_M = 0 \), in which \( \mathcal{M} \) is maximally flat:

\[
\mathcal{M} \cong \{ y = b \}.
\]

**16 Local lie group structure**

These theoretical preliminaries justify to work only with submanifolds of solutions that are finitely nondegenerate both with respect to parameters and to variables.

As is well known in the context of CR geometry, the group of local automorphisms of an arbitrary CR submanifold of \( \mathbb{C}^N \) may be not finite-dimensional, and hence, it is necessary to
impose nondegeneracy conditions. A similar phenomenon occurs in our setting. As in the preceding sections, two independent nondegeneracy conditions are needed, instead of the single finite nondegeneracy condition for CR manifolds.

Now, let us state and present a new general result, not contained in [14].

**Theorem 16.1** Let $\mathcal{M} \subset \mathbb{K}^{n+1} \times \mathbb{K}^{m+1}$ be a hypersurface which is $k$-nondegenerate with respect to parameters and $l$-nondegenerate with respect to variables. Then its (pseudo) group of self-transformations close to the identity:

$$\text{Aut} (\mathcal{M}) := \{(F, \Phi) \in \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}} : (F, \Phi)(\mathcal{M}) \subset \mathcal{M}\},$$

is a local Lie group of dimension:

$$\dim \text{Aut}(\mathcal{M}) \leq (n + 1) \left( \frac{n + 1 + 2k + 2l}{n + 1} \right) + (m + 1) \left( \frac{m + 1 + 2k + 2l}{m + 1} \right).$$

The two nondegeneracy conditions are necessary to apply the Implicit Function Theorem at many steps in the proof of jet parametrizations of $F$ and $\Phi$, cf. [14, Section 11]. In fact, it follows from Theorem 15.2 above that if $\mathcal{M}$ (connected) is either not finitely nondegenerate with respect to variables, or not finitely nondegenerate with respect to parameters (at a Zariski-generic point), then $\text{Aut}(\mathcal{M})$ is infinite-dimensional (at such points).

Note that in general, a further condition on submanifolds, that is the condition of minimality [9,10,14], is needed for finiteness of the dimension of $\text{Aut}(\mathcal{M})$. In our case of a hypersurface $\mathcal{M}$, however, minimality follows from either one of the two finite nondegeneracy conditions.

**Proof** Now, by taking inspiration from [10], let us present ideas of the proof. Recall that a local Lie transformation group consists of a family of local analytic diffeomorphisms $x' = \tau(x; g)$, where $x = (x_1, \ldots, x_n)$ and $x' = (x_1', \ldots, x_n')$ are source and target space coordinates, and where $g = (g_1, \ldots, g_r)$ are group parameters, with $0' = \tau(0; 0)$, satisfying:

1. $(\tau(x; g); g') = \tau(x; \mu(g, g'))$, for some local analytic (multiplication) map $\mu$ with $\mu(0, 0) = 0$, with $\mu(g, 0) = g = \mu(0, g)$, which enjoys associativity:
   $$\mu(g, \mu(g', g'')) = \mu(\mu(g, g'), g''');$$

2. $\tau(x; 0) = x$ and $\tau(\tau(x; g); \iota(g)) = x$, for some local analytic (inversion) map $g \mapsto \iota(g)$ with $\iota(0) = 0$ satisfying:
   $$\mu(g, \iota(g)) = g = \mu(\iota(g), g).$$

These properties are supposed to hold near 0, up to some neighborhood shrinking. The identity transformation $x \mapsto x = x'$ corresponds to $g = 0$, and considerations are localized near $x = 0$ and $x' = 0'$.

Next, abbreviate:

$$G_{v, p} := \text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}},$$

and, for small $\varepsilon > 0$, with the nondegeneracy invariants $k, l$ of Theorem 16.1, introduce a space of self-maps of $\mathcal{M}$ which are close to the identity map $Id$ in the jet sense:

$$\text{Aut}_{\varepsilon}(\mathcal{M}) := \{(F, \Phi) \in G_{v, p} : (F, \Phi)(\mathcal{M}) \subset \mathcal{M},
\quad \left\| j^{2k+2l}_{x,y} F - j^{2k+2l}_{x,y} Id \right\| < \varepsilon, bignorm_{a,b} j^{2k+2l}_{a,b} \Phi - j^{2k+2l}_{a,b} Id \right\| < \varepsilon \}. \quad \blacksquare$$
where order $\kappa$ jets are:

$$j^k_{x,y}F := \left( \partial_x^\beta \partial_y^\gamma F \right)_{|\beta|+|\gamma| \leq \kappa}, \quad j^k_{a,b}\Phi := \left( \partial_x^\beta \partial_y^\gamma \Phi \right)_{|\gamma|+|\gamma| \leq \kappa}.$$ 

In our particular case of a hypersurface $M$ in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, the type $(\mu, \mu^*)$ of $M$ as defined in [14, Definition 10.21] is always $(2, 2)$ (see [14, Section 10]), and the jet parametrization reads as follows.

**Theorem 16.2** [14, Theorem 11.6] Suppose that the hypersurface $\mathcal{M}$ is $k$-nondegenerate with respect to parameters and $l$-nondegenerate with respect to variables. Then there exist two local analytic maps:

\[
H: \mathbb{K}^{n+1+(n+1)(n+1+2k+2l)}_{2k+2l} \rightarrow \mathbb{K}^{n+1}, \\
\Pi: \mathbb{K}^{m+1+(m+1)(m+1+2k+2l)}_{2k+2l} \rightarrow \mathbb{K}^{n+1},
\]

which are constructed only from the defining functions $Q, P$ of $\mathcal{M}$ so that the maps $F, \Phi$ satisfy:

\[
F(x, y) \equiv H(x, y, j^k_{x,y}F(0)), \\
\Phi(a, b) \equiv \Pi(a, b, j^k_{a,b} \Phi(0)),
\]

as power series.

This theorem is in fact the para-CR analog of the jet parametrization theorem for CR manifolds in [10, Theorem 6.4]. It is also clear from the jet parametrizations of $F$ and $\Phi$ that the jet order $\kappa$ in the definition of $\text{Aut}_x(\mathcal{M})$ is $\kappa = 2k + 2l$.

Since Theorem 16.2 concerns any self-map of $\mathcal{M}$ sufficiently close to the identity, it applies to compositions of such maps (provided both are sufficiently close to the identity), and then this theorem shows that the composition is again of the same form, with the same parametrizing functions $H, \Pi$. Also, the inverse of any self-map of $\mathcal{M}$ is again of this parametrized form.

So Theorem 16.2, provides exactly Lie’s original definition of a local (Lie) transformation group [4, pp. 3–4], [5, Section 3.1], cf. also [6], hence Lie’s general theory applies. This would be enough to conclude the proof, but it yet remains to bound $\text{dim Aut } \mathcal{M}$, and this will show other aspects.

Taking either $R := Y - Q(x, a, b)$ or $R := b - P(a, x, y)$, or even any other definining equation for the hypersurface $\mathcal{M}$:

$$\mathcal{M}: \quad 0 = R(x, y, a, b),$$

the assumption that $(F, \Phi)$ is a local self-map of $\mathcal{M}$ (close to the identity, up to shrinking neighborhoods), reads as:

$$(x, y, a, b) \in \mathcal{M} \implies (F(x, y), \Phi(a, b)) \in \mathcal{M},$$

that is, it expresses as the identical vanishing of the following convergent power series in the variables $(x, a, b)$:

$$0 \equiv R\left( F(x, Q(x, a, b)), F(a, b) \right).$$

The key act is then to plug the two jet parametrization formulas of Theorem 16.2 in this identity:

$$0 \equiv R\left( H(x, Q(x, a, b)), j^k_{x,y} F(0), \Pi(a, b, j^k_{a,b} \Phi(0)) \right).$$
and then to power-expand:
\[ 0 \equiv \sum_{\beta \in \mathbb{N}^n} \sum_{\gamma \in \mathbb{N}^m} \sum_{j \in \mathbb{N}} x^\beta a^\gamma b^j C_{\beta,\gamma,j} \left( j_{x,y}^{2k+2l} F(0), j_{a,b}^{2k+2l} \Phi(0) \right), \]
getting an infinite collection of equations in the finite-order jets of \( F \) and \( \Phi \):
\[ 0 = C_{\beta,\gamma,j} \left( j_{x,y}^{2k+2l} F(0), j_{a,b}^{2k+2l} \Phi(0) \right) \quad (\forall \beta, \forall \gamma, \forall j), \]
which define a certain analytic subvariety of \( \mathbb{K}^{N_{n,m,k,l}} \), with the following integer counting all such jets:
\[ N := N_{n,m,k,l} := (n + 1) \left( \frac{n + 1 + 2k + 2l}{n + 1} \right) + (m + 1) \left( \frac{m + 1 + 2k + 2l}{m + 1} \right). \]
By Noetherianity of the local ring of \( \mathbb{K} \)-analytic functions, only a finite number of equations suffices.

By arguing as in [10, Lemma 6.5], one can show that this analytic subvariety is in fact geometrically smooth at the origin, as a consequence that compositions and inversions of elements of \( \text{Aut}_\varepsilon(M) \) are still of the form given by Theorem 16.2. But this was already done by Engel and Lie in [4] in their general theory, what the authors of [10] were not aware of.

Thus, we certainly have the announced dimension bound:
\[ \dim \mathcal{M} \leq (n + 1) \left( \frac{n + 1 + 2k + 2l}{n + 1} \right) + (m + 1) \left( \frac{m + 1 + 2k + 2l}{m + 1} \right). \]

However, let us close this section by mentioning that analyzing these equations \( C_{\beta,\gamma,j} = 0 \) is the ‘hard core’ of the equivalence/classification problem, which is very complicated and very ramified even in small dimensions. By mastering Cartan’s method of equivalence, Hill and Nurowski [12] were able to understand and to analyze the ‘hard core’ of quite a number of para-CR structures of specific dimensions and codimensions.

Anyway, in the para-CR context, this Theorem 16.1 seems to appear nowhere in the literature.

Also, Theorem 16.1 generalizes easily to para-CR structures of any codimension, with the supplementary assumption of minimality, or of covering property, see [14, Section 9]. The proof is the same as in the hypersurface case (to which this paper restricts itself), with the same proof, since the jet parametrization Theorem 16.2 was already stated and proved in any codimension as Theorem 11.6 in [14].

17 Generic \((2, 2, 1)\) para-CR structures

Let us therefore examine the special case \( n = m = 2 \), still in codimension \( c = 1 \). The rest of the paper will in fact not touch anymore the general theory. Hence, we allow ourselves to change notation from now on, and to write out the two graphed equations of \( \mathcal{M} \) as:
\[ z = Q(x, y, a, b, c) \iff c = P(a, b, x, y, z). \]

We already saw that when \( \text{Levi}_{\text{var}} \equiv 0 \equiv \text{Levi}_{\text{par}} \), there is an equivalence to the (very) flat model \( \{ y = b \} \).

At the other extreme lies the case where the (identical) rank(s) of the (two) Levi form(s) is (are) equal to 2. This case has already been treated by Hachtroudi [11] in 1937.

What about intermediate cases?
Since we are working at a generic point, we have therefore to examine the remaining branch:

\[ \text{genrank} ( \text{Levi}_{\text{var}}(Q)) = 1 = \text{genrank} ( \text{Levi}_{\text{par}}(P)). \]

But remind that in the previous paragraphs, the two Levi forms have been inserted in the wider, more adequate and more general concept of jets of leaves of the two invariant foliations on \( \mathcal{M} \). More precisely, recall that according to Corollary 11.1:

\[ \text{rank} (J_x^1 \mathcal{Q}_{a,b}) = 3 + \text{rank} (\text{Levi}_{\text{var}}(Q)) = 3 + \text{rank} (\text{Levi}_{\text{par}}(P)) = \text{rank} (J_a^1 \mathcal{P}_{x,y}), \]

whence we are looking at the branch:

\[ \text{genrank} (J_x^1 \mathcal{Q}_{a,b}) = 4 = \text{genrank} (J_a^1 \mathcal{P}_{x,y}). \]

Now, remind that Proposition 13.3 has shown that second order (generic) jet ranks can either increase, or stabilize, hence it may happen that they jump to the maximal value 5, or stay at 4:

\[ \text{genrank} (J_x^2 \mathcal{Q}_{a,b}) = \begin{cases} 4, & \text{and } \text{genrank} (J_a^2 \mathcal{P}_{x,y}) = \begin{cases} 4, \\ 5. \end{cases} \end{cases} \]

Consequently, it is obvious that we have to examine three degenerate cases:

- **Case I**
  \[
  \begin{align*}
  4 &= \text{genrank} (J_x^1 \mathcal{Q}_{x,y}) \quad \text{and} \quad 4 &= \text{genrank} (J_a^2 \mathcal{Q}_{x,y}), \\
  4 &= \text{genrank} (J_x^1 \mathcal{P}_{x,y}) \quad \text{and} \quad 4 &= \text{genrank} (J_a^2 \mathcal{P}_{x,y}).
  \end{align*}
  \]

- **Case II**
  \[
  \begin{align*}
  4 &= \text{genrank} (J_x^1 \mathcal{Q}_{x,y}) \quad \text{and} \quad 4 &= \text{genrank} (J_a^2 \mathcal{Q}_{x,y}), \\
  4 &= \text{genrank} (J_x^1 \mathcal{P}_{x,y}) \quad \text{while} \quad 5 &= \text{genrank} (J_a^2 \mathcal{P}_{x,y}).
  \end{align*}
  \]

- **Case III**
  \[
  \begin{align*}
  4 &= \text{genrank} (J_x^1 \mathcal{Q}_{x,y}) \quad \text{while} \quad 5 &= \text{genrank} (J_a^2 \mathcal{Q}_{x,y}), \\
  4 &= \text{genrank} (J_x^1 \mathcal{P}_{x,y}) \quad \text{and} \quad 4 &= \text{genrank} (J_a^2 \mathcal{P}_{x,y}).
  \end{align*}
  \]

plus one nondegenerate case:

- **Case IV**
  \[
  \begin{align*}
  4 &= \text{genrank} (J_x^1 \mathcal{Q}_{x,y}) \quad \text{while} \quad 5 &= \text{genrank} (J_a^2 \mathcal{Q}_{x,y}), \\
  4 &= \text{genrank} (J_x^1 \mathcal{P}_{x,y}) \quad \text{while} \quad 5 &= \text{genrank} (J_a^2 \mathcal{P}_{x,y}).
  \end{align*}
  \]

Serendipitously, Theorem 15.2 offers without work

**Proposition 17.1** *At a generic point:*

- **Case I** \( \mathcal{M} \cong \{ z = Q(x, \emptyset, a, \emptyset, c) \} \) or \( \{ c = P(a, \emptyset, x, \emptyset, z) \} \),
- **Case II** \( \mathcal{M} \cong \{ z = Q(x, y, a, \emptyset, c) \} \) or \( \{ c = P(a, \emptyset, x, y, z) \} \),
- **Case III** \( \mathcal{M} \cong \{ z = Q(x, \emptyset, a, b, c) \} \) or \( \{ c = P(a, b, x, \emptyset, z) \} \).

These three degenerate cases could be treated separately, but we prefer to study only Case IV. In this case, also according to Theorem 15.2, we can assume that \( \mathcal{M} \) is both 2-nondegenerate with respect to parameters and 2-nondegenerate with respect to variables.
18 Vanishing of the two $3 \times 3$ Levi determinants

Thus, recall that we assume that the two Levi forms are of rank 1 at every point, which expresses as:

$$0 \equiv \det \text{Levi}_{\text{par}}(Q) = \begin{vmatrix} -Q_c Q_{ab} + Q_a Q_{bc} & -Q_c Q_{ac} + Q_a Q_{bc} \\ Q_c Q_{ab} + Q_a Q_{bc} & -Q_c Q_{ac} + Q_a Q_{bc} \\ Q_c Q_{ab} + Q_a Q_{bc} & -Q_c Q_{ac} + Q_a Q_{bc} \end{vmatrix},$$

$$0 \equiv \det \text{Levi}_{\text{var}}(P) = \begin{vmatrix} -P_z P_{ab} + P_a P_{bz} & -P_z P_{ay} + P_a P_{zy} \\ P_z P_{ab} + P_a P_{bz} & -P_z P_{ay} + P_a P_{zy} \\ P_z P_{ab} + P_a P_{bz} & -P_z P_{ay} + P_a P_{zy} \end{vmatrix}.$$

Moreover, the general relation (known from the end of Sect. 8):

$$\text{Levi}_{\text{par}}(Q) = -P_z^T \text{Levi}_{\text{var}}(P),$$

gives after taking determinants, using $Q_c P_z = 1$:

$$\det \text{Levi}_{\text{par}}(Q) = P_z P_z \det \text{Levi}_{\text{var}}(P) \iff Q_c Q_c \det \text{Levi}_{\text{par}}(Q) = \det \text{Levi}_{\text{var}}(P).$$

By what precedes, the two degeneracies of the two Levi forms are linked with the degeneracies of the two ranks of the two (reduced) first jet maps:

$$(a, b, c) \mapsto (Q, Q_x, Q_y) \quad \text{and} \quad (x, y, z) \mapsto (P, P_a, P_b),$$

whose respective Jacobian determinants are:

$$L_{3 \times 3}(Q) := \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} P_x & P_y & P_z \\ P_{xa} & P_{ya} & P_{za} \\ P_{xb} & P_{yb} & P_{zb} \end{vmatrix} := L_{3 \times 3}(P),$$

and indeed here, a direct check (just develop determinants) convinces that:

$$\det \text{Levi}_{\text{par}}(Q) = \frac{1}{Q_c Q_c Q_c} L_{3 \times 3}(Q) \quad \text{and} \quad \det \text{Levi}_{\text{var}}(P) = \frac{1}{P_z P_z P_z} L_{3 \times 3}(P).$$

As a byproduct, it comes:

$$L_{3 \times 3}(Q) = Q_c Q_c Q_c \det \text{Levi}_{\text{par}}(Q) = Q_c \det \text{Levi}_{\text{var}}(P) = \frac{1}{P_z P_z P_z} L_{3 \times 3}(P),$$

which shows (again) that the vanishings of these two $3 \times 3$ determinants are equivalent. In summary, we are assuming in Case IV that:

$$0 \equiv \begin{vmatrix} Q_a & Q_b & Q_c \\ Q_{xa} & Q_{xb} & Q_{xc} \\ Q_{ya} & Q_{yb} & Q_{yc} \end{vmatrix} \iff 0 \equiv \begin{vmatrix} P_x & P_y & P_z \\ P_{xa} & P_{ya} & P_{za} \\ P_{xb} & P_{yb} & P_{zb} \end{vmatrix}.$$

19 Local graphs for doubly 2-nondegenerate submanifolds

Assume therefore (Case IV) that $\mathcal{M}$, represented in coordinates $(x, y, z, a, b, c)$ by two graphed equations:

$$z = Q(x, y, a, b, c) \quad \text{and} \quad c = P(a, b, x, y, z),$$
is simultaneously 2-nondegenerate with respect to parameters and with respect to variables. If either \((x, y, a, b, c)\) or \((a, b, x, y, z)\) are taken as (horizontal) coordinates on \(\mathcal{M}\), we have two choices of fundamental vector fields:

\[
\begin{align*}
\mathcal{K}_x &:= \frac{\partial}{\partial x}, \\
\mathcal{K}_y &:= \frac{\partial}{\partial y}, \\
\mathcal{L}_a &:= \frac{\partial}{\partial a} - \frac{Q_a}{Q_c} \frac{\partial}{\partial c}, \\
\mathcal{L}_b &:= \frac{\partial}{\partial b} - \frac{Q_b}{Q_c} \frac{\partial}{\partial c},
\end{align*}
\]

\[
\text{or} \quad
\begin{align*}
\mathcal{K}_x &:= \frac{\partial}{\partial x} - \frac{P_x}{P_z} \frac{\partial}{\partial z}, \\
\mathcal{K}_y &:= \frac{\partial}{\partial y} - \frac{P_y}{P_z} \frac{\partial}{\partial z}, \\
\mathcal{L}_a &:= \frac{\partial}{\partial a}, \\
\mathcal{L}_b &:= \frac{\partial}{\partial b}.
\end{align*}
\]

Corollary 10.2 has already shown that there exist coordinates with:

\[c \equiv Q(0, 0, a, b, c) \equiv Q(x, y, 0, 0, c) \quad \text{and} \quad z \equiv P(0, 0, x, y, z) \equiv P(a, b, 0, 0, z),\]

in which the two graphing functions \(P\) and \(Q\) have normalized second-order terms:

\[
z = c + xa + O_{x,y,a,b}(3) + c O_{x,y,a,b,c}(2),
\]

\[
z = z - ax + O_{a,b,x,y}(3) + z O_{a,b,x,y,z}(2).
\]

Next, specifying the homogeneous order 3 terms, we can write:

\[z = c + xa + Q_3(x, y, a, b) + c Q_2^- (x, y, a, b) + O_{x,y,a,b,c}(4),\]

with \(Q_3\) and \(Q_2^-\) homogeneous of degrees 3 and 2, independent of \(c\). In the \(2 \times 2\) matrix \(\text{Levi}_{\text{par}}(Q)\), it is easy to see that the \((1, 2)\) and the \((2, 1)\) entries are \(O_1\), while the \((2, 2)\)-entry is:

\[
-\frac{Q_c Q_y b + Q_b Q_y c}{Q_c Q_c} = \frac{(-1 + O_1 \left(Q_{3,yb} + c Q_{2,yb}^-\right) + O_2}{(-1 + O_1)^2} = \frac{-Q_{3,yb} - c Q_{2,yb}^- + O_2}{O_1} = O_1,
\]

and therefore, the identical vanishing:

\[
0 \equiv \begin{vmatrix} -1 + O_1 & O_1 \\ 0 & -Q_{3,yb} - c Q_{2,yb}^- + O_2 \end{vmatrix},
\]

yields an interesting annihilation [the second one will not be used]:

\[
0 \equiv Q_{3,yb} \quad \text{and} \quad 0 \equiv Q_{2,yb}^-.
\]

Then, expanding the cubic homogeneous polynomial \(Q_3\) without pure \((x, y)\)-terms or \((a, b)\)-terms, all of which monomials divisible by \(yb\) should be absent, it remains:

\[
z = c + xa + \alpha xx a + \alpha xaa + \beta xxxb + \beta yaa + \gamma xya + \gamma xab + \delta yya + \delta xbb + c O_{x,y,a,b}(2) + O_{x,y,a,b,c}(4).
\]

But by replacing:

\[
x + \alpha xx + \gamma xy + \delta yy =: \ x',
\]

\[
a + \alpha aa + \gamma ab + \delta bb =: \ a',
\]
and dropping primes, we come to:

\[ z = c + xa + \beta xxb + ya + c O_{x,y,a,b}(2) + O_{x,y,a,b,c}(4). \]

**Assertion 19.1** \( \mathcal{M} \) is 2-non-degenerate with respect to parameters if and only if:

\[ \beta \neq 0. \]

**Proof.** We have to guarantee that the rank at 0 of the second jet map:

\[ (a, b, c) \mapsto \left( Q, Q_x, Q_y, Q_{xx}, Q_{xy}, Q_{yy} \right), \]

is maximal equal to 3. But after all these normalizations, it is the map:

\[ (a, b, c) \mapsto \left( c + O_2, a + O_2, O_{2a}, 2\beta b + O_2, O_{2b}, O_{2c} \right) \]

whose Jacobian determinant at the origin (after dropping its useless components 3, 5, 6) is:

\[
\det \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 2\beta & 0
\end{pmatrix} = 2 \beta. \]

\[ \square \]

It is important to realize (repeat) that another, different, condition of 2-non-degeneracy exists in the context of submanifolds of solutions of PDE systems, contrary to the CR case in which both 2-non-degeneracy conditions are complex-conjugate one to another, hence reduce to a single condition.

**Assertion 19.2** \( \mathcal{M} \) is 2-non-degenerate with respect to variables if and only if:

\[ \beta \neq 0. \]

**Proof.** As we know, the implicit function theorem enables to solve \( c \) as \( c = P(a, b, x, y, z) \), and the result writes out as:

\[ c = z - ax - bxx - \beta xxb - \beta aay + z O_{a,b,x,y}(2) + O_{a,b,x,y,z}(4). \]

Similarly, we have to guarantee that the rank at 0 of the second jet map:

\[ (x, y, z) \mapsto \left( P, P_a, P_b, P_{aa}, P_{ab}, P_{bb} \right), \]

is maximal equal to 3, and this is also clear from its expression:

\[ (x, y, z) \mapsto \left( -z + O_2, -x + O_2, O_{2a}, -2\beta y + O_2, O_{2b}, O_{2c} \right). \]

\[ \square \]

**Proposition 19.3** If \( \mathcal{M} \subset \mathbb{K}^{2+1} \times \mathbb{K}^{2+1} \) has Levi form(s) of constant rank 1 and is 2-non-degenerate both with respect to parameters and to variables, then there exist normalized coordinates \( (x, y, z, a, b, c) \) in which its two equations read as:

\[ z = c + xa + xxb + yaa + c O_{x,y,a,b}(2) + O_{x,y,a,b,c}(4), \]

\[ c = z - ax - aay - bxx + z O_{a,b,x,y}(2) + O_{a,b,x,y,z}(4). \]

**Proof** Indeed, plain dilations make \( \beta = 1 = \beta. \)
20 Determinantal expressions of the two 2-nondegeneracy assumptions

Now, and because it will be regularly used in what follows, it is advisable to express in a concrete manner the two 2-nondegeneracy assumptions:

\[
\begin{vmatrix}
Q_a & Q_b & Q_c \\
Q_{xa} & Q_{xb} & Q_{xc} \\
Q_{xxa} & Q_{xxb} & Q_{xxc}
\end{vmatrix} \neq 0 \quad \text{and} \quad 0 \neq \begin{vmatrix}
P_x & P_y & P_z \\
P_{ax} & P_{ay} & P_{az} \\
P_{aax} & P_{aay} & P_{aaz}
\end{vmatrix}.
\]

Let us therefore abbreviate:

\[
\Delta(Q) := \begin{vmatrix}
Q_a & Q_b & Q_c \\
Q_{xa} & Q_{xb} & Q_{xc} \\
Q_{xxa} & Q_{xxb} & Q_{xxc}
\end{vmatrix} \quad \text{and} \quad \square(P) := \begin{vmatrix}
P_x & P_y & P_z \\
P_{ax} & P_{ay} & P_{az} \\
P_{aax} & P_{aay} & P_{aaz}
\end{vmatrix}.
\]

We repeat here that the two hypotheses \( \Delta(Q) \neq 0 \) and \( \square(P) \neq 0 \) are independent, as show the two examples:

\[
\begin{align*}
z &= c + xa + xxb, \\
\Delta(Q) &= 2, \\
\square(P) &= 0
\end{align*}
\]

\[
\begin{align*}
z &= c + ax + aay, \\
\Delta(Q) &= 0, \\
\square(P) &= 2.
\end{align*}
\]

21 PDE system and dual PDE system associated to doubly 2-nondegenerate \((2, 2, 1)\) para-CR structures

Now, in the equation \(z = Q(x, y, a, b, c)\) of \(\mathcal{M}\), we may view \(z = z(x, y)\) as a function of \((x, y)\), and differentiate it with respect to \(x, xx:\)

\[
\begin{align*}
z &= Q(x, y, a, b, c), \\
z_x &= Q_x(x, y, a, b, c), \\
z_{xx} &= Q_{xx}(x, y, a, b, c).
\end{align*}
\]

The Jacobian of the map:

\[
(a, b, c) \mapsto \left( Q(x, y, a, b, c), Q_x(x, y, a, b, c), Q_{xx}(x, y, a, b, c) \right),
\]

being precisely:

\[
\Delta(Q) \neq 0,
\]

we can solve by means of the implicit function theorem these three equations for \((a, b, c)\):

\[
\begin{align*}
z &= Q(x, y, a, b, c) \\
z_x &= Q_x(x, y, a, b, c) \\
z_{xx} &= Q_{xx}(x, y, a, b, c)
\end{align*} \iff \begin{align*}
a &= A(x, y, z, z_x, z_{xx}) \\
b &= B(x, y, z, z_x, z_{xx}) \\
c &= C(x, y, z, z_x, z_{xx}),
\end{align*}
\]

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and then replace these solutions in the two equations obtained firstly by differentiating with respect to $y$:

$$z_y = Q_y(x, y, a, b, c)$$

$$= Q_y(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx}))$$

$$=: F(x, y, z, z_x, z_{xx}),$$

and secondly with respect to $xxx$:

$$z_{xxx} = Q_{xxx}(x, y, a, b, c)$$

$$= Q_{xxx}(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx}))$$

$$=: H(x, y, z, z_x, z_{xx}).$$

Diagrammatically, we may represent these two derivatives $z_y \leftrightarrow (0, 1)$ and $z_{xxx} \leftrightarrow (3, 0)$, and generally, all derivatives $z_{x^k y^l}$ by pairs of integers $(k, l) \in \mathbb{N} \times \mathbb{N}$. And then beyond, all other derivatives $z_{x^k y^l}$ with either $k \geq 3$ or $l \geq 1$ express in terms of the horizontal $(x, y, z, z_x, z_{xx})$, for instance:

$$z_{xy} = F_x + F_y + z_x F_z + z_{xx} F_{zx} + z_{xxx} F_{zxx}$$

$$= F_x + F_y + z_x F_z + z_{xx} F_{zx} + H F_{zxx}$$

$$=: \mathcal{F}_{1,1}(x, y, z, z_x, z_{xx}),$$

that is to say generally:

$$z_{x^k y^l} = \mathcal{F}_{k,l}(x, y, z, z_x, z_{xx}) \quad (k \geq 3 \text{ or } l \geq 1).$$

Later, we will see how to transfer equivalences of submanifolds of solutions to equivalences of associated PDE systems.

For now, let us point out that we have used only one assumption of 2-nondegeneracy, $\Delta(Q) \neq 0$. Unfortunately, we do not see what is happening with the other $\Box(P) \neq 0$.

At least, this $\Box(P) \neq 0$ is useful to set up a certain dual PDE system. Indeed, in $c = P(a, b, x, y, z)$, we can similarly view $c$ as a function of $(a, b)$, then solve by means of the implicit function theorem:

$$[c = P(a, b, x, y, z) \quad \iff \quad x = X(a, b, c, c_a, c_{aa})$$

$$c_a = P_a(a, b, x, y, z)$$

$$c_{aa} = P_{aa}(a, b, x, y, z) \quad \iff \quad y = Y(a, b, c, c_a, c_{aa})$$

$$z = Z(a, b, c, c_a, c_{aa}),$$

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thanks to the fact that the Jacobian determinant of the map:

\[(x, y, z) \mapsto \left( P(a, b, x, y, z), P_a(a, b, x, y, z), P_{aa}(a, b, x, y, z) \right)\]

is precisely:

\[\square(P) \neq 0,\]

and then replace—very similarly!—in the other two relevant partial derivatives:

\[c_b = P_b(a, b, x, y, z)\]
\[= P_b\left( a, b, X(a, b, c, c_a, c_{aa}), Y(a, b, c, c_a, c_{aa}), Z(a, b, c, c_a, c_{aa}) \right)\]
\[= E(a, b, c, c_a, c_{aa}),\]
\[c_{aaa} = P_{aaa}(a, b, x, y, z)\]
\[= P_{aaa}\left( a, b, X(a, b, c, c_a, c_{aa}), Y(a, b, c, c_a, c_{aa}), Z(a, b, c, c_a, c_{aa}) \right)\]
\[= G(a, b, c, c_a, c_{aa}).\]

However, the symmetric question arises here: how to view the second 2-nondegeneracy condition \(\Delta(Q) \neq 0\) in this dual PDE system?

The (symmetric) answer to both questions is accessible, but it needs little preliminaries.

### 22 Transfer of derivations \(\mathcal{M} \leftrightarrow \text{PDE}_{\text{var}}(\mathcal{M})\) and \(\mathcal{M} \leftrightarrow \text{PDE}_{\text{par}}(\mathcal{M})\)

We shall treat only the transfer of derivations (of vector fields, of frames) from the solution space \(\mathcal{M}\) equipped with coordinates \((x, y, a, b, c)\) to the jet space equipped with coordinates \((x, y, z, z_x, z_{xx})\):

\[(x, y, a, b, c) \longleftrightarrow (x, y, z, z_x, z_{xx}).\]

Given a function \(G = G(x, y, a, b, c)\) on the left space and a function \(F = F(x, y, z, z_x, z_{xx})\) on the right space, the transfer of basic coordinate vector fields is obtained by differentiating the composition identity:

\[F(x, y, z, z_x, z_{xx}) \equiv G \left( x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx}) \right),\]

with respect to all five variables:

\[
\begin{align*}
\frac{\partial F}{\partial x} &= \frac{\partial G}{\partial x} + A_x \frac{\partial G}{\partial a} + B_x \frac{\partial G}{\partial b} + C_x \frac{\partial G}{\partial c}, \\
\frac{\partial F}{\partial y} &= \frac{\partial G}{\partial y} + A_y \frac{\partial G}{\partial a} + B_y \frac{\partial G}{\partial b} + C_y \frac{\partial G}{\partial c}, \\
\frac{\partial F}{\partial z} &= A_z \frac{\partial G}{\partial a} + B_z \frac{\partial G}{\partial b} + C_z \frac{\partial G}{\partial c}, \\
\frac{\partial F}{\partial z_x} &= A_{zx} \frac{\partial G}{\partial a} + B_{zx} \frac{\partial G}{\partial b} + C_{zx} \frac{\partial G}{\partial c}, \\
\frac{\partial F}{\partial z_{xx}} &= A_{zxx} \frac{\partial G}{\partial a} + B_{zxx} \frac{\partial G}{\partial b} + C_{zxx} \frac{\partial G}{\partial c}.
\end{align*}
\]
Yet, the appearing coefficients \(A_*, B_*, C_*\) are not expressed in terms of the left coordinates \((x, y, a, b, c)\).

To re-express them as required, from the three identically satisfied equations:

\[
A \equiv A(x, y, Q(x, y, a, b, c), Q_x(x, y, a, b, c), Q_{xx}(x, y, a, b, c)),
\]
\[
b \equiv B(x, y, Q(x, y, a, b, c), Q_x(x, y, a, b, c), Q_{xx}(x, y, a, b, c)),
\]
\[
c \equiv C(x, y, Q(x, y, a, b, c), Q_x(x, y, a, b, c), Q_{xx}(x, y, a, b, c)),
\]
differentiations with respect to \(x, y, a, b, c\) provide firstly:

\[
0 = A_x + Q_x A_z + Q_{xx} A_{zx} + Q_{xxx} A_{zxx},
\]
\[
0 = A_y + Q_y A_z + Q_{xy} A_{zx} + Q_{xxy} A_{zxx},
\]
\[
1 = Q_a A_z + Q_{xa} A_{zx} + Q_{xxa} A_{zxx},
\]
\[
0 = Q_b A_z + Q_{xb} A_{zx} + Q_{xxb} A_{zxx},
\]
\[
0 = Q_c A_z + Q_{xc} A_{zx} + Q_{xxc} A_{zxx},
\]

secondly:

\[
0 = B_x + Q_x B_z + Q_{xx} B_{zx} + Q_{xxx} B_{zxx},
\]
\[
0 = B_y + Q_y B_z + Q_{xy} B_{zx} + Q_{xxy} B_{zxx},
\]
\[
0 = Q_a B_z + Q_{xa} B_{zx} + Q_{xxa} B_{zxx},
\]
\[
1 = Q_b B_z + Q_{xb} B_{zx} + Q_{xxb} B_{zxx},
\]
\[
0 = Q_c B_z + Q_{xc} B_{zx} + Q_{xxc} B_{zxx},
\]

and thirdly:

\[
0 = C_x + Q_x C_z + Q_{xx} C_{zx} + Q_{xxx} C_{zxx},
\]
\[
0 = C_y + Q_y C_z + Q_{xy} C_{zx} + Q_{xxy} C_{zxx},
\]
\[
0 = Q_a C_z + Q_{xa} C_{zx} + Q_{xxa} C_{zxx},
\]
\[
0 = Q_b C_z + Q_{xb} C_{zx} + Q_{xxb} C_{zxx},
\]
\[
1 = Q_c C_z + Q_{xc} C_{zx} + Q_{xxc} C_{zxx}.
\]

Resolutions of the three \(3 \times 3\) linear systems consisting each time of the last three lines provide:

\[
A_z = \frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix}, \quad A_{zx} = -\frac{1}{\Delta(Q)} \begin{vmatrix} Q_b & Q_{xxb} \\ Q_c & Q_{xxc} \end{vmatrix}, \quad A_{zxx} = \frac{1}{\Delta(Q)} \begin{vmatrix} Q_b & Q_{xb} \\ Q_c & Q_{xc} \end{vmatrix},
\]
\[
B_z = -\frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix}, \quad B_{zx} = \frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_{xxa} \\ Q_c & Q_{xxc} \end{vmatrix}, \quad B_{zxx} = -\frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_{xa} \\ Q_c & Q_{xc} \end{vmatrix},
\]
\[
C_z = \frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix}, \quad C_{zx} = -\frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_{xxa} \\ Q_b & Q_{xxb} \end{vmatrix}, \quad C_{zxx} = \frac{1}{\Delta(Q)} \begin{vmatrix} Q_a & Q_{xa} \\ Q_b & Q_{xb} \end{vmatrix}.
\]
and consequently, the transfer of the last three (among five) vector fields is:

\[
\begin{align*}
\frac{\partial}{\partial z} &= \begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial a} - \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial b} + \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_x} &= -\begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial a} + \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial b} - \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix} \frac{\partial}{\partial c}, \\
\frac{\partial}{\partial z_{xx}} &= \begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial a} - \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix} \frac{\partial}{\partial b} + \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix} \frac{\partial}{\partial c}. 
\end{align*}
\]

**Proposition 22.1** If the submanifold of solutions \( \mathcal{M} \subset \mathbb{K}^2 \times \mathbb{K}^1 \times \mathbb{K}^2 \times \mathbb{K}^1 \) has (degenerate) Levi form of constant rank 1 and if it is 2-nondegenerate with respect to parameters, then in its associated system \( \text{PDE}_{\text{var}}(\mathcal{M}) \):

\[
\begin{align*}
z_y &= F(x, y, z, z_x, z_{xx}), \\
z_{xxx} &= H(x, y, z, z_x, z_{xx}),
\end{align*}
\]

the function \( F \) is independent of \( z_{xx} \):

\[ 0 \equiv F_{z_{xx}}. \]

**Proof.** By construction:

\[
F(x, y, z, z_x, z_{xx}) =: Q_y(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx})),
\]

whence a differentiation with respect to \( z_{xx} \) makes re-appear the Levi determinant \( L_{3\times3}(Q) \equiv 0 \):

\[
F_{z_{xx}} = A_{z_{xx}} Q_{xa} + B_{z_{xx}} Q_{yb} + C_{z_{xx}} Q_{yc}
\]

\[
= \begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix} Q_{ya} - \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix} Q_{yb} + \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix} Q_{yc}
\]

\[
= \frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xb} & Q_{xxb} \\ Q_{xc} & Q_{xxc} \end{vmatrix} Q_{ya} - \frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xc} & Q_{xxc} \end{vmatrix} Q_{yb} + \frac{1}{\Delta(Q)} \begin{vmatrix} Q_{xa} & Q_{xxa} \\ Q_{xb} & Q_{xxb} \end{vmatrix} Q_{yc}
\]

\[ = 0. \]

Of course, we have in a symmetric way for \( \text{PDE}_{\text{par}}(\mathcal{M}) \):

\[ 0 \equiv E_{c_{xx}}. \]

One must observe that if, on the contrary, one would have:

\[ F_{z_{xx}} \neq 0, \]

then in a neighborhood of a generic point near which \( F_{z_{xx}} \neq 0 \), one would be able to solve for \( z_{xx} \) in the first partial differential equation \( z_y = F \):

\[ z_{xx} = \Lambda(x, y, z, z_x, z_y), \]
and further replacements and differentiations would show that one would come to a PDE system of the kind already studied by Hachtroudi [11]:

\[
\begin{align*}
  z_{xx} &= F_{2,0}(x, y, z, z_x, z_y), \\
  z_{xy} &= F_{1,1}(x, y, z, z_x, z_y), \\
  z_{yy} &= F_{0,2}(x, y, z, z_x, z_y).
\end{align*}
\]

This is equivalent to the observation that, on a hypersurface \( M^5 \subset \mathbb{C}^3 \) whose Levi form is not identically degenerate, at a generic point, the reduction to an \( \{e\} \)-structure and to a Cartan connection was already studied by Chern-Moser [3]. So we will definitely assume \( F_{zxx} \equiv 0 \) up to the end of our considerations.

### 23 The two kernels of the two Levi forms

Recall that with the two transversal vector fields:

\[
\begin{align*}
  \mathcal{T}_b &= \frac{\partial}{\partial b} \quad \text{and} \quad \mathcal{U}_y = \frac{\partial}{\partial y},
\end{align*}
\]

we have introduced two 1-forms \( \rho \) and \( \sigma \) on \( \mathcal{M} \) which satisfy:

\[
\rho(\mathcal{T}_b) \equiv 1 \quad \text{and} \quad \sigma(\mathcal{U}_y) \equiv 1,
\]

and that:

\[
\text{Levi}_{\text{var}}(\mathcal{K}, \mathcal{L}) = \rho([\mathcal{K}, \mathcal{L}]) \quad \text{and} \quad \text{Levi}_{\text{par}}(\mathcal{L}, \mathcal{K}) = \sigma([\mathcal{L}, \mathcal{K}]).
\]

By assumption, the two matrices of these two Levi forms:

\[
\begin{bmatrix}
  -Q_c Q_{xy} + Q_a Q_{xc} & -Q_c Q_{xb} + Q_a Q_{xc} \\
  -Q_c Q_{ya} + Q_a Q_{yc} & -Q_c Q_{yb} + Q_a Q_{yc}
\end{bmatrix}
\]

\[(x, y, a, b, c)\] and

\[
\begin{bmatrix}
  -P_c P_{a} + P_b P_{ac} & -P_c P_{bc} + P_a P_{bc} \\
  -P_c P_{by} + P_b P_{by} & -P_c P_{by} + P_a P_{by}
\end{bmatrix}
\]

\[(a, b, x, y, z),\]

have rank 1 at every point, and their two upper-left \((1, 1)\)-entries are nowhere vanishing, since we have already normalized:

\[
Q = c + xa + O_3 \iff P = -z + ax + O_3,
\]

or because this can always be achieved after an allowed change of coordinates.

**Definition 23.1** The two kernels of the two Levi forms are:

\[
\begin{align*}
  \text{Ker } \text{Levi}_{\text{var}} &:= \left\{ \mathcal{K} \in \Gamma(T^{\text{var}}\mathcal{M}) : 0 = \text{Levi}_{\text{var}}(\mathcal{K}, \mathcal{L}), \forall \mathcal{L} \in \Gamma(T^{\text{par}}\mathcal{M}) \right\}, \\
  \text{Ker } \text{Levi}_{\text{par}} &:= \left\{ \mathcal{L} \in \Gamma(T^{\text{par}}\mathcal{M}) : 0 = \text{Levi}_{\text{par}}(\mathcal{L}, \mathcal{K}), \forall \mathcal{K} \in \Gamma(T^{\text{var}}\mathcal{M}) \right\}.
\end{align*}
\]

Since the above two Levi matrices have constant rank 1 all over \( \mathcal{M} \), their two kernels define two analytic (smooth) rank 1 distributions. Two vector field generators can be explicitly written in terms of two quotients of the entries of the first rows of the two Levi matrices above, as follows.
Lemma 23.2 Two natural generators of these two Levi forms kernels are the two vector fields:

\[ \mathcal{K}_{\text{ker}}^{\text{var}} := -P_z P_{az} + P_y P_{az} \left( \frac{\partial}{\partial x} - P_x \frac{\partial}{\partial z} \right) + \frac{\partial}{\partial y} - P_z \frac{\partial}{\partial z}, \]

\[ \mathcal{K}_{\text{ker}}^{\text{par}} := -Q_c Q_{xb} + Q_b Q_{xc} \left( \frac{\partial}{\partial a} - Q_a \frac{\partial}{\partial c} \right) + \frac{\partial}{\partial b} - Q_b \frac{\partial}{\partial c}. \]

Proof. It suffices to verify that:

\[ \left[ \frac{\partial}{\partial a}, \mathcal{K}_{\text{ker}}^{\text{var}} \right] \equiv 0 \equiv \left[ \frac{\partial}{\partial b}, \mathcal{K}_{\text{ker}}^{\text{var}} \right], \]

which is done by a direct computation, the first equality to zero being true for free, while the second one comes from the Levi degeneracy assumption \( L_{3 \times 3}(P) \equiv 0. \)

The other pair of annihilations is checked in a quite symmetric manner:

\[ \left[ \frac{\partial}{\partial x}, \mathcal{K}_{\text{ker}}^{\text{par}} \right] \equiv 0 \equiv \left[ \frac{\partial}{\partial y}, \mathcal{K}_{\text{ker}}^{\text{par}} \right]. \]

For simplicity, we will sometimes write:

\[ \mathcal{K}_{\text{ker}}^{\text{var}} = k \mathcal{K}_x + \mathcal{K}_y \quad \text{and} \quad \mathcal{K}_{\text{ker}}^{\text{par}} = l \mathcal{L}_a + \mathcal{L}_b, \]

after giving a name to the two Levi entries quotients in question (mind the two minus signs):

\[ k := -\frac{P_z P_{az} + P_y P_{az}}{-P_z P_{ax} + P_x P_{az}}, \quad \text{and} \quad l := -\frac{-Q_c Q_{xb} + Q_b Q_{xc}}{-Q_c Q_{xa} + Q_a Q_{xc}}. \]

24 Reexpressions of \( F_{zx} \) in terms of \( Q \) and in terms of \( P \)

By differentiating with respect to \( z_x \) the function \( F \) of \( \text{pde}_{\text{var}}(\mathcal{M}) \) from its definition:

\[ F(x, y, z, z_x, z_{xx}) := Q_y(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx})), \]

we get:

\[ F_{zx} = A_{zx} Q_ya + B_{zx} Q_yb + C_{zx} Q_yc \]

\[ = - \frac{Q_b Q_{xb}}{\Delta(Q)} Q_ya + \frac{Q_a Q_{xa}}{\Delta(Q)} Q_yb - \frac{Q_a Q_{xa}}{\Delta(Q)} Q_yc \]

\[ = \frac{Q_a Q_ya Q_b Q_c}{Q_xa Q_xb Q_xc}. \]

So we have a quotient of two determinants, one in the denominator which is nowhere vanishing, and one in the numerator in which only the second row differs. A surprising (and useful) simplification occurs.

Lemma 24.1 One has in fact:

\[ F_{zx} = -Q_c Q_ya + Q_a Q_yc. \]
Proof An expansion of both determinants along their last (identical) rows gives:

$$F_{zx} = \frac{Q_{xxa} | Q_b Q_c - Q_{xxb} | Q_{ya} Q_{yc} + Q_{xxc} | Q_a Q_b}{Q_{xxa} | Q_b Q_c - Q_{xxb} | Q_{xa} Q_{xc} + Q_{xxc} | Q_a Q_{xb}}$$

Then by eliminating (cross-producting) the denominators in this equality under question, one recovers a multiple of $L_{3\times3}(Q) \equiv 0$.

**Proposition 24.2** One has:

$$F_{zx} = -P_z P_{ay} + P_y P_{az} - P_z P_{ax} + P_x P_{az} = -k.$$  

**Proof.** Indeed, the very useful Lemma 8.1 applies:

$$F_{zx} = \frac{-Q_c Q_{ya} + Q_a Q_{yc}}{-Q_c Q_{xa} + Q_a Q_{xc}}.$$  

Lastly, two definitions of invariant higher order Levi forms, analogous to the Freeman form [8] in CR Geometry [19, Section 9], exist, but because the jet theory is more general, we will dispense ourselves of introducing these two concepts. Indeed, the nondegeneracy of one of these two ‘Freeman forms’ could then be expressed as the nonvanishing $\frac{\partial}{\partial a} (k) \neq 0$, entirely analogous to the nonvanishing condition $\mathcal{J}^1(k) \neq 0$ which was central in [18], but we recover one of our two favorite nonvanishing $3 \times 3$ determinants anyway.

**Lemma 24.3** One has the nowhere vanishing invariant expression:

$$\frac{\partial}{\partial a} (F_{zx}) = P_z \frac{P_x P_y P_z}{P_{ax} P_{ay} P_{az}} \frac{1}{P_{ax} P_{ay} P_{az} - P_z P_{ax} + P_x P_{az}} \neq 0.$$  

**Proof.** This just amounts to differentiate and to reorganize properly:

$$\frac{\partial}{\partial a} \left[ -P_z P_{ay} + P_y P_{az} \right]. \square$$  

In summary, the system PDE$_{var}(\mathcal{M})$ and in its dual system PDE$_{par}(\mathcal{M})$:

$$z_y = F(x, y, z, z_x, z_{xx}), \quad \text{and} \quad c_b = E(a, b, c, c_a, c_{aa}),$$

$$z_{xxx} = H(x, y, z, z_x, z_{xx}), \quad \text{and} \quad c_{aaa} = G(a, b, c, c_a, c_{aa}),$$

which come from a submanifold of solutions $\mathcal{M}$ whose two Levi forms have constant rank 1, and which is 2-nondegenerate both with respect to parameters and to variables, must satisfy:

$$F_{cxx} \equiv 0 \quad \text{and} \quad 0 \equiv E_{c_{aa}}.$$
25 Reexpression of the hypothesis of 2-nondegeneracy with respect to variables as \( F_{zxzx} \neq 0 \)

To conclude this preliminary trip, we must answer the question raised in Sect. 21, which we can now formulate a bit more precisely (of course, a symmetric question can also be formulated).

**Question 3** How to view the hypothesis of 2-nondegeneracy with respect to variables in the system \( \text{PDE}_{\text{var}} (\mathcal{M}) \)?

The thing is that the hypothesis of 2-nondegeneracy with respect to parameters (and not variables!) has already been used to set up this system \( \text{PDE}_{\text{var}} (\mathcal{M}) \).

**Proposition 25.1** The system \( \text{PDE}_{\text{var}} (\mathcal{M}) \) coming from an \( \mathcal{M} \) which is 2-nondegenerate with respect to parameters is also 2-nondegenerate with respect to variables if and only if:

\[
F_{zxzx} \neq 0.
\]

**Proof.** Start by rewriting, in the variables \((x, y, a, b, c)\) instead of in the variables \((x, y, z, zx, zxx)\), the functional equality which was used in Sect. 22 to perform a transfer of derivations:

\[
F(x, y, Q(x, y, a, b, c), Q_x(x, y, a, b, c), Q_{xx}(x, y, a, b, c)) \equiv G(x, y, a, b, c),
\]

and differentiate it with respect to \(a, b, c\) to get alternative formulas for vector fields (derivations):

\[
Q_a \frac{\partial}{\partial z} + Q_{xa} \frac{\partial}{\partial zx} + Q_{xxa} \frac{\partial}{\partial zxx} = \frac{\partial}{\partial a},
\]

\[
Q_b \frac{\partial}{\partial z} + Q_{xb} \frac{\partial}{\partial zx} + Q_{xxb} \frac{\partial}{\partial zxx} = \frac{\partial}{\partial b},
\]

\[
Q_c \frac{\partial}{\partial z} + Q_{xc} \frac{\partial}{\partial zx} + Q_{xxc} \frac{\partial}{\partial zxx} = \frac{\partial}{\partial c}.
\]

The path of computations is to eliminate \(\frac{\partial}{\partial z}\) from lines 1 and 3:

\[
0 + (Q_c Q_{xa} - Q_a Q_{xc}) \frac{\partial}{\partial z_x} + (Q_c Q_{xxa} - Q_a Q_{xxc}) \frac{\partial}{\partial z_{xx}} = Q_c \frac{\partial}{\partial a} - Q_a \frac{\partial}{\partial c},
\]

to apply this derivation to the identity of Proposition 24.2:

\[
F_{zx}(x, y, z, z_x, z_{xx}) = \frac{-P_z P_{ay} + P_y P_{az}}{-P_z P_{ax} + P_x P_{az}} (a, b, x, y, z),
\]

taking advantage of two facts: firstly that \(F_{zx} \equiv 0\) by degeneracy of the Levi form; secondly that \(P\) is independent of \(c\); this conducts to observe that two terms, one in each side, disappear:

\[
(Q_c Q_{xa} - Q_a Q_{xc}) F_{zxz_x} + (Q_c Q_{xxa} - Q_a Q_{xxc}) F_{zxz_{xx}} = Q_c \frac{\partial}{\partial a} \left( \frac{-P_z P_{ay} + P_y P_{az}}{-P_z P_{ax} + P_x P_{az}} \right) - Q_a \frac{\partial}{\partial c} \left( \text{same} \right);
\]
it then suffices to solve to conclude:

\[
F_{\partial x z} = \frac{Q_c}{Q_c Q_xa - Q_a Q_xc} \frac{\partial}{\partial a} \left( \frac{-P_z P_{ay} + P_y P_{az}}{-P_z P_{ax} + P_x P_{az}} \right)
\]

\[
= -\frac{Q_c}{-Q_c Q_xa - Q_a Q_xc} \frac{P_z}{\left( -P_z P_{ax} + P_x P_{az} \right)^2} \left| \begin{array}{ccc}
P_x & P_y & P_z \\
P_{ax} & P_{ay} & P_{az} \\
P_{aax} & P_{aay} & P_{aaz}
\end{array} \right| \neq 0.
\]

### 26 Complete integrability

By construction, in the system \(z_y = F, z_{xxx} = H\), we have \(F = Q_y\) and \(H = Q_{xxx}\), hence from \((Q_y)_{xxx} = (Q_{xxx})_y\), we deduce a compatibility constraint on \(F\) and \(H\):

\[
D_x (D_x (D_x (F))) = D_y (H),
\]

in terms of the two total differentiation operators:

\[
D_x := \frac{\partial}{\partial x} + z_x \frac{\partial}{\partial z} + z_{xx} \frac{\partial}{\partial z_x} + H \frac{\partial}{\partial z_{xx}},
\]

\[
D_y := \frac{\partial}{\partial y} + F \frac{\partial}{\partial z} + D_x (F) \frac{\partial}{\partial z_x} + D_x (D_x (F)) \frac{\partial}{\partial z_{xx}}.
\]

The converse holds true, and is an elementary consequence of the Frobenius theorem [14].

**Theorem 26.1** If two analytic functions \(F\) and \(H\) of \((x, y, z, z_x, z_{xx})\) satisfy:

\[
D_x (D_x (D_x (F))) = D_y (H),
\]

then the analytic PDE system \(z_y = F, z_{xxx} = H\) is completely integrable in the sense that there exists a \(C^\infty\) family of solutions:

\[
z = Q(x, y, z(0, 0), z_x(0, 0), z_{xx}(0, 0)),
\]

parametrized by initial conditions:

\[
Q(0, 0, z(0, 0), z_x(0, 0), z_{xx}(0, 0)) \equiv z(0, 0),
\]

\[
Q_x(0, 0, z(0, 0), z_x(0, 0), z_{xx}(0, 0)) \equiv z_x(0, 0),
\]

\[
Q_{xx}(0, 0, z(0, 0), z_x(0, 0), z_{xx}(0, 0)) \equiv z_{xx}(0, 0).
\]

### 27 Initial G-structure for equivalences \(\mathcal{M} \sim \mathcal{M}'\)

Suppose given an equivalence in \(\text{Diff}_{\text{var}} \times \text{Diff}_{\text{par}}:\)

\[
(F, \Phi) = (f, g, \varphi, \psi): \mathcal{M} \rightarrow \mathcal{M}'.
\]
Work in coordinates \((x, y, a, b, c)\) on \(\mathcal{M}\) and \((x', y', a', b', c')\) on \(\mathcal{M}'\), and take the two frames of vector fields (now written in this precise order):

\[
\begin{align*}
\mathcal{T}_c &= \frac{\partial}{\partial c}, \\
\mathcal{L}_a &= \frac{\partial}{\partial a} - \frac{Q_a}{Q_c} \frac{\partial}{\partial c}, \\
\mathcal{L}_b &= \frac{\partial}{\partial b} - \frac{Q_b}{Q_c} \frac{\partial}{\partial c}, \\
\mathcal{K}_x &= \frac{\partial}{\partial x}, \\
\mathcal{K}_y &= \frac{\partial}{\partial y},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{T}'_{c'} &= \frac{\partial}{\partial c'}, \\
\mathcal{L}'_{a'} &= \frac{\partial}{\partial a'} - \frac{Q'_{a'}}{Q'_{c'}} \frac{\partial}{\partial c'}, \\
\mathcal{L}'_{b'} &= \frac{\partial}{\partial b'} - \frac{Q'_{b'}}{Q'_{c'}} \frac{\partial}{\partial c'}, \\
\mathcal{K}'_{x'} &= \frac{\partial}{\partial x'}, \\
\mathcal{K}'_{y'} &= \frac{\partial}{\partial y'}.
\end{align*}
\]

On \(\mathcal{M}\) and on \(\mathcal{M}'\), the two pairs of Levi kernel direction fields are generated by:

\[
\begin{align*}
\mathcal{K}_{\text{ker}} &= k \mathcal{K}_x + \mathcal{K}_y, \\
\mathcal{L}_{\text{ker}} &= l \mathcal{L}_a + \mathcal{L}_b, \\
\mathcal{K}'_{\text{ker}} &= k' \mathcal{K}'_{x'} + \mathcal{K}'_{y'}, \\
\mathcal{L}'_{\text{ker}} &= l' \mathcal{L}'_{a'} + \mathcal{L}'_{b'},
\end{align*}
\]

and Lemma 8.1 enables to reexpress \(k\) in terms of \(Q\):

\[
\begin{align*}
k &= -\frac{P_z P_y + P_y P_z}{P_z P_x + P_x P_z}, \\
l &= -\frac{Q_x Q_a + Q_a Q_x}{Q_x Q_a + Q_a Q_x}.
\end{align*}
\]

with similar expressions for \(k'\) and \(l'\).

**Observation 27.1** Through an equivalence \((F, \Phi): \mathcal{M} \rightarrow \mathcal{M}'\), Levi-kernel directions transfer one to another:

\[
\begin{align*}
(F, \Phi)_* (T_{\text{ker}} \mathcal{M}) &= T_{\text{ker}} \mathcal{M}', \\
(F, \Phi)_* (T'_{\text{ker}} \mathcal{M}') &= T'_{\text{ker}} \mathcal{M}.
\end{align*}
\]

**Proof** This comes from \((F, \Phi)_* (T_{\text{var}} / \text{par} \mathcal{M}) = T_{\text{var}} / \text{par} \mathcal{M}'\), from the definitions of Levi kernels, and from the invariancy of Levi forms expressed by the matrix identities of Sect. 9.

As an important consequence, there are certain functions \(f_3\) and \(h_3\) such that, after unwritten push-forward:

\[
\begin{align*}
l' \mathcal{L}'_{a'} + \mathcal{L}'_{b'} &= f_3 \cdot (l \mathcal{L}_a + \mathcal{L}_b), \\
k' \mathcal{K}'_{x'} + \mathcal{K}'_{y'} &= h_3 \cdot (k \mathcal{K}_x + \mathcal{K}_y).
\end{align*}
\]

Consequently, there exist 11 functions on \(\mathcal{M}\) such that:

\[
\begin{pmatrix}
\mathcal{T}'_{c'} \\
\mathcal{L}'_{a'} \\
l' \mathcal{L}'_{a'} + \mathcal{L}'_{b'} \\
\mathcal{K}'_{x'} + \mathcal{K}'_{y'}
\end{pmatrix}
= \begin{pmatrix}
a & b_1 & b_2 & c_1 & c_2 \\
0 & f_1 & f_2 & 0 & 0 \\
0 & 0 & f_3 & 0 & 0 \\
0 & 0 & 0 & h_1 & h_2 \\
0 & 0 & 0 & 0 & h_3
\end{pmatrix}
\begin{pmatrix}
\mathcal{T}_c \\
\mathcal{L}_a \\
l \mathcal{L}_a + \mathcal{L}_b \\
k \mathcal{K}_x + \mathcal{K}_y
\end{pmatrix}.
\]
The initial $G$-structure for equivalences of such $\mathcal{M}$ (in terms of vector fields) is therefore represented by such (invertible) matrices. Before starting Cartan’s method, it remains only to re-express this in terms of differential 1-forms.

The coframe dual to the frame on $\mathcal{M}$ (in this order):

$$\{ \mathcal{L}_c, L_a, L_b, \mathcal{X}_x, kaux \mathcal{X}_x + \mathcal{X}_y \}$$

is:

$$\left\{ \frac{Q_a}{Q_c} da + \frac{Q_b}{Q_c} db + dc, \quad da - ldb, \quad db, \quad dx - kdy, \quad dy \right\}.$$ A plain transposition of the above $5 \times 5$ matrix then yields with new functions:

$$\begin{pmatrix}
    Q'_a da' + Q'_b db' + Q'_c dc' \\
    da' - l' db' \\
    db' \\
    dx' - k' dy' \\
    dy'
\end{pmatrix}
= \begin{pmatrix}
    a & 0 & 0 & 0 & 0 \\
    b_1 & f_1 & 0 & 0 & 0 \\
    b_2 & f_2 & f_3 & 0 & 0 \\
    c_1 & 0 & h_1 & 0 & 0 \\
    c_2 & 0 & h_2 & h_3 & 0
\end{pmatrix}
= \begin{pmatrix}
    Q_a da + Q_b db + dc \\
    da - ldb \\
    db \\
    dx - kdy \\
    dy
\end{pmatrix}.$$ 

28 Triangular initial $G$-structure for equivalences

$$\text{PDE}_{\text{var}}(\mathcal{M}) \xrightarrow{\sim} \text{PDE}_{\text{var}}(\mathcal{M}^\prime)$$

By what precedes, on $\mathcal{M}$ equipped with coordinates $(x, y, a, b, c)$, there is a natural coframe:

$$\left\{ \frac{Q_a}{Q_c} da + \frac{Q_b}{Q_c} db + dc, \quad da - ldb, \quad db, \quad dx - kdy, \quad dy \right\},$$
on which the two invariant plane fields $T^{\text{var}}\mathcal{M}$ and $T^{\text{par}}\mathcal{M}$ and the two invariant Levi-kernel direction fields $T^{\text{var}}\ker \mathcal{M}$ and $T^{\text{par}}\ker \mathcal{M}$ are visible. The next goal is to transmit this initial geometry to the associated system $\text{PDE}_{\text{var}}(\mathcal{M})$.

As is known, the transfer:

$$(x, y, a, b, c) \longleftrightarrow (x, y, z, z_x, z_{xx}),$$
is represented by a known map accompanied with its inverse:

$$x = x, \quad y = y, \quad a = A(x, y, z, z_x, z_{xx}), \quad (28.1)$$

$$Q(x, y, a, b, c) = z, \quad b = B(x, y, z, z_x, z_{xx}), \quad Q_x(x, y, a, b, c) = z_x, \quad c = C(x, y, z, z_x, z_{xx}).$$

Next, on the system $\text{PDE}_{\text{var}}(\mathcal{M})$:

$$z_y = F(x, y, z, z_x, z_{xx}) = Q_y(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx})), \quad (28.1)$$

$$z_{xxx} = H(x, y, z, z_x, z_{xx}) = Q_{xxx}(x, y, A(x, y, z, z_x, z_{xx}), B(x, y, z, z_x, z_{xx}), C(x, y, z, z_x, z_{xx})), \quad (28.1)$$
there is a natural coframe consisting of 5 differential 1-forms, the first 3 being contact forms pulled-back to the PDE system:

\[
\begin{align*}
\lambda & := dz - z_x dx - F dy, \\
\mu_1 & := dz_x - z_{xx} dx - D_x(F) dy, \\
\mu_2 & := dz_{xx} - H dx - D_x(D_x(F)) dy, \\
v_1 & := dx, \\
v_2 & := dy.
\end{align*}
\]

We want to relate these 1-forms to the coframe on \(\mathcal{M}\) introduced above. More precisely, in some (possibly more) appropriate initial coframe, we want to determine the initial \(G\)-structure for point equivalences:

\[
\begin{pmatrix}
(\lambda',\mu'_1,\mu'_2) \\
v'_1 \\
v'_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\lambda \\
v_1 \\
v_2
\end{pmatrix}
\]

that transfer a system \(z_x = F, z_{xxx} = H\), to a similar one \(z'_y = F', z'_x x' x' = H'\), having of course the same geometric features.

First of all, since contact forms of any fixed jet order must be sent to contact forms of the same jet order, there is a triangular \(3 \times 3\) matrix of functions such that, after (unwritten) pullback:

\[
\begin{pmatrix}
\lambda' \\
\mu'_1 \\
\mu'_2
\end{pmatrix}
= 
\begin{pmatrix}
a & b_1 & b_2 \\
0 & f_1 & f_2 \\
0 & 0 & f_3
\end{pmatrix}
\begin{pmatrix}
\lambda \\
v_1 \\
v_2
\end{pmatrix}. 
\]

Also, since the considered transformations are punctual:

\[
(x, y, z) \mapsto (x'(x, y, z), y'(x, y, z), z'(x, y, z)),
\]

and since the coframes \(\{\lambda, v_1, v_2\}\) and \(\{dz, dx, dy\}\) have the same span, there are functions such that:

\[
\begin{pmatrix}
\lambda' \\
v'_1 \\
v'_2
\end{pmatrix}
= 
\begin{pmatrix}
a & c_1 & c_2 \\
0 & h_1 & h_2 \\
0 & 0 & h_3
\end{pmatrix}
\begin{pmatrix}
\lambda \\
v_1 \\
v_2
\end{pmatrix}. 
\]  \hspace{1cm} (28.2)

But we already saw that the hypothesis that \(\mathcal{M}\) is also 2-nondegenerate with respect to variables conducts to choose:

\[
\{dx - k dy, dy\} \quad \text{instead of} \quad \{dx, dy\},
\]

and since there is in (28.1) plainly \(dx = dx, dy = dy\), and because we know already that from Proposition 24.2:

\[-k = F_{z_x},\]

a more natural coframe to start with later on when running Cartan’s method is (notice the modification of \(v_1\)) is the following:

\[
\begin{align*}
\lambda & := dz - z_x dx - F dy, \\
\mu_1 & := dz_x - z_{xx} dx - D(F) dy, \\
\mu_2 & := dz_{xx} - H dx - D_x(D_x(F)) dy, \\
v_1 & := dx + F_{z_x} dy, \\
v_2 & := dy.
\end{align*}
\]
so that initial equivalences would take the triangular form:

\[
\begin{pmatrix}
\lambda' \\
\mu_1' \\
\mu_2' \\
v_1' \\
v_2'
\end{pmatrix} = \begin{pmatrix}
a & 0 & 0 & 0 & 0 \\
b_1 & f_1 & 0 & 0 & 0 \\
b_2 & f_2 & f_3 & 0 & 0 \\
c_1 & 0 & h_1 & 0 & 0 \\
c_2 & 0 & 0 & h_2 & h_3
\end{pmatrix} \begin{pmatrix}
\lambda \\
\mu_1 \\
\mu_2 \\
v_1 \\
v_2
\end{pmatrix},
\]

with \( f_4 = h_4 = 0 \) disappearing. This is under the assumption that \( F_{z x z x} \neq 0 \) only!

In fact, without assuming this, namely starting from \( \nu_1 = dx, \nu_2 = dy \) and from a nontriangular initial \( G \)-structure, one can easily re-find [16] that \( F_{z x z x} \) is a relative differential invariant, and then assume again \( F_{z x z x} \neq 0 \).

Before launching Cartan’s method, let us end up by examining a bit what the transfer \( \mathcal{M} \rightarrow \text{PDE}_{\text{var}}(\mathcal{M}) \) can tell us about \( f_4 \) and \( h_4 \). Indeed, we want to confirm the fact that \( \mu_1' \) is a linear combination of \( \{\lambda, \mu_1\} \), without \( \mu_2 \), a fact which the general theory of contact forms already gave, and which we want to see as a consequence of the triangular form of the \( G \)-structure for equivalences \( \mathcal{M} \rightarrow \mathcal{M}' \).

Abbreviate the considered jet space as:

\[
J^2_{\text{hor}} := \{(x, y, z, z_x, z_{xx})\}.
\]

Equivalences between submanifolds of solutions are in one-to-one correspondence with equivalences between systems of partial differential equations:

\[
\begin{array}{c}
J^2_{\text{hor}} \leftrightarrow J^2_{\text{hor}}' \\
\mathcal{M} \leftrightarrow \mathcal{M}',
\end{array}
\]

All steps of Cartan’s method act parallely! Then the basic contact 1-form transfers as:

\[
\lambda = dz - z_x dx - F dy \\
= dQ - Q_x dx - Q_y dy \\
= Q_x dx + Q_y dy + Q_a da + Q_b db + Q_c dc - Q_x dx - Q_y dy \\
= Q_a da + Q_b db + Q_c dc,
\]

hence up to a nowhere vanishing factor (absorbed anyway in the \( G \)-structure matrix), we recover our 1-form \( \frac{Q_a}{Q_c} da + \frac{Q_b}{Q_c} db + dc \). Next:

\[
\mu_1 = dz_x - z_{xx} dx - D_x(F) dy \\
= d(Q_x) - Q_{xx} dx - D_x(Q_y) dy \\
= Q_{xx} dx + Q_{xy} dy + Q_{xa} da + Q_{xb} db + Q_{xc} dc - Q_{xx} dx - Q_{xy} dy \\
= Q_{xa} da + Q_{xb} db + Q_{xc} dc,
\]

and similarly:

\[
\mu_2 = dz_{xx} - H dx - D_x(D_x(F)) dy \\
= d(Q_{xx}) - Q_{xxx} dx - Q_{xyy} dy \\
= Q_{xxa} da + Q_{xxx} db + Q_{xxc} dc.
\]
In summary, we have the formulas which provide the transfer:

\[
\{\lambda, \mu_1, \mu_2\} \\
\{dc, db, da\},
\]

namely:

\[
\lambda = Q_a da + Q_b db + Q_c dc, \\
\mu_1 = Q_{xa} da + Q_{xb} db + Q_{xc} dc, \\
\mu_2 = Q_{xxa} da + Q_{xxb} db + Q_{xxc} dc,
\]

with invertible determinant \(\Delta(Q) \neq 0\)—by assumption!—so that \(\{dc, db, da\}\) can inversely be expressed in terms of \(\{\lambda, \mu_1, \mu_2\}\).

If we eliminate \(dc\) from lines 1 and 2:

\[
\mu_1 - \frac{Q_{xc}}{Q_c} \lambda = -\frac{Q_c Q_{xa} + Q_a Q_{xc}}{Q_c} \left\{ da + \frac{-Q_c Q_{xb} + Q_b Q_{xc}}{-Q_c Q_{xa} + Q_a Q_{xc}} db \right\} = \text{nonzero} \cdot \{ da - l db \},
\]

we recognize the 1-form whose kernel, in \(T^{\text{par}} \mathcal{M}\), spans the Levi kernel bundle \(T^{\text{par}}_{\ker \mathcal{M}}\).

To finish with these considerations, remembering that we showed that:

\[
Q'_a da' + Q'_b db' + Q'_c dc' \in \text{Span} \left\{ Q_a da + Q_b db + Q_c dc \right\}, \\
da' - l' db' \in \text{Span} \left\{ Q_a da + Q_b db + Q_c dc, da - l db \right\},
\]

we deduce that on the PDE side:

\[
\lambda' \in \text{Span} \{ \lambda \}, \\
\mu'_1 \in \text{Span} \{ \lambda, \mu_1 \},
\]

which means, as predicted by the property that the jet order is preserved by pullbacks of contact forms, that \(f_4 = 0, \text{always}\).

Without assuming \(h_4 = 0\), we may launch Cartan’s equivalence method for PDE systems of the form \(z_y = F, z_{xxx} = H\), with \(11 + 1\) independent group variables, and with the lifted coframe:

\[
\begin{pmatrix}
\lambda \\
\mu_1 \\
\mu_2 \\
v_1 \\
v_2
\end{pmatrix} = 
\begin{pmatrix}
a & 0 & 0 & 0 & 0 \\
b_1 & f_1 & 0 & 0 & 0 \\
b_2 & f_2 & f_3 & 0 & 0 \\
e_1 & 0 & 0 & h_1 & h_4 \\
e_2 & 0 & 0 & h_2 & h_3
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\mu_1 \\
\mu_2 \\
v_1 \\
v_2
\end{pmatrix}.
\]

This preliminary paper may now stop, because starting from the basic point reached here, advanced and non-straightforward Cartan-type computations are conducted in [16,17].

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