Graph IRF Models and Fusion Rings

DORON GEPNER*

Division of Physics, Mathematics and Astronomy
Mail Code 452–48
California Institute of Technology
Pasadena, CA 91125

ABSTRACT

Recently, a class of interaction round the face (IRF) solvable lattice models were introduced, based on any rational conformal field theory (RCFT). We investigate here the connection between the general solvable IRF models and the fusion ones. To this end, we introduce an associative algebra associated to any graph, as the algebra of products of the eigenvalues of the incidence matrix. If a model is based on an RCFT, its associated graph algebra is the fusion ring of the RCFT. A number of examples are studied. The Gordon–generalized IRF models are studied, and are shown to come from RCFT, by the graph algebra construction. The IRF models based on the Dynkin diagrams of A-D-E are studied. While the A case stems from an RCFT, it is shown that the $D - E$ cases do not. The graph algebras are constructed, and it is speculated that a natural isomorphism relating these to RCFT exists. The question whether all solvable IRF models stems from an RCFT remains open, though the $D - E$ cases shows that a mixing of the primary fields is needed.

* On leave of absence from the Weizmann Institute. Incumbent of the Soretta and Henry Shapiro Chair.
Recently, this author has put forward four categorical isomorphisms among important problems that arise in two dimensional physics [1]. These are integrable $N = 2$ supersymmetric models, rational conformal field theories (RCFT), fusion interaction round the face models (IRF), and integrable soliton systems. It has been shown that the latter three categories are equivalent, and evidence was given for the equivalence with the first category.

The purpose of this note is to further explore these equivalences. In particular, many solvable IRF models are known to be connected to certain (very special) graphs. The graphs are the admissibility conditions for the state variables that are allowed to be on the same link on the lattice. One could try to enlarge the isomorphisms mentioned above to all solvable IRF lattice models. This would mean that the graph must be obtained from the fusion ring of some rational conformal field theory. Specifically, this raises the question: is an IRF model solvable if and only if its admissibility graph arises from the fusion ring of some RCFT? In this note we wish investigate this question. In particular, for each graph we will associate a commutative algebra, which is essentially the multiplication table for the eigenvalues of the incidence matrix. If our conjecture holds, this very same ring must be the fusion ring of some rational conformal field theory. We shall explore a variety of examples.

Let the pair $(S, K)$ where $K \in S \times S$ be a graph, where $S$ is the set of points of the graph and $K$ is the incidence matrix. More generally, we shall allow oriented graphs with multiple links, more conveniently described by the incidence matrix $M_{i,j}$, whose non–negative integer entries describe the number of links from the point $i$ to $j$ where $i, j \in S$. For such a graph we can associate (ambiguously) an interaction round the face (IRF) lattice model. The partition function of the model, which is defined on a square lattice, is given by

\[ Z = \sum_{\text{states}} \prod_{\text{faces}} w \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \] (1)

where $a, b, c, d$ are the state variables, and $a$ and $b$ are allowed to be on the same
link iff $a$ and $b$ are connected by the graph, denoted by $a \sim b$. $w \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is some Boltzmann weight, which is still undefined, and can be considered as the parameters of the model. For multiple links, the Boltzmann weight depends on the particular link, in an obvious generalization. For some graphs and some choices of Boltzmann weights the models are solvable, in the sense that two other Boltzmann weights $w'$ and $w''$ can be found, obeying the same admissibility condition, such that the following relation holds,

$$\sum_c w \begin{pmatrix} b & d \\ a & c \end{pmatrix} w' \begin{pmatrix} a & c \\ g & f \end{pmatrix} w'' \begin{pmatrix} c & c \\ f & e \end{pmatrix} = \sum_c w'' \begin{pmatrix} a & b \\ g & c \end{pmatrix} w' \begin{pmatrix} b & d \\ c & e \end{pmatrix} w \begin{pmatrix} c & e \\ g & f \end{pmatrix}$$

This relation is called the star–triangle equation (STE). It is a very powerful tool in the calculation of the partition function eq. (1), and forms the basis for the solvability of the model.

This raises the important question of which graphs and which choices of Boltzmann weights lead to solvable IRF models. In fact, study have shown, that only for very special graphs such solvable Boltzmann weights exists at all, and then they are more or less unique (for a review, see e.g., [2]). One might speculate that such a solution exists if and only if the graph in question corresponds to the fusion ring of some RCFT, and then the Boltzmann weights are described uniquely by the braiding matrices of the RCFT. Actually, there are obvious counter examples to this conjecture. However, these IRF models do not have second order phase transition points and thus can be considered as ‘bad’ models in the aforementioned sense. Precisely put: does all solvable IRF models with a second order fixed point stem from an RCFT, in the above sense?

Let us thus delve into the definition of fusion IRF models. Let $\mathcal{O}$ be a rational conformal field theory, and let $x$ be a field, typically primary, in the theory. For
an explanation of these notions see for example [1]. In such a theory, the fusion of
the primary fields defines a commutative semi–simple ring,

\[ [p] \times [q] = \sum_r N_{p,q}^r [r], \quad (3) \]

where \([p], [q], [r]\) denote the primary fields, and \(N_{p,q}^r\) are the structure constants,
which are non–negative integers. Now, for any such ring we can associate a family
of graphs in the following fashion. We let the points of the graph be the primary
fields of the theory, and we identify the incidence matrix \(M_{p,q}\) with the structure
constants with respect to a fixed field in the theory \([x]\), \(M_{p,q} = N^q_{x,p}\). Now, given
such a pair \((\mathcal{O}, x)\), we can define an IRF model based of the fusion graph of the
field \(x\), i.e., \(p \sim q\) iff \(N^q_{x,p} > 0\). We denote the resulting lattice model by IRF\((\mathcal{O}, x)\).
It was shown in ref. [1] that indeed all such models, termed fusion IRF models are
solvable, and that Boltzmann weights satisfying the STE, eq. (2), can be found.
The Boltzmann weights are extensions of the braiding matrices of the corresponding
RCFT. The questions is then, is the converse true and all such solvable models are
fusion IRF?

In any event, we can examine known solvable IRF models, to determine if their
admissibility graph comes from an RCFT. If this is the case, such graphs has to
obey some very special properties that are nearly enough to settle the question,
case by case, as well as to determine the specific RCFT.

It was shown in ref. [3] that the fusion ring in an RCFT is connected to a
unitary matrix which is the matrix of modular transformations. The important
thing about \(S\) is that it diagonalizes the eigenvalues of the fusion ring. Namely, if
we define

\[ \{i\} = \sum_j \frac{S_{i,j}}{S_{i,0}} [j], \quad (4) \]

then \(\{i\}\) obeys the fusion product

\[ \{i\} \times \{j\} = \delta_{ij} \{i\}. \quad (5) \]
As the matrix $S$ is unitary, this determines it uniquely from the fusion ring, up to a permutation of the rows, as it is simply the matrix that diagonalizes the fusion ring. (More precisely, it is the point basis in the affine variety defined by the ring [4]). However, not all rings lead to sensible $S$ matrices, and those that do are very special. The reason is that in RCFT, the $S$ matrix needs to be symmetric, and not just unitary, $S = S^t$. This alone is a very strong constraint on the allowed fusion rings. A further restriction arises from the fact that every such ring must admit a non-degenerate symmetric bi-linear form $(a, b)$, where $a$ and $b$ are primary fields, defined by $(a, b) = 1$ iff $N_{a b}^1 = (a, b)$, where 1 stands for the unit in the ring (which is a primary field). Further $(a, b)$ must be either 0 or 1 for all the primary fields $a$ and $b$, and for each primary $a$, $(a, b)$ is zero, for all $b$ except for a unique choice. Thus the bi-linear form defines a unique conjugate for each field, $\bar{a}$ which is the unique field for which $(a, \bar{a}) = 1$.

Thus, the question whether a given graph stems from an RCFT can be examined on the basis of whether the above properties holds for the graph, and its associated fusion ring. Let $(S, K)$ be an arbitrary graph then, with the incidence matrix $M_{i,j}$. Denote the eigenvalues of $M$ by $v_\alpha^i$, i.e.,

$$\sum_j M_{i,j} v_\alpha^j = \sum_\gamma \lambda^\alpha v_i.$$  \hspace{1cm} (6)

We can normalize the eigenvalues to unity, $\sum_j v_\alpha^j v_\alpha^j = 1$. The eigenvalues are thus uniquely defined (up to a phase). We can now write down a commutative associative algebra associated to the eigenvalues. We do so by specifying a unique choice for the ‘unit’ element, denoted by say 1. Further, we define the product of the elements $\alpha$ and $\beta$ to be,

$$[\alpha] \times [\beta] = \sum_\gamma N_{\alpha,\beta}^\gamma [\gamma],$$  \hspace{1cm} (7)

5
where the structure constants \( N^\gamma_{\alpha,\beta} \) are defined by

\[
\frac{v^\alpha_j v^\beta_j}{v^1_j v^1_j} = \sum_\gamma N^\gamma_{\alpha,\beta} \frac{v^\gamma_j}{v^1_j},
\]

(8)

for all \( j \). Since the eigenvectors are linearly independent, \( N \) is so uniquely defined.

The eigenvalues can be normalized, in which case the matrix \( v^h_j \) is unitary,

\[
\sum_j v^h_j (v^p_j)^* = \delta_{h,p},
\]

(9)

where \( h \) and \( p \) are any two exponents. We thus find from eq. (8), the following form for the structure constants,

\[
N^\gamma_{p,q} = \sum_j \frac{v^p_j v^q_j (v^\gamma_j)^*}{v^1_j}.
\]

(10)

If \( v^h_j \) is a modular matrix of an RCFT [5], eq. (10) gives the fusion coefficients [5], according to the formula of ref. [3].

For a non RCFT, since the matrix of eigenvalues, \( v^p_j \), is inherently non–symmetric, we can define a transposed algebra, based on the nodes of the diagram, instead, in a similar fashion. The structure constants of the algebra are then given by

\[
M^k_{i,j} = \sum_h \frac{v^i_h v^j_h (v^k_h)^*}{v^1_h},
\]

(11)

where the structure constants, \( M^k_{i,j} \), describe the product of the nodes of the graph,

\[
\frac{v^i_h v^j_h}{v^1_h v^1_h} = \sum_k M^k_{i,j} \frac{v^k_h}{v^1_h}.
\]

(12)

For an RCFT the two algebras are, of course, the same. The algebra so defined suffers from a number of ambiguities. First, the phase of the eigenvectors was
arbitrary. However, this is simply a redefinition of the basis elements. More importantly the choice for the unit field ‘1’ was arbitrary, and for each such choice a different algebra is found. To summarize, for a pair of any graph and a point in it we defined uniquely an algebra, denoted by $A(G,p)$, where $G$ is the graph and $p$ is the point. Further, the algebra has a unique preferred basis, up to a phase.

Now, if the graph in question stems from a fusion ring, then the graph algebra, so defined, is identical with the fusion ring of the theory, provided that we take for the preferred point the unit field of the fusion ring. Further, up to a phase, the preferred basis of the graph algebra is the primary field basis of the fusion ring, up to the phase ambiguity mentioned above.

It follows that the question whether an IRF model stems from an RCFT boils down to the question of whether its associated graph algebra is a fusion ring. In light, of the many properties of such fusion rings, only very special graphs can be candidates for fusion rings. Further, the RCFT may be constructed from the fusion ring itself. Thus by studying the graph algebra, the question raised in the introduction can be settled.

Let us illustrate this construction by an example. A class of solvable IRF models called the Gordon–Generalized (GG) hierarchy has been found [6]. The state variables in these models take the values $a = 0, 1, \ldots, k - 1$, where $a$ are the state variables, and $k$ is any integer. The admissibility condition for the graph is

$$a \sim b \quad \text{iff} \quad a + b \leq k - 1.$$  

(13)

It was shown in ref. [6] that the models so obtained are solvable, and Boltzmann weights satisfying the STE were found. The case of $k = 2$ is the well known hard hexagon model [7]. Now, let us construct the graph algebra associated to this graph, for any $k$. For the unit field we choose the element $[k - 1]$. It is a straightforward calculation that the algebra so obtained assumes the form,

$$[i] \times [j] = \sum_{m = |i-j|}^{2k-1-i-j} [m],$$  

(14)

$$m \equiv i-j \mod 2$$

7
where we identified $[2k - 1 - i] \equiv [i]$. It can be checked that this algebra has all the properties of a fusion ring. In fact, this is the known fusion ring of the RCFT $SU(2)_{2k-1}/SU(2)_{1/(2k-1)}$, described in ref. [1]. The graph itself is obtained from the field $x = [k - 1]$. We conclude that the GG hierarchy is the fusion IRF model $IRF(SU(2)_{2k-1}/SU(2)_{1/(2k-1)}, [k - 1])$. It can be further checked that the Boltzmann weights described in ref. [6] are indeed the extensions of the braiding matrices of this RCFT.

It is quite straightforward to see directly that the admissibility condition for the GG hierarchy models, eq. (13), is indeed precisely what is obtained by fusion with respect to the field $[k - 1]$ in the theory $G = SU(2)_{2k-1}/SU(2)_{1/(2k-1)}$. We identify the state $(\sigma)$ of the GG hierarchy model with the primary field $[k - 1 - \sigma]$ in the theory $G$. We can now compute the fusion with respect to the field $[k - 1]$. We find, according to eq. (14),

$$[k - 1] \times [k - 1 - \sigma] = [\sigma] + [\sigma + 2] + \ldots + [2k - 2 - \sigma] = \sum_{\rho=0}^{k-1-\rho} [k - 1 - \rho],$$

(15)

where we used the identification of fields, $[\sigma] = [2k - 1 - \sigma]$, which holds in the theory $G$. As the state $(\sigma)$ is identified with the primary field $[k - 1 - \sigma]$, eq. (15) implies precisely the GG admissibility condition, eq. (13). This concludes the proof that the GG models are fusion IRF.

The theory $G$ may be constructed explicitly, ref. [1], as a sub-sector of the theory $SU(2)_{2k-1} \times (E_7)_1$, with an extended algebra (for an example). The case of $k = 1$ corresponds to $(G_2)_1$ current algebra.

Consider now, as another example, the KAW hierarchy of models defined in ref. [8]. The state variables of the models along with their admissibility condition are given by,

$$\sigma_i = 0, 1, 2, \ldots, k - 1, \quad k - 2 \leq \sigma_i + \sigma_j \leq k,$$

(16)

where $k$ is some integer, and $\sigma_i$ and $\sigma_j$ are any two neighboring states. Let $G \equiv SU(2)_k$ be the current algebra theory associated to $SU(2)_k$. Denote as before by
the field with the isospin $j/2$, and let $p = [k - 1] + [k]$ be a field which is a mixture of two primary fields. We can compute the fusion with respect to the field $p$, according to the usual rules of $SU(2)_k$, eq. (24), and we find,

$$p \times [\sigma_i] = [\sigma_i] \times [k] + \sigma_i \times [k - 1] = \sum_{k - 2 \leq \sigma_i + \sigma_j \leq k} [\sigma_j],$$

(17)

which are precisely the fusion admissibility conditions of the KAW hierarchy, eq. (16). Thus, we conclude that the GAW model is the fusion lattice model IRF($SU(2)_k, [k] + [k - 1], [k] + [k - 1]$). This is an example of a model based on a mixture of primary fields. Such models where considered in ref. [1], and their Boltzmann weights given as an extension of the conformal braiding matrices of the RCFT. It would be an interesting exercise to compare the Boltzmann weights given in ref. [1], based on the fusion properties, and those given in ref. [8], by a direct solution, and to show that they indeed coincide.

Let us turn now to another example. Consider the so called grand hierarchy of solvable IRF lattice models, discussed in refs. [8,9]. These models are described by

$$l_i = 0, 1, \ldots, k, \quad l_i - l_j = -N, -N + 2, \ldots, N, \quad l_i + l_j = N, N + 2, \ldots, k - N,$$

(18)

where $l_i$ are the state variables, $l_i$ and $l_j$ are adjacent, and $k$ and $N$ are arbitrary integers. For each $k$ and $N$, a solvable model was found [9], using compositions of the eight vertex model. In fact, eq. (18), is exactly the well known fusion rules of $SU(2)_k$. It is rather evident from these fusion rules, eq. (24), that $l_i$ is admissible to $l_j$, if and only if $N^l_{i,j,p} \geq 0$, where $p = [N]$ primary field, in the previous notation. We conclude that the grand hierarchy is exactly the fusion IRF models IRF($SU(2)_k, [N], [N]$), for any $k$ and any $N$. Again, it would be interesting to verify that the Boltzmann weights coming from RCFT [1], and those computed directly in ref. [9], are identical. For $N = 1$ (the Andrews–Baxter–Forrester model
this was done in ref. [1], and the results indeed agree. For larger $N$ this verification is left to further work.

Before proceeding, let us discuss one subtlety in the logic of identifying the graph algebra with the fusion rules. Recall that the primary fields were identified with the eigenvalues of the incidence matrix, and that this identification was unambiguous up to a phase. However, in case the incidence matrix has degenerate eigenvalues, we can no longer distinguish which mixture of these eigenvectors are the primary fields, and additional information may be needed.

Let us now proceed to another interesting family of solvable IRF lattice models. These are the lattice models based on the simple Lie algebras which are $A_n$ [10] and $D_n$, $E_6$, $E_7$ and $E_8$ [11]. The states of the ADE IRF models are in one-to-one correspondence with the simple roots of the respective Lie algebra. Similarly, the admissibility graph is the Dynkin diagram of the algebra. Thus, the incidence matrix of the model is given by $M_{ab} = 2\delta_{ab} - C_{ab}$ where $C_{ab}$ is the Cartan matrix of the algebra. (For a review on simple Lie algebras see, e.g., [12].)

The Boltzmann weights of the ADE models [11] have the relatively simple graph state form (see, e.g., [1] and ref. therein),

$$w \left( \begin{array}{c|c} a & b \\ \hline c & d \end{array} \right| u \right) = \sin(\lambda - u)\delta_{bc} + \left( \frac{\psi_b\psi_c}{\psi_a\psi_d} \right)^{\frac{1}{2}} \sin u,$$

(19)

where $u$ labels the different Boltzmann weights satisfying the STE, eq. (2), which are given by the values $u$, $u + v$ and $v$, for $w$, $w'$ and $w''$, respectively, for any complex $u$ and $v$. $\psi_a$ is the eigenvector with the largest eigenvalue of the incidence matrix (the so called Perron–Frobenius vector),

$$\sum_{b} M_{ab}\psi_b = 2\cos \lambda \psi_a,$$

(20)

where $\beta = 2\cos \lambda$ is the maximal eigenvalue of the incidence matrix, given by

$$\lambda = \frac{\pi}{g},$$

(21)
Table 1.

| Algebra | Coxeter Number | Exponents          |
|---------|----------------|--------------------|
| $A_n$   | $n + 1$        | $1, 2, \ldots, n$  |
| $D_n$   | $2n - 2$       | $1, 3, 5, \ldots, 2n - 3, n - 1$ |
| $E_6$   | 12             | $1, 4, 5, 7, 8, 11$ |
| $E_7$   | 18             | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$   | 30             | $1, 7, 11, 13, 17, 19, 23, 29$ |

where $g$ is the Coxeter number of the algebra. The entire set of eigenvalues of the incidence matrix is given by

$$\lambda_h = 2 \cos(\pi h/g), \quad (22)$$

where $h$ is any of the exponents of the Lie algebra (which can be degenerate). The exponents, along with the Coxeter number are described in table (1).

Now, in light of the conjecture raised in the introduction, we would like to examine if the ADE IRF models are fusion IRF models, i.e. if they arise from a conformal field theory. To this end, let us proceed with constructing the graph algebras associated with the Dynkin diagrams of simple Lie algebras. If our conjecture is correct, this should be the fusion ring of some RCFT. To do so, we first need to calculate the eigenvectors of the Cartan matrix of each Lie algebra, and then insert these into eq. (10).

We shall skip the $A_n$ cases (ABF models), as these have already been demonstrated to be the fusion IRF models associated with the RCFT $SU(2)_{n-1}$ [1]. For completeness sake, the eigenvectors for the $A_n$ graph are given by

$$v_{ij} = \left(2 \right)^{\frac{1}{2}} \sin\left(\frac{\pi ij}{n+1}\right), \quad (23)$$

where $i$ labels the simple roots and $j$ labels the exponents, and $i, j = 1, 2, \ldots, n$. This is non–else but the toroidal modular matrix of the RCFT $SU(2)_k$ [5], showing
that the graph algebra is identical to the fusion ring of the model, which has the product rule [5],

\[ [i] \times [j] = \sum_{\substack{\ell = |i-j| \\ l-i-j=0 \mod 2}} \min(2k-i-j,i+j) [\ell], \]  

(24)

where \([\ell]\) labels the \(l\)th primary field. The Boltzmann weight, eq. (10), can be seen to give at the limit \(u \to i\infty\) the braiding matrix of the respective RCFT [1], concluding the proof that the ABF model is a fusion IRF.

Let us turn now to the \(D_n\) algebras. From table (1) the exponents of the algebra are, 1, 3, 5, \ldots, \(2n-3\), \(n-1\), and thus are all different for odd \(n\), and have a twofold degeneracy at \(n-1\) for even \(n\). The eigenvalues of the incidence matrix are given by eq. (22), \(\lambda_h = 2 \cos(\pi h/(2n-2))\), where \(h\) is any of the exponents. The eigenvectors of the incidence matrix are readily computed and are found to be,

\[ v^h_j = \begin{cases} \sqrt{\frac{2}{n-1}} \sin\left(\frac{\pi hj}{2(n-2)}\right) & \text{for } j \leq n-2 \\ \frac{(-1)^j}{\sqrt{2}(n-1)} & \text{for } j = n-1, n-2, \end{cases} \]  

(25)

for the odd exponents \(h = 1, 3, 5, \ldots, 2n-3\). For the exceptional exponent, \(h = n-1\), we find,

\[ v^{n-1}_j = \begin{cases} 0 & \text{for } j \leq n-2, \\ \frac{(-1)^j}{\sqrt{2}} & \text{for } j \geq n-1. \end{cases} \]  

(26)

We have normalized the eigenvectors to have the absolute value one, and thus \(v^h_j\) is the unitary matrix which diagonalizes the incidence matrix.

We next proceed to calculate the graph algebra, using eq. (10). Denote by \([h]\), \(h = 1, 3, \ldots 2n-3\), the elements of the algebra associated to the regular exponents, and by \(z = [n-1]\) the element associated to the exceptional one. Then the graph
algebra can be computed from eq. (10), and we find,

\[
[p] \times [q] = \sum_{r=|p-q|+1 \mod 2}^{\min(p+q-1,4n-5-p-q)} [r],
\]

\[
z \times [p] = (-1)^{(p-1)/2} z,
\]

\[
z \times z = \frac{1}{2(n-1)} \sum_{r=1 \mod 2}^{2n-3} (-1)^{(r-1)/2} [r].
\]

(27)

It is striking that up to a trivial rescaling of \( z \), \( z \rightarrow z \sqrt{2(n-1)} \), all the structure constants are integers. This implies that the graph algebra of \( D_n \), any \( n \), is actually a commutative ring with a unit. As no two eigenvectors are the same, the ring is a semi–simple one, which is a finite dimensional algebra, with vanishing nil and Jacobson radicals. It is straight forwards to present this ring in terms of generators and relations. Let \( T_n(x) \) be the Chebyshev polynomial of the second kind, defined by \( T_n(2 \cos \phi) = \frac{\sin((n+1)\phi)}{\sin \phi} \). Then the generators of the ring may be taken to be \( z \) and \( x = [3] \), along with the relations \( p_1(x) = p_2(z,x) = p_3(z,x) = 0 \), where

\[
p_1(x) = T_{2n-4}(\sqrt{1+x}) + T_{2n-2}(\sqrt{1+x}),
\]

\[
p_2(x,z) = z^2 - \frac{1}{2(n-1)} \sum_{h=1 \mod 2}^{2n-3} (-1)^{(h-1)/2} [h],
\]

\[
p_3(z,x) = zx + x,
\]

\[
[h] = T_{h-1}(\sqrt{1+x}),
\]

(28)

where \([h]\) expresses the basis element \([h]\) as a polynomial in \( x \), and is used to express \( p_2 \) as a polynomial in \( x \) and \( z \). In other words,

\[
\mathcal{R} \approx \frac{P[x,z]}{(p_1,p_2,p_3)},
\]

(29)

where \( \mathcal{R} \) denotes the graph algebra, \( P[x,z] \) is the algebra of polynomials in \( x \) and \( z \), and \((p_1,p_2,p_3)\) is the ideal in it generated by the three polynomials \( p_1, p_2 \) and \( p_3 \).
It remains to be seen, now, if this graph algebra satisfies any of the properties of a fusion ring of an RCFT, in accordance with the conjecture raised in the introduction. Quite evidently the answer is no! There are a number of problems. 1) Some of the structure constants are negative integers. 2) There is no appropriate symmetric bilinear form. 3) In a related way, the $S$ matrix cannot be made symmetric. To be more precise let $p^I_h$ be the alleged symmetric bi–linear form, where since the matrix $p$ has a unique 1 in each row and column, it is actually a permutation, expressing a map between exponents $h$, denoted by $k(j)$, where $j$ is a Dynkin node, and $k(j)$ is an exponent, and the nodes of the Dynkin diagram. From the properties of RCFT, the modular matrix must be symmetric, when $p$ is used to lower the index. Thus, RCFT requires that,

$$v^k(j) = v^l_{k(l)}.$$  \hspace{1cm} (30)

More generally, we could allow for a change of normalizations of the eigenvectors, which are defined only up to a phase, in which case, eq. (30) assumes the form,

$$\sum_h v^h_j p_{hl} = \sum_h v^h_l p_{hj}.$$ \hspace{1cm} (31)

It is readily seen that eq. (31), has no solutions for $p_{hj}$, when $v^h_j$ is taken to be the eigenmatrix of $D_n$, eqs. (25-26). Thus, this graph algebra is not the fusion ring of any RCFT.

This appears to be quite a catastrophe for the original conjecture we raised, and in fact, a counter example for it. The question is if there is a way in which the conjecture we raised could be relaxed, and that this graph algebra can be related to an RCFT? More precisely, we assumed in our entire discussion, that the nodes of the graph are the primary fields of the RCFT. There is actually no reason for this assumption, as we can well build a fusion IRF model based on non–primary fields [1]. Clearly, this entails a change of basis for the graph algebra. Thus, the question becomes whether the graph algebra is isomorphic to a fusion ring of an
RCFT. Unfortunately, this is a rather meaningless question, since it is well known that two finite dimensional algebras are isomorphic if they are both semi–simple, and have the same dimension. Thus, any graph algebra is isomorphic to any fusion algebra, provided they have the same number of nodes, respectively, primary fields. Clearly, this is too weak a criteria to be of much use.

We conclude the discussion of the $D_n$ cases by noting one particular basis in which things look rather close to an RCFT. We form the basis,
\[
\alpha^+_h = \frac{1}{2}([h] + [2n - 2 - h]),
\]
\[
\alpha^-_h = (-1)^{(h-1)/2} \frac{1}{2}([h] - [2n - 2 - h]),
\]
where $h = 1, 3, \ldots, n - 3$. In this basis, the algebra decouples to a direct sum of two subalgebras. These are the subalgebras generated by $\alpha^+_h$ (along with $z$, for even $n$), and $\alpha^-_h$ (along with $z$, for odd $n$). Denote these two subalgebras by $A$ and $B$. Then $AB = 0$, and $G \approx A \oplus B$. The question now is any of $A$ and $B$ are the fusion rings of an RCFT? The fusion rules in this basis become,
\[
\alpha^+ \alpha^\pm = \sum_{s=|r-t|+1, \ t=1 \bmod 2}^{r+t-1} \alpha^\pm_s,
\]
\[
z\alpha^+_h = \begin{cases} 
0 & \text{odd } n, \\
z & \text{even } n,
\end{cases}
\]
\[
z\alpha^-_h = \begin{cases} 
0 & \text{even } n, \\
z & \text{odd } n,
\end{cases}
\]
\[
z^2 = \begin{cases} 
\sum_{h=1 \bmod 2}^{n-2} \alpha^+_h & \text{odd } n, \\
\sum_{h=1 \bmod 2}^{n-2} \alpha^-_h & \text{even } n.
\end{cases}
\]
It can now be seen that the subalgebra $A$ (generated by $\alpha^+_h$, for odd $n$, and by $\alpha^-_h$, for even $n$), gives rise to a symmetric $S$ matrix. Namely, the eigenvectors matrix
of this algebra, may be written as,

\[ S_{h,t} = C \sin \left( \frac{\pi ht}{2n-2} \right), \quad (34) \]

for \( h, t = 1, 3, \ldots n - 2 \), and where \( C \) is some constant. Clearly \( S \) is a symmetric unitary matrix, and the question remains whether it corresponds to an RCFT.

Note, that if we take \( n \) to be half integral, this is exactly the \( S \) matrix of the RCFT \( SU(2)_{2n-4}/SU(2)_{1/(2n-4)} \). Unfortunately, for an integral \( n \), it can be seen, except for the trivial, \( n = 4 \), not to correspond to an RCFT, as the equation \((ST)^3 = 1\) has no solutions with a diagonal matrix \( T \). As this is a necessary condition for \( S \) to be a modular matrix, we conclude that the above \( S \) is not the modular matrix of any RCFT. For \( n = 5, 7 \), the entire eigenvalue matrix is symmetric, but still does not appear to stem from an RCFT. Other basis in which the \( S \) matrix is symmetric can be found. The significance of these observations remains to be studied. At this point we conclude that the relation of the \( D_n \) models with RCFT is moot, though short of a counter example to our conjecture.

Let us turn now to the case of the exceptional algebras \( E_n \), for \( n = 6, 7, 8 \). The exponents of the respective algebras are listed in Table. 1. The eigenvalues are thus, \( \lambda_h = 2 \cos(\pi h/g) \), where \( h \) is any of the exponents. The eigenvectors are found to be,

\[
\begin{align*}
    v_j &= \sin \left( \frac{\pi j h}{g} \right) \\
    v_n &= \sin \left( \frac{3\pi h}{g} \right) \\
    v_{n-2} &= \sin \left( \frac{\pi (n-2)h}{g} \right) - \frac{\sin(\frac{\pi (n-3)h}{g})}{2 \cos(\frac{\pi h}{g})}, \\
    v_{n-1} &= \frac{v_{n-2}}{2 \cos(\frac{\pi h}{g})},
\end{align*}
\]

(35)

The eigenvectors are further normalized to have absolute value one, \( v_j^h \to v_j^h / \sqrt{\sum_j (v_j^h)^2} \).

Consider now the case of \( E_6 \). It is more convenient to define here the transposed graph algebra associated to the nodes, eq. (11). We find that all the structure constants are positive integers. Denoting by \([j]\) the element of the algebra associated
to the \(j\)th node, we find that the transpose ring is given by
\[
R \equiv \frac{P[x,y]}{(y^3 - 2y, x^2 - xy - 1)},
\]
where the Dynkin nodes basis elements of the algebra are given by,
\[
[1] = 1, \quad [2] = x, \quad [3] = xy, \quad [4] = x(y^2 - 1), \quad [5] = y^2 - 1, \quad [6] = y. \quad (36)
\]

All the products may be computed from the two relations, \(y^3 - 2y = x^2 - xy - 1 = 0\).
The first question now is whether the above graph algebra is a fusion ring. It is easy to see that this is not the case, and that the matrix of eigenvectors \(v^h_j\) cannot be symmetrized. Thus, although close to the notion of a fusion IRF model, the \(E_6\) IRF model does not stem from an RCFT.

As in some of the \(D_n\) cases we can form combinations of the Dynkin nodes that give rise to a symmetric matrix of eigenvalues. These are the combinations \([1] \pm [5], [2] \pm [4], [3] \pm [6]\) (up to normalizations). The combination \([1] - [5], [2] - [4]\), generates a subalgebra which decouples from the rest of the algebra, i.e., the fusion algebra is the direct sum of the two algebras, \(A\) generated by \([1] - [5]\) and \([2] - [4]\), and the algebra \(B\) generated by the rest, \(AB = 0\). The corresponding matrix of eigenvalues is symmetric, and thus a candidate for a modular matrix of an RCFT. The fusion ring \(A\) is seen to be isomorphic to that of \(SU(2)_1\) and thus is coming from an RCFT, \(R \equiv P[x]/(x^2 - 1)\). However, the structure constants of \(B\) cannot be made integral, and thus it cannot be the fusion ring of any RCFT.

Again, the significance of this observation is unclear.

In the case of \(E_7\) we find the transpose graph algebra which is,
\[
R \equiv \frac{P[x]}{(x^7 - 6x^5 + 9x^3 - 3x)}, \quad (37)
\]
where the elements associated to the Dynkin nodes are given by,
Again, all the structure constants are non-negative integers, in terms of the Dynkin basis elements, $M_{ij}^k \geq 0$.

In the case of $E_8$, we find the transposed graph ring again has all the structure constants as positive integers. The ring is given by

$$R \equiv \frac{P[x]}{(1 - 8x^2 + 14x^4 - 7x^6 + x^8)}.$$  \hspace{1cm} (39)

The Dynkin elements are now given by,

$$[1] = 1,$$
$$[2] = x,$$
$$[3] = -1 + x^2,$$
$$[4] = -2x + x^3,$$
$$[5] = 1 - 3x^2 + x^4,$$
$$[6] = -2x + 9x^3 - 6x^5 + x^7,$$
$$[7] = -2 + 9x^2 - 6x^4 + x^6,$$
$$[8] = 5x - 13x^3 + 7x^5 - x^7.$$  \hspace{1cm} (40)

The polynomials we find for $E_n$, $n = 6, 7, 8$, may be considered as the generalizations of the Chebishev polynomials, which arise for $A_n$, to the exceptional algebras.
A very interesting property of all the $E_n$ graph algebras we find is that the product with $x = [2]$ gives back the incidence matrix of the graph,

$$x \times [n] = \sum_m M_{nm}[m],$$

(41)

where $M_{nm}$ is the incidence matrix of the graph, which is the Dynkin diagram of the respective algebra. Further, this rule alone, determines uniquely the entire algebra. Thus, the rings we found are exactly those giving the Dynkin graph as an admissibility condition. Namely, the $E_n$ models can be thought of as the models \(\text{IRF}(R, x)\), where $R$ stands for the graph ring, and $x = [2]$ is the element used in the admissibility condition. The fact that we managed to lift the admissibility relation to a full ‘fusion ring’ with positive integer structure constants is non–trivial, and may be connected with the solvability of the model.

In all the $E_n$ cases, the matrix of eigenvalues is non–symmetric, and thus does not correspond directly to an actual RCFT. As noted earlier, they are certainly isomorphic to fusion rings of RCFT, but this is a somewhat meaningless fact. As in the $D_n$ case, we can also form the graph algebra, based on the exponents, eq. (10). Forming as in the $D_n$ case, the combinations, $A_\pm = \frac{1}{2}([h] - [g - h])$ and $A_\pm = \frac{1}{2}([h] + [g - h])$, where $h$ is any of the exponents, we find that the two subalgebras decouple, $A_-A_+ = 0$. For $E_7$ this leads also to a symmetric eigenvalue matrix, and fusion rules which are integers, and thus are full candidates for an RCFT. It remains to explore further whether an RCFT based on this can be built. For $E_6$ and $E_8$ we find non–symmetric eigenvalue matrix, indicating that the two subalgebras do not represent an RCFT.

In conclusion, we studied here a variety of examples of solvable IRF models, to judge if they stem from an RCFT. We formed graph algebras based on the eigenvalues of the admissibility conditions. If a theory stems from an RCFT its graph algebra must be the fusion rules of the model. This also gives immediately the solutions to the STE, eq. (2), by an extension of the braiding matrices of the theory, as described in ref. [1]. We studied here two families of examples,
the Gordon generalized (GG), KAW and grand hierarchies, and the Pasquier $D-E$ models. Remarkably, it was shown that all the hierarchy models model stem from some RCFT related to $SU(2)$ current algebra. On the other hand, the $D-E$ models were seen not to correspond directly to an RCFT. It remains to be studied, whether these models stem from a mixture of primary fields in an RCFT. While, not ruling the possibility out, we found that, in general, there does not seem to be a natural way to relate these models to an RCFT. The most likely conclusion to be drawn is that while most solvable IRF models studied to date come from some RCFT, other solutions to the STE exist, which do not appear to be related to an RCFT in the manner described in ref. [1], with the $D$ and $E$ models as examples. The question certainly requires further study.

We hope that we have further illuminated here the connection between RCFT and solvable lattice models. A great host of models, erratically constructed previously, all stem from the unified construction described in ref. [1]. While few exceptions were found, it remains to study how these can be fitted into the general framework.

**ACKNOWLEDGEMENTS**

I thank P. Di–Francesco, W. lerche and J.B. Zuber for helpful comments. While writing this work, I received [13], which somewhat relates to the present work.
REFERENCES

1. D. Gepner, “Foundations of rational conformal field theory, I”, Caltech preprint, CALT–68–1825, November (1992).

2. R.J. Baxter, “Exactly solved models in statistical mechanics”, Academic Press, London, 1982.

3. E. Verlinde, Nucl. Phys. B300 (1988) 360

4. D. Gepner, Comm. Math. Phys. 141 (1991) 381.

5. D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493.

6. A. Kuniba, Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn 55 (1986) 1092, Phys. Lett. A 116 (1986) 382, Phys. Lett. A 117 (1986) 358; R.J. Baxter and G.E. Andrews, J. Stat. Phys. 44(1986) 249, 371.

7. R.J. Baxter, J. Phys. A 13 (1980) L61.

8. A. Kuniba, Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. 55 (1986) 2605.

9. Y. Akutsu, A. Kuniba and M. Wadati, J. Phys. Soc. Jpn. 55 (1986) 2907; E. Date, M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 12 (1986) 209; Phys. Rev. B35 (1987) 2105; E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. B290 [FS20] (1987) 231

10. G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193

11. V. Pasquier, J. Phys. A20 (1986) L217, L221

12. Humphreys, Introduction to Lie algebras and representation theory, Springer–Verlag, New–York (1972)

13. P. Di Francesco, F. Lesage and J.B. Zuber, Saclay preprint SPhT 93/057, June (1993)