THE PATH GROUP CONSTRUCTION
OF LIE GROUP EXTENSIONS

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Abstract. We present an explicit realization of abelian extensions of infinite dimensional Lie groups using abelian extensions of path groups, by generalizing Mickelsson’s approach to loop groups and the approach of Losev-Moore-Nekrasov-Shatashvili to current groups. We apply our method to coupled cocycles on current Lie algebras and to Lichnerowicz cocycles on the Lie algebra of divergence free vector fields.

1. Introduction

In infinite dimensions Lie’s third theorem is not valid: there exist Lie algebras which do not correspond to any Lie group. In particular given a connected infinite dimensional Lie group $G$, not every abelian Lie algebra extension $\mathfrak{a} \to \hat{\mathfrak{g}} \to \mathfrak{g}$ of its Lie algebra $\mathfrak{g}$ comes from an abelian Lie group extension of $G$. The obstructions determined in [15] involve the fundamental and the second homotopy groups of $G$. For instance if $G$ is simply connected, the integrability condition for the Lie algebra extension $\hat{\mathfrak{g}}$ described by the Lie algebra 2-cocycle $\omega$ on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathfrak{a}$, reduces to the discreteness of the period group $\Pi_\omega \subset \mathfrak{a}$ of the cocycle $\omega$. Under this assumption, for any discrete subgroup $\Gamma$ of the space $\mathfrak{a}^G$ of $G$-invariant elements of $\mathfrak{a}$, containing the period group, there exists a corresponding abelian Lie group extension $A \to \hat{G} \to G$ of $G$ by the $G$-module $A = \mathfrak{a}/\Gamma$.

Much studied is the central extension of the loop group $G = C^\infty(S^1, H)$ of a simple Lie group $H$. With a suitable multiple $\kappa$ of the Killing form of the Lie algebra $\mathfrak{h}$ of $H$, a Lie algebra 2-cocycle on the loop algebra $\mathfrak{g} = C^\infty(S^1, \mathfrak{h})$ is

$$\omega(X, Y) = 2 \int_{S^1} \kappa(X, dY).$$

When the simple Lie group $H$ is simply connected then $G$ is also simply connected, and the period group is $\Pi_\omega = \mathbb{Z}$, so $\omega$ is integrable. Explicit constructions of the corresponding central extension $\mathbb{T} \to \hat{G} \to G$ of the loop group $G$ by the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be found in Chapter 4 of [18], in [11], [13], and in Chapter 4 of [12].

The approach in the book of Pressley-Segal [18] is very general: one considers a simply connected prequantizable manifold $(M, \Omega)$ (i.e. $\Omega$ is a closed integral 2-form on $M$ and there is a principal circle bundle $P$ over $M$ with a principal connection having curvature $\Omega$), together with a $G$-action preserving $\Omega$ and one pulls back Kostant’s prequantization central extension $\mathbb{Z}$ to $G$ by this action. More precisely the resulting extension $\hat{G}$ is the group of all fiber preserving diffeomorphisms of $P$ which preserve the connection 1-form and cover an element of $G$. A concrete
description of the central extension of \( G \) obtained in this way, using paths in \( M \), is given there. Central Lie group extensions associated to Hamiltonian actions on a prequantizable manifold \((M, \Omega)\) are considered in [17]. A generalization of this approach to abelian extensions is the subject of [21].

A second approach [11] and [12] is due to Mickelsson: the central extension of the loop group is explicitly realized as a quotient group of a central extension of the group of currents on the 2-disk \( D \). The last central extension is given by the group 2-cocycle \( c(f, g) = \int_D \delta^l f \wedge \delta^r g \), where \( \delta^l \) and \( \delta^r \) denote the left and right logarithmic derivative. This construction is generalized to central extensions of current groups on Riemann surfaces in [3]. The construction of central extensions of current groups on arbitrary compact manifolds [9] is due to Losev-Moore-Nekrasov-Shatashvili explicitly using the path group of the current group.

In this paper we generalize this last approach, obtaining a path group method for the construction of abelian Lie group extensions. We consider a connected Lie group \( G \) and the exact sequence of Lie groups \( \Omega_0G \to PG \to \tilde{G} \) where \( \tilde{G} \) is the universal covering group, \( PG \) the path group and \( \Omega_0G \) its subgroup of null-homotopic loops. We also consider a 2-cocycle \( \omega \) on \( g \) with values in the \( g \)-module \( a \), having a discrete period group, and the discrete subgroup \( \Gamma \supset \Pi_\omega \) of \( a^G \) with \( A = a/\Gamma \). An abelian Lie group extension of \( \tilde{G} \) by \( A \) integrating \( a \rtimes \omega \) \( g \) is obtained as the quotient of an abelian Lie group extension \( A \rtimes \omega \) \( PG \), by the graph of a map \( \lambda : \Omega_0G \to A \). The group cocycle \( c : PG \times PG \to A \), as well as the map \( \lambda \) are explicitly given by formulas (Theorem 1). A geometric construction of abelian Lie group extensions using the path group is presented in [5]. We show that the two group extensions are isomorphic.

Our method works well in concrete settings, in spite of the heaviness of the formula for \( c \). In Section 4 we specialize to loop groups and current groups. Here a symmetrization procedure applied to \( c \) and \( \lambda \) simplify considerably the result, thus recovering the constructions in [12] and [9]. In Section 5 we treat the coupled cocycle on current Lie algebras defined by Neeb in [16]. Finally we use this construction in Section 6 to explicitly realize central extensions of the group of volume preserving Lichnerowicz cocycles.

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2. The path group

Let \( G \) be a Lie group, \( g \) its Lie algebra, \( a \) a smooth \( G \)-module (i.e. the action map \( \rho_a : G \times a \to a \) is smooth) and \( a^G \) the subspace of \( G \)-invariant elements. The continuous \( a \)-valued Lie algebra 2-cocycle \( \omega \) on \( g \) defines a closed equivariant \( a \)-valued 2-form \( \omega^{eq} \) on \( G \). This means \( \rho_a(g) \circ \omega^{eq} = \lambda^g \omega^{eq} \), \( \lambda \) denoting the left translation by \( g \) in \( G \). The period homomorphism of the 2-cocycle \( \omega \) is by definition

\[
\text{per}_\omega : \pi_2(G) \to a^G, \quad \text{per}_\omega([\tau]) = \int_{S^2} \tau^* \omega^{eq}
\]  

for \( \tau : S^2 \to G \) a piecewise smooth representative of a free homotopy class (a spherical 2-cycle). The image \( \Pi_\omega \subset a^G \) of \( \text{per}_\omega \) is called the period group of \( \omega \).

We present van Est’s method [2] for obtaining group 2-cocycles by integrating the closed 2-form \( \omega^{eq} \) over suitable triangles. Let \( G \) be a simply connected Lie
group and \( \{a_g\}_{g \in G} \) a given family of smooth paths, \( a_g : I = [0, 1] \to G \) from \( e \) to \( g \). For each \( f, g \in G \) let \( \Sigma_{f,g} \) be a piecewise smooth 2-simplex in \( G \) with boundary \( fa_g - a_{fg} + af \) and let \( C \) be the map

\[
C : G \times G \to a, \quad C(f, g) = \int_{\Sigma_{f,g}} \omega^{eq}. \tag{2}
\]

Because \( \theta_{f,g,h} = f\Sigma_{g,h} - \Sigma_{fg,h} + \Sigma_{f,gh} - \Sigma_{f,g} \) is a spherical 2-cycle in \( G \) for any \( f, g, h \in G \), and because \( \omega^{eq} \) is equivariant, the map \( C \) satisfies the relation

\[
fC(g, h) - C(fg, h) + C(f, gh) - C(f, g) = \int_{\theta_{f,g,h}} \omega^{eq} \in \Pi_\omega. \tag{3}
\]

If the period group \( \Pi_\omega \) of the Lie algebra 2-cocycle \( \omega \) is discrete and \( \Gamma \supseteq \Pi_\omega \) is any discrete subgroup of the space \( \mathfrak{d}^G \) of \( G \)-invariant elements, we denote the abelian Lie group and smooth \( G \)-module \( a/\Gamma \) by \( A \) and the quotient map by \( \exp : a \to A \). Then \( c = \exp \circ C : G \times G \to A \) is a group 2-cocycle independent of the choice of the 2-cycles \( \Sigma_{f,g} \). When the paths \( a_g \) are carefully chosen \( [15] \), then \( c \) is smooth in an identity neighborhood and \( \omega \) is the associated Lie algebra cocycle, i.e.

\[
d^2c_{(e,e)}(X, Y) - d^2c_{(e,e)}(Y, X) = \omega(X, Y) \quad \text{for all } X, Y \in \mathfrak{g}.
\]

**Remark 1.** Given a smoothly contractible Lie group \( G \), each smooth retraction \( h : I \times G \to G \) of \( G \) to \( \{e\} \) provides a family of smooth paths \( \{a_g\} \) from \( e \) to \( g \) by \( a_g(s) = h(s, g), s \in I \). Any continuous Lie algebra 2-cocycle \( \omega \in Z^2(\mathfrak{g}, a) \) is integrable to a smooth \( \mathfrak{a} \)-valued van Est cocycle \( C \) on \( G \) given by \( \Theta_2 \), depending only on the retraction \( h \). One can choose for instance \( \Sigma_{f,g}(s, t) = h(s, fh(t, g)) \), \( s, t \in I \).

Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \). The group of smooth paths in \( G \) starting at the identity,

\[
PG = \{g \in C^\infty(I, G) \mid g(0) = e\},
\]
called the path group, is a smoothly contractible Lie group with Lie algebra

\[
P_{\mathfrak{g}} = \{X \in C^\infty(I, \mathfrak{g}) \mid X(0) = 0\},
\]

the path Lie algebra.

Each \( \mathfrak{a} \)-valued Lie algebra 2-cocycle \( \omega \) on \( \mathfrak{g} \) defines an \( \mathfrak{a} \)-valued Lie algebra 2-cocycle \( \Theta_2 \) on the path Lie algebra \( P_{\mathfrak{g}} \), the evaluation map \( \text{ev}_1 : P_{\mathfrak{g}} \to \mathfrak{g} \) being a Lie algebra homomorphism. Via the group homomorphism \( \text{ev}_1 : PG \to G \), the \( G \)-module \( \mathfrak{a} \) becomes a \( PG \)-module. With Remark[1] we can integrate \( P\omega \) to a smooth group 2-cocycle \( C \) on the contractible group \( PG \). One can write down an explicit formula for this cocycle using the left logarithmic derivative \( \delta f \in P_{\mathfrak{g}} \) for \( f \in PG \).

**Proposition 1.** A smooth group 2-cocycle on the path group \( PG \) integrating the Lie algebra cocycle \( P\omega \) is

\[
C(f, g) = \int_0^1 \left( \int_0^s \rho_a(f(s)g(t)) \omega(\Ad(g(t)^{-1})) \delta f(s, \delta g(t)) \, dt \right) \, ds. \tag{4}
\]

**Proof.** A smooth retraction of \( PG \) to \( e \) is obtained by reparametrisation of paths, namely \( h : I \times PG \to PG, h(s, g)(t) = g(st) \) for \( g \in PG \) and \( s, t \in I \). The system of paths \( \{a_g\}_{g \in PG} \) in \( PG \) defined by \( h \) is \( a_g(s)(t) = g(st) \). It has the property \( a_f a_g = a_{fg} \) for all \( f, g \in PG \), so we choose a particular 2-simplex \( \Sigma_{f,g} \) in \( PG \) with
integrating $P \omega$ on $G$ a discrete subgroup of $\sigma$ obtained by integrating the closed equivariant 2-form on $P \omega$ using the system of paths $s$, $t$.

The 2-simplex $\sigma_{f,g} = ev_1 \circ \Sigma_{f,g}$ on $G$ is given by $\sigma_{f,g}(s,t) = f(s)g(st)$ for $(s,t) \in I \times I$, hence

$$C(f,g) = \int_{\sigma_{f,g}} \omega = \int_0^1 \int_0^1 \omega^q(f(s)g(st) + tf(s)g(st), sf(s)g(st))dt ds$$

$$= \int_0^1 \int_0^1 \rho_a(f(s)g(st))\omega(\text{Ad}(g(st)^{-1})\delta f(s) + t\delta g(st), s\delta g(st))dt ds$$

$$= \int_0^1 \int_0^s \rho_a(f(s)g(t))\omega(\text{Ad}(g(t)^{-1})\delta f(s), \delta g(t))dt ds,$$

using the $G$-equivariance of $\omega^q$ at step 3 and a change of variable at step 4.

Remark 2. Another group cocycle on $PG$ integrating the Lie algebra 2-cocycle $P\omega$ is $C'(f,g) = -C(g^{-1}, f^{-1})$. It can be seen as the van Est cocycle associated to the symmetrization procedure

By the symmetrization procedure

$$C_{sym}(f,g) = \frac{1}{2}(C(f,g) + C'(f,g)) = \frac{1}{2}(C(f,g) - C(g^{-1}, f^{-1}))$$

we get a new group cocycle $C_{sym}$ on $PG$ integrating $P\omega$ and having the property

$$C_{sym}(g^{-1}, f^{-1}) = -C_{sym}(f,g).$$

Remark 3. When $a$ is a trivial $G$-module and $\omega^l$ the left invariant 2-form on $G$ defined by $\omega \in Z^2(g,a)$, then a formula for the van Est group 2-cocycle $C$ on $PG$ integrating $P\omega$ is

$$C_{sym}(f,g) = \frac{1}{2}(C(f,g) + C'(f,g)) = \frac{1}{2}(C(f,g) - C(g^{-1}, f^{-1}))$$

we get a new group cocycle $C_{sym}$ on $PG$ integrating $P\omega$ and having the property

$$C_{sym}(g^{-1}, f^{-1}) = -C_{sym}(f,g).$$

Remark 4. Given a continuous Lie algebra $p$-cocycle $\omega$ on $g$ with values in the smooth $G$-module $a$, a group $p$-cocycle $C$ on $PG$ integrating $P\omega$ is

$$C(g_1, \ldots, g_p) = \int_0^1 \left( \int_0^{t_1} \cdots \int_0^{t_{p-1}} \rho_a(g_1(t_1) \cdots g_p(t_p)) \cdot \omega(\text{Ad}(g_2(t_2)g_3(t_3) \cdots g_p(t_p))^{-1}\delta g_1(t_1), \text{Ad}(g_3(t_3) \cdots g_p(t_p))^{-1}\delta g_2(t_2), \ldots,\right.$$

$$\left. \text{Ad}(g_p(t_p))^{-1}\delta g_{p-1}(t_{p-1}), \delta g_p(t_p))dt_p \right) \cdots dt_2 dt_1,$$

obtained by integrating the closed equivariant $p$-form $\omega^q$ on $G$ over the $p$-simplex $\sigma_{g_1, \ldots, g_p} : (t_1, \ldots, t_p) \in I^p \mapsto g_1(t_1)g_2(t_1t_2) \cdots g_p(t_1t_2 \cdots t_p)$.

3. Construction of abelian Lie group extensions via path groups

Let $\omega$ be an $a$-valued Lie algebra $2$-cocycle on the Lie algebra $g$ of the connected Lie group $G$. We assume that its period group $\Pi_\omega \subset a^G$ is discrete and $\Gamma \supset \Pi_\omega$ is a discrete subgroup of $a^G$, and we denote by $exp : a \to A = a/\Gamma$ the quotient map.
The smooth $G$-action $\rho_a$ on $a$ descends to a smooth $G$-action $\rho_A$ on $A$, because $G \subset a^G$.

In this section we explicitly realize an abelian extension of the universal covering group $\tilde{G}$ of $G$ by the abelian Lie group $A$. It is done by factorising an abelian extension of the path group $PG$ (defined with a van Est cocycle) by the graph of a smooth map.

The subgroup $\Omega G \subset PG$ of loops based at $e$ has a subgroup $\Omega_0G \subset \Omega G$ of null-homotopic loops based at $e$. Both have as Lie algebra the Lie algebra $\Omega g$ of loops in $g$ based at $0$, moreover $\Omega_0G$ is the identity component of $\Omega G$. The following two sequences of Lie groups are exact:

\[
\begin{array}{ccc}
\Omega G & \longrightarrow & PG \\
\uparrow & & \uparrow \\
\Omega_0G & \longrightarrow & PG \\
& \longrightarrow & \tilde{G}
\end{array}
\]

and have the same exact sequence of Lie algebras $\Omega g \to P g \to g$.

Let $C : PG \times PG \to a$ be the van Est 2-cocycle integrating the Lie algebra 2-cocycle $P\omega$ on $P g$. The cocycle $c = \exp \circ C : PG \times PG \to A$ is given by

\[
c(f, g) = \exp \int_{\sigma_{f,g}} \omega^q = \exp \int_0^1 \left( \int_0^s \rho_a(f(s)g(t))\omega(\text{Ad}(g(t)^{-1})\delta^l f(s), \delta^l g(t))dt \right) ds,
\]

where $\sigma_{f,g}(s, t) = f(s)g(st)$, $s, t \in I$ for $f, g \in PG$. The Lie algebra cocycle $P\omega$ vanishes on $\Omega g$. The next proposition will show that the restriction of the group cocycle $c$ to $\Omega_0G$ is a coboundary (both $a$ and $A$ are trivial $\Omega_0G$-modules).

We consider the smooth map

\[
\Lambda : \Omega_0\tilde{G} \to a, \quad \Lambda([\tilde{g}]) = -\int_{\tilde{g}} \omega^q,
\]

viewing the path $\tilde{g}$ from $e$ to $g$ in $\Omega_0G$ as a map $\tilde{g} : I \times I \to G$. It is well defined since the integral of the closed 2-form $\omega^q$ over two homotopic paths in $\Omega_0G$ (hence homotopic maps from $I \times I$ to $G$) is the same.

**Remark 5.** Identifying $\pi_2(G)$ with $\pi_1(\Omega_0G) \subset \Omega_0\tilde{G}$, the restriction of $\Lambda$ to $\pi_1(\Omega_0G)$ equals $-\text{per}_{\omega}$, the opposite of the period map [1]. Indeed, a loop at $e$ of loops in $G$ determines a spherical 2-cycle in $G$. In particular the map $\Lambda$ is well defined on $\Omega_0G$ when considered modulo $\Pi_\omega$, hence it descends to a well defined map

\[
\lambda : \Omega_0G \to A, \quad \lambda(g) = \left( \exp \int_{\tilde{g}} \omega^q \right)^{-1},
\]

i.e. $\lambda(g)$ does not depend on the chosen path $\tilde{g}$ in $\Omega_0G$ from $e$ to $g$.

**Proposition 2.** The identity

\[
c(f, g) = \lambda(fg)\lambda(f)^{-1}\lambda(g)^{-1}
\]

holds for all $f, g \in \Omega_0G$. In particular $c$ restricted to $\Omega_0G$ is the coboundary of $\lambda^{-1}$.

**Proof.** Let $\tilde{f}$ and $\tilde{g}$ be paths in $\Omega_0G$ with $f = \tilde{f}(1)$ and $g = \tilde{g}(1)$. The 2-chains

\[
\sigma_{f,g} : I \times I \to G, \quad \sigma_{f,g}(s, t) = f(s)g(st)
\]
and \( f + g - fg \) have the same boundary \( f + g - fg \), so they determine a spherical 2-cycle \( \tau_{f,g} = \sigma_{f,g} - f - g + fg \) in \( G \). Hence the van Est cocycle \( C(f, g) = \int_{\tau_{f,g}} \omega^{eq} \) on \( \Gamma \) satisfies

\[
C(f, g) + \Lambda([f]) + \Lambda([g]) - \Lambda([fg]) = \int_{\tau_{f,g}} \omega^{eq} \in \mathbb{I}_\omega \subseteq \Gamma.
\]  

(12)

The projection of this relation to \( A \) gives the requested identity. \( \square \)

Remark 6. The stronger relation

\[
C(f, g) + \Lambda([f]) + \Lambda([g]) - \Lambda([fg]) = 0, \quad f, g \in \Omega_0 G
\]  

(13)

also holds, because there always exists a bordism from the 2-cycle \( \tau_{f,g} \) to \( e \) given by \( (r, t, s) \in I \times I \times I \mapsto f(r, s)g(r, st) - f(rt, s) - g(rs, t) + f(rt, s)g(rt, s) \in G \), for \( f \) and \( g \) paths in \( \Omega_0 G \). Still the \( \mathfrak{a} \)-valued van Est cocycle \( C \) restricted to \( \Omega_0 G \) is not a coboundary in general: \( \Lambda \) does not descend to a well defined \( \mathfrak{a} \)-valued map on \( \Omega_0 G \). Anyway, following [9], if (12) is satisfied we say that the map \( \Lambda \) resolves the 2-cocycle \( C \).

Remark 7. The cocycles \( C' \) and \( C_{sym} \) defined in Remark 2 also possess resolving maps. The map

\[
\Lambda'([g]) = -\Lambda([g^{-1}]) = \int_{g^{-1}} \omega^{eq}
\]

(14)

resolves the cocycle \( C' \) and \( C_{sym} \) defined in Remark 2. In particular we have that \( \Lambda_{sym}([g^{-1}]) = -\Lambda_{sym}([\bar{g}]) \).

Lemma 1. Let \( H \) be a normal split Lie subgroup of the Lie group \( G \) and \( A \) a smooth \( G \)-module, trivial as an \( H \)-module. Let \( c \) be an \( A \)-valued group 2-cocycle on the group \( G \) whose restriction to \( H \) is the coboundary of \( \lambda^{-1} \) for a given smooth map \( \lambda : H \to A \). When one of the following two equivalent conditions:

(i) \( c(g, h)c(gh^{-1}, g)^{-1} = (\rho_A(g)\lambda(h^{-1}))\lambda(ghg^{-1}) \) for all \( g \in G \) and \( h \in H \)

(ii) the graph of \( \lambda \) is a normal subgroup of \( A \rtimes_c G \)

is satisfied, then the quotient group \( (A \rtimes_c G)/\text{Graph}(\lambda) \) is an abelian Lie group extension of \( G/H \) by \( A \).

Proof. The graph of \( \lambda \) coincides with the image of the map \( \varphi : h \in H \mapsto (\lambda(h), h) \in A \rtimes_c G \). Because \( c \) is the coboundary of \( \lambda \):

\[
c(h_1, h_2) = \lambda(h_1 h_2)\lambda(h_1)^{-1}\lambda(h_2)^{-1} \text{ for all } h_1, h_2 \in H,
\]

(15)

\( \varphi \) is a group homomorphism. This follows from

\[
\varphi(h_1)\varphi(h_2) = (\lambda(h_1)\lambda(h_2)c(h_1, h_2), h_1 h_2) = \varphi(h_1 h_2).
\]

Hence the graph of \( \lambda \) is a subgroup of \( A \rtimes_c G \).

Let \( g \in G \) and \( h \in H \). The conjugate in \( A \rtimes_c G \) of the element \( (\lambda(h), h) \in \text{Graph}(\lambda) \) is (see for instance Lemma 2.1 in [15])

\[
(a, g)(\lambda(h), h)(a, g)^{-1} = ((\rho_A(g)\lambda(h))c(g, h)c(ghg^{-1}, g)^{-1}, ghg^{-1}).
\]

It belongs to the graph of \( \lambda \), i.e. it equals \( (\lambda(ghg^{-1}), ghg^{-1}) \), if and only if the identity (i) holds.

The kernel of the projection homomorphism \( (a, g) \in (A \rtimes_c G)/\text{Graph}(\lambda) \mapsto gH \in G/H \) is isomorphic to \( A \), hence \( (A \rtimes_c G)/\text{Graph}(\lambda) \) is an abelian group extension of \( G/H \) by \( A \), for the natural \( G/H \)-module structure on \( A \). It is a Lie group since \( \text{Graph}(\lambda) \) is a split Lie subgroup of \( A \rtimes_c G \), \( H \) being a split Lie subgroup of \( G \). \( \square \)
Theorem 1. Let $G$ be a connected Lie group, $a$ a smooth $G$-module and $\omega \in Z^2(g,a)$ a continuous Lie algebra 2-cocycle with discrete period group $\Pi_\omega$. Let $\Gamma \supset \Pi_\omega$ be a discrete subgroup of $a^\omega$, $\exp: a \to A = a/\Gamma$ the quotient map and $c$ the cocycle defined in (8).

Then the graph of the smooth map $\lambda$ defined in (10) is a normal subgroup of $A \rtimes_c PG$ and the quotient group $(A \rtimes_c PG)/\text{Graph}(\lambda)$ is an abelian Lie group extension of the universal covering group $\tilde{G}$ by $A$, integrating the Lie algebra extension $a \rtimes_c \omega$.

Proof. To apply Lemma 1 to the normal split Lie subgroup $\Omega_0G$ of $PG$, we verify relation (i) for the $A$-valued 2-cocycle $c$ on $PG$ and the map $\lambda: \Omega_0G \to A$.

The boundary of the 2-chain $\sigma_{f,g}$ for $f,g \in PG$ is $f(1)g - fg + f$ and the boundary of the 2-chain $\tilde{f}$ for $f \in \Omega_0G$ is $f$. Let $g \in PG$ and $h \in \Omega_0G$ with $\tilde{h}$ a path in $\Omega_0G$ from $e$ to $h$. The 2-chains $\sigma_{ghg^{-1},g} - \sigma_{g,h}$ and $g(h^{-1} - (1)\tilde{h})$ in $G$ have the same boundary, namely $ghg^{-1} - g(1)h$. Integrating $\omega^eq$ over the spherical 2-cycle $\nu_{g,h} = \sigma_{ghg^{-1},g} - \sigma_{g,h} - ghg^{-1} + g(1)\tilde{h}$ and using the $G$-equivariance of $\omega^eq$ we obtain

$$C(ghg^{-1},g) - C(g,h) + \Lambda(h^{-1}g^{-1}) - \rho_A(g(1))\Lambda(\tilde{h}) = \int_{\nu_{g,h}} \omega^eq \in \Pi_\omega \subseteq \Gamma.$$ 

The projection of this identity to $A$ gives (i), showing that the graph of $\lambda$ is a normal subgroup of $A \rtimes_c PG$.

The abelian Lie group extension $A \rtimes_c PG$ of $PG$ integrates $P\omega = \text{ev}^1_\Gamma \omega$, hence the quotient group $(A \rtimes_c PG)/\text{Graph}(\lambda)$ is an abelian Lie group extension of the universal covering group $\tilde{G} = PG/\Omega_0G$ integrating $\omega$.

The rows and the last column of the following diagram are exact sequences of Lie groups

$$\begin{array}{ccc}
A & \longrightarrow & (A \rtimes_c PG)/\text{Graph}(\lambda) \\
\uparrow & & \uparrow \\
A & \longrightarrow & A \rtimes_c PG \\
\uparrow & & \uparrow \\
A & \longrightarrow & A \rtimes_c \Omega_0G \\
\uparrow & & \uparrow \\
\tilde{G} & \longrightarrow & PG \\
\uparrow & & \uparrow \\
\Omega_0G & \longrightarrow & \Omega_0G.
\end{array}$$

Remark 8. There is a geometric construction of an abelian extension of $\tilde{G}$ using the path group $PG$ presented in [8]. One considers the following equivalence relation on $A \rtimes_c PG$, where $c$ is given by (8):

$$(a_1,g_1) \sim (a_2,g_2) \iff g_1(1) = g_2(1), g_1 - g_2 = \partial \sigma, a_2 = a_1 \exp \int_\sigma \omega^eq, \quad (16)$$

the second condition on the right meaning that $\sigma$ is any 2-chain in $G$ having as boundary the loop $g_1 - g_2$. Then $(A \rtimes_c PG)/\sim$ is an abelian extension of $\tilde{G}$ integrating $\omega$. We show it is isomorphic to the abelian extension in Theorem 1.

A pair $(a,g)$ is equivalent to the identity element if and only if $(a,g)$ belongs to $\text{Graph}(\lambda)$. Moreover, two pairs $(a_1,g_1)$ and $(a_2,g_2)$ are equivalent if and only if the composition $(a_1,g_1)^{-1}(a_2,g_2)$ taken in $A \rtimes_c PG$ belongs to $\text{Graph}(\lambda)$. Indeed, for
h ∈ Ω₀G and ˜h a path from e to h in Ω₀G,
\[(a₁, g₁)(\lambda(h), h) = \left( a₁ \left( \exp \int_{g₁(1)h} \omega^{eq} \right)^{-1} \exp \int_{\sigma_{g₁, h}} \omega^{eq}, g₁h \right) \]
\[= \left( a₁ \exp \left( \int_{\sigma_{g₁, h} - g₁(1)h} \omega^{eq} \right), g₁h \right),\]
so \((a₁, g₁)(\lambda(h), h) = (a₂, g₂)\) if and only if \(g₂ = g₁h\) and \(a₂ = a₁ \exp \int_{\sigma} \omega^{eq}\), where \(\sigma\) is any 2-cycle in \(G\) such that \(\partial \sigma = \partial(\sigma_{g₁, h} - g₁(1)h) = g₁ - g₁h = g₁ - g₂\).

Hence the abelian extension \((A \ltimes_c PG)/\text{Graph}(\lambda)\) from Theorem 4 and the abelian extension \((A \ltimes_e PG)/\sim\) are isomorphic.

4. Current groups

Let \(M\) be a compact manifold and \(H\) a finite dimensional connected Lie group with Lie algebra \(\mathfrak{h}\). The current group \(C^\infty(M, H)\) with pointwise multiplication is a Lie group with Lie algebra the current algebra \(\mathfrak{g} = C^\infty(M, \mathfrak{h})\) (as in [8] Section 42). By \(G\) we denote the identity component \(C^\infty(M, H)_0\) of the current group.

We consider an invariant symmetric bilinear form
\[
\kappa : \mathfrak{h} \times \mathfrak{h} \to V.
\]
Defining \(a = Ω¹(M, V)/dC^\infty(M, V)\), there is a continuous Lie algebra 2-cocycle on the current algebra,
\[
\omega : \mathfrak{g} \times \mathfrak{g} \to a, \quad \omega(X, Y) = [\kappa(X, dY) - \kappa(Y, dX)] = 2[\kappa(X, dY)]. \tag{17}
\]
In the loop group case \(M = S¹\) the space \(a\) can be identified with \(V\), so the cocycle on the loop algebra \(\mathfrak{g} = C^\infty(S¹, \mathfrak{h})\) can be taken as
\[
\omega : \mathfrak{g} \times \mathfrak{g} \to V, \quad \omega(X, Y) = 2 \int_{S¹} \kappa(X, dY). \tag{18}
\]

Remark 9. When \(H\) is simply connected and simple, the loop group \(G\) is simply connected. If \(\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}\) is the suitably normalized Killing form, \(V = \mathbb{R}\) and the period group is \(\Pi_ω = \mathbb{Z} \subset \mathbb{R}\). In Chapter 4 of [12] is presented the construction of the central extension of \(G\) by the circle \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\), integrating \(ω\). It is explicitly realized as a quotient group of the central extension of the group of currents on the 2-disk \(D\) given by the group 2-cocycle \(c(f, g) = \int_D \delta^1 f \wedge \kappa, \delta^1 g\).

The generalization of this result to current groups can be found in [9]. The bilinear form \(\kappa\) is a multiple of the Killing form of the simple Lie algebra \(\mathfrak{h}\), and the space \(a\) is \(Ω¹(M)/dC^\infty(M)\). When \(\kappa\) is suitably normalized, then the period group is the discrete subgroup \(\Pi_ω = Z^2_c(M)/dC^\infty(M)\) of \(a\) consisting of all cohomology classes with integral periods. The central extension of the universal covering group \(\tilde{G}\) by \(A = a/\Pi_ω = \tilde{Ω¹}(M)/Z^2_c(M)\) integrating \(ω\) is constructed in [9] as a quotient group of the central extension of the path group of \(G\) given by the group cocycle \(c(f, g) = [\int_I \delta^1 f \wedge \kappa, \delta^1 g], f, g : I \times M \to H\). Adapting this construction one gets also the central extension of the gauge group of automorphisms of a nontrivial vector bundle \([9]\).

Remark 10. The period group of \(ω\) is not always discrete; an example with non-discrete period group can be found in Remark II.10 \([10]\). The reduction theorem I.6 \([10]\) shows that given a \(V\)-valued invariant symmetric form \(\kappa\), the discreteness of the period group for \(M = S¹\) implies the discreteness of the period group for any compact manifold \(M\).
Remark 11. Considering the natural action of $\mathfrak{h}$ on the symmetric power $S^2(\mathfrak{h})$ induced by the adjoint action, the quotient space $V(\mathfrak{h}) = S^2(\mathfrak{h})/(\mathfrak{h} \cdot S^2(\mathfrak{h}))$ comes with a universal invariant symmetric bilinear form $\kappa_\mathfrak{h} : \mathfrak{h} \times \mathfrak{h} \to V(\mathfrak{h})$. For any invariant symmetric bilinear form $\kappa$ on $\mathfrak{h}$ with values in $V$, there is a unique linear map $\varphi : V(\mathfrak{h}) \to V$ such that $\kappa = \varphi \circ \kappa_\mathfrak{h}$. The continuous Lie algebra 2-cocycle $\omega_\mathfrak{h}(X,Y) = 2[\kappa_\mathfrak{h}(X,dY)]$ on the current algebra with values in $\Omega^1(M,V(\mathfrak{h}))/dC^\infty(M,V(\mathfrak{h}))$ has a discrete period group $\Pi_\omega$ contained in the subspace $H^1_{dR}(M,V(\mathfrak{h}))$. \[10\]

Assuming the period group $\Pi_\omega \subset \mathfrak{a}$ of the continuous Lie algebra 2-cocycle (12) is discrete, we apply Theorem 11 to the identity component of the current group to explicitly realize a central extension of its universal covering group integrating $\omega$.

The path group of $G = C^\infty(M,H)$ and the path Lie algebra of $\mathfrak{g} = C^\infty(M,\mathfrak{h})$ are (by $[8]$ Section 42):

$$PG = \{g \in C^\infty(I \times M,H) | g(0,x) = e, \forall x \in M \} = PC^\infty(M,H)$$

$$P\mathfrak{g} = \{X \in C^\infty(I \times M,\mathfrak{h}) | X(0,x) = 0, \forall x \in M \},$$

because the path group of a Lie group coincides with the path group of its identity component. Denoting by $d_xX$ the exterior differential of $X$ on $M$ and by $\delta^i_xg$ the logarithmic derivative of $g$ on $M$ (viewing $t \in I$ as a parameter), we get

$$dX \land_\kappa dY = dt \land (\kappa(\frac{d}{dt}X,d_xY) - \kappa(d_xX,\frac{d}{dt}Y)) + d_xX \land_\kappa d_xY.$$ 

An expression for the pullback 2-cocycle $P\omega = \text{ev}_1^*\omega$ on $P\mathfrak{g}$ in this case is

$$P\omega(X,Y) = 2 \left[ \int_0^1 \frac{d}{dt} \kappa(X,d_xY) dt \right] = 2 \left[ \int dX \land_\kappa dY \right].$$

A 2-cocycle on the path group $PG$ integrating $P\omega$ can be obtained from (11) using the invariance of $\kappa$, formulas from the appendix and the fact that $\delta^i_xg(0,x) = 0$ for $g \in PG$ and $x \in M$ by the following computation:

$$C(f,g) \equiv 2 \int_0^1 \left( \int_0^s [\kappa(\Ad(g(t,x)^{-1})i_{i\partial_t}\delta^i_t f(s,x),d_xi_{i\partial_t}\delta^i_t g(t,x))] dt \right) ds$$

$$= 2 \int_0^1 \left( \int_0^s [\kappa(i_{i\partial_x}\delta^i_t f(s,x),\Ad(g(t,x))\frac{d}{dt}\delta^i_x g(t,x)) + [i_{i\partial_x}\delta^i_t g(t,x),\delta^i_x g(t,x)]] dt \right) ds$$

$$= 2 \int_0^1 \left[ \kappa(i_{i\partial_x}\delta^i_t f(s,x),\Ad(g(s,x))\delta^i_x g(s,x)) ds = 2 \int_0^1 [\kappa(i_{i\partial_x}\delta^i t f,\delta^i_x g)] ds.\right]$$

The group $\Omega_0G$ of null-homotopic loops in the current group $G$ can be identified with the space of those smooth maps $g \in C^\infty(I \times M,H)$ with $g(0,\cdot) = g(1,\cdot) = e$ for which there exists a smooth homotopy $(s,t,x) \in I \times I \times M \to \tilde{g}(s,t,x) \in H$ with $\tilde{g}(0,\cdot,\cdot) = e, \tilde{g}(1,\cdot,\cdot) = g$ and $\tilde{g}(\cdot,0,\cdot) = \tilde{g}(\cdot,1,\cdot) = e$. The homotopy class of $\tilde{g}$ is identified with an element of $\Omega_0G$. The resolving map for the cocycle $C$ defined on $\Omega_0G$ is:

$$\Lambda([\tilde{g}]) = -\int_{\tilde{g}} \omega^1 = -2 \int_0^1 \int_0^1 \kappa(i_{i\partial_s}\delta^i_t \tilde{g},d_{i\partial_s}\delta^i_t \tilde{g}) ds dt]$$
These are not yet the type of expressions expected from \[12\] and \[9\]. To get them, we will apply the symmetrization procedure from Remark 2. A 2-cocycle on the path group of the current group \(G\), integrating \(P\omega\) is

\[
C_{\text{sym}}(f, g) = \frac{1}{2}(C(f, g) - C(g^{-1}, f^{-1}))
\]

\[
= \int_0^1 \left[ \kappa(i_{\partial_0} \delta^i f, \delta^x g) \right] ds - \int_0^1 \left[ \kappa(i_{\partial_0} \delta^i f, \delta^x g) \right] ds = \left[ \int_\Gamma \delta^i f \wedge \kappa \delta^r g \right],
\]

obtained by fiber integrating the 2-form \(\delta^i f \wedge \kappa \delta^r g \in \Omega^2(I \times M, V)\). At the last step we used the identity \(\delta^i f \wedge \kappa \delta^r g = dt \wedge (\kappa(i_{\partial_0} \delta^i f, \delta^x g) - \kappa(i_{\partial_0} \delta^i f, \delta^x g)) + \delta^x f \wedge \kappa \delta^r g \)

obtained from the relation \(\delta^i f = (i_{\partial_0} \delta^i f)dt + \delta^i f \in \Omega^1(M \times I, h)\).

We define \(\eta = \frac{1}{2} \theta^I \wedge \kappa (\theta^I \wedge \kappa \theta) \in \Omega^3(H, V)\), where \(\theta^I \in \Omega^1(H, h)\) is the left Maurer-Cartan form. Like the Cartan 3-form on a simple Lie group, \(\eta(X, Y, Z) = \kappa(X, [Y, Z])\) for \(X, Y, Z \in h\). The resolving map for the cocycle \(C_{\text{sym}}\) is

\[
\Lambda_{\text{sym}}([\tilde{g}]) = \frac{1}{2} \Lambda([\tilde{g}]) + \left[ \int_0^1 \int_0^1 \kappa(\text{Ad}(\tilde{g})i_{\partial_0} \delta^i \tilde{g}, d\text{Ad}(\tilde{g})i_{\partial_0} \delta^i \tilde{g}) ds dt \right]
\]

\[
= \left[ \int_0^1 \int_0^1 \kappa(i_{\partial_0} \delta^i \tilde{g}, [i_{\partial_0} \delta^i \tilde{g}, \delta^x \tilde{g}]) ds dt \right]
\]

\[
= -\frac{1}{6} \left[ \int_{I \times I} \delta^i \tilde{g} \wedge \kappa (\delta^i \tilde{g} \wedge \kappa \delta^r \tilde{g}) \right] = -\left[ \int_{I \times I} \tilde{g}^* \eta \right],
\]

for the homotopy \(\tilde{g} : I \times I \times M \to H\).

This gives a constructive proof for the following slight generalization of the result from \[9\] mentioned in Remark 9.

**Theorem 2.** Assuming the period group \(\Pi_{\omega} \subset a\) of the continuous Lie algebra 2-cocycle \(\{\tau\}\) is discrete, let \(\Gamma \subset a\) be a discrete set containing \(\Pi_{\omega}\) and \(\exp : a \to A = a/\Gamma\). The \(A\)-valued group cocycle \(c_{\text{sym}} = \exp \circ C_{\text{sym}}\) on \(PG\),

\[
c_{\text{sym}}(f, g) = \exp \left[ \int_{\Gamma} \delta^i f \wedge \kappa \delta^r g \right],
\]

integrates the Lie algebra cocycle \(\omega\). The restriction of \(c_{\text{sym}}\) to \(\Omega_0 G\) is the coboundary of the inverse of \(\lambda_{\text{sym}} = \exp \circ \Lambda_{\text{sym}} : \Omega_0 G \to A\),

\[
\lambda_{\text{sym}}(g) = \left( \exp \left[ \int_{I \times I} \tilde{g}^* \eta \right] \right)^{-1}.
\]

The quotient group \((A \times c_{\text{sym}} PG)/\text{Graph}(\lambda_{\text{sym}})\) is an abelian Lie group extension of the universal covering group \(\tilde{G}\) of the current group \(G\) by \(A\), integrating \(\omega\).

**Remark 12.** We apply this theorem to the special case treated in \[9\], where \(\kappa\) is a multiple of the Killing form of the simple Lie algebra \(h\), so \(\eta\) is a multiple of the Cartan 3-form, the constant factor being chosen such that \(\eta\) is integral. From the relation \(\Pi_{\omega} = \Lambda_{\text{sym}}(\pi_1(\Omega_0 G))\) in Remark 3 applied to \(\Lambda_{\text{sym}}([\tilde{g}]) = -\left[ \int_{I \times I} \tilde{g}^* \eta \right]\)

follows that the period group \(\Pi_{\omega} = Z_2(M)/dC^\infty(M)\), as mentioned in Remark 9.

Denoting by \(m : H \times H \to H\) the group multiplication and by \(pr_1, pr_2 : H \times H \to H\) the canonical projections, the Polyakov-Wiegmann formula is

\[
m^* \eta = pr_1^* \eta + pr_2^* \eta - d(pr_1^* \theta^I \wedge \kappa pr_2^* \theta^I).
\]
Integrating the pullback of the Polyakov-Wiegmann formula by the map \((\tilde{f}, \tilde{g}) : I \times I \times M \to H \times H\) over \(I \times I\), provides another proof that \(\Lambda_{\text{sym}}\) resolves \(C_{\text{sym}}\).

5. Coupled cocycle

The coupled cocycle on the current Lie algebra \(g = C^\infty(M, \mathfrak{h})\) was defined in [16]. It is built with a continuous invariant symmetric bilinear form \(\kappa : \mathfrak{h} \times \mathfrak{h} \to V\) whose image under the Cartan map

\[
\Gamma : S^2(\mathfrak{h}, V)^h \to Z^2(\mathfrak{h}, V), \quad \Gamma(\kappa)(X, Y, Z) = \kappa([X, Y], Z)
\]
is a coboundary, i.e. there is a 2-cochain \(\beta \in C^2(\mathfrak{h}, V)\) such that

\[
\Gamma(\kappa) = d_\kappa \beta.
\]

The corresponding coupled cocycle is

\[
\omega : g \times g \to \Omega^1(M, V), \quad \omega(X, Y) = \kappa(X, dY) - \kappa(Y, dX) - d(\beta(X, Y)),
\]
a Lie algebra 2-cocycle on \(g\) with values in the trivial module \(\Omega^1(M, V)\).

In [16] it is shown that the period map \(\text{per}_{\omega} : \pi_2(G) \to \Omega^1(M, V)\) of the coupled cocycle vanishes for \(G = C^\infty(M, H)\). Hence there exists a central extension of \(\tilde{G}\) by \(\Omega^1(M, V)\) integrating \(\omega\). We explicitly realize this Lie group extension with the path group method.

Remark 13. The coupled cocycle is a lift to \(\Omega^1(M, V)\) of the cocycle

\[
g \times g \to \Omega^1(M, V)/C^\infty(M, V) = a, \quad (X, Y) \mapsto 2[\kappa(X, dY)]
\]
studied in the previous section. In this special case when \(\Gamma(\kappa)\) is exact, i.e. a coboundary in \(B^2(\mathfrak{h}, V)\), the period map of this \(a\)-valued cocycle also vanishes.

The pullback cocycle \(P \omega\) on \(PG\) of the coupled cocycle \(\omega\) integrates to a group cocycle on the path group \(PG\) because the path group is contractible. A computation starting from [13], similar to the one in the previous section, together with the symmetrization procedure, gives the \(\Omega^1(M, V)\)-valued cocycle \(C_{\text{sym}}\) on \(PG\) as the sum of two cocycles, one has values in the subspace of exact \(V\)-valued 1-forms on \(M\), the other one is \((f, g) \mapsto \int_I \delta^1 f \wedge \kappa \delta^1 g\).

More precisely, for \(f, g \in PG\) we define

\[
m(s, t) = \kappa(\delta^1_t g(t, \cdot), \delta^1_s f(s, \cdot)) \in C^\infty(M, V)
\]
and

\[
C_{\beta}(f, g) = \int_0^1 \left( \int_0^s \beta(\text{Ad}(g(t, \cdot)^{-1})) \delta^1_s f(s, \cdot), \delta^1_t g(t, \cdot) dt \right) ds.
\]

\(C_{\beta}\) is not a group cocycle in general, nevertheless we consider its symmetrized version \(C_{\beta,\text{sym}}\) as in [16]. Then, by a computation which can be found in the appendix,

\[
C(f, g) = 2 \int_0^1 \kappa(\delta^1_s g(s, \cdot), \delta^1_t f(s, \cdot)) ds - ds \left( C_{\beta}(f, g) + \int_0^1 \int_0^s m(s, t) dt ds \right),
\]
so

\[
C_{\text{sym}}(f, g) = \int_I \delta^1 f \wedge \kappa \delta^1 g - ds \left( C_{\beta,\text{sym}}(f, g) + \frac{1}{2} \int_0^1 \int_0^s (m(s, t) - m(t, s)) dt ds \right).
\]
Let \( g : I \times M \to H \) in \( \Omega_0G \) and \( \bar{g} \in C^\infty(I \times I \times M, H) \) any homotopy with \( \bar{g}(0, \cdot, \cdot) = e, \bar{g}(1, \cdot, \cdot) = g \) and \( \bar{g}(\cdot, 0, \cdot) = \bar{g}(\cdot, 1, \cdot) = e \). The computation of the associated resolving map \( \Lambda \) presented in the appendix gives

\[
\Lambda(\bar{g}) = \int_0^1 \kappa(\delta^1_{z_2}g, \delta^1_{l_1}g)dt + \int_0^1 \beta(\delta^1_{r_2}g, \delta^1_{l_1}g)dt.
\]

(23)

It follows that

\[
\Lambda_{sym}(\bar{g}) = \frac{1}{2} \int_0^1 \beta(\delta^1_{z_2}g, \delta^1_{l_1}g)dt - \frac{1}{2} \int_0^1 \beta(\delta^1_{r_2}g, \delta^1_{r_1}g)dt = \Lambda_{sym}(g),
\]

is a map depending only on the endpoint \( g \) of the homotopy class \([\bar{g}]\). Hence the map \( \Lambda_{sym} \) descends to \( \Omega_0G \) and \( \Pi_\omega = \Lambda_{sym}(\pi_1(\Omega_0G)) = 0 \), this giving another proof that the period group of the coupled cocycle \( \omega \) vanishes. The restriction of \( C_{sym} \) to the subgroup \( \Omega_0G \) of null-homotopic loops based at \( e \) is the coboundary of \(-\Lambda_{sym} : \Omega_0G \to \Omega^1(M, V)\).

**Theorem 3.** The quotient group \((\Omega^1(M, V) \times_{C_{sym}} PG)/\text{Graph}(\Lambda_{sym})\) is a central extension of the universal cover \( \tilde{G} \) of the current group by \( \Omega^1(M, V) \), integrating the coupled cocycle \( \omega \). Here the group cocycle \( C_{sym} \) on \( PG \) is

\[
C_{sym}(f, g) = \int_I \delta^1 f \wedge \kappa \delta^2 g - dx\left( C_{\beta, sym}(f, g) + \frac{1}{2} \int_0^1 \int_0^t (m(s, t) - m(t, s))dt ds \right)
\]

for \( m \) and \( C_\beta \) given by (20) and (21), and the smooth map \( \Lambda_{sym} : \Omega_0G \to \Omega^1(M, V) \) is

\[
\Lambda_{sym}(g) = \frac{1}{2} \int_0^1 \beta(\delta^1_{z_2}g, \delta^1_{l_1}g)dt - \frac{1}{2} \int_0^1 \beta(\delta^1_{r_2}g, \delta^1_{l_1}g)dt.
\]

6. **The group of volume preserving diffeomorphisms**

On the compact manifold \( M \) we consider an integral volume form \( \mu \). Let \( G = \text{Diff}_\mu(M) \) be the connected component of the group of volume preserving diffeomorphisms and \( g = \chi_\mu(M) \) its Lie algebra, the Lie algebra of divergence free vector fields [8] Section 43. Its subgroup, the group of exact volume preserving diffeomorphisms, is a Lie group with Lie algebra \( \{ X \in \chi(M) : i_X\mu \text{ is an exact differential form} \} \), kernel of a flux homomorphism [1].

Given \( \eta \) a closed integral 2-form on \( M \), the Lichnerowicz cocycle

\[
\omega : g \times g \to \mathbb{R}, \quad \omega(X, Y) = \int_M \eta(X, Y)\mu
\]

is a Lie algebra 2-cocycle on the Lie algebra of divergence free vector fields. Indeed, by the closedness of \( \eta \),

\[
\sum_{\text{cyl}} \int_M \eta([X_1, X_2], X_3)\mu = \sum_{\text{cyl}} L_{X_1}(\eta(X_2, X_3))\mu = 0.
\]

The Lichnerowicz cocycle integrates to a central Lie group extension of the subgroup of exact volume preserving diffeomorphisms [6] [4]. It integrates also to a central Lie group extension of the universal cover \( \tilde{G} \) of the group of volume preserving diffeomorphisms: the existence is proved in [15] and a construction with Kostant’s prequantization central extension is given in [20]. In this section we use the method of Section 3 to explicitly realize this central Lie group extension.
Because $\eta$ is integral, there exists a principal circle bundle $q : (P, \theta) \to (M, \eta)$ with connection 1-form $\theta$ and curvature 2-form $\eta$, so that $q^* \eta = d\theta$. There is a natural volume form on $P$ determined by $\mu$ and $\theta$, namely $\tilde{\mu} = \theta \wedge q^* \mu$. It has the property that for any $f \in C^\infty(M)$, $\int_{\mathcal{P}} (q^* f) \tilde{\mu} = \int_M f \mu$. Considering the principal $\mathbb{T}$-action on $P$, each $\mathbb{T}$-invariant divergence free vector field in $\mathfrak{X}_\mu(P)^\mathbb{T}$ projects to a divergence free vector field in $\mathfrak{X}_\mu(M)$. Every vector field $X \in \mathfrak{X}(M)$ has a horizontal lift to $P$, denoted by $X^{\text{hor}}$, uniquely defined by $\theta(X^{\text{hor}}) = 0$ and $q_*X^{\text{hor}} = X$. Moreover, if $X$ is divergence free w.r.t. $\mu$, then $X^{\text{hor}}$ is divergence free w.r.t. $\tilde{\mu}$, hence the horizontal lift provides a section of the abelian Lie algebra extension

$$0 \to C^\infty(M) \to \mathfrak{X}_\mu(P)^\mathbb{T} \to \mathfrak{X}_\mu(M) \to 0.$$ (24)

Let $g$ be a volume preserving diffeotopy of $M$ starting at the identity. One can lift it to the volume preserving diffeotopy $g^{\text{hor}}$ of $P$, starting at the identity and defined by

$$\delta^r(g^{\text{hor}}) = (\delta^r g)^{\text{hor}}.$$ (25)

The diffeotopy $g^{\text{hor}}$ consists of $\mathbb{T}$-equivariant diffeomorphisms of $P$. For $X \in \mathfrak{X}_\mu(M)$ and $t \in I$ the vector field $g^{\text{hor}}(t)^*X^{\text{hor}}$ on $P$ descends to the vector field $g(t)^*X$, but is not necessarily horizontal. The failure of horizontality is measured by

$$g^{\text{hor}}(t)^*X^{\text{hor}} - (g(t)^*X)^{\text{hor}} = q_* (\theta(g^{\text{hor}}(t)^*X^{\text{hor}})) \in C^\infty(M).$$

Here the $\mathbb{T}$-equivariance of $g^{\text{hor}}(t)$ together with the $\mathbb{T}$-invariance of $X^{\text{hor}}$ assure that $\theta(g^{\text{hor}}(s)^*X^{\text{hor}})$ is the pullback of a function on $M$, function denoted by $q_* (\theta(g^{\text{hor}}(t)^*X^{\text{hor}}))$. The computation of a group 2-cocycle $C$ on $\text{PG}$ integrating $P\omega$, using the formula \cite{1}, gives $C$ as the integral over $M$ of an expression of this type.

Indeed, using the fact that the adjoint action in Diff($M$) is $\text{Ad}(g)X = (g^{-1})^*X = Tg \circ X \circ g^{-1}$ in the first step, the relation $\delta^r g = \text{Ad}(g^{-1})\delta^r g = g^* \delta^r g$ following from \cite{11} in the second step, and the fact that the horizontal lift of a divergence free vector field is again divergence free in step 4, we get

$$C(f, g) = \int_0^1 \int_0^s \left( \int_M \eta(g(t)^* \delta^r f(s), \delta^r g(t)) \mu \right) dt ds$$

$$= \int_0^1 \int_0^s \left( \int_M (g(t)^{-1} \eta)(\delta^r f(s), \delta^r g(t)) \mu \right) dt ds$$

$$= \int_0^1 \int_0^s \left( \int_P (dg^{\text{hor}}(t)^{-1})^* \theta)(\delta^r f(s)^{\text{hor}}, \delta^r g(t)^{\text{hor}}) \tilde{\mu} \right) dt ds$$

$$= \int_0^1 \int_0^s \left( \int_P \theta(g^{\text{hor}}(t)^* L_{\delta^r g(t)^{\text{hor}}} \delta^r f(s)^{\text{hor}}) \tilde{\mu} \right) dt ds$$

$$= \int_0^1 \int_0^s \left( \int_P \theta(\frac{d}{dt}(g^{\text{hor}}(t)^* \delta^r f(s)^{\text{hor}})) \tilde{\mu} \right) dt ds$$

$$= \int_0^1 \left( \int_M q_* (\theta(g^{\text{hor}}(s)^* \delta^r f(s)^{\text{hor}})) \mu \right) ds.$$

Let $\bar{g} \in \tilde{\Omega}_0 G$ be the homotopy class of a path in $\Omega_0 G$ viewed as a map $\bar{g} : I \times I \to G$ and $\tilde{g} : I \times I \times M \to M$, $\tilde{g}(s, t, x) = \bar{g}(s, t)(x)$. Let $p : I \times I \times M \to M$ denote
defines a map \( \hat{\Lambda}(g) \) to
\[
\Lambda([g]) - \int_0^1 \omega^q - \int_0^1 \int_0^1 \left( \int_M \eta(i_{\partial_s} \delta^t \hat{g}, i_{\partial_t} \delta^t \hat{g}) \mu \right) ds dt = - \int_0^1 \int_0^1 \left( \int_M \eta \wedge i_{\partial_s} \delta^t \hat{g}, i_{\partial_t} \delta^t \hat{g} \mu \right) ds dt - \int_{I \times I \times M} p^* \eta \wedge \hat{g}^* \mu.
\]

**Remark 14.** The restriction of \( \Lambda \) to \( \pi_2(G) = \pi_1(\Omega_0 G) \subset \Omega_0 G \) is minus the period map of \( \omega \) by Remark [5]. One can now easily see that the period group of \( \omega \) is discrete. Each piecewise smooth representative \( \tau : S^2 \to G \) of a class \( [\tau] \in \pi_2(G) \) defines a map \( \tilde{\tau} : S^2 \times M \to M \). Then per_\( \omega([\tau]) = -\Lambda([\tau]) = \int_{S^2 \times M} p^* \eta \wedge \tilde{\tau}^* \mu \in \mathbb{Z} \), because both \( \eta \) and \( \mu \) are integral forms on \( M \). Hence the period group \( \Pi_\omega \) is contained in \( \mathbb{Z} \).

Let \( \exp : \mathbb{R} \to \mathbb{T} \cong \mathbb{R}/\mathbb{Z}, c = \exp \circ C \) and \( \lambda = \exp \circ \Lambda \). The construction of Theorem 1 yields a central extension integrating the Lichnerowicz cocycle:

**Theorem 4.** The quotient group \( (\mathbb{T} \times_c PG)/\text{Graph}(\lambda) \) is a central extension of the universal covering group \( \hat{G} \) of \( G = \text{Diff}_\mu(M)_0 \) by the circle \( \mathbb{T} \), integrating the Lichnerowicz cocycle \( \omega(X, Y) = \int_M \eta(X, Y) \mu \) defined with the closed integral 2-form \( \eta \) on \( M \). Here the group cocycle \( c : PG \times PG \to \mathbb{T} \) is
\[
e(f, g) = \exp \int_0^1 \left( \int_M q_*(\theta(g^{\text{hor}}(t)^* \delta^t f(t)^{\text{hor}})) \right) dt,
\]
and the smooth map \( \lambda : \Omega_0 G \to \mathbb{T} \) is
\[
\lambda(g) = \left( \exp \int_{I \times I \times M} p^* \eta \wedge \hat{g}^* \mu \right)^{-1}.
\]

If \( M \) is 2-dimensional, then \( \mu \) can be viewed as a symplectic form and \( g = \theta(X) \) as the Lie algebra of symplectic vector fields. The kernel of the infinitesimal flux homomorphism \( \text{flux}_\mu : X \in \mathfrak{X}_\mu(M) \mapsto [i_X \mu] \in H^1_{\text{dR}}(M) \) is the Lie algebra of Hamiltonian vector fields. The Lie algebra cohomology class \( [\omega] \in H^2(\mathfrak{g}) \) of the Lichnerowicz cocycle is the pullback by \( \text{flux}_\mu \) of a multiple of the skew-symmetric pairing \( (a, b) \mapsto \int_M a \wedge b \) on \( H^1_{\text{dR}}(M) \). This is a consequence of the fact that \( \omega \) is cohomologous to a multiple of the 2-cocycle \( (X, Y) \mapsto \int_M \mu(X, Y) \mu \) on \( \mathfrak{g} \), whose restriction to the Lie algebra of Hamiltonian vector fields is trivial.

For an arbitrary compact symplectic manifold \( M \), Lie algebra 2-cocycles on the Lie algebra of Hamiltonian vector fields, having non-zero cohomology classes, are associated to closed 1-forms on \( M \) [19]. It seems that this path method does not work for them.

**APPENDIX: LOGARITHMIC DERIVATIVES**

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). The left logarithmic derivative of a smooth map \( g : M \to G \) is the 1-form \( \delta^l g \in \Omega^1(M, \mathfrak{g}) \), \( \delta^l g(X_x) = g(x)^{-1} T_x g.X_x \) for any \( X_x \in T_x M \). It is the pull-back of the left Maurer-Cartan form \( \theta^l \in \Omega^1(G, \mathfrak{g}) \) by the map \( g \). When \( M = I = [0, 1] \), we identify the left logarithmic derivative of a smooth curve \( g : I \to G \) with the curve \( i_{\partial_t} \delta^l g(t) = g(t)^{-1} \dot{g}(t) \), usually also denoted by \( \delta^l g : I \to \mathfrak{g} \).

The right logarithmic derivative is defined similarly and
\[
\delta^r g = \text{Ad}(g) \delta^l g = -\delta^l(g^{-1}). \tag{A1}
\]
For $f, g : M \to G$, $\xi : M \to \mathfrak{g}$ smooth maps and $\Phi : G \to H$ a Lie group homomorphism with derivative $\varphi : \mathfrak{g} \to \mathfrak{h}$, the following formulae hold:

$$\delta'(fg) = \delta'(g) + \text{Ad}(g)^{-1}\delta'(f) \quad (A2)$$

$$d\delta'g + \frac{1}{2}\delta'g \wedge [\cdot, \cdot] = 0 \quad (A3)$$

$$d\text{Ad}(g)\xi = \text{Ad}(g)[d\xi + \text{ad}(\delta'g)\xi] \quad (A4)$$

$$\delta'(\Phi \circ g) = \varphi \circ \delta'g \quad (A5)$$

The right Maurer-Cartan equation \([A3]\) applied to $X, Y \in \mathfrak{X}(M)$ becomes

$$(d\delta'g)(X, Y) = [\delta'g(Y), \delta'g(X)]. \quad (A6)$$

For $g : I \times M \to G$, $X \in \mathfrak{X}(M)$ and $Y = \partial_t$ it implies

$$d_x(\delta^t_l g) = \frac{d}{dt} \delta^t_l g + [\delta^t_l g, \delta^t_l g]. \quad (A7)$$

Here $\delta^t_l$ and $d_x$ denote the logarithmic derivative and the differential on $M$ considering $t \in I$ as a parameter, and $\delta^t_l g = t_{\partial_t} \delta^t_l g$.

From \([A4]\) and \([A7]\) follows that

$$\frac{d}{dt} \delta^t_r g = \text{Ad}(g)d_x\delta^t_l g. \quad (A8)$$

The right logarithmic derivative satisfies the left Maurer-Cartan equation, so the analogue of \([A7]\) for the right logarithmic derivative is

$$d_x(\delta^t_r g) = \frac{d}{dt} \delta^t_r g - [\delta^t_r g, \delta^t_r g] \quad (A9)$$

and from \([A4]\) follows

$$d_x\text{Ad}(g^{-1})\xi = \text{Ad}(g^{-1})d_x\xi - [\delta^t_l g, \text{Ad}(g^{-1})\xi]. \quad (A10)$$

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