Discrete Dirac system: rectangular Weyl functions, direct and inverse problems

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Abstract

A transfer matrix function representation of the fundamental solution of the general-type discrete Dirac system, corresponding to rectangular Schur coefficients and Weyl functions, is obtained. Connections with Szegö recurrence, Schur coefficients and structured matrices are treated. Borg-Marchenko-type uniqueness theorem is derived. Inverse problems on the interval and semiaxis are solved.

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1 Introduction

In this paper we deal with a discrete Dirac-type (or simply Dirac) system:

\[ y_{k+1}(z) = (I_m + izjC_k)y_k(z) \quad (k \in \mathbb{N}_0), \quad (1.1) \]

where \( \mathbb{N}_0 \) stands for the set of non-negative integer numbers, \( I_m \) is the \( m \times m \) identity matrix, "i" is the imaginary unit \((i^2 = -1)\) and the \( m \times m \) matrices \( \{C_k\} \) are positive and \( j \)-unitary:

\[ C_k > 0, \quad C_kjC_k = j, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad (m_1 + m_2 = m, \quad m_1, m_2 \neq 0). \quad (1.2) \]
Discrete systems are of great interest and their study is sometimes more complicated than the study of the corresponding continuous systems (see, e.g., [1–3, 7, 12] and references therein). The subcase \( m_1 = m_2 \) of system (1.1) (satisfying (1.2)) corresponds to the self-adjoint Dirac-type systems, which were studied in [14] (and the subcase \( j = I_m \) of system (1.1) corresponds to the skew-self-adjoint Dirac-type systems, an important subclass of which was investigated in [20, 23]). The analogies between system (1.1) and continuous Dirac-type systems are also discussed in [14, 20, 23] in detail. Here we follow the paper [15] on the continuous case, where \( m_1 \) does not necessarily equals \( m_2 \) and the \( m_2 \times m_1 \) Weyl matrix functions are, correspondingly, rectangular.

It is essential that Dirac system (1.1), (1.2) is equivalent to the very well-known Szegő recurrence (see, e.g., [10, 28]). This connection is discussed in detail in Section 2. Inverse problems for the subcase of the scalar Schur (or Verblunsky) coefficients were studied, for instance, in [5, 28] (see also various references therein), and here we deal with the rectangular matrix Schur coefficients.

In this paper \( \operatorname{Im} \) denotes image of a matrix (or operator), \( \sigma(A) \) stands for the spectrum of \( A \) and ”span” stands for the linear span.

2 Dirac system and Szegő recurrence

The next simple proposition is essential for our future research and could be of independent interest in the theory of functions (and powers, in particular) of matrices, which is developed in a series of works (see, e.g., [6, 29] and references therein).

**Proposition 2.1** Let an \( m \times m \) matrix \( C \) satisfy relations

\[
C > 0, \quad CjC = j \quad (j = j^* = j^{-1}).
\]  

Then the following relations hold for all \( s \in \mathbb{R} \):

\[
C^s > 0, \quad C^sjC^s = j.
\]

**Proof.** Since \( C > 0 \), it admits a representation

\[
C = u^*Du,
\]
where $D$ is a diagonal matrix and
\begin{equation}
D > 0, \quad u^* u = uu^* = I_m. \tag{2.4}
\end{equation}

We substitute (2.3) into the second equality in (2.1) to derive
\begin{equation}
u^* D u_j u^* D u = j,
\end{equation}
or, equivalently,
\begin{equation}
D J D^* = J, \quad J = J^* = J^{-1} := u_j u^*. \tag{2.5}
\end{equation}

Formula (2.5) yields $D^{-1} = J D J$ and, taking power $s$ of the both parts of this equality, we obtain
\begin{equation}
D^{-s} = J D^s J, \quad D^s J D^s = J. \tag{2.6}
\end{equation}

Finally, using (2.4)–(2.6) we have
\begin{equation}
u^* D^s u_j u^* D^s u = j. \tag{2.7}
\end{equation}

We substitute $s = 1/2$ and apply Proposition 2.1 to matrices $C_k$ in order to obtain the next proposition.

**Proposition 2.2** Let matrices $C_k$ satisfy (1.2). Then they admit representations
\begin{align}
C_k &= 2 \beta(k)^* \beta(k) - j, \quad \beta(k) j \beta(k)^* = I_{m_1}, \tag{2.8} \\
C_k &= j + 2 \gamma(k)^* \gamma(k), \quad \gamma(k) j \gamma(k)^* = - I_{m_2}, \tag{2.9}
\end{align}
where $\beta(k)$ and $\gamma(k)$ are $m_1 \times m$ and $m_2 \times m$ matrices given by (2.10) and (2.11), respectively.

**Proof.** We note that matrices $C_k$ satisfy conditions of Proposition 2.1 and so (2.2) holds for $C = C_k$. Next we put
\begin{equation}
\beta(k) := [I_{m_1} \ 0] C_k^{1/2} \tag{2.10}
\end{equation}
and take into account the equality
\[ C_k = C_k^{1/2} \left( 2 \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_1} \end{bmatrix} - j \right) C_k^{1/2}. \]

Now, representation (2.8) is apparent from (2.2) taken with \( s = 1/2 \). In a similar way, formula (2.2) and equality
\[ I_m = j + 2 \begin{bmatrix} 0 & I_{m_2} \\ I_{m_2} & 0 \end{bmatrix} \] imply
representation (2.9) for \( \gamma(k) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} C_k^{1/2}. \) (2.11)

Now, we will consider interrelations between Dirac system (1.1), (1.2) and Szegö recurrence, which is given by the formula
\[ X_{k+1}(\lambda) = D_k H_k \begin{bmatrix} \lambda I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} X_k(\lambda), \] (2.12)
where
\[ H_k = \begin{bmatrix} I_{m_1} & \rho_k \\ \rho_k^* & I_{m_2} \end{bmatrix}, \quad D_k = \text{diag} \left\{ (I_{m_1} - \rho_k \rho_k^*)^{-\frac{1}{2}}, (I_{m_2} - \rho_k^* \rho_k)^{-\frac{1}{2}} \right\}, \] (2.13)
and the \( m_1 \times m_2 \) matrices \( \rho_k \) are strictly contractive, that is,
\[ \| \rho_k \| < 1. \] (2.14)

**Remark 2.3** When \( m_1 = m_2 = 1 \), one easily removes the factor \((1 - |\rho_k|^2)^{-1/2}\) in (2.12) to obtain systems as in [4, 5], where direct and inverse problems for the case of scalar strictly pseudo-exponential potentials have been treated. The square matrix version (i.e., the version where \( m_1 = m_2 \)) of Szegö recurrence, its connections with Schur coefficients and applications are discussed in [8, 9] (see also references therein). For the rectangular matrices \( \rho_k \) see, for instance, [10]. We note that \( D_k H_k \) is the so called Halmos extension of \( \rho_k \) (see [10, p. 167]), and that the matrices \( D_k \) and \( H_k \) commute (which easily follows, e.g., from [10, Lemma 1.1.12]). The matrix \( D_k H_k \) is \( j \)-unitary and positive, that is,
\[ D_k H_k j H_k D_k = H_k D_k j D_k H_k = j, \] (2.15)
\[ D_k H_k > 0. \] (2.16)
According to [11, Theorem 1.2], any \( j \)-unitary matrix \( C \) admits a representation, which is close to Halmos extension. More precisely, partitioning \( C \) into blocks \( C = \{ c_{ik} \}_{i,k=1}^{2} \) we see that the \( m_1 \times m_1 \) block \( c_{11} \) and the \( m_2 \times m_2 \) block \( c_{22} \) are invertible. Then, putting

\[
\rho = c_{12}c_{22}^{-1} = (c_{11}^{-1})^*c_{21}^*, \quad u_1 = \left( I_{m_1} - \rho \rho^* \right)^{1/2}c_{11}, \quad u_2 = \left( I_{m_2} - \rho^* \rho \right)^{1/2}c_{22},
\]

we have the representation:

\[
C = \mathcal{D}H \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}, \quad u_i^*u_i = u_i^*u_i = I_{m_i}, \quad H = \begin{bmatrix} I_{m_1} & \rho \\ \rho^* & I_{m_2} \end{bmatrix}, \quad (2.17)
\]

\[
\mathcal{D} = \text{diag}\left\{ \left( I_{m_1} - \rho \rho^* \right)^{-1/2}, \left( I_{m_2} - \rho^* \rho \right)^{-1/2} \right\}, \quad \rho^* \rho < I_{m_2}. \quad (2.18)
\]

Although relations (2.15)-(2.17) are well-known, we could not find in the literature a statement, which is converse to (2.15), (2.16). Hence, we prove it below.

**Proposition 2.4** Let an \( m \times m \) matrix \( C \) be \( j \)-unitary and positive. Then it admits a representation

\[
C = \mathcal{D}H, \quad (2.19)
\]

where \( H \) and \( \mathcal{D} \) are of the form (2.17) and (2.18) (i.e., the last factor on the right-hand side of the first equality in (2.17) is removed).

**Proof.** Recall that \( C \) admits representation (2.17). We fix a unitary matrix \( \tilde{U} \) such that \( \mathcal{D}H = \tilde{U}\tilde{D}\tilde{U}^* \), where \( \tilde{D} \) is a diagonal matrix, \( \tilde{D} > 0 \). Then, relations \( C = C^* \) and (2.17) yield the equality

\[
\tilde{U}\tilde{D}\tilde{U}^* \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} = \begin{bmatrix} u_1^* & 0 \\ 0 & u_2^* \end{bmatrix} \tilde{U}\tilde{D}\tilde{U}^*,
\]

which we rewrite in the form

\[
\tilde{D}\tilde{U} = \tilde{U}^*\tilde{D}, \quad \tilde{U} := \tilde{U}^* \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \tilde{U}. \quad (2.20)
\]
According to (2.20), \( \tilde{D} \tilde{U} \) is a selfadjoint matrix, and so \( \tilde{D}^{1/2} \tilde{U} \tilde{D}^{-1/2} \) is a selfadjoint matrix too, that is, there is a representation

\[
\tilde{D}^{1/2} \tilde{U} \tilde{D}^{-1/2} = \tilde{U} \tilde{D}_1 \tilde{U}^*,
\]

(2.21)

where \( \tilde{U} \) and \( \tilde{D}_1 = \tilde{D}_1^* \) are unitary and diagonal matrices, respectively. The definition of \( \tilde{U} \) in (2.20) implies that \( \tilde{U} \) is unitary. Therefore, in view of (2.21), \( \tilde{D}_1 \) is linear similar to a unitary matrix, that is, its entries are \( \pm 1 \).

Moreover \( \tilde{D}_1 > 0 \), since \( C > 0 \) and formulas (2.17), (2.20) and (2.21) yield

\[
C = \tilde{U} \tilde{D} \tilde{U}^* = \tilde{U} \tilde{D}^{1/2} \tilde{U} \tilde{D}_1 \tilde{U}^* \tilde{D}^{1/2} \tilde{U}^*.
\]

(2.22)

From the inequality \( \tilde{D}_1 > 0 \) and the fact that the entries of \( \tilde{D}_1 \) equal either 1 or \(-1\), we have \( \tilde{D}_1 = I_m \). Thus, the last equality in (2.22) implies \( C = \tilde{U} \tilde{D} \tilde{U}^* \), that is, (2.19) holds.

\[\blacksquare\]

Proposition 2.4 completes Propositions 2.1 and 2.2 on representations and properties of \( C_k \). Taking into account (2.15), (2.16) and Proposition 2.4 we rewrite Szegő recurrence (2.12) in an equivalent form

\[
X_{k+1}(\lambda) = \tilde{C}_k \begin{bmatrix} \lambda I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} X_k(\lambda), \quad k \in \mathbb{N}_0,
\]

(2.23)

\[
\tilde{C}_k > 0, \quad \tilde{C}_k j \tilde{C}_k = j.
\]

(2.24)

Using (2.24) we see that the matrix functions \( U_k \), which are given by the equalities

\[
U_0 := I_m, \quad U_{k+1} := i U_k \tilde{C}_k j = \prod_{r=0}^{k} (i \tilde{C}_r j) \quad (k \geq 0),
\]

(2.25)

are also \( j \)-unitary. From (2.24) and (2.25) we have

\[
(i + z) U_{k+1}(I_m + izj) \tilde{C}_k \begin{bmatrix} \frac{z+1}{z+1} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} (I_m + izj)^{-1} U_k^{-1}
\]

\[
= I_m + iz U_{k+1} j U_k^{-1}.
\]

(2.26)
In view of (2.26), the function $y_k$ of the form

$$y_k(z) = (i + z)^k U_k(I_m + izj)X_k \left( \frac{z - i}{z + 1} \right)$$  \hspace{1cm} (2.27)

satisfies (1.1), where $y_0(z) = (I_m + izj)X_0(z)$ and $C_k = jU_{k+1}U_{k+1}^{-1}$. Since $U_{k+1}$ is $j$-unitary, we rewrite $C_k$ as

$$C_k = jU_{k+1}U_k^*,$$  \hspace{1cm} (2.28)

and so (1.2) holds. Because of (2.25), (2.28) and $j$-unitarity of $U_k$, we have

$$jU_k^*C_kU_kj = \tilde{C}_k^2,$$

that is,

$$\tilde{C}_k = (jU_k^*C_kU_kj)^{1/2}.$$  \hspace{1cm} (2.29)

The following theorem describes interconnections between systems (1.1) and (2.23).

**Theorem 2.5** Dirac systems (1.1), (1.2) and Szegö recurrences (2.23), (2.24) are equivalent. The transformation $\mathcal{M} : \{\tilde{C}_k\} \to \{C_k\}$ of Szegö recurrence into Dirac system, and the transformation of their solutions, are given, respectively, by formulas (2.28) and (2.27), where matrices $\{U_k\}$ are defined in (2.25). The mapping $\mathcal{M}$ is bijective, and the inverse mapping is obtained by applying (2.29) (and substitution of the result into (2.25)) for the successive values of $k$.

**Proof.** It is proved already above that the formulas (2.28) and (2.27) describe a mapping of Szegö recurrence and its solution into Dirac system and its solution, respectively. Moreover, the mapping $\mathcal{M}$ is injective, since we can successively and uniquely recover $\tilde{C}_k$ and $U_{k+1}$ from $C_k$ and $U_k$ using formulas (2.29) and (2.25), respectively.

Next, we prove that $\mathcal{M}$ is surjective. Indeed, given an arbitrary sequence $\{C_k\}$ satisfying (1.2), let us apply to the matrices from this sequence relation (2.29) (and substitute the result into (2.25)) for the successive values of $k$. In this way we construct a sequence $\{\tilde{C}_k\}$. Since the matrices $jU_k^*C_kU_kj$ are positive and $j$-unitary, we see, from (2.29) and Proposition 2.1 that the
matrices $\tilde{C}_k$ are also positive and $j$-unitary. Next, we apply to $\{\tilde{C}_k\}$ the mapping $\mathcal{M}$. Taking into account (2.25) and (2.29), we derive

$$jU_{k+1}^{*}U_{k+1}^{*}j = jU_{k}\tilde{C}_k^2U_{k}^{*}j = jU_{k}(jU_{k}^{*}C_kU_{k}j)U_{k}^{*}j = C_k,$$

(2.30)

that is, $\mathcal{M}$ maps the constructed sequence $\{\tilde{C}_k\}$ into the initial sequence $\{C_k\}$. Recall that we started from an arbitrary $\{C_k\}$ satisfying (1.2). Hence, $\mathcal{M}$ is surjective. ■

3 Weyl theory: direct problems

In this section we introduce Weyl functions for matricial discrete Dirac systems (1.1). Next we prove the Weyl function’s existence and, moreover, give a procedure to construct it (direct problems). Finally, we construct the $S$-node, which corresponds to system (1.1), and the transfer matrix function representation of the fundamental solution $W_k$. (See, e.g., [25–27] on the $S$-nodes and the transfer matrix functions in Lev Sakhnovich sense.)

The fundamental $m \times m$ solution $\{W_k\}$ of (1.1) we normalize by the condition

$$W_0(z) = I_m.$$  
(3.1)

Similar to the continuous analog of (1.1) in [15,16] (see also canonical system case [27, p. 7]), the Weyl functions of system (1.1) on the interval $[0, r]$ (i.e., system (1.1) considered for $0 \leq k \leq r$) are defined by the Möbius (linear-fractional) transformation:

$$\varphi_r(z, P) = [0\quad I_{m_2}] W_{r+1}(z)^{-1}P(z) \left( [I_{m_1}\quad 0] W_{r+1}(z)^{-1}P(z) \right)^{-1},$$  
(3.2)

where $P(z)$ are nonsingular $m \times m_1$ matrix functions with property-$j$. That is, $P(z)$ are meromorphic in $\mathbb{C}_+$ matrix functions such that

$$P(z)^*P(z) > 0, \quad P(z)^*jP(z) \geq 0$$  
(3.3)

for all points in $\mathbb{C}_+$ (excluding, possibly, a discrete set). The first inequality in (3.3) means non-singularity (non-degeneracy) of $P$ and the second
inequality is called property-\(j\). Since \(\mathcal{P}\) is meromorphic, property-\(j\) almost everywhere in \(\mathbb{C}_+\) and the first inequality in (3.3) at some \(z_0 \in \mathbb{C}_+\) suffice for the conditions on \(\mathcal{P}\) to hold.

It is apparent from (1.1) and (3.1) that

\[
W_{r+1}(z) = \prod_{k=0}^{r} (I_m + izjC_k). \tag{3.4}
\]

In view of (2.9) and (3.4) we obtain

\[
W_{r+1}(i) = (-2)^{r+1} \prod_{k=0}^{r} \left(j\gamma(k)^*\gamma(k)\right). \tag{3.5}
\]

Hence, \(\det W_{r+1}(i) = 0\), and we don’t consider \(z = i\) in this section.

**Remark 3.1** We note that the behavior of Weyl functions in the neighborhood of \(z = i\) is essential for the inverse problems that are dealt with in the next section. Therefore, unlike the Weyl disc case (see Notation 3.4), in the definition (3.2) of the Weyl functions on the interval we assume that \(\mathcal{P}\) is not only nonsingular with property-\(j\) but has also an additional property. Namely, it is well-defined and nonsingular at \(z = i\). We don’t use this additional property in this section, though, in important cases, it could be obtained via multiplication by a scalar function.

The lemma below shows that transformations \(\varphi_r(z, \mathcal{P})\) are well-defined.

**Lemma 3.2** Fix any \(z \in \mathbb{C}_+\) such that the inequalities \(\det W_r(z) \neq 0\) and (3.3) hold. Then we have the inequality

\[
\det \left( \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^{-1} \mathcal{P}(z) \right) \neq 0. \tag{3.6}
\]

**Proof.** Using (1.2) and (2.9) we obtain

\[
(I_m + izjC_k)^*j(I_m + izjC_k) = (1 + i(z - \overline{z}) + |z|^2)j + 2i(z - \overline{z})\gamma(k)^*\gamma(k) \leq (1 - 2\Re(z) + |z|^2)j, \quad (1 - 2\Re(z) + |z|^2) > 0 \quad \text{for} \quad z \neq i. \tag{3.7}
\]

Since the equality (3.4) holds, formula (3.7) implies that

\[
(W_{r+1}(z)^{-1})^*j W_{r+1}(z)^{-1} \geq (1 - 2\Re(z) + |z|^2)^{-r-1}j \quad (z \in \mathbb{C}_+, \ z \neq i). \tag{3.8}
\]
Because of (3.3) and (3.8), we see that 
\[ \tilde{P} := W_{r+1}(z)^{-1}P(z) \]
 satisfies the inequality \[ \tilde{P}^* \tilde{P} \geq 0. \] It is apparent that the same inequality holds for the matrix \[ [I_{m_1} \ 0]^* \]. In other words, \( \text{Im} \ W_{r+1}(z)^{-1}P(z) \) and \( \text{Im} \ [I_{m_1} \ 0]^* \) are maximal \( j \)-nonnegative subspaces. Therefore, the inequality (3.6) follows in a standard way from \( j \)-theoretic considerations (see, e.g., the proof of (3.45) or the proof of [14, inequality (5.6)] for such considerations). ■

**Corollary 3.3** The following inequalities hold for the fundamental solution \( W_{r+1} \) of (1.1) (where \( \{C_k\} \) satisfy (1.2)):

\[ \det W_{r+1}(z) \neq 0, \quad W_{r+1}(z)^{-1} = (1 + z^2)^{-r-1}jW_{r+1}(z^*)^*j \quad (z \neq \pm i). \] (3.9)

**Proof.** Relations (3.7) and (3.4) imply that \( W_{r+1}(z)^*jW_{r+1}(z) = (1 + z^2)^{r+1}j \), \( z = \overline{z} \).

Hence, using analyticity considerations, we obtain

\[ W_{r+1}(z)^*jW_{r+1}(z) \equiv (1 + z^2)^{r+1}j, \] (3.10)

and (3.9) is apparent. ■

**Notation 3.4** The set of values of matrices \( \varphi_r(z, P) \), which are given by the transformation (3.2) where parameter matrices \( P(z) \) satisfy (3.3), is denoted by \( \mathcal{N}(r, z) \) (or, sometimes, simply \( \mathcal{N}(r) \)).

Usually, \( \mathcal{N}(r, z) \) is called the Weyl disk.

**Corollary 3.5** The sets \( \mathcal{N}(r, z) \) are embedded (i.e., \( \mathcal{N}(r, z) \subseteq \mathcal{N}(r-1, z) \)) for all \( r > 0 \) and \( z \in \mathbb{C}_+, \ z \neq i \). Moreover, for all \( \varphi_k \) \((k \geq 0)\) we have

\[ \varphi_k(z)^*\varphi_k(z) \leq I_{m_1}. \] (3.11)

**Proof.** It follows from Corollary 3.3 that the matrices \( W_{r+1}(z), \ W_r(z) \) and \( (I_m + izjC_r)^{-1} \) are invertible. Hence formulas (3.3) and (3.7) imply that \( \tilde{P} := (I_m + izjC_r)^{-1}P(z) \) satisfies (3.3). Therefore, we rewrite (3.2) in the form

\[ \varphi_r(z, P) = \begin{bmatrix} 0 & I_{m_2} \\ I_{m_2}^{-1} & 0 \end{bmatrix} W_r(z)^{-1}\tilde{P}(z) \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_r(z)^{-1}\tilde{P}(z) \]^{-1}, \] (3.12)
and see that $\varphi_r(z) \in \mathcal{N}(r - 1, z)$ ($r > 0$). Inequality (3.11) is obtained for the matrices from $\mathcal{N}(0, z)$ via substitution of $r = 0$ into (3.12). □

Weyl functions of system (1.1) on the semiaxis $\mathbb{N}_0$ of non-negative integers are defined in a different and more traditional way (in terms of summability), see definition below. We will show also that the definitions of Weyl functions on the interval and semiaxis are interrelated.

**Definition 3.6** The Weyl-Titchmarsh (or simply Weyl) function of Dirac system (1.1) (which is given on the semiaxis $0 \leq k < \infty$ and satisfies (1.2)) is an $m_2 \times m_1$ matrix function $\varphi(z)$ ($z \in \mathbb{C}_+$), such that the following inequality holds:

$$
\sum_{k=0}^{\infty} q(z)^k \left[ I_{m_1} \varphi(z)^* \right] W_k(z)^* C_k W_k(z) \left[ I_{m_1} \varphi(z) \right] < \infty, \quad (3.13)
$$

$$
q(z) := (1 + |z|^2)^{-1}. \quad (3.14)
$$

**Lemma 3.7** If $\varphi_r(z) \in \mathcal{N}(r, z)$, we have the inequality

$$
\sum_{k=0}^{r} q(z)^k \left[ I_{m_1} \varphi_r(z)^* \right] W_k(z)^* C_k W_k(z) \left[ I_{m_1} \varphi_r(z) \right] \leq \frac{1 + |z|^2}{i(z - \bar{z})} \times (I_m - \varphi_r(z)^* \varphi_r(z)). \quad (3.15)
$$

**Proof.** Because of (1.1) and (1.2) we have

$$
W_{k+1}(z)^* j W_{k+1}(z) = W_k(z)^* \left( I_m - i\tau C_k j \right) j \left( I_m + iz C_k \right) W_k(z)
= q(z)^{-1} W_k(z)^* j W_k(z) + i(z - \bar{z}) W_k(z)^* C_k W_k(z). \quad (3.16)
$$

Using (3.1) and (3.16), we derive a summation formula, which is similar to the formula for the case that $m_1 = m_2$, see [14, formula (4.2)]:

$$
\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) = \frac{1 + |z|^2}{i(z - \bar{z})} \left( j - q(z)^* W_{r+1}(z) \right) W_{r+1}(z)^* j W_{r+1}(z). \quad (3.17)
$$

On the other hand, it follows from (3.2) that

$$
\begin{bmatrix} I_{m_1} \\ \varphi_r(z) \end{bmatrix} = W_{r+1}(z)^{-1} P(z) \left( \begin{bmatrix} I_{m_1} & 0 \\ W_{r+1}(z)^{-1} P(z) \end{bmatrix} \right)^{-1}, \quad (3.18)
$$
and so formula (3.3) yields
\[
\begin{bmatrix}
I_{m_1} & \varphi_r(z)^* \\
\varphi_r(z)
\end{bmatrix}
W_{r+1}(z)^* jW_{r+1}(z) \begin{bmatrix}
I_{m_1} \\
\varphi_r(z)
\end{bmatrix} \geq 0.
\] (3.19)

Formulas (3.17) and (3.19) imply (3.15). ■

Now, we are ready to prove the main direct theorem.

**Theorem 3.8** There is a unique Weyl function of the discrete Dirac system (1.1), which is given on the semi-axis \(0 \leq k < \infty\) and satisfies (1.2). This Weyl function \(\varphi\) is analytic and non-expansive (i.e., \(\varphi^* \varphi \leq I_{m_1}\)) in \(\mathbb{C}_+\).

**Proof.** The proof consists of 3 steps. First, we show that there is an analytic and non-expansive function
\[
\varphi_\infty(z) \in \bigcap_{r \geq 0} \mathcal{N}(r, z).
\] (3.20)

Next, we show that \(\varphi_\infty(z)\) is a Weyl function. Finally, we prove the uniqueness.

**Step 1.** This step is similar to the corresponding part of the proof of [16, Proposition 2.2]. Indeed, from Corollary 3.5 we see that the set of functions \(\varphi_r(z, \mathcal{P})\) of the form (3.2) is uniformly bounded in \(\mathbb{C}_+\). So, Montel’s theorem is applicable and there is an analytic matrix function, which we denote by \(\varphi_\infty(z)\) and which is a uniform limit of some sequence
\[
\varphi_\infty(z) = \lim_{i \to \infty} \varphi_{r_i}(z, \mathcal{P}_i) \quad (i \in \mathbb{N}, \ r_i \uparrow, \ \lim_{i \to \infty} r_i = \infty)
\] (3.21)
on all the bounded and closed subsets of \(\mathbb{C}_+\). Clearly, \(\varphi_\infty\) is non-expansive. Since \(r_i \uparrow\), the sets \(\mathcal{N}(r, z)\) are embedded and equality (3.18) is valid, it follows that the matrix functions
\[
\mathcal{P}_{ij}(z) := W_{r_{i+1}}(z) \begin{bmatrix}
I_{m_1} \\
[\varphi_{r_j}(z, \mathcal{P}_j)]
\end{bmatrix} \quad (j \geq i)
\]
satisfy relations (3.3). Therefore, using (3.21) we derive that (3.3) holds for
\[
\mathcal{P}_{i, \infty}(z) := W_{r_{i+1}}(z) \begin{bmatrix}
I_{m_1} \\
[\varphi_\infty(z)]
\end{bmatrix},
\]
(3.22)
which implies that we can substitute $P = P_{i,\infty}$ and $r = r_i$ into (3.2) to obtain

$$\varphi_\infty(z) \in \mathcal{N}(r_i, z).$$  \hspace{1cm} (3.23)

Since (3.23) holds for all $i \in \mathbb{N}$, we see that (3.20) is fulfilled.

**Step 2.** Because of (3.20), the function $\varphi_\infty$ satisfies condition of Lemma 3.7. Hence, (3.13) holds for any $r \geq 0$ and $\varphi_r = \varphi_\infty$, which implies (3.13). Therefore, $\varphi_\infty$ is a Weyl function.

**Step 3.** It is apparent from (2.8) that

$$W_k(z)^*C_kW_k(z) \geq W_k(z)^*(-j)W_k(z).$$  \hspace{1cm} (3.24)

Using (3.16) we derive also

$$q(z)^kW_k(z)^*(-j)W_k(z) \geq q(z)^{k-1}W_{k-1}(z)^*(-j)W_{k-1}(z).$$  \hspace{1cm} (3.25)

Formulas (3.1), (3.24) and (3.25) yield the basic for Step 3 inequality

$$q(z)^kW_k(z)^*C_kW_k(z) \geq -j.$$  \hspace{1cm} (3.26)

Therefore, the following equality is immediate for any $g \in \mathbb{C}^{m_2}$:

$$\sum_{k=0}^\infty g^*[0 \ I_{m_2}]q(z)^kW_k(z)^*C_kW_k(z) \left[ \begin{array}{c} 0 \\ I_{m_2} \end{array} \right] g = \infty.$$  \hspace{1cm} (3.27)

It was shown in Step 2 that $\varphi = \varphi_\infty$ satisfies (3.13). According to (3.13) and (3.27), the dimension of the subspace $L \in \mathbb{C}^m$ of vectors $h$ such that

$$\sum_{k=0}^\infty h^*q(z)^kW_k(z)^*C_kW_k(z)h < \infty$$  \hspace{1cm} (3.28)

equals $m_1$. Now, suppose that there is a Weyl function $\bar{\varphi} \neq \varphi_\infty$. Then we have

$$\text{Im} \left[ \begin{array}{c} I_{m_1} \\ \varphi_\infty(z) \end{array} \right] \subseteq L, \quad \text{Im} \left[ \begin{array}{c} I_{m_1} \\ \bar{\varphi}(z) \end{array} \right] \subseteq L.$$

Therefore, $\dim L > m_1$ (for those $z$, where $\bar{\varphi}(z) \neq \varphi_\infty(z)$) and we arrive at a contradiction. \hfill \blacksquare

Finally, let us construct representations of $W_{r+1}$ ($r \geq 0$) via S-nodes. First, recall that matrices $\{C_k\}$ generate via formula (2.11) a set $\{\gamma(k)\}$ of
the $m_2 \times m$ matrices $\gamma(k)$. Using $\{\gamma(k)\}$, we introduce $m_2(r+1) \times m$ matrices $\Gamma_r$ and $m_2(r+1) \times m_2(r+1)$ matrices $K_r$ ($0 \leq r < \infty$):

$$\Gamma_r := \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \vdots \\ \gamma(r) \end{bmatrix}; \quad K_r := \begin{bmatrix} \zeta_r(0) \\ \zeta_r(1) \\ \vdots \\ \zeta_r(r) \end{bmatrix},$$

$$\zeta_r(k) := i \gamma(k) j \begin{bmatrix} \gamma(0)* & \ldots & \gamma(k-1)* & \gamma(k)*/2 & \ldots & 0 \end{bmatrix}. \quad (3.30)$$

It is apparent from (3.29) and (3.30) that the identity

$$K_r - K_r^* = i \Gamma_r j \Gamma_r^* \quad (3.31)$$

holds. The $m_2(r+1) \times m_2(r+1)$ matrices $A_r$ are introduced by the equalities:

$$A_r = \{a_{p-k}\}_{k,p=0}^r, \quad a_n = \begin{cases} 0 & \text{for } n > 0, \\ (i/2)I_{m_2} & \text{for } n = 0, \\ iI_{m_2} & \text{for } n < 0. \end{cases} \quad (3.32)$$

**Proposition 3.9** Matrices $K_r$ and $A_r$ are linear similar:

$$K_r = E_r A_r E_r^{-1}. \quad (3.33)$$

Moreover, the similarity transformations $E_r$ can be constructed so that

$$E_r = \begin{bmatrix} E_{r-1} & 0 \\ X_r & e_r \end{bmatrix} \quad (r > 0), \quad E_r^{-1} \Gamma_r,2 = \Phi_r,2, \quad \Phi_r,2 := \begin{bmatrix} I_{m_2} \\ \ldots \\ I_{m_2} \end{bmatrix}, \quad (3.34)$$

$$E_0 = e_0 = \gamma_2(0), \quad (3.35)$$

where $\Gamma_{r,p}$ are $m_2(r+1) \times m_p$ blocks of $\Gamma_r = [\Gamma_{r,1} \Gamma_{r,2}]$ and $\gamma_p(k)$ are $m_2 \times m_p$ blocks of $\gamma(k) = [\gamma_1(k) \gamma_2(k)]$.

**Proof.** It follows from (2.9), (3.29), (3.30) and (3.32) that

$$K_0 = A_0 = -(i/2)I_{m_2}, \quad \det \gamma_2(0) \neq 0, \quad (3.36)$$

$$\zeta_r(r) = i \begin{bmatrix} \gamma(r) j \gamma(0)* & \ldots & \gamma(r) j \gamma(r-1)* & -I_{m_2}/2 \end{bmatrix}. \quad (3.37)$$
We see that (3.35) and (3.36) imply (3.33) for \( r = 0 \). Next, we prove (3.33) by induction. Assume that \( K_{r-1} = E_{r-1}A_{r-1}E_{r-1}^{-1} \) and let \( E_r \) have the form (3.34), where \( \det e_r^- \neq 0 \). Then we obtain

\[
E_r^{-1} = \begin{bmatrix}
E_{r-1}^{-1} & 0 \\
-(e_r^-)^{-1}X_r E_{r-1}^{-1} & (e_r^-)^{-1}
\end{bmatrix},
\]

(3.38)

and, in view of (3.29), (3.32), (3.34), (3.37), it is necessary and sufficient (for (3.33) to hold) that

\[
\left( \begin{bmatrix} X_r A_{r-1} & -(i/2)e_r^- 
\end{bmatrix} - ie_r^- \begin{bmatrix} I_{m_2} & \ldots & I_{m_2} & 0 \end{bmatrix} \right) \begin{bmatrix} I_{r m_2} \\
-(e_r^-)^{-1}X_r \end{bmatrix} E_{r-1}^{-1} = i\gamma(r)j \begin{bmatrix} \gamma(0)^* & \ldots & \gamma(r-1)^* \end{bmatrix}.
\]

(3.39)

We can rewrite (3.39) in the form

\[
X_r \left( A_{r-1} + (i/2)I_{r m_2} \right) = i\gamma(r)j \begin{bmatrix} \gamma(0)^* & \ldots & \gamma(r-1)^* \end{bmatrix} E_{r-1}^{-1} + ie_r^- \begin{bmatrix} I_{m_2} & \ldots & I_{m_2} \end{bmatrix}.
\]

(3.40)

We partition \( X_r \, (r > 1) \) into two \( m_2 \times m_2 \) and \( m_2 \times (r - 1)m_2 \), respectively, blocks

\[
X_r = \begin{bmatrix} x_r^- & \tilde{X}_r \end{bmatrix},
\]

(3.41)

and we will need also partitions of the matrices \( A_{r-1} + (i/2)I_{r m_2} \) and \( E_{r-1} \), which follow (for \( r > 1 \)) from (3.32) and (3.34):

\[
\left( A_{r-1} + (i/2)I_{m_2} \right) = \begin{bmatrix} 0 & 0 \\
(A_{r-2} - (i/2)I_{(r-1)m_2}) & 0 \end{bmatrix}, \quad E_{r-1} \begin{bmatrix} 0 & 0 \\
I_{m_2} & e_{r-1}^- \end{bmatrix} = \begin{bmatrix} 0 & 0 \\
I_{m_2} & e_{r-1}^- \end{bmatrix}.
\]

(3.42)

Using (3.41) and (3.42) we see that (3.40) is equivalent to the relations

\[
e_r^- = -\gamma(r)j\gamma(r-1)^*e_{r-1}^- \quad \text{for} \quad r \geq 1;
\]

\[
\tilde{X}_r = i\left( \begin{bmatrix} \gamma(0)^* & \ldots & \gamma(r-1)^* \end{bmatrix} E_{r-1} + e_r^- \begin{bmatrix} I_{m_2} & \ldots & I_{m_2} \end{bmatrix} \right)
\times \begin{bmatrix} 0 & 0 \\
(A_{r-2} - (i/2)I_{(r-1)m_2})^{-1} & 0 \end{bmatrix} \quad \text{for} \quad r > 1.
\]

(3.43, 3.44)
Hence, if $e_r^-$ and $X_r$ satisfy (3.43) and (3.44), respectively, and $\det e_r^- \neq 0$, the similarity relation (3.33) holds. The inequalities $\det e_r^- \neq 0$ are apparent (by induction) from (3.35), (3.43) and the inequalities

$$\det(\gamma(r)\gamma(r-1)^*) \neq 0, \quad (3.45)$$

and it remains to prove (3.45). Indeed, let $\gamma(r)\gamma(r-1)^*g = 0$, $g \neq 0$. Then, the subspaces $\text{Im} \gamma(r)^*$ and $\text{span} \gamma(r-1)^*g$ are $j$-orthogonal. The second equality in (2.9) (taken for $k = r$ and $k = r - 1$) implies that these subspaces are also $j$-negative, have zero intersection and have dimensions $m_2$ and 1, respectively. Thus, $\text{span} (\gamma(r-1)^*g \cup \text{Im} \gamma(r)^*)$ is an $m_2 + 1$-dimensional $j$-negative subspace, which does not exist. Therefore, the relation (3.45), and so also equality (3.33), is proved.

Formula (3.35) shows that the second equality in (3.34) holds for $r = 0$. Now, we choose $X_r$ (for $r = 1$) and $x_r^-$ (for $r > 1$) so that the second equality in (3.34) holds in the case that $r > 0$. Taking into account (3.38), (3.41) and using induction, we see that this equality is valid when

$$X_1 = \gamma_2(1) - e_1^-, \quad x_r^- = \gamma_2(r) - e_r^- - \tilde{X}_r \Phi_{r-2,2} \quad (r > 1). \quad (3.46)$$

We note that inequalities, which are similar to (3.36) and (3.45), are often required in the study of completion problems and Weyl theory. Therefore, the next proposition, which is easily proved using the same considerations as in the proof of (3.45), could be of more general interest.

**Proposition 3.10** Let the $m \times m$ matrix $J$ satisfy equalities $J = J^* = J^{-1}$ and have $m_1 > 0$ positive eigenvalues. Let $m \times m_1$ matrices $\vartheta$ and $\tilde{\vartheta}$ satisfy inequalities

$$\vartheta^* \vartheta > 0, \quad \vartheta^* J \vartheta > 0, \quad \tilde{\vartheta}^* \tilde{\vartheta} > 0, \quad \tilde{\vartheta}^* J \tilde{\vartheta} \geq 0. \quad (3.47)$$

Then we have

$$\det \vartheta^* J \tilde{\vartheta} \neq 0. \quad (3.48)$$
Let us substitute (3.33) into (3.31) to derive
\[ E_r A_r E_r^{-1} - (E_r^*)^{-1} A_r^* E_r = i \Gamma_r j \Gamma_r^* \]  \hspace{1cm} (3.49)

Multiplying both sides of (3.49) by \( E_r^{-1} \) and \( (E_r^*)^{-1} \) from the left and right, respectively, we obtain the operator identity
\[ A_r S_r - S_r A_r^* = i \Pi_j \Pi_r^* = i(\Phi_{r,1} \Phi_{r,1}^* - \Phi_{r,2} \Phi_{r,2}^*) \]  \hspace{1cm} (3.50)

where
\[ S_r := E_r^{-1} (E_r^*)^{-1}, \quad \Pi_r := E_r^{-1} \Gamma_r = [\Phi_{r,1} \Phi_{r,2}] \]  \hspace{1cm} (3.51)

**Definition 3.11**: The triple of matrices \( \{A_r, S_r, \Pi_r\} \) forms a symmetric S-node if the operator (matrix) identity (3.50) holds, \( S_r = S_r^* \) and \( \det S_r \neq 0 \).

The transfer matrix function (in Lev Sakhnovich form), which corresponds to the S-node, is given by the formula
\[ w_A(r, \lambda) = I_m - i \Pi_r S_r^{-1} (A_r - \lambda I_{(r+1)m_2})^{-1} \Pi_r \]  \hspace{1cm} (3.52)

**Remark 3.12**: A symmetric S-node corresponding to Dirac system (1.1) (which satisfies (1.2)) on the interval \( 0 \leq k \leq r \) is constructed using formulas (3.32) and (3.51), where \( \Gamma_r \) is given in (3.29).

Recall that S-nodes, transfer matrix functions \( w_A \) and the method of operator identities are introduced and studied in [24–27] (see also references therein).

For \( r > 0 \) introduce projectors:
\[ P_1 := [I_{rm_2} \ 0], \quad P_2 = P := [0 \ldots 0 \ I_{m_2}] \]  \hspace{1cm} (3.53)

Since \( E_r^{-1} \) is a block lower triangular matrix, we easily derive from (3.38) and (3.51) that
\[ P_1 S_r P_1^* = E_{r-1}^{-1} (E_{r-1}^*)^{-1} = S_{r-1}, \quad P_1 \Pi_r = \Pi_{r-1} \]  \hspace{1cm} (3.54)

It is apparent that
\[ \det S_{r-1} \neq 0, \quad P_1 A_r P_1^* = A_{r-1}. \]  \hspace{1cm} (3.55)
In view of (3.54) and (3.55), the factorization Theorem 4 from [25] (see also [27, p. 188]) yields

\[
 w_A(r, \lambda) = \left( I_m - i\Pi_r S_r^{-1} P^* (PA_r P^* - \lambda I_{m_2})^{-1} (PS_r^{-1} P^*)^{-1} PS_r^{-1} \Pi_r \right) \times w_A(r - 1, \lambda).
\]  

(3.56)

**Proposition 3.13** The fundamental solution \( W \) of the system (1.1), where \( W \) is normalized by the condition (3.1) and the potential \( \{C_k\} \) satisfies (1.2), admits representation

\[
 W_{r+1}(z) = (1 + iz)^{r+1} w_A(r, (2z)^{-1}).
\]  

(3.57)

**Proof.** Formulas (1.1) and (2.9) imply the following equalities

\[
 W_{r+1}(z) = (1 + iz) \left( I_m + 2iz(1 + iz)^{-1} j\gamma(r)^*\gamma(r) \right) W_r(z) \quad (r \geq 0). \]

(3.58)

On the other hand, we easily derive from (3.29), (3.32), (3.34) and (3.51) that

\[
 (PA_r P^* - \lambda I_{m_2})^{-1} = - (\lambda + i/2)^{-1} I_{m_2}, \quad S_r^{-1} = E_r^* E_r,
\]

\[
 PS_r^{-1} P^* = (e_r)^* e_r, \quad PS_r^{-1} \Pi_r = PE_r^* \Gamma_r = (e_r)^* \gamma(r).
\]

(3.59)

(3.60)

We substitute (3.59) and (3.60) into (3.56) to obtain

\[
 w_A(r, \lambda) = \left( I_m + \frac{2i}{2\lambda + 1} j\gamma(r)^*\gamma(r) \right) w_A(r - 1, \lambda) \quad (r \geq 1).
\]

(3.61)

In a similar way, we rewrite (3.52) (for the case that \( r = 0 \)) in the form

\[
 w_A(0, \lambda) = I_m + \frac{2i}{2\lambda + 1} j\gamma(0)^*\gamma(0).
\]

(3.62)

Finally, we compare (3.58) with (3.61) and (3.62) (and take into account (3.1)) to see that \( W_1(z) = (1 + iz) w_A(0, (2z)^{-1}) \) and iterative relations for the left- and right-hand sides of (3.57) coincide.
4 Weyl theory: inverse problems

The values of $\varphi$ and its derivatives at $z = i$ will be of interest in this section. Therefore, using (3.9) we rewrite (3.2) in the form

$$
\varphi_r(z, P) = - [0 \ I_{m_2}] W_{r+1}(\overline{z})^*P(z) \left( [I_{m_1} \ 0] W_{r+1}(\overline{z})^*P(z) \right)^{-1}, \quad (4.1)
$$

where $P$ in (4.1) differs from $P$ in (3.2) by the factor $j$ (and so this $P$ is also a nonsingular matrix function with property-$j$).

**Definition 4.1** Weyl functions of Dirac system (1.1) (which is given on the interval $0 \leq k \leq r$ and satisfies (1.2)) are $m_2 \times m_1$ matrix functions $\varphi(z)$ of the form (4.1), where $P$ are nonsingular matrix functions with property-$j$ such that $P(i)$ are well-defined and nonsingular.

It is apparent that (4.1) is equivalent to

$$
\begin{bmatrix}
I_{m_1} \\
\varphi_r(z, P)
\end{bmatrix} = jW_{r+1}(\overline{z})^*P(z) \left( [I_{m_1} \ 0] W_{r+1}(\overline{z})^*P(z) \right)^{-1}. \quad (4.2)
$$

**Lemma 4.2** Let $P$ satisfy conditions from Definition 4.1. Then we have the inequality

$$
\det \left( [I_{m_1} \ 0] W_{r+1}(-i)^*P(i) \right) \neq 0. \quad (4.3)
$$

**Proof.** First note that in view of (2.8) we obtain

$$
I_m + C_{kj} = 2\beta(k)^*\beta(k)j. \quad (4.4)
$$

Formulas (3.4) and (4.4) imply

$$
[I_{m_1} \ 0] W_{r+1}(-i)^*P(i) = 2^{r+1} \left( [I_{m_1} \ 0] \beta(0)^* (\beta(0)j\beta(1)^*) \ldots \times (\beta(r-1)j\beta(r)^*)(\beta(r)jP(i)) \right). \quad (4.5)
$$

Using Proposition 3.10 (and the second equality in (2.8)) and putting, correspondingly, $\vartheta = \beta(k)^*$ and $\vartheta = \beta(k+1)^*$ or $\vartheta = P(i)$, we derive inequalities

$$
\det(\beta(k)j\beta(k+1)^*) \neq 0 \quad \text{and} \quad \det(\beta(r)jP(i)) \neq 0, \quad (4.6)
$$

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respectively. In the same way we obtain \( \det \left( \begin{bmatrix} I_m & 0 \end{bmatrix} \beta(0)^* \right) \neq 0 \). Now, inequality (4.3) follows from (4.5). ■

Our next proposition is proved similar to Corollary 3.5.

**Proposition 4.3** Suppose \( \varphi \) is a Weyl function of Dirac system (1.1) on the interval \( 0 \leq k \leq r \), where the potential \( \{C_k\} \) satisfies (1.2). Then \( \varphi \) is a Weyl function of the same system on all the intervals \( 0 \leq k \leq \tilde{r} \) (\( \tilde{r} \leq r \)).

**Proof.** Clearly, it suffices to show that the statement of the proposition holds for \( \tilde{r} = r - 1 \) (if \( r > 0 \)). That is, in view of Definition 4.1, we should prove that \( \widetilde{P}(i) \) is well-defined, that the non-singularity of \( \widetilde{P}(i) = (I_m + C_r j)P(i) \) at \( z = i \) always holds (i.e., \( \widetilde{P}(i) \) is nonsingular), if only \( P \) has these properties.

Indeed, since we have
\[
(I_m - izC_r j)^* j (I_m - izC_r j) = (1 + |z|^2) j + i(z - z) j C_r j \geq (1 + |z|^2) j,
\]
the matrix function \( \widetilde{P} \) has property-\( j \). The non-singularity of \( \widetilde{P}(i) = (I_m + C_r j)P(i) \) is apparent from (4.4) and (4.6). ■

**Theorem 4.4** Suppose \( \varphi \) is a Weyl function of Dirac system (1.1) on the interval \( 0 \leq k \leq r \), where the potential \( \{C_k\} \) satisfies (1.2). Then \( \{C_k\} \) is uniquely recovered from the first \( r + 1 \) Taylor coefficients of \( \varphi \left( i \frac{1}{1+z} \right) \) at \( z = 0 \).

If \( \varphi \left( i \frac{1}{1+z} \right) = \sum_{k=0}^{r} \phi_k z^k + O(z^{r+1}) \), then matrices \( \Phi_{k,1} \) are recovered via the formula
\[
\Phi_{k,1} = - \begin{bmatrix}
\phi_0 \\
\phi_0 + \phi_1 \\
\vdots \\
\phi_0 + \phi_1 + \cdots + \phi_k
\end{bmatrix}.
\]

Using \( \Phi_{k,1} \) we easily recover consecutively \( \Pi_k = [\Phi_{k,1} \quad \Phi_{k,2}] \) (where \( \Phi_{k,2} \) is given in (3.34)) and \( S_k \), which is the unique solution of the matrix identity \( A_k S_k - S_k A_k^* = i \Pi_k j \Pi_k^* \). Next, we construct
\[
\gamma(k)^* \gamma(k) = \Pi_k S_k^{-1} P^* (P S_k^{-1} P^*)^{-1} P S_k^{-1} \Pi_k, \quad P = \begin{bmatrix} 0 & \ldots & 0 & I_{m_2} \end{bmatrix}.
\]
Finally, we use $\gamma(k)^*\gamma(k)$ to recover $C_k$ via (2.9).

**Proof.** Put

$$A(z) := |1 + z^2|^{-2(r+1)} \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} W_{r+1}(z)^* j W_{r+1}(z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix}. \quad (4.10)$$

According to (3.9) and (4.2) we have

$$A(z) = \begin{pmatrix} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^* \mathcal{P}(z) \end{pmatrix}^{-1} \mathcal{P}(z)^* j \mathcal{P}(z) \\ \times \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^* \mathcal{P}(z) \end{pmatrix}^{-1}. \quad (4.11)$$

From (4.3) and (4.11) we see that $A$ is bounded in the neighbourhood of $z = i$:

$$\|A(z)\| = O(1) \quad \text{for} \quad z \to i. \quad (4.12)$$

Let us include into considerations the $S$-node (corresponding to Dirac system), which is constructed in accordance with Remark 3.12. Substitute (3.57) into (4.10) to obtain

$$A(z) = \begin{pmatrix} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^* \mathcal{P}(z) \end{pmatrix}^{-1} \mathcal{P}(z)^* j \mathcal{P}(z) \\ \times \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W_{r+1}(z)^* \mathcal{P}(z) \end{pmatrix}^{-1} \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix}, \quad (4.13)$$

where $I = I_{(r+1)m_2}$. Here we used the important equality

$$w_{A}(r, \lambda)^* j w_{A}(r, \bar{\lambda}) = j - i(\lambda - \bar{\lambda}) \Pi_r (A_r^* - \bar{\lambda}I)^{-1} S_r^{-1} \left( A_r - \frac{1}{2z} I \right)^{-1} \Pi_r, \quad (4.14)$$

which follows from (3.50) and (3.52) (see, e.g., [21, 25]).

Notice that $S_r > 0$. Hence, formulas (4.11), (4.12) and (4.13) imply that

$$\left\| \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \right\| = O(1) \quad \text{for} \quad z \to i. \quad (4.15)$$

Using the block representation $\Pi_r = \begin{bmatrix} \Phi_{r,1} & \Phi_{r,2} \end{bmatrix}$ from (3.51) and multiplying both sides of (4.15) by $\left\| \begin{bmatrix} \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,2} \\ \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,1} \end{bmatrix}^{-1} \right\|$ we rewrite the result:

$$\left\| \varphi(z) + \begin{bmatrix} \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,2} \\ \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,1} \end{bmatrix} \right\| = O \left( \left\| \begin{bmatrix} \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,2} \\ \Phi_{r,2}^* (A_r - \frac{1}{2z} I)^{-1} \Phi_{r,1} \end{bmatrix} \right\| \right) \quad \text{for} \quad z \to i. \quad (4.16)$$
In order to obtain (4.16) we applied also the matrix (operator) norm inequality
\[ \|X_1 X_2\| \leq \|X_1\| \|X_2\| . \]

The resolvent \((A - \lambda I)^{-1}\) is easily constructed explicitly (see, for instance, formula (1.10) in [22]). In particular, we derive
\[ \Phi_{r,2}^*(A - \frac{1}{2z} I)^{-1} = -\frac{2z}{1 + i z} [\tilde{q}(z)^r \tilde{q}(z)^{r-1} \ldots I_{m_2}] , \quad \tilde{q} := \frac{1 - i z}{1 + i z} I_{m_2} . \] (4.17)

From (4.17) we see that
\[ \Phi_{r,2}^*(A - \frac{1}{2z} I)^{-1} \Phi_{r,2} = i \left( 1 - \left( \frac{1 - i z}{1 + i z} \right)^{r+1} \right) I_{m_2} . \] (4.18)

Partitioning \(\Phi_{r,1}\) into \(m_2 \times m_1\) blocks \(\Phi_{r,1}(k)\) and using (4.16)-(4.18) we obtain
\[ \varphi \left( i \frac{1 - z}{1 + z} \right) + \frac{1 - z}{1 - z^{r+1}} \sum_{k=0}^{r} z^k \Phi_{r,1}(k) = O(z^{r+1}) \quad \text{for} \quad z \to 0, \]
which can be easily transformed into
\[ \varphi \left( i \frac{1 - z}{1 + z} \right) + (1 - z) \sum_{k=0}^{r} z^k \Phi_{r,1}(k) = O(z^{r+1}) \quad \text{for} \quad z \to 0, \] (4.19)
and (4.8) follows for \(k = r\). Since \(\sigma(A_r) \cap \sigma(A_r^*) = \emptyset\) the matrix \(S_r\) is uniquely recovered from the matrix identity (3.50). Finally, (4.9) for the case, where \(k = r\), is apparent from (3.60). From Proposition 4.3 we see that \(\varphi\) is a Weyl function of our Dirac system on all the intervals \(0 \leq k \leq \tilde{r} (\tilde{r} \leq r)\) and so all \(C_{\tilde{r}}\) are recovered in the same way as \(C_r\).

The next corollary is a discrete version of Borg-Marchenko-type uniqueness theorems. The active study of such theorems was triggered by the seminal papers by F. Gesztesy and B. Simon [18,19].

**Corollary 4.5** Suppose \(\varphi\) and \(\tilde{\varphi}\) are Weyl functions of two Dirac systems with potentials \(\{C_k\}\) and \(\{\tilde{C}_k\}\), which are given on the intervals \(0 \leq k \leq r\) and \(0 \leq k \leq \tilde{r}\), respectively. We suppose that matrices \(\{C_k\}\) and \(\{\tilde{C}_k\}\) are positive and \(j\)-unitary. Moreover, we assume that
\[ \varphi \left( i \frac{1 - z}{1 + z} \right) - \tilde{\varphi} \left( i \frac{1 - z}{1 + z} \right) = O(z^{p+1}) , \quad z \to \infty , \quad p \in \mathbb{N}_0 , \quad p \leq \min (r, \tilde{r}) . \] (4.20)
Then we have $C_k = \tilde{C}_k$ for all $0 \leq k \leq p$.

Proof. According to Proposition 4.3 both functions $\varphi$ and $\tilde{\varphi}$ are Weyl functions of the corresponding Dirac systems on the same interval $[0, p]$. From (4.20) we see that the first $p + 1$ Taylor coefficients of $\varphi(i\frac{1-z}{1+z})$ and $\tilde{\varphi}(i\frac{1-z}{1+z})$ coincide. Hence, the uniqueness of the recovery of the potential from Taylor coefficients in Theorem 4.4 yields $C_k = \tilde{C}_k$ ($0 \leq k \leq p$). ■

Taking into account (4.8), we derive that the first $r + 1$ Taylor coefficients of $\varphi_r(i\frac{1-z}{1+z})$ at $z = 0$ (for any Weyl function $\varphi_r$ of a fixed Dirac system) can be uniquely and in the same way recovered from the matrix $\Phi_r$, which, in turn, can be constructed as proposed in Remark 3.12. Therefore, the next theorem is apparent.

**Theorem 4.6** Let Dirac system (1.1), where matrices $C_k$ satisfy (1.2), be given on the interval $0 \leq k \leq r$. Then all the functions $\varphi_d(z) = \varphi_r(i\frac{1-z}{1+z}, P)$, where $\varphi_r$ are Weyl functions of this Dirac system, are non-expansive in the unit disk and have the same first $r + 1$ Taylor coefficients $\{\phi_k\}_0$ at $z = 0$.

Step 1 in the proof of Theorem 3.8 shows that the Weyl function $\varphi_\infty$ of Dirac system on the semi-axis can be constructed as a uniform limit of Weyl functions $\varphi_r$ on increasing intervals. Hence, using Theorem 4.6 we obtain the following corollary.

**Corollary 4.7** Let $\varphi(z)$ be the Weyl function of some Dirac system (1.1), which is given on the semi-axis and satisfies (1.2). Assume that $\varphi_r$ is a Weyl function of the same system on the finite interval $0 \leq k \leq r$. Then the first $r + 1$ Taylor coefficients of $\varphi(i\frac{1-z}{1+z})$ and $\varphi_r(i\frac{1-z}{1+z})$ coincide. Therefore, the system can be uniquely recovered from $\varphi$ via procedure from Theorem 4.4.

### 5 Operator identities and interpolation problems

One can easily derive (see, e.g., [17, p. 474]) that the equality

$$s_{k+1,p+1} - s_{kp} = Q_{kp} + Q_{k+1,p+1} - Q_{k+1,p} - Q_{k,p+1}, \quad -1 \leq k, p \leq r - 1$$

(5.1)
holds for the blocks $s_{kp}$ and $Q_{kp}$ of the block matrices $S_r = \{s_{kp}\}_{k,p=0}^r$ and $Q_r = \{Q_{kp}\}_{k,p=0}^r$, respectively, which satisfy the operator identity

$$A_r S_r - S_r A_r^* + iQ = 0,$$  \hspace{1cm} (5.2)

where $A_r$ is given by (3.32). Here we add sometimes commas between the indices of blocks and put also

$$s_{-1,p} = s_{k,-1} = Q_{-1,p} = Q_{k,-1} = 0.$$  \hspace{1cm} (5.3)

For the case that $S_r$ corresponds to Dirac system, we rewrite (5.1) below in an equivalent form and obtain the structure of $S_r$.

**Proposition 5.1** Let $S_r$ satisfy (3.50), where $A_r, \Phi_{r,1}$ and $\Phi_{r,2}$ are given by (3.32), (4.8) and the last equality in (3.34), respectively. Then $S_r$ has the following structure:

$$s_{00} = I_{m_2} - \phi_0 \phi_0^* \quad \text{and} \quad s_{k+1,p+1} - s_{kp} = \phi_{k+1} \phi_{p+1}^*$$  \hspace{1cm} (5.4)

for $-1 \leq k, p \leq r - 1, \quad k + p + 2 > 0$.

The following statement is immediate from Theorem 4.6 and Proposition 5.1.

**Theorem 5.2** Let Dirac system (1.1), where matrices $C_k$ satisfy (1.2), be given on the interval $0 \leq k \leq r$. Then all the functions $\varphi_d(z) = \varphi_r \left( \frac{1 - z}{1 + z}, P \right)$, where $\varphi_r$ are given by (3.2), matrix functions $P(z)$ in (3.2) have property-j and matrices $P(i)$ are non-singular, are non-expansive in the unit disk and have the same first $r + 1$ Taylor coefficients $\{\phi_k\}_{0}^{r}$ at $z = 0$. The matrix $S_r$ determined by these coefficients via (5.4) is positive.

On the other hand, if we assume only that the coefficients $\{\phi_k\}_{0}^{r}$ are fixed and $S_r$ given (5.4) is positive, two related interpolation problems appear.

**Interpolation problem I.** Describe all the analytic and non-expansive in the unit disk matrix functions $\varphi_d$ such that the coefficients $\{\phi_k\}_{0}^{r}$ are their first $r + 1$ Taylor coefficients.

**Interpolation problem II.** Describe all the positive continuations of $S_r$, which preserve the structure given by (5.4).
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