ENERGY SCATTERING FOR THE FOCUSING FRACTIONAL GENERALIZED HARTREE EQUATION

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(Communicated by Tohru Ozawa)

Abstract. This note studies the asymptotics of radial global solutions to the non-linear fractional Schrödinger equation

\[ i\dot{u} - (\Delta)^s u + |u|^{p-2}(I_\alpha * |u|^p)u = 0. \]

Indeed, using a new method due to Dodson-Murphy [10], one proves that, in the inter-critical regime, under the ground state threshold, the radial global solutions scatter in the energy space.

1. Introduction. This work studies the focusing fractional generalized Hartree problem

\[ \begin{cases} 
  i\dot{u} - (\Delta)^s u = -|u|^{p-2}(I_\alpha * |u|^p)u, \\
  u(0, \cdot) = u_0. 
\end{cases} \quad (1.1) \]

Here and hereafter \( u \) is a complex valued function of the variable \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), for some \( N \geq 2 \). The Riesz-potential is defined on \( \mathbb{R}^N \) by

\[ I_\alpha : x \mapsto \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N-\alpha}{2}}|x|^{N-\alpha}}, \quad 0 < \alpha < N. \]

The fractional Laplacian operator is

\[ \mathcal{F}[(-\Delta)^s u] := |\cdot|^{2s} \mathcal{F} u, \quad s \in (0, 1). \]

The fractional Schrödinger equation is a fundamental model of fractional quantum mechanics. It was first proposed by Laskin [20, 21] as a result of extending the Feynman path integral, from the Brownian-like to Levy-like quantum mechanical paths.

The above equation, for \( s = 1 \) and \( p = 2 \), is called Hartree equation. It can be considered as a classical limit of a field equation describing a quantum mechanical non-relativistic many-boson [14]. If \( s = \frac{1}{2} \) and \( p = 2 \), the above equation is known as the pseudo-relativistic Hartree equation, which models the dynamics of boson stars [11, 22].

2000 Mathematics Subject Classification. Primary: 35Q55; Secondary: 35B35.
Key words and phrases. Fractional Hartree equation, global existence, scattering.
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This equation was used, for \( p = 2 \), by Choquard \([24]\) to characterize an electron trapped in its own hole. It was also proposed in order to explain self-gravitating matter in a software where the reduction of the quantum state is interpreted as a gravitational phenomenon \([26]\).

If \( u \) is a solution to (1.1), so is the family
\[
u_\lambda = \lambda \frac{2s+\alpha}{2(p-1)} u(\lambda^{2s} \cdot, \lambda \cdot), \quad \lambda > 0.
\]

One calls the critical Sobolev index,
\[
s_c := \frac{N}{2} - \frac{2s + \alpha}{2(p - 1)}
\]
which is the unique real number satisfying the dilatation norm invariance
\[
\|u_0\|_{\dot{H}^{s_c}} = \|u_\lambda(0, \cdot)\|_{\dot{H}^{s_c}}.
\]

In all this note, one considers the inter-critical regime \( 0 < s_c < s \).

The fractional generalized Hatree problem (1.1) was studied recently in many directions. Indeed, the existence of energy local solutions was proved in \([28]\). Using variational methods, the existence and some properties of ground states were obtained in \([9, 30]\). The orbital stability versus strong instability of standing waves respectively in the mass-subcritical and energy super-critical regimes were established in \([13, 33, 27]\). Because there is no variance identity for fractional Schrödinger equations, the existence of non-global solutions is not completely understood. Using a localized variance identity \([2]\), a partial analysis of blow-up solutions was given in \([12, 28]\).

Let us give some related works. The global existence and scattering of the focusing fractional Schrödinger equation with an energy-critical local source term was treated in \([15]\). See also \([25]\) for an inhomogeneous non-linearity. The global well-posedness versus finite time concentrations of the energy-critical solutions to the fractional Hartree equation were considered in \([4, 5]\).

It is the aim of this note to revisit the scattering of radial global solutions to the Choquard problem (1.1). Indeed, using the concentration-compactness method proposed by \([18]\), the scattering of energy global solutions under the ground state threshold was proved recently by the author \([29]\). Now, using a new method due to \([10]\), one proves the same scattering result. The proof is based on a scattering criterion and a Morawetz estimate in the spirit of \([32, 2]\).

The plan of this paper is as follows. The next section gathers the main result and some useful estimates. Section 3 contains a variational analysis. The fourth section is devoted to proving a Morawetz estimate. In section five, one establishes a scattering criterion. The last section contains a proof of the scattering of global solutions.

Let us close this section with some notations. \( C \) denotes a constant which may vary from line to another. \( A \lesssim B \) means that \( A \leq CB \), if \( A, B > 0 \). Take the Lebesgue and Sobolev spaces
\[
L^r := L^r(\mathbb{R}^N), \quad H^s := H^s(\mathbb{R}^N), \quad H^s_{rd} := \{ u \in H^s, \: u(x) = u(|\cdot|) \}.
\]

Let also the usual norms
\[
\| \cdot \|_r := \| \cdot \|_{L^r}, \quad \| \cdot \| := \| \cdot \|_2;
\]
\[
\| \cdot \|_{H^s} := \left( \| \cdot \|^2 + \|(-\Delta)^{\frac{s}{2}} \cdot \|^2 \right)^{\frac{1}{2}}.
\]
Moreover $T^* > 0$ denotes the lifespan for an eventual solution to (1.1). Finally, one uses the standard convention by summing over repeated indices, e.g., $x; y_i \equiv \sum_{i=1}^{N} x_i y_i$.

2. **Background and main result.** In this section, one gives the main contribution and some useful estimates.

2.1. **Notations.** Here and hereafter, one calls, respectively, the mass critical and energy critical exponents

$$p_* := 1 + \frac{\alpha + 2s}{N}, \quad p^* := 1 + \frac{\alpha + 2s}{N - 2s}.$$ 

Let also the positive real numbers

$$B := \frac{Np - N - \alpha}{s}, \quad A := 2p - B.$$ 

Let the stationary problem associated to (1.1),

$$\phi + (-\Delta)^s \phi = |\phi|^{p-2}(I_\alpha * |\phi|^p)\phi, \quad 0 \neq \phi \in H^s. \quad (2.1)$$

**Remarks 2.1.**

1. If $\phi$ satisfies (2.1), then, $e^{it}\phi$ is a global solution to (1.1), called standing wave;

2. the existence of solutions to (2.1) was obtained in [9, 27].

Let $\phi$ satisfy (2.1) and (2.3) and $u$ be a solution to (1.1). Inspired by [17], the following scale invariant quantities describe the dichotomy of global/non-global existence of solutions [29].

$$\mathcal{M}E[u] := \left(\frac{E[u]}{E[\phi]}\right)^{\frac{s_c}{s}} \left(\frac{M[u]}{M[\phi]}\right)^{s - s_c};$$

$$\mathcal{M}G[u] := \left(\frac{\|(-\Delta)^\frac{s}{2}u\|}{\|(-\Delta)^{\frac{s}{2}}\phi\|}\right)^{\frac{s_c}{s}} \left(\frac{\|u\|}{\|\phi\|}\right)^{s - s_c}.$$ 

One will denote by a radial smooth function $0 \leq \psi \leq 1$ satisfying for $R > 0$,

$$\psi \in C_0^\infty(\mathbb{R}^N), \quad supp(\psi) \subset \{|x| < 1\}, \quad \psi = 1 \text{ on } \{|x| < \frac{1}{2}\}, \quad \psi_R := \psi\left(\frac{\cdot}{R}\right).$$

Finally $x_{N,s,\alpha}$ is the non-negative root of the polynomial function

$$Q(X) := 2s(N - 2)X^2 + (N(1 + 2s) - 6s)X - \alpha - 2s.$$ 

2.2. **Preliminary.** The following Gagliardo-Nirenberg type inequality adapted to (1.1) was obtained in [27].

**Proposition 2.2.** Let $N \geq 2, 0 < \alpha < N, s \in (0,1)$ and $1 + \frac{\alpha}{N} < p < p^*$. Then,

1. there is $C_{N,p,s,\alpha} > 0$, such that $\forall u \in H^s$,

$$\int_{\mathbb{R}^N} |u|^p (I_\alpha * |u|^p) \, dx \leq C_{N,p,s,\alpha} \|u\|^A \|(-\Delta)^{\frac{s}{2}}u\|^B; \quad (2.2)$$

2. there exist $\phi$ a solution to (2.1) satisfying

$$C_{N,p,s,\alpha} = \left(\frac{A}{B}\right) \frac{2p}{A\|\phi\|^{2(p-1)}}. \quad (2.3)$$

The problem (1.1) has a local solution in the energy space [27].
Proposition 2.3. Under the assumption \( N \geq 2, \frac{N}{2N-1} < s < 1 \), \( \max\{0, N-4s\} < \alpha < N \) and \( 2 \leq p < p^* \). The Schrödinger problem (1.1) has a unique local solution in the energy space \( C([0, T], H^s_{rd}) \).

Moreover, the following mass and energy conservation are conserved
\[
M[u(t)] := \|u(t)\|^2 = M[u_0];
\]
\[
E[u(t)] := \|(-\Delta)^{\frac{s}{2}} u(t)\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u(t)|^p (I_\alpha * |u(t)|^p) \, dx = E[u_0].
\]

Remarks 2.4. 1. If \( 1 + \frac{N}{2} < p < p^* \), using (2.2), the energy is well-defined for \( u \in H^s \);
2. the restriction \( \alpha \geq N - 4s \) is due to the regularity assumption \( p \geq 2 \);
3. the parameters of this paper will be assumed to satisfy the following conditions
\[
N \geq 3, \quad \frac{N}{2} \leq s < 1 \quad \text{and} \quad \max\{0, N-4s\} < \alpha < N. \quad (2.4)
\]

The main result of this paper is given in the following subsection.

2.3. Main result. The contribution of this note is the following scattering result.

Theorem 2.5. Let \((N, \alpha, s)\) satisfying (2.4), \( \max\{p, 1 + x_{N,s,\alpha}\} < p < p^* \) such that \( p \geq 2 \) and \( \phi \) be a solution to (2.1) and (2.3). Take \( u_0 \in H^s_{rd} \) satisfying
\[
\max \left\{ \mathcal{ME}[u_0], \mathcal{MG}[u_0] \right\} < 1.
\]

Then, there exist a global solution to (1.1) which scatters. Precisely, there exist \( \psi^\pm \in H^s \) satisfying
\[
\limsup_{t \to \pm \infty} \|e^{-it(-\Delta)^{\frac{s}{2}}} \psi^\pm - u(t)\|_{H^s} = 0.
\]

Remarks 2.6. 1. The global existence of solutions was proved in [29];
2. the restriction \( N \geq 3 \) is required in the proof of the scattering criterion;
3. the radial assumption is required in the proofs of Morawetz estimate and the scattering criterion;
4. the set of \( p \) satisfying the above conditions is nonempty because
\[
Q(p^* - 1) = \frac{2s(N-2)(\alpha + N)(\alpha + 2s)}{(N-2s)^2} > 0;
\]
5. thanks to Pohozaev identities, the quantities \( \mathcal{ME}[\phi] \) and \( \mathcal{MG}[\phi] \), do not depend on the choice of \( \phi \).

In the next subsection, some technical estimates are given.

2.4. Tools. First, let us give a Hardy-Littlewood-Sobolev inequality [23].

Lemma 2.7. Take \( N \geq 1 \).
1. Let \( 0 < \lambda < N \) and \( 1 < r, s < \infty \) satisfying \( 2 = \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} \). Thus,
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)v(y)|}{|x-y|^\lambda} \, dx \, dy \leq C_{N,s,\lambda} \|u\|_{L^r} \|v\|_s, \quad \forall u \in L^r, \forall v \in L^s.
\]
2. Let \( 0 < \alpha < N \) and \( 1 < r, s, q < \infty \) satisfying \( 1 + \frac{\alpha}{N} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s} \). Thus,
\[
\|(I_\alpha * u)v\|_{L^r} \leq C_{N,s,\alpha} \|u\|_{L^s} \|v\|_q, \quad \forall u \in L^s, \forall v \in L^q.
\]

The following Sobolev injections [1, 7] will be useful.
Lemma 2.8. Take $N \geq 2$ and $0 < s < 1$. Then,
1. $H^s \hookrightarrow \mathcal{L}^q$, is a continuous injection, for any $q \in [2, \frac{2N}{N-2})$;
2. if $s \geq m$ and $s - \frac{N}{p} \geq m - \frac{N}{q}$, then $W^{s,p} \hookrightarrow W^{m,q}$ with a continuous injection;
3. the following fractional radial Sobolev inequality holds for all $\frac{1}{2} < \mu < \frac{N}{2}$,
   \[
   \sup_{x \neq 0} |x|^{\frac{1}{2} - \mu} |u(x)| \leq C(N, \mu) \|(-\Delta)^{\frac{s}{2}} u\|, \quad \forall u \in H^s_{rd}.
   \]

The next fractional chain rule [3] will be useful.

Lemma 2.9. Let $N \geq 1$, $0 < s \leq 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $i = 1, 2$. Then,
1. if $F \in C^1(\mathbb{C})$, then
   \[
   \|(-\Delta)^{\frac{s}{2}} F(u)\|_p \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{p_i} \|F'(u)\|_{p_i};
   \]
2. and
   \[
   \|(-\Delta)^{\frac{s}{2}} (u v)\|_p \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{p_1} \|v\|_{q_1} + \|(-\Delta)^{\frac{s}{2}} v\|_{p_2} \|u\|_{q_2}.
   \]

The next result gives a Leibniz rule for fractional derivatives [19].

Lemma 2.10. Let $N \geq 1$, $s_1 + s_2 := s \in (0, 1)$, such that $0 \leq s_1, s_2 \leq s$ and $1 < p, p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Thus,
   \[
   \|(-\Delta)^{\frac{s_1}{2}} (u v) - u(-\Delta)^{\frac{s_2}{2}} v - v(-\Delta)^{\frac{s_1}{2}} u\|_p \lesssim \|(-\Delta)^{\frac{s_1}{2}} v\|_{p_1} \|(-\Delta)^{\frac{s_2}{2}} u\|_{p_2}.
   \]
Moreover, for $s_1 = 0$, the value $q_1 = \infty$ is allowed.

The following absorption result will be useful.

Lemma 2.11. Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that
   \[
   X \leq a + bX^\theta \text{ on } [0, T],
   \]
where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})(\theta b)^{\frac{1}{\theta}}$ and $X(0) \leq (\theta b)^{\frac{1}{\theta}}$. Then
   \[
   X \leq \frac{\theta}{\theta - 1} a \text{ on } [0, T].
   \]

Proof. The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (\theta b)^{\frac{1}{\theta}}]$ and increasing on $[(\theta b)^{\frac{1}{\theta}}, \infty)$. The assumptions imply that $f((\theta b)^{\frac{1}{\theta}}) < 0$ and $f(\frac{\theta}{\theta - 1} a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (\theta b)^{\frac{1}{\theta}}$, we conclude the proof by a continuity argument.

Now, let us collect some standard estimates related to the Schrödinger problem.

Proposition 2.12. Letting the free operator associated to the Schrödinger equation
   \[
   \mathcal{F}^{-1}(e^{-it|\cdot|^2} \ast u) := e^{-it(-\Delta)^r} u,
   \]
one has the energy solution to the problem (1.1),
   \[
   e^{-it(-\Delta)^r} u_0 + i \int_0^t e^{-i(t-\tau)(-\Delta)^r} \|u|^{p-2} (I_\alpha \ast |u|^p) u \, d\tau.
   \]

Definitions 2.13.
1. A pair $(q, r)$ is said admissible if $q, r \geq 2$ and
   \[
   \frac{4N + 2}{2N - 1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2N - 1}{r} \leq N - \frac{1}{2};
   \]
or
   \[
   2 \leq q \leq \frac{4N + 2}{2N - 1}, \quad \frac{2}{q} + \frac{2N - 1}{r} < N - \frac{1}{2};
   \]
2. denote, for short, \((q,r) \in \Gamma\) if
\[N(\frac{1}{2} - \frac{1}{r}) = \frac{2s}{q} + \gamma.\]
Moreover, \(\Gamma := \Gamma_0;\)

3. let \(I \subset \mathbb{R}\) be an interval, one denotes the Strichartz spaces
\[S^r(I) := \cap_{(q,r) \in \Gamma} L^q(I,L^r), \quad S(I) := S^0(I).\]

Finally, let us give some Strichartz estimates \([16,6].\)

**Proposition 2.14.** Let \(N \geq 2, \gamma \in \mathbb{R}\) and \((\tilde{q},\tilde{r}), (q,r)\) two admissible couples satisfying \((N,\tilde{q},\tilde{r}) \neq (2,2,\infty)\). Then, there exists \(C > 0\) such that if \(u_0 \in L^2_{\text{rad}}\), one has

1. \(\|e^{-it(-\Delta)^{\gamma}} u_0\|_{S^r(I)} \leq C \|(\Delta)^{\gamma} u_0\|;\)
2. \(\|u - e^{-it(-\Delta)^{\gamma}} u_0\|_{S^r(I)} \leq C \inf_{(\tilde{q},\tilde{r}) \in \Gamma_{\gamma}} \|iu - (-\Delta)^{\gamma} u\|_{L^q(I),L^r(I,\nu)};\)
3. if \(\frac{N}{2N-1} < s \leq 1\), then, \(\|u\|_{S^s(I)} \leq C(\|u_0\| + \inf_{(\tilde{q},\tilde{r}) \in \Gamma} \|iu - (-\Delta)^{\gamma} u\|_{L^q(I),L^r(I,\nu)}).\)

3. **Variational analysis.** In this section, one collects some estimates needed in the proof of the scattering of global solutions to the focusing Choquard problem \((1.1).\)

**Lemma 3.1.** Take \((N,\alpha,s)\) satisfying \((2.4)\) and \(p_* < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^s\) satisfying
\[
\max \left\{ \mathcal{M}[u_0], \mathcal{M}[u_0] \right\} < 1.
\]
Then, there exists \(\delta > 0\) such that the solution \(u \in C(\mathbb{R},H^s)\) to \((1.1)\) satisfies
\[
\max \left\{ \sup_{t \in \mathbb{R}} \mathcal{M}[u(t)], \sup_{t \in \mathbb{R}} \mathcal{M}[u(t)] \right\} < 1 - \delta.
\]

**Proof.** The inequality \(\mathcal{M}[u_0] < 1\) gives the existence of \(0 < \epsilon < 1\) satisfying
\[
1 - \epsilon > \left( \frac{M(u_0)}{M(\phi)} \right) \frac{\|u\|_{E(\phi)}}{\|\phi\|_{E(\phi)}}.
\]

This gives the existence of \(\delta > 0\) such that the solution \(u \in C(\mathbb{R},H^s)\) to \((1.1)\) satisfies
\[
\max \left\{ \sup_{t \in \mathbb{R}} \mathcal{M}[u(t)], \sup_{t \in \mathbb{R}} \mathcal{M}[u(t)] \right\} < 1 - \delta.
\]

Thus, by Proposition 2.2,
\[
1 - \epsilon > \frac{B - 2}{B} \|(-\Delta)^{\gamma} \phi\|^2 = \frac{B - 2}{A} \|\phi\|^2 = E(\phi).
\]

Thus, by Proposition 2.2,
With the hypothesis of the previous Lemma, there exists

In particular, there exists

Take the real function defined on \( [0, T] \) near to zero such that

such that for any \( t < T \), one gets

\[
1 - \epsilon > \frac{B}{B - 2} \frac{M(u_0)^{\frac{2-p}{p-2}}}{M(\phi)^{\frac{2-p}{p-2}}}(\frac{\|\phi\|}{\|\phi\|_2})^B \frac{\|\phi\|^{2(1-\|\phi\|_2)}}{\|\phi\|^{2(1-\|\phi\|_2)}}\frac{\|\phi\|}{\|\phi\|_2}
\]

Using the equalities \( \frac{\|\phi\|}{\|\phi\|_2} = \frac{2^{1-\|\phi\|_2}}{2^{1-\|\phi\|_2}} \) and \( \frac{B}{\lambda} = \frac{\|\phi\|^{2(1-\|\phi\|_2)}}{\|\phi\|^{2(1-\|\phi\|_2)}} \), one has

\[
1 - \epsilon > \frac{B}{B - 2} \frac{M(u_0)^{\frac{2-p}{p-2}}}{M(\phi)^{\frac{2-p}{p-2}}}(\frac{\|\phi\|}{\|\phi\|_2})^B \frac{\|\phi\|^{2(1-\|\phi\|_2)}}{\|\phi\|^{2(1-\|\phi\|_2)}}\frac{\|\phi\|}{\|\phi\|_2}
\]

Taking the real function defined on \([0, 1]\) by \( f(x) := \frac{B}{B - 2} x^2 - \frac{2}{B - 2} x^B \), with first derivative \( f'(x) = \frac{2B}{B - 2} x(1 - x^{B - 2}) \). Thus, following the variations of \( f \) and the continuity of \( t \to X(t) := \frac{\|u_0\|^{\frac{2-p}{p-2}}}{\|\phi\|^{\frac{2-p}{p-2}}}(\frac{\|\phi\|}{\|\phi\|_2})^B \frac{\|\phi\|^{2(1-\|\phi\|_2)}}{\|\phi\|^{2(1-\|\phi\|_2)}}\frac{\|\phi\|}{\|\phi\|_2} \), it follows that \( X(t) < 1 \) for any \( t < T^* \). Thus, \( T^* = \infty \) and there exists \( 0 < \delta < 1 \) near to zero such that \( X(t) \in f \)(\( (0, 1 - \epsilon) \)) = \( (0, 1 - \delta) \). This finishes the proof.

Let us prove a coercivity estimate on centered balls with large radius.

**Lemma 3.2.** With the hypothesis of the previous Lemma, there exists \( R_0 := R_0(\delta, M(u), \phi) > 0 \) such that for any \( R > R_0 \),

\[
\sup_{t \in R} \|\psi_R u(t)\|^{s-\kappa} \|(-\Delta)\hat{\psi}(\psi_R u(t))\|^{\kappa} < (1 - \delta) \|\phi\|^{s-\kappa} \|(-\Delta)\hat{\phi}\|^{\kappa}.
\]

In particular, there exists \( \delta' > 0 \) such that

\[
\|(-\Delta)\hat{\psi}(\psi_R u)\|^2 \geq \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\psi_R u|^p)|\psi_R u|^p dx \geq \delta' \|\psi_R u\|^{2-\frac{2\kappa}{Np}}.
\]

**Proof.** Taking account of Proposition 2.2, one gets

\[
E[u] = \|(-\Delta)\hat{\psi} u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p (I_\alpha * |u|^p) dx \geq \|(-\Delta)\hat{\psi} u\|^2 \left( \frac{C_{N,p,s,\alpha}}{p} \|u\|^{s-\kappa} \|(-\Delta)\hat{\psi} u\|^{B-2} \right) = \|(-\Delta)\hat{\psi} u\|^2 \left( \frac{C_{N,p,s,\alpha}}{p} \|u\|^{s-\kappa} \|(-\Delta)\hat{\psi} u\|^{2-\frac{2\kappa}{Np}} \right).
\]
So, with the previous Lemma
\[ E[u] \geq \|(-\Delta)\hat{x}u\|^2 \left(1 - (1 - \delta) \frac{2}{A} \frac{1}{B} \frac{A}{B} \frac{1}{p} \|\phi\|^{-2(p-1)} \|\phi\|^{2(p-1)} \right) \]
\[ = \|(-\Delta)\hat{x}u\|^2 \left(1 - (1 - \delta) \frac{2}{A} \frac{1}{B} \frac{A}{B} \frac{1}{p} \|\phi\|^{-2(p-1)} \|\phi\|^{2(p-1)} \right) \]
\[ = \|(-\Delta)\hat{x}u\|^2 \left(1 - (1 - \delta) \frac{2}{B} \frac{1}{\|(-\Delta)\hat{x}\phi\|^{p-2}} \|\phi\|^{1+B-2} \right) \]
\[ = \|(-\Delta)\hat{x}u\|^2 \left(1 - (1 - \delta) \frac{2}{B} \right). \]

Thus, using Sobolev injections with the fact that \( p < p^* \), one gets
\[ \|(-\Delta)\hat{x}u\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} |u|^p(I_\alpha * |u|^p) \, dx \geq \delta \|(-\Delta)\hat{x}u\|^2 \geq \delta \|u\|^2 \frac{2Np}{Np + 2N}. \]

This gives the second part of the claimed Lemma provided that the first point is proved. A direct computation \cite{31} gives
\[ \|(-\Delta)\hat{x}(\psi_Ru)\|^2 - \|\psi_R(-\Delta)\hat{x}u\|^2 \leq C(u_0, \phi)R^{-1}. \]

Then, one gets the proof of the first point and so the Lemma. \( \square \)

4. Morawetz estimate. In this section, one proves the next result.

Proposition 4.1. Take \((N, \alpha, s)\) satisfying \((2.4)\) and \( p_* < p < p^* \) such that \( p \geq 2 \). Let \( u_0 \in H^s_{\text{rad}} \) satisfying
\[ \max \left\{ \mathcal{M}\mathcal{E}[u_0], \mathcal{M}\mathcal{G}[u_0] \right\} < 1. \]

Then, the global solution to \((1.1)\) satisfies
\[ \int_0^T \|u(t)\|_{L_{\frac{2Np}{Np + 2}}}^2 \, dt \leq CT^{\frac{1}{4+\alpha}}, \quad \text{for any} \quad T > 0. \]

Proof. Take a smooth real function such that \( 0 \leq f'' \leq 1 \) and
\[ f : r \to \begin{cases} \frac{r^2}{T}, & \text{if} \ 0 \leq r \leq 1; \\ 1, & \text{if} \ r \geq 2. \end{cases} \]

Moreover, for \( R > 0 \), let the smooth radial function defined on \( \mathbb{R}^N \) by \( f_R := R^2 f(\frac{\cdot}{R}) \).

One can check the properties
\[ 0 \leq f''_R \leq 1, \quad f'(r) \leq r, \quad N \geq \Delta f_R. \]

Denote the localized variance by
\[ M_R[u] := 23 \int_{\mathbb{R}^N} u \nabla f_R \nabla u \, dx := 23 \int_{\mathbb{R}^N} \bar{u} \partial_k f_R \partial_k u \, dx. \]

Let us denote the differential operator by
\[ \Gamma_R = -i \left[ \nabla((\nabla f_R)\cdot) + \nabla f_R \nabla \cdot \right], \]
which satisfies
\[ <u, \Gamma_R u> = M_R[u]. \]

Moreover, for \( m > 0 \), one defines the family of functions
\[ u_m := \sqrt{\frac{\sin(\pi s)}{\pi}} \frac{1}{m - \Delta} u. \]
Integrating by parts, one gets

\[
\frac{d}{dt} MR[u(t)] = \langle u(t), -|u|^{p-2}(I_\alpha * |u|^p), i\Gamma_R|u(t) \rangle + \langle u(t), (-\Delta)\alpha, i\Gamma_R|u(t) \rangle,
\]

where the commutator of $X$ and $Y$ is $[X,Y] := XY - YX$. Thanks to computation done in [2], one writes

\[
(A) := \langle u(t), (-\Delta)\alpha, i\Gamma_R|u(t) \rangle
\]

\[
= \int_0^\infty m^s \int_{\mathbb{R}^N} \left( 4\partial_k\partial_t u \partial_k f \partial_t u - \Delta^2 f |u_m|^2 \right) dx
dm
\]

\[
= 4 \int_0^\infty m^s \int_{|x|<R} |\nabla u_m|^2 \, dx
dm

+ \int_0^\infty m^s \int_{|x|<2R} \partial f(x) |\nabla u_m|^2 \, dx
dm

\]

Let us compute

\[
(N) := \langle u, -|u|^{p-2}(I_\alpha * |u|^p), i\Gamma_R|u \rangle
\]

\[
= -\langle u, -|u|^{p-2}(I_\alpha * |u|^p) \nabla f \nabla u \rangle - \langle u, |u|^{p-2}(I_\alpha * |u|^p) \nabla (u \nabla f) \rangle

+ \langle u, \nabla f \nabla |u|^{p-2}(I_\alpha * |u|^p) u \rangle

+ \langle u, \nabla f \nabla u |u|^{p-2}(I_\alpha * |u|^p) u \rangle

= -2 \langle u, |u|^{p-2}(I_\alpha * |u|^p) \nabla f \nabla u \rangle + 2 \langle u, \nabla f \nabla u |u|^{p-2}(I_\alpha * |u|^p) u \rangle
\]

\[
= 2 \int_{\mathbb{R}^N} |u|^2 \nabla f \nabla |u|^{p-2}(I_\alpha * |u|^p) \, dx
\]

Integrating by parts, one gets

\[
(N) = 2 \int_{\mathbb{R}^N} |\nabla (|u|^2) \nabla f |u|^{p-2}(I_\alpha * |u|^p) \, dx
\]

\[
= -2 \int_{\mathbb{R}^N} \nabla (|u|^2) \nabla f |u|^{p-2}(I_\alpha * |u|^p) \, dx + |u|^2 \Delta f |u|^{p-2}(I_\alpha * |u|^p) \, dx
\]

\[
= -2 \int_{\mathbb{R}^N} \nabla (|u|^2) \nabla f |u|^{p-2}(I_\alpha * |u|^p) \, dx - 2 \int_{\mathbb{R}^N} \Delta f |u|^2(I_\alpha * |u|^p) \, dx
\]

\[
= -2 \int_{\mathbb{R}^N} \Delta f |u|^p(I_\alpha * |u|^p) \, dx - \frac{4}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) \nabla f \nabla |u|^p \, dx
\]

\[
= \frac{4}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) \nabla f |u|^p \, dx + 2 \left( \frac{2}{p} - 1 \right) \int_{\mathbb{R}^N} \Delta f |u|^p(I_\alpha * |u|^p) \, dx.
\]

Now, let us define the following subsets of $\mathbb{R}^N \times \mathbb{R}^N$,

\[
\Omega := \{(x,y), \ R < |x| < 2R \} \cup \{(x,y), \ R < |y| < 2R \};

\Omega' := \{(x,y), \ |x| > 2R, |y| < R \} \cup \{(x,y), \ |x| < R, |y| > 2R \}.
\]

Consider the term

\[
(I) := \int_{\mathbb{R}^N} \nabla f R \left( \frac{x}{|x|^2} I_\alpha * |u|^p \right) |u|^p \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y) \frac{|u(y)u(x)|^p (\nabla f R(x) - \nabla f R(y))(x-y) \, dx \, dy.
\]
\[
\left( I_\alpha(x-y) \right) := 1 \\
\lambda \left| u(y)u(x) \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy
\]

Compute
\[
(a) := \int_{\Omega'} \left( I_\alpha(x-y) \right) \left| u(y)u(x) \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy
\]

\[
\begin{aligned}
&= \int_{\{|x|>2R, |y|<R\}} \left( I_\alpha(x-y) \right) \left| \frac{u(y)u(x)}{|x-y|^2} \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy \\
&+ \int_{\{|y|>2R, |x|<R\}} \left( I_\alpha(x-y) \right) \left| \frac{u(y)u(x)}{|x-y|^2} \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy \\
&= \int_{\{|x|>2R, |y|<R\}} \left( I_\alpha(x-y) \right) \left| \frac{u(y)u(x)}{|x-y|^2} \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy \\
&= \int_{\{|y|>2R, |x|<R\}} \left( I_\alpha(x-y) \right) \left| \frac{u(y)u(x)}{|x-y|^2} \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy.
\end{aligned}
\]

Moreover, since for large \( R \gg 1 \), on \( \{|x| > 2R, |y| < R\} \), \( |x-y| \approx |x| > 2R \gg |y| \), one has

\[
(a) \lesssim \int_{\{|x|>2R, |y|<R\}} \left( y(y-x) \left| \frac{I_\alpha(x-y)}{|x-y|^2} \right|^{p} \right) \, dx \, dy
\]

\[
\lesssim \int_{\{|x|>2R, |y|<R\}} \left( \frac{|y||x-y|}{|x-y|^2} \left| \frac{I_\alpha(x-y)}{|x-y|^2} \right|^{p} \right) \, dx \, dy
\]

\[
\lesssim \int_{\{|x|>2R, |y|<R\}} \left( I_\alpha(x-y) \right) \left| u(y)u(x) \right|^{p} \, dx \, dy
\]

\[
\lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( I_\alpha(x-y) \chi_{|x|>2R} \right) \left| u(y)u(x) \right|^{p} \, dx \, dy
\]

Taking account of Lemma 2.7 via Hölder and Strauss estimates and Sobolev injections via the fact that \( p_* < p < p^* \), write

\[
(a) \lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( I_\alpha(x-y) \chi_{|x|>2R} \right) \left| u(y)u(x) \right|^{p} \, dx \, dy
\]

\[
\lesssim \left\| |u|^p \right\|_{L^{\frac{2Np}{N+2p}}(|x|>2R)} \left\| |u|^p \right\|_{L^{\frac{2Np}{N+2p}}}
\]

\[
\lesssim \left( \int_{\{|x|>2R\}} \left| u \right|^{\frac{2Np}{N+2p}} \, dx \right)^{\frac{N+2p}{N+2}}
\]

\[
\lesssim \left( \int_{\{|x|>2R\}} \left| u \right|^{2} \left( |x|^{-\frac{N+2p}{2N}} \left| (-\Delta)^{\frac{N}{2}} u \right| \right)^{-\frac{N+2p}{N+2}} \, dx \right)^{\frac{N+2p}{N+2}}
\]

\[
\lesssim \|u\|^{\frac{N+2p}{N+2}} \left( \frac{1}{R^{\frac{N(N-2p)}{2N}+\frac{2p}{N}}} \right) \lesssim R^{-\frac{p(N-2p)}{2(N+2)}}.
\]

Furthermore,

\[
(b) := \frac{1}{2} \int_{\{|y|<R, |x|<R\}} \left( \frac{I_\alpha(x-y)}{|x-y|^2} \right) \left| u(y)u(x) \right|^{p} \nabla f_R(x)(x-y) \, dx \, dy
\]

\[
= \frac{1}{2} \int_{\{|y|<R, |x|<R\}} \left( (x-y)(x-y) \right) \left( \frac{I_\alpha(x-y)}{|x-y|^2} \right) \left| u(y)u(x) \right|^{p} \, dx \, dy
\]
Now, \[
\begin{align*}
\frac{1}{2} \int_{|y'|<R,|x'|<R} (I_{\alpha}(x-y)|u(y)u(x)|^p) \, dx \, dy.
\end{align*}
\]

Thus, with the above calculus
\[
(c) := \int_{\{R<|x'|<2R\} \cap \mathbb{R}^N} \left( \frac{I_{\alpha}(x-y)}{|x-y|^2} |u(y)u(x)|^p \nabla f_R(x)(x-y) \right) \, dx \, dy
\]
\[
= \int_{\{R<|x'|<2R,|y'|>\frac{R}{2}\} \cap \mathbb{R}^N} \left( \frac{I_{\alpha}(x-y)}{|x-y|^2} |u(y)u(x)|^p \nabla f_R(x)(x-y) \right) \, dx \, dy
\]
\[
+ \int_{\{R<|x'|<2R,|y'|<\frac{R}{2}\} \cap \mathbb{R}^N} \left( \frac{I_{\alpha}(x-y)}{|x-y|^2} |u(y)u(x)|^p \nabla f_R(x)(x-y) \right) \, dx \, dy
\]
\[
= O\left( \int_{\{|x'|>R\}} |u|^p(I_{\alpha} \ast |u|^p) \, dx \right).
\]

Thus, with the above calculus
\[
\frac{d}{dt} M_R[u(t)] \geq 4 \int_0^\infty m^s \int_{|x'|<R} |\nabla u_m|^2 \, dx \, dm + 4 \int_0^\infty m^s \int_{R<|x'|<2R} \partial^2 f_R(\frac{x}{R}) |\nabla u_m|^2 \, dx \, dm
\]
\[
- \int_0^\infty m^s \int_{\mathbb{R}^N} \Delta^2 f_R |u_m|^2 \, dx \, dm + 2(\frac{2}{p} - 1) \int_{\mathbb{R}^N} \Delta f_R |u|^p(I_{\alpha} \ast |u|^p) \, dx
\]
\[
- \frac{2(N-\alpha)}{p} \int_{|x'|<R} |u(x)|^p(I_{\alpha} \ast |u|^p) \, dx + O\left( R^{-2s} \right).
\]

Moreover, by Lemma 2.1 in [2], one has
\[
\left| \int_0^\infty m^s \int_{\mathbb{R}^N} \Delta^2 f_R |u_m|^2 \, dx \, dm \right| \lesssim R^{-2s}.
\]

Thus,
\[
\frac{d}{dt} M_R[u(t)] \geq 4 \int_0^\infty m^s \int_{|x'|<R} |\nabla u_m|^2 \, dx \, dm + 2N(\frac{2}{p} - 1) \int_{|x'|<R} |u|^p(I_{\alpha} \ast |u|^p) \, dx
\]
\[
- \frac{2(N-\alpha)}{p} \int_{|x'|<R} |u|^p(I_{\alpha} \ast |u|^p) \, dx + 2N(\frac{2}{p} - 1)
\]
\[
\times \int_{|x'|>R} |u|^p(I_{\alpha} \ast |u|^p) \, dx + O\left( R^{-2s} \right)
\]
\[
\geq 4 \int_0^\infty m^s \int_{|x'|<R} |\nabla u_m|^2 \, dx \, dm - \frac{2sB}{p} \int_{|x'|<R} |u|^p(I_{\alpha} \ast |u|^p) \, dx + O\left( R^{-2s} \right).
\]

Let us write using computation done in the proof of Lemma 4.8 in [31],
\[
\int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla (\psi_R u_m)|^2 \, dx \, dm = \int_0^\infty m^s \int_{\mathbb{R}^N} \psi^2_R |\nabla u_m|^2 \, dx \, dm + O(R^{-1})
\]
\[
\leq \int_0^\infty m^s \int_{|x'|<R} |\nabla u_m|^2 \, dx \, dm + O(R^{-1}).
\]
Moreover,
\[
\int_{|x|<R} |u|^p (I_\alpha * |u|^p) \, dx \\
= \int_{\mathbb{R}^N} \psi_R u|^p (I_\alpha * |u|^p) \, dx + \int_{\frac{2}{R} < |x| < R} (1 - \psi_R^p) |u|^p (I_\alpha * |u|^p) \, dx \\
= \int_{\mathbb{R}^N} \psi_R u|^p (I_\alpha * |u|^p) \, dx + O(R^{-2s}).
\]

So, taking account of Lemma 3.2, one gets
\[
d\frac{d}{dt} M_R[u(t)] \\
\geq 4 \int_0^\infty m^s \int_{|x|<R} |\nabla u_n|^2 \, dx \, dm - \frac{2sB}{p} \int_{|x|<R} |u|^p (I_\alpha * |u|^p) \, dx + O(R^{-2s}) \\
\geq 4 \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla (\psi_R u_n)|^2 \, dx \, dm - \frac{2sB}{p} \int_{\mathbb{R}^N} (I_\alpha * |\psi_R u|^p) |\psi_R u|^p \, dx + O(R^{-2s}) \\
\geq 8\delta'' \|\psi_R u\|_R^2 \frac{2Np}{N+n} + O(R^{-2s}).
\]

Then,
\[
\sup_{t \in [0,T]} |M_R[u(t)]| \geq \int_0^T \|\psi_R u\|_R^2 \frac{2Np}{N+n} \, ds + O(R^{-2s}) T \\
\geq \int_0^T \|u\|_{L^\frac{2Np}{N+n}(|x|<\frac{R}{2})}^2 \, ds + O(R^{-2s}) T.
\]

Thus, with previous computation via Lemma A.1 in [2] and the assumption \( s > \frac{1}{2} \), one gets
\[
\int_0^T \|u(t)\|_R^2 \frac{2Np}{N+n} \, dt \leq C \left( \sup_{[0,T]} |M_R| + T(R^{-2s} + R^{-\frac{sB(N-2s)}{2N}}) \right) \leq C \left( R + TR^{-2s} \right).
\]

Taking \( R = T^{\frac{1}{s+s'}} >> 1 \), one gets the requested estimate
\[
\int_0^T \|u(t)\|_R^2 \frac{2Np}{N+n} \, dt \leq CT^{\frac{s}{s'}}.
\]

For \( 0 < T << 1 \), the proof follows with Sobolev injections. \(\square\)

As a consequence, one has the following energy evacuation.

**Lemma 4.2.** Take \((N,\alpha, s)\) satisfying (2.4) and \( p_s < p < p^* \) such that \( p \geq 2 \). Let \( u_0 \in H^s_{rad} \) satisfying
\[
\max \left\{ \mathcal{M}_u[u_0], \mathcal{M}_u^2[u_0] \right\} < 1.
\]

Then, there exists a sequence of real numbers \( t_n \to \infty \) such that the global solution to (1.1) satisfies
\[
\lim_n \int_{|x|<R} |u(t_n, x)|^2 \, dx = 0, \quad \text{for all} \quad R > 0.
\]

**Proof.** By the previous Lemma there exists a sequence of real numbers \( t_n \to \infty \) satisfying \( \|u(t_n)\|_R^2 \frac{2Np}{N+n} \to 0 \). Indeed, otherwise, there exist \( a, b > 0 \) such that
\[
\|u(t)\|_R^2 \frac{2Np}{N+n} > a, \quad \forall t > b.
\]
This gives the contradiction for $T \gg 1$,

$$T^{\frac{1}{1+sp}} \gtrsim \int_0^T \|u(t)\|_{\frac{2Np}{N+sp}} \, dt \gtrsim T.$$ 

Thus, by Hölder estimate

$$\int_{\{|x|<R\}} |u(t_n, x)|^2 \, dx \leq R^{\frac{2p}{p+1}} \|u(t_n)\|_{\frac{2Np}{N+sp}}^2 \to 0. \quad \Box$$

5. Scattering criterion. In this section one proves the next result.

**Proposition 5.1.** Take $(N, \alpha, b)$ satisfying (2.4) and $\max\{p, 1 + x_{N,s,\alpha}\} < p < p^*$ such that $p \geq 2$. Let $u \in C(\mathbb{R}, H^s_{rd})$ be a global radial solution to (1.1). Assume that

$$0 < \sup_{t \geq 0} \|u(t)\|_{H^s} := E < \infty.$$ 

There exist $R, \epsilon > 0$ depending on $E, N, p, b, \alpha$ such that if

$$\lim_{t \to +\infty} \inf_{|x|<R} \int |u(t, x)|^2 \, dx < \epsilon,$$

then, $u$ scatters for positive time.

**Proof.** By Lemma 3.1, $u$ is bounded in $H^s$. Take $\epsilon > 0$ near to zero, $R(\epsilon) \gg 1$ to be fixed later and $I$ a time slab. Let us give a technical result.

**Lemma 5.2.** Let $(N, \alpha, s)$ satisfying (2.4) and $p_s < p < p^*$ satisfying $p \geq 2$. Then, there exist $(a, r) \in \Gamma_{s, c}$ and $(q, r) \in \Gamma$ such that the global solution to (1.1) satisfies

1. $\|u - e^{-it(-\Delta)^{r/2}} u_0\|_{S^c(I)} \lesssim \|u\|_{L^p(I, L^{2r})}^{2p-1};$
2. $\|(1 + |\nabla|^s)(u - e^{-it(-\Delta)^{r/2}} u_0)\|_{S^c(I)} \lesssim \|u\|_{L^p(I, L^{2r})}^{2p-1} \|\nabla^s u\|_{L^r(I, L^r)}.$

**Proof.** Take the real numbers

$$a := \frac{2sp}{s - s_c}, \quad d := \frac{2sp}{s + (2p - 1)s_c};$$

$$q := \frac{2sp}{s + s_c(p - 1)}; \quad r := \frac{2Np}{\alpha + N} = \frac{2Np}{(N - 2s_c)p - 2(s - s_c)}.$$

This gives

$$(q, r) \in \Gamma, \quad (a, r) \in \Gamma_{s, c}, \quad (d, r) \in \Gamma_{-s, c} \quad \text{and} \quad (2p - 1)d' = a.$$ 

Moreover, the equality $1 + \frac{a}{N} = \frac{2p}{r}$ gives using Hardy-Littlewood-Sobolev and Hölder estimates

$$\|u|^{p-2}(I_\alpha * |u|^p) u\|_{L^{2r'}(I, L^{2r'})} \lesssim \|u\|_{L^p(I, L^r)}^{2p-1} \|u\|_{L^p(I, L^r)}^{2p-1} \lesssim \|u\|_{L^p(I, L^r)}^{2p-1}.$$

Moreover, since $\frac{1}{q'} = \frac{2(p-1)}{a} + \frac{1}{q},$ one gets using Hardy-Littlewood-Sobolev and Hölder estimates

$$\|u|^{p-2}(I_\alpha * |u|^p) u\|_{L^{2r'}(I, L^{2r'})} \lesssim \|u\|_{L^p(I, L^r)}^{2p-1} \|u\|_{L^p(I, L^r)}^{2p-1} \|u\|_{L^p(I, L^r)}^{2p-1} \|u\|_{L^p(I, L^r)}^{2p-1}.$$

The proof is achieved via Strichartz estimates. \hfill \Box

The key of the proof of the scattering criterion is the next result.
Proposition 5.3. Take \((N, \alpha, b)\) satisfying \((2.4)\) and \(\max\{p_*, 1 + x_{N,s,\alpha}\} < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^s_{p,d}\) satisfying 
\[
\max \left\{ \mathcal{M} \mathcal{E}[u_0], \mathcal{M} \mathcal{G}[u_0] \right\} < 1.
\]
Then, for any \(\varepsilon > 0\), there exist \(T, \mu > 0\) satisfying
\[
\|e^{-i(t-T)(-\Delta)^s} u(T)\|_{L^\infty((T, \infty), L^\mu)} \lesssim \varepsilon^\mu.
\]
Proof. Let \(\beta > 0\) and \(T > \varepsilon^{-\beta} > 0\). By the integral formula
\[
e^{-i(t-T)(-\Delta)^s} u(T) = e^{-i(t-T)(-\Delta)^s} u_0 + i \int_0^T e^{-i(t-\tau)(-\Delta)^s} [|u|^{p-2} (I_\alpha \ast |u|^p) u] \, d\tau
\]
\[
e^{-i(t-T)(-\Delta)^s} u_0 + i \left( \int_0^{T-\varepsilon^{-\beta}} + \int_{T-\varepsilon^{-\beta}}^T \right) e^{-i(t-\tau)(-\Delta)^s} [|u|^{p-2} (I_\alpha \ast |u|^p) u] \, d\tau
\]
:= \(e^{-i(t-T)(-\Delta)^s} u_0 + F_1 + F_2\).

- The linear term. Take the real number \(\frac{1}{2} := \frac{1}{2} + \frac{s}{N}\). Since \((a, b) \in \Gamma\), by Strichartz estimate and Sobolev injections, one has
\[
\|e^{-i(t-T)(-\Delta)^s} u_0\|_{L^\infty((T, \infty), L^\mu)} \lesssim \|\nabla e^{-i(t-T)(-\Delta)^s} u_0\|_{L^\infty((T, \infty), L^\mu)} \lesssim \|u_0\|_{H^s}.
\]
- The term \(F_2\). By Strichartz estimate via Lemma 5.2, one has
\[
\|F_2\|_{L^\infty((T, \infty), L^\mu)} \lesssim \|u|^{p-1} (I_\alpha \ast |u|^p)\|_{L^\infty((T-\varepsilon^{-\beta},T), L^{2p-1})} \lesssim \|u\|_{L^\infty((T-\varepsilon^{-\beta},T), L^{2p-1})}^2
\]
\[
\lesssim \varepsilon^{-\frac{(2p-1)\beta}{a}} \|u\|_{L^\infty((T-\varepsilon^{-\beta},T), L^{2p-1})}^2.
\]
Now, by Lemma 4.2, one has for \(T > \varepsilon^{-1}\),
\[
\int_{\mathbb{R}^N} \psi_R(x)|u(T, x)|^2 \, dx < \varepsilon^2.
\]
Moreover, a computation with use of \((1.1)\) gives
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x)|u(t, x)|^2 \, dx = 2 \int_{\mathbb{R}^N} \psi_R(x) \Re[\bar{u}(t, x) \bar{\bar{u}}(t, x)] \, dx
\]
\[
= 2 \int_{\mathbb{R}^N} \psi_R(x) \Im[\bar{\Delta}^s u(t, x) \bar{\bar{u}}(t, x)] \, dx.
\]
Thus,
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R|u|^2 \, dx = 2 \Delta \int_{\mathbb{R}^N} (\bar{\Delta}^s \psi_R \bar{u})(\bar{\Delta}^s \bar{u}) \, dx
\]
\[
= 2 \Delta \int_{\mathbb{R}^N} (\bar{\Delta}^s \psi_R \bar{u})(\bar{\Delta}^s \bar{u}) \, dx
\]
\[
+ 2 \Delta \int_{\mathbb{R}^N} (\bar{\Delta}^s u)(\bar{\Delta}^s (\psi_R \bar{u}) - (\bar{\Delta}^s \psi_R \bar{u}) - (\bar{\Delta}^s \bar{u}) \psi_R \bar{u}) \, dx.
\]
So, by Lemma 2.10 with \(\psi_R\) rather than \(v\), \(p_1 > 1\) and \(s_1 > \frac{s}{2} + \frac{N}{p_1}\), one has via Sobolev injections
\[
\frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x)|u(t, x)|^2 \, dx \lesssim R^{-\frac{s}{p_1}}.
\]
Then, for any $T - \varepsilon^{-\beta} \leq t \leq T$ and $R > \varepsilon^{-\frac{2\alpha\beta}{p}}$, yields

$$\|\psi_R u(t)\| \leq \left( \int_{R^N} \psi_R(x)|u(T, x)|^2 \, dx + C \frac{T - t}{R^2} \right)^{\frac{1}{2}} \leq C \varepsilon.$$

This gives, for $R > \varepsilon^{-\frac{N(\alpha+1)}{p-1}}$,

$$\|u\|_{L^\infty((T, \infty), L^r)} \leq \|\psi_R u\|_{L^\infty((T, \infty), L^r)} + \|(1 - \psi_R)u\|_{L^\infty((T, \infty), L^r)} \lesssim \|\psi_R u\|_{L^\infty((T, \infty), L^r)}^{\frac{N-2\alpha}{2N-2\alpha}} \|\psi_R u\|_{L^\infty((T, \infty), L^\infty)}^{\frac{2\alpha}{2N-2\alpha}} + \|(1 - \psi_R)u\|_{L^\infty((T, \infty), L^\infty)}^{\frac{N-2\alpha}{2N-2\alpha}} \lesssim \varepsilon^{\frac{N-2\alpha}{2N-2\alpha}}.$$ 

So,

$$\|F_2\|_{L^\infty((T, \infty), L^r)} \lesssim \varepsilon^{-\frac{(2p-1)\beta}{2}} \|u\|_{L^\infty((T, \infty), L^r)}^{2-\frac{1}{p}} \lesssim \varepsilon^{-\frac{(2p-1)\beta}{2}} \varepsilon^{\frac{N-2\alpha}{2N-2\alpha}} \lesssim \varepsilon^{\frac{N-2\alpha}{2N-2\alpha}}.$$ 

The term $F_1$. Take $\frac{1}{\beta} = \frac{\lambda}{6}$. By interpolation

$$\|F_1\|_{L^\infty((T, \infty), L^r)} \lesssim \|F_1\|_{L^\infty((T, \infty), L^\infty)}^{1-\frac{\lambda}{r}} \|F_1\|_{L^\infty((T, \infty), L^\lambda)}^{\frac{\lambda}{r}} \lesssim \varepsilon^{-\frac{(2r-1)\beta}{2}} \varepsilon^{\frac{N-2\alpha}{2N-2\alpha}} \lesssim \varepsilon^{\frac{N-2\alpha}{2N-2\alpha}}.$$ 

With the free fractional Schrödinger operator dispersive estimate [8, 31],

$$\|e^{-it(-\Delta)^s} \|_{r} \lesssim \frac{C}{t^{N(\frac{1}{2} - \frac{s}{r})\frac{1-\beta}{2}}} \|(-\Delta)^{(1-s)(\frac{1}{2} - \frac{1}{r})} \|_{r'}, \quad \forall r \geq 2,$$

one has for $T - t$,

$$\|F_1\| \lesssim \int_0^{T-t-\beta} \frac{1}{(t-s)^\frac{1}{2}} \||-\Delta\|^2 \|u\|_{L^\infty} \|u\|_{L^\infty} \, ds \lesssim \int_0^{T-t-\beta} \frac{1}{(t-s)^\frac{1}{2}} \|u\|_{L^\infty} \|u\|_{L^\infty} \|u\|_{L^\infty} \, ds \lesssim \int_0^{T-t-\beta} \frac{1}{(t-s)^\frac{1}{2}} \|u\|_{L^\infty} \|u\|_{L^\infty} \, ds \lesssim \frac{1}{(t-s)^\frac{1}{2}} \|u\|_{L^\infty} \|u\|_{L^\infty} \, ds \lesssim \frac{1}{(t-s)^\frac{1}{2}} \|u\|_{L^\infty} \|u\|_{L^\infty} \, ds \lesssim \alpha \frac{1}{N} \frac{1}{t} + \frac{2(p-1)}{d} \frac{1}{\alpha},$$

One used the Hardy-Littlewood-Sobolev inequality via the fractional Chain rule and Sobolev injections in Lemmas 2.7-2.9 and assumes that

$$1 + \frac{\alpha}{N} = \frac{1}{e} + \frac{2(p-1)}{d}; \quad 2 \leq d \leq \frac{2N}{N - 2s}; \quad s \geq N(1 - s); \quad s - \frac{N}{2} \geq N(1-s) - \frac{N}{e}.$$
This is equivalent to

\[ 1 + \frac{\alpha}{N} = \frac{1}{e} + \frac{2(p-1)}{d}; \]

\[ 2 \leq d \leq \frac{2N}{N-2s}; \]

\[ s \geq \frac{N}{N+1}; \]

\[ \frac{1}{e} \geq \frac{3}{2} - s(1 + \frac{1}{N}). \]

Take the choice \( d := \frac{2N}{N-2s} \). Thus, one needs

\[ \frac{1}{e} = 1 + \frac{\alpha}{N} - \frac{2(p-1)}{d} = 1 + \frac{\alpha - (p-1)(N-2s)}{N} \geq \frac{3}{2} - s(1 + \frac{1}{N}). \]

This gives

\[ (p-1)(N-2s) - \alpha \leq s(1+N) - \frac{N}{2}. \]

So,

\[ p \leq 1 + \frac{1}{N-2s} \left( \alpha + s(1+N) - \frac{N}{2} \right). \]

This is satisfied since since \( p < p^* \) and \( s > \frac{N}{2(N-1)} \). It follows that if \( \frac{N}{2} - 1 - \frac{1}{\alpha} > 0 \),

\[ \| F_1 \|_{L^s((T,\infty),L^r)} \lesssim \| F_1 \|_{L^s((T,\infty),L^\infty)} \lesssim \left( \int_{T}^{\infty} (t-T+\varepsilon^{-\beta})^{s[1-N]} \, dt \right)^{\frac{1}{2}} \]

\[ \lesssim \varepsilon^{(1-\lambda)\beta[N^2-1-\frac{1}{2}].} \]

The above condition is equivalent to

\[ sp(N-2) > s - s_c = s - \frac{N}{2} + \frac{\alpha + 2s}{2(p-1)}. \]

Denoting \( x := p-1 \), one gets

\[ 2s(N-2)x(1+x) > x(2s-N) + \alpha + 2s. \]

This gives

\[ Q(x) := 2s(N-2)x^2 + (N(1+2s)-6s)x - \alpha - 2s > 0. \]

Take the real number

\[ Q(p^*-1) = 2s(N-2) \left( \frac{\alpha + 2s}{N-2s} \right)^2 + (N(1+2s)-6s) \left( \frac{\alpha + 2s}{N-2s} \right) - \alpha - 2s \]

\[ = \frac{\alpha + 2s}{(N-2s)^2} \left[ 2s(N-2)(\alpha+2s) + (N(1+2s)-6s)(N-2s) - (N-2s)^2 \right] \]

\[ = \frac{\alpha + 2s}{(N-2s)^2} \left[ 2s(N-2)(\alpha+2s) + 2s(N-2)(N-2s) \right] \]

\[ = \frac{2s(N-2)(\alpha + N)(\alpha + 2s)}{(N-2s)^2}. \]

Since \( N \geq 3 \), one concludes the proof by collecting the previous estimates. \( \square \)

Now, one proves the scattering criterion. Taking account of Duhamel formula via Proposition 5.3, there exists \( \mu > 0 \) such that

\[ \| e^{-it(-\Delta)^{s}} u(T) \|_{L^\infty(0,\infty),L^r} \equiv \| e^{-i(t-T)(-\Delta)^{s}} u(T) \|_{L^\infty((T,\infty),L^r)} \lesssim \varepsilon^\mu. \]

\[ d \leq \frac{2N}{N-2s}; \]

\[ s \geq \frac{N}{N+1}; \]

\[ \frac{1}{e} \geq \frac{3}{2} - s(1 + \frac{1}{N}). \]
So, with Lemma 5.2 via the absorption result Lemma 2.11, one gets
\[ \|u\|_{L^p((T,\infty),L^r)} \lesssim e^{t'}. \]
Take \( t, t' \gg 1 \) and write using Lemma 5.2 and Strichartz estimates
\[ \|u(t') - u(t)\|_{H^s} \lesssim \|(1 + |\nabla|^s)[|u|^{p-2}(I_\alpha * |u|^p)u]\|_{S(t',t)} \]
\[ \lesssim \|u\|_{L^p((t,t'),L^r)} \|u\|_{L^q((t',t),W^{s,r})} \to 0. \]
Indeed, thanks to Lemma 5.2 via the absorption result Lemma 2.11, one has
\[ u \in L^q(\mathbb{R},W^{s,r}). \]
The proof is achieved via the Cauchy criterion.

6. Proof of Theorem 2.5. The scattering of energy global solutions to the focusing problem (1.1) follows with Proposition 5.1 via Lemma 4.2.

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Received February 2021; revised June 2021; early access July 2021.

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