The excitation of a charged string passing through a shock wave in a charged Aichelburg-Sexl spacetime

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Abstract

We investigate how much a first-quantized charged bosonic test string gets excited after crossing a shock wave generated by a charged particle with mass $\tilde{M}$ and charge $\tilde{Q}$. On the basis of Kaluza-Klein theory, we pay attention to a closed string model where charge is given by a momentum along a compactified extra-dimension. The shock wave is given by a charged Aichelburg-Sexl (CAS) spacetime where $\tilde{Q} = 0$ corresponds to the ordinary Aichelburg-Sexl one. We first show that the CAS spacetime is a solution to the equations of motion for the metric, the gauge field, and the axion field in the low-energy limit. Secondly, we compute the mass expectation value of the charged test string after passing through the shock wave in the CAS spacetime. In the case of small $\tilde{Q}$, gravitational and Coulomb forces are canceled out each other and hence the excitation of the string remains very small. This is independent of the particle mass $\tilde{M}$ or the strength of the shock wave. In the case of large $\tilde{Q}$, however, every charged string gets highly excited by quantum fluctuation in the extra-dimension caused by both the gauge and the axion fields. This is quite different from classical “molecule”, which consists of two electrically charged particles connected by a classical spring.

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1 Introduction

One of the most significant progress in general relativity is the discovery of the singularity theorem [1]. The theorem states that spacetime has an incomplete causal geodesic generically, provided that some suitable conditions are satisfied. This means that a classical test particle along the geodesic cannot evolve for an infinite time. By analogy with the definition of classical singularities, it might be interesting to examine them by a first-quantized test string from a string theoretical point of view instead of a classical test particle because each excited mode of the fundamental string represents any kind of particles, including gravitons.

Orbifolds are simple examples which represent the difference between behaviors of a classical test particle and a first-quantized test string. They have conical singularities at the center of discrete symmetry. Although a trajectory of a classical particle ends for a finite time (i.e. geodesically incomplete), Dixon et. al. showed that a first-quantized string is well defined on orbifolds [2]. Like these examples, there are spacetimes where a first-quantized test string has well-behavior while the spacetimes are singular in a classical sense. That is, such spacetimes are singular in classical theory of relativity but they seem non-singular when we prove these spacetimes by the test string. Meanwhile there may be spacetimes with singularities where behaviors of both the test particle and the test string become irrelevant. Hence, it can be expected that we obtain some new informations about the “strength” of the classical singularities by considering the test string motion.

As examples of strong gravitational fields, it is interesting to investigate the plane-fronted waves (pp waves) because they are not only solutions to the vacuum Einstein equations but also solutions to the classical equations of motion for the metric in string theory [3, 4, 5], and also the geometries around the black hole event horizons of Gibbons-Maeda solutions can be approximated by pp waves in the near-extreme case [6]. Since the solutions contain arbitrary functions of time, we can construct singular spacetimes in pp waves in the sense of classical general relativity [3]. Therefore, several studies have been made on calculating the physical value, in particular, the mass expectation value of a test string in pp waves [3, 4, 5, 8, 10].

Amati and Klimcik investigated the first-quantized test string in spacetimes with impulsive gravitational shock wave [7]. The calculation was extended to more general time-dependent background fields by Horowitz and Steif and it was shown that the mass expectation value diverges infinitely in strong gravitational fields [8]. As pointed out, however, by de Vega and Sánchez [1, 4, 5, 12, 13], the mass expectation value is finite when the grav-
itational field consists of a shock wave generated by a localized gravitational source. The difference between these works originates in the transverse size of the gravitational wave front, i.e., infinite size of the wave front was adopted in the former, while finite size was used in the later [4]. Since the divergence of the mass expectation value is not caused by the singularities but by the infinite transverse size of the gravitational wave front, we cannot examine the singularities themselves if we consider the shock wave with infinite size. Hence, we shall pay attention to the shock wave generated by a localized gravitational source.

So, several efforts has been made on studying the gravitational interaction between a first-quantized test string and strong gravitational fields as mentioned above. What seems to be lacking, however, is the study of a charged test string interacting with both the gravitational and gauge fields such as electromagnetic field. Let us consider a classical “molecule” which consists of two electrically charged particles with mass $m$ and charge $e$, connected by a classical spring. In the Newtonian picture, gravitational and Coulomb forces from a charged particle source with mass $M$ and charge $Q$ works to the classical “molecule” like $mM/R^2$ and $eQ/R^2$, respectively, where $R$ represents a distance between the molecule and the source. Then, the molecule satisfying the condition $eQ = mM$ can pass through the strong gravitational field generated by the source without any additional excitation, due to the cancellation of both forces! This leads us to examine whether there is a charged string passing through strong gravitational and gauge fields with small excitation.

Motivated by this, as a first step, we generalize the work of Ref. [11] and calculate how much a charged bosonic test string gets excited after passing through a shock wave generated by a charged particle source with mass $\tilde{M}$ and $U(1)$ charge $\tilde{Q}$. In the neutral case ($\tilde{Q} = 0$), the shock wave is represented by Aichelburg-Sexl spacetime [14], which is one of vacuum solutions in $pp$ waves and the mass expectation value is proportional to the mass of the particle source [11, 12, 13]. The generalization of the Aichelburg-Sexl metric to include a charge has been performed in Ref. [15].

A charged bosonic string model is classified into the following two types: One is an open string model where an open string has a charge at each edge. Some properties of open bosonic strings in a background Abelian gauge field are investigated in [16]. The other is a closed string model based on Kaluza-Klein theory where momenta along compactified extra-dimensions correspond to charges. In this paper, we adopt the latter case and construct a charged Aichelburg-Sexl (CAS) spacetime where the shock wave is generated by the $U(1)$
charged particle source in the sense of Kaluza-Klein theory. The spacetime has a covariantly constant null vector field \( l^\mu \), i.e., \( l_\mu l^\mu = 0 \), \( \nabla_\mu l^\nu = 0 \), just like the \( pp \) waves. Thanks to this property, we can quantize a closed test string crossing the shock wave under the light cone gauge \( \Box \).

This paper is organized as follows. Firstly, we show that CAS spacetime is a solution to string theory in the low-energy limit in section II. Secondly, we derive the classical motion of a charged test string passing through the CAS spacetime in section III. Next, we first quantize the charged test string under the light cone gauge and calculate the expectation value of the mass of the string in section IV. Finally, discussions are devoted to section V.

2 A charged Aichelburg-Sexl spacetime

The coupling of a closed bosonic \( D = 26 \) string to a general metric \( G_{MN} \), axion \( B_{MN} \), and dilaton \( \Phi \) is given by

\[
S_p = -\frac{1}{8\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma \sqrt{-\det(h_{ab})} \left[ h^{ab}G_{MN}\partial_a X^M \partial_b X^N + \epsilon^{ab}B_{MN}\partial_a X^M \partial_b X^N + \alpha' R^{(2)} \Phi \right],
\]

(1)

where \( X^M(\tau, \sigma) \) is the embedding of the world sheet in \( D = 26 \) spacetime, \( h_{ab} \) is the two-dimensional world-sheet metric, \( R^{(2)} \) is the Ricci scalar of \( h_{ab} \), and \( \alpha' \) is the inverse string tension. We set the axion field strength \( H_{MNK} \) as \( H_{MNK} := 3\nabla_{[M}B_{NK]} \). The low-energy effective action is as follows,

\[
S = \int d^{26}X e^{-2\Phi} \sqrt{-\det(G_{MN})} \left[ R - 4\nabla_M \Phi \nabla^M \Phi + \frac{1}{12} H_{MNK} H^{MNK} \right],
\]

(2)

which yields the equations of motion,

\[
R_{MN} - 2\nabla_M \nabla_N \Phi + \frac{1}{4} H_{MKL} H^{KL} = 0,
\]

(3)

\[
\nabla^K H_{KMN} - 2 (\nabla^K \Phi) H_{KMN} = 0,
\]

(4)

\[
4 \nabla_K \nabla^K \Phi - 4(\nabla \Phi)^2 - R - \frac{1}{12} H_{MNK} H^{MNK} = 0.
\]

(5)

\(^1\)The coefficient in front of the integration should be replaced by \(-1/4\pi\alpha'\) if the period of \( \sigma \) is \( \pi \).
We will begin by considering the following \( pp \) waves metric

\[
d s_{26}^2 = -dudv + F(u, x, y) du^2 + dx^2 + dy^2 + \sum_{q=4}^{25} dx^q dx^q,
\]  

where \( u, v, x, y \) are our ordinary four-dimensional coordinates. The Riemann curvature \[5\] is

\[
R_{KLMN} = -2l_K(\partial_L l_M F) l_N, \tag{7}
\]

where \( l_M \) is a covariantly constant vector such as

\[
\nabla_M l_N = 0, \quad l^M l_M = 0. \tag{8}
\]

It is worthy to note that the curvature in Eq. (7) is orthogonal to \( l_M \) in all indices. For this property, Einstein vacuum solutions under the metric (8) are also solutions in all higher order in \( \alpha' \) terms in the equations of motion in string theory \[3, 4, 5\]. When \( \Phi = H_{MKN} = 0 \), it is easily verified that Eqs. (3) and (5) are automatically satisfied. The only non-vanishing equation in Eqs. (3) is

\[
(\partial^2_x + \partial^2_y) F(u, x, y) = 0. \tag{9}
\]

The solution of which source is a point particle is simply given by

\[
F(u, x, y) = \tilde{M} D(x, y) W(u), \quad D(x, y) = \ln (x^2 + y^2), \tag{10}
\]

where \( W \) is an arbitrary function of \( u \) and \( \tilde{M} \) is mass of the particle. In the case of the shock wave \( W(u) = \delta(u) \), the spacetime is called Aichelburg-Sexl geometry in Ref. \[14\]. The mass expectation value of a test string passing through the shock wave is explicitly calculated and it is proportional to \( \tilde{M} \) \[11\].

Now, we shall consider a closed charged test string propagating a shock wave generated by a charged particle. A closed bosonic string model with charge can be constructed on the basis of Kaluza-Klein theory. First, we shall briefly review Kaluza-Klein theory in five-dimensional spacetime. Next we explore a generalized solution of which the shock wave consists of the gravitational field \( F(u, x, y) \) and \( U(1) \) gauge field \( A_u(u, x, y) \) created by a charged particle.

The field equations of Kaluza-Klein theory are derived from the five-dimensional Einstein-Hilbert action

\[
S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g_5} R^{(5)}, \tag{11}
\]
where $G_5$ is the five-dimensional gravitational constant. We assume that the five-dimensional metric

$$
 ds_5^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu + (dx^4 + A_\mu dx^\mu)^2 \quad \mu, \nu = 0, 1, 2, 3
$$

has a closed killing orbit $\partial/\partial_4$, which means that all functions are independent of the coordinate $x^4$. The five-dimensional scalar curvature $R^{(5)}$ is decomposed as

$$
 R^{(5)} = R^{(4)} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
$$

where $F_{\mu\nu} = 2 \partial[\mu A_\nu]$. This means that the action (11) is reduced to the ordinary action for the four-dimensional Einstein-Maxwell system. The motion of a particle with mass $m$ is given by varying the action

$$
 I = -m \int \sqrt{|g^{(4)}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + (\dot{x}^4 + A_\mu \dot{x}^\mu)^2|} \, d\tau,
$$

where $\tau$ is the peculiar time and a dot denotes the derivative with respect to $\tau$. For the existence of the killing field $\partial_4$, the momentum $p_4$ conjugate to $x^4$ is constant;

$$
 p_4 = m(\dot{x}^4 + A_\mu \dot{x}^\mu) = \text{const.}
$$

Applying the action principle to $x^\mu$ again, we obtain the following four-dimensional equations

$$
 \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \frac{p_4}{m} F^\mu_{\nu} \dot{x}^\nu,
$$

where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol of the four-dimensional metric $g^{(4)}_{\mu\nu}$. If we set $p_4 = q$, Eq. (16) represents the usual four-dimensional motion of a particle with mass $m$ and charge $q$ accelerated by the electromagnetic field $F_{\mu\nu}$.

Next, in the spirit of Kaluza-Klein theory, we shall consider the following generalized metric of Eq. (6),

$$
 ds_{26}^2 = - du dv + [F(u, x, y) + A_u^2(u, x, y)] du^2 + 2 A_u(u, x, y) du dx^4 + dx^2 + dy^2 + (dx^4)^2 + \sum_{p=5}^{25} dx^p dx^p,
$$

where $A_u$ is interpreted as $U(1)$ gauge field. We can easily check that the metric also satisfies the property (8). It is worth noting that in supergravity, this type of metric admits solutions which have unbroken spacetime supersymmetries [18]. By defining the following quantities

$$
 2 \hat{A}_u = F + A_u^2, \quad \hat{A}_4 = A_u, \quad \hat{F}_{MN} = 2 \partial_{[M} \hat{A}_{N]},
$$

(18)
the only non-vanishing components of Ricci tensor \( R_{u4} \) are
\[
R_{u4} = \frac{1}{2} \partial^i \hat{F}_{4i},
\]
\[
R_{uu} = \partial^i \hat{F}_{ui} + \frac{1}{4} \hat{F}_{ij} \hat{F}^{ij}, \quad x^i = x, y, x^4, x^p,
\] (19)
which means that \( R = 0 \). Let us seek a solution of \( \Phi = 0 \) for simplicity. Setting the axion field like
\[
B_{4u} = A_u(u, x, y), \quad \text{the others} = 0,
\] (20)
we have \( H_{MNK} H^{MNK} = 0 \) and hence Eq. (19) is automatically satisfied. By solving Eq. (4) and substituting \( H_{MNK} \) into Eq. (3), we can obtain the following time-dependent solutions,
\[
A_u(u, x, y) = \tilde{Q} D(x, y) W(u)
\] (21)
together with \( F(u, x, y) \) in Eq. (10). When \( W(u) = \delta(u) \), we shall call the metric (17) charged Aichelburg-Sexl (CAS) spacetime since a charged particle with mass \( \tilde{M} \) and charge \( \tilde{Q} \) creates a shock wave in the sense of Kaluza-Klein theory. Hereafter, we shall simply call \( A_u \) the gauge field.

3 The classical motion of a charged test string under the light cone gauge

In this section, we will construct the light cone gauge Hamiltonian and derive the equations for the classical motion of a charged test string passing through the shock wave in CAS spacetime.

Under the conformal gauge \( h_{ab} = e^{\phi} \eta_{ab} \) the equations of the motion of a string are given by
\[
\partial_a \partial^a X^M + \Gamma^M_{NK} \partial_a X^N \partial^a X^K - \frac{1}{2} H^M_{NK} \partial_a X^N \partial_b X^K \epsilon_{ab} = 0
\] (22)
and the constraint equations are
\[
T_{ab} := \partial_a X^M \partial_b X^N G_{MN} - \frac{1}{2} \eta_{ab} \partial_c X^M \partial_c X^N G_{MN} = 0.
\] (23)
Hereafter, we will use \( X^M = (u, v, x^i) = (u, v, x, y, x^4, x^p) \) for simplicity. Thanks to the existence of the covariantly constant vector \( l^M \), the equation for \( u \) is simply
\[
\partial_a \partial^a u = 0. \tag{24}
\]
This implies that we can take the light-cone gauge \( u = p \tau \) (See, Ref. [5]). By substituting \( u = p \tau \) into Eqs. (22), the constraint equations (23) are reduced to the following two equations,
\[
p \dot{v} = p^2 (F + A_u^2) + 2p A_u \dot{x}^4 + \sum_{i=2}^{25} \left[ (\dot{x}^i)^2 + (x''^i)^2 \right], \tag{25}
\]
\[
p v' = 2p A_u x'^4 + 2 \sum_{i=2}^{25} \dot{x}^i x'^i, \tag{26}
\]
where a dot and a dash mean the derivatives with respect to \( \tau \) and \( \sigma \), respectively. We can easily check that the equation (22) for \( v \) is automatically satisfied, provided that the equations for \( x^i \) and the above constraint equations (25) and (26) are satisfied. This indicates that the only independent variables are \( x^i \).

Now, let us start with the construction of the light-cone gauge Hamiltonian. If we set \( \alpha' = 1/2 \) and take the light cone gauge \( u = p \tau \), the action (11) is reduced to
\[
S_p = \int d\tau \int_0^{2\pi} d\sigma L
= \frac{1}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \left\{ -p \dot{v} + p^2 (F + A_u^2) + 2p A_u (\dot{x}^4 + x'^4) + \sum_{i=2}^{25} \left[ (\dot{x}^i)^2 + (x''^i)^2 \right] \right\}. \tag{27}
\]
By introducing the momentum conjugate to \( x^i \),
\[
P_i (i \neq 4) = \frac{\dot{x}^i}{2\pi}, \quad P_4 = \frac{p A_u + \dot{x}^4}{2\pi}, \tag{28}
\]
we obtain the following canonical Hamiltonian
\[
H = H_0 + H_{\text{int}} = \int_0^{2\pi} [H_0 + H_{\text{int}}] d\sigma, \tag{29}
\]
where \( H_0 \) and \( H_{\text{int}} \) are defined by
\[
H_0 = \pi \sum_{i=2}^{25} P_i P_i + \frac{1}{4\pi} \sum_{i=2}^{25} x'^i x''^i, \tag{30}
\]
\[
H_{\text{int}} = -p A_u \left( P_4 + \frac{x'^4}{2\pi} \right) - \frac{p^2}{4\pi} F, \tag{31}
\]
7
respectively. It is worth noting that $H_0$ and $H_{\text{int}}$ represent the free part and the interaction part of the Hamiltonian $H$, respectively.

Thus, obeying the Hamilton’s principle, the classical motions of a test string for $x^i$ are derived as

\[ \dot{P}_4 = -\frac{p}{2\pi} (x' A_{u, x} + y' A_{u, y}) + \frac{x'^{\prime\prime}}{2\pi}, \]

\[ \dot{P}_x = \frac{p}{2\pi} (2\pi P_4 + x^i) A_{u, x} + \frac{x''}{2\pi} + \frac{p^2}{4\pi} F_x, \]

\[ \dot{P}_y = \frac{p}{2\pi} (2\pi P_4 + x^i) A_{u, y} + \frac{y''}{2\pi} + \frac{p^2}{4\pi} F_y, \]

\[ \dot{P}_p = \frac{x^{\prime\prime}}{2\pi}. \]

For CAS spacetime, $W(u) = \delta(u)$. However, as seen from the above equations, the integration of Eqs. (32)-(34) by $\tau$ is ill-defined. To avoid this difficulty, we consider the sequences of regular functions (see, Ref. [12])

\[ \delta_\epsilon(u) = \begin{cases} 
1 & (-\epsilon < u < \epsilon) \\
0 & (|u| \geq \epsilon)
\end{cases} \]  

and finally take a limit $\epsilon \to 0$. By integrating Eq. (32), we obtain the following equation for $|\tau| < \epsilon/p$,

\[ P_4(\tau, \sigma) = \frac{p\tilde{Q}}{4\pi\epsilon} D(x, y) + \frac{\dot{x}^4(\tau, \sigma)}{2\pi} \]

\[ = \frac{\dot{x}^4}{2\pi} - \frac{p\tilde{Q}}{4\pi\epsilon} \left( \tau + \frac{\epsilon}{p} \right) D_{\sigma}(x_0) + O(\epsilon), \]

where $x_0 = (x_0, y_0) = (x(0, \sigma), y(0, \sigma))$. Hereafter, for an arbitrary function $f(\tau, \sigma)$, we simply denote $f(\tau = +\epsilon/p, \sigma)$ and $f(\tau = -\epsilon/p, \sigma)$ by $f_>$ and $f_<$, respectively.

Integrating again the above equation, we find that

\[ x^4(\tau, \sigma) = x^4_< + \frac{p\tilde{Q}}{2\epsilon} \left( \tau + \frac{\epsilon}{p} \right) D(x_0) + O(\epsilon). \]

Note that $P_{4<} = \dot{x}_{4<}/2\pi$ and $P_{4>} = \dot{x}_{4>}/2\pi$ by Eqs. (28) and (34). Hence, by taking a limit $\epsilon \to 0$, $\dot{x}_{4<}$ and $\dot{x}_{4>}$ satisfy the following equation,

\[ \dot{x}_{4>} = \dot{x}_{4<} - \tilde{Q} D_{\sigma}(x_0), \]
which means that
\[ \int_0^{2\pi} \dot{x}^4 d\sigma = \int_0^{2\pi} \dot{x}^4 d\sigma. \] (40)

Eqs. (37) and (40) show that charge of a classical closed string is conserved while the string passes the shock wave. By substituting Eqs. (37) and (38) into Eq. (33) and (34) and taking a limit, we find the following equations,

\[ \dot{x}_> - \dot{x}_< = \left[ \tilde{Q}(\dot{x}_4 + x^{4'}) + \frac{pM}{2} - \tilde{Q}^2 D_\sigma(x_0) \right] D_x(x_0), \] (41)

\[ \dot{y}_> - \dot{y}_< = \left[ \tilde{Q}(\dot{x}_4 + x^{4'}) + \frac{pM}{2} - \tilde{Q}^2 D_\sigma(x_0) \right] D_y(x_0). \] (42)

The \( x^{4'} \) terms reflect the antisymmetric tensor \( B_{MN} \). The third terms in the square bracket come from the mixing of the excitation in the compactified extra-dimension \( x^4 \) and the excitation in the \( x, y \) dimensions. As easily checked, the above equations are reduced to the motion of a classical charged test particle when we neglect the third terms and \( \sigma \) dependence.

Thus, the particle with a momentum \( p_4 = -\frac{pM}{2} \tilde{Q} \) feels no shock in the \( x-y \) plane because \( \dot{x}_> = \dot{x}_< \) and \( \dot{y}_> = \dot{y}_< \). Furthermore, by neglecting \( \sigma \) dependence in Eqs. (25) and (39) again, we can see that the classical molecule constructed from two particles with \( p_4 = -\frac{pM}{2} \tilde{Q} \) feels no shock in any dimensional direction in CAS spacetime. In the next section, we first-quantize a test string passing through CAS spacetime and calculate its mass expectation value.

4 Excitation of a first-quantized test string in CAS spacetime

Let us start with the following Schrödinger equation
\[ i \frac{\partial}{\partial \tau} |\psi_S\rangle = H(\tau) |\psi_S\rangle. \] (43)

To change from the Schrödinger picture to the interaction one, we define a state \( |\psi_I\rangle \) by
\[ |\psi_I\rangle = e^{iH_0 \tau} |\psi_S\rangle. \] (44)
By substituting Eq. (44) into Eq. (43), we obtain the following equation

\[ i \frac{\partial}{\partial \tau} |\psi_I\rangle = H_I(\tau) |\psi_I\rangle, \]  

(45)

where

\[ H_I(\tau) := e^{iH_0\tau} H_{\text{int}} e^{-iH_0\tau}. \]  

(46)

It is noteworthy that $H_{\text{int}}$ is the interaction part of the Hamiltonian in Schrödinger picture. Under the interaction picture, we can expand $x^i$ coordinates such as

\[ x^i = q_i + p_i \tau + \frac{i}{\sqrt{2}} \sum_{n \neq 0} e^{-in\sigma} \left[ \alpha_n^i e^{in\sigma} + \tilde{\alpha}_n^i e^{-in\sigma} \right], \]

(47)

where the winding number with respect to the compactified extra-dimension is taken to be zero for simplicity. By setting canonical commutation relations

\[ [P_i(\tau, \sigma), x^j(\tau, \sigma')] = -i\delta(\sigma - \sigma')\delta^i_j, \]  

(48)

\[ [x^i(\tau, \sigma), x^j(\tau, \sigma')] = [P_i(\tau, \sigma), P_j(\tau, \sigma')] = 0, \]  

(49)

at equal $\tau$, we obtain the usual commutation relations

\[ [q_i, p_j] = i\delta_{ij}, \]  

(50)

\[ [\alpha_m^i, \alpha_n^j] = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m \delta_{m+n,0} \delta^i_j, \quad [\alpha_m^i, \tilde{\alpha}_n^j] = 0. \]  

(51)

Let us solve Eq. (45) iteratively. Then, $|\psi_I\rangle$ is expanded as

\[ |\psi_I\rangle = \left[ 1 + (-i) \int_{\tau_0}^\tau d\tau' H_I(\tau') + (-i)^2 \int_{\tau_0}^\tau d\tau' \int_{\tau_0}^{\tau'} d\tau'' H_I(\tau') H(\tau'') + \cdots 
+ (-i)^n \int_{\tau_0}^\tau d\tau' \int_{\tau_0}^{\tau'} \cdots \int_{\tau_0}^{\tau(n-1)} d\tau^{(n)} H_I(\tau') H_I(\tau'') \cdots H_I(\tau^{(n)}) + \cdots \right] |\psi_0\rangle, \]  

(52)

where $|\psi_0\rangle$ is defined by the initial state of $|\psi_I\rangle$ at $\tau = \tau_0$. By taking time ordering

\[ T[H_I(\tau') H_I(\tau'') \cdots H_I(\tau^{(n)})] = H_I(\tau') H_I(\tau'') \cdots H_I(\tau^{(n)}) \]  

\[ \tau' \geq \tau'' \geq \cdots \geq \tau^{(n)}, \]  

(53)

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\[ |\psi_I\rangle \text{ can be reduced to the following equation,} \]
\[
|\psi_I\rangle = \left[ 1 - i \int_{\tau_0}^{\tau} d\tau' H_I(\tau') + \frac{1}{2!} (-i)^2 \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau} d\tau'' T[H_I(\tau')H_I(\tau'')] + \cdots 
+ \frac{1}{n!} (-i)^n \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau} d\tau'' \cdots \int_{\tau_0}^{\tau} d\tau^{(n)} T[H_I(\tau')H_I(\tau') \cdots H_I(\tau^{(n)})] + \cdots \right] |\psi_0\rangle
\]
\[
= T \exp \left[ -i \int_{\tau_0}^{\tau} d\tau' H_I(\tau') \right] |\psi_0\rangle
\]
\[
=: U(\tau, \tau_0) |\psi_0\rangle. \tag{54}
\]

Therefore, the expectation value of a physical operator \( O_I \) with respect to the state \( |\psi_I\rangle \) is represented by
\[
\langle \psi_I | O_I | \psi_I \rangle = \langle \psi_0 | U^\dagger O_I U | \psi_0 \rangle = \langle \psi_0 | O_H | \psi_0 \rangle, \tag{55}
\]
where \( O_H := U^\dagger O_I U \) is the Heisenberg operator. Now, let us define in-operator \( O_{H<} \) and out-operator \( O_{H>} \) as \( \lim_{\epsilon \to 0} O_H(\tau = -\epsilon) \) and \( \lim_{\epsilon \to 0} O_H(\tau = \epsilon) \), respectively. Then, by setting \( \tau = \epsilon \) and \( \tau_0 = -\epsilon \), we obtain the following relation
\[
O_{H>} = \exp [iG] O_{H<} \exp [-iG], \tag{56}
\]
where \( G \) is defined by
\[
G := \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} H_I(\tau) \, d\tau. \tag{57}
\]

Under the interaction picture, the difficulty in the integration of Eqs. (32)-(34) we ran up against in Sec. III does not arise because all operators evolve as Eq. (47) in any time and the interaction operator \( P_{4I} \) corresponds to \( \dot{x}^4/2\pi \) in Eq. (47). Hence substituting \( P_{4I} \) and Eq. (47) for \( i = 4 \) into Eq. (31), we get \( G \) as
\[
G = -\frac{1}{4\pi} \int_0^{2\pi} d\sigma \left[ p\tilde{M} + 2\tilde{Q}(p^4 + \sqrt{2} \sum_{n \neq 0} \tilde{\alpha}^4_{n<} e^{-i\sigma}) \right] \int d^2k \, \phi(k) : \exp [i k \cdot x_0] :, \tag{58}
\]
where :: stands for normal ordering with respect to in-vacuum \( |0<\rangle \) and each \( \phi(k) \) is defined by the Fourier transformation of the function \( D(x, y) \) in Eq. (10):
\[
\phi(k) = \frac{1}{(2\pi)^3} \int d^2x \exp [-i k \cdot x] D(x). \tag{59}
\]
Hereafter, we simply denote the Heisenberg operator \( O_H \) at \( \tau = 0+ (\tau = 0-) \) by \( O_> (O_<). \)
As shown in the Appendix A, we obtain the following commutation relations for \( \alpha^l_m \) \((l = x, y)\),

\[
[G, \alpha^l_m] = \frac{1}{4\sqrt{2\pi}} \int_0^{2\pi} d\sigma e^{-im\sigma} \left[ p\bar{M} + 2\tilde{Q}\left(p_{4<} + \sqrt{2} \sum_{n \neq 0} \tilde{\alpha}^4_{n<} e^{-in\sigma}\right) \right]
\times \int d^2k k^l \phi(k) : \exp[i k \cdot x_0] : ,
\]

\( (60) \)

\[
[G, [G, \alpha^l_m]] = -\frac{i \tilde{Q}^2}{\sqrt{2\pi}^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\xi \sum_{n=1}^{\infty} n \sin n(\xi - \sigma) e^{-im\xi}
\times \int d^2k \phi(k) : \exp[i k \cdot x_0(\sigma)] : \int d^2\eta \eta^l \phi(\eta) : \exp[in \cdot x_0(\xi)] : .
\]

\( (61) \)

We should remind that Eq. \((60)\) does not include the right oscillators \(\tilde{\alpha}^4_{n<}\), as suggested in the first term in the interaction part of the Hamiltonian density \((31)\). This fact raises the next commutation relation \((61)\). By using the commutation relation in Eq. \((60)\), we can easily see that

\[
[G, [G, [G, \alpha^l_m]]] = \cdots = [G, [G, \cdots, [G, \alpha^l_m] \cdots]] = 0.
\]

\( (62) \)

The commutation relations for \(\tilde{\alpha}^l_m\) is also simply given by replacing \(m \rightarrow -m\) in Eq. \((60)\) and Eq. \((61)\).

For \(\alpha^4_m\) and \(\tilde{\alpha}^4_m\), the commutations with \(G\) are given by

\[
[G, \alpha^4_m] = 0, \quad [G, \tilde{\alpha}^4_m] = \frac{m\tilde{Q}}{\sqrt{2\pi}} \int_0^{2\pi} d\sigma \int d^2k \phi(k) : \exp[i k \cdot x_0] : e^{im\sigma}.
\]

\( (63) \)

These indicate that all higher commutations with \(G\) exactly vanish. The antisymmetry between \(\alpha^4_m\) and \(\tilde{\alpha}^4_m\) reflects the effect of the antisymmetric second rank tensor \(B_{MN}\). By using the formula

\[
\exp[iG] \mathcal{O} \exp[-iG] = \mathcal{O} + i[G, \mathcal{O}] + \frac{i^2}{2} [G, [G, \mathcal{O}]] + \cdots,
\]

out-operators, \(p_{4>}, \alpha^l_{m>},\) and \(\alpha^4_{m>}\) are described by in-operators \(p_{4<}, \alpha^l_{m<},\) and \(\alpha^4_{m<}\) as follows:

\[
p_{4>} = p_{4<},
\]

\( (65) \)

\[
\alpha^l_{m>} - \alpha^l_{m<} = \frac{i}{4\sqrt{2\pi}} \int_0^{2\pi} d\sigma e^{-im\sigma} \left[ p\bar{M} + 2\tilde{Q}\left(p_{4<} + \sqrt{2} \sum_{n \neq 0} \tilde{\alpha}^4_{n<} e^{-in\sigma}\right) \right]
\]

\( (66) \)
\[
\times \int d^2k \phi(k) : \exp [ik \cdot x_0] : \\
+ \frac{i \tilde{Q}^2}{2 \sqrt{2 \pi}^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\xi \sum_{n=1}^{\infty} n \sin n(\xi - \sigma) e^{-im\xi} \\
\times \int d^2k \phi(k) : \exp [ik \cdot x_0(\sigma)] : \int d^2\eta \eta^l \phi(\eta) : \exp [i\eta \cdot x_0(\xi)] : ,
\]
\[\tag{66}\]
\[
\tilde{\alpha}_{m>}^l - \tilde{\alpha}_{m<}^l = \alpha_{m>}^l - \alpha_{m<}^l ,
\]
\[\tag{67}\]
\[
\alpha_{m>}^4 - \alpha_{m<}^4 = 0 ,
\]
\[\tag{68}\]
\[
\tilde{\alpha}_{m>}^4 - \tilde{\alpha}_{m<}^4 = \frac{i \tilde{m} \tilde{Q}}{\sqrt{2 \pi}} \int_0^{2\pi} d\sigma \int d^2k \phi(k) : \exp [ik \cdot x_0] : e^{im\sigma} ,
\]
\[\tag{69}\]
Eq. (65) indicates that charge of a first-quantized test string is conserved before and after the shock wave in CAS spacetime. If we drop the normal orderings in Eqs. (66)-(69), the operators should be reduced to the classical coefficients \(\alpha_{n}^i\) at \(\tau = \pm \epsilon/p\) (\(\tilde{\alpha}_{n}^i\) at \(\tau = \pm \epsilon/p\)) in Eq. (67). We can easily check that if we integrate Eqs. (66)-(69) with respect to \(k\) together with Eq. (69), the relations between the coefficients in Eqs. (66)-(69) coincides with those obtained from the classical Eqs. (41) and (42).

Now, we can explicitly calculate the mass expectation value of a first-quantized test string by using the formula (69). We define the in-vacuum \(|0<\rangle\) as
\[
\alpha_{m<}^i |0<; p_4\rangle = \tilde{\alpha}_{m<}^i |0<; p_4\rangle = 0
\]
for \(m > 0\) and
\[
p_{4<} |0<; p_4\rangle = p_4 |0<; p_4\rangle .
\]
Eq. (71) means that the closed test string has a momentum \(p_4\) as an initial state. Then, by Eqs. (66) and (67), we obtain
\[
\sum_{l=x,y} \langle 0<; p_4 | \alpha_{m>}^{l\dagger} \alpha_{m>}^l + \tilde{\alpha}_{m>}^{l\dagger} \tilde{\alpha}_{m>}^l | 0<; p_4 \rangle \\
= \left( \frac{pM + 2p_4 \tilde{Q}}{4\pi} \right)^2 \int_0^{2\pi} d\sigma \int_0^{2\pi} d\xi \int d^2k \phi^2(k) k^2 e^{im(\sigma - \xi)} |2 \sin \left( \frac{\sigma - \xi}{2} \right)|^{-k^2} \\
+ \frac{\tilde{Q}^2}{4\pi^2} \sum_{n=1}^{\infty} n \int_0^{2\pi} d\sigma \int_0^{2\pi} d\xi \int d^2k \phi^2(k) k^2 \cos m(\sigma - \xi) e^{-im(\sigma - \xi)} |2 \sin \left( \frac{\sigma - \xi}{2} \right)|^{-k^2} \\
- \frac{\tilde{Q}^4}{4\pi^2} \prod_{i=1}^{4} \left[ \int_0^{2\pi} d\sigma_i \right] \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} ns \sin (\sigma_2 - \sigma_1) \sin (\sigma_4 - \sigma_3) \cos m(\sigma_2 - \sigma_4) \\
\times \prod_{i=1}^{4} \int d^2k_i \phi(k_i) \right] (k_2 \cdot k_4) \prod_{1 \leq i < j \leq 4} |2 \sin \left( \frac{\sigma_i - \sigma_j}{2} \right) k_i k_j \delta^2(k_1 + \cdots + k_4) ,
\]
\[\tag{72}\]
where we have used the symmetry
\[ \phi(-k) = \phi(k) \] (73)
and the fact
\[ \left[ \prod_{i=1}^{3} \int_{0}^{2\pi} d\sigma_{i} \right] \cos m(\sigma_{1} - \sigma_{2}) \sin n(\sigma_{3} - \sigma_{2}) \prod_{1 \leq i < j \leq 3} \left| 2 \sin \left( \frac{\sigma_{i} - \sigma_{j}}{2} \right) \right|^k_{i,j} = 0. \] (74)

The second and third terms in r.h.s. of Eq. (72) stem from quantum fluctuations by extra-dimension \( x^4 \). Similarly, by Eqs. (68) and (69) we also obtain
\[ \langle 0_1; p_4 | (\tilde{\alpha}_{m}^4 \tilde{\alpha}_{m}^4 + \tilde{\alpha}_{m}^4 \tilde{\alpha}_{m}^4) | 0_1; p_4 \rangle = -\frac{m^2 \tilde{Q}^2}{2\pi^2} \int_{0}^{2\pi} d\sigma \int_{0}^{2\pi} d\xi \int d^2 k \phi^2(k) e^{-i m(\sigma - \xi)} |2 \sin \left( \frac{\sigma - \xi}{2} \right)|^{-k^2}. \] (75)

As shown in Appendix B, Eq. (72) is rewritten as
\[
\sum_{l=x,y} \langle 0_1; p_4 | (\tilde{\alpha}_{m}^l \tilde{\alpha}_{m}^l + \tilde{\alpha}_{m}^l \tilde{\alpha}_{m}^l) | 0_1; p_4 \rangle = \frac{(p \tilde{M} + 2p_4 \tilde{Q})^2}{4\pi} \int d^2 k \phi^2(k) k^2 \sin \left( \frac{1}{2} \pi k^2 \right) \frac{\Gamma(1 - k^2) \Gamma(m + \frac{1}{2} k^2)}{\Gamma(1 + m - \frac{1}{2} k^2)}
+ \frac{\tilde{Q}^2}{\pi} \sum_{n=1}^{\infty} n \int d^2 k \phi^2(k) k^2 \sin \left( \frac{1}{2} \pi k^2 \right) \Gamma(1 - k^2) \frac{\Gamma(m + n + \frac{1}{2} k^2)}{\Gamma(1 + m + n - \frac{1}{2} k^2)} + \frac{\Gamma(m - n + \frac{1}{2} k^2)}{\Gamma(1 + m - n - \frac{1}{2} k^2)}
- \frac{\tilde{Q}^4}{4\pi^2} \left[ \prod_{i=1}^{4} \int_{0}^{2\pi} d\sigma_{i} \right] \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} n s \sin n(\sigma_2 - \sigma_1) \sin s(\sigma_4 - \sigma_3) \cos m(\sigma_2 - \sigma_4)
\times \left[ \prod_{i=1}^{4} \int d^2 k_{i} \phi(k_{i}) \right] (k_{2} \cdot k_{4}) \prod_{1 \leq i < j \leq 4} \left| 2 \sin \left( \frac{\sigma_{i} - \sigma_{j}}{2} \right) \right|^k_{i,j} \delta^2(k_{i} + \cdots + k_{j}), \] (76)

where \( k^2 = k \cdot k \). Similarly, Eq. (73) is rewritten as
\[
\langle 0_1; p_4 | (\tilde{\alpha}_{m}^4 \tilde{\alpha}_{m}^4 + \tilde{\alpha}_{m}^4 \tilde{\alpha}_{m}^4) | 0_1; p_4 \rangle = -\frac{2m^2 \tilde{Q}^2}{\pi} \int d^2 k \phi^2(k) \sin \left( \frac{1}{2} \pi k^2 \right) \frac{\Gamma(1 - k^2) \Gamma(m + \frac{1}{2} k^2)}{\Gamma(1 + m - \frac{1}{2} k^2)}. \] (77)

For the second line in Eq. (76), we use the following formula [14]
\[
\sum_{n=1}^{\infty} \frac{\Gamma(n + b)}{\Gamma(n + a)} = -\frac{b}{b - a + 1} \frac{\Gamma(b)}{\Gamma(a)}, \quad a > b + 1. \] (78)

The summation of the l.h.s. of Eq. (78) does not converge in the region \( a \leq b + 1 \). Hereafter, we use the r.h.s of Eq. (78) in all regions \( (a, b) \) in the sense of analytic continuation. Thus,
we can sum up the first term of the second line in Eq. (76) by $n$ as follows,

\[
\sum_{n=1}^{\infty} \frac{n\Gamma(n + m + \frac{1}{2}k^2)}{\Gamma(1 + n + m - \frac{1}{2}k^2)} = \sum_{n=1}^{\infty} \frac{\Gamma(1 + n + m + \frac{1}{2}k^2)}{\Gamma(1 + n + m - \frac{1}{2}k^2)} \left( m + \frac{1}{2}k^2 \right) \frac{\Gamma(n + m + \frac{1}{2}k^2)}{\Gamma(1 + n + m - \frac{1}{2}k^2)} - \left( m + \frac{1}{2}k^2 \right) \frac{\Gamma(2 + m + \frac{1}{2}k^2)}{\Gamma(1 + m - \frac{1}{2}k^2)} + \frac{m + k^2/2}{k^2} \frac{\Gamma(1 + m + \frac{1}{2}k^2)}{\Gamma(1 + m - \frac{1}{2}k^2)} - \frac{1}{k^2(1 + k^2)} \frac{\Gamma(2 + m + \frac{1}{2}k^2)}{\Gamma(1 + m - \frac{1}{2}k^2)} - \frac{1}{k^2} \frac{\Gamma(1 + m + \frac{1}{2}k^2)}{\Gamma(1 + m - \frac{1}{2}k^2)} \tag{79}
\]

Similarly, we can easily confirm that

\[
\sum_{n=1}^{\infty} \frac{n\Gamma(n - m + \frac{1}{2}k^2)}{\Gamma(1 + n - m - \frac{1}{2}k^2)} = \sum_{n=1}^{\infty} \frac{n\Gamma(n + m + \frac{1}{2}k^2)}{\Gamma(1 + n + m - \frac{1}{2}k^2)} \tag{80}
\]

Now, we can get the vacuum expectation value of the mass of a test string. In and out-mass operators are defined by

\[
M^2_{<(|)} := -p_{M_{<(|)}p^M_{<(|)}} + (p_{4_{<(|)}})^2 = 2 \sum_{m=1}^{\infty} \sum_{i}(\alpha_{m_{<(|)}}^{ij} \alpha_{m_{<(|)}}^{ij} + \tilde{\alpha}_{m_{<(|)}}^{ij} \tilde{\alpha}_{m_{<(|)}}^{ij}) + (p_{4_{<(|)}})^2 - 2. \tag{81}
\]

To see how much a test string gets excited after passing through the shock wave, we shall define the following mass excitation operator by using mass operators:

\[
\delta M^2 := M^2_{>} - M^2_{<}. \tag{82}
\]

Therefore, by using the formula Eq. (78) again, we can obtain the excitation as follows,

\[
\langle 0_{<}; p_4 | \delta M^2 | 0_{<}; p_4 \rangle = - \left( \frac{p\tilde{M} + 2p_4\tilde{Q}}{4\sqrt{\pi}} \right)^2 \int dk^2 k^2 2^{-k^2} \phi^2(k) \frac{\tan(\frac{\pi k^2}{2}) \Gamma(\frac{1}{2}k^2)}{\Gamma(\frac{1}{2} + \frac{1}{2}k^2)} + \tilde{Q}^2 \frac{2}{1 + k^2} \int \frac{d^2k}{2\sqrt{\pi}} \phi^2(k) \frac{\tan(\frac{\pi k^2}{2}) \Gamma(\frac{1}{2}k^2)}{\Gamma(\frac{1}{2} + \frac{1}{2}k^2)} - \frac{\tilde{Q}^4}{2\pi^2} \left[ \prod_{i=1}^{4} \int_{0}^{2\pi} ds_i \right] \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \sum_{m=1}^{\infty} ns \sin(n(\sigma_2 - \sigma_1) \sin(s(\sigma_4 - \sigma_3) \cos(m(\sigma_2 - \sigma_4))) \cos(m(\sigma_2 - \sigma_4))) \times \left[ \prod_{i=1}^{4} \int d^2k_i \phi(k_i) \right] \left( k_2 \cdot k_4 \right) \prod_{1 \leq i < j \leq 4}^{4} 2 \sin \left( \frac{\sigma_i - \sigma_j}{2} \right) \delta^2(k_1 + \cdots + k_4), \tag{83}
\]

where we have used the fact

\[
\sum_{m=1}^{\infty} m^2 \frac{\Gamma(m + \frac{1}{2}k^2)}{\Gamma(1 + m - \frac{1}{2}k^2)} = 0 \tag{84}
\]
with the help of Eq. (78). The above formula means that the vacuum expectation value from each left oscillator $\tilde{\alpha}_m^4$ exactly vanishes (the summation of Eq. (77) by $m$ is exactly zero).

The first two terms have simple poles at $k^2 = 1, 3, 5, \cdots$ and the integral seems to diverge. However, we can obtain finite values by taking a principal value prescription. The details and physical interpretations are devoted to Ref. [11]. It is noteworthy that when $\tilde{Q} = 0$, Eq. (83) explicitly coincides with the calculation in the Aichelburg-Sexl geometry in Ref. [11]. It should be noticed that the $\tilde{Q}^4$ term appears. We can see the origin of this term in Eqs. (60), (61), and (63). The second term in Eq. (60) does not include the right oscillators $\alpha_{n<}^4$ because of the existence of the axion field $B_{4u}$. This results in the next commutation relation (61). On the other hand, the quantum fluctuation in the compactified extra-dimension $x^4$ comes from the gauge field $A_{4u}$, as seen in Eq. (63). Thus, $\tilde{Q}^4$ term arises from quantum fluctuation in the extra-dimension caused by both the gauge field and the axion field.

In the case of $\tilde{Q} \ll 1$, the excitation value in Eq. (83) with an initial state

$$\left| 0; -\frac{p\tilde{M}}{2\tilde{Q}} \right>$$

is small enough independently of the value of the particle mass $\tilde{M}$. This indicates that a first-quantized test string with charge $\sim -p\tilde{M}/2\tilde{Q}$ can pass through the strong shock wave ($\tilde{M} \gg 1$) with small excitation. This is essentially due to the force cancellation between the gravitational and Coulomb forces, just like the classical molecule discussed in Sec. III. On the other hand, in the case of $\tilde{Q} \gg 1$, the excitation with the initial state (83) becomes quite large by the $\tilde{Q}^4$ term. In other words, all strings get excited strongly by the quantum fluctuation in the extra-dimension $x^4$ caused by both the gauge field and the axion field.
5 Discussions

We have investigated how much first-quantized charged bosonic test strings get excited after crossing the shock wave generated by a charged particle with mass $\tilde{M}$ and $\tilde{Q}$. We considered a closed string model where $U(1)$ charge is given by the conjugate momentum to a compactified extra-coordinate $x^4$ based on Kaluza-Klein theory. Solutions to classical equations of motion for the metric can be obtained in the low-energy limit by the CAS spacetime, accompanied by the axion field $B_{4u}$.

As we have shown in Sec. 4, in the case of small $\tilde{Q}$, the first-quantized strings with charge $\sim -p\tilde{M}/2\tilde{Q}$ hardly get excited, even though $\tilde{M}$ is quite large. This is due to the force balance between the gravitational and Coulomb forces. And hence the similar picture to the classical “molecule” mentioned in Sec. 3 is realized. Furthermore, since the singularity does not prevent the test string from passing through it, this type of singularity is rather mild. Also in the case of $\tilde{Q} \sim \tilde{M} \gg 1$, the fluctuations which appear for the neutral test string can be canceled by a part of the fluctuations which are charged string origin. However, the other parts of the fluctuations, which also originate in the gauge field $A_{4u}$ and the axion field $B_{4u}$, grow significantly. As a result, it is impossible to make the excitations small even if we take into account the gauge interaction between the test string and the background fields.

From our analysis, it is found that the mass expectation value of the test string becomes arbitrarily large in the neutral and the latter ($\tilde{Q} \sim \tilde{M}$) cases when the mass of the source particle of the shock wave becomes large. How is this result interpreted? Since arbitrarily large mass state can be created, one may think that some singular processes occur in passing through the shock wave. It should be noted, however, that the energy scale of the test string is much larger than the string mass scale $(\sqrt{\alpha'})^{-1}$. Hence, the back reaction to the fields has to be considered, i.e., full treatment is needed. If we perform this, we may define the “singularity” in string theory in physically reasonable sense, although there is a possibility (and most of the physicists hope) that all classical singularities are excluded by full string analysis. Unfortunately, it is far from our present knowledge.

It is noteworthy that the exact Weyl invariance of a two-dimensional $\sigma$ model except the point where a source particle exists would break down in the case of non-zero $\tilde{Q}$ because the classical solutions in Eqs. (3)-(5) are the solutions only in the low-energy limit, although $\tilde{Q} = 0$ case has the exact Weyl invariance. It is unlikely, however, that $\tilde{Q}^4$ terms in Eq. (33)
completely vanish in the solutions satisfying the invariance of the two-dimensional $\sigma$ model. This motivates us to consider that the above picture holds.

Most promising theory of all fundamental interactions and matters is superstring theory. Therefore, it is natural to include the degrees of freedom of fermion as well as boson. The scattering of spin particles and superstrings (without gauge interaction) in the Aichelburg-Sexl spacetime has been discussed in Refs. [20]. If we take the fermion into account in our model, a test string may not get excited infinitely. This is a future investigation.

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A Calculation of Commutation relations

Let us denote the annihilation and creation parts of $x_0^i$ by $x_0^{i(+)}$ and $x_0^{i(-)}$, respectively. Then, the following commutation relations

$$[\exp[i k \cdot x_0(\sigma)], \alpha_m^l] = -\frac{1}{2} k^l e^{-i m \sigma} : \exp[i k \cdot x_0(\sigma)] :,$$

(86)

$$[\exp[i k \cdot x_0(\sigma)], \tilde{\alpha}_m^l] = -\frac{1}{2} k^l e^{i m \sigma} : \exp[i k \cdot x_0(\sigma)] :,$$

(87)

are found with the help of Eq. (64). Furthermore, the commutator

$$[x_0^{l(+)}(\sigma), x_0^{l(-)}(\sigma')] = -\delta_{ll'} \ln|2 \sin \left(\frac{\sigma - \sigma'}{2}\right)|$$

(88)

leads us to the formula

$$\langle 0_\lessgtr \prod_{i=1}^n [\exp[i k_i \cdot x_0(\sigma_i)] : | 0_\lessgtr \rangle = \prod_{1 \leq i < j \leq n} 2 \sin \left(\frac{\sigma_i - \sigma_j}{2}\right) k_i k_j \delta^2(k_1 + \cdots + k_n).$$

(89)

Then, Eqs. (60) and (61) can be directly computed by using Eqs. (86) and (87), noting

$$[\exp[i k \cdot x_0(\sigma)] :, \exp[i k' \cdot x_0(\sigma')] :] = 0 \quad \text{for all} \quad \sigma, \sigma',$$

(90)

and

$$\sum_{n \neq 0} \sum_{s \neq 0} \left[\tilde{\alpha}_n^l e^{-i n \sigma}, \tilde{\alpha}_s^l e^{-i s \xi}\right] = 2i \sum_{n=1}^\infty n \sin n(\xi - \sigma).$$

(91)

B Representation by Gamma function

Under the change of coordinates

$$\eta = \sigma - \xi, \quad \rho = \sigma + \xi,$$

(92)

we calculate the following integration as

$$\int_0^{2\pi} \int_0^{2\pi} d\sigma \ d\xi \ e^{im(\sigma - \xi)} \left|2 \sin \left(\frac{\sigma - \xi}{2}\right)^{-k^2}\right|$$

$$= \frac{1}{2} \int_{-2\pi}^{0} d\eta \int_{\pi}^{4\pi + \eta} d\rho \ e^{im\eta} \left|2 \sin \left(\frac{\eta}{2}\right)^{-k^2}\right| + \frac{1}{2} \int_{2\pi}^{0} d\eta \int_{\eta}^{4\pi - \eta} d\rho \ e^{im\eta} \left|2 \sin \left(\frac{\eta}{2}\right)^{-k^2}\right|$$

$$= 8\pi \int_0^{\pi/2} d\eta \cos(2m\eta) \left|2 \sin \eta\right|^{-k^2}$$

$$= 4\pi \sin \left(\frac{\pi}{2} k^2\right) \frac{\Gamma(1 - k^2) \Gamma \left(m + \frac{1}{2} k^2\right)}{\Gamma \left(1 + m - \frac{1}{2} k^2\right)},$$

(93)
To obtain the last equation, we used the following analytic continuation for $\mu$ \cite{[19]},

\[
\int_0^{\pi/2} dx \ (2 \sin x)^{-2\mu} \cos(2mx) = \frac{1}{2} \sin \pi \mu \frac{\Gamma(1 - 2\mu)\Gamma(m + \mu)}{\Gamma(1 + m - \mu)},
\]

where the r.h.s. of Eq. (94) is finite except poles. Similar calculation shows that

\[
\int_0^{2\pi} \int_0^{2\pi} d\sigma d\xi \ e^{i(m-\xi)} \cos m(\sigma - \xi) \left|2 \sin \left(\frac{\sigma - \xi}{2}\right)\right|^{-k^2}
= 4\pi \int_0^{\pi/2} d\eta \cos[2(m+n)\eta] \left|2 \sin \eta\right|^{-k^2} + 4\pi \int_0^{\pi/2} d\eta \cos[2(m-n)\eta] \left|2 \sin \eta\right|^{-k^2}
= 2\pi \sin \left(\frac{\pi}{2} k^2\right) \frac{\Gamma(1 - k^2)\Gamma \left(n + m + \frac{1}{2} k^2\right)}{\Gamma \left(1 + n + m - \frac{1}{2} k^2\right)} + 2\pi \sin \left(\frac{\pi}{2} k^2\right) \frac{\Gamma(1 - k^2)\Gamma \left(m - n + \frac{1}{2} k^2\right)}{\Gamma \left(1 + m - n - \frac{1}{2} k^2\right)}.
\]

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