Conditional Information Inequalities and Combinatorial Applications

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Abstract—We provide a new relaxation of conditions under which the inequality \( H(A|B,X) + H(A|B,Y) \leq H(A|B) \) holds for jointly distributed random variables \( A, B, X, Y \).

Then we present two applications of our result. The first one is the following easy-to-formulate combinatorial theorem: Assume that edges of a bipartite graph are partitioned into \( K \) matchings so that for each pair (left vertex \( x \), right vertex \( y \)) there is at most one matching in the partition that involves both \( x, y \). Assume further that the degree of each left vertex is at least \( L \) and the degree of each right vertex is at least \( R \). Then \( K \geq LR \).

The second application is a new method to prove lower bounds for equalities; non-Shannon-type information inequalities defined above can be relaxed. Then we show that the relaxation of the conditional inequality \([KR11]\) is closely related to some combinatorial properties of bipartite graphs.

Let us notice that assuming these conditions we can essentially simplify \([1]\). Indeed, assuming \([3]\) and \([4]\), Ingleton’s inequality rewrites to

\[
I(A : B) \leq I(A : B|X) + I(A : B|Y),
\]

and assuming \([3]\) and \([4]\) the same inequality becomes equivalent to

\[
H(A|B,X) + H(A|B,Y) \leq H(A|B).
\]

Notice that the conditions \([2], [3], \) and \([4]\) are in some sense “degenerate properties” of the distribution; they are very fragile, and an infinitesimal perturbation would destroy them.

A noteworthy fact is that the conditional inequalities \([ZY97]\) and \([KR11]\) cannot be obtained as a direct implication of any unconditional linear inequality for Shannon’s entropy. More precisely, whatever pair of reals \( \lambda_1, \lambda_2 \) we take, the inequalities

\[
I(A : B) \leq I(A : B|X) + I(A : B|Y) + \lambda_1 I(X : Y) + \lambda_2 I(X : Y|A)
\]

and

\[
I(A : B) \leq I(A : B|X) + I(A : B|Y) + I(X : Y) + \lambda_1 I(X : Y|A) + \lambda_2 H(A|X,Y)
\]

(the natural unconditional reformulations of \([ZY97]\) and \([KR11]\) respectively) do not hold for some distributions, see \([7]\). Thus, the inequalities \([ZY97]\) and \([KR11]\) are, so to say, essentially conditional.

A remarkable property of the constraints in \([2], [3], \) and \([4]\) is that they involve only the distribution of the triple \( A, X, Y \), while the implying inequality involves another random variable \( B \). We believe that these constraints imply some “structural” properties of the distribution \( A, X, Y \), and these properties in turn imply Ingleton’s inequality. These structural properties should be interesting per se, and they can have other important implications. However, we still do not know how to formulate these properties, and the combinatorial nature of all these conditions remains not well understood.

In the present paper we move toward a better understanding of the “combinatorial” meaning of essentially conditional inequalities. We show that the conditions for both conditional inequalities defined above can be relaxed. Then we show that relaxation of the conditional inequality \([KR11]\) is closely related to some combinatorial properties of bipartite graphs;
in particular, it can be used to estimate the minimal size of a biclique covering (and non-deterministic communication complexity of some functions).

II. Notation

To simplify the formulas, we use the following notation: $p(a)$ denote $\Pr[A = a]$, $p(a, x) = \Pr[A = a, X = x]$, $p(a|x) = \Pr[A = a | X = x]$, $p(a, y) = \Pr[A = a, Y = y]$ and so on.

III. Relaxation of Conditional Inequalities

Now we present our relaxations. We generalize the [KR11] conditional inequality by proving that (6) holds under condition (4) and the following condition (weaker than (3)).

\[
\begin{align*}
\text{for each triple } x, y, a \text{ of values of } X, Y, A \quad & \quad \left\{ \begin{array}{l}
p(a, x) > 0 \\
p(a, y) > 0 \implies p(a, x, y) > 0.
\end{array} \right. \\
(7)
\end{align*}
\]

Notice that condition (4) means that
\[
p(a, x) \cdot p(a, y) = p(a, x, y) \cdot p(a)
\]
for all $a, x, y$. The condition (7) requires only that both sides of this equality vanish at the same points of the probabilistic space.

Moreover, we show that conditions (4) and (7) can be further relaxed to a single condition:

\[
\text{for each quadruple } a, a', x, y, \text{ if the probabilities } p(a, x), p(a, y), p(a', x), p(a', y) \text{ are all positive, then } a = a'.
\]

\[
(8)
\]

Remark 1. It is essential to require this property hold for all quadruples $a, a', x, y$, involving those where $x$ and $y$ are incompatible, i.e., $\Pr[X = x, Y = y] = 0$.

The conditions (4), (7) and (8) depend only on the support of the distribution of $A, X, Y$ and are thus less vulnerable than condition (3) used in (7). One can say that in some sense (7) is a condition of “general position” for a distribution, while condition (3) is a kind of “degenerate case”. The most robust among all these conditions is the property (8): it can be violated only by adding a new triple in the support of the distribution. This implies that it is relativizable:

Lemma 1. If a random variable $(A, X, Y)$ satisfies (8), then for each event $E$ having positive probability the conditional random variables of $(A, X, Y) | E$ satisfy (8).

The proof of Lemma 2. Assume that the four events
\[
\begin{align*}
|X = x, A = a | E, |Y = y, A = a | E, \\
\end{align*}
\]

have positive probability. Then the unconditional probability of each of these events is positive as well and hence $a = a'$ by (8).

The relativizability of Condition (8) explains why the inequality (5) holds under a condition that involves only $A, X, Y$: the Condition (8) implies that for every fixed value $b$ of $B$ the conditional distribution $A, X, Y | B = b$ also satisfies Condition (8).

The next lemma clarifies the relations between the conditions (3), (4), (7) and (8).

Lemma 2. (3) implies (7). Conditions (4) and (7) together imply (8). Finally, (5) implies (4).

The proof of Lemma 2. (3) ⇒ (7): Condition (3) means that
\[
p(a, x) \cdot p(a, y) = p(a, x, y) \cdot p(a)
\]
for all $x, y, a$. If the left hand side of this equality is positive then so is its right hand side.

(7) and (4) ⇒ (8): Assume (7) and (4). The assumptions of condition (8) and (7) imply that both events $p(a, x, y) > 0$ and $p(a', x, y) > 0$. Hence by (4) we have $a = a'$.

(8) ⇒ (4): Assume (8). Assume further than for certain $x, y, a, a'$ we have $p(a, x, y) > 0$ and $p(a', x, y) > 0$. Then the assumptions of (4) hold and hence $a = a'$.

For the inequality (5) we provide a less robust relaxation. We show that (5) remains valid provided the condition (7) holds and
\[
p(a, x) \cdot p(a, y) \cdot p(a, x) \leq p(a) \cdot p(x) \cdot p(y) \cdot p(a, x, y).
\]

This inequality may be re-written as follows:
\[
p(a) \cdot p(x) | a \cdot p(y) | a \leq p(x) \cdot p(y) \cdot p(a | x, y)
\]
(undefined values are considered to be zero). Under condition (7) the formulas in both sides of this inequality define probability distributions over triples $(a, x, y)$. Therefore the inequality can be valid only when its left hand side and its hand side coincide for all $a, x, y$.

The conditions (7) and (8) easily follow from (3) and (4) but the converse is not true. In other words, these conditions are relaxations of (3) and (4).

IV. New Conditional Inequalities

A. The method of [2] (reminder)

We first outline the method of [2] that was used to prove the known conditional versions of inequalities (5) and (6). To this end, we prove a lemma that provides unconditional versions of both inequalities with some error terms. Then, we discuss under which conditions these error terms vanish.

Lemma 3. We have
\[
H(A | X, B) + H(A | Y, B) \leq H(A | B) + \Gamma
\]
and
\[
I(A : B) \leq I(A : B | X) + I(A : B | Y) + \Delta
\]
for all $A, B, X, Y$. Here
\[
\Gamma = \log \left( \sum_{a,b,x,y: p(a, x, y) > 0, p(a, b, y) > 0} \frac{p(b, x) p(b, y)}{p(b)} \right).
\]
\[
\Delta = \log_2 \left( \sum_{a, b, x, y : \ p(a, b, x) > 0, \ p(a, b, y) > 0} \frac{p(a, x)p(a, y)p(b, x)p(b, y)}{p(a)p(x)p(y)p(b)} \right).
\]

**Remark 2.** Even though the conditional inequalities [ZY97] and [KR11] cannot be obtained directly from any unconditional linear inequality for entropies (see [1]), they can be deduced from some unconditional non-linear inequalities for entropies. (In fact, some interesting examples of non-linear inequalities for Shannon’s entropy are known, [8].) However, in Lemma 3 we do not achieve this goal; the “error terms” \( \Gamma \) and \( \Delta \) are functions of the involved distributions but not of their entropies.

**Proof:** The important properties of both inequalities (5) and (6) are the following: both inequalities do not have terms that contain both \( X \) and \( Y \) and after expressing in both inequalities all the terms through unconditional Shannon entropy both terms \( H(A, B, X) \) and \( H(A, B, Y) \) fall in the left hand side of the sign \( \leq \). This common features allows to treat them in a similar way.

Let us start with inequality (11). The inequality means that the average value of the logarithm of the ratio
\[
\frac{p(b, x)p(b, y)p(a, b)}{p(a, b, x)p(a, b, y)p(b)} \tag{13}
\]
is at most \( \Gamma \). The average is computed with respect to the distribution \( p(a, b, x, y) \). Computing the average, we take into account only those quadruples \( a, b, x, y \) with positive probability. For such quadruples, both the numerator and denominator of the ratio (13) are positive and hence its logarithm is well defined.

Consider a new distribution \( p' \), where
\[
p'(a, b, x, y) = \begin{cases} 
\frac{p(a, b, x)p(a, b, y)}{p(a, b)} & \text{if } p(a, b) > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Random variables distributed according to \( p' \) are generated by the following process: generate first a pair \( a, b \) using the original distribution of \( (A, B) \). Then generate independently \( x \) using the conditional distribution \( x|a,b \) and \( y \) using the conditional distribution \( y|a,b \).

Notice that \( p'(a, b, x, y) \) is positive if so is \( p(a, b, x, y) \) but not the other way around. However the ratio (13) is still well defined and positive for all quadruples \( a, b, x, y \) with positive \( p'(a, b, x, y) \). Therefore we can compute the average value of the logarithm of (13) using the distribution \( p' \) in place of \( p \). Moreover, changing the distribution does not affect the average. This follows from the equalities \( p'(a, b, x) = p(a, b, x) \) and \( p'(a, b, y) = p(a, b, y) \).

Indeed, the logarithm of (13) is the sum of logarithms of its factors. Thus it suffices to show that the average of the logarithm of each factors is not affected when we replace \( p \) by \( p' \). Let us prove this, say, for the factor \( 1/p(a, b, x) \).

This factor does not depend on \( y \). Therefore the average of its logarithm does not depend on how \( p(a, b, x) \) is split among \( p(a, b, x, y) \) for different values \( y \); we just sum up \( \log 1/p(a, b, x) \) over all \( a, b, x \) with weights \( p(a, b, x) \). As \( p(a, b, x) = p'(a, b, x) \), summing with weights \( p'(a, b, x) \) will yield the same result.

By Jensen’s inequality the average value of the logarithm of the ratio (13) with respect to the distribution \( p' \) is at most
\[
\log \left( \sum_{a, b, x, y : p'(a, b, x, y) > 0} \frac{p(b, x)p(b, y)}{p(b)} \right) = \Gamma.
\]

The inequality (12) means that the average value of the logarithm of the ratio
\[
\frac{p(a, b)p(a, x)p(b, y)p(b)}{p(a)p(x)p(b, y)p(b)} \tag{14}
\]
is at most \( \Delta \). Again, when computing this average we can change the distribution \( p \) to the newly defined \( p' \). Thus (12) follows from Jensen’s inequality, \( \Gamma \) and the definition of \( \Delta \).

**B. A generalized version of the [KR11] conditional inequality**

**Theorem 1.** If the condition (8) holds then the inequality (6) is true.

**Proof:** There are two ways to prove this theorem.

The first way: we first prove the unconditional version of the theorem (that is, without \( B \) in condition) and then note that the general case reduces to this special case. Indeed, if \( B \) is constant, then
\[
\Gamma = \log \left( \sum_{x, y : a : p(a, x) > 0, p(a, y) > 0} p(x)p(y) \right)
\]
The condition (8) guarantees that for each \( x, y \) there is at most one \( a \) with \( p(a, x) > 0, p(a, y) > 0 \) and hence
\[
\Gamma \leq \log \left( \sum_{x, y} p(x)p(y) \right) = \log 1 = 0.
\]

If \( B \) is not trivial then for every possible value \( b \) of \( B \) Lemma 1 guarantees that (8) remains valid conditional to event \( B = b \). As we just have shown, this implies
\[
H(A|X, B = b) + H(A|Y, B = b) \leq H(A|B = b).
\]

Taking the average over all \( b \), we get (6).

The second way: we directly estimate \( \Gamma \). By Condition (8) and Lemma 1 we have
\[
\Gamma \leq \log \left( \sum_{b, x, y : p(b, x) > 0, p(b, y) > 0} \frac{p(b, x)p(b, y)}{p(b)} \right)
\]
It is easy to see that \( \frac{p(b, x)p(b, y)}{p(b)} \) is a probability distribution over triples \( b, x, y \) and hence the sum here equals 1 and \( \Gamma \leq \log 1 = 0 \).

**Remark 3.** Theorem 1 is nontrivial even for a constant \( B \). Indeed, with a degenerate \( B \) the inequality (6) rewrites to to
\[
H(A|X) + H(A|Y) \leq H(A).
\]

1We need Jensen’s inequality for logarithmic function: let \( p_1, \ldots, p_n \) be positive numbers that sum up to 1; then \( p_1 \log x_1 + \cdots + p_n \log x_n \leq \log(p_1 x_1 + \cdots + p_n x_n) \).
The difference between the right hand side and the left hand side of this inequality is equal to
\[ I(X : Y) - I(X : Y|A) - H(A|X, Y). \]

Given that \( H(A|X, Y) = 0 \), the inequality (15) means that the mutual information of the triple \( I(X : Y : A) \) is non-negative. The mutual information of three random variables is defined by either of the four equivalent expressions:
\[ I(X : Y) - I(X : Y|A) = I(A : X) - I(A : X|Y) = I(A : Y) - I(A : Y|X) = H(X, Y, A) + H(X) + H(Y) + H(A) - H(X, Y) - H(X, A) - H(Y, A). \]

The quantity \( I(X : Y : A) \) is obviously non-negative under condition (3) since in this case it coincides with \( I(X : Y) \). Morally, Theorem 2 suggests that \( I(X : Y : A) \geq 0 \) holds under some more general assumption than (3).

Remark 4. Combining Lemma 3 and Theorem 1 we get the original version of the [KR11] conditional inequality from [7]. Our proof of Theorem 1 essentially follows the proof of [7] Theorem 1. The novelty is that we have explicitly stated a more general condition (5) and verified that it is sufficient for inequality (6). In what follows we discuss some connection of Theorem 1 with combinatorial properties of bipartite graphs.

C. A relaxation of the condition in the [ZY97] inequality

Theorem 2. Let \( A, B, X, Y \) be jointly distributed random variables. Assume that condition (7) holds. Then
\[ I(A : B) \leq I(A : B|X) + I(A : B|Y) + \Delta', \]
where \( \Delta' \) stands for the maximum of the logarithm of the ratio
\[ \frac{p(a, x)p(a, y)p(x, y)}{p(a)p(x)p(y)p(a, x, y)}. \]

In particular, if and for all \( x, y, a \)
\[ p(a, x)p(a, y)p(x, y) \leq p(a)p(x)p(y)p(a, x, y), \quad (16) \]
then \( I(A : B) \leq I(A : B|X) + I(A : B|Y) \).

The statement of this theorem is rather technical, and we make several remarks clarifying it.

Remark 5. As already mentioned, under condition (7), if the inequality (16) holds for all \( a, x, y \) then it is actually an equality.

Remark 6. Inequality (16) implies that \( I(A : X : Y) \leq 0 \) (and even \( I(A : X : Y) = 0 \) if we assume (7)). Indeed, \( I(A : X : Y) \) is the average of the logarithm of the ratio of the left hand side and right hand side of (16). However, the inequality
\[ I(A : B) \leq I(A : B|X) + I(A : B|Y) - I(A : X : Y) \]
is false in general case (let, say \( A = X = Y \) be a random bit, and \( B \) a constant random variable).

Remark 7. Condition (16) can be rewritten as
\[ \frac{p(a)}{p(a|x)} \cdot \frac{p(a)}{p(a|y)} = \frac{p(a)}{p(a|x, y)}. \]
In a sense, it means that the knowledge of \( X \) and \( Y \) bring “independent” parts of information about the value of \( A \).

Remark 8. The condition in Theorem 2 is indeed more general than the previously known condition of [ZY97]. In fact, there exists a distribution \( (A, X, Y) \) such that the conditions (7) and \( p(ax)p(ay)p(xy) = p(axy)p(a)p(x)p(y) \) are satisfied, but \( I(X : Y) \neq 0 \), see Section [IV-D].

Proof of Theorem 2. By Lemma 3 we have
\[ I(A : B) \leq I(A : B|X) + I(A : B|Y) + \Delta \]
where
\[ \Delta = \log_2 \left( \sum_{a, b, x, y} \frac{p(a, x)p(a, y)p(b, y)}{p(a)p(x)p(y)p(b)} \right). \]
The condition (7) implies that \( p(x, y) \) is positive for all \( a, b, x, y \) involved in the above sum. And the definition of \( \Delta' \) implies that the fraction \( \frac{p(a)p(x)p(y)p(a, x, y)}{p(a, x)p(a, y)p(x, y)} \) is at most \( 2^{\Delta} p(a|x, y) \). Hence the sum is at most
\[ 2^{\Delta'} \sum_{b, x, y} \frac{p(b, x)p(b, y)}{p(b)} \left( \sum_a p(a|x, y) \right) = 2^{\Delta'} \sum_{b, x, y} \frac{p(b, x)p(b, y)}{p(b)} = 2^{\Delta'}. \]
Hence \( \Delta \leq \Delta' \).

D. Why Theorem 2 is stronger than [ZY97].

Let \( F \) be a field of cardinality \( q = 2^k \). We split the field in two equal halves by letting \( F = F' \cup F'' \), and \( F'' \cap F'' = \emptyset \), and split \( F^2 \) into two “half-planes” as \( F^2 = (F' \times F') \cup (F'' \times F) \). Then, we define random variables \( (X, Y, A) \):
- let \( A \) be a uniformly chosen polynomial \( a(t) = a_0 + a_1 t \) of degree at most 1;
- let values of \( X \) and \( Y \) be randomly chosen points in the line defined by \( A \) such that the value of \( X \) belongs to \( F' \times F \) and the value of \( Y \) belongs to \( F'' \times F \). We do not specify this distribution precisely and only require that the first coordinate of \( X \) is uniformly distributed on \( F' \), and the first coordinate of \( Y \) is uniformly distributed on \( F'' \);
- \( X \) and \( Y \) are not independent.

For this triple of random variables the conditions (7) and (16) are satisfied. Indeed, \( p(x) = p(y) = 2/q^2 \), \( p(a) = 1/q^2 \), \( p(a, x) = p(a, y) = 2/q^3 \), \( p(x, y) = p(a, x, y) = 4/q^4 \), and both sides of (16) are equal to \( 16/q^{10} \). However, \( I(X : Y) > 0 \). So we can apply Theorem 2 but not the more conventional (and more restrictive) statement of [ZY97].
From Theorem 1, we can derive the following combinatorial statement:

**Corollary 1.** Assume that edges of a bipartite graph are partitioned into \( K \) matchings so that for each pair \((\text{left vertex } x, \text{right vertex } y)\) there is at most one matching in the partition that involves both \( x, y \). Assume further that the degree of each left vertex is at least \( L \) and the degree of each right vertex is at least \( R \). Then \( K \geq LR \).

**Proof:** Let \( M_1, \ldots, M_K \) denote the given matchings. Consider the uniform distribution on the set of edges of the graph. Denote by \((X, Y, A)\) the following triple of jointly distributed random variables:

- \( X = \) [the left end of the edge],
- \( Y = \) [the right end of the edge],
- \( A = \) [the index \( i \) of the matching \( M_i \) containing the edge].

The conditions of the corollary imply that the triple \((A, X, Y)\) satisfies (8): if \( p(a, x) > 0 \) and \( p(a, y) > 0 \), then both \( x \) and \( y \) are involved in the matching \( M_a \), and hence such a matching \( M_a \) is unique. Therefore by Theorem 1 we have \( H(A|X) + H(A|Y) \leq H(A) \) (see also Remark 3 on p. 9).

On the other hand, \( H(A|X) \geq \log L \) since conditional on \( X = x \), the value of \( A \) is the number of the matching that contains a randomly (and uniformly) chosen edge from the vertex \( x \). Similarly, we have \( H(A|Y) \geq \log R \). Hence, \( H(A) \geq \log L + \log R \). It follows that the range of \( A \) is at least \( LR \).
In Appendix we present several examples of distributions that do not satisfy the conditions of the inequality [KR11] and show how [KR11] can be employed in a proof of a lower bound for biclique coverings of graphs. Though we are not aware of any nontrivial result on biclique coverings proven by this technique, we believe this technique worth a more detailed investigation.

A. Why condition \(\text{(2)}\) is essential for Theorem \(\text{(1)}\)

In this section we show that \(\text{(4)}\) without \(\text{(7)}\) does not imply \(\text{(6)}\). Let us fix an integer \(n\) and introduce the uniform distribution on the set of triples \((x, y, a)\), where

- \(x\) and \(y\) are two distinct elements in \(\{1, \ldots, n\}\),
- \(a = \{x, y\}\) is the (non-ordered) two-elements set that consists of \(x\) and \(y\).

Thus, we obtain a triple of jointly distributed random variables that we denote \((X, Y, A)\).

For this distribution we have

\[
H(A) = \log \left( \frac{n}{2} \right) = \log(q(q-1)) - 1
\]

(since we have \(\binom{n}{2}\) equiprobable two-elements sets), and

\[
H(A|X) = H(A|Y) = \log(q-1)
\]

(to specify a set \(a\) given one of its elements, we need to specify the second elements; so we have \(n-1\) equiprobable variants). We see that this distribution does not satisfy \(\text{(6)}\):

\[
H(A|X) + H(A|Y) \not\leq H(A)
\]

as

\[
2\log(q-1) \not\leq \log(q(q-1)) - 1.
\]

Why the conditional inequality [KR11] does not work in this case? The reason is that condition \(\text{(7)}\) is violated: each value \(a = \{a_1, a_2\}\) of \(A\) is separately compatible with the events \(X = a_1\) and \(Y = a_1\), but not with both of them together. (The weaker condition \(\text{(8)}\) is also violated for a similar reason.)

B. From distinct points to disjoint sets.

In this section we slightly generalize the construction from Appendix A. Let \(n, k\) be integers, \(k \leq n/2\). We introduce the uniform distribution on the set of triples \((x, y, a)\), where

- \(x\) and \(y\) are two disjoint \(k\)-elements subsets of the universe \(\{1, \ldots, n\}\),
- \(a = x \cup y\) is the \(2k\)-elements union of \(x\) and \(y\).

We denote the constructed triple of random variables by \((X, Y, A)\). For the defined distribution we have

\[
H(A) = \log \left( \frac{n}{2k} \right)
\]

(all \(2k\)-elements sets are equiprobable) and

\[
H(A|X) = H(A|Y) = \log \left( \frac{n-k}{k} \right)
\]

(given \(k\) elements of \(a\), it remains to specify \(k\) other elements in the universe).

If \(k \ll q\), then this distribution does not satisfy \(\text{(6)}\), since

\[
2\log \left( \frac{n-k}{k} \right) \not\leq \log \left( \frac{n}{2k} \right).
\]

Moreover, if \(n \gg k\), then the imbalance between the LHS and the RHS is close to

\[
\log \frac{(2k)!}{k!k!} \approx 2k.
\]

Why is \(\text{(6)}\) violated for this distribution? Why [KR11] cannot be applied? The reason is that the constraint \(\text{(2)}\) does not hold. Indeed, \(a\) can be separately compatible with some value \(x\) of \(X\) and with some value \(y\) of \(Y\), but not with both these values together. This happens when \(x\) and \(y\) are not disjoint (i.e., incompatible with each other). Notice that the weaker condition \(\text{(8)}\) is also violated.

C. Towards a bound for biclique coverings.

Let us discuss in more detail the combinatorial properties of the distribution \((X, Y, A)\) from the previous section. This distribution can be represented as follows. Define a bipartite graph \(G_{n,k} = (V_1, V_2, E)\), where both parts \(V_1\) and \(V_2\) consist of \(k\)-elements subsets in \(\{1, \ldots, n\}\), and the set of edges \(E = V_1 \times V_2\) consists of all pairs of disjoint sets. We also assign to each edge \((x, y)\) a “color” defined as \(x \cup y\). That is, we have \(\binom{n}{2k}\) different colors (the number of \(2k\)-elements subsets in the \(n\)-elements universe), and each color is assigned to \(\binom{2k}{k}\) edges (this is the number of ways to split a \(2k\)-elements set into two disjoint \(k\)-elements parts).

The distribution \((X, Y)\) defined in Appendix B corresponds to the uniform distribution on the set of edges \(E\) of this graph: \(X\) and \(Y\) are the ends of a randomly chosen edge, and \(A\) is the color of this edge. Notice that the condition \(\text{(4)}\) is satisfied for this distribution (given the ends of the edge we can uniquely reconstruct the color of this edge). On the contrary, the conditions \(\text{(7)}\) and \(\text{(8)}\) are not satisfied.

We are going to apply the inequality from Section V-C to estimate some combinatorial parameters of the defined graph. Let us remind the well known notion of a biclique covering:

Definition 1. For any bipartite graph \(G = (V_1, V_2, E)\) (with the set of vertices \(V_1\) and \(V_2\) and a set of edges \(E \subset V_1 \times V_2\)) its biclique covering number \(\text{bcc}(G)\) is defined as the minimal number of bicliques (complete bipartite subgraphs) that cover all edges of \(G\).

Biclique covering plays an important role in communication complexity. Specifically, the non-deterministic communication complexity (see [9]) of a predicate

\[
P : U \times U \rightarrow \{0, 1\}
\]

can be defined as \(\log \text{bcc}(G)\) for the bipartite graph \(G = (V_1, V_2, E)\), where \(V_1 = V_2 = U\), and \(E\) is the set of all pairs \((x, y)\in U \times U\) such that \(P(x, y) = 1\). In particular, \(\log \text{bcc}(G_{n,k})\) for the graph define above is non-deterministic
communication complexity of the classic $k$-disjointness problem.

Assume that this graph $G$ can be covered by $t$ biclique subsets of $C_1, \ldots, C_t$. Then, we extend the distribution $(X, Y, A)$ and add another random variable: we define $Z$ as the index of a biclique $C_i$ that covers the edge $(X, Y)$. If an edge belongs to several bicliques $C_i$, then we choose one of them at random, with equal probabilities. Notice that $Z$ ranges over $\{1, \ldots, t\}$, so $H(Z) \leq \log t$.

The crucial point is that for a fixed value $Z$ in the conditional distribution (in a distribution restricted to a biclique) the conditions (7) is satisfied (by definition of a biclique, if vertices $x \in V_1$ and $y \in V_2$ are involved in $C_i$, then the edge $(x, y)$ belongs to $C_i$). Hence, the inequality [KR11] is valid for each conditional distribution $(A, X, Y)|Z = i$, and we get

$$H(A|X, Z) + H(A|Y, Z) \leq H(A|Z).$$

It follows that

$$H(A|X) - H(Z) + H(A|Y) - H(Z) \leq H(A).$$

Thus, we obtain $t \geq 2^H(Z) \geq 2^{\frac{1}{2}H(A|X)+H(A|Y)-H(A)}$. Combining this bound with (17) and (18) we get

$$t^2 \geq \frac{n}{2(\log 2k)^2}.$$

If $n \gg k$, the RHS of this inequality is close to $\frac{1}{2}(2k)^2$, and $\log t$ cannot be much less than $\frac{1}{2}\log \frac{2k}{k} \approx k$. Thus, we proved the lower bound $\Omega(k)$ for non-deterministic communication complexity of the predicate Disjointness on $k$-elements subsets of $\{1, \ldots, n\}$ (for $n \gg k$).

Of course, this bound in itself is of no interest; the simple and standard fooling set technique proves for this graph a better bound $\text{bcc}(G) \geq \log \frac{1}{2}(2k)^2$. However, the simple example shown above illustrates the connection between biclique covering and conditional information inequalities. Perhaps a similar technique can imply stronger bounds in less trivial examples.

D. Comparison of the technique from Appendix C with the fooling sets method

The standard technique of lower bounding of biclique covering number is so-called fooling set method. The fooling sets method uses the following property of the graph $G_{n, k}$ (defined in Appendix C).

(*) for every biclique $C$ and every color $a$ there is at most one edge $(x, y) \in C$ with color $a$. (Indeed, if a $2k$-element set $a$ is split into disjoint $k$-element sets in two different ways, $x \cup y = a$ and $x' \cup y' = a$, then either $x$ intersects $y'$ or $x'$ intersects $y$.)

Using this property we may fix a color $a$ (a $2k$-element set) and conclude that every biclique has at most one edge of color $a$. Hence the number of bicliques in any covering must be at least the number of edges with color $a$, which equals $\binom{2k}{k}$.

In the argument in Appendix C we bounded $\text{bcc}(G_{n, k})$ using not (*) but the following (weaker) combinatorial property (**) for every biclique $C$ and for every triple of edges $(x, y)$, $(x', y)$, and $(x, y')$, if the latter two edges have the same color $a$, then the color of the edge $(x, y)$ also equals $a$.

The following theorem provides a possible formalization of our method.

Theorem 3. Let $G = (V_1, V_2, E)$ be an edge-colored bipartite graph with some probability distribution on its edges. Assume that $G$ has the above property (**). Denote by $(X, Y, A)$ the random variables corresponding to the distribution on the set of edges, where

- $X = \text{the left end of the edge}$,
- $Y = \text{the right end of the edge}$,
- $A = \text{the color of the edge}$.

Then $\text{bcc}(G) \geq 2^H(A|X)+H(A|Y)-H(A))$.

Remark 9. We would like to notice that using the property (*) we can obtain the bound $\text{bcc}(G_{n, k}) \geq \log \frac{1}{2}(2k)^2$ (without factor $\frac{1}{2}$ in the exponent) also by our method. Indeed let $C_1, \ldots, C_t$ be a biclique covering of $G_{n, k}$. Consider the conditional inequality

$$H(A|X, Y) = 0 \Rightarrow H(A) \leq H(X, Y).$$

By property (*), for every subset $C$ of every biclique $C_i$ we have $H(A|X, Y, (X, Y) \in C) = 0$, where $(A, X, Y)$ are random variables defined in Theorem 3 for $G = G_{n, k}$.

Hence, $H(A|Z) \leq H(X, Y|Z)$ where $Z$ denotes the number $i$ of a biclique $C_i$ that contains the edge $(X, Y)$ (if there are several such $i$ then pick any of them). Therefore $H(A|Z) \leq H(X, Y|Z)$ and hence $H(A) - H(Z) \leq H(X, Y)$. On the other hand $H(A) = \log \frac{n}{2k}$ and $H(X, Y) = 2\log \left(\frac{n}{k}\right)$ and therefore

$$\log t \geq H(Z) \geq 2\log \left(\frac{n}{k}\right) - \log \frac{n}{2k} = \log \left(\frac{2k}{k}\right).$$

\footnote{Actually in this case $x = x'$ and $y = y'$, but our goal was to formulate a most general property of a graph making our technique applicable.}