A general update rule for Lyapunov-based adaptive control of mobile robots with wheel slip

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Summary
In this article, we introduce a novel family of Lyapunov-based adaptive kinematic control laws developed to solve the trajectory tracking problem for a differential-drive mobile robot under the influence of both longitudinal and lateral wheel slip. Each adaptive controller in this family is constructed by augmenting a nonadaptive nominal controller, originally designed for the slip-free case, with an update rule capable of estimating the longitudinal slip. In the absence of lateral slip and under constant longitudinal slip, we establish the convergence of the trajectory tracking error to zero and, assuming a persistent excitation condition, we also demonstrate the convergence of the slip estimate error to zero. When lateral slip is present, we analyse a particular control law from our family of adaptive controllers. This law ensures the trajectory tracking error is uniformly ultimately bounded around the origin, demonstrating the robustness of our adaptive control scheme in dealing with time-varying longitudinal and lateral slip. The validity of our approach is assessed through comprehensive numerical simulations.

KEYWORDS
adaptive control, kinematic models, mobile robots, nonlinear control, trajectory tracking problem, wheel slip

1 | INTRODUCTION

In recent years, control design techniques for mobile robots have gained significant attention due to their growing importance in various socioeconomic activities. These areas include forestry, mining, agriculture, search and rescue, medicine, housework, entertainment, industry, and space exploration.1,2 The successful deployment of mobile robots in these domains requires effective solutions to the robot motion control problem, a task that presents interesting theoretical challenges. Among the primary motion control problems for wheeled mobile robots explored in the literature are point stabilization,3 path following,4 and trajectory tracking,5 all of which have been extensively investigated. This article focuses on the latter issue of trajectory tracking, specifically the stabilization of the robot's state around a time-varying reference trajectory.

A significant challenge in this context is trajectory tracking for nonholonomic differential-drive mobile robots subject to wheel slip. The ideal scenario, in which wheel slip is absent, is a classical problem for which numerous control laws have been proposed.6,7 In particular, Lyapunov-based kinematic controller designs for reference tracking have successfully addressed the motion problem in the absence of wheel slip.8,9 However, the performance of such controllers significantly deteriorates whenever wheel slip occurs, leading in many cases to system instability. One viable solution to this issue is
to explicitly incorporate the slip effects into the controller design. Controller design methods for differential-drive mobile robots are typically based on either kinetic or kinematic models. In each of these modeling paradigms, wheel slip is treated differently. In kinetic models, which relate robot accelerations to wheel torques, modeling the slip phenomenon is generally complex, accounting for factors such as wheel temperature, thread pattern, and camber angle. A simpler approach that avoids the complexities of robot kinetics is to work directly with kinematic models, which relate robot planar velocities to wheel angular speeds.

In kinematic models, slip is considered as an unknown parameter that needs to be estimated. Following this approach, researchers have proposed controllers aimed at solving the trajectory tracking problem by estimating slip parameters. This typically involves designing a controller that directly uses the slip estimates obtained from an estimation algorithm. In these cases, as tracking and estimation problems are addressed separately, closed-loop stability analysis is usually not pursued. Instead, performance and stability are assessed experimentally. For estimation, previous studies have employed variants of the Kalman filter and observer-based methods. An alternative approach is to treat slip as a disturbance in the robot velocities. As a pioneering result, Reference extended the stability analysis from Reference to ensure trajectory tracking for a time-varying reference input.

This work proposes an adaptive control design method to address longitudinal and lateral wheel slip while guaranteeing closed-loop stability and robustness. Rather than focusing on designing a single controller, our approach introduces a family of Lyapunov-based adaptive controllers. This is achieved through an adaptive augmentation technique that transforms any nominal nonadaptive controller (designed for the slip-free problem) into an adaptive controller capable of addressing wheel slip. More specifically, our main result demonstrates that any nonadaptive Lyapunov-based control law designed to solve the trajectory tracking problem without wheel slip can, under mild conditions, be transformed into an adaptive control law that effectively solves the tracking problem when the robot wheels experience longitudinal slip. It is important to note that these adaptive control laws guarantee asymptotic trajectory tracking for a time-varying reference input when longitudinal slip is constant and lateral slip is zero. Furthermore, it is shown that a persistent excitation (PE) condition on the reference trajectory velocities ensures that these control laws can consistently estimate the constant longitudinal slip parameters affecting the wheels. This constructive adaptive augmentation procedure can be readily applied to existing nonadaptive control laws found in the literature.

Our second major contribution examines the properties of the proposed adaptive control laws under simultaneous time-varying longitudinal and lateral wheel slip. By applying our adaptive augmentation scheme to a nonadaptive controller found in the literature, we demonstrate that the trajectory tracking error of the resulting adaptive closed-loop system around the origin is a uniformly ultimately bounded solution, provided that longitudinal slip is slowly varying and that lateral slip is sufficiently small. Hence, this work contributes to the literature on Lyapunov-based controllers for wheeled mobile robots by (i) providing a general adaptive augmentation scheme for nonadaptive controllers that deal with wheel slip, based on the fact that the tracking error dynamics of a differential-drive mobile robot is affine in the longitudinal wheel slip parameter; (ii) ensuring asymptotic trajectory tracking and slip estimation for constant longitudinal slip; (iii) proving the robustness of the control scheme for time-varying longitudinal and lateral slip; (iv) providing a comprehensive analysis of the closed-loop stability properties of a smooth adaptive kinematic control method for differential-drive robots. To illustrate these contributions, we provide numerical simulations that compare a nonadaptive controller with its adaptive augmentation using our scheme.

This article is organized as follows. Section 2 presents the kinematic differential equations describing the motion of a wheeled mobile robot with longitudinal and lateral wheel slip. Section 3 introduces the family of adaptive tracking controllers capable of compensating for slip and provides proof of closed-loop stability. Section 4 presents numerical results obtained using the proposed adaptive controllers.

2 KINEMATIC MODEL OF A WHEELED MOBILE ROBOT

This section presents the kinematic model of the wheeled mobile robot, as depicted in Figure 1, considering both longitudinal and lateral slip. It is assumed that the robot can be represented by a rigid body with two independently actuated wheels. The robot’s state is described by its position \((x, y)\) and its orientation \(\theta\) in an inertial coordinate frame \(F_0(x_0, y_0)\). The robot’s position corresponds to the coordinate of its geometric center \(O\), which also serves as the origin for the local coordinate frame \(F_1(x_1, y_1)\). The robot’s orientation is defined as the angle between the \(x_0\)-axis and the \(x_1\)-axis. The distance between the centerlines of the two wheels is denoted as \(b > 0\). The robot’s motion is described by its rotational velocity \(\omega = \dot{\theta}\) and its translational velocity \(v\). The translational velocity can be decomposed into the forward velocity \(v_x\)
along the $x_1$-axis and the lateral velocity $v_y$ along the $y_1$-axis. The angle between $v$ and $v_x$ is $\alpha$, which is zero in the absence of lateral slip. The lateral wheel slip is commonly defined\textsuperscript{12,20} as

$$\sigma(t) = \tan \alpha(t) = \frac{v_y(t)}{v_x(t)},$$

where $\sigma(t)$ is referred to as the lateral slip parameter.

Considering the presence of lateral wheel slip, the kinematic model of the wheeled mobile robot can be described as

$$\dot{q}(t) = S_\sigma(q)\eta(t),$$

where the robot state $q(t)$ and velocity vector $\eta(t)$ are given by

$$q(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}, \quad \eta(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \\ \omega(t) \end{bmatrix},$$

and the matrix $S_\sigma(q(t)) = S_\sigma(q_1(t), q_2(t), q_3(t))$ is given by

$$S_\sigma(q(t)) = \begin{bmatrix} \cos q_3(t) - \sigma(t) \sin q_3(t) & 0 \\ \sin q_3(t) + \sigma(t) \cos q_3(t) & 0 \\ 0 & 1 \end{bmatrix},$$

where the third element of $q(t)$ is $q_3(t) = \theta(t)$.

While the robot velocities $v_x(t)$ and $\omega(t)$ are commonly used for control design, in practice, it is more realistic to define the control input in terms of the angular velocities $\omega_l(t)$ and $\omega_r(t)$ of the left and right wheels, respectively. To establish a relationship between these two types of velocities, the terms $v_x(t)$ and $\omega(t)$ are first expressed as follows:

$$v_x(t) = \frac{v_l(t) + v_r(t)}{2} \quad \text{and} \quad \omega(t) = \frac{v_r(t) - v_l(t)}{b},$$

where $v_l(t)$ and $v_r(t)$ are the linear velocities of the left and right wheels, respectively.

When considering longitudinal wheel slip, the linear velocities $v_l(t)$ and $v_r(t)$ are given by

$$v_l(t) = \rho \frac{\omega_l(t)}{a_l(t)}, \quad 1 \leq a_l(t),$$

$$v_r(t) = \rho \frac{\omega_r(t)}{a_r(t)}, \quad 1 \leq a_r(t),$$

where $\rho$ is a positive constant.
where \( \rho > 0 \) represents the radius of the wheel, and \( a_l(t) \) and \( a_r(t) \) denote the longitudinal slip parameters of the left and right wheels. A longitudinal slip parameter of unity implies pure rolling, while an infinite slip parameter indicates pure longitudinal slipping without rolling.

By collecting the wheel angular velocities in the vector

\[
\xi(t) = \begin{bmatrix} \omega_l(t) \\ \omega_r(t) \end{bmatrix}
\]

and substituting (5) into (4), we obtain

\[
\eta(t) = \Theta A^{-1}(t) \xi(t),
\]

(6)

where the matrices \( \Theta \) and \( A(t) \) are defined as

\[
\Theta = \frac{\rho}{2b} \begin{bmatrix} b & b \\ -2 & 2 \end{bmatrix}, \quad A(t) = \text{diag}(a(t)) = \begin{bmatrix} a_l(t) & 0 \\ 0 & a_r(t) \end{bmatrix}, \quad a(t) = \begin{bmatrix} a_l(t) \\ a_r(t) \end{bmatrix}.
\]

(7)

3 | ADAPTIVE KINEMATIC CONTROL LAWS

This section introduces a family of adaptive kinematic control laws to solve the trajectory tracking problem for the robot depicted in Figure 1 under longitudinal and lateral wheel slip. Section 3.1 formulates the trajectory tracking control problem in all generality. Section 3.2 demonstrates how this problem is exactly solved in the simplified slip-free case, where wheel slip is absent. Section 3.3 introduces a family of adaptive control laws tailored to the general case with both lateral and longitudinal wheel slip. Additionally, this section derives the closed-loop tracking error dynamics in a form suitable for stability analysis. The family of control laws is constructed using nonadaptive control laws that solve the slip-free problem. Section 3.4 shows that any member of this family can solve exactly the trajectory tracking problem in the case where longitudinal slip is constant and lateral slip is zero. Finally, Section 3.5 concentrates on a particular adaptive control law within this family, showing that it effectively addresses the trajectory tracking problem under time-varying longitudinal and lateral slip, ensuring that the tracking error remains uniformly ultimately bounded around the origin.

3.1 | Trajectory tracking problem

The goal of the trajectory tracking control problem is to provide an input to the robot so that the robot state \( q(t) \) tracks a given reference trajectory \( q_{\text{ref}}(t) \). The reference trajectory is generated from a reference velocity vector \( \eta_{\text{ref}}(t) \) using the kinematic model

\[
\dot{q}_{\text{ref}}(t) = S_0(q_{\text{ref}})\eta_{\text{ref}}(t),
\]

(8)

where

\[
q_{\text{ref}}(t) = \begin{bmatrix} x_{\text{ref}}(t) \\ y_{\text{ref}}(t) \\ \theta_{\text{ref}}(t) \end{bmatrix}, \quad \eta_{\text{ref}}(t) = \begin{bmatrix} v_{\text{ref}}(t) \\ \omega_{\text{ref}}(t) \end{bmatrix},
\]

and \( S_0(\cdot) \) is the matrix given by (3) with \( \sigma = 0 \). Thus, the reference trajectory is generated using a model that neglects slip. Formally, the trajectory tracking problem is solved when

\[
\lim_{t \to \infty}(q_{\text{ref}}(t) - q(t)) = 0
\]

(9)

is satisfied. To design a controller to solve this problem, the tracking error

\[
e(t) = \begin{bmatrix} e_x(t) \\ e_y(t) \\ e_\theta(t) \end{bmatrix}^T
\]
is defined in the local coordinate frame as
\[ e(t) = R^T(\theta(t))(q_{ref}(t) - q(t)), \] (10)
where
\[ R(\theta(t)) = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 \\ \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
is the orthogonal rotation matrix. Clearly, the robot state \( q(t) \) converges to the reference trajectory \( q_{ref}(t) \) whenever the tracking error \( e(t) \) converges to zero. The behavior of \( e(t) \) can be studied by analyzing the tracking error dynamics, which is obtained in terms of the velocity vectors \( \eta(t) \) and \( \eta_{ref}(t) \) by differentiating (10) and using the models (2) and (8). Following the derivation in Appendix A.1, the tracking error dynamics is given by
\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_\sigma(e)\eta(t), \] (11)
where the matrix \( M_\sigma(e(t)) \) is defined as
\[ M_\sigma(e(t)) = \begin{bmatrix} -1 & e_y(t) \\ -\sigma(t) & -e_x(t) \\ 0 & -1 \end{bmatrix}. \] (12)

When longitudinal slip is present, the robot velocity vector \( \eta(t) \) is given by (6). Substituting (6) into (11), one obtains
\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_\sigma(e)\Theta A^{-1}(t)\xi(t) \] (13)
which describes the tracking error dynamics in terms of the reference velocity vector \( \eta_{ref}(t) \), the longitudinal slip parameters \( a_l(t) \) and \( a_r(t) \), the lateral slip parameter \( \sigma(t) \), and the wheel velocity vector \( \xi(t) \). The latter is the control input to the system.

The trajectory tracking control design problem essentially involves the task of designing a control law for \( \xi(t) \) so that \( e(t) \to 0 \) as \( t \to \infty \). When both longitudinal and lateral slip are time-varying, and no information is available about the process that generates the slip parameters in (13), solving the design problem exactly becomes challenging, if not impossible. However, the following sections present a general adaptive controller design for \( \xi(t) \) that: i) guarantees \( e(t) \to 0 \) when the longitudinal slip is constant and the lateral slip is zero, and ii) ensures \( e(t) \) converges to a neighborhood of the origin when either longitudinal or lateral slip is time-varying. To introduce this design, the simplified slip-free problem is reviewed first.

### 3.2 The slip-free problem

This section reviews how the trajectory tracking problem is solved without longitudinal and lateral slip, in which case a control law for \( \xi(t) \) can always be found such that \( e(t) \to 0 \). In the absence of lateral slip (\( \sigma = 0 \)) and in the absence of longitudinal slip, one has \( A(t) = I \), where \( I \) represents the identity matrix. Thus it follows from (13) that
\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_0(e)\theta(t). \] (14)
Since \( \Theta \) is an invertible matrix with known constants, related to the robot dimensions, the trajectory tracking problem can be tackled using the control law for \( \xi(t) \) given by
\[ \xi(t, e) = \Theta^{-1}\eta_e(t, e), \] (15)
where \( \eta_e(t) \) is an auxiliary control law. Setting \( \xi(t) = \xi_e(t, e) \) in (14), the trajectory tracking problem is solved when the closed-loop error dynamics
\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_0(e)\eta_e(t, e) \] (16)
has a uniformly asymptotically stable origin.
Notice that the auxiliary control law \( \eta_c(t, e) \) can also be interpreted as a control law for the robot velocity vector \( \eta(t) \), and thus its components can be denoted accordingly as

\[
\eta_c(t, e) = \begin{bmatrix} v_c(t, e) \\ \omega_c(t, e) \end{bmatrix}.
\]  

(17)

With this interpretation, the slip-free trajectory control problem has known solutions in the literature. Of particular interest to this work are Lyapunov-based solutions satisfying the next assumption.

**Assumption 1.** Assume the reference velocity vector \( \eta_{ref}(t) \) is piecewise continuous and bounded. Assume that \( v_{ref}(t) \geq \mu \) for some \( \mu > 0 \). Define \( D \subset \mathbb{R}^3 \) as a domain containing the origin. The control law \( \eta_c(t, e) \), given by (17), is assumed to satisfy the following properties:

1. The function \( \eta_c(t, e) \) is piecewise continuous in \( t \) and locally Lipschitz in \( e \), uniformly in \( t \).
2. For all \( t \geq 0 \), it holds that \( \eta_c(t, 0) = \eta_{ref}(t) \). Furthermore, \( \eta_c(t, e) \) converges to \( \eta_{ref}(t) \) as \( e \to 0 \).
3. There exists a positive definite differentiable function \( V : D \to \mathbb{R} \), having locally Lipschitz partial derivatives, such that for all \( t \geq 0 \) and all \( e \in D \), the following inequality is satisfied:

\[
\frac{\partial V(e)}{\partial e} (S_0(e)\eta_{ref}(t) + M_0(e)\eta_c(t, e)) \leq -W(e),
\]  

(18)

where \( W(e) \) is a continuous, positive definite function defined on \( D \).

It is straightforward to verify that for any \( \eta_c(t, e) \) satisfying Assumption 1, the origin of the closed-loop error dynamics (16) for the slip-free problem is a uniformly asymptotically stable equilibrium point (see Theorem 4.9 in Reference 21). In particular, condition 2 ensures the origin \( e = 0 \) is an equilibrium point, since

\[
S_0(0)\eta_{ref}(t) + M_0(0)\eta_c(t, 0) = 0
\]  

(19)

while condition 3 ensures that \( V(e) \) is negative definite along the trajectories of (16).

Control laws satisfying Assumption 1 have been previously published in the literature. An example is the following commonly used control law first proposed in Reference 8:

**Example 1.** Let \( \eta_{ref}(t) = [v_{ref}(t) \quad \omega_{ref}(t)]^T \) be a piecewise continuous and bounded reference velocity vector. Then, the control law \( \eta_c(t, e) = [v_c(t, e) \quad \omega_c(t, e)]^T \) with

\[
v_c(t, e) = v_{ref}(t) \cos e_\theta - k_2 e_\theta \omega_c(t, e) + k_1 e_x,
\]

\[
\omega_c(t, e) = \omega_{ref}(t) + \frac{v_{ref}(t)}{2} \left( k_2 (e_y + k_3 e_\theta) + \frac{1}{k_3} \sin e_\theta \right), \quad k_i > 0
\]  

(20)

satisfies Assumption 1. To see this, note that \( \eta_c(t, e) \) is piecewise continuous in \( t \) and locally Lipschitz in \( e \). Moreover, it can be verified by inspection that \( v_c(t, 0) = v_{ref}(t) \) and \( \omega_c(t, 0) = \omega_{ref}(t) \). Thus Conditions 1 and 2 are satisfied. Finally, the Lyapunov function

\[
V(e) = \frac{1}{2} e_x^2 + \frac{1}{2} (e_y + k_3 e_\theta)^2 + \frac{(1 - \cos e_\theta)}{k_2}
\]  

(21)

is positive definite on \( D = \{ e \in \mathbb{R}^3 \mid -\pi < e_\theta < \pi \} \), and its derivative

\[
V(t, e) = \frac{\partial V(e)}{\partial e} (S_0(e)\eta_{ref}(t) + M_0(e)\eta_c(t, e))
\]

\[
= -k_1 e_x^2 - \frac{v_{ref}(t)}{2} k_2 k_3 (e_y + k_3 e_\theta)^2 - \frac{v_{ref}(t)}{2k_3} \sin^2 e_\theta
\]

\[
\leq -W(e), \quad W(e) = k_1 e_x^2 + \mu k_2 k_3 (e_y + k_3 e_\theta)^2 + \frac{\mu}{2} k_3 \sin^2 e_\theta
\]

is negative definite on \( D \) for \( v_{ref}(t) \geq \mu > 0 \).
An obvious consequence of the results in this section, which has not been emphasized in the literature, is that any kinematic control law \( \eta_c(t, e) \) that stabilizes the origin of the closed-loop error dynamics (16) for the slip-free problem, can also be used to stabilize the origin of the error dynamics (13) without lateral slip, as long as the longitudinal slip is precisely known, that is, \( A(t) \) is known. In this case, it suffices to select the control law (15) as \( \xi_c(t, e) = A(t)^{-1} \eta_c(t, e) \). However, when the longitudinal slip is unknown or lateral slip occurs, a more involved solution is required as shown in the following sections.

3.3 A family of adaptive control laws for trajectory tracking under wheel slip

This section introduces a family of adaptive control laws to solve the trajectory tracking problem with longitudinal and lateral wheel slip. This family is constructed by exploiting any nonadaptive control laws satisfying Assumption 1, that is, any control law designed to solve the trajectory tracking problem in the slip-free case of Section 3.2. For that purpose, the nonadaptive control laws are augmented with a general update rule to estimate the longitudinal slip.

Let \( \eta_c(t, e) \) in (17) be a control law satisfying Assumption 1, and let

\[
\hat{\alpha}(t) = \begin{bmatrix} \hat{\alpha}_l(t) \\ \hat{\alpha}_r(t) \end{bmatrix}, \quad \hat{A}(t) = \text{diag}(\hat{\alpha}(t)),
\]

denote an estimate of the longitudinal slip parameters and a diagonal matrix containing those estimates, respectively. Then, an adaptive control law for \( \xi(t) \) in (13) is defined by

\[
\xi_c(t, e) = \hat{A}(t)\Theta^{-1} \eta_c(t, e)
\]

(22a)

with the update rule

\[
\dot{\hat{\alpha}}(t) = -\Gamma \Psi^T(t, e) M_0^T \frac{\partial V(e)}{\partial e} \Psi(t, e) \quad \text{dV(e)}
\]

(22b)

in which \( V(e) \) is the Lyapunov function from Assumption 1 associated with \( \eta_c(t, e) \). In the update rule above, \( \Gamma \) is the constant diagonal positive definite matrix

\[
\Gamma = \text{diag}(\gamma_1, \gamma_2) > 0
\]

(23)

and \( \Psi(t, e) \) is the matrix given by

\[
\Psi(t, e) = \frac{1}{4b} \begin{bmatrix} b \kappa_-(t, e) & b \kappa_+(t, e) \\ -2\kappa_-(t, e) & 2\kappa_+(t, e) \end{bmatrix}, \quad \kappa_+(t, e) = 2\nu_c(t, e) + b\omega_c(t, e), \quad \kappa_-(t, e) = 2\nu_c(t, e) - b\omega_c(t, e).
\]

(24)

It is important to note that (22) defines a family of adaptive control laws since \( \eta_c(t, e) \) can be any auxiliary (nonadaptive) control law that satisfies Assumption 1. In other words, each auxiliary control law \( \eta_c(t, e) \), along with its associated Lyapunov function \( V(e) \), defines an adaptive control law of the form (22).

3.3.1 Closed-loop error dynamics

When the control input \( \xi(t) \) of the error dynamics (13) is given by the adaptive control law input \( \xi_c(t, e) \) from (22), the result is a closed-loop control system representing the dynamics of the robot in the presence of longitudinal and lateral slip. Figure 2 shows a detailed schematic representation of this closed-loop system. The numbering inside the blocks indicates the corresponding equation number.

The following sections analyze the properties of this closed-loop system in detail. For that purpose, it is useful to define the estimation error

\[
\hat{a}(t) = \hat{a}(t) - a(t)
\]

(25)
which is appended to the tracking error $e(t)$ to form the following augmented error state vector:

$$e_a(t) = \begin{bmatrix} e(t) \\ \dot{a}(t) \end{bmatrix}.$$

The following Proposition 1 is an important auxiliary result that allows us to write the augmented error dynamics in a form suitable for stability analysis.

**Proposition 1.** Consider the system described by (13). When the adaptive control law input is defined as $\xi(t) = \xi_c(t, e)$, given by (22), the closed-loop error dynamics can be succinctly expressed as:

$$\dot{e}(t) = S_0(e)\eta_{rel}(t) + M_0(e)\eta_c(t, e) + M_0(e)\Psi(t, e)A^{-1}(t)\ddot{a}(t)$$  \hfill (27a)

$$\dot{\dot{a}}(t) = -\Gamma\Psi^T(t, e)M_0^T(e)\frac{\partial V(e)}{\partial e} - \ddot{a}.$$  \hfill (27b)

**Proof.** See Appendix A.2.

The next sections analyze the properties of the closed-loop error dynamics under two different sets of assumptions on the slip parameters.

### 3.4 Asymptotic stability analysis under constant longitudinal slip and zero lateral slip

This section shows that any adaptive control law from the family (22) can solve the trajectory tracking problem in the particular case where the longitudinal slip parameters $a_l$ and $a_r$ are unknown, but constant, and the lateral slip parameter is zero, $\sigma = 0$.

In this scenario, as $A(t) = A$ is now constant, and thus $a(t) = a$ and $\dot{a} = 0$, it follows from Proposition 1 that the augmented error dynamics becomes

$$\dot{e}(t) = S_0(e)\eta_{rel}(t) + M_0(e)\eta_c(t, e) + M_0(e)\Psi(t, e)A^{-1}(t)\ddot{a}(t).$$  \hfill (28a)

$$\dot{\dot{a}}(t) = -\Gamma\Psi^T(t, e)M_0^T(e)\frac{\partial V(e)}{\partial e}. \hfill (28b)$$

It is worth emphasizing that the above dynamics (28) shows that, under these particular conditions when the lateral slip is zero and the longitudinal slip is constant, the adaptive control law (22) follows the certainty equivalence principle."
This is because from (25), making \( \dot{a}(t) = a(t) \) would result in \( \dot{a}(t) = 0 \), in which case (28a) becomes (16), whose origin is uniformly asymptotically stabilized by our assumption on \( \eta_c(t,e) \).

The following Theorem 1 proves that the family of control laws (22) solves the trajectory tracking problem under constant longitudinal slip and zero lateral slip.

**Theorem 1.** Let Assumption 1 hold. Then, all solutions \( e_a(t) \) of the closed-loop error dynamics (28) starting sufficiently close to the origin \( e_a = 0 \) are bounded, and satisfy \( e(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let us denote the augmented error dynamics (28) as

\[
\dot{e}_a(t) = f(t, e_a)
\]

with \( e_a(t) \) given by (26) and \( f(t, e_a) \) given by the right-hand side of (28). The analysis relies on the classical result of Theorem 8.4 from Reference 21. To apply that result, it is first required that \( f(t, 0) = 0 \) holds from (19) and from the fact that \( \partial V(e)/\partial e|_{e=0} = 0 \). Second, it is required that for some \( D_a \subset \mathbb{R}^5 \) containing the origin, there exists a positive definite Lyapunov function \( V_a : D_a \to \mathbb{R} \) such that \( V_a(t, e_a) \leq -W_a(e_a) \) for some positive semidefinite function \( W_a(e_a) \) on \( D_a \). For that purpose, consider the Lyapunov function

\[
V_a(e_a) = V(e) + \frac{1}{2} \dot{a}^T \Gamma^{-1} A^{-1} \dot{a},
\]

where \( V : D \subset \mathbb{R}^3 \to \mathbb{R} \) is taken from Assumption 1. Defining \( D_a = D \times \mathbb{R}^2 \), it follows from Assumption 1 and the fact that \( \Gamma^{-1} A^{-1} \) is positive definite, that \( V_a \) is positive definite on \( D_a \). Now, from (28a) and the fact that \( \Gamma^{-1} A^{-1} \) is diagonal, one has

\[
\dot{V}_a(t, e_a) = \frac{\partial V(e)}{\partial e} \left( S_0(t) \eta_{ref}(t) + M_0(t) \eta_\delta(t,e) + M_0(t) \Psi(t,e) A^{-1} \dot{a}(t) \right) + \dot{a}^T(t) \Gamma^{-1} A^{-1} \dot{a}(t).
\]

Using (18) from Assumption 1, the following inequality is obtained

\[
\dot{V}_a(t, e_a) \leq -W_a(e_a) + \frac{\partial V(e)}{\partial e} M_0(t) \Psi(t,e) A^{-1} \dot{a}(t) + \dot{a}^T(t) \Gamma^{-1} A^{-1} \dot{a}(t).
\]

Finally, using (28b) it follows that

\[
V_a(t, e_a) \leq -W_a(e_a), \quad W_a(e_a) = W(e).
\]

Therefore, \( V_a(t, e_a) \) is bounded above by \( -W_a(e_a) \), with \( W_a(e_a) \) a positive semidefinite function on \( D_a \). Thus, all conditions of Theorem 8.4 in Reference 21 are satisfied, which implies that all solutions of (29) starting sufficiently close to the origin \( e_a = 0 \) are bounded and satisfy \( W_a(e_a) = W(e) \to 0 \) as \( t \to \infty \). Furthermore, given that \( W(e) \) is positive definite on \( D \), it follows that \( e(t) \) approaches zero as \( t \to \infty \).

3.4.1 Convergence of the longitudinal slip estimate error

This section establishes conditions under which the slip estimation errors \( \hat{a}_l(t) \) and \( \hat{a}_s(t) \), as defined by (28), converge to zero as \( t \to \infty \). The concept of persistent excitation, crucial for ensuring the convergence of these parameter estimates, is presented in the following definition.

**Definition 1** (Definition 2.2 in Reference 24). A matrix function \( Q(t) \in \mathbb{R}^{n \times n} \) is said to be **persistently exciting** (PE) if there exist \( \bar{t} > 0 \), \( \rho_1 > 0 \), and \( \rho_2 > 0 \) such that for all \( t \geq 0 \),

\[
\rho_1 t \leq \int_t^{t+\bar{t}} Q(\tau) Q^T(\tau) \, d\tau \leq \rho_2 I.
\]

The following Lemma 1 is also presented as an important auxiliary result.
Lemma 1. Let $\eta_{\text{ref}}(t)$ and $\eta_e(t, e(t))$ be bounded functions for all $t \geq 0$. Assume that $\eta_e(t, e(t))$ converges to $\eta_{\text{ref}}(t)$ as $t \to \infty$. Define $\xi(t) = 2v_{\text{ref}}(t) \pm b\omega_{\text{ref}}(t)$. Under these conditions, if both $\xi(t)$ and $\zeta(t)$ are persistently exciting (PE), then the matrix $\Psi^T(t, e(t))$ is also PE.

Proof. The proof is provided in Appendix A.3.

The convergence of the slip estimates can now be stated. It relies on a Lemma from Reference 24, which is reproduced as Lemma 2 in Appendix B.

Theorem 2. Given that Assumption 1 holds, and assuming further that:

1. $\eta_{\text{ref}}(t)$ is bounded and differentiable with a bounded derivative.
2. $\eta_e(t, e)$ is bounded for all $t \geq 0$ and for all bounded $e$.
3. For all $t \geq 0$ and for all bounded $e$, the partial derivatives of $\eta_e(t, e(t))$ with respect to $t$ and $e$ are bounded. *
4. The signals $\xi(t) = 2v_{\text{ref}}(t) \pm b\omega_{\text{ref}}(t)$ are both persistently exciting.

Then, all solutions $e_o(t)$ of the closed-loop error dynamics (28) originating sufficiently close to the origin are bounded. Furthermore, these solutions satisfy $e(t) \to 0$ and $\dot{\alpha}(t) \to 0$ as $t \to \infty$.

Proof. We prove this result by applying Lemma 2 from Appendix B, which ensures that $\dot{\alpha}(t) \to 0$ as $t \to \infty$, in addition to the result already established by Theorem 1. To begin, we rewrite (28) as

$$\dot{e}(t) = h_1(t) + G(t)\dot{\alpha}(t),$$
$$\dot{\alpha}(t) = h_2(t),$$

(30)

where $h_1(t)$ is given by

$$h_1(t) = S_0(e(t))\eta_{\text{ref}}(t) + M_0(e(t))\eta_e(t, e(t)) + (M_0(e(t)) - M_0(0))(\Psi(t, e(t))A^{-1}\dot{\alpha}(t)),$$

(31)

$h_2(t)$ is defined by the right-hand side of (28b), specifically:

$$h_2(t) = -\Gamma\Psi^T(t, e(t))M_0^T(e(t))\frac{\partial V(e(t))}{\partial e}^T$$

and $G(t)$ is given by

$$G(t) = M_0(0)\Psi(t, e(t))A^{-1}.$$  

(32)

Since (30) is in the form (B1) with $z_1(t) = e(t)$ and $z_2(t) = \dot{\alpha}(t)$, Lemma 2 can be applied. To verify the first condition of Lemma 2, we observe that for all solutions $e_o(t) = [e^T(t), \dot{\alpha}^T(t)]^T$ starting sufficiently close to the origin, Theorem 1 ensures that $e_o(t)$ is bounded and that $e(t) \to 0$ as $t \to \infty$. Consequently, $S_0(e(t)) \to S_0(0)$ and $M_0(e(t)) \to M_0(0)$, owing to the continuity of $S_0$ and $M_0$. By the hypotheses from the theorem, $\eta_e(t, e(t))$ remains bounded as long as $e(t)$ is bounded. Hence, it can be deduced that $\eta_e(t, e(t))$ and, by extension, $\Psi(t, e(t))$, as defined in (24), also remain bounded. Moreover, since $\eta_{\text{ref}}(t)$ is also bounded, it follows that $h_1(t)$ is bounded for all $t \geq 0$. Considering (19) and the observation that $(M_0(e(t)) - M_0(0)) \to 0$ as $t \to \infty$, this leads to the conclusion that $h_1(t) \to 0$ as $t \to \infty$. Furthermore, given Assumption 1, and the fact that $V(e)$ is a positive definite differentiable function with locally Lipschitz partial derivatives, one can deduce that $\partial V(e(t))/\partial e \to 0$ as $e(t) \to 0$. Given that both $M_0(e(t))$ and $\Psi(t, e(t))$ remain bounded for $t \geq 0$, it can be concluded that $h_2(t) \to 0$ as $t \to \infty$. Consequently, the first condition of Lemma 2 is satisfied. Moreover, since $\partial V(e(t))/\partial e$ is locally Lipschitz and $e(t)$ is bounded, $h_2(t)$ is also bounded.

To verify the second condition of Lemma 2, first notice that $G(t)$ is bounded since $\Psi(t, e(t))$ is bounded for all $t \geq 0$. Now, observe that due to the relation: first, between $\dot{G}(t)$ and $\dot{\Psi}(t, e(t))$; secondly, between $\dot{\Psi}(t, e(t))$ and $\dot{\xi}(t, e(t))$; and thirdly, between $\dot{\xi}(t, e(t))$ and $\dot{\eta}_e(t, e) = (\dot{v}_e(t, e), \dot{\omega}_e(t, e))^T$, the boundedness of $\dot{G}(t)$ is equivalent to the boundedness of $\dot{\eta}_e(t, e(t))$. It is noticed that $\dot{e}(t)$ in (30) is bounded, since it was already shown that $h_1(t)$,
G(t), and \( \dot{\gamma}(t) \) are bounded. Additionally, the theorem assumes that \( \partial \eta_c / \partial t \) and \( \partial \eta_c / \partial e \) are bounded for all \( t \geq 0 \) and for all bounded \( e \), thus \( \eta_c(t, e(t)) \) is bounded. Consequently, \( \dot{G}(t) \) is also bounded, verifying the second condition of Lemma 2.

Thus, to complete the application of Lemma 2, it only remains to show that \( G(t) \) is persistently exciting (PE). From (12) and (32), it follows that

\[
G(t)G(t) = A^{-1}\Psi^T(t, e(t))M_0^T(0)M_0(0)\Psi(t, e(t))A^{-1} = A^{-1}\Psi^T(t, e(t))\Psi(t, e(t))A^{-1}
\]

and, since \( A^{-1} \) is a constant positive definite matrix, it follows that \( G(t) \) is PE if and only if \( \Psi^T(t, e(t)) \) is PE. Since by Assumption 1, it follows that \( \eta_{ref}(t) \) is bounded for all \( t \geq 0 \), \( \eta_c(t, e(t)) \) converges to \( \eta_{ref}(t) \) as \( t \to \infty \), and both \( \zeta_{ex}(t) = 2\nu_{ref}(t) \pm b\nu_{ref}(t) \) are PE, Lemma 1 establishes that \( G(t) \) is also PE, thereby proving our result.

### 3.5 Ultimate boundedness analysis under time-varying longitudinal and lateral slip

This section addresses the reference tracking problem under time-varying longitudinal and lateral slip. The closed-loop error dynamics for this general case was already derived in Section 3.3.1 and is given by (27) from Proposition 1. This section analyzes a particular member of the family of adaptive control laws given by (22). This adaptive control law ensures that the trajectory tracking error \( e(t) \), from the closed-loop dynamics, converges to a bounded region containing the origin. This convergence occurs as long as the longitudinal and lateral slip parameters are slowly varying in time.

The particular adaptive control law \( \xi_c(t, e) \) considered in this section is derived from (22) using the auxiliary control law \( \eta_c(t, e) \) and the associated Lyapunov function \( V(e) \) from Example 1, which satisfy Assumption 1. The adaptive control law input, defined in (22a), is thus given by

\[
\xi_c(t, e) = \dot{A}(t)\Theta^{-1}\eta_c(t, e)
\]

with the auxiliary law \( \eta_c(t, e) = (v_c(t, e), \omega_c(t, e))^T \) given by (20). The adaptive control law update rule, defined in (22b), can be computed using \( V(e) \) in (21). In Appendix A.4, it is shown that this update rule is given by

\[
\dot{\gamma}_1(t) = \gamma_1\left(v_c(t, e) - \frac{b}{2}\omega_c(t, e)\right)\left(\frac{e_x}{2} - k_3\left( e_\theta(e_x + k_3) + e_y \right) - \frac{\sin e_\theta}{b}k_2 \right),
\]

\[
\dot{\gamma}_2(t) = \gamma_2\left(v_c(t, e) + \frac{b}{2}\omega_c(t, e)\right)\left(\frac{e_x}{2} + k_3\left( e_\theta(e_x + k_3) + e_y \right) + \frac{\sin e_\theta}{b}k_2 \right)
\]

with the auxiliary law \( \eta_c(t, e) = (v_c(t, e), \omega_c(t, e))^T \) given by (20).

It can be verified by inspection that the adaptive control law (33) is precisely the control law originally proposed in Reference 18. Hence, the family of control laws (22) and the convergence analysis in the previous sections constitute a major generalization of previous results by allowing different choices of auxiliary laws \( \eta_c(t, e) \) and by identifying the PE condition, which establish the sufficient conditions to ensure convergence of the longitudinal slip parameter estimates.

In particular, Theorem 1 shows that the adaptive control law (33) solves the trajectory tracking problem with constant longitudinal slip and zero lateral slip. Under those same conditions, Theorem 2 shows that (33) is also capable of asymptotically estimating the longitudinal slip as long as the PE assumption holds.

In the remainder of this section, we further extend our previous analyses that lead to the adaptive control law (33) above by focusing on the particular choices of \( \eta_c(t, e) \) and \( V(e) \) given by (20) and (21), respectively. In the case that either longitudinal or lateral slip is time-varying, we show that the tracking error \( e(t) \) converges to a small neighborhood of the origin as long as the longitudinal slip parameters vector \( a(t) \) and the reference velocity vector \( \eta_{ref}(t) \) are bounded and vary slowly in time, and as long as the lateral slip parameter \( \sigma(t) \) remains sufficiently small. To that end, we show that the augmented error \( e_a(t) \) in the closed-loop error dynamics (27) is uniformly ultimately bounded around the origin. Uniform ultimate boundedness means that \( e_a(t) \) reaches a neighborhood of the origin in finite time and stays in that neighborhood thereafter. As a result, both the reference trajectory \( q_{ref}(t) \) and the true slip parameters \( a(t) \) are approximately tracked by the robot state \( q(t) \) and the slip estimates \( \dot{a}(t) \), respectively. For that purpose, the next Assumption 2 is assumed.
**Assumption 2.** The parameters $b, k_1, k_2, k_3, \gamma_1,$ and $\gamma_2$ are all finite positive constants. Furthermore:

1. $v_{ref}(t) \geq \mu > 0$, for some $\mu > 0$.
2. $|2v_{ref}(t) + b\omega_{ref}(t)| \geq \psi > 0$, for some $\psi > 0$.
3. $v_{ref}(t), \omega_{ref}(t), \dot{a}_l(t), a_r(t),$ and $\sigma(t)$ are differentiable and bounded on $[0, \infty)$.
4. $v_{ref}(t), \omega_{ref}(t), \dot{a}_l(t), a_r(t)$ are bounded on $[0, \infty)$.

Notice that condition 2 in Assumption 2 is a stronger version of the PE condition of Theorem 2. The next theorem states the main result of this section.

**Theorem 3.** Let $\eta_\ell(t, e)$ be given by (20) and $V(e)$ be given by (21). Then the adaptive control law (22) is given by (33). Furthermore, under Assumption 2, for sufficiently small bounds on $v_{ref}(t), \omega_{ref}(t), \dot{a}_l(t), a_r(t),$ and $\sigma(t)$, and for a sufficiently small choice of $\gamma_1 > 0$ and $\gamma_2 > 0$, the solution $e_a(t)$ of the closed-loop error dynamics (27) is uniformly ultimately bounded, that is, for $e_a(t_0)$ sufficiently close to the origin, there exist positive constants $a$, $\beta$, $u_b$, and $T$ such that

$$\|e_a(t)\| \leq a \exp \left[ -\beta(t - t_0) \right] \|e_a(t_0)\|, \quad \forall t \leq t_0 + T$$

and

$$\|e_a(t)\| \leq u_b, \quad \forall t \geq t_0 + T.$$ 

**Proof.** The derivation of the adaptive control law, given by (33), is shown in Appendix A.4. Thus it remains to prove uniform ultimate boundedness of

$$e_a(t) = \left[ e^T(t) \quad \ddot{a}^T(t) \right]^T.$$ 

To prove this result, the closed-loop error dynamics (27) with $\eta_\ell(t, e)$ given by (20) and $V(e)$ given by (21) is rewritten in terms of a nominal system that is affected by a vanishing and a nonvanishing perturbation, as follows:

$$\dot{e}_a(t) = f_a(t, e_a) + g(t, e_a) + u_a(t, e_a)$$

(34)

with the nominal term $f_a(t, e_a)$ given by

$$f_a(t, e_a) = \begin{bmatrix}
v_{ref}(t) \cos e_\theta - \Omega_2 \Omega_4 \Delta_l + \Omega_1 \Omega_3 \Delta_r \\
v_{ref}(t) \sin e_\theta + e_\theta(\Omega_2 \Delta_l - \Omega_1 \Delta_r) / b \\
\omega_{ref}(t) + (\Omega_2 \Delta_l - \Omega_1 \Delta_r) / b \\
\gamma_1 \Omega_2 (bk_2 e_x - 2k_2k_3(e_\theta(e_x + k_3) + e_y) - 2 \sin e_\theta) / (2bk_2) \\
\gamma_2 \Omega_1 (bk_2 e_x + 2k_2k_3(e_\theta(e_x + k_3) + e_y) + 2 \sin e_\theta) / (2bk_2)
\end{bmatrix},$$

(35)

where

$$\Delta_r = \left( 1 + \frac{\ddot{a}_r}{a_r(t)} \right), \quad \Omega_1 = \kappa_+(t, e) / 2, \quad \Omega_3 = \left( \frac{e_y}{b} - \frac{1}{2} \right),$$

$$\Delta_l = \left( 1 + \frac{\ddot{a}_l}{a_l(t)} \right), \quad \Omega_2 = \kappa_-(t, e) / 2, \quad \Omega_4 = \left( \frac{e_y}{b} + \frac{1}{2} \right),$$

the vanishing perturbation term $g(t, e_a)$ given by

$$g(t, e_a) = \begin{bmatrix} -\sigma(t) \left( \frac{1}{2} \frac{\ddot{a}_r}{a_r(t)} \Omega_2 + \frac{1}{2} \frac{\ddot{a}_l}{a_l(t)} \Omega_3 - k_3 e_\theta \omega_\ell(t, e) + k_1 e_x \right) \\
0 \\
0 \\
0
\end{bmatrix},$$

and

$$u_a(t, e_a) = \begin{bmatrix} 0 \\
0 \\
0 \\
0
\end{bmatrix}.$$
and the nonvanishing perturbation term \( g_n(t, e_a) \) given by

\[
g_n(t, e_a) = \begin{bmatrix}
0 \\
-\sigma(t)v_{ref}(t) \cos e_\theta \\
0 \\
-\dot{a}_l(t) \\
-\dot{a}_r(t)
\end{bmatrix}
\]

with \( v_c(t, e) \) and \( \omega_c(t, e) \) given by (20), and \( \kappa_-(t, e) \) and \( \kappa_+(t, e) \) given by (24). Notice that since \( \dot{a} = \dot{\hat{a}} - \dot{a} \), the two last rows of the augmented error dynamics (34) follow directly from the particular control law (33). For a derivation of the first three rows of (34) related to \( \dot{e}(t) \), see Appendix A.5. Note that \( g(t, e_a) \) vanishes at the origin \( e_a = 0 \), while \( g_n(t, e_a) \) does not.

The key remaining steps in the proof are as follows: first, we show that the origin \( e_a(t) = 0 \) of the so-called nominal system

\[
\dot{e}_a(t) = f_a(t, e_a)
\]

is locally exponentially stable as long as the derivatives of \( v_{ref}(t) \), \( \omega_{ref}(t) \), \( a_l(t) \), and \( a_r(t) \) are bounded by sufficiently small numbers, that is, as long as those parameters vary slowly; second, we use the robustness properties of exponentially stable equilibrium points to ensure that the perturbed system has uniformly ultimately bounded solutions near the origin. Proving the local exponential stability of the nominal system (36) is the most involved step. The detailed demonstration is provided in Appendix A.6. Then, since the origin \( e_a(t) = 0 \) of (36) is exponentially stable, it is possible to show that the origin \( e_a(t) = 0 \) of the perturbed system

\[
\dot{e}_a(t) = f_a(t, e_a) + g(t, e_a),
\]

where \( g(t, e_a(t)) \) is a vanishing perturbation, is also exponentially stable. To prove this, Lemma 9.1 from Reference 21 is applied. This lemma requires the existence of a Lyapunov function for the nominal system (36) satisfying

\[
\begin{aligned}
c_1\|e_a(t)\|^2 &\leq V(t, e_a(t)) \leq c_2\|e_a(t)\|^2, \\
\frac{\partial V(t, e_a(t))}{\partial t} + \frac{\partial V(t, e_a(t))}{\partial e_a(t)} f(t, e_a(t)) &\leq -c_3\|e_a(t)\|^2,
\end{aligned}
\]

with \( f = f_a \) for all \((t, e_a(t)) \in [0, \infty) \times D \) for some positive constants \( c_1, c_2, c_3 \), and \( c_4 \) and some domain \( D \) containing the origin. By exponential stability of the origin of the nominal system (36), the existence of a sufficiently small \( d > 0 \) and of a Lyapunov function satisfying (38)-(40) with \( D = \{ e_a \in \mathbb{R}^5 \mid \|e_a(t)\| < d \} \) is ensured by the Lyapunov converse Theorem 4.14 from Reference 21, provided that \( \partial f_a / \partial e_a \) is bounded on \( D \), uniformly in \( t \). This is indeed the case, since \( f_a(t, e_a) \) is composed of polynomial and sinusoidal functions of \( e_a \) and continuous functions of time that are bounded for all \( t \geq 0 \). Thus to successfully apply Lemma 9.1 from Reference 21, it remains to show that

\[
\|g(t, e_a(t))\| \leq \gamma\|e_a(t)\|. \quad \forall \ t \geq 0, \ \forall \ e_a(t) \in D
\]

and

\[
\gamma < \frac{c_1}{c_4}
\]

are satisfied. To verify this, first write \( g(t, e_a) = \sigma(t)\bar{g}(t, e_a) \). The vector function \( \bar{g}(t, e_a) \) is composed of sums and products of polynomials and sinusoidal in the elements of \( e_a \), as well as time-dependent functions which are
bounded for all $t \geq 0$, and thus $\|\bar{g}/\partial e_a\| \leq \bar{b}$ on $[0, \infty) \times \mathbb{D}$ for some $\bar{b} > 0$. Thus $\bar{g}(t, e_a)$ is Lipschitz in $e_a$ on the domain $\mathbb{D}$, uniformly in $t$, with Lipschitz constant $\bar{b}$. Since $\bar{g}(t, 0) = 0$ and $\bar{g}(t, e_a)$ is Lipschitz in $e_a$, it follows that $\|g(t, e_a)\| \leq \bar{b}\|\sigma(t)\| \|e_a\|$, and condition (41) is verified with $\gamma = \epsilon \bar{b}$, where $\epsilon$ is a bound on $|\sigma(t)|$. This also implies that, for a sufficiently small bound $\epsilon$, condition (42) is verified. Thus, the origin of the nominal system, perturbed only by the vanishing term, given by (37), is also exponentially stable, provided that the magnitude of the lateral slip $\sigma(t)$ is bounded by a sufficiently small value $\epsilon$.

Since the origin $e_a(t) = 0$ is an exponentially stable equilibrium point of the system (37), Lemma 9.2 of Reference 21 can be applied to show that the solution $e_a(t)$ of the perturbed system (34) is uniformly ultimately bounded. For this purpose, the perturbed system (34) is rewritten as

$$\dot{e}_a(t) = f(t, e_a) + g(t, e_a)$$

with $f(t, e_a) = f_a(t, e_a) + g(t, e_a)$. To use the aforementioned Lemma 9.2, it is necessary to find a Lyapunov function $V(t, e_a(t))$ for the system (37) that satisfies conditions (38)–(40) with $f = f_a + g$. Considering the same $\mathbb{D} = \{e_a \in \mathbb{R}^3 \mid \|e_a(t)\| < d\}$ from before, such a Lyapunov function indeed exists since the Lyapunov function for the system (36) is also a Lyapunov function for the system (37). This fact is shown in the proof of Lemma 9.1 of Reference 21. To conclude the application of Lemma 9.2 of Reference 21, it remains to show that

$$\|g(t, e_a(t))\| \leq \delta < \frac{c_1}{c_4} \sqrt{c_2} \theta d$$

is satisfied for some $\theta < 1$. The proof of this fact uses similar reasoning used to prove that conditions (41) and (42) are satisfied. Indeed, it only suffice to choose $\delta = \epsilon$, the bound on $\sigma(t)$, which can be made sufficiently small. This concludes the proof of Theorem 3.

4 | NUMERICAL RESULTS

This section presents numerical simulations of the proposed adaptive control strategy given in Section 3. These simulations aim to illustrate our theoretical results and give an idea of the performance of the adaptive kinematic controller (AKC) given by (33). To illustrate how adaptation affects performance, the nonadaptive kinematic controller (NKC) given by (20) is also simulated, as one can see it as a version of the adaptive scheme where adaptation is “turned off”.

In all simulations, the physical parameters of the wheeled robot model, taken from Reference 25, are given by $b = 0.1624$ m and $\rho = 0.0825$ m. All simulations are performed with the total time $t_f = 45$ s. Two reference trajectories are used to evaluate the performance of the AKC scheme. The first reference trajectory, which we call the s-curve, alternates between straight lines and curved paths. The second reference trajectory, which we call the lemniscate-curve, has a periodic time-varying curvature.

The s-curve reference trajectory used in the simulations is generated by numerical integration of the kinematic model (8) with the initial condition $q_{\text{ref}}(0) = (0.0, 0.0)^T$ and the parameterized reference input $\eta_{\text{ref}}(t) = (v_{\text{ref}}(t), \omega_{\text{ref}}(t))^T$, adapted from Reference 26, given by

$$0 \leq t < t_0 : \quad v_{\text{ref}}(t) = 5A_0(1 - \cos(\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = 0,$$
$$t_0 \leq t < 4t_0 : \quad v_{\text{ref}}(t) = 10A_0 \quad \text{and} \quad \omega_{\text{ref}}(t) = 0,$$
$$4t_0 \leq t < 5t_0 : \quad v_{\text{ref}}(t) = 5A_0(1 + \cos(\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = 0,$$
$$5t_0 \leq t < 6t_0 : \quad v_{\text{ref}}(t) = 3\pi A_0(1 - \cos(2\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = -v_{\text{ref}}(t)/(3A_0),$$
$$6t_0 \leq t < 7t_0 : \quad v_{\text{ref}}(t) = 3\pi A_0(1 - \cos(2\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = v_{\text{ref}}(t)/(3A_0),$$
$$7t_0 \leq t < 8t_0 : \quad v_{\text{ref}}(t) = 3\pi A_0(1 - \cos(2\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = -v_{\text{ref}}(t)/(3A_0),$$
$$8t_0 \leq t < 9t_0 : \quad v_{\text{ref}}(t) = 3\pi A_0(1 - \cos(2\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = v_{\text{ref}}(t)/(3A_0),$$
$$9t_0 \leq t < 10t_0 : \quad v_{\text{ref}}(t) = 5A_0(1 + \cos(\pi t/t_0)) \quad \text{and} \quad \omega_{\text{ref}}(t) = 0,$$
$$10t_0 \leq t : \quad v_{\text{ref}}(t) = 10A_0 \quad \text{and} \quad \omega_{\text{ref}}(t) = 0$$

(44)
with $t_0 = 3.2$ s and $A_0 = 0.05$ m/s. Figure 3 shows the s-curve $(x_{ref}(t), y_{ref}(t))$ generated using the input $\eta_{ref}(t) = (v_{ref}(t), \omega_{ref}(t))^T$ given by (44). Note that the s-curve can be divided into three parts. The first part consists of a straight line path that occurs on the interval $0 \leq t < 5t_0$; the second part consists of a curved path with a time-varying radius of curvature that occurs on the interval $5t_0 \leq t < 9t_0$; and the third part consists of a straight line path that occurs on the interval $t \geq 9t_0$.

The lemniscate-curve reference trajectory used in the simulations is generated by numerical integration of the kinematic model (8) with the initial condition $q_{ref}(0) = (3/2, 0, \pi/2)^T$ and the parameterized reference input $\eta_{ref}(t) = (v_{ref}(t), \omega_{ref}(t))^T$, adapted from Reference 27, given by

$$v_{ref}(t) = \sqrt{x_{ref}(t)^2 + y_{ref}(t)^2} \quad \text{and} \quad \omega_{ref}(t) = \frac{\dot{y}_{ref}(t)x_{ref}(t) - \dot{x}_{ref}(t)y_{ref}(t)}{x_{ref}(t)^2 + y_{ref}(t)^2},$$

(45)

where $x_{ref}(t), y_{ref}(t), \dot{x}_{ref}(t), \dot{y}_{ref}(t)$ are obtained from

$$x_{ref}(t) = B_0 \cos(\omega_0 t) \quad \text{and} \quad y_{ref}(t) = B_0 \sin(2\omega_0 t)$$

with $B_0 = 1.5$ m and $\omega_0 = 2\pi/t_f$ rad/s. Figure 4 shows the lemniscate-curve $(x_{ref}(t), y_{ref}(t))$ generated using the input $\eta_{ref}(t) = (v_{ref}(t), \omega_{ref}(t))^T$ given by (45). Notice that the reference trajectory $q_{ref}(t)$ returns to its initial configuration when $t = t_f$.

To simulate the controlled robot, it is also necessary to select controller gains. A standard criterion described in Appendix C for choosing optimal controller gains provides $(k_1, k_2, k_3) = (0.20, 9.24, 0.10)$ and $(\gamma_1, \gamma_2, k_1, k_2, k_3) = (7.28, 7.28, 1.08, 14.87, 1.37)$ for the NKC scheme and the AKC scheme, respectively. Once the controller gains have been selected, to verify the performance of the NKC and AKC schemes, the robot under both schemes is subjected to a slip profile chosen so as to vary significantly in time. The longitudinal slip parameters $a_l(t)$ and $a_r(t)$ for the robot on the s-curve and the lemniscate-curve are chosen as the following nonlinear time-varying signals

$$a_l(t) = 1/(0.7 + 0.3 \exp(-0.13t) \cos(1.3t)),$$

$$a_r(t) = 1/(0.4 + 0.6 \exp(-0.18t) \sin^2(0.04t^2)).$$

Figure 5 shows the longitudinal slip parameters $a_l(t)$ and $a_r(t)$ for the robot on the s-curve and the lemniscate-curve. It is expected that a high variation of the longitudinal slip occurs when the robot is accelerating. This occurs, for example, when the robot starts its motion from rest. Notice that $a_l(t)$ and $a_r(t)$ converge to constants values, and their derivatives $\dot{a}_l(t)$ and $\dot{a}_r(t)$ converge to zero. Thus, this longitudinal slip profile allows to investigate the behavior of the longitudinal slip estimator error under constant and time-varying conditions.
The lateral slip parameter $\sigma(t)$ for the robot on the s-curve is chosen as the following vanishing oscillatory signal

$$\sigma(t) = 0.8 \exp(-0.03t)\text{sinc}(t - 22.4).$$

For the robot on the lemniscate-curve, we choose a different lateral slip parameter $\sigma(t)$ given by the nonvanishing oscillatory signal

$$\sigma(t) = 0.1 \sin(t).$$

Figure 6A,B shows the lateral slip parameter $\sigma(t)$ for the robot on the s-curve and the lemniscate-curve, respectively. This choice of lateral slip profile is motivated as follows: it is expected that a high variation of the lateral slip occurs when the robot moves along a curve. For the robot on the s-curve, it is assumed that a high value of the lateral slip occurs in the middle of the interval $5t_0 \leq t < 9t_0$, where the reference trajectory is a curved path. During the straight-line motion, it is expected that the lateral slip is zero, but different values of longitudinal slip may cause a small lateral slip. For the robot on the lemniscate-curve, it is assumed that the lateral slip, taken from Reference 27, is a periodic signal with constant amplitude and frequency.
Lateralslip parameter $\sigma(t)$. (a) Slip profile for the robot on the s-curve. (b) Slip profile for the robot on the lemniscate-curve.

Figures 7–9 show the simulations using the AKC and NKC schemes with the optimal gains. Figure 7 shows the robot trajectory obtained using the NKC and AKC schemes. The dotted line, the dashed line, and the solid line stand, respectively, for the reference trajectory (RT), the robot trajectory obtained using the NKC scheme, and the robot trajectory obtained using the AKC scheme. The initial conditions of the robot, indicated by a black circle, are $q(0) = (1/2, -3/4, -\pi/6)^T$ and $q(0) = (5/4, 1/5, \pi/4)^T$ for the robot on the s-curve and the lemniscate-curve, respectively. Note that the robot with the AKC scheme is able to follow the reference trajectory with a small error on both s-curve and lemniscate-curve. On the other hand, the NKC scheme is not able to compensate for the slip, and consequently, the robot trajectory diverges with respect to the reference trajectory.

Figure 8 shows the estimation errors $\hat{a}_l$ (dashed) and $\hat{a}_r$ (solid) for the robot on the s-curve and the lemniscate-curve. The initial conditions of the update rule were taken as $\hat{a}_l(0) = 1.6$ and $\hat{a}_r(0) = 1.2$, which differ from the true values $a_l(0) = 1$ and $a_r(0) = 2.5$. Note that an estimation error occurs at the beginning of the trajectory. However, this is not surprising since convergence is only guaranteed for constant slip parameters.
Figure 8 shows the estimation errors $\hat{a}_l(t)$ and $\hat{a}_r(t)$.

Figure 9 shows the tracking error $e = (e_x, e_y, e_\theta)^T$ using the AKC scheme for the robot on the s-curve and the lemniscate-curve. The dotted line, dashed line, and solid line stand for the tracking errors $e_x$, $e_y$, and $e_\theta$, respectively. In the s-curve, since the lateral slip $\sigma(t)$ and the longitudinal slip derivatives $\dot{a}_l(t)$ and $\dot{a}_r(t)$ tend to zero for $t \geq 9t_0$ s, the errors $e_x$, $e_y$, and $e_\theta$ tend to zero. Results for the robot on the lemniscate-curve show that although the nonvanishing lateral slip directly affects the tracking errors, the proposed adaptive controller is able to ensure bounded errors. The results demonstrate competitive performance when compared to other methods. 27

5 | CONCLUSION

This article introduces a family of adaptive kinematic controllers, designed to guarantee reference tracking for wheeled mobile robots subject to wheel slip. This family of control laws is generated by augmenting existing nonadaptive nominal controllers with a novel update rule for estimating longitudinal wheel slip. We have proven that a particular control
law in this family ensures that the augmented error, which includes the tracking error and the longitudinal slip estimation error, is uniformly ultimately bounded under slowly time-varying longitudinal and lateral slip. Moreover, we have demonstrated that any member of this family of control laws ensures convergence of the tracking error to zero, under the assumption of constant longitudinal slip and zero lateral slip. Furthermore, under these conditions, the longitudinal slip estimation error also converges to zero, provided that the time-varying reference velocities satisfy a persistent excitation condition. A direction for future work involves extending the adaptive update rule to accommodate arbitrary robot geometries. Due to the generality of our approach, it is expected that the proposed adaptive law can be integrated with machine learning-based controllers, provided a Lyapunov function can be derived for the slip-free scenario. In summary, this article provides a systematic framework for adaptive control of wheeled robots under slippery conditions, with theoretical guarantees on tracking performance.

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ENDNOTE
*If the only explicit time-dependence of $\eta_c(t, e(t))$ is due to $\eta_{ref}(t)$, then this assumption can be relaxed to $\eta_c(t, e(t))$ being locally Lipschitz in $e$ and $\eta_{ref}$.

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REFERENCES
1. Choi JH, Nam K, Oh S. High-accuracy driving control of a stone-throwing mobile robot for curling. IEEE Trans Autom Sci Eng. 2022;19(4):3210-3221.
2. Klančar G, Zdešar A, Blažič S, Škrjanc I. Wheeled mobile robotics: from fundamentals towards autonomous systems. Butterworth-Heinemann; 2017.
3. Urakubo T. Feedback stabilization of a nonholonomic system with potential fields: application to a two-wheeled mobile robot among obstacles. Nonlinear Dyn. 2015;81:1475-1487.
4. Ibrahim F, Abouelsoud AA, Elbab AMRF, Ogata T. Path following algorithm for skid-steering mobile robot based on adaptive discontinuous posture control. Auton Robot. 2019;33(9):439-453.
5. Morin P, Samson C. Chapter Trajectory Tracking for Non-holonomic Vehicles. Lecture Notes in Control and Information Sciences. Springer-Verlag; 2006.
6. Wu J, Xu G, Yin Z. Robust adaptive control for a nonholonomic mobile robot with unknown parameters. J Control Theory Appl. 2009;7(2):212-218.
7. Zhai JY, Song ZB. Adaptive sliding mode trajectory tracking control for wheeled mobile robots. Int J Control. 2019;92(10):2255-2262.
8. Kim DH, Oh JH. Globally asymptotically stable tracking control of mobile robots. Proceedings of the IEEE International Conference on Control Applications. IEEE; 1998:1297-1301.
9. Panahandeh P, Alipour K, Tarvirdizadeh B, Hadi A. A kinematic Lyapunov-based controller to posture stabilization of wheeled mobile robots. Mech Syst Signal Process. 2019;134:1-19.
10. Alipour K, Robat AB, Tarvirdizadeh B. Dynamics modeling and sliding mode control of tractor-trailer wheeled mobile robots subject to wheels slip. Mech Mach Theory. 2019;138:16-37.
11. Korayem MH, Ghobadi N, Dekordi SF. Designing an optimal control strategy for a mobile manipulator and its application by considering the effect of uncertainties and wheel slipping. Optimal Control Appl Methods. 2021;42(5):1487-1511.
12. Zhou B, Peng Y, Han J. UKF based estimation and tracking control of nonholonomic mobile robots with slipping. IEEE International Conference on Robotics and Biomimetics. IEEE; 2007:2058-2063.
13. Michalek M, Dutkiewicz P, Kielczenewski M, Pazderski D. Trajectory tracking for a mobile robot with skid-slip compensation in the vector-field-orientation control system. Int J Appl Math Comput Sci. 2009;19(4):547-559.
14. Chen M. Disturbance attenuation tracking control for wheeled mobile robots with skidding and slipping. IEEE Trans Ind Electron. 2017;64(4):3359-3368.
15. Biswas K, Kar I. Validating observer based on-line slip estimation for improved navigation by a mobile robot. Int J Intell Robotics Appl. 2022;6(3):564-575.
16. Yu X, Liu L. Target enclosing and trajectory tracking for a mobile robot with input disturbances. IEEE Control Syst Lett. 2017;1(2):221-226.
17. Shafiei MH, Monfared F. Design of a robust tracking controller for a nonholonomic mobile robot with side slipping based on Lyapunov redesign and nonlinear $H_{\infty}$ methods. Syst Sci Control Eng. 2019;7(1):1-11.
18. Iossaqui JG, Camino JF. Wheeled robot slip compensation for trajectory tracking control problem with time-varying reference input. Proceedings of the 9th International Workshop on Robot Motion and Control. IEEE; 2013:167-173.
19. Iossaqui JG, Camino JF, Zampieri DE. A nonlinear control design for tracked robots with longitudinal slip. Proceedings of the 18th IFAC World Congress. Elsevier; 2011:5932-5937.
20. Cui M, Huang R, Liu H, Liu X, Sun D. Adaptive tracking control of wheeled mobile robots with unknown longitudinal and lateral slipping parameters. Nonlinear Dyn. 2014;78:1811-1826.
21. Khalil HK. Nonlinear Systems. Prentice-Hall; 2002.
22. Aström KJ, Wittenmark B. Adaptive Control. 2nd ed. Dover Publications; 2008.
23. Krstic M, Kokotovic PV, Kanellakopoulos I. Nonlinear and Adaptive Control Design. John Wiley & Sons; 1995.
24. Besançon G. Remarks on nonlinear adaptive observer design. Syst Control Lett. 2000;41(4):271-280.
25. Ryu JC, Agrawal SK. Differential flatness-based robust control of mobile robots in the presence of slip. Int J Robotics Res. 2011;30(4):463-475.
26. Kanayama Y, Kimura Y, Miyazaki F, Noguchi T. A stable tracking control method for an autonomous mobile robot. IEEE International Conference on Robotics and Automation. IEEE; 1990:384-389.
27. Mera M, Rios H, Martinez EA. A sliding-mode based controller for trajectory tracking of perturbed unicycle mobile robots. Control Eng Pract. 2020;102:104548.
28. Rosenbrock HH. The stability of linear time-dependent control systems. J Electron Control. 1963;15(1):73-80.
29. Gantmacher FR. The Theory of Matrices II. Chelsea Publishing Company; 1959.

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APPENDIX A. PROOFS

A.1 Derivation of the error dynamics

The derivative of (10) is obtained using (2) and (8) as follows:

$$\dot{e}(t) = \hat{R}^T(\theta(t))(q_{\text{ref}}(t) - q(t)) + R^T(\theta(t))(S_\sigma(q_{\text{ref}}(t))\eta_{\text{ref}}(t) - S_\sigma(q(t))\eta(t)).$$

It can be verified that

$$R^T(\theta(t))S_\sigma(q_{\text{ref}}(t)) = S_\sigma(e(t)) \quad \text{and} \quad -R^T(\theta(t))S_\sigma(q(t)) = M_\sigma(0),$$

where the matrix $M_\sigma(\cdot)$ is given by (12). Furthermore, given that $\omega(t) = \dot{\theta}(t)$ and $R(\theta(t))e(t) = q_{\text{ref}}(t) - q(t)$ (as per (10)) the following can be deduced:

$$\hat{R}^T(\theta(t))(q_{\text{ref}}(t) - q(t)) = \dot{R}^T(\theta(t))R(\theta(t))e(t)$$

$$= \dot{R}^T(\theta(t)) \begin{bmatrix} -\sin \theta(t) & \cos \theta(t) & 0 \\ -\cos \theta(t) & -\sin \theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 \\ \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\sigma \end{bmatrix} = \begin{bmatrix} 0 & e_y \\ 0 & -e_x \end{bmatrix} \eta(t).$$

Based on the aforementioned results, the following equation is obtained:

$$\dot{e}(t) = S_\sigma(e(t))\eta_{\text{ref}}(t) + \begin{bmatrix} 0 & e_y \\ 0 & -e_x \end{bmatrix} \eta(t) + M_\sigma(0)\eta(t)$$

$$= S_\sigma(e(t))\eta_{\text{ref}}(t) + M_\sigma(e(t))\eta(t)$$

which is exactly (11).
A.2 Proof of Proposition 1

Substituting the relation

\[ \dot{\hat{a}}(t) = \hat{a}(t) - \hat{a}(t) \]

into (22b) yields the expression in (27b).

Considering the error dynamics, substituting \( \xi(t) = \xi_c(t, e) \), as specified in (22a), into (13) leads to

\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_\varepsilon(e)\Theta A^{-1}(t)\hat{A}(t)\Theta^{-1} \eta_c(t, e). \]

Subsequently, substituting \( \hat{A}(t) = A(t) + \bar{A}(t) \) into this equation yields

\[ \dot{e}(t) = S_0(e)\eta_{ref}(t) + M_\varepsilon(e)\eta_c(t, e) + M_\varepsilon(e)\Theta A^{-1}(t)\bar{A}(t)\Theta^{-1} \eta_c(t, e). \]  

(A1)

Noticing that

\[ \Theta A^{-1}(t) = \frac{\rho}{2ba_l(t)a_r(t)} \begin{bmatrix} ba_r(t) & ba_l(t) \\ -2a_r(t) & 2a_l(t) \end{bmatrix} \quad \text{and} \quad \bar{A}(t)\Theta^{-1} \eta_c(t, e) = \frac{1}{2\rho} \begin{bmatrix} \kappa_{-}(t, e)\bar{a}_r \\ \kappa_{+}(t, e)\bar{a}_r \end{bmatrix} \]

one finally obtains

\[ \Theta A^{-1}(t)\bar{A}(t)\Theta^{-1} \eta_c(t, e) = \frac{1}{4ba_l(t)a_r(t)} \begin{bmatrix} ba_r(t)\kappa_{-}(t, e)\bar{a}_r + ba_l(t)\kappa_{+}(t, e)\bar{a}_r \\ -2a_r(t)\kappa_{-}(t, e)\bar{a}_r + 2a_l(t)\kappa_{+}(t, e)\bar{a}_r \end{bmatrix} = \Psi(t, e)A^{-1}(t)\bar{a}, \]  

(A2)

where \( \Psi(t, e), \kappa_{-}(t, e) \) and \( \kappa_{+}(t, e) \) are defined in (24) and \( \eta_c(t, e) \) in (17). Therefore, (27a) follows by combining (A1) and (A2).

A.3 Proof of Lemma 1

**Proof.** For the matrix \( \Psi^T(t, e(t)) \) to be persistently exciting (PE), there must exist strictly positive constants \( \nu^l, \nu^r, \) and \( T^r \) such that, for \( t \geq 0 \), the following inequality is satisfied:

\[ \nu^l I \leq \int_{t}^{t+T^r} \Psi^T(\tau, e(\tau))\Psi(\tau, e(\tau)) \, d\tau \leq \nu^r I. \]  

(A3)

It is observed that the boundedness of the signals immediately implies the existence of the upper bound \( \nu^r \). Therefore, it remains necessary to establish the existence of a lower bound \( \nu^l > 0 \) and a finite constant \( T^r > 0 \).

This proof is divided into two parts. First, we demonstrate that if \( \zeta_\pm(t) \) satisfies the PE condition

\[ \delta_\pm^l \leq \int_{t}^{t+T^r_\pm} \zeta_\pm^2(\tau) \, d\tau \leq \delta_\pm^r \]  

(A4)

with \( \delta_\pm^l, \delta_\pm^r, \) and \( T^r_\pm \) strictly positive constants, then \( \kappa_\pm(t, e(t)), \) defined in (24), must satisfy the PE condition

\[ \epsilon_\pm^l \leq \int_{t}^{t+T^r_\pm} \kappa_\pm^2(\tau, e(\tau)) \, d\tau \leq \epsilon_\pm^r \]  

(A5)

with \( \epsilon_\pm^l, \epsilon_\pm^r, \) and \( T^r_\pm \) strictly positive constants. Then, in the second step, we show that if \( \kappa_\pm(t, e(t)) \) is PE, it implies that the matrix \( \Psi^T(t, e(t)) \) must satisfy the PE condition (A3).

To prove the first assertion, we focus on the case with the + sign, that is, \( \zeta_+(t) \) being PE implies that \( \kappa_+(t, e(t)) \) is also PE. The case with the − sign is analogous. To establish the existence of \( \epsilon_+^l > 0 \), first notice that the boundedness of the functions involved and the convergence of \( \nu_\varepsilon(t, e(t)) \) and \( \omega_{ref}(t) \) to \( \nu_{ref}(t) \) and \( \omega_{ref}(t) \), respectively, imply the convergence of \( \kappa_\pm^2(t, e(t)) \) to \( \zeta_\pm^2(t) \). For any given \( \epsilon > 0 \), there exists \( \Gamma_\varepsilon \) such that for \( t > \Gamma_\varepsilon \), \( |\kappa_\pm^2(t, e(t)) - \zeta_\pm^2(t)| < \epsilon \). Therefore, for \( t > \Gamma_\varepsilon \), the following inequality holds:

\[ \kappa_\pm^2(t, e(t)) > \zeta_\pm^2(t) - \epsilon. \]
Integrating this inequality over the interval \([t, t + T^*]\) yields

\[
\int_{t}^{t+T^*} \kappa_+^2(\tau, e(\tau)) \, d\tau > \int_{t}^{t+T^*} \zeta_+^2(\tau) \, d\tau - \epsilon T^*_+.
\]

Define \(\Gamma\) as a sufficiently large threshold such that \(\Gamma > \Gamma_e\), and choose \(T^*_+\) such that \(T^*_+ > T^*_+\). Considering the condition (A4) from the lemma, specifically applied to the case with the + sign, it follows that for \(t \geq \Gamma\), the following holds:

\[
\int_{t}^{t+T^*} \kappa_+^2(\tau, e(\tau)) \, d\tau > \delta^+_+ - \epsilon T^*_+.
\]

Select \(\epsilon\) as \(\epsilon = \delta^+_+/(2T^*_+)\), and define \(\epsilon^+_+ = \delta^+_+/2 > 0\). Therefore, for \(t \geq \Gamma\), the following holds:

\[
\int_{t}^{t+T^*} \kappa_+^2(\tau, e(\tau)) \, d\tau \geq \epsilon^+_+.
\]

For \(0 \leq t < \Gamma\), choose \(T^*_+ \geq \max(\Gamma, T^*_+)\) to ensure the integral is bounded away from zero. This choice of \(T^*_+\) also satisfies the conditions for \(t \geq \Gamma\), resulting in a single \(T^*_+\) satisfying the inequality for \(t \geq 0\). This completes the first part of the proof, that is, \(\zeta_+(t)\) being PE implies that \(\kappa_+(t, e(t))\) satisfies the PE condition (A5).

To prove the second part, we need to establish the existence of the lower bound \(\nu^l > 0\) and of the finite constant \(T^* > 0\) in (A3). Computing \(\Psi^T(t, e(t))\Psi(t, e(t))\) and rearranging terms, gives

\[
\Psi^T(t, e(t))\Psi(t, e(t)) = \frac{1}{16b^2} \left[ \begin{array}{cc} (b^2 + 4)\kappa_+^2(t, e(t)) & (b^2 - 4)\kappa_-(t, e(t))\kappa_+(t, e(t)) \\ (b^2 - 4)\kappa_-(t, e(t))\kappa_+(t, e(t)) & (b^2 + 4)\kappa_+^2(t, e(t)) \end{array} \right].
\]

Integrating over \([t, t + T^*]\), yields

\[
\Sigma := \int_{t}^{t+T^*} \Psi^T(\tau, e(\tau))\Psi(\tau, e(\tau)) \, d\tau = \frac{1}{16b^2} \left[ \begin{array}{cc} (b^2 + 4)\beta_+ & (b^2 - 4)\beta_- \\ (b^2 - 4)\beta_- & (b^2 + 4)\beta_+ \end{array} \right],
\]

where

\[
\beta_+ = \int_{t}^{t+T^*} \kappa_+^2(\tau, e(\tau)) \, d\tau, \quad \beta_- = \int_{t}^{t+T^*} \kappa_-^2(\tau, e(\tau)) \, d\tau, \quad \beta_{-+} = \int_{t}^{t+T^*} \kappa_-(\tau, e(\tau))\kappa_+(\tau, e(\tau)) \, d\tau.
\]

The leading principal minors of \(\Sigma\) are

\[
\mu_1 = (b^2 + 4)\beta_-/(16b^2), \quad \mu_2 = \det(\Sigma) = \left( (b^2 + 4)^2\beta_-\beta_+ - (b^2 - 4)^2\beta_{-+}^2 \right)/(256b^4).
\]

From Sylvester’s criterion, for \(\Sigma\) to be positive definite all of its leading principal minors must be positive. Recognizing that \(\kappa_-(t, e(t))\) is PE with \(T^* = T_-\), it follows that \(\mu_1 \geq (b^2 + 4)^2\epsilon^l/(16b^2) > 0\). It now remains to demonstrate that \(\mu_2\) is also positive.

Substituting the Cauchy–Schwarz inequality

\[
\left| \int_{t}^{t+T^*} \kappa_-(\tau, e(\tau))\kappa_+(\tau, e(\tau)) \, d\tau \right|^2 \leq \left( \int_{t}^{t+T^*} \kappa_+^2(\tau, e(\tau)) \, d\tau \right) \left( \int_{t}^{t+T^*} \kappa_-^2(\tau, e(\tau)) \, d\tau \right) = \beta_-\beta_+ \quad \Rightarrow \quad \beta_{-+}^2 \leq \beta_-\beta_+
\]

into \(\mu_2\) yields

\[
256b^4\mu_2 \geq (b^2 + 4)^2\beta_-\beta_+ - (b^2 - 4)^2\beta_{-+}^2 = 16b^2\beta_-\beta_+.
\]
Since $\kappa_\pm(t, e(t))$ is PE with $T^* = \max\{T^*_+, T^*_c\}$, it follows that $\beta_-\beta_+ \geq \epsilon := e^\top e > 0$ and consequently $\mu_2 \geq \epsilon/(16b^2) > 0$. This, in combination with $\mu_1 > 0$, satisfies Sylvester’s criterion for the positive definiteness of $\Sigma$. Therefore, $\Sigma$ is positive definite for every $t \geq 0$. Thus, the proof is concluded.

**A.4 Derivation of the particular adaptive control law**

To begin, the term $M_0(e)\Psi(t, e)$ is obtained by direct calculation using (12) and (24) as follows

$$M_0(e)\Psi(t, e) = \frac{1}{4b} \begin{bmatrix}
-\kappa_-(t, e)(b + 2e_y) & -\kappa_+(t, e)(b - 2e_y) \\
2\kappa_-(t, e)e_x & -2\kappa_+(t, e)e_x \\
2\kappa_-(t, e) & -2\kappa_+(t, e)
\end{bmatrix}.$$  \hspace{1cm} (A6)

Using (21), the Jacobian of $V(e)$ is given by

$$\frac{\partial V(e)}{\partial e} = \begin{bmatrix} e_x & e_y + k_3e_\theta & k_3(e_y + k_3e_\theta) + \sin(e_\theta)/k_2 \end{bmatrix}. \hspace{1cm} (A7)$$

It now follows from (A6) and (A7) that

$$\Psi^T(t, e)M_0^T(e)\frac{\partial V(e)}{\partial e} = -\frac{1}{2} \left[ \kappa_-(t, e) \left( \frac{e_x}{b} + \frac{b}{2} \right) e_x - \frac{k_3}{b} e_y - \frac{1}{bk_2} \sin e_\theta \right].$$  \hspace{1cm} (A8)

Using (A8) and (23), the update rule (22b) is then given by

$$\hat{a}_l = \gamma_1 \frac{-\kappa_-(t, e)}{2} \left( e_x - \frac{k_3}{b} \left( e_\theta(e_x + k_3) + e_y \right) - \frac{1}{bk_2} \sin e_\theta \right),$$

$$\hat{a}_r = \gamma_2 \frac{-\kappa_+(t, e)}{2} \left( e_x + \frac{k_3}{b} \left( e_\theta(e_x + k_3) + e_y \right) + \frac{1}{bk_2} \sin e_\theta \right), \hspace{1cm} (A9)$$

where the terms in $e_x e_y$ have been canceled. Finally, the particular update law (33b) is recovered by substituting $\kappa_-(t, e)$ and $\kappa_+(t, e)$ in (A9) by their definition in terms of $\nu_2(t, e)$ and $\omega_2(t, e)$, given by (24).

**A.5 Derivation of closed-loop dynamics as a perturbed system**

First notice that by direct calculation from (24), we have

$$\eta_k(t, e) = \Psi(t, e) \mathbb{I}, \quad \text{with} \quad \mathbb{I} := \begin{bmatrix} 1 \\
1 \end{bmatrix}$$

and $\eta_k(t, e) = (\nu_2(t, e), \omega_2(t, e))^T$ as defined previously. Thus it follows that (27a) can be rewritten equivalently as

$$\dot{e}(t) = S_0(e)\eta_k(t) + M_\sigma(e)\Psi(t, e) \left( \mathbb{I} + A^{-1}(t)\hat{a} \right). \hspace{1cm} (A10)$$

The term $M_\sigma(e)\Psi(t, e)$ is obtained by direct calculation from (12) and (24). It is given by

$$M_\sigma(e)\Psi(t, e) = \begin{bmatrix}
-\kappa_-(t, e) \left( \frac{e_x}{b} + \frac{b}{2} \right) & -\kappa_+(t, e) \left( \frac{e_x}{b} - \frac{1}{2} \right) \\
\kappa_-(t, e) \left( \frac{e_x}{b} - \frac{b}{2} \right) & -\kappa_+(t, e) \left( \frac{e_x}{b} + \frac{1}{2} \right)
\end{bmatrix}. \hspace{1cm} (A11)$$
Now, recall that $\eta_{\text{ref}} = (v_{\text{ref}}, \omega_{\text{ref}})^T$ and $A(t) = \text{diag}(a(t))$, then substituting $S_\theta(e)$ given by (2) and $M_\sigma(e)\Psi(t,e)$ given by (A11) in (A10) produces

$$
\dot{e}(t) = \begin{bmatrix}
v_{\text{ref}}(t) \cos\theta - \frac{\kappa_-(t,e)}{2} \left( \frac{e_y}{b} + \frac{1}{2} \right) \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) + \frac{\kappa_+(t,e)}{2} \left( \frac{e_y}{b} - \frac{1}{2} \right) \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right) \\
v_{\text{ref}}(t) \sin\theta + \frac{\kappa_-(t,e)}{2} \left( \frac{e_y}{b} - \frac{\sigma(t)}{2} \right) \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) - \frac{\kappa_+(t,e)}{2} \left( \frac{e_y}{b} + \frac{\sigma(t)}{2} \right) \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right) \\
\omega_{\text{ref}}(t) + \frac{\kappa_+(t,e)}{2b} \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) - \frac{\kappa_+(t,e)}{2b} \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right)
\end{bmatrix}
$$

which can be rewritten, noticing that $\kappa_-(t,e) + \kappa_+(t,e) = 4\nu_c(t,e)$, as follows

$$
\dot{e}(t) = \begin{bmatrix}
v_{\text{ref}}(t) \cos\theta - \frac{\kappa_-(t,e)}{2} \left( \frac{e_y}{b} + \frac{1}{2} \right) \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) + \frac{\kappa_+(t,e)}{2} \left( \frac{e_y}{b} - \frac{1}{2} \right) \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right) \\
v_{\text{ref}}(t) \sin\theta + \frac{\kappa_-(t,e)}{2} \left( \frac{e_y}{b} - \frac{\sigma(t)}{2} \right) \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) - \frac{\kappa_+(t,e)}{2} \left( \frac{e_y}{b} + \frac{\sigma(t)}{2} \right) \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right) \\
\omega_{\text{ref}}(t) + \frac{\kappa_+(t,e)}{2b} \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right) - \frac{\kappa_+(t,e)}{2b} \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right) \\
- \frac{\sigma(t)}{4} \left[ \kappa_-(t,e) \frac{\hat{d}_l}{a_l(t)} + \kappa_+(t,e) \frac{\hat{d}_r}{a_r(t)} \right] - \sigma(t) \nu_c(t,e)
\end{bmatrix}.
(A12)
$$

Using the definitions

$$
\Delta_r = \left( 1 + \frac{\hat{d}_r}{a_r(t)} \right), \quad \Omega_1 = \kappa_+(t,e)/2, \quad \Omega_3 = \left( \frac{e_y}{b} - \frac{1}{2} \right), \\
\Delta_l = \left( 1 + \frac{\hat{d}_l}{a_l(t)} \right), \quad \Omega_2 = \kappa_-(t,e)/2, \quad \Omega_4 = \left( \frac{e_y}{b} + \frac{1}{2} \right),
$$

and replacing $\nu_c(t,e)$ given by (20) in (A12) results in

$$
\dot{e}(t) = \begin{bmatrix}
v_{\text{ref}}(t) \cos\theta - \Omega_2 \Omega_4 \Delta_l + \Omega_1 \Omega_3 \Delta_r \\
v_{\text{ref}}(t) \sin\theta + e_x(\Omega_2 \Delta_l - \Omega_1 \Delta_r)/b \\
\omega_{\text{ref}}(t) + (\Omega_2 \Delta_l - \Omega_1 \Delta_r)/b \\
- \sigma(t) \left[ \frac{1}{2} \Omega_1 \frac{\hat{d}_l}{a_l(t)} + \frac{1}{2} \Omega_2 \frac{\hat{d}_r}{a_r(t)} - k_3 e_y \omega_c(t,e) + k_1 e_x \right] - \sigma(t) v_{\text{ref}}(t) \cos\theta
\end{bmatrix}
$$

which corresponds to the first three rows of the dynamics (34).

### A.6 Proof of exponential stability of the nominal system

The local exponential stability of the origin of the nominal system (36) can be shown by linearizing (36) around $e_a = 0$. The resulting linear system is given by

$$
\dot{e}_a(t) = J(t)e_a,
(A13)
$$

where $J(t)$ is the time-varying Jacobian of (35) evaluated at the origin,

$$
J(t) = \left. \frac{\partial f_a(t, e_a)}{\partial e_a} \right|_{e_a=0}.
$$

It is a well-known fact (see Theorem 4.13 in Reference 21) that if the origin of the linear time-varying nominal system (A13) is exponentially stable, then the origin of the nonlinear nominal system (36) is locally exponentially stable. Hence, we now prove the former claim by analyzing the spectrum of $J(t)$. 
To avoid a lengthy derivation of $J(t)$, it can be verified by inspection that the nominal system vector field $f_{\alpha}(t, e_{\alpha})$ in (35) is precisely the vector field in equations (10)–(14) from Reference 18, with the difference that in (35) the longitudinal slip parameters $a_{i}(t)$ and $a_{r}(t)$ are time-varying, while in Reference 18 they were assumed to be constant. It follows that $J(t)$ is given by the same expression as equation (17) in Reference 18, but with time-varying $a_{i}(t)$ and $a_{r}(t)$:

$$J(t) = \begin{bmatrix} J_{11}(t) & J_{12}(t) \\ J_{21}(t) & J_{22}(t) \end{bmatrix},$$

where

$$J_{11}(t) = \begin{bmatrix} -k_1 & \omega_{\text{ref}}(t) & k_3\omega_{\text{ref}}(t) \\ -\omega_{\text{ref}}(t) & 0 & v_{\text{ref}}(t) \\ 0 & -\frac{k_2}{2}v_{\text{ref}}(t) & -\frac{k_4}{2k_5}v_{\text{ref}}(t) \end{bmatrix},$$

$$J_{12}(t) = \frac{1}{4ba_{r}(t)a_{r}(t)} \begin{bmatrix} ba_{r}(t)v_{2}(t) & -ba_{r}(t)v_{1}(t) \\ 0 & 0 \\ -2a_{r}(t)v_{2}(t) & -2a_{r}(t)v_{1}(t) \end{bmatrix},$$

$$J_{21}(t) = \frac{1}{4bk_{2}} \begin{bmatrix} -bk_{2}v_{2}(t) & 2k_{2}k_{3}v_{1}(t) & 2\gamma_{1}k_{4}v_{2}(t) \\ bk_{2}v_{1}(t) & 2k_{2}k_{3}v_{1}(t) & 2\gamma_{2}k_{4}v_{1}(t) \end{bmatrix},$$

$$J_{22}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $v_{1}(t) = b\omega_{\text{ref}}(t) + 2v_{\text{ref}}(t)$, $v_{2}(t) = b\omega_{\text{ref}}(t) - 2v_{\text{ref}}(t)$, and $k_{4} = 1 + k_{2}k_{3}^{2}$.

Following the main idea from section II.B in Reference 18, we can use Rosenbrock’s Theorem$^{28}$ (reproduced as Theorem 2 in Reference 18) to show that the origin of (A13) is exponentially stable. For that purpose, all the entries $J_{ik}$ of $J(t)$ must be bounded in time ($i, k = 1, \ldots, 5$), which is true by Assumption 2. Rosenbrock’s Theorem then states that if there exists an $\varepsilon > 0$ such that

$$\text{Re}[\lambda(J(t))] \leq -\varepsilon < 0 \quad (A14)$$

for all $t \geq 0$, then there is a bound $\delta > 0$ such that if all $|\frac{d}{dt}J_{ik}(t)| < \delta$, the linear system (A13) will have an exponentially stable origin. The main difficulty here lies in proving that (A14) is indeed satisfied for some $\varepsilon > 0$. If this can be proven, then $|\frac{d}{dt}J_{ik}(t)| < \delta$ can be satisfied for any $\delta$ as long as $\dot{a}_{i}(t)$, $\dot{a}_{r}(t)$, $\dot{\omega}_{\text{ref}}(t)$ and $\dot{v}_{\text{ref}}(t)$ are bounded by a sufficiently small number, in which case the origin of (A13) is exponentially stable.

The remainder of the section deals with showing (A14). Following the strategy proposed in Reference 18, one can use the stability criterion of Liénard and Chipart,$^{29}$ which deals with the characteristic polynomial of $J(t)$, to establish the condition (A14). However, departing from the analysis in Reference 18, who resorted to numerical methods (and assumed constant slip), one can provide a rigorous proof that (A14) is satisfied. Consider the characteristic polynomial of matrix $J(t)$, which is given by

$$p(s) = s^5 + a_{1}(t)s^4 + a_{2}(t)s^3 + a_{3}(t)s^2 + a_{4}(t)s + a_{5}(t), \quad (A15)$$

whose coefficients $a_{i}(t)$ are given by

$$a_{1}(t) = k_1 + \frac{k_4}{2k_3}v_{\text{ref}}(t),$$

$$a_{2}(t) = \frac{1}{16a_{r}(t)a_{r}(t)b^{2}k_{2}k_{3}} \left\{ a_{r}(t)\left[ \gamma_{2}v_{2}^{2}(t)k_{3}(b_{2}k_{2} + 4k_{4}) + 8a_{r}(t)b^{2}k_{2}\left( v_{\text{ref}}(t)(k_{1}k_{4} + k_{2}k_{3}v_{\text{ref}}(t)) + 2k_{3}\omega_{\text{ref}}^{2}(t) \right) \right] + a_{r}(t)v_{2}^{2}(t)k_{3}(b_{2}k_{2} + 4k_{4}) \right\},$$

$$a_{3}(t) = \frac{1}{32a_{r}(t)a_{r}(t)b^{2}k_{2}k_{3}} \left\{ a_{r}(t)\left[ 16a_{r}(t)b^{2}k_{2}v_{\text{ref}}(t)(k_{1}k_{4} + k_{2}k_{3}v_{\text{ref}}(t) + \omega_{\text{ref}}^{2}(t)) \right] \right\},$$
where

\[ \alpha \]

expression in \( \gamma \) satisfied. Furthermore, these second condition holds true by noticing that the expression

\[ \alpha \]

Notice also that

\[ \alpha \]

as

\[ \alpha \]

is bounded below by

\[ \alpha \]

By Assumption 2, it can be verified by inspection that \( 0 < \delta_1 \leq \alpha_i(t) \), for \( i = 1, \ldots, 5 \), and hence the first condition is satisfied. Furthermore, the second condition holds true by noticing that the expression

\[ \alpha \]

is positive.

It now remains to show the third condition. For that purpose, first notice that the expression for \( \alpha_3(t) \) can be rearranged as

\[ \alpha \]

and that \( \alpha_3(t) \) is bounded below by

\[ \alpha \]

This last inequality directly implies that

\[ \alpha \]

Notice also that \( \alpha_3(t) \alpha_4(t) \) is bounded below as follows

\[ \alpha \]

where \( \mathcal{A} = \sup \{ \alpha_i(t), \alpha_i(t) \} \) is finite by Assumption 2. Likewise, the term \( \overline{p}(t) \) in (A16) can also be written as a polynomial expression in \( \gamma_1 \) and \( \gamma_2 \) as follows

\[ \overline{p}(t) = p_{11}(t) \gamma_1 \gamma_2 + p_{20}(t) \gamma_2^2 + p_{02}(t) \gamma_2^2 + p_{21}(t) \gamma_1^2 \gamma_2 + p_{12}(t) \gamma_1 \gamma_2^2 + \cdots, \]
where the coefficients $p_i(t)$ are bounded by Assumption 2. Thus, for sufficiently small $\gamma_1$ and $\gamma_2$, the first two terms on the right-hand side of (A17) dominate all the remaining terms in the expression for $c_3(t)$ given by (A16), and hence there is a $\delta_3 > 0$ such that $c_3(t) \geq \delta_3$ for all $t \geq 0$. This concludes our proof, and we have shown that $\text{Re}[\lambda_j(J(t))] \leq -c < 0$.

**APPENDIX B. SUPPORTING RESULTS**

**Lemma 2** (Lemma A.1 in Reference 24). *Given a system of the following form:*

\[
\begin{align*}
\dot{z}_1(t) &= h_1(t) + G(t)z_2(t), \\
\dot{z}_2(t) &= h_2(t).
\end{align*}
\]

**Assume that**

1. $\lim_{t \to \infty} \|z_1(t)\| = 0; \lim_{t \to \infty} \|h_1(t)\| = 0; \lim_{t \to \infty} \|h_2(t)\| = 0$;
2. $G(t), \dot{G}(t)$ are bounded, and $G(t)^T$ is persistently exciting.

*Then, $\lim_{t \to \infty} \|z_2(t)\| = 0$.*

**APPENDIX C. OPTIMAL CONTROLLER GAINS**

A standard criterion for choosing optimal controller gains involves minimizing the cost function given by

\[
F = \int_0^T e^T Q e + (\xi - \xi_{\text{ref}})^T R (\xi - \xi_{\text{ref}}) \, dt
\]

with $R > 0$ and $Q > 0$, representing weights on the tracking error and control actions, while the robot is moving. This criterion is adopted to select gains for both the NKC and AKC schemes, using $R = 0.05I_2$ and $Q = I_3$, where $I_2$ and $I_3$ are respectively $2 \times 2$ and $3 \times 3$ identity matrices. Notice that the term $\xi - \xi_{\text{ref}}$ represents the difference between the controlled wheel speeds and the reference wheel speeds resulting from the reference trajectory $\eta_{\text{ref}}$. This difference depends on the longitudinal slip parameters $a_1(t)$ and $a_2(t)$, which also need to be provided for the optimization problem. Since the NKC is not designed to take into account the slip, for the purposes of gain selection, the robot controlled by the NKC is subjected to zero slip. Thus, for the NKC, $\xi - \xi_{\text{ref}} = \Theta^{-1}(\eta_c(t) - \eta_{\text{ref}})$, where $\Theta$ is given by (7) and $a_1 = a_2 = 1$. The AKC, on the other hand, has to be able to deal with the slip. For this reason, to select the gains for the AKC, a “training” slip profile given by $a_1(t) = H(t) + 2/3(H(t - 10) - H(t - 30))$ and $a_2(t) = H(t) + 3/2(H(t - 20) - H(t - 40))$ is used, where $H(\cdot)$ denotes the Heaviside step function. Using this slip, the controlled and reference wheel speeds for the AKC are given by (22a) and $\xi_{\text{ref}} = A(t)\Theta^{-1}\eta_{\text{ref}}$, respectively.

To select controller gains for the NKC and AKC schemes, a grid search over the gains $k_i$ ($i = 1, 2, 3$) and $\gamma_j$ ($j = 1, 2$) were performed, and those gains that yielded the smallest $F$ were selected. The grid was chosen such that $k_1, k_2, k_3$, and $\gamma_1 = \gamma_2$ range over 30 equally spaced points in logarithmic scale between $10^{-1}$ and $10^2$. For the AKC scheme, the values $\gamma_1$ and $\gamma_2$ were assumed equals for convenience.

The minimum value of $\log_{10}(F)$ using the AKC scheme was obtained when $(\gamma_1, \gamma_2, k_1, k_2, k_3) = (7.28, 7.28, 1.08, 14.87, 1.37)$. Using the NKC scheme, the gains $(k_1, k_2, k_3) = (0.20, 9.24, 0.10)$ provided the smallest value for $\log_{10}(F)$.