Spiky Strings on I-brane

Sagar Biswas
Department of Physics and Meteorology, 
Indian Institute of technology Kharagpur, 
Kharagpur-721 302, INDIA
biswas.sagar09iitkgp@gmail.com

Kamal L. Panigrahi
Department of Physics and Meteorology, 
Indian Institute of Technology Kharagpur, 
Kharagpur-721302, INDIA,
and
The Abdus Salam International Centre for Theoretical Physics, 
Strada Costiera 11, Trieste, ITALY
E-mail: panigrahi@phy.iitkgp.ernet.in

Abstract: We study rigidly rotating strings in the near horizon geometry of the 1+1 dimensional intersection of two orthogonal stacks of NS5-branes, the so called I-brane background. We solve the equations of motion of the fundamental string action in the presence of two form NS-NS fluxes that the I-brane background supports and write down general form of conserved quantities. We further find out two limiting cases corresponding to giant magnon and single spike like strings in various parameter space of solutions.

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1. Introduction

The AdS/CFT duality \cite{1} relates the spectrum of free strings in the bulk to that of field theory operators living on the boundary of AdS. Though finding the full string theory spectrum is extremely difficult it has been observed that in certain limits, for example, in large angular momentum limit the theory becomes tractable \cite{2} and it is interesting to study the corresponding field theory in detail. It has been noticed that in this region one can use the semiclassical approximation to find the string spectrum as well \cite{3}. Further an interesting observation is that the $\mathcal{N} = 4$ SYM in planar limit can be described by an integrable spin chain model where the anomalous dimension of the gauge invariant operators were found in \cite{4, 5, 6, 7, 8}, and it was also noticed that the string theory is integrable in the semiclassical limit, see for example \cite{9, 10, 11, 12}.

Study of rigidly rotating string in semiclassical approximation has been one of the interesting areas of research in the last few years because of its elegance. In this connection the so called Hofman-Maldacena (HM) limit\footnote{The HM limit: $J \to \infty$, $\lambda =$ fixed, $p =$ fixed, $E - J =$ fixed, where $J$ is one of the SO(6) charges and $p$ is the magnon momentum.} \cite{13} simplifies considerably the problem of finding out the spectrum on both sides of the duality. The spectrum consists of an elementary excitation known as magnon which propagates with a conserved momentum $p$ along the spin chain. Further, a more general class of rotating string solution in $AdS_5$ is the spiky string which describes the higher twist operators from dual field theory view point \cite{14} and magnon solutions can be thought of as a subspace of these spike solutions. Infact it was further determined in \cite{15} that if one solves the most general form of equations of motion...
for a rigidly rotating string on a sphere one encounters two set of solutions, corresponding precisely to the giant magnon and single spike solutions.

To understand the AdS/CFT like dualities in more general backgrounds, it is instructive to study rigidly rotating string in the gravity side which will teach us the corresponding operators in the dual field theory side. In this connection, magnon like dispersion relations have been found out in backgrounds which arise from intersecting branes in supergravities and which asymptotes to AdS and non-AdS backgrounds, see for example [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. One of such background is the so-called I-brane background [34] that arises in the 1+1 dimensional intersection of two orthogonal stacks of NS5-branes, with one set of branes lying along \((x_0, x_1, \cdots, x_5)\), and other set of lying along \((x_0, x_1, x_6, \cdots, x_9)\) directions. When all five branes are coincident, in the S-dual picture, the near horizon geometry is given by

\[ R^{2,1} \times R_\phi \times SU(2)_{k_1} \times SU(2)_{k_2}, \]

where, \(R_\phi\) is one combination of the radial directions away from the two sets of NS5-branes, and the coordinates of \(R^{2,1}\) are \(x^0, x^1\) and another combination of the two radial directions. The two \(SU(2)\)s describe the angular three-spheres corresponding to \((R^4)_{2345}\) and \((R^4)_{6789}\). As mentioned in [34] this geometry has the interesting property that it exhibits a higher symmetry than the full brane configuration itself. In particular, the combination of radial directions away from the intersection that enters in \(R^{2,1}\) appears symmetrically with the other spatial direction, and the background has a higher Poincare symmetry, ISO(2, 1), than the expected ISO(1, 1) and twice as many supercharges one might expect. Furthermore, in [35, 36, 37, 38], these interesting properties were studied from the point of view of D1-brane probe. It was shown that the enhancement of the near horizon geometry has clear impact on the worldvolume dynamics of the D1-brane probe dynamics.

In this paper, we would like to study the solutions of rigidly rotating strings in the near horizon geometry of this interesting background with an emphasis on finding out spiky strings. Earlier in [27] a class of such rigidly rotating strings were studied and some interesting solutions were obtained. In this paper we wish to find out solutions corresponding to the usual giant magnon and single spike solutions on \(R_t \times S^3\) and \(R \times S^3 \times S^3\) in the presence of various background NS-NS fluxes that it supports. Even though we can not compare our results to the dual field theory which is not completely known, but a knowledge of the bulk solutions gives us idea weather to look for operators in dual theory. The rest of the paper is organized as follows. In section-2 we give a brief overview of I-brane background and write down the near horizon geometry in a parametrization which we will use in the subsequent analysis. In section-3 we write the Nambu-Goto action for the Fundamental string in the near horizon geometry of I-brane, write down the conserved charges and compute the general equations of motion of the rigidly rotating string. Further we find out, in some parameter space of solution, the two limiting cases corresponding to giant magnon and single spike like strings corresponding to open string boundary condition. Finally in section-4 we conclude with some remarks.
2. Review of near horizon geometry of I-brane

In this section we would like to give a brief review of the near horizon geometry of the I-brane which arises from the 1+1 dimensional intersection of two orthogonal stacks of NS5-branes. This background arises when $k_1$ number of NS5-branes lying along $(0, 1, \cdots, 5)$ intersect $k_2$ number of NS5-branes lying along $(0, 1, 6, \cdots, 9)$ directions in $(0, 1)$-plane. If the branes are coincident, then the supergravity solution is given by [34]

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + H_1(y) \sum_{a=2}^5 (dy^a)^2 + H_2(z) \sum_{p=6}^9 (dz^p)^2, \quad e^{2\phi} = H_1(y)H_2(z),
\]

\[
H_{\alpha\beta\gamma} = -\epsilon_{\alpha\beta\gamma\delta} \partial^\delta H_1(y), \quad H_{mn\rho} = -\epsilon_{mn\rhoq} \partial^q H_2(z), \quad H_1 = 1 + \frac{k_1 l_s^2}{y^2}, \quad H_2 = 1 + \frac{k_2 l_s^2}{z^2},
\]

\[
y = \sqrt{\sum_{a=2}^5 (y^a)^2}, \quad z = \sqrt{\sum_{p=6}^9 (z^p)^2}.
\]

In the near horizon limit the 1 in the Harmonic functions above can be neglected and the metric and NS-NS ($B_{\mu\nu}$) fields are given by

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + k_1 l_s^2 \frac{dr_1^2}{r_1^2} + k_1 l_s^2 d\Omega_1^2 + k_2 l_s^2 \frac{dr_2^2}{r_2^2} + k_2 l_s^2 d\Omega_2^2,
\]

\[
B_{\phi_1\psi_1} = 2k_1 l_s^2 \sin^2 \theta_1, \quad B_{\phi_2\psi_2} = 2k_2 l_s^2 \sin^2 \theta_2,
\]

where

\[
d\Omega_1^2 = d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + \cos^2 \theta_1 d\psi_1^2, \quad d\Omega_2^2 = d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + \cos^2 \theta_2 d\psi_2^2.
\]

are the volume elements on the sphere along $(y^2, \cdots, y^5)$ and $(z^6, \cdots, z^9)$ directions respectively. To proceed further we make the following change of variables (we choose $l_s = 1, k_1 = k_2 = N$)

\[
\rho_1 = \ln \frac{r_1}{\sqrt{N}}, \quad \rho_2 = \ln \frac{r_2}{\sqrt{N}}, \quad x^0 = \sqrt{N}t, \quad x^1 = \sqrt{N}y.
\]

The final form of metric and the background NS-NS fields are given by

\[
ds^2 = N(-dt^2 + dy^2 + d\rho_1^2 + d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + \cos^2 \theta_1 d\psi_1^2 + d\rho_2^2 + d\theta_2^2 +
\]

\[
+ \sin^2 \theta_2 d\phi_2^2 + \cos^2 \theta_2 d\psi_2^2), \quad B_{\phi_1\psi_1} = 2N \sin^2 \theta_1, \quad B_{\phi_2\psi_2} = 2N \sin^2 \theta_2.
\]

We are interested in studying the rigidly rotating strings and finding out spiky strings in this background.\(^3\)

\(^2\)This is also supported by a dilaton whose explicit form we will not need in the calculation of this paper.

\(^3\)Recently in [35] a PP-wave background has been obtained by applying Penrose limit on this geometry.
3. Rigidly rotating strings in I-brane background

In this section we would like to study the solutions to the fundamental string equations of motion in the background discussed in the previous section. The Nambu-Goto action for the string in the background of (2.4) is written as

\[ S = -\frac{\sqrt{\lambda}}{2\pi} \int dt \sigma L, \]  

(3.1)

where the Lagrangian \( L \) is given by

\[
L = \left( \left[ -\dot{t}^2 + \dot{y}^2 + \dot{\rho}_1^2 + \dot{\rho}_2^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + \sin^2 \theta_1 \phi_1^\prime + \sin^2 \theta_2 \phi_2^\prime \\
+ \cos^2 \theta_1 \dot{\psi}_1^\prime + \cos^2 \theta_2 \dot{\psi}_2^\prime \right]^2 - \left( -\dot{t}^2 + \dot{y}^2 + \dot{\rho}_1^2 + \dot{\rho}_2^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + \sin^2 \theta_1 \phi_1^\prime \\
+ \sin^2 \theta_2 \phi_2^\prime + \cos^2 \theta_1 \dot{\psi}_1^\prime + \cos^2 \theta_2 \dot{\psi}_2^\prime \right) \left( -\dot{t}^2 + \dot{y}^2 + \dot{\rho}_1^2 + \dot{\rho}_2^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 \\
+ \sin^2 \theta_1 \phi_1'^2 + \sin^2 \theta_2 \phi_2'^2 + \cos^2 \theta_1 \psi_1'^2 + \cos^2 \theta_2 \psi_2'^2 \right) \right]^{1/2} \\
+ 2 \sin^2 \theta_1 (\dot{\phi}_1 \dot{\psi}_1' - \dot{\psi}_1 \dot{\phi}_1') + 2 \sin^2 \theta_2 (\dot{\phi}_2 \dot{\psi}_2' - \dot{\psi}_2 \dot{\phi}_2') 
\]  

(3.2)

and \( \sqrt{\lambda} = N \) is the 't Hooft coupling constant. For studying the rigidly rotating strings we choose the following ansatz

\[
t = \kappa \tau, \ y = \nu \tau, \ \rho_i = m_i \tau, \ \theta_i = \theta_i(\sigma), \ \phi_i = \nu_i \tau + \sigma, \ \psi_i = \omega_i \tau + \psi_i(\sigma), \ i = 1, 2 ,
\]  

(3.3)

The Euler-Lagrangian equations derived from the above action are given by,

\[
\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{t}} + \frac{\partial}{\partial y} \frac{\partial L}{\partial \dot{y}} + \frac{\partial}{\partial \rho_1} \frac{\partial L}{\partial \dot{\rho}_1} + \frac{\partial}{\partial \rho_2} \frac{\partial L}{\partial \dot{\rho}_2} + \frac{\partial}{\partial \theta_1} \frac{\partial L}{\partial \dot{\theta}_1} + \frac{\partial}{\partial \theta_2} \frac{\partial L}{\partial \dot{\theta}_2} + \frac{\partial}{\partial \phi} \frac{\partial L}{\partial \dot{\phi}} + \frac{\partial}{\partial \psi} \frac{\partial L}{\partial \dot{\psi}} = \frac{\partial}{\partial t} \frac{\partial L}{\partial t} + \frac{\partial}{\partial y} \frac{\partial L}{\partial y} + \frac{\partial}{\partial \rho} \frac{\partial L}{\partial \rho} + \frac{\partial}{\partial \theta} \frac{\partial L}{\partial \theta} + \frac{\partial}{\partial \phi} \frac{\partial L}{\partial \phi} + \frac{\partial}{\partial \psi} \frac{\partial L}{\partial \psi}. 
\]  

(3.4)

Next we have to solve these equations by the ansatz that we have proposed. Solving for \( t, \psi_1 \) and \( \psi_2 \) we get the following

\[
[C_1 \nu_1^2 \sin^2 \theta_1 + C_1 \nu_2^2 \sin^2 \theta_2 + C_1 \omega_1^2 \cos^2 \theta_2 + 2 \kappa \nu_1 \omega_1 \sin^2 \theta_1 - \alpha^2 C_1 - \kappa \omega_1 C_4] \psi_1^\prime \cos^2 \theta_1 \\
= [\kappa C_4 - 2 \kappa \nu_1 \sin^2 \theta_1 + C_1 \omega_1 \cos^2 \theta_1] \omega_2 \psi_2' \cos^2 \theta_2 + \nu_1 \sin^2 \theta_1 + \nu_2 \sin^2 \theta_2],
\]  

(3.9)
To make things look simpler, let us define the following constants

\[ \rho, P, K \]

and

\[ \kappa \]

one sphere, effectively making the string dynamics on \( R_t \times S^3 \). In this section we wish to study the case when the string is constrained to rotate only one direction.

\[ C \]

that the Lagrangian is invariant under the following transformation of the fields,

\[ E \]

where \( C_1, C_4 \) and \( C_5 \) are integration constants and \( \alpha = \sqrt{\kappa^2 - \nu^2 - m_1^2 - m_2^2} \). Note also that the Lagrangian is invariant under the following transformation of the fields,

\[ t'(\tau, \sigma) = t(\tau, \sigma) + \epsilon_t, \quad y'(\tau, \sigma) = y(\tau, \sigma) + \epsilon_y, \quad \rho'_1(\tau, \sigma) = \rho_1(\tau, \sigma) + \epsilon_{\rho_1}, \quad \rho'_2(\tau, \sigma) = \rho_2(\tau, \sigma) + \epsilon_{\rho_2}, \quad \psi'_1(\tau, \sigma) = \psi_1(\tau, \sigma) + \epsilon_{\psi_1}, \quad \psi'_2(\tau, \sigma) = \psi_2(\tau, \sigma) + \epsilon_{\psi_2}, \]

where \( \epsilon_t, \epsilon_y, \epsilon_{\rho_1}, \epsilon_{\rho_2}, \epsilon_{\psi_1}, \epsilon_{\psi_2} \) are constants. Hence the rigidly rotating string has a number of conserved charges, namely the total energy \( E \), angular momenta \( J_1 \), \( J_2 \), \( K_1 \) and \( K_2 \) corresponding to the translation along \( \phi_1 \), \( \phi_2 \), \( \psi_1 \) and \( \psi_2 \) directions respectively. The Conserved charge \( P_y, D_1 \) and \( D_2 \) are due to the translational invariance of \( y, \rho_1 \) and \( \rho_2 \) coordinates respectively. Now looking at (3.11), it is very hard to solve the equation in complete generality. Below we will look for solutions in some limits of \( \theta_1 \) and \( \theta_2 \).

### 3.1 For \( \theta_2 = 0 \) and \( \theta_1 = \theta \)

In this section we wish to study the case when the string is constrained to rotate only one one sphere, effectively making the string dynamics on \( R_t \times S^3 \). In this case the equation of motion simplifies considerably. We have to essentially solve the following two equations

\[ \begin{align*}
[C_1 \nu_1^2 \sin^2 \theta_1 + C_1 \omega_1^2 + 2\kappa \nu_1 \omega_1 \sin^2 \theta - \alpha^2 C_1 - \kappa \omega_1 C_4] \psi'_1 \cos^2 \theta_1 \\
= [\kappa C_4 - 2\kappa \nu_1 \sin^2 \theta + C_1 \omega_1 \cos^2 \theta] [\omega_2 \psi'_2 + \nu_1 \sin^2 \theta],
\end{align*} \]

\[ \begin{align*}
[C_1 \nu_1^2 \sin^2 \theta + C_1 \omega_1^2 \cos^2 \theta - \alpha^2 C_1 - \kappa \omega_2 C_5] \psi'_2 = [\kappa C_5 + C_1 \omega_2] [\omega_1 \psi'_1 \cos^2 \theta + \nu_1 \sin^2 \theta],
\end{align*} \]

To make things look simpler, let us define the following constants

\[ a = C_1 \nu_1^2 \sin^2 \theta - C_1 \alpha^2, \quad b = C_1 \omega_1^2 + 2\kappa \nu_1 \omega_1 \sin^2 \theta - \kappa \omega_1 C_4, \quad c = C_1 \omega_1^2 \cos^2 \theta - \kappa \omega_2 C_5, \quad d = \nu_1 \sin^2 \theta, \quad e = \kappa C_4 - 2\kappa \nu_1 \sin^2 \theta + C_1 \omega_1 \cos^2 \theta, \quad f = \kappa C_5 + C_1 \omega_2.
\]
With the above identifications, (3.9) and (3.10) reduce to
\[(a + b)ψ' \cos^2 θ = e(ω₂ψ'' + d),\]
\[(a + c)ψ_2 = f(ω₁ψ'_1 \cos^2 θ + d).\]

From these we can easily solve for \(ψ'_1\) and \(ψ'_2\), and they are given by,
\[ψ'_1 = \frac{de(a + c + ω₂f)}{\cos^2 θ[(a + b)(a + c) - ω₁ω₂ef]},\]  \hspace{1cm} (3.16)
\[ψ'_2 = \frac{df(a + b + ω₁e)}{[(a + b)(a + c) - ω₁ω₂ef]}.\]  \hspace{1cm} (3.17)

Further (3.11) is now given by
\[θ² = \frac{κ² - C_1²}{C_1²} \left(\nu_1 \sin^2 θ + ω₁ψ'_1 \cos^2 θ + ω₂ψ'_2\right) - \sin^2 θ - ψ'_1² \cos^2 θ - ψ'_2².\]  \hspace{1cm} (3.18)

Below we will discuss two limiting cases that correspond to single spike and giant magnon solutions.

3.1.1 Single Spike solution

In the limit \(θ' → 0\) as \(θ → \frac{π}{2}\), we obtain the condition
\[C_1 = \frac{κν_1}{\sqrt{α^2 - ω_2^2}},\] \hspace{1cm} (3.19)
and \(ψ'_1 → 0\) as \(θ → \frac{π}{2}\) implies \(C_4 = 2ν_1\) also \(ψ'_2 → 0\) as \(θ → \frac{π}{2}\) implies \(C_5 = -\frac{C_1ω_4}{κ}\). Using these values we get,
\[ψ'_1 = \frac{(ω_1 + 2√(α^2 - ω_2^2))ν_1 \sin^2 θ}{ν_1² \sin^2 θ - 2ω_1 √(α^2 - ω_2^2) \cos^2 θ + ω_2^2 - α^2},\] \hspace{1cm} (3.20)
\[ψ'_2 = 0,\] \hspace{1cm} (3.21)
and
\[θ' = \frac{3(ω_2^2 - ν_1²)(α^2 - ω_2^2) \cos θ \sin θ √(sin² θ - sin² θ₀)}{ν_1² \sin^2 θ - 2ω_1 √(α^2 - ω_2^2) \cos^2 θ + ω_2^2 - α^2},\] \hspace{1cm} (3.22)
where
\[sin² θ₀ = \frac{α^2 + 4ω_2^2 - ω_2^2 + 4ω_1 √(α^2 - ω_2^2)}{3(ω_2^2 - ν_1²)}.\]

Now we can calculate the conserved quantities as,
\[E = -2T \int_{θ₀}^{\frac{π}{2}} dθ \frac{∂L}{∂θ'} = \frac{2κT(α^2 - ν_1² - ω_2^2)}{(α^2 - ω_2^2) √(3(ω_2² - ν_1²))} \int_{θ₀}^{\frac{π}{2}} dθ \frac{sin θ}{cos θ √(sin² θ - sin² θ₀)},\] \hspace{1cm} (3.23)
\[P_y = 2T \int_{θ₀}^{\frac{π}{2}} dθ \frac{∂L}{∂y'} = \frac{2νT(α^2 - ν_1² - ω_2^2)}{(α^2 - ω_2²) √(3(ω_2² - ν_1²))} \int_{θ₀}^{\frac{π}{2}} dθ \frac{sin θ}{cos θ √(sin² θ - sin² θ₀)},\] \hspace{1cm} (3.24)
\[
D_1 = 2T \int_{\theta_0}^{\pi} \frac{d\theta}{\theta'} \frac{\partial \mathcal{L}}{\partial \dot{\rho}_1} = \frac{2m_1 T (\alpha^2 - \nu_1^2 - \omega_2^2)}{(\alpha^2 - \omega_2^2) \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_0}^{\pi} \frac{d\theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \quad (3.25)
\]

\[
D_2 = 2T \int_{\theta_0}^{\pi} \frac{d\theta}{\theta'} \frac{\partial \mathcal{L}}{\partial \dot{\rho}_2} = \frac{2m_2 T (\alpha^2 - \nu_1^2 - \omega_2^2)}{(\alpha^2 - \omega_2^2) \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_0}^{\pi} \frac{d\theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \quad (3.26)
\]

and finally the angular momentum \(K_2\) is given by,

\[
K_2 = 2T \int_{\theta_0}^{\pi} \frac{d\theta}{\theta'} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{2\omega_2 T (\alpha^2 - \nu_1^2 - \omega_2^2)}{(\alpha^2 - \omega_2^2) \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_0}^{\pi} \frac{d\theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}. \quad (3.27)
\]

They all diverge, their combination however can be written in as,

\[
\sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} = \frac{2T (\alpha^2 - \nu_1^2 - \omega_2^2)}{\sqrt{3(\omega_1^2 - \nu_1^2)}(\alpha^2 - \omega_2^2)} \int_{\theta_0}^{\pi} \frac{d\theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}. \quad (3.28)
\]

The so called deficit angle \(\Delta \phi\) also diverges, but the quantity

\[
\Delta \phi' = \Delta \phi + \frac{1}{T} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} = 2\theta_0 - \pi
\]

is finite. The regularised angular momenta \(J_1\) and \(K_1\) are given by,

\[
J_1 = J_1 - \frac{2\nu_1 (\omega_1 + 2\sqrt{\alpha^2 - \omega_2^2})}{(\alpha^2 - \nu_1^2 - \omega_2^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} = -\frac{2\nu_1 T (4\omega_1 + 5\sqrt{\alpha^2 - \omega_2^2})}{3(\alpha^2 - \omega_2^2)(\omega_1^2 - \nu_1^2)} \cos \theta_0 , \quad (3.30)
\]

\[
K_1 = K_1 - 2\sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} = \frac{2T (4\nu_1^2 + 5\omega_1 \sqrt{\alpha^2 - \omega_2^2})}{3(\alpha^2 - \omega_2^2)(\omega_1^2 - \nu_1^2)} \cos \theta_0 . \quad (3.31)
\]

If we define \(\tilde{J}_1 = -J\), then it is easy to show that the angular momenta \(\tilde{J}\) and \(K_1\) satisfy the following relation (for the parameters \(\omega_2 = 0, \alpha = \nu_1\)),

\[
\tilde{J} = \sqrt{\tilde{K}_1 - \frac{3\lambda}{\pi^2} \sin^2 \left(\frac{\pi}{2} - \theta_0\right)}.
\]

Note that this expression matches exactly with the ones derived in [33]. We wish to stress that in the present case the problem essentially reduces to that of rigidly rotating open string in the NS5-brane background.

### 3.1.2 Giant Magnon solution

In the opposite limiting case we get the condition \(\alpha^2 = \nu_1^2 + \omega_1^2\), and by putting \(C_4 = 2\nu_1\) and \(C_5 = -\frac{C_1 \omega_1}{\kappa}\), we obtain the following

\[
\psi' = \frac{(C_1 \omega_1 + 2\kappa \nu_1) \sin^2 \theta}{(C_1 \nu_1 + 2\kappa \omega_1) \cos^2 \theta} , \quad \psi'' = 0 , \quad (3.33)
\]

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and
\[ \theta' = \frac{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}}{\sin \theta \cos \theta} \],

(3.34)

where
\[ \sin \theta_1 = \frac{C_1 \nu_1 + 2 \kappa \omega_1}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} . \]

Now the conserved quantities become
\[ E = \frac{-2T(\kappa^2 - C_1^2)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} ; \]

(3.35)
\[ P_y = \frac{-2
\frac{\kappa}{T}(\kappa^2 - C_1^2)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} ; \]

(3.36)
\[ D_1 = \frac{-2m_1T(\kappa^2 - C_1^2)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} ; \]

(3.37)
\[ D_2 = \frac{-2m_2T(\kappa^2 - C_1^2)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} ; \]

(3.38)
\[ K_2 = \frac{-2\omega_2T(\kappa^2 - C_1^2)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} . \]

(3.39)

All these quantities diverges and their combination is given by,
\[ \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} = \frac{2\sqrt{\alpha^2 - \omega_1^2 T(\kappa^2 - C_1^2)}}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \frac{\sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} . \]

(3.40)

The deficit angle
\[ \Delta \phi = 2T \int_{\theta_1}^{\frac{\pi}{2}} \frac{d\theta}{\theta} = \pi - 2\theta_1 \]

(3.41)
is however finite. The regularised \( J_1 \) is given by,
\[ J_1 = J_1 + \frac{(5\kappa^2 \nu_1 + 2C_1 \kappa \omega_1 - C_1^2 \nu_1)}{\sqrt{\alpha^2 - \omega_1^2 (\kappa^2 - C_1^2)}} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} \]
\[ = \frac{2T(4C_1 \omega_1 + 5\kappa \nu_1)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \cos \theta_1 . \]

(3.42)

Angular momenta \( K_1 \) is once again finite and is given by,
\[ K_1 = \frac{-2T(4C_1 \nu_1 + 5\kappa \omega_1)}{\kappa \sqrt{3(\omega_1^2 - \nu_1^2)}} \cos \theta_1 \]

(3.43)

To know a precise relation between the energy, angular momentum and the distance between two end point of the string, we rescale the expression \( \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} \) as follows,
\[ \tilde{E} = \frac{(5\kappa^2 \nu_1 + 2C_1 \kappa \omega_1 - C_1^2 \nu_1)}{\sqrt{\alpha^2 - \omega_1^2 (\kappa^2 - C_1^2)}} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2 - K_2^2} . \]

(3.44)
If we define $J = -J_1$, then by looking at the form of various quantities, it is easy to derive (for $C_1 = \kappa$)

$$
\tilde{E} - J = \sqrt{K_1^2 - \frac{3\lambda}{\pi^2} \sin^2 \frac{\Delta \phi}{2}}. 
$$

(3.45)

Once again, this relationship was derived in [33], where $\Delta \phi$ was identified as the worldsheet momentum along the string worldsheet.

### 3.2 For $\theta_1 = \theta_2 = \theta$

In this subsection we wish to study the solution of the rotating string when it rotates around both the spheres. To make our calculations doable and simple, we will restrict ourselves to the case when $\theta_1 = \theta_2 = \theta$. To this end, let us define

$$
\begin{align*}
    a &= C_1(\nu_1^2 + \nu_2^2)\sin^2 \theta - C_1\alpha^2, \\
    b &= C_1\omega_2^2\cos^2 \theta + 2\kappa\nu_1\omega_1\sin^2 \theta - \kappa\omega_1C_4, \\
    c &= C_1\omega_1^2\cos^2 \theta + 2\kappa\nu_2\omega_2\sin^2 \theta - \kappa\omega_2C_5, \\
    d &= (\nu_1 + \nu_2)\sin^2 \theta, \\
    e &= \kappa C_4 - 2\kappa\nu_1\sin^2 \theta + C_1\omega_1\cos^2 \theta, \\
    f &= \kappa C_5 - 2\kappa\nu_2\sin^2 \theta + C_1\omega_2\cos^2 \theta.
\end{align*}
$$

(3.46)

With the above identifications, first two equations (3.9) and (3.10) reduce to

$$
\begin{align*}
    (a + b)\psi_1'\cos^2 \theta &= e(\omega_2\psi_2'\cos^2 \theta + d), \\
    (a + c)\psi_2'\cos^2 \theta &= f(\omega_1\psi_1'\cos^2 \theta + d).
\end{align*}
$$

(3.47)

From these we can easily solve for $\psi_1'$ and $\psi_2'$, and they are given by,

$$
\begin{align*}
    \psi_1' &= \frac{de(a + c + \omega_2f)}{\cos^2 \theta((a + b)(a + c) - \omega_1\omega_2ef)}, \\
    \psi_2' &= \frac{df(a + b + \omega_1e)}{\cos^2 \theta((a + b)(a + c) - \omega_1\omega_2ef)}.
\end{align*}
$$

(3.48)

(3.49)

Further (3.11) is now given by

$$
\theta'^2 = \frac{\kappa^2 - C_1^2}{2C_1^2} \left[\frac{(\nu_1 + \nu_2)\sin^2 \theta + (\omega_1\psi_1' + \omega_2\psi_2')\cos^2 \theta}{\alpha^2 - (\nu_1^2 + \nu_2^2)\sin^2 \theta - (\omega_1^2 + \omega_2^2)\cos^2 \theta} \right]^2 - \sin^2 \theta - \frac{1}{2} \left(\psi_1'^2 + \psi_2'^2\right)\cos^2 \theta
$$

(3.50)

Below we will discuss the different limits that correspond to the single spike and giant magnon solutions.

#### 3.2.1 Single Spike solution

In the limit $\theta' \to 0$ as $\theta \to \frac{\pi}{2}$, we obtain the condition

$$
C_1 = \frac{\kappa(\nu_1 + \nu_2)}{\alpha_1},
$$

(3.51)
where $\alpha_1 = \sqrt{2\alpha^2 - (\nu_1 - \nu_2)^2}$ and $\psi'_1 \to 0$ and $\psi'_2 \to 0$ as $\theta \to \frac{\pi}{2}$ implies $C_4 = 2\nu_1$ and $C_5 = 2\nu_2$ respectively. Using these values we get,

$$
\psi'_1 = \frac{(C_1\omega_1 + 2\kappa\nu_1)(\nu_1 + \nu_2)\sin^2 \theta}{C_1(\nu_1^2 + \nu_2^2) \sin^2 \theta - C_1\alpha^2 - 2\kappa\alpha^2 \cos^2 \theta}, \tag{3.52}
$$

and

$$
\psi'_2 = \frac{(C_1\omega_2 + 2\kappa\nu_2)(\nu_1 + \nu_2)\sin^2 \theta}{C_1(\nu_1^2 + \nu_2^2) \sin^2 \theta - C_1\alpha^2 - 2\kappa\alpha^2 \cos^2 \theta}, \tag{3.53}
$$

where $\alpha_2 = \nu_1 \omega_1 + \nu_2 \omega_2$. Now the equation for $\theta$ can be written down as

$$
\theta' = \frac{\sqrt{\frac{h}{\pi}} \sin \theta \cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}{(\nu_1 + \nu_2)(\nu_1^2 + \nu_2^2) \sin^2 \theta - 2\alpha_1\alpha_2 \cos^2 \theta - (\nu_1 + \nu_2)\alpha^2}, \tag{3.54}
$$

where

$$
\sin^2 \theta_0 = \frac{2}{h} \left[(\nu_1 + \nu_2)\alpha^2 + 2\alpha_1\alpha_2\right]^2, \tag{3.55}
$$

with

$$
h = (\nu_1 + \nu_2)^2[(\nu_1^2 + \nu_2^2)(2\alpha^2 - 4\alpha_1^2) - \alpha_1^2(\omega_1^2 + \omega_2^2)] + 8\alpha_1^2\alpha^2 + 4\alpha_1\alpha_2(\nu_1 + \nu_2)(\nu_1 - \nu_2)^2. \tag{3.56}
$$

Now we can calculate the conserved quantities as

$$
E = -2T \int_{\theta_0}^{\pi} \frac{d\theta \partial L}{\partial \theta'} \frac{\partial L}{\partial \dot{\theta}} = \frac{4\kappa T(\nu_1 + \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2)}{\sqrt{\frac{h}{2}\alpha_1}} \int_{\theta_0}^{\pi} \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \tag{3.57}
$$

$$
P_y = 2T \int_{\theta_0}^{\pi} \frac{d\theta \partial L}{\partial \theta'} \frac{\partial L}{\partial \dot{y}} = \frac{4\nu T(\nu_1 + \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2)}{\sqrt{\frac{h}{2}\alpha_1}} \int_{\theta_0}^{\pi} \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \tag{3.58}
$$

$$
D_1 = 2T \int_{\theta_0}^{\pi} \frac{d\theta \partial L}{\partial \theta'} \frac{\partial L}{\partial \dot{p}_1} = \frac{4m_1 T(\nu_1 + \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2)}{\sqrt{\frac{h}{2}\alpha_1}} \int_{\theta_0}^{\pi} \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \tag{3.59}
$$

$$
D_2 = 2T \int_{\theta_0}^{\pi} \frac{d\theta \partial L}{\partial \theta'} \frac{\partial L}{\partial \dot{p}_2} = \frac{4m_2 T(\nu_1 + \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2)}{\sqrt{\frac{h}{2}\alpha_1}} \int_{\theta_0}^{\pi} \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}, \tag{3.60}
$$

All the conserved quantities $E$, $P_y$, $D_1$ and $D_2$ described above are infinite. For our later use, we define the following rescaled quantity,

$$
\sqrt{E^2 - P_y^2 - D_1^2 - D_2^2} = \frac{4\alpha T(\nu_1 + \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2)}{\sqrt{\frac{h}{2}\alpha_1}} \int_{\theta_0}^{\pi} \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_0}}. \tag{3.61}
$$

Now the Deficit angle is defined by $\Delta \phi = 2 \int_{\theta_0}^{\pi} \frac{d\theta}{\theta'}$ is infinite but

$$
\Delta \phi' = \Delta \phi + \frac{\alpha_1}{2\alpha T} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2} = 2\theta_0 - \pi \tag{3.62}
$$
is finite. Similarly the regularized $J_1$, $J_2$, $K_1$, and $K_2$ are given by

\[
\tilde{J}_1 = J_1 - \frac{[(\nu_1 - \nu_2)(\alpha^2 - \nu_1^2 - \nu_2^2) + 2\alpha_1\omega_1(\nu_1 + \nu_2) + 4\alpha_1^2\nu_1]}{2\alpha(\alpha^2 - \nu_1^2 - \nu_2^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2}
\]

\[
= -\frac{2T(\nu_1 + \nu_2)\cos \theta_0}{\alpha_1 \sqrt{\frac{2}{\pi}}} \left[(\nu_2 - \nu_1)(\nu_1^2 + \nu_2^2) + 2\alpha_1\omega_1(\nu_1 + \nu_2) + 2\nu_1(\alpha^2 + 2\alpha_1^2) + 2\alpha_1\alpha_2 \right],
\]

(3.63)

\[
\tilde{J}_2 = J_2 - \frac{[(\nu_2 - \nu_1)(\alpha^2 - \nu_1^2 - \nu_2^2) + 2\alpha_1\omega_2(\nu_1 + \nu_2) + 4\alpha_1^2\nu_2]}{2\alpha(\alpha^2 - \nu_1^2 - \nu_2^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2}
\]

\[
= -\frac{2T(\nu_1 + \nu_2)\cos \theta_0}{\alpha_1 \sqrt{\frac{2}{\pi}}} \left[(\nu_1 - \nu_2)(\nu_1^2 + \nu_2^2) + 2\alpha_1\omega_2(\nu_1 + \nu_2) + 2\nu_2(\alpha^2 + 2\alpha_1^2) + 2\alpha_1\alpha_2 \right],
\]

(3.64)

\[
\tilde{K}_1 = \frac{2T\cos \theta_0}{\sqrt{\frac{2}{\pi}}} \left[(\nu_1 + \nu_2)(\alpha_1\omega_1 + 2(2\nu_1^2 + \nu_1\nu_2 + \nu_2^2)) + 4\alpha_1\alpha_2 \right],
\]

(3.65)

\[
\tilde{K}_2 = \frac{2T\cos \theta_0}{\sqrt{\frac{2}{\pi}}} \left[(\nu_1 + \nu_2)(\alpha_1\omega_2 + 2(2\nu_1^2 + \nu_1\nu_2 + \nu_2^2)) + 4\alpha_1\alpha_2 \right]
\]

(3.66)

where $\tilde{K}_i = K_i - \frac{\alpha^2}{4\alpha} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2}$, with $i = 1, 2$. Now if define $\tilde{J} = \tilde{J}_1 - \tilde{J}_2$ and $\tilde{K} = \tilde{K}_1 - \tilde{K}_2$ then we obtain,

\[
\tilde{J} = \sqrt{\tilde{K}^2 + \frac{2g(\nu_1 + \nu_2)^2 \lambda}{h \alpha_1^2 \pi^2 \sin^2(\frac{\pi}{2} - \theta_0)}},
\]

(3.67)

where

\[
g = \left[2(\nu_2 - \nu_1)(\alpha^2 - \nu_1^2 - \nu_2^2 + 2\alpha_1^2) - 2\alpha_1(\omega_1 - \omega_2)(\nu_1 + \nu_2) \right]^2
\]

\[
- \left[2\alpha_1(\nu_1^2 - \nu_2^2) + \alpha_1^2(\omega_1 - \omega_2) \right]^2
\]

\[
h = (\nu_1 + \nu_2)^2 [(\nu_1^2 + \nu_2^2)(2\alpha^2 - 4\alpha_1^2) - \alpha_1^2(\omega_1^2 + \omega_2^2)] + 8\alpha_1^2 \alpha_2^2
\]

\[
+ 4\alpha_1\alpha_2(\nu_1 + \nu_2)(\nu_1 - \nu_2)^2.
\]

(3.68)

Few comments are in order. The dispersion relation obtained has complicated prefactor multiplied with $\lambda \alpha^2$ can be attributed to the fact that this dispersion relation is not of a single spike when the string rigidly rotates around the sphere, but is supported by a large number of other charges including that of NS-NS B-field which supports the background and the fundamental string knows the presence of such charges through the open string boundary condition. Furthermore the presence of charges like $D_1$ and $D_2$ do not imply any new interpretation of the dispersion relation. It simply reflects the fact that the motion of the string in the radial direction in the near horizon geometry of I-brane is free.
3.2.2 Giant Magnon solution

In the opposite limit corresponding to the one just described above, we get

$$\alpha^2 = \nu_1^2 + \nu_2^2,$$

and by putting $C_4 = 2\nu_1$ and $C_5 = 2\nu_2$ we get

$$\psi'_1 = -\frac{(\nu_1 + \nu_2)(C_1\omega_1 + 2\kappa\nu_1)\sin^2 \theta}{\beta \cos^2 \theta},$$

$$\psi'_2 = -\frac{(\nu_1 + \nu_2)(C_1\omega_2 + 2\kappa\nu_2)\sin^2 \theta}{\beta \cos^2 \theta},$$

where $\beta = 2\kappa\nu_1\omega_1 + 2\kappa\nu_2\omega_2 + C_1\alpha^2$ and

$$\theta' = \frac{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}}{\sin \theta_1 \cos \theta},$$

where $\sin^2 \theta_1 = \frac{2\beta^2}{h_1}$ and

$$h_1 = (\nu_1 + \nu_2)^2[(\kappa^2 - C_1^2)(\alpha^2 - \omega_1^2 - \omega_2^2) - (C_1\omega_1 + 2\kappa\nu_1)^2 - (C_1\omega_2 + 2\kappa\nu_2)^2] + 2\beta^2.$$

The conserved quantities are

$$E = -\frac{2T(\nu_1 + \nu_2)(\kappa^2 - C_1^2)}{\sqrt{h_1}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \sin \theta \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}},$$

$$P_y = -\frac{2\nu T(\nu_1 + \nu_2)(\kappa^2 - C_1^2)}{\kappa \sqrt{h_1}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \sin \theta \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}},$$

$$D_1 = -\frac{2m_1 T(\nu_1 + \nu_2)(\kappa^2 - C_1^2)}{\kappa \sqrt{h_1}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \sin \theta \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}},$$

$$D_2 = -\frac{2m_2 T(\nu_1 + \nu_2)(\kappa^2 - C_1^2)}{\kappa \sqrt{h_1}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \sin \theta \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}}.$$

Here also the conserved quantities $E$, $P_y$, $D_1$ and $D_2$ are all infinite and the rescaled combination can be written as,

$$\sqrt{E^2 - P^2_y - D^2_1 - D^2_2} = \frac{2\nu T(\nu_1 + \nu_2)(\kappa^2 - C_1^2)}{\kappa \sqrt{h_1}} \int_{\theta_1}^{\frac{\pi}{2}} d\theta \sin \theta \frac{d\theta \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}}.$$  

The deficit angle $\Delta \phi$ is however finite and is given by

$$\Delta \phi = 2 \int_{\theta_1}^{\frac{\pi}{2}} \frac{d\theta}{\theta'} = 2 \int_{\theta_1}^{\frac{\pi}{2}} \frac{d\theta \sin \theta_1 \cos \theta}{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \theta_1}} = \pi - 2\theta_1.$$
The regularized \( J_1 \) and \( J_2 \) are given by,

\[
\tilde{J}_1 = J_1 + \frac{(5\kappa^2 \nu_1 + 2C_1 \kappa \omega_1 - C_1^2 \nu_1)}{\alpha(\kappa^2 - C_1^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2}
\]

\[
= \frac{2T \cos \theta_1}{\kappa \sqrt{\frac{h_1}{2}}} \left[ C_1 \beta + (\nu_1 + \nu_2)(5\kappa^2 \nu_1 + 2C_1 \kappa \omega_1 - C_1^2 \nu_1) \right],
\]

(3.79)

and

\[
\tilde{J}_2 = J_2 + \frac{(5\kappa^2 \nu_2 + 2C_1 \kappa \omega_2 - C_1^2 \nu_2)}{\alpha(\kappa^2 - C_1^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2}
\]

\[
= \frac{2T \cos \theta_1}{\kappa \sqrt{\frac{h_1}{2}}} \left[ C_1 \beta + (\nu_1 + \nu_2)(5\kappa^2 \nu_2 + 2C_1 \kappa \omega_2 - C_1^2 \nu_2) \right],
\]

(3.80)

The other two angular momenta are finite and are given by,

\[
K_1 = -\frac{2T \cos \theta_1}{\sqrt{\frac{h_1}{2}}} [(\nu_1 + \nu_2)(\kappa \omega_1 + 2C_1 \nu_1) + 2\beta],
\]

(3.81)

and

\[
K_2 = -\frac{2T \cos \theta_1}{\sqrt{\frac{h_1}{2}}} [(\nu_1 + \nu_2)(\kappa \omega_2 + 2C_1 \nu_2) + 2\beta],
\]

(3.82)

Similarly as before, if we define \( \tilde{J} = \tilde{J}_1 - \tilde{J}_2 \), \( K = K_1 - K_2 \) and \( J = J_2 - J_1 \) then we obtain,

\[
\tilde{E} - J = \sqrt{K^2 + \frac{2g_1 (\nu_1 + \nu_2)^2 \lambda}{\kappa^2 \pi^2 \sin^2 \frac{\Delta \phi}{2}},
\]

where

\[
\tilde{E} = \frac{2\kappa C_1 (\omega_1 - \omega_2) + (5\kappa^2 - C_1^2)(\nu_1 - \nu_2)}{\alpha(\kappa^2 - C_1^2)} \sqrt{E^2 - P_y^2 - D_1^2 - D_2^2},
\]

\[
g_1 = [2\kappa C_1 (\omega_1 - \omega_2) + (5\kappa^2 - C_1^2)(\nu_1 - \nu_2) - [\kappa^2 (\omega_1 - \omega_2) + 2C_1 \kappa (\nu_1 - \nu_2)]^2],
\]

\[
h_1 = (\nu_1 + \nu_2)^2 [(\kappa^2 - C_1^2)(\alpha^2 - \omega_1^2 - \omega_2^2) - (C_1 \omega_1 + 2\kappa \nu_1)^2 - (C_1 \omega_2 + 2\kappa \nu_2)^2] + 2\beta^2.
\]

(3.84)

Once again we would like to stress that this dispersion relation must be compared with that in the presence of other background fields. It will be really a challenge to find the dual states in the corresponding field theory side.
4. Conclusions

We have studied in this paper the rigidly rotating strings in the near horizon geometry of a 1+1 dimensional intersection of two stacks of orthogonal branes, the so called I-brane background. We have studied the solutions of fundamental string action in this background in the presence of various NS-NS B-fields. Because the fundamental string couple to the background NS-NS field, the string dynamics knows the presence of these fluxes and the dispersion relation also reflects this fact. As the background in question is too difficult to solve exactly we have assumed first the motion of the string constrained only on one sphere and found out two solutions corresponding to single spike and giant magnons. This procedure effectively means as if we are considering the motion of the string in the near horizon geometry of one stack of NS5-branes. Further we have generalized the solution to include motion on both the spheres and write down corresponding dispersion relation for the giant magnon and single spike solutions. We have also remarked that these solutions should be compared to the ones obtained in the presence of other background fields that couple to the string. We have used the open string boundary condition to explicitly observe the effect on the dispersion relation among various conserved charges.

There are various questions that can be persued further. As we know, the fact that I-brane background is an exact solution of string theory equations of motion, helps us in obtaining more information about the intersecting brane system itself which is otherwise not possible via gauge theory. It will definitely be interesting to see from a probe brane analysis what more information we get about the system which is apriori unknown from the field theory. Further it will be interesting to tell more about the existence of some operators corresponding to the spiky strings from the point of view of dual theory, whose exact nature in not known in detail. The supersymmetry of such states can be checked by following [26] in the bulk. Hence it will be interesting to see whether similar kind of states exist in the dual theory. It would also be interesting to whether such spiky strings exist as solutions in the worldvolume of D-branes.

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