A novel four-parameter log-logistic model: mathematical properties and applications to breaking stress, survival times and leukemia data

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Abstract
In this paper, we introduce a new continuous log-logistic extension. Several of its properties are established. A numerical analysis for skewness and kurtosis is presented. The new failure rate can be "bathtub" or "U shaped", "increasing", "decreasing-constant", "J shaped", "constant" and "decreasing". Many bivariate and multivariate type distributions are derived using the Clayton Copula and the Morgenstern family. To assess of the finite sample behavior of the estimators, we performed a graphical simulation. Some useful applications are considered for supporting the new model.

Key Words: Burr XII Distribution; Log-Logistic Model; Maximum Likelihood; Leukemia Data; Modeling; Simulation.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and motivation
A continuous random variable (RV) $Y$ is said to have the log-logistic (LL) model if its survival/reliability function (SF) be written as

$$S_\beta(y) = 1 - S_\beta(y) = (1 + ye^\beta)^{-1},$$

where $\beta > 0$ is a shape parameter and $S_\beta(y)$ refer to the cumulative distribution functions (CDF) of the LL model. The density function (PDF) due to (1) is given as

$$s_\beta(y) = \beta y e^{\beta - 1} (1 + ye^\beta)^{-2}.$$ (2)

The CDF and PDF in (1) and (2) is a sub-model of the Burr model from the type-XII (BUXII) model (Burr (1942, 1968 and 1973), Tadikamalla (1980) and Rodriguez (1977)). Due to Yousof et al. (2018), the CDF of the new Weibull Generalized log-logistic (WG-LL) is defined as

$$F(y) = 1 - e^{-a[(1 + ye^\beta)^{\beta} - 1]^{\alpha}},$$

where $y > 0$, $\alpha, \beta, a > 0$ and its corresponding PDF is given by

$$f(y) = a\beta a\beta e^{\beta - 1} (1 + ye^\beta)^{\beta - 1} \left[(1 + ye^\beta)^{\beta} - 1\right]^{\alpha - 1} e^{-a[(1 + ye^\beta)^{\beta} - 1]^{\alpha}}.$$ (4)

Figure 1 below gives some graphical results for the novel PDF and its corresponding HRF for the WG-LL model. Due to Fig. 1(left chart) it is seen that the novel PDF of the WG-LL model can be monotonical and unimodal PDF, symmetric PDF or negative skewed PDF. From Figure 1(right chart) the HRF can be "bathtub" ($\alpha = 0.85, \beta =$
0.5, \( \alpha = 0.001, \beta = 0.55 \) or “increasing (monotone-HRF)” \( (\alpha = 0.95, \beta = 7, \alpha = 0.0001, \beta = 0.55) \) or "decreasing-constant" \( (\alpha = 0.85, \beta = 3, \alpha = 0.05, \beta = 0.5) \) or "J-shaped" \( (\alpha = 2, \beta = 5, \alpha = 0.01, \beta = 1.5 ) \) or "constant" \( (\alpha = 1, \beta = 1, \alpha = 1, \beta = 1 ) \) or "decreasing (monotone-HRF)" \( (\alpha = 1, \beta = 0.5, \beta = \alpha = 1 ) \).

Due to Yousof et al. (2018), the new PDF in (4) can be re-expressed as

\[
f(y) = \sum_{j_{\alpha}=0}^{\infty} C[j_{\alpha}^\alpha]s_{[\phi,(1+j_{\alpha})]}(y),
\]

where
\[
s_{[\phi,(1+j_{\alpha})]}(y) = (1+j_{\alpha}) \beta y^{-1}(1+y^\beta)^{-2-j_{\alpha}}
\]
is the LL PDF with \( \beta \) and \( (1+j_{\alpha}) \) where
\[
C[j_{\alpha}] = \sum_{j_{1}+j_{2}+j_{3}+j_{4}=0}^{\infty} (-1)^{1+j_{1}+j_{2}+j_{3}+j_{4}} \frac{1}{j_{1}!j_{2}!j_{3}!j_{4}!} \beta^{j_{1}+j_{2}+j_{3}+j_{4}} \\alpha^{j_{1}} \left( \frac{\beta-j_{1} \alpha}{j_{2}} \right) \left( \frac{\beta-j_{1} \alpha}{j_{3}} \right) ^{j_{3}} \left( \frac{\beta-j_{1} \alpha}{j_{4}} \right) ^{j_{4}}.
\]

Analogously, the new CDF (3) can be re-expressed as

\[
F(y) = \sum_{j_{\alpha}=0}^{\infty} C[j_{\alpha}]s_{[\phi,(1+j_{\alpha})]}(y),
\]

where
\[
s_{[\phi,(1+j_{\alpha})]}(y) = -(1+y^\beta)^{-1-j_{\alpha}} + 1,
\]
represents the CDF of the well-known LL model with parameters \( \beta \) and \( (1+j_{\alpha}) \). This work can be motivated via the following applied justifications:

i. The new model in its novel pattern will be useful in mathematical modeling of the engineering real-life datasets such as the “monotonically-increasing HRF” engineering breaking stress real-life dataset.

ii. The novel model in its novel version can be used for statistical modeling of the reliability real-life datasets such as the “monotonically-increasing HRF” reliability real-life dataset.

iii. The new current version of the log-logistic version can be used in modeling the medical real-life datasets such as the “U- HRF” medical real-life dataset.
2. Properties

Moments and generating function

The m̅th ordinary moment of Y is given by

$$\mu_{m,Y} = E(Y^m) = \int_{-\infty}^{\infty} f(y) y^m dy.$$  

Then, we obtain

$$\mu_{m,Y} = \sum_{j_4=0}^{\infty} C_{[j_4]}(1 + \hat{j}_4)B \left( 1 + \frac{m}{\hat{j}_4}, (1 + \hat{j}_4) - \frac{m}{\hat{j}_4} \right) |_{m<(1+\hat{j}_4)\theta}.$$  

where

$$B(d_1, d_2) = \int_{0}^{\infty} t^{d_1-1}(1+t)^{-(d_1+d_2)} dt,$$

is the second type of beta function. By fixing \( m = 1 \), we obtain the mean of the model. The m̅th incomplete moment (\( I_{n}(t) \)) of Y can be expressed from (3) as

$$I_{m,Y}(t) = \int_{-\infty}^{t} y^m f(y) dy = \sum_{j_4=0}^{\infty} C_{[j_4]}(1 + \hat{j}_4)B \left( t^{\theta}; 1 + \frac{m}{\hat{j}_4}, (1 + \hat{j}_4) - \frac{m}{\hat{j}_4} \right) |_{m<(1+\hat{j}_4)\theta},$$

where

$$B(q; a_1, a_2) = \int_{0}^{q} t^{a_1-1}(1+t)^{-(a_1+a_2)} dt$$

is the second type of the incomplete beta function. The moment generating function (mgf) \( M_Y(t) = E(exp(tY)) \) of Y can be derived from (3) as

$$M_Y(t) = \sum_{j_4=0}^{\infty} \frac{t^n}{n!} C_{[j_4]}(1 + \hat{j}_4)B \left( 1 + \frac{n}{\hat{j}_4}, (1 + \hat{j}_4) - \frac{n}{\hat{j}_4} \right) |_{n<(1+\hat{j}_4)\theta},$$

Probability weighted moments (PWMs)

The (m,r)th PWM of Y can be expressed as

$$P_{m,r,Y} = \sum_{r=0}^{\infty} C_{[r]}(1 + r)B \left( \frac{m}{\hat{j}_4} + 1, (1 + r) - \frac{m}{\hat{j}_4} \right) |_{m<(1+r)\theta},$$

where

$$C_{[r]} = \alpha \beta a \sum_{(i,j_1,j_2,j_3)=0}^{\infty} (-1)^{i+j_1+j_2+j_3} \frac{x(1+i)^{i_1}}{i(1+r)B(1+\hat{j}_4)^{-1}(r)_{j_3}} \times (1+\hat{j}_4)(1+\hat{j}_4)_{r-1}$$

and \((c_1)c_2 = c_1(c_1-1)\ldots(1+c_1-c_2)\) is the common factorial for the descending king and \( c_2 \) should be integer and also positive.

Reversed residual life Moment (MRRL) function

The m̅th MRRL, say

$$A_{m,Y}(t) = E[(t - Y)^{m} |_{y \geq t, t > 0, m=1,2,...}],$$

Then, we have

$$A_{m}(t) = F^{-1}(t) \int_{0}^{t} (t - y)^{m} dF(y).$$

Then, the m̅th MRRL of Y becomes

$$A_{m,Y}(t) = F^{-1}(t) \sum_{r=0}^{\infty} C_{[r]}^{(m)} (1 + \hat{j}_4)B \left( t^{\theta}; 1 + \hat{j}_4 - \frac{m}{\hat{j}_4}, 1 + \frac{m}{\hat{j}_4} \right),$$

Where

$$C_{[r]}^{(m)} = C_{[r]} \sum_{r=0}^{m} (-1)^{r} \binom{m}{r} t^{m-r}.$$
3. Checking flexibility numerically

In this section, the impacts of the parameters on the model mean ($\mu'_1$), variance of the model ($V(X)$), skewness of the model ($S(X)$) and kurtosis of the model ($K(X)$) are given below in Table 1. The impacts of $\beta$ for the standard LL model on the $\mu'_1$, $V(X)$, $S(X)$ and $K(X)$ are provided in Table 2.

For the new WG-LL model, $S(x) \in (-1.08107, 16.7425)$. However, for the LL model, $S(X) \in (0.0872, 2.4853)$. For the novel WG-LL model, $K(x) \in (3.2451, 702.5)$. However, for the LL model, $K(X) \in (3.7409, 29.5562)$.

| $\alpha$ | $\beta$ | $a$ | $b$ | $\mu'_1$ | $V(X)$ | $S(X)$ | $K(X)$ |
|---------|---------|-----|-----|---------|-------|-------|-------|
| 0.35    | 1       | 1   | 1   | 5.029144| 399.0459| 16.74246| 702.4982|
| 0.5     | 1       | 1   | 1   | 5.029144| 399.0459| 16.74246| 702.4982|
| 1       | 1       | 1   | 1   | 5.029144| 399.0459| 16.74246| 702.4982|
| 2       | 0.8862269| 0.2146018| 0.6311107| 702.4982|
| 10      | 0.9513508| 0.01310046| -0.6376371| 702.4982|
| 35      | 0.984295 | 0.001249773| -0.978317 | 702.4982|
| 50      | 0.9884442| 0.0006253426| -1.024853 | 702.4982|
| 75      | 0.9924775| 0.0001603049| -1.081074 | 702.4982|
| 100     | 0.9943259| 0.0001603049| -1.081074 | 702.4982|

Table 1: $\mu'_1$, $V(X)$, $S(X)$, $K(X)$ for the WG-LL model.
4. Copula

**Bivariate WG-LL via Morgenstern family**

First, consider the CDF for Morgenstern model

\[ F_\lambda(y_1, y_2)_{|\lambda| |\delta_1|} = F_1(y_1)F_2(y_2)[1 + \lambda[1 - F_1(y_1)][1 - F_2(y_2)]]. \]

Let

\[ F_1(y_1) = 1 - e^{-\alpha_1 \left(1 + y_1^{\delta_1} \right)^{\beta_1 - 1} \gamma_1} \]

and

\[ F_2(y_2) = 1 - e^{-\alpha_2 \left(1 + y_2^{\delta_2} \right)^{\beta_2 - 1} \gamma_2} \]

then we have new bivariate model as

\[ F_\lambda(y_1, y_2)_{|\lambda| |\delta_1|} = \left(1 - e^{-\alpha_1 \left(1 + y_1^{\delta_1} \right)^{\beta_1 - 1} \gamma_1}\right) \left(1 - e^{-\alpha_2 \left(1 + y_2^{\delta_2} \right)^{\beta_2 - 1} \gamma_2}\right) \times \left\{1 + \lambda \left[e^{-\alpha_1 \left(1 + y_1^{\delta_1} \right)^{\beta_1 - 1} \gamma_1}\right] \left[e^{-\alpha_2 \left(1 + y_2^{\delta_2} \right)^{\beta_2 - 1} \gamma_2}\right] \right\}. \]

**Bivariate WG-LL via Clayton Copula**

Consider the following Clayton Copula

\[ C(u, v) = \left[u^{-(\delta_1 + \delta_2)} + v^{-(\delta_1 + \delta_2)} - 1\right]^{-\frac{1}{\delta_1 + \delta_2}}. \]

Then, setting

\[ u = 1 - e^{-\alpha_1 \left(1 + y_1^{\delta_1} \right)^{\beta_1 - 1} \gamma_1} \]
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and

\[ v = 1 - e^{-\alpha z[(1+\beta)\beta_2 - 1]^{\alpha_2}} \]

The bivariate CDF can be written as

\[ C(u, v) = \left( 1 - e^{-\alpha z[(1+x_1^{\beta_1} - 1)]^{\alpha_1}} \right)^{(\delta_1 + \delta_2)} + \left( 1 - e^{-\alpha z[(1+y_1^{\beta_2} - 1)]^{\alpha_2}} \right)^{(\delta_1 + \delta_2)} - 1 \]

\[ \frac{1}{\delta_1 + \delta_2} \]

The Multivariate extension via Copula of Clayton

The d-dimensional model can be expressed as

\[ H(x_i) = \left( \sum_{i=1}^{d} \left( 1 - e^{-\alpha z[(1+x_i^{\beta_i} - 1)]^{\alpha_i}} \right)^{(\delta_1 + \delta_2)} + 1 - d \right)^{\frac{1}{\delta_1 + \delta_2}} \]

where \( x_1, x_2, \ldots, x_d \). For other copulas see Ali et al. (2020a), Ali et al. (2020b), Elgohari et al. (2021), Elgohari and Yousof (2020a,b and 2021).

5. Simulations

Assessing the behavior of the maximum likelihood estimations (MLEs) is discussed in this section. For this purpose, consider the following active algorithm:

1) Use

\[ y_{hi} = \left( \left( -\frac{1}{\alpha} \ln(1 - U) \right)^{\frac{1}{\beta}} + 1 \right)^{\frac{1}{\delta}} - 1 \]

for generating 5000 group of size \( n \) from the WG-LL model.

2) Get the MLEs for the 5000 groups.

3) Using the inverting the observed information matrix, compute the standard errors (SEs) of the MLEs for the 1000 samples, where the SEs.

4) Compute the biases (\( B_h(n) | h = \alpha, \beta, a, \delta \) and mean squared errors (\( MSE_h(n) | h = \alpha, \beta, a, \delta \)) given for \( h = \alpha, \beta, a, \delta \). These steps must be repeated for \( n = 50, 100, ..., 5000 \) with \( 1 = a = \delta = \beta = a \) for getting the values of biases and the values of the MSEs for \( a, \beta, a \) and \( n = 50, 100, ..., 5000 \).

Figure 2, Figure 3, Figure 4 and Figure 5 (left charts) show how the four biases of \( \alpha, \beta, a \) and \( \delta \) vary with respect to \( n | n = 50, 100, ..., 5000 \). Figures 2, 3, 4 and 5 (right charts) show how the four MSEs of \( a, \beta, a \) and \( \delta \) vary with respect to \( n | n = 50, 100, ..., 5000 \). From Figure 2, 3 Figure, Figure 4 and Figure 5, the biases f decrease to zero as \( n \to \infty \), the valued of the obtained MSEs of \( a, \beta, a \) and \( \delta \) decrease to zero as \( n \to \infty \). Based on this assessment, the ML method performs well and can be used in estimating the model parameter. The following Section provide some useful real data applications using the ML method for comparing the competitive models.
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Figure 2: Bias and MSE for the parameter $\alpha$.

Figure 3: Bias and MSE for the parameter $\beta$. 

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Figure 4: Bias and MSE for the parameter $a$.

Figure 5: Bias and MSE for the parameter $b$. 

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6. Real data modeling and analysis

In this part, some different real-life data sets are modeled and analyzed for demonstrating applied importance, applicable potentiality, and wide flexibility of the WG-LL model. For these data, we compare the WG-LL distribution, with other models such as the Burr model of the kind XII (BUXII), the WLL, Marchall-Oklan-BUXII (MOBUXII), Topp-Leone-BUXII (TLBUXII), Zografos-Baklakrishnan-BUXII (ZOBUXII), the beta-BUXII which has five parameters (FBBUXII), Beta BUXII bBUXII, Beta-exponentiated-BUXII (BBBUXII), five-parameters Kumaraswamy-BUXII (FKwBUXII), Kumaraswamy-BUXII (KwBUXII) and Weibull-log-logistic (WLL). All versions are recently modeled by Altun et al. (2018a) Yousof et al. (2018 and 2019) and Altun et al. (2018b). Other real-life datasets are in Aryal and Yousof (2017), Yousof et al. (2017), Hamedani et al. (2017, 2018 and 2019), Merovci et al. (2017 and 2020), Korkmaz et al. (2018a), Korkmaz et al. (2018b), Nascimento et al. (2019), Alizadeh et al. (2020a,b), Korkmaz et al. (2020) and Karamikabir et al. (2020).

Real-life data set I: the breaking stress data. It consists of 100 observations of breaking stress of carbon fibers (in Gba) (see Nichols and Padgett (2006)). Real-life data set II: the survival times in days of 72 pigs from guinea which was infected with the virulent tubercle bacilli (Bjerkedal (1960)). Real-life data set III: the leukemia data. It represents the times of survival, in weeks, of 33 patients suffering from the well-known acute myelogenous leukemia.

The total time test (TTT) plots (Aarset(1987)) for the three real data sets are presented in Figure 2. It is seen that the HRFs of data sets I, II are monotonically increasing and U-HRF for data set III. We consider the following goodness-of-fit statistics: the Akaike-criterion ($T_1$), Bayesian-criterion ($T_2$), consistent-criterion ($T_3$) and Hannan-Quinn criterion ($T_4$). Generally, the smaller these statistics are, the better the fit. Tables 3, 4 and 5 give the MLEs, standard errors (SEs), confidence interval (CI95%) with for the data set I, II and III. Tables 6, 7 and 8 give the statistics $T_1$, $T_2$, $T_3$, and $T_4$ values for the data set I, II and III. Due to Table 6, Table 7 and Table 8 and Figure 3-6 the WG-LL model has the best results with small values of the $T_1$, $T_2$, $T_3$, and $T_4$.

Table 3: Estimation results for the data set I.

| Model       | Estimates, SEs and CI(95%) |
|-------------|----------------------------|
| BUXII($\beta,\alpha$) | Estimates 5.941, 0.187 |
|             | SEs (1.2792), (0.0443)    |
|             | CI(95%) (3.431,8,455), (0.100,0.277) |
| MOBUXII ($\beta,\alpha,\gamma$) | Estimates 1.1921, 4.8394, 838.713 |
|             | SEs (0.9522), (4.8961), (229.341) |
|             | CI(95%) (0, 3.064), (0, 14.438), (389.224,1288.246) |
| TLBUXII($\beta,\alpha,\gamma$) | Estimates 1.3501, 1.0614, 13.7284 |
|             | SEs (0.3788), (0.3844), (8.405) |
|             | CI(95%) (0.614, 2.09), (0.313,1.811), (0, 30.199) |
| KwBUXII ($\lambda,\theta,\alpha$) | Estimates 48.1032, 79.5162, 0.3519, 2.7305 |
|             | SEs (19.340), (58.185), (0.998), (1.080) |
|             | CI(95%) (10.10,86.038), (0.193,568), (0,16.0,547), (0.62,4.845) |
| BBUXII($\lambda,\theta,\alpha$) | Estimates 359.7, 260.1, 0.1752, 1.1233 |
|             | SEs (57.9), (132.4), (0.013), (0.24) |
|             | CI(95%) (246,473), (0.96,519,24), (0.14,0.204), (0.65,1.66) |
| BEBUXII($\lambda,\theta,\alpha,\gamma$) | Estimates 0.3813, 11.9492, 0.9372, 33,4026, 1.7057 |
|             | SEs (0.0783), (4.6353), (0.2674), (6.2871), (0.4789) |
|             | CI(95%) (0.2,0.535), (2.86,215), (0.41,1.55), (21,457), (0.82,6.7) |
| FBBUXII($\lambda,\theta,\beta,\alpha,\gamma$) | Estimates 0.4212, 0.834, 6.2, 1.67, 3.452 |
|             | SEs (0.01), (0.94), (2.31), (0.23), (1.96) |
|             | CI(95%) (0.4,0.44), (0.27), (1.57,10.7), (1.23,2.1), (0, 7) |
| FKwBUXII($\lambda,\theta,\beta,\alpha,\gamma$) | Estimates 0.542, 4.223, 5.313, 0.411, 4.152 |
|             | SEs (0.1375), (1.88), (2.32), (0.497), (1.9954) |
|             | CI(95%) (0.3, 0.88), (0.53,7.96), (0.9,9.5), (0, 1.74), (0,2,8,5) |
| ZOBBUXII($\lambda,\beta,\alpha$) | Estimates 123, 0.368, 139.23 |
|             | SEs (243), (0.34), (319) |
|             | CI(95%) (0, 599,401), (0, 1.043), (0, 763,599) |
| LL(a)       | Estimates 1.6292359 |
|             | SEs (0.128801) |
### Table 4: Estimation results for the data set II.

| Model               | Estimates, SEs and CI(95%)                     |
|---------------------|------------------------------------------------|
| BUXII(β, α)         | CI(95%) (1.3611, 1.847) |
|                     | Estimates 5.2214, 2.31971 |
|                     | SEs (0.5981, 0.14966) |
| MOBUXII(β, α, γ)    | CI(95%) (4.2, 6.19), (2, 2.59) |
|                     | Estimates 1.1159, 0.709 |
|                     | SEs (6.511, 4.1259) |
| TLBUXII(β, α, γ)    | CI(95%) (0, 16.09), (0, 8.938) |
|                     | Estimates 2.206, 0.667, 0.137, 1.581 |
|                     | SEs (2.509, 0.424), (0.351), (1.822) |
| KwBUXII (λ, β, α)   | CI(95%) (0, 7.5), (0, 1.5), (0, 0.84), (0, 5.1) |
|                     | Estimates 14.1094, 7.42, 0.53, 2.273 |
|                     | SEs (10.81), (11.85), (0.28), (0.991) |
| BBUXII(λ, β, α)     | CI(95%) (0.35, 0.37), (0, 1.1), (0.53, 4.26) |
|                     | Estimates 6.0580, 1.80, 0.29434 |
|                     | SEs (1.86), (10.39), (0.96), (0.47) |
| BEBUXII(λ, β, α, γ) | CI(95%) (0.63), (0.265), (0.37), (0.12) |
|                     | Estimates 1.8829, 1.781, 1.342, 0.573 |
|                     | SEs (0.09), (1.73), (0.7), (0.82), (0.33) |
| FBBUXII(λ, β, α, γ) | CI(95%) (1.72, 1), (0.64), (0.40, 3.19), (0, 3), (0, 1.22) |
|                     | Estimates 0.62, 0.553, 0.383, 1.38, 1.67 |
|                     | SEs (0.54), (1.01), (2.79), (2.31), (0.44) |
| FKwBUXII(λ, β, α, γ)| CI(95%) (0, 1.73), (0, 2.53), (0, 9.31), (0, 5.99), (0.8, 4.55) |
|                     | Estimates 0.5583, 0.308, 3.9991, 2.1312, 1.4754 |
|                     | SEs (0.44), (0.3143), (2.08), (1.83), (0.364) |
| LL(a)               | CI(95%) (0, 1.29), (0, 0.89), (0, 3.12), (0, 5.69), (0.76, 2.3) |
|                     | Estimates 2.27532 |
|                     | SEs (0.22327) |
| ExpLL(β, a)         | CI(95%) (1.92, 2.78) |
|                     | Estimates 1.9513, 2.254 |
|                     | SEs (0.2288), (0.20679) |
| WLL(β, a)           | CI(95%) (1.51, 2.33), (2.14, 2.90) |
|                     | Estimates 0.78545, 1.25401 |
|                     | SEs (0.00), (0.00) |
| WG-LL(β, α, b, a)   | CI(95%) (0, 2.33), (0, 0.852), (0, 3.879), (0, 9.1) |
|                     | Estimates 0.869, 0.328, 0.879, 3.486 |
|                     | SEs (0.729), (0.262), (1.477), (2.886) |

### Table 5: Estimation results for the data set III.

| Model               | Estimates, SEs and CI(95%)                     |
|---------------------|------------------------------------------------|
| BUXII(β, α)         | CI(95%) (0.76110, 0.0062) |
|                     | Estimates 58.71110, 0.0062 |
|                     | SEs (42.3821), (0.0046) |
| MOBUXII(β, α, γ)    | CI(95%) (0, 141.782), (0, 0.014) |
|                     | Estimates 11.8381, 0.0783, 12.2510 |
\[ \text{SEs} \quad (4.3681, 0.0129, (7.771) \]
\[ \text{CI}(95\%) \quad (0, 141.8), (0, 0.011), (0, 27.50) \]

**TLBUXII(\(\beta,\alpha,\gamma\))**

Estimates 0.2814, 1.8823, 50.2147
SEs (0.29), (2.40), (176.5)
CI(95\%) (0, 0.855), (0, 6.599), (0, 396)

**KwBUXII(\(\lambda,\theta,\beta,\alpha\))**

Estimates 9.201, 36.428, 0.242, 0.941
SEs (10.06), (35.651), (0.168), (1.0455)
CI(95\%) (0, 28.9123), (0, 106.3), (0, 0.57), (0, 3)

**BBUXII(\(\lambda,\theta,\beta,\alpha\))**

Estimates 96.1, 52.12, 0.104, 1.2278
SEs (41.2), (33.49), (0.02), (0.33)
CI(95\%) (15.4, 177), (0, 118), (0.6, 0.15), (0.592)

**BEBUXII(\(\lambda,\theta,\beta,\alpha,\gamma\))**

Estimates 0.087, 5.01, 1.56, 31.27, 0.323
SEs (0.09), (3.85), (0.012), (12.94), (0.0344)
CI(95\%) (0, 0.33), (0, 12.66), (1.5, 1.66), (5.9, 56.58), (0.3, 0.44)

**FBBUXII(\(\lambda,\theta,\beta,\alpha,\gamma\))**

Estimates 15.1943, 32.0482, 0.233, 0.5801, 21.8555
SEs (11.59), (9.868), (0.092), (0.07), (35.55)
CI(95\%) (0, 37.77), (12.751), (0.05.4), (0.45, 0.7), (0, 91.5)

**FKwBUXII(\(\lambda,\theta,\beta,\alpha,\gamma\))**

Estimates 14.73, 15.28, 0.29, 0.84, 0.0342
SEs (12.39), (18.87), (0.22), (0.05), (0.08)
CI(95\%) (0.39), (0.52.3), (0, 0.7), (0.2), (0.2)

**ZOBBUXII(\(\lambda,\beta,\alpha,\gamma\))**

Estimates 41.9733, 0.1572, 44.2632
SEs (38.785), (0.0824), (47.65)
CI(95\%) (0, 118), (0, 0.32), (0, 137.65)

**LL(\(\alpha\))**

Estimates 0.507323
SEs (0.07090)
CI(95\%) (0.366, 0.644)

**ExpLL(\(\beta,\alpha\))**

Estimates 5.5941, 0.76478
SEs (1.18343), (0.0933)
CI(95\%) (3.1, 7.99), (0.5, 0.888)

**WLL(\(\beta,\alpha\))**

Estimates 1.0699, 0.2245
SEs (0.000), (0.000)
CI(95\%) --

**WG-LL(\(\beta,\alpha,b,a\))**

Estimates 0.643, 0.160, 0.091, 6.701
SEs (0.253), (0.076), (0.048), (0.009)
CI(95\%) (0.1, 1.1), (0.02, 0.3), (0.0186), (6.68, 6.718)

| Model | \(T_1\), \(T_2\), \(T_3\), and \(T_4\) values for the data set I. |
|-------|---------------------------------------------------------------|
| BUXII | 382.942, 388.155, 383.063, 385.053                           |
| WLL  | 510.693, 515.911, 510.820, 512.844                            |
| TLBUXII | 323.524, 331.354, 323.771, 326.701                     |
| MOBUXII | 305.781, 313.614, 306.030, 308.960                       |
| FBBUXII | 304.260, 317.311, 304.892, 309.564                       |
| ExpLL | 325.931, 331.144, 326.066, 328.040                        |
| BBUXII | 305.644, 316.060, 306.063, 309.853                        |
| KwBUXII | 303.761, 314.201, 304.185, 308.001                      |
| BEBUXII | 305.822, 318.845, 306.462, 311.093                      |
| LL    | 469.633, 472.233, 469.670, 470.680                         |
| FKwBUXII | 305.503, 318.551, 306.145, 310.802                      |
| ZOBBUXII | 302.960, 310.781, 303.214, 306.130                     |
| WG-LL | **290.750, 301.173, 291.174, 294.970**                  |

Table 4: \(T_1\), \(T_2\), \(T_3\), and \(T_4\) values for the data set I.
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Table 6: $T_1$, $T_2$, $T_3$, and $T_4$ values for the data set II

| Model   | $T_1$, $T_2$, $T_3$, and $T_4$               |
|---------|---------------------------------------------|
| BUXII   | 209.600, 214.154, 209.771, 211.401           |
| LL      | 231.850, 234.134, 231.914, 232.765           |
| FBBUXII | 206.803, 218.204, 207.716, 211.303           |
| ExpLL   | 207.833, 212.385, 208.003, 209.644           |
| MOBXII  | 209.743, 216.564, 210.091, 212.443           |
| TLBUXII | 211.803, 218.633, 212.154, 214.520           |
| WLL     | 270.441, 274.981, 270.602, 272.230           |
| KwBUXII | 208.760, 217.865, 209.366, 212.380           |
| BBUXII  | 210.443, 219.545, 211.036, 214.060           |
| FKwBUXII| 206.501, 217.905, 207.413, 211.000           |
| BEBUXII | 212.103, 223.503, 213.001, 216.600           |
| WGLL    | 205.862, 214.970, 206.460, 209.490           |

Table 8: $T_1$, $T_2$, $T_3$, and $T_4$ values for the data set III.

| Model   | $T_1$, $T_2$, $T_3$, and $T_4$               |
|---------|---------------------------------------------|
| BUXII   | 328.203, 331.193, 328.601, 329.194           |
| LL      | 362.711, 364.200, 362.833, 363.215           |
| MOBXII  | 315.541, 320.010, 316.370, 317.043           |
| TLBUXII | 316.264, 320.733, 317.092, 317.761           |
| BBUXII  | 316.461, 322.454, 318.893, 318.471           |
| KwBUXII | 317.363, 323.301, 318.793, 319.343           |
| ExpLL   | 315.4652, 318.450, 315.850, 316.47           |
| FBBUXII | 317.863, 325.342, 320.081, 320.364           |
| BEBUXII | 317.580, 325.062, 319.801, 320.090           |
| FKwBUXII| 317.760, 325.213, 319.982, 320.260           |
| ZOBUXII | 313.862, 318.354, 314.39, 315.3603           |
| WLL     | 378.8212, 381.7902, 379.27, 379.84           |
| WGLL    | 310.900, 316.891, 312.332, 312.923           |
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7. Concluding remarks
In the present paper, we introduced a novel continuous log-logistic model. Several of its main characteristic properties such as the moments, the generating function, the weighted moments, the reversed residual life are mathematically derived. Numerical analysis for the skewness and the kurtosis is presented and useful comments are added. For the new log-logistic model, the skewness ∈ (−1.081, 16.74). However, for the standard log-logistic model, the skewness
\(\varepsilon(0.087, 2.4853)\), hence, the novel model can be negative skewed and positive skewed while the standard model can only be negative skewness. For the new log-logistic model, kurtosis \(\varepsilon(3.245089, 702.498)\). However, for the standard log-logistic model, kurtosis \(\varepsilon(3.741, 29.56)\). The new PDF can be unimodal, symmetric, or left skewed. The new failure rate can be "bathtub or U-failure rate", "increasing failure rate", "decreasing-constant failure rate", "J-failure rate", "constant failure rate " and "decreasing failure rate".

Many bivariate and extensions are derived. To assess the estimators, we performed a graphical simulation. Three different real-life data are modeled under some statistical tests. For all these real-life datasets, we compare the novel function with many relevant extensions. The new model is better than all other competitive models in modeling breaking stress data, survival times data and leukemia data.

Future points:

1. Presenting a novel discrete model for modeling count real-life data (see Aboraya et al. (2020), Chesneau et al. (2021), Ibrahim et al. (2021) and Yousof et al. (2021) for more details).
2. Applying the Nikulin-Rao-Robson and Bagdonavičius-Nikulin tests (see Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a,b,c,d,e,f), Yadav et al. (2020), Goual and Yousof (2020) and Aidi et al. (2021), among others.).

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