Quantum Entanglement and Generalized Uncertainty Relations

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The Robertson’s formulation of the uncertainty relation is the most widely accepted form of the Heisenberg uncertainty relation (HUR). It gets modified when we consider it for entangled particles. But this formulation does not consider the measurement process itself. There are reformulation of the uncertainty relations called the generalized uncertainty relations by including the measurement process into the uncertainty relation. HUR gets modified for the case of entangled particles. Here it is shown that the limit of the Generalized Uncertainty Relation (GUR) also reduces for entangled particles. So, GUR also shows similar trend as that of the HUR for entangled particles. Also as entanglement increases, the uncertainty reduces and measurement becomes more precise.

1 Introduction

Robertson’s relation is the most widely accepted form of the uncertainty relation. For two observables $A$ and $B$ on a state $\psi$ it is defined as

$$\sigma(A, \psi) \sigma(B, \psi) \geq \frac{|\langle \psi | [A, B] | \psi \rangle|^2}{2}$$  \hspace{1cm} (1)
where \( \sigma \) is the standard deviation, also known as the uncertainty in the measurement of \( \hat{A} \), is defined as 
\[
\sigma(A) = \left( \langle (A - \langle A \rangle)^2 \rangle \right)^{\frac{1}{2}}.
\]
Left side of equation (1) usually known as the uncertainty in the simultaneous measurement of \( \hat{A} \) and \( \hat{B} \) on a particle with state \( \psi \). If there are more particles, the state \( \psi \) is a product state or entangled state of the particles. Using quantum covariance function Rigolin [2] has explained how the entanglement between particles affects the uncertainty relation. It is shown that entanglement reduces the uncertainty in the simultaneous measurement of \( \hat{A} \) and \( \hat{B} \). It is also shown that right side of equation (1) tends to zero with large number of entangled particles.

The Robertson’s relation does not consider the measurement process, it only depends on the state of the system. In realistic situation measurement will introduce noise in the measured value disturbance to other quantities.

Measurement noise and the corresponding disturbance on the system are not included Robertson’s uncertainty relation. Also there is experimental violation of the usual uncertainty relations[3]. Ozawa[4] proposed a universal Generalized Uncertainty Relations (GUR) which are the reformulations of the usual uncertainty relation by considering the measurement process also. It was further modified by Fujikawa [5].

In this paper we are studying how the generalized uncertainty relations are affected by the build-up of entanglement on the system. We consider (GUR) by Fujikawa in presence two particles. The discussion can extended to \( n \) entangled particles.

2 Noise and Disturbance

The root-mean-square noise \( \varepsilon(A) \) of a measuring device measuring \( A \) is given as[6]
\[
\varepsilon(A)^2 = \langle (M^{out} - A^{in})^2 \rangle = \langle \psi \otimes \zeta | (U^\dagger (I \otimes M) U - (A \otimes I))^2 | \psi \otimes \zeta \rangle \tag{2}
\]
where \( |\psi\rangle \) is the state of the system of particles and \( |\zeta\rangle \) is the state of the measuring device (probe) and the unitary operator \( U \) on \( \psi \otimes \zeta \) gives the time evolution of the composite system (system+probe) during their interaction. \( M \) is the probe observable to be detected from the state after the measuring interaction. From now on \( \langle \ldots \rangle \) stands for \( \langle \psi \otimes \zeta | \ldots | \psi \otimes \zeta \rangle \). The root-mean-square disturbance \( \eta(B) \) is defined as[6]
\[
\eta(B)^2 = \langle (B^{out} - B^{in})^2 \rangle = \langle \psi \otimes \zeta | (U^\dagger (B \otimes I) U - (B \otimes I))^2 | \psi \otimes \zeta \rangle \tag{3}
\]
It is assumed that the measuring device measuring \( B \) is noiseless. so

\[
U^\dagger (B \otimes I)U = U^\dagger (I \otimes M_B)U
\]

and then it is only affected by the measurement of \( A \) during the interaction.

GUR due to Ozawa is

\[
\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{|\langle \psi | [A, B] | \psi \rangle|^2}{2} \tag{4}
\]

where \( \varepsilon \) is the noise in the measurement of \( A \) and \( \eta \) is the corresponding disturbance on \( B \). It was further modified by Fujikawa \[5\] as

\[
\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) + \sigma(A)\sigma(B) \geq \frac{|\langle \psi | [A, B] | \psi \rangle|^2}{2} \tag{5}
\]

In the absence of noise and disturbance, above relation reduces to Robertson’s relation \[1\]. To investigate many particle effect on Fujikawa relation we consider a system of two distinguishable particles.

For a two particle system let us define position operator

\[
Q(1, 2) = Q_1 \otimes I_2 + I_1 \otimes Q_2 \tag{6}
\]

for particles at \( q_1 \) and \( q_2 \) and measurement operator

\[
M_{Q(1, 2)} = M_{Q_1} \otimes I_2 + I_1 \otimes M_{Q_2} \tag{7}
\]

Where \( M_{Q_1} \) is measurement operator representing measurement \( Q_1 \) on particle 1 and \( M_{Q_2} \) is measurement operator representing measurement \( Q_2 \) on particle 2. Corresponding to this, noise on the measurement of \( Q(1, 2) \) is

\[
\varepsilon(Q(1, 2))^2 = \langle (U^\dagger (I \otimes M_{Q(1, 2)})U - (Q(1, 2) \otimes I))^2 \rangle
\]

\[
= \langle (U^\dagger (I \otimes M_{Q_1} \otimes I_2 + I \otimes I_1 \otimes M_{Q_2})U
\]

\[
- (Q_1 \otimes I_2 \otimes I + I_1 \otimes Q_2 \otimes I))^2 \rangle
\]

\[
= \langle (U^\dagger (I \otimes M_{Q_1} \otimes I_2)U - (Q_1 \otimes I_2 \otimes I))^2 \rangle +
\]

\[
\langle (U^\dagger (I \otimes I_1 \otimes M_{Q_2})U - (I_1 \otimes Q_2 \otimes I))^2 \rangle +
\]

\[
2\langle (U^\dagger (I \otimes M_{Q_1} \otimes I_2)(I \otimes I_1 \otimes M_{Q_2})U
\]

\[
+ 2\langle (Q_1 \otimes I_2 \otimes I)(I_1 \otimes Q_2 \otimes I) \rangle -
\]

\[
2\langle U^\dagger (I \otimes M_{Q_1} \otimes I_2)(I_1 \otimes Q_2 \otimes I) \rangle -
\]

\[
2\langle U^\dagger (I \otimes I_1 \otimes M_{Q_2})U(Q_1 \otimes I_2 \otimes I) \rangle \]
Since the operators are commuting, we could rearrange and obtain it as
\[ \varepsilon(Q(1,2))^2 = \varepsilon(Q_1)^2 + \varepsilon(Q_2)^2 + 2[\langle (\dot{U}^\dagger I \otimes M_{Q_1} \otimes I_2)U \rangle - \langle (Q_1 \otimes I_2 \otimes I) \rangle (\dot{U}^\dagger I \otimes I_1 \otimes M_{Q_2}U) - \langle (I_1 \otimes Q_2 \otimes I) \rangle] \] (8)

For the simultaneous measurement of position of both the particles using a symmetric experimental setup, the noise is not the sum of noise of measurement on the first particle and the second particle,
\[ \varepsilon(Q(1,2))^2 \neq \varepsilon(Q_1)^2 + \varepsilon(Q_2)^2. \] There is an additional term. If the measurement on the first particle is noiseless, then we have \( \varepsilon(Q_1)^2 = \varepsilon(Q_2)^2 \). So, the additional term should contain the noise of the measurement on the first particle and also on the second particle. Now we define
\[ \varepsilon(Q(1,2)) = \langle \dot{U}^\dagger (I \otimes M_{Q(1,2)})U - (Q(1,2) \otimes I) \rangle \] (9)

By definition, note that
\[ \varepsilon(Q(1))^2 \neq \varepsilon(Q(1))\varepsilon(Q(1)) \] (10)

Here \( \varepsilon(Q(1))^2 \) is the noise in the measurement of \( Q(1) \) in presence of \( Q(2) \). Then we could write the equation (8) as
\[ \varepsilon(Q(1,2))^2 = \varepsilon(Q_1)^2 + \varepsilon(Q_2)^2 + 2\varepsilon(Q_1)\varepsilon(Q_2) \] (11)

Evidently when \( \varepsilon(Q_1) = 0 \) (\( \varepsilon(Q_2) = 0 \)) we get \( \varepsilon(Q(1,2)) = \varepsilon(Q_1) \) (\( \varepsilon(Q(1,2)) = \varepsilon(Q_2) \)). Now we could similarly consider the disturbance caused by this position measurements on the system. We may assume that disturbance caused is to momentum. As in the case of position, we define the momentum operator as
\[ P(1,2) = P_1 \otimes I_2 + I_1 \otimes P_2 \]

By taking \( B = P(1,2) \) in equation (5) we get root-mean-square disturbance \( \eta(P(1,2)) \) as
\[ \eta(P(1,2))^2 = \eta(P_1)^2 + \eta(P_2)^2 + 2\eta(P_1)\eta(P_2) \] (12)

3 Entanglement and GUR

Now assume that using symmetric experimental techniques, we made the noise and disturbance on the first particle is equal to that of the second particle. That is
when \(Q_1 = Q_2\) and \(P_1 = P_2\), we would have

\[
\varepsilon(Q)^2 = 2\varepsilon(Q_1)^2 + 2[\varepsilon(Q_1)]^2 \tag{13}
\]

\[
\eta(P)^2 = 2\eta(P_1)^2 + 2[\eta(P_1)]^2 \tag{14}
\]

Consider that our two particle system is prepared in such a way that the two particles are entangled. When measurement occurs, due to the interaction with the measuring devices the entanglement between the particles get destroyed. So while calculating the noise and disturbance due to measurement on these particles we could treat them as separable states. Then for the separable states

\[
\varepsilon(Q_1)^2 = [\varepsilon(Q_1)]^2 \quad \text{and} \quad \eta(P_1)^2 = [\eta(P_1)]^2 \tag{15}
\]

and then we get

\[
\varepsilon(Q) = 2\varepsilon(Q_1) \quad \text{and} \quad \eta(P) = 2\eta(P_1) \tag{16}
\]

The uncertainty in position and momentum for the preparation is given by the standard deviations of position and momentum in the state. The standard deviations depend only on the state of the system and it is independent of the property of the measuring apparatus or measurement. That is measurement does not affect the standard deviation. So the standard deviation for two entangled particles is given as

\[
\sigma_Q = \sqrt{\langle Q(1,2)^2\rangle - \langle Q(1,2)\rangle^2}
\]

Now substituting equation \[6\]

\[
\sigma_Q = \sqrt{(\Delta Q_1)^2 + (\Delta Q_2)^2 + 2\langle Q_1 \otimes Q_2\rangle - 2\langle Q_1\rangle \langle Q_2\rangle}
\]

Using the quantum covariant function\[1\], for entangled particles we could arrive at\[2\]

\[
\sigma_Q = \sqrt{(\Delta Q_1)^2 + (\Delta Q_2)^2 + (\Delta Q_1)^2 + (\Delta Q_2)^2} \tag{17}
\]

If the preparation of our system is carried out in a way that, the deviation in position and momentum is same for the first and second particles. Then we would get

\[
\sigma_Q = 2\Delta Q_1 = 2\sigma_{Q_1} \tag{18}
\]

Using similar assumption

\[
\sigma_P = 2\Delta P_1 = 2\sigma_{P_1} \tag{19}
\]

5
For our two particle system we know that

\[ Q(1, 2), P(1, 2) = [Q_1, P_1] + [Q_2, P_2] = 2i\hbar \]  

(20)

Then by substituting equations (16), (18), (19) and (20) on to the generalized uncertainty relation by Ozawa, we could get

\[
\epsilon(Q)\eta(P) + \epsilon(Q)\sigma(P) + \sigma(Q)\eta(P) \geq \frac{|\langle \psi| [Q, P]|\psi\rangle|}{2}
\]

\[
2\epsilon_{Q_1}\eta_{P_1} + 2\epsilon_{Q_1}\sigma_{P_1} + 2\sigma_{Q_1}\eta_{P_1} \geq \frac{2\hbar}{2}
\]

\[
4(\epsilon_{Q_1}\eta_{P_1} + \epsilon_{Q_1}\sigma_{P_1} + \sigma_{Q_1}\eta_{P_1}) \geq \hbar
\]

\[
\epsilon_{Q_1}\eta_{P_1} + \epsilon_{Q_1}\sigma_{P_1} + \sigma_{Q_1}\eta_{P_1} \geq \frac{\hbar}{4}
\]  

(21)

The usual limit in the uncertainty is \(\frac{\hbar}{2}\), when the particles gets entangled the limit reduces to half of the traditional one.

Similarly for the Generalized Uncertainty Relation by Fujikawa we have

\[
\epsilon(Q)\eta(P) + \epsilon(Q)\sigma(P) + \sigma(Q)\eta(P) + \sigma(Q)\sigma(P) \geq |\langle \psi| [Q, P]|\psi\rangle|
\]

\[
\epsilon_{Q_1}\eta_{P_1} + \epsilon_{Q_1}\sigma_{P_1} + \sigma_{Q_1}\eta_{P_1} + \sigma_{Q_1}\sigma_{P_1} \geq \frac{\hbar}{2}
\]  

(22)

This is also only half of the traditional limit. Thus entanglement causes the limit of the uncertainty to become smaller. Similar to that of the Robertson’s relation, both generalized uncertainty relations by Ozawa and Fujikawa also reduces for the case of entangled particles. It can be easily shown that as the number of entangled particles increases the uncertainty reduces even further. That is, as entanglement increases the uncertainty reduces even further. It causes our measurements to become more precise. So when we consider more entanglement between the particles, the system is becoming more classical.

### 4 Conclusions

The limit of generalized uncertainty relation reduces when we are observing an entangled system. Our measurement becomes more precise in the case of entangled particles, which is similar to that of Robertson’s relation. Uncertainty in the system reduces as entanglement develops in the system. Also as the number of
entangled particles increases the uncertainty reduces further, it results the system to become more classical. So it is predicted that entanglement holds the key to the transition from quantum realm to classical realm.

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