MODELS FOR $q$-COMMUTATIVE TUPLES OF ISOMETRIES

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Abstract. A pair of Hilbert space linear operators $(V_1, V_2)$ is said to be $q$-commutative, for a unimodular complex number $q$, if $V_1V_2 = qV_2V_1$. A concrete functional model for $q$-commutative pairs of isometries is obtained. The functional model is parametrized by a collection of Hilbert spaces and operators acting on them. As a consequence, the collection serves as a complete unitary invariance for $q$-commutative pairs of isometries. A $q$-commutative operator pair $(V_1, V_2)$ is said to be doubly $q$-commutative, if in addition, it satisfies $V_2V_1^* = qV_1^*V_2$. Doubly $q$-commutative pairs of isometries are also characterized. Special attention is given to doubly $q$-commutative pairs of shift operators. The notion of $q$-commutativity is then naturally extended to the case of general tuples of operators to obtain a similar model for tuples of $q$-commutative isometries.

1. Introduction

A stepping stone to the development of model theory for contractive Hilbert space operators is what is known as the Wold decomposition: every isometric operator $V$ acting on a Hilbert space $H$ is unitarily equivalent to the direct sum $S \oplus W$, where $W$ is a unitary operator and $S$ is a shift operator, i.e., $S$ is an isometry with $S^*n \to 0$, in the strong operator topology, as $n \to \infty$. This is due to [9, 13] and [25]. There has been numerous generalizations of this classical decomposition theorem. For example, see [2, 22] for development in the commutative setting and [19, 23] for doubly commutative setting; also see [4, 5, 8, 14, 15, 16, 17, 24] and references therein for results in this direction.

The objective of this paper is to further extend these decomposition results in the $q$-commutative and doubly $q$-commutative settings.

Definition 1.1. A pair $(V_1, V_2)$ of operators is said to be $q$-commutative, if

$$V_1V_2 = qV_2V_1.$$  

Such pairs seem to be of significant importance in the area of quantum theory, see [6, 12, 18]. Recently, $q$-commutative operators have been studied by some operator theorists. To mention some of these works, Bhat and Bhattacharyya [3] studied $q$-commutative row contractions $((T_1, T_2, \ldots, T_d)$ (i.e., $T_iT_j = q(i, j)T_jT_i$ for each $i, j$ and $\sum_{i=1}^d T_iT_i^* \leq I$) in quest of its model. Later, Dey [7] studied $q$-commutative row contractions for its dilation theory. In contrast to the consideration in this paper, $q(i, j)$ were allowed to be any non-zero complex numbers in both the papers [3, 7].
Recently, Keshari and Mallick [11] showed by a commutant lifting approach, that any $q$-commutative pair of contractive operators has a $q$-commutative unitary dilations, where $q$ is a unimodular complex number. Thus this is an extension of Andô’s dilation theorem [1] and that of Sebestyén [21], where the result was proved for the case $q = -1$.

First, we note that unlike the commutative case, $q$-commutativity is ‘order-sensitive’, i.e., if $(V_1, V_2)$ is $q$-commutative, then $(V_2, V_1)$ is $\overline{q}$-commutative. However, it follows from the definition that if $(V_1, V_2)$ is $q$-commutative, then so is $(V_1^*, V_2^*)$. For a concrete example of a $q$-commutative pair of isometries, let us choose a unimodular complex number $q$ and define the rotation operator $R_q$ on $H^2(\mathbb{D}^d)$, the Hardy space over the $d$-disk, as

$$R_q f(\bar{z}) := f(q \bar{z}) \quad \text{for all} \quad f \in H^2(\mathbb{D}^d),$$

where for $\bar{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{D}^d$, $q \bar{z} := (q z_1, q z_2, \ldots, q z_d)$. For each $j = 1, 2$, let $M_{z_j}$ denote the multiplication by ‘$z_j$’ operator on $H^2(\mathbb{D})$. Consider the pair on $H^2(\mathbb{D}^2)$

$$(V_1, V_2) = (R_q M_{z_1}, M_{z_2}) \quad \text{or,} \quad (M_{z_1} R_q, M_{z_2}).$$

(1.2)

It is easy to verify that $(V_1, V_2)$ is a $q$-commutative pair of isometries. Let us note that if $R_q$ is the rotation operator on $H^2(\mathbb{D})$ (simply denoted by $H^2$ in the sequel), then the rotation operator on $H^2(\mathbb{D}^d)$ is given by taking the $d$-fold tensor product of $R_q$. With a slight abuse of notation, we use the same notation $R_q$ regardless of the dimension of the polydisk. It follows easily that the rotation operator $R_q$ does not commute with $M_z$, the multiplication by ‘$z$’ operator on $H^2$. Indeed, for every $f \in H^2$,

$$R_q M_z f(z) = q z f(qz) = q M_z R_q f(z).$$

Thus $(R_q, M_z)$ is actually $q$-commutative.

For a Hilbert space $\mathcal{H}$, the standard notation $\mathcal{B}(\mathcal{H})$ is used to denote the algebra of bounded linear operators on $\mathcal{H}$. Among several generalizations of the classical Wold decomposition, perhaps the most appealing is the one obtained by Berger, Coburn and Lebow [2, Theorem 3.1]. We extend the Berger-Coburn-Lebow program to the decomposition, perhaps the most appealing is the one obtained by Berger, Coburn bounded linear operators on $\mathcal{H}$.

A commutative pair $(\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2)$ in $\mathcal{B}(\mathcal{F})$, and a $q$-commutative pair of unitaries $(W_1, W_2)$ in $\mathcal{B}(\mathcal{K}_u)$, the pair

$$\begin{bmatrix}
R_q \otimes P^\perp U + M_z R_q \otimes PU & 0 \\
0 & W_1
\end{bmatrix}, \begin{bmatrix}
R_{\overline{q}} \otimes U^* P + R_{\overline{q}} M_z \otimes U^* P^\perp & 0 \\
0 & W_2
\end{bmatrix}$$

(1.3)

on $[H^2 \otimes \mathcal{F}]_{\mathcal{K}_u}$ is a $q$-commutative pair of isometries. And most importantly, for every $q$-commutative pair $(V_1, V_2)$ of isometries, there exists a collection $\{\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2\}$ of Hilbert spaces and operators as above such that $(V_1, V_2)$ is jointly unitarily equivalent to the model $(1.3)$. This is the content of Theorem 2.2. Moreover, the correspondence between $q$-commutative pairs of isometries and the parameters $\{\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2\}$ is one-to-one in the sense explained in Theorem 2.3.

Recall that a commutative pair $(V_1, V_2)$ is said to be doubly commutative, if in addition, $V_2 V_1^* = V_1^* V_2$. Let $(W_1, W_2)$ be a $q$-commutative pair of unitaries, i.e., $W_1 W_2 = q W_2 W_1$. On multiplying $W_1^*$ from left and right successively, we see that $q$-commutativity of $(W_1, W_2)$ is equivalent to $W_2 W_1^* = q W_1^* W_2$. In view of this, the following definition comes as a natural analogue of double commutativity.
Definition 1.2. A $q$-commutative pair of operators $(V_1, V_2)$ is said to be \emph{doubly $q$-commutative}, if in addition, it satisfies

$$V_2 V_1^* = q V_1^* V_2. \quad (1.4)$$

We remark that if $V_1$ and $V_2$ are isometries satisfying just $V_2 V_1^* = q V_1^* V_2$, then an easy computation shows that $(V_1 V_2 - q V_2 V_1)^* (V_1 V_2 - q V_2 V_1) = 0$ and thus $V_1 V_2 = q V_2 V_1$. Thus condition $(1.4)$ implies $q$-commutativity of $(V_1, V_2)$, if $V_1, V_2$ are isometries. The pair $(V_1, V_2)$ where each $V_j$ is as defined in $(1.2)$ is an example of a doubly $q$-commutative pairs of isometries on $H^2(\mathbb{D}^2)$. However, it can be shown that the same pair when restricted to the space $H^2(\mathbb{D}^2) \oplus \{\text{constants}\}$, is not doubly $q$-commutative; this is explained in §4, where we discuss several other simple examples to illustrate the model theory. Theorem 3.1 characterizes doubly $q$-commutative pairs of isometries.

As an application of the model theory, we exhibit a passage between commutative and $q$-commutative pairs of isometries. Similarly, we exhibit a way to go back and forth between the classes of doubly commutative and doubly $q$-commutative pairs of isometries. See Theorem 2.6 and Theorem 3.3 for these connections. As a consequence of these correlations, we show in Corollary 3.4 that given a doubly $q$-commutative pair of shift operators $(V_1, V_2)$, there is a unitary $s_q$ on $H^2(\mathbb{D}^2)$ such that $(V_1, V_2)$ is jointly unitarily equivalent to $(M_{z_1} s_q, M_{z_2})$ on $H^2(\mathbb{D}^2)$. This is an analogue of Słociński [22] who showed that every doubly commutative pair of shift operators is unitarily equivalent to $(M_{z_1}, M_{z_2})$ on $H^2(\mathbb{D}^2)$.

The notion of $q$-commutativity is naturally extended to the case of general tuples of operators, see Definition 5.1. This model theory for the pair case is then applied to the case of a general $d$-tuple ($d > 2$) of $q$-commutative isometries to obtain a similar model – see Theorem 5.2. In §6 we show that every $q$-commutative tuple of isometries $(X_1, X_2, \ldots, X_d)$ can be extended to a $q$-commutative tuple of unitaries $(Y_1, Y_2, \ldots, Y_d)$ (and hence doubly $q$-commutative) such that the unitary $Y_1 Y_2 \cdots Y_d$ is the minimal unitary extension of the isometry $X_1 X_2 \cdots X_d$. This both improves and gives a new proof of the ‘dilation’ result of [10] where it was shown that every doubly $q$-commutative tuple of isometries extends to a doubly $q$-commutative tuple of unitaries.

2. Functional models for $q$-commutative pairs of isometries

We begin with the following lemma which will be used in our quest for a functional model of $q$-commutative pairs of isometries. For a contraction $T$, we use the following standard notations for the \emph{defect operator} and the \emph{defect space} of $T$:

$$D_T = (I - T^* T)^{1/2} \quad \text{and} \quad D_T^* = \overline{\text{Ran} \ D_T}.$$ 

Lemma 2.1. Let $(V_1, V_2)$ be a $q$-commutative pair of isometries on a Hilbert space $\mathcal{H}$ and $V = V_1 V_2$. Then

(i)

$$\left\{ \begin{bmatrix} D_{V_1} V_2^* \\ D_{V_2} \end{bmatrix} h : h \in \mathcal{H} \right\} = \left[ \begin{bmatrix} D_{V_1}^* \\ D_{V_2} \end{bmatrix} \right] = \left\{ \begin{bmatrix} D_{V_2}^* \\ D_{V_2} V_1^* \end{bmatrix} h : h \in \mathcal{H} \right\}; \quad (2.1)$$

$$D_{V_1} V_2^* = q V_1^* V_2.$$
(ii) the defect space $D_{V^*}$ is unitarily equivalent to $\begin{bmatrix} D_{V_1^*} & D_{V_2^*}^* \\ D_{V_2^*} & V_1^* \end{bmatrix}$ via the unitary

$$D_{V^*} h \mapsto \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} V_1^* \end{bmatrix} h; \quad \text{and}$$

(iii) for every $j \geq 1$,

$$V^{*j} V_1 = q^{j-1} V_2^* V^{*j-1} \quad \text{and} \quad V^{*j} V_2 = \overline{q}^j V_1^* V^{*j-1}.$$

Proof. We only establish the first equality in (2.1), the proof of the second equality is similar. We use the general fact that if $V$ is an isometry, then $D_{V^*}$ is the projection onto $\text{Ran} \ V^\perp$. Let $f \oplus g \in D_{V_1^*} \oplus D_{V_2^*}$ be such that

$$\langle D_{V_1^*} V_2^* h + D_{V_2^*} f, f \oplus g \rangle = 0 \text{ for all } h \in H.$$ 

This is equivalent to $\langle D_{V_1^*} V_2^* h, f \rangle + \langle D_{V_2^*} h, g \rangle = 0$ for all $h \in H$, or, equivalently, $\langle h, V_2 f \rangle + \langle h, g \rangle = 0$ for all $h \in H$. Consequently, $g = -V_2 f$, which implies that $g = D_{V_2^*} g = -(I - V_2^* V_2) V_2 f = 0$ and since $V_1$ is an isometry $f$ must also be 0. This proves (i).

For (ii), we note that

$$D_{V^*}^2 = I - V V^* = I - V_1 V_1^* + V_1 V_1^* V_2 V_2^* V_1^* = D_{V_1^*}^2 + V_1 D_{V_2^*} V_1^*$$

$$= I - V_2 V_2^* + V_2 V_2^* V_1 V_1^* V_2^* = V_2 D_{V_1^*}^2 + V_2^2 + D_{V_2^*}^2. \quad \text{(2.4)}$$

This implies that for every vector $h \in H$,

$$\|D_{V^*} h\|^2 = \|D_{V_1^*} V_2^* h\|^2 + \|D_{V_2^*} h\|^2 = \|D_{V_1^*} h\|^2 + \|D_{V_2^*} V_1^* h\|^2. \quad \text{(2.6)}$$

Therefore to show that $D_{V^*}$ is isomorphic to $\begin{bmatrix} D_{V_1^*} & D_{V_2^*}^* \\ D_{V_2^*} & V_1^* \end{bmatrix}$, we can consider the map

$$D_{V^*} h \mapsto \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} V_1^* \end{bmatrix} h \text{ for every } h \in H. \quad \text{(2.7)}$$

Note that this is an isometry by (2.6) and surjective by part (ii) of the lemma.

For the intertwining relations (2.3), we see that for every $j \geq 1$,

$$V^{*j} V_1 = (V_2^* V_1^*)^j V_1 = V_2^* (V_1^* V_2^*)^j = q^{j-1} V_2^* V^{*j-1} \quad \text{(2.8)}$$

and

$$V^{*j} V_2 = (V_2^* V_1^*)^j V_1 = \overline{q}^j (V_1^* V_2^*)^j V_2 = \overline{q}^j V_1^* (V_2^* V_1^*)^j = \overline{q}^j V_1^* V^{*j-1}. \quad \text{(2.9)}$$

This completes the proof of the lemma. □

Now for the main theorem of this section, let us recall that the rotation operator $R_q$ is the unitary defined on $H^2$ as

$$R_q : f(z) \mapsto f(qz).$$

**Theorem 2.2.** Let $V_1$ and $V_2$ be two operators acting on a Hilbert space $H$. Then the following are equivalent.

(i) **$q$-commutativity:** The pair $(V_1, V_2)$ is $q$-commutative;
(ii) **BCL-1 q-model:** There exist Hilbert spaces $\mathcal{F}$ and $\mathcal{K}_u$, a projection $P$ and a unitary $U$ in $\mathcal{B}(\mathcal{F})$, and a pair $(W_1, W_2)$ of $q$-commuting unitaries in $\mathcal{B}(\mathcal{K}_u)$ such that $(V_1, V_2)$ is unitarily equivalent to

$$\left( \begin{bmatrix} R_q \otimes P^\perp U + M_z R_q \otimes PU & 0 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} \bar{R}_{\bar{q}} \otimes U^* P + \bar{R}_{\bar{q}} M_z \otimes U^* P^\perp & 0 \\ 0 & W_2 \end{bmatrix} \right) \text{ on } \begin{bmatrix} H^2 \otimes \mathcal{F} \\ \mathcal{K}_u \end{bmatrix}.$$  \tag{2.10}

Moreover, the tuple $(\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2)$ can be chosen to be such that

$$\begin{align*}
\mathcal{F} &= \left[ \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix}, \mathcal{K}_u \right] = \bigcap_{m \geq 0} (V_1 V_2)^n \mathcal{H}, \quad P : \left[ \begin{bmatrix} f \\ g \end{bmatrix} \right] \mapsto \left[ \begin{bmatrix} f \\ g \end{bmatrix} \right],
U : \left[ \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} \right] \mapsto \left[ \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} \right] \quad \text{and} \quad (W_1, W_2) = (V_1, V_2)|_{\mathcal{K}_u},
\end{align*}$$

and the unitary operator $\tau_{\text{BCL}} : \mathcal{H} \rightarrow \begin{bmatrix} H^2 \otimes \mathcal{F} \\ \mathcal{K}_u \end{bmatrix}$ can be chosen to be such that

$$\tau_{\text{BCL}} h = \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} (I - z V^*)^{-1} h \otimes \lim_{n \to \infty} (V_1 V_2)^n (V_2^* V_1^*)^n h;$$  \tag{2.12}

(iii) **BCL-2 q-model:** There exist Hilbert spaces $\mathcal{F}_1$ and $\mathcal{K}_{u\uparrow}$, a projection $P_1$ and a unitary $U_1$ in $\mathcal{B}(\mathcal{F}_1)$, and a pair $(W_1, W_2)$ of $q$-commuting unitaries in $\mathcal{B}(\mathcal{K}_{u\uparrow})$ such that $(V_1, V_2)$ is unitarily equivalent to

$$\left( \begin{bmatrix} R_q \otimes P_1^\perp U_1 + M_z R_q \otimes PU_1 & 0 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} \bar{R}_{\bar{q}} \otimes P_1^\perp U_1 + \bar{R}_{\bar{q}} M_z \otimes P_1^\perp U_1 & 0 \\ 0 & W_2 \end{bmatrix} \right) \text{ on } \begin{bmatrix} H^2 \otimes \mathcal{F}_1 \\ \mathcal{K}_{u\uparrow} \end{bmatrix}.$$  \tag{2.13}

Moreover, the tuple $(\mathcal{F}_1, \mathcal{K}_{u\uparrow}; P_1, U_1, W_1, W_2)$ can be chosen to be such that

$$\begin{align*}
(\mathcal{F}_1, \mathcal{K}_{u\uparrow}; P_1, U_1, W_1, W_2) &= (\mathcal{F}, \mathcal{K}_u; P, U^*, W_1, W_2),
\end{align*}$$

where $(\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2)$ is as in item (i) above, and the unitary operator $\tau_{\downarrow} : \mathcal{H} \rightarrow \begin{bmatrix} H^2 \otimes \mathcal{F}_1 \\ \mathcal{K}_{u\uparrow} \end{bmatrix}$ can be chosen to be as in (2.12).

**Proof of (i) $\Leftrightarrow$ (ii).** We first show that the pair in (2.10) is a $q$-commuting pair of isometries. To that end, let $\xi$ be in $\mathcal{F}$ and $n$ be a non-negative integer. We see that

$$\begin{align*}
(R_q \otimes U^* P + R_{\bar{q}} M_z \otimes U^* P^\perp)(z^n \otimes \xi) &= \bar{q}^n z^n \otimes U^* P \xi + \bar{q}^{n+1} z^{n+1} \otimes U^* P^\perp \xi
\end{align*}$$

and therefore

$$\begin{align*}
(R_q \otimes P^\perp U + M_z R_q \otimes PU)(R_q \otimes U^* P + R_{\bar{q}} M_z \otimes U^* P^\perp)(z^n \otimes \xi)
&= (R_q \otimes P^\perp U + M_z R_q \otimes PU)(\bar{q}^n z^n \otimes U^* P \xi + \bar{q}^{n+1} z^{n+1} \otimes U^* P^\perp \xi)
&= z^{n+1} \otimes P^\perp \xi + z^{n+1} \otimes P \xi = (M_z \otimes I_{\mathcal{F}})(z^n \otimes \xi).
\end{align*}$$  \tag{2.15}

On other hand, we have

$$\begin{align*}
(R_q \otimes P^\perp U + M_z R_q \otimes PU)(z^n \otimes \xi) &= q^n z^n \otimes P^\perp U \xi + q^n z^{n+1} \otimes PU \xi
\end{align*}$$

and hence

$$\begin{align*}
(R_q \otimes U^* P + R_{\bar{q}} M_z \otimes U^* P^\perp)(R_q \otimes P^\perp U + M_z R_q \otimes PU)(z^n \otimes \xi)
&= (R_q \otimes U^* P + R_{\bar{q}} M_z \otimes U^* P^\perp)(q^n z^n \otimes P^\perp U \xi + q^n z^{n+1} \otimes PU \xi)
&= \bar{q} z^{n+1} \otimes U^* PU \xi + \bar{q} z^{n+1} \otimes U^* P^\perp U \xi = \bar{q} (M_z \otimes I_{\mathcal{F}})(z^n \otimes \xi).
\end{align*}$$  \tag{2.16}
From equations (2.15) and (2.16) therefore follows the $q$-commutativity of the BCL-1 $q$-model (2.10). It remains to show that the entries of the BCL-1 $q$-model are isometries. But this is clear because the BCL-1 $q$-model is of the form

$$
\left( \begin{bmatrix} M_{(P^\perp Z + P)U} & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_q & 0 \\ 0 & I_{K_u} \end{bmatrix}, \begin{bmatrix} R_{\bar{q}} & 0 \\ 0 & I_{K_u} \end{bmatrix} \begin{bmatrix} M_{U^*(P^\perp Z P)} & 0 \\ 0 & W_2 \end{bmatrix} \right),
$$

and that the operators (neither $q$-commutative nor $q$-commutative)

$$
\begin{bmatrix} M_{(P^\perp Z + P)U} & 0 \\ 0 & W_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_{U^*(P^\perp Z P)} & 0 \\ 0 & W_2 \end{bmatrix}
$$

are isometries. Now it follows from the fact that the product of an isometry and a unitary is always an isometry. Therefore $(ii) \Rightarrow (i)$.

We now establish the direction $(i) \Rightarrow (ii)$. Let us denote the isometry $V := V_1 V_2 = qV_2 V_1$. By Wold decomposition $V$ is unitarily equivalent to

$$
\begin{bmatrix} M_z & 0 \\ 0 & W \end{bmatrix} : \begin{bmatrix} H^2(D_{V^*}) \\ K_u \end{bmatrix} \rightarrow \begin{bmatrix} H^2(D_{V^*}) \\ K_u \end{bmatrix}
$$

via the unitary

$$
h \mapsto \begin{bmatrix} D_{V^*}(I - zV^*)^{-1} h \\ \lim_n V^n V^{*n} h \end{bmatrix}. \quad (2.17)
$$

Here $W$ is a unitary operator on $K_u = \cap_{n \geq 0} V^n H$. We first note that the subspace $K_u$ is invariant under both $V_1$ and $V_2$. We make use of the following $q$-intertwining relations, which are easy to establish:

$$
V_1 V^n = q^n V^n V_1 \quad \text{and} \quad V_2 V^n = \bar{q}^n V^n V_2 \quad \text{for every} \quad n \geq 1.
$$

Let us suppose that for every $n \geq 0$, $g = V^n h_n$ for some $h_n \in H$. Then

$$
V_1 g = V_1 V^n h_n = V^n (q^n V_1 h_n) \quad \text{and} \quad V_2 g = V_2 V^n h_n = V^n (\bar{q}^n V_2 h_n).
$$

Therefore $K_u$ is jointly $(V_1, V_2)$-invariant. So for each $j = 1, 2$, the avatar of $V_j$ on $H^2(D_{V^*}) \oplus K_u$ is of the form

$$
\bar{V}_j = \begin{bmatrix} V_{1j}^j & 0 \\ V_{2j}^j & V_{22}^j \end{bmatrix}.
$$

Note that $(V_{12}^2, V_{22}^2)$ is a pair of $q$-commuting isometries such that

$$
W = V_{12}^2 V_{22}^2 = q V_{22}^2 V_{22}^1.
$$

Since $W$ is a unitary, the pair $(V_{12}^1, V_{22}^1)$ must be a $q$-commutative pair of unitaries – a fact that trivially follows from (2.6) when applied to the pair $(V_{22}^2, V_{22}^2)$. Therefore, each $\bar{V}_j$ must be a block diagonal matrix. Consequently, it is enough to assume – as we do for the rest of the proof – that $V = V_1 V_2$ is a shift. Therefore the operator $\tau_{\text{BCL}} : \mathcal{H} \rightarrow H^2 \left( \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} \right)$ defined as

$$
\tau_{\text{BCL}} h = \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} h + z \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} V^* h + z^2 \begin{bmatrix} D_{V_1^*} \\ D_{V_2^*} \end{bmatrix} V^{*2} h + \cdots \quad (2.18)
$$
is a unitary and satisfies $\tau_{BCL} V = M_z \tau_{BCL}$. To establish the unitary equivalence in part (ii) of the theorem, we use \((2.3)\) to first note that for every $h \in \mathcal{H}$

$$
\tau_{BCL} V_1 h = \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V_1 h + z \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + z^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* V_1^* h + \ldots
$$

is the same as

$$
\left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h + z \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] h + qz \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + q^2 z^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots,
$$

which we split in two parts as

$$
\left( \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h + qz \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + q^2 z^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots \right)
$$

$$
+ \left( \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] h + qz \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + (qz)^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots \right),
$$

which is equal to \((R_q \otimes P^\perp U + M_z R_q \otimes PU) \tau h\), where $P$ and $U$ are as describe in \((2.11)\) because for every $h \in \mathcal{H}$

$$
P^\perp U \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] h = P^\perp \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] h = \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h \quad \text{and} \quad PU \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] h = \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] h.
$$

It remains to show that

$$
\tau_{BCL} V_2 = \left( R_{\bar{V}} \otimes U^* P + R_{\bar{V}} M_z \otimes U^* P^\perp \right) \tau_{BCL}.
$$

For this we again use the relations \((2.3)\) to note that

$$
\tau_{BCL} V_2 h = \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} \end{array} \right] V_2 h + z \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_2 h + z^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_2 h + \ldots
$$

$$
= \left( I_{H^2} \otimes U^* \right) \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} \end{array} \right] V_2 h + z \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_2 h + z^2 \left[ \begin{array}{c} D_{V_1^*}^1 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_2 h + \ldots
$$

$$
= \left( I_{H^2} \otimes U^* \right) \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h + \bar{V}z \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + (\bar{V}z)^2 \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots
$$

As before, we split the last term in two parts as

$$
\left( I_{H^2} \otimes U^* \right) \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h + \bar{V}z \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + (\bar{V}z)^2 \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots
$$

$$
+ \left( I_{H^2} \otimes U^* \right) \bar{V}z \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} \end{array} \right] h + \bar{V}z \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* h + (\bar{V}z)^2 \left[ \begin{array}{c} D_{V_1^*}^0 \\ D_{V_2^*} V_1^* \end{array} \right] V^* V_1^* h + \ldots
$$

which is essentially equal to \((R_{\bar{V}} \otimes U^* P + R_{\bar{V}} M_z \otimes U^* P^\perp) \tau_{BCL} h\), where $P$ and $U$ are as describe in \((2.11)\). This establishes the equivalence of (i) and (ii). The equivalence of (i) with (iii) can be established in a similar way.

**Definition 2.3.** For a $q$-commutative pair of isometries the tuples \((\mathcal{F}, K_u; P, U, W_1, W_2)\) as in item (i) and \((\mathcal{F}, K_u; P, U, W_1, W_2)\) as in item (ii) of Theorem 2.2, will be referred to as the BCL-1 and BCL-2 $q$-tuples of \((V_1, V_2)\), respectively, and (as is indicated in the statement) the models as in \((2.10)\) and \((2.13)\) will be called the BCL-1 and BCL-2 $q$-models of \((V_1, V_2)\), respectively.

**Remark 2.4.** Note that the BCL-2 $q$-model can be obtained from the BCL-1 $q$-model by the following transformation of the BCL $q$-tuples

\[(\mathcal{F}, K_u; P, U, W_1, W_2) \mapsto (\mathcal{F}, K_u; U^* PU, U^*, W_1, W_2).\]

This indicates that it is enough to work with either of the model.
It was observed in [2] that a commutative pair of isometries is uniquely determined by the data set \((F, K_u; P, U, W_1, W_2)\). The same remains true in the case of \(q\)-commutativity also.

**Theorem 2.5.** Let \((V_1, V_2)\) and \((V'_1, V'_2)\) be two \(q\)-commutative pairs of isometries with \((F, K_u; P, U, W_1, W_2)\) and \((F', K'_u; P', U', W'_1, W'_2)\) as their respective BCL-1 \(q\)-tuples. Then \((V_1, V_2)\) and \((V'_1, V'_2)\) are unitarily equivalent if and only if there exist unitary operators \(\omega : F \to F'\) and \(\omega_u : K_u \to K'_u\) such that

\[
\omega(P, U) = (P', U')\omega \quad \text{and} \quad \omega_u(W_1, W_2) = (W'_1, W'_2)\omega_u.
\]

The statement remains true in case of BCL-2 \(q\)-tuples also.

**Proof.** The easier direction is the ‘if’ part. Note that if \((2.19)\) is true, then the unitary \[
[I_{H^2} \otimes \omega \quad 0 \atop 0 \quad \omega_u] : [H^2 \otimes F \atop K_u] \to [H^2 \otimes F' \atop K'_u]
\]
intertwines the BCL-1 (and BCL-2) \(q\)-models of \((V_1, V_2)\) and \((V'_1, V'_2)\). For the converse part, suppose that the BCL-1 \(q\)-models

\[
(V_1, V_2) = \left( \begin{bmatrix} M_{p+q\pm p} & 0 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} R_q & 0 \\ 0 & I_{K_u} \end{bmatrix}, \begin{bmatrix} M_{U^*-(p+q\pm p)} & 0 \\ 0 & W_2 \end{bmatrix} \right)
\]

and

\[
(V'_1, V'_2) = \left( \begin{bmatrix} M_{p+q\pm p} & 0 \\ 0 & W'_1 \end{bmatrix}, \begin{bmatrix} R_q & 0 \\ 0 & I_{K_u} \end{bmatrix}, \begin{bmatrix} M_{U^*-(p+q\pm p)} & 0 \\ 0 & W'_2 \end{bmatrix} \right)
\]

are unitarily equivalent via, say,

\[
\tau = \begin{bmatrix} \tau' & \tau_{12} \\ \tau_{21} \omega_u \end{bmatrix} : [H^2(F) \atop K_u] \to [H^2(F') \atop K'_u].
\]

Adopting the notations \(W := W_1W_2\) and \(W' = W'_1W'_2\), we see that \(\tau\) must satisfy

\[
[\tau' \atop \tau_{21} \omega_u] \begin{bmatrix} M_z & 0 \\ 0 & W \end{bmatrix} = \tau V_1V_2 = V'_1V'_2\tau = [\tau' \atop \tau_{21} \omega_u],
\]

equivalently, \(\tau\) must satisfy

\[
\tau' M_z = \tau' M_z, \quad \omega_u W = W' \omega_u \quad \text{and} \quad \tau_{12} W = M_z \tau_{12}, \quad \tau_{21} M_z = W' \tau_{21}.
\]

We now use the general functional analysis result that if \(X\) is any operator that satisfies \(XU = M_z X\) for some unitary \(U\), then \(X = 0\). Therefore from \((2.22)\), we see that \(\tau_{12} = 0\). Since \(\tau\) is a unitary that satisfies \((2.20)\), it must also satisfy

\[
[\tau' \atop \tau_{21} \omega_u] \begin{bmatrix} M_z^* & 0 \\ 0 & W'^* \end{bmatrix} = [\tau' \atop \tau_{21} \omega_u],
\]

comparing the \((12)\)-entries of which we get \(\tau_{21} M_z^* = W'^* \tau_{21}\). Since \(W'\) is unitary, \(\tau_{21} = 0\). Therefore the unitary \(\tau\) reduces to the block diagonal matrix \(\text{diag}(\tau', \omega_u)\).

From the first equation in \((2.21)\) we see that \(\tau' = I_{H^2} \otimes \omega\) for some unitary \(\omega : F \to F'\). Remembering that \(\tau\) intertwines \((V_1, V_2)\) and \((V'_1, V'_2)\), we readily have the second equality in \((2.19)\) and for the first equality we note that \(w\) must satisfy

\[
wP^\perp U = P'^\perp U' \omega \quad \text{and} \quad \omega PU = P'U' \omega.
\]

Adding these two equations we get \(\omega U = U' \omega\), which then implies that \(\omega P = P' \omega\). The proof for the case of BCL-2 \(q\)-tuples is along the same line as above. This completes the proof. \(\Box\)
The rest of this section is devoted to finding a connection between commutativity and \( q \)-commutativity. Let \((V_1, V_2)\) be a \( q \)-commutative pair of isometries on \( \mathcal{H} \) such that \( V = V_1 V_2 \) is a shift. Note that in this case the space \( \mathcal{K}_u \) in BCL-1 \( q \)-tuple will be zero, and hence by Theorem 2.2, \((V_1, V_2)\) is unitarily equivalent to
\[
(M_{(P_{++} Semester}) U R_q, R_{\mathcal{F} U^* (P_{++} Semester)})
\]
via the unitary similarity
\[
\tau_{\text{BCL}} : h \mapsto \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] h + z \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] V^* h + z^2 \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] V^* h + \cdots .
\]
(2.23)
Let us denote the unitary
\[
r_q := \tau_{\text{BCL}}^* R_q \tau_{\text{BCL}} : \mathcal{H} \to \mathcal{H}.
\]
(2.24)
To compute the unitary \( r_q \) explicitly, proceed as follows. For \( h, k \in \mathcal{H} \),
\[
\langle r_q h, k \rangle = \langle \tau_{\text{BCL}}^* R_q \tau_{\text{BCL}} h, k \rangle = \langle \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] (I - qz V^*)^{-1} h, \tau_{\text{BCL}} k \rangle
\]
\[
= \sum_{n \geq 0} q^n \langle \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] V^n h, \left[ \frac{D_{V_1^*}}{D_{V_2^*} V_1^*} \right] V^n k \rangle
\]
\[
= \sum_{n \geq 0} q^n \langle D_{V^*} V^n h, D_{V^*} V^n k \rangle \quad \text{[using Lemma 2.1 part (ii)]}
\]
\[
= \sum_{n \geq 0} q^n \langle V^n D_{V^*} V^n h, k \rangle.
\]
Thus
\[
r_q h = D_{V^*} h + q V D_{V^*} V^n h + \cdots q^n V^n D_{V^*} V^n h + \cdots .
\]
(2.25)
As a consequence of this observation and Theorem 2.2, we get the following connection between commutativity and \( q \)-commutativity of a pair of isometries.

**Theorem 2.6.** Let \( V_1 \) and \( V_2 \) be isometric operators such that \( V = V_1 V_2 \) is a shift operator. Then with the unitary \( r_q \) as defined in (2.24),

1. \((V_1, V_2)\) is commutative if and only if \((V_1 r_q, r_q V_2)\) is \( q \)-commutative;
2. \((V_1, V_2)\) is \( q \)-commutative if and only if \((V_1 r_q, r_q V_2)\) is commutative.

**Proof.** We prove only part (1) because it implies part (2). Suppose \((V_1, V_2)\) is a commutative pair of isometries and \((\mathcal{F}; P, U)\) is a BCL-1 tuple of \((V_1, V_2)\). Then applying Theorem 2.2 for the \( q = 1 \) case,
\[
\tau_{\text{BCL}}(V_1, V_2) = (M_{(P_{++} Semester)} U, M_{U^* (P_{++} Semester)}) \tau_{\text{BCL}}
\]
(2.26)
via the unitary similarity \( \tau_{\text{BCL}} \) as in (2.23) above. In view of (2.24) and (2.26),
\[
(M_{(P_{++} Semester)} U R_q, R_{\mathcal{F} U^* (P_{++} Semester)}) = (M_{(P_{++} Semester)} U \tau_{\text{BCL}} r_q^* \tau_{\text{BCL}}^*, \tau_{\text{BCL}}^* \tau_{\text{BCL}} r_q^* \tau_{\text{BCL}}^* M_{U^* (P_{++} Semester)})
\]
\[
= \tau_{\text{BCL}}(V_1 r_q, r_q V_2) \tau_{\text{BCL}}^*.
\]
By the equivalence of (1) and (2) of Theorem 2.2, the pair
\[
(M_{(P_{++} Semester)} U R_q, R_{\mathcal{F} U^* (P_{++} Semester)})
\]
is \( q \)-commutative, and thus so is the pair \((V_1 r_q, r_q V_2)\). \( \square \)
In view of the fact that \((R_q, M_z)\) is \(q\)-commutative, the following is an abstract version of Theorem \[2.6\]  

**Theorem 2.7.** Suppose that \(V_1, V_2\) are some operators acting on a Hilbert space \(\mathcal{H}\), \(r\) is a unitary operator on \(\mathcal{H}\) such that for a uni-modular \(q\),  
\[ r V_1 V_2 = q \cdot V_1 V_2 r. \]  
(2.27)  

Then \((V_1, V_2)\) is commutative if and only if \((V_1 r, r^* V_2)\) is \(q\)-commutative.  

**Proof.** Let us denote \((W_1, W_2) = (V_1 r, r^* V_2)\). Suppose that \((V_1, V_2)\) is commutative and compute  
\[ W_1 W_2 = V_1 r^* r V_2 = V_1 V_2 =: V \]  
while  
\[ W_2 W_1 = r^* V_2 V_1 r = r^* V r = \overline{q} \cdot V = \overline{q} \cdot W_1 W_2. \]  

So \((W_1, W_2)\) is \(q\)-commutative. Conversely, suppose \(W_1 W_2 = q \cdot W_2 W_1\), i.e., \(V_1 V_2 = q \cdot r^* V_2 V_1 r\). By (2.27), this is same as \(q \cdot r^* V_1 V_2 r = q \cdot r^* V_2 V_1 r\). This implies \(V_1 V_2 = V_2 V_1\). \(\Box\)  

### 3. Doubly \(q\)-commutative pairs of isometries

Let us recall that a \(q\)-commutative pair of operators \((V_1, V_2)\) is said to be **doubly \(q\)-commutative**, if in addition, it satisfies \(V_2 V_1^* = q V_1^* V_2\). Note that if \((V_1, V_2)\) is doubly \(q\)-commutative, then so is \((V_1^*, V_2^*)\). Then next result is a characterization of doubly \(q\)-commutative pairs of isometries.  

**Theorem 3.1.** Let \((V_1, V_2)\) be a pair of \(q\)-commutative isometries with BCL-1 and BCL-2 \(q\)-tuples as \((\mathcal{F}, \mathcal{K}_u; P, U, W_1, W_2)\) and \((\mathcal{F}_1, \mathcal{K}_{u1}; P_1, U_1, W_{11}, W_{21})\), respectively. Then the following are equivalent:

1. \((V_1, V_2)\) is doubly \(q\)-commutative;
2. \(PU P^\perp = 0\); and
3. \(P_1^\perp U_1 P_1^\perp = 0\).

**Proof.** By Theorem \[2.2\], we can assume without loss of generality that \((V_1, V_2)\) is either the BCL-1 \(q\)-model \[2.10\] or the BCL-2 \(q\)-model \[2.13\]; to prove (1) \(\Leftrightarrow\) (2), we work with the BCL-1 \(q\)-model. Since, \(q\)-commutativity of a pair of unitaries implies its doubly \(q\)-commutativity, we disregard the unitary part \((W_1, W_2)\) in the model \[2.10\] and suppose that  
\[ (V_1, V_2) = (R_q \otimes P^\perp U + M_z R_q \otimes P U, R_\overline{q} \otimes U^* P + R_\overline{q} M_z \otimes U^* P^\perp) \]  
on \(H^2 \otimes \mathcal{F}\).  

We shall make use of the following identities concerning the two operators \(R_q\) and \(M_z\) on \(H^2\). We do not prove these relations as the proofs are elementary. For every \(n \geq 1\),  
\[
R_\overline{q} M_z R_\overline{q}(z^n) = \overline{q}^{2n+1} z^n, \quad R_\overline{q} M_z R_\overline{q} M_z^*(z^n) = \overline{q}^{2n-1} z^n \\
R_\overline{q} M_z^* R_\overline{q}(z^n) = \overline{q}^{2n-1} z^{n-1}, \quad R_\overline{q} M_z^* R_\overline{q} M_z(z^n) = \overline{q}^{2n+1} z^n.
\]  

With the above relations in mind, we compute  
\[
V_2 V_1^* = (R_\overline{q} \otimes U^* P + R_\overline{q} M_z \otimes U^* P^\perp)(R_\overline{q} \otimes U^* P^\perp + R_\overline{q} M_z^* \otimes U^* P) \\
= R_\overline{q}^2 \otimes U^* P U^* P^\perp + R_\overline{q}^2 M_z^* \otimes U^* P U^* P + R_\overline{q} M_z R_\overline{q} \otimes U^* P^\perp U^* P^\perp \\
+ R_\overline{q} M_z R_\overline{q} M_z^* \otimes U^* P^\perp U^* P
\]
and
\[ V_1^*V_2 = (R_{\psi} \otimes U^* P^\perp + R_{\psi} M_z^* \otimes U^* P)(R_{\psi} \otimes U^* P + R_{\psi} M_z \otimes U^* P^\perp) \]
\[ = R_{\psi}^2 \otimes U^* P^\perp U^* P + R_{\psi}^2 M_z \otimes U^* P^\perp U^* P^\perp + R_{\psi} M_z^* M_{\psi} \otimes U^* P U^* P \]
\[ + R_{\psi} M_z^* R_{\psi} M_z \otimes U^* P U^* P^\perp. \]

Suppose \( n \geq 1 \) and \( \xi \in \mathcal{F} \). Then
\[ V_2 V_1^* (z^n \otimes \xi) = \overline{q}^{2n} z^n \otimes U^* P U^* P^\perp \xi + \overline{q}^{2n-2} z^{n-1} \otimes U^* P U^* P \xi \]
\[ + \overline{q}^{2n+1} z^{n+1} \otimes U^* P^\perp U^* P^\perp \xi + \overline{q}^{2n-1} z^n \otimes U^* P^\perp U^* P \xi \]
and
\[ V_1^* V_2 (z^n \otimes \xi) = \overline{q}^{2n} z^n \otimes U^* P^\perp U^* P^\perp \xi + \overline{q}^{2n+2} z^{n+1} \otimes U^* P^\perp U^* P \xi \]
\[ + \overline{q}^{2n-1} z^{n-1} \otimes U^* P^\perp U^* P^\perp \xi + \overline{q}^{2n+1} z^n \otimes U^* P^\perp U^* P \xi. \]

From the above expressions of \( V_2 V_1^* (z^n \otimes \xi) \) and \( q V_1^* V_2 (z^n \otimes \xi) \), one readily observes that
\[ V_2 V_1^* (z^n \otimes \xi) = q V_1^* V_2 (z^n \otimes \xi) \]
whenever \( n \geq 1 \) and \( \xi \in \mathcal{F} \).

We now compute
\[ V_2 V_1^* (1 \otimes \xi) = U^* P U^* P^\perp \xi + \overline{q} z \otimes U^* P^\perp U^* P \xi \]
and
\[ V_1^* V_2 (1 \otimes \xi) = U^* P^\perp U^* P \xi + \overline{q}^2 z \otimes U^* P^\perp U^* P^\perp \xi + \overline{q} U^* P U^* P^\perp \xi. \]

Therefore \( V_2^* V_1 = q V_1^* V_2 \) if and only if for every \( \xi \in \mathcal{F} \),
\[ V_2 V_1^* (1 \otimes \xi) = V_1^* V_2 (1 \otimes \xi), \]
which, in view of the above computation, is true if and only if
\[ P U P^\perp = 0. \]

This completes the proof of \( (1) \iff (2) \). To complete the proof of the theorem, one can either work with the BCL-2 \( q \)-model in (2.13) and proceed as before to prove \( (1) \iff (3) \),
or, simply apply Remark 2.4 and establish the equivalence of (2) and (3).

We wish to establish a connection between double commutativity and \( q \)-double commutativity in analogue of Theorem 2.6. We first observe the following.

**Lemma 3.2.** The BCL-1 model
\[ (M_{(P^\perp + z P) U} \oplus W_1, M_{U^* (P^\perp + z P)} \oplus W_2) \]
of a commutative pair of isometries is doubly commutative if and only if \( P U P^\perp = 0 \).

**Proof.** Since a commuting pair of unitaries is automatically doubly commuting, we only investigate the doubly commutativity of the pair
\[ (V_1, V_2) = (I_{H^2} \otimes P^\perp U + M_z \otimes PU, I_{H^2} \otimes U^* P + M_z \otimes U^* P^\perp). \]

We note that
\[ V_2^* V_1 = (I_{H^2} \otimes PU + M_z^* \otimes P^\perp U)(I_{H^2} \otimes P^\perp U + M_z \otimes PU) \]
\[ = I_{H^2} \otimes PU P^\perp U + M_z \otimes PU PU + M_z^* \otimes P^\perp U P^\perp U + I_{H^2} \otimes P^\perp U P^\perp U. \]
and
\[ V_1V_2^* = (I_{H^2} \otimes P^1U + M_z \otimes PU)(I_{H^2} \otimes PU + M_z^* \otimes P^1U) \]
\[ = I_{H^2} \otimes P^2U PittU + M_z \otimes PUPU + M_z^* \otimes P^1UP^1U + M_zM_z^* \otimes PUP^1U. \]

From the above two expressions, we see after cancellation of common terms that
\[ V_2^*V_1 - V_1V_2^* = (I - M_zM_z^*) \otimes PUP^1U. \]

Since \( I_{H^2} - M_zM_z^* \) is the projection of \( H^2 \) on the constant functions in \( H^2 \), we see that \( V_1 \) double commutes with \( V_2 \) exactly when \( PUP^1U = 0 \), or, equivalently, \( PUP^1 = 0 \). \( \square \)

**Theorem 3.3.** Let \( V_1 \) and \( V_2 \) be isometries such that \( V = V_1V_2 \) is a shift, and \( \tau_q \) be the unitary as in (2.23). Then

1. \((V_1, V_2)\) is doubly commutative if and only if \((V_1\tau_q, \tau_qV_2)\) is doubly \( q \)-commutative;
2. \((V_1, V_2)\) is doubly \( q \)-commutative if and only if \((V_1\tau_q^*, \tau_q^*V_2)\) is doubly commutative.

**Proof.** The proof is similar to that of Theorem 2.6. For part (1), suppose \((V_1, V_2)\) is a commutative and \((F; P, U)\) is a BCL-1 tuple of \((V_1, V_2)\). By Theorem 2.2
\[ \tau_{BCL}(V_1, V_2) = (M_{(P+\overline{z}P)}U, M_{U^*(P+\overline{z}P)}) \tau_{BCL}. \]
where \( \tau_{BCL} : \mathcal{H} \to H^2(\mathcal{F}) \) is the unitary as in (2.23). Suppose that \((V_1, V_2)\) is doubly commutative. Hence by Lemma 3.2, we have \( PUP^1 = 0 \). By Theorem 3.1, this is equivalent to the BCL-1 \( q \)-model \((M_{(P+\overline{z}P)}U, R_q, R_q^*M_{U^*(P+\overline{z}P)})\) being doubly \( q \)-commutative. But as observed in the proof of Theorem 2.6,
\[ (M_{(P+\overline{z}P)}U, R_q, R_q^*M_{U^*(P+\overline{z}P)}) = (V_1^*\tau_{BCL}\tau_q^*\tau_{BCL}, \tau_{BCL}\tau_q^*\tau_{BCL}V_2^*) = \tau_{BCL}(V_1\tau_q, \tau_q^*V_2)\tau_{BCL}. \]
Therefore equivalently, the pair \((V_1\tau_q, \tau_q^*V_2)\) must also be doubly \( q \)-commutative. Now part (1) implies part (2) and therefore the proof is complete. \( \square \)

Słociński [22] proved that any pair of doubly commuting shift operators is unitarily equivalent to \((M_{z_1}, M_{z_2})\) on \( H^2(\mathbb{D}^2) \). As a corollary to Theorem 3.3, we get the following analogue of Słociński’s result in the \( q \)-commutative setting.

**Corollary 3.4.** A pair of shift operators \((V_1, V_2)\) is doubly \( q \)-commutative if and only if it is unitarily equivalent to \((M_{z_1}, M_{z_2})\) on \( H^2(\mathbb{D}^2) \) for some unitary \( s_q \) on \( H^2(\mathbb{D}^2) \).

**Proof.** Suppose that \((V_1, V_2)\) is doubly \( q \)-commutative pair of shift operators. It is a general fact that if \((V_1, V_2)\) is \( q \)-commutative pair of isometries with one of the entries a shift, then the product \( V = V_1V_2 \) is also a shift. To see this we shall use the general fact that if \((T_1, T_2)\) is \( q \)-commutative, then with \( T = T_1T_2 \),
\[ T^n = q^{x_n}T_1^nT_2^n = q^{y_n}T_2^nT_1^n \quad \text{for every } n \geq 1, \]
where the sequences \( \{x_n\}_{n \geq 1} \) and \( \{y_n\}_{n \geq 1} \) are given by the iterative relations
\[ x_1 = 0, \quad x_n = x_{n-1} + n - 1, \quad \text{and } y_1 = 1, \quad y_n = y_{n-1} + n. \]

We omit the proof of (3.2) as it is routine. Applying this fact to the \( q \)-commutative pair \((V_1, V_2)\) of shift operators, we see that
\[ V^{*n} = q^{x_n}V_2^{*n}V_1^{*n} \to 0 \quad \text{as } n \to \infty. \]
Invoking part (2) of Theorem 3.3 we get \((V_1 \tau_q^*, r_q V_2)\) is doubly commutative, where \(r_q\) is the unitary as in (2.24). By Ślociński’s characterization of doubly commutative pair of shifts, there exist a unitary \(\tau_S: \mathcal{H} \to H^2(\mathbb{D}^2)\) such that
\[
\tau_S(V_1 \tau_q^*, r_q V_2) = (M_{z_1}, M_{z_2})\tau_S.
\]

The first component of (3.3) gives \(\tau_S V_1 = M_{z_1} \tau_S r_q = M_{z_1} \tau_S r_q \tau_S^* \tau_S\). This and a similar treatment for the second component give
\[
\tau_S(V_1, V_2) = (M_{z_1} \tau_S r_q \tau_S^* \tau_S, \tau_S r_q \tau_S^* M_{z_2}) \tau_S,
\]
which readily implies that with
\[
\mathcal{S}_q := \tau_S r_q \tau_S^* = \tau_S \mathcal{S}_{BCL} R_q \mathcal{S}_{BCL} \tau_S^* \tau_S
\]
the doubly \(q\)-commutative pair \((V_1, V_2)\) is unitarily equivalent to \((M_{z_1} \mathcal{S}_q, \mathcal{S}_q M_{z_2})\).

\[
\square
\]

4. Examples

It is interesting to work with some concrete examples to illustrate the model theory. First we exhibit a simple example of a pair of isometric operators that is doubly \(q\)-commutative.

Example 4.1. Consider the pair \((V_1, V_2) = (R_q, M_z)\) on the Hardy space \(H^2\). We have seen in the introduction that this pair is \(q\)-commutative. To see that this is doubly \(q\)-commutative, we prove the general fact that if a pair \((T_1, T_2)\) is \(q\)-commutative and \(T_1\) is unitary, then it is doubly \(q\)-commutative. For this we simply multiply \(T_1 T_2 = q T_2 T_1\) by \(T_1^*\) from right and left successively, to get \(T_2 T_1^* = q T_1^* T_2\). It is interesting to note that if, instead of \(T_1, T_2\) is unitary, then \((T_1, T_2)\) would be doubly \(\overline{q}\)-commutative.

Below we illustrate the equivalence of (1) and (2) of Theorem 2.2 for this particular example. First we compute explicitly the BCL-1 \(q\)-tuple for this pair.

Let us first note that if \((V_1, V_2) = (R_q, M_z)\), then
\[
\begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
0 & P_C \\
H^2 & H^2
\end{bmatrix} \to \begin{bmatrix}
H^2 & H^2
\end{bmatrix},
\]
and therefore \(\mathcal{F} = \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
0 \\
C
\end{bmatrix}\),

where \(P_C\) is the orthogonal projection of \(H^2\) onto the constant functions. Let \(f(z) = a_0 + za_1 + \cdots + z^n a_n + \cdots\) be in \(H^2\). We note that
\[
D_{V_1^*} V_2^* f = 0 \text{ and } D_{V_2^*} V_1^* f = D_{M_z} R_q = a_0.
\]

Therefore
\[
U : \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
0 \\
a_1
\end{bmatrix} \mapsto \begin{bmatrix}
0 \\
a_0
\end{bmatrix} = \begin{bmatrix}
D_{V_1^*} V_2^* \\
D_{V_2^*}
\end{bmatrix}
\]
is essentially \(I_{C^2}\). It is interesting to note that if \(P\) is the projection of \(D_{V_1^*} \oplus D_{V_2^*}\) onto \(D_{V_1^*}\) (which is zero), then \(P\) is essentially \([0 0] [0 0]\), while \(P^\perp = [0 0] [0 0]\). Since \((R_q, M_z)\) is \(q\)-commutative, applying (3.2) to the pair \((R_q, M_z)\), we get with \(V = R_q M_z\)
\[
V^m f = \overline{q}^m R_q M_z^m f = \overline{q}^m (a_n, \overline{q}^n a_{n+1}, \overline{q}^{2n} a_{n+2}, \ldots).
\]

Since \(V = V_1 V_2\) on \(H^2\) is a shift operator, the general unitary identification \(\tau_{BCL}\) from \(H^2\) onto \(H^2 \otimes \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix}\), which is of the form (as shown in (2.13))
\[
\tau_{BCL} h = \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} h + z \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} V^* h + z^2 \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} V^{*2} h + \cdots
\]

Therefore...
is given in this case as
\[
qf \mapsto [q^{0}_0] + zq^{0}_1 + \cdots + z^nq^{0}_{a_n} + \cdots.
\] (4.1)
Therefore
\[
(P^\perp U + M_z PU)R_q\tau_{BCL}f(z) = \left([0^{0}_0] + z[0^{0}_1] + \cdots + z^n[0^{0}_{a_n}] + \cdots\right)
\]
= \left([0^{0}_0] + zq^{0}_1 + \cdots + z^nq^{0}_{a_n} + \cdots\right) = R_q\tau_{BCL}f(z).

Similar computation for the intertwining relation \(R_q(U^*P+M_zU^*P^\perp)\tau_{BCL} = R_qM_z\tau_{BCL}\).

In view of Theorem 3.1 that the pair \((R_qM_z, M_z)\) is doubly \(q\)-commutative is reflected in the fact that
\[
PU^2 = [0^{0}_0][0^{0}_0][0^{0}_0] = [0^{0}_0].
\]

Next we find an example of a pair of shift operators that is \(q\)-commutative but not doubly \(q\)-commutative.

**Example 4.2.** Consider the pair \((V_1, V_2) = (R_qM_z, M_z)\) on \(H^2\). Then for every \(f \in H^2\),
\[
V_1V_2f(z) = R_qM_z^2f(z) = q^2z^2f(qz) \quad \text{while,}
\]
\[
V_2V_1f(z) = M_zR_qM_zf(z) = M_zqzf(qz) = qz^2f(qz),
\]
showing that \((V_1, V_2)\) is a \(q\)-commutative pair. However, it should be noted that the pair is not doubly \(q\)-commutative. One way to see this is that
\[
V_2V_1^*(1) = M_zM_z^*R_q(1) = 0 \quad \text{but} \quad V_1^*V_2(1) = M_z^*R_qM_z(1) = \overline{q}.
\]
Therefore \(V_2V_1^* \neq qV_1^*V_2\). To see that \(V_1 = R_qM_z\) is actually a shift operator, we apply (3.2) to the \(q\)-commutative pair \((R_q, M_z)\) to note
\[
V_1^n = (M_z^*R_q^n)\overline{q} = q^nR_q^nM_z^n \to 0 \quad \text{in the strong operator topology as} \ n \to \infty.
\]

Below we compute the BCL-1 \(q\)-tuple corresponding to the pair \((V_1, V_2) = (R_qM_z, M_z)\).

Let us first note that \(D_{V_1^*} = I - V_1V_1^* = I - R_qM_z^*R_{\overline{q}} = R_qD_{M_z^*}R_{\overline{q}}\), which is essentially the same as \(D_{M_z^*} = P_C\). Therefore
\[
\begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
P_C \\
P_C
\end{bmatrix} : \begin{bmatrix}
H^2 \\
H^2
\end{bmatrix} \to \begin{bmatrix}
H^2 \\
H^2
\end{bmatrix} , \quad \text{and therefore} \quad \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
P_C \\
P_C
\end{bmatrix}.
\]
Let \(f(z) = a_0 + za_1 + \cdots + z^n a_n + \cdots \) be in \(H^2\). We note that
\[
D_{V_1^*}V_2^*f = D_{M_z^*}M_z^*f = a_1 \quad \text{and} \quad D_{V_1}V_1^*f = D_{M_z^*}M_z^*R_{\overline{q}} = qa_1.
\]
Therefore
\[
U : \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix} = \begin{bmatrix}
a_0 \\
a_0
\end{bmatrix} \mapsto \begin{bmatrix}
a_1 \\
0
\end{bmatrix} = \begin{bmatrix}
D_{V_1^*}V_2^* \\
D_{V_2^*}V_1^*
\end{bmatrix}
\]
is given by \(U = \begin{bmatrix}
0 \\
0
\end{bmatrix} \overline{q}\). Next we note that since \((R_q, M_z)\) is \(q\)-commutative, \((R_q, M_z^2)\) is \(q^2\)-commutative, and therefore applying (3.2) we get with \(V = V_1V_2\)
\[
V^{*n}f = \overline{q}^{2n}R_{\overline{q}}M_z^{2n}f = \overline{q}^{2n}(a_{2n}, \overline{q}^n a_{2n+1}, \overline{q}^{2n} a_{2n+2}, \cdots).
\]
Since \(V\) is a shift operator, the unitary identification \(\tau_{BCL}\) from \(H^2\) onto \(H^2 \otimes \begin{bmatrix}
D_{V_1^*} \\
D_{V_2^*}
\end{bmatrix}\) in this case is given by
\[
f \mapsto \begin{bmatrix}
a_0 \\
0
\end{bmatrix} + z\overline{q}^{a_2} \begin{bmatrix}
a_3 \\
a_0
\end{bmatrix} + \cdots + z^n\overline{q}^{2n} \begin{bmatrix}
0 \\
a_0
\end{bmatrix} + \cdots.
\] (4.2)
To demonstrate that this $\tau_{\text{BCL}}$ intertwines $(V_1, V_2)$ and $(M_{(P_{\perp+}z)UR_q, R_q^\perp M_{U^*}(P_{\perp+}zP_{\perp})})$, we compute

$$P_{\perp}UR_q^\perp \tau_{\text{BCL}}f = P_{\perp}U \left( [\frac{a_0}{q_1}] + z\tau [\frac{a_2}{q_{a_2}}] + \cdots + z^n\frac{a_{2n}}{q_{a_{2n}}} [\frac{a_{2n+1}}{q_{a_{2n+1}}} + \cdots] \right)$$

$$= P_{\perp} \left( [\frac{a_0}{q_1}] + z\tau [\frac{a_1}{q_{a_1}}] + \cdots + z^n\frac{a_{2n}}{q_{a_{2n}}} [\frac{a_{2n+1}}{q_{a_{2n+1}}} + \cdots] \right)$$

$$= [\frac{a_0}{q}] + z\tau [\frac{a_1}{q}] + \cdots + z^n\tau^2y^n - n [\frac{a_n}{q}] + \cdots$$

and (using the action of $UR_q^\perp \tau_{\text{BCL}} f$ from the above computation)

$$M_zPUPR_q^\perp \tau_{\text{BCL}}f = M_zP \left( [\frac{a_0}{q_1}] + z\tau [\frac{a_2}{q_{a_2}}] + \cdots + z^n\frac{a_{2n}}{q_{a_{2n}}} [\frac{a_{2n+1}}{q_{a_{2n+1}}} + \cdots] \right)$$

$$= z [\frac{a_0}{q}] + z^2\tau [\frac{a_2}{q_2}] + \cdots + z^{n+1}\tau^2y^n - n [\frac{a_n}{q}] + \cdots.$$ 

Therefore

$$(P_{\perp}U + M_z PU)R_q^\perp \tau_{\text{BCL}}f = [\frac{a_0}{q}] + z [\frac{a_1}{q}] + \cdots + z^n\tau^2y^n - n [\frac{a_n}{q}] + \cdots. \quad (4.3)$$

We note that

$$V_1f(z) = R_qM_zf = za_0q + z^2a_1q^2 + \cdots + z^n a_{n-1}q^n + \cdots = \sum_{n \geq 0} z^n b_n.$$ 

Therefore replacing $f$ by $V_1 f$ in the expression (4.2) of $\tau_{\text{BCL}}$, we get $\tau_{\text{BCL}}V_1f$ the same as $(P_{\perp}U + M_z PU)R_q^\perp \tau_{\text{BCL}}f$. Similar computation for the other intertwining relation.

In view of Theorem 3.1 that the pair $(R_qM_z, M_z)$ is not doubly $q$-commutative is reflected in the fact that

$$PU^{\perp} = [\frac{1}{0} \frac{1}{0}] [\frac{0}{q} \frac{0}{1}] [\frac{0}{q} \frac{0}{1}] = [\frac{0}{q} \frac{0}{1}] \neq [\frac{0}{q} \frac{0}{1}].$$ 

The following couple of examples are interesting to note.

**Example 4.3.** Consider the pair $(V_1, V_2) = (R_qM_{z_1}, M_{z_2})$ on $H^2(D^2)$. We have noticed in the Introduction that this is indeed $q$-commuting. The computation below shows that it is actually doubly $q$-commutative.

$$V_2V_1^* f(z_1, z_2) = M_{z_2}M_{z_1}^* R_q f(z_1, z_2) = M_{z_2}M_{z_1}^* f(\overline{q}z_1, \overline{q}z_2) = M_{z_1}^* M_{z_2} f(\overline{q}z_1, \overline{q}z_2) \text{ and}$$

$$V_1^* V_2 f(z_1, z_2) = M_{z_1}^* R_q M_{z_2} f(z_1, z_2) = \overline{q} M_{z_1}^* M_{z_2} f(\overline{q}z_1, \overline{q}z_2).$$

Consider the subspace $H_\circ := H^2(D^2) \ominus \{\text{constants}\}$. Just like commutativity, it is trivial that $q$-commutativity is hereditary, i.e., the restriction of a $q$-commutative pair is $q$-commutative. However, the restriction $(V_1', V_2') = (R_qM_{z_1}, M_{z_2})|_{H_\circ}$ is not doubly $q$-commutative as the following computation reveals:

$$V_2'V_1'' f(z_1) = M_{z_2}M_{z_1}^* R_q f(z_1) = 0 \neq \overline{q}z_2 = M_{z_1}^* M_{z_2} f(\overline{q}z_1, \overline{q}z_2) = q M_{z_2} M_{z_2}^* f(z_1) = qV_1'' V_2'(z_1).$$

It is interesting to have an example of a pair of isometries which is not $q$-commutative for any complex number $q$. Let $\alpha, \beta$ are two distinct numbers in $\mathbb{T}$. Consider

$$V_1 = [\frac{\alpha}{0} \frac{0}{\beta} ] \text{ and } V_2 = \left[ \frac{\overline{\alpha}}{\sqrt{2}} \frac{\sqrt{2}}{\overline{\beta}} \right].$$

Then clearly

$$V_1 V_2 = \left[ \frac{\overline{\alpha}}{\sqrt{2} \beta - \sqrt{2} \beta} \right] \neq q \left[ \frac{\overline{\alpha}}{\sqrt{2} \beta - \sqrt{2} \beta} \right] = q V_2 V_1$$

for any number $q$, because $\alpha$ and $\beta$ are distinct.
5. The tuple case

In this section, we use the model for the pair case to exhibit a parallel model for tuples \((V_1, V_2, \ldots, V_d)\) of \(q\)-commutative isometries. We first define \(q\)-commutativity for tuples of operators.

**Definition 5.1.** Let \(q : \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d\} \to \mathbb{T}\) be a function such that \(q(i, i) = 1\) and \(q(i, j) = q(j, i)\) for each \(i, j = 1, 2, \ldots, d\). A \(d\)-tuple \((V_1, V_2, \ldots, V_d)\) of operators is said to be \(q\)-commutative, if

\[
V_i V_j = q(i, j) V_j V_i \quad \text{for each} \quad i, j = 1, 2, \ldots, d.
\]

As an example of a \(q\)-commutative tuple of isometries, let us define \(V_j\) on \(H^2(\mathbb{D}^d)\), the Hardy space of the \(d\)-disk, as

\[
V_j = R_{q^{d-j}} M_{z_j} \quad \text{or} \quad M_{z_j} R_{q^{d-j}} \quad \text{for each} \quad j = 1, 2, \ldots, d, \tag{5.1}
\]

and \(q : \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d\} \to \mathbb{T}\) as \(q(i, j) = q^{j-i}\). To see that \((V_1, V_2, \ldots, V_d)\) is \(q\)-commutative, we compute

\[
V_i V_j f(z) = R_{q^{d-i}} M_{z_i} R_{q^{d-j}} M_{z_j} f(z) = q^{d-j} R_{q^{d-i}} z_i z_j f(q^{d-j} z) = q^{3d-2i-j} z_i z_j f(q^{2d-i-j} z)
\]

while \(V_j V_i f(z) = q^{3d-i-2j} z_i z_j f(q^{2d-i-j} z)\) (obtained by just switching \((i, j)\) to \((j, i)\) in the above expression).

Let us denote

\[
V_{(i)} := V_1 \cdots V_{i-1} V_{i+1} \cdots V_d.
\]

A key observation that makes it possible to apply the results for the pair case to the general case, is that if \((V_1, V_2, \cdots, V_d)\) is \(q\)-commutative, then for each \(i = 1, 2, \ldots, d\), the pair \((V_i, V_{(i)})\) is \(q_i\)-commutative, where

\[
q_i := \prod_{j=1}^{d} q(i, j). \tag{5.2}
\]

This is because for each \(i\),

\[
V_i V_{(i)} = V_i V_1 V_2 \cdots V_{i-1} V_{i+1} \cdots V_d = \prod_{i \neq j=1}^{d} q(i, j) V_{(i)} V_i = q_i V_{(i)} V_i,
\]

where we used the fact that \(q(i, i) = 1\). This observation makes it easy to obtain a Berger–Coburn–Lebow-type model for any \(q\)-commutative tuples of isometries \((V_1, V_2, \ldots, V_d)\). Indeed, the idea is to just apply Theorem 2.2 to each of the \(q_i\)-commutative pairs \((V_i, V_{(i)})\). However, unlike the pair case, a BCL-1 and BCL-2 \(q\)-models need not in general be \(q\)-commutative. This will happen when the BCL-1 and BCL-2 \(q\)-tuples satisfy some compatibility conditions.

**Theorem 5.2.** Let \((V_1, V_2, \ldots, V_d)\) be a \(d\)-tuple of \(q\)-commutative isometries. Then

1. **BCL-1 \(q\)-model:** there exist Hilbert spaces \(\mathcal{F}\) and \(\mathcal{K}_u\), projections \(P_1, P_2, \ldots, P_d\) and unitaries \(U_1, U_2, \ldots, U_d\) in \(\mathcal{B}(\mathcal{F})\), and a \(q\)-commutative tuple \((W_1, W_2, \ldots, W_d)\) of unitaries in \(\mathcal{B}(\mathcal{K}_u)\) such that for each \(i = 1, 2, \ldots, d\), \(V_i\) is unitarily equivalent to

\[
\begin{bmatrix}
R_{q_i} \otimes P_i^+ U_i + M_{z_i} R_{q_i} \otimes P_i U_i & 0 \\
0 & W_i
\end{bmatrix} \text{ on } \begin{bmatrix} H^2 \otimes \mathcal{F} \\ \mathcal{K}_u \end{bmatrix}, \tag{5.3}
\]
and $V(i)$ is unitarily equivalent to
\[
\begin{bmatrix}
R_{q_i} \otimes U_{i}^* P_i + R_{q_i} M_i \otimes U_{i}^* P_i + R_{q_i} M_i \otimes U_{i}^* P_i^i & 0
\end{bmatrix}
\text{ on } \begin{bmatrix} H^2 \otimes F \end{bmatrix}.
\]

Moreover, the tuple $(F, K_{u^i}; P_i, U_i, W_i)_{i=1}^d$ can be chosen to be such that
\[
F = D_{V_i^*} \oplus D_{V_2} \oplus \cdots \oplus D_{V_d}, \quad K_{u^i} = \bigcap_{n \geq 0} (V_1 V_2 \cdots V_d)^n \mathcal{H},
\]
\[
(W_1, W_2, \ldots, W_d) = (V_1, V_2, \ldots, V_d)|_{K_{u^i}}, \quad P_i = \text{projection onto } D_{V_i^*}, \quad \text{and}
\]
\[
U_i : D_{V_i^*} \oplus \Delta_i D_{V_i^*} \rightarrow D_{V_i^*} V_i^* \oplus \Delta_i D_{V_i^*} \text{ for some unitary }
\]
\[
\Delta_i : D_{V_i^*} \rightarrow \bigoplus_{i \neq j=1} D_{V_j^*} \text{ given explicitly in (5.12) below,}
\]

and

(2) BCL-2 q-model: there exist Hilbert spaces $F_i$ and $K_{u^i}$, projections $P_i$ and a unitary $U_i$ in $B(F_i)$, and a tuple $(W_1, W_2, \ldots, W_d)$ of $q$-commutative unitaries in $B(K_{u^i})$ such that for each $i = 1, 2, \ldots, d$, $V_i$ is unitarily equivalent to
\[
\begin{bmatrix}
R_{q_i} \otimes U_{i}^* P_i + R_{q_i} M_i \otimes U_{i}^* P_i + R_{q_i} M_i \otimes U_{i}^* P_i^i & 0
\end{bmatrix}
\text{ on } \begin{bmatrix} H^2 \otimes F_i \end{bmatrix}.
\]

and $V(i)$ is unitarily equivalent to
\[
\begin{bmatrix}
R_{q_i} \otimes P_i U_i + R_{q_i} M_i \otimes P_i U_i & 0
\end{bmatrix}
\text{ on } \begin{bmatrix} H^2 \otimes F_i \end{bmatrix}.
\]

Moreover, the tuple $(F_i, K_{u^i}; P_i, U_i, W_i)_{i=1}^d$ can be chosen to be such that
\[
(F_i, K_{u^i}; P_i, U_i, W_i)_{i=1}^d = (F, K_{u^i}; P_i, U_i, W_i)_{i=1}^d \quad \text{for each } i,
\]

where $(F, K_{u^i}; P_i, U_i, W_i)_{i=1}^d$ is as in part (1) above.

Proof. As in the pair case, we only do the analysis for part (1), as a similar analysis works for part (2). The first step is to fix $i = 1, 2, \ldots, d$ and apply the implication (1) $\Rightarrow$ (2) of Theorem 2.2 to the $q_i$-commutative pair $(V_i, V(i))$. This will give us Hilbert spaces $F_i$, $K_{iu}$, a projection $P_i$, a unitary $U_i$ in $B(F_i)$, and a pair $(W_i, W_i')$ of $q_i$-commuting unitaries in $B(K_{iu})$ such that $(V_i, V(i))$ is unitarily equivalent to
\[
\left(\begin{bmatrix} R_{q_i} \otimes P_i U_i + R_{q_i} M_i \otimes P_i U_i & 0 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} R_{q_i} \otimes U_i^* P_i + R_{q_i} M_i \otimes U_i^* P_i^i & 0 \\
0 & 0 \end{bmatrix}\right) \text{ on } \begin{bmatrix} H^2 \otimes F_i \end{bmatrix},
\]

where by (2.11) the parameters $(F_i, K_{iu}; P_i, U_i, W_i, W_i')$ can be chosen to be
\[
\begin{cases}
F_i = \begin{bmatrix} D_{V_i^*} \\ D_{V_i^*} \\ D_{V_i^*} \\ D_{V_i^*} \end{bmatrix}, & K_{iu} = \bigcap_{n \geq 0} (V_1 V_2 \cdots V_d)^n \mathcal{H}, \quad P_i : [f] \mapsto [f], \\
U_i : \begin{bmatrix} D_{V_i^*} \\ D_{V_i^*} \\ D_{V_i^*} \end{bmatrix} \mapsto \begin{bmatrix} D_{V_i^*} \\ D_{V_i^*} \end{bmatrix} \quad \text{and } (W_i, W_i') = (V_i, V(i))|_{K_{iu}}.
\end{cases}
\]

Let us first note that by definition of $V(i)$ it follows that
\[
W_i' = \prod_{i \neq j=1}^d W_j = W(i).
\]

Next we note that for each $i = 1, 2, \ldots, d$,
\[
K_{iu} = \bigcap_{n \geq 0} (V_1 V_2 \cdots V_d)^n \mathcal{H} = q(i, 1) q(i, 2) \cdots q(i, i-1) \bigcap_{n \geq 0} V^n \mathcal{H} =: \mathcal{K}_u,
\]
where \( V = V_1 V_2 \cdots V_d \) and we used the fact that for every \( i \),
\[
V_i V(i) = V_i V_1 V_2 \cdots V_{i-1} V_{i+1} \cdots V_d = q(i, 1) q(i, 2) \cdots q(i, i-1) V.
\]

We next argue that for each \( i = 1, 2, \ldots, d \), \( F_i = D_{V_i^*} \oplus D_{V_2^*} \oplus \cdots \oplus D_{V_d^*} \). By the expression of \( F_i \) as given in (5.10), this will be achieved if we can show that
\[
D_{V_i^*} \quad \text{is unitarily equivalent to} \quad \oplus_{i \neq j=1}^d D_{V_j^*}.
\]

For (5.11), we define the map \( \Delta_i : D_{V_i^*} h \mapsto \oplus_{i \neq j=1}^d D_{V_j^*} \) by
\[
\Delta_i : D_{V_i^*} h \mapsto D_{V_i^*} V_2^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \oplus D_{V_2^*}^* V_3^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \\
\quad \oplus \cdots \oplus D_{V_{d-1}^*} V_d^* h \oplus D_{V_d^*} h.
\]

Using the general fact that for a contraction \( T \), \( \| D_T h \|^2 = \| h \|^2 - \| Th \|^2 \), we see that
\[
\| D_{V_1^*} V_2^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \|^2 + \| D_{V_2^*} V_3^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \|^2 \\
\quad + \cdots + \| D_{V_{d-1}^*} V_d^* h \|^2 + \| D_{V_d^*} h \|^2
\]

is a telescopic sum and is equal to
\[
\| h \|^2 - \| V_1^* V_2^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \|^2 = \| D_{V_i^*} h \|^2.
\]

Therefore \( \Delta_i \) is an isometry. We claim that
\[
\{ D_{V_1^*} V_2^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \oplus D_{V_2^*} V_3^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \\
\quad \oplus \cdots \oplus D_{V_{d-1}^*} V_d^* h \oplus D_{V_d^*} h : h \in \mathcal{H} \} = \oplus_{i \neq j=1}^d D_{V_j^*}.
\]

We follow the same technique as used to prove Lemma 2.1; we show that the orthocomplement of the space on the left-hand side in \( \oplus_{i \neq j=1}^d D_{V_j^*} \) is zero. Let \( \oplus_{i \neq j=1}^d f_j \in \oplus_{i \neq j=1}^d D_{V_j^*} \) be such that for every \( h \in \mathcal{H} \),
\[
0 = \langle \oplus_{i \neq j=1}^d f_j, D_{V_1^*} V_2^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \oplus D_{V_2^*} V_3^* \cdots V_{i-1}^* V_{i+1}^* \cdots V_d^* h \\
\quad \oplus \cdots \oplus D_{V_{d-1}^*} V_d^* h \oplus D_{V_d^*} h \rangle.
\]

This implies that for every \( h \in \mathcal{H} \)
\[
\langle h, f_d + V_d f_{d-1} + \cdots + V_d V_{d-1} V_{i+1} V_{i-1} \cdots V_d f_1 \rangle = 0,
\]
which means that
\[
f_d + V_d f_{d-1} + \cdots + V_d V_{d-1} V_{i+1} V_{i-1} \cdots V_d f_1 = 0.
\]
Since \( D_{V_i^*} f_d = f_d \) and \( D_{V_i^*} V_d = 0 \), we conclude by applying \( D_{V_d^*} \) on the vector above that \( f_d = 0 \). A similar analysis yields that each of the vectors \( f_{d-1}, \ldots, f_{i+1}, f_i-1, \ldots, f_1 \) are zero vectors. Consequently, \( \Delta_i \) is a unitary. Hence claim (5.11) is proved.

Remark 5.3. As in the pair case, for a tuple of \( q \)-commutative isometries, the BCL \( q \)-tuples uniquely determine a tuple of \( q \)-commutative isometries in the sense that is explained for the pair case in the statement of Theorem 2.5. The proof is similar.
6. \( q \)-COMMUTATIVE UNITARY EXTENSION OF \( q \)-COMMUTATIVE ISOMETRIES

Just as in the commutative case, every \( q \)-commutative tuple of isometries can be extended to a \( q \)-commutative tuple of unitaries. Moreover, as the following theorem shows, this unitary extension can be made so as to have some additional structure.

**Theorem 6.1.** Every \( d \)-tuple \((X_1, X_2, \ldots, X_d)\) of \( q \)-commutative isometric operators has a \( q \)-commutative unitary extension \((Y_1, Y_2, \ldots, Y_d)\). Moreover, there is an extension \((Y_1, Y_2, \ldots, Y_d)\) such that \( Y = Y_1 Y_2 \cdots Y_d \) is the minimal unitary extension of \( X = X_1 X_2 \cdots X_d \).

**Proof.** Let us suppose without loss of generality that the \( q \)-commutative isometric tuple \((X_1, X_2, \ldots, X_d)\) is given exactly in the BCL-1 \( q \)-model (5.3). Consider the tuple \((Y_1, Y_2, \ldots, Y_d)\) given for each \( i = 1, 2, \ldots, d \), by

\[
Y_i = \begin{bmatrix} R_{q_i} \odot P_{U_i} + M_i R_{q_i} \odot P_{U_i} & 0 \\ 0 & W_i \end{bmatrix} \quad \text{on} \quad L^2 \odot F \oplus K_u. \tag{6.1}
\]

Here \( L^2 \) denotes the usual \( L^2 \) space over \( \mathbb{T} \) with respect to the arc-length measure. It is a routine computation that the tuple \((Y_1, Y_2, \ldots, Y_d)\) above is a \( q \)-commutative tuple of unitary operators. Moreover, it extends the model in (2.13) in view of the natural embedding of \((H^2 \otimes F) \oplus K_u\) into \((L^2 \otimes F) \oplus K_u\):

\[
\begin{bmatrix} z^n \otimes \xi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} z^n \otimes \xi \\ \eta \end{bmatrix} \quad \text{for} \quad \xi \in F, \eta \in K_u \quad \text{and} \quad n \geq 0.
\]

For the second part of the lemma, we note that

\[
X = X_1 X_2 \cdots X_d = X_1 X_1 = M_z \oplus W_1 W_2 \cdots W_d \quad \text{on} \quad H^2(F_+^1) \oplus K_u
\]

and

\[
Y = Y_1 Y_2 \cdots Y_d = M_\zeta \oplus W_1 W_2 \cdots W_d \quad \text{on} \quad L^2(F_+^1) \oplus K_u.
\]

Therefore it follows from the classical theory that \( Y \) as above is indeed the minimal unitary extension of \( X \). \( \square \)

Let us say that a \( q \)-commutative tuple \((X_1, X_2, \ldots, X_d)\) is doubly \( q \)-commutative, if in addition, it satisfies

\[
X_j X_i^* = q(i, j) X_i^* X_j \quad \text{for each} \quad i, j = 1, 2, \ldots, d.
\]

As in the pair case, a \( q \)-commutative tuple of unitaries is automatically doubly \( q \)-commutative. A doubly \( q \)-commutative version of Theorem 6.1 can be easily derived.

**Corollary 6.2** (See also §6 of [10]). Every doubly \( q \)-commutative tuple of isometries extends to a doubly \( q \)-commutative tuple of unitaries.

**Proof.** This follows from Theorem 6.1 and the fact that a \( q \)-commutative tuple of unitaries is doubly \( q \)-commutative. \( \square \)

7. MODELS FOR \( q \)-COMMUTATIVE CONTRACTIONS

Let \((T_1, T_2)\) be a pair of operators acting on a Hilbert space \( \mathcal{H} \). Let us call a pair \((U_1, U_2)\) of operators acting on \( \mathcal{K} \supset \mathcal{H} \) a dilation of \((T_1, T_2)\), if

\[
T_1^m T_2^n = P_H U_1^m U_2^n \mid \mathcal{H} \quad \text{for every non-negative integers} \quad m \quad \text{and} \quad n,
\]
where $P_H$ is the orthogonal projection of $K$ onto $H$. Andô’s dilation theorem \cite{Ando} states that every pair of commutative Hilbert space operators has a dilation to a pair of commutative unitary operators. Thus, a natural generalization of Andô’s dilation theorem is whether every $q$-commutative pair of contractions has a dilation to a $q$-commutative unitary operators. This question is beautifully answered in affirmative very recently in \cite{KeshariMallick} using a commutant lifting approach. In an upcoming paper, we plan to give two constructive proofs of this $q$-dilation theorem and use the Berger–Coburn–Lebow-type model proved in this paper to consequently produce functional models for $q$-commutative pairs of contractions; the $q = 1$ case is done in \cite{Sau}. 

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