UNIFORM BOUNDS AND ULTRAPRODUCTS OF CYCLES

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Abstract. This paper is about the question whether a cycle in the l-adic cohomology of a smooth projective variety over $\mathbb{Q}$, which is algebraic over almost all finite fields $\mathbb{F}_p$, is also algebraic over $\mathbb{Q}$. We use ultraproducts respectively nonstandard techniques in the sense of A. Robinson, which the authors applied systematically to algebraic geometry in [BS07] and [BS08]. We give a reformulation of the question in form of uniform bounds for the complexity of algebraic cycles over finite fields.

1. Introduction

Let $X$ be a smooth and projective variety over $\mathbb{Q}$, and let $l$ be a prime. Let $\eta \in H^{2i}(X, \mathbb{Q}_l(i))$ be a class in the geometric l-adic cohomology. This paper concerns the question whether $\eta$ is algebraic over $\mathbb{Q}$, i.e. whether there is an $\alpha \in CH^i(X) \otimes \mathbb{Z} \otimes \mathbb{Q}_l$ such that $cl_Q(\alpha) = \eta$, where

\[ cl_Q : CH^i(X) \otimes \mathbb{Q}_l \to H^{2i}(X, \mathbb{Q}_l(i)) \]

is the cycle class map into l-adic cohomology.

Over an open dense part $\mathcal{B}$ of Spec($\mathbb{Z}$) the variety $X$ has a smooth and projective model $\mathcal{X}$. Therefore by smooth and proper base change in étale cohomology, for almost all primes $p$ we can identify

\[ H^{2i}_{\text{ét}}(X_{\mathbb{Q}}, \mathbb{Q}_l(i)) \cong H^{2i}_{\text{ét}}(\mathcal{X}_{\mathbb{F}_p}, \mathbb{Q}_l(i)). \]

Further there is a specialization map for Chow groups

\[ CH^i(\mathcal{X}_{\mathbb{Q}}) \to CH^i(\mathcal{X}_{\mathbb{F}_p}) \]

such that the diagram

\[
\begin{array}{ccc}
CH^i(\mathcal{X}_{\mathbb{Q}}) \otimes \mathbb{Z} \otimes \mathbb{Q}_l & \xrightarrow{cl_Q} & CH^i(\mathcal{X}_{\mathbb{F}_p}) \otimes \mathbb{Z} \otimes \mathbb{Q}_l \\
\downarrow{cl_{\mathbb{Q}}} & & \downarrow{cl_{\mathbb{F}_p}} \\
H^{2i}_{\text{ét}}(\mathcal{X}_{\mathbb{Q}}, \mathbb{Q}_l(i)) & \xrightarrow{\sim} & H^{2i}_{\text{ét}}(\mathcal{X}_{\mathbb{F}_p}, \mathbb{Q}_l(i))
\end{array}
\]

commutes. Therefore we know that if $\eta$ is algebraic over $\mathbb{Q}$, then for almost all primes $p$ there is an $\alpha_p \in CH^i(\mathcal{X}_{\mathbb{F}_p}) \otimes \mathbb{Q}_l$ with $cl_{\mathbb{F}_p}(\alpha_p) = \eta$, i.e. $\eta \in H^{2i}_{\text{ét}}(\mathcal{X}_{\mathbb{F}_p}, \mathbb{Q}_l(i))$ is algebraic over $\mathbb{F}_p$. It is a conjecture that the converse is also true:

1.1. Conjecture. In the above situation $\eta \in H^{2i}_{\text{ét}}(X_{\mathbb{Q}}, \mathbb{Q}_l(i))$ is algebraic over $\mathbb{Q}$ if and only if it is algebraic over $\mathbb{F}_p$ for almost all primes $p$.

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This conjecture would for example follow from the Tate conjecture for $X$.

If $\eta$ is indeed algebraic over $\mathbb{Q}$, we know that all the $\alpha_p \in CH^i(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_l$ are specializations of one single $\alpha \in CH^i(X) \otimes \mathbb{Q}_l$. And this tells us something about the complexity of the $\alpha_p$. To make this more precise, we define:

1.2. Definition. Let $X$ be a projective scheme over a field $k$ with a fixed projective embedding $X \hookrightarrow \mathbb{P}^n_k$, and let $\alpha \in CH^i(X) \otimes \mathbb{Q}_l$. We say that the complexity of $\alpha$ is bounded by $c \in \mathbb{N}$ if we can write $\alpha$ as $\alpha = \sum_{i=1}^r a_i[Z_i]$ with $a_i \in \mathbb{Q}_l$ and $Z_j \hookrightarrow X$ such that the following holds:

$$r < c,$$

and for all $i = 1, \ldots, r$ we have $|a_i|_l < c$ and $\text{deg}(Z_i) < c$.

Now the complexity of every $\alpha \in CH^i(X) \otimes \mathbb{Q}_l$ is obviously bounded by some $c_0 \in \mathbb{N}$. It can be seen easily that the complexity of the resulting family of specializations $\alpha_p$ is then uniformly bounded by $c_0$ as well. The main result of this paper is the converse of this observation:

1.3. Theorem (cf. corollary 3.8). In the above situation let $\eta \in H^2_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(i))$. We assume that there is a $c_0 \in \mathbb{N}$ such that for almost all primes $p$ there is an $\alpha_p \in CH^i(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_l$ with complexity bounded by $c_0$ and with $cl_{\mathbb{F}_p}(\alpha_p) = \eta$. Then there is an $\alpha \in CH^i(X) \otimes \mathbb{Q}_l$ with $cl_{\mathbb{Q}}(\alpha) = \eta$, i.e. $\eta$ is algebraic over $\mathbb{Q}$.

In particular this means that in order to decide the question whether a cohomology class is algebraic over $\mathbb{Q}$, it is enough to understand the case of all finite fields uniformly well enough.

In the integral case, where we consider the image of the cycle class map

$$CH^i(X) \to H^2_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l),$$

we prove a stronger version for $i = 1$, which does not need the notion of complexity (cf. theorem 3.10):

1.4. Theorem. In the above situation let $\eta \in H^2_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(1))$. We assume that for almost all primes $p$ there is a $z_p \in CH^1(\mathcal{X}_p \otimes \mathbb{F}_p)$ such that

$$cl_{\mathbb{F}_p}(z_p) = \eta.$$

Then there is a field extension $K/\mathbb{Q}$ and an element $z \in CH^1(\mathcal{X}_K)$ with

$$cl_K(z) = \eta \text{ in } H^2_{\acute{e}t}(\mathcal{X}_K, \mathbb{Z}_l(1)).$$

The basic idea for the proofs is the following: By our assumption there is a cofinite set $S \subset \mathbb{P}$ of the set of primes $\mathbb{P}$ such that for all $p \in S$ there are cycles $\alpha_p \in CH^i(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_l$ with $cl_{\mathbb{F}_p}(\alpha_p) = \eta$. Now we choose an ultrafilter $\mathcal{U}$ on $S$ and consider the ultraproduct of these cycles

$$(\alpha_p)_{p \in S, \mathcal{U}} \in \prod_{p \in S, \mathcal{U}} (CH^i(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_l).$$
In our papers [BS08] we studied how Chow groups and étale cohomology behave under such ultraproducts. In particular in loc. cit. we constructed morphisms

\[ CH^i \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p \right) \to \prod_{S, \mathcal{U}} \left( CH^i(X_{F_p}) \right) \]

and

\[ H^{2i}_{\text{ét}} \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p, \mathbb{Z}/l^n \mathbb{Z}(i) \right) \to \prod_{S, \mathcal{U}} H^{2i}_{\text{ét}} \left( X_{F_p}, \mathbb{Z}/l^n \mathbb{Z}(i) \right) \]

and described the image of these morphisms: The image of (1) can be described by the complexity of the occurring cycles, whereas morphism (2) is an isomorphism in our situation. Furthermore, for all \( n \in \mathbb{N} \) we have a commutative diagram

\[
\begin{array}{ccc}
\prod_{S, \mathcal{U}} \left( CH^i(X_{F_p}) \right) & \xrightarrow{\Pi^{cl}} & \prod_{S, \mathcal{U}} H^{2i}_{\text{ét}} \left( X_{F_p}, \mathbb{Z}/l^n \mathbb{Z}(i) \right) \\
\downarrow & & \uparrow \\
CH^i \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p \right) & \xrightarrow{cl} & H^{2i}_{\text{ét}} \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p, \mathbb{Z}/l^n \mathbb{Z}(i) \right) .
\end{array}
\]

Now \( \prod_{p \in S, \mathcal{U}} F_p \) is a field of characteristic zero, so an extension of \( \mathbb{Q} \), and the assumptions imply that our cycle \( \eta \in H^{2i}_{\text{ét}}(X_{\mathbb{Q}}, \mathbb{Z}(i)) \) lies in the image of

\[ cl : CH^i \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p \right) \otimes_{\mathbb{Z}} \mathbb{Q}_{l} \to H^{2i}_{\text{ét}} \left( X \otimes \mathbb{Z} \prod_{p \in S, \mathcal{U}} F_p, \mathbb{Z}(i) \right) . \]

Instead of using ultraproducts directly, we use the more general method of enlargements in the sense of A. Robinson, which the authors applied to modern algebraic geometry in [BS07] and [BS08].

All results of this paper remain true (with minor modifications) if we start with a smooth projective variety over an arbitrary number field. The proofs need only little modifications in this case. Yet we decided to stick to the case of a smooth projective variety over \( \mathbb{Q} \) to make notations easier and the paper more readable.

In section 2 we give the basic definitions we need and introduce an appropriate notion of complexity of cycles.

In section 3 we give different versions of our main result and discuss the integral case for divisors without using the notion of complexity.

## 2. Basic notations

First we fix some notations. For a smooth, proper and connected scheme \( X \) over a field \( k \), we denote by \( Z_i(X) \) the free abelian group generated by those integral subschemes \( Y \hookrightarrow X \) of \( X \) with \( \dim(Y) = i \). By \( Z_i^{\text{eff}}(X) \) we denote the submonoid of \( Z_i(X) \) of effective cycles, i.e. elements of the form \( \sum_j \alpha_j [Z_j] \) with \( \alpha_j \in \mathbb{N} \). By \( CH_i(X) \) we denote the quotient of \( Z_i(X) \) by rational equivalence, and by \( CH_i^{\text{eff}}(X) \) we denote the submonoid of \( CH_i(X) \)
generated by elements of $Z_{i}^{eff}(X)$. We also want to use the grading by codimension, and so we denote

$$
Z^{i}(X) := Z_{\dim(X)-i}(X), \\
Z_{eff}^{i}(X) := Z_{\dim(X)-i}^{eff}(X), \\
CH^{i}(X) := CH_{\dim(X)-i}(X), \\
CH_{eff}^{i}(X) := CH_{\dim(X)-i}^{eff}(X).
$$

For 0-cycles we denote by

$$
\deg : CH^{0}(X) = CH^{\dim(X)}(X) \to \mathbb{Z}
$$

the degree map which is defined by $\sum_{j} \alpha_{j}[Z_{j}] \mapsto \sum_{j} \alpha_{j}[\kappa(Z_{j}) : k]$. For higher dimensional cycles we first fix an ample line bundle $L$ on $X$ and denote by $c_{1}(L)$ the first Chern class of $L$. With that we define the degree in general as

$$
\deg_{L} : CH^{i}(X) = CH^{\dim(X)-i}(X) \to \mathbb{Z} \\
\quad x \quad \mapsto \deg(c_{1}(L)^{i} \cap x).
$$

2.1. **Remark.** If $X \hookrightarrow \mathbb{P}^{n}$ is a closed embedding and $L := O_{\mathbb{P}^{n}}(1)|_{X}$, then the above notion gives the ordinary degree of integral subschemes as subschemes of $\mathbb{P}^{n}$. For details we refer to [Ful84][section 2.5].

Next consider the cycle class map

$$
cl_{k} : CH^{i}(X) \to H^{2i}_{\text{et}}(X_{\overline{\mathbb{F}}}, \mathbb{Z}_{l}(i)),
$$

where the right hand side is the l-adic cohomology

$$
H^{2i}_{\text{et}}(X_{\overline{\mathbb{F}}}, \mathbb{Z}_{l}(i)) := \lim_{\leftarrow n} H^{2i}_{\text{et}}(X_{\overline{\mathbb{F}}}, \mu_{l}^{\otimes i}).
$$

2.2. **Lemma.** There is a degree map

$$
\deg_{L}^{\text{et}} : H^{2i}_{\text{et}}(X, \mathbb{Z}_{l}) \to \mathbb{Z}_{l}
$$

such that we have $\deg_{L}^{\text{et}} \circ cl = \deg_{L}$.

**Proof.** The follows from the fact that the cycle map is compatible with products. q.e.d.

2.3. **Definition.** Let $k$ be a field, $X$ a proper, smooth, connected scheme over $k$ and $c \in \mathbb{N}$ a natural number.

(i) We say that an element $x \in CH^{i}(X)$ has complexity less than $c$ and write $\text{compl}(x) < c$ iff we can write $x$ as a difference $x = x_{0} - x_{1}$ of two effective cycles $x_{0}, x_{1} \in CH^{i}(X)$ with

$$
\deg(x_{0}) < c \text{ and } \deg(x_{1}) < c.
$$

(ii) We say that an element $x \in CH^{i}(X) \otimes \mathbb{Q}$ has complexity less than $c \in \mathbb{N}$ and write $\text{compl}(x) < c$ iff there is an $x' \in CH^{i}(X)$ with $\text{compl}(x') < c$ and an $\alpha \in \mathbb{N}$ with $\alpha < c$ such that $\alpha \cdot x = x'$. 
We say that an element \( x \in CH^i(X) \otimes \mathbb{Z}_l \) has complexity less than \( c \in \mathbb{N} \) and write \( \text{compl}(x) < c \) iff we can write it as \( x = \sum_{i=1}^n \alpha_i x_i \) with \( \alpha_i \in \mathbb{Z}_l \) and \( x_i \in CH^i(X) \) such that for all \( i = 1, \ldots, n \) we have

\[
\text{compl}(x_i) < \frac{c}{n}.
\]

We say that an element \( x \in CH^i(X) \otimes \mathbb{Q}_l \) has complexity less than \( c \in \mathbb{N} \) and write \( \text{compl}(x) < c \) iff there is an \( x' \in CH^i(X) \otimes \mathbb{Z}_l \) with \( \text{compl}(x') < c \) and an \( \alpha \in \mathbb{Q}_l \) with \( |\alpha|_l < c \) such that \( \alpha \cdot x' = x \).

2.4. Remark. The reason why (ii) and (iv) seems to be inverse to each other is the following: \( \mathbb{Z} \subset \mathbb{Q} \) is discrete and unbounded in the archimedian metric and \( \mathbb{Z}_l \subset \mathbb{Q}_l \) is not discrete and bounded in the \( l \)-adic topology.

2.5. Remark. Part (iv) of definition 2.3 gives essentially the same notion of complexity as definition 1.2 from the introduction. For more details we refer to remark 2.8.

2.6. Remark. It would be nice to be able to give the complexity only in terms of the degree of cycles, but that is in general not possible. To see why, consider a smooth, projective surface \( F \) over \( \mathbb{C} \) with \( p_g \neq 0 \). By [Mum68], there is no \( n \in \mathbb{N} \) such that the map

\[
\text{Sym}^n F \times \text{Sym}^n F \to CH_0(X)
\]

is surjective. Here \( CH_0(X) \) denotes 0-cycles of degree zero, and \( \text{Sym}^n \) denotes the \( n \)-th symmetric power. Furthermore, \( x \in CH_0(X) \) is a cycle with \( \text{compl}(x) < n \) if and only if \( x \) lies in the image of (3).

2.7. Remark. The behavior of this notion of complexity under intersection products is studied in [BS08, theorem 5.11].

By \( *: \hat{S} \to \hat{W} = \hat{S} \) we always denote an enlargement of the superstructure \( \hat{S} \) in the sense of nonstandard analysis (cf. for example [LW00, section I.2]). We rigorously use the concept of enlarging categories, schemes, cycles and étale cohomology, which the authors developed in [BS05, BS07] and [BS08]. We always assume that our fixed superstructure \( \hat{S} \) is large enough, so that all needed categories are \( \hat{S} \)-small in the sense of [BS05].

2.8. Remark.

(i) Let \( K \) be an internal field, \( X \) a *proper, *smooth, *connected *scheme with a *ample \( O_X * \)-module \( L \). Then by transfer we get the degree map

\[
* \text{deg}_L: *CH^i(X) \to *\mathbb{Z}
\]

and a corresponding notion of *complexity. Then the complexity of a *cycle on \( X \) can be bound by an element of \( \mathbb{N} \subset *\mathbb{N} \) in the sense of definition 1.2 if and only if it can be bound by an element of \( \mathbb{N} \subset *\mathbb{N} \) in the sense of definition 2.3.

(ii) If we compare part (i) of definition 2.3 with definition [BS08, 5.1] the same as in (i) is true.
If \( \mathcal{X} \to \mathcal{B} \) and \( \text{Spec}(A) \to \mathcal{B} \) are morphisms of schemes, we denote the fibre product by \( \mathfrak{X}_A := \mathfrak{X} \times_{\mathcal{B}} \text{Spec}(A) \). Also, if \( \mathfrak{X} \to \mathcal{B} \) and \( \text{Spec}(A) \to \mathcal{B} \) are morphisms of schemes, we denote the fibre product by \( \mathfrak{X}_A := \mathfrak{X} \times_{\mathcal{B}} \text{Spec}(A) \). Finally, for a field (respectively internal field) \( k \), we denote a separable closure (respectively *separable *closure) of \( k \) by \( \overline{k} \).

### 3. Lifting cycles modulo homological equivalence

In this section we consider mainly the following situation: Let \( \mathcal{B} \subset \text{Spec}(\mathcal{Z}) \) be an open, non-empty subset, \( \mathfrak{X} \to \mathcal{B} \) a smooth and proper morphism and \( \mathcal{L} \) an ample sheaf on \( \mathfrak{X} \). For a point \( s \in \mathcal{B} \) we consider the l-adic cohomology groups of the geometric fibre

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mathbb{Z}_l(i)) := \lim_{\rightarrow n} H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mu_{l^n}^{\otimes i}),
\]

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mathbb{Q}_l(i)) := H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mathbb{Z}_l(i)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.
\]

By the smooth and proper base change theorems we know that the specialization homomorphisms

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mu_{l^n}^{\otimes i}) \to H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mu_{l^n}^{\otimes i}),
\]

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mathbb{Z}_l(i)) \to H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mathbb{Z}_l(i))
\]

and

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(s)}, \mathbb{Q}_l(i)) \to H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mathbb{Q}_l(i))
\]

are isomorphisms. Also, if \( K/\mathbb{Q} \) is a field extension with a chosen embedding \( \overline{\mathbb{Q}} \to \overline{\kappa} \), the pullback homomorphisms

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mathbb{Z}_l(i)) \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Z}_l(i))
\]

and

\[
H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mathbb{Q}_l(i)) \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Q}_l(i))
\]

are isomorphisms. In what follows, we identify these groups without further mentioning.

If \( K \) is a field and \( \text{Spec}(K) \to \mathcal{B} \) is a morphism, we consider the following cycle class maps

(i) \( cl_K : CH^i(\mathfrak{X}_K) \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Z}_l(i)) \),

(ii) \( cl_K : CH^i(\mathfrak{X}_K) \otimes \mathbb{Q} \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Q}_l(i)) \).

(iii) \( cl_K : CH^i(\mathfrak{X}_K) \otimes \mathbb{Z}_l \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Z}_l(i)) \),

(iv) \( cl_K : CH^i(\mathfrak{X}_K) \otimes \mathbb{Q}_l \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Q}_l(i)) \).

In each case we investigate the question whether a cohomology class in the l-adic cohomology, which is algebraic over the finite fields in \( \mathcal{B} \), is also algebraic over a field of characteristic zero. For that we use our notion of complexity of section 2.

(i) **The case** \( cl_K : CH^i(\mathfrak{X}_K) \to H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Z}_l(i)) \)

### 3.1. Theorem.** Let as above \( \mathfrak{X} \to \mathcal{B} \) be a smooth, proper morphism, \( \mathcal{B} \subset \text{Spec}(\mathcal{Z}) \) open and non empty, \( \mathcal{L} \) an ample sheaf on \( \mathfrak{X} \) and \( \eta \in H_{\text{ét}}^{2i}(\mathfrak{X}_{\overline{\kappa}}, \mathbb{Z}_l(i)) \) a cohomology class of the geometric generic fibre of \( \mathfrak{X} \). We assume that there is a constant \( c \in \mathbb{N} \) such that there are infinitely many closed points \( b \in \mathcal{B} \) with:
There is a \( z_b \in CH^i(\mathfrak{X}_{\kappa(b)}) \) with \( cl(z_b) = \eta \) and \( compl(z_b) < c \) with respect to the ample sheaf \( \mathcal{L}|_{\mathfrak{X}_{\kappa(b)}} \).

Then there is a field extension \( K/\mathbb{Q} \) and an element \( z_K \in CH^i(\mathfrak{X}_K) \) with \( cl(z_K) = \eta \) in \( H_{\text{ét}}^{2i}(\mathfrak{X}_K, \mathbb{Z}_l(i)) \).

For the proof we use the following lemma:

3.2. Lemma. Let \( K \) be an internal field, \( X \) a proper scheme over \( K \) and \( \mathcal{L} \) an ample sheaf on \( X \). Then the image of the morphism

\[
N : CH^i(X) \to {}^*CH^i(N(X))
\]

consists exactly of those elements in \( {}^*CH^i(N(X)) \) whose \(^*\text{complexity is less than} c \) for a natural number \( c \in \mathbb{N} \).

Proof. This follows from lemma 5.2 in \[BS08\] and remark 2.8 (ii). q.e.d.

Proof of theorem 3.1. By transfer there are an \( b \in {}^*\mathfrak{B} \), corresponding to an infinite prime \( P \in {}^*\mathfrak{P} \), and \( x_b \in {}^*CH^i({}^*\mathfrak{X}_{\kappa(b)}) \) with

\[
(4) \quad {}^*\text{compl}(x_b) < c
\]

and

\[
(5) \quad {}^*cl(x_b) = {}^*\eta.
\]

The field \( \kappa(b) \) is externally of characteristic zero. By definition (cf. [BS07]) we have \( N({}^*\mathfrak{X}_{\kappa(b)}) = {}^*\mathfrak{X}_{\kappa(b)} \). Therefore condition (1) and lemma 3.2 imply that \( x_b \) lies in the image of

\[
N : CH^i(\mathfrak{X}_{\kappa(b)}) \to {}^*CH^i({}^*\mathfrak{X}_{\kappa(b)}),
\]

i.e. there is an \( z_{\kappa(b)} \in CH^i(\mathfrak{X}_{\kappa(b)}) \) with \( N(z_{\kappa(b)}) = x_b \). For all \( n \in \mathbb{N} \) the above morphism fits into the following commutative diagram

\[
\begin{array}{ccc}
CH^i(\mathfrak{X}_{\kappa(b)}) & \xrightarrow{N} & {}^*CH^i({}^*\mathfrak{X}_{\kappa(b)}) \\
\downarrow & & \downarrow \\
H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(b)}, \mu_l^{\otimes i}) & \xrightarrow{N} & {}^*H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(b)}, \mu_l^{\otimes i}) \\
\uparrow & & \uparrow \\
H_{\text{ét}}^{2i}(\mathfrak{X}_Q, \mu_l^{\otimes i}) & \xrightarrow{\sim} & {}^*H_{\text{ét}}^{2i}(\mathfrak{X}_Q, \mu_l^{\otimes i})
\end{array}
\]

The upper square commutes by [BS08], the lower square by the following lemma \[3.3\] and the horizontal morphism in the middle

\[
N : H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(b)}, \mu_l^{\otimes i}) \to {}^*H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(b)}, \mu_l^{\otimes i})
\]

is an isomorphism by [BS08][Cor.2.14].

Thus we see that for all \( n \in \mathbb{N} \) we have

\[
cl(z_{\kappa(b)}) = \eta \text{ in } H_{\text{ét}}^{2i}(\mathfrak{X}_{\kappa(b)}, \mu_l^{\otimes i})
\]
and therefore
\[ cl(z_{\kappa(b)}) = \eta \text{ in } H^{2i}_{\text{ét}}(X_{\kappa(b)}, \mathbb{Z}_l(i)). \]

q.e.d.

3.3. **Lemma.** In the above situation we have the following commutative diagram:

\[
\begin{array}{ccc}
H^{2i}_{\text{ét}}(X_{\kappa(b)}, \mu_{l^n}^{\otimes i}) & \xrightarrow{N} & *H^{2i}_{\text{ét}}(X_{\kappa(b)}, *\mu_{l^n}^{\otimes i}) \\
\downarrow i & & \downarrow i \\
H^{2i}_{\text{ét}}(X_{\kappa(b)}, \mu_{l^n}^{\otimes i}) & \xrightarrow{*} & *H^{2i}_{\text{ét}}(X_{\kappa(b)}, *\mu_{l^n}^{\otimes i}).
\end{array}
\]

*Proof. Consider the base extension
\[
X \otimes \mathbb{Z} \to \text{Spec}(\mathbb{Z}).
\]
The left arrow of the diagram can be identified with the inverse of the base change homomorphism of the two points \((0) \in \mathbb{Z}^*\) and \(b \in \text{Spec}(\mathbb{Z})\) (note that internal prime ideals are in particular external ones). The lemma then follows from the compatibility of \(N\) for étale cohomology with the specialization homomorphism (cf. [BS08]). q.e.d.

(ii) **The case** \(cl_K : CH^i(X_K) \otimes \mathbb{Q} \to H^{2i}_{\text{ét}}(X_{\kappa(b)}, \mathbb{Q}_l(i))\)

3.4. **Theorem.** Let as above \(X \to \mathcal{B}\) be a smooth and proper morphism, \(\mathcal{B} \subset \text{Spec}(\mathbb{Z})\) open and non-empty, \(\mathcal{L}\) an ample sheaf on \(X\) and \(\eta \in H^{2i}_{\text{ét}}(X_{\kappa(b)}, \mathbb{Q}_l(i))\) a cohomology class of the geometric generic fibre of \(X\). We assume that there is a constant \(c \in \mathbb{N}\) such that there are infinitely many closed points \(b \in \mathcal{B}\) with:

There is a \(z_b \in CH^i(X_{\kappa(b)}) \otimes \mathbb{Q}\) with \(cl(z_b) = \eta\) and \(\text{compl}(z_b) < c\) with respect to the ample sheaf \(\mathcal{L}|_{X_{\kappa(b)}}\).

Then there are a field extension \(K/\mathbb{Q}\) and an element \(z_K \in CH^i(X_K) \otimes \mathbb{Q}\) with \(cl(z_K) = \eta\) in \(H^{2i}_{\text{ét}}(X_K, \mathbb{Q}_l(i))\).

*Proof. By transfer there are a \(b \in \mathbb{B}^*\), corresponding to an infinite prime \(P \in \mathbb{P}^*\), and \(x_b \in *CH^i(X_{\kappa(b)}) \otimes \mathbb{Q}\) with \(*\text{compl}(x_b) < c\) and \(*cl(x_b) = *\eta\).

By the definition of complexity there are an \(\alpha \in \mathbb{N}^*\) with \(\alpha < c\), in particular \(\alpha \in \mathbb{N}\), and \(x'_b \in *CH^i(X_{\kappa(b)})\) with \(*\text{compl}(x'_b) < c\) such that \(\alpha \cdot x_b = x'_b\). As in the proof of theorem 3.1 we see that there is a \(z_b \in CH^i(X_{\kappa(b)})\) with

\[ N(z_b) = x'_b \]

and that

\[ cl_{\kappa(b)}(z_b) = \alpha \cdot \eta, \]

and therefore we have

\[ cl_{\kappa(b)}(\frac{1}{\alpha} \cdot z_b) = \eta \]

as desired. q.e.d.
3.5. **Remark.** One prominent example, where one would like to know whether a cohomology class lies in the image of
\[ CH^i(X) \otimes \mathbb{Q} \to H_{\text{et}}^{2i}(\mathcal{X}, \mathbb{Q}(i)) , \]
is the case of the Künneth components. Over finite fields it is known by [KM74] that the Künneth components are algebraic. Unfortunately one uses the Frobenius morphism to construct the cycles, and it is not possible to use this representation to find a uniform bound for the complexity.

(iii) **The case** \( cl_K : CH^i(\mathcal{X}_K) \otimes \mathbb{Z}_l \to H_{\text{et}}^{2i}(\mathcal{X}, \mathbb{Z}_l(i)) \)

3.6. **Theorem.** Let as above \( \mathcal{X} \to \mathbb{B} \) be a smooth proper morphism, \( \mathbb{B} \subset \text{Spec}(\mathbb{Z}) \) open and non-empty, \( L \) an ample sheaf on \( \mathcal{X} \) and \( \eta \in H_{\text{et}}^{2i}(\mathcal{X}, \mathbb{Z}_l(i)) \) a cohomology class of the geometric generic fibre of \( \mathcal{X} \). We assume that there is a constant \( c \in \mathbb{N} \) such that there are infinitely many closed points \( b \in \mathbb{B} \) with:

There is an \( z_b \in CH^i(\mathcal{X}_{n(b)}) \otimes \mathbb{Z}_l \) with \( cl(z_b) = \eta \) and \( \text{compl}(z_b) < c \) with respect to the ample sheaf \( L|_{\mathcal{X}_{n(b)}} \).

Then there are a field extension \( K/\mathbb{Q} \) and an element \( z_K \in CH^i(\mathcal{X}_K) \otimes \mathbb{Z}_l \) with \( cl(z_K) = \eta \) in \( H_{\text{et}}^{2i}(\mathcal{X}, \mathbb{Z}_l(i)) \).

**Proof.** By transfer there are a \( b \in *\mathbb{B} \) corresponding to an infinite prime \( P \in *\mathbb{P} \) and \( x_b \in *CH^i(\mathcal{X}_{n(b)}) \otimes *\mathbb{Z}_l \) with

\[ \text{compl}(x_b) < c \]

and

\[ cl(x_b) = *\eta. \]

Now by the definition of the complexity of elements in \( *CH^i(\mathcal{X}_{n(b)}) \otimes \mathbb{Z}_l \), we can write \( x_b = \sum_{i=1}^n \alpha_i x_i \) with \( \alpha_i \in *\mathbb{Z}_l \) and \( x_i \in *CH^i(\mathcal{X}_{n(b)}) \) with \( \text{compl}(x_i) < \frac{\alpha_i}{n} \). A priori we have \( n \in *\mathbb{N} \), but if we assume \( n \in *\mathbb{N} - \mathbb{N} \) then \( \frac{\alpha_i}{n} \) would be infinitesimal and \( \text{compl}(x_i) < \frac{\alpha_i}{n} \) would imply \( x_i = 0 \). So we assume that \( n \in \mathbb{N} \) and that the sum \( x_b = \sum_{i=1}^n \alpha_i x_i \) is finite. By lemma 3.2 as in the proof of theorem 3.1 there are \( z_{i,b} \in CH^i(\mathcal{X}_{n(b)}) \) such that \( cl(z_{i,b}) = x_i \). Now \( \mathbb{Q}_l \) is complete with respect to the \( l \)-adic norm, and \( \mathbb{Z}_l = \{ \alpha \in \mathbb{Q}_l ||\alpha|| \leq 1 \} \). Therefore we have the standard part map

\[ st : *\mathbb{Z}_l \to \mathbb{Z}_l \]

with the property that for all \( \alpha \in *\mathbb{Z}_l \), the difference \( \alpha - st(\alpha) \) is infinitesimally small. In particular this means that for all standard \( n \in \mathbb{N} \subset *\mathbb{N} \)

\[ \alpha = st(\alpha) \text{ in } *\mathbb{Z}_l/l^n*\mathbb{Z}_l = \mathbb{Z}/l^n\mathbb{Z}. \]

We can thus define

\[ z_b := \sum_{i=1}^n st(\alpha_i) \cdot z_{i,b}, \]
use $\mathcal{S}$ and argue as in the proof of theorem 3.1 to show that
\[ cl_{\kappa(b)}(z_b) = \eta. \]
\text{q.e.d.}

(iv) The case $cl_K : CH^i(\mathcal{X}_K) \otimes \mathbb{Q}_l \rightarrow H^{2i}_{\text{et}}(\mathcal{X}_K, \mathbb{Q}_l(i))$

3.7. Theorem. Let as above $\mathcal{X} \rightarrow \mathcal{B}$ be a smooth and proper morphism, $\mathcal{B} \subset \text{Spec} \,(\mathbb{Z})$ open and non-empty, $\mathcal{L}$ an ample sheaf on $\mathcal{X}$ and $\eta \in H^{2i}_{\text{et}}(\mathcal{X}_K, \mathbb{Q}_l(i))$ a cohomology class of the geometric generic fibre of $\mathcal{X}$. We assume that there is a constant $c \in \mathbb{N}$ such that there are infinitely many closed points $b \in \mathcal{B}$ with:

There is a $z_b \in CH^i(\mathcal{X}_{\kappa(b)}) \otimes \mathbb{Q}_l$ with $cl(z_b) = \eta$ and $\text{compl}(z_b) < c$ with respect to the ample sheaf $\mathcal{L} |_{\mathcal{X}_{\kappa(b)}}$.

Then there is a field extension $K/\mathbb{Q}$ and an element $z_K \in CH^i(\mathcal{X}_K) \otimes \mathbb{Q}_l$ with $cl(z_K) = \eta$ in $H^{2i}_{\text{et}}(\mathcal{X}_K, \mathbb{Q}_l(i))$.

Proof. By transfer there are an $b \in \ast \mathcal{B}$, corresponding to an infinite prime $P \in \ast \mathcal{P}$, and $x_b \in \ast CH^i(\ast \mathcal{X}_{\kappa(b)}) \otimes \ast \mathbb{Q}_l$ with $\ast \text{compl}(x_b) < c$ and $\ast cl(x_b) = \ast \eta$.

By the definition of complexity there are an $\alpha \in \ast \mathbb{Q}_l$ and $x'_b \in \ast CH^i(\ast \mathcal{X}_{\kappa(b)}) \otimes \ast \mathbb{Z}_l$ such that $|\alpha|_l < c$ and $x_b = \alpha_b \cdot x'_b$. The claim then follows as in the proof of the previous theorem. \text{q.e.d.}

3.8. Corollary. Let $\mathcal{X}$, $\mathcal{B}$ and $\mathcal{L}$ be as in the theorem. Now we assume that there is a constant $c \in \mathbb{N}$, such that for almost all closed points $s \in \mathcal{B}$ we have:

There is a $z_b \in CH^i(\mathcal{X}_{\kappa(b)}) \otimes \mathbb{Q}_l$ with $cl(z_b) = \eta$ and $\text{compl}(z_b) < c$ with respect to the ample sheaf $\mathcal{L} |_{\mathcal{X}_{\kappa(b)}}$.

Then there is an element $z_Q \in CH^i(X) \otimes \mathbb{Q}_l$ such that
\[ cl(z_Q) = \eta. \]

Proof. The corollary follows from the theorem, the density theorem of Chebotarev and the next lemma. \text{q.e.d.}

3.9. Lemma. Let $k$ be a field and $X$ a smooth, projective scheme over $k$. Let $\eta \in H^{2i}_{\text{et}}(X_K, \mathbb{Q}_l(i))$ be a cohomology class which is invariant under the Galois group $Gal(\overline{k}/k)$. Assume further that there is a field extension $K/k$, such that
\[ \eta \in H^{2i}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_l(i)) = H^{2i}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_l(i)) \]
is in the image of
\[ cl : CH^i(X_K) \otimes \mathbb{Q}_l \rightarrow H^{2i}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_l(i)). \]

Then $\eta$ is in the image of
\[ CH^i(X) \otimes \mathbb{Q}_l \rightarrow H^{2i}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_l(i)). \]
Proof. By rigidity for Chow groups with finite coefficients, the map

\[ CH^i(X_{\overline{k}})/l^n \rightarrow CH^i(X_{\overline{k}})/l^n \]

is bijective. So we can assume that \( K = \overline{k} \). Now let \( \eta = cl(x) \) with \( x \in CH^i(X_{\overline{k}}) \otimes \mathbb{Q}_l \).

Let \( k' \) be a finite field extension of \( k \) with \( k \subset k' \subset \overline{k} \), such that \( x \) is already defined over \( k' \). By the Galois invariance of \( \eta \) we have

\[
cl\left( \sum_{\sigma \in Gal(k'/k)} \sigma \cdot x \right) = [k':k]\eta,
\]

and the claim follows from Galois decent for Chow groups. q.e.d.

The next theorem states that for divisors, we do not need the notion of complexity.

3.10. Theorem. Let \( \mathfrak{X} \to \mathfrak{B} \subset Spec(\mathbb{Z}) \) be a smooth and proper morphism with geometrically reduced fibers, \( L \) an ample line bundle on \( \mathfrak{X} \), and let \( \eta \in H^2_{\acute{e}t}(X \otimes \mathbb{Q}, Z_l(i)) \) be a cohomology class of the generic fibre. We assume that for infinitely many closed points \( s \in \mathfrak{B} \) there is \( z_s \in CH^1(\mathfrak{X} \otimes \mathfrak{B}(s)) \) such that

\[ cl(z_s) = \eta. \]

Then there are a field extension \( K/\mathbb{Q} \) and an element \( z \in CH^1(\mathfrak{X}_K) \) with

\[ cl(z) = \eta \text{ in } H^2_{\acute{e}t}(\mathfrak{X}_K, Z_l(i)). \]

Proof. Main parts of the proof are the same as in the proof of theorem 3.1.

By transfer there is an \( s \in ^*\mathfrak{B} \), corresponding to an infinite prime, such that there is a \( z_s \in ^*CH^1(\mathfrak{X}_s) \) with

\[ ^*cl(z_s) = ^*\eta. \]

Now the Hilbert polynomial of \( z_s \) can be calculated in the étale cohomology and is therefore finite. Therefore the theorem follows from Corollary \([BS08][5.4]\) q.e.d.

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