Asymptotic methods and their applications in nonlinear fracture mechanics: a review

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Abstract. Asymptotic methods, perturbation theory techniques and their applications in nonlinear fracture mechanics and continuum damage mechanics are reviewed. I analysed asymptotic stress, strain and damage fields near the crack tip for power-law materials and the influence of the damage accumulation processes on the stress-strain state in the vicinity of the crack tip. I reviewed fundamental results obtained in nonlinear fracture mechanics by means of asymptotic methods and perturbation theory approaches. Particular attention is focused on power-law materials and asymptotic stress and strain fields in the vicinity of the crack in both non-damaged and damaged materials under mixed mode loadings. I discussed asymptotic elastic-plastic crack-tip fields derived by Hutchinson, Rice and Rosengren as a singular dominant term of the asymptotic expansion for the stress field in a power-law hardening material. I presented and discussed new solutions of nonlinear fracture mechanics and continuum damage mechanics obtained by perturbation methods.

1. Introduction

Asymptotic methods and perturbation theory are promising and effective approaches to the derivation of approximate or even exact solutions. Nowadays the perturbation theory techniques are applied to a wide variety of static and dynamic solid mechanics problems. A detailed review of asymptotic methods in nonlinear fracture mechanics and continuum damage mechanics is presented in this paper. The asymptotic analysis of the stress and strain distributions in the vicinity of the wedge-shaped domain is one of the most fundamental problems both in linear and nonlinear fracture mechanics. In linear fracture mechanics very important results have been obtained using the methods of asymptotic analysis. A review of the most important contributions in this field is provided by Carpinteri and Paggi [1]. These authors focused on the epistemological steps towards full appreciation of the mathematical and engineering relevance of stress singularities. They provided a detailed review of geometrical configurations and mechanical conditions that can relieve or remove singularities, including: re-entrant corners, power-law hardening constitutive laws, fractal cracks, multi-material junctions and wedges in nonhomogeneous materials. In [1] these main geometrical configurations and mechanical conditions that can be used to effectively relieve the power of the stress-singularities with respect to the well-known square-root singularity typical of Linear Elastic Fracture Mechanics are considered. These
The fundamental asymptotic solution by Hutchinson, Rice and Rosengren

The power law material response is described as [2-4]

\[ \varepsilon / \varepsilon_0 = \alpha (\sigma / \sigma_0)^n, \]

where \( \alpha \) is the material constant, \( \sigma_0 \) is the reference yield strength, \( n \) is the strain hardening exponent, \( \varepsilon_0 = \sigma_0 / E \) is the reference yield strain. The crack tip fields can be obtained as [2-4]

\[ \sigma_y(r, \theta) = \sigma_0 \left( k / r \right)^{1/(n+1)} \bar{\varepsilon}_y(\theta, n), \]

\[ \varepsilon_y(r, \theta) = \alpha \sigma_0 \left( k / r \right)^{n/(n+1)} \bar{\varepsilon}_y(\theta, n), \]

\[ u_1(r, \theta) = \alpha \sigma_0 r^{n/(n+1)} \bar{u}_1(\theta, n), \]

where \( J \) is the path-independent integral, \( I_n \) is the dimensionless \( J \)-integral (the integration constant depends on \( n \)), \( k = J / (\alpha \sigma_0 \varepsilon_0 I_n) \). The asymptotic fields (2) are referred to as the Hutchinson-Rice-Rosengren (HRR) fields in the vicinity of the crack tip in power-law materials.

2.1. Basic equations

The equilibrium equations for the generalized plane stress and plane strain conditions are always satisfied regardless of the choice of the Airy stress function \( \chi(r, \theta) \)

\[ \sigma_{rr}(r, \theta) = \chi_{rr} / r + \chi_{r0} / r^2, \quad \sigma_{\theta \theta}(r, \theta) = \chi_{\theta \theta}, \quad \sigma_{r \theta}(r, \theta) = -\left( \chi_{r0} / r \right), \]

where the origin of cylindrical coordinates \( r, \theta \) is at the right end of the crack. According to [2], the non-dimensional stress function and coordinates are presented in terms of the dimensional quantities as \( \chi = 1/(\sigma_r L^2) \), \( r = \tau / L \), \( L \) is the half length of the crack. The compatibility condition has the form

\[ 2 \left( r e_{r0,\theta} \right)_r = e_{r0,\theta} - r e_{r0,\theta} - r e_{r0,\theta}, \]

For plane strain conditions the constitutive relations can be written as

\[ e_{rr} = -e_{\theta \theta} = 3\alpha \sigma^{-1} (\sigma_{rr} - \sigma_{\theta \theta}) / 4, \quad e_{r0} = 3\alpha \sigma^{-1} \sigma_{r0} / 2, \quad \sigma_e = \left( \sqrt{3} / 2 \right) \sqrt{(\sigma_{rr} - \sigma_{\theta \theta})^2 + 4\sigma_{r0}^2}. \]

For plane stress conditions the constitutive equations have the form

\[ e_{rr} = \alpha \sigma^{-1} (2\sigma_{rr} - \sigma_{\theta \theta}) / 2, \quad e_{\theta \theta} = \alpha \sigma^{-1} (2\sigma_{\theta \theta} - \sigma_{rr}) / 2, \quad e_{r0} = 3\alpha \sigma^{-1} \sigma_{r0} / 2, \]

\[ \sigma_e = \sqrt{\sigma_{rr}^2 + \sigma_{\theta \theta}^2 - \sigma_{rr} \sigma_{\theta \theta} + 3\sigma_{r0}^2}. \]

An asymptotic expansion of the solution was presented as

\[ \chi(r, \theta) = r^{k+1} f_1(\theta) + r^{k+1} f_2(\theta) + ... \]
where $\lambda < \gamma$, if the first term is the dominant one. The authors of [2] focused their attention only to the dominant (leading) term in this expression, $\chi(r, \theta) = r^{\lambda+1} f(\theta)$. The ordinary differential equation following from Eqs. (3) and (4) is homogeneous with respect to $f(\theta)$ and is associated with homogeneous boundary conditions. It has the form of the nonlinear eigenvalue equation for $\lambda$:

$$f_1^{\lambda} f^{\nu} \left( n-1 \right) \left[ (1-\lambda^2) f + f' \right]^2 + f_2^{\lambda} = C_1 \left[ (1-\lambda^2) f + f' \right]^2 + C_2 f^\nu + (n-1)(n-3) \times$$

$$\times \left\{ (1-\lambda^2) f + f' \right\} \left[ (1-\lambda^2) f + f' \right]^2 + 4\lambda^2 f^\nu \left[ (1-\lambda^2) f + f' \right]^2 + C_1 f^\nu +$$

$$+ (n-1) f_2^{\lambda} \left[ (1-\lambda^2) f + f' \right]^2 + (1-\lambda^2) f + f' \right\} (1-\lambda^2) f + f' + 4\lambda^2 \left( f^{\nu+2} + f^\mu \right) \times$$

$$\times \left[ (1-\lambda^2) f + f' \right] + 2(n-1) f_2^{\lambda} \left[ (1-\lambda^2) f + f' \right]^2 + 4\lambda^2 f^\nu \left[ (1-\lambda^2) f + f' \right] +$$

$$+ C_1 (n-1) f_2^{\lambda} \left[ (1-\lambda^2) f + f' \right]^2 + (1-\lambda^2) f + f' \right\} f + f_1^{\lambda} (1-\lambda^2) f = 0,$$

where $f_1^{\lambda} = \left[ (1-\lambda^2) f + f' \right]^2 + 4\lambda^2 f^{\nu+2}$, $C_1 = 4\lambda (\alpha - 1)n + 1$, $C_2 = (\alpha - 1)n (\lambda - 1)n + 2$.

Eq. (5) was derived for plane strain conditions. The boundary conditions for it are the vanishing condition of stress components at the crack planes

$$f(\theta = \pm \pi) = 0, \quad f'(\theta = \pm \pi) = 0.$$  

(6)

The corresponding nonlinear ordinary differential equation for plane stress conditions has the form

$$f^{\nu} f_1^{\lambda} \left( n-1 \right) \left( (\lambda + 1)(2-\lambda) f + 2 f' \right)^2 + 2 f_2^{\lambda} + 6 \left( \lambda - 1 \right) (n+1) \lambda \left( n-1 \right) f_2^{\nu} h^2 + f_4^{\nu} + (n-1)(n-3) \times$$

$$\times \left\{ (\lambda + 1)(2-\lambda) f + 2 f' \right\} \left( n-1 \right) f_2^{\lambda} \left( (\lambda + 1)(2-\lambda) f + 2 f' \right)^2 + (\lambda + 1)(2-\lambda) f +$$

$$+ (\lambda + 1)^2 \lambda^2 ( f^{\nu+2} + f^{\nu} ) - (\lambda + 1)^2 \lambda^2 f^{\nu+2} f^{\nu} / 2 - \left( (\lambda + 1)(2-\lambda) f \right)^2 + f_1^{\nu} (\lambda + 1)(2-\lambda) f = 0.$$  

(7)

where

$$f_2^{\nu} = \sqrt{ (\lambda + 1)(2-\lambda) f + f' } + (\lambda + 1) f + f'$$

$$+ (\lambda + 1)^2 \lambda^2 ( f^{\nu+2} + f^{\nu} ) - [ (\lambda + 1)(2-\lambda) f ]^2 + 3 \lambda^2 ( f^{\nu+2} + f^{\nu} ) - h \left( \lambda + 1 \right) f + f' \times$$

$$\times \left[ (\lambda + 1)(2-\lambda) f + f' \right] (\lambda + 1)(2-\lambda) f + f' \right\} (\lambda + 1)(2-\lambda) f = 0,$$

The order of the stress singularity is controlled by the eigenvalue and the angular variations of the field quantities are controlled by the eigenfunctions. The HRR solution is one of the most important achievements in the nonlinear fracture mechanics. Eqs. (5) and (7) give the angular distribution function of the solution. Function $f(\theta)$ is inferred from the solution of a rather complicated nonlinear differential equation of the fourth order.

### 2.2. Nonlinear eigenvalue problems

As follows from the previous analysis, the eigenfunction expansion method results in a nonlinear eigenvalue problem: it is necessary to find eigenvalues $\lambda$ leading to nontrivial solutions to Eqs. (5) and (7) satisfying boundary conditions (6). The eigenvalues corresponding to the HRR problem are known as $\lambda = n/(n+1)$. For these eigenvalues the numerical solution can be easily obtained [13]. The eigenfunctions and angular distributions of the stress components for different values of the hardening exponent $n$ are shown in Figures 1 and 2.

### 2.3. Eigenspectra and orders of stress singularity at the Mode I crack tip for a power-law medium

Further development of fracture mechanics required the analysis of eigenspectra and orders of singularity at a crack tip for power-law materials [7 - 20]. The authors of [9] highlighted the need to introduce higher or lower order singular terms for accurate description of the asymptotic fields in the vicinity of crack tip. The coordinate perturbation technique was used to study the eigenspectra of a creeping body. An appropriate numerical scheme was developed. As a result, the number of singularities, and their orders, as well as the angular distributions of stresses were obtained.
Figure 1. Circumferential distributions of the stress components in the vicinity of the Mode I crack tip (plane strain conditions).

Figure 2. Circumferential distributions of the stress components in the vicinity of the Mode II crack tip (plane strain conditions).

In [10,11] eigenspectra and orders of singularity of the stress field near the Mode I crack tip in a power-law material are discussed. The perturbation theory technique was used to obtain the required asymptotic solution. The whole set of eigenvalues was obtained. It is shown that the eigenvalues for the nonlinear problem are fully determined by the corresponding eigenvalues of the linear problem and the hardening exponent $n$. The Airy stress function is presented as [12]

$$\chi(r,\theta) = r^{\lambda+1} f^{(0)}(\theta) + r^{\lambda+1} f^{(1)}(\theta) + r^{\lambda+1} f^{(2)}(\theta) + ...$$

The authors of [12] looked for new eigenvalues different from those for the classical HRR problem. The results for the Mode I crack for plane strain conditions are shown in Table 1 [12].

| $n$ | $\lambda$ | $f^{*(0)}$ |
|-----|-----------|------------|
| 2   | 0         | -0.5       |
| 3   | 0.228641  | -0.436714  |
| 4   | 0.331913  | -0.408663  |
| 5   | 0.382535  | -0.397996  |
| 6   | 0.410325  | -0.394479  |
| 7   | 0.427209  | -0.393828  |
| 8   | 0.438319  | -0.394328  |
| 9   | 0.446089  | -0.395275  |

2.4. Nonlinear eigenvalue problems for mixed mode loading

As follows from the previous analysis, the eigenfunction expansion method results in a nonlinear eigenvalue problem: it is necessary to find eigenvalues $\lambda$ leading to nontrivial solutions to Eqs. (5) or (7) satisfying boundary conditions (6). Therefore, the order of stress singularity is the eigenvalue and the angular variations of the field quantities correspond to the eigenfunctions. When Mode I or Mode
II loading conditions were used, symmetry or antisymmetry requirements of the problem with respect to the crack plane at \( \theta = 0 \) were used. Due to symmetry the solution is sought for one of the half-planes. In analyzing the crack problem under mixed-mode loading conditions the symmetry or antisymmetry arguments can’t be used and it is necessary to seek for the solution in the whole plane \(-\pi \leq \theta \leq \pi\). To find the numerical solution one has to take into account the value of the mixity parameter \( M^p \) [13-16]. In this case, in the framework of the proposed technique Eq. (7) is solved numerically at \([0, \pi]\) and the two-point boundary value problem is reduced to the problem with the initial conditions reflecting the value of the mixity parameter \( M^p \) [13-16]. At the next step the numerical solution of Eq. (4) is found at \([-\pi, 0]\). One can find a whole set of eigenvalues for plane stress conditions from the requirement of the continuity of the radial stress components at \( \theta = 0 \). Following [15, 16] the spectrum of eigenvalues \( \lambda \) is found numerically. The results of computations are shown in Table 2. The angular distributions of the stress components are shown in Figure 3.

### Table 2. Eigenvalues \( \lambda \), initial values \( f^*(0) \), \( f^*(\pi) \), \( f^*(-\pi) \) and \( f^*(-\pi) \) obtained for different values of the mixity parameter \( M^p \) for plane stress conditions \( (n = 2) \).

| \( M^p \) | \( \lambda \) | \( f^*(0) \) | \( f^*(\pi) \) | \( f^*(-\pi) \) | \( f^*(-\pi) \) |
|--------|--------|--------|--------|--------|--------|
| 0.9    | -0.30032000 | -0.25428500 | -0.52319280 | 0.36781000 | 0.41793000 |
| 0.8    | -0.28609000 | 0.30988600 | -0.65543910 | -0.14222000 | 1.23657500 |
| 0.7    | -0.26789000 | -0.40297913 | -0.80444475 | -0.37921000 | 0.54939150 |
| 0.6    | -0.26093000 | -0.46493199 | 1.03847110 | -0.54340000 | 0.46094200 |
| 0.5    | -0.25233200 | -0.52217930 | -1.40075019 | -0.72780000 | 0.42459230 |
| 0.4    | -0.24369800 | -0.57136233 | -1.98711539 | -0.97155000 | 0.40989380 |
| 0.3    | -0.23701900 | -0.61089207 | -2.95625279 | -1.35116000 | 0.41294900 |
| 0.2    | -0.23247900 | -0.64000914 | -4.76598544 | -2.08610169 | 0.44429335 |
| 0.1    | -0.22987230 | -0.65774480 | -9.82544937 | -4.26300089 | 0.57184713 |

**Figure 3.** Circumferential distributions of stresses in the vicinity of the mixed mode crack tip.

2.5. **Perturbation method to find the whole set of eigenvalues**

Perturbation methods based on artificial small parameters for solving the nonlinear eigenvalue problems arising in fracture mechanics problems have been used by several authors [19 - 21]. This approach has been used for determining the higher order fields at a notch or a crack tip in a power-law hardening material under Mode III loading [19]. A closed form analytical solution for eigenvalues determining the asymptotic behavior of the crack tip fields has based on perturbation methods. It was shown [19] that the eigenvalues for the nonlinear problem depend only on the eigenvalues for the...
corresponding linear problem and on the hardening exponent. This referred to all three combinations of homogeneous boundary conditions considered in [19]. Basic equations used for a sharp notch with an opening angle $\alpha$ in an infinite region under the longitudinal shear were the equilibrium and compatibility equations. The constitutive behavior is described by the power-law. Remembering that singular stresses are expected at the notch tip the stress function was presented as an asymptotic expansion with increasing powers to in $r$: $\chi(r, \theta) = r^nf(\theta)$. Introducing the stress function into the compatibility equation one can find the nonlinear ordinary differential equation $f^*(nf^{*2} + s^2f^3) + f(C_1f^{*2} + C_2f^{*3}) = 0$, where $C_1 = s(n-1)(2s-1) + s^2$, $C_2 = s^3(n-1)(s-1) + s^4$. Together with the boundary conditions this equation describes a nonlinear eigenvalue problem, where the unknown eigenvalue $s$ and the eigenfunction $f(\theta)$ depend on the boundary conditions and the hardening exponent. An analytical expression for the eigenvalues of the nonlinear equation can be derived based on the application of the perturbation method. In this case, the eigenvalue can be presented as $s = s_0 + \epsilon$, where $s_0$ refers to the “undisturbed” linear problem and $\epsilon$ is the deviation taking into account the effect of nonlinearity. The hardening exponent $n$ and the stress function are represented as power series

$$n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \ldots = \sum_{j=0}^{\infty} \epsilon^j n_j, \quad f(\theta) = f_0(\theta) + \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \epsilon^3 f_3(\theta) + \ldots = \sum_{j=0}^{\infty} \epsilon^j f_j(\theta),$$

where $n_0 = 1$ and $f_0(\theta)$ refer to the linear solution. The homogeneous problem for function $f_0(\theta)$ has the nontrivial solution. The inhomogeneous problems for functions $f_j(\theta)$, $j > 0$ will not have a solution unless a solvability condition is satisfied [20, 21]. The solvability condition allows us to find all the coefficients of the asymptotic series for the hardening exponent. Thus, it is possible to find all the coefficients $n_j$ and to present the asymptotic expansion in the form [19]

$$n = 1 + s_0 / (s_0 - s_x) \sum_{j=1}^{\infty} (-\epsilon / (s_0 - s_x))^j - 1 / (s_0 - 1) \sum_{j=1}^{\infty} (-\epsilon / (s_0 - 1))^j = s / (s - s_x) - 1 / (s - 1), \quad s_x = s_0^2 / (2s_0 - 1).$$

Thus, the whole set of eigenvalues is found.

3. Higher order asymptotic analysis of the crack tip fields

3.1. Higher order asymptotic analysis of the crack tip fields under creeping conditions

The HRR field is the leading or the first order term in an asymptotic analysis of a crack in a power-law hardening material. In the general case, the application of a single fracture parameter from the leading term may not be sufficient to describe the influence of the specimen geometry, loading level as well as material mismatch. This is usually referred to as the constraint effect [22]. To quantify fracture higher order terms from an asymptotic analysis one needs to describe the crack tip stress field accurately and then characterizes the fracture event [22]. According to Hoff’s analogy between power-law hardening material and power-law creeping material, the first term for a creeping crack with power law elastic-creep also satisfies the HRR singularity. Recently, various constraint parameters for Mode I creep crack and mixed mode crack have been proposed [22-25]:

$$\sigma_0(r, \theta, t) / \sigma_0 = A_1(t) \bar{r}^{s_1} \bar{\sigma}^{(1)}(\theta) + A_2(t) \bar{r}^{s_2} \bar{\sigma}^{(2)}(\theta) + A_3(t) \bar{r}^{s_3} \bar{\sigma}^{(3)}(\theta),$$

where $A_i(t), (k = 1, 2, 3, \ldots)$ are the $k$ th undermined constants depending on the creep time and subscript $k$ indicates the $k$ th term, respectively. Functions $\bar{\sigma}^{(k)}(\theta)$ are the dimensionless angular functions corresponding to the $k$ th term, $s_i$ is the power exponent for the $k$ th term, $\bar{r} = r / L$ and $L$ here is the characteristic length. A methodology for finding higher order terms in the asymptotic expansion of the crack tip fields for the problem of quasi-static steady-state crack growth in an elastic-nonlinear-viscous solid was developed [23]. The creep strain rate was assumed to be proportional to
the stress raised to some power. A possibility of separation of variables with respect to \( r \) and \( \theta \) in each term of the asymptotic expansion (\( r, \theta \) are polar coordinates at the crack tip) was suggested, and the first two terms in the asymptotic series were obtained. Anti-plane shear (Mode-III) and plane strain solutions for a crack in a homogeneous material and a crack lying along the interface between an elastic-viscous solid and a rigid substrate were obtained [23]. In all cases both the first and the second order terms of the asymptotic expansion of the crack tip stress fields had singularity at \( r \to 0 \), when the creep exponent was in the range \( 3 < n < 4 \). For the case of steady-state crack growth along the interface between an elastic-nonlinear-viscous solid and a rigid substrate, it was found [23] that both the amplitude of the leading term and the crack tip “mode-mix” were determined by the asymptotic solution, when \( n \geq 3 \). For each value of the creep exponent, there were two distinct solutions with very different mode-mixes at the interfacial line ahead of the crack: one of the solutions was Mode-I-like, whereas the other had a substantial Mode-II component. The asymptotic analysis developed in [22] indicates that the three terms in the solution are sufficient to characterize the full mechanics field of Mode II creep crack under various conditions. Some typical fracture parameters under creep conditions were extended and discussed. The higher order asymptotic solutions presented in [23-25] contributed to further understanding of fracture behavior of cracks in power-law creeping solids.

3.2. Analysis of stress and strain fields near the tip of a steady-state growing crack in an elastic-viscous medium

An asymptotic solution for a steadily slowly growing crack in an elastic–nonlinear viscous medium was suggested by Hui and Riedel [20]. For uniaxial tension their constitutive equations have the following form

\[
\dot{\varepsilon} = \frac{\sigma}{E} + \alpha \sigma^n.
\]

The first term shows the elastic strain rate (\( E \) is Young’s modulus) and the second term describes the power law secondary creep where \( \alpha \) is the creep coefficient and \( n \) is the creep exponent. The mechanics of a plane strain Mode I growing crack under transient conditions was investigated using self-similar solutions in [7]. For small crack extensions, the evolution of the near crack tip stress field of Hui and Riedel (HR) was considered as a singular perturbation problem. For small crack extensions, the HR field, the HRR field and the elastic \( K_I \) field coexist near the crack tip, one inside the other. The regions of the dominance of these fields were found. An approximate solution was found for the singular perturbation problem. The transition time predicted by Riedel and Rice [5], which is exact for stationary cracks, remains accurate for growing cracks provided that crack extension is small. Also, the effect of crack growth rate on the small scale yielding assumption was studied. It was shown that for fast crack growth, creep relaxation can be ignored and the results for steady-state analysis can be modified to describe the near tip stress fields. Also, the application of the results to non-self-similar crack growth was discussed.

The authors of [26] focused on the the asymptotic stress-strain field around a crack tip, steadily propagating in viscous materials in antiplane conditions. The stress and strain fields near the tip of a steady-state growing crack are examined for elastic-viscous materials. A solution to this problem was originally obtained by Hui and Riedel [20]. This solution, however, contained some inconsistencies and contradictions. For example, it did not predict the crack growth rate. This solution predicted an autonomous crack growth (independent on the loading state) in the general case. In [26, 27] this problem was revisited using a multiscale asymptotic analysis based on the method of matched asymptotic expansions [28-30]. A two-scale match asymptotic analysis was suggested [26,27] to overcome these inconsistencies. The scale factor was determined by material properties. A small parameter \( \varepsilon \), proportional to the crack growth rate, was introduced to switch from the inner solution (close to the crack tip) to the outer one (far field), using an asymptotic expansion of the solution. The outer solution was equivalent to the nonlinear elastic HRR field as the first order term while the viscosity appeared as the second order term. The inner scale was considered as a boundary layer, where the stress field was described by the serial Fourier analysis. The inner solution coincides with
the Hui and Riedel solution [20]. The second order asymptotic solutions to the near-tip field for a crack propagating under steady-state are presented [27]. The first-order solutions for this problem were originally obtained by Hui and Riedel [20]. For the values of the power-law exponent $n < 3$, the second-order solution was found to satisfy a linear, inhomogeneous differential equation. For $n > 3$, however, the problem is reduced to an eigenvalue problem. In contrast to the first-order solution for $n > 3$, the second order solution contains an undetermined parameter the value of which must be found from the remote boundary conditions. The matching conditions allow the authors to link the far and close fields, and to correct the paradox whereby the crack velocity should not depend to the far field controlled by the loading.

4. Asymptotic analysis in Continuum Damage Mechanics

4.1. On the effect of the damage accumulation processes on the stress-strain state in the vicinity of the crack tip

Though the HRR field [2-4] has been obtained for an ideal discrete crack in intact non-linear hardening materials, the fracture process in usual ductile materials is brought about by nucleation, growth and coalescence of distributed microscopic cavities in front of the crack tip, and this damage field has significant influence on stress field near the crack tip. Analyses of material damage on the stress and strain fields in the vicinity of the crack tip in nonlinear materials provide very important problems [31 - 46]. In this context these problems have been discussed in a number of papers [31 - 46]. In spite of it systematic information on the effect of material damage on the asymptotic crack-tip field is not available from the analysis. Thus, the paper [32] was one of the first work where the effect of damage accumulation process has been elucidated. Asymptotic fields of stress, creep strain rate and damage of a mode I creep crack in steady-state growth are analyzed on the basis of CDM by means of a semi-inverse method. In [36] an asymptotic analysis of the near-tip field is presented in terms of the coordinate perturbation technique for fast crack propagation in an elastic-plastic-viscoplastic materials with damage. A damage variable is incorporated in the constitutive relation based upon the strain-equivalence principle of damage mechanics. The damage evolution law used is a quasi-brittle type, in which both equivalent and hydrostatic stresses are involved. A non-singular stress field is obtained, as the damage has substantial influence on the material behavior that the high stresses are relaxed at the crack tip. An analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained in [39, 40]. The perturbation technique for solving the nonlinear eigenvalue problem is used. The method allows us to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law. It is shown that the eigenvalues of the nonlinear eigenvalue problem are fully determined by the exponents of the damage evolution law. In the paper [39] the third - order (four-term) asymptotic expansions of the angular functions determining the stress and continuity fields in the neighborhood of the crack tip are given. The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem considered.

The general constitutive equations, including the damage coupled stress-strain relations of isotropic linear elastic materials and the damage evolution are used for analysis of fatigue crack growth:

$$\sigma_{ij} = (1 + \nu)\varepsilon_{ij} / (E\nu) - (\nu / E)\sigma_{ii} \delta_{ij} / \psi,$$

where is the Young modulus, is the Poisson ratio, where $\psi$ is an integrity parameter. It is assumed that the continuity parameter evolves according to the following damage evolution equation

$$d\psi / dN = \left\{ \begin{array}{ll}
-c(\sigma / \psi)^m \psi^{-n}, & \text{if } (\sigma \geq \sigma_c) \\
0, & \text{if } (\sigma < \sigma_c) \end{array} \right.$$

The asymptotic solution to the problem in the vicinity of the crack tip is presented in the separable form $\Phi(r, \theta) = \alpha r^{\psi/2} f(\theta)$. The asymptotic behaviour of the continuity parameter in the neighborhood of the crack tip is either sought in the separable form $\psi(r, \theta) = \beta r^m f(\theta)$. Among the theoretical
methods of solving many nonlinear problems of applied mathematics, mechanics, physics and modern technology asymptotic methods are those that recently deserve special attention [39, 40]. The perturbation theory method used in [39, 40] is based on introducing small artificial parameter 
\[ \varepsilon = \mu - \mu_0 \]
reflecting the effect of nonlinearity in the damage evolution law. The small parameter \( \varepsilon \) quantitatively describes the nearness of the eigenvalue of the nonlinear eigenvalue problem for the nonlinear damage evolution law and the eigenvalue \( \mu_0 \) corresponding to the linear “undisturbed” problem for the linear damage evolution law. The multi-term asymptotic approximation of the solution to the nonlinear eigenvalue problem is constructed using the following asymptotic ansatz for solution

\[
\mu = \mu_0 + \varepsilon, \quad \lambda = \sum_{j=0}^{\infty} \varepsilon^j \lambda_j, \quad n = \sum_{j=0}^{\infty} \varepsilon^j n_j, \quad m = \sum_{j=0}^{\infty} \varepsilon^j m_j, \quad f(\theta) = \sum_{j=0}^{\infty} \varepsilon^j f_j(\theta), \quad g(\theta) = \sum_{j=0}^{\infty} \varepsilon^j g_j(\theta).
\]

The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem considered. It should be noted that the numerical integration of the nonlinear differential equations by the Runge-Kutta-Felhberg method in conjunction with the shooting method becomes multiparametric since three parameters are required to choose as part of numerical solution. Consequently, the numeric results still require further investigations and verification (especially for the case when the system includes the singular disturbed as \( \theta \to 0 \) equations. The approach proposed based on the perturbation theory enables to overcome difficulties caused by the singular disturbed equation. The results [39,40] show that the perturbation techniques effectively constitutes the closed-form solution of the nonlinear eigenvalue problem. Also, the higher order fields in the vicinity of the fatigue growing crack were obtained, and the analytical expressions of the dominant and second order exponents and angular distribution functions of the near tip stress and continuity fields were derived. Further theoretical and numerical efforts are needed in the respects of appropriate determination of the amplitude coefficients as functions of particular specimen geometry and loading. Effects of the damage on the near crack tip quantities are discussed in detail [39,40]. It is found that damage leads to the nonsingular stress field. Therefore, it is seen that damage has important effect on the angular distribution functions of the near tip stress and strain fields.

### 4.2. The coupled statement of the creep-damage crack problems

In [41] the creep crack problem in damaged materials under mixed mode loading under creep-damage coupled formulation is considered. The class of the self-similar solutions to the plane creep crack problems in a damaged medium under mixed-mode loading is given. With the similarity variable and the self-similar representation of the solution for a power-law creeping material and the power-law damage evolution equation the near crack-tip stresses, creep strain rates and continuity distributions for plane stress conditions are obtained. The self-similar solutions are based on the hypothesis of the existence of the completely damaged zone near the crack tip. It is shown that the asymptotical analysis of the near crack-tip fields gives rise to the nonlinear eigenvalue problems. The technique permitting to find all the eigenvalues numerically is proposed and numerical solutions of the nonlinear eigenvalue problems arising from the mixed-mode crack problems in a power-law medium under plane stress conditions are obtained. Using the approach developed the eigenvalues different from the eigenvalues corresponding to the HRR problem are found. Having obtained the eigenspectra and eigensolutions the geometry of the completely damaged zone in the vicinity of the crack tip is found for all values of the mixity parameter. By noting that the creep damage is brought about by the development of microscopic voids in creep process, L.M. Kachanov [33] represented the damage state by a scalar integrity variable \( \psi \) \((0 \leq \psi \leq 1)\), where \( \psi = 1 \) and \( \psi = 0 \) signify the initial undamaged state and the final completely damaged state, respectively. Kachanov described the damage development by means of an evolution equation

\[ \dot{\psi} = -\lambda(\sigma / \psi)^m. \]

In CDM [33], the damage state at an arbitrary point in the material is represented by a properly defined integrity variable \( \psi(r, \theta) \). The integrity parameter reaches its critical value at fracture.
According to this notion, a crack in a fracture process can be modeled with the concept of a completely damaged zone in the vicinity of the crack tip. The crack can be represented by a region where the damage state has attained to its critical state $\psi = \psi_{cr}$, i.e., by the completely damaged zone (CDZ). Then the development of the crack and its preceding damage can be elucidated by analyzing the local states of stress, strain and damage. The CDZ may be interpreted as the zone of critical decrease in the effective area due to damage development. Inside the completely damaged zone the damage involved reaches its critical value (for instance, the damage parameter reaches unity) and a complete fracture failure occurs. In view of material damage stresses are relaxed to vanishing \[15,16,40,41,47,48\]. Therefore, one can assume that the stress components in the CDZ equal zero. Therefore asymptotic remote boundary conditions have the form

$$\sigma_{ij}(r \to \infty, \theta, t) = (kr)^{l(n+1)} \sigma_{ij}^{*}(\theta, n), \quad k = C / (B l_n^*). \tag{8}$$

Dimensional analysis shows the damage mechanics equations must have similarity solutions:

$$\sigma_{ij}(r, \theta, t) = \left( \frac{At}{m} \right)^{1/m} \sigma_{ij}^{*}(R, \theta), \quad \psi(r, \theta, t) = \tilde{\psi}(R, \theta), \quad R = r \left( \frac{At}{m} \right)^{-1/m} B l_n^* / C^*,$$

$R$ is the similarity variable. It should be noted that the remote boundary conditions can be formulated in a more general form compared with Eq. (8): $\sigma_{ij}(r \to \infty, \theta, t) = \tilde{C} r^s \sigma_{ij}^{*}(\theta, n)$, where the stress singularity exponent $s$ is unknown and has to be determined as a part of solution, $\tilde{C}$ is the amplitude of the stress field at infinity defined by the specimen configuration and loading conditions. For the power-law constitutive relations, the power damage evolution law and the more general remote boundary conditions the self-similar variable $R = r (kr)^{1/(sm)}$, $k = A \tilde{C}^{sm}$ can be introduced.

### 4.3. The geometry of the totally damaged zone

The configuration of the CDZ modelled near the crack tip is determined by the equation

$$\psi(R, \theta) = 1 - \sum_{j=0}^{k} R^j \gamma_j(\theta) = 0, \quad k = 1, 2, 3, 4, 5...$$

One can compare the boundaries of the CDZ given by the multi-term asymptotic expansions of the integrity parameter. The new stress asymptotic behavior results in the contours of the CDZ which converge to the limit contour. The new far field stress asymptotic can be interpreted as the intermediate stress asymptotics valid for times and distances at which effects of the initial and boundary conditions on the stress and damage distributions are lost. The geometry of the completely damage zone for different values of the mixity parameter $M^p$ is shown in Figure 4.

### 5. Conclusion

Recent activity in the analysis of crack-tip stress and strain fields for stationary and growing cracks in power-law materials is surveyed. Some of the main subjects to further progress in nonlinear fracture mechanics obtained by perturbation methods are discussed. In the paper the detailed review of asymptotic solutions to crack problems obtained for the power law constitutive equations is presented. The review can’t be considered as a full and exhaustive one. However, one can elucidate the characteristic features of the research performed in nonlinear fracture mechanics. In nonlinear fracture mechanics, one often needs to solve nonlinear differential equations to find eigenfunction and eigenvalue. Many nonlinear eigenvalue equations have multiple solutions. It is well known that multiple solutions of nonlinear boundary value problems are not easy to gain by means of numerical techniques such as the shooting method. The perturbation and asymptotic approximations of nonlinear problems often break down as nonlinearity becomes strong \[49,50\]. Therefore, they are only valid for weakly nonlinear ordinary differential equations and partial differential equations in general.

The homotopy analysis method (HAM) is an analytic approximation method for highly nonlinear problems \[49,50\]. Unlike perturbation techniques, the HAM is independent of any small or large physical parameters at all. Secondly, different from all of other analytic techniques, the HAM provides
us a convenient way to guarantee the convergence of solution series so that it is valid even if nonlinearity becomes rather strong. Thus, in fracture mechanics HAM may play a significant role in solving nonlinear eigenvalue problems arising in nonlinear fracture mechanics.

Figure 4. The geometry of the totally damaged zone in the vicinity of the crack tip given by the $k+1$–term asymptotic expansion of the continuity parameter $\psi$.

Results obtained in [49] show the great potential and validity of the for highly nonlinear eigenvalue equations with multiple solutions and singularity. The HAM provides one of the promising approach for nonlinear eigenvalue problem arising in fracture mechanics. The present review shows that asymptotic solutions of fracture mechanics problem will be connected either with derivation of multi-term asymptotic series expansions for the crack-tip fields using effective computer algorithms and procedures [45-51]. The further development of CDM will probably be connected with experimental determination of active damage accumulation zone in the vicinity of the crack tip via interference-optic methods [45, 52] and acoustic emission methods [53].

6. References

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