Strong version of Andrica’s conjecture

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Abstract:
A strong version of Andrica’s conjecture can be formulated as follows: Except for \( p_n \in \{3, 7, 13, 23, 31, 113\} \), that is \( n \in \{2, 4, 6, 9, 11, 30\} \), one has \( \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{2} \). While a proof is far out of reach I shall show that this strong version of Andrica’s conjecture is unconditionally and explicitly verified for all primes below the location of the 81st maximal prime gap, certainly for all primes \( p < 1.836 \times 10^{19} \). Furthermore this strong Andrica conjecture is slightly stronger than Oppermann’s conjecture — which in turn is slightly stronger than both the strong and standard Legendre conjectures, and the strong and standard Brocard conjectures. Thus the Oppermann conjecture, and strong and standard Legendre conjectures, are all unconditionally and explicitly verified for all primes \( p < 1.836 \times 10^{19} \). Similarly, the strong and standard Brocard conjectures are unconditionally and explicitly verified for all primes \( p < \sqrt{1.836 \times 10^{19}} \approx 4.285 \times 10^9 \).

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1 Introduction

The distribution of the primes is a fascinating topic that continues to provide many subtle and significant open questions [1–22]. In the current article I will consider strong, standard, and weak versions of the Andrica conjecture, and the closely related Oppermann, Legendre, and Brocard conjectures. These conjectures all impose constraints on the prime gaps of the form

\[ g_n := p_{n+1} - p_n = O \left( \sqrt{p_n} \right). \] (1.1)

Specifically, consider the strengthened version of the usual Andrica conjecture presented below.

**Conjecture 1 (Strong Andrica conjecture)**

Except for \( p_n \in \{3, 7, 13, 23, 31, 113\} \), that is \( n \in \{2, 4, 6, 9, 11, 30\} \), one has

\[ \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{2}; \quad \text{equivalently} \quad g_n := p_{n+1} - p_n < p_n^{1/2} + \frac{1}{4}. \] (1.2)

We shall soon see that this strong Andrica conjecture, like the usual Andrica conjecture [21], has the virtue that it can easily be verified on suitable intervals by inspecting a table of known maximal prime gaps [22–25].
The specific choice of the constant $\frac{1}{2}$ in the left sub-equation of (1.2), which leads to the coefficient unity in front of the $p_n^{1/2}$ in the right sub-equation of (1.2), was (with hindsight) carefully arranged to be just strong enough to imply the Oppermann conjecture below. The specific offset $\frac{1}{4}$ in the right sub-equation is merely $(\frac{1}{2})^2$ and is not particularly important, more on this later. In counterpoint, Oppermann’s conjecture [1–4, 26] can be cast in either of the two equivalent forms given below.

**Conjecture 2** (Oppermann conjecture [1–4, 26])

(1) For all integers $m \geq 2$ there is at least one prime in each of the intervals

$$\left( m(m - 1), m^2 \right); \quad \text{and} \quad \left( m^2, m(m + 1) \right). \quad (1.3)$$

(2) For all integers $m \geq 1$ there is at least one prime in each of the intervals

$$\left( m^2, m(m + 1) \right) \quad \text{and} \quad \left( m(m + 1), (m + 1)^2 \right). \quad (1.4)$$

No proof of Oppermann’s conjecture is known as of February 2019, so one must instead resort to verifying it on certain (hopefully large) intervals. For this purpose it is useful to note that the strong variant of Andrica’s conjecture introduced above implies Oppermann’s conjecture.

Other weaker conjectures closely related to Oppermann’s conjecture are:

**Conjecture 3** (Strong Legendre conjecture)

There are at least two primes in the interval

$$\left( m^2, (m + 1)^2 \right). \quad (1.5)$$

**Conjecture 4** (Standard Legendre conjecture [27–29])

There is at least one prime in the interval

$$\left( m^2, (m + 1)^2 \right). \quad (1.6)$$

**Conjecture 5** (Strong Brocard conjecture)

There are at least $2g_n := 2(p_{n+1} - p_n)$ primes in the interval

$$\left( p_n^2, p_{n+1}^2 \right). \quad (1.7)$$
Conjecture 6 (Standard Brocard conjecture [1–3, 30])
There are at least four primes in the interval
\[(p_n^2, p_{n+1}^2)\].  

(1.8)

For completeness we also define:

Conjecture 7 (Standard Andrica conjecture [1–4, 21])
Either of these two equivalent forms
\[\forall n \geq 1 : \sqrt{p_{n+1}} - \sqrt{p_n} < 1; \quad g_n := p_{n+1} - p_n < 2\sqrt{p_n} + 1.\]  

(1.9)

Conjecture 8 (Weakened versions of the Andrica conjecture)
A weakened version of the Andrica conjecture can be presented in either of these two equivalent forms
\[\forall n \geq 1 : \sqrt{p_{n+1}} - \sqrt{p_n} < 2; \quad \text{equivalently} \quad g_n := p_{n+1} - p_n < 4\sqrt{p_n} + 4.\]  

(1.10)

We could try to be more general and conjecture
\[\forall n \geq 1, \forall c > 1 : \text{equivalently} \quad \sqrt{p_{n+1}} - \sqrt{p_n} < c; \quad g_n := p_{n+1} - p_n < 2c\sqrt{p_n} + c^2,\]  

(1.11)

but the specific choice \(c = 2\) is more useful in that we shall soon see that it is easily related to the standard Legendre conjecture. An even weaker conjecture would be to merely assert that \(\sqrt{p_{n+1}} - \sqrt{p_n}\) is bounded.

2 Verifying the strong Andrica conjecture for primes \(p < 1.836 \times 10^{19}\)

The argument is a minor variant of that given for the standard Andrica conjecture in reference [22]. Consider the maximal prime gaps: Following the notation of reference [22], let the triplet \((i, g_i^*, p_i^*)\) denote the \(i^{th}\) maximal prime gap; of width \(g_i^*\), starting at the prime \(p_i^*\). (See see the sequences A005250, A002386, A005669, A000101, A107578.) 80 such maximal prime gaps are currently known [23–25], up to \(g_80^* = 1550\) and \(p_{80}^* = 18,361,375,334,787,046,697 > 1.836 \times 10^{19}\). Now consider the interval \([p_i^*, p_{i+1}^* - 1]\), from the lower end of the \(i^{th}\) maximal prime gap to just below the beginning of the \((i + 1)^{th}\) maximal prime gap. Then everywhere in this interval
\[\forall p_n \in [p_i^*, p_{i+1}^* - 1] \quad g_n \leq g_i^*; \quad \sqrt{p_i^*} + \frac{1}{4} \leq \sqrt{p_n} + \frac{1}{4}.\]  

(2.1)
Therefore, if the strong Andrica conjecture holds at the beginning of the interval $p_n \in [p^*_i, p^*_{i+1} - 1]$, then it certainly holds on the entire interval. Explicitly checking a table of maximal prime gaps [23], the strong Andrica conjecture certainly holds on the interval $[p^*_7, p^*_81 - 1]$, from $p^*_7 = 523$ up to just before the beginning of the 81st maximal prime gap, $p^*_81 - 1$, even if we do not yet know the value of $p^*_81$. Then explicitly checking the primes below $p^*_7 = 523$ the strong Andrica conjecture holds for all primes $p$ less than $p^*_81$ except $p \in \{3, 7, 13, 23, 31, 113\}$.

Since we do not explicitly know $p^*_81$ a safe fully explicit statement is to use the known value of $p^*_80$ to verify the strong Andrica conjecture for all primes $p < 1.836 \times 10^{19}$. (This argument also automatically verifies the standard and weak Andrica conjectures over the same domain.)

3 Verifying the Oppermann conjecture for primes $p < 1.836 \times 10^{19}$ and integers $m < 4.285 \times 10^9$

**Theorem 1** The strong Andrica conjecture implies the Oppermann conjecture.

Proof:
(1) Note $\lfloor \sqrt{113} \rfloor = 10$, so taking $m \geq 12$ means we safely avoid the exceptional cases in the strong Andrica conjecture.

(2) Pick some fixed $m \geq 12$ and let $p_n$ be the largest prime less than $m^2$.
Then by construction $p_n < m^2 < p_{n+1}$ and by the strong Andrica conjecture we have
\[
    p_{n+1} := p_n + g_n < p_n + p^n_{n}\frac{1}{2} + \frac{1}{4} < m^2 + m + \frac{1}{4}.
\]  
(3.1)

But since $p_{n+1} \in \mathbb{N}$ this implies $p_{n+1} \leq m(m + 1)$.
But since $p_{n+1} \in \mathbb{P}$ this implies $p_{n+1} < m(m + 1)$.
(3) Pick some fixed $m \geq 12$ and let $p_{n+1}$ be the smallest prime greater than $m^2$.
Then by construction $p_{n+1} > m^2 > p_n$ and by the strong Andrica conjecture we have
\[
    p_n := p_{n+1} - g_n > p_{n+1} - p^n_{n}\frac{1}{2} - \frac{1}{4} > m^2 - m - \frac{1}{4}.
\]  
(3.2)

But since $p_n \in \mathbb{N}$ this implies $p_n \geq m(m + 1)$.
But since $p_n \in \mathbb{P}$ this implies $p_n > m(m + 1)$.

(4) Check the cases $m < 12$ by explicit computation.

Now while the proof that the strong Andrica conjecture implies the Oppermann conjecture is unconditional, we have only verified the strong Andrica conjecture up to the

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location of the 81st maximal prime gap \( p < p_{81}^* \). Consequently we can only verify the Oppermann conjecture up to \( m \leq \lfloor \sqrt{p_{81}} \rfloor \). Since we do not yet know \( p_{81}^* \) the best explicit verification range is \( m \leq \lfloor \sqrt{p_{80}} \rfloor \approx 4.285 \times 10^9 \).

4 Verifying other weaker conjectures

First, note that the Oppermann conjecture implies the strong Legendre conjecture which in turn implies the standard Legendre conjecture. To see this write

\[
(m^2, (m + 1)^2) = (m^2, m(m + 1)) \cup \{m(m + 1)\} \cup (m(m + 1), (m + 1)^2).
\]  

Then assuming the Oppermann conjecture, \((m^2, m(m + 1))\) contains a prime. Also \((m(m + 1), (m + 1)^2)\) contains a prime, and \(m(m + 1)\) is not a prime. So there are at least two primes in \((m^2, (m + 1)^2)\). Furthermore, automatically there is at least one prime in \((m^2, (m + 1)^2)\).

Second, note that the strong Legendre conjecture implies the strong Brocard conjecture, which in turn implies the standard Brocard conjecture. To see this, split the interval \((p_n^2, p_{n+1}^2)\) into \(g_n := p_{n+1} - p_n\) sub-intervals of the form \(((p_n + i)^2, (p_n + i + 1)^2)\) for \(0 \leq i \leq g_n - 1\). Assuming the strong Legendre conjecture, each of these \(g_n\) sub-intervals contains at least 2 primes, so \((p_n^2, p_{n+1}^2)\) contains at least \(2g_n\) primes, which is the strong Brocard conjecture. Since \(g_n \geq 2\) this automatically implies the standard Brocard conjecture.

Third, the Oppermann conjecture implies the standard Andrica conjecture. To see this define the two integers \(m = \lfloor p_n^{1/2} \rfloor\) and \(\tilde{m} = \lfloor p_{n+1}^{1/2} \rfloor\).

- If \(m = \tilde{m}\) then \(p_n > m^2\) and \(p_{n+1} < (m+1)^2\). So \(\sqrt{p_{n+1}} - \sqrt{p_n} < (m+1) - m = 1\).

- If \(m \neq \tilde{m}\) then \(m^2 < p_n < \tilde{m}^2 < p_{n+1}\). Then, assuming the Oppermann conjecture, we have both \(p_n > \tilde{m}^2 - \tilde{m}\) and \(p_{n+1} < \tilde{m}^2 + \tilde{m}\), and so \(g_n < 2\tilde{m}\). But then

\[
p_n > \tilde{m}(\tilde{m} - 1) = (\tilde{m} - 1/2)^2 - 1/4 > (\tilde{m} - 1/2)^2,
\]  

so \(\tilde{m} < 1/2 + \sqrt{p_n}\), and so \(2\tilde{m} < 2\sqrt{p_n} + 1\). Finally this implies \(g_n < 2\sqrt{p_n} + 1\).

Fourth, the standard Andrica conjecture implies the standard Legendre conjecture. Pick some fixed \(m\) and let \(p_n\) be the largest prime less than \(m^2\). Then by construction \(p_{n+1} > m^2 > p_n\), and assuming the standard Andrica conjecture we have

\[
p_{n+1} = p_n + g_n < p_n + 2\sqrt{p_n} + 1 < m^2 + 2m + 1 = (m + 1)^2.
\]  

\[\text{– 5 –}\]
Fifth, the standard Legendre conjecture implies the weak \((c = 2)\) Andrica conjecture. To see this pick some prime \(p_n\) and let \(m = \lceil \sqrt{p_n} \rceil\), then \(m^2 < p_n < (m + 1)^2\) and by the standard Legendre conjecture there is at least one more prime in \(((m + 1)^2, (m + 2)^2)\). Then

\[
g_n := p_{n+1} - p_n < (m + 2)^2 - m^2 = 4m + 4 < 4\sqrt{p_n} + 4, \tag{4.4}
\]

which is the weak Andrica conjecture.

In view of what we have already seen for the Oppermann conjecture, we now see that the strong and weak Legendre conjectures are certainly verified for integers up to \(m \leq \lceil \sqrt{p_{81}} \rceil\). Since we do not yet know \(p_{81}^*\) the best explicit verification range is for integers \(m \leq \lceil \sqrt{p_{80}} \rceil \approx 4.285 \times 10^9\). Similarly, the strong and weak Brocard conjectures are certainly verified for primes up to \(p \leq \lceil \sqrt{p_{81}} \rceil\). Since we do not yet know \(p_{81}^*\) the best explicit verification range is for primes \(p \leq \lceil \sqrt{p_{80}} \rceil \approx 4.285 \times 10^9\).

\section{Discussion}

While it is reasonably well known that the Andrica, Oppermann, Legendre, and Brocard conjectures are closely related, little work seems to have gone into using these interrelations to find suitably large regions where these conjectures can all be verified to be true. By setting up a strong version of the Andrica conjecture one can easily demonstrate that all of the strong, standard, and weak Andrica conjectures are certainly valid for primes \(p < 1.836 \times 10^{19}\). By proving that the strong Andrica conjecture implies the Oppermann conjecture, which in turn implies the strong and standard Legendre conjectures, and the strong and standard Brocard conjectures, one can demonstrate that the Oppermann, and strong and standard Legendre conjectures, are likewise certainly all valid for primes \(p < 1.836 \times 10^{19}\), corresponding to integers \(m \leq 4.285 \times 10^9\). Similarly the strong and weak Brocard conjectures are certainly valid for primes \(p \leq 4.285 \times 10^9\). The key item in these bounds is the location of the highest-known maximal prime gap, so updating these bounds will be automatic as new maximal prime gaps are identified.

In counterpoint, what would it take for all of these conjectures to suddenly fail at the next opportunity, the 81st maximal prime gap? One would need \(g_{81}^* > \sqrt{p_{81}} + \frac{1}{4} > \sqrt{p_{80}} \sim 4.285 \times 10^9\). That is, since \(g_{80} = 1550\), one would need the next maximal prime gap to suddenly change from order \(10^3\) to order \(10^9\). While, given current knowledge, this cannot be entirely ruled out — it does at the very least look suggestively unlikely.

Finally, while the strong, standard, and weak versions of the Andrica conjecture, (and the closely related Oppermann, Legendre, and Brocard conjectures) all impose con-
straints on the prime gaps of the form
\[ g_n := p_{n+1} - p_n = O\left(\sqrt{p_n}\right), \quad (5.1) \]
these are by no means the most stringent conjectures one might plausibly make. For instance, consider the following.

**Conjecture 9** (*Square root conjecture*)
*Except for \( p_n \in \{3, 7, 13, 23, 31, 113\} \), that is \( n \in \{2, 4, 6, 9, 11, 30\} \), one has
\[ g_n := p_{n+1} - p_n < p_n^{1/2}. \quad (5.2) \]

This is equivalent to asserting
\[ \sqrt{p_{n+1}} - \sqrt{p_n} < \sqrt{p_n} \left(\sqrt{1 + \frac{1}{\sqrt{p_n}}} - 1\right). \quad (5.3) \]
This square root conjecture can also be easily verified to certainly hold for all primes less than \( p_{81}^* \). The price paid here is that while the conjecture looks somewhat simpler when phrased in terms of prime gaps, (no \( \frac{1}{4} \) offset term), the statement in terms of \( \sqrt{p_{n+1}} - \sqrt{p_n} \) is more complicated and less “Andrica-like” in flavour. Note that
\[ \sqrt{p_n} \left(\sqrt{1 + \frac{1}{\sqrt{p_n}}} - 1\right) < \frac{1}{2}, \quad (5.4) \]
indeed
\[ \sqrt{p_n} \left(\sqrt{1 + \frac{1}{\sqrt{p_n}}} - 1\right) = \frac{1}{2} - \frac{1}{8\sqrt{p_n}} + O\left(\frac{1}{p_n}\right), \quad (5.5) \]
so this square root conjecture asymptotically approaches the strong Andrica conjecture for large primes. The current best unconditional result along these lines is the Baker–Harman–Pintz [31] result that for sufficiently large \( x \) the interval \([x - x^{0.525}, x]\) always contains primes — so that for sufficiently large primes \( p_{n+1} - p_n \leq O\left((p_n)^{0.525}\right) \). Note the exponent has not yet been (unconditionally) reduced to \( \frac{1}{2} \), and the implied constant in the phrase “sufficiently large” is still undetermined.

Furthermore, observe that in references [1–3] the author mentions the “open problem” as to whether
\[ \lim_{n \to \infty} \left(\sqrt{p_{n+1}} - \sqrt{p_n}\right) = 0? \quad (5.6) \]
If this limit exists, (and it is easy to see that \( \lim_{n \to \infty} \left( \sqrt{p_{n+1}} - \sqrt{p_n} \right) = 0 \), this is for instance a special case of the discussion in reference [21], so it is only the existence of the limit that is in question), then for any specified \( \epsilon > 0 \) the inequality \( \sqrt{p_{n+1}} - \sqrt{p_n} < \epsilon \) can be violated only a finite number of times. This observation can be linked back to the finite “exception list” we needed to invoke in setting up the strong Andrica conjecture.

Finally the Cramér conjecture [6] (and closely related conjectures such as the Firoozbakht conjecture [1–3, 32–35]) impose significantly stronger constraints on the prime gaps of the form

\[
g_n := p_{n+1} - p_n = O \left( (\ln p_n)^2 \right).
\]  

(5.7)

I will not say anything further regarding the Cramér and Firoozbakht conjectures in the current article, but hope to turn to this topic in future work.

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