Abstract

We consider the problem of firefighting to save a critical subset of nodes. The firefighting game is a turn-based game played on a graph, where the fire spreads to vertices in a breadth-first manner from a source, and firefighters can be placed on yet unburnt vertices on alternate rounds to block the fire. In this work, we consider the problem of saving a critical subset of nodes from catching fire, given a total budget on the number of firefighters.

We show that the problem is para-NP-hard when parameterized by the size of the critical set. We also show that it is fixed-parameter tractable on general graphs when parameterized by the number of firefighters. We also demonstrate improved running times on trees and establish that the problem is unlikely to admit a polynomial kernelization (even when restricted to trees). Our work is the first to exploit the connection between the firefighting problem and the notions of important separators and tight separator sequences.

Finally, we consider the spreading model of the firefighting game, a closely related problem, and show that the problem of saving a critical set parameterized by the number of firefighters is W[2]-hard, which contrasts our FPT result for the non-spreading model.

1 Introduction

The problem of Firefighting [17] formalizes the question of designing inoculation strategies against a contagion that is spreading through a given network. The goal is to come up with a strategy for placing firefighters on nodes in order to intercept the spread of fire. More precisely, firefighting can be thought of as a turn-based game between two players, traditionally the fire and the firefighter, played on a graph \(G\) with a source vertex \(s\). The game proceeds as follows.

1. At time step 0, fire breaks out at the vertex \(s\). A vertex on fire is said to be burned.
At every odd time step \( i \in \{1, 3, 5, \ldots \} \), when it is the turn of the firefighter, a firefighter is placed at a vertex \( v \) that is not already on fire. Such a vertex is permanently protected.

At every even time step \( j \in \{2, 4, 6, \ldots \} \), the fire spreads in the natural way: every vertex adjacent to a vertex on fire is burned (unless it was protected).

The game stops when the fire cannot spread any more. A vertex is said to be saved if there is a protected vertex on every path from \( s \) to \( v \). The natural algorithmic question associated with this game is to find a strategy that optimizes some desirable criteria, for instance, maximizing the number of saved vertices [4], minimizing the number of rounds, the number of firefighters per round [6], or the number of burned vertices [13, 4], and so on. These questions are well-studied in the literature, and while most variants are NP-hard, approximation and parameterized algorithms have been proposed for various scenarios. See the excellent survey [14] as well as references within for more details.

In this work, we consider the question of finding a strategy that saves a designated subset of vertices, which we shall refer to as the critical set. We refer to this problem as SAVING A CRITICAL SET (SACS) (we refer the reader to Section 2 for the formal definitions). This is a natural objective in situations where the goal is to save specific locations as opposed to saving some number of them. This version of the problem has been studied by [6, 18, 7] and is known to be NP-hard even when restricted to trees.

Our Contributions and Methodology. We initiate the study of SAVING A CRITICAL SET from a parameterized perspective. We first show that the problem is para-NP-hard when parameterized by the size of the critical set, by showing that SAVING A CRITICAL SET is NP-complete even on instances where the size of the critical set is one. It is already clear from known results that SAVING A CRITICAL SET is para-NP-hard also when parameterized by treewidth. A third natural parameter is the number of firefighters deployed to save the critical set. Our main result is that SAVING A CRITICAL SET is FPT when parameterized by the number of firefighters, although it is not likely to have a polynomial kernel.

Our FPT algorithm is a recursive algorithm that uses the structure of tight separator sequences. The notion of tight separator sequences was introduced in [19] and has several applications [16, 20, 21] (some of which invoke modified definitions). A tight separator sequence is, informally speaking, a sequence of minimal separators such that the reachability set of \( S_i \) is contained in the reachability set of \( S_{i+1} \). Note that any firefighting solution is a \( s - C \) separator, where \( s \) is the source of the fire, and \( C \) is the critical subset of vertices. We also obtain faster algorithms on trees by using important separators.

As is common with such approaches, we do not directly solve SACS, but an appropriately generalized form, which encodes information about the behavior of some solution on the “border” vertices, which in this case is the union of all the separators in the tight separator sequence.
Related Work.

The Firefighting problem has received much attention in recent years. It has been studied in the parameterized complexity setting [4, 7, 10, 2] but mostly by using the number of vertices burnt or saved as parameters. King et.al. [18] showed that for a tree of degree at most 3, it is NP-hard to save a critical set with budget of one firefighter per round, but is polynomial time when the fire starts from a vertex of degree at most 2. Chopin [7] extended the hardness result of [18] to a per-round budget \( b \geq 1 \) and to trees with maximum degree \( b + 3 \). Chalermsook et.al. [6] gave an approximation to the number of firefighters per round when trying to protect a critical set.

Anshelevich et.al. [1] initiated the study of the the spreading model, where the vaccination also spreads through the network. In Section 4 we study this problem in the parameterized setting.

2 Preliminaries

In this section, we introduce the notation and the terminology that we will need to describe our algorithms. Most of our notation is standard. We use \([k]\) to denote the set \([1, 2, \ldots, k]\), and we use \([k]_1\) and \([k]_2\), respectively, to denote the odd and even numbers in the set \([k]\).

Graphs, Important Separators and Tight Separator Sequences. We introduce here the most relevant definitions, and use standard notation pertaining to graph theory based on [9, 11]. All our graphs will be simple and undirected unless mentioned otherwise. For a graph \( G = (V, E) \) and a vertex \( v \), we use \( N(v) \) and \( N[v] \) to refer to the open and closed neighborhoods of \( v \), respectively. The distance between vertices \( u, v \) of \( G \) is the length of a shortest path from \( u \) to \( v \) in \( G \); if no such path exists, the distance is defined to be \( \infty \). A graph \( G \) is said to be connected if there is a path in \( G \) from every vertex of \( G \) to every other vertex of \( G \). If \( U \subseteq V \) and \( G \setminus U \) is connected, then \( U \) itself is said to be connected in \( G \). For a subset \( S \subseteq V \), we use the notation \( G \setminus S \) to refer to the graph induced by the vertex set \( V \setminus S \).

The following definitions about important separators and tight separator sequences will be relevant to our main FPT algorithm. We first define the notion of the reachability set of a subset \( X \) with respect to a subset \( S \).

**Definition 1 (Reachable Sets).** Let \( G = (V, E) \) be an undirected graph, let \( X \subseteq V \) and \( S \subseteq V \setminus X \). We denote by \( R_G(X, S) \) the set of vertices of \( G \) reachable from \( X \) in \( G \setminus S \) and by \( NR_G(X, S) \) the set of vertices of \( G \) not reachable from \( X \) in \( G \setminus S \). We drop the subscript \( G \) if it is clear from the context.

We now turn to the notion of an \( X-Y \) separator and what it means for one separator to cover another.

**Definition 2 (Covering by Separators).** Let \( G = (V, E) \) be an undirected graph and let \( X, Y \subseteq V \) be two disjoint vertex sets. A subset \( S \subseteq V \setminus (X \cup Y) \) is called an \( X-Y \) separator in \( G \) if \( R_G(X, S) \cap Y = \emptyset \), or in other words, there is no path from \( X \) to \( Y \) in the graph \( G \setminus S \). We denote by \( \lambda_G(X, Y) \) the size of the smallest \( X-Y \) separator in \( G \). An \( X-Y \) separator \( S_1 \) is said to cover an \( X-Y \) separator \( S \) with respect to \( X \) if \( R(X, S_1) \supset R(X, S) \). If the set \( X \) is clear from the context, we just say that \( S_1 \) covers \( S \). An \( X-Y \) separator is said to be inclusionwise minimal if none of its proper subsets is an \( X-Y \) separator.
We are now ready to recall the notion of tight separator sequences introduced in [19].

**Lemma 7.** [8] Let $G = (V, E)$ be an undirected graph and let $X, Y \subset V$ be two disjoint vertex sets. An $X - Y$ separator $S_1$ is said to dominate an $X - Y$ separator $S$ with respect to $X$ if $|S_1| \leq |S|$ and $S_1$ covers $S$ with respect to $X$. If the set $X$ is clear from the context, we just say that $S_1$ dominates $S$.

We finally arrive at the notion of important separators, which are those that are not dominated by any other separator:

**Definition 4 (Important Separators [8]).** Let $G = (V, E)$ be an undirected graph, $X, Y \subset V$ be disjoint vertex sets and $S \subseteq V \setminus (X \cup Y)$ be an $X - Y$ separator in $G$. We say that $S$ is an important $X - Y$ separator if it is inclusionwise minimal and there does not exist another $X - Y$ separator $S_1$ such that $S_1$ dominates $S$ with respect to $X$.

It is useful to know that the number of important separators is bounded as an FPT function of the size of the important separators.

**Lemma 5.** [8] Let $G = (V, E)$ be an undirected graph, $X, Y \subset V$ be disjoint vertex sets of $G$. For every $k \geq 0$ there are at most $4^k$ important $X - Y$ separators of size at most $k$. Furthermore, there is an algorithm that runs in time $O(4^k k(m + n))$ which enumerates all such important $X - Y$ separators, where $n = |V|$ and $m = |E|$.

We are now ready to recall the notion of tight separator sequences introduced in [19]. However, the definition and structural lemmas regarding tight separator sequences used in this paper are closer to that from [21]. Since there are minor modifications in the definition as compared to the one in [21], we give the requisite proofs for the sake of completeness.

**Definition 6.** Let $X$ and $Y$ be two subsets of $V(G)$ and let $k \in \mathbb{N}$. A tight $(X, Y)$-reachability sequence of order $k$ is an ordered collection $\mathcal{H} = \{H_0, H_1, \ldots, H_q\}$ of sets in $V(G)$ satisfying the following properties:

- $X \subseteq H_1 \subseteq V(G) \setminus N[Y]$ for any $0 \leq i \leq q$;
- $X = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_q$;
- for every $0 \leq i \leq q$, $H_i$ is reachable from $X$ in $G[H_i]$ and every vertex in $N(H_i)$ can reach $Y$ in $G - H_i$ (implying that $N(H_i)$ is a minimal $(X, Y)$-separator in $G$);
- $|N(H_i)| \leq k$ for every $1 \leq i \leq q$;
- $N(H_i) \cap N(H_j) = \emptyset$ for all $1 \leq i, j \leq q$ and $i \neq j$;
- For any $0 \leq i \leq q - 1$, there is no $(X, Y)$-separator $S$ of size at most $k$ where $S \subseteq H_{i+1} \setminus N[H_i]$ or $S \cap N[H_q] = \emptyset$ or $S \subseteq H_1$.

We let $S_i = N(H_i)$, for $1 \leq i \leq q$, $S_{q+1} = Y$, and $S = \{S_0, S_1, \ldots, S_q, S_{q+1}\}$. We call $S$ a tight $(X, Y)$-separator sequence of order $k$.

**Lemma 7.** (see for example [21]) There is an algorithm that, given an $n$-vertex $m$-edge graph $G$, subsets $X, Y \subset V(G)$ and an integer $k$, runs in time $O(kmn^2)$ and either correctly concludes that there is no $(X, Y)$-separator of size at most $k$ in $G$ or returns the sets $H_0, H_1, H_2 \setminus H_1, \ldots, H_q \setminus H_{q-1}$ corresponding to a tight $(X, Y)$-reachability sequence $\mathcal{H} = \{H_0, H_1, \ldots, H_q\}$ of order $k$. 

If $X = \{x\}$ is a singleton, then we abuse notation and refer to a $x - Y$ separator rather than a $\{x\} - Y$ separator. A separator $S_1$ dominates $S$ if it covers $S$ and is not larger than $S$ in size:
We are now ready to define the parameterized problem that is the focus of this work. If there is no X-S separator of size at most k, we stop and return the set R(X, S) as the only set in a tight (X, Y)-reachability sequence. Otherwise, we recursively compute a tight (X, S)-reachability sequence \( P = \{P_0, \ldots, P_r\} \) of order k and define \( \Omega = \{P_0, \ldots, P_r, R(X, S)\} \) as a tight (X, Y)-reachability sequence of order k. It is straightforward to see that all the properties required of a tight (X, Y)-reachability sequence are satisfied. Finally, since the time required in each step of the recursion is \( O(kmn) \) and the number of recursions is bounded by \( n \), the number of vertices, the claimed running time follows.

**Proof.** The algorithm begins by checking whether there is an X-Y separator of size at most k. If there is no such separator, then it simply outputs the same. Otherwise, it uses the algorithm of Lemma 5 to compute an arbitrary important X-Y separator \( S \) of size at most k such that there is no X-Y separator of size at most k that covers \( S \).

Although the algorithm of Lemma 5 requires time \( O(4^k k(m+n)) \) to enumerate all important X-Y separators of size at most k, one important separator of the kind described in the previous paragraph can in fact be computed in time \( O(kmn) \) by the same algorithm.

**Saving a Critical Set.** We now turn to the definition of the firefighting problem. The game proceeds as described earlier: we are given a graph \( G \) with a vertex \( s \in V(G) \). To begin with, the fire breaks out at \( s \) and vertex \( s \) is burning. At each step \( t \geq 1 \), first the firefighter protects one vertex not yet on fire - this vertex remains permanently protected - and the fire then spreads from burning vertices to all unprotected neighbors of these vertices. The process stops when the fire cannot spread anymore. In the definitions that follow, we formally define the notion of a firefighting strategy.

**Definition 8.** [Firefighting Strategy] A k-step firefighting strategy is defined as a function \( h : [k] \to V(G) \). Such a strategy is said to be valid in \( G \) with respect to \( s \) if, for all \( i \in [k] \), when the fire breaks out in \( s \) and firefighters are placed according to \( h \) for all time steps up to \( 2(i-1) - 1 \), the vertex \( h(i) \) is not burning at time step \( 2i - 1 \), and the fire cannot spread anymore after timestep \( 2k \). If \( G \) and \( s \) are clear from the context, we simply say that \( h \) is a valid strategy.

**Definition 9 (Saving C).** For a vertex \( s \) and a subset \( C \subseteq V(G) \setminus \{s\} \), a firefighting strategy \( h \) is said to save \( C \) if \( h \) is a valid strategy and \( V(G) \setminus \cup_{i=1}^{k} h(i) \) is a \( \{s\} \)-C separator in \( G \), in other words, there is no path from \( s \) to any vertex in \( C \) if firefighters are placed according to \( h \).

We are now ready to define the parameterized problem that is the focus of this work.

| SAVING A CRITICAL SET (SACS) | Parameter: k |
|------------------------------|--------------|
| Input: An undirected n-vertex graph \( G \), a vertex \( s \), a subset \( C \subseteq V(G) \setminus \{s\} \), and an integer \( k \). | Question: Is there a valid k-step strategy that saves \( C \) when a fire breaks out at \( s \)? |

**Parameterized Complexity.** We follow standard terminology pertaining to parameterized algorithms based on the monograph [9]. Here we define a known technique to prove kernel lower bounds, called cross composition. Towards this, we first define polynomial equivalence relations.
Definition 10 (polynomial equivalence relation [3]). An equivalence relation \( \mathcal{R} \) on \( \Sigma^* \), where \( \Sigma \) is a finite alphabet, is called a polynomial equivalence relation if the following holds: (1) equivalence of any \( x, y \in \Sigma^* \) can be checked in time polynomial in \(|x| + |y|\), and (2) any finite set \( S \subseteq \Sigma^* \) has at most \( \max_{x \in S} |x|^{O(1)} \) equivalence classes.

Definition 11 (cross-composition [3]). Let \( L \subseteq \Sigma^* \) and let \( Q \subseteq \Sigma^* \times \mathbb{N} \) be a parameterized problem. We say that \( L \) cross-composes into \( Q \) if there is a polynomial equivalence relation \( \mathcal{R} \) and an algorithm which, given \( t \) strings \( x_1, x_2, \ldots, x_t \) belonging to the same equivalence class of \( \mathcal{R} \), computes an instance \((x^*, k^*) \in \Sigma^* \times \mathbb{N}\) in time polynomial in \( \sum_{i=1}^t |x_i| \) such that: (i) \((x^*, k^*) \in Q \iff x_i \in L \) for some \( 1 \leq i \leq t \) and (ii) \( k^* \) is bounded by a polynomial in \( \max_{1 \leq i \leq t} |x_i| + \log t \).

The following theorem allows us to rule out the existence of a polynomial kernel for a parameterized problem.

Theorem 12 ([3]). If an NP-hard problem \( L \subseteq \Sigma^* \) has a cross-composition into the parameterized problem \( Q \) and \( Q \) has a polynomial kernel then \( NP \subseteq \text{coNP}/\text{poly} \).

3 The Parameterized Complexity of Saving a Critical Set

In this section, we describe the FPT algorithm for saving a critical set and our cross-composition construction for trees. The starting point for our FPT algorithm is the fact that every solution to an instance \((G, s, C, k)\) of SACS is in fact a \( s \cdot C \) separator of size at most \( k \). Although the number of such separators may be exponential in the size of the graph, it is a well-known fact that the number of important separators is bounded by \( 4^k n^{O(1)} \) \[\text{[3]}\]. For several problems, one is able to prove that there exists a solution that is in fact an important separator. In such a situation, an FPT algorithm is immediate by guessing the important separator.

In the SACS problem, unfortunately, there are instances where none of the solutions are important separators. However, this approach turns out to be feasible if we restrict our attention to trees, leading to improved running times. This is described in greater detail in Section 3.2. Further, in Section 3.3, we also show that we do not expect SACS to admit a polynomial kernel under standard complexity-theoretic assumptions. We establish this by a cross-composition from SACS itself, using the standard binary tree approach, similar to [2].

We describe our FPT algorithm for general graphs in Section 3.1. This is an elegant recursive procedure that operates over tight separator sequences, exploiting the fact that a solution can never be contained entirely in the region “between two consecutive separators”. Although the natural choice of measure is the solution size, it turns out that the solution size by itself cannot be guaranteed to drop in the recursive instances that we generate. Therefore, we need to define an appropriate generalized instance, and work with a more delicate measure. We now turn to a detailed description of our approach.

We note that the SACS problem is para-NP-complete when parameterized by the size of the critical set, by showing that the problem is already NP-complete when the critical set has only one vertex.

Theorem 13. SACS is NP-complete even when the critical set has one vertex.

Proof. Let \((G, k)\) be an instance of \( k \) – CLIQUE problem. We construct a graph \( G' \) as follows. For each edge \((u, v) \in E(G)\) create a node \( s_{uv} \). We denote this set of nodes by \( E \). For each
node $v \in V(G)$ create a node $v \in G'$ and denote this set of nodes by $V$. Add two nodes $s$ and $t$, where $s$ is the node on which the fire starts and $t \in C$ (CRITICAL SET), the node to be saved. Connect $t$ to all the nodes in set $E$. Connect $s$ to each node $v \in V$ by a path of length $k$. Let us refer these nodes, which are on the paths from $s$ to $V$, as $V_{-1}$. Create $(m - \binom{k}{2}) - 1$ copies of set $V$. Denote these copies as $V_1, V_2, ..., V_{m-\binom{k}{2}-1}$. Let $V_m = \cup_{i=1}^{m-\binom{k}{2}-1} V_i$ and $V$ be denoted by $V_0$. Add an edge $(v_i, v_j)$ for $v_i \in V_l$ and $v_j \in V_{j+1}$ for all $0 \leq j < (m - \binom{k}{2} - 1)$. For each edge $e = (u, v)$ in $E(G)$ add an edge $(u, s_{u,v})$ and $(v, s_{u,v})$ where $u, v \in V_{m-\binom{k}{2}-1}$ and $s_{u,v} \in E$. Now set $k' = k + m - \binom{k}{2}$.

Lemma 14. At most $k$ firefighters can be placed on the nodes in set $(V_{-1} \cup V_0)$ in $G'$.

Proof. As per the construction of $G'$, each node $v \in V_0$ is connected to $s$ with a path of length $k$ and each node $u \in V_{-1}$ is at a distance of length less than $k$. Thus, there are at most $k$ time steps at which firefighters can be employed on the nodes in $V_{-1} \cup V_0$.

Lemma 15. Protecting less than $k$ nodes from the set $H_k = (V_{-1} \cup V_0 \cup V_m)$ in $G'$ is not a successful strategy to save $t$.

Proof. To save $t$, all the nodes in $E (|E| = m)$ must either be protected or saved. Suppose we protect $\gamma < k$ nodes in set $H_k$. Observe that, protecting $\gamma$ nodes in $H_k$ can save at most $\binom{\gamma}{2}$ nodes in $E$. Therefore, in order to save $t$ we need to protect/save remaining $m - \binom{\gamma}{2}$ nodes in $E$. Given the constraint on the budget as well as the number of time instances we are left with, we can protect $m - \binom{\gamma}{2} + (k - \gamma)$ more nodes to save $t$. If $\alpha = m - \binom{\gamma}{2} + (k - \gamma)$ and $\beta = m - \binom{\gamma}{2}$ then, the claim is $\beta - \alpha > 0$.

\[
\beta - \alpha = m - \binom{\gamma}{2} - \left( m - \binom{k}{2} + (k - \gamma) \right)
\]

\[
= \binom{k}{2} - \binom{\gamma}{2} + (\gamma - k) = \frac{k(k - 1)}{2} - \frac{\gamma(\gamma - 1)}{2} + (\gamma - k)
\]

\[
= \frac{k^2 - k - \gamma^2 + \gamma}{2} + (\gamma - k)
\]

\[
= \frac{(k + \gamma)(k - \gamma) - (k - \gamma)}{2} + (\gamma - k)
\]

\[
= (k - \gamma) \left[ \frac{k + 1}{2} - \frac{1}{2} - 1 \right] > 0
\]

Therefore, it is not possible to save $t$ if we protect $\gamma < k$ nodes in $H_k$, as this requires $m - \binom{\gamma}{2}$ more nodes to be protected, which is not feasible.

We claim that SAVING A CRITICAL SET with $|C| = 1$ on $(G', k', s, C = t)$ is a YES-instance if and only if $k - \text{CLIQUE}$ is a YES-instance in $(G, k)$.

Suppose $G$ has a $k - \text{clique}$ denoted as $K$. Then the firefighting strategy is to protect the vertices $v_i \in V_0$ corresponding to the vertices $v_i \in K$. Protecting these $k$ vertices guarantees to save $\binom{k}{2}$ vertices in the set $E$. Also, from Lemma 14 it follows that protecting these $k$
vertices is valid with respect to time constraints. The remaining \( m - \binom{k}{2} \) in \( E \) can be protected at each allowed time step after placing the \( k \) firefighters in \( V_0 \). This as well is valid with respect to the time constraints as the set of nodes \( E \) are at a distance \( m - \binom{k}{2} \) from the set of nodes \( V_0 \).

Suppose that \( G' \) has a valid firefighting strategy \( S = \{u_1, u_2, \ldots, u_{k+m-\binom{k}{2}}\} \) with \( k + m - \binom{k}{2} \) firefighters. If a firefighter is placed on any node \( u \in V_{-1} \) or on any node \( w \in V_m \), then it is equivalent to placing a firefighter on the node \( v_i \in V_0 \) to which the nodes \( u \) and \( w \) have a path. Therefore, in the firefighting strategy \( S \) if there is a firefighter on node \( u \in V_{-1} \) then it is pushed to \( v_i \in V_0 \) such that \( u - v_i \) is a path, and if there is a firefighter on node \( w \in V_m \), then it is pulled back to \( v_i \in V_0 \) such that \( w - v_i \) is a path. Also, it follows from Lemma 15 that for a successful firefighting strategy, \( k \) firefighters must be placed in \( H_k = (V_{-1} \cup V_0 \cup V_m) \).

Therefore, these \( k \) nodes in \( V_0 \) on which the firefighters are placed (by pushing from \( u \in V_{-1} \) or pulling from \( w \in V_m \)), must form a clique in \( G \) with the \( \binom{k}{2} \) edges corresponding to the nodes saved in the set \( E \). If these \( k \) nodes do not form a clique, then with the remaining \( m - \binom{k}{2} \) nodes it won't be possible to save/cover all the vertices in \( E \).

\[ \square \]

### 3.1 The FPT Algorithm

Towards the FPT algorithm for SACS, we first define a generalized firefighting problem as follows. In this problem, in addition to \((G, s, C, k)\), we are also given the following:

- \( P \cup Q \subseteq [2k]_0 \), a set of available time steps,
- \( Y \subseteq V(G) \), a subset of predetermined firefighter locations, and
- a bijection \( \gamma : Q \rightarrow Y \), a partial strategy for \( Q \).

The goal here is to find a valid partial \( k \)-step firefighting strategy over \((P \cup Q)\) that is consistent with \( \gamma \) on \( Q \) and saves \( C \) when the fire breaks out at \( s \). We assume that no firefighters are placed during the time steps \([2k]_0 \setminus (P \cup Q)\). For completeness, we formally define the notion of a valid partial firefighting strategy over a set.

\[ \text{Definition 16 (Partial Firefighting Strategy). A partial} \ k\text{-step firefighting strategy on} \ X \subseteq [2k]_0 \text{is defined as a function} \ h : X \rightarrow V(G) \text{. Such a strategy is said to be valid in} \ G \text{ with respect to} \ s \text{if, for all} \ i \in X \text{, when the fire breaks out in} \ s \text{and firefighters are placed according to} \ h \text{for all time steps upto} [i-1]_0 \cap X \text{, the vertex} \ h(i) \text{is not burning at time step} i \text{. If} \ G \text{and} \ s \text{are clear from the context, we simply say that} \ h \text{is a valid strategy over} \ X \text{.} \]

What it means for partial strategy to save \( C \) is also analogous to what it means for a strategy to save \( C \). The only difference here is that we save \( C \) despite not placing any firefighters during the time steps \( j \) for \( j \in [2k]_0 \setminus X \).

\[ \text{Definition 17 (Saving} \ C \text{with a Partial Strategy). For a vertex} \ s \text{and a subset} \ C \subseteq V(G) \setminus \{s\} \text{, a partial firefighting strategy} \ h \text{over} \ X \text{is said to save} \ C \text{if} \ h \text{is a valid strategy and} \cup_{i \in X} h(i) \text{is a} \ s - C \text{separator in} \ G \text{, in other words, there is no path involving only burning vertices from} \ s \text{to any vertex in} \ C \text{if the fire starts at} \ s \text{and firefighters are placed according to} \ h \text{.} \]

The intuition for considering this generalized problem is the following: when we recurse, we break the instance \( G \) into two parts, say subgraphs \( G' \) and \( H \). An optimal strategy for \( G \)
We now describe our algorithm for solving an instance \( I \). We use \( p \) we compute a tight sequence associated with \( S \). More precisely, if \( h \) is a valid partial firefighting strategy over \( P \cup Q \) that is consistent with \( \gamma \) on \( Q \) and that saves \( C \) when a fire breaks out at \( s \), then the algorithm returns \( \text{YES} \) or \( \text{NO} \) as appropriate.

If \( p > 0 \) and \( s \) is already separated from \( C \) in \( G \setminus Y \), then we return \( \text{YES} \), since any arbitrary partial strategy over \( P \cup Q \) that is consistent with \( \gamma \) on \( Q \) is a witness solution.

If we have a non-trivial instance, then our algorithm proceeds as follows. To begin with, we compute a tight \( s - C \) separator sequence of order \( p \) in \( G \setminus Y \). Recalling the notation of Definition 6, we use \( S_0, \ldots, S_{q+1} \) to denote the separators in this sequence, with \( S_0 \) being the set \( \{s\} \) and \( S_{q+1} = C \). We also use \( W_0, W_1, \ldots, W_q, W_{q+1} \) to denote the reachability regions between consecutive separators. More precisely, if \( \mathcal{F} \) is the tight \( s - C \) reachability sequence associated with \( \delta \), then we have:
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\[ W_i := H_i \setminus N[H_{i-1}] \text{ for } 1 \leq i \leq q, \]

while \( W_{q+1} \) is defined as \( G \setminus (N[H_q] \cup C) \). We will also frequently employ the following notation:

\[ S = \bigcup_{i=1}^{q} S_i \text{ and } W = \bigcup_{i=1}^{q+1} W_i. \]

This is a slight abuse of notation since \( S \) is also used to denote the sequence \( S_0, \ldots, S_{q+1} \), but the meaning of \( S \) will typically be clear from the context.

We first observe that if \( q > k \), the separator \( S_q \) can be used to define a valid partial firefighting strategy. The intuition for this is the following: since every vertex in \( S_q \) is at a distance of at least \( k \) from \( s \), we may place firefighters on vertices in \( S_q \) in any order during the available time steps. Since \( |S_q| \leq p \) and \( S_q \) is a \( s - C \) separator, this is a valid solution. Thus, we have shown the following:

\[ \text{Lemma 18. If } G \text{ admits a tight } s - C \text{ separator sequence of order } q \text{ in } G \setminus Y \text{ where } q > k, \text{ then } J \text{ is a YES-instance.} \]

Therefore, we return \text{YES} if \( q > k \) and assume that \( q \leq k \) whenever the algorithm proceeds to the next phase.

This concludes the pre-processing stage.

**Phase 1 — Recursion.** Our first step here is to guess a partition of the set of available time steps, \( P \), into \( 2q + 1 \) parts, denoted by \( A_0, \ldots, A_q, A_{q+1} \) and \( B_1, \ldots, B_{q+1} \). The partition of the time steps represents how a solution might distribute the timings of its firefighting strategy among the sets in \( S \) and \( W \). The set \( A_i \) denotes our guess of \( \bigcup_{v \in S_i} h^{-1}(v) \) and \( B_j \) denotes our guess of \( \bigcup_{v \in W_j} h^{-1}(v) \). Note that the number of such partitions is \((2q + 1)^p \leq (2k + 1)^{k}\). We define \( g_0(k) := (2k + 1)^k \). We also use \( T_1(P) \) to denote the partition \( A_0, \ldots, A_q \) and \( T_2(P) \) to denote \( B_0, \ldots, B_{q+1} \).

We say that the partition \((T_1(P), T_2(P))\) is non-trivial if none of the \( B_i \)'s are such that \( B_i = P \).

Our algorithm only considers non-trivial partitions — the reason this is sufficient follows from the way tight separator sequences are designed, and this will be made more explicit in Lemma 20 in due course.

Next, we would like to guess the behavior of a partial strategy over \( P \) restricted to \( S \). Informally, we do this by associating a signature with the strategy \( h \), which is a labeling of the vertex set with labels corresponding to the status of a vertex in the firefighting game when it is played out according to \( h \). Every vertex is labeled as either a vertex that had a firefighter placed on it, a burned vertex, or a saved vertex. The labels also carry information about the earliest times at which the vertices attained these statuses. More formally, we have the following definition.

\[ \text{Definition 19. Let } h \text{ be a valid } k \text{-step firefighting strategy (or a partial strategy over } X). \]

The signature of \( h \) is defined as a labeling \( L_h \) of the vertex set with labels from the set:
\[ \mathcal{L} = (\{f\} \times X) \cup (\{b\} \times [2k\varepsilon]) \cup \{p\}, \]

where:

\[ \mathcal{L}_b(v) = \begin{cases} (f, t) & \text{if } h(t) = v, \\ (b, t) & \text{if } t \text{ is the earliest time step at which } v \text{ burns}, \\ p & \text{if } v \text{ is not reachable from } s \text{ in } G \setminus (\{h(i) \mid i \in [2k\varepsilon]\}) \end{cases} \]

We use array-style notation to refer to the components of \( \mathcal{L}(v) \), for instance, if \( \mathcal{L}(v) = (b, t) \), then \( \mathcal{L}(v)[0] = b \) and \( \mathcal{L}(v)[1] = t \). The algorithm begins by guessing the restriction of \( \mathcal{L}_b \) on \( S \), that is, it loops over all possible labelings:

\[ \mathcal{T} : S \to (\{f\} \times P) \cup (\{b\} \times [2k\varepsilon]) \cup \{p\}. \]

The labeling \( \mathcal{T} \) is called legitimate if, for any \( u \neq v \), whenever \( \mathcal{T}(u)[0] = \mathcal{T}(v)[0] = f \), we have \( \mathcal{T}(u)[1] \neq \mathcal{T}(v)[1] \). We say that a labeling \( \mathcal{T} \) over \( S \) is compatible with \( \mathcal{T}_1(P) = (A_0, \ldots, A_q) \) if we have:

- for all \( 0 \leq i \leq r \), if \( v \in S_i \) and \( h(v)[0] = f \), then \( h(v)[1] \in A_i \).
- for all \( 0 \leq i \leq r \), if \( t \in A_i \), there exists a vertex \( v \in S_i \) such that \( h^{-1}(f, t) = v \).

The algorithm considers only legitimate labelings compatible with the current choice of \( \mathcal{T}_1(P) \). By Lemma 18, we know that any tight \( s - C \) separator sequence considered by the algorithm at this stage has at most \( k \) separators of size at most \( p \) each. Therefore, we have that the number of labelings considered by the algorithm is bounded by \( g_1(k) := (p + k + 1)^{(kp)} \leq (3k)^{O(k^2)} \leq k^{O(k^2)} \).

We are now ready to split the graph into \( q + 1 \) recursive instances. For \( 1 \leq i \leq q + 1 \), let us define \( G_i = G[S_{i-1} \cup W_i \cup S_i \cup Y] \). Also, let \( \mathcal{T}_i := \mathcal{T} |_{V(G_i) \cap S} \). Notice that when using \( G_i \)'s in recursion, we need to ensure that the independently obtained solutions are compatible with each other on the non-overlapping regions, and consistent on the common parts. We force consistency by carrying forward the information in the signature of \( h \) using appropriate gadgets, and the compatibility among the \( W_i \)'s is a result of the partitioning of the time steps.

Fix a partition of the available time steps \( P \) into \( \mathcal{T}_1(P) \) and \( \mathcal{T}_2(P) \), a compatible labeling \( \mathcal{T} \) and \( 1 \leq i \leq q + 1 \). We will now define the SACS-R instance \( J(i, \mathcal{T}_1(P), \mathcal{T}_2(P), \mathcal{T}_i) \). Recall that \( J = (G, s, C, k, g, P, Q, Y, \gamma) \). To begin with, we have the following:

- Let \( X_i := A_{i-1} \cup A_i \) and let \( P_i = B_i \).
- Let \( Q_i := X_i \cup Q \) and \( Y_i := Y \cup X_i \). We define \( \gamma_i \) as follows:

\[ \gamma_i(t) = \begin{cases} \gamma(t) & \text{if } t \in Q, \\ v & \text{if } t \in X_i \text{ and } \mathcal{T}_i(v) = (f, t) \end{cases} \]

Note that \( \gamma_i \) is well-defined because the labeling was legitimate and compatible with \( \mathcal{T}_1(P) \). We define \( H_i \) to be the graph \( \chi(G_i, \mathcal{T}_i) \), which is described below.
To begin with, $V(H_i) = V(G_i) \cup \{s^*, t^*\}$

Let $v \in V(G_i)$ be such that $\Gamma_i(v)[0] = b$. Use $\ell$ to denote $\Gamma_i(v)[1]/2$. Now, we do the following:

- Add $k + 1$ internally vertex disjoint paths from $s^*$ to $v$ of length $\ell + 1$, in other words, these paths have $\ell - 1$ internal vertices.
- Add $k + 1$ internally vertex disjoint paths from $v$ to $g$ of length $k - \ell - 1$.

Let $v \in V(G_i)$ be such that $\Gamma_i(v) = p$. Add an edge from $v$ to $t^*$.

Also, the strategy is well-defined, since $h$ steps. Next, we will demonstrate that $h$ is indeed a valid strategy that saves $I$ instance in the former case, let $h[I] = \gamma$. Add an edge from $v$ to $t^*$.

We also make $k + 1$ copies of the vertices $t^*$ and all vertices that are labeled either burned or saved. This ensures that no firefighters are placed on these vertices.

For $1 \leq i \leq q + 1$, the instance $J(i, T_1(P), T_2(P), \Gamma_i)$ is now defined as $(\chi(G_i, \Gamma_i), s^*, C = \{t^*\}, k, g, P_i, Q_i, Y_i, \gamma_i)$.

**Phase 2 — Merging.** Our final output is quite straightforward to describe once we have the $h[I]_i$, $i$'s. Consider a fixed partition of the available time steps $P$ into $T_1(P)$ and $T_2(P)$, and a labeling $\Gamma$ of $S$ compatible with $T_1(P)$. If all of the $(q + 1)$ instances $J(i, T_1(P), T_2(P), \Gamma_i), 1 \leq i \leq q + 1$ return YES, then we also return YES, and we return NO otherwise. Indeed, in the former case, let $h[i, T_1(P), T_2(P), \Gamma]$ denote a valid partial firefighting strategy for the instance $J(i, T_1(P), T_2(P), \Gamma_i)$. We will show that $h^*$, described as follows, is a valid partial firefighting strategy that saves $C$.

- For the time steps in $Q$, we employ firefighters according to $\gamma$.
- For the time steps in $T_1(P)$, we employ firefighters according to $\Gamma$. This is a well-defined strategy since $\Gamma$ is a compatible labeling.
- For all remaining time steps, i.e., those in $T_2(P) = \{B_1, \ldots, B_{q+1}\}$, we follow the strategy given by $h[i, T_1(P), T_2(P), \Gamma]$.

It is easily checked that the strategy described above agrees with $h[i, T_1(P), T_2(P), \Gamma]$ for all $i$. Also, the strategy is well-defined, since $T_1(P)$ and $T_2(P)$ form a partition of the available time steps. Next, we will demonstrate that $h^*$ is indeed a valid strategy that saves $C$, and also analyze the running time of the algorithm.

**Algorithm 1:** Solve-SACS-R($J$)

**Input:** An instance $(G, s, C, k, g, P, Q, Y, \gamma)$, $p := |P|$

**Result:** YES if $J$ is a YES-instance of SACS-R, and NO otherwise.

1. if $p = 0$ and $s$ and $C$ are in different components of $G \setminus Y$ then return YES;
2. else return NO;
3. if $p > 0$ and $s$ and $C$ are in different components of $G \setminus Y$ then return YES;
4. if there is no $s - C$ separator of size at most $p$ then return NO;
5. Compute a tight $s - C$ separator sequence $S$ of order $p$.
6. if the number of separators in $S$ is greater than $k$ then return YES;
7. else
8. for a non-trivial partition $T_1(P), T_2(P)$ of $P$ into $2q + 1$ parts do
9. for a labeling $\Gamma$ compatible with $T_1(P)$ do
10. if $\bigwedge_{i=1}^{q+1} (\text{Solve-SACS-R}(J(i, T_1(P), T_2(P), \Gamma_i)))$ then return YES;
11. return NO
Correctness of the Algorithm

We first show that the quantity \( p \) always decreases when we recurse.

\[ \text{Lemma 20.} \quad \text{Let } J' := J(i, I_1(P), I_2(P), \bar{\mathcal{I}}) = (\chi(G_i, \bar{\mathcal{I}}), s^*, C = \{t^*,k,g,p_i,Q_i,Y_i,\gamma_i\}) \text{ be an arbitrary but fixed instance constructed by } \text{Solve-SACS-R}(J). \text{ Then, } |P_i| < |P|. \]

\[ \text{Proof.} \quad \text{The claim follows from the fact that } P_i = B_i, \text{ and } (I_1(P), I_2(P)) \text{ is a non-trivial partition of } P. \]

\[ \text{Lemma 21.} \quad \text{If } J \text{ is a } \text{YES instance of } \text{SACS-R, then our algorithm returns } \text{YES}. \]

\[ \text{Proof.} \quad \text{(Sketch.)} \quad \text{The statement follows by induction on } p, \text{ which allows us to assume the correctness of the output of the recursive calls. The correctness of the base cases is easily checked. If } J \text{ is a } \text{YES instance admits a solution } h, \text{ then it induces a partition } I_1(P), I_2(P) \text{ of } P. \text{ We argue that this is always a non-trivial partition and is therefore considered by the algorithm. Indeed, suppose not. This would imply that } h \text{ places all its firefighters in } W_i \text{ for some } 1 \leq i \leq q + 1. \text{ However, this implies that } D := \cup_{t \in P} h(t) \subseteq W_i \text{ is a } s - C \text{ separator in } G \text{ that is entirely contained in } W_i, \text{ which contradicts the definition of a tight } s - C \text{ separator sequence.} \]

Now, define the labeling \( \mathcal{I} \) by projecting the signature of \( h \) on \( S \). Clearly, this labeling is compatible with \( I_1(P) \) (by definition). We claim that all the instances \( J(i, I_1(P), I_2(P), \bar{\mathcal{I}}) \) are \( \text{YES instances.} \) Indeed, it is easy to check that the projection of \( h \) on the subgraph \( G_i \) is a valid solution for the instance \( J(i, I_1(P), I_2(P), \bar{\mathcal{I}}) \). It follows that the algorithm returns \( \text{YES} \) since these \((q + 1)\) instances return \( \text{YES} \) by the induction hypothesis.

For the remainder of our discussion on correctness, our goal will be to prove the following reverse claim.

\[ \text{Lemma 22.} \quad \text{If the output of the algorithm is } \text{YES, then } J \text{ is a } \text{YES instance of } \text{SACS-R.} \]

This lemma is shown by establishing that \( h^* \) is indeed a valid solution for \( \text{SACS}. \) We arrive at this conclusion by a sequence of simple claims. We fix the accepting path in the algorithm, that is, an appropriate partition of \( P \) and the labeling \( \mathcal{I} \) on \( S \) that triggered \( \text{YES} \) output. Our first claim says that the recursive instances respect the behavior dictated by the labeling that they were based on.

\[ \text{Lemma 23.} \quad \text{Let } 1 \leq i \leq q + 1 \text{ and } v \in S \cap H_i. \text{ If } \mathcal{I}_i(v) = (b,t), \text{ then in any partial strategy employed on the instance } J(i, I_1(P), I_2(P), \mathcal{I}_i) \text{ that saves the critical set, the vertex } v \text{ burns exactly at time step } t. \]

\[ \text{Proof.} \quad \text{Let } h \text{ be an arbitrary but fixed valid partial strategy for } J(i, I_1(P), I_2(P), \mathcal{I}_i) \text{ that saves the critical set. Assume that } v \text{ burns with respect to } h. \text{ Let } t' \text{ be the earliest time step at which } v \text{ burns with respect to } h'. \text{ The graph } \chi(G_i, \mathcal{I}_i) \text{ contains } (k + 1) \text{ vertex-disjoint paths from the source to } v \text{ of length } t/2, \text{ which ensures that } t' \leq t. \text{ However, if } t' < t, \text{ then the vertex } g \text{ catches fire at time step } 2k - 1 \text{ because of the } (k + 1) \text{ vertex-disjoint paths of length } k - t/2 - 1 \text{ that are present from } v \text{ to } g. \text{ This implies that } t' \text{ cannot be saved if } v \text{ burns earlier than } t. \text{ The other case is that } v \text{ does not burn with respect to } h'. \text{ The only way for this to happen is if a firefighter is placed on } v, \text{ however, since the instance has } k + 1 \text{ copies of } v, \text{ we have that at least one copy of } v \text{ burns, and the claim follows.} \]
Our next claim is that vertices that are labeled saved never burn with respect to $h^\ast$.

**Lemma 24.** For any $v \in S$, if $\Xi(v) = p$, then $v$ does not burn with respect to $h^\ast$.

**Proof.** We prove this by contradiction. Let $P$ be a path from $s$ to $v$ where all vertices on $P$ are burning. Let $v'$ be the first vertex on this path which is such that $\Xi(v') = p$, and let $u$ be the last vertex on this path which is before $v'$ and that intersects a separator. Observe that $u$ is well-defined unless $v' \in S_1$, which is a special case that we will deal with separately. Note that $u$ must be a vertex that is labeled as burned (by the choice of $v'$). Therefore, the path from $u$ to $v$ is present in a recursive instance, where we know that $v'$ is adjacent to a critical vertex, which contradicts the fact that we defined $h^\ast$ based on valid strategies that save critical sets in the recursive instances. If $v' \in S_1$ and $u$ is not well-defined, then observe that there is a direct path from $s$ to $v'$ in the first recursive instance, and the same argument applies.

We finally show, over the next two claims, that the function $h^\ast$ is a valid strategy that saves the critical set.

**Lemma 25.** The function $h^\ast$ is a valid partial strategy over $(P \cup Q)$ for the instance $J$.

**Proof.** We prove this by contradiction, and we also assume that Lemma 22 holds for all recursive instances (by induction on $p$). If $h^\ast$ is not a valid strategy, then there exists some $1 \leq i \leq q + 1$ for which there is a vertex $v \in W_i$ and a time step $t \in B_i$ such that $h^\ast(t) = v$ and $v$ is burning at time step $t$. Consider the path $P$ from $s$ to $v$. Let $u$ be the last vertex on the path $P$ that intersects $S_{i-1} \cup S_i$. Observe that $u$ is a vertex that is either labeled by $(b, t)$ or $p$. This leads to the following two scenarios:

- In the first case, the part of the path from $u$ to $v$ is present in the instance $J(i, \mathcal{T}_1(P), \mathcal{T}_2(P), \Xi_i)$, because of the agreement of $h^\ast$ and $h[i, \mathcal{T}_1(P), \mathcal{T}_2(P), \Xi]$ and Lemma 23. Therefore, $h[i, \mathcal{T}_1(P), \mathcal{T}_2(P), \Xi]$ was not a valid firefighting strategy.
- The second situation implies that a vertex labeled protected is burning in $J$ when $h^\ast$ is employed, which contradicts Lemma 24.

**Lemma 26.** The partial strategy $h^\ast$ saves the set $C$ in the instance $J$.

**Proof.** (Sketch.) The proof of this is similar to the proof of Lemma 25. Any burning path $P$ from $s$ to any vertex in $C$ must intersect $S_q$. We let $v$ be the last vertex from $S_q$ on $P$, and observe that $v$ must have a $(b, t)$. Therefore, the path from $v$ to the vertex in $C$ exists in $J(q + 1, \mathcal{T}_1(P), \mathcal{T}_2(P), \Xi_i)$, and also burns in the same fashion, since $h^\ast$ agrees with $h[q + 1, \mathcal{T}_1(P), \mathcal{T}_2(P), \Xi]$. Again, this contradicts the accuracy of the algorithm on the recursive instance.

The proof of Lemma 22 now follows from Lemmas 25 and 23.
Running Time Analysis

We show that our algorithm runs in time $f(k)p^2O(n^2m)$, where $f(k) = k^{O(k)}$. Indeed, observe that the running time of the algorithm is governed by the following recurrence:

$$T(n, m, k, p) \leq O(n^2mp) + (p + k + 1)^kp \sum_{i=1}^{q+1} T(n_i, m_i, k, p_i),$$

where the term $(p + k + 1)^kp$ denotes an upper bound on the product of the total number of non-trivial partitions of $P$ into $(2q + 1)$ parts and the total number of legitimate labelings of $S$ compatible with $T_1(P)$. The first term accounts for checking the base cases and the running time of computing a tight separator sequence of order $p$ (see Lemma 7). Since $p$ is always at most $k$, we rewrite the recurrence as

$$T(n, m, k, p) \leq O(n^2mp) + f(k)p^{q+1} \sum_{i=1}^{q+1} T(n_i, m_i, k, p_i),$$

where $f(k) = k^{O(k)}$. Furthermore, observe that $\sum_{i=1}^{q+1} n_i \leq n + pk$. The first inequality follows from the fact that every vertex in $S$ appears in at most two recursive instances. The first occurrence is counted in $n$, while all the second occurrences combined amount to at most $pk$.

Finally, recall that $p_i < p$ by Lemma 20 and $p \leq k$ by definition (since $P \subseteq [2k]$). Therefore, the depth of the recursion is bounded by $p$, and the time spent at each level of recursion is proportional to $k^{O(k)p}n^2m$. This implies the claimed running time.

Based on the analysis above and Lemmas 21 and 22, we have thus demonstrated the main result of this section.

\textbf{Theorem 27.} SACS is FPT and has an algorithm with running time $f(k)O(nm^2)$, where $f(k) = k^{O(k^3)}$.

3.2 A Faster Algorithm For Trees

In this section we consider the setting when the input graph $G$ is a tree. WLOG, we consider the vertex $s$ to be the root of the tree. We first state an easy claim that shows that WLOG, we can consider the critical set to be the leaves. The proof of the following lemma follows from the fact that the firefighting solution has to be a $s-C$ separator.

\textbf{Lemma 28.} When the input graph $G$ is a tree, if there exists a solution to SACS, there exists a solution such that all firefighter locations are on nodes that are on some path from $s$ to $C$.

Given the above claim, our algorithm to construct a firefighting solution is the following—exhaustively search all the important $s-C$ separators that are of size $k$. For each vertex $v$ in a separator $Y$, we place firefighters on $Y$ in the increasing order of distance from $s$ and check whether this is a valid solution. The following lemma claims that if there exists a firefighting solution, the above algorithm will return one.
Lemma 29. Solving the SACS problem for input graphs that are trees takes time \(O^*(4^k)\).

Proof. Using Lemma 28, it is enough to consider the subtree \(T\) that contains nodes only on \(s - C\) paths. The critical set \(C\) is then a subset of the leaves of \(T\). Suppose \(Y \subset V(T)\) contains the locations for a solution to the firefighter problem. WLOG, \(Y\) is an important \(s - C\) separator. Consider \(I\) which is an important separator that dominates \(Y\). Clearly, \(|I| \leq |Y| \leq k\), and \(I\) is also a minimal \(s - C\) separator.

For each \(x \in I\), define \(S_x\) to be the set of \(y \in Y\) such that \(y\) lies on the (unique) \(s - x\) path. Note that each \(y \in Y\) must belong to some \(s - C\) path, and all \(s - C\) paths have some node \(x \in I\). Furthermore, \(R(s, Y) \subseteq R(s, I)\), which means that each \(y \in Y\) lies on some \(s - I\) path. It follows that \(Y \subseteq \bigcup_{x \in I} S_x\). Finally, by the minimality of \(Y\), each \(S_x\) satisfies \(|S_x| \leq 1\), since otherwise we could remove one of the nodes of \(S_x\) from \(Y\). Hence,

\[|Y| \leq \big| \bigcup_{x \in I} S_x \big| \leq \sum_{x \in I} |S_x| \leq |I| \leq |Y|\]

Thus \(\big| \bigcup_{x \in I} S_x \big| = \sum_{x \in I} |S_x|\), and it follows that \(S_x \cap S_u = \emptyset\) for all \(x \neq u\). From \(\sum_{x \in I} |S_x| = |I|\) and \(|S_x| \leq 1\) for each \(x\) it also follows that for each \(x\), \(|S_x| = 1\).

We then design a firefighting solution using \(I\) in the following manner. For each node \(x \in I\), the firefighter in location \(x\) is placed whenever the original solution (using \(Y\)) placed a firefighter on the unique node in \(S_x\). Since the node in \(S_x\) is closer to \(s\) than \(x\), the location \(x\) is still available at this step. Hence this is a valid strategy. Note that if there is any valid placement ordering for \(I\), then the placement ordering according to increasing distance from \(s\) is also valid.

The claim then follows by noting that enumerating all the important separators of size at most \(k\) takes time \(O^*(4^k)\).

3.3 No Polynomial Kernel, Even on Trees

Given that there is a FPT algorithm for SACS when restricted to trees, in this section we show that SACS on trees has no polynomial kernel. As mentioned before, the proof technique used here is on the similar lines of the proof showing no polynomial kernel for SAVING ALL BUT \(k\) VERTICES by Bazgan et. al.[2].

Theorem 30. SACS when restricted to trees does not admit polynomial kernel, unless \(NP \subseteq coNP/poly\).

To prove this theorem, we will use the Definitions (10, 11) mentioned in section 2. We use Theorem 12, for which we consider SACS on trees as analogous to language \(L\), which is shown to be NP-complete when the critical set \(C\) is the set of all leaves [18]. First we give a lemma that we will be using in the proof.

Lemma 31. For a given instance of SACS \((T, r, C, k)\), where \(T\) is a full binary tree with height \(h\) and \(k = h\), if more than one vertex is protected at a depth \(d \leq h\), then more than one leaf burns.
Proof. Consider a case when more than one firefighter is placed at a depth $d \leq h$. It is easy to see that at most 1 firefighter can be placed at depth 1. Therefore, more than 1 firefighter will always be at depth $d \geq 2$. Also, at any depth $d \geq 1$, there are $2^d$ nodes, which is strictly greater than the number of odd time steps at which the firefighters are allowed to be placed. This says that, all the nodes at a particular depth $d \geq 1$ cannot be protected by firefighters. Therefore, at any depth $d \geq 2$, there are at least two subtrees which are unprotected and can be burnt. And thus, given the constraint that at most one firefighter can be placed at any allowed time step, there is at least one subtree which is burnt and can never be completely protected/saved by the firefighters.

However, one of the strategies to let only one leaf burn, is to fix a path from root to a leaf and keep protecting the siblings of the nodes on that path. This will be more clearer as we go ahead in the proof of the current theorem.

Lemma 32. The unparameterized version of SACS restricted to trees cross composes to SACS restricted to trees when parameterized by the number of firefighters.

Proof. We take an appropriate equivalence relation $R$ such that $R$ puts all the malformed instances into one class and all the well formed instances are grouped into equivalence classes according to the number of vertices of the tree, and number of firefighters (parameter $k$) required to save the critical set $C$. We assume that we are given a sequence of $t$ instances $(T_i, s_i, C_i, k)_{i=1}^{t}$ of the unparameterized version of SACS restricted to trees, each rooted at $s_i$. Note that each of the $t$ instances belong to the same equivalence class i.e. for all the instances $k$ is same. Also, consider that $t = 2^h$, for some $h \geq 1$, else, we duplicate some instances and the duplication at most doubles the number of instances.

Using these $t$ instances, we create a new full binary tree $T'$ as follows. Let $T'$ be rooted at $s'$, and $h$ be the height of $T'$ ($2^h = t$). For each leaf $i \in \{1, \ldots, t\}$, replace the $i^{th}$ leaf by the instance $(T_i, s_i, C_i, k)$ and now, set $k' = k + h = k + \log t$. Observe that, for $T'$ the set of all leaves is the union of the leaves of the instances $T_i$ i.e. $\cup_{i=1}^{t} C_i$.

To prove the correctness, we show that the tree $T'$ formed by the composition of $t$ instances saves all the leaves with $k'$ firefighters if and only if there exits at least one instance $T_i$ for $i \in \{1, 2, \ldots, t\}$, that saves its critical set $C_i$ (i.e. the set of all leaves) with $k$ firefighters.

Suppose, $T'$ has a successful firefighting strategy. Then, from lemma 31, it follows that, the firefighting strategy that saves all but one root $s_i$ of $T_i$ is the one that protects exactly one vertex at each depth of the tree. This costs exactly $\log(t) = h$ firefighters. Thus, after $2h$ time steps, there is exactly one vertex $s_i$ (root of instance $T_i$) which is on fire. And the critical set $C_i$ for the instance $(T_i, s_i, C_i, k)$ is saved using $k$ firefighters.

Now suppose, there is an instance $(T_i, s_i, C_i, k)$ that has a successful firefighting strategy with $k$ firefighters. Thus, the leaves $C_i$ of this instance are saved by the firefighting strategy with $k$ firefighters. Now, the goal is to save all other leaves $(\cup_{i=1}^{t} C_i) \setminus C_i$. $T'$ being a tree, there is an unique path from the root $s$ to the node $s_i$ (i.e. the root of the instance $T_i$). Denote the path as $P = \{s, v_1, v_2, \ldots, v_{\log(t)}−1, s_i\}$. Note that each node $v_j$ is at depth $j$ and $s_i$ is at depth $\log(t)$. Let $u_1, u_2, \ldots, u_{\log(t)}−1, s_j$ be the siblings of the nodes $v_1, v_2, \ldots, v_{\log(t)}−1, s_i$ on the path $P$ respectively. Now, the firefighting strategy that protects the sibling $u_j$ of node $v_j$ at time step $2j − 1$ and sibling $s_j$ of the node $s_i$ at time step $2h − 1 = 2\log(t) − 1$ saves all other leaves of $T'$.
The Spreading Model

The spreading model for firefighters was defined by Anshelevich et al. [1] as “Spreading Vaccination Model”. In contrast to the firefighting game described in Section 1, in the spreading model, the firefighters (vaccination) also spread at even time steps as similar to that of the fire. That is, at any even time step if there is a firefighter at node \( v_i \), then the firefighter extends (vaccination spreads) to all the neighbors of \( v_i \) which are not already on fire or are not already protected by a firefighter. Consider a node \( v_i \) which is not already protected or burning at time step \( 2j \). If \( u_i \) and \( w_i \) are neighbors of \( v_i \), such that, \( u_i \) was already burning at time step \( 2j - 1 \) and \( w_i \) was protected at time step \( 2j - 1 \), then at time step \( 2j \), \( v_i \) is protected. That is, in the spreading model the firefighters dominate or win over fire. For the spreading model, the firefighting game can be defined formally as follows:

- At time step 0, fire breaks out at the vertex \( s \). A vertex on fire is said to be burned.
- At every odd time step \( i \in \{1, 3, 5, \ldots \} \), when it is the turn of the firefighter, a firefighter is placed at a vertex \( v \) that is not already on fire. Such a vertex is permanently protected.
- At every even time step \( j \in \{2, 4, 6, \ldots \} \), first the firefighter extends to every adjacent vertex to a vertex protected by a firefighter (unless it was already protected or burned), then the fire spreads to every vertex adjacent to a vertex on fire (unless it was already protected or burned). Needless to say, the vertices protected at even time steps are also permanently protected.

In the following theorem, we show that in spite of the spreading power that the firefighters have, SACS is hard.

\[ \textbf{Theorem 33.} \text{ In the spreading model, SACS is as hard as } k\text{-DOMINATING SET}. \]

\[ \text{Proof.} \text{ Let } (G, k) \text{ be an instance of } k\text{-DOMINATING SET problem. We construct a graph } G' \text{ as follows. Add 2 copies } V^{(1)} \text{ and } V^{(2)} \text{ of the nodes } V(G) \text{ in } G', \text{ i.e. for each node } v_i \in V(G) \text{ add nodes } v_i^{(1)} \text{ and } v_i^{(2)}. \text{ Add a vertex } s, \text{ the vertex from where the fire breaks out. For each vertex } v_i^{(1)} \in V^{(1)}, \text{ add a path of length } k \text{ from } s \text{ (i.e. a path from } s \text{ to } v_i^{(1)} \text{ would be like } (s, u_{i_1}, \ldots, u_{i_{k-1}}, v_i^{(1)})). \text{ Similarly, for each edge } (v_i, v_j) \in E(G), \text{ add a path of length } k \text{ from } v_i^{(1)} \text{ to } v_j^{(2)}, \text{ and a path of length } 2k \text{ from } v_i^{(2)} \text{ to } v_j^{(2)} \text{ in } G'. \text{ Also, for each } v_i \in V(G) \text{ add a path of length } 2k \text{ from } v_i^{(1)} \text{ to } v_i^{(2)}. \text{ Let } C = V^{(2)} \text{ be the critical set and } k' = k. \]

We claim that, SACS on \((G', s, k', C = V^{(2)})\) is a Yes-instance if and only if \( k\text{-DOMINATING SET} \) is a Yes instance on \((G, k)\).

Suppose that, G has a \( k\text{-DOMINATING SET} \) \( K \). Then the strategy that saves the critical set \( C \) is the one that protects the vertices \( v_i^{(1)} \in V^{(1)} \) corresponding to the vertices \( v_i \) in the dominating set \( K \). The nodes \( v_i \in K \) being the dominating set, the corresponding nodes \( v_i^{(1)} \in V^{(1)} \) dominate all the nodes \( v_i^{(2)} \in V^{(2)} \). And thus, the firefighters on the nodes \( v_i^{(1)} \in V^{(1)} \) corresponding to the dominating set \( K \), eventually extend firefighters to all the nodes in \( V^{(2)} \) before fire reaches to any node \( v_i^{(2)} \in V^{(2)} \). Also, protecting the vertices \( v_i^{(1)} \in V^{(1)} \) corresponding to \( v_i \in K \) is a valid firefighting strategy as per the Definition 8.

Suppose that \( h : [k] \rightarrow V(G') \) is an optimal firefighting strategy for \((G', s, k', C = V^{(2)})\). Let \( S = \{u_1, \ldots, u_k\} \) be the set of vertices which are protected by the firefighting strategy, i.e. \( h(i) = u_i \text{ for } i \in [k]. \) Note that, as per the definition of the firefighting game, \( i\text{th} \) firefighter is placed at time step \( 2i - 1 \). Let's denote the paths from the nodes in \( V^{(1)} \) to the nodes in \( V^{(2)} \) as...
P_q for q ∈ [1, 2m + n]. Observe that, an optimal firefighting strategy would not have (1) two distinct firefighters u_i and u_j for i ≠ j on the same path P_q and (2) two distinct firefighters u_i and u_j for i ≠ j on the paths P_q and P_q that are incident on the same node v_i^{(1)} ∈ V^{(1)}. As for both the conditions, it is better to have one firefighter on the path v_i^{(1)} ∈ V^{(1)} on which the paths incident. Hence, we can assume that at most one firefighter is placed on any path P_q and at most one firefighter is placed on the paths incident on vertex v_i^{(1)}. Moreover, if a firefighter is placed at any node p_i on a path P_q between v_i^{(1)} ∈ V^{(1)} and v_j^{(2)} ∈ V^{(2)}, then, in 2k time steps it can extend protections to none other than v_i^{(1)} ∈ V^{(1)}, v_j^{(2)} ∈ V^{(2)}, and the intermediate nodes on the path P_q. It can be said that any node on a path P_q has exactly one vertex v_i^{(1)} ∈ V^{(1)} within in a diameter of 2k. Thus, placing a firefighter on any path P_q is equivalent to placing a firefighter on the vertex v_i^{(1)} on which the path P_q is incident. Therefore, in the firefighting strategy S, if u_i is a node on a path P_q which is between v_i^{(1)} ∈ V^{(1)} and v_j^{(2)} ∈ V^{(2)}, we push back the firefighter to v_i^{(1)} ∈ V^{(1)}. As the fire reaches V^{(1)} in 2k time steps, there can be at most k nodes v_i^{(1)} ∈ V^{(1)} on which the firefighters are placed at the allowed time steps. Therefore, there is at least one vertex v_i^{(1)} ∈ V^{(1)} that is burnt. And thus, if the firefighters are not placed on the vertices v_i^{(1)} ∈ V^{(1)} that form a dominating set in G, then there is at least one path to a node v_i^{(2)} ∈ V^{(2)} at which the firefighters cannot extend protection.

Proof. Let (G, k) be an instance of k-DOMINATING SET problem. We construct a graph G’ as follows. Add 2 copies V^{(1)} and V^{(2)} of the nodes V(G) in G’, i.e. for each node v_i ∈ V(G) add nodes v_i^{(1)} and v_i^{(2)}. Add a vertex s, the vertex from where the fire breaks out. For each vertex v_i^{(1)} ∈ V^{(1)}, add a path of length k from s (i.e. a path from s to v_i^{(1)} would be like (s, u_{i1}, . . . , u_{ik−1}, v_i^{(1)})). Similarly, for each edge (v_i,v_j) ∈ E(G), add a path of length k from v_i^{(1)} to v_j^{(2)}, and a path of length 2k from v_i^{(2)} to v_j^{(1)} in G’. Also, for each v_i ∈ V(G) add a path of length 2k from v_i^{(1)} to v_i^{(2)}. Let C = V^{(2)} be the critical set and k’ = k.

We claim that, SACS on (G’, s, k’, C = V^{(2)}) is a YES-instance if and only if k-DOMINATING SET is a YES instance on (G, k).

Suppose that, G has a k-dominating set K. Then the strategy that saves the critical set C is the one that protects the vertices v_i^{(1)} ∈ V^{(1)} corresponding to the vertices v_i in the dominating set K. The nodes v_i ∈ K being the dominating set, the corresponding nodes v_i^{(1)} ∈ V^{(1)} dominate all the nodes v_i^{(2)} ∈ V^{(2)}. And thus, the firefighters on the nodes v_i^{(1)} ∈ V^{(1)} corresponding to the dominating set K, eventually extend firefighters to all the nodes in V^{(2)} before fire reaches to any node v_j^{(2)} ∈ V^{(2)}. Also, protecting the vertices v_i^{(1)} ∈ V^{(1)} corresponding to v_i ∈ K is a valid firefighting strategy as per the Definition 8.

Suppose that h : [k] → V(G’) is an optimal firefighting strategy for (G’, s, k’, C = V^{(2)}). Let S = {u_{i1}, . . . , u_{ik}} be the set of vertices which are protected by the firefighting strategy, i.e. h[i] = u_i for i ∈ [k]. Note that, as per the definition of the firefighting game, i^{th} firefighter is placed at time step 2i − 1. Let’s denote the paths from the nodes in V^{(1)} to the nodes in V^{(2)} as P_q for q ∈ [1, 2m + n]. Observe that, an optimal firefighting strategy would not have (1) two distinct firefighters u_i and u_j for i ≠ j on the same path P_q and (2) two distinct firefighters u_i and u_j for i ≠ j on the paths P_q and P_q that are incident on the same node v_i^{(1)} ∈ V^{(1)}. As for both the conditions, it is better to have one firefighter on the node v_i^{(1)} ∈ V^{(1)} on which the paths incident. Hence, we can assume that at most one firefighter is placed on any path P_q and at most one firefighter is placed on the paths incident on vertex v_i^{(1)}. Moreover,
if a firefighter is placed at any node $p_i$ on a path $P_{pq}$ between $v^{(1)}_i \in V^{(1)}$ and $v^{(2)}_j \in V^{(2)}$, then, in $2k$ time steps it can extend protections to none other than $v^{(1)}_i, v^{(2)}_j$, and the intermediate nodes on the path $P_{pq}$. It can be said that any node on a path $P_{pq}$ has exactly one vertex $v^{(1)}_i \in V^{(1)}$ within a diameter of $2k$. Thus, placing a firefighter on any path $P_{pq}$ is equivalent to placing a firefighter on the vertex $v^{(1)}_i$ on which the path $P_{pq}$ is incident. Therefore, in the firefighting strategy $S$, if $u_i$ is a node on a path $P_{pq}$ which is between $v^{(1)}_x \in V^{(1)}$ and $v^{(2)}_y \in V^{(2)}$, we push back the firefighter to $v^{(1)}_x \in V^{(1)}$. As the fire reaches $V^{(1)}$ in $2k$ time steps, there can be at most $k$ nodes $v^{(1)}_i \in V^{(1)}$ on which the firefighters are placed at the allowed time steps. Therefore, there is at least one vertex $v^{(1)}_i \in V^{(1)}$ that is burnt. And thus, if the firefighters are not placed on the vertices $v^{(1)}_i \in V^{(1)}$ that form a dominating set in $G$, then there is at least one path to a node $v^{(2)}_j \in V^{(2)}$ at which the firefighters cannot extend protection.

5 Summary and Conclusions

In this work we presented the first FPT algorithm, parameterized by the number of firefighters, for a variant of the Firefighter problem where we are interested in protecting a critical set. We also presented a faster algorithms on trees. In contrast, we also show that in the spreading model protecting a critical set is W[2]-hard. Our algorithms exploit the machinery of important separators and tight separator sequences. We believe that this opens up an interesting approach for studying other variants of the Firefighter problem.

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