Internal wave boluses as coherent structures in a continuously stratified fluid

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Abstract

Ocean internal waves transport energy and momentum, but generally material transport is a second-order effect. One exception to this occurs when shoaling internal waves break and form boluses, materially coherent vortices that trap and transport material up the continental slope with the wave. The global extent of bolus transport is unknown. Previous studies of bolus formation primarily focused on systems consisting of two layers of uniform density, which do not incorporate the presence of pycnoclines of finite thickness in the oceans. Our numerical simulations model the density stratification with hyperbolic tangent profiles to demonstrate the impact of the pycnocline on the bolus. We use a spectral clustering method to identify boluses as Lagrangian coherent structures. Our method quantifies bolus transport by determining the material advected by the bolus. The size and displacement upslope of the bolus are examined as a function of the pycnocline thickness, incoming wave energy, density change in the pycnocline, and topographic slope. The dependence of bolus transport on the pycnocline thickness demonstrates that boluses in continuous stratifications tend to be larger and transport material further than corresponding two-layer stratifications. The results revealed by the present study can lead to a better understanding of bolus-induced transport of particles and biota in the oceans.

1 Introduction

Temperature and salinity variations stratify the oceans (i.e. density increases with depth), which enables the propagation of density disturbances via internal waves. These waves are often generated by tides passing over the ocean’s topography or as a result of surface storms [Alford, 2003; Wunsch and Ferrari, 2004] and play an important role in transferring energy and momentum throughout the ocean [Munk and Wunsch, 1998]. Internal waves are particularly prominent in the pycnocline, the region of sharp transition between the upper mixed layer and the deep ocean, where the density gradients are strongest. These waves have amplitudes from tens to hundreds of meters [Duda et al., 2004; Susanto et al., 2005; Helfrich and Melville, 2006], and they can travel hundreds to thousands of kilometers from their sources [Osborne and Burch, 1980; Ray and Mitchum, 1996] before breaking on the continental slope [Troy and Koseff, 2005; Lamb, 2014]. The turbulence resulting from internal wave breaking plays an important role in ocean mixing and energy dissipation [Sandstrom and Oakey, 1995; Inall et al., 2000; Moum et al., 2003], but the transport induced is not fully known.

The propagation of internal waves above the continental slope towards the coastline causes wave shoaling and ultimately results in the breaking of the internal wave. This wave focusing can result in the formation of a bolus, a moving vortex capable of transporting water and suspended particulate up the slope [Helfrich and Melville, 1986; Helfrich, 1992; Venayagamoorthy and Fringer, 2007; Fructus et al., 2009; Aghsaee et al., 2010; Lamb, 2014]. Boluses have been observed in the ocean [Carter et al., 2005; Moum et al., 2007; Lamb and Farmer, 2011; Walter et al., 2012; Alford et al., 2015], and in some cases can span half the water column height [Klymak and Moum, 2003].

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It is not known whether boluses transport a significant amount of bio-matter or sediments up the continental slope globally, but it has been proposed that the upwelling and turbulent mixing supported by this phenomenon could be vital for transporting nutrient-rich fluid into coastal ecosystems [Wang et al. 2007; Klymak and Mouni 2003]. On the Scotian Shelf, for example, mixing resulting from internal tide breaking is believed to act as a nutrient pump from the deeper waters to the euphotic zone [Sandstrom and Elliott 1984]. In the South China Sea, pilot whales are known to track and follow internal waves to forage on prey aggregations [Moore and Lien 2007].

It has also been hypothesized that shoaling internal waves are effective agents for sediment resuspension because the wave destabilization and subsequent turbulence drive sediments out of the bottom boundary layer and further up into the water column [Hosegood et al. 2004; Quaresma et al. 2007; Stastna and Lamb 2008; Bourgault et al. 2014], and strong turbulent mixing and sediment resuspension associated to the run-up of internal wave boluses have been observed in the Otsuchi Bay [Masunaga et al. 2015]. An estimate of the influence of boluses on the transport of nutrients or their impact on resuspension, however, remains to be determined.

To better understand the breaking process and potential transport by boluses, laboratory experiments have been conducted to study incident internal waves on a slope [Cacchione and Wunsch 1974; Helfrich 1992; Dauxois et al. 2004], as well as bolus formation and propagation in approximately two-layer stratified systems [Helfrich 1992; Michallet and Ivey 1999; Boegman et al. 2005; Sutherland et al. 2013; Moore et al. 2016]. The two-layer systems used in these experiments correspond to an idealized ocean stratification. In the two-layer system, solitary waves or wave trains propagate along the pycnocline and shoal onto a constant slope topography, potentially generating boluses. Helfrich [1992] investigated the interaction of a solitary wave of depression with a sloping bottom and described the break up of the incoming wave into several boluses. Using a similar system, Michallet and Ivey [1999] described the breaking of large-amplitude internal waves, identifying the first sign of breaking as a gravitational instability. Using particle image velocimetry to get the velocity field, they visualized the vortex created at the breaking and suggested that this mechanism could sweep material originally close to the slope far offshore.

While experiments provide reliable measurements and allow for a description of the physics of the breaking process and bolus propagation, numerical simulations furnish a more complete representation of the bolus, the breaking mechanism and their characteristics [Lamb 2003; Legg and Adcroft 2003; Venayagamoorthy and Fringer 2006, 2007; Aghaee et al. 2010; Arthur and Fringer 2014, 2016; Arthur et al. 2017; Venayagamoorthy and Fringer 2007] performed two and three-dimensional laboratory-scale numerical simulations in a linearly stratified system, to study the formation of nonlinear boluses produced at a shelf break and the bolus propagation onshelf. They demonstrated that the three-dimensional bolus structure is stable in the transverse direction, therefore justifying the use of two-dimensional simulations as an accurate representation of the bolus dynamics. More recently, Arthur and Fringer [2016] investigated transport due to breaking internal waves on slopes by incorporating particle-tracking to three-dimensional simulations, demonstrating onshore and offshore transport within the bolus, as well as lateral particle transport away from the topography due to turbulence developed by the breaking. Masunaga et al. [2017] modeled sediment resuspension due to boluses with a transport equation for suspended sediment, obtaining resuspension processes that are in good agreement with observational data and investigating the effect of varying the topographic slope. Fringer and Street [2003] and Arthur et al. [2017] depart from the two-layer stratification to investigate the impact of a finite pycnocline thickness on mixing, dissipation and turbulence due to breaking internal waves on a slope.

While previous work on bolus characterization has considered either two layers of constant density or a linearly stratified fluid, no previous study has analyzed the impact of the pycnocline thickness on the generation of boluses and the resulting transport. From salinity and temperature measurements taken during the World Ocean Circulation Experiment at over 18,000 stations worldwide, we found that an improved description of the upper ocean stratification can be achieved by using a hyperbolic tangent profile [Maderich et al. 2004] and incorporating the parameter $\delta$, the pycnocline thickness, which describes the thickness of the density transition layer. The limiting cases of high and low $\delta$ correspond to the linearly stratified and two-layer density models, respectively. Figure 1 presents the potential density profiles $\rho_0(z)$ for four stations within 300 km of the coastline, where boluses are prone to form. The density profiles in figure 1 have values of $\delta$ ranging from 50 m in the Labrador Sea, south of Greenland (figure 1a), to 459 m in the East China Sea (figure 1f), where the profile is nearly linear. Even for
Figure 1: Potential density profiles near the continental shelf obtained from the World Ocean Circulation Experiment Hydrographic Program. (a)-(d) Observational data (black dots) and the best fit line using a hyperbolic tangent function (color line). The pycnocline thickness $\delta$ describes the width of the transition in density. (e) The geolocation of each density profile is identified with a marker corresponding to the color of the best fit line: (a) red, (b) green, (c) blue, and (d) yellow.

the thinnest transition identified, the two-layer density model does not provide a close approximation of the observed profile. These examples highlight the fact that stratification profiles, and in particular the pycnocline thicknesses, vary worldwide with the geographical location and the season [Aikman III, 1984; Liu et al., 2001; Sigman et al., 2004], and such variation may have an impact on the formation and propagation of boluses.

Another essential approach for understanding the impact of boluses is the quantification of bolus transport characteristics, which requires an objective definition of the bolus itself. In two-layer systems, the bolus can be naturally visualized as a propagating front of denser fluid, but such a definition is not straightforward in continuously stratified systems. For a linear stratification, Venayagamoorthy and Fringer [2007] propose two ways to define the bolus speed from the density field: the speed at which the bolus front travels as observed via isopycnals, and the speed minimizing the time rate of change of the density field in a co-moving reference frame. Both methods describe the bolus dynamics from an Eulerian perspective, directly from the density field, and therefore do not necessarily represent transport of fluid elements. Arthur and Fringer [2016] more accurately quantify transport by incorporating a particle-tracking model to the simulations. However, the question of how much of the resulting transport is due to boluses or to the general breaking dynamics remains undetermined.

In this work, we identify boluses as elliptic Lagrangian coherent structures [Froyland and Padberg-Gehle, 2015; Allshouse and Peacock, 2015], which are regions of the fluid that do not significantly mix with the rest of the domain. These objectively defined structures identify materially coherent vortices in the Lagrangian frame [Haller and Beron-Vera, 2013; Haller et al., 2016; Serra et al., 2017]. This characterization provides a precise description of the phenomenon from a transport perspective because it exclusively captures the material transported by the bolus. We will identify boluses by applying a clustering algorithm to the trajectories of passive tracers that are advected by the breaking dynamics,
based on the method presented by Hadjighasem et al. [2016], which can identify vortex-like coherent structures.

In this paper, we investigate the impact of the pycnocline thickness on the dynamics and transport properties of internal wave boluses. We use coherent structures to quantify the transport properties of internal wave boluses produced, and our numerical simulations demonstrate the dependence of bolus transport on pycnocline thickness, the incoming wave energy, the density change in the pycnocline, and the topographic slope. The computational approach and a sample simulation of a bolus forming as the internal wave breaks on a constant slope is presented in §2. The characterization of the bolus from the Lagrangian coherent structure perspective, the transport metrics, and a comparison of the results between two- and three-dimensional models are presented in §3. The dependence of bolus characteristics to the stratification, wave properties and topography are presented in §4. Finally, the conclusions, potential applications, and possible extensions of this work are discussed in §5.

2 Numerical model

This section discusses the numerical simulations of the internal wave breaking on a constant slope and the resulting formation and propagation of boluses. In §2.1, the governing equations, numerical domain, system forcing, relevant system parameters and measured quantities are presented. The breaking dynamics and bolus propagation up the slope for a sample simulation are illustrated in §2.2 from an Eulerian perspective.

2.1 Computational approach, domain and setup

The Navier-Stokes equations in an inertial frame, using the Boussinesq approximation for an inhomogeneous, incompressible fluid subject to gravity along the vertical direction \( z \) are used to simulate the laboratory-scale system. The stability of the bolus structure in the transverse direction as it propagates up the slope [Venayagamoorthy and Fringer, 2007] gives credence to using the two-dimensional model. However, we will verify the coherent structure analysis of a three-dimensional simulation produces a similar bolus to the two-dimensional case in §3.3. The Boussinesq approximation neglects effects of density variation except in the buoyancy term, and is appropriate for buoyancy-driven flows with weak relative density variations around a reference value \( \rho_{00} \). This approximation has been extensively used in ocean models and in two-layer density systems with small relative density change [Long, 1965; Helfrich and Melville, 2006; Pedlosky, 2013]. The system of equations is

\[
\nabla \cdot \mathbf{u} = 0, \tag{1}
\]

\[
\partial_t u + (\mathbf{u} \cdot \nabla) u = -\frac{1}{\rho_{00}} \partial_z p + \nu \nabla^2 u, \tag{2}
\]

\[
\partial_t w + (\mathbf{u} \cdot \nabla) w = -\frac{1}{\rho_{00}} \partial_z p + \nu \nabla^2 w - \frac{\rho}{\rho_{00}} g, \tag{3}
\]

\[
\partial_t \rho + (\mathbf{u} \cdot \nabla) \rho = \kappa \nabla^2 \rho, \tag{4}
\]

where \( \mathbf{u} = (u, w) \) is the velocity field, \( p \) is the pressure, \( \rho \) is the local density, \( \rho_{00} = 1000 \text{ kg/m}^3 \) is the reference density, \( \nu = 1 \cdot 10^{-6} \text{ m}^2/\text{s} \) is the kinematic viscosity and \( \kappa = 1.4 \cdot 10^{-7} \text{ m}^2/\text{s} \) is the thermal diffusivity for sea water [Kunze, 2003], and \( g = 9.8 \text{ m/s}^2 \) is the gravity acceleration. It is assumed in \( \rho \) that density diffusion in the system is uniquely driven by thermal diffusion.

The system of equations \( \text{[1]-[4]} \) is solved numerically for \( u, w, p \) and \( \rho \). The numerical simulations are performed using CDP 2.4, an unstructured, finite-volume based, large eddy simulation code that implements a fractional-step time-marching scheme [Ham and Iaccarino, 2004; Mahesh et al., 2004]. All subgrid-scale modelings are turned off, and the code is modified to include the buoyancy term in \( \text{[3]} \) and solve \( \text{[4]} \) along with \( \text{[1]-[3]} \). This code has previously been used to simulate internal waves and has been validated with experiments [King et al., 2009; Dettner et al., 2013; Lee et al., 2014; Paoletti et al., 2014; Zhang and Swinney, 2014; Allshouse et al., 2016; Lee et al., 2018].

An illustration of the domain is presented in figure 2. The system dimensions correspond to an available system used for experimental studies [Allshouse et al., 2016], with similar length scales as in previous experimental [Sutherland et al., 2013; Moore et al., 2016] and numerical [Venayagamoorthy et al., 2007] studies.
Figure 2: Schematic diagram of the computational domain, dimensions, mesh resolution, density profile (in red) and fundamental mode forcing velocity profile (in blue). The distance from the inlet boundary on the left to the midpoint of the slope $L = 2.87 \text{ m}$ and the domain height $H = 0.4 \text{ m}$ are kept constant in all simulations, while $\delta$ and $s$ are varied in the parametric study. A sample background density profile $\rho_0(z)$ (red) and inlet velocity profile $U(z)$ (blue) are presented corresponding to the sample simulation ($\delta = 0.2 \text{ m}$, $s = 0.176$, $\Delta \rho = 20 \text{ kg/m}^3$, presented in §2.2). The mesh resolution of the sample simulation is presented in a gray logarithmic scale with the highest resolution $\Delta x = 2 \times 10^{-4} \text{ m}$ in the breaking region (black) and the lowest resolution $3 \times 10^{-3} \text{ m}$ in the evanescent region (light gray).

and Fringer, 2007; Arthur and Fringer, 2014 bolus investigations. The primary reference frame $O_{xz}$ is positioned at the left-bottom corner of the domain. A constant slope topography is positioned at the right-end of the system. At the left-end, a wave-maker generates an internal wave that propagates towards the slope. The topography, with constant slope $s$, is positioned such that the mid-depth line ($z = H/2$) intersects the slope at a distance $L = 2.87 \text{ m}$ from the left boundary. This point on the slope is used as the origin for the rotated reference frame $\hat{O}_{xz}$, where $\hat{x}$ and $\hat{z}$ represent the coordinates along and normal to the slope, respectively. Because the location of $\hat{O}$ is held constant, there is a minimum slope of $s = 0.070$ below which the topography would reach the inlet. For there to be a constant depth development region, the minimum slope studied here is $0.105$.

In order to efficiently obtain the necessary accuracy for a direct simulation, unstructured triangular multi-block meshes with different resolution zones were adopted. The variable mesh resolution is illustrated in figure 2. While elements are more sparse in regions where the weak background density gradients dampen out internal wave activity, higher resolutions are used in regions of internal wave propagation ($z \in [0.1, 0.3] \text{ m}$), and the highest mesh resolution is used in the overturning and breaking region on slope. For the representative mesh in figure 2 the mesh size defined as the local averaged distance between cell centers, varies from $\Delta x = 3 \times 10^{-3} \text{ m}$ in the sparse regions, to $1 \times 10^{-3} \text{ m}$ at the internal wave propagation zone, to a maximum resolution of $2 \times 10^{-4} \text{ m}$. The meshes used for the simulations presented in this work contain between 3.8 and 5.5 million cells. Approximately 70% of these cells are in the breaking zone (corresponding to the darkest zone in figure 2), where the complex velocity field needs to be accurately resolved. Simulations with the narrowest pycnocline thickness required a higher resolution mesh to provide converged results, with a total of 12.6 million cells and smallest cell size in the breaking zone reduced to $5 \times 10^{-5} \text{ m}$.

Convergence studies to define the required spatial and temporal resolutions have been conducted varying both the mesh resolution in the breaking zone and the time step used in the simulations. The results obtained in the breaking region were compared for successively finer grids and smaller time steps. The results were considered converged for a given resolution when further refinements did not affect the interpolated density and velocity fields. This study resulted in the use of a constant time step of $5 \times 10^{-4} \text{ s}$ for all simulations. With velocity magnitudes below $0.05 \text{ m/s}$ and the smallest mesh size of $5 \times 10^{-5} \text{ m}$, the maximum Courant number is 0.5. The sea water values for the kinematic viscosity and thermal diffusivity [Kunze, 2003] correspond to a Prandtl number of $Pr = \nu/\kappa = 7.14$. The Reynolds number at the bolus forming region can be estimated using a characteristic length $L_c = 0.02 \text{ m}$ and a characteristic speed $U_c = 0.03 \text{ m/s}$ based on the bolus average size and speed, respectively, which
corresponds to $Re = U_c L_c/\nu = 600$. Therefore, the Kolmogorov length scale in the breaking region is $\eta = L_c Re^{-3/4} \approx 1.65 \cdot 10^{-4}$ m. With a default mesh resolution of $\Delta x = 2 \cdot 10^{-4}$ m in the breaking zone, of the same order as the microscale $\eta_k$, and $5 \cdot 10^{-5}$ m for the sharpest pycnocline thickness case, the breaking region is sufficiently resolved to capture the bolus dynamics.

The background density stratification is a continuous, decreasing function of $z$. For all simulations, a hyperbolic tangent profile of variable pycnocline thickness is used to provide a smooth transition between the densities at the top and bottom, representative of what is observed in the oceans (figure 1). The stratification is modeled as

$$\rho_0(z) = \rho_{H/2} - \frac{\Delta \rho}{2} \tanh \left[ \frac{2(z - H/2)}{\delta} \tanh^{-1} (0.95) \right], \quad (5)$$

for $z \in [0, H]$, where $\Delta \rho \approx \rho_0(0) - \rho_0(H)$ is the density change and $\rho_{H/2} = \rho_0(H/2)$ is the mid-depth unperturbed density value. The parameter $\delta$ is here defined as the pycnocline thickness for the hyperbolic tangent profile. It corresponds to the transition height in which the density varies by 95% of $\Delta \rho$, as illustrated in figure 3(a). The hyperbolic tangent profile shape has been previously used to model sharp density stratifications in the limiting case of an almost two-layer density fluid (corresponding to small $\delta$) because of its stability properties compared to a discontinuous profile [Thorpe, 1971; Fringer and Street, 2003; Troy and Koseff, 2005; Arthur et al., 2017]. Because the pycnocline is located at the center of the domain, two-layer theory does not predict a mode-1 internal solitary wave propagation [Long, 1956]. Unlike Arthur and Fringer [2014], we will not initialize the wave with a form similar to the solitary wave solution, and instead force the system with a vertical mode-1 profile.

At $t = 0$, the unperturbed system is at rest ($u = w = 0$), the density field is $\rho = \rho_0(z)$ and the hydrostatic pressure field is $p = p_0(z)$. As illustrated on the left boundary of the domain in figure 2, the system is perturbed from its quiescent state by forcing the horizontal velocity $u$ of the vertical mode-1 plane wave of frequency $\omega = 0.628$ rad/s (10s-period). The forcing mechanism reproduces numerically what would be obtained experimentally by an oscillating plate wave-maker [Mercier et al., 2010]. The mode-1 wave is particularly interesting as it is known that internal wave generation mechanisms create internal waves with a majority of the energy in the lowest modes, with the first mode being the one with the largest wavelength and lowest shearing stress, and therefore the least affected by viscous dissipation [Gerkema and Zimmerman, 2008]. The finite-difference approach for determining the vertical
mode-1 profile (and subsequent modes) of frequency \( \omega \) for an arbitrary stratification is presented in the Supplementary Material.

For \( \rho_0(z) \) in the form \( \text{(5)} \), the vertical mode \( U(z) \) is a function of \( \omega, H, \Delta \rho \) and \( \delta \). Figure \( \text{(3) b} \) presents how the fundamental mode profiles \( U(z) \) differ for density profiles \( \rho_0(z) \) varying the value of \( \delta \) and \( \Delta \rho \). Note that smoother density profiles correspond to smoother velocity profiles. Thinner pycnoclines are associated with a stronger velocity shear, \( dU/dz \), around \( z = H/2 \). The left boundary condition for \( u \) is given by

\[
    u(x = 0, z, t) = \begin{cases} 
    U(z) \sin(\omega t) & \text{for } t \in [0, 2\pi/\omega], \\
    0 & \text{for } t > 2\pi/\omega. 
\end{cases}
\]  

(6)

The system is forced for a single period, because we are only interested in the shoaling of the first, leading wave. After one period, the velocity at the left boundary is set to zero. Because the system is being forced with a mode-1 wave, we are not able to independently modify properties of the wave such as wave speed, amplitude or wavelength. Additionally, the amplitude of forcing is large enough to cause the propagating wave to be nonlinear, further complicating the relationship between these properties and the wave forcing parameters.

Changing the stratification profile modifies the shape of the velocity profile \( U(z) \) forced on the left boundary (as illustrated in figure \( \text{(3)} \)), but it does not provide a scaling for the wave amplitude, max\_z\_[\( U(z) \)]. To compare similar waves while varying parameters, we want the kinetic energy of the breaking waves to be held constant, and the amplitude of \( U(z) \) is thus determined by the amount of energy present in the resulting breaking wave. The wave front kinetic energy through the vertical transect at the horizontal position \( x \) and time \( t \) is quantified as:

\[
    E_k(x, t) = \int_0^H \frac{1}{2} \rho(x, z, t) [u^2(x, z, t) + w^2(x, z, t)] \, dz,
\]  

(7)

which has units of energy per unit area. As the wave travels through the domain, it experiences dissipation and dispersion, changing shape, amplitude and speed even before it arrives at the start of the slope. Such changes depend on the stratification profile and on the amplitude of the forcing, in such a way that injecting waves at the inlet with the same maximum front kinetic energy \( E_k(0, \pi/\omega) \) was found not to be equivalent to obtaining waves of same front kinetic energy at the breaking. In order to get the same kinetic energy at the breaking, preliminary simulations are run in a constant depth channel through an iterative process to determine the right scaling of \( U(z) \). This process guarantees that the generated wave fronts have the same instantaneous kinetic energy at the breaking point \( x = L \), at the time \( t_c(L) \) when the leading wave crest is at \( x = L \). The value used for the kinetic energy as defined in \( \text{(7)} \) is \( E_k(L, t_c(L)) = E_{k,0} = 6.734 \cdot 10^{-3} \text{J/m}^2 \) for all cases.

The boundary conditions for the top, bottom and sloping boundaries are no-slip for velocity, \( u = w = 0 \), and no-flux for density. Imposing a no-slip condition at the top boundary, which corresponds to the water surface, does not significantly impact the dynamics because internal wave perturbations are strongest around mid-depth and decrease exponentially towards top and bottom, so that induced vertical velocities decay towards \( z = 0 \) and \( z = H \). To ensure numerical convergence and complement the inlet boundary condition on the left, a small (5 \cdot 10^{-3} \text{m-high}) outlet is added to the top-right of the domain, which guarantees mass conservation within the domain. Studies were performed to ensure that the presence, size, and position of the outlet did not influence the dynamics in the breaking region.

### 2.2 Density perturbation field and bolus formation onslope

Before analyzing the simulation results from a Lagrangian perspective, we present a typical bolus simulation to establish a basic intuition about the life cycle of the bolus. The density perturbation field, \( \rho'(x, z, t) = \rho(x, z, t) - \rho_0(z) \), provides an Eulerian description of the wave propagation, breaking, bolus formation and propagation onslope. This field primarily corresponds to density fluctuations resulting from internal wave propagation, and because the bolus corresponds to higher density fluid moving up the slope, it will be highlighted by this field. The sample simulation parameters are \( \Delta \rho = 20 \text{kg/m}^3 \), \( \delta = 0.2 \text{m} \) and \( s = 0.176 \). The mode-1 profile for this case has a velocity amplitude of 0.0126 m/s, corresponding to oscillating plate displacements up to 0.02 m at frequency \( \omega = 0.628 \text{rad/s} \). This amplitude produces a front kinetic energy at the breaking site of \( E_k(L, t_c(L)) = E_{k,0} \).
Figure 4: Instantaneous density perturbation field $\rho'$ of the sample simulation ($\delta = 0.2 \text{ m}$) at $t = 55 \text{ s}$ for the (a) full domain and (b) breaking region. Positive values (red) represent fluid displaced upward and negative values (blue) represent fluid displaced downward. The solid gray lines in (a) represent isopycnals. The isopycnal $\rho = 1005 \text{ kg/m}^3$ is drawn in black and used to highlight the bolus front boundary in (b) and (c). The breaking region presented in (b) corresponds to the region surrounded by the thick black box in (a). (c) Velocity field corresponding to the region surrounded by the thick black box in (b), which contains the bolus front. (Video of the evolution of (a) and (b) is available in the Supplementary Material.)

The density perturbation field for the sample simulation at time $t = 55 \text{ s}$ is presented in figure 4. In this figure, white indicates regions of negligible density perturbation, red indicates positive density perturbation (i.e., denser fluid elements perturbed upwards from their equilibrium position), and blue indicates negative density perturbation (i.e., less dense fluid elements perturbed downwards). Figure 4(a) presents a view of the full fluid domain: the generated waves (with alternating crests and troughs indicated in red and blue) have propagated to the right from the inlet and reached the slope. Note that despite exciting the system for a single period, several waves of decreasing amplitude are produced as a result of the initial forcing and persist even after the left boundary conditions are set to zero perturbation (for $t > 2\pi/\omega = 10 \text{ s}$). Small asymmetries in the generated waves (the maximum perturbations are not perfectly centered at $z = H/2$) demonstrate that the velocity amplitude is large enough to produce weakly nonlinear waves, even before they reach the sloping region. The solid gray lines in 4 represent isopycnals, and the isopycnal $\rho = 1005 \text{ kg/m}^3$ (black solid line) highlights the bolus front. The negligible density perturbation above and below the pycnocline demonstrates how the perturbation amplitudes decrease exponentially in the vertical direction within the weakly stratified, evanescent regions.

The bolus resulting from the internal wave breaking consists of dense fluid moving upslope and is highlighted in figure 4(b) as a positive (red) density perturbation. The results are presented in the rotated frame $\hat{O}_{xy}$, and the domain in figure 4(b) corresponds to the bold, black rectangle (0.9 m-long by 0.02 m-tall) in 4(a). The Eulerian results for the bolus propagation are qualitatively similar to those numerically produced by Venayagamoorthy and Fringer [2007]. The internal wave breaking forms the bolus, which corresponds to a propagating counter-clockwise vortex, as illustrated in figure 4(c) by the velocity field around the bolus front. The bolus dynamics as it enters the zone of less dense fluid is well described by the theory of gravity currents propagating in a stratified fluid [Benjamin, 1968, Maxworthy et al., 2002, White and Helfrich, 2008]. The propagation of the dense front entering a zone of lower density fluid is subject to the induced shear and large difference in density between the bolus and the surrounding...
fluid, which results in vortex shedding and mixing that eventually stops the bolus from moving upslope. Towards the end of the bolus propagation, the bolus dynamics are impacted by other boluses produced by the following waves reaching the slope. These secondary boluses will not be addressed in this manuscript. We focus on the dynamics of the first bolus entering a quiescent region with no previous flow that will impact the bolus propagation.

While the density perturbation field and the velocity field provide good intuition for the bolus dynamics, a non-zero value of $\rho'$ at a given point only means that there is no local perturbation at that instant. This approach does not track fluid elements moving with the bolus, nor does this approach provide accurate information about how fluid is transported as the breaking happens. This is most acutely highlighted by the fact that the initial internal wave does not transport fluid from the inlet to the sloping boundary. At some point during the breaking process, the internal wave does begin to transport fluid (see the Supplementary Material for a demonstration video), and identifying when, how, and what fluid is transported is the purpose of the Lagrangian analysis.

3 Boluses as Lagrangian coherent structures

To investigate material transport, a Lagrangian approach is more appropriate than an Eulerian approach. This section presents how to use the Eulerian results from the numerical simulations to perform an objective, Lagrangian-based characterization of the bolus, which is here defined as a materially coherent region of the fluid that does not significantly mix with the rest of the domain. The steps for identifying the bolus, based on the spectral clustering approach presented by Hadjighasem et al. [2016], are presented in §3.1. In §3.2, transport quantities of the identified bolus are introduced and observations of the relationship between the bolus and the shoaling wave are discussed. Finally, a comparison between results for two- and three-dimensional models is presented in §3.3.

3.1 Lagrangian characterization of the bolus

Lagrangian coherent structures provide a robust means for identifying the key underlying transport features of a given flow [Haller, 2002]. One particular type of structure that can be detected are materially coherent vortices [Abernathey and Haller, 2018]. These features are unique in that as they move through the domain, the fluid inside the region deforms but does not significantly mix with the fluid outside the perimeter. Identifying this type of feature in the breaking region will determine the fluid that is being advected by the bolus. Materially coherent features can be identified using Cauchy-Green based metrics [Haller and Beron-Vera, 2013], transport operator methods [Froyland et al., 2010], and graph Laplacian methods [Froyland and Junge, 2018]. We use the clustering based approach based on Hadjighasem et al. [2016] with minor modifications.

Central to coherent structure detection is the analysis of trajectories, which are computed from the velocity field $\mathbf{u} = (u, w)$ obtained by numerically solving (1)-(4). The velocity $\hat{\mathbf{u}}$ in the rotated frame is used to advect massless fluid elements referred to as passive tracers, which have no inertia and move according to

$$\frac{d\hat{x}_i}{dt} = \hat{\mathbf{u}}(\hat{x}_i(t), t)$$

where $\hat{x}_i(t)$ is the trajectory of tracer $i$ with initial position $\hat{x}_i(t_0)$. Numerical integration of (8) is performed using a 4th-order Runge–Kutta method with a time-step of $5 \cdot 10^{-3}$ s. The breaking region is initially covered by a rectangular grid of passive tracers with along-slope dimensions that vary based on the slope and injected energy (grids used for each simulation are in the Supplementary Material). The tracer spacing is $\Delta \hat{x}_p = 5 \cdot 10^{-3}$ m and $\Delta \hat{z}_p = 2 \cdot 10^{-4}$ m, resulting in $n = 18600$ to $40000$ tracers. The advection time window starts at time $t_0$, which is just before the shoaling wave arrives at the breaking region, and ends at time $t_f$, which corresponds to when the initial Eulerian density perturbation has stopped propagating up the slope, an upper bound for when the bolus stopped moving upslope.

Having computed the $n$ trajectories for the interval $[t_0, t_f]$, we perform the coherent structure analysis based on the spectral clustering method of Hadjighasem et al. [2016]. The clustering problem attempts to partition the domain into clusters such that trajectories within the same cluster are similar and trajectories from different clusters are dissimilar. The spectral analysis relies on the construction of a
similarity graph that quantifies the pairwise similarity of trajectories. The similarity metric \( w_{ij} \) is the inverse of the time-averaged distance between these trajectories. This calculation is performed for all pairs of trajectories, but only values of \( w_{ij} \) greater than a user defined value are retained in order to sparsify the similarity matrix, \( W \). Here, only values of \( w_{ij} > 1/r^* \), with \( r^* = 0.07 \text{ m} \), are retained, resulting in an approximately 90\%-sparse matrix for the sample simulation discussed in §2.2. The clustering results were verified to be consistent for different choices of \( r^* \) in a neighborhood of this prescribed value. Based on the similarity matrix \( W \), the diagonal degree matrix \( D \) is produced, where each diagonal element is equal to the sum of the elements in the corresponding row of \( W \). From the sparse similarity matrix and degree matrix, the unnormalized graph Laplacian \( L = D - W \) is computed.

To create the clustering partition, we must identify characteristics of the set of trajectories. To characterize the trajectories, the next step of the algorithm is to compute the first generalized eigenvectors of the generalized eigenproblem

\[
Lq = ADq. \tag{9}
\]

It has been shown that the generalized eigenvalues of \( \{9\} \) satisfy \( 0 = \lambda_1 \leq \ldots \leq \lambda_n \), and the normalized dominant eigenvectors \( q_1, q_2, \ldots, q_k \) differentiate properties in the graph and facilitate the clustering process \cite{von Luxburg2007}. The dominant eigenvectors corresponding to the smallest eigenvalues reveal the most important characteristics of the flow. Each trajectory is characterized by a value within each eigenvector, and this characterization is used to group similar trajectories together. While all of the eigenvectors provide information, the dominant ones highlight the most important patterns.

The final step of the spectral clustering algorithm builds the eigenvector matrix \( Q \in \mathbb{R}^{n \times k} \) containing the \( k \) dominant eigenvectors \( q_1, q_2, \ldots, q_k \) as columns. While Hadjighasem et al. \cite{Hadjighasem2016} use the eigengap heuristic to determine \( k \), this heuristic is not ideal given the small amount of mixing outside of the bolus, so we adopted here a different heuristic based on the form of the dominant eigenvector to select \( k \). Let \( y_i \in \mathbb{R}^k \) be the characterization vector corresponding to the \( i \)-th row of \( Q \), which contains condensed differentiating information for trajectory \( i \) (note that \( k \ll n \)). The characterization vectors \( (y_i)_{1 \leq i \leq n} \) are clustered with a \( K \)-means algorithm, assigning each vector \( y_i \) and the corresponding trajectory to a cluster. We use \( k + 1 \) clusters to partition the domain, where an extra cluster is added to account for the incoherent cluster, as suggested by Hadjighasem et al. \cite{Hadjighasem2016}.

Figure 3(a) presents the initial position of the partitions assigned to each one of the seven clusters identified for the sample simulation discussed in §2.2. The dark blue cluster is identified by the method as the objective bolus: the one cluster composed of tracers that move upslope eventually entraining in a propagating vortex. The bolus cluster also corresponds to the cluster of maximum displacement of the center of gravity in the \( \hat{x} \) direction. Figure 3(b) presents how the bolus cluster propagates in time, from \( t_0 \) to \( t_f \), and it is worth noting that tracers maintain their cluster membership throughout the duration of the time interval. The bolus consists of tracers that are initially spread horizontally along the slope. As the wave arrives, those tracers get lifted, are trapped in the compact vortex, move along with the vortex for approximately 0.4 m up the slope, and eventually the front tracers stagnate and the trailing tracers start to recede down slope. All the elements identified as part of the bolus end up trapped inside of the vortex in intermediate times. While not the case for this example, it is possible that no bolus is detected by the algorithm, indicating that the shoaling internal wave does not result in effective transport.

### 3.2 Bolus transport properties

An objective identification of the material being transported with the bolus and its trajectory is achieved by the spectral clustering method described in §3.1. This objective quantification of the bolus makes it possible to measure bolus transport properties such as size, shape, position of center of mass and velocity. Important bolus properties are here defined, applied to the sample simulation, and will be used in the parametric studies in §4. Position and velocity of the Lagrangian bolus propagating up the slope are also compared to the unbroken portion of the shoaling internal wave.

Let \( I_b \) be the set of tracers identified as part of the bolus, so that \(|I_b| = n_b\) is the number of passive tracers identified as the bolus. The tracer positions \( \{x_i\} \) are known in the uniformly sampled time interval \([t_0, t_f]\). For each time instance \( t \), averaging the position of the tracers gives the position of the bolus center of gravity

\[
x_{CG}(t) = \frac{1}{n_b} \sum_{i \in I_b} x_i(t), \tag{10}
\]
Figure 5: (a) Sample simulation clustering result for elements in the initial uniform grid. Seven clusters have been identified, with the Lagrangian bolus cluster represented in dark blue. (b) The time evolution for the bolus cluster from $t_0 = 19.25\, \text{s}$ to $t_f = 65\, \text{s}$, which is the entire bolus lifespan. (Video of the evolution of the figure is available in the Supplementary Material.)

which we will use to quantify the bolus trajectory for $t \in [t_0, t_f]$. The position history of the horizontally foremost and trailing tracers inside the bolus, $x_+(t)$ and $x_-(t)$ respectively, are also recorded. The foremost and trailing positions will correspond to different tracers as time evolves and overturning inside the bolus takes place. These properties allow us to track the bolus center of gravity and its horizontal extent from a Lagrangian perspective.

To understand the relationship between the bolus and the shoaling wave that produces it, the wave trough and crest are also tracked. Because the unbroken components of the shoaling do not transport material, the displacement of isopycnals is tracked to locate the wave crest and trough. While the propagating wave amplitude is maximized at the center of the pycnocline, other isopycnals throughout the water column are also deformed accordingly with these deformations vertically aligned. To track the unbroken shoaling wave, we track an isopycnal that is well above the breaking region, which makes it possible to track the shoaling wave propagation smoothly even after the bolus is generated. The first propagating minimum and maximum of the isopycnal correspond to the leading wave trough and crest horizontal position. Tracking the position of the isopycnal minimum and maximum provide a full description of the horizontal position of trough and crest as a function of time.

By tracking the extent of the bolus and the wave crest and trough, we can unveil how the location of the bolus relative to the wave evolves with time. Bolus and wave kinematics are compared in figure 6. A magnified version of the tracked isopycnal is presented in figure 6(a) at three different time instances, together with the corresponding boluses, demonstrating that the front of the bolus is always between the wave crest and trough. Figure 6(b) presents the $\left( x(t), t \right)$ curves for the horizontal extent of the bolus and the shoaling wave. For the bolus, the trajectories presented are those of the center of gravity, foremost and trailing point, with the horizontal extent of the bolus in gray. For the leading wave, the trough and crest’s trajectories are presented. As the wave trough approaches, the trailing point of the bolus initially recedes while the front remains unaffected. The wave crest then starts to approach, pushing the rear of the bolus forward and ultimately creates a vortex, which entrains all members of the bolus in the vortex.
Figure 6: Kinematic comparison between the bolus and a co-propagating non-breaking isopycnal above the breaking region for $t \in [t_0, t_f]$. (a) Bolus and amplified isopycnal for three time instances. (b) The horizontal position as a function of time and (c) the horizontal speed as a function of horizontal position for the leading wave trough (dashed blue), the leading wave crest (dashed red), the bolus center of gravity (bold black), bolus trailing point (thin black) and foremost point (bold dark red). The isopycnal used to track the crest and trough in this plot was $\rho = 1000.05 \text{ kg/m}^3$, which has a neutrally buoyant height at $z = 0.3635 \text{ m}$. The tick marks in (b) correspond to the times for which the bolus and isopycnals are plotted in (a). The wave vertical positions and amplitudes in (a) were modified for illustration purposes.

As can be seen at $t = 46 \text{ s}$ in figure 6(a). Finally, as depicted at $t = 52 \text{ s}$, the bolus extent begins to increase as some of the tracers are shed from the vortex, while the foremost tracers continue moving upslope with the decelerating wave crest. While the leading edge tracers remain ahead of the wave crest, the ejection of tracers from the vortex and their recession downslope causes the center of gravity to be left behind. Eventually the trailing points begin to move back up slope, at $t \approx 62 \text{ s}$, as the next shoaling wave arrives.

The horizontal speeds of the bolus center of gravity, foremost point and the leading crest are obtained from the horizontal position plots and presented in figure 6(c). While the center of gravity moves slower than the crest, the correlated motion of the wave crest and the leading points is evident in the similar speed profiles with both undergoing the same deceleration resulting from the narrowing water column. Such a strong correlation between the Lagrangian bolus and the wave crest is not necessarily a result one would expect: the crest position is directly extracted from the density field, well above the breaking region, and the bolus is quantified by the movement of passive tracers identified as a coherent structure with the spectral clustering method. Nevertheless, the kinematics of both structures seem to consistently match, not only for the sample case presented, but for all parameter sweeps, which is an indicator that shoaling internal wave signatures close to the surface may possibly be used to estimate the bolus location and speed. Given the consistency of the correlation between the crest and the bolus leading edge throughout
the different cases analyzed, we will not demonstrate it for each parameter sweep.

Instead, for a direct comparison between simulations with different parameters, we quantify how far upslope material is transported by using the trajectory of the bolus center of gravity in the rotated reference frame, $\hat{x}_{CG}(t)$. The maximum displacement upslope relative to the initial position, $D_b$, is

$$D_b = \max_{t_0 \leq t \leq t_f} \{ \hat{x}_{CG}(t) - \hat{x}_{CG}(t_0) \}.$$  \hspace{1cm} (11)

For the sample case in figure 5(b), $D_b$ equals 0.402 m and is achieved at $t = 56$ s. One could, alternatively, define this distance based on the position of the bolus foremost point, and that definition would provide a similar displacement upslope (0.363 m for the sample case). We consider the center of gravity to provide a better description of the Lagrangian bolus dynamics as it accounts for the complete coherent structure describing material trapped in the bolus.

Another quantity of interest is the amount of material being transported, which we quantify by the approximate area of the bolus. The bolus size, $S_b$, is estimated using the number $n_b$ of tracers in the bolus cluster:

$$S_b = n_b \Delta \hat{x}_p \Delta \hat{z}_p,$$  \hspace{1cm} (12)

where $\Delta \hat{x}_p = 5 \cdot 10^{-3}$ m and $\Delta \hat{z}_p = 2 \cdot 10^{-4}$ m are the initial distances between tracers. Because the number of tracers defining the bolus is time-invariant, the bolus size $S_b$ is not a function of time, as is required for the incompressible flow. In figure 5(a), the dark blue cluster representing the bolus has size 5.14 $\cdot$ $10^{-4}$ m$^2$.

The initial shape of the bolus at $t_0$ is important as it describes the portion of the fluid domain transported up the slope. For the presented sample case, it is initially a thin sliver of fluid just above the boundary with an aspect ratio of approximately 60:1. However, there is a “hook” on top of the sliver, localized around (0.05, 0.003)m, that is also entrained into the vortex and carried upslope as presented in figure 5(b). Both the size and the shape of the bolus vary with system parameters, as will be first demonstrated in §4.1.

### 3.3 Comparison to three-dimensional analysis

While Venayagamoorthy and Fringer [2007] demonstrated that a three-dimensional simulation yielded similar results to its two-dimensional counterpart, we must verify that the coherent structure analysis is not subject to greater coherence due to unrealistic two-dimensional turbulence [Arthur and Fringer 2014]. To do so, an equivalent three-dimensional simulation was performed with identical physical parameters. To perform this simulation, it was necessary to decrease the spatial resolution along the shelf to $\Delta x_p = 1 \cdot 10^{-3}$ m. The three-dimensional domain was 0.15 m across the shelf with a spatial resolution of 3 $\cdot$ $10^{-3}$ m in the $y$ direction, corresponding to 51 mesh layers. Free-slip boundary conditions were applied to the lateral walls, and the total number of mesh cells for this simulation was 19.8 million. For the coherent structure analysis, the tracer spacing along the shelf was doubled to $\Delta \hat{x}_p = 0.01$ m. The across shelf tracer spacing was also 0.01 m.

The results of the three-dimensional coherent structure analysis are presented in figure 7. The initial position of the bolus tracers for the three-dimensional simulation align closely to the contour representing the two-dimensional cluster as demonstrated in figure 7(a). The three-dimensional bolus is slightly bigger, with an average cross-section area of $S_b = 5.48 \cdot 10^{-4}$ m$^2$ as compared to the 5.14 $\cdot$ $10^{-4}$ m$^2$ in the two-dimensional case. In the three-dimensional case, the “hook” of the bolus becomes less pronounced. The clustered tracers demonstrate minimal lateral variability as only two initial locations in the same in-plane position are not uniformly members of the bolus. The propagation of these tracers over time allows the lateral variability to appear. The bolus tracers at $t = 55$ s are presented in figure 7(b). The lateral coordinate of the three-dimensional tracers are represented in color, from blue to yellow, while the corresponding two-dimensional result is represented in red. While the front of the three-dimensional bolus is slightly further upslope, there is little difference in the distribution of the two- and three-dimensional tracers. The maximum displacement upslope is approximately the same in both cases: $D_b = 0.399$ m for the three-dimensional case, compared to 0.402 m for the the two-dimensional case.

The similarities of the two- and three-dimensional results aligns well with the results of Venayagamoorthy and Fringer [2007]. In their case, they use a background linear stratification which results in lower reduced gravity effects, similar to the broader pycnoclines in our study. In their study, they...
identify lateral variation of the bolus primarily when the speed of the bolus slows to below the carrying wave speed. In the present case, the water column continues to decay and the carrying wave and bolus decelerate at similar rates as was observed in figure 6(c). We speculate that this is the reason why we see a delayed onset of the lobe and cleft instability common to gravity currents flowing over a no-slip boundary. While this lateral variability is minimal for the first bolus, there is significant across shelf variation as anticipated by Aghsae et al. [2010]. This limits a two-dimensional study to just the first bolus.

4 Bolus dependence on pycnocline thickness and other parameters

The coherent structure methods enables an objective measurement of the bolus and can be used to understand the relationship between transport by boluses and system parameters such as stratification, wave energy and slope. Central to our investigation is the impact of the pycnocline thickness $\delta$ on bolus characteristics. Therefore, while secondary system parameters such as the wave energy, the density change and the slope are varied one at a time, the effect of the pycnocline thickness is analyzed for all cases. Results demonstrating the importance and consequences of incorporating the pycnocline thickness $\delta$ to the model are presented in §4.1. The effects of secondary parameters on bolus formation and propagation are presented as follows: the incoming wave kinetic energy in §4.2, the density change between top and bottom in §4.3 and the topographic slope in §4.4. The relationship between the bolus transport characteristics and relevant dimensionless parameters is investigated in §4.5. A summary of the simulations performed and the parameters used is presented in table 1.

| Section | Figure | Simulation # | $\delta$ (m) | $E_k/E_{k,0}$ | $\Delta \rho$ (kg/m$^3$) | $s$ |
|---------|--------|--------------|--------------|----------------|------------------------|-----|
| 4.1     | 8      | 1 - 9        | 0.025 - 0.4  | 1              | 20                     | 0.176 |
| 4.2     | 10     | 10 - 27      | 0.025, 0.2, 0.4 | 1/8 - 4 | 20 | 0.176 |
| 4.3     | 11     | 28 - 51      | 0.025 - 0.25 | 1              | 10 - 80                | 0.176 |
| 4.4     | 12(a-c)| 52 - 75      | 0.025, 0.2, 0.4 | 1              | 20                     | 0.105, 0.176, 0.231 |
| 4.4     | 12(d-f)| 76 - 102     | 0.025 - 0.4  | 1              | 20                     | 0.105 - 0.231 |

Table 1: The bolus simulation cases considered in the respective sections and figures are presented here. The pycnocline thickness $\delta$, the energy at the breaking point $E_k$, the change in density $\Delta \rho$, and the topographic slope $s$ are independently varied for each of the parameter studies. (An expanded version of this table is available in the Supplementary Material.)

Figure 7: (a) Projected three-dimensional bolus tracers (dark blue) and the two-dimensional bolus boundary (red). (b) Bolus tracer positions at time $t = 55$ s, with the lateral coordinate indicated in color (blue to yellow) for the three-dimensional results. Two-dimensional bolus tracers are superposed as transparent red dots.
4.1 Pycnocline thickness variation

While past internal wave bolus studies have focused on two-layer density stratifications, the impact of the pycnocline thickness parameter \( \delta \) that controls how smoothly the density changes is investigated here. Nine different pycnocline thicknesses, 

\[
\delta = 0.025, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4 \text{ m},
\]

are considered. Small \( \delta \) values represent a thin, sharp transition of density and larger values represent broader, smoother transitions. The case \( \delta = 0.025 \text{ m} \) is our approximate of the two-layer density system. A finer transition would have required a higher resolution mesh and more computational time to be directly simulated. The other extreme pycnocline thickness case, \( \delta = 0.4 \text{ m} \), matches the height of the simulation domain. The case \( \delta = 0.2 \text{ m} \), presented in §2 and §3 is the intermediate value between these two extremes. While the pycnocline thickness varies, the boluses are consistently generated by equally energetic waves, with instantaneous kinetic energy at the breaking point \( E_{k,0} \) (as discussed in §4.1). The trend for constant energy injected at the inlet is presented in the Supplemental Materials. \( E_{k,0}, \Delta \rho \) and \( s \) are held constant for this study. As \( \delta \) increases, the wave amplitude increases from \( a = 0.012 \text{ to } 0.016 \text{ m} \) and the wavelength remains approximately constant, \( \lambda = 0.68 \pm 0.01 \text{ m} \). The Iribarren number \( Ir = s/\sqrt{a/\lambda} \), which compares the steepness of the slope and the incoming wave, varies from 1.2 to 1.6 as \( \delta \) increases.

The pycnocline thicknesses impact on the resulting boluses dynamics is presented in figure 8. Snapshots of the bolus in the rotated frame \( \hat{O}_{xz} \) are presented for selected time steps within the respective time span \([t_0, t_f] \). For the sample case, in figure 8(c), the trajectory of the center of gravity is presented in gray and the maximum displacement upslope \( D_b \) defined in (11) is illustrated. The values of \( D_b \) and the bolus lifetime \( t_f - t_0 \) both grow monotonically with the values of \( \delta \) considered. A more complex relationship is observed between the bolus size \( S_b \) and \( \delta \): boluses are smaller for the approximate two-layer system, their sizes initially increase with \( \delta \), reach a maximum, then shrink from this maximum as \( \delta \) approaches the broadest pycnocline case. It is also worth noting from figure 8 that, as \( \delta \) increases, the wave breaking happens later in time and the fluid trapped is therefore initially located further up the slope.

To quantify bolus transport in terms of the bolus size \( S_b \) and displacement upslope \( D_b \), the results visualized in figure 8 are presented in figure 9 for the full \( \delta \) range. The maximum \( S_b \) is observed for \( \delta = 0.15 \text{ m} \), while the displacement \( D_b \) monotonically increases with \( \delta \) for the values considered. The largest bolus is approximately twice the size of the one observed for the approximate two-layer simulation and five times as large as the bolus produced in the broadest pycnocline stratification. The one obtained for the broadest pycnocline case, however small, travels about three times as far as the two-layer bolus, with a lifetime more than twice as long. An upper and lower layer has also been presented in figure 9(b) representing the range of tracer displacement in the bolus. This range is smallest for the narrowest pycnoclines and gradually grows until \( \delta = 0.3 \text{ m} \). Both the upper and lower bound are monotonically increasing.

A jump in \( S_b \) and \( D_b \) occurs between boluses formed with \( \delta = 0.25 \text{ m} \) and \( \delta = 0.3 \text{ m} \). This result relates to a difference in the shape of the bolus for each case. We refer to the shape of the bolus based on the geometric properties describing the initial position of the bolus tracers. In this paper, bolus shapes are divided into three categories: ball, hook and sliver, which are illustrated in figure 10. The ball category, illustrated in figure 10(a), is characterized by having a convex geometry, and is typical of thin transitions, for which the breaking is more abrupt. Increasing the thickness of the pycnocline causes the initial bolus shape to take on the shape of a “hook” of material, which is pulled inside of the vortex during the bolus propagation. This category is well illustrated by the case \( \delta = 0.25 \text{ m} \) in figure 10(b), in which the hook is located around \((0.1, 0.003)\text{m}\). As the pycnocline thickness grows even more, only a horizontal stripe of tracers remain part of the bolus, and the hook is lost. The single horizontal stripe pattern defines the sliver bolus category, illustrated in figure 10(c). The loss of the hook is a result of the propagating vortex not being strong enough to trap the material located in that area. In fact, the maximum vorticity felt by the bolus is monotonically decreasing as \( \delta \) increases. Note also that the transition from hook to sliver occurs, for the cases presented here, between two consecutive data points: \( \delta = 0.25 \text{ m} \) and \( \delta = 0.3 \text{ m} \). This transition results in the bolus size decreasing by more than half. This significant decrease in size may be the reason for the smaller range in displacements observed for \( \delta = 0.3 \text{ m} \) compared to \( \delta = 0.25 \text{ m} \).

Drastic changes in the bolus size and displacement are found to be related to transitions in the bolus geometric properties described by these three categories. For this reason, for figure 9 and all the \( S_b \) and
Figure 8: Time evolution of the boluses, from \( t_0 \) to \( t_f \) indicated by the time values on the left and right, for stratifications with pycnocline thicknesses (a) \( \delta = 0.025 \text{ m} \), (b) \( 0.1 \text{ m} \), (c) \( 0.2 \text{ m} \), (d) \( 0.25 \text{ m} \) and (e) \( 0.3 \text{ m} \). The trajectory of the bolus center of gravity for \( \delta = 0.2 \text{ m} \) and the displacement upslope \( D_b \) are illustrated in (c). (Video of the evolution of the figure is available in the Supplementary Material.)

\( D_b \) plots in the following sections, distinct bolus shapes are plotted using different symbols (as introduced in figure 10), so that transitions in shape are easily identified. The jump in size in figure 9 is related to the transition from a hook-shaped bolus to a sliver-shaped bolus. The corresponding jump in \( D_b \) is related to the fact that the tracers that form the hook component are typically the first ones to be ejected from the vortex.

### 4.2 Shoaling wave energy variation

In the previous section, waves reached the slope with the same kinetic energy \( E_{k,0} \). Here, the influence of the shoaling wave energy on the bolus properties is investigated. Preliminary simulations in a constant depth channel were run to determine the injected energy necessary to produce shoaling waves with the energy scale factors

\[
E_k/E_{k,0} = 1/8, 1/4, 1/2, 1, 2, 4. \tag{14}
\]

The upper bound for energy scale factors considered here is set by computational limitations of direct simulations in highly energetic cases, where the decreasing Kolmogorov length scale at the breaking
Figure 9: (a) Bolus size $S_b$ and (b) maximum displacement upslope $D_b$ as a function of the pycnocline thickness, $\delta$, for mode-1 waves breaking with constant energy. The error bars represent the range of displacement within the bolus tracers. Different markers are used to represent bolus shape categories: ball (circle), hook (triangle) and sliver (square).

Figure 10: (a) Ball, (b) hook and (c) sliver bolus shape categories represented by the bolus tracers position at time $t_0$. Shapes are obtained for $\delta = 0.025$ m (blue), 0.25 m (green) and 0.3 m (red). The aspect ratio is approximately 30:1. The markers in the top right of each plot are used to represent bolus shape categories.

region requires mesh resolutions beyond the available resources. The lower bound attempts to capture low-energy cases in which boluses do not form. To understand how the wave energy impacts the bolus propagation as a function of the pycnocline thickness, three thicknesses ($\delta = 0.025, 0.2$, and $0.4$ m) are considered for each scale factor. For all simulations, $\Delta \rho$ and $s$ are held constant. The wave amplitude increases as the energy increases, and the wavelength remains approximately constant for all simulations. The lowest energy with $\delta = 0.025$ m produces a wave with amplitude of $0.004$ m and $Ir = 2.19$, and the highest energy with $\delta = 0.4$ m case produces $a = 0.033$ m and $Ir = 0.80$.

Figure 11 presents the relationship between the shoaling wave energy and the properties of the bolus. The bolus size dependence is presented in figure 11(a) and the displacement dependence in figure 11(b). When no bolus is identified, the properties for that case are plotted in the graph as crosses with $S_b = 0$ m$^2$, $D_b = 0$ m. When the energy scale factor is 1/8, none of the three pycnocline thicknesses produce a bolus. For $E_k/E_{k,0} = 1/4$, the only case producing a bolus is $\delta = 0.2$ m. For energy factors of 1/2 and higher, all three cases of $\delta$ produce a bolus, and the bolus size increases with the wave energy. The bolus size sensitivity for the two-layer case is the smallest, which is possibly related to the more abrupt breaking and quick mixing characteristic of this stratification. The pycnocline thickness that produces the largest and smallest bolus depends on the amount of energy at the breaking point.

Varying the wave energy also plays a role in transitioning between bolus shape categories, and tran-
sitions happen between different energy factors for different pycnocline thicknesses. In particular, for $\delta = 0.2 \text{ m}$, a transition from sliver-shaped to hook-shaped boluses occurs when the energy factor is increased from 1/2 to 1, while this same transition for $\delta = 0.4 \text{ m}$ is observed between the energy factors 1 to 2. This result is related to the fact that at higher energy the vortex has a greater circulation and is more capable of retaining particles that are moving with the bolus. The resulting jumps in $S_b$ and $D_b$ produce counter-intuitive results: for example, for $\delta = 0.2 \text{ m}$, the bolus displacement is larger when the incoming wave has half of the energy compared to the baseline energy. The energy factor $E_k/E_{k,0} = 1/2$ produces a sliver bolus that is approximately half the size of the hook bolus of energy $E_{k,0}$, so this bolus consists of fewer tracers that travel a longer distance. Such hook-sliver transitions explain why there are two clearly different growth rates observed for $S_b$ and $D_b$ for $\delta = 0.2$ and 0.4 m as the smaller sliver-shaped boluses travel longer distances compared to the larger hook-shaped boluses.

### 4.3 Density change variation

In our previous studies, the stratification was varied by changing the pycnocline thickness parameter $\delta$ while the density change $\Delta \rho$ remained 20 kg/m$^3$. Variations in $\Delta \rho$ have not been investigated in previous bolus studies; however, when considering typical ocean stratification profiles, $\Delta \rho$ is not constant and its value may have an impact on the resulting boluses. Given the way gravity current dynamics depend on the reduced gravity, which relates to the intensity of $\Delta \rho$, it seems likely that the density change will impact the bolus dynamics. Here, we vary the stratification by imposing density changes

$$\Delta \rho = 10, 20, 40, 80 \text{ kg/m}^3.$$  \hspace{1cm} (15)

The choice of not using density changes higher than 80 kg/m$^3$ is related to the limitations of the Boussinesq approximation when describing stronger density variations. For the lower bound, choosing smaller changes than 10 kg/m$^3$, for the fixed value of $\omega$ used for all the simulations, would result in a stratification with buoyancy frequency $N(z)$ less than $\omega$ for all $z$, resulting in evanescent waves. To understand how the sensitivity to density change varies across different pycnocline thicknesses, each density change is applied to the values of $\delta \leq 0.25 \text{ m}$.

The combination of different density changes and pycnocline thicknesses has a significant impact on the wave speed of the leading wave, which varies from approximately 0.07 m/s for the broadest pycnocline thickness with the smallest density change to a wave speed of 0.25 m/s for the thinnest pycnocline thickness with the greatest density change. The wave amplitude, measured in terms of the vertical displacement of the mid-depth isopycnal, is largest for the smallest $\Delta \rho$ and largest $\delta$ ($\alpha = 0.024 \text{ m}$) and the
Figure 12: (a) Bolus size $S_b$ and (b) maximum displacement upslope $D_b$ as a function of the pycnocline thickness, for variable density change, $\Delta \rho$. Simulations were performed with density changes $\Delta \rho = 10$ (red), 20 (black), 40 (blue) and 80 kg/m$^3$ (green). When no bolus is identified, a cross is plotted at $S_b = 0 \text{ m}^2$ and $D_b = 0 \text{ m}$. Marker shapes are used to represent each bolus shape categories: ball (circle), hook (triangle) and sliver (square).

$$
\begin{array}{cccc}
\Delta \rho (\text{kg/m}^3) & \delta (\text{m}) & \theta(z=H/2) & s_\theta = \tan(\theta) \\
10 & 0.2 & 41.5^\circ & 0.885 \\
20 & 0.15 & 24.0^\circ & 0.445 \\
40 & 0.1 & 13.6^\circ & 0.242 \\
80 & 0.1 & 9.5^\circ & 0.167 \\
\end{array}
$$

Table 2: Critical angle at mid-depth, $\theta(z=H/2)$, and corresponding critical slope $s_\theta$ for parameter combinations $(\Delta \rho, \delta)_{\text{max}}$ that maximize the bolus size $S_b$.

amplitude increases with $\delta$. The wavelength increases with $\Delta \rho$. In all cases, the energy of the shoaling wave is $E_{k,0}$ and the slope is held constant.

The resulting bolus size and displacement upslope are presented in figure 12. Figure 12(a) demonstrates that the pycnocline thickness that maximizes the bolus size depends on the density change. Pycnocline thicknesses resulting in maximum bolus size shift from $\delta = 0.1 \text{ m}$ for $\Delta \rho = 80 \text{ kg/m}^3$ to $\delta = 0.2 \text{ m}$ for $\Delta \rho = 10 \text{ kg/m}^3$. As the density change is decreased, the pycnocline thickness that produces the maximum size increases for the range of values studied. For a strongly stratified system, the wave essentially reflects from the sloping wall, forming either a small bolus or no bolus, as is the case for $\Delta \rho = 80 \text{ kg/m}^3$ and $\delta = 0.025 \text{ m}$. This case corresponds to the highest Iribarren number observed, $Ir = 3.0$. Figure 12(b) in turn just confirms the trend previously observed in figure 9(b): the maximum displacement $D_b$ increases monotonically with the pycnocline thickness for every $\Delta \rho$ value considered.

The observed relationship between the bolus size and the stratification profile goes against the intuition that transport would maximize at the internal wave critical angle. Internal waves generated by an oscillating tidal flow on a topographic slope have highest amplitudes when the topography slope matches the angle of propagation of the internal wave beam $\theta$, $\sin \theta = \omega/N$ [Zhang et al., 2008]. Therefore it would be reasonable to conjecture that transport by internal waves shoaling on a constant slope topography in the system considered could also be maximized when the topographic slope equals $s_\theta = \tan(\theta(z=H/2))$, the internal wave beam angle at mid-depth, which varies with $d\rho/\,dz(H/2)$. This would imply thin pycnoclines for small $\Delta \rho$ and broad pycnoclines for large $\Delta \rho$. Instead, boluses are larger for sharper pycnoclines when $\Delta \rho$ is larger and smaller for broader pycnoclines when $\Delta \rho$ is smaller. The beam angles at mid-depth, $\theta(H/2)$, for the four optimal combinations $(\Delta \rho, \delta)_{\text{peak}}$ maximizing the bolus size, are
Finally, to investigate the influence of the underlying topography, the slope $s$, held at 0.176 for the previous studies, is varied. Keeping a constant density change $\Delta \rho = 20 \text{ kg/m}^3$ and shoaling wave energy $E_{k,0}$, the topographic slopes considered are

$$s = 0.105, 0.123, 0.141, 0.158, 0.176, 0.194, 0.213, 0.231,$$

(16)

which correspond to slope angles of $6^\circ$ to $13^\circ$, respectively. To ensure that the wave travels the same distance $L$ before breaking upslope in all cases, the domain geometry is adjusted so that the horizontal plane $z = H/2$ intersects the topography at $x = L$, as presented in figure 2. Such a geometric constraint increases the domain size (and therefore the number of grid cells) for smaller slopes, with increasing computational costs. The tested slopes are also constrained so that the topography does not intersect the domain inlet, and for these two reasons $s$ is not varied below 0.105. To identify the relationship between bolus transport, slope and pycnocline thickness, the three thicknesses ($\delta = 0.025, 0.2$ and 0.4 m) are tested with each of the slopes. Three slopes corresponding to the extremes and baseline cases ($s = 0.105, 0.176$ and 0.231) are also tested with all nine pycnocline thicknesses. The slope variation results are presented in figure 13(a-f).

Initially, we examine the direct dependence on the slope $s$ for three values of $\delta$ (figure 13a-c). The bolus size $S_b$ dependence on $s$ is presented in figure 13(a). For $\delta = 0.025$ and 0.4 m, maxima in bolus size are obtained at approximately $s = 0.123$. For $\delta = 0.2$ m, there is a monotonic decrease in the bolus size as the topographic slope is increased. Based on the behavior observed for the extreme cases, there is a slope shallower than $s = 0.105$ that maximizes the size of the bolus for the case $\delta = 0.2$ m. This conclusion is supported by the fact that there is no bolus generated when there is no slope. With respect to the displacement upslope $D_b$, presented in figure 13(b), steeper slopes result in boluses whose displacement along the slope decrease monotonically with $s$ for the range of slopes studied. This same trend is not observed if, instead of the displacement upslope $D_b$, the quantity plotted is the effective vertical displacement $\Delta z_b$. While $D_b$ decreases with $s$, the vertical displacement $\Delta z_b$ is observed to be nearly constant as $s$ is varied. This nearly constant $\Delta z_b$ behavior indicates that the stratification alone plays a major role in setting $\Delta z_b$, and $D_b$ is the projection of $\Delta z_b$ along the slope, resulting in longer displacements for shallower slopes and weaker stratifications.

Finally, the direct dependence of $S_b$, $D_b$ and $\Delta z_b$ on $\delta$ for three values of the slope $s$ is presented in figure 13(d-f). The bolus size dependence in figure 13(d) demonstrates that for all three values of $s$, the maximum $S_b$ is obtained for $\delta = 0.15$ m. That means that the stratification corresponding to maximum bolus size is independent of the topographic slope for the range of values considered. This is further evidence against the relationship between maximum transport and a topographic slope that matches the internal wave beam slope. The displacement upslope is presented in figure 13(e) and reveals that the displacement increases nearly monotonically with $\delta$. Figure 13(f) presents the corresponding vertical displacement $\Delta z_b$, and a close correspondence between the different $s$ cases is observed.

It is important to notice that the topographic slope plays a role in determining when the ball-hook or hook-sliver bolus category transitions occur with respect to the pycnocline thicknesses. Also, note that for $s = 0.105$ and $\delta = 0.3$ m we are, for the only time in all the cases analyzed in this paper, very close to the transitioning point between hook and sliver bolus categories, and both outputs are plotted with empty symbols. The output from the method as described in 3.1 corresponds to the hook case, but when we verify the robustness of the analysis by adding an extra cluster in the $K$-means step, the analysis produces a sliver-shaped bolus. This was the only case where this robustness analysis modified the bolus.
4.5 Dimensionless analysis

The parametric studies presented in §§4.1-4.4 focus on varying one physical parameter at a time and evaluating how that parameter, in conjunction with the pycnocline thickness, impacts the properties of the bolus. Previous efforts have utilized dimensionless parameters to provide arguments about how the bolus will break, properties of the bolus, and the amount of mixing induced by the breaking process [Venayagamoorthy and Fringer 2007; Aghsaee et al. 2010; Sutherland et al. 2013; Moore et al. 2016]. Most of these studies focused on shoaling internal solitary waves of depression in a two-layer or nearly-two-layer density stratification, where the forcing mechanism and the resulting wave properties can be directly controlled. The mode-1 generated, nonlinear internal waves in a symmetric stratification simulated here do not lend themselves to a simple isolation of the dimensionless numbers or the wave properties. Nevertheless, it is possible to recast our physical studies in a dimensionless perspective to relate our findings to previous studies.

The relevant dimensionless quantities for the addressed problem depend on wave characteristics that are computed here using the characteristics of the mid-depth isopycnal at time $t_0$: the amplitude $a$, defined as the crest vertical displacement, the wavelength $\lambda$, distance between the two zero isopycnal displacement points surrounding the crest, the wavenumber $k = 2\pi/\lambda$, and the instantaneous wave speed $c_x$, computed by tracking the wave crest. The typically investigated dimensionless numbers are the topographic slope $s$, the wave steepness $ka$, the Iribarren number $Ir = s/\sqrt{a/\lambda}$ comparing the steepness of the slope and the incoming wave [Sutherland et al. 2013], the wave Reynolds number $Re_w = c_x ka^2/\nu$, and...
the wave Richardson number $Ri_w = k\delta/(ka)^2$ [Thorpe, 1968] and the Froude number $Fr = \omega a/(g'H/4)^{1/2}$, with $g' = g\Delta\rho/\rho_H/2$ [Moore et al., 2016]. For each simulation, the wave properties and the corresponding dimensionless values are presented in the Supplementary Material.

The size and displacement dependence on pairs of dimensionless values is presented in figure 14. The first comparison evaluates the relationship between $Ir$ and $k\delta$ in figure 14(a-b). Here, the size of the bolus decreases with increasing $Ir$ and there is an optimal $k\delta$ that produces the largest bolus. The optimal $k\delta$ appears to decrease as $Ir$ increases. The displacement of the bolus increases for increasing $k\delta$ and decreasing $Ir$. This trend is observed in figure 14(c-d) as $s$ is correlated to $\delta$. This results in an optimal $ka$ for bolus size. The displacement upslope increases with $ka$ and decreases as the topographic slope increases. However, the trend in figure 14(c) indicates that the vertical displacement of the bolus has a more complicated relationship with the slope. Next, the $Re_w$ and $Re_w$ are compared in figure 14(e-f). These two numbers are most strongly varied by changing the energy at the breaking point $E_b$. Based on the simulations performed, the size of the bolus increases with $Re_w$, but there is a non-monotonic trend with $Re_w$. The displacement of the bolus grows with both $Re_w$ and $Re_w$. It is interesting to note that for $Re_w < 80$ no bolus is produced. Finally, the dependence with $Fr$ is presented in figure 14(g-h). These figures demonstrate that the bolus size grows with $Fr$ but there is limited change in displacement.

The calculation of these dimensionless numbers allows us to relate the bolus breaking to those observed in other studies. Aghsaee et al. [2010] numerically studied the breaking of two-layer waves for a range of topographic and wave slopes. Our topographic slope ranges from $s = 0.105-0.231$ and the wave slope ranges from 0.017-0.025. Based on their study, these boluses will be classified as “surging.” A “surging” event is identified by minimal breaking of the incoming wave interface, surging of the wave upslope, and the production of a bolus. This aligns well with what is being observed in our simulations. As the wave slope increases, the classification will transition to collapsing, which reflects a more abrupt impact of the shoaling wave on the slope. Interestingly, the most abrupt bolus generation occurs for smaller $\delta$, which corresponds to relatively shallower slopes. This discrepancy may be a result in the differences of our waves of elevation versus their waves of depression. Alternatively, this may be a byproduct of the larger pycnocline thickness. While Arthur and Fringer [2016] consider different pycnocline thicknesses and have classified their waves, the minimum wave steepness they consider is 0.07 where our maximum steepness is 0.025. It is worth noting that they do identify this case as surging.

5 Conclusions

We have presented how the pycnocline thickness impacts internal wave bolus dynamics by applying Lagrangian coherent structure techniques to objectively identify and quantify their transport properties. By modelling the continuous density stratification as a hyperbolic tangent profile with a tunable pycnocline thickness, the dynamics of the bolus and the resulting property trends reveal significant differences in transport properties when compared to two-layer density stratifications models. To identify the bolus in a continuous stratification, a spectral clustering method was used to detect the coherent structure. The results obtained indicate that boluses tend to be larger and travel for longer distances in continuously stratified systems like those found in the ocean. The parametric study reveals the relationship between transport properties of internal wave boluses and the pycnocline thickness, incoming wave energy, density change in the pycnocline and topographic slope. Non-monotonic trends of the bolus size with these parameters indicate that there may be situations where the bolus size can be maximized.

For a laboratory-scale system with continuous density stratification controlled by the pycnocline thickness $\delta$, the corresponding fundamental vertical mode internal wave was numerically forced to generate an internal wave that propagates towards a constant-slope topography. Direct simulations of the wave shoaling onto the topography produced the Eulerian velocity field describing the bolus formation and propagation up the slope. This velocity field was used to compute Lagrangian trajectories of passive tracers, and those trajectories were clustered using a spectral clustering method [Hadjighasem et al., 2016] to objectively identify the bolus. Position and velocity of the Lagrangian bolus propagating up the slope were compared to isopycnals located well above the breaking, and it was determined that the leading-edge of the bolus propagates immediately in front of the wave crest driving the bolus up the slope. Such an objective detection of tracers composing the bolus enabled the measurement of transport-related quantities such as size and displacement upslope. These quantities were then used as a basis of comparison for the parametric study.
Figure 14: Relationship between dimensionless parameters and bolus size $S_b$ and distance upslope $D_b$. Marker sizes represent the magnitude of the plotted quantities, and colors represent different studies: pycnocline thickness variation (gray), energy variation (green), density change variation (blue), slope variation (red). Crosses represent cases for which no bolus is formed.
When considering incoming waves of the same energy, the pycnocline thickness was observed to play an important role in determining the bolus size and displacement upslope. A maximum bolus size was obtained for an intermediate thickness case, not for the extreme two-layer density or broadest pycnocline stratified fluid cases. In addition to changing transport properties, the transition thickness also impacted the shape of the bolus. As $\delta$ increases, the shape transitions from ball to hook to sliver with the hook producing the largest bolus and the sliver propagating the furthest. This pattern was observed for all parameter studies with constant energy. When varying the wave energy, intuition would suggest that, as the internal wave energy is increased, the bolus size and displacement would also increase. While this is generally the case, our study also reveals transitions in the bolus shape that correspond to a non-monotonic trend. These transitions occur at different pycnocline thicknesses for different wave energies. The method also successfully identified the transition to a scenario where no bolus is formed for low energetic waves. The impact of the stratification density change was also investigated, and it was observed that boluses in stronger stratifications are bigger for thinner pycnoclines, while boluses in weaker stratifications are bigger for broader thicknesses. A variation of the topographic slope concluded that, for the range of slopes studied, shallower slopes allow for boluses to propagate for longer distances up the slope. However, the vertical displacement of the bolus appears to be driven by the particular stratification and is not dependent on the slope itself. For boluses formed on different slopes, the bolus size was maximal for the same pycnocline thickness case, independent of the slope. The simulation results were recast in dimensionless parameters revealing a non-monotonic relationship between $k\delta$ and the size of the bolus. The displacement upslope increased with $k\delta$, $ka$, $Ri_w$, $Re_w$, and decreased with $s$ and $Ir$. Finally, based on the dimensionless numbers, the boluses would fall into the surging category based on Aghsae et al. [2010], which matches the qualitative behaviour.

The results presented in this paper demonstrate that the incorporation of a more representative model for the density stratification, which accounts for the pycnocline thickness, will lead to different transport characteristics than the ones obtained using either the two-layer density or broadest pycnocline stratification models. The results show, in particular, that both of those simplified models could be underestimating the amount of material being transported by internal wave boluses on the continental slopes. In §4.1, for instance, it was observed that the optimal pycnocline thickness produces a bolus twice as large as the nearly two-layer stratification, and five times as large as the broadest pycnocline stratification. Larger differences were obtained for the variable energy and variable density change cases. There were cases for which boluses were not formed for the nearly two-layer or broadest pycnocline stratification cases, but were formed for continuous stratifications. While the study presented here is based on a laboratory-scale system, the trends established in this paper may help identify when and where boluses may be important in the ocean. For instance, high energy waves in a stratification with relatively small density change, shoaling in gradual sloping regions, may produce large boluses that propagate long distances depending on the pycnocline thickness.

Extending our results to ocean scale systems is a natural next step. An ocean model will require the incorporation of a turbulence model, Coriolis forces, a background tidal flow and realistic, more irregular topography geometries to estimate the velocity field. A suitable ocean model should also incorporate uncertainty, and the robustness of the spectral clustering method implemented to uncertainties is still an open question. Also, accounting for the inertia of the tracers can have an impact on the final set of trajectories and affect our conclusion for the amount of transport induced by boluses. Another extension of this work would be to study the shoaling of internal waves and bolus propagation on a bed of sediments. The bolus corresponding to a high shear event, it induces a large upwelling velocity from the topography at the front of the bolus, which could result in high amounts of resuspension and sediment transport. Accounting for a deformable boundary and resuspension, using a topography that consists of loosely-packed sediments that rearrange with the flow, could yield insight into the gradual transformation of the underlying topography. While internal wave induced resuspension and sediment transport has been observed in the ocean [Hosegood et al., 2004; Quaresma et al., 2007] and numerically reproduced [Stastna and Lamb, 2008; Bourgault et al., 2014], an estimate of the influence of boluses remains to be determined.

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References

Abernathey, R. and Haller, G. (2018). Transport by lagrangian vortices in the eastern pacific. *J. Phys. Oceanogr.*, 48:667–685.

Aghsaee, P., Boegman, L., and Lamb, K. G. (2010). Breaking of shoaling internal solitary waves. *Journal of Fluid Mechanics*, 659:289–317.

Aikman III, F. (1984). Pycnocline development and its consequences in the middle atlantic bight. *Journal of Geophysical Research: Oceans*, 89(C1):685–694.

Alford, M. H. (2003). Redistribution of energy available for ocean mixing by long-range propagation of internal waves. *Nature*, 423:159–162.

Alford, M. H., Peacock, T., MacKinnon, J. A., Nash, J. D., Buijsman, M. C., Centurioni, L. R., Chao, S.-Y., Chang, M.-H., Farmer, D. M., Fringer, O. B., et al. (2015). The formation and fate of internal waves in the south china sea. *Nature*, 521(7550):65.

Allshouse, M. R., Lee, F. M., Morrison, P. J., and Swinney, H. L. (2016). Internal wave pressure, velocity, and energy flux from density perturbations. *Phys. Rev. Fluids*, 1:014301.

Allshouse, M. R. and Peacock, T. (2015). Lagrangian based methods for coherent structure detection. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 25(9):097617.

Arthur, R. S. and Fringer, O. B. (2014). The dynamics of breaking internal solitary waves on slopes. *Journal of Fluid Mechanics*, 761:360–398.

Arthur, R. S. and Fringer, O. B. (2016). Transport by breaking internal gravity waves on slopes. *Journal of Fluid Mechanics*, 789:93–126.

Arthur, R. S., Koseff, J. R., and Fringer, O. B. (2017). Local versus volume-integrated turbulence and mixing in breaking internal waves on slopes. *Journal of Fluid Mechanics*, 815:169–198.

Benjamin, T. B. (1968). Gravity currents and related phenomena. *Journal of Fluid Mechanics*, 31(2):209–248.

Boegman, L., Ivey, G. N., and Imberger, J. (2005). The degeneration of internal waves in lakes with sloping topography. *Limnol. Oceanogr.*, 50:1620–1637.

Bourgault, D., Morsilli, M., Richards, C., Neumeier, U., and Kelley, D. E. (2014). Sediment resuspension and nepheloid layers induced by long internal solitary waves shoaling orthogonally on uniform slopes. *Continental Shelf Research*, 72:21–33.

Cacchione, D. and Wunsch, C. (1974). Experimental study of internal waves over a slope. *Journal of Fluid Mechanics*, 66(2):223–239.

Carter, G. S., Gregg, M. C., and Lien, R.-C. (2005). Internal waves, solitary-like waves, and mixing on the monterey bay shelf. *Continental Shelf Research*, 25(12-13):1499–1520.

Dauxois, T., Didier, A., and Falcon, E. (2004). Observation of near-critical reflection of internal waves in a stably stratified fluid. *Physics of Fluids*, 16(6):1936–1941.

Dettner, A., Paoletti, M. S., and Swinney, H. L. (2013). Internal tide and boundary current generation by tidal flow over topography. *Phys. Fluids*, 25:1–13.

Duda, T. F., Lynch, J. F., Irish, J. D., Beardsley, R. C., Ramp, S. R., Chiu, C.-S., Tang, T. Y., and Yang, Y.-I. (2004). Internal tide and nonlinear internal wave behavior at the continental slope in the northern south china sea. *IEEE Journal of Oceanic Engineering*, 29(4):1105–1130.

Fringer, O. B. and Street, R. L. (2003). The dynamics of breaking progressive interfacial waves. *Journal of Fluid Mechanics*, 494:319–353.
Froyland, G. and Junge, O. (2018). Robust fem-based extraction of finite-time coherent sets using scattered, sparse, and incomplete trajectories. *SIAM Journal of Dynamical Systems*, 17(2):1891–1924.

Froyland, G. and Padberg-Gehle, K. (2015). A rough-and-ready cluster-based approach for extracting finite-time coherent sets from sparse and incomplete trajectory data. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 25(8):087406.

Froyland, G., Santitissadeekorn, N., and Monahan, A. (2010). Transport in time-dependent dynamical systems: Finite-time coherent sets. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 20:043116.

Fructus, D., Carr, M., Grue, J., Jensen, A., and Davies, P. A. (2009). Shear-induced breaking of large internal solitary waves. *Journal of Fluid Mechanics*, 620:1–29.

Gerkema, T. and Zimmerman, J. T. F. (2008). An introduction to internal waves. *Lecture Notes, Royal NIOZ, Texel*, 207.

Hadjighasem, A., Karrasch, D., Teramoto, H., and Haller, G. (2016). Spectral-clustering approach to lagrangian vortex detection. *Physical Review E*, 93(6):063107.

Haller, G. (2002). Lagrangian coherent structures from approximate velocity data. *Phys. Fluids A*, 14:1851–1861.

Haller, G. and Beron-Vera, F. J. (2013). Coherent lagrangian vortices: The black holes of turbulence. *Journal of Fluid Mechanics*, 731.

Haller, G., Hadjighasem, A., Farazmand, M., and Huhn, F. (2016). Defining coherent vortices objectively from the vorticity. *Journal of Fluid Mechanics*, 795:136–173.

Ham, F. and Iaccarino, G. (2004). Energy conservation in collocated discretization schemes on unstructured meshes. In *Annual Research Briefs*, pages 3–14. Stanford University.

Helfrich, K. R. (1992). Internal solitary wave breaking and run-up on a uniform slope. *Journal of Fluid Mechanics*, 243:133–154.

Helfrich, K. R. and Melville, W. K. (1986). On long nonlinear internal waves over slope-shelf topography. *Journal of Fluid Mechanics*, 167:285–308.

Helfrich, K. R. and Melville, W. K. (2006). Long nonlinear internal waves. *Annu. Rev. Fluid Mech.*, 38:395–425.

Hosegood, P., Bonnin, J., and van Haren, H. (2004). Solibore-induced sediment resuspension in the faeroe-shetland channel. *Geophysical Research Letters*, 31(9).

Inall, M. E., Rippeth, T. P., and Sherwin, T. J. (2000). Impact of nonlinear waves on the dissipation of internal tidal energy at a shelf break. *Journal of Geophysical Research: Oceans*, 105(C4):8687–8705.

King, B., Zhang, H. P., and Swinney, H. L. (2009). Tidal flow over three-dimensional topography in a stratified fluid. *Phys. Fluids*, 21:116601.

Klymak, J. M. and Moum, J. N. (2003). Internal solitary waves of elevation advancing on a shoaling shelf. *Geophysical Research Letters*, 30(20).

Kunze, E. (2003). A review of oceanic salt-fingering theory. *Progress in Oceanography*, 56(3-4):399–417.

Lamb, K. G. (2003). Shoaling solitary internal waves: on a criterion for the formation of waves with trapped cores. *Journal of Fluid Mechanics*, 478:81–100.

Lamb, K. G. (2014). Internal wave breaking and dissipation mechanisms on the continental slope/shelf. *Annual Review of Fluid Mechanics*, 46:231–254.

Lamb, K. G. and Farmer, D. (2011). Instabilities in an internal solitary-like wave on the oregon shelf. *Journal of Physical Oceanography*, 41(1):67–87.
Lee, F. M., Allshouse, M. R., Swinney, H. L., and Morrison, P. J. (2018). Internal wave energy flux from density perturbations in nonlinear stratifications. *J. Fluid Mech.*, 856:898–920.

Lee, F. M., Paoletti, M. S., Swinney, H. L., and Morrison, P. J. (2014). Experimental determination of radiated internal wave power without pressure field data. *Phys. Fluids*, 26:046606.

Legg, S. and Adcroft, A. (2003). Internal wave breaking at concave and convex continental slopes. *Journal of Physical Oceanography*, 33(11):2224–2246.

Liu, Q., Jia, Y., Liu, P., Wang, Q., and Chu, P. C. (2001). Seasonal and intraseasonal thermocline variability in the central south china sea. *Geophysical Research Letters*, 28(23):4467–4470.

Long, R. R. (1956). Solitary waves in the one- and two-fluid system. *Tellus*, 8(4):460–471.

Long, R. R. (1965). On the buoysnesq approximation and its role in the theory of internal waves. *Tellus*, 17(1):46–52.

Maderich, V. S., Van Heijst, G. J. F., and Brandt, A. (2001). Laboratory experiments on intrusive flows and internal waves in a pycnocline. *Journal of Fluid Mechanics*, 432:285–311.

Mahesh, K., Constantinescu, G., and Moin, P. (2004). A numerical method for large-eddy simulation in complex geometries. *J. Comput. Phys.*, 197:215–240.

Masunaga, E., Arthur, R. S., Fringer, O. B., and Yamazaki, H. (2017). Sediment resuspension and the generation of intermediate nepheloid layers by shoaling internal bores. *Journal of Marine Systems*, 170:31–41.

Masunaga, E., Homma, H., Yamazaki, H., Fringer, O. B., Nagai, T., Kitade, Y., and Okayasu, A. (2015). Mixing and sediment resuspension associated with internal bores in a shallow bay. *Continental Shelf Research*, 110:85–99.

Maxworthy, T., Leilich, J. S. J. E., Simpson, J. E., and Meiburg, E. H. (2002). The propagation of a gravity current into a linearly stratified fluid. *Journal of Fluid Mechanics*, 453:371–394.

Mercier, M. J., Martinand, D., Mathur, M., Gostiaux, L., Peacock, T., and Dauxois, T. (2010). New wave generation. *J. Fluid Mech.*, 657:308–334.

Michallet, H. and Ivey, G. N. (1999). Experiments on mixing due to internal solitary waves breaking on uniform slopes. *Journal of Geophysical Research: Oceans*, 104(C6):13467–13477.

Moore, C. D., Koseff, J. R., and Hult, E. L. (2016). Characteristics of bolus formation and propagation from breaking internal waves on shelf slopes. *Journal of Fluid Mechanics*, 791:260–283.

Moore, S. E. and Lien, R.-C. (2007). Pilot whales follow internal solitary waves in the south china sea. *Marine mammal science*, 23(1):193–196.

Moun, J. N., Farmer, D. M., Smyth, W. D., Armi, L., and Vagle, S. (2003). Structure and generation of turbulence at interfaces strained by internal solitary waves propagating shoreward over the continental shelf. *Journal of Physical Oceanography*, 33(10):2093–2112.

Moun, J. N., Klymak, J. M., Nash, J. D., Perlin, A., and Smyth, W. D. (2007). Energy transport by nonlinear internal waves. *Journal of Physical Oceanography*, 37(7):1968–1988.

Munk, W. and Wunsch, C. (1998). Abyssal recipes II: Energetics of tidal and wind mixing. *Deep Sea Res., Part I*, 45:1977–2010.

Osborne, A. R. and Burch, T. L. (1980). Internal solitons in the andaman sea. *Science*, 208(4443):451–460.

Paoletti, M. S., Drake, M., and Swinney, H. L. (2014). Internal tide generation in nonuniformly stratified deep oceans. *J. Geophys. Res.*, 119:1943–1956.

Pedlosky, J. (2013). *Geophysical fluid dynamics*. Springer Science & Business Media.
Quaresma, L. S., Vitorino, J., Oliveira, A., and da Silva, J. (2007). Evidence of sediment resuspension by nonlinear internal waves on the western portuguese mid-shelf. *Marine Geology*, 246(2-4):123–143.

Ray, R. D. and Mitchum, G. T. (1996). Surface manifestation of internal tides generated near hawaii. *Geophysical Research Letters*, 23(16):2101–2104.

Sandstrom, H. and Elliott, J. A. (1984). Internal tide and solitons on the scotian shelf: A nutrient pump at work. *Journal of Geophysical Research: Oceans*, 89(C4):6415–6426.

Sandstrom, H. and Oakey, N. S. (1995). Dissipation in internal tides and solitary waves. *Journal of Physical Oceanography*, 25(4):604–614.

Serra, M., Sathe, P., Beron-Vera, F., and Haller, G. (2017). Uncovering the edge of the polar vortex. *Journal of the Atmospheric Sciences*, 74(11):3871–3885.

Sigman, D. M., Jaccard, S. L., and Haug, G. H. (2004). Polar ocean stratification in a cold climate. *Nature*, 428(6978):59.

Stastna, M. and Lamb, K. G. (2008). Sediment resuspension mechanisms associated with internal waves in coastal waters. *Journal of Geophysical Research: Oceans*, 113(C10).

Susanto, R., Mitnik, L., and Zheng, Q. (2005). Ocean internal waves observed. *Oceanography*, 18(4):80.

Sutherland, B. R., Barrett, K. J., and Ivey, G. N. (2013). Shoaling internal solitary waves. *Journal of Geophysical Research: Oceans*, 118(9):4111–4124.

Thorpe, S. A. (1968). On the shape of progressive internal waves. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 263(1145):563–614.

Thorpe, S. A. (1971). Experiments on the instability of stratified shear flows: miscible fluids. *Journal of Fluid Mechanics*, 46(2):299–319.

Troy, C. D. and Koseff, J. R. (2005). The instability and breaking of long internal waves. *Journal of Fluid Mechanics*, 543:107–136.

Venayagamoorthy, S. K. and Fringer, O. B. (2006). Numerical simulations of the interaction of internal waves with a shelf break. *Physics of Fluids*, 18(7):076603.

Venayagamoorthy, S. K. and Fringer, O. B. (2007). On the formation and propagation of nonlinear internal boluses across a shelf break. *Journal of Fluid Mechanics*, 577:137–159.

von Luxburg, U. (2007). A tutorial on spectral clustering. *Statistics and computing*, 17(4):395–416.

Walter, R. K., Woodson, C. B., Arthur, R. S., Fringer, O. B., and Monismith, S. G. (2012). Nearshore internal bores and turbulent mixing in southern monterey bay. *Journal of Geophysical Research: Oceans*, 117(C7).

Wang, Y.-H., Dai, C.-F., and Chen, Y.-Y. (2007). Physical and ecological processes of internal waves on an isolated reef ecosystem in the south china sea. *Geophysical Research Letters*, 34(18).

White, B. L. and Helfrich, K. R. (2008). Gravity currents and internal waves in a stratified fluid. *Journal of Fluid Mechanics*, 616:327–356.

Wunsch, C. and Ferrari, R. (2004). Vertical mixing, energy and the general circulation of the oceans. *Annu. Rev. Fluid Mech.*, 36:281–314.

Zhang, H. P., King, B., and Swinney, H. L. (2008). Resonant generation of internal waves on a model continental slope. *Physical review letters*, 100(24):244504.

Zhang, L. and Swinney, H. L. (2014). Virtual seafloor reduces internal wave generation by tidal flow. *Phys. Rev. Lett.*, 112:104502.