Abstract. We introduce a homogeneous multigrid method in the sense that it uses the same HDG discretization scheme for Poisson’s equation on all levels. In particular, we construct a stable injection operator and prove optimal convergence of the method under the assumption of elliptic regularity. Numerical experiments underline our analytical findings.

1. Introduction

While hybridizable discontinuous Galerkin (HDG) methods have become popular in mathematics and applications over the last years, literature on efficient solution of the resulting discrete systems is still scarce. In this article, we propose to our knowledge the first multigrid preconditioner for such methods which is homogeneous in the sense that it uses the same discretization scheme on all levels. Such methods are important, since they have the same mathematical properties on all levels. They can also be advantageous from a computational point of view, since their data structures and execution patterns are more regular.

HDG methods have been gaining popularity in the last decade. Originally, they were analyzed for the Laplacian, see for instance [12] for an overview. Meanwhile, they have been applied to stationary [11, 14, 15, 17, 28, 29] and instationary [25] Stokes systems, to the locking-free discretization of problems in elasticity [19], as well as to plates [9, 24] and beams [7]. However, only few results have been achieved for solving the large systems of equations arising from this type of numerical method, while multigrid [21] and domain decomposition [18] solvers for earlier discontinuous Galerkin (DG) schemes have been available for many years.

The first multigrid method for HDG discretizations was introduced in [10, 31]. It is basically a two-level method, where the “coarse space” consists of the piecewise linear, conforming finite element space. The coarse grid solver consists of a conforming multigrid method for lowest
order elements. In [22], similar results have been applied to hybridized mixed (e.g. Raviart–Thomas (RT)) elements.

A BPX preconditioner for non-standard finite element methods, including hybridized RT, BDM, the weak Galerkin, and Crouzeix–Raviart methods, is analyzed in [26] and a two level algorithm for HDG methods for the diffusion problem is presented in [27]. All of the aforementioned methods utilize the piecewise linear conforming finite element space as the auxiliary space.

Two-level analysis of HDG methods seems equally rare. A method for high order HDG discretizations is introduced in [30]. Here, the authors focus on showing that standard $p$-version domain decomposition techniques can be applied and prove condition number estimates on tetrahedral meshes polylogarithmic in $p$. More recently, a domain decomposition for a hybridized Stokes problem was presented in [1], but it does not discuss a coarse space. Moreover, [23] discusses local Fourier analysis of interior penalty based multigrid methods for tensor-product type polynomials in two spatial dimensions.

The methods known so far are heterogeneous in the sense that the multigrid cycle is not performed on the HDG discretization itself, but on a surrogate scheme. In view of future generalization beyond second order elliptic problems, we decided to devise a homogeneous method which uses the same discretization scheme on all levels, and thus only employs a single finite element method. The analysis uses the abstract framework developed for noninherited forms in [16]. It is based on arguments found in [20, 31]. Nevertheless, the coarse grid operator is genuinely HDG and of the same type as the fine grid operator. Only the injection operator from coarse to fine level uses continuous interpolation in an intermediate step.

Since we focus on new coarse spaces and intergrid operators, we rely on standard smoothers and analyze a standard Poisson problem with elliptic regularity to present the basic ideas of our method. However, the regularity assumption could be weakened utilizing the ideas of [4], of [5] (which analyzes nonconforming multigrid methods), of [21] (which deals with multigrid methods for DG), and of [22] at the cost of more technicalities within our proofs. As for robust smoothers for higher order methods, this will be subject to further research.

The remainder of this paper is structured as follows: In Section 2, we review the HDG method for elliptic PDEs. Furthermore, an overview over the used function spaces, scalar products, and operators is given. Section 3 is devoted to the definition of the multigrid and states its main convergence result. Sections 4 and 5 verify the assumptions of the main convergence result, while Section 6 underlines its validity by numerical experiments. Short conclusions wrap up the paper.
2. Model equation and discretization

We consider the standard diffusion equation in mixed form defined on a polygonally bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial \Omega$. We assume homogeneous Dirichlet boundary conditions on $\partial \Omega$. Thus, we approximate solutions $(u, q)$ of

$$\nabla \cdot q = f \quad \text{in } \Omega,$$

$$q + \nabla u = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

for a given function $f$. In the analysis, we will assume elliptic regularity, namely $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, such there is a constant $c > 0$ for which holds

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}.$$ (2.2)

Here, and in the following, $L^2(\Omega)$ denotes the space of square integrable functions on $\Omega$ with inner product and norm

$$(u, v) := \int_{\Omega} uv \, dx, \quad \text{and } \|u\|_{0}^2 := (u, u)_0.$$ (2.3)

The space $H^k(\Omega)$ is the Sobolev space of $k$-times weakly differentiable functions with derivatives in $L^2(\Omega)$. We note that the assumption of homogeneous boundary data was introduced for simplicity of presentation and can be lifted by standard arguments.

2.1. Spaces for the HDG multigrid method. Starting out from a subdivision $\mathcal{T}_0$ of $\Omega$ into simplices, we construct a hierarchy of meshes $\mathcal{T}_\ell$ for $\ell = 1, \ldots, L$ recursively by refinement, such that each cell of $\mathcal{T}_{\ell-1}$ is the union of several cells of mesh $\mathcal{T}_\ell$. We assume that the mesh is regular, such that each facet of a cell is either a facet of another cell or on the boundary. Furthermore, we assume that the hierarchy is shape regular and thus the cells are neither anisotropic nor otherwise distorted. We call $\ell$ the level of the quasi-uniform mesh $\mathcal{T}_\ell$ and denote by $h_\ell$ the characteristic length of its cells. We assume that refinement from one level to the next is not too fast, such that there is a constant $c_{\text{ref}} > 0$ with

$$h_\ell \geq c_{\text{ref}} h_{\ell-1}.$$ (2.4)

By $\mathcal{F}_\ell$ we denote the set of faces of $\mathcal{T}_\ell$. The subset of faces on the boundary is

$$\mathcal{F}_\ell^0 := \{ F \in \mathcal{F}_\ell : F \subset \partial \Omega \}.$$ (2.5)

Moreover, we define $\mathcal{F}_\ell^T := \{ F \in \mathcal{F}_\ell : F \subset \partial T \}$ as the set of faces of a cell $T \in \mathcal{T}_\ell$. On the set of faces, we define the space $L^2(\mathcal{F}_\ell)$ as the
space of square integrable functions with the inner product
\[ \langle \lambda, \mu \rangle \ell = \sum_{T \in \mathcal{T}_\ell} \int_{\partial T} \lambda \mu \, d\sigma, \]
and its induced norm \[ ||\mu||_2^\ell = \langle \mu, \mu \rangle \ell. \]
Note that interior faces appear twice in this definition such that expressions like \[ \langle \lambda, \mu \rangle \ell \] with possibly discontinuous \( u|_T \in H^1(T) \) for all \( T \in \mathcal{T}_\ell \) and \( \mu \in L^2(F_\ell) \) are defined without further ado. Additionally, we define an inner product commensurate with the \( L^2 \)-inner product in the bulk domain, namely
\[ \langle \lambda, \mu \rangle \ell = \sum_{T \in \mathcal{T}_\ell} \frac{|T|}{|\partial T|} \int_{\partial T} \lambda \mu \, d\sigma \approx \sum_{F \in \mathcal{F}_\ell} h_F \int_{F} \lambda \mu \, d\sigma. \]
Its induced norm is \[ ||\mu||_2^\ell = \langle \mu, \mu \rangle \ell. \]

Let \( p \geq 1 \) and \( P_p \) be the space of (multivariate) polynomials of degree up to \( p \). Then, we define the space of piecewise polynomials on the skeleton by
\[ M_\ell := \left\{ \lambda \in L^2(\mathcal{F}_\ell) \mid \lambda|_F \in P_p \quad \forall F \in \mathcal{F}_\ell, \quad \lambda|_{F_\ell} = 0 \quad \forall F \in F_\ell \right\}. \]

The HDG method involves a local solver on each mesh cell \( T \in \mathcal{T}_\ell \), producing cellwise approximations \( u_T \in V_T \) and \( q_T \in W_T \) of the functions \( u \) and \( q \) in equation (2.1), respectively. We choose \( V_T = P_p \). Then, choosing \( W_T = P^d_p \) yields the so-called hybridizable local discontinuous Galerkin (LDG-H) scheme. Our current analysis is in fact limited to this case and other choices require a modification of Lemma 4.7. We will also use the concatenations of the spaces \( V_T \) and \( W_T \), respectively, as a function space on \( \Omega \), namely
\[ V_\ell := \left\{ v \in L^2(\Omega) \mid v|_T \in V_T, \quad \forall T \in \mathcal{T}_\ell \right\}, \]
\[ W_\ell := \left\{ q \in L^2(\Omega; \mathbb{R}^d) \mid q|_T \in W_T, \quad \forall T \in \mathcal{T}_\ell \right\}. \]

2.2. Hybridizable discontinuous Galerkin method for the diffusion equation. The HDG scheme for (2.1) on a mesh \( \mathcal{T}_\ell \) consists of a local solver and a global coupling equation. The local solver is defined cellwise by a weak formulation of (2.1) in the discrete spaces \( V_T \times W_T \) and defining suitable numerical traces and fluxes. Namely, given \( \lambda \in M_\ell \) find \( u_T \in V_T \) and \( q_T \in W_T \), such that
\[ \int_T q_T \cdot p_T \, dx - \int_T u_T \nabla \cdot p_T \, dx = -\int_{\partial T} \lambda p_T \cdot \nu \, d\sigma \]
(2.10a)
\[ -\int_T q_T \cdot \nabla v_T \, dx + \int_{\partial T} (q_T \cdot \nu + \tau_i u_T) v_T \, d\sigma = \tau_i \int_{\partial T} \lambda v_T \, d\sigma \]
(2.10b)
hold for all \( v_T \in V_T \), and all \( p_T \in W_T \), and for all \( T \in \mathcal{T}_\ell \). Here, \( \nu \) is the outward unit normal with respect to \( T \) and \( \tau_i > 0 \) is the penalty coefficient. While the local solvers are implemented cell by cell, it is
helpful for the analysis to combine them by concatenation. Thus, the local solvers define a mapping
\[ M_\ell \to V_\ell \times W_\ell \]
\[ \lambda \mapsto (U_\ell \lambda, Q_\ell \lambda), \]
where for each cell \( T \in T_\ell \) holds \( U_\ell \lambda = u_T \) and \( Q_\ell \lambda = q_T \). In the same way, we define operators \( U_\ell f \) and \( Q_\ell f \) for \( f \in L^2(\Omega) \), where now the local solutions are defined by the system
\[
\int_T q_T \cdot p_T \, dx - \int_T u_T \nabla \cdot p_T \, dx = 0 \quad (2.12a)
\]
\[
- \int_T q_T \cdot \nabla v_T \, dx + \int_{\partial T} (q_T \cdot \nu + \tau_\ell u_T) v_T \, d\sigma = \int_T f v_T \, dx. \quad (2.12b)
\]

Once \( \lambda \) has been computed, the HDG approximation to (2.1) on mesh \( T_\ell \) will be computed as
\[
u_\ell = U_\ell \lambda + U_\ell f
q_\ell = Q_\ell \lambda + Q_\ell f
\]

The global coupling condition is derived through a discontinuous Galerkin version of mass balance and reads: Find \( \lambda \in M_\ell \), such that for all \( \mu \in M_\ell \)
\[
\sum_{T \in T_\ell} \sum_{F \in F(T)} \int_F (q_\ell \cdot \nu + \tau_\ell(u_\ell - \lambda)) \, \mu \, d\sigma = 0. \quad (2.14)
\]

In [12], it is shown that \( \lambda \in M_\ell \) is the solution of the coupled system from (2.10) to (2.14) if and only if it is the solution of
\[
a_\ell(\lambda, \mu) = b_\ell(\mu) \quad \forall \mu \in M_\ell,
\]
with
\[
a_\ell(\lambda, \mu) = \int_\Omega \mathcal{Q}_\ell \lambda \mathcal{Q}_\ell \mu \, dx + \sum_{T \in T_\ell} \int_{\partial T} \tau_\ell(U_\ell \lambda - \lambda)(U_\ell \mu - \mu) \, d\sigma, \quad (2.15b)
\]
\[
b_\ell(\mu) = \int_\Omega U_\ell f \, dx. \quad (2.15c)
\]

Furthermore, the bilinear form \( a_\ell(\lambda, \mu) \) is symmetric and positive definite. Thus, it induces a norm
\[
\| \mu \|_{a_\ell}^2 = a_\ell(\mu, \mu),
\]
We close this subsection by associating an operator \( A_\ell : M_\ell \to M_\ell \) with the bilinear form \( a_\ell(\cdot, \cdot) \) by the relation
\[
\langle A_\ell \lambda, \mu \rangle_\ell = a_\ell(\lambda, \mu) \quad \forall \mu \in M_\ell.
\]
2.3. The injection operator $I_\ell$. The difficulty of devising an “injection operator” $I_\ell : M_{\ell - 1} \to M_\ell$ originates from the fact that the finer mesh has edges which are not refinements of the edges of the coarse mesh. In [31], several possible injection operators are discussed, but turn out to be unstable. In order to assign reasonable values to these edges, we construct the injection operator in three steps. First, introduce the continuous finite element space

$$V_\ell^c := \{ u \in H^1_0(\Omega) \mid u|_T \in P_p(T) \quad \forall T \in \mathcal{T}_\ell \}. \quad (2.18)$$

We assume that the shape function basis on each mesh cell $T$ is defined through a Lagrange interpolation condition with respect to support points $x$ located on vertices, edges, and in the interior of the cell. Thus, a function in $V_\ell^c$ is uniquely determined by the values in these support points. We now define the continuous extension operator

$$\mathcal{U}_\ell^c : M_\ell \to V_\ell^c, \quad (2.19)$$

by the interpolation conditions

$$[\mathcal{U}_\ell^c \lambda](x) = \begin{cases} \overline{\lambda}(x) & \text{if } x \text{ is on the boundary of a face}, \\ \lambda(x) & \text{if } x \text{ is in the interior of a face}, \\ [\mathcal{U}_\ell \lambda](x) & \text{if } x \text{ is in the interior of a cell}. \end{cases} \quad (2.20)$$

Here, $\overline{\lambda}$ is the arithmetic mean of the values attained by $\lambda$ on different faces meeting in $x$.

The spaces $V_\ell^c$ are nested, such that there is a natural injection operator

$$I_\ell^c : V_{\ell - 1}^c \to V_\ell^c$$

$$u \mapsto u. \quad (2.21)$$

On $V_\ell^c$ the trace on edges is well defined, such that we can write

$$\gamma_\ell : V_\ell^c \to M_\ell$$

$$u \mapsto \gamma_\ell u. \quad (2.22)$$

Using these, we define the injection operator $I_\ell$ as the concatenation of extension, natural injection, and trace, namely

$$I_\ell : M_{\ell - 1} \to M_\ell$$

$$\lambda \mapsto \gamma_\ell I_\ell^c \mathcal{U}_{\ell - 1}^c \lambda. \quad (2.23)$$

By its definition, $I_\ell$ is the operator such that this diagram commutes:
Moreover, if \( \tau_\ell \sim h_\ell^{-1} \), the following Lemma can be proved similarly to [8, Thms. 3.6 and 3.8]:

**Lemma 2.1 (Boundedness).** The injection operator \( I_\ell \) is bounded in the sense that

\[
a_\ell(I_\ell \lambda, I_\ell \lambda) \lesssim a_{\ell-1}(\lambda, \lambda) \quad \forall \lambda \in M_{\ell-1}.
\]

(2.24)

### 2.4. Operators for the multigrid method and analysis.

After the discrete operator \( A_\ell \) and the injection operator \( I_\ell \) have been defined, we introduce the remaining operators here. First, there are two operators from \( M_\ell \) to \( M_{\ell-1} \), which replace the \( L^2 \) projection and the Ritz projection of conforming methods, respectively. They are \( \Pi_{\ell-1} \) and \( P_{\ell-1} \) defined by the conditions

\[
\Pi_{\ell-1} : M_\ell \to M_{\ell-1}, \quad \langle \Pi_{\ell-1} \lambda, \mu \rangle_{\ell-1} = \langle \lambda, I_\ell \mu \rangle_\ell \quad \forall \mu \in M_{\ell-1}. \quad (2.25)
\]

\[
P_{\ell-1} : M_\ell \to M_{\ell-1}, \quad a_{\ell-1}(P_{\ell-1} \lambda, \mu) = a_\ell(\lambda, I_\ell \mu) \quad \forall \mu \in M_{\ell-1}. \quad (2.26)
\]

The operator \( \Pi_{\ell-1} \) is used in the implementation, while \( P_{\ell-1} \) is key to the analysis.

The multigrid operator for preconditioning \( A_\ell \) will be defined in Section 3.1. It will be referred to as

\[
B_\ell : M_\ell \to M_\ell. \quad (2.27)
\]

It relies on a smoother

\[
R_\ell : M_\ell \to M_\ell, \quad (2.28)
\]

which can be defined in terms of Jacobi or Gauss-Seidel iterations, respectively. Denote by \( R_\ell^j \) the adjoint operator of \( R_\ell \) with respect to \( \langle \cdot, \cdot \rangle_\ell \) and define \( R_\ell^i \) by

\[
R_\ell^i = \begin{cases} 
R_\ell & \text{if } i \text{ is odd,} \\
R_\ell^j & \text{if } i \text{ is even.}
\end{cases} \quad (2.29)
\]

At this point, we have defined HDG versions of all operators involved in standard multigrid analysis. Additionally, we define the averaging operator

\[
I_{\ell}^{\text{avg}} : V_\ell \to V^c_\ell. \quad (2.30)
\]

Analog to (2.20), it is defined by interpolation in the support points \( x \) of the shape functions of the space \( V^c_\ell \), namely

\[
[I_{\ell}^{\text{avg}} u](x) = \overline{u}(x). \quad (2.31)
\]

Here \( \overline{u} \) is the arithmetic mean of the values \( u(x) \) from all mesh cells meeting at \( x \). For \( x \in \partial \Omega \), we let \( [I_{\ell}^{\text{avg}} u](x) = 0 \).

A summary of the different operators connecting the spaces can be found in Figure 1.
3. Multigrid method and main convergence result

We consider a standard (symmetric) V-cycle multigrid method for (2.15) (cf. [4, 16]) for which we conduct convergence analysis (as done in [16]). Thus, Section 3.1 is devoted to illustrating the multigrid method (cf. [16] where the algorithm is also taken from), while Section 3.2 states the main convergence result.

Let us begin citing an estimate for eigenvalues and condition numbers of the matrices $A_\ell$.

**Lemma 3.1.** Suppose that $\mathcal{T}_\ell$ is quasiuniform. Then, there are positive constants $C_1$ and $C_2$ independent of $\ell$ such that

$$C_1 \|\lambda\|_\ell^2 \leq a_\ell(\lambda, \lambda) \leq \beta_\ell C_2 h_\ell^{-2} \|\lambda\|_\ell^2, \quad \forall \lambda \in M_\ell,$$

where $\beta_\ell := 1 + (\tau_\ell h_\ell)^2$.

**Proof.** This is a Corollary of Theorem 3.2 in [10].

This implies that for the stiffness matrix, we can bound the condition number $\kappa_\ell$ by

$$\kappa_\ell \lesssim \beta_\ell h_\ell^{-2}$$

which implies that for all choices of $\tau_\ell$ satisfying $\tau_\ell \lesssim h_\ell^{-1}$ the condition number grows like $h_\ell^{-2}$. Here and in the following, $\lesssim$ has the meaning of smaller than or equal to up to a constant only dependent on the regularity constant of the mesh family and $c_{\text{ref}}$.

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**Figure 1.** Sketch of the different operators connecting function spaces of refinement levels $\ell$ and $\ell - 1$. Here, the operators needed to implement the multigrid preconditioner $B_\ell$ are depicted red, while operators only appearing in the analysis are in blue and spaces are black. The dashed arrows commute, while in general $\mathcal{U}_{\ell-1} \neq I_{\ell-1}^{\text{avg}} \circ \mathcal{U}_{\ell-1}$. 

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3.1. Multigrid algorithm. Let \( m \in \mathbb{N} \setminus \{0\} \) be the number of fine-level smoothing steps. We recursively define the multigrid operator of the refinement level \( \ell \)

\[
B_{\ell} : M_\ell \rightarrow M_\ell, \quad (3.3)
\]

by the following steps. Let \( B_0 = A_0^{-1} \). For \( \ell > 0 \), let \( x^0 = 0 \in M_\ell \). Then for \( \mu \in M_\ell \),

(1) Define \( x^i \in M_\ell \) for \( i = 1, \ldots, m \) by

\[
x^i = x^{i-1} + R_\ell^i (\mu - A_\ell x^{i-1}). \quad (3.4)
\]

(2) Set \( y^0 = x^m + I_\ell q \), where \( q \in M_{\ell-1} \) is defined as

\[
q = B_{\ell-1} \Pi_{\ell-1} (\mu - A_{\ell} x^m). \quad (3.5)
\]

(3) Define \( y^i \in M_\ell \) for \( i = 1, \ldots, m \) as

\[
y^i = y^{i-1} + R_{\ell}^{i+m} (\mu - A_{\ell} y^{i-1}). \quad (3.6)
\]

(4) Let \( B_{\ell} \mu = y^m \).

3.2. Main convergence result. The analysis of the multigrid method is based on the framework introduced in \([16]\). There, convergence is traced back to three assumptions. Let \( \lambda_{\ell}^A \) be the largest eigenvalue of \( A_{\ell} \), and

\[
K_{\ell} := (1 - (1 - R_{\ell} A_{\ell})(1 - R_{\ell}^\dagger A_{\ell})) A_{\ell}^{-1}. \quad (3.7)
\]

Then, there exists constants \( C_1, C_2, C_3 > 0 \) independent of the mesh level \( \ell \), such that there holds

- Regularity approximation assumption:

\[
|a_{\ell}(\lambda - I_{\ell} P_{\ell-1} \lambda, \lambda)| \leq C_1 \frac{\|A_{\ell} \lambda\|_A^2}{\lambda^A} \quad \forall \lambda \in M_{\ell}. \quad (A1)
\]

- Stability of the “Ritz quasi-projection” \( P_{\ell-1} \) and injection \( I_{\ell} : \)

\[
\|\lambda - I_{\ell} P_{\ell-1} \lambda\|_{a_{\ell}} \leq C_2 \|\lambda\|_{a_{\ell}} \quad \forall \lambda \in M_{\ell}. \quad (A2)
\]

- Smoothing hypothesis:

\[
\frac{\|\lambda\|^2_{\ell}}{\lambda^A} \leq C_3 (K_{\ell} \lambda, \lambda)_\ell. \quad (A3)
\]

Theorem 3.1 in \([16]\) reads

**Theorem 3.2.** Assume that \((A1), (A2), \) and \((A3)\) hold. Then for all \( \ell \geq 0 \),

\[
|a_{\ell}(\lambda - B_{\ell} A_{\ell} \lambda, \lambda)| \leq \delta a_{\ell}(\lambda, \lambda), \quad (3.8)
\]

where

\[
\delta = \frac{C_1 C_3}{m - C_1 C_3}. \quad (3.9)
\]

with the number of smoothing steps \( m > 2C_1 C_3 \).

Thus, in order to prove uniform convergence of the multigrid method, we will now set out to verify these assumptions.
4. Proof of (A1)

The main statement of this section is

**Theorem 4.1.** If (2.1) has elliptic regularity and \( \tau_{\ell} \approx h_{\ell}^{-1} \), then (A1) is satisfied.

4.1. Preliminaries. We begin the analysis with some basic results on the injection operator \( I_\ell \) and the “Ritz quasi-projection” \( P_{\ell-1} \). Afterwards, we deal with “quasi-orthogonality”, before we close the section analyzing the “reconstruction approximation”.

**Lemma 4.2** (Stability). The “Ritz quasi-projection” \( P_{\ell-1} : M_\ell \to M_{\ell-1} \) is stable in the sense that for all \( \lambda \in M_\ell \), we have

\[
\| P_{\ell-1} \lambda \|_{a_{\ell-1}} \lesssim \| \lambda \|_{a_\ell}. \tag{4.1}
\]

**Proof.** From (2.26), we can deduce that

\[
\| P_{\ell-1} \lambda \|_{a_{\ell-1}}^2 = a_\ell(\lambda, I_\ell P_{\ell-1} \lambda) \leq \| \lambda \|_{a_\ell} \| I_\ell P_{\ell-1} \lambda \|_{a_\ell} \lesssim \| \lambda \|_{a_\ell} \| P_{\ell-1} \lambda \|_{a_{\ell-1}}, \tag{4.2}
\]

where we used the Cauchy–Schwarz inequality for \( a_\ell(.,.) \) and the boundedness of \( I_\ell \) from Lemma 2.1. □

**Lemma 4.3.** The DG reconstructions of the injection operator admits the estimate

\[
\| U_{\ell-1} \mu - U_\ell I_\ell \mu \|_0 \lesssim h_\ell \| \mu \|_{a_{\ell-1}}, \quad \forall \mu \in M_{\ell-1}. \tag{4.3}
\]

**Proof.** We introduce intermediate approximations such that

\[
U_{\ell-1} \mu - U_\ell I_\ell \mu = (U_{\ell-1} \mu - I_{\ell-1}^{\text{avg}} U_{\ell-1} \mu) + (I_{\ell-1}^{\text{avg}} U_{\ell-1} \mu - U_{\ell-1} \mu) + (U_{\ell-1} \mu - U_\ell I_\ell \mu), \tag{4.4}
\]

and use triangle inequality. For \( \Xi_1 \), one utilizes the average operator approximation property \([2, (2.28)]\) and the definition of \( \| \cdot \|_{a_\ell} \) to obtain

\[
\| \Xi_1 \|_0^2 \lesssim h_{\ell-1} \| U_{\ell-1} \mu - \mu \|_{a_{\ell-1}}^2 \lesssim h_{\ell-1}^2 \frac{\| \mu \|_{a_{\ell-1}}^2}{\tau_{\ell-1}}. \tag{4.5}
\]

For the second term, similar to \([8, \text{Lem. 3.1}]\) we have

\[
\| \Xi_2 \|_0 \lesssim h_{\ell-1}^{1/2} \| U_{\ell-1} \mu - \mu \|_{a_{\ell-1}} \lesssim h_{\ell-1}^{1/2} \frac{\| \mu \|_{a_{\ell-1}}^{1/2}}{\tau_{\ell-1}^{1/2}}. \tag{4.6}
\]

If \( p = 1 \), \( \Xi_3 = 0 \), since \( I_\ell \mu \) is the restriction of a continuous function and \( U_\ell \) simply recovers \( U_{\ell-1} \mu \) in this case. If \( p \geq 2 \), we can use \([8, \text{Theo. 3.8}]\) to conclude

\[
\| \Xi_3 \|_0 \lesssim h_{\ell}^2 \| \Delta U_{\ell-1} \mu \|_0 \lesssim h_{\ell}^2 \| \nabla U_{\ell-1} \mu \|_0 \lesssim h_{\ell} \| \mu \|_{a_{\ell-1}}, \tag{4.7}
\]

where the last inequality is \([8, \text{Lem. 3.4}]\). □
In the following lemmas, we use the continuous linear finite element space
\[
\nabla_c^\ell := \{ u \in C(\Omega) : u|_T \in P_1(T) \ \forall T \in \mathcal{T}_\ell \text{ and } u = 0 \text{ on } \partial\Omega \}
\] (4.8)
to show quasi-orthogonality and a reconstruction approximation property of the method.

**Lemma 4.4** (Quasi-orthogonality). For all \( \lambda \in M_\ell \), we have that
\[
(\nabla w, Q_\ell \lambda - Q_{\ell-1}P_{\ell-1}\lambda)_0 = 0, \quad \forall w \in \nabla_c^{\ell-1}. 
\] (4.9)

**Proof.** For \( w \in \nabla_c^{\ell-1} \) define \( \mu := \gamma_{\ell-1}w \in M_{\ell-1} \). By definition, we immediately obtain
\[
I_\ell \mu = \gamma_\ell w, \\
Q_{\ell-1}\mu = Q_\ell I_\ell \mu = -\nabla w, \\
U_{\ell-1} \mu = U_\ell I_\ell \mu = w.
\] (4.10)

From the definitions of \( a_\ell \) and \( a_{\ell-1} \), we obtain
\[
a_{\ell-1}(P_{\ell-1}\lambda, \mu) = (Q_{\ell-1}P_{\ell-1}\lambda, Q_{\ell-1}\mu)_0 \\
+ \tau_{\ell-1} \langle [U_\ell P_{\ell-1}\lambda - P_{\ell-1}\lambda, U_{\ell-1} \mu - \mu]_{\ell-1}, a_\ell(\lambda, I_\ell \mu)_0 + \tau_\ell \langle [U_\ell \lambda - \lambda, U_\ell I_\ell \mu - I_\ell \mu]_{\ell-1}, (4.11)
\]
\[
a_{\ell}(\lambda, I_\ell \mu) = (Q_\ell \lambda, Q_\ell I_\ell \mu)_0 + \tau_\ell \langle [U_\ell \lambda - \lambda, U_\ell I_\ell \mu - I_\ell \mu]_{\ell-1}, (4.12)
\]

By definition of \( P_{\ell-1} \) in (2.26), these two terms are equal and thus we obtain the claimed result as their difference. \( \square \)

**Lemma 4.5** (Reconstruction approximation). If (2.1) has elliptic regularity and \( \tau_\ell \equiv h_\ell^{-1} \), then for all \( \lambda \in M_\ell \) there exists an auxiliary function \( \pi \in \nabla_c^{\ell-1} \) such that
\[
\|Q_\ell \lambda + \nabla \pi\|_0 + \|Q_{\ell-1}P_{\ell-1}\lambda + \nabla \pi\|_0 \lesssim h_\ell \|A_\ell \lambda\|_\ell. 
\] (4.13)

The proof of this lemma is based on an explicit construction of \( \pi \) based on the techniques in [10]. It is conducted in the following subsection.

**4.2. Proof of reconstruction approximation.** Following [10], we construct several auxiliary quantities in order to define \( \pi \) in Lemma 4.5. First, we define extension operators \( S_T \) on \( M_\ell \) for each cell \( T \) and each of its faces \( F_i \) into \( P_{p+1} \) on \( T \) by the interpolation conditions
\[
\langle S_T \lambda, \eta \rangle_{F_i} = \langle \lambda, \eta \rangle_{F_i}, \quad \forall \eta \in P_{p+1}(F_i), \\
\langle S_T \lambda, v \rangle_T = (U_\lambda, v)_T, \quad \forall v \in P_p(T). 
\] (4.14)
These are used to define the auxiliary inner product
\[
\langle \lambda, \mu \rangle_{\ell} = \sum_{T \in T_\ell} \frac{1}{\# F_\ell^T} \sum_{i=1}^{\# F_\ell^T} \int_T S_{T;i}\lambda S_{T;i}\mu \, dx,
\] (4.16)
and its corresponding norm \(\| \cdot \|_{\ell}\). Then, for \(\lambda \in M_\ell\) (which is fixed in this section, while \(u\) and \(\bar{u}\) depend on it) let \(\phi_\lambda \in M_\ell\) be defined by
\[
\langle \phi_\lambda, \mu \rangle_{\ell} = a_\ell(\lambda, \mu) = \langle A_\ell \lambda, \mu \rangle_\ell, \quad \mu \in M_\ell.
\] (4.17)
Thus, \(\phi_\lambda\) represents \(A_\ell \lambda\) in \(M_\ell\). Similarly, \(f_\lambda = \mathcal{U}_\ell \phi_\lambda \in V_\ell\) represents \(A_\ell \lambda\) on the whole domain. Based on these representations, we define \(\tilde{u} \in H_0^1(\Omega)\) by
\[
(\nabla \tilde{u}, \nabla v)_0 = (f_\lambda, v)_0, \quad \forall v \in H_0^1(\Omega),
\] (4.18)
and \(\tilde{\lambda}_\ell \in M_\ell\) by
\[
a_\ell(\tilde{\lambda}_\ell, \mu) = (f_\lambda, \mathcal{U}_\ell \mu)_0 \quad \forall \mu \in M_\ell.
\] (4.19)
In the remainder of this subsection, we show that \(\overline{\pi} = \overline{\mathcal{P}}_{\ell-1} \lambda_\ell\) can be used in Lemma 4.5. Here, the Ritz quasi-projection \(\overline{\mathcal{P}}_{\ell-1}\) to the cellwise linear coarse space \(\overline{V}_{\ell-1}\) is defined by
\[
\overline{\mathcal{P}}_{\ell-1} : M_\ell \to \overline{V}_{\ell-1}, \quad (\nabla \overline{\mathcal{P}}_{\ell-1} \lambda, \nabla w)_0 = a_\ell(\lambda, \gamma_{\ell} w) \quad \forall w \in \overline{V}_{\ell-1}.
\] (4.20)
We begin with an approximation result for \(\lambda_\ell\):

**Lemma 4.6.** If (2.1) has elliptic regularity, for all \(\lambda \in M_\ell\), we have
\[
\| \lambda - \tilde{\lambda}_\ell \|_{a_\ell} \lesssim h_\ell \| A_\ell \lambda \|_\ell \quad \text{and} \quad \| f_\lambda \|_0 \lesssim \| A_\ell \lambda \|_\ell.
\] (4.21)

**Proof.** This is [31, Lemma 5.10]. \(\square\)

We now prove the estimate for the second norm in Lemma 4.5. To this end, we introduce the auxiliary function \(\tilde{\lambda}_{\ell-1} \in M_{\ell-1}\), which is defined like \(\tilde{\lambda}_\ell\) in equation (4.19), but on level \(\ell - 1\). Then,
\[
\mathcal{Q}_{\ell-1} P_{\ell-1} \lambda + \nabla \overline{\mathcal{P}}_{\ell-1} \tilde{\lambda}_\ell
= \mathcal{Q}_{\ell-1} P_{\ell-1} \lambda - \mathcal{Q}_{\ell-1} P_{\ell-1} \tilde{\lambda}_\ell + \mathcal{Q}_{\ell-1} P_{\ell-1} \tilde{\lambda}_\ell - \mathcal{Q}_{\ell-1} \tilde{\lambda}_{\ell-1}
+ \mathcal{Q}_{\ell-1} \tilde{\lambda}_{\ell-1} + \nabla \tilde{u} + (\nabla \tilde{u} + \nabla \overline{\mathcal{P}}_{\ell-1} \lambda_\ell). \tag{4.22}
\]
To obtain the result, we now have to bound all four norms \(\| \Xi_1 \|_0\) to \(\| \Xi_4 \|_0\). First,
\[
\| \Xi_1 \|_0 \leq \| P_{\ell-1} \lambda - P_{\ell-1} \tilde{\lambda}_\ell \|_{a_{\ell-1}} \lesssim \| \lambda - \tilde{\lambda}_\ell \|_{a_\ell} \lesssim h_\ell \| A_\ell \lambda \|_\ell,
\] (4.23)
where the first inequality follows directly from the definition of \(\| \cdot \|_{a_{\ell-1}}\), the second inequality is Lemma 4.2, and the last inequality is Lemma
4.6. Using the definition of $\hat{\lambda}_{\ell-1}$, (2.26), and (4.19), we obtain for $e_{\ell-1} := P_{\ell-1} \hat{\lambda}_\ell - \hat{\lambda}_{\ell-1}$ that
\begin{equation}
\alpha_{\ell-1}(e_{\ell-1}, \mu) = (f_\lambda, U_{\ell} I_{\ell} I_{\ell} - U_{\ell-1} I_{\ell-1} I_{\ell-1} \mu) \quad \forall \mu \in M_{\ell-1}.
\end{equation}
Choosing $\mu = e_{\ell-1}$, reusing that $\|Q_{\ell-1} \cdot \|_0 \leq \| \cdot \|_{\alpha_{\ell-1}}$ and exploiting the Cauchy–Schwarz inequality, we get
\begin{equation}
\|\Xi_2\|_{0}^2 \leq \alpha_{\ell}(e_{\ell-1}, e_{\ell-1}) = (f_\lambda, U_{\ell} I_{\ell} I_{\ell-1} - U_{\ell-1} I_{\ell-1} I_{\ell-1})_0 \leq \|f_\lambda\|_0 \|U_{\ell-1} I_{\ell-1} I_{\ell-1} - U_{\ell-1} I_{\ell-1} I_{\ell-1}\|_0.
\end{equation}
The correct bound for $\|\Xi_2\|_0$ can now be deduced, since Lemma 4.3 implies that
\begin{equation}
\|U_{\ell-1} I_{\ell-1} I_{\ell-1} - U_{\ell} I_{\ell} I_{\ell-1}\|_0 \lesssim h_\ell \|e_{\ell-1}\|_{\alpha_{\ell-1}}
\end{equation}
which allows to extract
\begin{equation}
\|e_{\ell-1}\|_{\alpha_{\ell-1}}^2 \lesssim h_\ell \|f_\lambda\|_0 \|e_{\ell-1}\|_{\alpha_{\ell-1}}
\end{equation}
from (4.25). Dividing this inequality by $\|e_{\ell-1}\|_{\alpha_{\ell-1}}$ yields
\begin{equation}
\|\Xi_2\|_0^2 \leq \|e_{\ell-1}\|_{\alpha_{\ell-1}}^2 \lesssim h_\ell^2 \|f_\lambda\|_0^2 \lesssim h_\ell^2 \|A_{\ell} \lambda\|_\ell^2,
\end{equation}
where the last inequality follows from Lemma 4.6. For $\|\Xi_3\|_0$, we utilize the convergence properties of the HDG method and that $Q_{\ell-1} \hat{\lambda}_{\ell-1} + Q_{\ell-1} f_\lambda$ approximates $\bar{q} = -\nabla \bar{u}$. Hence, we can deduce that
\begin{equation}
\|Q_{\ell-1} \bar{\lambda}_{\ell-1} + Q_{\ell-1} f_\lambda + \nabla \bar{u}\|_0 \lesssim h_{\ell-1} \|\bar{u}\|_{H^2(\Omega)} \lesssim h_{\ell-1} \|f_\lambda\|_0 \lesssim h_\ell \|A_{\ell} \lambda\|_\ell.
\end{equation}
Here, the elliptic regularity of (2.1) enters together with Lemma 4.6 — which is also needed to deduce
\begin{equation}
\|Q_{\ell-1} f_\lambda\|_0 \lesssim h_{\ell-1} \|f_\lambda\|_0 \lesssim h_\ell \|A_{\ell} \lambda\|_\ell,
\end{equation}
where the first inequality is [8, Lem. 3.7]. (4.30) implies the desired properties for $\|\Xi_3\|_0$. For the last term, we observe that
\begin{equation}
(\nabla P_{\ell-1} \hat{\lambda}_\ell, \nabla w)_0 = a_{\ell} \gamma_{\ell} \gamma_{\ell} w = (f_\lambda, U_{\ell} \gamma_{\ell} w)_0 = (f_\lambda, w)_0
\end{equation}
for all $w \in V_{\ell-1}^e$. Thus, we can exploit the approximation property of (continuous) linear finite elements to obtain the result for $\|\Xi_4\|_0$ similar to (4.30a).

The inequality for the first term can be obtained analogously substituting (4.22) by
\begin{equation}
Q_{\ell-1} \lambda + \nabla P_{\ell-1} \hat{\lambda}_\ell = Q_{\ell-1} \lambda - Q_{\ell-1} \hat{\lambda}_\ell + Q_{\ell-1} \hat{\lambda}_\ell + \nabla \bar{u} - \nabla \bar{u} + \nabla P_{\ell-1} \hat{\lambda}_\ell.
\end{equation}
Additionally, we use similar techniques as above to prove the following lemma which will be used in the proof of the main convergence result.
Lemma 4.7. If for all \( \ell, \tau_\ell h_\ell \leq c \) for some \( c > 0 \), we have for all \( \lambda \in M_\ell, \mu_\ell \in M_\ell \), and \( \mu_{\ell+1} \in M_{\ell+1} \) that
\[
\tau_\ell \| U\lambda - \lambda \|_\ell^2 \lesssim \| Q_\ell \lambda + \nabla T_{\ell-1} \mu_{\ell} \|_0^2
\]
\[
\tau_\ell \| U\lambda - \lambda \|_\ell^2 \lesssim \| Q_\ell \lambda + \nabla T_{\ell} \mu_{\ell+1} \|_0^2.
\] (4.33)

Proof. Obviously, we have \( U_\ell \gamma_\ell P_{\ell-1} \mu_\ell = P_{\ell-1} \mu_\ell \) and \( Q_\ell \gamma_\ell P_{\ell-1} \mu_\ell = -\nabla P_{\ell-1} \mu_\ell \). Thus
\[
\tau_\ell \| U\lambda - \lambda \|_\ell^2 = \tau_\ell \| U\lambda - \gamma_\ell P_{\ell-1} \mu_\ell \|_\ell^2
\]
\[
\lesssim \| Q_\ell (\lambda - \gamma_\ell P_{\ell-1} \mu_\ell) \|_0^2 = \| Q_\ell \lambda + \nabla P_{\ell-1} \mu_\ell \|_0^2,
\] (4.34)
where the inequality is [31, p. 68]. The second inequality can be obtained analogously. \( \square \)

4.3. Proof of regularity approximation. With these preliminaries given, we can now state the proof of Theorem 4.1. At first, we recognize that it suffices to show that
\[
|a_\ell(\lambda - I_\ell P_{\ell-1} \lambda, \lambda)| \lesssim h_\ell^2 \| A_\ell \lambda \|^2,
\] (4.35)
since this is sufficient for (A1) to hold (cf. [31, Thm. 3.6]). Using the bilinearity of \( a_\ell \) and (2.26), we immediately obtain
\[
a_\ell(\lambda - I_\ell P_{\ell-1} \lambda, \lambda) = a_\ell(\lambda, \lambda) - a_{\ell-1}(P_{\ell-1} \lambda, P_{\ell-1} \lambda)
\]
\[
= (Q_\ell \lambda, Q_\ell \lambda) - (Q_{\ell-1} P_{\ell-1} \lambda, Q_{\ell-1} P_{\ell-1} \lambda)
\]
\[
+ \tau_\ell \| U\lambda - \lambda \|_\ell^2 - \tau_{\ell-1} \| U_{\ell-1} P_{\ell-1} \lambda - P_{\ell-1} \lambda \|_\ell^2.
\] (T1) (T2)
where the second equation is due to the definition of the bilinear forms \( a_\ell \) and \( a_{\ell-1} \). Now, we estimate both terms (T1) and (T2) separately. For the first, we use binomial factoring and quasi-orthogonality to obtain
\[
(T1) = (Q_\ell \lambda + Q_{\ell-1} P_{\ell-1} \lambda, Q_\ell \lambda - Q_{\ell-1} P_{\ell-1} \lambda)
\]
\[
= (Q_\ell \lambda + 2\nabla \overline{\pi} + Q_{\ell-1} P_{\ell-1} \lambda, Q_\ell \lambda - Q_{\ell-1} P_{\ell-1} \lambda). \] (4.36) (4.37)
Here, \( \overline{\pi} \) is from Lemma 4.5. Thus,
\[
(T1) \leq \left( \| Q_\ell \lambda + \nabla \overline{\pi} \|_0 + \| \nabla \overline{\pi} + Q_{\ell-1} P_{\ell-1} \lambda \|_0 \right) \| Q_\ell \lambda - Q_{\ell-1} P_{\ell-1} \lambda \|_0.
\] (4.38)
Due to Lemma 4.5, we can further estimate
\[
(T1) \lesssim h_\ell \| A_\ell \lambda \| \left( \| Q_\ell \lambda + \nabla \overline{\pi} \|_0 + \| \nabla \overline{\pi} + Q_{\ell-1} P_{\ell-1} \lambda \|_0 \right) \] (4.39)
Application of Lemma 4.5 gives the desired result for (T1).

For (T2), we exploit that both summands have exactly the same form and can be treated analogously. Thus, we only demonstrate the procedure for the second summand:
\[
\tau_{\ell-1} \| U_{\ell-1} P_{\ell-1} \lambda - P_{\ell-1} \lambda \|_{\ell-1}^2 \lesssim \| Q_{\ell-1} P_{\ell-1} \lambda + \nabla \overline{\pi} \|_0^2 \lesssim h_\ell^2 \| A_\ell \lambda \|^2,
\] (4.40)
where the first inequality is Lemma 4.7 and the second inequality is Lemma 4.5.

5. Proof of (A2) and (A3)

The proof of (A2) is a simple consequence of Lemma 2.1 with $P_{\ell-1}\lambda$ instead of $\lambda$ i.e., we have

\[
a_\ell(\lambda - I_\ell P_{\ell-1}\lambda, \lambda - I_\ell P_{\ell-1}\lambda) = a_\ell(\lambda, \lambda) - 2a_\ell(\lambda, I_\ell P_{\ell-1}\lambda) + a_\ell(I_\ell P_{\ell-1}\lambda, I_\ell P_{\ell-1}\lambda) \leq 0 \]

by Lemma 4.2, (5.3)

For the proof of (A3), we heavily rely on [3] (where (A3) is denoted (2.11)). Theorems 3.1 and 3.2 of [3] ensure that (A3) holds if the subspaces satisfy a “limited interaction property” which holds, because each degree of freedom (DoF) only “communicates” with other DoFs which are located on the same face as the DoF or on the other faces of the two adjacent elements.

6. Numerical experiments

For the numerical evaluation of our multigrid method for HDG, we consider the following Poisson problem on the unit square $\Omega = (0,1)^2$:

\[
-\Delta u = f \quad \text{in } \Omega, \quad (6.1a)
\]
\[
u = 0 \quad \text{on } \partial \Omega, \quad (6.1b)
\]

where $f$ is chosen as one. The first mesh is shown in Figure 2 and it is successively refined in our experiments. The implementation is based on the FFW toolbox from [6] and employs the Gauss–Seidel smoother. It uses a Lagrange basis and the Euclidean inner product in the coefficient space instead of the inner product $\langle \cdot, \cdot \rangle_\ell$. These two inner products are equivalent up to a factor of $h_\ell^2$. Supposing that the
matrix form of (6.1) is \( Ax = b \) we stop the iteration for solving the linear system of equations if
\[
\frac{\| b - A x_{\text{iter}} \|_2}{\| b \|_2} < 10^{-6}.
\] (6.2)

The initial value \( x \) on mesh level \( \ell \) is the solution on level \( \ell - 1 \), thus we perform a nested iteration. The numbers of iteration steps needed are shown in Table 1 for one and two pre- and post-smoothing steps on each level, respectively. Clearly, these numbers are independent of the mesh level, as predicted by our analysis. Additionally, the numbers are fairly small, such that we can conclude that we actually have an efficient method. If we employ two smoothing steps instead of one, the number of steps is almost divided by two, such that both options will result in similar numerical effort. Finally, we see that the choice of \( \tau \in \{ \frac{1}{h}, 1 \} \) does not significantly influence the number of iterations.

We ran experiments for polynomial degrees one, two, and three, where iteration counts remain well bounded; nevertheless, we expect rising counts for higher degrees, as we use a point smoother.

Additionally, we tested the correctness of our implementation by employing a right hand side leading to the solution \( u = \sin(2\pi x) \sin(2\pi y) \). The estimated orders of convergence (EOC) of the primary unknown \( u \) computed as
\[
\text{EOC} = \log \left( \frac{\| u - u_{\ell-1} \|_{L^2(\Omega)}}{\| u - u_\ell \|_{L^2(\Omega)}} \right) / \log(2),
\] (6.3)

and the secondary unknown \( q \) of the HDG method are reported in Table 2; they coincide well with the orders predicted in [13]. Iteration counts are almost identical to those in Table 1, such that we do not report them here. We see that the choice \( \tau = \frac{1}{h} \) is suboptimal as compared to \( \tau = 1 \), as the error in the secondary unknown \( q \) converges.
Table 2. Estimated orders of convergence (EOC) for primary unknown \( u \) and secondary unknown \( q \) when the polynomial degree of the HDG method is \( p \) and the exact solution is \( u = \sin(2\pi x) \sin(2\pi y) \).

| mesh | EOC | 2 | 3 | 4 | 5 | 6 | 7 |
|------|-----|---|---|---|---|---|---|
| \( p = 1 \) | \( \tau = \frac{1}{h} \) | 1.4 | 1.2 | 1.9 | 1.8 | 2.0 | 1.7 | 2.0 | 1.4 | 2.0 | 1.2 | 2.0 | 1.0 |
| | \( \tau = 1 \) | 1.5 | 1.3 | 2.0 | 1.9 | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 |
| \( p = 2 \) | \( \tau = \frac{1}{h} \) | 3.4 | 3.0 | 3.1 | 2.9 | 3.0 | 2.6 | 3.0 | 2.4 | 3.0 | 2.1 | 3.0 | 2.0 |
| | \( \tau = 1 \) | 3.4 | 3.1 | 3.1 | 2.9 | 3.0 | 3.0 | 3.0 | 3.0 | 3.0 | 3.0 | 3.0 | 3.0 |
| \( p = 3 \) | \( \tau = \frac{1}{h} \) | 3.9 | 2.8 | 4.4 | 3.7 | 4.2 | 3.5 | 4.1 | 3.2 | 4.0 | 3.1 | 4.0 | 3.0 |
| | \( \tau = 1 \) | 2.8 | 2.8 | 3.9 | 3.9 | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 | 4.0 |

slower by one order. This is why we included results for \( \tau = 1 \) in Table 1 albeit a theoretical justification is still missing.

7. Conclusions

We proposed a homogeneous multigrid method for HDG. We proved analytically that this method converges independently of the mesh size. Numerical examples have shown that the condition numbers are not only independent of the mesh size but also reasonably small. As a consequence, we have been enabled to efficiently solve linear systems of equations arising from HDG discretizations of arbitrary order. Our proofs apply to stabilization terms \( \tau \sim h^{-1} \), but numerical experiments suggest optimal convergence also for \( \tau \sim 1 \).

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