Radial and orbital Regge trajectories in heavy quarkonia

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The spectra of heavy quarkonia are studied in two approaches: with the use of the Afonin-Pusenkov representation of the Regge trajectory for the squared excitation energy $E^2(nl)$ (ERT), and using the relativistic Hamiltonian with the universal interaction. The parameters of the ERTs are extracted from experimental data differences and their values in bottomonium: the intercept $a(bb) = 0.131 \text{ GeV}^2$, the slope of the orbital ERT $b_l(bb) = 0.50 \text{ GeV}^2$, and the slope of the radial ERT $b_n(bb) = 0.724 \text{ GeV}^2$, appear to be smaller than those in charmonium, where $a(cc) = 0.381 \text{ GeV}^2$, $b_l(cc) = 0.686 \text{ GeV}^2$, and the radial slope $b_n(cc) = 1.074 \text{ GeV}^2$, which value is close to that in light mesons, $b_n(q\bar{q}) = 1.11(1) \text{ GeV}^2$. For the resonances above the $D\bar{D}$ threshold the masses of the $\chi_{c0}(nP)$ with $n = 2, 3, 4$, equal to 4218 MeV, 4503 MeV, 4754 MeV, are obtained, while above the $B\bar{B}$ threshold the resonances $\Upsilon(3\,3')$ with the mass 10693 MeV and $\chi_{b1}(4\,3')$ with the mass 10756 MeV are predicted.

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I. INTRODUCTION

In recent years in heavy quarkonia (HQ) a large number of new resonances were observed [1-8] and among them the resonances $X(4500)$ and $X(4700)$ with $J^{PC} = 0^{++}$ [8] are particularly interesting, being the highest excitations in the meson sector. The discovery of these resonances has stimulated new theoretical studies [8, 10] and different conceptions about their nature were presented, including diquark-antidiquark $c\bar{s}\bar{s}$ types of tetraquarks [10-13]. However, even within the tetraquark $c\bar{s}\bar{s}$ picture different interpretations were suggested. Also the conventional $c\bar{c}$ structure of these resonances was studied [14, 18], which implies that the $c\bar{c}$ component dominates in the wave function (w.f.) of a resonance, but does not exclude that other components, like diquark-antidiquark or meson-meson, can also be present in the wave function (w.f.) [14]. For decades the spectra and other properties of HQ were studied in different potential models (PMs), both non-relativistic and relativistic [19-27], which allow successfully to describe low-lying HQ states. However, the masses of the high excitations strongly depend on the $Q\bar{Q}$ interaction at large distances, as well as on the heavy quark mass used, and their values can differ by $\sim (100 - 150)$ MeV (see the compilation in Ref. [23]). This happens because using in PMs several fitting parameters, the first two or three excitations can be easily described with a good accuracy, while the masses of the high excitations appear to be very sensitive to behavior of the $Q\bar{Q}$ potential at large distances. For example, in Ref. [18], where the screened confining potential is used, the resonance $X(4140)$ is considered as a candidate to $\chi_{c1}(3P)$, while within a similar model the mass $M_{c1}(3P)$, larger by $\sim 140$ MeV, is obtained [14], and this state is identified as $X(4274)$.

The spectra of HQ were also studied via the radial Regge trajectories (RTs) with the parameters determined either in dynamical calculations [21-27, 28], or in the analysis of the experimental masses [29, 30], where high charmonium excitations are described by a linear radial RT, similar to those found in light mesons [31],

$$M^2(nl) = M^2(n = 0, l) + \mu^2_{c\bar{c}} n, \quad (n = n_r).$$ (1)

Here the slope $\mu^2_{c\bar{c}}$ has a large value, $\mu^2_{c\bar{c}}(c\bar{c}) \sim (2.8 - 3.4) \text{ GeV}^2$ [19, 21, 25, 27], and slightly depends on the angular momentum $l$ [21]. In bottomonium a larger slope $\mu^2(bb)$ was obtained [21, 27, 28], which lies in the range $(4 - 7) \text{ GeV}^2$ for different sets of parameters of the potential $V_0(r)$ (see Eq. [7] and the $b$-quark mass $m_b$.

A different representation of high HQ excitations was suggested by Afonin and Pusenkov [32, 33], who introduced a new type of radial RT in HQ, henceforth denoted as ERT, referring to the squared excitation energy $E^2(nl)$, which
TABLE I: The experimental mass differences (in MeV) in light mesons, charmonium, and bottomonium

| $\Delta$ | $n\bar{n}$ | cc | $b\bar{b}$ |
|----------|-----------|----|---------|
| $M(2^3S_1) - M(1^3S_1)$ | 690(25) | 589(2) | 563(1) |
| $M_{cog}(2S) - M_{cog}(1S)$ | 700(20) | 605(2) | 567(1) |
| $M(3^3S_1) - M(2^3S_1)$ | 415(55) | 353(1) | 332(1) |
| $M(2^3P_1) - M(1^3P_1)$ | 424(49) | 361(2) | 363(1) |

is defined as $E(nl) = M(nl) - 2m_Q$. Moreover, the authors assumed that there exists a universal radial ERT,

$$E^2(nl) = a + b_n n,$$  \(2\)

which can be applied to all unflavoured vector mesons, including $\rho$, $\phi$, charmonium, and bottomonium and therefore the mass $M(n^3S_1; q\bar{q})$, given by

$$M(n^3S_1) = 2m_Q + \sqrt{a + b_n n},$$  \(3\)

would have the same slope for these mesons and the values $b_n = 1.1$ GeV$^2$ and $a = 0.57$ GeV$^2$ were chosen in Ref. \cite{33}. As seen from Eq. (3), the HQ mass $M(n^3S_1)$ depends on the quark mass $m_Q$ taken; the values $m_c = 1.17$ GeV and $m_b = 4.33$ GeV were taken there.

It also follows from Eq. (3) that the ERT with the universal slope $b$ and the intercept $a$ have equal mass differences,

$$M(2^3S_1) - M(1^3S_1) = \sqrt{a + b_n n} - \sqrt{a}; \quad M(3^3S_1) - M(2^3S_1) = \sqrt{a + 2b_n n} - \sqrt{a + b_n n},$$  \(4\)

both for the heavy and the light vector mesons and a given radial quantum number $n = n_r$. However, this statement does not agree with the experimental values of the mass differences, which can differ by $\sim 100$ MeV (see Table I). In Table I we give also the experimental numbers for the mass difference, $M(2^3P_1) - M(1^3P_1)$, which will be used later in the analysis of the orbital ERT. From Table I one can see that the mass difference between the first excited state and the ground state does not change, if instead of the masses $M(n^3S_1)$ one takes the centroid masses $M_{cog}(nS)$, i.e., it does not depend on spin effects, being in light mesons larger than in charmonium and bottomonium, which values that differ only by 26 MeV. Such close values of the mass differences could be partly explained by the existence of the universal potential, $V_0(r) = V_\alpha(r) + V_{ge}$, which allows to describe the low-lying states of all mesons \cite{21, 21, 22, 25}, however, this choice is not sufficient to obtain equal slopes of ERTs for heavy and light mesons (see below).

Notice that in HQ the mass formula is more simple than in a light meson, where it includes the self-energy and the string corrections \cite{34, 32}, which are small and can be neglected in HQ \cite{35}. However, the masses of heavy mesons, which have small sizes, are strongly affected by the gluon-exchange (GE) interaction and for them the asymptotic freedom (AF) behavior of the strong coupling has to be taken into account, in contrast to light mesons, where the GE potential can be presented as the Coulomb potential with the coupling $\alpha_{(eff.)} = $ const. and the AF behavior is important only for the $1S$ and $1P$ states \cite{34}.

In the present paper we study orbital and radial ERTs of HQ in the $(E^2, n)$ and $(E^2, nl)$ planes, having in mind three goals: (i) to extract the slope of the radial RTs $\beta_n(Q\bar{Q})$ from experiment; (ii) to determine the slope of orbital RTs $\beta_l(Q\bar{Q})$ for the HQ mesons and show that the slopes $\beta_n$ and $\beta_l$ are different in charmonium and bottomonium and smaller than those in light mesons; (iii) to show that the generalized ERT with the $E^2(nl) = a + b_l l + b_n n$, with different slopes $b_l, b_n$ can be introduced in charmonium and bottomonium.

II. THE REGGE TRAJECTORIES IN THE $(E^2, n)$ AND $(E^2, l)$ PLANES IN BOTTOMONIUM

Bottomonium has a large number of levels below the open flavour threshold and provides the unique possibility to extract the parameters of the ERT in the $(E^2, n)$- and $(E^2, l)$-planes from experiment with high accuracy. For that it is sufficient to use the mass differences (see Eqs. (4), $M(\Upsilon(2S)) - M(\Upsilon(1S)) = \sqrt{a + b_n n} - \sqrt{a} = 0.563(1)$ MeV and $M(\Upsilon(3S)) - M(\Upsilon(2S)) = \sqrt{a + 2b_n n} - \sqrt{a + b_n n} = 0.355(5)$ MeV and also the definition of the ground state mass, $M(\Upsilon(1S)) = \sqrt{a} + 2m_b = 9.460(1)$ GeV. From these relations the following values of the slope and the intercept of the radial ERT in the $(E^2, n)$-plane are calculated,

$$a = 0.1307 \text{ GeV}^2, \quad b_n(l = 0) = 0.7242 \text{ GeV}^2.$$  \(5\)
TABLE II: The experimental masses $M(\Upsilon(nS))$ (in MeV) \cite{1}, the masses $M(\Upsilon(nS))$, defined by the ERT, Eq. (2), with the parameters from Eqs. (5,6), and the solutions of Eq. (13), $M(\Upsilon(nS)) = M_{\text{cog}}(nS) + 1/4 \delta_{\text{hf}}(nS)$

| State | from ERT | $M_{\text{cog}}(nS) + 1/4 \delta_{\text{hf}}$ | experiment [1] |
|-------|----------|---------------------------------|-----------------|
| $\Upsilon(1S)$ | 9460 | 9465 | 9460.3(3) |
| $\Upsilon(2S)$ | 10023 | 10017 | 10023.3(3) |
| $\Upsilon(3S)$ | 10355 | 10359 | 10355.2(5) |
| $\Upsilon(4S)$ | 10616 | 10635 | 10579(1.2) |
| $\Upsilon(5S)$ | 10838 | 10884 | 10891(4) |
| $\Upsilon(6S)$ | 11035 | 11093 | 10987(11) |
| $3^3D_1$ | 10693 | 10701 | abs. |
| $4^3D_1$ | 10901 | 10933 | abs. |

Note, that the slope $b_n(\Upsilon)$ appears to be smaller than the slope of the $\rho(n^3S_1)$ trajectory, $b_n(\rho) = \mu^2 \approx 1.45$ GeV$^2$, which follows from the experimental values of the masses $M(\rho(nS))$ \cite{1}. As the next important step, knowing the intercept $a$ and the ground state mass $M(\Upsilon(1S)) = 9.460$ GeV, we extract the quark mass $m_b$,

$$m_b = 4.5492 \text{ GeV}. \quad (6)$$

This mass appears to be not a fitting parameter, but just coincides with the one-loop pole mass $m_b(1\text{-loop}) = 1.086 \tilde{m}_b = 4.550$ GeV, if the conventional current mass $\tilde{m}_b = 4.18(1)$ GeV and the QCD constant $\Lambda_{\overline{MS}}(n_f = 5) = 200$ MeV (or $\alpha_s(\tilde{m}_b) = 0.20$) \cite{38,39} are adopted.

The masses of $\Upsilon(nS)$, defined by the radial ERT, Eq (3) with the parameters from Eqs. (4,6), are presented in Table II where also the masses of the same states, calculated with the spinless Salpeter equation (SSE),

$$M_{\text{S}}(\Upsilon(nS)) = 1.2 \times M_{\text{cog}}(nS) + 1/4 \delta_{\text{hf}}(nS) \text{ (for } \delta_{\text{hf}} \text{ see Ref. [5])}$$

Table II shows the experimental masses

$$M(\Upsilon(nS)) = M_{\text{cog}}(nS) + 1/4 \delta_{\text{hf}}(nS)$$

which are not fitting parameters but defined on fundamental grounds and therefore the eigenvalues (e.v.s) $M_{\text{cog}}(nl)$ of Eq. (4) do not depend on any fitting parameters. Only one extra parameter, $\alpha_{\text{hf}}$, is present in the hyperfine correction to the masses $M(\Upsilon(nS)) = M_{\text{cog}}(nS) + 1/4 \delta_{\text{hf}}(nS)$ (for $\delta_{\text{hf}}$ see Ref. [5])

From Table II one can see that the masses of $\Upsilon(nS)$ with $n = 0, 1, 2$ are exactly equal to the experimental values, while the masses of the states with $n = 3, 4, 5$, which lie above the $B\bar{B}$ threshold, have mass shifts, e.g. the $\Upsilon(4S)$ is shifted down by 38 MeV. The situation with $\Upsilon(5S)$ and $\Upsilon(6S)$ is more complicated, because the ERT gives the mass of $\Upsilon(5S)$ by ~ 40 MeV smaller than the experimental value, i.e., this resonance does not show the typical mass shift up. This may occur because of the nearby located $3^3D_1$ and $3^3D_3$ resonances (see Table II and the calculations below), and also because the threshold $B\bar{B}$ (its mass $M_{\text{thres}} = 10831(1)$ MeV) is close by. Thus, here we face the channel-coupling problem, where a shift of one resonance up and of another resonance down is possible. The same many-channel situation takes place in the region near 11 GeV, where a mass shift down of the $\Upsilon(6S)$ is possible due to the $S - D$ mixing of $\Upsilon(6S)$ and $\Upsilon(4D)$ \cite{10}.

To describe the orbital excitations we introduce the generalized ERT,

$$E^2(nl) = a + b_n n + b_l l,$$

where the parameters $a$ and $b_n$, as well as $m_b$, are already defined and given in Eqs. (5,6). To find the slope $b_l$ one can use the experimental mass of $\chi_{b1}(1P)$ (with $l = 1, n = 0$): $M(\chi_{b1}(1P)) = 9.893(1)$ GeV = $2m_Q + \sqrt{a + b_l}$. It gives $b_l(b\bar{b}) = 0.50$ GeV$^2$.

When higher excitations with $l \neq 0$ are considered in bottomonium, the radial slope, extracted from the mass difference $M(\chi_{b1}(2P)) - M(\chi_{b1}(1P)) = 0.362$ GeV, appears to be a bit smaller than that in the $\Upsilon(nS)$-family, namely, $\beta_n(l \neq 0) = 0.7060$ GeV$^2$. This situation is similar to the one in light mesons, where the radial slope of the $\rho(nS)$-trajectory is larger than that for $a_1(nP)$ and the $\rho(nD)$ mesons.

Then the complete set of parameters of the generalized ERT, Eq. (10) is:

$$a(b\bar{b}) = 0.1307 \text{ GeV}^2, \quad b_n(b\bar{b}, l = 0) = 0.7242 \text{ GeV}^2, \quad b_n(l \neq 0) = 0.7060 \text{ GeV}^2, \quad b_l(b\bar{b}) = 0.50 \text{ GeV}^2, \quad m_b = 4.5492 \text{ GeV}. \quad (10)$$
TABLE III: The masses of the $\chi_{b1}(nP)$ and $\Upsilon(n^3D_1)$ (in MeV), calculated from ERT Eq. 8 and the solutions of the SSE [1] and experiment [1]

| State  | ERT       | Solutions of SSE | Experiment [1] |
|--------|-----------|------------------|----------------|
| $\chi_{b1}(1P)$ | 9892      | 9880             | 9893(1)        |
| $\chi_{b1}(2P)$ | 10262     | 10246            | 10255(1)       |
| $\chi_{b1}(3P)$ | 10540     | 10541            | 10512(2)       |
|           | 10933     |                  |                |
| $\chi_{b1}(4P)$ | 10772     | 10793            | abs.           |
| $\Upsilon(1D)$ | 10161     | 10141            | 10164(1)       |
| $\Upsilon(2D)$ | 10460     | 10440            | abs.           |
| $\Upsilon(3D)$ | 10704     | 10701            | abs.           |
| $\Upsilon(4D)$ | 10915     | 10933            | abs.           |
| $\Upsilon(5D)$ | 11105     | 11145            | abs.           |

The masses of $\chi_{b1}(nP)$ and $\Upsilon(n^3D_1)$, calculated with the use of this ERT, Eq. 9,10, are given in Table III.

Here we pay attention to the fact that this ERT predicts the correct value of the mass of the $\Upsilon(1^3D_1)$ state, while the solution of the SSE is $\sim 20$ MeV smaller. Also the ERT gives the mass of $\chi_{b1}(3P)$ between the values observed in the experiments of LHCb [6] and Belle [41].

III. THE REGGE TRAJECTORIES IN THE $(E^2, n)$- AND $(E^2, l)$-PLANES IN CHARMONIUM

In charmonium there are only three multiplets $(1S, 2S, 1P)$ below the $D\bar{D}$ threshold and these experimental data do not allow to extract exact values of the $c$-quark mass as well as all parameters of the generalized ERT, Eq. 9; nevertheless two mass differences, known from experiment,

$$M(\psi(2S)) - M(J/\psi) = 0.589(1) \text{ GeV}, \quad M(\chi_{c1}(1P)) - M(J/\psi) = 0.414(1) \text{ GeV},$$

put restrictions on the parameters of the ERT. Our analysis shows that the main uncertainty comes from a variation of the $c$-quark mass, entering the relation Eq. 3. Varying $m_c$ in the range $(1.2 - 1.4) \text{ GeV}$, the best agreement in the description of the charmonium spectrum is reached for $m_c = (1.22 - 1.28) \text{ GeV}$. Note that this value of $m_c$ coincides with the current $c$-quark mass, $m_c(\bar{m}_c) = 1.26(6) \text{ GeV}$ [35,37]. Here we take $m_c = 1.24 \text{ GeV}$. Then by definition,

$$a(\psi(nS)) = (M(J/\psi) - 2.48 \text{ GeV})^2 = 0.3807 \text{ GeV}^2,$$

while the slope of the radial ERT is extracted from the mass difference, $M(\psi(2S)) - M(J/\psi) = 0.589 \text{ GeV}$,

$$\sqrt{a + b_n} - \sqrt{a} = 0.589 \text{ GeV},$$

which gives

$$b_n(l = 0) = 1.0738 \text{ GeV}^2, \quad (m_c = 1.24 \text{ GeV}).$$

This value is smaller than the radial slope of the $\rho(nS)$-trajectory, $\beta_n(n\bar{n}, l = 0) = 1.45(5) \text{ GeV}^2$, but close to the value of the radial slope $\beta_n(n\bar{n}, l \neq 0) = 1.14(3) \text{ GeV}^2$ for the $a_1(nP)$ and $\rho(n^3D_1)$ trajectories in light mesons.

Correspondingly, from the mass difference $M(\chi_{c1}(1P)) - M(\psi(1S)) = \sqrt{a + b_1} - \sqrt{a} = 0.414 \text{ GeV}$ the slope of the orbital ERT,

$$\sqrt{b_1(m_c = 1.24 \text{ GeV})} = 0.6863 \text{ GeV}^2,$$

is extracted, which appears to be significantly smaller than the slope $b_1(n\bar{n}) = 1.1(1) \text{ GeV}^2$ in light mesons.

Then all charmonium states (with the exception of the $\chi_{c0}(nP)$, see below) are described by the renormalized ERT,

$$E^2(\psi)(\text{in GeV}^2) = 0.3807 + 1.0738 n + 0.6863 l, \quad m_c = 1.240 \text{ GeV},$$

(16)

The ERT of $\chi_{c0}(nP)$ needs to be considered separately, because of its large fine-structure splitting, which gives a smaller mass difference $M(\chi_{c0}(1P)) - 2m_c = 0.318 \text{ GeV}$ than the value $0.414 \text{ GeV}$ for $\chi_{c1}(1P)$. Therefore a smaller slope $b_1(\chi_{c0}) = 0.4935 \text{ GeV}^2$ is extracted and for $\chi_{c0}(nP)$ its generalized ERT is

$$E^2(\chi_{c0}(nP))(\text{in GeV}^2) = 0.3807 + 1.070 n + 0.4935 l,$$

(17)
In Table III we give the masses \( M \) calculated according to the ERT and also the solutions of the SSE (including the fine-structure corrections).

\[ M_{\text{observed by the LHCb Collaboration}} \] 10 MeV larger than those defined by the ERT. This interesting fact shows that in the ERT the flattening effect is taken into account. To see how the ERT parameters change when decreasing the quark mass and, moreover, in charmonium the radial slope is smaller than the one in the conventional radial RT with a given \( \mu \). This coincidence can be considered as an indication that these resonances corresponding ERT has a a smaller orbital slope, \( b \) in the static potential, flattening effect is taken into account.

From Table IV one can see that in our calculations the masses of high excitations with \( J^P = 0^+ \), \( M(\chi_{c0}(4P)) = 4754 \text{MeV} \) are close to those of the \( \chi(4500) \) and \( \chi(4700) \) resonances with \( J^P = 0^+ \), observed by the LHCb Collaboration [6]. This coincidence can be considered as an indication that these resonances could have a large \( cc \) component. Notice that due a large fine-structure shift down of \( \chi_{c0}(4P) \) (\( \approx -110 \text{ MeV} \)), the corresponding ERT has a smaller orbital slope, \( b_l(\chi_{c0}) = 0.4935 \text{ GeV}^2 \), than that of \( b_l(\chi_{c1}) = 0.6863 \text{ GeV}^2 \).

In Table IV we give also the masses, defined as the solutions of the SSE plus spin-dependent corrections, where in the static potential \( V_0(r) \) the linear confining potential is taken at all distances, i.e., the flattening effect at large distances was neglected. For that reason the higher \( nl \) resonances with \( n = 3, 4, 5 \), determined by the SSE, appear to be by (100 – 200) MeV larger than those defined by the ERT. This interesting fact shows that in the ERT the flattening effect is taken into account.

It is also worth to underline that the ERTs present the physical picture in HQ in a clear way and one can see how the ERT parameters change when decreasing the quark mass and, moreover, in charmonium the radial slope is almost already equal to that in light mesons [12], where the generalized ERT,

\[ M^2(nl, n\bar{n})(\text{in GeV}^2) = 0.60 + 1.13(5)(n + l), \quad (l \neq 0) \quad m_q = 0, \]

was obtained. Notice that in charmonium the radial slope of the ERT, \( b_n(cc) = 1.0738 \text{ GeV}^2 \) is about three times smaller than the one in the conventional radial RT with a given \( l \),

\[ M^2(cc, n) = M^2(n = 0) + \mu^2 n, \]

where the radial slope, \( \mu^2(cc) \sim (2.8 – 3.5) \text{ GeV}^2 \), has large value [26 29].

**IV. DISCUSSION AND CONCLUSIONS**

We have studied the HQ spectra in two approaches: with the use of the relativistic Hamiltonian with the universal interaction and via the generalized ERTs, defined by the excitation energy, \( E(nl) = M(nl) - 2m_Q \). We have shown

| State     | SSE Eq. | ERT Eqs. | Experiment |
|-----------|---------|----------|------------|
| \( J/\psi \) | 3100    | 3007     | 3007       |
| \( \psi(2S) \) | 3685    | 3686     | 3686       |
| \( \psi(3S) \) | 4100    | 4070     | 4039(1)    |
| \( \psi(4S) \) | 4455    | 4378     | 4346(6) or 4421(4) |
| \( \psi(5S) \) | 4760    | 4642     | 4643(9)    |
| \( \psi(6S) \) | 5043    | 4878     | abs.       |
| \( \chi_{c1}(1P) \) | 3500    | 3513     | 3.510.7(1) |
| \( \chi_{c1}(2P) \) | 3949    | 3943     | 3871.7(2)  |
| \( \chi_{c1}(3P) \) | 4319    | 4273     | 4274(8)    |
| \( \chi_{c1}(4P) \) | 4642    | 4549     | abs.       |
| \( \chi_{c1}(5P) \) | 4933    | 4796     | abs.       |
| \( \chi_{c1}(6P) \) | 5201    | 5017     | abs.       |
| \( \chi_{c1}(1P) \) | 3435    | 3415     | 3414.8(3)  |
| \( \chi_{c0}(2P) \) | 3929    | 3876     | 3918(2)*   |
| \( \chi_{c0}(3P) \) | 4289    | 4218     | abs.       |
| \( \chi_{c0}(4P) \) | 4622    | 4504     | 4506(25)*  |
| \( \chi_{c0}(5P) \) | 4920    | 4754     | 4704+24*   |
| \( \psi(1^3D_1) \) | 3802    | 3804     | 3773.1(4)  |
| \( \psi(2^3D_1) \) | 4.188   | 4161     | 4191(5)    |
| \( \psi(3^3D_1) \) | 4.521   | 4455     | abs.       |
| \( \psi(4^3D_1) \) | 4.821   | 4710     | abs.       |
| \( \psi(5^3D_1) \) | 5095    | 4939     | abs.       |

* not yet identified in PDG as \( \chi_{c0}(nP) \) resonance.
that the orbital and the radial ERT have different slopes, both in charmonium and bottomonium, and in bottomonium (charmonium) the intercept of the radial and the orbital ERT is the same, being smaller in bottomonium. This fact allows to introduce the generalized ERT, Eq. (9), which determines the HQ masses of the large number of states with $l = 0, 1, 2, 3$. The parameters of the ERT, as well as $m_Q$, were extracted from the experimental mass differences and their values are collected in Table V together with those of the $\rho(3S_1)$ and $\rho(3D_1)$ trajectories [42].

From Table V one can see how the intercept, the orbital and the radial slopes are increasing with a decreasing quark mass. In bottomonium the resonances $\chi_b(4P)$ with the mass 10756 MeV and $\Upsilon(3S_1)$ with the mass 10700 MeV are predicted, while in charmonium the masses of the resonances $\chi_c(nP)$ with $J^{PC} = 0^{++}$ and $n = 3, 4$, equal to 4504 MeV, 4754 MeV respectively, are obtained. These masses appear to be close to those of the $X(4500)$ and $X(4700)$ resonances and this fact can be considered as an indication that $X(4500)$ and $X(4700)$ have a large $c\bar{c}$ component in their wave function.

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