STRATEGIES FOR OPTIMAL SINGLE-SHOT DISCRIMINATION OF QUANTUM MEASUREMENTS

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ABSTRACT. In this work we study the problem of single-shot discrimination of von Neumann measurements. We associate each measurement with a measure-and-prepare channel. There are two possible approaches to this problem. The first one, which is simple, does not utilize entanglement. We focus only on discrimination of classical probability distribution, which are outputs of the channels. We find necessary and sufficient criterion for perfect discrimination in this case. A more advanced approach requires the usage and entanglement. We quantify the distance of the two measurements in terms of the diamond norm (called sometimes the completely bounded trace norm). We provide an exact expression for the optimal probability of correct distinction and relate it to the discrimination of unitary channels. We also state a necessary and sufficient condition for perfect discrimination and a semidefinite program which checks this condition. Our main result, however, is a cone program which calculates the distance of these measurements and hence provides an upper bound on the probability of their correct distinction. As a by-product the program also finds a strategy (input state) which achieves this bound. Finally, we provide a full description for the cases of Fourier matrices and mirror isometries.

1. Introduction

The state of a quantum system is inherently non-observable. Despite this, quantum states have been the focus of quantum theory since its beginning as they provide a way of computing the value of any observable. The picture changes when we consider two quantum states and ask about their distance. This quantity can, in principle, be measured and provides an upper bound on the probability of discriminating these states. This was shown by Helstrom \cite{helstrom1976}. Such problems are fundamental in quantum information science and quantum physics, and have attracted a lot of attention in recent years. These range from experimental studies \cite{teo2010, teo2012, rohrlich2014}, theoretical considerations of finite-dimensional random quantum states \cite{kippenberg2015} to asymptotic properties of random quantum states \cite{kippenberg2015}. This approach can be extended to quantum
channels via the Choi-Jamiolkowski isomorphism \cite{7,8}. Helstrom’s result can be easily extended to this case and once again we obtain a simple expression for the upper bound for the probability of discriminating two quantum channels. There is, however, one additional feature in this setting, which is the input state. This input state is what we call the strategy for discriminating quantum channels. Due to the complicated structure of the set of quantum channels, the problem has been studied in the limit of large input and output dimensions \cite{9}. In this paper we focus on the problem of discriminating quantum measurements which are viewed as a subclass of quantum channels.

The problem of discriminating quantum measurements is of the utmost importance in modern quantum information science. Imagine we have an unknown measurement device, a black-box. The only information we have is that it performs one of two measurements, say $S$ and $T$. Our goal is twofold. First, we want to tell whether it is possible to discriminate $S$ and $T$ perfectly, i.e. with probability equal to one. If this is not the case, we would like to know the upper bound of such a probability. Secondly, we need to devise an optimal strategy for this process, which means finding an optimal input state that achieves the highest possible probability of discrimination.

This issue has already attracted a lot of attention from the scientific community. In \cite{10}, authors have presented the scheme for complete local discrimination for various kinds of unitary operations. The results in \cite{11} indicate that it is possible to perfectly distinguish projective measurements with the help of measurement–unitary operation–measurement scheme. A single-shot scenario was studied in \cite{12} for $m$ measurements and $n$ outcomes. The authors have also managed to show that ancilla-assisted discrimination can outperform ancilla-free for perfect distinguishability. The case when the black box can be used multiple times was investigated by the authors of \cite{13}, who have also proven that the use of entanglement can improve the discrimination. This issue was also studied in \cite{14}, where it was shown that entanglement in general improves quantum measurements for either precision or stability. According to the authors of \cite{15}, the optimal strategy for discrimination of two unknown unitary channels is closely related to problem of discriminating pure states. They also postulate that entanglement is a key factor in designing an optimal experiment for a comparison. In the work of A. Jenčová and M. Plávala, \cite{16}, the optimality conditions for testers in distinguishability of quantum channels were obtained by the use of semidefinite programming. The optimal strategies with the use of either entangled or not entangled states for the discrimination of Pauli channels were compared by M. Sacchi in \cite{17}.

In this work we study the problem of discriminating von Neumann positive operator valued measures (POVMs). We associate a POVM with a quantum channel and study the distinguishability of these channels. These channels output classical probability distributions, hence we first apply known results for distinguishing classical probability distributions. The results are applicable for the case when we are not able to utilize entangled states to perform discrimination. This, somewhat limited, approach gives us a good starting
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point towards our main result. We obtain that entanglement-assisted discrimination of von Neumann POVMs is related to discrimination of unitary channels. This allows us to find a simple condition for perfect discrimination of measurements. Additionally, we are able to write this result as a semidefinite program (SDP) which is numerically efficient. The problem gets more complex in the case when the probability of correct discrimination is strictly less than one. In this case we have a convex program which calculates the maximum probability of correct discrimination. Furthermore, it gives us the optimal input state for this case.

This paper is organized as follows. In Section 2 we formulate our problem by introducing necessary concepts concerning discrimination of measurements with and without the assistance of entanglement. Mathematical framework necessary for stating our results is introduced in Section 3. In Section 4 we consider the case of discrimination without the assistance of entanglement and provide a necessary and sufficient criterion for perfect discrimination of two von Neumann measurements in this case. Entanglement-assisted discrimination of two von Neumann measurements is analyzed in Section 5. In this section we state an exact expression for the optimal probability of correct distinction of two measurements and relate it to the discrimination probability of unitary channels. We provide a necessary and sufficient condition for perfect discrimination of two von Neumann measurements as well as a semidefinite program which is able to check this condition. We also state a simple necessary and a simple sufficient conditions for perfect discrimination. Finally, we formulate a convex program which provides the optimal input state for discrimination of two von Neumann measurements. In Section 6 we analyze special cases, that is we consider the discrimination problem of measurement in the Fourier basis of any dimension and a measurement in the computational basis. We derive the optimal input state for this task and identify the cases when entanglement is (not) necessary. Similarly, we consider mirror isometries and provide a full description of this case. Concluding remarks are presented in the final Section 7, while proofs of main theorems are relegated to Appendix A.

2. FORMULATION OF THE PROBLEM

Consider the following scenario. There is an unknown measurement device and the only thing we know about it is that it performs one of two known measurements, call them $S$ and $T$. We input a state into the device and our aim is to decide which of the measurements is performed. We aim to identify the assumptions needed for perfect discrimination of $f$ and $T$. Further, want to construct the optimal discrimination scheme for this task. In the case when perfect distinctions is not possible, we would like to bound from above the probability of correct discrimination as well as derive a scheme which enables a correct guess with the optimal probability.

The second field of our interest is finding the optimal strategy for the discrimination. In other words, we would like to know which state should be used to provide the greatest possible probability of correct discrimination.

In the simplest approach, we may think of measurements $S$ and $T$ as measure-and-prepare channels outputting diagonal states, that is classical
probability distributions, see Fig. 1. This notion will be formalized in later sections. Thus, the simplest approach to this problem is to consider the distance between probability distributions. We can use the distance of these distribution as an upper bound on the probability of correct discrimination. In this setting it is also straightforward to find the optimal state for discrimination.

Of course there is another possibility. As we are dealing with quantum states, we can utilize entanglement. Hence, we input one part of the entangled state into the unknown measurement device and later use the other part to strengthen the inference. The scheme of this process is presented in Fig. 2.

3. Mathematical framework

Let us introduce the following notation. We denote the matrices of dimension $d_1 \times d_2$ over the field $\mathbb{C}$ as $M_{d_1,d_2}$. To simplify, square matrices will be denoted $M_d$. The subset of $M_d$ consisting of Hermitian matrices of dimension $d$ will be denoted by $\mathcal{H}_d$ while the set of positive semidefinite matrices of dimension $d$ by $\mathcal{H}_d^+$. The set of quantum states $\rho$, that is positive semidefinite operators of dimension $d$ such that $\text{Tr} \rho = 1$, will be denoted $\Omega_d$. The set of unitary matrices of size $d$ will be denoted by $\mathcal{U}_d$, and its subset of
diagonal unitary matrices of dimension $d$ we denote by $DU_d$. We will also need a linear mapping transforming $M_{d_1}$ into $M_{d_2}$. It will be denoted

$$\Phi : M_{d_1} \to M_{d_2}.$$  

Finally, we introduce a special subset of all mappings $\Phi$, called quantum channels. These are the mappings which are completely positive and trace preserving. In other words, the first condition reads

$$\forall A \in H^+_{d_1}, \ (\Phi \otimes 1 l)(A) \in H^+_{d_2 d_1},$$  

while the second one implies $\forall X \in M_{d_1} \ tr \Phi(X) = tr(X)$.

The most general form of describing quantum measurements utilizes the notion of positive operator valued measures (POVMs). In this case a measurement $\mathcal{T}$ is given by a set of positive operators $\{T_1, \ldots, T_n\}$, for which we impose the condition $\sum_i T_i = 1 l$. Each $T_i \in H^+_{d}$ is called an effect associated with the label $i$.

While performing a measurement on some quantum state $\rho \in \Omega_d$, the probabilities of obtaining each of the outcomes $i$ are $p_i = tr \rho T_i$. Such measurements can be considered as measure-and-prepare channels. The action of a channel $\mathcal{T}$ is given by

$$\mathcal{T}(\rho) = \sum_{i=1}^n p_i |i\rangle \langle i|.$$  

We will be interested in projective rank-one measurements. In this case we have $n = d$. We will denote such measurements as $P_U$. Here $U \in U_d$ and the effects are $P_i = |u_i\rangle \langle u_i|$, where $|u_i\rangle = U |i\rangle$, i.e. the $i^{th}$ column of $U$. We arrive at

$$P_U(\rho) = \sum_{i=1}^d \langle u_i|\rho|u_i\rangle |i\rangle \langle i|.$$  

Now we introduce the bijection between linear operators and vectors in the form of the vectorization operation $|X\rangle \rangle$. It is defined for base operators as $|(i|j)\rangle \rangle = |i\rangle |j\rangle$ and uniquely extended from linearity. We also recall the well-known equality

$$\langle A \otimes B |X\rangle \rangle = |AXB^\top\rangle \rangle,$$

where $A \in M_{d_1,d_2}$, $B \in M_{d_3,d_4}$ and $X \in M_{d_3,d_1}$. For any square matrix $C$ we denote by $\text{diag}(C)$ the linear operation which gives the diagonal of the matrix $C$ and its conjugate operation $\text{diag}^\dagger(v)$, which gives a square diagonal matrix with vector $v$ on the diagonal.

Let us now consider linear mappings transforming square into square matrices i.e. $\Phi : M_{d_1} \to M_{d_2}$. It is well known that quantum channels are a special subclass of such mappings.

**Definition 1.** Consider $\Phi : M_{d_1} \to M_{d_2}$. We define its completely bounded trace norm, also known as a diamond norm as

$$\|\Phi\|_d = \max_{\|X\|_1 = 1} \| (\Phi \otimes 1 l)(X) \|_1.$$
It can be shown \[18\], that for Hermiticity-preserving \(\Phi\) we may restrict maximization to rank-1 orthogonal projectors of the form \(|x\rangle \langle x|\).

There exists a linear bijection between linear mappings \(\Phi : M_{d_1} \rightarrow M_{d_2}\) and matrices \(M_{d_1 \times d_2}\) which was discovered by Choi \[7\] and Jamiołkowski \[8\]. The operator corresponding to quantum channel \(\Phi\), denoted \(J(\Phi)\), can be explicitly obtained as

\[
J(\Phi) = \sum_{i,j=1}^{d_1} \Phi(|i\rangle \langle j|) \otimes |i\rangle \langle j|. 
\]

It has the following properties:

1. \(\Phi\) is Hermiticity-preserving if and only if \(J(\Phi) \in \mathcal{H}_{d_1 \times d_2}\),
2. \(\Phi\) is completely positive if and only if \(J(\Phi) \in \mathcal{H}_{d_1 \times d_2}^+\),
3. \(\Phi\) is trace-preserving if and only if \(\text{Tr}_1 J(\Phi) = 1\).

From these properties it follows that every completely positive \(\Phi\) is necessarily Hermiticity-preserving. Moreover, the difference of completely positive mappings is Hermiticity-preserving. Finally, \(\Phi\) is a quantum channel when it has properties 2 and 3.

Note that in case of a measurement \(T\), \(J(T)\) takes the form of a block diagonal matrix with transposed effects on the diagonal, that is \(J(T) = \sum_{i=1}^d |i\rangle \langle i| \otimes T_i^T\).

For Hermiticity preserving \(\Phi\), we have the following well-known bounds for the diamond norm \[9, 10, 18\]

\[
\frac{1}{d_1} \|J(\Phi)\|_1 \leq \|\Phi\|_\diamond \leq \|\text{Tr}_1 |J(\Phi)|\|.
\]

The celebrated result by Helstrom \[1\] gives an upper bound on the probability of correct distinction between two quantum channels \(\Phi\) and \(\Psi\) in terms of their distance with the use of diamond norm

\[
p \leq \frac{1}{2} + \frac{1}{4} \|\Phi - \Psi\|_\diamond.
\]

The above inequality can be saturated with an appropriate choice of measurements on the output space.

Furthermore, for Hermiticity-preserving \(\Phi\), we have the following alternative formula for the diamond norm \[18\]

\[
\|\Phi\|_\diamond = \max\{\|(|\mathbb{1} \otimes \sqrt{\rho})J(\Phi)(|\mathbb{1} \otimes \sqrt{\rho})\|_1 : \rho \in \Omega_{d_1}\}.
\]

The state \(\rho\), such that \(\|\Phi\|_\diamond = \|(|\mathbb{1} \otimes \sqrt{\rho})J(\Phi)(|\mathbb{1} \otimes \sqrt{\rho})\|_1\) will be called a discriminator.

To complete the mathematical introduction let us recall the definition of total variational distance of probability vectors.

**Definition 2.** Given two discrete probability distributions, represented by vectors \(p, q \in \mathbb{R}^d\), their total variation distance is defined as

\[
\|p - q\|_1 = \sum_{i=1}^d |p_i - q_i| = 2 \sum_{\Delta \subseteq \{1, \ldots, d\}} \max_{a \in \Delta} \left(\sum_{a \in \Delta} p_a - q_a\right).
\]
4. Discrimination without entanglement

4.1. Discrimination of classical probability distributions. Let us consider a simple approach to the discrimination of measurements. The idea is to distinguish discrete random variables, with distributions given by probability vectors obtained after performing the measurements on some state $\rho$. The following corollary states the upper bound for correct discrimination between two measurements in the case we do not use entanglement.

**Proposition 1.** Let $S, T$ be two measure-and-prepare channels with effects $\{S_i\}_{i=1}^{n}$ and $\{T_i\}_{i=1}^{n}$ respectively. It holds that the probability $p$ of their correct discrimination, without the usage of entangled states, is upper bounded by the value

$$p \leq \frac{1}{2} + \frac{1}{4} \max_{\rho} \| \text{diag} [(S - T)(\rho)] \|_1$$

(12)

$$= \frac{1}{2} + \frac{1}{2} \min_{\Delta \subseteq \{1, \ldots, d\}} \left\| \sum_{i \in \Delta} (S_i - T_i) \right\|_1.$$

**Proof.** We can note that

$$\| \text{diag} [(S - T)(\rho)] \|_1 = \max_{\rho} \sum_i |\text{Tr} (\rho (S_i - T_i))| = \max_{\psi} \sum_i |\langle \psi | (S_i - T_i) |\psi \rangle|$$

(13)

$$= 2 \min_{\Delta \subseteq \{1, \ldots, d\}} \left\| \sum_{i \in \Delta} (|i\rangle \langle i| - |u_i\rangle \langle u_i|) \right\|_1.$$

In the case of projective measurements $P_V$ and $P_U$, without loss of generality, we assume that one measurement can be performed in the computational basis, i.e. $V = \mathbb{1}$. We have the following fact

**Corollary 1.** Let $P_{\mathbb{1}}$ and $P_U$ be two projective measurements such that $U \in \mathcal{U}_d$ for arbitrary $d$. Then the bound from Corollary 1 reads

$$p \leq \frac{1}{2} + \frac{1}{2} \min_{\Delta \subseteq \{1, \ldots, d\}} \left\| \sum_{i \in \Delta} (|i\rangle \langle i| - |u_i\rangle \langle u_i|) \right\|_1$$

(14)

$$= \frac{1}{2} + \frac{1}{2} \sqrt{1 - \min_{\Delta \subseteq \{1, \ldots, d\}} \sigma_{\min}^2 (U_\Delta)},$$

where $\sigma_{\min}$ denotes minimal singular value and $U_\Delta = \{U_{ij}\}_{ij \in \Delta}$ is a principal submatrix of matrix $U$, with rows and columns taken from the subset $\Delta$.

**Proof.** Proof follows from Proposition 1 and the result concerning singular values of the difference of projectors [19].
Remark 1. From the above Corollary we see that $P_1$ and $P_U$ are perfectly distinguishable without entanglement if and only if there exists a rank-deficient principal submatrix of matrix $U$.

Remark 2 (Optimal strategy for discrimination of measurements without entanglement). The optimal input state is the normalized leading eigenvector $(ev_1(\cdot))$ of matrix $\left| \sum_{i \in \Delta} (S_i - T_i) \right|$, i.e.

$$|\psi_{opt}\rangle = ev_1\left( \left| \sum_{i \in \Delta} (S_i - T_i) \right| \right)$$

for a subset $\Delta$ which maximizes eq. (13). In the case of projective measurements it reads

$$|\psi_{opt}\rangle = ev_1\left( \left| \sum_{i \in \Delta} (|i\rangle\langle i| - |u_i\rangle\langle u_i|) \right| \right).$$

4.2. Discrimination of unitary channels. Before we proceed to present our main results, we need to briefly discuss the problem of discrimination of unitary channels. This can be done without the usage of entangled input. In order to formulate the condition for perfect discrimination of unitary channels we introduce the notion of numerical range of a matrix $A \in M_d$, denoted by $W(A) = \{\langle x|A|x \rangle : |x\rangle \in \mathbb{C}^d, \langle x|x\rangle = 1\}$. The celebrated Hausdorff-Töplitz theorem [20, 21] states that $W(A)$ is a convex set and therefore $W(A) = \{\text{tr} \, A \sigma : \sigma \in \Omega_d\}$. Let us now recall the well-known [18] result for the distinguishability of unitary channels.

Proposition 2. Let $U \in U_d$ and $\Phi_U : \rho \mapsto U\rho U^\dagger$ be a unitary channel. Then

$$\|\Phi_U - \Phi_1\| = 2\sqrt{1 - \nu^2},$$

where $\nu = \min \{ |x| : x \in W(U^\dagger) \}$.

From the above proposition it follows that unitary channels $\Phi_U, \Phi_1$ are perfectly distinguishable if and only if $0 \in W(U^\dagger)$. The above can also be formulated as: there exists a density matrix $\sigma$, such that $\text{tr} \, U^\dagger \sigma = 0$.

5. Entanglement assisted discrimination

A more sophisticated idea for discriminating quantum measurements requires the use of an entangled state. We put one part of the state into the measurement device and later use the other part to improve the probability of correct discrimination. Our goal is to show how the discrimination of projective measurements is connected with the problem of discrimination of unitary channels. Finally, we would like to state the analytical form of the optimal discriminator $\rho$.

Proposition 3. Let $\Phi$ be Hermiticity-preserving and $\rho \in \Omega_{d_1,d_2}$ be a discriminator of $\Phi$ such that $\text{rank}(\rho) = k$. Then it is possible to obtain the value of the diamond norm on a channel extended by a $k$-dimensional identity channel. If the state $\rho$ is rank-one, then the optimal discrimination can be performed without the use of entanglement.
Proof. Let us take the Schmidt decomposition of $|\sqrt{\rho}^\top\rangle\rangle = \sum_{i=1}^{k} \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle = \|\Phi\|_\diamond = \| (1 \otimes \sqrt{\rho}) J(\Phi) (1 \otimes \sqrt{\rho}) \|^1_1$

\[
\|\Phi\|_\diamond = \| (\Phi \otimes 1_d) \left( (1 \otimes \sqrt{\rho}^\top) \langle\langle \sqrt{\rho}^\top | \right) \|^1_1
\]

where $V$ is a unitary matrix such that for the Schmidt decomposition of $|\sqrt{\rho}^\top\rangle\rangle$ we have $(1 \otimes \sqrt{\rho}^\top) = \sum_{i=1}^{k} \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle$. Thus $(\Phi \otimes 1_d) (|\sqrt{\rho}^\top\rangle\rangle \langle\langle \sqrt{\rho}^\top |) \) admits a block structure. Neglecting all zeros we can obtain the same value of the trace norm for a pure state with the second subsystem of dimension $k$.

The following theorem gives us a simple condition that lets us decide whether $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable. This condition is one of the main results of our work and its proof is postponed to Appendix A.3

**Theorem 1.** Let $U, V \in \mathcal{U}_d$ and let $\mathcal{P}_U$ and $\mathcal{P}_V$ be two projective measurements and $\mathcal{D}U_d$ be the set of diagonal unitary matrices of dimension $d$. Then

\[
\|\mathcal{P}_U - \mathcal{P}_V\|_\diamond = \min_{E \in \mathcal{D}U_d} \| \Phi_{UE} - \Phi_V \|_\diamond.
\]

Theorem 1 gives us a potentially easy method to calculate the diamond norm. A simple observation is that if we build projections $U |i\rangle\langle i | U^\dagger$ from unitary matrix $U$, then the same projections will be built from matrix $UE$, where $E \in \mathcal{D}U_d$. This means that matrices $UE$ form an equivalence class of matrix $U$. The interesting thing is that a “properly-rotated” matrix gives us an easy way of calculating value of the diamond norm $\|\mathcal{P}_U - \mathcal{P}_1\|_\diamond$ - it is enough to utilize Proposition 2. Since all unitary channels of the form $\Phi_{UE}$ are coherifications of channel $\mathcal{P}_U$, the above theorem gives us that the value of completely bounded trace norm is the minimal value of the norm on the difference between coherified channels.

The case of perfect distinguishability can be formulated, by the use of Theorem 1, as a Corollary

**Corollary 2.** Let $U \in \mathcal{U}_d$. Then $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable if and only if for all $E \in \mathcal{D}U_d$, unitary channels $\Phi_{UE}$ are perfectly distinguishable from the identity channel $\Phi_1$.

The above condition together with Proposition 2 gives us that perfect distinguishability is equivalent to the fact that $\forall E \in \mathcal{D}U_d : \text{tr} E^\dagger U^\dagger \rho = 0$. In fact, the above is equivalent to $\exists \rho : \forall E \in \mathcal{D}U_d : \text{tr} E^\dagger U^\dagger \rho = 0$, which at first glance seems to be much stronger. Of course the latter statement can be rewritten as $\exists \rho : \text{tr} (U^\dagger \rho) = 0$. We state this algebraic condition for perfect distinguishability in the next theorem whose proof is postponed to Appendix A.1

**Theorem 2.** Let $U \in \mathcal{U}_d$. Then $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable if and only if there exists $\rho \in \Omega_d$ such that

\[
\text{diag}(U^\dagger \rho) = 0.
\]
We would like to perfectly discriminate the measurements with the lowest possible amount of entanglement. This translates to the lowest possible rank of $\rho$. We are especially interested in the case when $\rho$ is a one-dimensional projection, so we do not need to use entanglement, see Remark 1 for necessary and sufficient condition in terms of matrix $U$.

In the general case, the diamond norm of a Hermiticity-preserving $\Phi : M_{d_1} \rightarrow M_{d_2}$ can be computed using the following semidefinite program [23].

$$
\begin{align*}
\text{Primal problem} & \quad \text{maximize: } \quad \text{Tr} X J(\Phi) \\
\text{subject to: } & \quad \begin{bmatrix} I_{d_2} \otimes \rho & X \\ X^* & I_{d_2} \otimes \rho \end{bmatrix} \succeq 0 \\
& \quad \rho \in \mathcal{H}_{d_1}^+ \\
& \quad X \in M_{d_1,d_2}(\mathbb{C})
\end{align*}

\begin{align*}
\text{Dual problem} & \quad \text{minimize: } \quad \| \text{Tr}_1 Y \|_\infty \\
\text{subject to: } & \quad \begin{bmatrix} Y & -J(\Phi) \\ -J(\Phi) & Y \end{bmatrix} \succeq 0 \\
& \quad Y \in \mathcal{H}_{d_1,d_2}^+.
\end{align*}

\text{PROGRAM 1. } \text{Semidefinite program for calculating diamond norm [23].}

This program allows us to compute the diamond norm for an arbitrary mapping $\Phi$. Regrettfully, it has one major drawback – very lengthy computations in practical applications. In theory, the complexity is polynomial in size of the input matrix $J(\Phi)$ which has the size of $d_1 d_2 \times d_1 d_2$. Due to this, the computational time and memory usage allow us to calculate the diamond norm only for $d_1, d_2 < 10$. 

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Dependence of the behavior of the numerical range of a matrix $UE \in \mathcal{U}_3$ on the eigenvectors of $U$. We start with a matrix $U$ with fixed eigenvalues and assign them different eigenvectors. The matrices above the arrows are the unistochastic matrices corresponding to these eigenvector matrices. The red shaded area is the numerical range of the matrix $UE$ for which $\min_{E \in D\mathcal{U}_d} \| \Phi_{UE} - \Phi_V \|_\diamond$ is achieved.}
\end{figure}
The result stated in Theorem 2 is in actuality a simple check whether \( P_U \) can be distinguished perfectly from \( P_1 \) and can also be used to find a state \( \rho \in \Omega_d \) for which \( \| P_1 - P_U \|_6 = 2 \). In the standard approach we would need to solve the semidefinite programming problem stated in Program 1.

To state the condition (20) formally as a semidefinite program we first introduce the notation

\[
A_0 = 1 \quad A_i = U|i\rangle\langle i| + |i\rangle\langle i|U^\dagger, \quad i = 1, \ldots, d \\
A_i = i \left( |i\rangle\langle i|U^\dagger - U|i\rangle\langle i| \right), \quad i = d + 1, \ldots, 2d.
\]

Hence we arrive at the primal and dual problems presented in Program 2.

**Primal problem**

maximize: \( \text{Tr} \rho A_0 \)

subject to: \( \text{Tr} \rho A_i = 0 \)

\( \text{Tr} \rho = 1 \)

\( \rho \in \mathcal{H}_d^+ \)

**Dual problem**

minimize: \( \langle 0|Y|0 \rangle \)

subject to: \( \sum_{i=0}^{2d} A_i Y_{ii} \geq 1 \)

\( Y \in \mathcal{H}_d \).

Program 2. Semidefinite program for checking perfect distinguishability of von Neumann measurements.

Note here that the maximization target is a trivial functional, as it reads \( \text{tr} \rho \) and later we constrain it to \( \text{tr} \rho = 1 \). Hence, the problem reduces to satisfying the constraints.

From [24, Theorem 3] we know that the primal problem of Program 2 has no solutions \( \rho \geq 0 \) if and only if

\[
\inf_{(x_0, \ldots, x_{2d}) \in \mathbb{R}^{2d+1}} e^{x_0} \text{tr} \left( e^{\sum_{i=1}^{2d} x_i A_i} \right) - x_0 = -\infty.
\]

This is equivalent to the condition that there exists a vector \((x_1, \ldots, x_{2d}) \in \mathbb{R}^{2d}\) such that \( \sum_{i=1}^{2d} x_i A_i < 0 \). In a general case, this is a complicated problem and no analytical methods of finding a solution are known. Nonetheless, there exist various algorithms, such as semidefinite programming, which approximate the solution [24, 25]. The above considerations can be summarized as a lemma.

**Lemma 1.** Let \( U \in \mathcal{U}_d \) and let \( \mathcal{P}_U, \mathcal{P}_1 \) be POVMs. Then \( \mathcal{P}_U \) and \( \mathcal{P}_1 \) are perfectly distinguishable if and only if for all real vectors \((x_1, \ldots, x_{2d}) \in \mathbb{R}^{2d}\) we have \( 0 \in W \left( \sum_{i=1}^{2d} x_i A_i \right) \).

**Proof.** The lemma follows directly from the fact that the solution of primal problem in Program 2 exists if and only if the real span of \( A_i \) contains only matrices without a determined sign.

The above considerations lead us to the following theorem, which proof is postponed to Appendix A.2.
Theorem 3. Let $U \in U_d$ and let $\mathcal{P}_U, \mathcal{P}_1$ be von Neumann’s POVMs. Then $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable if and only if for all diagonal matrices $D$ we have $0 \in W(U D + D^\dagger U^\dagger)$.

As the above conditions for perfect discrimination require solving a semi-definite problem, here we state a simple necessary and a simple sufficient conditions based only on the absolute values of the diagonal elements of the unitary matrix $U$. These turn out to be conclusive in the 3-dimensional case.

Theorem 4. Let $U \in U_d$ and $E \in DU_d$ such that $\langle i | U E | i \rangle \geq 0$. Then the following holds:

1. if $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable, then $\text{Tr}(U E) \leq d - 2$
2. if $\text{Tr}(U E) \leq 1$, then $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable for odd $d \geq 3$.

In particular, if $d = 3$, then $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable if and only if $\text{Tr}(U E) \leq 1$.

Proof. Assume that $\mathcal{P}_U$ and $\mathcal{P}_1$ are perfectly distinguishable. This implies that $0 \in W(U E)$. Consider a set of possible eigenvalues of $U E$ that maximizes $\text{Tr}(U E)$ . It can be either $\{\lambda, -\lambda, 1, \ldots, 1\}$ or $\{\lambda_1, \lambda_2, \lambda_3, 1, \ldots, 1\}$, where $0 \in \text{conv}(\lambda_1, \lambda_2, \lambda_3)$. To finish this part of the proof it is enough to note that $\lambda_1 + \lambda_2 + \lambda_3 \leq 1$.

Let now $E \in DU_d$. We note that $|\text{Tr}(U E)| \leq 1$. Assume $0 \notin W(U E)$.

Let $d = 2k + 1$ and $\lambda(U E) = \{\lambda_1, \ldots, \lambda_d\}$ be a set of eigenvalues written in an angular order. It is enough to see that

$$1 = |\lambda_{k+1}| < \sum_{i=k}^{k+2} |\lambda_i| < \sum_{i=k-1}^{k+3} |\lambda_i| < \ldots < \sum_{i=1}^{d} |\lambda_i|,$$

where the first inequality comes from the fact that if we consider unit vectors on a semicircle, then the absolute value of $\lambda_{k+1}$ can only increase when added to the sum $\lambda_k + \lambda_{k+2}$, which cannot be zero as $0 \notin W(U E)$. Other inequalities follow from similar reasoning. Thus $|\text{Tr}(U E)| > 1$, which finishes the proof. \qed

Now, we are ready to state the convex program for calculating diamond norm of the difference of two von Neumann measurements.

Using Proposition 2 the value of diamond norm from Theorem 1 can be rewritten as

$$\|\mathcal{P}_U - \mathcal{P}_1\|_\diamond = \min_{E \in DU_d} \|\Phi_{U E} - \Phi_1\|_\diamond = \min_{E \in DU_d} 2 \sqrt{1 - \min_{\rho \in \Omega_d} |\text{Tr}\rho U E|^2}$$

$$= 2 \sqrt{1 - \max_{E \in DU_d} \min_{\rho \in \Omega_d} |\text{Tr}\rho U E|^2}.$$

As shown in Appendix A.3 we may exchange the minimization with the maximization and obtain

$$\nu := \max_{E \in DU_d} \min_{\rho \in \Omega_d} |\text{Tr}\rho U E| = \min_{\rho \in \Omega_d} \max_{E \in DU_d} |\text{Tr}\rho U E| = \min_{\rho \in \Omega_d} \sum_i |\langle i | \rho U | i \rangle|.$$
Now we note that values \( \langle i| \rho U|i \rangle = \text{tr} \rho U|i \rangle \langle i| \) are the coefficients of a projection, in the Hilber-Schmidt space, of \( \rho \) onto a subspace \( \mathcal{L}_U \) spanned by unit orthogonal vectors \( \{ U|i \rangle \langle i| \}_i \). Therefore, the value \( \nu \) is a minimal taxi-cab norm of a projection of density matrix \( \rho \) onto a subspace \( \mathcal{L}_U \) calculated in the basis \( \{ U|i \rangle \langle i| \}_i \). The simplified geometrical sketch of this is presented in Fig. 4.

![Figure 4](image-url)

**Figure 4.** Sketch of the Hilber-Schmidt space with the cone of positive semidefinite matrices and its intersection with the affine plane \( \text{Tr}(\cdot) = 1 \). The optimal density matrix \( \rho_0 \) is marked together with its projection \( \pi(\rho_0) \) onto a plane \( \mathcal{L} \) spanned by orthonormal vectors \( \{ U|i \rangle \langle i| \}_i \). The taxicab distance to the origin of the projection gives the value \( \nu \) which in turn determines the diamond norm.

The value \( \nu \) can be calculated using cone programming and we get the SDP shown in Program 3. The minimum value \( \nu \) of this program gives us

\[
\begin{align*}
\text{Primal problem} \\
\text{minimize:} \quad & \| \text{diag}(U^\dagger \rho) \|_1 \\
\text{subject to:} \quad & \text{tr} \rho = 1, \\
& \rho \geq 0.
\end{align*}
\]

**Program 3.** Convex program for calculation of the diamond norm of the difference of two nov Neumann measurements.

the value of the diamond norm as

\[
\| P_U - P_1 \|_o = 2\sqrt{1 - \nu^2}.
\]
We use a state $\rho$ which minimizes objective function in Program 3 to construct the input state for discrimination scheme. The input state $|\psi\rangle$ is a purification of $\rho$, thus its rank is equal to the dimension of additional subsystem needed for optimal procedure. This program is polynomial in the size of the input matrix $U$.

6. Special cases

In this section we will present several examples of projective measurements, which can be perfectly distinguished from a measurement in the computational basis.

6.1. Fourier matrices. First, we consider the Fourier unitary matrices $F_2 \in U_2$ and $F_3 \in U_3$. We note that unitary channels $\Phi_{F_2}, \Phi_{F_3}$ are perfectly distinguishable from the corresponding identity channels. On the other hand, it is not possible to perfectly distinguish $P_{F_2}, P_{F_3}$ from the $P_{1_2}$ and $P_{1_3}$, which follows from Theorem 4.

In the case of higher dimensions, we have the following corollary.

**Corollary 3.** Let $d \geq 4$ and $F_d \in U_d$ be a Fourier matrix. Then $P_{F_d}$ is perfectly distinguishable from $P_{1_d}$. The perfect discrimination may be performed with an entangled input state with additional subsystem of dimension 2. Moreover, if $d = m^2n$ for $m, n \in \mathbb{N}$, $m > 1$, then the discrimination can be done without entangled input, while this is not possible for prime dimension.

**Proof.** Define a matrix $X \in M_d$

\[
X = \begin{bmatrix}
4 \cos \frac{2\pi}{d} & -2 \cos \frac{2\pi}{d} & 0 & \ldots & 0 & -2 \cos \frac{2\pi}{d} \\
-2 \cos \frac{2\pi}{d} & 1 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 \\
-2 \cos \frac{2\pi}{d} & 1 & 0 & \ldots & 0 & 1
\end{bmatrix},
\]

which is positive semidefinite for $d \geq 4$. Direct calculations show that

\[
\text{diag}(F_d^\dagger X) = 0
\]

and $\text{rank}(X) \leq 2$, so, as stated in Theorem 2 we have perfect distinguishability. In the case of $d = m^2n$ we take

\[
X' = (|0\rangle - |mn\rangle)(\langle 0| - \langle mn|),
\]

then it holds that $\text{diag}(F_d^\dagger X') = 0$. To check that there does not exist rank-one perfect discriminator for prime dimension we need to check if among principal submatrices of a Fourier matrix there does not exist a rank-deficient one (Remark 1). The Chebotarev theorem on roots of unity states that such a submatrix does not exist, see e.g. [26] or Theorem 4 in [27].

The optimal input states for discrimination scheme are purifications of matrices $X$ in the proof.
6.2. Reflection matrices. Now, we will consider a unitary matrix given by a mirror isometries.

**Corollary 4.** Let $\mathcal{U}_d \ni U = \mathbb{1} - 2|x\rangle\langle x|$. Then $\mathcal{P}_U$ is perfectly distinguishable from $\mathcal{P}_1$ if and only if $\omega = \max_i |x_i|^2 \leq \frac{1}{2}$. It is also possible to use a discriminator $\rho \in \Omega_d$ such that $\text{rank}(\rho) \leq 2$. Moreover, we can find $\rho$ such that $\text{rank}(\rho) = 1$ if and only if

$$ (30) \quad \exists \Delta \subset \{0,1,\ldots,d-1\} : \sum_{i \in \Delta} |x_i|^2 = \frac{1}{2}. $$

In the case when $\omega > \frac{1}{2}$, we have $\|\mathcal{P}_U - \mathcal{P}_1\|_\diamond = 2\sqrt{1 - 4(\omega - \frac{1}{2})^2}$.

**Proof.** If $\omega \leq \frac{1}{2}$, we provide a construction

$$ (31) \quad \rho = \frac{1}{2} |x\rangle\langle x| + \frac{1}{2} |y\rangle\langle y|, $$

where

$$ (32) \quad y_i = |x_i| e^{i\alpha_i} $$

such that

$$ (33) \quad \langle y|x \rangle = 0 = \sum_i |x_i|^2 e^{i(\arg(x_i) - \alpha_i)}. $$

By the polygon inequality we know that such phases $\alpha_i$ do exist, and therefore we receive $\text{diag}(U^\dagger \rho) = 0$.

Next, we can note that the existence of a set $\Delta \subset \{0,1,\ldots,d-1\}$, such that $\sum_{i \in \Delta} |x_i|^2 = \frac{1}{2}$, is equivalent to the fact that principal submatrix $V = \{U_{ij}\}_{i,j \in \Delta}$ is rank-deficient. Thus, the third statement follows from Remark 1.

Now, we assume that $\omega = |x_0|^2 > \frac{1}{2}$. The case when $\omega = 1$ is trivial, so we assume $\omega < 1$. Let $E' = \mathbb{1} - 2|0\rangle\langle 0|$. Direct calculation gives us

$$ (34) \quad \lambda(E') = \{2|x_0|^2 - 1 \pm 2|x_0|\sqrt{1 - |x_0|^2}, 1, \ldots, 1\}. $$

Eigenvectors corresponding to outlying eigenvalues have the form

$$ (35) \quad |\lambda_{\pm} \rangle = |x\rangle + (-x_0 \pm \frac{x_0}{|x_0|} \sqrt{1 - |x_0|^2})|0\rangle $$

and from this form we can see that $|\lambda_{+,i}| = |\lambda_{-,i}|$, and according to proof of Theorem 1 we have

$$ (36) \quad \max_{E \in \mathcal{E}_d} \min_{\rho \in \Omega_d} |\text{Tr}UE\rho| = \min_{\rho \in \Omega_d} |\text{Tr}UE'\rho| = 2|x_0|^2 - 1. $$

Utilizing Theorem 1 and Proposition 2 we obtain

$$ (37) \quad \|\mathcal{P}_U - \mathcal{P}_1\|_\diamond = 2\sqrt{1 - 4(\omega - \frac{1}{2})^2}. $$

□
7. Final remarks

In this work we have studied the problem of single shot discrimination of two von Neumann measurements with finitely many outcomes. Our aim was to design an optimal strategy for the discrimination in both cases: with and without the assistance of entanglement. We have parametrized both measurements with a single unitary matrix $U$ and expressed the results using the properties of $U$. In the first case, when we do not use entanglement, the optimal probability can be expressed as a function of minimal singular value of a submatrix of unitary matrix $U$, see Corollary 1. We have also provided a construction of an optimal input state which enables performing optimal discrimination strategy in this scenario. In the second case of entanglement-assisted discrimination, the optimal probability is a function of minimal taxicab norm of a projection (in the Hilbert-Schmidt space) of a density matrix on a plane spanned by vectors $U|i⟩⟨i|$, see Theorem 1 and discussion below. Moreover, we have provided a convex program for calculating this optimal probability and deriving the optimal input state for entanglement-assisted discrimination scheme. Finally, we have considered special cases of Fourier matrices and mirror isometries.

Appendix A. Proofs

A.1. Proof of Theorem 2

Proof of Theorem 2 Let $ρ ∈ Ω_d$ be a discriminator. Then

$$
\|P_1 - P_U\|_f = \left\| \sum_{i=1}^{d} i |i⟩⟨i| \otimes (\sqrt{p}|i⟩⟨i| - |u_i⟩⟨u_i|) \sqrt{p} \right\|_f
$$

(38)

$$
= \sum_{i=1}^{d} tr \left| \sqrt{p}|i⟩⟨i| \sqrt{p} - \sqrt{p}|u_i⟩⟨u_i| \sqrt{p} \right|
$$

$$
= \sum_{i=1}^{d} \sqrt{\left( ⟨i|ρ|i⟩ + ⟨u_i|ρ|u_i⟩ \right)^2 - 4 |⟨i|ρ|u_i⟩|^2},
$$

where the last equality follows from the singular value decomposition for rank-two matrices.

Assume that $\|P_1 - P_U\|_f = 2$. If for any state $ρ$, the condition (20) is not satisfied, i.e. $\forall ρ \exists_i ⟨i|ρ|u_i⟩ \neq 0$, then

(39)

$$
\sum_{i=1}^{d} \sqrt{\left( ⟨i|ρ|i⟩ + ⟨u_i|ρ|u_i⟩ \right)^2 - 4 |⟨i|ρ|u_i⟩|^2} < \sum_{i=1}^{d} \left( ⟨i|ρ|i⟩ + ⟨u_i|ρ|u_i⟩ \right) = 2,
$$

which gives a contradiction.

Next, assume that there exists a state $ρ$ such that $⟨i|ρ|u_i⟩ = 0$ for all $i$. From eq. (38) we have $\|P_1 - P_U\|_f = 2$. □

A.2. Proof of Theorem 3

Proof of Theorem 3 Perfect distinguishability between $P_U$ and $P_1$ means, by Theorem 2, that for some density matrix $ρ ∈ Ω_d$ we have

(40) $\text{diag}(U^\dagger ρ) = 0$. 

If this condition is satisfied, we also have $\text{diag}(D^\dagger U^\dagger \rho) = 0$ for any diagonal matrix $D$, and therefore $0 \in W(U D + D^\dagger U^\dagger)$.

Now, let us assume that for all diagonal matrices $D$ we have $0 \in W(U D + D^\dagger U^\dagger)$. We define a matrix
\begin{equation}
D = \text{diag}(x_1 - i x_{d+1}, x_2 - i x_{d+2}, \ldots, x_d - i x_{2d}).
\end{equation}
Thus, there exists a nonzero, $x$-dependent state $|\psi\rangle$, such that
\begin{equation}
\langle \psi | \left( U D + D^\dagger U^\dagger \right) | \psi \rangle = 0.
\end{equation}
This can be equivalently expressed as
\begin{equation}
\langle \psi | \sum_i x_i A_i | \psi \rangle = 0.
\end{equation}
Using Lemma 1 we arrive at our result. □

A.3. Proof of Theorem 1

Proof of Theorem 1: First, we consider the case of perfect distinguishability, i.e. we will prove Corollary 2.

Let us assume that $P_U$ is perfectly distinguishable from $P_1$. Then, from Theorem 2 there exists a density matrix such that
\begin{equation}
\text{diag}(U^\dagger \rho) = 0.
\end{equation}
Hence, for all $E \in DU_d$ we have $\text{diag}(E^\dagger U^\dagger \rho) = 0$. Therefore $0 \in W(E^\dagger U^\dagger)$, and thus unitary channel $\Phi_{UE}$ is perfectly distinguishable from the identity channel.

Now, we assume that for all $E \in DU_d$ we have $0 \in W(E^\dagger U^\dagger)$. We will show that for any diagonal matrix $D$ (not necessarily unitary), we have $0 \in W(U D + D^\dagger U^\dagger)$ (see Theorem 3). One may assume that $D$ is invertible as otherwise we would have $\langle \psi | \left( U D + D^\dagger U^\dagger \right) | \psi \rangle = 0$ for $|\psi\rangle \in \ker(D)$. We write
\begin{equation}
UD = U E D_+,
\end{equation}
where $E \in DU_d$ and $D_+$ is a strictly positive diagonal matrix. Let $V$ be a unitary matrix such that
\begin{equation}
UE = V \text{diag}^\dagger(\lambda)V^\dagger,
\end{equation}
where $\lambda$ denotes eigenvalues of $UE$. From our assumption we have that there exists a probability vector $p$, such that
\begin{equation}
\sum_i \lambda_i p_i = 0.
\end{equation}
Now we define a density matrix
\begin{equation}
\sigma = V \text{diag}^\dagger(q)V^\dagger,
\end{equation}
where
\begin{equation}
q_i = c^{-1} \frac{p_i}{\langle i | V^\dagger D_+ V | i \rangle}; \quad c = \sum_j \frac{p_j}{\langle j | V^\dagger D_+ V | j \rangle}.
\end{equation}
Using this we obtain
\begin{equation}
\text{tr} U D \sigma = c^{-1} \sum_i \lambda_i p_i = 0.
\end{equation}
Thus $0 \in W(UD)$ and therefore $0 \in W(UD + D^1 U^\dagger)$. This finishes the proof of Theorem \ref{thm:trace_norm} in the case of perfect distinguishability.

Now, we will show the remaining part in the case when $\min_{\mathcal{D} \in \mathcal{L}_d} \|\Phi_{UE} - \Phi_1\|_\diamond < 2$. First we show that
\begin{equation}
\|P_U - P_1\|_\diamond \leq \|\Phi_{UE} - \Phi_1\|_\diamond.
\end{equation}
Let $\rho^\top$ be a discriminator of $P_U - P_1$. Thus
\begin{equation}
\|P_U - P_1\|_\diamond = \left\| \left( \mathbb{1} \otimes \sqrt{\rho^\top} \right) J_{P_U - P_1} \left( \mathbb{1} \otimes \sqrt{\rho^\top} \right) \right\|_1 = \left\| \sum_i |i\rangle \langle i| \otimes \sqrt{\rho^\top} M_i^\top \sqrt{\rho^\top} \right\|_1,
\end{equation}
where $M_i = |i\rangle \langle i| - |u_i\rangle \langle u_i| = |i\rangle \langle i| - UE|i\rangle \langle i|E^\dagger U^\dagger$. Now, using the operational definition of the trace norm ($\|A\|_1 = \max_{V \in \mathcal{L}_d} \|\text{tr}(AV)\|$) and the fact that the matrix is in a block form, and get
\begin{equation}
\left\| \sum_i |i\rangle \langle i| \otimes \sqrt{\rho^\top} M_i^\top \sqrt{\rho^\top} \right\|_1 = \text{tr}(\sqrt{\rho} M_i \sqrt{\rho} V_i)
\end{equation}
where $V_i$ is a unitary matrix, which is optimal for $i$th block. Next we note that
\begin{equation}
\text{tr} \left( \sum_{ij} |i\rangle \langle i| \otimes \sqrt{\rho} \left( |i\rangle \langle i| - UE|i\rangle \langle i|E^\dagger U^\dagger \right) \sqrt{\rho} \right) \left( \sum_j |i\rangle \langle i| \otimes V_j \right)
\end{equation}
\begin{equation}
\leq \max_{V \in \mathcal{L}_d(\mathbb{1}^2)} \left\| \sum_{ij} |i\rangle \langle j| \otimes \sqrt{\rho} \left( |i\rangle \langle j| - UE|i\rangle \langle j|E^\dagger U^\dagger \right) \sqrt{\rho} \right\|_1 \|V\|
\end{equation}
\begin{equation}
\leq \|\Phi_{(UE)^\top} - \Phi_1\|_\diamond = \|\Phi_{UE} - \Phi_1\|_\diamond.
\end{equation}
For the case when $\min_{E \in DUd} \|\Phi_{UE} - \Phi_1\|_\diamond = 2$, that is in the case of perfect distinguishability, the proof was already done. Let $\min_{E \in DUd} \|\Phi_{UE} - \Phi_1\|_\diamond < 2$. We have
\begin{equation}
\min_{E \in DUd} \|\Phi_{UE} - \Phi_1\|_\diamond = \min_{E \in DUd} \frac{2}{\sqrt{1 - \min_{\rho \in \Omega_d} \|\text{tr} \rho UE\|^2}}
\end{equation}
\begin{equation}
= 2 \sqrt{1 - \max_{E \in DUd, \rho \in \Omega_d} \min_{\rho \in \Omega_d} \|\text{tr} \rho UE\|^2}.
\end{equation}
In the case of $\rho_0 \in \Omega_d$ and $E_0 \in DU_d$ which saturate $\min_{E \in DU_d} \|\Phi_{UE} - \Phi_1\|_2$, we have that $0 \not\in W(UE_0)$.

Let $D^1_d$ be the set of diagonal matrices $E$ such that $|E_{ii}| \leq 1$. The set of density matrices and the set $D^1_d$ are both compact and convex. Moreover, the sets $\{E \in D^1_d : \Re(\Tr(\rho UE)) = \max_{D \in D^1_d} \Re(\Tr(\rho UD))\}$ and $\{\rho \in \Omega_d : \Re(\Tr(\rho UE)) = \min_{\sigma \in \Omega_d} \Re(\Tr(\sigma UE))\}$ are convex. Since all assumptions of the Theorem 3 in [28] are fulfilled, we obtain the existence of saddle points, and therefore

$$\min_{\rho \in \Omega_d} \max_{E \in D^1_d} \Re(\Tr(\rho UE)) = \max_{E \in D^1_d} \min_{\rho \in \Omega_d} \Re(\Tr(\rho UE)).$$

One can note that it implies that for a saddle point $(\rho_0, E_0)$ we have $\Re(\Tr(\rho_0 UE_0)) = \Tr \rho_0 U E_0 = |\Tr \rho_0 U E_0|$. Moreover, $\max_{E} |\Tr \rho_0 UE| = \sum_i |\langle i | \rho_0 U | i \rangle| = \Tr \rho_0 U E_0$ and $\Tr \rho_0 U E_0 = \min_{\rho} |\Tr \rho U E_0|$. That means $(\rho_0, E_0)$ is the saddle point of $|\Tr \rho U E|$ and

$$\min_{\rho \in \Omega_d} \max_{E \in D^1_d} |\Tr(\rho UE)| = \max_{E \in D^1_d} \min_{\rho \in \Omega_d} |\Tr(\rho UE)|.$$

Let us write $E_0 = F_0 D$, where $F_0 \in DU_d$ and $D$ is a diagonal matrix with $0 \leq D_{ii} \leq 1$. We will show that we have the saddle point also for $(\rho_0, F_0)$. First of all, we will observe that for arbitrary $U \in U_d$

$$\min_{\rho} |\Tr \rho U| \geq \min_{\rho} |\Tr \rho UD|,$$

For the case when $0 \in W(U)$, for some probability vector $p$ we have $\sum_i \lambda_{i} p_i = 0$, where $\lambda_i$ are eigenvalues of $U$. If there exists $i$ such that $|\langle i | D | i \rangle| = 0$, then $|\Tr |\lambda_i\rangle \langle \lambda_i| UD| \rangle| = 0$. Otherwise, we can take the state $\rho = \sum_i q_i |\lambda_i\rangle \langle \lambda_i|$, where $q_i = \frac{p_i}{(\lambda_i | D | \lambda_i)}$ and notice that $0 \not\in W(UD)$. In the case, when $0 \not\in W(U)$ for the most distant eigenvalues $\lambda_1, \lambda_d$, using Toeplitz-Hausdorff theorem we have inclusion of an interval in a numerical range

$$[\Tr |\lambda_1\rangle \langle \lambda_1| UD| \rangle, \Tr |\lambda_d\rangle \langle \lambda_d| UD| \rangle] \subset W(UD).$$

Using optimality condition we receive $\min_{\rho} |\Tr \rho UF_0| = \min_{\rho} |\Tr \rho U F_0 D|$. Now we are ready to check if $(\rho_0, F_0)$ is the saddle point. We write

$$|\Tr \rho_0 U F_0| \leq \max_{E \in D^1_d} |\Tr \rho_0 UE| = |\Tr \rho_0 U E_0| = \min_{\rho} |\Tr \rho U F_0 D|$$

$$= \min_{\rho} |\Tr \rho U F_0| \leq |\Tr \rho_0 U F_0|.$$

The above gives us information that

$$|\Tr \rho_0 U F_0| = \min_{\rho} |\Tr \rho U F_0| = \max_{E \in D^1_d} |\Tr \rho_0 UE|.$$

Denote by $\lambda$ and $\bar{\lambda}$ two most distant eigenvalues of $V = UF_0$, (we may write it like this because we can add global phase without changing the objective function). By $P_1, P_2$ we denote projectors onto subspaces spanned by eigenvectors associated with $\lambda$ and $\bar{\lambda}$ respectively. Since $\rho_0$ gives minimum of the $|\Tr \rho V|$, thus $\rho_0$ is supported on the subspace spanned by range of $P_1$ and $P_2$, i.e.

$$\rho_0 = P \rho_0 P \text{ for } P = P_1 + P_2.$$
We may write
\[(62)\quad \rho_0 = P\rho_0 P = P_1\rho_0 P_1 + P_2\rho_0 P_2 + P_1\rho P_2 + P_2\rho_0 P_1\]
and define
\[
\rho_1 = P_1\rho_0 P_1, \\
\rho_2 = P_2\rho_0 P_2, \\
\rho_{12} = P_1\rho_0 P_2, \\
\rho_{21} = P_2\rho_0 P_1.
\]
Note that the optimality forces \(\text{tr}\ \rho_1 = \text{tr}\ \rho_2 = \frac{1}{2}\). Now we write
\[(63)\quad z_i = \langle i|\rho_0 V|i\rangle = \lambda\langle i|\rho_1|i\rangle + \bar{\lambda}\langle i|\rho_2|i\rangle + 2\text{Re}(\lambda\langle i|\rho_{21}|i\rangle).
\]
We have \(\sum_i z_i = \frac{\lambda + \bar{\lambda}}{2}\). If elements \(z_i\) have different phases, then by additional unitary matrix one can increase the value of the sum and contradict the fact that \((\rho_0, F_0)\) is a saddle point. Therefore, we conclude that all elements have the same phase and therefore we obtain that
\[(64)\quad \langle i|\rho_1|i\rangle = \langle i|\rho_2|i\rangle \quad \text{for all} \ i.
\]
Now we define a new state
\[(65)\quad \tau = \rho_1 + \rho_2\]
and calculate \(\| (\mathbb{1} \otimes \sqrt{\tau})J(P_1 - P_{UF_0})(\mathbb{1} \otimes \sqrt{\tau}) \|_1\) according to eq. (38). Direct calculation gives us
\[(66)\quad \sum_{i=1}^d \sqrt{(\langle i|\tau|i\rangle + \langle u_i|\tau|u_i\rangle)^2 - 4\langle i|\tau|u_i\rangle^2} = 2\sqrt{1 - \frac{\lambda + \bar{\lambda}}{2}},
\]
where \(\frac{\lambda + \bar{\lambda}}{2} = |\text{Tr} \tau U F_0|\). To end this proof we write
\[(67)\quad 2\sqrt{1 - \frac{\lambda + \bar{\lambda}}{2}} = \| (\mathbb{1} \otimes \sqrt{\tau})J(P_1 - P_{U})(\mathbb{1} \otimes \sqrt{\tau}) \|_1
\]
\[(68)\quad \leq \| P_{U} - P_{1}\|_\diamond \leq \min_{E \in \mathcal{B} \mathcal{M}} \| \Phi_{UE} - \Phi_1 \|_\diamond = 2\sqrt{1 - \frac{\lambda + \bar{\lambda}}{2}}.
\]
\[\square\]

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