Solutions of the Dirichlet-Schrödinger problems with continuous data admitting arbitrary growth property in the boundary

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Abstract
By using the modified Green-Schrödinger function, we consider the Dirichlet problem with respect to the stationary Schrödinger operator with continuous data having an arbitrary growth in the boundary of the cone. As an application of the modified Poisson-Schrödinger integral, the unique solution of it is also constructed.

Keywords: modified Green-Schrödinger potential; modified Poisson-Schrödinger integral; Dirichlet-Schrödinger problem

1 Introduction and main theorem
We denote the n-dimensional Euclidean space by \( \mathbb{R}^n \), where \( n \geq 2 \). The sets \( \partial E \) and \( E \) denote the boundary and the closure of a set \( E \) in \( \mathbb{R}^n \). Let \( |V - W| \) denote the Euclidean distance of two points \( V \) and \( W \) in \( \mathbb{R}^n \), respectively. Especially, \( |V| \) denotes the distance of two points \( V \) and \( O \) in \( \mathbb{R}^n \), where \( O \) is the origin of \( \mathbb{R}^n \).

We introduce a system of spherical coordinates \((\tau, \Lambda) \), \( \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \), in \( \mathbb{R}^n \) which are related to the Cartesian coordinates \((y_1, y_2, \ldots, y_{n-1}, y_n)\) by

\[
y_1 = \frac{1}{\Gamma(\frac{n}{2})} \sin \lambda_1 \left( \prod_{j=1}^{n-1} \frac{\sin \lambda_j}{\Gamma(\frac{n-j}{2})} \right) \quad (n \geq 2), \quad y_n = \tau \cos \lambda_1,
\]

and if \( n \geq 3 \), then

\[
y_{n-m+1} = \tau \left( \prod_{j=m}^{n-1} \frac{\sin \lambda_j}{\Gamma(\frac{n-j}{2})} \cos \lambda_m \right) \quad (2 \leq m \leq n - 1),
\]

where \( 0 \leq \tau < +\infty, -\frac{1}{2} \pi \leq \lambda_{n-1} < \frac{3}{2} \pi \), and if \( n \geq 3 \), then \( 0 \leq \lambda_j \leq \pi \) \((1 \leq j \leq n - 2)\).

Let \( B(V, \tau) \) denote the open ball with center at \( V \) and radius \( \tau \) in \( \mathbb{R}^n \), where \( \tau > 0 \). Let \( S^{n-1} \) and \( S_{+}^{n-1} \) denote the unit sphere and the upper half unit sphere in \( \mathbb{R}^n \), respectively. The surface area \( 2\pi^{n/2}(\Gamma(n/2))^{-1} \) of \( S^{n-1} \) is denoted by \( w_n \). Let \( \Xi \subset S^{n-1} \), \( \Lambda \) and \( \Xi \) denote a point \((1, \Lambda)\) and the set \( \{ \Lambda; (1, \Lambda) \in \Xi \} \), respectively. For two sets \( \Lambda \subset \mathbb{R}_+ \) and \( \Xi \subset S^{n-1} \), we denote

\[
\Lambda \times \Xi = \{ (\tau, \Lambda) \in \mathbb{R}^n; \tau \in \Lambda, (1, \Lambda) \in \Xi \},
\]
The corresponding eigenfunctions are denoted by \( P_j \) and non-increasing cases, as \( P_j \) are denoted by \( Q_j \) (\( j = 1, 2, 3, \ldots \)), respectively, for the increasing and non-increasing cases, as \( \tau \to +\infty \), which is normalized under the condition \( P_j(1) = 1 \).
where for \( j = 0, 1, 2, 3 \ldots \).

It is well known (see [6]) that in the case under consideration the solutions to equation (1.1) have the asymptotics

\[
P_j(\tau) \sim d_1 \tau^{\zeta_{+}}, \quad Q_j(\tau) \sim d_2 \tau^{\zeta_{-}}, \quad \text{as } \tau \to \infty,
\]

where \( d_1 \) and \( d_2 \) are some positive constants.

The Green-Schrödinger function \( G(\Xi; a)(V, W) \) (see [4], Chap. 11) has the following expansion:

\[
G(\Xi; a)(V, W) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} P_j(\min(\tau, i)) \left(\max(\tau, i)\right) \left(\sum_{v=1}^{p_j} \varphi_{jv}(\Lambda) \varphi_{jv}(\Phi)\right),
\]

for \( a \in A_a \), where \( V = (\tau, \Lambda), W = (i, 1), \tau > i, \) and \( \chi'(s) = w(Q_j(\tau), P_j(\tau))\) are their Wronskian. The series converges uniformly if either \( \tau \leq s \) or \( \tau \leq s \) \((0 < s < 1)\).

For a nonnegative integer \( m \) and two points \( V = (\tau, \Lambda), W = (i, \Upsilon) \in H_n(\Xi) \), we put

\[
K(\Xi; a, m)(V, W) = \left\{ \begin{array}{ll}
0 & \text{if } 0 < i < 1, \\
\tilde{K}(\Xi; a, m)(V, W) & \text{if } 1 \leq i < \infty,
\end{array} \right.
\]

where

\[
\tilde{K}(\Xi; a, m)(V, W) = \sum_{j=0}^{m} \frac{1}{\chi'(1)} P_j(\tau) Q_j(i) \left(\sum_{v=1}^{p_j} \varphi_{jv}(\Lambda) \varphi_{jv}(\Phi)\right).
\]

The modified Green-Schrödinger function can be defined as follows (see [4], Chap. 11):

\[
G(\Xi; a, m)(V, W) = G(\Xi; a)(V, W) - K(\Xi; a, m)(V, W)
\]

for two points \( V = (\tau, \Lambda), Q = (i, \Upsilon) \in H_n(\Xi) \), then the modified Poisson-Schrödinger case on cones can be defined by

\[
P_\Pi(\Xi; a, m)(V, W) = \frac{\partial G(\Xi; a, m)(V, W)}{\partial \mathcal{N}_W}
\]

accordingly, which has the following growth estimates (see [7]):

\[
|P_\Pi(\Xi; a, m)(V, W)| \leq M(n, m, s) P_{m+1}(\tau) \left(\begin{array}{c}
Q_{m+1}(i) \\
\varphi_1(\Lambda)
\end{array}\right) \frac{\partial \varphi_1(\Upsilon)}{\partial \mathcal{N}_\Upsilon} \quad (1.2)
\]
for any $V = (\tau, \Lambda) \in H_\eta(\Xi)$ and $W = (\iota, \Gamma) \in S_\eta(\Xi)$ satisfying $\tau \leq s$ $(0 < s < 1)$, where $M(n, m, s)$ is a constant dependent of $n, m, \text{and} s$.

We remark that

$$\mathbb{P}(\Xi; a, 0)(V, W) = \mathbb{P}(\Xi; a)(V, W).$$

In this paper, we shall use the following modified Poisson-Schrödinger integrals (see [7]):

$$\mathbb{P}^a(m, f)(V) = \int_{S_\eta(\Xi)} \mathbb{P}(\Xi; m, W)f(W) d\sigma_W,$$

where $f(W)$ is a continuous function on $\partial H_\eta(\Xi)$ and $d\sigma_W$ is the surface area element on $S_\eta(\Xi)$.

For more applications of modified Green-Schrödinger potentials and modified Poisson-Schrödinger integrals, we refer the reader to the papers (see [7, 8]).

Recently, Huang and Ychussie (see [7]) gave the solutions of the Dirichlet-Schrödinger problem with continuous data having slow growth in the boundary.

**Theorem A.** If $f$ is a continuous function on $\partial H_\eta(\Xi)$ satisfying

$$\int_{S_\eta(\Xi)} \frac{|f(\iota, \Gamma)|}{1 + P_{m+1}(\iota)^{m-1}} d\sigma_W < \infty,$$

then the modified Poisson-Schrödinger integral $\mathbb{P}^a(m, f)$ is a solution of the Dirichlet-Schrödinger problem in $H_\eta(\Xi)$ with $f$ satisfies

$$\lim_{\tau \to \infty, V = (\tau, \Lambda) \in H_\eta(\Xi)} \pi^{1/2 + k}\pi^{-1}(\Lambda)^m \mathbb{P}^a(m, f)(V) = 0.$$

It is natural to ask if the continuous function $f$ satisfying (1.3) can be replaced by continuous data having the arbitrary growth property in the boundary. In this paper, we shall give an affirmative answer to this question. To do this, we also construct a modified Poisson-Schrödinger kernel. Let $\phi(l)$ be a positive function of $l \geq 1$ satisfying

$$P(2)\phi(1) = 1.$$

Denote the set

$$\{l \geq 1; -\zeta_{j/k}\log 2 = \log (P^{-1}\phi(l))\}$$

by $\pi_\Xi(\phi, j)$. Then $1 \in \pi_\Xi(\phi, j)$. When there is an integer $N$ such that $\pi_\Xi(\phi, N) \neq \Phi$ and $\pi_\Xi(\phi, N + 1) = \Phi$, denote

$$J_\Xi(\phi) = \{j; 1 \leq j \leq N\}$$

of integers. Otherwise, denote the set of all positive integers by $J_\Xi(\phi)$. Let $l(j) = l_\Xi(\phi, j)$ be the minimum elements $l$ in $\pi_\Xi(\phi, j)$ for each $j \in J_\Xi(\phi)$. In the former case, we put $l(N + 1) = \cdots$
Then \( l(1) = 1 \). The kernel function \( \tilde{K}(\Xi; a, \phi)(V, W) \) is defined by

\[
\tilde{K}(\Xi; a, \phi)(V, W) = \begin{cases} 
0 & \text{if } 0 < t < 1, \\
K(\Xi; a, j)(V, W) & \text{if } l(j) \leq t < l(j + 1) \text{ and } j \in J_{\Xi}(\phi),
\end{cases}
\]

where \( V \in H_\nu(\Xi) \) and \( W = (i, \Upsilon) \in S_n(\Xi) \).

The new modified Poisson-Schrödinger kernel \( \mathbb{P}(\Xi; a, \phi)(V, W) \) is defined by

\[
\mathbb{P}(\Xi; a, \phi)(V, W) = \mathbb{P}(\Xi; a)(V, W) - \tilde{K}(\Xi; a, \phi)(V, W),
\]

where \( V \in H_\nu(\Xi) \) and \( W \in S_n(\Xi) \).

As an application of modified Poisson-Schrödinger kernel \( \mathbb{P}(\Xi; a, \phi)(V, W) \), we have the following.

**Theorem** Let \( g(V) \) be a continuous function on \( S_n(\Xi) \). Then there is a positive continuous function \( \phi_\nu(l) \) of \( l \geq 1 \) depending on \( g \) such that

\[
\mathbb{P}(\Xi; a, g)(V) = \int_{S_n(\Xi)} \mathbb{P}(\Xi; a, \phi_\nu)(V, W) g(W) d\sigma_W,
\]

is a solution of the Dirichlet-Schrödinger problem in \( H_\nu(\Xi) \) with \( g \).

**2 Main lemmas**

**Lemma 1** Let \( \phi(l) \) be a positive continuous function of \( l \geq 1 \) satisfying

\[
(2) \phi(1) = 1.
\]

Then

\[
\mathbf{P}(2)\phi(l) \leq M\phi(l)
\]

for any \( V = (\tau, \Lambda) \in H_\nu(\Xi) \) and any \( W = (i, \Upsilon) \in S_n(\Xi) \) satisfying

\[
\tau > \max \{-1, \Lambda\}.
\]

**Proof.** We can choose two points \( V = (\tau, \Lambda) \in H_\nu(\Xi) \) and \( W = (i, \Upsilon) \in S_n(\Xi) \), satisfying (2.1). Moreover, we also can choose an integer \( j = j(V, W) \in J_{\Xi}(\Upsilon) \) such that

\[
l(j - 1) \leq i < l(j).
\]

Then

\[
\tilde{K}(\Xi; a, \phi)(V, W) = \tilde{K}(\Xi; a, j - 1)(V, W).
\]

Hence we have from (1.2), (2.1), and (2.2)

\[
\mathbf{P}(\Xi; a)(V, W) - \tilde{K}(\Xi; a, \phi)(V, W) \leq M2^{-l(j)} \leq M\phi(l),
\]

which is the conclusion. \( \square \)
Lemma 2 (see [9]) Let $g(V)$ be a continuous function on $S_n(\Xi)$ and $\widehat{V}(V, W)$ be a locally integrable function on $S_n(\Xi)$ for any fixed $V \in H_n(\Xi)$, where $W \in S_n(\Xi)$. Define

$$\widehat{W}(V, W) = P(\Xi; a)(V, W) - \widehat{V}(V, W)$$

for any $V \in H_n(\Xi)$ and any $W \in S_n(\Xi)$.

Suppose that the following two conditions are satisfied:

(I) For any $Q' \in S_n(\Xi)$ and any $\epsilon > 0$, there exists a neighborhood $B(Q')$ of $Q'$ such that

$$\int_{S_n(\Xi; (R, \infty))} |\widehat{W}(V, W)| d\sigma_W < \epsilon$$

for any $V = (\tau, \Lambda) \in H_n(\Xi) \cap B(Q')$, where $R$ is a positive real number.

(II) For any $W' \in S_n(\Xi)$, we have

$$\limsup_{V \to W', V \in H_n(\Xi) \cap S_n(\Xi; (0, R))} \int_{S_n(\Xi; (R, \infty))} |\widehat{V}(V, W)| d\sigma_W = 0$$

for any positive real number $R$.

Then

$$\limsup_{V \to W', V \in H_n(\Xi) \cap S_n(\Xi; (0, R))} \int_{S_n(\Xi; (R, \infty))} \widehat{W}(V, W) u(W) d\sigma_W < \epsilon(W')$$

for any $W' \in S_n(\Xi)$.

3 Proof of Theorem

Take a positive continuous function $\phi(t)$ $(t \geq 1)$ such that

$$\phi(1)V(2) = 1$$

and

$$\int_{\Xi} \phi(t) g(t, \Upsilon) d\sigma_\Upsilon \leq \frac{L}{\tau^s}$$

for $t > 1$, where

$$L = \int_{\Xi} |g(1, \Upsilon)| d\sigma_\Upsilon.$$

For any fixed $V = (\tau, \Lambda) \in H_n(\Xi)$, we can choose a number $R$ satisfying $R > \max\{1, 4r\}$.

Then we see from Lemma 1 that

$$\int_{S_n(\Xi; (R, \infty))} |P(\Xi; a, \phi_{\Xi})(V, W)| d\sigma_W$$

$$\leq M \int_R^\infty \left( \int_{\Xi} |g(1, \Upsilon)| d\sigma_\Upsilon \right) \phi(t) t^{s-2} dt$$

$$\leq ML \int_R^\infty t^{s-2} dt$$

$$< \infty.$$  

(3.1)
Obviously, we have

\[
\int_{S_n(\mathbb{Z},(0,R))} \left| \mathbb{P}(\mathbb{Z};a,\phi_g)(V,W) \right| |g(W)| d\sigma_W < \infty,
\]

which gives

\[
\int_{S_n(\mathbb{Z})} \left| \mathbb{P}(\mathbb{Z};a,\phi_g)(V,W) \right| |g(W)| d\sigma_W < \infty.
\]

To see that \( \mathbb{P}_{\mathbb{Z}}(\phi_g,g)(V) \) is a harmonic function in \( H_\nu(\mathbb{Z}) \), we remark that \( \mathbb{P}_{\mathbb{Z}}(\phi_g,g)(V) \) satisfies the locally mean-valued property by Fubini’s theorem.

Finally we shall show that

\[
\lim_{V \in H_\nu(\mathbb{Z}), V \rightarrow \bar{W}} \mathbb{P}_{\mathbb{Z}}(\phi_g,g)(V) = g(\bar{W})
\]

for any \( \bar{W} = (\iota', \tau') \in \partial H_\nu(\mathbb{Z}) \). Setting

\[
V(V,W) = \tilde{K}(\mathbb{Z};a,\phi_g)(V,W)
\]

in Lemma 2, which is locally integrable on \( S_n(\mathbb{Z}) \) for any fixed \( V \in H_\nu(\mathbb{Z}) \). Then we apply Lemma 2 to \( g(V) \) and \( -g(V) \).

For any \( \epsilon > 0 \) and a positive number \( \delta \), by (3.3) we can choose a number \( R (> \max\{1, 2(\iota' + \delta)\}) \) such that (2.2) holds, where \( \nu \in H_\nu(\mathbb{Z}) \cap B(\bar{W}', \delta) \).

Since

\[
\lim_{\Lambda \rightarrow \Phi} \phi_\iota(\Lambda) = 0 \quad (\iota = 1, 2, 3 \ldots)
\]

as \( V = (\tau, \Lambda) \rightarrow \bar{W}' = (\iota, \nu) \in S_n(\mathbb{Z}) \), we have

\[
V \rightarrow \nu \quad Y(\mathbb{Z};a,\phi_g)(V,W) = 0,
\]

where \( W \in S_n(\mathbb{Z}) \) and \( \bar{W}' \in S_n(\mathbb{Z}) \). Then (2.3) holds.

Thus we complete the proof of the theorem.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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