In this work, we present basic results and applications of Stepanov pseudo-almost periodic functions with measure. Using only the continuity assumption, we prove a new composition result of $\mu$-pseudo-almost periodic functions in Stepanov sense. Moreover, we present different applications to semilinear differential equations and inclusions with weak regular forcing terms in Banach spaces. We prove the existence and uniqueness of $\mu$-pseudo-almost periodic solutions (in the strong sense) to a class of semilinear fractional inclusions and semilinear evolution equations respectively, provided that the nonlinear forcing terms are only Stepanov $\mu$-pseudo-almost periodic in the first variable and not a uniformly strict contraction with respect to the second argument. Our results are obtained using the Meir–Keeler principle and the Banach fixed point principle respectively. Some examples of fractional and nonautonomous partial differential equations illustrating our theoretical results are also presented.

**KEYWORDS**

$\mu$-pseudo-almost periodic solutions, fractional inclusions, Meir and Keeler fixed point argument, nonautonomous reaction-diffusion equations, semilinear evolution equations, Stepanov $\mu$-pseudo-almost periodic functions

**MSC CLASSIFICATION**

34G10; 47D06

1 **INTRODUCTION**

The notion of almost periodicity was introduced by Bohr around 1925 and later generalized by many others in different contexts, namely, almost automorphic functions due to Bochner, Stepanov almost periodic functions, and pseudo-almost periodic functions are other different interesting generalizations. Unlike the classical almost periodic functions, Stepanov almost periodic functions are only locally integrable and not necessarily bounded or continuous. Recently, Blot et al have introduced the concept of $\mu$-pseudo-almost periodic functions defined as perturbation of almost periodic functions by ergodic terms (i.e., a class of bounded continuous functions with mean vanishing at infinity; see Appendix A1), while in the mentioned paper, the ergodicity is defined through positive measures in which the previous concepts of pseudo-almost periodic functions and weighted pseudo-almost periodic functions are just particular cases. Lately, in Baroun et al and Es-sebbar and Ezzinbi introduced a more general concept of $\mu$-pseudo-almost periodic functions in Stepanov sense, saying Stepanov almost periodic functions perturbed by general locally integrable ergodic terms. Moreover, the main interesting properties about the spaces of such functions as completeness and composition results were already established. However, for completeness results, we refer the reader to Zheng and Ding in the weighted pseudo-almost periodic case and to Blot et al for the general $\mu$-pseudo-almost periodic functions in the classical senses.
For \( \mu \)-pseudo-almost periodic functions in Stepanov sense, a completeness result has also been proven in Baroun et al.; see Appendix A1 for more details.

In the literature, we found several works devoted to the existence and uniqueness of \( \mu \)-pseudo-almost periodic solutions to semilinear evolution equations and inclusions; we quote previous studies. In that case, a solution (at least in a mild sense) is usually represented by an integral operator (this holds under a certain decay of the associated linear system). More specifically, for given an integral operator solution, namely,

\[
(Su)(t) = \int_{\mathbb{R}} G(t,s)f(s,u(s))ds, \quad t \in \mathbb{R},
\]

where the input parameters, \((G(t,s))_{t \geq 0}\) which represents the Green function (resp. the resolvent operator), is assumed to be bi-almost periodic, and \(f\) is the nonlinearity. In the case where \(f\) is uniformly Lipschitzian with respect to the second variable and \(\mu\)-pseudo-almost periodic (resp. Stepanov \(\mu\)-pseudo-almost periodic) in \(t\), it was shown (see previous studies) that the output (the corresponding solution) \(u\) is \(\mu\)-pseudo-almost periodic. Extensively, the existence and uniqueness results established so far are obtained in view of some composition results and the Banach strict contraction principle. However, in the instance where the input \(f\) belongs to a broader class of Stepanov \(\mu\)-pseudo-almost periodic functions and not necessarily uniformly Lipschitzian, to our knowledge, there are no consistent results in the literature devoted to the existence (and/or the uniqueness) of \(\mu\)-pseudo-almost periodic solutions, since the composition results for Stepanov \(\mu\)-pseudo-almost periodic functions established in the literature up to here require the uniform Lipschitz condition.

Among the main objectives of this work, on the one hand, is to extend the results of the literature and to prove a new composition result of Stepanov \(\mu\)-pseudo-almost periodic functions using only the continuity condition (more specifically, we do not use the uniform Lipschitz assumption). That is, for a given function \(f : \mathbb{R} \times X \to Y\) which is Stepanov \(\mu\)-pseudo-almost periodic with respect to \(t\) and continuous with respect to \(x\) and for given a function \(u : \mathbb{R} \to X\) which is \(\mu\)-pseudo-almost periodic for any Banach spaces \(X\) and \(Y\), \(f(\cdot, u(\cdot))\) is Stepanov \(\mu\)-pseudo-almost periodic. On the other hand, we take advantage of the obtained composition result, and we prove the existence and uniqueness of solutions to a class of semilinear fractional inclusions and a class of semilinear nonautonomous evolution equations in Banach spaces, respectively, without assuming the uniform Lipschitz assumption on the nonlinear forcing terms.

First, using Meir and Keeler fixed point argument, we prove the existence and uniqueness of \(\mu\)-pseudo-almost periodic solutions to the following inclusion:

\[
D^\gamma_{t,+} u(t) \in Au(t) + f(t, u(t)), \quad t \in \mathbb{R},
\]

where \(D^\gamma_{t,+}\) denotes the Riemann–Liouville fractional derivative of order \(\gamma \in (0, 1)\) and \(f : \mathbb{R} \times X \to X\) is Stepanov \(\mu\)-pseudo-almost periodic in \(t \in \mathbb{R}\) and satisfies certain properties with respect to \(x \in X\). Here, the operator solution (1.1) has the following form:

\[
(Su)(t) = \int_{-\infty}^{t} R_\gamma(t-s)f(s, u(s))ds, \quad t \in \mathbb{R},
\]

where \((R_\gamma(t))_{t \geq 0}\) is the associated operator resolvent. In contrast of the Banach contraction principle, the main result obtained here does not require that \(S\) be necessarily a uniformly strict contraction mapping to obtain the existence and uniqueness of \(\mu\)-pseudo-almost periodic solutions; see Theorem 3.6. This is the advantage of the application of Meir and Keeler fixed point theorem in our contribution. To our knowledge, the obtained result is the first application and exploitation of the advantage of the Meir and Keeler fixed point principle in the context of functional differential equations. Indeed, our version of Meir and Keeler principle consists of a new functional fixed point principle; see Section 3.1. As application, we study a class of semilinear fractional Poisson heat equations in \(L^2\)-setting; see Section 3.2.

Second, by the Banach contraction principle, we prove the existence and uniqueness of \(\mu\)-pseudo-almost periodic solutions (see Theorem 3.16 for \(1 < p < \infty\) and Theorem 3.17 for \(p = 1\)) to the following evolution equation:

\[
x'(t) = A(t)x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R},
\]

where \((A(t), D(A(t)))\), \(t \in \mathbb{R}\) is a family of closed linear operators on a Banach space \(X\) that generates an evolution family \((U(t,s))_{t \geq s}\) which have an exponential dichotomy on \(\mathbb{R}\), the nonlinear term \(f : \mathbb{R} \times X \to X\) is assumed to be Stepanov...
\( \mu \)-pseudo-almost periodic (of order \( 1 \leq p < \infty \)) in \( t \in \mathbb{R} \) and only Lipschitzian in bounded sets with respect to \( x \in X \); here, we refer to Chávez et al.\(^{26} \) for similar results in the case of weighted pseudo-almost periodic functions. Besides, we provide an application to a class of reaction-diffusion equations describing the behavior of bounded solutions of a one-species intraspecific competition Lotka–Volterra model. Notice that in general models of competition-interaction population dynamics, nonlinear terms are generally not of uniformly Lipschitz type but rather only Lipschitzian in bounded sets.

The work is organized as follows. Section 2 is devoted to a new composition result of Stepanov \( \mu \)-pseudo-almost periodic functions. In Section 3, we prove the existence and uniqueness of \( \mu \)-pseudo-almost periodic solutions to the fractional evolution inclusion (1.2) (see Section 3.1) and the nonautonomous evolution Equation (1.4) (see Section 3.2), respectively. Finally, all the preliminary materials about the concepts of \( \mu \)-pseudo-almost periodic functions in the classical and in Stepanov senses, respectively, are gathered in Appendix A1.

### 1.1 | Notations

Throughout this work, \((X, \| \cdot \|)\) and \((Y, \| \cdot \|_Y)\) are two complex Banach spaces; \((\mathcal{L}(X), \| \cdot \|_{\mathcal{L}(X)})\) stands for the Banach algebra of bounded linear operators in \( X \). \( BC(\mathbb{R}, X) \) equipped with the sup-norm \( \| \cdot \|_{\infty} \) is the Banach space of bounded continuous functions \( f : \mathbb{R} \to X \). Moreover, for \( 1 \leq p < \infty \), \( q \) denotes its conjugate exponent defined by \( 1/p + 1/q = 1 \), if \( p \neq 1 \), and \( q = \infty \), if \( p = 1 \). Let \( I \subseteq \mathbb{R} \) be any interval, and by \( L^p_{\text{loc}}(I, X) \) (resp. \( L^p(I, X, \| \cdot \|_p) \)), we designate the space (resp. the Banach space) of all equivalence classes of measurable functions \( f : I \to X \) such that \( \| f(\cdot) \|_{L^p} \) is locally integrable (resp. integrable). By \( \rho(A) \), we denote the resolvent set of \((A, D(A))\), defined by \( \rho(A) := \{ \lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ exists in } \mathcal{L}(X) \} \subseteq \mathbb{C} \); the spectrum \( \sigma(A) \) of \((A, D(A))\) is defined by \( \sigma(A) := \mathbb{C} \setminus \rho(A) \). For \( \lambda \in \rho(A) \), the resolvent operator \( R(\lambda, A) \) is defined by \( R(\lambda, A) := (\lambda - A)^{-1} \). By \( B(\mathbb{R}) \), we denote the collection of all Lebesgue measurable subsets of \( \mathbb{R} \), and by \( \mathcal{M} \), we denote the set of all positive measures \( \mu \) on \( B(\mathbb{R}) \) satisfying \( \mu([a, b]) = +\infty \) and \( \mu([a, b)) < +\infty \) for all \( a, b \in \mathbb{R} \) with \( a \leq b \).

### 2 | NEW COMPOSITION RESULTS FOR STEpanov \( \mu \)-PSEUDO-ALMOST PERIODIC FUNCTIONS

In this section, we establish new composition results of \( \mu \)-pseudo-almost periodic functions in Stepanov sense of order \( 1 \leq p < \infty \).

**Theorem 2.1** (Khalil 27). Let \( 1 \leq p < +\infty \) and \( f \in APS^p U(\mathbb{R} \times X, Y) \). Assume that \( x \in AP(\mathbb{R}, X) \). Then, \( f(\cdot, x(\cdot)) \in APS^p(\mathbb{R}, Y) \).

**Remark 2.2.** In Kostić,17 theorem 2.7.2 we have used condition that \( x \in APS^p(\mathbb{R}, X) \), and there exists a set \( E \subseteq \mathbb{R} \) with \( m(E) = 0 \) such that the set \( K = \{x(t) : t \in \mathbb{R} \setminus E\} \) is relatively compact in \( Y \) (this condition is clearly satisfied if \( x \in AP(\mathbb{R}, X) \); here, \( m(\cdot) \) denotes the Lebesgue measure). In this case, we have \( f(\cdot, x(\cdot)) \in APS^p(\mathbb{R}, Y) \) provided that there exists a finite Lipschitz constant \( L \geq 1 \) such that

\[
\| f(t, x) - f(t, y) \| \leq L \| x - y \|, \quad t \in \mathbb{R}, x, y \in X. \tag{2.1}
\]

As Theorem 2.1 shows, we do not need Lipschitz-type assumption (2.1) if we assume that \( x \in AP(\mathbb{R}, X) \).

In order to prove our main composition theorem of Stepanov \( \mu \)-pseudo-almost periodic functions, we need to prove the following substantial preliminary results.

**Theorem 2.3.** Let \( 1 \leq p < +\infty \) and \( \mu \in \mathcal{M} \). If \( f \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu) \) and \( x \in BC(\mathbb{R}, X) \) such that \( K = \{x(t) : t \in \mathbb{R}\} \) is compact in \( X \) (i.e., with relatively compact range), then \( f(\cdot, x(\cdot)) \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \).

**Proof.** Let \( f \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu) \) and \( K = \{x(t) : t \in \mathbb{R}\} \subseteq X \) a compact subset. Then, for every \( \varepsilon > 0 \), there exists \( \delta_{\varepsilon, x} > 0 \) such that (A5) holds. Since \( K \) is compact, there exists a finite subset \( \{x_1, \ldots, x_n\} \subseteq K \) (\( n \in \mathbb{N} \)) such that \( K \subseteq \bigcup_{i=1}^n B(x_i, \delta_{\varepsilon, x}) \). Therefore, for every \( t \in \mathbb{R} \), there exists \( i(t) \in \{1, \ldots, n\} \) such that \( \|x(t) - x_{i(t)}\| \leq \delta \). Furthermore,
\[
\left( \int_t^{t+1} \|f(s,x(s))\|_Y^p \, ds \right)^{1/p} \leq \left( \int_t^{t+1} \|f(s,x(s)) - f(s,x(t))\|_Y^p \, ds \right)^{1/p} + \left( \int_t^{t+1} \|f(s,x(t))\|_Y^p \, ds \right)^{1/p} \leq \epsilon + \sum_{i=1}^n \left( \int_t^{t+1} \|f(s,x_i)\|_Y^p \, ds \right)^{1/p}, \quad t \in \mathbb{R}.
\]

(2.2)

Since \( f(., x_i) \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \) for \( i = 1, \ldots, n \), we have

\[
\frac{1}{\mu([-r, r])} \int_{-r}^r \left( \int_t^{t+1} \|f(s,x(s))\|_Y^p \, ds \right)^{1/p} \, d\mu(t) \leq \epsilon + \frac{1}{\mu([-r, r])} \sum_{i=1}^n \int_{-r}^r \left( \int_t^{t+1} \|f(s,x_i)\|_Y^p \, ds \right)^{1/p} \, d\mu(t),
\]

for \( r > 0 \) large enough. Consequently,

\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \left( \int_t^{t+1} \|f(s,x(s))\|_Y^p \, ds \right)^{1/p} \, d\mu(t) \leq \epsilon.
\]

(2.3)

Since \( \epsilon > 0 \) was arbitrary, (2.3) yields

\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \left( \int_t^{t+1} \|f(s,x(s))\|_Y^p \, ds \right)^{1/p} \, d\mu(t) = 0.
\]

Corollary 2.4. Let \( \mu \in M \). Assume that \( x \in \mathcal{A}P(\mathbb{R}, X) \) and \( f \in \mathcal{E}^pU(\mathbb{R} \times X, Y, \mu) \). Then, \( f(., x(\cdot)) \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \).

Proof. From \( x \in \mathcal{A}P(\mathbb{R}, X) \), we deduce that \( x \in BS^p(\mathbb{R}, X) \) and \( K = \{x(t) : t \in \mathbb{R}\} \) is a compact subset of \( X \). Hence, conditions and hypotheses of Theorem 2.3 are satisfied.

Lemma 2.5 (Blot et al\(^5\)). Let \( \mu \in M \) and \( f \in BC(\mathbb{R}, X) \). Then, \( f \in \mathcal{E}(\mathbb{R}, X, \mu) \) if and only if for all \( \epsilon > 0 \)

\[
\lim_{r \to +\infty} \frac{\mu(M_{\epsilon,r}(f))}{\mu([-r, r])} = 0,
\]

(2.4)

where \( M_{\epsilon,r}(f) := \{t \in [-r, r] : \|f(t)\| \geq \epsilon\} \).

The proof of our result related to the composition of \( S^p \)-\( \mu \)-pseudo-almost periodic functions is based on the following lemma due to Schwartz.\(^{28, p109}\)

Lemma 2.6. Let \( \Phi \in C(X,Y) \). Then, for each compact set \( K \subseteq X \) and for each \( \epsilon > 0 \), there exists \( \delta_{K,\epsilon} > 0 \) such that for any \( x_1, x_2 \in X \), we have

\[
x_1 \in K \text{ and } \|x_1 - x_2\| \leq \delta_{K,\epsilon} \Rightarrow \|\Phi(x_1) - \Phi(x_2)\|_Y \leq \epsilon.
\]

Theorem 2.7. Let \( 1 \leq p < +\infty \) and \( \mu \in M \). Assume the following:

(i) \( f : \mathbb{R} \times X \to Y \) be a function such that \( f = \tilde{f} + \varphi \in \mathcal{AP}S^pU(\mathbb{R} \times X, Y, \mu) \) with \( \tilde{f} \in \mathcal{AP}S^pU(\mathbb{R} \times X, Y) \) and \( \varphi \in \mathcal{E}^pU(\mathbb{R} \times X, Y, \mu) \);

(ii) \( x = x_1 + x_2 \in \mathcal{AP}(\mathbb{R}, X, \mu) \), where \( x_1 \in \mathcal{A}P(\mathbb{R}, X) \) and \( x_2 \in \mathcal{E}(\mathbb{R}, X, \mu) \); and

(iii) for every bounded subset \( B \subseteq X \), we have \( \sup_{x \in B} \|f(., x)\|_{S^p} < \infty \).

Then, \( f(., x(\cdot)) \in \mathcal{P}AP^p(\mathbb{R}, Y, \mu) \).
Proof. We have the following decomposition:

\[
  f(t, u(t)) = \tilde{f}(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + \varphi(t, x_1(t))
\]

\[= F(t) + F(t) + \Psi(t), \ t \in \mathbb{R}. \tag{2.5} \]

Using Theorem 2.1, it follows that \( \tilde{F} \in \text{APSP}^p(\mathbb{R}, Y) \) and by using Corollary 2.4, we deduce that \( \Psi \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \). Now, it suffices to prove that \( F \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \). In view of Lemma 2.5, we have

\[
\lim_{r \to +\infty} \frac{\mu(M_{r, r}(\{x_2\}))}{\mu([-r, r])} = 0, \ \varepsilon > 0. \tag{2.6} \]

Let \( \varepsilon > 0 \). Then, for \( r > 0 \) large enough, we have

\[
\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left( \int_{t}^{t+1} \|F(s)\|^p_Y ds \right)^{\frac{1}{p}} d\mu(t) \leq \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \int_{t}^{t+1} \|F(s)\|^p_Y ds \right)^{\frac{1}{p}} d\mu(t) + \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \int_{t}^{t+1} \|F(s) - f(s, x_1(s))\|^p_Y ds \right)^{\frac{1}{p}} d\mu(t) \leq \frac{\mu(M_{r, r}(\{x_2\}))}{\mu([-r, r])} + \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \int_{t}^{t+1} \|f(s, x(s)) - f(s, x_1(s))\|^p_Y ds \right)^{\frac{1}{p}} d\mu(t). \tag{2.7} \]

Let \( K := \{x_1(t) : t \in \mathbb{R}\} \). From \( x_1 \in \text{AP}(\mathbb{R}, X) \), we assert that \( K \) is a compact subset of \( X \). Define

\[
\Phi : X \to \text{APSP}^p(\mathbb{R}, Y) \text{ through } x \mapsto f(\cdot, x).
\]

Since \( f \in \text{PAPSP}^p(U(\mathbb{R} \times X, Y, \mu)) \), using Proposition A.17, we may deduce that the restriction of \( \Phi \) on any compact \( K \) of \( X \) is uniformly continuous, which is equivalent to saying that the function \( \Phi \) is continuous on \( X \). If we apply Lemma 2.6 on \( \Phi \), we get that, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for every \( t \in \mathbb{R} \) and \( \xi_1, \xi_2 \in X \), we have

\[
\xi_1 \in K \text{ and } \|\xi_1 - \xi_2\| \leq \delta \Rightarrow \left( \int_{t}^{t+1} \|f(s, \xi_1) - f(s, \xi_2)\|^p_Y ds \right)^{\frac{1}{p}} \leq \varepsilon.
\]

Since \( x(t) = x_1(t) + x_2(t) \) and \( x_1(t) \in K \), we have

\[
t \in \mathbb{R} \text{ and } \|x_2(s)\| \leq \delta \text{ for } s \in [t, t+1] \Rightarrow \left( \int_{t}^{t+1} \|f(s, x(s)) - f(s, x_1(s))\|^p_Y ds \right)^{\frac{1}{p}} \leq \varepsilon.
\]

Therefore, by the fact that \( x_2 \in \mathcal{E}(\mathbb{R}, X, \mu) \), we have

\[
\limsup_{r \to +\infty} \frac{\mu(M_{r, r}(\{x_2\}))}{\mu([-r, r])} = 0.
\]
Using (2.7), we obtain

$$\limsup_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left( \int_{t}^{t+1} \|F(s)\|_{Y}^{p} ds \right)^{\frac{1}{p}} d\mu(t) \leq \varepsilon$$

to all \( \varepsilon > 0 \).

Consequently,

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left( \int_{t}^{t+1} \|F(s)\|_{Y}^{p} ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$ 

\( \square \)

**Remark 2.8.** Notice that condition (iii) is only needed to prove that \( f(\cdot, u(\cdot)) \in B^p(\mathbb{R}, Y) \) (for \( 1 \leq p < \infty \)). Further, condition (iii) is satisfied for a wide large class of functions \( f: \mathbb{R} \times X \to Y \); for more details, see Remark 3.2 and Lemma 3.15.

Keeping in mind Theorem 2.7 and Remark 2.8, we obtain the following corollary.

**Corollary 2.9.** Let \( 1 \leq p < +\infty \) and \( \mu \in \mathcal{M} \). Assume that \( f: \mathbb{R} \times X \to Y \) satisfies the following:

(i) \( f = f + \varphi \in \text{PAPS}^p U(\mathbb{R} \times X, Y, \mu) \) with \( f \in \text{APS}^p U(\mathbb{R} \times X, Y) \) and \( \varphi \in \text{E}^p U(\mathbb{R} \times X, Y, \mu) \);

(ii) \( x = x_1 + x_2 \in \text{PAP}(\mathbb{R}, X, \mu) \), where \( x_1 \in \text{AP}(\mathbb{R}, X) \) and \( x_2 \in \text{E}(\mathbb{R}, X, \mu) \); and

(iii) there exists a nonnegative scalar function \( L(\cdot) \in B^p(\mathbb{R}) \) such that

$$\|f(t, x) - f(t, y)\|_{Y} \leq L(t)\|x - y\|, \quad x, y \in X, \quad t \in \mathbb{R}.$$ 

Then, \( f(\cdot, x(\cdot)) \in \text{PAPS}^p(\mathbb{R}, Y, \mu) \).

### 3 Applications to the Abstract Volterra Integrodifferential Inclusions and Nonautonomous Semilinear Evolution Equations

In this section, we shall apply our theoretical results proved so far in the qualitative analysis of bounded solutions for various kinds of abstract semilinear evolution equations in Banach spaces with applications to reaction-diffusion equations.

#### 3.1 Pseudo-almost periodic solutions of the fractional semilinear inclusions

Consider the following semilinear evolution inclusion:

$$D^\gamma_{t, t+} u(t) \in A u(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where \( D^\gamma_{t, t+} \) denotes the Riemann–Liouville fractional derivative of order \( \gamma \in (0, 1) \) and \( f: \mathbb{R} \times X \to X \) is Stepanov \( \mu \)-pseudo-almost periodic in \( t \in \mathbb{R} \) and satisfies certain properties with respect to \( x \in X \). We assume that the multivalued linear operator \( A \) satisfies the following condition:

(P1) There exist \( c, M > 0 \) and \( \beta \in (0, 1] \) such that

$$\Sigma := \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \geq -c(\text{Im}(\lambda)) + 1 \} \subseteq \rho(A).$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{(1 + |\lambda|)^\beta}$$

for all \( \lambda \in \Sigma \).

It is clear that any sectorial operator in the sense of Engel and Nagel\(^{20}\), definition 4.1 satisfies (P1); see Favini and Yagi\(^{30}\), remark 1 of proposition 3.6 for more details about this concept, and for more further results, see also the monograph.\(^{17}\) Let \( (R(t))_{t \geq 0} \) be the operator family considered in Kostic.\(^{17}\) Then, we know that

$$\|R(t)\| = O(t^{\gamma-1}), \quad t \in (0, 1] \quad \text{and} \quad \|R(t)\| = O(t^{\gamma-1}), \quad t \geq 1.$$  \hspace{1cm} (3.1)
It is said that a continuous function \( u : \mathbb{R} \to X \) is a mild solution of (1.2) if and only if

\[
u(t) = \int_{-\infty}^{t} R_s (t-s) f (s, u(s)) \, ds, \quad t \in \mathbb{R}.
\] (3.2)

In the subsequent application of Theorem 2.7, we will use the following result of Meir and Keeler, who employed the so-called condition of weakly uniformly strict contraction:

**Lemma 3.1.** Suppose that \((X, d)\) is a complete metric space and \( T : X \to X \) satisfies that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for each \( x, y \in X \), we have

\[
\varepsilon \leq d(x, y) \leq \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon.
\]

Then, the mapping \( T \) has a unique fixed point \( \xi \), and moreover, \( \lim_{n \to +\infty} T^n x = \xi \) for any \( x \in X \).

Therefore, we provide the following hypotheses on \( f \):

(P2) There exists \( L \geq 0 \) such that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) satisfying

\[
\varepsilon \leq \|x - y\| < \varepsilon + \delta \implies \sup_{t \in \mathbb{R}} \left( \int_{t}^{t+1} \| f(s, x) - f(s, y) \|^p ds \right)^{\frac{1}{p}} < L \varepsilon \text{ for all } x, y \in X.
\]

(P3) For every bounded subset \( B \subseteq X \), we have

\[
\sup_{x \in B} \sup_{t \in \mathbb{R}} \left( \int_{t}^{t+1} \| f(s, x) \|^p ds \right)^{\frac{1}{p}} < \infty.
\]

**Remark 3.2.** Notice that any function \( f : \mathbb{R} \times X \to X \) with \( f(\cdot, x_0) \in BS^p(\mathbb{R}, X) \) for some \( x_0 \in X \), and it is Lipschitzian with respect to the second argument, that is, there exists a nonnegative scalar function \( L(\cdot) \in BS^p(\mathbb{R}) \) (for \( 1 \leq p < \infty \)) such that

\[
\| f(t, x) - f(t, y) \| \leq L(t) \| x - y \|, \quad x, y \in X, \quad t \in \mathbb{R},
\]

in particular satisfies the hypothesis (P3).

In the next, we show that a function satisfying (P2) is not necessarily a strict contraction, but using Lemma 3.1, it has a unique fixed point. We introduce two examples: the first is in the scalar case, and the second one is in the Banach-valued setting.

**Example 3.3.**

1. Set the Banach space \( X = (\mathbb{R}, \| \cdot \|) \) and define the scalar-valued function \( g : X \to X \) given by

\[
g(x) = \frac{|x|}{1 + |x|}.
\]

Let \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( \varepsilon \leq |x - y| < \delta + \varepsilon \) for all \( x, y \in \mathbb{R} \). Then,

\[
|g(x) - g(y)| = \left| \frac{|x|}{1 + |x|} - \frac{|y|}{1 + |y|} \right| \\
\leq \frac{|x| - |y|}{1 + |x| + |y| + |x||y|} \\
\leq \frac{1}{1 + |x| + |y|}.
\] (3.3)
Moreover, by assumptions and (3.3), we have
\[ |g(x) - g(y)| < \frac{(\varepsilon + \delta)}{1 + \varepsilon} = \frac{(\varepsilon + \varepsilon^2)}{1 + \varepsilon} = \varepsilon \] (by choosing \( \delta = \varepsilon^2 \)).

Thus, (P2) holds with \( L = 1 \) and \( f(s, x) = g(x) \) for all \( s \in \mathbb{R} \), and from (3.3), we obtain that
\[ |g(x) - g(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}. \]

If we assume that \( g \) is a strict contraction mapping, then there exists \( 0 < \hat{K} < 1 \) such that
\[ |g(x) - g(y)| \leq \hat{K}|x - y| \text{ for all } x, y \in \mathbb{R}. \]

Let \( 0 \neq x \in X \) and \( y = 0 \). Then, \( |g(x) - g(0)| = g(x) \leq \hat{K}|x| \) which implies that
\[ \frac{1}{|x|} \frac{|x|}{|x| + 1} = \frac{1}{|x| + 1} \leq \hat{K}. \]

Therefore, by letting \( x \to 0 \), we obtain that \( 1 \leq \hat{K} \) which yields a contradiction. Hence, \( g \) is not a strict contraction.

2. Let the Banach space \( X := (L^2(\Omega), \| \cdot \|) \) (equipped with its usual norm), where \( \Omega \subset \mathbb{R}^n \) is any bounded open set. Define the function \( f : \mathbb{R} \times X \to X \) by
\[ f(t, \varphi)(x) = K(t) \frac{\| \varphi \|}{1 + \| \varphi \|} Q(x) + H(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega, \]
where \( K : \mathbb{R} \to (0, \infty), \ Q \in X, \ Q \geq 0 \) with \( \| K \|_{\text{ess}} \| Q \| \geq 1 \) and \( H : \mathbb{R} \times \Omega \to [0, \infty) \). The function \( f \) satisfies (P2), but it is not a strict contraction. In fact, let \( \varphi_1, \varphi_2 \in X \) and \( \varepsilon, \delta > 0 \) such that \( \varepsilon \leq \| \varphi_1 - \varphi_2 \| < \varepsilon + \delta \). Then, a straightforward calculation yields:
\[ |f(t, \varphi_1)(x) - f(t, \varphi_2)(x)| = K(t) \left( \frac{\| \varphi_1 \|}{1 + \| \varphi_1 \|} - \frac{\| \varphi_2 \|}{1 + \| \varphi_2 \|} \right) Q(x) \]
\[ \leq K(t) \frac{\| \varphi_1 \| - \| \varphi_2 \|}{1 + \| \varphi_1 \| + \| \varphi_2 \| + \| \varphi_1 \| \| \varphi_2 \|} Q(x) \]
\[ < K(t) \frac{\varepsilon + \varepsilon}{1 + \varepsilon} Q(x) \]
\[ := K(t) \frac{\varepsilon^2 + \varepsilon}{1 + \varepsilon} Q(x) \]
\[ = K(t) Q(x) \varepsilon, \quad t \in \mathbb{R}, \quad x \in \Omega. \]

Using (3.4), we have
\[ \| f(t, \varphi_1) - f(t, \varphi_2) \|^2 = \int_\Omega |f(t, \varphi_1)(x) - f(t, \varphi_2)(x)|^2 dx \]
\[ < K(t)^2 \int_\Omega Q(x)^2 dx \varepsilon^2 \]
\[ = K(t)^2 \| Q \|^2 \varepsilon^2, \quad t \in \mathbb{R}. \]

Finally,
\[ \left( \int_t^{t+1} \| f(s, \varphi_1) - f(s, \varphi_2) \|^p ds \right)^{\frac{1}{p}} < \| K \|_{\text{ess}} \| Q \| \varepsilon \text{ for all } t \in \mathbb{R}. \]

This proves the result with \( L : = \| K \|_{\text{ess}} \| Q \|. \) To show that \( f \) is not necessarily a strict contraction, we assume the converse, that is, there exists \( 0 < \tilde{K} < 1 \) such that for all \( \varphi_1, \varphi_2 \in X \), we have
\[ \| f(t, \varphi_1) - f(t, \varphi_2) \| \leq \tilde{K} \| \varphi_1 - \varphi_2 \|, \quad t \in \mathbb{R}. \]
Proposition 3.5. Let $\mu \in M$ satisfy $(M)$. Suppose that $f(\cdot, u(\cdot)) \in \text{PAPS}^p(\mathbb{R}, X, \mu)$. Then, the mapping given by

$$(F_0 u)(t) := \int_{-\infty}^t R_r(t-s) f(s, u(s)) \, ds, \quad t \in \mathbb{R}$$

maps $\text{PAP}(\mathbb{R}, X, \mu)$ into $\text{PAP}(\mathbb{R}, X, \mu)$. 

Proof. For any $u \in \text{PAP}(\mathbb{R}, X, \mu)$, we set

$$(F_0 u)(t) := \int_{-\infty}^t R_r(t-s) f(s, u(s)) \, ds, \quad t \in \mathbb{R}.$$ 

Since $s \mapsto f(s, u(s)) \in \text{BS}^p(\mathbb{R}, X)$, we obtain that

$$
\|F_0 u(t)\| \leq \int_{-\infty}^t \|R_r(t-s)\| \cdot \|f(s, u(s))\| \, ds = \int_0^\infty \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \, ds \\
= \int_0^1 \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \, ds + \int_1^\infty \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \, ds \\
\leq \left( \int_0^1 q^{(\gamma-1)} \left( \int_0^1 \|f(t-s, u(t-s))\|^p \, ds \right)^{\frac{1}{p}} + \sum_{k \geq 1} k^{\gamma-1} \left( \int_k^{k+1} \|f(t-s, u(t-s))\|^p \, ds \right)^{\frac{1}{p}} \right) \\
= (1 - q(\gamma - 1)) \left( \int_0^1 \|f(t-s, u(t-s))\|^p \, ds \right)^{\frac{1}{p}} + S_r \left( \int_1^{k+1} \|f(t-s, u(t-s))\|^p \, ds \right)^{\frac{1}{p}} \\
\leq \left( 1 - q(\gamma - 1) + S_r \right) \|f(\cdot, u(\cdot))\|_{\text{BS}^p}, \quad t \in \mathbb{R}.
$$
Thus, $F_0$ is well defined and yields a continuous bounded mapping. Furthermore, from the fact that $s \mapsto f(s, u(s)) \in \text{PAPS}^p(\mathbb{R}, X, \mu)$, we obtain by definition $f(s, u(s)) = \tilde{f}(s, u(s)) + \psi(s, u(s))$, where $s \mapsto \tilde{f}(s, u(s)) \in \text{APS}^p(\mathbb{R}, X)$ and $s \mapsto \psi(s, u(s)) \in \mathcal{E}^p(\mathbb{R}, X, \mu)$. Then, we obtain

$$(F_0 u)(t) := \int_{-\infty}^t R_r(t-s)f(s, u(s))\,ds + \int_{-\infty}^t R_r(t-s)\psi(s, u(s))\,ds, \quad t \in \mathbb{R}.$$ 

Let $\varepsilon > 0$, since $s \mapsto \tilde{f}(s, u(s)) \in \text{APSP}^p(\mathbb{R}, X)$, there exists $l_\epsilon > 0$ such that each interval of length $l_\epsilon$ contains an element $r$ such that

$$\left(\int_{t}^{t+1} \|\tilde{f}(s + r, u(s + r)) - \tilde{f}(s, u(s))\|^p\,ds\right)^{\frac{1}{p}} < \varepsilon / ((1 - q^((r - 1) \frac{1}{r})) + S_r)$$

uniformly in $t \in \mathbb{R}$. Hence, by Hölder inequality, we have

$$\|F_0 u(t + \tau) - F_0 u(t)\| \leq \int_{0}^{\infty} \|R_r(s)\| \cdot \|\tilde{f}(t + \tau - s, u(t + \tau - s)) - \tilde{f}(t - s, u(t - s))\| \,ds$$

$$= \int_{0}^{1} \|R_r(s)\| \cdot \|\tilde{f}(t + \tau - s, u(t + \tau - s)) - \tilde{f}(t - s, u(t - s))\| \,ds$$

$$+ \int_{1}^{\infty} \|R_r(s)\| \cdot \|\tilde{f}(t + \tau - s, u(t + \tau - s)) - \tilde{f}(t - s, u(t - s))\| \,ds$$

$$\leq \left(\int_{0}^{1} s^{q(r-1)} \left(\int_{0}^{1} \|\tilde{f}(t + \tau - s, u(t + \tau - s)) - \tilde{f}(t - s, u(t - s))\|^p \,ds\right)^{\frac{1}{p}} \right)^{\frac{1}{q}}$$

$$+ \sum_{k \geq 1} k^{-r-1} \left(\int_{k}^{k+1} \|\tilde{f}(t + \tau - s, u(t + \tau - s)) - \tilde{f}(t - s, u(t - s))\|^p \,ds\right)^{\frac{1}{p}}$$

$$= (1 - q(r - 1)) \left(\int_{0}^{1} \|f(t + \tau - s, u(t + \tau - s)) - f(t - s, u(t - s))\|^p \,ds\right)^{\frac{1}{p}}$$

$$+ S_r \left(\int_{k}^{k+1} \|f(t + \tau - s, u(t + \tau - s)) - f(t - s, u(t - s))\|^p \,ds\right)^{\frac{1}{p}}$$

$$\leq \varepsilon, \quad t \in \mathbb{R}.$$ 

Therefore, it suffices to prove that $t \mapsto \int_{-\infty}^{t} R_r(t-s)\psi(s, u(s))\,ds \in \mathcal{E}(\mathbb{R}, X, \mu)$. Indeed, let $r > 0$. Then, by the Hölder inequality, we obtain

$$\frac{1}{\mu([-r, r])} \int_{-r}^{r} \|F_0 u(t)\| \,d\mu(t)$$

$$\leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{-\infty}^{1} \|R_r(t-s)\| \cdot \|f(s, u(s))\| \,ds \,d\mu(t)$$

$$= \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{0}^{1} \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \,ds \,d\mu(t)$$

$$= \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{0}^{1} \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \,ds \,d\mu(t)$$

$$+ \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{1}^{\infty} \|R_r(s)\| \cdot \|f(t-s, u(t-s))\| \,ds \,d\mu(t)$$

$$\leq (1 - q(r - 1)) \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_{0}^{1} \|f(t-s, u(t-s))\|^p \,ds\right)^{\frac{1}{p}} \,d\mu(t)$$

$$+ \frac{S_r}{\mu([-r, r])} \int_{-r}^{r} \left(\int_{k}^{k+1} \|f(t-s, u(t-s))\|^p \,ds\right)^{\frac{1}{p}} \,d\mu(t)$$

$$\to 0 \quad \text{as} \quad r \to +\infty.$$ 

We recall from Proposition A.11 (i) that the set $\mathcal{E}^p(\mathbb{R}, X, \mu)$ is translation invariant. Hence, the result follows immediately.
Theorem 3.6. Let $1 < p < +\infty$ and $\mu \in \mathcal{M}$ satisfy (M). Assume that $f \in \text{PAP}^p U(\mathbb{R} \times X, X, \mu)$ such that (P2) and (P3) hold with $L^q_{\mu} \leq 1$ where

$$S^q_{\mu} := \sum_{k \geq 1} k^{-\gamma - 1} + (1 - q(\gamma - 1))^\frac{1}{\gamma}.$$

Then, the inclusion (1.2) has a unique $\mu$-pseudo-almost periodic mild solution given by the integral representation (3.2).

Proof. Let $u \in \text{PAP}(\mathbb{R}, X, \mu)$. From (P3), we have that the function $s \mapsto f(s, u(s))$ belongs to the space $\text{PAP}^p(\mathbb{R}, X, \mu)$ in view of Theorem 2.1. Then, by Proposition 3.5, $F_0$ maps $\text{PAP}(\mathbb{R}, X, \mu)$ into itself. Therefore, it suffices to prove that $F_0$ has a unique fixed point in $\text{PAP}(\mathbb{R}, X, \mu)$ using Lemma 3.1. Let $\epsilon > 0$ and $\delta > 0$ be determined from (P2). For all $u, v \in \text{PAP}(\mathbb{R}, X, \mu)$, $\epsilon \leq \|u(t) - v(t)\| < \epsilon + \delta$ for all $t \in \mathbb{R}$. Then, the hypothesis (P2) yields:

$$\|F_0 u(t) - F_0 v(t)\| \leq \int_{-\infty}^{t} \|R_{\gamma}(t-s)\| \cdot \|f(s, u(s)) - f(s, v(s))\| \, ds$$

$$= \int_{0}^{1} \|R_{\gamma}(s)\| \cdot \|f(t-s, u(t-s)) - f(t-s, v(t-s))\| \, ds$$

$$= \int_{0}^{1} \|R_{\gamma}(s)\| \cdot \|f(t-s, u(t-s)) - f(t-s, v(t-s))\| \, ds$$

$$+ \int_{1}^{\infty} \|R_{\gamma}(s)\| \cdot \|f(t-s, u(t-s)) - f(t-s, v(t-s))\| \, ds$$

$$\leq \left( \int_{0}^{1} s^{\gamma(1-\gamma)} \left( \int_{0}^{1} \|f(t-s, u(t-s)) - f(t-s, v(t-s))\|^p \, ds \right)^{\frac{1}{p}} \right)^{\frac{1}{1-\gamma}}$$

$$+ \sum_{k \geq 1} k^{-\gamma - 1} \left( \int_{k}^{k+1} \|f(t-s, u(t-s)) - f(t-s, v(t-s))\|^p \, ds \right)^{\frac{1}{p}}$$

$$= (1 - q(\gamma - 1))^\frac{1}{\gamma} \left( \int_{0}^{1} \|f(t-s, u(t-s)) - f(t-s, v(t-s))\|^p \, ds \right)^{\frac{1}{p}}$$

$$+ \sum_{k \geq 1} k^{-\gamma - 1} \left( \int_{k}^{k+1} \|f(t-s, u(t-s)) - f(t-s, v(t-s))\|^p \, ds \right)^{\frac{1}{p}}$$

$$< S^q_{\mu} \epsilon \text{ for all } t \in \mathbb{R}.$$

Hence, by the assumption $S^q_{\mu} \epsilon \leq 1$, we obtain that

$$\|F_0 u - F_0 v\|_{\infty} < \epsilon.$$

The result follows immediately from Lemma 3.1. \qed

In order to visualize the advantage of our result (Theorem 3.6), in the next theorem, we will prove the existence and uniqueness of $\mu$-pseudo-almost periodic solutions to the inclusion (1.2) under the following Lipschitz assumption:

(Q) There exists a nonnegative scalar function $L(\cdot) \in \text{BS}^p(\mathbb{R})$ (for $1 < p < \infty$) such that

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \ x, \ y \in \mathbb{X}, \ t \in \mathbb{R}.$$

Theorem 3.7. Let $1 < p < +\infty$ and $\mu \in \mathcal{M}$ satisfy (M). Assume that $f \in \text{PAP}^p U(\mathbb{R} \times X, X, \mu)$ such that (Q) holds with $\|L\|_{\text{BS}^p} S^q_{\mu} < 1$. Then, the inclusion (1.2) has a unique $\mu$-pseudo-almost periodic mild solution given by the integral representation (3.2).
Proof. From Theorem 3.6 and Corollary 2.9, it suffices to prove that the mapping $F_0$ has a unique fixed point. Indeed, let $u, v \in PAP(\mathbb{R}, X, \mu)$. Then, by (Q), we get

$$
\|F_0u(t) - F_0v(t)\| \leq \int_{-\infty}^{t} \|R_s(t - s)\| \cdot \|f(s, u(s)) - f(s, v(s))\| \, ds
$$

$$
= \int_{0}^{\infty} \|R_s\| \cdot \| f(t - s, u(t - s)) - f(t - s, u(t - s))\| \, ds
$$

$$
= \int_{0}^{1} \|R_s\| \cdot \| f(t - s, u(t - s)) - f(t - s, v(t - s))\| \, ds
$$

$$
+ \int_{1}^{\infty} \|R_s\| \cdot \| f(t - s, u(t - s)) - f(t - s, v(t - s))\| \, ds
$$

$$
\leq \left( \int_{0}^{1} s^{p(t-1)} \right)^{\frac{1}{p}} \left( \int_{0}^{1} \|f(t - s, u(t - s)) - f(t - s, v(t - s))\|^p \, ds \right)^{\frac{1}{p}}
$$

$$
+ \sum_{k \geq 1} k^{-q-1} \left( \int_{k}^{k+1} L(t - s)p \, ds \right)^{\frac{1}{p}} \|u - v\|_{\infty}
$$

$$
\leq (1 - q(t - 1))^{\frac{1}{p}} \left( \int_{0}^{1} L(t - s)p \, ds \right)^{\frac{1}{p}} \|u - v\|_{\infty}
$$

$$
+ \sum_{k \geq 1} k^{-q-1} \left( \int_{k}^{k+1} L(t - s)p \, ds \right)^{\frac{1}{p}} \|u - v\|_{\infty}
$$

$$
\leq S^q_\gamma \|L\|_{BSp} \|u - v\|_{\infty}, \quad t \in \mathbb{R}.
$$

Hence, we obtain that

$$
\|F_0u - F_0v\|_{\infty} \leq S^q_\gamma \|L\|_{BSp} \|u - v\|_{\infty}.
$$

Then, the result follows from the Banach strict contraction principle, since by assumption $S^q_\gamma \|L\|_{BSp} < 1$. \qed

Remark 3.8. It is very important to state that, in Theorem 3.6, under the assumptions (P2) and (P3), the condition $\|L\|_{BSp}S^q_\gamma = 1$ (i.e., $\|L\|_{BSp} = (S^q_\gamma)^{-1}$) yields the existence and uniqueness of $\mu$-pseudo-almost periodic mild solutions to the inclusion (1.2) in view of Meir and Keeler fixed theorem. However, the existence and uniqueness result does not hold in this case when $f$ satisfies (Q); see Theorem 3.7.

3.1.1 Example

Consider the following semilinear fractional Poisson heat equation in the $L^2$-setting, namely,

$$
\begin{cases}
D_t^{\gamma}(m(x)v(t, x)) = \Delta v(t, x) + g(t, v(t, x)) + H(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega, \\
v(t, x)|_{\partial \Omega} = 0; \quad t \in \mathbb{R}, \quad x \in \partial \Omega,
\end{cases}
$$

(3.5)

where $\gamma \in (0, 1)$, $\Omega \subset \mathbb{R}^N$ an open bounded subset with sufficiently smooth boundary $\partial \Omega$ and $m \in L^\infty(\Omega)$, $m \geq 0$. Here, $H : \mathbb{R} \times \Omega \to \mathbb{R}$ is $S^2 - \mu$-pseudo-almost periodic function. Let $X = L^2(\Omega)$ be the Lebesgue space of square integrable functions on $\Omega$ equipped with its usual norm denoted by $\| \cdot \|$. Define the operator $A$ on $X$ by

$$
A\varphi := \Delta M^{-1} \varphi,
$$

with $\varphi \in D(A)$ (the maximal domain of $A$) where $\Delta$ is the Dirichlet Laplacian on $X$ and $M$ is the multiplication operator by $m$ on $X$. In addition, we set $f : \mathbb{R} \times X \to X$ given by

$$
f(t, \varphi)(x) = g(t, \varphi(x)) + H(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega.
$$
Then, our model (3.5) is equivalent to the inclusion (1.2). It is clear that $\mathcal{A}$ satisfies (P1); see Kostić and references therein. Moreover, for $\varphi \in X$, we define
\[ g(t, \varphi(x)) := K(t) \frac{R(x)}{1 + \|\varphi\|}, \quad t \in \mathbb{R}, x \in \Omega, \]
where $K : \mathbb{R} \to (0, \infty)$ is $S^2-$μ-pseudo-almost periodic and $R \in X, R \geq 0$. Then, we have

**Lemma 3.9.** The function $f$ satisfies (P2) with $L = \|K\|_{BS^2} \|R\|$.

**Proof.** Let $\varphi_1, \varphi_2 \in X$, and let $\epsilon > 0$ and $\delta > 0$ such that $\epsilon \leq \|\varphi_1 - \varphi_2\| < \epsilon + \delta$. So
\[ |f(t, \varphi_1(x)) - f(t, \varphi_2(x))| = K(t) \left| \frac{R(x)}{1 + \|\varphi_1\|} - \frac{R(x)}{1 + \|\varphi_2\|} \right| \leq K(t) \frac{\|\varphi_1 - \varphi_2\|}{1 + \|\varphi_1\| + \|\varphi_2\|} R(x), \quad t \in \mathbb{R}, \quad x \in \Omega. \]

Then, using (3.6), we get
\[ \int_{\Omega} |f(t, \varphi_1(x)) - f(t, \varphi_2(x))|^2 dx \leq K(t)^2 \frac{\|\varphi_1 - \varphi_2\|^2}{(1 + \epsilon)^2} \int_{\Omega} R(x)^2 dx \leq K(t)^2 \frac{(\delta + \epsilon)^2}{(1 + \epsilon)^2} \|R\|^2 \leq K(t)^2 \|R\|^2 \epsilon^2, \quad t \in \mathbb{R} \text{ (by taking } \delta := \epsilon^2). \]

Hence, for all $\varphi_1, \varphi_2 \in X$, we have
\[ \left( \int_{t}^{t+1} \|f(s, \varphi_1) - f(s, \varphi_2)\|^2 ds \right)^{\frac{1}{2}} < \|K\|_{BS^2} \|R\| \epsilon \text{ for all } t \in \mathbb{R}. \]

This proves the result with $L := \|K\|_{BS^2} \|R\|$.

**Lemma 3.10.** The function $f$ is Lipschitzian with respect to the second argument with Lipschitz constant $L(\cdot) := K(\cdot) \|R\|$.

Moreover, $f$ satisfy (P3).

**Proof.** Let $\varphi_1, \varphi_2 \in X$. By the proof of Lemma 3.9, we assert that
\[ \|f(t, \varphi_1) - f(t, \varphi_2)\| \leq K(t) \|R\| \|\varphi_1 - \varphi_2\|, \quad t \in \mathbb{R}. \]

Hence, the result yields from Remark 3.2.

Then, we have the following main result.

**Theorem 3.11.** Assume that
- $K \in PAPS^2(\mathbb{R}, (0, \infty), \mu)$ and $R \in X$ such that $R \geq 0$.
- $H(\cdot, x) \in PAPS^2(\mathbb{R}, \mu)$ for each $x \in \hat{\Omega}$.

Then, the fractional Poisson heat equation (3.5) has a unique $\mu$-pseudo-almost periodic integral solution provided that
\[ \|K\|_{BS^2} \|R\| \leq (S^2)^{-1}, \]
where $\mu$ is the measure given by (3.18).

**Proof.** The result follows from Theorem 3.6 since hypotheses (P1)–(P3) are all satisfied.
3.2 | Pseudo-almost periodic solutions of Equation (1.4)

In this subsection, we consider the existence and uniqueness of \( \mu \)-pseudo-almost periodic solutions of the following semilinear nonautonomous evolution equations:

\[
x'(t) = A(t)x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R}.
\]

Let \((A(t), D(A(t))), t \in \mathbb{R}\) be a family of linear closed operators on a Banach space \(X\). Of concern is the following linear Cauchy problem:

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u'(t) = A(t)u(t), & t \geq s, \\
u(s) = x & \in X.
\end{array} \right.
\end{aligned}
\]  

(3.7)

Here, we assume that \((A(t), D(A(t))), t \in \mathbb{R}\), generates an evolution family, which solves the problem (3.7), that is, a two-parameter family \((U(t, s))_{t \geq s}\) of linear bounded operators in \(X\) such that the map \((t, s) \mapsto U(t, s) \in \mathcal{L}(X)\) is strongly continuous, \(U(t, s)U(s, r) = U(t, r)\) and \(U(t, t) = I\) for \(t \geq s \geq r\). A (mild) solution to problem (3.7) is \(u(t) = U(t, s)x\) for \(t \geq s\). In particular, if \((A(t))\) is time independent, that is, \(A(t) = A\) for all \(t \in \mathbb{R}\), then \(U(t, s) = T(t - s)\), where \((T(t))_{t \geq 0}\) is a semigroup of bounded linear operators on \(X\). Notice that, in general, the domains \(D(A(t))\) of the operators \(A(t)\) are not necessarily dense in \(X\) and may change with respect to \(t\). Unlike semigroups, there is no necessary and sufficient spectral criteria for \((A(t), D(A(t))), t \in \mathbb{R}\), to generate an evolution family. In the parabolic context, we refer to previous studies and references therein for some results obtained so far in this direction. If \(A(t)\) has a constant domain \(D(A(t)) = D, t \in \mathbb{R}\), then we have the following result:

(C1) Let \((A(t), D), t \in \mathbb{R},\) be the generators of analytic semigroups \((T(t))_{t \geq 0}\) on \(X\) of the same type \((N, \omega)\); that is, \(\|T(t)\| \leq N e^{\omega t}\) (uniformly in \(t\)). Assume that \(A(t)\) is invertible for all \(t \in \mathbb{R}\), \(\sup_{t \in \mathbb{R}} \|A(t)A(s)^{-1}\| < \infty\) and there exist constants \(\omega \in \mathbb{R}, L \geq 0\), and \(0 < \mu \leq 1\) such that

\[
\|\langle A(t) - A(s)\rangle R(\omega, A(\tau))\| \leq L|t - s|^\mu \quad \text{for } t, s, \tau \in \mathbb{R}.
\]  

(3.8)

In this case, the map \((t, s) \mapsto U(t, s) \in \mathcal{L}(X)\) is continuously differentiable for \(t > s\) with respect to the variable \(t\), \(U(t, s)\) maps \(X\) into \(D(A(t))\), and we have \(\partial U(t, s)/\partial t = A(t)U(t, s)\). Moreover, \(U(t, s)\) and \((t - s)A(t)U(t, s)\) are exponentially bounded.

Now, we recall the notion of exponential dichotomy of an evolution family (for more details, see Engel and Nagel\(^{29}\) and Schnaubelt\(^{34}\)):

**Definition 3.12.** An evolution family \((U(t, s))_{t \geq s}\) on \(X\) is said to have an exponential dichotomy in \(\mathbb{R}\) if there exists a family of projections \(P(t) \in \mathcal{L}(X), t \in \mathbb{R},\) being strongly continuous with respect to \(t\), and constants \(\delta, N > 0\) such that

(i) \(U(t, s)P(s) = P(t)U(t, s),\)

(ii) \(U(t, s)Q(s)X \rightarrow Q(t)X\) is invertible with the inverse \(\tilde{U}(t, s)\) (i.e., \(U(t, s) = U(s, t)^{-1}\)), and

(iii) \(\|U(t, s)P(s)\| \leq N e^{-\delta(t-s)}\) and \(\|\tilde{U}(s, t)Q(t)\| \leq N e^{-\delta(t-s)}\),

for all \(t, s \in \mathbb{R}\) with \(t \geq s\), where \(Q(t) := I - P(t)\). If this is the case, then we also say that the evolution family \((U(t, s))_{t \geq s}\) is hyperbolic.

Given a hyperbolic evolution family \((U(t, s))_{t \geq s}\), then its associated Green function is defined by

\[
G(t, s) := \begin{cases} 
U(t, s)P(s), & t, s \in \mathbb{R}, \ s \leq t, \\
-\tilde{U}(t, s)Q(s), & t, s \in \mathbb{R}, \ s > t.
\end{cases}
\]  

(3.9)

The exponential dichotomy can be characterized in many cases; for more details, see Henry.\(^{35}\) From Schnaubelt,\(^{34}\) the exponential dichotomy holds in the following case:

(C2) Assume that (C1) holds, and the semigroups \((T(t))_{t \geq 0}\) are hyperbolic with projections \(P_t\) and constants \(N, \delta > 0\) such that \(\|A(t)T^\tau P_t\| \leq \psi(\tau)\) and \(\|A(t)T^\tau Q_t\| \leq \psi(-\tau)\) for \(\tau > 0\) and a function \(\psi\) such that the mapping \(\mathbb{R} \ni s \mapsto \psi(s) := |s|^\mu \psi(s)\) is integrable with \(L\|\psi\|_{L^1(\mathbb{R})} < 1\).

Now, we give our main hypotheses:

(H1) The operator \(A(t), t \in \mathbb{R}\), generates a strongly continuous evolution family \((U(t, s))_{t \geq s}\) on \(X\).
The evolution family \((U(t,s))_{t \geq s}\) has an exponential dichotomy on \(\mathbb{R}\) with constants \(N, \delta > 0\), projections \(P(t), t \in \mathbb{R}\) and Green's function \(G(\cdot, \cdot)\).

\((H2)\) \(R(\omega, A(\cdot))\) is almost periodic for some \(\omega \in \mathbb{R}\).

\((H4)\) \(f\) is Lipschitzian in bounded sets with respect to the second argument, that is, for each \(\rho > 0\), there exists a nonnegative scalar function \(L_\rho(\cdot) \in BSP(\mathbb{R})\) (for \(1 \leq p < \infty\)) such that

\[
\|f(t,x) - f(t,y)\| \leq L_\rho(t)\|x - y\|, \; x, y \in B(0, \rho), \; t \in \mathbb{R}.
\]

Moreover, we assume that \(f(t,0) \neq 0\) for all \(t \in \mathbb{R}\).

**Remark 3.13.** (a) Notice if \((C2)\) is satisfied, then hypotheses \((H1)-(H2)\) hold. (b) Schnaubelt \(^{34}\) have proved that if \(R(\omega, A(\cdot))\) is almost periodic for some \(\omega \in \mathbb{R}\), then the associated Green function is bi-almost periodic.

By a mild solution of (1.4), we mean any continuous function \(u : \mathbb{R} \to X\) which satisfies the following variation of constants formula:

\[
u(t) = U(t, \sigma)u(\sigma) + \int_\sigma^t U(t, s)f(s, u(s))ds \quad \text{for all } t \geq \sigma.
\]

(3.10)

In particular, we analyze the existence and uniqueness of \(\mu\)-pseudo-almost periodic solutions of the following linear inhomogeneous equations:

\[
u'(t) = A(t)\nu(t) + h(t) \quad \text{for all } t \in \mathbb{R}.
\]

(3.11)

**Theorem 3.14** (Theorem 3.5 of Akkad et al\(^{11}\)). Let \(\mu \in \mathcal{M}\) satisfy \((M)\) and \(h \in PAPS(\mathbb{R}, X, \mu)\). Assume that \((H1)-(H3)\) hold. Then, the abstract Cauchy problem (3.11) has a unique \(\mu\)-pseudo-almost periodic mild solution \(u : \mathbb{R} \to X\), given by

\[
u(t) = \int_{\mathbb{R}} G(t, s)h(s)ds \quad \text{for all } t \in \mathbb{R}.
\]

(3.12)

Now, we give our main result about the existence and uniqueness of \(\mu\)-pseudo-almost periodic solutions to Equation (1.4). We need the following lemma:

**Lemma 3.15.** Let for each \(x \in X, f(\cdot, x) \in PAPS(\mathbb{R}, X, \mu)\) (for \(1 \leq p < \infty\)). Assume that \(f\) satisfies hypothesis \((H4)\). Then, the statement Theorem 2.7 (iii) holds for \(f\).

**Proof.** Let \(B\) be any bounded set in \(X\), that is, there exists \(M \geq 0\) such that \(\|x\| \leq M\) for all \(x \in B\). Let \(x \in B\), and by assumptions on \(f\), we have

\[
\int_t^{t+1} \|f(s, x)\|^p ds \leq \int_t^{t+1} L_M(s)^p ds \|x\|^p + \int_t^{t+1} \|f(s, 0)\|^p ds \\
\leq M^p \|L_M\|^p_{BSP} + \|f(\cdot, 0)\|^p_{BSP}, \; t \in \mathbb{R}.
\]

Thus,

\[
\sup_{x \in B} \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(s, x)\|^p ds \right) \leq M \|L_M\|^p_{BSP} + \|f(\cdot, 0)\|^p_{BSP} < \infty.
\]

This proves the result. \(\square\)

Now, let \(u \in PAP(\mathbb{R}, X, \mu)\). Then, using Lemma 3.15, it follows in view of Theorem 2.1 that \(h := f(\cdot, u(\cdot)) \in PAPS(\mathbb{R}, X, \mu)\) for all \(1 \leq p < \infty\). Hence, the integral mapping defined by (3.12) belongs to \(PAP(\mathbb{R}, X, \mu)\) and the map \(F : PAP(\mathbb{R}, X, \mu) \to PAP(\mathbb{R}, X, \mu)\) given by
\[ Fu(t) = \int_{\mathbb{R}} G(t, s) f(s, u(s)) ds, \quad t \in \mathbb{R} \]

is well defined. Moreover, we distinguish two cases: \( p = 1 \) and \( 1 < p < \infty \). For \( 1 < p < \infty \), we have the following existence result.

**Theorem 3.16.** Let \( \mu \in \mathcal{M} \) satisfy (M) and \( f(\cdot, x) \in \text{PAP}^p(\mathbb{R}, X, \mu) \) for each \( x \in X \). Suppose that (H1)--(H4) hold and there exists \( \rho > 0 \) such that

\[
\rho > \left( 2N \left( \frac{2}{q \delta} \frac{\frac{1}{q} e^\frac{s}{\delta} - 1}{e^\frac{s}{\delta} - 1} \right) \right) \| f(\cdot, 0) \|_{\text{BSF}}, \tag{3.13}
\]

\[
\| L_\rho \|_{\text{BSF}} \leq \left( 2N \left( \frac{2}{q \delta} \frac{\frac{1}{q} e^\frac{s}{\delta} - 1}{e^\frac{s}{\delta} - 1} \right) \right)^{-1} \rho^{-1} \| f(\cdot, 0) \|_{\text{BSF}}. \tag{3.14}
\]

Then, Equation (1.4) has a unique \( \mu \)-pseudo-almost periodic solution \( u \) with

\[ \| u \|_\infty \leq \rho. \]

**Proof.** Define the mapping \( F : \Lambda^\mu_\rho \to \text{PAP}(\mathbb{R}, X, \mu) \) by

\[ Fu(t) = \int_{\mathbb{R}} G(t, s) f(s, u(s)) ds, \quad t \in \mathbb{R}, \]

where \( \Lambda^\mu_\rho := \{ v \in \text{PAP}(\mathbb{R}, X, \mu) : \| v \|_\infty \leq \rho \} \). We show that \( FA^\mu_\rho \subset \Lambda^\mu_\rho \). Let \( u \in \Lambda^\mu_\rho \). Then, by (H4), we obtain that

\[
\| Fu(t) \| \leq \int_{\mathbb{R}} \| G(t, s) f(s, u(s)) \| ds
\]

\[
\leq N \int_{-\infty}^t e^{-\delta(t-s)} \| f(s, u(s)) - f(s, 0) \| ds + N \int_t^{+\infty} e^{-\delta(t-s)} \| f(s, 0) \| ds
\]

\[
+ N \int_{-\infty}^t e^{-\delta(t-s)} \| f(s, u(s)) - f(s, 0) \| ds + N \int_t^{+\infty} e^{-\delta(t-s)} \| f(s, 0) \| ds
\]

\[
\leq \rho N \int_{-\infty}^t e^{-\delta(t-s)} L_\rho(s) ds + N \int_{-\infty}^t e^{-\delta(t-s)} \| f(s, 0) \| ds
\]

\[
+ \rho N \int_t^{+\infty} e^{-\delta(t-s)} L_\rho(s) ds + N \int_t^{+\infty} e^{-\delta(t-s)} \| f(s, 0) \| ds
\]

\[
\leq N \left( \int_{-\infty}^t e^{-q \frac{1}{2}(t-s)} ds \right)^\frac{1}{q} \left[ \rho \left( \int_{-\infty}^t e^{-p \frac{1}{2}(t-s)} \| L_\rho(s) \| ds \right)^\frac{1}{p} + \left( \int_{-\infty}^t e^{-p \frac{1}{2}(t-s)} \| f(s, 0) \| ds \right)^\frac{1}{p} \right]
\]

\[
+ N \left( \int_t^{+\infty} e^{-q \frac{1}{2}(t-s)} ds \right)^\frac{1}{q} \left[ \rho \left( \int_t^{+\infty} e^{-p \frac{1}{2}(t-s)} \| L_\rho(s) \| ds \right)^\frac{1}{p} + \left( \int_t^{+\infty} e^{-p \frac{1}{2}(t-s)} \| f(s, 0) \| ds \right)^\frac{1}{p} \right]
\]

\[
\leq N \left( \int_{-\infty}^t e^{-q \frac{1}{2}(t-s)} ds \right)^\frac{1}{q} \sum_{k=1}^{+\infty} e^{-k \frac{1}{2}} \left[ \rho \left( \int_{t-k}^{t-k+1} \| L_\rho(s) \| ds \right)^\frac{1}{p} \left( \int_{t-k}^{t-k+1} \| f(s, 0) \| ds \right)^\frac{1}{p} \right]
\]

\[
+ N \left( \int_t^{+\infty} e^{-q \frac{1}{2}(t-s)} ds \right)^\frac{1}{q} \sum_{k=1}^{+\infty} e^{-k \frac{1}{2}} \left[ \rho \left( \int_{t-k}^{t-k+1} \| L_\rho(s) \| ds \right)^\frac{1}{p} \left( \int_{t-k}^{t-k+1} \| f(s, 0) \| ds \right)^\frac{1}{p} \right]
\]

\[
\leq 2N \left( \frac{2}{q \delta} \frac{\frac{1}{q} e^\frac{s}{\delta} - 1}{e^\frac{s}{\delta} - 1} \right) (\rho \| L_\rho \|_{\text{BSF}} + \| f(\cdot, 0) \|_{\text{BSF}})
\]

\[ \leq \rho, \quad t \in \mathbb{R}. \]
Hence, $FA_p^\text{PAP} \subset \Lambda_p^\text{PAP}$. Let $u, v \in \Lambda_p^\text{PAP}$. Then, a straightforward calculation yields

$$
\| Fu(t) - Fv(t) \| \leq \int_{\mathbb{R}} \| G(t, s) [ f(s, u(s)) - f(s, v(s)) ] \| ds \\
\leq N \int_{-\infty}^{t} e^{-\delta(t-s)} \| f(s, u(s)) - f(s, v(s)) \| ds \\
+ N \int_{t}^{+\infty} e^{-\delta(s-t)} \| f(s, u(s)) - f(s, v(s)) \| ds \\
\leq N \left( \int_{-\infty}^{t} e^{-\delta(t-s)} L_\rho(s) ds + \int_{t}^{+\infty} e^{-\delta(s-t)} L_\rho(s) ds \right) \| u - v \|_{\infty} \\
\leq N \sum_{k \geq 1} e^{-k\frac{\delta}{2}} \left[ \left( \int_{-\infty}^{t} e^{-q(s-t)} ds \right)^{\frac{1}{q}} \left( \int_{t-k}^{t} L_\rho(s) ds \right)^{\frac{1}{p}} \right] \| u - v \|_{\infty} \\
\leq 2N \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}} - 1} \| L_\rho \|_{\text{BS}^1} \| u - v \|_{\infty}, \ t \in \mathbb{R}.
$$

Therefore, by hypothesis $\| f(\cdot, 0) \|_{\text{BS}^1} > 0$ (see (H4)) which yields in view of (3.13) and (3.14) that

$$
\left( 2N \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}} - 1} \| L_\rho \|_{\text{BS}^1} \right) \| u - v \|_{\infty} \leq 1 - \rho^{-1} \| f(\cdot, 0) \|_{\text{BS}^1} < 1.
$$

Henceforth, the mapping $F$ is a strict contraction in $\Lambda_p^\text{PAP}$. Consequently, by the Banach contraction principle, we obtain the existence and uniqueness of a solution $u \in \Lambda_p^\text{PAP}$. This proves the result.

For $p = 1$, we have:

**Theorem 3.17.** Let $\mu \in \mathcal{M}$ satisfy (M) and $f(\cdot, x) \in \text{PAPS}^1(\mathbb{R}, X, \mu)$ for each $x \in X$. Suppose that (H1)–(H4) hold and there exists $\rho > 0$ such that

$$
\rho > \left( 2N \frac{e^{\delta}}{e^{\delta} - 1} \right) \| f(\cdot, 0) \|_{\text{BS}^1}, \quad \text{(3.15)}
$$

$$
\| L_\rho \|_{\text{BS}^1} \leq \left( 2N \frac{e^{\delta}}{e^{\delta} - 1} \right)^{-1} - \rho^{-1} \| f(\cdot, 0) \|_{\text{BS}^1}. \quad \text{(3.16)}
$$

Then, Equation (1.4) has a unique $\mu$-pseudo-almost periodic solution $u$ with

$$
\| u \|_{\infty} \leq \rho.
$$

**Proof.** We define the mapping $F : \Lambda_p^\mu \to \text{PAP}(\mathbb{R}, X, \mu)$ by

$$
Fu(t) = \int_{\mathbb{R}} G(t, s) f(s, u(s)) ds, \ t \in \mathbb{R}.
$$
Let \( u \in \Lambda_p^n \). Arguing similarly as in the proof of Theorem 3.17 (without using Hölder inequality), we get that

\[
\| F(u)(t) \| \leq \int_{\mathbb{R}} \| G(t, s) f(s, u(s)) \| \, ds \\
\leq N \int_{-\infty}^{t} e^{-\delta(t-s)} \| f(s, u(s)) - f(s, 0) \| \, ds + N \int_{t}^{\infty} e^{-\delta(t-s)} \| f(s, 0) \| \, ds \\
+ N \int_{t}^{\infty} e^{-\delta(t-s)} \| f(s, u(s)) - f(s, v(s)) \| \, ds \\
\leq \rho N \int_{-\infty}^{t} e^{-\delta(t-s)} L_{\rho}(s) \, ds + N \int_{t}^{\infty} e^{-\delta(t-s)} \| f(s, 0) \| \, ds \\
+ \rho N \int_{t}^{\infty} e^{-\delta(t-s)} L_{\rho}(s) \, ds + \int_{t}^{\infty} e^{-\delta(t-s)} L_{\rho}(s) \, ds \\
\leq 2 N \frac{e^\delta}{e^\delta - 1} \left( \rho \| L_{\rho} \|_{BS^1} + \| f(\cdot, 0) \|_{BS^1} \right) \\
\leq \rho, \quad t \in \mathbb{R}.
\]

Hence, \( \Lambda_p^{PAP} \subset \Lambda_p^{PAP} \). Furthermore, we have

\[
\| F(u)(t) - F(v)(t) \| \leq \int_{\mathbb{R}} \| G(t, s) \left[ f(s, u(s)) - f(s, v(s)) \right] \| \, ds \\
\leq N \int_{-\infty}^{t} e^{-\delta(t-s)} \| f(s, u(s)) - f(s, v(s)) \| \, ds \\
+ N \int_{t}^{\infty} e^{-\delta(t-s)} \| f(s, u(s)) - f(s, v(s)) \| \, ds \\
\leq N \left( \int_{-\infty}^{t} e^{-\delta(t-s)} L_{\rho}(s) \, ds + \int_{t}^{\infty} e^{-\delta(t-s)} L_{\rho}(s) \, ds \right) \| u - v \|_{\infty} \\
\leq 2 N \frac{e^\delta}{e^\delta - 1} \| L_{\rho} \|_{BS^1} \| u - v \|_{\infty}, \quad t \in \mathbb{R}.
\]

So using the fact that \( \| f(\cdot, 0) \|_{BS^1} \neq 0 \), by hypothesis (H4), it follows from (3.15) and (3.16) that

\[
\left( 2 N \frac{e^\delta}{e^\delta - 1} \right) \| L_{\rho} \|_{BS^1} \leq 1 - \rho^{-1} \| f(\cdot, 0) \|_{BS^1} < 1.
\]

Hence, the mapping \( F \) is a strict contraction in \( \Lambda_p^{PAP} \), and then, the result follows by the Banach contraction principle.

\[\square\]

### 3.2.1 Example

Consider the following time-dependent parameters reaction-diffusion equation describing the behavior of bounded solutions to a one-species intraspecific competition Lotka–Volterra model, namely,

\[
\begin{cases}
  v_t(t, x) = \Delta v(t, x) - a(t)v(t, x) + b(t)v(t, x)^2 + C(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega, \\
  v(t, x)|_{\partial \Omega} = 0; \quad t \in \mathbb{R}, \quad x \in \partial \Omega,
\end{cases}
\]

(3.17)

where

- \( \Omega \subseteq \mathbb{R}^N (N \geq 1) \) is an open bounded subset with a sufficiently smooth boundary.
- \( \Delta \) is the Laplace operator on \( \Omega \), here the diffusion parameter equals 1.
- \( a \in AP(\mathbb{R}, [0, \infty)) \) with \( 0 < a_0 := \inf_{t \in \mathbb{R}} a(t) \leq a(t) \leq \sup_{t \in \mathbb{R}} a(t) = a_1 < \infty \). It is assumed to be Hölder continuous with constant \( L = 1 \) and exponent 1.
- The nonlinear term \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
g(t, v(t,x)) = b(t)v^2(t,x) + C(t,x), \quad x \in \bar{\Omega},
\]

where \( b \in APS^1(\mathbb{R}, [0, \infty)) \).

- \( C : \mathbb{R} \times \bar{\Omega} \to (0, \infty) \) is locally integrable with respect to \( t \) and continuous with respect to \( x \). More precisely, we take \( C(t, x) = H(x) + c(t) \) where \( c : \mathbb{R} \to (0, \infty) \) is \( \mu \)-pseudo-almost periodic with \( \mu \) be the measure with the Radon–Nikodym derivative \( \theta \) defined by

\[
\theta(t) = \begin{cases} 
    e^t \text{ for } t \leq 0, \\
    1 \text{ for } t > 0.
\end{cases}
\]

The term \( H : \bar{\Omega} \to [0, \infty) \) is continuous. In order to transform our model (3.17) to the abstract form, that is, Equation (1.4), we define the Banach space \( X = C(\bar{\Omega}) \) equipped with the sup-norm. Set the linear operators \( (A(t), D(A(t))) \), \( t \in \mathbb{R} \) by

\[
\begin{align*}
A(t) \varphi & := \Delta + a(t), \\
D(A(t)) & := \{ \varphi \in C(\bar{\Omega}) \cap H^1(\Omega) : \varphi(\Delta) \in C(\bar{\Omega}) \} := D.
\end{align*}
\]

Here, \( \Delta \) is the Laplacian in the sense of distributions on \( \Omega \). It is well-known (see Da Prato and Sinestrari\(^{16, \text{proposition 14.6}} \)) that \((\Delta, D)\) generates a contraction analytic semigroup \((e^{\Delta})_{s \geq 0}\) on \( X \), with angle \( \phi \in \left( \frac{\pi}{2}, \pi \right) \) such that

\[
\| e^{s \Delta} \| \leq 1, \quad s \geq 0
\]

and

\[
\| R(\lambda, \Delta) \| \leq 1 / \lambda, \quad \lambda > 0.
\]

Moreover, the semigroup is exponentially stable, and we have

\[
\| e^{s \Delta} \| \leq e^{-\lambda_1 s}, \quad s \geq 0,
\]

where \( \lambda_1 \) is the smallest eigenvalue of \(-\Delta\). Therefore, by assuming that \( a_0 < 0 \), we obtain that for each \( t \in \mathbb{R}, (A(t), D) \) generates an analytic semigroup \((T(s))_{s \geq 0}\) of type \((1, \omega) := a_0 + \lambda_1 > 1\) on \( X \). Therefore, each semigroup \((T(s))_{s \geq 0}\) is exponentially stable. Moreover, we have

\[
\sup_{t, s \in \mathbb{R}} \| A(t)A(s)^{-1} \| \leq 1 + 2a_1 \| A(s)^{-1} \| \leq 1 + \frac{2a_1}{a_0 + \lambda_1}, \quad t, s \in \mathbb{R}.
\]

Thus,

\[
\sup_{t, s \in \mathbb{R}} \| A(t)A(s)^{-1} \| < \infty.
\]

Furthermore, by our assumption on \( a \), we obtain that

\[
\|(A(t) - A(s))A(r)^{-1} \| = |a(t) - a(s)||A(r)^{-1} \| \leq \omega^{-1}|t - s| \quad \text{for } t, s, r \in \mathbb{R}. \tag{3.23}
\]

Since \( \omega^{-1} \int_0^\infty e^{-\omega t} d\tau = \omega^{-2} < 1 \), we obtain from (C2) that \((A(t), D)_{t \in \mathbb{R}}\) generates an exponentially stable analytic evolution family with exponent \( 0 < \delta < \sqrt{\omega^{-1}} \) which yields that hypotheses (H1) and (H2) hold (see Remark 3.13 (a)). Furthermore, the fact that \( a \in \text{AP}(\mathbb{R}) \) implies \( A(\cdot)^{-1} \in \text{AP}(\mathbb{R}, L(X)) \). Indeed, let \( t, r \in \mathbb{R} \). So we have

\[
A(t + \tau)^{-1}\varphi - A(t)^{-1}\varphi = A(t + \tau)^{-1}(A(t + \tau) - A(t))A(t)^{-1}\varphi = (a(t + \tau) - a(t))A(t + \tau)^{-1}A(t)^{-1}\varphi, \quad \varphi \in X.
\]

Thus, the result follows from the next inequality:

\[
\| A(t + \tau)^{-1}\varphi - A(t)^{-1}\varphi \| \leq |a(t + \tau) - a(t)||A(t + \tau)^{-1}||A(t)^{-1}\varphi\|_D \leq C|a(t + \tau) - a(t)||\varphi|, \quad \varphi \in X.
\]
Therefore, (H3) is also satisfied. In order to check (H4), we define the superposition operator \( f : \mathbb{R} \times X \to X \) by
\[
f(t, \varphi)(x) := g(t, \varphi(x)), \quad x \in \tilde{\Omega}.
\]

Hence, we obtain the following result

**Proposition 3.17.** The function \( f \) satisfies hypothesis (H4) with \( L_\rho(\cdot) = 2\rho b(\cdot) \).

**Proof.** Let \( \varphi, \psi \in X \) and let \( \rho > 0 \) be such that \( \|\varphi\|, \|\psi\| \leq \rho \). Then, we have
\[
|g(t, \varphi(x)) - g(t, \psi(x))| = [b(t)]|\varphi(x) - \psi(x)||\varphi(x) + \psi(x)| \\
\leq [b(t)](\|\varphi\| + \|\psi\|)\|\varphi - \psi\| \\
\leq 2\rho [b(t)]\|\varphi - \psi\|, \quad t \in \mathbb{R}, \ x \in \tilde{\Omega}.
\]

Hence,
\[
\|f(t, \varphi) - f(t, \psi)\| \leq 2\rho [b(t)]\|\varphi - \psi\|.
\]

Since \( b \in BS^1(\mathbb{R}, [0, \infty)) \), it follows that (H4) holds with \( L_\rho = 2\rho b(\cdot) \).

As a consequence of Theorem 3.17, we obtain the following main result.

**Theorem 3.18.** Assume that:

- \( a \in AP(\mathbb{R}, [0, \infty)) \) with \( 0 < a_0 := \inf_{t \in \mathbb{R}} a(t) \leq a(t) \leq \sup_{t \in \mathbb{R}} a(t) = a_1 < \infty \) such that it is Hölder continuous with constant \( L = 1 \) and exponent \( 0 < a \leq 1 \) (we take \( a = 1 \) for simplicity).
- \( b \in APS^1(\mathbb{R}, [0, \infty)) \) such that \([b]_{BS} < \frac{1}{2}\) and \( e^\delta > (1 - 2[b]_{BS})^{-1} \).
- \( c \in PAPS^1(\mathbb{R}, (0, \infty), \mu) \) and \( H \in C(\tilde{\Omega}) \) are chosen small in norm such that \((|c|_{BS} + \|H\|_{\infty})\frac{2e^\delta}{e^\delta - 2[b]_{BS}e^\delta - 1} \leq 1\).

Then, Equation (3.19) has a unique \( \mu \)-pseudo-almost periodic solution with respect to \( t \) satisfying \( \sup_{(t,x) \in \mathbb{R} \times \tilde{\Omega}} |v(t,x)| \leq \rho \) for some \( \rho > 0 \) satisfying
\[
\rho \geq \frac{2e^\delta}{e^\delta - 2[b]_{BS}e^\delta - 1} (|c|_{BS} + \|H\|_{\infty}), \quad (3.24)
\]
where \( \mu \) is the measure given by (3.18).

**Proof.** It is clear that \( \mu \) given by (3.18) satisfies (M), see Example A.18. Moreover, by construction, the function \( f \) given by
\[
f(t, \varphi)(x) := g(t, \varphi(x)), \quad x \in \tilde{\Omega},
\]
yields that for each \( \varphi \in X \), \( f(\cdot, \varphi) \in PAPS^1(\mathbb{R}, X, \mu) \). On the other hand, by assumptions on \( b \), we have
\[
\frac{2e^\delta}{e^\delta - 2[b]_{BS}e^\delta - 1} (|c|_{BS} + \|H\|_{\infty}) > \frac{2e^\delta}{e^\delta - 1} (|c|_{BS} + \|H\|_{\infty}).
\]

Therefore, from (3.24), we obtain that (3.15) and (3.16) hold, respectively. Notice that hypotheses (H1)–(H4) are all satisfied. Finally, we conclude using Theorem 3.17.

### 4 Conclusion and Future Research

The results obtained so far in the current work are focused on a theoretical framework about the existence and uniqueness of \( \mu \)-pseudo-almost periodic solutions to large classes of semilinear fractional inclusions and semilinear nonautonomous evolution equations, respectively, under the weak assumptions that the forcing terms are only \( \mu \)-pseudo-almost periodic in Stepanov sense with respect to the second variable and not necessarily uniformly Lipschitzian with respect to the second argument. A key fact in our approach is the new composition result established in Section 2. It is well-known from the literature that composition results for Stepanov \( \mu \)-pseudo-almost periodic functions require the uniform Lipschitz continuity with respect to the second argument. That is, our new composition result needs only the continuity assumption with respect to the second variable.
In what follows, we present some of the open problems that arise in connection with the topics we have addressed and analyzed within this work:

- **Stochastic nonlinear boundary evolution equations.** Consider the following nonautonomous boundary stochastic evolution equations in a real separable Hilbert space $H$

$$du(t) = A(t)u(t)dt + f(t, u(t))dt + \gamma(t)dW(t), \quad t \in \mathbb{R},$$

(4.1)

where $f$ and $\gamma$ are $\mu$-pseudo-almost periodic processes. $W(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. We are interested in the study of existence and uniqueness of $\mu$-pseudo-almost periodic (resp. $\mu$-pseudo-almost automorphic) solutions; see Gu et al.$^{37}$ and references therein for more details.

- **Time-fractional semilinear boundary evolution equations.** Consider the following autonomous boundary time-fractional evolution equations in a complex Banach space $X$

$$D_\gamma^\alpha u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

(4.2)

where $1 < \alpha \leq 2$, $(A, D(A))$ is a densely defined sectorial operator acting in $X$ and $D_\gamma^\alpha$ is the Riemann–Liouville or Caputo fractional derivatives of order $\gamma$. The function $f$ is $\mu$-pseudo-almost periodic (resp. $\mu$-pseudo-almost automorphic). We are interested in the study of existence and uniqueness of positive $\mu$-pseudo-almost periodic solutions using fixed point arguments in cones; see Seemab et al.$^{38}$ and references therein for more details. Notice that the positivity of solutions is required especially when the states functions defined a density or a specific rate.

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**CONFLICT OF INTEREST**

The current work does not have any conflicts of interest.

**ORCID**

Kamal Khalil

https://orcid.org/0000-0003-0666-2219

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APPENDIX A: $\mu$-PSEUDO-ALMOST PERIODIC FUNCTIONS

In this section, we provide preliminaries about the concept of $\mu$-pseudo-almost periodic functions, in Bohr sense and Stepanov sense, respectively, needed to elaborate the main results of this work.
A.1 | Almost periodic functions

A continuous function \( f : \mathbb{R} \to X \) is said to be almost periodic if for every \( \varepsilon > 0 \), there exists \( l_\varepsilon > 0 \) such that for every \( a \in \mathbb{R} \), there exists \( \tau \in [a, a + l_\varepsilon] \) satisfying \( \| f(t + \tau) - f(t) \| < \varepsilon \), \( t \in \mathbb{R} \). The space of all such functions is denoted by \( AP(\mathbb{R}, X) \).

Let \( 1 \leq p < \infty \). A function \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) is said to be Stepanov \( p \)-bounded if

\[
\sup_{t \in \mathbb{R}} \left( \int_{[t,t+1]} \| f(s) \|^p ds \right)^{\frac{1}{p}} = \sup_{t \in \mathbb{R}} \left( \int_{[0,1]} \| f(t+s) \|^p ds \right)^{\frac{1}{p}} < \infty.
\]

The space of all such functions is denoted by \( BS^p(\mathbb{R}, X) \); equipped with the norm

\[
\| f \|_{BS^p} := \sup_{t \in \mathbb{R}} \left( \int_{[t,t+1]} \| f(s) \|^p ds \right)^{\frac{1}{p}} = \sup_{t \in \mathbb{R}} \| f(t + \cdot) \|_{L^p([0,1], X)}.
\]

\( BS^p(\mathbb{R}, X) \) is a Banach space. The following inclusions hold:

\[
BC(\mathbb{R}, X) \subseteq BS^p(\mathbb{R}, X) \subseteq L^p_{\text{loc}}(\mathbb{R}, X).
\]

**Definition A.1** (Bochner transform). Let \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) for \( 1 \leq p < \infty \). The Bochner transform of \( f(\cdot) \) is the function \( f^b : \mathbb{R} \to L^p([0,1], X) \), defined by

\[
(f^b(t))(s) := f(t + s) \quad \text{for} \quad s \in [0,1], \ t \in \mathbb{R}.
\]

Now, we recall the definition of Stepanov \( p \)-almost periodicity:

**Definition A.2.** Let \( 1 \leq p < \infty \). A function \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) is said to be Stepanov \( p \)-almost periodic (\( SP^p \)-almost periodic, for short), if for every \( \varepsilon > 0 \), there exists \( l_\varepsilon > 0 \), such that for every \( a \in \mathbb{R} \), there exists \( \tau \in [a, a + l_\varepsilon] \) satisfying

\[
\left( \int_{[t,t+1]} \| f(s + \tau) - f(s) \|^p ds \right)^{\frac{1}{p}} < \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.
\]

The space of all such functions is denoted by \( APS^p(\mathbb{R}, X) \).

**Remark A.3** (Previous studies\(^7,17,39\)).

(i) Every (Bohr) almost periodic function is \( SP^p \)-almost periodic for \( 1 \leq p < \infty \). The converse is not true in general.

(ii) For all \( 1 \leq p_1 \leq p_2 < \infty \), if \( f \) is \( SP^{p_2} \)-almost periodic, then \( f \) is \( SP^{p_1} \)-almost periodic.

(iii) The Bochner transform of an \( X \)-valued function is an \( L^p([0,1], X) \)-valued function. Moreover, a function \( f(\cdot) \) is \( SP^p \)-almost periodic if and only if \( f^b(\cdot) \) is (Bohr) almost periodic.

(iv) A function \( \varphi(t, s) \), defined for \( t \in \mathbb{R}, s \in [0,1] \), is the Bochner transform of a function \( f(\cdot) \) (i.e., \( \exists f : \mathbb{R} \to X \) such that \( f^b(t)(s) = \varphi(t, s), \ t \in \mathbb{R}, s \in [0,1] \)) if and only if \( \varphi(t + \tau, s - \tau) = \varphi(t, s) \) for all \( t \in \mathbb{R}, s \in [0,1] \) and \( \tau \in [s - 1, s] \).

A sufficient condition for a Stepanov almost periodic function to be Bohr almost periodic is given in the next theorem:

**Theorem A.4** (Amerio and Prouse\(^39\)). Let \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) for \( 1 \leq p < \infty \). If \( f \) is \( SP^p \)-almost periodic and uniformly continuous, then \( f \) is almost periodic.

**Definition A.5.** Let \( 1 \leq p < \infty \). A function \( f : \mathbb{R} \times X \to Y \) such that \( f(\cdot, x) \in L^p_{\text{loc}}(\mathbb{R}, Y) \) for each \( x \in X \) is said to be \( SP \)-almost periodic in \( t \) uniformly with respect to \( x \) in \( X \) if for each compact set \( K \) in \( X \) and for each \( \varepsilon > 0 \), there exists
\( l_{a,k} > 0 \) such that for every \( a \in \mathbb{R} \), there exists \( \tau \in [a, a + l_{a,k}] \) satisfying

\[
\sup_{x \in K} \left( \int_{[t,t+1]} \| f(s + \tau, x) - f(s, x) \|^p \, ds \right)^{\frac{1}{p}} < \varepsilon \text{ for all } t \in \mathbb{R}.
\]

The space of all such functions is denoted by \( A P S^p_U(\mathbb{R} \times X, Y) \).

### A.2 \( \mu \)-Pseudo-almost periodic functions

In this subsection, we provide the main properties of \( \mu \)-ergodic functions and (Stepanov) \( \mu \)-pseudo-almost periodic functions. We will use the following assumption on the measure \( \mu \in \mathcal{M} \):

(M) For all \( \tau \in \mathbb{R} \), there exist a number \( \beta > 0 \) and a bounded interval \( I \) such that

\[
\mu([a + \tau : a \in A]) \leq \beta \mu(A), \quad \text{provided } A \in B(\mathbb{R}) \text{ and } A \cap I = \emptyset.
\]

**Remark A.6.** In particular, the Lebesgue measure \( \lambda \) on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) belongs to \( \mathcal{M} \) and it satisfies the hypothesis (M).

**Definition A.7** (Blot et al\(^5\)). Let \( \mu \in \mathcal{M} \). A bounded continuous function \( f : \mathbb{R} \to X \) is said to be \( \mu \)-ergodic if

\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \| f(t) \| \, d\mu(t) = 0.
\]

The space of all such functions is denoted by \( \mathcal{E}(\mathbb{R}, X, \mu) \).

**Example A.8** (Blot et al\(^5\)). (1) Any ergodic function that belongs to the space \( P A P_0(\mathbb{R}, X) \), introduced by Zhang\(^4\), is nothing else but a \( \mu \)-ergodic function in the particular case when \( \mu \) is the Lebesgue measure. (2) Let \( \rho : \mathbb{R} \to [0, +\infty) \) be a Lebesgue measurable function. We define the positive measure \( \mu \) on \( B(\mathbb{R}) \) by

\[
\mu(A) := \int_A \rho(t) \, dt \quad \text{for } A \in B(\mathbb{R}),
\]

where \( dt \) denotes the Lebesgue measure. The measure \( \mu \) is absolutely continuous with respect to \( dt \) and the function \( \rho \) is called the Radon–Nikodym derivative of \( \mu \) with respect to \( dt \). In that case, \( \mu \in \mathcal{M} \) if and only if the function \( \rho \) is locally Lebesgue integrable on \( \mathbb{R} \) and satisfies

\[
\int_{\mathbb{R}} \rho(t) \, dt = +\infty.
\]

**Definition A.9** (Es-sebbar and Ezzinbi\(^8\)). Let \( \mu \in \mathcal{M} \). A function \( f \in B S^p(\mathbb{R}, X) \) for \( 1 \leq p < \infty \) is said to be Stepanov \( \mu \)-ergodic (\( \mu \)-S\(^p\)-ergodic, for short) if

\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \int_{[t,t+1]} \| f(s) \|^p \, ds \right)^{\frac{1}{p}} \, d\mu(t)
\]

\[
= \lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \| f^b(t) \|^p \, d\mu(t) = 0.
\]

The space of all such functions is denoted by \( \mathcal{E}^p(\mathbb{R}, X, \mu) \).

**Remark A.10.** Using (A2), we obtain that \( f \in \mathcal{E}^p(\mathbb{R}, X, \mu) \) if and only if \( f^b \in \mathcal{E}(\mathbb{R}, L^p([0, 1], X), \mu) \).

**Proposition A.11** (Es-sebbar and Ezzinbi\(^8\)). Let \( 1 \leq p < \infty \) and \( \mu \in \mathcal{M} \) satisfy (M). Then, the following holds:

(i) \( \mathcal{E}^p(\mathbb{R}, X, \mu) \) is translation invariant.
Let \( \mu \in \mathcal{M} \) and \( 1 \leq p < \infty \). A function \( f : \mathbb{R} \times X \to Y \) such that \( f(\cdot, x) \in BS^p(\mathbb{R}, Y) \) for each \( x \in X \) is said to be \( \mu \)-\( S^p \)-ergodic in \( t \) uniformly with respect to \( x \) in \( X \) if the following holds:

(i) For all \( x \in X \), \( f(\cdot, x) \in \mathcal{E}^p(\mathbb{R}, Y, \mu) \).

(ii) \( f \) is \( S^p \)-uniformly continuous with respect to the second argument on each compact subset \( K \) in \( X \), namely, for every \( \epsilon > 0 \), there exists \( \delta_{K, \epsilon} > 0 \) such that for all \( x_1, x_2 \in K \), we have

\[
\|x_1 - x_2\| \leq \delta_{K, \epsilon} \Rightarrow \left( \int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|^p ds \right)^{\frac{1}{p}} \leq \epsilon \quad \text{for all } t \in \mathbb{R}. \tag{A3}
\]

Denote by \( \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu) \) the set of all such functions.

Now, we give the important properties of \( \mu \)-pseudo-almost periodic functions and \( S^p \)-\( \mu \)-pseudo-almost periodic functions.

**Definition A.13** (Blot et al\(^5\)). Let \( \mu \in \mathcal{M} \). A continuous function \( f : \mathbb{R} \to X \) is said to be \( \mu \)-pseudo-almost periodic if \( f \) can be decomposed in the form:

\[ f = g + \varphi, \]

where \( g \in AP(\mathbb{R}, X) \) and \( \varphi \in \mathcal{E}(\mathbb{R}, X, \mu) \). The space of all such functions is denoted by \( PAP(\mathbb{R}, X, \mu) \).

**Proposition A.14** (Blot et al\(^5\)). Let \( \mu \in \mathcal{M} \) satisfy (M). Then, the following holds:

(i) The decomposition of a \( \mu \)-pseudo-almost periodic in the form \( f = g + \varphi \), where \( g \in AP(\mathbb{R}, X) \) and \( \varphi \in \mathcal{E}(\mathbb{R}, X, \mu) \), is unique.

(ii) \( PAP(\mathbb{R}, X, \mu) \) equipped with the sup-norm is a Banach space.

(iii) \( PAP(\mathbb{R}, X, \mu) \) is translation invariant.

**Definition A.15**. Let \( \mu \in \mathcal{M} \). A function \( f : \mathbb{R} \times X \to Y \) such that \( f(\cdot, x) \in BS^p(\mathbb{R}, Y) \) for each \( x \in X \) is said to be \( S^p \)-\( \mu \)-pseudo-almost periodic in \( t \) uniformly with respect to \( x \) if \( f \) can be decomposed in the form:

\[ f = g + \varphi, \]

where \( g \in APS^p U(\mathbb{R} \times X, Y) \) and \( \varphi \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu) \). The space of all such functions will be denoted by \( PAPS^p U(\mathbb{R}, X, \mu) \).

Furthermore, we need the following preliminary results obtained in Khalil.\(^27\)

**Lemma A.16**. Let \( 1 \leq p < +\infty \) and \( f : \mathbb{R} \times X \to Y \) be such that \( f(\cdot, x) \in L^p_{loc}(\mathbb{R}, Y) \) for each \( x \in X \). Then, \( f \in APS^p U(\mathbb{R} \times X, Y) \) if and only if the following holds:

(i) For each \( x \in X \), \( f(\cdot, x) \in APS^p(\mathbb{R}, Y) \).

(ii) \( f \) is \( S^p \)-uniformly continuous with respect to the second argument on each compact subset \( K \) in \( X \) in the following sense: for all \( \epsilon > 0 \), there exists \( \delta_{K, \epsilon} > 0 \) such that for all \( x_1, x_2 \in K \), one has

\[
\|x_1 - x_2\| \leq \delta_{K, \epsilon} \Rightarrow \left( \int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|^p ds \right)^{\frac{1}{p}} \leq \epsilon \quad \text{for all } t \in \mathbb{R}. \tag{A4}
\]

It is clear that Lemma A.16 implies the following:

**Proposition A.17**. Let \( \mu \in \mathcal{M} \) and \( f \in PAPS^p U(\mathbb{R} \times X, Y, \mu) \), for \( 1 \leq p < +\infty \). Then, the following holds:

(i) for each \( x \in X \), \( f(\cdot, x) \in PAPS^p(\mathbb{R}, Y, \mu) \);
(ii) \( f \) is \( S^p \)-uniformly continuous with respect to the second argument on each compact subset \( K \) in \( X \); namely, for each \( \varepsilon > 0 \) and for each compact set \( K \) in \( X \), there exists \( \delta_{K, \varepsilon} > 0 \) such that for all \( x_1, x_2 \in K \), we have

\[
\|x_1 - x_2\| \leq \delta_{K, \varepsilon} \Rightarrow \left( \int_0^{t+1} \| f(s, x_1) - f(s, x_2) \|^p ds \right)^{\frac{1}{p}} \leq \varepsilon \text{ for all } t \in \mathbb{R}.
\] (A5)

Next, we provide some examples of Stepanov \( \mu \)-pseudo-almost periodic functions of order \( 1 \leq p < \infty \).

**Example A.18.** Let \( X \) be any Banach space and let \( \mu \) be a measure with the Radon–Nikodym derivative \( \theta \) defined by

\[
\theta(t) = \begin{cases} 
\varepsilon^t & \text{for } t \leq 0, \\
1 & \text{for } t > 0.
\end{cases}
\] (A6)

From Blot et al.\(^5\), example 3.6 the measure \( \mu \) satisfies the hypothesis \( (M) \). Consider the function \( \Phi : \mathbb{R} \to \mathbb{R} \) given by \( \Phi(t) := \Phi_1(t) + \Phi_2(t) \) with \( \Phi_2(t) = \left( \arctan(t) - \frac{\pi}{2} \right) \) and

\[
\Phi_1(t) = \sum_{n \geq 1} \beta_n(t),
\]

such that for every \( n \geq 1 \),

\[
\beta_n(t) = \sum_{i \in P_n} H(n^2(t - i)),
\]

with \( P_n = 3^n(2\mathbb{Z} + 1) \) and \( H \in C_0^\infty(\mathbb{R}, \mathbb{R}) \) with support in \( \left( -\frac{1}{2}, \frac{1}{2} \right) \) such that

\[
H \geq 0, \ H(0) = 1 \text{ and } \int_{-\frac{1}{2}}^{\frac{1}{2}} H(s) ds = 1.
\]

By the proof in Blot et al.\(^5\), section 5 the function \( t \mapsto \arctan(t) - \frac{\pi}{2} \) belongs to \( \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu) \). Otherwise, from Tarallo,\(^40\) \( \Phi_1 \in C^\infty(\mathbb{R}, \mathbb{R}) \), but \( \Phi_1 \not\in AP(\mathbb{R}) \) since it is not bounded. However, \( \Phi_1 \in APS^1(\mathbb{R}) \).

Let \( \Psi_1 : \mathbb{R} \to \mathbb{C} \) be any essentially bounded function which is \( S^p \)-almost periodic \((1 \leq p < \infty)\) but not almost periodic; for example, if \( \alpha, \beta \in \mathbb{R} \) and \( \alpha \beta^{-1} \) is a well-defined irrational number, then we can take

\[
\Psi_1(t) = \sin \left( \frac{1}{2 + \cos \alpha t + \cos \beta t} \right), \ t \in \mathbb{R},
\]

or

\[
\Psi_1(t) = \cos \left( \frac{1}{2 + \cos \alpha t + \cos \beta t} \right), \ t \in \mathbb{R}.
\]

The function \( \Psi_1 \) is bounded continuous but not uniformly continuous; see Basit and Günzler\(^41\) for more details. Set \( \Psi : \mathbb{R} \to \mathbb{C} \) defined such that \( \Psi(t) = \Psi_1(t) + \Phi_2(t), t \in \mathbb{R} \). Thus, \( \Psi \) yields a \( S^p \)-\( \mu \)-pseudo-almost periodic scalar function (for all \( 1 \leq p < \infty \)). Further on, let \( h : X \to X \) be any continuous function. Then, the functions \( f(t, x) := \Psi(t)h(x) \) and \( g(t, x) := \Phi(t)h(x) \) for \( t \in \mathbb{R} \) and \( x \in X \) define two examples of (purely) Stepanov \( \mu \)-pseudo-almost periodic \( X \)-valued functions. In particular, \( f \) is \( S^1 \)-\( \mu \)-pseudo-almost periodic and \( g \) is \( S^2 \)-\( \mu \)-pseudo-almost periodic.