Invariant Vectors for Weak Endoscopic and Saito-Kurokawa Lifts to GSp(4)

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Let $\mathcal{A}$ be the ring of adele over a totally real number field $F/\mathbb{Q}$. For cohomological cuspidal automorphic irreducible representations of $GSp(4, \mathcal{A})$ coming from weak endoscopic or Saito-Kurokawa Lifts, we determine the local invariant spaces under the first principal congruence subgroup at the non-archimedean places of $F$. For $F = \mathbb{Q}$ this gives rise to dimension formulas regarding certain subspaces of the inner cohomology of the genus two Shimura variety corresponding to the principal congruence subgroup of level $N = 2$. We prove the conjectures made by Bergström, Faber and van der Geer in a recent paper.

Introduction

Fix a totally real number field $F$ with adele ring $\mathcal{A} = \prod_v F_v$. Suppose $\sigma$ is a generic irreducible admissible representation of $GL(2, F_v)$ at a non-archimedean place $v$. By a result of Casselman there is a unique ideal $p_v^n \subseteq o_v$, called the level of $\sigma$, such that the space of invariants in $\sigma_v$ under the congruence subgroup

$$K_0(p_v^n) = \{g \in GL(2, o_v) \mid g \equiv (\ast \ast) \mod p_v^n\}$$

is one-dimensional [3]. B. Roberts and R. Schmidt have determined the corresponding invariant spaces for paramodular subgroups of $GSp(4, F_v)$ [13]. In this article we discuss results obtained in [14] concerning invariant spaces of $GSp(4, F_v)$-representations under the first principal congruence subgroup

$$K(p_v) = \ker(GSp(4, o_v) \to GSp(4, o_v/p_v)).$$

For a cuspidal automorphic irreducible representation $\pi = \pi_{\infty}\pi_{\text{fin}}$ of $GSp(4, \mathcal{A})$ we assume there are cuspidal automorphic irreducible representations $\sigma_1, \sigma_2$ of $GL(2, \mathcal{A})$ with equal central character such that for almost every place $v$

$$L(\pi_v, s) = L(\sigma_{1,v}, s) \cdot L(\sigma_{2,v}, s).$$
then we call \( \pi \) a weak endoscopic lift attached to \( \sigma = (\sigma_1, \sigma_2) \), compare \cite[§5.2]{26}. We also consider the case where \( \pi \) arises via the Saito-Kurokawa lift in the sense of \cite{10} from some cuspidal automorphic irreducible representation \( \sigma \) of \( PGL(2, \mathcal{A}) \). For both lifts we address the question: How can the local invariant-space \( \pi_v^{K(p_v)} \) be described as a representation of \( GSp(4, o_v/p_v) \)? This means we study the image of \( \pi \) under the parahoric restriction functor of taking \( K \)-parahoric restriction.

To an admissible complex linear representation of \( GSp \) the functor \( F_{K GSp} \) is preserved under the action of \( K \)-parahoric restriction functor of taking \( GSp \) a representation of \( GSp \) reducible representations of \( GSp \) arises via the Saito-Kurokawa lift in the sense of \cite{10}

Then we call \( \pi \) lifts we address the question: How can the local invariant-space \( \pi_v^{K(p_v)} \) be described as a representation of \( GSp(4, o_v/p_v) \)? This means we study the image of \( \pi \) under the parahoric restriction functor of taking \( K(p_v) \)-invariants:

\[
\mathcal{F}_{p_v} : \text{Rep}(GSp(4, F_v)) \rightarrow \text{Rep}(GSp(4, o_v/p_v))
\]

\[
(\pi, V_\pi) \mapsto (\pi|_{GSp(4, o_v)}, V_\pi^{K(p_v)}).
\]

To an admissible complex linear representation of \( GSp(4, F_v) \) it assigns the space of \( K(p_v) \)-invariants. Since \( K(p_v) \subseteq GSp(4, o_v) \) is a normal subgroup, this invariant space is preserved under the action of \( GSp(4, o_v) \) and defines a representation of \( GSp(4, o_v/p_v) \). The functor \( \mathcal{F}_{p_v} \) is additive and exact because \( K(p_v) \) is compact. For non-cuspidal irreducible representations of \( GSp(4, F_v) \) at non-archimedean local number fields \( F_v \) the representation \( \mathcal{F}_{p_v}(\pi, V_\pi) \) has been determined explicitly \cite[Tab. 2.2]{14}. The dimension of the representation \( \mathcal{F}_{p_v}(\pi_v) \) has also been determined whenever \( \pi_v = \theta_-(\sigma_v) \) is the anisotropic theta-lift of an irreducible representation \( \sigma_v \) of \( GSO_{4,0}(F_v) \) \cite[Thm. 3.41]{14}. For non-cuspidal representations at non-archimedean places \( v \) of odd residue characteristic this functor has previously been studied by Breeding \cite{3}.

In Section 2 we recall the local endoscopic L-packets of admissible irreducible representations of \( GSp(4, F_v) \). For representations \( \pi_v \) in a local endoscopic L-packet at a non-archimedean place \( v \) we determine the representation \( \mathcal{F}_{p_v}(\pi_v) \) (Prop. 2.2). Under the condition that \( \sigma_{1,v} \) and \( \sigma_{2,v} \) admit non-zero invariants under \( K_0^{(1)}(p_v) \subseteq GL(2, o_v) \) this implies that \( \pi_v \) admits non-zero invariants under the modified principal congruence subgroup \( K(p_v) \) by Cor. 2.3. The global endoscopic L-packets attached to a cuspidal automorphic irreducible representation \( \sigma \) of \( M(\mathcal{A}) = GSO_{2,2}(\mathcal{A}) \) are introduced in Section 3. They consist of cuspidal automorphic irreducible representations \( \pi \) of \( GSp(4, \mathcal{A}) \) that satisfy

\[
L(\pi_v, s) = L(\sigma_{1,v}, s) \cdot L(\sigma_{2,v}, s)
\]

at almost every place \( v \). The local factors \( \pi_v \) are then contained in the local L-packets attached to \( \sigma_v \) \cite[Thm. 5.2]{26}. We determine the action of \( GSp(4, \mathcal{A}) \) on the invariant space \( \pi^{K(S)} \) explicitly. For the modified principal congruence subgroup we obtain the corresponding global version (Theorem 3.3). This can be applied to construct Yoshida lifts like in \cite[Thm. 7.2]{28} with principal congruence subgroup level \( N = \text{lcm}(N_1, N_2) \) attached to cuspidal newforms \( f_i \) with squarefree level \( N_i \). Tehrani \cite{22} has obtained related results, he has proved the existence of non-zero invariant vectors in \( \pi \) under paramodular subgroups.

In Section 4 we make the analogous computations for Saito-Kurokawa lifts in the sense of Piatetski-Shapiro \cite{10}. Let \( \sigma \) be a cuspidal automorphic irreducible representation of \( GL(2, \mathcal{A}) \) and let \( S \) be a finite set of places \( v \) where \( \sigma_v \) is in the discrete series such
that $(-1)^{#S} = \epsilon(\sigma, 1/2)$. Let $\sigma_S \in \text{Irr}(GL(2, A))$ be the irreducible constituent of $1_A \times 1_A$ such that $\sigma_{S,v}$ is in the discrete series if and only if $v \in S$. Via local theta-lifts one constructs an irreducible automorphic representation $\pi$ of $GSp(4, \mathbb{A})$ with spiner $L$-function

$$L(\pi_v, s) = L(\sigma_v, s) \cdot L(\sigma_{S,v}, s)$$

at almost every place $v$. This $\pi$ is cuspidal if and only if $S$ is non-empty or $L(\sigma, 1/2) = 0$. Under the condition that $\sigma_v$ admits non-zero invariants under the Iwahori subgroup of $GL(2, F_v)$, this implies that $\pi_v$ admits non-zero invariants under the first principal congruence subgroup $K'(p_v) \subseteq GSp(4, \mathfrak{o}_v)$. In a more classical language this means we can control the principal congruence subgroup level of Saito-Kurokawa Lifts from elliptic cuspforms of squarefree level to Siegel cuspforms of genus two (Cor. 4.5). We can completely determine the action of $GSp(4, \mathfrak{o}_v)$ on the space of invariants in $\pi_v$ under the principal congruence subgroup $K(p_v)$ (Prop. 4.3).

In Section 5 we consider the Shimura variety

$$X_K = GSp(4, \mathbb{Q}) \backslash GSp(4, \mathbb{A}) / KK_\infty$$

over $F = \mathbb{Q}$, where $K_\infty$ is the stabilizer of $i \cdot I_2 \in \mathbb{H}_2$ in the Siegel upper halfspace $\mathbb{H}_2$ of genus two and $K \subseteq GSp(4, \mathbb{Z})$ is the principal congruence subgroup of level $N = 2$. For the local system $V_\lambda$ over $GSp(4, \mathbb{Q})$ with weight $\lambda = (\lambda_1, \lambda_2)$ we can decompose the endoscopic and the Saito-Kurokawa part of the inner cohomology $H^*(X_K, V_\lambda)$ in terms of $Sp(4, \mathbb{Z}/2\mathbb{Z})$-representations. This proves the conjectures made by Bergström, Faber and van der Geer [1]. Indeed, the first equation of Corollary 5.4 together with the natural isomorphism

$$H^{(3,0)}_!(X_K, V_\lambda) \cong S^{(2)}_{(k_1-k_2, k_2)}(\Gamma(2)(2))$$

proves Conjecture 6.4 of loc. cit. and that implies Conjecture 6.1. Note that Conjecture 6.1 asks for the equality of $L$-factors at every non-archimedean place, this is implied by Prop. 3.4. The second equation of Corollary 5.4 proves Conjecture 7.1. The analogous question for Saito-Kurokawa lifts is answered by Corollary 5.7, proving Conjecture 6.6 and Conjecture 7.4 of loc. cit. For the case $\lambda_1 > \lambda_2$ Conjecture 7.2 of loc. cit. is answered by Corollary 5.5. For $\lambda_1 = \lambda_2$ we have Corollary 5.8 describing the sum of the dimensions of Saito-Kurokawa parts in the inner cohomology.

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1 Preliminaries

For a totally real number field $F/\mathbb{Q}$ with ring of integers $\mathcal{o}$ let $\mathbb{A} = \prod_v F_v$ be its adele ring. At the non-archimedean places $v$ let $\mathcal{o}_v$ and $p_v$ denote the ring of $F_v$-integers and its maximal ideal. The compact ring $\prod_{v < \infty} \mathcal{o}_v \subseteq \mathbb{A}$ will also be denoted $\mathfrak{o}$. A non-archimedean place is sometimes called odd or even depending on its residue characteristic being odd or even, respectively. The valuation character of $F_v^\times$ is also denoted by $\nu_v(x) = |x|_v$. Arbitrary nontrivial quadratic characters of $F_v^\times$ are denoted by $\xi$. We also write $\xi_u$ for the unramified and $\xi_t$ for one of the two tamely ramified quadratic characters, such a $\xi_t$ only exists for odd residue characteristic. The group of symplectic similitudes $GSp(2n)$ over $\mathcal{o}$ is

$$GSp(2n) = \{ g \in \text{Mat}_{2n \times 2n} \mid gJg^t = \lambda J \text{ for } \lambda \in GL(1) \} \text{ with } J = (-I_n, I_n)$$

with similitude character $\text{sim} : g \mapsto \lambda$. For each non-archimedean place $v$ let

$$K(p_v) := \{ g \in GSp(4, \mathcal{o}_v) \mid g \equiv I_4 \mod p_v \} = \ker(GSp(4, \mathcal{o}_v) \to GSp(4, \mathcal{o}_v/p_v))$$

be the first principal congruence subgroup of $GSp(4, \mathcal{o})$ and let

$$K'(p_v) := \{ g \in GSp(4, \mathcal{o}_v) \mid \exists \lambda \in \mathcal{o}_v^\times : g \equiv (I_2, I_2 \lambda I_2) \mod p_v \}$$

be the first modified principal congruence subgroup. The modified principal congruence subgroups satisfy the requirement $\text{sim}(K'(p_v)) = \mathcal{o}_v^\times$ for strong approximation. The analogous subgroup of $GL(2, \mathcal{o}_v)$ will be denoted $K^{(1)}(p_v) = \ker(GL(2, \mathcal{o}_v) \to GL(2, \mathcal{o}_v/p_v))$. For the classical notion of level in the sense of [5] depends on the congruence subgroup $K_0^{(1)}(p_v^\#) = \{ g \in GL(2, \mathcal{o}_v) \mid g \equiv (*, *) \mod p_v^\# \}$.

By a representation $\pi$ of $GSp(2n, F_v)$ at a non-archimedean $v$ we will always mean an admissible complex linear representation. The twist of $\pi$ by a character $\chi : F_v^\times \to \mathbb{C}^\times$ will be denoted by $\chi \cdot \pi := (\chi \circ \text{sim}) \otimes \pi$. For non-cuspidal irreducible representations of $GSp(4, F_v)$ we use the notation of Roberts and Schmidt [13]. The Steinberg representation of $GSp(2n, F_v)$ will be denoted $\text{St}_{GSp(2n, F_v)}$ and for the trivial representation we write $1_{GSp(2n, F_v)}$. Suppose $(\pi, V_\pi)$ is a representation of $GSp(4, F_v)$. The restriction of $\pi$ to $GSp(4, \mathcal{o}_v)$ stabilizes the space $V^{K(p_v)}_\pi$ of $K(p_v)$-invariant vectors. This defines the representation

$$F_{p_v}(\pi, V_\pi) := (\pi|_{GSp(4, \mathcal{o}_v)}, V^{K(p_v)}_\pi) \quad (1)$$

of the group $GSp(4, \mathbb{F}_q) \cong GSp(4, \mathcal{o}_v/p_v)$. The functor of parahoric restriction

$$F_{p_v} : \text{Rep}(GSp(4, F_v)) \to \text{Rep}(GSp(4, \mathbb{F}_q)) \quad (2)$$

is additive, exact and it commutes with parabolic induction [14, Cor. 2.19]. It maps cuspidal irreducible representations to cuspidal irreducible ones or to zero [14, Thm. 2.23]. For the non-cuspidal irreducible representation $\pi$ of $GSp(4, F_v)$ the representation
The finite field of order $q$ with odd $q$ have been classified by Shimoda [19]. For even $q$ there is an isomorphism $GSp(4, \mathbb{F}_q) \cong Sp(4, \mathbb{F}_q) \times \mathbb{F}_q^\times$, so every irreducible representation $\rho$ of $GSp(4, \mathbb{F}_q)$ is a product of its central character $\omega_\rho$ and its restriction to $Sp(4, \mathbb{F}_q)$. For representations of $Sp(4, \mathbb{F}_q)$ with even $q$ we use the notation of Enomoto [7]. In [1] a different notation for irreducible representations of $Sp(4, \mathbb{F}_q)$ is used: Fix an isomorphism $Sp(4, \mathbb{F}_2) \cong \Sigma_6$ to the symmetric group in six letters and label the irreducible representations by partitions of 6, which describe the corresponding Young-Tableaux [23, §110]. The dictionary is given by Table 11. The correspondence is only unique up to the outer automorphism of $\Sigma_6$. 

| $Sp(4, \mathbb{F}_2)$ | $\theta_0$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\chi_5(1)$ | $\chi_8(1)$ | $\chi_9(1)$ | $\chi_{12}(1)$ | $\chi_{13}(1)$ |
|----------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\Sigma_6$           | [6]       | [4, 2]    | [3, 2]    | [3, 2, 1] | [1^6]     | [2^2, 1^2]| [3^2]     | [2, 1^4]  | [4, 1^2]  | [3, 1^3]  |
| dim                  | 1         | 9         | 5         | 5         | 16        | 1         | 9         | 5         | 5         | 10        | 10        |

Table 1: Irreducible representations of $Sp(4, \mathbb{F}_2) \cong \Sigma_6$. 

$\mathcal{F}_v(\pi)$ has been determined in [14, Table 2.2]. A representation $\pi$ is called spherical, if it admits non-zero invariants under $GSp(2n, \mathfrak{o}_v)$. For a generic irreducible representation $\sigma_v$ of $GL(2, F_v)$ with unramified central character $\omega_v$ recall that its level is $p_v^n$ for the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $\sigma_v$ admits a non-zero subspace of $K_v(1)(p_v^n)$-invariants. If the level of $\sigma_v$ is $p_v$, then $\sigma_v$ is either $St_{GL(2, F_v)}$ or $\xi_v \cdot St_{GL(2, F_v)}$ [14, Prop. 5.21]. For each real place $v$, the discrete series representation of $PGL(2, F_v)$ with lowest weight $k \in 2\mathbb{Z}_{>0}$ is denoted $D(k - 1)$. There are two notions of $L$- and $\epsilon$-factors for irreducible representations of $GSp(4, F_v)$, the standard factors of degree 5 and the spinor factors of degree 4; we will use the spinor factors constructed by Piatetski-Shapiro [11]. We also consider the group 

$$M := GSO_{2,2} = GL(2) \times GL(2)/\Delta GL(1)$$

over $\mathfrak{o}$, where $\Delta GL(1)$ is the image of the antidiagonal embedding $GL(1) \hookrightarrow GL(2) \times GL(2)$. An irreducible representation $\sigma$ of $M(F_v)$ corresponds to a pair of two irreducible representations $\sigma_1, \sigma_2$ of $GL(2, F_v)$ with equal central character $\omega_{\sigma_1} = \omega_{\sigma_2}$. Let $\sigma^* := (\sigma_2, \sigma_1)$ be the representation obtained by switching the two factors. 

The set of isomorphism classes of cuspidal automorphic irreducible representations of $GSp(4, \mathfrak{o})$ is denoted $\mathcal{A}_0(GSp(2n, \mathfrak{o}))$. Irreducible representations $\pi, \pi'$ of $GSp(2n, \mathfrak{o})$ are weakly equivalent, if their local factors $\pi_v, \pi'_v$ are isomorphic at almost all places. A cuspidal automorphic irreducible representation $\pi$ of $GSp(2n, \mathfrak{o})$ is CAP, if it is weakly equivalent to a constituent of a globally parabolically induced representation. Among the automorphic cuspidal irreducible representation of $GSp(4, \mathfrak{o})$ the CAP representations are distinguished by the fact that their spinor $L$-function has poles [10, Thm. 2.2]. The factorization of an irreducible representation $\pi$ of $GSp(4, \mathfrak{o})$ into its archimedean and non-archimedean part is denoted by $\pi = \pi_{\infty} \otimes \pi_{\fin}$. 

The finite field of order $q$ is denoted $\mathbb{F}_q$. Irreducible representations of $GSp(4, \mathbb{F}_q)$ with odd $q$ have been classified by Shinoda [19]. For even $q$ there is an isomorphism $GSp(4, \mathbb{F}_q) \cong Sp(4, \mathbb{F}_q) \times \mathbb{F}_q^\times$, so every irreducible representation $\rho$ of $GSp(4, \mathbb{F}_q)$ is a product of its central character $\omega_\rho$ and its restriction to $Sp(4, \mathbb{F}_q)$. For representations of $Sp(4, \mathbb{F}_q)$ with even $q$ we use the notation of Enomoto [7]. In [1] a different notation for irreducible representations of $Sp(4, \mathbb{F}_q)$ is used: Fix an isomorphism $Sp(4, \mathbb{F}_2) \cong \Sigma_6$ to the symmetric group in six letters and label the irreducible representations by partitions of 6, which describe the corresponding Young-Tableaux [23, §110]. The dictionary is given by Table 11. The correspondence is only unique up to the outer automorphism of $\Sigma_6$. 

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The space of elliptic cuspforms with congruence group $\Gamma$ and weight $k$ is denoted $S_k(\Gamma)$. For the subset of newforms, which are eigenforms of the Hecke algebra, we write $S_k(\Gamma)^{\text{new}}$. The space of genus two Siegel cuspforms with congruence group $\Gamma \subseteq Sp(4, \mathbb{Z})$ and type $\text{Sym}^r(\text{std}) \otimes \det^k$ is $S^{(2)}_{(r,k)}(\Gamma)$.

2 The Local Endoscopic Lift

Fix a place $v$ of the totally real number field $F$. The local endoscopic character lift $r : R(M(F_v)) \to R(GSp(4, F_v))$ is defined on Grothendieck groups of virtual representations [26, p. 152]. For every preunitary generic irreducible representation $\sigma_v$ of $M(F_v)$, its endoscopic lift $r(\sigma_v)$ is a linear combination of finitely many irreducible constituents. The set of these constituents is the local $L$-packet attached to $\sigma_v$ [26, Def. 4.5]. The local $L$-packets of $\sigma_v = (\sigma_1,v,\sigma_2,v)$ and $\sigma_v^\ast = (\sigma_2,v,\sigma_1,v)$ coincide [26, Lem. 4.5].

Theorem 2.1 (Weissauer/Shelstad). Let $\sigma_v = (\sigma_1,v,\sigma_2,v)$ be a generic preunitary irreducible representation of $M(F_v)$. Fix even integers $r_1 > r_2 \geq 2$ and let $(k_1,k_2) = \left(\frac{1}{2}(r_1+r_2),\frac{1}{2}(r_1-r_2+4)\right)$.

i) If $\sigma_1,v$ and $\sigma_2,v$ are both in the discrete series, the local $L$-packet attached to $\sigma_v$ consists of a generic representation $\Pi_+(\sigma_v)$ and a non-generic representation $\Pi_-(\sigma_v)$.

a) For non-archimedean $v$ the representation $\Pi_+(\sigma_v) = \theta_+(\sigma_v)$ is the isotropic theta-lift of $\sigma_v$ and $\Pi_-(\sigma_v) = \theta_-(\sigma_v)$ is the anisotropic theta-lift of $\sigma_v$.

b) For archimedean $v$ suppose $\sigma_1,v$ is in the discrete series with weight $r_1$. The representation $\Pi_-(\sigma_v)$ is holomorphic with lowest $K$-type of weight $(k_1,k_2)$ and $\Pi_+(\sigma_v)$ is non-holomorphic with lowest $K$-type of weight $(k_1,2-k_2)$.

ii) If $\sigma_2,v = \mu_1 \times \mu_2$ is Borel induced from a pair of smooth characters $\mu_1,\mu_2$ of $F_v^\times$, then the local $L$-packet attached to $\sigma_v$ contains only the generic irreducible representation $\Pi_+(\sigma_v) = (\mu_1^{-1} \cdot \sigma_1,v) \rtimes \mu_1$.

Proof. For archimedean places i) is implied by Cor. 4.2 and Remark 4.7 in [26] and the proof is due to Shelstad [18]. For the $K$-types see [25, p. 68]. For non-archimedean $v$ i) is Thm. 4.5 in [26]. ii) is implied by Lemma 4.27 in [26].

The isotropic theta-lift $\theta_+(\sigma_v)$ is constructed in [12, §1(e)], it is denoted by $\Pi(\sigma)$ there. For the anisotropic theta-lift $\theta_-(\sigma_v)$ see [26, Def. 4.7]. The local $L$-packets at non-archimedean $v$ are given explicitly by [26, Thm. 4.5], compare Table 2. The Table contains all possible cases, since $\Pi_+(\sigma_1,v,\sigma_2,v) = \Pi_+(\sigma_2,v,\sigma_1,v)$. 

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In this section, the space \(\pi_K\) (Tables 2 and 3) contains principal congruence subgroups. Suppose \(\mu\) for Table 3 we additionally assume that the field \(o\) of \(K\) is non-zero. Proposition 2.2. See [14, Tables 4.1 and 4.2 and Thm. 4.11].

**Table 2: Local \(L\)-packets attached to generic irreducible representations \(\sigma_v\) of \(M(F_v)\) at non-archimedean \(v\).**

| \(GL(2, F_v)\)       | \(GL(2, F_v)\)       | \(GSp(4, F_v)\)       | \(GSp(4, F_v)\)       |
|-----------------------|-----------------------|-----------------------|-----------------------|
| \(\sigma_{1, v}\)     | \(\sigma_{2, v}\)     | \(\Pi_+(\sigma_v) = \theta_+(\sigma_v)\) | \(\Pi_-(\sigma_v) = \theta_-(\sigma_v)\) |
| \(\mu_1 \times \mu_2\) | \(\mu_3 \times \mu_4\) | \(\mu_1 \mu_3^{-1} \times \mu_2 \mu_3^{-1} \times \mu_3\) | —                      |
| \(\mu_1 \cdot \text{St}_{GL(2,F_v)}\) | \(\mu_3 \times \mu_4\) | \(\mu_1 \mu_3^{-1} \cdot \text{St}_{GL(2,F_v)} \times \mu_3\) | —                      |
| \(\rho_1\)            | \(\mu_3 \times \mu_4\) | \(\mu_3^{-1} \rho_1 \times \mu_3\) | —                      |
| \(\xi \mu \cdot \text{St}_{GL(2,F_v)}\) | \(\mu \cdot \text{St}_{GL(2,F_v)}\) | \(\delta(\text{St}_{GL(2,F_v)}, \nu_v^{-1/2} \times \mu_v^{-1/2})\) | cuspidal               |
| \(\mu \cdot \text{St}_{GL(2,F_v)}\) | \(\mu \cdot \text{St}_{GL(2,F_v)}\) | \(\tau(S, \nu_v^{-1/2} \mu)\) | \(\tau(T, \nu_v^{-1/2} \mu)\) |
| \(\rho_1\)            | \(\rho_2(\equiv \rho_1)\) | \(\tau(S, \rho_1)\) | \(\tau(T, \rho_1)\) |
| \(\rho_1\)            | \(\rho_2(\neq \rho_1)\) | cuspidal               | cuspidal               |

2.1 Invariant Vectors for local \(L\)-packets

Fix a non-archimedean place \(v\) with residue field \(F_q \cong o_v/p_v\) for the rest of this section. Recall that \(K(p_v) \subseteq GSp(4, o_v)\) and \(K^{(1)}(p_v) \subseteq GL(2, o_v)\) are the first principal congruence subgroups. Suppose \(\pi_v\) is an irreducible representation of \(GSp(4, F_v)\) in the local \(L\)-packet attached to a preunitary generic irreducible representation \((\sigma_v, V_{o_v})\) of \(M(F_v)\). In this section, the space \(\pi_v^{(1)}(p_v)\) of invariant vectors under the principal congruence subgroup \(K(p_v) \subseteq GSp(4, o)\) will be determined.

**Proposition 2.2.** Let \(\sigma_v\) be a generic irreducible representation of \(M(F_v)\) and let \(\pi_v\) be a constituent of the local \(L\)-packet of \(GSp(4, F_v)\) attached to \(\sigma_v\). Then

\[
\begin{align*}
\sigma_{1, v}^{K^{(1)}(p_v)} \neq 0 \text{ and } \sigma_{2, v}^{K^{(1)}(p_v)} \neq 0 \quad &\implies \quad \mathcal{F}_{p_v}(\pi_v) \text{ is given by Table 3} \\
\sigma_{1, v}^{K^{(1)}(p_v)} = 0 \text{ or } \sigma_{2, v}^{K^{(1)}(p_v)} = 0 \quad &\implies \quad \mathcal{F}_{p_v}(\pi_v) = 0.
\end{align*}
\]

**Proof.** See [14, Tables 4.1 and 4.2 and Thm. 4.11].

**Notation 2.3** (Tables 2 and 3). For \(j = 1, 2, 3, 4\) let \(\mu_j\) be smooth characters of \(F_v^\times\) and \(\rho_1\) and \(\rho_2\) be cuspidal irreducible representations of \(GL(2, F_v)\). The condition \(\omega_{\rho_1} = \omega_{\rho_2}\) is implicitly imposed.

For Table 3 we additionally assume that \(\sigma_{i, v}\) admits non-zero invariants under the first principal congruence subgroup \(K^{(1)}(p_v) \subseteq GL(2, o_v)\). That means the \(\mu_j\) are at most tamely ramified and the restriction of \(\mu_j\) to \(o_v^\times\) defines a character \(\tilde{\mu}_j\) of the residue field \(o_v^\times/(1 + p_v) \cong F_q^\times\). To the cuspidal irreducible representations \(\rho_i\) of \(GL(2, F)\) with non-zero \(K^{(1)}(p_v)\)-invariants we attach a character \(\Lambda_i\) of \(F_q^\times\) as follows: The action of \(GL(2, o_v)\) on \(\tilde{\rho}_i^{K^{(1)}(p_v)}\) defines a cuspidal irreducible representation \(\tilde{\rho}_i = \mathcal{F}_{p_v}(\rho_i)\) of
Table 3: Invariants of $\Pi_{\pm}(\sigma_v)$ under $K(p_v)$ for $\sigma^{K(1)}_{i,v}(p_v) \neq 0$.

$GL(2, F_q)$ [14, Thm. 2.23]. This $\hat{\rho}_i$ is parametrized by a character $\Lambda_i$ of $F_q^\times$, unique up to conjugation, such that $\text{tr}(\hat{\rho}_i) \left( \frac{1}{q-1} \right) = -\Lambda_i(\alpha) - \Lambda_i^q(\alpha)$ for $\alpha \in F_q^\times \backslash F_q^\times$. Each character $\Lambda$ of $F_q^\times$ with $\Lambda^{q+1} = 1$ factors over a character $\omega_{\Lambda}$ of the kernel of the norm $N_{F_q^2/F_q}: F_q^2 \to F_q$ such that $\Lambda(x) = \omega_{\Lambda}(x^{q-1})$. The quadratic character $\xi$ can be either $\xi_u$ or $\xi_t$.

For even $q$ fix an injective character $\hat{\theta} : F_q^\times \to \mathbb{C}^\times$ and let $\hat{\gamma}$ and $\hat{\eta}$ be its restriction to $F_q^\times$ and $\ker N_{F_q^2/F_q}$, respectively. Let $k_j \in \mathbb{Z}/(q-1)\mathbb{Z}$ be such that $\hat{\gamma}^{k_j} = \hat{\mu}_j$. We consider the natural embedding and projection

$$\kappa : \mathbb{Z}/(q + 1)\mathbb{Z} \hookrightarrow \mathbb{Z}/(q^2 - 1)\mathbb{Z}, \quad \kappa^* : \mathbb{Z}/(q^2 - 1)\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(q + 1)\mathbb{Z},$$

$$x \mapsto (q + 1)x \quad \text{ and } \quad x \mapsto x.$$

For the cuspidal representations let $l_i \in \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ be such that $\hat{\theta}^{l_i} = \Lambda_i$. If the central character of $\hat{\rho}_i$ is trivial, $\kappa^{-1}(l_i)$ is well-defined and satisfies $\omega_{\Lambda_i} = \hat{\eta}^{\kappa^{-1}(l_i)}$. Finally, for $\Lambda_1|_{F_q^2} = \Lambda_2|_{F_q^2}$ let $\tilde{k}_1 = \frac{q+2}{2}(\kappa^*(l_1 + l_2))$ and $\tilde{k}_2 = \frac{q+2}{2}(\kappa^*(l_1 - l_2))$.

**Corollary 2.4.** For a preunitary generic irreducible representation $\sigma_v$ of $M(F_v)$ let $\pi_v = \Pi_{\pm}(\sigma_{1,v}, \sigma_{2,v})$ be in the local $L$-packet attached to $\sigma_v$. Then we have:

- $\sigma_{1,v}$ and $\sigma_{2,v}$ spherical $\iff$ $\pi_v$ spherical,
- $\sigma_{1,v}^{K(1)}(p_v) \neq 0$ and $\sigma_{2,v}^{K(1)}(p_v) \neq 0$ $\implies$ $\pi_v^{K(p_v)} \neq 0$,
- $\pi_v(p_v) \neq 0$ $\iff$ $\pi_v^{K(p_v)} \neq 0$. 

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Proof. In the first line of Table 3, \( \pi_v \) is spherical if and only if the \( \mu_j \) are all unramified. The other cases are not spherical, compare [14, Tables 1.3 and 1.7].

\( \sigma_{i,v} \) admits non-zero invariants under \( K_0(p_v) \) if and only if \( \sigma_{i,v} \) is either unramified or isomorphic to \( \mu \cdot \text{St}_{GL(2)} \) for an unramified character \( \mu \) of \( \kappa^\times \). For the corresponding cases Table 3 implies \( \pi_{K_0(p_v)} \neq 0 \).

The subspace of \( K_0(p_v) \)-invariants in \( \pi_v \) is the subspace of \( \{ \text{diag}(1,1,* ,*) \} \)-invariants in \( \tilde{\pi}_v \), this is determined in [14, Tables 1.4 and 1.8].

3 The Global Weak Endoscopic Lift

In this section the weak endoscopic lift is recalled and its invariant spaces under principal congruence subgroups of squarefree level are described. We fix a generic cuspidal automorphic irreducible representation \( \sigma = (\sigma_1, \sigma_2) \in \mathcal{A}_0(M(\mathbb{A})) \) with \( \sigma_1 \not\sim \sigma_2 \).

Definition 3.1. A cuspidal automorphic irreducible representation \( \pi \in \mathcal{A}_0(GSp(4, \mathbb{A})) \) is called a weak endoscopic lift attached to \( \sigma \) if for almost every place \( v \) its local spinor \( L \)-factor is

\[
L(\pi_v, s) = L(\sigma_{1,v}, s)L(\sigma_{2,v}, s).
\]

The set of isomorphism classes of weak endoscopic lifts attached to \( \sigma \) is the global \( L \)-packet attached to \( \sigma \).

The representation \( \pi_+(\sigma) := \bigotimes_v \Pi_+(\sigma_v) \) is a weak endoscopic lift attached to \( \sigma \), so the \( L \)-packet is not empty [26, Thm. 4.3, p. 194]. A weak endoscopic lift attached to \( \sigma \) is also attached to \( \sigma^* = (\sigma_2, \sigma_1) \), but to no other cuspidal automorphic irreducible representation of \( M(\mathbb{A}) \) [26, Prop. 5.2].

Theorem 3.2 (Weissauer). Suppose \( \sigma = (\sigma_1, \sigma_2) \in \mathcal{A}_0(M(\mathbb{A})) \) is an automorphic cuspidal irreducible representation of \( M(\mathbb{A}) \) with \( \sigma_1 \not\sim \sigma_2 \). An irreducible representation \( \pi = \bigotimes_v \pi_v \) of \( GSp(4, \mathbb{A}) \) belongs to the global \( L \)-packet attached to \( \sigma \) if and only if

(i) \( \pi_v \cong \Pi_-(\sigma_v) \) for an even number of places \( v \) where \( \sigma_v \) is in the discrete series,

(ii) \( \pi_v \cong \Pi_+(\sigma_v) \) for every other place \( v \).

Then \( \pi \) occurs with multiplicity 1 in the discrete spectrum and is cuspidal, but not CAP.

Proof. Any weak endoscopic lift \( \pi \) attached to \( \sigma \) is not CAP by [26, Lem. 5.2] and so we can apply [26, Thm. 5.2] to show the other properties. Conversely, an irreducible representation \( \pi \) of \( GSp(4, \mathbb{A}) \) satisfying (i) and (ii) is weakly equivalent to \( \pi_+(\sigma) \). Therefore \( \pi \) must be cuspidal and not CAP, else \( \pi_+(\sigma) \) would be CAP. Using Table 2 one can show that (3) holds at least for every non-archimedean place \( v \) where \( \sigma_v \) is spherical. Finally, by [26, Thm. 5.2.4] \( \pi \) is automorphic with multiplicity one. Therefore \( \pi \) is a weak endoscopic lift attached to \( \sigma \).
The global $L$-packet attached to $\sigma$ is equal to the set of equivalence classes of automorphic irreducible representations of $GSp(4, A)$ which are weakly equivalent to $\pi_+ (\sigma)$, compare [26, Def. 5.2]. Let $d(\sigma) \in \mathbb{Z}_{\geq 0}$ be the number of places where $\sigma_v$ is in the discrete series. If $d(\sigma) \leq 1$, then the global $L$-packet attached to $\sigma$ contains only $\pi_+$. For $d(\sigma) \geq 2$ the global $L$-packet contains $2^{d(\sigma) - 1}$ isomorphism classes of representations, compare [26, Thm. 5.2.5]. Indeed, by Thm. 3.2 any choice of an even number of places $v$ where $\sigma_v$ is in the discrete series gives rise to a weak endoscopic lift.

### 3.1 Invariant Vectors for weak endoscopic Lifts

For automorphic irreducible representations of $GL(2, \mathbb{A})$ and $GSp(4, \mathbb{A})$ we now look at invariant spaces under the groups

$$K^{(1)}_0 = \{ g \in GL(2, \hat{\mathbb{A}}) \mid g_v \in K^{(1)}_0 \text{ for all } v \in S \},$$

$$K^{(1)} = \{ g \in GL(2, \hat{\mathbb{A}}) \mid g_v \in K^{(1)} \text{ for all } v \in S \},$$

$$K = \{ g \in GSp(4, \hat{\mathbb{A}}) \mid g_v \in K \text{ for all } v \in S \},$$

$$K' = \{ g \in GSp(4, \hat{\mathbb{A}}) \mid g_v \in K' \text{ for all } v \in S \}$$

for finite sets $S$ of non-archimedean places of $F$.

**Theorem 3.3.** Suppose $\sigma = (\sigma_1, \sigma_2) \in \mathcal{A}_0(M(\mathbb{A}))$ is a cuspidal automorphic irreducible representation of $M(\mathbb{A})$ with $\sigma_1 \not\cong \sigma_2$. Let $\pi = \pi_{\infty} \otimes \pi_{\text{fin}} \in \mathcal{A}_0(GSp(4, \mathbb{A}))$ be a cuspidal automorphic irreducible representation in the global $L$-packet attached to $\sigma$. Let $S = S_1 \cup S_2$ be finite sets of non-archimedean places.

(i) If $K^{(1)}_1(S_1) \neq 0$ and $K^{(1)}_2(S_2) \neq 0$, then the action of $GSp(4, \hat{\mathbb{A}})$ on $\pi_{\text{fin}}^{K(S)}$ is isomorphic to the representation $\bigotimes_{v \in S} F_v(\pi_v)$ as given by Table 3. Especially, for $\sigma^{K(S)}_1(S_1) \neq 0$ and $\sigma^{K(S)}_2(S_2) \neq 0$ we have $\pi_{\text{fin}}^{K(S)} \neq 0$.

(ii) If $\sigma^{K(S)}_1(S_1) = 0$ or $\sigma^{K(S)}_2(S_2) = 0$, then $\pi_{\text{fin}}^{K(S)} = 0$.

**Proof.** For each non-archimedean place $v$ the local factors $\pi_v$ are given by Thm. 3.2. Recall that $\dim \pi_v^{GSp(4, \mathbb{A})} = 1$ for spherical $\pi_v$ and so these factors can be dropped from the tensor product $\pi_{\text{fin}}^{K(S)}$. Now Cor. 2.3 and Prop. 2.2 imply (i) and (ii). \qed

**Proposition 3.4.** Let $F = \mathbb{Q}$. Suppose $\sigma = (\sigma_1, \sigma_2)$ is a cuspidal automorphic irreducible representation of $M(\mathbb{A})$ with $\sigma_1 \not\cong \sigma_2$ and trivial central character $\omega_{\pi}$. Fix a cuspidal automorphic irreducible representation $\pi$ of $GSp(4, \mathbb{A})$ in the global $L$-packet attached to $\sigma$. Then for every nonarchimedean place $v$ we have:

$$L(\sigma_{1,v}, \sigma_{2,v}, s) = L(\sigma_v, s) \quad \text{and} \quad \epsilon(\sigma_{1,v}, s) \epsilon(\sigma_{2,v}, s) = \epsilon(\pi_v, s).$$

(4)
Proof. Each non-archimedean factor \( \pi_v \) is given by Table 2. For non-cuspidal \( \pi_v \) the local \( L \)- and \( \epsilon \)-factors are given by Roberts and Schmidt [13, Tables A.8 and A.9]. Now assume \( \pi_v \cong \theta_\pi(\sigma_v) \) is generic and cuspidal, then \( \sigma_{1,v} \) and \( \sigma_{2,v} \) are both cuspidal and so \( L(\pi_v, s) = L(\pi, s)L(\sigma_2, s) \) by [20, Prop. 3.9]. The corresponding equation of \( \gamma \)-factors has been shown by Soudry and Piatetski-Shapiro [12, Thm. 3.1] and this implies (4) by the local functional equation. Next, for non-generic and cuspidal \( \pi_v \cong \theta_- (\sigma_v) \) at a place \( v \) with odd residue characteristic, (4) is implied by [6, Thm. 4.4 and Cor. 4.5] and by the fact the local Jacquet-Langlands-correspondence preserves \( L \)- and \( \epsilon \)-factors.

By Riemann-Roch there are cuspidal automorphic irreducible representations \( \sigma'_1 \not\cong \sigma'_2 \) of \( GL(2, \mathbb{A}) \), unramified at \( v = 2 \) but both ramified at some other non-archimedean place, whose archimedean factors are \( \sigma'_{i,\infty} \cong \sigma_{i,\infty} \). Now the previous arguments apply to \( \sigma'_i \) at every non-archimedean place. Hence Eq. (4) holds for any lift \( \pi' \) of \( \sigma \) with \( \pi_{\infty} \cong \pi'_{\infty} \). By the functional equation for \( \sigma'_i \) and \( \pi' \), the local \( \gamma \)-factors must satisfy \( \gamma(\sigma_{1,\infty}, s)\gamma(\sigma_{2,\infty}, s) = \gamma(\pi_{\infty}, s) \), and by the one for \( \sigma \) and \( \pi \) we have then \( \gamma(\sigma_{1,2}, s)\gamma(\sigma_{2,2}, s) = \gamma(\pi_{2}, s) \). The implication ([3, Thm. 4.4] \Rightarrow [6, Cor. 4.5]) holds for \( v = 2 \), so the \( \gamma \)-factors uniquely determine the \( L \)- and \( \epsilon \)-factors of \( \pi_2 \).

The weak endoscopic lift defines a lifting from pairs of elliptic cuspidal eigenforms to Siegel cuspidal eigenforms of genus two. There is already a similar result by R. Schmidt and A. Saha [17, Prop. 3.1] under the restriction that the Atkin-Lehner-eigenvalues of \( f_1 \) and \( f_2 \) coincide at every place dividing \( \gcd(N_1, N_2) \).

Corollary 3.5 (Yoshida-Lift). For \( i = 1, 2 \) suppose \( f_i \in \mathcal{S}_{r_i}(\Gamma_0(N_i)) \) are elliptic newforms with squarefree \( N_i \) and weight \( r_1 > r_2 \geq 2 \), eigenforms under the Hecke-algebra, such that \( \gcd(N_1, N_2) > 1 \). Then for \( N = \text{lcm}(N_1, N_2) \) there is a genus two Siegel cuspidal eigenform \( f \) with type \( \text{Sym}^{r_2-2} \otimes \det^{(r_1-r_2)/2+2} \), congruence group \( \Gamma^{(2)}(N) \) and with spinor \( L \)-factor

\[
L_p(f, s) = L_p(f_1, s)L_p(f_2, s + \frac{1}{2}(r_2 - r_1)) \quad \forall p < \infty.
\]  

(5)

Proof. For \( F = \mathbb{Q} \) let \( \sigma \) be the cuspidal automorphic irreducible representation of \( GL(2, \mathbb{A}) \) generated by \( f_1 \) and let \( S_i = \{ v < \infty \mid p_v \nmid N_i \} \) be the set of non-archimedean places \( v \) where \( \sigma_{i,v} \) is in the discrete series. Fix a prime \( p_0 \) dividing \( N_1 \) and \( N_2 \), then by Thm. 3.3 there is a weak endoscopic lift \( \pi \), attached to \( \sigma \), that is locally generic at every place except \( p_0 \) and \( \infty \). By Thm. 2.4 the archimedean factor \( \pi_{\infty} = \Pi_{-}(\sigma_{\infty}) \) is holomorphic. We have \( \sigma_{i,\text{fin}}^{K(S)}(S_i) \not= 0 \) by strong approximation, so Thm. 3.3 implies \( \pi_{\text{fin}}^{K(S)} \not= 0 \) for \( S = S_1 \cup S_2 \). Hence there is a non-zero adelic automorphic form \( \phi \) in \( \pi \) invariant under \( K' \). By strong approximation \( \phi \) corresponds to a Siegel modular form \( f \) of weight \( \text{Sym}^{r_2-2} \otimes \det^{(r_1-r_2)/2+2} \) invariant under the principal congruence subgroup \( K'(S) \cap Sp(4, \mathbb{Q}) = \Gamma^{(2)}(N) \). Finally, Prop. 3.4 implies (5). \( \square \)
4 The Saito-Kurokawa Lift

The classical Saito-Kurokawa Lift has been constructed by Maass, Andrianov and Zagier, compare [29]. To an elliptic cuspform \( f \in S_{2k-2}(SL(2, \mathbb{Z})) \) with even \( k \geq 10 \) it attaches a scalar-valued Siegel cuspform \( \tilde{f} \in S^{(2)}_{(0,k)}(Sp(4, \mathbb{Z})) \) with spinor \( L \)-function

\[
L(\tilde{f}, s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s),
\]

where \( \zeta \) denotes the Riemann Zeta-Function. A representation theoretic description of this lift in terms of the corresponding automorphic representations has been given by Piatetski-Shapiro [10] using results of Waldspurger [24]. In order to define the local theta-lifts, one has to fix a non-trivial additive character \( \psi \) of \( F \). The lift is independent of \( \psi \) and so \( \psi \) is suppressed in the notation. For a given generic irreducible representation \( \sigma_v \) of \( PGL(2, F_v) \) we consider two possible cuspidal irreducible representations \( \tilde{\pi}_v \) of the double cover \( \widetilde{SL}(2, F_v) \) of \( SL(2, F_v) \). One possibility is \( \tilde{\pi}_v = \theta'_1(\sigma_v) \) via the \( \theta'_1 \)-lift between \( PGL(2, F_v) \) and \( \widetilde{SL}(2, F_v) \). For \( \sigma_v \) in the discrete series there is another possible lift, \( \tilde{\pi}_v = \theta'_2(\sigma_v^{JL}) \). The Jacquet-Langlands-correspondence \( \sigma_v \mapsto \sigma_v^{JL} \) gives rise to an irreducible representation of \( PD^\times(F_v) \), the group of units in the quaternion division algebra over \( F_v \) modulo center. Then \( \sigma_v^{JL} \) corresponds to an irreducible representation \( \tilde{\pi}_v \) via the \( \theta'_2 \)-lift from \( PD^\times(F_v) \) to \( \widetilde{SL}(2, F_v) \). For both cases attached to \( \tilde{\pi}_v \) there is a unique irreducible representation of \( PGSp(4, F_v) \) via the \( \theta \)-correspondence described by Piatetski-Shapiro [10]. These liftings can be visualized by the following non-commutative diagram of partial one-to-one functions:

\[
\begin{align*}
\text{Irr}(PGL(2, F_v)) \xleftarrow{\theta'_1} & \text{Irr}(\widetilde{SL}(2, F_v)) \xrightarrow{\theta'} \text{Irr}(PGSp(4, F_v)) \\
\text{Irr}(PD^\times(F_v)) \xrightarrow{\theta'_2} & \text{Irr}(PGL(2, F_v))
\end{align*}
\]

An explicit construction of this lifting can be found in [10], compare [15, 16]. Fix a finite set \( S \) of \( F \)-places where \( \sigma_v \) is in the discrete series, and let \( \sigma_{S,v} \) be

\[
\sigma_{S,v} := \begin{cases} 
1_{GL(2, F_v)} & v \notin S, \\
St_{GL(2, F_v)} & v \in S.
\end{cases}
\]

Then \( \sigma_S = \bigotimes_v \sigma_{S,v} \) is a constituent of the parabolically induced representation \( 1_{A^\times} \times 1_{A^\times} \) of \( GL(2, A) \). Since \( \sigma_S \) has trivial central character, it defines a non-cuspidal automorphic irreducible representation of \( PGL(2, A) \).

**Definition 4.1.** For a generic cuspidal automorphic irreducible representation \( \sigma \) of \( PGL(2, A) \) and for \( S \) as above, the *local Saito-Kurokawa Lift at \( v \)* is the irreducible representation of \( PGSp(4, F_v) \) given by

\[
\Pi(\sigma_v, \sigma_{S,v}) := \begin{cases} 
\theta \circ \theta'_1(\sigma_v) & v \notin S, \\
\theta \circ \theta'_2(\sigma_v^{JL}) & v \in S.
\end{cases}
\]
The global Saito-Kurokawa Lift is $\Pi(\sigma, \sigma_S) = \bigotimes_v \Pi(\sigma_v, \sigma_{S,v})$. 

The local factors $\pi_v = \Pi(\sigma_v, \sigma_{S,v})$ are given in [15, p. 24] and also in Table 4. The lift $\theta \circ \theta'_2(\sigma_{v}^{JL})$ coincides with the anisotropic lift $\theta_-(\sigma_v, \text{St})$ [15, Prop. 5.8)]. Each factor $\Pi(\sigma_v, \sigma_{S,v})$ is non-generic.

**Theorem 4.2** (Piatetski-Shapiro/Schmidt/Waldspurger). Suppose $\sigma$ is an irreducible cuspidal automorphic representation of $\text{PGL}(2, \mathbb{A})$ and $S$ is a finite set of places where $\sigma_v$ is in the discrete series. If

$$(-1)^{\#S} = \epsilon(\sigma, 1/2),$$

then $\pi = \Pi(\sigma, \sigma_S)$ is an irreducible automorphic representation of $\text{PGSp}(4, \mathbb{A})$ with spinor factors

$$L(\pi_v, s) = L(\sigma_v, s)L(\sigma_{S,v}, s) \quad \text{and} \quad \epsilon(\pi_v, s) = \epsilon(\sigma_v, s)\epsilon(\sigma_{S,v}, s)$$

at almost every $v$. The representation $\pi$ is cuspidal if and only if $L(\sigma, 1/2) = 0$ or $S \neq \emptyset$.

**Proof.** The existence and automorphy of $\pi$ is Thm. 3.1 from [15]. The equation of $L$-functions is given by Remark 3.2 in [15] depending on the Local Langlands Correspondence [8]. Cuspidality of $\pi$ is implied by Thm. 3.1 of Schmidt [15], compare Thm. 2.6 of Piatetski-Shapiro [10].

For non-archimedean places the local Saito-Kurokawa lift is given by the third column of Table 4. At the archimedean places for $k \geq 3$ we have $\Pi(D(2k-3), \text{St}) = \theta_-(\nu^{1/2}D(2k-3), \nu^{-1/2})$, a holomorphic discrete series representation with $K$-type of weight $(k, k)$, which appears in the cohomology with Hodge types $(3, 0)$ and $(0, 3)$. The non-tempered Langlands quotient $\Pi(D(2k-3), 1_{\text{PGL}(2,F_v)}) = L(\nu^{1/2}D(2k-3), \nu^{-1/2})$ has with minimal $K$-type of weight $(k-1, 1-k)$ and contributes to cohomology with Hodge-type $(1, 1)$ and $(2, 2)$ [2, 7.3.ii)]. This Langlands quotient has been denoted $\pi_{\lambda}^{2,\pm}$ in [21]. For additional information on the Saito-Kurokawa-lift, see [15, Sect. 4].

Every global Saito-Kurokawa lift $\pi$ is weakly equivalent to a globally Siegel induced representation, this is directly implied by Table 4. For $F = \mathbb{Q}$ equation (7) holds at every non-archimedean place $v$, the proof is analogous to Prop. 3.4.
The reader can easily construct the global statement analogous to Theorem 3.3. For a generic irreducible representation \( \sigma_v \) of \( \text{PGL}(2, F_v) \) at a non-archimedean place \( v \) let \( \pi_v \) be a local Saito-Kurokawa lift of \( \sigma_v \). Then:

\[
\begin{align*}
\sigma_v^{K(1)(p_v)} &\neq 0  \quad \implies \quad \pi_v^{K(p_v)} \text{ is given by Table } 4, \\
\sigma_v^{K(1)(p_v)} &\neq 0  \quad \implies \quad \pi_v^{K(p_v)} = 0, \\
\sigma_v \text{ spherical} &\iff \pi_v \text{ spherical}, \\
\sigma_v^{K(1)(p_v)} &\neq 0  \quad \implies \quad \pi_v^{K(p_v)} \neq 0, \\
\pi_v^{K(p_v)} &\neq 0  \quad \iff \quad \pi_v^{K(p_v)} \neq 0.
\end{align*}
\]

**Proof.** For non-cuspidal \( \pi_v \) the representation \( \tilde{\pi}_v = \pi_v^{K(p_v)} \) is determined in [14, Tab. 2.2]. The cuspidal \( \pi_v \) are given by the anisotropic theta-lift \( \theta_{-}(\sigma_v, \text{St}) \) and \( \tilde{\pi}_v \) has been determined in [14, Table 4.1]. Only the first line of Table 4 for unramified \( \mu \) contains spherical representations. Non-zero \( K(1)(p_v) \)-invariants are only possible for \( \sigma_v \cong \text{St} \), for \( \sigma_v \cong \xi_u \cdot \text{St} \) and for spherical \( \sigma_v \), so the Table implies \( \pi_v^{K(p_v)} \neq 0 \). For the dimension of \( \text{diag}(1, 1, *, *) \)-invariants in \( \tilde{\pi}_v \), see [14, Tables 1.4 and 1.8].

The reader can easily construct the global statement analogous to Theorem 5.3.

**Notation 4.4 (Table 4).** The notation is analogous to Table 3. Representations of \( \text{PGSp}(4) \) are representation of \( \text{GSp}(4) \) with trivial central character. For an at most tamely ramified character \( \mu \) of \( F_v^\times \) we write \( \tilde{\mu} \) for its restriction to \( \sigma_v^\times / (1 + p_v) \). Each cuspidal irreducible representation \( \rho \) of \( \text{GL}(2, F_v) \) with \( \rho^{K(1)(p_v)} \neq 0 \) defines a cuspidal irreducible representation \( \tilde{\rho} \) of \( \text{GL}(2, F_q) \), which in turn defines a character \( \Lambda \) of \( F_q^\times \). The central character of \( \rho \) is unramified, so \( \Lambda(x) = \omega_\Lambda(x^{q-1}) \) factors over a character \( \omega_\Lambda \) of
ker $\mathbb{F}_{q^2}^\times$/$\mathbb{F}_q^\times$. By $\xi_u$ we denote the non-trivial unramified quadratic character of $F_v^\times$ and $\xi_t$ is one of the tamely ramified quadratic characters. For even $q$ let $k \in \mathbb{Z}/(q + 1)\mathbb{Z}$ be such that $\tilde{\mu} = \hat{\cdot}^k$ as an $F_v^\times$-character and let $l \in \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ be such that $\Lambda = \hat{\theta}^l$, then $\omega_{\Lambda} = \hat{\cdot}^{k-1}(l)$.

Proposition 4.3 can be applied to determine the principal congruence subgroup level of Saito-Kurokawa lifts in the sense of Piatetski-Shapiro. The following corollary has already been shown by R. Schmidt [16, Thm. 5.2.ii)] for even $k$ and $M = \prod q^\epsilon_p = -1$.

Corollary 4.5 (Classical Saito-Kurokawa Lift). Suppose $f \in S_{2k-2}(\Gamma_0(N))$ is an elliptic cuspidal newform of squarefree level $N$ and weight $2k - 2 \geq 4$ with Atkin-Lehner-eigenvalues $\epsilon_p$ at $p | N$, an eigenform of the Hecke-algebra. For any divisor $M$ of $N$ with

$$(-1)^{\# \{p | M\}} = (-1)^k \prod_{p | N} \epsilon_p,$$

there is a scalar-valued Siegel cuspform $\tilde{f} \in S_{(2)}^{(2)}(\Gamma^{(2)}(N))$, an eigenform for the Hecke-algebra, with spinor $L$-function

$$L(\tilde{f}, s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s) \prod_{p | M} \frac{(1 - p^{-s+k-1})(1 - p^{-s+k-2})}{(1 + \epsilon_p p^{-s+k-2})},$$

where $\zeta$ denotes the Riemann zeta function.

Proof. For $F = \mathbb{Q}$ set $S := \{p \mid M\} \cup \{\infty\}$ and let $\sigma$ be the cuspidal automorphic irreducible representation of $GL(2, \mathbb{A})$ generated by $f$. The lift $\pi = \Pi(\sigma, \sigma_S)$ is automorphic because of Eq. (8) and $\epsilon(\sigma, 1/2) = \epsilon(f, k - 1) = (-1)^{k-1} \prod_{p | N} \epsilon_p$. It is cuspidal because $S$ is not empty. Prop. 4.3 implies $\pi^K_{s'}(p) \neq 0$ for $K'(p) = K'(p_x)$. By strong approximation each $K'(\{p \mid N\})$-invariant Hecke-eigenvector in $\pi$ gives rise to a Siegel cuspidal eigenform $\tilde{f}$, invariant under $K' \cap Sp(4, \mathbb{Z}) = \Gamma^{(2)}(N)$. Since $F = \mathbb{Q}$, (7) holds at every non-archimedean place by an argument similar to Prop. 3.4.

5 The Inner Cohomology

From now on we only consider $F = \mathbb{Q}$ and fix a neat compact open subgroup $K \subseteq GSp(4, \mathbb{Z}) = \prod_{p < \infty} GSp(4, \mathbb{Z}_p)$. Every principal congruence subgroup of principal congruence subgroup level $N \geq 3$ is neat. Consider the Shimura variety

$$X_K = GSp(4, \mathbb{Q}) \backslash GSp(4, \mathbb{A})/KK_{\infty},$$

where $K_{\infty}$ is the stabilizer of $iI_2 \in \mathbb{H}^2$ in the Siegel upper half plane. Fix a pair of integers $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$. Let $V_{\lambda}$ be the finite dimensional representation
of $GSp(4,\mathbb{Q})$ with central character $t \mapsto t^{\lambda_1 + \lambda_2}$ whose restriction to $Sp(4,\mathbb{Q})$ has weight $(\lambda_1, \lambda_2)$. We consider the local coefficient system

$$\mathcal{V}_\lambda : GSp(4,\mathbb{Q})/(GSp(4,\mathbb{A})/\mathcal{K}K_\infty \times V_\lambda) \rightarrow X_K.$$  

For this local system let $H^\bullet(X_K, \mathcal{V}_\lambda)$ be the étale cohomology and $H^\bullet_c(X_K, \mathcal{V}_\lambda)$ be the étale cohomology with compact support. The inner cohomology is

$$H^\bullet_c(X_K, \mathcal{V}_\lambda) = \text{Image}(H^\bullet_c(X_K, \mathcal{V}_\lambda) \rightarrow H^\bullet(X_K, \mathcal{V}_\lambda)),$$

it is denoted $H^\bullet_c$ in [21]. Attached to the projective limit $X := \varprojlim X_K$ over neat compact open subgroups there is the inner cohomology

$$H^\bullet(X, \mathcal{V}_\lambda) = \lim_K H^\bullet(X_K, \mathcal{V}_\lambda).$$

For neat $K$ we get back the inner cohomology of $X_K$ by taking $K$-invariants

$$H^\bullet(X, \mathcal{V}_\lambda)^K = H^\bullet(X_K, \mathcal{V}_\lambda).$$

By this equation we can define $H^\bullet(X_K, \mathcal{V}_\lambda)$ for every compact open subgroup $K$ of $GSp(4,\mathbb{Z})$. The cohomology in degree 3 admits the following decomposition [21]:

$$H^3(X, \mathcal{V}_\lambda) \cong H^{0,3}(X, \mathcal{V}_\lambda) \cong \bigoplus_{\pi_\infty \pi_{\text{fin}} \in A_0(GSp(4,\mathbb{A}))} \pi_{\text{fin}},$$

The sum is over the cuspidal automorphic irreducible representations $\pi$, whose archimedean component is isomorphic to the holomorphic discrete series representation of weight $(k_1, k_2) = (\lambda_1 + 3, \lambda_2 + 3)$. For the non-holomorphic case we have

$$H^2(X, \mathcal{V}_\lambda) \cong H^{1,2}(X, \mathcal{V}_\lambda) \cong \bigoplus_{\pi_\infty \pi_{\text{fin}} \in A_0(GSp(4,\mathbb{A}))} \pi_{\text{fin}},$$

the sum is over cuspidal automorphic irreducible representations $\pi$ with generic discrete series $\pi_\infty$ belonging to the same local $L$-packet as $\pi^{H^1}_{(k_1, k_2)}$. For $\lambda_1 > \lambda_2 > 0$ we have $H^i(X_K, \mathcal{V}_\lambda) = 0$ for every $i \neq 3$ [21], so this describes $H^1_c(X, \mathcal{V}_\lambda)$ completely.

There is a subspace $H^\bullet_{\text{c}}(X, \mathcal{V}_\lambda)$ of the inner cohomology $H^\bullet(X, \mathcal{V}_\lambda)$, which is maximal with respect to the property that its automorphic irreducible constituents are those representations of $GSp(4,\mathbb{A})$ which are weakly equivalent to constituents of globally parabolically induced representations. The archimedean factor of each constituent of $H^\bullet_{\text{c}}(X, \mathcal{V}_\lambda)$ is in the discrete series and belongs to the Arthur packet attached to weight $(k_1, k_2) = (\lambda_1 + 3, \lambda_2 + 3)$. Each constituent is either a residue of an Eisenstein series or it is CAP. The residues of Eisenstein series only occur with Hodge type (1,1) and (2,2) [21]. The CAP-representations are those given by Thm. 4.2 and can occur with
Hodge types (1, 1), (2, 2), (3, 0) and (0, 3). By [27, Lemma 3] we have $H_{i,E}^\bullet (X, \mathcal{V}_\lambda) = 0$ for $\lambda_1 > \lambda_2 > 0$.

The orthocomplement of $H_{i,E}(X, \mathcal{V}_\lambda)$ with respect to the cup-product in $H_i(X, \mathcal{V}_\lambda)$ is composed of the cuspidal automorphic irreducible representations $\pi$ which are not CAP and whose archimedean factor $\pi_\infty$ belongs to the local $L$-packet of the holomorphic discrete series representation of $GSp(4, \mathbb{R})$ with weight $(k_1, k_2)$. It splits into the direct sum $H_{i,\text{endo}}^\bullet (X, \mathcal{V}_\lambda) \oplus H_{i,00}^\bullet (X, \mathcal{V}_\lambda)$. The endoscopic part $H_{i,\text{endo}}^\bullet (X, \mathcal{V}_\lambda)$ is composed of cuspidal automorphic representations which are global weak endoscopic lifts, compare Def. [3.1]. It contributes with Hodge numbers (3, 0), (2, 1), (1, 2), (0, 3). The other part $H_{i,00}^\bullet (X, \mathcal{V}_\lambda)$ is composed of cuspidal irreducible representations that are neither CAP nor weak endoscopic lifts [27].

### 5.1 Endoscopic Part

**Proposition 5.1.** Suppose $\lambda_1 \geq \lambda_2 \geq 0$ and let $r_1 = \lambda_1 + \lambda_2 + 4$ and $r_2 = \lambda_1 - \lambda_2 + 2$. Let $K \subseteq GSp(4, \mathbb{Z})$ be an open compact subgroup, then the endoscopic part of the inner cohomology of $X_K$ is given by

\[
H_{i,\text{endo}}^{(3,0)} (X_K, \mathcal{V}_\lambda) = \bigoplus_{\sigma_1, \sigma_2 \in \mathcal{A}_0(\text{GL}(2, \mathbb{A}))} \bigoplus_{\sigma_1, \sigma_2 \in \mathcal{D} (r_1 - 1)} \bigoplus_{S_{ng} \text{ odd}, v < \infty} \bigoplus_{S_{ng} \text{ even}, v < \infty} \pi_v^{K(p_v)},
\]

\[
H_{i,\text{endo}}^{(2,1)} (X_K, \mathcal{V}_\lambda) = \bigoplus_{\sigma_1, \sigma_2 \in \mathcal{A}_0(\text{GL}(2, \mathbb{A}))} \bigoplus_{\sigma_1, \sigma_2 \in \mathcal{D} (r_1 - 1)} \bigoplus_{S_{ng} \text{ odd}, v < \infty} \bigoplus_{S_{ng} \text{ even}, v < \infty} \pi_v^{K(p_v)}.
\]

The first sum runs over the cuspidal automorphic irreducible $\text{GL}(2, \mathbb{A})$-representations $\sigma_i$ with equal central character, whose archimedean factor is in the holomorphic discrete series of lowest weight $r_i$. The second sum is over the subsets $S_{ng}$ of the finite subsets of non-archimedean places $v$ where both $\sigma_{1,v}$ and $\sigma_{2,v}$ are in the discrete series. The factor $\pi_v$ is given by $\pi_v = \Pi_-(\sigma_v)$ for $v \in S_{ng}$ and $\pi_v = \Pi_+ (\sigma_v)$ for $v \notin S_{ng}$.

**Proof.** This is implied by Thm. [3.2] and Eqs. ([9], [10]). ☐

**Corollary 5.2.** Fix $\lambda_1 \geq \lambda_2 \geq 0$ and let $K = K(\{p_0\}) \subseteq GSp(4, \mathbb{Z})$ be the principal congruence subgroup of prime level $p_0$. Then the endoscopic part of the inner cohomology is

\[
H_{i,\text{endo}}^{(3,0)} (X_K, \mathcal{V}_\lambda) \cong \bigoplus_{\sigma_1, \sigma_2} \Pi_- (\sigma_1, \sigma_2, l)^{K(\{p_0\)}),
\]

\[
H_{i,\text{endo}}^{(2,1)} (X_K, \mathcal{V}_\lambda) \cong \bigoplus_{\sigma_1, \sigma_2} \Pi_+ (\sigma_1, \sigma_2, l)^{K(\{p_0\)}).
\]

The sum runs over the cuspidal automorphic irreducible representations $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A})$ such that each archimedean $\sigma_{i,\infty}$ belongs to the discrete series of weight $r_i$ and that $\sigma_{i,v}$ is spherical at the non-archimedean places $v$ apart from $v = p_0$.  

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Proof. For \( \Pi_{\pm}(\sigma_v) \) in order to be spherical at every non-archimedean \( v \neq p_0 \) it is necessary and sufficient by Cor. [24] that the \( \sigma_{\pm,v} \) are both spherical. The only possibilities for \( S_{sg} \) in Prop. [5.1] are \( \{ p_0 \} \) for the first equation and the empty set for the second equation, respectively. \( \square \)

Note that the sum is only over those automorphic representations \( \sigma \) whose central character factors over \( (\mathbb{Z}/p_0\mathbb{Z})^\times \). For \( p_0 = 2 \) the central character must be trivial. Before we give the endoscopic contribution more explicitly, we first need to describe irreducible \( GL(2, \mathbb{Q}_2) \)-representations of level 4:

Lemma 5.3. Let \( (\sigma, V_\sigma) \) be an irreducible representation of \( GL(2, \mathbb{Q}_2) \) of level \( N = 4 \). Then \( (\sigma, V_\sigma) \) is cuspidal, \( \dim V_\sigma^{K^{(1)(2)(2)}} = q - 1 \) for the first principal congruence subgroup \( K^{(1)(2)(2)} \subseteq GL(2, \mathbb{Z}_2) \) and \( \sigma \) is uniquely determined by its central character. Its local factors are \( \epsilon(\sigma, s) = -1 \) and \( L(\sigma, s) = 1 \).

Proof. By assumption the representation \( (\sigma, V_\sigma) \) admits non-zero invariants under the subgroup \( \{(a/b) \in GL(2, \mathbb{Z}_2) \mid c \equiv 0 \mod 4 \} \), which is conjugate to the principal congruence subgroup \( K^{(1)(2)(2)} \subseteq GL(2, \mathbb{Z}_2) \). The action of \( GL(2, \mathbb{Z}_2) \) on the space of \( K^{(1)(2)(2)} \)-invariants defines a non-zero representation \( \tilde{\sigma} \) of \( GL(2, \mathbb{Z}_2)/K^{(1)(2)(2)} \cong GL(2, \mathbb{F}_2) \). If \( \tilde{\sigma} \) was non-cuspidal, then it would contain non-zero Borel-invariant vectors and so \( \sigma \) would admit non-zero invariants under the standard Iwahori subgroup \( K_0^{(1)(2)} \subseteq GL(2, \mathbb{Z}_2) \). That is only possible for level \( N \leq 2 \), and so \( \tilde{\sigma} \) must be cuspidal. Now \( \tilde{\sigma} \) is a sum of irreducible cuspidal representations of \( GL(2, \mathbb{F}_2) \), so by [14, Thm. 2.23] \( \sigma \) itself must be cuspidal. Since \( \sigma \) is irreducible, the same theorem implies that \( \tilde{\sigma} \) must be irreducible. Up to isomorphy there is only one cuspidal irreducible representation \( \tilde{\sigma} \) of \( GL(2, \mathbb{F}_2) \) and by [14, Thm. 2.23] we have \( \sigma \cong c\text{Ind}_{Z SL(2, \mathbb{Z}_2)}^{GL(2, \mathbb{Q}_2)}(\omega_\sigma \boxtimes \tilde{\sigma})|_{SL(2, \mathbb{Z}_2)} \), so \( \sigma \) is uniquely determined by \( \omega_\sigma \).

Fix an additive character \( \psi : \mathbb{Q}_2 \to \mathbb{C}^\times \) with \( 2\mathbb{Z}_2 \subseteq \ker(\psi) \), which gives rise to the non-trivial additive character \( \hat{\psi} : \mathbb{Z}_2/2\mathbb{Z}_2 \to \mathbb{C}^\times \). The \( \epsilon \)-factor of \( \sigma \) is

\[
\epsilon(\sigma, s, \psi) = 2^{2\ell(\sigma)}(\frac{1}{2} - s) \frac{\tau(\Xi, \psi)}{(\mathfrak{A} : \mathfrak{Q}^{n+1})^{1/2}},
\]

in the notation of [4, 25.2, Theorem]. It is clear that \( n = \ell(\sigma) = 0 \) since \( \sigma^{K^{(1)(2)}} \neq 0 \) and that \( (\mathfrak{A} : \mathfrak{Q}) = 16 \) because \( \mathfrak{A} = \text{Mat}_2(\mathbb{Z}_2) \) and \( \mathfrak{Q} = I_2 + \text{Mat}_2(2\mathbb{Z}_2) \). The cuspidal irreducible \( \tilde{\sigma} \) corresponds to a non-trivial character \( \Lambda \) of \( \mathbb{F}_4^\times \) with \( \Lambda^2 \neq \Lambda \). This \( \Lambda \) is denoted by \( \theta \) in [4, §6.4]. By (25.4.1) and the first equation in Section 23.7 in loc. cit. we have

\[
\tau(\Xi, \psi) = -2 \sum_{x \in \mathbb{F}_4^\times} \overline{\Lambda(x)} \psi(x + x^2) = -4.
\]

The local \( L \)-factor is \( L(\sigma, s) = 1 \) by [4, §24.5]. \( \square \)

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Let \( \tau_{N,i} := \dim S_{r_i}(\Gamma_0(N))^{\text{new}} \) denote the dimension of the space of newforms of weight \( r_i \) and level \( N \). Furthermore, let \( \tau_i^\pm \) denote the dimension of the subspace of level \( N = 2 \) newforms of weight \( r_i \) and with Atkin-Lehner eigenvalue \( \pm 1 \). Recall that the principal congruence subgroup \( K = K(\{2\}) \) of level 2 is a normal subgroup of \( GSp(4, \mathbb{Z}) \) that satisfies the conditions for strong approximation. We use the notation of Enomoto \cite{Enomoto}.

**Corollary 5.4.** Suppose \( K \subseteq GSp(4, \mathbb{Z}) \) is the principal congruence subgroup of level 2 and let \( \lambda_1 \geq \lambda_2 \geq 0 \). Then the endoscopic part of the inner cohomology is given by

\[
H_{t, \text{endo}}^{(3,0)}(X_K, V_\lambda) = (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \cdot \theta_0 + (\tau_1^+ \tau_2^+ + \tau_1^- \tau_2^-) \cdot \theta_2 + \tau_4 \tau_4,2 \cdot \chi(1) \quad \text{and}
\]

\[
H_{t, \text{endo}}^{(2,1)}(X_K, V_\lambda) = \tau_{4,1} \tau_{4,2} \cdot \chi(1) + (\tau_{4,1} \tau_{2,2} + \tau_{2,1} \tau_{4,2}) \cdot \chi(1)
\]

\[
+ (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \cdot (\theta_1 + \theta_4 + (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \cdot (\theta_3 + \theta_4)
\]

\[
+ (\tau_1 \tau_{4,2} + \tau_4 \tau_{1,2}) \cdot (\chi(1) + \chi(1))
\]

\[
+ (\tau_1 \tau_{2,2} + \tau_2 \tau_{1,2}) \cdot (\theta_1 + \theta_3 + \theta_4)
\]

\[
+ \tau_1 \tau_{1,2} \cdot (\theta_0 + 2 \theta_1 + \theta_2 + \theta_3 + \theta_4).
\]

**Proof.** We begin with the first equation. By Corollary 5.2 for each given \( \sigma = (\sigma_1, \sigma_2) \) we consider exactly one holomorphic lift with non-archimedean part

\[
\pi_{\text{fin}} = \Pi_{-}(\sigma_{1,2}, \sigma_{2,2}) \otimes \bigotimes_{2 < \nu < \infty} \Pi_{+}(\sigma_{1,v}, \sigma_{2,v}).
\]

and thus \( \pi^K_{\text{fin}} \cong (\Pi_{-}(\sigma_{1,2}, \sigma_{2,2}))^{K_2} \). Therefore by Prop. 2.2 \( \sigma_{1,2} \) and \( \sigma_{2,2} \) must both admit non-zero invariants under the principal congruence subgroup \( K^{(1)} \) in \( GL(2, \mathbb{Z}) \) of principle congruence subgroup level two, else \( \pi^K_{\text{fin}} \) would be zero. The cuspidal \( \sigma_{1,2} \) with non-zero \( K^{(1)}(2 \mathbb{Z}_2) \)-invariants are exactly those with level \( N = 4 \) by Lemma 5.3 so the number of relevant automorphic irreducible representations \( \sigma \) with cuspidal \( \sigma_{1,2} \) is exactly \( \tau_{4,1} \). The number of \( \sigma_i \) with \( \sigma_{i,2} \cong \text{St}_{GL(2)} \) and \( \sigma_i^{K^{(1)}(2 \mathbb{Z}_2)} \not= 0 \) is \( \tau_i^- \) while the corresponding number for \( \sigma_{1,2} \cong \xi_u \cdot \text{St}_{GL(2)} \) is \( \tau_i^+ \), where \( \xi_u \) is the unramified non-trivial quadratic character of \( \mathbb{Q}_\nu^2 \) \cite{Prop. 5.21}. Now Proposition 2.2 describes the invariant spaces \( \pi^K \) as representations of \( GSp(4, \mathbb{F}_2) \).

The proof for the second equation is analogous. We use the following decompositions into irreducibles: \( \chi(0, 0) = \theta_0 + 2 \theta_1 + \theta_2 + \theta_3 + \theta_4 \) and \( \chi(1) = \chi(1) + \chi(1) \) and \( \lambda_{10}(0) = \theta_1 + \theta_3 + \theta_4 \).

For \( K = K(\{2\}) \) and \( \lambda_1 = \lambda_2 \) we have \( H_t^{(3,0)}(X_K, V_\lambda) = 0 \), since \( S_2(\Gamma_0(2)) = 0 \).

**Corollary 5.5.** Suppose \( \lambda_1 \geq \lambda_2 \geq 0 \) and \( K \subseteq GSp(4, \mathbb{Z}) \) is the principal congruence subgroup of level two. Then

\[
\dim H_{t, \text{endo}}^{(2,1)}(X_K, V_\lambda) - \dim H_{t, \text{endo}}^{(3,0)}(X_K, V_\lambda) = 5 \cdot \dim S_{r_1}(\Gamma_0(4)) \cdot \dim S_{r_2}(\Gamma_0(4)).
\]
we now look at the special case where $K \subseteq GSp(4, \mathbb{Z})$ is the principal congruence subgroup of level 2. As before, let $\tau_{N,r} := \dim \mathcal{S}_r(\Gamma_0(N))$ and let $\tau_r^\pm$ denote the dimension of the subspace of $\mathcal{S}_r(\Gamma_0(2))$ with Atkin-Lehner eigenvalue $\pm 1$.

**Proof.** Table [3] implies for generic irreducible representations $\sigma_{i,2}$ of $GL(2, \mathbb{Q}_2)$:

$$\dim \Pi_i((\sigma_{1,2}, \sigma_{2,2})^K_2) - \dim \Pi_0((\sigma_{1,2}, \sigma_{2,2})^K_2) = 5 \cdot \dim \sigma_{1,2}^{K(2)} \cdot \dim \sigma_{2,2}^{K(2)}$$

compare [14, Cor. 4.12]. The dimension of $\sigma_{1,2}^{K(2)}$ is given by [14, Table 2.1]. The right hand side of (13) equals $5 \cdot (\tau_{1,1} \cdot 3 + \tau_{2,1} \cdot 2 + \tau_{4,1} \cdot 1) \cdot (\tau_{1,2} \cdot 3 + \tau_{2,2} \cdot 2 + \tau_{4,2} \cdot 1)$ by Atkin-Lehner theory.

Note that by [24, Thm. 1] the subspaces $H^{(3,0)}(X_K, \nu_\lambda)$ and $H^{(2,1)}(X_K, \nu_\lambda)$ are isomorphic. For $\lambda_1 > \lambda_2 > 0$ we have $H^{(1,0)}(X_K, \nu_\lambda) = 0$, so in this case:

$$\dim H^{(2,1)}(X_K, \nu_\lambda) - \dim H^{(3,0)}(X_K, \nu_\lambda) = 5 \cdot \dim \mathcal{S}_{r_1}(\Gamma_0(4)) \cdot \dim \mathcal{S}_{r_2}(\Gamma_0(4)). \quad (14)$$

### 5.2 Saito-Kurokawa Part

We now consider the subspace $H^{*}_{SK}(X, \nu_\lambda) \subseteq H^{*}(X, \nu_\lambda)$ that is composed of automorphic irreducible representations with trivial central character coming from Saito-Kurokawa Lifts in the sense of Piatetski-Shapiro, compare Thm. [4.2]. These lifts are all weakly equivalent to globally parabolically induced representations from the Borel or Siegel parabolic [10, Thm. 2.2], so they are either CAP or not cuspidal. We have already mentioned that $H^{*}_{E}(X_K, \nu_\lambda) = 0$ for $\lambda_1 > \lambda_2 > 0$, so we assume $\lambda_1 = \lambda_2$.

**Proposition 5.6.** Suppose $\lambda_1 = \lambda_2 \geq 0$, let $k = \lambda_1 + 3$ and $r = 2k - 2$. For an arbitrary open compact subgroup $K \subseteq GSp(4, \mathbb{Z})$, the Saito-Kurokawa part of the inner cohomology of $X_K$ is given by

$$H^{(3,0)}_{i,SK}(X_K, \nu_\lambda) \cong H^{(0,3)}_{i,SK}(X_K, \nu_\lambda) \cong \bigoplus_{\sigma \in \mathcal{A}_0 (PGL(2, \mathbb{A}))} \bigotimes_{\sigma \in \mathcal{D}(r-1)} \Pi(\sigma, \sigma^S_{v,v})^{K(p, v)}, \quad (15)$$

$$H^{(1,1)}_{i,SK}(X_K, \nu_\lambda) \cong H^{(2,2)}_{i,SK}(X_K, \nu_\lambda) \cong \bigoplus_{\sigma \in \mathcal{A}_0 (PGL(2, \mathbb{A}))} \bigotimes_{\sigma \in \mathcal{D}(r-1)} \Pi(\sigma, \sigma^S_{v,v})^{K(p, v)} \quad (16)$$

The sum is over the finite subsets $S$ of non-archimedean places where $\sigma_v$ is in the discrete series satisfying the condition $(-1)^{# S} = \epsilon(\sigma, 1/2)$.

**Proof.** By construction, the sum is over the Saito-Kurokawa lifts given by Thm. [4.2] with archimedean factor in the holomorphic discrete series. Those with Hodge-type $(3, 0)$ and $(0, 3)$ are the ones with holomorphic archimedean factor, hence we need to assume $\infty \in S$. The Hodge-types $(1, 1)$ and $(2, 2)$ belong to the $\sigma$ with non-tempered archimedean factor, i.e. $\infty \notin S$.

We now look at the special case where $K \subseteq GSp(4, \mathbb{Z})$ is the principal congruence subgroup of level 2. As before, let $\tau_{N,r} := \dim \mathcal{S}_r(\Gamma_0(N))$ and let $\tau_r^\pm$ denote the dimension of the subspace of $\mathcal{S}_r(\Gamma_0(2))$ with Atkin-Lehner eigenvalue $\pm 1$. 

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Corollary 5.7. For \( \lambda_1 = \lambda_2 \geq 0 \) let \( k = \lambda_1 + 3 \) and \( r = 2k - 2 \). For the principal congruence subgroup \( K \subseteq \text{GSp}(4, \mathbb{Z}) \) of level 2, the Saito-Kurokawa-part of the inner cohomology of \( X_K \) is given by

\[
H^{(3,0)}_{\text{SK}}(X_K, \mathcal{V}_\lambda) \cong H^{(0,3)}_{\text{SK}}(X_K, \mathcal{V}_\lambda)
= \begin{cases} 
\tau^+ \cdot \theta_1 + \tau^- \cdot \theta_2 + \tau_{1,r} \cdot (\theta_0 + \theta_1 + \theta_2) & \text{for } k \text{ even,} \\
\tau_{4,r} \cdot \chi_8(1) + \tau^+ \cdot \theta_5 + \tau^- \cdot \theta_3 & \text{for } k \text{ odd,}
\end{cases}
\]

\[
H^{(1,1)}_{\text{SK}}(X_K, \mathcal{V}_\lambda) \cong H^{(2,2)}_{\text{SK}}(X_K, \mathcal{V}_\lambda)
= \begin{cases} 
\tau_{4,r} \cdot \chi_8(1) + \tau^+ \cdot \theta_5 + \tau^- \cdot \theta_3 & \text{for } k \text{ even,} \\
\tau^+ \cdot \theta_1 + \tau^- \cdot \theta_2 + \tau_{1,r} \cdot (\theta_0 + \theta_1 + \theta_2) & \text{for } k \text{ odd.}
\end{cases}
\]

**Proof.** We begin with \( H^{(3,0)}_{\text{SK}}(X_K, \mathcal{V}_\lambda) \). Suppose some automorphic Saito-Kurokawa lift \( \pi \) gives a non-zero contribution to this space, then the central character \( \omega_\pi \) must factor over \( (\mathbb{Z}/2\mathbb{Z})^* \), hence \( \omega_\pi = \omega_\pi = 1 \). We need \( \pi_\infty \) to be isomorphic to the holomorphic discrete series representation of weight \((k, k)\) so we only consider those \( \sigma_k \) in the discrete series of \( PGL(2, \mathbb{R}) \) with lowest weight \( k \). At the non-archimedean places \( v \neq 2 \) the representation \( \pi_v \) must be spherical and by Prop. 4.3 \( \pi_v \) must then also be spherical. By Prop. 4.3 the representation \( \pi_2 \) can admit non-zero invariants under \( K_2 = K(2\mathbb{Z}_2) \) only if \( \sigma_2 \) admits non-zero \( K^{(1)}(2\mathbb{Z}_2) \)-invariants.

The non-cuspidal possibilities for \( \sigma_2 \) are \( \sigma_2 = \text{St}_{GL(2, \mathbb{Q}_2)} \) and \( \sigma_2 = \xi \cdot \text{St}_{GL(2, \mathbb{Q}_2)} \) and the principal series representation \( \sigma_2 = \mu \times \mu^{-1} \), where \( \mu : \mathbb{Q}_2^* \rightarrow \mathbb{C}^* \) is a smooth character. Non-zero \( K^{(1)}(2\mathbb{Z}_2) \)-invariant vectors in \( \sigma_2 \) only occur for at most tamely ramified \( \mu \) and \( \xi \) and for \( \mathbb{Q}_2^* \) that already implies unramified. For even \( k \) condition 4 permits the lifts \( \pi_2 = \Pi(\text{St}_{GL(2, \mathbb{Q}_2)} \times \text{st}_{GL(2, \mathbb{Q}_2)}) \) and \( \pi_2 = \Pi(\xi \cdot \text{St}_{GL(2, \mathbb{Q}_2)} \times \text{1}_{GL(2, \mathbb{Q}_2)}) \) and \( \pi_2 = \Pi(\mu \times \mu^{-1} \times \text{1}_{GL(2, \mathbb{Q}_2)}) \), which occur with multiplicity \( \tau^- \), \( \tau^+ \) and \( \tau_{1,r} \), respectively. For odd \( k \) by condition 6 there are the lifts \( \pi_2 = \Pi(\text{St}_{GL(2, \mathbb{Q}_2)} \times \text{1}_{GL(2, \mathbb{Q}_2)}) \) and \( \pi_2 = \Pi(\xi_\mu \cdot \text{St}_{GL(2, \mathbb{Q}_2)} \times \text{1}_{GL(2, \mathbb{Q}_2)}) \) occurring with multiplicities \( \tau^- \) and \( \tau^+ \). Prop. 4.3 describes the invariant space \( \pi_2 \cong \mathcal{F}_{\pi_2}(\pi_2) \) as a representation of \( GSp(4, \mathbb{Z}_2/2\mathbb{Z}_2) \cong Sp(4, \mathbb{F}_2) \); note that \( \chi_6(0) = \theta_0 + \theta_1 + \theta_2 \).

Now consider the case where \( \sigma_2 \) is cuspidal and \( \sigma_2^{K^{(1)}(2\mathbb{Z}_2)} \neq 0 \). The central \( \epsilon \)-value is \( \epsilon(\sigma, 1/2) = (-1)^k \) by Lemma 5.3. For even \( k \) there is a Saito-Kurokawa-lift \( \pi \) with \( \pi_2 = \Pi(\sigma_2, \text{st}_{GL(2, \mathbb{Q}_2)}) \). By Prop. 4.3 this lift does not admit non-zero \( K_2 \)-invariants. For odd \( k \) we have the lift \( \pi_2 = \Pi(\sigma_2, \text{1}_{GL(2, \mathbb{Q}_2)}) \) and the invariant space is isomorphic to \( \chi_8(1) \) as a representation of \( Sp(4, \mathbb{Z}_2/2\mathbb{Z}_2) \).

The proof for the second equation is analogous. Note that the lifts are not necessarily cuspidal anymore. 

\( \square \)
Corollary 5.8. Suppose $\lambda_1 = \lambda_2 \geq 0$ and let $r = 2k - 2$. For the principal congruence subgroup $K \subseteq GSp(4, \mathbb{Z})$ of level 2 we have the equation

$$\dim H^{(1,1)}_{\text{SK}}(X_K, V_{\lambda}) + \dim H^{(3,0)}_{\text{SK}}(X_K, V_{\lambda}) = 5 \cdot \dim S_r(\Gamma_0(4)).$$

(17)

Proof. The left hand side is given by Corollary 5.7 and Table 1. The proof is analogous to Corollary 5.5.

References

[1] J. Bergström, C. Faber, and G. van der Geer. Siegel Modular Forms of Genus 2 and Level 2: Cohomological Computations and Conjectures. International Mathematics Research Notices, 2008, 2008.

[2] D. Blasius and J.D. Rogawski. Zeta Functions of Shimura Varieties. In Motives, volume 55, II of AMS Proc. Symp. Pure Math., 1994.

[3] J. Breeding. Irreducible non-cuspidal characters of $GSp(4, \mathbb{F}_q)$. PhD thesis, University of Oklahoma, 2011.

[4] C. J. Bushnell and G. Henniart. The Local Langlands Conjecture for $GL(2)$. Grundlehren der Mathematischen Wissenschaften. Springer, 2006.

[5] W. Casselman. On some results of Atkin and Lehner. Math. Ann., 201:301–314, 1973.

[6] Y. Danisman. $L$-Factors of Supercuspidal Representations of $p$-adic $GSp(4)$. PhD thesis, The Ohio State University, 2011.

[7] H. Enomoto. The characters of the finite symplectic group $Sp(4, q)$, $q = 2^f$. Osaka J. Math., 9:75–94, 1972.

[8] W. T. Gan and S. Takeda. The Local Langlands Conjecture for $GSp(4)$. Ann. of Math., 173(3):1841–1882, 2011.

[9] S. S. Gelbart. Automorphic Forms on Adele Groups. Number 83 in Annals of mathematics studies. Princeton University Press, 1975.

[10] I. Piatetski-Shapiro. On the Saito-Kurokawa-Lifting. Inv. Math., pages 309–338, 1983.

[11] I. Piatetski-Shapiro. $L$-functions for $GSp(4)$. Pacific J. Math., 181(3):259–275, 1997.

[12] I. Piatetski-Shapiro and D. Soudry. The $L$ and $\varepsilon$ Factors for $GSp(4)$. J. Fac. Sci. Univ. Tokyo, Sect. 1a, 28:505–530, 1981.

[13] B. Roberts and R. Schmidt. Local Newforms for $GSp(4)$, volume 1918 of Lecture Notes in Mathematics. Springer, 1 edition, October 2007.

[14] M. Rösner. The anisotropic Theta lift for $GSp(4, F)$. Preprint, 2014. URL http://www.mathi.uni-heidelberg.de/~mroesner/Aniso_Theta_Roesner.pdf

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[15] R. Schmidt. The Saito-Kurokawa lifting and functoriality. *Amer. J. Math.*, 127:209–240, 2005.

[16] R. Schmidt. On classical Saito-Kurokawa-Liftings. *J. Reine Angew. Math.*, 604:211–236, 2007.

[17] R. Schmidt and A. Saha. Yoshida Lifts and Simultaneous non-vanishing of dihedral twists of modular $L$-functions. *J. London Math. Soc*, 2013.

[18] D. Shelstad. L-Indistinguishability for real groups. *Math. Ann.*, 259:385–430, 1982.

[19] K.-I. Shinoda. The characters of the finite conformal symplectic group $\text{CSp}(4, q)$. *Communications in Algebra*, 10(13):1369–1419, 1982.

[20] R. Takloo-Bighash. $L$-functions for the $p$-adic $\text{GSp}(4)$. *Amer. J. Math.*, 122:1085–1120, 2000.

[21] R. Taylor. On the $l$-adic cohomology of Siegel threefolds. *Inv. Math.*, 114:289–310, 1993.

[22] S.S. Tehrani. *Non-holomorphic cuspidal automorphic forms of $\text{GSp}(4, \mathbb{A})$ and the Hodge structure of Siegel threefolds*. PhD thesis, University of Toronto, 2012.

[23] B. L. van der Waerden. *Algebra*. Springer Verlag, 1964.

[24] J.-L. Waldspurger. Correspondances de Shimura et quaternions. *For. Math.*, 3:219–307, 1991.

[25] R. Weissauer. Four dimensional Galois representations. volume 302 of *Asterisque*, pages 67–149. Société mathematique de France, 2005.

[26] R. Weissauer. *Endoscopy for $\text{GSp}(4)$ and the Cohomology of Siegel Modular Threefolds*, volume 1968 of *Lecture notes in Mathematics*. Springer, 2009.

[27] R. Weissauer. The Trace of Hecke operators on the space of classical holomorphic Siegel modular forms of genus two. arXiv:0909.1744v1, 2009.

[28] H. Yoshida. Siegel’s Modular Forms and the Arithmetic of Quadratic Forms. *Inv. Math.*, 60:193–248, 1980.

[29] D. Zagier. Sur la Conjecture de Saito-Kurokawa (d’après Hans Maass). *PM*, 1219, 1981.