Characteristics of Minimal Effective Programming Systems*

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Abstract. The Rogers semilattice of effective programming systems (\(\text{eps}\)) is the collection of all effective numberings of the partial computable functions ordered such that \(\theta \leq \psi\) whenever \(\theta\)-programs can be algorithmically translated into \(\psi\)-programs. Herein, it is shown that an \(\text{eps}\) \(\psi\) is minimal in this ordering if and only if, for each translation function \(t\) into \(\psi\), there exists a computably enumerable equivalence relation (ceer) \(R\) such that (i) \(R\) is a subrelation of \(\psi\)'s program equivalence relation, and (ii) \(R\) equates each \(\psi\)-program to some program in the range of \(t\). It is also shown that there exists a minimal \(\text{eps}\) \(\psi\) for which no single such \(R\) does the work for all such \(t\). In fact, there exists a minimal \(\text{eps}\) \(\psi\) such that, for each ceer \(R\), either \(R\) contradicts \(\psi\)'s program equivalence relation, or there exists a translation function \(t\) into \(\psi\) such that the range of \(t\) fails to intersect infinitely many of \(R\)'s equivalence classes.

Keywords: computably enumerable equivalence relation, Friedberg numbering, minimal effective programming system, Rogers semilattice

1 Introduction

Let \(\mathbb{N}\) be the set of natural numbers, i.e., \(\{0, 1, 2, \ldots\}\). An effective programming systems (\(\text{eps}\)) is a partial computable function \(\lambda p, x, \psi_p(x)\) mapping \(\mathbb{N}^2\) to \(\mathbb{N}\), and having the following property. For each partial computable function \(\zeta\) mapping \(\mathbb{N}\) to \(\mathbb{N}\), there exists a \(p\) such that \(\psi_p = \zeta\). Effective programming systems abstract the notion of programming language in the following sense. One can think of \(p\) as a program, and of \(\psi_p\) as the partial computable function denoted by \(p\) within some programming language corresponding to \(\psi\).

Rogers [Rog58] introduced the following ordering on \(\text{eps}\)es. For \(\text{eps}\)es \(\theta\) and \(\psi\), \(\theta \leq \psi\) if there exists a computable function \(t: \mathbb{N} \rightarrow \mathbb{N}\) such that, for each \(p\), \(\theta_p = \psi_{t(p)}\). Intuitively, \(\theta \leq \psi\) whenever \(\theta\)-programs can be algorithmically translated into \(\psi\)-programs. Moreover, an \(\text{eps}\) \(\psi\) is minimal in this ordering if and only if the ability to algorithmically translate \(\theta\)-programs into \(\psi\)-programs implies having the ability to algorithmically translate \(\psi\)-programs into \(\theta\)-programs, for each \(\text{eps}\) \(\theta\).

Arguably, the most well studied collection of minimal \(\text{eps}\)es is that of the Friedberg numberings [Fri58, Kum90]. Recall that a Friedberg numbering is an \(\text{eps}\) that is 1-1, i.e., for each \(p\) and \(q\), \(\psi_p = \psi_q\) implies \(p = q\). Examples of works that make use of this concept include [Lav77, MWY78, Ric81, FKW82, Sch82, Roy87, Kum89, Spr90, GYY93, HK94, JST11].

In [PE64], Pour-El asked whether every minimal \(\text{eps}\) is equivalent to some Friedberg numbering. Ershov [Ers68, §5] showed that there exists a minimal effective numbering of the computably enumerable sets that is not equivalent to any 1-1 numbering. Shortly thereafter, his student, Khutoretskii, established the analogous result for the partial computable functions, thereby answering Pour-El's question.

Theorem 1 (Khutoretskii [Khu69a, Ex. 1 and Cor. 4]). There exists a minimal \(\text{eps}\) that is not equivalent to any Friedberg numbering.

For the purposes of this paper, Theorem 1 is best viewed through the following folklore theorem. (For completeness we give a proof of this result.)

Theorem 2 (Folklore). For each \(\text{eps}\) \(\psi\), \(\psi\) is equivalent to a Friedberg numbering iff \(\psi\)'s program equivalence relation is computable.

* This is an expanded version of [Moe12].
Proof. Let \( \psi \) be given.

(\( \Rightarrow \)) Suppose that \( \psi \) is equivalent to a Friedberg numbering \( \eta \), and that \( t : \mathbb{N} \to \mathbb{N} \) witnesses \( \psi \leq \eta \). Then, clearly, for each \( p \) and \( q \),

\[
\psi_p = \psi_q \iff \eta(p) = \eta(q) \iff t(p) = t(q).
\]

Thus, since \( \lambda p, q. [t(p) = t(q)] \) is computable, \( \psi \)'s program equivalence relation is computable.

(\( \Leftarrow \)) Suppose that \( \psi \)'s program equivalence relation is computable. Let \( M \) be the set of minimal programs in \( \psi \), i.e., \( M = \{m_0, m_1, \ldots \} \) where, for each \( i \), \( m_i \) is least such that

\[
\psi_{m_i} \not\in \{\psi_{m_0}, \ldots, \psi_{m_{i-1}}\}.
\]

Note that, since \( \psi \)'s program equivalence relation is computable, \( M \) is computable. Let \( \eta \) be such that, for each \( i \),

\[
\eta_i = \psi_{m_i}.
\]

Using the fact the \( M \) is computable, it is straightforward to verify that \( \eta \) is a Friedberg numbering, and that \( \psi \equiv \eta \). \( \square \) (Theorem 2)

In light of Theorem 2, Theorem 1 may be restated as: there exists a minimal \( \epsilon \psi \)'s program equivalence relation is not computable. On the other hand, as noted in the proof of Theorem 1, the constructed \( \epsilon \psi \)'s program equivalence relation is computably enumerable. (In particular, exactly one such equivalence class is a simple set \([\psi \text{Khum67}, \text{§8.1}]\), and all others a singletons.) Thus, one has the following.

**Theorem 3 (Khutoretskii, corollary of Thm. 2 and proof of Thm. 1).** There exists an \( \epsilon \psi \)'s program equivalence relation is computably enumerable, but not computable.

Subsequent to the above, Khutoretskii showed the following.

**Theorem 4 (Khutoretskii, corollary of [Khu69b, Thm. 1]).** There exists a minimal \( \epsilon \psi \)'s program equivalence relation is computably enumerable.

Clearly, Theorems 3 and 4 can be viewed as a sharpening of Theorem 1. Herein, we sharpen Khutoretskii’s results even further.

To facilitate the statement of our results, we first give a few definitions. Suppose that \( \psi \) is an \( \epsilon \psi \). For each \( t : \mathbb{N} \to \mathbb{N} \), we say that \( t \) is a translation function into \( \psi \) iff there exists an \( \epsilon \psi \) \( \theta \) such that \( t \) witnesses \( \theta \leq \psi \). The following definition is equivalent. For each \( t : \mathbb{N} \to \mathbb{N} \), \( t \) is a translation function into \( \psi \) iff \( t \) is computable and the partial function \( \lambda p, x. \psi_{t(p)}(x) \) is an \( \epsilon \psi \).

**Definition 5.** Suppose that \( \psi \) is an \( \epsilon \psi \), and that \( t \) is a translation function into \( \psi \). Then, for each equivalence relation \( R \), (a) and (b) below.

(a) \( R \) strongly ties \( t \) into \( \psi \) iff \( R \) satisfies (i) and (ii) just below\(^2\)

(i) \( R \) is a subrelation of \( \psi \)'s program equivalence relation.

(ii) The range of \( t \) intersects each of \( R \)'s equivalence classes.

(b) \( R \) weakly ties \( t \) into \( \psi \) iff \( R \) satisfies (i) just above and (ii\(^*\)) just below\(^2\)

(ii\(^*\)) The range of \( t \) intersects all but finitely many of \( R \)'s equivalence classes.

Thus, if equivalence relation \( R \) strongly ties translation function \( t \) into \( \epsilon \psi \), then \( R \) equates each \( \psi \)-program to some program in the range of \( t \). If \( R \) merely weakly ties \( t \) into \( \psi \), then there may be infinitely many \( \psi \)-programs that \( R \) does not equate to any program in the range of \( t \). However, those infinitely many such \( \psi \)-programs will form only finitely many equivalence classes.

Our first main result is that the minimal \( \epsilon \psi \)es may be characterized as follows.

**Theorem 6.** For each \( \epsilon \psi \psi \), (a)-(c) below are equivalent.

(a) \( \psi \) is minimal.

\(^1\) In some places, we omit the phrase “into \( \psi \)” when it is clear from context.

\(^2\) See footnote 4
For each translation function $t$, there exists a computably enumerable equivalence relation (ceer) that strongly ties each translation function into $\psi$.

For each translation function $t$ into $\psi$, there exists an eps $\psi$ that satisfies (a) and (b) below.

(a) $\psi$ is minimal.
(b) For each ceer $R$, there exists a translation function $t$ into $\psi$ such that $R$ does not weakly tie $t$ into $\psi$.

Continuing with this line of thought, one finds that the strong and weak notions of Definition 5 separate when one considers single equivalence relations.

There exists an eps $\psi$ and a ceer $R$ satisfying (a) and (b) below.

(a) For each translation function $t$ into $\psi$, $R$ weakly ties $t$ into $\psi$.
(b) For each ceer $R'$, there exists a translation function $t$ into $\psi$ such that $R'$ does not strongly tie $t$ into $\psi$.

Clearly, if $\psi$ is an eps, and $\psi$'s program equivalence relation is computably enumerable, then there exists a single ceer $R$ that strongly ties each translation function into $\psi$, i.e., $R$ is $\psi$'s program equivalence relation. Thus, one might ask: does the converse hold? Theorem 9, just below, establishes that it does not.

There exists an eps $\psi$ and a ceer $R$ satisfying (a) and (b) below.

(a) For each translation function $t$ into $\psi$, $R$ strongly ties $t$ into $\psi$.
(b) $\psi$'s program equivalence relation is not computably enumerable.

Figure 1 summarizes the results mentioned in this section. The remainder of this paper is organized as follows. Section 2 covers preliminaries. Section 3 gives complete proofs of Theorems 6 through 9.

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We pronounce ceer like the first syllable of “series”. Computably enumerable equivalence relations are of interest in their own right. Gao and Gerdes [GG01] give an excellent survey.
2 Preliminaries

Computability-theoretic concepts not covered below are treated in [Rog67].

Lowercase math-italic letters (e.g., i, p, x), with or without decorations, range over elements of \( \mathbb{N} \), unless stated otherwise. Uppercase math-italic letters (e.g., I, P, X), with or without decorations, range over subsets of \( \mathbb{N} \), unless stated otherwise. For each non-empty \( X \), \( \min X \) denotes the minimum element of \( X \). min\( \emptyset \) \( \triangleq \) \( \infty \). For each non-empty, finite \( X \), max\( X \) denotes the maximum element of \( X \). max\( \emptyset \) \( \triangleq \) \(-1\). \( \text{fin} \) denotes the collection of all finite subsets of \( \mathbb{N} \).

\( \langle \cdot, \cdot \rangle \) denotes any fixed pairing function, i.e., a 1-1, onto, computable function of type \( \mathbb{N}^2 \rightarrow \mathbb{N} \) [Rog67 page 64]. For each \( x, y \), and \( z \), \( \langle x, y, z \rangle \triangleq \langle \langle x, y \rangle, z \rangle \). For each \( X \) and \( Y \), \( X \times Y \triangleq \{ \langle x, y \rangle \mid x \in X \land y \in Y \} \).

Every partial function considered herein maps \( \mathbb{N} \) to \( \mathbb{N} \), unless stated otherwise. For each partial function \( \zeta \), and each \( x \), \( \zeta(x)\downarrow \) denotes that \( \zeta(x) \) converges; whereas, \( \zeta(x)\uparrow \) denotes that \( \zeta(x) \) diverges. We use \( \uparrow \) to denote the value of a divergent computation. For the sake of some subsequent proofs, it is convenient to have the following notation. For each \( i \) and \( n \),

\[
i^{<n} \triangleq \lambda x. \begin{cases} i, & \text{if } x < n; \\ \uparrow, & \text{otherwise.} \end{cases}
\]

Thus, \( i^{<n} \) is the partial function that maps each value less than \( n \) to \( i \), and that diverges everywhere else. For each partial function \( \zeta \), \( \text{rng}(\zeta) \) denotes the range of \( \zeta \), i.e., \( \text{rng}(\zeta) \triangleq \{ y \mid (\exists x)[\zeta(x) = y] \} \). \( \text{PartComp} \) denotes the set of all partial computable functions (mapping \( \mathbb{N} \) to \( \mathbb{N} \)).

\( \varphi \) denotes any fixed acceptable (i.e., maximal) \( \text{eps} \) [Rog58 Rog67 MWY78 Ric81 Roy87]. For each \( p \), \( W_p \triangleq \{ x \mid \varphi_p(x) \downarrow \} \). For each \( p \) and \( s \), the following.

\[
\varphi_p^s \triangleq \lambda x. \begin{cases} \varphi_p(x), & \text{if } x < s \text{ and } \varphi_p(x) \text{ converges in fewer than } s \text{ steps;} \\ \uparrow, & \text{otherwise.} \end{cases}
\]

\[
W_p^s \triangleq \{ x \mid \varphi_p^s(x)\downarrow \}.
\]

For each \( \text{eps} \) \( \psi \), Equiv(\( \psi \)) denotes \( \psi \)’s program equivalence relation, i.e.,

\[
\text{Equiv}(\psi) \triangleq \{ \langle p, q \rangle \mid \psi_p = \psi_q \}.
\]

For each equivalence relation \( R \), \( \text{Classes}(R) \) denotes the set of \( R \)’s equivalence classes, i.e., \( \text{Classes}(R) \) is the set of exactly those \( E \) satisfying (a)-(c) below.

(a) \( E \neq \emptyset \).

(b) \( \forall p, q \in E \) \( \langle p, q \rangle \in R \).

(c) \( \forall p \in E \) \( \forall q \notin E \) \( \langle p, q \rangle \notin R \).

3 Results

This section recounts our main results (Theorem 6 through 9), and gives their complete proofs.

Our first main result is that the minimal \( \text{eps} \) may be characterized as per Theorem 8, restated just below. Recall from Definition 5 that if equivalence relation \( R \) strongly ties translation function \( t \) into \( \text{eps} \) \( \psi \), then (i) \( R \) is a subrelation of \( \psi \)’s program equivalence relation, and (ii) the range of \( t \) intersects each of \( R \)’s equivalence classes. On the other hand, if \( R \) merely weakly ties \( t \) into \( \psi \), then the range of \( t \) need only intersect all but finitely many of \( R \)’s equivalence classes.

**Theorem 6.** For each \( \text{eps} \) \( \psi \), (a)-(c) below are equivalent.

(a) \( \psi \) is minimal.

(b) For each translation function \( t \) into \( \psi \), there exists a \( \text{ceer} \) that strongly ties \( t \) into \( \psi \).

(c) For each translation function \( t \) into \( \psi \), there exists a \( \text{ceer} \) that weakly ties \( t \) into \( \psi \).
Proof. Let $\psi$ be given.

(a) $\Rightarrow$ (b): Suppose that $\psi$ is minimal. Let $t$ be any translation function into $\psi$, and let $\theta$ be such that $t$ witnesses $\theta \leq \psi$. Since $\psi$ is minimal, there exists a $t': \mathbb{N} \to \mathbb{N}$ witnessing $\psi \leq \theta$. Let $R$ be the reflexive, symmetric, transitive closure of

$$\{ (p, (t \circ t')(p)) \mid p \in \mathbb{N} \}.$$  

Clearly, $R$ is a ceer and $R \subseteq \text{Equiv}(\psi)$. It remains to show that, for each $E \in \text{Classes}(R)$, $\text{rng}(t) \cap E \neq \emptyset$. So, let $E \in \text{Classes}(R)$ be given, and let $p \in E$ be arbitrary. Then, clearly, $(t \circ t')(p) \in \text{rng}(t) \cap E$.

(b) $\Rightarrow$ (c): Immediate.

(c) $\Rightarrow$ (a): Suppose (c). Further suppose that $\theta$ is an eps, and that $t : \mathbb{N} \to \mathbb{N}$ witnesses $\theta \leq \psi$. Then, by (c), there exists a ceer $R \subseteq \text{Equiv}(\psi)$ such that, for all but finitely many $E \in \text{Classes}(R)$, $\text{rng}(t) \cap E \neq \emptyset$. Let $n$ be the number of elements of $\text{Classes}(R)$ that do not intersect $\text{rng}(t)$, and let $E_0, \ldots, E_{n-1}$ be those elements. Choose $q_0, \ldots, q_{n-1}$ such that, for each $i < n$ and $p \in E_i$, $\theta_{q_i} = \psi_p$. Note that, for each $p$, either $R$ equates $p$ to some element of $\text{rng}(t)$, or $p \in E_i$, for some $i < n$. It follows that the function $t' : \mathbb{N} \to \mathbb{N}$, defined next, is computable.

$$t' = \lambda p. \begin{cases} q_i, \text{ where } q_i \text{ is first found such that } \langle p, t(q_i) \rangle \in R, & \text{if such a } q_i \text{ exists;} \\ q_i, \text{ otherwise, where } i \text{ is such that } p \in E_i. & \end{cases}$$ 

It is straightforward to verify that $t'$ witnesses $\psi \leq \theta$. \hfill \Box (Theorem 6)

Theorem[?] restated just below, is our second main result. It establishes that there exists a minimal eps $\psi$ such that, for each ceer $R$, either $R$ contradicts $\psi$’s program equivalence relation, or there exists a translation function $t$ into $\psi$ such that the range of $t$ fails to intersect infinitely many of $R$’s equivalence classes.

**Theorem 7.** There exists an eps $\psi$ satisfying (a) and (b) below.

(a) $\psi$ is minimal.

(b) For each ceer $R$, there exists a translation function $t$ into $\psi$ such that $R$ does not weakly tie $t$ into $\psi$.

The proof of Theorem[?] makes use of the following lemma.

**Lemma 10.** Let $J_0, \ldots, J_{n-1}$ be any finite collection of computably enumerable sets. Then, there exists an infinite, computable set $X$, and a finite set $L \subseteq \{0, \ldots, n-1\}$, such that, for each $x \in X$ and $\ell < n$, $x \in J_\ell$ iff $\ell \in L$.

**Proof.** Let $J_0, \ldots, J_{n-1}$ be as stated. The set $X$ is the set $X_n$, constructed as follows. Set $X_0 = \mathbb{N}$. Then, for each $\ell < n$, act according to the following conditions.

- **Cond. (a)** $|J_\ell \cap X_\ell|$ is infinite. Set $X_{\ell+1}$ to any infinite, computable subset of $J_\ell \cap X_\ell$.
- **Cond. (b)** $|J_\ell \cap X_\ell|$ is finite. Set $X_{\ell+1} = \{ x \in X_\ell \mid x > \max(J_\ell \cap X_\ell) \}$.

The set $L$ is such that

$$L = \{ \ell \mid \text{cond. (a) applies for } \ell \}.$$ 

Clearly, $X$ is infinite and computable. Further note that

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n.$$ 

It is easily seen that, for each $\ell < n$: if $\ell \in L$, then $J_\ell \supseteq X_{\ell+1}$; whereas, if $\ell \notin L$, then $J_\ell \cap X_{\ell+1} = \emptyset$. It then follows from [11] that, for each $x \in X_n$ and $\ell < n$, $x \in J_\ell$ iff $\ell \in L$. \hfill \Box (Lemma 10)

**Proof of Theorem[?]** The eps $\psi$ is constructed below, following some necessary definitions. Let $\text{Aux} \subseteq \text{PartComp}$ be such that

$$\text{Aux} = \text{PartComp \setminus \{ (i,j)^{<k+1} \mid i,j \in \mathbb{N} \land k < 2^t \}}.$$ 

It is straightforward to show that $\text{Aux}$ is 1-1, computably enumerable. So, let $(\alpha_\ell)_{\ell \in \mathbb{N}}$ be a 1-1, effective numbering of $\text{Aux}$.

As is common, $\psi$ is constructed in stages, i.e., $\psi$ is the union of $\psi^0 \subseteq \psi^1 \subseteq \cdots$. In conjunction with $\psi$, four computable predicates are constructed: $\lambda i,s. [i \in R\text{-flags}^s]$, $\lambda i,j,\ell,s. [(i,j,\ell) \in t\text{-flags}^s]$, $\lambda \ell,s.[\ell \in \text{Src}^s]$, and $\lambda p,s.[p \in \text{Dst}^s]$. The purposes of these predicates are as follows.
The $R$-flags predicate keeps track of which $i$ are such that $W_i$ contradicts $\psi$'s program equivalence relation. More precisely, for each $i$, if there exists an $s$ such that $i \in R$-flags$^s$, then $W_i \not\subseteq \text{Equiv}(\psi)$.

The $t$-flags predicate helps to keep track of which $\ell$ may be such that $\varphi_\ell$ is a translation function into $\psi$. It will turn out that: if $i$ and $\ell$ are such that $W_i \subseteq \text{Equiv}(\psi)$ and $\varphi_\ell$ is a translation function into $\psi$, then, for each $j$, and all but finitely many $s$, $(i, j, \ell) \in t$-flags$^s$.

The Src predicate keeps track of which $\ell$ are such that $\alpha_\ell$ has not yet been assigned to any $\psi$-program. In particular, if $\ell$ and $s$ are such that $\ell \in \text{Src}^s$ and $\alpha_\ell \notin \lambda x.\uparrow$, then, for each $p$, $\psi_p^s \neq \alpha_\ell$.

The Dst predicate keeps track of which $\psi$-programs have not yet been used. More precisely, if $p$ and $s$ are such that $p \notin \text{Dst}^s$, then $\psi_p^s = \lambda x.\uparrow$.

For each $i$ and $s$, $i \in R$-flags$^{s+1}$ iff $i \in R$-flags$^s$, unless stated otherwise. Analogous statements apply to the $t$-flags, Src, and Dst predicates, as well. The following will be clear from the construction of $\psi$, for each $s$.

$$R\text{-flags}^s \subseteq R\text{-flags}^{s+1}. \quad (13)$$

$$t\text{-flags}^s \subseteq t\text{-flags}^{s+1}. \quad (14)$$

$$\text{Src}^s \supseteq \text{Src}^{s+1}. \quad (15)$$

$$\text{Dst}^s \supseteq \text{Dst}^{s+1}. \quad (16)$$

Let $\text{height} : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each $i$, $j$, and $s$,

$$\text{height}_{i,j}^s = |\{\ell \mid (i, j, \ell) \in t\text{-flags}^s\}|. \quad (17)$$

It will be clear from the construction of $\psi$ that, for each $i$, $j$, $\ell$, and $s$,

$$(i, j, \ell) \in t\text{-flags}^s \Rightarrow \ell < i. \quad (18)$$

Thus, for each $i$, $j$, and $s$,

$$\text{height}_{i,j}^s \leq i. \quad (19)$$

Let $\text{num} : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each $i$, $j$, and $s$,

$$\text{num}_{i,j}^s = 2i - h, \text{ where } h = \text{height}_{i,j}^s. \quad (20)$$

Let $f : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each $i$, $j$, and $k$,

$$f_{i,j}(k) = 2(i \cdot 2^{i+1} + k). \quad (21)$$

For each $i$, $j$, and $k < \text{num}_{i,j}^s$, let $E_{i,j,k}^s \subseteq \text{Fin}$ and $\tilde{E}_{i,j,k}^s \subseteq \text{Fin}$ be as follows, with $h = \text{height}_{i,j}^s$.

$$E_{i,j,k}^s = \{f_{i,j}(k \cdot 2^{h+1}), \ldots, f_{i,j}(k \cdot 2^{h+1} + 2^h - 1)\}. \quad (22)$$

$$\tilde{E}_{i,j,k}^s = \{f_{i,j}(k \cdot 2^{h+1} + 2^h), \ldots, f_{i,j}((k + 1) \cdot 2^{h+1} - 1)\}. \quad (23)$$

Note that, for each $i$, $j$, and $s$, if one lets $h = \text{height}_{i,j}^s$, and it happens that $\text{height}_{i,j}^{s+1} = h + 1$, then, for each $k < \text{num}_{i,j}^{s+1}$,

$$E_{i,j,k}^{s+1} = \{f_{i,j}(k \cdot 2^{h+2}), \ldots, f_{i,j}(k \cdot 2^{h+2} + 2^{h+1} - 1)\}$$

$$= \{f_{i,j}(k \cdot 2^{h+2}), \ldots, f_{i,j}(k \cdot 2^{h+2} + 2^h - 1)\} \cup \{f_{i,j}(2k \cdot 2^{h+1} + 2^h), \ldots, f_{i,j}(2k \cdot 2^{h+1} + 2^h - 1)\} \quad (24)$$

$$= \{f_{i,j}(2k \cdot 2^{h+1}), \ldots, f_{i,j}(2k \cdot 2^{h+1} + 2^h - 1)\} \cup \{f_{i,j}(2k \cdot 2^{h+1} + 2^h), \ldots, f_{i,j}((2k + 1) \cdot 2^{h+1} - 1)\}$$

$$= E_{i,j,2k}^s \cup \tilde{E}_{i,j,2k}^s.$$ 

It can be shown that, under the same conditions,

$$\tilde{E}_{i,j,k}^{s+1} = E_{i,j,2k+1}^s \cup \tilde{E}_{i,j,2k+1}^s. \quad (25)$$
- **Stage** \( s = -1 \). Do the following.
  - Set \( R \)-flags\(^0\) = \( \emptyset \).
  - Set \( t \)-flags\(^0\) = \( \emptyset \).
  - Set \( \text{Src}^0\) = \( \mathbb{N} \).
  - Set \( \text{Dst}^0\) = \( 2\mathbb{N} + 1 \).
  - For each \( i, j \), and \( k < 2^i \), set \( \psi_{i,j}^0(2k) = \psi_{i,j}^0(2k+1) = (i,j)_{<k+1} \).
  - For each \( p \in 2\mathbb{N} + 1 \), set \( \psi_p^0 = Ax_t \).

- **Stage** \( s \in (0, \ell) \). If \( \ell \in \text{Src}^s \), then do the following.
  - Set \( \text{Src}^{s+1} = \text{Src}^s \setminus \{ \ell \} \).
  - Set \( \text{Dst}^{s+1} = \text{Dst}^s \setminus \{ \min \text{Dst}^s \} \).
  - Set \( \psi_{\min \text{Dst}^s} = \alpha \ell \).

- **Stage** \( s = (i+1, 0, -) \). Determine whether there exist \( j \) and \( k \) satisfying conditions (a)-(c) just below.
  (a) \( i \not\in R \)-flags\(^s\).
  (b) \( k < \text{num}^s_{i,j} \).
  (c) \( \psi_{i,j}^s \cap (E_{i,j,k} \times \bar{E}_{i,j,k}) \neq \emptyset \).

  If such \( j \) and \( k \) exist, then do the following.
  - Set \( R \)-flags\(^{s+1} = R \)-flags\(^s \cup \{ i \} \).
  - Choose any \( \ell, m \in \text{Src}^s \) such that \( \ell \neq m \) and \( (i,j)_{<2^\ell} \subseteq \alpha \ell \cap \alpha m \).
  - Let \( d : \mathbb{N} \to \mathbb{N} \) be any 1-1, computable function such that \( \text{rng}(d) \) is computable, \( \text{rng}(d) \subseteq \text{Dst}^s \), and \( \text{Dst}^s \setminus \text{rng}(d) \) is infinite.
  - Set \( \text{Src}^{s+1} = \text{Src}^s \setminus \{ \ell, m \} \).
  - Set \( \text{Dst}^{s+1} = \text{Dst}^s \setminus \text{rng}(d) \).
  - For each \( j \), each \( k < \text{num}^s_{i,j} \), and each \( p \in E_{i,j,k} \), set \( \psi_{p}^{s+1} = \alpha \ell \).
  - For each \( j \), each \( k < \text{num}^s_{i,j} \), and each \( q \in \bar{E}_{i,j,k} \), set \( \psi_{q}^{s+1} = \alpha m \).
  - For each \( j \) and \( k < \text{num}^s_{i,j} \), set \( \psi_{d(n+k)}^{s+1} = (i,j)_{<(k+1)-2^h} \), where \( n = \sum_{j \subseteq \text{num}^s_{i,j}} \text{num}^s_{i,j} \) and \( h = \text{height}^s_{i,j} \).

- **Stage** \( s = (i+1, j+1, \ell, -) \). Let \( h = \text{height}^s_{i,j} \). Determine whether conditions (i)-(iv) just below are satisfied.
  (i) \( \ell < i \).
  (ii) \( i \not\in R \)-flags\(^s\).
  (iii) \( \langle i, j, \ell \rangle \not\in t \)-flags\(^s\).
  (iv) For each \( k < \text{num}^s_{i,j} \), \( \text{rng}(\varphi_{i,j}^s) \cap (E_{i,j,k} \cup \bar{E}_{i,j,k}) \neq \emptyset \).

If so, then do the following.
  - Set \( t \)-flags\(^{s+1} = t \)-flags\(^s \cup \{ \langle i, j, \ell \rangle \} \). (Note that this implies \( \text{height}^{s+1} = \text{height}^s_{i,j} + 1 \).)
  - Let \( n = \text{num}^{s+1}_{i,j} \). (Note that, by the just previous step, \( n = \text{num}^s_{i,j} + 2 \).)
  - Let \( \{ q_0 < q_1 < \cdots < q_{n-1} \} \) be the \( n \) least elements of \( \text{Dst}^s \).
  - Set \( \text{Dst}^{s+1} = \text{Dst}^s \setminus \{ q_0, q_1, \ldots, q_{n-1} \} \).
  - For each \( k < n \) and \( p \in (E_{i,j,k} \cup \bar{E}_{i,j,k}) \), set \( \psi_{p}^{s+1} = (i,j)_{(2k+2)-2^h} \).
  - For each \( k < n \), set \( \psi_{d(n+k)}^{s+1} = (i,j)_{<(2k+1)-2^h} \).

---

**Fig. 2.** The construction of \( \psi \) in the proof of Theorem 7. The symbols height, num, \( f \), \( E \), and \( \bar{E} \) are defined in [17], [20], [21], [22], and [23], respectively.
The partial function ψ is constructed in Figure 2. To help to give some of the intuition behind the construction, Figure 3 depicts what could happen with respect to the ψ-programs of the form \( f_{3,j}(k) \), where \( j \) is arbitrary and \( k < 16 \) (see text).

Note that by (14), (17), and (19), the following function \( \text{height}^\infty : \mathbb{N}^2 \rightarrow \mathbb{N} \) is well-defined. For each \( i \) and \( j \),

\[
\text{height}^\infty_{i,j} = \max \{ \text{height}^s_{i,j} \mid s \in \mathbb{N} \}.
\]

(26)

For each \( i \) and \( j \), let \( \text{num}^\infty_{i,j} \) be defined in a manner analogous to (20), but with \( h = \text{height}^\infty_{i,j} \). For each \( i, j, \) and \( k < \text{num}^\infty_{i,j} \), let \( E^\infty_{i,j,k} \) and \( \overline{E}^\infty_{i,j,k} \) be defined in a manner analogous to (22) and (23) (respectively), but with \( h = \text{height}^\infty_{i,j} \).

Claim 7.1 below establishes that \( \psi \) is an \( \text{eps} \). Claim 7.7 below establishes that \( \psi \) satisfies (a) in the statement of the theorem, i.e., that \( \psi \) is minimal. Claim 7.8 below establishes that \( \psi \) satisfies (b) in the statement of the theorem, i.e., that for each \( \text{ceer} R \), there exists a translation function \( t \) into \( \psi \) such that \( R \) does not weakly tie \( t \) into \( \psi \).

**Claim 7.1.** \( \psi \) is an \( \text{eps} \).

**Proof of Claim.** Clearly, \( \psi \) is partial computable. Thus, it suffices to show that, for each \( \zeta \in \text{PartComp} \), there exists a \( p \) such that \( \psi_p = \zeta \). So, let \( \zeta \in \text{PartComp} \) be given. Consider the following cases.

**Case \([ \zeta \in \text{Aux} \).** Let \( \ell \) be such that \( \alpha_\ell = \zeta \), and let \( s = (0, \ell) \). Then, the following are easily verifiable from the construction of \( \psi \).

- If \( \ell \in \text{Src}^* \), then there exists a \( p \) of the form \( f_{i,j}(k) \), for some \( i, j \), and \( k \), such that \( \psi^*_p = \zeta \).
- If $\ell \in \text{Src}^s$, then $\psi_{\text{min}}^{p+1} \text{Dst}^s = \zeta$.

Case $[\zeta \notin \text{Aux}]$. Let $i, j, k$, and $h$ be such that $\zeta = (i, j)^{(2k+1)^{-2h}}$. Then, the following are easily verifiable from the construction of $\psi$.

- If $\text{height}_{i,j}^{s} \leq h$ and $(\forall s)[i \notin \text{R-flags}^s]$, then, for each $p$,
  $$\quad p \in \{ f_{i,j}((2k+1) \cdot 2^{h+1} - 2), f_{i,j}((2k+1) \cdot 2^{h+1} - 1) \} \Rightarrow \psi_p = \zeta. \quad (27)$$

- If $\text{height}_{i,j}^{s} > h$ or $(\exists s)[i \in \text{R-flags}^s]$, then there exists a $p \in \text{Dst}^0 (= 2h + 1)$ such that $\psi_p = \zeta$.

□ (Claim 7.1)

Claim 7.2. Suppose that $i$ is such that $(\forall s)[i \notin \text{R-flags}^s]$. Then, for each $j$, each $k < \text{num}_{i,j}^{s}$, and each $p$,

$$\quad p \in (E_{i,j,k}^s \cup \bar{E}_{i,j,k}^s) \Leftrightarrow \psi_p = (i, j)^{(k+1) \cdot 2^h}, \quad (28)$$

where $h = \text{height}_{i,j}^{s}$.

Proof of Claim. Easily verifiable from the construction of $\psi$. □ (Claim 7.2)

Claim 7.3. Suppose that $i$ is such that $(\exists s)[i \in \text{R-flags}^s]$. Let $s_{\text{min}}$ be least such that $i \in \text{R-flags}^{s_{\text{min}}+1}$.

Then, there exist distinct $\ell$ and $m$ such that (a) and (b) below.

(a) For each $p$,
  $$\quad p \in \bigcup \{ E_{i,j,k}^s | \quad j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\text{min}}} \} \Leftrightarrow \psi_p = \alpha_{\ell}. \quad (29)$$

(b) For each $q$,
  $$\quad q \in \bigcup \{ \bar{E}_{i,j,k}^s | \quad j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\text{min}}} \} \Leftrightarrow \psi_q = \alpha_{m}. \quad (30)$$

Proof of Claim. Easily verifiable from the construction of $\psi$. □ (Claim 7.3)

Claim 7.4. For each $p \in \text{Dst}^0 (= 2h + 1)$ and $q$, if $\psi_p = \psi_q$, then $p = q$.

Proof of Claim. Easily verifiable from the construction of $\psi$. □ (Claim 7.4)

Claim 7.5. Suppose that $i, j, \ell$, and $s$ are such that $(i, j, \ell) \in \text{t-flags}^s$. Then,

$$\quad \text{rng}(\varphi_{\ell}) \cap E_{i,j,k}^s \neq \emptyset \land \text{rng}(\varphi_{\ell}) \cap \bar{E}_{i,j,k}^s \neq \emptyset. \quad (31)$$

Proof of Claim. Suppose that $i, j, \ell$, and $s$ are as stated. Let $s_{\text{min}}$ be least such that

$$\quad (i, j, \ell) \in \text{t-flags}^{s_{\text{min}}+1}. \quad (32)$$

Thus, $s > s_{\text{min}}$. By the construction of $\psi$, for each $k' < \text{num}_{i,j}^{s_{\text{min}}}$,

$$\quad \text{rng}(\varphi_{\ell}) \cap (E_{i,j,k'}^{s_{\text{min}}} \cup \bar{E}_{i,j,k'}^{s_{\text{min}}}) \neq \emptyset. \quad (33)$$

It follows from (24) and (32) that, for each $s > s_{\text{min}}$ and $k < \text{num}_{i,j}^{s}$, there exists a $k' < \text{num}_{i,j}^{s_{\text{min}}}$ such that

$$\quad E_{i,j,k'}^{s_{\text{min}}} \cup \bar{E}_{i,j,k'}^{s_{\text{min}}} \subseteq E_{i,j,k}^s. \quad (34)$$

Similarly, it follows from (25) and (32) that, for each $s > s_{\text{min}}$ and $k < \text{num}_{i,j}^{s}$, there exists a $k' < \text{num}_{i,j}^{s_{\text{min}}}$ such that

$$\quad E_{i,j,k'}^{s_{\text{min}}} \cup \bar{E}_{i,j,k'}^{s_{\text{min}}} \subseteq \bar{E}_{i,j,k}^s. \quad (35)$$

Formula (31) is implied by (33), (34), and (35). □ (Claim 7.5)
For each $i$, $j$, and $s$, act according to the following computable conditions. (Note that cond. (a) is computable, in part, because there are only finitely many $i \leq \ell$)

- **COND. (a)** $[\text{height}^s_i < \text{height}^{s+1}_i \land i \leq \ell \land (\forall s)[i \not\in R\text{-flags}]]$. For each $k < \text{num}_{i,j}^{s+1}$ and $p, q \in (E^s_{i,j,k} \cup \bar{E}^s_{i,j,k})$, list $(p, q)$ into $R$.

- **COND. (b)** $[\text{height}^s_i < \text{height}^{s+1}_i \land i > \ell]$. For each $k < \text{num}_{i,j}^{s+1}$ and $p, q \in E^s_{i,j,k}$, list $(p, q)$ into $R$. Similarly, for each $p, q \in E^{s+1}_{i,j,k}$, list $(p, q)$ into $R$.

For each $i$, act according to the following partial computable condition.

- **COND. (c)** $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\text{min}}$ be least such that $i \in R\text{-flags}^{s_{\text{min}} + 1}$, and do the following. For each $p, q \in \bigcup\{E_{i,j,k}^{s_{\text{min}}} \mid j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\text{min}}}\}$, list $(p, q)$ into $R$. Similarly, for each $p, q \in \bigcup\{\bar{E}_{i,j,k}^{s_{\text{min}}} \mid j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\text{min}}}\}$, list $(p, q)$ into $R$.

**Claim 7.6.** Suppose that $i$ is such that $W_i \subseteq \text{Equiv}(\psi)$. Then, $(\forall s)[i \not\in R\text{-flags}^s]$.

**Proof of Claim.** The proof is by contrapositive. Suppose that $i$ is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\text{min}}$ be least such that $i \in R\text{-flags}^{s_{\text{min}} + 1}$. Then, by the construction of $\psi$, there exist $j$ and $k$ such that

$$W_i^{s_{\text{min}}} \cap (E_{i,j,k}^{s_{\text{min}}} \times \bar{E}_{i,j,k}^{s_{\text{min}}}) \neq \emptyset. \quad (36)$$

Furthermore, by Claim 7.3 $\Rightarrow$, there exist distinct $\ell$ and $m$ such that (a) and (b) below.

(a) For each $p \in E_{i,j,k}^{s_{\text{min}}}$, $\psi_p = \alpha_{\ell}$.

(b) For each $q \in E_{i,j,k}^{s_{\text{min}}}$, $q = \alpha_{m}$.

Since $\alpha$ is 1-1 and $\ell \neq m$, $\alpha_{\ell} \neq \alpha_{m}$. Thus, by (36) and (a) and (b) just above, $W_i \not\subseteq \text{Equiv}(\psi)$. □ (Claim 7.6)

**Claim 7.7.** $\psi$ satisfies (a) in the statement of the theorem, i.e., $\psi$ is minimal.

**Proof of Claim.** Let $t$ be any translation function into $\psi$, and let $\ell$ be such that $\varphi_{\ell} = t$. To show the claim, a ceer $R$ is exhibited such that $R$ strongly ties $t$ into $\psi$. Initially, $R$ consists of $\{(p, p) \mid p \in \mathbb{N}\}$. Then, pairs are added to $R$ as in Figure 4.

Clearly, $R$ is a ceer. That $R \subseteq \text{Equiv}(\psi)$ follows from the $\Rightarrow$ directions of Claims 7.2 and 7.3. It remains to show that, for each $E \in \text{Classes}(R)$, $\text{rng}(t) \cap E \neq \emptyset$. It is straightforward to verify that each $E \in \text{Classes}(R)$ is of one of the following four types.

- **TYPE I.** $E$ is of the form

$$E^{\infty}_{i,j,k} \cup \bar{E}^{\infty}_{i,j,k}, \quad (37)$$

where: $i \leq \ell$, $(\forall s)[i \not\in R\text{-flags}^s]$, $j$ is arbitrary, and $k < \text{num}_{i,j}^{\infty}$. (Intuitively, $E$ is the result of one or more invocations of cond. (a) in Figure 4.)
- Type II. Either \( E \) is of the form
\[
E_{i,j,k}^\infty
\] (38)
or \( E \) is of the form
\[
\bar{E}_{i,j,k}^\infty
\] (39)
where: \( i > \ell, (\forall s)[i \not\in R\text{-flags}^s], j \) is arbitrary, and \( k < \text{num}_{i,j}^\infty \). (Intuitively, \( E \) is the result of one or more invocations of cond. (b) in Figure 4.)

- Type III. Either \( E \) is of the form
\[
\bigcup \{ E_{i,j,k}^{s_{\min}} \mid j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\min}} \}
\] (40)
or \( E \) is of the form
\[
\bigcup \{ \bar{E}_{i,j,k}^{s_{\min}} \mid j \in \mathbb{N} \land k < \text{num}_{i,j}^{s_{\min}} \}
\] (41)
where: \( i \) is such that \( (\exists s)[i \in R\text{-flags}^s] \), and \( s_{\min} \) is least such that \( i \in R\text{-flags}^{s_{\min}+1} \). (Intuitively, \( E \) is the result of zero or more invocations of cond. (b) in Figure 4 followed by a single invocation of cond. (c).)

- Type IV. \( E = \{ p \}, \) for some \( p \in \text{Dst}^0 \) (= \( 2N + 1 \)).

Let \( E \in \text{Classes}(R) \) be given. If \( E \) is of type I, then it follows from Claim 7.2 \( \Rightarrow \) that \( \text{rng}(t) \cap E \neq \emptyset \). If \( E \) is of type III, then it follows from Claim 7.3 \( \Rightarrow \) that \( \text{rng}(t) \cap E \neq \emptyset \). If \( E \) is of type IV, then it follows from Claim 7.4 that \( \text{rng}(t) \cap E \neq \emptyset \).

So, suppose that \( E \) is of type II. Let \( i, j, \) and \( k \) be such that \( E = E_{i,j,k}^\infty \) or \( E = \bar{E}_{i,j,k}^\infty \), as appropriate. Further suppose, by way of contradiction, that \( \text{rng}(t) \cap E = \emptyset \). Thus,
\[
\text{rng}(t) \cap E_{i,j,k}^\infty = \emptyset \lor \text{rng}(t) \cap \bar{E}_{i,j,k}^\infty = \emptyset.
\] (42)
Note that by Claim 7.2 \( \Leftrightarrow \), for each \( k' < \text{num}_{i,j}^\infty \) and \( p \),
\[
\psi_p = \langle i, j \rangle^{(k'+1)\cdot 2^h} \Rightarrow p \in (E_{i,j,k'}^\infty \cup \bar{E}_{i,j,k'}^\infty),
\] (43)
where \( h = \text{height}_{i,j}^\infty \). Thus, since \( t \) is a translation function into \( \psi \), it must be the case that, for each \( k' < \text{num}_{i,j}^\infty \),
\[
\text{rng}(t) \cap (E_{i,j,k'}^\infty \cup \bar{E}_{i,j,k'}^\infty) \neq \emptyset.
\] (44)
Choose \( s \) such that \( s \) is of the form \( \langle i+1, j+1, \ell, - \rangle \), \( \text{height}^s_{i,j} = \text{height}_{i,j}^\infty \), and, for each \( k' < \text{num}_{i,j}^\infty \),
\[
\text{rng}(\varphi^s_p) \cap (E_{i,j,k'}^\infty \cup \bar{E}_{i,j,k'}^\infty) \neq \emptyset.
\] (45)
Note that by (42) and Claim 7.5 \( (i, j, \ell) \not\in t\text{-flags}^s \). It follows that all of the conditions of stage \( s \) are satisfied. Thus, \( (i, j, \ell) \in t\text{-flags}^{s+1} \). But then
\[
\text{height}^{s+1}_{i,j} > \text{height}^s_{i,j} = \text{height}_{i,j}^\infty
\] (46)
— a contradiction.  \( \square \) (Claim 7.7)

**Claim 7.8.** \( \psi \) satisfies (b) in the statement of the theorem, i.e., for each ceer \( R \), there exists a translation function \( t \) into \( \psi \) such that \( R \) does not weakly tie \( t \) into \( \psi \).

**Proof of Claim.** Suppose that ceer \( R \) is such that
\[
R \subseteq \text{Equiv}(\psi).
\] (47)
Let \( i \) be such that \( W_i = R \). Note that by Claim 7.6
\[
(\forall s)[i \not\in R\text{-flags}^s].
\] (48)
For each \( \ell < i \), let \( J_\ell \) be as follows.
\[
J_\ell = \{ j \mid (\exists s)[(i, j, \ell) \in t\text{-flags}^s] \}.
\] (49)
Clearly, for each $\ell < i$, $J_i$ is computably enumerable. Thus, by Lemma 10, there exists an infinite, computable set $X$, and a finite set $L \subseteq \{0, ..., i-1\}$, such that, for each $x \in X$ and $\ell \in L$, $x \in J_i$ iff $\ell \in L$. Thus, for each $x \in X$,

$$L = \{\ell \mid x \in J_i\} = \{\ell \mid x \in \{j \mid (3s)[(i, j, \ell) \in t\text{-flags}^*]\}\} = \{\ell \mid (3s)[(i, x, \ell) \in t\text{-flags}^*]\}.$$

It follows that, for each $x \in X$, $\text{height}_{i,x}^\infty = |L|$ and $\text{num}_{i,x}^\infty = 2^{i-|L|}$. Let $t$ be any computable function such that

$$\text{rng}(t) = \mathbb{N} \setminus \bigcup\{E_{i,x,0}^\infty \mid x \in X\}.$$  \hspace{1cm} (50)

It is straightforward to show that that $t$ is a translation function into $\psi$. On the other hand, it is clearly the case that, for each $x \in X$,

$$\text{rng}(t) \cap E_{i,x,0}^\infty = \emptyset.$$  \hspace{1cm} (51)

Thus, to complete the proof, it suffices to show that, for each $E \in \text{Equiv}(R)$ and $x \in X$,

$$E \cap E_{i,x,0}^\infty \neq \emptyset \Rightarrow E \subseteq E_{i,x,0}^\infty.$$  \hspace{1cm} (52)

By way of contradiction, suppose otherwise, as witnessed by $E$ and $x$, i.e.,

$$E \cap E_{i,x,0}^\infty \neq \emptyset \land E \not\subseteq E_{i,x,0}^\infty.$$  \hspace{1cm} (53)

By (47), (48), (53), and Claim 7.2 (both directions), it must be the case that

$$E \cap E_{i,x,0}^\infty \neq \emptyset.$$  \hspace{1cm} (54)

Thus, by the first conjunct of (53), and by (54), there exists a stage $s$ of the form $(i+1, 0, –)$ in which all of the conditions of that stage are satisfied. Thus, $i \in R\text{-flags}^{s+1}$. But this contradicts (48). \hspace{1cm} \Box \hspace{1cm} \text{(Claim 7.8)}

\hspace{1cm} \Box \hspace{1cm} \text{(Theorem 7)}

Theorem 8, restated just below, is our third main result. It establishes that the strong and weak notions of Definition 5 separate when one considers single equivalence relations.

**Theorem 8.** There exists an $\epsilon$-flags $\psi$ and a ceer $R \subseteq \text{Equiv}(\psi)$ satisfying (a) and (b) below.

(a) For each translation function $t$ into $\psi$, $R$ weakly ties $t$ into $\psi$.

(b) For each ceer $R'$, there exists a translation function $t$ into $\psi$ such that $R'$ does not strongly tie $t$ into $\psi$.

**Proof.** The proof is essentially a modification to the proof of Theorem 7. Intuitively, one eliminates all uses of $j$ in that proof. So, for example, for each $i$, rather than start with infinitely many pairs of equivalence classes,

$$\{(E_{i,j,0}^0, E_{i,j,k}^0) \mid j \in \mathbb{N} \land k < 2^i\},$$  \hspace{1cm} (55)

one instead starts with just $2^i$ many such pairs,

$$\{(E_{i,k}^0, E_{i,k}^0) \mid k < 2^i\}.$$  \hspace{1cm} (56)

This has the effect of invalidating Claim 7.8 (and of making Lemma 10 unnecessary).

Let $\text{Aux} \subseteq \text{PartComp}$ be such that

$$\text{Aux} = \text{PartComp} \setminus \{i^{<k+1} \mid i \in \mathbb{N} \land k < 2^i\}.$$  \hspace{1cm} (57)

Let $(\alpha_i)_{i \in \mathbb{N}}$ be a 1-1, effective numbering of $\text{Aux}$.

In conjunction with $\psi$, four computable predicates are constructed: $\lambda i, s, [(i \in R\text{-flags}^*)$, $\lambda i, \ell, s, [(i, \ell) \in t\text{-flags}^*$], $\lambda \ell, s, [\ell \in \text{Src}^*$], and $\lambda p, s, [p \in \text{Dst}^*]$. The purposes of these predicates are similar to those in the proof of Theorem 7 (Note, however, the difference in the type of the $t$-flags predicate.)

Let $f : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each $i$ and $k$,

$$f_i(k) = 2 \cdot (2^{i+1} + k - 2).$$  \hspace{1cm} (58)

The following symbols are defined in a manner analogous to the proof of Theorem 7.

---

\[\text{In (50), we chose to use } \bigcup\{E_{i,x,0}^\infty \mid x \in X\}. \text{ But the proof can be completed using } \bigcup\{E_{i,x,k}^\infty \mid x \in X\} \text{ or } \bigcup\{E_{i,x,k} \mid x \in X\}, \text{ for any } k < \min\{\text{num}_{i,x}^\infty \mid x \in X\}.\]
height : \( \mathbb{N}^2 \to \mathbb{N} \) and \( \text{height}^\infty : \mathbb{N} \to \mathbb{N} \)
num : \( \mathbb{N}^2 \to \mathbb{N} \) and \( \text{num}^\infty : \mathbb{N} \to \mathbb{N} \)

The following symbols are defined similarly, but with \( f \) as in (58).

\(- E : \mathbb{N}^3 \to \text{fin} \) and \( E^\infty : \mathbb{N}^2 \to \text{fin} \)
\(- \bar{E} : \mathbb{N}^3 \to \text{fin} \) and \( \bar{E}^\infty : \mathbb{N}^2 \to \text{fin} \)

Suppose that \( i \) and \( s \) are such that \( \text{height}^s_{i+1} = \text{height}^s_i + 1 \). Then, by reasoning in a manner analogous to (24), it can be shown that, for each \( k < \text{num}^s_i+1 \), the following.

\[
\begin{align*}
E^s_{i,k+1} &= E^s_{i,2k} \cup \bar{E}^s_{i,2k} \\
\bar{E}^s_{i,k+1} &= E^s_{i,2k+1} \cup \bar{E}^s_{i,2k+1}.
\end{align*}
\]

(59)

The partial function \( \psi \) is constructed in Figure 5. One can show Claims 8.1 through 8.6 below. The proofs are similar to those of Claims 7.1 through 7.6 (respectively).

Claim 8.1. \( \psi \) is an eps.

Claim 8.2. Suppose that \( i \) is such that \((\forall s)[i \not\in R\text{-flags}^s]\). Then, for each \( k < \text{num}^{s+1}_i \), and each \( p \),

\[
p \in (E^s_{i,k} \cup \bar{E}^s_{i,k}) \iff \psi_p = i^{(k+1) \cdot 2^h}.
\]

(60)

where \( h = \text{height}^{s+1}_i \).

Claim 8.3. Suppose that \( i \) is such that \((\exists s)[i \in R\text{-flags}^s]\). Let \( s_{\min} \) be least such that \( i \in R\text{-flags}^{s_{\min}+1}_s \). Then, there exist distinct \( \ell \) and \( m \) such that (a) and (b) below.

(a) For each \( p \),

\[
p \in \bigcup \{E^s_{i,k} \mid k < \text{num}^{s_{\min}}_i\} \iff \psi_p = \alpha_\ell.
\]

(61)

(b) For each \( q \),

\[
q \in \bigcup \{ar{E}^s_{i,k} \mid k < \text{num}^{s_{\min}}_i\} \iff \psi_q = \alpha_m.
\]

(62)

Claim 8.4. For each \( p \in \text{Dst}^0 (= 2\mathbb{N} + 1) \) and \( q \), if \( \psi_p = \psi_q \), then \( p = q \).

Claim 8.5. Suppose that \( i, \ell, \) and \( s \) are such that \((i, \ell) \in t\text{-flags}^s \). Then,

\[
\text{rng}(\varphi_\ell) \cap E^s_{i,k} \neq \emptyset \land \text{rng}(\varphi_\ell) \cap \bar{E}^s_{i,k} \neq \emptyset.
\]

(63)

Claim 8.6. Suppose that \( i \) is such that \( W_i \subseteq \text{Equiv}(\psi) \). Then, \((\forall s)[i \not\in R\text{-flags}^s]\).

The relation \( R \) consists initially of \( \{\langle p, p \rangle \mid p \in \mathbb{N}\} \). Then, pairs are added to \( R \) as in Figure 6.

Clearly, \( R \) is a ceeer. That \( R \subseteq \text{Equiv}(\psi) \) follows from the \((\Rightarrow)\) directions of Claims 8.2 and 8.3.

Claim 8.7 below establishes that \( \psi \) and \( R \) satisfy (a) in the statement of the theorem, i.e., that for each translation function \( t \) into \( \psi \), \( R \) weakly ties \( t \) into \( \psi \). Claim 8.8 below establishes that \( \psi \) satisfies (b) in the statement of the theorem, i.e., that for each ceeer \( R' \), there exists a translation function \( t \) into \( \psi \) such that \( R' \) does not strongly tie \( t \) into \( \psi \).
\begin{itemize}
  \item \textsc{Stage} $s = -1$. Do the following.
    \begin{itemize}
      \item Set $R$-flags$^0 = \emptyset$.
      \item Set $t$-flags$^0 = \emptyset$.
      \item Set Src$^0 = \emptyset$.
      \item Set Dst$^0 = 2N + 1$.
      \item For each $i$ and $k < 2^i$, set $\psi_{\ell_i(2k)}^0 = \psi_{\ell_i(2k+1)}^0 = \kappa^{<k+1}_i$.
      \item For each $p \in 2N + 1$, set $\psi_p^0 = \lambda x_i.1$.
    \end{itemize}
  \item \textsc{Stage} $s = (0, \ell)$. If $\ell \in \text{Src}^s$, then do the following.
    \begin{itemize}
      \item Set $\text{Src}^{s+1} = \text{Src}^s \setminus \{\ell\}$.
      \item Set $\text{Dst}^{s+1} = \text{Dst}^s \setminus \{\text{min} \text{Dst}^s\}$.
      \item Set $\psi_{\min \text{Dst}^s}^1 = \alpha_\ell$.
    \end{itemize}
  \item \textsc{Stage} $s = (i + 1, 0, -)$. Determine whether there exists a $k$ satisfying conditions (a)-(c) just below.
    \begin{itemize}
      \item (a) $i \notin R$-flags$^s$.
      \item (b) $k < \text{num}_s^i$.
      \item (c) $W_s^i \cap (E_{i,k}^s \times E_{i,k}^s) \neq \emptyset$.
    \end{itemize}
    If such a $k$ exists, then do the following.
    \begin{itemize}
      \item Set $R$-flags$^{s+1} = R$-flags$^s \cup \{i\}$.
      \item Choose any $\ell, m \in \text{Src}^s$ such that $\ell \neq m$ and $i^{<\ell} \subseteq \alpha_\ell \cap \alpha_m$.
      \item Let $n = \text{num}_s^i$.
      \item Let $\{p_0 < p_1 < \cdots < p_{n-1}\}$ be the $n$ least elements of Dst$^s$.
      \item Set $\text{Src}^{s+1} = \text{Src}^s \setminus \{\ell, m\}$.
      \item Set $\text{Dst}^{s+1} = \text{Dst}^s \setminus \{p_0, p_1, \ldots, p_{n-1}\}$.
      \item For each $k < n$ and $p \in E_{i,k}^s$, set $\psi_p^i = \alpha_\ell$.
      \item For each $k < n$ and $q \in E_{i,k}^s$, set $\psi_q^{i+1} = \alpha_m$.
      \item For each $k < n$, set $\psi_p^{i+1} = i^{<(k+1)}2^h$, where $h = \text{height}_s^i$.
    \end{itemize}
  \item \textsc{Stage} $s = (i + 1, \ell + 1, -)$. Let $h = \text{height}_s^i$. Determine whether conditions (i)-(iv) just below are satisfied.
    \begin{itemize}
      \item (i) $\ell < i$.
      \item (ii) $i \notin R$-flags$^s$.
      \item (iii) $\langle i, \ell \rangle \notin t$-flags$^s$.
      \item (iv) For each $k < \text{num}_s^i$, $\text{rng}(p_i^s) \cap (E_{i,k}^s \cup E_{i,k}^s) \neq \emptyset$.
    \end{itemize}
    If so, then do the following.
    \begin{itemize}
      \item Set $t$-flags$^{s+1} = t$-flags$^s \cup \{(i, \ell)\}$. (Note that this implies height$^{s+1}_s = \text{height}_s^i + 1$.)
      \item Let $n = \text{num}_s^{s+1}$. (Note that, by the just previous step, $n = \text{num}_i^s/2$.)
      \item Let $\{q_0 < q_1 < \cdots < q_{n-1}\}$ be the $n$ least elements of Dst$^s$.
      \item Set Dst$^{s+1} = \text{Dst}^s \setminus \{q_0, q_1, \ldots, q_{n-1}\}$.
      \item For each $k < n$ and $p \in (E_{i,k}^s \cup E_{i,k}^s)$, set $\psi_p^{s+1} = i^{(2k+2)}2^h$.
      \item For each $k < n$, set $\psi_p^{s+1} = i^{<(2k+1)}2^h$.
    \end{itemize}
\end{itemize}

\textbf{Fig. 5.} The construction of $\psi$ in the proof of Theorem \ref{thm}.
For each $i$ and $s$, act according to the following computable condition.

- **COND. (a)** $[\text{height}^*_i < \text{height}^{i+1}_i]$. For each $k < \text{num}^{i+1}_i$ and
  \[ p, q \in E^{i+1}_{i,k}, \]
  list $(p, q)$ into $R$. Similarly, for each
  \[ p, q \in \bar{E}^{i+1}_{i,k}, \]
  list $(p, q)$ into $R$.
  
  For each $i$, act according to the following partial computable condition.

- **COND. (b)** $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\text{min}}$ be least such that $i \in R\text{-flags}^{s_{\text{min}}+1}$, and do the following. For each
  \[ p, q \in \bigcup \{E^{s_{\text{min}}}_{i,k} \mid k < \text{num}^{s_{\text{min}}}_i\}, \]
  list $(p, q)$ into $R$. Similarly, for each
  \[ p, q \in \bigcup \{\bar{E}^{s_{\text{min}}}_{i,k} \mid k < \text{num}^{s_{\text{min}}}_i\}, \]
  list $(p, q)$ into $R$.

\begin{figure}[h]
\centering
\textbf{Fig. 6.} The construction of $R$ in the proof of Theorem 8.
\end{figure}

**Claim 8.7.** $\psi$ and $R$ satisfy (a) in the statement of the theorem, i.e., for each translation function $t$ into $\psi$, $R$ weakly ties $t$ into $\psi$.

**Proof of Claim.** It is straightforward to verify that each $E \in \text{Classes}(R)$ is of one of the following three types.

- **Type I.** Either $E$ is of the form
  \[ E^\infty_{i,k} \] (64)
  or $E$ is of the form
  \[ \bar{E}^\infty_{i,k} \] (65)

where: $i$ is such that $(\forall s)[i \not\in R\text{-flags}^s]$, and $k < \text{num}^{\infty}_i$. (Intuitively, $E$ is the result of one or more invocations of cond. (a) in Figure 5.)

- **Type II.** Either $E$ is of the form
  \[ \bigcup \{E^{s_{\text{min}}}_{i,k} \mid k < \text{num}^{s_{\text{min}}}_i\} \] (66)
  or $E$ is of the form
  \[ \bigcup \{\bar{E}^{s_{\text{min}}}_{i,k} \mid k < \text{num}^{s_{\text{min}}}_i\} \] (67)

where: $i$ is such that $(\exists s)[i \in R\text{-flags}^s]$, and $s_{\text{min}}$ is least such that $i \in R\text{-flags}^{s_{\text{min}}+1}$. (Intuitively, $E$ is the result of zero or more invocations of cond. (a) in Figure 6 followed by a single invocation of cond. (b).)

- **Type III.** $E = \{p\}$, for some $p \in \text{Dst}^0$ ($= 2\mathbb{N} + 1$).

Let $t$ be any translation function into $\psi$, and let $\ell$ be such that $\varphi_\ell = t$. Note that there are only finitely many $E \in \text{Classes}(R)$ of type I for which $i \leq \ell$, where $i$ is such that $E = E^\infty_{i,k}$ or $E = \bar{E}^\infty_{i,k}$, as appropriate. Thus, to show the claim, it suffices to show that, for each $E \in \text{Classes}(R)$: if $E$ is of type II or III, then $\text{rng}(t) \cap E \neq \emptyset$; whereas, if $E$ is of type I, then $\text{rng}(t) \cap E \neq \emptyset$ or $i \leq \ell$ (where $i$ is as just mentioned).

So, let $E \in \text{Classes}(R)$ be given. If $E$ is of type II, then it follows from Claim 8.3 $\Leftarrow$ that $\text{rng}(t) \cap E \neq \emptyset$. If $E$ is of type III, then it follows from Claim 8.4 that $\text{rng}(t) \cap E \neq \emptyset$.

So, suppose that $E$ is of type I, and that $\text{rng}(t) \cap E = \emptyset$. Let $i$ and $k$ be such $E = E^\infty_{i,k}$ or $E = \bar{E}^\infty_{i,k}$, as appropriate. To show that $i \leq \ell$, one first assumes otherwise, by way of contradiction. One then proceeds in a manner analogous to the proof of Claim 7.7, beginning just before (42).

(\textbf{Claim 8.7})

**Claim 8.8.** $\psi$ satisfies (b) in the statement of the theorem, i.e., for each $\text{ceer} R'$, there exists a translation function $t$ into $\psi$ such that $R'$ does \textit{not} strongly tie $t$ into $\psi$. 

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Proof of Claim. Suppose that \( \text{ceer } R' \subseteq \text{Equiv}(\psi) \). Let \( i \) be such that \( W_i = R' \). Let \( t \) be any computable function such that
\[
\text{rng}(t) = \mathbb{N} \setminus E^\infty_{i,0}.
\]
(68)
It is straightforward to show that \( t \) is a translation function into \( \psi \). On the other hand, it is clearly the case that
\[
\text{rng}(t) \cap E^\infty_{i,0} = \emptyset.
\]
(69)
Thus, to complete the proof, it suffices to show that, for each \( E \in \text{Equiv}(R) \),
\[
E \cap E^\infty_{i,0} \neq \emptyset \Rightarrow E \subseteq E^\infty_{i,0}.
\]
(70)
This can be shown in a manner analogous to the proof of Claim 7.8, beginning just after (52).
\[\square\]
(Claim 8.8)
\[\square\]
(Theorem 8)

Theorem 9, restated just below, is our final main result. It establishes that there can exist a single ceer that strongly ties each translation function into an eps, yet that eps’s program equivalence relation can fail to be computably enumerable.

**Theorem 9.** There exists an eps \( \psi \) and a ceer \( R \subseteq \text{Equiv}(\psi) \) satisfying (a) and (b) below.

(a) For each translation function \( t \) into \( \psi \), \( R \) strongly ties \( t \) into \( \psi \).
(b) \( \text{Equiv}(\psi) \) is not computably enumerable.

**Proof.** The eps \( \psi \) is constructed below, following some necessary definitions. Let \( \text{Aux} \subseteq \text{PartComp} \) be such that
\[
\text{Aux} = \text{PartComp} \setminus (\{i \leq j + 1 \mid i, j \in \mathbb{N}\} \cup \{\lambda x.i \mid i \in \mathbb{N}\}).
\]
(71)
Let \( (\alpha_k)_{k \in \mathbb{N}} \) be a 1-1, effective numbering of \( \text{Aux} \).

In conjunction with \( \psi \), the following six computable predicates are constructed.

- \( \lambda i, s.[i \in R\text{-flags}^s] \)
- \( \lambda i, j, s.[(i, j) \in t\text{-flags}^s] \)
- \( \lambda j, s.[j \in \text{Src}^s] \)
- \( \lambda p, s.[p \in \text{Dst}^s] \)
- \( \lambda p, i, s.[p \in E_i^s] \)
- \( \lambda q, i, s.[q \in \bar{E}_i^s] \)

The purposes of these predicates are similar to those in the proofs of Theorems 7 and 8. Note, however, that in the proofs of Theorems 7 and 8 the \( E \) and \( \bar{E} \) predicates were calculated; whereas, in this proof, they are constructed. The following will be clear from the construction of \( \psi \), for each \( i \) and \( s \).
\[
E_i^s \subseteq E_i^{s+1}.
\]
(72)
\[
\bar{E}_i^s \subseteq \bar{E}_i^{s+1}.
\]
(73)

For each \( i \), let \( E_i^\infty \) and \( \bar{E}_i^\infty \) be as follows.
\[
E_i^\infty = \bigcup \{E_i^s \mid s \in \mathbb{N}\}.
\]
(74)
\[
\bar{E}_i^\infty = \bigcup \{\bar{E}_i^s \mid s \in \mathbb{N}\}.
\]
(75)

The partial function \( \psi \) is constructed in Figure 7. Claim 9.1 below establishes that \( \psi \) is an eps.

**Claim 9.1.** \( \psi \) is an eps.

**Proof of Claim.** Clearly, \( \psi \) is partial computable. Thus, it suffices to show that, for each \( \zeta \in \text{PartComp} \), there exists a \( p \) such that \( \psi_p = \zeta \). So, let \( \zeta \in \text{PartComp} \) be given. Consider the following cases.

CASE \([\zeta \in \text{Aux}]\). Let \( k \) be such that \( \alpha_k = \zeta \), and let \( s = (0, k) \). Then, the following are easily verifiable from the construction of \( \psi \).
- **Stage** \(s = -1\). Do the following.
  - Set \(R\)-flags\(^0\) = \(\emptyset\).
  - Set \(t\)-flags\(^0\) = \(\emptyset\).
  - Set Src\(^0\) = \(\mathbb{N}\).
  - Set Dst\(^0\) = \(3\mathbb{N} + 2\).
  - For each \(i\), set \(E_i^0 = \bar{E}_i^0 = \emptyset\).
  - For each \(i\) and \(j\), set \(\psi_{3(i,j)}^0 = i < 2j + 1\).
  - For each \(i\) and \(j\), set \(\psi_{3(i,j) + 1}^0 = i < 2j + 2\).
  - For each \(p \in 3\mathbb{N} + 2\), set \(\psi_p^0 = \lambda x + \uparrow\).

- **Stage** \(s = (0, k)\). If \(k \in \text{Src}^s\), then do the following.
  - Set \(\text{Src}^{s+1} = \text{Src}^s \setminus \{k\}\).
  - Set \(\text{Dst}^{s+1} = \text{Dst}^s \setminus \{\min \text{Dst}^s\}\).
  - Set \(\psi_{\min \text{Dst}^s} = \alpha_k\).

- **Stage** \(s = (i + 1, 0, -)\). If \(i \not\in \text{R-flags}^s\) and \(W^s_i \cap (E^s_i \times E^s_i) \neq \emptyset\), then do the following.
  - Set \(R\)-flags\(^{s+1}\) = \(R\)-flags\(^s\) \(\cup \{i\}\).
  - Let \(n\) be least such that, for each \(p \in (E^s_i \cup \bar{E}_i^s)\), \(\psi_p \subseteq i < n\).
  - Choose any \(k, \ell \in \text{Src}^s\) such that \(k \neq \ell\) and \(i < n\) \(\subseteq\) \(\alpha_k \cap \alpha_\ell\).
  - Set \(\text{Src}^{s+1} = \text{Src}^s \setminus \{k, \ell\}\).
  - Set \(\text{Dst}^{s+1} = \text{Dst}^s \setminus \{\min \text{Dst}^s\}\).
  - For each \(p \in E^s_i\), set \(\psi_p^{s+1} = \alpha_k\).
  - For each \(q \in E^s_i\), set \(\psi_q^{s+1} = \alpha_\ell\).
  - Set \(\psi_{\min \text{Dst}^s} = \lambda x + \uparrow\).

- **Stage** \(s = (i + 1, j + 1, -)\). Determine whether conditions (i)-(iii) just below are satisfied.
  1. \(i \not\in \text{R-flags}^s\).
  2. \((i, j) \not\in \text{t-flags}^s\).
  3. \(\{3(i, j), 3(i, j) + 1\} \subseteq \text{rng}(\phi_j^s)\).
  If so, then do the following.
  - Set \(t\)-flags\(^{s+1}\) = \(t\)-flags\(^s\) \(\cup \{(i, j)\}\).
  - Set \(E_i^{s+1} = E_i^s \cup \{3(i, j)\}\).
  - Set \(\bar{E}_i^{s+1} = \bar{E}_i^s \cup \{3(i, j) + 1\}\).
  - Let \(n\) be least such that, for each \(p \in (E_i^{s+1} \cup \bar{E}_i^{s+1})\), \(\psi_p^{s+1} \subseteq i < n\).
  - For each \(p \in (E_i^{s+1} \cup \bar{E}_i^{s+1})\), set \(\psi_p = i < n\).
  - Let \(\{q_0 < q_1\}\) be the two least elements of \(\text{Dst}^s\).
  - Set \(\text{Dst}^{s+1} = \text{Dst}^s \setminus \{q_0, q_1\}\).
  - Set \(\psi_{q_0}^{s+1} = i < 2j + 1\).
  - Set \(\psi_{q_1}^{s+1} = i < 2j + 2\).

---

Fig. 7. The construction of \(\psi\) in the proof of Theorem 9
(b) For each $q$, there exist infinitely many stages of the form $\psi_i = \zeta$.

Case $[\zeta \notin \mathbb{A}_k \land (\exists i,j)[\zeta = i^{<2^j+1}]]$. Let $i$ and $j$ be as in the case. Then, the following are easily verifiable from the construction of $\psi$.

- If $([v]s)[(i,j) \notin t\text{-flags}^a]$, then $\psi_{3(i,j)} = \zeta$.
- If $([v]s)[(i,j) \notin t\text{-flags}^a]$, then there exists a $p \in \text{Dst}^0 (= 3N + 2)$ such that $\psi_p = \zeta$.

Case $[\zeta \notin \mathbb{A}_k \land (\exists i,j)[\zeta = i^{<2^j+2}]]$. Similar to the previous case. □ (Claim 9.1)

The relation $R$ is defined as follows.

$$R = \{ \langle p, p \rangle \mid p \in \mathbb{N} \} \cup \{ \langle p, q \rangle \mid p, q \in E_i^\infty \land i \in \mathbb{N} \} \cup \{ \langle p, q \rangle \mid p, q \in E_i^\infty \land i \in \mathbb{N} \}. \quad (76)$$

Clearly, $R$ is a c.eer. That $R \subseteq \text{Equiv}(\psi)$ follows from the $(\Rightarrow)$ directions of Claims 9.2 and 9.3.

Claim 9.6 below establishes that $\psi$ and $R$ satisfy (a) in the statement of the theorem, i.e., that for each translation function $t$ into $\psi$, $R$ strongly ties $t$ into $\psi$. Claim 9.7 below establishes that $\psi$ satisfies (b) in the statement of the theorem, i.e., that $\text{Equiv}(\psi)$ is not computably enumerable.

**Claim 9.2.** Suppose that $i$ is such that $([v]s)[i \notin R\text{-flags}^a]$. Then, (a) and (b) below.

(a) Each of $E_i^\infty$ and $E_i^{\infty}$ is infinite.
(b) For each $p, p \in (E_i^\infty \cup E_i^{\infty}) \Leftrightarrow \psi_p = \lambda x. i$.

*Proof of Claim.* Suppose that $i$ is such that $([v]s)[i \notin R\text{-flags}^a]$. Note that there exist infinitely many $j$ such that $\text{rng}(\varphi_j) = \mathbb{N}$. Thus, there exist infinitely many $j$ such that $\{3(i,j), 3(i,j) + 1 \} \subseteq \text{rng}(\varphi_j^a)$, for all but finitely many $s$. It follows that there exist infinitely many stages of the form $(i + 1, j + 1, \bar{v})$ such that all of the conditions of those stages are satisfied. Given this fact, both (a) and (b) are easily verifiable from the construction of $\psi$. □ (Claim 9.2)

**Claim 9.3.** Suppose that $i$ is such that $([v]s)[i \in R\text{-flags}^a]$. Let $s_{\text{min}}$ be least such that $i \in R\text{-flags}^{s_{\text{min}}+1}$. Then, (a) and (b) below.

(a) $E_i^\infty = E_i^{s_{\text{min}}}$ and $E_i^{\infty} = E_i^{s_{\text{min}}}$.
(b) There exist distinct $k$ and $\ell$ such that (i) and (ii) below.

(i) For each $p, p \in E_i^\infty \Leftrightarrow \psi_p = \alpha_k$.
(ii) For each $q, q \in E_i^{\infty} \Leftrightarrow \psi_q = \alpha_\ell$.

*Proof of Claim.* Easily verifiable from the construction of $\psi$. □ (Claim 9.3)

**Claim 9.4.** For each $p$ such that $p \notin \bigcup\{E_i^\infty \cup E_i^{\infty} \mid i \in \mathbb{N} \}$, and, for each $q$, if $\psi_p = \psi_q$, then $p = q$.

*Proof of Claim.* Easily verifiable from the construction of $\psi$. □ (Claim 9.4)

**Claim 9.5.** Suppose that $i$ is such that $W_i \subseteq \text{Equiv}(\psi)$. Then, $([v]s)[i \notin R\text{-flags}^a]$.

*Proof of Claim.* The proof is by contrapositive. Suppose that $i$ is such that $([v]s)[i \in R\text{-flags}^a]$. Let $s_{\text{min}}$ be least such that $i \in R\text{-flags}^{s_{\text{min}}+1}$. Then, by the construction of $\psi$,

$$W_i^{s_{\text{min}}} \cap (E_i^{s_{\text{min}}} \times E_i^{s_{\text{min}}}) \neq \emptyset. \quad (78)$$

Furthermore, by Claim 9.3 $\Rightarrow$, there exist distinct $k$ and $\ell$ such that (a) and (b) below.

(a) For each $p \in E_i^{s_{\text{min}}}$, $\psi_p = \alpha_k$.
(b) For each $q \in E_i^{s_{\text{min}}}$, $\psi_q = \alpha_\ell$.

Since $\alpha$ is 1-1 and $k \neq \ell$, $\alpha_k \neq \alpha_\ell$. Thus, by (78) and (a) and (b) just above, $W_i \not\subseteq \text{Equiv}(\psi)$. □ (Claim 9.5)
Claim 9.6. \( \psi \) and \( R \) satisfy (a) in the statement of the theorem, i.e., for each translation function \( t \) into \( \psi \), \( R \) strongly ties \( t \) into \( \psi \).

**Proof of Claim.** It is straightforward to verify that each \( E \in \text{Classes}(R) \) is of one of the following three types.

- **Type I.** Either \( E \) is of the form \( E_\infty^i \) or \( E \) is of the form \( \bar{E}_\infty^i \) where: \( i \) is such that \((\forall s)[i \notin R\text{-flags}^s] \).
- **Type II.** Either \( E \) is of the form \( E_\infty^i \) or \( E \) is of the form \( \bar{E}_\infty^i \) where: \( i \) is such that \((\exists s)[i \in R\text{-flags}^s] \).
- **Type III.** \( E = \{p\} \), for some \( p \) such that

\[
P \notin \bigcup\{E_\infty^i \cup \bar{E}_\infty^i \mid i \in \mathbb{N}\}.
\] (79)

Let \( t \) be any translation function into \( \psi \), and let \( E \in \text{Classes}(R) \) be given. If \( E \) is of type I, then it follows from Claim 9.2(\( \Leftarrow \)) that \( \text{rng}(t) \cap E \neq \emptyset \). If \( E \) is of type II, then it follows from Claim 9.3(\( \Leftarrow \)) that \( \text{rng}(t) \cap E \neq \emptyset \). If \( E \) is of type III, then it follows from Claim 9.4 that \( \text{rng}(t) \cap E \neq \emptyset \). \( \square \) (Claim 9.6)

Claim 9.7. \( \psi \) satisfies (b) in the statement of the theorem, i.e., \( \text{Equiv}(\psi) \) is not computably enumerable.

**Proof of Claim.** By way of contradiction, let \( i \) be such that

\[
W_i = \text{Equiv}(\psi).
\] (80)

By (80) and Claim 9.5,

\[
(\forall s)[i \notin R\text{-flags}^s].
\] (81)

By (81) and Claim 9.2(a), each of \( E_\infty^i \) and \( \bar{E}_\infty^i \) is infinite, and, thus,

each of \( E_\infty^i \) and \( \bar{E}_\infty^i \) is non-empty. (82)

By (81) and Claim 9.2(b)(\( \Rightarrow \)), for each \( p \in (E_\infty^i \cup \bar{E}_\infty^i) \),

\[
\psi_p = \lambda x. \_.
\] (83)

By (80), (82), and (83),

\[
W_i \cap (E_\infty^i \times \bar{E}_\infty^i) \neq \emptyset.
\] (84)

By (81) and (84), there exists a stage \( s \) of the form \( \langle i+1, 0, - \rangle \) in which all of the conditions of that stage are satisfied. But then \( i \in R\text{-flags}^{s+1} \), contradicting (81). \( \square \) (Claim 9.7)

\( \square \) (Theorem 9)

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