The conjugation thin inclusions problem in elastic bodies with crack

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Abstract. A problem of pairing thin elastic and rigid inclusions with possible delamination in elastic bodies is considered. On the crack and at the intersection of the crack with a hard inclusion, boundary conditions are imposed in the form of inequalities, excluding mutual penetration of the crack faces and thin inclusions. The existence and uniqueness of the solution of the problem is established. It is shown that a corresponding variational statement is equivalent to a differential setting. A passage to the limit as stiffness parameter of thin inclusions tends to infinity is investigated.

1. Introduction
The topicality of studying boundary value problems of equilibrium of the elastic bodies containing thin inclusions and cracks is due to the widespread use of composite materials in technology. The thin inclusion term is used in the cases where the dimension of the inclusion is less than that of the body. A crack can also be considered as a thin inclusion of zero rigidity. It is known that the classical approach to the description of cracks in deformable bodies is characterized by linear boundary conditions on the crack faces. However, from the standpoint of physics, these linear models have some shortcoming: The opposite crack faces may penetrate into each other. Systematic study of equilibrium boundary value problems concerned the elastic bodies with cracks under boundary conditions of mutual nonpenetration of faces was initiated in 1992. Presently, there is a wide range of studies where the mathematical formulation of this class of problems has been developed [1–7].

The purpose of the present work is to study the equilibrium problems of two-dimensional elastic bodies with a crack that crosses a thin inclusion. It is assumed that the crack at the intersection point divides the inclusion into two parts. As a result, conjugation of these parts occurs. The two types of thin inclusions are considered: rigid and elastic Bernoulli–Euler ones. On the crack faces, some nonlinear boundary conditions describe the mutual nonpenetration of the crack faces. Taking into account the nonpenetration condition at the point of intersection of the crack and thin inclusion, we obtain a complete set of boundary conditions that are satisfied on the crack as well as on a part of the inclusion boundaries. Applying the direct variational method of minimizing the energy functionals, we prove the existence and uniqueness of solutions to the problems under consideration. For the variational formulations of the problems, the equivalent differential formulations are found. The passage to the limit with respect to the rigidity parameter of a thin elastic inclusion is investigated. It is shown that, as the rigidity
parameter tends to infinity, the limit problem precisely describe the equilibrium problem of an elastic body with a rigid inclusion.

2. Equilibrium problems

Let $\Omega$ be a bounded domain with a smooth boundary $\Gamma$ in the space $\mathbb{R}^2$. We assume that $\gamma \subset \Omega$, $\Gamma_c \subset \Omega$, be some smooth curves without self-intersections which intersect at the point $P : \gamma \cap \Gamma_c = \{P\} = \{(0, 0)\}$. Suppose that there exist some extensions of $\Gamma_c$ and $\gamma$ crossing the boundary $\Gamma$ and subdividing $\Omega$ into the four subdomains $D_1, D_2, D_3$ and $D_4$ with Lipschitz boundaries $\partial D_i$, where $\text{meas}(\Gamma \cap \partial D_i) > 0$, $i = 1, 2, 3, 4$. To simplify the notation, we denote the normals to $\Gamma_c$ and $\gamma$ by $\nu = (\nu_1, \nu_2)$; $\nu_0$ is the normal at $P$ whose direction coincides with that of the normal $\nu$ to $\Gamma_c$ (see Figure 1). The direction of the normals $\nu$ determines the positive and negative sides of these curves. Put $\Omega_\gamma^c = \Omega \setminus (\gamma \cup \Gamma_c)$. In the sequel, the domain $\Omega_\gamma^c$ represents the elastic body in the natural state, a thin inclusion is located at $\gamma$, and the crack described by the curve $\Gamma_c$. Note that the crack subdivides the thin rigid inclusion into the two parts so that $\gamma = \gamma_1 \cup \gamma_2 \cup \{P\}$, where $\gamma_1$ and $\gamma_2$ smooth curves.

In this section, we will consider three equilibrium problems for elastic bodies containing the following types of thin inclusion: 1) rigid, 2) elastic, 3) rigid and elastic in conjugation.

2.1. Delaminated rigid inclusion

We consider the case when $\gamma$ corresponds to a thin rigid inclusion with delamination. Assume that a delamination of the rigid inclusion takes place at $\gamma^+$, thus we have a crack between the elastic body and the thin inclusion. In our model, inequality type boundary conditions will be considered to prevent an mutual penetration between the crack faces. Displacements of the inclusion should coincide with the displacements of the elastic body at $\gamma^-$. 

Introduce the space of infinitesimal rigid displacements

$$R(\gamma_i) = \{\rho_i = (\rho_{i1}, \rho_{i2}) | \rho_i(x_1, x_2) = b_i(-x_2, x_1) + (c_{i1}, c_{i2}) ; b_i, c_{i1}, c_{i2} = \text{const}, (x_1, x_2) \in \gamma_i\},$$

$i = 1, 2$. Below, in order to simplify the representation, we use the following notation:

$\rho|_{\gamma_1} = \rho_1$, $\rho|_{\gamma_2} = \rho_2$, $\rho_i \in R(\gamma_i)$, $i = 1, 2$.

Formulation of the equilibrium problem for a two-dimensional elastic body with a crack $\Gamma_c$ and thin rigid inclusions $\gamma_1$ and $\gamma_2$ with delamination is as follows. We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, $\rho_0^i \in R(\gamma_i)$ such that

$$-\text{div} \sigma = f, \quad \sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_\gamma^c,$$

$$u = 0 \text{ on } \Gamma,$$
\[ \langle \rho^0(P) \rangle \nu_0 \geq 0, \quad u^- = \rho^0_i \text{ on } \gamma_i, \quad i = 1, 2, \quad (3) \]
\[ [u_\nu] \geq 0, \quad \sigma_\nu \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u_\nu] = 0 \text{ on } \Gamma_c, \quad (4) \]
\[ \int \langle \sigma \nu \rangle \rho_1 + \int \langle \sigma \nu \rangle \rho_2 \leq 0 \quad \forall \rho_1 \in R(\gamma_i), \quad i = 1, 2, \quad \langle \rho(P) \rangle \nu_0 \geq 0, \quad (6) \]
\[ \int \langle \sigma \nu \rangle \rho_0^1 + \int \langle \sigma \nu \rangle \rho_2^0 = 0. \quad (7) \]

Here \([v] = v^+ - v^-\) is a jump of a function \(v\) on \(\Gamma_c\), where \(v^\pm = v|_{\Gamma^\pm}\) correspond to the values of the function \(v\) on the curve \(\Gamma_c\) with respect to the normal \(\nu\). The same is true for the functions \(\nu\) on \(\gamma\). The angular brackets \(\langle \rho \rangle = \rho^+ - \rho^-\) mean the jump of \(\rho\) at the point \(P\), where \(\rho^+\) is the values of the functions \(\rho\) on the curves \(\gamma_1\) and \(\gamma_2\) at the point \(P\) relative to the normal \(\nu_0\).

\[ \langle \rho^0(P) \rangle \nu_0 = \langle \rho^0_1(P) - \rho^0_2(P) \rangle \nu_0 \geq 0 \text{ and } \langle \rho(P) \rangle \nu_0 = \langle \rho_1(P) - \rho_2(P) \rangle \nu_0 \geq 0; \quad \varepsilon = \{\varepsilon_{ij}\} \text{ is the strain tensor, } \varepsilon_{ij}(u) = \frac{1}{2}(u_{ij} + u_{ji}), \quad i, j = 1, 2. \]

By \(A = \{a_{ijkl}\}\), \(i, j, k, l = 1, 2\), \(a_{ijkl} = a_{jikl} = a_{klij}, \quad a_{ijkl} \in L^\infty(\Omega), \quad a_{ijkl} \xi_{kl} \xi_{ij} \geq c_0 |\xi|^2, \quad \forall \xi_{ij} = \xi_{ji}, \quad c_0 = \text{const} > 0.\)

In our setting are also we have

\[ \text{div} \sigma = (\sigma_{11j}, \sigma_{22j}), \quad \sigma_\nu = \sigma_{ij} \nu_j \nu_i, \quad \sigma \nu = (\sigma_{1j} \nu_j, \sigma_{2j} \nu_j), \quad i, j = 1, 2, \quad (\tau_1, \tau_2) = (\nu_2, -\nu_1), \quad \sigma_\tau = \sigma \nu \cdot \tau, \quad u_\nu = u \cdot \nu. \]

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices. For the convenience of presentation, throughout the text we omit the differential of the integration variable in the integrals.

Relations (1) are the equilibrium equations for the elastic body and Hooke’s law. The boundary condition (2) establishes fixing of the body on the boundary \(\Gamma\). Relation (3) indicate the nature of displacements on \(\gamma_1\) and \(\gamma_2\). Inequalities (4) provide a mutual non-penetration between of he crack faces \(\Gamma_\varepsilon\) with zero friction. Note that the nonpenetration condition \([u_\nu] \geq 0\) is fulfilled almost everywhere on \(\Gamma_c\), and, in this sense, it may fail to be satisfied at \(P\). Thus, the first condition in (3) ensures the mutual nonpenetration of thin rigid inclusions \(\gamma_1\) and \(\gamma_2\) at \(P\). Nonlinear conditions (5) describes the delamination of a thin rigid inclusion. Conjugation conditions (6) and (7) are the realization of the principle of virtual displacements.

Now we provide a variational formulation of the problem (1)–(7). To this end, we introduce the Sobolev space

\[ H^1_0(\Omega^\varepsilon_1) = \left\{ v \in H^1(\Omega^\varepsilon_1) \mid v = 0 \text{ almost everywhere on } \Gamma \right\}, \]

where \(H^1_0(\Omega^\varepsilon_1)\) is the Sobolev space of the functions that are square summable in \(\Omega^\varepsilon_1\) together with the first derivatives. Define the set of admissible displacements which is convex and closed in \(H^1_0(\Omega^\varepsilon_1)^2\):

\[ K_1 = \left\{ v = (v_1, v_2) \in H^1_0(\Omega^\varepsilon_1)^2 \mid u_{\nu_1} \in R(\gamma_i); \langle \rho(P) \rangle \nu_0 \geq 0; \quad [u_\nu] \geq 0 \text{ on } \Gamma_c \text{ and } \gamma_i, \quad i = 1, 2 \right\}. \]

Consider the problem of minimizing the potential energy

\[ \inf_{u \in K_1} \Pi(u), \text{ where } \Pi(u) = \frac{1}{2} \int_{\Omega^\varepsilon_1} \sigma(u) \varepsilon(u) - \int_{\Omega^\varepsilon_1} fu. \quad (8) \]
Here $\sigma(u) = \sigma$ are determined from (1), that is $\sigma(u) = A\varepsilon(u)$. For simplicity, we use the notation $\sigma(u)\varepsilon = \sigma_{ij}(u)\varepsilon_{ij}(u)$, $fu = f_{i}u_{i}$. Problem (8) has a solution since the functional $\Pi(u)$ possesses the properties of coercivity and weak lower semicontinuity. The solution of this problem satisfies the variational inequality

$$u \in K_{1}, \quad \int_{\Omega_{c}}\sigma(u)\varepsilon(\bar{u} - u) \geq \int_{\Omega_{c}}f(\bar{u} - u) \quad \forall \bar{u} \in K_{1}. \quad (9)$$

**Theorem 1.** Formulations (1)–(7) and (9) are equivalent under assumption that the solutions of these problems are sufficiently smooth.

2.2. Delaminated elastic inclusion

Now we consider the case when $\gamma = \gamma_{1} \cup \gamma_{2} \cup \{(0, 0)\}$ corresponds to a thin elastic inclusion with delamination on $\gamma^{+}$; $\gamma_{1} = (0, 1) \times \{0\}$, $\gamma_{2} = (-1, 0) \times \{0\}$ (see Figure 1). As before, the curve $\Gamma$ corresponds to a crack. Let us formulate the equilibrium problem in this case: find functions $u = (u_{1}, u_{2})$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, $v^{i}$, $w^{i}$ defined in $\Omega^{\varepsilon}_{1}$, $\Omega^{\varepsilon}_{2}$, $\gamma_{i}$, $\gamma_{l}$, $i = 1, 2$, such that

$$-\text{div} \sigma = f, \quad \sigma = A\varepsilon(u) \quad \text{in} \quad \Omega^{\varepsilon}_{c}, \quad (10)$$

$$v_{xxxx}^{i} = [\sigma_{\nu}], \quad -w_{xx}^{i} = [\sigma_{\tau}] \quad \text{on} \quad \gamma_{i}, \quad i = 1, 2, \quad (11)$$

$$u = 0 \quad \text{on} \quad \Gamma, \quad (12)$$

$$[u_{\nu}] \geq 0, \quad \sigma_{\nu} \leq 0, \quad [\sigma_{\nu}] = 0, \quad \sigma_{\tau} = 0, \quad \sigma_{\nu} \cdot [u_{\nu}] = 0 \quad \text{on} \quad \Gamma_{c}, \quad (13)$$

$$\left\langle (w(0), v(0))\nu_{0} \geq 0 \right\rangle \quad \text{for} \quad x = 0, \quad (14)$$

$$v^{i} = u_{\nu}^{i}, \quad w^{i} = u_{\tau}^{i} \quad \text{on} \quad \gamma_{i}, \quad i = 1, 2, \quad (15)$$

$$[u_{\nu}] \geq 0, \quad \sigma_{\nu}^{+} \leq 0, \quad \sigma_{\tau}^{+} = 0, \quad \sigma_{\nu}^{+} \cdot [u_{\nu}] = 0 \quad \text{on} \quad \gamma_{i}, \quad i = 1, 2, \quad (16)$$

$$v_{xx}^{1} = v_{xxx}^{1} = w_{x} = 0 \quad \text{for} \quad x = 1, \quad v_{xx}^{2} = v_{xxx}^{2} = w_{x} = 0 \quad \text{for} \quad x = -1, \quad (17)$$

$$v_{xx}^{1} = 0, \quad v_{xx}^{2} = 0 \quad \text{for} \quad x = 0, \quad (18)$$

$$(v_{xxx}^{1}(0) - (u_{x}w_{x}^{1}(0)) - (w_{x}^{2}v_{x}^{2}(0)) + (u_{x}^{2}w_{x}^{2}(0)) \geq 0 \quad (19)$$

$$\forall (\bar{v}^{i}, \bar{w}^{i}) \in H^{2}(\gamma_{i}) \times H^{1}(\gamma_{i}), \quad i = 1, 2, \quad \left\langle (\bar{v}(0), \bar{v}(0))\nu_{0} \geq 0 \right\rangle, \quad (20)$$

Here the functions defined on $\gamma$ are identified with the functions of $x_{1}$: $u_{\nu} = w_{\nu}$, $u_{\tau} = w_{\tau}$; $v|_{\gamma_{1}} = v^{1}$ and $v|_{\gamma_{2}} = v^{2}$. The same is true for the function $w$. And $\left\langle (w(0), v(0))\nu_{0} \geq 0 \right\rangle$, $\left\langle (\bar{w}(0), \bar{v}(0))\nu_{0} \geq 0 \right\rangle$.

Equations (11) correspond to fourth and second order differential equations for the displacements of thin elastic inclusions in the framework of the Bernoulli–Euler beam model. Moreover, the right-hand sides of these relations describe the influence of elastic medium on $\gamma_{i}$, $i = 1, 2$. Relation (15) ensure the coincidence of the vertical (along the $x_{2}$ axis) and horizontal (along the $x_{1}$ axis) displacements of the elastic body with the displacements of the inclusions on both $\gamma_{i}$, $i = 1, 2$. The boundary conditions (17) ensure zero moments, zero shear forces, and zero tangential forces for the elastic inclusion for $x = -1$ and $x = 1$. Relations (19) and (20) are the agreement conditions for $x = 0$ whose meaning is as follows: The work of internal forces on the admissible displacements of the body points is no less than the work of external forces, while on true displacements the work vanishes.
Let us show that problem (10)–(20) also admits a variational formulation. To this end, we introduce the set of admissible displacements

\[ K_2 = \{(u, v, w) \in H^1(\Omega_1)^2 \times H^2(\gamma_1 \cup \gamma_1) \times H^1(\gamma_1 \cup \gamma_1) \mid v^i = u^i_\nu, \ w^i = u^i_\tau \text{ on } \gamma_i, \ i = 1, 2; \ [u_\nu] \geq 0 \text{ on } \Gamma_c; \ [u_\tau] \geq 0 \text{ on } \gamma_i, \ i = 1, 2; \ (w(0), v(0))\nu_0 \geq 0 \} \]

and consider the minimization problem

\[
\inf_{(u,v,w) \in K_2} \left\{ \frac{1}{2} \int_{\Omega_1} \sigma(u)\varepsilon(u) - \int_{\Omega_1} fu + \frac{1}{2} \int_{\gamma_1} (v^1_{xx} + v^1_{xx})^2 + \frac{1}{2} \int_{\gamma_2} (w^1_x)^2 + \frac{1}{2} \int_{\gamma_2} (w^2_x)^2 \right\}. \tag{21}
\]

Problem (21) is solvable and can be written as the variational inequality

\[
(u, v, w) \in K_2, \quad \int_{\Omega_1} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_1} f(\bar{u} - u) + \int_{\gamma_1} v^1_{xx}(\bar{v}^1_{xx} - v^1_{xx}) + \int_{\gamma_2} w^1_x(\bar{w}^1_x - w^1_x) + \int_{\gamma_2} v^2_{xx}(\bar{v}^2_{xx} - v^2_{xx}) + \int_{\gamma_2} w^2_x(\bar{w}^2_x - w^2_x) \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K_2. \tag{22}
\]

**Theorem 2.** Formulations (10)–(20) and (22) are equivalent under assumption that the solutions of these problems are sufficiently smooth.

2.3. **The interaction of rigid inclusion with elastic inclusion.**

Consider the case in which \( \gamma_1 \) and \( \gamma_2 \) correspond to thin elastic and rigid inclusions (see Figure 1). Suppose that there is a delamination of the hard inclusion on \( \gamma_2^- \). The formulation of the equilibrium problem is as follows. We have to find functions \( \rho^0 \in R(\gamma_2), \ u, \sigma, v, w, \) such that

\[
-\text{div } \sigma = f, \quad \sigma = \Lambda \varepsilon(u) \quad \text{in } \Omega_1^0, \tag{23}
\]

\[
v_{xxx} = [\sigma_\nu], \quad -w_{xx} = [\sigma_\tau] \quad \text{on } \gamma_1, \tag{24}
\]

\[
u|_{\gamma_2^-} = \rho^0, \quad [u_\nu] \geq 0, \quad \sigma_\nu^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+ \cdot [u_\nu] = 0 \quad \text{on } \gamma_2, \tag{25}
\]

\[
[u_\nu] \geq 0, \quad \sigma_\nu \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau = 0, \quad \sigma_\nu \cdot [u_\tau] = 0 \quad \text{on } \Gamma_c, \tag{26}
\]

\[
v = u_\nu, \quad w = u_\tau \quad \text{on } \gamma_1, \tag{27}
\]

\[
((w(0), v(0)) - \rho^0(0))\nu_0 \geq 0 \quad \text{for } x = 0, \tag{28}
\]

\[
v_{xx} = v_{xxx} = w_x = 0 \quad \text{for } x = 1, \quad v_{xx} = 0 \quad \text{for } x = 0, \tag{29}
\]

\[
(v_{xxx} v(0) - (w_x w)(0) - \int_{\gamma_2} [\sigma\nu] \rho \geq 0 \quad \forall \rho \in R(\gamma_2), \tag{30}
\]

\[
(\bar{v}, \bar{w}) \in H^2(\gamma_1) \times H^1(\gamma_1), \quad ((\bar{w}(0), \bar{v}(0)) - (0))\nu_0 \geq 0, \tag{31}
\]

\[
(v_{xxx} v(0) - (w_x w)(0) - \int_{\gamma_2} [\sigma\nu] \rho^0 = 0. \tag{32}
\]
The inequality (29) ensures that the thin elastic inclusion $\gamma_1$ and the thin hard inclusion $\gamma_2$ do not penetrate at the point $x = 0$. Relations (31)–(32) represent the conjugation conditions for $x = 0$. Consider the minimization problem:

$$
\inf_{(u,v,w) \in K_3} \left\{ \frac{1}{2} \int_{\Omega_1^c} \sigma(u) \varepsilon(u) - \int_{\Omega_1^c} f u + \frac{1}{2} \int_{\gamma_1} (v_{xx})^2 + \frac{1}{2} \int_{\gamma_1} (w_x)^2 \right\}, \tag{33}
$$

where

$$
K_3 = \{(u, v, w) \in H^1(\Omega_1^c)^2 \times H^2(\gamma_1) \times H^1(\gamma_1) \mid v = u_{\nu}, \ w = u_{\tau} \text{ on } \gamma_1; \ [u_{\nu}] \geq 0 \text{ on } \Gamma_c \gamma_2; \ u|_{\gamma_2} \in R(\gamma_2); \ ((w(0), v(0)) - \rho^0(0))v_0 \geq 0\}.
$$

The solution to the problem (33) is unique and satisfies the variational inequality

$$(u, v, w) \in K_3,$$

$$
\int_{\Omega_1^c} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_1^c} f(\bar{u} - u) + \int_{\gamma_1} v_{xx}(\bar{v}_{xx} - v_{xx}) + \int_{\gamma_1} w_x(\bar{w}_x - w_x) \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K_3. \tag{34}
$$

**Theorem 3.** Formulations (23)–(32) and (34) are equivalent under assumption that the solutions of these problems are sufficiently smooth.

We also investigated the limiting transitions in problems (10)–(20) and (23)–(32) with respect to the stiffness parameter of a thin inclusion when this parameter tends to infinity.

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