Highest weight representations of the $N = 1$ Ramond algebra

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ABSTRACT

We analyse the highest weight representations of the $N = 1$ Ramond algebra and show that their structure is richer than previously suggested in the literature. In particular, we show that certain Verma modules over the $N = 1$ Ramond algebra contain degenerate (2-dimensional) singular vector spaces and that in the supersymmetric case they can even contain subsingular vectors. After choosing a suitable ordering for the $N = 1$ Ramond algebra generators we compute the ordering kernel, which turns out to be two-dimensional for complete Verma modules and one-dimensional for $G$-closed Verma modules. These two-dimensional ordering kernels allow us to derive multiplication rules for singular vector operators and lead to expressions for degenerate singular vectors. Using these multiplication rules we study descendant singular vectors and derive the Ramond embedding diagrams for the rational models. We give all explicit examples for singular vectors, degenerate singular vectors, and subsingular vectors until level 3. We conjecture the ordering kernel coefficients of all (primitive) singular vectors and therefore identify these vectors uniquely.

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1 Introduction

The extension of string theory to supersymmetric string theories identified the $N = 1$ Ramond algebra and the $N = 1$ Neveu-Schwarz algebra as the symmetry algebras of closed superstrings as first introduced by Ademollo et al. $^1$. Therefore, the Ramond algebra and the Neveu-Schwarz algebra are the most important symmetry algebras with respect to superstring theory. Even though this significance of the Ramond algebra is known since more than a decade, the representations of the Ramond algebra are still very poorly understood and results given in the literature on its representation theory are largely incomplete as we shall show in this paper.

Due to the conformal invariance on the string world-sheet only superconformal extensions of the Virasoro algebra are suitable symmetry algebras of superstrings. In the case of one fermionic current one obtains only two possible super extensions of the Virasoro algebra$^{16}$: the $N = 1$ Ramond algebra and the $N = 1$ Neveu-Schwarz algebra. Whilst the latter corresponds to antiperiodic boundary conditions of the fields living on the world-sheet of the closed superstrings, the Ramond algebra corresponds to periodic boundary conditions. The space of states of a superstring theory will therefore decompose in irreducible highest weight representations of the Ramond algebra and Neveu-Schwarz algebra. Certain aspects of the highest weight representations of both algebras have been studied by many authors in the past. But so far only in the Neveu-Schwarz case one has achieved all the desirable results about the structure of its representations$^2$. The Kac-determinant formulae play a crucial rôle for the analysis of these representations. In the Ramond case it has first been conjectured by Friedan, Qiu, and Shenker$^{13}$ whilst in the Neveu-Schwarz case the formula has first been given by Kac$^{17}$. Both formulae have been proven by Meurman and Rocha-Caridi$^{20}$.

One can construct all irreducible representations from Verma modules by considering quotients of the Verma modules divided by their largest proper submodules. Submodules of Verma modules are spanned by null-vectors which themselves are generated by singular and subsingular vectors. In this context, singular vectors are the vectors with lowest conformal weight of such submodules. Once the Kac-determinant formulae are known, one can obtain information on null-vectors and singular vectors by analysing the determinant’s roots. The largest proper submodule and thus all null-vectors are given by the kernel of the inner product matrix. Hence, null-vectors are orthogonal on the whole space of states and therefore decouple from the string theory. In this sense they lead us to differential equations for the $n$-point functions and therefore describe the dynamics of the string theory. In the Virasoro case$^{12}$ and in the Neveu-Schwarz case$^2$ subsingular vectors do not exist and therefore their highest weight representations can solely be analysed using singular vectors. Embedding diagrams of singular vectors have been conjectured for the Ramond algebra only for the unitary cases by assuming the non-existence of subsingular vectors$^{19}$. However, for the Ramond algebra this property has never been proven. Furthermore, it was assumed that singular vectors with the same weight would always be linearly dependent, a fact that is true for the Virasoro case and has been proven to be true for the Neveu-Schwarz case$^2$. We will show that both assumptions fail for the Ramond algebra. The $N = 1$ Ramond algebra contains both degenerate singular vectors (linearly independent singular vectors with the same weights) as well as subsingular vectors. The latter occur only in the supersymmetric case$^b$ $\Delta = \frac{c}{24}$ whilst the former appear even for some of the minimal models, however not for the unitary cases. We therefore show that the general Ramond embedding diagrams do not follow the simple rules as for the Virasoro case and consequently the connexion

$^a$For simplicity we shall call the $N = 1$ Ramond algebra simply the Ramond algebra and similarly for the $N = 1$ Neveu-Schwarz algebra.

$^b$The cases with ground states of conformal weight $\Delta = \frac{c}{24}$ are called supersymmetric since the global supersymmetry is unbroken$^{13}$. 
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between the Verma modules and the irreducible characters contains more complications.

Motivated by similar findings for the $N = 2$ superconformal algebras in Ref. 9 a method has been developed to derive upper limits for the degree of degeneracy of singular vectors. Using this method one easily finds for the case of the Virasoro algebra and also for the $N = 1$ Neveu-Schwarz algebra that both algebras do not allow any degeneracy at all\textsuperscript{18,9}. However for the $N = 2$ algebras degenerate singular vector spaces have been found for all 4 types of $N = 2$ superconformal algebras\textsuperscript{9,10}. The main result of this so-called adapted ordering procedure is the ordering kernel. The number of elements of the ordering kernel sets an upper limit on the linearly independent singular vectors with the same weight. Surprisingly enough, we will find that the $N = 1$ Ramond algebra also has an ordering kernel of size 2 and therefore also allows degenerate levels for which we will give explicit examples. Degenerate levels complicate very much the analysis of the embedding structure of submodules - so-called embedding diagrams - since the simple comparison of levels is not enough to decide whether or not two singular vectors are proportional. This information, however, is crucial for the corresponding character formulae. The ordering kernel helps even in this matter as has been shown in Ref. 9 that the elements of the ordering kernel are sufficient to uniquely identify a singular vector. By comparing their coefficients with respect to the ordering kernel it is hence easy to identify linearly independent singular vectors. Furthermore, we can obtain product expressions for singular vector operators that easily reveal the vanishing of a descendant singular vector. In Ref. 11 it will be shown that the vanishing product of two singular vector operators may lead to subsingular vectors which therefore can also be found using the ordering kernel.

The $N = 1$ Ramond algebra is a subalgebra of the $N = 2$ twisted superconformal algebra. Its highest weight representations inherit features that are very similar to those of the latter one. The $N = 2$ twisted superconformal highest weight representations have been studied in Ref. 10. Considering these strong similarities of their representation theories it is rather puzzling that the $N = 1$ Ramond algebra is so extremely important for string theory whilst the $N = 2$ twisted superconformal algebra has so far obtained very little attention in string theory. Despite these similarities, the representation theories also have a few important differences which will become clear in section 5.

The multiplication rules derived in the present paper give the necessary basis for the analysis of descendant singular vectors and consequently the derivation of the Ramond embedding diagrams. We discuss the most interesting embedding diagrams including the minimal models and indicate the limitations of the embedding diagrams of the unitary cases by Kiritsis\textsuperscript{19}. We point out the main differences of Ramond embeddings to Virasoro embeddings. A full set of Ramond embedding diagrams will be given in Ref. 4.

Some necessary introductory remarks about the Ramond algebra will be given in section 2. In section 3, we first review the main ideas and implications of the adapted ordering method introduced in Ref. 9. We then define an adapted ordering for the Ramond algebra and compute its ordering kernel. In section 4, we thus deduce the singular dimensions and derive multiplication rules for vectors identified by their ordering kernel coefficients. In section 5, we compute necessary conditions on the ordering kernel coefficients of (primitive) singular vectors, which allows us to conjecture these coefficients for all (primitive) singular vectors. We hence compute all cases of degenerate singular vectors. Using the multiplication rules of section 4 we give in section 6 a first analysis of the most interesting embedding diagrams, the rational models. A complete discussion of the Ramond embedding diagrams is beyond the scope of this work and shall be given in a future publication\textsuperscript{4}. $G$-closed Verma modules and $G$-closed singular vectors are considered in section 7 and 8 respectively. $G$-closed Verma modules also lead to the simplest type of subsingular vectors. In section 7 we will also show that there are no further subsingular vectors that are related to the vanishing of singular vector operator products. Examples of all
singular vectors, degenerate singular vector spaces, and subsingular vectors for levels \( \leq 3 \) will be given in the appendix. We conclude the paper with some final remarks in section 9.

2 The \( N = 1 \) Ramond algebra

Periodic boundary conditions for the fields on the world-sheet of a closed string require the fermionic current to have modes with integer scaling dimensions. Therefore, the symmetry algebra is the super-extension of the Virasoro algebra by integer moded odd generators \( G_m \).

**Definition 2.A** The \( N = 1 \) Ramond algebra consists of the Virasoro algebra generators \( c_m \), \( m \in \mathbb{Z} \), and the fermionic current generators \( G_n \), \( n \in \mathbb{Z} \) with the commutation relations:

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_n] &= \left(\frac{m}{2} - n\right)G_{m+n}, \\
\{G_m, G_n\} &= 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}, \\
[L_m, C] &= [G_n, C] = 0,
\end{align*}
\]

for \( m, n \in \mathbb{Z} \).

Let us note that the Ramond algebra is a subalgebra of the twisted \( N = 2 \) superconformal algebra. Comparing with the results obtained in Ref. 10 we shall see that this fact leads to strong similarities between the Ramond highest weight representations and the highest weight representations of the twisted \( N = 2 \) superconformal algebra. The squares of the fermionic operators can be expressed in terms of Virasoro operators, as one can easily obtain from the commutation relations Eqs. (1):

\[
G_m^2 = L_{2m} - \delta_{m,0} \frac{C}{24}, m \in \mathbb{Z}.
\]

For applications in physics one usually extends the Ramond algebra by the (fermion) parity operator \((-1)^F\). It commutes with the operators \( L_m \) and \( C \) but anticommutes with \( G_n \):

\[
\begin{align*}
\langle (-1)^F, L_m \rangle &= \langle (-1)^F, C \rangle = 0, m \in \mathbb{Z}, \\
\{ (-1)^F, G_n \} &= 0, n \in \mathbb{Z}.
\end{align*}
\]

The operator \((-1)^F\) therefore distinguishes fermionic states from bosonic states in the space of states and is present in most applications in physics. We will therefore in the following always consider the extended Ramond algebra. Later in this section we will show that this does not at all restrict our results to the extended algebra but they can easily be transferred to the unextended algebra. We hence give the following definition.

**Definition 2.B** The Ramond algebra Eqs. (1) extended by the fermion parity operator \((-1)^F\) of Eq. (3) shall simply be called the Ramond algebra \( R_1 \). We will explicitly say the unextended Ramond algebra, denoted by \( R_1^0 \) whenever we want to refer to the algebra of Eqs. (1) only.

\({}^c\)We use the notation \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), and \( \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\} \).

\({}^d\)The \( N = 1 \) Ramond algebra satisfies the same commutation relations as the \( N = 1 \) Neveu-Schwarz. The only difference is that for the latter algebra the fermionic current has modes with half-integer indices.

\({}^e\)Note that due to Eq. (2) the fermion number \( F \) is not well defined, however, the parity \((-1)^F\) is.
The central term $C$ commutes with all other operators and can therefore be fixed as $c \in \mathbb{C}$. $\mathcal{H}_{R_1} = \text{span}\{L_0, (-1)^F, C\}$ defines a Cartan subalgebra of $R_1$ which elements can hence be diagonalised simultaneously. Generators with positive index span the set of positive operators $R_1^+$ of $R_1$ and likewise generators with negative index span the set of negative operators $R_1^-$ of $R_1$:

\begin{align*}
R_1^+ &= \text{span}\{L_m, G_m : m \in \mathbb{N}\}, \\
R_1^- &= \text{span}\{L_{-m}, G_{-m} : m \in \mathbb{N}\}.
\end{align*}

The zero modes are spanned by $R_1^0 = \text{span}\{L_0, G_0, (-1)^F, C\}$ which contain besides $\mathcal{H}_{R_1}$ also the operator $G_0$.

For eigenvectors of $\mathcal{H}_{R_1}$ we denote the $L_0$-eigenvalue by $\Delta$ (the conformal weight), the $C$-eigenvalue by $c$, and the $(-1)^F$ eigenvalue simply by $\pm$ (the parity) for $\pm 1$. The $C$-eigenvalue $c$ (the conformal anomaly) shall for simplicity be suppressed in the notation. Highest weight vectors, Verma modules and singular vectors are defined in the usual way.

**Definition 2.C** An eigenvector $|\Delta\rangle$ of $\mathcal{H}_{R_1}$ with vanishing $R_1^+$ action is called a highest weight vector. By convention we define a highest weight vector $|\Delta\rangle$ to have positive parity. Additional zero-mode vanishing conditions are possible with respect to the operator $G_0$. We write $|\Delta\rangle^G$ if $G_0 |\Delta\rangle^G = 0$ and call it a $G$-closed highest weight vector. For $|\Delta\rangle^G$ we necessarily have $\Delta = \frac{c}{24}$.

**Definition 2.D** For a given highest weight vector $|\Delta\rangle$ the Verma module $\mathcal{V}_\Delta$ is the left-module

\[ \mathcal{V}_\Delta = U(R_1) \otimes_{\mathcal{H}_{R_1} \oplus R_1^+} |\Delta\rangle, \]

where $U(R_1)$ denotes the universal enveloping algebra of $R_1$. For $G$-closed highest weight vectors $|\frac{c}{24}\rangle^G$ we define $G$-closed Verma modules as

\[ \mathcal{V}_{\frac{c}{24}}^G = U(R_1) \otimes_{\mathcal{H}_{R_1} \oplus R_1^+ \oplus \text{span}\{G_0\}} \left| \frac{c}{24} \right\rangle^G. \]

For an eigenvector $\Psi^p_1 \in \mathcal{H}_{R_1}$ in $\mathcal{V}_\Delta$ the conformal weight is $\Delta + l$ and the parity is $p$ with $l \in \mathbb{N}_0$ and $p \in \{\pm\}$ which we denote as level $|\Psi^p_1\rangle_L = l$ and parity $|\Psi^p_1\rangle_F = p$. We can hence define the notion of singular vectors which - as we shall explain later - correspond to vectors of lowest conformal weight in a given subrepresentation of a Verma module and conversely every proper subrepresentation of a Verma module needs to contain at least one singular vector.

**Definition 2.E** An eigenvector $\Psi^\pm_1 \in \mathcal{H}_{R_1}$ in the Verma module $\mathcal{V}_\Delta$ is called singular vector if it is not proportional to $|\Delta\rangle$ but is still annihilated by $R_1^+$. $G$-closed singular vectors are denoted as $\Psi^\pm_1^G$ and satisfy necessarily $\Delta + l = \frac{c}{24}$. For $\Psi^\pm_1$ singular in $\mathcal{V}_\Delta$ there exists a unique operator $\theta^\pm_1 \in U(R_1 \oplus \{G_0\})$ such that $\Psi^\pm_1 \theta^\pm_1 = \theta^\pm_1 |\Delta\rangle$ called the singular vector operator. Similarly for $\Psi^\pm_1^G$ where $\theta^\pm_1^G \in U(R_1^-)$. If $\theta^\pm_1 |\Delta\rangle$ can be written as $\theta_1 \theta_2 |\Delta\rangle$ with two singular vector operators $\theta_1$ and $\theta_2$, $\theta_1$ not proportional to $G_0$, then $\Psi^\pm_1$ is called a secondary singular vector or a descendant singular vector of $\theta_2 |\Delta\rangle$. Otherwise it is called a primitive singular vector.

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*There is the usual historical confusion: what physicists call highest weight vector is in fact a vector of lowest weight in the Verma module.

*c.f. Ref. 9, definition 3.A.*
In the definition of secondary singular vectors we had to exclude operators $\theta_1$ that are multiples of $G_0$ simply because, for most singular vectors, $G_0$ interpolates between two singular vectors at the same level but of different parity. The only exceptions appear for levels $l$ with $\Delta + l = \frac{c}{24}$ for which the action of $G_0^2$ vanishes and therefore $G_0$ cannot interpolate between two states. Surely, if $G_0$ interpolates between a pair of singular vectors then both vectors are either secondary or primitive.

Before we continue, we shall now compare Verma modules over the (extended) Ramond algebra $R_1$ with Verma modules over the unextended Ramond algebra. For the unextended algebra $R_{1}^{X}$ one chooses the fermionic generator $G_0$ in the Cartan subalgebra $H_{R_{1}^{X}}$ together with $L_0$ and $C$. We hence define highest weight vectors of the unextended algebra with respect to their $G_0$ eigenvalue.

**Definition 2.F** An (unextended) highest weight vector $|\lambda\rangle^X$ of the unextended algebra $R_{1}^{X}$ is an eigenvector of $G_0$ that satisfies

\begin{align}
G_0 |\lambda\rangle^X &= \lambda |\lambda\rangle^X , \\
L_0 |\lambda\rangle^X &= (\lambda^2 + \frac{c}{24}) |\lambda\rangle^X , \\
R_{1}^{+} |\lambda\rangle^X &= 0 .
\end{align}

An (unextended) Verma module $V_{\lambda}^X$ is defined by

$$ V_{\lambda}^X = U(R_{1}) \otimes \mathcal{H}_{R_{1}^{X}} \otimes R_{1}^{+} |\lambda\rangle^X , $$

Except for the supersymmetric case $\Delta = \frac{c}{24}$, a Verma module $V_{\Delta}$ is always reducible to two Verma modules $V_{\Delta,\pm\lambda}^X$ of the unextended Ramond algebra, with $\pm\lambda = \pm \sqrt{\Delta - \frac{c}{24}}$. We should note that for Lie superalgebras *Schur’s lemma* does not hold any more in the sense that the Cartan subalgebra can always be diagonalised on irreducible representations. Instead we can obtain grade spaces for which only the square of the negative parity operator $G_0$ is diagonal but not necessarily $G_0$ itself. But except for the supersymmetric case $\Delta = \frac{c}{24}$ we can always build a $R_{1}$ Verma module on an eigenvector $|\lambda\rangle^X$ of $G_0$. Suppose that $|\lambda\rangle^X$ was not an eigenvector of $G_0$ but of $L_0$ with eigenvalue $\Delta = \lambda^2 + \frac{c}{24}$, then the combinations $G_0 |\lambda\rangle^X \pm \lambda |\lambda\rangle^X$ are two eigenvectors of $G_0$ with eigenvalues $\pm\lambda$ that generate together the module built on $|\lambda\rangle^X$ and form themselves two $R_{1}^{X}$ Verma modules built on highest weight vectors $|\pm\lambda\rangle^X$ as we shall see. Therefore, the above definition 2.F of (unextended) highest weight vectors is justified. The fact that odd generators of the Cartan subalgebra may not be diagonal in an irreducible representation is another reason why for applications in physics we may want to choose a Lie algebra as Cartan subalgebra rather than a Lie superalgebra. Therefore the extended algebra $R_{1}$ is a more natural choice. Nevertheless, let us show here that we can easily relate extended and unextended Verma modules and in most cases the unextended Verma modules embedded in an extended Verma module diagonalise $G_0$.

If $\Delta \neq \frac{c}{24}$ it is easy to see that the two vectors $|\pm\lambda\rangle^X = G_0 |\Delta\rangle \pm \sqrt{\Delta - \frac{c}{24}} |\Delta\rangle$ are eigenvectors of $G_0$ with eigenvalues $\pm\lambda = \pm \sqrt{\Delta - \frac{c}{24}}$. These are the highest weight vectors of the two unextended Verma modules $V_{\pm\lambda}^X$ embedded in $V_{\Delta}$. The sum of $V_{\pm\lambda}^X$ and $V_{\Delta}^X$ is direct as we shall show in the following theorem. This is essentially because the operator $(-1)^F$ - which is not contained in the unextended algebra - interpolates between the two vectors.
Theorem 2.6 The Verma module $V_\Delta$ contains for $\Delta \neq \frac{c}{24}$ the two (unextended) highest weight vectors $|\pm \lambda|^\times$ with $\Delta = \lambda^2 + \frac{c}{24}$ (i.e. $\lambda \neq 0$). The sum

$$V_\Delta = V_\lambda^\times \oplus V_{-\lambda}^\times$$

(12)

is direct and we have $(-1)^F$ interpolating between the two highest weight vectors: $(-1)^F |\pm \lambda|^\times = |\mp \lambda|^\times$.

Proof: The vectors $|\pm \lambda|^\times$ are given by $G_0 |\Delta\rangle \pm \sqrt{\Delta - \frac{c}{24}} |\Delta\rangle$ as previously discussed. Assume $\Psi \in V_\lambda^\times$ and $\Psi \in V_{-\lambda}^\times$, we hence find operators $\theta^+, \theta^-, \gamma^+, \gamma^- \in U(R^-)$ such that

$$\Psi = (\theta^+ + \gamma^+ G_0) |\lambda\rangle = (\theta^- + \gamma^- G_0) |\pm \lambda\rangle.$$ (13)

We can write this in the standard basis for $V_\Delta$ and easily obtain $(\theta^+ - \theta^- + \lambda \gamma^+ + \lambda \gamma^-) G_0 |\Delta\rangle + \lambda(\theta^+ - \theta^- + \lambda \gamma^+ - \lambda \gamma^-) |\Delta\rangle = 0$. Taking into account that in the non-supersymmetric case $|\Delta\rangle$ and $G_0 |\Delta\rangle$ do not satisfy any vanishing conditions in $V_\Delta$ we therefore find that the operators acting on these two basis vectors must vanish identically. However, this results in $\theta^+ \lambda \gamma^+ = 0$ and hence $\Psi = (\theta^+ \lambda \gamma^+) |\lambda\rangle = 0$. Therefore the sum is direct. Furthermore, the highest weight vector $|\Delta\rangle = \frac{|\lambda\rangle - |\pm \lambda\rangle}{\sqrt{2}\lambda}$ is contained in the sum $V_\lambda^\times \oplus V_{-\lambda}^\times$ and therefore $V_\Delta \subset V_\lambda^\times \oplus V_{-\lambda}^\times$. By construction the vectors $|\pm \lambda\rangle$ are contained in $V_\Delta$ and thus also $V_\lambda^\times \oplus V_{-\lambda}^\times \subset V_\Delta$. Finally, one easily checks that $(-1)^F |\pm \lambda|^\times$ has $G_0$ eigenvalue $\mp \lambda$. \Box

Let us now try to diagonalise $G_0$ on the $L_0$-grade spaces of $V_\lambda^\times$. We define

$$\lambda_l = \sqrt{\Delta + l - \frac{c}{24}}.$$ (14)

We assume that $\lambda_l \neq 0$ and take any operator $\theta_l \in U(R^- \cup \{G_0\})$. It is easy to see that the vectors

$$\Psi^{\pm \lambda_l}_{\theta_l, \lambda} = (G_0 \theta_l \pm \lambda_l \theta_l) |\lambda\rangle,$$ (15)

are - if non-trivial - eigenvectors of $G_0$ with eigenvalues $\pm \lambda_l$. Again, we have $(-1)^F$ interpolating between the vectors $\Psi^{\pm \lambda_l}_{\theta_l, \lambda}$ and $\Psi^{\mp \lambda_l}_{\theta_l, -\lambda}$ for $\lambda \neq 0$.

An $L_0$-grade space in $V_\lambda^\times$ with $l$ such that $\lambda_l \neq 0$ has therefore also a diagonal $G_0$ action. With respect to the Verma module $V_\Delta$ this change of basis involves the 4 vectors $\theta_l |\Delta\rangle$, $G_0 \theta_l |\Delta\rangle$, $\theta_l G_0 |\Delta\rangle$, and $G_0 \theta_l G_0 |\Delta\rangle$ transforming them into the 4 $G_0$-eigenvectors $\Psi^{\pm \lambda_l}_{\theta_l, \lambda}$. Surely, in the cases $\Delta = \frac{c}{24}$ or $\Delta + l = \frac{c}{24}$ two of these vectors would obviously be trivial and the basis transformation breaks down. The case where some of these 4 vectors are proportional plays a very important rôle for our later considerations. If we assume that $\theta_l G_0 |\Delta\rangle = \kappa G_0 \theta_l |\Delta\rangle$ then we easily obtain $G_0 \theta_l G_0 |\Delta\rangle = \kappa \lambda^2 \theta_l |\Delta\rangle$. If the first relation is true on $|\Delta\rangle$, then it is also true on $G_0 |\Delta\rangle$ and thus $\lambda^2 \theta_l |\Delta\rangle = \kappa G_0 \theta_l G_0 |\Delta\rangle$. Comparing the two last conditions finally results in $\lambda^2 \kappa^2 = \lambda^2$. Hence, only two proportionality factors are allowed by grouping the vectors in two proportional pairs. In this case, however, the set of 4 vectors $\theta_l |\Delta\rangle$, $G_0 \theta_l |\Delta\rangle$, $\theta_l G_0 |\Delta\rangle$, and $G_0 \theta_l G_0 |\Delta\rangle$ defines only a 2-dimensional vector space whose basis transforms to

$$\Psi^{\pm \lambda_l}_{\theta_l, \lambda} = \pm (\kappa \lambda_l \pm \lambda) (G_0 \theta_l \pm \lambda \theta_l) |\Delta\rangle,$$ (16)

$$\Psi^{\pm \lambda_l}_{\theta_l, -\lambda} = \pm (\kappa \lambda_l \mp \lambda) (G_0 \theta_l \pm \lambda \theta_l) |\Delta\rangle.$$ (17)

Hence, in either case of the two possible proportionality factors $\kappa$ one of the vectors $\Psi^{\pm \lambda_l}_{\theta_l, \lambda}$ would be trivial and also one of the vectors $\Psi^{\pm \lambda_l}_{\theta_l, -\lambda}$ would vanish. Therefore, the two-dimensional
space defined by $\theta_1|\Delta\rangle$, $G_0\theta_1|\Delta\rangle$, $\theta_1G_0|\Delta\rangle$, and $G_0\theta_1G_0|\Delta\rangle$ would be transformed in a one-dimensional $G_0$-eigenspace in $V^X_\lambda$ and a one dimensional $G_0^\perp$-eigenspace in $V^{-\lambda}_\lambda$, one of them having $G_0$-eigenvalue $\lambda_l$ and $-\lambda_l$ for the other one. In the case that $\lambda_l = 0$ we find that half of the operators $\theta_1$ cannot be diagonalised with respect to $G_0$. The corresponding vectors $G_0\theta_1|\lambda\rangle^X$ are hence eigenvectors of $G_0$ with eigenvalue 0. The vectors generated by the other half of operators cannot be transformed into a basis of $G_0$-eigenvectors and only the square of $G_0$ is diagonal on them.

**Theorem 2.H** The operator $G_0$ can be diagonalised on the $L_0$-grade spaces of $V^X_\lambda$ if $\lambda_l \neq 0$. The eigenvalues are $\pm \lambda_l$ which both appear except for the case of proportionality among pairs of the vectors $G_0\theta_1|\Delta\rangle$ and $\theta_1G_0|\Delta\rangle$ for some $\theta_1 \in U(R_1 \cup \{G_0\})$. In this case the eigenvalues $\pm \lambda_l$ will be shared among the Verma modules $V^X_{\pm\lambda}$. If $\lambda_l = 0$ then $G_0$ cannot be diagonalised on the corresponding $L_0$-grade space of $V^X_{\pm\lambda}$.

The fact that $G_0$ cannot be diagonalised appears in the supersymmetric case already at level 0. For $\Delta = \frac{\lambda}{24}$ the Verma module $V^X_{\pm\lambda}$ contains one unextended Verma module generated by $|0\rangle^X = G_0 |\frac{\lambda}{24}\rangle$. $G_0$ can easily be diagonalised on the whole Verma module $V^X_0$ as explained above as $\lambda_l \neq 0$, ($l > 0$). However, $V^X_{\pm\lambda}$ consists of vectors outside $V^X_0$, $|\frac{\lambda}{24}\rangle$ being one example.

**Theorem 2.1** The Verma module $V^X_{\pm\lambda}$ contains an (unextended) Verma module $V^X_0$. The vectors in $V^X_{\pm\lambda}$ which lie outside $V^X_0$ do not belong to any (unextended) Verma module built on an (unextended) highest weight vector (i.e. $G_0$-eigenvector).

Conversely, if we are now given a Verma module of the unextended Ramond algebra with (only one) highest weight vector $|\lambda\rangle^X$ with $G_0$ eigenvalue $\lambda \neq 0$ and hence $L_0$-eigenvalue $\Delta = \lambda^2 + \frac{\lambda}{24}$ we can easily obtain $V^X_{\Delta}$ by considering $|\lambda\rangle^X$ together with its partner $|\lambda\rangle^X = (-1)^F |\lambda\rangle^X$ with $G_0$-eigenvalue $-\lambda$ and imposing $|\Delta\rangle = \frac{|\lambda\rangle^X - |\lambda\rangle^X}{2\lambda}$. This automatically implies $G_0 |\Delta\rangle = \frac{|\lambda\rangle^X + |\lambda\rangle^X}{2\lambda}$. In the case of $\lambda = 0$ we simply have to add $\chi$ with $G_0\chi = |0\rangle^X$ in order to obtain $V^X_{\Delta}$ from $V^X_0$. The notion of singular vectors is the same in extended and unextended Verma modules and furthermore $G_0$ applied to a singular vector produces another singular vector if non-trivial. Therefore, if a highest weight representation of the extended $R_1$ algebra is irreducible (i.e. it has no singular vectors), then the two subrepresentations of the unextended algebra are obviously also both irreducible. Conversely, if we have an irreducible representation of the unextended algebra built on $|\lambda\rangle^X$, $\lambda \neq 0$, then the Verma module of $R_1$ built on $|\Delta\rangle$ constructed from $|\lambda\rangle^X$ and $|\lambda\rangle^X$ is also irreducible. Since it is rather simple to transform highest weight representations of the extended Ramond algebra to highest weight representations of the unextended algebra and conversely, it is justified to restrict ourselves to the extended Ramond algebra $R_1$ which turns out to be more suitable for our considerations as the Cartan subalgebra is just a Lie algebra. Furthermore, applications in physics usually require to distinguish bosonic from fermionic states and therefore the extended algebra is more important for physics.

In the Verma module one usually defines an inner product by defining

$$L_m^+ = L_m \quad G_m^+ = G_m \quad \forall m \in \mathbb{Z},$$

and obvious linear hermitian extensions to the universal enveloping algebra. One sets $langle \Delta | L_0 = \Delta \langle \Delta \rangle$ and $\langle \Delta | R_1^\pm \equiv 0$. This consequently defines an inner product of the vectors $X |\Delta\rangle$ and $Y |\Delta\rangle$, $X,Y \in U(R_1)$ via

$$\langle \Delta | X^+ Y | \Delta \rangle,$$

\[\text{Note that } V^X_{\pm\lambda} \text{ is also a } R_1 \text{ module, which is not the case for } V^X_{\Delta} \text{ with } \Delta \neq \pm\lambda.\]
where we define $\langle \Delta | G_0 | \Delta \rangle = 0$ since $(-1)^F$ is diagonal on the two states. We obtain a \textit{(pseudo-)norm} of the vector $X | \Delta \rangle$, $X \in U(R_1)$ as $\langle \Delta | X^+ X | \Delta \rangle$. A singular vector and the whole embedded Verma module built on it have obviously vanishing \textit{(pseudo-)norm}. Furthermore, every vector contained in the largest proper submodule of $V_\Delta$ has vanishing \textit{(pseudo-)norm}. These vectors form the kernel of the inner product matrix formed via Eq. (19) as it is shown in Ref. 11. They therefore decouple from the field theory. They are usually called \textit{null-vectors}.

The quotient module where all null-vectors are set to 0 is hence an irreducible highest weight representation. Conversely, all irreducible highest weight representations can be constructed in this way. Singular vectors and their descendants do not necessarily span the whole submodule of null-vectors. The quotient module of the Verma module divided by the submodule spanned by all singular vectors may again contain new singular vectors, called subsingular vectors.

\textbf{Definition 2.1} Let $H_\Delta$ be the submodule generated by all singular vectors of a Verma module $V_\Delta$. The quotient module

$$Q_\Delta = \frac{V_\Delta}{H_\Delta}$$

may or may not be irreducible. In the case that $Q_\Delta$ is reducible it contains new singular vectors $\Upsilon$ that were not singular in $V_\Delta$. $\Upsilon$ is called subsingular vector in $V_\Delta$. Continuing with this process one may find subsingular vectors in $Q_\Delta$ and further quotient modules until one obtains an irreducible representation.

For the Virasoro algebra and for the $N = 1$ Neveu-Schwarz algebra there are no subsingular vectors$^{12,2}$, however, they have been discovered for the $N = 2$ superconformal algebras by Gato-Rivera et al. in Refs. 15, 14, 10. At first, it seems that $N = 1$ superconformal symmetry is not large enough to allow subsingular vectors. However, that this assumption does not hold for one particular case of the extended Ramond algebra will be shown in section 7 of this paper. The fact that the largest proper submodule is equivalent to the kernel of the inner product matrix assigns an important rôle to the determinant of the inner product matrix, usually called the \textit{Kac-determinant}. If we choose an $H_{R_1}$ graded basis, then the inner product matrix is block diagonal and the Kac-determinant factorises in determinants of these blocks. The lowest grade for which the determinant vanishes indicates the existence of a singular vector. For the $N = 1$ Ramond algebra the determinant formula has been first conjectured by Friedan, Qiu, Shenker$^{13}$ and has later been proven by Meurman and Rocha-Caridi$^{20}$. For level zero the determinant formulae are $\det M_0^+ = 1$ and $\det M_0^- = \Delta - \frac{c}{24}$. For the central term $c$ at level $l \in \mathbb{N}$ and parity $\pm$ we use the parametrisation$^{22,13}$ in $t \in \mathbb{C}$, $(t \neq 0)$,

$$c(t) = \frac{15}{2} - \frac{3}{t} - 3t \ .$$

We can then give the determinant formulae as

$$\det M_l^{\pm}(\Delta, t) = (\Delta - \frac{c(t)}{24})^{p_R(t)} \prod_{1 \leq p, q \leq 2l \atop p, q \in \mathbb{N}, p - q \text{ odd}} [\Delta - \Delta_{p,q}(t)]^{p_R(l - \frac{pq}{2})} \ .$$

The functions $\Delta_{p,q}$ can hence be written as

$$\Delta_{p,q} = \frac{(q - tp)^2 - (t - 1)^2}{8t} + \frac{1}{16} = \frac{(q - tp)^2}{8t} + \frac{c}{24} \ ,$$

$^{1}$If the pseudo-norm is positive semi-definite, then all vectors with vanishing norm are null-vectors. However, if the pseudo-norm is not positive semi-definite there are vectors with vanishing pseudo-norm that are not null-vectors$^{11}$ and therefore do not decouple from the field theory.
where $p$ and $q$ are positive integers and $p - q$ is odd. Finally, the partition function $P_R(l)$ is given by

$$
\sum_{i=0}^l x^i P_R(i) = \prod_{k=1}^{l} \frac{1 + x^k}{1 - x^k}, \quad (24)
$$

and satisfies $P_R(0) = 1$. Let us note that $P_R(l)$ is always even for $l \geq 1$. For most $c$ the parametrisation Eq. (21) has two solutions $t^1$ and $t^2$. In which case we obtain $t^1 = 1/t^2$. We note that $\Delta_{p,q}(t) = \Delta_{q,p}(1/t)$ and $c(t) = c(1/t)$ and therefore the set of curves $\Delta_{p,q}(t)$ is invariant under the choice of solution $t^1$ and $t^2$. Hence the parametrisation is well-defined. In fact, the Verma modules $V_{\Delta_{p,q}}(t)$ and $V_{\Delta_{q,p}}(1/t)$ are identical. From the determinant we know that following the curves $\Delta_{p,q}(t)$ we obtain singular vectors which turn out to be generically primitive.

**Theorem 2.1** Along the curve $\Delta_{p,q}(t)$ we find (at least) two singular vectors $\Psi^\pm_{p,q}(t)$ both at level $\frac{pq}{2}$ and with parity $\pm$, where $p, q \in \mathbb{N}$ and $p - q$ is odd. Both vectors are generically primitive.

**Proof:** For a given $\Delta_{p,q}(t)$ which does not intersect with any other curve $\Delta_{p',q'}(t)$ where $\frac{pq}{2} \leq \frac{p'q'}{2}$ the claim is trivial. Intersections of such curves are only given by discrete points $t = \pm \frac{-p}{p-q}$. Continuity arguments easily show that $\Psi^\pm_{p,q}$ exists also for these discrete points (if it happens to vanish identically, then at least one derivative would be singular) and these are the only points for which $\Psi^\pm_{p,q}$ could be secondary. Hence $\Psi^\pm_{p,q}$ is generically primitive. $\square$

$G_0$ interpolates between the two parity sectors, provided $G_0$ does not act on any $G$-closed vector. Therefore, the two sectors have exactly the same determinant expressions. Starting with a given singular vector $\Psi^+_l = \theta^+_l |\Delta\rangle \in V_\Delta$ with singular operator $\theta^+_l$ we can construct two negative parity singular vectors:

$$
\Psi^-_{G,l} = \theta^+_l G_0 |\Delta\rangle, \quad (25)
$$

$$
G \Psi^-_l = G_0\theta^+_l |\Delta\rangle. \quad (26)
$$

Likewise, $G_0$ constructs two positive parity singular vectors from the singular vector $\Psi^-_l = \theta^-_l |\Delta\rangle \in V_\Delta$:

$$
\Psi^+_{G,l} = \theta^-_l G_0 |\Delta\rangle, \quad (27)
$$

$$
G \Psi^+_l = G_0\theta^-_l |\Delta\rangle. \quad (28)
$$

This seems as if the (primitive) vectors $\Psi^\pm_{p,q}$ always come at least in pairs of 4 unless the corresponding vectors of Eqs. (25)-(28) are proportional. It is the Ramond partition function that requires the latter.

**Theorem 2.1** For the singular vectors $\Psi^\pm_{p,q}$ the vectors $\Psi^\pm_{G,p,q}$ and $G \Psi^\pm_{p,q}$ are proportional as well as $\Psi^-_{G,p,q}$ and $G \Psi^-_{p,q}$.

**Proof:** The entries of the inner product matrix are polynomials in $\Delta$ and $c$. It is easy to show\(^1\) that if a determinant with polynomial entries vanishes for $\Delta = \Delta(t)$, then the multiplicity of

\(^1\)In the case of $G$-closed Verma modules or $G$-closed singular vectors some of these vectors may be trivial.
the factor \((\Delta - \Delta(t))\) in the determinant sets an upper limit for the dimension of the kernel of the inner product matrix. Generically, the multiplicity of \(\Delta_{p,q}(t)\) at level \(l\) is \(P_l(0) = 1\) for each parity sector. As \(\Psi_{p,q}^\pm\) has to be in the kernel, there will generically be exactly one singular vector for each parity at level \(l\) along the curve \(\Delta_{p,q}(t)\) and hence the claimed proportionalities have to be true generically. Due to continuity this extends all along the curves \(\Delta_{p,q}(t)\). Note that the fact that there is just one singular vector along \(\Delta_{p,q}(t)\) at level \(l\) for each parity must be true generically but may not be true for the discrete intersection points of the curves \(\Delta_{p,q}(t)\).

We conclude this section by introducing convenient notation that will be used in the following section.

**Definition 2.2M** For \(Y \in U(R_1)\) of the form

\[
Y = L_{-m_L} \cdots L_{-m_1} G_{-n_G} \cdots G_{-n_1} L_{-1}^{n_1} G_0^{r_2},
\]

where \(r_1, r_2 \in \{0, 1\}\) and for all its reorderings we define the level \(|Y|_L = \sum_{j=1}^L m_j + \sum_{j=1}^G n_j + n + r_1\), the parity \(|Y|_F = (-1)^{1+r_1+r_2}\), and their length \(|Y| = L + G\). For the trivial case we put \(|1|_L = |1|_F = 0\) and \(|1|_F = +1\). Further, we set \((l, n, k) \in \mathbb{N}_0\)

\[
\begin{align*}
L_l &= \{ Y = L_{-m_L} \cdots L_{-m_1} : m_L \geq \ldots \geq m_1 \geq 2, |Y|_L = l \}, \\
G_l &= \{ Y = G_{-n_G} \cdots G_{-n_1} : n_G > \ldots > n_1 \geq 2, |Y|_L = l \}, \\
L_0 &= \{ T_0 = G_0 = \{1\} \},
\end{align*}
\]

\[
S_k^\pm = \{ Y = LG : L \in L_m, G \in G_n, |Y|_L = k = m + n, |Y|_F = \pm 1 = |G|_F, m, n \in \mathbb{N}_0 \},
\]

\[
C_k^\pm = \{ S_{m,p} L_{-1}^{k-m-r_1} G_0^{r_2} : S_{m,p} \in S_k^p, m \in \mathbb{N}_0, r_1, r_2 \in \{0, 1\}, \\
k - m - r_1 \geq 0, p(-1)^{r_1+r_2} = \pm 1 \}.
\]

Elements of \(C_k^\pm\) are therefore of the form of Eq. (29) with \(r_1, r_2 \in \{0, 1\}\), \(|Y|_L = k\), and \(|Y|_F = \pm 1\). We will always use \(C_k^\pm\) in order to define the basis of a Verma module \(V_\Delta\):

**Definition 2.2N** The standard basis for \(V_\Delta\) is

\[
B_\Delta = \{ X | \Delta \} : X \in C_k^\pm, k \in \mathbb{N}_0 \},
\]

which is naturally \(\mathbb{N}_0 \otimes \{\pm 1\}\) graded with respect to their \(L_0\) and \((-1)^F\) eigenvalues relative to the highest weight vector. This grading is inherited from the grading on \(U(R_1)\). The decomposition of \(\Psi^p_i \in V_\Delta\) with respect to the standard basis

\[
\Psi^p_i = \sum_{X \in C^p_i} c_X X | \Delta \rangle,
\]

is called the normal form of \(\Psi^p_i\). \(X \in C^p_i\) are the terms of \(\Psi^p_i\) and the coefficients \(c_X\) its coefficients. Terms \(X\) with non-trivial coefficients \(c_X\) are called non-trivial terms of \(\Psi^p_i\).
In the case of \( G \)-closed Verma modules the basis is obviously smaller and will be defined similarly through

\[
\mathcal{B}_G^G = \left\{ X \left| \frac{e}{24} \right| : X \in \mathcal{C}_k^\pm G, k \in \mathbb{N}_0, \right\},
\]

with

\[
\mathcal{C}_k^\pm G = \left\{ S_{m,p} L^{k-m-r_1} G^{r_1} : S_{m,p} \in \mathcal{S}_m, m \in \mathbb{N}_0, r_1 \in \{0,1\}, k-m-r_1 \geq 0, p(-1)^{r_1} = \pm 1 \right\}, \quad k \in \mathbb{N}_0.
\]

Whilst Verma modules \( V_\Delta \) may contain both, singular vectors and \( G \)-closed singular vectors, \( G \)-closed Verma modules \( V_G^G \) can only contain singular vectors but not \( G \)-closed singular vectors. Similar phenomena are true for the topological and Neveu-Schwarz \( N = 2 \) algebras as shown in Refs. 15, 10. To be precise, for the topological and Neveu-Schwarz \( N = 2 \) algebras there are no chiral singular vectors in chiral Verma modules whereas for the twisted \( N = 2 \) algebra there are no \( G \)-closed singular vectors in \( G \)-closed Verma modules.

3 Ordering kernels for Ramond Verma modules

Verma modules of the Virasoro algebra contain at most one singular vector at a given level. This is a consequence of the embedding diagrams proven by Feigin and Fuchs\(^{12} \). On the other hand, if one knows in advance that there can be at most one singular vector for a given level, then the analysis of descendant singular vectors comes down to a detailed analysis of the roots of the determinant formula. Whenever one finds that the level of two secondary singular vectors agree then these singular vectors need to be the same and the generated submodules would be identical. This also allows us to derive the character formulae of the irreducible highest weight representations. However, for Verma modules where such a uniqueness of singular vectors at a given level is not true the procedure of deriving the embedding patterns is much more complicated. Consequently the derivation of the characters via embedding diagrams also needs more information in this case. This has been demonstrated in Ref. 7 for the minimal models of the \( N = 2 \) Neveu-Schwarz algebra. To solve such problems there are hence two questions of major interest. First, which are the maximal dimensions a space of singular vectors can have for a given weight - the so-called singular dimensions - and secondly, if the singular dimension is greater than 1, how can we decide if two singular vectors at the same weight are proportional? Both questions can be answered with a rather simple but powerful method, the adapted ordering method, introduced in Ref. 9. There, this method has first been used for the 3 isomorphic \( N = 2 \) superconformal algebras (the topological \( N = 2 \) algebra, the \( N = 2 \) Neveu-Schwarz algebra, and the \( N = 2 \) Ramond algebra) and just recently also for the twisted \( N = 2 \) algebra in Ref. 10. In the latter case the adapted ordering method shows strong similarities to what we will find for \( R_1 \). \textit{A priori} the connexion between the following definition and the answers to the questions above does not seem very obvious. The power of this method, however, lies in two theorems which were proven in Ref. 9 and which we will review after introducing the notion of adapted orderings in the case of the Ramond algebra. For convenience, let us first define the following notation.

\textbf{Definition 3.A} For \( X \in \mathcal{C}_l^p \) and \( \Gamma \in R_1 \) we set

\[
\Omega_X^\Gamma(\Delta) = \left\{ \text{non-trivial terms of } \Gamma X |\Delta) \right\}.
\]

As the set of all non-trivial terms of \( \Gamma X |\Delta) \) in its normal form, \( \Omega_X^\Gamma(\Delta) \) is - if non-trivial - a subset of \( \mathcal{C}_{l+|\Gamma|}^{l_+|\Gamma|} \).
In this notation we can give the definition of an adapted ordering in a particularly suitable form:

**Definition 3.B** Let $O$ be a total ordering on $C^p_i$. $K = R^+_1$ is the set of annihilation operators. $O$ is said to be adapted to the subset $C^{pA}_i \subset C^p_i$ in $V_\Delta$ with $K = R^+_1$, if for any $X \in C^{pA}_i$ at least one $\Gamma \in K$ exists for which

$$\Omega^\Gamma_X(\Delta) \not\subset \bigcup_{Y \in C^p_i, Y \neq X} \Omega^\Gamma_Y(\Delta).$$

The complement of $C^{pA}_i$, $C^{pK} = C^p_i \setminus C^{pA}_i$ is the (ordering) kernel with respect to the ordering $O$ in the Verma module $V_\Delta$.

In other words, for each $X \in C^{pA}_i$ there exists an annihilation operator $\Gamma$ such that $\Gamma X |\Delta\rangle$ contains at least one non-trivial term that is not contained in $\Gamma Y |\Delta\rangle$ for all $Y$ that are strictly $O$-larger than $X$. The significance of this definition lies in the fact that if we know that $\Gamma \Psi^p_i \equiv 0$ for $\Psi^p_i \in V_\Delta$ and we assume that $X$ is the $O$-smallest non-trivial term in $\Psi^p_i$, then there will be at least one non-trivial term in $\Gamma \Psi^p_i$ which contradicts $\Gamma \Psi^p_i \equiv 0$. Hence only elements of the ordering kernel $C^{pK}_i$ can be $O$-smallest non-trivial terms of a singular vector $\Psi^p_i$. These rather simple thoughts result in the following two theorems that were proven in a more general setting in Ref. 9.

**Theorem 3.C** If the ordering kernel $C^{pK}_i$ of an adapted ordering at level $l$, parity $p$, in $V_\Delta$ and with annihilation operators $K = R^+_1$ has $n$ elements, then there are at most $n$ linearly independent singular vectors $\Psi^p_i$ in $V_\Delta$ at level $l$ with parity $p$.

**Theorem 3.D** Let $O$ denote an adapted ordering at level $l$, parity $p$, with ordering kernel $C^{pK}_i$ for a given Verma module $V_\Delta$ and annihilation operators $K = R^+_1$. If the normal form of two vectors $\Psi^{p,1}_i$ and $\Psi^{p,2}_i$ in $V_\Delta$ at the same level $l$ and parity $p$, both satisfying the highest weight conditions, have $c^p_X = c^p_Y$ for all terms $X \in C^{pK}_i$, then

$$\Psi^{p,1}_i \equiv \Psi^{p,2}_i. \quad (41)$$

Surely, every total ordering on $C^p_i$ would be adapted at least to the empty set $\emptyset \subset C^p_i$. This, however, is not at all useful as the adapted kernel would be the whole set $C^p_i$. The aim is to find orderings that lead to very small ordering kernels. Coming back to the two questions set in the introduction of this section, theorem 3.C answers the first one: the size of the adapted kernel sets an upper limit for the dimension of a space of singular vectors for a given weight. Theorem 3.D answers the second question: in order to decide if two singular vectors with the same weight are proportional we do not need to compare all their coefficients, in fact we do not even need to know all their coefficients, it is already enough to compare the few coefficients with respect to the ordering kernel. These coefficients completely identify a singular vector.

This fact has been used for the $N = 2$ algebra in Refs. 9, 10 and in a simpler version already in Ref. 6 for the $N = 2$ Neveu-Schwarz case. As the trivial vector has vanishing coefficients for all terms it follows that a singular vector needs to have at least one non-trivial term of the ordering kernel. In the following we will compute ordering kernels for the Ramond algebra $R_1$ which will prove to be the smallest possible by explicit examples given in the appendix.

We obtain the corresponding definition of adapted orderings for $G$-closed Verma modules by replacing in definition 3.B $C^p_i$ by $C^{pG}_i$, $C^{pA}_i$ by $C^{pGA}_i$, $V_\Delta$ by $V^G_\Delta$, and $B_\Delta$ by $B^G_\Delta$. Theorem
3.C and theorem 3.D hold accordingly. The same is true for $G$-closed singular vectors for which we simply have to extend the set of annihilation operators to $\mathcal{K} = \mathbb{R}_+^\ast \oplus \{G_0\}$.

The adapted ordering we are now going to introduce coincides formally with the adapted ordering given in Ref. 9 for the topological $N = 2$ superconformal algebra, restricted to the fewer fermionic and bosonic generators in the $N = 1$ Ramond case. Even though our later results will be similar to the twisted $N = 2$ case, the ordering for the twisted $N = 2$ case\(^{10}\) is very different in the sense that the special rôle of the translation operator $L_{-1}$ has been taken over by another $N = 2$ bosonic operator. We first introduce an ordering on the sets\(^k\) $L_n$ and $G_n$.

**Definition 3.E** We take $Y_i \in L_n$, for $i = 1, 2$ which we call case $\Upsilon = L$ or $Y_i \in G_n$, for $i = 1, 2$ which we denote by case $\Upsilon = G$. Hence $Y_i$ is of the form

$$Y_i = Z_{-m^i_{\Upsilon,Y}}^i \cdots Z_{-m^i_1}^i,$$

with $|Y_i|_L = n^i$ ($n^i \in \mathbb{N}_0$), or $Y_i = 1$, $i = 1, 2$, with $Z_{-m^i_j}^i$ being either $L_{-m^i_j}$ in case $L$ or $G_{-m^i_j}$ in case $G$. We compute the index $j_0 = \min\{j : m^i_1 - m^i_j \neq 0, j = 1, \ldots, \min(|Y_1|, |Y_2|)\}$. If non-trivial, $j_0$ is the index for which the levels of the generators in $Y_1$ and $Y_2$ first disagree when read from the right to the left. For $j_0 > 0$ we then define

$$Y_1 <_\Upsilon Y_2 \quad \text{if} \quad m^1_{j_0} < m^2_{j_0}.$$

If, however, $j_0 = 0$, we set

$$Y_1 <_\Upsilon Y_2 \quad \text{if} \quad ||Y_1|| > ||Y_2||.$$

Throughout this definition, $\Upsilon$ either stands for $L$ or for $G$.

With the help of definition 3.E we now introduce a suitable adapted ordering for Ramond Verma modules. It is easy to see that the following definition really defines a total ordering on $\mathcal{C}_l^p$ with global minimum $L^1_{-1}$ for $\mathcal{C}_l^+$ and $L^1_{-1}G_0$ for $\mathcal{C}_l^-$. $L^1_{-1}G_{-1}G_0$ and $L^1_{-1}G_{-1}$ are the smallest elements after the minimum elements respectively.

**Definition 3.F** On the set $\mathcal{C}_l^p$, $l \in \mathbb{N}$, $p \in \{\pm\}$ we introduce the total ordering $\mathcal{O}$. For two elements $X_1, X_2 \in \mathcal{C}_l^p$, $X_1 \neq X_2$ with $X_1 = L^nG^n L^r_1 G^r_{-1} G^n_0$, $L^i \in L_{m^i}$, $G^1 \in G_k$, for some $m^i, n^i, k^i \in \mathbb{N}_0$, $r^i_1, r^i_2 \in \{0, 1\}$, $i = 1, 2$, we define

$$X_1 <_\mathcal{O} X_2 \quad \text{if} \quad n^1 > n^2.$$

For $n^1 = n^2$ we set

$$X_1 <_\mathcal{O} X_2 \quad \text{if} \quad r^1_1 > r^2_1.$$

If $r^1_1 = r^2_1$ then we set

$$X_1 <_\mathcal{O} X_2 \quad \text{if} \quad G^1 <_\mathcal{O} G^2.$$

In the case that $G^1 = G^2$ we then define

$$X_1 <_\mathcal{O} X_2 \quad \text{if} \quad L^1 <_\mathcal{O} L^2.$$

For $X_1 = X_2$ we define $X_1 <_\mathcal{O} X_2$ and $X_2 <_\mathcal{O} X_1$.

\(^{10}\)Examples of definition 3.E can be found in Refs. 9, 10.

\(^k\)For subsets of $\mathbb{N}$ we define $\min\emptyset = 0$. 
As explained before, the main result we intend to extract from an adapted ordering is the ordering kernel as the number of elements of the ordering kernel defines an upper limit for the singular dimensions. Secondly, the terms in the ordering kernel identify singular vectors uniquely according to theorem 3.C and theorem 3.D. We find the following result.

**Theorem 3.G** The ordering kernels of $O$ on $C^p_l$ for the Verma module $V_\Delta$ are given for $l \in \mathbb{N}$, $p \in \{\pm\}$ and for all central terms $c \in \mathbb{C}$ in the following tables.

| $(-1)^\mathbf{F}$ | ordering kernel |
|-------------------|-----------------|
| $+$               | $\{L_{-1}^l, L_{-1}^{r_1}G_{-1}G_0\}$ |
| $-$               | $\{L_{-1}^lG_0, L_{-1}^{r_1}G_{-1}\}$ |

**Tab. a** Ordering kernels for $O$, annihilation operators $R^+_1$.

| $(-1)^\mathbf{F}$ | ordering kernel |
|-------------------|-----------------|
| $+$               | $\{L_{-1}^l\}$  |
| $-$               | $\{L_{-1}^lG_0\}$ |

**Tab. b** Ordering kernels for $O$, annihilation operators $R^+_1$ and $G_0$.

For the proof of theorem 3.G we follow the lines of the proof for the topological $N = 2$ algebra given in Ref. 9.

**Proof:** The most general term $X_0$ at level $l$, parity $p$, in $C^p_l$, $l \in \mathbb{N}$, $p \in \{\pm\}$ is

$$X_0 = L^0 G^0 L^n_{-1} G^{r_1}_{-1} G^2_0 \in C^p_l,$$

with

$$L^0 = L_{-m||L^0||} \ldots L_{-m_1} \in L_m,$$

$$G^0 = G_{-k||G^0||} \ldots G_{-k_1} \in G_k,$$

for $m, n, k \in \mathbb{N}_0$, $r_1, r_2 \in \{0, 1\}$, such that $l = m + n + k + r_1$ and $p = (-1)^{||G^0||+r_1+r_2}$. We then construct the vector $\Psi^0 = X_0 |\Delta\rangle$.

In the case $G^0 \neq 1$ we look at $G_{-k_1}$ which necessarily has $k_1 > 1$ and consider the positive operator $G_{k_{1-1}}$. The commutation relations of $G_{-k_1}$ and $G_{k_{1-1}}$ create a generator $L_{-1}$. Therefore, $\Omega^{G_{k_{1-1}}}_{X_0}$ contains the term

$$X^G = L^0 G^0 L^{n+1}_{-1} G^{r_1}_{-1} G^2_0,$$

$$\tilde{G}^0 = G_{-k||G^0||} \ldots G_{-k_2},$$

or $\tilde{G}^0 = 1$ in the case $||G^0|| = 1$. For any other term $Y \in C^p_l$ which also contains $X^G$ in $\Omega^{G_{k_{1-1}}}_{Y}$ the action of $G_{k_{1-1}}$ also creates at least one generator $L_{-1}$ in order to obtain $X^G$ or $Y$ has more generators $L_{-1}$ than $X_0$. In the latter case $Y <_{\bigcirc} X_0$. In the first case, however, the action of one positive operator can according to the commutation relations create at most one new generator. We can therefore restrict ourselves to terms $Y$ of the form

$$Y = L^Y G^Y L^{n+1}_{-1} G^{r_1}_{-1} G^2_0 \in C^p_l,$$

with the same number $n$ of generators $L_{-1}$ than $X_0$. $G_{k_{1-1}}$ commuting through operators of $L^Y$ can only create operators of the form $G_{-k}$ with $k \leq k_1$. Hence, this additional operator $L_{-1}$
can only be created by commutation of $G_k$ with operators in $G^Y$ or with $G_{r1}^Y$. In the first case the operator with smallest (absolute) index in $G^Y$ must have an index $k_1^Y \leq k$ and consequently $Y \subset X_0$ (note that in this case $r_i^Y = r_i$). In the latter case, however, we necessarily obtain $r_i^Y = 1 > r_1$ and again $Y \subset X_0$. Hence, we have shown that $X^G \in \Omega_{Y}^{G_{k1-1}}$ implies $Y \subset X_0$ and therefore

$$X^G \notin \bigcup_{y \in C^{p}_{X_0 \subset Y, Y \neq X_0}} \Omega_{X_0}^{G_{k1-1}},$$

but

$$X^G \in \Omega_{X_0}^{G_{k1-1}}.$$  (54)

Let us now continue with terms $X_0$ of the form

$$X_0 = L^0 L_{-1}^{m_1} G_{-1}^{r1} G_0^{r2} \in C^{p}_{l}.$$  (55)

If $L^0 \neq 1$ we consider the positive operator $L_{m1-1}$. $\Omega_{X_0}^{L_{m1-1}}$ contains

$$X^L = \hat{L}^0 L_{-1}^{m1} G_{-1}^{r1} G_0^{r2},$$

$$\hat{L}^0 = L_{-m1} g_1 \ldots L_{-m2},$$

or $\hat{L} = 1$ in the case $||L^0|| = 1$. For any other term $Y \in C^{p}_{l}$ which also contains $X^L$ in $\Omega_{Y}^{L_{m1-1}}$ we again find that the action of $L_{m1-1}$ creates exactly one additional $L_{-1}$ in order to obtain $X^L$ or we have automatically $Y \subset X_0$. If the additional $L_{-1}$ is created by commutation with $G_{-1}^Y$ we would as before find $Y \subset X_0$. We can hence concentrate on terms $Y$ of the form

$$Y = L^Y L_{-1}^{m1} G_{-1}^{r1} G_0^{r2} \in C^{p}_{l},$$

where also $r_2^Y = r_2$ due to parity equality with $X_0$. The only way of creating $L_{-1}$ from $Y$ under the action of $L_{m1-1}$ is via commutation with generators in $L^Y$. Therefore $L^Y$ needs to contain an operator $L_{-m}$ with $m \leq m_1$ and therefore again $Y \subset X_0$. Thus

$$\Omega_{X_0}^{L_{m1-1}} \notin \bigcup_{y \in C^{p}_{X_0 \subset Y, Y \neq X_0}} \Omega_{Y}^{L_{m1-1}},$$

This completes the proof of the ordering kernels of table Tab. a.

Finally, if we also have $G_0$ as annihilation operator, we can then act with $G_0$ on terms $X_0$ of the form

$$X_0 = L_{-1}^{n} G_{-1} G_0^{r2} \in C^{p}_{l},$$

which creates a non-trivial term

$$X' = L_{-1}^{n+1} G_0^{r2} \in C^{p}_{l}.$$  (60)

Obviously, the only way of creating $L_{-1}$ by commuting $G_0$ with products of generators in $C^{p}_{l}$ is by commuting $G_0$ directly with $G_{-1}$. Therefore any other term $Y$ that creates $X'$ under the action of $G_0$ and is not $O$-smaller than $X_0$ would also need to create one $L_{-1}$ and thus also have $r_i^Y = r_i = 1$, besides $n^Y = n$. Trivially, we find in this case that $Y = X_0$. This proves the
Highest weight representations of the $N = 1$ Ramond algebra

ordering kernel of table Tab. b which finally completes our proof of theorem 3.G. □

Obviously $\mathcal{O}$ also defines a total ordering on $\mathcal{C}^p_G$. We can set $r_2 \equiv 0$ everywhere in the above proof and obtain the same reasoning for the $G$-closed Verma modules $\mathcal{V}^G_\Pi$. Hence, we can already give the ordering kernels for $G$-closed Verma modules in the following theorem.

**Theorem 3.H** The ordering kernels of $\mathcal{O}$ on $\mathcal{C}^p_G$ for the Verma module $\mathcal{V}^G_\Pi$ are given for $l \in \mathbb{N}$, $p \in \{\pm\}$ and for all central terms $c \in \mathbb{C}$ by:

| $(-1)^p$ | ordering kernel |
|-----------|----------------|
| +         | $\{L^l_{-1}\}$ |
| -         | $\{L^l_{-1}G_{-1}\}$ |

Tab. c Ordering kernels for $\mathcal{O}$ on $\mathcal{C}^p_G$, annihilation operators $R_i^\dagger$.

As mentioned earlier, $G$-closed Verma modules do not contain any $G$-closed singular vectors. Therefore, theorem 3.H does not consider orderings on $\mathcal{C}^p_G$ with annihilation operators $R_1^\dagger$ and $G_0$. Knowing the ordering kernels, we can now use our powerful theorems 3.C and 3.D to find upper limits for singular dimensions and to identify singular vectors. Both will be demonstrated in the following section.

4 Singular dimensions and singular vectors in Verma modules over the $N = 1$ Ramond algebra

Theorem 3.C tells us that the dimension of spaces of singular vectors for a given level $l$ and parity $p$ is bounded by the number of elements of the corresponding ordering kernel found in the previous section. In appendix A we give all the singular vectors until level 3. In particular, at level 3 we give explicit examples of two-dimensional singular vector spaces in (complete) Verma modules. The upper limits for the singular dimensions given by the ordering kernels turn out to be maxima. We can therefore state that the ordering we have chosen is the best possible since the results we have found cannot be improved by choosing any other ordering. We easily deduce the following theorem from 3.G and 3.H which is surprisingly similar to the corresponding results for the twisted $N = 2$ superconformal algebra.

**Theorem 4.A** For singular vectors $\Psi^p_l$ and $G$-closed singular vectors $\Psi^{pG}_l$ in the Verma module $\mathcal{V}_\Delta$, or singular vectors $\Psi^p_l$ in the $G$-closed Verma module $\mathcal{V}^G_\Pi$, one finds the following maximal singular dimensions for singular vectors at the same level $l \in \mathbb{N}_0$ and with the same parity $p \in \{\pm\}$.

| Vector | $p = +$ | $p = -$ |
|--------|---------|---------|
| $\Psi^p_l \in \mathcal{V}_\Delta$ | 2 | 2 |
| $\Psi^p_l \in \mathcal{V}^G_\Pi$ | 1 | 1 |
| $\Psi^{pG}_{\Delta -} \in \mathcal{V}_\Delta$ | 1 | 1 |
| $\Psi^{pG}_{\Pi} \in \mathcal{V}^G_\Pi$ | 0 | 0 |

Tab. d Maximal dimensions for singular spaces of the $N = 1$ Ramond algebra.
Using theorem 3.D every singular vector in its normal form can be uniquely identified by its components with respect to elements in the ordering kernel. All the other coefficients of a singular vector with respect to the standard basis are hence fixed once the coefficients on the ordering kernel are known. However, it may be extremely hard to actually compute them especially at very high levels. For the Ramond algebra the so-called fusion method has been used by Watts\textsuperscript{22} to compute certain classes of Ramond singular vectors explicitly.

**Definition 4.B** If $\Psi^\pm_\ell = \theta^\pm_\ell |\Delta\rangle$ is a singular vector\textsuperscript{24} in $\mathcal{V}_\Delta$ at level $l \in \mathbb{N}_0$ with parity $\{\pm\}$ then the normal form of $\Psi^\pm_\ell$ is completely determined by the coefficients of $L^{-1}_l$ and $L^{-1}_l G_0$. Alternatively the normal form of $\Psi^\mp_\ell$ is determined by the coefficients of $L^{-1}_l G_0$ and $L^{-1}_l G_0$. We therefore introduce the following notation

\begin{align}
(a, b)_\ell^+ &= \theta^+_\ell = a L^{-1}_l + b L^{-1}_l G_0 + \ldots , \\
(a, b)_\ell^- &= \theta^-_\ell = a L^{-1}_l G_0 + b L^{-1}_l G_0 + \ldots ,
\end{align}

for the singular vector operators $\theta^\pm_\ell$ given in their normal form, where $a, b \in \mathbb{C}$. Similarly, $G$-closed singular vectors $\Psi^\pm_G \equiv \theta^\pm_G |\frac{c}{24} - l\rangle$ are identified by their $L^{-1}_l$ or $L^{-1}_l G_0$ component for parity $+$ or $-$ respectively. We use the notation

\begin{align}
(a)_\ell^G &= \theta^G_\ell = a L^{-1}_l + \ldots , \\
(a)_\ell^-G &= \theta^-G_\ell = a L^{-1}_l G_0 + \ldots .
\end{align}

Finally, singular vectors $\Psi^\pm_\ell = \theta^\pm_\ell |\frac{c}{24}\rangle^G$ in $G$-closed Verma modules $\mathcal{V}^G_{\frac{c}{24}}$ can be identified in the following way:

\begin{align}
(a)_\ell^+ &= \theta^+_\ell = a L^{-1}_l + \ldots , \\
(a)_\ell^- &= \theta^-_\ell = a L^{-1}_l G_0 + \ldots .
\end{align}

In the introduction we mentioned that in the Virasoro case it was sufficient to compare the level of two singular vectors in the same Verma module in order to deduce whether they are proportional. This important feature simplified the derivation of the embedding structure of submodules\textsuperscript{12} and finally the connexion to the Virasoro irreducible characters. Our examples in the appendix show that in the Ramond case the simple comparison of weights is not sufficient in order to decide if singular vectors are proportional and therefore the derivation of the embedding structure and character formul\ae{} seems almost unmanageable if we do not know the singular vectors explicitly. However, theorem 3.D reduces this problem to the knowledge of the coefficients with respect to the ordering kernel only. In the case of $G$-closed singular vectors or singular vectors in $G$-closed Verma modules this single coefficient can always be normalised to 1 - if the vector is non-trivial - and therefore for these cases we can almost proceed as in the Virasoro case. For singular vectors that are not $G$-closed in complete Verma modules we need to find the two leading coefficients. This will be done in the following section. However, in order to analyse secondary singular vectors we need to find product formul\ae{} for singular vector operators. This can easily be achieved by taking into account all possible contributions to leading terms. For example

\begin{equation}
(a_1 L^{-1}_l + b_1 L^{-1}_l G_0 + \ldots)(a_2 L^{-1}_l + b_2 L^{-1}_l G_0 + \ldots) = \\
a_1 a_2 L^{-1+l}_{l-1} + (a_1 b_2 + a_2 b_1 + 2b_1 b_2) L^{-1+l}_{l-1} G_0 + \ldots ,
\end{equation}

\textsuperscript{m}Note that this definition only makes sense if $\Psi^\pm_\ell$ is a singular vector. Certainly for most pairs $a, b \in \mathbb{C}$ singular vectors do not exist.
leads to the multiplication rule for \((a_1, b_1)_{l_1}^+(a_2, b_2)_{l_2}^+\) provided the conformal weights satisfy \(\Delta_2 + l_2 = \Delta_1\). In the same way we can easily find similar rules for other types of singular vector operators.

**Theorem 4.C** \(\theta_{\mathfrak{p}_1}^a\) and \(\theta_{\mathfrak{p}_2}^b\) denote two singular vector operators for the Verma modules \(V_{\Delta_1}\) and \(V_{\Delta_2}\) with \(\Delta_1 + l_1 = \Delta_2\) then \(\theta_{\mathfrak{p}_1}^a \theta_{\mathfrak{p}_2}^b |\Delta_2\rangle\) is either trivial or singular in \(V_{\Delta_2}\) at level \(l_1 + l_2\) with parity \(p_1 p_2\). Depending on the parities the resulting singular vector operator is:

\[
(a_1, b_1)_{l_1}^+ (a_2, b_2)_{l_2}^+ |\Delta_2\rangle = (a_1 a_2, a_1 b_2 + a_2 b_1 + 2 b_1 b_2)_{l_1 + l_2}^+ |\Delta_2\rangle,
\]
\[
(a_1, b_1)_{l_1}^+ (a_2, b_2)_{l_2}^- |\Delta_2\rangle = (a_1 a_2, a_1 b_2 + a_2 b_1 |\Delta_2 - \frac{c}{24}| + 2 b_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle,
\]
\[
(a_1, b_1)_{l_1}^- (a_2, b_2)_{l_2}^+ |\Delta_2\rangle = (a_1 a_2 + 2 a_1 b_2, 1\l l_2 a_1 a_2 + a_2 b_1 - a_1 b_2 |\Delta_2 - \frac{c}{24}|}_{l_1 + l_2}^- |\Delta_2\rangle,
\]
\[
(a_1, b_1)_{l_1}^- (a_2, b_2)_{l_2}^- |\Delta_2\rangle = (a_1 a_2 |\Delta_2 - \frac{c}{24}| + 2 a_1 b_2, a_2 b_1 + 1\l l_2 a_1 a_2 - a_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle.
\]

If, however, \(\theta_{\mathfrak{p}_2}^b\) denotes a singular vector in the \(G\)-closed Verma module \(V_{\Delta_1}^G\) and \(\Delta_1 = \frac{c}{24} + l_2\) then \(\theta_{\mathfrak{p}_1}^a \theta_{\mathfrak{p}_2}^b |\Delta_2\rangle^G\) is either trivial or singular in \(V_{\Delta_1}^G\) at level \(l_1 + l_2\) with parity \(p_1 p_2\). The multiplication rules for these cases are:

\[
(a_1, b_1)_{l_1}^+ (a_2, b_2)_{l_2}^+ |\Delta_2\rangle^G = (a_1 a_2)_{l_1 + l_2}^+ |\Delta_2\rangle^G,
\]
\[
(a_1, b_1)_{l_1}^+ (a_2, b_2)_{l_2}^- |\Delta_2\rangle^G = (a_1 b_2 + 2 b_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle^G,
\]
\[
(a_1, b_1)_{l_1}^- (a_2, b_2)_{l_2}^+ |\Delta_2\rangle^G = \left(\frac{1}{2} l_2 a_1 a_2 + a_2 b_1\right)_{l_1 + l_2}^- |\Delta_2\rangle^G,
\]
\[
(a_1, b_1)_{l_1}^- (a_2, b_2)_{l_2}^- |\Delta_2\rangle^G = (2 a_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle^G.
\]

Finally, if \(\theta_{\mathfrak{p}_1}^a\) denotes a \(G\)-closed singular vector operator of \(|\Delta_1 - l_1\rangle\) and \(\theta_{\mathfrak{p}_2}^b\) a singular vector operator of \(|\Delta_2\rangle\) with \(\Delta_2 + l_1 + l_2 = \frac{c}{24}\) then \(\theta_{\mathfrak{p}_1}^a \theta_{\mathfrak{p}_2}^b |\Delta_2\rangle\) is either trivial or a \(G\)-closed singular in \(V_{\Delta_1 - l_1 - l_2}^G\) at level \(l_1 + l_2\) with parity \(p_1 p_2\).

\[
(a_1)_{l_1}^+ (a_2, b_2)_{l_2}^+ |\Delta_2\rangle = (a_1 a_2)_{l_1 + l_2}^+ |\Delta_2\rangle,
\]
\[
(a_1)_{l_1}^+ (a_2, b_2)_{l_2}^- |\Delta_2\rangle = (a_1 a_2)_{l_1 + l_2}^- |\Delta_2\rangle,
\]
\[
(a_1)_{l_1}^- (a_2, b_2)_{l_2}^+ |\Delta_2\rangle = (a_1 a_2 + 2 a_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle,
\]
\[
(a_1)_{l_1}^- (a_2, b_2)_{l_2}^- |\Delta_2\rangle = (-a_1 a_2 (l_1 + l_2) + 2 a_1 b_2)_{l_1 + l_2}^- |\Delta_2\rangle.
\]

In exactly the same way, we can also find the leading coefficients of the vectors \(\Psi_{G,l}^\pm = (a, b)^\pm |\Delta\rangle\) and \(G\Psi_{l}^\pm = G(a, b)^\pm |\Delta\rangle\), introduced in Eqs. (25)-(28). For example, the product

\[
(a L_{l-1} + b L_{l-1}^- G_{-1} G_0 + \ldots) G_0 = a L_{l-1}^- G_0 + b L_{l-1}^- G_{-1} (L_0 - \frac{c}{24}) + \ldots,
\]

leads to \(a, b\)\(\Psi_{G,l}^\pm\). Using similar products one easily obtains the following theorem.

**Theorem 4.D** If the vector \(\Psi_{l}^\pm\) is given by \((a, b)^\pm\), then the vectors \(\Psi_{G,l}^\pm = (a, b)^\pm |\Delta\rangle\) and \(G\Psi_{l}^\pm = G(a, b)^\pm |\Delta\rangle\) are:

\[
G (a, b)^-_l |\Delta\rangle = (a + 2 b, -b |\Delta - \frac{c}{24}| + \frac{1}{2} l(a)^-_l |\Delta\rangle,
\]

leads to \((a, b)\)\(\Psi_{G,l}^\pm\). Using similar products one easily obtains the following theorem.
\[
(a, b)_{G,l}^\pm |\Delta\rangle = (a, b[\Delta - \frac{c}{24}]_l^\pm |\Delta\rangle ,
\]
(82)
\[
G(a, b)_l^\pm |\Delta\rangle = (a[\Delta - \frac{c}{24}] + 2b, \frac{1}{2}l(a - b)_l^\pm |\Delta\rangle ,
\]
(83)
\[
(a, b)_{G,l}^\pm |\Delta\rangle = (a[\Delta - \frac{c}{24}], b)_l^\pm |\Delta\rangle .
\]
(84)

For \((a, b)_l^\pm\) being \(G\)-closed we require \(G(a, b)_l^\pm \equiv 0\), which is necessary and sufficient. From the above multiplication formulae we easily compute that in both cases non-trivial solutions exist only for the known condition \(\Delta + l = \frac{c}{24}\). For these solutions we obtain \((a, b)_l^\pm\) is \(G\)-closed as given in the following theorem.

**Theorem 4.E** (a, \(-\frac{a}{2}\))\_l^+ = (a)_l^+\(\[\Delta\] + l = \frac{c}{24}\) and \((\frac{8}{p}, a)_l^- = (a)_l^-\(\[\Delta\] + l = \frac{c}{24}\) \((l \neq 0)\) are the only possibilities for \(G\)-closed singular vectors in \(V_\Delta\) with \(\Delta = \frac{c}{24} - l\).

5 All Ramond singular vectors.

In the previous section, we were able to derive the ordering kernel coefficients for the vectors \(\Psi_{G,l}^\pm\) and \(G\Psi_{l}^\pm\) provided the ones of \(\Psi_{l}^\pm\) were known. In theorem 2.L, we showed that in the case of the primitive singular vectors \(\Psi_{p,q}^\pm\), the corresponding singular vectors \(\Psi_{G,l}^\pm\) and \(G\Psi_{l}^\pm\) are proportional. This proportionality requirement restricts the coefficients of \(\Psi_{p,q}^\pm\).

**Theorem 5.A** The singular vectors \(\Psi_{p,q}^\pm\) in \(V_\Delta\) at level \(\frac{c}{24}\) are either given by
\[
(q - tp, -q)_l^\pm \text{ or } (q - tp, tq)_l^\pm.
\]
(85)
Similarly, the singular vectors \(\Psi_{p,q}^\pm\) in \(V_\Delta\) at level \(\frac{c}{24}\) are either
\[
(-\frac{8t}{q}, q - tp)_l^- \text{ or } (\frac{8}{p}, q - tp)_l^-.
\]
(86)

Proof: Requiring \((a, b)_{G,l}^\pm \propto G(a, b)_{l}^\pm\) has for \(\Delta \neq \frac{c}{24}\) exactly two solutions as one can easily see: \(\frac{a}{b} = -\frac{q}{q - tp}\) or \(\frac{a}{b} = \frac{tp}{q - tp}\). The cases for which \(\Delta = \frac{c}{24}\) intersects with \(\Delta = \Delta_{p,q}\) are only the discrete points at \(t = \frac{2}{p}\). Continuity arguments for the coefficients of polynomial type of the singular vectors lead to the claim for parity +. One easily verifies the parity − case in the same way.

We note that the pairs of possible solutions given in theorem 5.A are symmetric under the operation \(p \leftrightarrow q\), \(t \leftrightarrow \frac{1}{t}\). Since the Verma modules \(V_{\Delta_{p,q}(t)}\) and \(V_{\Delta_{p,q}(\frac{1}{t})}\) are identical\(^n\) we find the following identity of singular vectors using the normalisation of equations Eqs. (85)-(86):
\[
\Psi_{p,q}^\pm(t) = -\frac{1}{t}\Psi_{p,q}^\pm(\frac{1}{t}).
\]
(87)

Due to this identity, all (primitive) singular vectors have been found once the vectors \(\Psi_{p,q}^\pm(t)\) for \(p, q \in \mathbb{N}\), \(q\) even and \(p\) odd have been given.

The examples in the appendix and computer exploration\(^{21}\) until level 6 show which of the two possibilities can be assigned to the primitive singular vectors \(\Psi_{p,q}^\pm(t)\) depending on \(p\) or \(q\) being even or odd. We can therefore give the following conjecture.

\(^n\)Note that \(c(t) = c(\frac{1}{t})\).
Conjecture 5.B The singular vectors $\Psi_{p,q}^\pm(t)$ at level $\frac{pq}{2}$ are given by

$$
\Psi_{p,q}^+(t) = (q-tp,-q)^{\frac{1}{2}} \Delta_{p,q}(t), \quad q \text{ even,}
$$

(88)

$$
\Psi_{p,q}^+(t) = (q-tp,tp)^{\frac{1}{2}} \Delta_{p,q}(t), \quad p \text{ even,}
$$

(89)

$$
\Psi_{p,q}^-(t) = (-\frac{8t}{q},q-tp)^{\frac{1}{2}} \Delta_{p,q}(t), \quad q \text{ even,}
$$

(90)

$$
\Psi_{p,q}^-(t) = (\frac{8}{p},q-tp)^{\frac{1}{2}} \Delta_{p,q}(t), \quad p \text{ even,}
$$

(91)

for $p, q \in \mathbb{N}$, $p - q$ odd.

According to theorem 5.B the singular vectors $\Psi_{p,q}^\pm$ only have two possible choices for their coefficients, therefore explicit computer calculations are a sufficiently convincing indication for conjecture 5.B. A final proof could follow in the line of Watts\textsuperscript{22}, using the fusion of conformal fields in order to derive expressions for singular vectors. But for our purpose only two coefficients are needed and therefore the construction of Watts should be applicable in a simplified form and may only be needed for a particular limit of the parameter $t$.

The expressions of 5.B reveal once more striking similarities to the twisted $N=2$ superconformal algebra. For the twisted $N=2$ algebra the singular vectors also follow a choice of two possible ordering kernel coefficients\textsuperscript{10}. They are also labelled by two integers $r, s$ where $r$ plays the rôle of $p$ and $s$ the one of $q$, however $s$ has to be odd. In the twisted $N=2$ case, the observed expressions for singular vectors always take an expression similar to Eq. (89) whilst the second possibility similar to Eq. (88) never appears. Considering the fact that $s$ has to be odd for the twisted $N=2$ case and taking into account that $q$ is odd for Eq. (89), we observe a rather close connexion between the $N=1$ Ramond case and the twisted $N=1$ case.

Let us note that the singular vectors of conjecture 5.B never vanish identically. This reveals a significant difference with respect to the singular vectors of the three isomorphic $N=2$ algebras, where discrete vanishings of the primitive singular vector expressions lead to the fact that the two-dimensional tangent space is spanned by two secondary\textsuperscript{6} linearly independent singular vectors. In contrast, the primitive singular vector expressions for the twisted $N=2$ algebra never vanish\textsuperscript{10}, like for the $N=1$ Ramond algebra.

Based on conjecture 5.B it is easy to compute the Verma modules with degenerate singular vectors, i.e. with singular spaces of dimension 2. For a given level $l \in \mathbb{N}$ there are normally multiple solutions to the number theoretical problem $pq = 2l$, $p,q \in \mathbb{N}$, $p - q$ odd. Solutions with $p$ even are said to be of type $\epsilon = +$ whilst for $q$ even we assign type $\epsilon = -$. In the general case one factorises the integer $2l$ in its prime factors

$$
2l = 2^n \prod_{i=1}^{P} \gamma_i^{n_i},
$$

(92)

with $\gamma_i > 2$ distinct primes and $n \in \mathbb{N}$, $n_i \in \mathbb{N}_0$. The type $\epsilon = -$ ($q$ even) solutions to $pq = 2l$ are then given by

$$
p_{\epsilon -} = \prod_{i=1}^{P} \gamma_i^{k_i},
$$

\textsuperscript{6}For example, in the case of the $N=2$ Neveu-Schwarz algebra the spaces of linearly independent singular vectors are always uncharged and are secondary singular vectors of charged singular vectors at lower levels.
\[ q_{\pi^-} = 2^n \prod_{i=1}^{P} \gamma_i^{n_i - k_i}, \]  
whilst for \( \epsilon = + \) (\( p \) even) we have

\[ p_{\pi^+} = 2^n \prod_{i=1}^{P} k_i, \]

\[ q_{\pi^+} = \prod_{i=1}^{P} \gamma_i^{n_i - k_i}, \]  
for each \( P \)-tuple \( \pi^\pm = (k_1, k_2, \ldots, k_P) \in \mathbb{N}_0^P \), with \( k_i \leq n_i, \forall i = 1, \ldots, P \) where the type of the solution is indicated with a superscript \( \pm \). Note that for each level \( l \in \mathbb{N} \) there exist at least two solutions: \( p = 2l, q = 1 \) and \( p = 1, q = 2l \). The number of solutions to \( pq = 2l, p, q \in \mathbb{N}, p - q \) odd is thus determined by the number of \( P \)-tuples \( \pi^\pm = (k_1, \ldots, k_P) \) with \( \pi^\pm \in \mathbb{N}_0^P \) and \( \pi^\pm \leq (n_1, \ldots, n_P) \) for the prime factorisation of \( 2l \). Hence, there are \( 3^0 \Pi_l = 2 \prod_{i=1}^{P} (n_i + 1) \) solutions to \( pq = 2l, p, q \in \mathbb{N}, p - q \) odd.

The proof of theorem 2.L sheds light on the fact that there are generically only one-dimensional singular spaces defined by the singular vectors \( \Psi_{p,q}^\pm \) (or \( \Psi_{p,q}^- \)) at level \( l = \frac{2l}{p} \) for all \( \Pi_l \) solutions \( p \) and \( q \). The most interesting Verma modules are those, where some of these \( \Pi_l \) solutions intersect and may hence lead to two-dimensional singular spaces provided the corresponding singular vectors are not proportional. Using conjecture 5.B we shall now investigate such cases of degeneration. Let us remark again that the singular dimension cannot be bigger than \( 2 \) (the size of the ordering kernel) even though a priori we would expect that even more than \( 2 \) solutions \( \pi_l \) could intersect.

Let us analyse the intersections of the conformal weights \( \Delta_{p,q}(t) \) of the \( \Pi_l \) solutions corresponding to \( pq = 2l, p, q \in \mathbb{N}, p - q \) odd. For fixed \( l \in \mathbb{N} \) we find the prime factorisation as given above. Let us now take two \( P \)-tuples \( \pi_1^+ \) and \( \pi_2^+ \) of assigned types \( \epsilon_1 \) and \( \epsilon_2 \). We write \( p_i \) for \( p_{\pi_i^+} \) and \( q_i \) for \( q_{\pi_i^+} \) which satisfy \( p_1 q_1 = p_2 q_2 = 2l \). Let us further assume that the corresponding conformal weights intersect: \( \Delta_{p_1, q_1} = \Delta_{p_2, q_2} \). This easily implies \( (q_1 - p_1 t)^2 = (q_2 - p_2 t)^2 \) for the intersection points which results in

\[ t = \pm \frac{q_2}{p_1} = \pm \frac{q_1}{p_2}. \]  

Using the prime factorisation the intersection values of the central parameter \( t \) can be given as

\[ t = \pm \beta_{\pi_1^+, \pi_2^+} \prod_{i=1}^{P} \gamma_i^{n_i - k_i^1 - k_i^2}, \]  
with

\[ \beta_{\pi_1^+, \pi_2^+} = \begin{cases} 2^n & \text{for } (\epsilon_1, \epsilon_2) = (-, -) \\ 1 & \text{for } (\epsilon_1, \epsilon_2) = (-, +), (+, -) \\ \frac{1}{2^n} & \text{for } (\epsilon_1, \epsilon_2) = (+, +) \end{cases} \]  

Therefore, the conformal weights \( \Delta_{p_1, q_1} \) and \( \Delta_{p_2, q_2} \) have exactly 2 intersection points where the two singular vectors \( \Psi_{p_1, q_1}^+ \) and \( \Psi_{p_2, q_2}^+ \) both exist as well as \( \Psi_{p_1, q_1}^- \) and \( \Psi_{p_2, q_2}^- \). The key question is whether these singular vectors are different and hence define two-dimensional singular spaces, or if they are proportional and thus lead to one-dimensional singular spaces instead.  

\[ ^p \text{We define the empty product as } \prod_{i=1}^{0} = 1. \]
For this purpose, let us take again two $P$-tuples $\pi_{1,i}$ and $\pi_{2,i}$. The corresponding conformal weights intersect for $t = \pm \beta_{n,q}^{\epsilon_1,\epsilon_2} \prod_{i=1}^{P} \gamma_{n_i}^{-1} = \pm \frac{q_1}{p_2} = \pm \frac{q_2}{p_1}$. From 5.B we obtain the corresponding singular vectors $\Psi_{p,j}^\pm$, $j = 1, 2$ as:

\[
\Psi_{p,j}^+ (t = \pm \frac{q_1}{p_2}) = (q_j + q_j, -q_j)^+ |_{\Delta_{p,j}^+} \epsilon_j = -, (q_j \text{ even}) ,
\]

\[
\Psi_{p,j}^+ (t = \pm \frac{q_1}{p_2}) = (q_j + q_j, \pm q_j)^+ |_{\Delta_{p,j}^+} \epsilon_j = +, (p_j \text{ even}) ,
\]

\[
\Psi_{p,j}^- (t = \pm \frac{q_1}{p_2}) = (\pm \frac{q_1}{p_2}, q_j + q_j)^+ |_{\Delta_{p,j}^-} \epsilon_j = -, (q_j \text{ even})
\]

\[
\Psi_{p,j}^- (t = \pm \frac{q_1}{p_2}) = (\frac{q_1}{p_2}, q_j + q_j)^+ |_{\Delta_{p,j}^-} \epsilon_j = +, (p_j \text{ even}) ,
\]

with $\bar{l} = 2$ and $\bar{2} = 1$. By considering the determinant of these coefficients we easily obtain the following result.

**Theorem 5.C** For a given level $l \in \mathbb{N}$ the singular vectors $\Psi_{p,j}^\pm$ with $l = \frac{p_1}{p_2}$ define generically one-dimensional singular spaces. There are two intersection points of $\Delta_{p-1,q} = \Delta_{p-1,q}$ for $p_1 q_1 = p_2 q_2 = 2l$ which are $t = \pm \beta_{n,q}^{\epsilon_1,\epsilon_2} \prod_{i=1}^{P} \gamma_{n_i}^{-1} = \pm \frac{q_1}{p_2} = \pm \frac{q_2}{p_1}$. At these points degenerate singular spaces of dimension 2 occur as given in the following table.

| $\epsilon_1$ | $\epsilon_2$ | $p_1$ | $p_2$ | $t$ | dimension |
|----------------|----------------|-------|-------|-----|-----------|
| -              | -              | odd   | odd   | $\pm 2^n \prod_{i=1}^{P} \gamma_{n_i}^{-1}$ | 2 |
| -              | +              | odd   | even  | $\pm \prod_{i=1}^{P} \gamma_{n_i}^{-1}$ | 1 |
| +              | -              | even  | odd   | $\pm \prod_{i=1}^{P} \gamma_{n_i}^{-1}$ | 1 |
| +              | +              | even  | even  | $\pm \frac{1}{q_2} \prod_{i=1}^{P} \gamma_{n_i}^{-1}$ | 2 |

**Tab. e Degenerate $N = 1$ Ramond cases.**

The values for $t$ at the intersection points are real and rational. However, at most 2 of these conformal weight curves can intersect in any common values of $t$.

**Proof:** The determinant of the ordering kernel coefficients of Eqs. (98)-(101) in the different cases are $\pm (q_2^2 - q_2^2)$ for $(\epsilon_1, \epsilon_2) = (-, -)$ and $\pm (q_1^2 - q_2^2)$ for $(\epsilon_1, \epsilon_2) = (+, +)$ by considering only the positive parity case. In the case of $(\epsilon_1, \epsilon_2) = (+, -)$ and $(\epsilon_1, \epsilon_2) = (-, +)$ the determinant always vanishes. Similar relations can easily be found for the negative parity cases. Hence for $\epsilon_1 \epsilon_2 = +$ (equal type singular vectors) we obtain 2-dimensional spaces, whilst for $\epsilon_1 \epsilon_2 = -$ (different type singular vectors) we have just 1-dimensional singular spaces defined by the two singular vectors. Let us assume now that the conformal weights of $\pi_{1,i}^{\epsilon_1}$ and $\pi_{2,i}^{\epsilon_2}$ intersect at $t = \pm \beta_{n,q}^{\epsilon_1,\epsilon_2} \prod_{i=1}^{P} \gamma_{n_i}^{-1} = \pm \frac{q_1}{p_2}$. In the same way the conformal weights of $\pi_{2,i}^{\epsilon_1}$ and $\pi_{3,i}^{\epsilon_3}$ intersect at $\bar{t} = \pm \beta_{n,q}^{\epsilon_2,\epsilon_3} \prod_{i=1}^{P} \gamma_{n_i}^{-1} = \pm \frac{q_2}{p_2}$. Assuming $t = \bar{t}$ obviously leads to $q_1 = q_3$ and hence also $p_1 = p_3$. Therefore, there are no values of $t$ for which more than two conformal weights $\Delta_{p,q}$ intersect at the same level $pq = 2l$.

Intersections of equal type singular vectors occur for the first time at level 3 when $\Delta_{1,6}$ intersects $\Delta_{3,2}$ and for $\Delta_{6,1}$ intersecting with $\Delta_{2,3}$ and therefore lead to degenerate primitive singular vectors. All other intersections at level 3 and below are intersections of different type.
of singular vectors and therefore do not lead to any degenerate levels. In appendix A we give all primitive singular vectors until level 3 and indicate explicitly the degenerate singular vectors. So far, degenerate singular vector spaces had only been discovered for the $N = 2$ superconformal algebras\textsuperscript{6,15,10}, and the fact that also the $N = 1$ Ramond algebra leads to degenerate singular vector spaces in its highest weight representations has been overlooked in the past. Let us stress that for the $N = 2$ isomorphic algebras (Neveu-Schwarz, topological and Ramond) the degenerate singular vectors found so far are always secondary whereas for the $N = 2$ twisted algebra and for the $N = 1$ Ramond algebra the degenerate singular vectors are generically primitive\textsuperscript{9}.

The multiplication rules of theorem 4.C together with the identification of the singular vectors with their ordering kernel coefficients supply the fundamental tools needed to investigate the structure of embedded singular vectors and descendant singular vectors and hence the embedding diagrams. In the following section, we show that theorem 4.C allows us to follow the lines of Feigin and Fuchs\textsuperscript{12}, who discussed the Virasoro case, to extend their approach to the Ramond algebra using the results of this section. In a first analysis we will comment on the main similarities for the rational models ($t \in \mathbb{Q}$) with the Virasoro case but we will also reveal certain major differences. A complete discussion and derivation of all Ramond embedding diagrams will be given in a forthcoming paper\textsuperscript{4}.

6 Descendant singular vectors.

In the previous sections we have discussed that a singular vector $\theta_{l_1} |\Delta_1\rangle$ of the Verma module $\mathcal{V}_{\Delta_1}$ generates a submodule homomorphic to $\mathcal{V}_{\Delta_1+l_1}$. On the other hand $\mathcal{V}_{\Delta_1+l_1}$ may again contain a singular vector $\theta_{l_2} |\Delta_1 + l_1\rangle$. In this section we want to focus on the resulting vector $\theta_{l_2} \theta_{l_1} |\Delta_1\rangle$, which - if non-trivial - is again a singular vector, called a descendant or secondary singular vector, as was explained before. Descendant singular vectors that are proportional are considered to be equal. The relative structure of the singular vectors and their descending patterns are usually given in so-called embedding diagrams. Embedding diagrams are important to find out the dimension of the null submodule of a Verma module and are hence crucial for the character formulae of the irreducible highest weight representations. Embedding diagrams include all singular vectors and usually - if applicable - also subsingular vectors. The diagrams indicate which singular vectors appear as descendant singular vectors of others and proportional singular vectors are indicated as one singular vector only.

For the Virasoro algebra the derivation of the embedding diagrams\textsuperscript{12} is merely a matter of solving the number theoretical problem for which levels the roots of the Kac determinant formula predict a singular vector and of comparing the relative levels of these singular vectors to reveal the embedding pattern. This procedure is rather simple due to the properties of Virasoro algebra that does not allow two or more Virasoro singular vectors at the same level in the same Verma module. Hence Virasoro singular vectors at the same level in the same Verma module are always proportional. As a remark, let us add that also for the Virasoro case for a given level $l \in \mathbb{N}_0$ we have different one-parameter families of singular vectors $\xi_{p,q}(t)$ that intersect for certain values of $t$, pairs $(p_1,q_1)$, and $(p_2,q_2)$ at the same level $p_1 q_1 = p_2 q_2$. However, for Virasoro Verma modules these singular vectors at such intersection points are always linearly dependent\textsuperscript{12,18,9}. Furthermore, for the Virasoro algebra the products of singular vector operators never vanish and also Virasoro Verma modules do not contain any subsingular vectors. All these non-trivial properties make the derivation of Virasoro embedding diagrams straightforward.

\textsuperscript{9}We are very grateful to B. Gato-Rivera for pointing this out to us.
For superconformal algebras most of these remarkable properties of the Virasoro algebra may not hold. Products of singular vector operators may vanish identically, a very common consequence of Lie superalgebras, singular spaces may be bigger than just one-dimensional and subsingular vectors may also arise. Such features have recently been discussed for the \( N = 2 \) superconformal algebras\textsuperscript{14, 15, 9, 10, 6, 5}. In this section we will show that the Ramond algebra also shares some of these difficulties and we will discuss the derivation of the most interesting embedding diagrams, the rational models. A full consideration of the Ramond embedding diagrams shall be given elsewhere\textsuperscript{4}. The multiplication rules of theorem 4.C and the expressions of 5.B will prove to be the main tools for the analysis of the Ramond embedding diagrams. With their help it will be possible to find out which descendant singular vectors are identical and which are trivial.

Like in the Virasoro case\textsuperscript{12} and the \( N = 2 \) Neveu-Schwarz case\textsuperscript{5} we first compute for a given Verma module \( \mathcal{V}_\Delta \) and a central parameter \( t \) the intercept \( a \) as

\[
a^2 = 8t\Delta - \frac{5t}{2} + 1 + t^2. \tag{102}
\]

The sign of \( a \) is not relevant but for real \( a \) we choose \( a \) positive. In the case that \( \Delta = \Delta_{p,q} \) for \( p, q \in \mathbb{N}, p - q \) odd, we obviously find \((q - pt)^2 = a^2\). A vanishing \( a \)-value corresponds to the supersymmetric case \( \Delta = \frac{s^2\tau}{24} \) which will be analysed in much detail in the following section. Therefore, \( a = 0 \) shall be excluded for the rest of this section. Conversely, if for given values of \( t \) and \( a \) we are looking for the integers \( p, q \) that solve \( \Delta = \Delta_{p,q} \), then we find as solutions all \( \hat{p}, \hat{q} \) with

\[
\hat{q} = \hat{p}t - a, \quad \hat{p}, \hat{q} \in \mathbb{Z}, \quad \hat{p} - \hat{q} \text{ odd}, \quad \hat{pq} > 0. \tag{103}
\]

For these values of \((\hat{p}, \hat{q})\) we have the required \( \Delta_{p,|q|} = \Delta \) with \( p = \hat{p}, q = |\hat{q}| \) and hence \( p - q \) odd. Note that we distinguish \( p, q \) from \( \hat{p}, \hat{q} \) as the latter are also allowed to be negative as long as their product stays positive. Just like in the Virasoro case\textsuperscript{12} the problem reduces to finding the integer pair solutions \((\hat{p}, \hat{q})\) on the straight line \( \Delta = \hat{\Delta} \) with \( \hat{p}q > 0 \) and \( \hat{p} - \hat{q} \) odd. In contrast to the Virasoro case we require here the difference \( \hat{p} - \hat{q} \) to be odd. The cases of irrational \( t \) are simple and will be included in a full discussion on the embedding diagrams in a forthcoming publication\textsuperscript{4}, the cases of rational \( t \) are however very interesting. These so-called rational models shall be analysed here. We therefore assume \( t \in \mathbb{Q} \) and find the unique pair of integers \((u, v)\) with \( u, v \) coprime and \( v \) positive such that \( t = \frac{u}{v} \).

If \( \mathcal{L} \) has no integer pair solutions at all or only integer pair solutions with \( \hat{p} - \hat{q} \) even, then the Verma module \( \mathcal{V}_{\Delta(a,t)} \) is obviously irreducible as the determinant formula is non-trivial for all levels. In the case that \( \mathcal{L} \) has at least one integer pair solution with \( \hat{p} - \hat{q} \) odd, then it obviously has infinitely many. In the notation of Feigin and Fuchs\textsuperscript{12} these cases are denoted by \( \text{III}_- \) for positive \( t \) and \( \text{III}_+ \) for negative\textsuperscript{5} values of \( t \). Our task is now to describe among these infinitely many solutions those solutions \((\hat{p}, \hat{q})\) with \( \hat{pq} > 0 \) and \( \hat{p} - \hat{q} \) odd. For this purpose it is essential to distinguish two different cases for which we shall add the label \( e \) for even or \( o \) for odd to \( \text{III}_\pm \). Thus \( \text{III}_{e,\pm} \) denotes the case where one of \( u \) or \( v \) is even (but obviously not both) whilst \( \text{III}_{o,\pm} \) stands for the cases with both \( u \) and \( v \) odd.

Let us first concentrate on \( t \) positive. The integer pair solution of \( \mathcal{L} \) with \( \hat{p} - \hat{q} \) odd, both \( \hat{p} \) and \( \hat{q} \) positive, and the smallest for \( \hat{p} \) is denoted by \((p_0, q_0)\). Likewise, the integer pair solution of \( \mathcal{L} \) with \( \hat{p} - \hat{q} \) odd, both \( \hat{p} \) and \( \hat{q} \) negative, and the smallest value of \( |\hat{p}| \) is denoted by \((-\tilde{p}_0, -\tilde{q}_0)\).

We can now easily give all other allowed solutions:

\[
p_n = p_0 + nv, \quad q_n = q_0 + nu, \quad \tilde{p}_n = -\tilde{p}_0 - nv, \quad \tilde{q}_n = -\tilde{q}_0 - nu, \tag{104}
\]

\[
\text{Note the opposite definition of the index of \text{III}_\pm and the sign of } t \text{ for historical reasons.}
\]
with \( n \in \mathbb{N}_0 \) for the case of \( \text{III}_{o-} \). However, for \( \text{III}_{e-} \) we only find every other point as a solution and hence \( n \in 2\mathbb{N}_0 \). Therefore, we can already state that the Verma module \( \mathcal{V}_{p_0,q_0} \) contains singular vectors at the levels \( \frac{p_0 q_0}{2} \) and \( \frac{\tilde{p}_0 \tilde{q}_0}{2} \) for all \( n \in \mathbb{N}_0 \) in the case \( \text{III}_{o-} \) but only for \( n \in 2\mathbb{N}_0 \) for \( \text{III}_{e-} \). For all the solutions of type \((p_n, q_n)\) we have \( q - tp = -a \) whilst for \((\tilde{p}_n, \tilde{q}_n)\) we obtain \( q - \tilde{t}p = a \) which fixes already one of the coefficients of \( 5.B \) for each case. We shall now investigate descendant singular vectors of these primitive singular vectors and shall use the multiplication rules of theorem 4.C to compare their ordering kernel coefficients.

Along this line, we first look at the singular vector \( \Psi^+_{p_0,q_0} \). Its highest weight \( \Delta_{p_0,q_0} + \frac{2p_0 q_0}{2} \) leads to a parameter \( a' \), which is related to the original \( a \) for \( \Delta_{p_0,q_0} \) through: \( a' = a + 2q_0 \). For the descendant singular vectors we hence have to solve exactly the same number theoretical problem, namely finding integer solutions on the straight line \( \tilde{q} = \tilde{p}t - a' \). This again resembles the Virasoro case\(^1\), however, a main difference in our case is that we have to take into account the implications of the two subcases \( \text{III}_{e-} \) and \( \text{III}_{o-} \).

Figure i shows that in the case of \( \text{III}_{o-} \) the two lowest level solutions of this shifted straight line are \((p_0 + v, -q_0 + u)\) (if \( q_0 \neq u \)) and \((-\tilde{p}_0, -\tilde{q}_0 - 2q_0)\) which again both satisfy that the difference of the \( p \)-value and the \( q \)-value is odd. In the case that \( q_0 = u \) the former solution changes to \((p_0 + 2v, u)\). This case shall be denoted by \( \text{III}_{o-}^{(u)} \). In the same way we can analyse the shifted straight line for the singular vector \( \Psi^+_{p_0,q_0} \), which results in the two lowest level solutions\(^8\) \((p_0 + 2\tilde{p}_0, q_0)\) and \((\tilde{p}_0 - v, -\tilde{q}_0 - u)\) (for \( \tilde{p}_0 \neq v \)). In the case of \( \tilde{p}_0 = v \) the latter changes to \((-v, -\tilde{q}_0 - 2u)\) which shall be denoted by \( \text{III}_{o-}^{(v)} \).

\[
\begin{align*}
\text{Fig. i} \quad \text{III}_{o-} \text{ solutions of } \tilde{q} &= \tilde{p}t - a' \\
\end{align*}
\]

Therefore, we find for the case \( \text{III}_{o-} \) the descendant singular vectors

\[
\begin{align*}
\Psi_a^+ &= \theta^+_{p_0+v, u-q_0} \theta^+_{p_0, q_0} \left| \Delta_{p_0, q_0} \right>, \\
\Psi_b^+ &= \theta^+_{p_0, q_0} + 2q_0 \theta^+_{p_0, q_0} \left| \Delta_{p_0, q_0} \right>, \\
\Psi_c^+ &= \theta^+_{p_0+2\tilde{p}_0, q_0} \theta^+_{p_0, q_0} \left| \Delta_{p_0, q_0} \right>, \\
\Psi_d^+ &= \theta^+_{v-\tilde{p}_0, -\tilde{q}_0 - u} \theta^+_{p_0, q_0} \left| \Delta_{p_0, q_0} \right>.
\end{align*}
\]

The vectors given are for the positive parity sector which can easily be transformed into the

\(^8\)For convenience we invert the line in the origin.
negative parity sector. We observe that the vectors $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^b_+ \mid \Delta_{p_0,q_0}^+\rangle$ are at the same level and the same is true for the pair $\Psi^c_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^d_+ \mid \Delta_{p_0,q_0}^+\rangle$. Using the multiplication rules of 4.C and taking into account the different possible types for the singular vector operators we easily find by explicit calculations that the singular vectors $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^b_+ \mid \Delta_{p_0,q_0}^+\rangle$ are proportional as well as $\Psi^c_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^d_+ \mid \Delta_{p_0,q_0}^+\rangle$. Performing the same procedure on the singular vectors $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^b_+ \mid \Delta_{p_0,q_0}^+\rangle$ using exactly the same method of shifting the straight line $\hat{p} = \hat{q}t - a$ appropriately, we consequently obtain a second list of descendant singular vectors of $\Psi^a_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle$ for the case $\text{III}_{0-}$.

\begin{align}
\Psi^c_+ &= \theta^+_{p_0+2v,q_0} \theta^+_{p_0+v,u-q_0} \theta^+_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle, \\
\Psi^d_+ &= \theta^+_{p_0,q_0+2v} \theta^+_{p_0+v,u-q_0} \theta^+_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle, \\
\Psi^e_+ &= \theta^+_{p_0,q_0+2v} \theta^+_{p_0+v,u-q_0} \theta^+_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle, \\
\Psi^f_+ &= \theta^+_{p_0,q_0+2v} \theta^+_{p_0+v,u-q_0} \theta^+_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle.
\end{align}

Again, the multiplication rules 4.C easily prove that $\Psi^c_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^d_+ \mid \Delta_{p_0,q_0}^+\rangle$ are proportional and also the pair $\Psi^e_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^f_+ \mid \Delta_{p_0,q_0}^+\rangle$. Furthermore, the results of the multiplication show that the latter pair of singular vectors is proportional to $\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$ whilst the former is proportional to $\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$. Therefore, after having reached singular vectors of the original straight line $\mathcal{L} (\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$) the whole procedure starts again and ultimately exhausts all (infinitely many) singular vectors. So far, this reminds us very much of the Virasoro case but we should keep in mind that differently to the Virasoro case the multiplication rules 4.C were crucial to prove this embedding structure. We will summarise this structure in the embedding diagrams given below in theorem 6.A. Before continuing to the case $\text{III}_{e-}$ we shall discuss what happens in the special cases $\text{III}_{0(u)}$ and $\text{III}_{0(v)}$.

According to the solutions on the shifted straight lines given above for the case $\text{III}_{0(u)}$ we find that the singular vector $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ has to be replaced by

\begin{equation}
\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle = \theta^+_{p_0+2v,u} \theta^+_{p_0,q_0} \mid \Delta_{p_0,q_0}^+\rangle.
\end{equation}

Taking into account that $p_0$ is even for $\text{III}_{0(u)}$ ($q_0 = u$) we hence find that $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ becomes proportional to $\Psi^b_+ \mid \Delta_{p_0,q_0}^+\rangle$ whilst $\Psi^c_+ \mid \Delta_{p_0,q_0}^+\rangle$ is already proportional to $\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$, again by using our multiplication rules. Therefore the above embedding pattern collapses in the way indicated in theorem 6.A. Similarly, for the cases $\text{III}_{0(v)}$ the vectors $\Psi^a_+ \mid \Delta_{p_0,q_0}^+\rangle$ and $\Psi^c_+ \mid \Delta_{p_0,q_0}^+\rangle$ become proportional whilst $\Psi^d_+ \mid \Delta_{p_0,q_0}^+\rangle$ is already proportional to $\Psi^1_+ \mid \Delta_{p_0,q_0}^+\rangle$, each resulting in the same embedding patterns as for $\text{III}_{0(u)}$.

Finally, if both conditions hold, $q_0 = u$ and $\tilde{p}_0 = v$, we obtain the proportionalities of both cases. This case shall be denoted by $\text{III}_{0-}$ following the notation of Feigin and Fuchs. Let us finally mention that for the cases $\text{III}_{a+}$ with negative, rational $t$ we find exactly the same patterns except that the embedding patterns come to an end just like in the Virasoro case. In figure i this corresponds to solutions in the regions with $\tilde{p}\tilde{q} < 0$ and therefore the number of solutions has to be finite. We have hence proven that the descendant singular vectors of the primitive singular vectors have the following embedding patterns.

**Theorem 6.A** For the subsets $\text{III}_{a\pm}$, $\text{III}_{0\pm}$, $\text{III}_{0(u)}$, $\text{III}_{0(v)}$, and $\text{III}_{0\pm}$ of rational models ($\Delta \neq \frac{2\pi}{24}$) we find the following embedding patterns for descendant singular vectors of the type $\Psi^1_{p,q} \mid \Delta_{p,q}^+\rangle$ in the positive parity sector. The negative parity sector leads to exactly the same diagrams. The zero mode $G_0$ interpolates between the embedding diagrams of the positive parity sector and the negative parity sector.

---

1Note that not all combinations of types are possible. If for example $\theta^+_{p_0,q_0}$ is of type $\epsilon = +$, then $\theta^+_{p_0+v,u-q_0}$ has to be of type $\epsilon = -$. Indeed, wrong type combinations do not lead to proportional singular vectors in some of the following cases.

2The fact that these are all singular vectors will be explained in Ref. 4.
descending solutions of $\Psi_p$ (but not both) is odd. In this case not every integral point pair on the straight line $v$ to a singular vector because the addition of $u$ from odd to even. Hence, only every other integral point pair on the straight line $p$. 

In the following we analyse in the same way the cases $\Psi_{1,q}$ which means that either $u$ or $v$ (but not both) is odd. In this case not every integral point pair on the straight line $L$ leads to a singular vector because the addition of $u$ and $v$ to $q_0$ and $p_0$ respectively changes $p-q$ from odd to even. Hence, only every other integral point pair on the straight line $L$ satisfies the condition $p-q$ odd. At first, this seems to complicate our analysis. However, it turns out that the only difference is that $v$ and $u$ need to be replaced by $2v$ and $2u$ respectively. Analysing descending solutions of $\Psi_{p_0,q_0}$ and $\Psi_{\tilde{p}_0,\tilde{q}_0}$ we obtain the following figure.

![Diagram](image.png)

**Fig. ii** $N=1$ Ramond embedding diagrams $\Pi_{\alpha}$, positive parity sector.

In the following we analyse in the same way the cases $\Pi_{\epsilon-}$ which means that either $u$ or $v$ (but not both) is odd. In this case not every integral point pair on the straight line $L$ leads to a singular vector because the addition of $u$ and $v$ to $q_0$ and $p_0$ respectively changes $p-q$ from odd to even. Hence, only every other integral point pair on the straight line $L$ satisfies the condition $p-q$ odd. At first, this seems to complicate our analysis. However, it turns out that the only difference is that $v$ and $u$ need to be replaced by $2v$ and $2u$ respectively. Analysing descending solutions of $\Psi_{p_0,q_0}$ and $\Psi_{\tilde{p}_0,\tilde{q}_0}$ we obtain the following figure.

![Diagram](image.png)

**Fig. iii** $\Pi_{\epsilon-}$ solutions of $\tilde{q} = \tilde{p}t - a$ at low levels.

It is worth noting that in the cases $\Pi_{\epsilon}$ the singular vectors of one embedding diagram are always of the same type whilst for $\Pi_{\alpha}$ they can be of different type. As before, we can easily define the descendant singular vectors of first and second order analogous to the Eqs. (106)-(113) simply by replacing $v$ and $u$ by $2v$ and $2u$ respectively everywhere in the index of the
vectors. In this way we obtain for example the singular vectors
\begin{equation}
\Psi^+_a = \theta^+_{p_0 + 2u, 2u - q_0} \theta^+_{p_0, q_0} |\Delta_{p_0,q_0}\rangle,
\end{equation}
\begin{equation}
\Psi^+_b = \theta^+_{p_0, q_0 + 2q_0} \theta^+_{p_0, q_0} |\Delta_{p_0,q_0}\rangle.
\end{equation}
We can again use the multiplication rules of 4.C to obtain exactly the same proportionality as above. The main difference, however, is that the embedding pattern does not collapse for \( q_0 = u \) and neither for \( \tilde{p}_0 = v \) as it does for \( \III_{0 \pm} \) and also for the Virasoro case. Nevertheless, the embedding pattern does simplify to the single line pattern for \( q_0 = 2u \) and for \( \tilde{p}_0 = 2v \). For example, in the former case the vector
\begin{equation}
\Psi^+_d = \theta^+_{p_0 + 4u, 2u} \theta^+_{p_0, q_0} |\Delta_{p_0,q_0}\rangle,
\end{equation}
becomes proportional to \( \Psi^+_{_{2u,2v}} \) and \( \Psi^+_d \) is proportional to \( \Psi^+_{p_0,q_0} \). We shall denote these special cases by \( \III^0_{_{2u}} \) and \( \III^0_{_{2v}} \) respectively. Let us note that \( q_0 = 2u \) and \( \tilde{p}_0 = 2v \) results in \( \frac{2u - q_0}{2u - \tilde{p}_0} = \frac{v}{u} \) and therefore either \( \tilde{q}_0 \) or \( p_0 \) must be even. This is related to the fact that the singular vector solutions for \( \III_{e \pm} \) are always of the same type. But since \( q_0 \) and \( \tilde{p}_0 \) are both even this leads to a contradiction. Therefore, cases of the type \( \III^0_{_{2u}}(2u)(2v) \) do not exist. We have therefore proven the results contained in the following theorem.

**Theorem 6.B** For the subsets \( \III_{e \pm}, \III^0_{e \pm}, \) and \( \III^0_{e \pm} \) of rational models (\( \Delta \neq \frac{c}{24} \)) we find the following embedding patterns for descendant singular vectors of the type \( \Psi^+_{p,q} \) in the positive parity sector. The negative parity sector leads to exactly the same diagrams. The zero mode \( G_0 \) interpolates between the embedding diagrams of the positive parity sector and the negative parity sector.

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![Fig. iv N = 1 Ramond embedding diagrams \( \III_{e}, \) positive parity sector.](image)

As explained, the main difference to the embedding patterns of the Virasoro case is that the cases \( \III_{e \pm} \) do not simplify to the single line embedding patterns for \( q_0 = u \) or \( \tilde{p}_0 = v \). By grouping these cases together with the types \( \III_{e \pm} \), we have defined our notation \( \III^0_{e \pm} \) in such a way that the resulting embedding patterns seem similar to the Virasoro case. Let us now analyse some of these special cases further. \( q_0 = u \) and \( \tilde{p}_0 = v \) easily implies \( (a \frac{v}{u} + v, u) \) and...
$(-v,-a-u)$ as the two lowest level solutions. The corresponding singular vectors are hence both at the level $av + uv$. Conversely, $p_0q_0 = \tilde{p}_0\tilde{q}_0$ implies that the two intercepts $-a$ and $\frac{q}{p}$ are both integers. Therefore, we have only two possibilities for each of the points $(p_0, q_0)$ and $(\tilde{p}_0, \tilde{q}_0)$ which for the former are for example $(a\frac{u}{a} + v, u)$ or $(a\frac{u}{a} + 2v, 2u)$. However, equality of the levels is only possible for $q_0 = u$ and $\tilde{p}_0 = v$. Therefore the subset of these special cases which satisfy both $q_0 = u$ and $\tilde{p}_0 = v$ correspond exactly to the cases where two curves of primitive singular vectors intersect at the same level. The values of $t$ for these cases were given in Eq. (95) as $t = \pm \frac{p_1}{p_2} = \pm \frac{q_2}{p_1}$. If the two intersecting primitive singular vectors are of different type $\epsilon$, then $u$ and $v$ would obviously be both odd and therefore the embedding diagram would go over to the straight line embedding pattern in accordance with our earlier result that intersections of singular vectors of different type do not lead to degenerate singular states. However, if the two intersecting singular vectors are of the same type then consequently $t = \pm \frac{p_2}{p_1} = \pm \frac{q_2}{q_1}$ implies that $u$ and $v$ cannot both be odd and therefore we obtain case $\text{III}_{\epsilon+}$. The corresponding embedding diagram does not simplify to the straight line pattern as in the Virasoro case, which clearly shows the degenerate singular vectors.

**Theorem 6.C** The Verma modules with degenerate singular vectors given by the intersection of 2 curves of primitive singular vectors of the same type at the same level are exactly the cases $\text{III}_{\epsilon+}$ with $q_0 = u$ and $\tilde{p}_0 = v$. The corresponding embedding diagrams do not simplify to a straight line pattern but stay as the crossed embedding pattern given in figure iv.

The embedding diagrams given above are equally valid for the positive parity sector and the negative parity sector with the only exception of the special case $\Delta = \frac{c}{2}$, i.e. $\Delta = \Delta_{p,q}$ with the integer pair $(p,q)$, $0 < p \leq m - 1$, $0 < q \leq m + 1$, $p - q$ odd. Obviously, the highest common factor of $m$ and $m + 2$ is 2 for $m$ even and 1 for $m$ odd. For $m$ even we therefore have $u = \frac{m}{2} + 1$ and $v = \frac{m}{2}$ whilst for $m$ odd we obtain $u = m + 2$ and $v = m$. Consequently, we find for $m$ even that the solutions $(p_0, q_0)$ and $(\tilde{p}_0, \tilde{q}_0)$ satisfy the conditions $0 < p_0, \tilde{p}_0 \leq m - 1 < 2v$, $0 < q_0, \tilde{q}_0 \leq m + 1 < 2u$ and for $m$ odd $0 < p_0, \tilde{p}_0 \leq m - 1 < v$, $0 < q_0, \tilde{q}_0 \leq m + 1 < u$. Therefore, the unitary minimal models for $\Delta = \frac{c}{2}$ ($a \neq 0$) are of type $\text{III}_{\epsilon-}$ for $m$ even and of type $\text{III}_{\epsilon-}$ for $m$ odd. The corresponding embedding diagrams are always of the crossed type. This finally proves for $\Delta \neq \frac{c}{2}$ the only $N = 1$ Ramond embedding diagrams that had been conjectured in the literature so far$^{10}$ for the unitary minimal models. Note that none of these cases has degenerate singular levels since $t = \frac{p_0}{p_0} = \frac{q_0}{\tilde{p}_0} = \frac{q_0}{\tilde{q}_0}$ implies $p_0 = \tilde{p}_0$ and $q_0 = \tilde{q}_0$ and thus $a = 0$. Therefore, we should stress that the conditions of the critical cases of $\text{III}_{\epsilon-}$ of theorem 6.C lead for the unitary minimal models to the excluded case $a = 0$. Hence, embedding diagrams of the unitary minimal supersymmetric cases, i.e. with $\Delta = \frac{c}{2}$ do not have the embedding diagrams of singular vectors given in the literature$^{19}$. Their embedding patterns change and in
addition we need to incorporate subsingular vectors as described in the following section. We summarise these results in the following theorem.

**Theorem 6.D** The unitary minimal models with \( \Delta \neq \frac{c}{24} \) \((a \neq 0)\) are of type \( \text{III}_{0-} \) for \( m \) odd and of type \( \text{III}_{c-} \) for \( m \) even.

### 7 The supersymmetric case and subsingular vectors.

The supersymmetric case \( \Delta = \frac{c}{24} \) corresponds to \( a = 0 \) which we had to exclude in our analysis in the previous section. In theorem 2.1 we have explained that for the supersymmetric case \( \Delta = \frac{c}{24} \) the structure of the Verma module \( V_\Delta \) changes compared to other values of \( \Delta \). In this case \( V_{\frac{c}{24}} \) contains just one (unextended) submodule \( V_0^c \), which in the language of the extended algebra can be written as the \( G \)-closed Verma module \( V_{\frac{c}{24}}^G = V_0^c \). Note that \( G_0 \) can be diagonalised on the whole \( G \)-closed module \( V_{\frac{c}{24}}^G \). Besides \( V_{\frac{c}{24}}^G \), the module \( V_{\frac{c}{24}} \) contains vectors outside \( V_{\frac{c}{24}}^G \) that are not contained in any (unextended) Verma module built on a \( G_0 \)-eigenvector. The submodule \( V_{\frac{c}{24}}^G \) is generated by the singular vector \( \Psi_0^- = G_0 | \frac{c}{24} \rangle \) at level 0 with parity \(-\). The quotient module

\[
V_{\frac{c}{24}}^G = \frac{V_{\frac{c}{24}}}{U(R_1)\Psi_0^-},
\]

is again the \( G \)-closed Verma module \( V_{\frac{c}{24}}^G \). This process of constructing the quotient module shall be considered in more detail in this section. At first, it seems that this quotient module is only one of many quotient modules one may want to consider. We could also consider quotients with respect to modules generated by singular vectors \( \Psi_{p,q}^\pm \) at levels \( \frac{pq}{24} \). However, the supersymmetric quotient module is special as we shall explain at the end of this section.

In a forthcoming publication\(^\text{11}\) it will be shown that the existence of singular vector operators with vanishing product leads to the appearance of subsingular vectors. In the supersymmetric case we have \( G_\frac{c}{24} | \frac{c}{24} \rangle \equiv 0 \) and therefore we expect the possibility of having subsingular vectors in \( V_{\frac{c}{24}} \). Via the canonical map defined for quotient spaces, singular vectors from \( V_{\frac{c}{24}} \) are either also singular in \( V_{\frac{c}{24}}^G \) or they lie in the kernel of the quotient and hence vanish in the quotient space. The latter happens if and only if the singular vector in question is a descendant vector of the singular vector \( \Psi_0^- \) in \( V_{\frac{c}{24}} \). Thus, singular vectors of \( V_{\frac{c}{24}}^G \) can either be inherited from \( V_{\frac{c}{24}} \) via the canonical quotient map or they appear for the first time in the quotient \( V_{\frac{c}{24}}^G \) as vectors that were not singular in \( V_{\frac{c}{24}} \). In the latter case these singular vectors of \( V_{\frac{c}{24}}^G \) are called subsingular vectors in \( V_{\frac{c}{24}} \). Whilst for the Virasoro algebra\(^\text{12}\) and for the \( N = 1 \) Neveu-Schwarz algebra\(^\text{2}\) there are no subsingular vectors, they have been found for all \( N = 2 \) superconformal algebras by Gato-Rivera et al.\(^\text{14,15,10}\). The fact that the \( N = 1 \) Ramond algebra also contains subsingular vectors - at least in the supersymmetric case - shall be demonstrated here. This may at first seem surprising and confirms the obvious problems encountered so far with \( N = 1 \) Ramond theories. However, we shall argue that subsingular vectors are likely to be restricted to the (extended) supersymmetric case which is though a very particular and highly interesting case.

Singular dimensions for \( G \)-closed Verma modules are less than or equal to 1 as shown in section 4. Hence, there cannot be any degenerate singular vectors in \( G \)-closed Verma modules and furthermore, two singular vectors in a \( G \)-closed Verma module at the same level and with the same parity must be proportional. A singular vector of a \( G \)-closed Verma module at level \( l \)
can therefore be identified by one coefficient only, which is the coefficient with respect to $L_{-1}$ or $L_{-1}^{-1}G_{-1}$ depending on the parity.

**Theorem 7.A** Two singular vectors in the G-closed Verma module $V_G^{\pm}$ at the same level and with the same parity are always proportional. If a vector satisfies the highest weight conditions at level $l$, with parity $+$ or parity $-$, and its coefficient for the term $L_{-1}^{-1}$ or $L_{-1}^{-1}G_{-1}$ vanishes, respectively, when written in the normal form, then this vector is trivial.

Besides the singular vector $\psi_0 = G_0 |\frac{c}{24}\rangle$ the Verma module $V_\varPsi^{\pm}$ contains also the singular vectors $\Psi_{\pm}^{\pm}$ at level $\frac{pt}{2}$ whenever $\Delta_{p,q}(t) = \frac{ct}{24}$. The latter equation has the solutions

$$t_{p,q} = \frac{q}{p}, p, q \in \mathbb{N}, p - q \text{ odd}.$$  \hfill (119)

For these values of $t$ the corresponding singular vectors are given by $\Psi_{p,q}^{+}(t_{p,q}) = (0, 1) |\frac{p}{2}\rangle$ and by $\Psi_{p,q}^{-}(t_{p,q}) = (1, 0) |\frac{p}{2}\rangle$. In the quotient module $V_G^{\pm} = \frac{V_\varPsi^{\pm}}{U(R_1)\psi_0}$ which is isomorphic to the G-closed module $V_G^{\pm}$; these singular vectors hence vanish due to theorem 7A. Therefore, the singular vectors $\Psi_{p,q}^{+}(t_{p,q})$ and $\Psi_{p,q}^{-}(t_{p,q})$ are both descendants of $\Psi_0^{-}$. Hence, these vectors can be written as $\Psi_{p,q}^{+}(t_{p,q}) = \tilde{\theta}_{p,q}^{-}G_0 |\frac{c}{24}\rangle$ and $\Psi_{p,q}^{-}(t_{p,q}) = \tilde{\theta}_{p,q}^{+}G_0 |\frac{c}{24}\rangle$ with singular vector operators $\tilde{\theta}_{p,q}^{\pm}$ where $\theta_{p,q}$ does not contain any operators $G_0$. We can now construct the vectors $\Upsilon_{p,q}^{\pm} = \tilde{\theta}_{p,q}^{\pm} |\frac{c}{24}\rangle$. Clearly, the vectors $\Upsilon_{p,q}^{\pm}(\frac{c}{24})$ are singular in the quotient module $\frac{V_\varPsi^{\pm}}{U(R_1)G_0 |\frac{c}{24}\rangle}$. In principle $\Upsilon_{p,q}^{\pm}(\frac{c}{24})$ could also be singular in $V_{\varPsi}^{\pm}$, a simple consideration, however, shows that this is not the case. **Theorem 7.A** shows that $\Upsilon_{p,q}^{\pm}(\frac{c}{24})$ contains the non-trivial term $L_{-1}^{\pm} |\frac{c}{24}\rangle$ or $L_{-1}^{-1}G_{-1} |\frac{c}{24}\rangle$ depending on its parity. Terms like $L_{-1}^{\pm}G_{-1} |\frac{c}{24}\rangle$ and $L_{-1}^{-1}G_{-1}G_{-1} |\frac{c}{24}\rangle$ are not contained in $\Upsilon_{p,q}^{\pm}(\frac{c}{24})$ by definition of $\tilde{\theta}_{p,q}^{\pm}$. Therefore, the action of $G_1$ on $L_{-1}^{\pm} |\frac{c}{24}\rangle$ creates a non-trivial term $L_{-1}^{\pm}G_0 |\frac{c}{24}\rangle$ which obviously cannot be created by any other non-trivial term of $\Upsilon_{p,q}^{\pm}(\frac{c}{24})$. Hence, $G_1\Upsilon_{p,q}^{\pm}(\frac{c}{24})$ is non-trivial in $V_\varPsi^{\pm}$. Similar considerations are valid for $G_1\Upsilon_{p,q}^{\pm}(\frac{c}{24})$. Therefore, the vectors $\Upsilon_{p,q}^{\pm}(t_{p,q})$ are both subsingular vectors in $V_{\varPsi}^{\pm}$ for $t_{p,q} = \frac{c}{24}$. We summarise our results in the following theorem.

**Theorem 7.B** For $t_{p,q} = \frac{c}{24}$ the singular vectors $\Psi_{p,q}^{\pm}(t)$ are contained in $V_{\varPsi}^{\pm}$ and are descendants of the singular vector $\Psi_0^{-} = G_0 |\frac{c}{24}\rangle$. They can therefore be written as $\Psi_{p,q}^{\pm}(t_{p,q}) = \tilde{\theta}_{p,q}^{\pm}G_0 |\frac{c}{24}\rangle$ with $\theta_{p,q}^{\pm} \in U(R_1)$. The vectors

$$\Upsilon_{p,q}^{\pm} = \tilde{\theta}_{p,q}^{\pm} |\frac{c}{24}\rangle$$  \hfill (120)

are then subsingular vectors in $V_{\varPsi}^{\pm}$ for $t_{p,q}$ at level $\frac{pt}{2}$ becoming singular in the quotient module $V_G^{\pm} = \frac{V_{\varPsi}^{\pm}}{U(R_1)\psi_0}$.

**Theorem 7.B** does not come as a surprise considering the fact that $U(R_1)\psi_0^{-1}$ and $V_G^{\pm} = \frac{V_{\varPsi}^{\pm}}{U(R_1)\psi_0}$ are in fact both isomorphic to the G-closed Verma module $V_G^{\pm}$ and therefore show the same structure. Singular vectors appear in the quotient therefore at exactly the grades where singular vectors disappear in the kernel of the quotient. Explicit examples of the subsingular vectors $\Upsilon_{p,q}^{\pm}$ are given in appendix B. The symmetry of $\Psi_{p,q}^{\pm}(t)$ and $\Psi_{q,p}^{\pm}(t)$ leads to the
proportionality of the operators $\tilde{\theta}_{p,q}^\pm, q \neq 0$ and $\tilde{\theta}_{q,p}^\pm, q \neq 0$. Taking into account that $c(t) = c(\frac{1}{t})$ consequently leads to $\Upsilon_{p,q}^\pm$ and $\Upsilon_{q,p}^\pm$ being proportional.

We have shown that the singular vectors $\Psi_+^\pm$ disappear in the kernel of the quotient module for the $G$-closed case. However, based on their singular vector operators we can construct subsingular vectors in $\mathcal{V}_{p,q}^G$ and thus the singular vectors in the $G$-closed Verma module $\mathcal{V}_{p,q}^G$. For the case that $\Psi_+^\pm$ leads to a two-dimensional singular space one would most likely obtain a two-dimensional singular space in the corresponding $G$-closed Verma module which would contradict theorem 7.A. Therefore, two-dimensional singular spaces defined by the intersection of vectors $\Psi_+^\pm$ are not allowed for $t = t_{p,q}$. This can easily be seen by requiring $t = \frac{p_1}{p_2}$ and $t = \frac{q_1}{q_2} = \frac{p_1}{p_2}$. The first condition takes care of the $G$-closed case and the second condition arises from the intersection of two such vectors. This leads to $(p_1, q_1) = (p_2, q_2)$ and therefore excludes the existence of degenerate cases for the $G$-closed Verma modules. Even more interestingly due to $p - q$ odd exactly one of the coprime factors $(u, v)$ of $t = \frac{u}{v}$ has to be even. Furthermore, since $a = 0$ the straight line $L$ is point symmetric in the origin and the negative solutions $(\tilde{p}_n, \tilde{q}_n)$ are exactly the same as the positive solutions $(p_n, q_n)$. They are simply given by $(p_n, q_n) = (v + n w, u + n u)$ with $n \in 2\mathbb{N}_0$. Since $(0, 0)$ is on the straight line $L$ these are the only possible reducible Verma modules having $a = 0$. Thus, reducible $G$-closed Verma modules always belong to type $\Pi_{l,-}$. However, their structure is quite different from the structure of the Verma modules of type $\Pi_{l,-}$ with $a \neq 0$. Therefore, we shall use the notation $\Pi_{l,-}^{\mp}$. The reason why their structure is different to $\Pi_{l,-}$ is mainly due to the fact that the positive and negative solutions are not only at the same level but they are the same and therefore do not lead to degenerate singular vectors as we would expect it from the cases $\Pi_{l,-}$. The degeneracy of null-vectors is for $\Pi_{l,-}^{\mp}$ obtained through the subsingular vectors of theorem 7.B.

In Ref. 11, it will be shown that for superconformal algebras the existence of subsingular vectors is often a consequence of additional discrete vanishing conditions on an embedded singular vector. To understand this, let us assume that we have given a singular vector $\Psi_l(t)$ at level $l$ along the curve $\Delta = \Delta(t)$. This singular vector generates an embedded Verma module built on $\Psi_l$. If for a discrete point $t_0$ the singular vector satisfies an additional vanishing condition $\theta_{l_0} \Psi_l \equiv 0$ at level $l_0$, then the embedded Verma module has to be smaller at this level compared to the generic cases along $\Delta = \Delta(t)$. Since null-vectors correspond to the kernel of the inner product matrix and the dimension of this kernel is upper semi-continuous, there must exist for this discrete case an additional null-vector at level $l + l_0$ outside the embedded module built on $\Psi_l$ in order to compensate for this embedded module shrinking at level $l + l_0$. This additional null-vector is usually a subsingular vector. As shown for the $N = 2$ superconformal algebras in some cases they may, however, also be ordinary singular vectors. This argument is explained in more detail in Ref. 11, where it is also shown that only singular vector operators need to be considered in order to find possible vanishing conditions on singular vectors. We therefore conclude this section by analysing the products of singular vector operators and the modules and grades for which the product of two singular vector operators may vanish in the Ramond case.

We consider products of the form $\Psi_{l_1,l_2}^{\mu_p,q_p,q_q} = \tilde{\theta}_{p_1,q_1}^\pm \tilde{\theta}_{p_2,q_2}^\pm |\Delta_{p_1,q_1} \Delta_{p_2,q_2}| + \frac{2q_2}{2} = \Delta_{p_1,q_1}$. There are four cases to be considered depending on the parities of the two singular vector operators indicated by $\mu$ as $++, --, -, -+$, or $-+$ for the parities of $\tilde{\theta}_{p_1,q_1}^\pm$ and $\tilde{\theta}_{p_2,q_2}^\pm$. We write $\theta_{p_1,q_1}^{\pm} = (a_i, b_i)_i^{\pm}$ in the notation of definition 4.B and use the product expressions of 4.C. Requiring that one of these product expressions vanishes identically, which means that both coefficients with respect to the ordering kernel vanish for the product, leads to the conditions

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$^v$See appendix B of Ref. 8.
$a_1 = 0$ and $a_2 + 2b_2 = 0$ or $a_2 = 0$ and $a_1 + 2b_1 = 0$ for the case of $\mu = ++$. One easily obtains similar relations for the other cases of $\mu$. Using the actual coefficients for the singular vector operators given in 5.B we again encounter four subcases for each value of $\mu$ depending on the type $\epsilon_i$ of the singular vector operators. The resulting equations need to be solved for $t$. In the case of $\mu = ++$ we easily find that the first condition given above ($a_1 = 0$) leads to $t = \frac{a_2}{p_1} > 0$, whilst the second ($a_2 + 2b_2 = 0$) results in $t = -\frac{a_2}{p_2} < 0$ independent of the types $\epsilon_i$. The second set of conditions for $\mu = ++$ leads to a similar contradiction on $t$. Hence, there are no vanishing products of singular vector operators in the case $\mu = ++$. We determine the solutions to all other cases of $\mu$ in exactly the same way. We easily obtain the following result.

**Theorem 7.C** The vector $\Psi_{p_{1,2},q_{1,2}}^\mu = \theta_{p_{1,2},q_{1,2}}^\pm |\Delta_{p_{1,2},q_{1,2}}\rangle$ for $\Delta_{p_{1,2},q_{1,2}} + \frac{pq}{2} = \Delta_{p_1,q_1},\ \mu \in \{++,-+,-+,+-\}$ is always non-trivial.

Finally, we can also allow one of the singular vector operators $\theta_i$ to be the operator $G_0$ which is a singular vector operator for $\Delta_i = \frac{c}{24}$. Inserting $\theta_i = (1,0)_0^\mu$ in the conditions above leads to the following cases for vanishing products.

**Theorem 7.D** The vector $\Psi_{i+1,l_2} = \theta_i^{\pm} \theta_l^\pm |\Delta_2\rangle$ for $\Delta_2 + l_2 = \Delta_1$, $\mu \in \{++,-+,-+,+-\}$, one of $\theta_i^{\pm}$ being of the form $\theta_{p,q}^{\pm}$ and the other one equals $G_0$, is trivial if and only if

$$\theta_1^{\pm} = \theta_{p,q}^{\pm}, \theta_2 = G_0 \ \Delta_2 = \frac{c}{24}, \text{ and } t = \frac{q}{p},$$

(121) or

$$\theta_1^- = G_0, \theta_2^\pm = \theta_{p,q}^\pm, \ \Delta_2 + \frac{pq}{2} = \frac{c}{24}, \text{ and } t = -\frac{q}{p},$$

(122)

Obviously the cases of Eq. (121) correspond exactly to the supersymmetric cases $\Delta_2 = \frac{c}{24}$. Hence, in this case the operator $\theta_{p,q}^\pm$ vanishes on $\Psi_0^-$. As expected, these are exactly the cases for which we have found the subsingular vectors given above in this section. Our earlier explanation that an additional vanishing condition on $\Psi_0^-$ is responsible for these subsingular vectors does, however, not apply in this case since the vanishing of $\theta_{p,q}^\pm\Psi_0^-$ is not an additional vanishing condition on $\Psi_0^-$ but arises simply from $G_0\Psi_0^- \equiv 0$ and the fact that $\theta_{p,q}^\pm$ can be written as $\theta_{p,q}^\pm G_0$ as shown earlier. Nevertheless, exactly these cases contain subsingular vectors. The other cases in theorem 7.D are cases when vectors $\Psi_{p,q}^{\pm}$ become $G$-closed and as we shall show in the following section that these are all such cases. In this context, one automatically raises the question whether there are subsingular vectors appearing for the case that $\Psi_{p,q}^{\pm}$ becomes $G$-closed. This interesting question will be analysed in the following section. Finally, let us remark that theorem 7.C states that besides the $G$-closed cases and the supersymmetric cases there are no trivial products of singular vector operators given by 5.B. However, if these singular vector operators form a 2-dimensional space it could well be that a linear combination of two such operators may lead to a trivial product by acting on another singular vector. By noting that the operators $(a,b)_f^\pm$ are elements of a (larger) algebra with operations 4.C in which all operators are invertible except for the $G$-closed and supersymmetric cases, we find that such vanishing products are only possible for these cases. We therefore do not have any further subsingular vectors arising in the way described above. Furthermore we can state that there are no subsingular vectors in Verma modules $\mathcal{V}_\Delta$ other than the ones found in $\mathcal{V}_\infty$. A rigorous proof for this claim will be presented in Ref. 4 using a detailed analysis of the multiplicities of the roots of the determinant together with the results of theorem 7.C and 7.D.

For the embedding structure of $III_\epsilon^\pm$ we find the singular vectors corresponding to $(p_n,q_n)$, $n \in \mathbb{N}_0$ in a single line diagram as descendants of $\Psi_0^-$. At first, it seems as if the descendant
singular vectors of $\Psi_{p_0,q_0}^\pm$ would form a two-dimensional singular space as the shifted line $\mathcal{L}$ has the intercept $a' \neq 0$ and is of type $\text{III}_c$. With negative and positive solutions being at the same levels. However, as explained above, one dimension of this 2-dimensional space is being projected out in the product on $\Psi_{p_0,q_0}^+$ and the resulting singular vector equals $\Psi_{p_2,q_2}^+$. This single line embedding pattern for the singular vectors is, however, complemented by the subsingular vectors and their descendants. Again, $G_0$ interpolates between identical embedding patterns for the different parity sectors.

To conclude this subsection we note that if we had started off with (unextended) Verma modules $\mathcal{V}_X^\lambda$, then we would have not found these subsingular vectors of the complete Verma module in the supersymmetric case $\lambda = 0$ as $\mathcal{V}_0^\lambda$ is simply the above quotient module. On the other hand, in our search for irreducible representations we might as well start off using this quotient module rather than the full (extended) Verma module for $\lambda = 0$. Hence we have shown that for the irreducible highest weight representations also in the supersymmetric case $\Delta = \frac{c}{27}$ the highest weight vector can be chosen to be an eigenvector of $G_0$. We can therefore state that for all irreducible highest weight representations of the Ramond algebra $R_1$ we may choose highest weight vectors that are eigenvectors of $G_0$ in both the extended and the unextended case. In the former case we find the condition $(-1)^F |\lambda \rangle = | -\lambda \rangle$. However, $G_0$ may not be completely diagonalisable on the highest weight module for $\lambda - \frac{c}{27} \in \mathbb{N}$.

## 8 G-closed singular vectors.

G-closed singular vectors $\Psi_i^\pm$ in $\mathcal{V}_\Delta$ can only appear at levels $l$ that satisfy $\Delta + l = \frac{c}{27}$. The embedded Verma module generated by $\Psi_i^\pm$ is homomorphic to $\mathcal{V}_\Delta^G$. This homomorphism is an isomorphism if there are no additional vanishing conditions on $\Psi_i^\pm$ except for $G_0 \Psi_i^\pm \equiv 0$ and surely for all descendants of $G_0$. In theorem 7.D we have analysed vanishing conditions coming from the singular vector operators $\theta_{p,q}^\pm$. We have found $\theta_{p,q}^\pm G_0 | \xi_l \rangle \equiv 0$ for $t = \frac{2}{p}$. In these cases however $\theta_{p,q}^\pm$ can be written as $\tilde{\theta}_{p,q} G_0$ as shown in the previous section. Therefore, these vanishing conditions simply arise from the vanishing condition for the operator $G_0$. The fact that we only need to look at singular vector operators and their descendants whilst dealing with additional vanishing conditions is proven in Ref. 11. Hence, the embedded modules generated by $\Psi_i^\pm$ are isomorphic\(^w\) to $\mathcal{V}_\Delta^G$.

We shall now analyse for which cases singular vectors become G-closed. Starting with the singular vectors $\Psi_{p,q}^\pm(t)$ we find that they can only become G-closed for $\Delta_{p,q} + \frac{pq}{2} = \frac{c}{27}$ which leads to:

$$\tilde{t}_{p,q} = -\frac{q}{p}. \tag{123}$$

Inserting $\tilde{t}_{p,q}$ into the expressions of 5.B for $\Psi_{p,q}^\pm(t)$ we obtain $\Psi_{p,q}^+ (\tilde{t}_{p,q}) = (2q, -q)^+_{\frac{pq}{2}}$ and $\Psi_{p,q}^- (\tilde{t}_{p,q}) = (\frac{p}{2}, 2q)^-_{\frac{pq}{2}}$. We use equations 4.D to act with $G_0$ on the vectors $\Psi_{p,q}^\pm(\tilde{t}_{p,q})$, which leads for both parities to the zero vector $(0,0)^{\pm}_{\frac{pq}{2}}$. Hence for all possible cases $t = \tilde{t}_{p,q}$ for which $\Psi_{p,q}^\pm(t)$ may be G-closed it turns out that it really is. Furthermore, the ordering kernels for G-closed singular vectors, given in table Tab. b, are just one-dimensional and therefore G-closed singular vectors are defined by one coefficient only.

\(^w\)Strictly speaking, we would first need to prove that we have really found all null-vectors which will become clear in our work on the embedding diagrams\(^4\).
The singular vectors $\Psi_{p,q}^\pm(t)$ always become $G$-closed for $t = \tilde{t}_{p,q} = -\frac{q}{p}$ and these are the only cases such that $\Delta_{p,q} + \frac{pq}{2} = \frac{c}{24}$. $G$-closed singular vectors $\Psi_{l}^\pm$ are uniquely defined by fixing the coefficient of $L_{-1}$ or $L_{-1}G_{-0}$ depending on the parity.

Because $t < 0$ and $t = -\frac{q}{p}$ these cases belong to the type $\text{III}_{l+}$. One easily finds that the straight line $\mathcal{L}$ has only one relevant solution $(p, q)$. The shifted line for descendant solutions goes through the origin and has $t < 0$ and therefore does not have any descendant solutions at all. Hence, the embedding diagram contains only the singular vectors $\Psi_{p,q}^\pm$ unconnected as they are both $G$-closed.

The amazing finding is that whenever an embedded singular vector $\Psi_{l}^\pm$ in $\mathcal{V}_\Delta$ happens to have the conformal weight $\Delta + l = \frac{c}{24}$ then it becomes $G$-closed. Hence the module generated by this singular vector is not isomorphic to the full Verma module $\mathcal{V}_{\frac{c}{24}}$ but only to the smaller $G$-closed module $\mathcal{V}_{G\frac{c}{24}}$.

As in the previous section it is evident that among the cases of $\Psi_{p_1,q_1}$ that become $G$-closed there are none of the degenerate cases. For the latter we require $t = \pm \frac{q_1}{p_2} = \pm \frac{q_2}{p_1}$, whilst the former need $t = -\frac{q_2}{p_2}$ and ultimately $(p_1, q_1) = (p_2, q_2)$. Hence, the degenerate cases do not correspond to $G$-closed singular vectors, i.e. two-dimensional singular spaces are never $G$-closed, not even partly.

9 Conclusions and prospects

By introducing an adapted ordering on the weight spaces of Ramond Verma modules we derived the ordering kernels of the $N = 1$ Ramond algebra. These ordering kernels contain in the most general cases two elements and consequently Ramond singular vectors are uniquely defined by just two particular coefficients. In addition, Ramond singular vector spaces can be degenerate (2-dimensional). We derived expressions for these two coefficients for all primitive Ramond singular vectors. It turned out that these two coefficients are closed under multiplication. The corresponding multiplication formulae therefore yield the whole Ramond embedding diagrams.

We argued that the structure of the highest weight representations of the (extended) Ramond algebra can easily be transformed into the corresponding representations of the unextended Ramond algebra. The only case where this transformation breaks down is the supersymmetric case $\Delta = \frac{c}{24}$ for which the unextended module is already the quotient module of the extended module divided by its level 0 singular vector. In some of these cases we find that the (extended) Verma module has subsingular vectors but not the unextended Verma module. We argued further that one should not expect any other subsingular vectors. This was related to the fact that the product of singular vector operators has proven to be always non-trivial with the only exception of the supersymmetric case $\Delta = \frac{c}{24}$. A rigorous proof of this claim will be given in Ref. 4 through a detailed analysis of the multiplicities of the roots of the determinant and their intersections.

The embedding diagram patterns proven in this paper reveal that the derivation of Ramond embedding diagrams is significantly more complicated than in the Virasoro case and the conjectures given in the literature\textsuperscript{19} for the Ramond unitary models can only be proven by using the multiplication rules for singular vector operators derived via the ordering kernel coefficients. We showed that degenerate singular vector spaces do appear for the Ramond algebra which arise from ordering kernels with two elements. The structure of the embedding diagrams is very similar to the Virasoro embedding diagrams, nevertheless the rules for embedding patterns to collapse to simpler patterns are very different than in the Virasoro case. As a consequence, we
have embedding diagrams where the two lowest level singular vectors are at the same level but
the embedding diagram still does not simplify to the straight line embedding diagram - as it
would do in the Virasoro case - but simply stays as the typical crossed embedding pattern of
conformal rational models.

We have once more demonstrated that the adapted ordering method is a very simple
mechanism that uniquely defines all singular vectors and reveals much information about the
embedding structure of the Verma modules. One easily manages even complications through
subsingular vectors and degenerate singular vectors and finally obtains the whole structure
of the embedding patterns. In principle, it should easily generalise to any other kind of Lie
(super)-algebra.

It was shown that the \( N = 1 \) Ramond algebra contains many of the interesting features that
had only been discovered for the \( N = 2 \) superconformal algebras so far. Surprisingly enough,
the results obtained for the \( N = 1 \) Ramond algebra resemble much more the corresponding
results for the twisted \( N = 2 \) algebra than for the Ramond \( N = 2 \) algebra. These facts are one
more reason to claim that Ramond field theories are still much less understood than Neveu-
Schwarz field theories. Nevertheless, the structure finally proven has still much in common with
the Virasoro structure and the \( N = 1 \) Neveu-Schwarz structure.

A Appendix: Level 1, level 2, and level 3 singular vectors.

Explicit expressions for the singular vectors \( \Psi_{1,q}^{\pm} \), which are the analogues of \( \Psi_{1,q}^{\pm} \) in \( V_{\lambda} \) for \( \lambda^2 = \Delta_{1,q} - \frac{c}{24} \), were given by Watts\(^{22} \) using the concept of fusion in conformal field theory. These
vectors can easily be transformed to the basis used in this paper via the basis transformation
discussed earlier. In this section of the appendix we shall give all singular vectors in
conformal rational models.

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vectors can easily be transformed to the basis used in this paper via the basis transformation
discussed earlier. In this section of the appendix we shall give all singular vectors in \( V_{\Delta_{p,q}} \) until
level 3. With the exception of \( \Psi_{3,2}^{\pm}(t) \), these vectors are of the type \( \Psi_{1,q}^{\pm} \) (and its symmetric
partners) given by Watts in the unextended basis. As mentioned earlier, at level 3 we observe
for the first time degenerate singular vectors arising from the intersection of \( \Psi_{1,6}^{\pm} \) and \( \Psi_{2,3}^{\pm} \) and
also from the intersection of \( \Psi_{1,6}^{\pm} \) and \( \Psi_{3,2}^{\pm} \). Let us note again that intersections of singular vectors of different type \( \epsilon \) (see theorem 5.C), such as \( \Psi_{2,1}^{\pm} \) and \( \Psi_{1,2}^{\pm} \) only lead to one-dimensional
singular spaces. By explicit calculations\(^{21} \) one finds the following singular vectors for parity +. 

Level 1:

\[
\Psi_{1,2}^{\pm}(t) = \left\{ (2-t)L_{-1} - 2G_{-1}G_0 \right\} \left[ -\frac{3}{16} (1 - \frac{2}{t}) \right], \quad (124)
\]

\[
\Psi_{2,1}^{\pm}(t) = \left\{ (1 - 2t)L_{-1} + 2tG_{-1}G_0 \right\} \left[ -\frac{3}{16} (1 - 2t) \right]. \quad (125)
\]

We recall the symmetry \( \Psi_{p,q}^{\pm}(t) = -\frac{1}{t} \Psi_{q,p}^{\pm}(\frac{1}{t}) \) in our normalisation, which we can observe
in Eqs. (124) and (125). In the following we shall therefore give only the results for \( q \) even.

Level 2:

\[
\Psi_{1,4}^{\pm}(t) = \left\{ (4-t)L_{-1} - 4L_{-1}G_{-1}G_0 + \left( \frac{3}{2} - \frac{6}{t} \right)L_{-2} \right\}
+ (1 + \frac{6}{t})G_{-2}G_0 \right\} \left[ \frac{11}{16} + \frac{15}{8t} \right]. \quad (126)
\]

Level 3:

\[
\Psi_{1,6}^{\pm}(t) = \left\{ (6-t)L_{-1} - 6L_{-1}G_{-1}G_0 + \left( 3 + \frac{24}{t} \right)G_{-2}L_{-1}G_0 + \left( \frac{13}{2} - \frac{39}{t} \right)L_{-2}L_{-1}
\]

\[+(3 - \frac{28}{t} + \frac{60}{t^2})L_{-3} + (3 + \frac{18}{t} - \frac{60}{t^2})G_{-3}G_0 + \left(\frac{1}{4} - \frac{3}{2t}\right)G_{-2}G_{-1}\]
\[+ \frac{15}{t}L_{-2}G_{-1}G_0\left\{\frac{1}{16}\left(\frac{70}{t} - 19\right)\right\},\]
\[\Psi_{3,2}^+(t) = \left\{(2 - 3t)L_{-1}^3 - 2L_{-1}G_{-1}G_0 + (1 + 8t)G_{-2}L_{-1}G_0 + (6t^2 - 4t + \frac{3}{2} - \frac{1}{t})L_{-2}L_{-1}\right.\]
\[\left.+ (-3t^3 + 11t^2 - 9t + 2)L_{-3} + (-2t^2 + 6t - 1)G_{-3}G_0 + (-4t + \frac{1}{t})L_{-2}G_{-1}G_0\right.\]
\[\left.+ (-6t^2 + 4t + \frac{3}{4} - \frac{1}{2t})G_{-2}G_{-1}\right\}\left|t - \frac{19}{16} + \frac{3}{8t}\right\rangle.\]

One easily verifies that these examples can be written using the notation introduced in definition 4.B, where we only need to consider the coefficients with respect to the ordering kernel.

\[\Psi_{1,2}^+(t) = (2 - t, -2)^+ \left|\frac{3}{16} (1 - \frac{2}{t})\right\rangle,\]
\[\Psi_{1,4}^+(t) = (4 - t, -4)^+ \left|\frac{11}{16} + \frac{15}{8t}\right\rangle,\]
\[\Psi_{1,6}^+(t) = (6 - t, -6)^+ \left|\frac{1}{16}\left(\frac{70}{t} - 19\right)\right\rangle,\]
\[\Psi_{3,2}^+(t) = (2 - 3t, -2)^+ \left|t - \frac{19}{16} + \frac{3}{8t}\right\rangle.\]

At the intersection point \(t = \pm 2\) the vectors \(\Psi_{1,6}^+(\pm 2)\) and \(\Psi_{3,2}^+(\pm 2)\) obviously define a two-dimensional singular space in \(V_{1\text{ or }3}\) for \(t = 2\) or \(t = -2\) respectively. The same is of course true for \(\Psi_{6,1}^+(\pm \frac{1}{2})\) and \(\Psi_{2,3}^+(\pm \frac{1}{2})\) for \(t = \pm \frac{1}{2}\). However, \(\Psi_{1,6}^+(\pm 1)\) and \(\Psi_{6,1}^+(\pm 1)\) are proportional as well as \(\Psi_{1,6}^+(\pm 3)\) and \(\Psi_{2,3}^+(\pm 3)\) and other combinations of this type.

Besides the vectors \(\Psi_{p,q}\) we should - of course - have at least also found the descendant singular vectors \(\theta_{p,q} \Psi_{p',q'}\) for \(\Delta_{p,q} = \Delta_{p',q'} + \frac{pq}{2}\) and descendants thereof. The latter equation simplifies to \(t = \pm \frac{q_1 - q_2}{p_2 - p_1}\). If the two vectors are of different type, then obviously the values for \(t\) in question have \(t = \frac{u}{v}\) with \(u\) and \(v\) both odd. Furthermore, at level 1, 2, and 3 all possible cases simplify to either \(\Pi_{0\pm}^{(u)}\) or \(\Pi_{0\pm}^{(v)}\) and hence the corresponding embedding patterns contain descendant singular vectors already among the pattern for the (generically) primitive singular vectors. In the case that the two singular vectors are of the same type, we obtain both numerator and denominator of \(t\) even and hence one of \(u\) or \(v\) may or may not be even. For the levels 1, 2, 3 these cases are \((p_1,q_1) = (1,4), (p_2,q_2) = (1,2)\) with the corresponding values for \(t = 1\) or \((p_1,q_1) = (1,2), (p_2,q_2) = (1,4)\) with \(t = -1\). Hence both cases again contain no additional descendant singular vectors at level 3. Therefore there are no additional descendant singular vectors until level 3. The first examples of descendant singular vectors not contained in the pattern of (generically) primitive singular vectors appear at level 4, for example for \((p_1,q_1) = (1,6), (p_2,q_2) = (1,2)\) with \(t = 2\) which is of type \(\Pi_{0\pm}\) and does not simplify to the straight line embedding pattern.

We conclude this section of the appendix with all singular vectors of negative parity at level 1, 2, and 3. The above remarks about degenerate singular vectors and descendant singular vectors are equally valid for these cases. Also for the negative parity vectors the condition \(\Psi_{p,-}(t) = -\frac{1}{t} \Psi_{q,p}(\frac{1}{t})\) holds and we hence give the singular vectors only for \(q\) even.
Level 1:

\[
\Psi_{1,2}(t) = \left\{ -4t L_{-1} G_0 + (2-t) G_{-1} \right\} \left| -\frac{3}{16} \left( 1 - \frac{2}{t} \right) \right>, \quad (133)
\]

Level 2:

\[
\Psi_{1,4}(t) = \left\{ -2t L_{-1}^2 G_0 + (4-t) L_{-1} G_{-1} + 3L_{-2} G_0 \right\}
+ \left( \frac{t}{4} + \frac{1}{2} - \frac{6}{t} \right) G_{-2} \left| -\frac{11}{16} + \frac{15}{8t} \right>. \quad (134)
\]

Level 3:

\[
\Psi_{1,6}(t) = \left\{ -\frac{4}{3} t L_{-1}^3 G_0 + (6-t) L_{-1}^2 G_{-1} + (1 + \frac{t}{2} - \frac{24}{t}) G_{-2} L_{-1} \right\}
+ \left( \frac{26}{3} - \frac{2}{t} \right) L_{-2} L_{-1} G_0 + (4 - \frac{40}{3t}) L_{-3} G_0 + (-\frac{28}{t} + \frac{t}{2} + \frac{60}{t^2}) G_{-3} \right.
+ \left( \frac{1}{3} G_{-2} G_{-1} G_0 + \frac{5}{2} - \frac{15}{t} \right) L_{-2} G_{-1} \left| \frac{1}{16} \left( \frac{70}{t} - 19 \right) \right>, \quad (135)
\]

\[
\Psi_{3,2}(t) = \left\{ -4t L_{-1}^3 G_0 + (2 - 3t) L_{-1}^2 G_{-1} + (12t^2 - \frac{13}{2} t - 1) G_{-2} L_{-1} \right\}
+ (2 + 8t^2) L_{-2} L_{-1} G_0 + (12t^2 - 4t - 4t^3) L_{-3} G_0 + (1 - \frac{15}{2} t + 11t^2 - 3t^3) G_{-3} \right.
+ \left( \frac{3}{2} - 6t^2 - \frac{1}{t} \right) L_{-2} G_{-1} \left| t - \frac{19}{16} + \frac{3}{8t} \right>. \quad (136)
\]

Once again, we can use the notation of definition 4.B to write these singular vectors with ordering kernel coefficients only.

\[
\Psi_{1,2}(t) = \left( -4t, 2-t \right)_1 \left| -\frac{3}{16} \left( 1 - \frac{2}{t} \right) \right>, \quad (137)
\]

\[
\Psi_{1,4}(t) = \left( -2t, 4-t \right)_2 \left| -\frac{11}{16} + \frac{15}{8t} \right>, \quad (138)
\]

\[
\Psi_{1,6}(t) = \left( -\frac{4t}{3}, 6-t \right)_3 \left| \frac{1}{16} \left( \frac{70}{t} - 19 \right) \right>, \quad (139)
\]

\[
\Psi_{3,2}(t) = \left( -4t, 2-3t \right)_3 \left| t - \frac{19}{16} + \frac{3}{8t} \right>. \quad (140)
\]

For \( t = \pm 2 \) and \( t = \pm \frac{1}{2} \) we find two-dimensional singular spaces with highest weight vectors \( |1\rangle \) or \( \left| -\frac{27}{8} \right> \) for \( +2 \) (and \( -\frac{1}{2} \)) or \( -2 \) (and \( -\frac{1}{2} \)) respectively.

### B Appendix: Examples of subsingular vectors in the supersymmetric case

We have shown in section 7 that for the supersymmetric case \( \Delta = \frac{c}{24} \) the singular vectors \( \Psi_{p,q}^\pm \) for \( \Delta = \Delta_{p,q} \) always lie in the embedded submodule built on \( G_0 \left| \frac{c}{24} \right> \), which is isomorphic to \( V_0^\gamma \). They therefore vanish in the \( G \)-closed quotient module \( \frac{V_0^\gamma}{U(R_1)G_0 \left| \frac{c}{24} \right>^\Gamma} \). The singular vectors, which we find in the quotient module are hence subsingular vectors in \( V_0^\gamma \). Since the quotient
module \( \mathcal{V}_{c}^{\mu} \big/ U(R_{1})G_{0} \big/ R_{2} \big/ \) is also isomorphic to \( \mathcal{V}_{0}^{\mu} \) it has singular vectors at the same levels as the embedded module built on \( G_{0} \big/ G_{1} \big/ G_{2} \big/ \). We shall give here all examples at levels 1, 2, and 3. Let us recall again that \( c(t) = c \left( \frac{1}{2} \right) \) and therefore the \( t \leftrightarrow \frac{1}{2} \) symmetric solutions of \( \Delta_{p,q}(t) = \frac{1}{2t} \) lead to the same value of \( \Delta \) and hence to the same vector.

Level 1 has subsingular vectors for \( t = 2 \):

\[
\begin{align*}
\mathcal{Y}_{1,2}^{+}(2) &= L_{-1}|0\rangle, \\
\mathcal{Y}_{1,2}^{-}(2) &= G_{-1}|0\rangle.
\end{align*}
\]

Level 2 has subsingular vectors for \( t = 4 \):

\[
\begin{align*}
\mathcal{Y}_{1,4}^{+}(4) &= \left\{ 8L_{-1}^{2} - 3L_{-2} \right\} \frac{7}{32}, \\
\mathcal{Y}_{1,4}^{-}(4) &= \left\{ 8L_{-1}G_{-1} - 5G_{-2} \right\} \frac{7}{32}.
\end{align*}
\]

Level 3 has subsingular vectors for \( t = 6 \) and for \( t = \frac{2}{3} \):

\[
\begin{align*}
\mathcal{Y}_{1,6}^{+}(6) &= \left\{ -24L_{-1}^{3} + 26L_{-2}L_{-1} + \frac{16}{3}L_{-3} + G_{-2}G_{-1} \right\} \frac{-11}{24}, \\
\mathcal{Y}_{1,6}^{-}(6) &= \left\{ -12L_{-1}^{2}G_{-1} + 5L_{-2}G_{-1} + \frac{26}{3}G_{-3} + 14G_{-2}L_{-1} \right\} \frac{-11}{24}, \\
\mathcal{Y}_{3,2}^{+}(\frac{2}{3}) &= \left\{ -36L_{-1}^{3} + 75L_{-2}L_{-1} + 20L_{-3} - \frac{69}{2}G_{-2}G_{-1} \right\} \frac{1}{24}, \\
\mathcal{Y}_{3,2}^{-}(\frac{2}{3}) &= \left\{ -12L_{-1}^{2}G_{-1} - 7L_{-2}G_{-1} + \frac{38}{3}G_{-3} + 38G_{-2}L_{-1} \right\} \frac{1}{24}.
\end{align*}
\]

These vectors are easily obtained from their singular vector counterpart of \( \mathcal{V}_{p,q}^{\mu} \). Evaluating for example the singular vector \( \Psi_{1,4}^{+} \) of Eq. (126) for \( t = 4 \) leads to

\[
\begin{align*}
\Psi_{1,4}^{+}(4) &= \left\{ -4L_{-1}G_{-1}G_{0} + \frac{5}{2}G_{-2}G_{0} \right\} \frac{-7}{32},
\end{align*}
\]

which can be rewritten as \( \Psi_{1,4}^{+}(4) = -\frac{1}{2}\left\{ 8L_{-1}G_{-1} - 5G_{-2} \right\} G_{0} \frac{-7}{32} \) and therefore \( \mathcal{Y}_{1,4}^{-} = \left\{ 8L_{-1}G_{-1} - 5G_{-2} \right\} \frac{-7}{32} \).

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References

[1] M. Ademollo, L. Brink, A. d’Adda, R. d’Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino. Supersymmetric strings and colour confinement. *Phys. Lett.*, B62:105, 1976.

[2] A.B. Astashkevich. On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras. *Commun. Math. Phys.*, 186:531, 1997.

[3] J.D. Cohn and D. Friedan. Super characters and chiral asymmetry in superconformal field theory. *Nucl. Phys.*, B296:779, 1988.

[4] M. Dörzzapf. work in progress.

[5] M. Dörzzapf. *Superconformal field theories and their representations*. PhD thesis, University of Cambridge, http://www.damtp.cam.ac.uk/user/md131/research/thesis.html, 205 pages, September 1995.

[6] M. Dörzzapf. Analytic expressions for singular vectors of the $N = 2$ superconformal algebra. *Commun. Math. Phys.*, 180:195, 1996.

[7] M. Dörzzapf. The embedding structure of unitary $N = 2$ minimal models. *Nucl. Phys.*, B529:639, 1998.

[8] M. Dörzzapf and B. Gato-Rivera. Determinant formula for the topological $N = 2$ superconformal algebra. *IMAFF-FM-99/07, NIKHEF-99-004, HUP-98/A055, DAMTP-99-32 preprint, hep-th/9905063*, 1999.

[9] M. Dörzzapf and B. Gato-Rivera. Singular dimensions of the $N = 2$ superconformal algebras. I. *hep-th/9807234, to appear in Commun. Math. Phys.*, 1999.

[10] M. Dörzzapf and B. Gato-Rivera. Singular dimensions of the $N = 2$ superconformal algebras. II. *DAMTP-99-19, IMAFF-FM-99/08, NIKHEF-99-06 preprint*, 1999.

[11] M. Dörzzapf and B. Gato-Rivera. Vanishing operator products and superconformal subsingular vectors. *work in progress*, 1999.

[12] B.L. Feigin and D.B. Fuchs. *Representations of Lie groups and related topics*. A.M. Vershik and A.D. Zhelobenko eds., Gordon & Breach, 1990.

[13] D. Friedan, Z. Qiu, and S. Shenker. Superconformal invariance in two dimensions and the tricritical Ising model. *Phys. Lett.*, B151:37, 1985.

[14] B. Gato-Rivera and J.I. Rosado. Chiral determinant formula and subsingular vectors for the $N = 2$ superconformal algebras. *Nucl. Phys.*, B503:447, 1997.

[15] B. Gato-Rivera and J.I. Rosado. Families of singular and subsingular vectors of the topological $N = 2$ superconformal algebra. *Nucl. Phys.*, B514:477, 1998.

[16] V.G. Kac. Lie superalgebras. *Adv. Math.*, 26:8, 1977.

[17] V.G. Kac. *Group theoretical methods in physics*, volume Lecture notes in physics 94. Springer, 1979.

[18] A. Kent. Singular vectors of the Virasoro algebra. *Phys. Lett.*, B273:56, 1991.
[19] E.B. Kiritsis. Character formulae and the structure of the representations of the $N = 1$, $N = 2$ superconformal algebras. *Int. J. Mod. Phys.*, A3:1871, 1988.

[20] A. Meurman and A. Rocha-Caridi. Highest weight representations of the Neveu-Schwarz and Ramond algebras. *Commun. Math. Phys.*, 107:263, 1986.

[21] Waterloo Maple Software. Programmed with Maple V, release 3.

[22] G.M.T. Watts. Null vectors of the superconformal algebra: The Ramond sector. *Nucl. Phys.*, B407:213, 1993.