An inverse source problem for hyperbolic equations and the Lipschitz-like convergence of the quasi-reversibility method

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Abstract

We propose in this paper a new numerical method to solve an inverse source problem for general hyperbolic equations. This is the problem of reconstructing sources from the lateral Cauchy data of the wave field on the boundary of a domain. In order to achieve the goal, we derive an equation involving a Volterra integral, whose solution directly provides the desired solution of the inverse source problem. Due to the presence of such a Volterra integral, this equation is not in a standard form of partial differential equations. We employ the quasi-reversibility method to find its regularized solution. Using Carleman estimates, we show that the obtained regularized solution converges to the exact solution with the Lipschitz-like convergence rate as the measurement noise tends to 0. This is one of the novelties of this paper since currently, convergence results for the quasi-reversibility method are only valid for purely differential equations. Numerical tests demonstrate a good reconstruction accuracy.

Key words: inverse source problem, quasi-reversibility method, regularized solution, Lipschitz stability, Carleman estimates, Volterra integral

AMS subject classification: 35L10, 35R30

1 Introduction

In this paper, we propose a rigorous numerical method to solve an inverse source problem (ISP) for a general hyperbolic equation. We demonstrate that a modified idea of the Bukhgeim-Klibanov method [14] works for the numerical solution of our ISP. The majority of the known methods to solve ISPs are based on the optimization approach. However, the theory of this approach does not guarantee convergence of regularized solutions to the exact one when the level of the noise in the data tends to zero. Motivated by this limitation, we establish here the Lipschitz-like convergence of regularized solutions of our ISP to the true ones. We verify our theory numerically.

The inverse source problem (ISP) is the problem of determining a source term from external information about solutions of the governing equations. The ISP has uncountable applications. The important fact is that the desired solutions can be used to directly detect the source even when the source is inactive after a certain time. We name here some examples about the applications of ISPs. In the case that the governing equation is hyperbolic, the ISP addresses ultrasonics imaging, photoacoustic tomography, seismic imaging [1] [13] [19] [29] [48] [51]. In the case of the parabolic equation, the ISP plays an important role in various applications [17]; for e.g., in identifying the pollution sources of a river or a lake, [2] [20] [21] [49], and in the case of elliptic equation, the ISP arises from electroencephalography, biomedical imaging [20] [4] [15]. Due to its real world applications, the ISP was studied intensively. We refer the reader to

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The ISP in this paper is the linearization of severely ill-posed and highly nonlinear coefficient inverse problems for hyperbolic equations; see e.g., [44, 45, 46]. In these references, the questions about uniqueness and stability of coefficient inverse problems are addressed via addressing similar questions for ISPs. We briefly discuss the linearization issue in Section 2. Hence, besides the direct application of finding the restoring force, the method of this paper has potential contributions in sonar imaging, geographical exploration, medical imaging, near-field optical microscopy, nano-optics, etc.

In their celebrated paper, Bukhgeim and Klibanov [14] in 1981 have paved the way, for the first time, to prove uniqueness theorems for a large class of inverse problems, including the ISP of this paper. Indeed, in [14] the powerful tool of Carleman estimates was introduced for the first time in the field of inverse problems. Since then, many important related uniqueness results were proved on the basis of the Bukhgeim-Klibanov method. In this regard, we refer to, e.g. [25, 27, 28, 32, 40]. In addition, the book [8] extends the Bukhgeim-Klibanov method to the case of inverse problems for some systems of PDEs. Some extensions of the Bukhgeim-Klibanov method allow one not only to prove uniqueness theorems but also to establish Hölder and Lipschitz stability estimates for inverse problems. It is worth mentioning that recent modifications of the Bukhgeim-Klibanov method are used to find numerical solutions of nonlinear coefficient inverse problems via the so-called “convexification” method, see, e.g. [36, 37]. Surveys of this method can be found in [6, 35]. Many stability results for ISPs were proved this way. We list here several important works. When the source is in the form of the separation of variables the Hölder stability result was obtained in [51]. In [25, 26, 29, 41, 45, 46] some Lipschitz stability estimates for ISPs for hyperbolic PDEs were obtained.

For the numerical solutions of ISPs, the widely used approach is optimization. We draw the reader’s attention to several important publications [12, 22, 23, 24, 46, 48], in some of which good numerical results using optimal control were obtained. In particular, in [29] the ISP, which is similar with the one of the current paper, is considered, a numerical method is proposed and implemented. The numerical method of [29] is based on the optimization approach. The convergence of regularized solutions to the exact one is not proved in [29]. To contribute to the field, we propose in this paper a numerical method, which is not difficult to implement, without using the straight forward optimal control approach. First, we derive an integro-differential equation involving a Volterra integral together with lateral Cauchy data. Next, we apply the quasi-reversibility method to solve that Cauchy problem numerically.

The quasi-reversibility method was first proposed by Lattès and Lions [12] in 1969. Since then it has been studied intensively [5, 9, 10, 11, 16, 17, 31, 33]. The application of Carleman estimates for proofs of convergence of those minimizers was first proposed in [39] for Laplace’s equation. In particular, [38] is the first publication where it was proposed to use Carleman estimates to obtain Lipschitz stability of solutions of hyperbolic equations with lateral Cauchy data. We draw the reader’s attention to the paper [35] that represents a survey of the quasi-reversibility method. Using a Carleman estimate, we prove Lipschitz-like convergence rate of regularized solutions generated by the quasi-reversibility method to the exact solution of that Cauchy problem. The convergence of regularized solutions is known for quasi-reversibility method for partial differential equations without integrals [35]. The current publication is the first one where this convergence is proven for the case of an integro-differential equation with a Volterra integral in it. This can be considered as an important contribution of this paper to the quasi-reversibility method.

The paper is organized as follows. In the next section, we state the inverse source problem. In Section 3, we derive an equation involving a Volterra integral leading to our reconstruction method. In Section 4, we discuss about the quasi-reversibility method. Section 5 is to prove the Lipschitz stability result for the quasi-reversibility method. Then, in Section 6, we present the implementation and numerical examples.
Section [7] is for concluding remarks.

2 The problem statement

The ISP we solve in this paper is stated as follows. Let \( d \geq 2 \) be the spatial dimension. Let \( a \in C^{[(d+1)/2]+3}(\mathbb{R}^d) \), \( B \in C^{[(d+1)/2]+3}(\mathbb{R}^d, \mathbb{R}^d) \) and \( c \in C^{[(d+1)/2]+3}(\mathbb{R}^d) \) be known coefficients where \([s]\) is the integer part of \( s \) for all \( s \in \mathbb{R} \). Assume that all functions \( c(x) - 1, a \) and \( B \) are compactly supported.

Consider the following problem

\[
\begin{aligned}
c(x)u_{tt} - \Delta u &= a(x)u + B(x) \cdot \nabla u + p(x)h(x, t) \quad (x, t) \in \mathbb{R}^d \times (0, \infty), \\
u(x, 0) &= f(x) \\
u_t(x, 0) &= g(x)
\end{aligned}
\]  

(2.1)

where \( f \) and \( g \) are functions in \( H^{[(d+1)/2]+5}(\mathbb{R}^d) \) whose supports are compact. The function \( h : \mathbb{R}^d \times [0, \infty) \) is such that \( h(\cdot, t) \in C^{[(d+1)/2]+1}(\mathbb{R}^d) \) for all \( t > 0 \) and \( h(x, \cdot) \in C^3([0, \infty)) \) for all \( x \in \mathbb{R}^d \). According to [25, 26], the function \( p(x) \) corresponds to a restoring force. The conditions imposed on \( c, a, B, p \) and \( h \) above guarantee that the solution \( u \) of (2.1) exists uniquely and belongs to \( C^4(\mathbb{R}^d) \), see [11] Proposition 4.1] and [34 Theorem 2.2]. These technical conditions, however, are not concerned in numerical studies.

Let \( T > 0 \) be large enough and let \( \Omega \) be a piecewise smooth bounded domain of \( \mathbb{R}^d \). We are interested in the problem below.

**Problem** (Inverse Source Problem (ISP)). Fix \( T > 0 \) and let \( u \) be the solution of (2.1). Assume that \( |h(x, 0)| > 0 \) for all \( x \in \Omega \). Determine the function \( p(x) \), \( x \in \Omega \), from the boundary measurements of the following lateral Cauchy data

\[
F(x, t) = u(x, t) \quad \text{and} \quad G(x, t) = \partial_\nu u(x, t) \quad (x, t) \in \partial \Omega \times [0, T].
\]  

(2.2)

As mentioned in the Introduction section, the uniqueness of this ISP was first proved by Bukhgeim and Klibanov in [13], also see [6, 7, 25, 30, 32, 41, 44]. Furthermore, the uniqueness for the ISP was established when the data \( F(x, t) \) and \( G(x, t) \) are measured only on a part of \( \partial \Omega \) in [7] and in a subdomain of \( \Omega \) in [25].

We would like to roughly show that ISP above serves as an important step in solving coefficient inverse problems, which are well-known to be severely ill-posed and highly nonlinear. Assume that we want to reconstruct the function \( c(x) \geq 1 \) from the measurement of \( u \) and \( \partial_\nu u \) on \( \partial \Omega \times [0, T] \) where \( u \) is the solution to the following problem

\[
\begin{aligned}
cu_{tt}(x, t) &= \Delta u(x, t) \quad (x, t) \in \mathbb{R}^d \times [0, T], \\
u(x, 0) &= f(x) \\
u_t(x, 0) &= g(x)
\end{aligned}
\]  

(2.3)

where \( f(x) \) and \( g(x) \) are known as the initial value and velocity, respectively, of the wave. The function \( f(x) \) is supposed to satisfy the condition \( \Delta f \) is non-zero everywhere in the closure of a domain \( \Omega \). Assume that an initial guess for \( c \) is available and denoted by \( c_0 \). Let \( u_0 \) denote the solution of (2.3) when \( c(x) \) is replaced by the function \( c_0(x) \). Write

\[
v(x, t) = u(x, t) - u_0(x, t).
\]

It is not hard to verify that

\[
\begin{aligned}
c_0(x)v_{tt}(x, t) &= \Delta v(x, t) + (c_0(x) - c(x))u_{tt}(x, t) \quad (x, t) \in \mathbb{R}^d \times [0, T], \\
v(x, 0) &= 0 \\
v_t(x, 0) &= 0
\end{aligned}
\]  

(2.4)

\]

where \( c(x) \) is replaced by the function \( c_0(x) \). Write

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It is not hard to verify that

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\begin{aligned}
c_0(x)v_{tt}(x, t) &= \Delta v(x, t) + (c_0(x) - c(x))u_{tt}(x, t) \quad (x, t) \in \mathbb{R}^d \times [0, T], \\
v(x, 0) &= 0 \\
v_t(x, 0) &= 0
\end{aligned}
\]  

(2.4)
Since $c$ is a small perturbation of $c_0$, roughly speaking, we can replace the function $u_{tt}(x, t)$ in the problem above by $(u_0)_{tt}(x, t)$ to obtain
\[
\begin{cases}
  c_0(x)v_{tt}(x, t) = \Delta v(x, t) + p(x)h(x, t) & (x, t) \in \mathbb{R}^d \times [0, T], \\
  v(x, 0) = 0 & x \in \mathbb{R}^d, \\
  v_t(x, 0) = 0 & x \in \mathbb{R}^d,
\end{cases}
\]
where $p(x) = c_0(x) - c(x)$ and $h(x, t) = (u_0)_{tt}(x, t)$. Note that
\[
h(x, 0) = (u_0)_{tt}(x, 0) = \Delta u_0(x, 0) = \Delta f(x),
\]
which is non-zero everywhere in $\Omega$ by our assumption. The boundary values of $v(x, t)|_{\partial \Omega \times [0,T]}$ and $\partial_x v(x, t)|_{\partial \Omega \times [0,T]}$ are computed from the knowledge of $u_0$ and $u$. The problem of finding $p(x) = c_0(x) - c(x)$ to obtain $c(x)$ is a particular case of our ISP. This is, actually, the linearization approach to enhance the accuracy for the solution of coefficient inverse problem.

In the next section, we will derive an integro-differential equation that leads to our numerical method.

### 3 A Volterra integro-differential equation

This section aims to establish an equation whose solution directly yields the solution of the ISP. Define
\[
v(x, t) = u_t(x, t) \quad (x, t) \in \Omega \times [0, T].
\]
It follows from (2.1) that
\[
c(x)v_{tt} - \Delta v = a(x)v + B(x) \cdot \nabla v + p(x)h_t(x, t) \quad (x, t) \in \Omega \times [0, T].
\]
The function $v$ satisfies the initial conditions
\[
v(x, 0) = u_t(x, 0) = g(x),
\]
for $x$ in $\Omega$. Let $\tilde{h} : \Omega \times [0, \infty)$ be a smooth function satisfying
\[
\tilde{h}(x, 0) = h(x, 0), \quad \tilde{h}_t(x, 0) = h_t(x, 0), \quad \tilde{h}(x, t) \neq 0
\]
for all $x \in \Omega$ and $t \in [0, T]$. An example for such a function is
\[
\tilde{h}(x, t) = h(x, 0) \exp(\theta h_t(x, 0)/h(x, 0)).
\]
Define the function
\[
w(x, t) = \frac{v_t(x, t)}{h(x, t)}, \quad \text{so that} \quad v_t(x, t) = \tilde{h}(x, t)w(x, t) \quad (x, t) \in \Omega \times [0, T].
\]
A straight forward calculation yields
\[
c(x)w_{tt} - \Delta w = \frac{(c(x)v_{tt} - \Delta v)_t}{h(x, t)} + c(x) \left( \frac{1}{\tilde{h}(x, t)} \right)_t v_t + 2c(x) \left( \frac{1}{\tilde{h}(x, t)} \right)_t v_{tt} - \Delta \left( \frac{1}{\tilde{h}(x, t)} \right) v_t - 2\nabla \left( \frac{1}{\tilde{h}(x, t)} \right) \nabla v_t.
\]
Therefore, by (3.2) and (3.6), we have
\[
c(x)w_t - \Delta w = \frac{a(x)\dot{h}(x, t)w + B(x) \cdot \nabla(\dot{h}(x, t)w) + p(x)h_t(x, t)}{h(x, t)} + c(x) \left( \frac{1}{h(x, t)} \right) \ddot{h}(x, t)w \\
+ 2c(x) \left( \frac{1}{h(x, t)} \right) (\dot{h}(x, t)w)_t - \Delta \left( \frac{1}{h(x, t)} \right) \ddot{h}(x, t)w - 2\nabla \left( \frac{1}{h(x, t)} \right) \nabla(\dot{h}(x, t)w).
\]  
(3.7)

At the time \( t = 0 \), by (3.2) and (3.4),
\[
c(x)w(x, 0) = \frac{c(x)v_t(x, 0)}{h(x, 0)} = \frac{\Delta f(x) + a(x)f(x) + B(x) \cdot \nabla f(x) + p(x)h(x, 0)}{h(x, 0)}.
\]

It follows from (3.5) that
\[
p(x) = c(x)w(x, 0) - \frac{\Delta f(x) + a(x)f(x) + B(x) \cdot \nabla f(x)}{h(x, 0)}.
\]  
(3.8)

Since
\[
w(x, 0) = w(x, t) - \int_0^t w_t(x, \tau) d\tau,
\]
we can rewrite (3.8) as
\[
p(x) = c(x)w(x, t) - c(x) \int_0^t w_t(x, \tau) d\tau - \frac{\Delta f(x) + a(x)f(x) + B(x) \cdot \nabla f(x)}{h(x, 0)}.
\]

Plugging this into (3.7), we derive an equation for \( w \)
\[
cw_t - \Delta w = L(w, \nabla w, w_t) - \frac{c(x)h_t(x, t)}{h(x, 0)} \int_0^t w_t(x, \tau) d\tau + F(x, t)
\]  
(3.9)

where
\[
L(w, \nabla w, w_t) = \frac{a(x)\dot{h}(x, t)w + B(x) \cdot \nabla(\dot{h}(x, t)w)}{h(x, t)} + c(x) \left( \frac{1}{h(x, t)} \right) \ddot{h}(x, t)w \\
+ 2c(x) \left( \frac{1}{h(x, t)} \right) (\dot{h}(x, t)w)_t - \Delta \left( \frac{1}{h(x, t)} \right) \ddot{h}(x, t)w - 2\nabla \left( \frac{1}{h(x, t)} \right) \nabla(\dot{h}(x, t)w)
\]

and
\[
F(x, t) = - \frac{h_{tt}(x, t)(\Delta f(x) + a(x)f(x) + B(x) \cdot \nabla f(x))}{h(x, 0)\dot{h}(x, t)}.
\]  
(3.10)

We are now in the position to derive some boundary and initial constraints for the function \( w \). It follows from (3.2), (3.3), (3.4) and (3.6), for all \( x \in \Omega \),
\[
w_t(x, 0) = \frac{v_{tt}(x, 0)}{h(x, 0)} - \frac{v_t(x, 0)\dot{h}(x, 0)}{h^2(x, 0)} = \frac{\Delta g(x) + a(x)g(x) + B(x) \cdot \nabla g(x) + p(x)h_t(x, 0)}{c(x)\dot{h}(x, 0)}
\]
\[
- \frac{[\Delta f(x) + a(x)f(x) + B(x) \cdot \nabla f(x) + p(x)h(x, 0)]\dot{h}(x, 0)}{c(x)h^2(x, 0)}.
\]
which is simplified as
\[ w_t(x, 0) = \frac{(\Delta g(x) + B(x) \cdot \nabla g(x))h(x, 0) - (\Delta f(x) + B(x) \cdot \nabla f(x))}{c(x)h^2(x, 0)}. \]

On the other hand, by (2.2), (3.1) and (3.6), we can find the boundary data for the function \( w \):
\[ w(x, t) = \frac{F_{\Omega}(x, t)}{h(x, t)} \quad \text{and} \quad \partial_\nu w(x, t) = \frac{G_{\Omega}(x, t)\hat{h}(x, t) - \partial_\nu \hat{h}(x, t)F_{\Omega}(x, t)}{h^2(x, t)} \quad (x, t) \in \partial\Omega \times [0, T]. \tag{3.11} \]

The arguments above are summarized as the following proposition.

**Proposition 3.1.** Define \( \hat{h} \) as in (3.5) and \( F \) as in (3.10). Let \( u \) be the solution of the hyperbolic problem (2.1). Then the function
\[ w(x, t) = \frac{u_t(x, t)}{h(x, t)} \quad \text{for all} \quad (x, t) \in \Omega \times [0, T] \tag{3.12} \]
satisfies
\[ Lw = F(x, t) \quad (x, t) \in \Omega \times [0, T] \tag{3.13} \]
where
\[ L\phi = c(x)\phi_{tt} - \Delta \phi - L(\phi, \nabla \phi, \phi_t) + \frac{c(x)h_t(x, t)}{h(x, t)} \int_0^t \phi_t(x, \tau)d\tau \tag{3.14} \]
for all functions \( \phi \). Furthermore,
\[ w_t(x, 0) = \Psi(x) \tag{3.15} \]
where
\[ \Psi(x) = \frac{(\Delta g(x) + B(x) \cdot \nabla g(x))h(x, 0) - (\Delta f(x) + B(x) \cdot \nabla f(x))}{c(x)h^2(x, 0)} \tag{3.16} \]
for all \( x \in \Omega \) and
\[ w(x, t) = \zeta(x, t) \quad \text{and} \quad \partial_\nu w(x, t) = \xi(x, t) \quad \text{for all} \quad (x, t) \in \partial\Omega \times [0, T] \tag{3.17} \]
where
\[ \zeta(x, t) = \frac{F_{\Omega}(x, t)}{h(x, t)} \quad \text{and} \quad \xi(x, t) = \frac{G_{\Omega}(x, t)\hat{h}(x, t) - \partial_\nu \hat{h}(x, t)F_{\Omega}(x, t)}{h^2(x, t)}. \tag{3.18} \]

**Remark 3.1.** Our method to find the solution of the ISP is based on a numerical method to solve (3.13), (3.15) and (3.17), for a function \( w_\alpha \). The knowledge of \( w_\alpha \) directly yields that of \( p(x) \) via (3.8). Involved a Volterra integral, equation (3.13) is not a standard partial differential equation. A theoretical method to solve it is not yet available. We solve it numerically by the quasi-reversibility method.

**Remark 3.2.** From now on, without loss of generality, we consider the functions \( \zeta(x, t) \) and \( \xi(x, t) \), \((x, t) \in \partial\Omega \times [0, T]\), as available data. They can be computed directly in terms of the measured data \( F(x, t) \) and \( G(x, t) \) via (3.18).

Throughout the paper, we consider the case when the given data \( F \) and \( G \) are noisy. We explain in Section 6.2 how to differentiate them. We require that those functions are admissible in the following sense.
Definition 3.1 (The set of admissible data). The functions $F$ and $G : \partial \Omega \times [0, T] \to \mathbb{R}$ are said to be admissible if and only if the set
\[
K = \left\{ \phi \in H^3(\Omega \times [0, T]) : \phi_t(x, 0) = \Psi(x) \text{ for all } x \in \Omega, \text{ and} \right. \\
\left. \phi(x, t) = \zeta(x, t) \text{ and } \partial_\nu \phi(x, t) = \xi(x, t) \text{ for all } (x, t) \in \partial \Omega \times [0, T] \right\} 
\] (3.19)
is not empty.

Remark 3.3. Let
\[
H = \left\{ \phi \in H^3(\Omega \times [0, T]) : \phi_t(x, 0) = 0 \text{ for all } x \in \Omega, \text{ and} \right. \\
\left. \phi(x, t) = 0 \text{ and } \partial_\nu \phi(x, t) = 0 \text{ for all } (x, t) \in \partial \Omega \times [0, T] \right\}. 
\] (3.20)
It is not hard to see that $H$ is a subspace of $H^3(\Omega \times [0, T])$ and
\[
K = \phi + H
\]
where $\phi$ is a particular element of $K$. Hence, $K$ has no boundary with respect to the topology of $H^3(\Omega \times [0, T])$.

In the next section, we propose a numerical method to solve the Cauchy problem \(3.13, 3.15\) and \(3.17\) with the presence of a Volterra integral.

4 The quasi-reversibility method

Throughout this section, we assume $F$ and $G$ are admissible so that $K$ is non-empty. We have the proposition.

Proposition 4.1. For each $\alpha > 0$, the functional
\[
J_\alpha(w) = \|\mathcal{L}w - F(x, t)\|^2_{L^2(\Omega \times [0, T])} + \alpha \|w\|^2_{H^3(\Omega \times [0, T])}
\]
has a unique minimizer $w_\alpha$ in $K$.

Proof. Fix $\alpha > 0$ and let $\{w_n\}_{n=1}^\infty \subset K$ be such that
\[
\lim_{n \to \infty} J_\alpha(w_n) = \inf_{w \in K} J_\alpha(w). \tag{4.1}
\]
We claim that $\{w_n\}_{n=1}^\infty$ is bounded in $H^3(\Omega \times [0, T])$. In fact, if $\{w_n\}_{n=1}^\infty$ has a unbounded subsequence in $H^3(\Omega \times [0, T])$, still named as $\{w_n\}_{n=1}^\infty$, then by \(4.1\)
\[
\infty = \limsup_{n \to \infty} \alpha \|w_n\|^2_{H^3(\Omega \times [0, T])}
\leq \limsup_{n \to \infty} \left( \|\mathcal{L}w_n - F(x, t)\|^2_{L^2(\Omega \times [0, T])} + \alpha \|w_n\|^2_{H^3(\Omega \times [0, T])} \right) = \inf_{w \in K} J_\alpha(w),
\]
which is impossible. Since $\{w_n\}_{n=1}^\infty$ is bounded, it has a subsequence, still called $\{w_n\}_{n=1}^\infty$, weakly converges to a function $w_0$ in $H^3(\Omega \times [0, T])$. Since $K$ is closed and convex in $H^3(\Omega \times [0, T])$, by [13]...
Theorem 3.7], $K$ is weakly closed in $H^3(\Omega \times [0, T])$. Thus, $w_0$ is in $K$. By the compact continuous embedding from $H^3(\Omega \times [0, T])$ to $H^2(\Omega \times [0, T])$, without lost of generality, we can assume that $\{w_n\}_{n=1}^{\infty}$ converges strongly to $w_0$ in $H^2(\Omega \times [0, T])$. The function $w_0$ is a minimizer of $J_\alpha$. In fact,

$$J_\alpha(w_0) = \|Lw_0 - F(x, t)\|_{L^2(\Omega \times [0, T])} + \alpha \|w_0\|_H^3(\Omega \times [0, T])$$

$$\leq \lim_{n \to \infty} \|Lw_n - F(x, t)\|_{L^2(\Omega \times [0, T])} + \alpha \limsup_{n \to \infty} \|w_n\|_H^3(\Omega \times [0, T])$$

$$= \limsup_{n \to \infty} J_\alpha(w_n) = \inf_{w \in K} J_{w \in K}(w).$$

On the other hand, since $J_\alpha$ is strictly convex and $K$ has no boundary (see Remark 3.3), $J_\alpha$ has only one minimizer.

\[ \square \]

**Definition 4.1** (Regularized solution [6, 50]). The unique minimizer $w_\alpha \in K$ of $J_\alpha$, $\alpha > 0$, is called the regularized solution of problem (3.13), (3.15) and (3.17).

**Remark 4.1.** The non-empty condition imposed on $K$ is necessary for the theoretical purpose. However, it is not a serious concern in computation. In fact, we find the minimizer $w_\alpha$ of $J_\alpha$ by directly solving the equation $D_j J_\alpha(w_\alpha) = 0$ where $D_j J_\alpha$ is the Fréchet derivative of $J_\alpha$. In the finite difference scheme, this equation and the constraint $w_\alpha \in K$ constitute a linear system, say for e.g., $\Lambda w_\alpha = b$. Since we do not check if $K$ is non-empty, that linear system might not have a solution. We, therefore, approximate $w_\alpha$ by the solution of $(A^T A + \epsilon I)w = A^T b$.

In the next section, we prove that the regularized solution obtained by the quasi-reversibility method converges to the true solution of (3.13), (3.15) and (3.17) as $\delta$, the noise in measurement, and $\alpha$, the regularized parameter, tend to 0. The convergence rate is $O(\delta + \sqrt{\alpha})$.

### 5 The convergence of the quasi-reversibility

In this section, we study the convergence of the regularized solution to the true solution as the noise level and the regularized parameter tend to 0.

#### 5.1 The main result

Let $R$ be a large positive number such that $\Omega \subseteq D = B(R)$. We impose the following condition on the function $c(x)$. Assume that there exists a point $x_0$ in $D \setminus \overline{\Omega}$ such that

$$\langle x - x_0, \nabla c(x) \rangle \geq 0 \quad \text{for all } x \in \mathbb{R}^d. \quad (5.1)$$

Recall that data for the ISP are given by the functions $F$ and $G$. As mentioned in Remark 3.2, we can calculate $\zeta$ and $\xi$ via (3.18) in terms of $F$ and $G$. These two functions serve as new data for the ISP. Let $F^*$ and $G^*$ be the noiseless “direct” data. Denote by $\zeta^*$ and $\xi^*$ the functions defined in (3.18) when $F$ and $G$ are replaced by $F^*$ and $G^*$ respectively. Since $K$ is non-empty, so is the set

$$\mathcal{H} = \left\{ \phi \in H^3(\Omega \times [0, T]) : \phi_t(x, 0) = 0 \text{ for all } x \in \Omega, \right. \right.$$

$$\left. \text{and } \phi(x, t) = \zeta(x, t) - \zeta^*(x, t) \text{ and } \partial_\nu \phi(x, t) = \xi(x, t) - \xi^*(x, t) \text{ for all } (x, t) \in \partial \Omega \times [0, T] \right\}. \quad (5.2)$$
Define the following quantity

$$\| \zeta - \zeta^*, \xi - \xi^* \|_H = \inf_{\phi \in H} \{ \| \phi \|_{H^3(\Omega \times [0,T])} \},$$

(5.3)

which is considered as the measured noise. By the trace theory, we can verify that the norm $\| \cdot \|_H$ is stronger than the $L^2$ norm in the following sense

$$\| \zeta - \zeta^*, \xi - \xi^* \|_{L^2(\partial \Omega \times [0,T])} \leq C \| \zeta - \zeta^*, \xi - \xi^* \|_H$$

for some constant $C > 0$. As a consequence, if the measurement noise is small with respect to $\| \cdot \|_H$, then it is small in the $L^2$ sense.

The following theorem is the main result of this paper.

**Theorem 5.1.** Suppose that $K$ is non-empty and that condition [5.1] holds true. Assume that

$$\| \zeta - \zeta^*, \xi - \xi^* \|_H \leq \delta$$

(5.4)

for some $0 \leq \delta < 1$. Fix $\alpha \in (0,1)$ and let $w_\alpha$ be the regularized solution of (3.13), (3.15) and (3.17). Let $w^* = \frac{w_{\|H\|}}{h}$ be the true solution of (3.13), (3.15) and (3.17). Then, if $T > \left( \max_{x \in \Omega} |x - x_0|^2 / \eta_0 \right)^{1/2}$, where $\eta_0$ is the number, that will be indicated in Lemma 5.1, the following estimate is true

$$\| w_\alpha - w^* \|^2_{H^1(\Omega \times [0,T])} \leq C \left( \delta^2 + \alpha \| w^* \|^2_{H^3(\Omega \times [0,T])} \right)$$

(5.5)

for some constant $C = C(D, x_0, \Omega, T, a, B, c, h)$. As a result,

$$\| p_{\delta, \alpha} - p^* \|^2_{L^2(\Omega)} \leq C \left( \delta^2 + \alpha \| w^* \|^2_{H^3(\Omega \times [0,T])} \right)$$

(5.6)

where $p_{\delta, \alpha}$ is computed via (3.8) when $w$ is replaced by $w_\alpha$ and $p^*$ is the true source.

We recall here the Carleman estimates for the reader’s convenience. It is important mentioning that these results play a crucial role for the proof of Theorem 5.1. Introduce the function

$$W(x, t) = \exp(\lambda(|x - x_0|^2 - \eta t^2)), \quad (x, t) \in \mathbb{R}^d \times [0, \infty)$$

(5.7)

where $\lambda$ and $\eta$ are two positive numbers. The function $W(x, t)$ is known as the Carleman weight function. For $\eta > 0$ and $\epsilon > 0$, define

$$D_{\eta, \epsilon} = \{(x, t) \in D \times [0, \infty) : |x - x_0|^2 - \eta t^2 > \epsilon \}.$$  

(5.8)

We have the lemmas.

**Lemma 5.1** (Lemma 6.1 in [35] and Theorem 1.10.2 in [6]). Assume condition [5.1] holds true. Then, there exists a sufficiently small number $\eta_0 = \eta_0 \left( x_0, D, \| c \|_{C^1(\overline{D})} \right) \in (0,1)$ such that for any $\eta \in (0, \eta_0]$, one can choose a sufficiently large number $\lambda_0 = \lambda_0(D, \eta, c, x_0) > 1$ and a number $C = C(D, \eta, c, x_0)$ such that for all $z \in C^2(\overline{D}_{\eta, \epsilon})$ and for all $\lambda \geq \lambda_0$, the following pointwise Carleman estimate holds true

$$|c(x)z_\eta(x, t) - \Delta z(x, t)|^2 W^2(x, t) \geq C \lambda \left( |\nabla z(x, t)|^2 + z_\eta^2(x, t) + \lambda^2 z^2(x, t) \right) W^2(x, t)$$

$$+ \nabla \cdot Z(x, t) + Y_t(x, t)$$

(5.9)

in $D_{\eta, \epsilon}$. The vector valued function $Z(x, t)$ and the function $Y(x, t)$ satisfy

$$|Z(x, t)| \leq C \lambda^3 (|\nabla z(x, t)|^2 + z_\eta^2(x, t) + z^2(x, t)) W^2(x, t)$$

(5.10)

$$|Y(x, t)| \leq C \lambda^3 (|\nabla z(x, t)|^2 + z_\eta^2(x, t) + z^2(x, t) + (|\nabla z(x, t)| + |z|) |z_t(x, t)|) W^2(x, t).$$

(5.11)

In particular, if either $z(x, 0) = 0$ or $z_t(x, 0) = 0$, then $Y(x, 0) = 0$.  

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Lemma 5.2 (Lemma 1.10.3 [39]). Let the function \( \varphi \in C^1[0,a] \) and \( \varphi'(t) \leq -b \) in \([0,a]\), where \( b = \text{const} > 0 \). For a function \( g \in L^2(-a,a) \), consider the integral

\[
I(g,\lambda) = \int_{-a}^{a} \left( \int_0^t g(\tau) d\tau \right)^2 \exp(2\lambda \varphi(t^2)) dt.
\]

Then,

\[
I(g,\lambda) \leq \frac{1}{4ab} \int_{-a}^{a} g^2(t) \exp(2\lambda \varphi(t^2)) dt.
\]

We refer the reader to [39, 35] for the proof of Lemmas 5.1 and 5.2.

5.2 The proof of Theorem 5.1

The arguments in this section follow the ideas of Klibanov in [6, 30, 35, 38], in which the Lipschitz stability was established when the operator \( \mathcal{L} \) does not involve the Volterra integral. Note that this stability estimate was first established in [38]. In this subsection, \( C \) denotes a generic constant depending only on known sets and functions: \( D, x_0, \Omega, T, a, B, c, h \). The number \( C \) might change from estimate to estimate.

Step 1. Let

\[
H = \{ \varphi \in H^3(\Omega \times [0,\infty)) : \varphi_t(x,0) = 0 \text{ for } x \in \Omega \text{ and } \varphi(x,t) = \partial_t \varphi(x,t) = 0 \text{ for } (x,t) \in \partial \Omega \times [0,T] \}
\]

be the set of test functions. Since the regularized solution \( w_\alpha \) is the minimizer of \( \mathcal{J}_\alpha \) on \( K \), for all \( \varphi \in H \), we have

\[
\langle \mathcal{L} w_\alpha - \mathcal{F}, \mathcal{L} \varphi \rangle_{L^2(\Omega \times [0,T])} + \alpha \langle w_\alpha, \varphi \rangle_{H^3(\Omega \times [0,T])} = 0.
\]

(5.12)

On the other hand, since \( w^* \) is the true solution of the Volterra integro-differential equation (3.13), we have

\[
\langle \mathcal{L} w^* - \mathcal{F}, \mathcal{L} \varphi \rangle_{L^2(\Omega \times [0,T])} + \alpha \langle w^*, \varphi \rangle_{H^3(\Omega \times [0,T])} = \alpha \langle w^*, \varphi \rangle_{H^3(\Omega \times [0,T])}.
\]

(5.13)

Subtracting (5.12) from (5.13), we have

\[
\langle \mathcal{L}(w_\alpha - w^*), \mathcal{L} \varphi \rangle_{L^2(\Omega \times [0,T])} + \alpha \langle w - w^*, \varphi \rangle_{H^3(\Omega \times [0,T])} = -\alpha \langle w^*, \varphi \rangle_{H^3(\Omega \times [0,T])}
\]

(5.14)

for any \( \varphi \in H \). By (5.3) and assumption (5.4), there is an “error” function \( \mathcal{E} \) in \( \mathcal{H} \) such that

\[
\| \mathcal{E} \|_{H^3(\Omega \times [0,T])} \leq C\delta.
\]

(5.15)

Since \( w_\alpha \in K \), by (5.2), it is obvious that

\[
z = w_\alpha - w^* - \mathcal{E}
\]

(5.16)

is in \( H \). Choosing \( \varphi = z \) as a test function for (5.14), we have

\[
\| \mathcal{L} z \|_{L^2(\Omega \times [0,T])}^2 + \alpha \| z \|_{H^3(\Omega \times [0,T])}^2 = -\langle \mathcal{L} \mathcal{E}, \mathcal{L} z \rangle_{L^2(\Omega \times [0,T])} - \alpha \langle \mathcal{E}, z \rangle_{H^3(\Omega \times [0,T])}^2 - \alpha \langle w^*, z \rangle_{H^3(\Omega \times [0,T])}.
\]

(5.17)

On the other hand, using the inequality \( ab \leq \frac{a^2}{8} + 2b^2 \) and (5.17) and noting that \( 0 < \alpha < 1 \), we have

\[
\| \mathcal{L} z \|_{L^2(\Omega \times [0,T])} + \alpha \| z \|_{H^3(\Omega \times [0,T])} \leq C \left( \delta^2 + \alpha \| w^* \|_{H^3(\Omega \times [0,T])}^2 \right).
\]

(5.18)
Step 2. Recall that
\[ P_1 = \min_{x \in \Omega} \{|x - x_0|\}, \quad P_2 = \max_{x \in \Omega} \{|x - x_0|\}. \]

Let \( \eta_0 \) be the number as in Lemma 5.1 and \( \epsilon \) be a small positive number. Introduce the cut-off function \( \chi(x, t) \) satisfying \( \chi_t(x, t) \leq 0 \) and

\[
\chi(x, t) = \begin{cases} 
1 & \text{if } |x - x_0|^2 - \eta_0 t^2 > 2\epsilon, \\
0 & \text{if } |x - x_0| - \eta_0 t^2 > \epsilon, \\
(0, 1) & \text{otherwise}
\end{cases}
\]

(5.19)

and define the set

\[ Q_\epsilon = \{(x, t) \in \Omega \times [0, T] : |x - x_0|^2 - \eta_0 t^2 > \epsilon\} = D_{\eta_0, \epsilon} \cap \Omega \times [0, T] \]

where \( D_{\eta_0, \epsilon} \) is as in (5.8). We next apply Lemma 5.1 to get (5.9) for the function \( z \). Multiply \( \chi(x, t) \) to both side of (5.9) and then integrate the resulting on \( Q_\epsilon \). We have

\[
\int_{Q_\epsilon} W(x, t) \chi(x, t)(z_t - \Delta z)^2 \, dx \, dt \geq C \lambda \int_{Q_\epsilon} \chi(x, t) (|\nabla z|^2 + z_t^2 + \lambda^2 z^2) W(x, t) \, dx \, dt \\
+ \int_{Q_\epsilon} \chi(x, t) \nabla \cdot Z \, dx \, dt + \int_{Q_\epsilon} \chi(x, t) Y_t \, dx \, dt
\]

(5.20)

where \( Z \) and \( Y \) satisfy (5.10) and (5.11) respectively.

Since \( z \in H \), it follows from (5.10) that \( Z(x, t) = 0 \) for all \( x \in \partial \Omega \) and \( t \in [0, \hat{t}_s(x)] \) where

\[
\hat{t}_s(x) = \sqrt{\frac{|x - x_0|^2 - s}{\eta_0}}, \quad s \in (0, P_1^2)
\]

(5.21)

Recall that

\[
\chi(x, t) \nabla \cdot Z = \nabla \cdot (\chi(x, t) Z(x, t)) - \nabla \chi(x, t) \cdot Z(x, t).
\]

Hence, by (5.19)

\[
\left| \int_{Q_\epsilon} \chi(x, t) \nabla \cdot Z \, dx \, dt \right| = \left| \int_\Omega \int_0^{\hat{t}_s(x)} \chi(x, t) \nabla \cdot Z \, dx \, dt \right| \\
\leq \left| \int_\Omega \int_0^{\hat{t}_s(x)} \nabla \cdot (\chi(x, t) Z) \, dx \, dt \right| - \left| \int_\Omega \int_0^{\hat{t}_s(x)} \nabla \chi(x, t) \cdot Z \, dx \, dt \right|.
\]

(5.22)

Since \( z(x, t) \in H \), both \( z(x, t) \) and \( \partial_t z(x, t) \) are identically 0 on \( \partial \Omega \times [0, T] \). By (5.10), \( Z = 0 \) on \( \partial \Omega \times [0, T] \). Hence, the first integral on the right hand side of (5.22) vanishes. By (5.10), we have

\[
\left| \int_{Q_\epsilon} \chi(x, t) \nabla \cdot Z \, dx \, dt \right| \leq C \lambda^3 \int_\Omega \int_0^{\hat{t}_s(x)} W(x, t)(|z|^2 + |\nabla z|^2 + |z_t|^2) \, dx \, dt.
\]

Noting that

\[
W(x, t) \leq \exp(4\lambda \epsilon) \quad \text{for all } x \in \Omega, t \in [\hat{t}_s(x), \hat{t}_s(x)],
\]

(5.23)

we obtain

\[
\left| \int_{Q_\epsilon} \chi(x, t) \nabla \cdot Z \, dx \, dt \right| \leq C \lambda^3 \exp(4\lambda \epsilon) \int_\Omega \int_0^{\hat{t}_s(x)} (|z|^2 + |\nabla z|^2 + |z_t|^2) \, dx \, dt.
\]

(5.24)
We next estimate the last term in (5.20). Since \( z_t(x, 0) = 0 \), it follows from (5.11) that \( Y(x, 0) = 0 \). We have

\[
\left| \int_{Q_\epsilon} \chi(x, t) Y_t(x, t) dx dt \right| = \left| \int_0^{\hat{t}_\epsilon(x)} \int_{Q_\epsilon} \chi(x, t) Y_t(x, t) dt dx \right|
\]

\[
= \left| \int_0^{\hat{t}_\epsilon(x)} \int_{Q_\epsilon} \left[ (\chi(x, t) Y(x, t)) - \chi_t(x, t) Y(x, t) \right] d\tau dx \right|
\]

By (5.19), \( \nabla \chi(x, t) = 0 \) for all \( t \in (0, \hat{t}_\epsilon(x)) \). Hence, by (5.11) and (5.23),

\[
\left| \int_{Q_\epsilon} \chi(x, t) Y_t dx dt \right| \leq C\lambda^3 \exp(4\lambda\epsilon) \int_0^{\hat{t}_\epsilon(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt.
\]  

(5.25)

It follows from (5.20), (5.24) and (5.25) that

\[
\int_{Q_\epsilon} W^2(x, t) \chi(x, t) (z_{tt} - \Delta z)^2 dx dt \geq C\lambda \int_{Q_{\Omega_\epsilon}} \chi(x, t) (|\nabla z|^2 + z_t^2 + \lambda^2 z^2) W^2(x, t) dx dt
\]

\[- C\lambda^3 \exp(4\lambda\epsilon) \int_0^{\hat{t}_\epsilon(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt.
\]  

(5.26)

It follows from (3.14) that

\[
\int_{Q_\epsilon} W^2(x, t) \chi(x, t) |\nabla z|^2 dx dt \geq \int_{Q_\epsilon} W^2(x, t) \chi(x, t) (z_{tt} - \Delta z)^2 dx dt
\]

\[- C \int_{Q_\epsilon} W^2(x, t) \chi(x, t) (|\nabla z|^2 + z_t^2 + z^2) dx dt - C \int_{Q_\epsilon} W^2(x, t) \chi(x, t) \left| \int_0^t z_t(x, \tau) d\tau \right|^2 dx dt.
\]

Using this, (5.26) and the fact that \( \chi(x, t) \leq 1 \), we obtain

\[
\int_{Q_\epsilon} W^2(x, t) \chi(x, t) |\nabla z|^2 dx dt \geq C(\lambda - 1) \int_{Q_\epsilon} W^2(x, t) \chi(x, t) (|\nabla z|^2 + z_t^2 + z^2) dx dt
\]

\[- C\lambda^3 \exp(4\lambda\epsilon) \int_0^{\hat{t}_\epsilon(x)} (|\nabla z|^2 + z_t^2 + z^2) dt dx - C \int_{Q_\epsilon} W^2(x, t) \left| \int_0^t z_t(x, \tau) d\tau \right|^2 dx dt.
\]  

(5.27)

**Step 3.** We next estimate the Volterra-integral

\[
\int_{Q_\epsilon} W^2(x, t) \left| \int_0^t z_t(x, \tau) d\tau \right|^2 dx dt
\]

in the right hand side of (5.27). We have

\[
\int_{Q_\epsilon} W^2(x, t) \left( \int_0^t z_t(x, \tau) d\tau \right)^2 dt dx
\]

\[
= \int_{\Omega} \exp(2\lambda|x - x_0|^2) \int_0^{\hat{t}_\epsilon(x)} \exp(-2\lambda\eta_0^2 t^2) \left( \int_0^t z_t(x, \tau) d\tau \right)^2 dt dx.
\]  

(5.28)
Letting $I$ denote the integral with respect to $t$, extending $z$ as an even function i.e. $z(x, -t) = z(x, t)$ and applying Lemma 5.2 with $\phi(t) = -\eta_0 t$, we have

$$I \leq \frac{1}{4\lambda\eta_0} \int_0^{t_0(x)} \exp(-2\lambda\eta_0 t^2)\chi(x, t)|z_t(x, t)|^2 dt.$$  

By (5.28)

$$\int_{Q_\epsilon} W^2(x, t) \left( \int_0^t z_t(x, \tau) d\tau \right)^2 dx dt \leq \frac{1}{4\lambda\eta_0} \int_{Q_\epsilon} W^2(x, t)|z_t|^2 dx dt.$$

Since $\chi(x, t) = 1$ for $0 < t < \hat{t}_{2\epsilon}(x)$, we have

$$\left| \int_{Q_\epsilon} W^2(x, t)(\chi(x, t) - 1)|z_t|^2 dx dt \right| \leq C\lambda^3 \exp(4\lambda\epsilon) \int_\Omega \int_{\hat{t}_{2\epsilon}(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt.$$

Hence, noting that $\lambda \gg 1$, we obtain by (5.27) that

$$\int_{Q_\epsilon} W^2(x, t)\chi(x, t)|Lz|^2 dx dt \geq C\lambda \int_{Q_\epsilon} W^2(x, t)\chi(x, t)(|\nabla z|^2 + z_t^2 + z^2) dx dt$$

$$- C\lambda^3 \exp(4\lambda\epsilon) \int_\Omega \int_{\hat{t}_{2\epsilon}(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt. \quad (5.29)$$

Step 4.

In this step, we estimate $\|z\|_{H^1(Q_{2\epsilon})}$. Note that

$$\chi(x, t) \leq 1 \text{ and } W(x, t) \leq \exp(2\lambda P_2^2) \text{ for all } (x, t) \in Q_\epsilon.$$

Using (5.18) gives

$$\int_{Q_\epsilon} W^2(x, t)\chi(x, t)|Lz|^2 dx dt \leq \exp(2\lambda P_2^2) \left( \|Lz\|^2_{L^2(\Omega \times [0, T])} + \alpha \|z\|^2_{H^3(\Omega \times [0, T])} \right)$$

$$\leq C \exp(2\lambda P_2^2) \left( \delta^2 + \alpha \|w^*\|^2_{H^3(\Omega \times [0, T])} \right).$$

By (5.29)

$$\lambda \int_{Q_\epsilon} W^2(x, t)\chi(x, t)(|\nabla z|^2 + z_t^2 + z^2) dx dt \leq C\lambda^3 \exp(4\lambda\epsilon) \int_\Omega \int_{\hat{t}_{2\epsilon}(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt$$

$$+ C \exp(2\lambda P_2^2) \left( \delta^2 + \alpha \|w^*\|^2_{H^3(\Omega \times [0, T])} \right).$$

Since $Q_{4\epsilon} \subset Q_\epsilon$ and

$$\chi(x, t) = 1 \text{ and } W^2(x, t) \geq \exp(8\lambda\epsilon) \text{ for all } (x, t) \in Q_{4\epsilon},$$

we have

$$\lambda \exp(8\lambda\epsilon) \int_{Q_{4\epsilon}} (|\nabla z|^2 + z_t^2 + z^2) dx dt$$

$$\leq C\lambda^3 \exp(4\lambda\epsilon) \int_\Omega \int_{\hat{t}_{2\epsilon}(x)} (|\nabla z|^2 + z_t^2 + z^2) dx dt + C \exp(\lambda P_2^2) \left( \delta^2 + \alpha \|w^*\|^2_{H^3(\Omega \times [0, T])} \right). \quad (5.30)$$
It follows from (5.30) that
\[
\int_{Q_{\epsilon}} (|\nabla z|^2 + z_t^2 + z^2)dxdt \leq \frac{\exp(-4\lambda\epsilon)}{\lambda} \int_{\Omega} \int_{t_2(x)}^{t_1(x)} (|\nabla z|^2 + z_t^2 + z^2)dxdt + C \exp(2\lambda(P_2^2 - 4\epsilon)) \left( \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])} \right). \tag{5.31}
\]

**Step 5.**
For all \( t \in [0, T] \), using (3.14), we have
\[
\int_{\Omega} z_t(z_{tt} - \Delta z + z)dx \leq C \left( \int_{\Omega} |z_t \mathcal{L}z|dx + \int_{\Omega} |z_t||z| + |z_t| + |\nabla z|dx + \int_{\Omega} |z_t| \int_0^t z_t d\tau \right) dx
\]
\[
= C \left( \int_{\Omega} |z_t \mathcal{L}z|dx + \int_{\Omega} |z_t||z| + |z_t| + |\nabla z|dx + \int_{\Omega} |z_t| |z - z(x, 0)| dx \right).
\]

Using (5.18) and the inequality \( 2ab \leq a^2 + b^2 \), we have for all \( t \in [0, T] \)
\[
\int_{\Omega} z_t(z_{tt} - \Delta z + z)dx \leq C \left( \int_{\Omega} |z| + |z_t|^2 + |\nabla z|^2 dx + \int_{\Omega} |z(x, 0)|^2 dx + \right.
\]
\[
+ \int_{\Omega} |\mathcal{L}z|^2 dx + \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])}. \tag{5.32}
\]

Integrating by parts for the left hand side of the inequality above, we have
\[
E'(t) \leq C \left( E(t) + \int_{\Omega} |z(x, 0)|^2 dx + \int_{\Omega} |\mathcal{L}z|^2 dx + \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])} \right) \tag{5.33}
\]
where
\[
E(t) = \int_{\Omega} (|z|^2 + |z_t|^2 + |\nabla z|^2)dx. \tag{5.33}
\]

Define a smooth cut-off function \( \chi_1(t) \) satisfying
\[
\chi_1(t) = \begin{cases} 
0 & t < \theta, \\
\in (0, 1) & \theta < t < 2\theta, \\
1 & t > 2\theta.
\end{cases}
\]

Multiplying \( \chi_1(t) \) to both sides of (5.32), we have
\[
(\chi_1(t)E(t))' \leq C\chi_1(t)E(t) + \chi_1'(t)E(t)(t)
\]
\[
+ C \left( \int_{\Omega} |z(x, 0)|^2 dx + \int_{\Omega} |\mathcal{L}z|^2 dx + \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])} \right). \tag{5.34}
\]

Due to the trace theory, \( \int_{\Omega} |z(x, 0)|^2 dx \) is bounded by \( C\|z\|^2_{H^1(Q_{\epsilon})} \). We deduce from (5.34) that
\[
(\chi_1(t)E(t))' \leq C\chi_1(t)E(t) + C \left( \|z\|^2_{H^1(Q_{\epsilon})} + |\chi_1'(t)E(t)| + \int_{\Omega} |\mathcal{L}z|^2 dx + \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])} \right). \tag{5.34}
\]

Using Grönwall’s inequality and noting that \( \chi_1(0) = 0 \), we have
\[
\chi_1(t)E(t) \leq C \left( \|z\|^2_{H^1(Q_{\epsilon})} + |\chi_1'(t)E(t)| + \int_{\Omega} |\mathcal{L}z|^2 dx + \delta^2 + \alpha \|w^*\|^2_{H^2(\Omega \times [0,T])} \right) t \in [0, T].
\]
Integrating the inequality above with respect to \( t \), we have
\[
\int_{\Omega \times [0,T]} \chi_1(\tau)(|z|^2 + |z_t|^2 + |\nabla z|^2) d\tau d\mathbf{x} \leq C \left( \|z\|_{H^1(Q_{4\epsilon})}^2 + \delta^2 + \alpha \|w^*\|_{H^3(\Omega \times [0,T])}^2 \right) + C \int_0^T |\chi'_1(\tau) E(\tau)| d\tau + C \int_{\Omega \times [0,T]} |Lz|^2 d\mathbf{x} d\tau. \tag{5.35}
\]
Since \( \chi'_1(\tau) = 0 \) for \( t > 2\theta \) and \( \Omega \times [0, 2\theta] \subset Q_{4\epsilon} \), it follows from (5.33) that
\[
\int_0^T |\chi'_1(\tau) E(\tau)| d\tau = \int_0^{2\theta} |\chi'_1(\tau) E(\tau)| d\tau \leq C \|z\|_{H^1(Q_{4\epsilon})}^2. \tag{5.36}
\]
Using (5.18) and (5.36), we deduce from (5.35) that
\[
\int_{\Omega \times [0,T]} \chi_1(\tau)(|z|^2 + |z_t|^2 + |\nabla z|^2) d\tau d\mathbf{x} \leq C \left( \|z\|_{H^1(Q_{4\epsilon})}^2 + \delta^2 + \alpha \|w^*\|_{H^3(\Omega \times [0,T])}^2 \right). \tag{5.37}
\]
It follows from (5.31) and (5.37) that
\[
\int_{\Omega \times [0,T]} \chi_1(\tau)(|\nabla z|^2 + |z_t|^2 + |z|^2) d\tau d\mathbf{x} \leq \frac{C \exp(-4\lambda \epsilon)}{\lambda} \int_{\Omega} \int_{I_{2\alpha}(x)} (|\nabla z|^2 + |z_t|^2 + |z|^2) d\mathbf{x} d\tau
\quad + C \exp(2\lambda (P_2^2 - 4\epsilon)) \left( \delta^2 + \alpha \|w^*\|_{H^3(\Omega \times [0,T])}^2 \right). \tag{5.38}
\]
Adding (5.31) and (5.38), we have
\[
\int_{\Omega \times [0,T]} (|\nabla z|^2 + |z_t|^2 + |z|^2) d\tau d\mathbf{x} \leq \frac{C \exp(-4\lambda \epsilon)}{\lambda} \int_{\Omega} \int_{I_{2\alpha}(x)} (|\nabla z|^2 + |z_t|^2 + |z|^2) d\mathbf{x} d\tau
\quad + C \exp(2\lambda (P_2^2 - 4\epsilon)) \left( \delta^2 + \alpha \|w^*\|_{H^3(\Omega \times [0,T])}^2 \right). \tag{5.39}
\]
Fix \( \lambda \) as a large number such that \( C \exp(-4\lambda \epsilon) < 1/2 \). We have
\[
\int_{\Omega \times [0,T]} (|\nabla z|^2 + |z_t|^2 + |z|^2) d\tau d\mathbf{x} \leq C \left( \delta^2 + \alpha \|w^*\|_{H^3(\Omega \times [0,T])}^2 \right). \tag{5.39}
\]
Combining (5.15), (5.16), (5.17) and (5.39), we obtain (5.5).

**Step 6.** The analog of (3.8) for the function \( p^* \) is
\[
p^*(\mathbf{x}) = c(\mathbf{x}) w^*(\mathbf{x}, 0) - \frac{\Delta f(\mathbf{x}) + a(\mathbf{x}) f(\mathbf{x}) + B(\mathbf{x}) \cdot \nabla f(\mathbf{x})}{h(\mathbf{x}, 0)}
\]
and the one for the function \( p_{\delta, \alpha} \) is
\[
p_{\delta, \alpha}(\mathbf{x}) = c(\mathbf{x}) w_{\alpha}(\mathbf{x}, 0) - \frac{\Delta f(\mathbf{x}) + a(\mathbf{x}) f(\mathbf{x}) + B(\mathbf{x}) \cdot \nabla f(\mathbf{x})}{h(\mathbf{x}, 0)}
\]
Using these identities, (5.5) and the trace theory, we obtain (5.6).

The proof is complete.

**Remark 5.1.** Up to the knowledge of the author, the available convergence results and the rates of the convergence are known for partial differential equation without the presence of Volterra integral; see [35]. This is the first time when this method is extended for integro-differential equations.
6 Numerical tests

In this section, we display some numerical examples when \( d = 2 \). The set \( \Omega \) is the cube \((-0.5, 0.5)^2\) and the time \( T \) is 1.

For the simplicity, we only consider the case \( c(x) = 1, a(x) = B(x) = 0 \) for which condition (5.1) holds true. In addition, we set the initial value and velocity \( f \) and \( g \) of the wave field \( u(x, t) \) identically equal zero. These simplifications do not weaken the result of the paper because the contributions of those functions are not important in our analysis. More precisely, in this section, we implement our method and display numerical solutions to the ISP for the problem

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
u_{tt} = \Delta u + p(x)h(x, t) & \quad (x, t) \in \mathbb{R}^2 \times [0, \infty), \\
u(x, 0) = u_t(x, 0) = 0 & \quad x \in \Omega.
\end{array}
\right.
\end{align*}
\]

6.1 Generating the simulated data

We solve the forward problem using the finite difference method. Set \( \Omega_1 = (-R, R)^2 \). In the computation, we choose \( R = 3 \). We create a \( N \times N, N = 500 \), grid \( G = \{ x_{m,n} = (-R + (m - 1)d_x, -R + (n - 1)d_x) : 1 \leq m, n \leq N \} \subset \Omega_1 \) where \( d_x = 2R/(N - 1) \). We also split the time interval into a uniform partition

\[
t_1 = 0 < t_2 < \cdots < t_{N_t} = T = 1
\]

where \( N_t = 120 \). The step size of the time variable is \( d_t = t_2 - t_1 = T/N_t \). Then, given \( u(x, 0) = u(x, d_t) = 0 \) for all \( x \in G \), we explicitly calculate \( u(x, t + d_t) \) by

\[
u(x, t + d_t) = 2u(x, t) - u(x, t - d_t) + d_t^2(\Delta u(x, t) + p(x)h(x, t)).
\]

The noiseless data \( F^*(x, t) = u(x, t) \) and \( G^*(x, t) = \partial_\nu u(x, t) \) for \( x \in \partial \Omega \) and \( t \in \{ t_1, \ldots, t_{N_t} \} \) can be extracted easily.

Let \( \delta > 0 \) denote the noise level. Noisy data are set to be

\[
F(x, t) = F^*(x, t)(1 + \delta(2\text{rand}(x, t) - 1)) \quad (6.1)
\]

and

\[
G(x, t) = G^*(x, t)(1 + \delta(2\text{rand}(x, t) - 1)). \quad (6.2)
\]

Here, \text{rand} is a Matlab function that generates uniformly distributed random numbers in the interval \([0, 1]\). We use \( 2\text{rand}(x, t) - 1 \) in (6.1) and (6.2) to create random numbers in the interval \([-1, 1]\). In this paper, we test our method when \( \delta \) takes values 2\%, 5\% and 10\%.

6.2 Differentiating noisy data by Tikhonov regularization

The first step to solve the ISP is to calculate the second derivatives of \( F(x, t) \) and \( G(x, t) \) with respect to \( t \). Since data are supposed to contain noise, see (6.1) and (6.2), they can not be differentiated using the finite difference method. We calculate the second derivative of data by the Tikhonov regularization method, which is widely used in the community. For each \( x \in \partial \Omega \), noting that

\[
F(x, 0) = F_t(x, 0) = 0
\]
we can write
\[
F(x, t_n) = \int_0^{t_n} \int_0^s F_{tt}(\sigma)d\sigma ds = d_t^2 \sum_{i=1}^{n} \sum_{j=1}^{i} F_{tt}(x, t_j), \quad n = 1, \ldots, N_t.
\] (6.3)

All equations in (6.3) constitute a linear system, say
\[
F(x) = A F_{tt}(x)
\] (6.4)
where \( F \) and \( F_{tt} \) are the vectors \((F(x, t_n))_{n=1}^{N_t} \) and \((F_{tt}(x, t_n))_{n=1}^{N_t} \) respectively and the matrix \( A \) is given by
\[
A = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
2 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
N_t & N_t - 1 & \ldots & 1
\end{bmatrix}.
\] (6.5)

The vector \( F_{tt}(x) \) is approximated by the solution of
\[
(A^T A + \varepsilon \text{Id}) F_{tt}(x) = F(x)
\]
where \( \text{Id} \) is the identity matrix and \( \varepsilon > 0 \) is a small number. In the computation, we choose \( \varepsilon = 10^{-5} \).

### 6.3 Implementation

In the previous section, we solved the forward problem in the domain \( \Omega_1 \), which was spitted to 500×500 grid. The mesh restricted on \( \Omega = (-0.5, 0.5)^2 \subset \Omega_1 \) is of size 85×85. We thus reset \( N = 85 \). The meshgrid for \( \Omega \times [0, T] \) is the set \( \{(x_m, n), t_j\}_{1 \leq m, n \leq N, 1 \leq j \leq N_t} \). Recall that \( N_t = 120 \). Without confusing, for any function \( \phi \) defined in \( \Omega \times [0, T] \), we identify \( \phi \) by the tensor \((\phi(x_m, n, t_j))_{1 \leq m, n \leq N, 1 \leq j \leq N_t}\).

We establish the tensor \( \mathcal{L} \) so that for all function \( \phi \), the function \( \phi - \Delta \phi - \phi(0) \frac{h_t}{h} \) is approximated by \( L\phi \). Based on finite difference, the entries of the tensor \( \mathcal{L} \) is given by
\[
\mathcal{L}_{m,n,j;m,n,j} = -\frac{2}{\Delta t^2} + \frac{4}{\Delta x^2} \quad \mathcal{L}_{m,n,j;m,n,j+1} = -\frac{1}{\Delta t^2} \quad \mathcal{L}_{m,n,j;m,n,j-1} = -\frac{1}{\Delta t^2} \\
\mathcal{L}_{m,n,j;m-1,n,j} = \frac{1}{\Delta x^2} \quad \mathcal{L}_{m,n,j;m+1,n,j} = \frac{1}{\Delta x^2} \quad \mathcal{L}_{m,n,j;m,n-1,j} = \frac{1}{\Delta t} \quad \mathcal{L}_{m,n,j;m,n+1,j} = \frac{1}{\Delta t} \\
\mathcal{L}_{m,n,j;m,n+1,j} = \frac{1}{\Delta x} \quad \mathcal{L}_{m,n,j;m,n+1,j} = -\frac{h_t(x_m, n, t_j)}{h(x_m, n, t_j)}
\]
for \( 2 \leq m, n \leq N - 1 \) and \( 2 \leq j \leq N_t - 1 \). The other entries of \( \mathcal{L} \) are 0.

We next implement the constraints in (3.15) with \( \Psi = 0 \) and (3.17).

1. The constraint \( w_t(x, 0) = 0 \) can be understood in the sense of finite difference as \( D_t w = 0 \) where \( (D_t)_{m,n,1} = -\frac{1}{\Delta t} \), \( (D_t)_{m,n,2} = \frac{1}{\Delta t} \) and the other entries of \( D_t \) are 0.

2. The constraint \( w(x, t) = \zeta(x, t) \) can be written as \( D w = \zeta(x, t) \) where
\[
D_{m,n,j;m,n,j} = \begin{cases}
1 & m \in \{1, N\} \text{ or } n \in \{1, N\} \\
0 & \text{otherwise}
\end{cases}
\]

Here, we need to extend \( \zeta(x, t) = 0 \) inside \( \Omega \).
3. The constraint \( \partial_r w(x, t) = \xi(x, t) \) can be written as \( \mathcal{N}w = \xi(x, t) \) where

\[
\begin{align*}
\mathcal{N}_{1,n,j} &= \frac{1}{dx} & \mathcal{N}_{2,n,j} &= -\frac{1}{dx} & 1 \leq n \leq N, \\
\mathcal{N}_{N,n,j} &= \frac{1}{dx} & \mathcal{N}_{N-1,n,j} &= -\frac{1}{dx} & 1 \leq n \leq N, \\
\mathcal{N}_{m,1,j} &= \frac{1}{dx} & \mathcal{N}_{m,2,j} &= -\frac{1}{dx} & 1 \leq m \leq N, \\
\mathcal{N}_{m,N,j} &= \frac{1}{dx} & \mathcal{N}_{m,N-1,j} &= -\frac{1}{dx} & 1 \leq m \leq N
\end{align*}
\]

The other entries of \( \mathcal{N} \) are 0. Here, we need to extend \( \xi(x, t) = 0 \) inside \( \Omega \).

Similarly, we approximate the derivatives of the function \( \phi \) by \( D_x, D_y \) and \( D_t \). Here,

\[
\begin{align*}
(D_x)_{m,n,j; m,n,j} &= -\frac{1}{dx} & (D_x)_{m,n;j; m+1,n,j} &= -\frac{1}{dx} & 1 \leq m \leq N - 1 \\
(D_y)_{m,n,j; m,n,j} &= -\frac{1}{dx} & (D_y)_{m,n;j; m+1,n,j} &= -\frac{1}{dx} & 1 \leq n \leq N - 1 \\
(D_t)_{m,n,j; m,n,j} &= -\frac{1}{dt} & (D_t)_{m,n;j; m+1,n,j} &= -\frac{1}{dt} & 1 \leq j \leq N_t - 1
\end{align*}
\]

Other entries of \( D_x, D_y, D_t \) are 0.

It is convenient to identify the 6–order tensors \( \mathcal{L}, \mathcal{D} \) and \( \mathcal{N} \) above by matrices by assigning the triple index \( m, n, j \) the single index

\[ i = (m - 1)NN_t + (n - 1)N_t + j. \]

By this, instead of approximating a function \( \phi \) by a 3–order tensor \( (\phi(x_m, t_j)) \), we consider function \( \phi \) by the vector \( \phi(x_i) \). This “line up” technique is employed because operators on multi-dimensional tensors are not supported in Matlab.

Hence a function \( w \) satisfies equation \((3.13), (3.15)\) and \((3.17)\) if and only if

\[ Cw = b \tag{6.6} \]

where

\[
C = \begin{bmatrix} D_t \\ D \\ \mathcal{N} \\ \mathcal{L} \end{bmatrix}, \quad b = \begin{bmatrix} \zeta \\ \zeta \\ \zeta \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{0} \text{ is the zero vector in } \mathbb{R}^{N^2N_t}.
\]

Since the data \( F \) and \( G \) are noisy, \((6.6)\) might have no solution. Instead, we solve

\[ (C^TC + \varepsilon_1I)w + \varepsilon_2(D_x^TD_x + D_y^TD_y + D_t^TD_t)w = b. \tag{6.7} \]

In our computation \( \varepsilon_1 = 3 \cdot 10^{-3} \) and \( \varepsilon_2 = 1.5 \cdot 10^{-4} \).

**Remark 6.1.** Solving \((6.7)\) is somewhat equivalent to finding the regularized solution of \((3.13), (3.15)\) and \((3.17)\). In fact, the minimizer of \( J_\alpha \) is the solution of the equation \( D\mathcal{J}_\alpha w = 0 \) and \( D\mathcal{J}_\alpha w \) is actually \( L^T L + \alpha L^T I \) where \( |Iw| \) gives the \( H^3 \) norm of \( w \). Here are some differences of the numerical implementation from the analysis in Sections 4 and 5:

1. A small difference of solving \((6.7)\) and minimizing \( J_\alpha \) is that we reduce the \( H^3 \) norm to be \( H^1 \) norm in the regularization term because this regularization term is already provide good reconstruction results.

2. In theory, we set the regularization parameter by a single number \( \alpha \) but in computation, we choose the regularization parameters as \( \varepsilon_1 \) and \( \varepsilon_2 \) above by a trial and error procedure.

3. The more important fact is that since \((6.7)\) is uniquely solvable, it is not necessary to verify the condition \( K \) is non-empty. See Remark 4.1.
6.4 Numerical examples

To illustrate the efficiency of the method, we display here some numerical results. Set

\[ h(x, t) = 1 + \exp(-(4 + |x|^2)t) \quad (x, t) \in \mathbb{R}^d \times [0, T]. \]

We test four (4) models, described below. In Figures 1–4 the notation \( p_{\text{comp}} \) indicates the computed source functions.

**Test 1.** The true function \( p^*(x) \) is given by

\[
 p^*(x) = \begin{cases} 
 1 & \text{if } 4(x - 0.15)^2 + y^2 / 8 \leq 0.1^2, \\
 -1 & \text{if } \max\{|x + 0.15|, |y|\} < 0.1, \\
 0 & \text{otherwise.} 
\end{cases}
\]

With this function, the source involves both negative and positive parts. The reconstructed results with various noise levels are displayed in Figure 1.

**Test 2.** The true function \( p^*(x) \) is given by

\[
 p^*(x) = \begin{cases} 
 1.5 & \text{if } (x - 0.25)^2 + y^2 < 0.12^2, \\
 1 & \text{if } 4x^2 + (y + 0.25)^2 < 0.15^2, \\
 -1 & \text{if } |x + 0.25| + |y| < 0.17, \\
 0 & \text{otherwise.} 
\end{cases}
\]

With this function, the source involves three “inclusions” with different values. The reconstructed results with various noise levels are displayed in Figure 2.

**Test 3.** The true function \( p^*(x) \) is a smooth function given by

\[
 p^*(x) = 3(1 - x)^2 \exp(-x^2 - (y + 1)^2) - 10 \left( \frac{x}{5} - x^3 - y^5 \right) \exp(-x^2 - y^2) - \frac{1}{3} \exp(-(x + 1)^2 - y^2)
\]

for \( x = (x, y) \in \Omega \). This function is named as “peaks” in Matlab, which has several local maxima and minima. The reconstructed results are displayed in Figure 3.

**Test 4.** We try a non-smooth function \( p^*(x) \) as the “negative” characteristic function of a letter “A” and the “positive” characteristic function of a letter L. It is interesting to see that our method recovers this step function very well, see Figure 4.

**Remark 6.2.** We observe that the higher noise level, the blurrier reconstruction is. However, the reconstructed value of the function is stable. This can be seen by the color bar in Figures 1–4. We list here the comparison of the minimum and maximum values of the true and reconstructed sources in Tables 1 as a part of the numerical proof for the stability of the reconstruction method.

7 Concluding remarks

We have established in this paper a robust numerical method to solve an inverse source problem for hyperbolic equations using the lateral Cauchy data. Our method consists of deriving an integro-differential equation involving a Volterra-like integral and then solving it by the well-known quasi-reversibility method. We have established the Lipschitz-like convergence rate of regularized solutions. This result is an extension of the known convergence result for quasi-reversibility method for the case when a pure
(a) The function $p_{\text{true}}$

(b) The function $p_{\text{comp}}$ when $\delta = 2\%$.

(c) The function $p_{\text{comp}}$ when $\delta = 5\%$.

(d) The function $p_{\text{comp}}$ when $\delta = 10\%$.

(e) The functions $p_{\text{comp}}$ (dash-dot) and $p_{\text{true}}$ (solid) on the line $y = 0$ when $\delta = 5\%$

(f) The functions $p_{\text{comp}}$ (dash-dot) and $p_{\text{true}}$ (solid) on the line $y = 0$ when $\delta = 10\%$

Figure 1: The true and reconstructed source functions in Test 1 where the data is noisy. It is evident that the shape and values of the source can be reconstructed with a high accuracy.
Figure 2: The true and reconstructed source functions in Test 2 where the data is noisy. Similarly to the case of Test 1, the shape and values of the source are reconstructed very well.
Figure 3: The true and reconstructed source functions in Test 3. The function $p^*$ has several local maxima and minima, which can be reconstructed efficiently.
Figure 4: The true and reconstructed source functions in Test 4 where the data is noisy. The reconstructions of nonconvex “inclusions” A and L are satisfactory even when $\delta = 10\%$. 

(a) The function $p_{\text{true}}$

(b) The function $p_{\text{comp}}$ when $\delta = 2\%$.

(c) The function $p_{\text{comp}}$ when $\delta = 5\%$.

(d) The function $p_{\text{comp}}$ when $\delta = 10\%$.

(e) The functions $p_{\text{comp}}$ (dash-dot) and $p_{\text{true}}$ (solid) on the line $y = 0$ when $\delta = 5\%$

(f) The functions $p_{\text{comp}}$ (dash-dot) and $p_{\text{true}}$ (solid) on the line $y = 0$ when $\delta = 10\%$. 

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Table 1: Correct and computed maximal and minimal values of source functions. $p_{\text{true}}$ means the correct source functions. $p_{\text{comp}}$ indicates the computed source functions. $\text{error}_{\text{rel}}$ denotes the relative error.

| Test id | noise level | $p_{\text{true}}$ | $p_{\text{comp}}$ (error$_{\text{rel}}$) | $p_{\text{true}}$ | $p_{\text{comp}}$ (error$_{\text{rel}}$) |
|---------|-------------|------------------|-----------------------------------|-----------------|-----------------------------------|
| 1       | 2%          | -2.0             | -1.99 (0.5%)                      | 2               | 2.00 (0.0%)                       |
| 2       | 2%          | -1.0             | -1.03 (3.0%)                      | 1.5             | 1.46 (2.7%)                       |
| 3       | 2%          | -6.5             | -5.84 (10.1%)                     | 8.1             | 7.36 (9.1%)                       |
| 4       | 2%          | -1.0             | -1.07 (7.0%)                      | 1               | 1.04 (4.0%)                       |
| 1       | 5%          | -2.0             | -2.14 (7.0%)                      | 2               | 2.14 (7.0%)                       |
| 2       | 5%          | -1.0             | -1.03 (3.0%)                      | 1.5             | 1.46 (2.7%)                       |
| 3       | 5%          | -6.5             | -5.53 (15.0%)                     | 8.1             | 7.36 (9.1%)                       |
| 4       | 5%          | -1.0             | -1.07 (7.0%)                      | 1               | 1.10 (10.0%)                      |
| 1       | 10%         | -2.0             | -2.33 (16.5%)                     | 2               | 2.24 (12.0%)                      |
| 2       | 10%         | -1.0             | -1.10 (10.0%)                     | 1.5             | 1.56 (4.0%)                       |
| 3       | 10%         | -6.5             | -6.24 (4.0%)                      | 8.1             | 7.30 (9.9%)                       |
| 4       | 10%         | -1.0             | -1.23 (23.0%)                     | 1               | 1.24 (24.0%)                      |

A hyperbolic partial differential equation is in place without a Volterra integral in it. A Carleman estimate is essential in the analysis. Accurate numerical results are obtained.

Studying the ISP in the cases when the data is measured only on a part of the boundary of the domain under consideration is reserved for a near future research. Moreover, also in the future research, we will extend this method to study ISPs for parabolic and elliptic equations.

As mentioned in Section 2, the governing equation for the ISP in this paper is the linearization of a coefficient inverse problem, which is highly nonlinear. The numerical method developed in this paper, therefore, can be used as a refinement step in solving that severely ill-posed and highly nonlinear problem. For example, one might refine numerical results obtained by the convexification globally convergent numerical method, see, e.g. [36, 37].

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