Abstract

We discuss the \( q \)-Virasoro algebra based on the arguments of the Noether currents in a two-dimensional massless fermion theory as well as in a three-dimensional nonrelativistic one. Some notes on the \( q \)-differential operator realization and the central extension are also included.
1 Introduction

Several years ago, much attention was paid to the study of $q$-deformation of the Virasoro algebra [1]-[3]. The $q$-deformed Virasoro algebra was first introduced by Curtright and Zachos (CZ) as a deformation for both commutators and structure constants [1]. Some other versions [2, 3] which can be transformed from the CZ deformation, conformal field theoretical analogues [4], matrix representation [5] and some approaches to discretized systems [6] have also been discussed. From the quantum group theoretical viewpoint [7], these developments of the CZ deformation are nothing more than analogies because a Hopf algebra structure has not been established for them.

Another type of deformed Virasoro algebra has been investigated lately [8]:

$$\left[ L_n^{(i)}, L_m^{(j)} \right] = \sum_{\epsilon=\pm1} C_{m\epsilon}^{n} i \epsilon^{i+\epsilon j} L_{n+m}^{(i+\epsilon j)}, \quad (1.1)$$

where the structure constants are

$$C_{m\epsilon}^{n} i \epsilon^{i+\epsilon j} = \frac{[n\epsilon-m\epsilon]}{2} q^{[(i+j)]q} \quad (1.2)$$

with

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (1.3)$$

This algebra is similar to the trigonometric algebra presented in [9], but is different from it as is pointed out in reference [11]. In this paper, we present some formulae based on this algebra from various points of view. We refer to this algebra as a (fermionic) $q$-Virasoro algebra. It possesses a Hopf algebra structure, which is a cocommutative one. However, its operator representation satisfies the quantum algebra $U_q(sl(2))$ incorporating the Virasoro zero mode operator. The Hopf algebra structure was found by introducing an additional set of generator indices [8] into the differential operator representation of the CZ deformation. In this sense, the differential operator representation plays an important role in deriving the situation. After this discovery, the oscillator representation [11] and the operator product representations [12, 13], relevance to a discretized Liouville model [9], were intensively developed. Further extensions of the $q$-Virasoro algebra have been considered: to the supersymmetric case [11] and to more general structure constants [10].
In spite of these successful developments, various unsolved questions remain about this algebra; what is the origin of this $q$-deformation or what physical situation embodies this algebra. As for its central extension, the general solution of the Jacobi identity constraint equation for central extension has yet to be found. We do not still know a unique supersymmetric algebra which includes ghost and superghost sectors. Moreover, is there any further nontrivial generalization which might lead us to a quantum Hopf algebra structure? These problems are important in dealing with physical situations utilizing the $q$-Virasoro algebra.

In this paper, we would like to focus our attention on the transformation properties of the $q$-Virasoro algebra from a point of the classical Noether current. In sect.2, starting from an analogy with the classical Noether current, we define a $q$-analogue of the canonical energy-momentum (EM) tensor. (Throughout this paper, we often refer to an analogous object deformed by the parameter $q$ as a $q$-object for short, e.g., $q$-EM tensor). In sect.3, applying the result to a two-dimensional chiral fermion theory, we define the Fourier mode operators of the $q$-EM tensor. It becomes clear that the mode operators coincide with the $q$-Virasoro generators. In this case, the $q$-EM tensor can be shown to be related to an analogue of conformal transformations. Next in sect. 4, from the point of a transformation law for a field, we discuss the relation between magnetic translations and $q$-conformal transformations in a nonrelativistic classical theory under constant magnetic field. In sect.5 and 6, some brief remarks on the differential operator associated with the $q$-conformal transformations are added. In the appendix, we describe a method for obtaining the central extension of the $q$-Virasoro algebra through the Jacobi identities.

2 $q$-Noether current

In this section, we discuss an analogue of the classical Noether current in conformity with the standard Noether current in order to give a hint to consider the origin or the reason why the $q$-Virasoro generators take a particular form as in $[1,2]$. Although we often refer to the commutator algebra $[1,1]$ (including central extensions), there arises
no confusion if we keep in mind the correspondence between the Poisson bracket and commutator.

In field theories (for any field $\phi_i$), a conserved current comes from the invariance of the action

$$
\delta S = \int \partial_\mu (\frac{\delta L}{\delta (\partial_\mu \phi_i)} \delta \phi_i - L_\epsilon \epsilon^\mu) d^D x,
$$

(2.1)

under an infinitesimal transformation

$$
x'^\mu = x^\mu - \epsilon^\mu(x),
$$

(2.2)

where $\delta \phi_i$ is the Lie derivative. If we require the invariance of $\phi_i$ under the transformation, i.e. $\phi'_i(x') = \phi_i(x)$, the Lie derivative can be written in the following form

$$
\delta \phi_i(x) = \phi'_i(x) - \phi(x) = \epsilon^\mu \partial_\mu \phi_i(x) + O(\epsilon^2),
$$

(2.3)

and we get the conserved current

$$
J^\mu = (\frac{\delta L}{\delta (\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta_\nu^\mu L) \epsilon^\nu = T^\mu_{\nu} \epsilon^\nu.
$$

(2.4)

The canonical EM tensor $T^\mu_{\nu}$ reflects the translational invariance of the system. In particular, the EM tensor of the conformal field theories corresponds to the generators of the Virasoro algebra and satisfies the conservation law

$$
\partial_z T(z) = \partial_{\bar{z}} \bar{T}(\bar{z}) = 0, \quad T_{zz} = T_{\bar{z}\bar{z}} = 0,
$$

(2.5)

where $T(z) = T_{zz}$ and $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$. Namely we obtain the generator

$$
L_n = \frac{1}{2\pi i} \oint dzz^{n+1} T(z),
$$

(2.6)

which satisfies

$$
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}.
$$

(2.7)

When constructing a $q$-analogue of some quantity, the invariance under the inversion $q \rightarrow q^{-1}$ may be useful (hereafter in this section, we denote $q$ by $Q$ ). We then insert $Q$ into the Lagrangian so that the action should be invariant under the replacement
$Q \to Q^{-1}$. Furthermore we assume that $Q$ is not related to the dynamics, i.e., the action should not depend on $Q$:

$$S = S(Q) = S(Q^{-1}), \quad \frac{dS}{dQ} = 0. \quad (2.8)$$

One of the possible ways to introduce the parameter $Q$ is in the following change of variables in the integrand,

$$S(Q) = Q^D \int \mathcal{L}(\phi, \partial \phi; xQ)d^Dx, \quad (2.9)$$

and

$$S(Q^{-1}) = Q^{-D} \int \mathcal{L}(\phi, \partial \phi; xQ^{-1})d^Dx. \quad (2.10)$$

The above requirements \((2.8)-(2.10)\) may be reinterpreted as follows. Hereafter we abbreviate the index $i$ of fields. The invariance of $S$ under $Q \to Q^{-1}$ can be understood as the invariance under the dilatation $xQ \to xQ^{-1}$. Namely, this dilatation determines the $\epsilon^\mu(x)$ defined in \((2.2)\) under the transformation $x \to xQ^{-2}$

$$\epsilon^\mu(x) = x^\mu(1 - Q^{-2}). \quad (2.11)$$

The invariance of $\phi$ under this transformation reads $\phi'(x) = Q^{2\Delta} \phi(xQ^2)$, where $\Delta$ means the canonical dimension of the field. If $\phi(x)$ is a regular function of $x$, $\phi(xQ)$ can be expressed as an infinite series in the derivatives of $\phi(x)$

$$\phi(xQ) = Q^{x\partial} \phi(x). \quad (2.12)$$

The Lie derivative of $\phi(x)$ is thus exactly written in the following form

$$\delta \phi(x) = Q^2 D^\Delta_\mu \phi(xQ) \epsilon^\mu(x), \quad (2.13)$$

where $D^\Delta_\mu$ is the $Q$-derivative defined by

$$(D^{(\Delta)}f)(ax) = \frac{1}{ax} Q^{\Delta + x\partial} - Q^{-\Delta - x\partial} f(ax). \quad (2.14)$$

If $Q$ is infinitesimally deviated from the unity, \((2.11)\) becomes an infinitesimal quantity and we get the following analogue of the conserved current,

$$J^\mu_Q = \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} Q^2 D^\Delta_\nu \phi(xQ) - \delta^\mu_\nu \mathcal{L} \right) \epsilon^\nu. \quad (2.15)$$
It should be noted here that when $q = 1$, (2.15) coincides with the dilatation current

$$j^\mu = \frac{\delta \mathcal{L}}{\delta (\partial^\mu \phi)} (\Delta + x^\nu \partial^\nu) \phi - x^\mu \mathcal{L}$$

and can be written in the form $x^\nu T^\mu_\nu + \Delta AB^\mu C$ where $B^\mu$ is $\partial^\mu$ (or $\gamma^\mu$) for a boson (fermion) field. Thus, the canonical EM tensor can be obtained by putting $\Delta = 0$ and dropping $x^\nu$ out from $j^\nu$. In the same way, we define an analogous ”EM tensor” from (2.15) by putting $\Delta = 0$ and dropping $\epsilon^\nu$,

$$J^\mu_\nu(x) = Q^2 \frac{\delta \mathcal{L}}{\delta (\partial^\mu \phi(x))} D^0_\nu \phi(xQ) - \delta^\mu_\nu \mathcal{L}. \quad (2.16)$$

This may be interpreted as a $q$-analogue of the EM tensor because it becomes the canonical EM tensor in the limit of $Q \to 1$. In general, (2.15) is no longer a conserved current in the case of $Q$ being finitely deviated from unity. On the other hand in conformal field theories, this current is trivially conserved because of the chiral decomposition of the theories. In spite of the triviality of conservation, the $q$-EM tensor plays an important role in generating deformed Virasoro algebras. For example, in the case of a massless fermion which has the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \bar{\psi} \partial \psi + \frac{1}{2} \psi \partial \bar{\psi}, \quad (2.17)$$

the components of $J_{\mu\nu}$ are as follows;

$$J_{zz} = J_{\bar{z}\bar{z}} = 0, \quad (2.18)$$

$$J(z) = J_{zz} = -\frac{1}{2} Q^2 \psi(z) D_z \psi(zQ), \quad (2.19)$$

$$\bar{J}(\bar{z}) = J_{\bar{z}\bar{z}} = -\frac{1}{2} Q^2 \bar{\psi}(\bar{z}) D_{\bar{z}} \bar{\psi}(\bar{z}Q). \quad (2.20)$$

The first equation of the above shows that $J_{\mu\nu}$ is traceless $J^\mu_\mu = 0$. The others indicate that $J_{zz}$ ($J_{\bar{z}\bar{z}}$) is a function depending only on $z$ ($\bar{z}$) and the conservation law of $J_{\mu\nu}$ is

$$\partial_z J(z) = \partial_{\bar{z}} \bar{J}(\bar{z}) = 0. \quad (2.21)$$

The $J(z)$ becomes $T(z)$ defined in (2.18) in the limit $q \to 1$ and the Fourier mode of $J(z)$ satisfies the $q$-Virasoro algebra (1.1) as well as its classical Poisson bracket algebra. This will be clear soon in the heading of the next section.
3 2-dimensional fermion current

Now let us define the Fourier mode expansion of the $q$-EM tensor (2.19)

$$L_n^{(k)} = \frac{1}{2\pi i} \oint dzz^{n+1} J^{(k)}(z)$$

in which

$$J^{(k)}(z) = \frac{2Q^{-2}}{Q + Q^{-1}} J(zQ^{-1}) \quad \text{with} \quad Q = q^{k/2} \quad (k \in \mathbb{Z})$$

Taking into account normal ordering in the quantum situation, the operator (3.1) satisfies the following centrally extended algebra [9] [12]

$$[L_n^{(i)}, L_m^{(j)}] = \sum_{\epsilon = \pm 1} C_{ij}^{(n)} \delta_n^{(i+\epsilon j)} + \frac{1}{2} C_{ij}^{(n)} \delta_n^{(m)} ,$$

with

$$C_{ij}^{(n)} = \frac{1}{[i]_q [j]_q} \sum_{k=1}^{n} \frac{(n + 1 - 2k)i}{2} \frac{(n + 1 - 2k)j}{2} \cdot$$

It is obvious that we can easily reduce the above statement to the classical situation by omitting the central term at any time.

Next, we show that our $q$-EM tensor $J(z)$ is related to an analogy of conformal transformations. The explicit expression for (3.1) is given in [12]

$$\frac{-1}{2\pi i} \oint dzz^n : \psi(zq^{-k/2}) q^{kz\partial - \frac{k}{2}z\partial} q^{k - q^{-k}} :$$

This can be rewritten as

$$\frac{-1}{2\pi i} \oint dzz^n \psi(zq^{k(n+1)/2}) q^{kz\partial - \frac{k}{2}z\partial} :$$

The factor $q^{k(n+1)/2}$ comes from the scaling of $z$. Furthermore, $L_n^{(k)}$ has the following property

$$L_n^{(k)} = L_n^{(-k)} ,$$

and thus (3.6) can be symmetrized on the upper index $k$

$$\frac{1}{2\pi i} \oint d\bar{z} \frac{1}{2} : \psi(z) z^n \frac{[kz\partial + k(n + 1)/2]}{[k]_q} \psi(z) :$$
Going back to the Noether current argument, the conserved charge for this chiral fermion theory should be

\[ Q_{\text{q-Vir}} = \frac{1}{2\pi i} \oint dz : \frac{\delta L}{\delta (\bar{\partial}\psi)} \delta \psi(z) : \]  

(3.9)

where \( \delta \psi \) means a Lie derivative for our certain particular transformation which might be called \( q \)-conformal transformations. Comparing RHSs between (3.8) and (3.9), we obtain the variation

\[ \delta \psi(z) = z^n \frac{[kz\bar{\partial} + k(n+1)/2]q}{[k]_q} \psi(z). \]  

(3.10)

This is nothing but a \( q \)-analogue of the conformal transformation \( \delta \psi = z^n(z\bar{\partial} + \frac{1}{2}(n+1))\psi \).

4 3-dimensional fermion current

A similar situation to the previous section exists in the following nonrelativistic fermion field theory in two-dimensional space under constant magnetic field

\[ \mathcal{L}_3 = i\Psi^\dagger \dot{\Psi} - \frac{1}{2} (D\Psi)^\dagger (D\Psi) \]  

(4.1)

where \( D_i = \partial_i - iA_i \). The equation of motion leads to the Schrödinger equation for the Landau motion. In this system, it is known that the usual translational invariance is modified into the so-called magnetic translational one \([16]\) defined by

\[ \Psi'(x, y) = exp(\epsilon b - \bar{\epsilon}b^\dagger)\Psi(x, y) = T(\epsilon, \bar{\epsilon})\Psi(x, y) , \]  

(4.2)

and thus

\[ \delta \Psi = (T_{\epsilon, \bar{\epsilon}} - 1)\Psi(x, y) \]  

(4.3)

where \( b \) and \( b^\dagger \) are the harmonic oscillators which commute with the Hamiltonian. In the gauge \( A = (-y/2, x/2) \),

\[ b = \frac{1}{2} \tilde{w} + \partial_w , \quad b^\dagger = \frac{1}{2} w - \bar{\partial}_w \quad w = x + iy \]  

(4.4)

Instead of (4.3), let us define a new transformation which is composed of the difference between two magnetic translations

\[ \delta \Psi = \hat{L}_n^{(k)}(w, \bar{w})\Psi = \frac{T_{(\epsilon, \bar{\epsilon})} - T_{(-\epsilon, \bar{\epsilon})}}{q^k - q^{-k}} \Psi. \]  

(4.5)
The classical conserved current and charges for this transformation are given by

\[ J_\mu = \frac{\delta L_3}{\delta (\partial_\mu \Psi)} \delta \Psi - \delta_0^\mu \mathcal{L}_3, \]  

(4.6)

\[ \mathcal{L}^{(k)}_n = \int d^2 w \Psi^\dagger(w, t) \frac{T_{(k, n\bar{\epsilon})} - T_{(-k, n\bar{\epsilon})}}{q^k - q^{-k}} \Psi(w, t). \]  

(4.7)

Eq. (4.7) satisfies the \( q \)-Virasoro algebra [18].

In order to see the similarity between (3.8) and (4.7), let us compare the transformation law (4.5) with the two-dimensional relativistic case (3.10) using dimensional reduction. We follow the method of ref. [17] to extract holomorphic parts from \( \hat{\mathcal{L}}^{(k)}_n(w, \bar{w}) \).

After moving all \( \bar{w} \) parts to the left of the \( w \) parts, we replace \( \bar{w} \rightarrow 2\partial \) and \( \bar{\partial} \rightarrow 0 \). For example,

\[ T_{(k, n\bar{\epsilon})} = \exp\left(\frac{k\epsilon}{2} \bar{w}\right) \exp(n\bar{\epsilon}\bar{\partial}) \exp\left(-\frac{n\bar{\epsilon}}{2} w\right) \exp(k\epsilon\partial) \rightarrow \exp(-n\bar{\epsilon}w/2) \exp(2k\epsilon\partial - \epsilon\bar{\epsilon}kn/2). \]

Further transformation is needed to see the coincidence with the form (3.10). Taking account of the coordinate transformation from a cylinder to the \( z \)-plane \( w = -\frac{2}{i} \ln z \) and of the parametrization \( q = e^{-\epsilon\bar{\epsilon}} \), we get a dimensionally reduced operator for \( \hat{\mathcal{L}}^{(k)}_n(w, \bar{w}) \) and so

\[ \delta \Psi \rightarrow z^n [kz\partial_z + nk/2]q^{[k]}_n \Psi. \]  

(4.8)

This expression is very similar to (3.10). This is a natural result judging from the fact that the theory (4.1) can be effectively described by a two-dimensional massless fermion theory [20].

In closing the section, we make a few remarks. First, the relations among \( \hat{\mathcal{L}}^{(j)}_0 \) can be easily found in this representation. All the \( \hat{\mathcal{L}}^{(j)}_0 \)’s are written as

\[ \hat{\mathcal{L}}^{(j)}_0 = \frac{k^j - k^{-j}}{q^j - q^{-j}}. \]  

(4.9)

Using this relation, we obtain

\[ \hat{\mathcal{L}}^{(2)}_0 = \frac{1}{[2]_q} \hat{\mathcal{L}}^{(1)}_0 \sqrt{4 + (q - q^{-1})^2 \hat{\mathcal{L}}^{(1)}_0} \]  

(4.10)

\[ \hat{\mathcal{L}}^{(3)}_0 = \frac{1}{[3]_q} \hat{\mathcal{L}}^{(1)}_0 \{ 1 + (q - q^{-1})[2]_q \hat{\mathcal{L}}^{(2)}_0 \}, \]  

(4.11)
and so on. Second, one may consider the dimensional reduction of other differential
operator algebras for the operators $T_{(k,n)}$ or $V^k_n$

$$T_{(k,n)} \rightarrow M^k_n \equiv z^n q^{k(z\partial + n/2)}$$ (4.12)

$$V^k_n = 2(T_{(k,n)} + T_{(-k,n)}) \frac{T_{(1,0)} - T_{(-1,0)}}{q - q^{-1}}$$

$$\rightarrow 2(M^k_n + M^{-k}_n)\frac{M^1_0 - M^{-1}_0}{q - q^{-1}}$$ (4.13)

where $T_{(k,l)}$ means $T_{(k\epsilon,l\bar{\epsilon})}$. The dimensionally reduced differential operators satisfy the
same algebras as those before reduction

$$[T_{(k,n)}, T_{(l,m)}] = (q^{\frac{mk-nl}{2}} - q^{-\frac{nl-mk}{2}})T_{(k+l,n+m)}$$ (4.14)

$$[V^j_m, V^k_n] = \sum_{\epsilon, \eta = \pm 1} C^m_n \epsilon k(\eta)V^{j+\epsilon k+\eta}_{m+n}, \quad C^m_n \epsilon k(r) = r \left[ \frac{r(m+j+r) - m(k+r)}{2} \right]_q.$$ (4.15)

The former is called the Moyal-sine algebra [19] and the latter the bosonic $q$-Virasoro
algebra [11]. However, this situation changes when they are put in a fermion bilinear
form. Once they are put into the fermion bilinear form like in (3.6), any operator $O(b, b^\dagger)$
is symmetrized as $O(b, b^\dagger) - O(-b, b^\dagger)$ because of the Grassmann property of the fermion
field. For example, the insertion of the above reduced operator (4.12) into the bilinear
form is equivalent to (3.6) up to some normalizations, and (3.6) satisfies not the original
Moyal-sine algebra (4.14) but the $q$-Virasoro algebra (3.3). Similarly, any other algebraic
relation composed of $T_{(k,n)}$ operators becomes different from the original algebra after
inserted in the bilinear integral. Only the (fermionic) $q$-Virasoro algebra (3.3) is preserved
in this reduction procedure in the bilinear form. In this sense, the appearance of the
fermionic $q$-Virasoro algebra in the 3-d system is nontrivial.

5 Differential operator algebra

Let us consider a little more about the differential operator in (4.8) (Although it is
slightly different from one in (3.10), the results after eq.(5.9) do not change).

$$\hat{L}^{(k)}_n = -\frac{1}{[k]} z^n [k(z\partial + \frac{n}{2})]$$ . (5.1)
This is known as a realization of the centerless $q$-Virasoro algebra \([1,1][8]\). It may be convenient to rewrite the operator \((5.1)\) as

$$\hat{L}_n^{(k)} = -\frac{1}{[k]} z^n \sum_{j=1}^{k} q^{(k-2j+1)(z\partial+\frac{n}{2})} [z\partial + \frac{n}{2}]. \quad (5.2)$$

Operating it on the basis

$$\hat{\phi}(z) = z^{-h}, \quad (h \geq 0) \quad (5.3)$$

it is obvious that

$$\hat{L}_n^{(k)} \hat{\phi}(z) = -z^n \frac{[k(h+\frac{n}{2})]}{[k]} \hat{\phi}(z). \quad (5.4)$$

Using this relation, we obtain the following formulae

$$\hat{L}_0^{(k)} \hat{\phi}(z) = \frac{[kh]}{[k]} \hat{\phi}(z) \quad (5.5)$$

and

$$\hat{L}_n^{(k)} \hat{\phi}(z) = \frac{[k(h+\frac{n}{2})]}{[k]} \hat{L}_1^{(1)} \hat{\phi}(z), \quad (5.6)$$

$$[\hat{L}_n^{(i)}, \hat{L}_m^{(j)}] \hat{\phi}(z) = \sum_{\epsilon = \pm 1} \frac{[nj-\epsilon mi]}{2} \frac{[(i+\epsilon j)(h-n-m+\frac{n}{2})]}{[i][j][h-n-m+\frac{n}{2}]} \hat{L}_{n+m}^{(1)} \hat{\phi}(z). \quad (5.7)$$

In order to write the analogy with conformal primary state vectors \([1,3]\), let us introduce the following operation rule of an arbitrary polynomial $D$ of $\hat{L}_n^{(k)}$ on the 'state'

$$D|\hat{\phi}(z)\rangle \equiv \lim_{z \to 0} \hat{\phi}(z)^{-1} D\hat{\phi}(z). \quad (5.8)$$

As a result of \((5.4)\) and \((5.3)\), we obtain

$$\hat{L}_0^{(k)} |\hat{\phi}\rangle = \frac{[kh]}{[k]} |\hat{\phi}\rangle \quad (5.9)$$

$$\hat{L}_n^{(k)} |\hat{\phi}\rangle = 0 \quad (n > 0), \quad (5.10)$$

which are similar to the definition of the Virasoro primary vectors \([1,3]\). Some other formulae for the above 'primary vectors' can be derived using \((1.1), (5.6)\) and \((5.7)\):

\(i\)

$$\hat{L}_n^{(k)} |\hat{\phi}\rangle = \frac{[k(h+\frac{n}{2})]}{[k]} \hat{L}_1^{(1)} |\hat{\phi}\rangle, \quad (5.11)$$
(ii) \( n > 0 \) and \( n + m \neq 0 \)

\[
\hat{L}_n^{(i)} \hat{L}_m^{(j)} |\hat{\phi}\rangle = \sum_{\epsilon = \pm 1} \frac{[nj-\epsilon mi]}{[i][j][h-n+\frac{m}{2}]} \hat{L}_{n+m}^{(1)} |\hat{\phi}\rangle, \quad (5.12)
\]

(iii) \( n > 0, m, l < 0, n + m > 0 \) and \( n + l > 0 \)

\[
\hat{L}_n^{(i)} \hat{L}_m^{(j)} \hat{L}_l^{(k)} |\hat{\phi}\rangle = \sum_{\epsilon, \eta = \pm 1} \frac{[nj-\epsilon mi]}{[i][j][k][h-n+\frac{m+l}{2}]} \hat{L}_{n+m+l}^{(1)} |\hat{\phi}\rangle. \quad (5.13)
\]

Although we do not write down further formulae for higher order of \( L_n^{(k)} \)

\[
|k_1 k_2 \cdots k_m; \hat{\phi}\rangle = \prod_{j=1}^{m} \hat{L}_{-n_j}^{(k_j)} |\hat{\phi}\rangle, \quad (5.14)
\]

it is clear that they can be also obtained straightforwardly. For example, the eigenvalue of (5.14) for \( L_0^{(k)} \) is given by

\[
\frac{1}{[k]} [k(h + \sum_{j=1}^{m} n_j)]. \quad (5.15)
\]

6 Primary fields

We rederive the formulae (5.9)-(5.15) presented in the previous section from the point of a field representation. Similarly to the Virasoro primary vectors, let us define the following primary vector

\[
|h\rangle = \lim_{z \to 0} \Phi(z) |0\rangle \quad (6.1)
\]

in which we introduce the vacuum vector defined by

\[
L_n^{(k)} |0\rangle = 0 \quad (n \geq -1). \quad (6.2)
\]

We assume that our primary field \( \Phi \) should satisfy the following commutator

\[
[L_n^{(k)}, \Phi(z)] = \frac{1}{[k]} z^n [k(z\partial + \frac{n}{2} + h)] \Phi(z) + A(n, h, k) z^n \Phi(z), \quad (6.3)
\]

where \( A(n, h, k) \) is an operator which satisfies the following conditions

\[
\lim_{q \to 1} A(n, h, k) = n(h - \frac{1}{2}), \quad (6.4)
\]

\[
A(n = 0, h, k) = A(n, h = \frac{1}{2}, k) = 0. \quad (6.5)
\]
It is obvious that the RHS of (6.3) becomes the usual commutator for the Virasoro primary field $z^n(z\partial + h(n+1))\Phi(z)$ in the limit $q \to 1$. If $h = 1/2$, (6.3) coincides with the case of a massless free fermion [12]

$$[L_n^{(k)}, \psi(z)] = \frac{1}{[k]} z^n[kz\partial + \frac{k}{2}(n+1)]\psi(z). \quad (6.6)$$

For other value of $h$, however at present, we have not found any realization of the same $q$-Virasoro generators even in bosonic field case. Under these assumptions, the second term on the RHS in (6.3) is not necessary for the derivation of the formulae (5.9)-(5.15) as will be shown below.

Now let us derive the formulae (5.9)-(5.15) from (6.3). First, the primarity conditions (5.9) and (5.10) can be verified from (6.3) as

$$L_0^{(k)}|h\rangle = \lim_{z \to 0}[L_0^{(k)}, \Phi(z)] = \left[\frac{kh}{[k]}\right]|h\rangle \quad (6.7)$$

$$L_n^{(k)}|h\rangle = \lim_{z \to 0}[L_n^{(k)}, \Phi(z)] = 0 \quad (n > 0). \quad (6.8)$$

Second, the formula (5.11) is obtained as follows. Rewriting (6.3) as

$$[L_n^{(k)}, \Phi(z)] = \frac{1}{[k]} \sum_{j=1}^k q^{(k-2j+1)(z\partial + \frac{h-n/2}{2})} \left([L_n^{(1)}, \Phi(z)] - A(n, h, 1)z^n\Phi(z)\right) + z^n A(n, h, k)\Phi(z),$$

and considering

$$\lim_{z \to 0}[L_n^{(k)}, \Phi(z)]|0\rangle = \frac{1}{[k]} \sum_{j=1}^k q^{(k-2j+1)(h-n/2)} \lim_{z \to 0}[L_n^{(1)}, \Phi(z)]|0\rangle,$$

we hence obtain

$$L_n^{(k)}|h\rangle = \left[\frac{k(n/2 - h)}{[k](n/2 - h)}\right] L_n^{(1)}|h\rangle. \quad (6.9)$$

Finally, the other formulae eqs.(5.12) and (5.13) follow from (1.1), (6.8) and (6.9)

$$L_n^{(i)}L_m^{(j)}|h\rangle = \sum_{\epsilon = \pm 1} \frac{[n-j-2m]}{[i][j][h - \frac{n+m}{2}]} [i + \epsilon j] (h - \frac{n+m}{2}) L_n^{(1)}|h\rangle \quad (n > 0, n + m \neq 0), \quad (6.10)$$

and so on.
As a simple application of the formula (6.9), we write a similar formula for the following commutation relation \[12\]

\[
[L_n^{(j)}, T^{(k)}(z)] = z^n \sum_{\epsilon = \pm 1} \frac{[k + \epsilon j]}{[j][k]} [\frac{j}{2}(z \partial + 2) + \frac{n}{2}(j + \epsilon k)]T^{(j+\epsilon k)}(z) + C_{jk}(n)z^{n-2} \tag{6.11}
\]

where

\[
T^{(k)}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n^{(k)}}{z^{n+2}}. \tag{6.12}
\]

Eq. (6.11) can be rewritten on the primary state using (6.9)

\[
[L_n^{(j)}, T^{(k)}(z)] |h\rangle = z^n \sum_{\epsilon = \pm 1} \frac{\frac{j}{2}(z \partial + 2) + \frac{n}{2}(j + \epsilon k)}{[j]} \frac{[(k + \epsilon j)(h + 1 + \frac{1}{2}z \partial)]}{[k(h + 1 + \frac{1}{2}z \partial)]} T^{(k)}(z) |h\rangle
\]

\[+ C_{jk}(n)z^{n-2} |h\rangle. \tag{6.13}\]

Integrating this formula, we can verify the algebra (3.3) on the state

\[
[L_n^{(i)}, L_m^{(j)}] |h\rangle = \frac{1}{2\pi i} \oint dz z^{m+1} [L_n^{(i)}, T^{(j)}(z)] |h\rangle = \sum_{\epsilon = \pm 1} \frac{\epsilon n j - \epsilon m i}{[i][\epsilon j]} L_n^{(i+\epsilon j)} |h\rangle + C_{ij}(n) |h\rangle.
\]

All the formula in this section are explicitly verified in the case of massless fermion with \( h = 1/2. \)

7 Conclusions

The \( q \)-Virasoro algebra may be understood as a natural extension of the quantum algebra \( U_q(sl(2)) \) \[21\] in spite of possessing an ordinary Hopf algebra structure and thus it is different from the \( W_{1+\infty} \) algebra \[22\] in this point. We have discussed various features of it in fermion systems and in the differential operator realization starting from deforming the canonical EM tensor.

We have shown that the \( q \)-EM tensor defined in sect.2 generates the \( q \)-Virasoro algebra in two-dimensional chiral fermion theory. However, we would like to mention here that there remain a few points which should be investigated as further problems. First, the \( q \)-EM tensor for higher dimensions is not conserved in the sense either of ordinary divergence
equation or of \(q\)-derivative equation. Second, we have not found any relevance of the \(q\)-EM tensor to the bosonic \(q\)-Virasoro algebra \([4,13]\) \([9,10,11]\) in contrast with the success for the fermionic case

\[
J(z) = Q^2 \partial \phi(z) D_z \phi(zQ), \quad (7.1)
\]

\[
\bar{J}(\bar{z}) = Q^2 \partial \bar{\phi}(\bar{z}) D_{\bar{z}} \bar{\phi}(\bar{z}Q). \quad (7.2)
\]

This situation might be related to the fact that an isomorphism between the fermionic \(q\)-Virasoro algebra and the bosonic one has not been found. From this point of view, improvement of the \(q\)-EM tensor of this paper must be an interesting topic as a further work.

We have attempted to formulate the \(q\)-Virasoro algebra as an expression of invariance under a deformed transformation, which is called the \(q\)-conformal transformation. We have considered the deformation of a conserved current in an undeformed field theory. In distinction to this case, there may exist a more fundamental approach for analyzing a conserved current after constructing a deformed field theory as well. Finally, we expect that our analysis may point the way to a more suitable formulation of the \(q\)-conformal transformation law of a fermion/boson field and to a clarification of the meaning of the \(q\)-Virasoro algebra.

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A Central extension

Although the central extension was found in [11] taking account of normal ordering of fermion oscillators, we present another method to determine the central extension through solving the Jacobi identity conditions. The central extension should be found from the Jacobi identity;

\[
\sum_{\epsilon = \pm 1} C^m_l \epsilon l C(n, m + l; i, j + \epsilon k) + \text{cycl perms.} = 0, \tag{A.1}
\]

where both indices \((i, j, k)\) and \((n, m, l)\) should be permuted simultaneously. \(C(n, m; i, j)\) is defined by

\[
[L^{(i)}_n, L^{(j)}_m] = \sum_{\epsilon = \pm 1} C^m_n \epsilon l C(n, m + l; i, (a + \epsilon)j) + C(n, m; i, j), \tag{A.2}
\]

and it satisfies the following relations

\[
C(n, m; i, j) = C(n, m; |i|, |j|), \tag{A.3}
\]

\[
C(n, m; i, j) = -C(m, n; j, i). \tag{A.4}
\]

Although eq.\((A.1)\) has the solution

\[
C(n, m; i, j) = C^m_j + C^m_{-j},
\]

it is trivially eliminated by the translation \(L^{(i)}_n \rightarrow L^{(i)}_n - 1\). As it is difficult to find the general solution of eq.\((A.1)\), we impose the condition

\[
C(n, 0; i, j) = C(1, -1; i, j) = 0. \tag{A.5}
\]

The above conditions reflect those satisfied by the ordinary Virasoro algebra. In order to find a solution of \((A.1)\), we fix some redundant variables to be \(n = -l - 1, m = 1, k = i\) and \(j = ai\) \((a\) is an integer). Using \((A.3)\) and \((A.4)\), eq.\((A.1)\) becomes

\[
\sum_{\epsilon} \{C^1_l ai C(-l - 1, l + 1; i, (a + \epsilon)i) + C^{-l - 1}_l ai C(l, -l; i, (a + \epsilon)i)\} = 0. \tag{A.6}
\]
Assuming
\[ C(n, m; i, j) = C(n, m; j, i), \]  
(A.7)
we can immediately find the simplest solution putting \( a = 1 \) in (A.6),
\[ C^l_i \delta(l + 1, -l - 1; 2i, 2i) = C^l_i \delta(l, -l; 2i, 2i). \]  
(A.8)
The solution of this equation is
\[ C(n, m; i, 2i) = \left[ \frac{i/2}{i/2} \right] \hat{C}(i) \delta_{n+m,0}. \]  
(A.9)
The \( \hat{C}(i) \) is \( C(2, -2; i, 2i) \) and corresponds to the central charge \( c/2 \) in the limit \( q \to 1 \).

We can adjust the normalization as in (3.4) choosing
\[ \hat{C}(i) = \left[ \frac{i/2}{2i} \right] 2c \]  
(A.10)
or making use of null state condition for the conformal weight \( 1/2 \) state [12].

All other solutions should be recursively determined from (A.6) as follows. First fix the values of \( C(n, -n; j, j) \) and \( C(n, -n; 2j, j) \) as the initial conditions of the recursive equations. Then all of the \( C(n, -n; i + 2j, j) \) can be determined after \( C(n, -n; i, j) \) \((1 \leq i \leq 2j)\) are obtained. Hence, all the solutions of (A.6) will be found recursively.

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