The rate of decay of the Wiener sausage in local Dirichlet space

Melanie Pivarski
(work with Lee Gibson)

July 26, 2012
Consider the path of a Brownian motion. The Wiener sausage for this path up to time $t$ is the set of all points within distance $\epsilon$ of the Brownian motion path.
**Question:** How does the negative exponential moment (with parameter $\nu$) of the Wiener sausage behave?

**Answer:** Donsker and Varadhan (1975): In $\mathbb{R}^d$ it is log asymptotically equivalent to $-k(\nu, d)t^{d/(d+2)}$, for a known value of $k(\nu, d)$:

$$
\lim_{t \to \infty} \frac{1}{t^{d/(d+2)}} \log \mathbb{E}^x [\exp (-\nu \mu (C_t^\epsilon))] = -k(\nu, d)
$$

$C_t^\epsilon$ is the union of the radius $\epsilon$ balls centered at the points of the sample path up to time $t$. 

Note: Discrete versions exist for $\mathbb{Z}^d$ (Donsker and Varadhan), Cayley graphs for finitely generated groups with polynomial volume growth (Erschler), and non-transitive graphs with assumptions on volume and heat kernels (Gibson).
**Question:** How does the negative exponential moment (with parameter $\nu$) of the Wiener sausage behave?

**Answer:** Donsker and Varadhan (1975): In $\mathbb{R}^d$ it is log asymptotically equivalent to $-k(\nu, d)t^{d/(d+2)}$, for a known value of $k(\nu, d)$:

$$\lim_{t \to \infty} \frac{1}{t^{d/(d+2)}} \log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C_t^\epsilon \right) \right) \right] = -k(\nu, d)$$

$C_t^\epsilon$ is the union of the radius $\epsilon$ balls centered at the points of the sample path up to time $t$.

Note: Discrete versions exist for $\mathbb{Z}^d$ (Donsker and Varadhan), Cayley graphs for finitely generated groups with polynomial volume growth (Erschler), and non-transitive graphs with assumptions on volume and heat kernels (Gibson).
Sznitman’s Method of Enlargement of Obstacles (1990’s) gives the same result for $\mathbb{R}^d$.

**Idea:** Avoid obstacles placed via Poisson process in $X$ with rate function $\nu d\mu$

Averaging over all realizations of the point process,

$$\mathbb{E}^x [\exp(-\nu \mu(C_s^\epsilon))] = \mathbb{E}^\nu [\mathbb{P}^x [T > s]]$$

where $T$ is the hitting time of the diffusion to the obstacle set $\{B(x_i, \epsilon)\}_i$, and $\mathbb{E}^\nu$ is the expectation with respect to $\mathbb{P}^\nu$. 

Melanie Pivarski (work with Lee Gibson)
**Question:** Can we use the method of Enlargement of Obstacles to move from $\mathbb{R}^d$ to a more general space?

**Answer:** Yes!

1. We need some assumptions on the space.
2. These will give us an asymptotic bound.

Note: $f(t) \simeq g(t)$ means there are positive constants $c$ and $C$ for which $cg(t) \leq f(t) \leq Cg(t)$ for all sufficiently large $t$. 
1. Assumptions on the metric measure space \((X, \mu, d)\):

- \(X\) is complete
- The positive radon measure \(\mu\) is \(\sigma\)-finite.
- Local Dirichlet space (as described in Sturm)
  - Fixed regular, strongly local, symmetric Dirichlet form, \(\mathcal{E}\)
  - \(\text{Dom}(\mathcal{E})\) is on the real Hilbert space \(L^2(X, \mu)\) with norm
    \[
    \|f\|_2 = \sqrt{\int_X f^2 \, d\mu}.
    \]
  - Dirichlet form, \(\mathcal{E}\), has an associated nonnegative semi-definite self-adjoint operator, \(\Delta\), on \(L^2(X, \mu)\).
  - For all \(f \in \text{Dom}(\Delta)\), \(g \in \text{Dom}(\mathcal{E})\):
    \[
    \mathcal{E}(f, g) = \langle \Delta f, g \rangle = \int_X \Delta f(x) g(x) \, d\mu(x)
    \]
  - There exists an associated energy measure \(d\Gamma(f, f)\) acting on \(f \in \text{Dom}(\mathcal{E})\).
2. Uniform Poincaré inequality with parameter $\beta$

For $f \in \text{Dom}(\mathcal{E})$,

$$
\int_B |f(x) - f_B|^2 d\mu(x) \leq P_0 r^\beta \int_B \Delta f(x)f(x) d\mu(x)
$$

where $f_B$ is the average of $f$ over $B$, $B = B(z, r)$ and $z \in X$. $P_0$ is a constant that depends on the space, $X$, but is independent of $z$ and $r$. 
3. Heat kernel estimates with parameter $\beta$ (or $\beta$ and 2)

The heat kernel, $h_t(x, y)$, is the fundamental solution to the heat equation $\partial_t u = \Delta u$.

There exist constants $c, C$ such that for all $x, y \in X$ and $t > 0$ we have both

- **Gaussian-style upper estimate**

  $$h_t(x, y) \leq \frac{C}{V(x, t^{1/\beta})} \exp \left( -c \left( \frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right)$$

- **Gaussian-style lower estimate**

  $$h_t(x, y) \geq \frac{c}{V(x, t^{1/\beta})} \exp \left( -C \left( \frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right)$$

We can also have a mixed set of parameters: 2 for $t \in (0, 1)$ and $\beta$ for $t \in [1, \infty)$. 
4. Volume estimates for $V(x, r) := \mu(B(x, r))$

- **Growth:** For some $c > 0$ and all $r > 1$

  $$cr < \inf_x V(x, r) \leq \sup_x V(x, r) < \infty$$

- **Relative sizes of nearby balls:**

  $$\liminf_{r \to \infty} V(x, r)^{-1} \inf_{y \in B(x, \beta r \log V(x, r))} V(y, r) > 0$$

Note that such an estimate will hold whenever $V(x, r) \sim f(r)$. When we combine these with the other estimates, we have a volume doubling space:

$$V(x, 2r) \leq C_D V(x, r)$$
Examples of spaces satisfying assumptions:

- \((\beta = 2)\) Complete Riemannian manifolds with non-negative Ricci curvature that are Ahlfors regular \((V(x, r) \sim r^\gamma \text{ for some } \gamma > 0)\)
- \((2 \text{ for } t \in (0, 1) \text{ and } \beta > 2 \text{ for } t \in [1, \infty])\) Tube-like fractals
Theorem

Under assumptions 1-4, the sample path of the diffusion process starting from $x \in X$ satisfies:

$$\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C_t^\epsilon \mathcal{V}(x, t) \right) \right) \right] \sim V(x, t),$$

where $\nu > 0$ and $C_t^\epsilon$ is the union of the radius $\epsilon$ balls centered at the points of the sample path up to time $t$. 
Theorem

Under assumptions 1-4, the sample path of the diffusion process starting from $x \in X$ satisfies:

$$\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C^\varepsilon_{t^\beta} V(x, t) \right) \right) \right] \simeq V(x, t),$$

where $\nu > 0$ and $C^\varepsilon_t$ is the union of the radius $\varepsilon$ balls centered at the points of the sample path up to time $t$.

In $\mathbb{R}^d$ (dropping constants):

$$\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C^\varepsilon_{t^2 t^d} \right) \right) \right] \simeq t^d,$$
Theorem

Under assumptions 1-4, the sample path of the diffusion process starting from \( x \in X \) satisfies:

\[
\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C^\varepsilon_t \nu(x,t) \right) \right) \right] \simeq V(x, t),
\]

where \( \nu > 0 \) and \( C^\varepsilon_t \) is the union of the radius \( \varepsilon \) balls centered at the points of the sample path up to time \( t \).

In \( \mathbb{R}^d \) (dropping constants):

\[
\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C^\varepsilon_{t^2 t^d} \right) \right) \right] \simeq t^d,
\]

Setting \( s = t^{2+d} \) means \( t = s^{1/(2+d)} \):

\[
\log \mathbb{E}^x \left[ \exp \left( -\nu \mu \left( C^\varepsilon_s \right) \right) \right] \simeq s^{d/(2+d)}.
\]

This is Donsker and Varadhan’s original asymptotic.
Sketch of proof (lower bound):
Let \( s = t^\beta V(x, t) \). For \( t \) sufficiently large,

\[
\mathbb{E}^x[\exp(-\nu \mu(C^\epsilon_s))] \geq \exp((-\nu - c)V(x, t))
\]
Sketch of proof (lower bound):
Let $s = t^\beta V(x, t)$. For $t$ sufficiently large,

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \exp((-\nu - c)V(x, t))
$$

Let $\tau_{B(x,r)}$ be the exit time of the diffusion process starting from $x$ from the ball $B(x, r)$.

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \mathbb{E}^x[\exp(-\nu V(x, t)); C_s^\epsilon \subset B(x, t)] \\
\geq \exp(-\nu V(x, t)) \mathbb{P}^x[\tau_{B(x,t-\epsilon)} > s].
$$
Sketch of proof (lower bound):
Let $s = t^\beta V(x, t)$. For $t$ sufficiently large,

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \exp((-\nu - c)V(x, t))$$

Let $\tau_{B(x, r)}$ be the exit time of the diffusion process starting from $x$ from the ball $B(x, r)$.

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \mathbb{E}^x[\exp(-\nu V(x, t)); C_s^\epsilon \subset B(x, t)]$$

$$\geq \exp(-\nu V(x, t)) \mathbb{P}^x[\tau_{B(x, t-\epsilon)} > s].$$

Lemma: For some constants $c, C', A > 0$

$$\mathbb{P}^x[\tau_{B(x, t-\epsilon)} > s] \geq \frac{ce^{-C'(t-\epsilon)}}{2V(x, (t - \epsilon)^{1/\beta})} \exp(-s\lambda(B(x, At-A\epsilon))).$$
Sketch of proof (lower bound):
Let $s = t^\beta V(x, t)$. For $t$ sufficiently large,

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \exp((\nu - c) V(x, t))$$

Let $\tau_{B(x,r)}$ be the exit time of the diffusion process starting from $x$ from the ball $B(x, r)$.

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \mathbb{E}^x[\exp(-\nu V(x, t)); C_s^\epsilon \subset B(x, t)] \geq \exp(-\nu V(x, t)) \mathbb{P}^x[\tau_{B(x,t-\epsilon)} > s].$$

Lemma: For some constants $c, C', A > 0$

$$\mathbb{P}^x[\tau_{B(x,t-\epsilon)} > s] \geq \frac{ce^{-C'(t-\epsilon)}}{2V(x, (t - \epsilon)^{1/\beta})} \exp(-s\lambda(B(x, At-A\epsilon))).$$

Lemma: $\lambda(B(x, At - A\epsilon)) \geq c/t^\beta$. 

Melanie Pivarski (work with Lee Gibson)
Sketch of proof (lower bound):
Let $s = t^\beta V(x, t)$. For $t$ sufficiently large,

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \exp((-\nu - c) V(x, t))
$$

Let $\tau_{B(x,r)}$ be the exit time of the diffusion process starting from $x$ from the ball $B(x, r)$.

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \geq \mathbb{E}^x[\exp(-\nu V(x, t)); C_s^\epsilon \subset B(x, t)] \\
\geq \exp(-\nu V(x, t)) \mathbb{P}^x[\tau_{B(x,t-\epsilon)} > s].
$$

Lemma: For some constants $c, C', A > 0$

$$
\mathbb{P}^x[\tau_{B(x,t-\epsilon)} > s] \geq \frac{ce^{-C'(t-\epsilon)} \exp(-s \lambda(B(x, At-A\epsilon)))}{2V(x, (t - \epsilon)^{1/\beta})}.
$$

Lemma: $\lambda(B(x, At - A\epsilon)) \geq c/t^\beta$.
Use $V(x, t) \geq ct$. 
Sketch of proof (upper bound): Let \( s = t^\beta V(x, t) \):

\[
\mathbb{E}^x [\exp(-\nu \mu(C^\epsilon_s))] \leq \exp((-\nu - c) V(x, t))
\]

Enlargement of obstacles: (\( T \) the hitting time of obstacle set)

\[
\mathbb{E}^x [\exp(-\nu \mu(C^\epsilon_s))] = \mathbb{E}^\nu [\mathbb{P}^x [T > s]]
\]

Finish via eigenvalue estimates and volume comparisons.
Sketch of proof (upper bound): Let $s = t^\beta V(x, t)$:

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s))] \leq \exp((-\nu - c)V(x, t))
$$

Enlargement of obstacles: ($T$ the hitting time of obstacle set)

$$
\mathbb{E}^x[\exp(-\nu \mu(C_s))] = \mathbb{E}^\nu[\mathbb{P}^x[T > s]]
$$

$$
\mathbb{P}^x[T > s] \leq \mathbb{P}^x[T \wedge \tau_{B(x,s)} > s] + \mathbb{P}^x[\tau_{B(x,s)} < s].
$$
Sketch of proof (upper bound): Let \( s = t^\beta V(x, t) \):

\[
\mathbb{E}^x[\exp(-\nu \mu(C_s^\varepsilon))] \leq \exp((-\nu - c) V(x, t))
\]

Enlargement of obstacles: (\( T \) the hitting time of obstacle set)

\[
\mathbb{E}^x[\exp(-\nu \mu(C_s^\varepsilon))] = \mathbb{E}^\nu[\mathbb{P}^x[T > s]]
\]

\[
\mathbb{P}^x[T > s] \leq \mathbb{P}^x[T \wedge \tau_{B(x,s)} > s] + \mathbb{P}^x[\tau_{B(x,s)} < s].
\]

Lemma:

\[
\mathbb{P}^x[\tau_{B(x,s)} < s] \leq C \exp(-ct^\beta V(x, t)).
\]
Sketch of proof (upper bound): Let $s = t^\beta V(x, t)$:

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] \leq \exp((-\nu - c)V(x, t))$$

Enlargement of obstacles: ($T$ the hitting time of obstacle set)

$$\mathbb{E}^x[\exp(-\nu \mu(C_s^\epsilon))] = \mathbb{E}^\nu[\mathbb{P}^x[T > s]]$$

$$\mathbb{P}^x[T > s] \leq \mathbb{P}^x[T \land \tau_{B(x,s)} > s] + \mathbb{P}^x[\tau_{B(x,s)} < s].$$

Lemma:

$$\mathbb{P}^x[\tau_{B(x,s)} < s] \leq C \exp(-ct^\beta V(x, t)).$$

Lemma:

$$\mathbb{P}^x[T \land \tau_{B(x,s)} > s] \leq \frac{C \exp(-s\lambda(B_s^\omega)/8)}{V(x, (s/2)^{1/\beta})} \cdot \frac{cV(x, s)}{V(x, (1/2)^{\beta})}.$$
Sketch of proof (upper bound): Let $s = t^\beta V(x, t)$:

$$\mathbb{E}^x [\exp(-\nu \mu(C^\epsilon_s))] \leq \exp((-\nu - c) V(x, t))$$

Enlargement of obstacles: ($T$ the hitting time of obstacle set)

$$\mathbb{E}^x [\exp(-\nu \mu(C^\epsilon_s))] = \mathbb{E}^\nu [\mathbb{P}^x [T > s]]$$

$$\mathbb{P}^x [T > s] \leq \mathbb{P}^x [T \land \tau_B(x,s) > s] + \mathbb{P}^x [\tau_B(x,s) < s].$$

Lemma:

$$\mathbb{P}^x [\tau_B(x,s) < s] \leq C \exp(-ct^\beta V(x, t)).$$

Lemma:

$$\mathbb{P}^x [T \land \tau_B(x,s) > s] \leq \frac{C \exp \left( -s \lambda (B^\omega_s) / 8 \right)}{V(x, (s/2)^{1/\beta})} \frac{cV(x, s)}{V(x, (1/2)^{\beta})}.$$ 

Finish via eigenvalue estimates and volume comparisons.
Thank you!

An early draft version of this work can be found on the arXiv at: http://arxiv.org/abs/1007.4987
E-mail me for an updated version.
Highly Abridged Bibliography:

- Donsker, M. D., Varadhan, S. R. S. (1975). Asymptotics for the Wiener sausage. Comm. Pure Appl. Math., 28(4) 525–565.

- Sturm, K-T. (1998). Diffusion processes and heat kernels on metric spaces. Ann. Probab. 26 no. 1 1–55.

- Sturm, K-T. (1998). How to construct diffusion processes on metric spaces. Potential Analysis 8 no. 2 149–161.

- Sznitman, A.S. (1998). Brownian Motion, Obstacles, and Random Media, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg.