The Hall property $D_{\pi}$ is inherited by overgroups of $\pi$-Hall subgroups

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Abstract

Let $\pi$ be a set of primes. We say that a finite group $G$ is a $D_{\pi}$-group if the maximal $\pi$-subgroups of $G$ are conjugate. In this paper, we give an affirmative answer to Problem 17.44(b) from “Kourovka notebook”, namely we prove that in a $D_{\pi}$-group an overgroup of a $\pi$-Hall subgroup is always a $D_{\pi}$-group.

1 Introduction

Throughout $G$ is a finite group, and $\pi$ is a set of primes. We denote by $\pi'$ the set of all primes not in $\pi$, by $\pi(n)$ the set of all prime divisors of a positive
integer \( n \), given a group \( G \) we denote \( \pi(|G|) \) by \( \pi(G) \). A natural number \( n \) with \( \pi(n) \subseteq \pi \) is called a \( \pi \)-number, while a group \( G \) with \( \pi(G) \subseteq \pi \) is called a \( \pi \)-group. A subgroup \( H \) of \( G \) is called a \( \pi \)-Hall subgroup, if \( \pi(H) \subseteq \pi \) and \( \pi(|G : H|) \subseteq \pi' \), i.e. the order of \( H \) is a \( \pi \)-number and the index of \( H \) is a \( \pi' \)-number.

Following [8], we say that \( G \) satisfies \( \mathcal{E}_\pi \) (or briefly \( G \in \mathcal{E}_\pi \)), if \( G \) has a \( \pi \)-Hall subgroup. If \( G \) satisfies \( \mathcal{E}_\pi \) and every two \( \pi \)-Hall subgroups of \( G \) are conjugate, then we say that \( G \) satisfies \( \mathcal{C}_\pi \) (\( G \in \mathcal{C}_\pi \)). Finally, \( G \) satisfies \( \mathcal{D}_\pi \) (\( G \in \mathcal{D}_\pi \)), if \( G \) satisfies \( \mathcal{C}_\pi \) and every \( \pi \)-subgroup of \( G \) is included in a \( \pi \)-Hall subgroup of \( G \). Thus \( G \in \mathcal{D}_\pi \) if a complete analogue of the Sylow theorems for \( \pi \)-subgroups of \( G \) holds. Moreover, the Sylow theorems imply that \( G \in \mathcal{D}_\pi \) if and only if the maximal \( \pi \)-subgroups of \( G \) are conjugate.

A group \( G \) satisfying \( \mathcal{E}_\pi \) (or \( \mathcal{C}_\pi \), \( \mathcal{D}_\pi \)) is also called an \( \mathcal{E}_\pi \)-group (respectively, a \( \mathcal{C}_\pi \)-group, a \( \mathcal{D}_\pi \)-group). Given set \( \pi \) of primes we denote by \( \mathcal{E}_\pi \), \( \mathcal{C}_\pi \), and \( \mathcal{D}_\pi \) the classes of all finite \( \mathcal{E}_\pi \)-, \( \mathcal{C}_\pi \)-, and \( \mathcal{D}_\pi \)-groups, respectively.

In the paper, we solve the following problem from “Kourovka notebook” [16]:

**Problem 1.** [16, Problem 17.44(b)] In a \( \mathcal{D}_\pi \)-group, is an overgroup of a \( \pi \)-Hall subgroup always a \( \mathcal{D}_\pi \)-group?

The analogous problem for \( \mathcal{C}_\pi \)-property (see [16, Problem 17.44(a)]) was answered in the affirmative (cf. [26, 27]). An equivalent formulation to this statement is: *in a \( \mathcal{C}_\pi \)-group \( \pi \)-Hall subgroups are pronormal*. Recall that a subgroup \( H \) of a group \( G \) is said to be pronormal if, for every \( g \in G \), \( H \) and \( H^g \) are conjugate in \( \langle H, H^g \rangle \).

According to [25], we say that \( G \) satisfies \( \mathcal{U}_\pi \), if \( G \in \mathcal{C}_\pi \) and every overgroup of a \( \pi \)-Hall subgroup of \( G \) satisfies \( \mathcal{D}_\pi \). We denote also by \( \mathcal{U}_\pi \) the class of all finite groups satisfying \( \mathcal{U}_\pi \). Thus Problem 1 can be reformulated in the following way:

**Problem 2.** Is it true that \( \mathcal{D}_\pi = \mathcal{U}_\pi \)?

The following main theorem gives an affirmative answer to Problems 1 and 2.

**Theorem 1.** (Main theorem) Let \( \pi \) be a set of primes. Then \( \mathcal{D}_\pi = \mathcal{U}_\pi \). In other words, if \( G \) satisfies \( \mathcal{D}_\pi \) and \( H \) is a \( \pi \)-Hall subgroup of \( G \), then every subgroup \( M \) of \( G \) with \( H \leq M \) satisfies \( \mathcal{D}_\pi \).

One can formulate this statement by using the concept of strong pronormality. According to [25], a subgroup \( H \) of a group \( G \) is said to be strongly pronormal if, for every \( g \in G \) and \( K \leq H \), there exists \( x \in \langle H, K^g \rangle \) such that \( K^gx \leq H \). Theorem 1 is equivalent to the following
Theorem 2. Let \( \pi \) be a set of primes. In a \( D_\pi \)-group \( \pi \)-Hall subgroups are strongly pronormal.

In [18, Theorem 7.7], it was proven that \( G \) satisfies \( D_\pi \) if and only if each composition factor of \( G \) satisfies \( D_\pi \). Using this result, an analogous criterion for \( U_\pi \) is obtained in [23].

Theorem 3. [23, Theorem 2] A finite group \( G \) satisfies \( U_\pi \) if and only if each composition factor of \( G \) satisfies \( U_\pi \).

In order to solve [16, Problem 17.44(a)], the pronormality of Hall subgroup in finite simple groups was proven in [26]. The strong pronormality of Hall subgroups in finite simple groups together with Theorem 3 would imply the main theorem. However, M. Nesterov in [17] showed that \( \text{PSp}_{10}(7) \) contains a \( \{2,3\} \)-Hall subgroup that is not strongly pronormal.

Theorem 3 reduces Problem 1 to a similar problem for simple \( D_\pi \)-groups. All simple \( D_\pi \)-groups are known: in pure arithmetic terms, necessary and sufficient conditions for a simple group \( G \) to satisfy \( D_\pi \) can be found in [19].

It was proved in [23] that if \( G \) is an alternating group, a sporadic group or a group of Lie type in characteristic \( p \) \( \pi \), then \( G \) satisfies \( U_\pi \). An affirmative answer to Problem 1 in case \( 2 \in \pi \) is obtained in [13]. In this paper, we consider the remaining case of \( D_\pi \)-groups of Lie type in characteristic \( p \) with \( 2, p \notin \pi \).

2 Notation and preliminary results

All groups in the paper are assumed to be finite. Our notation is standard and agrees with that of [4] and [11]. By \( A : B \) and \( A . B \) we denote a split extension and an arbitrary extension of a group \( A \) by a group \( B \), respectively. Symbol \( A \times B \) denotes the direct product of \( A \) and \( B \). If \( G \) is a group and \( S \) is a permutation group, then \( G \wr S \) is the permutation wreath product of \( G \) and \( S \). We use notations \( H \leq G \) and \( H \trianglelefteq G \) instead of “\( H \) is a subgroup of \( G \)” and “\( H \) is a normal subgroup of \( G \)” respectively. For \( M \leq G \) we set \( M^G = \{ M^g \mid g \in G \} \). The subgroup generated by a subset \( M \) is denoted by \( \langle M \rangle \). The normalizer and the centralizer of \( H \) in \( G \) are denoted by \( N_G(H) \) and \( C_G(H) \), respectively, while \( Z(G) \) is the center of \( G \). The generalized Fitting subgroup of \( G \) is denoted by \( F^*(G) \). For a group \( G \), we denote by \( \text{Aut}(G) \) and \( \text{Out}(G) \) the automorphism group and the outer automorphism group, respectively. Denote a cyclic group of order \( n \) by \( n \), and an arbitrary solvable group of order \( n \) by \( \{n\} \). Recall that, for a group \( X \) and a prime \( t \), a \( t \)-rank \( m_t(X) \) is the maximal rank of elementary abelian \( t \)-subgroups of \( X \).
Throughout, $F_q$ is a finite field of order $q$ and characteristic $p$. By $\eta$ we always denote an element of the set $\{+, -\}$ and we use $\eta$ instead of $\eta_1$ as well. In order to make uniform statements and arguments, we use the following notations $GL_n^+(q) = GL_n(q)$, $GL_n^-(q) = GU_n(q)$, $SL_n^+(q) = SL_n(q)$, $SL_n^-(q) = SU_n(q)$, $PSL_n^+(q) = PSL_n(q)$, $PSL_n^-(q) = PSU_n(q)$, $E_6^+(q) = E_6(q)$, $E_6^-(q) = 2E_6(q)$. If $G$ is a group of Lie type, then by $W(G)$ we denote the Weyl group of $G$.

The integral part of a real number $x$ is denoted by $[x]$. For integers $n$ and $m$, we denote by $\gcd(n, m)$ and $\lcm(n, m)$ the greatest common divisor and the least common multiple, respectively. If $\pi$ is a set of primes, then $\min(\pi)$ is the smallest prime in $\pi$. If $n$ is a positive integer, then $n_\pi$ is the largest divisor $d$ of $n$ with $\pi(d) \subseteq \pi$. If $g$ is an element of a group then there are elements $g_\pi$ and $g_{\pi'}$ in $\langle g \rangle$ such that $g = g_\pi g_{\pi'}$ and $|g_\pi|$ is a $\pi$-number, while $|g_{\pi'}|$ is a $\pi'$-number.

If $r$ is an odd prime and $k$ is an integer not divisible by $r$, then $e(k, r)$ is the smallest positive integer $e$ with $k^e \equiv 1 \pmod{r}$. So, $e(k, r)$ is the multiplicative order of $k$ modulo $r$. In particular, if $e = e(k, r)$, then

$$e(k^a, r) = \frac{e}{\gcd(e, a)}.$$ 

For a natural number $e$ set

$$e^* = \begin{cases} 2e & \text{if } e \equiv 1 \pmod{2}, \\ e & \text{if } e \equiv 0 \pmod{4}, \\ e/2 & \text{if } e \equiv 2 \pmod{4}. \end{cases}$$

The next result may be found in [28].

**Lemma 1.** ([28], Lemmas 2.4 and 2.5) Let $r$ be an odd prime, $k$ an integer not divisible by $r$, and $m$ a positive integer. Denote $e(k, r)$ by $e$.

Then the following identities hold.

$$(k^m - 1)_r = \begin{cases} (k^e - 1)_r(m/e)_r & \text{if } e \text{ divides } m, \\ 1 & \text{if } e \text{ does not divide } m; \end{cases}$$

$$(k^m - (-1)^m)_r = \begin{cases} (k^{e^*} - (-1)^{e^*})_r(m/e^*)_r & \text{if } e^* \text{ divides } m, \\ 1 & \text{if } e^* \text{ does not divide } m. \end{cases}$$

$$\prod_{i=1}^{m}(k^i - 1)_r = (k^e - 1)_r^{[m/e]}([m/e]!)_r$$

$$\prod_{i=1}^{m}(k^i - (-1)^i)_r = (k^{e^*} - (-1)^{e^*})_r^{[m/e^*]}([m/e^*]!)_r.$$
In Lemma 2, we collect some known facts about $\pi$-Hall subgroups in finite groups.

**Lemma 2.** Let $G$ be a finite group, $A$ a normal subgroup of $G$.

(a) If $H$ is a $\pi$-Hall subgroup of $G$ then $H \cap A$ is a $\pi$-Hall subgroup of $A$ and $HA/A$ is a $\pi$-Hall subgroup of $G/A$. In particular, a normal subgroup and a homomorphic image of an $E_\pi$-group satisfy $E_\pi$. (see [8, Lemma 1])

(b) If $M/A$ is a $\pi$-subgroup of $G/A$, then there exists a $\pi$-subgroup $H$ of $G$ with $M = HA$. (see [1, Lemma 2.1])

(c) An extension of a $C_\pi$-group by a $C_\pi$-group satisfies $C_\pi$. (see [8, C1 and C2] or [25, Proposition 5.1])

(d) If $2 \not\in \pi$ then $E_\pi = C_\pi$. In particular, if $2 \not\in \pi$ then a group $G$ satisfies $E_\pi$ if and only if each composition factor of $G$ satisfies $E_\pi$. (see [3, Theorem A], [6, Theorem 2.3], [25, Theorem 5.4])

(e) If $G$ possesses a nilpotent $\pi$-Hall subgroup then $G$ satisfies $D_\pi$. (see [8, Theorem A], [7, Theorem 2.3], [25, Theorem 6.2])

(f) A group $G$ satisfies $D_\pi$ if and only if both $A$ and $G/A$ satisfy $D_\pi$. Equivalently, $G \in D_\pi$ if and only if each composition factor of $G$ satisfies $D_\pi$. (see [18, Theorem 7.7], [25, Collorary 6.7])

**Lemma 3.** (see [8, Theorem 3], [25, Theorem 6.9]) Let $S$ be a simple group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$. Suppose $2 \not\in \pi$ and $|\pi \cap \pi(S)| \geq 2$. Then $S$ satisfies $D_\pi$ if and only if the pair $(S, \pi)$ satisfies one of the Condition I-IV below.

**Condition I.** Let $p \in \pi$ and $\tau = (\pi \cap \pi(S)) \backslash \{p\}$. We say that $(S, \pi)$ satisfies Condition I if $\tau \subseteq \pi(q - 1)$ and every number from $\pi$ does not divide $|W(S)|$.

**Condition II.** Suppose that $S$ is not isomorphic to $2B_2(q), 2G_2(q), 2F_4(q)'$ and $p \not\in \pi$. Set $r = \min(\pi \cap \pi(S))$ and $\tau = (\pi \cap \pi(S)) \backslash \{r\}$. Denote by $\alpha$ the number $e(q, \tau)$. We say that $(S, \pi)$ satisfies Condition II if there exists $t \in \tau$ with $b = e(q, t) \neq a$ and one of the following holds.

(a) $S \cong A_{n-1}(q), a = r - 1, b = r, (q^{r-1} - 1) = r, \left\lceil \frac{n}{r} \right\rceil = \left\lceil \frac{n}{r} \right\rceil + 1$, and both $e(q, s) = b$ and $n < bs$ hold for every $s \in \tau$.

(b) $S \cong A_{n-1}(q), a = r - 1, b = r, (q^{r-1} - 1) = r, \left\lceil \frac{n}{r} \right\rceil = \left\lceil \frac{n}{r} \right\rceil + 1$, $n \equiv -1 \pmod r$ and both $e(q, s) = b$ and $n < bs$ hold for every $s \in \tau$. 

5
(c) $S \simeq 2A_{n-1}(q)$, $r \equiv 1 \pmod{4}$, $a = r - 1$, $b = 2r$, $(q^{r-1} - 1)_r = r$,
\[
\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right] \quad \text{and} \quad e(q, s) = b \quad \text{for every} \ s \in \tau.
\]

(d) $S \simeq 2A_{n-1}(q)$, $r \equiv 3 \pmod{4}$, $a = \frac{r - 1}{2}$, $b = 2r$, $(q^{r-1} - 1)_r = r$,
\[
\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{2} \right] \quad \text{and} \quad e(q, s) = b \quad \text{for every} \ s \in \tau.
\]

(e) $S \simeq 2A_{n-1}(q)$, $r \equiv 1 \pmod{4}$, $a = r - 1$, $b = 2r$, $(q^{r-1} - 1)_r = r$,
\[
\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{r} \right] + 1, \ n \equiv -1 \pmod{r} \quad \text{and} \quad e(q, s) = b \quad \text{for every} \ s \in \tau.
\]

(f) $S \simeq 2A_{n-1}(q)$, $r \equiv 3 \pmod{4}$, $a = \frac{r - 1}{2}$, $b = 2r$, $(q^{r-1} - 1)_r = r$,
\[
\left[ \frac{n}{r - 1} \right] = \left[ \frac{n}{2} \right] + 1, \ n \equiv -1 \pmod{r} \quad \text{and} \quad e(q, s) = b \quad \text{for every} \ s \in \tau.
\]

(g) $S \simeq 2D_n(q)$, $a \equiv 1 \pmod{2}$, $n = b = 2a$ and for every $s \in \tau$ either
\[e(q, s) = a \quad \text{or} \quad e(q, s) = b.
\]

(h) $S \simeq 2D_n(q)$, $b \equiv 1 \pmod{2}$, $n = a = 2b$ and for every $s \in \tau$ either
\[e(q, s) = a \quad \text{or} \quad e(q, s) = b.
\]

In cases (e)-(h), a $\pi$-Hall subgroup of $S \simeq 2D_n(q)$ is cyclic.

**Condition III.** Suppose that $S$ is not isomorphic to $2B_2(q), 2G_2(q), 2F_4(q)'$ and $p \notin \pi$. Set $r = \min(\pi \cap \pi(S))$ and $\tau = (\pi \cap \pi(S)) \setminus \{r\}$. Denote by $c$ the number $e(q, r)$. We say that $(S, \pi)$ satisfies Condition III if $e(q, t) = c$ for every $t \in \tau$ and one of the following holds.

(a) $S \simeq A_{n-1}(q)$ and $n < ct$ for every $t \in \tau$.

(b) $S \simeq 2A_{n-1}(q)$, $c \equiv 0 \pmod{4}$ and $n < ct$ for every $t \in \tau$.

(c) $S \simeq 2A_{n-1}(q)$, $c \equiv 2 \pmod{4}$ and $2n < ct$ for every $t \in \tau$.

(d) $S \simeq 2A_{n-1}(q)$, $c \equiv 1 \pmod{2}$ and $n < 2ct$ for every $t \in \tau$.

(e) $S$ is isomorphic to one of the groups $B_n(q)$, $C_n(q)$ or $2D_n(q)$, $c$ is even and $2n < ct$ for every $t \in \tau$.

(f) $S$ is isomorphic to one of the groups $B_n(q)$, $C_n(q)$ or $D_n(q)$, $c$ is odd and $n < ct$ for every $t \in \tau$.

(g) $S \simeq D_n(q)$, $c$ is even and $2n \leq ct$ for every $t \in \tau$. 

6
(h) $S \simeq 2D_n(q)$, $c$ is odd and $n \leq ct$ for every $t \in \tau$.

(i) $S \simeq 3D_4(q)$.

(j) $S \simeq E_6(q)$ and if $r = 3$ and $c = 1$ then $5, 13 \notin \tau$.

(k) $S \simeq 2E_6(q)$ and if $r = 3$ and $c = 2$ then $5, 13 \notin \tau$.

(l) $S \simeq E_7(q)$; if $r \neq 3$ and $c \neq 1$ then $5, 7, 13, 1R, \tau$.

(m) $S \simeq E_8(q)$; if $r \neq 3$ and $c \neq 2$ then $5, 7, 13, 1R, \tau$.

(n) $S \simeq G_2(q)$.

(o) $S \simeq F_4(q)$ and if $r = 3$ and $c = 1$ then $13 \notin \tau$.

Condition IV. We say that $(S, \pi)$ satisfies Condition IV if one of the following holds.

(a) $S \simeq 2B_2(2^{2m+1}), \pi \cap \pi(G)$ is contained in one of the sets $\pi(2^{2m+1} - 1), \pi(2^{2m+1} \pm 2^{m+1} + 1)$.

(b) $S \simeq 2G_2(3^{2m+1}), \pi \cap \pi(G)$ is contained in one of the sets $\pi(3^{2m+1} - 1) \setminus \{2\}, \pi(3^{2m+1} \pm 3^{m+1} + 1) \setminus \{2\}$.

(c) $S \simeq 2F_4(2^{2m+1}), \pi \cap \pi(G)$ is contained in one of the sets $\pi(2^{2(2m+1)} \pm 1), \pi(2^{2m+1} \pm 2^{m+1} + 1), \pi(2^{2(2m+1)} + 2^{3m-2} + 2^{m+1} - 1), \pi(2^{2(2m+1)} \pm 2^{3m+2} + 2^{2m+1} + 2^{m+1} - 1)$.

In the next three lemmas, we recall some preliminary results about $U_\pi$-property.

Lemma 4. [23, Theorem 4] If $G \in D_\pi$ is either an alternating group, or a sporadic simple group, or a simple group of Lie type in characteristic $p \in \pi$, then $G$ satisfies $U_\pi$.

Lemma 5. [13, Lemma 3] The following statements are equivalent.

(a) $D_\pi = U_\pi$.

(b) In every simple $D_\pi$-group $G$, all maximal subgroups containing a $\pi$-Hall subgroup of $G$ satisfy $D_\pi$.

Lemma 6. [13, Theorem 1] If $2 \in \pi$ then $D_\pi = U_\pi$.
In view of Lemma [3], we consider the case when \( \pi \) is a set of odd primes.

**Lemma 7.** [24, Theorem 1] Let \( G \) be a group of Lie type in characteristic \( p \). Suppose that \( 2, p \notin \pi \) and \( H \) is a \( \pi \)-Hall subgroup of \( G \). Set \( r = \min(\pi \cap \pi(G)) \) and \( \tau = (\pi \cap \pi(G)) \setminus \{r\} \). Then \( H \) has a normal abelian \( \tau \)-Hall subgroup.

In spite of Lemma [4], we say that a pair \( (G, \pi) \) satisfies (\[4\]) if

\[
\text{every } \pi\text{-subgroup of } G \text{ has a normal abelian } \tau\text{-Hall subgroup,}
\]

where \( r = \min(\pi \cap \pi(G)) \) and \( \tau = (\pi \cap \pi(G)) \setminus \{r\} \). 

(\[*\])

Suppose that \( G \in \mathcal{E}_\pi \) is a group of Lie type in characteristic \( p \) and \( 2, p \notin \pi \). If \( G \in \mathcal{D}_\pi \), then every \( \pi \)-subgroup of \( G \) is contained in a \( \pi \)-Hall subgroup of \( G \), and hence by Lemma [4] we have that \( (G, \pi) \) satisfies (\[4\]). The following lemma gives sufficient conditions for the validity of the converse statement.

**Lemma 8.** [24, Theorem 5] Let \( G \) be a group of Lie type with the base field \( \mathbb{F}_q \) of characteristic \( p \), and \( G \) is not isomorphic to \( 2B_2(q), 2G_2(q), 2F_4(q) \)'.

Suppose that \( 2, p \notin \pi \) and \( G \in \mathcal{E}_\pi \). Assume further that \( e(q, t) = e(q, s) \) for every \( t, s \in \pi \cap \pi(G) \). Then \( G \in \mathcal{D}_\pi \) if and only if \( (G, \pi) \) satisfies (\[4\]).

The next lemma says, when a simple group \( S \) satisfies \( \mathcal{E}_\pi \) and does not satisfy \( \mathcal{D}_\pi \).

**Lemma 9.** ([3, Theorem 1.1], [3, Theorem 6.14], [20, Lemmas 5-7]) Let \( S \) be a simple group. Suppose that \( 2 \notin \pi \) and \( S \in \mathcal{E}_\pi \setminus \mathcal{D}_\pi \). Set \( r = \min(\pi \cap \pi(S)) \) and \( \tau = (\pi \cap \pi(S)) \setminus \{r\} \). Then one of the following holds.

(I) \( S \cong O'N \) and \( \pi \cap \pi(S) = \{3, 5\} \).

(II) \( S \) is a group of Lie type with the base field \( \mathbb{F}_q \) of characteristic \( p \) and either (A) or (B) below is true:

(A) \( p \in \pi, \ p \) divides \( |W(S)| \), every \( t \in (\pi \cap \pi(S)) \setminus \{p\} \) divides \( q - 1 \) and does not divide \( |W(S)| \).

(B) \( p \notin \pi \) and one of (a)-(i) below holds.

(a) \( S \cong \text{PSL}_n(q), \ e(q, r) = r - 1, (q^{r-1} - 1)_r = r, \left[\frac{n}{r-1}\right] = \left[\frac{q^n}{r}\right] \) and for every \( t \in \tau \) we have \( e(q, t) = 1 \) and \( n < t \).

(b) \( S \cong \text{PSU}_n(q), r \equiv 1 \pmod{4}, e(q, r) = r - 1, (q^{r-1} - 1)_r = r, \left[\frac{n}{r-1}\right] = \left[\frac{q^n}{r}\right] \) and for every \( t \in \tau \) we have \( e(q, t) = 2 \) and \( n < t \).
(c) $S \cong \text{PSU}_n(q)$, $r \equiv 3 \pmod{4}$, $e(q, r) = \frac{r-1}{2}$, $(q^{r-1} - 1)_r = r$, $\left[ \frac{r}{q} \right] = \left[ \frac{2}{q} \right]$ and for every $t \in \tau$ we have $e(q, t) = 2$ and $n < t$.

(d) $S \cong E_6(q)$, $\pi \cap \pi(S) \subseteq \pi(q-1)$, $3, 13 \in \pi \cap \pi(S)$, $5 \notin \pi \cap \pi(S)$.

(e) $S \cong {E}_6(q)\, ^2$, $\pi \cap \pi(S) \subseteq \pi(q+1)$, $3, 13 \in \pi \cap \pi(S)$, $5 \notin \pi \cap \pi(S)$.

(f) $S \cong E_7(q)$, $\pi \cap \pi(S)$ is contained in one of the sets $\pi(q-1)$ or $\pi(q+1)$, $3, 13 \in \pi \cap \pi(S)$, $5, 7 \notin \pi \cap \pi(S)$.

(g) $S \cong E_8(q)$, $\pi \cap \pi(S)$ is contained in one of the sets $\pi(q-1)$ or $\pi(q+1)$, $3, 13 \in \pi \cap \pi(S)$, $5, 7 \notin \pi \cap \pi(S)$.

(h) $S \cong E_8(q)$, $\pi \cap \pi(S)$ is contained in one of the sets $\pi(q-1)$ or $\pi(q+1)$, $5, 31 \in \pi \cap \pi(S)$, $3, 7 \notin \pi \cap \pi(S)$.

(i) $S \cong F_4(q)$, $\pi \cap \pi(S)$ is contained in one of the sets $\pi(q-1)$ or $\pi(q+1)$, $3, 13 \in \pi \cap \pi(S)$.

Remark on Lemma 9. Consider simple classical groups in characteristic $p$. Suppose that $2, p \notin \pi$. If a simple classical group satisfies $\mathcal{E}_\pi$ and does not satisfy $\mathcal{D}_\pi$, then it must be linear or unitary by Lemma 9. Thus if $S \in \mathcal{E}_\pi$ is a simple orthogonal or sympletic group, then $S \in \mathcal{D}_\pi$. However, there are isomorphisms amongst the classical groups, and it may happen that a simple orthogonal or sympletic group $S \in \mathcal{E}_\pi$ is isomorphic to a linear or unitary group $S_1$ (see Proposition 2.9.1]. One can check in this case that $S_1 \in \mathcal{D}_\pi$. For instance, suppose that $S = \Omega^+_{n}(q) \cong \text{PSL}^±_n(q) = S_1$. Assume also that $S_1 \in \mathcal{E}_\pi \setminus \mathcal{D}_\pi$, and therefore $S_1$ satisfies one of items (I(B)a), (I(B)b), or (I(B)c) of Lemma 8. If $r = \min(\pi \cap \pi(S_1))$, we have that $r \leq n = 4$ (see Lemma 9 below) and so $r = 3$. Then $\left[ \frac{r}{q} \right] \neq \left[ \frac{2}{q} \right]$ and none of conditions (I(B)a), (I(B)b), or (I(B)c) holds. Hence we conclude that $S_1 = \text{PSL}^±_n(q) \in \mathcal{D}_\pi$. Thus if $S$ is a simple orthogonal or sympletic group in characteristic $p$, $\pi$ is a set of primes with $2, p \notin \pi$ and $S \in \mathcal{E}_\pi$, then $S \in \mathcal{D}_\pi$.

We consider $\text{GL}_n^\pi(q)$ as the set $\{(a_{ij}) \in \text{GL}_n(q^2) \mid (a_{ij})^\pi = ((a_{ij})^{-1})^\pi\}$, where $(a_{ij})^\pi = (a_{ji})^\pi$ is the transposed of $(a_{ij})$. In the following statement, we specify the structure of $\pi$-Hall subgroup in $\text{GL}_n^\pi(q)$ in case $2, p \notin \pi$ and $\text{GL}_n^\pi(q)$ does not satisfy $\mathcal{D}_\pi$.

Lemma 10. Let $G = \text{GL}_n^\pi(q)$, where $q = p^m$ and $p$ is a prime. Denote by $D$ the subgroup of all diagonal matrices in $G$, so that $D \cong (q - \eta)^n$, and by $P$ the subgroup of permutation matrices of $G$, so that $P \cong S_n$ and $P$ normalizes $D$. Suppose that $2, p \notin \pi$ and $G \in \mathcal{E}_\pi \setminus \mathcal{D}_\pi$. Set $r = \min(\pi \cap \pi(G))$ and $\tau = (\pi \cap \pi(G)) \setminus \{r\}$. Then the following statements hold.

(a) $r$ does not divide $q - \eta$, $\pi \subseteq \pi(q - \eta)$, and a $\pi$-Hall subgroup $T$ of $D$ is isomorphic to $(q - \eta)^n$.
(b) $P$ is a $\pi'$-group, a $\pi$-Hall subgroup of $P$ is nontrivial and coincides with a Sylow $r$-subgroup $R$ of $P$ and $R$ is elementary abelian of order $r^{[n/r]}$.

d) Consider the automorphism $\varphi : (a_{ij}) \mapsto (a_{ij}^p)$ of $G$. Then $\varphi$ normalizes $T$ and centralizes $R$. In particular, $\varphi$ normalizes $TR$.

e) Let $d = \left[ \frac{r}{p} \right]$ and $n = dr + k$. Then $C_{TR}(R) \simeq (q - \eta)^{d+k} \times R$ and $m_t(C_{TR}(R)) = d + k$ for every $t \in \pi$.

Proof. Since $G \in \mathcal{E}_\pi \setminus \mathcal{D}_\pi$ and $2 \notin \pi$, Lemma 2 implies that the unique nonabelian composition factor $S = \text{PSL}_n^q(q)$ of $G$ lies in $\mathcal{E}_\pi \setminus \mathcal{D}_\pi$. Since $p \notin \pi$, we obtain that $S$ satisfies one of items (II(B)a-II(B)c) of Lemma 3. Observe that $\pi(G) = \pi(S)$. Denote $e(q, r)$ by $e$.

(a) Since $D \simeq (q - \eta)^n$ is abelian, $D$ satisfies $\mathcal{D}_\pi$. Items (II(B)a-II(B)c) of Lemma 3 imply $\tau \subseteq \pi(q - \eta)$. It suffices, therefore, to prove that $r$ does not divide $q - \eta$.

If $\eta = +$, then $S = \text{PSL}_n(q)$ satisfies item (II(b)a) of Lemma 3. It now follows that $e = r - 1 > 1$, and so $r$ does not divide $q - \eta$.

If $\eta = -$, then $S = \text{PSU}_n(q)$ satisfies either (II(b)b) or (II(b)c) of Lemma 3. It is easy to see that in both cases $e$ is not equal to 2, and so $r$ does not divide $q + 1$. Observe also that in both cases $e^* = r - 1$. Thus $r \notin \pi(q - \eta)$, as required.

(b) It follows from items (II(B)a-II(B)c) of Lemma 3 that $n < t$ for every $t \in \tau$, and hence $P \simeq S_n$ is a $\pi'$-group. Then $|\pi \cap \pi(P)| \leq 1$, and $P$ satisfies $\mathcal{D}_\pi$ by the Sylow theorems. As we have seen above, if $\eta = +$ then $e = r - 1$, and if $\eta = -$ then $e^* = r - 1$. Lemma 4 implies that

$$\left| \text{GL}_n(q) \right|_r = (q^e - 1)^{\left[ \frac{n}{e} \right]} \left( \left\lfloor \frac{n}{e} \right\rfloor \right)_r,$$

$$\left| \text{GU}_n(q) \right|_r = (q^{e^*} - (-1)^{e^*})^{\left[ \frac{n}{e^*} \right]} \left( \left\lfloor \frac{n}{e^*} \right\rfloor \right)_r.$$

Hence

$$\left| \text{GL}_n(q) \right|_r = (q^{r-1} - 1)^{\left[ \frac{n}{r-1} \right]} \left( \left\lfloor \frac{n}{r-1} \right\rfloor \right)_r. \quad (1)$$

Since $r$ divides the order of $G$, we have that $n \geq r - 1$. Also, since $\left[ \frac{n}{r-1} \right] = \left[ \frac{n}{r} \right]$, we have that $n \geq r$, and so a Sylow $r$-subgroup $R$ of $P$ is nontrivial. Then $d = \left[ \frac{n}{r} \right] > 0$ and it follows from $d = \left[ \frac{n}{r-1} \right]$ that

$$n = dr + k = d(r - 1) + (d + k) \text{ and } 0 < d + k < r - 1. \quad (2)$$
In particular, \( d < r - 1 \) and a Sylow \( r \)-subgroup \( R \) of \( P \) is isomorphic to \( r^d \) by [11, 11.3.1, Example III]. Moreover

\[
    n = d(r - 1) + (d + k) \leq d(r - 1) + r - 2 \leq r(r - 2).
\]

(c) Since \( T \) is a characteristic subgroup of \( D \), we conclude that \( R \) normalizes \( T \), and so \( TR \) is a \( \pi \)-subgroup of \( G \). To prove that \( TR \) is a \( \pi \)-Hall subgroup of \( G \), it suffices to show that \( |G|_\pi = |TR| = (q - \eta)_r^{n/[r]}n/r \).

Items [11(B)a] and [11(B)c] of Lemma 3 yield \( (q^{r-1} - 1)_r = r \) and we have seen that \( [n/(r - 1)] = d < r - 1 \). Thus,

\[
    ([n/(r - 1)]!)_r = 1.
\]

In view of (c), we conclude that \( |G|_\pi = r^{n/[r-1]} = r^{n/r} \), as required.

Calculate \( t \)-part of order of \( G \) for every \( t \in \tau \). It follows from items [11(B)a] and [11(B)c] of Lemma 3 that \( e(q, t) = e(q, s) \) for every \( t, s \in \tau \). Denote \( e(q, t) \) by \( f \).

Observe that if \( \eta = + \) then \( f = 1 \), and if \( \eta = - \) then \( f = 2 \) and \( f^* = 1 \). Since \( n < t \), Lemma 4 yields

\[
    |GL_n(q)|_t = (q^f - 1)_t^{[n/f]} ([n/f]!)_t = (q - 1)_t^n (n!)_t = (q - 1)_t^n
\]

and

\[
    |GU_n(q)|_t = (q^{f^*} - (-1)^{f^*})_t^{[n/f^*]} ([n/f^*]!)_t = (q + 1)_t^n (n!)_t = (q + 1)_t^n
\]

Thus \( |G|_\pi = (q - \eta)_r^n \), as required.

(d) Clearly, \( \varphi \) normalizes \( D \) and centralizes \( P \). In particular, \( \varphi \) also centralizes \( R \). Since \( T \) is a characteristic subgroup of \( D \), we have that \( \varphi \) normalizes \( T \).

(e) We will denote by \( D_1 \) the subgroup of \( D \) consisting of all matrices \( \text{diag}(1, \ldots, \alpha, \ldots, 1) \) where \( \alpha \) is an element of the corresponding field such that \( \alpha^q = \alpha \) and \( \alpha \) is placed on the \( i \)th position. Let \( T_i \) be the unique \( \tau \)-Hall subgroup of \( D_1 \). It is clear that both \( P \) and \( R \) act on the set

\[
    \Omega = \{T_1, \ldots, T_n\}
\]

via conjugation. Since \( R \) is a Sylow \( r \)-subgroup of \( P \simeq S_n \), and in view of (c), \( R \) has \( d \) orbits of size \( r \) and \( k \) orbits of size 1 on \( \Omega \) (see [11, 11.3.1, Example III]). It is easy to see that if \( \Delta \) is an orbit of \( R \) on \( \Omega \) then

\[
    C_{\langle \Delta \rangle}(R) \simeq (q - \eta)_r \quad \text{and}
\]

\[
    C_T(R) = \langle C_{\langle \Delta \rangle}(R) \mid \Delta \text{ is an orbit of } R \text{ on } \Omega \rangle \simeq (q - \eta)_r^{d+k}.
\]
Since $R$ is abelian, we obtain that

$$C_{TR}(R) \cong (q - \eta)^{d+k} \times R.$$  

Thus, for every $t \in \tau$, the maximal rank of elementary abelian $t$-subgroup

$m_t(C_{TR}(R))$ is equal to $d + k$. 

Since $\text{PSL}_n^\eta(q)$ is a unique nonabelian composition factor of $\text{GL}_n^\eta(q)$, as a consequence of (a) and the proof of (b) we obtain

**Lemma 11.** Let $S = \text{PSL}_n^\eta(q)$, where $q = p^m$, and let $2, p \notin \pi$. Set $r = \min(\pi \cap \pi(S))$. Then $S \in \mathcal{E}_r \setminus \mathcal{D}_r$ implies $\gcd(n, q - \eta) = 1$ and $r \leq n \leq r(r-2)$.

**Lemma 12.** Let $G = \text{GL}_n^\eta(q)$, where $q = p^m$ and $p$ is a prime. Consider the automorphism $\varphi : (a_{ij}) \mapsto (a_{ij}^{\eta})$ of $G$. Suppose that $2, p \notin \pi$ and $G \in \mathcal{E}_r \setminus \mathcal{D}_r$. Set $r = \min(\pi \cap \pi(G))$ and $\tau = (\pi \cap \pi(G)) \setminus \{r\}$. Take arbitrary $t \in \tau$. Then $G$ contains an $r$-subgroup $R$ and an elementary abelian $t$-subgroup $K$ such that

(a) $R$ is a Sylow $r$-subgroup of $G$ centralized by $\varphi_2$;

(b) $K$ is $\varphi$-invariant, $K \leq C_G(R)$, and the rank of $K$ equals $2d + k$, where $d$ and $k$ are defined by $d = \lceil n/r \rceil$ and $k = n - dr$;

(c) if $m$ is divisible by $t$ and $\psi \in \langle \varphi \rangle$ is of order $t$, then $K$ in (II) can be chosen such that $K \leq C_G(\psi) \cap C_G(R)$.

**Proof.** It follows from Lemma [1](b) that $d = \lceil n/r \rceil > 0$. The equality $n = d(r-1) + d + k$ implies that $G$ contains a subgroup of block-diagonal matrices

$$X = \underbrace{\text{GL}_n^\eta(q) \times \ldots \times \text{GL}_n^\eta(q)}_{d \text{ times}} \times \underbrace{\text{GL}_1^\eta(q) \times \ldots \times \text{GL}_1^\eta(q)}_{k+d \text{ times}}.$$

This subgroup is $\varphi$-invariant. Every $\text{GL}_n^\eta(q)$ contains a subgroup $\text{GL}_n^\eta(p)$ centralized by $\varphi_2$. Consider a subgroup

$$Y = \underbrace{\text{GL}_n^\eta(p) \times \ldots \times \text{GL}_n^\eta(p)}_{d \text{ times}} \times \underbrace{\text{GL}_1^\eta(p) \times \ldots \times \text{GL}_1^\eta(p)}_{k+d \text{ times}}$$

of $X$. It follows from Fermat’s little theorem that $|\text{GL}_n^\eta(p)|_r > 1$. So, $Y$ contains an elementary abelian $r$-subgroup $R$ of order $r^d$ and $R$ is centralized by $\varphi_2$. By Lemma [1] we have that $|G|_r = r^d$ and $R$ is a Sylow $r$-subgroup.
of $G$, as required. Observe that $Z(X) \simeq (q - \eta)^{2d+k}$. Lemma [10](a) implies that $t$ divides $q - \eta$. Thus the unique maximal elementary abelian $t$-subgroup $K$ of $Z(X)$ is a desired subgroup.

If $m$ is divisible by $t$, then $q = q_0$ where $q_0 = p^{m/t}$. By Fermat’s little theorem $q_0 \equiv q \pmod{t}$ and since $q \equiv \eta \pmod{t}$ we obtain that

$$q_0 \equiv \eta \pmod{t}.$$ 

Consider a subgroup $X_0$ such that $Y \leq X_0 \leq X$ and

$$X_0 = \prod_{r=1}^{d} \left( \prod_{r=1}^{k} \left( \prod_{r=1}^{d} \left( \prod_{r=1}^{k} \right) \right) \right).$$

Clearly, $X_0 \leq \text{GL}_n(q) = C_G(\psi)$. It is easy to see that $Z(X_0) \leq Z(X)$ and since $t$ divides $q_0 - \eta$, we conclude that $K \leq Z(X_0) \leq C_G(\psi)$. □

We also need some information about automorphisms of groups of Lie type. Let $S$ be a simple group of Lie type. Definitions of diagonal, field and graph automorphisms of $S$ agree with that of [21]. The group of inner-diagonal automorphisms of $S$ is denoted by $p_S$. By [21, 3.2], there exists a field automorphism $\rho$ of $S$ such that every automorphism $\sigma$ of $S$ can be written $\sigma = \beta \rho \gamma$, with $\beta$ and $\gamma$ being an inner-diagonal and a graph automorphisms, respectively, and $l \geq 0$. The group $\langle \rho \rangle$ is denoted by $\Phi_S$. In view of [3, 7-2], the group $\Phi_S$ is determined up to $\hat{S}$-conjugacy. Since $S$ is centerless, we can identify $S$ with the group of its inner automorphisms.

**Lemma 13.** [21, 3.3, 3.4, 3.6] Let $S$ be a simple group of Lie type over $\mathbb{F}_q$ of characteristic $p$. Set $A = \text{Aut}(S)$ and $\hat{A} = \hat{S}\Phi_S$. Then the following statements hold.

(a) $S \leq \hat{S} \leq \hat{A} \leq A$ is a normal series for $A$.

(b) $\hat{S}/S$ is abelian; $\hat{S} = S$ for the groups $E_6(q), F_4(q), G_2(q), 3D_4(q)$, in other cases the order of $\hat{S}/S$ is specified in Table 1.

| $S$      | $|\hat{S}/S|$ |
|----------|--------------|
| $A_1(q)$ | gcd($l+1, q-1$) |
| $2A_1(q)$ | gcd($l+1, q+1$) |
| $B_1(q), C_1(q), E_7(q)$ | gcd($2, q-1$) |
| $D_1(q)$ | gcd($4, q^l-1$) |
| $2D_1(q)$ | gcd($4, q^l+1$) |
| $E_6(q)$ | gcd($3, q-1$) |
| $2E_6(q)$ | 13, gcd($3, q+1$) |
(c) $A = \hat{A}$ with the exceptions: $A/\hat{A}$ has order 2 if $S$ is $A_l(q)$ ($l \geq 2$), $D_l(q)$ ($l \geq 5$) or $E_6(q)$, or if $S$ is $B_2(q)$ or $F_4(q)$ and $q = 2^{2n+1}$, or if $S$ is $G_2(q)$ and $q = 3^{2n+1}$; $A/\hat{A}$ is isomorphic to $S_3$ if $S$ is $D_4(q)$.

Below we need an information about maximal subgroups of groups of Lie type. For classical groups we use the Aschbacher theorem [11, Theorem 1.2.1] and for the information about subgroups lying in the Aschbacher classes we refer to [1]. Maximal subgroups of exceptional groups of Lie type are specified in Lemmas [14] and [15].

Let $G$ be a finite exceptional simple group of Lie type over $\mathbb{F}_q$, where $q = p^s$. Then by [22] there is a simple adjoint algebraic group $\overline{G}$ over the algebraic closure of $\mathbb{F}_q$, and a surjective endomorphism $\sigma$ of $\overline{G}$ such that $G = O^\sigma(\overline{G}_\sigma)$, the subgroup of $\overline{G}_\sigma$ generated by all its $p$-elements.

\textbf{Lemma 14.} [13, Theorem 2] Let $G = O^\sigma(\overline{G}_\sigma)$ be a finite exceptional group of Lie type, $G_1$ is chosen so that $G \leq G_1 \leq \text{Aut}(G)$, and let $M$ be a maximal subgroup of $G_1$ such that $G \leq M$. Then either $F^*(M)$ is simple, or one of the following holds.

(a) $M = N_{G_1}(D_\sigma)$, where $D$ is a $\sigma$-stable closed connected subgroup and $D$ is either parabolic or reductive of maximal rank.

(b) $M = N_{G_1}(E)$, where $E$ is an elementary abelian $s$-subgroup with $s$ prime and $E \leq \overline{G}_\sigma$; the pair $(G, E)$ is as in Table 2, in each case $s \neq p$.

(c) $M$ is the centralizer of a graph, field, or graph-field automorphism of $G$ of prime order.

(d) $\overline{G} = E_8$, $p > 5$ and $F^*(M) \in \{\text{PSL}_2(5) \times \text{PSL}_2(9), \text{PSL}_2(5) \times \text{PSL}_2(q)\}$.

(e) $F^*(M)$ is as in Table 3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$G$ & $E$ & $N_{\overline{G}}(E)$ & Conditions \\
\hline
$G_2(p)$ & $2^3$ & $2^3 \cdot \text{SL}_3(2)$ & \\
$2G_2(3)'$ & $2^3$ & $2^3 \cdot 7$ & \\
$F_4(p)$ & $3^3$ & $3^3 \cdot \text{SL}_3(3)$ & $p \geq 5$ \\
$E_6(p)$ & $3^3$ & $3^{3+3} \cdot \text{SL}_3(3)$ & $p = \eta \mod 3, p \geq 5$ \\
$E_7(q)$ & $2^2$ & $(2^2 \times (\text{PO}_6^+(q) \cdot 2^2)) \cdot S_3$ & $\text{PO}_6^+(q) \cdot 2^2 = D_4(q)$ \\
$E_8(p)$ & $2^5$ & $2^{5+10} \cdot \text{SL}_5(2)$ & \\
$E_8(p^n)$ & $5^3$ & $5^3 \cdot \text{SL}_3(5)$ & $p \neq 2, 5$; $a = \begin{cases} 1, & \text{if } 5 \mid p^2 - 1 \\ 2, & \text{if } 5 \mid p^2 + 1. \end{cases}$ \\
$2E_6(2)$ & $3^2$ & $\text{NG}(E) = (3^2 \cdot [8]) \times (\text{PSL}_3(3) \cdot 2)$ & \\
$E_7(3)$ & $2^2$ & $\text{NG}(E) = (2^2 : 3) \times 3^1$ & \\
\hline
\end{tabular}
\end{table}
Table 3

| $G$ | $F^*(M)$ |
|-----|----------|
| $F_4(q)$ | $\text{PSL}_2(q) \times G_2(q)$ (\(p \geq 3, q \geq 5\)) |
| $E_6^+(q)$ | $\text{PSL}_2(q) \times G_2(q)$, $\text{PSU}_3(q) \times G_2(q)$ (\(q \geq 3\)) |
| $E_7(q)$ | $\text{PSL}_2(q) \times \text{PSL}_2(q)$ (\(p \geq 5\)), $\text{PSL}_2(q) \times G_2(q)$ (\(p \geq 3, q \geq 5\)), $G_2(q) \times F_4(q)$ (\(q \geq 4\)), $G_2(q) \times \text{PSp}_6(q)$ |
| $E_8(q)$ | $\text{PSL}_2(q) \times \text{PSL}_2(q)$ (\(p \geq 5\)), $\text{PSL}_2(q) \times G_2(q^2)$ (\(p \geq 3, q \geq 5\)), $G_2(q) \times F_4(q)$, $\text{PSL}_2(q) \times G_2(q) \times G_2(q)$ (\(p \geq 3, q \geq 5\)) |

To simplify our proof of Theorem 1 we need a list of maximal subgroups of $^2F_4(q)$.

**Lemma 15.** [14, Main Theorem] Every maximal subgroup of $G = ^2F_4(q)$, $q = 2^{2m+1}$, $m \geq 1$, is isomorphic to one of the following.

(a) $[q^{11}] : (A_1(q) \times (q-1))$.
(b) $[q^{10}] : (2B_2(q) \times (q-1))$.
(c) $\text{SU}_3(q) : 2$.
(d) $((q+1) \times (q+1)) : \text{GL}_2(3)$.
(e) $((q - \sqrt{2q} + 1) \times (q - \sqrt{2q} + 1)) : [96]$ if $q > 8$.
(f) $((q + \sqrt{2q} + 1) \times (q + \sqrt{2q} + 1)) : [96]$.
(g) $(q^2 - \sqrt{2qq} + q - \sqrt{2q} + 1) : 12$.
(h) $(q^2 + \sqrt{2qq} + q + \sqrt{2q} + 1) : 12$.
(i) $\text{PGU}_3(q) : 2$.
(j) $^2B_2(q) : 2$.
(k) $B_2(q) : 2$.
(l) $^2F_4(q_0)$, if $q_0 = 2^{2k+1}$ with $(2m + 1)/(2k + 1)$ prime.
3 Proof of the main theorem

In view of Lemma 6, we may assume that $2 \not\in \pi$. By Lemma 5 it is sufficient to prove the following statement

in each simple nonabelian $D_\pi$-group $G$,

all maximal subgroups containing a $\pi$-Hall subgroup of $G$ satisfy $D_\pi$. (3)

Statement (3) is true for alternating and sporadic simple groups, and simple groups of Lie type, if the characteristic $p$ lies in $\pi$, by Lemma 4. Thus we remain to consider simple groups of Lie type in characteristic $p$ with $p \not\in \pi$.

So we assume that $G$ is a simple $D_\pi$-group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$ and $2, p \not\in \pi$.

In view of the Sylow theorems, we suppose that $|\pi \cap \pi(G)| \geq 2$. So $G$ satisfies one of Conditions II-IV of Lemma 3.

Throughout this section, let $H$ be a $\pi$-Hall subgroup of $G$, $M$ a maximal subgroup of $G$ with $H \leq M$. Clearly, $H$ is a $\pi$-Hall subgroup of $M$, in particular $M \in \mathcal{E}_\pi$. We proof (3) if we show that $M$ satisfies $D_\pi$.

Assume by contradiction that $M$ does not satisfy $D_\pi$. Since $M \in \mathcal{E}_\pi$, Lemma 2(d) implies that every composition factor of $M$ satisfies $\mathcal{E}_\pi$. Since $M \not\in D_\pi$, Lemma 2(4) implies that $M$ has a nonabelian composition factor $S \in \mathcal{E}_\pi \setminus D_\pi$.

Recall that $r = \min(\pi \cap \pi(G))$ and $\tau = (\pi \cap \pi(G)) \setminus \{r\}$. We proceed in a series of steps.

Step 1. The following statements hold.

(a) $|\pi \cap \pi(S)| \geq 2$;

(b) $r \in \pi(S)$;

(c) $(S, \pi)$ satisfies (4);

(d) $S \simeq \text{PSL}^\eta_{n_1}(q_1)$ for some $q_1, n_1$ and $\eta$; $S$ satisfies one of items [I(B)a-I(B)c] of Lemma 4. In particular, $\tau \cap \pi(S) \subseteq \pi(q_1 - \eta)$, and $t > n_1$ for every $t \in \tau \cap \pi(S)$;

(e) $r \not\in \pi(q_1 - \eta)$, $\gcd(n_1, q_1 - \eta) = 1$ and $r \leq n_1 \leq r(r - 2)$;

(f) $\text{Out}(S)$ is an $r'$-group.

Note: As we mentioned in Remark on Lemma 3, $S$ cannot be isomorphic to an orthogonal or sympletic group.
(a) If $|\pi \cap \pi(S)| \leq 1$ then $S \in \mathcal{D}_\pi$ by the Sylow theorems. Consequently, $|\pi \cap \pi(S)| \geq 2$.

(b), (c) Since $G$ satisfies $\mathcal{D}_\pi$, every $\pi$-subgroup of $G$ is contained in a $\pi$-Hall subgroup of $G$. It follows from Lemma 4 that $(G, \pi)$ satisfies $\mathcal{A}$, i.e. every $\pi$-subgroup of $G$ has a normal abelian $\tau$-Hall subgroup. Lemma 2(3) implies that every $\pi$-subgroup of $S$ is a homomorphic image of a $\pi$-subgroup of $M$ and hence of $G$. Thus every $\pi$-subgroup of $S$ possesses a normal abelian $\tau$-Hall subgroup, in particular, a $\pi$-Hall subgroup of $S$ possesses a normal abelian $\tau$-Hall subgroup.

If $r \notin \pi(S)$ then a $\pi$-Hall subgroup of $S$ is abelian. So by Lemma 2(4) we have $S \in \mathcal{D}_\pi$, a contradiction. Therefore, we conclude $r \in \pi(S)$, as required. So $r = \min(\pi \cap \pi(S))$ and $(\pi \cap \pi(S)) \cap \{r\} = \tau \cap \pi(S)$. Since every $\pi$-subgroup of $S$ possesses a normal abelian $\tau$-Hall subgroup, we obtain that $(S, \pi)$ satisfies $\mathcal{B}$.

(d) Since $2 \notin \pi$ and $S \in \mathcal{E}_\pi \setminus \mathcal{D}_\pi$, the possibilities for $S$ are determined in Lemma 3.

Suppose that $S$ satisfies item (I) of Lemma 4. Then $S \cong O'\mathrm{N}$ and $\pi \cap \pi(S) = \{3, 5\}$. But a $\{3, 5\}$-Hall subgroup of $O'\mathrm{N}$ does not possess a normal Sylow 5-subgroup (see proof of [4, Theorem 6.14]), hence $(S, \pi)$ does not satisfy $\mathcal{B}$ and this case is impossible.

Consequently, $S$ is a group of Lie type with a base field $\mathbb{F}_{q_1}$ of a characteristic $p_1$. Assume first that $S$ satisfies item II(A) of Lemma 4 and so $p_1 \in \pi$. If $p_1 \neq r$ then $r \in (\pi \cap \pi(S)) \setminus \{p_1\}$ and $r$ does not divide $|W(S)|$. Since $\pi(|W(S)|) = \pi(l!)$ for some natural $l$, we obtain that $l < r < p_1$ and it contradicts the fact that $p_1$ divides $|W(S)|$. Suppose now that $p_1 = r$. Denote by $U$ a Sylow $p_1$-subgroup of $S$. In view of [4, Theorem 3.2], a Borel subgroup $B = N_S(U)$ contains a $\pi$-Hall subgroup $H_0$ of $S$. Since a $\tau$-Hall subgroup $Q$ of $H_0$ is normal (in $H_0$), we obtain that $H_0 = U \times Q$. So $H_0$ is nilpotent, and $S \in \mathcal{D}_\pi$ by Lemma 2(4), a contradiction.

Hence $S$ satisfies item II(B) of Lemma 4 in particular, $p_1 \notin \pi$. If $S$ satisfies one of items II(B)c-II(B)c, then $\pi \cap \pi(S) \subseteq \pi(q_1 \pm 1)$, and therefore $e(q_1, t) = e(q_1, s)$ for every $t, s \in \pi \cap \pi(S)$. Recall that $2, p_1 \notin \pi$ and $S \in \mathcal{E}_\pi$.

Now $(S, \pi)$ satisfies $\mathcal{B}$ by Step 1(c), so Lemma 3 implies that $S$ satisfies $\mathcal{D}_\pi$, a contradiction. Thus $S$ satisfies one of items II(B)c-II(B)c of Lemma 4 in particular,

$$S \cong \mathrm{PSL}_{n_1}^\eta(q_1)$$

for some $q_1, n_1$ and $\eta$.

Now the rest of statement (d) follows from items II(B)c-II(B)c of Lemma 3.

(e) The statements follow from Lemma 4.

(f) In view of (e) and Lemma 13, it is sufficient to prove that $|\Phi_S|$ is
an $r'$-group. If $r$ divides $|\Phi_S|$ then $q_1 = q_0^r$ for some $q_0$ and this equality contradicts the conclusion $(q_1^{r-1} - 1)_r = r$ in $\text{II}(\text{B})\text{a} - \text{II}(\text{B})\text{c}$ of Lemma 9.

Indeed, suppose that $S \cong \text{PSU}_{n_1}(q_1)$ and $r \equiv 1 \pmod{4}$ or $S \cong \text{PSL}_{n_1}(q_1)$ i.e. $S$ satisfies $\text{II}(\text{B})\text{a} - \text{II}(\text{B})\text{b}$ of Lemma 9. Under these conditions $e(q_1, r) = r - 1$. Since $q_1^{r-1} - 1$ is divisible by $q_0^{r-1}$ for every $i$, we conclude that $e(q_0, r) = r - 1$. Now

\[ q_0^{r-1} - 1 = \left( \frac{r-1}{q_0^{r-1}} - 1 \right) \left( \frac{r-1}{q_0^{r-1}} + 1 \right) \]

implies that $q_0^{(r-1)/2} + 1$ is divisible by $r$, i.e. $q_0^{(r-1)/2} \equiv -1 \pmod{r}$. Therefore

\[ \sum_{i=0}^{r-1} (-1)^{r-1-i} q_0^{r-1+i} \equiv r \pmod{r} \]

and we obtain that

\[ q_1^{r-1} - 1 = \left( \frac{r-1}{q_1^{r-1}} - 1 \right) \left( \frac{r-1}{q_0^{r-1}} + 1 \right) \left( \sum_{i=0}^{r-1} (-1)^{r-1-i} q_0^{r-1+i} \right) \]

is divisible by $r^2$; a contradiction.

Now, suppose that $S \cong \text{PSU}_{n_1}(q_1)$ and $r \equiv 3 \pmod{4}$, i.e. $S$ satisfies $\text{II}(\text{B})\text{c}$ of Lemma 9. Then $e(q_1, r) = (r - 1)/2$ and $e(q_0, r) = (r - 1)/2$. This implies that $q_0^{(r-1)/2} \equiv 1 \pmod{r}$ and

\[ \sum_{i=0}^{r-1} q_0^{r-1+i} \equiv r \pmod{r}. \]

Hence

\[ q_1^{r-1} - 1 = \left( \frac{r-1}{q_1^{r-1}} + 1 \right) \left( \frac{r-1}{q_0^{r-1}} - 1 \right) \left( \sum_{i=0}^{r-1} q_0^{r-1+i} \right) \]

is divisible by $r^2$; a contradiction again.

**Step 2.** $M$ is not almost simple.

Assume that $M$ is an almost simple group. Therefore, $S$ is a unique non-abelian composition factor of $M$ and we may assume that $S \leq M \leq \text{Aut}(S)$. Since $M$ contains a $\pi$-Hall subgroup $H$ of $G$, we arrive at a contradiction with $G \in \mathcal{D}_\pi$ if we find a $\pi$-subgroup of $M$ (and hence of $G$) which is not isomorphic to any subgroup of $H$. 
In order to prove Step 2, first, for every \( t \in \tau \cap \pi(S) \), we estimate \( m_t(C) \), where \( C \) is the centralizer in \( H \) of a Sylow \( r \)-subgroup of \( H \) (and of both \( M \) and \( G \), of course) and, second, we find an elementary abelian \( t \)-subgroup \( E \) of \( M \), which centralizes a Sylow \( r \)-subgroup \( R_0 \) of \( H \) and whose rank is greater than \( m_t(C) \). It is clear that \( ER_0 \) is not isomorphic to any subgroup of \( H \). In particular, \( ER_0 \) is not conjugate in \( G \) to any subgroup of \( H \).

Consider the group \( GL^n_{n_1}(q_1) \) first. Recall that

\[
GL^n_{n_1}(q_1) = \{(a_{ij}) \in GL_n(q_1^2) \mid (a_{ij}) = ((a_{ij})^{-1})^T\},
\]

where \((a_{ij})^T = (a_{ji})\) is the transposed of \((a_{ij})\). Let \( \varphi \) be an automorphism of \( GL^n_{n_1}(q_1) \) defined by \( \varphi : (a_{ij}) \mapsto (a_{ij}^q) \), where \( p_1 \) is the characteristic of \( \mathbb{F}_{q_1} \).

Let \( T \) be a \( \pi \)-Hall subgroup of the subgroup of all diagonal matrices in \( GL^n_{n_1}(q_1) \), and \( R \) a Sylow \( r \)-subgroup of the subgroup of permutation matrices of \( GL^n_{n_1}(q_1) \). Denote by \( \pi \) the \( \pi \)-part of \( \varphi \). Then by Lemma 2 we have that \( TR \) is a \( \pi \)-Hall subgroup of \( GL^n_{n_1}(q_1) \) and \( H_t = TR \langle \chi \rangle \) is a \( \pi \)-Hall subgroup of \( GL^n_{n_1}(q_1) \).

Now consider the natural homomorphism

\[ \pi : GL^n_{n_1}(q_1) \langle \varphi \rangle \to B, \text{ where } B = GL^n_{n_1}(q_1) \langle \varphi \rangle / Z(GL^n_{n_1}(q_1)). \]

Observe that \( B \) is isomorphic to \( \hat{S} \Phi_S \), where \( \hat{S} = PGL^n_{n_1}(q_1) \) and \( \Phi_S \) are defined in Lemma 3. By Lemma 3 we see that \( \overline{TR} \) is a \( \pi \)-Hall subgroup of \( \hat{S} \) and \( \overline{H_1} \) is a \( \pi \)-Hall subgroup of \( B \). Since \( |\hat{S} : S| = \gcd(n_1, q_1 - \eta) \) is a \( \pi \)-number by Step 1(e), we obtain \( \overline{TR} \leq S \) and \( \overline{H_1} \cap S = \overline{TR} \) is a \( \pi \)-Hall subgroup of \( S \). In particular, \( \overline{R} \) is a Sylow \( r \)-subgroup of \( S \).

Note that \( H \) is a \( \pi \)-subgroup of \( \text{Aut}(S) \). It follows from Lemma 3 that \( |\text{Aut}(S)/B| \in \{1, 2\} \). Since \( 2 \not\equiv 1 \pmod{\pi} \), we conclude that \( H \) is contained in \( B \). Since \( H \) is a \( \pi \)-Hall subgroup of \( M \), we have that \( H \cap S \) is a \( \pi \)-Hall subgroup of \( S \). Lemma 3 yields that \( S \in \mathcal{C}_\pi \). Therefore we may assume that \( H \cap S \) and \( \overline{TR} \) coincide.

Step 1(f) implies that every Sylow \( r \)-subgroup of \( S \) is a Sylow \( r \)-subgroup of \( \text{Aut}(S) \). In particular, \( \overline{R} \) is a Sylow \( r \)-subgroup of \( H \) and \( H / (H \cap S) \) is a \( \tau \)-group. Recall that \( r \) does not divide \( q_1 - \eta \) by Step 1(e). Therefore, \( \gcd(|\overline{R}|, |Z(GL^n_{n_1}(q_1))|) = 1 \) and \( R \simeq \overline{R} \). It now follows from [3, 3.28] that \( C_{\tau}(\overline{R}) = \overline{C_{\tau}(R)} \). Thus,

\[
C_{H \cap S}(\overline{R}) = C_{\overline{TR}}(\overline{R}) = \overline{RC_{\tau}(R)} = \overline{RC_{\tau}(R)} = \overline{C_{\tau}(R)}.
\]

Since \( C_{\tau}(R) \simeq (q_1 - \eta)^{d+k} \times R \) by Lemma 10(e), where \( d \) and \( k \) are defined by \( d = [n/r] \) and \( k = n - dr \), we obtain that \( C_{\tau}(R) \simeq (q_1 - \eta)^{d+k-1} \times R \) and \( m_t(C_{H \cap S}(\overline{R})) = d + k - 1 \) for every \( t \in \tau \cap \pi(S) \).
As we have seen above, \(|\widehat{S} : S|\) is a \(\pi'-\text{number}\). Thus

\[
H/(H \cap S) = H/(H \cap \widehat{S}) \cong H\widehat{S}/\widehat{S} \leq B/\widehat{S} \cong \langle \varphi \rangle
\]

and \(H/(H \cap S)\) is cyclic. Therefore, if \(t \in \tau \cap \pi(S)\), then

\[
m_t(C_H(\overline{R})) - m_t(C_{H \cap S}(\overline{R})) \leq 1
\]

and \(m_t(C_H(\overline{R}))\) is equal to either \(d + k - 1\) or \(d + k\). The Sylow theorems imply that the same statement holds for the centralizer in \(H\) of an arbitrary Sylow \(r\)-subgroup of \(H\).

Take some \(t \in \tau \cap \pi(S)\). As we have noted above, we complete Step 2 if we find a subgroup \(E\) in \(HS \leq M\) such that \(E\) is an elementary abelian \(t\)-group of rank greater than \(m_t(C_H(\overline{R}))\) and \(E\) centralizes a Sylow \(r\)-subgroup \(R_0\) of \(HS\).

Lemma \[2\] implies that there is a subgroup \(R_1 \times K\) in \(\text{GL} \!\!\!\!\!_n(q_t)\) such that \(R_1\) is a Sylow \(r\)-subgroup of \(\text{GL} \!\!\!\!\!_n(q_t)\) centralized by \(\varphi_2\) and \(K\) is a \(\varphi\)-invariant elementary abelian \(t\)-subgroup of rank \(2d + k\). A subgroup \(\overline{K}\) is an elementary abelian \(t\)-subgroup of \(\overline{S} = \text{GL} \!\!\!\!\!_n(q_t)\). Since \(\overline{S} : S|\) is a \(\pi'-\text{number}\), we conclude that \(\overline{K} \leq \overline{S}\).

If \(m_t(C_H(\overline{R}))\) equals \(d + k - 1\), then \(E = \overline{K}\) is a desired subgroup. Indeed, the rank of \(\overline{K}\) is equal to \(2d + k - 1\) and is greater than \(d + k - 1\), since \(d = [n_1/r] > 0\) in view of Step 1(e). Moreover, \(\overline{K}\) centralizes the Sylow \(r\)-subgroup \(R_0 = \overline{R_1}\) of both \(S\) and \(HS\).

If \(m_t(C_H(\overline{R}))\) equals \(d + k\), then \(|C_H(\overline{R})/(C_{H \cap S}(\overline{R}))| > 1\) and \(C_H(\overline{R})\) contains an element \(h\) of order \(t\) such that \(h \notin S\). Moreover, \(h \notin \overline{S}\), since \(\overline{S} : \overline{S}|\) is a \(\pi'-\text{number}\). In view of \([3\ (7-2)]\) we obtain that \(\langle h \rangle = \langle \psi \rangle^\delta\) where \(\psi \in \langle \varphi \rangle\) is of order \(t\) and \(\delta\) is an element in \(\overline{S}\). By Lemma \[2(\[3])\], we can assume that \(K\) is centralized by \(\psi\). The subgroup \(E = \langle \overline{K}, \psi \rangle = \langle \overline{K}, \psi \rangle\) is an elementary abelian \(t\)-subgroup of \(HS\). The rank of \(E\) is equal to \(2d + k\) and \(E\) centralizes the Sylow \(r\)-subgroup \(R_0 = \overline{R_1}\) of both \(S\) and \(HS\). So, \(E\) is a desired subgroup. This completes the proof of Step 2.

**Step 3.** \(G\) is not a classical group.

Assume that \(G\) is a classical group, and so \(G\) satisfies either Condition II or Condition III of Lemma \[3\]. If \(G\) satisfies either item (g) or item (h) of Condition II, then a \(\pi\)-Hall subgroup \(H\) of \(G\) is cyclic. Since \(H \leq M\), it follows from Lemma \[4(\[3])\] that \(M\) satisfies \(D_\pi\), a contradiction. Therefore, \(G\) satisfies either Condition III or one of items (a)-(f) of Condition II, in particular \(\epsilon(q, t) = \epsilon(q, s)\) for every \(t, s \in \tau\).
Set
\[ a = e(q, r) \text{ and } b = e(q, t) \] for every \( t \in \tau \).

Since \( M \) is not almost simple by Step 2, the famous Aschbacher’s theorem \([\text{1}]\) implies that \( M \) belongs to one of Aschbacher’s classes \( C_1 - C_8 \). The structure of members of Aschbacher’s classes is specified in \([\text{11}]\). Recall that by Step 1(b,d) \( M \) possesses a composition factor \( S \cong \text{PSL}_{n_1}(q_1), r \in \pi(S) \) and \( e(q_1, t) = e(q_1, s) \) for every \( t, s \in \tau \cap \pi(S) \). Set
\[ a_1 = e(q_1, r) \text{ and } b_1 = e(q_1, t) \] for every \( t \in \tau \cap \pi(S) \).

Assume that \( q_1 = q \). Then \( a_1 = a \) and \( b_1 = b \). Since \( S \) satisfies one of items (II(b),c) of Lemma \([\text{3}]\) by Step 1(d), we have that \( a \neq b \) and \( b \leq 2 \). Consequently, \( G \) cannot satisfy Condition III, and so one of items (a)-(f) of Condition II holds for \( G \). This implies that \( b \geq r > 2 \), a contradiction. Thus we conclude that \( q_1 \neq q \).

We now consider Aschbacher’s classes to specify all possibilities for \( M \) to have a composition factor \( S \) isomorphic to \( \text{PSL}_{n_1}(q_1) \) with \( q_1 \neq q \) (recall that \( S \) cannot be isomorphic to orthogonal or sympletic groups). The structure of members of Aschbacher’s classes \( C_1 - C_8 \) is presented in \([\text{11}], \text{Chapter 4}\). By using this information, we check below that, in every case when \( M \) is an element of corresponding Aschbacher’s class \( C_1 - C_8 \), there is at most one such possibility for \( M \).

\( C_1 \): The structure of members of \( C_1 \) is presented in \([\text{11}], \text{\S} 4.1\). The unique possibility for \( M \) appears in \([\text{11}], \text{Proposition 4.1.18}\):

(a) \( G = \text{PSU}_n(q), M \cong \left[ q^{m(2n-3m)} \right] : \left[ c / \gcd(q + 1, n) \right] \cdot (\text{PSL}_m(q^2) \times \text{PSU}_{n-2m}(q)) \cdot [d] \), where \( 1 \leq m \leq [n/2] \),

\[ c = |\{(\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{F}_{q^2}, \lambda_2^{q+1} = 1, \lambda_1^{m(q-1)} \lambda_2^{-2m} = 1\}|, \]

\[ d = (q^2 - 1) \gcd(q^2 - 1, m) \gcd(q + 1, n - 2m)/c. \]

In this case, \( S \cong \text{PSL}_{n_1}(q_1) \) with \( n_1 = m \) and \( q_1 = q^2 \).

\( C_2 \): The structure of members of \( C_2 \) is presented in \([\text{11}], \text{\S} 4.2\). The unique possibility for \( M \) appears in \([\text{11}], \text{Proposition 4.2.4}\):

(b) \( G = \text{PSU}_n(q), M \cong \left[ \frac{(q-1) \gcd(q+1, \frac{q}{2})}{\gcd(q+1, n)} \right] \cdot \text{PSL}_{n_1/2}(q^2) \cdot \left[ \frac{\gcd(q^2-1, \frac{q}{2})}{\gcd(q+1, \frac{q}{2})} \right] \cdot 2. \)

In this case, \( S \cong \text{PSL}_{n_1}(q_1) \) with \( n_1 = n/2 \) and \( q_1 = q^2 \).

\( C_3 \): The structure of members of \( C_3 \) is presented in \([\text{11}], \text{\S} 4.3\). The unique possibility for \( M \) appears in \([\text{11}], \text{Proposition 4.3.6}\):

21
(c) $G = \text{PSL}_n^\eta(q)$, $M \simeq c \cdot \text{PSL}_m^u(q^u) \cdot d \cdot u$, where $n = mu$, $u$ is prime
(if $\eta = -$, $u \geq 3$), $c = \frac{\gcd(q^{u-\eta},m)}{\gcd(q^u,\eta)}$, $d = \frac{\gcd(q^{u-\eta},m)}{\gcd(q^u,\eta)}$.

In this case, $S \simeq \text{PSL}_n^\eta(q_1)$ with $n_1 = m$, $q_1 = q^u$ and $\eta$ is the same for
$G$ and $S$.

$C_4, C_7$: The structure of members of $C_4$ and $C_7$ presented in [11, §4.4 and §4.7]
implies that if a composition factor of members of $C_4$ or $C_7$ is isomorphic
to $\text{PSL}_n^\eta(q_1)$ then $q_1 = q$.

$C_5$: The structure of members of $C_5$ is presented in [11, §4.5]. The unique
possibility for $M$ appears in [11, Proposition 4.5.3]:

(d) $G = \text{PSL}_n^\eta(q)$, $M$ is a normal subgroup in $\text{PGL}_n^\eta(q_1)$ of index
$\frac{lcm(q-1, \frac{q^\eta - q^{\eta-1}}{gcd(q-1,q^\eta)})}{gcd(q-\eta,n)}$, where $q = q_1^u$, $u$ is prime and $u \geq 3$ if $\eta = -$.

In this case, $S \simeq \text{PSL}_n^\eta(q_1)$ with $n_1 = n$ and $\eta$ is the same for $G$ and $S$.

$C_6$: The structure of members of $C_6$ presented in [11, §4.6] implies that if a
composition factor of members of $C_6$ is isomorphic to $\text{PSL}_n^\eta(q_1)$ then
$q_1 = q$.

$C_8$: The structure of members of $C_8$ is presented in [11, §4.8]. The unique
possibility for $M$ appears in [11, Proposition 4.8.5]:

(e) $G = \text{PSL}_n(q)$, $M \simeq \text{PSU}_n(q_1) : \left[ \frac{gcd(q_1+1,n)c}{gcd(q-1,n)} \right]$, where $q = q_1^2$ and
$c = \frac{q-1}{lcm(q_0+1, \frac{q^\eta - q^{\eta-1}}{gcd(q-1,q^\eta)})}$.

In this case, $S \simeq \text{PSL}_n^\eta(q_1)$ with $n_1 = n$.

In cases (d) and (e), $M$ is almost simple and we exclude them in view of
Step 2.

Now we exclude the remaining cases (a)–(c). Recall some statements from
Step 1 which hold for $S$.

| $S \simeq \text{PSL}_n(q_1)$ | $S \simeq \text{PSU}_n(q_1)$ |
|--------------------------|--------------------------|
| $a_1 = r - 1$          | $r \equiv 1 \pmod{4}$ and $a_1 = r - 1$, or $r \equiv 3 \pmod{4}$ and $a_1 = \frac{r-1}{2}$ |
| $b_1 = 1$              | $b_1 = 2$                |
| $(q_1^{r-1} - 1)_{a_1} = r$ | $(q_1^{r-1} - 1)_{a_1} = r$ |
| $r \leq n_1 \leq r(r-2)$ | $r \leq n_1 \leq r(r-2)$ |

If $S \simeq \text{PSL}_n(q_1)$, we see that $a_1$ is even and $a_1 > 1$. If $S \simeq \text{PSU}_n(q_1)$,
we see that either $a_1 \equiv 0 \pmod{4}$, or $a_1 \equiv 1 \pmod{2}$. So, in the case where
$S$ is unitary, $a_1$ cannot equal $2k$ with $k$ odd, in particular, $a_1 \neq 2$. 

22
In the rest of Step 3, we fix some \( t \in \tau \cap \pi(S) \).

Cases (a) and (b). In these cases \( S \cong \text{PSL}_{m_1}(q^2) \). Since \( e(q^2, t) = b_1 = 1 \), we conclude \( b = e(q, t) \) is equal to 1 or 2. This implies that \( G \) cannot satisfy Condition II, and so \( G \) satisfies Condition III; in particular \( a = b \). Therefore, \( a = e(q, r) \) equals 1 or 2, and \( a_1 = e(q^2, r) = 1 \), which is a contradiction with the fact that \( a_1 > 1 \).

Case (c). In this case \( S \cong \text{PSL}_{m}(q^u) \), where \( mu = n \) and \( u \) is prime (if \( \eta = -1, u \geq 3 \)). Show that \( u = r \) and, in particular, \( r \) divides \( n \).

Assume first that \( \eta = + \), i. e. \( G = \text{PSL}_n(q) \) and \( S \cong \text{PSL}_{m}(q^u) \). Since \( a_1 = e(q^u, r) > 1 \), we have that \( a = e(q, r) \geq a_1 > 1 \) and \( a \neq u \). Since \( b_1 = e(q^u, t) = 1 \), we obtain that \( b = e(q, t) = \gcd(b, u)b_1 = \gcd(b, u) \) divides \( u \). Therefore, \( b \) is equal to 1 or \( u \). Hence \( G \) cannot satisfy Condition III where \( a = b \), since \( a \) cannot equal 1 or \( u \). Consequently, \( G \) satisfies either item (a) or item (b) of Condition II, and \( b = r = u \).

Assume now that \( \eta = -1 \), i. e. \( G = \text{PSU}_n(q) \) and \( S \cong \text{PSU}_{m}(q^u) \). Since

\[
   a = \gcd(a, u)a_1 \quad \text{and} \quad a_1 \not\equiv 2 \pmod{4},
\]

we have that \( a \not\equiv 2 \pmod{4} \). In particular \( a \neq 2 \) and \( a \neq 2u \) (recall that \( u \geq 3 \) is prime in this case). It follows from \( b = \gcd(b, u)b_1 \) and \( b_1 = 2 \) that \( b \) is equal to \( 2 \) or \( 2u \). Consequently, \( a \neq b \) and \( G \) satisfies one of items (c)-(f) of Condition II. This implies \( b = 2r = 2u \).

Thus we have that \( r = u \), and so \( r \) divides \( n \). Therefore, \( G \) cannot satisfy items (b), (e) and (f) of Condition II, where \( n \equiv -1 \pmod{r} \). Hence \( G \) satisfies one of items (a), (c) or (d) of Condition II. Now it follows that

\[
   \left\lfloor \frac{n}{r-1} \right\rfloor = \left\lfloor \frac{n}{r} \right\rfloor = \frac{n}{r} = m.
\]

These equalities yield that

\[
   n = mr = m(r-1) + m
\]

and \( m < r - 1 \), which is a contradiction with the fact that \( m = n_1 \geq r \).

Thus in all cases (a)-(e) we obtain a contradiction, and so \( G \) cannot be a classical group, as wanted. To prove the statement (3) it remains to show that \( G \) cannot be an exceptional group.

**Step 4.** \( G \) is not an exceptional group.

Assume that \( G \) is an exceptional group, and so \( G \) satisfies either Condition III or Condition IV of Lemma 3. The description of \( \pi \)-Hall subgroups...
in the exceptional groups in characteristic $p$ with $2, p \not\equiv \pi$ is given in \cite{24}. Recall that by Step 1 $M$ possesses a composition factor $S \cong \text{PSL}^n_{n_1}(q_1)$, $r \leq n_1 \leq r(r-2)$ and $|\tau \cap \pi(S)| \geq 1$. Also if $\eta = +$ then $e(q_1,t) = 1$, and if $\eta = -$ then $e(q_1,t) = 2$ for every $t \in \tau \cap \pi(S)$, in particular, $q_1 \geq 4$.

Suppose $G$ satisfies Condition III first. So $G$ is not isomorphic to $2B_2(q)$, $2G_2(q)$ or $2F_4(q)^\prime$. Hence by \cite{24} Lemmas 7-13 we have that a $\pi$-Hall subgroup $H$ of $G$ is abelian or $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$. If $H$ is abelian then $M$ satisfies $D_\pi$ by Lemma 3\cite{3}, a contradiction. Consequently $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$. If $q_1 = q^n$ for some natural $n$, it follows from $\pi \cap \pi(S) \subseteq \pi(q \pm 1)$ that $\pi \cap \pi(S) \subseteq \pi(q^n \pm 1) = \pi(q_1 \pm 1)$, and so $e(q_1,t) = e(q_1,s)$ for every $t, s \in \pi \cap \pi(S)$. Since $(S, \pi)$ satisfies (3) by Step 1(c), Lemma 8\cite{8} implies that $S$ satisfies $D_\pi$, a contradiction. Thus we conclude that $q_1$ is not a power of $q$.

Let $\overline{G}$ be a adjoint simple algebraic group and $\sigma$ a surjective endomorphism of $\overline{G}$ such that $G = O^\sigma(\overline{G})$. All maximal subgroups of $G$ are presented in Lemma 14\cite{14}. Since $M$ is not almost simple, we consider all possibilities for $M$ according to items (a)-(e) of Lemma 14\cite{14}.

Case (a): $M = N_G(D_\sigma)$, where $D$ is a $\sigma$-stable closed connected subgroup and $D$ is either parabolic or reductive subgroup of maximal rank. If $D$ is parabolic, then there are no composition factors of $M$ isomorphic to $\text{PSL}^n_{n_1}(q_1)$ with $q_1 \neq q^n$. If $D$ is reductive subgroup of maximal rank, then $M$ is a subgroup of maximal rank in sense of \cite{12}. Since $G \subseteq \overline{G}_\sigma$, we have that $S$ is a composition factor of $N_{\overline{G}_\sigma}(D_\sigma)$. According to Tables 5.1 and 5.2 from \cite{12}, we obtain that $S$ is isomorphic to one of the following groups $\text{PSL}_2(5)$, $\text{PSL}_3(2)$ or $\text{PSU}_4(2)$. If $S \cong \text{PSL}_2(5)$ then $n_1 = 2$, which is a contradiction with the fact that $n_1 \geq r > 2$. If $S$ is isomorphic to $\text{PSL}_3(2)$ or $\text{PSU}_4(2)$ then $q_1 = 2$ and it contradicts the fact $q_1 \geq 4$.

Case (b): $M = N_G(E)$, where $E$ is an elementary abelian $s$-subgroup with $s$ prime and $E \subseteq \overline{G}_\sigma$ (see Table 2). Since $G \subseteq \overline{G}_\sigma$, we have that $S$ is a composition factor of $N_{\overline{G}_\sigma}(E)$. According to Table \cite{2} we obtain that $S$ is isomorphic to one of the following groups $\text{PSL}_3(2)$, $\text{PSL}_3(3)$, $\text{PSU}_3(3)$, $\text{PSL}_5(5)$ or $\text{PSL}_5(2)$. If $S$ is isomorphic to $\text{PSL}_3(2)$, $\text{PSL}_3(3)$, $\text{PSU}_3(3)$ or $\text{PSL}_5(2)$, then $q_1 < 4$, a contradiction. If $S \cong \text{PSL}_3(5)$, then there is no odd prime $q$ with $e(q_1,t) = 1$, and so $\tau \cap \pi(S) = \emptyset$, which is a contradiction with the fact that $|\tau \cap \pi(S)| \geq 1$.

Case (c): $M$ is the centralizer of a graph, field, or graph-field automorphism of $G$ of prime order. The structure of $M$ is presented in \cite{8}. Theorem 4.5.1, Theorem 4.7.3, Propositions 4.9.1 and 4.9.2]. As we mentioned in Remark on Lemma 1\cite{1}. $S$ cannot be isomorphic to an orthogonal or sympletic group. So we see that there are no centralizers of a graph, field, or graph-field
automorphism of $G$ of prime order with a composition factor isomorphic to $\text{PSL}_{n_1}(q_1)$ where $q_1 \neq q^n$.

Cases (d) and (e): either $G = E_8$, $p > 5$ and $F^*(M)$ is one of groups $\text{PSL}_2(5) \times \text{PSL}_2(9)$, $\text{PSL}_2(5) \times \text{PSL}_2(q)$ or $F^*(M)$ is as in Table 3. The maximality and the structure of $M$ implies that $M$ is a subgroup of $\text{Aut}(F^*(M))$, in particular, $M/F^*(M)$ is solvable. Hence case (d) holds and $S$ is isomorphic to one of the following groups $\text{PSL}_2(5)$, $\text{PSL}_2(9)$. Now we have $n_1 = 2$, a contradiction with the fact that $n_1 \geq r > 2$.

Thus in all cases (a)-(e) we obtain a contradiction, and so $G$ cannot satisfy Condition III. Consequently, we conclude that $G$ satisfies Condition IV, and $G$ is isomorphic to one of groups $2^2 B_2(q)$, $2^2 G_2(q)$ or $2^2 F_4(q)'$. Since $2 \neq \pi$, in view of [3], 6.13 Corollary $2^2 F_4(2)'$ does not satisfy $E_\pi$, and therefore $G$ cannot be isomorphic to $2^2 F_4(2)'$. Since $2^2 F_4(q)' = 2^2 F_4(q)$ with $q > 2$, further we write $2^2 F_4(q)$ instead of $2^2 F_4(q)'$.

By [24], Lemma 14, if $G$ is isomorphic to $2^2 B_2(q)$ or $2^2 G_2(q)$, or if $G$ is isomorphic to $2^2 F_4(q)$ and $3 \neq \pi$, then $H$ is abelian. Since $H \leq M$, it now follows form Lemma [3] that $M$ satisfies $D_\pi$, a contradiction. Consequently, we deduce that $G$ is isomorphic to $2^2 F_4(q)$ and $3 \in \pi$, and therefore $r = 3$. All maximal subgroups of $G$ are specified in Lemma [13]. Since $M$ has a composition factor $S \simeq \text{PSL}_{n_1}(q_1)$ and $r < n_1 < r(r - 2)$, we obtain that $n_1 = 3$ and $S \simeq \text{PSU}_3(q_1)$ with $q_1 = q = 2^{2m+1}$. By Step 1(d) $S$ satisfies II(B) of Lemma [4]. In particular,

$$e(q_1, r) = \frac{r - 1}{2} = 1.$$ 

But $q_1 = 2^{2m+1} \equiv -1 \pmod{3}$. So we obtain a contradiction with the fact that $e(q_1, 3) = e(q_1, r) = 1$.

Thus we conclude that $G$ cannot be an exceptional group, and so the main theorem is proved.

References

[1] Aschbacher M.: On the maximal subgroups of the finite classical groups. Inventiones mathematicae 76 (3) 469—514 (1984).

[2] Borel A. and Institute for Advanced Study (Princeton, N.J.): Seminar on algebraic groups and related finite groups. Lecture notes in mathematics, Springer-Verlag (1970).
[3] Gorenstein D., Lyons R.: The local structure of finite groups of characteristic 2 type. Vol. 42, American Mathematical Society (1983).

[4] Gorenstein D., Lyons R., Solomon R.: The classification of the finite simple groups. Number 3, American Mathematical Soc., Providence, RI (1998).

[5] Gross F.: On a conjecture of Philip Hall. Proc. London Math. Soc. s3-52 (3), 464—494 (1986).

[6] Gross F.: Conjugacy of odd order Hall subgroups. Bull. London Math. Soc. 19 (4), 311—319 (1987).

[7] Gross F.: Odd order Hall subgroups of the classical linear groups. Mathematische Zeitschrift 220 (1), 317—336 (1995).

[8] Hall P.: Theorems like Sylow’s. Proceedings of the London Mathematical Society s3-6 (2), 286—304 (1956).

[9] Isaacs I.M.: Finite group theory. Graduate Studies in Mathematics, Vol. 92, American Mathematical Soc., Providence, RI (2008).

[10] Kargapolov M.I., Merzljakov Ju.I.: Fundamentals of the theory of groups. Graduate Texts in Mathematics, 62. Springer-Verlag, New York-Berlin (1979).

[11] Kleidman P.B., Liebeck M.W.: The subgroup structure of the finite classical groups. Vol. 129, Cambridge University Press (1990).

[12] Liebeck M.W., Saxl J., Seitz G.M.: Subgroups of maximal rank in finite exceptional groups of Lie type. Proc. London Math. Soc 65 (3) 297—325 (1992).

[13] Liebeck M.W., Seitz G.M.: Maximal subgroups of exceptional groups of Lie type, finite and algebraic. Geom. Dedicata 36, 353—387 (1990).

[14] Malle G.: The maximal subgroups of $^2F_4(q^2)$. J.Algebra 139 (1), 52—69 (1991).

[15] Manzaeva N.Ch.: Heritability of the property $D_π$ by overgroups of $\pi$-Hall subgroups in the case where $2 \in \pi$. Algebra and Logic 53 (1), 17—28 (2014).

[16] Mazurov V.D., Khukhro E.I. (Eds.): The Kourovka notebook. Unsolved problems in group theory. 17th Edition, Russian Academy of Sciences Siberian Division Institute of Mathematics, Novosibirsk (2010).
[17] Nesterov M.N.: On pronormality and strong pronormality of Hall subgroups. Siberian Mathematical Journal 58 (1), 128—133 (2017).

[18] Revin D.O., Vdovin E.P.: Hall subgroups of finite groups. Contemporary Mathematics 402 229—265 (2006).

[19] Revin D.O.: The $D_\pi$-property in finite simple groups. Algebra and Logic 47 (3), 210—227 (2008).

[20] Revin D.O.: Around a conjecture of P. Hall. Siberian Electronic Mathematical Reports 6, 366—380 (2009) (in Russian).

[21] Steinberg R.: Automorphisms of finite linear groups. Canad. J. Math 12 (4), 606—616 (1960).

[22] R. Steinberg, Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80 (1968).

[23] Vdovin E.P., Manzaeva N.Ch., Revin D.O.: On the heritability of the property $D_\pi$ by subgroups. Proceedings of the Steklov Institute of Mathematics 279 (1) 130—138 (2012).

[24] Vdovin E.P., Revin D.O.: Hall subgroups of odd order in finite groups. Algebra and Logic 41 (1), 8–29 (2002).

[25] Vdovin E.P., Revin D.O.: Theorems of Sylow type. Russian Math. Surveys 66 (5), 829—870 (2011).

[26] Vdovin E.P., Revin D.O.: Pronormality of Hall subgroups in finite simple groups. Siberian Math. J. 53 (3), 419—430 (2012).

[27] Vdovin E.P., Revin D.O.: On the pronormality of Hall subgroups. Siberian Math. J. 54 (1), 22—28 (2013).

[28] Weir A.J.: Sylow $p$-subgroups of the classical groups over finite fields with characteristic prime to $p$. Proc. AMS 6 (4) 529—533 (1955).

[29] Wielandt H.: Zum Satz von Sylow. Math. Z. 60 (1), 407—408 (1954).