Gravitational backreaction in cosmological spacetimes

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Abstract

We develop a new formalism for the treatment of gravitational backreaction in the cosmological setting. The approach is inspired by projective techniques in non-equilibrium statistical mechanics. We employ group-averaging with respect to the action of the isotropy group of homogeneous and isotropic spacetimes (rather than spatial averaging), in order to define effective FRW variables for a generic spacetime. Using the Hamiltonian formalism for gravitating perfect fluids, we obtain a set of equations for the evolution of the effective variables; these equations incorporate the effects of backreaction by the inhomogeneities. Specializing to dust-filled spacetimes, we find regimes that lead to a closed set of backreaction equations, which we solve for small inhomogeneities. We then study the case of large inhomogeneities in relation to the proposal that backreaction can lead to accelerated expansion. In particular, we identify regions of the gravitational state space that correspond to effective cosmic acceleration. Necessary conditions are (i) a strong expansion of the congruences corresponding to comoving observers, and (ii) a large negative value of a dissipation variable that appears in the effective equations (i.e., an effective “anti-dissipation”).

1 Introduction

1.1 Preamble

The fundamental postulate of modern cosmology is the assumption of a homogeneous and isotropic universe. The spacetime must then possess a six-dimensional group of spacelike isometries, i.e., it must be of the Friedmann-Robertson-Walker (FRW) type. However, isotropy and homogeneity refer to a coarse-grained level of description: there is significant inhomogeneity at short length-scales.

Since homogeneity is approximate, one may inquire how inhomogeneities affect the evolution of the FRW variables. This question is of foundational interest, as it touches upon the domain of validity of the fundamental assumption of modern cosmology [1, 2]. Moreover, both in early universe cosmology (inflation in particular) and in present-epoch cosmology, gravitational backreaction effects may play a significant role in the evolution of the universe. In particular, it has been proposed that the backreaction of spatial inhomogeneities may be responsible for the apparent cosmic acceleration,

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so that one would not have to invoke the existence of dark energy [3, 4, 5]—see also the reviews [6, 7] and critique [8].

In this paper we develop a systematic approach for the study of gravitational backreaction with an emphasis on the cosmological context. Our starting point is the observation that the issue of gravitational backreaction has many analogues in non-equilibrium statistical mechanics. The description of a spacetime by a single quantity (the scale factor), obtained from the “averaging” of a generic metric, resembles the description of many-body systems through coarse-grained bulk variables (the mean field approximation in particular) [9, 10, 11, 12]. A key problem in the study of gravitational backreaction is the consistency of the approximation scheme. Problems of this type are specifically addressed by the techniques of non-equilibrium statistical mechanics. A transfer of ideas from this field of research for the study of gravitational perturbations could prove highly fruitful.

A second problem in treatments of gravitational backreaction arises from the issue of gauge invariance. A definition of effective FRW variables involves spatial averaging of the inhomogeneities, and such averaging can be implemented covariantly only for spatial scalars. For this reason, studies of backreaction often restrict to averaging the Hamiltonian constraint, which is a spatial scalar; the averaging cannot be implemented at the level of the tensor-field-valued equations of motion. Moreover, even if one restricts to scalars, the results depend on the choice of foliation, i.e., on the family of spatial surfaces upon which one integrates. Lack of gauge invariance may also pose a problem in another stage, namely, in the implementation of the dynamics. If the dynamics of gravitational perturbations are described through gauge-fixing, then there is the danger that that the study of back-reaction will lead to results that depend on the chosen gauge [8].

The formalism we develop in this paper is gauge-invariant. The main difference from previous approaches is that we employ group averaging over the isometry group of the FRW spacetime, rather than spatial averaging. Group averaging is defined covariantly for any tensor field, and it reduces to spatial averaging for scalar quantities. As a matter of fact, the properties of the group-averaging calculus allow all group averages that appear in this paper to be reduced to spatial averages of scalar quantities.

In order to avoid the problems related to gauge-fixing, we work within the Hamiltonian formalism for general relativity. In this paper, we also restrict the matter content to perfect fluids. In this case, the solution of the constraints can be implemented in a fully geometrical way (i.e., without gauge fixing) at the level of the Lagrangian. The reduction entails a solution of the diffeomorphism constraints: this corresponds in an implicit ”selection” of a class of foliations tied to the perfect fluid. The only ”gauge choice” that remains is that of a time variable. The formalism allows the derivation of backreaction equations for any such choice, as long as it can be made consistently over the system’s state space; we choose time as measured by observers comoving with the fluid.

1.2 Backreaction and non-equilibrium statistical mechanics

A common approach for the study of backreaction effects in a cosmological spacetime employs a perturbation expansion around the FRW solution. This involves solving the linearized Einstein equations around the classical FRW solution; the perturbations are then employed for the construction of an effective stress-energy tensor, which, when inserted into the Einstein equations, provides the corrections of the FRW evolution. In general, this procedure suffers from consistency problems. In particular, the construction of the effective stress-energy tensor is gauge-dependent—hence the result-
ing backreaction equations are also gauge-dependent [8].

Moreover, when large perturbations are taken into account, the accuracy of such an approximation scheme degenerates rapidly with time: the solution of the equations of motion with backreaction diverges cumulatively from the FRW solution. This means that the perturbations around the FRW equations capture less and less of the physics of the system as time increases. Outside the gravitational context, such treatments are known to misrepresent backreaction-induced effects such as dissipation and diffusion.

The consistent treatment of backreaction is a major ingredient in most techniques developed in the field of non-equilibrium statistical mechanics. The methodology of such treatments varies according to the system under consideration. However, all treatments follow a common pattern, which is abstractly and compactly described in the language of the so-called projection formalism [10][13]. This pattern can be described as a sequence of three steps.

The first step is the specification of the level of description, namely, of a set of variables that provide a coarse-grained description of the system under consideration. For example, in quantum Brownian motion, one studies a selected particle interacting with a heat bath of harmonic oscillators (environment). The level of description corresponds to the degrees of freedom of the selected particle. In Boltzmann’s treatment of the rare gas, the system is a collection of weakly interacting particles, and the level of description is defined by a probability density on the phase space of a single particle.

In general, the level of description corresponds to a subspace of the space of functions \( F(\Gamma) \) on the system’s state space \( \Gamma \). It is represented by a projective map \( P \) on \( F(\Gamma) \). We shall call the variables that lie within the range of \( P \) relevant variables and ones that lie on the range of \( 1 - P \) non-relevant variables. For the cosmological perturbations considered here, the relevant variables correspond to homogeneous and isotropic field configurations.

The second step involves a splitting of the dynamical evolution into components in accordance with the chosen level of description. Let \( L_t \) be the evolution operator on the space of states (the propagator of the Liouville equation in a Hamiltonian system).

- \( P L_t P \) describes the self-evolution of the relevant variables.
- \( (1 - P) L_t (1 - P) \) describes the self-evolution of the non-relevant variables.
- \( (1 - P) L_t P \) describes the coupling between relevant and non-relevant variables.

The splitting above allows for the derivation of a set of evolution equation for the relevant variables: this contains the evolution terms \( P L_t P \) for the relevant variables and backreaction terms that arise from the coupling \( (1 - P) L_t P \) of relevant to non-relevant variables. The backreaction terms depend on the state of the non-relevant variables and on their self-evolution in terms of \( (1 - P) L_t (1 - P) \). In general, the set of evolution equations for the relevant variables is not autonomous, or not-closed, and bears an explicit dependence on the initial state of the non-relevant variables.

The third step is the derivation of a closed set of equations (e.g., a Fokker-Planck-type equation for quantum Brownian motion or Boltzmann’s equation for the rare gas). To this end, one introduces additional assumptions about the state and evolution of the non-relevant variables. For example, one may assume that the state of the non-relevant variables is not significantly affected by the evolution of the relevant variables (Born approximation), or that the memory effects in the evolution are negligible (Markov approximation), or that the non-relevant variables are “fast” in relation to the relevant ones,
or that the correlations between relevant variables are insignificant (e.g., in the truncation of the Bogolubov-Born-Kirkwood-Green-Yvon hierarchy in kinetic theory), and so on. Such assumptions are necessary for the closure of the set of evolution equations and they often involve the introduction of semi-phenomenological parameters that describe the properties of the irrelevant variables.

To summarize, the general procedure for the consistent construction of backreaction equations involves:

1. a specification of the level of description.
2. a splitting of the dynamics and the construction of equations for the relevant variables
3. additional assumptions about the state of the irrelevant variables that allow for the closure of the system of effective equations.

1.3 Our approach

In this paper, we apply the reasoning above to the treatment of the backreaction from cosmological inhomogeneities. The rationale is that the FRW variables provide a coarse-grained level of description for a cosmological spacetime, which is, in a sense, analogous to the description of statistical systems in terms of mean-field theory. The backreaction of inhomogeneities is then conceptually similar to the incorporation of the effects of the second- and higher-order correlation functions into the mean-field evolution.

Our primary aim in this paper is to set the basis of a general procedure for the treatment of inhomogeneities. The projective formalism can, in principle, be applied to spacetimes with arbitrary matter content. Specializing to the case of a gravitating perfect fluid allows us to solve the diffeomorphism constraints in a gauge-invariant way. We then analyze in detail the special case of backreaction in a dust-filled spacetime.

We do not assume a specific form for the "true" spacetime metric, which is to be approximated by an FRW spacetime. Rather, we construct backreaction equations in terms of a small number of variables that are defined for any spatial geometry, i.e., the backreaction variables are functions defined over the full gravitational state space. Hence, the formalism can, in principle, accommodate any choice for the "true" spacetime metric. Each choice corresponds to different a region of the gravitational phase space, in which the backreaction variables take different values and may generate qualitatively different evolutions.

The method developed here implements steps 1 and 2 described in Sec. 1.2, in a gauge-invariant way. The level of description is the subspace of the gravitational phase space with initial data invariant under the action of the six-dimensional isometry group of the FRW spacetimes, and the projector $P$ is constructed by the group averaging of observables. Evolution equations for the relevant variables are obtained through the Hamiltonian formalism.

The third step involves assumptions about the nature of cosmological perturbations, i.e., it requires the specification of a region on the gravitational state space. This ought to be an observational, rather than a theoretical input, because the intuitions from non-equilibrium statistical mechanics are not

\footnote{We also note that other techniques from non-equilibrium statistical mechanics (mainly functional methods for quantum fields) have been employed in the context of early Universe cosmology–see [12] and references therein.}
directly relevant to the cosmological context. For small perturbations, we find generic regimes that lead to a closed set of backreaction equations, which can solve explicitly.

For large perturbations, additional variables appear in the effective evolution equations. It is, therefore, more difficult to obtain a closed set of backreaction equations. For this reason, in this paper, we only study the kinematics of backreaction in the large perturbations regime. In particular, we examine whether large backreaction effects can lead to an effective accelerated expansion. We find that there exists a plausible regime in the gravitational state space that manifests effective cosmic acceleration. This regime corresponds to initial data (present era) in the gravitational state space that satisfy a small number of conditions: these conditions are rather restrictive but they are generic, in the sense that they do not require a ”fine-tuning” of parameters. We find that acceleration necessitates a strong expansion of the congruences corresponding to comoving observers (i.e., an ”intrinsic” expansion of the inhomogeneous regions, in addition to the Hubble expansion), and a large negative value of a ”dissipation” variable that appears in the effective equations.

However, the physical relevance of this regime remains an open issue: it is necessary to demonstrate that the evolution of initial data in this regime correspond to a cosmological histories compatible with observations. Such a demonstration requires a full dynamical treatment for large perturbations, and it will be taken up in another work.

It is important to emphasize that in our approach the relevant variables are determined through integration over the spacetime group of isometries. In group-averaging the whole set of points of a Cauchy surface of a cosmological spacetime is involved; specific subsets cannot be isolated in a coordinate-independent way. For this reason, the method, in its present form, cannot provide a definition of an ”average geometry” for a generic spatial region. The scope is, therefore, different from that of the approaches reviewed in [7]. In particular, we do not aim to provide an answer to questions such as ”how does the Universe look at different scales?” [1].

1.4 Structure of the paper

In Sec. 2, we introduce the notion of group averaging in the context of general relativity, we study its properties and develop the calculational tools needed in the remaining of the paper. In Sec.3 we present the Hamiltonian formalism of perfect fluids (in the Lagrangian rather than the Eulerian picture), following the treatment of Ref. [15]; we show that the diffeomorphism constraints can be implemented without gauge-fixing.

In Sec. 4, we define the relevant variables for the treatment of backreaction, we elaborate on the Hamiltonian equations of motion, and we construct the backreaction equations. In Sec. 5, we study the evolution of the backreaction parameters in the regime of small perturbations. We specialize to dust-filled spacetimes and we identify regimes that lead to closed sets of backreaction equations. Large perturbations are taken up in Sec. 6: we find the corresponding backreaction equations, when spatial curvature effects are negligible, and we show that there are regimes that correspond to accelerated expansion. In the final section, we summarize our results, and briefly discuss possible extensions.
2 Group averaging

As we explained in the introduction, our aim is to incorporate the backreaction of the inhomogeneities into the evolution of homogeneous and isotropic FRW metrics. To this end, we must construct a map that takes generic inhomogeneous tensorial variables (for example, a Riemannian metric) into variables of the same type compatible with homogeneity and isotropy. A naive way to proceed would be to integrate the tensor field over a Cauchy surface and divide by the volume. However, coordinate-invariant integration can only be defined for scalar fields: spatial integration of a generic tensor field is not an invariant procedure. In particular, it is not possible to define an average metric through this method.

The method we develop here is based on group-averaging rather than spatial averaging. The method is suggested by the fact that the defining feature of the FRW metric is the existence of a six-dimensional group of isometries.

2.1 The basic construction

Let \( G \) be a compact Lie group acting on a compact three-manifold \( \Sigma \). This means that there exists a smooth map \( f : G \rightarrow Diff(\Sigma) \), such that \( f_{g_1} \circ f_{g_2} = f_{g_1g_2} \) for all \( g_1, g_2 \in G \).

For any tensor field \( A^{i_1\ldots i_n}_{j_1\ldots j_m} \) on \( \Sigma \), we define the group-averaged tensor field \( \langle A^{i_1\ldots i_n}_{j_1\ldots j_m} \rangle \) as

\[
\langle A^{i_1\ldots i_n}_{j_1\ldots j_m} \rangle(x) = \int d\mu[g] (f^*_g A)^{i_1\ldots i_n}_{j_1\ldots j_m}(x),
\]

where \( d\mu[g] \) is the left-invariant Haar measure on \( G \) (normalized to unity).

The invariance of \( d\mu[g] \) implies that

\[
[f^*_g \langle A^{i_1\ldots i_n}_{j_1\ldots j_m} \rangle](x) = \langle A^{i_1\ldots i_n}_{j_1\ldots j_m} \rangle(x).
\]

It follows that there exist vector fields \( X_A \) on \( \Sigma \) that correspond to elements \( A \) of the Lie algebra of \( G \), such that

\[
\mathcal{L}_{X_A} \langle A^{i_1\ldots i_n}_{j_1\ldots j_m} \rangle = 0.
\]

Let \( \Gamma \) be the state space of Hamiltonian initial data \( (h_{ij}, \pi^{ij}) \) for the gravitational field on a space-time with Cauchy surfaces of topology \( \Sigma \), where \( h_{ij} \) is a Riemannian metric on \( \Sigma \) and \( \pi^{ij} \) the conjugate momentum (a tensor density). \( \Gamma = T^*Riem(\Sigma) \), where \( Riem(\Sigma) \) is the space of Riemannian metrics on \( \Sigma \).

A tensorial variable \( A \) that is a functional of \( h_{ij} \) and \( \pi^{ij} \) corresponds to a family of functions on the state space \( \Gamma \). The map \( P : F(\Gamma) \rightarrow F(\Gamma) \), defined as \( P[A] = \langle A \rangle \) is a projector on \( F(\Gamma) \) and can be used to define the relevant variables with respect to the action of the group \( G \) on \( \Gamma \).

We also note that the map \( \Pi : \Gamma \rightarrow \Gamma \), defined as

\[
\Pi[(h_{ij}, \pi^{ij})] = [\langle h_{ij} \rangle, \langle \pi^{ij} \rangle],
\]

projects onto the submanifold \( \Gamma_0 \) of \( \Gamma \), which consists of initial data invariant under the action \( f \) of the group \( G \). If \( G \) is the 6-dimensional group characterizing homogeneous and isotropic spacetimes, \( \Gamma_0 \) consists of all constant curvature metrics on \( \Sigma \) and the corresponding conjugate momenta.
2.2 Comments

1. The construction above applies for a generic (compact) Lie group $G$ and it does not specifically require isotropy and homogeneity. For example, $G$ can be a group corresponding solely to special homogeneity (Bianchi models). The only requirement is that the group $G$ has a smooth action on the three-manifold $\Sigma$.

2. The group $G$ was assumed compact. (When $G$ implements the symmetry of homogeneity and isotropy, $\Sigma$ also must be compact: $\Sigma = S^3$.) However, the idea can also be applied to non-compact groups, provided they possess a left-invariant measure $d\mu[g]$: Let $O_n$ be a sequence of open subspaces of $G$ with compact support, such that $O_{n-1} \subset O_n$ and $\bigcup_n O_n = G$. Then, for any tensor field $A^{i_1 \ldots i_n j_1 \ldots j_m}$ on $\Sigma$ we define

$$\langle A^{i_1 \ldots i_n j_1 \ldots j_m} \rangle(x) = \lim_{n \to \infty} \frac{1}{\mu(O_n)} \int_{O_n} d\mu[g](f_g^* A)^{i_1 \ldots i_n j_1 \ldots j_m}(x).$$

If this limit exists, and if it is independent of the choice of the sequence $O_n$, then the group averaged tensor field $\langle A^{i_1 \ldots i_n j_1 \ldots j_m} \rangle$ is invariant under the action of the group $G$. The properties presented in Sec. 2.1 then follow.

In the following, we shall assume that the group $G$ is compact, keeping in mind that with suitable conditions the results can also be applied to the non-compact case.

3. From the physical point of view there is an ambiguity in the construction of the projector above. The action of a group $G$ on $\Sigma$ is unique most up to diffeomorphisms. If $f_g$ is an action of $G$ on $\Sigma$, and $F$ a diffeomorphism (that does not coincide with any of the diffeomorphisms $f_g$), then $F \circ f_g \circ F^{-1}$ is a different group action on $\Sigma$.

Group-averaging is equivariant with respect to the action of the diffeomorphism group $Diff(\Sigma)$, i.e.

$$F^* \langle A^{i_1 \ldots i_n j_1 \ldots j_m} \rangle = \langle (F^* A)^{i_1 \ldots i_n j_1 \ldots j_m} \rangle,'$$

where $\langle \cdot \rangle'$ denotes the average with respect to the group action $F \circ f_g \circ F^{-1}$.

Hence, if $\Gamma_0$ is the submanifold of initial data invariant under the $f$-action of $G$ on $\Sigma$, $F^* \Gamma_0$ is the submanifold invariant under the $F \circ f \circ F^{-1}$ action of $G$. The effective description of the group-averaged quantities admits the group $Diff(\Sigma)$ as a gauge symmetry. It is therefore necessary to reduce the system by removing the gauge degrees of freedom corresponding to the $Diff(\Sigma)$-symmetry. This is equivalent to the selection of a specific group action among the class of diffeomorphic equivalent actions. We shall see in the next section, that the presence of a perfect fluid allows for a gauge-invariant reduction of the $Diff(M)$ symmetry.

2.3 The group averaging calculus

Let $A$ be a tensor field invariant under the action of the group of isometries, i.e., $f_g^* A = A$ for all $g \in G$. Then for any tensor field $B$

$$\langle A \otimes B \rangle = \int d\mu[g] f_g^* (A \otimes B) = \int d\mu[g] f_g^* A \otimes f_g^* B = \int d\mu[g] A \otimes f_g^* B = A \otimes \int d\mu[g] f_g^* B = A \otimes \langle B \rangle.$$
This means that an invariant tensor can be taken out of the group averaging operation. The same holds if some indices of $A$ and $B$ are contracted.

Another important property of group averaging is that in a spacetime characterized by homogeneity and isotropy, the group-averaging of a scalar field equals the spatial average of the field on the three-surface $\Sigma$. The proof is the following.

We consider the case of compact three-surface $\Sigma$ and group $G$. Let $G_x$ be the stability group of $x \in \Sigma$, i.e. the subset of $G$ of all elements $g \in G$, such that $f_g(x) = x$. The quotient $G/G_x$ coincides with $\Sigma$, so that $G$ forms a fiber bundle over $\Sigma$ with fiber $G_x$. Considering a local trivialization of the bundle $\phi_x : G \rightarrow \Sigma \times G_x$, such that $\phi_x(g) = (f_g(x), g')$, where $g' \in G_x$. Then the measure $d\mu(g)$ splits as $d\mu(\Sigma)d\mu_{G_x}(g')$, $y = f_g(x) \in \Sigma$, where $d\mu_\Sigma$ is the G- invariant measure on $\Sigma$. Hence

$$\langle \phi \rangle(x) = \int d\mu(g)\phi[f_g^{-1}(x)] = \int d\mu_\Sigma(y)\phi(y) \left( \int d\mu_{G_x} \right) = c \int d\mu_\Sigma(y)\phi(y),$$

where $c = (\int d\mu_{G_x})$ is a constant (it does not depend on $\phi$). Since for the constant function $\phi(x) = 1$, $\langle \phi \rangle(x) = 1$, $c = 1/V$, where $V$ is the volume of $\Sigma$ with respect to the invariant metric. We have therefore shown that the group average of a scalar function equals its spatial average over $\Sigma$ with respect to the group-invariant measure.

The only properties of group averaging we will use in this paper are the identity (7) and group-averaging of scalar fields. The reason is that we will only encounter group averages of tensors with two indices, which, when contracted with the group-invariant FRW metric $\bar{h}_{ij}$, lead to group averages of scalars. It turns out that these are the only averages that appear explicitly in the backreaction equations.

3 The Hamiltonian description of fluids

The Hamiltonian formalism is well suited for dealing with the problem of backreaction, because the level of description associated with homogeneity and isotropy corresponds to a submanifold of the canonical state space of a gravity theory. Moreover, in spacetimes with a perfect fluid, the gauge symmetry of spatial diffeomorphisms can be factored out completely, due to the special properties of the perfect fluid’s Lagrangian. Perfect fluid spacetimes suffice for many cosmological applications. Dust-filled spacetimes, in particular, are relevant for examining the issue of backreaction-induced cosmic acceleration.

In this section, we present the Lagrangian and Hamiltonian formalism for relativistic gravitating perfect fluids. There exist several different approaches [14]; here, we follow an adaptation of the formalism presented in Ref. [15].

3.1 Perfect fluid Lagrangian

Thermodynamics. The thermodynamic properties of a fluid are encoded in the internal energy (Gibbs) functional $e(V, S)$, which expresses the internal energy per particle $e$ as a function of the specific volume $V$ and the specific entropy $S$. The first law of thermodynamics takes the form

$$de = -PdV + TdS,$$
where \( P \) is the pressure and \( T \) the temperature.

We consider the special case that the internal energy is a function of the specific volume \( V \) only—i.e., we ignore thermal effects. The internal energy can be written as \( e(1/n) \), where \( n = 1/V \) is the number density. The energy density is \( \rho = e/V = ne \). We note that for an equation of state \( P = w\rho \), the Gibbs functional is

\[
e = cn^w,
\]

where \( c \) a constant. In particular, for \( w = 0 \) (dust), we obtain \( e = c \), where \( c \) has dimension of mass and hence \( \rho = cn \).

The matter space. Let \( Z \) be the matter space, i.e., a three-dimensional manifold, whose points correspond to material particles. These particles are distinguishable, in the sense that each point of \( Z \) corresponds to a particle of definite identity. The configuration of the fluid is fully determined, if, for every point \( X \) of the spacetime \( M \), one specifies a particle \( z \in Z(X) \), whose worldline passes through \( X \). Hence, a configuration of the fluid is represented by an on-to mapping \( \zeta : M \rightarrow Z \). Given a coordinate system in \( Z \), the mapping is described by three functions \( \zeta_i(X) \).

Globally, if \( M = \Sigma \times \mathbb{R} \), where \( \Sigma \) is a three-manifold, \( Z \) must be diffeomorphic to \( \Sigma \). \( Z \) should be a homogeneous space, namely it should carry the transitive action of a Lie group. \( Z \) is also equipped with a volume three-form \( \nu \)

\[
\nu = \nu(z) \, dx^1 \wedge dx^2 \wedge dx^3,
\]

which measures the number of particles \( n(D) \) within any given region \( D \) in \( Z \): \( n(D) = \int_D \nu \).

The pullback \( \zeta^*\nu \) is a three-form on the spacetime \( M \). It is closed \( (d(\zeta^*\nu) = 0) \), since \( d\nu = 0 \) on the three-dimensional manifold \( Z \). This implies that the corresponding vector density

\[
j^\mu = -\nu \, \epsilon_{\mu\nu\rho} \partial_\nu \zeta^1 \partial_\rho \zeta^2 \partial_\sigma \zeta^3,
\]

satisfies the conservation equation

\[
\partial_\mu j^\mu = 0.
\]

The vector density \( j^\mu \) is the conserved particle-number current. It can be split into a normalized velocity vector field \( u_\mu \) and a particle number density \( n \) as

\[
j^\mu = \sqrt{-g} n u^\mu,
\]

whence

\[
n = \frac{1}{\sqrt{-g}} \sqrt{-g} j^\mu j^\nu g_{\mu\nu}.
\]
First-order Lagrangian. Having expressed the particle density as a function of the first derivatives of the fields $z^i(X)$, we write a first-order Lagrangian that reproduces the fluid’s equations of motion

$$L = \sqrt{-g}\rho[n] = \sqrt{-g}ne[1/n].$$

The three variational equations of $L$ with respect to $\zeta^i$ together with the equation (13) lead to the conservation equation for the stress-energy tensor

$$\nabla_\mu T^{\mu\nu} = 0,$$

where

$$T^{\mu\nu} = -\sqrt{-g}[\rho u^\mu u^\nu + P(g^{\mu\nu} + u^\mu u^\nu)].$$

When the gravitational field is included, the total action

$$S[g, \zeta] = \int d^4X \sqrt{-g} \left( \frac{1}{\kappa} R - \rho[n] \right)$$

leads to the Einstein equations $G^{\mu\nu} = \frac{8\pi}{\kappa} T^{\mu\nu}$, where $\kappa = 16\pi G$.

3.2 3+1 splitting

We next perform a change of variables in the action (19). This change of variables corresponds to a 3+1 splitting of spacetime with respect to the matter space $Z$. Let $t(X)$ be a time function on the spacetime $(M, g)$, i.e., a scalar function on $M$, such that the condition $t(X) = t$ defines Cauchy surfaces $\Sigma_t$, and $\bigcup_t \Sigma_t = M$.

We define a diffeomorphism from $M$ to $Z \times R$ as $X \rightarrow [\zeta(X), t(X)]$. The inverse map $E : Z \times R \rightarrow M$, $(x^i, t) \rightarrow X^\mu = E^\mu(X)$ is a functional of $\zeta$ and allows for the standard 3+1 decomposition of the spacetime metric $g_{\mu\nu}$.

Using the vector fields

$$t^\mu(X) = \frac{\partial E^\mu}{\partial t}[t(X), \zeta(X)]$$
$$\mathcal{E}^\mu_i(X) = \frac{\partial E^\mu}{\partial x^i}[t(X), \zeta(X)]$$

we express the Riemannian three-metric $h_{ij}$ on $\Sigma_t$ (obtained from the pull-back of $g_{\mu\nu}$ to $\Sigma_t$ under $\mathcal{E}$) as

$$h_{ij}(t, x) = \mathcal{E}_i^\mu(t, x)\mathcal{E}_j^\nu(t, x)g_{\mu\nu}[\mathcal{E}(t, x)].$$

Let $n^\mu$ be the unit normal on the surfaces $\Sigma_t$; then $n_\mu \mathcal{E}^\mu_i = 0$. The lapse function $N$ and the shift vector $N^i$ are defined as

$$t^\mu = N n^\mu + \mathcal{E}_i^\mu N^i.$$

The extrinsic curvature tensor on $\Sigma_t$ equals

$$K_{ij} := \mathcal{E}_i^\mu \mathcal{E}_j^\nu \nabla_\mu n_\nu = \frac{1}{2N}(\dot{h}_{ij} - 3\nabla_i N_j - 3\nabla_j N_i),$$

where

$$\dot{h}_{ij} := \frac{\partial h_{ij}}{\partial t}.$$
with $\nabla_i$ the covariant derivative on $Z$ compatible with the three-metric $h_{ij}$—the dot denotes derivative with respect to $t$. We also note that $\sqrt{-g} = N\sqrt{h}$, where $h = \text{det} h_{ij}$.

The current density $j^\mu$ of Eq. (12) takes the form

$$j^\mu = \nu t^\mu. \quad (25)$$

Substituting Eq. (25) into Eq. (14), we obtain

$$n = \nu \sqrt{1 - h_{ij} \tilde{N}^i \tilde{N}^j \sqrt{h}}, \quad (26)$$

where $\tilde{N}^i = N^i/N$.

The action (19) then equals (up to boundary terms)

$$S = \int dt \int d^3z N \sqrt{h} \left( K_{ij} K^{ij} - K^2 + 3R - \kappa ne[n] \right), \quad (27)$$

where $K = K_{ij} h^{ij}$ and $3R$ is the Ricci scalar on $Z$ associated to the three-metric $h_{ij}$.

The action (19) is a functional on a space $Q$ spanned by the 13 independent field variables $g_{\mu\nu}, \zeta^i$. However, after the redefinition of the variables involved in the 3+1 splitting, the action takes the form (27) that depends on the 10 variables $h_{ij}, N^i, N$. The change of variables allowed for the separation of the physical from the gauge degrees of freedom. The action (27) is defined on the space $Q_{\text{red}}$ spanned by the field variables $(h_{ij}, N^i, N)$.

This change of variables is physically equivalent to the imposition of the comoving coordinate conditions. However, no gauge-fixing was involved. The reduction took place at the Lagrangian level: the variables $h_{ij}, N, N^i$ are functionals of the original variables $g_{\mu\nu}, \zeta^i$. They are coordinates on the space $Q$, invariant, by construction, under the ‘gauge symmetry’ of spatial diffeomorphisms. It follows that (at least locally) $Q_{\text{red}}$ can be identified with the quotient space $Q/\text{Diff}(\Sigma)$.

The construction above does not apply to generic matter Lagrangians: the perfect fluid Lagrangian is special in that it allows a gauge-invariant way to select a spatial coordinate system tied on the matter degrees of freedom.

### 3.3 The Hamiltonian description

We next perform the Legendre transform of the action (27). The momenta conjugate to $h_{ij}$ are

$$\pi^{ij} = \frac{\sqrt{h}}{\kappa} \left( K^{ij} - Kh^{ij} \right), \quad (28)$$

while the variables conjugate to $N$ and $N_i$ vanish identically (they correspond to primary constraints).

The Legendre transform yields the Hamiltonian

$$H = \int d^3z N \left( \mathcal{H} + \tilde{N}^i \mathcal{H}_i + \sqrt{h} \rho \left[ \nu \sqrt{1 - h_{ij} \tilde{N}^i \tilde{N}^j \sqrt{h}} \right] \right), \quad (29)$$
where

\[ \mathcal{H} = \frac{\kappa}{\sqrt{h}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - \sqrt{h} \frac{3}{\kappa} R \]  
(30)

\[ \mathcal{H}^i = -2 \nabla_j \pi^{ij}. \]  
(31)

From now on we drop the prefix \(^3\) from the spatial covariant derivative.

Variation with respect to \( \tilde{N}^i \) yields

\[ \tilde{H}^i = -2 \nabla_j \pi^{ij}. \]  
(32)

which can be solved for \( \tilde{N}^i \):

\[ \tilde{N}^i = \frac{\tilde{H}^i}{\sqrt{\nu^2 F^2 + \tilde{H}^i \tilde{H}_i}}, \]  
(33)

where \( F(x) \) is a function of \( x = \mathcal{H}^i \mathcal{H}_i / \nu^2 \) defined implicitly by the algebraic equation

\[ F(x) = \rho' \left( \frac{\nu}{\sqrt{h}} \frac{1}{\sqrt{1 + x/F(x)^2}} \right). \]  
(34)

Substituting Eq. (33) back to the Hamiltonian we obtain

\[ H = \int d^3z N \left[ \mathcal{H} + \frac{\mathcal{H}_i \mathcal{H}^i}{\sqrt{\nu^2 F^2 + \tilde{H}^i \tilde{H}_i}} + \sqrt{h} \rho \left( \frac{\nu/\sqrt{h}}{1 + \mathcal{H}^i \mathcal{H}_i / \nu^2 F^2} \right) \right] \]  
(35)

Variation with respect to the lapse \( N \) yields the constraint equation

\[ \mathcal{H} + \frac{\mathcal{H}_i \mathcal{H}^i}{\sqrt{\nu^2 F^2 + \tilde{H}^i \tilde{H}_i}} + \sqrt{h} \rho \left( \frac{\nu/\sqrt{h}}{1 + \mathcal{H}^i \mathcal{H}_i / \nu^2 F^2} \right) = 0. \]  
(36)

The gravitating perfect fluid with the variables above is then a first-class constrained system, with the Hamiltonian vanishing on the constraint surface. We see that the use of coordinates tied on the matter state have led effectively to an automatic implementation of the spatial diffeomorphism constraints. Active diffeomorphisms are not gauge symmetries of the system described by Eqns. (35 and 36).

For a dust-filled spacetime \( (\rho = cn) \), we obtain \( F(x) = c \) and the Hamiltonian simplifies:

\[ H = \int d^3z N \left[ \mathcal{H} + \sqrt{\mu^2 + \tilde{H}^i \tilde{H}_i} \right], \]  
(37)

where \( \mu = c \nu \).
4 The effective description

4.1 The symmetry surface

Having obtained the Hamiltonian for a gravitating perfect fluid, we proceed to the description of the effective dynamics in terms of a homogeneous and isotropic metric.

The Hamiltonian (35) is defined on the space $\Gamma = T^* \text{Riem}(Z)$ spanned by the variables $h_{ij}(x)$ and $\pi^{ij}(x)$. $\Gamma$ carries a symplectic form

$$\Omega = \int_Z d^3x \delta \pi^{ij}(x) \wedge \delta h_{ij}(x).$$

(38)

For concreteness, we set $Z = S^3$. Let $0 h_{ij}$ be the metric of the unit sphere

$$0 h_{ij} dx^i dx^j = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2).$$

(39)

This metric is our standard for isotropy and homogeneity in the matter space. It has six linearly independent Killing vectors. These vectors generate an action $f$ of the group $G$ on $Z$. This action extends to an action on the space $\text{Riem}(Z)$ of three-metrics on $Z$ and on the phase space $\Gamma$. We will employ this group action in order to define group averaging for tensors on $Z$ as described in section 2.

Let $\Gamma_0$ be the submanifold of $\Gamma$ invariant under the action $f$; we will refer to $\Gamma_0$ as the symmetry surface. It consists of pairs $(\bar{h}_{ij}, \bar{\pi}^{ij})$ of the form $\bar{h}_{ij} = \alpha^2 0 h_{ij}$ and $\bar{\pi}^{ij} = \frac{p}{12\pi^2 \alpha} \sqrt{0 h} 0 h_{ij}$. $\Gamma_0$ is a symplectic submanifold of $\Gamma$, in which the variables $\alpha$ and $p$ define a symplectic chart, i.e., the restriction of $\Omega$ on $\Gamma_0$ is

$$\Omega_{\Gamma_0} = dp \wedge d\alpha.$$  

(40)

We also assume that the number density three-form $\nu$ is invariant under the group action on $Z$. This implies that $\nu = \nu_0 \sqrt{0 h}$, where $\nu_0$ is a constant.

The submanifold $\Gamma_0$ defines the level of description in terms of homogeneous and isotropic variables. We shall project the dynamics of the gravitating fluid onto $\Gamma_0$. The group action $f$ allows us to define group averaging for the variables $h_{ij}, \pi^{ij}$ through Eq. (1), and thus to construct a projector $\Pi : \Gamma \to \Gamma_0$, such that

$$\Pi(h_{ij}, \pi^{ij}) = (\langle h_{ij} \rangle, \langle \pi^{ij} \rangle).$$

(41)

Hence, the coordinates of any point of $\Gamma$ split uniquely as

$$(h_{ij}, \pi^{ij}) = (\bar{h}_{ij} + \delta h_{ij}, \bar{\pi}^{ij} + \delta \pi^{ij}),$$

(42)

where $\bar{h}_{ij} = \langle h_{ij} \rangle$, $\bar{\pi}^{ij} = \langle \pi^{ij} \rangle$ and $\langle \delta h_{ij} \rangle = 0$, $\langle \delta \pi^{ij} \rangle = 0$.

4.2 Choice of time variable

In order to write the evolution equations corresponding to the Hamiltonian (35) and the constraint (36), one needs to choose a time variable, i.e., to specify the time function $t(X)$ employed in the...
3+1 decomposition of the Lagrangian—see Sec. 3.2. To do so we must identify a family of timelike curves spanning the spacetime \( M \), such that \( t \) coincides with their proper time. The equations of motion obtained from the Hamiltonian (35) hold for any choice of time, as long as we take into account that the definition of the canonical data depends explicitly in the choice of the time function \( t^2 \).

The choice of time variable must be made consistently over the whole gravitational phase space. This is made possible by the presence of a perfect fluid, because there is a natural choice of time associated to comoving observers. Comoving observers correspond to the integral curves of the fluid’s four-velocity \( u^\mu \)—see Eq. (14). In matter coordinates, Eq. (14) implies that

\[
\frac{du^\mu}{\sqrt{-g}g^\mu_\nu \dot{t}_\nu} = \frac{1}{N \sqrt{1 - \dot{N}^i \dot{N}_i}} \left( \frac{\partial}{\partial t} \right)^\mu .
\]

The parameter \( t \) corresponds to the proper time along the comoving observers only if

\[
N = \frac{1}{\sqrt{1 - \dot{N}^i \dot{N}_i}}.
\]

The decomposition (23) of \( t^\mu \) implies that \( N^i = u^i \), where \( u^i \) is the spatial component of the three-velocity for the comoving observers; \( u^i \) corresponds to the fluid-particles’ deviation velocity from the uniform cosmological expansion. \( \dot{N}^i \) coincides with \( \dot{v}_i = u^i / \sqrt{1 + u^k u_k} \), namely, the three-velocity of observers along worldlines normal to the hypersurfaces \( \Sigma_t \).

Eq. (33) implies that

\[
u \dot{u}^i = \mathcal{H}^i - \dot{\mathcal{H}} / (\mathcal{H}^k \mathcal{H}_k / \nu^2).
\]

We note that for a FRW spacetime \( \mathcal{H}^i = 0 \); hence, \( u_i = 0 \): on a homogeneous and isotropic spacetime, the worldlines of the comoving observers are normal to the homogeneous and isotropic hypersurfaces. Moreover, they are geodesics (since \( N = 1 \)).

We next consider a spacetime that is not homogeneous and isotropic. The non-vanishing of the vector field \( u_i \) measures the degree of deviation of the comoving observers from being normal to the homogeneous and isotropic hypersurfaces and from being geodesics.

Test particles move on geodesics of the spacetime metric. This follows under the assumption that the particle’s backreaction to the geometry is insignificant. However, when we consider a large number of particles, we have to take their mutual interaction into account. This results into ‘forces’: the motion of an individual particle is no more a free-fall, and as such it does not correspond to a geodesic of the metric.

When we model an inhomogeneous spacetime by an FRW solution, we essentially assume that the world-lines of the particles are geodesics. Hence, we ignore the ‘inter-particle interaction’ and we assume that individual particles evolve under the average fields generated by the whole matter content of the universe. In this sense, the FRW description is conceptually similar to the mean field

\footnote{This means that the time function \( t(X) \) must be a functional of the configuration space variables \( g_{\mu\nu} \) and \( \zeta^i \). Hence, the foliation defined from \( t(X) \) and \( \zeta(X) \) is a functional of the configuration variables. Such “foliation functionals” have been introduced in Ref. \[16\]. The specific functional corresponding to time measured by comoving observers is equivariant with respect to spacetime diffeomorphisms.}
approximation in statistical mechanics. To move beyond the mean field approximation, it is necessary to consider fluctuations. In many-body systems, the effect of fluctuations is substantial if they exhibit strong (long-range) correlation. In analogy, the incorporation of back-reaction in the cosmological setting might result in a substantial change over the FRW predictions if there are persistent correlations between the properties of the inhomogeneous regions.

4.3 Equations of motion

Having chosen the time variable \( t \) to be the one measured by comoving observers, we write now the equations of motion following by the Hamiltonian (29). We recall that with this choice \( N^i = u^i = \mathcal{H}^i / \nu F(\mathcal{H}^k \mathcal{H}_k / \nu^2) \) and \( N = \sqrt{1 + u^i u^i} \).

\[
\dot{h}_{ij} = \frac{2N\kappa}{\sqrt{h}} (\pi_{ij} - \frac{1}{2} \pi h_{ij}) + \mathcal{L}_u h_{ij} \tag{46}
\]

\[
\dot{\pi}^{ij} = - \frac{N\sqrt{h}}{\kappa} (R^{ij} - \frac{1}{2} R h^{ij}) - \frac{2N\kappa}{\sqrt{h}} (\pi^{ik} \pi_k^j - \frac{1}{2} \pi^2) - \frac{N\kappa}{2\sqrt{h}} h^{ij} (\pi^{kl} \pi_{kl} + \frac{1}{2} \pi^2) + \sqrt{h} (\nabla^i \nabla^j N - h^{ij} \nabla^k \nabla_k N) + \mathcal{L}_u \pi^{ij} + \frac{1}{2} N\sqrt{h} \{ h^{ij} P(n) + u^i u^j [\rho(n) - P(n)] \}, \tag{47}
\]

where \( P = \rho - n \frac{\partial \rho}{\partial n} \) is the pressure, \( n = \frac{\nu}{\sqrt{h}} / \sqrt{1 + u^k u_k} \) and the Lie derivatives \( \mathcal{L}_u \) read explicitly

\[
\mathcal{L}_u h_{ij} = \nabla_i u_j + \nabla_j u_i \tag{48}
\]

\[
\mathcal{L}_u \pi^{ij} = \pi^{ik} \nabla_k u^j + \pi^{kj} \nabla_k u^i - u^k \nabla_k \pi^{ij} - u^{ij} \nabla_k u^k, \tag{49}
\]

The initial data satisfy the constraint equation

\[
\Phi = \frac{\kappa}{h} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) - \frac{1}{\kappa} R + [u^k u_k (\rho(n) - P(n)) + \rho(n)] = 0. \tag{50}
\]

Note that \( \Phi \) is a scalar on \( \Sigma \).

**Symmetric solutions** The symmetry surface \( \Gamma_0 \) is invariant under the Hamiltonian evolution. Setting \( h_{ij} = \alpha^2 \delta_{ij} \) and \( \pi^{ij} = \frac{p}{12\pi^2 \alpha} \sqrt{\alpha} \delta^{ij} \), we obtain \( u^i = 0, N = 1 \) and reduced Hamilton equations,

\[
\frac{d}{dt} \left( \frac{p}{\alpha} \right) = \left( \frac{p^2}{48\pi^2 \alpha^3} + \frac{2\pi^2}{\kappa \alpha} + 6\pi^2 P \alpha \right) \tag{52}
\]

\[
\frac{\dot{\alpha}}{\alpha} = -\frac{\kappa}{12} (\rho + 3P). \tag{54}
\]

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4.4 Effective equations

We next construct the effective evolution equations for homogeneous and isotropic metrics \( \bar{h}_{ij} \) and conjugate momenta \( \bar{\pi}^{ij} \). This is equivalent to a projection of the Hamilton equations from the full phase space to the symmetry surface \( \Gamma_0 \). Let the system be at point \((h_{ij}, \pi_{ij})\) of the constraint surface at time \( t \). The corresponding relevant variables are \((\bar{h}_{ij}, \bar{\pi}^{ij}) = (\langle h_{ij} \rangle, \langle \pi^{ij} \rangle)\); they define a point of the symmetry surface. At time \( t + \delta t \), the system lies at \((h_{ij} + \delta t \dot{h}_{ij}, \pi_{ij} + \delta t \dot{\pi}^{ij})\). The effective equation of motion is obtained by projecting \((h_{ij} + \delta t \dot{h}_{ij}, \pi_{ij} + \delta t \dot{\pi}^{ij})\) to \( \Gamma_0 \). Hence,

\[
\frac{d}{dt} \bar{h}_{ij} = \langle \dot{h}_{ij} \rangle \tag{55}
\]

\[
\frac{d}{dt} \bar{\pi}^{ij} = \langle \dot{\pi}^{ij} \rangle, \tag{56}
\]

where \( \dot{h}_{ij} \) and \( \dot{\pi}^{ij} \) are given by (46) and (47) respectively.

Equations (55,56) are exact: we made no assumptions about the form of the canonical variables \( h_{ij}, \pi^{ij} \). In order to derive a meaningful effective description, we must specify a region in the system’s phase space. This is equivalent to a choice of an approximation scheme, according to which the right-hand-side of Eqs. (55,56) will be evaluated.

Here, we employ the following approximations.

1. Perturbation expansion. We first effect of a perturbation expansion around the symmetry surface. We point out that we do not perturb around the solutions of the FRW equations, for in this case it would be impossible to obtain consistent backreaction equations.

The first-order terms in the perturbation expansion vanish. This is easy to see for ultra-local terms in the perturbations (for example ones that involve \( \langle \delta h_{ij} \rangle \)), because these vanish by definition. There are however non-ultralocal first-order terms, such as \( \nabla_k \delta h_{ij} \), which do not vanish identically. It turns out that the contribution of all such terms reduces to integrals over a total divergence and hence vanishes for compact spacetimes.

(This property follows from the high degree of symmetry of the FRW cosmologies. If, for example, we had considered expansion around the symmetry surface for a three-dimensional isometry group, the first-order contributions would be non-vanishing.)

Hence, the dominant terms in the back-reaction are second order to the perturbations. We note that to second order \( N \approx 1 + \frac{1}{2} \bar{h}^{ij} u_i u_j \), i.e., deviation velocities are ”non-relativistic”.

2. Ignore ultralocal terms. . The second assumption in our derivation of the backreaction equations is that the fluctuations \( \delta h_{ij}(z) \) and \( \delta \pi^{ij}(z) \) are small with respect to the averages \( \bar{h}_{ij}(z) \) and \( \bar{\pi}^{ij}(z) \) in an ultralocal sense. By this we mean that the trace norm of, say, the matrix \( \delta h_{ij}(z) \) must be a fraction \( \epsilon << 1 \) of the trace norm of the matrix \( \bar{h}_{ij}(z) \) at almost all points of \( z \). Note that there is no ambiguity in the notion of the trace norm, because \( h_{ij}(x) \) is defined with reference to the comoving co-ordinate system and there is no ”gauge” freedom from active diffeomorphisms. However, the perturbations are functions of \( x \): if their characteristic scale is of order \( l << 1 \), the action of a derivative operator leads to a term of order \( \epsilon/l >> \epsilon \). This implies that the ultralocal terms in the backreaction

\[\text{3The coordinates on } Z \text{ are dimensionless corresponding to the unit sphere. In terms of distances, the condition } l << 1 \text{ translates to fluctuations at a scale much smaller than the scale factor } \alpha.\]
equation are much smaller than the contain derivatives of $\delta h_{ij}, \delta \pi^{ij}$. We therefore drop all terms that involve only ultralocal terms of the perturbations.

Using the approximations above, we consider the special case of a dust-filled spacetime $\rho = cn$, where $c$ a constant. We write $\mu = cv$, and $\mu_0 = cv_0$, so that $\mu = \mu_0 \sqrt{\bar{h}}$; hence $\rho = c(\mu/\sqrt{\bar{h}})\sqrt{1 + u_ku^k}$.

The velocity field $u_i$ satisfies $u_i = \mathcal{H}_i/\mu$.

We define the tensor field $C_{jk}^i$ through the equation

$$\nabla_i V^j - \bar{\nabla}_i V^j = C_{jk}^i V^k,$$

(57)

where $\bar{\nabla}_i$ is the covariant derivative associated to $h_{ij} = \langle h_{ij} \rangle$. From this definition, we obtain

$$C_{jk}^i = \frac{1}{2} \bar{h}^{il} \left( \bar{\nabla}_j \delta h_{il} + \bar{\nabla}_k \delta h_{jl} - \bar{\nabla}_l \delta h_{jk} \right).$$

(58)

We shall also use the variables

$$\chi_i := \bar{h}^{kl} C_{kl}^i = -\frac{1}{\sqrt{\bar{h}}} \bar{\nabla}_k (\sqrt{\bar{h}} h^{ik}),$$

(59)

$$\kappa_i := C_{ji}^j.$$  

(60)

With the definitions above, $\mathcal{H}^i = \bar{\nabla}_j \delta \pi^{ij} + \pi^{kl} C_{kl}^i$. Since a term $\delta \pi^{ij} C_{jk}^i$ is of a higher order and does not appear in second-order expansion

$$\mathcal{H}^i = -2 (\bar{\nabla}_j \delta \pi^{ij} + \pi^{kl} C_{kl}^i) = -2 (\bar{\nabla}_j \delta \pi^{ij} + \frac{p}{12\pi^2 \alpha^2} \sqrt{\bar{h}} \chi_i).$$

(61)

Expanding Eqs. (55-56) to leading order (second) in perturbations around the symmetry surface and dropping the sub-dominant ultralocal terms we obtain

$$\frac{d}{dt} \alpha = -\frac{\kappa p}{24\pi^2 \alpha} \left( 1 + \frac{1}{2} \overline{u^2} \right) - \frac{1}{3} \Gamma \alpha,$$

(62)

$$\frac{d}{dt} \left( \frac{p}{\alpha} \right) = \frac{\kappa p^2}{48\pi^2 \alpha^3} \left( 1 + \frac{1}{2} \overline{u^2} \right) + \frac{12\pi^2}{\kappa \alpha} \left( 1 + \frac{1}{2} \overline{u^2} \right) + \frac{2\pi^2 \alpha}{\kappa} \Delta - 2\pi^2 \frac{h_0}{\alpha^2 \overline{u^2}} - \frac{2}{3} \Gamma \frac{p}{\alpha},$$

(63)

The quantity $\Delta$ is obtained from the perturbations of the scalar curvature and equals

$$\Delta = \bar{h}_{ij} \langle \sqrt{\bar{h}} (R^{ij} - \frac{1}{2} h^{ij} R) \rangle.$$ 

(64)

The quantity $\overline{u^2}$ is the mean-square deviation velocity

$$\overline{u^2} = \bar{h}^{ij} \langle u_i u_j \rangle,$$

(65)

and

$$\Gamma := \langle \chi^i u_i \rangle,$$

(66)
plays the role of a time dependent "dissipation" coefficient. We also average over the constraint Eq. (50). Keeping only the second-order non ultralocal terms, we obtain

\[ \frac{\kappa p^2}{96\pi^4\alpha^4} + \frac{6}{\kappa\alpha^3} + \frac{1}{\kappa} \delta R - \frac{\mu_0}{\alpha^2} (1 + \frac{1}{2} u^2) = 0, \]  

(67)

In Eq. (67) \( \delta R \) is the perturbation of the Ricci scalar \( \delta R = \langle R \rangle - \bar{R} \), where \( \bar{R} \) is the Ricci scalar associated to the metric \( \bar{h}_{ij} \). We find,

\[ \delta R = \langle \xi^i \xi_i - C^{ijk} C_{kij} \rangle. \]  

(68)

Eqs. (62, 63, 67) form a system of equations for the effective evolution on the symmetry surface \( \Gamma_0 \). These equations depend on four functions of time \( \Gamma(t), \overline{u^2}(t), \Delta(t), \delta R(t) \), whose form depends explicitly on the state of the non-relevant degrees of freedom; they are also functionally dependent on the relevant variables \( \alpha(t) \) and \( p(t) \). Since we have three differential equations determining the motion in a two-dimensional surface, one of them can be viewed as a compatibility condition between the four functions of time. It is convenient to use Eq. (63) for the determination of the independent function \( \Delta(t) \). Then the effective dynamics on \( \Gamma_0 \), can be described by Eqs. (62) and (67). These take the form

\[ \dot{\alpha} = \xi (1 + \frac{1}{2} u^2) - \frac{1}{3} \Gamma \alpha \]  

(69)

\[ \xi^2 + \frac{\alpha^2 \delta R}{6} = \frac{\kappa p^2}{6\alpha^2} (1 + \frac{1}{2} u^2). \]  

(70)

where we introduced

\[ \xi = - \frac{\kappa p}{24\pi^2\alpha}. \]  

(71)

In order to study the properties of the backreaction Eqs. (69) and (70), we must find explicit forms for the functions \( \Gamma(t), \overline{u^2}(t) \) and \( \delta R(t) \). We explore different regimes for them in the following sections.

5 The regime of small perturbations

5.1 Time evolution of the backreaction parameters

Eqs. (69) and (70) provide the evolution of the relevant (FRW) variables including the backreaction terms. They are valid at each moment of time. However, they are not, as yet, dynamical equations. The reason is that the coefficients \( \Gamma, \overline{u^2} \) and \( \delta R \) that appear in these equations depend on the variables \( \alpha \) and \( \xi \). We must find their explicit functional relation, in order to obtain a closed set of dynamical equations for the level of description.

To compute the variables \( \Gamma = \langle \lambda^i u_i \rangle \) and \( \overline{u^2} = \langle u_k u^k \rangle \), it is necessary to find the evolution equations for \( \lambda^i \) and \( u_i \). In this section, we do so, to leading (second) order in the perturbations, ignoring the ultralocal terms.
Evolution of $u^2$. Using Eqs. (31), (37) and the fact that $u_i = \mathcal{H}^i/ \mu$ we find that on the constraint surface

$$\dot{u}_i = -\nabla_i (u_k u^k).$$

Eq. (72) is \textit{exact}, i.e., no approximation has been employed in its derivation. We note that the right-hand-side in Eq. (72) is of higher order to the perturbations than the left-hand-side. Hence, to leading order $u^i$ is a constant. We do not expect this to be the case in general; however, this result suggests that for sufficiently small deviations from the FRW evolution, the deviation velocity vary slowly with time.

To leading-order in the perturbations $u^2 = \overline{h}^{ij}(t) \langle u_i u_j \rangle$, and since $u_i$ is constant

$$u^2(t) = \left( \frac{\alpha(t_0)}{\alpha(t)} \right)^2 u^2(t_0).$$

Evolution of $\Gamma$. In order to calculate $\lambda'$, we use Eqs. (59) and (46). We obtain

$$\dot{\lambda}^i = -3 \frac{\dot{\alpha}}{\alpha} \lambda^i - \frac{2\kappa}{\sqrt{h}} \nabla_k \pi^{ki} - \frac{\kappa}{2\sqrt{h}} \nabla_k (\pi h^{ki}) + \frac{1}{\sqrt{h}} \nabla_k [\sqrt{h} (\nabla^i u^k + \nabla^k u^i - h^{ki} \nabla_l u^l)].$$

(74)

Keeping first-order perturbation terms around the symmetry surface and using Eq. (61), we obtain

$$\dot{\lambda}^i = -3 \frac{\dot{\alpha}}{\alpha} \lambda^i - \kappa \frac{\mu u^i}{\sqrt{h}} - \frac{\kappa p}{24 \pi^2 \alpha^2} \lambda^i - \frac{\kappa}{2} \frac{\pi}{\sqrt{h}} \nabla_k (\nabla^i u^k + \nabla^k u^i - \overline{h}^{ki} \nabla_l u^l).$$

(75)

Then, to second order in perturbations

$$\dot{\Gamma} = \frac{\xi - 3 \dot{\alpha}}{\alpha} \Gamma - \kappa \rho u^2 + \frac{\kappa}{2} \psi + 3 \Theta^2$$

(76)

where we wrote

$$\psi = \langle \frac{\pi}{\sqrt{h}} \nabla_k u^k \rangle,$$

(77)

$$\Theta^2 = \frac{1}{3} ((\Theta^k_k)^2 - 2 \Theta_{ij} \Theta^{ij}),$$

(78)

and

$$\rho = \frac{\mu_0}{\alpha^3}$$

(79)

is the matter density of the corresponding homogeneous and isotropic solution.

The tensor $\Theta_{ij}$ is defined as

$$\Theta_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i).$$

(80)

$\Theta_{ij}$ is the expansion tensor for the (typically non-geodesic) congruence of comoving observers, defined with respect to the homogeneous and isotropic metric.
In linear response theory for non-relativistic fluids, the tensor $\Theta_{ij}$ is proportional to the fluid’s pressure tensor [9]. This suggests that $\Theta_{ij}$ incorporates the effects of pressure generated by the ”gravitational interaction” of the inhomogeneities. It vanishes for a homogeneous and isotropic solution.

In the regime that $u_i$ is constant, $\Theta_{ij}$ is constant, too. It follows that to leading-order in perturbations

$$\Theta^2(t) = \left(\frac{\alpha(t_0)}{\alpha(t)}\right)^4 \Theta^2(t_0). \quad (81)$$

Eq. (76) is a differential equation for $\Gamma$ in terms of $\bar{u}^2$ and $\Theta^2$, which are known functions of time, $\frac{\dot{\alpha}}{\alpha}$ and $\xi/\alpha$, which is described by the evolution equations and $\psi$, which is, as yet, undetermined. In effect, $\psi$ does not allow for the existence of a closed system of equations at our level of description. The derivation of a closed system of equations requires additional assumptions.

To this end, we consider different types of perturbations and how the evolution of $\Gamma$ differs according to the type.

**a. Dominant conformal perturbations.** In this regime, the perturbations of the metric and conjugate momentum are of the form $\delta h_{ij} \simeq b\bar{h}_{ij}$ and $\delta \pi^{ij} \simeq b'\sqrt{\bar{h}}\bar{h}^{ij}$, for some scalar functions $b$ and $b'$. Then, the term $\nabla_k(\pi h^{ik})$ in Eq. (75) equals $3\nabla_k \pi^{ki}$, and this leads to the following equation

$$\dot{\Gamma} = \frac{\xi - 3\dot{\alpha}}{\alpha} \Gamma - \frac{1}{4\kappa \rho u^2} + 3\Theta^2. \quad (82)$$

**b. Perturbations not affecting the extrinsic curvature scalar.** This regime includes the case of transverse-traceless tensor perturbations and transverse vector perturbation for both variables. In these cases, $\delta(\pi/\sqrt{h}) = 0$ and hence, $\psi = 0$. Then,

$$\dot{\Gamma} = \frac{\xi - 3\dot{\alpha}}{\alpha} \Gamma - \kappa \rho u^2 + 3\Theta^2 \quad (83)$$

**c. Vector perturbations.** In this regime, the terms corresponding to $\lambda^i$ and $\nabla_k \delta \pi^{ki}$ dominate in the fluctuations of the metric and momentum variables respectively. A two-tensor $A^{ij}$ corresponds to vector perturbations if it can be written in the form

$$A^{ij}(x) = \sum_q \left( \frac{q^i q^j}{q^2} + \frac{q^j q^i}{q^2} - (\beta \cdot q) \frac{q^i q^j}{q^4} \right) \alpha_q(x), \quad (85)$$

where $q$ denotes the wave-vectors, corresponding to eigenvalues of the Laplace equation for the metric $0h_{ij}$ and $\beta^i$ a vector-valued function of $q$. Assuming that $\delta \pi^{ij}$ and $\delta(\sqrt{h}h^{ij})$ are of this form, we find

$$\delta(\pi/\sqrt{h}) = -\nabla_i \left( \frac{\bar{\rho} u^i}{2} + \frac{8\xi}{\kappa \alpha} \lambda^i \right). \quad (86)$$
It then follows that
\[ \psi = \langle u^i P_{ij} \bar{\rho} u^j \rangle, \]  
(87)
where \( P_{ij} = \bar{\nabla}^i \bar{\nabla}^j \) is the projector to the longitudinal part of the vector fluctuations.

If the vector perturbations are purely longitudinal, then
\[ \psi = \bar{\rho} u^2. \]  
(88)
At the other end, if the vector perturbations are purely transverse, then we fall back to case b) and \( \psi = 0 \).

d. Mixed perturbations. In the previous three cases, the perturbations of the metric \( \delta h_{ij} \) and of the momentum \( \delta \pi^{ij} \) were of the same type. In principle, other cases are possible. For example, the perturbations \( \delta \pi^{ij} \) may be purely conformal and the perturbations \( \delta h_{ij} \) longitudinal. This is an acceptable region of the gravitational state space, since the only restriction is that the metric \( h_{ij} \) and the momentum \( \pi_{ij} \) satisfy the Hamiltonian constraint. For small perturbations, this regime is not dynamically stable (i.e., the system does not spend much time in the corresponding phase space region), because the linearized equations of motion couple perturbations of the same type. However, this regime might contain be stabilized when large perturbations are considered.

An interesting case is that of traceless perturbations \( \delta \pi^{ij} \) and longitudinal perturbations \( \delta h_{ij} \), for which
\[ \psi = \frac{2 \kappa}{\kappa \alpha} \langle u^i P_{ij} \lambda^j \rangle. \]  
(89)

Evolution of \( \delta R \). The scalar curvature \( R \) is a function on the state space, and under Hamilton’s equations evolves as
\[ \dot{R} = \frac{2N}{\sqrt{h}} R_{ij} (\pi^{ij} - \frac{1}{2} \pi h^{ij}) + u^i \nabla_i R. \]  
(90)
To second-order in the perturbation and ignoring the ultra-local terms, we find
\[ \overline{\dot{\delta R}} = -\frac{2 \kappa}{\kappa \alpha} \delta R + \frac{6 \xi}{\kappa^2 \alpha^3} u^2 + \langle \delta R \nabla_i u^i \rangle + \frac{2 \xi}{\kappa \alpha} \langle \lambda^i k_i - \kappa^i k_i \rangle \\
+ \frac{2}{\kappa} \langle \kappa_i \nabla_j \delta \left( \frac{\delta \pi^{ij} - \frac{1}{2} \pi h^{ij}}{\sqrt{h}} \right) \rangle - \frac{2}{\kappa} \langle C_k^{ij} \nabla_k \delta \left( \frac{\delta \pi^{ij} - \frac{1}{2} \pi h^{ij}}{\sqrt{h}} \right) \rangle. \]  
(91)

5.2 Autonomous backreaction equations
In the previous section, we derived the evolution equations for the functions \( \Gamma(t), \bar{u}^2(t) \) and \( \overline{\delta R}(t) \) that enter the backreaction equations \((69)\) and \((70)\) for the FRW variables \( \alpha \) and \( \xi \). If these equations contained no other variables other than \( \Gamma(t), \bar{u}^2(t) \) and \( \overline{\delta R}(t) \), we would obtain a closed dynamical
description of backreaction in terms of a small number of variables. We would then have effectively
substituted the infinite-dimensional dynamical system described by the Hamilton equations (46) and
(47) with a finite-dimensional one.

However, the set of equations for the variables $\Gamma(t), \overline{u^2}(t)$ and $\delta R(t)$ is not autonomous. It depends
on histories of additional variables, and these histories can be determined by deriving additional evolu-
tion equations. These in turn will contain additional variables and so on. This is a general feature in
the derivation of effective equations in systems with a large number of degrees of freedom. A com-
mon practice in non-equilibrium statistical mechanics is to enforce closure on a system of effective
equations for relevant variables by considering specific regimes, making simplifying assumptions, or
introducing phenomenological parameters.

In ordinary statistical mechanical systems, simplifications occur because we have some knowl-
gedge for the system that allows reasonable estimations about the state of the non-relevant variables,
like, for example, the assumption of local equilibrium or of molecular chaos in the treatment of gases.
However, in cosmology our present knowledge of the statistical behavior of the inhomogeneities is
not detailed enough for this purpose.

A simplification that allows for the derivation of a set of closed backreaction equations is to
assume that the term involving the perturbation $\delta R$ of the scalar curvature in Eq. (70 is negligible in
comparison to the other backreaction terms. There is no justification for this assumption, other than
the simplification it entails to the backreaction equations; in particular, it does not follow from any
observed properties of cosmological perturbations. However, it suffices to require that the contribution
of $\delta R$ to the effective equations is much smaller than only one of the other backreaction terms. This
terms dominates over the others as it does over $\overline{\delta R}$. It is not necessary to specify which one of
these terms is dominant: we keep them all in the solution of the equations. We must remember this
restriction when the issue of the validity of the approximations involved is to be considered. A

Setting $\delta R \simeq 0$, the backreaction equations under consideration are

$$\dot{\alpha} = \xi (1 + \frac{1}{2} \overline{u^2}) - \frac{1}{3} \Gamma \alpha$$

$$\xi^2 = \frac{\kappa \mu_0}{6 \alpha} (1 + \frac{1}{2} \overline{u^2}) - 1$$

$$\dot{\Gamma} = \frac{\xi - 3 \dot{\alpha}}{\alpha} \Gamma - \frac{\kappa \mu_0}{4 \alpha^3} \overline{u^2} + \frac{\kappa}{2} \psi + 3 \overline{\Theta^2},$$

where $\overline{u^2}$ and $\overline{\Theta^2}$ are functions of time given by Eqs. (73) and (81) respectively.

This set of equations can be made closed only if $\psi$ can be written as a function of the other
variables. We showed in section 5.1 that this is possible for the specific cases a) and b). We shall
also show that there are physically reasonable approximations that allow closure also for the case of
vector fluctuations.

For vector perturbations it is reasonable to consider an interpolation between the purely longitudi-
nal and the purely transverse cases. The simplest interpolation involves a simple parameter $\epsilon \in [0, 1],
which takes the value $\epsilon = 0$ for transverse vector perturbations and the value $\epsilon = 1$ for purely longi-

In fact, it is possible to find regimes leading to a closed set of backreaction equation without assuming that $\overline{\delta R}$
vanishes, but these regimes involve many additional conditions with no clear physical or geometric meaning.
tudinal perturbations. The parameter $\epsilon$ is introduced by postulating an approximation

$$\langle u^i P_{ij} \bar{p} u^j \rangle \simeq \epsilon \langle u^i \bar{h}_{ij} \bar{p} u^j \rangle,$$

from which we obtain

$$\psi = \frac{\epsilon}{2} \bar{u}^2.$$

In order to understand the meaning of the approximation (95) and to provide an interpretation for the parameter $\epsilon$, we next consider a simple model for the perturbations.

5.2.1 A model for perturbations.

Let us assume that perturbations $\delta h_{ij}(x)$ and $\delta \pi_{ij}(x)$ are concentrated in $N$ non-overlapping regions $U_n$ of compact support on $Z$, and that they vanish elsewhere. This implies that $u^i(x) = \sum_n u^i_{(n)}(x)$, where $u^i_{(n)}(x)$ is a field that has support only in the region $U_n$. We also assume that the regions $U_n$ are much smaller than the curvature radius of the homogeneous metric $\bar{h}_{ij}$, so that they are adequately described by Cartesian coordinate systems corresponding to the geodesics of $\bar{h}_{ij}$. We denote by $V_n$ the volume of each region $U_n$ with respect to the metric $\bar{h}_{ij}$ and by $v_n = V_n/V$ the corresponding relative volume ($V$ is the volume of $\Sigma$).

‘Rigid’ perturbations. The simplest approximation follows from the assumption that the regions $U_n$ are ‘rigid’, in the sense that they can be accurately described by the spatial averages $\bar{u}^i_{(n)}$ of the fields $u^i_{(n)}(x)$. (The spatial average is taken with respect to the local Cartesian coordinate system.) This means that $u^i_{(n)}(x) \simeq \bar{u}^i_{(n)}(x) \chi_{U_n}(x)$, where $\chi_{U_n}$ is a smeared characteristic function of $U_n$. We note that this case corresponds to $\Theta_{ij} \simeq 0$.

In this approximation

$$\psi = \frac{1}{N} \sum_{n=1}^N v_n L \bar{u}^i_{(n)} \frac{\bar{p} \bar{u}^i_{(n)}}{2},$$

where

$$L \bar{u}^i_{(n)} = \frac{1}{V_n} \int d^3x P^i_j u^j_{(n)}(x) \chi_{U_n}(x) = \frac{1}{V_n} \int \frac{d^3k}{(2\pi)^3} \frac{k^i k_j}{k^2} |\tilde{\chi}_{U_n}(k)|^2,$$

where $\tilde{\chi}_{U_n}(k)$ is the Fourier transform of $\chi_{U}$.

The value of $L \bar{u}^i_{(n)}$ depends on the shape of the region $U_n$. For a region with no preferred direction (i.e., a ball) $\tilde{\chi}_{U_n}(k)$ is a function only of $k^2$ and Eq. (98) yields

$$L \bar{u}^i_{(n)} = \frac{1}{3} \bar{u}^i_{(n)}.$$  

Then Eq. (97) leads to Eq. (95) with $\epsilon = 1/3$.

\footnote{A smeared characteristic function corresponds to a boundary that is not sharply defined. It is convenient in order to avoid problems arising from differentiation.}
A second interesting case is that of fluctuation regions $U_n$ with one preferred direction. Let $r_{(n)}^i$ be the unit vector corresponding to this direction; $\tilde{\chi}_{U_n}$ is a function of $r_{(n)} \cdot k$ and $k^2$. Eq. (98) yields

$$L \bar{u}_{(n)}^i = \frac{1}{2} - s_n \bar{u}_{(n)}^i + \frac{3s_n - 1}{2} (\bar{u}_{(n)}^i r_{(n)}^j) n_{i(n)},$$

(100)

where $s_n(x) \in [0, 1]$ is defined as

$$s_n = \frac{1}{V_n} \int \frac{d^3k}{(2\pi)^3} \frac{r_{(n)} \cdot k}{k^2} |\tilde{\chi}_{U_n} (r_{(n)} \cdot k, k^2)|^2$$

(101)

In general, $s_n$ depends on the relative size of the preferred direction: it takes values close to 1 for disk-shaped regions and close to 0 for rod-shaped ones.

Different assumptions about the distribution of the directions of $r_{(n)}^i$ corresponds to different values of $\epsilon$. For example, we may assume that the directions $r_{(n)}^i$ and the shape parameters $s_n$ are not statistically correlated (i) with each other, (ii) with $\bar{u}_{(n)}^i$, and (iii) along different regions $U_n$. Then, Eq. (97) yields (95) with $\epsilon = \frac{1}{3}$. Hence, the shape of the inhomogeneity regions does not affect the value of $\epsilon$ if its determining parameters can be treated as uncorrelated random variables.

In presence of correlations, the value of $\epsilon$ differs. For example, if the mean velocity of the inhomogeneous regions tends to align along the axis $r_{(n)}^i$, then $L \bar{u}_{(n)}^i = s_n \bar{u}_{(n)}^i$. Substituting into Eq. (97) (and assuming no correlations for the value of $s_n$ in different regions), we find $\epsilon = \frac{1}{N} \sum_n \bar{s}_n$, i.e., $\epsilon$ can take any value in the interval $[0, 1]$, depending on the distribution of shapes for the inhomogeneity regions. Similarly, if $u_{(n)}^i$ tends to lie on the plane normal to $r_{(n)}^i$, we find that $\epsilon \simeq 1 - \frac{1}{N} \sum_n \bar{s}_n$.

**Expanding (contracting) perturbations.** If the mean velocities $\bar{u}_{(n)}^i$ of the regions $U_n$ are small compared to the values of $u_{(n)}^i(x)$, then the regions $U_n$ are ‘at rest’ in the comoving coordinates and the field $u_{(n)}^i(x)$ primarily contributes to the change of their shape. The longitudinal part corresponds to change in volume, while the transverse part corresponds to volume-preserving (‘tidal’) deformations of the regions’ shape. The parameter $\epsilon$ then corresponds to the fraction of the (non-relativistic) kinetic energy of the fluid that corresponds to expansion (or contraction) of the region. In this light, the approximation (95) is equivalent to the assumption that the tidal deformations in the regions $U_n$ are not persistent and they vanish on the mean—hence, that $\psi$ receives contributions only from the fraction of the velocity field that corresponds to changes of volume in the perturbed regions.

With similar arguments, we can show that for the case of asymmetric perturbations—Eq. (89), the approximation $\langle u^i P_{ij} \lambda^j \rangle \simeq \langle u^i \tilde{h}_{ij} \lambda^j \rangle$ yields

$$\psi = \frac{2\epsilon}{\kappa \alpha} \Gamma$$

(102)

The parameter $\epsilon$ is therefore introduced as a phenomenological quantity that characterizes the state of the perturbations (i.e. the actual three- metric) at a moment of time $t$. In a sense, it is analogous to similar phenomenological quantities that are introduced in non-equilibrium thermodynamics, e.g., coefficients of viscosity, heat transfer and so on. The approximation (95) is not expected to be meaningful over the whole of the gravitational state space, but only in a specific region, corresponding, to
the cases considered above. $\epsilon$ is expected to change in time, as the perturbations evolve. One then needs also assume that such changes occur in a time-scale much larger than the ones relevant to the resulting effective equations.

Either with the introduction of the $\epsilon$ parameter, or by restricting to the specific regimes considered earlier, Eqs. (92-94) together with Eqs. (73) and (81) provide a consistent and closed set of backreaction equations to leading order in perturbations around the symmetry surface. These equations hold if the perturbation terms are significantly smaller to the ones of FRW evolution. Applied to cosmology, such backreaction terms could perhaps lead to corrections on the order of a few percent from the FRW evolution.

5.3 Solutions

We next solve the backreaction equations (92-94) for the scale factor, assuming that the perturbation parameters $\Gamma$, $\overline{u^2}$ and $\overline{\Theta^2}$ are small so that only their first-order contributions to their evolution equations are significant. We assume that the geometry is approximately flat so that in Eq. (94) $\xi >> 1$ and $\sqrt{\kappa_{\mu0}} \alpha >> 1$; this assumption allows for an analytic solution of the differential equations.

Using Eqs. (73), (81) and (94) we find

$$\dot{\Gamma} = -2\frac{\dot{\alpha}}{\alpha}\Gamma - \frac{f\kappa_{\mu0}u_0^2\alpha_0^2}{\alpha^5} + 3\frac{\theta_0^2\alpha_0^4}{\alpha^4},$$

(103)

where we set $\alpha(t_0) = \alpha_0$, $u_0^2(t_0) = u_0^2$ and $\overline{\Theta^2}(t_0) = \theta_0^2$, for some time $t_0$. The parameter $f$ takes the values $\frac{1}{4}$, $1$ and $1 - \frac{\epsilon}{4}$ for the cases a, b and c of Sec. 5.2, respectively.

Using Eq. (92) and keeping terms of leading order to the perturbations we obtain

$$\frac{d\Gamma}{d\alpha} = -2\frac{\dot{\alpha}}{\alpha}\Gamma - \sqrt{\frac{6\kappa_{\mu0}fu_0^2\alpha_0^2}{a^9/2}} + \frac{3\theta_0^2\alpha_0^4\sqrt{\frac{f}{\kappa_{\mu0}}}}{\alpha^7/2}$$

(104)

This is a linear inhomogeneous equation with solution

$$\Gamma = \left(\frac{\alpha_0}{\alpha}\right)^2 \left\{ \Gamma_0 + b_1 \left[ \left(\frac{\alpha_0}{\alpha}\right)^{3/2} - 1 \right] - b_2 \left[ \left(\frac{\alpha_0}{\alpha}\right)^{1/2} - 1 \right] \right\},$$

(105)

where $b_1 = \sqrt{\frac{6\kappa_{\mu0}}{\alpha_0}} \frac{\epsilon u_0^2}{3}$ and $b_2 = 6\theta_0^2 \sqrt{\frac{6\kappa_{\mu0}}{\alpha_0}}$, and $\Gamma_0 = \Gamma(t_0)$.

Substituting into Eq. (92) and changing into the variable $u = (\alpha/\alpha_0)^{3/2}$, we find

$$\frac{2}{3} \frac{d}{du} = \sqrt{\frac{\kappa_{\mu0}}{6}} \left[ 1 + A u^{-4/3} + B u^{-2/3} + C u^{-1/3} \right],$$

(106)

in terms of the constants

$$A = \frac{u_0^2}{\alpha_0^{3/2}} \left( \frac{3}{4} - \frac{4f}{3} \right)$$

(107)

$$B = 12 \frac{\theta_0^2 \alpha_0^{3/2}}{\kappa_{\mu0}}$$

(108)

$$C = -\sqrt{\frac{2}{3\kappa_{\mu0}}} \Gamma_0 + \frac{4u_0^2}{3\alpha_0^{3/2}} - 12 \frac{\theta_0^2 \alpha_0^{3/2}}{\kappa_{\mu0}}.$$
Figure 1: The normalised scale factor $\alpha/\alpha_0$ as a function of time $t$ in units of the Hubble time $H_0^{-1}$; $H_0 = \dot{\alpha}/\alpha(t_0)$, for $\theta_0^2 = 0.01, u_0 = 0.0001, \Gamma_0 = -0.1$. The dashed line shows the FRW solution.

We then obtain the solution to the backreaction equations

$$
\int_1^{(\alpha/\alpha_0)^{3/2}} \frac{du}{1 + Au^{-4/3} + Bu^{-2/3} + Cu^{-1/3}} = \frac{3}{2} \sqrt{\frac{\kappa \mu_0}{6}} (t - t_0). \tag{110}
$$

In Fig. 1 a plot is shown of a solution to Eq. (110) in comparison to the FRW-solution. The two curves diverge at early times. However, in this regime the backreaction parameters are large; for example $\Theta^2$ grows at small $\alpha$ as $\sim \alpha^{-4}$. The approximation employed in the derivation of (110) is therefore not reliable at very early times and the divergence may be absent in the regime of large perturbations.

The approximation is preserved for later times, and then we see that $\alpha(t)$ essentially coincides with the FRW solution. Indeed, let us denote by $\alpha_{FRW}(t)$ the FRW solution corresponding to $\alpha(t_0) = \alpha_0$ ($A = B = C = 0$), and by $\alpha(t)$ the solution corresponding to $\alpha(t_0) = \alpha_0$ for non-zero but small values of the parameters $A, B$ and $C$. The difference $\delta \alpha(t) = \alpha(t) - \alpha_{FRW}(t)$ has the following asymptotic behavior

$$
|\delta \alpha(t)| \sim C \alpha(t). \tag{111}
$$

This implies that the change in the Hubble parameter $H(t) = (\dot{\alpha}/\alpha)(t)$ is $|\delta H(t)| \sim C$, i.e., the FRW solution is stable with respect to small perturbations.

This result holds for symmetric perturbations (e.g., cases a, b and c of Sec. 5.2). If, however, the perturbations are mixed (case d of Sec. 5.2), the conclusions may be different. The solution to the backreaction equations for $\psi$ given by Eq. (102) can be obtained through a similar procedure to the one leading to Eq. (110):

$$
\int_1^{(\alpha/\alpha_0)^{3/2}} \frac{du}{1 + A'u^{-4/3} + B'u^{-2/3} + C'u^{-1/3}} = \frac{3}{2} \sqrt{\frac{\kappa \mu_0}{6}} (t - t_0), \tag{112}
$$

in terms of constants $A', B'$ and $C'$ that depend on the values of $\theta_0, u_0$ and $\Gamma_0$. In this case, the deviation $\delta \alpha$ from the FRW evolution behaves asymptotically as

$$
|\delta \alpha(t)| \sim C' [\alpha(t)]^{1+\epsilon}. \tag{113}
$$
Hence, for large values of $t$, the change in the Hubble parameter grows as $\delta H(t) \sim [\alpha(t)]^\epsilon$. Even a small value of $\epsilon$ produces an evolution that diverges from the FRW prediction. However, an assumption in this derivation is that the value of $\epsilon$ at time $t_0$ remains constant in time, which, in the linearized treatment of perturbations, is only plausible for very special conditions at $t = t_0$. Nonetheless, this result strongly suggests that even small perturbations may make problematic the extrapolation of the FRW evolution into the asymptotic future.

6 Large perturbations

6.1 The approximation scheme

Part of the motivation for this work is the idea that the cosmic acceleration, deduced from the supernova data, can be accounted by backreaction effects rather than dark energy. This is only possible if the perturbations are large: the corresponding terms must be of the same order of magnitude as the ones of FRW evolution. The approximation used in the previous section is therefore not sufficient for this case: terms of third and higher order to the perturbations $\delta h_{ij}, \delta \pi^{ij}$ from the symmetry surface $\Gamma_0$ are expected to become significant.

The effective equations describing the backreaction of large perturbations depend strongly on the region of the gravitational phase space to which they correspond. Inevitably, in order to construct such a set of equations for the cosmological setting, it is necessary to have substantial information about the nature and behavior of inhomogeneities in the present-day Universe. In absence of such information, any model describing backreaction of large perturbations must proceed by more or less ad hoc assumptions about the nature of inhomogeneities. Such assumptions are necessary both for the implementation of a consistent approximation scheme (i.e., which effective variables can be thought of as ”large”?) and for the closure of the system of effective equations.

In this section, we will explore one regime of large perturbations, which corresponds to a rather natural generalization of the perturbation scheme developed in the previous section. We do not construct a closed set of backreaction equations for the general case as in the previous section; the problem is substantially more involved. Our study is mainly kinematical: we explore the plausibility of the idea that large perturbations around an FRW universe can be held responsible for the observed acceleration of the scale factor.

To simplify our calculations we assume that the deviation velocities are very small, namely that $u^2 \ll 1$. Indeed, this condition is born out by observations, if we identify $u_i$ with velocities of galaxy clusters—see [17] and references therein. The term $u^2$ cannot be, therefore, responsible for the substantial divergence from the FRW prediction that would be necessary, in order to account cosmic acceleration in terms of backreaction effects. We will drop such terms from our calculations. As in the previous section, we shall also ignore the the perturbations of the curvature scalar and consider almost flat three-geometries. Consequently, the only large backreaction variable in the effective equations (69–70) is $\Gamma$.

In order to construct backreaction equations for large perturbations, we employ the following approximation scheme. We assume that the perturbations $\delta h_{ij}$ and $\delta \pi^{ij}$ are small in the ultra-local sense, but that they may be large when acted upon by derivatives. This means that the backreaction terms may become large because of the strong spatial variation of the inhomogeneities. In particular,
we assume that the perturbations $\delta h_{ij}$ and $\delta \pi^{ij}$ are a fraction of order $\epsilon$ of the averaged variables $\bar{h}_{ij}$ and $\bar{\pi}^{ij}$ (with respect to a matrix norm), and that the variation of the perturbations takes place at a scale $l$ on the spatial surface. Expanding to the $n$th order of perturbations around the symmetry surface, a term that involves $m$ derivatives will be of order $\epsilon^n/l^m$. The approximation scheme we shall use in this section involves keeping all terms such that $m - n \geq 0$, which means that $\epsilon$ is of the same order of magnitude with $l$.

### 6.2 The effective equations

In the approximation scheme described above, the backreaction equations (69–70) remain unchanged. The higher order backreaction terms are characterized by negative values of $m - n$. Again, as in the previous section, one of these equations can be used to determine the functional dependence of the curvature perturbations $\delta R$ in terms of the other parameters. Dropping the contribution of the $u^2$ terms, Eqs. (69–70) take a simple form

$$\dot{\alpha} = \frac{1}{3} \Gamma \alpha$$

$$\xi^2 + 1 = \frac{\kappa \mu_0}{6 \alpha}$$

Eq. (74) for the evolution of $\lambda^i$ is valid also within this scheme. Together with (72), it leads to the following equation for $\Gamma$

$$\dot{\Gamma} = \frac{\xi - 3 \dot{\alpha}}{\alpha} \Gamma + 3 \theta^2 + \omega$$

Compared to Eq. (76), terms $u^2$ have been dropped (in particular $\psi \approx 0$) and there appears a new term.

$$\omega = \langle \bar{\nabla}_i \lambda^i u_k u^k \rangle + 2 \langle \bar{\nabla}_k u^i C^{iik} u_l \rangle - \langle \bar{\nabla}_i u^i \lambda^k u_k \rangle.$$

In the derivation of Eq. (116), we dropped ultralocal terms $\delta h_{ij}$ when they are additive to terms $\bar{h}_{ij}$: for example, we approximated $\langle \bar{\nabla}_i \lambda^i u_k u^k h^{kl} \rangle \approx \langle \bar{\nabla}_i \lambda^i u_k u_l \bar{h}^{kl} \rangle$.

To simplify Eq. (117), we further assume that the tensor $\Theta_{ij} = \frac{1}{2} (\bar{\nabla}_i u_j + \bar{\nabla}_j u_i)$ is isotropic, i.e., that

$$\Theta_{ij} = \Theta \bar{h}_{ij},$$

where $\Theta$ is a scalar function on $Z$. This means that expansion dominates over shear in the congruence of comoving observers.

Condition (118) implies that

$$\omega = -3 \langle \Theta \lambda^i u_i \rangle.$$
The right-hand-side of (119) corresponds to a spatial integral over $Z$. Schwarz’ inequality then applies

$$|\omega|^2 \leq 9 \langle \Theta^2 \rangle \left( \langle (\lambda^k u_k)^2 \rangle \right).$$  \hspace{1cm} (120)

We define the deviation $\delta \Gamma$ for $\lambda^k u_k$ as

$$(\delta \Gamma)^2 = \langle (\lambda^k u_k)^2 \rangle - \Gamma^2,$$  \hspace{1cm} (121)

Eq. (120) then can be written as

$$|\omega| \leq 3 \sqrt{\Theta^2 \Gamma^2 + (\delta \Gamma)^2}.$$  \hspace{1cm} (122)

This implies that

$$\omega = 3 \eta \sqrt{\Theta^2 \Gamma \sqrt{1 + (\delta \Gamma/\Gamma)^2}},$$  \hspace{1cm} (123)

for $|\eta| < 1$. The parameter $\eta$ is, in general, a function of time. We may view it as a phenomenological parameter characterizing the present state of fluctuations. It characterizes the strength of correlations between the backreaction parameters $\Theta^2$ and $\Gamma$. If $\eta = 0$, then there is no correlation and $\omega = \langle \Theta \rangle \langle \lambda^k u_k \rangle = 0$, since $\langle \Theta \rangle = \frac{1}{3} \langle \nabla_k u^k \rangle = 0$. If $\eta = \pm 1$ then the Schwarz inequality (120) is saturated and $\Theta = a \lambda^k u_k$ for some constant $a$ at all points of $Z$.

### 6.3 Cosmological models with backreaction

With the assumptions state above, the set of equations for the backreaction consists of Eqs. (114, 115, 116, 123) and the expressions for $\psi$ provided in the previous section. In general, this does not constitute a closed set of equations, because we have not provided an evolution equation for $\Theta^2$ that appears in Eq. (123). The set of equations only closes for the case of longitudinal ‘rigid’ perturbations considered in section 5.3. However, these expressions suffice for establishing the plausibility that cosmic acceleration is possible in the regime of large perturbations we have considered here.

We assume that the effective FRW spacetime is almost flat, i.e. that Eq. (115) implies that

$$\xi = \sqrt{\frac{\kappa \mu_0}{6 \alpha}}.$$  \hspace{1cm} (124)

The following equations then apply

$$\left( \frac{\dot{\alpha}}{\alpha} + \frac{1}{3} \Gamma \right)^2 = \frac{\kappa \bar{\rho}}{6}$$  \hspace{1cm} (125)

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa \bar{\rho}}{12} - \frac{1}{6} \Gamma \dot{\alpha} + \frac{1}{18} \Gamma^2 - \frac{1}{3} \dot{\Gamma},$$  \hspace{1cm} (126)

$$\ddot{\Gamma} = -2 \Gamma \dot{\alpha} + \frac{1}{3} \Gamma^2 + 3 \left( \Theta^2 + \eta \sqrt{\Theta^2 \Gamma \sqrt{1 + (\delta \Gamma/\Gamma)^2}} \right),$$  \hspace{1cm} (127)

Substituting (127) into (126) we obtain an equation for the acceleration $\ddot{\alpha}$:

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa \bar{\rho}}{12} + \frac{1}{2} \Gamma \ddot{\alpha} - \frac{1}{18} \Gamma^2 - \left[ \Theta^2 + \eta \sqrt{\Theta^2 \Gamma \sqrt{1 + (\delta \Gamma/\Gamma)^2}} \right]$$  \hspace{1cm} (128)
Comparing Eqs. (125–128) to the standard FRW equations, we note the following. First, the constraint equation (125) is dissimilar to the FRW equations $H^2 = \frac{\kappa}{6} \rho$ irrespective of the theory’s matter content. Dark energy or a cosmological constant or any other form of matter will be added to the energy density term. The backreaction term $\Gamma$ is additive to the Hubble parameter $H$, which appears squared in the FRW equations. A negative-valued $\Gamma$ implies that the actual value of the Hubble parameter is larger than the one predicted from the matter content of the FRW solution.

Moreover, the presence of the $\Gamma$-dependent terms does not allow us to bring Eq. (128) into an FRW-like form

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa}{12} (\bar{\rho} + 3P_{\text{eff}})$$

in terms of some “effective pressure” some $P_{\text{eff}}$. Again, this indicates how the backreaction terms differ in their structure to the ones that can be obtained from any form of matter. The presence of $\Theta$-dependent terms in the right-hand of Eq. (128) would suggest that at least this part of the acceleration could be interpreted as some sort of bulk viscosity pressure generated by the “gravitational interaction” between the inhomogeneities—see [18] for a possible relation of bulk viscosity to cosmic acceleration. However, the dependence of the acceleration on pressure is quadratic on $\Theta$, instead of linear as would be expected from an ordinary viscous fluid. Overall, the structure of the backreaction terms are very different from what one would expect from any form of matter.

We next consider Eq. (125) at the present moment of time $t = t_0$. Defining the Hubble constant $H_0 = (\dot{\alpha}/\alpha) (t_0)$ and $\Omega_m = \frac{\kappa \rho(t_0)}{6H_0^2}$ we obtain

$$1 + \frac{\Gamma(t_0)}{3H_0} = \sqrt{\Omega_m}.$$ 

(130)

If $\Omega_m < 1$, then $\Gamma(t_0) < 0$. We introduce the parameter $\gamma_0 := -\Gamma(t_0)/(3H_0)$; substituting into (126) we calculate the deceleration parameter $q_0 = -\frac{\ddot{\alpha}}{\alpha}(t_0)$ as

$$q_0 = \frac{1}{2} + 2\gamma_0 + \gamma_0^2 + 3(\theta_0^2 - \eta \theta_0 \gamma_0 \sqrt{1 + (\delta \gamma_0/\gamma_0)^2}),$$

(131)

where $\theta_0 = \sqrt{\Theta^2}(t_0)$, and we also defined $\delta \gamma_0 := -(\delta \Gamma)(t_0)/(3H_0)$. 

6.4 An exactly solvable regime

Before proceeding to an examination of Eq. (131), we note that in the regime of “rigid” perturbations described in Sec. (5.2), the system of backreaction equations is closed. In this case, $\Theta^2 = 0$, and the set of backreaction equations (114, 115, 116) reduces to

$$\dot{\alpha} = \sqrt{\frac{\kappa H_0}{6\alpha}} - \frac{1}{3} \Gamma \alpha$$

(132)

$$\dot{\Gamma} = -2\frac{\dot{\alpha}}{\alpha} \Gamma + \frac{1}{3} \Gamma^2.$$ 

(133)

This system of equations can be integrated to determine the functional dependence between $\Gamma$ and $\alpha$. 

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Figure 2: Plots of the decelerations parameter $q_0$ as a function of $\Omega_m$, for $\eta = 0.95$, $\theta_0 = 0.4$ and different values of $\delta \gamma_0$: from top to bottom: $\delta \gamma_0 = 0.8$, $\delta \gamma_0 = 1.0$, $\delta \gamma_0 = 1.2$.

$$\frac{\Gamma}{H_0} \left( \frac{\alpha}{\alpha_0} \right)^{3/2} = \sqrt{\Omega_m} + (\Gamma/\Gamma_0)^{3/2}(\alpha/\alpha_0)^3(\Gamma_0 - \sqrt{\Omega_m})$$

(134)

Eq. (134) is an algebraic equation that can be implicitly solved to determine $\Gamma$ as a function of $\alpha$. Substituting $\Gamma(\alpha)$ into Eq. (132) then provides a closed differential equation for the scale factor $\alpha$.

Since $\Gamma_0$ is negative, Eq. (133) implies that it was still negative and with larger negative value in the past. Then Eq. (128) implies that the acceleration ratio $\dot{\alpha}/\alpha$ was negative and with larger absolute value in the past. It follows that Eqs. (132) and (133) describe a universe that has expanded more slowly than the FRW solution. Hence, this regime does not correspond to an evolution compatible with the supernova data.

### 6.5 Cosmic acceleration

We return to Eq. (131), and examine whether there is a regime in which acceleration is possible ($q_0 < 0$). We note the following

- For $\theta_0 = 0$ acceleration is not possible. For acceleration, it is necessary that the $\theta_0$-dependent term is negative-valued, i.e., that $\eta > 0$ and that $0 < \theta_0 < \eta \gamma_0 \sqrt{1 + (\delta \gamma_0/\gamma_0)^2}$.

- A necessary condition for acceleration ($q_0 < 0$) is that $\eta \delta \gamma_0 > \frac{2}{3}$; this implies, in particular, that if $\delta \gamma_0 \simeq 0$, no acceleration is possible.

Plots of $q_0$ as a function of $\gamma_0$ are provided in Fig. 2 for specific values of $\theta_0$, $\eta$ and $\delta \gamma_0$. The region of the gravitational state space region that gives rise to negative $q_0$ is characterized by negative value for the parameter $\Gamma = \langle \lambda^k u_k \rangle$. Moreover, the field $\Theta = \frac{1}{3} \nabla_i u^i$ must be non-zero and positive-valued (by virtue of Eqns. (119), (123) and of the fact that $\eta > 0$). This means that the expansion of the congruence of comoving observers must be significant. In other words there must be a strong expansion of the inhomogeneous regions, in addition to the Hubble expansion. A substantial correlation of the fields $\Theta$ and $\lambda^k u_k$ is also necessary.
The above are strong restrictions, but they do not require any fine-tuning of parameters. They correspond to a fairly generic region of the gravitational state space.

The conclusion above only involved a kinematical study of backreaction. To argue that cosmic acceleration is a consequence of backreaction, a dynamical study is necessary. Namely, one must demonstrate that the phase space region corresponding to acceleration can be reached from sufficiently generic (i.e., not fine-tuned) and physically reasonable initial conditions. To do so one needs to write a closed set of effective backreaction equations and study the properties of its solutions. This will be the topic of future work.

6.6 Special choices for the background metric

Many studies of cosmic acceleration as arising from the inhomogeneous nature of the universe have considered as a true spacetime various forms of the Lemaitre-Tolman-Bondi (LTB) family of solutions [3]. These describe a spherically-symmetric cosmological spacetime. In particular, void LTB models, namely, models describing a large underdense central region surrounded by a flat-matter dominated spacetime provide remarkable agreement with a large fraction of cosmological date, provided we are located near the void’s center.

In principle, our method can be straightforwardly applied to LTB-type solutions to Einstein’s equations. In the canonical formulation, such solutions correspond to three-metrics of the form

\[ ds^2 = f(\chi)d\chi^2 + g(\chi)(d\theta^2 + \sin^2 \theta d\phi^2), \]  

and momenta

\[ \pi^{ij} = \text{diag}[p_f(\chi), \frac{1}{2}p_g(\chi), \frac{1}{2}p_g(\chi)/\sin^2 \theta] \sin \theta, \]

where \( p_f, p_g \) are the conjugate momenta of the variables \( f \) and \( g \) respectively, and the canonical variables satisfy the Hamiltonian constraint in Eq. (36). We assumed that the Cauchy surfaces are three-spheres and that the perfect fluid is dust.

The group-averaging calculus yields allows us to define the variables \( \alpha \) and \( \xi \) of the effective description, through equations \( \langle h_{ij}^0 h^{ij} \rangle = 3\alpha^2 \) and \( \langle \pi^{ij} h_{ij}/\sqrt{\det h} \rangle = -3\kappa \xi \). We find

\[ \alpha^2 = \frac{2}{3\pi} \int_0^\pi d\chi [f(\chi) \sin^2 \chi + 2g(\chi)] \]  

\[ \xi = -\frac{2}{3\pi\kappa} \int_0^\pi d\chi [p_f(\chi) + p_g(\chi) \sin^2 \chi]. \]

A study of backreaction for the LTB models would then proceed along the lines described in sections 4 and 6. The state space of the LTB model is infinite-dimensional, and a full study of backreaction would be beyond the scope of this paper. However, there are some points we can make in relation to the approximation schemes we have employed.

In particular, we have relied on the approximation that the ultra-local terms to the perturbations are negligible. This means that the magnitude of the metric perturbations \( \delta f = f - \alpha^2, \delta g = g - \alpha^2 \sin^2 \chi \) must be much smaller than \( \alpha^2 \) and \( \alpha^2 \sin^2 \chi \) respectively. The same holds for the perturbations to the momenta \( \delta p_f \) and \( \delta p_g \). For LTB solutions that do not satisfy this condition, the approximation scheme employed in sections 4 and 6 fails, and a different scheme should be developed.
The derivation of Eqs. (125–128) employed additional assumptions. First, a negligible value of the mean scalar curvature \( \langle R \rangle \) and of the mean-square deviation velocities \( \langle u^2 \rangle \). These conditions are restrictive, but it is possible to find configurations that satisfy them. However, the large degree of symmetry of the LTB solutions does not allow for the isotropy of the tensor \( \Theta_{ij} \). The deviation velocity field \( u^i \) is radial (along the \( \partial/\partial \chi \) direction) and depends only on \( \chi \). As a result the only non-zero component of \( \Theta_{ij} \) is \( \Theta_{\chi \chi} \). It follows that Eqs. (125–128) do not hold in an LTB spacetime, because of its high degree of symmetry.

It is sometimes suggested that the use of a spherically symmetric spacetime as a cosmological model can be motivated by the assumption of an implicit averaging of observations over the celestial sphere. In light of our analysis, it is necessary to point out that such an averaging would most probably misrepresent backreaction. An averaging over the celestial sphere could be viewed in our formalism as a group averaging over an action of the group \( G = SO(3) \). The resulting distribution of deviation velocities would correspond to a tensor \( \Theta_{ij} \), whose only non-zero component would be \( \Theta_{\chi \chi} \) as above, which may be substantially different from the "true" tensor \( \Theta_{ij} \), and as such to lead to qualitatively different behavior for backreaction. Void models, in particular, need not refer to spherical symmetry, and it is to be expected that a non-spherically symmetric void (especially with respect to the distribution of deviation velocities) would lead to different predictions from the LTB ones.

In fact, if we have reasons to believe that a family of solutions to the Einstein equations with a high degree of symmetry are good approximations to the "true" spacetime metric, it would be more convenient to work directly with these solutions. The method we developed here is intended to be used for generic "true" spacetimes: its aim is to identify generic variables that drive backreaction, and, subsequently, to relate them to observed quantities. In particular, our method would be suitable for dealing with "Swiss-cheese"-like models, which have also been studied in relation to cosmological backreaction [5]. In fact, the models for the distribution of the perturbations in Sec. 5.2.1 could provide a starting point for such a study, without the assumption of spherical symmetry for the void regions.

Finally, we note that Eq. (125) leads to a relation between the Hubble factor \( \frac{\dot{a}}{a} \) and the quantity \( z = \alpha_0/\alpha - 1 \), of the form

\[
H(z) = H_0 \left[ \gamma(z) + \sqrt{\Omega_m(1 + z)^3} \right],
\]

(139)

where \( \gamma(z) = -\Gamma/(3H_0) \) is a function of \( z \), such that \( \gamma(0) = 1 - \sqrt{\Omega_m} \). In effect \( \gamma(z) \) plays the role of a redshift-dependent "anti-dissipation" coefficient.

In contrast, the relation between the Hubble factor and the redshift for dark energy is of the form

\[
H(z) = H_0 \sqrt{1 - \Omega_m} f(z) + \Omega_m(1 + z)^3,
\]

(140)

for some function \( f(z) \). It is important to emphasize the structural difference between the two expressions for \( H(z) \): an "anti-dissipation" term \( \Gamma \) leads to a luminosity-distance relation that differs strongly from the one obtained by most reasonable energy-density terms. For this reason, it is conceivable that good fits to the supernova data may be provided by simple functional expressions for \( \gamma(z) \) that do not correspond to accelerated expansion.

The consideration of perturbations of the scalar curvature would lead to a modified expression of Eq. (139) of the form

\[
H(z) = H_0 \left[ \gamma(z) + \sqrt{\Omega_m(1 + z)^3 + r(z)} \right],
\]

(141)
where \( r(z) = -\delta R/(6H_0^2) \) incorporates the effect of curvature perturbations. Unlike the \( \gamma(z) \) term, the curvature perturbations are additive to the energy density for dust; their presence in \( H(z) \) is like that of a dark energy term.

However, one should be careful before using the variable \( z \) as a measure of the observable redshift. The relation between redshift and the scale factor should also take into account the presence of inhomogeneities. Averaging over the inhomogeneities is expected to introduce additional terms in the relation between the physical redshift and the scale factor. The derivation of such a relation, through group-averaging of a generic spacetime is the necessary next step, before to attempt to relate the dynamical equations derived from the present method to observable quantities. This implies that the relation between the Hubble factor and redshift will be of the form \( (140) \), where \( z \) will be a functional of the physical redshift. This would seem to imply an even stronger divergence from the predictions of dark energy models.

7 Conclusions

In this paper, we developed a general procedure for the treatment of backreaction in cosmological spacetimes. The key ingredients to the formalism has been the averaging with respect to the isometry group of FRW cosmologies. This allowed for the construction of a projective map in the gravitational phase space for the gravitating fluid, which was used in order to provide a consistent and gauge covariant treatment of gravitational backreaction. For dust-filled spacetimes we identified and solved the backreaction equations for small perturbations and we identified regions of phase space, in which accelerated expansion is possible if the perturbations are large. A dynamical study of the case of large perturbations (aiming to construct explicit solution of backreaction equations) will be undertaken in a latter publication.

Some comments on the potential generalizations of the method are in order. We have exploited here the properties of the Lagrangian formalism of perfect fluids, in order to factor out the gauge freedom corresponding to spatial diffeomorphisms. In principle, the method can be applied to spacetimes with matter content other than a perfect fluid, by taking into account the group-average of (combinations of) the diffeomorphism constraint. There are, however, issues related to the gauge-invariance of the perturbation expansion around the symmetry surface that need to be explored.

Here, we described backreaction in terms of an autonomous set of evolution equations, constructing an effective dynamical system with a small number of variables. An alternative approach is to use probabilistic arguments: the effective equations would then involve a degree of stochasticity due to our ignorance of the detailed state of the perturbations. Models, such as those of Sec. 5.2.1, may be useful in this approach, the behavior of perturbations being encoded in a small number of parameters characterizing an effective probability distribution for the properties of localized perturbations.

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