Nowhere-uniform continuity of the solution map of the Camassa-Holm equation in Besov spaces

Jinlu Li$^1$, Y anghai Yu$^{2,*}$ and Weipeng Zhu$^3$

$^1$ School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China
$^2$ School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China
$^3$ School of Mathematics and Big Data, Foshan University, Foshan, Guangdong 528000, China

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Abstract: In the paper, we gave a strengthening of our previous work in [32] (J. Differ. Equ. 269 (2020)) and proved that the data-to-solution map for the Camassa-Holm equation is nowhere uniformly continuous in $B^{s}_{p,r}(\mathbb{R})$ with $s > \max\{1 + 1/p, 3/2\}$ and $(p, r) \in [1, \infty] \times [1, \infty)$. The method applies also to the b-family of equations which contain the Camassa-Holm and Degasperis-Procesi equations.

Keywords: Camassa-Holm equation, Nowhere-uniform continuity, Solution map, Besov spaces

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1 Introduction

In this paper, we are concerned with the Cauchy problem for the classical Camassa-Holm (CH) equation

\[
\begin{cases}
  u_t - uu_{xx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
  u(x, t = 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]  

(1.1)

Here the scalar function $u = u(t, x)$ stands for the fluid velocity at time $t \geq 0$ in the $x$ direction.

Setting $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$, then $\Lambda^{-2}f = G * f$ where $G(x) = \frac{1}{2}e^{-|x|}$ is the kernel of the operator $\Lambda^{-2}$. Thus, we can transform the CH equation (1.1) equivalently into the following transport type equation

\[
\begin{cases}
  \partial_t u + u\partial_x u = P(u), & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
  u(x, t = 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]  

(1.2)

where

\[P(u) = P(D)\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) \quad \text{with} \quad P(D) = -\partial_x \Lambda^{-2}.
\]  

(1.3)

*E-mail: lijinlu@gnnu.edu.cn; yuyanghai214@sina.com (Corresponding author); mathzwp2010@163.com
The CH equation (1.1) was firstly proposed in the context of hereditary symmetries studied in [17] and then was derived explicitly as a water wave equation by Camassa-Holm [3]. Many aspects of the mathematical beauty of the CH equation have been exposed over the last two decades. Particularly, The CH equation (1.1) is completely integrable [3, 7] with a bi-Hamiltonian structure [6, 17] and infinitely many conservation laws [3, 17]. Also, it admits exact peaked soliton solutions (peakons) of the form \( u(x, t) = ce^{-|x-c|t} \) with \( c > 0 \), which are orbitally stable [14]. Another remarkable feature of the CH equation is the wave breaking phenomena: the solution remains bounded while its slope becomes unbounded in finite time [5, 10, 11]. It is worth mentioning that the peaked solitons present the characteristic for the travelling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, see Refs. [8, 12, 13, 40] for the details. Due to these interesting and remarkable features, the CH equation has attracted much attention as a class of integrable shallow water wave equations in recent twenty years. Its systematic mathematical study was initiated in a series of papers by Constantin and Escher, see [9–13].

After the CH equation (1.1) was derived physically in the context of water waves, there are a large amount of literatures devoted to studying the well-posedness of the Cauchy problem (1.1) (see Molinet’s survey [38]). Li and Olver [37] proved that the Cauchy problem (1.1) is locally well-posed with the initial data \( u_0 \in H^s(\mathbb{R}) \) with \( s > 3/2 \) (see also [39]). Danchin [15, 16] proved the local existence and uniqueness of strong solutions to (1.1) with initial data in \( B^s_{p,r} \) if \( (p, r) \in [1, \infty] \times [1, \infty) \), \( s > \max \{1 + 1/p, 3/2\} \) and \( B^{3/2}_{2,1} \). Meanwhile, he [15] only obtained the continuity of the solution map of (1.1) with respect to the initial data in the space \( C([0,T]; B^{s'}_{p,r}) \) with any \( s' < s \). Li-Yin [34] proved the continuity of the solution map of (1.1) with respect to the initial data in the space \( C([0,T]; B^{s'}_{p,r}) \) with \( r < \infty \). In particular, they [34] proved that the solution map of (1.1) is weak continuous with respect to initial data \( u_0 \in B^s_{p,\infty} \). For the endpoints, Danchin [16] obtained that the data-to-solution map is not continuous by using peakon solution, which implies the ill-posedness of (1.1) in \( B^{3/2}_{2,\infty} \). Guo-Liu-Molinet-Yin [18] showed the ill-posedness of (1.1) in \( B^{1+1/p}_{1,p}(\mathbb{R} \text{ or } \mathbb{T}) \) with \( (p, r) \in [1, \infty] \times (1, \infty] \) (especially in \( H^{3/2} \)) by proving the norm inflation. Very recently, Guo-Ye-Yin [19] obtained the ill-posedness for the CH equation in \( B^{1}_{\infty,1}(\mathbb{R}) \) by proving the norm inflation. In our recent papers [35, 36], we established the ill-posedness for (1.1) in \( B^{s}_{p,\infty}(\mathbb{R}) \) by proving the solution map to the Camassa-Holm equation starting from \( u_0 \) is discontinuous at \( t = 0 \) in the metric of \( B^{s}_{p,\infty}(\mathbb{R}) \).

Continuity properties of the solution map is an important part of the well-posedness theory. In fact, the non-uniform continuity of data-to-solution map suggests that the local well-posedness cannot be established by the contraction mappings principle since this would imply Lipschitz continuity for the solution map. After the phenomenon of non-uniform continuity for some dispersive equations was studied by Kenig et al. [30], the issue of non-uniform dependence on the initial data has been a fascinating object of research in the recent past. Naturally, we may wonder which regularity assumptions are relevant for the initial data \( u_0 \) such that the Cauchy problem to (1.1) is not uniform dependent on initial data, namely, the dependence of solution on the initial data associated with this equation is not uniformly continuous. Himonas-Misiolek [23] obtained the first result on the non-uniform dependence for (1.1) in \( H^s(\mathbb{T}) \) with \( s \geq 2 \) using explicitly constructed travelling wave solutions, which was sharpened to \( s > \frac{3}{2} \) by Himonas-Kenig [21] on the real-line and Himonas-Kenig-Misiolek [22] on the circle. Recently, Ye-Yin-Guo [41] proved the uniqueness and continuous dependence of the Camassa-Holm type equations in critical Besov spaces \( B^{1+1/p}_{p,1} \) with \( p \in [1, \infty) \). In [32, 33], we proved the non-uniform dependence on initial data for the Camassa-Holm equations
under the framework of Besov spaces $B^s_{p,r}$. In particular, we established

**Theorem 1.1 ([32])** Assume that $(s, p, r)$ satisfies

\[ s > \max \left\{ \frac{3}{2}, 1 + \frac{1}{p} \right\} \quad \text{and} \quad (p, r) \in [1, \infty] \times [1, \infty). \]

Denote $U_R = \{ u_0 \in B^s_{p,r} : \|u_0\|_{B^s_{p,r}} \leq R \}$ for any $R > 0$. Then the data-to-solution map of the Cauchy problem (1.2)-(1.3)

\[ S_t : U_R \to B^s_{p,r}(\mathbb{R}), \; u_0 \to S_t(u_0) \]

is not uniformly continuous from any bounded subset $U_R$ in $B^s_{p,r}$ into $C([0, T]; B^s_{p,r})$. More precisely, there exists two sequences of solutions $S_t(f_n + g_n)$ and $S_t(f_n)$ such that

\[ \|f_n\|_{B^s_{p,r}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|g_n\|_{B^s_{p,r}} = 0 \]

but

\[ \liminf_{n \to \infty} \|S_t(f_n + g_n) - S_t(f_n)\|_{B^s_{p,r}} \geq t, \quad \forall \; t \in [0, T_0], \]

with small time $T_0$.

In this paper we shall present a stronger version of Theorem 1.1. The main result of this paper is the following theorem.

**Theorem 1.2 (Nowhere uniformly continuous)** Assume that $(s, p, r)$ satisfies

\[ s > \max \left\{ \frac{3}{2}, 1 + \frac{1}{p} \right\} \quad \text{and} \quad (p, r) \in [1, \infty] \times [1, \infty). \tag{1.4} \]

Denote by $U \subset B^s_{p,r}(\mathbb{R})$ the set of initial values $u_0$. Then the data-to-solution map of the Cauchy problem (1.2)-(1.3)

\[ S_t : U \to B^s_{p,r}(\mathbb{R}), \; u_0 \to S_t(u_0) \]

is nowhere uniformly continuous from $B^s_{p,r}$ into $C([0, T]; B^s_{p,r})$. More precisely, let $u_0 \in B^s_{p,r}$ be a given initial data, there exists two sequences of solutions $S_t(u_0 + f_n + g_n)$ and $S_t(u_0 + f_n)$ such that

\[ \|f_n\|_{B^s_{p,r}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|g_n\|_{B^s_{p,r}} = 0 \]

but

\[ \liminf_{n \to \infty} \|S_t(u_0 + f_n + g_n) - S_t(u_0 + f_n)\|_{B^s_{p,r}} \geq t, \quad \forall \; t \in [0, T_0], \]

with small time $T_0$.

**Remark 1.1** Theorem 1.1 implies that the data-to-solution map $S_t$ has not the property to be uniformly continuous on bounded sets while Theorem 1.2 means that for any non-empty $V \subseteq U$ the restriction $S_t|_V$ is not uniformly continuous. In this sense, Theorem 1.2 improves the previous results in [32].
Remark 1.2 We also mention that, using a geometric approach, Inci obtained a series of nowhere-uniform continuity results includes the b-family of equations [26], the two component b-family of equations [28], the hyperelastic rod equation [27], and the incompressible Euler equation [29]. Bourgain and Li [2] showed that the data-to-solution map for the incompressible Euler equations is nowhere-uniform continuity in $H^s(\mathbb{R}^d)$ with $s \geq 0$ by using an idea of localized Galilean boost. However, our method in this paper is pure analytic, and seems more simpler and robust.

Remark 1.3 Holm and Staley [24] introduced the following b-family equation (see [4, 20, 25] etc.):

\[
\begin{cases}
\partial_t u + uu_x = -\partial_x(1 - \partial_x^2)^{-1}\left(\frac{b}{2}u^2 + \frac{3-b}{2}u^3\right), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\] (1.5)

It should be emphasized that the Camassa-Holm equation corresponds to $b = 2$ and Degasperis-Procesi equation corresponds to $b = 3$. Since the concrete values of the parameter $b$ have no impact on the proof of Theorem 1.2, thus Theorem 1.2 also holds for the b-family equations (1.5). Due to the fact that the Besov space $B^s_{2,2}$ coincides with the Sobolev space $H^s$, Theorem 1.2 covers the previous result given by Inci [26] who proved that the corresponding solution map of b-family of equations in the Sobolev spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ is nowhere locally uniformly continuous.

When $b = 3$, (1.5) reduces to the Degasperis-Procesi (DP) equation

\[
\begin{cases}
\partial_t u + uu_x = -3\partial_x(1 - \partial_x^2)^{-1}u^2, & (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\end{cases}
\] (1.6)

In particular, we have

**Theorem 1.3** Assume that $(s, p, r)$ satisfies

\[s > 1 + \frac{1}{p}, \ p \in [1, \infty], \ r \in [1, \infty) \quad \text{or} \quad s = \frac{1}{p} + 1, \ p \in [1, \infty), \ r = 1.\]

Then the DP equation (1.6) is nowhere uniformly continuous from $B^s_{p,r}$ into $C([0, T]; B^s_{p,r})$.

**Remark 1.4** Following the procedure in the proof of Theorem 1.2 with suitable modification, we can prove Theorem 1.3. Here we will omit the details.

**Organization of our paper.** In Section 2, we list some notations and recall some useful results which will be used in the sequel. In Section 3, we establish two crucial Propositions and then prove Theorem 1.2.

## 2 Preliminaries

### 2.1 Notations

We will use the following notations throughout this paper. Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. We shall use the simplified notation $\| f, \ldots, g \|_X = \| f \|_X + \cdots + \| g \|_X$ if there is no confusion. For $I \subset \mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$. The symbol $A \lesssim B$ means that there is a uniform positive constant $C$ independent of $A$ and $B$ such that $A \leq CB$. 


2.2 Littlewood-Paley analysis

Next, we will recall some facts about the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties (see [1] for more details).

There exists a couple of smooth functions \((\chi, \varphi)\) valued in \([0, 1]\), such that \(\chi\) is supported in the ball \(B = \{\xi \in \mathbb{R} : |\xi| \leq \frac{1}{4}\}\), and \(\varphi\) is supported in the ring \(C = \{\xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}\). Moreover,

\[\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}.\]

For every \(u \in S'(\mathbb{R})\), the inhomogeneous dyadic blocks \(\Delta_j\) are defined as follows

\[
\Delta_j u = \begin{cases} 
0, & \text{if } j \leq -2; \\
\chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F} u), & \text{if } j = -1; \\
\varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}) \mathcal{F} u), & \text{if } j \geq 0.
\end{cases}
\]

With this we give the definition of nonhomogeneous Besov space \(B^s_{p, r}(\mathbb{R})\).

**Definition 2.1 (see [1])** Let \(s \in \mathbb{R}\) and \((p, r) \in [1, \infty]^2\). The nonhomogeneous Besov space \(B^s_{p, r}(\mathbb{R})\) is defined by

\[B^s_{p, r}(\mathbb{R}) := \left\{ f \in S'(\mathbb{R}) : \|f\|_{B^s_{p, r}(\mathbb{R})} < \infty \right\},\]

where

\[
\|f\|_{B^s_{p, r}(\mathbb{R})} = \begin{cases} 
\left( \sum_{j \geq -1} 2^{jsr} \|\Delta_j f\|_{L^r(\mathbb{R})}^r \right)^{1/r}, & \text{if } 1 \leq r < \infty, \\
\sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^r(\mathbb{R})}, & \text{if } r = \infty.
\end{cases}
\]

We recall the following product estimate and interpolation inequality which will be used in the later of the paper.

**Lemma 2.1 (see [1])** Let \((p, r) \in [1, \infty]^2\) and \(s > 0\). For any \(u, v \in B^s_{p, r}(\mathbb{R}) \cap L^\infty(\mathbb{R})\), we have

\[
\|uv\|_{B^s_{p, r}(\mathbb{R})} \leq C(\|u\|_{B^s_{p, r}(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} + \|v\|_{B^s_{p, r}(\mathbb{R})}\|u\|_{L^\infty(\mathbb{R})}).
\]

**Lemma 2.2 (see [1])** If \(s_1\) and \(s_2\) are real numbers such that \(s_1 < s_2\), \(\theta \in (0, 1)\), and \((p, r)\) is in \([1, \infty]\), then we have

\[
\|u\|_{B^{s_1}_{p, r}} \leq \|u\|_{B^{s_2}_{p, r}}^{\theta} \|u\|_{B^{s_2}_{p, r}}^{1-\theta} \quad \text{and}
\]

\[
\|u\|_{B^{s_1}_{p, r}} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \|u\|_{B^{s_2}_{p, r}}^{\theta} \|u\|_{B^{s_2}_{p, r}}^{1-\theta}.
\]

Finally, let us end this section with the key estimates.
Lemma 2.3 (see [1,31]) Let \((p, r) \in [1, \infty)^2\) and \(\sigma \geq -\min\{\frac{1}{p}, 1 - \frac{1}{p}\}\). Assume that \(f_0 \in B_{p,r}^\sigma(\mathbb{R})\), \(g \in L^1([0, T]; B_{p,r}^\sigma(\mathbb{R}))\) and

\[
\partial_t u \in \begin{cases} L^1([0, T]; B_{p,r}^{\sigma-1}(\mathbb{R})), & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \sigma = 1 + \frac{1}{p}, \ r = 1; \\
L^1([0, T]; B_{p,r}^{\sigma}(\mathbb{R})), & \text{if } \sigma = 1 + \frac{1}{p}, \ r > 1; \\
L^1([0, T]; B_{p,\infty}^{1/p}(\mathbb{R}) \cap L^\infty(\mathbb{R})), & \text{if } \sigma < 1 + \frac{1}{p}.
\end{cases}
\]

If \(f \in L^\infty([0, T]; B_{p,r}^{\sigma}(\mathbb{R})) \cap C([0, T]; S'(\mathbb{R}))\) solves the following linear transport equation:

\[
\partial_t f + u \partial_x f = g, \quad f|_{t=0} = f_0.
\]

1. Then there exists a constant \(C = C(p, r, \sigma)\) such that the following statement holds

\[
\|f(t)\|_{B_{p,r}^\sigma(\mathbb{R})} \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^\sigma(\mathbb{R})} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B_{p,r}^\sigma(\mathbb{R})}d\tau\right),
\]

where

\[
V(t) = \begin{cases} \int_0^t \|\partial_x u(\tau)\|_{B_{p,r}^{\sigma-1}(\mathbb{R})}d\tau, & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \sigma = 1 + \frac{1}{p}, \ r = 1; \\
\int_0^t \|\partial_x u(\tau)\|_{B_{p,r}^{\sigma}(\mathbb{R})}d\tau, & \text{if } \sigma = 1 + \frac{1}{p}, \ r > 1; \\
\int_0^t \|\partial_x u(\tau)\|_{B_{p,\infty}^{1/p}(\mathbb{R}) \cap L^\infty(\mathbb{R})}d\tau, & \text{if } \sigma < 1 + \frac{1}{p}.
\end{cases}
\]

2. If \(\sigma > 0\), then there exists a constant \(C = C(p, r, \sigma)\) such that the following statement holds

\[
\|f(t)\|_{B_{p,r}^\sigma(\mathbb{R})} \leq \|f_0\|_{B_{p,r}^\sigma(\mathbb{R})} + \int_0^t \|g(\tau)\|_{B_{p,r}^\sigma(\mathbb{R})}d\tau
\]

\[
+ C \int_0^t \left(\|f(\tau)\|_{B_{p,r}^\sigma(\mathbb{R})}\|\partial_x u(\tau)\|_{L^\infty(\mathbb{R})} + \|\partial_x u(\tau)\|_{B_{p,r}^{\sigma-1}(\mathbb{R})}\|\partial_x f(\tau)\|_{L^\infty(\mathbb{R})}\right)d\tau.
\]

3 Proof of Theorem 1.2

In this section, our aim is to proving Theorem 1.2.

Firstly, we need to establish the useful Lemma.

Lemma 3.1 Let \(1 \leq p, r \leq \infty\) and \(s > \max\{1 + 1/p, 3/2\}\). Suppose that we are given

\((u, v) \in \left(L^\infty ([0, T]; B_{p,r}^s) \cap C ([0, T]; B_{p,r}^{s-1})\right)^2\)

two solutions of (CH) with initial data \(u_0, v_0 \in B_{p,r}^s\). Letting \(w := u - v\), then we have, for every \(t \in [0, T]\) and some constant \(C = C(s, p, r)\)

\[
\|w(t)\|_{B_{p,r}^{s-1}} \leq \|w_0\|_{B_{p,r}^{s-1}} \exp \left( C \int_0^t \|(u, v)(\tau)\|_{B_{p,r}^s}d\tau \right)
\]

and

\[
\|w(t)\|_{B_{p,r}^{s}} \leq \left(\|w_0\|_{B_{p,r}^{s}} + C \int_0^t \|w\|_{B_{p,r}^{s-1}}\|\partial_x v\|_{B_{p,r}^{s-1}}d\tau\right)\exp \left( C \int_0^t \|(u, v)(\tau)\|_{B_{p,r}^s}d\tau \right).
\]
Proposition 3.1

Under the assumptions of Theorem 1.2, we have

\[ ||S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})||_{B_{p,r}^s} \leq C ||(\text{Id} - S_n) u_0||_{B_{p,r}^s}. \]

Proof. The local well-posedness result (see Lemma 3.2) tells us that \( S_t(u_0 + v_{0,m_n}^{n,\omega}), S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) \in C([0, T]; B_{p,r}^s) \) and has common lifespan \( T \approx 1 \). Moreover, there holds

\[ ||S_t(u_0 + v_{0,m_n}^{n,\omega}), S_t(S_n u_0 + v_{0,m_n}^{n,\omega})||_{L^\infty(B_{p,r}^s)} \leq C. \]  

(3.9)
Using Lemma 3.1 with \( u = S_t(u_0 + v_{0,m_n}^{n,\omega}) \) and \( v = S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) \) yields that

\[
\|S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \\
\leq C\|\text{Id} - S_n\|_{B_{p,r}^1} \\
\leq C2^{-n}\|\text{Id} - S_n\|_{B_{p,r}^1} \tag{3.10}
\]

and

\[
\|S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \\
\leq \|\text{Id} - S_n\|_{B_{p,r}^1} + C \int_0^t \|S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \, d\tau \\
+ C2^n \int_0^t \|S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \, d\tau \text{ by (3.10)} \\
\leq C\|\text{Id} - S_n\|_{B_{p,r}^1} + C \int_0^t \|S_t(u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0 + v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \, d\tau.
\]

Applying Gronwall’s inequality gives the desired result of Proposition 3.1.

The following proposition plays a key role in the proof of Theorem 1.2.

**Proposition 3.2** Under the assumptions of Theorem 1.2, we have

\[
\|S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0) - S_t(v_{0,m_n}^{n,\omega})\|_{B_{p,r}^1} \leq C2^{n(1-\theta)} e^{2\theta^2 F_n} \text{ with } \theta = \frac{1}{s + 1},
\]

where we denote

\[
F_n = \int_0^t \|S_t(S_n u_0) S_t(v_{0,m_n}^{n,\omega}) - \partial_x^2 (S_t(S_n u_0) S_t(v_{0,m_n}^{n,\omega})) - \partial_x S_t(S_n u_0) \partial_x S_t(v_{0,m_n}^{n,\omega})\|_{L^p} \, d\tau.
\]

**Proof.** For the sake of simplicity, we set \( w = S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) - S_t(S_n u_0) - S_t(v_{0,m_n}^{n,\omega}) \).

By the interpolation inequality (see Lemma 2.2), we obtain

\[
\|w\|_{B_{p,r}^1} \leq C\|w\|_{B_{p,\infty}^0}^{\theta} \|w\|_{L^p}^{1-\theta} \leq C2^{n(1-\theta)} \|w\|_{L^p}^{\theta} \tag{3.11}
\]

Next, we need to estimate the \( L^p \)-norm of \( w \). It is obvious that \( w \) solves

\[
\begin{cases}
\partial_t w + S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) \partial_x w = -\sum_{i=1}^3 F_i - w \partial_x \left(S_t(S_n u_0) + S_t(v_{0,m_n}^{n,\omega})\right), \\
w(0, x) = 0,
\end{cases} \tag{3.12}
\]

where

\[
F_1 = \partial_x (S_t(S_n u_0) S_t(v_{0,m_n}^{n,\omega})), \\
F_2 = 2\mathcal{B}(S_t(S_n u_0), S_t(v_{0,m_n}^{n,\omega})), \\
F_3 = \mathcal{B}(w, S_t(S_n u_0 + v_{0,m_n}^{n,\omega}) + S_t(S_n u_0) + S_t(v_{0,m_n}^{n,\omega})).
\]
Taking the inner product of Eq. (3.12) with $|w|^{p-2}w$ with $p \in [1, \infty)$, we obtain

$$
\frac{1}{p} \frac{d}{dt} \|w\|^p_{L^p} = - \int_\mathbb{R} \sum_{i=1}^3 F_i |w|^{p-2}w \cdot dx - \int_\mathbb{R} \partial_x \left( S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o}) \right) |w|^p \cdot dx
+ \frac{1}{p} \int_\mathbb{R} \partial_x S_i (S_n u_0 + v_{0,m_i}^{n,o}) |w|^p \cdot dx
\leq \left( \|\partial_x S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o})\|_{L^p} + \frac{1}{p}\|\partial_x S_i (S_n u_0 + v_{0,m_i}^{n,o})\|_{L^p} \right) \|w\|^p_{L^p} + \sum_{i=1}^3 \|F_i\|_{L^p} \|w\|_{L^p}^{p-1}
\leq \|w\|^p_{L^p} + \|\partial_x w\|_{L^p} \|w\|_{L^p}^{p-1} + \sum_{i=1}^3 \|F_i\|_{L^p} \|w\|_{L^p}^{p-1},
$$

(3.13)

where we have used that

$$
\|\mathcal{B}(f, g)\|_{L^p} \leq C(\|f\|_{L^p} + \|\partial_x f, \partial_x g\|_{L^p}).
$$

Then (3.13) reduces to

$$
\frac{d}{dt} \|w\|^p_{L^p} \leq C \|w\| \|\partial_x w\|_{L^p} + \|F_1, F_2\|_{L^p}.
$$

(3.14)

Setting $v = \partial_x w$, then we have

$$
\left\{ \begin{array}{l}
\partial_x v + S_i (S_n u_0 + v_{0,m_i}^{n,o}) \partial_x w = - \sum_{i=1}^3 \partial_x F_i - \nu \partial_x \left( S_i (S_n u_0 + v_{0,m_i}^{n,o}) + S_i (v_{0,m_i}^{n,o}) \right)
\end{array} \right.
$$

(3.15)

Taking the inner product of Eq. (3.15) with $|v|^{p-2}v$, we obtain

$$
\frac{1}{p} \frac{d}{dt} \|v\|^p_{L^p} = - \int_\mathbb{R} \sum_{i=1}^3 \partial_x F_i |v|^{p-2}v \cdot dx - \int_\mathbb{R} \partial_x \left( S_i (S_n u_0 + v_{0,m_i}^{n,o}) + S_i (v_{0,m_i}^{n,o}) \right) |v|^p \cdot dx
+ \frac{1}{p} \int_\mathbb{R} \partial_x S_i (S_n u_0 + v_{0,m_i}^{n,o}) |v|^p \cdot dx - \int_\mathbb{R} \nu \partial_x^2 \left( S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o}) \right) |v|^{p-2}v \cdot dx
\leq \left( \sum_{i=1}^3 \|\partial_x F_i\|_{L^p} \|v\|_{L^p}^{p-1} + \|\partial_x S_i (S_n u_0), \partial_x S_i (v_{0,m_i}^{n,o})\|_{L^p} \|v\|_{L^p} \|v\|_{L^p}^{p-1} \right)
+ \|\partial_x^2 \left( S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o}) \right)\|_{L^p} \|v\|_{L^p} \|v\|_{L^p}^{p-1}
\leq \|v\|^p_{L^p} + 2^n\|v\|_{L^p} \|v\|_{L^p}^{p-1} + \sum_{i=1}^3 \|\partial_x F_i\|_{L^p} \|v\|_{L^p}^{p-1},
$$

(3.16)

where we have used the fact due to $s-1 > \max\{\frac{1}{2}, \frac{1}{p}\}$

$$
\|\partial_x^2 \left( S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o}) \right)\|_{L^p} \leq \|\partial_x^2 \left( S_i (S_n u_0) + S_i (v_{0,m_i}^{n,o}) \right)\|_{L^p} \leq 2^n.
$$

Then (3.16) reduces to

$$
\frac{d}{dt} \|v\|_{L^p} \leq 2^n\|v\|_{L^p} + \|v\|_{L^p} + \|\partial_x F_1, \partial_x F_2\|_{L^p}.
$$

(3.17)
Combining (3.14) and (3.17) yields that
\[
\|w, \partial_s w\|_{L^p} \leq e^{2\nu t} \int_0^t \sum_{i=1}^2 \|F_i, \partial_s F_i\|_{L^p} \, dt \leq e^{2\nu} F_n. \tag{3.18}
\]

Particularly, we would like to emphasize that (3.18) holds for \( p = \infty \) (just letting \( p \to \infty \) in (3.18)). Inserting (3.18) into (3.11), then we complete the proof of Proposition 3.2.

With Propositions 3.1–3.2 in hand, we can prove Theorem 1.2 by dividing it into three steps.

**Step 1.** By Theorem 1.1, we can constructing two sequences of initial data \( f_n \) and \( f_n + g_n \) which satisfy \( \|f_n\|_{b^{p,r}} \leq 1 \) and \( \lim_{n \to \infty} \|g_n\|_{b^{p,r}} = 0 \) and find two sequences of solutions \( S_t(f_n + g_n) \) and \( S_t(f_n) \) which satisfy \( \|S_t(f_n + g_n), S_t(f_n)\|_{b^{p,r}} \leq C \), but for short time \( t \in [0, T_0] \)
\[
\lim_{n \to \infty} \inf \|S_t(f_n + g_n) - S_t(f_n)\|_{b^{p,r}} \geq c_0 t,
\]
where \( c_0 \) is some positive constant.

**Step 2.** Notice that \( S_t(u^{\omega,0}_{n,m}) = S_t(f_n + \omega g_n)(t, x - m_n) := u^{\omega}_n(t, x - m_n) \), for fixed \( n \) and any \((t, x)\), thanks to the smoothness and decay, we have
\[
\begin{align*}
&\lim_{|\nu| \to \infty} S_t(S_n u_0(x)S_t(f_n + \omega g_n)(t, x - y)) = 0, \\
&\lim_{|\nu| \to \infty} \partial^2_{x} S_t(S_n u_0(x)S_t(f_n + \omega g_n)(t, x - y)) = 0, \\
&\lim_{|\nu| \to \infty} \partial_\nu S_t(S_n u_0(x))\partial_\nu S_t(f_n + \omega g_n)(t, x - y) = 0.
\end{align*}
\]
Also, we have
\[
\begin{align*}
|S_t(S_n u_0(x)S_t(f_n + \omega g_n)(t, x - y))| &\leq M|S_t(S_n u_0(x))|, \\
|\partial^2_{x} S_t(S_n u_0(x))|\partial^{\nu}_\nu S_t(f_n + \omega g_n)(t, x - y)) &\leq M|\partial^2_{x} S_t(S_n u_0(x))|, \\
|\partial_\nu S_t(S_n u_0(x))\partial_\nu S_t(f_n + \omega g_n)(t, x - y)| &\leq M|\partial_\nu S_t(S_n u_0(x))|.
\end{align*}
\]
By the Lebesgue Dominated Convergence Theorem, one has for \( \omega = 0, 1 \)
\[
\lim_{|\nu| \to \infty} \int_0^t \|S_t(S_n u_0) u^{\omega,0}_n(t, x - y), \partial^2_{x} (S_t(S_n u_0) u^{\omega,0}_n(t, x - y)), \partial_\nu S_t(S_n u_0)\partial_\nu u^{\omega,0}_n(t, x - y)|_{L^p} \, dt = 0.
\]
By Proposition 3.2, for fixed \( n \) and any \((t, x)\), one can find \( m_n \) with \( |m_n| \) sufficiently large such that
\[
\|S_t(S_n u_0 + v^{\omega,0}_{0,m_n}) - S_t(S_n u_0) - S_t(v^{\omega,0}_{0,m_n})\|_{b^{p,r}} \leq \frac{c_0 t}{4}, \quad \omega = 0, 1.
\]

**Step 3.** We decompose the difference of \( S_t(u_0 + v^{1,0}_{0,m_n}) \) and \( S_t(u_0 + v^{0,0}_{0,m_n}) \) as follows
\[
\begin{align*}
S_t(u_0 + v^{1,0}_{0,m_n}) - S_t(u_0 + v^{0,0}_{0,m_n}) &= S_t(v^{1,0}_{0,m_n}) - S_t(v^{0,0}_{0,m_n}) \\
&= \left\{ \begin{array}{l}
S_t(u_0 + v^{1,0}_{0,m_n}) - S_t(S_n u_0 + v^{1,0}_{0,m_n}) + S_t(S_n u_0 + v^{1,0}_{0,m_n}) - S_t(S_n u_0) - S_t(v^{1,0}_{0,m_n}) \\
\quad =: I_1 \\
S_t(u_0 + v^{0,0}_{0,m_n}) - S_t(S_n u_0 + v^{0,0}_{0,m_n}) + S_t(S_n u_0 + v^{0,0}_{0,m_n}) - S_t(S_n u_0) - S_t(v^{0,0}_{0,m_n}) \\
\quad =: I_2 \\
S_t(u_0 + v^{0,0}_{0,m_n}) - S_t(S_n u_0 + v^{0,0}_{0,m_n}) + S_t(S_n u_0 + v^{0,0}_{0,m_n}) - S_t(S_n u_0) - S_t(v^{0,0}_{0,m_n}) \\
\quad =: I_3 \\
\end{array} \right.
\end{align*}
\]
Hence, from Proposition 3.1 and Step 2 we deduce

\[
\|S_t(u_0 + v_{0,m}^{n,1}) - S_t(u_0 + v_{0,m}^{n,0})\|_{B_{p,r}^s} \geq \|S_t(v_{0,m}^{n,1}) - S_t(v_{0,m}^{n,0})\|_{B_{p,r}^s} - \sum_{i=1}^{4} \|I_i\|_{B_{p,r}^s}
\]

\[
\geq \|S_t(v_{0,m}^{n,1}) - S_t(v_{0,m}^{n,0})\|_{B_{p,r}^s} - C\|(\text{Id} - S_n)u_0\|_{B_{p,r}^s} - \frac{c_0 t}{2},
\]

which follows from Step 1 that for \(t\) small enough

\[
\liminf_{n \to \infty} \|S_t(u_0 + v_{0,m}^{n,1}) - S_t(u_0 + v_{0,m}^{n,0})\|_{B_{p,r}^s} \geq \frac{c_0 t}{2}.
\]

We also notice that

\[
\lim_{n \to \infty} \|v_{0,m}^{n,1} - v_{0,m}^{n,0}\|_{B_{p,r}^s} = \lim_{n \to \infty} \|g_{n,m}^s\|_{B_{p,r}^s} = 0.
\]

This completes the proof of Theorem 1.2.

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**Data Availability**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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