THE MINIMAL CREMONA DEGREE OF QUARTIC SURFACES

MASSIMILIANO MELLA

Dedicated to Ciro Ciliberto with admiration

Abstract. Two birational projective varieties in $\mathbb{P}^n$ are Cremona Equivalent if there is a birational modification of $\mathbb{P}^n$ mapping one onto the other. The minimal Cremona degree of $X \subset \mathbb{P}^n$ is the minimal integer among all degrees of varieties that are Cremona Equivalent to $X$. The Cremona Equivalence and the minimal Cremona degree is well understood for subvarieties of codimension at least 2 while both are in general very subtle questions for divisors. In this note I compute the minimal Cremona degree of quartic surfaces in $\mathbb{P}^3$. This allows me to show that any quartic surface of elliptic ruled type has non trivial stabilizers in the Cremona group.

Introduction

Birational geometry and birational maps are one of the most peculiar aspects of Algebraic Geometry. Among the many interests of Ciro in all realms of Algebraic Geometry, and actually Mathematics, an important spot has to be reserved for birational arguments, not only for their intrinsic interests but also for their link to the Italian school of geometry and to projective geometry.

The most studied birational object is certainly the Cremona Group, that is the group of birational self-maps of the projective space

$$Cr_n := \{ f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \mid \text{birational map} \}.$$ 

This group is wild, from almost all points of view, see [Ca18] for a nice introduction. In this note I will focus my attention on a problem that is related to the wildness of $Cr_n$, the so called Cremona Equivalence.

Let $X, Y \subset \mathbb{P}^N$ be irreducible and reduced birational varieties. The subvariety $X$ and $Y$ are said to be Cremona Equivalent if there is a birational modification $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ such that $\varphi(X) = Y$ and $\varphi$ is an isomorphism of the generic point of $X$. The Cremona Equivalence problem has an old history that I will resume in Section II and a quite recent evolution thanks to the modern tools of birational geometry inherited from Minimal Model Program and Sarkisov program.

As a matter of fact any pair of birational projective varieties is Cremona equivalent as long as their codimension is at least two. This is a quite surprising result proved in [MP09] and improved in [CCMRZ16]. Note that this forces $Cr_n$ to contain, as a set, all groups of birational modifications of its subvarieties.

Date: May 2021.

2020 Mathematics Subject Classification. Primary 14E25 ; Secondary 14E05, 14N05, 14E07.

Key words and phrases. Birational maps; Cremona equivalence; embeddings; hypersurfaces.
The divisorial case is quite intricate. It is easy to give examples of non Cremona Equivalent divisors, [MP09], but it is quite hard to understand the divisors that are Cremona Equivalent to a given one. For instance rational divisor in $\mathbb{P}^n$ Cremona Equivalent to a hyperplane are only known for $\mathbb{P}^2$ and in a less precise way $\mathbb{P}^3$.

A natural notion arising from Cremona Equivalence, is that of minimal Cremona degree. see Definition 1.1. The complete classification of minimal Cremona degree plane curves is known, [MP12] and [CC10], and Ciro together with Alberto Calabri completed also the classification of minimal Cremona degree linear system of plane curves, [CC10].

Recently in a series of papers, [MP12] [Me13] [Me20], I tried to shed some light on the surface case and here I present the classification of minimal Cremona degree surfaces of degree at most 4, see Theorem 2.9. This is done following the main ideas in [Me20] and plugging in the detailed description of singularities of non rational quartic surfaces obtained in a series of papers by Umezu and Urabe, [Um81] [Um84] [Ur86]. This is the real bottleneck of my methods: the need of a complete understanding of the singularities of divisors I am considering. This prevents me to extend this classification to surfaces of higher degrees.

I want to finish the introduction thanking Ciro for all he taught me during our long friendship and for all the nice moments we shared both in life and in mathematics. I am in debt, more than you ever thought.

1. History and background

Let $C \subset \mathbb{P}^2$ be an irreducible and reduced plane curve. It is natural to ask what is the minimal degree of curves that are equivalent to $C$ via a Cremona modification. This is a classical problem studied since the XIXth century by Cremona and Noether. More generally one can introduce the notion of minimal Cremona degree as follows.

**Definition 1.1.** Let $X \subset \mathbb{P}^n$ be an irreducible and reduced hypersurface. The minimal Cremona degree of $X$ is

$$\min\{d| X \text{ is Cremona Equivalent to a hypersurface of degree } d\}.$$  

The divisor $X$ is of minimal Cremona degree if its degree is equal to the minimal Cremona degree. That is it not possible to lower its degree with a Cremona modification in $\text{Cr}_n$.

As I said the case of plane curves has been widely treated in the old times, [Ju888] [Ca890] [Ca891]; see also the beautiful books of Coolidge [Coo28] and Conforto [Con39] for a complete account of the result proved by that time. More recently the subject has been studied with the theory of log pairs, [Na60], [Ii99], [KM83] and finally with a mixture of old and new techniques a complete classification of minimal Cremona degree irreducible plane curves has been achieved in [CC10], and [MP12].

As a matter of fact the Cremona equivalence of a plane curve is dictated by its singularities but, unfortunately, its minimal Cremona degree cannot be guessed without a partial resolution of those. [MP12, Example 3.18]. Due to this it is quite hard even in the plane curve case to determine the minimal Cremona degree of a fixed curve simply by its equation. The main tool developed in the XXth century and improved by Ciro and Alberto is the theory of adjoint linear systems. Let
D ⊂ P\textsuperscript{N} be a divisor and f : X → P\textsuperscript{N} a log resolution of (P\textsuperscript{N}, D), with D\textsubscript{X} the strict transform divisor. The adjoint linear system, with m ≥ n, is

\[ \text{adj}_{n,m}(D) = f_* (|nD\textsubscript{X} + mK\textsubscript{X}|). \]

S. Kantor first noticed that the dimension of adjoint linear systems is invariant under Cremona modifications. It is easy to see that \( \text{adj}_{n,m}(D) \) is independent of the resolution \( f \), as long as \( m ≥ n \), therefore a divisor of minimal Cremona degree 1 has all adjoint linear systems empty.

It is quite natural to ask whether the opposite is true and actually for plane curves this is one of the main results obtained at the beginning of XX\textsuperscript{th} century.

\textbf{Theorem 1.2 (CE900).} An irreducible and reduced curve \( C ⊂ P\textsuperscript{2} \) is Cremona Equivalent to a line if and only if all of its adjoint vanish.

In modern terms, also related to the Abhyankar–Moh problem [AM75], we may rephrase this result saying that a plane curve \( C \) is Cremona Equivalent to a line if and only if its log Kodaira dimension is negative. Pushing the theory of adjoint linear systems Ciro and Alberto were able to classify minimal Cremona degree curves, minimal Cremona degree linear systems and the contractibility of configurations of lines, [CC10, CC17].

It is then natural to investigate surfaces in \( P\textsuperscript{3} \), keeping in mind that, quite often, the numerical invariants related to the canonical class and its log variants are not subtle enough in higher dimensions. Think of the beautiful Castelnuovo’s rationality Theorem and the wild rationality behavior of Fano 3-folds.

For the Cremona Equivalence of surfaces it is useful to adopt the \( \sharp \)-Minimal Model Program, developed in [Me02] or minimal model program with scaling [BCHM10]. In this way a criterion for detecting surfaces Cremona equivalent to a plane has been given in [MP12]. The criterion, inspired by the previous work of Coolidge on curves Cremona equivalent to lines [Coo28], allows to determine all rational surfaces that are Cremona equivalent to a plane, [MP12 Theorem 4.15]. Unfortunately, worse than in the plane curve case, the criterion requires not only the resolution of singularities but also a control on different log varieties attached to the pair \( (P\textsuperscript{3}, S) \).

Let us start to enter in the Cremona Equivalence problem for surfaces with some notations and definitions.

\textbf{Definition 1.3.} Let \( (T, H) \) be a \( \mathbb{Q} \)-factorial uniruled 3-fold and \( H \) an irreducible and reduced Weil divisor on \( T \). Let

\[ \rho = \rho_H = \rho(T, H) =: \sup \{ m ∈ \mathbb{Q} | H + mK_T \text{ is an effective } \mathbb{Q} \text{-divisor } \} ≥ 0, \]

be the (effective) threshold of the pair \( (T, H) \).

\textbf{Remark 1.4.} The threshold is not a birational invariant of pairs and it is not preserved by blowing up. Consider a plane \( H ⊂ P\textsuperscript{3} \) and let \( Y → P\textsuperscript{3} \) be the blow up of a point in \( H \) then \( \rho(Y, H_Y) = 0 \), while \( \rho(P\textsuperscript{3}, H) = 1/4 \). For future reference note that both are less than one.

In [MP12], to overcome this problem it was introduced the notion of good models and of sup threshold.

\textbf{Definition 1.5.} Let \( (Y, S_Y) \) be a 3-fold pair. The pair \( (Y, S_Y) \) is a birational model of the pair \( (T, S) \) if there is a birational map \( φ : T → Y \) such that \( φ \) is well defined on the generic point of \( S \) and \( φ(S) = S_Y \). A good model, [MP12], is a pair \( (Y, S_Y) \) with \( S_Y \) smooth and \( Y \) terminal and \( \mathbb{Q} \)-factorial.
Remark 1.6. Let \((T, S)\) be a pair, to produce a good model it is enough to consider a log resolution of \((T, S)\). Clearly there are infinitely many good models for any pair and running a directed MMP one can find the one that is more suitable for the needs of the moment.

The threshold allowed to produce an equivalent condition to being Cremona Equivalent to a plane, [MP12 Theorem 4.15], but unfortunately it is almost impossible to check this condition on specific examples. More recently, [Mc20], a numerical trick allowed to simplify the criteria and provided an effective test for a large class of rational surfaces.

Lemma 1.7 ([Me20]). Let \((T, S)\) and \((T_1, S_1)\) be birational models of a pair. Assume that \((T, S)\) has canonical singularities. If \(\rho(T, S) = a \geq 1\) then \(\rho(T_1, S_1) \geq a\).

As a direct consequence of Lemma 1.7 one can reformulate the condition of being Cremona Equivalent to a plane as follows.

Corollary 1.8 ([Me20]). A rational surface \(S \subset \mathbb{P}^3\) is Cremona equivalent to a plane if and only if there is a good model \((T, S_T)\) of \((\mathbb{P}^3, S)\) with \(0 < \rho(\mathbb{P}^3, S_T) < 1\).

There is a class of divisor that are always Cremona Equivalent to a hyperplane.

Remark 1.9. Let \(S \subset \mathbb{P}^3\) be a monoid, that is an irreducible and reduced surface of degree \(d\) with a point, say \(p\), of multiplicity \(d-1\). Then \(S = (x_3F_{d-1} + F_d = 0)\), consider the linear system

\[ L := \{(F_{d-1}x_0 = 0), (F_{d-1}x_1 = 0), (F_{d-1}x_2 = 0), S\}. \]

Then \(\varphi_L : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3\) is a birational modification and \(\varphi_L(S)\) is a plane. That is any monoid is Cremona Equivalent to a plane.

As a warm up I apply Corollary 1.8 and Remark 1.9 to determine the minimal Cremona degree of all surfaces of degree at most 3.

Proposition 1.10. Let \(S \subset \mathbb{P}^3\) be an irreducible and reduced surface of degree at most 3 and \(\sigma\) its minimal Cremona degree. Then \(\sigma \in \{1, 3\}\) and

\(\sigma = 1\) if and only if \(S\) is rational

\(\sigma = 3\) if and only if \(S\) is not rational, i.e. \(S\) is a cone over an elliptic curve.

Proof. The statement is immediate in degree 2 by Remark 1.9. Let \(S\) be a rational cubic. If \(S\) is smooth then \((\mathbb{P}^3, S)\) is a good model with \(\rho(\mathbb{P}^3, S) = 3/4\), hence we conclude by Corollary 1.8. If \(S\) has a double point, then it is a monoid and Remark 1.9 allows to conclude. If \(S\) is a cone I conclude by [Me13].

My aim is to improve this result determining the minimal Cremona degree of quartic surfaces in \(\mathbb{P}^3\).

The case of quartics is, as usual, more subtle due to their own intrinsic complexity. Smooth quartic surfaces are the only smooth hypersurfaces with automorphisms not coming from linear automorphisms of \(\mathbb{P}^3\), [MM63]. In a recent paper K. Oguiso produced examples of isomorphic smooth quartic surfaces that are not Cremona Equivalent, [Og17]. It is a long standing problem to determine which quartic surfaces are stabilized by non trivial subgroups in \(Cr_3\), that is for which quartic surface \(S \subset \mathbb{P}^3\) there is a Cremona modification \(\omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3\) such that \(\omega\) is not an isomorphism and \(\omega(S) = S\). The above problem has been studied by Enriques [En906] and Fano [Fa906] and also by Sharpe and coauthors in a series of papers, [MS913].
and [SS91], at the beginning of the XXth century. More recently Araujo-Corti-Massarenti continued the study of mildly singular quartic surfaces admitting a non trivial stabilizers in the Cremona Group, [ACM], in the context of Calabi-Yau pairs preserving symplectic forms.

On the other hand the singularities of quartic surfaces are completely classified, [De90], and there are a few hundreds of non isomorphic rational quartic surfaces, [Jes91]. This allowed, quite surprisingly, to prove the following result.

**Theorem 1.11 ([Me20]).** Let $S \subset \mathbb{P}^3$ be a rational quartic surface then $S$ is Cremona Equivalent to a plane.

This shows that any rational quartic has a huge stabilizer in the Cremona group disregarding the type of singularity it may have. Indeed it is amazing that, even if there are hundreds of non isomorphic families of rational quartics the Cremona group of $\mathbb{P}^3$ is playable enough to smooth any of them to a plane. In the next section I determine the minimal Cremona degree of an arbitrary quartic surface, adapting the techniques used to prove Theorem 1.11 to an arbitrary quartic surface.

**2. Minimal Cremona degree of quartics**

My aim is to study the minimal Cremona degree of an arbitrary quartic. The main tool I use, beside the $\sharp$-Minimal Model techniques, is the complete classification of singularities of quartic surfaces, see [Jes91] [De90] and in particular the detailed analysis of the singular locus given in [Uns81] [Uns84] [Ur86].

Let us start treating quartic cones.

**Lemma 2.1.** Let $S$ be a quartic cone in $\mathbb{P}^3$. Then $S$ is of minimal degree if and only if its sectional genus is at least 2.

**Proof.** A surface of degree less than 4 is either rational or an elliptic cone. By [Me13, Corollary 2.7] two surface cones are Cremona Equivalent if and only if their hyperplane sections are birational. □

Next I study non normal quartics.

**Lemma 2.2.** Let $S \subset \mathbb{P}^3$ be a non normal quartic, which is not a cone, then $S$ is not of minimal Cremona degree.

**Proof.** If $S$ is rational I apply Theorem 1.11. Then I assume that $S$ is not rational. Let $Y \subset S$ be the singular locus of $S$. Then by the classification of [Ur86] I have that either $Y$ is a pair of skew lines or $Y$ is a line and the general hyperplane section, say $H$, has an $A_3$ singularity in $H \cap Y$.

Here I mimic part of the proof in [Me20] Proposition 2.4] Let $L \subset Y$ be a line and $x \in S$ a general point. Consider the linear system $\Lambda$ of quadrics through $L$ and $x$. Let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ be the map associated to the linear system $\Lambda$. I have $\varphi(\mathbb{P}^3) = Z \cong \mathbb{P}^1 \times \mathbb{P}^2$, embedded via the Segre map, and $\varphi(L) = \hat{S}$ is a divisor of type $(3, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Note that divisors of type $(1, 0)$ are planes and divisors of type $(0, 1)$ are quadrics, then I have $\deg \hat{S} = 3 + 4 = 7$.

Claim 1. The surface $\hat{S}$ is singular along a smooth conic.

**Proof of the Claim.** If $Y = L \cup R$ is a pair of lines then $L \cap R = \emptyset$ and clearly $f(R)$ is a smooth conic, singular for $\hat{S}$. Assume that $Y = L$ and let $\nu : T \to \mathbb{P}^3$ be the blow up of $L$, with exceptional divisor $E$. Then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and since
the general hyperplane section of $S$ has a singular point of type $A_3$ I have that $\nu^{-1}(S) \cap E$ is a conic and $\nu^{-1}(S)$ is singular along this conic. This is enough to conclude.

Let $y \in \text{Sing}(\tilde{S})$ be a general point and $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ the projection from $y$. Then $\pi|_{Z}$ is a birational map, $Y := \pi(Z) \subset \mathbb{P}^4$ is a quadric of rank $4$, and $S_Q := \pi(\tilde{S})$ is a surface of degree $7 - 2 = 5$.

**Claim 2.** The vertex of the quadric is a smooth point of $S_Q$.

**Proof.** The surface $\tilde{S}$ is a divisor of type $(3, 2)$ in $Z$ and it is singular in $y$. Let $l$ and $P$, respectively, be the line and the plane passing through $x$ in $Z$. The general choice of $x \in S_y$ yields $l \not\subset \tilde{S}$. The line $l$ is mapped to the vertex of the quadric and $\tilde{S}_l = 2x + p$ for some point $p$. This shows that $S_Q$ contains the vertex of the quadric and it is smooth there. □

The 3-fold $Q$ is a quadric cone and $S_Q$ is singular along a line. Let $z \in \text{Sing}(S_Q)$ be a point. By the Claim 2 $z$ is not the vertex of $Q$. Thus the projection from $z$ produces a birational model of $(Q, S_Q)$, say $(\mathbb{P}^3, Z)$, with $Z$ a cubic surface. Therefore $S$ is not of minimal degree. □

**Remark 2.3.** Incidentally note that Lemma 2.2 gives a different proof of [Ur86, Proposition 2.6], where it is proven that a non normal quartic birational to a ruled surface over a curve of genus 2 is a cone.

Finally I treat the case of normal quartics. Let us first recall the following well known result, [Um81, Proposition 8].

**Proposition 2.4.** The minimal resolution of a normal quartic surface $S \subset \mathbb{P}^3$ is one of the following:

i) a $K3$ surface

ii) a rational surface

iii) birationally equivalent to an elliptic ruled surface

iv) a ruled surface of genus 3.

It is immediate that quartic surfaces in i) and iv) are of minimal Cremona degree, see the proof of Theorem 2.9 for the details. Theorem 1.11 treats surfaces in ii). Then I am left to study surfaces in iii). That is surfaces of elliptic ruled type. The main tool I use for this type of surfaces is the detailed description of their singularities contained in [Um84]. I summarize what I need in the following Theorem.

**Theorem 2.5.** (Um84, Corollary pg 134) Let $S \subset \mathbb{P}^3$ be a normal quartic of elliptic ruled type. Then the set of irreducible components of a minimal resolution of its singular locus contains either two disjoint elliptic curves or an elliptic curve, say $E$, and one rational curve intersecting $E$. In particular a minimal resolution has always at least two irreducible components with at least one elliptic curve.

I am ready to complete the analysis.

**Lemma 2.6.** Let $S \subset \mathbb{P}^3$ be a normal quartic of elliptic ruled type. Then $S$ is not of minimal Cremona degree.
Proposition 2.4. Let $S$ be a quartic with a singular point of type $(a, a \in \{1, 2\})$, and let $\Lambda_a \subset |O(2)|$ be the linear system of quadrics having multiplicity $a+1$ on the valuation associated to the double line. Then it is easy to check that the map
\[
\varphi_{\Lambda_a} : \mathbb{P}^3 \dashrightarrow X_a \subset \mathbb{P}^{7-a}
\]
is birational.

As observed in [Me20, Proposition 2.4] we have two cases:

1. a double point with an infinitely near double line
   \[
   [x_0^2x_1^2 + x_0x_1Q(x_2, x_3) + F_4(x_1, x_2, x_3) = 0],
   \]
2. a tachnode with an infinitely near double line
   \[
   [x_0^2x_1^2 + x_0(x_1^3 + x_1Q_2(x_2, x_3)) + F_4(x_1, x_2, x_3) = 0].
   \]

Let $S$ be a quartic with a singular point of type $(a, a \in \{1, 2\})$, and let $\Lambda_a \subset |O(2)|$ be the linear system of quadrics having multiplicity $a+1$ on the valuation associated to the double line. Then it is easy to check that the map
\[
\varphi_{\Lambda_a} : \mathbb{P}^3 \dashrightarrow X_a \subset \mathbb{P}^{7-a}
\]
is birational.

As observed in [Me20, Proposition 2.4] we have two cases:

$S_1$ $X_1 \subset \mathbb{P}^6$ is the cone over the Veronese surface
$S_2$ $X_2 \subset \mathbb{P}^5$ is the cone over the cubic surface $C \subset \mathbb{P}^4$, where $C$ is the projection of the Veronese surface, say $V$, from a point $z \in V$.

The main point here is that in both cases I have $S_a := \varphi_{\Lambda_a}(S) \subset |O_{\mathbb{P}^7-a}(2)|$, in particular deg $X_a = 5 - a$ and deg $S_a = 10 - 2a$.

Case 2.7 $(S_2)$. Assume that $S$ has a point of type $(2)$. Then the pair $(\mathbb{P}^3, S)$ is birational to $(X_2, S_2)$. The surface $S_2 \subset X_2 \subset \mathbb{P}^5$ has degree 6 and $X_2$ has degree 3. Let $x \in S_2$ be a general point and $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ the projection from $x$. Then $\pi(X_2) = Q$ is a quadric cone and $S_x := \pi(S_2)$ is a surface of degree 5. Hence there is a a cubic hypersurface $D \subset \mathbb{P}^4$ such that
\[
D|_Q = S_x + H,
\]
for some plane $H$. Let $y \in S_x$ be a general point and $\pi_y : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ the projection from $y$.

Claim 3. $\tilde{S} := \pi_y(S_x)$ is a quartic surface singular along a line.

Proof. The point $y$ is general therefore deg $\tilde{S} = 4$. The map $\pi_y|_Q$ is birational and it contract the embedded tangent cone $T_yQ \cap Q = \Pi_1 \cup \Pi_2$ to a pair of lines $l_1 \cup l_2$. Up to reordering I may assume that $H \cap \Pi_1$ is the vertex of the cone. Therefore $\tilde{S} \cap \Pi_1$ is a cubic passing through $y$. Hence $\tilde{S}$ has multiplicity 2 along $l_1$. \qed

In particular the surface $S$ is Cremona Equivalent to a non normal quartic.

Case 2.8 $(S_1)$. Assume that $S$ has a point of type $(1)$. Then $(\mathbb{P}^3, S)$ is birational to $(X_1, S_1)$. First I prove that $S_1$ is always singular and on the smooth locus of $X_1$.

Claim 4. $S_1$ is in the smooth locus of $X_1$ and $S_1$ is singular.

Proof. I need to describe deeper the map $\varphi := \varphi_{\Lambda_1} : \mathbb{P}^3 \dashrightarrow X_1$, following [Me20, Proposition 2.4].

Let $S \subset \mathbb{P}^3$ be the cubic, I may assume that there is an irrational singular point of type $(1)$ in $p \equiv [1, 0, 0, 0] \in S$ and the equation of $S$ is
\[
(x_0^2x_1^2 + x_0x_1Q + F_4 = 0) \subset \mathbb{P}^3.
\]

Let $\epsilon : Y \rightarrow \mathbb{P}^3$ be the weighted blow up of $p$, with weights $(2, 1, 1)$ on the coordinates $(x_1, x_2, x_3)$, and exceptional divisor $E \cong \mathbb{P}(1, 1, 2)$. Then I have:
- $\epsilon^*(x_1 = 0) = H + 2E$, $\epsilon|_H : H \to (x_1 = 0)$ is an ordinary blow up and $H|_E$ is a smooth rational curve;
- $\epsilon^*(S) = S_Y + 4E$,
- $S_Y|_E$ has at most two irreducible components and in this case both curves are rational,
- $S_Y|_H$ is a union of four smooth disjoint rational curves.

In particular:
- both $H$ and $S_Y$ are on the smooth locus of $E$ and hence on the smooth locus of $Y$;
- The surface $H$ is ruled by, the strict transforms of, the lines in the plane $(x_1 = 0)$ passing through the point $p$.
- $S_Y$ has not further singularities along $H$.
- by Theorem 2.5 the surface $S_Y$ is not a resolution of singularities of $S$. That is $S_Y$ is singular.

Let $l_Y$ be a general curve in the ruling and $\Lambda_Y = \epsilon^{-1}_*(\Lambda_1)$ the strict transform linear system. Then $E \cdot l_Y = 1$ and by a direct computation I have
- $\Lambda_Y \cdot l_Y = (\epsilon^* (O(2)) - 2E) \cdot l_Y = 0$,
- $S_Y \cdot l_Y = (\epsilon^* (O(4)) - 4E) \cdot l_Y = 0$,
- $K_Y \cdot l_Y = (\epsilon^* (O(-4)) + 3E) \cdot l_Y = -1$,
- $H \cdot l_Y = (\epsilon^* (O(1)) - 2E) \cdot l_Y = -1$.

Then $H$ can be blown down to a smooth rational curve with a birational map $\mu : Y \to X_1$ and by construction $S_Y = \mu^* S_1$. This shows that the unique singularity of $X_1$ is the singular point in $E$ and the surface $S_1$ is singular.

Let $x \in \text{Sing}(S_1)$ be a singular point. Set $\pi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$ be the projection from $x$. Then $\pi|_{X_1} : X_1 \dashrightarrow X_2$ is birational and $\pi(S_1) \in |O_{X_2}(2)|$. I am therefore back to case (S2). This shows that, also in this case, $(\mathbb{P}^3, S)$ is Cremona Equivalent to a non normal quartic.

To conclude observe that $S$ is of elliptic ruled type and it is birational to a non normal quartic, say $V$. If $V$ is a cone I conclude by Lemma 2.4. If $V$ is not a cone I apply Lemma 2.2.

I am ready to compute the minimal Cremona degree of quartic surfaces.

**Theorem 2.9.** Let $S \subset \mathbb{P}^3$ be a quartic surface and $\sigma$ its minimal Cremona degree. Then $\sigma \in \{1, 3, 4\}$ and

- $\sigma = 1$ if and only if $S$ is rational
- $\sigma = 3$ if and only if it is of elliptic ruled type, i.e. it is birational to a ruled surface over an elliptic curve
- $\sigma = 4$ in all other cases, i.e. $S$ has at most rational double points or it is a cone of sectional genus at least 2.

**Proof.** If $S$ is a cone I conclude by Lemma 2.4. If $S$ is rational I conclude by Theorem 1.11. By [MP09, Lemma 3.1] if $S$ is not of minimal degree the pair $(\mathbb{P}^3, S)$ has worse than canonical singularities. Therefore I may assume that $S$ is not rational, is not a cone and $(\mathbb{P}^3, S)$ has worse then canonical singularities. If $S$ is not normal, by Lemma 2.2 it is not of minimal Cremona degree and being not rational it has $\sigma = 3$ and it is of elliptic ruled type. If $S$ is normal, by Proposition 2.4 it is of elliptic ruled type. By Lemma 2.6 the surface $S$ is not of minimal degree and
being not rational it has \( \sigma = 3 \). On the other hand all surfaces of degree at most 2 are rational and non rational surfaces of degree 3 are elliptic cones. Therefore \( \sigma = 3 \) if and only if \( S \) is of elliptic ruled type.

Thanks to the detailed classification of singularities I am able to easily characterize the minimal Cremona degree of quartic surfaces with isolated singularities.

**Corollary 2.10.** Let \( S \subset \mathbb{P}^3 \) be a quartic surface with isolated singularities and \( \sigma \) its minimal degree. Then

\[
\sigma = 1 \text{ if and only if there is a unique elliptic singularity,} \\
\sigma = 3 \text{ if and only if there are either two elliptic singularities or one singular point of genus 2,} \\
\sigma = 4 \text{ if and only if it is a cone or has only rational double points.}
\]

**Proof.** Immediate by classification in [De90].

**Remark 2.11.** A similar result for non isolated singularities is possible, thanks to [Ur86], but it is not as neat as the one in Corollary 2.10.

It is hopeless to look for a similar statement in higher degrees. The singularities of surfaces of degree greater than 4 are not classified. Even the Cremona Equivalence of rational surfaces is not easy to tackle due to the lack of classification of rational surfaces of degree greater than 4. The most intriguing problem is to determine whether the vanishing of adjoints is equivalent to the Cremona Equivalence to a plane, like in the plane curve case.

Thanks to Theorem 2.9 I am able to prove that any quartic surface whose minimal Cremona degree is less than 4 has a non trivial stabilizer in the Cremona Group. I start with the following probably known result that I prove for lack of an adequate reference.

**Lemma 2.12.** Let \( X \subset \mathbb{P}^n \) be a cubic hypersurface, then its stabilizer in \( \text{Cr}_n \) is non trivial.

**Proof.** Let \( T \subset \mathbb{P}^{n+1} \) be a cubic hypersurface with a double point in \([0, \ldots, 0, 1]\) and containing \( X \) has the hyperplane section \( x_{n+1} = 0 \). If \( (F_3 = 0) = X \subset \mathbb{P}^n \) it is enough to consider

\[
T = (x_{n+1}Q + F_3 = 0),
\]

for \( Q \in \mathbb{C}[x_0, \ldots, x_n] \) general. Then the projection

\[
\pi : T \dashrightarrow \mathbb{P}^n = (x_{n+1} = 0) \subset \mathbb{P}^{n+1}
\]

from the point \([0, \ldots, 0, 1]\) is a birational map such that \( \pi(X) = X \). Fix a general point \( p \in X \) and let \( \tau_p : T \dashrightarrow T \) be the involution induced by \( p \). That is \( \tau_p(q) \) is the third point of intersection of the line spanned by \( p \) and \( q \). By construction \( \tau(X) = X \). Hence \( \pi \circ \tau \circ \pi^{-1} \) is a non trivial element in \( \text{Cr}_n \) that stabilizes \( X \).

**Corollary 2.13.** Let \( S \subset \mathbb{P}^3 \) be a quartic surface of minimal Cremona degree different from 4. Then \( S \) has a non trivial stabilizer in \( \text{Cr}_3 \).

**Proof.** Let \( \sigma \) be the minimal Cremona degree of \( S \), then \( \sigma \in \{1, 3\} \) by Theorem 2.9. If \( \sigma = 1 \) the result is immediate. If \( \sigma = 3 \) then \( S \) is Cremona equivalent to a cubic cone and I apply Lemma 2.12 to conclude.
Remark 2.14. Note that for quartic surfaces with minimal Cremona degree 4 the situation is completely different and not much is known, see [ACM] for a modern reference.

REFERENCES

[AM75] Abhyankar, S. S., Moh, T. T.: Embeddings of the line in the plane. J. Reine Angew. Math. 276, 148–166 (1975)

[ACM] Araujo, C., Corti, A., Massarenti, A.: Pliability of Calabi-Yau pairs and subgroups of the Cremona group of $E^3$ in preparation.

[BCHM10] Birkar, C., Cascini, P., Hacon, C., McKernan, J.: Existence of minimal models for varieties of log general type. Journal of the A.M.S. 23 2010, 405-468

[BM97] Bruno, A., Matsuki, K.: Log Sarkisov program. Internat. J. Math. 8, no. 4, 451–494 (1997)

[CC10] Calabri, A., Ciliberto C.: Birational classification of curves on rational surfaces. Nagoya Math. Journal 199, 43-93 (2010)

[CC17] Calabri, A., Ciliberto, C. On the Cremona contractibility of unions of lines in the plane. Kyoto J. Math. 57 (2017), 55-78

[Ca18] Cantat, S. The Cremona group. Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., 97.1, Amer. Math. Soc., 2018 101-142

[Ca890] Castelnuovo, G: Massima dimensione dei sistemi lineari di curve piane di dato genere. Ann. Mat. (2) 18 (1890), 119-128.

[Ca891] Castelnuovo G., Ricerche generali sopra i sistemi lineari di curve piane. Mem. R. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (2) 42 (1890–1891), 137-188.

[CE900] Castelnuovo, G., Enriques, F. Sulle condizioni di razionalit’a dei piani doppi. Rend. Circ. Mat. Palermo 14 (1900), 290–302

[CCMRZ16] Ciliberto, C., Cueto, M. A., Mella, M., Ranestad, K., Zwiernik, P. Cremona linearizations of some classical varieties. From classical to modern algebraic geometry, 375–407, Trends Hist. Sci., Birkhäuser/Springer, Cham, 2016

[Con39] Conforto F., Le superficie razionali. Zanichelli, Bologna, 1939.

[Coo28] Coolidge, J.L.: A treatise of algebraic plane curves. Oxford Univ. Press. Oxford (1928)

[CM04] A.Corti and M. Mella, Birational geometry of terminal quartic 3-folds I. American Journ. of Math. 126 (2004), 739-761.

[De90] Degtyarev, A.I. Classification of surfaces of degree four having a non-simple singular point. USSR Izvestiya 35 (1990) 607-627.

[En906] Enriques, F. Sulle superficie algebriche, che ammettono una serie discontina di trasformazioni birazionali. Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 15 (1906), pp. 665-669.

[Fa906] Fano, G. Sopra alcune superficie del 4-ordine rappresentabili sul piano doppio. Rendiconti del Reale Istituto Lombardo, ser. 2, vol. 39 (1906), pp. 1071- 1086.

[Ii99] Iitaka, S.: Birational geometry of plane curves. Tokyo J. Math. 22, no. 2, 289-321 (1999)

[Jessop] Jessop, C. M. Quartic surfaces with singular points. Cambridge [Eng.] University Press 1916, available at https://archive.org/details/cu31924062545383

[Ju888] Jung, G.: Ricerche sui sistemi lineari di genere qualunque e sulla loro riduzione all’ordine minimo. Annali di Mat. (2) 16 (1888), 291-327.

[Ka891] Kantor S., Sur une thorie des courbes et des surfaces admettant des correspondances univoques, C. R. Acad. Sci. Paris 100 (1885), 343-345.

[Ka891] Kantor S., Premiers fondements pour une th’orie des transformations p’eriodiques univoques, Atti Accad. Sci. Fis. Mat. Napoli (2) 4 (1891), 1-335.

[KM83] Kumar, N.M., Murthy, M.P.: Curves with negative self intersection on rational surfaces. J. Math. Kyoto Univ. 22, 767–777 (1983)

[MM63] Matsumura, H., Monsky, P. On the automorphisms of hypersurfaces. J. Math. Kyoto Univ. 3 (1963), no. 3, 347–361.

[Me02] Mella, M.: #-Minimal Model of uniruled 3-folds. Mat. Zeit. 242, 187–207 (2002)

[Me13] Mella, M. Equivalent birational embeddings III: Cones. Rend. Semin. Mat. Univ. Politec. Torino 71 (2013), no. 3-4, 463–472.
[Me20] Mella, M. Birational geometry of rational quartic surfaces J. Math. Pures Appl. (9) 141 (2020) 89-98.

[MP09] Mella, M., Polastri, E.: Equivalent Birational Embeddings Bull. London Math. Soc. 41, N.1 89–93 (2009)

[MP12] Mella, M., Polastri, E.: Equivalent Birational Embeddings II: divisors Mat. Zeit. 270, Numbers 3-4 (2012), 1141-1161, DOI: 10.1007/s00209-011-0845-3

[MS913] Morgan, F. M., Sharpe, F. R. Quartic surfaces invariant under periodic transformations Ann. of Math. (2) 15 (1913/14), no. 1-4, 84–92.

[Na60] Nagata, M.: On rational surfaces. I. Irreducible curves of arithmetic genus 0 or 1. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. 32, 351–370 (1960)

[Og17] Oguiso, K. Isomorphic Quartic K3 Surfaces in the View of Cremona and Projective Transformations Taiwanese J. Math. 21 (2017), no. 3, 671–688

[SS914] Sharpe, F. R.; Snyder, V. Birational transformations of certain quartic surfaces Trans. Amer. Math. Soc. 15 (1914), no. 3, 266–276.

[Um81] Umezu, Y. On normal projective surfaces with trivial dualizing sheaf Tokyo J. Math. 4 (1981), no. 2, 343–354.

[Um84] Umezu, Y. Quartic Surfaces of Elliptic Ruled Type Transactions of the American Mathematical Society , 283 (1984), 127-143

[Ur86] Urabe, T. Classification of Non-normal Quartic Surfaces Tokyo J. Math. 9 (1986) 265-295

Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli 30, 44121 Ferrara, Italia

Email address: mll@unife.it