Analogue Gravity Models for Planar Black Holes

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Abstract

An explicit Lagrangian for a scalar field is presented for which the perturbation equations are analogues of a scalar field propagating in a planar black hole space-time. This is valid for all planar black holes conformal to a Painlevé-Gullstrand type metric.

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1 Introduction

Certain condensed matter systems, respectively the Lagrangians describing these systems, have a property that small perturbations around a given background are described by the equations of motion of a field propagating in curved space-time. Thus, these systems may serve as ‘analogues’ of phenomena in gravitational physics and could, in principle, be employed to simulate gravity in tabletop experiments. Though already known in theory since the 1980s \cite{1,2}, it is only in recent years that this approach in simulating gravity has attracted more attention, mostly due to new technologies – in particular in dealing with BECs or cold atom systems - having been developed and making these kinds of experiments more accessible \cite{3,4,5,6,7,8,9}.

However, it should be noted that not all interesting geometries can be mimicked by analogue geometries. For example, counting the available degrees of freedom, in general relativity (GR) in $3 + 1$ dimensions we have four degrees of freedom per point in space-time: 10 independent components of a general metric minus 6 owing to the 6 independent Einstein equations. Whereas an analogue metric basically depends on two independent functions, which are the scalar potential $\theta$ that generates the flow velocities and the speed of sound $c$. Though, additional degrees of freedom enter by coupling to an external potential, which is assumed to be freely tunable. Thus, while a simple analogue gravity setup can not reproduce all possible metrics of GR, we can mimic the most important phenomena such as black holes, FRW cosmology, and even some aspects of semiclassical quantum gravity.

As these phenomena are all of central importance in gravity physics, it is desirable to extend the class of analog gravity systems to as many metrics as possible. Besides astronomic observations, analog gravity provides the only way to experimentally test such predictions in a lab environment. In this paper, we extend the applicability of analog gravity, by demonstrating that it can potentially capture all phenomena described by a field propagating in any space-time that is conformal to a rather generic stationary planar black hole. This provides a generalization and adds to the examples of planar space-times that have already been found to have an analog dual \cite{10,11,12}.

The paper is organized as follows. In section \ref{sec:geometry} we define the geometry and its conformally rescaled metric. In section \ref{sec:fluid} we outline a field theory description of a fluid. In section \ref{sec:mapping} we show how a planar black hole metric can be mapped to the effective geometry of a fluid in which acoustic perturbations propagate. Concluding remarks are given in section \ref{sec:conclusions}.

We adopt a convention in which the speed of light and Planck constant $\hbar$ are set to unity, $c$ denotes the speed of sound, and the metric signature is ‘mostly plus’, i.e., $\{-,+,...,+,\}$. 

1
2 Conformal Rescaling

Consider a space-time in $n + 1$ dimensions conformal to a rather generic stationary planar black hole metric, which for later convenience we take to be parameterized as

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{\Omega^2(t,x,z)}{\sqrt{1-\gamma(z)}} \left[ -\gamma(z)dt^2 + \frac{dz^2}{\gamma(z)} + dx^2 \right].$$  \hspace{1cm} (1)

If there is a horizon located at $z = z_0$, where $\gamma(z_0) = 0$, the outside region is characterized by $\gamma > 0$. A canonical scalar field $\varphi$ propagating in this background with effective mass $m_{\text{eff}}$ satisfies the equation of motion,

$$\Box \varphi - m_{\text{eff}}^2 \varphi \equiv \frac{1}{\sqrt{|G|}} \partial_\mu \left( \sqrt{|G|} G^{\mu\nu} \partial_\nu \varphi \right) - m_{\text{eff}}^2 \varphi = 0.$$  \hspace{1cm} (2)

Via a rescaling $\varphi = \Omega^{1-n/2} \tilde{\varphi}$, this equation is equivalent to the conformally rescaled equation of motion [13, 14]

$$\Box \tilde{\varphi} - \tilde{m}_{\text{eff}}^2(t,z,x) \tilde{\varphi} = 0,$$  \hspace{1cm} (3)

where the rescaled field $\tilde{\varphi}$ is propagating in the background geometry with conformally rescaled line element

$$d\tilde{s}^2 = \Omega^2(t,x,z)^{-2} ds^2 = \tilde{G}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{\sqrt{1-\gamma(z)}} \left[ -\gamma(z)dt^2 + \frac{dz^2}{\gamma(z)} + dx^2 \right].$$  \hspace{1cm} (4)

and effective mass squared

$$\tilde{m}_{\text{eff}}^2 = \Omega^2 m_{\text{eff}}^2 + \Omega^{1/2-n/2} \Box \Omega^{n/2-1/2}.$$  \hspace{1cm} (5)

Building up on [15] we now proceed to show that an acoustic perturbation in a fluid – the dynamics of which is described by an explicit field theory Lagrangian – can be realized as a scalar field propagating in the background [4].

3 The Lagrangian

Consider a Lagrangian of the form

$$L = F(\chi) - V(\theta, t, z, x),$$  \hspace{1cm} (6)

where $\theta$ is a dimensionless scalar field, and $F$ is an arbitrary function of the kinetic energy, which we denote by

$$\chi = -g^{\mu\nu} \theta_\mu \theta_\nu.$$  \hspace{1cm} (7)
The energy-momentum tensor corresponding to (6) is
\[ T_{\mu \nu} = 2L_{\chi} \theta_{,\mu} \theta_{,\nu} + \mathcal{L} g_{\mu \nu}, \]  
where the subscript $\chi$ denotes a partial derivative with respect to $\chi$. For $\chi > 0$, this energy-momentum tensor will describe a perfect fluid if we identify the pressure and energy density as
\[ p = \mathcal{L}, \]  
\[ \rho = 2\chi \mathcal{L} - \mathcal{L}, \]
and the fluid velocity vector as
\[ u_{\mu} = \frac{\theta_{,\mu}}{\sqrt{\chi}}. \]
This equation describes the so-called 'potential flow' and the field $\theta$ is referred to as the velocity potential. Furthermore, we can also identify the particle number density and specific enthalpy
\[ n = 2\sqrt{\chi} L_{\chi}. \]
\[ w \equiv \frac{p + \rho}{n} = \sqrt{\chi}. \]
This is consistent with the Gibbs relation
\[ dp = n dw - nT ds = L_{\chi} d\chi + L_{\theta} d\theta, \]
when a functional relationship $s = s(\theta)$ is assumed, with $s$ being the specific entropy, i.e. the entropy per particle.

Thus, we have constructed a field theory description of a fluid. Following [15], the ideal irrotational fluid will satisfy the Euler equation – i.e. the energy momentum conservation – if, in addition to the potential flow equation (11), the field satisfies the equation of motion
\[ (2L_{\chi} g^{\mu \nu} \theta_{,\nu})_{,\mu} + \frac{\partial \mathcal{L}}{\partial \theta} = 0. \]
Using (11) and (12) this equation can be written as
\[ (nw_{\mu})_{,\mu} = \frac{\partial V}{\partial \theta}. \]
When considering a small perturbation $\theta = \theta_0 + \delta \theta$, the equation of motion for the perturbation $\delta \theta$ around the background $\theta_0$ can be written as
\[ \frac{1}{\sqrt{|G|}} \partial_{\mu} \left( \sqrt{|G|} \tilde{G}^{\mu \nu} \partial_{\nu} \delta \theta \right) - m_{\text{eff}}^2 \delta \theta = 0, \]
where
\[ m^2 \sqrt{|\tilde{G}|} \tilde{G}^{\mu \nu} = \left. \frac{\partial^2 F}{\partial \theta_{\nu} \partial \theta_{\mu}} \right|_{\theta_0}, \tag{18} \]
and
\[ m^2 \sqrt{|\tilde{G}|} m_{\text{eff}}^2 = \left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta_0}, \tag{19} \]
as is well known in analogue gravity (see e.g., Ref. [2]). Here \( \tilde{G}^{\mu \nu} \) is the inverse of the metric \( \tilde{G}_{\mu \nu} \) with determinant \( \tilde{G} \) and an arbitrary mass parameter \( m \) is introduced to make the metric dimensionless.

### 4 Relativistic Acoustic Metric

In the following we show that an acoustic perturbation propagating in a fluid described by the Lagrangian of the form (6) represents an analogue dual of a scalar field propagating in the background (4). In other words, if a fluid is described by the Lagrangian (6), the dynamics of acoustic perturbations, described by (17-19), will have the form of the the Klein-Gordon equation (3) in a curved space-time described by the line element (4).

The first step is to bring the metric to a form that can be compared to the relativistic acoustic metric [16] with components
\[ \tilde{G}_{\mu \nu} = \frac{n}{m^2 \omega_c} \left[ g_{\mu \nu} + (1 - c^2) u_{\mu} u_{\nu} \right]. \tag{20} \]
For this purpose, consider a simple coordinate transformation,
\[ t = \tilde{t} + f(z), \quad z = g(\tilde{z}). \tag{21} \]
such that the line element from (4) takes the form
\[ ds^2 = \frac{1}{\sqrt{1 - \gamma}} \left\{ -d\tilde{t}^2 + d\tilde{z}^2 + dx^2 + \left[ (1 - \gamma)d\tilde{t}^2 - 2\sqrt{(1 - \gamma)(c^2 - \gamma)}d\tilde{t}d\tilde{z} + (c^2 - \gamma)d\tilde{z}^2 \right] \right\}, \tag{22} \]
where
\[ \frac{dg}{d\tilde{z}} = c, \quad \frac{df}{d\tilde{z}} = \frac{\sqrt{(1 - \gamma)(c^2 - \gamma)}}{c\tilde{t}}. \tag{23} \]
Comparing to (20) allows to read off the non-vanishing components of the 4-velocity
\[ u^\tilde{t} = \sqrt{\frac{1 - \gamma}{1 - c^2}}, \quad u^\tilde{z} = -\sqrt{\frac{c^2 - \gamma}{1 - c^2}}. \tag{24} \]
Next, assuming a potential flow

\[ w u_\mu = \theta_{,\mu}, \quad (25) \]

we derive closed expressions for \( w, n, \) and \( c \) in terms of the variable \( \tilde{z} \). Since the metric is stationary, the velocity potential must be of the form

\[ \theta = m \tilde{t} + h(z), \quad (26) \]

where \( m \) is an arbitrary mass and \( h(z) \) is a function of \( \tilde{z} \) through \( z = g(\tilde{z}) \). The specific enthalpy is then given by

\[ w = \frac{m}{u_\tau} = m \sqrt{1 - \frac{c^2}{1 - \gamma}} \quad (27) \]

and the function \( h(z) \) is determined through

\[ \frac{dh}{dz} = -\frac{m}{c} \sqrt{\frac{c^2 - \gamma}{1 - \gamma}} \quad (28) \]

Furthermore, in view of (20) and (22) the particle number density is

\[ n = \frac{m^2 wc}{\sqrt{1 - \gamma}} = \frac{m^3 c \sqrt{1 - c^2}}{1 - \gamma}. \quad (29) \]

From the definition of the sound speed

\[ c^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_s = \left. \frac{n \partial w}{w \partial n} \right|_s = \frac{n w_{,\tilde{z}}}{w n_{,\tilde{z}}} \quad (30) \]

it follows that \( c \) must satisfy a differential equation

\[ \frac{\partial}{\partial \tilde{z}} c^2 = \left( c^2 - \frac{1}{2} \right) \frac{\partial}{\partial \tilde{z}} \ln(1 - \gamma), \quad (31) \]

which is solved by

\[ c^2 = c_1 (1 - \gamma) + \frac{1}{2} \quad (32) \]

Then we find

\[ w = m \sqrt{\frac{1}{2(1 - \gamma)} - c_1}, \quad (33) \]
\[ n = m^3 \sqrt{\frac{1}{4(1-\gamma)^2} - c_1^2} = m^2 w \sqrt{\frac{1}{2(1-\gamma)} + c_1} \]  

(34)

In principle, \( c_1 \) could be an arbitrary function of \( s \). However, since \( w \) and \( s \) are considered as independent variables, the right hand side of (33) admits no explicit \( s \)-dependence and hence a consistent choice is \( c_1 \equiv \text{const.} \) From (33) and (34) it follows

\[ n \frac{\partial w}{\partial \tilde{z}} = m^2 \sqrt{\frac{1}{2(1-\gamma)} + c_1} \frac{\partial^2 w}{\partial \tilde{z}^2} = m^4 \frac{\partial}{\partial \tilde{z}} \left( \frac{1}{2(1-\gamma)} + c_1 \right)^{3/2}. \]  

(35)

Then, according to (14) the pressure reads

\[ p = m^4 \left( \frac{1}{2(1-\gamma)} + c_1 \right)^{3/2} - c_2(s), \]  

(36)

where \( c_2(s) \) is an arbitrary function of \( s \). In view of (33) the pressure can also be expressed as

\[ p = m^4 \left( \frac{w^2}{m^2} + 2c_1 \right)^{3/2} - c_2(s). \]  

(37)

This expression is precisely of the form (6) in which

\[ F(\chi) = m^4 \left( \frac{\chi}{m^2} + 2c_1 \right)^{3/2}, \]  

(38)

\( c_2 \) is identified with \( V \) and the specific enthalpy with \( \sqrt{\chi} \) as in (13).

Therefore, we have shown that the Lagrangian (6) with (38) can be used to construct an analogue model for a scalar propagating in the metric (4). However, we have not specified the potential \( V \) yet. This will be done in the following.

4.1 The potential

Recall that we are considering a scalar field \( \theta = \theta_0 + \delta \theta \), i.e. a small acoustic perturbation \( \delta \theta \) around a fixed background \( \theta_0 \). If the equation of motion of this perturbation is to consistently be an analog model of a particle propagating in the curved space-time, the potential \( V \) has to meet a requirement that its first derivative, when evaluated on the background (26), is determined by the equation of motion (16).

In applications where one wishes to not only simulate a specific metric but also a specific effective mass\(^{1}\), then (19) demands to also impose conditions on the second derivative of \( V \). Thus, the potential \( V \) has to be chosen such that

\[ \frac{\partial V}{\partial \theta} \bigg|_{\theta = \theta_0} = (nd^\mu)_{\mu}, \quad \frac{\partial^2 V}{\partial \theta^2} \bigg|_{\theta = \theta_0} = \sqrt{G} m^2 \bar{m}_{\text{eff}}(\tilde{z}). \]  

(39)

\(^{1}\)such as e.g. in [17]
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In principle, one could satisfy these conditions in many ways. Quite generally, a suitable potential can be written as

\[ V = \alpha(\tilde{z})f_1(\theta/\theta_0) + \beta(\tilde{z})\theta^2 f_2(\theta/\theta_0) \]  

(40)

where \( f_1(x) \) and \( f_2(x) \) are arbitrary functions which at \( x = 1 \) (i.e., when \( \theta = \theta_0 \)) satisfy

\[ (xf_1(x))'' \big|_{x=1} = 0, \quad (x^2 f_2(x))' \big|_{x=1} = 0. \]  

(41)

and \( \alpha(\tilde{z}), \beta(\tilde{z}) \) are chosen to match (39). Therefore, the potential \( V \) will generally have to be chosen coordinate dependent. This would present no real obstacle from a practical point of view, as experimental setups for analogue gravity with moving and oscillating horizons are already being conducted, e.g., \([5, 6]\), and time and position dependent external potentials could be simulated with the same setup.

From a theoretical point of view, there could be some caveat that limits the choice of potentials. That comes from the condition that the Gibbs relation (14) must hold. At first sight it may seem a bit odd how the relation containing only two degrees of freedom could be satisfied with a generic potential \( V(\theta,t,z,x) \). However, one has to keep in mind that the functional identities in section 3 are independent of the specific coordinate dependence of the potential and the crucial point is that the Gibbs relation (14) has to hold as an on-shell functional identity. This is to say that it must be possible to express the pressure \( p \) as a functional depending on two variables \( w \) and \( s \) which are defined on the function space of solutions to the equations of motion. This reduces the effective number of degrees of freedom.\(^2\) In practice, however, it could be rather non-trivial to check (14) explicitly, and following construction might be more convenient.

Assume a Lagrangian with no explicit coordinate dependence of the form

\[ L = F(\chi) - V(\theta). \]  

(42)

The Gibbs relation (14) is now automatically satisfied and for a solution \( \theta_0 \) of the equations of motion (16) the analogue metric and effective mass for a perturbation follow from (18,19). In this situation one can proceed to construct a potential \( V \) that reproduces the desired analog metric analogously to \([15]\), where it has been worked out for the case of a planar black hole in AdS space-time. In order to then explicitly match the effective mass to a desired value, consider the Lagrangian (42) changed by an \( O(\theta - \theta_0)^2 \) deformation around the found background solution \( \theta_0 \), e.g.

\[ L' = F(\chi) - V(\theta) - \frac{a(\theta,t,x,z)}{2}(\theta - \theta_0)^2 \]  

(43)

By construction, \( \theta_0 \) is still a solution to the equations of motion and all identities from section 3 will hold identically when evaluated for \( \theta_0 \), with the exception of (19), which, as

\(^2\)Note that the Gibbs relation need not hold for a generic field that does not satisfy the equations of motion.
only quantity in the perturbation equations, depends on second order derivatives of the Lagrangian with respect to $\theta$. Thus, the effective mass changes to

$$
(m'_{\text{eff}})^2 = m^2_{\text{eff}} + \frac{a}{\sqrt{|G|}}.
$$

(44)

Therefore, by a suitable choice of $a(\theta,t,x,z)$, any $m'_{\text{eff}}$ can be reproduced without changing the analog metric.

What of course has to be maintained is that (42) has to remain an analog model when considering deviations around $\theta_0$, including the Gibbs relation (14), which is the most crucial for the analog gravity construction to work. This however follows directly from the theorem of implicit functions, if (42) is an analog model and $\theta_0$ is not a degenerate point in the space of solutions.

5 Conclusions

Using the formalism of analogue gravity for the case of nonisentropic fluids from [15], we have shown that by a suitable transformation of variables and choice of parametrization, a Lagrangian of the form (6) is an analogue model for a scalar field propagating in a space-time that is conformal to a static stationary planar black hole space-time. Furthermore, we have also demonstrated how, with a suitable adjustment of the external potential that couples to the analog Lagrangian, it is possible, for any given analog metric, to simulate an arbitrary effective mass for the perturbation. These results vastly extend the class of phenomena in gravity physics that can be simulated in condensed matter systems via the analogue gravity formalism.

This class has now been shown to include most non-rotating planar black hole metrics considered in the literature – as well as several cosmological space-times of particular interest. A particular emphasis was put on planar black hole geometries due to their prominence in applications and investigation on how analog gravity interlinks with gauge/gravity duality and condensed matter physics in the last years [20] [21] [22] [23] [24] [17]. A generalization to geometries with spherical or axial symmetry is possible and relatively straightforward, but will be left for future work.

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3 for reviews see e.g. [18] [19]
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