Existence of values on reachability games with inductive functions

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Abstract. In this study, we prove some basic facts on reachability games. We show that there exists a memoryless mixed optimal strategy for Player II. Our main contribution is a proof of the existence of memoryless ε-optimality for Player I in any reachability games. Actually, this result for reachability games was shown by Chatterjee et al. [3] in a slightly different setting. We show that every reachability game is determined and give a simple expression of value for this game by defining the notion of limit value inductively of finite-step games.

1. Introduction
We investigate a game with reachability objective, where the goal of Player I is to force the game to reach a specified set of target states, and the objective of the opponent to avoid the game from reaching the target state. We assign value 1 to any plays that reach the target state, and value 0 if otherwise. Since there are two players on whose decisions the probability depends, we talk about the highest probability that the Player I can achieve against any opponent’s strategy, and otherwise for Player II. If these two quantities are equal, we call it the value of the game and the game is determined. An optimal strategy for Player I is a strategy that guarantees the value of the game from each position. In this study, we give answers to the following questions. Are the games determined or can we derive the value of game? Is it possible to show optimal and ε-optimal strategies in some way? Then, we are interested in the question of what type of optimal strategies exists for both players.

2. Games, Strategies, and Values

Definition 1 A (two-player simultaneous infinite) game is a quadruple \( G = (S, A_I, A_{II}, \delta) \), where

- \( S, A_I \) and \( A_{II} \) are nonempty finite sets,
- \( \delta \) is a function from \( S \times A_I \times A_{II} \) into \( S \), called a transition function.

Elements of \( S \) are called states, elements of \( A_I \) and \( A_{II} \) are called actions or moves of Player I and Player II, respectively.

Definition 2 A path or a play of a game \( G = (S, A_I, A_{II}, \delta) \) is a finite or infinite sequence \( s_0 \delta_1 s_2 \ldots \) of states in \( S \) such that for all \( n \in \mathbb{N} \), there exist \( a_n \in A_I \) and \( b_n \in A_{II} \) where
\[ \delta(s_n, a_n, b_n) = s_{n+1}. \] Infinite paths of \( G \) are sometimes called \textit{runs}. We write \( \Omega(G) \) for the set of all infinite plays; and \( \Omega^\text{fin}(G) \) for the set of all finite plays of \textit{non-zero length}.

### 2.1. Strategies

Informally, a strategy for a player in the game is a rule that specifies the next move of the player for a given finite play.

**Definition 3** A (mixed) strategy of Player I in \( G \) is any function \( \sigma : \Omega^\text{fin}(G) \to D(A_I) \). We write \( \Sigma_I^G \) or \( \Sigma_I \) for the set of all strategies of Player I. We denote \( \tau : \Omega^\text{fin}(G) \to D(A_{II}) \) as a strategy for Player II, and we use \( \Sigma_{II}^G \) or \( \Sigma_{II} \) for the set of all strategies of Player II.

**Definition 4** A strategy \( \sigma \) of Player I is called \textit{memoryless} if \( \sigma(p) = \sigma(q) \) holds whenever \( p, q \in \Omega^\text{fin}(G) \) satisfy \( p(|p|−1) = q(|q|−1) \). A memoryless strategy of Player II is defined similarly. We write \( \Sigma_I^M \) and \( \Sigma_{II}^M \) for the set of all memoryless strategies of Player I and Player II, respectively.

### 2.2. Expected values and optimal strategies

A probability measure \( P_s^{\sigma, \tau} \) on \( \Omega_s = \{ w \in \Omega : w(0) = s \} \) is defined as follows.

**Definition 5** For a pair \( (\sigma, \tau) \in \Sigma_I^G \times \Sigma_{II}^G \) of strategies and a state \( s \in S \), \( P_s^{\sigma, \tau} \) denotes the probability measure on \( \Omega_s \) defined by

\[
P_s^{\sigma, \tau}([p]) = \prod_{n \in \{1, \ldots, |p|−1\}} \sum_{a, b \in \delta^{-1}(p(n))} (\sigma(p \upharpoonright n)(a)\tau(p \upharpoonright n)(b): (p(n−1), a, b) ∈ \delta^{-1}(p(n)))
\]

for any \( p \in \Omega^\text{fin} = \{ q \in \Omega^\text{fin} : q(0) = s \} \), where \([p] = \{ w \in \Omega : p ⊂ w \} \).

Let \( F : \Omega(G) → [0, 1] \) satisfy that \( P_s^{\sigma, \tau}(F) \) exists for any \( \sigma \in \Sigma_I^G, \tau \in \Sigma_{II}^G \) and \( s \in S \). We call such a function \( F \) a \textit{payoff function} of \( G \). The value of Player I is the supremum of expected value which Player I can ensure. Formally, it is \( \sup_{\sigma \in \Sigma_I^G} \inf_{\tau \in \Sigma_{II}^G} P_s^{\sigma, \tau}(F) \). Let \( \neg F \) be a function defined by \( \neg F(w) = 1 − F(w) \). The value of Player II is defined as \( \sup_{\sigma \in \Sigma_I^G} \inf_{\tau \in \Sigma_{II}^G} P_s^{\sigma, \tau}(\neg F) \).

This value is equal to \( 1 − \inf_{\tau \in \Sigma_{II}^G} \sup_{\sigma \in \Sigma_I^G} P_s^{\sigma, \tau}(F) \). We call the game \( G(F) \) is \textit{determinate} if

\[
\sup_{\sigma \in \Sigma_I^G} \inf_{\tau \in \Sigma_{II}^G} P_s^{\sigma, \tau}(F) + \sup_{\tau \in \Sigma_{II}^G} \inf_{\sigma \in \Sigma_I^G} P_s^{\sigma, \tau}(\neg F) = 1
\]

true for all \( s \in S \). Or equivalently, the game \( G(F) \) is determinate if and only if

\[
\sup_{\sigma \in \Sigma_I^G} \inf_{\tau \in \Sigma_{II}^G} P_s^{\sigma, \tau}(F) = \inf_{\tau \in \Sigma_{II}^G} \sup_{\sigma \in \Sigma_I^G} P_s^{\sigma, \tau}(F)
\]

holds for each \( s \in S \). In this case, we write \( \text{val}_I^G(F) \) or \( \text{val}_s(F) \) instead of \( \sup_{\sigma \in \Sigma_I^G} \inf_{\tau \in \Sigma_{II}^G} P_s^{\sigma, \tau}(F) \), and call it the \textit{value} at \( s \) in the game \( G(F) \).

The following is well-known theorem obtained by Martin.

**Theorem 1 (Martin [6])** Let \( G \) be a game and let \( F : \Omega(G) → [0, 1] \) a Borel measurable function. Then the game \( G(F) \) is determinate.

A strategy \( \sigma \in \Sigma_I \) of Player I is optimal if and only if

\[
\inf_{\tau \in \Sigma_{II}} P_s^{\sigma, \tau}(F) = \text{val}_s(F)
\]
holds for all \( s \in S \). Likewise, \( s \) strategy \( \tau \in \Sigma_I \) is optimal if and only if

\[
\sup_{\sigma \in \Sigma_I} P^s_\sigma,\tau(F) = \val_s(F)
\]

(4)

holds for all \( s \in S \).

When \( G(F) \) is determinate and \( \varepsilon \) is a positive real number, then \( \varepsilon \)-optimal strategies of Player I and Player II always exist by the definition. However, there are some cases that Player I or Player II has no optimal strategy.

**Theorem 2 (von Neumann [7])** In any one-step game, both players have their optimal strategies.

### 3. Values and Optimality on Reachability games

We first show every reachability game is determined and prove the existence of value by using the notion of limit value for the \( n \)-step game. This result leads us to the existence of optimal strategy (memoryless and mixed) for Player II.

**Definition 6** For a target state \( T \subset S \), we define \( R^G,T : \Omega(G) \to \{0, 1\} \) by

\[
R^G,T(w) = \begin{cases} 
1 & \text{if } (\exists k \in \mathbb{N})[w(k) \in T], \\
0 & \text{otherwise}, 
\end{cases}
\]

A game of the form \( G(R^G,T) \) is called a reachability game.

**Definition 7** For any \( s \in S \) and \( n \in \mathbb{N} \), we define inductively \( V_n^G,T : S \to [0, 1] \) by

\[
V_0^G,T(s) = \begin{cases} 
1 & \text{if } s \in T, \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
V_{n+1}^G,T(s) = \begin{cases} 
1 & \text{if } s \in T, \\
\val_s(V_n^G,T) & \text{otherwise}. 
\end{cases}
\]

(5)

We let \( V^G,T(s) = \lim_{n \to \infty} V_n^G,T(s) \) for all \( s \), as the limit value at \( s \).

For any \( n \in \mathbb{N} \), we define \( R_n^G,T : \Omega(G) \to \{0, 1\} \) by \( R_n^G,T(w) = 1 \) if there exists \( m \leq n \) with \( w(m) \in T \) and \( R_n^G,T(w) = 0 \) otherwise. Games of the form \( G(R_n^G,T) \) are called \( n \)-step reachability games. In this game, it requires Player I to visit the target set \( T \) during the first \( n \)-step.

The following result shows there exist an optimal strategy in \( n \)-step games.

**Theorem 3** For any \( n \in \mathbb{N} \), both players have their optimal strategy in the game \( G(R_n^G,T) \), and the equality \( V_n^G,T(s) = \val_s(R_n^G,T) \) holds for all \( s \in S \).

**Proof.** We define \( \sigma^*_n \) and \( \tau^*_n \) inductively. Let \( \sigma^*_0 \) and \( \tau^*_0 \) be any strategies. Now suppose that we have constructed \( \sigma^*_n \) and \( \tau^*_n \). Choose \( \sigma \) and \( \tau \) as optimal strategies of Player I and II respectively in the one-step game \( G(V_n^G,T) \). Define \( \sigma^*_{n+1}(s) = \sigma(s) \) and \( \sigma^*_{n+1}(s) = \sigma^*_n(\rho) \) for any \( s \in S \) and any \( \rho \neq \emptyset \) with \( s \rho \in \Omega^{\text{fin}} \). Similarly, define \( \tau^*_{n+1}(s) = \tau(s) \) and \( \tau^*_{n+1}(s) = \tau^*_n(\rho) \) for any \( s \in S \) and any \( \rho \neq \emptyset \) with \( s \rho \in \Omega^{\text{fin}} \). It is easy to see by induction on \( n \) that \( \sigma^*_n \) and \( \tau^*_n \) satisfy the equalities \( V_n^G,T(s) = \inf_{\tau \in \Sigma_I} P^s_\sigma,\tau(R_n^G,T) \) holds.

This equalities imply that the \( \sigma^*_n \) and \( \tau^*_n \) are optimal strategies in the game \( G(R_n^G,T) \) and \( V_n^G,T(s) = \val_s(R_n^G,T) \) holds.
Theorem 4 For all \( s \in S \), the equation \( V^{G,T}(s) = \text{val}_s(R^{G,T}) \) holds.

Proof. It is enough to show that the following inequalities:

\[
\inf_{\tau \in \Sigma_I} \sup_{s \in S} P^s_{\sigma,\tau}(R^{G,T}) \leq V^{G,T}(s) \leq \sup_{\sigma \in \Sigma_I} \inf_{\tau \in \Sigma_{II}} P^s_{\sigma,\tau}(R^{G,T}). \tag{6}
\]

To show the first inequality, choose an optimal strategy \( \tau^* \) of Player II in the one-step game \( G(V^{G,T}) \). We may see \( \tau^* \) as a memoryless strategy of Player II in the reachability game \( G(R^{G,T}) \).

We show that \( \tau^* \) satisfies the inequality \( \sup_{\sigma \in \Sigma_I} P^s_{\sigma,\tau^*}(R^{G,T}) \leq V^{G,T}(s) \) for any \( s \in S \). (Thus, if we prove the second inequality, then we can say this \( \tau^* \) is, in fact, an optimal strategy of Player II in the game \( G(R^{G,T}) \).)

It is enough to show that \( \sup_{\sigma \in \Sigma_I} P^s_{\sigma,\tau^*}(R^{G,T}) \leq V^{G,T}(s) \) for any \( s \in S \) and \( n \in \mathbb{N} \). We show this by induction on \( n \). The case \( n = 0 \) is clear. Suppose that \( \sup_{\sigma \in \Sigma_I} P^s_{\sigma,\tau^*}(R^{G,T}) \leq V^{G,T}(s) \) holds for any \( s \in S \) as an induction hypothesis. Fix \( s \in S \). If \( s \in T \), then it is obvious that the inequality holds for \( s \). Otherwise, we have the equality \( P^s_{\sigma,\tau^*}(R^{G,T}_{n+1}) = \sum_{s' \in S} P^s_{\sigma,\tau^*}(s') V^{G,T}_{n+1} \) for any \( \sigma \in \Sigma_I \). By the induction hypothesis, we know that \( P^s_{\sigma,\tau^*}(R^{G,T}_{n+1}) \leq \sum_{s' \in S} P^s_{\sigma,\tau^*}(s') V^{G,T}_{n+1} \).

Hence the equalities

\[
\sup_{\sigma} P^s_{\sigma,\tau^*}(R^{G,T}_{n+1}) \leq \sup_{\sigma} \sum_{s' \in S} P^s_{\sigma,\tau^*}(s') V^{G,T}(s) = V^{G,T}(s) \tag{7}
\]

hold by the optimality of \( \tau^* \) in the one-step game. Let us now show the second inequality. We have \( P^s_{\sigma,\tau^*}(R^{G,T}_{n+1}) \leq P^s_{\sigma,\tau^*}(R^{G,T}_{n}) \) since \( R^{G,T}_{n+1}(w) \leq R^{G,T}_{n+1}(w) \) for any \( w \in \Omega \). Hence \( \sup_{\sigma} \inf_{\tau} P^s_{\sigma,\tau^*}(R^{G,T}_{n}) \leq \sup_{\sigma} \inf_{\tau} P^s_{\sigma,\tau^*}(R^{G,T}) \) holds. By Theorem 3, \( V^{G,T}_{n+1}(s) = \text{val}_s(R^{G,T}_{n+1}) \).

Note that the strategy \( \tau^* \) constructed in the proof is memoryless and randomized optimal strategy because the game has a value in every state. So, we have a following corollary.

Corollary 1 Player II has a memoryless and randomized optimal strategy in any reachability game.

The next theorem shows that Player I has no such a strategy in reachability games and must settle for \( \epsilon \)-optimality.

Theorem 5 There is no memoryless and randomized optimal strategy for Player I in reachability game.

Remark 1 Consider the following simultaneous reachability game as shown in Figure 1. Let \( S = \{s_0, s_1, s_2\} \), \( A_I = \{x_1, x_2\} \) and \( A_{II} = \{y_1, y_2\} \). Define a transition function \( \delta \) by \( \delta(s_0, x_1, y_1) = s_0 \), \( \delta(s_0, x_2, y_2) = s_2 \), \( \delta(s_0, x_1, y_2) = \delta(s_0, x_2, y_1) = s_1 \) and \( \delta(s_i, x, y) = s_i \) for any \( i \in \{1, 2\} \). Now consider the reachability game \( G(R_{\{s_1\}}) \).
One can prove that $\text{val}_{\Omega}(R_{\{s_1\}}) = 1$. We show that Player I has no optimal strategy in the reachability game $G(R_{\{s_1\}})$. Fix a strategy $\sigma \in \Sigma_I$. We construct $\tau \in \Sigma_H$ such that $P^{\sigma,\tau}_{s_0}(R_{\{s_1\}}) < 1$. For $\rho \in \Omega^\infty(s)$, define $\tau(\rho)(y_1) = 1$ if $\sigma(\rho)(x_1) = 1$, and define $\tau(\rho)(y_2) = 1$ otherwise. It is clear that $P^{\sigma,\tau}_{s_0}(R_{\{s_1\}}) < 1$ by the definitions of $G$ and $\tau$.

The next theorem says that, given a reachability game, Player I always has a memoryless $\varepsilon$-optimal strategy in this game for any positive real number $\varepsilon$. In fact, this result for reachability games was shown by Chatterjee et al. [3] in a slightly different setting. We essentially use their method to prove our theorem.

**Theorem 6** In every reachability game $G(\mathcal{G})$, there exist an $\varepsilon$-optimal memoryless strategy of Player I for any $\varepsilon > 0$.

**Proof.** Let $\mathcal{G} = (S, A_I, A_H, \delta)$ be a game and let $T \subset S$ be set of target states of Player I. Without loss of generality, we may assume that if $s \in T$ or $\text{val}_s(\mathcal{G}) = 0$, then $\delta(s, x, y) = s$ holds for any $(x, y) \in A_I \times A_H$. We may call state $s$ is absorbing state.

Fix a positive real $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that for any $s \in S$, the inequality $V_{n-1}(s) \geq \text{val}_s(\mathcal{G}) - \varepsilon$ holds, and $\text{val}_s(\mathcal{G}) > 0$ implies $V_n(\mathcal{G}) > 0$.

For $m \leq n$, choose $\sigma_m \in \Sigma_I$ such that $\sigma_m$ is an optimal strategy of Player I in the one-step game $\mathcal{G}(V_{m-1})$. We define a strategy $\sigma^* \in \Sigma_I$ by $\sigma^*(s) = \sigma_m(s)$ for any $s \in S$, where $m$ is the least number $m \leq n$ such that $V_m(s) = V_n(s)$. By the definition, $V_m(s) = \inf_{T} \Sigma \in \Sigma_H P_s^{\sigma,\tau}(V_{m-1})$ holds for any $s \in S \setminus T$. Now choose a strategy $\tau^* \in \Sigma_H$ such that $P_s^{\sigma^*,\tau^*}(\mathcal{G}) = \inf_{\Sigma} P_s^{\sigma^*,\tau}(\mathcal{G})$ for all $s \in S$.

Fix a $s \in S \setminus T$ with $V_m(s) > 0$. Suppose that $V_n(s) \geq V_n(s')$ holds for any $s' \in S$ with $P_s^{\sigma^*,\tau^*}([s']) > 0$. We have

$V_{m-1}(s') = V_n(s')$

for any $s' \in T$ with $P_s^{\sigma^*,\tau^*}([s']) > 0$ since $V_n(s') = V_m(s')$. $V_{m-1}(s') \leq V_n(s')$ and $V_m(s') \leq P_s^{\sigma^*,\tau^*}(V_{m-1})$ hold. Therefore, if $s' \in S$ satisfies $P_s^{\sigma^*,\tau^*}([s'])$, then $m > m-1$. As a result, we know that for any $s \in S \setminus T$ there exists $s'$ with $P_s^{\sigma^*,\tau^*}([s']) > 0$ such that

$V_n(s') < V_n(s')$ or $m > m-1$.

Note that $\{V_n(s) : s \in S \setminus T\}$ is finite, and $m = 0$ implies $s \in T$ or $V_n(s) = 0$. Here $V_n(s) = 0$ implies $\text{val}_s(\mathcal{G}) = 0$. Hence for any $s \in S$ there exists $\rho \in \Omega^\infty(s)$ such that $P_s^{\sigma^*,\tau^*}([\rho]) > 0$ and $\rho(|\rho| - 1) \in T$ or $\text{val}_\rho(|\rho| - 1)(\mathcal{G}) = 0$. 

$$
\text{Fig. 1. An illustration of reachability game}
$$
As a conclusion, we have $P_{\sigma^*,\tau^*}(A) = 0$ for any $s \in S$, where $A = \{w \in \Omega : (\forall n \in \mathbb{N})[w(n) \in T \& \text{val}_w(n)(\mathcal{R}^G_T) > 0]\}$. Thus, the sum

$$\sum \left\{ V_{n,T}^G(\rho(|\rho| - 1))P_{\sigma^*,\tau^*}^s(\rho) : \rho \in \Omega_s^{\text{fin}} \& |\rho| = k \right\}$$

tends to $P_{\sigma^*,\tau^*}^G(\mathcal{R}^G_T)$ as $k \to \infty$. It is easy to see by induction on $k \in \mathbb{N}$ that

$$\sum \left\{ V_{n,T}^G(\rho(|\rho| - 1))P_{\sigma^*,\tau^*}^s(\rho) : \rho \in \Omega_s^{\text{fin}} \& |\rho| = k \right\} \geq V_{n-1,T}^G(s)$$

holds for any $k \in \mathbb{N}$. Hence we have $P_{\sigma^*,\tau^*}^G(\mathcal{R}^G_T) \geq V_{n-1,T}^G(s) \geq \text{val}_n(\mathcal{R}^G_T) - \varepsilon$.

We provide a following example in order to understand easily the existence of $\varepsilon$-optimal strategy $\sigma_\varepsilon$ for Player I. Let observe again the reachability game (Figure 1) as shown before.

**Example 1** Let $S = \{s_0, s_1, s_2\}$ be a finite set of states, and let $A_I = \{x_1, x_2\}$ and $A_{II} = \{y_1, y_2\}$ be the set of actions for Player I and Player II, respectively. Now consider the reachability game $G(\mathcal{R}_{\{s_1\}})$ where a transition function $\delta$ at initial state $s_0$ is given by $\delta(s_0, x_1, y_1) = s_0$, $\delta(s_0, x_2, y_2) = s_2$, $\delta(s_0, x_1, y_2) = s_1$, and $\delta(s_1, x, y) = s_i$ for any $i \in \{1, 2\}$ and $(x, y) \in A_I \times A_{II}$.

For any $\varepsilon > 0$, let consider a memoryless and randomized strategy $\sigma_\varepsilon$ for Player I, such that

$$\sigma(s_0)(x_1) = 1 - \varepsilon; \quad \sigma(s_0)(x_2) = \varepsilon.$$ 

Intuitively, at state $s_0$, Player I choose action $x_1$ with probability $1 - \varepsilon$ and choose action $x_2$ with probability $\varepsilon$. Let fix any reals $\beta > 0$. Thus, for each round of the play, Player II choose an action $y_1$, the game moves to state $s_1$ or stays in $s_0$ with probability $\beta$, and Player II chooses action $y_2$ succeeds to $s_1$ or $s_2$ with probability $1 - \beta$. In particular,

$$\tau^*(s_0)(y_1) = \beta \text{ such that } \delta(s_0, x_2, y_1) = s_1,$$

$$\tau^*(s)(y_1) = 1 - \beta \text{ such that } \delta(s, x_1, y_1) = s.$$ 

Similarly, the actions $y_2$ is chosen by Player II defined as follows

$$\tau^*(s_0)(y_2) = 1 - \beta \text{ such that } \delta(s_0, x_1, y_2) = s_1,$$

$$\tau^*(s)(y_2) = \beta \text{ such that } \delta(s, x_2, y_2) = s_2.$$ 

Let $1 - \varepsilon := E$ and $\varepsilon := 1 - E$. We show that the strategy $\sigma_\varepsilon$ is $\varepsilon$-optimal such that the expected value of Player I is satisfied with a probability within $\varepsilon$ difference the value of the game, against every strategy $\tau^*$ as follows. Note that $\mathcal{R}_{\{s_1\}}$ is disjoint union sets $\rho_n$ of plays such that $w \in \rho_n$ if and only if $w(i) = s_0$ for each $i < n$ and $w(n) = s_1$, where $n \in \mathbb{N}$. Since

$$P_{\sigma^*,\tau^*}^G(\rho_n) = (E\beta)^n(E(1 - \beta) + (1 - E)\beta),$$

we obtain the following equation

$$P_{\sigma^*,\tau^*}^G(\mathcal{R}_{\{s_1\}}) = \sum_{n \in \mathbb{N}} (E\beta)^n(E(1 - \beta) + (1 - E)\beta).$$

By a distributive law of multiplication and addition of numbers, we have

$$(E(1 - \beta) + (1 - E)\beta) \sum_{n \in \mathbb{N}} (E\beta)^n,$$
and the following equation

\[ (E \beta^{n+1} + 1) (1 - E \beta^{n}) = 1 - r^{n+1} \]

is obtained by the following general equation

\[ \sum_{n \in \mathbb{N}} r^n = (1 - r)^{-1} \]

where \(-1 < r < 1\). To get this general equation, notice that

\[ (1 - r)(1 + r + r^2 + \ldots + r^n) = 1 - r^{n+1} \]

holds for all \( r \in \mathbb{R} \) and all \( n \in \mathbb{N} \). Taking a limit, we have

\[ (1 - \sum_{n \in \mathbb{N}} r^n) = 1 \]

when \(-1 < r < 1\). Finally we have

\[ P_{s_0 \rightarrow s} (\mathcal{R}_{\{s_1\}}) = (E + \beta - 2E\beta)(1 - E\beta)^{-1}. \]

Furthermore, we obtain the following equivalences.

\[ P_{s_0 \rightarrow \tau} (\mathcal{R}_{\{s_1\}}) \geq E \iff (E + \beta - 2E\beta)(1 - E\beta)^{-1} \geq E \]
\[ \iff E + \beta - 2E\beta \geq E - E^2\beta \]
\[ \iff (E^2 - 2E + 1)\beta \geq 0 \]
\[ \iff E^2 - 2E + 1 \geq 0 \]
\[ \iff (E - 1)^2 \geq 0 \]
\[ \iff \text{True} \]

Hence, for all reals \( \varepsilon > 0 \), there exist strategy \( \sigma_\varepsilon \) such that for every strategy \( \tau^* \), satisfies

\[ P_{s_0 \rightarrow \tau^*} (\mathcal{R}_{\{s_1\}}) \geq 1 - \varepsilon. \]

4. Concluding Remarks

We proved the game has a value by defining a limit value and show such a value is equal to the value of the game. This proof is a simplified version of the proof for turn-based stochastic games [4] and [5]. In particular, we investigate the existence of optimal (\( \varepsilon \)-optimal) strategy for both player in reachability games. We showed that there exist a memoryless mixed optimal strategy for Player II, while Player I must settle for \( \varepsilon \)-optimality (memoryless). We provide an easy example of reachability game in showing such strategy exists for Player I. Actually, we already observed games with more complex objectives, that is, games with Büchi objectives (the determinacy result can be found in [2]). All these games are zero-sum games and note that, there are also many contributions on non-zero-sum simultaneous (turn-based as well) games, where the objectives of the players (at least two) are not necessarily conflicting. The problem of interests are the existence and effective computability of Nash equilibria for various classes of players’ objectives. For example, in [1] we formulated a network security problem and studied a mixed Nash equilibria for stochastic strategies of games on graphs.

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