Critical region for an Ising model coupled to causal dynamical triangulations

J. Cerda-Hernández

a Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão, 1010, São Paulo, CEP 05508-090, Brazil

Abstract

A lower and an upper bound are established upon a critical curve for the (annealed) Ising model coupled to two-dimensional causal dynamical triangulations. Using the Fortuin-Kasteleyn (FK) representation of quantum Ising models via path integrals, we determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve can be located. Moreover, this approach serves to outline a region where the infinite-volume Gibbs measure exist and is unique and a region where the finite-volume Gibbs measure has no weak limit. We also provide lower and upper bounds for the infinite-volume free energy.

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1 Introduction. A review of related results

Models of planar random geometry appear in physics in the context of two-dimensional quantum gravity and provide an interplay between mathematical physics and probability theory.

A causal dynamical triangulation (CDT), introduced by Ambjørn and Loll (see [6]), together with its predecessor a dynamical triangulation (DT), constitute attempts to provide a meaning to formal expressions appearing in the path integral quantisation of gravity (see [4], [5] for an overview). The idea is to regularise the path integral by approximating the geometries emerging in the integration by DTs. As a result, the path integral over geometries is replaced with a summation over strip triangulations each weighted by a discrete form of the original action. In this paper we work exclusively in the framework of CDTs coupled with an Ising spin system.

Putting a spin system on triangulations is interpreted as a coupling gravity with matter; it is a subject of persistent interest in physics since the successful application of matrix integral methods to the Ising model on random lattice (see [22] and [13]). In this paper we will look in detail at an Ising model coupled to Lorentzian CDTs.

Lorentzian CDTs are given in the form of an ensemble of triangulated spacetimes (see [2], [12], [3] and [20] where an initial progress has been made). It is well known that the CDTs ensemble is more regular than that of the DTs (see [6], [15]). However, the CDT ensemble still carries enough randomness to allow for a back-reaction of the spin system with the quantum geometry. Monte Carlo’s numerical simulations (see [2], [12] and [3]) give a strong evidence that critical exponents of the Ising model coupled to Lorentzian CDTs are identical to the Onsager values. These simulations used a representation of the partition function of the Ising model at a high temperature (a high-T expansion). However, the causality constraints still make it difficult to find an analytical solution of the Ising model coupled to a CDT.

For the Ising model coupled to a CDT some progress has been recently made on existence of Gibbs measures and phase transitions (see [20] for details). Using transfer matrix methods and the Krein-Rutman theory of positivity-preserving operators, a region in the quadrant of parameters $\beta, \mu > 0$ as determined where the infinite-volume free energy has a limit, providing results on convergence and asymptotic properties of the partition function and the Gibbs measure.

Since FK models were introduced by Fortuin and Kasteleyn (see [16]), these models have become an important tool in the study of phase transi-
tion for the Ising and Potts model. The goal of this paper is to use the FK representation of a quantum Ising model coupled to the CDT via a path integral (see [1], [21] for an overview). We join results of [20] to establish a region where the N-strip Gibbs measure has no limit as $N \to \infty$. The aforementioned FK representation uses a family of Poisson point processes and the Lie-Trotter product formula to interpret exponential sums of operators as random operator products. This representation was originally derived in [1].

The results presented in this paper, together with results from [20], enable us to determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve can be located. Moreover, the FK representation serves to find lower and upper bounds for the infinite-volume free energy.

The paper is organized as follows. In Section 2.1-2.3 we review the main features of Lorentzian CDTs and we define the Ising model coupled to the CDT. In Section 2.4 we present the main results (Theorems 1 and 2). In Section 3 we give the FK representation of quantum Ising model coupled to the CDT via a path integral. Section 4 contains the proof of Theorems 1 and 2.

2 The notation and results

2.1 Two-dimensional Lorentzian CDTs

We work with rooted causal dynamic triangulations of the cylinder $C_N = S \times [0, N]$, $N = 1, 2, \ldots$, which have $N$ bonds (strips) $S \times [j, j+1]$. Here $S$ stands for a unit circle. The definition of a causal triangulation starts with a connected graph $G$ embedded in $C_N$ with the property that all faces of $G$ are triangles (using the convention that an edge incident to the same face on two sides counts twice; see [30] for more details). A triangulation $\mathfrak{t}$ of $C_N$ is a pair formed by a graph $G$ with the above property together with the set $F$ of all its (triangular) faces: $\mathfrak{t} = (G, F)$.

**Definition 2.1.** A triangulation $\mathfrak{t}$ of $C_N$ is called a causal dynamic triangulation (CDT) if the following conditions hold:

- each triangular face of $\mathfrak{t}$ belongs to some strip $S \times [j, j+1]$, $j = 1, \ldots, N-1$, and has all vertices and exactly one edge on the boundary $(S \times \{j\}) \cup (S \times \{j+1\})$ of the strip $S \times [j, j+1]$;

- the number of edges on $S \times \{j\}$ should be finite for any $j = 0, 1, \ldots, N-1$: let $n^j = n^j(\mathfrak{t})$ be the number of edges on $S \times \{j\}$, then $1 \leq n^j < \infty$ for all $j = 0, 1, \ldots, N-1$. 


Definition 2.2. A CDT \( t \) of \( C_N \) is called rooted if it has a root. The root in the triangulation \( t \) is represented by a triangular face \( t \) of \( t \), called the root triangle, with an anticlock-wise ordering on its vertices \( (x, y, z) \) where \( x \) and \( y \) belong to \( S \times \{0\} \). The vertex \( x \) is identified as the root vertex and the (directed) edge from \( x \) to \( y \) as the root edge.

Definition 2.3. Two rooted CDTs of \( C_N \), say \( t = (G, F) \) and \( t' = (G', F') \), are equivalent if there exists a self-homeomorphism of \( C_N \) which (i) transforms each slice \( S \times \{j\}, j = 0, \ldots, N - 1 \) to itself and preserves its direction, (ii) induces an isomorphism of the graphs \( G \) and \( G' \) and a bijection between \( F \) and \( F' \), and (iii) takes the root of \( t \) to the root of \( t' \).

It is convenient to introduce the notion of “up” and “down” triangles (see Figure 2). We call a triangle \( t \in t(i) \) an up-triangle if it has an edge on the slice \( S \times \{i\} \) and a down-triangle if it has an edge on the slice \( S \times \{i + 1\} \). By Definition 2.1, every triangle is either of type up or down. Let \( n_{up}(t(i)) \) and \( n_{do}(t(i)) \) stand for the number of up- and down-triangles in the triangulation \( t(i) \).

A rooted CDT \( t \) of \( C_N \) is identified with a compatible sequence

\[
\mathbf{t} = (t(0), t(1), \ldots, t(N - 1)),
\]

where \( t(i) \) is a triangulation of the strip \( S \times [i, i + 1] \). The compatibility means that

\[
n_{up}(t(i + 1)) = n_{do}(t(i)), \quad i = 0, \ldots, N - 2. \tag{2.1}
\]

The triangles forming \( t(i) \) are denoted by \( t(i, j) \), \( 1 \leq j \leq n(t(i)) \), where \( n(t(i)) \) stands for the number of triangles in triangulation \( t(i) \). The enumeration of these triangles starts with what we call the root triangle in \( t(i) \); it is determined recursively as follows. First, we have the root triangle \( t(0, 1) \) in \( t(0) \) (see Definition 2.2). Take the vertex of the triangle \( t(0, 1) \) which lies on the slice \( S \times \{1\} \) and denote it by \( x' \). This vertex is declared the root vertex for \( t(1) \). Next, the root edge for \( t(1) \) is the one incident to \( x' \) and lying on \( S \times \{1\} \), so that if \( y' \) is its other end and \( z' \) is the third vertex of the corresponding triangle then \( x', y', z' \) lists the three vertices anticlock-wise. Accordingly, the triangle with the vertices \( x', y', z' \) is called the root triangle for \( t(1) \). This construction can be iterated, determining the root vertices, root edges and root triangles for \( t(i) \), \( 0 \leq i \leq N - 1 \) (see Figure 1(b)).

Note that for any edge lying on the slice \( S \times \{i\} \) belongs to exactly two triangles: one up-triangle from \( t(i) \) and one down-triangle from \( t(i - 1) \). This provides the following relation: the number of triangles in the triangulation
\( t \), denoted by \( n(t) \), is twice the total number of edges on the slices. More precisely, remind that \( n^i \) is the number of edges on slice \( S \times \{i\} \). Then, for any \( i = 0, 1, \ldots, N-1 \),

\[
    n(t(i)) = n_{\text{up}}(t(i)) + n_{\text{do}}(t(i)) = n^i + n^{i+1},
\]

implying that

\[
    n(t) = \sum_{i=0}^{N-1} n(t(i)) = 2 \sum_{i=0}^{N-1} n^i. \tag{2.3}
\]

There is another useful property regarding the counting of triangulations. Let us fix the number of edges \( n_i \) and \( n_i+1 \) in the slices \( S \times \{i\} \) and \( S \times \{i+1\} \). The number of possible rooted CTs of the slice \( S \times [i, i+1] \) with \( n_i \) up- and \( n_i+1 \) down-triangles is equal to

\[
    \begin{pmatrix}
    n^i + n^{i+1} - 1 \\
    n^i - 1
    \end{pmatrix} = \begin{pmatrix}
    n(t(i)) - 1 \\
    n_{\text{up}}(t(i)) - 1
    \end{pmatrix} = \begin{pmatrix}
    n(t(i)) - 1 \\
    n_{\text{do}}(t(i)) - 1
    \end{pmatrix}. \tag{2.4}
\]

### 2.2 Transfer matrix formalism for pure CDTs

We begin by discussing the case of pure CDTs, as was first introduced in [6] (see also [26] for a mathematically rigorous account).

For technical reasons it will be convenient to consider triangulations with periodical spatial boundary conditions, i.e. the strip \( t(N-1) \) is compatible with \( t(0) \). Let \( \mathbb{L}_N \) denote the set of causal triangulations on the cylinder \( C_N \) with this boundary condition. Thus the partition function for rooted CTs in the cylinder \( C_N \) with periodical spatial boundary conditions and for the value of the cosmological constant \( \mu \) is given by

\[
    Z_N(\mu) = \sum_{t(0), \ldots, t(N-1)} e^{-\mu n(t)} = \sum_{t(0), \ldots, t(N-1)} \exp\left\{ -\mu \sum_{i=0}^{N-1} n(t(i)) \right\}. \tag{2.5}
\]

Using the properties (2.3) and (2.4) we can represent the partition function (2.5) in the following way

\[
    Z_N(\mu) = \sum_{n^0 \geq 1, \ldots, n^{N-1} \geq 1} \exp\left\{ -2\mu \sum_{i=0}^{N-1} n^i \right\} \prod_{i=0}^{N-1} \left( \frac{n^i + n^{i+1} - 1}{n^i - 1} \right). \tag{2.6}
\]

Moreover, the periodical spatial boundary condition on the CDTs permits to write the partition function \( Z_N(\mu) \) in a trace-related form

\[
    Z_N(\mu) = \text{tr} \left( U^N \right). \tag{2.7}
\]
This gives rise to a transfer matrix $U = \{u(n,n')\}_{n,n'=1,2,...}$ describing the transition from one spatial strip to the next one. It is an infinite matrix with positive entries

$$u(n,n') = \binom{n+n'-1}{n-1} e^{-\mu(n+n')}.$$  

(2.8)

The entry $u(n,n')$ yields the number of possible triangulations of a single strip (say, $S \times [0,1]$) with $n$ lower boundary edges (on $S \times \{0\}$) and $n'$ upper boundary edges (on $S \times \{1\}$) (see Figure 1(a)). The asymmetry in $n$ and $n'$ is due to the fact that the lower boundary is marked while the upper one is not.

Using the $N$-strip partition function for pure CDTs with periodical boundary condition, defined by the formula (2.5), we define the $N$-strip Gibbs probability distribution for pure CDTs

$$Q_{N,\mu}(t) = \frac{1}{Z_N(\mu)} e^{-\mu n(t)}.$$  

(2.9)

The transfer-matrix formalism suggests that, as $N \to \infty$, the partition function is controlled by the largest eigenvalue $\Lambda$ of the transfer matrix (2.8):

$$Z_N(\mu) = \text{tr} U^N \sim \Lambda^N.$$  

(2.10)
Here
\[ \Lambda := \Lambda(\mu) = \left[ \frac{1 - \sqrt{1 - 4 \exp(-2\mu)}}{2 \exp(-\mu)} \right]^2 , \] (2.11)
and we suppose that \( \mu > \ln 2 \) (this yields a subcritical region). The following properties hold.

**Theorem 1, [26].** For any \( \mu > \ln 2 \) the following relation holds true:
\[ \lim_{N \to \infty} \frac{1}{N} \ln Z_N(\mu) = \ln \Lambda(\mu) \] (2.12)
with \( \Lambda(\mu) \) given in (2.11). Further, the \( N \)-strip Gibbs measure \( Q_{N,\mu} \) converges weakly to a limiting measure \( Q_\mu \).

**Proposition 5, [26].** For any \( \mu < \ln 2 \), the \( N \)-strip partition function \( Z_N(\mu) \) exists only if
\[ \mu > \ln \left( 2 \cos \frac{\pi}{N + 1} \right) . \] (2.13)

### 2.3 Ising model coupled to the CDT: definitions and transfer matrix formalism

Henceforth, for simplicity in notation and exposure of the following sections, we shall denote a triangle of any triangulation \( t \) doing without put the indices \( i, j \) as was done in previous section.

Let \( \Delta(t) \) and \( \Delta(t(i)) \) denotes the set of triangles of the triangulation \( t \) and the set of triangles of the strip \( t(i) \), respectively. With any triangle from a triangulation \( t \) we associate a spin taking values \( \pm 1 \). An \( N \)-strip configuration of spins is represented by a collection
\[ \sigma = (\sigma(0), \sigma(1), \ldots, \sigma(N - 1)) \]
where \( \sigma(i) \equiv \sigma(t(i)) := \{ \sigma(t) \}_{t \in \Delta(t(i))} \) is a configuration of spins \( \sigma(t) \) over triangles \( t \) forming a triangulation \( t(i) \). We say that a single-strip configuration of spins \( \sigma(i) \) is supported by a triangulation \( t(i) \) of strip \( S \times [i, i + 1] \).

We consider a usual (ferromagnetic) Ising-type energy where two spins \( \sigma(t) \) and \( \sigma(t') \) interact if their supporting triangles \( t, t' \) share a common edge; such triangles are called nearest neighbors, and this property is reflected
in the notation \( \langle t, t' \rangle \). Thus, in our model each spin has three neighbors. Formally, the Hamiltonian of the (annealed) model reads:

\[
h(\sigma) = - \sum_{\langle t, t' \rangle} \sigma(t)\sigma(t'). \tag{2.14}
\]

We use the following decomposition:

\[
h(\sigma) = \sum_{i=0}^{N-1} h(\sigma(i)) + \sum_{i=0}^{N-1} v(\sigma(i), \sigma(i+1)), \tag{2.15}
\]

where we assume that \( \sigma(0) \equiv \sigma(N) \) (the periodic spatial boundary condition). Here \( h(\sigma(i)) \) represents the energy of the configuration \( \sigma(i) \):

\[
h(\sigma(i)) = - \sum_{\langle t, t' \rangle, t, t' \in t(i)} \sigma(t)\sigma(t'). \tag{2.16}
\]

Further, \( v(\sigma(i), \sigma(i+1)) \) is the energy of interaction between neighboring triangles belonging to the adjacent strips \( S \times [i, i+1] \) and \( S \times [i+1, i+2] \):

\[
v(\sigma(i), \sigma(i+1)) = - \sum_{t \in t(i), t' \in t(i+1)} \sigma(t)\sigma(t'). \tag{2.17}
\]

The partition function for the \( N \)-strip Ising model coupled to CDT, at the inverse temperature \( \beta > 0 \) and the cosmological constant \( \mu \), is given by

\[
\Xi_N(\beta, \mu) = \sum_{(t(0), \ldots, t(N-1))} \exp\left\{ -\mu \sum_{i=0}^{N-1} n(t(i)) \right\} \tag{2.18}
\]

\[
\times \sum_{(\sigma(0), \ldots, \sigma(N-1))} \prod_{i=0}^{N-1} \exp \left\{ -\beta h(\sigma(i)) - \beta v(\sigma(i), \sigma(i+1)) \right\}. \]

Here \( n(t(i)) \) stands for the number of triangles in the triangulation \( t(i) \). The formula

\[
\Xi_N(\beta, \mu) = \text{tr} K^N \tag{2.19}
\]

gives rise to a transfer matrix \( K \) with entries \( K((t, \sigma), (t', \sigma')) \). The entries are labelled by pairs \( (t, \sigma), (t', \sigma') \) representing triangulations of a single
strip (say, $S \times [0,1]$) and their supported spin configurations positioned next to each other. Formally,

$$K((t, \sigma), (t', \sigma')) = 1_{t \sim t'} \exp\left\{ -\frac{\mu}{2} (n(t) + n(t')) \right\}$$

$$\times \exp\left\{ -\frac{\beta}{2} (h(\sigma) + h(\sigma')) - \beta v(\sigma, \sigma') \right\}.$$  (2.20)

As earlier, $n(t)$ and $n(t')$ are the numbers of triangles in single-strip triangulations $t$ and $t'$. The indicator $1_{t \sim t'}$ means that $t, t'$ have to be compatible, see (2.1). Equivalently, it means that the pair $(t, t')$ forms a CDT for the strip $S \times [0,2]$. In this paper the operator determined by matrix $K$ is denoted by the same symbol, and this action is considered in the Hilbert space $\ell^2_{T-C}$ (the subscript T-C refers to triangulations and spin-configurations) of square-summable functions

$$(t, \sigma) \mapsto \phi(t, \sigma), \text{ with } \sum_{t, \sigma} |\phi(t, \sigma)|^2 < \infty,$$

where the argument $(t, \sigma)$ run over single-strip triangulations and supported configurations of spins, with the scalar product $\langle \phi', \phi'' \rangle_{T-C} = \sum_{t, \sigma} \phi'(t, \sigma) \phi''(t, \sigma)$ and the induced norm $\|\phi\|_{T-C}$.

We introduce the $N$-strip Gibbs probability distribution associated with Eqn (2.18):

$$P_N^{\beta, \mu}(t, \sigma) = P_N^{\beta, \mu}((t(0), \sigma(0)), \ldots, (t(N-1), \sigma(N-1)))$$

$$= \frac{1}{\Xi_N(\beta, \mu)} e^{-\mu n(t) - \beta h(\sigma)}$$

$$= \frac{1}{\Xi_N(\beta, \mu)} \prod_{i=0}^{N-1} \exp\left\{ -\mu n(t(i)) - \beta h(\sigma(i)) - \beta v(\sigma(i), \sigma(i+1)) \right\}.  \hspace{1cm} (2.21)$$

We denote by $G_{\beta, \mu}$ the set of Gibbs measures given by the closed convex hull of the set of weak limits:

$$P^{\beta, \mu} = \lim_{N \to \infty} P_N^{\beta, \mu},$$

and define the domain of parameters where the weak limit Gibbs distribution exists

$$\Gamma = \{ (\beta, \mu) \in \mathbb{R}_+^2 : G_{\beta, \mu} \neq \emptyset \}.  \hspace{1cm} (2.22)$$
The critical curve $\gamma_{cr}$ for the Ising model coupled to CDT is defined as follows:

$$\gamma_{cr} = \partial \Gamma \cap \mathbb{R}_+^2.$$  \hfill (2.24)

Set:

$$\lambda(\beta, \mu) = c^2 (m^2 + 1) (\cosh 2\beta) \left( 1 + \sqrt{1 - \frac{1}{(\cosh 2\beta)^2} \frac{(m^2 - 1)^2}{(m^2 + 1)^2}} \right).$$  \hfill (2.25)

where $c$ and $m$ are determined by

$$c = \frac{\exp(\beta - \mu)}{e^{2\beta}(1 - \exp(\beta - \mu))^2 - e^{-2\mu}}$$  \hfill (2.26)

$$m = e^{2\beta} + (1 - e^{4\beta}) \exp(-(\beta + \mu)).$$  \hfill (2.27)

and define the strictly increasing function

$$\psi(\beta) = \inf\{\mu \in \mathbb{R}^+: \lambda(\beta, \mu) < 1\}, \quad \text{for} \quad \beta > 0.$$  \hfill (2.28)

Using the transfer matrix methods and Krein-Rutman theory of positivity preserving operators, in paper [20] a region where the infinite-volume free energy converges was determined, yielding results on the convergence to and asymptotic properties of the (unique) infinite-volume Gibbs measure. Consequently, it determines an upper bound for the critical curve for the Ising model coupled to CDT:

**Theorem 4.2, [20].** For any $\beta, \mu > 0$ such that $\mu > \psi(\beta)$ (or equivalently $\lambda(\beta, \mu) < 1$) the following limit holds:

$$\lim_{N \to \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) = \ln \Lambda(\beta, \mu),$$  \hfill (2.29)

where $\Lambda(\beta, \mu)$ is the maximal eigenvalue of $K$ and $K^T$ in $\ell^2_{T-C}$. Moreover, as $N \to \infty$, the $N$-strip Gibbs measure $\mathbb{P}_N^{\beta,\mu}$ converges weakly to a limiting probability distribution $\mathbb{P}^{\beta,\mu}$. Furthermore, the following upper bound for the critical curve is satisfied: If $(\beta, \mu) \in \gamma_{cr}$ the $\mu < \psi(\beta)$, for all $\beta > 0$ (see dotted curve I in Figure 2).

### 2.4 The main results

This subsection contains the statement of the main theorems of the present article.
Define the set
\[ \Sigma = \left\{ (\beta, \mu) \in \mathbb{R}^2 : \mu < -\frac{3}{2} \beta + 2 \ln 2 \right\} \cup \left\{ (\beta, \mu) \in \mathbb{R}^2 : \mu < -\frac{3}{2} \beta + \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) + \ln 2 \right\}. \]

Let \( t_1, \ldots, t_k \) be triangulations of a single strip \( S \times [0, 1] \) and \( \sigma_1, \ldots, \sigma_k \) be their corresponding spin configurations. Given \( 0 \leq i_1 < \cdots < i_k \leq N - 1 \) we define the finite-dimensional cylinder \( C_{i_1, \ldots, i_k} = C^{(t_1, \sigma_1), \ldots, (t_k, \sigma_k)}_{i_1, \ldots, i_k} \) as follows
\[
C_{i_1, \ldots, i_k} = \left\{ (t, \sigma) : (t(i_1), \sigma(i_1)) = (t_1, \sigma_1), \ldots, (t(i_k), \sigma(i_k)) = (t_k, \sigma_k) \right\}
\]

\[ (2.30) \]

**Theorem 1.** If \((\beta, \mu) \in \Sigma\) then there exists \( N_0 \in \mathbb{N} \) such that the partition function \( \Xi_N(\beta, \mu) = +\infty \) whenever \( N > N_0 \). Moreover, the Gibbs distribution \( \mathbb{P}_N^{\beta, \mu} \) with periodic boundary conditions cannot be defined by using the standard formula with \( \Xi_N(\beta, \mu) \) as a normalising denominator, consequently, there is no limiting probability measure \( \mathbb{P}^{\beta, \mu} \) as \( N \to \infty \). Furthermore, for any finite-dimensional cylinder \( C_{i_1, \ldots, i_k} \) we obtain \( \mathbb{P}_N^{\beta, \mu}(C_{i_1, \ldots, i_k}) = 0 \) whenever \( N > N_0 \geq \max\{i_1, \ldots, i_k\} \).

Let \( \beta_1^*, \beta_2^* \) be positive solution of equations
\[
-\frac{3}{2} \beta + 2 \ln 2 = -\frac{3}{2} \beta + \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) + \ln 2 \tag{2.31}
\]
and
\[
\frac{3}{2} \beta + 2 \ln 2 = \psi(\beta), \tag{2.32}
\]
respectively. Together with results from [20], Theorem 1 provides two-side bounds for the critical curve.

**Theorem 2.** The critical curve \( \gamma_{cr} \) satisfies the following inequalities.

1. If \((\beta, \mu) \in \gamma_{cr} \) and \( 0 < \beta < \beta_1^* \), then
\[
-\frac{3}{2} \beta + 2 \ln 2 \leq \mu < \psi(\beta).
\]
The above bound implies that: For any sequence \( \{ (\beta_k, \mu_k) \} \subset \gamma_{cr} \) such that \( \beta_k \to 0 \), then \( \lim_{k \to \infty} \mu_k = 2 \ln 2 \).
2. If \((\beta, \mu) \in \gamma_{cr}\) and \(\beta_1^* \leq \beta < \beta_2^*\), then
\[
\frac{3}{2} \ln(e^{2\beta} - 1) - \frac{3}{2} \beta + \ln 2 \leq \mu < \psi(\beta).
\]

3. If \((\beta, \mu) \in \gamma_{cr}\) and \(\beta_2^* \leq \beta < \infty\), then
\[
\frac{3}{2} \ln(e^{2\beta} - 1) - \frac{3}{2} \beta + \ln 2 \leq \mu < \frac{3}{2} \beta + 2 \ln 2.
\]

In the present article, as a by-product of the proof of Theorems 1 and 2, using the FK representation we also find a lower and upper bound for the infinite-volume free energy.

**Corollary 1.** If \(\mu > \frac{3}{2} \beta + 2 \ln 2\), then the free energy for the infinite-volume Ising model coupled to CDT is finite and satisfies the following inequalities.

1. If \(0 < \beta < \frac{1}{3} \ln 2\), then
\[
\ln \Lambda \left(\mu + \frac{3}{2} \beta - \ln 2\right) \leq \lim_{N \to \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) \leq \ln \Lambda \left(\mu - \frac{3}{2} \beta - \ln 2\right).
\]

2. If \(\frac{1}{3} \ln 2 \leq \beta < \infty\), then
\[
\ln \Lambda \left(\mu - \frac{3}{2} \beta\right) \leq \lim_{N \to \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) \leq \ln \Lambda \left(\mu - \frac{3}{2} \beta - \ln 2\right).
\]

Here \(\Lambda(s)\) is given by \([2.17]\).

For each \(N \in \mathbb{N}\), we define the follow set in \(\mathbb{R}^2_+\)
\[
\Gamma_N = \{(\beta, \mu) \in \mathbb{R}^2_+ : \mathbf{K}^N \text{ is of trace class in } \ell^2_{T-C}\}; \quad (2.33)
\]
\[
\Gamma^- = \bigcap_{N \in \mathbb{N}} \Gamma_N \quad \text{and} \quad \Gamma^+ = \bigcup_{N \in \mathbb{N}} \Gamma_N. \quad (2.34)
\]

Obviously, \(\Gamma^- \subset \Gamma_N \subset \Gamma^+, \) for any \(N \geq 1,\) and \(\mathbb{P}^\beta_\mu\) there exist on \(\Gamma_N.\) In order to each \(N \geq 1,\) we define the \(N\)-strip functions \(f_N\) associated with the partition function for the \(N\)-strip Ising model coupled to CDT as
\[
f_N(\beta) = \inf\{\mu \in \mathbb{R}^2_+ : (\beta, \mu) \in \Gamma_N\} \quad \text{for} \quad \beta > 0. \quad (2.35)
\]

According to Theorem 4.2 in \([20]\), Theorem 2 and Proposition 6, given in the Section 3 implies a similar version of Theorem 4.2, \([20]\), as following.
Theorem 3. For \((\beta, \mu) \in \Gamma^+ = \{ (\beta, \mu) \in \mathbb{R}_+^2 : \mu > f_{T-C}(\beta) \}\), the following limit holds:

\[
\lim_{N \to \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) = \ln \Lambda(\beta, \mu),
\]

where \(\Lambda(\beta, \mu)\) is the maximal eigenvalue of \(K\) and \(K^T\) in \(\ell^2_{T-C}\) and \(f_{T-C}\) is pointwise limit of the family of functions \(\{f_N\}\). Consequently, as \(N \to \infty\) the N-strip Gibbs measure \(P^\beta_N\) converges weakly to a limiting probability distribution \(P^\beta\).

3 The model dressed as quantum

3.1 The quantum Ising model

In this section we write the classical partition function, over a given Lorentzian triangulation, by using ingredients of the quantum Ising model.
Let $\mathbf{t} = (t(0), t(1), \ldots, t(N - 1))$ be a Lorentzian CDT of $C_N$ with periodical spatial boundary condition (see Figure 1 (b)). We define $\Omega_{\mathbf{t}}$ to be the set of all spin configurations supported by the triangles of $\mathbf{t}$. Let $\mathcal{Z}_{\mathbf{t}}^\beta$ be the partition function of the Ising model on the CDT $\mathbf{t}$, at inverse temperature $\beta > 0$

$$\mathcal{Z}_{\mathbf{t}}^\beta = \sum_{\sigma \in \Omega_{\mathbf{t}}} \exp\{-\beta h(\sigma)\}, \quad (3.1)$$

where $h(\sigma)$ represents the energy of configuration $\sigma \in \Omega_{\mathbf{t}}$, defined by the formula (2.14).

The quantum Ising model on a Lorentzian triangulation $\mathbf{t}$ is defined as follows.

Let

$$\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.2)$$

be the Pauli matrix with their corresponding eigenvectors

$$\phi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.3)$$

To each triangle $t \in \Delta(\mathbf{t})$ we associate a spin taking values $\phi_{+1}$ and $\phi_{-1}$. Thus, the space of all such spin configurations on $\mathbf{t}$ is define as the real vector space $X_{\mathbf{t}} = \otimes_{t \in \Delta(\mathbf{t})} \mathbb{R}^2$, where $\otimes$ stands for the tensor product. Note that $X_{\mathbf{t}}$ is a real vector space of dimension 2 to the $n(\mathbf{t})$ power.

For each configuration $\sigma \in \Omega_{\mathbf{t}}$ we associate the quantum configuration as tensor products $\phi_\sigma := \otimes_{t \in \Delta(\mathbf{t})} \phi_{\sigma(t)}$, where $\sigma(t)$ is the spin supported by the triangle $t \in \Delta(\mathbf{t})$. Note that there is a one-one correspondence between $\Omega_{\mathbf{t}}$ and the collection $\{\phi_\sigma\}_{\sigma \in \Omega_{\mathbf{t}}}$. Moreover, the collection of quantum configurations is a complete orthonormal basis of $X_{\mathbf{t}}$ with respect to the following scalar product

$$\langle \phi_\sigma | \phi_{\sigma'} \rangle := \prod_{t \in \Delta(\mathbf{t})} \langle \phi_{\sigma(t)}, \phi_{\sigma'(t)} \rangle_2,$$

where $\langle \cdot, \cdot \rangle_2$ is the usual scalar product of $\mathbb{R}^2$. With each triangle $t \in \Delta(\mathbf{t})$ we associate a linear self-adjoint operator $\hat{\sigma}^z_t : X_{\mathbf{t}} \rightarrow X_{\mathbf{t}}$ which acts as a copy of Pauli matrix $\hat{\sigma}^z$ on the coordinate of $\phi_\sigma$ associated to the triangle $t$ of $\mathbf{t}$. That is, for each $\sigma \in \Omega_{\mathbf{t}},$

$$\hat{\sigma}^z_t \phi_\sigma = \phi_{\sigma(t_1)} \otimes \cdots \otimes \left( \hat{\sigma}^z \phi_{\sigma(t)} \right) \otimes \cdots = \sigma(t) \phi_\sigma. \quad (3.4)$$
Note that
\[ \hat{\sigma}^z_t \hat{\sigma}^z_{t'} \phi = \sigma(t) \sigma(t') \phi. \] (3.5)

The Hamiltonian \( H_t \) of the quantum Ising model is a linear self-adjoint operator on \( \mathcal{X}_t \):
\[ H_t = -\sum_{\langle t, t' \rangle} \hat{\sigma}^z_t \hat{\sigma}^z_{t'}, \] (3.6)
where two operators \( \hat{\sigma}^z_t \) and \( \hat{\sigma}^z_{t'} \) interact if their supporting triangles \( t, t' \in \Delta(t) \) are neighbors.

Note that \( H_t \phi = \mathfrak{h}(\sigma) \phi \). This allows write the classical partition function \( Z^\beta_N \) for Ising model, at inverse temperature \( \beta > 0 \) associated with triangulation \( \mathfrak{t} \), as follows
\[ Z^\beta_N = \text{tr} \left( e^{-\beta H_t} \right) = \sum_{\phi \in \Omega_t} \langle \phi | e^{-\beta H_t} | \phi \rangle. \] (3.7)

Finally, using the quantum representation (3.7), the partition function for the \( N \)-strip Ising model coupled to CDT, at the inverse temperature \( \beta > 0 \) and for the cosmological constant \( \mu \), can be written as follows
\[ \Xi_N(\beta, \mu) = \sum_{\mathfrak{t}} e^{-\mu n(\mathfrak{t})} \text{tr} \left( e^{-\beta H_t} \right). \] (3.8)

### 3.2 FK representation for Ising model coupled to CDT

In order to calculate \( \text{tr} \left( e^{-\beta H_t} \right) \) in (3.8) we will use the FK representation for the Ising model via path integrals, see [1, 21]. By this representation the trace \( \text{tr} \left( e^{-\beta H_t} \right) \) may be expressed in terms of a type of path integral with respect to the continuous random-cluster model on \( \Delta(t) \times [0, \beta] \) for any Lorentzian triangulation \( \mathfrak{t} \), see Proposition [1].

For any pair \( t, t' \in \Delta(\mathfrak{t}) \) of neighbor triangles we associate a Poisson process \( \xi_{\langle t, t' \rangle}(s) \) on the time interval \( [0, \beta] \) with intensity 2. We refer to the process \( \xi_{\langle t, t' \rangle} \) as process of arrivals of operator \( K_{\langle t, t' \rangle} \) on the interval \( [0, \beta] \), where
\[ K_{\langle t, t' \rangle} = \frac{I + \hat{\sigma}^z_t \hat{\sigma}^z_{t'}}{2}. \] (3.9)

Let \( \xi \) be the collection of independent Poisson processes \( \xi_{\langle t, t' \rangle} : \xi(s) = \{ \xi_{\langle t, t' \rangle}(s) \}_{\langle t, t' \rangle \in \mathcal{E}_t} \) where \( \mathcal{E}_t \) is the set of all pairs of neighbor triangles: \( \mathcal{E}_t = \{ \langle t, t' \rangle : t, t' \in \Delta(t) \} \).
Let $P_{\beta, t}$ denote the probability measure associated with the family of Poisson process $\xi$. We shall abuse notation by using $\xi$ to denote a realization of process of arrivals $\xi(s), \ s \in [0, \beta]$. By independence there are no simultaneous arrivals $P_{\beta, t}$-a.s. Thus, a realization $\xi$ of process of arrivals can be represented by a collection of arrival times $\{s_i\}_{i=1,\ldots,N_\xi}$ contained in $[0, \beta]$ and its corresponding arrival types $L(s_i) \in E_t$, $\xi \equiv \{s_i, L(s_i)\}_{i=1,\ldots,N_\xi}$, where $N_\xi$ is the total number of arrivals during the time $[0, \beta]$.

With a fixed realization $\xi$ we associated a family of all possible piecewise constant right-continuous functions $\psi_\xi = \{\phi : [0, \beta] \to \{\phi_{\sigma}\}\}$, having jumps only at arrival times of $\xi$. Since $X_t$ is finite dimensional and there are $P_{\beta, t}$-a.s. finite number of arrivals, we have $|\psi_\xi| < \infty, P_{\beta, t}$-a.s., where $|\psi_\xi|$ is the total number of functions in the set $\psi_\xi$.

**Proposition 1.** The matrix elements of the linear operator $e^{-\beta H_t}$ with respect to the basis $\{\phi_{\sigma}\}$ are given by

$$\frac{\langle \phi_{\sigma} | e^{-\beta H_t} | \phi_{\sigma'} \rangle}{\exp \left( \frac{\beta}{2} n(t) \right)} = \int P_{\beta, t}(d\xi) \sum_{\varphi \in \psi_\xi} \prod_{s \in \xi} \langle \varphi(s^-) | K_{L(s)} | \varphi(s) \rangle, \quad (3.10)$$

for all $t \in LT_N$.

Formula (3.10) is proved in [1] and [21] for any finite general graph.

With any realization $\xi$ we associate a graph $G_\xi = (\Delta_\xi, E_\xi)$, where the set of vertices is $\Delta_\xi = \Delta(t)$ and the set of edges $E_\xi \subseteq E_t$ is defined by following rule: an edge $e \in E_\xi$ belong to $E_\xi$ if and only if there exist the arrival type $e$ into the realization $\xi$.

We say that two triangulations $t$ and $t'$ are connected, denoted by $t \leftrightarrow t'$, if and only if there exist a path in $G_\xi$ connecting $t$ and $t'$. We suppose that for any $t \in \Delta(t)$ $t \leftrightarrow t$. A subset $C \subseteq \Delta(t)$ we call a cluster (maximal connected component) if for any $t, t' \in C$ then $t \leftrightarrow t'$, and $t \not\leftrightarrow t'$ for any $t \in C$ and $t' \not\in C$ (see Figure 3). Thus, any realization $\xi$ of the Poisson process splits $\Delta(t)$ into the disjoint union of maximal connected components, i.e. for any realization $\xi$ there exists $c = c(\xi) \in \{1, \ldots, n(t)\}$ and sets $C_1 = C_1(\xi) \ldots, C_{c(\xi)} = C_{c(\xi)}(\xi) \subseteq \Delta(t)$ such that

$$\Delta(t) = \bigcup_{k=1}^{c(\xi)} C_k,$$

where $c(\xi)$ is the number of clusters defined by the relation $\leftrightarrow$. 

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Let $\sigma, \sigma' \in \Omega(t)$ be two configurations and let $\phi_\sigma, \phi_{\sigma'}$ be the corresponding quantum configuration. Then, for any $\langle t, t' \rangle \in E_t$

\[
\langle \phi_\sigma | K_{\langle t, t' \rangle} | \phi_{\sigma'} \rangle = \delta_{\{\sigma = \sigma'\}} \delta_{\{\sigma(t) = \sigma(t')\}}.
\]

Relation (3.11) implies that, for any realization $\xi$ only constant functions of $\psi_\xi$ contribute to the sum inside the integral (3.10). Additionally, arrival of an operator $K_{\langle t, t' \rangle}$ at some arrival time $s \in [0, \beta]$ imposes the additional condition $\sigma(t) = \sigma(t')$ for contribute to the sum in (3.10). Define

\[
\Omega(t, \xi) = \{ \sigma \in \Omega(t) : \sigma \text{ have same sign in each cluster } C_k(\xi) \}.
\]

Note that $|\Omega(t, \xi)| = 2^{c(\xi)}$, and

\[
\sum_{\varphi \in \psi_{\xi}} \prod_{s \in \xi} \langle \varphi(s^-) | K_{L(s)} | \varphi(s) \rangle = \begin{cases} 
1 & \text{if } \sigma \in \Omega(t, \xi) \\
0 & \text{if } \sigma \notin \Omega(t, \xi)
\end{cases}
\]

(3.12)

As an elementary consequence of (3.12) the following representation for $Z_{\beta,t}^\beta$ holds.

**Proposition 2.** Let $t \in \mathbb{LT}_N$ and $\beta > 0$. We have that

\[
Z_{\beta,t}^\beta = e^{\frac{3}{4} \beta n(t)} \int 2^{c(\xi)} P_{\beta, t}(d\xi).
\]

(3.13)
Using the $N$-strip Gibbs probability distribution $Q_{N,\mu}$, introduced in Eqn (2.9), for pure CDTs with periodical boundary condition and substituting (3.13) on the right-hand side of (3.8) we obtain the FK representation of partition function for the $N$-strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ and for the cosmological constant $\mu$

$$\Xi_N(\beta, \mu) = Z_N(r) \sum_{t \in \mathbb{LT}_N} \left\{ \int 2^{c(\xi)} \mathbb{P}_{\beta, t}(d\xi) \right\} Q_{N,r}(t),$$  

(3.14)

where $r = \mu - \frac{3}{2} \beta$ and $Z_N(\cdot)$ is defined by (2.5).

4 The proof of Theorem 1 and 2

The proof is based on finding of upper and lower bounds for the functions $f_N$ using the FK representation (3.14) and the asymptotic behaviour of the partition function $Z_N(\cdot)$ for pure CDTs with periodical boundary condition. These bounds with the Proposition 3, Proposition 4 and Proposition 5, established bounds for the critical curve.

4.1 Proof of Theorem 1

In order to prove Theorem 1 we need two preparatory results. Let $t$ be a Lorentzian CDT on cylinder $C_N$. For any $1 \leq k \leq n(t)$, we define the sets

$$\Pi_k = \{ \text{all realization } \xi \text{ of process } \{ \xi_{(t,t')} \} \text{ such that } c(\xi) = k \}.$$  

(4.1)

Thus, we have the following representation of (3.13)

$$Z_N^{\beta,t} = e^{\frac{3}{2} \beta n(t)} \sum_{k=1}^{n(t)} 2^k \mathbb{P}_{\beta, t}(\Pi_k).$$  

(4.2)

Let $\xi \in \Pi_k$ and let $\{C_l\}_{l=1}^k$ be the corresponding cluster decomposition of the set $\Delta(t)$. Let $\eta_l = \eta(C_l)$ and $\kappa_l = \kappa(C_l)$ denote the number of vertices (triangles) in cluster $C_l$ and the number of edges in $C_l$, respectively. Note that $\kappa_l$ depends on the geometry of cluster $C_l$.

Probability that two neighbor triangles $t, t'$ are linked is $\mathbb{P}_{\beta, t}(\exists s_i : L(s_i) = \langle t, t' \rangle) = 1 - e^{-2\beta}$. Then, denoting $p := 1 - e^{-2\beta}$ we obtain the following
representation for the probability of the set $\Pi_k$,

$$P_{\beta, t}(\Pi_k) = \sum_{C_1, \ldots, C_k \subseteq \Delta(t)} p^{\sum_{l=1}^k \kappa_l} (1 - p)^{\frac{3}{2} n(t) - \sum_{l=1}^k \kappa_l} \left( \frac{p}{1 - p} \right)^{\sum_{l=1}^k \kappa_l}. \quad (4.3)$$

Combining (4.3) with (4.2), we have the representation by cluster of the partition function of Ising model supported by the triangulation $t$:

$$Z_{\beta, N} = e^{-\frac{3}{2} \beta n(t)} \sum_{k=1}^{n(t)} 2^k \sum_{C_1, \ldots, C_k \subseteq \Delta(t)} \left( e^{2\beta} - 1 \right)^{\sum_{l=1}^k \kappa_l}. \quad (4.4)$$

In order to obtain lower bounds for the critical curve, we employ the representation (4.4) and consider several particular cases of interest.

**The case $k = n(t)$**: In this case there exists a unique way to decompose the set $\Delta(t)$ in $n(t)$ maximal connected components, considering clusters as isolated vertices $C_t = \{ t \}, t \in \Delta(t)$, and $1 \leq l \leq n(t)$. This decomposition implies that $\kappa_l = \kappa(C_t) = 0$. Thus, by (4.4), we obtain

$$Z_N^\beta \geq e^{\left( -\frac{3}{2} \beta + \ln 2 \right) n(t)}. \quad (4.5)$$

and, see (3.14),

$$\Xi_N(\beta, \mu) \geq Z_N \left( \mu + \frac{3}{2} \beta - \ln 2 \right). \quad (4.6)$$

Thus, using estimation (2.13) we obtain that the partition function $\Xi_N(\beta, \mu)$ exists if

$$\mu > -\frac{3}{2} \beta + 2 \ln 2 + \ln \left( \cos \frac{\pi}{N + 1} \right).$$

Letting $N \to \infty$ we obtain the following

**Proposition 3.** If $(\beta, \mu) \in \mathbb{R}^2_+$ such that $\mu < -\frac{3}{2} \beta + 2 \ln 2$ then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by using the standard formula with $\Xi_N(\beta, \mu)$ as a normalising denominator, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \to \infty$. Furthermore, for any finite-dimensional cylinder $C_{i_1, \ldots, i_k}$ we obtain $\mathbb{P}_N^{\beta, \mu}(C_{i_1, \ldots, i_k}) = 0$ whenever $N > N_0 \geq \max \{ i_1, \ldots, i_k \}$. 

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The case \( k = n(t) - 1 \): This case is discussed here for an illustrative purpose. Note that in this case there exists \( \frac{3}{2} n(t) \) ways to decompose the set \( \Delta(t) \) in \( n(t) - 1 \) maximal connected components: \( n(t) - 1 \) isolated vertices (triangles) and one cluster of two neighbor vertices (triangles). That is, if \( C \) is a cluster, then \( \eta(C) = 1 \) or \( 2 \). Moreover, for each decomposition \( C_1, \ldots, C_{n(t) - 1} \) we have that \( \sum_{l=1}^{n(t) - 1} \kappa_l = 1 \). This implies the following inequality

\[
Z_{\beta,t}^{\beta,t} > \frac{1}{2} e^{-\frac{3}{2} \beta + \ln 2} n(t) \sum_{C_1, \ldots, C_{n(t) - 1}} \left( e^{2\beta} - 1 \right)^{\sum_{l=1}^{n(t) - 1} \kappa_l}
\]

\[
= \frac{3}{4} \left( e^{2\beta} - 1 \right) n(t) e^{-\frac{3}{2} \beta + \ln 2} n(t)
\] (4.7)

\[
> \frac{3}{4} \left( e^{2\beta} - 1 \right) e^{-\frac{3}{2} \beta + \ln 2} n(t), \quad \text{as } n(t) \geq 1.
\]

Thus, we obtain another lower bound for the partition function of \( N \)-strip Ising model coupled to CDTs

\[
\Xi_N(\beta, \mu) \geq \frac{3}{4} \left( e^{2\beta} - 1 \right) Z_N \left( \mu + \frac{3}{2} \beta - \ln 2 \right).
\] (4.8)

Therefore, in this case we get the same inequality that in Proposition 3.

It would be interesting to analyse a general case \( k = n(t) - l \), but it seems that it won’t yield a better bound.

The case \( k = 1 \): Consider the following subset of \( \Pi_1 \):

\[
\Pi_1^{(0)} = \left\{ \text{number of edges in cluster is } \kappa_1 = \frac{3}{2} n(t) \right\} \cap \Pi_1.
\]

The probability of \( \Pi_1^{(0)} \) is easy to calculate

\[
P_{\beta,t}(\Pi_1^{(0)}) = \left( 1 - e^{-2\beta} \right)^{\frac{3}{2} n(t)}.
\]

Then by (4.4)

\[
Z_{\beta,t}^{\beta,t} > 2 e^{-\frac{3}{2} \beta n(t)} \left( e^{2\beta} - 1 \right)^{\frac{3}{2} n(t)}
\]

\[
= 2 \exp \left( - \left( \frac{3}{2} \beta - \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) \right) n(t) \right).
\]

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Thus
\[
\Xi_N(\beta, \mu) > \sum_t e^{-\mu n(t)} 2 \exp \left( - \left( \frac{3}{2} \beta - \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) \right) n(t) \right)
= 2Z_N \left( \mu + \frac{3}{2} \beta - \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) \right),
\]
and, as before, by estimation (2.13) the partition function exists if
\[
\mu > -\frac{3}{2} \beta + \frac{3}{2} \ln \left( e^{2\beta} - 1 \right) + \ln \left( 2 \cos \frac{\pi}{N+1} \right).
\]
Letting \( N \to \infty \) we obtain the following proposition.

**Proposition 4.** If \((\beta, \mu) \in \mathbb{R}_+^2\) such that \(\mu < -\frac{3}{2} \beta + \frac{3}{2} \ln(e^{2\beta} - 1) + \ln 2\) then there exists \(N_0 \in \mathbb{N}\) such that the partition function \(\Xi_N(\beta, \mu) = +\infty\) whenever \(N > N_0\). Moreover, the Gibbs distribution \(\mathbb{P}_N^{\beta, \mu}\) with periodic boundary conditions cannot be defined by using the standard formula with \(\Xi_N(\beta, \mu)\) as a normalising denominator, consequently, there is no limiting probability measure \(\mathbb{P}^{\beta, \mu}\) as \(N \to \infty\). Furthermore, for any finite-dimensional cylinder \(C_{i_1, \ldots, i_k}\) we obtain \(\mathbb{P}_N^{\beta, \mu}(C_{i_1, \ldots, i_k}) = 0\) whenever \(N > N_0 \geq \max\{i_1, \ldots, i_k\}\).

**Proof of Theorem 1** The proof follows immediately from the Proposition 3 and Proposition 4.

### 4.2 Proof of Theorem 2

The proof of Theorem 2 relies on two aditional observations. These are:

1. Upper bounds for the functions \(f_N\) and existence of the pointwise limit \(\lim_N f_N = f_{T-C}\).

2. The fact that graph of \(f_{T-C}\) provides an upper bound for the critical curve.

Consequently, we obtain as by-product of [20] the following assertions.

**Proposition 5.** For all \(N \in \mathbb{N}\), the following property of functions \(f_N\) is fulfilled:

1. If \(0 < \beta < \beta_2^*\), then
   \[
   f_N(\beta) \leq \psi(\beta),
   \]
   where \(\beta_2^*\) is positive solution of Eqn (2.32) and function \(\psi\) is introduced in Eqn (2.28).

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2. If $\beta^*_2 \leq \beta < \infty$, then

$$f_N(\beta) \leq \frac{3}{2} \beta + 2 \ln 2.$$  \hfill (4.11)

Proposition 6. There exist the pointwise limit of the family of functions $\{f_N\}$, i.e.

$$f_{T-C}(\beta) := \lim_{N \to \infty} f_N(\beta) \quad \text{for} \quad \beta > 0.$$ \hfill (4.12)

Combining (4.10), (4.11) with Proposition 6 and letting $N \to \infty$, we obtain the desired upper bound for the limit function $f_{T-C}$

$$\begin{align*}
\begin{cases}
  f_{T-C}(\beta) \leq \psi(\beta) & \text{if } 0 < \beta < \beta^*_2 \\
  f_{T-C}(\beta) \leq \frac{3}{2} \beta + 2 \ln 2 & \text{if } \beta^*_2 \leq \beta < \infty.
\end{cases}
\end{align*}$$ \hfill (4.13)

As $f_{T-C}$ is a upper bound for the critical curve then, the terms on the right-hand side of the inequality (4.13) is a upper bound for the critical curve.

Proof of Theorem 2. The upper bound of Theorem 2 is consequence of Eqn (4.13). The lower bound is consequence of Proposition 3 and 4. This concludes the proof of Theorem 2.

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