Geometrical Unification of Gravitation and Dark Energy: 
The Universe as a Relativistic Particle

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The Lagrangian, the Hamilton–Jacobi equation and the Schrödinger, Dirac and Klein–Gordon equations for the Friedmann–Robertson–Walker–Quintessence (FRWQ) system are presented and solved exactly for different interesting scenarios. The classical Lagrangian reproduces the usual two (second order) dynamical equations for the radius of the Universe and for the scalar field as well as the (first order) constraint equation. The approach naturally unifies gravity and dark energy, which may be related to the tlaplon (scalar torsion potential). The Lagrangian and the equations of motion are those of a relativistic particle moving on a two dimensional spacetime where the conformal metric factor is related to the dark energy scalar field potential. This allows us to quantize the system, obtaining a Klein-Gordon equation when the Universe is considered as a spinless particle, and a Dirac equation when the Universe is thought as a relativistic spin particle.

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I. INTRODUCTION

Quintessence is the name of one model put forward in order to explain that the rate of expansion of the Universe increases. The model modifies the equations of General Relativity by adding a Lagrangian density for a massless scalar (quintessence) field rolling down a potential minimally coupled to the usual Einstein Hilbert Lagrangian density.

The equations of motion for the Friedmann–Robertson–Walker–Quintessence (FRWQ) system, obtained from Einstein’s equations modified by the addition of the quintessence field, simplified by symmetry considerations (which transforms them from partial to ordinary differential equations) consist of a set of two ordinary dynamical second order differential equations which govern the evolution of the dynamical variables (the radius of the universe and the scalar quintessence field) and one ordinary first order differential equation which constraints the initial values and velocities of the dynamical variables. It is worth to mention that a similar Lagrangian to the one defined in quintessence models was put forward in a proposal to consider a modified form of torsion field (generated as the gradient of a scalar field) in Einstein–Cartan theory [1]. The torsion generating scalar field was called tlaplon.

Some of the articles which deal with the subject, write down a Lagrangian formulation for the FRWQ system. Nevertheless, to the best of our knowledge, all of the work published up to now is based on a classical Lagrangian which gives rise to the two dynamical equations but does not yield the constraint equation.

We show that the FRWQ system may be completely described in terms of a Lagrangian similar to that of a relativistic particle moving on a two dimensional gravitational field. The conformally flat two dimensional metric conformal factor is a function of the radius of the universe and of the scalar quintessence (tlaplon) field which naturally play the role of coordinates in this two dimensional (mini) superspace.

II. LAGRANGIAN FOR FRWQ SYSTEM

In general, the Lagrangian density \( \mathcal{L} \) for the evolution of the spacetime metric \( g_{\mu\nu}(x^\alpha) \) in interaction with a massless scalar field \( \phi(x^\alpha) \) (which could be quintessence) is written as

\[
\mathcal{L} = \sqrt{-g} \left( \frac{R}{2K} + \mathcal{L}_\phi \right),
\]

where \( g \) stands for the determinant of the metric, \( K = 8\pi G/c^4 \) (with \( G \) as the gravitational constant and \( c \) the speed of light), and \( \mathcal{L}_\phi \) is the Lagrangian density for the massless scalar field

\[
\mathcal{L}_\phi = \epsilon \left( \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right),
\]

where \( V(\phi) \) is (up to now) an unspecified potential for the scalar field \( \phi \) whereas \( \epsilon \) is a parameter that classifies the nature of the scalar field, such that \( \epsilon = 1 \) for usual scalar fields and \( \epsilon = -1 \) defines the quintessence field.
As it is well known, the Lagrangian density $L$ defined by (1) is singular. The action $S$

$$S = \int L \, d^4x,$$

(3)
gives rise (upon variation with respect to the metric tensor $g_{\mu\nu}$) to gauge invariant (generally covariant) and constrained (Einstein) field equations coupled to matter, i.e.,

$$G^{\mu\nu} = KT^{\mu\nu},$$

(4)
where $G^{\mu\nu}$ is the Einstein tensor and $T^{\mu\nu}$ is the energy–momentum tensor of matter

$$T_{\mu\nu} = g_{\mu\nu}L_\phi - 2\frac{\delta L_\phi}{\delta g_{\mu\nu}}.$$

(5)

Variation of the action $S$ with respect to $\phi$ yields the Klein–Gordon equation for the massless scalar field

$$\Box \phi + \frac{dV(\phi)}{d\phi} = 0.$$  

(6)

Now, let us take the line element for an isotropic and homogeneous Friedmann–Robertson–Walker (FRW) spacetime, with the metric defined by

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right],$$

(7)
where $a(t)$ is the (time dependent) radius of the Universe and $k = -1, 0, +1$ measure its negative, zero or positive curvature.

When Einstein equations coupled to matter are written in terms of the line element (7) and the scalar field $\phi$ is assumed to depend on time only, they reduce to two (second order) dynamical and one (first order) constraint. The dynamical equations are (choosing $K = 1$, for convenience)

$$2\frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} - \epsilon \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right) = 0,$$

(8)
and

$$\ddot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0,$$

(9)
while the constraint equation is

$$3 \left( \frac{\dot{a}}{a} \right)^2 + 3\frac{k}{a^2} + \epsilon \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) = 0.$$  

(10)

Also, another useful equation can be obtained manipulating (10) and (8)

$$\ddot{a} = \frac{\epsilon}{3} \left( \dot{\phi}^2 - V \right),$$

(11)

Notice how the set (8)-(11) becomes the FRWQ system for $\epsilon = -1$. This system has been already studied and solved for quintessence by Capozziello and Roshan [2] for different scenarios and configurations of matter. Here, our aim is to explore the analogy of this system with a relativistic particle which implies, as we will show, a geometric unification. With this purpose in mind we focus our attention to the fact that the Lagrangian $L$ which gives rise to the dynamical equations (8) and (9) is

$$L = 3a\ddot{a}^2 - 3ka + a^3 \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right).$$

(12)

However, it is important to emphasize that the Lagrangian (12) does not produce the constraint equation (10), which is equivalent to imposing that the Hamiltonian $H$ associated to $L$ vanishes, i.e.,

$$H \equiv \frac{\partial L}{\partial \dot{a}} \dot{a} + \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L = 0.$$  

(13)

Despite of that, notice now that it is a remarkable fact that the change of variables

$$r = \frac{2\sqrt{6}}{3} a^{3/2}, \quad \theta = \frac{3}{2\sqrt{6}} \phi,$$

(14)
recasts the Lagrangian $L$ in a “kinetic energy minus potential energy ($T - V$)” form (disregarding the fact that both $T$ and $V$ have the “wrong signs” in the $\theta$ associated terms, in the case of quintessence)

$$\tilde{L} = \frac{1}{2} (\dot{r}^2 + cr^2 \dot{\theta}^2) - \tilde{V}(r, \theta),$$

(15)
where $\tilde{V}(r, \theta)$ is a general potential

$$\tilde{V}(r, \theta) = 3 \left( \frac{3}{8} \right)^{1/4} kr^{2/3} + \frac{3}{8} r^2 V(\theta).$$

(16)

Note that in the Lagrangian $\tilde{L}$ defined in (15), which describes the evolution of a FRWQ Universe in the presence of geometry (represented by $r$ or $a$) and dark energy (represented by $c = -1$, $\theta$ or $\phi$), shows that the Universe evolves as a relativistic particle moving on a two dimensional surface under the influence of the potential (16), thus geometrically unifying gravity and dark energy.

Nevertheless, the Lagrangian (15) does not produce the constraint equation (10). In order to construct a Lagrangian, which in fact gives rise to all three equations (8), (9) and (10) it is enough to recall that [3] Maupertuis and Fermat principles give rise to identical equations of motion in classical mechanics and geometrical (ray) optics except for the fact that Fermat principle also produces a constraint equation. It is worth mentioning that exactly the same results are reached for the case of a relativistic particle moving on a two dimensional conformally flat spacetime.

From now on, we just focus in the case of $\epsilon = -1$ to be the most relevant case for the dark energy (quintessence) scenario.
The description of the FRWQ system in terms of a Fermat-type Lagrangian is established by defining the relation between the potential \( \bar{V}(r, \theta) \) and the conformal factor \( n(r, \theta) \)

\[
\bar{V}(r, \theta) \equiv -\frac{1}{2} [n(r, \theta)]^2 ,
\]

and the constraint

\[
\bar{H} \equiv \frac{\partial \bar{L}}{\partial \dot{r}} \dot{r} + \frac{\partial \bar{L}}{\partial \dot{\theta}} \dot{\theta} - \bar{L} = 0 .
\]

The Fermat-like Lagrangian \( L_F \) which gives rise to all three equations [3, 4] and \( \lambda \) is, in principle, an arbitrary parameter. Thus, the Lagrangian [10] may be appropriately rewritten as

\[
L_F = n(r, \theta) \sqrt{\left( \frac{dr}{d\lambda} \right)^2 - r^2 \left( \frac{d\theta}{d\lambda} \right)^2} .
\]

To reproduce the relativistic equations of motion, \( \lambda \) is defined by Luneburg’s parameter choice [3, 3]

\[
\sqrt{\left( \frac{dr}{d\lambda} \right)^2 - r^2 \left( \frac{d\theta}{d\lambda} \right)^2} = n(r, \theta) .
\]

III. QUANTIZATION

Having the description of a relativistic particle for the Universe, we can go further in our scheme and attempt to quantize this theory. First, let us rewrite the Lagrangian [20] as

\[
L_F = \sqrt{\bar{V}(\xi, \theta) e^{2\xi} \left( \dot{\theta}^2 - \dot{\xi}^2 \right)} ,
\]

where we have introduced the new variable \( \xi = \ln r \), and then \( \bar{V} \equiv \bar{V}(\xi, \theta) = \bar{V}(r, \theta) \) and \( \theta = d\theta/d\lambda, \dot{\theta} = d\xi/d\lambda \).

We can notice that the quintessence field acts as a Super-time in this new description where the particle is moving in a two-dimensional conformally flat spacetime. The conformally flat metric becomes

\[
g_{00} = \Omega^2 , \quad g^{00} = \frac{1}{\Omega^2} , \quad g_{11} = -\Omega^2 , \quad g^{11} = -\frac{1}{\Omega^2} ,
\]

where we introduce the notation \( \Omega \equiv \sqrt{\bar{V}} e^\xi \), which will appear frequently in this work.

In order to avoid with the procedure of canonical quantization of the FRWQ system, we restrict ourselves to the cases where \( \bar{V} > 0 \), and to consider only static manifolds [3, where there exist a family of spacelike surfaces which are always orthogonal to a timelike Killing vector. This implies that \( \partial_\theta g_{\mu\nu} = 0 \), or

\[
\frac{\partial \bar{V}}{\partial \theta} = 0 ,
\]

implying that the original potential \( V(\theta) \) is a constant. This means that for the current quantization process, \( V(\theta) \) can only be the cosmological constant.

Classically, the Hamiltonian for the system described in the previous section is

\[
H = m\sqrt{g_{00}} \sqrt{1 - g^{11} \pi^2} = \sqrt{g_{00}} \sqrt{1 + \frac{\pi^2}{\Omega^2}} ,
\]

where we have included \( m \) as a new parameter (representing the analogue of a mass) the \( \pi \) is the canonical momentum.

We use this Hamiltonian to construct the quantum theory for the FRWQ system. The quantum equation will be in the form

\[
i\hbar \frac{\partial \Psi}{\partial \theta} = 1\mathcal{H} \Psi ,
\]

where \( 1 \) is the unit matrix and the Hamiltonian operator is

\[
\mathcal{H} = \Omega^{1/2} \sqrt{1 + \bar{\rho}^2} \Omega^{1/2}
\]

where \( \bar{\rho} \) is the momentum operator. This Hamiltonian \( \mathcal{H} \) is constructed to avoid problems with the ordering.

In the following we proceed to quantize the FRWQ theory for two different and specific cases. First, we quantize the system as it were a spin-0 particle. This will produce a Klein-Gordon equation for the wavefunction of the Universe. The second case correspond to the quantization of the FRWQ system as a spin-1/2 particle. Despite of the possible arguments against this path, we perform this kind of quantization in an exploratory spirit, and because currently we do not know if the Universe has spin or not.

A. Quantization of the FRWQ system as a spin-0 particle

One way to canonically quantize the relativistic spinless particle can be obtained following Gavrilov and Gitman’s method [3]. This procedure is consistent as construct the quantum theory along the Dirac’s theory for gauges and constraints [7, 8].

We will not reproduce here the calculations for the quantization of relativistic spin-0 particle, but just show that the quantization for the FRWQ system (considered as a spinless relativistic particle) produces the quantum equation [3]
where $\Psi$ is the spinor
\[ \Psi = \begin{pmatrix} \chi \\ \psi \end{pmatrix}, \] (29)

and $\hat{h}$ is the matrix Hamiltonian
\[ \hat{h} = \begin{pmatrix} 0 & -\partial_\xi^2 + m^2\Omega^2 \\ 1 & 0 \end{pmatrix}, \] (30)

where $m$ is anew a quantity analogue to a "mass" for the FRWQ system. From now on we will take $\hbar = 1$ for convenience.

Eq. (28) gives rise to the equation
\[ 0 = \frac{\partial^2 \chi}{\partial \theta^2} - \frac{\partial^2 \psi}{\partial \xi^2} + m^2\Omega^2 \psi, \] (31)

where we have used that $i\theta \psi = \chi$. This is a Klein-Gordon equation for the wavefunction $\psi$ with a mass-corrected term due to the metric. This general equation represents the quantization of the FRWQ system as a spinless particle.

It can be simplified assuming the dependence $\psi = \phi e^{i\theta}$, with $\partial_\theta \phi = 0$. Thus, we can find the equation
\[ \frac{\partial^2 \phi}{\partial \xi^2} = (m^2\Omega^2 + E^2)\phi. \] (32)

For the simplest case of $k = 0$ and $\Omega^2 = \lambda e^{4\xi}$ (where we have chosen $V(\theta) = -8\lambda/3$, with a constant positive $\lambda$), the general solution is
\[ \psi(\xi) = C_1(-1)^{-E/4}I_{-E/2} \left( \frac{m\sqrt{\lambda}}{2} e^{2\xi} \right) \Gamma \left( 1 - \frac{E}{2} \right) + C_2(-1)^{E/4}I_{E/2} \left( \frac{m\sqrt{\lambda}}{2} e^{2\xi} \right) \Gamma \left( 1 + \frac{E}{2} \right), \] (33)

where $C_1$ and $C_2$ are constants, $\Gamma$ is the Euler gamma function, and $I_n$ is the modified Bessel function of the first kind of order $n$. Notice that $\lambda$ is related to the cosmological constant.

A solution for $k \neq 0$ is not simple to obtain.

As a final point, we would like to remark an interesting consequence of this quantization scheme. We can write the Eq. (31) in a Super-Hamiltonian formalism, being less restrictive with the assumption that the potential $V(\theta)$ is constant. For a closed universe $k = 1$ (and the case quintessence $\epsilon = -1$) it is possible to rewrite Eq. (31) as
\[ \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\partial^2 \chi}{\partial \varphi^2} + (m^2 \varphi^2 e^{6\alpha} - e^{4\alpha}) \psi \equiv \mathcal{H}\psi = 0 \] (34)

where $\alpha = \ln a$, and $\varphi = 2\theta/3$. To obtain this equation we chose $m^2 = 1/18$ and $V(\theta) = 3\tilde{m}^2 \varphi^2$, where $\tilde{m}$ is the mass of the field. Here, $\mathcal{H}$ is usually called the Wheeler-DeWitt Super-Hamiltonian (10). Thus, the quantization of the FRWQ system as a spinless relativistic particle proposed here can reproduce known results of quantization using the Super-Hamiltonian to construct the equivalent of Schrödinger equation.

B. Quantization of the FRWQ system as a spin-1/2 particle

We can now propose a different way to quantize the FRWQ system. The problem is to solve the square root in the Hamiltonian (27). This can be done as in the previous section, or with Dirac matrices (as it is done for the Dirac equation). This means that the FRWQ system is considered in an analogue way to a spin-1/2 particle. There is no restriction to this ansatz.

The momentum operator in (27), $\hat{p} = \sqrt{-g^{\alpha\beta}} \hat{\pi} = \hat{\pi}/\Omega$, must be defined such that
\[ \hat{\pi} = -i\frac{\partial}{\partial \xi}. \] (35)

In a way similar for to the Dirac equation, we propose that the square-root in the Hamiltonian (27) can be solved using the Dirac matrices $\alpha$ and $\beta$. This will give us the Hamiltonian
\[ \mathcal{H} = \Omega^{1/2} (\alpha \cdot \hat{p} + m\beta) \Omega^{1/2}, \] (36)

where $m$ is still a parameter associated to a "mass" of the FRWQ system. This Hamiltonian allows us to quantize a FRWQ Universe as a spin particle. The quantum equation is Eq. (26), where now $\Psi$ is a bi-spinor. Explicitly, using the operator (35) in the Hamiltonian, the quantum equation reads
\[ i\frac{\partial \Psi}{\partial \theta} = -i\alpha^\xi \left[ \frac{\partial}{\partial \xi} + \frac{1}{2} \frac{\partial \ln \Omega}{\partial \xi} \right] \Psi + m\beta \Omega \Psi, \] (37)

where $\alpha^\xi$ stands for any of the matrices $\alpha^1$, $\alpha^2$ or $\alpha^3$. Multiplying the equation by $\gamma^0 = \beta$ ($\gamma^0\gamma^0 = 1$) and remembering that $\gamma^\xi = \gamma^0\alpha^\xi$, we have
\[ i\gamma^0 \frac{\partial \Psi}{\partial \theta} + i\gamma^\xi \left[ \frac{\partial}{\partial \xi} + \frac{1}{2} \frac{\partial \ln \Omega}{\partial \xi} \right] \Psi = m\Omega \Psi. \] (38)

It is shown in the Appendix that this equation can be obtained directly from the known form of the Dirac equation in curved spacetimes. This gives validity to our quantization scheme.

Eq. (38) represents the quantized FRWQ system a spin particle. To study it, we can simplify it using the ansatz
\[ \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} e^{-f(1/2)\theta_0 \ln \Omega \ddot{\xi}}, \] (39)

where $\psi$ and $\chi$ are spinors. Then, from Eq. (38), we get
\[ \frac{i}{2} \frac{\partial \psi}{\partial \theta} + i\sigma^\xi \frac{\partial \chi}{\partial \xi} = m\Omega \psi, \] (40)
\[ \frac{i}{2} \frac{\partial \chi}{\partial \theta} + i\sigma^\xi \frac{\partial \psi}{\partial \xi} = -m\Omega \chi, \] (41)

where $\sigma^\xi$ are the Pauli matrices.
One form to solve the previous set of equations is to assume that \( \psi = \psi e^{iE\theta} \) and \( \chi = \chi e^{iE\theta} \), with \( \partial_\theta \psi = 0 = \partial_\theta \chi \). Then we find

\[
\begin{align*}
  i\sigma^1 \frac{\partial \chi}{\partial \xi} &= (E + m\Omega)\dot{\psi}, \\
  i\sigma^2 \frac{\partial \psi}{\partial \xi} &= (E - m\Omega)\dot{\chi},
\end{align*}
\]

which completely solved the problem if we can find the solution to the following equation

\[
\frac{\partial}{\partial \xi} \left( \frac{1}{E - m\Omega} \frac{\partial \dot{\psi}}{\partial \xi} \right) + (E + m\Omega)\dot{\psi} = 0. \tag{44}
\]

This equation has a general solution for the case \( E = 0 \)

\[
\dot{\psi}(\xi) = C_1 e^{m \int \Omega(\xi) d\xi'} + C_2 e^{-m \int \Omega(\xi) d\xi'}. \tag{45}
\]

However, a solution for \( E \neq 0 \) is not simple to be obtained.

Finally, we notice that if the previous assumptions are not taken, then we can find the set of coupled second order equations directly from Eqs. (40) and (41). These equations are

\[
\begin{align*}
  \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial^2 \psi}{\partial \xi^2} + im \frac{\partial \Omega}{\partial \xi} \sigma^1 \chi + m^2 \Omega^2 \psi &= 0, \tag{46} \\
  \frac{\partial^2 \chi}{\partial \theta^2} - \frac{\partial^2 \chi}{\partial \xi^2} - im \frac{\partial \Omega}{\partial \xi} \sigma^1 \psi + m^2 \Omega^2 \chi &= 0, \tag{47}
\end{align*}
\]

from where we can see that the spinors \( \psi \) and \( \chi \) do not satisfy simple Klein-Gordon equations. In the FRWQ model, the spinors are coupled to metric of the spacetime.

\[\text{Appendix A: Dirac equation in curved spacetimes}\]

The curved-space Dirac equation is

\[
ie^\mu d^\gamma_a \left( \partial_\mu + \frac{1}{8} \omega_{abcd} [\gamma^a, \gamma^d] \right) \Psi = m \Psi, \tag{A1}\]

where we defined the vierbein as

\[
g_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \eta_{ab}, \tag{A2}\]

with the flat-spacetime metric \( \eta_{ab} \). Notice that \( e^\mu_d \) is the inverse vierbein in the sense that \( e^{a}_{\mu} e^{b}_{\nu} \delta^a_{b} \). Also we define the spin connection \( \omega_{\alpha\mu} = \eta_{ac} \omega^{c}_{b\mu} \), with

\[
\omega^{c}_{b\mu} = e^{c}_{\nu} e^{d}_{b,\mu} + e^{c}_{\nu} e^{\sigma}_{b} \Gamma^{\nu}_{\sigma\mu}, \tag{A3}\]

where \( \Gamma^{\nu}_{\sigma\mu} \) are the Christoffel symbols. Because the antisymmetry of the spin connection in its first two indices, we have \( \omega_{\alpha\mu} [\gamma^\alpha, \gamma^\beta] = 2\omega_{\alpha\beta} \gamma^\alpha \gamma^\beta \).

In our two-dimensional case of the FRWQ system, the vierbeins are

\[
e_0 = \Omega, \quad e_1 = \Omega, \quad e_0 = \frac{1}{\Omega}, \quad e_1 = \frac{1}{\Omega}. \tag{A4}\]

Thus, the Dirac equation becomes

\[
i \gamma^0 \left( \partial_0 + \frac{1}{4} \omega_{abcd} \gamma^a \gamma^b \right) \Psi + i \gamma^1 \left( \partial_1 + \frac{1}{4} \omega_{abcd} \gamma^a \gamma^b \right) \Psi = m \Omega \Psi. \tag{A5}\]

Also we have that \( \omega_{ab0} \gamma^a \gamma^b = 2\omega_{010} \gamma^0 \gamma^1 \), \( \omega_{ab1} \gamma^a \gamma^b = 2\omega_{011} \gamma^0 \gamma^1 \), and

\[
\omega_{010} = \Gamma^0_{10} = \partial_\xi \ln \Omega, \quad \omega_{011} = \Gamma^0_{11} = 0, \tag{A6}\]

for the two-dimensional case of conformally flat metric.

Then, the Dirac equation becomes

\[
i \gamma^0 \left( \partial_0 + \frac{1}{2} \frac{\partial \ln \Omega}{\partial \xi} \gamma^0 \gamma^1 \right) \Psi + i \gamma^1 \partial_\xi \Psi = m \Omega \Psi, \tag{A7}\]

or better written as

\[
i \gamma^0 \partial_0 \Psi + i \gamma^1 \left( \partial_\xi + \frac{1}{2} \frac{\partial \ln \Omega}{\partial \xi} \right) \Psi = m \Omega \Psi, \tag{A8}\]

which is exactly the same than Eq. (53) that we obtained proposing a quantization method (with \( \partial_0 = \partial_{\theta} \)).

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