Realistic teleportation with linear optical elements

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We calculate the highest possible information gain in a measurement of entangled states when employing a beamsplitter. The result is used to evaluate the fidelity, averaged over all unknown inputs, in a realistic teleportation protocol that takes account of the imperfect detection of Bell states. Finally, we introduce a probabilistic teleportation scheme, where measurements are made in a partially entangled basis.

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In the recent years some striking features of quantum entanglement have been unveiled in the context of quantum information [1]. One of them is the possibility of teleporting an unknown state between two distant locations [2]. Teleportation has been realised experimentally for qubits [3,4]. A fundamental ingredient in the teleportation protocol is a Bell measurement, i.e. a projection onto the four Bell states

\[ |\psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) , \]
\[ |\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) . \] (1)

In this paper we study teleportation and Bell measurements with realistic resources. The teleportation channel, which consists ideally of a maximally entangled state as in (1), can nowadays be readily provided experimentally by entangled photons [5–7]. Implementing teleportation with photons means having to perform a Bell measurement, i.e. a joint measurement on two photons. Using non-linear devices this would be in principle easy to do, but the efficiency of present day non-linear optical elements is not sufficient to implement Bell measurements. Standard optical tools, like e.g. a beamsplitter, act in a linear way on photons.

It has been shown in [8] that it is impossible to distinguish the four Bell states unambiguously with linear optical elements. In the case of a beamsplitter with 50% transmission this is easy to understand, as both |\phi^+\rangle and |\phi^-\rangle as input state will lead to two identically polarised photons in one of the outgoing paths. The topics we address in this letter are outlined as follows: First, we ask what is the highest information one can retrieve from a Bell measurement with a linear optical element, namely a beamsplitter? This question has also been discussed recently in [9], for a maximally entangled basis. We will in addition consider the case of measurements in a partially entangled basis, which can be more successful than in the case of maximal entanglement, depending on the parameters of the beamsplitter. We will then explain a strategy for teleportation, using such imperfect measurements, and calculate the best teleportation fidelity. In the case of measurements in a partially entangled basis teleportation can be implemented probabilistically. Note that our approach uses a maximally entangled state shared between Alice and Bob, and is therefore different from the scheme considered in [10].

Let us first study the gain of information in a Bell measurement using a beamsplitter. Our notation is the same as in [8], and we also use their method to represent the Bell states in matrix form. We denote the photon modes with \(a^\dagger\), where \(a\) and \(b\) stand for two different directions of propagation, and the indices \(H\) and \(V\) denote horizontal and vertical polarisation. Thus, the Bell states are

\[ |\psi^\pm\rangle = \frac{1}{\sqrt{2}} (a_H^\dagger b_V^\dagger \pm a_V^\dagger b_H^\dagger) |0\rangle , \]
\[ |\phi^\pm\rangle = \frac{1}{\sqrt{2}} (a_H^\dagger b_H^\dagger \pm a_V^\dagger b_V^\dagger) |0\rangle . \] (2)

The action of a realistic beamsplitter is given by [11]

\[
\begin{pmatrix}
    a_H^\dagger' \\
    a_V^\dagger' \\
    b_H^\dagger' \\
    b_V^\dagger'
\end{pmatrix} = \begin{pmatrix}
    \cos\theta_H e^{i\varphi_H} & 0 & \sin\theta_H e^{i\chi_H} & 0 \\
    0 & \cos\theta_V e^{i\varphi_V} & 0 & \sin\theta_V e^{i\chi_V} \\
    -\sin\theta_H & 0 & \cos\theta_H & 0 \\
    0 & -\sin\theta_V & 0 & \cos\theta_V
\end{pmatrix} \begin{pmatrix}
    a_H^\dagger \\
    a_V^\dagger \\
    b_H^\dagger \\
    b_V^\dagger'
\end{pmatrix}. \] (3)

Here \(\cos\theta_H(V)\) is the transmission coefficient for horizontal (vertical) polarisation, and the phases of the transmitted (reflected) beam are given by \(\varphi(\chi)\). These parameters can be set by the experimentalist.

The gain of information when making a Bell measurement is given, due to Bayes theorem, as follows:
\[ \Delta S = S_i - S_f = -\sum_i p_i \log p_i + \sum_{i,j} p(\psi_i, \kappa_j) \log p(\psi_i | \kappa_j) \ . \] (4)

All logarithms in this paper are taken to base 2. Here \( \psi_i \) enumerates the four Bell states, and \( \kappa_j \) denotes the 10 possible different product states in the two-photon manifold, for example \( a_H^\dagger b_H^\dagger |0\rangle \). These product states can in principle be distinguished unambiguously. In our calculations the initial probabilities \( p_i \) for the four Bell states are taken to be identical.

Calculating \( \Delta S \) numerically as a function of \( \theta_H \) and \( \theta_V \) leads to the results presented in Fig. 3. The highest possible information gain is \( \Delta S_{\text{max}} = 1.5 \), which is reached by a beamsplitter with 50% transmission for both polarizations.

Which setting of the beamsplitter leads to the highest gain of information for less than maximally entangled states? We denote a basis of partially entangled states as

\[
\begin{align*}
|\psi^-\rangle' &= (x a_H^\dagger b_V^\dagger - \sqrt{1-x^2} a_V^\dagger b_H^\dagger)|0\rangle, & |\psi^+\rangle' &= (\sqrt{1-x^2} a_H^\dagger b_V^\dagger + x a_V^\dagger b_H^\dagger)|0\rangle, \\
|\phi^-\rangle' &= (x a_H^\dagger b_H^\dagger - \sqrt{1-x^2} a_V^\dagger b_V^\dagger)|0\rangle, & |\phi^+\rangle' &= (\sqrt{1-x^2} a_H^\dagger b_H^\dagger + x a_V^\dagger b_V^\dagger)|0\rangle.
\end{align*}
\] (5)

Repeating the calculation from above one now finds the information gain as a function of the degree of entanglement \( x \). An example for \( x^2 = 0.1 \) is shown in Fig. 2. Here the minima correspond to equal reflection and transmission. The maximum information gain is higher than for Bell states, and will reach \( \Delta S_{\text{max}} = 2 \) for \( x = 0 \). In the example \( x^2 = 0.1 \) we find \( \Delta S_{\text{max}} = 1.52 \).

How can one use such imperfect measurements for realistic teleportation, and what is the fidelity one can achieve? In order to answer this question one has to develop a strategy that copes with the cases where the Bell states have not been detected unambiguously. As usually in teleportation, Alice and Bob share a maximally entangled state, and Alice wants to teleport an unknown state \( |\tau\rangle \) to Bob. The protocol we suggest progresses as follows:

- Alice fixes the parameters for her beamsplitter and prepares a look-up table that tells her the conditional probabilities \( p(\psi_i | \kappa_j) \).
- Alice does an imperfect “Bell measurement” on the unknown state \( |\tau\rangle \) and her part of the entangled state, and finds a detector coincidence \( \kappa_j \).
- Alice uses her look-up table to determine for which \( i \) the conditional probability \( p(\psi_i | \kappa_j) \) is maximal, and informs Bob via classical communication about the result.
- Bob rotates his state according to Alpes most likely Bell state \( \psi_i \).

With this protocol different input states will be teleported with different fidelity. If one does not want to restrict the set of possible inputs, a meaningful figure of merit will be an averaged fidelity, namely

\[ \bar{F} = \int_\tau d\tau F(|\tau\rangle) \ , \] (6)

where \( d\tau \) symbolizes an appropriate integration measure for the input state. In this paper, the fidelity is weighted with a constant probability for each input state.

Performing the numerics for this strategy leads to the results given in Fig. 3, the maximal averaged fidelity is again reached for a beamsplitter with 50% transmission, and is \( F_{\text{max}} \approx 0.88 \). Remember that the value for the fidelity for “teleportation” without using entanglement, i.e. a projection measurement by Alice and subsequent state preparation by Bob, is \( F_{\text{class}} = 2/3 \). Therefore, even with one linear optical element it is in principle possible to demonstrate teleportation of an unknown quantum state in an experiment that takes all Bell states into account. Using our protocol one can, in principle, arrive at higher fidelities than in the existing experiments, which reach a fidelity of \( F = 0.82 \pm 0.01 \) [12], with an efficiency of 25%.

In the remainder of this paper we will discuss teleportation with measurements in a partially entangled basis. This kind of teleportation works probabilistically, i.e. one succeeds with a certain probability and then knows to have succeeded. The unsuccessful runs have to be discarded. To our knowledge, this idea has not yet been studied elsewhere. It is motivated by the fact that partially entangled states are easier to distinguish than maximally entangled states, as shown in Figs. 4 and 5.

In order to present the idea of probabilistic teleportation, for simplicity we will go back to the following notation of a basis of partially entangled states:

\[
\begin{align*}
|\psi^-\rangle' &= x|0\rangle - \sqrt{1-x^2}|10\rangle, & |\psi^+\rangle' &= \sqrt{1-x^2}|01\rangle + x|10\rangle, \\
|\phi^-\rangle' &= x|00\rangle - \sqrt{1-x^2}|11\rangle, & |\phi^+\rangle' &= \sqrt{1-x^2}|00\rangle + x|11\rangle.
\end{align*}
\] (7)
In this basis we can rewrite the total state before Alice’s measurement as

\[
(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2xy}} \left[ |\psi^+\rangle'(\alpha x|0\rangle + \beta y|1\rangle) + |\psi^-\rangle'(\alpha y|0\rangle - x\beta|1\rangle) + |\phi^+\rangle'(\beta x|0\rangle + \alpha y|1\rangle) + |\phi^-\rangle'(\beta y|0\rangle - x\alpha|1\rangle) \right],
\]

where we have introduced \( y = \sqrt{1 - x^2} \). Without loss of generality we assume \( x \geq y \). After Alice finds one of the four basis states (here we assume the detection to be perfect), she tells Bob the result, and he performs a positive operator measurement (POVM). Let us consider the case where Alice finds \( |\psi^+\rangle' \), the other cases work in an analogous way. For this case the POVM elements are given by

\[
A_1 = \left( \begin{array}{ccc} \frac{y}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad A_2 = \left( \begin{array}{ccc} 1 - \frac{y}{x} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\]

With probability \( p = 1/(2x^2) \) this procedure will lead to the correct state on Bob’s side.

At the end of this article we formulate the problem of choosing the best degree of entanglement for the basis of Alice’s measurement: there will be a trade-off between the information gain (which is higher for less entangled states) and the probability of successful teleportation (which is higher for more entangled states). It remains an open question which degree of entanglement one has to choose in a realistic experiment in order to achieve the highest teleportation fidelity.

In conclusion, we have found that a beamsplitter with 50% transmission leads to the highest information gain in a Bell measurement, and discussed the optimal parameters of a beamsplitter for measuring partially entangled states. We have then presented a strategy for realistic teleportation with imperfect Bell measurements, using the “most likely” Bell state. The fidelity, averaged over all possible inputs to be teleported, was found to be \( F_{\text{max}} \approx 0.88 \) with our protocol. Finally, we introduced the concept of probabilistic teleportation, using measurements in a partially entangled basis, with a subsequent POVM on Bob’s side.

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FIG. 1. Information gain for Bell states as a function of the transmission coefficient for vertical and horizontal polarization. The maxima correspond to transmission equals reflection for both polarizations.

FIG. 2. Information gain for partially entangled states with $x^2 = 0.1$ as a function of the transmission coefficient for vertical and horizontal polarization. The minima correspond to transmission equals reflection for both polarizations.

FIG. 3. Averaged fidelity for teleportation of an unknown state, as a function of the transmission coefficient for vertical and horizontal polarization.
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