Strong-Field QED and the Inverse Mellin Transform

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We introduce the technique of inverse Mellin transform in a problem of strong-field QED. We show that the moments of pair production width in a uniform background magnetic field are proportional to the derivatives of photon polarization function at the zero momentum. Hence, the pair-production width or the absorptive part of the photon polarization function can be computed as well. Therefore the analytic property of the photon polarization function in all energy range is obtained. We also discuss briefly the possible extensions of this technique to other problems.

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I. INTRODUCTION

The wave function and energy quantization of a relativistic charged particle in a uniform background magnetic field is well understood. For a relativistic electron (positron), one solves for the Dirac equation in a background magnetic field. The solution is characterized by the energy levels:

\[ E_{n,s_z}^2 = m^2_e + p_z^2 + eB(2n + 1 + 2s_z), \]  

with \( s_z \) the electron spin projection along the +z direction, and the correspondent wave function

\[ \psi_{n,p_y,p_z,s_z}^{\pm}(\vec{r},t) = \exp(-iE_n t + ip_y y + ip_z z) \mathbf{F}_{n,p_z,s_z}^{\pm}(x'), \]  

where \( \mathbf{F}_{n,p_z,s_z}^{\pm}(x') \) is a four-component spinor with \( x' = x - p_y/eB \). The detailed form for \( \mathbf{F}_{n,p_z,s_z}^{\pm} \) can be found, for example, in [1].

In spite of our knowledge in the above energy quantization and wave function, it is often non-trivial to compute a physical process occurring in a background magnetic field, particularly, if there are more than one charged fermion involving in the reaction. A well-known example is the pair production process, \( \gamma \rightarrow e^+ e^- \), which is relevant to the gamma-ray attenuation in the neutron star [2], and was studied first by Toll [3] and Klepikov [4] independently. Both authors computed the pair production width by squaring the \( \gamma \rightarrow e^+ e^- \) matrix elements using exact electron and positron wave functions in a background magnetic field, and summing over all available final states consistent with the initial photon energy. The most updated calculation using this approach was given in [5]. We note that all of the above works assume \( q^2 = 0 \) for the photon momentum, despite the presence of background magnetic field. To apply the precise photon dispersion relation in the calculation of pair-production width, one needs to study the photon polarization function in a background magnetic field. In fact, a consistent calculation of pair-production width require the knowledge of both absorptive and dispersive parts of photon polarization function. In this regard, Tsai and Erber [6] obtained the absorptive part of the one-loop photon polarization function in the asymptotic limit \( \omega \gg 2m_e \) and \( B \ll B_c \equiv m^2_e/e \) with the assumption \( q^2 = 0 \). Their result was shown [6] to agree with that of Toll and Klepikov. However, in the above asymptotic limit, the threshold behavior of the pair-production width is completely lost. Later on, Shabad [7] obtained the absorptive part (and the dispersive part as well) of one-loop photon polarization function for a general photon energy and magnetic-field strength. In that work, the threshold behavior of the pair-production width was worked out explicitly. We remark that Refs. [6,7] employed Schwinger’s proper-time representation for the electron and positron Green’s functions [8] which make up the photon polarization function. To our knowledge, Ref. [7] is the first work demonstrating that the proper-time representation for the photon polarization function gives equivalent pair-production width to that given by squaring the \( \gamma \rightarrow e^+ e^- \) amplitude directly. Unfortunately, the

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manipulations of Ref. [3] are rather involved and the substantial details of them were given in some other unpublished preprints [3]. It is not very clear how one can generalize the approach of Ref. [3] to other processes.

In this report, we provide an alternative derivation of the pair-production width (or equivalently the absorptive part of the photon polarization function) from the proper-time representation of the photon polarization function. We shall also outline the procedure of obtaining the dispersive part of the polarization function, which is required for computing the photon index of refraction. It will be clear that our approach is very straightforward and physically intuitive. This report is organized as follows: In Section 2, we derive the sum rule that relates the moments of the pair-production width to the derivatives of the photon polarization function at the zero momentum. In Section 3, we apply this sum rule to the photon polarization function in an asymptotic limit, i.e., $\omega \gg 2m_e$ and $B \ll B_c$. A general analysis valid for arbitrary $\omega$ and $B$ is given in Section 4. We conclude in Section 5.

II. THE SUM RULE

We are interested in the properties of photon polarization function in the background magnetic field, $\Pi_{\mu\nu}$. Since a photon has two different polarization states in a background magnetic field, one may choose the polarization states as $\epsilon_\parallel \equiv (0, \hat{e}_\parallel)$ and $\epsilon_\perp \equiv (0, \hat{e}_\perp)$ where $\hat{e}_\parallel$ is lying on the plane spanned by the photon momentum $q$ and the magnetic field vector $B$, while $\hat{e}_\perp$ is perpendicular to the plane. It is then convenient to define the scalar functions $\Pi_{\parallel,\perp} = \epsilon_\parallel^{\mu} \Pi_{\mu\nu} \epsilon_\perp^\nu$ which govern the behaviors of polarization states $\epsilon_\perp^{\mu}$ in the background magnetic field. The sum rule for $\Pi_{\parallel,\perp}$ is derived by considering the following contour integral [10]:

$$I_n = \int_C \frac{d\omega^2}{2\pi i} \frac{\Pi_{\parallel,\perp}(\omega^2)}{(\omega^2 + \omega_0^2)^{n+1}},$$

where the integration contour $C$ is shown in Fig. 1. We have implicitly assume $q^2 = 0$ in the above equation in order to compare with previous results [3]. This assumption will be relaxed later. For convenience, let us take $B$ to be along $+z$ direction, while the photon momentum $q$ is on the $XZ$ plane making an angle $\theta$ to $B$. Hence $q$ can be parameterized as $q^\mu = (\omega, \omega \sin \theta, 0, \omega \cos \theta)$. We note that the integral $I_n$ may be evaluated in two different ways. One computes $I_n$ either by the residue theorem or by a direct integration along the contour $C$ with the realization that the contribution from the outer circle vanishes. The equivalence of two integration procedures gives rise to the relation:

$$\frac{1}{n!} \left( \frac{d^n}{d(\omega^2)^n} \Pi_{\parallel,\perp} \right) \bigg|_{\omega^2 = -\omega_0^2} = \frac{1}{\pi} \int_{M_{\parallel,\perp}^2}^\infty d\omega^2 \frac{\text{Im} \Pi_{\parallel,\perp}(\omega^2)}{(\omega^2 + \omega_0^2)^{n+1}},$$

where $M_{\parallel,\perp}^2$ are the threshold energies for pair productions, given by

$$M_\parallel^2 \sin^2 \theta = 4m_e^2, \quad M_\perp^2 \sin^2 \theta = m_e^2 \left( 1 + \sqrt{1 + 2\frac{B}{B_c}} \right)^2.$$  

(5)

Since the absorption coefficient (pair production width) is given by $\kappa_{\parallel,\perp} = \text{Im} \Pi_{\parallel,\perp}/\omega$, Eq. (4) relates the real part of photon polarization function to the absorption coefficient. One can further simplify Eq. (4) by taking $\omega_0^2 = 0$ such that

$$\frac{1}{n!} \left( \frac{d^n}{d(\omega^2)^n} \Pi_{\parallel,\perp} \right) \bigg|_{\omega^2 = 0} = \frac{M_{\parallel,\perp}^{1-2n}}{\pi} \int_0^1 dy_{\parallel,\perp} \cdot y_{\parallel,\perp}^{-1} \cdot (\kappa_{\parallel,\perp} y_{\parallel,\perp}^{-1/2})^n,$$

(6)

with $y_{\parallel,\perp} = M_{\parallel,\perp}^2 / \omega^2$. One notes that the absorptive part of $\Pi_{\parallel,\perp}(\omega^2)$ vanishes for the range $0 \leq \omega^2 \leq M_{\parallel,\perp}^2$ [11]. Therefore one can effectively set the integration range of Eq. (3) as from $y_{\parallel,\perp} = 0$ to $y_{\parallel,\perp} = \infty$. As mentioned before, the above sum rule is derived by assuming $q^2 = 0$. For a general $q^2$, the sum rule becomes [12]:

$$\frac{1}{n!} \left( \frac{d^n}{d(q^2)^n} \Pi_{\parallel,\perp} \right) \bigg|_{q^2 = 0} = \frac{m_{\parallel,\perp}^{1-2n}}{\pi} \int_0^\infty du_{\parallel,\perp} \cdot u_{\parallel,\perp}^{-1} \cdot (\omega \kappa_{\parallel,\perp}),$$

(7)

where $m_\parallel^2 = 4m_e^2$, $m_\perp^2 = m_e^2(1 + \sqrt{1 + 2B/B_c})^2$, $q_\parallel^2 = \omega^2 - q_\perp^2$, and $u_{\parallel,\perp} = m_{\parallel,\perp}^2 / q_\parallel^2$. Both of Eqs. (3) and (7) indicate that the moments of absorption coefficients are proportional to the derivatives of photon polarization functions at
the zero value of $q^2_\parallel \equiv \omega^2 - q^2_z$. Hence the absorption coefficients can be calculated from the derivatives of photon polarization functions at the zero value of $q^2_\parallel$ by the inverse Mellin transform.

**FIG. 1.** The integration contour for $I_n$ and the analytic structure of $\Pi_{\parallel,\perp}$. In actual calculations, we take the radius of the circle to infinity.

### III. THE BEHAVIORS OF $\Pi_{\parallel,\perp}$ IN THE ASYMPTOTIC LIMIT

To study the behaviors of $\Pi_{\parallel,\perp}$, let us begin with the proper-time representation of photon polarization function $\Pi_{\mu\nu}$ in a uniform background magnetic field [13]:

$$
\Pi_{\mu\nu}(q) = -\frac{e^3 B}{(4\pi)^2} \int_0^\infty ds \int_{-1}^{+1} dv \{ e^{-i s \phi_0} [(q^2_\perp g_{\mu\nu} - q_\mu q_\nu)N_0 \\
- (q^2_\parallel g_{\mu\nu} - q_\mu q_\nu)N_{\parallel} + (q^2_\perp g_{\perp\mu\nu} - q_{\perp\mu} q_{\perp\nu})N_{\perp}] \\
- e^{-i s m_e^2 v^2} (1 - v^2)(q^2 g_{\mu\nu} - q_\mu q_\nu) \},
$$

(8)

where the photon momentum has been decomposed into $q^\mu_\parallel \equiv (\omega, 0, 0, q_z)$ and $q^\mu_\perp \equiv (0, q_x, q_y, 0)$. The phase $\phi_0$ and the functions $N_0$, $N_{\parallel}$, and $N_{\perp}$ are given by

$$
\phi_0 = m_e^2 \frac{1 - v^2}{4} q^2_\parallel - \frac{\cos(zv) - \cos(z) q^2_\perp}{2z \sin(z)} q^2_\perp
$$

(9)

with $z = eBs$, and

$$
N_0 = \frac{\cos(zv) - v \cot(z) \sin(zv)}{\sin(z)},
$$

$$
N_{\parallel} = -\cot(z) \left( 1 - v^2 + \frac{v \sin(zv)}{\sin(z)} \right) + \frac{\cos(zv)}{\sin(z)},
$$

$$
N_{\perp} = -\frac{\cos(zv)}{\sin(z)} + \frac{v \cot(z) \sin(zv)}{\sin(z)} + 2 \frac{\cos(zv) - \cos(z)}{\sin^3(z)}.
$$

(10)
In the limit \( B \ll B_c \) with \( q^2 = 0 \), we write the derivative of \( \Pi_{||,\perp} \) into an asymptotic series in \( B/B_c \):

\[
\frac{1}{n!} \left( \frac{d^n}{d\omega^2} \Pi_{||,\perp} \right)_{\omega^2=0} = \frac{2\alpha e^2}{\pi} \left( \frac{B^2 \sin^2 \theta}{3B_c^2 m_e^2} \right)^n \frac{\Gamma(3n-1)\Gamma^2(2n)}{\Gamma(n)\Gamma(4n)} \times \left( \frac{6n+1, 3n+1}{4n+1} \right) + \cdots,
\]

(11)

where the disregarded terms are of higher order in \( B/B_c \). From the sum rule in Eq. (9), we obtain the absorption coefficients \( \kappa_{||,\perp} \) via the inverse Mellin transform \( [10] \):

\[
\kappa_{||}(\lambda') = \frac{ams_e^2}{2\pi i} \int_{-\infty}^{1+\infty+a} ds \Gamma(3s)\Gamma^2(2s) \frac{1}{\Gamma(s)\Gamma(4s)} \frac{1}{3s-1} \times \frac{6s+1}{4s+1} \]
\[
\kappa_{\perp}(\lambda'') = \frac{ams_e^2}{2\pi i} \int_{-\infty}^{1+\infty+a} ds \Gamma(3s)\Gamma^2(2s) \frac{1}{\Gamma(s)\Gamma(4s)} \frac{1}{3s-1} \times \frac{3s+1}{4s+1},
\]

(12)

where \( a \) is any real number greater than 1/3; while \( \lambda' = (\omega \sin \theta B/\sqrt{3}m_e B_c) \) and \( \lambda'' = \lambda' \cdot (1 + \sqrt{1 + 2B/B_c})/2 \). These absorption coefficients were shown \( [10] \) to agree with previous results by Tsai and Erber \( [6] \) who obtained \( \kappa_{||,\perp} \) to agree with previous results of Eq. (11). As it is easily

\[
T_{||,\perp}(\lambda) = \frac{4\sqrt{3}}{\pi a} \int_0^1 dv \cos(v - v^2)^{-1} \left[ (1 - \frac{1}{3}v^2), \frac{1}{2} + \frac{1}{6}v^2 \right] K_{2/3} \left( \frac{4}{\lambda} \frac{1}{1 - v^2} \right),
\]

(13)

with \( K_{2/3} \) the modified Bessel function.

It is important to point out that both of \( \kappa_{||} \) and \( \kappa_{\perp} \) are smooth functions of the photon energy. This behavior actually does not describe the real physical situation where new absorption peaks keep on emerging as the photon energy increases. One attributes this problem to the asymptotic expansion performed in Eq. (11). As it is easily

\[
\Pi_{||,\perp}(\omega, q^2, \theta) = -\kappa_{||} q_{||} / 4\pi_2, \quad \Pi_{||,\perp}(\omega, q^2, \theta) = -\kappa_{\perp} q_{\perp} / 4\pi_2 \Pi_{||,\perp}, \quad \text{with}
\]

\[
\Pi_{||} = \int_0^\infty dz \int_{-1}^{1+1} dv \exp[-is\phi_0]N_{||},
\]
\[
\Pi_{\perp} = \int_0^\infty dz \int_{-1}^{1+1} dv \exp[-is\phi_0]N_{\perp},
\]

(14)

where \( q_{||,\perp}^2 = q_{||,\perp}^2 + q_{||,\perp}^2 \) and \( \theta \) is the angle between the photon momentum \( q \) and the magnetic field vector \( B \). The calculations of \( \Pi_{||} \) and \( \Pi_{\perp} \) are non-trivial. For illustrations, we will only show the details of computing one particular term in \( \Pi_{||} \). Let us consider the following integral

\[
\Pi_{||}^4 = \int_0^\infty dz \int_{-1}^{1+1} dv \exp[-is\phi_0] \frac{\cos(zv)}{\sin(z)},
\]

(15)

where \( \cos(zv)/\sin(z) \) is the last term in \( N_{||} \). To apply the sum rule given by Eq. (10), we rotate the integration contour \( s \to -is \) and write \( \Pi_{||}^4 \) in an infinite series of \( q_{||}^2 \), i.e.,
\[ \Pi_{\|}^A(q_\|^2, q_\perp^2) = \sum K_{\text{imp}}^A(q_\perp^2) \left( \frac{1}{B'} \right)^n \left( \frac{q_\perp^2}{n-l+1} \right)^{n-l} \int_0^\infty dz \ z^{n-l} \exp[-z\beta] \times \int_0^1 dv (1 - v^2)^{n-l} \left( \exp[\alpha z v] + \exp[-\alpha z v] \right), \] (16)

with \( \sum = \sum_{n=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \sum_{p',p=0}^l \), \( q_\|^2 = q_\perp^2/4m_e \), \( B' = B/B_c \), \( q_\perp^2 = q_\perp^2/4m_e \), \( \alpha = p' - p + 1 \), \( \beta = p + p' + 2m + 1 + 1/B' \), and

\[ K_{\text{imp}}^A(q_\perp^2) = \frac{2(-1)^{l+p+p'} (2q_\perp^2)^l \Gamma(l + m + 1)}{(l-p)!p!(l-p')!\Gamma(m+1)}. \] (17)

Note that the various indices in the summation arise as follows: the index \( n \) comes from the photon-energy expansion, \( l \) arises from the binomial expansion of \( ((1 - v^2)\omega^2 + 2(\cosh(zv) - \cosh(z))q_\perp^2/z \sinh(z))^n \), \( p \) and \( p' \) arises from writing \( (\cosh(zv) - \cosh(z))^l \) as a sum of exponential functions, and \( m \) is due to expansions such as \( \sinh^{-l} z = 2^l \exp[-lz] \sum_{m=0}^\infty C_{m+m-1} \exp[-2mz] \). After performing the integrals in Eq. (16) and replacing \( n - l + 1 \) by \( n \), we arrive at

\[ \Pi_{\|}^A = -\frac{\alpha \omega^2 \sin^2 \theta \Pi_{\|}}{4\pi} = \frac{\alpha \omega^2 \sin^2 \theta}{\sqrt{n^2}} \sum_{l=0}^{\infty} \sum_{m=0}^{n-l} \sum_{p',p=0}^{l} K_{\text{imp}}^A(q_\perp^2) (B')^{1-l} \left( \frac{q_\perp^2}{B'} \right)^n \times \frac{\Gamma(n)}{\Gamma(n + \frac{1}{2})} 2F_1 (n, n + 1, \frac{1}{2}; n + 1, \frac{1}{2}, \beta^2). \] (18)

It is easy to calculate the derivatives of \( \Pi_{\|}^A \). Let us define \( F_{\|}^A(n, q_\perp^2, B) = m_\| \frac{1}{n!} \frac{\partial^n}{\partial q_\perp^{2n}} \Pi_{\|}^A |_{q_\perp^2=0} \). We have

\[ \kappa_{\|} = \frac{1}{2i\omega} \int_C ds F_{\|}^A(s, q_\perp^2, B) u_{\parallel}^{-s}, \] (19)

by inverting the sum rule in Eq. (10). The above integral transform is evaluated using

\[ \frac{1}{2\pi i} \int_C ds x^{-s} \frac{\Gamma(s)}{\Gamma(s + \frac{1}{2})} 2F_1 \left( s, \frac{s + 1}{2}; s + 1, \frac{1}{2}; z^2 \right) = \frac{1}{\sqrt{\pi}} \frac{\Theta(1 - x + \frac{x^2}{4})}{\sqrt{1 - x + \frac{x^2}{4}}}. \] (20)

We arrive at

\[ \kappa_{\|} = -\frac{\alpha e B' \omega^2 \sin^2 \theta}{4q_\perp^2} \sum_{n} K_{\text{imp}}^A(q_\perp^2) \Theta \left( 1 - \frac{\beta B'}{q_\perp^2} + \frac{\alpha B'}{2q_\perp^2} \right) \frac{B^d}{\sqrt{\left( 1 - \frac{\beta B'}{q_\perp^2} + \frac{\alpha B'}{2q_\perp^2} \right)^2}}, \] (21)

where \( \sum'' = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p',p=0}^{l} \). To understand the structure of \( \kappa_{\|} \), we rewrite the denominator of the above equation as

\[ \sqrt{(1 - \frac{\beta B'}{q_\perp^2}) + (\frac{\alpha B'}{2q_\perp^2})^2} = \frac{1}{q_\perp^2} \sqrt{q_\perp^4 - \left( \frac{1}{B'} + l_1 + l_2 \right) B' q_\perp^2 + \frac{1}{4} (l_1 - l_2)^2 B'^2}, \] (22)

with \( l_1 = (\beta + \alpha - 1/B')/2 \) and \( l_2 = (\beta - \alpha - 1/B')/2 \). It is easy to verify that this denominator has a root at \( q_\perp^2 = m_e^2 (\sqrt{1 + 2(1/B')^2 + \sqrt{1 + 2(1/B')^2} - 1}/2 B'^2 \), which is precisely the threshold value for \( q_\perp^2 \) to produce an electron-positron pair at the Landau levels \( l_1 \) and \( l_2 \) respectively. This threshold behavior is also indicated by the step function in the numerator.

Having worked out the \( q_\perp^2 \) dependence of \( \kappa_{\|} \), we proceed to perform the summation in \( \Sigma'' \). We first perform the summation over the index \( l \) in \( K_{\text{imp}}^A(q_\perp^2) \):

\[ \sum_{l=0}^{\infty} \sum_{p',p=0}^{l} \frac{x^l \Gamma(l + m + 1)}{(l-p)!/(l-p')} = \sum_{p',p=0}^{\infty} x^p e^{\gamma} \Gamma(p + m + 1) L_{p+m}^{p+m}(-x), \] (23)
where \( x = -2q_{\perp}^2/B' \equiv q_{\perp}^2/(2\epsilon B) \), \( \hat{p} = \max\{p, p'\} \), \( \hat{\rho} = \min\{p, p'\} \), and \( L_{\rho+p}^{\rho-p} \) is the Laguerre polynomial. We then rewrite the left-over summation \( \sum_{p, p'=0}^{\infty} \sum_{m=0}^{\infty} \) as \( \sum_{l_1=1}^{\infty} \sum_{l_2=0}^{\infty} \sum_{m=0}^{\infty} \) with \( l = \min\{l_1 - 1, l_2\} \) using the relation \( l_1 = (\beta + \alpha - 1/B')/2 = p' + m + 1 \) and \( l_2 = (\beta - \alpha - 1/B')/2 = p + m \). The summation over the index \( m \) can now be performed as follows:

\[
\sum_{m=0}^{l} x^{-m} \frac{1}{\Gamma(l - m + 1)} \frac{1}{\Gamma(l + m + 1)} = \frac{x^{-\hat{l}}}{\Gamma(\hat{l} + 1)} L_{\hat{l}}^{\hat{l}}(-x),
\]

(24)

where \( \hat{l} = \max\{l_1 - 1, l_2\} \). With the above summations over \( l \) and \( m \), we arrive at

\[
\kappa_{||}^{A} = \frac{\alpha_{\epsilon}B'}{2q_{||}^2} \omega \sin^2 \theta \sum_{l_1=1}^{\infty} \sum_{l_2=0}^{\infty} T_{l_1 l_2}^{A}(x) \frac{\Theta \left( 1 - \frac{\beta B'}{q_{||}^2} \right) + \left( \frac{\alpha B'}{2q_{||}^2} \right)^2}{\sqrt{1 - \left( \frac{\beta B'}{q_{||}^2} \right)^2} + \left( \frac{\alpha B'}{2q_{||}^2} \right)^2},
\]

(25)

with

\[
T_{l_1 l_2}^{A}(x) = (-1)^{l_1 + r_A} e^x e^{x^2} \frac{\Gamma(\lambda_A + 1)}{\Gamma(\lambda_A + r_A + 1)} L_{\lambda_A}^{r_A}(-x) L_{\lambda_A}^{r_A}(x),
\]

(26)

where \( r_A \equiv \hat{l} - \hat{l} = |l_1 - l_2 - 1| \), and \( \lambda_A \equiv \hat{l} = (l_1 + l_2 - |l_1 - l_2 - 1| - 1)/2 \).

Apart from the factor \( \omega \sin^2 \theta \), the absorption coefficient \( \kappa_{||}^{A} \) is a function of \( x = -q_{\perp}^2/(2\epsilon B) \) and \( q_{||}^2 \) where the latter determines the threshold behaviors of the absorption coefficient. Now that we have obtained the absorptive part of the polarization function \( \Pi_{||}^{A} \), the dispersive part can be calculated using the Kramers-Kronig relation. It is convenient to recall the relation \( \Pi_{||}^{A} = -(\alpha_{\epsilon} \omega^2 \sin^2 \theta/4\pi) \Pi_{||}^{A}(q_{\perp}^2, q_{||}^2) \). Hence

\[
\text{Re}\Pi_{||}^{A}(q_{\perp}^2, q_{||}^2) = \frac{P}{\pi} \int_{1}^{\infty} dt \frac{\text{Im}\Pi_{||}^{A}(t, q_{||}^2)}{t - q_{||}^2},
\]

(27)

where \( P \) stands for evaluating the principal part of the integral. We point out that the refractive index for the polarization state, \( \epsilon_{||}^{A} \), can be calculated through the equation \( q^2 + \text{Re}\Pi_{||} = 0 \). Hence

\[
1 - n_{||}^2(q_{||}^2, q_{\perp}^2, \theta) = \frac{\alpha_{\epsilon} \omega^2 \sin^2 \theta}{4\pi^2} P \int_{1}^{\infty} dt \frac{\text{Im}\Pi_{||}(t, q_{||}^2)}{t - q_{||}^2}.
\]

(28)

It is important to note that the index of refraction \( n_{||} \) also appears on the R.H.S. of the above equation, through relations among the quantities, \( q_{\perp}^2, q_{||}^2 \), and the angle \( \theta \). Thus the calculation of \( n_{||} \) is in principle nontrivial even if the polarization function \( \text{Re}\Pi_{||} \) is known. Let us consider a simplified case where the magnetic-field strength is super-critical, i.e., \( B >> B_c \). In this case, the quantity \( 1 - n_{||}^2 \) to the leading order in \( B_c/B \) is given by the lowest Landau-level contribution to the dispersion integral in Eq. (28). Hence, we arrive at \( [12] \)

\[
n_{||}^2(q_{||}^2, q_{\perp}^2, \theta) = 1 - \frac{\alpha_{\epsilon} B'/B}{2\pi q_{||}^2} \left( \frac{1}{q_{||}^2(1 - q_{||}^2)} \arctan \frac{q_{\perp}^2}{1 - q_{||}^2} - 1 \right) \exp\left(-\frac{2q_{\perp}^2}{B'}\right).
\]

(29)

It is instructive to rewrite \( \exp(-2q_{||}^2/B') \) as \( \exp(-2\omega^2 \sin^2 \theta \Theta n_{||}^2/B') \) with \( \omega' = \omega/2m_{c} \). Clearly, for \( q_{||}^2 \) not very close to \( 4m_{c}^2 \) (the threshold energy for the pair production), one can safely set \( \exp(-2q_{||}^2/B') = 1 \). The index of refraction \( n_{||}^2 \) with this approximation then agrees with the previous result \([12] \). However, as \( q_{||}^2 \) is so close to 1 that \( \alpha_{\epsilon}/\sqrt{1 - q_{||}^2} \) becomes greater than unity, one can no longer neglects the exponential factor \( \exp(-2q_{||}^2/B') \). In fact,

\[1\]The absorption coefficient is not written in a symmetrized form for saving the space. Nevertheless, the symmetrization with respect to \( l_1 \) and \( l_2 \) can be easily done as one wishes.
in the limit $q^2 \to 1$, one finds that $q^2 \to \infty$, as pointed in the previous literature \[1\]. We note that the behaviors of another polarization state, $q^2\perp\to \infty$, can be studied in the similar manner. The details are given in Ref. [2].

For consistency, it is desirable to compare our absorption coefficients with those obtained by squaring the $\gamma \to e^+e^-$ amplitude directly\[2]. Our results reduce to that of Daugherty and Harding in the special limit $q^2 = 0$, which is an assumption made by these authors in their calculations. Our finding is similar to what Shabad has demonstrated in Ref. [3] as he compared his result with that of Klepikov \[4\] in the same limit for $q^2 = 0$. Our approach differs from that of Refs. [4,5] in that Shabad performs the calculation in the beyond-threshold energy where the algebraic manipulations are rather involved and precautions are required, whereas we take the advantage of inverse Mellin transform which permits us to calculate the polarization function near the zero longitudinal momentum with a convenient momentum expansion.

V. CONCLUSION

In conclusion, we have developed an integral-transform technique to compute the absorptive part of photon polarization function in a background magnetic field, while the dispersive part can be obtained via the Kramers-Kronig relation. We emphasize that this technique is rather powerful for the current problem in which there are infinite numbers of singularity occurring in the real positive axis of $q^2$, each corresponds to an $e^+ - e^-$ system in a specific combination of Landau levels. This technique has many other applications. To name a few, one may analyze the photon polarization function in a general background electromagnetic field, or study other current-current correlation functions under the same external condition. To our knowledge, the vector-axial vector correlation function relevant to weak interaction processes has not been analyzed as detailed as the fashion presented in this work. We shall report the results of such analysis as well as other relevant studies in a forthcoming publication \[16\].

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\[2\] We make comparisons with the most updated results in Ref. [5].