Remarks on local regularity of axisymmetric solutions to the 3D Navier–Stokes equations

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ABSTRACT
In this article, a new local regularity criterion for the axisymmetric solutions to the 3D Navier–Stokes equations is investigated. It is slightly supercritical and implies an upper bound for the oscillation of $C_{\Gamma}: \text{ for any } 0 < \tau < 1$, there exists a constant $c > 0$, $|\Gamma( r, x_3, t) | \leq N e^{-c\ln r}$, $0 < r \leq \frac{1}{4}$.

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1. Introduction

In this article, we discuss potential singularities of axisymmetric solutions to the incompressible 3D Navier–Stokes equations. Roughly speaking, we would like to show that if slightly supercritical quantities of an axisymmetric solution are bounded, then such a solution is smooth. In order to present it precisely, let us first recall the basic notions from the mathematical theory of the Navier–Stokes equations.

In Cartesian coordinates, the incompressible 3D Navier–Stokes equations are given by:

$$
\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla \Pi = 0, \quad \nabla \cdot u = 0,
$$

here $u(x, t)$ and $\Pi(x, t)$ denote the fluid velocity field and the pressure, respectively. A global weak solution with finite energy was constructed by Leray [1] and Hopf [2]. However, the uniqueness and regularity of such weak solution is still one of the most challenging open problems in the field of mathematical fluid mechanics. One essential work is usually referred as Ladyzhenskaya–Prodi–Serrin conditions (see [3–7] and the references therein), that is, if the weak solution $u$ satisfies

$$
u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty,
$$
then the weak solution is regular in \((0, T]\). Its endpoint \(q = 3\) is proved in the celebrated article [3] of Escauriaza, Seregin, and Šverák. In a remarkable recent development [8], Tao used a new approach to provide the explicit quantitative estimates for solutions of the Navier-Stokes equations belonging to the critical space \(L^\infty(0, T; L^3(\mathbb{R}^3))\). As a consequence of these quantitative estimates, Tao showed that if the solution \(u\) first blows up at \(T_* > 0\), then for some absolute constant \(\alpha > 0\),

\[
\limsup_{t \to T_*} \frac{\|u(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{(\ln \ln \frac{1}{T_* - t})^\alpha} = \infty.
\]  

(1.3)

Barker and Prange [9] and Palasek [10] made progress on removing some logarithms from the blow-up rate.

A particular class of weak solutions to (1.1) called suitable weak solutions is introduced by Caffarelli, Kohn and Nirenberg in their celebrated article [11]. A simple proof is also given by Lin in ref. [12]; see Ladyzhenskaya and Seregin [13] for revision.

**Definition 1.1.** We say that \((u, \Pi)\) is a suitable weak solution of (1.1) in an open domain \(\Omega_T = \Omega \times (-T, 0)\), \(T > 0\), if

1. \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))\) and \(\Pi \in L^3(\Omega_T)\);
2. is satisfied in the sense of distributions;
3. the local energy inequality holds: for any nonnegative test function \(\phi \in C_c^\infty(\Omega_T)\) and \(t \in (-T, 0)\),

\[
\int_\Omega |u(x, t)|^2 \phi \, dx + 2 \int_{-T}^t \int_\Omega |\nabla u|^2 \phi \, dx \, ds \\
\leq \int_{-T}^t \int_\Omega |u|^2 (\partial_t \phi + \Delta \phi) + u \cdot \nabla \phi (|u|^2 + 2\Pi) \, dx \, ds.
\]  

(1.4)

We introduce the conventional notations in local regularity theory for suitable weak solutions. Let \(B(x_0, R)\) be the ball in \(\mathbb{R}^3\) with center at \(x_0\) and radius \(R\); \(Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)\) with \(z_0 = (x_0, t_0)\); and \(L^p,q(Q(z_0, R)) = L^q(t_0 - R^2, t_0; L^p(B(x_0, R)))\). For a given solution \((u, \Pi)\), let

\[
A(z_0, R) = \sup_{t_0 - R^2 < t < t_0} \frac{1}{R} \int_{B(x_0, R)} |u(x, t)|^2 \, dx, \quad E(z_0, R) = \frac{1}{R} \int_{Q(z_0, R)} |\nabla u|^2 \, dx \, dt,
\]

\[
C(z_0, R) = R^{-2} \int_{Q(z_0, R)} |u|^3 \, dx \, dt, \quad D(z_0, R) = R^{-2} \int_{Q(z_0, R)} |\Pi|^2 \, dx \, dt,
\]

\[
\mathcal{E}(z_0, R) = A(z_0, R) + E(z_0, R) + D(z_0, R).
\]

(1.5)

Gustafson, Kang and Tsai [14] proved general \(\varepsilon\)-regularity criteria for Navier–Stokes equations. They showed that a suitable weak solution \((u, \Pi)\) is regular at \(z_0\) if for some small \(\varepsilon > 0\),

\[
\limsup_{R \to 0} R^{1 - \frac{3}{q} - \frac{3}{p}} \|u\|_{L^p,q(Q(z_0, R))} \leq \varepsilon,
\]

(1.6)

where \(\frac{3}{p} + \frac{3}{q} \leq 2\).
In our standing assumption, it is supposed that a suitable weak solution \((u, \Pi)\) of Navier–Stokes equations (1.1) is axially symmetric with respect to the axis \(x_3\). It means that in the corresponding cylindrical coordinate system,

\[
u(x, t) = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^z(r, x_3, t)e_3, \quad \Pi(x, t) = \Pi(r, x_3, t)
\]

where \(x = (x_1, x_2, x_3), r = \sqrt{x_1^2 + x_2^2}\), and

\[
\begin{align*}
e_r &= \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), & e_\theta &= \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), & e_3 &= (0, 0, 1). 
\end{align*}
\]

The angular velocity \(u^\theta\) is usually called the swirl. For the axisymmetric solution \((u, \Pi)\), we can equivalently reformulate (1.1) as

\[
\begin{cases}
\partial_t u^r + (b \cdot \nabla) u^r - \left(\Delta - \frac{1}{r^2}\right) u^r - \frac{(u^\theta)^2}{r} + \partial_r \Pi = 0, \\
\partial_t u^\theta + (b \cdot \nabla) u^\theta - \left(\Delta - \frac{1}{r^2}\right) u^\theta + \frac{u^\theta u^r}{r} = 0, \\
\partial_t u^3 + (b \cdot \nabla) u^3 - \Delta u^3 + \partial_3 \Pi = 0,
\end{cases}
\]

where \(b = u^r e_r + u^z e_3\). Define the quantity \(\Gamma = ru^\theta\), which satisfies

\[
\partial_t \Gamma + (b \cdot \nabla) \Gamma - \left(\Delta - \frac{2}{r} \partial_r\right) \Gamma = 0.
\]

The global well-posedness was firstly investigated under no swirl assumption (i.e. \(u^\theta = 0\)), independently by Ladyzhenskaya [15] and Ukhovskii and Yudovich [16], see also Leonard, and so on, [17] for a refined proof. When the swirl \(u^\theta\) is not trivial, it is still open. There are many articles on regularity of axisymmetric solutions, please refer to [10, 18–28] and the references therein. For the convenience of readers, we list some related results here.

In regard to the regularity criteria only involving swirl component \(u^\theta\), one of the primary results is given by Chen, Fang and Zhang [20]: the solution \(u\) is smooth in \((0, T)\), provided that

\[
r^d u^\theta \in L^q((0, T); L^p(\mathbb{R}^3)),
\]

where \(\frac{2}{q} + \frac{3}{p} \leq 1 - d, \ 0 \leq d < 1, \ \frac{3}{2 - d} < p \leq \infty, \ \frac{2}{1 - d} \leq q \leq \infty\). Lei and Zhang [24] obtained the regularity of the solution under the condition

\[
r^d |u^\theta| \leq N |\ln r|^{-2}, \quad 0 < r \leq \frac{1}{2},
\]

here \(N\) is the general constant throughout this article. Wei [28] improved it to

\[
r^d |u^\theta| \leq N \ln r^{-\frac{1}{2}}, \quad 0 < r \leq \frac{1}{2},
\]

Chen, Strain, Yau and Tsai [18, 19] and Koch, Nadirashvili, Seregin and Sverak [22] proved that the suitable weak solution is smooth if the solution \(u\) satisfies

\[
|u| \leq Nr^{-1+\varepsilon} r^{-\frac{1}{2}},
\]

where \(0 \leq \varepsilon \leq 1\). Lei and Zhang [23] obtained a similar results if
\[ b \in L^\infty((0, T); BMO^{-1}(\mathbb{R}^3)). \]  

(1.14)

A local regularity condition for axisymmetric solutions is proved by Seregin [26, Theorem 2.1], which says that suitable weak solution \( (u, \Pi) \) is regular at \( z_0 = (0, x_3, t) \) if

\[
\min \left\{ \limsup_{R \to 0} A(z_0, R), \limsup_{R \to 0} E(z_0, R), \limsup_{R \to 0} C(z_0, R) \right\} < \infty. 
\]

(1.15)

Compared with the \( \varepsilon \)-regularity (1.6), the one here in axisymmetric system does not need the smallness of those scale-invariant energy quantities. It reads that the axisymmetric solution have no Type I singularities.

Recently, Pan [25, Theorem 1.1] proved the regularity of the solution under a slightly supercritical assumption

\[
r|u| \leq N\left( \ln \ln \frac{100}{r} \right)^{0.028}, \quad 0 < r \leq \frac{1}{2}. 
\]

(1.16)

Seregin [27] obtained another type of supercritical regularity criterion:

\[
R^{-\frac{1}{2}}\|u\|_{L^3, r(Q(z_0, R))} + R^{-\frac{1}{2}}\|u\|_{L^p, r(Q(z_0, R))} \leq N\left( \ln \ln \frac{100}{R} \right)^{\frac{1}{32}}, 
\]

(1.17)

for any \( 0 < R < 1 \) and \( z_0 = (0, x_0, 3, t_0) \in \mathbb{R}^3 \times (0, T) \). The key ingredient is that under the assumption (1.16) or (1.17), the bound for oscillation of \( \Gamma(x, t) \) contained a logarithmic factor.

Using the strategy of quantitative estimates introduced by Tao [8], Palasek [10, Theorem 2] showed that if the axisymmetric solution \( u \) first blows-up at \( T_\ast > 0 \) and \( 2 < p < \infty \), then for a constant \( \delta > 0 \) depending only on \( p \),

\[
\limsup_{t \to T_\ast} \left( \frac{r^{1-\frac{1}{2}}\|u\|_{L^p(\mathbb{R}^3)}}{\ln \ln \frac{100}{T_\ast-t}} \right)^{\frac{1}{2}} = \infty. 
\]

(1.18)

Let \( Q(r) = Q(0, r) \) for \( 0 \in \mathbb{R}^4 \) and \( r > 0 \). Let the weight

\[
\omega(R) = \left( \ln \ln \frac{100}{R} \right)^{-1}. 
\]

In this article, we obtain the following key proposition.

**Proposition 1.2.** Assume that \( (u, \Pi) \) is an axisymmetric suitable weak solution to the Navier–Stokes equations in \( Q(1) \) and there exist constants \( \beta \in (0, \frac{1}{8}) \) and \( K > 0 \) such that

\[
A(z_0, R)\omega(R)^\beta \leq K, 
\]

(1.19)

for all \( 0 < R \leq \frac{1}{4} \), for some \( z_0 = (0, x_0, 3, t_0) \in Q(\frac{1}{8}) \). Then for any \( 0 < \tau < 1 \), there exists a constant \( c = c(K, \beta, \tau) \) such that

\[
\text{OSC}_{(x, t) \in Q(z_0, \rho)} \Gamma(x, t) \leq e^{-c\left( \frac{1}{\ln \ln \frac{100}{\rho}} \right)} \cdot \text{OSC}_{(x, t) \in Q(z_0, R)} \Gamma(x, t), 
\]

(1.20)

for \( 0 < \rho < R \leq \frac{1}{4} \). Here the oscillation \( \text{OSC}_{(x, t) \in Q(z_0, R)} \Gamma(x, t) = \sup_{(x, t) \in Q(z_0, R)} \Gamma(x, t) - \inf_{(x, t) \in Q(z_0, R)} \Gamma(x, t) \).

If in addition, assuming that (1.19) holds for all \( z_0 = (0, x_0, 3, t_0) \in Q(\frac{1}{8}) \), then we have that for \( 0 < r \leq \frac{1}{4}, |x_3| < \frac{1}{8}, -\frac{1}{64} < t < 0, \)
\[ |\Gamma(r, x_3, t)| \leq N e^{-c \ln r}. \] (1.21)

It should be pointed out that decay in (1.21) is an improvement of the result in ref. [25, Theorem 1.2].

**Remark.** When this article was near completion, Professor Seregin posted a similar result [29] to arXiv. His assumption is still (1.17) and his decay exponent \( \tau \) is 1/4. Our \( \tau \) is limited to \( \tau < 1 \), weaker than the Hölder continuity case \( \tau = 1 \).

The following theorems are corollaries of Proposition 1.2.

Fix \( 1 < p, q \leq \infty \) with \( 3/p + 2/q = 2 - \gamma, 0 < \gamma < 1 \). Denote

\[ G(z_0, R) = R^{1/2 - 3/4} \|b\|_{L^q(Q(z_0, R))}, \quad G_z(z_0, R) = G(z_0, R) \omega(R)^a, \] (1.22)

where \( 0 < \alpha < \alpha_0 = \frac{7}{48 + 16\gamma} \).

**Theorem 1.3.** Assume that \((u, \Pi)\) is an axisymmetric suitable weak solution to the Navier–Stokes equations in \(Q(1)\). If there exist a positive constant \(G\) such that

\[ G_z(z_0, R) \leq G, \] (1.23)

for all \( z_0 = (0, x_0, 0) \in Q(1/8) \) and \( 0 < R \leq \frac{1}{4} \) then the solution is regular at \((0, 0)\).

The constant \( \alpha_0 = \frac{7}{48 + 16\gamma} \) is not optimal and may be improved if choosing \( p, q \) specifically. We leave it to the interested readers. The regularity criterion (1.23) would imply criteria (1.16), (1.17), and (1.18), if one could increase the exponent \( \alpha \).

More supercritical regularity criteria for the stream function \( \psi^0 \) with \( b = \nabla \times (\psi^0 e_0) \) or \( \omega^0 = \partial_3 u^\theta - \partial_r u^3 \) or one component \( u^3 \) can be obtained, if we make some minor modifications to Lemma 3.2, see [14, 20]. We also leave it to the interested readers. Here we give a regularity criterion for \( b \) in a weaker critical space \( \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3) \). To avoid unessential issues, we consider the problem of regularity in the whole space \( \mathbb{R}^3 \).

**Theorem 1.4.** Let \((u, \Pi)\) be a classical axisymmetric solution to Navier–Stokes equations in \(\mathbb{R}^3 \times (-1, 0)\) which blows up at time \(t = 0\). Then

\[ \limsup_{t \to 0} \frac{\|b(\cdot, t)\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}}{(\ln \ln \frac{100}{t})^a} = \infty, \] (1.24)

where \(a\) is any positive constant smaller than \(a\).

If \( \alpha = 0 \), we can deduce the following result, which covers the regularity criteria (1.13) and (1.15). Its proof is directly from Lemma 4.1 and regularity criterion (1.15).

**Theorem 1.5.** Assume that \((u, \Pi)\) is an axisymmetric suitable weak solution to the Navier–Stokes equations in \(Q(1)\). The solution is regular at \(z_0\) provided that

\[ \limsup_{R \to 0} G(z_0, R) < \infty. \] (1.25)

### 2. Proof of Proposition 1.2

In this section, we prove Proposition 1.2 by a De Giorgi–Nash–Moser type method. Without loss of generality, we assume \(z_0 = (0, 0)\).
Lemma 2.1 (local maximum estimate). For $0 < R < 1$,

$$
\sup_{Q(\frac{1}{2}R)} |\Gamma| \leq N \left( \frac{1 + A(R)}{R} \right) \frac{\|\Gamma\|_{L^2(Q(R))}}{\|u\|_{L^2(Q(R))}}. \tag{2.1}
$$

Proof. Let $\frac{1}{2} \leq \sigma_2 < \sigma_1 \leq 1$ be arbitrary constants and $\varepsilon > 0$ be a sufficient small constant. We introduce cutoff functions $\varphi(x, t) = \psi(|x|)\eta(t)$ satisfying

$$
\begin{align*}
\text{supp } \psi &\subset B(\sigma_1 R), \psi = 1 \text{ in } B(\sigma_2 R), 0 \leq \psi \leq 1, \\
\text{supp } \eta &\subset (-(\sigma_1 R)^2, 0], \eta = 1 \text{ in } (-(\sigma_2 R)^2, 0], 0 \leq \eta \leq 1, \\
|\eta'| + |\nabla^2 \eta| &\leq \frac{N}{(\sigma_1 - \sigma_2)^2 R^2}, \quad \left| \frac{\nabla \eta}{\sqrt{\psi}} \right| \leq \frac{N}{(\sigma_1 - \sigma_2) R},
\end{align*}
$$

and $\xi = \xi(r)$ satisfying $0 \leq \xi(r) \leq 1$, $\xi(r) = 0$ if $r \leq \varepsilon$, $\xi(r) = 1$ if $r \geq 2\varepsilon$ and $|\xi^{(k)}(r)| \leq \frac{N}{r^2}$ for $k = 1, 2$. 

Step 1. Lift of regularity. We denote the truncation $\Gamma_n = \max(\min(\Gamma, n), -n)$ with $n > 100$. Note that $\Gamma$ and $b$ are smooth in $\supp \varphi \cap \supp \xi$ when $t < 0$. Multiplying (1.9) by $2m|\Gamma_n|^{2m-2}\Gamma_n \varphi^2 \xi$ with $m \geq 1$ and integrating by parts, we have that

$$
\begin{align*}
\int_{B(1)} |\Gamma_n|^{2m} \varphi^2(t) \xi \, dx + \frac{4m - 2}{m} \int_{-1}^{t} \int_{B(1)} |\nabla |\Gamma_n|^{m}|^2 \varphi^2 \xi \, dx \, dt \\
= \int_{-1}^{t} \int_{B(1)} |\Gamma_n|^{2m} \left( \partial_t + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) (\varphi^2 \xi) \, dx \, dt \\
- 2m \int_{B(1)} (\Gamma - \Gamma_n)|\Gamma_n|^{2m-2}\Gamma_n(t) \varphi^2 \xi \, dx \\
+ 2m \int_{-1}^{t} \int_{B(1)} (\Gamma - \Gamma_n)|\Gamma_n|^{2m-2}\Gamma_n \left( \partial_t + \Delta + \frac{2}{r} \partial_r \right) (\varphi^2 \xi) \, dx \, dt \\
\leq \int_{-1}^{t} \int_{B(1)} |\Gamma_n|^{2m} \left( \partial_t + \Delta + \frac{2}{r} \partial_r \right) (\varphi^2 \xi) \, dx \, dt \\
+ 2m \int_{-1}^{t} \int_{B(1)} (\Gamma - \Gamma_n)|\Gamma_n|^{2m-2}\Gamma_n \left( \partial_t + \Delta + \frac{2}{r} \partial_r \right) (\varphi^2 \xi) \, dx \, dt.
\end{align*}
$$

By $|\Gamma_n| \leq |\Gamma| \leq 2\varepsilon|u|$ in $\supp \xi'(r)$ and $u, b \in L^\infty(Q(1))$, we can pass the limit $\varepsilon \to 0$ in the inequality (2.2) and obtain that
\[
\int_{B(1)} |\Gamma_n|^{2m}(t) \varphi^2 \, dx + 2 \int_{-1}^{t} \int_{B(1)} |\nabla |\Gamma_n|^{m}|^2 \varphi^2 \, dx \, ds \\
\leq \int_{-1}^{t} \int_{B(1)} |\Gamma_n|^{2m} \left( \partial_s + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) \varphi^2 \, dx \, ds \\
+ 2m \int_{-1}^{t} \int_{B(1)} (\Gamma - \Gamma_n) |\Gamma_n|^{2m-2} \Gamma_n \left( \partial_s + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) \varphi^2 \, dx \, dt.
\]

Therefore, using \( b \in L^\infty(Q(1)) \), we have

\[
\|\Gamma_n\|^{2m} L^{2m}(Q(\sigma R)) \leq \frac{Nm\delta(1)^\frac{1}{2}}{(\sigma_1 - \sigma_2)^2 R^2} \|\Gamma_n, \Gamma\|^{2m} L^{2m}(Q(\sigma R))
\]

for all \( n \) and the same estimate for \( \Gamma \). By an argument of iteration, we have \( \Gamma \in L^p(Q(R)) \) for any \( 1 \leq p < +\infty \) and \( 0 < R < 1 \).

Step 2. Moser’s iteration. Passing the limit \( n \rightarrow +\infty \) in (2.3), the last integral of (2.3) vanishes and we obtain

\[
\int_{B(1)} |\Gamma|^{2m}(t) \varphi^2 \, dx + 2 \int_{-1}^{t} \int_{B(1)} |\nabla |\Gamma|^m|^2 \varphi^2 \, dx \, ds \\
\leq \int_{-1}^{t} \int_{B(1)} |\Gamma|^{2m} \left( \partial_s + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) \varphi^2 \, dx \, ds \\
\leq \frac{N}{(\sigma_1 - \sigma_2)^2 R^2} \int_{-1}^{t} \int_{B(\sigma R)} |\Gamma|^{2m} \, dx \, ds \\
+ 2 \int_{-1}^{t} \int_{B(1)} (|\Gamma|^m)^\frac{1}{2} (|\Gamma|^m \varphi)^\frac{1}{2} \|b \cdot \nabla \varphi\| \, dx \, ds \\
\leq \frac{N}{(\sigma_1 - \sigma_2)^2 R^2} \int_{-1}^{t} \int_{B(\sigma R)} |\Gamma|^{2m} \, dx \, ds \\
+ \frac{N(RA(R))^\frac{1}{2}}{(\sigma_1 - \sigma_2)^2 R} \int_{-1}^{t} \|\Gamma|^m\|_{L^2(B(\sigma R))}\|\nabla (|\Gamma|^m \varphi)\|_{L^2(\mathbb{R}^3)} \, ds \\
\leq N \frac{(1 + A(R))^\frac{1}{2}}{(\sigma_1 - \sigma_2)^2 R^2} \int_{-1}^{t} \int_{B(\sigma R)} |\Gamma|^{2m} \, dx \, ds + \int_{-1}^{t} \int_{B(1)} |\nabla |\Gamma|^m|^2 \varphi^2 \, dx \, ds.
\]

Accordingly, absorbing the last term in (2.5) by its first line, and using Lemma A.1 on imbedding,

\[
\|\Gamma|^m\|_{L^2(Q(\sigma R))} \leq N \frac{(1 + A(R))}{(\sigma_1 - \sigma_2)^2 R} \|\Gamma|^m\|_{L^2(Q(\sigma R))}.
\]
Picking $\mu_j = \frac{1}{2} + 2^{-j-1}$ and $m_j = 2(\frac{3}{2})^j$ for integer $j \geq 0$, we have that

$$
\| \Gamma \|_{L^{m_j+1}(Q(\mu_j, R))} \leq \left( \frac{2^{2j+2}N(1 + A(R))}{R} \right)^{\frac{R}{(\mu_j + 1)}} \| \Gamma \|_{L^{m_j}(Q(\mu_j R))}.
$$

(2.7)

Thus, we have

$$
\| \Gamma \|_{L^{m_j}(Q(R))} \leq \| \Gamma \|_{L^{m_j}(Q(\mu_j R))} \leq N \left( \frac{(1 + A(R))}{R} \right)^{\frac{R}{(\mu_j + 1)}} \| \Gamma \|_{L^{m_0}(Q(\mu_j R))}.
$$

(2.8)

Note $\mu_0 = 1$ and $m_0 = 2$. Letting $j \to \infty$, we obtain (2.1).

For $0 < R \leq \frac{1}{4}$, we define

$$
m_R = \inf_{Q(R) \setminus S} \Gamma, \quad M_R = \sup_{Q(R) \setminus S} \Gamma, \quad J_R = M_R - m_R,
$$

(2.9)

and

$$
h(x, t) = \begin{cases} 
2(M_R - \Gamma), & \text{if } M_R > -m_R, \\
2(\Gamma - m_R), & \text{else}. 
\end{cases}
$$

(2.10)

Hence for $(x, t) \in Q(R) \setminus S$, $h(x, t)$ satisfies $0 \leq h(x, t) \leq 2$ and

$$
\partial_t h + (b \cdot \nabla)h - \left( \Delta - \frac{2}{r} \partial_r \right) h = 0.
$$

(2.11)

For $(0, x_3, t) \in Q(1) \setminus S$, $h(0, x_3, t)$ equals a constant $a \geq 1$ since $\Gamma(0, x_3, t) = 0$.

**Lemma 2.2** (initial lower bound). There exists a constant $0 < N_4 < 1$ such that

$$
R^{-5} \| h \|_{L^1(B(\frac{1}{4} R) \times (-R^2, -\frac{3}{8} R^2))} \geq N_4 (1 + A(R))^{-1},
$$

(2.12)

for $0 < R \leq \frac{1}{4}$.

**Proof.** We introduce the cutoff function $\tilde{\phi}(x, t) = \tilde{\psi}(|x|)\tilde{\eta}(t)$ satisfying

$$
\begin{align*}
\text{supp } \tilde{\psi} &\subset B \left( \frac{1}{2} R \right), \quad \tilde{\psi} = 1 \text{ in } B \left( \frac{1}{4} R \right), \quad 0 \leq \tilde{\psi} \leq 1, \\
\text{supp } \tilde{\eta} &\subset \left( -R^2, -\frac{1}{4} R^2 \right), \quad \tilde{\eta} = 1 \text{ in } \left( -\frac{7}{8} R^2, -\frac{3}{8} R^2 \right), \quad 0 \leq \tilde{\eta} \leq 1, \\
|\nabla \tilde{\psi}| &\leq \frac{N}{R}, \quad |\nabla^2 \tilde{\psi}| + |\tilde{\eta}'| \leq \frac{N}{R^2}.
\end{align*}
$$

Let $\tilde{\xi}(x)$ be as in the proof of Lemma 2.1. Multiplying (2.11) by $\tilde{\phi} \tilde{\xi}$ and integrating by parts, we have that

$$
0 = \int_{-R^2}^{\frac{1}{4} R^2} \int_{B(\frac{1}{4} R)} h(\partial_s + b \cdot \nabla)(\tilde{\phi} \tilde{\xi}) - \nabla h \cdot \nabla(\tilde{\phi} \tilde{\xi}) - \frac{2}{r} \partial_r h \cdot (\tilde{\phi} \tilde{\xi}) \ dx ds.
$$
Passing the limit \( \varepsilon \to 0 \) using \( h \in L^\infty \cap L^2 H^1 \), integrating by parts and using the fact that \( 0 \leq h \leq 2 \), we have
\[
0 = \int_{-\frac{1}{2} R^2}^{\frac{1}{2} R^2} \int_{B(\frac{1}{2} R)} h \left( \partial_s + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) \bar{\phi} \, dx ds + 4 \pi a \int_{-\frac{1}{2} R^2}^{\frac{1}{2} R^2} \int_{-\frac{1}{2} R}^{\frac{1}{2} R} \bar{\phi}(0, x_3, s) \, dx_3 ds.
\]
Thus,
\[
\pi R^3 \leq 4 \pi a \int_{-\frac{1}{2} R^2}^{\frac{1}{2} R^2} \int_{-\frac{1}{2} R}^{\frac{1}{2} R} \bar{\phi}(0, x_3, s) \, dx_3 ds
\]
\[
= - \int_{-\frac{1}{2} R^2}^{\frac{1}{2} R^2} \int_{B(\frac{1}{2} R)} h \left( \partial_s + \Delta + b \cdot \nabla + \frac{2}{r} \partial_r \right) \bar{\phi} \, dx ds
\]
\[
\leq NR^{-2} \| h \|_{L^1(B(\frac{1}{2} R) \times (-R^2, -\frac{1}{4} R^2))} + NR^{-1} \int_{-\frac{1}{2} R^2}^{\frac{1}{2} R^2} \| h \|_{L^2(B(\frac{1}{2} R))} \| b \|_{L^2(B(\frac{1}{2} R))} \, ds
\]
\[
\leq NR^2 \| h \|_{L^1(B(\frac{1}{2} R) \times (-R^2, -\frac{1}{4} R^2))} \left( 1 + A(R)^\frac{2}{3} \right).
\]
Hence (2.12).

Denote the cut-off function \( \zeta(x) = \kappa(|x|) \) satisfies
\[
\text{supp} \, \kappa \subset [0, 1], \quad \kappa = 1 \in \left[ 0, \frac{1}{2} \right], \quad -N \leq \kappa' \leq 0, \quad \int \zeta^2 \, dx = 1,
\]
and \( \zeta_R(x) = R^{-\frac{3}{2}} \zeta \left( \frac{x}{R} \right) \) with \( \int_{B(R)} \zeta_R^2 \, dx = 1. \)

**Lemma 2.3** (weak Harnack inequality).
\[
-\int_{R^3} \ln h(x, t) \cdot \zeta_R^2(x) \, dx \leq N(1 + A(R))^3,
\]
for \( -\frac{1}{2} R^2 \leq t < 0 \) with \( 0 < R \leq \frac{1}{4} \).

**Proof.** Denote \( h_\delta = h + \delta \) and \( H_\delta = -\ln \frac{h_\delta}{x} > 0 \) for \( 0 < \delta < 1 \) a small constant. It is easy to see that \( H_\delta(t) \) solves the equation
\[
\partial_t H_\delta + b \cdot \nabla H_\delta - \left( \Delta - \frac{2}{r} \partial_r \right) H_\delta + |\nabla H_\delta|^2 = 0.
\]
(2.15)

Let \( \bar{\zeta}(x) \) be as in the proof of Lemma 2.1. Multiplying the equation (2.15) with \( \zeta_R^2 \bar{\zeta} \) and integrating by parts, we have that for \( -R^2 \leq t_0 < t < 0, \)
\[
\int_{B(R)} H_\delta(t) \zeta_R^2 \bar{\zeta} \, dx + \int_{t_0}^t \int_{B(R)} |\nabla H_\delta|^2 \zeta_R^2 \bar{\zeta} \, dx ds
\]
\[
= \int_{B(R)} H_\delta(t_0) \zeta_R^2 \bar{\zeta} \, dx - \int_{t_0}^t \int_{B(R)} b \cdot \nabla H_\delta \cdot \left( \zeta_R^2 \bar{\zeta} \right) + \nabla H_\delta \cdot \nabla \left( \zeta_R^2 \bar{\zeta} \right) + \frac{2}{r} \partial_r H_\delta \cdot \left( \zeta_R^2 \bar{\zeta} \right) \, dx ds.
\]
Let \( H_\delta = \int_{B(R)} H_\delta \zeta_R^2 \, dx \in C(-R^2, 0) \). Passing \( \varepsilon \to 0 \), we have
\[
\int_{B(R)} H_\delta(t) \zeta^2_R \, dx + \int_{t_0}^t \int_{B(R)} |\nabla H_\delta|^2 \zeta^2_R \, dx \, ds
\]

\[
= \int_{B(R)} H_\delta(t_0) \zeta^2_R \, dx - \int_{t_0}^t \int_{B(R)} \mathbf{b} \cdot \nabla H_\delta \cdot \zeta^2_R + \nabla H_\delta \cdot \nabla \zeta^2_R + \frac{2}{r} \partial_r H_\delta \cdot \zeta^2_R \, dx \, ds.
\]

(2.16)

Since

\[
-\int_{B(R)} \mathbf{b} \cdot \nabla H_\delta \cdot \zeta^2_R + \nabla H_\delta \cdot \nabla \zeta^2_R \, dx \leq NR^{-2}(1 + A(R)) + \frac{1}{4} \int_{B(R)} |\nabla H_\delta|^2 \zeta^2_R \, dx,
\]

and by \( H_\delta |_{r=0} \leq \ln 3 \) and Lemma A.3 (weighted Poincaré inequality),

\[
\int_{B(R)} -\frac{2}{r} \partial_r H_\delta \cdot \zeta^2_R \, dx = \int_{B(R)} -\frac{2}{r} \partial_r (H_\delta - \bar{H}_\delta) \cdot \zeta^2_R \, dx
\]

\[
= 4\pi \int_{-R}^R (H_\delta - \bar{H}_\delta) \zeta^2_R |_{r=0} \, dx_3 + 2 \int_{B(R)} (H_\delta - \bar{H}_\delta) \frac{2}{r} \partial_r \zeta^2_R \, dx
\]

\[
\leq NR^{-2} - \bar{H}_\delta R^{-2} + NR^{-1} \left( \int_{B(R)} |H_\delta - \bar{H}_\delta|^2 \zeta^2_R \, dx \right)^{1/2}
\]

\leq NR^{-2} - \bar{H}_\delta R^{-2} + \frac{1}{4} \int_{B(R)} |\nabla H_\delta|^2 \zeta^2_R \, dx,
\]

we have that

\[
\bar{H}_\delta(t) \leq \bar{H}_\delta(t_0) + R^{-2} \int_{t_0}^t \left( N \left( 1 + A(R) \right) - \bar{H}_\delta \right) \, ds
\]

\[
- \frac{1}{2} \int_{t_0}^t \int_{B(R)} |\nabla H_\delta|^2 \zeta^2_R \, dx \, ds,
\]

(2.17)

which implies that

\[
\bar{H}_\delta(t) \leq \bar{H}_\delta(t_0) + N(1 + A(R)).
\]

(2.18)

Applying Lemma A.2 (Nash inequality) with \( f = \frac{h_\delta}{3} \), \( d\mu = \zeta^2_R \, dx \) and Lemma A.3 (weighted Poincaré inequality), one has

\[
\left| \ln \int_{B(R)} h_\delta \zeta^2_R \, dx + \bar{H}_\delta \right|^2 \left( \int_{B(R)} h_\delta \zeta^2_R \, dx \right)^2 \leq N \int_{B(R)} |H_\delta - \bar{H}_\delta|^2 \zeta^2_R \, dx
\]

\[
\leq NR^2 \int_{B(R)} |\nabla H_\delta|^2 \zeta^2_R \, dx.
\]

(2.19)

Thereby we obtain that
\[
\begin{align*}
\tilde{H}_\delta(t) & \leq \tilde{H}_\delta(t_0) + R^{-2} \int_{t_0}^t \left( N(1 + A(R)) - \tilde{H}_\delta \right) ds \\
& - N R^{-2} \int_{t_0}^t \ln \int_{B(R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx + \tilde{H}_\delta \left( \int_{B(R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx \right)^2 ds.
\end{align*}
\] (2.20)

Let \( \chi(s) \) be the characteristic function of the set
\[
W = \left\{ s \in \left( -R^2, -\frac{1}{4} R^2 \right) : \| h_\delta \|_{L^1(B(\frac{3}{2}R))} \geq N_4 R^3 (1 + A(R))^{-1} \right\},
\]
where \( N_4 \) is the constant in Lemma 2.2. We assert that \( |W| > \frac{N_4 (1 + A(R))^{-1} R^2}{8} \). In fact, if \( |W| \leq \frac{N_4 (1 + A(R))^{-1} R^2}{8} \), then
\[
\| h_\delta \|_{L^1(B(\frac{3}{2}R))} \leq \int_W \| h_\delta \|_{L^1(B(\frac{3}{2}R))} \, ds + \int_{(-R^2, -\frac{3}{2} R^2) \setminus W} \| h_\delta \|_{L^1(B(\frac{3}{2}R))} \, ds
\]
\[
\leq \frac{\pi}{6} R^3 |W| \cdot \| h_\delta \|_{L^\infty} + \frac{3 N_4 R^5 (1 + A(R))^{-1}}{4}
\]
\[
\leq \left( \frac{\pi}{16} + \frac{3}{4} \right) N_4 R^5 (1 + A(R))^{-1},
\]
which contradicts Lemma 2.2. Thus, one has by (2.20) that for \( -R^2 \leq t_0 < t < 0 \),
\[
\tilde{H}_\delta(t) \leq \tilde{H}_\delta(t_0) + R^{-2} \int_{t_0}^t \left( N_5 (1 + A(R)) - \tilde{H}_\delta \right) ds
\]
\[
- N_6 R^{-2} (1 + A(R))^{-2} \int_{t_0}^t \ln \int_{B(R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx + \tilde{H}_\delta \left( \int_{B(R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx \right)^2 \chi(s) \, ds.
\] (2.21)

We claim that for \( -\frac{1}{4} R^2 \leq t < 0 \),
\[
\tilde{H}_\delta(t) \leq N(1 + A(R))^3,
\] (2.22)
which implies (2.14) directly by passing \( \delta \to 0 \).

If for some \( t_0 \in [-R^2, -\frac{1}{4} R^2) \) we have
\[
\tilde{H}_\delta(t_0) \leq 2 \ln \frac{1 + A(R)}{N_4} + N_5 (1 + A(R)) + 100,
\]
then by (2.18), we have (2.22). Otherwise, we have
\[
\tilde{H}_\delta(s) > 2 \ln \frac{1 + A(R)}{N_4} + N_5 (1 + A(R)) + 100,
\]
for all \( s \in [-R^2, -\frac{1}{4} R^2) \). For \( s \in W \cap (-R^2, -\frac{1}{4} R^2) \),
\[
\ln \int_{B(R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx \geq \ln \int_{B(\frac{3}{2} R)} \frac{h_\delta}{3} \frac{r^2}{s_R} \, dx \geq \ln \left( \frac{N_4}{3 (1 + A(R))} \right) \geq -\frac{1}{2} \tilde{H}_\delta.
\]
Therefore, for \(-R^2 \leq t_0 < t \leq -\frac{1}{4} R^2\), (2.21) gives (noting its first integral is nonpositive)

\[
\hat{H}_\delta(t) \leq \hat{H}_\delta(t_0) - \frac{1}{4} N_6 R^{-2} (1 + A(R))^{-2} \int_{t_0}^t \hat{H}_\delta^2(s) ds.
\]

By comparison with the solution of \(g(t) = g(t_0) - C \int_{t_0}^t g^2(s) \, ds\), \(g(t)^{-1} = g(t_0)^{-1} + C \int_{t_0}^t g(s) \, ds\), we get

\[
\hat{H}_\delta\left(-\frac{1}{4} R^2\right) \leq \frac{4}{N_6 R^{-2} (1 + A(R))^{-2} \int_{-R^2}^{1/4} g^2(s) ds}
\]

\[
\leq N(1 + A(R))^3.
\]

By (2.18) with \(t_0 = -\frac{1}{4} R^2\), we have (2.22). \(\square\)

**Lemma 2.4** (strong Harnack inequality). Let \(0 < \beta < \frac{1}{8}\), \(0 < \tau < 1\), and \(\omega(R) = (\ln \ln \frac{100}{R})^{-1}\). If \(A(R)\) satisfies that for all \(0 < R \leq \frac{1}{4}\),

\[
A(R) \omega(R) \beta \leq K,
\]

then for \(0 < R \leq \frac{1}{4}\),

\[
\inf_{Q(\frac{1}{4} R) \setminus S} h \geq \frac{1}{2} \lambda(R),
\]

where \(\lambda(R) = N_7 (\ln \frac{100}{R})^{-\tau}\) and \(0 < N_7 = N_7(\tau) < 1\) is a sufficiently small constant.

**Proof.** By (2.14) of Lemma 2.3, one has that for \(-\frac{1}{4} R^2 \leq t < 0\)

\[
-\int_{h \leq \lambda(R)} \ln h \cdot s^2 \, dx \leq N(1 + A(R))^3,
\]

which implies that

\[
\left| \left\{ x \in B \left( \frac{1}{2} R \right) \mid h \leq \lambda(R) \right\} \right| \leq \frac{NR^3 (1 + A(R))^3}{-\ln \lambda(R)}.
\]

By a similar argument as Step 2 in the proof of Lemma 2.1, using \((\lambda(R) - h)_+ = 0\) on \(r = 0\), we have the same conclusion of Lemma 2.1 for \(\Gamma = (\lambda(R) - h)_+\) that

\[
\sup_{Q(\frac{1}{4} R) \setminus S} (\lambda(R) - h)_+ \leq N \left( \frac{1 + A(R)}{R} \right)^{\frac{3}{2}} \| (\lambda(R) - h)_+ \|_{L^2(Q(\frac{1}{4} R))}.
\]

By (2.26),

\[
\sup_{Q(\frac{1}{4} R) \setminus S} (\lambda(R) - h)_+ \leq N \lambda(R) \frac{(1 + A(R))^4}{\sqrt{-\ln \lambda(R)}}
\]

\[
\leq \frac{NK^4}{(-\ln N_7)^{\frac{1}{2} - 4\beta}} (1 - \tau)^{4\beta} \lambda(R),
\]

which implies that
\[
\inf_{Q(\frac{1}{4}R)^{i}} h \geq \hat{\lambda}(R) \left( 1 - \frac{NK^4}{(- \ln N)^{\frac{1}{1-4\beta}}} (1 - \tau)^{4\beta} \right).
\]

Thus, we obtain (2.25) if we pick \( N_\tau \) sufficiently small such that
\[
1 - \frac{NK^4}{(- \ln N)^{\frac{1}{1-4\beta}}} (1 - \tau)^{4\beta} \geq \frac{1}{2}.
\]

(2.27)

\( \square \)

Now, we are in the position to prove Proposition 1.2.

**Proof of Proposition 1.2.** Noting that \( m_R \leq m_{\frac{1}{2}R} \leq M_{\frac{3}{2}R} \leq M_R \) (2.10) and (2.25), we obtain that
\[
J_{{\frac{1}{4}R}} \leq \left( 1 - \frac{\hat{\lambda}(R)}{4} \right) J_R,
\]
for \( 0 < R \leq \frac{1}{4} \). By standard iteration and using \( \ln (1 - t) < -t \) for \( 0 < t < 1 \), we have that for \( j \geq 1 \),
\[
J_{{\frac{1}{2^{j-1}}R}} \leq \exp \left( \sum_{k=0}^{j-1} \ln \left( 1 - \frac{N_{\frac{1}{2^{j-k}}}}{4} \left( \ln \left( \frac{100 \cdot 2^{k+j}}{R} \right)^{\frac{\tau-1}{2}} \right) \right) \right) J_R
\]
\[
\leq \exp \left( - \frac{N_{\frac{1}{2^{j-1}}}}{4} \sum_{k=0}^{j-1} \left( \ln \left( \frac{100 \cdot 2^{k+j}}{R} \right)^{\frac{\tau-1}{2}} \right) \right) J_R
\]
\[
\leq \exp \left( - \frac{N_{\frac{1}{2^{j-1}}}}{4} \int_{0}^{j} \left( \ln \left( \frac{100 \cdot 2^{k+j}}{R} + s \ln 4 \right)^{\frac{\tau-1}{2}} \right) ds \right) J_R
\]
\[
\leq e^{-\frac{N_{\frac{1}{2^{j-1}}}}{4} \int_{0}^{j} \left( \ln \left( \frac{100}{R} + s \ln 4 \right)^{\frac{\tau-1}{2}} - \left( \ln \left( \frac{100}{R} \right)^{\frac{\tau-1}{2}} \right) \right) J_R,
\]
which implies that for \( 0 < \rho < R \leq \frac{1}{4} \),
\[
J_\rho \leq e^{-\frac{N_{\frac{1}{2^{j-1}}}}{4} \int_{0}^{j} \left( \ln \left( \frac{100}{R} + s \ln 4 \right)^{\frac{\tau-1}{2}} - \left( \ln \left( \frac{100}{R} \right)^{\frac{\tau-1}{2}} \right) \right) J_R
\]
\[
\leq e^{-\frac{N_{\frac{1}{2^{j-1}}}}{4} \int_{0}^{j} \left( \ln \left( \frac{100}{R} \right)^{\frac{\tau-1}{2}} - \left( \ln \left( \frac{100}{R} \right)^{\frac{\tau-1}{2}} - 2 \right) \right) J_R.
\]

(2.29)

Hence (1.20).

Finally, if (1.19) holds for all \( z_0 = (0, x_0, 3, t_0) \in Q(\frac{1}{3}) \), then we have (1.20) for all such \( z_0 \). By Lemma 2.1, \( \| \Gamma \|_{L^\infty(Q(3/8))} \leq N(1 + A(1))^3 < \infty \). We can deduce (1.21) from (1.20) and the fact that \( \Gamma(0, x_3, t) = 0 \) for a.e. \( (0, x_3, t) \in Q(\frac{1}{3}) \).

\( \square \)

### 3. Local energy estimates

In this section, we will give some useful local energy estimates. Let \((u, \Pi)\) be a suitable weak solution of (1.1) in \( Q(1) \). Recall the quantities \( A, E, C, D, E \) defined in (1.5) and \( G \) and \( G_2 \) in (1.22).
Lemma 3.1. For \( z_0 = (x_0, t_0) \in Q(\frac{1}{3}) \) and \( 0 < \rho \leq \frac{1}{4} \), we have
\[
A(z_0, \rho) + E(z_0, \rho) \leq N(1 + C(z_0, 2\rho) + D(z_0, 2\rho)).
\] (3.1)

**Proof.** By choosing a suitable test function \( \varphi \) in the local energy inequality (1.4), we get
\[
A(z_0, \rho) + E(z_0, \rho) \leq N \left( C(z_0, 2\rho)^{\frac{3}{2}} + C(z_0, 2\rho) + \frac{1}{\rho^2} \| u \|_{L^1(Q(z_0, 2\rho))} \| \Pi \|_{L^1(Q(z_0, 2\rho))} \right)
\leq N \left( C(z_0, 2\rho)^{\frac{3}{2}} + C(z_0, 2\rho) + C(z_0, 2\rho)^{\frac{3}{2}}D(z_0, 2\rho)^{\frac{3}{2}} \right)
\leq N(1 + C(z_0, 2\rho) + D(z_0, 2\rho)).
\]

\[ \square \]

Lemma 3.2. For \( 0 < \rho \leq R \leq \frac{1}{4} \), we have
\[
C(z_0, \rho) \leq N \left( \frac{R}{\rho} \right)^2 \mathcal{E}(z_0, R)^{1 - \frac{2}{3} + \frac{1}{3}} + N \mathcal{E}(1)^{\frac{3}{2}} \left( \frac{R}{\rho} \right)^2 E(z_0, R)^{\frac{3}{2}}.
\] (3.2)

**Proof.** Set \( \frac{1}{p_1} = \frac{3 - \frac{1}{2} - \frac{3}{2}}{6 - \gamma} \) and \( \frac{1}{q_1} = \frac{3 - \frac{1}{2} - \frac{3}{2}}{6 - \gamma} \). Thus, we have that \( \frac{3}{p_1} + \frac{2}{q_1} = \frac{3}{2} \) with \( 2 \leq p_1 \leq 6 \). Using Lemma 2.1, Lemma A.1 and the Hölder inequality, we have that
\[
C(z_0, \rho) \leq N \rho^{-2} \int_{Q(z_0, R)} |b|^3 \, dx \, dt + N \rho^{-2} \int_{Q(z_0, R)} |u|^3 \, dx \, dt
\leq N \rho^{-2} R^{\frac{5}{3} + \frac{2}{3}} \| b \|_{L^p(Q(z_0, R))} \| b \|_{L^q(Q(z_0, R))} + N \mathcal{E}(1)^{\frac{3}{2}} \rho^{-2} \int_{Q(z_0, R)} \left| \frac{u}{r} \right|^3 \, dx \, dt
\leq N \left( \frac{R}{\rho} \right)^2 \mathcal{E}(z_0, R)^{1 - \frac{2}{3} + \frac{1}{3}} + N \mathcal{E}(1)^{\frac{3}{2}} \left( \frac{R}{\rho} \right)^2 E(z_0, R)^{\frac{3}{2}}.
\]

\[ \square \]

Lemma 3.3. Denote \( B(t_0, R) = R^{-\frac{1}{2}} \| b \|_{L^2(B(t_0 - R^2, t_0; \mathbb{R}^3))} \). For \( 0 < 2\rho \leq R \leq \frac{1}{4} \), we have
\[
C(z_0, \rho) \leq N \left( \frac{R}{\rho} \right)^2 \left( B(t_0, R)^{\frac{3}{2}} + \mathcal{E}(1)^{\frac{3}{2}} \right) \mathcal{E}(z_0, R)^{\frac{3}{2}}.
\] (3.3)

**Proof.** By [30, Theorem 2.42] and [31, Lemma 2.2], we have that
\[
\| b \|_{L^2(B(x_0, \frac{1}{3} R))} \leq N \| b \|_{B^{-1}_{\infty, \infty}(\mathbb{R}^3)} \left( \| \nabla b \|_{L^2(B(x_0, R))} + R^{-1} \| b \|_{L^2(B(x_0, R))} \right)^{\frac{1}{2}}.
\] (3.4)

Therefore, by Lemma 2.1 and (3.4), we have
We can pick a sufficient small constant 0 < \vartheta \leq \frac{1}{4} such that
\begin{align*}
C(z_0, \rho) \leq & N \rho^{-2} \int_0^1 \left( \frac{1}{2} \right) |b|^3 \, dx \, dt + N \rho^{-2} \int_{Q(z_0, R)} |u|^3 \, dx \, dt \\
\leq & N \rho^{-2} R^2 \int_{t_0 - R^2}^{t_0} \left\| b \right\|^3_{L^2(B(x_0, 4\rho))} \, ds + N E(1)^{\frac{3}{2}} \rho^{-2} \int_{Q(z_0, R)} \left\| \frac{u}{r} \right\|^3 \, dx \, dt \\
\leq & N \rho^{-2} R^2 \int_{t_0 - R^2}^{t_0} \left\| b \right\|^3_{B_t^{1, \infty}(R^1)} \left( \left\| \nabla b \right\|^2_{L^2(B(x_0, 4\rho))} + R^{-1} \left\| b \right\|^2_{L^2(B(x_0, 4\rho))} \right)^{\frac{3}{2}} \, ds + N E(1)^{\frac{3}{2}} \left( \frac{R}{\rho} \right)^2 E(z_0, R)^{\frac{3}{2}}.
\end{align*}

Hence (3.3). \qed

Lemma 3.4. [14, Lemma 3.4] For 0 < 2\rho \leq R \leq \frac{1}{\rho} we have
\begin{equation}
D(z_0, \rho) \leq N \left( \frac{\rho}{R} \right) D(z_0, R) + N \left( \frac{R}{\rho} \right)^2 C(z_0, R). \tag{3.5}
\end{equation}

4. Proof of main theorems

Lemma 4.1. Assume that \((u, \Pi)\) is a suitable weak solution to the Navier–Stokes equations in \(Q(1)\), which satisfies that for all 0 < R \leq \frac{1}{4},
\begin{equation}
G_3(z_0, R) = G(z_0, R) \omega(R) \leq G. \tag{4.1}
\end{equation}

There exists a constant \(N_0\) such that for all 0 < R \leq \frac{1}{4},
\begin{equation}
\mathcal{E}_\beta(z_0, R) = \mathcal{E}(z_0, R) \omega(R) \beta \leq N_0 \left( 1 + \mathcal{E}(1)^{18} + G^{\frac{6+2\beta}{\alpha}} \right), \tag{4.2}
\end{equation}
where \(\alpha = \frac{7}{6+2\beta}\). \(\beta\).

Proof. By Lemma 3.1, Lemma 3.2, and Lemma 3.4, we have that for 0 < 4\rho \leq R \leq \frac{1}{4},
\begin{align*}
\mathcal{E}(z_0, \rho) \leq & N(1 + C(z_0, 2\rho) + D(z_0, 2\rho)) + D(z_0, \rho) \\
\leq & N + N \left( \frac{R}{\rho} \right)^2 \left( \mathcal{E}(z_0, R)^{1+\frac{3}{2}} G(z_0, R)^{1+\frac{3}{2}} + \mathcal{E}(1)^{\frac{3}{2}} E(z_0, R)^{\frac{3}{2}} \right) + N \left( \frac{\rho}{R} \right) \mathcal{E}(z_0, R).
\end{align*}

Since \(\omega(R)\) is non-decreasing for 0 < R \leq \frac{1}{4}, we have that
\begin{align*}
\mathcal{E}_\beta(z_0, \rho) \leq & N \left( 1 + \left( \frac{R}{\rho} \right)^8 \mathcal{E}(1)^{18} \right) + N \left( \frac{R}{\rho} \right)^2 \mathcal{E}_\beta(z_0, R)^{1+\frac{3}{2}} G(z_0, R)^{1+\frac{3}{2}} \\
& + \left( \frac{1}{8} + N \left( \frac{\rho}{R} \right) \right) \mathcal{E}_\beta(z_0, R) \tag{4.3}
\end{align*}
\begin{align*}
& \leq N_1 \left( 1 + \left( \frac{R}{\rho} \right)^8 \mathcal{E}(1)^{18} + \left( \frac{R}{\rho} \right)^{\frac{12}{7}} G^{\frac{6+2\beta}{\alpha}} \right) + \left( \frac{1}{4} + N_1 \frac{\rho}{R} \right) \mathcal{E}_\beta(z_0, R).
\end{align*}

We can pick a sufficient small constant 0 < \vartheta \leq \frac{1}{4} such that
Thus, for any $0 < R \leq \frac{1}{4}$, we have
\[
\mathcal{E}_\beta(z_0, R) \leq \frac{1}{2} \mathcal{E}_\beta(R) + N_2,
\]
where $N_2 = N_1 \left( 1 + \vartheta^{-8} \mathcal{E}(1)^{18} + \vartheta^{-12} G^{\frac{8+2\vartheta}{7}} \right)$. By standard iterations, we have that for all $0 < R \leq \frac{1}{4}$,
\[
\mathcal{E}_\beta(z_0, R) \leq 100 \vartheta^{-2} \mathcal{E}(1) + 2N_2. \tag{4.4}
\]
Hence (4.2).

Lemma 4.2. Assume that $(u, \Pi)$ is a suitable weak solution to the Navier–Stokes equations in $\mathbb{R}^3 \times (-1, 0)$. If there exists a constant $B$ such that for all $0 < R \leq \frac{1}{4}$,
\[
B(t_0, R) \omega(R) \leq B, \tag{4.5}
\]
then for all $0 < R \leq \frac{1}{4}$ and $x_0 \in \mathbb{R}^3$,
\[
\mathcal{E}_\beta(z_0, R) = \mathcal{E}(z_0, R) \omega(R)^\beta \leq N \left( 1 + \mathcal{E}(1)^{18} + B^6 \right). \tag{4.6}
\]

Proof. By Lemma 3.1, Lemma 3.3, and Lemma 3.4, we have that for $0 < 8 \rho \leq R \leq \frac{1}{4}$,
\[
\mathcal{E}(z_0, \rho) \leq N \left( 1 + C(z_0, 2 \rho) + D(z_0, 2 \rho) \right) + D(z_0, \rho)
\leq N + N \left( \frac{R}{\rho} \right)^2 \left( B(t_0, R) \right)^2 + \mathcal{E}(1)^{18} \mathcal{E}(z_0, R)^2 + N \left( \frac{\rho}{R} \right) \mathcal{E}(z_0, R).
\]
Since $\omega(R)$ is non-decreasing for $0 < R \leq \frac{1}{4}$, we have that
\[
\mathcal{E}_\beta(z_0, \rho) \leq N \left( \frac{R}{\rho} \right)^8 \left( 1 + \mathcal{E}(1)^{18} + B^6 \right) + \left( \frac{1}{4} + N \frac{\rho}{R} \right) \mathcal{E}_\beta(z_0, R). \tag{4.7}
\]
The rest is analogous to the proof of Lemma 4.1 and we omit the details.

Now we are in a position to prove Theorems 1.3 and 1.4.

By (1.23), Lemma 4.1 and then Proposition 1.2, we obtain that for $0 < r \leq \frac{1}{4}$,
\[
|\Gamma(r, x_3, t)| \leq N |\ln r|^{-2}, \tag{4.8}
\]
which implies Theorem 1.3 by a localized version of [24, Corollary 1.3]. The localization argument is standard, see for example [32].

Theorem 1.4 is proved by reductio ad absurdum. We assume that for some $0 < \beta < \frac{1}{8}$,
\[
\|b\|^\beta \left( B_{\infty, \infty}(\mathbb{R}^3) \right) \leq N \left( \ln \ln \frac{100}{-t} \right)^{\frac{\beta}{2}}, \quad -\frac{1}{2} < t < 0. \tag{4.9}
\]
Notice that for $-\frac{1}{2} < t_0 < 0$ and $0 < R \leq \frac{1}{4}$,
\[
B(t_0, R)\omega(R)^{\frac{\theta}{2}} \leq NR^{-\frac{1}{2}}\left(\int_{t_0-R^2}^{t_0} \left(\ln \ln \frac{100}{t} \right)^{\beta} \, dt\right)^{\frac{1}{2}} \omega(R)^{\frac{\theta}{2}} \\
\leq N\left(\int_{0}^{1} \left(\ln \ln \frac{100}{t_0+R^2s} \right)^{\beta} \, ds\right)^{\frac{1}{2}} \omega(R)^{\frac{\theta}{2}} \\
\leq N\left(\int_{0}^{1} \left(\ln \ln \frac{100}{s} + \ln \ln \frac{1}{R^2s} \right)^{\beta} \, ds\right)^{\frac{1}{2}} \omega(R)^{\frac{\theta}{2}} \\
\leq N.
\]

Therefore, by Lemma 4.2 and then Proposition 1.2, we obtain (4.8), which implies that the solution is regular in \(\mathbb{R}^3 \times (-\frac{1}{2}, 0]\). It leads to a contradiction. Hence, Theorem 1.4.

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Appendix A

Lemma A. 1. For \( \frac{3}{p} + \frac{2}{q} = \frac{2}{2} \) with \( 2 \leq p \leq 6 \), there exists a constant \( N \) such that

\[
\|f\|_{L^{p,q}(Q(z_0,R))} \leq N \left( \|f\|_{L^{\infty}(Q(z_0,R))} + \|\nabla f\|_{L^2(Q(z_0,R))} \right).
\] (A.1)

The space \( L^{p,q} \) is defined above (1.5).

Proof. Without loss of generality, we may assume \( z_0 = (0,0) \). Define the integral mean value \( (f)_{B(R)} = \int_{-B(R)} f(y,t) \, dy \). By Hölder’s inequality and Poincaré inequality, we have

\[
\|f\|_{L^p(B(R))} \leq \|f - (f)_{B(R)}\|_{L^p(B(R))} + \|f\|_{L^p(B(R))}
\leq N\|f\|_{L^2(B(R))}^{\frac{2}{2} - \frac{3}{p}} \|\nabla f\|_{L^2(B(R))} \frac{3}{p} + N R^\frac{2}{2} \|f\|_{L^2(B(R))}.
\]

Therefore

\[
\|f\|_{L^{p,q}(Q(z_0,R))} \leq N\|f\|_{L^{\infty}(Q(z_0,R))}^{\frac{2}{2} - \frac{3}{p}} \|\nabla f\|_{L^2(Q(z_0,R))} + N\|f\|_{L^{\infty}(Q(z_0,R))}.
\]

Hence (A.1). \( \square \)

Lemma A.2. (Nash inequality, see Section 5.3 in [18]). Let \( M \geq 1 \) be a constant and \( \mu \) be a probability measure. Then for all \( 0 \leq f \leq M \), there holds

\[
|\ln \int fd\mu - \int \ln fd\mu| \leq \frac{M\|f\|_{L^p(d\mu)}}{\int fd\mu},
\]

where \( g = \ln f - \int \ln fd\mu \).

Lemma A.3. (weighted Poincaré inequality). In \( B(\mathbb{R}^3) \subset \mathbb{R}^3 \), let the weight \( \Lambda_R(x) = R^{-3} \Lambda \left( \frac{x}{R} \right) \) where \( \Lambda(x) = \Lambda(|x|) \) is non-increasing function with \( 0 \leq \Lambda \leq 1 \) and \( \int_{B(1)} \Lambda \, dx = 1 \). For \( 1 \leq p < \infty \) and \( f \in W^{1,p}(B(R)) \), we have

\[
\int_{B(R)} \left| \int_{B(R)} f \cdot \Lambda_R \, dy \right|^p \cdot \Lambda_R \, dx \leq 2^{p+6} \cdot R^6 \int_{B(R)} |\nabla f|^p \cdot \Lambda_R \, dx.
\]

Remark. By the same proof, this lemma remains valid if we relax the condition on \( \Lambda \leq 0 \leq \Lambda \leq C_1 \), \( \int_{B(1)} \Lambda = 1 \), and

\[
\min_{0 \leq t \leq 1} \Lambda(x + t(y - x)) \geq C_2 \min(\Lambda(x), \Lambda(y)), \quad \forall x, y \in B(1),
\]

with the constant \( 2^{p+6} \) multiplied by \( C_1C_2 \). It needs not be monotone or radial.
Proof. Define \( g(x) = f(Rx) \). It is sufficient to prove that
\[
\int_{B(1)} |g - \int_{B(1)} g \cdot \Lambda \, dy|^{p} \cdot \Lambda \, dx \leq 2^{p+6} \int_{B(1)} \left| \nabla g \right|^{p} \cdot \Lambda \, dx. \tag{A.2}
\]

We first show that for \( y \in B(1) \),
\[
\int_{B(1)} |g(x) - g(y)|^{p} \cdot \Lambda(x)\Lambda(y) \, dx \leq \frac{2^{p+2}}{p+2} \int_{B(1)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dz. \tag{A.3}
\]

Since
\[
|g(x) - g(y)| \leq \int_{0}^{1} \left| \nabla g(y + t(x - y)) \right| \cdot |x - y| \, dt,
\]
and \( \Lambda(x)\Lambda(y) \leq \min(\Lambda(x), \Lambda(y)) \cdot \max \Lambda \leq \Lambda(y + t(x - y)) \cdot 1 \), we have
\[
\int_{B(1)} |g(x) - g(y)|^{p} \cdot \Lambda(x)\Lambda(y) \, dx \leq \int_{0}^{1} \int_{B(1)} \left| \nabla g(y + t(x - y)) \right|^{p} \cdot |x - y|^{p} \cdot \Lambda(y + t(x - y)) \, dx \, dt
\]
\[
\leq \int_{0}^{1} \int_{B(1) \cap \partial B(y,t)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dS(z) \, dt
\]
\[
= \int_{0}^{1} \int_{B(1) \cap \partial B(y,t)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dS(z) \, dt
\]
\[
= \int_{0}^{1} \int_{B(1) \cap \partial B(y,t)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dS(z) \, dt
\]
\[
= \frac{2^{p+2}}{p+2} \int_{B(1) \cap \partial B(y,t)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dz
\]
\[
\leq \frac{2^{p+2}}{p+2} \int_{B(1) \cap \partial B(y,t)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dz.
\]

Now we proceed to prove (A.2). By (A.3), we have
\[
\int_{B(1)} \left| g - \int_{B(1)} g \cdot \Lambda \, dy \right|^{p} \cdot \Lambda \, dx \leq \int_{B(1)} \int_{B(1)} |g(x) - g(y)|^{p} \cdot \Lambda(x)\Lambda(y) \, dx \, dy
\]
\[
\leq \frac{2^{p+2}}{p+2} \int_{B(1)} \int_{B(1)} \left| \nabla g(z) \right|^{p} \cdot |z - y|^{-2} \cdot \Lambda(z) \, dz \, dy
\]
\[
= \frac{2^{p+2}}{p+2} \int_{B(1)} \int_{B(1)} \left| \nabla g(z) \right|^{p} \cdot \Lambda(z) \int_{B(1)} |z - y|^{-2} \, dy \, dz
\]
\[
\leq 2^{p+6} \int_{B(1)} \left| \nabla g(z) \right|^{p} \cdot \Lambda(z) \, dz.
\]
\[\square\]