Treelike families of multiweights

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Abstract

Let $T = (T, w)$ be a weighted finite tree with leaves $1, \ldots, n$. For any $I := \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, let $D_I(T)$ be the weight of the minimal subtree of $T$ connecting $i_1, \ldots, i_k$; the $D_I(T)$ are called $k$-weights of $T$. Given a family of real numbers parametrized by the $k$-subsets of $\{1, \ldots, n\}$, $\{D_I\}_{I \in \binom{\{1, \ldots, n\}}{k}}$, we say that a weighted tree $T = (T, w)$ with leaves $1, \ldots, n$ realizes the family if $D_I(T) = D_I$ for any $I$.

In [13] Pachter and Speyer proved that, if $3 \leq k \leq (n + 1)/2$ and $\{D_I\}_{I \in \binom{\{1, \ldots, n\}}{k}}$ is a family of positive real numbers, then there exists at most one positive-weighted essential tree $T$ with leaves $1, \ldots, n$ that realizes the family ("essential" means that there are no vertices of degree 2). We say that a tree $P$ with leaves $1, \ldots, n$ is an $r$-pseudostar if any edge of $P$ divides $\{1, \ldots, n\}$ into two sets such that at least one of them has cardinality $\leq r$. Here we show that, if $3 \leq k \leq n - 1$ and $\{D_I\}_{I \in \binom{\{1, \ldots, n\}}{k}}$ is a family of real numbers realized by some weighted tree, then there is exactly one weighted essential $(n - k)$-pseudostar $P = (P, w)$, with leaves $1, \ldots, n$ and without internal edges of weight 0, that realizes the family; moreover we describe how any other weighted tree realizing the family can be obtained from $P$. We point out that the unicity statement in the case of positive-weighted trees has been already proved (even if not stated), in fact it follows from Theorem 6 in [11]. Furthermore, we associate to any $(n - k)$-pseudostar with leaves $1, \ldots, n$ a hierarchy on $\{1, \ldots, n\}$ with clusters of cardinality between 2 and $n - k$ and, by using this association, we give a characterization of the families of real numbers that are realized by some weighted tree. Finally we examine the range of the total weight of the weighted trees realizing a fixed family.

1 Introduction

For any graph $G$, let $E(G)$, $V(G)$ and $L(G)$ be respectively the set of the edges, the set of the vertices and the set of the leaves of $G$. A weighted graph $G = (G, w)$ is a graph $G$ endowed with a function $w : E(G) \to \mathbb{R}$. For any edge $e$, the real number $w(e)$ is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that

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We can wonder when a family of real numbers is the family of the weighted tree and of some subset of the set of its vertices. If \{a, b, c, d\} have negative weights is the tropical Grassmannian \(G\). More precisely, they proved that, for any set of real numbers \(\{a, b, c, d\}\) for any order of \(i_1, ..., i_k\). We call the \(D_{i_1, ..., i_k}(T)\) the \(k\)-weights of \(T\) and we call a \(k\)-weight of \(T\) for some \(k\) a multiweight of \(T\).

If \(S\) is a subset of \(V(T)\), the \(k\)-weights give a vector in \(\mathbb{R}^P\). This vector is called \(k\)-dissimilarity vector of \((T, S)\). Equivalently, we can speak of the family of the \(k\)-weights of \((T, S)\).

We can wonder when a family of real numbers is the family of the \(k\)-weights of some weighted tree and of some subset of the set of its vertices. If \(S\) is a finite set, \(k \in \mathbb{N}\) and \(k < \#S\), we say that a family of real numbers \(\{D_I\}_{I \in \binom{S}{k}}\) is treelike (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted) tree \(T = (T, w)\) and a subset \(S\) of the set of its vertices such that \(D_I(T) = D_I\) for any \(k\)-subset \(I\) of \(S\). If in addition \(S \subset L(T)\), we say that the family is \(l\)-treelike (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike).

A criterion for a metric on a finite set to be nn-l-treelike was established in [4], [16], [18]:

**Theorem 2.** Let \(\{D_I\}_{I \in \binom{\{1, 2, ..., n\}}{k}}\) be a set of positive real numbers satisfying the triangle inequalities. It is p-treelike (or nn-l-treelike) if and only if, for all distinct \(i, j, k, h \in \{1, ..., n\}\), the maximum of

\[
\{D_{i,j} + D_{k,h}, D_{i,k} + D_{j,h}, D_{i,h} + D_{k,j}\}
\]

is attained at least twice.

In terms of tropical geometry, the theorem above can be formulated by saying that the set of the 2-dissimilarity vectors of weighted trees with \(n\) leaves and such that the internal edges have negative weights is the tropical Grassmannian \(G_{2,n}\) (see [17]).

In [2], Bandelt and Steel proved a result, analogous to Theorem 2 for general weighted trees; more precisely, they proved that, for any set of real numbers \(\{D_I\}_{I \in \binom{\{1, 2, ..., n\}}{k}}\), there exists a weighted tree \(T\) with leaves \(1, ..., n\) such that \(D_I(T) = D_I\) for any \(I \in \binom{\{1, 2, ..., n\}}{k}\) if and only if, for any \(a, b, c, d \in \{1, ..., n\}\), at least two among \(D_{a,b} + D_{c,d}, D_{a,c} + D_{b,d}, D_{a,d} + D_{b,c}\) are equal. For higher \(k\) the literature is more recent. In 2004, Pachter and Speyer proved the following theorem (see [13]).
Theorem 3. (Pachter-Speyer) Let \( k, n \in \mathbb{N} \) with \( 3 \leq k \leq (n + 1)/2 \). A positive-weighted tree \( T \) with leaves \( 1, \ldots, n \) and no vertices of degree 2 is determined by the values \( D_I(T) \), where \( I \) varies in \( \binom{1 \ldots n}{k} \).

In \([3]\) Bocci and Cools gave a characterization of 3-dissimilarity vectors of trees with \( n \) leaves in term of the tropical Grassmannian \( G_{3,n} \) and in \([10]\) and \([12]\) Iriarte Giraldo and Manon proved that \( k \)-dissimilarity vectors of trees with \( n \) leaves are contained in the tropical Grassmannian \( G_{k,n} \).

In \([7]\), the authors gave the following characterization of the ip-l-treelike families of positive real numbers:

Theorem 4. (Herrmann, Huber, Moulton, Spillner) If \( n \geq 2k \), a family of positive real numbers \( \{D_I\}_{I \in \binom{1 \ldots n}{k}} \) is ip-l-treelike if and only if the restriction to every \( 2k \)-subset of \( \{1, \ldots, n\} \) is ip-l-treelike.

Furthermore, they studied when a family of positive real numbers indexed by \( \binom{1 \ldots n}{k} \) is ip-l-treelike in the case \( k = 3 \).

In \([14]\) and \([15]\), the author gave an inductive characterization of the families of real numbers that are indexed by the subsets of \( \{1, \ldots, n\} \) of cardinality greater than or equal to 2 and are the families of the multiweights of a tree with \( n \) leaves.

Let \( n, k \in \mathbb{N} \) with \( n > k \). In \([1]\) we studied the problem of the characterization of the families of positive real numbers, indexed by the \( k \)-subsets of an \( n \)-set, that are p-treelike in the “border” case \( k = n - 1 \). In particular we got:

Theorem 5. (Baldisserri-Rubei) Let \( n \in \mathbb{N} \), \( n \geq 3 \) and let \( \{D_I\}_{I \in \binom{[n]}{n-1}} \) be a family of positive real numbers. Let us denote \( D_{i-1,i+1,\ldots,n} \) by \( D_i \).

(a) There exists a positive-weighted tree \( T = (T, w) \) with at least \( n \) vertices, \( 1, \ldots, n \), and such that \( D_i(T) = D_i \) for any \( i = 1, \ldots, n \) if and only if

\[
(n - 2)D_i \leq \sum_{j \in [n] - \{i\}} D_j \tag{1}
\]

for any \( i \in [n] \) and at most one of the inequalities (1) is an equality.

(b) There exists a positive-weighted tree \( T = (T, w) \) with at least \( n \) leaves, \( 1, \ldots, n \), and such that \( D_i(T) = D_i \) for any \( i \in [n] \) if and only if

\[
(n - 2)D_i < \sum_{j \in [n] - \{i\}} D_j \tag{2}
\]

for any \( i \in [n] \).

Moreover we studied the analogous problem for graphs. See \([9]\), \([8]\) for other results on graphs and see the introduction of \([1]\) for a survey; finally we quote \([6]\) for some results on graphs with minimal total weight among the ones realizing a given metric space.

Here we examine the case of trees for general \( k \). To state one of the main results we need the following definition.
Definition 6. Let \( r \in \mathbb{N} - \{0\} \). We say that a tree \( P \) is an \( r \)-pseudostar if any edge of \( P \) divides \( L(P) \) into two sets such that at least one of them has cardinality less than or equal to \( r \).

![Figure 1: A 2-pseudostar](image)

With the intent of generalizing Theorem 3, we prove that, if \( 3 \leq k \leq n - 1 \), given a \( l \)-treelike family of real numbers, \( \{D_I\}_{I \in \binom{\{1,\ldots,n\}}{k}} \), there exists exactly one internal-nonzero-weighted \((n-k)\)-pseudostar \( P \) with leaves \( 1, \ldots, n \) and no vertices of degree 2 such that \( D_I(P) = D_I \) for any \( I \); it is positive-weighted if the family is \( p \)-l-treelike. Moreover any other tree realizing the family \( \{D_I\}_I \) and without vertices of degree 2 is obtained from the pseudostar by a certain kind of operations we call “OI operations” and by inserting some internal edges of weight 0 (see Definition 14 and Theorem 17). We point out that the unicity statement in the case of positive-weighted trees has been already proved (even if not stated), in fact it follows from the following theorem in [11].

Theorem 7. (Levy-Yoshida-Pachter) Let \( T = (T, w) \) be a positive-weighted tree with \( L(T) = [n] \). For any \( i, j \in [n] \), define

\[
S(i, j) = \sum_{Y \in \binom{[n]-\{i,j\}}{k-2}} D_{i,j,Y}(T).
\]

Then there exists a positive-weighted tree \( T' = (T', w') \) such that the quartet system of \( T' \) is contained in the quartet system of \( T \) and, defined \( T_{\leq s} \) the subforest of \( T \) whose edge set consists of edges whose removal results in one of the components having size at most \( s \), we have \( T_{\leq n-k} \simeq T'_{\leq n-k} \). There is an invertible linear map between the weights of the edges in \( T_{\leq n-k} \) and the weights of the edges in \( T'_{\leq n-k} \).

Levy, Yoshida and Pachter also describe explicitly this linear map, recovering the weights of the edges in \( T_{\leq n-k} \) from the weights of the edges in \( T'_{\leq n-k} \). Furthermore, we associate to any \((n-k)\)-pseudostar with leaves \( 1, \ldots, n \) a hierarchy on \( \{1, \ldots, n\} \) with clusters of cardinality between 2 and \( n - k \) and, by using this association and by pushing forward the ideas in [14] and [15], in §3 we get a theorem (Theorem 21) characterizing the
l-treelike dissimilarity families; consequently, we obtain also a characterization of p-l-treelike dissimilarity families (see Remark 22). Finally, in §4, given a p-l-treelike family \( \{ D_I \}_{I \in \binom{[n]}{k}} \) in \( \mathbb{R}_+ \), we examine the range of the total weight of the trees realizing it and we show that the \((n-k)\)-pseudostar realizing it has maximum total weight (see Theorem 25); then we study the analogous problem for l-treelike families in \( \mathbb{R} \) (see Theorem 26).

2 Notation and some remarks

**Notation 8.** • Let \( \mathbb{R}_+ = \{ x \in \mathbb{R} | x > 0 \} \).

• We use the symbols \( \subset \) and \( \subseteq \) respectively for the inclusion and the strict inclusion.

• For any \( n \in \mathbb{N} \) with \( n \geq 1 \), let \( [n] = \{ 1, ..., n \} \).

• For any set \( S \) and \( k \in \mathbb{N} \), let \( \binom{S}{k} \) be the set of the \( k \)-subsets of \( S \).

• For any \( A, B \subseteq [n] \), we will write \( AB \) instead of \( A \cup B \). Moreover, we will write \( a, B \), or even \( aB \), instead of \( \{a\} \cup B \).

• Throughout the paper, the word “tree” will denote a finite tree.

• We say that a vertex of a tree is a node if its degree is greater than 2.

• Let \( F \) be a leaf of a tree \( T \). Let \( N \) be the node such that the path \( p \) between \( N \) and \( F \) does not contain any node apart from \( N \). We say that \( p \) is the twig associated to \( F \). We say that an edge is internal if it is not an edge of a twig.

• We say that two weighted trees \( T = (T, w) \) and \( T' = (T', w') \) are equivalent if each of them can be obtained from the other by a sequence of operations of the following kind: replacing two edges, \( e = \{x, y\} \) and \( e' = \{y, z\} \), with \( y \) vertex of degree 2, with only one edge \( e'' = \{x, z\} \) of weight \( w(e) + w(e') \), or the converse operation.

• We say that a tree is essential if it has no vertices of degree 2.

• Let \( T \) be a tree and let \( \{i, j\} \in E(T) \). We say that a tree \( T' \) is obtained from \( T \) by contracting \( \{i, j\} \) if there exists a map \( \varphi : V(T) \to V(T') \) such that:

\[
\varphi(i) = \varphi(j),
\]

\( \varphi^{-1}(y) \) is a set with only one element for any \( y \neq \varphi(i) \),

\[
E(T') = \{ \varphi(a), \varphi(b) \} | \{a, b\} \in E(T) \text{ with } \varphi(a) \neq \varphi(b) \}.
\]

We say also that \( T \) is obtained from \( T' \) by inserting an edge.

• Let \( T \) be a tree and let \( S \) be a subset of \( L(T) \). We denote by \( T|_S \) the minimal subtree of \( T \) whose set of vertices contains \( S \). If \( T = (T, w) \) is a weighted tree, we denote by \( T|_S \) the tree \( T|_S \) with the weight induced by \( w \).

• For simplicity, the vertices of trees will be often named with natural numbers.

• Let \( T = (T, w) \) be a weighted tree. We denote \( w(T) \) by \( D_{\text{tot}}(T) \) and we call it total weight of \( T \).

• Let \( n, k \in \mathbb{N}, \ n \geq 3 \) and \( 1 < k < n \). Given a family of real numbers \( \{ D_I \}_{I \in \binom{[n]}{k}} \), we say that a weighted tree \( T = (T, w) \) with \( L(T) = [n] \) realizes the family \( \{ D_I \}_{I \in \binom{[n]}{k}} \) if \( D_I(T) = D_I \) for any \( I \in \binom{[n]}{k} \).
Definition 9. Let $T$ be a tree.
We say that two leaves $i$ and $j$ of $T$ are neighbours if in the path from $i$ to $j$ there is only one node; furthermore, we say that $C \subset L(T)$ is a cherry if any $i, j \in C$ are neighbours.
We say that a cherry is complete if it is not strictly contained in another cherry.
The stalk of a cherry is the unique node in the path with endpoints any two elements of the cherry.
Let $C$ be a cherry in $T$. We say that a tree $T'$ is obtained from $T$ by pruning $C$ if it is obtained from $T$ by “deleting” all the twigs associated to leaves of $C$ (more precisely, by contracting all the edges of the twigs associated to leaves of $C$).
We say that a cherry $C$ in $T$ is good if it is complete and, if $T'$ is the tree obtained from $T$ by pruning $C$, the stalk of $C$ is a leaf of $T'$. We say that a cherry is bad if it is not good.
Let $i, j, l, m \in L(T)$. We say that $\langle i, j | l, m \rangle$ holds if in $T|_{\{i, j, l, m\}}$ we have that $i$ and $j$ are neighbours, $l$ and $m$ are neighbours, and $i$ and $l$ are not neighbours; in this case we denote by $\gamma_{i,j,l,m}$ the path between the stalk of $\{i, j\}$ and the stalk of $\{l, m\}$ in $T|_{\{i, j, l, m\}}$. The symbol $\langle i, j | l, m \rangle$ is called Buneman's index of $i, j, l, m$.

Example. In the tree in Figure 2 the only good cherries are $\{1, 2, 3\}$ and $\{6, 7\}$.

![Figure 2: Good cherries and bad cherries](image)

Remark 10. (i) A 1-pseudostar is a star, that is, a tree with only one node.
(ii) Let $r \in \mathbb{N} - \{0\}$. In any $r$-pseudostar that is not a star, all the complete cherries, except at most one, have cardinality less than or equal to $r$.
(iii) Let $r, n \in \mathbb{N} - \{0\}$. If $\frac{n}{2} \leq r$, then every tree with $n$ leaves is a $r$-pseudostar, in fact if we divide a set with $n$ elements into two parts, at least one of them has cardinality less than or equal to $\frac{n}{2}$, which is less than or equal to $r$.

Definition 11. Let $S$ be a set. We say that a set system $\mathcal{H}$ of $S$ is a hierarchy over $S$ if, for any $H, H' \in \mathcal{H}$, we have that $H \cap H'$ is one among $\emptyset, H, H'$. We say that $\mathcal{H}$ covers $S$ if $S = \bigcup_{H \in \mathcal{H}} H$.

Definition 12. Let $r, n \in \mathbb{N}$ with $2 \leq r \leq n - 2$. Let $P$ be a $r$-pseudostar with $L(P) = [n]$. 
Let \( P^1 \) be the tree obtained from \( P \) by pruning all the good cherries of cardinality \( \leq r \).
Let \( P^2 \) be the tree obtained from \( P^1 \) by pruning all the good cherries of cardinality \( \leq r \) and so on. We call \( P \) also \( P^0 \).
We say that \( x \in [n] \) descends from \( y \in L(P^s) \) for some \( s \) if the path between \( x \) and \( y \) in \( P \) contains no leaf of \( P^s \) apart from \( y \). For any \( Y \subseteq L(P^s) \), let \( \partial Y \) denote the subset of the elements of \([n]\) descending from any element of \( Y \).
We can define over \([n]\) a hierarchy \( \mathcal{H} \) (depending on \( r \)) as follows:
- we say that a cherry \( C \) of \( P \) is in \( \mathcal{H} \) if and only if \( C \) is good and \( \#C \leq r \);
- if \( C \) is a cherry of \( P^s \) for some \( s \), we say that \( \partial C \) is in \( \mathcal{H} \) if and only if \( C \) is good and \( \#\partial C \leq r \);
- if, for some \( s \), we have that \( L(P^s) \) is the union of two complete cherries, \( C_1 \) and \( C_2 \), and both have cardinality less than or equal to \( r \), we put in \( \mathcal{H} \) only \( \partial C_i \) for \( i \) such that \( \partial C_i \) contains the minimum of \( \partial C_1 \cup \partial C_2 \).
- The elements of \( \mathcal{H} \) are only the ones above. We call the elements of \( \mathcal{H} \) “\( \mathcal{H} \)-clusters”.
- For any \( H \in \mathcal{H} \), we define \( e_H \) as follows: let \( H = \partial C \) for some \( C \) cherry of \( P^s \); we call \( e_H \) the twig of \( P^{s+1} \) associated to the stalk of \( C \).
- For any \( i \in [n] \), we call \( e_i \) the twig associated to \( i \).
Observe that the set of the leaves of a star is a bad cherry; so, according to our definition of \( \mathcal{H} \), if for some \( s \in \mathbb{N} \) we have that \( P^s \) is a star, we do not consider \( \partial L(P^s) \), which is \([n]\), a cluster of \( \mathcal{H} \). So, for instance, the hierarchy of a star is empty.

**Examples.** Let \( P \) be the 6-pseudostar in Figure 3(a). The associated hierarchy over \([12]\) is
\[
\mathcal{H} = \{\{4, 5, 6\}, \{7, 8, 9\}, \{1, 2, 3, 4, 5, 6\}\}.
\]
Let \( Q \) be the 5-pseudostar in Figure 3(b). The associated hierarchy over \([10]\) is
\[
\mathcal{H} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}.
\]
Let \( R \) be the 5-pseudostar in Figure 3(c). The associated hierarchy over \([12]\) is
\[
\mathcal{H} = \{\{3, 4, 5\}, \{6, 7\}, \{8, 9\}, \{1, 2, 3, 4, 5\}\}.
\]

**Remark 13.** It is easy to reconstruct the pseudostar \( P \) (up to the equivalence defined in Notation \( \mathbb{N} \)) from the hierarchy \( \mathcal{H} \):
Let \( \mathcal{H} \) be a hierarchy on \([n]\) such that its clusters have cardinality between 2 and \( r \). Let us consider a star \( B \) with \( L(B) = [n] - \cup_{H \in \mathcal{H}} H \) and call \( O \) its stalk. For any \( M \) maximal element of \( \mathcal{H} \), we add an edge \( e_M \) with endpoint \( O \); let \( V_M \) be the other endpoint of \( e_M \). Then we add a cherry with stalk \( V_M \) and leaves \( M - \cup_{H \in \mathcal{H}, H \subseteq M} H \); for every element \( M' \) of \( \mathcal{H} \) strictly contained in \( M \) which is maximal among the elements of \( \mathcal{H} \) strictly contained in \( M \), we add an edge with endpoint \( V_M \) and we call \( V_M' \) the other endpoint and so on. When we arrive at a minimal element \( N \) of \( \mathcal{H} \), we add a cherry with stalk \( V_N \) and set of leaves \( N \).
**Example.** Let $r = 6$. Consider the following hierarchy over $[12]$:

$$
H = \{\{1, 2, 3, 4, 5, 6\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \{7, 8\} \}.
$$

The associated 6-pseudostar is the one in Figure 4 in fact: $L(B) = \{11, 12\}$, the maximal elements of $H$ are $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9, 10\}$; for $M = \{1, 2, 3, 4, 5, 6\}$, the set $M - \cup_{H \subseteq M} H$ is $\{1, 2, 3\}$ and the only element of $H$ strictly contained in $M$ is $\{4, 5, 6\}$, which is minimal in $H$; for $M = \{7, 8, 9, 10\}$ the set $M - \cup_{H \subseteq M} H$ is empty and the only elements of $H$ strictly contained in $M$ are $\{7, 8\}$ and $\{9, 10\}$, which are minimal in $H$.

**Definition 14.** Let $r, n \in \mathbb{N} - \{0\}$. Let $T = (T, w)$ be a weighted tree with $L(T) = [n]$. Let $e$ be an edge of $T$ with weight $y$ and dividing $[n]$ into two sets such that each of them has strictly more than $r$ elements. Contract $e$ and add $y/(n-r)$ to the weight of every twig of the tree. We call this operation an $r$-IO operation on $T$ and we call the inverse operation an $r$-OI operation.

**Remark 15.** It is easy to check that, if $T = (T, w)$ and $T' = (T', w')$ are weighted trees with $L(T) = L(T') = [n]$ and $T'$ is obtained from $T$ by a $r$-IO operation on an edge $e$ of weight

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**Figure 3:** Pseudostars and hierarchies

**Figure 4:** How to recover the pseudostar from the hierarchy
y, we have that \( T \) and \( T' \) have the same \((n - r)\)-dissimilarity vector. Furthermore, if \( T \) is positive-weighted we have that \( D_{\text{tot}}(T') > D_{\text{tot}}(T) \), precisely

\[
D_{\text{tot}}(T') = D_{\text{tot}}(T) + \frac{r}{n - r} y.
\]

3 Existence and uniqueness of a pseudostar realizing a treelike family

Proposition 16. Let \( k, n \in \mathbb{N} \) with \( 2 \leq k \leq n - 2 \). Let \( \mathcal{P} = (P, w) \) be a weighted tree with \( L(P) = [n] \).

1) Let \( i, l \in [n] \).
   (1.1) If \( i, l \) are neighbours, then \( D_{i,X}(P) - D_{l,X}(P) \) does not depend on \( X \in \binom{[n] - \{i,l\}}{k-1} \).
   (1.2) If \( \mathcal{P} \) is an internal-nonzero-weighted essential \((n - k)\)-pseudostar, then also the converse is true.

2) Let \( i, j, l, m \in [n] \).
   (2.1) If \( \langle i, j | l, m \rangle \) holds or \( P|_{i,j,l,m} \) is a star, then
   \[
   D_{i,m,R}(P) + D_{j,l,R}(P) = D_{i,l,R}(P) + D_{j,m,R}(P)
   \]
   for any \( R \in \binom{[n] - \{i,j,l,m\}}{k-2} \).
   (2.2) Let \( k \geq 4 \) and \( \mathcal{P} \) be an internal-nonzero-weighted essential \((n - k)\)-pseudostar. We have that \( \langle i, j|l, m \rangle \) holds if and only if at least one of the following conditions holds:
   (a) \( \{i, j\} \) and \( \{l, m\} \) are complete cherries in \( \mathcal{P} \),
   (b) there exists \( R \in \binom{[n] - \{i,j,l,m\}}{k-2} \) such that
   \[
   D_{i,j,R}(P) + D_{l,m,R}(P) \neq D_{i,m,R}(P) + D_{j,l,R}(P).
   \]

   and there exists \( S \in \binom{[n] - \{i,j,l,m\}}{k-2} \) such that
   \[
   D_{i,j,S}(P) + D_{m,l,S}(P) \neq D_{i,m,S}(P) + D_{j,l,S}(P).
   \]

(2.3) (Case \( k = 3 \).) Let \( n \geq 5 \). Let \( \mathcal{P} \) be essential and internal-nonzero-weighted. We have that \( \langle i, j|l, m \rangle \) holds if and only if at least one of the following conditions holds:
   (a) there exists \( r \in [n] - \{i, j, l, m\} \) such that the inequality
   \[
   D_{i,j,l}(P) + D_{m,r,l}(P) \neq D_{i,r,l}(P) + D_{m,j,l}(P)
   \]
   holds and the inequalities obtained from this by swapping \( i \) with \( j \) and/or \( l \) with \( m \) hold.
   (b) for any \( r \in [n] - \{i, j, l, m\} \), the following inequalities hold:
   \[
   D_{i,j,r}(P) + D_{m,l,r}(P) \neq D_{i,m,r}(P) + D_{j,l,r}(P),
   \]
   \[
   D_{i,j,r}(P) + D_{m,l,r}(P) \neq D_{i,l,r}(P) + D_{j,m,r}(P).
   \]
Proof. (1.1) Obvious.

(1.2) Let \( P \) be as in the assumptions and suppose that \( D_{i,X}(P) - D_{l,X}(P) \) does not depend on \( X \in ([n] - \{i,l\}) \). For every \( \delta \in [n] \), let \( \overline{\delta} \) be the node on the path from \( i \) to \( l \) such that

\[
path(i,l) \cap path(i,\delta) = path(i,\overline{\delta}).
\]

Suppose, contrary to our claim, that \( i \) and \( l \) are not neighbours. Therefore, on the path between \( i \) and \( l \) there are at least two nodes. For any \( a, b \) nodes in the path between \( i \) and \( l \), we say that \( a \leq b \) if and only if \( path(i,a) \subseteq path(i,b) \). Let \( x, y \) be two nodes on the path between \( i \) and \( l \) such that there is no node in the path between \( x \) and \( y \) apart from \( x \) and \( y \); thus in the path between \( x \) and \( y \) there is only one edge since \( P \) is essential. Suppose \( x < y \), see Figure 5.

We can divide \([n]\) into two disjoint subsets:

\[
X = \{ \delta \in [n] \mid \delta \leq x \},
\]

\[
Y = \{ \delta \in [n] \mid \delta \geq y \}.
\]

Since \( P \) is an \((n-k)\)-pseudostar, then either \(#X \geq k\) or \(#Y \geq k\); suppose \(#X \geq k\) (we argue analogously in the other case); let \( \gamma_1, \ldots, \gamma_{k-1} \) be distinct elements of \( X - \{i\} \) with \( \overline{\gamma_{k-1}} = x \).

Up to interchanging the names of \( \gamma_1, \ldots, \gamma_{k-2} \) (and correspondingly the names of \( \overline{\gamma_1}, \ldots, \overline{\gamma_{k-2}} \)), we can suppose \( \overline{\gamma_1} \leq \overline{\gamma_2} \leq \ldots \leq \overline{\gamma_{k-1}} \). Let \( \eta \in Y - \{l\} \) such that \( \overline{\eta} = y \).

\[
\begin{align*}
D_{i,\gamma_1,\ldots,\gamma_{k-1}} - D_{i,\gamma_1,\ldots,\gamma_{k-1}} &= w(path(i,\overline{\gamma_1})) - w(path(l,\overline{\gamma_{k-1}})) \\
D_{i,\gamma_1,\ldots,\gamma_{k-2},\eta} - D_{i,\gamma_1,\ldots,\gamma_{k-2},\eta} &= w(path(i,\overline{\gamma_1})) - w(path(l,\overline{\eta})).
\end{align*}
\]

Since the first members of the equalities above are equal by assumption, we have that

\[
w(path(l,\overline{\gamma_{k-1}})) = w(path(l,\overline{\eta})),
\]

that is

\[
w(path(l,x)) = w(path(l,y)),
\]

Figure 5: Neighbours in pseudostars
thus the weight of the edge \( \{x, y\} \) must be 0, which contradicts the assumption.

If \( k = 2 \), we have:

\[
D_{i, \gamma_1} - D_{l, \gamma_1} = w(path(i, \gamma_1)) - w(path(l, \gamma_1)) = w(path(i, x)) - w(path(l, x))
\]

\[
D_{i, \eta} - D_{l, \eta} = w(path(i, \eta)) - w(path(l, \eta)) = w(path(i, y)) - w(path(l, y)).
\]

Since the first members of the equalities above are equal, we must have that the weight of the edge \( \{x, y\} \) must be 0, which contradicts the assumption.

(2.1) Let \( R \in \binom{[n]-\{i,j,l,m\}}{k-2} \), \( A = P|_{i,j,l,m,R} \) and \( A' = P|_{i,j,l,m} \). Suppose \( \langle i, j | l, m \rangle \) holds. Call \( x \) the stalk of the cherry \( \langle i, j \rangle \) and \( y \) the stalk of the cherry \( \langle l, m \rangle \) in \( A' \). Let us denote the set of the connected components of \( A - A' \) by \( C_{A-A'} \). For any \( H \in C_{A-A'} \), let \( v_H \) be the vertex that is both a vertex of \( H \) and a vertex of \( A' \). Then \( D_{i,m,R}(P) \) is equal to

\[
D_{i,m}(P) + \sum_{H \in C_{A-A'}} w(H) + w \left( \bigcup_{H \in C_{A-A'} \text{ s.t. } v_H \in V(path(i,x))} path(v_H, x) \right) + w \left( \bigcup_{H \in C_{A-A'} \text{ s.t. } v_H \in V(path(l,y))} path(v_H, y) \right).
\]

Analogous formulas hold for \( D_{i,l,R}(P), D_{j,m,R}(P), D_{j,l,R}(P) \) and we can easily prove our statement. If \( A' \) is a star, we argue analogously.

(2.2) \( \Leftarrow \) Obviously (a) implies \( \langle i, j | l, m \rangle \). Suppose (b) holds; then, if \( P|_{i,j,l,m} \) were a star or \( \langle i, l | j, m \rangle \) or \( \langle i, m | j, l \rangle \) held, from (2.1) we would get a contradiction of the assumptions. Thus \( \langle i, j | l, m \rangle \) holds.

\( \Rightarrow \) Let us consider the path between \( i \) and \( l \). We use the same notation as in (1.2). By assumption \( \overline{j} < \overline{m} \). Let \( m' \in [n] \) be such that \( \overline{m'} \) is the maximum node strictly less than \( \overline{m} \) and let \( j' \in [n] \) be such that \( \overline{j'} \) is the minimum node strictly greater than \( \overline{j} \). We could possibly have \( j' = m \) and \( m' = j \) or \( j' = m' \). In Figure 6 we sketch the situation in case \( j' < m' \).

```
Figure 6: \( j' \) and \( m' \)
```

Since \( P \) is a \((n-k)\)-pseudostar, we have:

\[
\#\{x \in [n] | \overline{x} \leq \overline{m'}\} \geq k \quad \text{or} \quad \#\{x \in [n] | \overline{x} \geq \overline{m}\} \geq k \quad (5)
\]

and

\[
\#\{x \in [n] | \overline{x} \leq \overline{j}\} \geq k \quad \text{or} \quad \#\{x \in [n] | \overline{x} \geq \overline{j'}\} \geq k. \quad (6)
\]
• First suppose that there exists \( s \in [n] - \{i, j\} \) such that \( \overline{x} \leq \overline{y} \) and there exists \( t \in [n] - \{l, m\} \) such that \( \overline{t} \geq \overline{m} \). From (3) and (6) we get

\[
\#\{x \in [n] | \overline{x} \leq \overline{m'} \} \geq k \quad \text{or} \quad \#\{x \in [n] | \overline{x} \geq \overline{y} \} \geq k.
\]

Suppose \( \#\{x \in [n] | \overline{x} \leq \overline{m'} \} \geq k \) (the other case is analogous). Let \( R \) be a \((k - 2)\)-subset of \( \{x \in [n] - \{i, j\}| \overline{x} \leq \overline{m'} \} \) such that, if \( m' \neq j \), then \( R \) contains \( s \) and \( m' \). Then

\[
D_{i,j,R}(P) - D_{l,j,R}(P) = w(path(i, min(j, R))) - w(path(l, max(j, R))) - w(path(i, min(R))) + w(path(l, m'),)
\]

\[
D_{i,m,R}(P) - D_{l,m,R}(P) = w(path(i, min(m, R))) - w(path(l, max(m, R))) + w(path(i, min(R))) - w(path(l, m)).
\]

So we get that \( D_{i,j,R}(P) - D_{l,j,R}(P) - D_{i,m,R}(P) + D_{l,m,R}(P) = -w(\overline{m'}, \overline{m}) \), which is nonzero by assumption. Thus \( D_{i,j,R}(P) + D_{i,m,R}(P) \neq D_{l,j,R}(P) + D_{l,m,R}(P) \); hence (1) holds.

• Now suppose that there exists \( s \in [n] - \{i, j\} \) such that \( \overline{s} \leq \overline{j} \) and there does not exist \( t \in [n] - \{l, m\} \) such that \( \overline{t} \geq \overline{m} \) (analogously if the converse holds). Then \( \#\{x \in [n] | \overline{x} \leq \overline{m'} \} \geq k \).

By taking \( R \) to be a \((k - 2)\)-subset of \( \{x \in [n] - \{i, j\}| \overline{x} \leq \overline{m'} \} \) such that, if \( m' \neq j \), then \( R \) contains \( s \) and \( m' \), we conclude as above that (1) holds.

• Finally, if there does not exist \( s \in [n] - \{i, j\} \) such that \( \overline{s} \leq \overline{j} \) and there does not exist \( t \in [n] - \{l, m\} \) such that \( \overline{t} \geq \overline{m} \), then (a) holds.

By considering the path between \( i \) and \( m \), we get analogously that either (3) holds or (a) holds.

(2.3) \Rightarrow Let \( x \) be the stalk of the cherry \( \{i, j\} \) in \( P_{i,j,l,m} \) and let \( y \) be the stalk of the cherry \( \{l, m\} \) in \( P_{i,j,l,m} \). Suppose first that in \( V(\gamma_{i,j,l,m}) \) there are some nodes of \( P \) different form \( x \) and \( y \); call \( c \) the node of \( P \) in \( V(\gamma_{i,j,l,m}) - \{x, y\} \) such that \( path(x, c) \subseteq path(x, c') \) for any \( c' \) node of \( P \) in \( V(\gamma_{i,j,l,m}) - \{x, y\} \) (that is \( c \) is the node in \( \gamma_{i,j,l,m} \) “nearest” to \( x \)). Let \( r \in [n] \) be such that \( path(x, y) \cap path(x, r) = path(x, c) \). For such an \( r \), we have the inequalities in (a), in fact the edge \( \{x, c\} \) has nonzero weight by assumption. Thus, if (a) does not hold, then there are no nodes of \( P \) in \( V(\gamma_{i,j,l,m}) - \{x, y\} \), hence \( \gamma_{i,j,l,m} \) is an edge; by assumption \( w(\gamma_{i,j,l,m}) \neq 0 \) and we can prove easily that (b) holds.

\( \Leftarrow \) We can easily prove that, if (a) holds or (b) holds, then \( P_{i,j,l,m} \) is not a star and \( \{i, m|j, l\} \) and \( \{i, l|j, m\} \) do not hold.

\( \square \)

**Theorem 17.** Let \( n, k \in \mathbb{N} \) with \( 3 \leq k \leq n - 1 \). Let \( \{D_I\}_{I \in \binom{\mathbb{N}}{k}} \) be a family of real numbers. If it is \( l \)-treelike, then there exists exactly one internal-nonzero-weighted essential \((n - k)\)-pseudostar \( P \) realizing the family. Any other weighted essential tree realizing the family \( \{D_I\}_I \) can be obtained from \( P \) by \( OI \) operations and by inserting internal edges of weight 0.

If the family \( \{D_I\}_{I \in \binom{\mathbb{N}}{k}} \) is \( p \)-l-treelike, then \( P \) is positive-weighted and any other positive-weighted essential tree realizing the family \( \{D_I\}_I \) can be obtained from \( P \) by \( OI \) operations.
Proof. Let \( \mathcal{T} = (T, w) \) be a weighted tree with \( L(T) = [n] \) and realizing the family \( \{D_I\}_{I \in \binom{[n]}{k}} \). Obviously we can suppose that it is essential. By \((n-k)\)-IO operations and contracting the internal edges of weight 0 we can change \( \mathcal{T} \) into an internal-nonzero-weighted essential \((n-k)\)-pseudostar \( \mathcal{P} \); it realizes the family \( \{D_I\}_I \) by Remark 15. If \( \mathcal{T} \) is positive-weighted, obviously also \( \mathcal{P} \) is positive-weighted.

If \( k = n - 1 \), it is easy to see that there exists at most a weighted essential star with leaves 1, ..., \( n \) realizing the family \( \{D_I\}_I \).

Suppose \( k \leq n - 2 \). By part 1 of Proposition 16 the \( D_I \) for \( I \in \binom{[n]}{k} \) determine the complete cherries of an internal-nonzero-weighted essential \((n-k)\)-pseudostar \( \mathcal{P} = (P, w) \) with \( L(P) = [n] \) and realizing the family \( \{D_I\}_I \), so, by part 2, they determine the Buneman’s indices, and then they determine \( P \), in fact it is well known that the Buneman’s indices of a tree determine the tree (see for instance [5]). We have to show that the \( D_I \) determine also the weights of the edges of \( \mathcal{P} \). The argument is completely analogous to the proof of the theorem in [13]; we sketch it for the convenience of the reader. Let \( e \) be an edge of \( P \) which is not a twig. Then there exist \( i, j, l, m \in [n] \) such that \( e = \gamma_{i,j,l,m} \); since \( P \) is an \((n-k)\)-pseudostar, there exists \( R \in \binom{[n]-\{i,j,l,m\}}{k-2} \) such that \( e \) is not an edge of \( P|_R \). Then

\[
2w(e) = D_{i,m,R}(\mathcal{P}) + D_{j,i,R}(\mathcal{P}) - D_{i,j,R}(\mathcal{P}) - D_{l,m,R}(\mathcal{P}),
\]

so \( w(e) \) is determined by the \( D_I \). For any \( I \in \binom{[n]}{k} \) we have that

\[
D_I(\mathcal{P}) = \sum_{e \in E(T|_I)} w(e) + \sum_{i \in I} w(e_i) = \sum_{e \notin T|_I} w(e) + \sum_{i \in I} w(e_i).
\] (7)

So, for any \( i, j \in [n] \) and for any \( S \in \binom{[n]-\{i,j\}}{k-1} \), we have:

\[
w(e_i) - w(e_j) = \sum_{l \in (iS)} w(e_l) - \sum_{l \in (jS)} w(e_l) = \left(D_{iS} - \sum_{e \in E(T|_{iS})} w(e)e \notin \text{not twig}\right) - \left(D_{jS} - \sum_{e \in E(T|_{jS})} w(e)e \notin \text{not twig}\right).
\]

Hence the difference of the weights of the twigs is determined by the \( D_I \). From the formula (7) we get the weight of every twig.

\[\square\]

4 Characterization of treelike families

Remark 18. Let \( n, k \in \mathbb{N} \) with \( 2 \leq k \leq n - 2 \). Let \( \mathcal{P} = (P, w) \) be a weighted \((n-k)\)-pseudostar with \( L(P) = [n] \). Let \( \mathcal{H} \) be a hierarchy induced by \( P \) over \([n]\) as in Definition 12. Observe that, for any \( J \in \mathcal{H} \) and any \( I \in \binom{[n]}{k} \), the subtree realizing \( D_I(\mathcal{P}) \) contains \( e_J \) if and only if \( I \cap J \neq \emptyset \) and \( I \not\subset J \).
Before formulating the characterization of treelike families we state two very technical lemmas, which will be useful only in the proof of the theorem. We suggest reading them after reading the theorem.

**Lemma 19.** Let \( k, n \in \mathbb{N} \) with \( 3 \leq k \leq n - 2 \). Let \( \mathcal{H} \) be a hierarchy on \([n]\) such that its clusters have cardinality less than or equal to \( n - k \) and greater than or equal to \( 2 \). Let \( i, l \in [n] \) and \( X \in \binom{[n]\setminus\{i,l\}}{k-1} \) satisfy the following conditions:

- if \( i, l \in \bigcup_{H \in \mathcal{H}} H \), then
  
  \( X \) contains an element \( \hat{i} \) of the minimal \( \mathcal{H} \)-cluster containing \( i \),
  
  \( X \) contains an element \( \hat{l} \) of the minimal \( \mathcal{H} \)-cluster containing \( l \);

- if \( i \in \bigcup_{H \in \mathcal{H}} H \) and \( l \not\in \bigcup_{H \in \mathcal{H}} H \), then
  
  \( X \) contains an element \( \hat{i} \) of the minimal \( \mathcal{H} \)-cluster containing \( i \),
  
  \( X \) contains an element \( \hat{i} \) that is not in the maximal \( \mathcal{H} \)-cluster containing \( i \);

- if \( l \in \bigcup_{H \in \mathcal{H}} H \) and \( i \not\in \bigcup_{H \in \mathcal{H}} H \), then
  
  \( X \) contains an element \( \hat{l} \) of the minimal \( \mathcal{H} \)-cluster containing \( l \),
  
  \( X \) contains an element \( \hat{l} \) that is not in the maximal \( \mathcal{H} \)-cluster containing \( l \).

Then, in the free \( \mathbb{Z} \)-module \( \bigoplus_{H \in \mathcal{H}} \mathbb{Z}H \),

\[
\sum_{H \in \mathcal{H}, H \cap (iX) \neq \emptyset, H \nless(iX)} H - \sum_{H \in \mathcal{H}, H \cap (lX) \neq \emptyset, H \nless(lX)} H = 0
\]

**Proof.** We have to show that, for every \( V \in \mathcal{H} \), we have that \( V \cap (iX) \neq \emptyset \) and \( V \nless (iX) \) if and only if \( V \cap (lX) \neq \emptyset \) and \( V \nless (lX) \). We have five possible cases:

- \( V \cap X = \emptyset \).
  
  In this case, we have that \( V \cap (iX) = \emptyset \), in fact: suppose on the contrary that \( V \cap (iX) \neq \emptyset \); then \( V \ni i \); therefore obviously \( i \in \bigcup_{H \in \mathcal{H}} H \) and, since the minimal \( \mathcal{H} \)-cluster containing \( i \) contains \( \hat{i} \) and is contained in \( V \), we have that \( V \) contains \( \hat{i} \); since, by assumption, \( X \ni \hat{i} \), we get that \( V \cap X \ni \hat{7} \), thus \( V \cap X \neq \emptyset \), which is absurd. Analogously, \( V \cap (lX) = \emptyset \) and we can conclude.

- \( V \cap X \neq \emptyset \), \( V \ni i, l \).
  
  In this case, we have obviously that \( V \cap (iX) \neq \emptyset \), \( V \cap (lX) \neq \emptyset \), \( V \nless (iX) \), \( V \nless (lX) \) and we can conclude.

- \( V \cap X \neq \emptyset \), \( V \ni i, l \).
  
  In this case, we have obviously that \( V \cap (iX) \neq \emptyset \) and \( V \cap (lX) \neq \emptyset \). Furthermore, \( \Vdash (iX) \) if and only if \( V \ni X \) and this holds if and only if \( V \ni (lX) \), so we can conclude.

- \( V \cap X \neq \emptyset \), \( V \ni i, V \nless l \).
  
  In this case, we have obviously that \( i \in \bigcup_{H \in \mathcal{H}} H \); moreover \( V \cap (iX) \neq \emptyset \) and \( V \cap (lX) \neq \emptyset \). Furthermore, \( V \nless (lX) \) since \( V \nless l \). So we have to prove that \( V \nless (iX) \). Suppose on the contrary that \( V \ni (iX) \); thus \( V \ni X \).
If \( l \not\in \bigcup_{H \in \mathcal{H}} H \), then, by assumption, \( X \ni i \); since \( V \ni X \), we have that \( V \ni \hat{i} \), and thus \( \hat{i} \) is in the maximal cluster containing \( i \), which is absurd.

If \( l \in \bigcup_{H \in \mathcal{H}} H \), then, by assumption, \( X \ni \tilde{T} \); since \( V \ni X \), we have that \( V \ni \tilde{T} \); therefore \( V \ni l \), which is absurd.

- \( V \cap X \neq \emptyset \), \( V \ni l \), \( V \not\ni i \).

Analogous to the previous case.

\( \square \)

**Lemma 20.** Let \( k, n \in \mathbb{N} \) with \( 4 \leq k \leq n - 2 \). Let \( \mathcal{H} \) be a hierarchy on \([n]\) such that its clusters have cardinality less than or equal to \( n - k \) and greater than or equal to \( 2 \). Let \( a, a' \in [n] \), \( J \in \mathcal{H} \) with \( a \in J \), \( a' \not\in J \). Let \( X, X' \in \left( [n] - \{a, a'\} \right) \) satisfy the following conditions:

1. if \( a' \in \bigcup_{H \in \mathcal{H}} H \), then
   1.1 \( X \) and \( X' \) contain an element \( b \) of the minimal cluster containing \( a' \) with \( b \neq a' \);
   1.2 \( X \) contains an element \( c \) which is not in the maximal cluster containing \( a' \) and \( X' \) contains an element \( c' \) which is not in the maximal cluster containing \( a' \);

2. if \( a' \not\in \bigcup_{H \in \mathcal{H}} H \), then \( X \) and \( X' \) contain an element \( d \) which is not in the maximal cluster containing \( J \);

3. if there exists \( \tilde{J} \) in \( \mathcal{H} \) with \( a \in \tilde{J} \subseteq J \), suppose that \( \tilde{J} \) is maximal among the \( \mathcal{H} \)-clusters with these characteristics; then \( X' \) contains an element of \( J - \tilde{J} \) and \( X' \cap \tilde{J} = \emptyset \); if there does not exist \( \tilde{J} \) in \( \mathcal{H} \) with \( a \in \tilde{J} \subseteq J \), then \( X' \cap J \neq \emptyset \);

4. \( X \cap J = \emptyset \); moreover, if there exists \( \tilde{J} \) in \( \mathcal{H} \) with \( J \subseteq \tilde{J} \), suppose that \( \tilde{J} \) is minimal among the \( \mathcal{H} \)-clusters with these characteristics; then \( X \) contains an element of \( \tilde{J} - J \);

Then, in the free \( \mathbb{Z} \)-module \( \bigoplus_{H \in \mathcal{H}} \mathbb{Z}H \),

\[
J = \sum_{H \in \mathcal{H}, \ H \cap \{aX\} \neq \emptyset, H \not\supset (aX)} H - \sum_{H \in \mathcal{H}, \ H \cap \{aX\} \neq \emptyset, H \supset (aX)} H - \sum_{H \in \mathcal{H}, \ H \cap (aX') \neq \emptyset, H \not\supset (aX')} H + \sum_{H \in \mathcal{H}, \ H \cap (aX') \neq \emptyset, H \supset (aX')} H
\]

(8)

**Proof.** In order to prove (8), we have to show that every \( \mathcal{H} \)-cluster \( V \) different from \( J \) does not appear in the second member of (8) and that \( J \) appears with coefficient 1. Let \( V \in \mathcal{H} \).

- Suppose \( V \not\ni a, a' \) (so \( V \neq J \)).

In this case \( V \) does not contain any of \( aX, a'X, aX', a'X' \) and we can conclude easily by considering the four possible cases:
- \( V \cap X \neq \emptyset, V \cap X' \neq \emptyset \),
- \( V \cap X = \emptyset, V \cap X' \neq \emptyset \),
- \( V \cap X \neq \emptyset, V \cap X' = \emptyset \),
- \( V \cap X = \emptyset, V \cap X' = \emptyset \).

- Suppose \( V \ni a' \) (so \( V \neq J \)).
Then \( a' \in \bigcup_{H \in \mathcal{H}} H \), therefore, by assumption (1), \( b \in X, X', c \in X, c' \in X' \). Moreover \( V \ni a' \), thus \( V \ni b \), so \( V \cap X \neq \emptyset \) and \( V \cap X' \neq \emptyset \). Since \( c \in X, c' \in X' \) and \( c, c' \notin V \), we have that \( X \not\ni V \) and \( X' \not\ni V \), so we can conclude.

- Suppose \( V \ni a, V \not\ni a' \).

There are at most three possible cases: \( V \subset J, V \supset J, V = J \).

  - If \( V \subset J \), then \( V \cap X = \emptyset \) and \( V \cap X' = \emptyset \) by assumptions (3) and (4), thus \( V \not\ni (a'X') \), \( V \not\ni (a'X') \), \( V \not\ni (aX) \), \( V \not\ni (a'X') \), \( V \cap (a'X') = \emptyset \) and \( V \cap (aX) = \emptyset \). Moreover, since \( V \ni a \), \( V \cap (aX) \neq \emptyset \) and \( V \cap (aX) \neq \emptyset \) and we conclude easily.

  - If \( V \supset J \), then \( V \cap X \neq \emptyset \) and \( V \cap X' \neq \emptyset \) since \( J \cap X = \emptyset \) and \( J \cap X' = \emptyset \) by assumptions (3) and (4).

Suppose \( a' \in \bigcup_{H \in \mathcal{H}} H \). Then, if \( V \) contained \( X \), then it would contain \( b \) and thus it would contain \( a' \), which is absurd, so \( V \) does not contain \( X \). Analogously \( V \) does not contain \( X' \). So \( V \not\ni (a'X') \), \( V \not\ni (a'X') \), \( V \not\ni (aX) \), and we conclude.

Suppose \( a' \notin \bigcup_{H \in \mathcal{H}} H \). Hence \( X \) and \( X' \) contain \( d \) by assumption (2). Then, if \( V \) contained \( X \), then it would contain \( d \), which is absurd since \( d \) is not in the maximal cluster containing \( J \); thus \( V \) does not contain \( X \). Analogously \( V \) does not contain \( X' \). So \( V \not\ni (a'X') \), \( V \not\ni (a'X') \), \( V \not\ni (aX) \), and we conclude.

- Finally consider the cluster \( J \). We have that \( J \cap X' \neq \emptyset \) by assumption (3) and \( J \ni a \), so \( J \cap (aX') \neq \emptyset \), \( J \cap (a'X') \neq \emptyset \), \( J \cap (a'X') \neq \emptyset \). Since \( a' \notin J \) and \( J \ni X = \emptyset \) by assumption (4), we have that \( J \cap (a'X) = \emptyset \). Moreover \( J \not\ni (aX) \), since \( J \cap X = \emptyset \), and \( J \not\ni (a'X) \) and \( J \not\ni (a'X) \), since \( J \not\ni a' \). Finally \( J \not\ni X' \), in fact: if \( a' \in \bigcup_{H \in \mathcal{H}} H \), then \( b \in X' \) by assumption (1), so, if \( J \) contained \( X' \), it would contain \( b \) and thus \( a' \), which is absurd; if \( a' \notin \bigcup_{H \in \mathcal{H}} H \), then \( d \in X' \) by assumption (2), so, if \( J \) contained \( X' \), it would contain \( d \), which is absurd; so \( J \not\ni (aX') \) and we can conclude.

\[ \square \]

**Theorem 21.** Let \( n, k \in \mathbb{N} \) with \( 5 \leq k \leq n - 1 \). Let \( \{D_I\}_{I \in \binom{[n]}{k}} \) be a family of real numbers.

If \( k \leq n - 2 \), the family \( \{D_I\}_{I \in \binom{[n]}{k}} \) is l-tree-like if and only if there exists a hierarchy \( \mathcal{H} \) over \([n]\) such that:

(i) the clusters of \( \mathcal{H} \) have cardinality between 2 and \( n - k \),

(ii) if \( \mathcal{H} \) covers \([n]\), then the number of the maximal clusters of \( \mathcal{H} \) is not 2,

(iii) for any \( q \in \{1, ..., n - 1\} \), \( s \in \{1, k - 1\} \) and for any \( W, W' \in \binom{[n]}{s} \)

\[ \sum_{i=1}^{q} D_{W, Z_i} - D_{W', Z_i} \]

does not depend on \( Z_i \in \binom{[n]-\{W', W\}}{k-s} \) under the condition that, in the free \( \mathbb{Z} \)-module \( \bigoplus_{H \in \mathcal{H}} \mathbb{Z}H \),

the sum

\[ \sum_{i=1}^{q} \left[ \sum_{H \in \mathcal{H}, H \cap (WZ_i) \neq \emptyset, H \not\ni (WZ_i)} H - \sum_{H \in \mathcal{H}, H \cap (W'Z_i) \neq \emptyset, H \not\ni (W'Z_i)} H \right] \]

does not change.
If $k = n - 1$, the family $\{D_I\}_I$ is always l-treelike.

Proof. If $k = n - 1$, it is easy to show that there exists a weighted star realizing the family $\{D_I\}_I$.

Suppose $k \leq n - 2$. If the family $\{D_I\}_I$ is l-treelike, then there exists a weighted $(n - k)$-pseudostar realizing it by Theorem 17. It induces a hierarchy over $[n]$ as in Definition 12 (with $r = n - k$) and it is easy to see that conditions (i) and (ii) hold; by Remark 18, we have also that condition (iii) holds.

Suppose now that there exists a hierarchy $\mathcal{H}$ over $[n]$ satisfying (i), (ii) and (iii). Let $P$ be the essential $(n - k)$-pseudostar determined by $\mathcal{H}$ (see Remark 13); observe that it is essential by condition (ii). For any $J \in \mathcal{H}$, let $e_J$ be defined as in Remark 13; we define

$$w(e_J) := D_{a,X} - D_{a',X} - D_{a,X'} + D_{a',X'},$$

(9)

for any $a, a' \in [n], X, X' \subset [n]$ such that $a, a' \not\in X, X'$ and

$$\sum_{H \in \mathcal{H}, H \cap (aX) \neq \emptyset, H \not\supset (aX)} H - \sum_{H \in \mathcal{H}, H \cap (a'X) \neq \emptyset, H \not\supset (a'X)} H - \sum_{H \in \mathcal{H}, H \cap (aX') \neq \emptyset, H \not\supset (aX')} H + \sum_{H \in \mathcal{H}, H \cap (a'X') \neq \emptyset, H \not\supset (a'X')} H$$

(10)

is equal to $J$. Let us check that the definition of $w(e_J)$ is a good definition:

- to see that it does not depend on $X$, it is sufficient to see that $D_{a,X} - D_{a',X}$ does not depend on $X$ under the condition that the sum in (10) does not depend on $X$; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap (aX) \neq \emptyset, H \not\supset (aX)} H - \sum_{H \in \mathcal{H}, H \cap (a'X') \neq \emptyset, H \not\supset (a'X')} H$ does not depend on $X$; so our assertion follows from condition (iii) by taking $q = 1, s = 1$, $W = \{a\}, W' = \{a'\}$ and $Z_1 = X$; in an analogous way we can see that it does not depend on $X'$;

- to see that it does not depend on $a$, it is sufficient to see that $D_{a,X} - D_{a,X'}$ does not depend on $a$ under the condition that the sum in (10) does not depend on $a$; obviously this is equivalent to the fact that $\sum_{H \in \mathcal{H}, H \cap (aX') \neq \emptyset, H \not\supset (aX')} H$ does not depend on $a$; so our assertion follows from condition (iii) by taking $q = 1, s = k - 1$, $W = X, W' = X'$, and $Z_1 = \{a\}$; in an analogous way we can see that it does not depend on $a'$.

Moreover, observe that, by Lemma 20, it is possible to find $a, a', X, X'$ as required.

For any $i \in [n]$, we define the weight of the twig $e_i$ as follows:

$$w(e_i) := \frac{1}{k} \left[ D_I + \sum_{l \in I} (D_{i,X(i,l)} - D_{i,X(i,l)\setminus i}) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \not\supset I} w(e_J) \right]$$

(11)

for any $I \in \binom{[n]}{k}, X(i,l) \in \binom{[n] \setminus \{i,l\}}{k-1}$ such that

$$\sum_{H \in \mathcal{H}, H \cap (i,X(i,l)) \neq \emptyset, H \not\supset (i,X(i,l))} H - \sum_{H \in \mathcal{H}, H \cap (i,X(i,l)) \neq \emptyset, H \not\supset (i,X(i,l))} H = 0.$$
Observe that, by Lemma 19, it is possible to find $X(i, l)$ as required. The definition of $w(e_t)$ does not depend on the choice of $X(i, l)$ by condition (iii); we have to show that it does not depend on $I$. Let $I = (a, Y)$ and $I' = (a', Y')$ for some distinct $a, a' \in [n]$, $Y \in (\binom{[n]}{k-1})$. We have to show that

$$D_a, Y + \sum_{i \in (a Y)} (D_i, X(i, l) - D_i, X(i, l)) - \sum_{J \in H, J \cap (a Y) \neq \emptyset, J \not\supset (a Y)} w(e_J) =$$

$$= D_{a'}, Y + \sum_{i \in (a' Y)} (D_i, X(i, l) - D_i, X(i, l)) - \sum_{J \in H, J \cap (a' Y) \neq \emptyset, J \not\supset (a' Y)} w(e_J),$$

that is

$$D_a, Y + D_{i, X(i, a)} - D_{a, X(i, a)} - \sum_{J \in H, J \cap (a Y) \neq \emptyset, J \not\supset (a Y)} w(e_J) = D_{a'}, Y + D_{i, X(i, a')} - D_{a', X(i, a')} - \sum_{J \in H, J \cap (a' Y) \neq \emptyset, J \not\supset (a' Y)} w(e_J).$$

Observe that $\{J \in H | J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\}$ can be written as disjoint union of the following sets:

$$\{J \in H | J \supseteq a, J \not\supset a', J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\}, \{J \in H | J \not\supset a, J \supseteq a', J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\},$$

$$\{J \in H | J \not\supset a, J \not\supset a', J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\}, \{J \in H | J \not\supset a, J \not\supset a', J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\},$$

that is, as disjoint union of

$$\{J \in H | J \supseteq a, J \not\supset a', J \not\supset Y\}, \{J \in H | J \not\supset a, J \supseteq a', J \not\supset Y\},$$

$$\{J \in H | J \not\supset a, J \not\supset a', J \not\supset Y\}, \{J \in H | J \not\supset a, J \not\supset a', J \not\supset Y\};$$

and then as disjoint union of

$$\{J \in H | J \supseteq a, J \not\supset a', J \cap Y \neq \emptyset, J \not\supset Y\}, \{J \in H | J \not\supset a, J \supseteq a', J \cap Y \neq \emptyset, J \not\supset Y\},$$

$$\{J \in H | J \not\supset a, J \not\supset a', J \cap Y = \emptyset\}, \{J \in H | J \not\supset a, J \not\supset a', J \cap Y = \emptyset\},$$

$$\{J \in H | J \not\supset a, J \not\supset a', J \not\supset Y\}, \{J \in H | J \not\supset a, J \not\supset a', J \not\supset Y\};$$

Analogously we can write $\{J \in H | J \cap (a Y) \neq \emptyset, J \not\supset (a Y)\}$. Let us take both $X(i, a)$ and $X(i, a')$ equal to a set $X$ satisfying the conditions of Lemma 19 for $i, a$, for $i, a'$ and for $a, a'$ (there exists since $k \geq 5$). By simplifying, the assertion becomes

$$D_a, Y - D_a, X - \sum_{J \in H \text{ and either } J \supseteq a, J \not\supset a', J \cap Y = \emptyset \text{ or } J \not\supset a', J \not\supset a, Y \subset J} w(e_J) = D_{a'}, Y - D_{a'}, X - \sum_{J \in H \text{ and either } J \supseteq a', J \not\supset a, J \cap Y = \emptyset \text{ or } J \not\supset a, J \not\supset a', Y \subset J} w(e_J).$$

For any $J \in H$ such that $J \supseteq a', J \not\supset a$, and $J \cap Y = \emptyset$ or $Y \subset J$, let $Z_J, Z'_J$ be such that the sum

$$\sum_{H \in H, H \cap (a' Z_J) \neq \emptyset} H - \sum_{H \in H, H \cap (a Z_J) \neq \emptyset} H - \sum_{H \in H, H \cap (a' Z'_J) \neq \emptyset} H + \sum_{H \in H, H \cap (a Z'_J) \neq \emptyset} H$$

18
is equal to $J$. By the definition in (9), we have that

$$w(e_J) = D_{a',Z_J} - D_{a,Z_J} - D_{a',Z'_J} + D_{a,Z'_J}.$$  

For any $J \in \mathcal{H}$ such that $J \ni a, J \not\ni a'$, and $J \cap Y = \emptyset$ or $Y \subset J$, let $R_J, R'_J$ be such that the sum

$$\sum_{H \in \mathcal{H}, H \cap (aR_J) \neq \emptyset} H - \sum_{H \in \mathcal{H}, H \cap (a'R_J) \neq \emptyset} H - \sum_{H \in \mathcal{H}, H \cap (aR'_J) \neq \emptyset} H + \sum_{H \in \mathcal{H}, H \cap (a'R'_J) \neq \emptyset} H$$

is equal to $J$; by the definition in (9), we have that

$$w(e_J) = D_{a,R_J} - D_{a',R_J} - D_{a,R'_J} + D_{a',R'_J}.$$  

So our assertion becomes

$$D_{a,Y} - D_{a',Y} = D_{a,X} - D_{a',X} - \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \not\ni a', J \not\ni a} (D_{a',Z_J} - D_{a,Z_J} - D_{a',Z'_J} + D_{a,Z'_J}) - \sum_{J \in \mathcal{H}, J \subset J \not\ni a, J \not\ni a} (D_{a,R_J} - D_{a',R_J} - D_{a,R'_J} + D_{a',R'_J}) + \sum_{J \in \mathcal{H}, J \subset Y = \emptyset, J \not\ni a, J \not\ni a} (D_{a,R'_J} - D_{a',R'_J} - D_{a,R'_J} + D_{a',R'_J}),$$

that is

$$\begin{pmatrix}
D_{a,Y} - D_{a',Y} \\
+ \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \not\ni a', J \not\ni a} (D_{a,Z_J} - D_{a',Z'_J}) \\
+ \sum_{J \in \mathcal{H}, J \subset J \not\ni a, J \not\ni a} (D_{a,R_J} - D_{a',R'_J}) \\
+ \sum_{J \in \mathcal{H}, Y \subset J \not\ni a, J \not\ni a} (D_{a,Z_J} - D_{a',Z'_J}) \\
+ \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \not\ni a, J \not\ni a} (D_{a,R'_J} - D_{a',R'_J})
\end{pmatrix} = \begin{pmatrix}
D_{a,X} - D_{a',X} \\
+ \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \not\ni a', J \not\ni a} (D_{a,Z_J} - D_{a',Z'_J}) \\
+ \sum_{J \in \mathcal{H}, J \subset J \not\ni a, J \not\ni a} (D_{a,R'_J} - D_{a',R'_J}) \\
+ \sum_{J \in \mathcal{H}, Y \subset J \not\ni a, J \not\ni a} (D_{a,Z'_J} - D_{a',Z'_J}) \\
+ \sum_{J \in \mathcal{H}, J \cap Y = \emptyset, J \not\ni a, J \not\ni a} (D_{a,R'_J} - D_{a',R'_J})
\end{pmatrix}.$$}

Observe that

$$\#(\{J \in \mathcal{H} | J \ni a', J \not\ni a\} \cup \{J \in \mathcal{H} | J \ni a, J \not\ni a'\}) \leq n - 2,$$

in fact: let

$$x := \#\{J \in \mathcal{H} | J \ni a', J \not\ni a\}, \quad y := \#\{J \in \mathcal{H} | J \ni a, J \not\ni a'\};$$

the set $\{J \in \mathcal{H} | J \ni a', J \not\ni a\}$ is a chain, so in its largest $\mathcal{H}$-cluster, call it $A$, there are at least $x + 1$ elements; analogously in the largest $\mathcal{H}$-cluster contained in $\{J \in \mathcal{H} | J \ni a', J \not\ni a\}$, call it $B$, there are at least $y + 1$ elements; since $A$ and $B$ are disjoint, we have that

$$(x + 1) + (y + 1) \leq n,$$
thus \( x + y \leq n - 2 \), as we wanted to prove. Hence the number of the terms at each member of (13) is at most \( n - 1 \). Therefore it is easy to see that our assertion (13) follows from condition (iii): write it as (12) and observe that the sum

\[
\sum_{H \in \mathcal{H}, H \cap (aX) \neq \emptyset, H \nsubseteq (a'X)} H - \sum_{H \in \mathcal{H}, H \cap (a'X) \neq \emptyset, H \nsubseteq (a'X)} H
\]

is 0 for the definition of \( X \).

So we have defined the weight of \( e_i \) for every \( i \in [n] \) and the weight of \( e_J \) for every \( J \in \mathcal{H} \). Let \( \mathcal{P} = (P, w) \), where \( w \) is the weight we have just defined. We have to show that \( D_I(\mathcal{P}) = D_I \) for any \( I \in \binom{[n]}{k} \). First we show that, for any \( i, j \in [n] \),

\[
w(e_i) - w(e_j) = D_{i,X(j,i)} - D_{j,X(j,i)}, \quad (14)
\]

for any \( X(i, j) \) such that

\[
\sum_{H \in \mathcal{H}, H \cap (j, X(j,i)) \neq \emptyset} H - \sum_{H \in \mathcal{H}, H \cap (i, X(j,i)) \neq \emptyset} H = 0.
\]

Let us choose the same \( I \) in the definition of \( w(e_i) \) and \( w(e_j) \) (see (11)) and let us choose it containing neither \( i \) nor \( j \); so we get

\[
w(e_i) - w(e_j) = \frac{1}{k} \left[ \sum_{t \in I} \left( D_{i,X(t,i)} - D_{t,X(t,i)} \right) - \sum_{t \in I} \left( D_{j,X(t,j)} - D_{t,X(t,j)} \right) \right] =
\]

\[
= \frac{1}{k} \left[ \sum_{t \in I} \left( D_{i,X(t,j)} - D_{t,X(t,i)} - D_j X(t,j) + D_j X(t,j) \right) \right].
\]

For any \( t \in I \), take \( X(t, i) \) and \( X(t, j) \) equal to a set \( X_t \) satisfying the conditions of Lemma 19 for the couple \( t, i \), for the couple \( t, j \) and for the couple \( i, j \) (there exists since \( k \geq 5 \)). So we get

\[
w(e_i) - w(e_j) = \frac{1}{k} \left[ \sum_{t \in I} \left( D_{i,X_t} - D_{j,X_t} \right) \right].
\]

Moreover, by condition (iii), we have that \( D_{j,X_t} - D_{i,X_t} = D_{j,X(j,i)} - D_{i,X(j,i)} \) for any \( t \in I \), since

\[
\sum_{H \in \mathcal{H}, H \cap (j, X_t) \neq \emptyset} H - \sum_{H \in \mathcal{H}, H \cap (i, X_t) \neq \emptyset} H = 0.
\]

Hence we get (14).

Obviously, for any \( I \in \binom{[n]}{k} \), we have that

\[
D_I(\mathcal{P}) = \sum_{t \in I} w(e_t) + \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset} w(e_J).
\]
So, for any \( i \in I \),

\[
 w(e_i) = \frac{1}{k} \left[ D_I(P) + \sum_{t \in I} (w(e_i) - w(e_t)) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \ni I} w(e_J) \right],
\]

which, by (11), is equal to

\[
 w(e_i) = \frac{1}{k} \left[ D_I(P) + \sum_{t \in I} (D_{i,X(t,i)} - D_{t,X(t,i)}) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \ni I} w(e_J) \right].
\]

On the other side we have defined \( w(e_i) \) to be

\[
 \frac{1}{k} \left[ D_I + \sum_{t \in I} (D_{i,X(t,i)} - D_{t,X(t,i)}) - \sum_{J \in \mathcal{H}, J \cap I \neq \emptyset, J \ni I} w(e_J) \right],
\]

so we get \( D_I(P) = D_I \) for any \( I \).

\[\square\]

**Remark 22.** Let \( n, k \in \mathbb{N} \) with \( 2 \leq k \leq n - 2 \). Let \( \{D_I\}_{I \in [n]} \) be a family of positive real numbers. Obviously the family \( \{D_I\}_I \) is p-l-treelike if and only if there exists a hierarchy \( \mathcal{H} \) over \([n]\) such that the conditions (i), (ii) and (iii) of Theorem 21 hold, and, in addition, the numbers in (9) and (11) are positive for any \( i \in [n], J \in \mathcal{H} \).

**Remark 23.** Let \( T \) be a tree with \( L(T) = [n] \). Let \( C \subset [n] \).

a By definition, we have that \( C \) is a cherry if and only if, for any \( i, j \in C \), \( i \) and \( j \) are neighbours, and this is true if and only if, for any \( i, j \in C \), there do not exist \( x, y \in [n] - \{i, j\} \) such that \( \langle i, x, j, y \rangle \) holds.

b Let \( C \) be a complete cherry. Let \( i, j \in C \). Then \( C \) is good if and only if, for any \( k, l \in [n] - C \), we have that \( \langle i, j, k, l \rangle \) holds.

Let \( r \in \mathbb{N} \) and let us define \( T^0 = T \) and \( T^r \) to be the tree obtained from \( T^{r-1} \) by pruning the good cherries of cardinality less or equal than \( r \). If \( J \) is a good cherry of \( T^r \), we denote the stalk of \( J \), which is a leaf of \( T^{r+1} \), by \( [\min(J)] \). Let \( C \subset L(T^r) \).

c By definition, we have that \( C \) is a cherry of \( T^r \) if and only if, for any \( [i], [j] \in C \), \( [i] \) and \( [j] \) are neighbours, and this is true if and only if, for any \( [i], [j] \in C \), there do not exist \( [x], [y] \in L(T^r) - \{[i], [j]\} \) such that \( \langle [i], [x], [j], [y] \rangle \) holds. This is true if and only if, for any \( [i], [j] \in C \), there do not exist \( [x], [y] \in L(T^r) - \{[i], [j]\} \) such that \( \langle [i], [x], [j], [y] \rangle \) holds.
Let $C$ be a complete cherry of $T^*$. Let $[i], [j] \in C$. Then $C$ is good if and only if, for any $[k], [l] \in L(T^*) - C$, we have that $\langle [i], [j] | [k], [l] \rangle$ holds. This is true if and only if for any $[k], [l] \in L(T^*) - C$, we have that $\langle i, j | k, l \rangle$ holds.

Remark 24. Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n - 2$. Let $P = (P, w)$ be an internal-nonzero-weighted essential $(n - k)$-pseudostar with $L(P) = [n]$ and let us denote $D_1(P)$ by $D_1$ for any $I \in \binom{[n]}{k}$. Observe that the hierarchy $\mathcal{H}$ over $[n]$ defined by $P$ as in Definition 12 can be recovered from the family $\{D_1\}_I$ as follows:

- by part 1 of Proposition 16 the complete cherries of $P$ are exactly the maximal subsets $C$ of $[n]$ such that $D_{i,X}(P) - D_{i,X}(P)$ does not depend on $X \in \binom{[n]}{k-1}$ for any $i, l \in C$;
- we can discover which of them are good (thus, which are the minimal elements of $\mathcal{H}$) by using Remark 23 (b) and part 2 of Proposition 16;
- we can discover which subsets of $L(P^*)$ are complete cherries by using Remark 23 (c) and part 2 of Proposition 16;
- we can discover which of them are good by using Remark 23 (d) and part 2 of Proposition 16.

So we could formulate Theorem 21 in a less elegant, but more practical, way by saying that the family $\{D_1\}_I$ is $l$-treelike if and only if the hierarchy defined by the family $\{D_1\}_I$ as above satisfies conditions (i), (ii), (iii). This can be useful if we want to make a program to decide if a family of real numbers indexed by $\binom{[n]}{k}$ is $l$-treelike.

## 5 The range of the total weight

Let $\{D_1\}_{I \in \binom{[n]}{k}}$ be a $p$-$l$-treelike family in $\mathbb{R}_+$. If $2 \leq k \leq (n + 1)/2$ we know that there exists a unique positive-weighted essential tree $T = (T, w)$ with $L(T) = [n]$ and realizing the family (see Theorem 3). On the other hand, for $k > (n + 1)/2$ this statement no longer holds and, if we call $U$ the set of all positive-weighted trees realizing the family $\{D_1\}_I$, we can wonder which is the range of the total weight of the weighted trees in $U$.

Theorem 25. Let $k, n \in \mathbb{N}$ with $3 \leq k \leq n - 1$. Let $\{D_1\}_{I \in \binom{[n]}{k}}$ be a $p$-$l$-treelike family of positive real numbers and let $U$ be the set of the positive-weighted trees with $[n]$ as set of leaves and realizing the family $\{D_1\}_I$. Call $P$ the unique essential $(n - k)$-pseudostar in $U$ (see Theorem 17). The following statements hold:

(i) $$\sup_{T \in U} \{D_{\text{tot}}(T)\} = D_{\text{tot}}(P)$$

and the supremum is attained only by $P$;

(ii) if $|U| > 1$, then $$\inf_{T \in U} \{D_{\text{tot}}(T)\} = D_{\text{tot}}(P) - (n - k) \cdot m$$

where $m$ is the minimum among the weights of the twigs of $P$; the infimum is not attained.
Proof. (i) Let \( T = (T, w) \) be a weighted tree in \( U \). Without changing the dissimilarity family and the total weight we can suppose that it is essential. By using several \((n - k)\)-IO operations, we can transform it into a \((n - k)\)-pseudostar. By Remark 15 the dissimilarity family does not change, so the \((n - k)\)-pseudostar we have obtained must be the unique essential \((n - k)\)-pseudostar in \( U \), that is \( P \). By Remark 15 we have that \( D_{tot}(T) \leq D_{tot}(P) \); furthermore, if \( T \) is different from \( P \), then \( D_{tot}(T) < D_{tot}(P) \).

(ii) Suppose \(#U > 1\). Then we can make a \((n - k)\)-OI operation on \( P \): we add an edge of weight \( kx \), where \( x < m \), in such a way that the edge divides the tree in two trees with more than \( n - k \) leaves, and we subtract \( x \) from the weight of every twig of \( P \). Let \( T \) be the tree we have obtained. We have

\[
D_{tot}(T) = D_{tot}(P) + k \cdot x - n \cdot x = D_{tot}(P) - (n - k) \cdot x
\]

Obviously, the limit of \( D_{tot}(T) \), as \( x \) approaches \( m \), is \( D_{tot}(P) - (n - k) \cdot m \).

Finally, let \( A \in U \). Without changing the dissimilarity family and the total weight we can suppose that it is essential. We can transform \( A \) into \( P \) by several \((n - k)\)-IO operations, contracting edges with weights \( y_1, y_2, ... y_s \) and adding \( \frac{y_1 + y_2 + ... + y_s}{k} \) to the weight of every twig. Then we get:

\[
D_{tot}(P) = D_{tot}(A) + \frac{n - k}{k}(y_1 + y_2 + ... + y_s).
\] (15)

Furthermore, since, to obtain \( P \) from \( A \), we have added \( \frac{y_1 + y_2 + ... + y_s}{k} \) to the weight of every twig, we have that

\[
m > \frac{y_1 + y_2 + ... + y_s}{k}.
\] (16)

Thus, from (15) and (16) we get:

\[
D_{tot}(A) = D_{tot}(P) - \frac{n - k}{k}(y_1 + y_2 + ... + y_s) > D_{tot}(P) - (n - k) \cdot m.
\]

The following theorem answers the analogous problem for general weighted trees.

**Theorem 26.** Let \( k, n \in \mathbb{N} \) with \( 3 \leq k \leq n - 1 \). Let \( \{D_I\}_{i \in [n]} \) be a \( t \)-treelike family of real numbers and let \( U \) be the set of weighted trees with \([n]\) as set of leaves and realizing the family \( \{D_I\}_t \).

(i) If in \( U \) there are only weighted \((n - k)\)-pseudostars (for instance if \( k \leq \frac{n}{2} \)), then \( D_{tot}(P) = D_{tot}(P') \) for any \( P, P' \in U \); in particular

\[
\inf_{P \in U} \{ D_{tot}(P) \} = \sup_{P \in U} \{ D_{tot}(P) \}.
\]

(ii) If in \( U \) there are weighted trees that are not \((n - k)\)-pseudostars, then

\[
\inf_{T \in U} \{ D_{tot}(T) \} = -\infty, \quad \sup_{T \in U} \{ D_{tot}(T) \} = +\infty.
\]
Proof. (i) Let $P$ and $P'$ be in $U$. Denote by $\hat{P}$ and $\hat{P}'$ the weighted trees obtained respectively from $P$ and $P'$ by contracting the internal edges of weight 0. Let $\overline{P}$ and $\overline{P}'$ be the weighted essential trees equivalent respectively to $\hat{P}$ and $\hat{P}'$. Obviously both $\overline{P}$ and $\overline{P}'$ are $(n-k)$-pseudostars and realize the family $\{D_I\}$; so, by Theorem 17, they are equal. Thus $D_{\text{tot}}(\overline{P}) = D_{\text{tot}}(\overline{P}')$, therefore $D_{\text{tot}}(P) = D_{\text{tot}}(P')$.

(ii) Let $T = (T, w)$ be an element of $U$ that is not a $(n-k)$-pseudostar; then there is an edge $\overline{e}$ dividing the tree into two trees such that each of them has more than $n-k$ leaves. Let $z \in \mathbb{R}$. We define on $T$ a new weight $w'$ as follows:

$$w'(\overline{e}) := w(\overline{e}) + z;$$

for every twig $t$,

$$w'(t) := w(t) - \frac{1}{k}z;$$

for any edge $e$ different from $\overline{e}$ and not contained in a twig, we define $w'(e) = w(e)$.

Let $T' = (T, w')$. We have that $D_I(T') = D_I$ for any $I \in \binom{[n]}{k}$, so $T' \in U$. Furthermore

$$D_{\text{tot}}(T') = D_{\text{tot}}(T) + z - \frac{n}{k}z = D_{\text{tot}}(T) - \frac{n-k}{k}z.$$

Hence $\lim_{z \to -\infty} D_{\text{tot}}(T') = +\infty$, while $\lim_{z \to +\infty} D_{\text{tot}}(T') = -\infty.$

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24
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