APPORXIMATION OF THE FIXED-PROBABILITY LEVEL FOR A COMPOUND RENEWAL PROCESS

Vsevolod K. Malinovskii

Abstract. Dealing with compound renewal process with generally distributed jump sizes and inter-renewal intervals, we focus on the approximation for the fixed-probability level, which is the core of inverse level crossing problem. We are developing an analytical technique presented in [15]–[17] and based on Kendall’s identity; this yields (see [18]) inverse Gaussian approximation in the direct level crossing problem. These issues are of great importance in risk theory.

1. Introduction

In this paper, we will be focused on $T_1, T_i \overset{d}{=} T, i = 2, 3, \ldots$, with p.d.f. $f_{T_1}$ and $f_T$, which are independent positive random variables, called intervals between renewals, and on $Y_i \overset{d}{=} Y$, $i = 1, 2, \ldots$, with p.d.f. $f_Y$, which are independent positive random variables called jump sizes at the moments of renewals. We assume that these sequences are mutually independent and $T_1 \overset{d}{=} T$, i.e., we confine ourselves to the ordinary renewal process $N_s := \max \{n > 0 : \sum_{i=1}^{n} T_i \leq s\}$, with $N_s := 0$, if $T_1 > s$. Moreover, we focus on p.d.f. $f_{T_1}$, $f_T$, and $f_Y$ bounded above by a finite constant; these restriction may be relaxed, but not in this paper.

Let us introduce the random process

$$R_s := u + cs - V_s, \quad s \geq 0,$$

where $u \geq 0$ and $c \geq 0$ are constants, $V_s := \sum_{i=1}^{N_s} Y_i$, with $V_s := 0$, if $T_1 > s$. The random process $V_s, s \geq 0$, is called compound (ordinary) renewal processes; its trajectories are piecewise linear.

By the direct level crossing problem we call the study of probability $P[\Upsilon_{u,c} \leq t]$, where

$$T_{u,c} := \inf \{s > 0 : V_s - cs > u\}$$

$$= \inf \{s > 0 : R_s < 0\},$$

or $+\infty$, if $V_s - cs \leq u$ for all $s > 0$, whereas the inverse level crossing problem is focused on the study of a solution (with respect to $u$) to the equation

$$P[T_{u,c} \leq t] = \alpha,$$

where $\alpha$ is positive and reasonably small, e.g., $\alpha = 0.05$. This solution is denoted by $u_{a,c}(c)$, $c \geq 0$, and is called fixed-probability level. It is easily seen that $P[T_{u,c} \leq t] = P[\inf_{0 \leq s \leq t} R_s < 0]$, and the left-hand side of (1.3) can be rewritten accordingly.

Key words and phrases. Compound renewal process, Time of first level crossing, Fixed-probability level, Kendall’s identity, Inverse Gaussian approximation, Generalized inverse Gaussian distribution.

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The fixed-probability level defined by equation (1.3) is an implicit function. Its analysis is based on a detailed study of the probability in the left-hand side of (1.3), i.e., on the direct level crossing problem. When the random variables $T$ and $Y$ are exponentially distributed with parameters $\delta > 0$ and $\rho > 0$, the random process $N_s, s \geq 0$, is a Poisson process with intensity $\delta$. In this case, $P\{V_r < x\}$, $E(V_r)$, $D(V_r)$, and $P\{Y_{u,c} \leq t\}$ are expressed in a closed form, using elementary and special functions such as the modified Bessel functions $I_\alpha(z)$ of the first kind of order $n$. Consequently, equation (1.3) is written explicitly and the study of an implicit function $u_{\alpha,\delta}(c), c \geq 0$, is carried out in [12, 14]; it goes along the road map set before in [11, 13] in the diffusion model.

In the case of generally distributed random variables $T$ and $Y$, to find a solution to the direct (let alone inverse) level crossing problem in terms of elementary and special functions seems impossible, except for a few very special cases, whence our attention to approximations of $P\{Y_{u,c} \leq t\}$, as $u \to \infty$ and $u, t \to \infty$, and of $u_{\alpha,\delta}(c), c \geq 0$, as $t \to \infty$.

We proceed investigating $u_{\alpha,\delta}(c), c \geq 0$, from the inverse Gaussian approximation obtained in [18, 15–17]. In a nutshell, we use Kendall’s identity for $P\{Y_{u,c} \leq t\}$, which expresses this probability through convolution powers of $f_T$ and $f_Y$; the central limit theory is then applied to them. This method is widely applicable, e.g., it allows us to find approximations for the first-order derivatives $\frac{\partial}{\partial u} P\{Y_{u,c} \leq t\}$ and $\frac{\partial}{\partial c} P\{Y_{u,c} \leq t\}$, and even for higher-order derivatives, such as $\frac{\partial^2}{\partial c^2} P\{Y_{u,c} \leq t\}$ and $\frac{\partial^2}{\partial u \partial c} P\{Y_{u,c} \leq t\}$. This issue is central (see Theorem 7.1) in the study of, e.g., monotony and convexity of the implicitly defined function $u_{\alpha,\delta}(c), c \geq 0$.

This approach is aimed at obtaining a large set of results using standard techniques. Such results include approximations and estimates of the rate of convergence, as well as various refinements, e.g., asymptotic expansions. The main focus is on the diversity and accuracy of the results, rather than the minimality of technical conditions, although the conditions in these results are close to minimal.

In applications, both direct and inverse level crossing problems are of a great importance. In risk theory, the function $u_{\alpha,\delta}(c), c \geq 0$, models a non-ruin capital that makes the probability of ruin over time $t$ equal to a predetermined value $\alpha$, chosen as an acceptable degree of insolvency. This academic concept is related to fundamental methods of insurance solvency’s regulation (see, e.g., [11, 6, 22, 26, 27]); in practice, they are mainly implemented by simulation. Analytically, as a problem of collective risk theory, the inverse level crossing problem was first investigated in [12] (see also [14]), where equitable solvent controls in a multi-period game model of risk were considered; as a partial single-period model, Lundberg’s model with exponentially distributed claim size was focussed in [12]; similar issue in the diffusion risk model was investigated in [13].

The rest of this paper is arranged as follows. In Section 2, we recall Kendall’s identity. In Section 3, we derive similar identities for derivatives $\frac{\partial}{\partial c} P\{Y_{u,c} \leq t\}$ and $\frac{\partial}{\partial u} P\{Y_{u,c} \leq t\}$. In Section 4, we outline the inverse Gaussian approximation (see [18]) in the direct level crossing problem; this result is presented in detail in [15, 16, 20]. In Section 5, we establish approximations for derivatives, using the same stages as in the proof of inverse Gaussian approximation. In Section 6, which is core of this paper, we focus on approximations in the inverse level crossing.
Making the change of variables \( y \delta > n \) exponentially distributed with parameters \( Y \) between these renewal processes.

\[ \mathbb{E}_i \delta > n \text{ exponentially distributed with parameters } Y \]

\[ t \text{ transitions and closed-form results presented in } [10] \]

\[ \text{finally with heuristic fixed-probability level and with elementary bounds on the fixed-probability level, which are a tool for numerical calculations.} \]

2. Kendall’s identity: a keystone result

Let us introduce \( M_k := \inf \left\{ k \geq 1 : \sum_{i=1}^{k} Y_i > x \right\} - 1, \quad x > 0, \]

(2.1)

which is a renewal process generated by the random variables \( Y_i, i = 1, 2, \ldots \). The following result is known as Kendall’s identity (see first \([10]\), and then \([2]-[3], [9], [24], [29], [31]\)).

Assertion 2.1 (Kendall’s identity). With \( 0 < v < t \), we have

\[ \mathbb{P}[v < T_{a,c} \leq t \mid T_1 = v] = \frac{u + cv}{u + cz} \mathbb{P}[M_{u+c} = n] f^n_T(z - v) \]

(2.2)

Proceeding from Assertion 2.1 and using the equality

\[ \mathbb{P}[T_{a,c} \leq t] = \int_0^t \mathbb{P}[u + cv - Y < 0] f_T(v) dv \]

(2.3)

we switch back to (unconditional) distribution of the level crossing time.

The identity (2.2) may be rewritten exclusively in terms of \( n \)-fold convolutions \( f^n_T \) and \( f^n_T \). Indeed, bearing in mind that \( Y_i, i = 1, 2, \ldots \), are i.i.d., we have

\[ \mathbb{P}[M_{u+c+c+} = n] = \mathbb{P}\left\{ \sum_{i=1}^{n} Y_i \leq u + cv + cy < \sum_{i=1}^{n+1} Y_i \right\} \]

\[ = \int_{u+c+c+}^{u+c+c+} f^n_T(u + cv + cy - z) \mathbb{P}[Y_{n+1} > z] dz. \]

Making the change of variables \( y = z - v \) in (2.2), we rewrite it as

\[ \mathbb{P}[v < T_{a,c} \leq t \mid T_1 = v] = \sum_{n=1}^{\infty} \int_0^{u+cv} \frac{u + cv}{u + cv + cy} \mathbb{P}[Y_{n+1} > z] \]

\[ \times f^n_T(u + cv + cy - z) f^n_T(y) dy dz. \]

Equality (2.2), or its copy (2.3), and equality (2.2) are fundamental in a series of approximations and closed-form results presented in \([12], [14]-[20]\). In particular, when \( T \) and \( Y \) are exponentially distributed with parameters \( \delta > 0 \) and \( \rho > 0 \), closed-form expressions for \( \mathbb{P}[T_{a,c} \leq t] \) follow from the next corollary of Assertion 2.1.
Corollary 2.1. For $Y$ exponentially distributed with parameter $\rho > 0$, we have

$$
\mathbb{P}[T_{u,c} \leq t] = \int_0^t e^{-\rho(u+cv)} \left( f_T(s) + \frac{1}{u+cs} \sum_{n=1}^{\infty} \left( \rho(u+cs) \right)^n n! \right) \times \int_0^t (u+cv) f_T^n(s-v) f_T(v) \, dv \, ds.
$$

(2.5)

Proof of Corollary 2.1 For $Y \overset{d}{=} Y_i$, $i = 1, 2, \ldots$, when $Y$ is exponentially distributed with parameter $\rho$, we have $\mathbb{P}[u + cv - Y < 0] = e^{-\rho(u+cv)}$ and

$$
\mathbb{P}[M_{u+cv} = n] = \frac{\rho(u+cs)^n}{n!}, \quad n = 1, 2, \ldots,
$$

whence equality (2.3) rewrites as (2.5). □

3. Derivatives of $\mathbb{P}[T_{u,c} \leq t]$ via Kendall’s identity

Let p.d.f. $f_T$ and $f_Y$ be differentiable. Kendall’s identity (2.2), or its copy (2.4), and equality (2.3) allow us to express the derivatives of $\mathbb{P}[T_{u,c} \leq t]$ with respect to $c$ and $u$ in a similar way.

3.1. Derivative $\frac{\partial}{\partial v} \mathbb{P}[T_{u,c} \leq t]$. Let us start with the derivative of $\mathbb{P}[T_{u,c} \leq t]$ with respect to $c$ and introduce the following expressions:

$$
\mathcal{C}^{[1]}_{u,c}(t \mid v) = - \sum_{n=1}^{\infty} \int_0^t \left( \frac{u(v+y)}{(u+cv+cy)^2} \right) \times \int_0^{u+cv+cy} \mathbb{P}[Y_{n+1} > z] f_T^n(u + cv + cy - z) \, dz \, f_T^n(y) \, dy,
$$

(3.1)

$$
\mathcal{C}^{[2]}_{u,c}(t \mid v) = \sum_{n=1}^{\infty} \int_0^t \frac{(u + cv)(v+y)}{u + cv + cy} \times \int_0^{u+cv+cy} \mathbb{P}[Y_{n+1} > z] \left( f_T(0)f_T^{n(n-1)}(u + cv + cy - z) + \int_0^{u+cv+cy} f_T(\xi)f_T^{n(n-1)}(u + cv + cy - z - \xi) \, d\xi \right) \, dz \, f_T^n(y) \, dy,
$$

$$
\mathcal{C}^{[3]}_{u,c}(t \mid v) = \sum_{n=1}^{\infty} f_T^n(0) \int_0^t \frac{(u+cv)(v+y)}{u+cv+cy} \mathbb{P}[Y_{n+1} > u + cv + cy] \times f_T^n(y) \, dy.
$$

Lemma 3.1. For $c > 0$, $u > 0$, $t > v > 0$, we have

$$
\frac{\partial}{\partial c} \mathbb{P}[T_{u,c} \leq t] = - \int_0^t f_Y(u + cv) \, v \, f_T(v) \, dv
$$

(3.2)

where

$$
\frac{\partial}{\partial c} \mathbb{P}[v < T_{u,c} \leq t \mid T_1 = v] = \mathcal{C}^{[1]}_{u,c}(t \mid v) + \mathcal{C}^{[2]}_{u,c}(t \mid v) + \mathcal{C}^{[3]}_{u,c}(t \mid v).
$$

(3.3)
Proof of Lemma 3.1. The proof is based on identities (2.3) and (2.4), i.e.,
\[ P[Y_{n,c} \leq t] = \int_0^t P[u + cv - Y < 0] f_T(v) \, dv \]
and
\[ P[v < Y_{n,c} \leq t \mid T_1 = v] = \sum_{n=1}^{\infty} \int_0^{c^{-v}} \frac{u + cv}{u + cv + cy} \int_0^{u + cy} P[Y_{n+1} > z] \]
\[ \times f_T^n(u + cv - z) f_T(v) \, dy \, dz. \]  
(3.5)

Differentiating (3.5), we have
\[ \frac{\partial}{\partial c} P[v < Y_{n,c} \leq t \mid T_1 = v] = \sum_{n=1}^{\infty} \int_0^{c^{-v}} \frac{u + cv}{u + cv + cy} \int_0^{u + cy} P[Y_{n+1} > z] \]
\[ \times f_T^n(u + cv + cy - z) f_T(v) \, dy \, dz. \]
(3.5)

The integrand is
\[ \frac{\partial}{\partial c} \left( \frac{u + cv}{u + cv + cy} \int_0^{u + cy} P[Y_{n+1} > z] f_T^n(u + cv - z) \, dz \right) \]
\[ = \frac{\partial}{\partial c} \left( \frac{u + cv}{u + cv + cy} \right) \int_0^{u + cy} P[Y_{n+1} > z] f_T^n(u + cv - z) \, dz \]
\[ + \frac{u + cv}{u + cv + cy} \frac{\partial}{\partial c} \left( \int_0^{u + cy} P[Y_{n+1} > z] f_T^n(u + cv - z) \, dz \right), \]
where
\[ \frac{\partial}{\partial c} \left( \frac{u + cv}{u + cv + cy} \right) = -\frac{uv}{(u + cv + cy)^2} \]
and
\[ \frac{\partial}{\partial c} \left( \int_0^{u + cy} P[Y_{n+1} > z] f_T^n(u + cv - z) \, dz \right) \]
\[ = \int_0^{u + cy} P[Y_{n+1} > z] \left( \frac{\partial}{\partial c} f_T^n(u + cv - z) \right) dz \]
\[ + (v + y) P[Y_{n+1} > u + cv + cy] f_T^n(0). \]

For \( n \geq 2 \), differentiation of \( n \)-fold convolutions yields
\[
\frac{\partial}{\partial c} f_T^n(u + cv + cy - z) = \frac{\partial}{\partial c} \int_0^{u + cy + cz} f_T((u + cv + cy - z) - \zeta) f_T^{(n-1)}(\zeta) \, d\zeta
\]
\[ = (v + y) \int_0^{u + cy + cz} f_T'((u + cv + cy - z) - \zeta) f_T^{(n-1)}(\zeta) \, d\zeta
\]
\[ + (v + y) f_T(0) f_T^{(n-1)}(u + cv + cy - z), \]
whence, by elementary calculations, the result. \( \square \)
3.2. Derivative $\frac{\partial}{\partial u} P[T_{u,c} \leq t]$. Let proceed with the derivative of $P[T_{u,c} \leq t]$ with respect to $u$ and introduce the following expressions:

\[
U_{u,c}^{[1]}(t \mid v) = \sum_{n=1}^{\infty} \int_0^{t-v} \frac{cy}{(u + cv + cy)^2} \times \int_0^{u+cv+cy} P[Y_{n+1} > z] f_y^n(u + cv - z) f_y^n(y) \, dy \, dz,
\]

\[
U_{u,c}^{[2]}(t \mid v) = \sum_{n=1}^{\infty} \int_0^{t-v} \frac{u + cv}{u + cv + cy} \times \int_0^{u+cv+cy} P[Y_{n+1} > z] \left\{ f_y(0) f_y^{(n-1)}(u + cv - z) + \int_0^{u+cv+cy} f_y'(\xi) f_y^{(n-1)}((u + cv - z) - \xi) \, d\xi \right\} \times f_y^n(y) \, dy \, dz,
\]

\[
U_{u,c}^{[3]}(t \mid v) = \sum_{n=1}^{\infty} f_y^n(0) \int_0^{t-v} \frac{u + cv}{u + cv + cy} P[Y_{n+1} > u + cv + cy] \times f_T^n(y) \, dy.
\]

**Lemma 3.2.** For $c > 0$, $u > 0$, $t > v > 0$, we have

\[
\frac{\partial}{\partial u} P[T_{u,c} \leq t] = - \int_0^t f_y(u + cv) v f_T(v) \, dv + \int_0^t \frac{\partial}{\partial u} P[v < T_{u,c} \leq t \mid T_1 = v] f_T(v) \, dv,
\]

where

\[
\frac{\partial}{\partial u} P[v < T_{u,c} \leq t \mid T_1 = v] = U_{u,c}^{[1]}(t \mid v) + U_{u,c}^{[2]}(t \mid v) + U_{u,c}^{[3]}(t \mid v).
\]

**Proof of Lemma 3.2** Differentiating identity (3.5), we have

\[
\frac{\partial}{\partial u} P[v < T_{u,c} \leq t \mid T_1 = v] = \sum_{n=1}^{\infty} \int_0^{t-v} \frac{u + cv}{u + cv + cy} \int_0^{u+cv+cy} P[Y_{n+1} > z] \times f_y^n(u + cv - z) \, dz f_T^n(y) \, dy.
\]

The integrand is

\[
\frac{\partial}{\partial u} \left( \frac{u + cv}{u + cv + cy} \int_0^{u+cv+cy} P[Y_{n+1} > z] f_y^n(u + cv - z) \, dz \right)
\]

\[
= \frac{\partial}{\partial u} \left( \frac{u + cv}{u + cv + cy} \right) \int_0^{u+cv+cy} P[Y_{n+1} > z] f_y^n(u + cv - z) \, dz
\]

\[
+ \frac{u + cv}{u + cv + cy} \frac{\partial}{\partial u} \left( \int_0^{u+cv+cy} P[Y_{n+1} > z] f_y^n(u + cv - z) \, dz \right).
\]
where \( \frac{\partial}{\partial u} \left( \frac{u + cv}{u + cv + cy} \right) = \frac{cy}{(u + cv + cy)^2} \) and
\[
\frac{\partial}{\partial u} \left( \int_0^{u + cv + cy} P[Y_{n+1} > z] f_Y^n(u + cv + cy - z) \, dz \right) = \int_0^{u + cv + cy} P[Y_{n+1} > z] \left( \frac{\partial}{\partial u} f_Y^n(u + cv + cy - z) \right) \, dz + P[Y_{n+1} = u + cv + cy] f_Y^n(0),
\]
whence, by elementary calculations, the result. \( \square \)

4. Approximations in direct level crossing problem

The probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of a Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \) are denoted by \( \Phi_{(\mu,\sigma^2)}(x) \) and \( \varphi_{(\mu,\sigma^2)}(x) \).

4.1. Core integral expressions. For \( t, u, c, M, \) and \( D^2 \) fixed positive constants, the elementary integral expressions are defined as
\[
I_{u,c}^{[b]}(t) := \int_0^t \frac{1}{(x+1)^k} \Phi_{(\mu(c+1)^2,\sigma^2(c+1)^2)}(x) \, dx, \quad k = 0, 1, 2, \ldots \quad (4.1)
\]
We write \( c^* := M^{-1} \) and \( I_u^{[e]}(t) := \lim_{c^* \to 0} I_{u,c}^{[b]}(t), I_{u,\infty}^{[b]}(t) := \lim_{c^* \to \infty} I_{u,c}^{[b]}(t), \) and the like.

These integral expressions can be expressed through c.d.f. \( F(x; \mu, \lambda, p) \) of a generalized inverse Gaussian distribution\(^6\), which depends on parameters \( \mu > 0, \lambda > 0, \) and \( p \in \mathbb{R}, \) and whose p.d.f. is\(^7\)
\[
f(x; \mu, \lambda, p) := \frac{e^{-\frac{x}{\mu}}}{2\mu^p K_p\left(\frac{\lambda}{\mu}\right)} x^{p-1} \exp\left\{ -\frac{\lambda(x - \mu)^2}{2\mu^2 x} \right\}
= \sqrt{2\pi} e^{-\frac{x}{\mu}} 2\mu^p K_p\left(\frac{\lambda}{\mu}\right) x^{p-1} \varphi_{(0,1)}\left(\sqrt{\frac{\lambda}{\mu}} \left(\frac{x}{\mu} - 1\right)\right), \quad x > 0.
\]

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\(^6\)There are some differences in terminology. In \([32]\), this distribution is called Wald’s distribution. Several authors (see \([21]\), \([29]\), \([33]\)) attribute the invention of generalized inverse Gaussian distributions to E. Halphen and use the term “Halphen Distribution System” or “Halphen’s laws”. The others, e.g., M.A. Chaudry and S.M. Zubair, refer to B. Jørgensen \([2]\) and attribute the invention of generalized inverse Gaussian distribution to I.J. Good \([8]\).

\(^7\)Note that the choice \( p = -1/2 \) yields the “ordinary” inverse Gaussian distribution.
where $K_p(z)$, $z > 0$, with $p \in \mathbb{R}$, denotes the modified Bessel function of the second kind. In particular, for $u > 0$, $t > 0$, we have

$$I_{u,c}^{[1]}(t) = \begin{cases} \left[ F(x + 1; \mu, \lambda, -\frac{t}{2}) ight]_{x = \frac{\mu}{\mu - t} = \frac{t}{\mu - t}}, & 0 < c < c^*, \\ \exp \left\{ -\frac{2 \lambda}{\mu} \right\} \left[ F(x + 1; \mu, \lambda, -\frac{t}{2}) ight]_{x = \frac{\mu}{\mu - t} = \frac{t}{\mu - t}}, & c > c^*, \end{cases}$$

(4.3)

and

$$I_{u,0}^{[1]}(t) = \Phi_{(0,1)} \left( \frac{M \sqrt{u}}{D} \right) - \Phi_{(0,1)} \left( \frac{M u - t}{D \sqrt{u}} \right),$$

$$I_{u,c}^{[1]}(t) = 2 \Phi_{(0,1)} \left( \frac{M \sqrt{u}}{D} \right) - \Phi_{(0,1)} \left( \frac{M u - t}{D \sqrt{u} + c^*} \right),$$

$$I_{u,0,c}^{[1]}(t) = 0.$$  

The c.d.f. of generalized inverse Gaussian distribution can be represented in terms of c.d.f. and p.d.f. of a standard Gaussian distribution, e.g.,

$$F(x; \mu, \lambda, -\frac{t}{2}) = \Phi_{(0,1)} \left( \sqrt{\frac{1 - \lambda}{\lambda} \left( \frac{x}{\mu} - 1 \right)} \right) + \exp \left\{ -\frac{2 \lambda}{\mu} \right\} \Phi_{(0,1)} \left( -\sqrt{\frac{1 - \lambda}{\lambda} \left( \frac{x}{\mu} + 1 \right)} \right), \quad x > 0,$$

(4.5)

and

$$F(x; \mu, \lambda, -\frac{t}{2}) = \Phi_{(0,1)} \left( \sqrt{\frac{1 - \lambda}{\lambda} \left( \frac{x}{\mu} - 1 \right)} \right) + \frac{\lambda^2 - 3 \lambda \mu + 3 \mu^2}{\lambda^2 + 3 \lambda \mu + 3 \mu^2} \exp \left\{ \frac{2 \lambda}{\mu} \right\} \times \Phi_{(0,1)} \left( -\sqrt{\frac{1 - \lambda}{\lambda} \left( \frac{x}{\mu} + 1 \right)} \right) + \frac{2 \sqrt{\lambda} \mu (\lambda + 3 \mu)}{\lambda^{3/2} (\lambda^2 + 3 \lambda \mu + 3 \mu^2)} \varphi_{(0,1)} \left( \sqrt{\frac{1 - \lambda}{\lambda} \left( \frac{x}{\mu} - 1 \right)} \right), \quad x > 0.$$  

**4.2. Inverse Gaussian approximation, as $u$ tends to infinity.** The inverse Gaussian approximation for $P(T_{u,c} \leq t)$ in the direct level crossing problem was studied in [15–18, 20]. For $c > 0$, $u > 0$, $0 < v < t$, $c^* := E Y / E T$, and for

$$M := E T / E Y, \quad D^2 := ((E T)^2 D Y + (E Y)^2 D T) / (E Y)^3$$

(4.6)

we write

$$M_{u,c}(t \mid v) := \int_0^{v - (t - v)} \frac{1}{x + 1} \varphi_{\left( M (x + 1), \frac{\delta^2}{\rho} (x + 1) \right)}(x) \, dx$$

and note that $M_{u,c}(t) := M_{u,c}(t \mid 0)$ equals $I_{u,c}^{[1]}(t) = \int_0^t \frac{1}{x + 1} \varphi_{\left( M (x + 1), \frac{\delta^2}{\rho} (x + 1) \right)}(x) \, dx.$

The following theorem for conditional distribution of $T_{u,c}$ (see the left-hand side of (2.2)) is fundamental.

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8We do not present here all the expressions for $F(x; \mu, \lambda, -\frac{t}{2})$, $F(x; \mu, \lambda, -\frac{t}{2})$, for the derivatives like $\frac{d}{dx}F(x; \mu, \lambda, -\frac{t}{2})$ and $\frac{d}{dx}F(x; \mu, \lambda, -\frac{t}{2})$, and for $I_{u,0}^{[1]}(t)$, $I_{u,c}^{[1]}(t)$, though they are available by means of direct calculations and similar to these presented: this would require dramatically more space. We leave this to the reader.

9For $T$ and $Y$ exponentially distributed with parameters $\delta > 0$ and $\rho > 0$, we have $E T = \delta^{-1}$, $E Y = \rho^{-1}$, $D T = \delta^{-2}$, $D Y = \rho^{-2}$, and $M = \rho / \delta$, $D^2 = 2 \rho / \delta^2$, whence $D M^2 = \sqrt{2\rho / \delta}$. 
Theorem 4.1. In the renewal model, let p.d.f. \( f_Y \) and \( f_Y^* \) be bounded above by a finite constant, \( D > 0, E(T^3) < \infty, E(V^3) < \infty \). Then for any fixed \( c > 0 \) and \( 0 < v < t \) we have

\[
\sup_{r > 0} \left| \mathbb{P} \{ v < T_{u,c} \leq t \mid T_1 = v \} - M_{u,c}(t \mid v) \right| = O \left( \frac{\ln (u + cv)}{u + cv} \right),
\]

where \( \alpha = 1/2 \), \( D \) and \( c \) are fixed.

The following results for non-conditional distribution of \( T_{u,c} \) is an easy corollary of Theorem 4.1 and equality (2.3).

Theorem 4.2. Suppose that conditions of Theorem 4.1 are satisfied. Then

\[
\sup_{r > 0} \left| \mathbb{P} \{ T_{u,c} \leq t \} - \int_0^t M_{u,c}(t \mid v) f_T(v) \, dv \right| = O \left( \frac{\ln u}{u} \right), \quad u \to \infty.
\]

We can replace the integral \( \int_0^t M_{u,c}(t \mid v) f_T(v) \, dv \) by the integral \( \int_0^t M_{u,c}(t - v) f_T(v) \, dv \), which is a convolution. It agrees with the probabilistic intuition about the role which plays the first time interval \( T_1 \) in the event of crossing a high level \( u \) within finite time \( t \); given \( T_1 = v \), the whole time length becomes \( t - v \), with no other changes.

Theorem 4.3. Suppose that conditions of Theorem 4.1 are satisfied, and that \( \mathbb{E} T_1 < \infty \). Then

\[
\sup_{r > 0} \left| \mathbb{P} \{ T_{u,c} \leq t \} - \int_0^t M_{u,c}(t - v) f_T(v) \, dv \right| = O \left( \frac{\ln u}{u} \right), \quad u \to \infty.
\]

4.3. Approximation, as \( u \) and \( t \) tend to infinity. Whereas the influence of \( T_1 \) in Theorem 4.3 can not be eliminated for \( t \) small and moderate, it becomes negligible for \( t \) large. Given that \( t \to \infty \), the integral \( \int_0^t M_{u,c}(t - v) f_T(v) \, dv \) in Theorem 4.3 can be approximated by \( M_{u,c}(t) \), whence the following result.

Theorem 4.4. Suppose that conditions of Theorem 4.1 are satisfied, and that \( \mathbb{E} T_1 < \infty \). Then

\[
\sup_{r > 0} \left| \mathbb{P} \{ T_{u,c} \leq t \} - M_{u,c}(t) \right| = O \left( \frac{\ln u}{u} + \frac{1}{u^{1/2}} \right), \quad t \to \infty.
\]

4.4. An outline of the proof. The proofs in [15], [16] are conducted in a uniform manner. In the proof of Theorem 4.1, Step 0 is the identity (2.2), or its copy (2.4), for the probability \( \mathbb{P} \{ v < T_{u,c} \leq t \mid T_1 = v \} \). Step 1 is a reduction of the range of integration in (2.2). This cutting off of unlikely events, such as when \( Y_{n+1} \) is excessively large compared to the whole sum \( \sum_{i=1}^n Y_i \), complies with intuition. Step 2 is a reduction of the range of summation in (2.2). This applies Nagaev’s inequalities for sums used to reject the summands in the range \( 0 < n < \epsilon n_{u,c} \), for which the probability of the event \( \{ M_{u+cv+cy} = n \} = \{ \sum_{i=1}^n Y_i < u + cv + cy < \sum_{i=1}^{n+1} Y_i \} \) is small, as \( u + cv + cy \) is large. This step is also intuitively clear. It relies on the fact that under mild technical assumptions the occurrence of few renewals in a long time interval is an unlikely event.

Step 3 consists in applying the Berry-Esseen bound with non-uniform remainder term to \( n \)-fold convolutions \( f_Y^{*n} \) and \( f_Y^{*n} \) in (2.4), where the ranges of integration and summation are reduced. It is the nub of the proof, where the full force of the central limit theory is applied.

Step 4 is an estimation of the remaining term, and Step 5 is an elaboration of the approximating term obtained in this way. In the course of this study, certain identities (see, e.g., Section 4.3 in [15]) are used, which allow us to represent sums in the form of integral sums.

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10With \( c \) and \( v \) fixed, \( u + cv \to \infty \) is trivially equivalent to \( u \to \infty \).
5. Approximation for derivatives

The interest in derivatives \( \frac{\partial}{\partial c} P[Y_{a,c} \leq t] \) and \( \frac{\partial}{\partial u} P[Y_{a,c} \leq t] \) is related, e.g., with Theorem 4.4. Equality (7.1) is the basis of a standard method for studying the monotony of an implicit function.

In the case we are considering, for each fixed \( u \), the derivative \( \frac{\partial}{\partial u} P[Y_{a,c} \leq t] \), considered as a function of \( c \), is monotone decreasing, as the variable \( c \geq 0 \) monotone increases; for each fixed \( c \), this function is monotone increasing, as the variable \( u \geq 0 \) monotone increases.

We will study these derivatives much deeper. First, we examine their approximations; this will allow us to draw conclusions about their magnitude. Secondly, we develop a technique that allows us to study higher order derivatives in the same way.

5.1. Approximation for derivative \( \frac{\partial}{\partial c} P[Y_{a,c} \leq t] \).

Theorem 5.1. In the renewal model, let p.d.f. \( f_T \) and differentiable \( f_Y \) be bounded above by a finite constant, \( D^2 > 0, E(T^2) < \infty, E(T^3) < \infty \). Then for any fixed \( c > 0 \) and \( 0 < v < t \) we have

\[
\frac{\partial}{\partial c} P[Y_{a,c} \leq t] = - \int_0^t f_T(u + cv) v f_T(v) \, dv + \int_0^t \frac{\partial}{\partial c} P[v < Y_{a,c} \leq t \mid T_1 = v] f_T(v) \, dv, \tag{5.1}
\]

where

\[
\sup_{v \leq v} \left| \frac{\partial}{\partial c} P[v < Y_{a,c} \leq t \mid T_1 = v] \right| - \frac{M(u + cv)}{c^2 D^2} \left( (1 - cM) I_{a,c}^{[0]}(t \mid v) - I_{a,c}^{[1]}(t \mid v) \right) + \frac{Mu}{c^2 D^2} \left( (1 - cM) I_{a,c}^{[1]}(t \mid v) - I_{a,c}^{[2]}(t \mid v) \right) + \frac{1}{c} I_{a,c}^{[1]}(t \mid v) - \frac{1}{c} I_{a,c}^{[2]}(t \mid v) = O \left( \frac{\ln(u + cv)}{u + cv} \right),
\]

for as \( u + cv \to \infty \).

The starting point in the proof of Theorems 4.1 was Kendall’s identity, whereas the starting point in the proof of Theorem 5.1 is Lemma 5.1. The proof of Theorem 5.1 follows the scheme outlined in Section 4.4. We leave the details to the reader.

In the same way as Theorem 4.4 follows from Theorem 4.1 as \( u, t \to \infty \), it follows from Theorem 5.1 that the derivative \( \frac{\partial}{\partial u} P[Y_{a,c} \leq t] \) is approximated, as \( u, t \to \infty \), by the expression

\[
F_{a,c}(t) = \frac{Mu}{c^2 D^2} \left( (1 - cM) I_{a,c}^{[0]}(t) - I_{a,c}^{[1]}(t) \right) - \frac{Mu}{c^2 D^2} \left( (1 - cM) I_{a,c}^{[1]}(t) - I_{a,c}^{[2]}(t) \right) - \frac{1}{c} I_{a,c}^{[1]}(t) + \frac{1}{c} I_{a,c}^{[2]}(t). \tag{5.3}
\]

Using the representation of \( I_{a,c}^{[0]}(t), I_{a,c}^{[1]}(t), I_{a,c}^{[2]}(t) \), such as (4.3), we can verify that

\[
(1 - cM) I_{a,c}^{[0]}(t) - I_{a,c}^{[1]}(t) = O(u^{-1}), \quad u \to \infty, \tag{5.4}
\]

and that

\[
(1 - cM) I_{a,c}^{[1]}(t) - I_{a,c}^{[2]}(t) = O(u^{-1}), \quad u \to \infty. \tag{5.5}
\]
Alternatively, this equality is written as
\[ M^{(0,1)}_{u,c}(t) = \frac{u}{c^3D^2} \left( (1 - cM) T^{[0]}_{u,c}(t) - \frac{1}{c} \frac{u (2 - cM)}{c^2D} + 1 \right) T^{[1]}_{u,c}(t) \]
\[ + \frac{u}{c^3D^2} T^{[2]}_{u,c}(t) + \frac{t}{u + ct} \varphi (cM(1+\psi), \frac{cM^2}{u+ct}(1+\psi); cM(1+\psi)) \left( \frac{ct}{u} \right). \]

This equality\(^{11}\) is suitable for comparison with equality (5.3). Both are illustrated in Fig. 1.

We conclude this analysis with the following summary. The proximity between \( F_{u,c}(t) \), i.e., “approximation of derivative”, and \( M^{(0,1)}_{u,c}(t) \), i.e., “derivative of approximation”, illustrated numerically in Fig. 1 can be proved rigorously using equalities (5.3) and (5.6), evaluated analytically. However, the “approximation of derivative” is one thing and the “derivative of approximation” is another. Their study requires a separate analysis; the naive belief that one can replace the other is largely groundless.

5.2. Approximation for derivative \( \frac{\partial}{\partial u} P[T_{u,c} \leq t] \).

Theorem 5.2. In the renewal model, let p.d.f. \( f_T \) and differentiable \( f_Y \) be bounded above by a finite constant, \( D^2 > 0 \), \( E(T^3) < \infty \), \( E(Y^3) < \infty \). Then for any fixed \( c > 0 \) and \( 0 < v < t \) we have
\[ \frac{\partial}{\partial u} P[T_{u,c} \leq t] = - \int_0^t f_Y(v + cv) f_T(v) dv \]
\[ + \int_0^t \frac{\partial}{\partial u} P[v < T_{u,c} \leq t \mid T_1 = v] f_T(v) dv, \tag{5.7} \]

\(^{11}\)Bear in mind the asymptotic relations (5.3) and (5.5).
where
\[
\sup_{v \in \Omega_T} \left| \frac{\partial}{\partial u} P[v < Y_{u,c} \leq t \mid T_1 = v] \right| = \frac{M}{c D^2} \left( (1 - cM) \mathcal{I}_{u,c}^{[1]}(t \mid v) - \mathcal{I}_{u,c}^{[2]}(t \mid v) \right) - \frac{1}{u} \left( \mathcal{I}_{u,c}^{[1]}(t \mid v) - \mathcal{I}_{u,c}^{[2]}(t \mid v) \right) = O \left( \frac{\ln (u + cv)}{(u + cv)^2} \right),
\] as \( u + cv \to \infty. \)

Remark 2. Let us compare (5.9) with \( M_{u,c}^{(1,0)}(t) := \frac{a}{c} \mathcal{M}_{u,c}(t) \). In other words, let us compare the “approximation of derivative” with the “derivative of approximation”.

For \( u > 0, t > 0, c > 0 \), we have by straightforward differentiation
\[
M_{u,c}^{(1,0)}(t) = -\frac{(1 - cM)^2}{2c^2 D^2} \mathcal{I}_{u,c}^{[0]}(t) + \frac{1}{2u} \mathcal{I}_{u,c}^{[1]}(t) + \frac{1 - cM}{c^2 D^2} \mathcal{I}_{u,c}^{[1]}(t) - \frac{1}{2c^2 D^2} \mathcal{I}_{u,c}^{[2]}(t) - \frac{ct}{u (u + ct)} \phi \left( cM(1 + \frac{u}{u}) - \frac{ct}{(1 + \frac{u}{u})} \bigg) \left( \frac{ct}{u} \right).\]

Bear in mind the asymptotic relation (5.5).
Alternatively, this equality is written as
\[
M_{\alpha,c}(t) = \frac{1}{c^2D^2} \left( (1 - cM) J_{\alpha,c}^{[1]}(t) - J_{\alpha,c}^{[2]}(t) \right) - \frac{1}{2c^2D^2} \left( (1 - cM)^2 J_{\alpha,c}^{[0]}(t) - J_{\alpha,c}^{[2]}(t) \right) + \frac{1}{2a} I_{\alpha,c}^{[1]}(t) - \frac{ct}{u(u + ct)} \phi \left( c(M(1 + \frac{\varphi}{\sqrt{u}}) + \frac{\varphi^2}{\sqrt{u}}) (1 + \frac{\varphi}{u}) \right) \left( \frac{ct}{u} \right).
\]
(5.10)

This equality is suitable for comparison with equality (5.9). Both are illustrated in Fig. 2.

Remark 3. At the beginning of Section 3 we noted that the derivative \( \frac{\partial}{\partial a} P[Y_{\alpha,c} \leq t] \) is negative for all \( u > 0 \). But neither the approximation \( G_{\alpha,c}(t) \) nor the expression \( M_{\alpha,c}(0)(t) \) is negative for all \( u ; \) for \( u \) small and moderate (see Fig. 2, both these expressions take positive values. This is not a flaw, because the approximation of Theorem 5.2 only works for \( u \) large.

5.3. Approximation for higher-order derivatives. The approximations for higher-order derivatives, such as \( \frac{\partial^2}{\partial a^2} P[Y_{\alpha,c} \leq t] \), \( \frac{\partial^2}{\partial a^2} P[Y_{\alpha,c} < t] \), and \( \frac{\partial^2}{\partial a^2} P[Y_{\alpha,c} \leq t] \) used in (7.2) are carried out as described above, and left to the reader.

6. Approximations in inverse level crossing problem

6.1. Structural results for fixed-probability level. The inverse level crossing problem when \( T \) and \( Y \) are exponentially distributed with parameters \( \delta > 0 \) and \( \rho > 0 \) was studied in [12] and [14]. In a diffusion model, the similar analysis was done in [13]; see also [11]. When \( T \) and \( Y \) are non-exponentially distributed, simulation analysis in the inverse level crossing problem was done in [19].

Going along the road map set in [11]–[14], we focus on the asymptotic structure of the fixed-probability level. We start with the following simple result.

Theorem 6.1. Assume that \( f_r(x) \) and \( f_y(x) \) are bounded above by a finite constant, \( D^2 > 0, E(T^3) < \infty, \) and \( E(Y^3) < \infty. \) Then for \( t \to \infty, \) we have
\[
u_{a,c}(0) = \frac{t}{M} + \frac{D}{M^{3/2}} \kappa_a \sqrt{T}(1 + \overline{a}(1)).
\]
(6.1)

Since \( u_{a,c}(0) \) is a solution to the equation \( P[Y_{a,0} < t] = a, \) where \( Y_{a,0} := \inf \{ s > 0 : V_s > u \}, \) and since the trajectories of the compound renewal process \( V_s, s \geq 0, \) are (a.s.) step functions with only jumps up, this equation rewrites as \( P[V_t > u] = a. \) Therefore, Theorem 6.1 is a direct corollary of the normal approximation for the distribution of compound renewal process \( V_t, \) which is well known: as \( t \to \infty, \) the probability \( P[V_t > u] \) is approximated by
\[
1 - \Phi_{0,1} \left( \frac{t - E(V_t)}{\sqrt{D(V_t)}} \right),
\]
where
\[
E(V_t) = (E(Y/E)T) t + E(Y(DT - (E(T))^2)/(2(E(T)))^2 + \overline{a}(1),
\]
\[
D(V_t) = \frac{((E(Y)^2 DT + (E(T)^2)DY)/(E(Y)^3) t + \overline{a}(t),
\]
(6.2)
whence (6.1).

The following theorem is a generalization of Theorem 2.2 in [14]; it is worthwhile to compare it with Theorem 1 in [13].
Theorem 6.2. Assume that \( f_T(x) \) and \( f_Y(x) \) are bounded above by a finite constant, \( D^2 > 0 \), \( E(T^3) < \infty \), and \( E(Y^3) < \infty \). Then for \( t \to \infty \), we have

\[
 u_{a,t}(c^* ) = \frac{D}{M^{3/2}} \kappa_{a/2} \sqrt{t} (1 + \varphi(1)).
\] (6.3)

Proof of Theorem 6.2. The left-hand side of equation (1.3) is (see (2.3))

\[
 P \{ \mathcal{T}_{a,c} \leq t \} = \int_0^t P \{ u + cv - Y < 0 \} f_T(v) \, dv + \int_0^\infty P \{ v < \mathcal{T}_{a,c} \leq t \mid T_1 = v \} f_{T_1}(v) \, dv,
\]

whence for \( c = c^* \),

\[
 \int_0^t P \{ u + c^* v - Y_1 < 0 \} f_T(v) \, dv + \int_0^\infty P \{ v < \mathcal{T}_{a,c} \leq t \mid T_1 = v \} f_{T_1}(v) \, dv = \alpha.
\] (6.4)

Bearing in mind the second equality in (6.4), the probability \( P \{ v < \mathcal{T}_{a,c} \leq t \mid T_1 = v \} \) is approximated by

\[
 2 \left( \phi_{(0,1)} \left( \sqrt{\frac{E(T(Y + u + EY))}{(EY)^2D^2}} \right) - \phi_{(0,1)} \left( \sqrt{\frac{E(T + v + EY)}{(EY)^2D^2}} \right) \right).
\]

Let us show that \( u_{a,t}(c^* ) \) in (6.3) is an asymptotic solution to equation (6.4). First, bearing in mind that \( E(T^3) < \infty \), \( E(Y^3) < \infty \), it is easily seen that

\[
 \int_0^t P \{ u + c^* v - Y_1 < 0 \} f_T(v) \, dv \to 0, \quad t \to \infty, \quad u \to \infty.
\]

Secondly, it is easy to see that

\[
 \phi_{(0,1)} \left( \sqrt{\frac{E(T(Y + u + EY))}{(EY)^2D^2}} \right) \to 1, \quad u \to \infty.
\]

Selecting in (6.3) \( u \) as \( O \left( t^{1/2} \right) \), we have

\[
 \frac{u + EY}{\sqrt{u + ET + tEY}} = \frac{u + EY}{\sqrt{u + ET}} (1 + \varphi(1)), \quad t \to \infty.
\]

Therefore, equation (6.4) reduces to

\[
 2 \left( 1 - \phi_{(0,1)} \left( \sqrt{\frac{E(T + v + EY)}{(EY)^2D^2}} \right) \right) = \alpha,
\]

whence the result.

The following theorem is a generalization of Theorem 2.2 in [14]. It is useful to compare it with Theorem 2 in [13], or Theorem 4.4 in [11].

Theorem 6.3. Assume that differentiable \( f_T(x) \) and \( f_Y(x) \) are bounded above by a finite constant, and \( D^2 > 0 \), \( E(T^3) < \infty \), \( E(Y^3) < \infty \). Then for \( c^* = M^{-1} \) we have

\[
 u_{a,t}(c ) = \begin{cases} 
 (c^* - c) t + \frac{D}{M^{3/2}} z_{a,t} \left( \frac{M^{3/2}(c^* - c)}{D} \sqrt{t} \right) \sqrt{t}, & 0 \leq c \leq c^*, \\
 \frac{D}{M^{3/2}} z_{a,t} \left( \frac{M^{3/2}(c^* - c)}{D} \sqrt{t} \right) \sqrt{t}, & c > c^*, 
\end{cases}
\] (6.5)

where for $t$ sufficiently large the function $Z_{a,t}(y)$, $y \in \mathbb{R}$, is continuous and monotone increasing, as $y$ increases from $-\infty$ to 0, and monotone decreasing, as $y$ increases from 0 to $\infty$, and such that\footnote{We recall that $0 < k_\alpha < k_{\alpha/2}$ for $0 < \alpha < \frac{1}{2}$.}

$$\lim_{y \to -\infty} Z_{a,t}(y) = 0, \quad \lim_{y \to \infty} Z_{a,t}(y) = k_\alpha$$

and $Z_{a,t}(0) = k_{\alpha/2}(1 + \overline{a}(1))$, as $t \to \infty$.

**Proof of Theorem 6.3.** This proof is carried out in two stages. In each stage we make a suitable change of variables. Its aim is to focus on the function $Z_{a,t}(y)$, $y \in \mathbb{R}$, and to check its monotony using the standard criterion based on the sign of its derivative; this is calculated by means of (see Theorem 7.1) the implicit function derivative theorem.

**Step 1.** Let us consider the case $0 < c < c^* = M^{-1}$. Regarding equation (1.3), we switch from the variables $u$ and $c$ in its left-hand side to the variables $z$ and $u$. Referring to the implicit function derivative theorem (see Theorem 7.1), we have

$$z = \frac{u}{DM^{-3/2}} - \frac{(M^{-1} - c) \sqrt{t}}{DM^{-3/2}} \in \mathbb{R}, \quad y = \frac{(M^{-1} - c) \sqrt{t}}{DM^{-3/2}} > 0. \quad (6.6)$$

The original equation (1.3) rewrites as

$$P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} = \alpha.$$

To prove that $Z_{a,t}(y)$, $y > 0$, is monotone decreasing, we have to prove that $\frac{d}{dy} Z_{a,t}(y) < 0$, $y > 0$. Referring to the implicit function derivative theorem (see Theorem 7.1), we have

$$\frac{d}{dy} Z_{a,t}(y) = -\left( \frac{\partial}{\partial y} \left( P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} \right) \right)_{t = Z_{a,t}(y)}.$$

The numerator is

$$\frac{\partial}{\partial y} \left( P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} \right) = \left( \frac{\partial}{\partial u} P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} \right) \cdot \frac{\partial}{\partial y} \left( \frac{D \sqrt{t}}{M^{3/2} (z+y)} \right) \cdot \left( \frac{1}{M^{3/2} \sqrt{t}} \right).$$

whose approximation, as $t \to \infty$, follows from (5.9), (5.3), (5.5), (5.4), for large $t$ this is obviously negative. The denominator is

$$\frac{\partial}{\partial c} \left( P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} \right) = \left( \frac{\partial}{\partial u} P(T_{u,c} \leq t) \bigg|_{u = \frac{\alpha}{M^{3/4} (z+y)}} \right) \cdot \frac{\partial}{\partial c} \left( \frac{D \sqrt{t}}{M^{3/2} (z+y)} \right).$$

$$\frac{\partial}{\partial y} \left( \frac{D \sqrt{t}}{M^{3/2} (z+y)} \right) \cdot \left( \frac{1}{M^{3/2} \sqrt{t}} \right).$$
whose approximation, as \( t \to \infty \), follows from (5.9), (5.5), (5.4); for large \( t \) this is obviously negative, whence the result.

**Step 2.** We continue the proof with investigating the case \( c > c^* = M^{-1} \). We switch from the variables \( u \) and \( c \) to the variables

\[
z = \frac{u}{DM^{-3/2} \sqrt{t}} > 0, \quad y = \frac{(M^{-1} - c) \sqrt{t}}{DM^{-3/2}} < 0,
\]

(6.10)

Let us rewrite the original equation (1.3) as

\[
P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} = \alpha.
\]

To prove that \( z_{u,c}(y) \), \( y < 0 \), is monotone increasing, we have to prove that \( \frac{d}{dy} z_{u,c}(y) > 0 \), \( y < 0 \). Referring to the implicit function derivative theorem (see Theorem 7.1), we have

\[
\frac{d}{dy} z_{u,c}(y) = \frac{\partial}{\partial y} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} = \frac{\partial}{\partial c} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} \frac{1}{M} - \frac{D}{M^{3/2} \sqrt{t}} y.
\]

The numerator is

\[
\frac{\partial}{\partial y} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} = \frac{\partial}{\partial c} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} \frac{1}{M} - \frac{D}{M^{3/2} \sqrt{t}} y.
\]

(6.11)

whose approximation, as \( t \to \infty \), follows from (5.3), (5.5), (5.4); for large \( t \) this is obviously negative. The denominator is

\[
\frac{\partial}{\partial z} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} = \frac{\partial}{\partial c} P(\tau_{u,c} \leq t) \bigg|_{u = \frac{\alpha \sqrt{t}}{M}, c = \frac{1}{M} - \frac{\alpha}{M} \sqrt{t}} \frac{D}{M^{3/2} \sqrt{t}} z.
\]

(6.12)

whose approximation, as \( t \to \infty \), follows from (5.9), (5.5), (5.4); for large \( t \) this is obviously negative, whence the result.

\[\square\]

6.2. Monotony and convexity of fixed-probability level. The fixed-probability level \( u_{a,t}(c) \), \( c \geq 0 \), for all \( t \) monotone decreases, as \( c \) increases. The following result, called weak-form convexity, differs in that it is established by our means only for \( t \) large.

**Theorem 6.4** (Weak-form convexity). Suppose that conditions of Theorem 6.3 are satisfied. Then for \( t > 0 \) sufficiently large, the function \( u_{a,t}(c) \), \( c \geq c^* \), is convex.

The proof of Theorem 6.4 requires dramatically large space and is left to the reader. Nevertheless, it is quite clearootnote{In the diffusion model, the proof of convexity was carried out with complete details in [13].}, one should check that for \( c > c^* \) the inequality \( \frac{d^2}{dc^2} u_{a,t}(c) > 0 \) holds for \( t \) sufficiently large. This starts with equality (7.2), proceeds with, first, calculation of the second-order derivatives as it is done in Section 8 and, second, approximating them as it is done in Section 5. The proof of positivity of the approximation for \( \frac{d^2}{dc^2} u_{a,t}(c) > 0 \), when \( t \) is large, brings the proof to a close.
6.3. Heuristic fixed-probability level. Let us introduce $u_{\alpha,t}^{[M]}(c), c \geq 0$, which is a positive solution to the equation\(^{15}\)

$$M_{\alpha,t}(t) = \alpha,$$  \hspace{1cm} (6.13)

whose right-hand side is expressed\(^{16}\) in a closed form. Plainly, to get (6.13), we replaced the left-hand side of the original equation \(1,4\) by an approximation found in Theorem 4.3.

**Theorem 6.5. For** $0 \leq c < Kc^*, 0 < K < 1$, **we have**\(^{16}\)

$$u_{\alpha,t}^{[M]}(c) = (c^* - c)t + \frac{D}{M^{3/2}} \kappa_0 \sqrt{7}(1 + \bar{\sigma}(1)), \quad t \to \infty,$$  \hspace{1cm} (6.14)

**Proof of Theorem 6.5.** For $0 \leq c < Kc^*, 0 < K < 1$, and $u_t(c) = (c^* - c)t + \frac{D}{M^{3/2}} \sqrt{7}z_t(c)$, where $z_t(c) = O(1)$, as $t \to \infty$, we have

$$M_{\alpha,t}(t) \bigg|_{u_t(c)} = \alpha \Phi_{(0,1)}(z_t(c)), \quad t \to \infty.$$

Therefore, the equation $M_{\alpha,t}(t) |_{u_t(c)} = \alpha$ rewrites as $1 - \Phi_{(0,1)}(z_t(c)) = \alpha(1 + \bar{\sigma}(1)), t \to \infty$, whose solution is $z_t(c) = K_0(1 + \bar{\sigma}(1)), t \to \infty$. 

**Theorem 6.6. For** $c_{t,\delta} = c^* - \frac{D}{M^{3/2}} \delta^{1/2}, 0 \leq \delta < K$, **we have**\(^{16}\)

$$u_{\alpha,t}^{[M]}(c_{t,\delta}) = \frac{D}{M^{3/2}} x_\alpha(\delta) \sqrt{7}(1 + \bar{\sigma}(1)), \quad t \to \infty,$$  \hspace{1cm} (6.15)

where $x_\alpha(\delta)$ is a solution to the equation

$$1 - \Phi_{(0,1)}(-\delta + x_{\alpha}) + \Phi_{(0,1)}(-\delta - x_{\alpha}) \exp\{2\delta x\} = \alpha.$$  \hspace{1cm} (6.16)

**Proof of Theorem 6.6.** For $c_{t,\delta} = c^* - \frac{D}{M^{3/2}} \delta^{1/2}, 0 \leq \delta < K$, and $u_{\alpha,t}^{[M]}(c_{t,\delta})$ defined in (6.15), we have

$$M_{\alpha,t}(t) \bigg|_{u_{\alpha,t}^{[M]}(c_{t,\delta})} \sim 1 - \Phi_{(0,1)}(-\delta + x_{\alpha}(\delta))$$

$$\quad + \Phi_{(0,1)}(-\delta - x_{\alpha}(\delta)) \exp\{2\delta x_{\alpha}(\delta)\} = \alpha, \quad t \to \infty,$$

whence the result. 

**Theorem 6.7. For** $c_{t,\delta} = c^* + \frac{D}{M^{3/2}} \delta^{1/2}, 0 \leq \delta < K$, **we have**\(^{16}\)

$$u_{\alpha,t}^{[M]}(c_{t,\delta}) = \frac{D}{M^{3/2}} x_{\alpha}(\delta) \sqrt{7}(1 + \bar{\sigma}(1)), \quad t \to \infty,$$  \hspace{1cm} (6.17)

where $x_\alpha(\delta)$ is a solution to the equation

$$1 - \Phi_{(0,1)}(\delta + x_{\alpha}) + \Phi_{(0,1)}(\delta - x_{\alpha}) \exp\{-2\delta x\} = \alpha.$$  \hspace{1cm} (6.18)

**Proof of Theorem 6.7.** For $c_{t,\delta} = c^* + \frac{D}{M^{3/2}} \delta^{1/2}, 0 \leq \delta < K$, and $u_{\alpha,t}^{[M]}(c_{t,\delta})$ defined in (6.17), we have

$$M_{\alpha,t}(t) \bigg|_{u_{\alpha,t}^{[M]}(c_{t,\delta})} \sim 1 - \Phi_{(0,1)}(\delta + x_{\alpha}(\delta))$$

$$\quad + \Phi_{(0,1)}(\delta - x_{\alpha}(\delta)) \exp\{-2\delta x_{\alpha}(\delta)\} = \alpha, \quad t \to \infty,$$

whence the result.

It is noteworthy that in both Theorems 6.6 and 6.7, the expression $x_\alpha(0)$ is a solution to the equation $1 - \Phi_{(0,1)}(x) = \alpha/2$, i.e., is equal to $k_0/2$.

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\(^{15}\)Recall that $M_{\alpha,t}(t)$ is an alternative notation for $\frac{1}{2} \int_0^t \frac{D}{M^{3/2}} \kappa_0 \sqrt{7}(1 + \bar{\sigma}(1)) dx$.

\(^{16}\)See (4.4) with $p = -1/2, (4.3)$, and (4.5).

\(^{17}\)For $c = 0$, asymptotic equality (6.18) coincides with (6.1).
From (6.20), we have $u_t(6.20)$, we have

$$\mu \text{ Fig. 3. Although }$$

when they are defined by equations close to each other. $\mu$

reason to expect that $u_t(c^*) = 80$. $\mu$

6.4. Elementary asymptotic bounds for the fixed-probability level. For $0 \leq c \leq c^*$, elementary asymptotic bounds

$$(c^* - c) t + \frac{D}{M^{3/2}} \kappa_a \sqrt{t} (1 + o(1)) \leq u_{a,t}(c)$$

$$\leq (c^* - c) t + \frac{D}{M^{3/2}} \kappa_a \sqrt{t} (1 + o(1)), \quad t \to \infty,$$
follow straightforwardly from (6.5).

For \( c > c^\ast \), elementary upper bounds, quite satisfactory for \( c > Kc^\ast \) with \( K > 1 \) large enough, are also straightforward in many cases of interest. In particular, when \( T \) and \( Y \) are exponentially distributed with parameters \( \delta \) and \( \rho \), we have \( c^\ast = \delta/\rho \) and (see, e.g., [25]) \( P[T_{nc} < \infty] = (1 - \kappa/\rho) e^{-u} \) for all \( u \geq 0 \), where \( \kappa = \rho - \delta/c \). This rewrites as \( P[T_{nc} < \infty] = (\delta/(c\rho)) \exp\{-\rho(\delta/c)u\} \); by simple calculations we have

\[
u_{\alpha,c}(c) \leq \max \left\{ 0, -\frac{\ln(\alpha c^\ast/\delta)}{\rho - \delta/c} \right\}, \quad c > c^\ast.
\]

When \( T \) is exponentially distributed with parameter \( \delta \) and the distribution of \( Y \) is light-tailed, but non-exponential, we have \( c^\ast = \delta \infty Y \) and (see, e.g., [25]) \( P[T_{nc} < \infty] = e^{-u} \) for all \( u > 0 \), where \( \kappa \) is a positive solution to the equation \( E \exp[\kappa Y] = 1 + c\kappa/\delta \). Therefore, we have

\[
u_{\alpha,c}(c) \leq -\ln \kappa/c, \quad c > c^\ast.
\]

When \( Y \) is exponentially distributed with parameter \( \rho \) and the distribution of \( T \) is arbitrary, we have \( c^\ast = 1/(\rho E T) \) and (see, e.g., [25]) \( P[T_{nc} < \infty] = (1 - \kappa/\rho) e^{-u} \) for all \( u > 0 \), where \( \kappa \) is a positive solution to the equation \( E \exp[-\kappa c T] = 1 - \kappa/\rho \). Bearing in mind that \( 1 - \kappa/\rho \leq 1 \), we have

\[
u_{\alpha,c}(c) \leq -\ln \kappa/c, \quad c > c^\ast.
\]

In this case, the elementary bounds for the fixed-probability level are shown in Fig.4.

7. Derivatives of implicit function

The derivatives of an implicit function defined by the equation \( F(x, y) = 0, x, y \in \mathbb{R} \), can be obtained (see, e.g., [30], Chapter I, § 5.2 and § 5.3) without finding this implicit function in closed form.

**Theorem 7.1.** Assume that the function \( F(x, y) \), \( x, y \in \mathbb{R} \), possesses partial derivatives up to second order, which are continuous in some neighborhood of a solution \((x_0, y_0)\) of the equation \( F(x, y) = 0 \). If \( \frac{\partial^2 F}{\partial y^2}(x_0, y_0) \neq 0 \), then there exists an \( \epsilon > 0 \) and a unique continuously differentiable function \( f \) such that \( f(x_0) = y_0 \) and \( F(x, f(x)) = 0 \) for \( |x - x_0| < \epsilon \). Moreover, for \( |x - x_0| < \epsilon \) we
and have
\[ f'(x) = -\frac{\partial}{\partial y} F(x, y) \bigg|_{y=f(x)}, \quad (7.1) \]
and
\[ f''(x) = -\left\{ \frac{\partial^2}{\partial x \partial y} F(x, y) - \frac{\partial^2}{\partial x \partial y} F(x, y) \frac{\partial}{\partial x} F(x, y) \frac{\partial}{\partial y} F(x, y) + \frac{\partial^2}{\partial y^2} F(x, y) \left( \frac{\partial}{\partial y} F(x, y) \right)^2 \right\} \bigg|_{y=f(x)}. \quad (7.2) \]

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Central Economics and Mathematics Institute (CEMI) of Russian Academy of Science, 117418, Nakhimovsky prosp., 47, Moscow, Russia

E-mail address: Vsevolod.Malinovskii@mail.ru, admin@actlab.ru

URL: http://www.actlab.ru