Fundamental dynamical equations for spinor wavefunctions: I. Lévy-Leblond and Schrödinger equations

R Huegele, Z E Musielak and J L Fry

Department of Physics, The University of Texas at Arlington, Arlington, TX 76019, USA
E-mail: randy.huegele@mavs.uta.edu, zmusielak@uta.edu and fry@uta.edu

Received 7 January 2012, in final form 3 March 2012
Published 22 March 2012
Online at stacks.iop.org/JPhysA/45/145205

Abstract
A search for fundamental (Galilean invariant) dynamical equations for two- and four-component spinor wavefunctions is conducted in Galilean spacetime. A dynamical equation is considered as fundamental if it is invariant under the symmetry operators of the group of the Galilei metric and if its state functions transform like the irreducible representations of the group of the metric. It is shown that there are no Galilean invariant equations for two-component spinor wavefunctions. A method to derive the Lévy-Leblond equation for a four-component spinor wavefunction is presented. It is formally proved that the Lévy-Leblond and Schrödinger equations are the only Galilean invariant four-component spinor equations that can be obtained with the Schrödinger phase factor. Physical implications of the obtained results and their relationships to the Pauli–Schrödinger equation are discussed.

PACS numbers: 11.10.–z, 31.15.xh, 03.65.Ta

1. Introduction

A physical theory of free particles is considered fundamental if its dynamical equations have the same form in all isometric frames of reference. All coordinate transformations that do not change a given metric of spacetime form a representation of the group of the metric. In order for two observers with the same metric to identify the same particle, a state function describing this particle must transform like one of the irreducible representations (irreps) of the group of the metric.

This definition was first formally introduced by Wigner [1], who determined all unitary irreps of the Poincaré group [2] and used them to classify the elementary particles in Minkowski spacetime. The original Wigner work was used by Bargmann and Wigner [3] to obtain Poincaré invariant dynamical equations associated with each representation. As a result, the Klein–Gordon [4, 5], Dirac [6] and Proca [7] equations were formally obtained. Local symmetries of these equations were discussed by Fushchich and Nikitin [8].
The situation is more complicated in Galilean spacetime because vector irreps of the Galilei group of the metric have no physical interpretation [9]. However, there is an infinite number of projective (ray) irreps, which are different from the vector irreps of the group [10]. Typically, these projective irreps are determined by the method of induced representations [11, 12], and they are characterized by a constant that enters a phase factor in defining the projective irreps [2]. The process of introducing the constant is the central extension of the Lie algebra and the corresponding group is called the extended Galilei group [2, 8].

This group played an important role in the work of Lévy-Leblond [13, 14], who considered free particles with arbitrary spin in Galilean spacetime and represented spin $1/2$ particles by a four-component spinor state function. He used the Bargmann–Wigner method to derive a dynamical equation that describes evolution of this function in time and space, and demonstrated that the derived equation was Galilean invariant. The obtained equation is now known as the Lévy-Leblond equation [8]. Moreover, Lévy-Leblond [14] also derived the Pauli–Schrödinger (PS) equation [15, 16] by adding the electromagnetic field to his Galilean invariant equation. The fields were made Galilean invariant by dropping the Maxwell term from Maxwell’s equations. He obtained the PS equation for a two-component spinor wavefunction, however, he discussed its Galilean invariance [14].

Extensive studies of the Lévy-Leblond and PS equations were performed by Fushchich and Nikitin [8], who derived those equations and discussed their local symmetries. The authors also ‘deduced’ the Schrödinger equation [17] by assuming a form of their partial differential operator that directly leads to this equation, and showed that the equation was Galilean invariant; however, see also [18–20]. Actually, they demonstrated that the Schrödinger equation was the only Galilean invariant equation for a scalar wavefunction.

In our work on free and spinless particles described by scalar and analytic state functions in Galilean spacetime, we used the principle of Galilean relativity and the principle of analyticity to formally derive Schrödinger-like equations [21]. The principle of relativity requires that all inertial observers formulate the same physical law and that they identify the same elementary particle. According to this principle, all dynamical equations describing evolution of the wavefunction in time and space must be invariant with respect to all transformations that leave the metric unchanged. The principle of analyticity demands that the wavefunction describing elementary particles is analytic.

With these two principles, we showed that a set of linear eigenvalue equations must be satisfied by any function transforming like an irrep of the extended Galilei group. These equations are valid in a given inertial frame of reference, but they are not Galilean invariant. However, they form a required starting point from which it is possible to deduce an invariant dynamical equation fully justified by the principle of relativity. This shows that our method greatly differs from that used by Fushchich and Nikitin [8]. Similarly, our formal proof of the uniqueness of the Schrödinger equation [22] is also significantly different from that presented by Fushchich and Nikitin. In our further development, we demonstrated how to formulate fundamental (Galilean invariant) theories of waves and particles without using the concept of classical mass [23].

More recently, de Montigny et al [24] considered Galilean invariant theories by constructing indecomposable finite-dimensional representations of the homogeneous Galilei group. They used these representations to derive a general Pauli anomalous interaction term and deduce wave equations that describe interaction of Galilei particles with an external electric field. Moreover, Niederle and Nikitin [25] derived Galilean invariant dynamical equations for massless and massive fields. They showed that their Galilean invariant equations can also be obtained by contraction of known and new relativistic wave equations.
The main objective of this paper is to search for Galilean invariant dynamical equations for spinor wavefunctions. The presented results are complementary to those obtained by Fushchich and Nikitin [8], de Montigny et al [24] and Niederle and Nikitin [25], however, our approach is significantly different and it is also more fundamental in the physical sense. The reasons are the following. First of all, we begin our search with a general partial differential equation whose form is justified by our previously obtained results [21–23], which are based on the principle of Galilean relativity and the principle of analyticity. Second, we search for dynamical equations that are symmetric under the transformations of the extended Galilei group [2, 8, 13]. Third, our approach allows for other than Schrödinger’s phase function [20]. Finally, we consider both two-component and four-component spinor wavefunctions. Our motivation for considering the former is that such spinors are now widely used in general relativity primarily through the work of Penrose [26, 27].

We define a dynamical equation to be fundamental if it has the following properties: (i) invariance under the symmetry operators of the group of the metric; (ii) no mixed time and space partial derivatives and (iii) state functions that transform like the irreps of the group of the metric. All the results presented in this paper are obtained with Schrödinger’s phase function [20]. Other phase functions are also possible and they lead to new equations that will be presented and discussed in the next paper of this series.

Among the results presented in this paper a special emphasis should be given to our formal proof that there are no Galilean invariant equations for two-component spinor wavefunctions. For four-component spinor wavefunctions, we derived the Lévy-Leblond equation by a different method than Lévy-Leblond [13, 14] and Fushchich and Nikitin [8]. Our proof that the Lévy-Leblond equation is the only first-order Galilean invariant equation is significantly different from that of Fushchich and Nikitin. We also proved that the Schrödinger equation is the only second-order fundamental dynamical equation in Galilean spacetime, and that there are no other higher order fundamental equations for two- and four-component spinors. Finally, we showed how to derive the PS equation from the Lévy-Leblond equation.

The outline of this paper is as follows: the basic properties of Galilean relativity and some previously obtained results, including Schrödinger’s phase function, are briefly described in section 2; fundamental dynamical equations for two- and four-component spinor wavefunctions are derived and discussed in section 3; relationship of the obtained results to the PS equation is described in section 4 and our conclusions are given in section 5.

2. Galilean relativity and Schrödinger’s phase function

2.1. Group of the Galilei metric

Galilean spacetime is defined by the Galilei metric: $d\mathbf{s}_1^2 = dx^2 + dy^2 + dz^2$ and $d\mathbf{s}_2^2 = dt^2$, where $x$, $y$, $z$ and $t$ are spatial and time coordinates. The metric is invariant under a set of transformations that forms the Galilei group. The group may be decomposed into subgroups such that

$$G = [T(1) \otimes R(3)] \otimes_s [T(3) \otimes B(3)],$$

where $T(1)$, $R(3)$, $T(3)$ and $B(3)$ are the subgroups of translation in time, rotations in space, translations in space and boosts, respectively. The direct product and semi-direct product are denoted $\otimes$ and $\otimes_s$.

The Galilean transformations can be used to relate the coordinate systems of two observers that are spatially rotated, translated and boosted relative to one another. A Galilean transformation can be defined by

$$\mathbf{x} \rightarrow \mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a} \quad \text{and} \quad t \rightarrow t' = t + b,$$
where \( R \) is a rotation matrix, \( \vec{v} \) is the velocity vector of a boost relating the two coordinate systems and \( \vec{a} \) is a spacial translation relating the two coordinate systems. The inverse Galilean transformation is

\[
\vec{x} = R^{-1} \vec{x}' - R^{-1} \vec{v} - R^{-1} (\vec{a} - \vec{b}) \quad \text{and} \quad t = t' - b. \tag{3}
\]

The chain rule can be used to determine how the differential operators transform under the Galilean transformation

\[
\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t'} \frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} - R^{-1}_{ij} v_j \frac{\partial}{\partial x^i}, \tag{4}
\]

and

\[
\frac{\partial}{\partial x'^i} = \frac{\partial t}{\partial x'^i} \frac{\partial}{\partial t} + \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = R^{-1}_{ij} \frac{\partial}{\partial x^j}, \tag{5}
\]

where \( i = 1, 2 \) and \( 3 \), and \( j = 1, 2 \) and \( 3 \).

It has been demonstrated for scalar wavefunctions that the Galilei group does not lead to any dynamical equations that satisfy the principles of analyticity and relativity [21]. The principle of analyticity requires that state functions are analytic and the principle of relativity demands that dynamical equations governing the state function are Galilean invariant. Therefore an additional symmetry \(|\psi^*\psi| = |\psi'^*\psi'|\) must be added to the group of the metric.

The expanded symmetry group, called the extended Galilei group, is the universal covering group of the Galilei group [11, 12]. The extended Galilei group exhibits structure that is similar to the Poincare group [2, 8, 13, 21]. The arguments used in [21] for scalar wavefunctions apply equally well to \( n \)-component state functions such as spinors and vectors. Consequently, we begin this work with the extended Galilei group, which has the following structure:

\[
G_e = [R(3) \otimes B(3)] \otimes [T(3 + 1) \otimes U(1)],
\]

where \( U(1) \) is a one-parameter unitary group. We consider only the proper isochronal subgroup \( G_e^+ \) of \( G_e \) that omits the space and time inversions, which can be treated separately.

Measurements of the norm of the scalar state function must produce the same results for all observers related by the Galilei transformations. Hence, the resulting transformation of the wavefunction \( \psi \) between two inertial frames of reference \((\vec{x}, t)\) and \((\vec{x}', t')\) is

\[
\psi(\vec{x}, t) = e^{i\phi(\vec{x}, t')} \psi(\vec{x}', t'), \tag{7}
\]

with \( \phi(\vec{x}', t') \) being a phase function to be determined.

### 2.2. Schrödinger’s phase function

In the previous work, Musielak and Fry [21] used the Galilei group of the metric and the principle of analyticity and the principle of Galilean relativity to formally derive Schrödinger-like equations. They concluded that the Galilei group was incomplete for forming a fundamental theory of free particles and that the necessary modifications of the group led to the extended Galilei group. The derived Schrödinger-like equation can be written in the following form:

\[
i \frac{\partial \psi}{\partial t} + \frac{\omega}{k^2} \nabla^2 \psi = 0, \tag{8}
\]

where \( \omega \) and \( k \) are the eigenvalues of the translation operators in time and space. Properties of the eigenvalue equations ensure that \( \omega/k^2 = 1/2M \) is a constant in all inertial frames of reference. \( M \) is referred to as the ‘wave mass’ and is related to the classical mass through the Planck constant \( m = hM \) (see [21–23]).
Galilean invariance of the Schrödinger-like equation under the transformations of the extended Galilei group requires a phase factor $e^{i\phi(x', t')}$ with the phase function given by

$$\phi(x', t') = m\vec{v} \cdot \vec{x}' + \frac{1}{2}mv^2t',$$  \hspace{1cm} (9)

where $\vec{v}$ is the constant velocity of one inertial frame of reference with respect to the other.

Now, when the state function is a spinor it has two or more components and each component must satisfy the Schrödinger-like equation given by equation (8), which means that the phase function given by equation (9) must be used (see [8, 20]). We will call $\phi(x', t')$ the Schrödinger phase function and our search for fundamental (Galilean invariant) equations for spinor state functions described in this paper will be exclusively based on this phase function. Our approach allows us to consider other phase functions and these functions lead to new fundamental dynamical equations that will be presented and discussed in the second paper of this series.

3. Dynamical equations for spinor state functions

3.1. Method

Let us consider the following first-order partial differential equation:

$$\left[ B_1 \frac{\partial}{\partial t} + B_2 \frac{\partial}{\partial x_j} + B_3 \right] \psi(x, t) = 0,$$  \hspace{1cm} (10)

where $B_1$, $B_2$, and $B_3$ are arbitrary $N \times N$ matrices that are assumed to be free of any dependence on the space and time coordinates, and $\psi$ is an $N$-component spinor state function. The motivation for this general form of the above equation is given by our previously obtained results that were based on the principle of Galilean relativity and the principle of analyticity [21, 22]. The results show that the first-order derivatives used in this equation are eigenoperators on the state function $\psi$, which transforms as an irrep of the extended Galilei group, and that a term with a constant matrix times $\psi$ can always be added to the equation if needed to obtain invariance for the assumed form of the spinor state function.

Dynamical equations must be invariant under Galilean transformations, so we will require Galilean invariance of the first-order differential equation in order to derive a set of restrictions on the matrices $B$. Applying Galilean transformations to equation (10) and regrouping the terms, we obtain

$$\left[ B_1' \frac{\partial}{\partial t} + (B_1' R_{jk} R_{ik} + B_2' R_{ij}) \frac{\partial}{\partial x_j} \right] \psi(x, t)$$
$$+ \left[ -\frac{i}{2}mv^2B_1' - imR_{jk}v_jR_{ik}B_2' + B_3' \right] \psi(x, t) = 0.$$  \hspace{1cm} (11)

For equation (10) to be invariant under Galilei transformation, equation (11) must be of the same form. This requirement leads to the following conditions on the set of matrices:

$$B_1 = GB_1G^{-1},$$  \hspace{1cm} (12)

$$B_2 = GB_1G^{-1}R_{jk}v_j + GB_2G^{-1}R_{ij}$$  \hspace{1cm} (13)

and

$$B_3 = -\frac{i}{2}mv^2GB_1G^{-1} - imvGB_2G^{-1} + GB_3G^{-1}.$$  \hspace{1cm} (14)

The conditions will be now used to search for fundamental dynamical equations for two- and four-component spinor wavefunctions.
3.2. Dynamical equations for two-component spinors

Our main result obtained for two-component spinor state functions is given by the following proposition.

**Proposition 1.** If a two-component spinor state function $\psi$ transforms as $\psi(\vec{x}, t) = e^{i\phi(\vec{x}, t')}\psi(\vec{x}', t')$, where $\phi$ is the Schrödinger phase function given by equation (9), and $(\vec{x}', t')$ are two inertial frames of reference, then there are no invariant dynamical equations describing evolution of this function in Galilean spacetime.

**Proof.** Applying rotations only (no boosts) in the conditions given by equations (12)–(14) constrains the matrices to the following forms: $B_1 = c_1 I$, $B_2 = c_2 \sigma_j$ and $B_3 = c_3 I$, where $c_1$, $c_2$ and $c_3$ are arbitrary constants. This demonstrates that there are first-order equations that are rotationally invariant. To be Galilean invariant the equation must also be boost invariant.

It turns out that it is not possible to construct Galilei boost operators for two-component spinors. In general, one may construct a boost matrix from the velocity parameters $v_i$ and boost generators $X_j$ as the exponential expression given by $B(v) = e^{v_j X_j}$. The generators of Galilei boosts must obey the following commutation relations of the Galilei group: $[X_0, X_0] = iX_0\epsilon_{ijk}$, $[X_i, X_0] = 0$ and $[X_i, X_j] = i\epsilon_{ijk}$.

Because Galilei boosts commute, they form an Abelian subgroup and a one-dimensional irrep exists. However, the composition of boosts and rotations is the result of a semi-direct product and requires boosts and rotations to obey the group composition law that can be written as

$$G(a, b, v, R(\theta)) = T(b)S(a)B(v)R(\theta) = G(a_2, b_2, v_2, R_2)G(a_1, b_1, v_1, R_1) = G(R_2 a_1 + a_2 - v_2 b_1, b_1 + b_2, R_2 v_1 + v_2, R_2 R_1),$$

which is composed of translations in time $T(b)$, translations in space $S(a)$, boosts $B(v)$ and rotations $R(\theta)$. For the $2 \times 2$ generators of rotations $X_0 = \sigma_i/2$, where $\sigma_i$ are the Pauli spin matrices, there are no $2 \times 2$ matrices that are able to satisfy the commutation relations as boost generators.

An interesting result is that this problem does not exist in the Minkowski spacetime [26, 28]. Hence, one may try to take the limit $c \to \infty$ of the Lorentz boost for two-component spinors. Following [29], the result is

$$B_{v_i} = \begin{pmatrix} \cosh v_i/c & \sinh v_i/c \\ \sinh v_i/c & \cosh v_i/c \end{pmatrix},$$

$$B_{v_i} = \begin{pmatrix} \cosh v_i/c & i \sinh v_i/c \\ -i \sinh v_i/c & \cosh v_i/c \end{pmatrix},$$

and

$$B_{v_i} = \begin{pmatrix} e^{v_i/c} & 0 \\ 0 & e^{-v_i/c} \end{pmatrix},$$

which shows that diverging matrix elements are obtained. This is not surprising since the Galilei spinor boosts cannot be represented with $2 \times 2$ matrices, which is a well-known result [24]. Lorentz boosts do not commute as their Galilei counterparts. As such the Lorentz group has a different universal covering group, SL(2,C) and it can be represented with $2 \times 2$ matrices [14]. The physical implications are that we cannot perform a Galilei boost of two component spinors thus there are no fundamental dynamical equations of any order for two component spinors. This concludes the proof of proposition 1. □
3.3. First-order equations for four-component spinors

After showing that there are no Galilean invariant dynamical equations for two-component spinor wavefunctions, we searched for fundamental dynamical equations describing evolution of four-component spinor wavefunctions in time and space. The obtained results are summarized by the following proposition.

Proposition 2. If a four-component spinor state function $\psi$ transforms as $\psi(\vec{x}, t) = e^{i\phi(\vec{x}', t')}\psi(\vec{x}', t')$, where $\phi$ is the Schrödinger phase function given by equation (9), and $(\vec{x}, t)$ and $(\vec{x}', t')$ are two inertial frames of reference, then there is a Galilean invariant first-order partial differential equation for this function and the equation is known as the Lévy-Leblond equation

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \frac{\partial}{\partial t} + \left(\begin{array}{cc} 0 & 0 \\ 0 & \sigma_j \end{array}\right) \frac{\partial}{\partial x_j} + \left(\begin{array}{cc} 0 & 2imI \\ 0 & 0 \end{array}\right)\right] \psi(\vec{x}, t) = 0, \quad (19)$$

where $j = 1, 2$ and 3, $\sigma_j$ are the $2 \times 2$ Pauli matrices and $I$ is the $2 \times 2$ identity matrix.

Proof. As in the case for two-component spinors, we seek a set of matrices that satisfy the conditions for invariance given by equations (12)–(14). Let $B_1$ be an arbitrary $4 \times 4$ matrix, then

$$B_1 = \begin{pmatrix} P & Q \\ S & T \end{pmatrix}, \quad (20)$$

where $P, Q, S$ and $T$ are arbitrary $2 \times 2$ matrices. Applying an arbitrary rotation in the condition given by equation (12) results in four conditions on the $2 \times 2$ matrices $P, Q, S$ and $T$. The conditions are

$$B_1 = RB_1R^{-1} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \begin{pmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} UPU^{-1} & 0 \\ 0 & UQU^{-1} \end{pmatrix} = \begin{pmatrix} UPU^{-1} & UQU^{-1} \\ USU^{-1} & UTU^{-1} \end{pmatrix}. \quad (21)$$

Individually these four conditions are identical in form to that given by equation (12) for two-component spinors and the results are the same. Therefore the $2 \times 2$ matrices must be diagonal with arbitrary constant coefficients $p, q, s$ and $t$ and

$$B_1 = \begin{pmatrix} pI & 0I \\ 0I & qI \end{pmatrix}. \quad (22)$$

For rotations only, the condition given by equation (14) has the same form as that given by equation (12). Therefore, the matrix $B_3$ is similarly constrained to

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \quad (23)$$

For rotations only, the condition given by equation (13) produces four $2 \times 2$ conditions with the same results as those found for two-component spinors. The matrices are constrained to

$$B_{2j} = \begin{pmatrix} \sigma_j & f \sigma_j \\ g \sigma_j & h \sigma_j \end{pmatrix}. \quad (24)$$

Applying boosts (without rotations), and the condition given by equation (12) leads to $q = 0$ and $t = p$, so that

$$B_1 = \begin{pmatrix} pI & 0 \\ 0I & pI \end{pmatrix}. \quad (25)$$
In the case of boosts again without rotations, the condition given by equation (13) leads to
\[ e = -ih = s \quad \text{and} \quad f = 0, \]
so that
\[ B_{2j} = \begin{pmatrix} s \sigma_j & 0 \\ g \sigma_j & -s \sigma_j \end{pmatrix}. \] (26)
Applying boosts without rotations in the condition given by equation (14) leads to
\[ b = 2i m s, \quad p = 0, \quad a = d \quad \text{and} \quad g = 0, \]
and the set of matrices become
\[ B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \] (27)
\[ B_{2j} = \begin{pmatrix} s \sigma_j & 0 \\ 0 & -s \sigma_j \end{pmatrix} \] (28)
and
\[ B_3 = \begin{pmatrix} aI & 2imsI \\ cI & al \end{pmatrix}. \] (29)
Applying rotations and boosts in the conditions leads to \( a = 0 \) and \( c = 0, \)
so that
\[ B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \] (30)
\[ B_{2j} = \begin{pmatrix} s \sigma_j & 0 \\ 0 & -s \sigma_j \end{pmatrix} \] (31)
and
\[ B_3 = \begin{pmatrix} 0 & 2imsI \\ 0 & 0 \end{pmatrix}. \] (32)
The constant \( s \) can now be factored out of the equation leaving the following first-order differential equation
\[ \left[ \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2i mI \\ 0 & 0 \end{pmatrix} \right] \psi(x, t) = 0, \] (33)
which is the Galilean invariant first-order dynamical equation for four-component spinors.

This concludes the proof of proposition 2. \( \square \)

In the literature, equation (33) is known as the Lévy-Leblond equation [8]. It is important to point out that our derivation of this equation presented in proposition 2 is different than that originally used by Lévy-Leblond (see [13] and [14]), and it also differs from the method used by Niederle and Nikitin [25]. Actually, the obtained equation can be cast into several different but equivalent forms. This can be done by similarity transformations, which corresponds to a change of basis. The equation can also be transformed into other representations such as momentum representation [13].

3.4. Higher order Lévy-Leblond equations

We also considered higher-order Lévy-Leblond equations, which were derived by raising the original Lévy-Leblond equation to higher powers. The following proposition summarizes the obtained results.

**Proposition 3.** The Lévy-Leblond equation with the operator raised to Nth power
\[ \left[ \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2i mI \\ 0 & 0 \end{pmatrix} \right]^N \psi(x, t) = 0 \] (34)
is Galilean invariant.
Proof. Let
\[ \mathcal{L} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2iml \\ 0 & 0 \end{pmatrix} \]
(35)
be the Lévy-Leblond operator. It has already been proven that the first-order equation is invariant (see proposition 2), therefore we have
\[ G\mathcal{L}G^{-1} \psi(\vec{x}, t) = G\mathcal{L} \mathcal{L}^{-1} \mathcal{L} \psi(\vec{x}, t) = e^{i\phi} \mathcal{L} \psi(\vec{x}, t) = 0. \]
(36)
This process can be repeated for each power of \( \mathcal{L} \) until the phase factor has been commuted fully to the left, and the result is
\[ G\mathcal{L}^N \mathcal{L}^{-1} \mathcal{L} \psi(\vec{x}, t) = G \mathcal{L}^{N-1} \mathcal{L} \mathcal{L}^{-1} \mathcal{L} \psi(\vec{x}, t) = e^{i\phi} \mathcal{L}^N \psi(\vec{x}, t) = 0. \]
(37)
This concludes the proof of proposition 3. \( \square \)

In the special case of \( N = 2 \), equation (34) gives the Schrödinger equation [17]. Hence, we can formulate the following corollary.

Corollary. The Lévy-Leblond equation with the operator raised to \( N = 2 \) power is equivalent to the Schrödinger equation.

The resulting Schrödinger equation can be written as
\[ \mathcal{L}^2 \psi(\vec{x}, t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2iml \\ 0 & 0 \end{pmatrix} \psi(\vec{x}, t) \]
\[ = \begin{pmatrix} 2im \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j, \sigma_k \end{pmatrix} \begin{pmatrix} 0 \\ \sigma_j, \sigma_k \end{pmatrix} \begin{pmatrix} \partial \frac{\partial}{\partial x_j} \partial \frac{\partial}{\partial x_k} \end{pmatrix} \psi(\vec{x}, t) \]
\[ = \begin{pmatrix} 2im \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\partial}{\partial t} + 2\delta_{jk} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \partial \frac{\partial}{\partial x_j} \partial \frac{\partial}{\partial x_k} \end{pmatrix} \psi(\vec{x}, t) \]
\[ = \begin{pmatrix} i \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \frac{\partial}{\partial t} \partial \psi(\vec{x}, t) = 0, \]
(38)
which shows that each component of the four-component spinor state function \( \psi(\vec{x}, t) \) obeys the Schrödinger equation, and that the latter does not mix the spinor components.

3.5. Fundamental dynamical equations for four-component spinors

We have already shown that the only Galilean invariant first-order dynamical equation for four-component spinor state functions is the Lévy-Leblond equation (see proposition 2). Furthermore there is an entire class of higher order Lévy-Leblond equations, which result from taking the \( N \)th power of the Lévy-Leblond operator, and these equations are Galilean invariant (see proposition 3). An interesting result is that among this class of equations, there is the Schrödinger equation [17], which is obeyed by all spinor components, however, it does not mix them.

We now present a formal proof that there are no other invariant dynamical equations for four-component spinor state functions in Galilean spacetime. This is a new result and one of the most important of this paper, so we present it as the following proposition.

Proposition 4. If a four-component spinor state function \( \psi \) transforms as \( \psi(\vec{x}, t) = e^{i\phi(\vec{x}', t')} \psi'(\vec{x}', t') \), where \( \phi \) is the Schrödinger phase function given by equation (9), and \( (\vec{x}, t) \) and \( (\vec{x}', t') \) are two inertial frames of reference, then the Lévy-Leblond and Schrödinger equations are the only fundamental (Galilean invariant) dynamical equations.
Proof. Let us first consider equations among the Lévy-Leblond class. The $N$th-order equation is obtained by raising the Lévy-Leblond operator to the $N$th power. The even and odd powers can be examined separately such that

$$\mathcal{L}^N = \begin{cases} \mathcal{L}^{2M} & N \text{ even,} \\ \mathcal{L}^{2Q+1} & N \text{ odd,} \end{cases} \quad (39)$$

where $M = N/2 \geq 1$ and $Q = (N - 1)/2 \geq 1$. Using the result given by equation (38) for $N = 2$ and expanding the binomial to power $M$ produces

$$\mathcal{L}^{2M} = \left[2im\partial_t + 2i\partial_j^3\right]^M = \sum_{p,q} \frac{M!}{p!q!} (2im\partial_t)^p (2i\partial_j)^q . \quad (40)$$

Since mixed partial derivatives are produced for every term where $p > 0$ and $q > 0$, and since such mixed derivatives are not allowed in fundamental equations (see section 1), then there are no fundamental equations for $M > 1$. When $N$ is odd, then

$$\mathcal{L}^{2Q+1} = \sum_{p,q} \frac{Q!}{p!q!} (2im\partial_t)^p (2i\partial_j)^q . \quad (41)$$

In this case, the mixed partial differentials result for every $Q > 0$. There is no fortuitous canceling of terms as it was found for $N = 2$, which gave the Schrödinger equation, so there are no other fundamental equations.

Knowing that there are no other fundamental equations among the Lévy-Leblond class does not rule out other second- and higher order fundamental equations. To prove that there are no other fundamental equations for four-component spinor state functions, we must consider the most arbitrary $N$th order differential equation

$$\sum_{a,b} D_{abj} \partial^a \partial_j^b \psi(\vec{x}, t) = 0 , \quad (42)$$

where $D_{abj}$ are $4 \times 4$ constant matrices and there is an implied summation over the index $j = 1, 2, 3$. Eliminating mixed partial differential terms, equation (42) becomes

$$\sum_{a} D_{a0j} \partial^a + \sum_{b} D_{b0j} \partial_j^b + D_{00j} \psi(\vec{x}, t) = 0 . \quad (43)$$

The Galilean transformation rule (see equations (4) and (5)) can be written

$$G\partial^a \partial^b G^{-1}e^{i\psi(\vec{x},t)} \psi(\vec{x}, t) = e^{i\psi(\vec{x},t)}\left[k_1 + \partial_1 + k_2\partial_2 \right]^a \left[k_3 + \partial_3\right]^b \psi(\vec{x}, t) , \quad (44)$$

where the introduced constants are $k_1 = -im\vec{v}/2$, $k_2 = R_{ji}(\vec{v})^2 = (R_{ji}(\vec{v}))^2 = v^2$, $k_3 = -imR_{ji}v_jR_{kji}$, $k_{\partial^2} = -(R_{ji}(\vec{v})^2)^2 = -m^2 \vec{v}^2$, and $k_{\partial_3} = R_{ji}$. According to this rule equation (43) transforms into

$$e^{i\psi(\vec{x},t)} \sum_{a} D'_{a0j} \left(k_1 + \partial_1 + k_2\partial_2 \right)^a \sum_{b} D'_{b0j} \left(k_3 + \partial_3\right)^b + D'_{00j} \psi(\vec{x}, t) = 0. \quad (45)$$

The trinomial and binomials can be expanded by their respective powers

$$e^{i\psi(\vec{x},t)} \sum_{a} D'_{a0j} \left(\sum_{p,q=r} P(k_{1}^a, 1^q, k_{2}^b) \partial^a \partial^b \right) + \sum_{b} D'_{b0j} \left(\sum_{u,v} P(k_{3}^a, k_{3}^b) \partial^a \right) + D'_{00j} \psi(\vec{x}, t) = 0 , \quad (46)$$
where the function $P()$ produces the permutation of its arguments. To be Galilean invariant this equation must be equal to the untransformed equation (see equation (43)) and terms with mixed partial differentials must vanish. The condition that must be met for the sum of mixed partial differential terms of like powers $e$ and $f$ to vanish is

$$0 < a \leq N \sum a D_a^0 \left( \sum_p P(k_1^p, 1^r, k_2^j) = a \sum_p P(k_1^p, 1^r, k_2^j) \right) = 0,$$

(47)

with $e > 0$ and $f > 0$.

The permuted terms do not vanish so the matrices must sum together to equal zero. Hence, the remaining conditions for invariance are

$$D_{e0} = \sum_{0 < a \leq N} a D_{a0} \left( \sum_p P(k_1^p, 1^r, k_2^j) \right),$$

(48)

$$D_{0fj} = \sum_{0 < a \leq N} a D_{a0} \left( \sum_p P(k_1^p, 1^r, k_2^j) \right) + \sum_{0 < b \leq N} b D_{0b} \left( \sum_a P(k_1^a, 1^r, k_2^j) \right),$$

(49)

and

$$D_{00} = \sum_{0 < a \leq N} a D_{a0} \left( P(k_1^a, 1^r, k_2^j) + \sum_{0 < b \leq N} b D_{0b} \left( P(k_1^a, 1^r, k_2^j) \right) + D_{00}^f. \right.$$

(50)

Setting $f = 0$ in equation (47) and combining with equation (48) proves that $D_{e0} = 0$. So there are no other Galilean invariant dynamical equations that are free of mixed partial differentials for four-component spinors. This concludes the proof of proposition 4.

3.6. Discussion

There are several important results of this paper. First, we demonstrated that there are no fundamental dynamical equations for two-component spinor wavefunctions in Galilean spacetime; this is consistent with a well-known result that $2 \times 2$ Galilean boost matrices do not exist [24]. Second, we derived the Lévy-Leblond equation for a four-component spinor wavefunction by using a better justified method than the original approach used by Lévy-Leblond [13, 14], Fushchich and Nikitin [8] and Niederle and Nikitin [25]. Third, we presented a formal proof that the Lévy-Leblond equation is the only first-order fundamental dynamical equation for four-component spinor wavefunctions in Galilean spacetime.

We also proved that the Schrödinger equation is the only second-order fundamental dynamical equation in Galilean spacetime, and that there are no other higher order fundamental equations for two- and four-component spinors; this result is consistent with the fact that the extended Galilei group has only one invariant Casimir operator, which depends explicitly on the Hamiltonian. Finally, we want to point out that all our results were obtained using the Schrödinger phase factor given by equation (9), and that these results have far reaching physical consequences that will be now discussed.

Our results demonstrate that the Schrödinger equation for four-component spinor state functions is Galilean invariant (see proposition 3 and corollary that follows it). As a result, each spinor component must obey this equation. On the other hand, the equation does not mix the spinor component, which means that it does not lead to any new results. Hence, our results allow us to conclude that the Lévy-Leblond equation for four-component spinor state functions is the only fundamental dynamical equation that correctly describes elementary particles with spin $1/2$ in Galilean spacetime.
4. PS equation

The fact that the Dirac, PS and Schrödinger equations are intimately related is well known [30–33]. The PS equation is an approximation to the Dirac equation for small electron velocities and the Schrödinger equation can be obtained from the PS equation by neglecting magnetic interaction of the spin. Since the PS equation describes evolution of a two-component spinor state function in Galilean spacetime, it is used to introduce spin in non-relativistic quantum mechanics. The equation can be formally derived from the Lévy-Leblond equation [14]. We now discuss the relationship between the obtained results and the PS equation.

Let us introduce the 4-potential 

$$ V, A_j $$

of the electromagnetic field, where $V$ and $A_j$ are the scalar and vector potentials, and make the following substitutions:

$$ i\partial_t \rightarrow i\partial_t - V(\vec{x}, t), \quad (51) $$

and

$$ -i\partial_j \rightarrow -i\partial_j - A_j(\vec{x}, t). \quad (52) $$

Maxwell’s equations break Galilean invariance but may be cast into a Galilean invariant form by the elimination of Maxwell’s term in the non-relativistic limit ($c \to \infty$). Performing the above substitution in the Lévy-Leblond equation and splitting the four-component state function into a pair of two-component state functions $\psi = (\phi, \chi)$ produces a pair of coupled equations of the form

$$ \sigma_j(i\partial_j + A_j)\phi - 2m\chi = 0, \quad (53) $$

and

$$ (i\partial_t - v)\phi + \sigma_j(i\partial_j + A_j)\chi = 0. \quad (54) $$

The PS equation is obtained by eliminating $\chi$ from the above pair of equations, and we have

$$ i\partial_t \phi = V\phi - \frac{1}{2m}[(i\partial_j + A_j)^2 + i\sigma \cdot (i\partial_j + A_j) \times (i\partial_j + A_j)]\phi. \quad (55) $$

The PS equation is a second-order differential equation governing the space and time evolution of a two-component spinor $\phi$, which is only half of the four-component spinor state function $\psi$.

According to Lévy-Leblond [14], a four-component spinor $\psi$ transforms as

$$ \psi(\vec{x}, t) \rightarrow e^{if(\vec{x}, t)} \begin{pmatrix} U_R & 0 \\ -U_R \sigma \cdot v / 2 & U_R \end{pmatrix} \psi(\vec{x}, t), \quad (56) $$

where the phase function $f(\vec{x}, t)$ is given by equation (9), and $U_R$ is the a $2 \times 2$ unitary rotation matrix. Under this transformation the PS equation has been shown to be Galilean invariant [8, 14, 33] and the top half of the transformation can be extracted to obtain a rule for how $\phi$ transforms by itself

$$ \phi(\vec{x}, t) \rightarrow e^{if(\vec{x}, t)} U_R \phi(\vec{x}, t). \quad (57) $$

This two-component transformation can be used to show that the PS equation is Galilean invariant [33].

The above result does not contradict the earlier result (see proposition 1) that there are no boosts for two-component spinors. In fact proposition 1 is correct because by itself this two-component transformation on $\phi$ does not conform to the group composition law and its generators do not obey the commutation relations of the extended Galilei group. Only in the context of the full four-component transformation (see equation (56)) is this transformation a valid representation of the group.
An extensive discussion of the PS equation was given by de Montigny et al [24], who used the so-called reduction approach and direct approach to formally derive the equation and also to determine all possible PS interactions for four-component spinor state functions, which are compatible with the Galilean invariance. The fact that the authors considered four-component spinor state functions make their PS equation Galilean invariant, and this result is consistent with our proposition 2 and also with equations (53) and (54) in this paper.

Since the PS equation is derived from the Lévy-Leblond equation, we conclude that the Lévy-Leblond and Schrödinger equations for four-component spinor wavefunctions are the only fundamental dynamical equation in Galilean spacetime with the Schrödinger phase factor (see proposition 4). This is an important result as it shows that only the Lévy-Leblond and Schrödinger equations are available to formulate quantum field theories of elementary particles described by four-component spinor state functions in Galilean spacetime.

5. Conclusions

A search for fundamental dynamical equations for two- and four-component spinor state functions was conducted in Galilean spacetime represented by the extended Galilei group. A dynamical equation was considered fundamental if it was invariant under the symmetry operators of the group of the Galilei metric, if the state functions transformed like the irreducible representations of the group of the metric, and if the dynamical equation did not have mixed time and space partial derivatives.

The main results obtained were: (i) there are no fundamental dynamical equations for two-component spinor wavefunctions in Galilean spacetime; (ii) the Lévy-Leblond equation for a four-component spinor wavefunction can be derived by using a different method than the one originally used by Lévy-Leblond; (iii) a formal proof that the Lévy-Leblond equation is the only first-order fundamental dynamical equation for four-component spinor wavefunctions in Galilean spacetime; (iv) the Schrödinger equation is the only second-order fundamental dynamical equation in Galilean spacetime; (v) there are no other higher order fundamental equations for two and four-component spinors and (vi) the Pauli–Schrödinger equation for two-component spinor wavefunctions can be derived from the Lévy-Leblond equation. Our conclusions (ii), (iii), (iv) and (vi) are similar to those reached by Fushchich and Nikitin [8], and more recently by de Montigny et al [24] and by Niederle and Nikitin [25], albeit by a different analysis.

All our results presented here were obtained with the Schrödinger phase function [20], however, other phase functions are also possible. It will be shown in the next paper of this series that such phase functions lead to new fundamental dynamical equations that have important physical implications.

Acknowledgments

We are grateful to two anonymous referees for their comments and suggestions, and specifically want to thank the second referee for his/her valuable comments on Galilean invariance of the Pauli–Schrödinger equation. We also thank Alex Weiss and Chris Jackson for discussions and their comments on this paper. ZEM acknowledges the support of this work by the Alexander von Humboldt Foundation.

References

[1] Wigner E P 1939 Ann. Math. 40 149
[2] Kim Y S and Noz M E 1986 Theory and Applications of the Poincaré Group (Dordrecht: Reidel)
