An efficient method to evaluate energy variances for extrapolation methods.

G. Puddu
Dipartimento di Fisica dell’Università di Milano,
Via Celoria 16, I-20133 Milano, Italy

December 21, 2013

Abstract

The energy variance extrapolation method consists in relating the approximate energies in many-body calculations to the corresponding energy variances and inferring eigenvalues by extrapolating to zero variance. The method needs a fast evaluation of the energy variances. For many-body methods that expand the nuclear wave functions in terms of deformed Slater determinants, the best available method for the evaluation of energy variances scales with the sixth power of the number of single-particle states.

We propose a new method which depends on the number of single-particle orbits and the number of particles rather than the number of single-particle states. We discuss as an example the case of $^4He$ using the chiral N3LO interaction in a basis consisting up to 184 single-particle states.

Pacs numbers: 21.60.De, 24.10.Cn, 27.10.+h
1 Introduction.

Both the Monte Carlo shell model (MCSM) (refs. [1]-[3]) and the Hybrid Multi-Determinant (HMD) method (refs. [4],[5]) approximate eigenstates of the nuclear Hamiltonian, although with a different parametrization and minimization method, with a linear combination of Slater determinants as a variational ansatz. Several years ago (refs.[6]-[8]) a robust method has been proposed whereby the error in the energy $E_{\text{approx}} - E_{\text{exact}}$ was shown to have a linear behavior as a function of the energy variance for wave-functions sufficiently close to the exact ones. By extrapolating to zero variance one can then infer the exact (or almost exact) energy eigenvalues. This idea was put forward for the first time in the context of shell model calculations where the energy variance is generated by the Lanczos diagonalization method. In ref. [6] a sequence of approximate shell model wave functions $|\psi>$ was obtained by truncation of number of possible excitations. Using the proportionality relation $<\psi|\hat{H}|\psi>-E_{\text{exact}} \propto <\psi|\hat{H}^2|\psi>-<\psi|\hat{H}|\psi>^2$, valid if the approximate wave function is sufficiently close to the exact one, and extrapolating to zero variance one can obtain the exact value of the energy, at least in principle. Within the Lanczos diagonalization method the variance is obtained without extra effort. For variational methods, although the extrapolation method is still valid, the evaluation of the energy variance is computationally very expensive. Note however that access to the full Hilbert space is not needed. Recently the extrapolation method has been revived within the Monte Carlo shell model method (refs.[9],[10]) and applied to ab-initio calculations. The cost of the method is proportional to the sixth power of number of single-particle states. The very same cost applies to the Hybrid Multi-Determinant method. Clearly for a systematic applicability of the method, the computational cost must be reduced. In this work
we propose an efficient method to evaluate the energy variance. More precisely we give an efficient recipe to evaluate $\langle \psi | \hat{H}^2 | \phi \rangle$ where $| \psi \rangle$ and $| \phi \rangle$ are two Slater determinants. The method which will be described in detail in the next section is based on a factorization of the density matrix and its computational costs depends on the number of particles and of single-particle orbits $n, l, j$ rather than on the number of single-particle states $n, l, j, m$. In section 3 we discuss the application of the method to the case of $^4\text{He}$ using the $N3LO$ interaction (ref.[11]) in an harmonic oscillator basis of 6, 7 and 8 major shells ($N_{ho} = 5, 6, 7$ respectively). We restrict ourselves to $l \leq 5$ and the largest single-particle space consists of 184 single-particle states. Rather than work with the bare N3LO interaction we renormalize the interaction with two methods. In the first method we renormalize the interaction with a sharp relative momentum cutoff $K_{\text{max}} = 2.5 \text{fm}^{-1}$ as done in $V_{\text{lowk}}$ (ref. [12]), then we renormalize once more to a specified number of harmonic oscillator shells using the Lee-Suzuki renormalization procedure (refs.[13],[14]). In the second method we use directly the Lee-Suzuki procedure on the N3LO interaction. The method we propose to evaluate the energy variance has been implemented on a personal computer with modest resources.

2 Evaluation of the energy variance.

In both the MCSM and in the HMD methods the variational ansatz for the eigenstates is

$$|\psi_N \rangle = \sum_{\alpha=1}^{N} g_{\alpha N} |U_{\alpha, N} \rangle,$$  

(1)
Only for simplicity we omit the projector to good angular momentum and parity. The $|U_{\alpha N}\rangle$ are variational Slater determinants or, more precisely, a product of a neutron and a proton Slater determinant. $N$ is the number of Slater determinants. If Slater determinants are added one by one the cost of generating and optimizing $N$ Slater determinants scales as $N^2$. Therefore it is computationally expensive to increase the accuracy in the evaluation of energies and observables after a large number of Slater determinants. Therefore an efficient and theoretically robust extrapolation method is highly valuable. The coefficients $g_{\alpha N}$ are determined by solving the generalized eigenvalue problem

$$
\sum_\beta <U_{\alpha N}|\hat{H}|U_{\beta N}> g_{\beta N} = E_N \sum_\beta <U_{\alpha N}|U_{\beta N}> g_{\beta N},
$$

(2)

$E_N$ being the approximate energy for a set of $N$ Slater determinants. We write the Hamiltonian as

$$
\hat{H} = \frac{1}{2} \sum_{ijkl} v_{ijkl} a^\dagger_i a_j a^\dagger_k a_l
$$

(3)

where $a^\dagger_i$ is the creation operator in the single-particle state $i$. For convenience we include the single-particle Hamiltonian in the anti-symmetric part of the two-body potential $\tau_{ijkl} = \frac{1}{2} (v_{ijkl} - v_{ijlk})$. The matrix elements of $\hat{H}$ and of $\hat{H}^2$ between two different Slater determinants $|U\rangle$ and $|V\rangle$ are given by

$$
\frac{<V|\hat{H}|U>}{<V|U>} = \tau_{ijkl} \rho_{ki} \rho_{lj}
$$

(4)

and

$$
\frac{<V|\hat{H}^2|U>}{<V|U>} = (\tau_{ijkl} \rho_{ki} \rho_{lj})^2 + 4\text{tr}(\nabla^2 F^\dagger V) + \tau_{ijkl} F_{ki} F_{kj} \tau_{pqrs} \rho_{ri} \rho_{sj}
$$

(5)

where $\rho_{ki} = <V|a^\dagger_i a_k|U>$ is the generalized density matrix, $F_{ik} = <V|a_i a^\dagger_k|U>$ and

$$
\nabla_{jl} = \tau_{ijkl} \rho_{ki}
$$

(6)
in eqs. (4)-(6), the sum over repeated indices is understood. Note that the indices of the density matrix refer only to identical particles, that is $\rho_{np} = 0$.

The use of the anti-symmetric part of the Hamiltonian $\tau$ reduces the number of terms since the exchange terms are opposite to the direct contributions. In eq. (5), the trace term is taken in the single-particle indices, i.e.

$$\text{tr}(\nabla F \nabla \rho) = \nabla_{ij} F_{jk} \nabla_{kl} \rho_{li}$$

(7)

In ref.[9], apart a slight change in the notations, the last term in eq. (5) is recast as,

$$Q = \tau_{(ij),(kl)}(FF)_{(kl),(pq)}\tau_{(pq),(rs)}(\rho\rho)_{(rs),(ij)}$$

(8)

and it is treated as two products of matrices of dimensions equal to $N_{sp}^2$, where $N_{sp}$ is the number of single-particle states. Hence the computational cost scales as $N_{sp}^6$.

We now proceed to modify these expressions in order to reduce the computational cost. The first two terms in eq. (5) pose no problem and we will focus entirely on the third term alone which we call $Q$. For readability we replace latin indices with numbers (e.g. $1 \equiv (n_1, l_1, j_1, m_1)$ etc. of dimension $N_{sp}$).

The Slater determinants $|U >$ and $|V >$ are identified by the single-particle wave functions $U_{1,\alpha}$ and $V_{1,\alpha}$ with and $\alpha = 1, 2, .. A$ where $A$ is the number of particles (we will treat explicitly neutrons and protons later). The expressions for $\rho$ and $F$ are given by

$$\rho_{12} = \sum_{\alpha} U_{1\alpha} W_{\alpha,2}$$

(9a)

and

$$F_{12} = \delta_{12} - \rho_{12}$$

(9b)

where $\delta$ is the Kronecker $\delta$, and

$$W_{\alpha,2} = \sum_{\beta} (V^\dagger U)^{-1}_{\alpha,\beta} V^\dagger_{\beta,2}$$

(10)
From eq.(9a) and eq.(10), one can see that \( \rho^2 = \rho \) and \( \text{tr} \rho = A \) and the matrix multiplications in the expression for \( \rho \) are of the type \((NA) \times (AA) \times (AN)\). The rectangular matrices \( U, W \) are stored at the beginning of the calculations. The various \( nn, pp \) and \( np \) contributions give

\[
Q = Q^{(nn)} + Q^{(pp)} + 4Q^{(np)}
\]

the factor of 4 comes from all possible exchanges between \( n \) and \( p \) indices, and \( Q^{(np)} \) contains only terms like \( < np|\vec{v}|np > \), for sake of argument, in this order.

Inserting eqs. (9a)-(10) in the last term of eq.(5) (the \( Q \) term) and using eq.(9b), we obtain

\[
Q(tt') \equiv \overline{v}_{1234} F_{35} F_{46} \rho_{71} \rho_{82} = C_2(tt') + C_{3a}(tt') + C_{3b}(tt') + C_4(tt')
\]

for \( (tt') = (nn), (pp), (np) \), with

\[
C_2(tt') = (W_{\gamma_1} W_{\delta_2} \overline{v}_{1234})(\overline{v}_{341'2'} U_{1'\gamma} U_{2'\delta})
\]

\[
C_{3a}(tt') = -(W_{\gamma_1} W_{\delta_2} \overline{v}_{1234} U_{43})(W_{\beta_4} \overline{v}_{34'1'2'} U_{1'\gamma} U_{2'\delta})
\]

\[
C_{3b}(tt') = -(W_{\gamma_1} W_{\delta_2} \overline{v}_{1234} U_{3a})(W_{\alpha_3} \overline{v}_{34'1'2'} U_{1'\gamma} U_{2'\delta})
\]

\[
C_4(tt') = (W_{\gamma_1} W_{\delta_2} \overline{v}_{1234} U_{3a} U_{43})(W_{\alpha_3} W_{\beta_4} \overline{v}_{34'1'2'} U_{1'\gamma} U_{2'\delta})
\]

Notice that in the case \( (tt') = (np) \), odd numbers/letters represent neutrons (e.g. 135, .., \( \alpha \gamma \)) while even numbers/letters represent protons (e.g. 2, 4, 6, .., \( \beta \delta \)). Again the sum over repeated indices is understood. In eq.(13a)-(13d) we grouped the various terms so that matrix multiplications can be efficiently performed. Notice that

\[
L_{\gamma_\delta,34} = W_{\gamma_1} W_{\delta_2} \overline{v}_{1234}
\]

and

\[
R_{34,\gamma_\delta} = \overline{v}_{341'2'} U_{1'\gamma} U_{2'\delta}
\]
enter in all contributions and can be very efficiently evaluated since $\overline{v}$ is very sparse. The various $C_2, C_3, C_4$ carry a label that identifies their 2-body, 3-body and 4-body character. Each term contained between the brackets can be evaluated from the previous equations and using eqs.(14a),(14b). Eqs. (11)-(14) improve the computation of the $Q$ term, since some of the indices are particle indices, rather than single-particle ones. Essentially we make use of the fact that $\rho$ is a low rank matrix. In order to see how the computational cost scales with the particle number and with the number of single-particle states, consider for example the case $(tt') = (nn)$ and let $N_n$ be the number of neutrons and $N_v$ the number of non-zero matrix elements of $\overline{v}$. It is easy to see that the computational cost for the evaluation once for all of the matrices $L$ and $R$ in eq.(14) scales as $N_n N_v^2$ and the evaluation of $C_2, C_3$ and $C_4$ scale as $N_n^2 N_{sp}^2$, $N_n^3 N_{sp}^2$ and $N_n^4 N_{sp}$ respectively. Let us remark that in ab-initio calculations $N_{sp} \gg N_n$.

Despite this improvement, we prefer to use the angular momentum coupled matrix elements of $\overline{v}$. The reason is two-fold. First of all for large single-particle spaces the number of uncoupled matrix elements of $\overline{v}$ is very large. In some of the calculations described in the next section we have used a personal computer of 1Gyb of RAM, Second, instead of dealing as in eqs.(13a)-(13d) with single-particle indices, we prefer to deal with orbits (i.e. $nlj$).

Let us first change slightly the notations: let us label orbits by numbers, i.e. $1 \equiv (n_1, l_1, j_1)$ etc., while single-particle indices are now represented as $(1, m_1)$ etc., and, as before, greek letters count particles. It is then straightforward to arrive at the following results. Define the angular momentum coupled quantities by Clebsh-Gordan coefficients

$$R_{34\gamma\delta}^{JM} = \overline{\rho}_{3412}^{J}(UU)^{JM}_{12\gamma\delta} \quad (16)$$
\[ L_{\gamma^3}^{JM} = (WW)_{\gamma^6}^{JM} \pi_{1234} \] (17)

with

\[ (WW)^{JM}_{\alpha\beta} = \sum_{m_1m_2} <j_1m_1j_2m_2|JM > W_{\alpha,1m_1}W_{\beta,2m_2} \] (18)

\[ (UU)^{JM}_{12\alpha\beta} = \sum_{m_1m_2} <j_1m_1j_2m_2|JM > U_{1m_1,\alpha}U_{2m_2,\beta} \] (19)

We stress again that in the case of the \((np)\) contribution, odd numbers/letters refer to neutrons and even numbers/letters refer to protons. Then the 2-body contribution to \(Q\) is given by

\[ C^2(tt') = \sum_{JM} L_{\alpha\beta}^{JM} R_{12\alpha\beta}^{JM} \] (20)

where we have shown explicitly the sum over \(JM\) only, and the sum over the remaining indices is implicit. All matrix multiplications have now small dimensions. For the contribution of 3-body type, define also

\[ W_{\beta 6}^{JM}(j_3m_3) = \sum_{m_6} <j_3m_3j_6m_6|JM > W_{\beta,6m_6} \] (21a)

\[ U_{4\beta}^{JM}(j_3m_3) = \sum_{m_4} <j_3m_3j_4m_4|JM > U_{4m_4,\beta} \] (21b)

and

\[ R_{\beta,\gamma\delta}(3, m_3) = \sum_{JM,6} W_{\beta,6}^{JM}(j_3, m_3) R_{30\gamma\delta}^{JM} \] (22a)

\[ L_{\gamma\delta,\beta}(3, m_3) = \sum_{JM,4} L_{\gamma\delta,4\beta}^{JM} U_{4\beta}^{JM}(j_3m_3) \] (22b)

Then

\[ C_{3a}(tt') = -\sum_{3m_3} L_{\gamma\delta,\beta}(3m_3) R_{\beta,\gamma\delta}(3m_3) \] (23)

In the case of \(C_{3b}\) define

\[ U_{3\alpha}^{JM}(j_4m_4) = \sum_{m_4} <j_3m_3j_4m_4|JM > U_{3m_3,\alpha} \] (24a)

\[ W_{\alpha 5}^{JM}(j_4m_4) = \sum_{m_5} <j_5m_5j_4m_4|JM > W_{\alpha,5m_5} \] (24b)

8
and

\[ \mathcal{L}_{\gamma \delta, \alpha}(4, m_4) = \sum_{JM} L_{\gamma \delta,34}^{JM} U_{3\alpha}^{JM}(j_4 m_4) \]  

(25a)

\[ \mathcal{R}_{\alpha, \gamma \delta}(4, m_4) = \sum_{J5M} W_{\alpha,5}^{JM}(j_4, m_4) R_{54, \gamma \delta}^{JM} \]  

(25b)

Then

\[ C_{3b}(tt') = - \sum_{4, m_4} \mathcal{L}_{\gamma \delta, \alpha} \mathcal{R}_{\alpha, \gamma \delta}(4m_4) \]  

(26)

Note the difference in the implicitly summed indices in this expression compared with the ones in eq.(23). The contribution of 4-body type has the simple expression

\[ C_4(tt') = M_{\alpha \beta, \gamma \delta} M_{\gamma \delta, \alpha \beta} \]  

(27)

with

\[ M_{\alpha \beta, \gamma \delta} = \sum_{JM} (WW)^{JM}_{\alpha \beta 12} R_{12, \gamma \delta}^{JM} \]  

(28)

In the np case the contributions \( C_{3a} \) and \( C_{3b} \) are not equal in general. They are equal in the case of \((tt') = (nn), (pp)\). Also it is worth to note that the four-fold sum in eq.(28) is over the particle indices only. As before, let us analyze how the computational cost scales the number of orbits and the number of particles. Let us consider the case of the \( nn \) contribution. The evaluation of eqs.(18) and (19) is straightforward because of the angular momentum selection rules. The evaluation of eqs.(16) and (17) scales as \( N_{JM} N_e(JM) N_n^2 \) where \( N_{JM} \) is the number of the possible \( JM \) values, \( N_e(JM) \) is the number of non-zero matrix elements \( \tau_{1234}^J \) (the total number of angular momentum coupled matrix elements \( \tau_{1234}^J \) can be a factor of 30 smaller than the number of uncoupled matrix elements of \( \tau \)). Once eqs.(16) and (17) have been evaluated for all \( JM \) values, the computational cost of \( C_2 \) scales as \( N_{JM} N_{orb}^2 N_n^2 \) where \( N_{orb} \) is the number of single-particle orbits. The evaluation of \( C_3 \) (cf. eq.(23)) scales as \( N_{sp} N_n^3 \) in addition to the cost of evaluation
of eqs. (22a) and (22b) which scales as $N_J N_{sp} N_{orb} N_n^3$ (note that the 3-body term retains one power of $N_{sp}$. The computational cost of the evaluation of the matrix $\mathcal{M}$ scales as $N_J N_{orb}^2 N_n^4$ and the evaluation of the 4-body contribution scales as $N_n^4$. Since the number of uncoupled matrix elements of the Hamiltonian can easily be of the order of $10^7$ we use only the angular momentum coupled representation.

3 A case study with $^4He$.

Let us apply the method derived in the previous section to $^4He$. We use the N3LO interaction (ref. [11]) as the nucleon-nucleon potential. This potential has a smooth energy cutoff at $\Lambda = 500\text{MeV}$ corresponding to a relative momentum $K_{\text{max}} = 2.534\text{fm}^{-1}$ The potential is however non-zero at higher relative momenta due to the smooth cutoff. In ref.[15], a rather large number of harmonic oscillator major shells was used in order to reach independence from the h.o. frequency and from the single-particle space, using the 'bare' potential. In order to be able to work with smaller spaces we renormalize the interaction with two methods. In the first method we first renormalize the interaction much in the same way it is done to obtain $V_{\text{lowk}}$ interactions (cf. ref. [12]) to a sharp relative momentum $K_{\text{max}} = 2.5\text{fm}^{-1}$ (essentially to get rid of the high momentum tail) and then we renormalize once more to a specified number of major shells with the Lee-Suzuki method, in order to reach some independence from the h.o. frequency with reasonably small single-particle spaces. In the second method we apply directly the Lee-Suzuki method to the N3LO interaction.
The rationale for the first choice is the following. For an interaction which has a sharp cutoff \( K_{max} \) and for a system of radius \( R \) the acceptable values of \( \hbar \Omega \) are given by the inequality (ref.[15])

\[
\hbar^2 K_{max}^2 / (mN_{ho}) < \hbar \omega < N_{ho} \hbar^2 / (mR^2)
\]

and in order to have a reasonable large interval in \( \hbar \omega \) we need large values of \( N_{ho} \). This is correct if we use the bare interaction. It is hoped that, by renormalizing the interaction one more time, we can reach the energies given by the bare interaction having a specified \( K_{max} \), with smaller values of \( N_{ho} \). The only problem with this argument is that the first renormalization to a sharp cutoff could give rise to induced many-body interactions. This is precisely the reason why we took a large value of \( K_{max} \), that is, to minimize the induced many-body interactions. In ref. [17], it was shown that in order to minimize the effect of induced many-body interactions, \( K_{max} \) should be as large as possible \( (K_{max} \approx 5 \text{ fm}^{-1}) \), this result has been obtained using the CD-Bonn interaction which contains larger energy scales than N3LO. In any case, if we use \( V_{lowk} \) interactions with a large cutoff, another renormalization step is nearly unavoidable, unless we are willing to work with very large single-particle spaces.

We have used the HMD method of type-a (described in ref. [16]) to generate a sequence of wave-functions \( |\psi_n> \) consisting of a \( n \) Slater determinants (up to a maximum of 100). We have used a projector to good parity and good \( z \)-component of the angular momentum. We considered several values of the harmonic oscillator frequency namely \( \hbar \Omega = (21, 24, 27, 30, 33, 36) \text{MeV} \) and \( N_{ho} = 5, 6 \). For \( N_{ho} = 7 \) we considered only \( \hbar \Omega = (27, 30) \text{MeV} \). Only states with \( l < 6 \) have been included in the single-particle basis. The largest number of single-particle states is 184 and the corresponding number of orbits is 32 for \( N_{ho} = 7 \).
Figure 1: Energy-variance plot for $N_{ho} = 7$. 
Among the several variants of the energy vs. variance extrapolation methods we considered the first one, used in ref.[6], where \( E_n = \langle \psi_n | \hat{H} | \psi_n \rangle \) is expanded as a function of the dimensionless variance defined as

\[ \Delta E_n = \frac{\sigma^2}{\langle \psi_n | \hat{H} | \psi_n \rangle^2} = \frac{\langle \psi_n | \hat{H}^2 | \psi_n \rangle - \langle \psi_n | \hat{H} | \psi_n \rangle^2}{\langle \psi_n | \hat{H} | \psi_n \rangle^2} \]

where \(| \psi_n \rangle\) is the wave-function obtained with the first \( n \) Slater determinants, and fitted with a linear function \( E_n = a + b \Delta E_n \). In the limit \( \Delta E_n = 0 \), \( a \) is the extrapolated ground-state energy.

For the purpose of the extrapolation we discard low values of \( n \) as done in ref. [9]. As a measure of the soundness of the linear fit we consider the quantity

\[ \mathcal{E}_{n_0} = \sqrt{\sum_{n>n_0} (E_n - a - b \Delta E_n)^2} \quad (29) \]
where the first \( n_0 \) many-body wave functions have been discarded, since they are a poor approximation to the exact eigenstate. The value of \( n_0 \) is selected so that \( E_{n_0} \) is of the order of 10\( K eV \) or smaller. In fig. 1 we show a few plots of the energy as a function of the variance \( \sigma^2 \). Typically variances for this nucleus and this interaction are much larger than the ones shown in ref. [9]. This is not very surprising since in our case the spectrum extends over very large values of the energy, while in the case study of ref. [9], where only one and two major shells were considered, the spectrum is much more compressed. In fig. 2 we show our results of the calculation and the the exact Faddeev-Yakubovski result. In fig. 2 we also show the results obtained using directly the Lee-Suzuki method for \( N_{ho} = 5 \). Interestingly enough, the double renormalization method performs better than the single L-S renormalization which shows a too marked variation as a function of the harmonic oscillator frequency. Note also the results for \( N_{ho} = 7 \) are very close to the ones for \( N_{ho} = 6 \).

In conclusion, in this work we have implemented an efficient method to evaluate energy variances for variational many-body calculations which are needed in order to apply the energy variance extrapolation method. This method avoids the need to determine a very large number of Slater determinants in order to improve the accuracy of observables. Since the computational cost of the evaluation of \( N \) Slater determinants scales as \( N^2 \), the method is a useful tool to decrease the cost of variational many-body calculations.

References

[1] M.Honma, T.Mizusaki, and T.Otsuka, Phys. Rev. Lett. 75, 1284 (1995).
[2] T. Otsuka, M. Honma, and T. Mizusaki, Phys. Rev. Lett. 81, 1588 (1998).

[3] T. Otsuka, M. Honma, T. Mizusaki, N. Shimizu, and Y. Utsuno.
   Prog. Part. Nucl. Phys. 47, 319 (2001).

[4] G. Puddu. J. Phys. G: Nucl. Part. Phys. 32, 321 (2006).

[5] G. Puddu. Eur. Phys. J. A 34, 413 (2007).

[6] T. Mizusaki and M. Imada. Phys. Rev. C 65, 064319 (2002).

[7] T. Mizusaki and M. Imada. Phys. Rev. C 67, 041301 (2003).

[8] T. Mizusaki, Phys. Rev. C 70, 044316 (2004).

[9] N. Shimizu, Y. Utsuno, T. Mizusaki, T. Otsuka, T. Abe, and M. Honma.
   Phys. Rev. C 82, 061305 (2010).

[10] T. Abe, P. Maris, T. Otsuka, N. Shimizu, Y. Utsuno and J. P. Vary.
    arXiv: 1107.1784 [nucl-th].

[11] D. R. Entem and R. Machleidt, Phys. Rev. C 68, 041001 (2003).

[12] J. D. Holt, T. T. S. Kuo, and G. E. Brown, Phys. Rev. C 69, 034428 (2004).

[13] K. Suzuki and S. Y. Lee. Prog. Theor. Phys. 64, 209, (1980).

[14] S. Fujii, R. Okamoto, and K. Suzuki. Phys. Rev. C 69, 034328 (2004).

[15] G. Hagen, T. Pappenbrock, D. J. Dean, and M. Hjorth-Jensen.
    Phys. Rev. C 82, 034330 (2010).

[16] G. Puddu. Eur. Phys. J. A 45, 233 (2010).
[17] S. Fujii, E. Epelbaum, H. Kamada, R. Okamoto, K. Suzuki, and W. Glockle. Phys. Rev. C 70, 024003 (2004).