Coupled Quantum Harmonic Oscillators and Feynman–Kac path integrals for Linear Diffusive Particles

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Abstract: We propose a new solvable class of multidimensional quantum harmonic oscillators for a linear diffusive particle and a quadratic energy absorbing well associated with a semi-definite positive matrix force. Under natural and easily checked controllability conditions, for non necessarily reversible models with possibly transient free particle diffusions, the ground state and the zero-point energy are explicitly computed in terms of the unique positive fixed point of a continuous time algebraic Riccati matrix equation. We also present an explicit solution of normalised and time dependent Feynman–Kac measures on path spaces in terms of a time varying linear dynamical system coupled with a differential Riccati matrix equation. A refined non asymptotic analysis of the stability of these models is developed based on a recently developed Floquet-type representation of time varying exponential semigroups of Riccati matrices. We provide explicit and non asymptotic estimates of the exponential decays to equilibrium of Feynman–Kac semigroups in terms of Wasserstein distances or Boltzmann-relative entropy. For reversible models we develop a series of functional inequalities including de Bruijn identity, Fisher’s information decays, log-Sobolev inequalities, and entropy contraction estimates. In this context, we also provide a complete and explicit description of all the spectrum and the excited states of the Hamiltonian, yielding what seems to be the first result of this type for this class of models. We illustrate these formulæ with the traditional harmonic oscillator associated with real time Brownian particles and Mehler’s formula. The analysis developed in this article can also be extended to solve time dependent Schrédinger equations equipped with time varying linear diffusions and quadratic potential functions.

1. Introduction

1.1. Description of the models. Fix \( r \geq 1 \) and let \( A, R, S \) be \((r \times r)\)-matrices with real entries. Further, let \( \mathcal{H} \) denote the Hamiltonian differential operator

\[ \mathcal{H} = -\mathcal{L} + V, \]  (1)
where $\mathcal{L}$ is the second order differential kinetic energy operator

$$\mathcal{L} := \sum_{1 \leq k,l \leq r} A_{k,l} x_k \partial x_l + \frac{1}{2} \sum_{1 \leq k,l \leq r} R_{k,l} \partial x_k \partial x_l$$  \hspace{1cm} (2)$$

and the potential energy, $V$, is given by

$$V(x) := \frac{1}{2} \sum_{1 \leq k,l \leq r} x_k S_{k,l} x_l.$$  \hspace{1cm} (3)$$

We shall assume throughout the article that $R$ and $S$ are positive semi-definite matrices and the pairs of matrices $(A, R^{1/2})$ and $(A', S^{1/2})$ are both controllable, in the sense that the $(r \times r^2)$-matrices

$$\left[ R^{1/2}, A R^{1/2} \ldots, A^{r-1} R^{1/2} \right] \quad \text{and} \quad \left[ S^{1/2}, A' S^{1/2} \ldots, (A')^{r-1} S^{1/2} \right]$$  \hspace{1cm} (4)$$

have rank $r$. We have used the notation $A'$ to denote the transpose of the matrix $A$. Note that the above condition holds trivially when $R$ and $S$ are positive matrices. Also note that the matrix $A$ may not be symmetric nor a stable (also called Hurwitz) matrix.

The time dependent Schrödinger equation and the imaginary time version associated with the Hamiltonian $\mathcal{H}$ are given, respectively, by the equations

$$i \partial_t \Psi_t(x) = \mathcal{H}(\Psi_t)(x) \quad \text{and} \quad -\partial_t \psi_t(x) = \mathcal{H}(\psi_t)(x),$$  \hspace{1cm} (5)$$

with prescribed initial conditions $(\Psi_0, \psi_0)$. In the above display, $i \in \mathbb{C}$ stands for the imaginary unit. The right-hand side equation is obtained via a formal time change by setting $\psi_t(x) = \Psi_{-it}(x)$, and can be equivalently written in the following form

$$\partial_t \psi_t(x) = \mathcal{L}(\psi_t)(x) - V(x) \psi_t(x),$$  \hspace{1cm} (6)$$

with initial condition $\psi_0$. The first term $\mathcal{L}$ represents the generator of a free linear diffusion process $X_t$ with drift matrix $A$ and diffusion matrix $R$. The stochastic differential equation associated with this free evolution process is given by

$$dX_t = AX_t dt + B dW_t,$$  \hspace{1cm} (7)$$

for some initial state $X_0$ with distribution $\eta_0$ on $\mathbb{R}^r$. Here, $W_t$ is an $r_1$-dimensional Brownian motion, for some $r_1 \geq 1$, which is independent of $X_0$, and $B$ is a $(r \times r_1)$-matrix such that $BB' = R$. Again, we emphasize that $A$ is not required to be a stable matrix so that $X_t$ can be a transient diffusion process that drifts exponentially fast to $\infty$. As we shall see later in the article, the controllability condition (4) ensures that the trapping force of the potential energy always compensates the delocalization kinetic energy of the diffusion.

For a twice differentiable function $\psi_0$, the solution of (6) is given by the Feynman–Kac path integral formula

$$\psi_t(x) = K_t(\psi_0)(x) := \int K_t(x, dy) \psi_0(y)$$

$$= \mathbb{E} \left( \psi_0(X_t) \exp \left( - \int_0^t V(X_s) \, ds \right) \mid X_0 = x \right).$$  \hspace{1cm} (8)$$
To facilitate the interpretation of the theoretical and numerical physics in the measure theoretical framework used in this article, we note that the Feynman–Kac propagator defined by the integral operator (8) is sometimes written in terms of the exponential of the Hamiltonian operator with the exponential-type symbol

$$K_t := e^{-tH}$$ or in the bra-kets formalism $$K_t(\psi_0) = |e^{-tH}|\psi_0\rangle.$$

The exponential notation is compatible with finite space models and the matrix notation of the continuous one-parameter semigroup for time homogenous models. The bra-ket (or Dirac) notation is also used to represent linear projection forms acting on Hilbert spaces associated with some reversible or some stationary measure, such as the Lebesgue measure for the harmonic oscillator.

The integral operator $$K_t$$ is sometimes called the Feynman–Kac propagator. For any $$s, t \geq 0$$ the integral operators $$K_t$$ satisfy the semigroup property

$$K_{s+t}(x, dz) = (K_sK_t)(x, dy) := \int K_s(x, dz)K_t(z, dy) \implies \psi_{s+t} = K_s(\psi_t).$$

In terms of left action bra-kets, defining $$\mu_p(dx) := \varphi(x)dx$$, Fubini’s theorem yields

$$\langle \varphi|e^{-sH}|\psi_t\rangle = \int dx \varphi(x)K_s(x, dy)\psi_t(dy) = (\mu_pK_s)(\psi_t)$$

$$= \mu_p((K_pK_s)(\psi_0)) = \mu_pK_{s+t}(\psi_0) = \langle \varphi|e^{-(s+t)H}|\psi_0\rangle.$$ 

Besides its mathematical elegance, the conditional expectations in (8) can rarely be solved analytically for general diffusions $$X_t$$ and/or potential functions $$V$$ that are not necessarily quadratic. Moreover, numerical solutions require extensive calculations, see for instance [20,21,33,35] and references therein.

Now consider a process $$X^c_t$$ starting from $$X^c_0 = X_0$$, evolving as the diffusion $$X_t$$ and killed with rate $$V(X^c_t)$$. We denote by $$\tau^c$$ the random killing time of the process. In this interpretation, the Feynman–Kac propagator discussed in (8) takes the following form

$$\psi_t(x) = K_t(\psi_0)(x) = E(\psi_0(X^c_t) 1_{\tau^c > t} | X^c_0 = x).$$ (9)

An important question arising in applied probability and rare event analysis is to study the long time behavior of the conditional probability of the process $$X^c_t$$ with respect to the non-absorption event and starting from a random variable $$X^c_0 = X_0$$ with distribution $$\eta_0$$. In a more synthetic form this distribution is given by the formula

$$\eta_t(dx) := \mathbb{P}_{\eta_0}(X^c_t \in dx \mid \tau^c > t).$$ (10)

In contrast with conventional Markov processes, the flow of conditional probability measures $$\eta_t$$ has a nonlinear evolution semigroup; that is, for any $$s \leq t$$ we have

$$\eta_t = \Phi_{t-s}(\eta_s),$$ (11)

for some nonlinear mapping $$\Phi_t$$ from the set of probability measures on $$\mathbb{R}^r$$ into itself. A more precise definition of these mappings is provided in Sect. 2.3.2. The stability analysis of the nonlinear evolution semigroups given by the composition of mappings $$\Phi_{t+s} = \Phi_t \circ \Phi_s$$ is closely related to the long time behavior of a particle evolving in the
ground state $h_0$, sometimes called the $h$-process, denoted $(X^h_t)_{t \geq 0}$ and defined by the stochastic differential equation
\[
dX^h_t = \left( AX^h_t + R \nabla \log h_0(X^h_t) \right) dt + BdW_t,
\] (12)
with initial distribution $X^h_0$, say. Analogously to (10), we let $\eta^h_t$ denote the distribution of the random states $X^h_t$, $K^h_t$ denote the transition probabilities
\[
K^h_t(x, dy) := \mathbb{P}(X^h_t \in dy \mid X^h_0 = x),
\] (13)
and $\mathcal{L}^h$ the generator associated to $X^h_t$.

Under our controllability conditions (4), we shall see that the flow of probability measures $\eta_t$ and $\eta^h_t$ converge exponentially fast as $t \to \infty$ towards a pair of unique limiting measures, $\eta_\infty$ and $\eta^h_\infty$. That is for any $t \geq 0$ we have
\[
\eta_\infty = \Phi_t(\eta_\infty) \quad \text{and} \quad \eta^h_\infty(dy) = (\eta^h_\infty K^h_t)(dy) := \int \eta^h_\infty(dx)K^h_t(x, dy). \tag{14}
\]
The measure $\eta_\infty$ satisfies a nonlinear fixed point equation and it is sometimes called a quasi-invariant probability measure.

Choosing $V = 0$, or equivalently $S = 0$, in (6) or (1), the Feynman–Kac propagator $K_t$ coincides with the Markov transition semigroup of the process $X_t$. That is, we have
\[
K_t(\psi_0)(x) = \mathbb{E}(\psi_0(X_t) \mid X_0 = x).
\]
In this context, we have
\[
X_t \quad \text{reversible} \iff AR = RA'.
\]
A detailed proof of the above assertion can be found in [46].

In this situation, the diffusion process $X_t$ defined in (7) is reversible with respect to the locally finite measure
\[
\nu(dx) := \exp(U(x)) \, dx \quad \text{with} \quad U(x) := x' R^{-1} A x.
\] (15)
The stochastic differential equations (7) and (12) resume to the overdamped Langevin diffusions
\[
dX_t = \frac{1}{2} R \nabla U(X_t) dt + BdW_t
\]
and
\[
dX^h_t = \frac{1}{2} R \nabla \left( U + \log h_0^2 \right)(X^h_t) dt + BdW_t.
\] (16)

One of the key ideas in this article is to exploit the relationship between these three models. Indeed, equation (9) shows how one can transfer between the linear diffusion $(X_t)_{t \geq 0}$ and the killed process $(X^c_t)_{t \geq 0}$. As hinted above, the latter process is then related to the Ornstein–Uhlenbeck process $(X^h_t)_{t \geq 0}$ via a change of measure, which will be discussed later in the article. The advantage of this is that, under the controllability conditions (4), $(X^h_t)_{t \geq 0}$ is stable, even when $(X_t)_{t \geq 0}$ is not, which makes it much more tractable to work with.
1.2. Literature review. One of the main questions of quantum mechanics is to find the quantum numbers $n$, the eigenstates $h_n$ and the energy levels $\lambda_n$ of the Hamiltonian operator introduced in (1); that is, to find a sequence of functions $h_n(x)$ in some Hilbert space and some energy levels $\lambda_n \in \mathbb{R}_+ := [0, \infty]$ satisfying for any quantum numbers $n$ the time independent Schrödinger equation

$$H(h_n)(x) = \lambda_n \ h_n(x) \iff \mathcal{K}_t(h_n)(x) = \exp(-\lambda_n t) \ h_n(x). \quad (17)$$

The complete answer to this question is rather well known for the conventional isotropic harmonic oscillator associated with a null matrix $A = 0$ and diagonal matrices $(R, S)$. The one dimensional case with $A = 0$ corresponds to the well known harmonic oscillator treated in any textbook in quantum mechanics, see for instance [31,49,59,79,86]. In the multidimensional case, the Hamiltonian resumes to the sum of independent operators in each dimension. The resulting energy levels coincide with the tensor product of energy levels in each dimension. The isotropic harmonic oscillator corresponds to the case where $S = \rho I$, for some constant $\rho > 0$. The case $A = 0$ and non diagonal matrices $S$ arise in the analysis of coupled harmonic oscillators, see for instance [26,30,63,65,67] and the more recent article [57]. Coupled harmonic oscillators arise in a variety of applications including quantum and nonlinear physics [45,68], quantum cryptography and communication [6,44], quantum teleportation [78], as well as in biophysics [73,76] and in molecular chemistry [42,55].

To the best of our knowledge the case $A \neq 0$ has not been considered in quantum mechanics literature, the hypothesis of universal Brownian particle velocities in real time is always in force in all the studies published in the field of quantum harmonic oscillators. Manifold-valued Brownian motions such as Brownian motion on the sphere [87] have nonlinear infinitesimal generators and cannot be written in terms of a linear diffusion. In other words, Riemannian Laplacian cannot be encapsulated in the class of linear second order differential kinetic energy operators (2) considered in this article.

There exists a rich literature on the micro-local analysis [50,56,80] and the semiclassical analysis [43,64,89] of self-adjoint Hamiltonian operators for general smooth potentials and Brownian particle free motions. These powerful mathematical tools provide a precise spectral asymptotic analysis by connecting the Schrödinger equation with the classical mechanics of point particles (also known as the Bohr correspondence principle) when the diffusion Planck constant tends to 0.

Non-asymptotic estimates for general models are rarely studied in the literature and they often rely on proving the existence of limiting unknown mathematical objects such as quasi-invariant measures, the zero-point energy and the corresponding ground state, see for instance [33,36,38,39] and the more recent articles of Champagnat and Villemonais [22–25,38].

We also mention the closely related notion of the $h$-process, as briefly introduced in the previous section. The $h$-process and the associated exponential change of probability measures are closely related to variational principles and the design of importance-sampling-type guiding waves to simulate quantum many-body systems, see for instance the articles [17–19,70] on this subject, as well as exercises 445 and 449–450 in the book [37] for a more probabilistic approach based on rather well-known exponential changes of measures.

The main objective of this article is to extend conventional quantum harmonic oscillators to linear drift-type particle diffusions and general potential functions associated with some quadratic form. This class of models differs from the damped quantum harmonic oscillators with Ornstein–Uhlenbeck stable diffusions in imaginary time discussed in the series or articles [1,5,14,28,29,32,58].
1.3. Outline of some main results. The main aims of this article are to study the spectral and stability properties of the Hamiltonian, $\mathcal{H}$, defined in (1). More precisely,

1. For matrices $(A, R, S)$ satisfying the controllability condition (4), we provide an explicit description of the zero-point energy $\lambda_0$ and the ground state $h_0$ of the Hamiltonian (see Theorem 3.1) in terms of the positive fixed point of a continuous time algebraic Riccati matrix equation (CARE).

2. When the process is not necessarily reversible, our second main objective is to explicitly compute the time varying distribution flow of survival probabilities $P_{\eta_0}(\tau^c > t)$ including the distributions of a non-absorbed particle (10); see for instance the Gaussian preserving property presented in Theorem 3.2 and the coupled equations (27). The distribution of a non-absorbed diffusion and a particle evolving in the ground state (a.k.a. $h$-process) are connected to each other by a Boltzmann-Gibbs transformation (a.k.a. Bayes’ rule or Doob’s $h$-transform, see for instance (42) and (40), as well as Theorem 7.1 in the context of path space models). For any initial Gaussian state we show that a non-absorbed particle remains distributed according to a Gaussian probability with a mean vector satisfying a coupled time-varying linear system depending of the solution of a time dependent Riccati differential equation (27).

3. One of the main objectives of this article is to construct an explicit closed form solution of the time dependent Schrödinger equation (5) for general Hamiltonian operators of the form (1) when $R > 0$ and $AR = RA'$, which corresponds to the reversible case, (see Theorem 4.3). In this context we provide a complete description of the entire spectrum of the Hamiltonian operator, including all the excited states in terms of the matrices $(A, R, S)$.

4. An important part of the article is concerned with the long time behavior of $h$-processes (12) and normalised Feynman–Kac measures (cf. Sect. 2.3.2), including the convergence of the conditional distribution of a non absorbed particle (10) towards the unique fixed point (a.k.a. quasi-invariant distribution) of a nonlinear semigroup in distribution space. In the reversible case, the density of these limiting distributions with respect to the reversible measure of the free particle coincides with the ground state of the Hamiltonian (51).

Our main contributions to the stability analysis of $h$-processes and normalised Feynman–Kac semigroups are twofold:

- Firstly, we provide explicit and non-asymptotic estimates of the exponential decays to equilibrium in terms of Wasserstein distances or Boltzmann-relative entropy. These results are summarised in Theorems 5.1 and 5.2. We emphasise that these theorems are valid for models that may not be reversible, even when the drift matrix $A$ is unstable, yielding what we believe to be the first result of this type for this class of Feynman–Kac particle absorption models.

- In the reversible case, we analyze the stability properties of the $h$-process with a series of functional inequalities including a de Bruijn identity (93), Fisher’s information decays, log-Sobolev inequalities, and entropy contraction estimates (cf. Theorem 5.6). We also deduce Poincaré inequality and variance-type exponential decays directly from the spectral theorem 4.1 (see also Corollary 4.2).

We also note that in “Appendix C”, we discuss several classes of McKean–Vlasov interpretations of the distribution of a non-absorbed particle. These probabilistic models and their mean field simulation are defined in terms of a nonlinear Markov process that depends on the distribution of the random states so that the flow of distributions of all random states coincides with the conditional distribution of a non-absorbed particle. For
a more thorough discussion on these nonlinear sampling methodologies we refer to the books [33,35–37] and the references therein.

In particular, “Appendix C.1” is dedicated to interacting jump interpretations. Their mean field interpretations coincides with conventional Quantum Monte Carlo methods currently used in numerical physics. Their path space version can be interpreted as the genealogical tree associated with the killing and the birth/duplication/selection of walkers. An alternative way of sampling the trajectories of a non absorbed particle backward is provided in “Appendix B.1” (see for instance Theorem B.1).

In “Appendix C.2” we present a new class of mean field samplers based on Ensemble Kalman filters and the novel feedback particle filter methodology introduced by Mehta and Meyn and their co-authors in a series of seminal articles [81–85]. To the best of our knowledge, this class of advanced Monte Carlo methods have not been used to solve ground state energies nor to sample non-absorbed particle processes. Their application to the models presented in this article serve the same purposes as in filtering and data assimilation, and can be applied in high dimensions which offers a new bridge between these different scientific disciplines.

With these objectives in mind, the rest of the article is laid out as follows. In the next section, we introduce some notation that will be used throughout the rest of the paper. Sections 3, 4 and 5 house the main results, corresponding to the above objectives. More precisely, the description of the ground state, \( h_0 \), and zero-point energy, \( \lambda_0 \) is given in Sect. 3, along with the Gaussian preserving property of \( \eta_t \) and \( \eta_{h_t} \). The spectral decompositions of the normalised Feynman Kac measures and the \( h \)-process are given in Sect. 4, as well as Poincaré type inequalities. Finally, the stability analysis, as outlined in the final point above, is housed in Sect. 5. Then, in Sect. 6, we consider our results in various contexts. In particular, in Sect. 6.1, we consider a one dimensional model, which shows that the trapping force of the potential energy always compensates the delocalisation kinetic energy of the diffusion even when \( A > 0 \) is very large. In Sect. 6.2 we show how to recover directly Mehler’s formula from our spectral decompositions. In Sect. 6.3, we discuss the multi-dimensional quantum harmonic oscillator, corresponding to \( A = 0 \) and diagonal matrices \( (R, S) \). Sections 7, 8 and 9 are dedicated to the proofs of Sects. 3, 4 and 5, respectively. Finally, the appendix contains several sections on relevant background material.

2. Background and Basic Notation

2.1. Norms and matrix spaces. We denote by \( \mathcal{M}_{r_1, r_2} \) the set of \((r_1 \times r_2)\)-matrices with real entries and \( r_1, r_2 \geq 1 \). When \( r = r_1 = r_2 \) we write \( \mathcal{M}_r \) instead of \( \mathcal{M}_{r, r} \) the set of square \((r \times r)\)-matrices.

A square root of a square matrix \( A \in \mathcal{M}_r \) is a (non unique) matrix \( A^{1/2} \) such that \( A^{1/2} A^{1/2} = A \). When \( A \) has positive eigenvalues, we choose the square root \( A^{1/2} \) that has positive eigenvalues. We let \( S_r \subseteq \mathcal{M}_r \) denote the subset of symmetric matrices, \( S^0_r \subseteq S_r \) the subset of positive semi-definite matrices, and \( S^+_r \subseteq S^0_r \) the subset of positive definite matrices. We also let \( S^-_r \) the set of negative definite matrices.

Given \( B \in S^0_r - S^+_r \) we denote by \( B^{1/2} \) a (non-unique) but symmetric square root of \( B \) (given by a Cholesky decomposition). When \( B \in S^+_r \) we choose the principal (unique) symmetric square root. We write \( A' \) to denote the transposition of a matrix \( A \), and \( A_{\text{sym}} = (A + A')/2 \) to denote the symmetric part of \( A \in \mathcal{M}_r \). We denote by \( \text{Spec}(A) \) the spectrum of \( A \) defined by

\[
\text{Spec}(A) := \{ \lambda \mid \lambda \text{ eigenvalue of } A \},
\]
where each eigenvalue is listed the number of times it occurs as a root of the characteristic polynomial of \( A \). We also denote by \( \mathcal{G}_{lr} \subset \mathcal{M}_r \) the general linear group of invertible matrices. The set \( \mathcal{M}_r \) is equipped with the spectral or the Frobenius norm (otherwise known as the Hilbert-Schmidt norm) defined by

\[
\| A \| = \sqrt{\lambda_{\text{max}}(AA')} \leq \| A \|_F := \sqrt{\text{Tr}(AA')} \leq \sqrt{r} \| A \|,
\]

where \( \lambda_{\text{max}}(\cdot) \) denotes the maximal eigenvalue. The minimal eigenvalue is denoted by \( \lambda_{\text{min}}(\cdot) \). Let \( \text{Tr}(A) = \sum_{1 \leq i \leq r} A(i,i) \) denote the trace operator. We also denote by \( \mu(A) = \lambda_{\text{max}}(A_{\text{sym}}) \) the logarithmic norm and by \( \varsigma(A) := \max_{\lambda \in \text{Spec}(A)} \{ \text{Re}(\lambda) \} \), the spectral abscissa. We recall that

\[
\| A \| \geq \mu(A) = \lambda_{\text{max}}(A_{\text{sym}}) \geq \varsigma(A) := \max \{ \text{Re}(\lambda) : \lambda \in \text{Spec}(A) \}.
\]

A matrix \( A \) is said to be stable, or Hurwitz, when \( \varsigma(A) < 0 \). We recall that

\[
\varsigma(A) < 0 \implies \exists \alpha, \beta > 0 : \forall t \geq 0 \ \| e^{tA} \| \leq \alpha e^{-\beta t}.
\]

The parameters \( (\alpha, \beta) \) can be made explicit in terms of the spectrum of the matrix \( A \). For instance, applying Coppel’s inequality (cf. Proposition 3 in [27]), for any \( t \geq 0 \) and any \( 0 < \gamma < 1 \), we can choose

\[
\alpha = (a/\gamma)^{r-1} \quad \text{and} \quad \beta = (1 - \gamma)\varsigma(A) \quad \text{with} \quad a := 2\| A \|/|\varsigma(A)|.
\]

In the case where \( \beta = -\mu(A) > 0 \), we can choose \( \alpha = 1 \).

2.2. Relative entropy and metrics. The \( n \)-th Wasserstein distance between two probability measures \( \nu_1 \) and \( \nu_2 \) on \( \mathbb{R}^r \) is defined for any parameter \( n \geq 1 \) by the formula

\[
\mathbb{W}_n(\nu_1, \nu_2) = \inf \left\{ \mathbb{E} \left( \| Z_1 - Z_2 \|_n^{1/n} \right) \right\},
\]

where the infimum is taken over all pairs of random variables \( (Z_1, Z_2) \) such that \( \text{Law}(Z_i) = \nu_i, \) for \( i = 1, 2 \). The expectation \( \mathbb{E}(\| Z_1 - Z_2 \|_n) \) is taken with respect to the distribution of the pair of variables \( (Z_1, Z_2) \).

We denote by \( \text{Ent}(\nu_1 \mid \nu_2) \) the Boltzmann-relative entropy, defined as

\[
\text{Ent}(\nu_1 \mid \nu_2) := \int \log \left( \frac{d\nu_1}{d\nu_2} \right) \, d\nu_1,
\]

whenever \( \nu_1 \ll \nu_2 \), and \( +\infty \) otherwise. Further, the Fisher information is defined by

\[
\mathcal{J}(\nu_1 \mid \nu_2) := \int \| \nabla \log \left( \frac{d\nu_1}{d\nu_2} \right) \|^2 \, d\nu_1,
\]

if \( \log d\nu_1/d\nu_2 \in L_2(\nu_1) \), and \( +\infty \) otherwise. The total variation distance between the measures \( \nu_1 \) and \( \nu_2 \) is defined by

\[
\| \nu_1 - \nu_2 \|_{tv} := \frac{1}{2} \sup \{ |\nu_1(f) - \nu_2(f)| : f \text{ s.t. } \| f \|_\infty \leq 1 \},
\]
with the uniform norm and Lebesgue integrals defined, respectively, by
\[ \|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|, \quad v_i(f) := \int v_i(dx) f(x). \]

Finally, given some locally finite measure \( \nu \) on \( \mathbb{R}^r \), for any \( n \geq 1 \) we denote by \( L_n(\nu) \) the Banach space of measurable functions \( f \) on \( \mathbb{R}^r \) equipped with the norm
\[ \|f\|_{n,\nu} := \nu\left(\left|f|^n\right)^{1/n}\right). \]

2.3. Feynman–Kac propagators. In this section, we discuss the Feynman Kac propagator, \( K_t \), defined in (8). To analyze these general stochastic models, we will use elementary and standard measure theory notation. For a given integral operator \( K_t(x, dy) \), we define the left action, \( \mu \mapsto \mu K_t \), which maps measures to measures by
\[ (\mu K_t)(dy) := \int \mu(dx) K_t(x, dy) \]
and the right action, \( f \mapsto K_t(f) \), which maps functions to functions by
\[ K_t(f)(x) := \int K_t(x, dy) f(y). \]

This notation is clearly compatible with finite space models and matrix notation, in which case the integrals are replaced by finite or countable sums, the integral operator \( K_t(x, dy) \) is replaced by a square matrix, the function \( f(y) \) by a column vector, and by duality, the measure \( \mu(dx) \) is represented by a row vector.

2.3.1. Unnormalised semigroups. Recall \( \psi_t \) defined in (6). Whenever the initial state \( X_0 \) is distributed according to some probability measure \( \eta_0 \) on \( \mathbb{R}^d \), by Fubini’s theorem and (8) we have
\[ \eta_0(\psi_t) := \int \eta_0(dx) \psi_t(x) = \int \eta_0(dx) \int K_t(x, dy) \psi_0(y) = \int \left( \int \eta_0(dx) K_t(x, dy) \right) \psi_0(y) = (\eta_0 K_t)(\psi_0). \]

This yields the formula
\[ \eta_0(\psi_t) = \gamma_t(\psi_0) \quad \text{with the measure} \quad \gamma_t := \eta_0 K_t. \]

Equivalently, the measure \( \gamma_t \) is defined by the unnormalised Feynman–Kac path integral
\[ \gamma_t(\psi_0) = \mathbb{E} \left[ \psi_0(X_t) \exp \left( - \int_0^t V(X_u) \, ds \right) \right] \quad \text{and} \quad \gamma_t(1) = \mathbb{E} \left[ \exp \left( - \int_0^t V(X_u) \, ds \right) \right]. \]

In the above display, 1 stands for the unit function. Observe that the evolution semigroup of \( \gamma_t \) is linear and given by the formulae
\[ \gamma_{t+s} = \eta_0 K_{s+t} = \eta_0 (K_s K_t) = (\eta_0 K_s) K_t = \gamma_s K_t. \]
Finally observe that
\[
\partial_t \gamma_t(\psi_0) = \mathbb{E}\left( L(\psi_0)(X_t) \exp\left( -\int_0^t V(X_u) \, ds \right) \right)
- \mathbb{E}\left( \psi_0(X_t) V(X_t) \exp\left( -\int_0^t V(X_u) \, ds \right) \right)
\]
This yields the evolution equation
\[
\partial_t \gamma_t(\psi_0) = \gamma_t(L(\psi_0)) - \gamma_t(\psi_0 V) = -\gamma_t(\mathcal{H}(\psi_0)),
\quad (20)
\]
where \( \mathcal{H} \) was defined in (1).

2.3.2. Normalised semigroups
We shall denote by \( \eta_t \) the normalised probability measures
\[
\eta_t(f) := \frac{\gamma_t(f)}{\gamma_t(1)} \implies \gamma_t(f) = \eta_t(f) \exp\left( -\int_0^t \eta_s(V) \, ds \right).
\quad (21)
\]

We check this claim using the formula
\[
-\partial_t \log \gamma_t(1) = \frac{1}{\gamma_t(1)} \mathbb{E}\left( V(X_t) \exp\left( -\int_0^t V(X_u) \, ds \right) \right) = \eta_t(V).
\]
In bra-ket notation, the semigroup property (19) and the probability measure \( \eta_t \) can be written in the form
\[
\eta_0 = \mu_\phi \implies \langle \phi | e^{-(s+t)\mathcal{H}} | \psi \rangle = \langle \phi | e^{-s\mathcal{H}} e^{-t\mathcal{H}} | \psi \rangle \quad \text{and} \quad \eta_t(\psi_0) = \frac{\langle \phi | e^{-t\mathcal{H}} | \psi_0 \rangle}{\langle \phi | e^{-t\mathcal{H}} | 1 \rangle}.
\]

Observe that for any \( s \leq t \) we have the correspondence principle
\[
\eta_t(f) = \frac{\gamma_t(f)}{\gamma_t(1)} = \frac{\gamma_s \mathcal{K}^{-s}_t(f)}{\gamma_s \mathcal{K}^{-s}_t(1)} = \frac{\eta_s \mathcal{K}^{-s}_t(f)}{\eta_s \mathcal{K}^{-s}_t(1)} =: \Phi^{t-s}_s(\eta_s)(f).
\quad (22)
\]
The semigroup \( \Phi^{t-s}_s(\eta_s) = \eta_t \) of the normalised measures described above is a nonlinear mapping from the set of probability measures on \( \mathbb{R}^r \) into itself.

There are two different ways to normalise the integral Feynman–Kac operators \( \mathcal{K}_t \). The first one is based on (21), which implies that
\[
\eta_t(f) = \gamma_t(f) \exp\left( \int_0^t \eta_s(V) \, ds \right).
\quad (23)
\]
This yields the formula
\[
\eta_t(f) = \mathbb{E}\left( f(X_t) \exp\left( -\int_0^t V_{\eta_s}(X_s) \, ds \right) \right) \quad \text{with} \quad V_{\eta_s}(x) = V(x) - \eta_s(V).
\]
This shows that the normalised measures \( \eta_t \) are defined as \( \gamma_t \) by replacing \( V \) by the time varying centered potential function \( V_{\eta_t} \). It is therefore natural to consider the normalised propagator defined below.
**Definition 2.1.** For any initial distribution $\eta_0$ and for any $t \geq 0$ we denote by $\mathcal{K}_t^{\eta_0}$ the integral operator

$$
\mathcal{K}_t^{\eta_0}(f)(x) := \mathbb{E}\left(f(X_t) \exp\left(-\int_0^t V_{\eta_s}(X_s)\,ds\right) \mid X_0 = x\right).
$$

Using (22), it is straightforward to see that

$$
\mathcal{K}_t^{\eta_0}(f)(x) = \exp\left(\int_0^t \Phi_s(\eta_0)(V)\,ds\right) \times \mathcal{K}_t(f)(x) = \frac{\mathcal{K}_t(f)(x)}{\eta_0 \mathcal{K}_t(1)}.
$$

From this we deduce that

$$
\eta_0 \mathcal{K}_t^{\eta_0}(f) = \frac{1}{\gamma_t(1)} \eta_0 \mathcal{K}_t(f) = \frac{\gamma_t(f)}{\gamma_t(1)} = \eta_0 \mathcal{K}_t^{\eta_0} = \eta_t.
$$

**Proposition 2.2.** For any $s, t \geq 0$ we have the evolution semigroup properties

$$
\mathcal{K}_s \mathcal{K}_t^{\eta_0} = \mathcal{K}_t^{\eta_0} \mathcal{K}_s^{\eta_t}, \quad \eta_{s+t} = \eta_s \mathcal{K}_t^{\eta_s} \quad \text{and} \quad \mathcal{K}_t^{\eta_\infty}(f) = e^{\lambda_0 t} \mathcal{K}_t(f)
$$

**Proof.** To prove the two semigroup properties, note that

$$
\mathcal{K}_t^{\eta_0}(f)(x) = \exp\left(\int_0^s \Phi_s(V)\,ds\right) \exp\left(\int_s^t \Phi_{t-s}(\eta_s)(V)\,du\right) \mathcal{K}_{s+t}(f)(x)
$$

$$
= \exp\left(\int_0^t \Phi_u(\eta_s)(V)\,du\right) \mathcal{K}_s^{\eta_0}(\mathcal{K}_t(f))(x) = \mathcal{K}_s^{\eta_0}(\mathcal{K}_t^{\eta_s}(f))(x).
$$

The second then follows from $\eta_0 \mathcal{K}_t^{\eta_0} = \eta_t$.

To check the right hand side, note that

$$
\eta_\infty(\mathcal{K}_t(1)) = \exp\left(-\int_0^t \Phi_s(\eta_\infty)(V)\,ds\right) = e^{-\eta_\infty(V)t} = e^{-\lambda_0 t}.
$$

The result then follows from (24). $\square$

Arguing as in (20) with $V$ replaced by $V_{\eta_t}$, we also find the evolution equation

$$
\psi_t := \mathcal{K}_t^{\eta_0}(\psi_0) \implies -\partial_t \psi_t = \mathcal{H}_{\eta_t}(\psi_t) \quad \text{and} \quad -\partial_t \eta_t(f) = \eta_t(\mathcal{H}_{\eta_t}(f)),
$$

with the normalised time varying Hamiltonian

$$
\mathcal{H}_{\eta_t} = -\mathcal{L} + (V - \eta_t(V)) \iff \partial_t \eta_t(f) = \eta_t(\mathcal{L}(f)) - \eta_t(fV) + \eta_t(f)\eta_t(V).
$$

A second strategy to normalise the Feynman–Kac propagator is to divide by its total mass.

**Definition 2.3.** We associate with $\mathcal{K}_t$ the normalised the Markov integral operator $\overline{\mathcal{K}}_t$ defined by the ratio formula

$$
\overline{\mathcal{K}}_t(f)(x) := \frac{\mathcal{K}_t(f)(x)}{\mathcal{K}_t(1)(x)} = \int f(y) \Phi_t(\delta_x)(dy).
$$

By (19) and (21) the normalising constant is given by

$$
\mathcal{K}_t(1)(x) = \delta_x \mathcal{K}_t(1) = \exp\left(-\int_0^t \Phi_s(\delta_x)(V)\,ds\right).
$$
2.4. Evolution equations. In Sect. 4, we shall see that if \( \eta_0 \sim \mathcal{N}(x, P) \), then \( \eta_t \) is also normally distributed for any \( t \geq 0 \). Moreover, in a sense to be made precise we have \( \eta_t \to \eta_\infty := \mathcal{N}(0, P_\infty) \) as \( t \to \infty \) for some matrix \( P_\infty \). A detailed convergence analysis is provided in Sect. 5 (cf. for instance Theorem 5.2 as well as Theorem 5.3 when \( \eta_0 \) is non necessarily Gaussian). In order to state and prove this result, we need to introduce a coupled process whose evolution is closely related to \( \eta_t \).

To this end, denote by \( \mathcal{N}(m, P) \) an \( r \)-dimensional Gaussian probability measure with mean \( m \in \mathbb{R}^r \) and covariance matrix \( P \). We also let \((\hat{X}_t, P_t) \in (\mathbb{R}^r \times \mathbb{R}^{r \times r}) \) be the solution of the coupled evolution equations given by the system

\[
\begin{align*}
\partial_t \hat{X}_t &= (A - P_t S) \hat{X}_t \\
\partial_t P_t &= \text{Ricc}(P_t),
\end{align*}
\]

(27)

where

\[
\text{Ricc}(P) := AP + PA' R - PSP
\]

and the initial conditions are given by \( \hat{X}_0 \in \mathbb{R}^r \) and \( P_0 \in S_r^+ \).

The controllability conditions (4) are well known in filtering theory, see for instance [13, 60] and the more recent articles [9, 12], and references therein. They ensure the existence of an unique pair \((P^-_\infty, P_\infty)\) of negative and positive fixed point matrices of the algebraic Riccati equation

\[
\text{Ricc}(P^-_\infty) = 0 = \text{Ricc}(P_\infty).
\]

(28)

In addition, the matrices

\[
A - P_\infty S \quad \text{and} \quad A' + (P^-_\infty)^{-1} R
\]

(29)

are stable. For a more thorough discussion on the above assertions we refer to [13, Chapter 3], [60] and the more recent articles [9, 12].

In addition, the pair of matrices \((P^-_\infty)^{-1}, P_\infty^{-1}\) satisfy the same fixed point equation as that of \((P^-_\infty, P_\infty)\) by replaced \((A, R, S)\) by \((-A', S, R)\). Similarly, the matrices \((Q^-_\infty, Q_\infty)\) defined by

\[
Q^-_\infty := -P_\infty^{-1} < 0 < Q_\infty := -(P^-_\infty)^{-1}
\]

(30)

satisfy the same fixed point equation as \(((P^-_\infty)^{-1}, P_\infty^{-1})\) by replacing \(A\) by \((-A)\). Thus, the matrices \((Q^-_\infty, Q_\infty)\) satisfy the algebraic Riccati matrix equation

\[
A' Q_\infty + Q_\infty A - Q_\infty R Q_\infty + S = 0.
\]

(31)

Now let us consider the evolution semigroup \((\hat{X}_t(x, P), \phi_t(P))\) of the equations (27). In this case, when started at \((x, P)\), the pair satisfies the coupled equations

\[
\begin{align*}
\partial_t \hat{X}_t(x, P) &= (A - \phi_t(P) S) \hat{X}_t(x, P) \\
\partial_t \phi_t(P) &= \text{Ricc}(\phi_t(P)).
\end{align*}
\]

(32)

The evolution semigroup \((\hat{X}_t(x, P), \phi_t(P))\) defines the mean and covariance matrices of the distribution \( \eta_t \) when \( \eta_0 \sim \mathcal{N}(x, P) \). We refer to Theorem 3.2, as well as
“Appendix A.2” and Proposition 5.4 for different ways of writing the evolution semigroup \( \left( \tilde{X}_t(x, P), \phi_t(P) \right) \).

Now let \( \mathcal{E}_{s,t}(P) \) be the exponential semigroup associated with the matrix flow \( u \mapsto (A - \phi_u(P))S) \); that is the solution for any \( 0 \leq s \leq t \) of the matrix evolution equations

\[
\partial_t \mathcal{E}_{s,t}(P) = (A - \phi_t(P)) \mathcal{E}_{s,t}(P) \quad \text{and} \quad \partial_s \mathcal{E}_{s,t}(P) = -\mathcal{E}_{s,t}(P) (A - \phi_s(P)) S, \tag{33}
\]

with \( \mathcal{E}_{s,s}(P) = I \) and where we often write \( \mathcal{E}_t(P) \) for \( \mathcal{E}_{0,t}(P) \). In this notation, the solution of the right hand side equation in (27) takes the form

\[
\tilde{X}_t(x, P) = \mathcal{E}_t(P)x \quad \text{and} \quad \mathcal{E}_{s,s+t}(P_\infty) = \mathcal{E}_t(P_\infty) = \exp (t(A - P_\infty S)). \tag{34}
\]

From a mathematical viewpoint, it is tempting to integrate sequentially the differential equations (33), to obtain an explicit description of \( \mathcal{E}_{s,t}(P) \) in terms of the Peano–Baker series [4,69], see also [16,48,53]. Another natural strategy is to express the semigroup as a true matrix exponential involving a Magnus series expansion of iterated integrals on the Lie algebra generated by the matrices \( (A - \phi_u(P))S) \), with \( s \leq u \leq t \). For more details on these exponential expansions we refer to [15,66]. In practical terms, the use of Peano–Baker and/or exponential Magnus series in the study of the stability properties of time-varying linear dynamical systems is rather limited.

We will also see that a similar Gaussian preserving property holds for the process \( (X^h_t)_{t \geq 0} \) defined in (12). As such, we also introduce the analogues of the evolution equations (27). In this setting, (12) can be equivalently written as

\[
dX_t^h = (A - RQ_\infty) X_t^h \, dt + BdW_t. \tag{35}
\]

Due to (29), the matrix \( (A - RQ_\infty) \) is Hurwitz so that \( X^h_t \) is a stable Ornstein–Uhlenbeck process even when \( A \) is unstable. This property ensures the existence of some parameters \( \alpha_h, \beta_h > 0 \) such that

\[
\| e^{(A-RQ_\infty) t} \| \leq \alpha_h \, e^{-\beta_h t} \quad \text{and} \quad \nu_t := \int_0^\infty \| e^{(A-RQ_\infty) s} \|^2 \, ds \leq \frac{\alpha_h^2}{2\beta_h}. \tag{36}
\]

For a more thorough discussion on the exponential decays of matrix exponential-type semigroups (a.k.a. fundamental matrices) we refer to Sect. 2.1.

In this case, the mean and covariances matrices of \( \eta^h_t \) (assuming \( \eta_0 \sim N(x, P) \)) are given, respectively, by

\[
\tilde{X}_t^h(x) = e^{(A-RQ_\infty)t}x, \quad \phi^h_t(P) := e^{(A-RQ_\infty)t} Pe^{(A-RQ_\infty)t} + \int_0^t e^{(A-RQ_\infty)s}Re^{(A-RQ_\infty)'s} \, ds. \tag{37}
\]

Notice that \( \phi^h_t(P) \) is the evolution semigroup of associated with the matrix valued differential equation

\[
\partial_t P_t^h = (A - RQ_\infty) P_t^h + P_t^h (A - RQ_\infty)' + R \quad \text{with} \quad P_0^h = P.
\]
3. Ground State Energy

In this section, we present our first two main results. The first provides provides an explicit description of the ground state energy of the Hamiltonian operator for general matrices \((A, R, S)\) in terms of the negative and positive fixed points of the algebraic Riccati equation \((28)\).

**Theorem 3.1.** For any matrices \((A, R, S)\) satisfying the rank condition \((4)\), the function

\[
h_0(x) = \exp \left( -\frac{1}{2} x' Q_\infty x \right)
\]

is the ground state of the Hamiltonian \(\mathcal{H}\) introduced in \((1)\). That is, we have

\[
\mathcal{H}(h_0) = \lambda_0 h_0 \quad \text{with} \quad \lambda_0 := \frac{1}{2} \text{Tr} (SP_\infty) = \frac{1}{2} \text{Tr} (R Q_\infty) > 0.
\]

In addition, we have the Feynman–Kac propagator formula

\[
\exp (\lambda_0 t) K_t (x, dy) = h_0(x) K^h_t (x, dy) h_0^{-1}(y),
\]

where we recall that \(K_t\) and \(K^h_t\) were defined in \((8)\) and \((13)\), respectively.

Recalling the absorption time, \(\tau\), we have

\[
-\frac{1}{t} \log \mathbb{P}_{\eta_0} (\tau^c > t) = \frac{1}{2t} \int_0^t \left( \hat{X}'_s S \hat{X}_s + \text{Tr}(SP_s) \right) ds.
\]

Thus, we have the following relationship between the zero-point energy level \(\lambda_0\) and the survival probability of a non-absorbed particle:

\[
\lambda_0 = \frac{1}{2} \text{Tr} (SP_\infty) = -\frac{1}{s} \log \mathbb{P}_{\eta_\infty} (\tau^c > s).
\]

Applying \((21)\) to the unit function yields \(\lambda_0 = \eta_\infty (V)\).

Now recall the law \(\eta^h_t\) of the random states \(X^h_t\) of the \(h\)-process defined in \((12)\) and \(\eta_t\), the distribution of the non-absorbed particle defined in \((10)\). The following result shows that these two distributions are related via the Boltzmann-Gibbs transformation, defined via

\[
\mathbb{B}_{h_0} (\eta_0)(dx) := \frac{1}{\eta_0(h_0)} h_0(x) \eta_0(dx),
\]

whenever \(\eta_0(h_0)\) is a well-defined positive normalising constant. Moreover, we show that \(\eta_t\) and \(\eta^h_t\) exhibit a Gaussian preserving property.

**Theorem 3.2.** For any matrices \((A, R, S)\), the laws \(\eta_t\) and \(\eta^h_t\) are connected by the Boltzmann-Gibbs transformation; that is, for any time horizon \(t \geq 0\) we have that

\[
\eta^h_t = \mathbb{B}_{h_0} (\eta_t) \quad \text{and} \quad \eta_t = \mathbb{B}_{h_0^{-1}} (\eta^h_t),
\]

where \(h_0^{-1}(x) := 1/h_0(x)\). In addition, we have the Gaussian preserving property

\[
\eta_0 = \mathcal{N} (\hat{X}_0, P_0) \quad \implies \forall t \geq 0 \quad \eta^h_t = \mathcal{N} (\hat{X}^h_t, P^h_t) \quad \text{and} \quad \eta_t = \mathcal{N} (\hat{X}_t, P_t)
\]

with the parameters \((\hat{X}_t, P_t)\) defined in \((27)\), and the covariance matrix and mean vector given, respectively, by

\[
P^h_t = (P_t^{-1} + Q_\infty)^{-1} \quad \text{and} \quad \hat{X}^h_t = P^h_t P_t^{-1} \hat{X}_t.
\]

The proofs of the above results are given in Sect. 7.
4. Spectral Decompositions

In this section, we present our main results related to the spectral properties of the Feynman–Kac propagator. We now assume that the matrices \((A, R, S)\) satisfy (4), that \(R > 0\) and that \(AR = RA'\).

Let us introduce some further notation. Let \(\Lambda^h\) be the matrix defined by
\[
\Lambda^h := -(P^h_\infty)^{-1/2}(A^2 + RS)^{1/2}(P^h_\infty)^{1/2},
\] (44)
where
\[
P^h_\infty := (P^{-1}_\infty + Q_\infty)^{-1}
\] (45)
We denote by \(Z := (z_1, \ldots, z_r)\) the orthogonal matrix with columns given by the orthonormal eigenvector \(z_i\) of the matrix \(\Lambda^h\) associated with an eigenvalue \(\lambda^h_i(\Lambda^h):=-\lambda^h_i(<0, i \in \{1, \ldots, r\}.

Further, set \(\lambda_0(\Lambda^h) = -\lambda_0^h := 0\).

We are now in position to state the main result of this section.

**Theorem 4.1.** For any time horizon \(t \geq 0\), we have the \(L^2(\eta^h_\infty)\)-spectral decomposition
\[
K^h_t(x, dy) = \sum_{n \in \mathbb{N}^r} e^{-\lambda^h_n t} \varphi^h_n(x) \varphi^h_n(y) \eta^h_\infty(dy),
\]
where the \(L^2(\eta^h_\infty)\) orthonormal basis eigenfunctions \(\varphi^h_n\) and corresponding eigenvalues \(\lambda^h_n\) are given, respectively, by
\[
\varphi^h_n(x) := \frac{1}{\sqrt{n!}} H^r_n \left(Z(P^h_\infty)^{-1/2}x\right), \quad \lambda^h_n := \sum_{1 \leq i \leq r} n_i \lambda^h_i.
\] (47)
Thus, for any $n \in \mathbb{N}$ and any $t \geq 0$ and $x \in \mathbb{R}^r$ we have the formulae
\[ K^h_t \left( \varphi^h_n \right) = e^{-\lambda^h_n t} \varphi^h_n \quad \text{and} \quad \mathcal{L}^h_t \left( \varphi^h_n \right) = -\lambda^h_n \varphi^h_n. \] (48)

Exponential decays to equilibrium can be extracted directly from the spectral decomposition. For instance we have the following estimates.

**Corollary 4.2.** For any time horizon $t \geq 0$, we have the exponential decays to equilibrium
\[ \| K^h_t (f) - \eta^h_\infty (f) \|_{2, \eta^h_\infty} \leq e^{-\lambda^h_1 t} \| f - \eta^h_\infty (f) \|_{2, \eta^h_\infty}. \] (49)

Equivalently, we have the Poincaré inequality
\[ \lambda^h_1 \| f - \eta^h_\infty (f) \|_{2, \eta^h_\infty}^2 \leq E_h (f, f) := -\eta^h_\infty (f \mathcal{L}^h (f)). \] (50)

Recall that the free particle $X_t$ is reversible with respect to $\nu$, cf. (15). In this case, the quasi-invariant distribution $\eta_\infty$ discussed above is related to the ground state $h_0$ by the Boltzmann–Gibbs formula
\[ \eta_\infty = B_{h_0} (\nu) \quad \text{and} \quad \eta^h_\infty = B_{h_0} (\eta^h_\infty) = B_{h_0^2} (\nu). \] (51)

The proof of the above assertion is given in section 8.1.

The left hand side assertion above indicates that the stationary density of a non-absorbed particle with respect to $\nu$ is proportional to the ground state, while the stationary distribution with respect to $\nu$ of a particle evolving in the ground state is proportional to the square of the ground state. These Boltzmann–Gibbs formulae yield the Hilbert space isometry
\[ \Upsilon_h : f \in \mathbb{L}_2 (\eta^h_\infty) \mapsto \Upsilon_h (f) := \sqrt{\eta^h_\infty (h_0^{-2})} f h_0 \in \mathbb{L}_2 (\nu) \] (52)

with the inverse
\[ \Upsilon_h^{-1} (f) = \sqrt{\nu (h_0^2)} f h_0^{-1}. \]

This a direct consequence of the fact that (51) implies
\[ \langle f, g \rangle_{2, \nu} := \nu (f g) = \nu (h_0^2) \langle h_0^{-1} f, h_0^{-1} g \rangle_{2, \eta^h_\infty} \quad \text{and} \quad \eta^h_\infty (h_0^{-2}) = 1 / \nu (h_0^2). \]

The Feynman–Kac propagator is connected to the semigroup of the particle evolving in the ground state $h_0$ via the operator formulae
\[ \exp (\lambda_0 t) K_t = \Upsilon_h \circ K^h_t \circ \Upsilon_h^{-1} \quad \text{and} \quad -\mathcal{L} + (V - \lambda_0) = \Upsilon_h \circ \mathcal{L}^h \circ \Upsilon_h^{-1}. \]

The Boltzmann–Gibbs mappings discussed in (42), (40) and (51) and the Hilbert space isometry (52) allow one to transfer directly any known regularity property at the level of the $h$-process $(K^h_t, \eta^h_t, \eta^h_\infty)$ to the Feynman–Kac model $(K_t, \eta_t, \eta_\infty)$, and vice versa.

In particular, it follows that the $\mathbb{L}_2 (\nu)$ orthonormal basis and those of $\mathbb{L}_2 (\eta^h_\infty)$ are linked to each other by the formulae
\[ \varphi_n = \Upsilon_h \left( \varphi^h_n \right) \quad \text{and} \quad \varphi^h_n = \Upsilon_h^{-1} (\varphi_n), \quad n \in \mathbb{N}_r. \]

Hence, rewritten in terms of Feynman–Kac propagators, Theorem 4.1 takes the following form.
Theorem 4.3. For any \( t \geq 0 \) we have the \( L_2(\nu) \) spectral decomposition

\[
K_t(x, dy) = \sum_{n \in \mathbb{N}^r} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \nu(dy),
\]

with the \( L_2(\nu) \) orthonormal basis given for any \( n \in \mathbb{N}^r \) by the eigenstates

\[
\varphi_n(x) = \left( \frac{1}{(2\pi)^r} \det \left( P^{-1}_\infty + Q_\infty \right) \right)^{1/4} h_0(x) \varphi^n_0(x),
\]

and corresponding eigenvalues \( \lambda_n := \lambda_0 + \lambda^n_h \).

Using the definitions of \( \lambda^n_h \) and \( \lambda_0 \) given in (47) and (39), respectively, along with the fact that

\[
Q_\infty = R^{-1} A + R^{-1} (A^2 + RS)^{1/2},
\]

it is straightforward to show that the energy levels \( \lambda_n \) given by the formulae

\[
\lambda_n = \frac{1}{2} \text{Tr}(A) + \sum_{1 \leq i \leq r} \left( n_i + \frac{1}{2} \right) \sqrt{|\lambda_i(A^2 + RS)|}, \quad n \in \mathbb{N}^r,
\]

where \( \lambda_i(A^2 + RS) \) stands for the non-negative eigenvalues of \( (A^2 + RS) \).

In addition, using the above spectral decomposition we check the formulae

\[
K_t(\varphi_n) = e^{-\lambda_n t} \varphi_n \quad \text{and} \quad L(\varphi_n) - V \varphi_n = -\lambda_n \varphi_n.
\]

Choosing \( \mu(dx) = \nu(dx)/\nu(h_0^2) \) and \( h_n(x) = \sqrt{\nu(h_0^2)} \varphi_n(x) \) we obtain the formulae (17).

After decomposing the initial state \( \psi_0 = \Psi_0 \) into the \( h_n \)-basis discussed above, we apply the time evolution at each energy level \( \lambda_n \). Reassembling all eigenstates we obtain the solution of both the time dependent Schrödinger equation and the imaginary time version discussed in (5); that is, we have that

\[
\Psi_t(x) = \sum_{n \in \mathbb{N}^r} e^{-i\lambda_n t} h_n(x) \int \Psi_0(y) h_n(y) \mu(dy) \quad \text{and} \quad \psi_t(x) = \Psi_{-i \lambda^n}(x).
\]

In addition, we have the Feynman–Kac propagator version of Corollary 4.2.

Corollary 4.4. For any time horizon \( t \geq 0 \) and any \( f \in L_2(\nu) \) we have the estimates

\[
\left\| e^{\lambda_0 t} K_t(f) - \frac{h_0}{\eta_\infty(h_0)} \eta_\infty(f) \right\|_{2, \nu} \leq e^{-\lambda_0^* t} \left( \nu(f^2) - \frac{\nu(h_0^2)}{\nu(h_0^2)} \eta_\infty(f^2) \right)^{1/2}.
\]

The proofs of the above theorems and corollaries are given in Sect. 4.

Note that if, in addition, we have \( R > 0 \) and the matrix \( A \) is Hurwitz, then we have \( \nu(1) < \infty \). In this situation, applying (56) to the unit function \( f = 1 \) yields

\[
\left\| e^{\lambda_0 t} K_t(1) - \frac{h_0}{\eta_\infty(h_0)} \eta_\infty(1) \right\|_{2, \nu} \leq e^{-\lambda_0^* t} \left( \nu(1) - \frac{\nu(h_0^2)}{\nu(h_0^2)} \right)^{1/2}.
\]
The above results are clearly unsatisfactory when \( A \) is not Hurwitz. For this situation, we will provide a non-asymptotic expansion in the following section.

We note that when \( AR \neq RA' \), we cannot expect to apply the spectral theorem. In this case, if the controllability condition (4) is met, besides the fact that we don’t have a complete spectral decomposition, the ground state and the zero-point energy are explicitly computed using (31) by solving an algebraic Riccati matrix equation.

5. Stability Analysis

This section is dedicated to the stability analysis of the \( h \)-process and normalised Feynman Kac semigroups. In the first part, we provide explicit, non-asymptotic estimates of the exponential decays to equilibrium in terms of Wasserstein distances and Boltzmann relative entropy. In the second part, we focus on the reversible case and analyse the stability of the \( h \)-process. The proofs of these results will be provided in Sect. 9.

5.1. Decay to equilibrium. As promised in the previous section, we first provide a non-asymptotic expansion of the Feynman Kac propagator.

**Theorem 5.1.** For any time horizon \( t \geq \delta > 0 \) and any \( f \in L_1(\eta_\infty) \) we have

\[
e^{\lambda_0 t} K_t(f)(x) = \frac{h_0(x)}{\eta_\infty(h_0)} (\eta_\infty(f) + \epsilon_t(f)(x)) k_t(x)
\]

where

\[
\epsilon_t(f)(x) := \Phi_t(\delta_x)(f) - \eta_\infty(f)
\]

and \( k_t \) is a function satisfying

\[
\exp \left( -c_\delta e^{-2\beta t} \right) \leq k_t(x) \leq \exp \left( c_\delta (1 + \|x\|^2) e^{-2\beta t} \right).
\]

In the above display \( \beta > 0 \) is a positive constant that depends on the model parameters \( (A, R, S) \) and \( c_\delta < \infty \) is a finite constant whose value only depends on \( \delta \).

The convergence of \( \eta_t \) to the limiting measure \( \eta_\infty \) introduced in (14) can be studied in terms of both the stationary properties of the \( h \)-process and the stability properties of the Riccati matrix flow \( P_t \) introduced in (27). Due to the exponential semigroup formula (34), the long time behavior of the mean vector \( \hat{X}_t \) is also directly related to \( P_t \). Thus, in appendix A we provide a brief discussion on Riccati matrix flows, including the Floquet-type theory developed in [12], as well as several Lipschitz type inequalities and exponential type decays to equilibrium for Riccati flows and their associated exponential semigroups. Applied to our context, these quantitative estimates allow one to prove a variety of non asymptotic convergence theorems.

To this end, consider the initial distributions

\[
\eta_0 = \mathcal{N}(x, P) \quad \text{and} \quad \mu_0 = \mathcal{N}(y, Q),
\]

for some \( x, y \in \mathbb{R}^r \) and some covariance matrices \( P, Q \in S_0^r \). Then we have the following result.
Theorem 5.2. There exists $\delta = \delta(P, Q) > 0$, which depends on the distance $P - Q$, such that for any $t \geq \delta$ we have
\[
\mathbb{W}_2(\Phi_t(\eta_0), \Phi_t(\mu_0)) \leq c_\delta e^{-\beta t} \left(\|x - y\| + ((\|x\| \wedge \|y\|) \vee 1) \|P - Q\|\right)
\]
and for sufficiently large time horizon we have
\[
\text{Ent}(\Phi_t(\eta_0) \mid \Phi_t(\mu_0)) \leq c_\delta e^{-2\beta t} \left(\|P - Q\| + \left(\|x\|^2 \|P - Q\|^2 + \|x - y\|^2\right)\right)
\]
for some finite constants $\beta, c_\delta > 0$ that were introduced in the previous theorem.
In particular, when $P = Q$, for any $n \geq 1$ we have
\[
\mathbb{W}_n(\Phi_t(\eta_0), \Phi_t(\mu_0)) \leq c_\delta e^{-\beta t} \|x - y\|
\]
and
\[
\text{Ent}(\Phi_t(\eta_0) \mid \Phi_t(\mu_0)) \leq c_\delta e^{-2\beta t} \|x - y\|^2.
\]
A precise description of the constant $c_\delta$ and the time horizon in the relative entropy estimates are given in Sect. 9 however, we prefer to omit them here in favour of readability.

Applying Theorem 5.2 to $\mu_0 = \eta_\infty = \mathcal{N}(0, P_\infty)$, for sufficiently large time horizon we have the exponential decays
\[
\mathbb{W}_2(\Phi_t(\eta_0), \eta_\infty) \leq c_\delta e^{-\beta t} \|x\|,
\]
as well as
\[
\text{Ent}(\Phi_t(\eta_0) \mid \eta_\infty) \leq c_\delta e^{-2\beta t} \left((1 \vee \|x\|) (1 + \|P - P_\infty\|)\right)^2
\]
with the finite constants $\beta, c_\delta > 0$ defined in Theorem 5.2.

Theorem 5.2 provides several ways of estimating the difference $\epsilon_t(f)(x)$ defined in Theorem 5.1. For instance, for any Lipschitz function $f$ with unit Lipschitz constant or for any bounded function $g$ with unit uniform norm, we have the estimates
\[
|\epsilon_t(f)(x)| \leq \mathbb{W}_1(\Phi_t(\delta_x), \eta_\infty) \quad \text{and} \quad |\epsilon_t(g)(x)| \leq 2 \|\Phi_t(\delta_x) - \eta_\infty\|_{tv}.
\]

Using Theorem 5.2 (see also the estimates presented in “Appendix A.3”) it follows that
\[
\Phi_t(\delta_x)(dy) = \mathcal{N}(\tilde{X}_t(x, 0), \phi_t(0)) \quad (dy) \quad \longrightarrow_{t \to \infty} \eta_\infty = \mathcal{N}(0, P_\infty).
\]
This yields the formula
\[
2\Phi_t(\delta_x)(V) = \tilde{X}_t(x, 0)'S\tilde{X}_t(x, 0) + \text{Tr}(S\phi_t(0)) = x'E_t(0)'S\mathcal{E}_t(0)x + \text{Tr}(S\phi_t(0)),
\]
from which we conclude that
\[
\mathcal{K}_t(1)(x) = \exp \left(-\frac{1}{2} x'E_t(0)x - \frac{1}{2} \int_0^t \text{Tr}(S\phi_t(s))ds\right),
\]
with
\[
\Delta_t(0) := \int_0^t \mathcal{E}_s(0)'S\mathcal{E}_s(0) \quad ds \quad \longrightarrow_{t \to \infty} \Delta_\infty(0) := \int_0^\infty \mathcal{E}_s(0)'S\mathcal{E}_s(0) \quad ds.
\]
In contrast with linear semigroups we have the nonlinear transport formula
\[
\Phi_t(\eta_0) = \mathbb{B}_{\mathcal{K}_t(1)}(\eta_0)\mathcal{K}_t \neq \eta_0\mathcal{K}_t,
\]
which brings us to the following theorem.
Theorem 5.3. For any probability measure \( \eta \) on \( \mathbb{R}^r \) and for any \( t \geq 0 \) we have the Boltzmann-Gibbs formula

\[
\Phi_t(\eta) = B_{\sigma_t}(\eta)K_t
\]

with the energy function

\[
\log \sigma_t(x) = -\frac{1}{2} x' \Delta_t(0) x \quad \text{as} \quad t \to \infty \quad \log \sigma_{\infty}(x) = -\frac{1}{2} x' \Delta_{\infty}(0) x.
\]

In addition, for any \( \delta > 0 \) there exists some constant \( c_\delta \) such that for any pair of probability measures, \( \eta \) and \( \mu \), on \( \mathbb{R}^r \) and for \( t \geq \delta > 0 \), we have

\[
\|\Phi_t(\eta) - \Phi_t(\mu)\|_W \leq \frac{c_\delta}{\eta(\sigma_{\infty}) \wedge \mu(\sigma_{\infty})} e^{-\beta t},
\]

where \( \beta > 0 \) is a constant that depends on the model parameters \((A, R, S)\).

The above result implies the uniqueness of the fixed point Gaussian distribution \( \eta_{\infty} \) introduced in (14). In addition, choosing \( \mu_0 = \eta_{\infty} \), the result also shows that for any initial distribution \( \eta_0 \), the probability measure \( \Phi_t(\eta_0) \) converges exponentially fast towards a Gaussian distribution as \( t \to \infty \).

Corollary 5.4. For any \( t \geq 0 \), we have the Gaussian preserving property

\[
\eta_0 = N(x, P) \implies \eta_t = N(\hat{X}_t(x, P), \phi_t(P))
\]

with the mean vector and covariance matrices given by

\[
\begin{align*}
\hat{X}_t(x, P) &= E_t(0) (P^{-1} + \Delta_t(0))^{-1} P^{-1} x \\
\phi_t(P) &= \phi_t(0) + E_t(0) (P^{-1} + \Delta_t(0))^{-1} E_t(0)'
\end{align*}
\]

5.2. Reversible models. We now turn to the stability analysis of the h-process in the reversible case. Thus, until the end of this section we shall assume that the matrices \((A, R, S)\) satisfy the rank condition (4), that \( R > 0 \) and \( AR = RA' \).

We now state our first result of this subsection.

Theorem 5.5. For any \( t \geq 0 \) we have the master equation

\[
\eta_{\infty}^h(dx) K_t^h(x, dy) = \eta_{\infty}^h(dy) K_t^h(y, dx)
\]

with the distribution \( \eta_{\infty}^h \) defined in (14). In addition, we have the density-transport formulae

\[
\eta_0^h(dx) := f_0(x) \eta_{\infty}^h(dx) \implies \eta_t^h(dx) = f_t(x) \eta_{\infty}^h(dx) \quad \text{with} \quad f_t(x) := K_t^h(f_0)(x).
\]

The master equation (61) is equivalent to the reversibility property

\[
u(dx) K_t(x, dy) = \nu(dy) K_t(y, dx).
\]

Now recall the Boltzmann relative entropy and the Fisher information defined in Sect. 2.2. We have the following estimates for \( \eta_t^h \) in these metrics.
Theorem 5.6. For any $t \geq 0$ we have the relative entropy exponential decays
\[ J \left( \eta_t^h \mid \eta_\infty^h \right) \leq \alpha_h^2 e^{-2\beta_h t} J \left( \eta_0^h \mid \eta_\infty^h \right) \text{ and } \text{Ent} \left( \eta_t^h \mid \eta_\infty^h \right) \leq e^{-t/\iota_h} \text{Ent} \left( \eta_0^h \mid \eta_\infty^h \right) \]
where the parameters $(\iota_h, \alpha_h, \beta_h)$ were introduced in (36).

From the practical point of view, the functional inequalities discussed above are rarely useful when the matrix $Q_\infty$ and thus the limiting measure $\eta_\infty^h$ is not explicitly known.

6. Some Illustrations

6.1. One dimensional models. When $r = 1$ the reversibility condition is trivially met and we have
\[ P_\infty = \frac{A + \sqrt{A^2 + RS}}{S}, \quad Q_\infty = \frac{A + \sqrt{A^2 + RS}}{R} \quad \text{and} \quad P_\infty^h = \frac{R}{2} \frac{1}{\sqrt{A^2 + RS}}. \]
In this situation, the ground state $h_0$ and corresponding energy level discussed in (38) and (39) are given by
\[ h_0(x) := \exp \left( - \frac{A + \sqrt{A^2 + RS}}{2R} x^2 \right) \quad \text{and} \quad \lambda_0 = \frac{1}{2} \left( A + \sqrt{A^2 + RS} \right). \]
In addition, the eigenfunctions defined in (47) are given, for any $n \geq 1$, by
\[ \varphi_n^h(x) := \frac{1}{\sqrt{n!}} H_n^1 \left( \sqrt{2\sqrt{A^2 + RS}} \right) \frac{x}{R} \quad \text{and} \quad \lambda_n^h = n \sqrt{A^2 + RS}. \]
For instance, we have
\[ \varphi_1^h(x) = \sqrt{2\sqrt{A^2 + RS}} \frac{x}{R} \quad \text{and} \quad \varphi_2^h(x) := \frac{1}{\sqrt{2}} \left( \frac{2(A^2 + RS)^{1/2}}{R} x^2 - 1 \right). \]
As expected the first excited state $\varphi_1$ of the Hamiltonian is null at the origin, while
\[ \varphi_2(x) = 0 \iff x = \pm \frac{R}{\sqrt{2\sqrt{A^2 + RS}}}. \]

6.2. Mehler’s formula. We now assume that $r \geq 1$, $A = 0$ and $R = S^{-1}$. In this situation, we have
\[ Q_\infty = S, \quad 2P_\infty^h = P_\infty = S^{-1} \quad \text{and} \quad \Lambda^h = -I. \]
Thus, we may choose $Z = I$. In this context, we readily check that
\[ \nu(dx) = dx, \quad \lambda_0 = \frac{r}{2} \quad \text{and} \quad \lambda_n = \sum_{1 \leq i \leq r} \left( n_i + \frac{1}{2} \right) = \left| n \right| + \frac{r}{2}. \]
The eigenstates are defined by the formulae

$$\varphi_n(x) = \left( \frac{1}{\pi^r} \det(S) \right)^{1/4} h_0(x) \varphi_n^h(x)$$

with the re-scaled Hermite polynomials

$$\varphi_n^h(x) := \frac{1}{\sqrt{n!}} H_n\left( \sqrt{2} S^{1/2} x \right),$$

and the ground state

$$h_0(x) = \exp\left(-\frac{1}{2} x'Sx\right).$$

We also have that

$$\hat{X}^h_t(x) = e^{-t} x \quad \text{and} \quad P^h_t = \frac{1}{2} \left( 1 - e^{-2t} \right) S^{-1}$$

which implies that

$$\det(P^h_t) = \frac{(1 - e^{-2t})^r}{2^r} \frac{1}{\det(S)}.$$

The spectral decomposition reduces to the formula

$$K^h_t(x, dy) = \frac{1}{\pi^{r/2}} \frac{\sqrt{\det(S)}}{(1 - e^{-2t})^{r/2}} \exp\left(-\frac{1}{(1 - e^{-2t})} (y - e^{-t} x)' S (y - e^{-t} x) \right) dy$$

$$= \sum_{n \in \mathbb{N}^r} e^{-|n|t} \varphi_n^h(x) \varphi_n^h(y) \eta^h_\infty(dy),$$

with $\varphi_n^h$ defined above and

$$\eta^h_\infty = \mathcal{N}(0, P^h_\infty) = \mathcal{N}\left(0, \frac{1}{2} S^{-1}\right).$$

Thus, we have

$$\frac{1}{(1 - e^{-2t})^{r/2}} \exp\left(-\frac{1}{(1 - e^{-2t})} (y - e^{-t} x)' S (y - e^{-t} x) \right) \exp(y'Sy)$$

$$= \sum_{n \in \mathbb{N}^r} e^{-|n|t} \frac{1}{\sqrt{n!}} H_n\left( \sqrt{2} S^{1/2} x \right) \frac{1}{\sqrt{n!}} H_n\left( \sqrt{2} S^{1/2} y \right).$$

Replacing $(S^{1/2}x, S^{1/2}y)$ by $(x, y)$ and $e^{-t}$ by $\rho$, we recover Mehler’s formula

$$\frac{1}{(1 - \rho^2)^{r/2}} \exp\left(-\frac{\rho^2}{1 - \rho^2} (\|x\|^2 + \|y\|^2) + \frac{2\rho}{1 - \rho^2} x'y \right)$$

$$= \sum_{n \in \mathbb{N}^r} \rho^{|n|} \frac{1}{\sqrt{n!}} H_n\left( \sqrt{2} x \right) \frac{1}{\sqrt{n!}} H_n\left( \sqrt{2} y \right).$$
The Feynman–Kac propagator takes the form
\[
K_t(x, dy) = \exp \left(-\frac{rt}{2}\right) \exp \left(-\frac{1}{2} x'Sx \right) K^h_t(x, dy) \exp \left(\frac{1}{2} y'Sy\right).
\]

On the other hand, we have
\[
-\frac{1}{2} x'Sx + \frac{1}{2} y'Sy - \frac{1}{1-e^{-2t}} (y - e^{-t} x)'(y - e^{-t} x)
= -y'Sy \left(\frac{1}{1-e^{-2t}} - \frac{1}{2}\right) - x'Sx \left(\frac{1}{2} + \frac{e^{-2t}}{1-e^{-2t}}\right) + 2x'Sy \frac{e^{-t}}{1-e^{-2t}}
= -\frac{1}{2} (x'Sx + y'Sy) \frac{1 + e^{-2t}}{1-e^{-2t}} + x'Sy \frac{2e^{-t}}{1-e^{-2t}}.
\]

Thus, we recover Mehler’s formulation of the Feynman–Kac propagator
\[
K_t(x, dy) = \sqrt{\det(S)} \left(\frac{1}{2\pi}\right)^{r/2} \exp \left(-\frac{\coth(t)}{2} (x'Sx + y'Sy) + \frac{x'Sy}{\sinh(t)}\right) dy.
\]

6.3. Quantum harmonic oscillator. For diagonal matrices \( R \) and \( S \) we can choose \( \mathcal{Z} = I \). When \( A = 0 \), the measure \( \nu \) coincides with the Lebesgue measure \( \nu(dx) = dx \). The \( r \)-dimensional quantum harmonic oscillator is associated with diagonal matrices \( R \) and \( S \) with diagonal entries
\[
S_{i,i} = \kappa_i = m\omega_i^2 \quad \text{and} \quad R_{i,i} = \frac{\hbar^2}{m}
\]
\[
\Rightarrow \sqrt{\lambda_i(rs)} = \sqrt{(rs)_{i,i}} = \hbar\omega_i \quad \text{and} \quad \left((SR^{-1})_{i,i}\right)^{1/2} = \frac{m\omega_i}{\hbar},
\]
for \( 1 \leq i \leq r \), where \( m \) is the mass of a particle, \( \hbar \) is Planck’s constant, and \( \omega_i = \sqrt{\kappa_i/m} \) is the angular frequencies for some non-negative parameters \( \kappa_i \). In this situation, have
\[
\lambda_n = \sum_{1 \leq i \leq r} \lambda_{n_i} \quad \text{and} \quad h_n(x) = \prod_{1 \leq i \leq r} h_{n_i}(x_i)
\]
with the energy
\[
\lambda_{n_i} := \left(n_i + \frac{1}{2}\right) \hbar\omega_i.
\]

In addition, the eigenfunctions are given by
\[
\varphi_{n_i}(x_i) := \frac{1}{\sqrt{n_i! \pi \hbar}} \left(\frac{m\omega_i}{\pi \hbar}\right)^{1/4} \exp \left(-\frac{1}{2} \frac{m\omega_i}{\hbar} x_i^2\right) \mathcal{H}_{n_i}\left(\sqrt{\frac{2m\omega_i}{\hbar}} x_i\right).
\]

The isotropic harmonic oscillator corresponds to the case \( \kappa = k \Rightarrow \omega_i = \omega \). In this case, the energy is given by
\[
\lambda_n := (|n| + \frac{r}{2}) \hbar\omega.
\]
7. Proofs of Results in Sect. 3

Recall the process $X_c$ defined in (9). In order to prove the results of Sect. 3, we first state and prove a result that makes the link between the $h$-process, $X^h$, and the non-absorbed process, $X^c$, explicit. From this result we will be able to deduce Theorems 3.1 and 3.2.

**Theorem 7.1.** For all $x \in \mathbb{R}^r$, we have

$$h_0^{-1} \mathcal{L}(h_0)(x) = V(x) - \lambda_0,$$

where $h_0$ was defined in (38) and $\lambda_0$ was defined in (39). In addition, for any time horizon $t \geq 0$, any measurable function $F_t$ on $\mathcal{C}([0, t], \mathbb{R}^r)$ and any starting point $x \in \mathbb{R}^r$, we have the Feynman–Kac path-integral formula

$$\mathbb{E}(F_t \left((X^c_s)_{s \in [0, t]}\right) 1_{t^c > t} \mid X_0 = x) = \exp\left(-\lambda_0 t\right) h_0^{-1}(x) \mathbb{E}(F_t \left((X^h_s)_{s \in [0, t]}\right) \mid X^h_0 = x),$$

where $X^h_t$ stands for the diffusion with generator

$$\mathcal{L}^h(f) = \mathcal{L}(f) + h_0^{-1} \Gamma_\mathcal{L}(h_0, f),$$

and where the carré-du-champ operator, $\Gamma_\mathcal{L}$ is given by

$$\Gamma_\mathcal{L}(h_0, f)(x) := (B \nabla h_0(x))^\prime (B \nabla f(x)) = -h_0(x) (R Q_\infty x)^\prime \nabla f(x).$$

**Proof.** From the definition of $h_0$ in (38), we have the gradient formulae

$$\nabla \log h_0(x) = -Q_\infty x \quad \text{and} \quad \nabla^2 \log h_0(x) = -Q_\infty,$$

which imply that

$$h_0^{-1}(x) \nabla h_0(x) = -Q_\infty x \quad \text{and} \quad h_0^{-1}(x) \nabla^2 h_0(x) = -Q_\infty + (Q_\infty x)^\prime (Q_\infty x)^\prime.$$

Hence

$$h_0^{-1} \mathcal{L}(h_0)(x) = h_0^{-1}(x) \left( (Ax)^\prime \nabla h_0(x) + \frac{1}{2} \text{Tr} \left( R \nabla^2 h_0 \right) \right) = -x^\prime A^\prime Q_\infty x + \frac{1}{2} \text{Tr} \left( R (Q_\infty x)(Q_\infty x)^\prime \right) - \frac{1}{2} \text{Tr} \left( R Q_\infty \right),$$

from which it follows that

$$h_0^{-1} \mathcal{L}(h_0)(x) - V(x) = -\frac{1}{2} x^\prime (A^\prime Q_\infty + Q_\infty A)x + \frac{1}{2} x^\prime (Q_\infty R Q_\infty)x - \frac{1}{2} x^\prime Sx - \frac{1}{2} \text{Tr} \left( R Q_\infty \right) = -\lambda_0,$$

where the last equality follows from the fact that $A^\prime Q_\infty + Q_\infty A - Q_\infty R Q_\infty + S = 0$, as in (31).
This yields the exponential change of probability formula

\[ e^{\lambda_0 t} \mathbb{E}(F_t((X^h_s)_{s \in [0,t]}) \ 1_{\tau \geq t}) = e^{\lambda_0 t} \mathbb{E}(F_t((X_s)_{s \in [0,t]}) \exp \left(-\int_0^t V(X_s) \, ds\right)) = \mathbb{E} \left(F_t((X_s)_{s \in [0,t]}) \frac{h_0(X_0)}{h_0(X_t)} \left(h_0(X_t) \exp \left(-\int_0^t (h_0^{-1} \mathcal{L}h_0)(X_s) \, ds\right)\right)\right) = h_0(x) \mathbb{E}(F_t((X^h_s)_{s \in [0,t]}) h_0^{-1}(X^h_t)), \]  

which ends the proof of the theorem. \( \square \)

Note that by choosing functions that only depend on the terminal time in the first equality in (67), we recover the Feynman–Kac propagator formula (8).

Combining this observation with the above result and Lemma A.1, Theorem 3.1 is now proved. In addition, Theorem 3.2 is a now a direct consequence of (65). Indeed, using (65) we verify that the law \( \eta^h_t \) of the random states \( X^h_t \) of the \( h \)-process defined in (35) and the distribution of the non-absorbed particle defined in (10) are connected by the Boltzmann-Gibbs transformation (42). The Gaussian preserving property of the linear diffusion process \( X^h_t \) is immediate. The formulae given in (43) are easily checked using the the Boltzmann-Gibbs transformation (42). Moreover, we can check that the pair \( (\hat{X}_t, P_t) \) given by (43) satisfies (27) using brute force calculations, or by checking that the Gaussian distributions with mean and covariance matrices \( (\hat{X}_t, P_t) \) solves the nonlinear equation (26).

8. Proofs of Results in Sect. 4

The main focus of this section are the proofs of Theorems 4.1 and 4.3, and the corresponding corollaries. Thus, in what follows, we assume that \( R > 0 \) and \( A' = R^{-1}AR \).

Before turning to the proofs, we first spend some time discussing the reversibility condition the spectrum of \( \mathcal{H} \).

8.1. Reversible \( h \)-processes. Under the reversibility conditions, the fixed points of the algebraic Riccati equation (28) are given by the formulae

\[ (P^{-1}_\infty)^{-1} = -R^{-1}A - R^{-1}(A^2 + RS)^{1/2} < 0 \quad P^{-1}_\infty = -R^{-1}A + R^{-1}(A^2 + RS)^{1/2}, \]  

with the square root \((A^2 + RS)^{1/2}\) that has all positive eigenvalues. A proof of the above result can be found in [12]. To check that this square root is well-defined, observe that

\[ A' = R^{-1}AR \implies A^2 + RS = R(A'R^{-1}A + S) = R((A')^2 + SR)R^{-1}, \]

which has positive eigenvalues. We also have the formulae

\[ Q_\infty = P^{-1}_\infty + 2R^{-1}A \quad \text{and} \quad A - RQ_\infty = -(A^2 + RS)^{1/2}, \]
which yields

$$\text{Spec}(A - RQ_\infty) = \left\{ -|\lambda|^{1/2} : \lambda \in \text{Spec}(A^2 + RS) \right\} = \{-\lambda^h_1, \ldots, -\lambda^h_r\} \subset \mathbb{R}.$$  

Note that This implies that the spectral abcissa satisfies,

$$\varsigma(A - RQ_\infty) = -\lambda^h_1 < 0.$$  

Observe that in general situations, even though $(A - RQ_\infty)$ and $(A - RQ_\infty)'$ have the same eigenvalues we have

$$(A - RQ_\infty)' = -(A')^2 + SR)^{1/2} \neq -(A^2 + RS)^{1/2} = (A - RQ_\infty).$$

Thus even when $\mu(A) < 0$ there are situations where

$$\varsigma(A - RQ_\infty) < 0 < \mu(A - RQ_\infty).$$

For a more thorough discussion on these situations, we refer the reader to section 4.1 in the article [40]. Using the formula (70), as well as the definitions of $h_0$ and $v$ given in (38) and (15), respectively, we have

$$v(dx \ h_0(x)) = \exp\left(-\frac{1}{2} x' P^{-1}_\infty x\right) \quad \text{and} \quad v(h_0) := \int v(dx \ h_0(x)) = \sqrt{\det(2\pi P_\infty)}.$$

Note that this implies the Boltzmann Gibbs formulae in (51).

The limiting covariance matrix of the $h$-process is given more explicitly by the formulae

$$P^h_\infty := \left(P^{-1}_\infty + Q_\infty\right)^{-1} = \frac{1}{2} \left(P^{-1}_\infty + R^{-1} A\right)^{-1} = \frac{1}{2} R(A^2 + RS)^{-1/2}, \quad (71)$$

where we have used (70) and (68).

Combining this with the second equality in (70), we obtain

$$Q_\infty - R^{-1} A = R^{-1} (A^2 + RS)^{1/2} = (2 P^h_\infty)^{-1} = \frac{1}{2} \left(P^{-1}_\infty + Q_\infty\right), \quad (72)$$

which implies that

$$v(h^2_0) = (2\pi)^{r/2}/\sqrt{\det(P^{-1}_\infty + Q_\infty)}.$$  

Finally notice that

$$AR = RA' \iff (A - RQ_\infty) P^h_\infty + P^h_\infty (A - RQ_\infty)' + R = 0. \quad (74)$$

Thus, our condition ensures the reversibility property (61) of the $h$-process.
Remark 8.1. Assume that $S > 0$ and $SA = A'S$. In this situation, the fixed point matrices $(P_\infty^-, P_\infty)$ are given by (cf. [12])

$$P_\infty^- = AS^{-1} - (A^2 + RS)^{1/2}S^{-1} < 0 < P_\infty = AS^{-1} + (A^2 + RS)^{1/2}S^{-1}. \quad (75)$$

Thus, whenever $S, R > 0$ and $SAS^{-1} = A' = R^{-1}AR$ we have

$$(A - RQ_\infty) = -(A^2 + RS)^{1/2} = (A - P_\infty S) \quad \text{and} \quad RQ_\infty = P_\infty S. \quad (76)$$

Using (101) we also have

$$(P_\infty^h)^{-1} = P_\infty^{-1} + Q_\infty = 2P_\infty^{-1} + R^{-1}A$$

$$= 2R^{-1}(A^2 + RS)^{1/2} = -2R^{-1}(A - P_\infty S)$$

$$\implies \det(P_\infty^{-1} + Q_\infty) = 2^{r} \sqrt{|\det(A^2 + RS)|}/\det(R).$$

This implies that

$$RQ_\infty = P_\infty S = RP_\infty^{-1} + 2A = A + (A^2 + RS)^{1/2}. \quad (77)$$

Whenever $S > 0$, up to a change of basis, there is no loss of generality to assume that $S = I$. More precisely the matrices $ar{P}_t := S^{1/2}P_tS^{1/2}$ satisfy the same Riccati equation as $P_t$ when we replace $(A, R, S)$ by the matrices

$$(\bar{A}, \bar{R}, \bar{S}) := (S^{1/2}AS^{-1/2}, S^{1/2}RS^{1/2}, I). \quad (77)$$

Now observe that (70) implies that

$$\Lambda^h = (P_\infty^h)^{-1/2}(A - RQ_\infty)(P_\infty^h)^{1/2} = -\frac{1}{2}(P_\infty^h)^{-1/2}R(P_\infty^h)^{-1/2} < 0. \quad (78)$$

In addition, we have

$${\text{Spec}}(\Lambda^h) = \{-|\lambda|^{1/2} : \lambda \in \text{Spec}(A^2 + RS)\} \subset \mathbb{R}_-. \quad (79)$$

Note that this justifies the claims regarding the eigenvalues in Sect. 4.

We denote by $z_i$ an orthonormal eigenvector of the matrix $\Lambda^h$ associated with the eigenvalue $\lambda_i(\Lambda^h) := -\lambda_i^h < 0$, with $i \in \{1, \ldots, r\}$, and we set

$$\mathcal{Z} := (z_1, \ldots, z_r) \quad \text{and} \quad \mathcal{E}_i^h(Q_\infty) := \exp(\Lambda^h t). \quad (80)$$

Note that $\mathcal{Z}'\mathcal{Z} = I$.

Before proceeding to the proofs of the spectral theorems, we first state and prove a technical lemma that will be used in the aforementioned proofs.

**Lemma 8.2.** For any $u, x \in \mathbb{R}^r$ and any $t \geq 0$ we have

$$\mathbb{E}\left(\mathbb{S}_u^r \left( (P_\infty^h)^{-1/2}X_t^h(x) \right) \right) = \mathbb{S}_u^r \mathcal{E}_i^h(Q_\infty)u \left( (P_\infty^h)^{-1/2}x \right), \quad (81)$$

where $\mathbb{S}_u^r$ was defined in (46).
Proof. We first show that for any \( t \geq 0 \) we have

\[
I - (P^h_\infty)^{-1/2} P^h_t (P^h_\infty)^{-1/2} = \mathcal{E}^h_t (Q_\infty)^2 \quad \text{and} \quad \mathcal{E}^h_t (Q_\infty)' = \mathcal{E}^h_t (Q_\infty).
\]

(82)

First observe that

\[
AR = RA' \implies R^{-1} (A - R Q_\infty) = (A - R Q_\infty)' R^{-1} \implies \forall n \geq 1 \ R^{-1} (A - R Q_\infty)^n R = ((A - R Q_\infty)')^n.
\]

This yields the formula

\[
P^h_t = P^h_\infty \left( I - R^{-1} e^{2(A-R Q_\infty) t} R \right) = P^h_\infty \left( I - e^{2(A-R Q_\infty)' t} \right)
\]

and therefore

\[
P^h_t = \left( I - e^{2(A-R Q_\infty)' t} \right) P^h_\infty.
\]

This implies that

\[
(P^h_\infty)^{-1/2} P^h_t (P^h_\infty)^{-1/2} = I - (P^h_\infty)^{-1/2} e^{2(A-R Q_\infty)' t} (P^h_\infty)^{1/2}.
\]

By (78), we have the commutative property

\[
(P^h_\infty)^{-1/2} (A - R Q_\infty) (P^h_\infty)^{1/2} = (P^h_\infty)^{1/2} (A - R Q_\infty)' (P^h_\infty)^{-1/2},
\]

which implies that

\[
P^h_\infty e^{t (A-R Q_\infty)'} = e^{t (A-R Q_\infty)} P^h_\infty \iff e^{t (A-R Q_\infty)'} = P^h_\infty e^{t (A-R Q_\infty)'} (P^h_\infty)^{-1}.
\]

Thus, we have

\[
(P^h_\infty)^{-1/2} e^{2(A-R Q_\infty)' t} (P^h_\infty)^{1/2}
\]

\[
= \left( (P^h_\infty)^{-1/2} e^{(A-R Q_\infty)' t} (P^h_\infty)^{1/2} \right) \left( (P^h_\infty)^{1/2} e^{(A-R Q_\infty)'} (P^h_\infty)^{-1/2} \right)
\]

and can conclude that

\[
\mathcal{E}^h_t (Q_\infty)^2 = \mathcal{E}^h_t (Q_\infty) \mathcal{E}^h_t (Q_\infty)'.
\]

Returning to the proof of the lemma, using the decomposition

\[
(P^h_\infty)^{-1/2} X^h_t (x) \overset{law}{=} \left( P^h_\infty \right)^{-1/2} \tilde{X}^h_t (x) + \left( P^h_\infty \right)^{-1/2} \left( P^h_t \right)^{1/2} W_1,
\]

where \( W_1 \sim \mathcal{N}(0, I) \), it follows that

\[
\log \mathbb{E} \left( \exp \left( u' \left( P^h_\infty \right)^{-1/2} X^h_t (x) - \frac{1}{2} u' u \right) \right)
\]

\[
= u' \left( P^h_\infty \right)^{-1/2} e^{(A-R Q_\infty)' t} x - \frac{1}{2} u' \left( I - (P^h_\infty)^{-1/2} P^h_t (P^h_\infty)^{-1/2} \right) u
\]

\[
= u' \left( P^h_\infty \right)^{-1/2} e^{(A-R Q_\infty)' t} (P^h_\infty)^{1/2} \left( P^h_\infty \right)^{-1/2} x - \frac{1}{2} u' \left( P^h_\infty \right)^{-1/2} e^{2(A-R Q_\infty)' t} (P^h_\infty)^{1/2} u,
\]
where we have used (37) to obtain the first equality and (82) to obtain the second. This implies that

\[
\log \mathbb{E} \left( \exp \left( u' (P^h_\infty)^{-1/2} X^h_t (x) - \frac{1}{2} u' u \right) \right) = \left( \mathcal{E}^h_t (Q_\infty) u \right)' (P^h_\infty)^{-1/2} x - \frac{1}{2} \left( \mathcal{E}^h_t (Q_\infty) u \right)' \left( \mathcal{E}^h_t (Q_\infty) u \right),
\]

\[(84)\]
from which the result now follows.

8.2. Proofs of the spectral results.

**Proof of Theorem 4.1** It suffices to show that

\[
\mathbb{E} \left( S'_u \left( Z' (P^h_\infty)^{-1/2} X^h_t (x) \right) \right) = S'_{e^{\overline{\Lambda}^h} u} \left( Z' (P^h_\infty)^{-1/2} x \right),
\]

where \( Z \) was defined in (80) and \( \overline{\Lambda}^h := Z' \Lambda^h Z = \text{Diag} (-\lambda^h_1, \ldots, -\lambda^h_r) \).

To this end, observe that for any \( x, u \in \mathbb{R}^r \), we have

\[
S'_u (Z' x) = \exp \left( u' Z' x - \frac{1}{2} u' Z' Z u \right) = S'_{Z u} (x).
\]

Also note that

\[
Z' e^{\Lambda^h} Z = e^{\overline{\Lambda}^h} \Rightarrow \mathcal{E}^h_t (Q_\infty) Z = Z e^{\overline{\Lambda}^h}.
\]

Thus, combining these observations with Lemma 8.2, we have

\[
\mathbb{E} \left( S'_u \left( Z' (P^h_\infty)^{-1/2} X^h_t (x) \right) \right) = \mathbb{E} \left( S'_{Z u} \left( (P^h_\infty)^{-1/2} X^h_t (x) \right) \right)
= S'_{\mathcal{E}^h_t (Q_\infty) Z u} \left( (P^h_\infty)^{-1/2} x \right)
= S'_{Z e^{\overline{\Lambda}^h} u} \left( (P^h_\infty)^{-1/2} x \right)
= S'_{e^{\overline{\Lambda}^h} u} \left( Z' (P^h_\infty)^{-1/2} x \right),
\]

as required.

\[\square\]

**Proof of Corollary 4.2** We first prove the estimate (49). Using the decomposition from Theorem 4.1, it is straightforward to show that for any function \( f \in \mathbb{L}_2(\eta^h_\infty) \) we have

\[
\| \mathcal{K}^h_t (f) - \eta^h_\infty (f) \|^2_{2, \eta^h_\infty} \leq e^{-2\lambda^h_t} \sum_{n \in \mathbb{N}^r - \{0\}} \eta^h_\infty (f \varphi^h_n)^2
\]
\[
= e^{-2\lambda^h_t} \sum_{n \in \mathbb{N}^r - \{0\}} \eta^h_\infty (f \varphi^h_n)^2
\]
\[
= e^{-2\lambda^h_t} \| f - \eta^h_\infty (f) \|^2_{2, \eta^h_\infty}.
\]
Now let us prove that this is equivalent to (50). For $t < (2\lambda_i h)^{-1}$, an elementary second order Taylor expansion of the exponential function yields
\[
e^{-2\lambda_i h t} \| f - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 = \left( 1 - 2\lambda_i h t \right) \| f - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 + o(t).
\]
In the same vein, we have
\[
\| K_i^h (f) - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 = \| f - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 + t \eta_\infty^h \left( \frac{1}{t} \left( K_i^h (f) - f \right) \left( K_i^h (f) + f \right) \right)
\]
\[
= \| f - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 - 2t E_h (f, f) + o(t).
\]
Using (49) we obtain the Poincaré inequality (50).

On the other hand, suppose (50) holds. The Markov transitions $K_i^h$ satisfy the Chapman-Kolmogorov evolution equation given in weak form by the formulae
\[
\partial_t K_i^h = K_i^h \mathcal{L}^h = \mathcal{L}^h K_i^h.
\]
This yields the Dirichlet form equation
\[
\partial_t \eta_\infty^h (f K_i^h (g))|_{t=0} = -E_h (f, g) := \eta_\infty^h (f \mathcal{L}^h (g)),
\]
and hence
\[
\partial_t \| K_i^h (f) - \eta_\infty^h (f) \|_{2, \eta_\infty^h}^2 = -2 E_h \left( K_i^h (f), K_i^h (f) \right).
\]
Combining this with (50) yields (49). $\square$

**Proof of Theorem 4.3** Recall that $K_i^h$ and $K_i$ are connected via
\[
e^{\lambda_i t} K_i = \Upsilon_i \circ K_i \circ \Upsilon_i,
\]
where the isometry $\Upsilon_i$ was defined in (52). Using this and the formula $\eta_\infty^h = \mathcal{B}_{h^2 \eta_0} (\nu)$ given in (51), it is straightforward to show that
\[
K_i (x, \delta y) = \sum_{n \in \mathbb{N}^r} e^{-\lambda_i n t} \varphi_n (x) \varphi_n (y) \nu(dy),
\]
where
\[
\varphi_n (x) = \Upsilon_i \left( \varphi_n^h \right) = \frac{h_0 (x)}{\sqrt{\nu (h_0^2)}} \varphi_n^h (x) = \frac{h_0 (x) \varphi_n^h (x)}{\sqrt{\nu (h_0) \eta_\infty (h_0)}} \quad \Longrightarrow \quad \varphi_0 (x) = \frac{h_0 (x)}{\sqrt{\nu (h_0^2)}}
\]
and $\lambda_n = \lambda_0 + \lambda_i n$. To complete the proof, note that from the definitions of $\eta_\infty$, $h_0$ and $\nu$ given in (14), (38) and (15), respectively, we observe that
\[
\eta_\infty (h_0) = (\det (I + P_\infty Q_\infty))^{-1/2} \quad \text{and} \quad \nu (h_0) = (\det (2\pi P_\infty))^{1/2}.
\]
This yields the formula
\[
\nu (h_0) \eta_\infty (h_0) = (2\pi)^{r/2} \left( \det \left( P_\infty^{-1} + Q_\infty \right) \right)^{-1/2},
\]
which ends the proof of the theorem. $\square$
Proof of Corollary 4.4  We have the decomposition

\[ e^{\lambda at} K_t(f)(x) - \frac{h_0(x)}{\eta_{\infty}(h_0)} \eta_{\infty}(f) = \sum_{n \in \mathbb{N} - \{0\}} e^{-\lambda n t} \varphi_n(x) \nu(\varphi_n f). \]  

(87)

The proof of (56) is now a direct consequence of the formulae \( \eta_{\infty} = \mathbb{E}_{h_0}(\nu) \) and

\[ \sum_{n \in \mathbb{N} - \{0\}} \nu(f \varphi_n)^2 = \| f \|_{L^2_{\nu}}^2 - \nu(\varphi_0 f)^2 = \nu(f^2) - \nu \left( \frac{h_0 f}{\sqrt{\nu(h_0^2)}} \right)^2. \]

\[ \square \]

9. Proof of Results in Sect. 5

9.1. Proofs of results in Sect. 5.1. In this section, we will prove the results presented in Sect. 5, starting with the proof of Theorem 5.1.

Proof of Theorem 5.1

First observe that

\[ K_t(h_0)(x) = e^{-\lambda at} h_0(x) \implies e^{\lambda at} K_t(1)(x) = \frac{h_0(x)}{K_t(h_0)(x)} = \frac{h_0(x)}{\eta_{\infty}(h_0)} k_t(x), \]

with

\[ k_t(x) := \eta_{\infty}(h_0)/K_t(h_0)(x) = \eta_{\infty}(h_0)/\Phi_t(\delta_x)(h_0). \]

Then, writing \( K_t(f) = K_t(1)(K_t(f)/K_t(1)) \), for any \( t \geq 0 \) and any \( f \in L_1(\eta_{\infty}) \) we have the decomposition

\[ e^{\lambda at} K_t(f)(x) = \frac{h_0(x)}{\eta_{\infty}(h_0)} \left( \eta_{\infty}(f) + \tilde{K}_t(f)(x) \right) k_t(x), \]

where

\[ \tilde{K}_t(f)(x) = \overline{K}_t(f)(x) - \eta_{\infty}(f) = \Phi_t(\delta_x)(f) - \eta_{\infty}(f). \]

It remains to prove the estimates (58). First note that from the definitions of \( \eta_{\infty}, h_0 \) and \( \Phi_t \) given in (14), (38) and (11), respectively, it follows that

\[ k_t(x) = \left( \frac{\det(I + \phi_t(0) Q_{\infty})}{\det(I + P_{\infty} Q_{\infty})} \right)^{1/2} \exp \left( \frac{1}{2} \left. \tilde{X}_t(x, 0) \right| I + Q_{\infty} \phi_t(0) \left. \right|^{-1} \tilde{X}_t(x, 0) \right). \]

(88)

This shows that

\[ k_t(x) \xrightarrow{|x| \to +\infty} +\infty \text{ and } k_t(x) \xrightarrow{t \to \infty} 1 \]

and hence one cannot expect a uniform upper bound with respect to the state variable.

We will now use the Lipschitz estimates presented in “Appendix A.3” to derive the estimates (58). Let us first consider the square root term on right-hand side of (88). Applying (110) in the appendix to \( P = 0 \) we have the formula

\[ (I + \phi_t(0) Q_{\infty})(I + P_{\infty} Q_{\infty})^{-1} = I - E_t(P_{\infty}) \mathbb{E}_t(0)^{-1} P_{\infty} E_t(P_{\infty})' (Q_{\infty}^{-1} + P_{\infty})^{-1}. \]
Then, by (105), for any \( t \geq \delta > 0 \) we have

\[
\| \mathcal{E}_t(P_\infty) \mathbb{E}_t(0)^{-1} P_\infty \mathcal{E}_t(P_\infty)' (Q_\infty^{-1} + P_\infty)^{-1} \| \leq \chi_\delta^3 \| \mathcal{E}_t(P_\infty) \|^2 \| P_\infty (Q_\infty^{-1} + P_\infty)^{-1} \|.
\]

Using (106), this implies that

\[
\| \mathcal{E}_t(P_\infty) \mathbb{E}_t(0)^{-1} P_\infty \mathcal{E}_t(P_\infty)' (Q_\infty^{-1} + P_\infty)^{-1} \| \leq \sqrt{r} \chi_\delta^3 \alpha^2 e^{-2\beta t} \| P_\infty Q_\infty (I + P_\infty Q_\infty)^{-1} \|.
\]

Now, note that for any \((r \times r)\)-matrix \( A \) with \( \| A \|_F < \frac{1}{2} \), we have

\[
| \log \det (I - A) | \leq \frac{3}{2} \| A \|_F. \tag{89}
\]

We refer the reader to [40] for a prove of the above result.

Using this fact, for any

\[
t > t_\delta := \frac{1}{2\beta} \log \left( 2 \sqrt{r} \chi_\delta^3 \alpha^2 \| P_\infty Q_\infty (I + P_\infty Q_\infty)^{-1} \| \right)
\]

we have

\[
\left| \log \left( \frac{\det(I + \phi_t(0) Q_\infty)}{\det(I + P_\infty Q_\infty)} \right)^{1/2} \right| \leq \frac{3}{4} \sqrt{r} \chi_\delta^3 \alpha^2 e^{-2\beta t} \| P_\infty Q_\infty (I + P_\infty Q_\infty)^{-1} \|.
\]

Finally, to deal with the exponential term in (88), we have

\[
\hat{X}_t(x, 0)' (I + Q_\infty \phi_t(0))^{-1} \hat{X}_t(x, 0) \leq \| \phi_t(0)^{-1} \| \| (\phi_t(0)^{-1} + Q_\infty)^{-1} \| \| \hat{X}_t(x, 0) \|^2.
\]

By (106) for any \( t \geq \delta > 0 \) we have

\[
\phi_t(0)^{-1} \leq \Pi_{-, \delta}^{-1} \quad \text{and} \quad (\phi_t(0) + Q_\infty)^{-1} \leq (\Pi_{-, \delta} + Q_\infty)^{-1}.
\]

Combining the above with the estimates (106) and (105) we check that

\[
\hat{X}_t(x, 0)' (I + Q_\infty \phi_t(0))^{-1} \hat{X}_t(x, 0) \leq \| \Pi_{-, \delta}^{-1} \| \| (\Pi_{-, \delta} + Q_\infty)^{-1} \| (\alpha \chi_\delta)^2 e^{-2\beta t} \| x \|^2,
\]

as required.

Proof of Theorem 5.2 Now let us turn to the proof of Theorem 5.2. We will split the proof into two parts, one focussing on the proof of the relative entropy bound and the other focussing on the Wasserstein distance bound. We will actually provide explicit formulae for the constant \( c_\delta \) stated in the theorem. To this end, for any \( n \geq 1 \) and \( \delta > 0 \) we set

\[
t_n, \delta := \delta \vee \frac{1}{2\beta} \log (2n\alpha_\delta) \quad \text{with} \quad \alpha_\delta := (\alpha \chi_\delta)^2 \left( r \lambda_{\max} (\Pi_{-, \delta}^{-1}) \text{ Tr}(\Pi_{-, \delta}^{-1}) \right)^{1/2},
\]

with the positive matrix \( \Pi_{-, \delta} \) and the parameters \((\alpha, \beta, \chi_\delta)\) defined in (106) and Theorem A.2. Then we have the following result for the relative entropy.
**Theorem 9.1.** For any initial conditions
\[ \eta_0 = \mathcal{N}(x, P) \] and \( \mu_0 = \mathcal{N}(y, Q) \) s.t. \( \|P - Q\| \leq n \),
and for any \( t \geq t_{\delta,n} \) with \( n \geq 1 \) and \( \delta > 0 \), we have the exponential decay estimate
\[
\text{Ent} \left( \Phi_t(\eta_0) \mid \Phi_t(\mu_0) \right) \\
\leq \left( \frac{5}{4} \alpha_\delta \|P - Q\| + \lambda_{\max}(\Pi_{-\delta}^{-1}) (\alpha_\delta)^2 \left( \chi_\delta \| \Delta_\infty \|_\infty \|x\|^2 \|P - Q\|^2 + \|x - y\|^2 \right) \right) e^{-2\beta t}.
\]

**Proof.** For initial conditions \( \eta_0 \) and \( \mu_0 \) as stated above, the Boltzmann-Kullback-Leibler relative entropy of \( \Phi_t(\eta_0) \) with respect to \( \Phi_t(\mu_0) \) has a closed form (see for instance the article [88] and formula (A.23) in [77]):
\[
\text{Ent} \left( \Phi_t(\eta_0) \mid \Phi_t(\mu_0) \right) \\
= \frac{1}{2} \left( \text{Tr} \left( \phi_t(P) \phi_t(Q)^{-1} - I \right) + \log \det \left( \phi_t(Q) \phi_t(P)^{-1} \right) \right) + \left( \tilde{X}_t(x, P) - \tilde{X}_t(y, Q) \right) \phi_t(Q)^{-1} \left( \tilde{X}_t(x, P) - \tilde{X}_t(y, Q) \right).
\]

(90)

We start by controlling the log det term on the right-hand side above. To this end, first note that, by (106), for any \( t \geq \delta > 0 \) we have
\[ 0 < \Pi_{+,\delta}^{-1} \leq \phi_t(Q)^{-1} \leq \Pi_{-,\delta}^{-1} \implies \text{Tr}(\phi_t(Q)^{-2}) \leq r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}). \]

Combining this with the Lipschitz estimates stated in Theorem A.3 we check that
\[ \|I - \phi_t(Q)^{-1} \phi_t(P)\|_F \leq \left( r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}) \right)^{1/2} (\alpha_\delta)^2 e^{-2\beta t} \|P - Q\|. \]

For any \( t \geq t_{n,\delta} \) we have \( t \geq \delta \) and
\[ e^{-2\beta t} \leq \frac{1}{2n\alpha_\delta} \quad \text{with} \quad \alpha_\delta := (\alpha_\delta)^2 \left( r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}) \right)^{1/2}. \]

Thus
\[ \|I - \phi_t(Q)^{-1} \phi_t(P)\|_F \leq \frac{1}{2n} \|P - Q\|. \]

Applying (89) to \( A = \phi_t(Q)^{-1} (\phi_t(Q) - \phi_t(P)) \), for any \( t \geq t_{n,\delta} \) and \( \|P - Q\| \leq n \) we have the estimate
\[
\left| \log \det \left( \phi_t(P) \phi_t(Q)^{-1} \right) \right| \leq \frac{3}{2} \|\phi_t(P) - \phi_t(Q)\|_F \|\phi_t(Q)^{-1}\|_F \\
\leq \frac{3}{2} \left( r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}) \right)^{1/2} (\alpha_\delta)^2 e^{-2\beta t} \|P - Q\|.
\]

Similarly, to control the first term on the right-hand side of (90), for any \( t \geq \delta \) we have
\[
\left| \text{Tr} \left( I - \phi_t(Q)^{-1} \phi_t(P) \right) \right| \leq \left\| \phi_t(Q)^{-1} \right\|_F \|\phi_t(P) - \phi_t(Q)\|_F \\
\leq \left( r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}) \right)^{1/2} \|\phi_t(P) - \phi_t(Q)\| \\
\leq \left( r \lambda_{\max}(\Pi_{-,\delta}^{-1}) \text{Tr}(\Pi_{-,\delta}^{-1}) \right)^{1/2} (\alpha_\delta)^2 e^{-2\beta t} \|P - Q\|. \]
For the final term, we note that
\[
\left| (\hat{X}_t(x, P) - \hat{X}_t(y, Q))' \phi_t(Q)^{-1} (\hat{X}_t(x, P) - \hat{X}_t(y, Q)) \right| \leq \lambda_{\max}(\Pi_{-\delta}^{-1}) \| (\hat{X}_t(x, P) - \hat{X}_t(y, Q)) \|^2.
\]
Applying Theorem A.5 for any \( t \geq \delta \) we check that
\[
\left| (\hat{X}_t(x, P) - \hat{X}_t(y, Q))' \phi_t(Q)^{-1} (\hat{X}_t(x, P) - \hat{X}_t(y, Q)) \right| \\
\leq \lambda_{\max}(\Pi_{-\delta}^{-1}) (\alpha \chi_\delta)^2 e^{-2\beta t} (\chi_\delta \| \Delta_{\infty} \| \| x \| \| P - Q \| + \| x - y \|)^2,
\]
which concludes the proof. \( \square \)

Applying the above theorem first to \( P = 0 \) and \( (y, Q) = (0, P_\infty) \) and then \( P = Q = 0 \) yields the following corollary.

**Corollary 9.2.** (i) For any \( x \in \mathbb{R}^r, t \geq t_{\delta,n} \) with \( n = \| P_\infty \| \) and \( \delta > 0 \), we have the exponential decay estimate
\[
\text{Ent} \left( \Phi_t(\delta_x) \mid \eta_\infty \right) \leq \left( \frac{5}{4} \gamma \| P_\infty \| + \lambda_{\max}(\Pi_{-\delta}^{-1})(\alpha \chi_\delta)^2 (1 + (\chi_\delta \| \Delta_{\infty} \| \| P_\infty \|)^2 \| x \|^2 \right) e^{-2\beta t}.
\]

(ii) For any \( t \geq \delta > 0 \) and any \( x, y \in \mathbb{R}^r \) we have
\[
\text{Ent} \left( \Phi_t(\delta_x) \mid \Phi_t(\delta_y) \right) \leq \frac{1}{2} \lambda_{\max}(\Pi_{-\delta}^{-1})(\alpha \chi_\delta)^2 e^{-2\beta t} \| x - y \|^2.
\]

Let us now turn to the proof of the bound on the Wasserstein distance given in Theorem 5.2. For any initial conditions of the form (59) we have
\[
\mathbb{W}_2(\Phi_t(\mu_0), \Phi_t(\eta_0)) \leq \| \hat{X}_t(x, P) - \hat{X}_t(y, Q) \|^2 + \| \phi_t(P)^{1/2} - \phi_t(Q)^{1/2} \|^2.
\]
For any \( P, Q \in \mathcal{S}_+^r \) we also have the Ando-Hemmen inequality
\[
\| P^{1/2} - Q^{1/2} \| \leq \left[ \frac{1}{2} \lambda_{\min}(P) + \lambda_{\min}(Q) \right]^{-1} \| P - Q \|,
\]
(91)
where \( \| \cdot \| \) represents any unitary invariant matrix norm, which includes the spectral and the Frobenius norms (see for instance Theorem 6.2 on page 135 in [52], as well as Proposition 3.2 in [2]).

Using (106) and Theorem A.3 for any \( t \geq \delta > 0 \) we check that
\[
\| \phi_t(P)^{1/2} - \phi_t(Q)^{1/2} \|^2 \leq \sqrt{\frac{2}{\gamma}} \left[ \frac{1}{2} \lambda_{\min}(P) \right]^{-1} (\alpha \chi_\delta)^2 e^{-2\beta t} \| P - Q \|.
\]
Using theorem A.5 we obtain the following theorem.

**Theorem 9.3.** For any \( t \geq \delta > 0 \) and any initial conditions
\[
\eta_0 = \mathcal{N}(x, P) \quad \text{and} \quad \mu_0 = \mathcal{N}(y, Q)
\]
we have the exponential decay estimate
\[
\mathbb{W}_2(\Phi_t(\mu_0), \Phi_t(\eta_0)) \\
\leq \alpha \chi_\delta e^{-\beta t} \left( \| x - y \| + \chi_\delta \left( \| \Delta_{\infty} \| \| x \| + \alpha \sqrt{\frac{2}{\gamma}} \left[ \frac{1}{2} \lambda_{\min}(P) \right]^{-1} e^{-\beta t} \right) \| P - Q \| \right).
\]
Proof of Theorem 5.3. We now turn to the proofs of Theorem 5.3 and Corollary 5.4. The first assertion in Theorem 5.3 comes from the fact that

\[ K_t(1)(x) = \omega_t(x) \exp \left( -\frac{1}{2} \int_0^t \text{Tr}(S\phi_s(0))ds \right) \Rightarrow B_{K_t}(1) = B_{\omega_t}. \]

To prove (60), observe that

\[ \Phi_t(\eta)(f) - \Phi_t(\mu)(f) = \int \mathbb{B}_{\omega_t}(\eta)(dx)\mathbb{B}_{\omega_t}(\mu)(dy) \left( K_t(f)(x) - K_t(f)(y) \right). \]

From Definition 2.3 and Theorem 5.2, for any function \( f \) such that \( \|f\| \leq 1 \), we have

\[ |K_t(f)(x) - K_t(f)(y)| \leq \|\Phi_t(\delta_x) - \Phi_t(\delta_y)\|_v \leq c_\delta e^{-\beta t} \|x - y\|. \]

This implies that

\[ \|\Phi_t(\eta) - \Phi_t(\mu)\|_v \leq 2c_\delta e^{-\beta t} \left( \int \mathbb{B}_{\omega_t}(\eta)(dx)\|x\| \vee \int \mathbb{B}_{\omega_t}(\mu)(dx)\|x\| \right). \]

On the other hand, we have

\[ \eta(\omega_t) \geq \eta(\omega_\infty) > 0 \quad \text{with} \quad \log \omega_\infty(x) = -\frac{1}{2} x' \Delta_\infty(0) x, \]

as well as the uniform estimate

\[ \int \eta(dx)\omega_t(x)\|x\| \leq \int \eta(dx)\omega_\infty(x)\|x\| \sup_{x \in \mathbb{R}} \|\omega_\infty(x)\| =: c'_\delta < \infty, \quad t \geq \delta > 0. \]

Thus

\[ \int \mathbb{B}_{\omega_t}(\eta)(dx)\|x\| = \frac{1}{\eta(\omega_t)} \int \eta(dx)\omega_t(x)\|x\| \leq \frac{c'_\delta}{\eta(\omega_\infty)}, \]

which completes the proof of the theorem.

For the proof of Corollary 5.4, observe that

\[ \eta = \mathcal{N}(x, P) \implies \left\{ \begin{array}{l} \mathbb{B}_{\omega_t}(\eta) = \mathcal{N} \left( (P^{-1} + \Delta_t(0))^{-1} P^{-1} x, \ (P^{-1} + \Delta_t(0))^{-1} \right) \\ \eta K_t = \mathcal{N} \left( \mathcal{E}_t(0)x, \phi_t(0) + \mathcal{E}_t(0) P \mathcal{E}_t(0)' \right). \end{array} \right. \]

Combining these two formulae yields the result. \(\square\)

9.2. Proofs of results in Sect. 5.2. We now turn to the proofs of the results presented in Sect. 5.2. Recall that we are in the reversible setting and so we assume that the matrices \((A, R, S)\) satisfy the rank condition (4), that \( R > 0 \) and \( AR = RA' \).
Proof of Theorem 5.5

Proof. In the reversible case, it is convenient to rewrite the generator of the $h$-process given by (16) in the divergence form

$$\mathcal{L}^h(f) = \frac{1}{2} e^{U_h} \sum_{1 \leq i \leq r} \partial_x_i \left( e^{-U_h} \partial_x_i f \right)$$

with $U_h(x) := \frac{1}{2} x' (P^h_{\infty})^{-1} x$.

Using the divergence form of the generator we check that for sufficiently smooth functions $f$ and $g$ for which we can perform integration by parts, we have

$$\eta^h_{\infty} \left( g \mathcal{L}^h(K^h_t(f_0)) \right) = -\frac{1}{2} \int \eta^h_{\infty}(dx) \sum_{1 \leq i \leq r} \partial_x_i (g)(x) \partial_x_i K^h_t(f_0)(x). \tag{92}$$

This yields for any $f, g \in L_2(\eta^h_{\infty})$ the formula

$$\eta^h_{\infty}(f K^h_t(g)) = \eta^h_{\infty}(K^h_t(f) g),$$

which is equivalent to (61). The density-transport formulae (62) is a direct consequence of the reversible property (61).

Proof of Theorem 5.6  By the density transport formula (62) we have

$$\eta^h_0(dx) = f_0(x) \eta^h_{\infty}(dx) \implies \partial_t \text{Ent} \left( \eta^h_t \mid \eta^h_{\infty} \right) = \int (1 + \log K^h_t(f_0)) \mathcal{L}^h(K^h_t(f_0)) d\eta^h_{\infty}.$$

Applying the integration by parts formula (92) to $g = 1 + \log K^h_t(f_0)$ we find the de Bruijn identity

$$\partial_t \text{Ent} \left( \eta^h_t \mid \eta^h_{\infty} \right) = -\frac{1}{2} \int \frac{||\nabla K^h_t(f_0)||^2}{K^h_t(f_0)} d\eta^h_{\infty} := -\frac{1}{2} \mathcal{J} \left( \eta^h_t \mid \eta^h_{\infty} \right). \tag{93}$$

Next, observe that using (37) we obtain

$$\nabla \hat{X}^h_t(x) = \exp \left( (A - R Q_{\infty})' t \right).$$

Also note that

$$\nabla K^h_t(f_0)(x) = \nabla \mathbb{E}(f_0(\hat{X}^h_t(x))) = \mathbb{E}(\nabla \hat{X}^h_t(x)(\nabla f_0)(\hat{X}^h_t(x))).$$

This yields the commutative property

$$\nabla K^h_t(f_0) = e^{(A - R Q_{\infty})' t} K^h_t(\nabla f_0)$$

$$\implies \mathcal{J} \left( \eta^h_t \mid \eta^h_{\infty} \right) = \int \frac{K^h_t(\nabla f_0)(x)'}{K^h_t(f_0)(x)} e^{(A - R Q_{\infty})' t} K^h_t(\nabla f_0)(x) \sqrt{K^h_t(f_0)(x)} \eta^h_{\infty}(dx).$$
Applying Cauchy Schwartz inequality we find that
\[
\mathcal{J} \left( \eta^h_t \mid \eta^h_\infty \right) \leq \| e^{(A-RQ_\infty)t} \|^2 \int \eta^h_\infty(dx) \frac{||K^h_t(\sqrt{f_0} (\nabla f_0/\sqrt{f_0}))(x)||^2}{K^h_t(f_0)(x)} \\
\leq \| e^{(A-RQ_\infty)t} \|^2 \int \eta^h_\infty(dx) K^h_t(||\nabla f_0||^2/f_0)(x) \\
= \| e^{(A-RQ_\infty)t} \|^2 \eta^h_\infty(||\nabla f_0||^2/f_0).
\]
This yields the Fisher information exponential decay
\[
\mathcal{J} \left( \eta^h_t \mid \eta^h_\infty \right) \leq \| \exp ((A - RQ_\infty)t) \|^2 \mathcal{J} \left( \eta^h_0 \mid \eta^h_\infty \right) \longrightarrow_{t \to \infty} 0,
\]
which proves the first inequality in the statement of the theorem.

Now, combining this with the de Bruijn identity, we have the log-Sobolev inequality
\[
\text{Ent} \left( \eta^h_0 \mid \eta^h_\infty \right) = \frac{1}{2} \int_0^\infty \mathcal{J} \left( \eta^h_s \mid \eta^h_\infty \right) ds \\
\leq \mathcal{J} \left( \eta^h_0 \mid \eta^h_\infty \right) \frac{1}{2} \int_0^\infty \| \exp ((A - RQ_\infty)s) \|^2 ds.
\]
Finally, applying the log-Sobolev inequality to \( \eta^h_t \), the de Bruijn identity now yields the free energy exponential decays
\[
\partial_t \text{Ent} \left( \eta^h_t \mid \eta^h_\infty \right) = -\frac{1}{2} \mathcal{J} \left( \eta^h_t \mid \eta^h_\infty \right) \leq - \left( \int_0^\infty \| e^{(A-RQ_\infty)s} \|^2 ds \right)^{-1} \text{Ent} \left( \eta^h_t \mid \eta^h_\infty \right).
\]
This ends the proof of the theorem. \( \square \)

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A Riccati Matrix Differential Equations

In this section, we discuss some of the theory behind Riccati matrix differential equations and present some results that are of use throughout the paper. This section is mainly taken from [12].
Recall that

\[ P_\infty - P^-_\infty = \Delta^{-1}_\infty \]  

(97)

is defined via the Gramian matrices

\[ \Delta_t := \int_0^t e^{s(A-P^-\infty S)'} S e^{s(A-P^-\infty S)} ds \rightarrow t \rightarrow \infty \Delta_\infty \]

\[ := \int_0^\infty e^{s(A-P^-\infty S)'} S e^{s(A-P^-\infty S)} ds \in S^+_r. \]  

(98)

Consider now the linear matrix functional

\[ \mathbb{F}_t : P \in S^0_r \mapsto \mathbb{F}_t(P) := \left( \Delta_t^{-1} - \Delta_\infty^{-1} \right) + (P - P^-_\infty) \Delta_t \in \mathcal{G}l_r. \]  

(99)

Rearranging and using (98) implies that

\[ \mathbb{F}_t(P) = I + (P - P^-_\infty) \Delta_t \quad \text{and} \quad \mathbb{F}_t(P^-_\infty) = I. \]  

(100)

Recall that \( Q_\infty \) is defined as \( P^-_\infty \) by replacing \((A, R, S)\) by \((A', S, R)\). In the same vein, \( \Delta^h_t \) is defined as \( \Delta_t \) by replacing \((A, R, S)\) and \( P^-_\infty \) by \((A', S, R)\) and \( Q_\infty \). Thus, by symmetry arguments and (30), we also have

\[ (\Delta^h_\infty)^{-1} = P^-_\infty^{-1} - (P^-_\infty)^{-1} = Q_\infty - Q^-_\infty \quad \text{and} \quad \left( \phi^h_t(0), P^h_\infty \right) = \left( \Delta^h_t, \Delta^h_\infty \right), \]  

(101)

with the Gramian matrices

\[ \Delta^h_t := \int_0^t e^{(A-RQ_\infty)\frac{s}{t}} R e^{(A-RQ_\infty)\frac{s}{t}} ds \rightarrow t \rightarrow \infty \Delta^h_\infty \]

\[ := \int_0^\infty e^{(A-RQ_\infty)\frac{s}{t}} R e^{(A-RQ_\infty)\frac{s}{t}} ds \in S^+_r. \]

The following lemma proves the second equality on the right hand side of (39).

\textbf{Lemma A.1.} \textit{For any \((A, R, S)\) satisfying the rank condition (4), we have}

\[ \text{Tr}(SP^-_\infty) = 2 \text{ Tr}(A) - \text{ Tr}(SP^-_\infty) = \text{ Tr}(Q_\infty R). \]  

(102)

\textbf{Proof.} \textit{The Gramian} \( \Delta_\infty \) satisfies the Sylvester equations given by

\[ (A - P^-_\infty S)' \Delta_\infty + \Delta_\infty (A - P^-_\infty S) + S = 0 = \Delta_\infty^{-1} (A - P^-_\infty S)' + (A - P^-_\infty S) \Delta_\infty^{-1} + \Delta_\infty^{-1} S \Delta_\infty^{-1}. \]

It then follows that

\[ \text{Tr} \left( \Delta_\infty \left( \Delta_\infty^{-1} (A - P^-_\infty S)' + (A - P^-_\infty S) \Delta_\infty^{-1} + \Delta_\infty^{-1} S \Delta_\infty^{-1} \right) \right) = 0 \]

\[ \implies 2 \text{ Tr}(SP^-_\infty) = 2 \text{ Tr}(A) + \text{ Tr}(S \Delta_\infty^{-1}) = 2 \text{ Tr}(A) + \text{ Tr}(SP^-_\infty) - \text{ Tr}(SP^-_\infty). \]
where we have used (97) to obtain the final equality. From this we obtain
\[ \text{Tr}(SP_{\infty}) = 2 \text{Tr}(A) - \text{Tr}(SP_{\infty}). \] (103)
In the same vein, we have
\[ \text{Tr}\left(P_{\infty}^{-1} \text{Ricc}(P_{\infty})\right) = 0 \implies \text{Tr}(SP_{\infty}) = 2 \text{Tr}(A) + \text{Tr}(P_{\infty}^{-1}R) \]
\[ \text{Tr}\left((P_{\infty}^{-1})^{-1} \text{Ricc}(P_{\infty})\right) = 0 \implies \text{Tr}(SP_{\infty}) = 2 \text{Tr}(A) + \text{Tr}((P_{\infty}^{-1})^{-1}R). \]
Combining the last assertion with (103) we conclude that
\[ \text{Tr}(P_{\infty}S) = -\text{Tr}((P_{\infty}^{-1})^{-1}R) = \text{Tr}(Q_{\infty}R), \]
as required. □

A.2 A Floquet-type representation. For any \( P \in S^0_r \) and \( \delta > 0 \) set
\[ \chi(P) := \|P_{\infty}\|^{-1} \left[ \|P_{\infty} - P_{\infty}\| + \|P - P_{\infty}\| \right] \quad \text{and} \quad \chi_\delta := \left[ \lambda_{\min}(\Delta_\delta) \lambda_{\min}(-P_{\infty}) \right]^{-1}. \]
The next theorem provides an explicit description of \( \mathcal{E}_t(P), \) defined in (33), in terms of the matrices \( (A, S, P_{\infty}). \) In what follows, \( \| \cdot \| \) stands for the spectral norm of matrices.

**Theorem A.2** [Floquet-type representation [12]]. For any time horizon \( t \geq 0 \) and any \( P \in S^0_r \) we have Riccati exponential semigroup formula
\[ \mathcal{E}_t(P) = e^{t(A-P_{\infty}S)} \mathcal{F}_t(P)^{-1} = \mathcal{E}_t(P_{\infty}) \mathcal{F}_t(P)^{-1}, \] (104)
where \( \mathcal{F}_t(P) \) was defined in (99). For any \( t \geq \delta > 0 \) we have the uniform estimates
\[ \|\mathcal{F}_t(P)^{-1}\| \leq \chi_\delta \quad \text{and} \quad \|\mathcal{E}_t(P)\| \leq \chi_\delta \|\mathcal{E}_t(P_{\infty})\|. \] (105)
In addition, for any \( t \geq 0 \) we have the exponential estimates
\[ \forall t \geq \delta > 0, \quad 0 \leq \Pi_{\_\delta} \leq \phi_t(P) \leq \Pi_{+_\delta} \quad \text{and} \quad \forall t \geq 0, \quad \|\mathcal{E}_t(P_{\infty})\| \leq \alpha e^{-\beta t}. \] (106)
for some positive matrices \( \Pi_{\_\delta}, \Pi_{+_\delta} \) and some \( \alpha, \beta > 0, \) all of which depend on the model parameters \( (A, R, S). \) Finally, we have the bounds
\[ \|\mathcal{F}_t(P)^{-1}\| \leq \chi(P) \quad \text{and} \quad \|\mathcal{E}_t(P)\| \leq \chi(P) \|\mathcal{E}_t(P_{\infty})\|. \] (107)
From this theorem, we can develop some useful identities. Indeed, using the decomposition
\[ \text{Ricc}(Q_1) - \text{Ricc}(Q_2) = (A - Q_1S)(Q_1 - Q_2) + (Q_1 - Q_2)(A - Q_2S)', \] (108)
for \( Q_1, Q_2 \in S^0_r, \) applying (104) we have the closed form Lipschitz type matrix formula
\[ \phi_t(Q_1) - \phi_t(Q_2) = \mathcal{E}_t(P_{\infty}) \mathcal{F}_t(Q_1)^{-1}(Q_1 - Q_2) \left( \mathcal{E}_t(P_{\infty}) \mathcal{F}_t(Q_2)^{-1} \right)'. \] (109)
Applying (109) with \( Q_2 = P_{\infty} \) and using (100), we recover the Bernstein-Prach-Tekinalp formula [71,72] given by
\[ \phi_t(P) = P_{\infty} + \mathcal{E}_t(P_{\infty}) \mathcal{F}_t(P)^{-1}(P - P_{\infty}) \mathcal{E}_t(P_{\infty})'. \] (110)
A.3 Lipschitz inequalities. Combining Theorem A.2, (106) and (109) we easily obtain the following result.

**Theorem A.3.** For any time horizon $t \geq \delta > 0$ and any $Q_1, Q_2 \in S_r^0$ we have the Lipschitz estimate

$$\|\phi_t(Q_1) - \phi_t(Q_2)\| \leq (\alpha \chi \delta)^2 e^{-2\beta t} \|Q_1 - Q_2\|$$

with the parameters $(\alpha, \beta, \chi, \delta)$ defined in (106) and Theorem A.2. In addition, for any $t \geq 0$ we have the local Lipschitz estimate

$$\|\phi_t(Q_1) - \phi_t(Q_2)\| \leq \alpha^2 \chi(P_1) \chi(P_2) e^{-2\beta t} \|Q_1 - Q_2\|,$$

with the parameters $\chi(P_i)$ defined in Theorem A.2.

Noting that

$$\mathcal{E}_t(Q_1) - \mathcal{E}_t(Q_2) = \mathcal{E}_t(P_\infty) \mathcal{F}_t(Q_1)^{-1} [\mathcal{F}_t(Q_2) - \mathcal{F}_t(Q_1)] \mathcal{F}_t(Q_2)^{-1}$$

$$= \mathcal{E}_t(P_\infty) \mathcal{F}_t(Q_1)^{-1} (Q_2 - Q_1) \Delta_t \mathcal{F}_t(Q_2)^{-1},$$

where $\Delta_t$ was defined in (98), we also obtain the following corollary.

**Corollary A.4.** For any time horizon $t \geq \delta > 0$ and any $Q_1, Q_2 \in S_r^0$ we have the Lipschitz estimate

$$\|\mathcal{E}_t(Q_1) - \mathcal{E}_t(Q_2)\| \leq \alpha \chi \delta^2 \|\Delta_t\| e^{-\beta t} \|Q_1 - Q_2\|$$

with the parameters $(\alpha, \beta, \chi, \delta)$ defined in (106) and Theorem A.2. In addition, for any $t \geq 0$ we have local Lipschitz estimate

$$\|\mathcal{E}_t(Q_1) - \mathcal{E}_t(Q_2)\| \leq \alpha \|\Delta_t\| \chi(Q_1) \chi(Q_2) e^{-\beta t} \|Q_1 - Q_2\|$$

with the parameter $\chi(Q_i)$ defined in Theorem A.2.

The first coordinate of the evolution semigroup (32) can be written as

$$\hat{X}_t(x, P_0) = \mathcal{E}_t(P_0)x$$

Using the decomposition

$$\hat{X}_t(x_1, Q_1) - \hat{X}_t(x_2, Q_2) = (\mathcal{E}_t(Q_1) - \mathcal{E}_t(Q_2))x_1 + \mathcal{E}_t(Q_2)(x_1 - x_2),$$

we readily check the following theorem.

**Theorem A.5.** For any time horizon $t \geq \delta > 0$ and any $Q_1, Q_2 \in S_r^0$ we have the estimate

$$\|\hat{X}_t(x_1, Q_1) - \hat{X}_t(x_2, Q_2)\| \leq \alpha \chi \delta e^{-\beta t} (\chi \delta \|\Delta_t\| \|x_1\| \|Q_1 - Q_2\| + \|x_1 - x_2\|)$$

with the parameters $(\alpha, \beta, \chi, \delta)$ defined in (106) and Theorem A.2. In addition, for any $t \geq 0$ we have the estimate

$$\|\hat{X}_t(x_1, Q_1) - \hat{X}_t(x_2, Q_2)\| \leq \alpha \chi(Q_2) e^{-\beta t} (\chi(Q_1) \|\Delta_t\| \|x_1\| \|Q_1 - Q_2\| + \|x_1 - x_2\|)$$

with the parameter $\chi(Q_i)$ defined in Theorem A.2.
B Path Integral Formulations

In this section, we discuss in further detail the $h$-process and its link to the absorbed particle process $(X^c_t)_{t \geq 0}$. We also discuss how one may extend the methodology used in this article to non-linear diffusions.

**B.1 Backward $h$-processes.** For a fixed time horizon $t \geq 0$, we let $\overline{X}$ be a random sample from $\mathcal{N}(\hat{X}_t, P_t)$. We also denote by $X^h_{t,s}(x)$, with $s \in [0, t]$, be the backward diffusion defined by

$$dX^h_{t,s}(x) = \left( AX^h_{t,s}(x) + RP_s^{-1}(X^h_{t,s}(x) - \hat{X}_s) \right) \, ds + BdW_s,$$

starting at $X^h_{t,t}(x) = x$ at time $s = t$. In the above display, $P_s$ stands for the solution of the Riccati matrix differential equation defined in (27). The backward semigroup property is given for any $u \leq s \in [0, t]$ by the mapping composition formulae

$$X^h_{t,u} = X^h_{s,u} \circ X^h_{t,s} \quad \text{and} \quad X^h_{t,t} = I, \quad \text{the identity function.}$$

We assume that $\overline{X}$, and $(W_s)_{s \leq t}$ are independent.

Rewritten in terms of the density $g_s$ of the Gaussian distribution $\mathcal{N}(\hat{X}_s, P_s)$, we have

$$\overline{X}^h_{t,s} := X^h_{t,s}(\overline{X}_s) \implies d\overline{X}^h_{t,s} = \left( A\overline{X}^h_{t,s} - R \nabla \log g_s(\overline{X}^h_{t,s}) \right) \, ds + B \, dW_s. \quad (111)$$

The following theorem, taken from [8] links the non-absorbed particle process with the above backward diffusion

**Theorem B.1** [ [8]]. Assume that $X_0 \sim \mathcal{N}(\hat{X}_0, P_0)$. In this situation, for any $t \geq 0$ we have the backward formulation of the Feynman–Kac path integral

$$\mathbb{E} \left( F \left( (X^c_s)_{s \in [0, t]} \right) \mid \tau^c \geq t \right) = \mathbb{E} \left( F \left( (\overline{X}^h_{t,s})_{s \in [0, t]} \right) \right).$$

The random state $\overline{X}^h_{t,s}$ is a Gaussian variable with a mean $\hat{\overline{X}}^h_{t,s}$ and covariance matrix $P^h_{t,s}$ satisfying the backward equations

$$\begin{cases} 
\partial_s \overline{X}^h_{t,s} = AX^h_{t,s} + RP_s^{-1}(\overline{X}^h_{t,s} - \hat{X}_s) \\
\partial_s P^h_{t,s} = (A + RP_s^{-1})P^h_{t,s} + P^h_{t,s}(A + RP_s^{-1})' - R
\end{cases}$$

with the terminal condition $(\hat{\overline{X}}^h_{t,t}, P^h_{t,t}) = (\hat{X}, P_t)$, where $(\hat{X}_s, P_s)$ is the solution to the forward equations described in (27).
B.2 Extensions to non-linear diffusions. The $h$-process methodology can be extended to more general generators $L$ and other choices of the potential function $V$. We now assume that $L$ is the generator of the diffusion equation

$$dX_t = A(X_t)dt + B(X_t)dW_t$$

(112)

for some drift function $A(x)$ and some diffusion matrix valued function $B(x)$ with appropriate dimensions. We also assume there exists some ground state $h_0$ associated with some energy $\lambda_0$; that is, we have that

$$h_0^{-1}L(h_0)(x) = V(x) - \lambda_0$$

In this situation, the $h$-process $X^h_t$ is a diffusion with generator defined by

$$L^h(f) = L(f) + h_0^{-1}\Gamma\mathcal{L}(h_0, f)$$

with the carré-du-champ operator

$$\Gamma\mathcal{L}(h_0, f)(x) := (B(x)\nabla h_0(x))^\prime (B(x)\nabla f(x)) = (R(x)\nabla h_0(x))^\prime \nabla f(x),$$

where we have defined $R(x) := B(x)B(x)^\prime$. Equivalently, the $h$-process is defined by the diffusion

$$dX^h_t = \left(A(X^h_t) + R(X^h_t)\nabla \log h_0(X^h_t)\right)dt + B(X^h_t)dW_t$$

Let $\overline{X}_t$ a random sample from the Feynman–Kac probability measures $\eta_t$ defined as in (21) for some potential function $V$.

Whenever it exists, let $g_s$ be the density of the normalised or unnormalised Feynman–Kac measures $\eta_s$ or $\gamma_s$. In this situation, following the analysis developed in [8], the assertion of Theorem B.1 remains valid with the backward diffusion

$$d\overline{X}^h_{t,s} = \left(A(\overline{X}^h_{t,s}) - \text{div}_R \log g_s(\overline{X}^h_{t,s})\right)ds + B(\overline{X}^h_{t,s})dW_s$$

(113)

with the terminal condition $\overline{X}^h_{t,t} = \overline{X}_t$ and the $R$-divergence $m$-column vector operator with $j$-th entry given by the formula

$$\text{div}_R(f)(x)^j := \sum_{1 \leq i \leq r} \partial_{x_i} \left(R_{i,j}(x) f(x)\right).$$

C McKean–Vlasov Interpretations

In this section, we discuss several classes of McKean-Vlasov interpretations of the distribution of a non-absorbed particle. These probabilistic models and their mean field simulation are defined in terms of a nonlinear Markov process that depends on the distribution of the random states so that the flow of distributions of all random states coincides with the conditional distribution of a non-absorbed particle. To the best of our knowledge, these particle models have not been applied to the models considered in this article, despite being applicable in a wide variety of situations. In particular, these methods can be applied in high dimension and even when $A$ is unstable.
C.1 Interacting jump processes. Let $\mathbf{X}_t$ be a nonlinear jump diffusion process with generator

$$\mathcal{L}_\eta_t(f)(x) = \mathcal{L}(f)(x) + V(x) \int (f(y) - f(x)) \eta_t(dy) \quad \text{where} \quad \eta_t := \text{Law}(\mathbf{X}_t).$$

The process starts at $\mathbf{X}_0 = X_0$. Recall that the second order differential kinetic energy operator $\mathcal{L}$ defined in (2) coincides with the generator of the process $X_t$ arising in the Feynman–Kac representation (8). Between the jumps the process $\mathbf{X}_t$ evolves as $X_t$. At rate $V(\mathbf{X}_t)$ the process jumps onto a new location randomly selected according to the distribution $\eta_t$. Observe that

$$\partial_t \eta_t(f) = \eta_t(L_\eta_t(f)) = \eta_t(L(f)) - \eta_t(fV) + \eta_t(f)\eta_t(V).$$

This shows that $\eta_t$ satisfies the same evolution equation as the one satisfied by $\eta_t$ given in (26). Thus, for any choice of the generator $\mathcal{L}$ and any choice of the potential function $V$ we have that

$$\eta_t(dx) = \eta_t(dx) := \mathbb{P}(X^c_t \in dx \mid \tau^c > t).$$

The mean field particle interpretation of the nonlinear process $\mathbf{X}_t$ is defined by a system of $N$ walkers, $\xi^i_t$, evolving independently as $X_t$ with a spatial jump rate $V_t$, for $1 \leq i \leq N$. At each jump time, the particle $\xi^i_t$ jumps onto a particle uniformly chosen in the pool. The occupation measure of system is given by the empirical measure

$$\eta^N_t = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi^i_t} \longrightarrow_{N \rightarrow \infty} \eta_t \longrightarrow_{t \rightarrow \infty} \eta_\infty.$$  \hspace{1cm} (114)

Mimicking (23) we also define the normalising constant approximations

$$\frac{1}{t} \int_0^t \eta^N_s(V)ds := -\frac{1}{t} \log \gamma^N_t(1) \longrightarrow_{N \rightarrow \infty} -\frac{1}{t} \log \gamma_t(1) \longrightarrow_{t \rightarrow \infty} \lambda_0 = \eta_\infty(V).$$  \hspace{1cm} (115)

Observe that the $N$ ancestral lines $\zeta^i_t := (\xi^i_{s,t})_{0 \leq s \leq t}$ of length $t$ of the above genetic-type process can also be seen as a system of $N$ path-valued particles evolving independently as the historical process $Y_t = (X^c_s)_{0 \leq s \leq t}$ of $X_t$, with a jump rate $V_t(X_t)$ that only depends on the terminal state $X_t$ of the ancestral line $Y_t$.

The interacting particle system discussed above belongs to the class of diffusion Monte Carlo algorithms, see for instance the series of articles [17–20], as well as the articles [21,38,61,62,74,75] in the context of ground state calculations and the series of articles [3,34,36–38] in the context of a general class of continuous time Feynman-Kac path integral formulae. The discrete time version of the interacting jump processes discussed above coincides with the genetic-type samplers and the Monte Carlo reconfiguration/selection methodologies discussed in [33,36,38], see also the pioneering articles in the mid-1980s by Hetherington [51] and further extended by Caffarel and his co-authors in the series of articles [17–19], see also Buonaura-Sorella [7] as well as the pedagogical introduction to quantum Monte Carlo by Caffarel-Assaraf [20].
C.2 Interacting diffusions. For any probability measure $\eta$ on $\mathbb{R}^r$ we let $\mathcal{P}_\eta$ denote the $\eta$-covariance

$$\eta \mapsto \mathcal{P}_\eta := \eta \left( (e - \eta(e))[(e - \eta(e))'] \right)$$  \hspace{1cm} (116)$$

where $e(x) := x$ is the identity function and $\eta(f)$ is a column vector whose $i$-th entry is given by $\eta(f_i)$ for some measurable function $f : \mathbb{R}^r \to \mathbb{R}^r$.

We now consider three different nonlinear McKean-Vlasov-type diffusion process,

(1) \hspace{.4cm} \begin{align*}
    d\bar{X}_t &= (A - \mathcal{P}_{\bar{\eta}_t} S) \bar{X}_t \, dt + \mathcal{P}_{\bar{\eta}_t} S^{1/2} \, d\mathcal{W}_t + B \, d\bar{\mathcal{W}}_t, \\
    \partial_t \bar{X}_t &= A \bar{X}_t - \frac{1}{2} \mathcal{P}_{\bar{\eta}_t} S \left( \bar{X}_t + \bar{\eta}_t(e) \right) \, dt + B \, d\bar{\mathcal{W}}_t, \quad (117)
\end{align*}

(2) \hspace{.4cm} \begin{align*}
    d\bar{X}_t &= \left( A \bar{X}_t - \frac{1}{2} \mathcal{P}_{\bar{\eta}_t} S \left( \bar{X}_t + \bar{\eta}_t(e) \right) \right) \, dt + B \, d\bar{\mathcal{W}}_t, \\
    \partial_t \bar{X}_t &= A \bar{X}_t - \frac{1}{2} \mathcal{P}_{\bar{\eta}_t} S \left( \bar{X}_t + \bar{\eta}_t(e) \right) + (R + M_t) \mathcal{P}_{\bar{\eta}_t}^{-1} \left( \bar{X}_t - \bar{\eta}_t(e) \right),
\end{align*}

for any skew symmetric matrix $M_t' = -M_t$ that may also depend on $\bar{\eta}_t$. In all three cases $(\mathcal{W}_t, \bar{\mathcal{W}}_t)$ are independent copies of $\mathcal{W}_0$; and $\bar{X}_0$ is an independent copies of $X_0$. We also assume that $(\mathcal{W}_t, \mathcal{W}_t, \bar{X}_0)$ are independent. In all three cases in (117), $\bar{\eta}_t$ stands for the probability distribution of $\bar{X}_t$; that is, we have that

$$\bar{\eta}_t := \text{Law}(\bar{X}_t).$$  \hspace{1cm} (118)$$

Observe that, in all three cases the stochastic processes discussed above depend in some nonlinear fashion on the law of the diffusion process itself.

**Theorem C.1.** In all the three cases presented in (117), for any $t \geq 0$ we have the Gaussian preserving property

$$\eta_0 = \mathcal{N}(\hat{\mathcal{X}}_0, P_0) = \bar{\eta}_0 \implies \bar{\eta}_t = \mathcal{N}(\hat{\bar{X}}_t, P_t) = \eta_t.$$

**Proof.** Let $\bar{X}_t$ be the process defined as in (1) by replacing $\mathcal{P}_{\bar{\eta}_t}$ by $P_t$. In this case, we have

$$d \left( \bar{X}_t - \mathbb{E}(\bar{X}_t) \right) = (A - P_t S) \left( \bar{X}_t - \mathbb{E}(\bar{X}_t) \right) \, dt + P_t S^{1/2} \, d\mathcal{W}_t + B \, d\bar{\mathcal{W}}_t.$$

Applying Ito’s formula and taking expectations we obtain

$$\partial_t \mathcal{P}_{\bar{\eta}_t} = (A - P_t S) \mathcal{P}_{\bar{\eta}_t} + \mathcal{P}_{\bar{\eta}_t} \left( A - P_t S \right)' + P_t S P_t + R.$$

This yields the linear system

$$\partial_t \left( \mathcal{P}_{\bar{\eta}_t} - P_t \right) = (A - P_t S)(\mathcal{P}_{\bar{\eta}_t} - P_t) + \mathcal{P}_{\bar{\eta}_t} \left( A - P_t S \right)'(\mathcal{P}_{\bar{\eta}_t} - P_t) \implies P_t = \mathcal{P}_{\bar{\eta}_t}.$$

We conclude that $\bar{X}_t$ is a linear diffusion with mean $\hat{\bar{X}}_t$ and covariance matrix $P_t$. The proof for the other two cases follows the same lines of arguments, thus we leave the details to the reader. \hfill \square
The mean-field particle interpretation of the first nonlinear diffusion process in (117) is given by the McKean–Vlasov type interacting diffusion process

\[ d\xi^i_t = (A - P^N_t S) \overline{X}_t \, dt + P^N_t S^{1/2} \, d\mathcal{W}^i_t + B \, d\overline{W}^i_t, \quad i = 1, \ldots, N, \quad (119) \]

where \((\mathcal{W}^i_t, \overline{W}^i_t, \xi^i_0)_{1 \leq i \leq N}\) are \(N\) independent copies of \((\mathcal{W}_t, \overline{W}_t, X_0)\). In the above display, the \(P^N_t\) are the rescaled empirical covariance matrices given by the formulae

\[ P^N_t := \left(1 - \frac{1}{N}\right)^{-1} \mathcal{P}^N_{\eta_t} = \frac{1}{N-1} \sum_{1 \leq i \leq N} \left(\xi^i_t - m^N_t\right)\left(\xi^i_t - m^N_t\right)', \quad (120)\]

with the empirical measures

\[ \eta^N_t := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi^i_t} \quad \text{and the sample mean} \quad m^N_t := \frac{1}{N} \sum_{1 \leq i \leq N} \xi^i_t. \]

Note that (119) is a set of \(N\) stochastic differential equations coupled via the empirical covariance matrix \(P^N_t\). The mean-field particle interpretation of the second and third nonlinear diffusion processes in (117) are defined as above by replacing \(\mathcal{P}_{\eta_t}\) by the sample covariance matrices \(P^N_t\). The quasi-invariant measure \(\eta_\infty\) and the parameter \(\lambda_0\) are computed using the limiting formulae (114) and (115).

The interacting diffusions discussed above belong to the class of Ensemble Kalman filters, see for instance the pioneering article by Evensen [47], the series of articles [9–11], as well as [40,41] and the references therein.

In contrast with the interacting jump process discussed in “Appendix C.1” none of the nonlinear diffusions discussed in (117) can be extended to more general generators \(L\) and other choices of the potential function \(V\).

We end this section with an application of the seminal feedback particle filter methodology recently developed by Mehta and Meyn and their co-authors [81–85] to Feynman–Kac models. Consider the diffusion

\[ d\overline{X}_t = (A(\overline{X}_t) + U_t(\overline{X}_t)) \, dt + B(\overline{X}_t) dW_t, \]

where \(U_t(x)\) is the solution of the Poisson equation

\[ \sum_{1 \leq i \leq r} \frac{1}{g_t(x)} \partial_{x_i} \left(U^i_t(x) g_t(x)\right) = (V(x) - \eta_t(V)), \quad t \geq 0. \]

In the above display \(g_t(x)\) stands for the density of the distribution \(\overline{\eta}_t\) of the random state \(\overline{X}_t\). The generator \(\mathcal{L}_{\overline{\eta}_t}\) of the above time varying diffusion satisfies the equation

\[ \overline{\eta}_t(\mathcal{L}_{\overline{\eta}_t}(f)) = \overline{\eta}_t(\mathcal{L}(f)) + \sum_{1 \leq i \leq r} \int U^i_t(x) \partial_{x_i} f(x) \, g_t(x) \, dx. \]

Integrating by part the last term we obtain the formula

\[ \overline{\eta}_t(\mathcal{L}_{\overline{\eta}_t}(f)) = \overline{\eta}_t(L(f)) - \int f(x) (V(x) - \overline{\eta}_t(V)) \, \overline{\eta}_t(dx), \]
from which we conclude that
\[ \overline{\eta}_t(L \eta_t(f)) = \overline{\eta}_t(L(f)) - \overline{\eta}_t(fV) + \eta_t(f) \overline{\eta}_t(V). \]

This shows that \( \eta_t = \text{Law}(\overline{X}_t) = \eta_t \) coincides with the normalised Feynman–Kac measures.

For linear-Gaussian models we have \( \eta_t = N(\hat{X}_t, P_t) \). Thus, the Poisson equation resumes to the formula
\[
\sum_{1 \leq i \leq r} \partial x_i U_i^t(x) - (x - \hat{X}_t)' P_t^{-1} U_t(x) = \frac{1}{2} (x'Sx - \hat{X}_t'S\hat{X}_t - \text{Tr}(SP_t))
\]
\[
= \frac{1}{2} (x - \hat{X}_t)' S(x - \hat{X}_t).
\]

The solution of the above equation is clearly given by
\[ U_t(x) = -\frac{1}{2} P_t S(x + \hat{X}_t) \implies \sum_{1 \leq i \leq r} \partial x_i U_i^t(x) = \text{Tr}(P_t S). \]

The resulting diffusion coincides with the second case in (117).

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