Determinantal Martingales
and Noncolliding Diffusion Processes

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Abstract

Noncolliding diffusion processes are dynamical extensions of random matrix models such that any spatio-temporal correlation function is expressed by a determinant specified by a single continuous function called the correlation kernel and such processes are said to be determinantal. The time-dependent correlation kernel is a functional of initial configuration of the stochastic process and in order to determine it, we have had to deal with multiple orthogonal functions, which should be prepared depending on the initial configuration. From the view point of probability theory, the noncolliding diffusion processes are interesting, since they are realized as the harmonic transforms of absorbing particle systems in the Weyl chambers. Determinantal structure of correlations has not been clear, however, from this view-point. In the present paper, we show direct connections between harmonic transforms and determinantal correlation functions. In this new approach, spatio-temporal correlation kernels can be determined without constructing any system of orthogonal functions. Key quantities are local martingales, which are stochastic processes preserving their expectations in time. By introducing integral transforms, we prove that the harmonic functions of diffusions, which are used for the harmonic transforms, give determinants of matrices whose elements are all martingales. We call them determinantal martingales. We demonstrate how to calculate them depending on processes and initial configurations and show a variety of spatio-temporal correlation kernels are readily derived from them. In special cases, the present martingales are expressed by using complex diffusion processes.

1 Introduction

Consider a standard Brownian motion (BM) in one-dimension $\Lambda = \mathbb{R}$, $V(t), t \geq 0$ on the probability space $(\Omega, \mathcal{F}, P)$, where the expectation is written as $E[\cdot]$. We introduce a filtration $\{F(t) : 0 \leq t < \infty\}$ defined by $F(t) = \sigma(V(s), s \in [0, t])$. The process $Y(t), t \geq 0$ on $(\Omega, \mathcal{F}, P)$ adapted to $\{F(t) : 0 \leq t < \infty\}$, is said to be a martingale if, for every $0 \leq s <
\( t < \infty, \ E[Y(t)|\mathcal{F}(s)] = Y(s) \) a.s.P. If there exists a nondecreasing sequence \( \{T_n : n \in \mathbb{N}\} \) of stopping times of \( \{\mathcal{F}(t) : 0 \leq t < \infty\} \), \( \mathbb{N} = \{1, 2, \ldots\} \), such that \( Y_n(t) \equiv Y(t \land T_n), t \geq 0 \) is a martingale for every \( n \in \mathbb{N} \) and \( P[\lim_{n \to \infty} T_n = \infty] = 1 \), then we say \( Y(t) \) is a local martingale. BM \( V(t), t \geq 0 \) is a local martingale, while its square \( V(t)^2, t \geq 0 \) is not, since Itô’s formula gives \( dV(t)^2 = 2V(t)dV(t) + dt \). This calculation implies that \( V(t)^2 - t, t \geq 0 \) is a local martingale for it has a stochastic integral representation \( \int_0^t 2V(t_1)dV(t_1) \). Moreover, we can show that

\[
m_n(t, V(t)) = \left(\frac{t}{2}\right)^{n/2} H_n \left(\frac{V(t)}{\sqrt{2t}}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}
\]

give a series of local martingales, where \( H_n(x) = \sum_{j=0}^{[n/2]}(-1)^j n! / \{j!(n-2j)!\} \{2x\}^{n-2j}, n \in \mathbb{N}_0 \) are the Hermite polynomials and thus (1.1) are the monic polynomials of \( V(t) \) of degrees \( n \in \mathbb{N}_0 \) with time-dependent coefficients. The proof is given by Itô’s formula combined with the differential equation for \( y = H_n(x); y'' - 2xy' + 2ny = 0 \) (Lemma 2.1). Actually, we find that (1.1) are the multiple Itô integrals of \( V(t); n! \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dV(t_n) \cdots dV(t_2)dV(t_1), n \in \mathbb{N}_0 \) (see, for instance, [10, 28]).

Next we consider the squared Bessel process with index \( \nu > -1 \) (BESQ\( ^{\nu} \)) in \( \Lambda = \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\} \), which is denoted as \( V^{(\nu)}(t), t \geq 0 \). For \( \nu = D/2 - 1, D \in \mathbb{N}, V^{(\nu)}(\cdot) \) can be defined as a sum of squares of \( D \) independent BMs, \( V_j(\cdot), 1 \leq j \leq D \) such as \( V^{(\nu)}(t) = \sum_{j=1}^D V_j(t)^2, t \geq 0 \). For general \( \nu > -1 \), it is given by the solution of the stochastic differential equation (SDE),

\[
V^{(\nu)}(t) = \int_0^t 2\sqrt{V^{(\nu)}(s)}dV(s) + 2(\nu + 1)t, \quad t \geq 0,
\]

where, if \(-1 < \nu < 0 \), a reflection wall is put at the origin. Even though \( V^{(\nu)}(\cdot) \) itself is not a local martingale, we can ask the question whether some monic polynomials of \( V^{(\nu)}(\cdot) \) with time-dependent coefficients can become local martingales. The answer is affirmatively given by

\[
m^{(\nu)}_n(t, V^{(\nu)}(t)) = (-1)^n n! (2t)^n L_n^{(\nu)} \left(\frac{V^{(\nu)}(t)}{2t}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0,
\]

where \( L_n^{(\nu)}(x) = \sum_{j=0}^{n} (-1)^j \Gamma(n + \nu + 1) / \{\Gamma(\nu + j + 1)(n-j)!\} j! x^j, n \in \mathbb{N}_0 \) are the Laguerre polynomials with index \( \nu \). Since \( L_n^{(\nu)}(x) \) satisfies the differential equation \( xy'' + (\nu + 1 - x)y' + ny = 0 \), Itô’s formula proves that (1.3) have stochastic integral representations (Lemma 2.1).

Let \( \hat{V}(t), t \geq 0 \) be BM or BESQ\( ^{\nu} \), \( \nu > -1 \) mentioned above. For each \( n \in \mathbb{N}_0 \), the polynomial of \( x \) of degree \( n \)

\[
\hat{m}_n(t, x) = x^n + \sum_{j=0}^{n-1} c_{n}^{(j)}(t)x^j, \quad t \geq 0
\]

is determined so that

\[
\hat{m}_n(\cdot, \hat{V}(\cdot)) \quad \text{is a local martingale, and} \\
c_{n}^{(j)}(0) = 0, \quad 0 \leq j \leq n - 1.
\]
If $\hat{V}(\cdot) = V(\cdot)$ (i.e., BM), $\hat{m}_n = m_n$ given by (1.1) and if $\hat{V}(\cdot) = V^{(\nu)}(\cdot)$ (i.e., BESQ$^{(\nu)}$), $\hat{m}_n = m_n^{(\nu)}$ given by (1.3), respectively. The transition probability density of $\hat{V}(\cdot)$ is denoted by $\hat{p}(t, y|x)$ for the transition from $x \in \Lambda$ to $y \in \Lambda$ in time duration $t \geq 0$ (see (2.1)-(2.3)). Put

$$
\theta = \begin{cases} 
\pi/2, & \text{if } \hat{V}(\cdot) = V(\cdot), \\
\pi, & \text{if } \hat{V}(\cdot) = V^{(\nu)}(\cdot),
\end{cases}
$$

(1.6)

and

$$
\hat{q}(t, y|x) = \hat{p}(t, y|e^{-i\theta}x), \quad x, y \in \Lambda, t \geq 0,
$$

(1.7)

where $i = \sqrt{-1}$ (see (2.9)-(2.11)). Then for $(t, x) \in [0, \infty) \times \Lambda$ we can prove that the integral transform of an integrable function $f$

$$
\hat{M}[f(W)|(t, x)] = \int_{\Lambda} dw \hat{q}(t, w|x)f(w),
$$

(1.8)

satisfies the equalities

$$
\hat{m}_n(t, x) = \hat{M}\left[(e^{i\theta}W)^n \, | \, (t, x)\right], \quad \forall n \in \mathbb{N}_0
$$

(1.9)

(Lemma 2.2). Then the following are concluded: Given any polynomial $f$,

$$
\hat{M}[f(e^{i\theta}W)|(t, \hat{V}(t))], t \geq 0 \text{ is a local martingale, and}
$$

$$
\hat{M}[f(e^{i\theta}W)|(0, \hat{V}(0))] = f(\hat{V}(0)).
$$

(1.10) (1.11)

Note that by setting $f \equiv 1$ in (1.11) we have $\hat{M}[1|(t, \hat{V}(t))] \equiv 1, t \geq 0$.

Now we extend the above results to multivariate cases. Let $N \in \mathbb{N}$ and $\{\hat{V}_j(t), t \geq 0 : 1 \leq j \leq N\}$ be a collection of $N$ independent diffusion processes (BM’s or BESQ$^{(\nu)}$’s). We consider the $N$-component vector-valued process $\hat{V}(t) = (\hat{V}_1(t), \ldots, \hat{V}_N(t)), t \geq 0$, for which the initial values are fixed to be $\hat{V}_j(0) = u_j \in \Lambda, 1 \leq j \leq N$ and the probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P}_\mathbf{u})$ with expectation $\mathbb{E}_\mathbf{u}[]$, $\mathbf{u} = (u_1, \ldots, u_N)$. The integral transform (1.8) is extended to the linear integral transform of functions of $\mathbf{x} \in \Lambda^N$ such that, if

$$
F^{(k)}(\mathbf{x}) = \prod_{j=1}^N f_j^{(k)}(x_j)
$$

with integrable functions $f_j^{(k)}, 1 \leq j \leq N, k = 1, 2$, then

$$
\hat{M}\left[F^{(k)}(W) \, | \, \{(t_\ell, x_\ell)\}_{\ell=1}^N\right] = \prod_{j=1}^N \hat{M}\left[f_j^{(k)}(W_j) \, | \, (t_j, x_j)\right], \quad k = 1, 2.
$$

(1.12)

and

$$
\hat{M}\left[c_1 F^{(1)}(W) + c_2 F^{(2)}(W) \, | \, \{(t_\ell, x_\ell)\}_{\ell=1}^N\right]
\quad = c_1 \hat{M}\left[F^{(1)}(W) \, | \, \{(t_\ell, x_\ell)\}_{\ell=1}^N\right] + c_2 \hat{M}\left[F^{(2)}(W) \, | \, \{(t_\ell, x_\ell)\}_{\ell=1}^N\right],
$$

(1.13)

$c_1, c_2 \in \mathbb{C}$, for $0 < t_j < \infty, 1 \leq j \leq N$, where $W = (W_1, \ldots, W_N) \in \Lambda^N$. In particular, if

$t_\ell = t, 1 \leq \ell \leq N$, we write $\hat{M}[]|(t, \mathbf{x})$ simply as $\hat{M}[]|(t, \mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_N)$. 

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In the present paper, we show that the martingales introduced above are very useful to characterize noncolliding diffusion processes, which are strongly correlated interacting particle systems. They give typical examples of one-dimensional log-gases [5, 17]. The noncolliding Brownian motion was introduced by Dyson as the eigenvalue process of Hermitian matrix-valued process [3] and has been also called Dyson’s Brownian motion model (with parameter $\beta = 2$) [31, 24, 7, 9, 23, 33, 12, 14, 15, 25, 18, 26]. The noncolliding BESQ $^{(\nu)}$ was studied by König and O’Connell [19], where it is considered as the generalization of the eigenvalue process of a matrix-valued process called the Laguerre process (or the complex Wishart process). See also [7, 12, 34, 13, 16]. Here we consider these noncolliding diffusion processes.

The important fact on the noncolliding BM and noncolliding $^{(\nu)}$, $\nu > -1$ with $N \in \mathbb{N}$ particles is that they are realized as the harmonic transforms ($h$-transform) of the $N$-component process $\hat{V}(t) = (\hat{V}_1(t), \ldots, \hat{V}_N(t))$, $t \geq 0$ in the Weyl chambers with absorbing walls on the boundaries [7, 9, 19, 12]. There the harmonic function is given by the Vandermonde determinant
\begin{equation}
    h(x) = \det_{1 \leq j, k \leq N} [x_j^{k-1}] = \prod_{1 \leq j < k \leq N} (x_k - x_j),
\end{equation}
and the $h$-transform is performed by multiplying the measures of absorbing processes in time period $[0, T], T < \infty$ by a factor
\begin{equation}
    \frac{h(\hat{V}(T))}{h(u)},
\end{equation}
when the processes start at positions $u = (u_1, \ldots, u_N) \in \Lambda^N$, provided that there is no multiple point; $u_j \neq u_k, 1 \leq j \neq k \leq N$ [7, 19, 12]. By multilinearity of determinant, $\det_{1 \leq j, k \leq N} [x_j^{k-1}]$ does not change by replacing $x_j^{k-1}$ by any monic polynomial of $x_j$ of degree $k - 1, 1 \leq j, k \leq N$. We use our polynomials $\hat{m}_{k-1}(t, x_j)$ with time-dependent coefficients, $1 \leq j, k \leq N$ to make (1.15) be martingale
\begin{equation}
    \frac{1}{h(u)} \det_{1 \leq j, k \leq N} [\hat{m}_{k-1}(T, \hat{V}_j(T))] = \frac{1}{h(u)} \hat{M} \left[ \det_{1 \leq j, k \leq N} [(e^{ijW_j})^{k-1}] (T, \hat{V}(T)) \right] = \hat{M} \left[ \frac{h(e^{ijW})}{h(u)} (T, \hat{V}(T)) \right],
\end{equation}
where $e^{ijW} = e^{ij(W_1, \ldots, W_N)} = (e^{ijW_1}, \ldots, e^{ijW_N})$. Let the initial configuration of the noncolliding diffusion process be expressed by
\begin{equation}
    \xi(\cdot) = \sum_{j=1}^{N} \delta_{u_j}(\cdot),
\end{equation}
where $\delta_{u_j}(\cdot)$ is a delta measure concentrated on a point $u_j \in \Lambda, 1 \leq j \leq N$. Now we use the identity [18]
\begin{equation}
    \frac{h(x)}{h(u)} = \det_{1 \leq j, k \leq N} [\Phi^{x_k}_{\xi}(x_j)], \quad x, u \in \mathbb{C}^N, u_j \neq u_k, 1 \leq j \neq k \leq N,
\end{equation}
where
\[ \Phi^v_\xi(z) = \prod_{r \in \text{supp } \xi \cap \{v\}^c} \frac{z - r}{v - r}, \quad z, v \in \mathbb{C} \]  
with supp \( \xi = \{u_j : 1 \leq j \leq N\} \). Note that
\[ \Phi^u_k(\xi)(u_j) = \delta_{jk}, \quad 1 \leq j, k \leq N. \]  
(See Appendix A for the identity (1.17).) Then the factor for the \( h \)-transform (1.15) is rewritten as
\[ \hat{\mathcal{M}}^u_\xi(T, \hat{\mathbf{V}}(T)) \equiv \hat{\mathcal{M}} \left[ \det_{1 \leq j, k \leq N} [\Phi^u_k(e^{i\theta} W_j)](T, \hat{\mathbf{V}}(T)) \right]. \]  
By the properties (1.12) and (1.13) of the integral transform \( \hat{\mathcal{M}}[] \), multilinearity of determinant, and the independence of \( \hat{V}_j(t), 1 \leq j \leq N \),
\[ \hat{\mathcal{M}}^u_\xi(t, \hat{\mathbf{V}}(t)) = \det_{1 \leq j, k \leq N} [\hat{\mathcal{M}}^u_k(t, \hat{V}_j(t))], \quad 0 < t \leq T < \infty \]  
with
\[ \hat{\mathcal{M}}^u_k(t, x) \equiv \hat{\mathcal{M}}[\Phi^u_k(e^{i\theta} W)(t, x)]. \]  
We should note that, for \( 0 < t \leq T < \infty \),
\[ \mathbb{E}_u \left[ \hat{\mathcal{M}}^u_k(T, \hat{V}_j(T)) \right] = \mathbb{E}_u \left[ \hat{\mathcal{M}}^u_k(t, \hat{V}_j(t)) \right] = \mathbb{E}_u \left[ \hat{\mathcal{M}}^u_k(0, \hat{V}_j(0)) \right] = \mathbb{E}_u \left[ \Phi^u_k(\hat{V}_j(0)) \right] = \Phi^u_k(u_j) = \delta_{jk}, \quad 1 \leq j, k \leq N, \]  
where the property (1.11) of \( \hat{\mathcal{M}}[] \) and (1.19) have been used. We call (1.20) the determinantal martingale in the present paper.

We show that the noncolliding diffusion processes are represented by the collections of independent one-dimensional processes, \( \hat{\mathbf{V}}(t) \), weighted by the determinantal martingales (Theorem 3.1). These determinantal-martingale representations are nontrivial extension of the harmonic transforms. There (1.21)-(1.23) imply reducibility property of determinantal martingale for matrix sizes (Lemma 2.4), while the expressions (1.18) for polynomials will become Weierstrass canonical products for entire functions [22] in proper limits to infinite particle systems. Then the determinantal martingale can be used to analyze also infinite particle systems [18]. (See Proposition 4.1 and examples in Section 4.1).

From the determinantal-martingale representations, we can prove that, when there is no multiple point in \( \xi \), these noncolliding diffusion processes are determinantal in the sense of [15] and the spatio-temporal correlation kernels are given by
\[ \bar{\mathcal{K}}_\xi(s, x; t, y) = \int_{\Lambda} \xi(dv)\tilde{p}(s, x|v)\hat{\mathcal{M}}^v_\xi(t, y) - 1(s > t)\tilde{p}(s - t, x|y), \quad (s, x), (t, y) \in [0, \infty) \times \Lambda, \]  
(1.24)
where $1(\omega)$ is the indicator of $\omega$; $1(\omega) = 1$ if $\omega$ is satisfied, and $1(\omega) = 0$ otherwise (Corollary 3.4). This type of asymmetric correlation kernel was first obtained by Eynard and Mehta [4, 23] and has been extensively studied in multi-matrix models of random matrix theory and stochastic models [24, 6, 33, 23, 2, 5]. Note that in determinantal point processes correlation kernels are usually symmetric [30, 29].

We will generalize the function (1.18) as following. Depending on the transition probability density of a process, $\hat{p}(s, x|v)$, $0 < s < \infty$, $x, v \in \Lambda$ we put

$$\hat{\phi}^\nu_\xi((s, x); z, \zeta) = \frac{\hat{p}(s, x|\zeta)}{\hat{p}(s, x|v)} \frac{1}{z - \zeta} \prod_{r \in \text{supp}\, \xi} \left( \frac{z - r}{\zeta - r} \right)^{\xi(r)}, \quad z, \zeta \in \mathbb{C}. \quad (1.25)$$

Let $\Gamma(\delta_v)$ be a closed contour on the complex plane $\mathbb{C}$ encircling a point $v$ on $\Lambda$ once in the positive direction and set

$$\hat{\Phi}^\nu_\xi((s, x); z) = \frac{1}{2\pi i} \oint_{\Gamma(\delta_v)} d\zeta \hat{\phi}^\nu_\xi((s, x); z, \zeta) = \text{Res} \left[ \hat{\phi}^\nu_\xi((s, x); z, \zeta); \zeta = v \right]. \quad (1.26)$$

This function is defined for any finite configuration $\xi$, in which there can be multiple points in general. (If there is no multiple point, (1.26) is reduced to (1.18).) Since (1.26) is a polynomial with respect to $z$, we can extend $\hat{\mathcal{M}}^\nu_\xi((t, y)|\nu)$ to

$$\hat{\mathcal{M}}^\nu_\xi((s, x)|(t, y)) = \hat{\mathcal{M}} \left[ \hat{\Phi}^\nu_\xi((s, x); e^{i\theta}W) \right]|(t, y), \quad (s, x), (t, y) \in [0, \infty) \times \Lambda. \quad (1.27)$$

Let

$$\xi_s(\cdot) = \sum_{v \in \text{supp}\, \xi} \delta_v(\cdot). \quad (1.28)$$

Then the correlation kernel (1.24) is generalized to

$$\hat{K}_\xi(s, x; t, y) = \int_{\Lambda} \xi_s(dv) \hat{p}(s, x|v) \hat{\mathcal{M}}^\nu_\xi((s, x)|(t, y)) - 1(s > t) \hat{p}(s - t, x|y), \quad (s, x), (t, y) \in [0, \infty) \times \Lambda. \quad (1.29)$$

By definition of the present martingales $\hat{\mathcal{M}}^\nu_\xi((\cdot, \cdot)|(\cdot, \cdot))$, it is written by double integral as

$$\hat{K}_\xi(s, x; t, y) = \frac{1}{2\pi i} \oint_{\Gamma(\xi)} d\zeta \hat{p}(s, x|\zeta) \int_{\Lambda} dw \hat{p}(t, w|e^{-i\theta}y) \frac{1}{e^{i\theta}w - \zeta} \prod_{r \in \text{supp}\, \xi} \left( \frac{e^{i\theta}w - r}{\zeta - r} \right)^{\xi(r)}$$

$$- 1(s > t) \hat{p}(s - t, x|y), \quad (s, x), (t, y) \in [0, \infty) \times \Lambda, \quad (1.30)$$

where $\Gamma(\xi)$ denotes a counterclockwise contour on $\mathbb{C}$ encircling the points in $\text{supp}\, \xi$ on $\Lambda$ but not point $e^{i\theta}w, w \in \Lambda$; $\Gamma(\xi) = \sum_{v \in \text{supp}\, \xi} \Gamma(\delta_v)$. In the previous papers [15, 16], we proved that the noncolliding BM and the noncolliding BESQ$^{(\nu)}$, $\nu > -1$ are determinantal for all

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finite deterministic initial configurations $\xi$ by deriving the double integral representations (1.30) for the spatio-temporal correlation kernels. There we used the multiple orthogonal polynomials [14, 15, 16] in order to obtain the expression (1.30). In the present paper, however, we do not need to use any multiple orthogonal polynomials (see Section 4).

In the special cases, the present martingales can be expressed by using complex processes. Let $W_j(t), t \geq 0, 1 \leq j \leq N$ be a collection of independent BM's on a probability space $(\Omega, \mathcal{F}, \mathcal{P}_0)$, where the expectation w.r.t. $\mathcal{P}_0$ is written as $\mathbb{E}_0[\cdot]$. Note that they are independent from $\hat{V}_j(t), t \geq 0, 1 \leq j \leq N$, and $W_j(0) = 0, 1 \leq j \leq N$. Let the integral transform (1.8) denoted as $M[\cdot; \cdot]$ for BM. Then we can show the following (see (2.21) below). For BM's, $\tilde{V}_j = V_j$, if we introduce a set of independent complex BM's

$$Z_j(t) = V_j(t) + iW_j(t), \quad 1 \leq j \leq N, \quad t \geq 0, \quad \text{(1.31)}$$

then

$$M \left[ \prod_{j=1}^{N} f_j(iW_j) \biggm\{ (t_j, V_j(t_j)) \biggm\} \right]^{N}_{j=1} \biggm| \mathcal{M} \left[ \prod_{j=1}^{N} f_j(Z_j(t_j)) \right] = \mathbb{E}_0 \left[ \prod_{j=1}^{N} f_j(Z_j(t_j)) \right] \quad \text{(1.32)}$$

for polynomials $f_j$'s, and then the determinantal martingale (1.20) is written as

$$\mathcal{M}_T^{\mathbb{u}}(T, V(T)) = \mathbb{E}_0 \left[ \det_{1 \leq j, k \leq N} \left[ \Phi_{\xi j}^{uk}(Z_j(T)) \right] \right] \quad \text{(1.33)}$$

The Bessel process with index $\nu$ (BES$^{(\nu)}$), $\tilde{V}^{(\nu)}(t), t \geq 0$, is defined by

$$\tilde{V}_j^{(\nu)}(t) \equiv \sqrt{V_j^{(\nu)}(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \quad \nu > -1, \quad \text{(1.34)}$$

where $V_j^{(\nu)}(t), t \geq 0$ is BESQ$^{(\nu)}$. Corresponding to this, the integral transform (1.8) for BESQ$^{(\nu)}$, which is denoted as $M[\nu; \cdot; \cdot]$, is converted into that for BES$^{(\nu)}$ expressed as $\tilde{M}[\nu; \cdot; \cdot]$ so that the following relation holds,

$$M[\nu; f(-W)|(t, V^{(\nu)}(t))] = \tilde{M}[\nu; f(iW)|\{ (t_j, \tilde{V}_j^{(\nu)}(t_j)) \}_{j=1}^{N}], \quad t \geq 0, \quad \text{(1.35)}$$

where $f$ and $\tilde{f}$ are polynomials with the relation $\tilde{f}(z) = f(z^2), z \in \mathbb{C}$. We introduce a set of independent complex diffusions

$$Z_j^{(\nu)}(t) = \tilde{V}_j^{(\nu)}(t) + iW_j(t), \quad 1 \leq j \leq N, \quad t \geq 0, \quad \text{(1.36)}$$

where $W_j(t), 1 \leq j \leq N$ are independent BM's on the probability space $(\Omega, \mathcal{F}, \mathcal{P}_0)$. For BES and BESQ with odd dimensions, $D = 2n + 3, n \in \mathbb{N}_0$, the indices are half-odds $\nu = D/2 - 1 = n + 1/2, n \in \mathbb{N}_0$. Let $\tilde{f}(z)$ be a polynomial of $z^2, z \in \mathbb{C}$. For $n \in \mathbb{N}_0$,

$$\tilde{M}[n+1/2; \prod_{j=1}^{N} \tilde{f}_j(iW_j) \biggm\{ (t_j, \tilde{V}_j^{(n+1/2)}(t_j)) \biggm\}^{N}_{j=1}] = \mathbb{E}_0 \left[ \prod_{j=1}^{N} Q_{t_j}^{(n+1/2)}(Z_j^{(n+1/2)}(t_j)) \tilde{f}_j(Z_j^{(n+1/2)}(t_j)) \right], \quad \text{(1.37)}$$
where
\[ Q_t^{(n+1/2)}(z) = \left( \frac{t}{2} \right)^n \frac{z}{(\Re z)^{2n+1}} \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!} \left( \frac{2(\Re z)}{t} \right)^k, \]
\( z \in \mathbb{C} \) (Lemma 2.3). Then we have
\[ \mathcal{M}_{\xi}^{(n+1/2),u}(T, V^{(n+1/2)}(T)) = \tilde{E} \left[ \det_{1 \leq j,k \leq N} \left[ Q_T^{(n+1/2)}(Z_j^{(n+1/2)}(T))\tilde{\Phi}_u^v(Z_j^{(n+1/2)}(T)) \right] \right], \]
where
\[ \tilde{\Phi}_u^v(z) = \prod_{r \in \text{supp} \xi \cap \{u,v\}} \frac{z^2 - r^2}{v^2 - r^2}, \quad z, v \in \mathbb{C}. \]

Following (1.33), the determinantal-martingale representation of the noncolliding BM is reduced to the complex BM representation reported in [18] (Corollary 3.2). Following (1.39) with (1.38), we have a new representation of the noncolliding BES\(^{(n+1/2)}\), \( n \in \mathbb{N}_0 \) by using the complex processes (1.36) (Corollary 3.3).

The paper is organized as follows. In Section 2 polynomial martingales and determinantal martingales are given and their properties are explained. Main results are shown in Section 3. In Section 4 we demonstrate the calculations of martingales and correlation kernels in the present approach. Section 5 is devoted to proofs of theorems. An appendix is given for the determinantal identity (1.17), which plays a key role in the present study.

Generalization of the present work to other processes will be an interesting future problem [28, 11].

2 Martingales

2.1 Transition probability densities

The transition probability densities of BM and BES\(^{(\nu)}\) are given by
\[ p(t, y|x) = \left\{ \begin{array}{ll}
p(t, y|x), & \text{for } \hat{V}(\cdot) = \hat{V}^{(\nu)}(\cdot), \\
p^{(\nu)}(t, y|x), & \text{for } \hat{V}(\cdot) = \hat{V}(\cdot), \\
\end{array} \right. \]
where
\[ p(t, y|x) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}, & t > 0, x, y \in \mathbb{R} \\
\delta(y-x), & t = 0, x, y \in \mathbb{R}, \\
\end{array} \right. \]
and
\[ p^{(\nu)}(t, y|x) = \left\{ \begin{array}{ll}
\frac{1}{2t} \left( \frac{y}{x} \right)^{\nu/2} e^{-\frac{x+y}{2t}} I_{\nu} \left( \frac{\sqrt{xy}}{t} \right), & t > 0, x > 0, y \in \mathbb{R}_+, \\
\frac{y^{\nu}}{(2t)^{\nu+1} \Gamma(\nu+1)} e^{-y/2t}, & t > 0, x = 0, y \in \mathbb{R}_+, \\
\delta(y-x), & t = 0, x, y \in \mathbb{R}_+, \\
\end{array} \right. \]
if $-1 < \nu < 0$, the origin is assumed to be reflecting. Here $I_\nu(x)$ is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}$$  \hspace{1cm} (2.4)$$

with the Gamma function $\Gamma(z) = \int_0^{\infty} e^{-u}u^{z-1}du$, $\Re u > 0$. The following integral representations are useful,

$$p(t, y|x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda e^{-\lambda t/2+i\lambda(y-x)}, \quad x, y \in \mathbb{R}, \quad t \geq 0, \hspace{1cm} (2.5)$$

$$p^{(\nu)}(t, y|x) = \frac{1}{4} \left(\frac{x}{y}\right)^{\nu/2} \int_{\mathbb{R}^+} d\lambda e^{-\lambda t/2} J_\nu(\sqrt{\lambda}x)J_\nu(\sqrt{\lambda}y), \quad x, y \in \mathbb{R}^+, \quad t \geq 0, \hspace{1cm} (2.6)$$

where $J_\nu(x)$ is the Bessel function defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{2n+\nu}. \hspace{1cm} (2.7)$$

For BES$(\nu)$, defined by (1.34), the transition probability density is obtained from (2.2) as

$$\tilde{p}^{(\nu)}(t, y|x) = p^{(\nu)}(t, y^2|x^2)2y$$

$$= \begin{cases} 
\frac{1}{2} \frac{y^{\nu+1}}{t^{x^\nu}} \exp \left( -\frac{x^2 + y^2}{2t} \right) I_\nu \left( \frac{xy}{t} \right), & t > 0, x > 0, y \in \mathbb{R}^+, \\
\frac{y^{2\nu+1}}{2^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}, & t > 0, x = 0, y \in \mathbb{R}^+, \\
\delta(y-x), & t = 0, x, y \in \mathbb{R}^+. 
\end{cases} \hspace{1cm} (2.8)$$

### 2.2 Polynomial martingales

**Lemma 2.1** The polynomials (1.1) of BM and those (1.3) of BESQ$(\nu)$, $\nu > -1$, are local martingales.

**Proof.** $m_0(t, V(t)) = m_0^{(\nu)}(t, V^{(\nu)}(t)) \equiv 1$. By Itô’s formula, for $n \geq 1$,

$$dm_n(t, V(t)) = \left[ \frac{n t^{n/2-1}}{2} H_n \left( \frac{V(t)}{\sqrt{2t}} \right) + \left( \frac{t}{2} \right)^{n/2} H_n' \left( \frac{V(t)}{\sqrt{2t}} \right) \left( -\frac{V(t)}{(2t)^{3/2}} \right) \right] dt$$

$$+ \left( \frac{1}{2} \right)^{n/2} H_n' \left( \frac{V(t)}{\sqrt{2t}} \right) \frac{1}{\sqrt{2t}} dV(t) + \frac{1}{2} \left( \frac{t}{2} \right)^{n/2} H_n'' \left( \frac{V(t)}{\sqrt{2t}} \right) \frac{1}{2t} dt$$

$$= \frac{1}{2} \left( \frac{t}{2} \right)^{(n-1)/2} H_n' \left( \frac{V(t)}{\sqrt{2t}} \right) dV(t) + A_n(t) dt.$$
Here we find
\[ A_n(t) = \frac{t^{n/2-1}}{2n^{3/2}} \left[ H''_n \left( \frac{V(t)}{\sqrt{2t}} \right) - \sqrt{\frac{2}{t}} V(t)H'_n \left( \frac{V(t)}{\sqrt{2t}} \right) + 2nH_n \left( \frac{V(t)}{\sqrt{2t}} \right) \right] = 0, \]
since \( H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0, n \geq 0. \)

By (1.2), the quadratic variation of BESQ\(^{(\nu)}\) is \( \langle V^{(\nu)} \rangle_t = 4 \int_0^t V^{(\nu)}(s)ds, t \geq 0, \nu > -1. \) Then, for \( n \geq 1, \) Itô’s formula gives
\[
dm^{(\nu)}(t, V^{(\nu)}) = \left( -1 \right)^n n! \left[ 2n^2t^{n-1}L^{(\nu)}_n \left( \frac{V^{(\nu)}(t)}{2t} \right) + (2t)^n L^{(\nu)'}_n \left( \frac{V^{(\nu)}(t)}{2t} \right) \left( -\frac{V^{(\nu)}(t)}{2t^2} \right) \right]
\]
\[ + (2t)^n L^{(\nu)'_n} \left( \frac{V^{(\nu)}(t)}{2t} \right) \left( 2\sqrt{\frac{V^{(\nu)}(t)}{2t}}V^{(\nu)}(t)dt + 2(\nu + 1)dt \right) \]
\[ + \frac{1}{2} \left( -1 \right)^n n! (2t)^n L^{(\nu)''_n} \left( \frac{V^{(\nu)}(t)}{2t} \right) \left( \frac{1}{(2t)^2}4V^{(\nu)}(t)dt \right) \]
\[ = \left( -1 \right)^n n! 2^n t^{n-1} \sqrt{\frac{V^{(\nu)}(t)}{2t}}L^{(\nu)'}_n \left( \frac{V^{(\nu)}(t)}{2t} \right) dV(t) + A_n^{(\nu)}(t)dt \]
with
\[ A_n^{(\nu)}(t) = \left( -1 \right)^n n! 2^n t^{n-1} \left[ \frac{V^{(\nu)}(t)}{2t}L^{(\nu)''_n} \left( \frac{V^{(\nu)}(t)}{2t} \right) \right. \]
\[ + \left( \nu + 1 - \frac{V^{(\nu)}(t)}{2t} \right) L^{(\nu)'}_n \left( \frac{V^{(\nu)}(t)}{2t} \right) + nL^{(\nu)}_n \left( \frac{V^{(\nu)}(t)}{2t} \right) \right]. \]

Since \( xL''_n(x) + (\nu + 1 - x)L'_n(x) + nL_n(x) = 0, A_n^{(\nu)}(t) = 0, n \geq 1. \) Then (1.1) and (1.3) are given by stochastic integrals
\[ m_n(t, V(t)) = \frac{1}{2(n+1)^{3/2}} \int_0^t s^{(n-1)/2}H'_n \left( \frac{V(s)}{\sqrt{2s}} \right) dV(s), \]
\[ m_n^{(\nu)}(t, V^{(\nu)}(t)) = \left( -1 \right)^n n! 2^n \int_0^t s^{n-1} \sqrt{\frac{V^{(\nu)}(s)}{2s}}L^{(\nu)'}_n \left( \frac{V^{(\nu)}(s)}{2s} \right) dV(s), \quad t \geq 0, \quad n \geq 1. \]
The proof is thus completed.

Relations between the Hermite polynomials and BM have been well studied. See [28] and references therein.

By (1.6) and (1.7), (2.1)-(2.3) give the following,
\[
\hat{q}(t, y|x) = \begin{cases} 
q(t, y|x), & \text{for } \hat{V}(\cdot) = V(\cdot), \\
q^{(\nu)}(t, y|x), & \text{for } \hat{V}(\cdot) = V^{(\nu)}(\cdot), 
\end{cases}
\]
with
\[ q(t, y|x) = \begin{cases} 
\frac{1}{\sqrt{2\pi t}} e^{-(ix+y)^2/2t}, & t > 0, x, y \in \mathbb{R} \\
\delta(y-x), & t = 0, x, y \in \mathbb{R}, 
\end{cases} 
\]
(2.10)
\[
q^{(\nu)}(t, y|x) = \begin{cases} 
\frac{1}{2t} \left( \frac{y}{x} \right)^{\nu/2} \exp \left( -\frac{(-x)+y}{2t} \right) J_\nu \left( \frac{\sqrt{xy}}{t} \right), & t > 0, x > 0, y \in \mathbb{R}_+, \\
\frac{y^\nu}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-y/2t}, & t > 0, x = 0, y \in \mathbb{R}_+, \\
\delta(y-x), & t = 0, x, y \in \mathbb{R}_+.
\end{cases}
\]

Here \(J_\nu(x)\) is the Bessel function (2.7). As usual we define \(z^\nu\) to be \(\exp(\nu \log z)\), where the argument of \(z\) is given by its principal value;

\[
z^\nu = \exp \left[ \nu \left\{ \log |z| + \sqrt{-1 \arg(z)} \right\} \right], \quad -\pi < \arg(z) \leq \pi.
\]

In order to obtain (2.11) from (2.3), we have used the relation

\[
I_\nu(z) = \begin{cases} 
e^{-\nu i/2} J_\nu(iz), & -\pi < \arg(z) \leq \pi/2, \\
\nu^{2}\pi i/2 J_\nu(iz), & \pi/2 < \arg(z) \leq \pi.
\end{cases}
\]

Therefore, for BM, \(V(t), t \geq 0\), the integral transform (1.8) is given by

\[
M[f(W)|(t,x)] = \int_{\mathbb{R}} dw q(t,w|x)f(w), \quad (t,x) \in [0,\infty) \times \mathbb{R},
\]

and for BESQ\(^{(\nu)}\), \(V^{(\nu)}(t), t \geq 0, \nu > -1\), it is given by

\[
M^{(\nu)}[f(W)|(t,x)] = \int_{\mathbb{R}^+} dw q^{(\nu)}(t,w|x)f(w), \quad (t,x) \in [0,\infty) \times \mathbb{R}_+,
\]

where \(f\) is an integrable function.

**Lemma 2.2** The integral transforms (2.12) with (2.10) and (2.13) with (2.11) satisfy the equalities

\[
M[(iW)^n|(t,x)] = m_n(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},
\]

\[
M^{(\nu)}[(-W)^n|(t,x)] = m_n^{(\nu)}(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}_+,
\]

\forall n \in \mathbb{N}_0, where \(m_n\) and \(m_n^{(\nu)}\), \(\nu > -1\) are given by (1.1) and (1.3), respectively.

**Proof.** Since \(p(t,|x|)\) solves the diffusion equation, \(q(t,|x|) = p(t,|e^{-i\theta}x|)\) satisfies \(\partial q/\partial t = (e^{-i\theta})^2(1/2)\partial^2 q/\partial x^2\). Then Itô’s formula implies

\[
dM[f(W)|(t,V(t))] = \int_{\mathbb{R}} dw dq(t,w|V(t))f(w)
= \left[ \int_{\mathbb{R}} dw \left\{ \frac{\partial q}{\partial t}(t,w|V(t)) + \frac{1}{2} \frac{\partial^2 q}{\partial x^2}(t,w|V(t)) \right\} f(w) \right] dt
+ \left\{ \int_{\mathbb{R}} dw \frac{\partial q}{\partial x}(t,w|V(t))f(w) \right\} dV(t)
= \left[ \int_{\mathbb{R}} dw \frac{\partial q}{\partial x}(t,w|V(t))f(w) \right] dV(t),
\]
if \((e^{-i\theta})^2 = -1 \iff \theta = \pm \pi/2\). Similarly, since \(q^{(\nu)}(t, \cdot | x) = p^{(\nu)}(t, \cdot | e^{-i\theta} x)\) satisfies
\[
\frac{\partial q^{(\nu)}}{\partial t} = e^{-i\theta} \left\{ 2x \frac{\partial^2 q^{(\nu)}}{\partial x^2} + 2(\nu + 1) \frac{\partial q^{(\nu)}}{\partial x} \right\},
\]
\(M^{(\nu)}[f(W)|(t, V^{(\nu)}(t))]\) is a local martingale, if \(e^{-i\theta} = -1 \iff \theta = \pi\). Therefore, the assignment (1.6) of the value \(\theta\) guarantees that \(\hat{M}[f(W)|(t, \hat{V}(t))\) is a local martingale. The Hermite polynomials have the following integral representations,
\[
H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} du e^{-(ix+u)^2} (iu)^n, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}, \quad \text{(2.16)}
\]
\[
= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} du e^{-u^2} (x+iu)^n. \quad \text{(2.17)}
\]
The formula (2.16) gives (2.14) with appropriate change of variables. Similarly the integral representations of the Laguerre polynomials in terms of Bessel functions (for instance, see Eq.(6.2.15) in [1]),
\[
L^{(\nu)}_n(x) = \frac{1}{n!} \frac{e^x}{x^{\nu/2}} \int_0^\infty du e^{-u^{\nu+1/2}} J_\nu(2\sqrt{ux}), \quad n \in \mathbb{N}_0, \quad \nu > -1, \quad x \in \mathbb{R}_+, \quad \text{(2.18)}
\]
give (2.15).

The integral formula (2.17) implies
\[
m_n(t, x) = \hat{E}_0[(x + iW(t))^n], \quad n \in \mathbb{N}_0, \quad (t, x) \in [0, \infty) \times \mathbb{R}, \quad \text{(2.19)}
\]
where \(\hat{E}_0\) denotes the expectation of BM, \(W(t), t \geq 0\) started at \(W(0) = 0\). Then, if we consider the complex BM,
\[
Z(t) = V(t) + iW(t), \quad t \geq 0, \quad \text{(2.20)}
\]
then
\[
m_n(t, V(t)) = M[(iW)^n|(t, V(t))] = \hat{E}_0[Z(t)^n], \quad t \geq 0, \quad n \in \mathbb{N}. \quad \text{(2.21)}
\]
As a matter of course, the map \(z \to z^n, z \in \mathbb{C}, n \in \mathbb{N}\) are analytic, and then \(Z(t)^n, t \geq 0, n \in \mathbb{N}_0\) are conformal maps of \(Z(t), t \geq 0\). Since the probability distribution of the complex BM is conformal invariant, \(Z(t)^n, t \geq 0, n \in \mathbb{N}\) are time changes of \(Z(t)\). In other words, \(Z(\cdot)^n, n \in \mathbb{N}\) are conformal local martingales (see Section V.2 of [27]). Since \(V(\cdot) = \Re Z(\cdot)\) and \(W(\cdot) = \Im Z(\cdot)\) are independent one-dimensional standard BM’s, \(m_n(\cdot, V(\cdot)), n \in \mathbb{N}\), which are obtained by taking the average over the imaginary parts of \(Z(\cdot)^n\) as (2.21) are also local martingales. This consideration will give another proof of (2.14) of Lemma 2.2. Moreover, from (2.21) we can conclude the relation (1.32) and (1.33) in Section 1.
For BES\((\nu)\), we set
\[
\tilde{q}^{(\nu)}(t, y|x) = q^{(\nu)}(t, y^2|x^2)2y \\
= \begin{cases} 
\frac{1}{t} \frac{y^{\nu+1}}{x^\nu} \exp \left( -\frac{(x^2) + y^2}{2t} \right) J_{\nu} \left( \frac{xy}{t} \right), & t > 0, x > 0, y \in \mathbb{R}_+, \\
\frac{y^{2\nu+1}}{2^\nu t^{\nu+1} \Gamma(\nu + 1)} e^{-y^2/2t}, & t > 0, x = 0, y \in \mathbb{R}_+, \\
\delta(y - x), & t = 0, x, y \in \mathbb{R}_+. 
\end{cases}
\] (2.22)
and define the integral transform for BES\((\nu), \nu > -1\) by
\[
\tilde{M}^{(\nu)}[f(W)|(t, x)] = \int_{\mathbb{R}_+} dw \tilde{q}^{(\nu)}(t, w|x)f(w), \quad (t, x) \in [0, \infty) \times \mathbb{R}_+ \tag{2.23}
\]
for an integrable function \(f\). Then the relation (1.35) is satisfied. For \(m \in \mathbb{N}_0\), the Bessel functions have the following expansions by the trigonometric functions,
\[
J_{2m+1/2}(x) = (-1)^m \sqrt{\frac{2}{\pi x}} \left[ \sin x \sum_{k=0}^{m} \frac{(-1)^k(2m + 2k)!}{(2k)!(2m - 2k)!} (2x)^{-2k} \right. \\
\left. + \cos x \sum_{k=0}^{m-1} \frac{(-1)^k(2m + 2k + 1)!}{(2k + 1)!(2m - 2k - 1)!} (2x)^{-(2k+1)} \right],
\]
\[
J_{2m+3/2}(x) = (-1)^m \sqrt{\frac{2}{\pi x}} \left[ -\cos x \sum_{k=0}^{m} \frac{(-1)^k(2m + 2k + 1)!}{(2k)!(2m - 2k + 1)!} (2x)^{-2k} \right. \\
\left. + \sin x \sum_{k=0}^{m} \frac{(-1)^k(2m + 2k + 2)!}{(2k + 1)!(2m - 2k)!} (2x)^{-(2k+1)} \right] . \tag{2.24}
\]
They are obtained from Eq.(4.6.12) in [1]. For example, if we set \(m = 0\) in (2.24), we have
\[
J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right).
\]
Assume that \(\tilde{f}(z)\) is a polynomial of \(z^2\), and thus
\[
\tilde{f}(-z) = \tilde{f}(z), \quad z \in \mathbb{C}. \tag{2.25}
\]
Then (2.23) with (2.22) gives
\[
\tilde{M}^{(1/2)} \left[ \tilde{f}(iW) \right](t, x) = \int_{\mathbb{R}_+} dw \sqrt{\frac{2}{\pi t x}} e^{-(-x^2+w^2)/2t} \sin(xw/t) \tilde{f}(iw) \\
= \frac{1}{\sqrt{2\pi t ix}} \int_{\mathbb{R}_+} dw w \left\{ e^{-(w-ix)^2/2t} - e^{-(w+ix)^2/2t} \right\} \tilde{f}(iw),
\]
for an integrable function \(f\). Then the relation (1.35) is satisfied. For \(m \in \mathbb{N}_0\), the Bessel functions have the following expansions by the trigonometric functions,
and
\[
\tilde{M}^{(3/2)} \left[ \tilde{f}(iW) \right] (t, x) = \left\{ \begin{aligned}
&= \int_{\mathbb{R}_+} dw \sqrt{\frac{2}{\pi t}} e^{-\frac{t}{x^2}} \sin(xw/t) - \cos(xw/t) \bigg\{ \frac{t}{xw} \sin(xw/t) - \cos(xw/t) \bigg\} \tilde{f}(iw) \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}_+} dw \left\{ e^{-(w-ix)^2/2t} - e^{-(w+ix)^2/2t} \right\} \\
&\quad - \frac{1}{x^2} \int_{\mathbb{R}_+} dw w^2 \left\{ e^{-(w-ix)^2/2t} + e^{-(w+ix)^2/2t} \right\} \tilde{f}(iw).
\end{aligned} \right.
\]

By the assumption (2.25), they are rewritten as
\[
\tilde{M}^{(1/2)} \left[ \tilde{f}(iW) \right] (t, x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} dw \frac{1}{(i,t)} x^3 \int_{\mathbb{R}_+} dw \left\{ e^{-(w+ix)^2/2t} \tilde{f}(iw) \right\}
\]
\[
\tilde{M}^{(3/2)} \left[ \tilde{f}(iW) \right] (t, x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}_+} dw \left\{ t(x + iu) \right\quad \frac{x^3}{x^2} + \frac{(x + iu)^2}{x^2} \right\} e^{-u^2/2t} \tilde{f}(x + iu),
\]

where we have changed the integral variables by \( u = w + ix \). Since the integrands are entire, \( \int_{-\infty+ix}^{\infty+ix} du \cdot z \) can be replaced by \( \int_{\mathbb{R}} du \cdot z \). Then we have the following expressions for the martingales (1.35) with \( \nu = 1/2 \) and 3/2
\[
\tilde{M}^{(1/2)} \left[ \tilde{f}(iW) \right] (t, \tilde{V}^{(1/2)}(t)) = \tilde{E}_0 \left[ \frac{Z^{(1/2)}(t)}{\mathbb{R}Z^{(1/2)}(t)} \tilde{f}(Z^{(1/2)}(t)) \right], \quad t \geq 0,
\]
\[
\tilde{M}^{(3/2)} \left[ \tilde{f}(iW) \right] (t, \tilde{V}^{(1/2)}(t)) = \tilde{E}_0 \left[ \left\{ \frac{tZ^{(3/2)}(t)}{(\mathbb{R}Z^{(3/2)}(t))^3} + \frac{(Z^{(3/2)}(t))^2}{(\mathbb{R}Z^{(3/2)}(t))^2} \right\} \tilde{f}(Z^{(3/2)}(t)) \right], \quad t \geq 0, \tag{2.26}
\]

where \( Z^{(\nu)}(t) = \tilde{V}^{(\nu)}(t) + iW(t) \), \( \nu = 1/2, 3/2 \). These calculations are generalized as follows.

**Lemma 2.3** Let \( \tilde{f}(z) \) be a polynomial of \( z^2 \). Then \( \tilde{M}^{(n+1/2)}[\tilde{f}(iW)](\cdot, \tilde{V}^{(n+1/2)}(\cdot)) \), \( n \in \mathbb{N}_0 \), are local martingales, and
\[
\tilde{M}^{(n+1/2)} \left[ \tilde{f}(iW) \right] (t, \tilde{V}^{(n+1/2)}(t)) = \tilde{E}_0 \left[ Q_t^{(n+1/2)}(Z^{(n+1/2)}(t)) \tilde{f}(Z^{(n+1/2)}(t)) \right], \quad t \geq 0, \tag{2.27}
\]
where
\[ Z^{(n+1/2)}(t) = \hat{\nu}^{(n+1/2)}(t) + iW(t), \quad t \geq 0, \]  
and \( Q_t^{(n+1/2)}(\cdot) \) is given by \((1.38)\).

From this lemma we can conclude the relation \((1.37)\). Note that the equalities \((2.21)\) with \((1.1)\) hold even if we replace the complex BM, \((2.20)\) by the present complex diffusion, \((2.28)\), since the imaginary parts are the same;
\[ \hat{E}_0[(Z^{(n+1/2)}(t))^k] = m_k(t, \hat{\nu}^{(n+1/2)}(t)), \quad t \geq 0, \quad n, k \in \mathbb{N}_0. \]  
Then if \( \tilde{f}(\cdot) \) are monomials, \((2.27)\) are also written as follows. For \( n, \ell \in \mathbb{N}_0, \)
\[ \hat{\mathcal{M}}^{(n+1/2)} \left[ (iW)^{2\ell} \left( (t, \hat{\nu}^{(n+1/2)}(t)) \right) \right] \]
\[ = \left( \frac{t}{2} \right)^n \frac{1}{(\hat{\nu}^{(n+1/2)}(t))^{2n+1}} \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!} \left( \frac{2\hat{\nu}^{(n+1/2)}(t)}{t} \right)^k \hat{E}_0[(Z^{(n+1/2)}(t))^{2\ell+k+1}] \]
\[ = \frac{1}{2^{2n+1}} \left( \frac{t}{2} \right)^{\ell} \left( \frac{\hat{\nu}^{(n+1/2)}(t)}{\sqrt{2t}} \right)^{-(2n+1)} \]
\[ \times \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!} \left( \frac{2\hat{\nu}^{(n+1/2)}(t)}{\sqrt{2t}} \right)^k H_{2\ell+k+1} \left( \frac{\hat{\nu}^{(n+1/2)}(t)}{\sqrt{2t}} \right). \]  

### 2.3 Reducibility of determinantal martingales

Assume that \( f_j, 1 \leq j \leq N \) are polynomials. Then by \((1.12)\) and \((1.13)\), the multivariate extension of integral transform has the following properties.
\[ \mathcal{M} \left[ f_j(e^{i\theta}W_j)(\cdot, \hat{V}_j(\cdot)) \right], 1 \leq j \leq N \quad \text{are independent local martingales}. \]  
\[ \mathbb{E}_u \left[ \mathcal{M} \left[ \prod_{j=1}^{N} f_j(e^{i\theta}W_j) \left\{ (\hat{V}_j(T), T) \right\}_{j=1}^{N} \right] \right] = \prod_{j=1}^{N} f_j(u_j), \quad 0 < T < \infty. \]  

Let \( 1 \leq N' \leq N \). With \( 0 < t_j \leq T < \infty \) and measurable functions \( g_j, 1 \leq j \leq N' \),
\[ \mathbb{E}_u \left[ \prod_{j=1}^{N'} g_j(\hat{V}_j(t_j)) \mathcal{M} \left[ \prod_{j=1}^{N} f_j(e^{i\theta}W_j) \left\{ (\hat{V}_j(T), T) \right\}_{j=1}^{N} \right] \right] \]
\[ = \mathbb{E}_u \left[ \prod_{j=1}^{N'} g_j(\hat{V}_j(t_j)) \mathcal{M} \left[ \prod_{j=1}^{N'} f_j(e^{i\theta}W_j) \left\{ (t_j, \hat{V}_j(t_j)) \right\}_{j=1}^{N'} \right] \right] \times \prod_{k=N'+1}^{N} f_k(u_k). \]  

For \( p \in \mathbb{N} \), let \( \mathbb{I}_p = \{1, 2, \ldots, p\} \). Let \( \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{C}^N \) and \( 1 \leq N' \leq N \). In this paper the cardinality of a finite set \( S \) is denoted by \#S. We write \( \mathbb{J} \subset \mathbb{I}_N, \#\mathbb{J} = N' \), if \( \mathbb{J} = \{j_1, \ldots, j_{N'}\}, 1 \leq j_1 < \cdots < j_{N'} \leq N \), and put \( \mathbf{x}_j = (x_{j_1}, \ldots, x_{j_{N'}}) \). In particular, we write \( \mathbf{x}_{N'} = \mathbf{x}_{i_{N'}}, 1 \leq N' \leq N \). By the properties \((2.33)\) and \((1.23)\) we have the following.
Lemma 2.4 Assume that $\xi(\cdot) = \sum_{j=1}^{N} \delta_{u_{j}}(\cdot)$. Let $1 \leq N' \leq N$. For $0 < t \leq T < \infty$ and a measurable function $G_{N'}$ on $\Lambda^{N'}$,  
\[
\sum_{J \subset I, \sharp J = N'} \mathbb{E}_{u} \left[ G_{N'}(\hat{V}_{J}(t)) \hat{\mathcal{M}}_{\xi}^{u}(T, \hat{V}(T)) \right] = \int_{\mathbb{W}^{N'}} \xi^{\otimes N'}(dv) \mathbb{E}_{v} \left[ G_{N'}(\hat{V}_{N'}(t)) \hat{\mathcal{M}}_{\xi}^{u_{N'}}(T, \hat{V}_{N'}(T)) \right]. \tag{2.34}
\]

Proof. By (1.21) with (1.22),  
\[
(\text{LHS}) \text{ of (2.34)} = \sum_{J \subset I, \sharp J = N'} \mathbb{E}_{u} \left[ G_{N'}(\hat{V}_{J}(t)) \det_{j,k \in I, N} \left[ \hat{\mathcal{M}}_{\xi}^{u_{k}}(T, \hat{V}_{j}(T)) \right] \right].
\]

We use the martingale properties (2.33) and (1.23) to find that it is equal to  
\[
\sum_{J \subset I, \sharp J = N'} \mathbb{E}_{u} \left[ G_{N'}(\hat{V}_{J}(t)) \det_{j,k \in I, N} \left[ \hat{\mathcal{M}}_{\xi}^{u_{k}}(T, \hat{V}_{j}(T)) \right] \right] = \int_{\mathbb{W}^{N'}} \xi^{\otimes N'}(dv) \mathbb{E}_{v} \left[ G_{N'}(\hat{V}_{N'}(t)) \det_{j,k \in I, N'} \left[ \hat{\mathcal{M}}_{\xi}^{u_{k}}(T, \hat{V}_{j}(T)) \right] \right].
\]

It is the RHS of (2.34). \qed

Lemma 2.4 shows the reducibility of the determinantal martingale in the sense that, if we observe a symmetric function depending on $N'$ particles, $N' \leq N$, then the size of determinantal martingale is reduced from $N$ to $N'$.

3 Main Results

3.1 Noncolliding diffusion processes

For $N \in \mathbb{N}$, we consider $N$-particle systems of BM’s, $X(t) = (X_{1}(t), X_{2}(t), \ldots, X_{N}(t))$, $t \geq 0$, and of BESQ with index $\nu > 1$, $X^{(\nu)}(t) = (X_{1}^{(\nu)}(t), X_{2}^{(\nu)}(t), \ldots, X_{N}^{(\nu)}(t))$, $t \geq 0$, both conditioned never to collide with each other particle. The former process, which is called the noncolliding BM, solves the following set of SDEs  
\[
dX_{j}(t) = dB_{j}(t) + \sum_{1 \leq k \leq N, k \neq j} \frac{dt}{X_{j}(t) - X_{k}(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \tag{3.1}
\]

with independent one-dimensional standard BMs, $B_{j}(t), 1 \leq j \leq N, t \geq 0$ [3, 31, 7, 9, 15, 25, 26]. The latter process, the noncolliding BESQ, does the following set of SDEs  
\[
dX_{j}^{(\nu)}(t) = 2\sqrt{X_{j}^{(\nu)}(t)}dB_{j}(t) + 2(\nu + 1)dt + 4X_{j}^{(\nu)}(t) \sum_{1 \leq k \leq N, k \neq j} \frac{dt}{X_{j}^{(\nu)}(t) - X_{k}^{(\nu)}(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \tag{3.2}
\]
where \( B_j(t), 1 \leq j \leq N, t \geq 0 \) are independent one-dimensional standard BMs different from \( B_j(t), 1 \leq j \leq N, t \geq 0 \), and, if \(-1 < \nu < 0\), the reflection boundary condition is assumed at the origin \([19, 16]\). Consider subsets of \( \mathbb{R}^N \), \( \mathcal{W}_N = \{ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < \cdots < x_N \} \), and \( \mathcal{W}_N^+ = \{ \mathbf{x} \in \mathbb{R}_N^+ : x_1 < \cdots < x_N \} \). The former is called the Weyl chamber of type \( A_{N-1} \). If we replace the condition \( \mathbf{x} \in \mathbb{R}_N^+ \) by \( \mathbf{x} \in (0, \infty)^N \) for the latter, it will be the Weyl chamber of type \( C_N \). It is proved that, provided \( \mathbf{X}(0) \in \mathcal{W}_N \) and \( \mathbf{X}(\nu)(0) \in \mathcal{W}_N^+ \), then the SDEs (3.1) and (3.2) guarantee that with probability one \( \mathbf{X}(t) \in \mathcal{W}_N \), and \( \mathbf{X}(\nu)(t) \in \mathcal{W}_N^+, \forall t > 0 \) [8]. That is, in both processes, at any positive time \( t > 0 \) there is no multiple point at which coincidence of particle positions \( X_j(t) = X_k(t) \) or \( X_j(\nu)(t) = X_k(\nu)(t) \) for \( j \neq k \) occurs. It is the reason why these processes are called noncolliding diffusion processes \([17]\). In general, however, we can consider them starting from initial configurations with multiple points. In order to describe a general initial configuration we express it by a sum of delta measures in the form \( \xi(\cdot) = \sum_{j=1}^N \delta_{x_j}(\cdot) \).

Let \( \mathcal{M} \) be the space of nonnegative integer-valued Radon measures on \( \Lambda \). Any element \( \xi \) of \( \mathcal{M} \) can be represented as \( \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) \) with a countable index set \( \mathbb{I} \), in which a sequence of points in \( \Lambda \), \( \mathbf{x} = (x_j)_{j \in \mathbb{I}} \) satisfying \( \xi(K) = \# \{ x_j, x_j \in K \} < \infty \) for any compact subset \( K \subset \mathbb{R} \). We write the restriction of configuration in \( \Lambda \subset \mathbb{R} \) as \( (\xi \cap \Lambda)(\cdot) = \sum_{j \in \mathbb{I}, x_j \in \Lambda} \delta_{x_j}(\cdot) \) and a square of configuration as \( \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j^2}(\cdot) \), respectively. Let \( \mathcal{M}_+ = \{ (\xi \cap \mathbb{R}_+) : \xi \in \mathcal{M} \} \).

We consider the noncolliding BM and the noncolliding BESQ as \( \mathcal{M} \)-valued and \( \mathcal{M}_+ \)-valued processes and write them as

\[
\Xi(t, \cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), \quad \Xi(\nu)(t, \cdot) = \sum_{j=1}^N \delta_{X_j(\nu)(t)}(\cdot), \quad t \geq 0, \tag{3.3}
\]

respectively \([15, 16] \). The probability law of \( \Xi(t, \cdot) \) starting from a fixed configuration \( \xi \in \mathcal{M} \) is denoted by \( \mathbb{P}_\xi \) and that of \( \Xi(\nu)(t, \cdot) \) from \( \xi \in \mathcal{M}_+ \) by \( \mathbb{P}_\xi(\nu) \), and the noncolliding diffusion processes specified by initial configurations are expressed by \( (\Xi(t), \mathbb{P}_\xi) \) and \( (\Xi(\nu)(t), \mathbb{P}_\xi(\nu)) \), \( \nu > -1 \). The expectations w.r.t. \( \mathbb{P}_\xi \) and \( \mathbb{P}_\xi(\nu) \) are denoted by \( \mathbb{E}_\xi[\cdot] \) and \( \mathbb{E}_\xi(\nu)[\cdot] \), respectively. The set of \( \mathcal{M}_+ \)-valued continuous functions defined on \([0, \infty)\) is denoted by \( C([0, \infty) \to \mathcal{M}_+) \) for \( \mathcal{M} = \mathcal{M} \) or \( \mathcal{M}_+ \). We introduce a filtration \( \{ \mathcal{F}(t) \}_{t \in [0, \infty)} \) on the space \( C([0, \infty) \to \mathcal{M}) \) defined by \( \mathcal{F}(t) = \sigma(\tilde{\Xi}(s), s \in [0, t]) \), where \( \tilde{\Xi}(\cdot) = \Xi(\cdot) \) for \( \mathcal{M} = \mathcal{M} \) and \( \tilde{\Xi}(\cdot) = \Xi(\nu)(\cdot) \) for \( \mathcal{M} = \mathcal{M}_+ \). Let \( \mathcal{C}_0(\Lambda) \) be the set of all continuous real-valued functions with compact supports on \( \Lambda = \mathbb{R} \) or \( \mathbb{R}_+ \). We set \( \mathcal{M}_0 = \{ \xi \in \mathcal{M} : \xi(\{x\}) \leq 1 \) for any \( x \in \Lambda \} \), which denotes collections of configurations without any multiple points.

In this paper, we shall set for the noncolliding BM (resp. BESQ(\( \nu \)), \( \nu > -1 \)) \( \dot{X}(\cdot) = X(\cdot) \) (resp. \( \mathbf{X}(\nu)(\cdot) \)), \( \dot{P}_\xi = \mathbb{P}_\xi \) (resp. \( \mathbb{P}_\xi(\nu) \)), \( \dot{E}_\xi[\cdot] = \mathbb{E}_\xi[\cdot] \) (resp. \( \mathbb{E}_\xi(\nu)[\cdot] \)), \( \dot{V}(\cdot) = V(\cdot) \) with \( \dot{V}_j(\cdot) = V_j(\cdot) \), \( 1 \leq j \leq N \) (resp. \( \mathbf{V}(\nu)(\cdot) \)) with \( \mathbf{V}_j(\nu)(\cdot) \), \( 1 \leq j \leq N \), \( \Lambda = \mathbb{R} \) (resp. \( \mathbb{R}_+ \)) and \( \mathcal{W}_N = \mathcal{W}_N^+ \) (resp. \( \mathcal{W}_N^+ \)).

For any integer \( M \in \mathbb{N} \), a sequence of times \( t = (t_1, \ldots, t_M) \) with \( 0 < t_1 < \cdots < t_M \leq T < \infty \), and a sequence of functions \( \mathbf{f} = (f_{t_1}, \ldots, f_{t_M}) \in \mathcal{C}_0(\Lambda)^M \), the moment generating
function of multitime distribution of \((\hat{\Xi}(t), \hat{\mathcal{P}}_\xi)\) is defined by
\[
\hat{\Psi}_\xi^t[f] \equiv \mathbb{E}_\xi \left[ \exp \left\{ \sum_{m=1}^M \int_{\Lambda} f_{tm}(x) \hat{\Xi}(t_m, dx) \right\} \right].
\] (3.4)

It is expanded w.r.t. \(\chi_{tm}(\cdot) = e^{f_{tm}(\cdot)} - 1, 1 \leq m \leq M\) as
\[
\hat{\Psi}_\xi^t[f] = \sum_{N_m \geq 0} \prod_{1 \leq m \leq M} \chi_{tm} \left( \prod_{j=1}^{N_m} x^{(m)}_j \right) \hat{\rho}_\xi \left( t_1, \ldots, t_M, x^{(M)}_N \right),
\] (3.5)

and it defines the spatio-temporal correlation functions \(\hat{\rho}_\xi(\cdot)\) for the process, where \(x^{(m)}_N = (x^{(m)}_1, \ldots, x^{(m)}_{N_m})\) denotes \((x^{(m)}_1, \ldots, x^{(m)}_{N_m})\) and \(d x^{(m)}_{N_m} = \prod_{j=1}^{N_m} dx_j^{(m)}, 1 \leq m \leq M\).

The noncolliding BES\(^{(\nu)}\), \(\nu > -1\) is denoted by \((\tilde{\Xi}^{(\nu)}(t), \tilde{\mathcal{P}}_\xi^{(\nu)}), t \geq 0, \nu > -1\), with the initial configuration \(\xi \in \mathcal{M}_+.\) It is defined by
\[
(\tilde{\Xi}^{(\nu)})^{(2)}(t, \cdot) = \Xi^{(\nu)}(t, \cdot), \quad t \geq 0,
\]
\[
\iff \quad (\tilde{\Xi}^{(\nu)}(t, \cdot) = \sum_{j=1}^N \delta \sqrt{X^{(\nu)}_j(t)}(\cdot), \quad t \geq 0,
\] (3.6)

where \(\Xi^{(\nu)}(t, \cdot), t \geq 0\) is the noncolliding BESQ\(^{(\nu)}\) given by (3.3) and \(\{X^{(\nu)}_j(\cdot)\}_{j=1}^N\) is a solution of the set of SDEs (3.2). By Itô’s formula, we can see that
\[
\tilde{X}^{(\nu)}_j(t) = \sqrt{X^{(\nu)}_j(t)}, \quad 1 \leq j \leq N, \quad t \geq 0.
\] (3.7)

solve the SEDs [12],
\[
d\tilde{X}^{(\nu)}_j(t) = dB_j(t) + \frac{2\nu + 1}{2} \frac{dt}{\tilde{X}^{(\nu)}_j(t)} \]
\[
+ \sum_{1 \leq k \leq N, k \neq j} \left\{ \frac{1}{\tilde{X}^{(\nu)}_j(t) - \tilde{X}^{(\nu)}_k(t)} + \frac{1}{\tilde{X}^{(\nu)}_j(t) + \tilde{X}^{(\nu)}_k(t)} \right\}, \quad 1 \leq j \leq N, t \geq 0,
\] (3.8)

where \(B_j(t), 1 \leq j \leq N, t \geq 0\) are independent one-dimensional standard BMs and, if \(-1 < \nu < 0\), the reflection boundary condition is assumed at the origin. The expectation w.r.t. \(\tilde{\mathcal{P}}_\xi^{(\nu)}\) is written as \(\tilde{\mathbb{E}}^{(\nu)}_\xi[\cdot]\).

### 3.2 Determinantal-martingale representations

The following is the main theorem of the present paper.
Theorem 3.1 Suppose that $N \in \mathbb{N}$ and $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \hat{\mathfrak{M}}_{0}$. Let $0 < t \leq T < \infty$. For any $\mathcal{F}(t)$-measurable bounded function $F$ we have

$$\hat{E}_\xi \left[ F \left( \hat{\Xi}(\cdot) \right) \right] = E_u \left[ F \left( \sum_{j=1}^{N} \delta_{V_j(\cdot)} \right) \hat{M}_\xi(T, \hat{V}(T)) \right].$$

(3.9)

In particular,

$$\hat{\Psi}_\xi^t[f] = \sum_{0 \leq p \leq N} \sum_{(J_m)_{m=1}^{M} \in \hat{T}_p} \int_{\mathcal{W}_p} \xi \otimes_p (dv) E_v \left[ \prod_{m=1}^{M} \prod_{j_m \in J_m} \chi_{tm}(\hat{V}_{jm}(t_m)) \hat{M}_\xi^{\nu_p}(T, \hat{V}_p(T)) \right],$$

(3.10)

where $\chi_{tm}(\cdot) = e^{h_m(\cdot)} - 1$, $1 \leq m \leq M$.

We call (3.9) the determinantal martingale representations of the noncolliding diffusion processes. The proof is given in Section 5.1.

Through the equality (1.33), which is concluded by (2.21), the determinantal martingale representation (3.9) is transformed into the following complex BM representation, which was given as Theorem 1.1 in [18]. Let $Z_j(t), 1 \leq j \leq N, t \geq 0$ be a set of independent complex BM's given by (1.31). If they start at $Z_j(0) = u_j \in \mathbb{R}, 1 \leq j \leq N$, the probability space is denoted by $(\Omega, \mathcal{F}, \mathbf{P}_\mathbf{u})$ with $\mathbf{u} = (u_1, \ldots, u_N)$. The space $(\Omega, \mathcal{F}, \mathbf{P}_\mathbf{u})$ is a product of two probability spaces $(\Omega, \mathcal{F}, \mathbf{P}_\mathbf{u})$ for $V_j(\cdot) = \Re Z_j(\cdot)$ and $(\tilde{\Omega}, \mathcal{F}, \mathbf{P}_0)$ for $W_j(\cdot) = \Im Z_j(\cdot), 1 \leq j \leq N$. The expectation w.r.t. $\mathbf{P}_\mathbf{u}$ is denoted by $E_{\mathbf{u}}[\cdot]$.

Corollary 3.2 Suppose that $N \in \mathbb{N}$ and $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \hat{\mathfrak{M}}_{0}$. Let $0 < t \leq T < \infty$. For any $\mathcal{F}(t)$-measurable bounded function $F$ we have

$$E_{\xi} \left[ F \left( \Xi(\cdot) \right) \right] = E_u \left[ F \left( \sum_{j=1}^{N} \delta_{\Re Z_j(\cdot)} \right) \det_{1 \leq j, k \leq N} \left[ \Phi_{\xi}^{u_k}(Z_j(T)) \right] \right].$$

(3.11)

In particular, the moment generating function (3.4) is given by

$$\Psi_{\xi}^t[f] = \sum_{0 \leq p \leq N} \sum_{(J_m)_{m=1}^{M} \in \hat{T}_p} \int_{\mathcal{W}_p} \xi \otimes_p (dv) \times E_v \left[ \prod_{m=1}^{M} \prod_{j_m \in J_m} \chi_{tm}(\Re Z_{jm}(t_m)) \det_{1 \leq j, k \leq p} \left[ \Phi_{\xi}^{u_k}(Z_j(T)) \right] \right].$$

(3.12)

By the equality (1.39), which was proved as Lemma 2.3, the following corollary of Theorem 3.1 is obtained for the noncolliding $\operatorname{BES}^{(n+1/2)}, n \in \mathbb{N}_0$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}^{(n+1/2)})$ be the probability space of the set of complex diffusions (1.36) with half-odd indices $\nu = n + 1/2, n \in \mathbb{N}_0$. Here the process starts at $\mathbf{u} = (u_1, \ldots, u_N) \in \mathbb{R}_+^N$, and the expectation is denoted as $E_{\mathbf{u}}^{(n+1/2)}[\cdot]$. 

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Corollary 3.3 Suppose that \( N \in \mathbb{N} \) and \( \xi = \sum_{j=1}^{N} \delta_{u_j} \in \hat{\mathfrak{M}}_{t,0} \). Let \( n \in \mathbb{N}_0 \) and \( 0 < t \leq T < \infty \). For any \( \mathcal{F}(t) \)-measurable bounded function \( F \) we have

\[
\tilde{\mathbb{E}}_{\xi}^{(n+1/2)} \left[ F \left( \Xi^{(n+1/2)}(\cdot) \right) \right] = \tilde{\mathbb{E}}_{\tilde{\xi}}^{(n+1/2)} \left[ F \left( \sum_{j=1}^{N} \delta_{\mathbb{R}Z_{j}^{(n+1/2)}}(\cdot) \right) \right] \det_{1 \leq j,k \leq N} \left[ Q_{T}^{(n+1/2)}(Z_{j}^{(n+1/2)}(T))\tilde{\Phi}_{\xi}^{(n+1/2)}(Z_{j}^{(n+1/2)}(T)) \right],
\]

(3.13)

where \( Q_t(\cdot) \) and \( \tilde{\Phi}_{\xi}(\cdot) \) are given by (1.38) and (1.40), respectively. In particular, the moment generating function is given by

\[
\Psi^{(n+1/2),t}[f] = \sum_{0 \leq p \leq N} \sum_{(j_{m})_{m=1}^{M} \in \mathfrak{I}_p} \prod_{m=1}^{M} \chi_{t_{j_{m}}} (\mathbb{R}Z_{j_{m}}^{(n+1/2)}(t_{m}))
\times \det_{1 \leq j,k \leq p} \left[ Q_{T}^{(n+1/2)}(Z_{j}^{(n+1/2)}(T))\tilde{\Phi}_{\xi}^{(n+1/2),u_k}(Z_{j}^{(n+1/2)}(T)) \right].
\]

(3.14)

3.3 Spatio-temporal correlation kernels

Let \( \chi_s, \chi_t \in C_0(\Lambda) \) and introduce the following measurable symmetric functions by

\[
g_s(\mathbf{x}) = \sum_{j=1}^{N} \chi_s(x_j), \quad g_t(\mathbf{x}) = \sum_{j=1}^{N} \chi_t(x_j).
\]

Assume \( 0 < s < t \leq T < \infty \). Then Theorem 3.1 gives the equality

\[
\hat{\mathbb{E}}_{\xi} \left[ g_s(\mathbf{X}(s))g_t(\mathbf{X}(t)) \right] = \mathbb{E}_{\tilde{\xi}} \left[ g_s(\tilde{\mathbf{V}}(s))g_t(\tilde{\mathbf{V}}(t))\hat{\mathfrak{M}}_{\xi}(T,\tilde{\mathbf{V}}(T)) \right].
\]

(3.15)

By (3.3), the LHS is written as

\[
(\text{LHS}) = \hat{\mathbb{E}}_{\xi} \left[ \int_{\Lambda} dx \chi_s(x)\tilde{\mathcal{E}}(s,dx) \int_{\Lambda} dy \chi_t(y)\tilde{\mathcal{E}}(t,dy) \right].
\]

If there exists an integrable continuous function \( \rho_{\xi}(s,x;t,y) \) such that the above is written as

\[
\int_{\Lambda} dx \int_{\Lambda} dy \chi_s(x)\chi_t(y)\rho_{\xi}(s,x;t,y),
\]

(3.16)

then \( \rho_{\xi}(s,x;t,y) \) is called the two-time and two-point spatio-temporal correlation function.
Now we study the RHS of (3.15).

\[
(RHS) = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbb{E}_u \left[ \chi_s(\hat{V}_j(s)) \chi_t(\hat{V}_k(t)) \hat{M}_\xi^u(T, \hat{V}(T)) \right] \\
= \sum_{1 \leq j, k \leq N, j \neq k} \mathbb{E}_u \left[ \chi_s(\hat{V}_j(s)) \chi_t(\hat{V}_k(t)) \hat{M}_\xi^u(T, \hat{V}(T)) \right] \\
+ \sum_{j=1}^{N} \mathbb{E}_u \left[ \chi_s(\hat{V}_j(s)) \chi_t(\hat{V}_j(t)) \hat{M}_\xi^u(T, \hat{V}(T)) \right].
\]

By the reducibility of determinantal martingale given by Lemma 2.4, it is equal to

\[
\int_{\mathcal{W}_2} \xi^{\otimes 2}(dv) \mathbb{E}_{(v_1, v_2)} \left[ \chi_s(\hat{V}_1(s)) \chi_t(\hat{V}_2(t)) \det \begin{pmatrix} \hat{M}_\xi^1(s, V_1(s)) & \hat{M}_\xi^1(t, \hat{V}_2(t)) \\ \hat{M}_\xi^2(s, V_1(s)) & \hat{M}_\xi^2(t, \hat{V}_2(t)) \end{pmatrix} \right] \\
+ \int_{\Lambda} \xi(dv) \mathbb{E}_v \left[ \chi_s(\hat{V}(s)) \chi_t(\hat{V}(t)) \hat{M}_\xi^u(t, \hat{V}(t)) \right].
\]

Let

\[
\hat{G}_{s,t}(x, y) = \int_{\Lambda} \xi(dv) \hat{p}(s, x|v) \hat{M}_\xi^u(t, y), \quad s, t \in [0, \infty), \quad x, y \in \Lambda. \tag{3.17}
\]

Using Fubini’s theorem, we find that

\[
(RHS) = \int_{\Lambda} dx \int_{\Lambda} dy \chi_s(x) \chi_t(y) \det \begin{pmatrix} \hat{G}_{s,s}(x, x) & \hat{G}_{s,t}(x, y) \\ \hat{G}_{t,s}(y, x) & \hat{G}_{t,t}(y, y) \end{pmatrix} \\
+ \int_{\Lambda} dx \int_{\Lambda} dy \chi_s(x) \chi_t(y) \hat{G}_{s,t}(x, y) \hat{p}(t - s, y|x) \\
= \int_{\Lambda} dx \int_{\Lambda} dy \chi_s(x) \chi_t(y) \det \begin{pmatrix} \hat{G}_{s,s}(x, x) & \hat{G}_{s,t}(x, y) \\ \hat{G}_{t,s}(y, x) & \hat{G}_{t,t}(y, y) - \hat{p}(t - s, y|x) \end{pmatrix}.
\]

Then the two-time and two-point spatio-temporal correlation function is determined as

\[
\hat{\rho}_\xi(s, x; t, y) = \det \begin{pmatrix} \hat{K}_\xi(s, x; s, x) & \hat{K}_\xi(s, x; t, y) \\ \hat{K}_\xi(t, y; s, x) & \hat{K}_\xi(t, y; t, y) \end{pmatrix}
\]

for \(0 < s < t \leq T < \infty\) with

\[
\hat{K}_\xi(s, x; t, y) = \hat{G}_{s,t}(x, y) - 1(s > t)\hat{p}(s - t, x|y) \tag{3.18}
\]

for any \(\xi \in \hat{\mathcal{M}}_0\).

The above calculation is generalized and the following is concluded from the present determinantal-martingale representations.
Corollary 3.4 For any initial configuration $\xi \in \hat{\mathcal{M}}_0$ with $\xi(\Lambda) = N \in \mathbb{N}$, the noncolliding BM, $(\Xi(t), \mathbb{P}_\xi)$ and the noncolliding BESQ($\nu$), $(\Xi^{(\nu)}(t), \mathbb{P}^{(\nu)}_\xi), \nu > -1$ are determinantal with the kernels (1.24) in the sense that the moment generating functions (3.4) are given by Fredholm determinants

$$
\hat{\Psi}_\xi^t[f] = \det_{(s,t) \in \{t_1,t_2,\ldots,t_M\}^2} \left[ \delta_{st}\delta_x(y) + \hat{K}_\xi(s,x;t,y)\chi_t(y) \right],
$$

(3.19)

and then all spatio-temporal correlation functions are given by determinants as

$$
\hat{\rho}_\xi(t_1,x^{(1)};t_2,x^{(2)};\ldots;t_M,x^{(M)}) = \det_{1 \leq j \leq N_m, 1 \leq k \leq N_n} \left[ \hat{K}_\xi(t_m,x^{(m)}_j;t_n,x^{(n)}_k) \right],
$$

(3.20)

$0 < t_1 < \cdots < t_M \leq T < \infty$, $0 \leq N_m \leq N$, $1 \leq m \leq M$.

The proof is given in Section 5.2. The statement will be generalized for $\xi \in \hat{\mathcal{M}}$ with $\xi(\Lambda) = N \in \mathbb{N}$ by replacing the correlation kernel (1.24) by (1.29).

4 Examples

Here we give some examples in order to demonstrate the calculations of local martingales and derivations of correlation kernels from them.

4.1 Martingales associated with infinite particle systems

In [15] and [16], useful sufficient conditions for initial configurations $\xi$ were given so that $(\Xi(t), \mathbb{P}_\xi)$ and $(\Xi^{(\nu)}(t), \mathbb{P}^{(\nu)}_\xi), \nu > -1$ are well-defined as determinantal processes even if $N = \xi(\Lambda) = \infty$. The families of $\xi$ satisfying the conditions are denoted by $\hat{\mathfrak{X}} = \mathfrak{X}$ for the noncolliding BM and $\hat{\mathfrak{X}} = \mathfrak{X}^+$ for the noncolliding BESQ($\nu$), respectively. Define $\hat{\mathfrak{X}}_0 = \hat{\mathfrak{X}} \cap \hat{\mathcal{M}}_0$. Assume $M \in \mathbb{N}$, $0 < t_1 < \cdots < t_M \leq T < \infty$, $\phi_m \in C_0(\Lambda), 1 \leq m \leq M$ and $G = G(\{x_m\}_{m=1}^M)$ is a polynomial on $\Lambda^M$. For $0 < t \leq T < \infty$, if an $\mathcal{F}(t)$-measurable function $F(\hat{\Xi}(\cdot))$ is represented as

$$
F(\hat{\Xi}(\cdot)) = G \left( \left\{ \int_\Lambda \phi_m(x)\hat{\Xi}(t_m, dx) \right\}_{m=1}^M \right),
$$

we say $F$ is polynomial.

Proposition 4.1 Suppose that $\xi \in \hat{\mathfrak{X}}_0$, $0 < t \leq T < \infty$, and $F$ is an $\mathcal{F}(t)$-measurable polynomial function. Then the determinantal-martingale representation (3.9) is valid for $(\hat{\Xi}(t), \hat{\mathbb{P}}_\xi)$ also in the case with $N = \xi(\Lambda) = \infty$.  

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For \((\Xi(t), \mathbb{P}_\xi)\), the similar statement was proved for the complex BM representation in [18] (Corollary 1.3). Here by the reducibility of the determinantal martingale given by Lemma 2.4, this proposition is readily concluded.

There are two interesting examples of local martingales for infinite particle systems. First we consider the configuration
\[
\xi_z(\cdot) = \sum_{j \in \mathbb{Z}} \delta_j(\cdot),
\]
that is, the configuration in which every integer point \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}\) is occupied by one particle. It is easy to confirm that \(\xi_z \in \mathcal{X}_0\) and the noncolliding BM started at \(\xi_z, (\Xi(t), \mathbb{P}_{\xi_z})\), is a determinantal process with an infinite number of particles [15]. Since
\[
\prod_{n \in \mathbb{N}} (1 - x^2/n^2) = \frac{\sin(\pi x)}{\pi x},
\]
we have
\[
\Phi_{\xi_z}(z) = \prod_{r \in \mathbb{Z}, r \neq u} \frac{z - r}{v - r} = \frac{\sin\{\pi(z - v)\}}{\pi(z - v)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda e^{i\lambda(z-v)}, \quad z, v \in \mathbb{C},
\]
(4.2)
Its integral transform is calculated as
\[
\hat{M} \left[ \Phi_{\xi_z}(iw) \right](t, x) = \int_{\mathbb{R}} dw q(t, w|x)\Phi_{\xi_z}(iw) \\
= \int_{-\infty}^{\infty} dw \frac{1}{\sqrt{2\pi t}} e^{-(ix+w)^2/2t}\Phi_{\xi_z}(iw) \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda e^{t\lambda^2/2 + i\lambda(x-v)}.
\]
(4.3)
Then we have local martingales
\[
\mathcal{M}^k_{\xi_z}(t, V_j(t)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \exp \left\{ \frac{\lambda^2}{2} t + i\lambda(V_j(t) - k) \right\}, \quad j, k \in \mathbb{Z}, \quad 0 < t \leq T < \infty.
\]
(4.4)
We see that
\[
E_{\xi_z} \left[ \mathcal{M}^k_{\xi_z}(t, V_j(t)) \right] = E_{\xi_z} \left[ \mathcal{M}^k_{\xi_z}(0, V_j(0)) \right] \\
= \begin{cases} 
\frac{\sin\{\pi(j - k)\}}{\pi(j - k)}, & j \neq k, \\
1, & j = k, \quad 0 < t \leq T < \infty.
\end{cases}
\]
(4.5)
If \(\nu > -1\), the Bessel function \(J_\nu(z)\) given by (2.7) has an infinite number of pairs of positive and negative zeros with the same absolute value, which are all simple. We write the positive zeros of \(J_\nu(z)\) arranged in ascending order of the absolute values as
\[
0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots.
\]
(4.6)
Then, $J_\nu(z)$ has the following infinite product expression [35],

$$J_\nu(z) = \frac{(z/2)\nu}{\Gamma(\nu + 1)} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2 \nu, j}\right). \quad (4.7)$$

For the noncolliding BESQ($\nu$), we consider the initial configuration in which every point of the squares of positive zeros of $J_\nu(z)$ is occupied by one particle, which is denoted as

$$\xi^{(2)}_{J_\nu}(\cdot) = \sum_{j=1}^{\infty} \delta_{j^2 \nu, j}(\cdot). \quad (4.8)$$

We can see that $\xi^{(2)}_{J_\nu} \in X^+_0$ and thus $(\Xi^{(\nu)}, P^{(\nu)}_{\xi^{(2)}_{J_\nu}})$ is a determinantal process with an infinite number of particles [16]. For $k \in \mathbb{N}$ we find that

$$\Phi^{(j_{\nu, k})^2}_{\xi^{(2)}_{J_\nu}}(z) = \left(\frac{(j_{\nu, k})^2}{z}\right)^{\nu/2} \frac{1}{(J_{\nu+1}(j_{\nu, k}))^2} \int_{0}^{1} d\lambda J_\nu(\sqrt{\lambda z}) J_\nu(\sqrt{\lambda j_{\nu, k}}), \quad (4.9)$$

and their integral transforms gives the martingales,

$$M^{(j_{\nu, k})^2}_{\xi^{(2)}_{J_\nu}}(t, V^{(\nu)}_j(t)) = \hat{M}^{(\nu)} \left[ \Phi^{(j_{\nu, k})^2}_{\xi^{(2)}_{J_\nu}}(-W) \left| (t, V^{(\nu)}_j(t)) \right. \right]$$

$$= \left(\frac{(j_{\nu, k})^2}{V^{(\nu)}_j(t)}\right)^{\nu/2} \frac{1}{(J_{\nu+1}(j_{\nu, k}))^2} \int_{0}^{1} d\lambda e^{M/2} J_\nu(\sqrt{\lambda V^{(\nu)}_j(t)}) J_\nu(\sqrt{\lambda j_{\nu, k}}),$$

$$j, k \in \mathbb{N}, \quad 0 < t \leq T \leq \infty. \quad (4.10)$$

We see that for $0 < t \leq T < \infty$,

$$E^{(\nu)}_{\xi^{(2)}_{J_\nu}} \left[ M^{(j_{\nu, k})^2}_{\xi^{(2)}_{J_\nu}}(t, V^{(\nu)}_j(t)) \right] = E^{(\nu)}_{\xi^{(2)}_{J_\nu}} \left[ M^{(j_{\nu, k})^2}_{\xi^{(2)}_{J_\nu}}(0, V^{(\nu)}_j(0)) \right]$$

$$= \left\{ \begin{array}{ll}
\left(\frac{j_{\nu, k}}{j_{\nu, j}}\right)^{\nu} \frac{2j_{\nu, k}}{(j_{\nu, k})^2 -(j_{\nu, j})^2} \frac{J_{\nu}(j_{\nu, j})}{J_{\nu+1}(j_{\nu, k})}, & j \neq k; \\
1, & j = k.
\end{array} \right. \quad (4.11)$$

Through Corollary 3.4 with the formula (1.24), these martingales determine the correlation kernels, which are denoted as $K_{\xi_{\nu}}$ and $K^{(\nu)}_{\xi^{(2)}_{J_\nu}}$. In the previous papers [15, 16], we showed

$$\lim_{\tau \to \infty} K_{\xi_{\nu}}(s + \tau, x; t + \tau, y) = K_{\sin}(t - s, y - x),$$

$$\lim_{\tau \to \infty} K^{(\nu)}_{\xi^{(2)}_{J_\nu}}(s + \tau, x; t + \tau, y) = \left(\frac{x}{y}\right)^{\nu/2} K_{J_\nu}(t - s, y|x), \quad (4.12)$$

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and proved that the noncolliding BM started from (4.1) and the noncolliding BESQ\(_{(\nu)}\) started from (4.8) converge in the long-term limit to the equilibrium determinantal processes governed by the extended sine kernel

\[
K_{\sin}(t, x) = \begin{cases} 
\int_0^1 d\lambda e^{\pi^2 \lambda^2 t/2} \cos(\pi \lambda x), & \text{if } t > 0 \\
\frac{\sin(\pi x)}{\pi x} & \text{if } t = 0 \\
-\int_1^\infty d\lambda e^{\pi^2 \lambda^2 t/2} \cos(\pi \lambda x), & \text{if } t < 0,
\end{cases}
\] (4.13)

and the extended Bessel kernel [5]

\[
K_{\text{J}_\nu}(t, y|x) = \begin{cases} 
\frac{1}{4} \int_0^1 d\lambda e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu(\sqrt{\lambda y}), & \text{if } t > 0 \\
\frac{J_\nu(\sqrt{x}) \sqrt{y} J'_\nu(\sqrt{y}) - \sqrt{x} J'_\nu(\sqrt{x}) J_\nu(\sqrt{y})}{2(x-y)}, & \text{if } t = 0 \\
-\frac{1}{4} \int_1^\infty d\lambda e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu(\sqrt{\lambda y}), & \text{if } t < 0,
\end{cases}
\] (4.14)

respectively. These relaxation phenomena of infinite particle systems are caused by the following properties of the present martingales,

\[
\lim_{\tau \to \infty} \sum_{k \in \mathbb{Z}} p(\tau, x|k) M_{\xi^k}(t + \tau, V(t)) = M_{\xi^k}(t, V(t)), \quad x \in \mathbb{R},
\]

\[
\lim_{\tau \to \infty} \sum_{k \in \mathbb{N}} \frac{4p^{(\nu)}(\tau, x|j_{\nu,k})^2}{(J_{\nu+1}(x))^2} \mathcal{M}_{\xi^{J_{\nu,k}}}^{(\nu,k)}(t + \tau, V^{(\nu)}(t)) = M_{\xi^{J_{\nu,k}}}^{(\nu,k)}(t, V^{(\nu)}(t))
\]

\[
= \frac{x^{\nu/2}}{(J_{\nu+1}(x))^2 (V^{(\nu)}(t)))^{\nu/2}} \int_0^1 d\lambda e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu(\sqrt{\lambda y}) (V^{(\nu)}(t))^{\nu/2}, \quad x \in \mathbb{R}^+,
\] (4.15)

for 0 < t ≤ T < ∞.

**4.2 Martingales for configurations with multiple points**

For a general configuration ξ ∈ \(\mathfrak{M}\) with multiple points, a local martingale \(\hat{\mathcal{M}}_{\xi}(t, \hat{V}(t))\) with (1.22) is extended by (1.27) as

\[
\hat{\mathcal{M}}_{\xi}^{(\nu)}((s, x)|(t, \hat{V}(t))) = \mathcal{M}\left[\Phi^{(\nu)}((s, x); e^{iW}) | (t, \hat{V}(t))\right], \quad 0 < t \leq T < \infty,
\] (4.16)

having parameters \((s, x) \in [0, T] \times \Lambda\), where \(\nu \in \text{supp} \xi\).
As an example, we consider the extreme case such that all $N$ points are concentrated on an origin,
\[ \xi = N\delta_0 \iff \xi_s = \delta_0 \text{ with } \xi(\{0\}) = N. \] (4.17)

In this case (1.25) and (1.26) become
\[
\hat{\Phi}^0_{N\delta_0}((s, x); z, \zeta) = \frac{1}{\hat{p}(s, x|0)} \sum_{\ell=0}^{N-1} z^{N-\ell-1} \frac{1}{2\pi i} \oint_{\Gamma(\delta_0)} d\zeta \frac{\hat{p}(s, x|\zeta)}{\zeta^{N-\ell}},
\]
(4.18)
and
\[
\hat{\Phi}^0_{N\delta_0}((s, x); z) = \frac{1}{\hat{p}(s, x|0)} \sum_{\ell=0}^{N-1} z^{N-\ell-1} \frac{1}{2\pi i} \oint_{\Gamma(\delta_0)} d\zeta \frac{\hat{p}(s, x|\zeta)}{\zeta^{N-\ell}},
\]
(4.19)

since the integrands are holomorphic when $\ell \geq N$, where we have assumed $\nu > -1$ for BESW\(^{(\nu)}\).

For BM with the transition probability density (2.2), (4.19) gives
\[
\Phi^0_{N\delta_0}((s, x); z) = \sum_{\ell=0}^{N-1} z^{N-\ell-1} \frac{1}{2\pi i} \oint_{\Gamma(\delta_0)} d\zeta \frac{e^{x\zeta/s - \zeta^2/2s}}{\zeta^{N-\ell}},
\]
(4.20)

where we have used the contour integral representation of the Hermite polynomials [32]
\[
H_n(x) = \frac{n!}{2\pi i} \oint_{\Gamma(\delta_0)} d\eta e^{2x\eta - \eta^2} \eta^{n+1}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}.
\]
(4.21)

Thus its integral transform is calculated as
\[
M[\Phi^0_{N\delta_0}((s, x); iW)| (t, y)]
= \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) \frac{1}{(2s)^{(N-\ell-1)/2}} M[(iW)^{N-\ell-1}| (t, y)]
= \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) \frac{1}{(2s)^{(N-\ell-1)/2}} M_{N-\ell-1}(t, y)
= \sum_{\ell=0}^{N-1} \frac{1}{(N-\ell-1)!2^{N-\ell-1}} \left( \frac{t}{s} \right)^{(N-\ell-1)/2} H_{N-\ell-1} \left( \frac{x}{\sqrt{2s}} \right) H_{N-\ell-1} \left( \frac{y}{\sqrt{2t}} \right),
\]
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where we have used (1.9) with (1.1). Then we obtain the following,

\[
\mathcal{M}_{N\delta_0}^\nu ((s, x) | (t, V(t))) = \sum_{n=0}^{N-1} \frac{1}{n!2^n} m_n(s, x) m_n(t, V(t))
\]

\[
= \sum_{n=0}^{N-1} \frac{1}{n!2^n} \left( \frac{t}{s} \right)^{n/2} H_n \left( \frac{x}{\sqrt{2s}} \right) H_n \left( \frac{V(t)}{\sqrt{2t}} \right)
\]

\[
= \sqrt{\pi} e^{x^2/4s + V(t)^2/4t} \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi_n \left( \frac{x}{\sqrt{2s}} \right) \varphi_n \left( \frac{V(t)}{\sqrt{2t}} \right),
\]  

(4.22)

where

\[
\varphi_n(x) = \frac{1}{\sqrt{\pi 2^n n!}} H_n(x) e^{-x^2/2t}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}.
\]

Similarly, for BESQ\(^{(\nu)}\), \(\nu > -1\) with the transition probability density (2.3), we obtain

\[
\Phi^{(\nu)}_{N\delta_0} ((s, x); z) = \frac{(2s)^\nu \Gamma(\nu + 1)}{\nu^{\nu/2}} \sum_{\ell=0}^{N-1} z^{-\nu-\ell-1} \frac{1}{2\pi i} \oint_{\Gamma(\delta_0)} d\zeta \frac{e^{-\zeta/2s}}{\zeta^{N-\nu-\ell/2}} I_\nu \left( \frac{\sqrt{x\zeta}}{s} \right)
\]

\[
= \Gamma(\nu + 1) \sum_{\ell=0}^{N-1} \left( \frac{-z}{2s} \right)^{N-\nu-\ell-1} \frac{1}{\Gamma(N - \ell + \nu)} L_{N-\nu-\ell-1}^{(\nu)} \left( \frac{x}{2s} \right),
\]  

(4.23)

where we used the contour integral representation of the Laguerre polynomials

\[
L_n^{(\nu)}(x) = \frac{\Gamma(n + \nu + 1)}{\nu^{\nu/2}} \frac{1}{2\pi i} \oint_{\Gamma(\delta_0)} d\eta \frac{e^{\eta}}{\eta^{n+1+\nu/2}} J_\nu(2\sqrt{\eta x})
\]  

(4.24)

with the relation \(I_\nu(iz) = (-1)^{\nu/2} J_\nu(z), -\pi < \arg(z) \leq \pi/2\). By using (1.9) with (1.3), we have

\[
\mathcal{M}^{(\nu)}_{N\delta_0} ((s, x) | (t, V^{(\nu)}(t)))
\]

\[
= \mathcal{M}^{(\nu)} \left[ \Phi^{(\nu)}_{N\delta_0} ((s, x); -W) \right] (t, V^{(\nu)}(t))
\]

\[
= \Gamma(\nu + 1) \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)2^{2n}} m_n^{(\nu)}(s, x) m_n^{(\nu)}(t, V^{(\nu)}(t))
\]

\[
= \Gamma(\nu + 1) \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \left( \frac{t}{s} \right)^n L_n^{(\nu)} \left( \frac{x}{2s} \right) L_n^{(\nu)} \left( \frac{V^{(\nu)}(t)}{2t} \right)
\]

\[
= \Gamma(\nu + 1) \left( \frac{x}{2s} \right)^{-\nu/2} \left( \frac{V^{(\nu)}(s)}{2s} \right)^{-\nu/2} e^{x/4s + V^{(\nu)}(t)/4t}
\]

\[
\times \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \left( \frac{t}{s} \right)^n \varphi_n^{(\nu)} \left( \frac{x}{2s} \right) \varphi_n^{(\nu)} \left( \frac{V^{(\nu)}(t)}{2t} \right),
\]  

(4.25)
where
\[ \varphi_n^{(\nu)}(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}} x^{\nu/2} L_n^{(\nu)}(x) e^{-x/2}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}_+. \]

The processes (4.22) and (4.25) are local martingales and

\[ E_0 \left[ \hat{\mathcal{M}}_{N_0}^0((s,x)|(t,\hat{V}(t))) \right] = E_0 \left[ \hat{\mathcal{M}}_{N_0}^0((s,x)|(0,\hat{V}(0))) \right] = 1 \quad (4.26) \]

for \( 0 < t \leq T < \infty, (s,x) \in [0,T] \times \Lambda. \)

By the formula (1.29), we obtain the correlation kernels as

\[ \mathbb{K}_{N_0}^{s,x}(s,x; t,y) = \frac{1}{\sqrt{2s}} \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi \left( \frac{x}{\sqrt{2s}} \right) \varphi \left( \frac{y}{\sqrt{2t}} \right) - 1(s > t)p(s-t,x|y), \quad (4.28) \]

and

\[ \mathbb{K}_{N_0}^{(\nu)}(s,x; t,y) = \frac{1}{\sqrt{2s}} \sum_{n=0}^{N-1} \left( \frac{t}{s} \right)^{n/2} \varphi \left( \frac{x}{\sqrt{2s}} \right) \varphi \left( \frac{y}{\sqrt{2t}} \right) - 1(s > t)p^{(\nu)}(s-t,x|y), \quad (4.30) \]

where \( \mathbb{K}_{N_0}^{\cdot,\cdot} \) and \( \mathbb{K}_{N_0}^{(\nu)}(\cdot,\cdot) \) are known as the extended Hermite and Laguerre kernels, respectively (see, for instance, [5]). Here we would like to emphasize the fact that these kernels have been derived here by not following any ‘orthogonal-polynomial arguments’ but by only using proper martingales associated with the chosen initial configuration (4.17). In this special case, they are expressed by the Hermite and Laguerre polynomials in the forms (4.22) and (4.25), respectively. In the present new approach, the martingale properties (4.26) and the reducibility (2.34) coming from the independence of diffusion processes play essential roles instead of orthogonality in the theory of orthogonal ensembles in random matrix theory [23, 5].

5 Proofs

5.1 Proof of Theorem 3.1

It is sufficient to consider the case that \( F \) is given as \( F(\hat{\Xi}(\cdot)) = \prod_{m=1}^{M} g_m(\hat{X}(t_m)) \) for \( M \in \mathbb{N}, 0 < t_1 < \cdots < t_M \leq T < \infty \) with symmetric bounded measurable functions \( g_m \).
on $\Lambda^N$, $1 \leq m \leq M$. We give a proof for the case with $M = 2$, since the generalization for $M \geq 3$ is straightforward by the Markov property of the processes: Here we prove

$$
\mathbb{E}_\xi \left[ g_1(\dot{X}(t_1))g_2(\dot{X}(t_2)) \right] = \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\dot{\mathcal{M}}_{\xi}(T, \dot{V}(T)) \right],
$$

(5.1)

for $\xi = \sum_{j=1}^{N} \delta_{u_j}, u = (u_1, \ldots, u_N) \in \mathbb{W}_N$, $0 < t_1 < t_2 \leq T < \infty$, where $g_1$ and $g_2$ are symmetric bounded measurable functions on $\Lambda^N$.

We showed in [18] that it equals to

$$
\mathbb{E}_u \left[ 1(\tau > t_2)g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{h(\dot{V}(t_2))}{h(u)} \right].
$$

We showed in [18] that it equals to

$$
\mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{h(\dot{V}(t_2))}{h(u)} \right],
$$

(5.2)

since all contributions with $\tau \leq t_2$ for the expectation are canceled out by the reflection principle of BM. By multilinearity of determinant, $h(x) = \det_{1 \leq j, k \leq N}[x_j^{k-1}]$ does not change by replacing $x_j^{k-1}$ by any monic polynomial of $x_j$ of degree $k - 1, 1 \leq j, k \leq N$. Then (5.2) is written as

$$
\mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{j}(t_2)^{k-1}] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{W}_{j}(t_2))] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{V}_{j}(t_2))] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{V}_{j}(t_2))] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{V}_{j}(t_2))] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{V}_{j}(t_2))] \right]
$$

$$
= \mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\frac{1}{h(u)}\det_{1 \leq j, k \leq N}[\dot{W}_{k-1}(t_2, \dot{V}_{j}(t_2))] \right]
$$

where we have used the properties (1.12) and (1.13) of the integral transform $\dot{\mathcal{M}}[\dot{\cdot}]$. Now we apply the identity (1.17) and the LHS of (5.1) is written as

$$
\mathbb{E}_u \left[ g_1(\dot{V}(t_1))g_2(\dot{V}(t_2))\dot{\mathcal{M}}[\dot{\cdot}] \right]
$$

(5.3)

By martingale property of $\dot{\mathcal{M}}_{\xi}(\dot{\cdot}, \dot{V}(\dot{\cdot}))$, it is equal to the RHS of (5.1).
Here we derive the expression (3.10) for the moment generating function defined by (3.4) for the case $M = 2$. By the relation
\[
\exp \left\{ \sum_{m=1}^{2} \sum_{j_m=1}^{N} f_{t_m}(x_{j_m}) \right\} = \prod_{m=1}^{2} \prod_{j_m=1}^{N} \left\{ \chi_{t_m}(x_{j_m}) + 1 \right\} = \sum_{J_1, J_2 \subset I_N} \prod_{m=1}^{2} \prod_{j_m \in J_m} \chi_{t_m}(x_{j_m}),
\]
(3.4) with $M = 2$ is written as
\[
\sum_{J_1, J_2 \subset I_N} \mathbb{E}\[ \prod_{m=1}^{2} \prod_{j_m \in J_m} \chi_{t_m}(\hat{V}_{j_m}(t_m)) \].
\]
Applying (5.1) with $g_{m}(x) = \sum_{J_m \subset I_N} \prod_{j_m \in J_m} \chi_{t_m}(x_{j_m})$, $m = 1, 2,$ we see that it equals
\[
\sum_{J_1, J_2 \subset I_N} \mathbb{E}\[ \prod_{m=1}^{2} \prod_{j_m \in J_m} \chi_{t_m}(\hat{V}_{j_m}(t_m)) \hat{M}_{u}(T, \hat{V}(T)) \].
\]
By the reducibility of determinantal martingale given by Lemma 2.4, it is equal to
\[
\sum_{1 \leq p \leq N} \sum_{J_1 \subset I_p, J_2 \subset I_p} \int_{\mathbb{W}_p} \xi^{o_p}(dv) \mathbb{E}\[ \prod_{m=1}^{2} \prod_{j_m \in J_m} \chi_{t_m}(\hat{V}_{j_m}(t_m)) \hat{M}_{u}\[ T, \hat{V}(T) \]] \],
\]
which is the RHS of (3.10) with $M = 2$. By the similar argument, (3.10) is concluded from (3.9) for any $M \geq 3$. Then the proof is completed.

5.2 Proof of Corollary 3.4

The following argument is the same as that given in [18] to prove Corollary 1.2 there, but some lines in the proof become simpler since the present martingales are more general than the conformal martingales in [18].

By definition, the Fredholm determinant (3.19) is given by
\[
= \sum_{N_m \geq 0} \prod_{m=1}^{M} \{ dx_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}(x_{j}^{(m)}) \} \det_{1 \leq j \leq N_m, 1 \leq k \leq N_n, 1 \leq m, n \leq M} [\hat{K}(t_m, x_{j}^{(m)}; t_n, x_{k}^{(n)})].
\]
(5.4)
Then (3.10) in Theorem 3.1 implies that, for proof of Corollary 3.4, it is enough to show that the following equality is established for any $M \in \mathbb{N}$, $(N_1, \ldots, N_M) \in \mathbb{N}^M$

$$
\int_{\prod_{m=1}^{M} W_{N_m}} \prod_{m=1}^{M} d\mathbf{x}_m \sum_{1 \leq j \leq N_m} \chi_{t_m, \mathbf{x}_m}(x_{m,j}) \left[ \det_{1 \leq m, n \leq M} \left[ R\xi(t_m, x_{m,j}; t_n, x_{n,k}) \right] \right] = \sum_{1 \leq p \leq N} \sum_{\ell \in \mathfrak{I}_{\ell=m}^{(m)}} \left( \int_{\mathfrak{I}_{\ell=m}^{(m)} \cap \mathfrak{I}_{\ell=m}^{(m)}} \prod_{m=1}^{M} \prod_{j \in \mathfrak{I}_{m}} \chi_{t_m, \mathbf{x}_m}(\hat{V}_{j_m}(t_m)) \right) \prod_{j \in \mathfrak{I}_{m}} \lambda_{\tau_j} \left[ \hat{M}_{\xi}^{-1} (T, \hat{V}_j(T)) \right].
$$

(5.5)

As a matter of fact, if we take the summation of (5.5) over all $0 \leq N_m \leq N, 1 \leq m \leq M$, the LHS gives (3.19) with (3.20) and the RHS does (3.10). Here we will prove (5.5) by fixing $M \in \mathbb{N}$, $(N_1, \ldots, N_M) \in \mathbb{N}^M$.

Let $\mathfrak{I}^{(1)} = \mathfrak{I}_{N_1}$ and $\mathfrak{I}^{(m)} = \mathfrak{I}_{\sum_{j=1}^{m} N_j} \setminus \mathfrak{I}_{\sum_{j=1}^{m-1} N_j}, 2 \leq m \leq M$. Put $n = \sum_{m=1}^{M} N_m$ and $\tau_j = \sum_{m=1}^{M} t_m 1 (j \in \mathfrak{I}^{(m)}), 1 \leq j \leq n$. Then the integrand in the LHS of (5.5) is simply written as

$$
\prod_{j=1}^{n} \chi_{t_j, \mathbf{x}_j}(x_{j,j}) \left[ \det_{1 \leq j, k \leq n} \left[ R\xi(\tau_j, x_{j,j}; \tau_k, x_{k,k}) \right] \right],
$$

and the integral $\int_{\prod_{m=1}^{M} W_{N_m}} \prod_{m=1}^{M} d\mathbf{x}_m (\cdot)$ can be replaced by $\{\prod_{m=1}^{M} N_m!\}^{-1} \int_{\mathfrak{I}_n} d\mathbf{x} (\cdot)$.

The determinant is defined using the notion of permutations. For a finite set $S$, we write the collection of all permutations of elements in $S$ as $S(S)$. In particular, we express $S_{\ell}(S)$ simply by $S_{\ell}$. Any permutation $\sigma \in S_n$ can be decomposed into a product of cycles. Let the number of cycles in the decomposition be $\ell(\sigma)$ and express $\sigma$ by

$$
\sigma = c_1 c_2 \cdots c_{\ell(\sigma)},
$$

where $c_\lambda$ denotes a cyclic permutation

$$
c_\lambda = (c_{\lambda}(1)c_{\lambda}(2) \cdots c_{\lambda}(q_\lambda)), \quad 1 \leq q_\lambda \leq n, \quad 1 \leq \lambda \leq \ell(\sigma).
$$

For each $1 \leq \lambda \leq \ell(\sigma)$, we write the set of entries $\{c_{\lambda}(j)\}_{j=1}^{q_\lambda}$ of $c_\lambda$ simply as $\{c_\lambda\}$, in which the periodicity $c_{\lambda}(j + q_\lambda) = c_{\lambda}(j), 1 \leq j \leq q_\lambda$ is assumed. By definition, for each $1 \leq \lambda \leq \ell(\sigma)$, $c_{\lambda}(j), 1 \leq j \leq q_\lambda$ are distinct indices chosen from $\mathfrak{I}_n$, $\{c_\lambda\} \cap \{c_{\lambda'}\} = \emptyset$ for $1 \leq \lambda \neq \lambda' \leq \ell(\sigma)$, and $\sum_{\lambda=1}^{\ell(\sigma)} q_\lambda = n$. The determinant $\det_{1 \leq j, k \leq n} \left[ R\xi(\tau_j, x_{j,j}; \tau_k, x_{k,k}) \right]$ is written as

$$
\sum_{\sigma \in S_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} R\xi(\tau_{c_\lambda(j)}, x_{c_\lambda(j)}; \tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)})
$$

$$
= \sum_{\sigma \in S_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} \left\{ \hat{g}_{\tau_{c_\lambda(j)}, \tau_{c_\lambda(j+1)}} x_{c_\lambda(j)} x_{c_\lambda(j+1)} \right\} - 1 (\tau_{c_\lambda(j)} > \tau_{c_\lambda(j+1)}) \hat{p}(\tau_{c_\lambda(j)} - \tau_{c_\lambda(j+1)}, x_{c_\lambda(j)} x_{c_\lambda(j+1)})
$$

(5.6)
where we use the notation (3.18) with (3.17).

We will perform binomial expansions in (5.6). In order to show the result, we introduce the following notations. For each cyclic permutation \( c_\lambda \), we consider a subset of \( \{ c_\lambda \} \),

\[
C(c_\lambda) = \left\{ c_\lambda(j) \in \{ c_\lambda \} : \tau_{c_\lambda(j)} > \tau_{c_\lambda(j+1)} \right\}.
\]

Choose \( M_\lambda \) such that \( \{ c_\lambda \} \setminus C(c_\lambda) \subset M_\lambda \subset \{ c_\lambda \} \), and define \( M_\lambda^c = \{ c_\lambda \} \setminus M_\lambda \). Therefore if we put

\[
G(c_\lambda, M_\lambda) = \int_{\Lambda(c_\lambda)} \prod_{j=1}^{q_\lambda} \left\{ dx_{c_\lambda(j)} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \beta(\tau_{c_\lambda(j)} - \tau_{c_\lambda(j+1)}; x_{c_\lambda(j)}|x_{c_\lambda(j+1)}) 1(c_\lambda(j) \in M_\lambda^c) \right\},
\]

the LHS of (5.5) is expanded as

\[
\frac{1}{\prod_{m=1}^{M} N_m!} \sum_{\sigma \in S_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \sum_{M^c_\lambda : \{ c_\lambda \} \setminus C(c_\lambda) \subset M_\lambda \subset \{ c_\lambda \}} (-1)^{2 M^c_\lambda} G(c_\lambda, M_\lambda). \tag{5.8}
\]

Using only the entries of \( M_\lambda \), we can define a subcycle \( \hat{c}_\lambda \) of \( c_\lambda \) uniquely as follows. For each \( 1 \leq j \leq q_\lambda \) with \( c_\lambda(j) \in M_\lambda \), we define

\[
\tilde{j} = \min\{ k > j : c_\lambda(k) \in M_\lambda \}, \quad \hat{j} = \max\{ k < j : c_\lambda(k) \in M_\lambda \}.
\tag{5.9}
\]

Since \( c_\lambda \) is a cyclic permutation, \( \hat{q}_\lambda \equiv 2 M_\lambda \geq 1 \). Let \( j_1 = \min\{ 1 \leq i \leq q_\lambda : c_\lambda(j) \in M_\lambda \} \). If \( \hat{q}_\lambda \geq 2 \), define \( j_{k+1} = j_k + 1 \leq k \leq \hat{q}_\lambda - 1 \). Then \( \tilde{c}_\lambda = (c_\lambda(j_1) c_\lambda(j_2) \cdots c_\lambda(j_{\hat{q}_\lambda})) \equiv (c_\lambda(j_1) c_\lambda(j_2) \cdots c_\lambda(j_{\hat{q}_\lambda})) \). Moreover, we decompose the set \( M_\lambda \) into \( M \) subsets, \( M_\lambda = \bigcup_{m=1}^{M} J_m^\lambda \), by letting

\[
J_m^\lambda = J_m^\lambda(c_\lambda, M_\lambda) = \left\{ c_\lambda(j) \in M_\lambda : \hat{j} < k \leq j, \text{ s.t. } c_\lambda(k) \in (m) \right\}, \quad 1 \leq m \leq M. \tag{5.10}
\]

Note that by definition \( J_m^\lambda \cap J_m^{\lambda'} \neq \emptyset, m \neq m' \) in general, and \( J_1^\lambda = I_{N_1} \cap M_\lambda = I_{N_1} \cap \{ c_\lambda \} \), \( J_m^\lambda \subset I_{\sum_{k=1}^{m} N_k} \) for \( 2 \leq m \leq M \), \( J_m^\lambda \cap I^{(k)} \subset J_k^\lambda \) for \( 1 \leq k < m \leq M \).

Now we prove the following lemma.

**Lemma 5.1** The quantity (5.7) is equal to

\[
\int_{\Lambda M^c_\lambda} \prod_{j : \hat{c}_\lambda(j) \in M_\lambda} \xi(dv_{c_\lambda(j)}) E_v \left[ \prod_{m=1}^{M} \prod_{j_m \in J_m^\lambda} \chi_{\tau_m}(\hat{V}_{j_m}(t_m)) \prod_{j=1}^{\hat{q}_\lambda} \hat{M}_t^{\hat{c}_\lambda(j)}(T, \hat{V}_{\hat{c}_\lambda(j+1)}(T)) \right]. \tag{5.11}
\]

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Proof of Lemma 5.1. We note that if we set

$$F(\{x_{c\lambda(k)} : c\lambda(k) \in M^c_{\lambda}\})$$

$$= \int_{\Lambda^{M\lambda}} \prod_{j : c\lambda(j) \in M\lambda} \left\{ dx_{c\lambda(j)} \chi_{c\lambda(j)}(x_{c\lambda(j)}) \hat{G}_{c\lambda(j), c\lambda(j+1)}(x_{c\lambda(j)}, x_{c\lambda(j+1)}) \right\}$$

$$\times \prod_{k : c\lambda(k) \in M^c_{\lambda}} \hat{p}(\tau_{c\lambda(k) - \tau_{c\lambda(k+1)}}, x_{c\lambda(k)} | x_{c\lambda(k+1)}), \tag{5.12}$$

which is the integral only over $\Lambda^{M\lambda}$, then (5.7) is obtained by performing the integral of it also over $\Lambda^{M^c\lambda} = \Lambda^{(c\lambda)} \setminus \Lambda^{M\lambda}$,

$$G(c\lambda, M\lambda) = \int_{\Lambda^{M^c\lambda}} \prod_{k : c\lambda(k) \in M^c_{\lambda}} \left\{ dx_{c\lambda(k)} \chi_{c\lambda(k)}(x_{c\lambda(k)}) \right\} F(\{x_{c\lambda(k)} : c\lambda(k) \in M^c_{\lambda}\}). \tag{5.13}$$

In (5.12), use the definition (3.17) for $\hat{G}_{c\lambda(j), c\lambda(j+1)}(x_{c\lambda(j)}, x_{c\lambda(j+1)})$ by putting the integral variables to be $v = v_{c\lambda(j)}$. We obtain

$$F(\{x_{c\lambda(k)} : c\lambda(k) \in M^c_{\lambda}\})$$

$$= \int_{\Lambda^{M\lambda}} \prod_{j : c\lambda(j) \in M\lambda} \xi(dv_{c\lambda(j)}) \int_{\Lambda^{M\lambda}} \prod_{j : c\lambda(j) \in M\lambda} \left\{ dx_{c\lambda(j)} \hat{p}(\tau_{c\lambda(j)}, x_{c\lambda(j)} | v_{c\lambda(j)}) \chi_{c\lambda(j)}(x_{c\lambda(j)}) \right\}$$

$$\times \prod_{j : c\lambda(j) \in M\lambda} M^c_{\xi}(\tau_{c\lambda(j+1)}, x_{c\lambda(j+1)}) \prod_{k : c\lambda(k) \in M^c_{\lambda}} \hat{p}(\tau_{c\lambda(k) - \tau_{c\lambda(k+1)}}, x_{c\lambda(k)} | x_{c\lambda(k+1)})$$

$$= \int_{\Lambda^{M\lambda}} \prod_{j : c\lambda(j) \in M\lambda} \xi(dv_{c\lambda(j)}) E_{\mathbf{v}} \left[ \prod_{j : c\lambda(j) \in M\lambda} \left\{ \chi_{c\lambda(j)}(\hat{V}_{c\lambda(j)}(\tau_{c\lambda(j)})) \right. \right.$$

$$\times M^c_{\xi}(\tau_{c\lambda(j+1)}, \hat{V}_{c\lambda(j+1)}(\tau_{c\lambda(j+1)})) 1(c\lambda(j+1) \in M\lambda)$$

$$\times M^c_{\xi}(\tau_{c\lambda(j+1)}, x_{c\lambda(j+1)}) 1(c\lambda(j+1) \in M^c_{\lambda}) \left. \right\}$$

$$\times \prod_{k : c\lambda(k) \in M^c_{\lambda}} \left\{ \hat{p}(\tau_{c\lambda(k) - \tau_{c\lambda(k+1)}}, x_{c\lambda(k)} | \hat{V}_{c\lambda(k+1)}(\tau_{c\lambda(k+1)})) 1(c\lambda(k+1) \in M^c_{\lambda}) \right.$$

$$\times \hat{p}(\tau_{c\lambda(k) - \tau_{c\lambda(k+1)}}, x_{c\lambda(k)} | x_{c\lambda(k+1)}) 1(c\lambda(k+1) \in M^c_{\lambda}) \right]. \]$$
We perform integration over $v$ w.r.t. $E$ coincides with the conditional expectation of $\int_{\Lambda}^{M_\lambda} \prod_{j,c_\lambda(j) \in M_\lambda} \xi(dv_{c_\lambda(j)}) E\left[ \prod_{j,c_\lambda(j) \in M_\lambda} \chi_{\tau_{c_\lambda(j)}}(\hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j)})) \times \prod_{j,c_\lambda(j),c_\lambda(j+1) \in M_\lambda} \hat{M}^{v_{c_\lambda(j)}}_{\xi}(\tau_{c_\lambda(j+1)}; \hat{V}_{c_\lambda(j+1)}(\tau_{c_\lambda(j+1)})) \right. \\
\times \int_{\Lambda}^{M_\lambda} \prod_{k,c_\lambda(k) \in M_\lambda} \left\{ dx_{c_\lambda(k)} \chi_{\tau_{c_\lambda(k)}}(x_{c_\lambda(k)}) \right\} \\
\times \prod_{k,c_\lambda(k),c_\lambda(k+1) \in M_\lambda} \hat{p}(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | \hat{V}_{c_\lambda(k+1)}(\tau_{c_\lambda(k+1)})) \\
\times \prod_{k,c_\lambda(k),c_\lambda(k+1) \in M_\lambda} \hat{p}(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | x_{c_\lambda(k+1)}) \\
\times \prod_{j,c_\lambda(j) \in M_\lambda, c_\lambda(j+1) \in M_\lambda} \hat{M}^{v_{c_\lambda(j)}}_{\xi}(\tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)}) \right]. \tag{5.14}

We perform integration over $x_{c_\lambda(k)}$'s for $c_\lambda(k) \in M_\lambda$ before taking the expectation $E\psi$. That is, integrals over $x_{c_\lambda(k)}$'s with indices in intervals $\lambda < k < j$ for all $j$, s.t. $c_\lambda(j) \in M_\lambda$ are done. For each $j$, s.t. $c_\lambda(j) \in M_\lambda$, if $\lambda < j < j - 1$,

$$
\chi_{\tau_{c_\lambda(j)}}(\hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j)})) \left\{ \prod_{k=\lambda+1}^{j-1} \int_{\Lambda} dx_{c_\lambda(k)} \chi_{\tau_{c_\lambda(k)}}(x_{c_\lambda(k)}) \right\} \\
\times \hat{p}(\tau_{c_\lambda(j-1)} - \tau_{c_\lambda(j)}, x_{c_\lambda(j-1)} | \hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j)})) \\
\times \prod_{l=\lambda+2}^{j-1} \hat{p}(\tau_{c_\lambda(l-1)} - \tau_{c_\lambda(l)}, x_{c_\lambda(l-1)} | x_{c_\lambda(l)}) \hat{M}^{v_{c_\lambda(l)}}_{\xi}(\tau_{c_\lambda(l+1)}, x_{c_\lambda(l+1)}) \\
\end{array}
$$

coincides with the conditional expectation of

$$
\prod_{k=\lambda+1}^{j-1} \chi_{\tau_{c_\lambda(k)}}(\hat{V}_{c_\lambda(k)}(\tau_{c_\lambda(k)})) \hat{M}^{v_{c_\lambda(k)}}_{\xi}(\tau_{c_\lambda(k+1)}, \hat{V}_{c_\lambda(k+1)}(\tau_{c_\lambda(k+1)})) \\
$$

w.r.t. $E\psi[\mid \hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j)})]$. Since

$$
\prod_{j,c_\lambda(j) \in M_\lambda} \hat{M}^{v_{c_\lambda(j)}}_{\xi}(\tau_{c_\lambda(j+1)}, \hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j+1)})) = \prod_{j,c_\lambda(j) \in M_\lambda} \hat{M}^{v_{c_\lambda(j)}}_{\xi}(\tau_{c_\lambda(j+1)}, \hat{V}_{c_\lambda(j)}(\tau_{c_\lambda(j+1)})),
$$

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(5.14) is equal to
\[
\int_{A^M} \prod_{j:\lambda(j) \in \Lambda^M} \xi(dv_{c\lambda(j)}) \\
\times \mathbb{E}_V \left[ \prod_{j:\lambda(j) \in \Lambda^M} \left\{ \prod_{k=j+1}^{j} \chi_{\tau_{c\lambda(k)}(\tau_{c\lambda(k+1)})} \hat{M}^{v_{c\lambda(j)}}_{\xi}(\tau_{c\lambda(j+1)}; \hat{V}_{c\lambda(j)}(\tau_{c\lambda(j+1)})) \right\} \right].
\]

Then, by definition (5.10), we arrive at the expression (5.11) of \( G(c_\lambda, M_\lambda) \), if we use the martingale property (1.23).

Let \( M = \bigcup_{\lambda=1}^{M} M_\lambda \). Since \( n - \sum_{\lambda=1}^{M} M_\lambda = \sharp M \), the LHS of (5.5), which is written above as (5.8), becomes now
\[
\frac{1}{\prod_{m=1}^{M} N_m!} \sum_{\sigma \in S_n} \sum_{\mathcal{M} : \mathcal{I}_n \setminus \bigcup_{\lambda=1}^{M} M_\lambda \subseteq \mathcal{I}_n} (-1)^{\sharp \mathcal{M} - \ell(\sigma)} \int_{A^M} \prod_{\lambda=1}^{M} \prod_{j: \lambda(j) \in \Lambda^M} \xi(dv_{c\lambda(j)}) \\
\times \mathbb{E}_V \left[ \prod_{\lambda=1}^{\ell(\sigma)} \left\{ \prod_{m=1}^{M} \prod_{j: \lambda(j) \in \Lambda^M} \chi_{t_m}(\hat{V}_{j_m}(t_m)) \hat{\mathcal{M}}_{\xi}^{v_{c\lambda(j)}}(T; \hat{V}_{c\lambda(j+1)}(T)) \right\} \right].
\]

We define
\[
\hat{\sigma} \equiv \hat{c}_1 \hat{c}_2 \cdots \hat{c}_{\ell(\sigma)}
\]
and
\[
\mathcal{J}_m \equiv \bigcup_{\lambda=1}^{\ell(\sigma)} \Lambda^M, \quad 1 \leq m \leq M.
\]

Note that \( \ell(\hat{\sigma}) = \ell(\sigma) \). The obtained \( (\mathcal{J}_m)_{m=1}^{M} \)'s form a collection of series of index sets satisfying the following conditions, which we write as \( \mathcal{J}(\{N_m\}_{m=1}^{M}) \):
\[
\mathcal{J}_1 = \mathbb{I}_{N_1}, \quad \mathcal{J}_m \subset \mathbb{I}_{\sum_{k=1}^{m-1} N_k} \quad \text{for} \quad 2 \leq m \leq M, \\
\mathcal{J}_m \cap \mathbb{I}_{\mathbb{I}_{(k)}(m)} \subset \mathcal{J}_k \quad \text{for} \quad 1 \leq k < m \leq M, \quad \text{and} \\
\sharp \mathcal{J}_m = N_m \quad \text{for} \quad 1 \leq m \leq M.
\]

For each \( (\mathcal{J}_m)_{m=1}^{M} \in \mathcal{J}(\{N_m\}_{m=1}^{M}) \), we put
\[
A_1 = 0 \quad \text{and} \quad A_m = \sharp \left( \mathcal{J}_m \cap \mathbb{I}_{\sum_{k=1}^{m-1} N_k} \right) = \sharp \left( \mathcal{J}_m \cap \bigcup_{k=1}^{m-1} \mathcal{J}_k \right), \quad 2 \leq m \leq M.
\]

Then, if we put \( M = \bigcup_{m=1}^{M} \mathcal{J}_m, \sharp M = \sum_{m=1}^{M} (N_m - A_m) \), which means that from the original index set \( \mathbb{I}_n = \bigcup_{m=1}^{M} \mathbb{I}(m) \) with \( \sharp \mathbb{I}(m) = N_m, 1 \leq m \leq M \), we obtain a subset \( M \) by eliminating \( A_m \) elements at each level \( 1 \leq m \leq M \). By this reduction, we obtain \( \hat{\sigma} \in S(M) \) from \( \sigma \in S_n \).
It implies that, for all \( \hat{\sigma} \in \mathcal{S}(M) \), the number of \( \sigma \)'s in \( \mathcal{S}_n \) which give the same \( \hat{\sigma} \) and \((J_m)^M_{m=1}\) by this reduction is given by \( \prod_{m=1}^M A_m! \), where 0! = 1. Then (5.15) is equal to

\[
\sum_{\text{max}_m \{N_m\} \leq \# M \leq (J_m)^M_{m=1} \in \mathcal{J}(\{N_m\}^M_{m=1})} \sum_{\text{max}_m \{N_m\} \leq \# M \leq (J_m)^M_{m=1} \in \mathcal{J}(\{N_m\}^M_{m=1})} \prod_{m=1}^M A_m! \prod_{m=1}^M N_m! \sum_{\hat{\sigma} \in \mathcal{S}(M)} (-1)^{\# M - \ell(\hat{\sigma})} \\
\times \# M! \int_{W_{2M}} \xi \otimes M (dv) E_v \left[ \prod_{m=1}^M \prod_{j \in J_m} \chi_{tm} (\hat{V}_{jm}(t_m)) \prod_{\lambda=1}^M \prod_{j \in J_m} \hat{M}_{\xi}(\hat{\lambda}) (T, \hat{V}_{j}(T)) \right] \\
= \sum_{\text{max}_m \{N_m\} \leq \# M \leq (J_m)^M_{m=1} \in \mathcal{J}(\{N_m\}^M_{m=1})} \sum_{\text{max}_m \{N_m\} \leq \# M \leq (J_m)^M_{m=1} \in \mathcal{J}(\{N_m\}^M_{m=1})} \# M! \prod_{m=1}^M A_m! \\
\times \int_{W_{2M}} \xi \otimes M (dv) E_v \left[ \prod_{m=1}^M \prod_{j \in J_m} \chi_{tm} (\hat{V}_{jm}(t_m)) \det_{j,k \in M} \left[ \hat{M}_{\xi}(T, \hat{V}_{j}(T)) \right] \right]. \tag{5.17}
\]

Assume 1 \( \leq p \leq N \), 0 \( \leq A_m \leq N_m \), 2 \( \leq m \leq M \) and set \( A_1 = 0 \). Consider

\[
\Delta_1 = \left\{ (J_m)^M_{m=1} \in \mathcal{J}(\{N_m\}^M_{m=1}) : \# \left( \bigcup_{m=1}^M J_m \right) = p, \right. \\
\# \left( J_m \cap \bigcup_{k=1}^{m-1} J_k \right) = A_m, 2 \leq m \leq M \right\},
\]

\[
\Delta_2 = \left\{ (J_m)^M_{m=1} : \# J_m = N_m, 1 \leq m \leq M, \bigcup_{m=1}^M J_m = I_p, \right. \\
\# \left( J_m \cap \bigcup_{k=1}^{m-1} J_k \right) = A_m, 2 \leq m \leq M \right\}.
\]

Since \( \hat{V}_j(\cdot) \)'s are i.i.d. in \( P_v \), the integral in (5.17) has the same value for all \((J_m)^M_{m=1} \in \Delta_1\) with \( \bigcup_{m=1}^M J_m = M \) and it is also equal to

\[
\int_{W_{p}} \xi \otimes p (dv) E_v \left[ \prod_{m=1}^M \prod_{j \in J_m} \chi_{tm} (\hat{V}_{jm}(t_m)) \det_{j,k \in I_p} \left[ \hat{M}_{\xi}(T, \hat{V}_{j}(T)) \right] \right] 
\]

for all \((J_m)^M_{m=1} \in \Delta_2\).

In \( \Delta_1 \), for each \( 2 \leq m \leq M \), \( A_m \) elements in \( J_m \) are chosen from \( \bigcup_{k=1}^{m-1} J_k \), in which \( \# (\bigcup_{k=1}^{m-1} J_k) = \sum_{k=1}^{m-1} (N_k - A_k) \), and the remaining \( N_m - A_m \) elements in \( J_m \) are from \( I^{(m)} \) with \( \# I^{(m)} = N_m \). Then

\[
\# \Delta_1 = \prod_{m=2}^M \left( \sum_{k=1}^{m-1} (N_k - A_k) \right) \left( \frac{N_m}{A_m} \right). 
\]
In $\Delta_2$, on the other hand, $N_1$ elements in $\mathbb{J}_1$ is chosen from $\mathbb{I}_p$, and then for each $2 \leq m \leq M$, $A_m$ elements in $\mathbb{J}_m$ are chosen from $\bigcup_{k=1}^{m-1} \mathbb{J}_k$ with $\#(\bigcup_{k=1}^{m-1} \mathbb{J}_k) = \sum_{k=1}^{m-1} (N_k - A_k)$ and the remaining $N_m - A_m$ elements in $\mathbb{J}_m$ are from $\mathbb{I}_p \setminus \bigcup_{k=1}^{m-1} \mathbb{J}_k$ with $\#(\mathbb{I}_p \setminus \bigcup_{k=1}^{m-1} \mathbb{J}_k) = p - \sum_{k=1}^{m-1} (N_k - A_k)$. Then

$$\#\Delta_2 = \binom{p}{N_1} \prod_{m=2}^{M} \left( \sum_{k=1}^{m-1} (N_k - A_k) \right) \left( p - \sum_{k=1}^{m-1} (N_k - A_k) \right).$$

Since $\sum_{m=1}^{M} (N_m - A_m) = p$, we see $\#\Delta_2/\#\Delta_1 = p! \prod_{m=1}^{M} A_m!/N_m!$. Then (5.17) is equal to the RHS of (5.5) and the proof is completed. 

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**A On the identity (1.17)**

The following determinantal identity was given as Lemma 2.2 in [20] and as Lemma 3 in [21] proved by Krattenthaler.

**Lemma A.1** Let $X_1, \ldots, X_N, A_2, \ldots, A_N$ and $B_2, \ldots, B_N$ be indeterminates. Then there holds

$$\det_{1 \leq j, k \leq N} \left[ (X_j + A_N)(X_j + A_{N-1}) \cdots (X_j + A_{k+1})(X_j + B_k)(X_j + B_{k-1}) \cdots (X_j + B_2) \right]$$

$$= \prod_{1 \leq j < k \leq N} (X_j - X_k) \prod_{2 \leq j \leq k \leq N} (B_j - A_k). \quad \text{(A.1)}$$

In the identity (A.1), set

$$X_j = x_j, \quad 1 \leq j \leq N,$$

$$A_j = -u_j, \quad B_j = -u_{j-1}, \quad 2 \leq j \leq N.$$ 

Then we find

$$H(u, x) \equiv \det_{1 \leq j, k \leq N} \left[ \prod_{1 \leq \ell \leq N, \ell \neq j} (u_\ell - x_k) \right] = (-1)^{N(N-1)/2} h(u) h(x).$$

Assume $\xi = \sum_{j=1}^{N} \delta_{u_j}$ and there is no multiple point; $u_j \neq u_k$ for $j \neq k$. Since

$$\det_{1 \leq j, k \leq N} [\Phi^u_\xi(x_j)] = \frac{H(u, x)}{\prod_{1 \leq j \leq N} \prod_{1 \leq k \leq N; k \neq j} (u_k - u_j)} = \frac{(-1)^{N(N-1)/2} h(u)^2}{H(u, x)},$$

the identity (1.17) is obtained as a special case of (A.1).
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