GLOBAL SOLUTIONS TO THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. An initial-boundary value problem of the three-dimensional incompressible magnetohydrodynamic (MHD) equations is considered in a bounded domain. The homogeneous Dirichlet boundary condition is prescribed on the velocity, and the perfectly conducting wall condition is prescribed on the magnetic field. The existence and uniqueness is established for both the local strong solution with large initial data and the global strong solution with small initial data. Furthermore, the weak-strong uniqueness of solutions is also proved, which shows that the weak solution is equal to the strong solution with certain initial data.

1. Introduction. Magnetohydrodynamics (MHD) concerns the motion of conducting fluids, such as gases, in an electromagnetic field. If a conducting fluid moves in a magnetic field, electric fields are induced and an electric current flow is developed. The magnetic field exerts forces on these currents which considerably modify the hydrodynamic motion of the fluid. On the other hand, the development of electric currents yields a change in the magnetic field. There is a complex interaction between the magnetic field and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. The equations for compressible magnetohydrodynamics consist of the Euler equations of gas dynamics coupled with the Maxwell’s equations of electromagnetic field. The applications of magnetohydrodynamics cover a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics and plasma physics. The equations of three-dimensional viscous incompressible magnetohydrodynamic flows

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have the following form (see [3, 19, 29]):

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) - \mu \triangle u + \nabla P &= (\nabla \times H) \times H, \\
\frac{\partial H}{\partial t} - \nabla \times (u \times H) &= -\nabla \times (\nu \nabla \times H), \\
\nabla \cdot u &= 0, \quad \nabla \cdot H = 0,
\end{align*}
\]

(1.1a, 1.1b, 1.1c)

where \( u = u(x,t) \in \mathbb{R}^3, H = H(x,t) \in \mathbb{R}^3 \) are the velocity field of the flow and the magnetic field respectively, \( x \in \mathbb{R}^3, t > 0, P = P(x,t) \) is a scalar function representing the pressure and \( \nabla P \) may be seen as the Lagrange multiplier associated to the constraint \( \nabla \cdot u = 0 \); \( \mu > 0 \) is the kinematic viscosity of the flow and is a constant, \( \nu > 0 \) is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and both are independent of the magnitude and direction of the magnetic field. The symbol \( \otimes \) denotes the usual Kronecker multiplication, i.e. \( (a \otimes b)_{ij} = a_i b_j \) for \( a, b \in \mathbb{R}^3 \). Usually, we refer to (1.1a) as the momentum conservation equation. It is well-known that the electromagnetic fields are governed by the Maxwell’s equations. In magnetohydrodynamics, the displacement current can be neglected ([18, 19]). As a consequence, (1.1b) is called the induction equation. An important feature of the MHD equations is the induction effect, which brings about the strong coupling of the magnetic field and velocity field. Therefore, the MHD equations are much more complex, and the main estimates depend strongly on each other for the magnetic field and velocity field. As for the constraint \( \nabla \cdot H = 0 \), it can be seen as a restriction on the initial value \( H_0 \) since \( (\nabla \cdot H)_t = 0 \).

We shall study the viscous incompressible magnetohydrodynamic flow in a connected bounded domain \( \Omega \subset \mathbb{R}^3 \) with boundary \( \partial \Omega \in C^{2+\varepsilon} \) for some \( \varepsilon > 0 \) by supplementing the system (1.1) with the initial conditions:

\[
(u, H) \big|_{t=0} = (u_0, H_0), \quad \text{for all } x \in \Omega, \tag{1.2}
\]

and the boundary conditions:

\[
\begin{align*}
\mathbf{u} \big|_{\partial \Omega} &= 0, \\
\mathbf{H} \cdot \mathbf{n} \big|_{\partial \Omega} &= 0, \quad (\nabla \times \mathbf{H}) \times \mathbf{n} \big|_{\partial \Omega} = 0, \tag{1.3, 1.4}
\end{align*}
\]

where \( \mathbf{n} \) denotes the unit outward normal on \( \partial \Omega \). The condition (1.3) is the so-called non-slip boundary condition, and the condition (1.4) is known as the perfectly conducting wall condition which describes the case where the wall of container is made of perfectly conductive materials. This boundary condition (1.4) is classical in the theory of magnetohydrodynamics.

There have been a lot of studies on MHD in both incompressible and compressible cases by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [6, 7, 8, 13, 14, 15, 16, 17, 18, 24, 26, 27, 28, 29, 31] and the references cited therein. When the magnetic field is absent (\( H \equiv 0 \)), the system (1.1) is reduced to the incompressible Navier-Stokes equations. It was shown in [10], by making use of the theory of fractional powers of the so-called Stokes operator and the theory of semi-groups of the operator through a Hilbert space approach, that there exists a unique local strong solution for any initial data \( u_0 \) belonging to the domain of the Stokes operator of order \( \frac{1}{4} \). Furthermore, under a certain smallness condition on the initial data \( u_0 \), the solution can be continued globally in time. There are some well-known results on the existence and uniqueness for weak (and strong) solutions of the incompressible MHD equations which can be found for instance in [7] where a global weak solution and the local
strong solution to the initial boundary value problem were constructed, and properties of such solutions have been examined in [29]. By use of techniques slightly different from those used in [12] for Navier-Stokes equations, it was proved in [29] that, if a solution of the system (1.1)-(1.4) belongs to $L^\infty(0,T;H^1_0(\Omega) \times H^1(\Omega))$, then it is unique and actually a global strong solution. Furthermore, some sufficient conditions for smoothness were presented for the weak solution to the MHD equations in [13] and the local behavior of the solutions (the uniform gradient estimates for smooth solutions and regularity of weak solutions) to the three-dimensional MHD equations were studied in [14]. And recently in [24], the well-posedness of the Cauchy problem for the multi-dimensional incompressible MHD system (1.1)-(1.2) were studied in the framework of Besov spaces. For the ideal MHD system, the existence and uniqueness of local strong solution was proved in a bounded domain [26, 28] and in the whole space [31] with analytic initial data. The inviscid and non-resistive limit of the viscous incompressible MHD solutions was proved in [6].

The existence of global weak solutions to the compressible magnetohydrodynamic equations was obtained in [8, 16, 17] in the spirit of Lions-Feireisl [9, 21] for the compressible Navier-Stokes equations. We refer to [5, 22, 23, 25, 30] and the references therein for more results on the compressible Navier-Stokes equations and [11, 20] for the incompressible case.

The aim of the present paper is to establish the existence and uniqueness of the global (in time) strong solution $(u, H, P)$ (up to an additive constant for $P$) in $(W^{2,q}(\Omega))^3 \times (W^{2,q}(\Omega))^3 \times W^{1,q}(\Omega)$ with $q > 3$ satisfying the incompressible MHD system (1.1) almost everywhere with the initial-boundary conditions (1.2)-(1.4). Under the constraint (1.1c), (1.1a) and (1.1b) reduce to

$$\frac{\partial u}{\partial t} - \mu \Delta u + \nabla P = -u \cdot \nabla u + (\nabla \times H) \times H,$$

$$\frac{\partial H}{\partial t} - \nu \Delta H = -u \cdot \nabla H + H \cdot \nabla u,$$

(1.5a)

(1.5b)

where the notation $u \cdot \nabla u$ is understood to be $(u \cdot \nabla)u$. Our strategy to consider (1.1) in $(W^{2,q}(\Omega))^3 \times (W^{2,q}(\Omega))^3 \times W^{1,q}(\Omega)$ is to linearize it as

$$\frac{\partial u}{\partial t} - \mu \Delta u + \nabla P = -v \cdot \nabla v + (\nabla \times B) \times B,$$  

(1.6a)

$$\frac{\partial H}{\partial t} - \nu \Delta H = -v \cdot \nabla B + B \cdot \nabla v,$$  

(1.6b)

$$\nabla \cdot u = 0, \quad \nabla \cdot H = 0,$$  

(1.6c)

for some given $v \in \mathbb{R}^3$ and $B \in \mathbb{R}^3$. One of the motivations of making such a linearization is that (1.1a) is the evolutionary incompressible Navier-Stokes equation with the source term $(\nabla \times H) \times H$ while (1.1b) is the parabolic system in terms of $H$, so that we can use the maximal regularity properties of the Stokes operator and the parabolic operator (see Theorem 3.1 and Theorem 3.2 below). Recently, in [15] a mixed discontinuous Galerkin (DG) method was proposed and analyzed for such a linear stationary incompressible magnetohydrodynamics model problem, stating a priori energy norm error estimates for the proposed DG method. We shall first establish the local existence and uniqueness of the strong solution with general initial data via an iteration scheme, then we prove the global existence for small initial data by establishing the global a priori estimates. The basic idea in the present paper for constructing the solutions is in the spirit of [4]. To overcome the difficulties arising from the strong coupling of the fluid and the magnetic field, we
need to develop new estimates for the magnetic field $H$. The global weak solution of the MHD system (1.1)-(1.4) was obtained in [29], but the uniqueness is still an open problem. We will show that, by conducting independent and elementary proofs in which the strong coupling of the magnetic field and velocity field has important effects, when the strong solution exists, any weak solution constructed in [29] must be equal to the unique strong solution, which is called the weak-strong uniqueness. See [4, 24] for related results.

The paper is organized as follows. In Section 2, we state our main results on local and global existence of strong solution, as well as the weak-strong uniqueness. In Section 3, we recall the maximal regularities for the Stokes operator and the parabolic operator, and also show some $L^\infty$ estimates. In Section 4, we give the proof of the local existence. In Section 5, we prove the global existence. Finally in Section 6, we show the weak-strong uniqueness.

2. Main results. Throughout this paper, the standard notations for Sobolev spaces $W^{s,q}(\Omega)$ (Hilbert spaces $H^s(\Omega)$ when $q = 2$) will be used, and we denote by $W_0^{s,q}(\Omega)(H_0^s(\Omega))$ the subspace of $W^{s,q}(\Omega)(H^s(\Omega))$ of functions vanishing on $\partial\Omega$.

For $p \in [1, +\infty]$, we denote by $L^p(0, T; X)$ the set of Bochner measurable $X$-valued time dependent functions $\varphi$ such that $t \mapsto \|\varphi\|_X$ belongs to $L^p(0, T)$, and the corresponding Lebesgue norm is denoted by $\|\cdot\|_{L^p_T(X)}$. Denote the Sobolev space $W^{1,p}(0, T; X) := \{\varphi \mid \varphi \in L^p(0, T; X), \varphi_t \in L^p(0, T; X)\}$. We use the letter $C$ to denote a generic constant that can be explicitly computed in terms of known quantities, and the exact value denoted by $C$ may therefore change from line to line in a given computation.

If $A = ((a_{ij}))$ and $B = ((b_{ij}))$ are $3 \times 3$ matrices, then denote

$$A : B = \sum_{i,j=1}^3 a_{ij}b_{ij} \quad \text{and} \quad |A| = (A : A)^{1/2} = \left(\sum_{i,j=1}^3 a_{ij}^2\right)^{1/2}.$$ 

$|u|_p$ = norm in the space $L^p(\Omega)$ with $p \geq 1$; $\|u\|_s = \text{norm in the Sobolev space } H^s(\Omega)$ of order $s$ on $L^2(\Omega)$; $\|u\|_{s,p} = \text{norm in the Sobolev space } W^{s,p}(\Omega)$ of order $s$ on $L^p(\Omega)$ (s is a real number). Also, denote

$$Q_T := \Omega \times [0, T], \quad \text{for any fixed time } T > 0.$$ 

Next, let us define the function spaces in which the existence of strong solution $(u, H, P)$ is going to be established.

**Definition 2.1.** For $T > 0$ and $1 < p, q < \infty$, we denote by $M_T^{p,q}$ the set of triplets $(u, H, P)$ such that

$$u \in C([0, T]; D_A^{1 - \frac{1}{p} + \frac{1}{p}}) \cap L^p(0, T; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^p(0, T; L^q(\Omega)), \quad \nabla \cdot u = 0,$$

$$H \in C([0, T]; B_{q,p}^{2(1 - \frac{1}{q})}) \cap L^p(0, T; W^{2,q}(\Omega)), \quad \frac{\partial H}{\partial t} \in L^p(0, T; L^q(\Omega)),$$

and

$$P \in L^p(0, T; W^{1,q}(\Omega)), \quad \int_\Omega Pdx = 0.$$
The corresponding norm is denoted by $\| \cdot \|_{M_{p,q}^{T}}$ and obviously, $M_{p,q}^{T}$ is a Banach space.

**Remark 2.1.** We note that the condition
\[ \int_{\Omega} P \, dx = 0 \]
in Definition 2.1 holds automatically if we replace $P$ by $P - \frac{1}{|\Omega|} \int_{\Omega} P \, dx$ in (1.1). Also, in the above definition, the space $D_{A_{q}}^{1-\frac{1}{p}}$ stands for non-homogeneous fractional domains of the Stokes operator in $L^{q}(\Omega)$ (cf. Section 2.3 in [4]). Roughly, the vector-fields of $D_{A_{q}}^{1-\frac{1}{p}}$ are vectors which have $2 - \frac{2}{p}$ derivatives in $L^{q}(\Omega)$, are divergence-free and vanish on $\partial \Omega$. The Besov space $B_{q,p}^{2(1-\frac{1}{p})}$ (for definition, see [2]) can be regarded as the interpolation space between $L^{q}(\Omega)$ and $W_{2,q}^{2}(\Omega)$, which is given by
\[ B_{q,p}^{2(1-\frac{1}{p})} = (L^{q}(\Omega), W_{2,q}^{2}(\Omega))_{1-\frac{1}{p},p}, \]
(cf. Theorem 6.24 in [2]). We note that, from Proposition 2.5 in [4],
\[ D_{A_{q}}^{1-\frac{1}{p}} \hookrightarrow B_{q,p}^{2(1-\frac{1}{p})} \cap X^{q}, \]
where for $1 < q < \infty$, $X^{q}$ is the completion in $L^{q}$ of the set of solenoidal vector-fields with coefficients in $C_{0}^{\infty}(\Omega)$ and it is well known (see [4, 30]) that for $C^{1}$ domains,
\[ X^{q} = \{ u \in L^{q}(\Omega)^{3} | \nabla \cdot u = 0 \text{ in } \Omega \text{ and } u \cdot n = 0 \text{ on } \partial \Omega \}. \] (2.1)

The local existence will be shown by using an iterative scheme, and if the initial data is sufficiently small in some suitable function spaces, the solution is indeed global in time. More precisely, our existence and uniqueness results read as follows:

**Theorem 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^{3}$ with $C^{2+\varepsilon}$ boundary, $\varepsilon > 0$. Assume $1 < p, q < \infty$ with $\frac{2}{p}(1 - \frac{2}{q}) \in (0, 1)$ and $u_{0} \in D_{A_{q}}^{1-\frac{1}{p}}$ and $H_{0} \in B_{q,p}^{2(1-\frac{1}{p})}$. Then,

1. There exists a $T_{0} > 0$, such that, the system (1.1) with the initial-boundary conditions (1.2)-(1.4) has a unique local strong solution $(u, H, P) \in M_{p,q}^{T}$ in $Q_{T_{0}}$;
2. Moreover, there exists a $\delta_{0} > 0$, such that, if the initial data satisfy
\[ \| u_{0} \|_{D_{A_{q}}^{1-\frac{1}{p}}} \leq \delta_{0}, \quad \| H_{0} \|_{B_{q,p}^{2(1-\frac{1}{p})}} \leq \delta_{0}, \]
then the system (1.1) with the initial-boundary conditions (1.2)-(1.4) has a unique global strong solution $(u, H, P) \in M_{p,q}^{T}$ in $Q_{T}$ for all $T > 0$.

According to [29], for the given initial-boundary conditions (1.2)-(1.4), we know only that there exists at least a weak solution to (1.1). More precisely, $(u, H)$ is called a weak solution to (1.1) with the initial-boundary conditions (1.2)-(1.4) in $Q_{T}$ if
\[ u \in L^{2}(0, T; H_{0}^{1}(\Omega)) \cap L^{\infty}(0, T; L^{2}(\Omega)) \]
and
\[ H \in L^{2}(0, T; H^{1}(\Omega)) \cap L^{\infty}(0, T; L^{2}(\Omega)) \]
satisfy system (1.1) in the sense of distributions, i.e., for all \( v \in C^0_0(0,T; H^1_0(\Omega)) \) with \( \nabla \cdot v = 0 \) and \( B \in C^0_0(0,T; H^1(\Omega)) \), we have

\[
- \int_0^T \int_{\Omega} u \cdot \frac{\partial v}{\partial t} \, dx \, dt + \int_0^T \int_{\Omega} \nabla u : \nabla v \, dx \, dt + \int_0^T \int_{\Omega} u \cdot \nabla v \, dx \, dt = \int_0^T \int_{\Omega} H \nabla H \cdot v \, dx \, dt,
\]

\[
- \int_0^T \int_{\Omega} H \cdot \frac{\partial B}{\partial t} \, dx \, dt + \int_0^T \int_{\Omega} (\nabla \times H) \cdot (\nabla \times B) \, dx \, dt + \int_0^T \int_{\Omega} u \cdot \nabla H \cdot B \, dx \, dt = \int_0^T \int_{\Omega} \nabla H \cdot H \cdot B \, dx \, dt,
\]

\( u(0) = u_0, \quad H(0) = H_0, \)

and also the energy inequality holds:

\[
\frac{1}{2} \int_{\Omega} (|u(t)|^2 + |H(t)|^2) \, dx + \int_0^T \int_{\Omega} (|\nabla u|^2 + |\nabla \times H|^2) \, dx \, ds \leq \frac{1}{2} \int_{\Omega} (|u_0|^2 + |H_0|^2) \, dx.
\] (2.2)

In this weak formulation, the pressure \( P \) can be determined as in the Navier-Stokes equations, see Galdi [11]. We state here the existence of weak solution in Theorem 3.1 of [29]:

**Proposition 2.1.** Assume that \( u_0 \in L^2(\Omega)^3 \) and \( H_0 \in L^2(\Omega)^3 \). Then the system (1.1) with the initial-boundary conditions (1.2)-(1.4) has a global weak solution \((u, H)\) such that

\[ u \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \]

and

\[ H \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \]

for all \( T > 0 \).

**Remark 2.2.** (1) It is not known whether all the weak solutions satisfy the above energy inequality (2.2) or whether the inequality (2.2) is actually an equality. On the other hand, we refer the readers to [27] for the upper and lower bounds for the rate of decay of the total energy and the magnetic energy of weak solutions to the magnetohydrodynamics equations. (2) The theory of uniqueness of weak solutions is not complete (see [7] and [29]): we do not know whether the weak solution is unique (or what further condition could perhaps make it unique).

Actually, for the same initial-boundary conditions, the relation between its weak solution and strong solution is that as long as a strong solution exists, it is unique in the class of weak solutions, which we state as follows:

**Theorem 2.2.** Assume that \( u_0 \in D_{A_q}^{1-\frac{1}{p}} \) and \( H_0 \in B_{q,p}^{2(1-\frac{1}{p})} \). Then its corresponding weak solution to (1.1) with the initial-boundary conditions (1.2)-(1.4) is unique and indeed is equal to its unique strong solution.

Usually, we call this kind of uniqueness as Weak-Strong Uniqueness. For the similar results on the compressible Navier-Stokes equations, we refer the readers to [5, 20].
3. Maximal regularity. In this section, we recall the maximal regularities for the parabolic operator and the Stokes operator, as well as some \(L^\infty\) estimates.

For \(T > 0\), \(1 < p, q < \infty\), denote

\[
W(0, T) := \left( W^{1,p}(0, T; L^q(\Omega)) \right)^3 \cap \left( L^p(0, T; W^{2,q}(\Omega)) \right)^3.
\]

We first recall the maximal regularity for the parabolic operator (cf. Theorem 4.10.7 and Remark 4.10.9 in [1]):

**Theorem 3.1.** Given \(1 < p, q < \infty\), \(\omega_0 \in B^{2(1 - \frac{1}{p})}_{q,p}\) and \(f \in \left( L^p(0, T; L^q(\Omega)) \right)^3\), the Cauchy problem

\[
\begin{cases}
\frac{d}{dt} \omega - \Delta \omega = f, & t \in (0, T), \\
\omega(0) = \omega_0,
\end{cases}
\]

has a unique solution \(\omega \in W(0, T)\), and

\[
\|\omega\|_{W(0, T)} \leq C \left( \|f\|_{L^p_t(L^q)} + \|\omega_0\|_{B^{2(1 - \frac{1}{p})}_{q,p}} \right),
\]

where \(C\) is independent of \(\omega_0\), \(f\) and \(T\). Moreover, there exists a positive constant \(c_0\) independent of \(f\) and \(T\) such that

\[
\|\omega\|_{W(0, T)} \geq c_0 \sup_{t \in (0, T)} \|\omega(t)\|_{B^{2(1 - \frac{1}{p})}_{q,p}}.
\]

**Remark 3.1.** As our boundary condition on the magnetic field \(H\) (see (1.4)) does not coincide with the space \(E_1\) (see [1]) where the prescribed boundary condition has been incorporated in, for the purpose of using the maximal regularity for the parabolic operator in our case, we only need to set

\[
E_0 = \{ H \in (L^q(\Omega))^3 \mid \nabla \cdot H = 0 \text{ in } \Omega \},
\]

and

\[
E_1 = X^{q'} \cap \{ H \in (W^{2,q}(\Omega))^3 \mid (\nabla \times H) \times n = 0 \text{ on } \partial \Omega \},
\]

where \(E_1\) is the domain of the linear operator \(\nabla \times (\nabla \cdot \cdot \cdot )\), endowed with its graph norm which is induced by \(E_0\), and \(X^q\) is defined as (2.1). We see that \((E_0, E_1)\) is a densely injected Banach couple corresponding to \(\nabla \times (\nabla \cdot \cdot \cdot )\).

Next, we recall the maximal regularity for the Stokes equations (cf. Theorem 3.2 in [4]):

**Theorem 3.2.** Let \(\Omega\) be a bounded domain with \(C^{2+\varepsilon}\) boundary of \(\mathbb{R}^3\) and \(1 < p, q < \infty\). Assume that \(u_0 \in D_A^{1 - \frac{1}{q}, p}\) and \(f \in (L^p(\mathbb{R}^3; L^q(\Omega)))^3\). Then the system

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \nabla P = f, & \int_\Omega P \, dx = 0, \\
\nabla \cdot u = 0, \\
u|_{t=0} = u_0, & u|_{\partial \Omega} = 0,
\end{cases}
\]

has a unique solution \((u, P)\) satisfying the following inequality for all \(T \geq 0\):

\[
\|u(T)\|_{D_A^{1 - \frac{1}{q}, p}} + \left( \int_0^T \left( \|\nabla P, \nabla^2 u, \frac{\partial u}{\partial t} \|_{q,p}^p \right)^{\frac{1}{p}} \, dt \right)^\frac{1}{p} \leq C \left( \|u_0\|_{D_A^{1 - \frac{1}{q}, p}} + \left( \int_0^T |f(t)|_{q,p}^p \, dt \right)^{\frac{1}{p}} \right)
\]

(3.1)
with \( C = C(q, p, \Omega) \).

**Remark 3.2.** We notice that (3.1) does not include the estimate for \( \| u \|_{L^p_t(L^q)} \). Indeed, since we consider only in bounded domain \( \Omega \), then there exists a constant \( C = C(q, d(\Omega)) \) such that
\[
\| u \|_{L^2_t} = \| \nabla^2 u \|_q + d(\Omega)^{-1} \| \nabla u \|_q + d(\Omega)^{-2} \| u \|_q \leq C \| \nabla^2 u \|_q,
\]
where \( d(\Omega) \) is the diameter to \( \Omega \), whenever \( u \in W^{2,q}(\Omega) \cap W^{1,q}(\Omega) \) (see Proposition 2.4 in [4]). Therefore, (3.1) can be rewritten as
\[
\| u(T) \|_{D_{\lambda_q}^{1-\frac{1}{p},p}} + \left( \int_0^T \left( \nabla P, u, \nabla^2 u, \frac{\partial u}{\partial t} \right)_{q}^p \, dt \right)^{\frac{1}{p}} 
\leq C \left( \| u_0 \|_{D_{\lambda_q}^{1-\frac{1}{p},p}} + \left( \int_0^T |f(t)|_q^p \, dt \right)^{\frac{1}{p}} \right). \tag{3.2}
\]

Finally, we state some very useful interpolation inequalities (cf. Lemma 4.1 in [4]) and the most of all are the \( L^\infty \) estimates in the spatial variable.

**Lemma 3.1.** Let \( 1 < p, q, r, s < \infty \) satisfy
\[
0 < \frac{p}{2} - \frac{3p}{2r} < 1, \quad \frac{s}{r} = \frac{s}{q} + \frac{1}{q}.
\]
The following inequalities hold true:
\[
\| \nabla f \|_{L^r_t(L^q)} \leq CT^{\frac{1}{s} - \frac{2}{p}} \| f \|_{L^p_t(D_{\lambda_r}^{1-\frac{1}{q},q})}^{1-\theta} \| f \|_{L^p_t(W^{2,r})}^\theta,
\]
\[
\| \nabla f \|_{L^q_t(L^s)} \leq CT^{\frac{1}{s} - \frac{2}{p}} \| f \|_{L^p_t(D_{\lambda_q}^{1-\frac{1}{s},s})}^{1-\theta} \| f \|_{L^p_t(W^{2,r})}^\theta,
\]
for some constant \( C \) depending only on \( \Omega, p, q \) and
\[
\frac{1 - \theta}{p} = \frac{1}{2} - \frac{3}{2r}.
\]

Similarly, we can prove the following inequality:

**Lemma 3.2.** Let \( 1 < p, q < \infty \) satisfy \( 0 < \frac{p}{2} - \frac{3p}{2q} < 1 \), then
\[
\| \nabla f \|_{L^q_t(L^s)} \leq CT^{\frac{1}{s} - \frac{2}{p}} \| f \|_{L^p_t(B_{q,p}^{2(1-\frac{1}{s})})}^{1-\theta} \| f \|_{L^p_t(W^{2,s})}^\theta,
\]
for some constant \( C \) depending only on \( \Omega, p, q \) and
\[
\frac{1 - \theta}{p} = \frac{1}{2} - \frac{3}{2q}.
\]

**Proof.** First, we notice that
\[
(B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{p}}, B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{q}})_{\theta,1} = B_{\infty, \infty}^{\theta} \quad \text{with} \quad \frac{1 - \theta}{p} = \frac{1}{2} - \frac{3}{2q},
\]
see Theorem 6.4.5 in [2]. Also the imbedding \( B_{\infty, \infty}^{\theta} \hookrightarrow L^\infty \) is true due to Theorem 6.2.4 in [2]. Hence,
\[
\| \nabla f \|_{L^\infty} \leq C \| \nabla f \|_{B_{\infty, \infty}^{\theta}} \leq C \| \nabla f \|_{B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{p}, \frac{1}{2}-\frac{2}{q}}}^{\theta} \| \nabla f \|_{B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{p}, \frac{1}{2}-\frac{2}{q}}}^{1-\theta}.
\tag{3.3}
\]
We remark that
\[
B_{q,p}^{2(1-\frac{1}{s})} \hookrightarrow B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{p}, \frac{1}{2}-\frac{2}{q}} \hookrightarrow B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{p}, \frac{1}{2}-\frac{2}{q}}, \quad W^{1,q} \hookrightarrow B_{q, \infty}^{1} \hookrightarrow B_{\infty, \infty}^{\frac{1}{2}-\frac{2}{q}},
\]
see Theorem 6.2.4 and Theorem 6.5.1 in [2]. Hence, according to (3.3) and by Hölder inequality, we deduce that

$$\|\nabla f\|_{L^p(B_\infty,\infty)} \leq C \left( \int_0^T \|f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^\theta \|\nabla f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^{1-\theta} \, dt \right)^\frac{1}{\theta}$$

$$\leq C \left( \int_0^T \|f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^\theta \|\nabla f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^{1-\theta} \, dt \right)^\frac{1}{\theta}$$

$$\leq C \left( \int_0^T \|f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^\theta \|\nabla f\|_{B^{\frac{3}{2}}(\frac{1}{2},T)}^{1-\theta} \, dt \right)^\frac{1}{\theta}$$

$$\leq CT^{\frac{1}{2} - \frac{3}{2}f} \|f\|_{L^p(B^{\frac{3}{2}}(\frac{1}{2},T),q)} \|f\|_{L^p(W^{2,q})}. \tag{3.4}$$

We can also obtain the $L^\infty$ estimate in time:

**Lemma 3.3.** For $f \in L^p(0,T;L^q(\Omega))$ and $\frac{\partial f}{\partial t} \in L^p(0,T;L^q(\Omega))$ with $f_0 = f(0) \in L^q(\Omega)$, we have, for all $t \in [0,T]$,

$$\|f\|_{L^p(0,T;L^q)} \leq C \left( |f_0| + \|f\|_{L^p(0,T;L^q)} + \|\frac{\partial f}{\partial t}\|_{L^p(0,T;L^q)} \right), \tag{3.4}$$

for some positive constant $C$ independent of $T$ and $f$.

**Proof.** Indeed, by Hölder inequality, we have

$$\|f(t)\|_{L^p} \leq \frac{\|f_0\|_{L^p}}{q} + \int_0^t \frac{d}{ds} \|f(s)\|_{L^p} \, ds$$

$$= \frac{\|f_0\|_{L^p}}{q} + \int_0^t \left( \|f(s)\|_{L^p}^{\frac{q}{p}} - \int_0^t \|f(s)\|_{L^p}^{q-2} f(s) \frac{\partial f}{\partial s} \, dx \right) \, ds$$

$$\leq \frac{\|f_0\|_{L^p}}{q} + \frac{\|f_0\|_{L^p}}{q} \int_0^t \|f(s)\|_{L^p}^{q-1} \frac{\partial f}{\partial s} \, ds$$

$$\leq \frac{\|f_0\|_{L^p}}{q} + \frac{\|f_0\|_{L^p}}{q} \left( \int_0^t |f(s)|^p \, ds \right)^{\frac{q}{p}} \left( \int_0^t \left( \frac{\partial f}{\partial s} \right)^p \, ds \right)^{\frac{1}{p}},$$

and consequently, (3.4) follows from Young’s inequality. \hfill \Box

4. **Local existence.** In this section, we prove the local existence and uniqueness of strong solution in Theorem 2.1. The proof will be divided into several steps, including constructing the approximate solutions by iteration, obtaining the uniform estimate, showing the convergence, consistency and uniqueness. For the sake of simplicity, we assume $\mu = \nu = 1$ throughout the following sections.

4.1. **Construction of approximate solutions.** We initialize the construction of approximate solutions by setting $u^0(x,t) := u_0(x)$ and $H^0(x,t) := H_0(x)$.

For $k = 0, 1, 2, \ldots$, the Stokes equations (1.6a) and the parabolic equations (1.6b) enable us to define

$$(u^{k+1}(x,t), H^{k+1}(x,t), P^{k+1}(x,t)).$$
inductively as the (global) solution of the linear equations

\[ \begin{align*}
\frac{\partial u_{k+1}}{\partial t} - \Delta u_{k+1} + \nabla p_{k+1} & = -u^k \cdot \nabla u^k + (\nabla \times H^k) \times H^k, \\
\frac{\partial H_{k+1}}{\partial t} - \Delta H_{k+1} & = -u^k \cdot \nabla H^k + H^k \cdot \nabla u^k, \\
\nabla \cdot u_{k+1} & = 0, \quad \nabla \cdot H_{k+1} = 0, \quad \int_{\Omega} p_{k+1} \, dx = 0,
\end{align*} \tag{4.1a} \]

with the initial-boundary conditions:

\[ \begin{align*}
& u_{k+1}|_{t=0} = u_0, \quad H_{k+1}|_{t=0} = H_0, \\
& u_{k+1}|_{\partial \Omega} = 0, \quad H_{k+1} \cdot n|_{\partial \Omega} = (\nabla \times H_{k+1}) \times n|_{\partial \Omega} = 0.
\end{align*} \]

According to Theorem 3.1 and Theorem 3.2, it is obvious that the iterations are well defined and the induction argument yields \( \{(u^k, H^k, P^k)\} \subset M^{p,q}_T \) for all \( T > 0 \).

4.2. Uniform bound for some small fixed time \( T^* \). In this section we aim at finding a positive time \( T^* \) independent of \( k \) for which \( \{(u^k, H^k, P^k)\} \) is uniformly bounded in the space \( M^{p,q}_{T^*} \). Indeed, fix a (large) reference time \( T \), applying Theorems 3.1-3.2, we obtain

\[ \begin{align*}
& \|u_{k+1}(T)\|_{L^{1-p}_p(D_{1-p}^{A_p})} + \left( \int_0^T \left( \nabla P_{k+1}, u_{k+1}, \nabla^2 u_{k+1}, \frac{\partial u_{k+1}}{\partial t} \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}} \\
& \leq C \left( \|u_0\|_{L^{1-p}_p(D_{1-p}^{A_p})} + \left( \int_0^T | -u^k \cdot \nabla u^k + (\nabla \times H^k) \times H^k |^{p} \, dt \right)^{\frac{1}{p}} \right), \tag{4.2} \\
& \|H_{k+1}(T)\|_{L^{2(1-\frac{1}{p})}_q(B_{q,p}^{2(1-\frac{1}{p})})} + \|H_{k+1}\|_{W(0,T)} \\
& \leq \left( \frac{1}{\epsilon_0} + 1 \right) \|H_{k+1}\|_{W(0,T)} \tag{4.3} \\
& \leq C \left( \|H_0\|_{L^{2(1-\frac{1}{p})}_q(B_{q,p}^{2(1-\frac{1}{p})})} + \| -u^k \cdot \nabla H^k + H^k \cdot \nabla u^k \|_{L^{p}_T(L^q)} \right).
\end{align*} \]

Define

\[ \begin{align*}
U^k(t) & := \|u^k\|_{L^{p}(D_{A_p}^{1-\frac{1}{p}})} + \|u^k\|_{L^{p}_2(W^{2,q})} + \|\frac{\partial u^k}{\partial t}\|_{L^{q}_T(L^q)} \\
& \quad + \|H^k\|_{L^{p}(B_{q,p}^{2(1-\frac{1}{p})})} + \|H^k\|_{W(0,t)} \quad (0 < t \leq T),
\end{align*} \]

and

\[ \begin{align*}
U^0 & := \|u_0\|_{D_{A_p}^{1-\frac{1}{p}}} + \|H_0\|_{B_{q,p}^{2(1-\frac{1}{p})}}.
\end{align*} \]
Hence, from (4.2) and (4.3), using Lemmas 3.1-3.3, we have,
\[
U^{k+1}(t) \leq C \left( \left\| u_0 \right\|_{L_{t_0}^{1,1}(\Omega_{\epsilon_q})} + \left\| u^k \cdot \nabla u^k + (\nabla \times H^k) \times H^k \right\|_{L_t^p(L^q)} + \left\| H_0 \right\|_{B_{q,p}^{2(1-\frac{1}{P})}} \\
\right. \\
+ \left\| u^k \cdot \nabla H^k + H^k \cdot \nabla u^k \right\|_{L_t^p(L^q)} \right)
\leq C \left( U^0 + \left\| u^k \right\|_{L_t^{\infty}(L^q)} \left\| \nabla u^k \right\|_{L_t^p(L^q)} + \left\| H^k \right\|_{L_t^{\infty}(L^q)} \left\| \nabla H^k \right\|_{L_t^p(L^q)} \\
+ \left\| u^k \right\|_{L_t^{\infty}(L^q)} \left\| \nabla H^k \right\|_{L_t^p(L^q)} + \left\| H^k \right\|_{L_t^{\infty}(L^q)} \left\| \nabla u^k \right\|_{L_t^p(L^q)} \right)
\leq C \left( U^0 + t^{\frac{1}{2} - \frac{1}{2p}}(U^0 + U^k(t))U^k(t) \right).
\]  
(4.4)

Furthermore, if we assume that \( U^k(t) \leq 4CU^0 \) on \([0, T_*]\) with
\[
0 < T_* \leq \left( \frac{3}{4C(4C + 1)U^0} \right)^{\frac{2}{2q-3}},
\]  
(4.5)

then a direct computation yields
\[
U^{k+1}(t) \leq 4CU^0 \text{ on } [0, T_*].
\]

From (4.2)-(4.4), we conclude that the sequence \( \{(u^k, H^k, P^k)\} \) is uniformly bounded in \( M_{T_*}^{p,q} \). More precisely, we have

**Lemma 4.1.** For all \( t \in [0, T_*] \) with \( T_* \) satisfying (4.5),
\[
U^k(t) \leq 4CU^0,
\]  
(4.6)

### 4.3. Convergence of the approximate sequence.

**Lemma 4.2.** There exists \( T_0 \), \( 0 < T_0 \leq T_* \) such that \( \{(u^k, H^k, P^k)\} \) is a Cauchy sequence in \( M_{T_0}^{p,q} \) and thus converges in \( M_{T_0}^{p,q} \).

**Proof.** For \( k = 0, 1, 2, \ldots \), define
\[
u^k := u^{k+1} - u^k, \quad H^k := H^{k+1} - H^k, \quad P^k := P^{k+1} - P^k,
\]
and
\[
\bar{U}^k(t) := \left\| \bar{u}^k \right\|_{L_t^{\infty}(D_{\epsilon_q}^{\frac{1}{2} - \frac{1}{2p}})} + \left\| \bar{u}^k \right\|_{L_t^p(W^{2,2})} + \left\| \frac{\partial \bar{u}^k}{\partial t} \right\|_{L_t^p(L^q)} \\
+ \left\| \bar{P}^k \right\|_{L_t^p(W^{1,2})} + \left\| \bar{H}^k \right\|_{L_t^p(B_{q,p}^{2(1-\frac{1}{P})})} + \left\| \bar{H}^k \right\|_{W(0,1)}.
\]  
(4.7)

Then it follows obviously that the triplet \((\bar{u}^k, \bar{H}^k, \bar{P}^k)\) satisfies
\[
\begin{cases}
\frac{\partial \bar{u}^k}{\partial t} - \Delta \bar{u}^k + \nabla \bar{P}^k = -u^{k-1} \cdot \nabla u^k - u^{k-1} \cdot \nabla u^{k-1} + (\nabla \times H^k) \times H^{k-1} \\
+ (\nabla \times H^k) \times H^{k-1} + \int_{\Omega} \bar{P}^k dx = 0,
\end{cases}
\]
\[
\begin{cases}
\frac{\partial \bar{H}^k}{\partial t} - \Delta \bar{H}^k = -u^{k-1} \cdot \nabla H^k - u^{k-1} \cdot \nabla H^{k-1} - \bar{H}^{k-1} \cdot \nabla u^k + H^{k-1} \cdot \nabla u^{k-1},
\end{cases}
\]
\[
\begin{cases}
\nabla \cdot \bar{u}^k = 0, \quad \nabla \cdot \bar{H}^k = 0,
\end{cases}
\]  
(4.8)

with the initial-boundary conditions:
\[
\bar{u}^k|_{t=0} = \bar{H}^k|_{t=0} = 0,
\]
\[
\bar{u}^k|_{\partial \Omega} = \bar{H}^k \cdot n|_{\partial \Omega} = (\nabla \times \bar{H}^k) \times n|_{\partial \Omega} = 0.
\]
For all \( t \in [0, T_*] \), using Lemmas 3.1-3.3, we get
\[
\| \nabla \bar{u}^k - \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \leq \| \nabla u^{k-1} \|_{L_t^p(L^q)} + \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} + C(U^0 + U^{k-1}(t)) \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \leq 4C U^0 \left( t^{\frac{1}{2} - \frac{n}{p}} \| \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \right),
\]
(4.9)
\[
\| \nabla \times \bar{H}^k \times \bar{H}^{k-1} + (\nabla \times \bar{H}^{k-1}) \times \bar{H}^{k-1} \|_{L_t^p(L^q)} = \| \bar{H}^{k-1} \cdot \nabla \bar{H}^k - (\nabla \bar{H}^k) \cdot \bar{H}^{k-1} + \bar{H}^{k-1} \cdot \nabla \bar{H}^{k-1} - (\nabla \bar{H}^{k-1}) \cdot \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq 2 \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{H}^k \|_{L_t^p(L^q)} + 2 \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq C t^{\frac{1}{2} - \frac{n}{p}} U^k(t) \| \bar{H}^{k-1} \|_{L_t^p(L^q)} + C(U^0 + U^{k-1}(t)) \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq 4C U^0 \left( t^{\frac{1}{2} - \frac{n}{p}} \| \bar{H}^{k-1} \|_{L_t^p(L^q)} + \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} \right),
\]
(4.10)
where \((\nabla \bar{H}^k) \cdot \bar{H}^{k-1} = (\nabla \bar{H}^k) \bar{H}^{k-1} \) and
\[
| - \nabla \bar{u}^{k-1} \cdot \nabla \bar{H}^k - \nabla \bar{u}^{k-1} \cdot \nabla \bar{H}^{k-1} + \bar{H}^{k-1} \cdot \nabla \bar{u}^{k-1} \bar{H}^{k-1} - (\nabla \bar{H}^{k-1}) \cdot \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq \| \bar{u}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{H}^k \|_{L_t^p(L^q)} + \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} + \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \leq C t^{\frac{1}{2} - \frac{n}{p}} U^k(t) \| \bar{u}^{k-1} \|_{L_t^p(L^q)} + C(U^0 + U^{k-1}(t)) \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} + C t^{\frac{1}{2} - \frac{n}{p}} U^k(t) \| \bar{H}^{k-1} \|_{L_t^p(L^q)} + C(U^0 + U^{k-1}(t)) \| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} \leq 4C U^0 \left( t^{\frac{1}{2} - \frac{n}{p}} \| \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \bar{H}^{k-1} \|_{L_t^p(L^q)} + \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} \right),
\]
(4.11)
Applying Theorems 3.1-3.2 to (4.8) and taking advantage of (4.9)-(4.11), we deduce that
\[
\bar{U}^k(t) \leq 8C U^0 \left( t^{\frac{1}{2} - \frac{n}{p}} \| \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \right)
\]
(4.12)
Moreover, one the one hand, bearing in mind Lemma 3.3, (4.7) implies that
\[
\| \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq C \bar{U}^{k-1}(t),
\]
(4.13)
and on the other hand, combining Lemmas 3.1-3.2, Young’s inequality and (4.7) again, we obviously have
\[
\| \nabla \bar{u}^{k-1} \|_{L_t^p(L^q)} + \| \nabla \bar{H}^{k-1} \|_{L_t^p(L^q)} \leq C t^{\frac{1}{2} - \frac{n}{p}} \bar{U}^{k-1}(t).
\]
(4.14)
Hence, for \( t \in [0, T_*] \), (4.12)-(4.14) lead to
\[
\bar{U}^k(t) \leq 16C U^0 t^{\frac{1}{2} - \frac{n}{p}} \bar{U}^{k-1}(t).
\]
If we choose a \( t = T_0 \in (0, T_*] \) such that the condition
\[
16C U^0 T_0^{\frac{1}{2} - \frac{n}{p}} \leq \frac{1}{2}
\]
(4.15)
is fulfilled, it is now clear that \( \{ (u^k, \mathbf{H}^k, P^k) \} \) is a Cauchy sequence in \( M_T^{p,q} \) and thus converges in \( M_T^{p,q} \).

Note that the time of existence \( T_0 \) depends (continuously) on the norms of the initial data, on the domain, and on the regularity parameters.

4.4. **Checking that the limit is a solution.** Let \( (u, \mathbf{H}, P) \in M_T^{p,q} \) be the limit of the sequence \( \{ (u^k, \mathbf{H}^k, P^k) \} \). We claim all those nonlinear terms in (4.1) converge to their corresponding terms in (1.1) in \( L^p(0, T_0; L^q(\Omega))^3 \). Indeed, using Lemma 3.1 and Lemma 3.3,

\[
\|u^k \cdot \nabla u^k - u \cdot \nabla u\|_{L_T^p(L^q)} \\
= \| (u^k - u) \cdot \nabla u^k + u \cdot \nabla (u^k - u) \|_{L_T^p(L^q)} \\
\leq \|u^k - u\|_{L_T^{\infty}(L^\infty)} \|\nabla u^k\|_{L_T^p(L^\infty)} + \|u\|_{L_T^{\infty}(L^\infty)} \|\nabla u^k - \nabla u\|_{L_T^p(L^\infty)} \\
\leq CT_0 \frac{1}{k^2} \|u^k\|_{M_T^{p,q}} + CT_0 \frac{1}{k^2} \|u\|_{M_T^{p,q}} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,
\]

follows from \( u^k \rightarrow u \) as \( k \rightarrow \infty \) in \( M_T^{p,q} \). Hence,

\[
u^k \cdot \nabla u^k \rightarrow u \cdot \nabla u \quad \text{in} \quad (L^p(0, T_0; L^q(\Omega))^3).
\]

Similarly, using Lemmas 3.1-3.3, we have

\[
(\nabla \times \mathbf{H}^k) \times \mathbf{H} \rightarrow (\nabla \times \mathbf{H}) \times \mathbf{H} \quad \text{in} \quad (L^p(0, T_0; L^q(\Omega))^3)
\]

\[
u^k \cdot \nabla \mathbf{H}^k \rightarrow u \cdot \nabla \mathbf{H} \quad \text{in} \quad (L^p(0, T_0; L^q(\Omega))^3)
\]

\[
\mathbf{H}^k \cdot \nabla u^k \rightarrow \mathbf{H} \cdot \nabla u \quad \text{in} \quad (L^p(0, T_0; L^q(\Omega))^3)
\]

Thus, passing to the limit in (4.1) as \( k \rightarrow \infty \), we conclude that the MHD system (1.1) holds in \( (L^p(0, T_0; L^q(\Omega))^3 \) and hence almost everywhere in \( Q_{T_0} \).

4.5. **Uniqueness.** Now we proceed to prove the uniqueness of the solution by the same procedure as that used for the proof that \( \{ (u^k, \mathbf{H}^k, P^k) \} \) is a Cauchy sequence in \( M_T^{p,q} \) in Lemma 4.2.

Let \( (u_1, \mathbf{H}_1, P_1) \) and \( (u_2, \mathbf{H}_2, P_2) \) be two solutions to (1.1) with the initial-boundary conditions (1.2)-(1.4). Denote

\[
u := u_1 - u_2, \quad \mathbf{H} := \mathbf{H}_1 - \mathbf{H}_2, \quad P := P_1 - P_2.
\]

Then, the triplet \( (\bar{\nu}, \bar{\mathbf{H}}, \bar{P}) \) satisfies the following system:

\[
\begin{cases}
\frac{\partial \bar{\nu}}{\partial t} - \Delta \bar{\nu} + \nabla \bar{P} = -\bar{\nu} \cdot \nabla u_1 - u_2 \cdot \nabla \bar{\nu} + (\nabla \times \mathbf{H}_1) \times \bar{\mathbf{H}} + (\nabla \times \bar{\mathbf{H}}) \times \mathbf{H}_2, \\
\frac{\partial \bar{\mathbf{H}}}{\partial t} - \Delta \bar{\mathbf{H}} = -\bar{\nu} \cdot \nabla \mathbf{H}_1 - u_2 \cdot \nabla \bar{\mathbf{H}} + \bar{\mathbf{H}} \cdot \nabla u_1 + \mathbf{H}_2 \cdot \nabla \bar{\nu}, \\
\nabla \cdot \bar{\nu} = 0, \quad \nabla \cdot \bar{\mathbf{H}} = 0, \quad \int_{\Omega} \bar{P} dx = 0,
\end{cases}
\]

(4.16)

with the initial-boundary conditions

\[
\bar{\nu}|_{t=0} = \bar{\mathbf{H}}|_{t=0} = 0, \\
\bar{\nu}|_{\partial \Omega} = \bar{\mathbf{H}} \cdot \mathbf{n}|_{\partial \Omega} = (\nabla \times \bar{\mathbf{H}}) \times \mathbf{n}|_{\partial \Omega} = 0.
\]
Moreover, Lemmas 3.1-3.3 imply that an accurate lower bound for the existence time.

Applying Theorems 3.1-3.2 and repeating the argument in (4.9)-(4.11), we obtain

\[ X(t) \leq 8C\mu_0 \left( t^2 - \frac{3}{8} \right) + \left( \| \nabla \mathbf{u} \|_{L^\infty_t(L^p)} + \| \nabla \mathbf{H} \|_{L^\infty_t(L^q)} \right) \]
\[ + \| \mathbf{P} \|_{L^1_t(W^{1,p})} + \| \mathbf{H} \|_{L_t^\infty(B_{q,p}^{2(1-\frac{1}{p})})} + \left( \| \mathbf{H} \|_{W(0,t)} \right) \]
\[ \leq 16C\mu_0 t^2 - \frac{3}{8} X(t) \]
\[ \leq \frac{1}{2} X(t). \]

Hence, \( X(t) = 0 \) for all \( t \in [0, T_0] \), which guarantees the uniqueness on \( [0, T_0] \).

5. Global existence. In this section, we prove that, if the initial data is sufficiently small, the local strong solution \((\mathbf{u}, \mathbf{H}, \mathbf{P})\) of (1.1)-(1.4) established in the previous section is indeed global in time. To this end, we first denote by \( T^* \) the maximal existence time for \((\mathbf{u}, \mathbf{H}, \mathbf{P})\) which means \((\mathbf{u}, \mathbf{H}, \mathbf{P})\) cannot be continued beyond \( T^* \) into a strong solution of (1.1)-(1.4).

We note that a lower bound for the existence time has already been obtained when proving the local existence and uniqueness of strong solution to (1.1)-(1.4) from the previous part (see (4.5) and (4.15), which insures that the existence time of a strong solution for (1.1)-(1.4) goes to infinity when \( u_0 \) (resp. \( H_0 \)) tends to 0 in \( D_{A_q}^{1-\frac{1}{p}, p} \) (resp. \( B_{q,p}^{2(1-\frac{1}{p})} \)). However, it is rather implicit. Now we give a more accurate lower bound for the existence time.

Define
\[ G(t) := \| \mathbf{u} \|_{L^p_t(D_{A_q}^{1-\frac{1}{p}, p})} + \| \nabla \mathbf{u} \|_{L^p_t(W^{2,p})} + \| \nabla \mathbf{H} \|_{L^\infty_t(L^p)} \]
\[ + \| \mathbf{P} \|_{L^1_t(W^{1,p})} + \| \mathbf{H} \|_{L_t^\infty(B_{q,p}^{2(1-\frac{1}{p})})} + \left( \| \mathbf{H} \|_{W(0,t)} \right), \]

and
\[ G_0 := \| u_0 \|_{D_{A_q}^{1-\frac{1}{p}, p}} + \| H_0 \|_{B_{q,p}^{2(1-\frac{1}{p})}}. \]

To extend the local solution, we need to control the maximal time \( T^* \) only in terms of the initial data. For this purpose, noticing that \( G(t) \) is an increasing and continuous function in \([0, T^*]\), and for all \( t \in [0, T^*] \), we have, applying Theorems 3.1-3.2,
\[ G(t) \leq C \left( G_0 + \| - \mathbf{u} \cdot \nabla \mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H} \|_{L^p_t(L^p)} + \| - \mathbf{u} \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{u} \|_{L^p_t(L^q)} \right). \]
(5.1)

Moreover, Lemmas 3.1-3.3 imply that
\[ \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{L^p_t(L^r)} \leq \| \mathbf{u} \|_{L^\infty_t(L^s)} \| \nabla \mathbf{u} \|_{L^p_t(L^\infty)} \leq C t^{\frac{1}{2} - \frac{1}{p}} (G_0 + G(t)) G(t) \]
(5.2)
Hence, we have new estimates for the terms on the right side of (5.1). Indeed, by the imbedding, sufficiently small, the solution exists globally in time. To this end, we need some other data approaches zero. More precisely, we can show that, if the initial data is sufficiently small, the maximal time of existence will go to infinity when the initial data approaches zero. This implies that the maximal time of existence will go to infinity when the initial data approaches zero. Hence, we have

\[ \| (\nabla \times \mathbf{H}) \times \mathbf{H} \|_{L^q_t(L^r)} \leq C \| \mathbf{H} \|_{L^r_t(L^q)} \| \nabla \mathbf{H} \|_{L^q_t(L^r)} \]

\[ \leq C t^{\frac{2}{2} - \frac{1}{q}} \left( |\mathbf{H}_0|_q + G(t) \right) G(t) \]

\[ \leq C t^{\frac{2}{2} - \frac{1}{q}} \left( G_0 + G(t) \right) G(t), \]

and

\[ \| - \mathbf{u} \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{u} \|_{L^q_t(L^r)} \]

\[ \leq \| \mathbf{u} \|_{L^r_t(L^q)} \| \nabla \mathbf{H} \|_{L^q_t(L^r)} + \| \mathbf{H} \|_{L^r_t(L^q)} \| \nabla \mathbf{u} \|_{L^q_t(L^r)} \]

\[ \leq C t^{\frac{2}{2} - \frac{1}{q}} \left( |\mathbf{u}_0|_q + G(t) \right) G(t) + C t^{\frac{1}{2} - \frac{1}{q}} \left( |\mathbf{H}_0|_q + G(t) \right) G(t) \]

\[ \leq C t^{\frac{1}{2} - \frac{1}{q}} \left( G_0 + G(t) \right) G(t). \]

The relations among (5.2)-(5.4) and (5.1) imply

\[ G(t) \leq C \left( G_0 + t^{\frac{1}{2} - \frac{1}{q}} \left( G_0 + G(t) \right) G(t) \right). \]

Assuming \( T_* \) to be the smallest number such that

\[ G(T_*) = 4CG_0, \]

which is possible because \( G(t) \) is an increasing and continuous function in time. Then,

\[ G(t) < 4CG_0, \quad t \in [0, T_*], \]

and from (5.5), we deduce that

\[ 3 \leq 4C(4C + 1)G_0 T_*^{\frac{1}{2} - \frac{1}{q}}. \]

Hence, we have

\[ T^* > T_* \geq \left( \frac{3}{4C(4C + 1)G_0} \right)^{\frac{2q}{q}}. \]

This implies that the maximal time of existence will go to infinity when the initial data approaches zero. More precisely, we can show that, if the initial data is sufficiently small, the solution exists globally in time. To this end, we need some other new estimates for the terms on the right side of (5.1). Indeed, by the imbedding

\[ W^{1,q}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{as} \quad q > 3, \]

and Lemma 3.3, we have

\[ \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{L^q_t(L^r)} \leq \| \mathbf{u} \|_{L^r_t(L^q)} \| \nabla \mathbf{u} \|_{L^q_t(L^r)} \]

\[ \leq C \left( |\mathbf{u}_0|_q + G(t) \right) \| \nabla \mathbf{u} \|_{L^q_t(W^{2,q})} \]

\[ \leq C \left( G_0 + G(t) \right) G(t). \]

Similarly, we have

\[ \| (\nabla \times \mathbf{H}) \times \mathbf{H} \|_{L^q_t(L^r)} \leq C \left( G_0 + G(t) \right) G(t), \]

and

\[ \| - \mathbf{u} \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{u} \|_{L^q_t(L^r)} \leq C \left( G_0 + G(t) \right) G(t). \]

Thus, (5.1) turns out to be

\[ G(t) \leq C \left( G_0 + \left( G_0 + G(t) \right) G(t) \right). \]

By the Cauchy-Schwarz inequality, (5.6) becomes

\[ G(t) \leq C \left( G_0 + \frac{1}{2} G_0^2 + \frac{3}{2} G^2(t) \right), \quad t \in [0, T^*]. \]
Now we take $G_0$ sufficiently small such that
\[ G_0^2 + 2G_0 \leq \frac{1}{3C^2} \] i.e., $G_0 \leq \delta_0 := \sqrt{1 + \frac{1}{3C^2}} - 1. \tag{5.8} \]

Then, under the assumption (5.8), we compute directly from (5.7) and the continuity of $G(t)$ that
\[ G(t) \leq \frac{1}{3C} \leq \sqrt{\frac{1}{3C^2}} (G_0^2 + 2G_0) \leq \frac{1}{3C}, \quad t \in [0, T^*). \tag{5.9} \]

In particular, this implies
\[ \| (u, H, P) \|_{M^p_T} \leq \frac{1}{3C} < \infty. \]

Hence, according to the local existence in the previous section, we can extend the solution in $[0, T^*)$ to some larger interval $[0, T^* + \tau)$ with $\tau > 0$. This is impossible since $T^*$ is already the maximal time of existence. Therefore, when the initial data satisfies (5.8), the strong solution is indeed global in time.

The proof of Theorem 2.1 is complete.

6. **Weak-strong uniqueness.** We here aim at showing **Weak-Strong Uniqueness** in Theorem 2.2. Before going into the heart of the proof, let us first obtain an energy equality for the strong solution to the system (1.1)-(1.4):

**Lemma 6.1.** Let $p, q$ satisfy the same conditions as Theorem 2.1 and $(u, H, P) \in M^p_T$ be the unique solution to (1.1)-(1.4) in $Q_T$. Then,
\[ \frac{1}{2} \int_\Omega (|u(t)|^2 + |H(t)|^2) \, dx + \int_0^t \int_\Omega (|\nabla u|^2 + |\nabla \times H|^2) \, dx \, ds \]
\[ = \frac{1}{2} \int_\Omega (|u_0|^2 + |H_0|^2) \, dx. \]

**Proof.** Due to the homogeneous incompressible character of the flows we are dealing with, the natural framework in which we shall work is that of the solenoidal vector field of $L^2(\Omega)^3$. Note that
\[ u \in C([0, T_0]; D^{1-\frac{1}{p},p}_{A_q} \cap L^p(0, T_0; W^{2,q}(\Omega)) \text{ with } q > 3, \]
and
\[ D^{1-\frac{1}{p},p}_{A_q} \hookrightarrow B^{2(1-\frac{1}{p})}_{q,p} \cap X^q \hookrightarrow B^{2(1-\frac{1}{p})}_{q,p} \cap L^q(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^2(\Omega), \]
then, when $1 < p < 2$, by the standard interpolation inequality
\[ L^\infty(0, T_0; L^q(\Omega)) \cap L^p(0, T_0; W^{2,q}(\Omega)) \subset L^2(0, T_0; H^{1+\alpha}(\Omega)), \]
where
\[ \frac{1}{2} = \frac{1 - \theta}{\infty} + \frac{\theta}{p} = \frac{1}{p} - \frac{1 + \alpha}{3} = (1 - \theta) \frac{1}{q} + \theta \left( \frac{1}{q} - \frac{2}{3} \right), \]
we have
\[ u \in C([0, T_0]; L^2(\Omega)) \cap L^2(0, T_0; H^{1+\alpha}(\Omega)); \tag{6.1} \]
when $2 \leq p < \infty$, due to
\[ W^{2,q}(\Omega) \hookrightarrow H^2(\Omega) \text{ as } q > 3, \]
(6.1) holds obviously.

Similarly,
\[ H \in C([0, T_0]; L^2(\Omega)) \cap L^2(0, T_0; H^{1+\alpha}(\Omega)). \]
Taking the $L^2$ scalar product in (1.5a) with $u$ and performing integration by parts, bearing in mind (1.1c) and the boundary conditions, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} u \cdot (\nabla u) u \, dx - \int_{\Omega} H \cdot (\nabla u) H \, dx, \tag{6.2}
\]
where $u \cdot (\nabla u) u = u^\top (\nabla u) u$. Likewise,
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |H|^2 \, dx + \int_{\Omega} |\nabla \times H|^2 \, dx = \int_{\Omega} H \cdot (\nabla H) u \, dx - \int_{\Omega} u \cdot (\nabla H) H \, dx. \tag{6.3}
\]
Noticing that
\[
\int_{\Omega} u \cdot (\nabla u) u = \frac{1}{2} \int_{\Omega} \nabla(|u|^2) \cdot u \, dx = -\frac{1}{2} \int_{\Omega} |u|^2 \nabla \cdot u \, dx = 0,
\]
\[
\int_{\Omega} H \cdot (\nabla H) u \, dx = \frac{1}{2} \int_{\Omega} \nabla(|H|^2) \cdot u \, dx = -\frac{1}{2} \int_{\Omega} |H|^2 \nabla \cdot u \, dx = 0,
\]
\[
\int_{\Omega} H \cdot (\nabla u) H \, dx = -\int_{\Omega} u \cdot (\nabla H) H \, dx,
\]
hence, adding (6.2) and (6.3) together, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|u|^2 + |H|^2) \, dx + \int_{\Omega} (|\nabla u|^2 + |\nabla \times H|^2) \, dx = 0.
\]
Integrating the above equality over time interval $[0, t]$, we obtain the energy equality of this lemma. \qed

Now, we recall that for the weak solution $(\tilde{u}, \tilde{H}, \Pi)$ obtained in [29], we have for (almost) all $t \in (0, T_0)$,
\[
\frac{1}{2} \int_{\Omega} (|\tilde{u}(t)|^2 + |\tilde{H}(t)|^2) \, dx + \int_0^t \int_{\Omega} (|\nabla \tilde{u}|^2 + |\nabla \times \tilde{H}|^2) \, dx ds \leq \frac{1}{2} \int_{\Omega} (|u_0|^2 + |H_0|^2) \, dx. \tag{6.4}
\]
We remark that, in view of the regularity of $u$, we deduce from the weak formulation of (1.1)-(1.4) the following equalities:
\[
\int_{\Omega} \tilde{u} \cdot u \, dx + \int_0^t \int_{\Omega} \nabla \tilde{u} : \nabla u \, dx ds + \int_0^t \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \cdot u \, dx ds \tag{6.5}
\]
and
\[
\int_{\Omega} \tilde{H} \cdot H \, dx + \int_0^t \int_{\Omega} (\nabla \times \tilde{H}) \cdot (\nabla \times H) \, dx ds + \int_0^t \int_{\Omega} \tilde{u} \cdot \nabla \tilde{H} \cdot H \, dx ds \tag{6.6}
\]
for a.e. $t \in (0, T_0)$.
Noticing that the following identity holds:
\[
\int_{\Omega} u \cdot \nabla v \cdot w \, dx = -\int_{\Omega} u \cdot \nabla w \cdot v \, dx, \tag{6.7}
\]
for any three differentiable vector functions \( u \in \mathbb{R}^3, v \in \mathbb{R}^3 \) and \( w \in \mathbb{R}^3 \), if \( \nabla \cdot u = 0 \) and \( v \mid_{\partial \Omega} = 0 \) or \( w \mid_{\partial \Omega} = 0 \) or \( u \cdot n \mid_{\partial \Omega} = 0 \). Then (6.5) becomes

\[
\int_{\Omega} \tilde{u} \cdot u \, dx + \int_{0}^{t} \int_{\Omega} \nabla \tilde{u} : \nabla u \, dxds = \int_{\Omega} |u_0|^2 \, dx + \int_{0}^{t} \int_{\Omega} \tilde{H} \cdot \nabla \tilde{H} \cdot u \, dxds + \int_{0}^{t} \int_{\Omega} \tilde{u} \cdot \left( \frac{\partial u}{\partial s} + \nabla u \cdot \tilde{u} \right) \, dxds, \tag{6.8}
\]

and (6.6) becomes

\[
\int_{\Omega} \tilde{H} \cdot H \, dx + \int_{0}^{t} \int_{\Omega} (\nabla \times \tilde{H}) \cdot (\nabla \times H) \, dxds = \int_{\Omega} |H_0|^2 \, dx - \int_{0}^{t} \int_{\Omega} \tilde{H} \cdot \nabla H \cdot \tilde{u} \, dxds + \int_{0}^{t} \int_{\Omega} \tilde{u} \cdot \nabla H \cdot H \, dxds \tag{6.9}
\]

Since \( H \) satisfies (1.5b), we substitute (1.5b) into (6.9) to obtain

\[
\int_{\Omega} \tilde{H} \cdot H \, dx + 2 \int_{0}^{t} \int_{\Omega} (\nabla \times \tilde{H}) \cdot (\nabla \times H) \, dxds = \int_{\Omega} |H_0|^2 \, dx - \int_{0}^{t} \int_{\Omega} \tilde{H} \cdot \nabla H \cdot \tilde{u} \, dxds + \int_{0}^{t} \int_{\Omega} \tilde{u} \cdot \nabla H \cdot H \, dxds - \int_{0}^{t} \int_{\Omega} u \cdot \nabla H \cdot H \, dxds + \int_{0}^{t} \int_{\Omega} H \cdot \nabla u \cdot H \, dxds. \tag{6.10}
\]

On the other hand, we can write the equation for \( u \) as

\[
\frac{\partial u}{\partial t} + \tilde{u} \cdot \nabla u - \Delta u + \nabla P = (\tilde{u} - u) \cdot \nabla u + (\nabla \times H) \times H. \tag{6.11}
\]

Multiplying (6.11) by \( \tilde{u} \) and integrating over \( Q_t \), we get

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial u}{\partial s} + \tilde{u} \cdot \nabla u \right) \cdot \tilde{u} \, dxds = -\int_{0}^{t} \int_{\Omega} \nabla u : \nabla \tilde{u} \, dxds + \int_{0}^{t} \int_{\Omega} (\tilde{u} - u) \cdot \nabla u \cdot \tilde{u} \, dxds + \int_{0}^{t} \int_{\Omega} \tilde{H} \cdot \nabla u \cdot \tilde{u} \, dxds. \tag{6.12}
\]

Substituting (6.12) into (6.8), we obtain

\[
\int_{\Omega} u \cdot \tilde{u} \, dx + 2 \int_{0}^{t} \int_{\Omega} \nabla u : \nabla \tilde{u} \, dxds = \int_{\Omega} |u_0|^2 \, dx + \int_{0}^{t} \int_{\Omega} \tilde{H} \cdot \nabla \tilde{H} \cdot u \, dxds + \int_{0}^{t} \int_{\Omega} (\tilde{u} - u) \cdot \nabla u \cdot \tilde{u} \, dxds + \int_{0}^{t} \int_{\Omega} H \cdot \nabla H \cdot \tilde{u} \, dxds. \tag{6.13}
\]
Also, according to Lemma 6.1, we have

\[
\frac{1}{2} \int_\Omega (|u(t)|^2 + |H(t)|^2) \, dx + \int_0^t \int_\Omega (|\nabla u|^2 + |\nabla \times H|^2) \, dx \, ds \\
= \frac{1}{2} \int_\Omega (|u_0|^2 + |H_0|^2) \, dx.
\]

(6.14)

Summing (6.4), (6.14) and subtracting the sum of (6.10) and (6.13), we obtain for (almost) all \( t \in (0, T_0) \),

\[
\frac{1}{2} \int_\Omega (|u(t) - \bar{u}(t)|^2 + |H(t) - \bar{H}(t)|^2) \, dx \\
+ \int_0^t \int_\Omega (|\nabla u - \nabla \bar{u}|^2 + |\nabla \times H - \nabla \times \bar{H}|^2) \, dx \, ds \\
\leq \int_0^t \int_\Omega (\bar{H} \cdot \nabla H \cdot \bar{u}) \, dx \, ds - \int_0^t \int_\Omega (\bar{u} \cdot \nabla \bar{H} \cdot \bar{H}) \, dx \, ds + \int_0^t \int_\Omega u \cdot \nabla H \cdot \bar{H} \, dx \, ds \\
- \int_0^t \int_\Omega H \cdot \nabla u \cdot \bar{H} \, dx \, ds - \int_0^t \int_\Omega \bar{H} \cdot \nabla \bar{H} \cdot u \, dx \, ds - \int_0^t \int_\Omega (\bar{u} - u) \cdot \nabla u \cdot \bar{u} \, dx \, ds \\
- \int_0^t \int_\Omega H \cdot \nabla \bar{H} \cdot \bar{u} \, dx \, ds \\
= - \int_0^t \int_\Omega (H - \bar{H}) \cdot \nabla \bar{H} \cdot \bar{u} + \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla H \cdot \bar{H} \, dx \, ds \\
- \int_0^t \int_\Omega (H - \bar{H}) \cdot \nabla u \cdot \bar{H} \, dx \, ds + \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla u \cdot \bar{u} \, dx \, ds \\
= \int_0^t \int_\Omega (H - \bar{H}) \cdot \nabla u \cdot (u - \bar{u}) \, dx \, ds + \int_0^t \int_\Omega (H - \bar{H}) \cdot \nabla u \cdot (H - \bar{H}) \, dx \, ds \\
- \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla H \cdot (H - \bar{H}) \, dx \, ds + \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla H \cdot \bar{H} \, dx \, ds \\
- \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla u \cdot (u - \bar{u}) \, dx \, ds + \int_0^t \int_\Omega (u - \bar{u}) \cdot \nabla u \cdot u \, dx \, ds,
\]

(6.15)

where we applied the identity (6.7) repeatly.

Note that

\[
\int_\Omega (u - \bar{u}) \cdot \nabla H \cdot H \, dx = \frac{1}{2} \int_\Omega \nabla (|H|^2) \cdot (u - \bar{u}) \, dx \\
= -\frac{1}{2} \int_\Omega |H|^2 \nabla \cdot (u - \bar{u}) \, dx = 0,
\]

and

\[
\int_\Omega (u - \bar{u}) \cdot \nabla u \cdot u \, dx = \frac{1}{2} \int_\Omega \nabla (|u|^2) \cdot (u - \bar{u}) \, dx \\
= -\frac{1}{2} \int_\Omega |u|^2 \nabla \cdot (u - \bar{u}) \, dx = 0.
\]
Then, by Hölder’s inequality and Cauchy’s inequality, (6.15) turns to be
\[
\frac{1}{2} \int_{\Omega} (|u(t) - \tilde{u}(t)|^2 + |H(t) - \tilde{H}(t)|^2) \, dx
\]
\[
+ \int_0^t \int_{\Omega} (|\nabla u - \nabla \tilde{u}|^2 + |\nabla \times H - \nabla \times \tilde{H}|^2) \, dx \, ds
\]
\[
\leq 2 \int_0^t |\nabla H|_{\infty} |u - \tilde{u}|_2 |H - \tilde{H}|_2 \, ds + \int_0^t |\nabla u|_{\infty} (|u - \tilde{u}|_2^2 + |H - \tilde{H}|_2^2) \, ds
\]
\[
\leq \int_0^t (|\nabla u|_{\infty} + |\nabla H|_{\infty}) (|u - \tilde{u}|_2^2 + |H - \tilde{H}|_2^2) \, ds.
\]
Notice that
\[
|\nabla u|_{\infty} + |\nabla H|_{\infty} \in L^1(0, T_0).
\]
Therefore, using (6.16) together with Gronwall’s inequality, we finally conclude that
\[
u = \tilde{u}, \quad H = \tilde{H} \quad \text{a.e.}
\]
and thus \(P = \Pi\) up to a constant in \(Q_{T_0}\).

The proof of Theorem 2.2 is complete.

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