Treewidth of graphs with balanced separations

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Abstract

We prove that if every subgraph of a graph $G$ has a balanced separation of order at most $a$ then $G$ has treewidth at most $105a$. This establishes a linear dependence between the treewidth and the separation number.

1 Introduction and results

Treewidth of a graph is an important graph parameter introduced by Robertson and Seymour [7], which expresses how “tree-like” a given graph is. The relation between the treewidth and other graph parameters, e.g. the maximum order of the tangle [9] and the size of the largest grid-minor [8, 10, 3], has been explored in a number of papers. (See [6] for a recent survey.) The goal of this paper is to establish a linear dependence between treewidth and another parameter, the separation number, which we now define.

A separation of a graph $G$ is a pair $(A, B)$ of subsets of $V(G)$ such that $A \cup B = V(G)$ and no edge of $G$ has one end in $A - B$ and the other in $B - A$. The order of the separation $(A, B)$ is $|A \cap B|$. A separation $(A, B)$ of a graph $G$ on $n$ vertices is balanced if $|A \setminus B| \leq 2n/3$ and $|B \setminus A| \leq 2n/3$. The separation number $\text{sn}(G)$ of a graph $G$ is a smallest number $s$ such that every subgraph of $G$ has a balanced separation of order at most $s$.

Let $\text{tw}(G)$ denote the treewidth of the graph $G$. (We will not need the definition of treewidth in this paper. We use a related parameter, the tangle.

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number, which is defined in Section 2 in our proofs, instead.) The relation between the separation number and the treewidth has been explored starting with Robertson and Seymour [8], who have shown that $\text{sn}(G) \leq \text{tw}(G) + 1$ for every graph $G$. On the other hand, Bodlaender et al. [1] have proved that $\text{tw}(G) \leq 1 + \text{sn}(G) \log(|V(G)|)$. Fox [5] stated without proof that $\text{sn}(G)$ and $\text{tw}(G)$ are tied, that is $\text{tw}(G) \leq f(\text{sn}(G))$ for some function $f$. Finally, Böttcher et al. [2] have investigated the relation between the separation number and the treewidth for graphs with bounded maximum degree. They have shown that for fixed $\Delta$ and a hereditary class of graphs with maximum degree at most $\Delta$, the treewidth is sublinear in the number of vertices if and only if the separation number is.

The following is our main result.

**Theorem 1.** $\text{tw}(G) \leq 105 \text{sn}(G)$ for every graph $G$.

Note that Theorem 1 implies the aforementioned result of Böttcher et al. without the restriction on the maximum degree.

The key ingredient of the proof of Theorem 1 is a result of Chen et al. [4] on the existence of confluent flows. We give the necessary definitions and prove the preliminary results in Section 2. The proof of Theorem 1 is completed in Section 3.

## 2 Clouds, tangles and confluent flows

In this section we introduce the tools and auxiliary results necessary for the proof of Theorem 1.

### 2.1 Tangles

A *g-separation* of a graph $G$ is a pair of subgraphs $(A, B)$ of $G$ such that $A \cup B = G$ and $E(A) \cap E(B) = \emptyset$. The *order* of a g-separation $(A, B)$ is $|V(A) \cap V(B)|$. A tangle $\mathcal{T}$ of order $\theta \geq 1$ in $G$ is a collection of g-separations of $G$, satisfying the following:

(i) for every g-separation $(A, B)$ of $G$ of order $< \theta$ either $(A, B) \in \mathcal{T}$, or $(B, A) \in \mathcal{T}$,

(ii) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$, and
The tangle number \( t_n(G) \) is the maximum order of a tangle in \( G \). The relation between the tangle number and the treewidth of a graph is captured in the following theorem of Robertson and Seymour.

**Theorem 2** (Robertson, Seymour [9]). Let \( G \) be a graph with \( t_n(G) \geq 2 \). Then the treewidth \( tw(G) \) of \( G \) satisfies

\[
 t_n(G) \leq tw(G) + 1 \leq \frac{3}{2} t_n(G).
\]

### 2.2 Confluent flows

Let \( G \) be a directed graph and let \( d : V(G) \to \mathbb{R}_+ \) be a function specifying the demand at vertices. The value \( d(V(G)) := \sum_{v \in V(G)} d(v) \) is called the total demand. Let \( S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G) \) be a set of sinks in \( G \). (A sink has no outgoing edges.) A flow \( f : E(G) \to \mathbb{R}_+ \) satisfies the flow conservation equation

\[
 \sum_{e=uv \in E(G)} f(e) - \sum_{e=uv \in E(G)} f(e) = d(v),
\]

for every \( v \in V(G) - S \). For a fixed flow \( f \) and a vertex \( v \in V(G) \) define \( in(v) \) to be the sum of the flow values on the edges into \( v \). The congestion at \( v \) is defined as \( in(v) + d(v) \). The congestion of a flow \( f \) is the maximum congestion over all the nodes of the graph. A flow is said to be confluent if \( G \) contains no directed cycle whose edges have positive flow values, and for every vertex \( v \in V(G) \) there exists at most one edge out of \( v \) that has a positive flow value. Note that the support of a confluent flow consists of vertex-disjoint arborescences \( T_1, T_2, \ldots, T_k \) such that \( T_i \) is directed towards \( v_i \).

We are now ready to state the necessary result from [4].

**Theorem 3** ([4, Theorem 22]). Let \( G \) be a directed graph and suppose that there exists a flow in \( G \) with demand function \( d \) and congestion \( c \). Then there exists a confluent flow in \( G \) with congestion at most \( c \) and demand function \( d' \) such that \( d'(v) \leq d(v) \) for every \( v \in V(G) \), and \( d'(V(G)) \geq d(V(G))/3 \).
2.3 Clouds

Let $G$ be a graph and let $W$ be a subset of $V(G)$. A $W$-cloud in $G$ is a forest $H \subseteq G$ such that every connected component of $H$ contains exactly one vertex of $W$. For a vertex $w \in W$, let $H_w$ denote the component of $H$ containing $w$. For a set $U \subseteq W$, let

$$n(H, U) = \sum_{w \in U} |V(H_w)|.$$  

For $s, \varepsilon > 0$, the $W$-cloud $H$ is $(s, \varepsilon)$-tame if $n(H, U) \geq s$ for every $U \subseteq W$ with $|U| \geq (1 - \varepsilon)|W|$. The $W$-cloud $H$ is strongly $(s, \varepsilon)$-tame if

$$n(H, U) \geq s + 3n(H, W \setminus U) \quad (1)$$

for every $U \subseteq W$ with $|U| \geq (1 - \varepsilon)|W|$. Intuitively, the $W$-cloud is tame if a large fraction of its components have substantial size, and it is strongly tame if it does not contain exceptionally large components containing most of the vertices of the cloud.

Lemma 4. Let $G$ be a graph, let $W$ be a subset of $V(G)$, and let $s, \varepsilon > 0$ be real numbers. If $G$ contains an $(s, 5\varepsilon)$-tame $W$-cloud, then $G$ also contains a strongly $(s, \varepsilon)$-tame $W$-cloud.

Proof. Let $H$ be an $(s, 5\varepsilon)$-tame $W$-cloud with the smallest number of vertices. Let $U$ be the smallest possible subset of $W$ such that $|U| \geq (1 - 5\varepsilon)|W|$ and $|V(H_u)| \leq |V(H_v)|$ for every $u \in U$ and $v \in W \setminus U$. Observe that all the components $H_u$ with $v \in W \setminus U$ have the same size $m$, since otherwise we can remove a leaf from largest such component without violating $(s, 5\varepsilon)$-tameness of $H$.

We claim that $H$ is strongly $(s, \varepsilon)$-tame. This is obvious when $\varepsilon < 1/|W|$, hence assume that $\varepsilon \geq 1/|W|$. Consider a subset $U_1 \subseteq W$ such that $|U_1| \geq (1 - \varepsilon)|W|$ and $n(H, U_1) - 3n(H, W \setminus U_1)$ is the smallest possible. Without loss of generality, we can assume that $U$ is a subset of $U_1$, and thus

$$n(H, U_1) - 3n(H, W \setminus U_1) = n(H, U) + m(|U_1 \setminus U|) - 3|W \setminus U_1|$$

$$\geq s + m(|U_1 \setminus U| - 3|W \setminus U_1|).$$

Since $|W \setminus U_1| \leq \varepsilon|W|$ and $|U_1 \setminus U| \geq 4\varepsilon|W| - 1 \geq 3\varepsilon|W|$, it follows that $|U_1 \setminus U| - 3|W \setminus U_1| \geq 0$, and the inequality (1) is satisfied. $\Box$
Let $G$ be a graph and let $W$ be a subset of $V(G)$. A separation $(A, B)$ of $G$ is $(s, \varepsilon)$-skewed with respect to $W$ if $W \subseteq A$, $|A| \leq 6s$ and $|A \cap B| \leq 6\varepsilon|W|$. Observe that if $G$ contains such a separation, then it cannot contain a $(6s + 1, 6\varepsilon)$-tame $W$-cloud, since the set $U = \{w \in W : V(H_w) \subseteq A\}$ has size at least $|W| - |A \cap B| \geq (1 - 6\varepsilon)|W|$, and $n(H, U) \leq |A| \leq 6s$. The following lemma shows an approximate converse to this observation.

**Lemma 5.** Let $G$ be a graph, let $W$ be a subset of $V(G)$, and let $s > 0$ and $0 < \varepsilon < 1/2$ be real numbers. If no separation of $G$ is $(s, \varepsilon)$-skewed with respect to $W$, then $G$ contains an $(s, \varepsilon)$-tame $W$-cloud.

**Proof.** Let $k = |W|$. Note that we can assume that $s \geq (1 - \varepsilon)k \geq \varepsilon k$, as otherwise the subgraph of $G$ consisting of the isolated vertices $W$ is an $(s, \varepsilon)$-tame $W$-cloud.

The key step in the proof of the lemma is an application of Theorem 3. Our first goal is constructing an appropriate flow. Consider an auxiliary directed graph $G'$ obtained from $G$ as follows: for each vertex $v \in V(G)$, the digraph $G'$ has two vertices $v_i$ and $v_o$ with an edge from $v_i$ to $v_o$; the capacity of this edge is $\frac{s}{\varepsilon k}$. For each edge $uv$ of $G$, the digraph $G'$ contains edges $(u_o, v_i)$ and $(v_o, u_i)$ of infinite capacity. Furthermore, $G'$ contains two additional vertices $a$ and $b$, edges of capacity 1 from $b$ to $v_i$ for all $v \in V(G)$ and edges of infinite capacity from $w_o$ to $a$ for each $w \in W$.

Let $C$ be an edge-cut in $G'$ separating $b$ from $a$ with the smallest capacity. Let $B \subseteq V(G)$ consist of the vertices $v \in V(G)$ such that $G' - C$ contains a path from $b$ to $v_i$, and let $A \subseteq V(G)$ consist of the vertices $v \in V(G)$ such that $G' - C$ does not contain a path from $b$ to $v_o$. Since $C$ has a finite capacity, $(A, B)$ is a separation in $G$ and $W$ is a subset of $A$. Note that the capacity of $C$ is at least $|A \setminus B| + \frac{s}{\varepsilon k}|A \cap B|$. Since the separation $(A, B)$ is not $(s, \varepsilon)$-skewed with respect to $W$, it follows that either $|A| \geq 6s$, in which case

$$|A \setminus B| + \frac{s}{\varepsilon k}|A \cap B| \geq |A \setminus B| + |A \cap B| = |A| \geq 6s,$$

or $|A \cap B| \geq 6\varepsilon k$, implying

$$|A \setminus B| + \frac{s}{\varepsilon k}|A \cap B| \geq \frac{s}{\varepsilon k}|A \cap B| \geq 6s.$$

Therefore the capacity of $C$ is at least $6s$.

It follows from the max-flow min-cut theorem that there exists a flow $f'$ in $G'$ (with the total demand 0) satisfying the capacity constraints specified
above, such that \( \text{in}(a) \geq 6s \). Note that without loss of generality, we may assume that \( f'(v,u) = 0 \) for every \( v \in W \) and \( u \in V(G') - \{a\} \). Let a digraph \( G'' \) be obtained from \( G \) by replacing every edge of \( G \) by two oppositely directed edges, and deleting the edges directed towards the vertices of \( W \), so that \( W \) is the set of sinks of \( G'' \). Let \( d : V(G) \to \mathbb{R}_+ \) be defined by \( d(v) := f'(bv_i) \). Let \( f \) be the flow in \( G'' \) corresponding to \( f' \) (defined by \( f(uv) := f'(u_o v_i) \) for every edge \( uv \) of \( G'' \)), where \( d \) is the demand function for \( f \). By the constraints on capacities, \( d(v) \leq 1 \) for every \( v \in V(G) \), \( d(V(G)) \geq 6s \), and the congestion of \( f \) is at most \( s \varepsilon k \).

By Theorem 3, the digraph \( G'' \) contains a confluent flow \( f'' \) with demand function \( d' \), such that \( d'(v) \leq 1 \) for every \( v \in V(G) \), \( d'(V(G)) \geq 2s \) and the congestion of \( f'' \) is at most \( s \varepsilon k \). We claim that the subgraph \( H \) of \( G \) corresponding to the support of \( f'' \) is an \((s, \varepsilon)\)-tame \( W \)-cloud. Indeed, consider a set \( U \subseteq W \) with \( |U| \geq (1 - \varepsilon)k \). We have

\[
\begin{align*}
n(H, U) &\geq \sum_{w \in U} \sum_{v \in V(H_w)} d'(v) = d'(V(G)) - \sum_{w \in W \setminus U} \sum_{v \in V(H_w)} d'(v) \\
&\geq 2s - \frac{s}{\varepsilon k} |W \setminus U| \geq s,
\end{align*}
\]

as desired.

Let us now give a brief sketch of the proof of Theorem 1 before going into technical details. We consider a hypothetical counterexample \( G \) with a small separation number \( a \) and with a large tangle (of order \( 70a + 1 \)). By repeatedly splitting the graph on balanced separations of order at most \( a \), we can obtain a separation \( (X, Y) \) of order at most \( 70a \) in the tangle where \( |X| \) is much larger than \( |Y| \). Thus, if we choose a separation \( (X, Y) \) of order at most \( 70a \) in the tangle so that \( Y \) is as small as possible, \( |X| \) will be much larger than \( |Y| \). Without loss of generality, we can assume \( |X \cap Y| = 70a \). A similar argument shows that actually a stronger claim holds: we cannot separate most of \( X \) from \( Y \) by a small cut, i.e., no separation in \( G[X] \) is \((3|Y|, 1/7)\)-skewed with respect to \( X \cap Y \). Let \( W = X \cap Y \). By Lemmas 4 and 5, this implies that \( G[X] \) contains a strongly \((3|Y|, 1/35)\)-tame \( W \)-cloud \( H \).

The subgraph \( G[Y] \cup H \) contains a balanced separation \( (A, B) \) of order at most \( a \). Since most components of \( H \) are of roughly the same size and fairly large compared to \( Y \), the sets \( A \cap W \) and \( B \cap W \) must have roughly equal size. However, then both separations \( (X \cup A, B) \) and \( (X \cup B, A) \) have
order less than $70a$, and since $A \cup B \cup X = V(G)$, the condition (ii) of the definition of a tangle implies that one of them belongs to the tangle. This contradicts the minimality of $Y$, since $A, B \not\subseteq Y$.

3 Proof of Theorem 1

Let $a := \text{sn}(G)$. We assume without loss of generality that $a \geq 1$. Suppose that $G$ has tree-width at least $105a$. By Theorem 2, $G$ contains a tangle $T$ of order at least $70a + 1$. We will write $(A, B) \in T$ for a separation $(A, B)$ of $G$, if the $g$-separation $(G[A], G[B] - E(G[A \cap B]))$ is in $T$. Note that the definition of a tangle implies that for every separation $(A, B)$ of $G$ of order at most $70a$, we have $(A, B) \in T$ or $(B, A) \in T$.

Let $(X, Y) \in T$ be a separation of order at most $70a$, such that $|Y|$ is as small as possible, and subject to that $|X|$ as large as possible. Clearly, $W = X \cap Y$ has order exactly $70a$. Let $G_0 := G[X]$, $\varepsilon = 1/7$ and $s = 3|Y|$.

Suppose first that $G_0$ has a separation $(A, B)$ that is $(s, \varepsilon)$-skewed with respect to $W$. Then, the separation $(B, Y \cup A)$ has order at most $6\varepsilon|W| = 60a$, and thus $(B, Y \cup A) \in T$, as $(Y \cup A, B) \in T$ contradicts condition (ii) in the definition of a tangle. Furthermore, $|A \cup Y| \leq |Y| + 6s = 19|Y|$.

We construct a sequence $(X_0, Y_0), (X_1, Y_1), \ldots, (X_8, Y_8) \in T$ of separations of $G$ with $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_8$, $Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_8$ and $|X_i \cap Y_i| \leq (60 + i)a$ for $0 \leq i \leq 8$, as follows. We set $(X_0, Y_0) = (B, A \cup Y)$. For $0 \leq i \leq 7$, let $(U_1, U_2)$ be a balanced separation of order at most $a$ in $G[Y_i]$. Note that $|X_i \cap Y_i| + a \leq 68a$, and since $(X_i, Y_i) \in T$ and $X_i \cup U_1 \cup U_2 = V(G)$, one of the separations $(X_i \cup U_1, U_2)$ and $(X_i \cup U_2, U_1)$ belongs to $T$, say the former one. We set $(X_{i+1}, Y_{i+1}) = (X_i \cup U_1, U_2)$. Since $(U_1, U_2)$ is a balanced separation of order at most $a$, we have $|U_2 \setminus U_1| \leq \frac{2}{3}|Y_i|$, and $|Y_{i+1}| \leq a + \frac{2}{3}|Y_i|$.

Therefore, $|Y_8| \leq \left(\frac{2}{3}\right)^8 |Y_0| + \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right) a < \left(\frac{5}{3}\right)^8 19|Y| + 3a \leq \frac{2}{3}|Y| + 3a$. Since $|Y| \geq 70a$, we conclude that $|Y_8| < |Y|$. However, this contradicts the choice of the separation $(X, Y)$ so that $|Y|$ is minimal.

Thus without loss of generality, we assume that no separation of $G_0$ is $(s, \varepsilon)$-skewed with respect to $W$. By Lemma 3, $G_0$ contains an $(s, \varepsilon)$-tame $W$-cloud, and by Lemma 4, $G_0$ also contains a strongly $(s, \varepsilon/5)$-tame $W$-cloud $H$. Let $F = G[Y] \cup H$.

Since $\text{sn}(G) = a$, there exists a balanced separation $(A, B)$ of $F$ of order
at most \( a \). Let \((A', B') = (A \cap Y, B \cap Y)\) be the corresponding separation in \( G[Y] \). If one of the parts of the separation, say \( B' \), is equal to \( Y \), then \( |A'| \leq a \), and, in particular,

\[
|A \cap X \cap Y| \leq a. \tag{2}
\]

Otherwise, since \((X, Y)\) was chosen so that \( |Y| \) is minimal, neither \((X \cup A', B')\) nor \((X \cup B', A')\) belongs to \( T \). Since \( V(G) = A' \cup B' \cup X \), the condition (ii) in the definition of a tangle implies that at least one of these separations (say the former) has order greater than \( 70a \). In particular,

\[
|(X \cup A') \cap B'| > |X \cap Y|.
\]

Since \((X \cup A) \cap B' = ((X \cap Y) \setminus (A' \cap X)) \cup (A \cap B')\), it follows that \(|A' \cap X| < |A' \cap B'| \leq a\). Hence, the inequality (2) holds in this case as well.

Let \( H_0 \) be the subgraph of \( H \) consisting of the components of \( H \) intersected by \( A \). Since at most \( a \) components of \( H \) contain a vertex of \( A \cap B \), by (2) we conclude that \( H_0 \) has at most \( 2a \) components. Let \( U = W \setminus V(H_0) \). Note that \(|U| \geq |W| - 2a = (1 - \frac{1}{35})|W| = (1 - \frac{\epsilon}{5})|W|\), and thus \( n(H, U) \geq s + 3|V(H_0)| \) by (1). We have \(|B \setminus A| \geq n(H, U) \) and \(|A| \leq |Y| + |V(H_0)| = s/3 + |V(H_0)|\). Therefore

\[
\frac{|B \setminus A|}{|V(F)|} = \frac{|B \setminus A|}{|A| + |B \setminus A|} = \frac{1}{1 + \frac{|A|}{|B \setminus A|}} \geq \frac{3}{4} > \frac{2}{3}.
\]

This contradiction to the fact that \((A, B)\) is a balanced separation of \( F \) finishes the proof.

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