A FIXED POINT THEOREM FOR COMMUTING FAMILIES OF RELATIONAL HOMOMORPHISMS. APPLICATIONS TO METRIC SPACES, ORIENTED GRAPHS AND ORDERED SETS

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Abstract. We extend to binary relational systems the notion of compact and normal structure, introduced by J.P. Penot for metric spaces, and we prove that for the involutive and reflexive ones, every commuting family of relational homomorphisms has a common fixed point. The proof is based upon the clever argument that J.B.Baillon discovered in order to show that a similar conclusion holds for bounded hyperconvex metric spaces and then refined by the first author to metric spaces with a compact and normal structure. Since the non-expansive mappings are relational homomorphisms, our result includes those of T.C.Lim, J.B.Baillon and the first author. We show that it extends the Tarski’s fixed point theorem to graphs which are retracts of reflexive oriented zigzags of bounded length. Doing so, we illustrate the fact that the consideration of binary relational systems or of generalized metric spaces are equivalent.

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Key words: Fixed-point, non-expansive mappings, normal structure, Chebyshev’s center, modular function space, relational homomorphisms, order-preserving maps, metric spaces, hyperconvex spaces, relational systems, ordered sets, graphs, fences, zigzags, retracts.

1. Introduction

Two results about fixed points are very much related. One is the famous Tarski’s theorem ([42], 1955): every order-preserving map on a complete lattice has a fixed point. The other is a theorem of R.Sine and P.M.Soardi ([39], [41], 1979): every non-expansive mapping on a bounded hyperconvex metric space has a fixed point. Indeed, as was shown by D.Misane and the second author (28, 1984, see also [32], 1985 and [17], 1986) if one considers a generalisation of metric spaces, where -instead of real numbers- the distance values are members of an ordered monoid equipped with an involution, then the Sine-Soardi’s theorem is still valid and for a particular ordered monoid, these generalized metric spaces and their non-expansive mappings translate into ordered sets and order-preserving maps and –as a matter of fact– hyperconvex spaces correspond to complete lattices.

Since A.Tarski obtained, in fact, that every commuting family of order-preserving maps on a complete lattice has a common fixed point, E. Jawhari et al [17] considered the question whether in this frame every commuting family of non-expansive mappings on a bounded hyperconvex space has a common fixed point, discovering that it was still unsettled in the frame of ordinary metric spaces. They got a positive answer for countable families; J.B.Baillon ([5], 1986) got a positive answer for arbitrary families acting on ordinary hyperconvex metric spaces. The Baillon’s proof is based upon a clever compactness argument. At first glance, this argument works with minor changes for generalized hyperconvex spaces considered in [17] and, on the other hand, with some extra work, it can be adapted to metric spaces endowed
with a compact and normal structure—as abstractly defined by J.P. Penot [31, 1977]—spaces which include the hyperconvex ones. This extension was done by the first author in [23].

In this paper we propose a generalization of the Penot’s notions in the frame of binary relational systems and their relational homomorphisms. Indeed, on one hand, the non-expansive mappings $f$ acting on an ordinary metric space, (or a generalized one), say $(E, d)$, with distance function $d$ from $E \times E$ into the set $\mathbb{R}^+$ of nonnegative reals (or into an ordered monoid $V$ equipped with an involution), are relational homomorphisms of the binary relational system $E := (E, \{\delta_v : v \in V \})$, where $\delta_v := \{(x, y) \in E \times E : d(x, y) \leq v\}$ for every $v$ belonging to $V$. On an other hand, the Penot’s notions are very easy to define in this frame. We prove that if a reflexive and involutive binary relational system has a compact and normal structure then every commuting family of relational homomorphisms has a common fixed point (Theorem 3.6). As an illustration, we get that on a graph which is a retract of a product of reflexive oriented zigzags of bounded length, every commuting family of preserving maps has a common fixed point (Theorem 5.25); Tarski’s result corresponds to the case of a retract of a power of a two-element zigzag. Characterizations of reflexive and involutive binary relational system with a compact and normal structure are left open. This paper is an other opportunity to go beyond the analogy between metric spaces and binary relational systems. We consider generalized metric spaces whose values of the distance belong to an involutive Heyting algebra (or involutive op-quantale) as it was initiated in [17]. In this context, the notion of one-local retract, which is the key in proving our main result, fits naturally with the parent notion of hole-preserving map. Our illustration with graphs fits in the case of bounded hyperconvex spaces.

After this introduction, this paper consists of four additional sections. Section 2 contains the notions of compact and normal structure for relational systems; an illustration with a fixed-point result is given. Section 3 contains the notion of one-local retract. The main property, Theorem 3.5 is stated; Theorem 3.6 is given a consequence. This property is proved in Section 4. Section 5 is an attempt to illustrate our main result. Subsection 5.2 contains the exact relationship between reflexive involutive binary systems and generalized metric spaces over an involutive monoid (e.g. Lemma 5.3 and Theorem 5.4). In Subsection 5.3, the notion of hyperconvexity is recalled. Notions of inaccessibility and boundedness insuring that hyperconvex spaces have a compact and normal structure are stated (Corollary 5.6). Spaces over a Heyting algebra with their main properties are presented (Theorems 5.8 and 5.9). Subsection 5.4 contains the relationship between one-local retract and hole-preserving maps. Subsection 5.5. rassembles the results for ordinary metric spaces. The case of ordered sets is treated in Subsection 5.6. It contains a characterization of posets with a compact structure. The case of directed graphs with the zigzag distance is treated in Subsection 5.7. It contains a characterization of graphs isometrically embeddable into a product of oriented zigzags (Theorem 5.21) and our fixed point theorem (Theorem 5.25).

2. Basic definitions, elementary properties and a fix-point result

2.1. Binary relations and metric notions. We adapt to binary relations and to binary relational systems the basic notions of the theory of metric spaces. The trick we use for this purpose consists to denote by $d(x, y) \leq r$ the fact that the pair $(x, y)$ belongs to the binary relation $r$, and to interpret $d$ as a distance, $d(x, y)$ and $r$ as numbers (a justification is given in Subsection 5.1).

The basic concepts about relational systems are the following. For a set $E$, a binary relation on $E$ is any subset $r$ of $E \times E$; the restriction of $r$ to a subset $A$ of $E$ is $r A := r \cap (A \times A)$. The inverse of $r$ is the binary relation $r^{-1} := \{(y, x) : (x, y) \in r\}$; the diagonal is $\Delta_E := \{(x, x) : x \in E\}$. We adapt to binary relations and to binary relational systems the basic notions of the theory of metric spaces. The trick we use for this purpose consists to denote by $d(x, y) \leq r$ the fact that the pair $(x, y)$ belongs to the binary relation $r$, and to interpret $d$ as a distance, $d(x, y)$ and $r$ as numbers (a justification is given in Subsection 5.1).

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x \in E \}$. A relation $r$ is symmetric if $r = r^{-1}$; the relation $r$ is reflexive if $\Delta_E \subseteq r$. A map $f : E \to E$ preserves $r$ if $(f(x), f(y)) \in r$ whenever $(x, y) \in r$; the map $f$ preserves a subset $A$ if $f(A) \subseteq A$ (this amounts to say that it preserves the unary relation $A$). The map $f$ preserves a set $\mathcal{E}$ of binary relations $r$ on $E$ if it preserves every member $r$ of $\mathcal{E}$. The pair $\mathbf{E} := (E, \mathcal{E})$ is a binary relational system; note that the maps which preserve $\mathcal{E}$ are in fact the relational homomorphisms of this system. We denote by $\text{End}(\mathbf{E})$ the collection of self-maps which preserve $\mathcal{E}$ (as far we only consider self-maps, no indexing of $\mathcal{E}$ by some index set is required). We set $\mathcal{E}^{-1} := \{ r \subseteq E \times E : r^{-1} \in \mathcal{E} \}$ and we say that $\mathbf{E} := (E, \mathcal{E})$ is involutive if $\mathcal{E} = \mathcal{E}^{-1}$; we say that $\mathbf{E}$ is reflexive, resp. symmetric, if each member $r \in \mathcal{E}$ is reflexive, resp. symmetric. For a subset $A$ of $E$, the restriction of $\mathcal{E}$ to $A$ is $\mathcal{E}_{|A} := \{ r_{|A} : r \in \mathcal{E} \}$ and the restriction of $\mathbf{E}$ to $A$ is the binary relational system $\mathbf{E}_{|A} := (A, \mathcal{E}_{|A})$. For a subset $\mathcal{E}'$ of binary relations on $A$, we set $\mathbf{P}^{-1}_A(\mathcal{E}') := \{ r \in \mathcal{E} : r_{|A} \in \mathcal{E}' \}$.

Let $r$ be a binary relation on $E$ and let $x \in E$; the ball of center $x$, radius $r$, is the set $B(x, r) := \{ y \in E : (x, y) \in r \}$. Let $\mathcal{E}$ be a set of binary relations on $E$. We denote by $\mathcal{B}_E$ the set of balls whose radius belong to $\mathcal{E}$ that is $\mathcal{B}_E := \{ B(x, r) : x \in E, r \in \mathcal{E} \}$, we denote by $\hat{\mathcal{B}}_E$ the set of all intersections of members of $\mathcal{B}_E$, and we set $\hat{\mathcal{B}}_E^* := \hat{\mathcal{B}}_E \setminus \{ \emptyset \}$. Note that, as the intersection over the empty set, $E \in \hat{\mathcal{B}}_E$. For a subset $A$ of $E$, the $r$-center is the set $C(A, r) := \{ x \in E : A \subseteq B(x, r) \}$. We set $\text{Cov}_E(A) := \bigcap \{ B \in \mathcal{B}_E : A \subseteq B \}$. The diameter of $A$ is the set $\delta_E(A) := \{ r \in \mathcal{E} : A \subseteq B(x, r) \}$; the radius of $A$ is the set $r_{\mathcal{E}}(A) := \{ r \in \mathcal{E} : A \subseteq B(x, r) \}$ for some $x \in A$; note that $\delta_E(\emptyset) = \mathcal{E}$ and $r_{\mathcal{E}}(\emptyset) = \emptyset$. If $\mathbf{E} := (E, \mathcal{E})$, we may replace the index $\mathcal{E}$ in the previous notations by $\mathbf{E}$, e.g. $\mathcal{B}_E, \hat{\mathcal{B}}_E^*, \text{Cov}_E(A), \text{End}(\mathbf{E})$, $\delta_E(A)$, and $r_{\mathbf{E}}(A)$ replace $\mathcal{B}_E, \hat{\mathcal{B}}_E^*, \text{Cov}_E(A), \text{End}(\mathbf{E})$, $\delta_E(A)$, and $r_{\mathbf{E}}(A)$.

Our notions of center and radius are inspired from the notions of Chebyshev’s center and radius.

The elementary properties about center, diameter and radius we need are given by the following proposition:

**Proposition 2.1.** Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system, $A \subseteq E$ and $r \subseteq E \times E$. Then:

1. $A \subseteq C(A, r)$ iff $r \in \delta(A)$ and moreover $r^{-1} \in \delta(A)$;
2. $C(A, r) = \cap \{ B(a, r^{-1}) : a \in A \}$;
3. If $r^{-1} \in \mathcal{E}$ then $C(A, r) \in \mathcal{B}_E$;
4. $C(A, r) = C(Cov_E(A), r)$ whenever $r \in \mathcal{E}$;
5. $r_{\mathcal{E}}(A) \subseteq r_{\mathbf{E}}(Cov_E(A))$ and if $A \neq \emptyset$, $\delta_E(A) \subseteq r_{\mathbf{E}}(A)$;
6. $\delta_{\mathbf{E}}(A) = \delta_E(Cov_E(A))$ provided that $\mathbf{E}$ is involutive.

**Proof.** (i). Immediate.

(ii). $x \in C(A, r)$ iff $A \subseteq B(x, r)$; this latter condition amounts to $x \in \{ B(a, r^{-1}) : a \in A \}$.

(iii). Follows immediately from (ii). (iv). From the definition of the $r$-center, $x \in C(A, r)$ means that $A \subseteq B(x, r)$; since $r \in \mathcal{E}$ this inclusion amounts to $Cov_E(A) \subseteq B(x, r)$. Again, from the definition of the $r$-center, this means $x \in C(Cov_E(A), r)$.

(v). Let $r \in r_{\mathbf{E}}(A)$ then $C(A, r) \cap A \neq \emptyset$. Since $C(A, r) = C(Cov_E(A), r)$ from (iv), we have $C(Cov_E(A), r) \cap Cov_E(A) \neq \emptyset$, hence $r \in r_{\mathbf{E}}(Cov_E(A))$. The second assertion is obvious.

(vi). Trivially $\delta_{\mathbf{E}}(Cov_E(A)) \subseteq \delta_E(A)$. Conversely, let $r \in \delta_E(A)$. Then $A \subseteq B(x, r)$ for every $x \in A$, that is $A \subseteq C(A, r)$. From (iv), this yields $A \subseteq C(Cov_E(A), r)$. Since $\mathbf{E}$ is involutive, $r^{-1} \in \mathcal{E}$, hence from (ii) we have $C(Cov_E(A), r) \subseteq \mathcal{B}_E$. Since $A \subseteq C(Cov_E(A), r)$ it follows $Cov_E(A) \subseteq C(Cov_E(A), r)$ that is $r \in \delta_E(Cov_E(A))$ by (i). \qed
2.2. Compact normal structure and retraction. We introduce the notion of compact and normal structure as Penot did for metric spaces ([31], 1977) and we prove a fix-point result.

**Definition 2.2.** A subset $A$ of a binary relational system $E$ is equally centered if $r_E(A) = \delta_E(A)$.

For an example, if $A$ is the empty set and $E$ nonempty then $A$ is not equally centered. If $A$ a singleton, say $\{a\}$, then $A$ is equally centered (indeed, by (v) of Proposition 2.1 we have $\delta_E(A) \subseteq r_E(A)$; if $r \in r_E(A)$ then $(x, x) \in r$ hence $r \in \delta_E(A)$). If in addition $E$ is reflexive and involutive, $\text{Cov}_E(A)$ is equally centered. Indeed, by (v) of Proposition 2.1 we have $\delta_E(\text{Cov}(A)) \subseteq r_E(\text{Cov}(A))$; now, if $r \in r_E(\text{Cov}(A))$ then, since $r$ is reflexive, $(x, x) \in r$ and thus $r \in \delta_E(A)$; since $\delta_E(A) = \delta_E(\text{Cov}(A))$ by (vi) of Proposition 2.1 $r \in \delta_E(\text{Cov}(A))$; hence $r_E(\text{Cov}(A)) = \delta_E(\text{Cov}(A))$ and $A$ is equally centered. A generalization of this fact is given in Lemma 3.3.

**Definition 2.3.** A binary relational system $E$ has a normal structure if no $A \in \hat{B}_E$ distinct from a singleton is equally centered. Equivalently, if $|A| \neq 1$ then $r_E(A) \neq \delta_E(A)$.

**Definition 2.4.** A binary relational system $E := (E, \mathcal{E})$ has a compact structure if $\hat{B}_E$ has the finite intersection property (f.i.p.) that is, for every family $F$ of members of $\hat{B}_E$, the intersection of $F$ is nonempty provided that the intersection of all finite subfamilies of $F$ are nonempty.

As it is easy to see, $B_E$ has the f.i.p. iff $\hat{B}_E$ has the f.i.p. (if $F$ is a family of members of $\hat{B}_E$ associate the family $\mathcal{G}$ made of balls which contain some member of $F$ and observe that $\cap F = \cap \mathcal{G}$. If all finite intersection of members of $F$ are nonempty, the same holds for $\mathcal{G}$.) Hence, if $B_E$ has the f.i.p., $\cap \mathcal{G} \neq \emptyset$. The equality $\cap F = \cap \mathcal{G}$ yields $F \neq \emptyset$, thus $\hat{B}_E$ has the f.i.p.

We have trivially:

**Lemma 2.5.** If a binary relational system $E := (E, \mathcal{E})$ has a compact structure then every chain of members of $\hat{B}_E$ has an infimum, namely the intersection of all members of that chain.

**Lemma 2.6.** Let $f$ be an endomorphism of an involutive binary relational system $E := (E, \mathcal{E})$.

Then:

(i) Every minimal member $A$ of $\hat{B}_E$ which is preserved by $f$ is equally centered;

(ii) If $E$ has a compact structure then every member of $\hat{B}_E$ preserved by $f$ contains a minimal one.

**Proof.** (i). Let $A \in \hat{B}_E$ and let $r \in r_E(A)$, then $A' := C(A, r) \cap A$ is nonempty. Indeed, by definition of $r_E(A)$ there is some $x \in A$ such that $A \subseteq B_E(x, r)$ and by definition of $C(A, r)$, $x \in C(A, r)$. This proves our assertion. Since $E$ is involutive, $r^{-1} \in \mathcal{E}$, hence from (iii) of Proposition 2.1 $C(A, r) \in \hat{B}_E$; hence $A' \in \hat{B}_E$. Assuming that $f$ preserves $A'$ it follows $A = A'$ from the minimality of $A$. This means $A \subseteq C(A, r)$, that is $r \in \delta_E(A)$. Hence $r_E(A) \subseteq \delta_E(A)$. Since $\delta_E(A) \subseteq r_E(A)$, this yields $r_E(A) = \delta_E(A)$. Thus, $A$ is equally centered as claimed.

In order to see that $f$ preserves $A'$, observes that $f$ preserves $C(A, r)$. Indeed, first, since $f$ is a relational homomorphism we have $f(C(A, r)) \subseteq C(f(A), r)$ (for $x \in C(A, r)$ we have $A \subseteq B_E(x, r)$) thus $f(A) \subseteq B_E(f(x), r)$ that is $f(x) \in C(f(A), r)$). Next, from (iv) of Proposition 2.1 we have $C(f(A), r) = C(\text{Cov}_E(f(A)), r)$. To conclude, it suffices to prove that $\text{Cov}_E(f(A)) = A$. This assertion follows from the minimality of $A$. Indeed, since $f(A) \subseteq$
A, we have \( \text{Cov}_E(f(A)) \subseteq \text{Cov}_E(A) = A \); it follows \( f(\text{Cov}_E(f(A))) \subseteq f(A) \subseteq \text{Cov}_E(f(A)) \) that is \( \text{Cov}_E(f(A)) \) is preserved by \( f \). The minimality of \( A \) proves our assertion.

(ii). The fact that \( \mathcal{B}_E \) has the f.i.p. implies that \( \mathcal{B}_E^* \), ordered by reverse of inclusion, is inductive (Lemma 3.2). The subset of \( \mathcal{B}_E^* \) made of \( A \) such that \( f(A) \subseteq A \) is inductive too. The conclusion follows from Zorn’s lemma.

\[
\square
\]

**Corollary 2.7.** If \( E \) involutive has a compact and normal structure then every endomorphism \( f \) has a fixed point.

For metric spaces, this is the result of Penot (1979) extending the result of Kirk (1965).

From Corollary 2.7, one can derives:

**Proposition 2.8.** If \( E \) involutive has a compact and normal structure then for every endomorphism \( f \), the restriction \( E_{f}\text{Fix}(f) \) to the set \( \text{Fix}(f) \) of fixed points of \( f \) has a compact and normal structure.

From this and the previous corollary, one deduces by an immediate recurrence that a finite set of commuting maps has a common fixed point. This leads to the question of what happens with infinitely many.

Behind the proof of the above proposition and the answer to the question is the notion of one-local retract.

### 3. ONE-LOCAL RETRACTS AND FIXED POINTS

An map \( g : E \to E \) is a *retraction* of \( E \) if \( g \) is an homomorphism of \( E \) such that \( g \circ g = g \). For a subset \( A \) of \( E \), we say that \( E_{f} \) is a retract of \( E \) if \( A \) is the image of \( E \) by some retraction of \( E \). We say that \( E_{f} \) is a one-retract if for every \( x \in E \), \( E_{f} \) is a retract of \( B_{E_{f}}(x) \).

**Lemma 3.1.** Let \( E := (E, \mathcal{E}) \) be a binary relational system and \( A \) be a subset of \( E \). If \( E_{f} \) is a one-retract then for every family of balls \( B_{E_{f}}(x_{i}, r_{i}) \), \( x_{i} \in A \), \( r_{i} \in \mathcal{E} \), the intersection over \( A \) is nonempty provided the intersection over \( E \) is nonempty. The converse holds provided that \( E \) is reflexive and involutive;

\[
\text{Proof.} \text{ Let } I \text{ be a set; let } \mathcal{B} := \{ B_{E}(x_{i}, r_{i}) : x_{i} \in A, r_{i} \in \mathcal{E} \} \text{ such that } B := \cap \mathcal{B} \neq \emptyset. \text{ Let } a \in B \text{ and let } h \text{ be a retraction from } E_{f} \text{ onto } E_{f}. \text{ The map } h \text{ fixes } A \text{ and preserves the relations induced by } \mathcal{E} \text{ on } A \cup \{a\}. \text{ We claim that } h(a) \in B. \text{ Indeed, let } i \in I. \text{ Since } a \in B_{E}(x_{i}, r_{i}) \text{ we have } (x_{i}, a) \in r_{i} \text{ and since } h \text{ preserves the relations induced by } \mathcal{E} \text{ on } A \cup \{a\}, (h(x_{i}), h(a)) \in r_{i}. \text{ Since } h(x_{i}) = x_{i}, \text{ we get } (x_{i}, h(a)) \in r_{i} \text{, hence } h(a) \in B_{E}(x_{i}, r_{i}) \text{. Our claim follows.} \]

Suppose that \( E \) is reflexive and involutive. We show that the ball’s property stated in the lemma implies that \( E_{f} \) is a one-local retract. Let \( a \in E \setminus A \). Let \( B := \{ B(u, r) : u \in A, a \in B(u, r), r \in \mathcal{E} \} \). We have \( a \in B := \cap B \) (note that \( B = E \) if \( B = \emptyset \)). Hence, \( B \neq \emptyset \). According to the ball’s property, \( B \cap A \neq \emptyset \). We claim that the map \( h : A \cup \{a\} \to A \) which is the identity on \( A \) and send \( a \) onto \( a' \) is a retraction of \( E_{f} \). Since \( h \) is the identity on \( A \), it suffices to check that for every \( r \in \mathcal{E} \) and \( u \in A \):

\[
\begin{align*}
(1) & \quad (u, a) \in r \text{ implies } (u, a') \in r; \\
(2) & \quad (a, u) \in r \text{ implies } (a', u) \in r; \\
(3) & \quad (a, a) \in r \text{ implies } (a', a') \in r.
\end{align*}
\]

The first item holds by our choice of \( a' \); the second item is equivalent to the first because \( E \) is involutive and the third item holds because \( E \) is involutif.

\[
\square
\]

**Lemma 3.2.** Let \( E = (E, \mathcal{E}) \) be a binary relational system and \( A \subseteq B \subseteq E \).
(i) If $E_{1A}$ is a one-local retract of $E$ then it is a one-local retract of $E_{1B}$.
(ii) If $E_{1A}$ is a one-local retract of $E_{1B}$ and $E_{1B}$ is a one-local retract of $E$ then $E_{1A}$ is a one-local retract of $E$.

Proof. The proof relies on the fact that $(E_{1B'})_{1A'} = E_{1A'}$ for every $A' \subseteq B'$.

(i). Immediate.

(ii). Let $x \in E \setminus A$. If $x \notin B$ then, since $E_{1A}$ is a one-local retract of $E_{1B}$, it is a retract of $E_{1A \cup \{x\}}$.

We deduce that

Proof of (4). By definition, $A$ above, we deduce that $\delta_{E}(E) = \delta_{E}(E_{1})$.

Lemma 3.3. Let $E_{1X}$ be a one-local retract of $E$. If $E$ has a compact structure then $E_{1X}$ too; if $E$ is involutive and has a normal structure then $E_{1X}$ too.

Proof. Let $E' := E_{1X}$ and $E' := E_{1X} := \{r \cap X \times X : r \in E\}$. We prove the first assertion. Let $B' := \{B_{E}(x_{i}^{r},r_{i}^{r}) : i \in I, r_{i}^{r} \in E'\}$ be a family of balls of $E'$ whose finite intersections are nonempty. For each $i \in I$, $r_{i}^{r} \cap (X \times X)$ for some $r_{i} \in E$. The family $B := \{B_{E}(x_{i}^{r},r_{i}) : i \in I\}$ of balls of $E$ satisfies the f.p. hence has a nonempty intersection. Let $x \in \cap B$. A retraction $g$ from $E_{1A \cup \{x\}}$ onto $E_{1X}$ will send $x$ into $\cap B'$, proving that this set is nonempty. We prove the second assertion.

Let $A \in B_{E}^{*}$. We claim that:

1. \[\delta_{E}(A) = \delta_{E}(\text{Cov}_{E}(A))\]

and

2. \[r_{E}(A) = r_{E}(\text{Cov}_{E}(A)).\]

Indeed, equality (1) is item (vi) of Proposition 2.1. Concerning equality (2), note that inclusion $r_{E}(A) \subseteq r_{E}(\text{Cov}_{E}(A))$ is item (v) of Proposition 2.1. For the converse, let $r \in r_{E}(\text{Cov}_{E}(A))$. Then, there is some $x \in \text{Cov}_{E}(A)$ such that $\text{Cov}_{E}(A) \subseteq B_{E}(x,r)$. Since $E'$ is a one-local retract of $E$, there is a retraction of $E_{1A \cup \{x\}}$ onto $E_{1X}$ which fixes $X$. Let $a := g(x)$. Since $A \subseteq \text{Cov}_{E}(A) \subseteq B_{E}(x,r)$, $A \subseteq B_{E}(x,r)$; since $g$ fixes $A$, $A \subseteq B_{E}(a,r)$. We claim that $a \in A$. Indeed, $A = r_{1}(B_{E}(x_{i}^{r},r_{i}) : i \in I)$ with $x_{i}^{r} \in X, r_{i}^{r} \in E_{1X}$. For each $i \in I$, choose $r_{i} \in E$ such that $r_{i}^{r} = r_{i} \cap X \times X$. Then $\text{Cov}_{E}(A) \subseteq A := r_{1}(B_{E}(x_{i}^{r},r_{i}) : i \in I)$. Since $x \in \text{Cov}_{E}(A)$, $x \in A$. Since $g$ fixes each $x_{i}$, $a := g(x) \in A$. Since $a \in X, a \in A$, proving our claim. Hence $r \in r_{E}(A)$.

From these two equalities, we obtain:

3. \[\delta_{E}(\text{Cov}(A)) = p_{X}^{-1}(\delta_{E}(A))\]

and

4. \[r_{E}(\text{Cov}(A)) = p_{X}^{-1}(r_{E}(A)).\]

Proof of (3). By definition $p_{X}^{-1}(\delta_{E}(A)) = \{r \in E : r_{1}X \in \delta_{E}(A)\} = \{r \in E : r_{1}X \subseteq r\} = \delta_{E}(A)$ since $A \subseteq X$. Equality (3) follows then from equality (1).

Proof of (4). By definition, $p_{X}^{-1}(r_{E}(A)) = \{r \in E : r_{1}X \in r_{E}(A)\} = \{r \in E : r_{1}X \subseteq B_{E}(a',r_{1})\}$ for some $a' \in A = \{r \in E : r_{1}X \subseteq B_{E}(a',r)\}$ for some $a' \subseteq A = r_{E}(A)$. Equality (4) follows from equality (2).

Suppose that $A$ is equally centered in $E'$, that is $\delta_{E}(A) = r_{E}(A)$. From the equations above, we deduce that $\delta_{E}(\text{Cov}(A)) = r_{E}(\text{Cov}(A))$. Hence $\text{Cov}_{E}(A)$ is equally centered. If $E$
Lemma 4.1. If $E$ is involutive, reflexive and has a compact and normal structure then for every homomorphism $f$, the set $\text{Fix}(f)$ of fixed points of $f$ is a one-local retract of $E$, thus $E_{|\text{Fix}(f)}$ has a compact and normal structure.

Proof. Let $B_E(x_i, r_i), x_i \in \text{Fix}(f)$, such that $A := \bigcap_i B_E(x_i, r_i)$ is nonempty. Since every $x_i$ belongs to $\text{Fix}(f)$, $f$ preserves $A$. According to (ii) of Lemma 2.6, since $A$ is an intersection of balls, $A$ contains an intersection of balls $A'$ which is equally centered and preserved by $f$. From the normality of $E$, $A'$ reduces to a single element, that is a fix-point of $f$. Consequently, $A \cap \text{Fix}(f) \neq \emptyset$. According to Lemma 3.1, $\text{Fix}(f)$ is a one-local retract. □

Theorem 3.5. If $E$ is involutive, reflexive and has a compact and normal structure then the intersection of every down-directed family of one-local retracts is a one-local retract.

We will prove Theorem 3.5 in the next section. From it, we derive easily our main result.

Theorem 3.6. If $E$ is involutive, reflexive and has a compact and normal structure then every commuting family $F$ of endomorphisms of $E$ has a common fixed point. Furthermore, the restriction of $E$ to the set $\text{Fix}(F)$ of common fixed points of $F$ is a one-local retract of $E$.

Proof. For a subset $\mathcal{F}'$ of $\mathcal{F}$, let $\text{Fix}(\mathcal{F}')$ be the set of fixed points of $\mathcal{F}'$.

Claim 3.7. For every finite subset $\mathcal{F}'$ of $\mathcal{F}$, $E_{|\text{Fix}(\mathcal{F}')} = E_{|\text{Fix}(\mathcal{F})}$ is a one-local retract of $E$.

Proof of Claim 3.7. Induction on $n := |\mathcal{F}'|$. If $n = 0$, there is no map, hence the set of fixed points is $E$, thus the conclusion holds. If $n = 1$, this is Theorem 2.8. Let $n \geq 1$. Suppose that the property holds for every subset $\mathcal{F}''$ of $\mathcal{F}$, such that $|\mathcal{F}'| < n$. Let $f \in \mathcal{F}'$ and $\mathcal{F}'' := \mathcal{F}' \setminus \{f\}$. From our inductive hypothesis, $E_{|\text{Fix}(\mathcal{F}''')} = E_{|\text{Fix}(\mathcal{F}'')}$ is a one-local retract of $E$. Thus, according to Lemma 3.3, $E_{|\text{Fix}(\mathcal{F}'')}$ has a compact and normal structure. Now since $f$ commutes with every member $g$ of $\mathcal{F}''$, $f$ preserves $\text{Fix}(\mathcal{F}'')$ (indeed, if $u \in \text{Fix}(\mathcal{F}'')$, we have $g(f(u)) = f(g(u)) = f(u)$, that is $f(u) \in \text{Fix}(\mathcal{F}'')$. Thus $f$ induces an endomorphism $f''$ of $E_{|\text{Fix}(\mathcal{F}'')}$, which commute with every member of $\mathcal{F}''$, and $f''$ is a one-local retract of $E$. □

Let $\mathcal{P} := \{\text{Fix}(\mathcal{F}') : |\mathcal{F}'| < \aleph_0\}$ and $P := \bigcap P$. According to Theorem 3.5, $E_{|P}$ is a one-local retract of $E$. Since $P = \text{Fix}(\mathcal{F})$ the conclusion follows.

4. Proof of Theorem 3.5

We recall the following basic fact about ordered sets (see [11], Proposition 5.9 p 33).

Lemma 4.1. Every down directed subset of a partially ordered set $P$ has an infimum if every totally ordered subset of $P$ has an infimum.

Let $E := (E, \mathcal{C})$ which is is involutive, reflexive and has a compact and normal structure. Let $P$ be the set, ordered by inclusion, of subsets $A$ of $E$ such that $E_{|A}$ is a one-local retract of $E$.

We will prove the following:
Lemma 4.2. The set $P$ is closed under intersection of every chain of its members.

We claim that with the help of Lemma 4.1 it follows that $P$ is closed under intersection of every down directed family of its members. This statement is Theorem 3.5.

Indeed, observe first that from Lemma 4.2 it follows that for every subset $X$ of $E$, the set $P_X := \{ A \in P : X \subseteq A \}$ is closed under intersection of every chain of its members (if $C$ is such a chain then $C := \bigcap_{i \in I} C_i \in P$ by Lemma 4.2 and trivially $X \subseteq C$, hence $C \in P_X$). Next, let $A$ be a down directed family of members of $P$ and let $X := \bigcap A$. Then $A \in P_X$. Since $P_X$ is closed under intersection of every chain of its members, Lemma 4.1 ensures that $A$ has an infimum in $P_X$. This infimum must be $X$.

In order to prove Lemma 4.2 we prove the following

Lemma 4.3. Let $E := (E, \mathcal{E})$ be a reflexive and involutive binary relational system with a compact and normal structure; let $\kappa$ be a cardinal. For every ordinal $\alpha, \alpha < \kappa$ let $B_\alpha, E_\alpha$ be subsets of $E$ such that:

1. $B_\alpha \supseteq B_{\alpha + 1}$ and $E_\alpha \supseteq E_{\alpha + 1}$ for every $\alpha < \kappa$;
2. $\cap_{\gamma < \alpha} B_\gamma = B_\alpha$ and $\cap_{\gamma < \alpha} E_\gamma = E_\alpha$ for every limit $\alpha < \kappa$;
3. $E_\alpha := \mathcal{E}_{\mid E_\alpha}$ is a one-local retract of $E$ and $B_\alpha$ is a nonempty intersection of balls of $E_\alpha$.

Then $B_\kappa := \cap_{\alpha < \kappa} B_\alpha \neq \emptyset$.

Before proving the lemma, let us deduce Lemma 4.2 from it. We argue by induction on the size of totally ordered families of one-local retracts of $E$. First we may suppose that $E$ has more than one element; next, we may suppose that these families are dually well ordered by induction. Thus, given an infinite cardinal $\kappa$, let $(E_{\mid E_\alpha})_{\alpha < \kappa}$ be a descending sequence of one-local retracts of $E$. From the induction hypothesis, we may suppose that the restriction of $E$ to $E'_\alpha := \cap_{\gamma < \alpha} E_\gamma$ is a one-local retract of $E$ for each limit ordinal $\alpha < \kappa$. Hence, we may suppose that $E_\alpha := \cap_{\gamma < \alpha} E_\gamma$ for each limit ordinal $\alpha < \kappa$. Since $E_\alpha$ is a one-local retract of $E$ and $E$ has a normal structure, $E_\alpha$ has a normal structure (Lemma 3.3). Hence, either $E_\alpha$ is a singleton, say $x_\alpha$, or $r_{E_\alpha}(E_\alpha) \setminus \delta_{E_\alpha}(E_\alpha) \neq \emptyset$. In both cases, $E_\alpha$ is a ball of $E_\alpha$ (since $E$ is reflexive, $(x_\alpha, x_\alpha) \in r$ for any $r \in \mathcal{E}$, hence the first case, $E_\alpha = B_{E_\alpha}(x_\alpha, r_{E_\alpha})$, whereas in second case, $E_\alpha \subseteq B_{E_\alpha}(x, r)$ for some $x \in E_\alpha, r \in r_{E_\alpha}(E_\alpha) \setminus \delta_{E_\alpha}(E_\alpha)$). Hence, Lemma 4.3 applies with $B_\alpha = E_\alpha$ and gives that $E_\alpha$ is nonempty. Let us prove that $E_0 := E_{\mid E_0}$ is a one-local retract of $E$. We apply Lemma 3.1. Let $(B_E(x_i, r_i))_{i \in I}, x_i \in E_\alpha, r_i \in \mathcal{E}$ be a family of balls such that the intersection is nonempty. Since $E_\alpha$ is a one-local retract of $E$, the intersection $B_\alpha := E_\alpha \cap \cap_{i \in I} B_E(x_i, r_i)$ is nonempty for every $\alpha < \kappa$. Now, Lemma 4.3 applied to the sequence $(E_\alpha, B_\alpha)_{\alpha < \kappa}$ tells us that $E_\kappa := E_\alpha \cap \cap_{i \in I} B_E(x_i, r_i)$ is nonempty. According to Lemma 3.1, $E_{\mid E_0}$ is a one-local retract of $E$.

4.1. Proof of Lemma 4.3 Let $\mathfrak{A}$ be the collection of all descending sequences $A := (A_\alpha)_{\alpha < \kappa}$ such that each $A_\alpha$ is a nonempty intersection of balls of $E_{\mid E_\alpha}$ included into $B_\alpha$. Set $E_\alpha := E_{\mid E_\alpha}$ and $\mathfrak{B} := \Pi_{\alpha < \kappa} B_{E_\alpha}^*$. The sequence $B := (B_\alpha)_{\alpha < \kappa}$ belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is included into $\mathfrak{B}$. The set $\mathfrak{B}$ is ordered as follows:

5. $(A'_\alpha)_{\alpha < \kappa} \leq (A''_\alpha)_{\alpha < \kappa}$ if $A'_\alpha \subseteq A''_\alpha$ for every $\alpha < \kappa$.

Since $E_\alpha$ is a one-local retract of $E$, $E_\alpha$ has a normal and compact structure (Lemma 3.3). Since it has a compact structure, every descending sequence in $B_{E_\alpha}^*$ has an infimum (Lemma 2.5). Thus, there is a minimal sequence $A := (A_\alpha)_{\alpha < \kappa}$ with $A \leq B$. 

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We prove first that the sequence of \( r_E(A_\alpha) \) is constant (see (a) of Sublemma 4.6). Let \( r \) be the common value. We prove next that \( \delta_E(A_\alpha) = r \) (see (b) of Sublemma 4.6). Since \( E \) has a normal structure, we deduce that each \( A_\alpha \) is a singleton. Since \( \mathcal{A} \) is decreasing, \( A_\kappa := \bigcap_{\alpha < \kappa} A_\alpha \) is a singleton too. Hence, \( B_\kappa \neq \emptyset \). This is the conclusion of the lemma.

The key argument for the proof of Sublemma 4.6 is the following.

**Sublemma 4.4.** Let \( \alpha < \kappa \) and \( A_\alpha \subseteq B_E(x_r, x) \), with \( r \in \mathcal{E} \) and \( x \in E_\alpha \). Then \( \alpha < \beta < \kappa \) for each \( \beta < \kappa \).

**Proof.** Set \( B := B_E(x_r, x) \). For \( \xi < \kappa \) we have \( A_\xi = A_\xi \cap B \) if \( \xi < \alpha \) and \( A_\xi' = A_\xi \) otherwise. The family \( \mathcal{A}' := \{ A_\xi' \}_{\xi < \kappa} \) belongs to \( \mathfrak{A} \) and satisfies \( \mathcal{A}' \subseteq \mathcal{A} \). Since \( \mathcal{A} \) is minimal, we get \( \mathcal{A}' = \mathcal{A} \), thus \( A_\xi = A_\xi \cap B \) for \( \xi < \alpha \) that is \( A_\xi \subseteq B \); since \( A_\xi \subseteq A_\alpha \subseteq B \) for \( \xi, \beta < \alpha \) it follows that \( A_\xi \subseteq B \).

Let \( \alpha < \kappa \). From the hypotheses of the lemma, there is a family \( \mathcal{B}' := (B_E(x_{r_i}, r_i))_{i \in I} \), with \( x_i \in E_\alpha \), \( r_i \in \mathcal{E}_{E_\alpha} \) such that \( A_\alpha = \bigcap \mathcal{B}' \). For each \( i \in I \), let \( i_i \) such that \( r_{i,i} \in E_\alpha = r_i \). Let \( B := (B_E(x_{r_i}, r_i))_{i \in I} \). Then \( A_\alpha = B \cap E_\alpha \).

From the sublemma above we deduce:

**Corollary 4.5.** Let \( \alpha < \kappa \). Then:

(a) \( A_\beta \subseteq B \cap E_\beta \) for every \( \beta < \alpha \);
(b) \( A_\alpha = \bigcap_{\beta < \alpha} A_\beta \) if \( \alpha \) is a limit ordinal;
(c) \( r_E(A_\alpha) \subseteq r_E(A_\beta) \) for every \( \beta < \alpha \);
(d) \( r_E(A_\beta) \subseteq r_E(A_\alpha) \) for every \( \beta < \alpha \).

**Proof.** (a). Follows directly from Sublemma 4.4. Indeed, we have \( A_\alpha \subseteq B_E(x_{r_i}, r_i) \) hence from Sublemma 4.4, \( A_\beta \subseteq B_E(x_r, x) \). This yields \( A_\beta \subseteq A_\alpha \).

(b). From \( \bigcap_{\gamma < \alpha} E_\gamma = E_\alpha \) and (a) we get

\[
A_\alpha = B \cap E_\alpha = \bigcap_{\beta < \alpha} B \cap E_\beta \supseteq \bigcap_{\beta < \alpha} A_\beta.
\]

This implies \( A_\alpha \supseteq \bigcap_{\beta < \alpha} A_\beta \).

Since \( \mathcal{A} \) is decreasing, we have \( A_\alpha \supseteq \bigcap_{\beta < \alpha} A_\beta \). Hence, \( A_\alpha = \bigcap_{\beta < \alpha} A_\beta \).

(c). Let \( r \in r_E(A_\alpha) \). Then \( A_\alpha \subseteq B_E(x_r, x) \) for some \( x \in A_\alpha \). According to Sublemma 4.4 we have \( A_\beta \subseteq B_E(x_r, x) \). Since \( A_\alpha \subseteq A_\beta \), \( x \in A_\beta \), hence \( r \in r_E(A_\beta) \).

(d). Let \( r \in r_E(A_\beta) \) and \( x \in A_\beta \) such that \( A_\beta \subseteq B_E(x_r, x) \). From (a) of Sublemma 4.4, we have \( A_\beta \subseteq B \) thus \( x \in B \cap \bigcap_{\alpha < \kappa} B \cap E_\alpha \). Since \( E_\alpha \) is a one-local retract, there is some \( y \in B \cap \bigcap_{\alpha < \kappa} E_\alpha \) \cap E_\alpha \). This means \( A_\alpha \subseteq B \cap E_\alpha \) which in turns implies \( r \in r_E(A_\alpha) \).

**Sublemma 4.6.** (a) \( r_E(A_\alpha) \) is independent of \( \alpha \);
(b) \( \delta_E(A_\alpha) = r_E(A_\alpha) \) for every \( \alpha < \kappa \).

**Proof.** (a). Follows from (c) and (d) of Corollary 4.5.

(b). Let \( r \) be the common value of all \( r_E(A_\alpha) \). Let \( r \in \mathfrak{A} \). Set \( C_r(A_\alpha) := \{ x \in E_\alpha : A_\alpha \subseteq B_E(x_r, x) \} \), \( A_r := A_\alpha \cap C_r(A_\alpha) \) and \( \mathcal{A}_r := (A_r)_\alpha < \kappa \).

**Claim 4.7.** (1) \( A_r \) is a nonempty intersection of balls of \( E_\alpha \);
(2) \( A_r \subseteq A_\alpha \);
(3) \( A_{r, \beta} \supseteq A_\alpha \) for \( \beta < \kappa \).


Proof of Claim 4.7 (1). Since \( r \in r_E(A_\alpha) \), \( A_\alpha \subseteq B_E(x, r) \) for some \( x \in A_\alpha \), hence \( x \in C_r(A_\alpha) \) proving that \( A'_\alpha \) is nonempty. Since \( E \) is involutive, \( r^{-1} \in E \), thus from (iii) of Proposition 2.1, \( C_r(A_\alpha) \) is an intersection of balls of \( E_{1:E_\alpha} \) with centers in \( A_\alpha \). Hence, \( A'_\alpha \) is a nonempty intersection of balls of \( E_{1:E_\alpha} \).

(2) Obvious.

(3). Let \( \beta < \alpha \). By construction of \( \mathcal{A} \), we have \( A_\beta \supseteq A_\alpha \). Let \( x \in A_\alpha \). By definition, we have \( A_\alpha \subseteq B_E(x, r) \). From Lemma 4.4 we have \( A_\beta \subseteq B_E(x, r) \). It follows that \( x \in C_r(A_\beta) \).

Since \( x \in A_\beta \), \( x \in A_\beta' \). This proves that (3) holds.

From Claim 4.7 and the minimality of \( \mathcal{A} \) we obtain \( \mathcal{A}' = \mathcal{A} \). From this it follows that \( A_\alpha \subseteq C_r(A_\alpha) \). Since this inclusion holds for every \( r \in r_E(A_\alpha) \) we get \( \delta_E(A_\alpha) = r_E(A_\alpha) \). This proves that (6) holds. This ends the proof of Sublemma 4.6.

5. Illustrations

5.1. Preservation. Let \( E \) be a set. For \( n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\} \), a map \( f : E^n \to E \) is an \( n \)-ary operation on \( E \), whereas a subset \( \rho \subseteq E^n \) is an \( n \)-ary relation on \( E \). Denote by \( \mathcal{O}^{(n)} \) (resp. \( \mathcal{R}^{(n)} \)) the set of \( n \)-ary operations (resp. relations) on \( E \) and set \( \mathcal{O} := \cup \{ \mathcal{O}^{(n)} : n \in \mathbb{N}^* \} \) (resp \( \mathcal{R} := \cup \{ \mathcal{R}^{(n)} : n \in \mathbb{N}^* \} \)). For \( n, i \in \mathbb{N}^* \) with \( i \leq n \), define the \( i \)-th \( n \)-ary projection \( e_i^n \) by setting \( e_i^n(x_1, \ldots, x_n) := x_i \) for all \( x_1, \ldots, x_n \in E \) and set \( \mathcal{P} := \{ e_i^n : i, n \in \mathbb{N}^* \} \). An operation \( f \in \mathcal{O} \) is constant if it takes a single value, it is idempotent provided \( f(x, \ldots, x) = x \) for all \( x \in E \). We denote by \( \mathcal{C} \) (resp. \( \mathcal{I} \)) the set of constant, (resp. idempotent) operations on \( E \).

Let \( m, n \in \mathbb{N}^* \), \( f \in \mathcal{O}^{(m)} \) and \( \rho \in \mathcal{R}^{(n)} \). Then \( f \) preserves \( \rho \) if:

(6) \( (x_1, \ldots, x_{i,n}) \in \rho, \ldots, (x_{m,1}, \ldots, x_{m,n}) \in \rho \implies (f(x_1, \ldots, x_{m,1}), \ldots, f(x_{m,n})) \in \rho \)

for every \( m \times n \) matrix \( X := (x_{i,j})_{i=1,...,m} \) of elements of \( E \).

If \( \rho \) is binary and \( f \) is unary, then \( f \) preserves \( \rho \) means:

(7) \( (x, y) \in \rho \implies (f(x), f(y)) \in \rho \)

for all \( x, y \in E \).

If \( \mathcal{F} \) is a set of operations on \( E \), let \( \text{Inv}(\mathcal{F}) \), resp. \( \text{Inv}_n(\mathcal{F}) \) be the set of relations, resp. \( n \)-ary relations, preserved by all \( f \in \mathcal{F} \). Dually, if \( \mathcal{R} \) is a set of relations on \( E \), let \( \text{Pol}(\mathcal{R}) \), resp. \( \text{Pol}_n(\mathcal{R}) \), be the set of operations, resp. \( n \)-ary operations, which preserve all \( \rho \in \mathcal{R} \). The operators Inv and Pol define a Galois correspondence. The study of this correspondence is the theory of clones [25].

5.2. Toward generalized metric spaces. We restrict our attention to the case of unary operations and binary relations. We recall that if \( \rho \) and \( \tau \) are two binary relations on the same set \( E \), then their composition \( \rho \circ \tau \) is the binary relation made of pairs \((x, y)\) such that \((x, z) \in \tau \) and \((z, y) \in \rho \). It is customary to denote it \( \tau \cdot \rho \).

The set \( \text{Inv}_2(\mathcal{F}) \) of binary relations on \( E \) preserved by all \( f \) belonging to a set \( \mathcal{F} \) of self maps has some very simple properties that we state below (the proofs are left to the reader). For the construction of many more properties by means of primitive positive formulas, see [40].

Lemma 5.1. Let \( \mathcal{F} \) be a set of unary operations on a set \( E \). Then the set \( \mathcal{R} := \text{Inv}_2(\mathcal{F}) \) of binary relations on \( E \) preserved by all \( f \in \mathcal{F} \) satisfies the following properties:

(a) \( \Delta_E \in \mathcal{R} \);

(b) \( \mathcal{R} \) is closed under arbitrary intersections; in particular \( E \times E \in \mathcal{R} \);

(c) \( \mathcal{R} \) is closed under arbitrary unions;
(d) If $\rho, \tau \in \mathcal{R}$ then $\rho \circ \tau \in \mathcal{R}$;
(e) If $\rho \in \mathcal{R}$ then $\rho^{-1} \in \mathcal{R}$.

Let $\mathcal{R}$ be a set of binary relations on a set $E$ satisfying items (a), (b), (d) and (e) of the above lemma (we do not require (c)). To make things more transparent, denote by 0 the set $\Delta_E$, set $\rho \circ \tau := \rho \cdot \tau$. Then $\mathcal{R}$ becomes a monoid. Set $\mathcal{R} := \rho^{-1}$, this defines an involution on $\mathcal{R}$ which reverses the monoid operation. With this involution $\mathcal{R}$ is an involutive monoid. With the inclusion order, that we denote $\leq$, this involutive monoid is an involutive complete ordered monoid.

With these definitions, we have immediately:

**Lemma 5.2.** Let $\mathcal{R}$ be an involutive complete ordered monoid of the set of binary relations on $E$ and let $d$ be the map from $E \times E$ into $\mathcal{R}$ defined by

$$d(x, y) := \bigcap \{\rho \in \mathcal{R} : (x, y) \in \rho\}.$$

Then, the following properties hold:

(i) $d(x, y) \leq 0$ iff $x = y$;
(ii) $d(x, y) \leq d(x, z) \oplus d(z, y)$;
(iii) $d(y, x) = d(x, y)$.

In [33], a set $E$ equipped with a map $d$ from $E \times E$ into an involutive ordered monoid $V$ and which satisfies properties (i), (ii), (iii) stated in Lemma 5.2 is called a $V$-distance, and the pair $(E, d)$ a $V$-metric space. This lemma could justify that we write $d(x, y) \leq \rho$ the fact that a pair $(x, y)$ belongs to a binary relation $\rho$ on the set $E$ and then uses notions borrowed to the theory of metric spaces.

**N.B.** From now on, we suppose that the neutral element of the monoid $V$ is the least element of $V$ for the ordering. In [13] (cf. p.82) the corresponding $V$-metric spaces are called generalized distance space and the maps $d$ are called generalized metric.

If $(E, d)$ is a $V$-metric space and $A$ a subset of $E$, the restriction of $d$ to $A \times A$, denoted by $d_{\upharpoonright A}$ is a $V$-distance and $(A, d_{\upharpoonright A})$ is a restriction of $(E, d)$. As in the case of ordinary metric spaces, if $(E, d)$ and $(E', d')$ are two $V$-metric spaces, a map $f : E \rightarrow E'$ is a non-expansive map (or a contraction) from $(E, d)$ to $(E', d')$ provided that $d'(f(x), f(y)) \leq d(x, y)$ holds for all $x, y \in E$ (and the map $f$ is an isometry if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in E$). The space $(E, d)$ is a retract of $(E', d')$, if there are two non-expansive maps $f : E \rightarrow E'$ and $g : E' \rightarrow E$ such that $g \circ f = id_E$ (where $id_E$ is the identity map on $E$). In this case, $f$ is a coretraction and $g$ a retraction. If $E$ is a subspace of $E'$, then clearly $E$ is a retract of $E'$ if there is a non-expansive map from $E'$ to $E$ such $g(x) = x$ for all $x \in E$. We can easily see that every coretraction is an isometry. We say that $(A, d_{\upharpoonright A})$ is a one-local retract if it is a retract of $(A \cup \{x\}, d_{\upharpoonright A \cup \{x\}})$ (via the identity map) for every $x \in E$.

Let $(E, d)$ be a $V$-metric space; for $x \in E$ and $v \in V$, the set $B(x, v) := \{y \in E : d(x, y) \leq v\}$ is a ball. One can define diameter and radius like in ordinary metric spaces, but for avoiding a problem with the existence of joins and meets, we suppose that $V$ is a complete lattice. The diameter $\delta(A)$ of a subset $A$ of $E$ is $\bigvee \{d(x, y) : x, y \in A\}$, while the radius $r(A)$ is $\bigwedge \{v \in V : A \subseteq B(x, v) \text{ for some } x \in A\}$. A subset $A$ of $E$ is equally centered if $\delta(A) = r(A)$. Following Penot, who defined the notions for ordinary metric spaces, a metric space $(E, d)$ has a compact structure if the collection of balls has the finite intersection property and it has a normal structure if for every intersection of balls $A$, either $\delta(A) = 0$ or $r(A) < \delta(A)$; this condition amounts to the fact that the only equally centered intersections of balls are singletons.
The correspondence between the notions defined for metric spaces and for binary relational systems is given in the lemma below.

**Lemma 5.3.** For \( v \in V \), set \( \delta_v := \{(x,y) : d(x,y) \leq v \} \) and \( E := (E, \{\delta_v : v \in V\}) \). Then \( E \) is reflexive and involutive. Furthermore:

(a) A self map \( f \) on \( E \) is nonexpansive iff this is an endomorphism of \( E \).
(b) \((E, d)\) has a compact structure iff \( E \) has a compact structure.
(c) For every subset \( A \) of \( E \), \( (A, d|_A) \) is a one-local retract of \((E, d)\) iff \( E|_A \) is a one-local retract of \( E \).
(d) For every subset \( A \) of \( E \), \( \delta(A) \) is the least element of the set of \( v \in V \) such that \( A \subseteq \delta_v \); equivalently \( \delta_E(A) = \{\delta_v : \delta(A) \leq v\} \). Also, \( r(A) = \bigwedge\{v \in V : \delta_v \in r_E(A)\} \).
(e) A subset \( A \) of \( E \) is equally centered w.r.t. the space \((E, d)\) iff it is equally centered w.r.t. the binary relational system \( E \).

**Proof.** The first three items are obvious.

Item (d). Let \( r := \delta(A) \). By definition, \( r = \bigvee\{d(x,y) : (x,y) \in A^2\} \). In particular, \( A \subseteq \delta_r \). Let \( v \) such that \( A \subseteq \delta_v \); this means \( d(x,y) \leq v \) for every \((x,y) \in A^2 \), hence \( r \leq v \). This proves that \( \delta(A) = \min\{v \in V : \delta_v \in \delta(E)\} \). The verification of the other assertions is immediate.

Item (e). By Item (d), \( r(A) = \bigwedge\{v \in V : \delta_v \in r_E(A)\} \) and \( \delta(A) = \min\{v \in V : \delta_v \in \delta_E(A)\} \). If \( r_E(A) = \delta_E(A) \), this implies immediately \( r(A) = \delta(A) \). Conversely, suppose that \( r(A) = \delta(A) \). In this case \( A \neq \emptyset \), hence \( \delta(E)(A) \subseteq r_E(A) \). If \( \delta(E)(A) \subseteq r_E(A) \) then since \( \delta(A) = \min\{v \in V : \delta_v \in \delta(E)\} \) and \( r(A) = \bigwedge\{v \in V : \delta_v \in r_E(A)\} \) it follows that \( r(A) < \delta(A) \), a contradiction. \( \square \)

With this lemma in hand, Theorem 3.6 becomes:

**Theorem 5.4.** If a generalized metric space \((E, d)\) has a has a compact and normal structure then every commuting family \( \mathcal{F} \) of non-expansive self maps has a common fixed point. Furthermore, the restriction of \((E, d)\) to the set \( \text{Fix}(\mathcal{F}) \) of common fixed points of \( \mathcal{F} \) is a one-local retract of \((E, d)\).

The fact that a space has a compact structure is an infinitistic property (any finite metric spaces enjoys it). A description of generalized metric spaces with a compact and normal structure eludes us. In the next subsection we describe a large class of generalized metric spaces with a compact and normal structure.

5.3. **Hyperconvexity.** We say that a generalized metric space \((E, d)\) is hyperconvex if for every family of balls \( B(x_i, r_i) \), \( i \in I \), with \( x_i \in E, r_i \in V \), the intersection \( \bigcap_{i \in I} B(x_i, r_i) \) is nonempty provided that \( d(x_i, x_j) \leq r_i \oplus r_j \) for all \( i, j \in I \). This property amounts to the fact that the collection of balls of \((E, d)\) has the 2-Helly property (that is an intersection of balls is nonempty provided that these balls intersect pairwise) and the following convexity property:

\begin{equation}
(8) \quad \text{Any two balls } B(x, r), B(y, s) \text{ intersect if and only if } d(x,y) \leq r \oplus s.
\end{equation}

An element \( v \in V \) is self-dual if \( \overline{v} = v \), it is accessible if there is some \( r \in V \) with \( v \notin r \) and \( v \leq r \oplus \overline{r} \) and inaccessible otherwise. Clearly, 0 is inaccessible; every inaccessible element \( v \) is self-dual (otherwise, \( \overline{v} \) is incomparable to \( v \) and we may choose \( r := \overline{v} \)). We say that a space \((E, d)\) is bounded if 0 is the only inaccessible element below \( \delta(E) \).

**Lemma 5.5.** Let \( A \) be an intersection of balls of \( E \). If \( \delta(A) \) inaccessible then \( A \) is equally centered; the converse holds if \((E, d)\) is hyperconvex.
Proof. Suppose that \( v := \delta(A) \) is inaccessible. According to (d) of Lemma 5.3, \( r(A) = \bigwedge r_E(A) \). Let \( r \in r_E(A) \). Then there is some \( x \in A \) such that \( A \subseteq B(x, r) \). This yields \( d(a, b) \leq d(a, x) \oplus d(x, b) \leq \overline{\Phi} \otimes r \) for every \( a, b \in A \). Thus \( v \leq \overline{\Phi} \otimes r \). Since \( v \) is inaccessible, \( v \leq r \), hence \( v \leq r(A) \). Thus \( v = r(A) \). Suppose that \( A \) is equally centered. Let \( r \) such that \( v \leq r \otimes \overline{\Phi} \). The balls \( B(x, r) \ (x \in A) \) pairwise intersect and intersect with each of the balls whose \( A \) is an intersection; since \( (E, d) \) is hyperconvex, these balls have a nonempty intersection. Any member \( a \) of this intersection is in \( A \) and verifies \( A \subseteq B(a, \overline{\Phi}) \), hence \( \overline{\Phi} \in r_E(A) \). Since \( A \) is equally centered \( r(A) = v \). Hence, \( v \leq \overline{\Phi} \). Since \( v \) is self-dual, \( v \leq r \). Thus \( v \) is inaccessible. \( \Box \)

This lemma with the fact that the 2-Helly property implies that the collection of balls has the finite intersection property yields:

**Corollary 5.6.** If a generalized metric space \( (E, d) \) is bounded and hyperconvex then it has a compact and normal structure.

From Theorem 5.4 we obtain:

**Theorem 5.7.** If a generalized metric space \( (E, d) \) is bounded and hyperconvex then every commuting family of non-expansive self maps has a common fixed point.

Hyperconvex spaces have a simple characterization provided that the set \( V \) of values of the distances satisfies the following distributivity condition:

\[
\bigwedge_{\alpha \in A, \beta \in B} u_\alpha \cdot v_\beta = \bigwedge_{\alpha \in A} \bigwedge_{\beta \in B} u_\alpha \cdot v_\beta
\]

for all \( u_\alpha \in V \ (\alpha \in A) \) and \( v_\beta \in V \ (\beta \in B) \).

In this case, we say that \( V \) is an involutive Heyting algebra or, better, an involutive op-quantale (see [30] about quantales).

On an involutive Heyting algebra \( V \), we may define a \( V \)-distance. This fact relies on the classical notion of residuation. Let \( v \in V \). Given \( \beta \in V \), the sets \( \{ r \in V : v \leq r \oplus \beta \} \) and \( \{ r \in V : v \leq \beta \oplus r \} \) have least elements, that we denote respectively by \( \lceil v \oplus -\beta \rceil \) and \( \lceil -\beta \oplus v \rceil \) and call the right and left quotient of \( v \) by \( \beta \) (note that \( \lceil -\beta \oplus v \rceil = \lceil v \oplus -\beta \rceil \)). It follows that for all \( p, q \in V \), the set

\[
D(p, q) := \{ r \in V : p \leq q \oplus r \quad \text{and} \quad q \leq p \oplus r \}
\]

has a least element. This last element is \( \lceil p \oplus -q \rceil \lor \lceil -p \oplus q \rceil \), we denote it by \( d_V(p, q) \).

As shown in [17], the map \( (p, q) \mapsto d_V(p, q) \) is a \( V \)-distance.

Let \( (E_i, d_i) \) be a family of \( V \)-metric spaces. The direct product \( \prod_{i \in I} (E_i, d_i) \), is the metric space \( (E, d) \) where \( E \) is the cartesian product \( \prod_{i \in I} E_i \) and \( d \) is the "sup" (or \( \ell^\infty \)) distance defined by \( d((x_i)_{i \in I}, (y_i)_{i \in I}) = \bigvee_{i \in I} d_i(x_i, y_i) \). We recall the following result of [17].

**Theorem 5.8.** \((V,d_V)\) is a hyperconvex \( V \)-metric space and every \( V \)-metric space embeds isometrically into a power of \((V,d_V)\).

This result is due to the fact that for every \( V \)-metric space \((E,d)\) and for all \( x,y \in E \) the following equality holds:

\[
d(x,y) = \bigvee_{z \in E} d_V(d(z,x),d(z,y)).
\]

A generalized metric space is an absolute retract if it is a retract of every isometric extension. The space \( E \) is injective if for all \( V \)-metric space \( E' \) and \( E'' \), each non-expansive map
f : E' → E and every isometry g : E' → E'' there is a non-expansive map h : E'' → E such that h ◦ g = f.

With this result follows the characterization given in [17].

**Theorem 5.9.** For metric spaces over an involutive Heyting algebra V, the notions of absolute retract, injective, hyperconvex and retract of a power of (V, dV) coincide.

Note that if v is accessible in V and V is an involutive Heyting algebra, then v is accessible in the initial segment \( \downarrow v \) of V (indeed, if \( v \leq r \oplus s \) then since by distributivity \( (r \wedge v) \oplus (\overline{r} \wedge v) = (r \oplus \overline{r}) \wedge (r \oplus v) \wedge (\overline{r} \oplus v) \) we have \( v \leq (r \wedge v) \oplus (\overline{r} \wedge v) \).

**5.4. One-local retracts and hole-preserving maps.** Let E and E' be two V-metric spaces. If f is a non-expansive map from E into E', and h is a map from E into V, the image of h is the map \( h_f \) from E' into V defined by \( h_f(y) : = \{ h(x) : f(x) = y \} \) (in particular \( h_f(y) = 1 \) where 1 is the largest element of V for every \( y \) not in the range of f. A hole of E is any map \( h : E \rightarrow V \) such that the intersection of balls \( B(x, h(x)) \) of E (\( x \in E \)) is empty. If h is a hole of E, the map \( f \) preserves h provided that \( h_f \) is a hole of E'. The map \( f \) is hole-preserving if the image of every hole is a hole.

Let \( B := \{ B(x_i, r_i) \}_{i \in I} \) be a family of balls of E. For every \( x \in E \), set \( V_B(x) = \{ r \in V : B(x_i, r_i) \subseteq B(x, r) \text{ for some } i \in I \} \) and \( h_B(x) := \bigwedge V_B(x) \). We have:

\[
\bigcap B = \bigcap_{x \in E} B(x, h_B(x)).
\]

**Proof.** Let \( z \in \bigcap \mathcal{B} \) and \( x \in E \). We claim that \( z \in B(x, h_B(x)) \). This amounts to \( d(x, z) \leq h_B(x) \); due to the definition of \( h_B(x) \), this amounts to \( d(x, z) \leq r \) for every \( r \in V \) such that \( B(x_i, r_i) \subseteq B(x, r) \) for some \( i \in I \). Let \( r \) and \( i \in I \) such that this property holds. Since \( z \in \bigcap \mathcal{B}, z \in B(x_i, r_i) \) and since \( B(x_i, r_i) \subseteq B(x, r) \), \( z \in B(x, r) \), that is \( d(x, z) \leq r \), as required. For the converse, let \( z \in \bigcap_{x \in E} B(x, h_B(x)) \) and \( i \in I \). By definition of \( h_B \), \( h_B(x_i) \leq r_i \); since \( z \in B(x_i, h(x_i)) \), \( z \in B(x_i, r_i) \); since this property holds for every \( i \in I \), \( z \in \bigcap \mathcal{B} \).

A hole h of E is finite if \( \bigcap_{x \in F} B(x, h(x)) = \emptyset \) for some finite subset F of E, otherwise it is infinite.

A poset is well-founded if every nonempty subset contains some minimal element. We recall that if a lattice is well-founded, every element \( x \) which is the infimum of some subset \( X \) is the infimum of some finite subset. In general, the order on the Heyting algebra V is not well-founded, still there are interesting examples (see Subsection 5.6 and 5.7).

The following lemma relates holes and compactness of the collection of balls (it contains a correction of Proposition II-4.9. of [17]):

**Lemma 5.10.** If a generalized space E has a compact structure then every hole is finite; the converse holds if V is well-founded.

**Proof.** Let h be a hole. Then, by definition, \( \bigcap_{x \in E} B(x, h(x)) = \emptyset \). Since E has a compact structure, \( \bigcap_{x \in E} B(x, h(x)) = \emptyset \) for some finite subset, hence h is finite. Conversely, let \( \mathcal{B} := \{ B(x_i, r_i) \}_{i \in I} \) be a family of balls of E such that \( \bigcap \mathcal{B} = \emptyset \). There are two ways of associating a finite hole to \( \mathcal{B} \). We may define \( h : E \rightarrow V \) be setting \( h(x) := \bigwedge \{ r_i : x_i = x \} \). We may also associate \( h_B \). By Formula (12), this is a hole. These hole are finite. We conclude by using \( h_B \).

Let F be some finite subset of E such that \( \bigcap_{x \in F} B(x, h_B(x)) = \emptyset \). Since V is well-founded, for each \( x \in E \), there is some finite subset \( V_x \) of \( V_B(x) = \{ r \in V : B(x_i, r_i) \subseteq B(x, r) \text{ for some } i \in I \} \) such that \( \bigwedge V_B(x) = \bigwedge V_x \). For each \( x \in F \), there is a finite subset \( I_x \) such that for each \( r \in V_x \),
Lemma 5.11. A non-expansive map \( f \) from a \( V \)-metric space \( E \) into a \( V' \)-metric space \( E' \) is hole-preserving iff \( f \) is an isometry of \( E \) onto its image and this image is a 1-local retract of \( E' \).

Proof. Let \( d \) and \( d' \) be the distances of \( E \) and \( E' \). Suppose that \( f \) is hole-preserving. We prove first that \( f \) is an isometry. Let \( a, b \in E \), \( a' := f(a), b' := f(b) \), \( r := d(a, b) \) and \( r' := d'(a', b') \). Our aim is to prove that \( r' = r \). Let \( h : E \to V \) defined by setting \( h(z) := 1 \) (where 1 is the largest element of \( V \)) if \( z \notin \{a, b\} \), \( h(z) = r' \) if \( z = a \) and \( h(z) := 0 \) (where 0 is the least element of \( V \)) if \( z = b \). The intersection of balls \( B(z', h_f(z')) \) of \( E' \) contains \( b' \), hence the map \( h_f \) is not a hole of \( E' \). Since \( f \) is hole-preserving, \( h \) is not a hole of \( E \). The intersection of balls \( B(z, h(z)) \) of \( E \) being included into \( \{b\} \) it is equal to \( \{b\} \). Hence, \( b \in B(a, r') \). It follows that \( r' = r \). Next, we prove that the range \( A' \) of \( f \) is a 1-local retract of \( E' \). We apply Lemma 3.1. Let \( B(x_i, r_i), i \in I \), where \( x_i \in A', r_i \in V \), be a family of balls of \( E' \) such that the intersection over \( E' \) is nonempty. Let \( a' \in \) this intersection and let \( h : E \to V \) defined by setting \( h(z) := d'(f(z), a') \). The intersection of balls \( B(z', h_f(z')) \) of \( E' \) contains \( a' \) hence \( h_f \) is not a hole of \( E' \). Since \( f \) is hole-preserving, \( h \) is not a hole. Hence, there is some \( a \) in the intersection of balls \( B(z, h(z)) \) of \( E \). Let \( b' := f(a) \). Then \( b' \) belongs to \( \bigcap_{i \in I} B(x_i, r_i) \). The conclusion that \( A' \) is a one-local retract follows from Lemma 3.1. The converse is immediate. Let \( h \) be a hole of \( E \) and \( h_f \) be its image. If \( h_f \) is not a hole of \( E' \), the intersection of balls \( B(z', h_f(z')) \) of \( E' \) contains some element \( a' \). Since \( A' \) is a one-local retract of \( E' \) there is a retraction fixing \( A' \) and sending \( a' \) onto some element \( b \in A' \). Let \( a \in E \) such that \( f(a) = b \). Then \( a \in \bigcap z \in EB(z, h(z)) \). Indeed, let \( z \in E \); we claim that \( d(z, a) \leq h(z) \) (indeed, since \( f \) is an isometry, \( d(z, a) = d'(f(z), b) \) and, via the retraction, \( d'(f(z), b) \leq d'(f(z), a) \leq h_f(f(z) \leq h(z)) \)). Hence \( h \) is not a hole of \( E \). Contradiction. \( \square \)

Replacing isometries by hole-preserving maps in the definition of absolute retracts and injectives, we have the notions of absolute retracts and injectives w.r.t. holes preserving maps.

We recall the following result of \cite{17}.

Theorem 5.12. On an involutive Heyting algebra \( V \), the absolute retracts and the injective w.r.t. hole-preserving maps coincide. The class \( \mathcal{H} \) of these objects is closed under product and retraction. Moreover, every metric space embeds into some member of \( \mathcal{H} \) by some hole-preserving map.

The proof relies on the introduction of the replete space \( H(E) \) of a metric space \( E \). The space \( E \) is a absolute retract or not depending wether \( E \) is a retract of \( H(E) \) or not. Furthermore, it allows to prove the transferability of holes preserving maps, that is the fact that for every non-expansive map \( f : E \to F \), hole-preserving map \( g : E \to G \) there are hole-preserving map \( g' F \to E' \) and non-expansive map \( f' : G \to E' \) such that \( g' \circ f = f' \circ g \). Indeed, one may choose \( E' = H(E) \).

We give the key ingredients.

Let \( E \) be a \( V \)-metric space. A weak metric form on \( E \) is any map \( h : E \to V \) such that \( d(x, y) \leq h(x) \oplus h(y) \) for all \( x, y \in E \). If in addition, \( h(x) \leq d(x, y) \oplus h(y) \) for all \( x, y \in E \), this is a metric form. We denote by \( C(E) \), resp. \( L(E) \), the set of weak metric form, resp. of metric forms.
Let $H(E)$ the subset of $L(E)$ consisting of metric forms $h$ such that the intersection of balls $B(x,h(x))$ for $x \in E$ is nonempty. If $V$ is an involutive Heyting algebra, we may equip $H(E)$ of the distance induced by the sup-distance on $(V,d_V)^E$. We call it the replete space.

We recall the following result of [17].

**Lemma 5.13.** If $V$ is an involutive Heyting algebra then $\overline{\delta}: E \to H(E)$ defined by $\overline{\delta}(x)(y) := d(y,x)$ is a hole-preserving map from $E$ into $H(E)$. Furthermore $H(E)$ is an absolute retract w.r.t. the hole-preserving maps (i.e. this is a retract of every extension by a hole-preserving map).

**Problems 1.** Let $E$ be a generalized metric space with a compact and normal structure.

(a) When a one-local retract of $E$ is a retract?

(b) When the set $\text{Fix}(f)$ of fixed point of a non-expansive self map a retract?

Note that if (a) has a positive answer then spaces with a compact and normal structure are absolute retracts w.r.t. hole-preserving maps. For these problems, it could be fruitful to consider the case of posets; there is a vast literature on fix-point and this type of questions (see [38, 4, 29]).

5.5. **The case of ordinary metric spaces.** Let $\mathbb{R}^+$ be the set of non negative reals with the addition and natural order, the involution being the identity. Let $V := \mathbb{R}^+ \cup \{+\infty\}$. Extend to $V$ the addition and order in a natural way. Then, metric spaces over $V$ are direct sums of ordinary metric spaces (the distance between elements in different components being $+\infty$). The set $V$ is an involutive Heyting algebra, the distance $d_V$ once restricted to $\mathbb{R}^+$ is the absolute value. The inaccessible elements are 0 and $+\infty$ since, if one deals with ordinary metric spaces, unbounded spaces in the above sense are those which are unbounded in the ordinary sense. If one deals with ordinary metric spaces, infinite products can yield spaces for which $+\infty$ is attained. On may replace powers with $\ell^\infty$-spaces (if $I$ is any set, $\ell^\infty_\mathbb{R}(I)$ is the set of families $(x_i)_{i \in I}$ of reals numbers, endowed with the sup-distance). With that, the notions of absolute retract, injective, hyperconvex and retract of some $\ell^\infty_\mathbb{R}(I)$ space coincide.

According to Corollary 5.6 a hyperconvex metric space has a normal structure iff its diameter is bounded. In fact, if a subset $A$ of a hyperconvex space is an intersection of balls, its radius is half the diameter. No description of metric spaces with a compact and normal structure seems to be known.

The existence of a fixed point for a non-expansive map on a bounded hyperconvex space is the famous result of Sine and Soardi. Theorem 3.6 applied to a bounded hyperconvex metric space is Baillon’s fixed point theorem. Applied to a metric space with a compact and normal structure, this is the result obtained by the first author [23].

5.6. **The case of ordered sets.** In this subsection, we consider posets as binary relational systems as well as metric spaces over an involutive Heyting algebra.

Let $P := (E, \leq)$ be an ordered set. Let $\mathcal{E} := \{\leq, \leq^{-1}\}$ and $E := (E, \mathcal{E})$. By definition, $E$ is reflexive and involutive. For $x \in E$, set $\uparrow x := \{y \in E : x \leq y\}$ and $\downarrow x := \{y \in E : y \leq x\}$; this sets are called the principal final, resp. initial, segment generated by $x$. With our terminology of balls of $E$, these sets are the balls $B(x,\leq)$ and $B(x,\leq^{-1})$.

Let $V$ be the following structure. The domain is the set $\{0,+,\,-,1\}$. The order is $0 \leq a, b \leq 1$ with $+$ incomparable to $-$; the involution exchange $+$ and $-$ and fixes 0 and 1; the operation $\oplus$ is defined by $p \oplus q := p \lor q$ for every $p, q \in V$. As it is easy to check, $V$ is an involutive Heyting algebra.

If $(E,d)$ is a $V$-metric space, then $P_d := (E,\delta_d)$, where $\delta_d := \{(x,y) : d(x,y) \leq +\}$, is an ordered set. Conversely, if $P := (E,\leq)$ be an ordered set, then the map $d : E \times E \to \mathbb{V}$
defined by \( d(x, y) := 0 \) if \( x = y, \) \( d(x, y) := + \) if \( a < b, \) \( d(x, y) := - \) if \( y < x \) and \( d(x, y) := 1 \) if \( x \) and \( y \) are incomparable. Clearly, \( (E, d) \) and \( (E', d') \) are two \( V \)-metric spaces, a map \( f : E \to E' \) is non-expansive from \( (E, d) \) into \( (E', d') \) iff it is order-preserving from \( P_d \) into \( P_{d'} \). Depending on the value of \( v \in V \), a \( V \)-metric space has four types of balls: singletons, corresponding to \( v = 0 \), the full space, corresponding to \( v = 1 \), the principal final segments, \( \uparrow x := \{ y \in E : x \leq y \} \), corresponding to balls \( B(x, +) \), and principal initial segments, \( \downarrow x := \{ y \in E : y \leq x \} \), corresponding to balls \( B(x, -) \). The set \( V \) can be equipped with the distance \( d_V \) given by means of the formula (10). The corresponding poset is the four element lattice \( \{ -, 0, 1, + \} \) with \( - < 0, 1 < + \). The retracts of powers of this lattice are all complete lattices. This is confirmed by the following fact.

**Proposition 5.14.** A metric space \((E, d)\) over \( V \) is hyperconvex iff the corresponding poset is a complete lattice.

**Proof.** Suppose that \((E, d)\) is hyperconvex. Let \( \leq := \delta_+ \) and \( P_d := (E, \delta_+) \). We prove that every subset \( A \) has a supremum in \( P_d \). This amounts to prove that \( A^+ := \{ y \in E : x \leq y \} \) for all \( x \in A \) has a least element. Since \((E, d)\) satisfies the convexity property, and \(+\sqrt{=1}, B(x', +) \cap B(x'', +) \neq \varnothing \) for every \( x', x'' \in E \); since \((E, d)\) satisfies the 2-Helly property, \( A^+ = \bigcap_{x \in A} B(x, +) \neq \varnothing \). Applying again the convexity and 2-Helly property, we get that the intersection of balls \( B(x, +) \) for \( x \in A \) and \( B(y, -) \), for \( b \in A^+ \) is nonempty. This intersection contains just one element, this is the supremum of \( A \). A similar argument yields the existence of the infimum of \( A \), hence \( P_d \) is a complete lattice. Conversely, let \( B(x_i, r_i), (i \in I) \), be a family of balls such that \( (x_i, x_j) \leq r_i \vee r_j \). We prove that \( C := \bigcap_{i \in I} B(x_i, r_i) \neq \varnothing \). If there is some \( i \in I \) such that \( r_i = 0 \), then \( x_i \in C \). If not, let \( A := \{ i \in I : r_i = + \}, B := \{ j \in I : r_j = - \}. \) Then \( x_i \leq x_j \) for all \( x_i \in A, x_j \in B \). Set \( c := \bigvee A \) and observe that \( c \in C \). \( \square \)

Since 0 is the only inaccessible element of \( V \), Theorem 5.7 applies: Every commuting family of order-preserving maps on a complete lattice has a common fixed point. This is Tarski’s theorem (in full).

Posets coming from \( V \)-metric spaces with a compact and normal structure are a bit more general than complete lattice, hence Theorem 3.6 on compact normal structure could say a bit more than Tarski’s fixed point theorem. In fact, for one order-preserving map, this is no more as Abian-Brown’s fixed-point theorem.

Indeed, let us recall that a poset \( P \) is chain-complete if every nonempty chain in \( P \) has a supremum and an infimum.

We prove below that:

**Proposition 5.15.** If the collection of intersection of balls of a poset \( P := (E, \leq) \) satisfies the f.i.p., that is \( \mathcal{B}_E \) is compact, then \( P \) is chain complete (converse false).

Abian-Brown’s theorem 11 asserts that in a chain-complete poset with a least or largest element, every order-preserving map has a fixed point.

The fact that the collection of intersection of balls of \( P \) has a normal structure means that every nonempty intersection of balls of \( P \) has either a least or largest element. Being the intersection of the empty family of balls, \( P \) has either a least element or a largest element.

Consequently, if \( P \) has a compact and normal structure, we may suppose without loss of generality that it has a least element. Since every nonempty chain have a supremum, it follows from Abian-Brown’s theorem that every order preserving map has a fixed point.

On another hand, a description of posets with a compact and normal structure has yet to come.

The proposition above follows from properties of gaps we assemble below.
A pair of subsets \((A, B)\) of \(E\) is called a gap of \(P\) if every element of \(A\) is dominated by every element of \(B\) but there is no element of \(E\) which dominates every element of \(A\) and is dominated by every element of \(B\) (cf. [15]). In other words: \((\cap_{x \in A} B(x, \leq)) \cap (\cap_{y \in B} B(y, \geq)) = \emptyset\) while \(B(x, \leq) \cap B(y, \geq) \neq \emptyset\) for every \(x \in A, y \in B\). A subgap of \((A, B)\) is any pair \((A', B')\) with \(A' \subseteq A, B' \subseteq B\), which is a gap. The gap \((A, B)\) is finite if \(A\) and \(B\) are finite, otherwise it is infinite. Say that an ordered set \(Q\) preserves a gap \((A, B)\) of \(P\) if there is an order-preserving map \(g\) of \(P\) to \(Q\) such that \((g(A), g(B))\) is a gap of \(Q\). On the preservation of gaps, see [29].

**Lemma 5.16.** Let \(P := (E, \mathcal{E})\) be a poset. Then:

(a) \(P\) is a complete lattice iff \(P\) contains no gap;

(b) An order-preserving map \(f : P \to Q\) is an embedding preserving all gaps of \(P\) iff it preserves all holes of \(P\) with values in \(V \setminus \{0\}\) iff \(f(P)\) is a one-local retract of \(Q\);

(c) \(\mathcal{B}_E\) satisfies the f.i.p. iff every gap of \(P\) contains a finite subgap iff every hole is finite.

**Proof.** (a). Let \((A, B)\) be a pair of subsets of \(E\) such that every element of \(A\) is dominated by every element of \(B\). Let \(A^\Delta := \{y \in E : x \leq y\ \text{for all} \ x \in A\}\). Then, trivially, \(A \subseteq A^\Delta\) and every element of \(A^\cap\) dominates every element of \(A\); furthermore \((A, A^\Delta)\) is not a gap iff \(A\) has a supremum. Thus, if \(P\) is a complete lattice, \(A\) has a supremum, hence \((A, A^\Delta)\) is not a gap and hence \((A, B)\) is not a gap. Conversely, if \(P\) contains no gap, \((A, A^\Delta)\) is not a gap and thus \(A\) has a supremum. It follows that \(P\) is a complete lattice.

(b). Suppose that \(f\) is an embedding preserving all gaps. Let \(h\) be a hole of \(P\) with values in \(V \setminus \{0\}\) and \(h_f\) be its image. Let \(A := \{x \in P : h(x) = +\}\) and \(B := \{y \in P : h(y) = -\}\). If there is some \(a \in A, b \in B\) such that \(a \not\leq b\) then, since \(f\) is an embedding, \(f(a) \not\leq f(b)\) and \(h_f\) is a hole of \(Q\). Otherwise, \(B \subseteq A^\Delta\). Since \(h\) is a hole in \(P\), \((A, B)\) is a gap of \(P\). Since \(f\) preserves all gaps of \(P\), \((f(A), f(B))\) is a gap of \(Q\). It turns out that \(h_f\) is a gap of \(Q\). For the converse, let \((A, B)\) be a gap of \(P\). We claim that \((f(A), f(B))\) is a gap of \(Q\). Since \(f\) is order preserving \((f(A) \subseteq f(B))^\Delta\). We only need to check that there is no element between \(f(A)\) and \(f(B)\). Let \(h : P \to V \setminus \{0\}\) defined by setting \(h(a) := +\) if \(a \in A, h(b) := -\) if \(b \in B\) and \(h(c) = 1\) if \(c \in E \setminus A \cup B\). Then, clearly, \(h\) is a hole of \(P\); since \(f\) preserves it, \(h_f\) is a hole of \(Q\). Hence \(\emptyset = \bigcap_{y \in E} B(y, h_f(y)) = \bigcap_{x \in A \cup B} f(B(x, h_f(x)))\). If follows that \((f(A), f(B))\) is a hole of \(Q\). The equivalence with the last assertion is essentially Lemma 5.11

(c). Suppose that \(\mathcal{B}_E\) satisfies the f.i.p. Let \((A, B)\) be a gap. If every finite pair \((A', B')\) with \(A' \subseteq A\) and \(B' \subseteq B\) is not a gap, then the finite intersections of \(\uparrow a \cap \downarrow b\) with \(a \in A, b \in B\) are nonempty. From the f.i.p. property, the whole intersection \(\bigcap_{a \in A \cap B} \uparrow a \cap \downarrow b\) is nonempty, contradicting the fact that \((A, B)\) is a gap. Conversely, let \(\mathcal{F}\) be a family of members of \(\mathcal{B}_E\) whose finite intersections are nonempty. Each member of \(\mathcal{F}\) being an intersection of balls, each of the form \(B(x, \leq)\) or \(B(y, \geq)\), we may in fact suppose that these members are of the form \(B(x, \leq)\) or \(B(y, \geq)\). Hence, we may suppose that there are two sets \(A\) and \(B\) such that \(\mathcal{F} := \{B(x, \leq) : x \in A\} \cup \{B(y, \geq) : y \in B\}\). Since \((A, B)\) contains no finite gap, the pair \((A, B)\) is not a gap, hence \(\bigcap \mathcal{F} \neq \emptyset\). The equivalence with the last assertion is Lemma 5.10

We only mention some examples.

Let \(\forall\) be the 3-element poset consisting of \(0, +, -\) with \(0 < +, -\) and \(+\) incomparable to \(-\). We denote by \(\wedge\) its dual. Then the reader will observe that retracts of powers of \(\forall\) have a compact and normal structure.

Theorem 5.10 above yields a fixed point theorem for a commuting family of order-preserving maps on any retract of power of \(\forall\) or of power of \(\wedge\). But this result says nothing about retract of products of \(\forall\) and \(\wedge\).
These two posets fit in the category of fences. A fence is a poset whose the comparability graph is a path. For example, a two-element chain is a fence. Each larger fence has two orientations, for example on the three vertices path, these orientations yield the $\lor$ and the $\land$.

From Theorem 5.25 proved in Subsection 5.3 it will follow:

**Theorem 5.17.** If a poset $Q$ is a retract of a product $P$ of finite fences of bounded length, every commuting set of order-preserving maps on $Q$ has a fixed point.

Since every complete lattice is a retract of the two-element chain, this result contains Tarski’s fixed point theorem.

### 5.7. The case of oriented graphs.

A directed graph $G$ is a pair $(E, \mathcal{E})$ where $\mathcal{E}$ is a binary relation on $E$. We say that $G$ is reflexive if $\mathcal{E}$ is reflexive and that $G$ is oriented if $\mathcal{E}$ is antisymmetric (that is $(x, y)$ and $(y, x)$ cannot be in $\mathcal{E}$ simultaneously except if $x = y$). If $\mathcal{E}$ is symmetric, we identify it with a subset of pairs of $E$ and we say that the graph is undirected.

If $G := (E, \mathcal{E})$ and $G' := (E', \mathcal{E}')$ are two directed graphs, an homomorphism from $G$ to $G'$ is a map $h : E \to E'$ such that $(h(x), h(y)) \in \mathcal{E}'$ whenever $(x, y) \in \mathcal{E}$ for every $(x, y) \in E \times E$.

Let us recall that a finite path is an undirected graph $L := (E, \mathcal{E})$ such that one can enumerate the vertices into a non-repeating sequence $v_0, \ldots, v_n$ such that edges are the pairs \{$(v_i, v_{i+1})$\} for $i < n$. A reflexive zigzag is a reflexive graph such that the symmetric hull is a path. If $L$ is a reflexive oriented zigzag, we may enumerate the vertices in a non-repeating sequence $v_0, \ldots, v_n$ such that $v_i$ is an edge and $\alpha_i := -1$ if $(v_{i+1}, v_i)$ is an edge.

We call such a sequence a word over the alphabet $\Lambda := \{+, -\}$. If the path has just one vertex, the corresponding word is the empty word, that we denote by $\Box$. Conversely, to a finite word $u := \alpha_0, \ldots, \alpha_{n-1}$ over $\Lambda$ we may associate the reflexive oriented zigzag $L_u := \{(0, \ldots, n), \mathcal{E}_u\}$ with end-points 0 and $n$ (where $n$ is the length $\ell(u)$ of $u$) such that $\mathcal{E}_u = \{(i, i+1) : \alpha_i = +\} \cup \{(i+1, i) : \alpha_i = -\} \cup \Delta_{\{0, \ldots, n\}}$.

### 5.8. The zigzag distance.

Let $G := (E, \mathcal{E})$ be a reflexive directed graph. For each pair $(x, y) \in E \times E$, the zigzag distance from $x$ to $y$ is the set $d_G(x, y)$ of words $u$ such that there is a non-expansive map $h$ from $L_u$ into $G$ which send 0 on $x$ and $\ell(u)$ on $y$.

This notion is due to Quilliot [34, 35] (Quilliot considered reflexive directed graphs, not necessarily oriented, and in defining the distance, considered only oriented paths). A general study is presented in [17]; some developments appear in [37] and [21].

Because of the reflexivity of $G$, every word obtained from a word belonging to $d_G(x, y)$ by inserting letters will be also into $d_G(x, y)$. This leads to the following framework.

Let $\Lambda^*$ be collection of words over the alphabet $\Lambda := \{+, -\}$. Extend the involution on $\Lambda$ to $\Lambda^*$ by setting $\overline{u} := \Box$ and $\overline{\alpha_0 \ldots \alpha_{n-1}} := \overline{\alpha_0} \ldots \overline{\alpha_{n-1}} \overline{\alpha_0}$ for every word in $\Lambda^*$. Order $\Lambda^*$ by the subword ordering, denoted by $\leq$. If $u := \alpha_0 \alpha_2 \ldots \alpha_m, v := \beta_1 \beta_2 \ldots \beta_n \in \Lambda^*$ set

$$u \leq v \text{ if and only if } \alpha_j = \beta_j \text{ for all } j = 1, \ldots, m \text{ with some } 1 \leq j_1 < \ldots < j_m \leq n.$$  

Let $F(\Lambda^*)$ be the set of final segments of $\Lambda^*$, that is subsets $F$ of $\Lambda^*$ such that $u \in F$ and $u \leq v$ imply $v \in F$. Setting $\overline{X} := \{\overline{u} : u \in X\}$ for a set $X$ of words, we observe that $\overline{X}$ belongs to $F(\Lambda^*)$. Order $\overline{F}(\Lambda^*)$ by reverse of the inclusion, denote by $0$ its least element (that is $\Lambda^*$), set $X \oplus Y$ for the concatenation $X \cdot Y := \{uv : u \in X, v \in Y\}$. Then, one immediately see that $H_\Lambda := (\overline{F}(\Lambda^*), \oplus, 0, -\lor)$ is an involutive Heyting algebra. This leads us to consider distances and metric spaces over $H_\Lambda$. There are two simple and crucial facts about the consideration of the zigzag distance (see [17]).
Lemma 5.18. A map from a reflexive directed graph $G$ into an other is a graph-homomorphism iff it is non-expansive.

Lemma 5.19. The distance $d$ of a metric space $(E, d)$ over $\mathcal{H}_\Lambda$ is the zigzag distance of a reflexive directed graph $G := (E, \mathcal{E})$ iff it satisfies the following property for all $x, y, z \in E$, $u, v \in \mathcal{F}(\Lambda^*)$: $u, v \in d(x, y)$ implies $u \in d(x, z)$ and $v \in d(z, y)$ for some $z \in E$. When this condition holds, $(x, y) \in \mathcal{E}$ iff $+ \in d(x, y)$.

Due to this later fact, the various metric spaces mentionned above (injective, absolute retracts, etc.) are graphs equipped with the zigzag distance; in particular, the distance $d_{\mathcal{H}_\Lambda}$ defined on $\mathcal{H}_\Lambda$ is the zigzag distance of some graph. This facts leads to a fairly precise description of absolute retracts in the category of reflexive directed graphs (see [21]). The situation of oriented graphs is different. These graphs cannot be modeled over a Heyting algebra (Theorem IV-3.1 of [17] is erroneous), but the absolute retracts in this category can be ([37]). The appropriate Heyting algebra is the MacNeille completion of $\Lambda^*$.

The MacNeille completion is in some sense the least complete lattice extending $\Lambda^*$.

We recall the important fact that sets of the form $W^\vee$ for $W$ nonempty coincide with nonempty finitely generated initial segments of $\Lambda^*$ (Jullien [18]). Hence:

Lemma 5.20. The set $\mathcal{N}(\Lambda^*) \setminus \{\emptyset\}$ is order isomorphic to the set $\mathcal{I}_{\omega}(\Lambda^*) \setminus \{\emptyset\}$ ordered by inclusion and made of finitely generated initial segments of $\Lambda^*$. In particular, $\mathcal{N}(\Lambda^*) \setminus \{\emptyset\}$ is a distributive lattice.

The concatenation, order and involution defined on $\mathcal{F}(\Lambda^*)$ induce an involutive Heyting algebra $\mathcal{N}_\Lambda$ on $\mathcal{N}(\Lambda^*)$ (see Proposition 2.2 of [6]). Being an involutive Heyting algebra, $\mathcal{N}_\Lambda$ supports a distance $d_{\mathcal{N}_\Lambda}$ and this distance is the zigzag distance of a graph $G_{\mathcal{N}_\Lambda}$. But it is not true that every oriented graph embeds isometrically into a power of that graph. For example, an oriented cycle cannot. The following result characterizes graphs which can be isometrically embedded, via the zigzag distance, into products of reflexive and oriented zigzags. It is stated in part in Subsection IV-4 of [17], cf. Proposition IV-4.1.

Theorem 5.21. For a directed graph $G := (E, \mathcal{E})$ equipped with the zigzag distance, the following properties are equivalent:

(i) $G$ is isometrically embeddable into a product of reflexive and oriented zigzags;
(ii) $G$ is isometrically embeddable into a power of $G_{\mathcal{N}_\Lambda}$;
(iii) The values of the zigzag distance between vertices of $E$ belong to $\mathcal{N}_\Lambda$. 

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii). The proof relies on the following:

Claim 5.22. Every finite reflexive oriented zigzag is isometrically embeddable into $G_{N_{\Lambda}}$.

Proof of Claim 5.22. Let $L$ be a finite reflexive oriented zigzag. Let $n$ be its number of vertices. There is a word $u := \alpha_{0}\cdots\alpha_{i-1} \in \Lambda^*$ such that $L$ is isomorphic to $L_u := \{(0,\ldots,n), L_u\}$. Let $\varphi : \{0,\ldots,n\} \to N(\Lambda^\ast)$ be the map defined by $\varphi(i) := u_{\leq i}$ where $u_{\leq i} := \emptyset$ if $i = 0$ and $u_{\leq i} := \alpha_0\cdots\alpha_{i-1}$ otherwise. We claim that $\varphi$ is an isometry from $L$ equipped with the zigzag distance into $(N_{\Lambda},d_{N_{\Lambda}})$, that is $d_L(i,j) = d_{N_{\Lambda}}(\varphi(i),\varphi(j))$ for all $i,j \leq n$. It suffices to check that this equality holds for $i < j$. In this case, $d_L(i,j) = \uparrow \alpha_i \cdots \alpha_{j-1}$. In $(N_{\Lambda},d_{N_{\Lambda}})$, we have:

$$d_{N_{\Lambda}}(v,v \oplus w) = w,$$

for all $v \in N_{\Lambda}, w \in N_{\Lambda} \setminus \{\emptyset\}$.

Indeed, due to the definition of the distance in $N_{\Lambda}$, we have $u + d_{N_{\Lambda}}(v,v \oplus w) = u + w$. As a monoid, $N_{\Lambda} \setminus \{\emptyset\}$ is cancellative (see Lemma 11 of [22]). Hence $d_{N_{\Lambda}}(v,v \oplus w) = w$. Thus, $d_{N_{\Lambda}}(\varphi(i),\varphi(j)) = d_{N_{\Lambda}}(\varphi(i),\varphi(i) \uparrow \alpha_i \cdots \alpha_{j-1}) = \uparrow \alpha_i \cdots \alpha_{j-1} = d_L(i,j)$, as required. Since $(N_{\Lambda},d_{N_{\Lambda}})$ is hyperconvex, the distance $d_{N_{\Lambda}}$ is the zigzag distance associated to the oriented graph $G_{N_{\Lambda}}$, hence the isometric embedding $\varphi$ induces a graph embedding. □

With Claim 5.22 we may embed isometrically any product of zigzags into a power of $G_{N_{\Lambda}}$. This proves that (ii) holds.

(ii) ⇒ (iii). If $G'$ is a product of graphs $G'_i$, the zigzag distance on $G'$ is the sup-distance on the product of the metric spaces $(G_i,d_{G_i})$. Thus, if $G$ isometrically embeds into a power of $G_{N_{\Lambda}}$, $(G,d_G)$ isometrically embeds into a power of $(N_{\Lambda},d_{N_{\Lambda}})$. Since the distance $d_{N_{\Lambda}}$ has values in $N_{\Lambda}$, $d_G$ has values in $N_{\Lambda}$ too hence (iii) holds.

(iii) ⇒ (i). The proof follows the same lines as the proof of Proposition IV.4.1 p.212 of [17].

We use the following property:

Claim 5.23. For each pair of vertices $x,y \in E$ and each word $u \in (d_G(x,y))^\vee$, let $L_u$ be a reflexive oriented path with end points 0 and $\ell(u)$ associated with $u$. The map carrying $x$ onto 0 and $y$ onto $\ell(u)$ extends to a non-expansive mapping $f_{x,y,u}$ from $G$ onto $L_u$.

Proof of Claim 5.23. The proof of the claim relies on two facts. First, $d_{L_u}(0,\ell(u)) = \uparrow u$. Since $u \in d_G(x,y)$, $u \leq d_G(x,y)$, hence the partial map carrying $x$ onto 0 and $y$ onto $\ell(u)$ is a non-expansive map from the subset $\{x,y\}$ of $G$ equipped with the zigzag distance $d_G$ into the space associated to the zigzag $L_u$. Next, such a partial map extends to $G$ to a non-expansive mapping. This is due to the fact that the space associated to $L_u$ is hyperconvex (it is trivially convex and since each ball in that space is an interval of its domain $\{0,\ldots,\ell(u)\}$, any collection of balls has the 2-Helly-property). For the fact that non-expansive maps with values into an hyperconvex space extend, see [17]. □

Let

$$G' := \Pi\{L_u : u \in (d_G(x,y))^\vee \text{ and } (x,y) \in E \times E\}.$$

For each $x,y \in E$ and each word $u \in (d_G(x,y))^\vee$, let $f_{x,y,u}$ be a non expansive mapping from $G$ onto $L_u$. We claim that the graph $G$ is isometrically embeddable into $G'$ by the map $f$ defined by setting for every $z \in E$,

$$f(z) := \{f_{x,y,u}(z) : u \in (d_G(x,y))^\vee \text{ and } (x,y) \in E \times E\}.$$
This map is an isometry; indeed first, by definition of the product, it is non-expansive; next, to conclude that it is an isometry, it suffices to check that for every \( v \in \Lambda^* \), if \( d_G(x, y) \neq v \) then \( d_G(f(x), f(y)) \neq v \), that is for some triple \( i = (x', y', u) \) one has \( d_G(f_i(x), f_i(y)) \neq v \). Let \( v \) and \( x, y \) such that \( d_G(x, y) \neq v \) (this amounts to \( v \neq d_G(x, y) \)). Since \( d_G(x, y) = (d_G(x, y))^\Lambda \) there is some \( u \in (d_G(x, y))^\Lambda \) such that \( u \neq v \). We may set \( i = (x, y, u) \).

We may note that the product can be infinite even if the graph \( G \) is finite. Indeed, if \( G \) consists of two vertices \( x \) and \( y \) with no value on the pair \( \{x, y\} \) (that is the underlying graph is disconnected) then we need infinitely many zigzags of arbitrarily long length.

\[\square\]

**Lemma 5.24.** Every element \( v \) of \( \mathcal{N}_\Lambda \setminus \{\Lambda^*, \emptyset\} \) is accessible.

**Proof.** Case 1. \( v = \uparrow u \). Then \( n := \ell(u) \neq 0 \) hence \( u = \alpha_0 \cdots \alpha_{n-1} \). Set \( u' := \alpha_0 \cdots \alpha_{n-2} \alpha_{n-1} \) and \( r := \uparrow u' \). Since \( u \neq u' \) \( v \neq r \). On another hand \( u \leq u' \uplus u' \) hence \( v = \uparrow u \leq \uparrow u' \uplus \uparrow u' = (\uparrow u') \uplus (\uparrow u') \). Hence \( v \) is accessible.

Case 2. If \( v \) is not of the form \( \uparrow u \) for some \( u \in \Lambda^* \). Since \( u \) is not the emptyset, it is a finite join of elements of the form \( \uparrow u \). Thus, we may suppose that \( v = v_1 \uplus v_2 \) where \( v_1 \uplus v_2 < v \) and \( v_2 < v \) and furthermore that \( v_1' \uplus v_2 < v \) for all \( v_1' < v_1 \). According to Case 1, there is some \( r_1 \) such that \( v_1 \neq r_1 \) and \( v_1 \leq r_1 \uplus r_1 \). Let \( r := v_1 \uplus v_2 \). We claim first that \( v \neq r \). Suppose the contrary, according to Lemma 5.20 \( \mathcal{N}_\Lambda \setminus \{\emptyset\} \) is a distributive lattice, hence from \( v \leq r \) we get \( v = v \land r = (v \land r_1) \lor (v \land v_2) = (v \land r_1) \lor v_2 \), contradicting the choice of \( r_1 \).

Next, we claim that \( v \leq r \uplus r \). The operation \( \land \) and \( \lor \) distribute (Theorem 10 in [22]), hence \( r \uplus r = (r_1 \lor v_2) \uplus (r_1 \lor v_2) = (r_1 \lor v_2) \lor (r_1 \lor v_2) = (r_1 \lor v_2) \lor (r_1 \lor v_2) \lor (v_2 \lor v_2) \), since \( v_1 \leq r_1 \uplus r_2 \), it follows that \( v \leq r \uplus r \). Hence \( v \) is accessible. \[\square\]

**Theorem 5.25.** If a graph \( G \), finite or not, is a retract of a product of reflexive and directed zigzags of bounded length then every commuting set of endomorphisms has a common fixed point.

**Proof.** We may suppose that \( G \) has more than one vertex. The diameter of \( G \) equipped with the zigzag distance belongs to \( \mathcal{N}_\Lambda \setminus \{\Lambda^*, \emptyset\} \). According to Lemma 5.24 it is accessible, hence as a metric space \( G \) is bounded. Being a retracts of a product of hyperconvex metric spaces it is hyperconvex. Theorem 5.7 applies. \[\square\]

### 5.9. Bibliographical comments

Generalizations of the notion of metric space are as old as the notion of ordinary metric space and arises from geometry, logic as well as probability. Ours, originating in [17], is one among several; the paper [17] contains 71 references, e.g. Blumenthal and Menger [7], [8], [9], as well as Lawvere [26], to mention just a few. It was motivated by the work of Quilliot on graphs and posets [34], [35]. It extended to metric spaces over an involutive Heyting algebra (more appropriately an involutive op-quantale) the characterization of hyperconvex spaces due to Aronszajn-Panitchpakdi [3] and the existence of injective envelope, obtained for ordinary metric spaces by Isbell [16]. It contained also a study of hole-preserving maps and a characterization of absolute retracts w.r.t. these maps by means of the replete space. For more recent developments, see [2] [6], [20], [21], [22].

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