Chord-arc curves and the Beurling transform

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Abstract

We study the relation between the geometric properties of a quasicircle $\Gamma$ and the complex dilatation $\mu$ of a quasiconformal mapping that maps the real line onto $\Gamma$. Denoting by $S$ the Beurling transform, we characterize Bishop-Jones quasicircles in terms of the boundedness of the operator $(I - \mu S)$ on a particular weighted $L^2$ space, and chord-arc curves in terms of its invertibility. As an application we recover the $L^2$ boundedness of the Cauchy integral on chord-arc curves.

Introduction

A global homeomorphism on the plane $\rho$ is called quasiconformal if it preserves orientation, belongs to the Sobolev class $W^{1,2}(\mathbb{C})$, and satisfies the Beltrami equation $\overline{\partial} \rho - \mu \partial \rho = 0$, where $\mu$ is a measurable function, called the complex dilatation, such that $\|\mu\|_{\infty} < 1$.

Conversely, the mapping theorem for quasiconformal mappings states that for each function $\mu \in L^\infty(\mathbb{C})$, $\|\mu\|_{\infty} < 1$, there exists an essentially unique quasiconformal mapping on the plane with dilatation $\mu$.

We will often use the notation $\rho_\mu$ and $\mu_\rho$ if we need to specify the relation between the mapping and its complex dilatation.

The image of the real line $\mathbb{R}$ under a global quasiconformal mapping $\rho$ is called a quasicircle. In general the restriction of the mapping $\rho$ to the real line does not

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satisfy any regularity conditions such as, for instance, absolute continuity. As well, the quasicircle \( \Gamma = \rho(\mathbb{R}) \) might not even be rectifiable, in fact its Hausdorff dimension, though less than 2, can be arbitrarily close to 2. In contrast, if \( \mu \) were zero in a neighbourhood of \( \mathbb{R} \) then the map would be analytic on \( \mathbb{R} \), and \( \Gamma \) would be a smooth curve.

The question that arises naturally when studying quasiconformal mappings is to determine conditions on the complex dilatation \( \mu \) that would reflect on the geometric properties of the corresponding quasicircle \( \Gamma = \rho(\mathbb{R}) \).

In this paper, we obtain global conditions on the complex dilatations \( \mu \) that generate chord-arc curves. Our approach will follow the setting developed by Semmes in [S] to study the chord-arc curves with small constant by applying the strong interactions between quasiconformal mappings and singular integrals.

A locally rectifiable curve \( \Gamma \) that passes through \( \infty \) is a chord-arc curve if \( \ell_{\Gamma}(z_1, z_2) \leq K|z_1 - z_2| \) for all \( z_1, z_2 \in \Gamma \), where \( \ell_{\Gamma}(z_1, z_2) \) denotes the length of the shortest arc of \( \Gamma \) joining \( z_1 \) and \( z_2 \). The smallest such \( K \) is called the chord-arc constant.

It is a well known fact that a chord-arc curve is the image of the real line under a bilipschitz mapping on the plane, that is, there exits a mapping \( \rho: \mathbb{C} \to \mathbb{C} \) such that \( \rho(\mathbb{R}) = \Gamma \) and \( C^{-1}|z - w| \leq |\rho(z) - \rho(w)| \leq C|z - w| \) for all \( z, w \in \mathbb{C} \). Bilipschitz mappings preserve Hausdorff dimension, and though they are a very special class of quasiconformal mappings, no characterization has been found in terms of their complex dilatation. See [MOV] for more results on this topic.

The dilatation coefficients whose associated quasicircles are chord-arc curves with small constant are well understood. The general idea is that that if \( \mu(z) \) tends to zero when \( z \) approaches \( \mathbb{R} \), then one expects some close-to-rectifiable behaviour on the quasicircle \( \Gamma \). It turns out that a right way to quantify the smallness of \( \mu \) is to consider the measure \( |\mu(z)|^2/|y| \, dm \), where \( y = \text{Im} \, z \) and \( dm \) denotes the Lebesgue measure on the plane.

**Theorem A** (see [AZ, MG, S]). A Jordan curve \( \Gamma \) is a chord-arc curve with small constant if and only if there is a quasiconformal mapping \( \rho: \mathbb{C} \to \mathbb{C} \) with \( \rho(\mathbb{R}) = \Gamma \) and such that the dilatation \( \mu \) satisfies that \( |\mu|^2/|y| \) is a Carleson measure with small norm.

For arbitrary constants, this result is no longer true. In fact, if no restriction on the Carleson norm of \( |\mu|^2/|y| \) is imposed, the quasicircle \( \Gamma \) might not even be rectifiable,
though in a sense, they are rectifiable most of the time on all scales. They are the so called Bishop-Jones quasicircles (BJ), introduced by Bishop and Jones in [BJ] and defined as follows:

A Jordan curve $\Gamma$ is a BJ curve if it is the boundary of a simply connected domain $\Omega$, and for any $z \in \Omega$, there is a chord-arc domain $\Omega_z \subset \Omega$ containing $z$ of “norm” $\leq k(\Omega)$, with diameter uniformly comparable (with respect to $z$) to $\text{dist}(z, \partial \Omega)$, and such that $\mathcal{H}^1(\Gamma \cap \partial \Omega_z) \geq c(\Omega) \text{dist}(z, \partial \Omega)$. Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, and $\Omega_z$ being a chord-arc domain means that its boundary is a chord-arc curve.

The result in [BJ] states that the boundary of a simply connected domain $\Omega$ is a BJ curve if and only if $\log \Phi' \in BMOA(\mathbb{R}^2_+)$, where $\Phi$ is the Riemann map from $\mathbb{R}^2_+$ onto $\Omega$.

A BJ curve which is a quasicircle is called a BJ quasicircle. A typical example is a variant of the snowflake where at each iteration step, one of sides of the triangle, for instance the left one, is left unchanged.

Theorem B (see [AZ, Mc]). A Jordan curve $\Gamma$ is a BJ quasicircle if and only if there is a quasiconformal mapping $\rho: \mathbb{C} \to \mathbb{C}$ with $\rho(\mathbb{R}) = \Gamma$, and such that the dilatation $\mu$ satisfies that $|\mu|^2/|y|$ is a Carleson measure.

Semmes proved Theorem A in [S] by applying $L^2$-estimates to a certain perturbed Cauchy integral operator defined by the pull-back under the quasiconformal mapping of the Cauchy integral on $\Gamma$. Those estimates were obtained by studying the operator $(\overline{\partial} - \mu \partial)^{-1}$ for a certain class of $\mu$’s.

The operator $(\overline{\partial} - \mu \partial)$ can be written as $(I - \mu S)\overline{\partial}$ where $S$ is the Beurling transform defined by

$$ Sf(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(w - z)^2} \, dm(w). $$

We will show that the boundedness of $(I - \mu S)$ on a certain $L^2$-weighted space characterizes BJ quasicircles, and that its invertibility characterizes chord-arc curves.

More precisely, set $L^2 \left( \frac{dm}{|y|} \right) = \left\{ f: \mathbb{C} \to \mathbb{C}; \int_{\mathbb{C}} \frac{|f(z)|^2}{|z|} \, dm(z) < \infty \right\}.$

Theorem 1. Let $\mu \in L^\infty(\mathbb{C})$, $\|\mu\|_\infty < 1$. Then $|\mu|^2/|y|$ is a Carleson measure if and only if the operator $\mu S$ is bounded on $L^2 \left( \frac{dm}{|y|} \right)$. Besides the Carleson norm and the norm of the operator are equivalent.
The proof of Theorem 1 involves showing the boundedness of the Beurling transform from the space $L^2\left(\frac{dm}{|y|}\right)$ to the space $L^2\left(\frac{|\mu|^2}{|y|} \, dm\right)$. This is a two weight problem, where the weight $1/|y|$ is not even locally integrable.

As an immediate consequence of Theorems 1 and 2, we get the following corollary:

**Corollary 1.** The curve $\Gamma$ is a BJ quasicircle if and only if there exists a quasiconformal mapping $\rho: \mathbb{C} \to \mathbb{C}$ with $\rho(\mathbb{R}) = \Gamma$, and such that its dilatation coefficient $\mu$ satisfies that the operator $(I - \mu S)$ is bounded in $L^2\left(\frac{dm}{|y|}\right)$.

**Theorem 2.** Let $\Gamma$ be a quasicircle analytic at $\infty$. Then $\Gamma$ is a chord-arc curve if and only if there exists a quasiconformal mapping $\rho: \mathbb{C} \to \mathbb{C}$ with $\rho(\mathbb{R}) = \Gamma$, and such that its dilatation coefficient $\mu$ is compactly supported and satisfies that the operator $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$.

We will show as part of the proof of Theorem 2 that $\rho$ being bilipschitz, with $\mu = \mu_\rho$ satisfying that $|\mu|^2/|y|$ is a Carleson measure, is a sufficient condition to assure the invertibility of the operator $(I - \mu S)$ in $L^2\left(\frac{dm}{|y|}\right)$. On the other hand, in the last section we construct a quasiconformal mapping that shows that the converse does not hold. The characterization of the quasiconformal mappings $\rho$ for which the operator $(I - \mu_\rho S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$ remains an open question.

Note that as a consequences of Theorems 1 and 2 we recover the result on chord-arc curves with small constant presented in Theorem A. For that, let the Carleson norm of the measure $|\mu|^2/|y|$ be small enough, then by Theorem 1, the norm of the operator $\mu S$ in $L^2\left(\frac{dm}{|y|}\right)$ is less than 1. Therefore $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$, and by Theorem 2 the associated quasicircle is chord-arc.

Our next result, that will be needed in the proof of Theorem 2, gives a new sufficient condition for a quasicircle to be rectifiable.

**Theorem 3.** Let $\rho: \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping, analytic at $\infty$, such that $\int_{\mathbb{C}} \frac{|\partial \rho|^2}{|y|} \, dm < \infty$. Then $\Gamma = \rho(\mathbb{R})$ is rectifiable and $\rho'_\mathbb{R} \in L^2_{loc}$.

Finally, as an application of Theorem 2 we will recover the following well known result due to G. David [D]:

**Corollary 2.** If $\Gamma$ is a chord-arc curve, the Cauchy integral on $\Gamma$ is a bounded operator on $L^2(\Gamma)$. 
Our main motivation to study these questions has been the open problem on the connectivity of the manifold of chord-arc curves. The topology on this manifold is defined by 
\[ d(\Gamma_1, \Gamma_2) = \| \log |\Phi'_1| - \log |\Phi'_2| \|_{\text{BMO}(\mathbb{R})}, \]
where \( \Phi_i, i = 1, 2 \) represent the corresponding Riemann mappings from \( \mathbb{R}^+ \) onto the domains \( \Omega_i \) bounded by \( \Gamma_i, i = 1, 2 \).

It was proved in [AZ] that, with this topology, the larger space of BJ quasicircles is connected. The idea was to show that \( \mu \rightarrow t\mu, 0 \leq t \leq 1 \) gives a continuous deformation. One has to be more careful in the case of chord-arc curves as Bishop showed in [B]. He constructed a quasiconformal map \( \rho \) of the disk to itself such that the quasiconformal mapping corresponding to the dilatation \( \frac{1}{2} \rho \), maps the circle to a curve of Hausdorff dimension \( > 1 \).

The characterization of chord-arc curves given in Theorem 2 provides a new approach to the connectivity problem by translating it into a question regarding the spectrum of a singular operator in a weighted \( L^2 \) space.

The paper is structured as follows: In Section 1 we review some definitions and basic facts. Section 2 is devoted to the proof of Theorem 1. In Section 3 we study some well-behaved quasiconformal mappings and use them to prove that the chord-arc condition implies the invertibility of the operator in Theorem 2. In Section 4 we prove Theorem 3, and describe Semmes’s approach to solve Theorem A. We show how these ideas are involved in proving the other implication in Theorem 2. We will finish with some remarks and proofs of the remaining results in Section 5.

In the paper, the letter \( C \) denotes a constant that may change at different occurrences. The notation \( A \simeq B \) means that there is a constant \( C \) such that \( 1/C.A \leq B \leq C.A \). The notation \( A \lesssim B \) (\( A \gtrsim B \)) means that there is a constant \( C \) such that \( A \leq C.A \) (\( A \geq C.B \)). Also, as usual, \( B(z_0, R) \) denotes a ball of radius \( R \) centered at the point \( z_0 \in \mathbb{C} \). If \( B \) is a ball, \( 2B \) denotes the ball with the same center as \( B \) and twice the radius of \( B \), and similarly for squares.

1 Basic facts and definitions

A locally integrable function \( f \) belongs to the space \( \text{BMO}(\mathbb{R}) \) if

\[ \| f \|_* = \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty \]
where \( I \) is any interval and
\[
f_I = \frac{1}{|I|} \int_I f(y) \, dy.
\]

If \( f \) is analytic on \( \mathbb{R}^2_+ \) with boundary values \( f(x) \in \text{BMO}(\mathbb{R}) \), we say that \( f \in \text{BMOA}(\mathbb{R}^2_+) \).

A positive measure \( \sigma \) on \( \mathbb{C} \) is called a Carleson measure (relative to \( \mathbb{R} \)) if for each \( R > 0 \) and \( x \in \mathbb{R} \), \( \sigma(\{w : |w - x| \leq R\}) \leq CR \). The smallest such \( C \) is called the Carleson norm of \( \sigma \), \( \|\sigma\|_C \). If we replace the condition \( x \in \mathbb{R} \) by \( x \in \Gamma \) for some fixed curve \( \Gamma \) then we say that \( \sigma \) is a Carleson measure with respect to \( \Gamma \).

Let \( \Gamma \) be an oriented rectifiable Jordan curve that passes through \( \infty \), and let \( \Omega_+ \) and \( \Omega_- \) denote its complementary regions. Given a function \( f \) on \( \Gamma \), define its Cauchy integral \( F(z) = C_\Gamma f(z) \) off \( \Gamma \) by
\[
F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} \, dw, \quad z \not\in \Gamma.
\]

The boundary values of \( F_\pm = F|_{\Omega_\pm} \) exist almost everywhere, with respect to the one-dimensional Hausdorff measure \( H^1 \). Denoting them by \( f_+ \) and \( f_- \), the classical Plemelj formula states that
\[
f_\pm(z) = \pm \frac{1}{2} f(z) + \frac{1}{2\pi i} \text{P.V.} \int_\Gamma \frac{f(w)}{w - z} \, dw, \quad H^1\text{-a.a. } z \in \Gamma.
\]

The singular integral is also called the Cauchy integral. In particular, the jump of \( F \) across \( \Gamma \), defined by \( f_+ - f_- \), is equal to \( f \). This property and the analyticity of \( F \) off \( \Gamma \) are expressed in the equations:
\[
\overline{\partial}F = f(z) \, dz
\]
\[
\partial F = F'_+ \chi_{\Omega_+} + F'_- \chi_{\Omega_-} - f(z) \, dz
\]
interpreted in the sense of distributions. Later on, we will consider expressions involving multiplication of the derivatives of the Cauchy transform by quasiconformal coefficients \( \mu \in L^\infty(\mathbb{C}) \). Since \( \mu \) is defined only a.e., we need to introduce the notation
\[
F' = \chi_{\Omega_+} F'_+ + \chi_{\Omega_-} F'_-.
\]
Thus, the expression \( \mu F' \) is meaningful in \( L^p(\Gamma), ~ 1 < p < \infty \). In other words, the index \( ' \) will mean no distributional term.
In the sequel, if $\Gamma = \mathbb{R}$, to simplify the notation we write $C f(z)$ instead of $C_{\mathbb{R}} f(z)$, and we let $C f(x)$ stand for the boundary values, that is

$$C f(x) = f_\pm(x) = \pm \frac{1}{2} f(x) + \frac{1}{2\pi i} \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{y-x} \, dx, \quad a.a. x \in \mathbb{R}$$

In this case, the $L^2$ estimate $\|C f\|_{L^2(\mathbb{R})} \leq c \|f\|_{L^2(\mathbb{R})}$ is a consequence of the Fourier transform and Plancherel’s theorem. On the other hand, for a general rectifiable curve $\Gamma$ this is unavailable because the Cauchy integral on $\Gamma$ is no longer a convolution operator.

A complete characterization of the curves for which the Cauchy integral is bounded has been obtained by G. David [D]. He showed that the Cauchy integral is bounded on $L^p(\Gamma)$, $1 < p < \infty$, if and only if $\Gamma$ is regular, that is, for all $z_0 \in \mathbb{C}$ and all $R > 0$, $\mathcal{H}^1(B(z_0, R) \cap \Gamma) \lesssim R$. The quasicircles which are regular curves are chord-arc curves, and vice versa.

Recall that the Beurling transform $S$ is defined by

$$S f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} \, dm(w)$$

The Fourier multiplier of $S$ is $\bar{\xi}/\xi$, thus $S$ represents an isometry on $L^2(\mathbb{C})$, i.e. $\|Sf\|_2 = \|f\|_2$. Moreover, $S$ is bounded on $L^p(\mathbb{C})$, $1 < p < \infty$.

In the study of the Beltrami equation $\overline{\partial} \rho - \mu \partial \rho = 0$, $\|\mu\|_\infty < 1$, there is another operator that plays a fundamental role: the Cauchy operator on the plane $T$

$$T f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{w-z} \, dm(w)$$

If $f \in L^p(\mathbb{C})$, $p > 2$ then $T f$ represents a continuous function in $\mathbb{C}$. Besides the relations

$$\overline{\partial}(T f) = f$$

$$\partial(T f) = S f$$

hold in the distributional sense.

Assuming that the complex dilatation $\mu$ has compact support, the solution to the Beltrami equation is given explicitly by the formula

$$\rho(z) = z + Th(z)$$

where the function $h(z) = \overline{\partial} \rho(z)$ is determined by the equation

$$(I - \mu S) h = \mu$$
2 Boundedness of the operator

We begin by stating a result in [CJS, p. 557] that will be needed in the proof of Theorem 1. For the sake of completeness we detail its proof.

**Lemma 1.** The operator $K$ defined by

$$Kf(z) = |\text{Im } z|^{1/2} \int_{\mathbb{R}^2} \frac{f(w)|\text{Im } w|^{1/2}}{|w-z|^3} \, dm(w), \quad z \in \mathbb{R}_+^2$$

represents a bounded operator from $L^2(\mathbb{R}^2_-)$ to $L^2(\mathbb{R}^2_+)$ and $\|K\|_{(L^2_-, L^2_+)} \leq 4\pi$.

**Proof.** Let $k(z,w)$ represent the kernel of the operator $K$,

$$k(z,w) = \frac{|\text{Im } z|^{1/2}|\text{Im } w|^{1/2}}{|w-z|^3}, \quad z \in \mathbb{R}_+^2, \ w \in \mathbb{R}_-^2.$$

Then, for $z \in \mathbb{R}_+^2$

$$\int_{\mathbb{R}_+^2} k(z,w) \, dm(w) \leq |\text{Im } z|^{1/2} \int_{|w-z| > |\text{Im } z|} \frac{dm(w)}{|w-z|^{3-1/2}}$$

$$= |\text{Im } z|^{1/2} 2\pi \int_{\text{Im } z}^{\infty} r^{-3/2} \, dr = 4\pi,$$

and therefore

$$\|Kf\|_\infty \leq 4\pi \|f\|_\infty.$$

Similarly, $\forall w \in \mathbb{R}_-^2$

$$\int_{\mathbb{R}_+^2} k(z,w) \, dm(z) \leq 4\pi$$

and

$$\|Kf\|_1 \leq \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+^2} k(z,w) f(w) \, dm(w) \right| \, dm(z)$$

$$\leq \int_{\mathbb{R}_+^2} |f(w)| \left( \int_{\mathbb{R}_+^2} k(z, w) \, dm(z) \right) \, dm(w)$$

$$\leq 4\pi \|f\|_1.$$

The lemma follows by interpolation or Shur’s Lemma.
Before proceeding to the proof of Theorem 1, let us recall its statement.

**Theorem 1.** Let \( \mu \in L^\infty(\mathbb{C}) \), \( \|\mu\|_\infty < 1 \). Then the following conditions are equivalent:

1. \( \nu = \frac{|\mu|^2}{|y|} \, dm \) is a Carleson measure with respect to \( \mathbb{R} \), i.e., there is \( c_1 > 0 \) such that
   \[
   \int_{B(x_0, r)} \frac{|\mu(z)|^2}{|y|} \, dm(z) \leq c_1 r, \quad \forall \ x_0 \in \mathbb{R}, \ r > 0 \ (y = \text{Im} \ z)
   \]

2. The operator \( \mu S \) is bounded on \( L^2 \left( \frac{dm}{|y|} \right) \), i.e., there is \( c_2 > 0 \) such that:
   \[
   \int_{\mathbb{C}} \frac{|\mu(z)|^2}{|y|} |Sf(z)|^2 \, dm(z) \leq c_2 \int_{\mathbb{C}} |f(z)|^2 \frac{dm}{|y|} \tag{2.1}
   \]
   for all (compactly supported) functions \( f \) with \( \int_{\mathbb{C}} |f|^2 \frac{dm}{|y|} < \infty \).

Besides the Carleson norm \( \|\nu\|_C \) and the norm of the operator \( \|\mu S\|_{L^2(\frac{dm}{|y|})} \) are comparable.

**Remark.** If \( f \in L^2 \left( \frac{dm}{|y|} \right) \) has compact support, say \( f \in B(0, M) \), then \( \int_{\mathbb{C}} |f|^2 \, dm \leq M \int_{\mathbb{C}} |f|^2 \frac{dm}{|y|} < \infty \), that is \( f \in L^2(\mathbb{C}) \), and therefore \( Sf \) is a well defined \( L^2 \)-function.

Since a general \( f \in L^2 \left( \frac{dm}{|y|} \right) \) can be approximated by compactly supported ones, by the usual density arguments, the statement (2.1) is equivalent to the boundedness of the operator \( \mu S \) on \( L^2 \left( \frac{dm}{|y|} \right) \).

**Proof.**

(1) \( \Rightarrow \) (2) Let \( B_0 \) be a ball of radius \( r > 0 \) centered at a point \( x_0 \in \mathbb{R} \). We shall apply the assumption on the boundedness of the operator \( \mu S \) to an appropriate function \( f \) to show that \( \nu(B_0) \leq cr \).

For that consider the ball \( \hat{B}_0(z_0, r) \) where \( z_0 = x_0 + i2r \), and the function \( f(z) = \chi_{\hat{B}_0}(z) \).

Note that \( \|f\|_{L^2(\frac{dm}{|y|})}^2 \simeq r \), and since

\[
Tf(z) = \begin{cases} 
\overline{z - z_0} & z \in \hat{B}_0 \\
\frac{r^2}{z - z_0} & z \notin \hat{B}_0
\end{cases}
\]
we get \((Sf)(z) = \frac{r^2}{(z-z_0)^2} \chi_{\mathbb{C}\setminus \tilde{B}_0}(z)\). Thus
\[
r \simeq \int_{\tilde{B}_0} \frac{1}{|y|} \, dm(z) \gtrsim \int_C \frac{|\mu(z)|^2}{|y|} |Sf(z)|^2 \, dm(z)
\]
\[
\geq \int_{B_0} \frac{|\mu(z)|^2}{|y|} \frac{r^4}{|z-z_0|^4} \, dm(z) \simeq \int_{B_0} \frac{|\mu(z)|^2}{|y|} \, dm(z).
\]
with comparison constants only depending on the norm of the operator. Therefore \(\nu\) is a Carleson measure with \(\|\nu\|_C \lesssim \|S\|_{L^2(\frac{dm}{|y|})}\).

(2) \Rightarrow (1) We can assume that \(\text{supp}(\mu) \subset \mathbb{R}_+^2\), otherwise write \(\mu = \mu\chi_{\mathbb{R}_2^+} + \mu\chi_{\mathbb{R}_2^-}\). We proceed to estimate \(\|S\|_{L^2(\frac{dm}{|y|})}\) when \(\nu = \frac{|w|^2}{|y|} \, dm\) is a Carleson measure.

For that, consider a Whitney decomposition of \(\mathbb{R}_+^2\), that is write \(\mathbb{R}_+^2\) as a disjoint union of cubes \(Q_k\) with \(\text{diam}(Q_k) = \frac{1}{2} \text{dist}(Q_k, \mathbb{R})\), and set \(Q_k^* = \frac{1}{2}Q_k\). Given \(z \in \mathbb{R}_+^2\), denote by \(Q_k(z)\) and \(Q_k^*(z)\) the corresponding cubes containing \(z\).

We can now express \(\|(\mu S)f\|_{L^2(\frac{dm}{|y|})}^2\) as the sum of the following integrals:
\[
\|(\mu S)f\|_{L^2(\frac{dm}{|y|})}^2 \simeq \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left(\int_C \frac{f(w)}{(w-z)^2} \, dm(w)\right)^2 \, dm(z)
\]
\[
\lesssim \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left(\int_{w \in \mathbb{R}_+^2} \frac{f(w)}{(w-z)^2} \, dm(w)\right)^2 \, dm(z)
\]
\[
+ \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left(\int_{w \in \mathbb{R}_+^2 \setminus Q_k^*(z)} \frac{f(w)}{(w-z)^2} \, dm(w)\right)^2 \, dm(z)
\]
\[
+ \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left(\int_{w \in Q_k^*(z)} \frac{f(w)}{(w-z)^2} \, dm(w)\right)^2 \, dm(z)
\]
\[
= I_1 + I_2 + I_3.
\]

Let us start by estimating \(I_1\). Since \(|\mu|^2/y\) is a Carleson measure and \(S(f\chi_{\mathbb{R}_2^+})\) represents an analytic function on \(\mathbb{R}_+^2\), it follows that:
\[
I_1 \lesssim \int_{\mathbb{R}} |S(f\chi_{\mathbb{R}_2^+})(x)|^2 \, dx.
\]
By Green’s formula, we can express this integral on the line as an integral on the upper half plane to obtain

\[ I_1 \lesssim \int_{\mathbb{R}_+^2} |(S(f\chi_{\mathbb{R}_+^2}))'(z)|^2 (\text{Im } z) \, dm(z) \]

\[ \lesssim \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+^2} \frac{f(w)}{(w-z)^3} \, dm(w) \right|^2 (\text{Im } z) \, dm(z) \]

\[ \leq \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+^2} k(z,w) |f(w)| \, \text{Im } w^{-1/2} \, dm(w) \right|^2 \, dm(z) \]

\[ \lesssim \int_{\mathbb{R}_+^2} \frac{|f(w)|^2}{|\text{Im } w|} \, dm(w) = \|f\|_{L^2(\frac{dy}{y})}^2. \]

The last inequality follows from Lemma [1]. Note that the comparison constants depend only on the Carleson norm \( \|\nu\|_C \).

To estimate \( I_2 \), write

\[ I_2 \leq \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left| \int_{\mathbb{R}_+^2 \setminus Q^*_k(z)} f(w) \left( \frac{1}{(w-z)^2} - \frac{1}{(w-\overline{z})^2} \right) \, dm(w) \right|^2 \, dm(z) \]

\[ + \int_{\mathbb{R}_+^2} \frac{|\mu(z)|^2}{y} \left| \int_{\mathbb{R}_+^2 \setminus Q^*_k(z)} f(w) \frac{1}{(w-\overline{z})^2} \, dm(w) \right|^2 \, dm(z) \]

\[ = I_{2,1} + I_{2,2}. \]

The second term on the right hand side \( I_{2,2} \) reduces to the previous case. For the first one, \( I_{2,1} \), note that \( \|\mu\|_\infty < 1 \) and

\[ \frac{1}{(w-z)^2} - \frac{1}{(w-\overline{z})^2} = 4i \frac{(w-x)y}{(w-z)^2(w-\overline{z})^2}, \quad z = x + iy. \]

Besides, for \( z \in \mathbb{R}_+^2 \) and \( w \in \mathbb{R}_+^2 \setminus Q^*_k(z) \)

\[ |w-\overline{z}| \leq |w-z| + 2y \leq c_0 |w-z| \]
for some universal constant $c_0 > 0$. Thus

$$I_{2,1} \lesssim \int_{\mathbb{R}^2_+} \frac{1}{y} \left| \int_{\mathbb{R}^2_+ \setminus Q^*_k(z)} f(w) \frac{(w-x)y}{(w-z)^2(w-\overline{z})^2} \, dm(w) \right|^2 \, dm(z)$$

$$\lesssim \int_{\mathbb{R}^2_+} \left| \int_{\mathbb{R}^2_+} \frac{|f(w)|}{|w-\overline{w}|^3} \, dm(w) \right|^2 \, dm(z)$$

$$= \int_{\mathbb{R}^2} |\text{Im } z| \left| \int_{\mathbb{R}^2_+} \frac{|f(w)| |\text{Im } w|^{-1/2}}{|w-z|^3} \, dm(w) \right|^2 \, dm(z)$$

$$\lesssim \|f\|^2_{L^2(\frac{dm}{|w|})}$$

by Lemma I.

Finally, to estimate $I_3$, write $\mathbb{R}^2_+ = \bigcup_k Q_k$ and use the fact that $S$ is an isometry on $L^2(\mathbb{C})$. So, if $z_k = x_k + iy_k$ denotes the center of $Q_k$, we get

$$I_3 = \sum_k \int_{Q_k} \frac{\mu(z)^2}{y} \left| \int_{w \in Q_k^*(z)} f(w) \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z)$$

$$\lesssim \sum_k \frac{1}{y_k} \int_{Q_k} \left| \int_{Q_k^*(w)} f(w) \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z)$$

$$\lesssim \sum_k \frac{1}{y_k} \|S(f \chi_{Q_k^*})\|^2_{L^2(\mathbb{C})} = \sum_k \frac{1}{y_k} \|f \chi_{Q_k^*}\|^2_{L^2(\mathbb{C})}$$

$$= \sum_k \frac{1}{y_k} \int_{Q_k^*} |f(w)|^2 \, dm(w) \simeq \int_{\mathbb{R}^2} \frac{|f(w)|^2}{\text{Im } w} \, dm(w)$$

since the set of $Q_k^*$'s are also Whitney cubes with finite overlap.

This concludes the proof of Theorem I. \qed

3 Chord-arc condition implies invertibility

Let us recall the statement of Theorem II.
**Theorem 2.** Let $\Gamma$ be a quasicircle analytic at $\infty$. Then the following conditions are equivalent:

1. $\Gamma$ is a chord-arc curve, i.e., there is $k > 0$ such that for any $z_1, z_2 \in \Gamma$
   \[ \ell_\Gamma(z_1, z_2) \leq k|z_1 - z_2| \]
   where $\ell_\Gamma(z_1, z_2)$ denotes the length of the shortest arc of $\Gamma$ joining $z_1$ and $z_2$.

2. There is a quasiconformal mapping $\rho: \mathbb{C} \to \mathbb{C}$ with $\Gamma = \rho(\mathbb{R})$ and such that $\mu = \mu_\rho$ has compact support, $|\mu|^2/|y|$ is a Carleson measure and satisfies that:
   \[(I - \mu S): L^2 \left( \frac{dm}{|y|} \right) \to L^2 \left( \frac{dm}{|y|} \right) \]
   is an invertible operator.

**Remark.** Note that by Theorem 1, $|\mu|^2/|y|$ being a Carleson measure is equivalent to the boundedness of the operator $(I - \mu S)$ on $L^2 \left( \frac{dm}{|y|} \right)$. We shall specify now what we mean by invertibility of $(I - \mu S)$.

Since $\|\mu\|_{\infty} < 1$ and $S$ is an isometry in $L^2(\mathbb{C})$, the operator $(I - \mu S)$ is invertible in $L^2(\mathbb{C})$. If $\Phi \in L^2(\mathbb{C})$ then $h = (I - \mu S)^{-1}(\Phi)$ is a well and uniquely defined element of $L^2(\mathbb{C})$. Moreover, $\|h\|_{L^2(\mathbb{C})} \leq c_0\|\Phi\|_{L^2(\mathbb{C})}$, i.e. $\|(I - \mu S)^{-1}(\Phi)\|_{L^2(\mathbb{C})} \leq c_0\|\Phi\|_{L^2(\mathbb{C})}$.

By saying that $I - \mu S$ is invertible on $L^2 \left( \frac{dm}{|y|} \right)$ or that

\[(I - \mu S)^{-1}: L^2 \left( \frac{dm}{|y|} \right) \to L^2 \left( \frac{dm}{|y|} \right) \]

we mean (by definition) that there is a constant $c_1 > 0$ such that if $\Phi \in L^2 \left( \frac{dm}{|y|} \right)$ has compact support (so $\Phi \in L^2(\mathbb{C})$!), by the remark in Section 2, then the uniquely determined element $h = (I - \mu S)^{-1}(\Phi) \in L^2(\mathbb{C})$ satisfies:

\[ \int_{\mathbb{C}} |h(z)|^2 \frac{dm}{|y|} \leq c_1 \int_{\mathbb{C}} |\Phi(z)|^2 \frac{dm(z)}{|y|}. \]

We write this as:

\[ \int_{\mathbb{C}} |(I - \mu S)^{-1}\Phi|^2 \frac{dm(z)}{|y|} \leq c_1 \int |\Phi|^2 \frac{dm(z)}{|y|}. \]

(3.1)
Let $\Gamma$ be a chord-arc curve analytic at $\infty$, and $\rho: \mathbb{C} \to \mathbb{C}$ a bilipschitz mapping with $\rho(\mathbb{R}) = \Gamma$. One would like to prove that for $\mu = \mu_\rho$ the operator $(I - \mu S)$ is invertible in $L^2 \left( \frac{dm}{|y|} \right)$. But before studying the invertibility we need to address the question of the boundedness, which is equivalent by Theorem 1 to $|\mu|^2/|y|$ being a Carleson measure. The following lemma due to Semmes ([S, Lemma 4.11]), based on a variant of the Ahlfors-Beurling extension, will provide a good candidate for the bilipschitz mapping $\rho$.

**Lemma 2.** Suppose that $r: \mathbb{R} \to \mathbb{C}$ is a bilipschitz mapping. Then $r$ can be extended to a quasiconformal mapping $\rho: \mathbb{C} \to \mathbb{C}$ which is also bilipschitz, and $\mu = \mu_\rho$ satisfies that $\nu = |\mu|^2/|y| \, dm$ is a Carleson measure.

**Proof of (1) $\Rightarrow$ (2) in Theorem 2.** Let $\Gamma$ be a chord-arc curve, and $r: \mathbb{R} \to \mathbb{C}$ a bilipschitz parametrization of $\Gamma$. Let $\mu$ be the dilatation coefficient of the bilipschitz mapping $\rho$ given by Lemma 2. Then, by Theorem 1 the operator $(I - \mu S)$ is bounded on $L^2 \left( \frac{dm}{|y|} \right)$. We will show that it is as well invertible.

Let $\Phi \in L^2 \left( \frac{dm}{|y|} \right)$ with compact support, we need to prove that the unique solution $h$ to the equation

$$(I - \mu S)h = \Phi$$

(3.2)

verifies that $\|h\|_{L^2 \left( \frac{dm}{|y|} \right)} \leq c\|\Phi\|_{L^2 \left( \frac{dm}{|y|} \right)}$, i.e. estimate (3.1) with $c = c(\Gamma, \|\nu\|_C)$.

By the usual density arguments, we can assume that $\Phi \in L^2 \left( \frac{dm}{|y|} \right) \cap L^p(\mathbb{C})$ for a fixed $p > 0$ such that $2 < p \leq 1 + \frac{1}{\|\mu\|_\infty}$. In this case $(I - \mu S)^{-1}: L^p(\mathbb{C}) \to L^p(\mathbb{C})$ [AIS], therefore $h \in L^p(\mathbb{C})$, where $p > 2$. Define

$$H(z) = Th(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(w)}{w - z} \, dm(w)$$

Then $H$ is continuous on $\mathbb{C}$, $\overline{\partial}H = h$ and $\partial H = Sh$. Thus (3.2) reads as

$$\overline{\partial}H - \mu \partial H = \Phi.$$

By applying the quasiconformal change of variables $u = H \circ \rho^{-1}$ as in [AIS], we get that $H = u \circ \rho$ and

$$h = \overline{\partial}H = (\partial u \circ \rho) \overline{\partial} \rho + (\overline{\partial u} \circ \rho) \overline{\partial} \rho$$

$$\partial H = (\partial u \circ \rho) \partial \rho + (\overline{\partial u} \circ \rho) \overline{\partial} \rho.$$
Since $\overline{\partial}\rho = \mu \partial\rho$, we obtain
\[
\Phi = \overline{\partial}H - \mu \partial H = (I - |\mu|^2)(\overline{\partial}u \circ \rho)\overline{\partial}\rho.
\]
Consequently
\[
(\overline{\partial}u \circ \rho)\overline{\partial}\rho = \frac{\Phi}{1 - |\mu|^2}.
\]
Since $\|\mu\|_\infty < 1$, this implies that
\[
\|(\overline{\partial}u \circ \rho)\overline{\partial}\rho\|_{L^2\left(\frac{dm}{y}\right)} \simeq \|\Phi\|_{L^2\left(\frac{dm}{y}\right)}
\]
with equivalence constants depending only on $\|\mu\|_\infty$. So, the estimate (3.1) will be proved if we can show that
\[
\int_C \frac{\partial u \circ \rho|^2}{|y|}|\mu|^2|\partial\rho|^2 \ dm \leq c \int_C \frac{|\partial u \circ \rho|^2}{|y|} |\partial\rho|^2 \ dm. \tag{3.3}
\]
Letting $w = \rho(z)$ and $v = \overline{\partial}u$, the above expression is equivalent to
\[
\int_C \frac{|\mu \circ \rho^{-1}(w)|^2}{\mathrm{dist}(\rho^{-1}(w), \mathbb{R})} |Sv(w)|^2 \ dm(w) \leq c \int_C \frac{|v(w)|^2}{\mathrm{dist}(\rho^{-1}(w), \mathbb{R})} \ dm(w). \tag{3.4}
\]
This setting is very similar to the one in Theorem 1. Since $\rho$ is bilipschitz, $\mathrm{dist}(\rho^{-1}(w), \mathbb{R}) \simeq \mathrm{dist}(w, \Gamma)$, and if we define $\tilde{\mu}(w) = \mu \circ \rho^{-1}(w)$, it can be easily checked that $\tau(w) = \frac{|\tilde{\mu}(w)|^2}{\mathrm{dist}(w, \Gamma)} \ dm$ represents a Carleson measure with respect to $\Gamma$ with $\|\tau\|_C \simeq \|\nu\|_C$. Thus, 3.4 is equivalent to the following claim:

**Claim.** The operator $(\tilde{\mu}S)$ is bounded on $L^2\left(\frac{dm}{\mathrm{dist}(w, \Gamma)}\right)$ whenever the measure $\tau = \frac{|\tilde{\mu}(w)|^2}{\mathrm{dist}(w, \Gamma)} \ dm$ is a Carleson measure with respect to $\Gamma$, i.e. if
\[
\int_{B(x_0, r)} \frac{|\tilde{\mu}(w)|^2}{\mathrm{dist}(w, \Gamma)} \ dm(w) \leq c_1 r; \quad \forall \ x_0 \in \Gamma, \ r > 0,
\]
then
\[
\int_C \frac{|\tilde{\mu}(w)|^2}{\mathrm{dist}(w, \Gamma)} |Sf(w)|^2 \ dm(w) \leq c_2 \int_C \frac{|f(w)|^2}{\mathrm{dist}(w, \Gamma)} \ dm(w)
\]
for all (compactly supported) functions $f$ with $\int_C \frac{|f(w)|^2}{\mathrm{dist}(w, \Gamma)} \ dm(w) < \infty$. 

15
To prove this claim we will follow precisely the same steps as in the proof of $(2) \Rightarrow (1)$ in Theorem 1.

Denote by $\Omega^\pm$ the two domains bounded by $\Gamma$. In this context, the following lemma will be the equivalent of Lemma 1. Since it can be proved in a similar way, we will omit its proof.

**Lemma 3.** The operator $\tilde{K}$ defined by

$$\tilde{K} f(z) = \left( \text{dist}(z, \Gamma) \right)^{1/2} \int_{\Omega^-} \frac{f(w)(\text{dist}(w, \Gamma))^{1/2}}{|w-z|^3} \, dm(w), \quad z \in \Omega^+$$

represents a bounded operator from $L^2(\Omega^-)$ to $L^2(\Omega^+)$.

Assuming that $\text{supp}(\tilde{\mu}) \subset \Omega^+$, we decompose the integral

$$\| (\tilde{\mu}S) f \|_{L^2(\frac{dm(\cdot)}{\text{dist}(\cdot, \Gamma)})}^2 = \frac{1}{\pi} \int_{\Omega^+} |\tilde{\mu}(z)|^2 \left| \int_C \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z) = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$$

where $\tilde{I}_i$ ($i = 1, 2, 3$) are the analogous of $I_i$ ($i = 1, 2, 3$) in (2.2), using in this case a Whitney decomposition of $\Omega^+$.

To estimate $\tilde{I}_1$, we apply the following result in [Z]: if $\Omega$ is a chord-arc domain, and $\sigma$ is a Carleson measure with respect to $\partial \Omega$, then for any function in the Hardy space $F \in H^2(\Omega)$

$$\int_{\partial \Omega} |F(\xi)|^2 |d\xi| \leq c(\|\sigma\|_{C^1}) \int_{\partial \Omega} |F(\xi)|^2 |d\xi|.$$

The next ingredient we need is a substitute for chord-arc domains of “Green’s formula” [JK]:

If $\Omega$ is a chord-arc domain and $F \in H^2(\Omega)$ then

$$\int_{\partial \Omega} |F(\xi)|^2 |d\xi| \simeq \int_{\Omega} |F'(w)|^2 \text{dist}(w, \Gamma) \, dm(w).$$

By applying these results together with Lemma 3 as in the estimate of $I_1$ in Theorem 1, we conclude:

$$\tilde{I}_1 \lesssim \int_{\Gamma} |S(f\chi_{\Omega^-})(\xi)|^2 |d\xi|$$

$$\simeq \int_{\Omega^+} |(S(f\chi_{\Omega^-})')(z)|^2 \text{dist}(z, \Gamma) \, dm(z)$$

$$\lesssim c \|f\|_{L^2(\frac{dm(\cdot)}{\text{dist}(\cdot, \Gamma)})}^2.$$
with comparison constants depending on $\Gamma$ and $\|\nu\|_C$.

To estimate $\tilde{I}_2$, we just replace in $I_2$ the conjugate of a point $z$, i.e. $\overline{z}$, by the quasiconformal reflection $r: \Omega^+ \to \Omega^-$, defined by $r(z) = \rho\left(\rho^{-1}(z)\right)$. Then $|z - r(z)| \simeq \text{dist}(z, \Gamma)$ and the desired result holds for $\tilde{I}_2$.

Similarly, to estimate $\tilde{I}_3$ we proceed as in $I_3$ using in this case the Whitney decomposition of $\Omega^+$, and the fact that $S$ is a bounded operator on $L^2(\mathbb{C})$.

This concludes the proof of (1) $\Rightarrow$ (2) in Theorem 2.

4 Invertibility implies chord-arc condition

In this section we will prove the remaining implication in Theorem 2. That is:

Let $\mu \in L^\infty(\mathbb{C})$ be compactly supported with $\|\mu\|_\infty \leq 1$, $\rho = \rho_\mu$ the associated quasiconformal mapping and $\Gamma = \rho_\mu(\mathbb{R})$. If $|\mu|^2/|y|$ is a Carleson measure and the operator $(I - \mu S): L^2\left(\frac{dm}{|y|}\right) \to L^2\left(\frac{dm}{|y|}\right)$ is invertible, then $\Gamma$ is a chord-arc curve.

By the results mentioned in the introduction, in particular Theorem B, we know that if $|\mu|^2/|y|$ is a Carleson measure then $\Gamma$ is a BJ quasicircle. We will use the estimate on the invertibility of the operator $(I - \mu S)$ given by (3.1), i.e.

$$\int_{\mathbb{C}} |(I - \mu S)^{-1}(\Phi)|^2 \frac{dm}{|y|} \leq c \int_{\mathbb{C}} |\Phi|^2 \frac{dm}{|y|}$$

to show that $\Gamma$ is also rectifiable and in fact chord-arc.

Firstly, we state the following lemma that we will be applied in the next result. It is a simple corollary of Fubini’s theorem, so we omit the proof.

**Lemma 4.** Assume $g$ has compact support with $g \in L^p(\mathbb{C})$ for some $p > 2$. Let

$$H(z) = T g(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(w)}{w - z} \, dm(w)$$

and suppose $h \in L^2(\mathbb{R})$ has compact support in $\mathbb{R}$. Then

$$\int_{\mathbb{R}} H(x)h(x) \, dx = 2i \int_{\mathbb{C}} g(z)C_h(z) \, dm(z).$$

(4.1)
Here note that $H$ is continuous and $\overline{\partial}H = g$.

The next result, that will be needed later, gives a sufficient condition for a quasicircle to be rectifiable.

**Theorem 3.** Let $\rho: \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping, analytic at $\infty$, such that $\int_{\mathbb{C}} |\overline{\partial}\rho|^2 dm < \infty$. Then $\Gamma = \rho(\mathbb{R})$ is locally rectifiable and $\rho'|_{\mathbb{R}} \in L^2_{\text{loc}}$.

**Proof.** Normalize $\rho$ so that $\rho(z) = z + O(1/z)$. To prove the theorem it is enough to show that the difference quotients $\frac{1}{h}(\rho(x + h) - \rho(x))$ are uniformly bounded in $L^2_{\text{loc}}$ (see for example Sect. 7.11 in [GT]). For that, we will use a duality argument. So, let $g$ be a test function in $L^2(\mathbb{R})$ with $\|g\|_{L^2(\mathbb{R})} = 1$. Since $T(\overline{\partial}\rho)(z) = \rho(z) - z$, by lemma 4

$$\int_{\mathbb{R}} \left( \frac{\rho(x + h) - \rho(x)}{h} - 1 \right) g(x) \, dx = \int_{\mathbb{C}} \frac{\overline{\partial}\rho(z + h) - \overline{\partial}\rho(z)}{h} C_g(z) \, dm$$

$$= \int_{\mathbb{C}} \frac{C_g(z - h) - C_g(z)}{h} \, dm$$

$$\leq \left( \int_{\mathbb{C}} \frac{|\overline{\partial}\rho(z)|^2}{|y|} \, dm \right)^{1/2} \left( \int_{\mathbb{C}} \left| \frac{C_g(z - h) - C_g(z)}{h} \right|^2 \, dm \right)^{1/2}$$

The first term is finite by hypothesis. To bound the second one note that

$$\frac{C_g(z - h) - C_g(z)}{h} = \frac{1}{h} \int_{\mathbb{R}} g(x) \left( \frac{1}{x - z + h} - \frac{1}{x - z} \right) \, dx$$

$$= -\frac{1}{h} \partial_z \int_{\mathbb{R}} g(x) \left( \log |x - z + h|^2 - \log |x - z|^2 \right) \, dx$$

$$= -\frac{2}{h} \partial_z \int_{\mathbb{R}} g(x) \log \frac{|x - z + h|}{|x - z|} \, dx.$$ 

So, by Green’s formula

$$\int_{\mathbb{C}} |y| \left| \frac{C_g(z - h) - C_g(z)}{h} \right|^2 \, dm = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x) \left\{ \frac{2}{h} \log \frac{|x - y + h|}{|x - y|} \right\} \, dx \, dy$$

$$= \|K_h * g\|_{L^2}^2$$

where $K_h(x) = \frac{1}{h} K(x/h)$ and $K(x) = 2 \log \frac{|1 + x|}{2}$. 

18
By Plancherel’s formula and the properties of the Fourier transform
\[ \|K_h * g\|_2 \leq \|\hat{K}_h\|_\infty \|\hat{g}\|_2 = \|\hat{K}_h\|_\infty \]
Since \( \hat{K}_h(\xi) = \hat{K}(h\xi) \), we obtain the uniform bound if \( \|\hat{K}\|_\infty < \infty \). An easy computation shows that \( \hat{K}(\xi) = c \frac{e^{2\pi i \xi} - 1}{|\xi|} \) where \( c \) is a complex constant, therefore \( \hat{K} \in L_\infty(\mathbb{R}) \).

We shall describe now some of the ideas developed by Semmes in [S] to prove Theorem A, and the \( L^2 \) boundedness of the Cauchy integral on chord-arc curves with small constant. They will play a fundamental role in the rest of this section.

Recall from Section 1 that if \( F = C_{\Gamma} f \), is the Cauchy integral of a function \( f \), then the expression \( \mu F' \) is well defined a.a. \( z \in \mathbb{C} \setminus \Gamma \) and it does not contain any distributional terms. Besides the jump of \( F \) across \( \Gamma \) is exactly \( f \).

The approach in [S] was to think of the Cauchy integral on \( \Gamma \) as a solution to a \( \overline{\partial} \)-problem relative to \( \Gamma \) and then, by a change of variables, reduce it to a \( (\overline{\partial} - \mu \partial) \) problem relative to \( \mathbb{R} \).

More precisely, let \( g \) be a function defined on \( \Gamma \) and let \( G = C_{\Gamma} g \). So \( G \) represents a holomorphic function on \( \mathbb{C} \setminus \Gamma \) with jump \( g \) on \( \Gamma \). Suppose that \( \rho \) is a quasiconformal mapping with \( \rho(\mathbb{R}) = \Gamma \), then \( f = g \circ \rho \) is now a function defined on \( \mathbb{R} \). Furthermore \( F(z) = G \circ \rho \) satisfies that \( \overline{\partial} F - \mu \partial F = 0 \) on \( \mathbb{C} \setminus \mathbb{R} \) and that its jump on \( \mathbb{R} \) is \( f \).

Consider now the function \( H = F - C_f \) defined on \( \mathbb{C} \setminus \mathbb{R} \). Since \( C_f \) is holomorphic off \( \mathbb{R} \) and its on jump on \( \mathbb{R} \) is \( f \) as well, the function \( H \) has no jump across \( \mathbb{R} \), and satisfies the equation
\[ \overline{\partial} H - \mu \partial H = \mu C'_f. \]
Because \( H \) has no distributional terms, one can think of this equation as holding on \( \mathbb{C} \) when integrated in the sense of distributions.

The problem of studying the \( L^2 \) boundedness of the Cauchy integral on \( \Gamma \) can be transferred in this way into a problem on the boundary values of \( F \) on \( \mathbb{R} \), or equivalently on finding estimates on the boundary values of \( H = (\overline{\partial} - \mu \partial)^{-1}(\mu C'_f) \) for appropriate dilatations \( \mu \). As we will show next, this problem is closely related to the invertibility of the operator \( (I - \mu S) \) on \( L^2 \left( \frac{dm}{|y|} \right) \).
Proposition 1. Let \( \mu \in L^\infty(\mathbb{C}) \), \( \|\mu\|_\infty < 1 \), and \( \nu = |\mu|^2/|y| \) \( dm \) a Carleson measure. If the operator \( (I - \mu S): L^2 \left( \frac{dm}{|y|} \right) \rightarrow L^2 \left( \frac{dm}{|y|} \right) \) is invertible then the following holds:

If \( f \in L^2(\mathbb{R}) \) and \( H \in W^{1,2}_{\text{loc}}(\mathbb{C}) \) satisfies:

\[
\overline{\partial} H - \mu \partial H = \mu C_f' \quad \text{a.a. } z \in \mathbb{C}
\] 

then the boundary values \( H \big|_\mathbb{R} \) belong to \( L^2(\mathbb{R}) \) and \( \| H \big|_\mathbb{R} \|_2 \leq c \| f \|_2 \) for some constant \( c > 0 \).

Proof. By standard density arguments, we can assume that the function \( f \) in (4.2) belongs to the following family of functions in \( L^2(\mathbb{R}) \)

\[
\mathcal{F} = \{ f \in L^2(\mathbb{R}); C'_f \in L^{p_0}(\mathbb{C}) \text{ for some } 2 < p_0 \leq 1 + \frac{1}{\|\mu\|_\infty} \},
\]

where \( C'_f(z) \) denotes the derivative of the holomorphic function \( C_f(z) \) defined off \( \mathbb{R} \), with no distributional term involved.

We know that \( (I - \mu S)^{-1}: L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C}) \) when \( 2 \leq p \leq 1 + \frac{1}{\|\mu\|_\infty} \) [AIS]. Therefore, if \( f \in \mathcal{F} \) then \( \overline{\partial} H = (I - \mu S)^{-1}(\mu C'_f) \in L^{p_0}(\mathbb{C}) \). Note also that, since \( C'_f \in L^{p_0}(\mathbb{C}) \)

\[
\mu(I - S\mu)^{-1}C'_f = (I - \mu S)^{-1}\mu C'_f = \overline{\partial} H.
\] 

In particular, \( \overline{\partial} H \) has compact support since \( \mu \) does.

To estimate the \( L^2 \) norm of the boundary values of the function \( H \) we use a duality argument, that is

\[
\| H \big|_\mathbb{R} \|_2 = \sup \left\{ \left| \int_\mathbb{R} H(x) h(x) \, dx \right| ; h \in L^2(\mathbb{R}); \| h \|_2 = 1 \text{ and } h \text{ has compact support} \right\}.
\]

Hence, by Lemma 4 and (4.3), we get

\[
\int_\mathbb{R} H(x) h(x) \, dx = 2i \int_\mathbb{C} \overline{\partial} H(z) C_h(z) \, dm(z)
\]

\[
= 2i \int_\mathbb{C} (I - \mu S)^{-1}(\mu C'_f)(z) C_h(z) \, dm(z)
\]

\[
= 2i \int_\mathbb{C} \mu(z)(I - S\mu)^{-1}(C'_f(z) C_h(z) \, dm(z)
\]

\[
= 2i \int_\mathbb{C} C'_f(z)(I - \mu S)^{-1}(\mu C_h(z) \, dm(z).
\]
Since $|\mu|/|y|$ is a Carleson measure, and the Cauchy integral is bounded on $L^2(\mathbb{R})$, 
\[
\int_{C} \left| \frac{\mu(z)}{|y|} \right|^2 |C_h(z)|^2 \, dm(z) \leq c \int_{\mathbb{R}} |C_h(x)|^2 \, dx \leq c \|h\|_{L^2(\mathbb{R})}^2 = c < \infty
\]

that is, $\mu C_h \in L^2\left(\frac{dn}{|y|}\right)$ with norm depending on $\|\nu\|_C$.

Applying the Cauchy-Schwarz inequality and the assumption on the invertibility of the operator $(I - \mu S)$ on $L^2\left(\frac{dn}{|y|}\right)$, we obtain
\[
\left| \int_{\mathbb{R}} H(x) h(x) \, dx \right| \lesssim \left( \int_{C} |C^\prime_f(z)|^2 |y| \, dm(z) \right)^{1/2} \left( \int_{C} \left| (I - \mu S)^{-1}(\mu C_h)(z) \right|^2 \, dm(z) \right)^{1/2}
\]
\[
\lesssim \left( \int_{C} |y| |C^\prime_f(z)|^2 \, dm(z) \right)^{1/2} \left( \int_{C} \left| \frac{\mu(z)}{|y|} \right|^2 |C_h(z)|^2 \, dm(z) \right)^{1/2}
\]
\[
\lesssim \|C_f(x)\|_2 \lesssim \|f\|_2.
\]

with comparison constants depending on $\|\nu\|_C$ and on the norm of the operator $(I - \mu S)^{-1}$.

The last two inequalities follow from Green’s formula and the boundedness of the Cauchy integral on $L^2(\mathbb{R})$, ending the proof of Proposition 1.

We are ready now to present the proof of the remaining implication in Theorem 2.

**Proof of (2) ⇒ (1) in Theorem 2**: Recall from Section 1 that if $\rho$ is the solution to the Beltrami equation $\overline{\partial} \rho - \mu \partial \rho = 0$, then
\[
\overline{\partial} \rho = (I - \mu S)^{-1}(\mu).
\]

Since $\mu$ has compact support and $|\mu|^2/|y|$ is a Carleson measure, the function $\mu$ belongs to the space $L^2\left(\frac{dn}{|y|}\right)$. The assumption on the invertibility of the operator $(I - \mu S)$ on $L^2\left(\frac{dn}{|y|}\right)$, yields that $\overline{\partial} \rho \in L^2\left(\frac{dn}{|y|}\right)$. Therefore, by Theorem 3, we know that the quasicircle $\Gamma = \rho(\mathbb{R})$ is rectifiable.

To show that $\Gamma$ is chord-arc, we will apply Proposition 1. The argument that follows is exactly the same as the one given by Semmes in [5]. For the sake of completeness, let us recall the main points in his proof.
Let \( f \in L^2(\mathbb{R}) \). Set \( F = C_\Gamma(f \circ \rho^{-1}) \circ \rho \) and as before, define the difference \( H = F - C_f \). Then \( H \) has no jump across \( \mathbb{R} \), and satisfies
\[
\overline{\partial} H - \mu \partial H = \mu C'_f.
\]
The boundary values of \( F \) are given by:
\[
F(x) = \pm \frac{1}{2} f(x) + \frac{1}{2\pi i} \text{P.V.} \int_\mathbb{R} \frac{f(y)}{\rho(y) - \rho(x)} d\rho(y), \quad x \in \mathbb{R}.
\]
Because \( \Gamma = \rho(\mathbb{R}) \) is rectifiable, \( \rho'(y) \) exists a.a. \( y \in \mathbb{R} \) and \( d\rho(y) = \rho'(y) dy \). The \( L^2 \)-estimate on \( H|_{\mathbb{R}} \) given by Proposition 1, and the boundedness of \( C_f \) on \( L^2(\mathbb{R}) \) imply that
\[
Kf(x) = \text{P.V.} \int_\mathbb{R} \frac{f(y)}{\rho(y) - \rho(x)} \rho'(y) dy, \quad x \in \mathbb{R},
\]
defines a bounded operator on \( L^2(\mathbb{R}) \). We use this fact to get estimates on \( |\rho'| \).

Fix an interval \( I = (a, b) \) on \( \mathbb{R} \) and set \( f = \bar{\rho}' X_I \). Then evaluate \( Kf \) on an interval \( J = (a', b') \) far away from \( I \) so that \( \rho(y) - \rho(x) \) is roughly constant if \( y \in J \) and \( x \in I \), but not too far away so that \( |\rho(y) - \rho(x)| \) is not too small. This can be done because \( \rho \) is quasiconformal. Using \( \|Kf\|_2 \leq c \|f\|_2 \) we will get estimates on \( \rho' \).

More precisely, fix \( I = (a, b) \). Then, by the distortion theorem for quasiconformal mappings, there are constants \( C_1, C_2 > 0 \), depending only on \( \|\mu\|_\infty \) and an interval \( J \), such that:

(a) \( \frac{1}{C_1} |J| \leq |I| \leq C_1 |J| \).

(b) \( \frac{1}{C_2} |I| \leq \text{dist}(I, J) \leq C_2 |I| \).

(c) \( |e^{i\alpha} - \frac{\rho(y) - \rho(x)}{|\rho(y) - \rho(x)|}| \leq \frac{1}{10} \) for some \( \alpha \in \mathbb{R} \).

Denote by \( c \) any constant depending on \( \|\mu\|_\infty, \|\mu\|_C \) and on the norm of \( (I - \mu S)^{-1} \).
Define $f \in L^2(\mathbb{R})$ by $f = \overline{\rho} X_{(a,b)}$. Then
\[
\int_I |\rho'(x)|^2 \, dx \geq c \|Kf\|_2^2
\]
\[
\geq c \int_J \left( \int_I \frac{\overline{\rho(y)} - \rho(x)}{\rho(y) - \rho(x)} \rho'(y) \, dy \right)^2 \, dx
\]
\[
\geq c \int_J \left( \int_I \frac{|\rho'(y)|^2}{|\rho(y) - \rho(x)|} \, dy \right)^2 \, dx
\]
\[
\geq c \frac{|a-b|}{|\rho(a) - \rho(b)|^2} \left( \int_I |\rho'(y)|^2 \, dy \right)^2.
\]

The third inequality is a consequence of (c). For the last one we have applied once more the quasiconformal distortion theorem: since $\text{dist}(I, J) \simeq |I| \simeq |J|$, we deduce that $\text{dist}(\rho(I), \rho(J)) \simeq \text{diam}(\rho(I))$. Thus,
\[
\int_I |\rho'(y)|^2 \, dy \leq c \frac{|\rho(a) - \rho(b)|^2}{|a-b|}.
\]

Therefore, by the Cauchy-Schwarz inequality we get
\[
\int_I |\rho'(y)| \, dy \leq c \text{diam}(\rho(I))
\]
for any interval $I \subset \mathbb{R}$, which is precisely the chord-arc condition on the curve $\Gamma$.

\[
\Box
\]

5 Applications and remarks

The results we have presented characterize the geometry of the curve and not its parametrization, in fact there are many parametrizations of chord-arc curves which are not bilipschitz. On the other hand, the proof of $(1) \Rightarrow (2)$ in Theorem 2 showed that if $\rho$ is bilipschitz and $|\mu_\rho|^2 / |y|$ is a Carleson measure then the operator $(I - \mu S)$ is invertible in $L^2 \left( \frac{dm}{|y|} \right)$. The next result shows that the converse does not hold.

**Proposition 2.** There exists a quasiconformal mapping $\rho$ which is not bilipschitz, and $\mu = \mu_\rho$ satisfies that $|\mu|^2 / |y|$ is a Carleson measure and the operator $(I - \mu S)$ is invertible in $L^2 \left( \frac{dm}{|y|} \right)$. 

23
Proof. Consider the following function \( f : \mathbb{R} \to \mathbb{R} \)

\[
f(x) = \begin{cases} x^{1/K} & x \geq 0 \\ -(x)^{1/K} & x \leq 0 \end{cases} \quad 1 < K < 2.
\]

Clearly \( f \) is not bilipschitz, but we will show that it exists \( \rho \), a quasiconformal extension of \( f \), verifying the required properties, that is if \( \mu = \mu_\rho \)

(i) \( |\mu|^2/|y| \) is a Carleson measure.

(ii) \((I - \mu S)\) is invertible in \( L^2 \left( \frac{dm}{|y|} \right) \).

To describe such an extension we define the sets: \( E_0 = \{ z \in \mathbb{C}; |\arg z| < \pi/4 \} \) and \( E_1 = -E_0 \). Set

\[
\rho(z) = \begin{cases} z^{1/K} & z \in E_0 \\ -(z)^{1/K} & z \in E_1 \end{cases}
\]

and extend \( \rho \) to \( \mathbb{C} \) so that for all \( z \in \mathbb{C} \), \( |\rho(z)| = |z|^{1/K} \) and, the argument of \( \rho(z) \), \( \arg \rho(z) \) is a piecewise linear function. Then \( \rho(z) : \mathbb{C} \to \mathbb{C} \) represents a homeomorphic extension of \( f \), i.e. \( \rho|_R = f \), which is not bilipschitz.

Moreover \( \mu = \mu_\rho \) is supported on the set \( \mathbb{C} \setminus (E_0 \cup E_1) \) and, as a small calculation shows, \( |\mu(z)| = c(K) \chi_{\mathbb{C} \setminus (E_0 \cup E_1)} \), \( |c(K)| < 1 \), where \( c(K) \) is a constant only depending on \( K \). Therefore \( \rho \) is quasiconformal and

\[
\frac{|\mu|^2}{|y|} \, dm \simeq \frac{1}{|z|} \chi_{\mathbb{C} \setminus (E_0 \cup E_1)} \, dm
\]

which is a Carleson measure w.r. to \( \mathbb{R} \).

It remains to prove that \((I - \mu S)\) is invertible in \( L^2 \left( \frac{dm}{|y|} \right) \), i.e. \((3.1)\). For that, set

\[
(I - \mu S)h = \Phi
\]

and proceed as in the proof of \((1) \Rightarrow (2)\) in Theorem \(2\) in Section \(3\). Recall that the key there was to perform a change of variables of the form \( u = H \circ \rho^{-1} \) where \( H(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(w)}{w - z} \, dm(w) \). Then, invertibility is proved if we can show \((3.3)\), that is:

\[
\int_{\mathbb{C}} \frac{|\partial u \circ \rho|^2}{|y|} |\mu|^2 |\partial \rho|^2 \, dm \leq c \int_{\mathbb{C}} \frac{|\partial \mu \circ \rho|^2}{|y|} |\partial \rho|^2 \, dm.
\]
In our case, the integral on the left is comparable to:

\[
\int_{\mathbb{C}\setminus(E_0\cup E_1)} |\partial u \circ \rho|^2 |\partial \rho| \frac{2 \, dm(z)}{|z|} \approx \int_{\mathbb{C}\setminus\Phi(E_0\cup E_1)} |\partial u(z)|^2 \frac{dm(z)}{|z|^K} \\
\leq \int_{\mathbb{C}} |\partial u(z)|^2 \frac{dm(z)}{|z|^K}. \tag{5.1}
\]

Since \(\|S\|_{L^2(\mathbb{C}) \to L^2(\mathbb{C})} = 1\) and \(\frac{1}{|z|^K}\) is an \(A_2\)-weight for \(1 < K < 2\), we deduce that (5.1) is bounded by

\[
c(K) \int_{\mathbb{C}} |\partial u(z)|^2 \frac{dm(z)}{|z|^K} = c(K) \int_{\mathbb{C}} |\partial u \circ \rho|^2 |\partial \rho|^2 \frac{dm(z)}{|z|} \\
\lesssim \int_{\mathbb{C}} |\partial u \circ \rho|^2 |\partial \rho|^2 \frac{dm(z)}{|y|},
\]

as we needed to show. \(\square\)

We end this section by recovering the theorem on the \(L^2\) boundedness of the Cauchy integral on chord-arc curves [D]. We follow the ideas in [S] where the result is proved in the small constant case. Let us recall the precise statement

**Corollary 3.** If \(\Gamma\) is a chord-arc curve, the Cauchy integral on \(\Gamma\) is a bounded operator in \(L^2(\Gamma)\).

**Proof.** Let \(\Gamma\) be a chord-arc curve, and \(\rho\) the quasiconformal mapping associated to \(\Gamma\) in Theorem [2] which in fact, it is the one given by Lemma [2]. Therefore, \(\rho\) is bilipschitz and \((I - \mu_\rho S)\) is invertible in \(L^2(\frac{dm}{|y|})\).

Given \(g \in L^2(\Gamma)\), let \(G(z) = C_\Gamma(g)\). Since bilipschitz mappings preserve \(L^2\), the pullback function \(f = g \circ \rho\) belongs to \(L^2(\mathbb{R})\). As it was explained in Section [4] if \(F(z) = G \circ \rho\), the function \(H = F - C_f\) satisfies

\[
\overline{\partial}H - \mu \partial H = \mu C_f'
\]

By Proposition [1], the boundary values \(H \big|_\mathbb{R} \in L^2(\mathbb{R})\) with \(\|H\|_{L^2(\mathbb{R})} \leq c\|f\|_2\). Thus, since \(C_f\) is bounded on \(L^2(\mathbb{R})\)

\[
\|F_\pm\|_{L^2(\mathbb{R})} \leq c\|f\|_{L^2(\mathbb{R})}.
\]

Again, since \(\rho\) is bilipschitz, we obtain that

\[
\|G_\pm\|_{L^2(\Gamma)} \leq c\|g\|_{L^2(\Gamma)}
\]

as we wanted to prove. \(\square\)
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