Green’s function of a finite chain and the discrete Fourier transform

S. Cojocaru

Dipartimento di Fisica “E.R. Caianiello”,
Universit`a degli Studi di Salerno, Italy
Institute of Applied Physics, Chişinău, Moldova

Abstract

A new expression for the Green’s function of a finite one-dimensional lattice with nearest neighbor interaction is derived via discrete Fourier transform. Solution of the Heisenberg spin chain with periodic and open boundary conditions is considered as an example. Comparison to Bethe ansatz clarifies the relation between the two approaches.
I. INTRODUCTION

Lattice Green functions have a broad spectrum of applications in physics and allow to study dynamic and statistical properties. [1, 2, 3, 4] Usually the lattice is assumed to be infinite while distances remain finite and discrete. [5, 6, 7] Often the range of relevant excitations allows to consider the system as continuous and apply field theoretic methods. Accordingly, the integral Fourier transform or infinite Fourier series are commonly used as powerful tools for such problems. It therefore appears natural to employ the discrete, or finite, Fourier transform (DFT) when the finite size is relevant. However, DFT has mainly served for numerical analysis or in fast Fourier transform calculations [9] and much less as an analytic method. Most of the known exact results on finite systems have been obtained in the Bethe ansatz approach [10] (see, e.g., [11, 12] for recent review) that is based on a different strategy. In [3] the Bethe ansatz solution is compared to the solution by Fourier series in the thermodynamic limit of the periodic quantum spin chain. However it is noted that the relation between them remains obscure. For instance, the bound two-magnon excitation is characterized by the same eigenenergy, but only one eigenstate is found in the Fourier approach, instead of the two solutions in Bethe ansatz. This discrepancy has been resolved in [13] by an approximate treatment of finite size corrections. In the present paper it is shown that the DFT approach recovers the exact solution. The derivation presents interest in itself, in view of further extensions. As another example, some results on open boundary conditions obtained by Bethe ansatz [14] are recovered.

The finite lattice sums are first expressed in terms of infinite Fourier series. For the nearest neighbor interaction the series represent an expansion in modified Bessel functions. Its Laplace transform results in analytic expressions for the finite lattice Green functions and allows to determine the corresponding discrete Fourier transform.

The standard definitions of the Fourier series for a function periodically continued from the interval $x \in [0; 2\pi)$ are

$$f(z; x) = \sum_{n=-\infty}^{\infty} c_n \exp (inx),$$  \hspace{1cm} (1)

$$c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(z; y) \exp (-iny) \, dy,$$

where the parameters $z$ and $\delta$ (below) take real values. The Fourier expansion of the lattice
sum
\[ \frac{1}{N} \sum_{m=0}^{N-1} f \left( z; \delta + \frac{2\pi m}{N} \right) \]
can then be written as
\[ \sum_{n=-\infty}^{\infty} c_n \exp \left( in\delta \right) \left[ \frac{1}{N} \sum_{m=0}^{N-1} \exp \left( i2\pi m \frac{n}{N} \right) \right] = \sum_{k=-(N-1)}^{\infty} c_{kN} \exp \left( i\delta kN \right). \]

It has been taken into account that the sum in the square brackets above gives either 1 or 0, provided \( n/N \) is an integer (\( k \)) or not. We note that the same property is essential for the definition of the DFT of a function \( a(X) \) defined on a set of \( N \) consecutive integers \( (0, 1, \ldots, N-1) \):
\[ a(X) = \frac{1}{N} \sum_{m=0}^{N-1} b \left( q = \frac{2\pi}{N} m \right) \exp \left( iqX \right). \] (2)

Then, with the expression of the Fourier coefficients \( c \) in (1), our lattice sum becomes
\[ \frac{1}{N} \sum_{m=0}^{N-1} f \left( z; \delta + \frac{2\pi m}{N} \right) = \sum_{k=-(N-1)}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f \left( z; y \right) \exp \left( ikN \left( \delta - y \right) \right) dy. \] (3)

Now the problem of finite summation has been reduced to integration and series summation.

To motivate a particular choice of \( f \left( z; y \right) \) in (3) we note that dynamics of different models on a finite d-dimensional lattice can be described in terms of Green functions of a generic form
\[ G \left( p; X, \ldots, Y \right) = \frac{1}{N^d} \sum_{Q_1} \ldots \sum_{Q_d} \frac{\exp \left( iQ_1X_1 + \ldots + iQ_dY_d \right)}{p - \varepsilon \left( Q_d, \ldots, Q_d \right)}, \] (4)

where \( p \) is the spectral parameter, \( \varepsilon \) is the lattice dispersion and \( (X_1, \ldots, X_d) \) are the discrete coordinates on a periodic lattice \( (X \in [0, 1, \ldots, N-1]) \). The conjugate "quasimomenta" \( (Q_1, \ldots, Q_d) \) defined as
\[ Q_i = \frac{2\pi}{N} m_i + \frac{\Delta}{N}; \quad m_i = 0, \ldots, N-1, \] (5)

are already present in the l.h.s. of (3) and their definition is explained in the example below.

For simple lattices the dispersion is a sum of cosines
\[ \varepsilon = \alpha_1 \cos Q_1 + \ldots + \alpha_d \cos Q_d. \] (6)

Therefore the Green function can be related to a Laplace transform of
\[ f \left( z; y \right) = \exp \left( z \left( \alpha_1 \cos y_1 + \ldots + \alpha_d \cos y_d \right) \right). \] (7)
II. LATTICE GREEN’S FUNCTION

Let us consider the consequences of (3) for the choice (7) of \( f(z; y) \) corresponding to the chain with nearest neighbor interaction. Integration gives the modified Bessel function \( I_{kN}(z) \) and \( (3) \) becomes

\[
\frac{1}{N} \sum_{m=0}^{N-1} \exp(z \cos Q_m) = \sum_{k=-\infty}^{\infty} I_{kN}(z) \exp(ik\Delta) = I_0(z) + 2 \sum_{k=0}^{\infty} I_{kN}(z) \cos(k\Delta). \quad (8)
\]

The parameters \( z \) and \( \Delta \) can now be continued to complex values, e.g., rewriting Eq. (8) in an equivalent form, we obtain a generalization of the Jacobi expansion \( [15] \):

\[
\frac{1}{N} \sum_{m=0}^{N-1} \exp \left( \frac{z}{2} \left( w \exp \left( \frac{2\pi m}{N} \right) + \frac{1}{w} \exp \left( -i \frac{2\pi m}{N} \right) \right) \right) = \sum_{k=-\infty}^{\infty} I_{kN}(z) \exp \left( \frac{\pi k}{2} \right) \exp(i\theta k). \quad (9)
\]

One can check that the expressions 9.6.33-9.6.40 in \( [16] \) are particular cases of \( (8) \) or \( (9) \) for \( N = 1, 2 \). For instance, taking \( \Delta = \theta + \frac{\pi}{2} \) in \( (8) \) one finds

\[
\frac{1}{N} \sum_{m=0}^{N-1} \exp \left( z \cos (\frac{\theta}{N} + \frac{\pi}{2} + 2\pi \frac{m}{N}) \right) = \sum_{k=-\infty}^{\infty} I_{kN}(z) \exp \left( \frac{i\pi k}{2} \right) \exp(i\theta k) \quad (10)
\]

\[
= I_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k I_{2kN}(z) \cos(2k\theta) + 2 \sum_{k=0}^{\infty} (-1)^k I_{(2k+1)N}(z) \sin((2k+1)\theta). 
\]

It is also not difficult to obtain a further generalization of these results by including the space dependent factor of the Green function \( (4) \) and a dispersion of the form:

\[
\varepsilon = b \cos Q + c \sin Q, \quad (11)
\]

so that the propagator can be written as

\[
G(p; X) = \frac{1}{N} \sum_{m=0}^{N-1} \frac{\exp(iQX)}{p - \cos(Q + \eta)}. \quad (12)
\]

It corresponds to the Laplace transform of the lattice sum

\[
L = \frac{1}{N} \sum_{m} \exp \left( z \cos \left( \frac{\Delta + 2\pi m}{N} + \eta \right) \right) \exp \left( i \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} \right) X \right). \quad (13)
\]

By denoting \( \xi \equiv \Delta + \eta N \) we have

\[
L = \exp(-i\eta X) \frac{1}{N} \sum_{m=0}^{N-1} \exp \left( z \cos \left( \frac{\xi}{N} + 2\pi \frac{m}{N} \right) \right) \exp \left( i \left( \frac{\xi}{N} + 2\pi \frac{m}{N} \right) X \right). \quad (14)
\]
Then substitute (14) into (3)

\[ L = \exp(-i\eta X) \sum_{k=-\infty}^{\infty} \exp(ik\xi) \frac{1}{2\pi} \int_0^{2\pi} \exp(z \cos y) \exp(-i(kN - X)y) \, dy, \]

and finally obtain

\[ \frac{1}{N} \sum_{m=0}^{N-1} \exp \left( z \cos \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} + \eta \right) \right) \exp \left( i \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} \right) X \right) = \exp(-i\eta X) \sum_{k=-\infty}^{\infty} I_{kN+X}(z) \exp(-ik(\Delta + \eta N)). \]  

(15)

(16)

The above expressions demonstrate a direct relation of the generalized Jacobi expansion to the finite lattice problem. However their main advantage becomes clear by realizing that the increasing orders of the Bessel function in these series correspond to the consecutive terms of the asymptotic \(N\)-expansion of the lattice propagators. I.e., the term with \(k = 0\) describes the thermodynamic limit \((N \rightarrow \infty)\) and the first finite-\(N\) correction is contained in the term with \(k = 1\). One can also evaluate the convergence of these series, that depends on lattice and on the spectral parameter \(p\). We will provide specific examples and explain the choice of "quasimomentum" \(Q\) and its apparent conflict with periodicity of the lattice.

The lattice Green function \((12)\) can now be obtained from the Laplace transform of \((16)\) with respect to \(z\) \(\int_0^{\infty} \exp(-pz) I_n(z) \, dz = \frac{\exp(-|n|v)}{\sinh v},\)

where \(n\) is an integer and we have introduced the notation \(p \equiv \cosh v\) (convergence of the Laplace transform requires \(\text{Re} \, p > 1\)). Thus

\[ \frac{\sinh v}{N} \sum_{m=0}^{N-1} \exp \left( i \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} \right) X \right) \cosh v - \cos \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} + \eta \right) \]

\[ = \sum_{k=-\infty}^{\infty} \exp(-v|kN+X|) \exp(-ik(\Delta + \eta N) - i\eta X), \]

leads to the following expression for the propagator \((12)\):

\[ \frac{1}{N} \sum_{m=0}^{N-1} \frac{\exp \left( i \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} \right) X \right)}{\cosh v - \cos \left( \frac{\Delta}{N} + 2\pi \frac{m}{N} + \eta \right)} = \frac{\exp(i\eta(N/2 - X_M) + i(M + 1/2)\Delta)}{2\sinh v} \]

\[ \times \left( \frac{\exp(v(N/2 - X_M))}{\sinh((vN + i(\Delta + \eta N))/2)} + \frac{\exp(-v(N/2 - X_M))}{\sinh((vN - i(\Delta + \eta N))/2)} \right). \]

(17)
The integer $M$ in the (17) defines a translation relating an arbitrary $X$ to $X_M$ from the main interval:

$$X_M = 0, 1, \ldots N - 1,$$

$$X = X_M + MN.$$  \hspace{1cm} (18)

We note the transformation properties following from the above definition,

$$X \rightarrow N + X : \quad X_M \rightarrow X_M, \quad M \rightarrow M + 1;$$
$$X \rightarrow -X : \quad X_M \rightarrow N - X_M, \quad M \rightarrow -(M + 1);$$
$$X \rightarrow N - X : \quad X_M \rightarrow N - X_M, \quad M \rightarrow -M.$$  \hspace{1cm} (19)

After the r.h.s. of (17) has been found, it is now easy to verify that the inverse DFT (i.e. taking the summation in $X$ ) indeed reproduces the function in the l.h.s. With (19) one also finds another useful form of DFT (17):

$$2 \sinh v \frac{\sum_{m=0}^{N-1} \cos \left( \left( \frac{N}{N} + 2\pi \frac{m}{N} \right) X \right)}{N} = \frac{\cosh \left( (v + i\eta) \left( \frac{N}{2} - X_M \right) + i \left( M + \frac{1}{2} \right) \Delta \right)}{\sinh \left( \frac{N}{2} (v + i\eta) + i\frac{\Delta}{2} \right)}$$
$$+ \frac{\cosh \left( (v - i\eta) \left( \frac{N}{2} - X_M \right) - i \left( M + \frac{1}{2} \right) \Delta \right)}{\sinh \left( \frac{N}{2} (v - i\eta) - i\frac{\Delta}{2} \right)}.$$

$$2i \sinh v \frac{\sum_{m=0}^{N-1} \sin \left( \left( \frac{N}{N} + 2\pi \frac{m}{N} \right) X \right)}{N} = \frac{\sinh \left( (v + i\eta) \left( \frac{N}{2} - X_M \right) + i \left( M + \frac{1}{2} \right) \Delta \right)}{\sinh \left( (v + i\eta) \frac{N}{2} + i\Delta/2 \right)}$$
$$- \frac{\sinh \left( (v - i\eta) \left( \frac{N}{2} - X_M \right) - i \left( M + \frac{1}{2} \right) \Delta \right)}{\sinh \left( (v - i\eta) \frac{N}{2} - i\Delta/2 \right)}.$$  \hspace{1cm} (20)

The condition $\cosh v \geq 1$ in our derivation was due to the use of the Laplace transform. It is however important to mention that the above expressions can be analytically continued to arbitrary complex values of the parameters $(v, \eta, \Delta)$, since the resonance conditions (zeros of the denominator in the l.h.s.) correspond to simple poles. For instance, a physically important possibility is the region $\text{Re } p < 1 \ (p \neq \cos Q)$, where respective solutions of the Schrödinger equation with $\text{Im } p = 0$ are known as scattered states.

III. THE HEISENBERG RING

The model of a cyclic chain of $N$ quantum spins $S = 1/2$ with ferromagnetic nearest neighbor interaction was the first to be solved by Bethe ansatz [10] and it offers a clear
test for the above formal results. The two-magnon subspace of the eigenfunctions of the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i \cdot S_j,$$

contains the main features of the interaction problem and is described by the amplitude of two spins flipped on the sites \( n_1 \) and \( n_2 \):

$$|\psi\rangle = \sum_{1 \leq n_1 < n_2 \leq N} A(n_1, n_2) S_{n_1}^- S_{n_2}^- |0\rangle.$$

Within the Bethe ansatz approach a phase shift parameter is introduced to account for nearest neighbor interaction in terms of a boundary condition problem. I.e., by matching this free parameter in the noninteracting particle form of the ansatz wave function at large separation to satisfy the Schrodinger equation when the two overturned spins are nearest neighbors. \[11\] An alternative approach is based on a representation of the Schrodinger equation in a form consisting of two contributions: free motion and interaction terms. \[3\]

$$[E - 2J] a(X) + J \cos \left( \frac{P}{2} \right) (a(X + 1) + a(X - 1)) =$$

$$J \left[ \cos \left( \frac{PX}{2} \right) a(0) - a(X) \right] (\delta_{X,1} + \delta_{X,N-1}),$$

where \( a(X) \) represents the amplitude of the relative position \( X \) of the flipped spins, \( E \) is the excitation energy over the ground state with all spins parallel and \( P \) is the total momentum associated to translation symmetry.

$$A(n_1, n_2) = \frac{\exp(iPR)}{\sqrt{N}} a(X);$$

$$R = \frac{n_1 + n_2}{2}; \quad X = (n_2 - n_1) = 1,...,N - 1.$$

The split form of \( [22] \) is convenient to set up a perturbation theory and is widely used in condensed matter, when the exact solution is not known. Then the solution is sought in terms of Fourier transform. The negative values of \( X \) correspond to transposition of the two overturned spins, i.e., to the same state. At the same time it is a common practice to chose the interval \( X \) symmetrically with respect to zero \((-N/2, (N - 1)/2\) and interpreting negative values as "left neighbor" and positive values as "right neighbor". The respective amplitudes do not necessarily coincide. Therefore here the definition in \( [18] \) is preferred to avoid this ambiguity and for a straightforward comparison to Bethe ansatz that follows the
same counting convention. For instance, if the ”right neighbor” is defined by $X$, then the ”left” one corresponds to $N - X$.

First we note that quantization of the total momentum $P = 2\pi k/N, k = 0, 1, .., N - 1$ in (23) follows from translation of the two spin complex as a whole $A(n_1, n_2) = A(n_1 + N, n_2 + N)$. Respectively, the quasimomenta $Q$ in (12), that are conjugate to relative distance $X$, should be determined by the boundary conditions on the displacement of separate spin flips. Thus the amplitude of relative motion in (23) should satisfy the condition

$$a(X) = \exp(i\pi k) a(X + N),$$

(25)

following from the translation $A(n_1, n_2) = A(n_1, n_2 + N)$. According to (25) the relative amplitude is periodic either on a length of the chain or on a double length, $2N$, depending on the parity of the total momentum quantum number $k = PN/2\pi$. In the latter case the amplitude changes sign after completing the first cycle and may be denoted as antisymmetric, $a$, to distinguish from the former, symmetric modes $s$, which completes the period after the first cycle. Together with the property $a(X) = a(-X)$ due to spin transposition symmetry, the above relations define the continuation of the solutions of (22) from the main interval (24) to arbitrary values of $X$. By applying these relations to the Fourier expansion

$$a(X) = \frac{1}{N} \sum_Q b(Q) \cos(QX),$$

(26)

we obtain the condition on $Q$

$$\exp \left( iN \left( \frac{P}{2} \pm Q \right) \right) = 1.$$

It determines the two sequences corresponding to the two types of modes

$$Q_s = \frac{2\pi l}{N}; \ l = 0, 1, ..., N - 1,$$

$$Q_a = \frac{2\pi l}{N} + \frac{\pi}{N}; \ l = 0, 1, ..., N - 1.$$

(27)

These quasimomenta correspond to $\eta = 0$ and $\Delta_s = 0, \Delta_a = \pi$ in the general expressions (5). The state counting convention for a one-dimensional lattice is to require that the second argument in the amplitude $A(n_1, n_2)$ is larger then the first. This leads to an additional relation $A(n_1, n_2) = A(n_2, n_1 + N)$ imposing a constraint on the amplitude for the main interval

$$a(X) = \exp(i\pi k) a(N - X).$$

(28)
For instance,
\[ a_a (X) = -a_a (N - X) = -a_a (N + X) = a_a (-X). \]  

(29)

One can check that (28) is automatically satisfied for the amplitude (26) and (27).

Solving (22) for the \( b(Q) \) in (26), we find
\[ b(Q) = C (P) \frac{\cos (Q)}{\cosh v - \cos Q}. \]

where the constant \( C (P) \) is
\[ C = -\frac{1}{N} \sum_{Q'} b(Q') \left[ \cos \left( \frac{P}{2} \right) - \cos Q' \right]. \]

The parameter \( v \) is related to the eigenenergy \( E = 2J (1 - \cos (P/2) \cosh v) \) and is determined by the compatibility equation
\[ 1 = \frac{1}{N \cos \left( \frac{P}{2} \right)} \sum_{Q'} \frac{\cos (Q') \left[ \cos Q' - \cos \left( \frac{P}{2} \right) \right]}{\cosh v - \cos Q'}. \]

From the DFT formulas (17), (20) and (27) we find the eigenenergy equation for each type of eigenstate. For instance,
\[ \frac{1}{N} \sum_{Q_a} \frac{\cos (Q_a)}{\cosh v_a - \cos Q_a} = \frac{1}{2} \frac{\sinh (v_a (N - 2)/2)}{\sinh (v_a) \cosh (v_a N/2)}. \]

After simple algebra we finally obtain
\[ \left( \cosh v_{a,s} - \cos \left( \frac{P}{2} \right) \right) \coth \left( \frac{N v_{a,s} - i \Delta_{a,s}}{2} \right) = \sinh v_{a,s}. \]  

(30)

Eqs. (30) indeed coincide with the Bethe ansatz result [10, 11]. Solutions with real values of \( v \), describing the two bound states mentioned earlier, merge in the thermodynamic limit to: \( v = -\ln (\cos P/2) \), \( E = J \sin^2 (P/2) \).

At any given \( P \) equation (30) has a number \( \sim N \) of solutions with imaginary values of \( v \), or Re \( p < 1 \), which correspond to scattered states.

These solutions are characterized by an oscillating space dependence, as is clearly seen from (20) (see also (31) below), unlike the smoothly decaying dependence of the bound states. At finite \( N \) they form two distinct classes of energy bands, shifted with respect to each other by \( \sim J/N \). All the solutions can be cast in a unified form by using the relation of \( \Delta \) to the total momentum \( (\Delta = P N/2) \):
\[ a (X) \sim \cosh \left( \frac{N}{2} - |X| \right) + i \frac{P N}{4}, \]  

(31)
where \( v = v(P) \) corresponds to a particular solution of (30) at fixed total momentum \( P \).

One can see that the phase shift of \( \pi/2 \) (stemming from \( PN/4 \)) between nearest values of \( P \) survives in the thermodynamic limit. This distinction in symmetry of eigenstates was missed in the previous Fourier series analysis [3, 18] because the effect is contained in the phase shifts of quasimomenta \( Q \) (27) that vanishes in the thermodynamic limit.

IV. OPEN BOUNDARY CONDITIONS

To consider another illustration of the formalism derived above we will include, according to [14], also arbitrary end fields \( (\mu \text{ and } \nu) \) into the anisotropic Hamiltonian on a chain of length \( L \) \( (i, j = 0, ..., L - 1) \). Only a single magnon excitation will be considered, as the interaction is already present due to noncyclic boundary conditions:

\[
H = -\sum_{i=0}^{L-2} \left( \eta S_i^z S_{i+1}^z + \frac{1}{2} \left( S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + \mu S_0^z + \nu S_{L-1}^z \right).
\]

The parameters are dimensionless, i.e., scaled with the exchange constant \( J_x \), and the energy of the state with all spins parallel is \( E_0 = -\eta (L - 1)/2 - (\mu + \nu)/2 \). The Schrödinger equation is represented in the form similar to (22) with the "interaction" on the r.h.s.:

\[
(E - E_0) a(j) - \left[ \eta a(j) - \frac{1}{2} a(j - 1) - \frac{1}{2} a(j + 1) \right]
\]

\[
= \delta_{j=L-1} \left( \frac{1}{2} a(L) - a(L - 1) \left( \frac{1}{2} \eta - \nu \right) \right)
\]

\[
+ \delta_{j=0} \left( \frac{1}{2} a(-1) - a(0) \left( \frac{1}{2} \eta - \mu \right) \right). \tag{32}
\]

As can be seen, the nonphysical amplitudes \( a(-1) \) and \( a(L) \) cancel each other and in the DFT approach are in fact defined by the Eq. (32) and the transformation properties (19). For instance, these amplitudes may differ from Bethe ansatz solution, as is the case of the Heisenberg ring. The two approaches, however, give the same solution for the physical interval. In DFT we need to consider an interval in \( X \) which can be periodically continued. However unlike the cyclic chain, here the last spin in the open chain is not at the same time the "left" nearest neighbor of the first spin, i.e., there can be no periodicity transformation: \( X \to X + L \). Instead, our chain can be viewed as a half of the cyclic chain (ring) of length \( N = 2L \) \( (j = 0, ..., 2L - 1) \) where the two links \( (2L - 1, 0) \) and \( (L - 1, L) \) are broken. With
this definition we can apply the DFT formulas (2) and (17) to solve (32). In this case \( \Delta = \eta = 0 \) and the quasimomentum is obviously defined as \( Q = 2\pi m/N, m = 0, ..., 2L - 1 \).

An immediate consequence of transformation properties (19) is that \( a(-1) = a(1) \), while \( a(L) \) has to be determined from (32). The Fourier amplitude resulting from Schrödinger equation is

\[
b(Q) = \frac{-C_1}{\cosh v - \cos Q} - C_2 \frac{\exp(-iQ(L - 1))}{\cosh v - \cos Q},
\]

where \( \cosh v \equiv \eta - (E - E_0) \) and the two constants are

\[
C_1 = \frac{1}{2} a(-1) - a(0) \left( \frac{1}{2} \eta - \mu \right),
\]

\[
C_2 = \frac{1}{2} a(L) - a(L - 1) \left( \frac{1}{2} \eta - \nu \right).
\]

Substituting the amplitude \( b(Q) \) into the definition of the constants one obtains a linear homogeneous system of equations

\[
C_1 \left(1 + \frac{1}{2N} \sum_{m=0}^{N-1} \frac{\exp(-iq(L - 1)) [\exp(-iq) - (\eta - 2\mu)]}{\cosh v - \cos q}\right)
\]

\[
= -C_2 \frac{1}{2N} \sum_{m=0}^{N-1} \frac{\exp(-iq(L - 1)) [\exp(-iq) - (\eta - 2\mu)]}{\cosh v - \cos q},
\]

\[
C_2 \left(1 + \frac{1}{2N} \sum_{m=0}^{N-1} \frac{\exp(iq(L - 1)) [\exp(iq) - (\eta - 2\nu)]}{\cosh v - \cos q}\right)
\]

\[
= -C_1 \frac{1}{2N} \sum_{m=0}^{N-1} \frac{\exp(iq(L - 1)) [\exp(iq) - (\eta - 2\nu)]}{\cosh v - \cos q},
\]

which defines the eigenenergies and eigenfunctions of the equation (32). From (17) we have

\[
\frac{1}{N} \sum_{m=0}^{N-1} \frac{\exp(-iq)}{\cosh v - \cos q} = \frac{\cosh(v(L - 1))}{\sinh v \sinh(vL)},
\]

\[
\frac{1}{N} \sum_{m=0}^{N-1} \frac{1}{\cosh v - \cos q} = \frac{\cosh(vL)}{\sinh v \sinh(vL)},
\]

\[
\frac{1}{N} \sum_{m=0}^{N-1} \frac{\exp(-iq(L - 1))}{\cosh v - \cos q} = \frac{\cosh(v)}{\sinh v \sinh(vL)},
\]

\[
\frac{1}{N} \sum_{m=0}^{N-1} \frac{\exp(-iq(L - 2))}{\cosh v - \cos q} = \frac{\cosh(2v)}{\sinh v \sinh(vL)}.
\]
With these expressions we obtain from (34) the eigenenergy equation of [14]:

\[
e^{2v(L-1)} = \frac{(e^{-v} - (\eta - 2\mu)) (e^{-v} - (\eta - 2\nu))}{(e^{v} - (\eta - 2\mu)) (e^{v} - (\eta - 2\nu))}.
\]  

(35)

The equation is invariant under \( v \leftrightarrow -v \) or \( \mu \leftrightarrow \nu \). By taking the ratio \( C_1/C_2 \) from (34) one obtains from (17) the space dependence of the amplitude:

\[
a(X) = \gamma \left[ \sinh(v(x+1)) - (\eta - 2\mu) \sinh(vx) \right].
\]  

(36)

The factor \( \gamma = \gamma(v,L) \) can be found from normalization. It is now easy to checked that (36) is equivalent to Bethe ansatz result provided \( v \) satisfies the eigenenergy equation (35).

Interchanging the end fields \( \mu \leftrightarrow \nu \) induces an obvious transformation (up to a phase factor) \( a(X) \rightarrow a(L-1-X) \) which amounts to changing the direction of counting the sites. As noted in, [14] the solution is particularly simple for the condition

\[(\eta - 2\mu)(\eta - 2\nu) = 1,
\]

when one gets \( e^{-2v} \) in the r.h.s. of (35) and consequently:

\[v = i\frac{\pi m}{L}, \quad m = 0, \ldots, 2L - 1.
\]

The wave function represents a superposition of two harmonic waves. Looking at the energy of the excitation

\[E - E_0 = \eta - \cos p,
\]

one can immediately see that in the case of anisotropic ferromagnetic interaction with \( \eta < 1 \) \((J_{z} < J_{x})\) the ground state is unstable against long wavelength magnons. The harmonic (purely oscillating) solution is also obtained in the limit of strong end fields \( \mu, \nu \rightarrow \infty \) (equivalent to fixed boundaries) since (35) becomes

\[e^{2v(L-1)} = 1 : v = ip, \quad p = \frac{\pi m}{L-1}, m = 0, \ldots, 2L - 3.
\]

Due to the symmetry of the chain mentioned above, only half of solutions is linearly independent. An additional solution corresponds to bound state recovering the total number of modes \( L \). For an arbitrary choice of parameters the solution will consist of scattered waves \((Rev = 0)\) and bound states \((Imv = 0)\).
V. CONCLUSIONS

A new expression for the finite 1D lattice Green function has been derived by the discrete Fourier transform approach, by establishing a generalization of the Jacobi expansion. We note its qualitative advantage compared to the approach based in infinite Fourier series that it make accessible a new region of parameters, not available in the thermodynamic limit, but having an important physical meaning (e.g., scattering wave solutions). In the Bethe ansatz approach the phase $\theta$ is introduced into the assumed form of the wave function and is then determined by substitution into the Schrödinger equation. As a result, the Bethe phase becomes a function of momentum (and consequently of energy) and contains all the information on the interaction in the system. In DFT the sequence is just opposite: no assumption is made on the wave function. The eigenenergy (Schrödinger) equation is first solved in the momentum space and then the wave function is obtained via Fourier transform into the direct lattice space. Then the identification of the Bethe phase is achieved by comparing the two functions, $\theta = ivN + \pi m$. The DFT allows to describe the scattering states on the same footing as the bound states, and at the same time to keep a physically transparent structure discussed above. It is known that the bound states are well separated from continuum of scattered states in the thermodynamic limit of $[21, 18]$ while the description of scattered states requires special treatment in a Lipmann-Schwinger type approach $[3]$. In the DFT approach it becomes clear that the qualitative change of the wave function is controlled by the effective magnon-magnon interaction described by the parameter $v$ (e.g., the r.h.s. of Eq. 22): the bound states are formed when this interaction is attractive, while the scattered states are due to repulsive effective interaction (see $[13]$). One can show that the critical line separating the two classes of states corresponds to the vanishing of interaction (i.e. the magnons are free if their dispersion crosses this line) and coincides with the lower boundary of the continuum of scattered states. The wave function of the antisymmetric solutions has nodes on the direct lattice and therefore the corresponding states are more loosely bound. At sufficiently long wavelength of the excitation the bound states are very close to scattered states and the antisymmetric ones can indeed become unstable, i.e., decay into scattered states. This mechanism is responsible for the so-called non-string behavior of the solutions of the Bethe equations at finite $N$. $[19]$ Thus also the physical interpretation of the Bethe solutions becomes more transparent. The same approach allows to reproduce
the BA solution for an open chain, which is amenable to Fourier expansion by doubling the length of the chain. This can be viewed as a cyclic ring with broken links (or impurities) introducing a scattering of magnon excitations and decay of the wave function with distance. The approach described in the paper presents interest for further development, since by its construction it is not limited to one space dimension.

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