EXISTENCE OF LARGE DEVIATIONS RATE FUNCTION FOR ANY $S$-UNIMODAL MAP

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Abstract. For an arbitrary negative Schwarzian unimodal map with a non-flat critical point, we establish the level-2 Large Deviation Principle for empirical distributions. We also give an example of a bimodal map for which the level-2 Large Deviation Principle does not hold.

1. Introduction

A main objective of the ergodic theory of smooth dynamical systems is to understand long-term behaviors of typical orbits for a majority of systems. Much effort has been dedicated to constructing physically relevant invariant measures which statistically predict typical asymptotic behaviors. A refined description of the dynamics requires the analysis of atypical, or transient behaviors before orbits settle down to equilibrium. The theory of large deviations is concerned with such rare events. The Large Deviation Principle (LDP) asserts the existence of the rate function which controls probabilities of rare events on exponential scale.

It is now classical that a transitive uniformly hyperbolic (Axiom A) attractor supports a unique Sinai-Ruelle-Bowen measure [2, 32], and Lebesgue almost every orbit in the basin of attraction is asymptotically distributed with respect to this measure. The LDP for Axiom A attractors was established by Kifer [20], Orey & Pelikan [25] and Takahashi [34]. For one-dimensional non-hyperbolic systems, after several progresses [7, 27, 28], a major advance was made in [8] which establishes the LDP for an arbitrary $C^{1+\alpha}$ multimodal map with non-flat critical points that is topologically exact. The aim of this paper is to treat what is left off in [8]: the LDP for renormalizable unimodal maps, including infinitely renormalizable ones. The conclusion is that the LDP holds for an arbitrary $S$-unimodal map with non-flat critical point.

We introduce our setting and results in more precise terms. Let $X \subset \mathbb{R}$ be a compact non-degenerate interval. A $C^1$ map $f: X \to X$ is called unimodal if it has a unique critical point $c$, which is contained in $\text{int}(X)$ and is an extremum, and satisfies $f(\partial X) \subset \partial X$. An $S$-unimodal map $f$ is a unimodal map of class $C^3$ on $X \setminus \{c\}$ with negative Schwarzian derivative $D^3 f / Df - (3/2)(D^2 f / Df)^2 < 0$ such that if $x \in \partial X$ is a fixed point of $f$ then $|Df(x)| > 1$. We say the critical point $c$ is non-flat if there exist $\ell > 1$ and $C^3$ diffeomorphisms $\varphi$ and $\psi$ defined on a neighborhood of $c$ and $f(c)$ respectively such that $\varphi(c) = 0 = \psi(f(c))$ and $|\varphi(x)|^\ell = |\psi(f(x))|$ for all $x$ near $c$.

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Let \( \mathcal{M} \) denote the space of Borel probability measures on \( X \) endowed with the weak* topology. The empirical measure at time \( n \) with initial point \( x \in X \) is the uniform probability distribution on the orbit \( \{ x, f(x), \ldots, f^{n-1}(x) \} \), denoted by

\[
\delta^n_x = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \in \mathcal{M},
\]

where \( \delta_{f^k(x)} \) denotes the unit point mass at \( f^k(x) \). For a Borel set \( A \subset X \), we write \( |A| \) for its Lebesgue measure. Our main result is stated as follows.

**Theorem A** (level-2 LDP). For an arbitrary \( S \)-unimodal map \( f: X \to X \) with a non-flat critical point, the level-2 LDP holds, namely, there exists a convex lower semicontinuous function \( I: \mathcal{M} \to [0, \infty] \) such that

(lower bound) \[ \liminf_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in \mathcal{G} \}| \geq -\inf_{\mathcal{G}} I \]

for any open set \( \mathcal{G} \subset \mathcal{M} \), and

(upper bound) \[ \limsup_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in \mathcal{C} \}| \leq -\inf_{\mathcal{C}} I \]

for any closed set \( \mathcal{C} \subset \mathcal{M} \).

Hereafter we follow the convention \( \sup \emptyset = -\infty \), \( \inf \emptyset = \infty \), \( \log 0 = -\infty \). The function \( I \) is called the rate function. Since \( \mathcal{M} \) is a metrizable space, the LDP determines the rate function uniquely [29, Theorem 2.13].

In order to explain the meaning of Theorem A, let us recall that an \( S \)-unimodal map \( f \) with a non-flat critical point is classified into the following mutually exclusive cases [15, 24]:

(I) \( f \) has an attracting periodic orbit.

(II) \( f \) is infinitely renormalizable.

(III) \( f \) is at most finitely renormalizable and has no attracting periodic orbit.

The dynamics is relatively simple in cases (I) or (II): the empirical measure along the orbit of Lebesgue almost every initial point converges in the weak* topology to the measure supported on the attracting periodic orbit or the attracting Cantor set. The dynamics in case (III) is much more complicated and displays a rich array of different statistical behaviors (see e.g., [5, 16, 18, 19]). As a prototypical example, consider the quadratic map \( x \in [0, 1] \mapsto ax(1-x) \) with \( 1 < a \leq 4 \). The following are well-known:

- The set of \( a \)-values corresponding to case (I) is open and dense in the parameter space [14].
- The set of \( a \)-values corresponding to case (II) is non-empty [12, 13], and has zero Lebesgue measure [21].
- The set of \( a \)-values corresponding to case (III) has positive Lebesgue measure [1, 18].

\(^1\)Guckenheimer [15] proved this classification for negative Schwarzian \( C^3 \) unimodal maps with non-degenerate critical points. The same holds in our slightly more general setting. For details, see [24, Chapter III, Section 4].
Irrespective of rich bifurcations, Theorem A states that the LDP continues to hold for an arbitrary parameter \( a \).

The rate function \( I \) in Theorem A is given as follows. Let \( \mathcal{M}(f) \) denote the set of elements of \( \mathcal{M} \) which are \( f \)-invariant. For each \( \mu \in \mathcal{M}(f) \), the limit
\[
\chi(x) = \lim_{n \to \infty} \frac{1}{n} \log |Df^n(x)|
\]
exists for \( \mu \)-almost every \( x \in X \) and belongs to \( \mathbb{R} \cup \{ -\infty \} \), because \( \log |Df| \) is uniformly bounded from above. We set
\[
(1.1) \quad \chi^+(x) = \max\{\chi(x), 0\} \in \mathbb{R}
\]
and define the Lyapunov exponent of \( \mu \) by
\[
(1.2) \quad \chi(\mu) = \int \chi^+(x) d\mu(x).
\]
Let \( h(\mu) \) denote the measure-theoretic entropy of \( \mu \) with respect to \( f \). Define a free energy \( F: \mathcal{M} \to [-\infty, 0] \) by
\[
(1.3) \quad F(\mu) = \begin{cases} h(\mu) - \chi(\mu) & \text{if } \mu \in \mathcal{M}(f), \\ -\infty & \text{otherwise.} \end{cases}
\]
The rate function \( I \) in Theorem A is defined to be the minus of the upper semicontinuous regularization of \( F \):
\[
(1.4) \quad I(\mu) = -\inf_{\mathcal{G} \ni \mu} \sup_{\mathcal{G}} F(\nu).
\]
Here, the infimum is taken over all open sets \( \mathcal{G} \) in \( \mathcal{M} \) containing \( \mu \). Note that the entropy and the Lyapunov exponent are upper semicontinuous as functions of measures. In cases (I) and (II), the Lyapunov exponent is in fact continuous (see Lemma 2.11) and thus \( I = -F \). In case (III), the Lyapunov exponent may fail to be lower semicontinuous [4, Section 2], which implies \( I \neq -F \).

The next theorem asserts that Theorem A cannot be extended to maps with multiple critical points.

**Theorem B** (Breakdown of the LDP). There exists a \( C^3 \) interval map with exactly two non-degenerate critical points for which the level-2 LDP does not hold.

The rest of this paper consists of four sections. In Section 2, we introduce several basic definitions and results on \( S \)-unimodal maps. In Section 3, we prove the lower bound for open sets in Theorem A. In Section 4, we prove the upper bound for closed sets in Theorem A. In Section 5 we prove Theorem B.

For only finitely renormalizable maps, the dynamics of typical orbits consist of two stages: transition to the deepest renormalization cycle, and circulation within that cycle. The LDP restricted to the deepest renormalization cycle mostly follows from the known result [3]. For infinitely renormalizable maps, orbits contained in sufficiently deep cycles are approximated by the post-critical measure supported on the attracting Cantor set. Therefore, all we have to do is to analyze the transitions between renormalization cycles. The main point of Section 3 is to prove that any \( f \)-invariant measure supported on a hyperbolic set is approximated with orbit
segments from finitely many subintervals of $X$ (see Proposition 3.1). This claim is standard if the measure is ergodic, and if non-ergodic then we glue orbit segments together which approximate its ergodic components.

The main technique used in Section 4 to derive the upper bound is a ‘coarse graining approach’. Here the set of orbits with prescribed time averages of continuous functions are coarse grained (see Section 4.3), and estimates on the resultant ‘clusters’ are transferred to the large deviations upper bound on empirical measures (see Propositions 4.1 and 4.4). The estimate on each cluster consists of contributions from the uniformly hyperbolic dynamics on each renormalization cycle and the transitions between them.

A counterexample of a bimodal map we construct in the proof of Theorem B is non-transitive and has two non-degenerate critical points, one of which is non-recurrent. With a slight modification of our construction, one can find a counterexample in polynomial maps of degree 3, see Remark 5.5. It is plausible that there is an $S$-unimodal map with a non-recurrent flat critical point for which the level-2 LDP fails. For a transitive $S$-unimodal map with a non-recurrent flat critical point and all periodic orbits hyperbolic repelling, the level-2 LDP was shown in [9] under an assumption that the criticality increases at some specific rates.

2. Preliminaries

This section introduces several basic definitions and results on the dynamics of $S$-unimodal maps and its renormalization. In Section 2.1 we classify attracting periodic orbits. In Section 2.2 we define renormalization of $S$-unimodal maps. In Section 2.3 we introduce basic structures associated with the renormalization. Sections 2.4, 2.5 and 2.6 are concerned with the dynamics on each renormalization cycle. In Section 2.7 we introduce some notation which will be frequently used later. In Section 2.8 we summarize a few results on infinitely renormalizable maps.

2.1. Classification of attracting periodic orbits. Let $f: X \to X$ be a unimodal map, and let $\{f^k(x)\}_{k=0}^{p-1}$ be a periodic orbit of $f$ with prime period $p$. We say $\{f^k(x)\}_{k=0}^{p-1}$ is: hyperbolic attracting if $|Df^p(x)| < 1$; neutral if $|Df^p(x)| = 1$; hyperbolic repelling if $|Df^p(x)| > 1$. The basin of the periodic orbit $\{f^k(x)\}_{k=0}^{p-1}$ is the set of points in $X$ whose omega-limit set is contained in the set $\{f^k(x)\}_{k=0}^{p-1}$. We say $\{f^k(x)\}_{k=0}^{p-1}$ is attracting if its basin contains an open set. In this case, the union of the connected components of the basin containing a point from $\{f^k(x)\}_{k=0}^{p-1}$ is called the immediate basin of $\{f^k(x)\}_{k=0}^{p-1}$. Each connected component of the immediate basin contains exactly one point from $\{f^k(x)\}_{k=0}^{p-1}$.

If $f: X \to X$ is $S$-unimodal, a neutral periodic point $x$ with prime period $p$ satisfies either

(a) $D^2f^p(x) \neq 0$, or (b) $D^2f^p(x) = 0$ and $D^3f^p(x)/Df^p(x) < 0$.

In case (a), the periodic point $x$ is locally attracting from only one side and in case (b) it is attracting from both sides. Hence a periodic point is attracting if and only if it is hyperbolic attracting or neutral. The immediate basin of an attracting periodic orbit contains the critical point [24, Chapter II, Lemma 6.1, Theorem 6.1]. Therefore, there is at most one attracting periodic orbit, denoted by $O(f)$. Let
2.2. Renormalization of $S$-unimodal maps. Let $f: X \to X$ be a unimodal map with a non-flat critical point $c$. A proper closed subinterval $J$ of $X$ is restrictive of period $p \geq 2$ if the following hold (c.f. [24] p.139):

- the interiors of $J, \ldots, f^{p-1}(J)$ are pairwise disjoint.
- $f^p(J) \subset J$ and $f^p(\partial J) \subset \partial J$.
- one of the intervals $J, \ldots, f^{p-1}(J)$ contains $c$ in its interior.
- $J$ is maximal with respect to these properties: if $J' \supset J$ is a closed interval which is strictly contained in $X$ and satisfies the previous three properties with the same integer $p$, then $J' = J$.

We define a strictly decreasing sequence of closed intervals

$$X = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$$

which contain the critical point $c$, and a strictly increasing sequence of integers

$$1 = p_0 < p_1 < p_2 < \cdots$$

so that $J_m$ is restrictive of period $p_m$ for $m \geq 1$, inductively as follows. Given $J_m$ and $p_m$ for some $m \geq 0$, then note that $f^{p_m}|_{J_m} : J_m \to J_m$ is a unimodal map. If $f^{p_m}|_{J_m}$ has a restrictive interval, then define $J_{m+1}$ to be the restrictive interval of $f^{p_m}|_{J_m}$ containing $c$ and with the smallest period $r_m$. We define $p_{m+1} = p_mr_m$. If $f^{p_m}|_{J_m}$ has no restrictive interval, then we do not define $J_{m+1}$ and stop the definition setting $\bar{m}(f) = m$. If this inductive definition continues for arbitrarily large $m$, we set $\bar{m}(f) = \infty$. We say $f$ is: (i) non-renormalizable if $\bar{m}(f) = 0$; (ii) renormalizable if $\bar{m}(f) \geq 1$; (iii) only finitely renormalizable if $\bar{m}(f) < \infty$; (iv) infinitely renormalizable if $\bar{m}(f) = \infty$.

We assume $f$ has negative Schwarzian derivative, and review the dynamics of the unimodal map $f^{p_m}|_{J_m}$, $m \geq 0$. We classify them into the following cases:

- (A) $f^{p_m}|_{J_m}$ is non-renormalizable and has no attracting fixed point.
- (B) $f^{p_m}|_{J_m}$ is renormalizable.
- (C) $f^{p_m}|_{J_m}$ is non-renormalizable and has an attracting fixed point.

Let $L_m$ denote the closed interval bordered by $f^{p_m}(c)$ and $f^{2p_m}(c)$, which is possibly a singleton (a degenerate closed interval).

In case (A), we have $f^{p_m}(L_m) = L_m$ and $f^{p_m}|_{L_m}$ is topologically exact: for any relatively open subset $U$ of $L_m$ there is an integer multiple $n \geq 1$ of $p_m$ such that $f^n(U) = L_m$. One can check this by combining [24] Theorem V.1.3 and [31] Theorem 2.19 and Proposition 2.34 for example.

In case (B), we have $c \in \text{int}(L_m)$ and $f^{p_m}(L_m) \subset L_m$, and the orbit of any point in $\text{int}(J_m)$ eventually falls into $L_m$. There exists a restrictive interval $J_{m+1}$ that contains the critical point $c$ and the set $\bigcup_{k=0}^{p_{m+1}/p_m} f^{k_{p_m}}(J_{m+1})$ is forward invariant with respect to $f^{p_m}$. There are two subcases.
Figure 1. The graphs of the renormalized unimodal maps
\( f_{p_m}^{|J_m}: J_m \to J_m \).

(B-i) \( p_{m+1}/p_m > 2 \), or
(B-ii) \( p_{m+1}/p_m = 2 \).

In case (C), \( m = \bar{m}(f) \) and \( J_m \) contains a unique point \( z \) from \( O(f) \). Indeed, if \( J_m \) contained more than two points from \( O(f) \), then one would find a restrictive interval for \( f_{p_m}^{|J_m} \) as an immediate basin of \( O(f) \). In particular, the point \( z \in O(f) \cap J_m \) is of prime period \( p_m \), and it is a fixed point of the unimodal map \( f_{p_m}^{|J_m} \). There are three subcases:

(C-i) \( z \in \text{int}(J_m) \).
(C-ii) \( z \in \partial J_m \) and \( O(f) \) is neutral.
(C-iii) \( z \in \partial J_m \) and \( O(f) \) is hyperbolic attracting.

See FIGURE 1. In case (C-i), \( O(f) \) is hyperbolic attracting and \( \text{int}(J_m) \) is the connected component of \( B(f) \) containing \( z \). For any closed interval \( J \) contained in \( \text{int}(J_m) \), the following uniform convergence holds:

\[
\lim_{n \to \infty} \sup_{x \in J} |f_{p_m}^n(x) - z| = 0.
\]
In cases (C-ii) and (C-iii), since \( f \) has negative Schwarzian derivative \( f'(c) \) lies in between \( z \) and \( c \) and the following uniform convergence on \( J_m \) holds:

\[
\lim_{n \to \infty} \sup_{x \in J_m} |f^{p_m n}(x) - z| = 0.
\]

If moreover \( z \) is two-sided attracting and \( m \geq 1 \), the dynamics of the previous renormalization \( f^{|J_{m-1}} \) is similar to that in case (C-i) from Lemma 2.1 below. For this reason, if \( 1 \leq \bar{m}(f) < \infty \) and \( \partial J_{\bar{m}(f)} \) contains a two-sided attracting fixed point of \( f^{\bar{m}(f)}|_{J_{\bar{m}(f)}} \), then we zoom out to the previous renormalization by setting

\[
m(f) = \bar{m}(f) - 1.
\]

In all other cases, we set

\[
m(f) = \bar{m}(f).
\]

In what follows, we will only be concerned with the \( m \)-th renormalization cycles with \( 1 \leq m \leq m(f) \).

**Lemma 2.1.** Let \( f: X \rightarrow X \) be a renormalizable \( S \)-unimodal map, and let \( J \) be a restrictive interval with period \( p \) containing \( c \) and not contained in any other restrictive interval (with period smaller than \( p \)). If \( z \in \partial J \) is periodic and two-sided attracting, then \( z \) is a fixed point of \( f \), \( p = 2 \) and its immediate basin coincides with \( \text{int}(X) \). Further, for any closed interval \( J \) contained in \( \text{int}(X) \), we have the uniform convergence \[2.1\] on \( J \) with \( p_m = 1 \).

**Proof.** Let \( B \) denote the connected component of the immediate basin of the periodic orbit of \( z \) that contains \( z \). Then we have \( f^p(B) \subset B \). Since the renormalized map \( f^p|_J \) belongs to case (C-ii) or (C-iii) and since \( z \) is two-sided attracting, the interval \( J \) is strictly contained in \( B \).

We claim that \( f(B) \subset B \). Suppose that this is not the case. Then, by the definition of immediate basin, we have that \( f(B) \cap B = \emptyset \). This implies that \( \text{cl}(B) \) is a restrictive interval and contradicts with the assumption of the lemma.

Since the immediate basin \( B \) contains only one point in the orbit of \( z \), the point \( z \) is a fixed point of \( f \). Then it follows that \( p = 2 \). Since \( f(\partial B) \subset \partial B \), it is immediate to see that \( B \) coincides with \( \text{int}(X) \). \( \square \)

**Remark 2.2.** To summarize, in the case \( m(f) < \infty \), the dynamics of the last renormalization \( f^{p_m(f)}|_{J_m(f)} \) is either in case (A), or

- (C-I) there is a two-sided attracting fixed point in \( \text{int}(J_m(f)) \) whose immediate basin contains \( \text{int}(J_m(f)) \).
- (C-II) there is a one-sided attracting fixed point in \( \partial J_m(f) \) whose immediate basin contains \( J_m(f) \).

### 2.3. Structures associated with renormalization.

Let \( f: X \rightarrow X \) be an \( S \)-unimodal map with a non-flat critical point \( c \) such that \( m(f) \geq 1 \). For each integer \( m \) with \( 0 \leq m < m(f) \) we define the \( m \)-th renormalization cycle \( K_m \) by

\[
K_m = \bigcup_{k=0}^{p_m-1} f^k(J_m).
\]
For $0 \leq m \leq m(f) - 2$, we denote by $\mathcal{P}_m$ the collection of connected components of $K_m \setminus \bigcup_{k=0}^{p_{m+1}-1} \text{int}(f^k(J_{m+1}))$.

In the case $m(f) < \infty$ we set

\begin{equation}
K_{m(f)} = \begin{cases} 
\bigcup_{k=0}^{p_m(f)-1} f^k(J_{m(f)}) & \text{if } f \text{ has an attracting periodic orbit}, \\
\bigcup_{k=0}^{p_m(f)-1} f^k(L_{m(f)}) & \text{if } f \text{ has no attracting periodic orbit}.
\end{cases}
\end{equation}

Moreover, we define $\tilde{\mathcal{P}}_{m(f)-1}$ to be the collection of connected components of the following sets: $K_{m(f)-1} \setminus \bigcup_{k=0}^{p_{m(f)}-1} \text{int}(f^k(J_{m(f)}))$ if $f$ has an attracting periodic orbit; $K_{m(f)-1} \setminus \bigcup_{k=0}^{p_{m(f)}-1} \text{int}(f^k(L_{m(f)}))$ if $f$ has no attracting periodic orbit.

The elements of $\mathcal{P}_m$ with $0 \leq m < m(f)$ are closed intervals in $K_m$, possibly singletons, and $\tilde{\mathcal{P}}_m$ has the Markov property: if $P, Q \in \mathcal{P}_m$ and $f(P) \cap Q \neq \emptyset$ then $f(P) \supset Q$. Let $\mathcal{P}_m$ denote the set of elements of $\mathcal{P}_m$ that are contained in $\bigcup_{k=0}^{p_{m-1}} f^k(L_m)$. We have $\mathcal{P}_m \neq \emptyset$.

Remark 2.3. Case (B-ii) is rather exceptional. The set $\mathcal{P}_m$ consists of the singletons $f^k(J_{m+1}) \cap f^{p_m+k}(J_{m+1})$ for $0 \leq k < p_m$, which form a hyperbolic repelling periodic orbit of prime period $p_m$.

For each $m \in \{0, \ldots, m(f) - 1\}$ we define

$$K_{m,m+1} = \text{cl}(K_m \setminus K_{m+1}).$$

We also define

$$\Gamma_m = \bigcap_{n=0}^{\infty} f^{-n} \left( \bigcup_{P \in \mathcal{P}_m} P \right).$$

Note that $\Gamma_m$ for $1 \leq m < m(f)$ may contain an attracting periodic orbit only if $m = m(f) - 1$, and this periodic orbit is one-sided attracting. In this case, $K_m$ is disjoint from the interior of the basin of the attracting periodic orbit. Set $\Gamma_{-1} = \partial X$. For convenience, in the case $m(f) < \infty$ we further define

$$\Gamma_{m(f)} = \begin{cases} 
O(f) & \text{if } f \text{ has a two-sided attracting periodic orbit}, \\
\bigcup_{k=0}^{p_{m(f)}-1} f^k(L_{m(f)}) & \text{otherwise}.
\end{cases}$$

The sets $\Gamma_m$ are non-empty, pairwise disjoint closed sets. For each $m \in \{-1, 0, \ldots, m(f)\}$ we set

$$\mathcal{M}_m(f) = \{ \mu \in \mathcal{M}(f) : \text{supp}(\mu) \subset \Gamma_m \},$$

where $\text{supp}(\mu)$ denotes the smallest closed subset of $X$ with full $\mu$-measure.

The sets $\mathcal{M}_m(f)$ are pairwise disjoint, and if $m(f) < \infty$ then the convex hull of $\bigcup_{m=1}^{m(f)} \mathcal{M}_m(f)$ is $\mathcal{M}(f)$. If $m(f) < \infty$ and $f$ has a one-sided attracting periodic orbit, then $L_{m(f)}$ is contained in the basin of the attracting periodic orbit, and so $\mathcal{M}_{m(f)}(f) = \emptyset$. 
2.4. Symbolic dynamics on each cycle. Let $f : X \to X$ be an $S$-unimodal map such that $m(f) \geq 1$. For each $m \in \{0, \ldots, m(f) - 1\}$, there is a topological Markov chain over the finite alphabet $\mathcal{P}_m$ determined by the transition matrix

$$(M_{PQ})_{P,Q \in \mathcal{P}_m}, \quad M_{PQ} = \begin{cases} 1 & \text{if } f(P) \supseteq Q, \\ 0 & \text{otherwise}. \end{cases}$$

Let $n \geq 2$ be an integer and let $P_0, P_1, \ldots, P_n \in \mathcal{P}_m$. The word $P_0P_1 \cdots P_n$ of length $n$ is admissible if $M_{P_k P_{k+1}} = 1$ holds for $0 \leq k \leq n - 2$. Let $E^n_m$ denote the set of admissible words of elements of $\mathcal{P}_m$ of length $n$. Let $\Sigma_m$ denote the set of one-sided infinite sequences $\{P_k\}_{k=0}^\infty$ of elements of $\mathcal{P}_m$ such that $P_0 \cdots P_n \in E^n_m$ holds for all $n \geq 2$. We endow $\Sigma_m$ with the restricted topology of the discrete topology of $\mathcal{P}_m$. Let $\sigma_m : \Sigma_m \to \Sigma_m$ denote the left shift: $\sigma_m(\{P_k\}_{k=0}^\infty) = \{P_k\}_{k=1}^\infty$. For each $P_0P_1 \cdots P_n \in E^n_m$, we set $I_{P_0P_1 \cdots P_n} = \bigcap_{k=0}^{n-1} f^{-k}(P_k)$, and define $\pi_m : \Sigma_m \to \Gamma_m$ by

$$\pi_m(\{P_k\}_{k=0}^\infty) \in \bigcap_{n=1}^\infty I_{P_0 \cdots P_n}. \quad (2.4)$$

By [22, Main Theorem], $f$ has no wandering interval, and so any homterval is contained in the basin of an attracting periodic orbit. Since $\bigcap_{n=1}^\infty I_{P_0 \cdots P_n}$ is not contained in the basin of an attracting periodic orbit, it is not a homterval, namely, a singleton. Hence, $\pi_m$ is well-defined by (2.4).

**Proposition 2.4.** Let $f : X \to X$ be an $S$-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each $m \in \{0, \ldots, m(f) - 1\}$, the restriction $f|_{\Gamma_m} : \Gamma_m \to \Gamma_m$ is topologically conjugate by the conjugacy map $\pi_m$ to the topological Markov chain $\sigma_m : \Sigma_m \to \Sigma_m$.

**Proof.** By definition, $\pi_m$ is continuous and surjective. Since the elements of $\mathcal{P}_m$ are pairwise disjoint, $\pi_m$ is injective and has a continuous inverse. For each $P_0P_1 \cdots P_n \in E^n_m$, the Markov property of $\mathcal{P}_m$ implies $f^{-k}(I_{P_0 \cdots P_n}) = I_{P_{k-1} \cdots P_n}$ for $1 \leq k \leq n$. Hence $f \circ \pi_m = \pi_m \circ \sigma_m$ holds. We have verified that $f|_{\Gamma_m}$ is topologically conjugate to $\sigma_m$ by $\pi_m$. \qed

2.5. Distortion estimate on each cycle. Let $Y$ be a non-gedegenerate compact interval in $X$ and let $g : Y \to X$ be a $C^1$ map. Let $J$ be a subinterval of $Y$ such that the restriction of $g$ to $J$ is a diffeomorphism onto its image. By a distortion of $g$ on $J$ we mean the quantity

$$\sup_{x,y \in J} \frac{|Dg(x)|}{|Dg(y)|},$$

We will frequently use the following distortion estimates on each cycle.

**Proposition 2.5.** Let $f : X \to X$ be an $S$-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each $m \in \{0, \ldots, m(f) - 1\}$ the following hold:

(a) if $\Gamma_m$ does not contain a neutral periodic orbit, then there exists a constant $\gamma_m \geq 1$ such that if $n \geq 1$ then the distortion of $f^n$ on any connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$ is bounded by $\gamma_m$. 

(b) if $\Gamma_m$ contains a neutral periodic orbit, then for any $\varepsilon > 0$ there exists $N \geq 1$ such that if $n \geq N$ then the distortion of $f^n$ on any connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$ is bounded by $e^{\varepsilon n}$.

Proof. Since $f$ is $C^2$, the restriction of $\log |Df|$ to the union of the elements of $\tilde{\mathcal{P}}_m$ not containing a point from an attracting periodic orbit is Lipschitz continuous. Moreover, by Mañé’s theorem [23, Theorem A], the maximal $f$-invariant set in this union is a hyperbolic set. Hence (a) holds.

As for (b), if $\Gamma_m$ contains a neutral periodic orbit, then $m = m(f) - 1$ and $\partial J_{m(f)}$ contains a point from the neutral periodic orbit. We need the following lemma, under the notation in Section 2.4.

Lemma 2.6. For each $m \in \{0, \ldots, m(f) - 1\}$ we have
\[
\lim_{n \to \infty} \sup \{|I_\omega| : \omega \in E_m^n \} = 0.
\]

Proof. Suppose there exist $\varepsilon > 0$ and an infinite subset $F$ of $\bigcup_{n=1}^\infty E_m^n$ such that $|I_\omega| > \varepsilon$ for all $\omega \in F$. Since $X$ has a finite diameter, we can choose a sequence $(\omega(n))_{n=1}^\infty$ in $F$ such that $(I_{\omega(n)})_{n=1}^\infty$ is a nested sequence. Since $\bigcap_{n=1}^\infty I_{\omega(n)}$ is not a singleton, it is a homterval. However, it is not contained in the basin of an attracting periodic orbit. We obtain a contradiction to [22, Main Theorem]. □

Let $\varepsilon > 0$, let $n \geq 2$ and let $W$ be a connected component of $\bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1})$. There exists $\omega \in E_m^n$ such that $W = I_\omega$. Since $K_{m,m+1}$ does not contain the critical point of $f$, the infimum of $|Df|$ over this set is positive. For all $x, y \in I_\omega$ we have
\[
\log \frac{|Df^n(x)|}{|Df^n(y)|} \leq \left( \sup_{K_{m,m+1}} \frac{|D^2f|}{|Df|} \right)^{n-1} \sum_{k=0}^{n-1} |f^k(\omega)| \leq \left( \sup_{K_{m,m+1}} \frac{|D^2f|}{|Df|} \right)^{n-1} \sum_{k=0}^{n-1} \sup_{\omega \in E_m^{n-k}} |I_\omega|.
\]

By Lemma 2.6 the last number is bounded by $\varepsilon n$ for a sufficiently large $n$. The proof of Proposition 2.5 is complete. □

2.6. Topological results. Let $Y$ be a non-degenerate interval in $X$ and $g : Y \to Y$ be a continuous map. Let $U$ be a non-degenerate subinterval of $Y$ and $n \geq 1$ an integer. Any connected component of $g^{-n}(U)$ is called a pullback of $U$ by $g^n$. If $V$ is a pullback of $U$ by $g^n$ and $g^n|_V$ is a diffeomorphism, then $V$ is called a diffeomorphic pullback of $U$ by $g^n$.

Results in this subsection will be used for the proof of the lower bound in Theorem A, in order to glue different orbits together to form one orbit with required properties.

Lemma 2.7. Let $f : X \to X$ be an $S$-unimodal map with a non-flat critical point such that $m(f) \geq 1$, and let $0 \leq m < m(f)$ be such that $p_m - p_{m+1} \neq 2$. For any open interval $J \subset L_m$ that intersects $\Gamma_m$, there exist $n \geq 1$ and a pullback $W$ of $L_m$ by $f^{m,n}$ that is contained in $J$.

Proof. Since $m < m(f)$, $f^{m,n}(c) \neq f^{2m,n}(c)$ and $L_m$ is a non-degenerate interval. Since $J$ intersects $\Gamma_m$, it is not contained in $K_{m+1}$. Since $\Gamma_m \cap B(f) = \emptyset$ and $f(\Gamma_m) = \Gamma_m$, $J$ is not a homterval of $f^{m,n}$. Hence $U = \bigcup \{f^{m,n}(J) : n \geq 0, c \in f^{m,n}(J)\}$ intersects $\Gamma_m$. If $n \geq 0$ and $c \in f^{m,n}(J)$, then $f^{m,n}(c)$ is contained in
the boundary of $f_{pn}^{(n+1)}(J)$. Hence, there exists an increasing sequence $\{n_i\}_{i \geq 1}$ of non-negative integers such that $c \in f_{pn}^{n_i}(J)$ for all $i$, and $\{f_{pn}^{(n_i+1)}(J)\}_{i \geq 1}$ is increasing and $f_{pn}(U) = \bigcup_{i \geq 1} f_{pn}^{(n_i+1)}(J)$.

We claim that $c \in \text{int}(f_{pn}^{k}(U))$ for some $k \geq 1$ implies $f_{pn}^{k}(U) \subset U$. Indeed, for each $x \in f_{pn}^{k}(U)$ there exists $n \geq 0$ such that $c \in f_{pn}^{n}(J)$ and $c, x \in f_{pn}^{(n+k)}(J)$. Hence $x \in U$, and the claim holds.

Set $r = \inf\{k \geq 1: c \in \text{int}(f_{pn}^{k}(U))\}$. Since $J$ is not a hominterval of $f_{pn}$, $f_{pn}(U)$ is not a hominterval of $f_{pn}$ either, and so $r$ is finite. By the above claim, if $r \geq 2$ then $f_{pn}$ is renormalizable with period $r$ and we have $U \subset K_{m+1}$, which is a contradiction. Hence $r = 1$, namely $c \in \text{int}(f_{pn}(U))$. This implies that the closed interval bordered by $c$ and $f_{pn}(c)$ is contained in $f_{pn}^{n}(J)$ for some $n \geq 0$, and hence $L_m \subset f_{pn}^{(n+1)}(J)$. This implies the existence of a pullback with the desired properties. 

\[ \square \]

**Corollary 2.8.** Let $f: X \to X$ be an $S$-unimodal map with a non-flat critical point such that $m(f) \geq 1$. For each $0 \leq m < m(f)$ such that $p_{m+1}/p_m \neq 2$, there exists an integer multiple $M_m \geq 1$ of $p_m$ such that any element of $\mathcal{P}_m$ that is contained in $L_m$ contains a pullback of $L_m$ by $f^{M_m}$.

**Proof.** Since $p_{m+1}/p_m \neq 2$, the elements of $\mathcal{P}_m$ are non-degenerate closed intervals. The statement is a consequence of Lemma 2.7. \[ \square \]

**Remark 2.9.** In the case $p_{m+1}/p_m = 2$, $L_m = f_{pn}^{m}(J_{m+1}) \cup f^{2p_m}(J_{m+1}) \subset K_{m+1}$.

2.7. **Notation.** Let $C(X)$ denote the set of real-valued continuous functions on $X$. For $\phi \in C(X)$ and an integer $n \geq 1$, we write $S_n \phi$ for the sum $\sum_{k=0}^{n-1} \phi \circ f^k$. For an integer $l \geq 1$ define

$$C(X)^l = \{\phi = (\phi_1, \ldots, \phi_l): \phi_j \in C(X) \text{ for every } j \in \{1, \ldots, l\}\}.$$ 

For $\phi = (\phi_1, \ldots, \phi_l) \in C(X)^l$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l$ and a measure $\mu \in \mathcal{M}$, the expression $\int \phi d\mu > \alpha$ indicates that $\int \phi_j d\mu > \alpha_j$ holds for every $j \in \{1, \ldots, l\}$. For an integer $n \geq 1$, we set

$$A_n(\phi, \alpha) = \left\{x \in X: \int \phi d\delta_x^n > \alpha \right\},$$

and

$$S_n \phi = (S_n \phi_1, \ldots, S_n \phi_l).$$

For $a \in \mathbb{R}$ and $\vec{v} = (v_1, \ldots, v_l) \in \mathbb{R}^l$, we write

$$\vec{\alpha} = (a, a, \ldots, a) \in \mathbb{R}^l \text{ and } \|\vec{v}\| = \max_{1 \leq j \leq l} |v_j|.$$ 

For $\vec{v}, \vec{w} \in \mathbb{R}^l$, the expression $\vec{v} \geq \vec{w}$ indicates that $v_j \geq w_j$ holds for every $j \in \{1, \ldots, l\}$. 


2.8. **Infinitely renormalizable maps.** Let \( f : X \to X \) be an infinitely renormalizable \( S \)-unimodal map with a non-flat critical point. The omega limit set of Lebesgue almost every initial point \( x \in X \) coincides with the closed invariant set

\[
\Lambda = \bigcap_{m=0}^{\infty} K_m
\]

and the empirical measure \( \delta^n_x \) converges to the post-critical measure \( \mu_\infty \) supported on \( \Lambda \). The restriction of \( f \) to \( \Lambda \) is uniquely ergodic and the post-critical measure \( \mu_\infty \) is the unique invariant measure on \( \Lambda \). Moreover, \( h(\mu_\infty) = \chi(\mu_\infty) = 0 \) holds (see [4, Theorem 3.4(b)]).

In order to ‘approximate’ \( \mu_\infty \), we will use the following lemma.

**Lemma 2.10.** ([35, Lemma 2.3]) Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point such that \( m(f) = \infty \). Let \( l \geq 1, \tilde{\phi} \in C(X)^l, \tilde{\alpha} \in \mathbb{R}^l \). For any \( \varepsilon > 0 \) there exist integers \( n_\star \geq 1 \) and \( N_\star \geq 1 \) such that the following hold:

(a) if \( n \geq N_\star \) and \( A_n(\tilde{\phi}, \tilde{\alpha}) \cap K_{m_\star} \neq \emptyset \) then

\[
\int \tilde{\phi} d\mu_\infty > \tilde{\alpha} - \varepsilon.
\]

(b) if \( n \geq N_\star \) then

\[
\sup_{K_{m_\star}} \left\| \frac{1}{n} S_n \tilde{\phi} - \int \tilde{\phi} d\mu_\infty \right\| < \varepsilon.
\]

For infinitely renormalizable maps, the Lyapunov exponent depends continuously on invariant measures.

**Lemma 2.11.** ([35, Lemma 2.6]) Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point such that \( m(f) = \infty \). Then \( \mu \in \mathcal{M}(f) \mapsto \chi(\mu) \) is continuous.

### 3. Large deviations lower bound

In this section we complete the proof of the lower bound in Theorem A. In Section 3.1 we show that this follows from a key lower bound stated in Proposition 3.1. The rest of this section is dedicated to the proof of Proposition 3.1.

#### 3.1. Key lower bound.** The next proposition allows us to approximate a given invariant measure with finitely many intervals in a particular sense.

**Proposition 3.1.** Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point. Let \( l \geq 1, \tilde{\phi} \in C(X)^l, \tilde{\alpha} \in \mathbb{R}^l \) and let \( \mu \in \mathcal{M}(f) \) satisfy \( \int \tilde{\phi} d\mu > \tilde{\alpha} \). For any \( \varepsilon > 0 \) there exists \( N \geq 1 \) such that for each integer \( n \geq N \), there exists a finite set \( \mathcal{A}_n \) of closed intervals in \( A_n(\tilde{\phi}, \tilde{\alpha}) \) with pairwise disjoint interiors such that

(a) \( \# \mathcal{A}_n > \exp((h(\mu) - \varepsilon)n), \) and

(b) \( |A| > \exp(-(\chi(\mu) + \varepsilon)n) \) for all \( A \in \mathcal{A}_n \).
In order to prove Proposition 3.1, we approximate the given measure \( \mu \) with a finite convex combination of measures each supported on some \( \Gamma_m \) or \( \Lambda \), and then for each of these measures construct a collection of finitely many intervals. We then glue orbits from these intervals together to construct a collection intervals with the desired properties.

The proof of Proposition 3.1 is given in Section 3.3. Below we deduce a corollary to Proposition 3.1 and complete the proof of the lower bound in Theorem A.

**Corollary 3.2.** Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point. Let \( l \geq 1, \bar{\phi} \in C(X)^l, \bar{\alpha} \in \mathbb{R}^l \). If \( \mu \in \mathcal{M}(f) \) satisfies \( \int \bar{\phi} d\mu > \bar{\alpha} \), then

\[
\liminf_{n \to \infty} \frac{1}{n} \log |A_n(\bar{\phi}, \bar{\alpha})| \geq F(\mu).
\]

**Proof.** From Proposition 3.1, for any \( \varepsilon > 0 \) there exists \( N \geq 1 \) such that \( |A_n(\bar{\phi}, \bar{\alpha})| > e^{(F(\mu) - 2\varepsilon)n} \) holds for all \( n \geq N \). Taking logarithms of both sides, dividing by \( n \) and letting \( n \to \infty \) and then \( \varepsilon \to 0 \), we obtain the desired inequality. \( \square \)

**Proof of the lower bound in Theorem A.** Subsets of \( \mathcal{M} \) of the form \( \{ \mu \in \mathcal{M} : \int \bar{\phi} d\mu > \bar{\alpha} \} \) with \( l \geq 1, \bar{\phi} \in C(X)^l, \bar{\alpha} \in \mathbb{R}^l \) constitute a base of the weak* topology of \( \mathcal{M} \). Therefore any non-empty open subset \( G \) of \( \mathcal{M} \) is written as the union \( G = \bigcup \lambda G_\lambda \) of sets \( G_\lambda \) of this form. For each \( G_\lambda \), Corollary 3.2 gives

\[
\liminf_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in G_\lambda \}| \geq \sup_{G_\lambda} F.
\]

Hence we obtain

\[
\liminf_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in G \}| \geq \sup_{\lambda} \sup_{G_\lambda} F = \sup_{G} F = - \inf I,
\]

as required. \( \square \)

### 3.2. Approximation of a measure on each cycle.

The rest of this section is entirely dedicated to a proof of Proposition 3.1. Therefore, \( l \geq 1, \bar{\phi} \in C(X)^l, \bar{\alpha} \in \mathbb{R}^l \) in the statement of Proposition 3.1 are fixed till the end of Section 3.3 and \( f : X \to X \) is assumed to be an \( S \)-unimodal map with a non-flat critical point \( c \).

The next lemma allows us to approximate an invariant measure on each single cycle with a finite collection of intervals.

**Lemma 3.3.** Let \( f : X \to X \) be such that \( m(f) \geq 1 \). Let \( 0 \leq m < m(f) \) and suppose \( p_{m+1}/p_m \neq 2 \). Let \( \mu \in \mathcal{M}(f) \) and let \( P \in \mathcal{P}_m \) be such that \( P \subseteq L_m \) and \( \mu(P) > 0 \). Let \( P' \) be a non-degenerate closed interval in \( L_m \). For any \( \varepsilon > 0 \), there exists \( N \geq 1 \) such that for each integer \( q \geq N \), there exists a finite collection \( \mathcal{B}_{p_{m+1}q}(P, P') \) of pullbacks of \( P' \) by \( f^p_{p_{m+1}q} \) which are contained in \( P \) such that

\[
(a) \quad \left| \frac{1}{p_{m+1}q} \log \# \mathcal{B}_{p_{m+1}q}(P, P') - h(\mu) \right| < \varepsilon, \quad \text{and}
\]

\[
(b) \quad \sup_{B \in \mathcal{B}_{p_{m+1}q}(P, P')} \sup_B \left\| \frac{1}{p_{m+1}q} S_{p_{m+1}q} \bar{\phi} - \int \bar{\phi} d\mu \right\| < \varepsilon.
\]
If moreover $P'$ does not contain $f^k(c)$ for all $1 \leq k \leq M_m$ where $M_m$ is the integer in Corollary 2.8 then any pullback $B \in \mathcal{B}_{pm}(P, P')$ is diffeomorphic and satisfies

$$(c) \quad \sup_B \frac{1}{p_m q} \log |Df| - \chi(\mu) < \varepsilon.$$ 

Proof. By Proposition 2.4, $f|_{r_m}$ is topologically conjugate to a topological Markov chain via the finite Markov partition $\mathcal{P}_m$. For an integer $r \geq 1$ let $\mathcal{B}_{pm}(P)$ denote the collection of pullbacks of elements of $\mathcal{P}_m$ by $f_{pm}^r$ that are contained in $P$. Lemma 2.6 gives $\lim_{r \to \infty} \sup_{B \in \mathcal{B}_{pm}(P)} |B| = 0$, and so for any continuous function $\varphi : K_{m, m+1} \to \mathbb{R}$ we have

$$(3.1) \quad \lim_{r \to \infty} \sup_{B \in \mathcal{B}_{pm}(P)} \sup_{x, y \in B} \frac{1}{p_m^r} |S_{pm} \varphi(x) - S_{pm} \varphi(y)| = 0.$$ 

We start the proof of Lemma 3.3 with the case where $\mu$ is ergodic. Let $\varepsilon > 0$. Since $\mu(L_m) = 1/p_m$, the normalized restriction of $\mu$ to $L_m$ is an $f_{pm}|_{L_m}$-invariant Borel probability measure with entropy $p_m h(\mu)$. For a sufficiently large integer $q \geq 1$, we set $r = q - M_m/p_m$. From (3.1) and Birkhoff’s ergodic theorem and Shannon-McMillan-Breiman’s theorem [11] for the normalized restriction, there exists a subset $\mathcal{B}_{pm}(P)$ of $\mathcal{B}_{pm}(P)$ for which the following hold:

$$\left| \frac{1}{p_m^r} \log \# \mathcal{B}_{pm}(P) - h(\mu) \right| < \varepsilon,$$

$$\sup_{B \in \mathcal{B}_{pm}(P)} \sup_B \left\| \frac{1}{p_m^r} S_{pm} \tilde{\varphi} - \int \tilde{\varphi} d\mu \right\| < \frac{\varepsilon}{2},$$

$$\sup_{B \in \mathcal{B}_{pm}(P)} \sup_B \left| \frac{1}{p_m^r} \log |Df| - \int \log |Df| d\mu \right| < \frac{\varepsilon}{2}.$$ 

By Corollary 2.8, $f^{M_m}(Q) = L_m$ holds for $Q \in \mathcal{P}_m$ that is contained in $L_m$. For each $B \in \mathcal{B}_{pm}(P)$, the interval $f_{pm}^r(B)$ is an element of $\mathcal{P}_m$ that is contained in $L_m$. We fix a connected component $W = W_B$ of $f_{pm}^r(B) \cap f^{-M_m}(P')$. If $f^k(c) \notin P'$ holds for all $1 \leq k \leq M_m$, $W$ is a diffeomorphic pullback of $P'$ by $f^{M_m}$. Let $B'$ denote the pullback of $W$ by $f_{pm}^r$ that is contained in $B$. Set

$$\mathcal{B}_{pm}(P, P') = \{B' : B \in \mathcal{B}_{pm}(P)\}.$$ 

Then clearly $\# \mathcal{B}_{pm}(P, P') = \# \mathcal{B}_{pm}(P)$, and the elements of $\mathcal{B}_{pm}(P, P')$ are pullbacks of $P'$ by $f_{pm}$. Therefore, for sufficiently large $n$ we obtain (a), (b) and (c) of Lemma 3.3 from the corresponding estimates for $\mathcal{B}_{pm}(P)$ above.

It is left to treat the case where $\mu$ is non-ergodic. Note that the correspondence $\mu \in \mathcal{M}_m(f) \mapsto \chi(\mu) \in \mathbb{R}$ is affine from the definition [11]. By virtue of the ergodic decomposition theorem and Jacobs’ theorem on the decomposition of entropy [17], for any $\varepsilon > 0$ there exist a finite number of ergodic measures $\mu_1, \ldots, \mu_s$ in $\mathcal{M}_m(f)$ and constants $\rho_1, \ldots, \rho_s$ in $(0, 1)$ for which $\sum_{i=1}^s \rho_i = 1$, such that the measure $\mu' = \sum_{i=1}^s \rho_i \mu_i$ in $\mathcal{M}_m(f)$ satisfies

$$|h(\mu) - h(\mu')| < \varepsilon, \quad \left\| \int \tilde{\varphi} d\mu - \int \tilde{\varphi} d\mu' \right\| < \varepsilon, \quad |\chi(\mu) - \chi(\mu')| < \varepsilon.$$
By this and $\mu(P) > 0$ as in the lemma, with no loss of generality we may assume that $\mu$ is written as a convex combination of ergodic measures $\mu_i \in \mathcal{M}_m(f)$:

$$\mu = \rho_1 \mu_1 + \rho_2 \mu_2 + \cdots + \rho_s \mu_s,$$

with $\rho_i \in (0, 1)$ satisfying $\sum_{i=1}^s \rho_i = 1$ and $\mu_1(P) > 0$. We fix $P_1, \ldots, P_s \in \mathcal{P}_m$ so that $P_i = P$, $P_i \subset L_m$ and $\mu_i(P_1) > 0$ for $1 \leq i \leq s$. Set $P_{s+1} = P'$.

Let $\varepsilon > 0$. Let $q \geq 1$ be a large integer, and write it as a sum of positive integers

$$q = q_1 + q_2 + \cdots + q_s,$$

with $|q_i - \rho_i n| \leq 1$ for $1 \leq i \leq s$. From the previous argument in the ergodic case, if $n$ is sufficiently large then for each $i \in \{1, \ldots, s\}$, there is a finite collection $\mathcal{B}_{p_m q_i}(P_i, P_{i+1})$ of diffeomorphic pullbacks of $P_{i+1}$ by $f_{p_m q_i}$ contained in $P_i$ such that the following inequalities hold:

$$\left| \frac{1}{p_m q_i} \log \# \mathcal{B}_{p_m q_i}(P_i, P_{i+1}) - h(\mu_i) \right| < \frac{\varepsilon}{2}, \quad (3.2)$$

$$\sup_{B \in \mathcal{B}_{p_m q_i}(P_i, P_{i+1})} \sup_B \left\| \frac{1}{p_m q_i} S_{p_m q_i} \vec{\phi} - \int \vec{\phi} d\mu_i \right\| < \frac{\varepsilon}{2}, \quad (3.3)$$

$$\sup_{B \in \mathcal{B}_{p_m q_i}(P_i, P_{i+1})} \sup_B \left\| \frac{1}{p_m q_i} \log |D f_{p_m q_i}| - \chi(\mu_i) \right\| < \frac{\varepsilon}{2}, \quad (3.4)$$

Let $\mathcal{B}_{p_m q}(P, P')$ denote the collection of pullbacks of $P'$ by the composition $g_s \circ g_{s-1} \circ \cdots \circ g_1$ of diffeomorphisms $g_i$ of the form $f_{p_m q_i}$ with $B \in \mathcal{B}_{p_m q_i}(P_i, P_{i+1})$. Using \[3.2\] for $1 \leq i \leq s$ we have

$$\left| \frac{1}{p_m q} \log \# \mathcal{B}_{p_m q}(P, P') - h(\mu) \right| \leq \frac{1}{p_m q} \sum_{i=1}^s \log \# \mathcal{B}_{p_m q_i}(P_i, P_{i+1}) - p_m q \sum_{i=1}^s \rho_i h(\mu_i)$$

$$\leq \frac{1}{p_m q} \sum_{i=1}^s \left| \log \# \mathcal{B}_{p_m q_i}(P_i, P_{i+1}) - p_m q_i h(\mu_i) \right|$$

$$+ \sum_{i=1}^s \left| \frac{q_i}{q} - \rho_i \right| h(\mu_i) < \frac{\varepsilon}{2} + \frac{s}{q} h_{\text{top}}(f) < \varepsilon,$$

where $h_{\text{top}}(f)$ denotes the topological entropy of $f$. For each $B \in \mathcal{B}_{p_m q}(P, P')$, using \[3.3\] for $1 \leq i \leq s$ we have

$$\sup_B \left\| \frac{1}{p_m q} S_{p_m q} \vec{\phi} - \int \vec{\phi} d\mu \right\| \leq \frac{1}{p_m q} \sum_{i=1}^s \sup_{B' \in \mathcal{B}_{p_m q_i}(P_i, P_{i+1})} \sup_{B'} \left\| S_{p_m q_i} \vec{\phi} - p_m q_i \int \vec{\phi} d\mu_i \right\|$$

$$+ \sum_{i=1}^s \left| \frac{q_i}{q} - \rho_i \right| \left\| \vec{\phi} \right\| < \frac{\varepsilon}{2} + \frac{s}{q} \left\| \vec{\phi} \right\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the above two estimates imply (a) and (b) in Lemma 3.3 respectively. We can prove (c) in the same way as (b), using \[3.4\] for $1 \leq i \leq s$ and noting that the orbits of points in elements of $\mathcal{B}_{p_m q}(P, P')$ up to time $p_m q$ is
uniformly bounded away from the critical point by a distance independent of $n$. This completes the proof of Lemma [3.3].

\section*{3.3. Proof of Proposition [3.1]}

Let $\mu \in \mathcal{M}(f)$ satisfy $\int \tilde{\phi} d\mu > \tilde{\alpha}$. Let $\varepsilon > 0$ be small enough so that

\begin{equation}
\int \tilde{\phi} d\mu - \varepsilon > \tilde{\alpha}.
\end{equation}

Let $P_{-1}$ denote the connected component of $X \setminus L_0$ that contains the fixed point of $f$ in $\partial X$. Let $\mu_{-1}$ denote the element of $\mathcal{M}(f)$ that is supported on the fixed point in $\partial X$. Put $p_{-1} = 1$.

In what follows we treat four cases separately. We start with the case where $f$ is infinitely renormalizable. Arguments for the remaining cases will proceed much in parallel to the infinitely renormalizable case.

**Case 1:** $m(f) = \infty$. Let $m_\ast, N_\ast$ be positive integers for which the conclusion of Lemma [2.10] holds with $\varepsilon$ replaced by $\varepsilon/3$. By the continuity of the Lyapunov exponent in Lemma [2.11], we may assume with no loss of generality that there exist constants $\rho_{-1}, \rho_0, \ldots, \rho_m \in (0, 1)$ with $\sum_{m=1}^{m_\ast} \rho_m = 1$, and for each $m \in \{-1, 0, \ldots, m_\ast - 1\}$ a measure $\mu_m \in \mathcal{M}_{m}(f)$ such that

$$\mu = \rho_{-1} \mu_{-1} + \rho_0 \mu_0 + \cdots + \rho_{m_\ast - 1} \mu_{m_\ast - 1} + \rho_m \mu_{\infty}.$$ 

For $m \geq 0$ with $p_{m+1}/p_m \neq 2$, we fix $P_m \in \mathcal{P}_m$ such that $P_m \subset L_m$ and $\mu_m(P_m) > 0$. For $m \geq 0$ with $p_{m+1}/p_m = 2$, we set $P_m = L_m$. We set $P_{m_\ast} = J_{m_\ast}$.

Let $q \geq 1$ be an integer. For $m \geq -1$ with $p_{m+1}/p_m \neq 2$, let $\mathcal{B}_{p_m q}(P_m, P_{m+1})$ denote the collection of diffeomorphic pullbacks of $P_{m+1}$ by $f^{p_m q}$ contained in $P_m$ which are obtained by applying Lemma [3.3] with $\mu = \mu_m$, $P = P_m$, $P' = P_{m+1}$ and $\varepsilon$ replaced by $\varepsilon/2$. For $m \geq -1$ with $p_{m+1}/p_m = 2$, let $B$ denote the diffeomorphic pullback of $f^{p_m q}(P_m) \cap P_{m+1}$ by $f^{p_m q}$ that is contained in $P_m$ and contains the periodic point in $\partial J_{m+1}$, and set $\mathcal{B}_{p_m q}(P_m, P_{m+1}) = \{B\}$.

We claim that for each $m \in \{-1, 0, \ldots, m_\ast - 1\}$, if $q \geq 1$ is sufficiently large then

\begin{equation}
\frac{1}{p_m q} \log \# \mathcal{B}_{p_m q_m}(P_m, P_{m+1}) - h(\mu_m) < \varepsilon/2.
\end{equation}

\begin{equation}
\sup_{B \in \mathcal{B}_{p_m q_m}(P_m, P_{m+1})} \sup_{B} \left\| \frac{1}{p_m q_m} S_{p_m q_m} \tilde{\phi} - \int \tilde{\phi} d\mu_m \right\| < \varepsilon/2,
\end{equation}

\begin{equation}
\sup_{B \in \mathcal{B}_{p_m q_m}(P_m, P_{m+1})} \sup_{B} \left| \frac{1}{p_m q_m} \log |Df^{p_m q_m}| - \chi(\mu_m) \right| < \varepsilon/2.
\end{equation}

Indeed, for $m = -1$, $\mu_m$ is the empirical measure on the orbit of the hyperbolic repelling periodic point in $\partial P_{-1}$. For $m \geq 0$ with $p_{m+1}/p_m = 2$, $\mu_m$ is the empirical measure on the orbit of the hyperbolic repelling periodic point in $\partial J_{m+1}$. So, (3.6), (3.7), (3.8) hold in these two cases. For $m \geq 0$ with $p_{m+1}/p_m \neq 2$, these are consequences of Lemma 3.3.
Let \( n \geq 1 \) be a large integer, and write it as a linear combination of non-negative integers \( q_{-1}, q_0, \ldots, q_{m-1}, r \) in the form

\[
n = p_{-1}q_{-1} + p_0q_0 + \cdots + p_{m-1}q_{m-1} + r,
\]

with \( |p_mq_m - \rho_m| \leq 1 \) for \(-1 \leq m \leq m-1\) and \( |r - \rho_m| \leq 1 \). In what follows we assume \( n \) is sufficiently large so that \( r > N_* \), and for each \( m \in \{-1, \ldots, m-1\} \), \( \mathcal{R}_{p_m} (P_m, P_{m+1}) \) satisfies (3.6) (3.7) (3.8). We have

\[
\sup_{K_m} \left\{ \frac{1}{r} S_r \phi - \int \phi d\mu_\infty \right\} \leq \frac{\varepsilon}{2}.
\]

Let \( \mathcal{A}_n \) denote the collection of pullbacks of \( P_m \) by maps \( h_{m-1} \circ \cdots \circ h_{-1} \), where \( h_m, m \in \{-1, \ldots, m-1\} \) is a diffeomorphism of the form \( f_{p_m} | B \) with \( B \in \mathcal{R}_{p_m} (P_m, P_{m+1}) \). By \# \mathcal{A}_n = \prod_{m=1}^{m-1} \# \mathcal{R}_{p_m} (P_m, P_{m+1}), (3.6) \) and \( h(\mu_\infty) = 0 \),

\[
\left| \frac{1}{n} \log \# \mathcal{A}_n - h(\mu) \right| < \varepsilon.
\]

A similar argument to the proofs of Lemma 3.3(b)(c) based on (3.7), (3.8) shows

\[
\sup_{A \in \mathcal{A}_n} \sup_A \left\{ \frac{1}{n} S_n \phi - \int \phi d\mu \right\} < \varepsilon.
\]

Since \( \chi(\mu) = \sum_{m=1}^{m-1} \rho_m \chi(\mu_m) \), we obtain

\[
\sup_{A \in \mathcal{A}_n} \sup_A \left| \frac{1}{n-r} \log |Df^{n-r}| - \chi(\mu) \right| < \varepsilon.
\]

Item (a) in Proposition 3.1 follows from (3.9). From (3.5) and (3.10), the elements of \( \mathcal{A}_n \) are contained in \( \mathcal{A}_n(\phi, \alpha) \). Item (b) in Proposition 3.1 follows from (3.11).

**Case 2:** \( m(f) < \infty \) and \( \text{int}(J_{m(f)}) \) contains an attracting periodic point (Case C-I in Remark 2.2). Recall that \( \delta_{O(f)} \) denotes the element of \( \mathcal{M}(f) \) supported on \( O(f) \). With no loss of generality we may assume there exist constants \( \rho_{-1}, \ldots, \rho_{m(f)} \in (0, 1) \) with \( \sum_{m=1}^{m(f)} \rho_m = 1 \), and for each \( m \in \{-1, \ldots, m(f) - 1\} \) a measure \( \mu_m \in \mathcal{M}(f) \) such that

\[
\mu = \rho_{-1} \mu_{-1} + \rho_0 \mu_0 + \cdots + \rho_{m(f)} \delta_{O(f)}.
\]

Similarly to Case 1, For \( m \geq 0 \) with \( p_{m+1}/p_m \neq 2 \), we fix \( P_m \in \mathcal{R} \) such that \( P_m \subset L_m \) and \( \mu_m(P_m) > 0 \). For \( m \geq 0 \) with \( p_{m+1}/p_m = 2 \), we set \( P_m = L_m \). We set \( P_m(f) = L_m(f) \).

Let \( q \geq 1 \) be an integer. For \( m \geq -1 \) with \( p_{m+1}/p_m \neq 2 \), let \( \mathcal{R}_{p_m} (P_m, P_{m+1}) \) denote the collection of diffeomorphic pullbacks of \( P_{m+1} \) by \( f_{p_m}^q \) contained in \( P_m \) which are obtained by applying Lemma 3.3 with \( \mu = \mu_m \), \( P = P_m \), \( P' = P_{m+1} \) and \( \varepsilon \) replaced by \( \varepsilon/2 \). For \( m \geq -1 \) with \( p_{m+1}/p_m = 2 \), let \( B \) denote the diffeomorphic pullback of \( f_{p_m}^q (P_m) \cap P_{m+1} \) by \( f_{p_m}^q \) that is contained in \( P_m \) and contains the periodic point in \( \partial J_{m+1} \), and set \( \mathcal{R}_{p_m} (P_m, P_{m+1}) = \{ B \} \).

Let \( n \geq 1 \) be an integer, and write it as a linear combination of positive integers \( q_1, \ldots, q_{m(f)-1} \) in the form

\[
n = p_{-1}q_{-1} + \cdots + p_{m(f)-1}q_{m(f)-1} + r,
\]
with $|p_m q_m - \rho_m n| \leq 1$ for $-1 \leq m \leq m(f) - 1$ and $|r - \rho_{m(f)} n| \leq 1$. In what follows we assume $n$ is sufficiently large so that for each $m \in \{-1, \ldots, m(f) - 1\}$, $\mathcal{B}_{p_m, q_m}(P_m, P_{m+1})$ satisfies (3.6), (3.7) and (3.13), and for the last $r$ iteration in $K_{m(f)}$,

$$\sup_{P_{m(f)}} \left\| \frac{1}{r} S_r \tilde{\phi} - \int \tilde{\phi} \, d\delta_{O(f)} \right\| < \frac{\varepsilon}{2}. \tag{3.13}$$

Let $\mathcal{A}_n$ denote the collection of pullbacks of $P_m(f)$ by maps $h_{m(f)}^{-1} \circ \cdots \circ h_0 \circ h_{-1}$, where $h_m, m \in \{-1, \ldots, m(f) - 1\}$ is a diffeomorphism of the form $f^{p_m q}B$ with $B \in \mathcal{B}_{p_m, q_m}(P_m, P_{m+1})$. From $\# \mathcal{A}_n = \prod_{m=-1}^{m(f)-1} \# \mathcal{B}_{p_m, q_m}(P_m, P_{m+1})$ and (3.6) we obtain Proposition 3.1(a). From (3.5), (3.7) and (3.13), the elements of $\mathcal{A}_n$ are contained in $A_n(\tilde{\phi}, \tilde{\alpha})$. Since the measure $\delta_{O(f)}$ is supported on $O(f)$, from the definition of the Lyapunov exponent (1.2) we have $\chi(\mu) = \sum_{m=-1}^{m(f)-1} \rho_m \chi(\mu_m)$. Hence (3.8) implies Proposition 3.1(b).

**Case 3:** $m(f) < \infty$ and $\partial J_{m(f)}$ contains an attracting periodic point (Case C-II in Remark 2.2). We may assume there exist constants $\rho_1, \ldots, \rho_{m(f)} \in (0, 1)$ with $\sum_{m=1}^{m(f)} \rho_m = 1$, and a measure $\mu_m \in \mathcal{M}_m(f)$ for each $m \in \{-1, \ldots, m(f) - 1\}$ such that $\mu$ is written as in (3.12). This case is treated by a slight modification of the argument for Case 2.

**Case 4:** $m(f) < \infty$ and $f$ has no attracting periodic orbit (Case A in Remark 2.2). We may assume there exist constants $\rho_1, \ldots, \rho_{m(f)} \in (0, 1)$ with $\sum_{m=1}^{m(f)} \rho_m = 1$, and for each $m \in \{-1, \ldots, m(f)\}$ a measure $\mu_m \in \mathcal{M}_m(f)$ such that

$$\mu = \rho_1 \mu_1 + \rho_0 \mu_0 + \cdots + \rho_{m(f)} \mu_{m(f)}.$$

For $m \geq 0$ with $p_m + 1/p_m \neq 2$, we fix $P_m \in \mathcal{P}_m$ such that $P_m \subset L_m$ and $\mu_m(P_m) > 0$. For $m \geq 0$ with $p_m + 1/p_m = 2$, we set $P_m = L_m$.

Let $q \geq 1$ be an integer. For $m \geq -1$ with $p_{m+1}/p_m \neq 2$, let $\mathcal{B}_{p_m, q_m}(P_m, P_{m+1})$ denote the collection of diffeomorphic pullbacks of $P_{m+1}$ by $f^{p_m q}B$ contained in $P_m$ obtained by applying Lemma 3.3 with $\mu = \mu_m$, $P = P_m$, $P' = P_{m+1}$ and $\varepsilon$ replaced by $\varepsilon/2$. For $m \geq -1$ with $p_{m+1}/p_m = 2$, let $B$ denote the diffeomorphic pullback of $f^{p_{m+1}}(P_m) \cap P_{m+1}$ by $f^{p_m q}B$ that is contained in $P_m$ and contains the periodic point in $\partial J_{m+1}$. Set $\mathcal{B}_{p_m, q_m}(P_m, P_{m+1}) = \{B\}$.

Set $g = f^{p_{m(f)}}|_{L_{m(f)}}$; and let $\nu$ denote the normalized restriction of $\mu_m(f)$ to $L_{m(f)}$. Let $h(g, \nu)$, $\chi(g, \nu)$ denote the entropy and Lyapunov exponent of the measure $\nu$ with respect to $g$. By (30) and (1.2), $\chi(g, \nu) = \int \log |Dg| \, d\nu$. As in Section 2.2, $g$ is topologically exact. From (8) and Lemma 2.2, for all sufficiently large $q$ there exist a closed interval $P_m(f)$ in $L_{m(f)}$ and a finite collection $\mathcal{B}_{p_{m(f)}}(P_m(f), P_m(f))$ of diffeomorphic pullbacks of $P_m(f)$ by $g^q$ which are contained in $P_m(f)$ such that

$$\# \mathcal{B}_{p_{m(f)}}(P_m(f), P_m(f)) > \exp \left(\left(h(g, \nu) - \frac{\varepsilon}{2}\right) q\right), \tag{3.14}$$

$$= \exp \left(\left(p_{m(f)} h(\mu_{m(f)}) - \frac{\varepsilon}{2}\right) q\right).$$
and for all \( B \in \mathcal{B}_{pm(q)}(P_{m(f)}, P_{m(f)}) \),

\[
\sup_B \left\| \frac{1}{q} \sum_{k=0}^{q-1} \tilde{\phi} \circ g^k - \int \tilde{\phi} d\nu \right\| = \sup_B \left\| \frac{1}{q} S_{pm(q)}\tilde{\phi} - p_{m(f)} \int \tilde{\phi} d\mu_{m(f)} \right\| < \frac{\varepsilon}{2},
\]

and

\[
|B| > \exp \left( - \left( \chi(g, \nu) + \frac{\varepsilon}{2} \right) q \right) = \exp \left( - \left( p_{m(f)} \chi(\mu_{m(f)}) + \frac{\varepsilon}{2} \right) q \right).
\]

Let \( n \geq 1 \) be a large integer which is written as a linear combination of non-negative integers \( q_1, \ldots, q_m \), \( r \) in the form

\[
n = p_1q_1 + \cdots + p_mq_m + p_{m(f)} - 1 + p_{m(f)}q_m + r,
\]

with \( |p_mq_m - \rho_m| \leq 1 \) for \( -1 \leq m \leq m(f) \) and \( 0 \leq r < p_{m(f)} \). We assume \( n \) is sufficiently large so that for each \( m \in \{-1, \ldots, m(f) - 1\} \), \( \mathcal{B}_{pm(q_m)}(P_m, P_{m+1}) \) satisfies (3.6), (3.7), (3.8), and \( \mathcal{B}_{pmq_m}(P_{m(f)}, P_{m(f)}) \) satisfying (3.14), (3.15), (3.16) exists. Set \( P_{m(f)+1} = P_{m(f)} \) for convenience. Define \( \mathcal{A}_n \) to be the collection of pullbacks of \( P_{m(f)} \) by maps \( h_m(f) \circ \cdots \circ h_0 \circ h^-1 \), where \( h_m, m \in \{-1, \ldots, m(f)\} \) is a diffeomorphism of the form \( f^{p_mq_m} \) with \( B \in \mathcal{B}_{pmq_m}(P_m, P_{m+1}) \). From \( \# \mathcal{A}_n = \prod_{m=1}^{m(f)} \# \mathcal{B}_{pmq_m}(P_m, P_{m+1}) \), (3.6) and (3.14) we obtain Proposition 3.1(a). From (3.5), (3.7) and (3.15), the elements of \( \mathcal{A}_n \) are contained in \( A_n(\tilde{\phi}, \tilde{\alpha}) \). Since \( \chi(\mu_{m(f)}) \geq 0 \), the definition of the Lyapunov exponent (1.2) gives \( \chi(\mu) = \sum_{m=-1}^{m(f)} \rho_m \chi(\mu_m) \). From (3.8) and (3.16) we obtain Proposition 3.1(b).

\[\square\]

4. Large deviations upper bound

In this section we complete the proof of the upper bound in Theorem A. In Section 4.1 we show that this follows from a key upper bound stated in Proposition 4.1. The rest of this section is dedicated to the proof of Proposition 4.1.

4.1. Key upper bound. Let \( l \geq 1, \tilde{\phi} \in C(X)^l, \tilde{\alpha} \in \mathbb{R}^l \). Define

\[
\tilde{A}_n(\tilde{\phi}, \tilde{\alpha}) = \left\{ x \in X : \int \tilde{\phi} d\delta^n_x \geq \tilde{\alpha} \right\} \supset \text{cl}(A_n(\tilde{\phi}, \tilde{\alpha})).
\]

**Proposition 4.1.** Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point. Let \( l \geq 1, \tilde{\phi} \in C(X)^l, \tilde{\alpha} \in \mathbb{R}^l \). For any \( \varepsilon > 0 \) there exists \( N \geq 1 \) such that if \( n \geq N \) and \( \tilde{A}_n(\tilde{\phi}, \tilde{\alpha}) \neq \emptyset \), then there exists a measure \( \mu \in \mathcal{M}(f) \) such that

(a) \[|\tilde{A}_n(\tilde{\phi}, \tilde{\alpha})| \leq \exp((F(\mu) + \varepsilon)n) \text{ and} \]

(b) \[
\int \tilde{\phi} d\mu > \tilde{\alpha} - \varepsilon.
\]

We finish the proof of the upper bound in Theorem A assuming Proposition 4.1.

**Proof of the upper bound in Theorem A.** Let \( C \) be a non-empty closed subset of \( \mathcal{M} \). Let \( G \) be an arbitrary open set containing \( C \). Since \( \mathcal{M} \) is metrizable and \( C \) is
compact, we can choose \( \varepsilon > 0 \) and finitely many closed sets \( C_i, 1 \leq i \leq s \) of the form \( C_i = \{ \mu \in \mathcal{M} : \int \phi_i d\mu \geq \bar{\alpha}_i \} \) with \( i \geq 1, \phi_i \in C(X)^l, \bar{\alpha}_i \in \mathbb{R}^l \) so that

\[
\mathcal{C} \subset \bigcup_{i=1}^s C_i \subset \bigcup_{i=1}^s C_i(\varepsilon) \subset \mathcal{G},
\]

where \( \mathcal{C}_i(\varepsilon) = \{ \mu \in \mathcal{M} : \int \tilde{\phi}_i d\mu > \bar{\alpha}_i - \varepsilon \} \). Since \( F(\mu) \leq -I(\mu) \) for \( \mu \in \mathcal{M} \), Proposition 4.1 implies

\[
\limsup_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in C_i \}| \leq -\inf_{\mathcal{C}_i(\varepsilon)} I + \varepsilon
\]

for each \( i \in \{1, \ldots, s\} \). These and the relation (4.1) give

\[
\limsup_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in \mathcal{C} \}| \leq \max_{1 \leq i \leq s} \left( -\inf_{\mathcal{C}_i(\varepsilon)} I \right) + \varepsilon \leq -\inf_{\mathcal{G}} I + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary and \( \mathcal{G} \) is an arbitrary open set containing \( \mathcal{C} \), it follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \log |\{ x \in X : \delta^n_x \in \mathcal{C} \}| \leq \inf_{\mathcal{G} \supseteq \mathcal{C}} ( -\inf_{\mathcal{C}} I ) = -\inf I,
\]

as required. The last equality is due to the lower semicontinuity of \( I \).

\( \square \)

**Standing hypotheses from Sections 4.2 to 4.4**: The rest of this section is entirely dedicated to a proof of Proposition 4.1. We assume \( f : X \to X \) is an \( S \)-unimodal map with a non-flat critical point \( c \), and \( l \geq 1, \tilde{\phi} \in C(X)^l, \bar{\alpha} \in \mathbb{R}^l \) in the statement of Proposition 4.1 are fixed till the end of Section 4.4

4.2. Escape estimate in a single renormalization cycle. Assume \( m(f) \geq 1 \).

For each \( m \in \{0, \ldots, m(f) - 1\}, \tilde{\beta} \in \mathbb{R}^l, P \subset K_{m,m+1} \) and \( n \geq 1 \), define

\[
A_n(\tilde{\phi}, \tilde{\beta}, P) = A_n(\tilde{\phi}, \tilde{\beta}) \cap P \cap \bigcap_{k=0}^{n-1} f^{-k}(K_{m,m+1}).
\]

**Lemma 4.2.** Assume \( m(f) \geq 1 \), and let \( m \in \{0, \ldots, m(f) - 1\} \). For any \( \varepsilon > 0 \) there exists \( N \geq 1 \) such that if \( \tilde{\beta} \in \mathbb{R}^l, P \in \mathcal{P}_m, n \geq N \) satisfy \( A_n(\tilde{\phi}, \tilde{\beta}, P) \neq \emptyset \), then there exists a measure \( \mu \in \mathcal{M}(f) \) such that

(a) \(|A_n(\tilde{\phi}, \tilde{\beta}, P)| \leq \exp ((F(\mu) + \varepsilon)n)|P| \) and

(b) \( \int \tilde{\phi} d\mu > \tilde{\beta} - \varepsilon \).

**Proof.** Let \( \varepsilon > 0 \) and let \( P \in \mathcal{P}_m \). If \( p_{m+1}/p_m = 2 \), we can verify the claim easily, taking the measure \( \mu \) to be the equidistribution on the orbit of the periodic point in \( \partial J_m \). Suppose \( p_{m+1}/p_m \neq 2 \). We split the rest of the proof of Lemma 4.2 into two cases, either \( P \in \mathcal{P}_m \) or \( P \in \mathcal{P}_m \setminus \mathcal{P}_m \).

**Case 1**: \( P \in \mathcal{P}_m \). We perform an escape estimate relative to \( P \). Let \( n \geq 1 \) be such that \( A_n(\tilde{\phi}, \tilde{\beta}, P) \neq \emptyset \). Let \( \mathcal{P}_{m,n}(P) \) denote the collection of diffeomorphic pullbacks of elements of \( \mathcal{P}_m \) by \( f^n \) which are contained in \( P \) and intersect \( A_n(\tilde{\phi}, \tilde{\beta}) \). The union of elements of \( \mathcal{P}_{m,n}(P) \) contain \( A_n(\tilde{\phi}, \tilde{\beta}, P) \). Using Corollary 2.8, we fix an
integer multiple \( M_m \geq 1 \) of \( p_m \) such that for any \( Q \in \mathcal{P}_m \) there is a diffeomorphic pullback of \( P \) by \( f^{M_m} \) that is contained in \( Q \). For each \( Q \in \mathcal{P}_m \), we pick such a pullback and denote it by \( W_Q \).

**Sublemma 4.3.** For any \( \varepsilon' > 0 \) there exists \( N' \geq 1 \) such that for any \( Q \in \mathcal{P}_m \), \( n \geq N' \) and any diffeomorphic pullback \( \bar{W} \) of \( Q \) by \( f^n \) contained in \( P \), there exists a diffeomorphic pullback \( Y \) of \( P \) by \( f^{n+M_m} \) that is contained in \( \bar{W} \) and satisfies

\[
|Y| \geq e^{-\varepsilon'n}|W|.
\]

**Proof.** Define \( Y \) to be the diffeomorphic pullback of \( W_Q \) by \( f^n \) that is contained in \( W \). Then \( Y \) is a diffeomorphic pullback of \( P \) by \( f^{n+M_m} \). By Proposition 2.5, if \( \varepsilon \) replaced by \( \varepsilon'/2 \), if \( n \) is sufficiently large we have

\[
\frac{|Y|}{|W|} \geq e^{-\varepsilon'n} \frac{|f^n(Y)|}{|f^n(W)|} \geq e^{-\varepsilon'n} \min_{Q \in \mathcal{P}_m} \frac{|W_Q|}{|Q|} \geq e^{-\varepsilon'n},
\]

as required. \( \square \)

In view of Sublemma 4.3, for each \( W \in \mathcal{P}_{m,n}(P) \) we fix a diffeomorphic pullback \( Y_W \) of \( P \) by \( f^{n+M_m} \) that is contained in \( W \) and satisfies \( |Y_W| \geq e^{-\varepsilon'n/3}|W| \). Set

\[
\mathcal{P}_{m,n}'(P) = \{ Y_W : W \in \mathcal{P}_{m,n}(P) \}.
\]

We have

\[
Y \subset A_{n+M_m} \left( \bar{\phi}, \bar{\beta}, -\frac{1}{2} \varepsilon \right) \quad \text{for} \quad Y \in \mathcal{P}_{m,n}'(P), \quad \text{and}
\]

\[
|A_n(\bar{\phi}, \bar{\alpha}, P)| \leq \sum_{W \in \mathcal{P}_{m,n}(P)} |W| \leq e^{\varepsilon'n} \sum_{Y \in \mathcal{P}_{m,n}'(P)} |Y|.
\]

Set \( n = n + M_m \). The restriction of \( f^\bar{\sigma} \) to \( \bigcup_{Y \in \mathcal{P}_{m,n}'(P)} Y \) induces a fully branched expanding Markov map onto \( P \) with finitely many branches. Let \( \Delta \subset \Gamma_m \) denote its maximal invariant set, namely

\[
\Delta = \bigcap_{k=0}^{\infty} (f^\bar{\sigma})^{-k} \left( \bigcup_{Y \in \mathcal{P}_{m,n}'(P)} Y \right).
\]

The map \( f^\bar{\sigma}|\Delta : \Delta \to \Delta \) is topologically conjugate to the full shift \( \sigma : \Sigma \to \Sigma \) over the finite alphabet \( \mathcal{P}_{m,n}'(P) \). We write \( \pi : \Sigma \to \Delta \) for the conjugacy map, and define the induced potential \( \Phi : \Sigma \to \mathbb{R} \) by \( \Phi(\omega) = -\log |Df^\bar{\sigma}(\pi(\omega))| \). From Lemma 2.6, \( \Phi \) is continuous with respect to the shift metric. The variational principle [2, p.40, 2.17] gives

\[
\sup_{\bar{\nu} \in \mathcal{M}(\sigma)} \left( h(\sigma, \bar{\nu}) + \int \Phi d\bar{\nu} \right) = \lim_{k \to \infty} \frac{1}{k} \log \left( \sum_{\omega \in \sigma^{-k}(\omega')} \exp \sum_{i=0}^{k-1} \Phi(\sigma^i(\omega')) \right)
\]

for any fixed \( \omega' \in \Sigma \), where \( \mathcal{M}(\sigma) \) denotes the space of \( \sigma \)-invariant Borel probability measures endowed with the weak* topology, and \( h(\sigma, \bar{\nu}) \) denotes the entropy of \( \bar{\nu} \).
with respect to $\sigma$. By the distortion estimates in Proposition 2.5, if $n \geq 1$ is sufficiently large then for all $Y \in \mathcal{P}_{m,n}(P)$ and $\omega \in \Sigma$ such that $\pi(\omega) \in Y$ we have
\[
\exp (\Phi(\omega)) \geq e^{-\frac{\varepsilon}{2n}} |Y| |P|.
\]
Hence, the series inside the logarithm in (4.4) is bounded from below as follows:
\[
\sum_{\omega \in \sigma^{-k}(\omega')} \exp \sum_{i=0}^{k-1} \Phi(\sigma^i\omega) \geq \left( \inf_{\omega' \in \Sigma} \sum_{\omega \in \sigma^{-1}(\omega')} \exp (\Phi(\omega)) \right)^k \geq \left( e^{-\frac{\varepsilon}{2n}} \sum_{Y \in \mathcal{P}_{m,n}(P)} |Y| |P| \right)^k.
\]
Taking logarithms of both sides, dividing by $k$ and letting $k \to \infty$, we have
\[
\lim_{k \to \infty} \frac{1}{k} \log \left( \sum_{\omega \in \sigma^{-k}(\omega')} \exp \sum_{i=0}^{k-1} \Phi(\sigma^i\omega) \right) \geq \log \sum_{Y \in \mathcal{P}_{m,n}(P)} \frac{|Y|}{|P|} - \frac{\varepsilon n}{2}.
\]
Plugging this inequality into (4.4) yields
\[
(4.5) \sup_{\nu \in \mathcal{M}(\sigma)} \left( h(\sigma, \nu) + \int \Phi d\nu \right) \geq \log \sum_{Y \in \mathcal{P}_{m,n}(P)} \frac{|Y|}{|P|} - \frac{\varepsilon n}{2}.
\]
Since $\mathcal{M}(\sigma)$ is compact in the weak* topology and the entropy is upper semicontinuous on it, there exists a measure $\tilde{\mu} \in \mathcal{M}(\sigma)$ which attains the supremum in (4.5). The measure $\tilde{\mu} \circ \pi^{-1}$ on $\Delta$ is $f^n$-invariant and its spread
\[
\mu = \frac{1}{\tilde{n}} \sum_{Y \in \mathcal{P}_{m,n}(P)} \sum_{k=0}^{\tilde{n}-1} (\tilde{\mu}|Y) \circ (f^k \circ \pi)^{-1}
\]
belongs to $\mathcal{M}_m(f)$. Although $\tilde{\mu}$ may not be the first return map to $\Delta$, Abramov’s formula, connecting the entropies of $\mu$ and $\tilde{\mu} \circ \pi^{-1}$, and Kac’s formula, connecting the integrals of $-\log |Df|$ and $\Phi$, still hold [26, Theorem 2.3] and we have
\[
(4.6) \quad h(\sigma, \tilde{\mu}) + \int \Phi d\tilde{\mu} = F(\mu) \tilde{n}.
\]
From (4.5) and (4.6) we have
\[
\sum_{Y \in \mathcal{P}_{m,n}(P)} |Y| \leq \exp \left( \left( F(\mu) + \frac{\varepsilon}{2} \right) \tilde{n} \right) |P|.
\]
From this inequality and (4.3), for all sufficiently large $n \geq 1$ we obtain
\[
|A_n(\bar{\phi}, \bar{\beta}, P)| \leq \exp \left( \frac{\varepsilon n}{3} \right) \sum_{Y \in \mathcal{P}_{m,n}(P)} |Y| \leq \exp \left( (F(\mu) + \varepsilon) n \right) |P|,
\]
as required in Lemma 4.2(a). Item (b) follows from (4.2).

Case 2: $P \in \mathcal{P}_m \setminus \mathcal{P}_m$. Let $z$ denote the periodic point of period $p_m$ in $\partial J_m$. Then $z$ is hyperbolic repelling, and the point in $\partial J_m$ other than $z$ is mapped to $z$ by $f^{p_m}$. For simplicity we assume $z \in P$. Otherwise, $z \in f^{p_m}(P)$ holds and so the argument is analogous.
Let \( n > 1 \) satisfy \( A_n(\vec{\phi}, \vec{\beta}, P) \neq \emptyset \). Our strategy is to use the simplest version of the coarse graining that will be formally introduced in Section 4.3: we begin by splitting \( A_n(\vec{\phi}, \vec{\beta}, P) \) into two subsets, one consisting of points which remain in \( \bigcup_{k=0}^{p_m-1} f^k(P) \) almost until time \( n \), and the complement of this set. For the first set, the influence of the dynamics near \( \Gamma_m \) is negligable, and a local analysis near the orbit of \( z \) suffices for the estimate of its Lebesgue measure. The second set is influenced by the dynamics near the orbit of \( z \) and the dynamics near \( \Gamma_m \). We estimate the Lebesgue measure of each set separately, and unify the estimates at the end to obtain the desired one in Lemma 4.2.

Let \( K'_m \subset K_m \) denote the union of elements of \( \mathcal{P}_m \). For each integer \( t \geq 1 \), put

\[
V_t = P \cap \left( f^{-t}(K'_m) \setminus \bigcup_{j=0}^{t-1} f^{-j}(K'_m) \right),
\]

and split \( |A_n(\vec{\phi}, \vec{\beta}, P)| = I + II \) where

\[
I = \sum_{t=1}^{n-1} |A_n(\vec{\phi}, \vec{\beta}, V_t)| \quad \text{and} \quad II = \sum_{t=n}^{\infty} |A_n(\vec{\phi}, \vec{\beta}, V_t)|.
\]

The set \( V_t \) is non-empty if and only if \( t \) is an integer multiple of \( p_m \). If \( V_t \neq \emptyset \), then \( V_t \) is an interval and satisfies

\[
|f^t(V_t)| \geq \min\{|Q| : Q \in \mathcal{P}_m\} > 0.
\]

Moreover, since \( f^t|_{V_t} \) extends to a diffeomorphism on an interval containing \( z \), the distortion of \( f^t \) on \( V_t \) is bounded by a constant \( C \) independent of \( t \). Then we have

\[
|V_t| \leq C \exp(F(\delta_{p_m}^t) n) |P|.
\]

By (4.8) we obtain

\[
II \leq \sum_{t=n}^{\infty} |V_t| \leq \frac{C \exp(F(\delta_{p_m}^t) n)}{1 - \exp(F(\delta_{p_m}^t) p_m)} |P|.
\]

We now estimate \( I \). For each \( t \in \{1, \ldots, n-1\} \), let \( M_{2,l}(t, \varepsilon \mathbb{Z}) \) denote the set of \( 2 \times l \) matrices \( B = (\beta_{ij})_{i \in \{0,1\}, j \in \{1,\ldots,l\}} \) with entries in \( \varepsilon \mathbb{Z} = \{\varepsilon a : a \in \mathbb{Z}\} \) for which the following hold for every \( j \in \{1, \ldots, l\} \):

\[
(4.9) \quad \beta_{0j} t + \beta_{1j} (n-t) > \left( \beta_j - \frac{\varepsilon}{3} \right) n,
\]

\[
(4.10) \quad \inf \phi_j - \varepsilon < \beta_{ij} \leq \sup \phi_j \quad \text{for} \ i \in \{0,1\}.
\]

For \( B = (\beta_{ij}) \in M_{2,l}(t, \varepsilon \mathbb{Z}) \) and \( i \in \{0,1\} \), put \( \vec{\beta}_i = (\beta_{i1}, \ldots, \beta_{il}) \). Consider the set

\[
A_n(t, B) = \left\{ x \in \bigcup_{k=0}^{n-1} f^{-k}(K_{m,m+1}) : \frac{1}{t} S_t \vec{\phi}(x) \geq \vec{\beta}_0 \text{ and } \frac{1}{n-t} S_{n-t} \vec{\phi}(f^t(x)) \geq \vec{\beta}_1 \right\}.
\]

We claim that

\[
A_n(\vec{\phi}, \vec{\beta}, V_t) \subset \bigcup_{B \in M_{2,l}(t, \varepsilon \mathbb{Z})} A_n(t, B) \cap V_t.
\]
Indeed, for each \( x \in A_n(\vec{\phi}, \vec{\beta}, V_i) \) and \( j \in \{1, \ldots, l\} \) there exist \( \beta_{0j}, \beta_{1j} \in \varepsilon \mathbb{Z} \) such that (4.9) holds and
\[
\frac{1}{t} S_t \phi_j(x) - \beta_{0j} \in [0, \varepsilon) \quad \text{and} \quad \frac{1}{n - t} S_{n-t} \phi_j(f^t(x)) - \beta_{1j} \in [0, \varepsilon).
\]
This implies (4.10), and that the matrix \( B = (\beta_{ij}) \) belongs to \( \mathbb{M}_{2l}(t, \varepsilon \mathbb{Z}) \), which verifies the claim.

Let \( N > 1 \) be an integer such that (a), (b) of Lemma 4.2 in the case \( P \in \mathcal{P}_m \) hold for all \( n \geq N \) with \( \varepsilon \) replaced by \( \varepsilon/3 \). Let \( n > N \) and let \( t \in \{1, \ldots, n - N\} \).

In view of the above claim, for each \( B \in \mathbb{M}_{2l}(t, \varepsilon \mathbb{Z}) \) with \( A_n(t, B) \cap V_i \neq 0 \) we estimate \( |A_n(t, B) \cap V_i| \) from above. Let \( \mathcal{P}_m(t, B) \) denote the collection of diffeomorphic pullbacks of elements of \( \mathcal{P}_m \) by \( f^{n-t} \) that intersect \( f^t(A_n(t, B) \cap V_i) \). The elements of \( \mathcal{P}_m(t, B) \) intersect \( A_{n-t}(\vec{\phi}, \vec{\beta}_1) \), and their union contains \( f^t(A_n(t, B) \cap V_i) \), because
\[
f^t(A_n(t, B) \cap V_i) \subset A_{n-t}(\vec{\phi}, \vec{\beta}_1) \cap \bigcap_{k=0}^{n-t-1} f^{-k}(K_{m,m+1}).
\]

From (a), (b) of Lemma 4.2 in the case \( P \in \mathcal{P}_m \), for each \( Q \in \mathcal{P}_m(t, B) \) there exists \( \mu_{t,Q} \in \mathcal{M}_m(f) \) such that
\[
\int \vec{\phi} d\mu_{t,Q} > \vec{\beta}_1 - \frac{1}{3} \varepsilon.
\]

Pick a measure \( \mu_{t,B} \in \{ \mu_{t,Q} : Q \in \mathcal{P}_m(t, B) \} \) which maximizes the free energy within this finite set. By (4.11) and \( \sum_{Q \in \mathcal{P}_m(t, B)} |Q| \leq |X| \) we have
\[
|f^t(A_n(t, B) \cap V_i)| \leq \sum_{Q \in \mathcal{P}_m(t, B)} |A_{n-t}(\vec{\phi}, \vec{\beta}_1) \cap Q| \leq \exp \left( \left( F(\mu_{t,B}) + \frac{\varepsilon}{3} \right) (n - t) \right) |X|.
\]

Combining this with (4.7),
\[
\frac{|A_n(t, B) \cap V_i|}{|V_i|} \leq C \frac{|f^t(A_n(t, B) \cap V_i)|}{|f^t(V_i)|} \leq \frac{C|X|}{|f^t(V_i)|} \exp \left( \left( F(\mu_{t,B}) + \frac{\varepsilon}{3} \right) (n - t) \right).
\]

Define a measure \( \nu_{t,B} \in \mathcal{M}(f) \) by
\[
\nu_{t,B} = \frac{t}{n} \delta^{P_m} + \left( 1 - \frac{t}{n} \right) \mu_{t,B}.
\]

We claim that
\[
\int \vec{\phi} d\nu_{t,B} > \vec{\beta} - \varepsilon.
\]
Indeed, if $t \sup \|\bar{\phi}\| < \varepsilon n$ then $(t/n) (\bar{\beta}_0 - \bar{\beta}_1) \leq (t/n) 2 \sup \|\bar{\phi}\| < \varepsilon/3$ by (4.10). Combining this with (4.9) and $\int \bar{\phi} d\mu_{t,B} > \bar{\beta}_1 - (1/3) \varepsilon$ from (4.12), we have

$$
\int \bar{\phi} d\nu_{t,B} = \int \bar{\phi} d\delta_{\bar{z}}^m + \int \bar{\phi} d\mu_{t,B} > -\frac{1}{6} \varepsilon + \bar{\beta}_1 - \frac{1}{3} \varepsilon
$$

$$
= \frac{t}{n} \bar{\beta}_0 + \frac{n-t}{n} \bar{\beta}_1 - \frac{t}{n} (\bar{\beta}_0 - \bar{\beta}_1) - \frac{1}{3} \varepsilon > \bar{\beta} - \varepsilon.
$$

Otherwise we have

$$
\int \bar{\phi} d\delta_{\bar{z}}^m = \int \bar{\phi} d\delta_{\bar{z}} > \bar{\beta}_0 - \frac{1}{3} \varepsilon,
$$

provided $n$ is sufficiently large. From this and (4.9) we obtain

$$
\int \bar{\phi} d\nu_{t,B} = \frac{t}{n} \int \bar{\phi} d\delta_{\bar{z}}^m + \frac{n-t}{n} \int \bar{\phi} d\mu_{t,B} > \frac{t}{n} \bar{\beta}_0 + \frac{n-t}{n} \bar{\beta}_1 - \frac{2}{3} \varepsilon \geq \bar{\beta} - \varepsilon.
$$

From (4.7), (4.8) and (4.13) we have

$$
|A_n(t, B) \cap V_t| \leq C^2 |X| \exp \left( \left( F(\nu_{t,B}) + \frac{\varepsilon}{3} \right) n \right) |P|.
$$

Let $\mu$ be a measure in $\{\nu_{t,B} : B \in \mathcal{M}_{2,l}(t, \varepsilon Z)\} \cup \{\delta_{\bar{z}}^m\}$ which maximizes the free energy within this finite set. Clearly we have

$$
\# \mathcal{M}_{2,l}(t, \varepsilon Z) \leq \prod_{j=1}^l \left( \frac{1}{\varepsilon} (\sup \phi_j - \inf \phi_j) + 2 \right)^2.
$$

If $n$ is sufficiently large, then (4.15) and (4.16) together imply

$$
|A_n(\bar{\phi}, \bar{\beta}, V_t)| \leq \sum_{B \in \mathcal{M}_{2,l}(t, \varepsilon Z)} |A_n(t, B) \cap V_t| \leq \exp \left( \left( F(\mu) + \frac{\varepsilon}{2} \right) n \right) |P|.
$$

To finish, from (4.8) and (4.17) we have

$$
I = \sum_{t=1}^{n-N} |A_n(\bar{\phi}, \bar{\beta}, V_t)| + \sum_{t=n-N+1}^{n-1} |A_n(\bar{\phi}, \bar{\beta}, V_t)|
$$

$$
\leq \frac{n}{p_m} \exp \left( \left( F(\mu) + \frac{\varepsilon}{2} \right) n \right) |P| + \frac{n}{p_m} C \exp \left( F(\mu)(n - N) \right) |P|.
$$

Combining this estimate with that of $II$, for all sufficiently large $n$ we obtain

$$
|A_n(\bar{\phi}, \bar{\beta}, P)| = I + II \leq \exp \left( (F(\mu) + \varepsilon)n \right) |P|,
$$

as required in Lemma 4.2(a). Item (b) is a consequence of (4.14). □

4.3. **Coarse graining decomposition into clusters.** In this subsection, we decompose the set $A_n(\bar{\phi}, \bar{\alpha})$ into a finite number of clusters consisting of points which share the same ‘itinerary’ up to time $n$, and share almost the same average values of $\bar{\phi}$ along the segments of orbits each contained in a single cycle.
Let $\varepsilon > 0$. If $m(f) = \infty$, then let $m_* \geq 1$ and $N_* \geq 1$ be integers for which the conclusion of Lemma 2.10 holds with $\varepsilon$ replaced by $\varepsilon/2$. We set

$$M = \begin{cases} m_* & \text{if } m(f) = \infty \\ m(f) & \text{if } 1 \leq m(f) < \infty. \end{cases}$$

For each $x \in X$, we define by induction two (possibly infinite) increasing sequences $\{\hat{n}_i(x)\}_{i=0}^{d_*(x)}$, $\{\hat{m}_i(x)\}_{i=0}^{d_*(x)}$ of non-negative integers that successively record the time and position at which the orbit of $x$ falls into deeper renormalization cycles. Start with

$$\hat{n}_0(x) = 0 \text{ and } \hat{m}_0(x) = \max\{m \in \{0, \ldots, M\} : x \in K_m\}.$$

Let $i \geq 0$ and suppose $\hat{n}_i(x)$ and $\hat{m}_i(x)$ are defined. If $\hat{m}_i(x) = M$, or if $\hat{m}_i(x) \leq M - 1$ and $f^{k}(x) \in K_{\hat{n}_i(x), \hat{m}_i(x)+1}$ for all $k > \hat{n}_i(x)$, then we stop the inductive definition by setting $d_*(x) = i$. Otherwise, we define

$$\hat{n}_{i+1}(x) = \min\{k \geq \hat{n}_i(x) + 1 : f^{k}(x) \notin K_{\hat{n}_i(x), \hat{m}_i(x)+1}\}$$

and

$$\hat{m}_{i+1}(x) = \max\{m \in \{\hat{m}_i(x) + 1, \ldots, M\} : f^{\hat{n}_{i+1}(x)}(x) \in K_m\}.$$

For two integers $n$, $d$ with $0 \leq d \leq \min\{n-1, M\}$ we define

$$I_n(d) = \left\{ t = ((n_0, m_0), (n_1, m_1), \ldots, (n_d, m_d)) \in (\mathbb{Z}^2)^{d+1} : \begin{align*} 0 &= n_0 < n_1 < \cdots < n_d < n, \\ 0 &\leq m_0 < m_1 < \cdots < m_d \leq M \end{align*} \right\}.$$

Let $t = ((n_0, m_0), (n_1, m_1), \ldots, (n_d, m_d)) \in I_n(d)$. For convenience, we set $n_{d+1} = n$ and $m_{d+1} = m_d$, and call $n_0, n_1, \ldots, n_{d+1}$ transition times. For each $i \in \{0, \ldots, d\}$ we set

$$t_i = n_{i+1} - n_i.$$

Let $R(t)$ denote the set of $x \in X$ such that $d_*(x) \geq d$, $(\hat{n}_i(x), \hat{m}_i(x)) = (n_i, m_i)$ for every $i \in \{0, \ldots, d\}$, and $\hat{n}_{d+1}(x) \geq n$ if $d_*(x) \geq d + 1$. We have

$$X \subset \bigcup_{d=0}^{M} \bigcup_{t \in I_n(d)} R(t).$$

We decompose the set $\bar{A}_n(\vec{\phi}, \vec{\alpha}) \cap R(t)$ with respect to coarse grained values at scale $\varepsilon$ of time averages of $\vec{\phi}$ between two consecutive transition times. For each $t \in I_n(d)$, let $M_{d+1, i}(t, \varepsilon \mathbb{Z})$ denote the set of $(d+1) \times I$ matrices $A = (\alpha_{ij})_{i \in \{0, \ldots, d\}, j \in \{1, \ldots, I\}}$ with entries in $\varepsilon \mathbb{Z}$ such that the following hold for every $j \in \{1, \ldots, I\}$:

$$\sum_{i=0}^{d} \alpha_{ij} t_i > \left( \alpha_j - \frac{\varepsilon}{3} \right) n,$$

$$\inf \phi_j - \varepsilon < \alpha_{ij} \leq \sup \phi_j \text{ for every } i \in \{0, \ldots, d\}.$$
Similarly to the argument in the proof of Lemma 4.2 in Case 2, we obtain
\[ \bar{A}_n(\vec{\phi}, \vec{\alpha}) \cap R(t) \subset \bigcup_{A \in \bar{M}_{d+1,t}(t, \varepsilon Z)} R(t, A). \]

4.4. Lebesgue measures of clusters. We are in position to state and prove a main technical estimate on the Lebesgue measure of each cluster $R(t, A)$.

**Proposition 4.4.** Assume $m(f) \geq 1$. For any $\varepsilon > 0$ there exists $N \geq 1$ such that if $n \geq N$, $t \in \bigcup_{d=0}^{M} I_n(d)$, $A \in \bar{M}_{d+1,t}(t, \varepsilon Z)$ and $R(t, A) \neq \emptyset$, then there exists a measure $\mu \in \mathcal{M}(f)$ satisfying

(a) \[ |R(t, A)| \leq \exp((F(\mu) + \varepsilon)n) \quad \text{and} \]

(b) \[ \int \vec{\phi} d\mu > \bar{\alpha} - \varepsilon. \]

**Proof.** Let $\varepsilon > 0$, and let $\delta > 0$ be such that

\[ \| \vec{\phi}(x) - \vec{\phi}(y) \| < \varepsilon \]

for $x, y \in X$ with $|x - y| < \delta$.

In view of Lemma 2.6, let $N_0 \geq 1$ be an integer such that for any $m \in \{0, \ldots, M-1\}$ and any connected component $W$ of $\bigcap_{k=0}^{N_0-1} f^{-k}(K_{m,m+1})$, we have $|W| < \delta$. The rest of the proof of Proposition 4.4 consists of three steps. In the course of the proof we will require $N_0$ to be sufficiently large.

**Step 1 (Estimate of error bound in transition).** Let $n \geq 1$, $d \in \{0, \ldots, M\}$, $t \in I_n(d)$, $A \in \bar{M}_{d+1,t}(t, \varepsilon Z)$ satisfy $R(t, A) \neq \emptyset$. We set $R_0(A) = K_{m_0,m_0+1}$, and let $\bar{R}_0(A)$ denote the collection of connected components of $R_0(A)$. For each $i \in \{1, \ldots, d+1\}$, let $\bar{R}_i(A)$ denote the collection of diffeomorphic pullbacks of the connected components of $K_{m_i}$ by $f^{m_i}$ which are contained in $R_0(A)$ and intersect $R(t, A)$. Let $R_i(A)$ denote the union of elements of $\bar{R}_i(A)$. Note that

\[ R(t, A) \subset R_{d+1}(A) \subset \cdots \subset R_1(A) \subset R_0(A). \]

Among all transition times, for our purpose it suffices to concentrate on consecutive ones which are well-separated.

**Lemma 4.5.** For any $i \in \{0, \ldots, d\}$ with $t_i > N_0$ we have

\[ f^{m_i}(R_{i+1}(A)) \subset \bar{A}_{t_i} \left( \vec{\phi}, \vec{\alpha}_i - \frac{1}{2}\varepsilon \right). \]

**Proof.** Let $R \in \bar{R}_{i+1}(A)$. Fix $x_0 \in R \cap R(t, A)$. The definition of $\bar{R}_{i+1}(A)$ gives

\[ S_{t_i} \vec{\phi}(f^{m_i}(x_0)) \geq t_i \vec{\alpha}_i. \]

For each $k \in \{0, \ldots, t_i - N_0 + 1\}$ we have $t_i - k \geq N_0 - 1$, and so $f^{m_i}(R) \subset \bigcap_{j=0}^{N_0-1} f^{-j}(K_{m_i,m_i+1})$. The choice of $N_0$ gives $|f^{m_i+k}(R)| < \delta$. By (4.21), for any $x \in R$ we have

\[ \|S_{t_i-N_0} \vec{\phi}(f^{m_i}(x)) - S_{t_i-N_0} \vec{\phi}(f^{m_i}(x_0))\| \leq \frac{\varepsilon}{3} (t_i - N_0). \]

For the remaining $N_0$ iterations, clearly we have

\[ \|S_{N_0} \vec{\phi}(f^{m_i+1-N_0}(x)) - S_{N_0} \vec{\phi}(f^{m_i+1-N_0}(x_0))\| \leq 2N_0 \sup \| \vec{\phi} \|. \]
Since $t_i > N_0$, combining these three displayed estimates we obtain
\[
\frac{1}{t_i} S_{t_i} \phi(f^{n_i}(x)) \geq \bar{\alpha}_i - \frac{1}{2}\varepsilon;
\]
promvided $N_0$ is sufficiently large. Since $x \in R$ is arbitrary and $R \in \mathcal{R}_{i+1}(A)$ is arbitrary, the desired inclusion follows.

\begin{proof}

\textit{Step 2 (Estimate of $i$-step conditional probability).} The next lemma provides estimates of conditional probabilities at well-separated transition times.

\begin{lemma}
For any $i \in \{0, \ldots, d\}$ with $t_i > N_0$, there exists a measure $\mu_i \in \mathcal{M}(f)$ such that
\[
(a) \quad \frac{|R_{i+1}(A)|}{|R_i(A)|} \leq \exp\left(\left(F(\mu_i) + \varepsilon\right) t_i\right) \quad \text{and}
\]
\[
(b) \quad \int \tilde{\phi} d\mu_i > \bar{\alpha}_i - \varepsilon.
\]
\end{lemma}

\begin{proof}

We first treat the case $m_i \leq M - 1$. Let $R \in \mathcal{R}_i(A)$. Let $\mathcal{P}_{m_i}(A, R)$ denote the collection of diffeomorphic pullbacks of elements of $K_{m_i}$ by $f^{t_i}$ that intersect $f^{n_i}(R \cap R_{i+1}(A))$. By the assumption $t_i > N_0$ and Lemma 4.5 each element $P \in \mathcal{P}_{m_i}(A, R)$ intersect $\tilde{A}_{t_i}(\tilde{\phi}, \bar{\alpha}_i - (1/2)\varepsilon)$, and by Lemma 4.2 there exists $\nu_{R, P} \in \mathcal{M}(f)$ such that
\[
\begin{align*}
& (4.23) \quad \left| A_{t_i}(\tilde{\phi}, \bar{\alpha}_i - \frac{1}{2}\varepsilon, P) \right| \leq \exp \left( \left( F(\nu_{R, P}) + \frac{\varepsilon}{2} \right) t_i \right) |P| \quad \text{and} \\
& (4.24) \quad \int \tilde{\phi} d\nu_{R, P} > \bar{\alpha}_i - \varepsilon.
\end{align*}
\]

Let $\mu_i \in \mathcal{M}(f)$ be a measure in \{\nu_{R, P}: $R \in \mathcal{R}_i(A), P \in \mathcal{P}_{m_i}(A, R)$\} which maximizes the free energy within this finite set. By (4.23) we have
\[
|f^{n_i}(R \cap R_{i+1}(A))| \leq \sum_{P \in \mathcal{P}_{m_i}(A, R)} \left| A_{t_i}(\tilde{\phi}, \bar{\alpha}_i - \frac{1}{2}\varepsilon, P) \right| \leq \sum_{P \in \mathcal{P}_{m_i}(A, R)} \exp \left( \left( F(\nu_{R, P}) + \frac{\varepsilon}{2} \right) t_i \right) |P| \leq \exp \left( \left( F(\mu_i) + \frac{\varepsilon}{2} \right) t_i \right) |f^{n_i}(R)|.
\]

If $i \geq 1$, then for $0 \leq j < i$, $f^{n_i}(R)$ is contained in $K_{n_j} \cap K_{n_{j+1}}$ for $n_j \leq n \leq n_{j+1} - n_j - 1$. By Proposition 2.5(a), the distortion of the composition $f^{n_i} = f^{t_{i+1}} \circ \cdots \circ f^{t_1} \circ f^{n_0}$ on $R$ is bounded by
\[
D_i = \gamma_{m_{i-1}} \cdots \gamma_{m_1} \gamma_{m_0}.
\]

From (4.25) we obtain
\[
\frac{|R \cap R_{i+1}(A)|}{|R|} \leq D_i \frac{|f^{n_i}(R \cap R_{i+1}(A))|}{|f^{n_i}(R)|} \leq \exp \left( \left( F(\mu_i) + \varepsilon \right) t_i \right),
\]

provided \( N_0 \) is sufficiently large. Since \( n_0 = 0 \), the same upper bound remains valid for \( i = 0 \). Consequently we obtain

\[
\frac{|R_{t+1}(A)|}{|R_t(A)|} \leq \max_{\nu \in \mathcal{S}(\lambda)} \frac{|R \cap R_{t+1}(A)|}{|R|} \leq \exp \left( (F(\mu_t) + \varepsilon) t_i \right),
\]

as required in Lemma 4.6(a). Lemma 4.6(b) is a consequence of (4.24).

To complete the proof of Lemma 4.6, it is left to treat the case \( m_i = M \). In this case we have \( i = d \), and there are four subcases.

\textbf{Case 1:} \( m(f) = \infty \). We have \( M = m_* \) by (4.18). Set \( \mu_d = \mu_\infty \). Clearly Lemma 4.6(a) holds since \( F(\mu_d) = 0 \). Since \( t_d > N_0 \) and \( f^{n_d}(R_{d+1}(A)) \subset \bar{A}_{t_d}(\phi, c_d - (1/2)\varepsilon) \cap K_M \) by Lemma 4.5, Lemma 4.6(b) follows from Lemma 2.10 if \( N_0 \) is sufficiently large.

\textbf{Case 2:} \( m(f) = M \) and \( \text{int}(J_{m(f)}) \) contains an attracting periodic point (Case C-I in Remark 2.2). Note that \( K_M = \text{cl}(B(f)) \) by (2.3) and Lemma 2.1. Let \( z \) denote the point from \( O(f) \) that is contained in \( \text{int}(J_M) \). Fix a closed interval \( J \subset J_M \) such that \( c, z \in \text{int}(J), f^{p_M}(J) \subset J \) and \( |K_M \setminus K_{M+1}| < \delta \) where \( K_{M+1} = \bigcup_{k=0}^{p_M-1} f^k(J) \). Let \( \nu \) denote the element of \( \mathcal{M}_{M-1}(f) \) supported on the orbit of the hyperbolic repelling periodic point in \( \partial J_M \).

The rest of the argument is similar in spirit to that in Case 2 in the proof of Lemma 4.2. For each \( R \in \mathcal{A}_d(A) \) and \( t \in \{0, \ldots, t_d\} \), define

\[
V_{R,t} = \begin{cases} 
  f^{n_d}(R) \cap K_{M+1} & \text{for } t = 0, \\
  f^{n_d}(R) \cap \left( f^{-t}(K_{M+1}) \setminus \bigcup_{j=0}^{t-1} f^{-j}(K_{M+1}) \right) & \text{for } t \in \{1, \ldots, t_d - 1\}, \\
  f^{n_d}(R) \setminus \bigcup_{j=0}^{t_d-1} f^{-j}(K_{M+1}) & \text{for } t = t_d.
\end{cases}
\]

We have \( f^{n_d}(R) = \bigcup_{t=0}^{t_d} V_{R,t} \). If \( V_{R,t} \neq \emptyset \), then \( V_{R,t} \) is the union of at most two intervals. Let \( W_{R,t} \) denote the pullback of \( V_{R,t} \) by \( f^{n_d} \) that is contained in \( R \). Define \( \nu_{R,t} \in \mathcal{M}(f) \) by

\[
\nu_{R,t} = \frac{t}{t_d} \nu + \left( 1 - \frac{t}{t_d} \right) \delta_{O(f)}.
\]

There exists a constant \( C_0 \geq 1 \) depending only \( f, \delta \) such that if \( x \in X \) and \( k \geq 1 \) satisfy \( x, \ldots, f^{k-1}(x) \in K_M \setminus K_{M+1} \) then \( |(f^k)'x| \geq C_0^{-1} e^{\lambda(\nu)k} \). Hence

\[
(4.27) \quad \frac{|V_{R,t}|}{|f^{n_d}(R)|} \leq C_0 \exp \left( -\lambda(\nu)t \right) = C_0 \exp (F(\nu)t).
\]

Since \( F(\delta_{O(f)}) = 0 \) we have \( F(\nu) \leq F(\nu_{R,t}) \). As in the proof for the first case, by Proposition 2.5 the distortion of \( f^{n_d} \) on \( R \) is bounded by the constant \( D_d \) in (4.26) so that

\[
(4.28) \quad \frac{|W_{R,t}|}{|R|} \leq C_0 D_d \exp (F(\nu_{R,t}) t_d).
\]

For each \( R \in \mathcal{A}_d(A) \) put \( T_R = \{ t \in \{0, \ldots, t_d\} : W_{R,t} \cap \mathcal{A}_{d+1}(A) \neq \emptyset \} \). Let \( \mu_d \) be a measure in \( \{ \nu_{R,t} : R \in \mathcal{A}_d(A), t \in T_R \} \) which maximizes the free energy within
this finite set. Since \( t_d > N_0 \), using (4.28) we obtain
\[
\frac{|R_{d+1}(A)|}{|R_d(A)|} \leq \max_{R \in \mathcal{R}_d(A)} \frac{|R \cap R_{d+1}(A)|}{|R|} \leq \max_{R \in \mathcal{R}_d(A)} \sum_{t \in T_R} \frac{|W_{R,t}|}{|R|}
\]
\[
\leq (t_d + 1) \exp \left( (F(\mu_d)) t_d \right) \leq \exp \left( (F(\mu_d) + \varepsilon) t_d \right),
\]
provided \( N_0 \) is sufficiently large. This implies Lemma 4.6(b).

For a proof of Lemma 4.6(b), it suffices to show that
\[
\int \hat{\phi} \, d\nu_{R,t} > \tilde{\alpha}_d - \bar{\varepsilon} \quad \text{for } R \in \mathcal{R}_d(A) \text{ and } t \in T_R.
\]

We may assume \( t_d - t \geq p_M \), for otherwise (4.29) holds if \( N_0 \) is sufficiently large. For each \( t \in T_R \) we fix \( x_{R,t} \in \bar{A}_d(\phi, \tilde{\alpha}_d) \cap V_{R,t} \subseteq K_M \), and let \( y_{R,t} \) denote the point from the periodic orbit supporting the measure \( \nu \) that is contained in the connected component of \( K_M \setminus K_{M-1} \) containing \( x_{R,t} \). Take \( k \in \{0, \ldots, p_M - 1\} \) with \( f^t(x_{R,t}) \in f^k(J) \). The choice of \( \delta \) in (4.21) and that of \( J \) imply
\[
S_t \hat{\phi}(x_{R,t}) \leq S_t \hat{\phi}(y_{R,t}) + \frac{t_d - t}{3} \bar{\varepsilon},
\]
and there exists a constant \( C_1 > 0 \) depending only on \( f, \delta, \phi \) such that
\[
\|S_{t_d-t} \hat{\phi}(f^t(x_{R,t})) - S_{t_d-t} \hat{\phi}(f^k(z))\| < \frac{t_d - t}{3} \bar{\varepsilon} + C_1.
\]
Write \( t_d - t = p_M q + r \) where \( q, r \) are non-negative integers with \( 0 \leq r \leq p_M - 1 \).

Since \( z \) is of period \( p_M \), (4.31) gives
\[
\int \hat{\phi} \, d\delta_{\tilde{O}(f)} = \frac{1}{p_M q} \left( S_{t_{d-t}} \hat{\phi}(f^k(z)) - S_r \hat{\phi}(f^{p_M q+k}(z)) \right)
\]
\[
> \frac{1}{p_M q} \left( S_{t_{d-t}} \hat{\phi}(f^t(x_{R,t})) - \frac{t_d - t}{3} \bar{\varepsilon} - C_1 \bar{\varepsilon} - S_r \hat{\phi}(f^{p_M q+k}(z)) \right).
\]

Meanwhile, (4.30) gives \( S_{t_{d-t}} \hat{\phi}(f^t(x_{R,t})) = S_{t_{d-t}} \hat{\phi}(x_{R,t}) - S_{t_{d-t}} \hat{\phi}(y_{R,t}) > t_d \tilde{\alpha}_d - S_{t_{d-t}} \hat{\phi}(y_{R,t}) - (t_d/3) \bar{\varepsilon} \). Plugging this inequality into (4.32) gives
\[
\int \hat{\phi} \, d\delta_{\tilde{O}(f)} > \frac{1}{p_M q} \left( t_d \tilde{\alpha}_d - S_{t_{d-t}} \hat{\phi}(y_{R,t}) - \frac{2t_d - t}{3} \bar{\varepsilon} - C_1 \bar{\varepsilon} - S_r \hat{\phi}(f^{p_M q+k}(z)) \right).
\]

Note that there exists a constant \( C_2 > 0 \) depending only on \( f, \delta, \phi \) such that
\[
\int \hat{\phi} \, d\nu > \frac{1}{t} (S_t \hat{\phi}(y_{R,t}) - C_2 \bar{\varepsilon}) \quad \text{for } t \in \{1, \ldots, t_d\}.
\]

The last two inequalities together imply (4.29) provided \( N_0 \) is sufficiently large.

Case 3: \( m(f) = M \) and \( \partial J_{m(f)} \) contains an attracting periodic point (Case C-II in Remark 2.2). Set \( \mu_d = \delta_{\tilde{O}(f)} \). Clearly Lemma 4.6(a) holds because \( F(\mu_d) = 0 \).

Lemma 4.5 gives \( f^{n_d}(R_{d+1}(A)) \in \bar{A}_d(\phi, \tilde{\alpha}_d - (1/2)\bar{\varepsilon}) \). The uniform convergence in (2.2) implies Lemma 4.6(b) if \( N_0 \) is sufficiently large.
Case 4: \( m(f) = M \) and \( f \) has no attracting periodic point (Case A in Remark 2.2).
Recall that \( K_M = \bigcup_{k=0}^{pM-1} f^k(L_M) \) by (2.3). Let \( R \in \mathcal{R}_d(A) \). Lemma 4.5 gives
\[
\frac{|\tilde{A}_d(\tilde{\phi}, \tilde{\alpha}_d - (1/2)\tilde{\varepsilon}) \cap K_M|}{|K_M|} \leq \exp \left( (F(\nu_R) + \frac{\varepsilon}{2}) t_d \right) \quad \text{and}
\]
(4.33)
\[
\int \tilde{\phi} d\nu_R > \tilde{\alpha}_d - \tilde{\varepsilon}.
\]
As in the proof for the first case, by Proposition 2.5 the distortion of \( f^n \) on \( R \) is bounded by the constant \( D_d \) in (4.26). Therefore
\[
\frac{|R \cap R_{d+1}(A)|}{|R|} \leq D_d \frac{|f^n_d(R \cap R_{d+1}(A))|}{|f^n_d(R)|} \leq D_d \frac{|\tilde{A}_d(\tilde{\phi}, \tilde{\alpha}_d - (1/2)\tilde{\varepsilon}) \cap K_M|}{|f^n_d(R)|} \leq \exp ( (F(\nu_R) + \varepsilon) t_d ).
\]
(4.34)
The last inequality holds if \( N_0 \) is sufficiently large, by the first inequality in (4.33) and the uniform lower bound \( |f^n_d(R)|/|K_M| \geq \inf_{j,k \in \{0,\ldots,pM-1\}} |f^j(J_M)|/|f^k(J_M)| \).

Let \( \mu_d \) be a measure in \( \{ \nu_R : R \in \mathcal{R}_d(A) \} \) which maximizes the free energy within this finite set. As required in Lemma 4.6(a), we have
\[
\frac{|R_{d+1}(A)|}{|R_d(A)|} \leq \max_{R \in \mathcal{R}_d(A)} \frac{|R \cap R_{d+1}(A)|}{|R|} \leq \exp ( (F(\mu_d) + \varepsilon) t_d ).
\]
Lemma 4.6(b) is a consequence of (4.34). The proof of Lemma 4.6 is complete. \( \square \)

Step 3 (Overall estimates). Set \( \ell(N_0) = \{ i \in \{0, \ldots, d\} : t_i > N_0 \} \), and let \( n > (d+1)N_0 \). For each \( i \in \ell(N_0) \), let \( \mu_i \) be a measure in \( \mathcal{M}(f) \) for which the conclusion of Lemma 4.6 holds with \( \varepsilon \) replaced by \( \varepsilon/2 \). Define a measure \( \mu \in \mathcal{M}(f) \) by
\[
\mu = \left( \sum_{i \in \ell(N_0)} t_i \right)^{-1} \sum_{i \in \ell(N_0)} t_i \mu_i.
\]
Since \( R_{i+1}(A) \subset R_i(A) \) for \( i \in \{0, \ldots, d\} \) as in (4.22) and \( (1/n) \sum_{i \in \ell(N_0)} t_i \rightarrow 1 \) as \( n \rightarrow \infty \), for all sufficiently large \( n \) we have
\[
\frac{|R_{d+1}(A)|}{|R_0(A)|} = \prod_{i=0}^d \frac{|R_{i+1}(A)|}{|R_i(A)|} \leq \prod_{i \in \ell(N_0)} \frac{|R_{i+1}(A)|}{|R_i(A)|} \leq \exp \left( \sum_{i \in \ell(N_0)} \left( F(\mu_i) + \frac{\varepsilon}{2} t_i \right) \right) \leq \exp \left( F(\mu)(n - (d+1)N_0) + \frac{\varepsilon}{2} n \right) < \exp ( (F(\mu) + \varepsilon) n ).
\]
which yields the argument of Case II-2 in the proof of Lemma 4.6. If \( \bar{\epsilon} \) combining the arguments developed so far. Let \( f \) the closed interval bordered by \( f(c) \) and \( f^2(c) \). Then \( f|_{L_+} : L \to L \) is topologically exact. The desired upper bound follows from the argument of Case II-4 in the proof of Lemma 4.6. The proof of Proposition 4.1 is complete. \( \square \)

4.5. Proof of Proposition 4.1. We are in position to prove Proposition 4.1 combining the arguments developed so far. Let \( f : X \to X \) be an \( S \)-unimodal map with a non-flat critical point. We begin with the case \( m(f) \geq 1 \). Let \( \epsilon > 0 \). Let \( N \geq 1 \) be an integer for which the conclusion of Proposition 4.4 holds with \( \epsilon \) replaced by \( \epsilon/4 \). Let \( n \geq N \) satisfy \( \bar{A}_n(\bar{\phi}, \bar{\alpha}) \neq \emptyset \). We have

\[
(4.35) \quad \bar{A}_n(\bar{\phi}, \bar{\alpha}) \subset \bigcup_{d=0}^{M} \bigcup_{t \in I_n(d)} \bigcup_{A \in \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z})} R(t, A).
\]

By Proposition 4.4, for \( t \in \bigcup_{d=0}^{M} I_n(d) \) and \( A \in \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z}) \) with \( R(t, A) \neq \emptyset \), there exists a measure \( \mu_{t, A} \in \mathcal{M}(f) \) such that

\[
(4.36) \quad |R(t, A)| \leq \exp \left( \left( F(\mu_{t, A}) + \frac{\epsilon}{4} \right) n \right) \quad \text{and} \quad \int \bar{\phi} d\mu_{t, A} > \bar{\alpha} - \frac{1}{4} \bar{\epsilon}.
\]

If \( n \) is sufficiently large, then we have

\[
\# \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z}) \leq \prod_{j=1}^{l} \left( \frac{1}{\epsilon} (\sup \phi_j - \inf \phi_j) + 2 \right)^{d+1} \leq e^{\frac{\epsilon}{4} n}, \quad \text{and}
\]

\[
\# I_n(d) \leq \binom{n}{d} \binom{M+1}{d+1} \leq n^d \binom{M+1}{d+1} \leq e^{\frac{\epsilon}{4} n}.
\]

Let \( \mu \in \mathcal{M}(f) \) be a measure in \( \{ \mu_{t, A} : t \in \bigcup_{d=0}^{M} I_n(d), A \in \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z}), R(t, A) \neq \emptyset \} \) which maximizes the free energy within this finite set. Combining the first inequality in (4.36) with the upper bound on the number of all itineraries yields

\[
|\bar{A}_n(\bar{\phi}, \bar{\alpha})| \leq \sum_{d=0}^{M} \sum_{t \in I_n(d)} \sum_{A \in \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z})} |R(t, A)|
\]

\[
\leq (M+1) \max_{0 \leq d \leq M} (\# I_n(d) \# \mathcal{M}_{d+1,1}(t, \epsilon \mathbb{Z})) \exp \left( \left( F(\mu) + \frac{\epsilon}{4} \right) n \right),
\]

which is less than \( \exp \left( (F(\mu) + \epsilon)n \right) \) as required in Proposition 4.1(a). Proposition 4.1(b) is a consequence of the second inequality in (4.36).

It is left to treat the case \( m(f) = 0 \). If \( \bar{m}(f) = 1 \), then \( f \) has a two-sided attracting neutral fixed point in \( \text{int}(X) \). The desired upper bound follows from the argument of Case II-2 in the proof of Lemma 4.6. If \( \bar{m}(f) = 0 \), then let \( L \) denote the closed interval bordered by \( f(c) \) and \( f^2(c) \). Then \( f|_L : L \to L \) is topologically exact. The desired upper bound follows from the argument of Case II-4 in the proof of Lemma 4.6. The proof of Proposition 4.1 is complete. \( \square \)
EXISTENCE OF LARGE DEVIATION RATE FUNCTION

5. BIMODAL MAPS FOR WHICH THE LEVEL-2 LDP DOES NOT HOLD

In this last section we prove Theorem B. We start with a post-critically finite map with two critical points, one periodic and the other not. After a small perturbation we obtain a map for which the level-2 LDP does not hold.

5.1. A one-parameter family of bimodal maps. Let $X = [0, 1]$ and $K = [1/2, 1]$. Let $f: X \to X$ be a $C^3$ map with negative Schwarzian derivative having only two critical points $c_0$ and $c_1$ with $0 < c_0 < 1/2 < c_1 < 1$, which are assumed to be non-degenerate: $f''(c_0)f''(c_1) \neq 0$. We assume the following conditions on $f$:

(i) $f(X) \subset K$ and $c_1$ is a local maximal point of $f$.

(ii) $c_1$ is periodic with prime period 3.

See Figure 2. Condition (ii) implies that the complement of the basin of the super-attracting periodic orbit of $c_1$ in $[f^2(c_1), f(c_1)]$ is a non-trivial hyperbolic basic set $\Lambda$. Set $p = f(c_0)$. Fix a periodic point $q \in \Lambda$. Further we assume

(iii) $p$ is a periodic point in $\Lambda$, and not contained in the periodic orbit of $q$.

Remark 5.1. From (i), we have $f^2(X) \subset [1/2, f(c_1)]$ and so $f^3(X) \subset [f^2(c_1), f(c_1)]$.

We consider a $C^3$ parametrized family of maps $f_a: X \to X$ ($a \in [-1, 1]$) such that $f_0 = f$ and

(iv) $f_a|_K$ does not depend on $a \in [-1, 1]$.

(v) the critical point $c_0$ of $f$ remains to be a non-degenerate critical point of $f_a$ for $a \in [-1, 1]$, and $\frac{d}{da}(f_a(c_0))|_{a=0} \neq 0$.

Let $\mu_p$ and $\mu_q$ denote the uniform probability distributions on the periodic orbits of $p$ and $q$ for $f$ respectively. Let $\chi_p$ and $\chi_q$ denote their Lyapunov exponents. From (iv), these do not depend on the parameter $a \in [-1, 1]$. For $x \in X$, $n \geq 1$ and $a \in [-1, 1]$, let $\delta_{a,x}^n$ denote the uniform probability distribution on the orbit $\{f_j^a(x)\}_{j=0}^{n-1}$. The next proposition is the main step in the proof of Theorem B.

Proposition 5.2. There exists a parameter $a \in [-1, 1]$ arbitrarily close to 0 such that for any $\varepsilon > 0$, there exist open sets $G_1$ and $G_2$ in $\mathcal{M}$ such that $\mu_p \in G_1 \subset$
the open
Let
Lemma 5.3.
observe the normal rate as in Lemma 5.4.
For many
µ
Abnormal and normal large deviations rates.
Proof. Set
δ
any
f
Therefore the level-2 LDP does not hold for
This contradicts the conclusion of Proposition 5.2 for sufficiently small
ε > 0.
Therefore the level-2 LDP does not hold for
a
proving Theorem B.

5.2. Abnormal and normal large deviations rates. For δ ∈ (0, 1), let
Oδ(p)
be the open δ-neighborhood of the periodic orbit of
p
and put
Gδ = \{μ ∈ M: μ(Oδ(p)) > 1 − δ\}.

This is an open neighborhood of
μp
and we have cl(Gδ) ⊂ Gδ for 0 < δ < δ < 1.
The next two lemmas provide large deviations rates for neighborhoods of
μp.
For many
a
, we observe an abnormal rate as in Lemma 5.3. For many other
a
, we observe the normal rate as in Lemma 5.4.

Lemma 5.3. If |a| is sufficiently small and
fa(m)(c0) = p
for some
m ≥ 1
, then for any δ ∈ (0, 1) we have
\liminf_{n \to \infty} \frac{1}{n} \log |\{x ∈ X: δ_{a,x}^n ∈ Gδ\}| ≥ −\frac{\chi_p}{2}.

Proof. Set γ = \sqrt{(fa(m)(c0))^{-1}δ}, and let ε > 0. The condition |x − c0| ≤ γe^{-(\chi_p + ε)/2}
for x ∈ X and sufficiently large
n ≥ 1
implies |fa(x) − fa(c0)| ≤ δe^{−n(\chi_p + ε)}, and so
δ_{a,x}^n \in Gδ.
Hence
\{|x ∈ X: |x − c0| < γe^{-(\chi_p + ε)/2}\} = 2γe^{-(\chi_p + ε)n/2} ≤ |\{x ∈ X: δ_{a,x}^n \in Gδ\}|.
This implies that the lower limit in the lemma is bounded from below by −(\chi_p + ε)/2. Since ε > 0 is arbitrary, the desired inequality follows.

□

Lemma 5.4. For any ε > 0 there exists δ* > 0 such that, if 0 < δ ≤ δ*, the following holds: If |a| is sufficiently small and
fa(m)(c0) = q
for some
m ≥ 1
, then
\limsup_{n \to \infty} \frac{1}{n} \log |\{x ∈ X: δ_{a,x}^n ∈ Gδ\}| ≤ −\chi_p + ε.

Proof. Let ε > 0. Since
f
coincides with
fa
on \([f^2(c1), f(c1)]\), we have the large deviation estimate
\limsup_{n \to \infty} \frac{1}{n} \log |\{x ∈ [f^2(c1), f(c1)]: δ_{a,x}^n \in Gδ\}| ≤ −\chi_p + \frac{ε}{2},
(5.1)
for $\delta > 0$ small enough. This is a simplest case in the proof of Theorem A, and basically a known result. Since $f$ is locally diffeomorphic except at the critical points, we see from Remark 5.1 that

\begin{equation}
\limsup_{n \to \infty} \frac{1}{n} \log |\{x \in X \setminus U : \delta_{a,x}^n \in \mathcal{G}_\delta\}| \leq -\chi_p + \frac{\varepsilon}{2}.
\end{equation}

for any open interval $U$ that contains $c_0$, provided that $|a|$ is sufficiently small. Below we fix a small neighborhood $U$ of $c_0$, and consider orbits starting from $U$.

Let $\rho > 0$. For an integer $k \geq 0$, let

$$J_k = (q - \rho e^{-\chi_0 k}, q + \rho e^{-\chi_0 k}).$$

We assume $\rho$ is small enough so that $J_0 \subset K$, and the following hold for all $k \geq 0$:

- The distortion of $f^k$ on $J_k$ is bounded by a constant independent of $k$.
- The length of any connected component of $f^k(J_k \setminus J_{k+1})$ is bounded from below by a positive constant independent of $k$.
- $f^\ell(J_k) \cap O_\delta(p) = \emptyset$ for all $\ell \in \{0, \ldots, k\}$.

Since $q$ is a hyperbolic repelling periodic point, all these conditions can be checked by taking a $C^2$ linearization of $f$ in a neighborhood of the orbit of $q$.

Below we suppose that $|a|$ is sufficiently small and $m \geq 1$ is the smallest integer with $f_m^a(c_0) = q$. Recall that $f_0^a|_K = f|_K$. The point $c_0$ is a non-degenerate critical point of $f_m^a$ because its orbit does not contain $c_1$. For $k \geq 0$, let $U_k$ be the connected component of $f_m^{-m}(J_k)$ that contains $c_0$. If $\rho > 0$ is sufficiently small, there exists a constant $C_1 = C_1(m) > 1$, depending on $m$, such that $C_1^{-1} e^{-\chi_0 k/2} < |U_k| < C_1 e^{-\chi_0 k/2}$ for $k \geq 0$. The restriction of $f_m^a$ to each connected component of $U_k \setminus U_{k+1}$ for $k \geq 0$ is a diffeomorphism onto a connected component of $J_k \setminus J_{k+1}$ with distortion bounded uniformly over all $k \geq 0$.

From the last property of $J_k$, if $x \in U_k$ and $m \leq \ell \leq m + k$ then $f_\ell^a(x) \notin O_\delta(p)$. Hence, if $\delta_{a,x}^n$ belongs to $\mathcal{G}_\delta$ for some $x \in U_k$ and $n \geq 0$, we have

\begin{equation}
m + \#\{\ell : m + k \leq \ell \leq n - 1, f_\ell^a(x) \in O_\delta(p)\} \geq (1 - \delta)n.
\end{equation}

Since the left-hand side is bounded by $n - k$, this happens only if $0 \leq k \leq \delta n$.

The length of the image $f_m^{m+k}(U_k \setminus U_{k+1}) = f_k^a(J_k)$ is bounded from below by a constant independent of $k$. Hence, combining the estimate (5.1) with the uniform distortion estimates for $f_k^a|_{J_k}$ and $f_m^a|_{U_k \setminus U_{k+1}}$, we conclude that there exists a constant $C_2 = C_2(m) > 1$ such that the Lebesgue measure of the set of points in $U_k \setminus U_{k+1}$ for $0 \leq k \leq \delta n$ satisfying (5.3) is bounded from above by $C_2 \exp((-\chi_p + \varepsilon/2)(1 - \delta)n)$ provided $n$ is sufficiently large. Summing this bound over all integers $k$ satisfying $0 \leq k \leq \delta n$ yields

$$\limsup_{n \to \infty} \frac{1}{n} \log \{|x \in U_0 : \delta_{a,x}^n \in \mathcal{G}_\delta\} \leq \left(-\chi_p + \frac{\varepsilon}{2}\right)(1 - \delta) \leq -\chi_p + \varepsilon.$$

The last inequality holds if $\delta > 0$ is sufficiently small. Since $\varepsilon > 0$ is arbitrary, this together with (5.2) gives the conclusion. \qed
5.3. **Proof of Proposition 5.2** Let \( \varepsilon > 0 \). Let \( \delta_\ast > 0 \) be as in Lemma 5.4 and let \( \delta, \delta' \in (0, \delta_\ast) \) satisfy \( \delta' < \delta \). Recall that the periodic points \( p, q \) of \( f_0 \) are contained in its basic set \( \Lambda \). From (iv) and (v), for \( n \geq 2 \) we have

\[
\frac{d}{da}(f_a^n(c_0)) = Df^{n-1}(f_a(c_0)) \frac{d}{da}(f_a(c_0)) \neq 0,
\]

provided that \( |a| \) is sufficiently small and \( f^\ell(f_a(c_0)) \neq c_1 \) for all \( \ell \in \{0, \ldots, n-2\} \).

By the transitivity of \( \Lambda \), if \( |a| \) is sufficiently small and \( f_a^m(c_0) = p \) (resp. \( f_a^m(c_0) = q \)) for some \( m \geq 1 \), we can find a parameter \( a' \) arbitrarily close to \( a \) such that \( f_a'^m(c_0) = q \) (resp. \( f_a'^m(c_0) = p \)) for some \( m' > m \).

We construct a sequence \( \{a(i)\}_{i=0}^\infty \) in \([-1, 1]\) converging to 0, a sequence \( \{\Delta(i)\}_{i=0}^\infty \) of positive reals converging to 0, and strictly increasing sequences \( \{m(i)\}_{i=0}^\infty \), \( \{n(i)\}_{i=0}^\infty \) of positive integers with the following properties:

(a) \( f_a^{m(i)}(c_0) = p \) for \( i \) even, and \( f_a^{m(i)}(c_0) = q \) for \( i \) odd.

(b) If \( a \in [-1, 1] \) and \( |a - a(i)| \leq \Delta(i) \) then

\[
\frac{1}{n(i)} \log \left| \{x \in X : \delta_a^{m(i)} \in G_{\delta'} \} \right| > -\frac{\chi_p}{2} - \frac{1}{i} \quad \text{for } i \text{ even}, \quad \text{and}
\]

\[
\frac{1}{n(i)} \log \left| \{x \in X : \delta_a^{m(i)} \in G_{\delta} \} \right| < -\chi_p + \varepsilon \quad \text{for } i \text{ odd}.
\]

(c) For all integers \( i, i' \) with \( 0 \leq i' < i \), \( |a(i') - a(i)| \leq \Delta(i') \).

Then, Proposition 5.2 holds for \( f_a \) with \( a = \lim_{i \to \infty} a(i) \), \( G_1 = G_{\delta'} \) and \( G_2 = G_\delta \).

The construction is inductive. Start with \( a(0') = 0 \), \( m(0) = n(0) = 1 \) and a small number \( \Delta(0) > 0 \) such that \( \frac{d}{da}(f_a(c_0)) \neq 0 \) when \( |a| < \Delta(0) \). Let \( j \geq 1 \) and suppose \( a(i), m(i), n(i), \Delta(i) \) have been constructed for \( 0 \leq i \leq j - 1 \). As in the remark after 5.4, we take \( a(j) \) close to \( a(j-1) \) and \( m(j) > m(j-1) \) so that (a) and (c) hold for \( i = j \). Further, we take a large integer \( n(j) > n(j-1) \) so that one of the two conditions in (b) holds for \( i = j \) and \( a = a(j) \), by Lemmas 5.3 or 5.4 depending on the parity of \( j \). Finally we take sufficiently small \( \Delta(j) > 0 \) so that one of the two conditions in (b) holds for any \( a \in [-1, 1] \) with \( |a - a(j)| < \Delta(j) \). Iterating this construction yields the four sequences with the required properties. \( \square \)

**Remark 5.5.** The counterexample in Theorem B can be found in polynomial maps of degree 3, by considering an appropriate one-parameter family \( f_a \) of polynomial of degree 3 and applying the argument parallel to the proof of Proposition 5.2 above. Below we explain the construction of the one-parameter family \( f_a \), but leave the details to the interested readers.

First we take a polynomial \( f \) of degree 3 so that there exist

- two non-degenerate critical points \( c_0 < c_1 \) and non-empty compact intervals \( K \) and \( X \) with \( K \subsetneq X \), \( c_0 \in X \setminus K \) and \( c_1 \in K \), and
- a non-trivial hyperbolic basic set \( \Lambda \subset K \), and a hyperbolic repelling periodic point \( q \in \Lambda \),

satisfying the conditions (i), (ii) and (iii) with \( p = f(c_1) \). This is possible by the combinatorial argument on the dynamics of continuous piecewise monotone maps [24, Chapter II, Sections 4-6].
Second we consider a one-parameter family \( f_a \) of polynomial of degree 3 with \( f_0 = f \). Clearly the condition (iv) is not true for \( f_a \) in this case, unless the family \( f_a \) is trivial. But this is not an essential problem as we put the condition (iv) just to make computations easier. The critical points \( c_0 < c_1 \) and periodic points \( p, q \) extend continuously for \( f_a \) with \( a \) close to 0, and we write \( c_0(a) < c_1(a) \) and \( p(a), q(a) \) for them respectively. In the place of the condition (v), we require
\[
\frac{d}{da}(f_a(c_0(a)) - p(a)) \bigg|_{a=0} \neq 0.
\]
Though this condition is likely to hold for generic family \( f_a \) with \( f_0 = f \), it is not easy to make sure of the condition rigorously for a family of polynomials of degree 3. Here we exploit the theory of Douady-Hubbard-Thurston on the dynamics of complex polynomials. Consider the two-parameter family
\[
f_{\alpha, \beta}(x) = f(x) + \alpha x + \beta
\]
and regard it as a family of complex dynamical systems with complex parameters \((\alpha, \beta) \in \mathbb{C}^2\). We apply [33, Main Theorem 1.1] to this family, restricting the parameter \((\alpha, \beta) \in \mathbb{C}^2\) to a small neighborhood \( V \) of \((0,0)\). To this end, we have to check that the family \( f_{\alpha, \beta} \) is a normalized family (see [33]), that is, no affine transformation conjugates two maps in the family \( f_{\alpha, \beta} \) with different parameters in \( V \). This is not difficult: compute \( L^{-1} \circ f_{a, \beta} \circ L \) for an affine map \( L(z) = A z + B \) and use the fact that the coefficients of \( f_{\alpha, \beta} \) with degree 2 and 3 do not depend on the parameter \((\alpha, \beta)\). As the conclusion, we obtain that the differential of
\[
\Psi : (\alpha, \beta) \in V \mapsto (f_{\alpha, \beta}(c_0(\alpha, \beta)) - p(\alpha, \beta), f^n_{\alpha, \beta}(c_1(\alpha, \beta)) - c_1(\alpha, \beta)) \in \mathbb{C}^2
\]
at \((0,0)\) is injective, where \( c_0(\alpha, \beta), c_1(\alpha, \beta), p(\alpha, \beta) \) denote the continuous extension of the non-degenerate critical points \( c_0, c_1 \) and the hyperbolic periodic point \( p \) for \( f_0 \) to those for \( f_{a, \beta} \) respectively. Clearly we can restrict the map \( \Psi \) to \( V \cap \mathbb{R}^2 \) and see that \( D\Psi(0,0) : \mathbb{R}^2 \to \mathbb{R}^2 \) is injective. Thus we can define the family \( f_a = f_{a(\alpha), \beta(\alpha)} \) for \( a \) with small modulus so that \( \Psi(\alpha(a), \beta(a)) = (a,0) \). Then the condition (5.5) holds for such a family \( f_a \).

For the family \( f_a \) constructed as above, we have
\[
\partial_a(f^n_a(c_0) - f^{n-1}_a(p(a))) = Df^{n-1}_a(f_a(c_0)) \cdot (\partial_a(f_a(c_0) - p(a)))
\]
at the parameter \( a = 0 \), correspondingly to (5.4). Since \( \partial_a(f^n_a(p(a))) \) is uniformly bounded with respect to \( n \), the condition (5.5) implies that the derivative \( \frac{d}{da}(f^n_a(c_0)) \) at \( a = 0 \) is comparable with \( Df^{n-1}(f(c_0)) = Df^{n-1}(p) \) and grows exponentially in \( n \). With this estimate and other conditions on \( f = f_0 \) discussed above, we can follow the argument in the proof of Proposition 5.2 and obtain the counterexample in Theorem B in the family \( f_a \).

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