VINOGRADOV THREE PRIME THEOREM WITH PIATETSKI-SHAPIRO PRIMES

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Abstract. We prove that for any $c_1, c_2, c_3 \in (1, 1.17)$, every sufficiently large odd $n$ can be represented as the sum of three primes $n = p_1 + p_2 + p_3$ with $p_i$ is of the form $\lfloor n^{c_i} \rfloor$ for $1 \leq i \leq 3$.

1. Introduction

The weak Goldbach conjecture asserts that every odd integer greater than 5 can be represented as the sum of three primes. The well-known Vinogradov three primes theorem says that the weak Goldbach conjecture is true for every sufficiently large odd integer. In 2013, Helfgott [9] completely confirmed Goldbach's weak conjecture.

Nowadays, Vinogradov’s three primes theorem has been extended to some primes of special forms. For each non-integral $c > 1$, let

$$N^c = \{\lfloor n^c \rfloor : n \in \mathbb{N}\}.$$

A prime $p$ is called Piatetski-Shapiro prime, if $p \in N^c$.

In 1992, Balog and Friedlander [2] generalized Vinogradov’s three primes theorem with the Piatetski-Shapiro primes. They obtained that for $1 < c < 1.05$, any sufficiently large odd number $N$ is the sum of three primes from $N^c$. The result of Balog and Friedlander was improved and extended in [1, 12, 13, 15, 19, 20, 27]. The best known result is given by Kumchev [13] which allows $1 < c < 1.06$ in Vinogradov’s three primes theorem with the Piatetski-Shapiro primes, by improving Type I information. In fact both of Balog-Friedlander and Kumchev studied the following Weak Balog-Friedlander condition

**Definition 1** (Weak Balog-Friedlander condition).

$$\sum_{p \leq N \atop p \in \mathcal{P} \cap N^c} cp^{1 - \frac{1}{c}} \log p \cdot e(p\theta) = \sum_{p \leq N \atop p \in \mathcal{P}} \log p \cdot e(n\theta) + O_A \left( \frac{N}{(\log N)^A} \right) \quad (1.1)$$

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So without new Type I and II information their results would be hard to improve by using the similar methods.

In 1953, Piatetski-Shapiro \cite{21} proved the asymptotic formula of the number of Piatetski-Shapiro primes for $1 < c < 1.1$. In fact they showed that the the number of Piatetski-Shapiro primes and the number of primes are comparable. i.e.

$$\pi_c(x) \sim \frac{x}{c \log x},$$

where $\pi_c(x) = \{p \in \mathbb{N}^c : p \leq x^c\}$. Subsequently, the upper bound for $c$ in this asymptotic formula was improved several times. The current best result is $c \in (1, 6121/5302)$ due to Rivat \cite{22}. Later a series of papers \cite{10, 11, 14, 23} used sieve methods to give a lower bound of the number Piatetski-Shapiro primes and get larger upper bound for $c$. The current best result is $c \in (1, 243/205)$ due to Rivat and Wu \cite{23}. These results imply that the upper bound $c$ for three primes theorem with Piatetski-Shapiro primes given by Kumchev \cite{13} is far from the best possible value.

In 1989, Bourgain \cite{3} introduced a strategy to estimate $L_q$ norm with non-integer $q > 0$ for a function which has a “Psedorandom” majorant. Enlighten by Bourgain’s work, we can give a new mean-value theorem for Piatetski-Shapiro primes which can replace $L_2$ norm in Kumchev’s proof and give a new upper bound $c \in (1, 73/64)$, see Theorem 2.1 and Corollary 2.2.

Green \cite{6} introduced a transference principle in proving Roth’s theorem in primes, which relied on $L_q$ norm with non-integer $q$, and also include some other ideas from additive combinatorics and harmonic analysis. The transference principle has been applied has been applied to the ternary Goldbach problems for several special types of primes. For example, Matomäki and Shao(MS) \cite{18} prove the three primes theorem with Chen primes. Matomäki, Maynard and Shao(MMS) \cite{17} proved Vinogradov theorem with almost equal summands. However the two papers utilize different techniques. In \cite{18} they studied the Fourier transform $\sum_n \rho^-(n)e(n\alpha)$, but In \cite{17} they studied the Fourier transform $\sum_n \rho^+(n)e(n\alpha)$ and need result of $\sum_{n \leq N} \rho^-(n)$.

In this paper, we shall prove a new mean value theorem by using Bourgain’s strategy and then combine MMS version of transference principle with the new mean value theorem and sieve methods, which then allow us to get a larger upper bound for $c$ in the Vinogradov theorem with the Piatetski-Shapiro primes.

**Theorem 1.1.** For any $c_1, c_2, c_3 \in (1, 1.17)$, every sufficiently large odd $n$ can be represented as

$$n = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes and $p_i \in \mathbb{N}^{c_i}$ for each $1 \leq i \leq 3$.

- The reason that we use MMS version of transference principle instead of using MS version is that for large $c$ we cannot understand the Fourier
transform of $\rho^-$ well. Namely, if we use the MS method, the largest $c$ is $8/7 < 1.17$. We cannot overcome the barrier $8/7$, since Kumchev’s estimation requires that $1/c < 7/8$ and also Balog-Firelander would give a narrow range of Type I sum so that we cannot use Buchstab identity twice to give an satisfying Fourier transform of lower bound sieve.

- The percentile in our result is the best possible value in current techniques. This is because [1] used sieve method to studied the lower bound of the number of Piatetski-Shapiro primes when $1 < c < 20/17 \approx 1.17$. And their lower bound result is

$$\pi_c(x) \geq \sum_{n \leq x^c} \rho^-(n) = \alpha^- \frac{x}{c\log x},$$

where $\alpha^-$ is the coefficient corresponding to $\rho^-$ and is around $1/4$. However when we use MMS methods, we need $3\alpha^- - \alpha^+ > 0$, where $\alpha^+$ is the coefficient corresponding to $\rho^+$ must be larger than 1. So in the case $c = \frac{20}{17}$ we have $3\alpha^- - \alpha^+ < 0$. The authors believe that the upper bound for $c$ can be slightly larger than 1.17. For example 1.17a with $a \leq 5$ is possible.

2. Outline

**Theorem 2.1.** Suppose that $n \leq N$, $C > 0$ is a constant, and $W \ll \log N$. Let $\rho(n)$ be a prime number indicator function and $\rho^+(n) \geq \rho(n)$ be a upper bound sieve. Let $Q = \log^A N$ and $T = \frac{N}{\log^B N}$ with $B > A$ sufficiently large positive numbers. We define the following Hardy-Littlewood decomposition:

$$\mathcal{M}_{a,q} = \{\alpha : |\alpha - a/q| \leq \frac{1}{qT}\}$$

$$\mathcal{M} = \bigcup_{a=0}^{q-1} \mathcal{M}_{a,q}$$

$$m = T \setminus \mathcal{M}.$$

Suppose that

$$f_{W,N,b}(n) = \log N \frac{\phi(W)}{W} (Wn + b)^{1-\frac{1}{\varepsilon}} \rho(Wn + b)$$

and

$$\nu_{W,N,b}(n) = \log N \frac{\phi(W)}{W} (Wn + b)^{1-\frac{1}{\varepsilon}} \rho^+(Wn + b)$$

satisfying the following conditions:

1. There exists $\varepsilon > 0$ such that for any $\theta \in \mathcal{M}_{a,q}$, we have $\hat{\nu}(\theta) \ll \frac{q^{-\varepsilon}}{1+N|\theta - a/q|} + O(\frac{N}{\log^A N})$.
(2) For any $\theta \in m$, we have $\hat{\nu}(\theta) = O\left(\frac{N}{\log^A N}\right)$.

Then for any $u > 2 + \frac{4(c - 1)}{2 - c}$, we have

$$\int_0^1 |\hat{f}_{W,N,B}(\alpha)|^u d\alpha \ll u^N N^{-1}.$$  \hspace{1cm} (2.3)

Theorem 2.1 has their own interests. For example we can get the following two corollaries

**Corollary 2.1.** Suppose that $c_1, c_2, c_3 \in (1, 6/5)$ satisfy the weak Balog-Friedlander condition. Then every sufficiently large odd $n$ can be represented as

$$n = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes and $p_i \in \mathbb{N}^{c_i}$ for each $1 \leq i \leq 3$.

**Proof.** Let $g(n) = cf_{1,N,0}(n) = cn^1 - \frac{1}{3} \log n 1_{n \in \mathbb{N} \cap \mathbb{P}}$ and $h(n) = \log n 1_{n \in \mathbb{P}}$. By weak Balog-Friedlander condition we have

$$\hat{g}(\theta) = \hat{h}(\theta) + O\left(\frac{N}{\log^A N}\right).$$

It is easy to verify that $\hat{h}(\theta)$ satisfying two conditions of Theorem 2.1 (Or one can refer to Green’s arguments [6]), so does $\hat{g}(\theta)$. So for any fixed $1 < c < 6/5$, there exists $2 < u < 3$ such that $\|\hat{g}\|_{u} \|\hat{h}\|_{u} \ll N^{u-1}$. Suppose $m \in [N, 2N]$ is a sufficiently large odd integer. Then by telescoping and Vinogradov’s three primes theorem, we have

$$\int_T \hat{g}^3 e(-m\theta) d\theta = \int_T \hat{h}^3 e(-m\theta) d\theta + \int_T \hat{h}^2 (\hat{g} - \hat{h}) e(-m\theta) d\theta$$

$$+ \int_T \hat{h} \hat{g} (\hat{g} - \hat{h}) e(-m\theta) d\theta + \int_T \hat{g}^2 (\hat{g} - \hat{h}) e(-m\theta) d\theta$$

$$\gg N^2 - o(N^2)$$

$$- \max\{|\hat{g} - \hat{h}|^{3-u}\} \left( \int_T |\hat{h}|^2 |\hat{g} - \hat{h}|^{u-2} + |\hat{h}\hat{g}| |\hat{g} - \hat{h}|^{u-2} + |\hat{g}|^2 |\hat{g} - \hat{h}|^{u-2} d\theta \right)$$

$$\gg N^2.$$  \hspace{1cm} \Box

The last inequality comes from Holder inequality by considering the $L_u$-norm.

In fact, Kumchev [13] improved the result of Balog and Friedlander, and showed that weak Balog-Friedlander condition is valid for every $1 < c < 73/64$. Hence
Corollary 2.2. For any $c_1, c_2, c_3 \in (1, 73/64)$, every sufficiently large odd $n$ can be represented as

$$n = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes and $p_i \in \mathbb{N}^c_i$ for each $1 \leq i \leq 3$.

Theorem 2.2. Let $N$ be a large positive integer, $\rho^+(n) \leq \rho(n) \leq \rho^+(n)$ for every $n \in [x]$. Let $\alpha^+, \alpha^-, \eta > 0$. Assume that the following conditions hold:

1. $\rho^+(n)$ satisfies two conditions in Theorem 2.1.
2. Let $W = \prod_{p \leq w} p$ where $w = 0.1 \log \log x$. For every residue class $l \pmod{W}$ with $(l, W) = 1$, and every $\theta \in \mathbb{T}$, we have

$$\left| \sum_{\substack{n \in [x] \cap \mathbb{N}^c \atop n \equiv l \pmod{W}}} cn^{1-\frac{1}{e}} \rho(n) e(n\theta) - \frac{\alpha^+}{\log x} \cdot \frac{W}{\phi(W)} \sum_{\substack{n \in [x] \atop n \equiv l \pmod{W}}} e(n\theta) \right| \leq \frac{\eta x}{\phi(W) \log x}.$$

3. For any $C > 0$, residue class $l \pmod{d}$ with $(l, d) = 1$ and $d \leq \log^C x$, and arithmetic progression $P \subset \{n \in [x] : n \equiv l \pmod{d}\}$ with length at least $\eta^2 x$, we have

$$\sum_{n \in \mathbb{P} \cap [x]} cn^{1-\frac{1}{e}} \rho(n) \geq \frac{d}{\phi(d)} \frac{|P|}{x \log x}.$$

If $\alpha^+ < 3\alpha^-$ and $\eta$ is small enough in terms of $3\alpha^- - \alpha^+$, then any odd integer in $[x/2, x]$ can be written as a sum of three primes in $\mathbb{N}^c$.

We record the following transference principle given by Matomäki, Maynard and Shao [17]:

Theorem 2.3. [Transference principle] Let $\epsilon, \eta \in (0, 1)$. Let $N$ be a positive integer and let $f_1, f_2, f_3 : [N] \to \mathbb{R}_{\geq 0}$ be functions, with each $f \in \{f_1, f_2, f_3\}$ satisfying the following assumptions:

1. For each arithmetic progression $P \subset [N]$ with $|P| \geq \eta N$ we have $\mathbb{E}_{n \in P} f(n) \geq 1/3 + \epsilon$.
2. There exists a majorant $\nu : [N] \to \mathbb{R}_{\geq 0}$ with $f \leq \nu$ pointwise, such that $\|\nu - \mathbf{1}_{[N]}\|_\infty \leq \eta N$.
3. We have $\|\hat{f}\|_q \leq KN^{1-1/q}$ for some fixed $q, K$ with $K \geq 1$ and $2 < q < 3$.

Then for each $n \in [N/2, N]$ we have

$$f_1 * f_2 * f_3(n) \geq (c(\epsilon) - O_{\epsilon, K, q}(\eta))N^2,$$

where $c(\epsilon) > 0$ is a constant depending only on $\epsilon$.

Proof. See [17, Proof of proposition 3.1].
Proof of Theorem 2.2 assuming Theorems 2.1 and 2.3 Let $\rho, \rho^+, \alpha^-, \alpha^+$ be as in the statement of Theorem 2.2. Let $n_0 \in [X, 2X]$ be odd, $W = \prod_{p \leq w} p$, where $w = 0.1 \log \log X$ and choose $b_1, b_2, b_3 \pmod{W}$ with $(b_i, W) = 1$ such that $b_1 + b_2 + b_3 \equiv n_0 \pmod{W}$. Let $N = \lfloor X/W \rfloor$

$$f_i(n) = \begin{cases} \ccdot \frac{\log X}{\alpha^+} \cdot \frac{\phi(W)}{W} (Wn + b_i)^{1 - \frac{1}{2} \rho} (Wn + b_i), & \text{if } Wn + b_i \in [X] \cap \mathbb{N}^c, \\ 0, & \text{otherwise.} \end{cases}$$

(2.4)

and

$$\nu_i(n) = \begin{cases} \ccdot \frac{\log X}{\alpha^+} \cdot \frac{\phi(W)}{W} (Wn + b_i)^{1 - \frac{1}{2} \rho^+} (Wn + b_i), & \text{if } Wn + b_i \in [X] \cap \mathbb{N}^c, \\ 0, & \text{otherwise.} \end{cases}$$

(2.5)

Then $f_i \leq \nu_i$ since $\rho \leq \rho^+$. To prove the Fourier uniformity of $\nu_i$, observe that

$$\sum_{n \in [N]} \nu(n) e(n\theta) = \ccdot \frac{\log X}{\alpha^+} \cdot \frac{\phi(W)}{W} \sum_{n \in [N]} (Wn + b_i)^{1 - \frac{1}{2} \rho^+} (Wn + b_i) e(n\theta)$$

$$= \c\cdot \frac{\log N}{\alpha^+} \cdot e\left(-\frac{b_i}{W} \theta\right) \frac{\phi(W)}{W} \left( \sum_{n \equiv b_i \pmod{W}} n^{1 - \frac{1}{2} \rho^+} e(n\theta/W) + O(1) \right)$$

and similarly

$$\sum_{n \in [N]} e(n\theta) = e\left(-\frac{b_i}{W} \theta\right) \sum_{n \equiv b_i \pmod{W}} e(n\theta/W) + O(1)$$

for any $\theta \in \mathbb{T}$. Comparing the two equations above and using the assumption (2) in Theorem 2.2 about the Fourier transform of $\rho^+$, we obtain

$$\left| \sum_{n \in [N]} \nu(n) e(n\theta) - \sum_{n \in [N]} e(n\theta) \right| \ll \eta N.$$ 

Condition (1) in Theorem 2.3 follows immediately.

On the other hand, for any arithmetic progression $P \subset [N]$ of length $\geq \eta N$, by assumption (3) in Theorem 2.2 we have the lower bound

$$\sum_{n \in P} f_i(n) = \c\cdot \frac{\log X}{\alpha^+} \cdot \frac{\phi(W)}{W} \sum_{n \in Q} n^{1 - \frac{1}{2} \rho} e(n\theta) \geq \frac{\alpha^-}{\alpha^+} |Q| = \frac{\alpha^-}{\alpha^+} |P|$$

Where $Q = \{Wn + b : n \in P\}$.

Finally, by the assumption (1) in Theorem 2.2 Theorem 2.1 holds, so does condition (3) in Theorem 2.3. We can apply Theorem 2.3 to find that for each
n ∈ [N/2, N], there exist \( n_1, n_2, n_3 \) with each \( n_i \) in the support of \( f_i \) such that

\[
n = n_1 + n_2 + n_3.
\]

In particular, each \( Wn_i + b_i \) is a prime in \( \mathbb{N}^c \), and we have the representation

\[
Wn + b_1 + b_2 + b_3 = Wn_1 + b_1 + Wn_2 + b_2 + Wn_3 + b_3 = p_1 + p_2 + p_3.
\]

Setting \( n = (n_0 - b_1 - b_2 - b_3)/W - 3N \).

The rest of the paper is organized as follows. In Section 3, we will prove Theorem 2.1 by van der Corput methods and Bourgain’s strategy. In Sections 4 and 5, we will use the known Type I and II estimates and Harman sieve method to construct the lower and upper bound sieves \( \rho^{-}(n) \), \( \rho^{+}(n) \) and \( \alpha^{-}, \alpha^{+} \) satisfying \( 3\alpha^{-} - \alpha^{+} > 0 \). In Section 6, we will study the Fourier transforms of upper bound sieve to give the “Pseudorandom properties” in Theorems 2.1 and 2.2.

3. Restrictive estimates

The following lemma is the well-known van der Corput inequality, which is very effective when dealing with the exponential sums estimates involving \( n^c \) with non-integer \( c \).

**Lemma 3.1.** Suppose that \( \Delta > 0 \) and

\[
|f''(x)| \asymp \Delta
\]

for any \( x \in [X, X + Y] \), where \( f \asymp g \) means \( f \ll g \ll f \). Then

\[
\sum_{X \leq n \leq X + Y} e(f(n)) \ll Y \Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{4}}.
\]

Let

\[
\psi(t) = \{t\} - \frac{1}{2}.
\]

**Lemma 3.2.** For each \( H \geq 2 \),

\[
\psi(t) = -\frac{1}{2\pi i} \sum_{0 < |h| \leq H} \frac{e(ht)}{h} + O\left( \min \left\{ 1, \frac{1}{H\|t\|} \right\} \right).
\]

Furthermore, we have

\[
\min \left\{ 1, \frac{1}{H\|t\|} \right\} = \sum_{h=-\infty}^{\infty} b_h e(ht),
\]

where

\[
b_h \ll \min \left\{ \log \frac{H}{h^2}, \frac{H}{h^2} \right\}.
\]
The next lemma claims that $cn^{1 - \frac{1}{c}} \mathbf{1}_{n \in \mathbb{N}^c}$ is “Pseudorandom” with a very good error term when $c < 2$. In fact we hope to use a “Pseudorandom” function as a majorant function to give a restriction estimate, e.g. Lemma 3.5.

**Lemma 3.3.** For each $\theta \in [0, 1)$,

$$
\sum_{n \leq N} cn^{1 - \frac{1}{c}} e(n\theta) = \sum_{n \leq N} e(n\theta) + O(N^{\frac{3}{2} - \frac{1}{c}} \log N). \quad (3.1)
$$

**Proof.** Since

$$
\sum_{n \leq N} cn^{1 - \frac{1}{c}} e(n\theta) = \sum_{k \geq 0} \sum_{\frac{N}{2^k+1} < n \leq \frac{N}{2^k}} cn^{1 - \frac{1}{c}} e(n\theta),
$$

we only need to prove that

$$
\sum_{n \sim N} cn^{1 - \frac{1}{c}} e(n\theta) = \sum_{n \sim N} e(n\theta) + O(N^{\frac{3}{2} - \frac{1}{c}} \log N) \quad (3.2)
$$

where $n \sim N$ means $N/2 < n \leq N$. It is easy to verify that $n \in \mathbb{N}^c$ if and only if

$$
[-n^{\frac{1}{c}}] - [(n+1)^{\frac{1}{c}}] = 1.
$$

Clearly

$$
[-n^{\frac{1}{c}}] = -n^{\frac{1}{c}} - \psi(-n^{\frac{1}{c}}) - \frac{1}{2}.
$$

So

$$
\sum_{n \sim N} cn^{1 - \frac{1}{c}} e(n\theta) = \sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}([-n^{\frac{1}{c}}] - [(n+1)^{\frac{1}{c}}])
$$

$$
= \sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}(\psi(-(n+1)^{\frac{1}{c}}) - \psi(-n^{\frac{1}{c}})) + \sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}((n+1)^{\frac{1}{c}} - n^{\frac{1}{c}})
$$

$$
= \sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}(\psi(-(n+1)^{\frac{1}{c}}) - \psi(-n^{\frac{1}{c}})) + \sum_{n \sim N} e(n\theta) + O(1).
$$

According to Lemma 3.2,

$$
\sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}(\psi(-(n+1)^{\frac{1}{c}}) - \psi(-n^{\frac{1}{c}}))
$$

$$
= -\frac{1}{2\pi i} \sum_{0 < |h| \leq H} \frac{1}{h} \sum_{n \sim N} e(n\theta) \cdot cn^{1 - \frac{1}{c}}(e(-hn^{\frac{1}{c}}) - e(-h(n+1)^{\frac{1}{c}}))
$$

$$
+ N^{1 - \frac{1}{c}} \cdot O\left(\sum_{n \sim N} \min\left\{1, \frac{1}{H\|n^{\frac{1}{c}}\|}\right\}\right).
$$
Clearly
\[ e(-h(n+1)^\frac{1}{c}) - e(-hn^\frac{1}{c}) = -\frac{2\pi i h}{c} \int_0^1 (n+u)^{\frac{1}{c}-1} e(-h(n+u)^\frac{1}{c}) du. \]

For \( u \in [0, 1) \), since
\[
\frac{d^2}{dx^2} (x \theta \pm h(x+u)^\frac{1}{c}) = \pm c \cdot \frac{h}{c^2} \cdot \frac{h}{(x+u)^{2-\frac{c}{c}}},
\]
by Lemma 3.1, we obtain that
\[
\sum_{n \sim N} e(n \theta - h(n+u)^\frac{1}{c}) \ll N \cdot \frac{h^{\frac{c}{2}}}{N^{1-\frac{c}{c}}} + \frac{N^{1-\frac{c}{c}}}{h^{\frac{c}{2}}}.\]

It follows that
\[
\sum_{n \sim N} \frac{n^{1-\frac{1}{c}}}{(n+u)^{1-\frac{1}{c}}} \cdot e(n \theta - h(n+u)^\frac{1}{c}) \ll N^{\frac{c}{2}} h^{\frac{c}{2}} + N^{1-\frac{c}{c}} h^{-\frac{1}{2}} + 1.
\]

Hence
\[
\sum_{0 < |h| \leq H} \sum_{n \sim N} e(n \theta) \cdot cn^{1-\frac{1}{c}} (e(-h(n+1)^\frac{1}{c}) - e(-hn^\frac{1}{c}))
\ll \int_0^1 \left( \sum_{0 < |h| \leq H} \sum_{n \sim N} \frac{n^{1-\frac{1}{c}}}{(n+u)^{1-\frac{1}{c}}} \cdot e(n \theta - h(n+u)^\frac{1}{c}) \right) du
\ll \sum_{0 < |h| \leq H} \left( N^{\frac{c}{2}} h^{\frac{c}{2}} + N^{1-\frac{c}{c}} h^{-\frac{1}{2}} + 1 \right)
\ll \leq N^{\frac{c}{2}} H^{\frac{c}{2}} + N^{1-\frac{c}{c}} H^{\frac{c}{2}} + H.
\]

On the other hand,
\[
N^{1-\frac{c}{2}} \sum_{n \sim N} \min \left\{ 1, \frac{1}{H \| n \|} \right\}
= N^{1-\frac{c}{2}} \sum_{h=-\infty}^{\infty} b_h \sum_{n \sim N} e(hn^\frac{1}{c}) \ll N^{1-\frac{c}{c}} \sum_{h=-\infty}^{\infty} |b_h| \cdot (h^{\frac{c}{2}} N^{\frac{c}{c}} + N^{1-\frac{c}{c}} h^{-\frac{1}{2}})
\ll \leq N^{1-\frac{c}{2}} \sum_{|h| < H} \log \frac{H}{H^2} \cdot (h^{\frac{c}{2}} N^{\frac{c}{c}} + N^{1-\frac{c}{c}} h^{-\frac{1}{2}}) + N^{1-\frac{c}{2}} \sum_{|h| \geq H} \frac{H}{h^2} \cdot (h^{\frac{c}{2}} N^{\frac{c}{c}} + N^{1-\frac{c}{c}} h^{-\frac{1}{2}})
\ll \leq N^{1-\frac{c}{2}} H^{\frac{c}{2}} \log H + N^{2-\frac{3c}{2c}} H^{-\frac{1}{2}} \log H + N^{1-\frac{c}{c}} H^{\frac{c}{2}} + N^{2-\frac{3}{2c}} H^{-\frac{1}{2}}.
\] (3.3)

Finally, letting \( H = N^{1-\frac{1}{c}} \), we obtain that
\[
\sum_{0 < |h| \leq H} \sum_{n \sim N} e(n \theta) \cdot cn^{1-\frac{1}{c}} (e(-h(n+1)^\frac{1}{c}) - e(-hn^\frac{1}{c})) \ll N^{\frac{c}{2} - \frac{c}{c}},
\]
and
\[ N^{1-\frac{1}{c}} \sum_{n \sim N} \min \left\{ 1, \frac{1}{H \|n^{-1}\|} \right\} \ll N^{\frac{3}{2} - \frac{1}{c}} \log N. \]

Thus (3.2) is derived. \hfill \Box

The following lemmas are motivated by Bourgain’s strategy [3].

For any \( \delta \in (0, 1) \), let
\[ R_\delta := \{ \alpha \in [0, 1) : |\hat{f}(\alpha)| > \delta N \}. \]

**Lemma 3.4.** For any \( \epsilon > 0 \), if
\[ \text{mes} (R_\delta) \ll \epsilon \frac{1}{\delta^{u+\epsilon} N}, \quad (3.4) \]
then for any \( u > v \)
\[ \int_\mathbb{T} |\hat{f}(\alpha)|^u d\alpha \ll_u N^{u-1}. \]

**Proof.** Let \( u > v + \epsilon_0 \)
\[ \int_0^1 |\hat{f}(\alpha)|^u d\alpha \leq \sum_{j \geq 0} \left( \frac{N}{2j-1} \right)^u \cdot \text{mes} \left( \left\{ \alpha \in [0, 1) : \frac{N}{2j} < |\hat{f}(\alpha)| \leq \frac{N}{2j-1} \right\} \right) \]
\[ \ll \epsilon_0 \sum_{j \geq 0} \left( \frac{N}{2j-1} \right)^u \cdot \frac{1}{(\frac{1}{2^j})^{v+\epsilon_0} N} = 2^u N^{u-1} \sum_{j \geq 0} \frac{1}{2^{(u-v-\epsilon)j}} \ll N^{u-1}. \quad (3.5) \]
\hfill \Box

**Lemma 3.5** (Bourgain-Simple). Let \( f, \nu : \mathbb{Z} \to \mathbb{C} \) be a function with \( |f| \leq \nu \). Assume the following assumptions hold:

1. \( \int_\mathbb{T} |\hat{f}|^{u_0} \ll KN^{u_0-1} \)
2. For any \( \theta \in \mathbb{T} \) and \( \epsilon > 0 \), we have \( \hat{\nu}(\theta) \ll \frac{N}{1 + N \|\theta\|} + o(NK^{-\frac{2}{u_1+\epsilon}}) \)
3. \( u_0 + u_1 \geq 2 \)

Then for any \( u > u_0 + u_1 \)
\[ \int_\mathbb{T} |\hat{f}(\alpha)|^u d\alpha \ll_u N^{u-1}. \]

**Proof.** Let \( R_\delta := \{ \alpha \in [0, 1) : |\hat{f}(\alpha)| > \delta N \} \) for any \( \delta \in (0, 1) \). According to Lemma 3.4, we only need to show that
\[ \text{mes} (R_\delta) \ll \epsilon \frac{1}{\delta^{u_0+u_1+\epsilon} N}, \quad (3.6) \]
for any \( \epsilon > 0 \).

Suppose that \( \theta_1, \ldots, \theta_R \in R_\delta \) are \( N^{-1} \)-spaced. Suppose that \( \delta < K^{-\frac{1}{u_1+\epsilon}} \).
Then

\[(\delta N)^{u_0} \cdot \text{mes} (R_\delta) \leq \int_T |\hat{f}(\alpha)|^{u_0} d\alpha \ll KN^{u_0-1}.\]

So

\[\text{mes} (R_\delta) \ll \frac{K}{\delta^{u_0}N} \leq \frac{1}{\delta^{u_0+u_1+\epsilon}N}.\]

Below we assume that \(\delta \geq K^{-\frac{1}{u_1+\epsilon}}\). Since \(\theta_r \in R_\delta\), we have

\[|\hat{f}(\theta_r)| \geq \delta N.\]

It follows that

\[R^2 \delta^2 N^2 \leq \left(\sum_{r=1}^R |\hat{f}(\theta_r)|\right)^2.\]

Since \(|f| \leq \nu\), for each \(n\), we may write

\[f(n) = a_n \nu(n)\]

with \(|a_n| \leq 1\). Further, for \(1 \leq r \leq R\), write

\[|\hat{f}(\theta_r)| = b_r \hat{f}(\theta_r)\]

where \(|b_r| = 1\). Then

\[R^2 \delta^2 N^2 \leq \left(\sum_{r=1}^R b_r \sum_{n \in \mathbb{Z}} a_n \nu(n)e(\theta_r n)\right)^2\]

\[\leq \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \nu(n)\right) \cdot \left(\sum_{n \in \mathbb{Z}} \nu(n) \left|\sum_{r=1}^R b_r e(\theta_r n)\right|^2\right)\]

\[\leq \left(\sum_{n \in \mathbb{Z}} \nu(n)\right) \cdot \sum_{1 \leq r, r' \leq R} b_r b_{r'} \sum_{n \in \mathbb{Z}} \nu(n)e((\theta_r - \theta_{r'})n)\]

\[\ll N \sum_{1 \leq r, r' \leq R} |\hat{\nu}(\theta_r - \theta_{r'})|.\]

Recall \(\delta \geq K^{-\frac{1}{u_1+\epsilon}}\). So we have

\[\delta^2 \ll \frac{1}{R^2} \sum_{1 \leq r, r' \leq R} \frac{1}{1 + N\|\theta_r \theta_{r'}\|}.\]
Let $\gamma = 1 + \frac{\epsilon}{2}$. By Holder inequality, we have

$$\delta^{2\gamma} \ll \frac{1}{R^2} \sum_{1 \leq r, r' \leq R} \frac{1}{(1 + N\|\theta_{r,r'}\|)^\gamma}$$

$$\ll \frac{1}{R^2} \left( \sum_{1 \leq r \neq r' \leq R} \frac{1}{(N\|\theta_{r,r'}\|)^\gamma} + R \right)$$

$$\ll \frac{1}{R^2} \sum_{1 \leq r' \leq R} \sum_{1 \leq r \leq R} \frac{1}{r^{\gamma}} + \frac{1}{R}$$

$$\ll \frac{1}{R}.$$

Hence

$$R \ll \frac{1}{\delta^{2+\epsilon}}$$

which implies that $\text{mes } (R_\delta) \ll \frac{1}{\delta^{u_0+u_1+\epsilon} N} \ll \frac{1}{\delta^{u_0+u_1+\epsilon} N}$ \qed

Next lemma is an application of the lemmas 3.3 and 3.5.

**Lemma 3.6.** For $c < 2$ and $n \leq N$, Let $\tau_{W,b}(n) := (Wn + b)^{1 - \frac{1}{c}}$. If $f : \mathbb{Z} \to \mathbb{C}$ be a function with $|f| \leq \tau_{W,b}$ and

$$u > 2 + \frac{4(c - 1)}{2 - c},$$

then

$$\int_0^1 |\hat{f}(\alpha)|^u d\alpha \ll_u N^{u-1}. \quad (3.7)$$

**Proof.** On the one hand, By Parseval’s identity

$$\int_{\mathbb{Z}} |\hat{f}(\alpha)|^2 d\alpha \leq \sum_n \tau_{W,b}(n)^2 \ll (WN)^{2 - \frac{1}{c}}.$$
On the other hand, by Lemma 3.3 for \( \theta \in [0, 1) \) we have
\[
\hat{\tau}(\theta) = \hat{\tau}_{W,b}^{W}(\theta) = 1 - \sum_{k=0}^{W-1} e \left( (\theta + k) \cdot \frac{b}{W} \right) \\
= \sum_{m \leq X} e \left( \theta \cdot \frac{m + b}{W} \right) + O(N^{\frac{3}{2} - \frac{1}{2} + o(1)}
\]
Thus in Lemma 3.5, let \( K = N^{1 - \frac{1}{c} + o(1)}, u_0 = 2, u_1 = \frac{4(c-1)}{2-c} \). Then our claim follows.

Next lemma is also attributed to Bourgain but with more complex techniques. Here we use Salmensuu version of this Lemma and give the necessary details, e.g. “Fejer analog”, in proving this Lemma.

**Lemma 3.7** (Bourgain-circle method). Let \( 0 < \kappa < 1, u_0 > 2/\kappa \) and \( u_1 > 0 \). Suppose that \( f, \nu : \mathbb{Z} \to \mathbb{C} \) be a function with \( |f| \leq \nu \). Let us define the following Hardy-Littlewood decomposition:
\[
\mathcal{M}_{a,q} = \{ \alpha : |\alpha - \frac{a}{q}| \leq \frac{1}{qT} \}
\]
\[
\mathcal{M} = \bigcup_{a,q} \mathcal{M}_{a,q} \\
\mathcal{M} = T \setminus \mathcal{M}
\]
where \( Q, T \) are some positive numbers with \( T > 2Q^2 \) and \( Q > C + K^{2/(\kappa u_1)} \) for some large constant \( C \).

Assume the following assumptions hold:

1. \( \int_T |\hat{f}|^{u_0} \ll KN^{u_0 - 1} \)
2. For any \( \theta \in \mathcal{M}_{a,q} \), we have \( \hat{\nu}(\theta) \ll \frac{q^{\kappa N}}{1+N\|\theta - a/q\|} + o(NK^{-\frac{2}{u_1}}) \)
3. For any \( \theta \in \mathcal{m} \), we have \( \hat{\nu}(\theta) = o(NK^{-\frac{2}{u_1}}) \)

Then for any \( u > u_0 + u_1 \)
\[
\int_T |\hat{f}(\alpha)|^u d\alpha \ll N^{u-1}.
\]
Proof. Similar to the proof of the above lemma. It is enough to prove that
$$\text{mes} (\mathcal{R}_\delta) \ll \epsilon \frac{1}{\delta^{u_0+u_1} N}$$
in the case
$$\delta \geq K^{-\frac{1}{u_1}}.$$
Suppose that $\theta_1, \ldots, \theta_R \in \mathcal{R}_\delta$ are $N^{-1}$-spaced. For $\gamma > 1$, by dual principle and
Holder inequality we have
$$R^2 \delta^{2\gamma} N^{2\gamma} \ll N^\gamma \sum_{1 \leq r, r' \leq R} |\hat{\nu}(\theta_{r,r'})|^\gamma.$$  
Recalling $\delta \geq K^{-\frac{1}{u_1}}$, we have
$$\sum_{1 \leq r, r' \leq R} |\hat{\nu}(\theta_{r,r'})|^\gamma = o(R^2 \delta^{2\gamma} N^\gamma).$$
Let $Q' = C + \delta^{-h}$, where $2/\kappa < h < 2/\kappa + u_1$. From (2) we have
$$\sum_{\theta_{r,r'} \in \mathcal{R}_\delta} |\hat{\nu}(\theta_{r,r'})|^\gamma = o(R^2 \delta^{2\gamma} N^\gamma).$$
Thus we have
$$R^2 \delta^{2\gamma} N^\gamma \ll \sum_{1 \leq r, r' \leq R} |\hat{\nu}(\theta_{r,r'})|^\gamma \ll \kappa \sum_{1 \leq a \leq q' \leq Q'} \sum_{1 \leq r, r' \leq R} N^\gamma q^{\frac{1}{2} - \kappa \gamma} \left(1 + \frac{N}{\|\theta_{r,r'} - \frac{a}{q}\|}\right)^\gamma.$$  
Let
$$F(\theta) = \frac{1}{(1 + N\|\theta\|)^\gamma}$$
and
$$G(\theta) := \sum_{1 \leq a \leq q} q^{-\kappa \gamma} \cdot F\left(\theta - \frac{a}{q}\right).$$
Evidently,
$$\delta^{2\gamma} R^2 \ll \sum_{1 \leq r, r' \leq R} G(\theta_{r,r'}). \quad (3.8)$$
And if $\|\theta - \theta'\| \leq N^{-1}$, then
$$F(\theta) \sim F(\theta'), \quad G(\theta) \sim G(\theta').$$
Let $\sigma$ be a function over $[0, 1)$ such that
(1) $\sigma(\theta) \in [0, C]$ for each $\theta \in [0, 1]$;
(2) $\text{supp } \sigma \subseteq [-N, N]$;
(3) \( \| \sigma \|_{L^1} \ll R/N \);

(4) \( \sigma(\theta) \geq 1 \) if \( \| \theta - \theta_r \| \leq 1/N \) for some \( 1 \leq r \leq R \).

Let

\[
\kappa(\theta) = \begin{cases} 
1, & \text{if } \theta \in (-\frac{1}{10N}, \frac{1}{10N}), \\
0, & \text{others}.
\end{cases}
\]

Clearly

\[ \sigma(\theta) \gg \sum_{r=1}^{R} \kappa(\theta - \theta_r). \]

Let

\[ \sigma_1(\theta) := \sigma(-\theta). \]

Then

\[
(\sigma * \sigma_1)(\theta) = \int_{0}^{1} \sigma(t)\sigma(t - \theta)dt \]
\[
\gg \int_{0}^{1} \left( \sum_{r=1}^{R} \kappa(t - \theta_r) \right) \cdot \left( \sum_{r=1}^{R} \kappa(t - \theta_r) \right) dt 
\]
\[
= \sum_{1 \leq r, r' \leq R} \int_{0}^{1} \kappa(t - \theta_r)\kappa(t - \theta_{r'}) dt 
\]
\[
\gg \sum_{1 \leq r, r' \leq R} \frac{\kappa(\theta - \theta_{r,r'})}{N}. 
\]

So

\[
\langle G, \sigma \ast \sigma_1 \rangle = \int_{0}^{1} G(\theta) \cdot (\sigma \ast \sigma_1)(\theta) d\theta \]
\[
\gg \frac{1}{N} \sum_{1 \leq r, r' \leq R} \int_{0}^{1} G(\theta) \cdot \kappa(\theta - \theta_{r,r'}) d\theta 
\]

Recall that \( G(\theta) \sim G(\theta_{r,r'}) \) if \( \kappa(\theta - \theta_{r,r'}) = 1 \). So, by (3.8),

\[
\langle G, \sigma \ast \sigma_1 \rangle \gg \frac{1}{N} \sum_{1 \leq r, r' \leq R} G(\theta_{r,r'}) \int_{0}^{1} \kappa(\theta - \theta_{r,r'}) d\theta \gg \frac{\delta^2 \gamma R^2}{N^2}. \quad (3.9)
\]
On the other hand, we have
\[
\hat{G}(k) = \int_0^1 G(\theta)e(k\theta)d\theta = \sum_{q \leq Q'} \sum_{q \leq Q' a=0} q^{-\kappa\gamma} \int_0^1 F\left(\theta - \frac{q}{a}\right)e(k\theta)d\theta
\]
\[
= \sum_{q \leq Q'} q^{-\kappa\gamma} \int_0^1 F(\theta)e(k\theta)\left(\sum_{a=0}^{q-1} e\left(\frac{ka}{q}\right)\right)d\theta = \hat{F}(k) \sum_{q \leq Q'} q^{1-\kappa\gamma}.
\]
So
\[
|\hat{G}(k)| \ll |\hat{F}(k)| \cdot d(k; Q') \leq \|F\|_{L^1} \cdot d(k; Q') \ll \frac{d(k; Q')}{N},
\]
where
\[
d(k; Q') := |\{q \leq Q : q \mid k\}|.
\]
Recall that supp $\hat{\sigma} \in [-N, N]$. It follows that
\[
\langle G, \sigma * \sigma_1 \rangle = \sum_{|k| \leq N} |\hat{G}(k)| \cdot |\hat{\sigma}(k)|^2 \ll \frac{1}{N} \sum_{|k| \leq N} |\hat{\sigma}(k)|^2 \cdot d(k; Q')
\]
\[
\leq \frac{1}{N}\left(Q'^r \sum_{|k| \leq N, d(k; Q') \leq Q'^r} |\hat{\sigma}(k)|^2 + \|\sigma\|_{L^1}^2 \sum_{|k| \leq N, d(k; Q') \geq Q'^r} d(k; Q')\right).
\]
Since $\sigma$ is bounded and $\|\sigma\|_{L^1} \ll R/N$, we have
\[
\sum_{|k| \leq N, d(k; Q') \leq Q'^r} |\hat{\sigma}(k)|^2 \leq \sum_{|k| \leq N} |\hat{\sigma}(k)|^2 = \int_0^1 \sigma(\theta)^2d\theta \leq \|\sigma\|_{L^\infty} \int_0^1 |\sigma(\theta)|d\theta \ll \frac{R}{N}.
\]
And according to \cite{3} Lemma 4.28],
\[
\sum_{|k| \leq N, d(k; Q') \geq Q'^r} d(k; Q') \leq Q' \sum_{|k| \leq N, d(k; Q') \geq Q'^r} 1 \leq C_{\tau, B} Q'^{1-B}
\]
So
\[
\frac{\delta^{2\gamma} R^2}{N^2} \ll \langle G, \sigma * \sigma_1 \rangle \ll \frac{Q'^r R}{N^2} + \frac{Q'^{1-B} R^2}{N^3}.
\]
If we choose $B$ sufficiently large depending on $\gamma$, then the second term of the RHS in above is negligible. Let $\gamma > 1/\kappa$ and a suitable $\tau$. We obtain that
\[
R \ll \delta^{-2\gamma} Q'^r \ll \delta^{-2\gamma} \delta^{-7h} \ll \delta^{u_0-u_1}.
\]
\]
Now we can prove Theorem 2.1
proof of Theorem 2.1. By Lemma 3.5, for any \( v_0 > 2 + \frac{4(c-1)}{2c-1} \) we have

\[
\int_{T} |\hat{f}(\alpha)|^{v_0} d\alpha \left( \frac{\phi(W)}{W} \cdot 2 \log N \right)^{v_0} \int_{T} |\hat{\tau}(\alpha)|^{v_0} d\alpha \ll (\log N)^{O(1)} N^{v_0}
\]

Let \( K = (\log N)^{O(1)} \) and \( u_0 = v_0 \). Note that for any \( u_1 > 0 \) our \( \nu \) is corresponding to the condition (2), (3) in Lemma 3.7.

\( \square \)

For corresponding to the same notations in many papers related to Shapiro primes, we let \( \gamma = \frac{1}{c} \) below. By dyadic argument, it is enough to prove the sieve result in the range \([X, 2X]\). So we will prove our sieve result in the range \([X, 2X]\) instead of \([1, X]\).

4. LOWER BOUND SIEVE

Lemma 4.1 (Type I sum). Let \( 16/19 + \epsilon \leq \gamma < 1 \), \( MN \asymp X \), \( H \leq X^{1-\gamma + 4\delta} \). Assume further that \( a(m), c(h) \) are complex numbers of modulus \( \leq 1 \), and \( M \) satisfies the condition

\[ M < X^{3\gamma - 2 - \epsilon}. \]

Then

\[ \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} c(h)a(m)e(h(mn)^\gamma) \ll X^{1-5\delta}. \]

Proof. See [14, Lemma 4]. \( \square \)

Lemma 4.2 (Type II sum). Let \( 16/19 + \epsilon \leq \gamma < 1 \), \( MN \asymp X \), \( H \leq X^{1-\gamma + 4\delta} \). Assume further that \( a(m), b(n), c(h) \) are complex numbers of modulus \( \leq 1 \), and \( M \) satisfies the condition

\[ X^{1-\gamma + \epsilon} < M \ll X^{2^{3\gamma - 2 - \epsilon}} \]

\[ \text{or} \quad X^{5-5\gamma + \epsilon} < M \ll X^{\gamma - \epsilon} \]

\[ \text{or} \quad X^{3-3\gamma + \epsilon} < M \ll X^{3\gamma - 2 - \epsilon}. \]

Then

\[ \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} c(h)a(m)b(n)e(h(mn)^\gamma) \ll X^{1-5\delta}. \]

Proof. See [14, Lemmas 1 and 3]. \( \square \)

Let \( X \) be a sufficiently large positive number and \( Y = \eta X \) with

\[ \eta \in [\exp(-c(\log X)^{1/8}), \exp(-(\log X)^{1/16})]. \]

We define

\[ \mathcal{A} = \{ X < n < X + Y : n \in \mathbb{N} \}, \]

and

\[ \mathcal{B} = \{ X < n < X + Y \}. \]

By using the above two lemmas, we can get our Type I and II sums which will be applied in Harman sieve method.
Lemma 4.3. Let $16/19 + \epsilon \leq \gamma < 1$. For
\[ M < X^{3\gamma-2-\epsilon}, \]
we have
\[ \sum_{m \sim M, \ mn \in A} \frac{1}{\gamma} a(m)(mn)^{1-\gamma} = \sum_{m \sim M, \ mn \in B} a(m) + O(X^{1-3\delta}). \] (4.2)

Moreover if
\[ X^{1-\gamma+\epsilon} < M < X^{5\gamma-4-\epsilon} \quad \text{or} \quad X^{5-5\gamma+\epsilon} < M < X^{\gamma-\epsilon} \]
or
\[ X^{3-3\gamma+\epsilon} < M < X^{3\gamma-2-\epsilon}, \]
then
\[ \sum_{m \sim M, \ mn \in A} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} = \sum_{m \sim M, \ mn \in B} a(m)b(n) + O(X^{1-3\delta}). \]

Proof. Let us consider the Type II sum. We have
\[ \sum_{m \sim M, \ mn \in A} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} = \sum_{m \sim M, \ mn \in B} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} \left( (mn+1)^\gamma - (mn)^\gamma \right) = \Sigma_1 + \Sigma_2, \]
where
\[ \Sigma_1 = \sum_{m \sim M, \ mn \in B} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} (mn+1)^\gamma - (mn)^\gamma \]
and
\[ \Sigma_2 = \sum_{m \sim M, \ mn \in B} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} \left( -(mn)^\gamma \right) - \left( -(mn+1)^\gamma \right). \]

Note that
\[ \Sigma_1 = \sum_{m \sim M, \ mn \in B} a(m)b(n) + O(X^{\delta}). \]

Applying a well-known reduction argument using the Fourier expansion of the function $\psi(t)$, Namely we use the Vaaler’s theorem (see [5, Theorem A.6]). There exists functions $\psi_1$ a $\psi_2$ such that
\[ \psi(t) = \psi_1(t) + O(\psi_2(t)). \]

Where
\[ \psi_1(t) = \sum_{1 \leq |j| \leq J} \beta_1(j)e(jt), \]
\[ \psi_2(t) = \sum_{|j| \leq J} \beta_2(j)e(jt), \]
\[ \beta_1(j) \ll j^{-1}, \ \beta_2(j) \ll J^{-1}. \]
Moreover, $\psi_2$ is non-negative. Let $f(k) = \frac{1}{\gamma}k^{1-\gamma} \sum_{m \sim M, \frac{m}{n} = k} a(m)b(n)$. Consequently,

$$\Sigma_2 = \sum_{k \in B} f(k)(\psi_1(-k^\gamma) - \psi_1(-(k+1)^\gamma)) + O(\sum_{k \in B} |f(k)|(\psi_2(-k^\gamma) + \psi_2(-(k+1)^\gamma)))$$

$$= \Sigma_3 + O(\Sigma_4).$$

We can dispense $\Sigma_4$ immediately. Note that $\gamma > \frac{1}{2}$. By Lemma 3.1 and taking $J = X^{1-\gamma+\delta}$, we have

$$\sum_{k \sim X} \psi_2(-k^\gamma) \ll \frac{1}{J} \sum_{|j| \leq J} |\sum_{k \sim X} e(jk^\gamma)|$$

$$\ll XJ^{-1} + J^{-1} \sum_{|j| \leq J} (X(jX^{\gamma-2})^{\frac{1}{2}} + (jX^{\gamma-2})^{-\frac{1}{2}})$$

$$\ll XJ^{-1} + J^{\frac{1}{2}}X^{\frac{\gamma}{2}} + J^{-\frac{1}{2}}X^{1-\frac{\gamma}{2}}$$

$$\ll X^{\gamma-4\delta}.$$

The same estimate holds if $k$ is replaced by $k+1$. Hence

$$\Sigma_4 \ll X^{1-3\delta}.$$

To complete the proof, it suffices to show that $\Sigma_3 \ll X^{1-3\delta}$. On writing

$$g_j(t) = 1 - e(j(t^\gamma - (t+1)^\gamma)),$$

By partial summation and Lemma 4.2, we have

$$\Sigma_3 = \sum_{1 \leq |j| \leq J} \beta_1(j) \sum_{k \sim X} f(k)(e(-jk^\gamma) - e(-j(k+1)^\gamma))$$

$$\ll \sum_{1 \leq j \leq J} \beta_1(j) \left| \sum_{k \sim X} f(k)g_j(k)e(-jk^\gamma) \right|$$

$$\ll \sum_{1 \leq j \leq J} j^{-1} \left| g_j(2X)(2X)^{1-\gamma} \sum_{k \sim X} \frac{f(k)}{k^{1-\gamma}}e(jk^\gamma) \right| + \int_X^{2X} \sum_{1 \leq j \leq J} j^{-1} \left| g_j(u)k^{1-\gamma} \sum_{X \leq k \leq u} \frac{f(k)}{k^{1-\gamma}}e(jk^\gamma) \right| du$$

$$\ll \max_{X \leq u \leq 2X} \sum_{1 \leq j \leq J} \left| \sum_{X \leq k \leq u} \frac{f(k)}{k^{1-\gamma}}e(jk^\gamma) \right| = \sum_{1 \leq j \leq J} c(j) \sum_{X \leq mn \leq 2X} a(m)b(n)e(j(mn)^\gamma)$$

$$\ll X^{1-3\delta}.$$

One can prove (4.2) similarly using Lemma 4.1 instead of using Lemma 4.2.

Next lemma is very useful to remove the joint condition, which is helpful when using Harman Sieve method.
Lemma 4.4 (Removing joint conditions). Let \( 16/19 + \epsilon \leq \gamma < 1, MN \asymp X \) and \( M \) satisfies one of the Lemma 4.2 conditions. Let \( I, J \) are integers and \( I_i, J_j \) are intervals for \( 1 \leq i \leq I, 1 \leq j \leq J \). Write

\[
a(m, n) = \sum_{k p_1 \cdots p_I = n} \sum_{p_i \in I_i} c(n) \sum_{l q_1 \cdots q_J = m} d(m)
\]

with \( |c(n)|, |d(m)| \leq 1 \) and \( p_1, \ldots, p_I \) and \( q_1, \ldots, q_J \) satisfying \( t \) joint conditions of the form

\[
p_i \leq q_j \quad \text{or} \quad q_j \leq p_i
\]

or

\[
p_i q_j \leq r \quad \text{(or analogous expression with more variables)}.
\]

Then

\[
\frac{1}{\gamma} \sum_{mn \in A} a(m, n)(mn)^{1-\gamma} = \sum_{mn \in B} a(m, n) + O(X^{1-\delta} L^t).
\]

Proof. Each joint condition as the above mentioned can be remove by the truncated Perron formula

\[
\frac{1}{\pi} \int_{-T}^{T} e^{iya} \frac{\sin(y\beta)}{y} dy = \begin{cases} 
1 + O(T^{-1}(\beta - |\alpha|)^{-1}), & \text{if } |\alpha| \leq \beta \\
O(T^{-1}(|\alpha| - \beta)^{-1}), & \text{if } |\alpha| > \beta
\end{cases} \quad (4.3)
\]

at the cost of an additional \( L \) factor in the error term.

We just proof the simplest case, \( p_u \leq q_v \) as an example, and the reader can prove other cases by using similar way and then by induction to get the result.

Let \( \alpha = \log p_i, \beta = \log(q_j + \frac{1}{2}) \). Obviously,

\[
\beta - \alpha \gg X^{-1}.
\]
We apply (4.3) with $T = X^2$ and get
\[
\sum_{\substack{mn \in A \\
m \sim M}} a(m, n)(mn)^{1-\gamma} = \frac{1}{\pi} \int_{-T}^{T} \sum_{\substack{mn \in A \\
m \sim M}} a(n)b(m)(mn)^{1-\gamma} p_i^{iy} \frac{\sin(y \log(q_v + \frac{1}{2}))}{y} dy + O\left(\frac{X \sum_{n \sim M} \tau(m) \tau(n)}{T} \right)
\]
\[
= \frac{1}{\pi} \int_{-T}^{T} \sum_{\substack{mn \in A \\
m \sim M}} a(n)b(m)(mn)^{1-\gamma} p_i^{iy} \frac{\sin(y \log(q_v + \frac{1}{2}))}{y} dy + O(X^\delta)
\]
\[
= \frac{1}{\pi} \int_{-T}^{T} \sum_{\substack{mn \in B \\
m \sim M}} a(n)b(m)p_i^{iy} \frac{\sin(y \log(q_v + \frac{1}{2}))}{y} dy + O(X^\delta)
\]
\[
+ O\left(X^{1-3\delta} \int_{-T}^{T} \left| \frac{\sin(y \log(q_v + \frac{1}{2}))}{y} \right| dy \right)
\]
\[
= \gamma \sum_{\substack{mn \in B \\
m \sim M}} a(m, n) + O(X^{1-3\delta} L)
\]

Let
\[
S(A_m, z) = \frac{1}{\gamma} \sum_{\substack{mn \in A}} \rho(n, z)(mn)^{1-\gamma}
\]
and
\[
S(B_m, z) = \sum_{\substack{mn \in B}} \rho(n, z)
\]

**Lemma 4.5** (Sieve lemmas). Let $u \geq 1$ with $\prod_{1 \leq k \leq u} p_k \leq X$ and $z = 6\gamma - 5 - 2\epsilon$. Suppose that for some $M$ satisfying
\[
X^{1-\gamma+\epsilon} < M < X^{5\gamma - 4 - \epsilon} \quad \text{or} \quad X^{5-5\gamma+\epsilon} < M < X^{\gamma-\epsilon}
\]

or
\[
X^{3-3\gamma+\epsilon} < M < X^{3\gamma - 2 - \epsilon},
\]
and there exists $D \subset \{1, \ldots, u\}$ such that
\[
\prod_{k \in D} p_k \sim M
\]
Then
\[
\sum_{z \leq p_1 \leq \cdots \leq p_u \leq (2X)^{1/2}} S(A_{p_1, \ldots, p_u}, p_1) = \sum_{z \leq p_1 \leq \cdots \leq p_u \leq (2X)^{1/2}} S(B_{p_1, \ldots, p_u}, p_1) + O(X^{1-\delta}).
\]

Furthermore, if $m \sim M$ with
\[
M < X^{3\gamma - 2 - \epsilon},
\]
then
\[
\sum_{m \sim M} a(m) S(A_m, z) = \sum_{m \sim M} a(m) S(B_m, z) + O(X^{1-\delta}).
\]

**Proof.** Let just give a brief proof of these two results. For the first one we use Lemmas 4.2 and 4.4 by removing possible joint conditions. For the first one we use Lemmas 4.1 and 4.4 by removing two possible joint conditions. For more details, see [14, Lemmas 9 and 10]. □

Now we choose \( \gamma = \frac{100}{117} + \epsilon \) as an example to give our \( \rho^{-}(n) \). For convenience we will remove \( \epsilon \) from the following exponents, so our \( z = \frac{15}{117} \). By Lemma 4.5 our Type I range is

\[ M \leq \frac{66}{117}. \]

Similarly, by Lemma 4.5 our Type II ranges are

\[ \frac{17}{117} < M < \frac{32}{117} \quad \text{or} \quad \frac{45}{117} < M < \frac{100}{117} \]

\[ \text{or} \quad \frac{51}{117} < M < \frac{66}{117}. \]

We apply Buchstab identity and get

\[
S(A, (2X)^{1/2}) = S(A, z) - \sum_{z < p \leq \frac{17}{117}} S(A_p, p) - \sum_{X^{17/117} < p \leq \frac{32}{117}} S(A_p, p) \\
- \sum_{X^{32/117} < p \leq \frac{51}{117}} S(A_p, p) - \sum_{X^{51/117} < p \leq (2X)^{1/2}} S(A_p, p) \\
= S(A, z) - \sum_{X^{17/117} < p \leq \frac{32}{117}} S(A_p, p) - \sum_{X^{32/117} < p \leq \frac{51}{117}} S(A_p, p) \\
- \sum_{X^{51/117} < p \leq (2X)^{1/2}} S(A_p, p) - \sum_{z < p \leq \frac{17}{117}} S(A_p, z) \\
+ \sum_{z < q \leq \frac{17}{117}} S(A_{pq}, q) + \sum_{pq \leq \frac{32}{117}} S(A_{pq}, q) \\
= S_1 - S_2 - S_3 - S_4 - S_5 + S_6 + S_7
\]
For $S_1, S_2, S_4, S_5$ and $S_6$, we use Lemma [1.3]. For $S_3$ we give further decomposition and for $S_7$ we discard it. We apply to $S_3$ the Buchstab identity and get

$$S_3 = \sum_{X^{32/117} < p \leq X^{51/117}} S(A_p, z) - \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q)$$

$$= \sum_{X^{32/117} < p \leq X^{51/117}} S(A_p, z) - \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) - \sum_{z < q < p} S(A_pq, q)$$

$$= S_{10} - S_{11} - S_{12} - S_{13}$$

For $S_8, S_{10}, S_{12}$ we use Lemma [4.5] and note that $S_{13} = 0$ since $pq^2 > X$. Now we consider $S_9.$

$$S_9 = \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) + \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) + \sum_{z < q < p} S(A_pq, q)$$

$$= \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) + \sum_{z < q < p} S(A_pq, q)$$

$$= S_{14} + S_{15} - S_{16} + S_{17}.$$

For $S_{14}, S_{15}$ and $S_{16}$, we use Lemma [1.3]. For $S_{17}$ we discard it. Similarly,

$$S_{11} = \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) + \sum_{X^{32/117} < p \leq X^{51/117}} S(A_pq, q) + \sum_{z < q < p} S(A_pq, q)$$

$$= S_{18} + S_{19} + S_{20}.$$

For $S_{19},$ we use Lemma [4.5]. For $S_{18}$ and $S_{20}$ we discard them.
Thus decomposition for

Now we define our

The remaining task is to calculate the contributions of $T_7, \ldots, T_{20}$ in $S(\mathcal{B}, (2X)^{1/2})$. Let us first define the corresponding regions. Let

$$D_7 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \left(\frac{16}{117}, \frac{17}{117}\right), \alpha_2 \in (z, \alpha_1), \alpha_1 + \alpha_2 \in \left(\frac{32}{117}, \frac{34}{117}\right)\},$$

$$D_{17} = \{(\alpha_1, \alpha_2) : \alpha_1 \in \left(\frac{32}{117}, \frac{51}{117}\right), \alpha_2 \in (z, \alpha_1), \alpha_1 + \alpha_2 \in \left(\frac{47}{117}, \frac{51}{117}\right), \alpha_1 + 2\alpha_2 \in \left(\frac{66}{117}, 1\right)\},$$

$$D_{18} = \{(\alpha_1, \alpha_2) : \alpha_1 \in \left(\frac{32}{117}, \frac{51}{117}\right), \alpha_2 \in (z, \alpha_1), \alpha_1 + \alpha_2 \in \left(\frac{66}{117}, \frac{85}{117}\right)\},$$

$$D_{20} = \{(\alpha_1, \alpha_2) : \alpha_1 \in \left(\frac{32}{117}, \frac{51}{117}\right), \alpha_2 \in \left(\frac{32}{117}, \alpha_1\right), \alpha_1 + \alpha_2 \in \left(\frac{66}{117}, \frac{85}{117}\right), \alpha_1 + 2\alpha_2 \in \left(\frac{98}{117}, 1\right)\}.$$

By prime number theorem

$$T_i = I_i S(\mathcal{B}, (2X)^{1/2}) + O(Y/(\log X)^4),$$

where

$$I_i = \int \int_{D_i} \omega \left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2} \right) \frac{1}{\alpha_2 \alpha_1} d\alpha_1 d\alpha_2.$$  

By simple calculation, we have $I_7 \leq 0.01551, I_{17} \leq 0.01459, I_{18} \leq 0.00987$ and $I_{20} \leq 0.14207$ Hence we have

$$S(\mathcal{A}, (2X)^{1/2}) = (1 - I_7 - I_{17} - I_{18} - I_{20}) S(\mathcal{B}, (2X)^{1/2}) + O(Y/(\log X)^4)$$

Thus

$$S(\mathcal{A}, (2X)^{1/2}) = \alpha^- S(\mathcal{B}, (2X)^{1/2}) + O(Y/(\log X)^4) \geq 0.8179 S(\mathcal{B}, (2X)^{1/2}).$$

Now we define our $\rho^-(n)$ as the following.

$$\rho^-(n) = \rho(n) - \rho_7(n) - \rho_{17}(n) - \rho_{18}(n) - \rho_{20}(n).$$

where $\rho_i(n)$ are corresponding to $S_i$ and $T_i$, for example,

$$\rho_7(n) = \sum_{n = mpq \atop z < q \leq X^{1/117} \atop pq > X^{32/117}} \rho(m, q).$$

Consequently our $\alpha^- \geq 0.8179$. 

Remark. By Siegel-Walfisz theorem, the above results can be extended to arithmetic progression cases with small common difference $d \ll \log^A X$ for any $A \geq 0$, which the assumption (3) in Theorem 2.2.

5. Upper bound Sieve

Here we use the Balog-Friedlander Type I and II sums.

Lemma 5.6 (Type I). Suppose that $5/6 + \varepsilon \leq \gamma < 1, \alpha \in \mathbb{T}, MN \asymp X, H \leq X^{1-\gamma+\varepsilon+\delta}$, and assume further that $a(m), c(h)$ are complex numbers of modulus $\leq 1$, and $M$ satisfies the condition

$$M < X^{4\gamma-3-\varepsilon}.$$  

Then

$$\sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} c(h)a(m)e(\alpha mn + h(mn)\gamma) \ll X^{1-\varepsilon-\delta}.$$  

Furthermore, we have

$$\frac{1}{\gamma} \sum_{m \sim M, mn \in A} a(m)(mn)^{1-\gamma} e(\alpha mn) = \sum_{m \sim M, mn \in B} a(m)e(\alpha mn) + O(X^{1-\varepsilon}).$$  

Proof. See [2, Proposition 3]

Lemma 5.7 (Type II). Suppose that $5/6 + \varepsilon \leq \gamma < 1, \alpha \in \mathbb{T}, MN \asymp X, H \leq X^{1-\gamma+\varepsilon+\delta}$, and assume further that $a(m), c(h)$ are complex numbers of modulus $\leq 1$, and $M$ satisfies the condition

$$X^{1-\gamma+\varepsilon} < M < X^{5\gamma-4-\varepsilon} \quad \text{or} \quad X^{5-5\gamma+\varepsilon} < M < X^{\gamma-\varepsilon}$$

Then

$$\sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} c(h)a(m)b(n)e(\alpha mn + h(mn)\gamma) \ll X^{1-\varepsilon-\delta}.$$  

Furthermore, we have

$$\frac{1}{\gamma} \sum_{m \sim M, mn \in A} a(m)b(n)(mn)^{1-\gamma} e(\alpha mn) = \sum_{m \sim M, mn \in B} a(m)b(n)e(\alpha mn) + O(X^{1-\varepsilon}).$$  

Proof. See [2, Proposition 2]
Motivated by the above Type I and II estimates, when \( \gamma = \frac{100}{117} + \epsilon \) we construct our \( \rho^+(n) \) as the following

\[
S(A, (2X)^{1/2}) = S(A, z) - \sum_{\frac{X}{117} < p < \frac{2X}{117}} S(A_p, p) - \sum_{\frac{2X}{117} < p < (2X)^{1/2}} S(A_p, p)
\]

\[
= S(A, z) - \sum_{\frac{X}{117} < p < \frac{2X}{117}} S(A_p, p) - \sum_{\frac{2X}{117} < p < (2X)^{1/2}} S(A_p, p)
- \sum_{\frac{X}{117} < p < \frac{2X}{117}, p^2 < \frac{32}{117}} S(A_p, z) + \sum_{\frac{X}{117} < p < \frac{2X}{117}, p^2 > \frac{32}{117}} S(A_p, q) - \sum_{\frac{2X}{117} < p < (2X)^{1/2}} S(A_p, p)
\]

\[
= S_1 - S_2 - S_3 - S_4 + S_5 - S_6.
\]

For \( S_1, S_2, S_4, S_5 \) we use Lemma 4.3. For \( S_3 \) and \( S_6 \) we discard them. Similarly, we have

\[
S(A, (2X)^{1/2}) = (1 + I_3 + I_6)S(B, (2x)^{1/2}) + O(Y/(\log X)^4).
\]

where \( I_3 < 1.05114, I_6 < 0.24277 \). Hence we have

\[
S(A, (2x)^{1/2}) \leq 2.294S(B, (2x)^{1/2}).
\]

Now we define \( \rho^+(n) = \rho(n) + \rho_3(n) + \rho_6(n) \), where \( \rho_i(n) \) is corresponding to \( T_i \) and \( S_i \), for example,

\[
\rho_3(n) = \sum_{\frac{X}{117} < p < (2X)^{1/2}} \rho(m, p).
\]

Consequently our \( \alpha^+ \leq 2.294 \), so \( \alpha^+ < 3\alpha^- \)

Remark. By the prime number theorem, the range \( [X, X + Y] \) of \( A \) and \( B \) can be replaced by \( [X, X + Y'] \) for any \( Y \leq Y' \leq X \) and the above results also hold.

6. Upper bound Fourier transform

We first note that in our case,

\[
\sum_{n \in [X, 2X] \cap \mathbb{N}^c, n \equiv l \pmod{W}} cn^{1 - \frac{1}{c}} \rho^+(n)e(n\theta)
\]

and

\[
\sum_{n \in [X, 2X] \cap \mathbb{N}^c, n \equiv l \pmod{W}} \rho^+(n)e(n\theta)
\]
are comparable, Since

\[
\sum_{n \in [X,2X]\cap \mathbb{N}^c} cn^{1-\frac{1}{r}} \rho^+(n)e(n\theta) = \sum_{n \in [X,2X]\cap \mathbb{N}^c} cn^{1-\frac{1}{r}} \rho^+(n)e(\theta n) \cdot \frac{1}{W} \sum_{k=1}^{W} e\left(\frac{n-k}{W}\right)
\]

\[
= \frac{1}{W} \sum_{k=1}^{W} e\left(\frac{-lk}{W}\right) \sum_{n \in [X,2X]\cap \mathbb{N}^c} cn^{1-\frac{1}{r}} \rho^+(n)e(n(\theta + \frac{k}{W}))
\]

\[
= \frac{1}{W} \sum_{k=1}^{W} e\left(\frac{-lk}{W}\right) \sum_{n \in [X,2X]} \rho^+(n)e(n(\theta + \frac{k}{W})) + O(X^{1-\epsilon})
\]

\[
= \sum_{n \in [X,2X]} \rho^+(n)e(n\theta) + O(X^{1-\epsilon})
\]

So next we just consider the approximation of \(\sum_{n \in [X,2X]} \rho^+(n)e(n\theta)\). We divide the Torus \(\mathbb{T}\) into so called Major arc and Minor arc. For the minor arc case, we just use the standard Type I and II estimates. For the major arc case we use the Siegel-Walfisz theorem.

Let \(P = L^B\) and \(Q = \frac{X}{L^{2B}}\). Then we define major arcs are

\[
\mathcal{M}_{a,q} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].
\]

Let

\[
\mathcal{M} = \bigcup_{(a,q)=1}^{q \leq P} \mathcal{M}_{a,q},
\]

and

\[
m = \mathbb{T} \setminus \mathcal{M}.
\]

For dealing with the minor arc case, we first note that \(\|q\theta\| \leq \frac{1}{q}\) for \(P \leq q \leq Q\) if \(\theta \in m\). And we just need to prove the following Type I and II exponent estimates.

**Lemma 6.8.** [Type I sum] Let \(M \leq X^{1-\epsilon}\) and \(MN \asymp X\) for any \(\epsilon > 0\). Then for all \(\alpha \in m\), we have

\[
| \sum_{m \sim M, n \sim N, mn \equiv c \pmod{d}} a_m e(mn\alpha) | \ll \frac{X}{\log^A X}.
\]
Proof. By Cauchy-Schwarz inequality

\[
| \sum_{m \sim M} a_n e(mn) | \ll \left( \sum_{m \sim M} |a_m|^2 \right)^{1/2} \left( \sum_{m \sim M} \sum_{n \sim N} e(mn) |^2 \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} M^{1/2} N^{1/2} \left( \sum_{m \sim M} \sum_{n \sim N} e(mn) |^2 \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} X^{1/2} \left( \sum_{m \sim M} \min \left\{ \frac{X}{d m}, \frac{1}{\|d \alpha\|} \right\} \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} X^{1/2} \left( \frac{X}{q} + dM + q \right)^{1/2}
\]

\[
\Box
\]

Lemma 6.9. [Type II sum] Let \( X^\epsilon \leq M \leq X^{1-\epsilon} \) and \( MN \sim X \) for any \( \epsilon > 0 \). Then for all \( \alpha \in m \), we have

\[
| \sum_{m \sim M} a_m b_n e(mn) | \ll \frac{X}{\log X}.
\]

Proof. By Cauchy-Schwarz inequality

\[
| \sum_{m \sim M} a_m b_n e(mn) | \ll \left( \sum_{m \sim M} |a_m|^2 \right)^{1/2} \left( \sum_{m \sim M} \sum_{n \sim N} b_n e(mn) |^2 \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} M^{1/2} \left( \sum_{n_1, n_2 \sim N} |b_{n_1} \overline{b}_{n_2}| \right) \left( \sum_{m \sim M} \sum_{mn \equiv c \pmod{d}} e(m(n_1 - n_2)\alpha) |^2 \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} M^{1/2} \left( \sum_{n_1 \sim N} |b_{n_1}|^2 \max_{k \sim N} \sum_{l \sim N} \min \left\{ \frac{M}{d}, \frac{1}{\|d \alpha\|} \right\} \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} X^{1/2} \left( \frac{dN}{q} + 1 \right) \left( \frac{X}{dN} + q \right)^{1/2}
\]

\[
\ll (\log X)^{O(1)} X^{1/2} \left( \frac{X}{q} + dN + \frac{X}{dN} + q \right)^{1/2}
\]

\[
\Box
\]

Lemma 6.10. For \( \alpha \in \mathfrak{M} \), the assumption (3) of Theorem 2.2 holds.
Proof. Since \( \rho^+(n) = 0 \) if \( n \) has factors less than \( N^{3/15} \), we have \( (n, qd) = 1 \). Thus

\[
\sum_{n \sim N} \rho^+(n)e(n\theta) = \sum_{n \equiv c \pmod{d}} \rho^+(n)e(n\frac{a}{q} + \lambda))
\]

\[
= \sum_{n \sim N} \rho^+(n)e(n\frac{a}{q} + \lambda))
\]

\[
= \sum_{r \equiv c \pmod{d}} e\left(\frac{ar}{q}\right) \sum_{n \equiv r \pmod{[d,q]}} \rho^+(n)e(\lambda n)
\]

Next we follow Green’s argument \([6]\) by dividing \( \{n : n \equiv r \pmod{[d,q]}, n \in [N, 2N]\} \) to small \( A \)'s \( X_i \) with \( |X_i| \sim \frac{N^{1/15}}{L^{2/3}} \).

So the inner sum is

\[
\sum_{n \sim N} \rho^+(n)e(\lambda n) = \sum_{i=1}^{T} \sum_{n \in X_i} \rho^+(n)e(\lambda n)
\]

\[
= \sum_{i=1}^{T} e(\lambda x_i) \sum_{n \in X_i} \rho^+(n) + \sum_{i=1}^{T} \sum_{n \in X_i} \rho^+(n)e(\lambda n) - e(\lambda x_i)
\]

\[
= \frac{\alpha^+}{\log N} \frac{[d, q]}{\phi([d, q])} \sum_{i=1}^{T} e(\lambda x_i)(1 + O(L^{-1}))[X_i] + O\left(N \frac{[d, q]}{\phi([d, q])} L^{-C-1}q^{-1}\right)
\]

\[
= \frac{\alpha^+}{\log N} \frac{[d, q]}{\phi([d, q])} \sum_{i=1}^{T} e(\lambda x_i)[X_i] + O\left(NL^{-1} \frac{1}{\phi([d, q])}\right) + O\left(N \frac{[d, q]}{\phi([d, q])} L^{-C-1}q^{-1}\right)
\]

\[
= \frac{\alpha^+}{\log N} \frac{[d, q]}{\phi([d, q])} \sum_{i=1}^{T} e(\lambda n) + O\left(NL^{-2} \frac{1}{\phi([d, q])}\right) + O\left(N \frac{[d, q]}{\phi([d, q])} L^{-C-1}q^{-1}\right)
\]
Thus we have
\[
\sum_{n \equiv c \pmod{d}} \rho^+(n)e(n\theta) = \frac{\alpha^+}{\log N} \sum_{n \equiv c \pmod{d}, n \sim N} e(\lambda n) + O(NL^{-2} \frac{\phi(q)}{\phi([d, q])}) + O(N \frac{\phi(q)[d, q]}{\phi([d, q])} L^{-c-1}q^{-1})
\]
\[
= \frac{\alpha^+}{\log N} \sum_{n \equiv c \pmod{d}, n \sim N} e(\lambda n) + O\left(\frac{N}{\phi(d) \log N}\right)
\]

Now let \( d = W \), and note that \([W, q]/\phi([W, q]) = (1 + o(1))W/\phi(W)\), since \( q \leq L^B \).

Also note that if \((n, q) \neq 1\), then \((n, q)\) is \( w \)-rough. The contribution from \( n \equiv c \pmod{W} \) with \((n, q) \neq 1\) is \( \sum_{p \mid q} \sum_{p \geq w} \frac{1}{p} = o\left(\frac{Y}{N}\right)\). Then Lemma 6.10 follows. \( \square \)

Next we will verify the minor arc and major arc estimates in Theorem 2.1. For the minor arc case, it is same to Lemmas 6.8 and 6.9. We just prove the major arc case.

**Lemma 6.11 (Major arc condition in Theorem 2.1).** The \( \nu \) involving our \( \rho^+ \) satisfies the major arc condition in Thm 2.2.

**Proof.** Let \( Y = \eta N \) and \( \eta \in [\exp(-c\log N)^{\frac{1}{2}}), \exp(-\log N)^{\frac{1}{2}}]\). By well-known Siegel Walfisz theorem, for any \( C > 0 \), we have

\[
\sum_{N \leq n \leq N+Y} \rho^+(Wn + l)e\left(\frac{a}{q}n\right) = \sum_{r=1}^{q} e\left(\frac{a}{q}r\right) \sum_{n \equiv r \pmod{q}, N \leq n \leq N+Y} \rho^+(Wn + l)
\]
\[
= \frac{\alpha^+ W}{\phi(Wq)} \int_N^{N+Y} \frac{1}{\log t} dt \sum_{r=1}^{q} e\left(\frac{a}{q}r\right) + O\left(\frac{Y}{\log^C N}\right).
\]

Note that one can replace \( \int_N^{N+Y} \frac{1}{\log t} dt \) by \( \frac{Y}{\log N} + O\left(\frac{Y}{\log^C N}\right)\). However we may deal with the long interval case below, so the integral form is convenient for our use. By the above equation, one can immediately get that

\[
\sum_{N \leq n \leq N+Y'} \rho^+(Wn + l)e\left(\frac{a}{q}n\right) = \frac{\alpha^+ W}{\phi(Wq)} \int_N^{N+Y'} \frac{1}{\log t} dt \sum_{r=1}^{q} e\left(\frac{a}{q}r\right) + O\left(\frac{Y'}{\log^C N}\right),
\]

for any \( Y \leq Y' \leq N \).
Then we can compare
\[
\sum_{n \sim N} \rho^+(Wn + l)e(\theta n)
\]
with
\[
\frac{\alpha^+ W}{\phi(Wq)} \sum_{r=1}^q e\left(\frac{a}{q} r\right) \sum_{n \sim N} \frac{e(\lambda n)}{\log n}
\]
For any \( \theta \in \mathcal{M}_{u,q} \) and \( \lambda = \theta - \frac{a}{q} \).

Let
\[
f(n) = \rho^+(Wn + l)e\left(\frac{a}{q} n\right) - \frac{\alpha^+ W}{\phi(Wq)} \log n \sum_{r=1}^q e\left(\frac{a}{q} r\right)
\]
By partial summation
\[
\sum_{n \sim N} f(n)e(\lambda n) = \int_N^{2N} e(\lambda t) d\left( \sum_{N \leq n < t} f(n) \right)
\]
\[
= e(2\lambda N) \sum_{n \sim N} f(n) - \int_N^{2N} \lambda e(\lambda t) \sum_{N \leq n \leq t} f(n) dt.
\]
Since \( \lambda \leq \frac{\log^B N}{N} \), we have
\[
\sum_{n \in N} f(n)e(\lambda n) \ll \frac{N}{\log^C N}.
\]
Thus
\[
\sum_{n \sim N} \rho^+(Wn + l)e(\theta n) = \frac{\alpha^+ W}{\phi(Wq)} \sum_{r=1}^q e\left(\frac{a}{q} r\right) \sum_{n \sim N} \frac{e(\lambda n)}{\log n} + O\left( \frac{N}{\log^C N} \right).
\]
Let \( G(t) = \sum_{n \leq t} e(\lambda n) \). By partial summation
\[
\sum_{n \sim N} \frac{e(\lambda n)}{\log n} = \sum_{n \sim N} \frac{1}{\log n} (G(n) - G(n - 1))
\]
\[
= \sum_{N \leq n \leq 2N - 1} G(n) \left( \frac{1}{\log n} - \frac{1}{\log (n + 1)} \right) + \frac{1}{\log 2N} G(2N) - \frac{1}{\log N} G(N - 1)
\]
\[
\ll \frac{1}{\log N} \min\{N, \lambda^{-1}\}.
\]
On the other hand, since \((l, W) = 1\), then
\[
\sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{(W r + l, W q) = 1} = \sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{d | (W r + l, W q)} \sum_{\mu(d) \neq 0} \sum_{d | W q} \mu(d) \sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{d | W r + l} = \sum_{d | W q} \mu(d) \sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{d | W r + l}.
\]

Note that
\[
\sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{d | W r + l} = \begin{cases} e(\frac{-W a}{q}) & d = q \text{ and } (W, q) = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Hence
\[
\sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{(W r + l, W q) = 1} = \sum_{d | W q} \mu(d) \sum_{r=1}^{q} \frac{e(\frac{a}{q} r)}{d | W r + l} = \begin{cases} \mu(q)e(\frac{-W a}{q}) & (W, q) = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

\[\square\]

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References

[1] R. C. Baker, G. Harman, J. Rivat, Primes of the form \([n^c]\), J. Number Theory 50 (1995), 261–277.
[2] A. Balog, J. Friedlander A hybrid of theorems of Vinogradov and Piatetski-Shapiro, Pacific J. Math. 156 (1992), 45-62.
[3] J. Bourgain, On \(\Lambda(p)\)-subsets of squares, Israel J. Math., 67 (1989), 291-311.
[4] Y.-C. Cai, A remark on the Piatetski-Shapiro-Vinogradov theorem, Acta Arith., 110 (2003), 73-75.
[5] S. W. Graham, G. Kolesnik, van der Corput’s method of exponential sums. London Mathematical Society Lecture Note Series, 126. Cambridge University Press, Cambridge, 1991.
[6] B. Green, Roth’s theorem in the primes, Ann. of Math. (2), 161 (2005), 1609-1636.
[7] L. Grumelt, Vinogradov’s Theorem with Fouvry-Iwaniec Primes, preprint, arXiv:1809.10008.
[8] D. R. Heath-Brown, The Pjateckii-Šapiro prime number theorem. J. Number Theory. 16 (1983), 242-266.
[9] H. Helfgott, The ternary Goldbach conjecture is true, preprint, arXiv:1312.7748.
[10] C. H. Jia, On Piatetski-Shapiro’s prime number theorem II, Sci. China, 36 (1993),913-926.
[11] C. H. Jia, On Piatetski-Shapiro’s prime number theorem I, Chinese Ann. Math. Ser. B, 15 (1994), 9-22.
[12] C.-H. Jia, On the Piatetski-Shapiro-Vinogradov theorem, Acta Arith., 73 (1995), 1-28.
[13] A. Kumchev, the Piatetski-Shapiro-Vinogradov theorem, J. Thé or. Nombres Bordeaux, 9 (1997), 11-23.
[14] A. Kumchev, *On the distribution of prime numbers of the form \([n^c]\)*, Glasg. Math. J. **41** (1999), 85-102.
[15] H.-Z. Li, *A hybrid of theorems of Goldbach and Piatetski-Shapiro*, Acta Arith., **107** (2003), 307-326.
[16] H.-Z. Li and H. Pan, *A density version of Vinogradov's three primes theorem*, Forum Math., **22** (2010), 699-714.
[17] K. Matomäki, J. Maynard and X. Shao, *Vinogradov’s theorem with almost equal summands*, Proc. Lond. Math. Soc., **115** (2017), 323-347.
[18] K. Matomäki and X. Shao, *Vinogradov’s three primes theorem with almost twin primes*, Compos. Math., **153** (2017), 1220-1256.
[19] X.-M. Meng and M.-Q. Wang, Mingqiang, *A hybrid of theorems of Goldbach and Piatetski-Shapiro*, Chinese Ann. Math. Ser. B, **27** (2006), 341-352.
[20] T. P. Peneva, *An additive problem with Piatetski-Shapiro primes and almost-primes*, Monatsh. Math., **140** (2003), 119-133.
[21] I. I. Piatetski-Shapiro, *On the distribution of prime numbers in sequences of the form \([f(n)]\)*, Mat. Sb., **33** (1953), 559-566.
[22] J. Rivat, *Autour d’un théorème de Pjateckii-Šapiro: (nombres premiers dans la suite \([n^c]\)]*, Thèse de Doctorat (Université Paris-Sud, 1992).
[23] J. Rivat and J. Wu, *Prime numbers of the form \([n^c]\)*, Glasg. Math. J., **43** (2001), 237-254.
[24] X. Shao, Xuancheng, *An \(L\)-function-free proof of Vinogradov’s three primes theorem*, Forum Math. Sigma, **2** (2014), e27, 26 pp.
[25] X. Shao, *A density version of the Vinogradov three primes theorem*, Duke Math. J., **163** (2014), 489-512.
[26] Q.-L. Shen, *The ternary Goldbach problem with primes in positive density sets*, J. Number Theory, **168** (2016), 334-345.
[27] X.-N. Wang and Y.-C. Cai, *An additive problem involving Piatetski-Shapiro primes*, Int. J. Number Theory, **7** (2011), 1359-1378.

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