INVOLUTIVE OPERATOR ALGEBRAS

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Abstract. Examples of operator algebras with involution include the operator $\ast$-algebras occurring in noncommutative differential geometry studied recently by Mesland, Kaad, Lesch, and others, several classical function algebras, triangular matrix algebras, (complexifications) of real operator algebras, and an operator algebraic version of the complex symmetric operators studied by Garcia, Putinar, Wogen and others. We investigate the general theory of involutive operator algebras, and give many applications. Much of our work is focused around ‘real positivity’ in the sense of several recent papers of the first author and collaborators referenced in our bibliography.

1. Introduction

An operator algebra is a closed subalgebra of $B(H)$, for a complex Hilbert space $H$. Here we study operator algebras with involution. Examples include the operator $\ast$-algebras occurring in noncommutative differential geometry studied recently by Mesland, Kaad, Lesch, and others (see e.g. [25, 24, 9] and references therein), (complexifications) of real operator algebras, and an operator algebraic version of the complex symmetric operators studied by Garcia, Putinar, Wogen and many others (see [20] for a survey, or e.g. [21]).

By an operator $\ast$-algebra we mean an operator algebra with an involution $\dagger$ making it a $\ast$-algebra with $\|[a_{ij}]\| = \|[a_{ij}]\|$ for $[a_{ij}] \in M_n(A)$ and $n \in \mathbb{N}$. Here we are using the matrix norms of operator space theory (see e.g. [29]). This notion was first introduced by Mesland in the setting of noncommutative differential geometry [25], who was soon joined by Kaad and Lesch [24]. In several recent papers by these authors and coauthors they exploit operator $\ast$-algebras and involutive modules in geometric situations. Subsequently we noticed very many other examples of operator $\ast$-algebras, and other involutive operator algebras, occurring naturally in general operator algebra theory which seem to have not been studied hitherto. It is thus natural to investigate the general theory of involutive operator algebras, and this is the focus of the present paper. We are able to include a rather large number of results since many proofs are similar to their operator algebra counterparts in the literature (see e.g. [11]). Thus we often need only discuss the new points that arise. However to follow some of the arguments the reader will need to have the operator algebra variant from the original papers in hand. It is worth saying that some of the arguments we are following are complicated, and so it is not clear in advance...
whether they have ‘involutive variants’. In fact some of the main theorems about operator algebras do not have operator $*$-algebra variants, so some work is needed to disentangle the items that do work. We make no attempt to be comprehensive for the sake of avoiding tedium. We will simply illustrate the main techniques and features, indicating what can be done. Many of the results are focused around ‘real positivity’ in the sense of several recent papers of the first author and collaborators referenced in our bibliography. Some related theory and several complementary results can be found in the second authors PhD thesis [32].

1.1. Structure of our paper. In the rest of this section we give some background, perspective, and notations. In Section 2 we give several general results. For example we prove some facts about involutions on nonselfadjoint operator algebras and their relationship to the $C^*$-algebras they generate. As an application of some ideas in the theory of complex symmetric operators we characterize the symmetric operator algebras introduced in [3]. This is a problem outstanding from the early years of operator space theory. Section 3 is devoted to examples of involutive operator algebras, for instance examples coming from operator space theory, subdiagonal algebras, model theory for contractions on a Hilbert space, and complex symmetric operators. In the remaining sections we restrict our focus, for specificity, to operator $*$-algebras. In Section 4 we discuss contractive approximate identities, Cohen factorization for operator $*$-algebras, multiplier operator $*$-algebras, dual operator $*$-algebras (by which we mean an operator $*$-algebra which is a dual operator space with weak* continuous involution), and involutive $M$-ideals. Section 5 has a common theme of hereditary subalgebras and ideals, noncommutative topology (e.g. open projections, support projections, and compact and peak projections), and peak interpolation, in the involutive setting. Thus we are finding the involutive variants of the operator algebra theory of these topics from e.g. the papers [14, 15, 16, 12, 8, 7].

1.2. Involutions, and notation. By an involution we mean at least a bijection $\tau : A \to A$ which is of period 2: $\tau^2(a) = a$ for $a \in A$. A $C^*$-algebra $B$ may have two kinds of extra involution: a period 2 conjugate linear $*\text{-anti}$automorphism or a period 2 linear $*\text{-anti}$automorphism. The former is just the usual involution $*$ composed with a period 2 $*$-automorphism of $B$. The latter is essentially the same as a ‘real structure’, that is if $\theta$ is the antiautomorphism then $B$ is just the complexification of a real $C^*$-algebra $D = \{x \in B : x = \tilde{x}\}$, where $\tilde{x} = \theta(x)^*$. We may characterize $x \mapsto \tilde{x}$ on $B$ very simply as the map $a + ib \mapsto a - ib$ for $a, b \in D$.

By way of contrast, there are four distinct natural kinds of ‘completely isometric involution’ on a general operator algebra $A$. Namely, period 2 bijections which are

1. conjugate linear antiautomorphisms $\dagger : A \to A$ satisfying $\|a_{ij}\| = \|a_{ji}\|$, 
2. linear antiautomorphisms $\theta : A \to A$ satisfying $\|a_{ij}\| = \|a_{ij}\|$, 
3. conjugate linear automorphisms $- : A \to A$ satisfying $\|a_{ij}\| = \|a_{ij}\|$, 
4. linear automorphisms $\pi : A \to A$ satisfying $\|a_{ij}\| = \|a_{ij}\|$.

Here $[a_{ij}]$ is a generic element in $M_n(A)$, the $n \times n$ matrices with entries in $A$, for all $n \in \mathbb{N}$. Class (1) is just the operator $*$-algebras mentioned earlier. In this paper we will call the algebras in class (2) operator algebras with linear involution $\theta$, and write $\theta(a)$ as $a^\theta$. We will not discuss (4) in this paper, these are well studied and are only mentioned here because most of the results in the present paper apply to all four
classes. We will just say that this class is in bijective correspondence with the unital completely symmetric projections on \( A \) in the sense of [13], this correspondence is essentially Corollary 4.2 there. Similarly, for the same reasons we will not discuss class (3) in this paper. By [30, Theorem 3.3], class (3) is essentially the same as ‘real operator algebra structure’, that is \( A \) is just the complexification of a real operator algebra \( D = \{ x \in B : x = \bar{x} \} \), and we may rewrite \( \bar{x} = a - ib \) if \( x = a + ib \) for \( a, b \in D \). Thus the variant of the main aspects of our paper in case (3) seem best treated within the theory of real operator algebras. However it is worth saying that the theory in our paper in case (3) may be viewed as a transliteration of a chapter in the theory of real operator algebras. We also remark that if \( A \) is unital or approximately unital then one can easily show using the Banach-Stone theorem for operator algebras (see e.g. [11, Theorem 4.5.13]) that the matrix norm equality in (3) and (4) (resp. (1) and (2)) force the ‘involution’ to be multiplicative (resp. anti-multiplicative).

If \( A \) is a \( C^* \)-algebra then classes (1) and (4) are essentially the same after applying the \( C^* \)-algebra involution \( \ast \). (Note that in this case the matrix norm equality in (1) or (4) follows from the same equality for \( 1 \times 1 \) matrices, that is that the involution is isometric. Indeed it is well known that \( \ast \)-isomorphisms of \( C^* \)-algebras are completely isometric.) Similarly classes (2) and (3) essentially coincide if \( A \) is a \( C^\ast \)-algebra.

We will mostly focus on class (1) for specificity. In fact most of the results in the present paper apply to all four classes, however it would be too tedious to state several cases of each result. Instead we leave it to the reader to state the matching results in cases (2)–(4). For example to get from case (1) to case (2) of results below one replaces \( a^\dagger \) by \( a^\theta \), and \( \dagger \)-selfadjoint elements, that is elements satisfying \( a^\dagger = a \), by elements with \( a^\theta = a \). We remark that if \( A \) is an operator algebra with linear involution \( \theta \), then \( \{ a \in A : a = a^\theta \} \) is a Jordan operator algebra in the sense of [18]. (We remark that these \( \theta \)-selfadjoint elements' need not generate \( A \), unlike for involutions of type (1).) Most of our discussion of class (2) involves finding interesting examples of such involutions. Indeed although classes (1)–(4) have similar theory from the viewpoint of our paper, the examples of algebras in these classes are quite different in general.

Because of the ubiquity of the asterisk symbol in our area of study, we usually write the involution on an operator \( \ast \)-algebra as \( \dagger \), and refer to, for example, \( \dagger \)-selfadjoint elements or subalgebras, and \( \dagger \)-homomorphisms (the natural morphisms for \( \ast \)-algebras).

A little more background and notation: A unital operator algebra has an identity of norm 1, and an approximately unital operator algebra has a contractive approximate identity (cai). For background on operator spaces and operator algebras from an operator space point of view we refer the reader to [11, 29]. Meyer’s theorem states that any operator algebra \( A \) has a unitization \( A^1 \) that is unique up to completely isometric isomorphism [11, Corollary 2.1.15]. If \( A \) is nonunital then \( A \) is of codimension 1 in the unital operator algebra \( A^1 \); otherwise set \( A^1 = A \). In this paper, all projections \( p \in A \) are orthogonal projections. If \( X \) and \( Y \) are sets then we write \( XY \) for the norm closure of the span of terms of the form \( xy \), for \( x \in X, y \in Y \). The second dual \( A^{**} \) of an operator algebra \( A \) is again an operator algebra, which is unital if \( A \) is approximately unital.
We recall that a $C^*$-cover $(B, j)$ of an operator algebra $A$ is a $C^*$-algebra $B$ and a completely isometric homomorphism $j : A \to B$ such that $j(A)$ generates $B$ as a $C^*$-algebra. Sometimes we simply call this a $C^*$-algebra generated by $A$. There is a ‘biggest’ and ‘smallest’ $C^*$-cover, $C^*_\text{max}(A)$ and $C^*_\text{e}(A)$ (see [11] Propositions 4.3.5 and 2.4.2)). For example $C^*_\text{max}(A)$ has the universal property that any completely contractive representation $\pi : A \to B(H)$ extends to a $*$-representation of $C^*_\text{max}(A)$ on $H$. Any completely isometric homomorphism $j : A \to B$ into a $C^*$-cover $B$ of $A$ generated by the copy of $A$, gives rise to a $*$-homomorphism $B \to C^*_\text{e}(A)$ which is ‘the identity’ on the copy of $A$.

Because of the uniqueness of unitization, for an operator algebra $A$ we can define unambiguously $\mathfrak{g}_A = \{ a \in A : \|1 - a\| \leq 1 \}$. Then $\mathfrak{g}_A = \{ a \in A : \|1 - 2a\| \leq 1 \} \subset \text{Ball}(A)$. Similarly, $\mathfrak{v}_A$, the real positive or accretive elements in $A$, is $\{ a \in A : a + a^* \geq 0 \}$, where the adjoint $a^*$ is taken in any $C^*$-cover of $A$. We write $\text{oa}(x)$ for the operator algebra generated by an operator $x$. By a symmetry we mean either a selfadjoint unitary operator, or a period 2 $*$-automorphism of a $C^*$-algebra, depending on the context.

2. INVOLUTIVE OPERATOR ALGEBRAS

We recall for any operator space $X$ the opposite and adjoint operator spaces $X^\circ$ and $X^\ast$ from 1.2.25 in [11]. Here $X^\circ$ is $X$ but with ‘transposed matrix norms’ $\| x_{ij} \| = \| x_{ji} \|$. Similarly $X^\ast$ is the set of formal symbols $x^\ast$ for $x \in X$, but with the same operator space structure as $\{ x^\ast \in B : x \in X \}$, if $X$ is (completely isometrically) a subspace of a $C^*$-algebra $B$.

If $A$ is an operator algebra we write $A^1$ for the unitization of $A$. If $X$ is an operator space then $I(X)$ and $T(X)$ are respectively the injective and ternary envelope of $X$ from e.g. [11, Chapter 4].

**Proposition 2.1.** If $X$ is an operator space and $A$ is an operator algebra then

1. $(A^1)^\circ = (A^\circ)^1$, $I(X^\circ) = I(X)^\circ$, $T(X^\circ) = T(X)^\circ$, $C^e(A^\circ) = C^e(A)^\circ$, and $C^\text{max}(A^\circ) = C^\text{max}(A)^\circ$.
2. $(A^1)^\ast = (A^\ast)^1$, $I(X^\ast) = I(X)^\ast$, $T(X^\ast) = T(X)^\ast$, $C^e(A^\ast) = C^e(A)^\ast$, and $C^\text{max}(A^\ast) = C^\text{max}(A)^\ast$.

**Proof.** Note that $(A^\circ)^1$ is a unital operator algebra containing $A^\circ$ as a codimension 1 subalgebra, so by Meyer’s theorem [11, Corollary 2.1.15] it must be the unitization. Similarly for $(A^1)^\ast$. The rest all follow by the universal properties defining these objects, and a diagram chase applying $\circ$ or $\ast$ to the maps in the diagrams. For example, such a strategy shows that $I(X)^\circ$ is injective. It contains $X^\circ$, and a similar strategy shows that it has the ‘rigidity’ property (or ‘essential’ property) characterizing the injective envelope.

**Remark.** There is a similar result for $X$ and $A$, which would be useful in treating class (3) mentioned early in the Introduction. Here $X = (X^\ast)^\circ$, and from this formula the proof of the result in this case is clear.

The following result, the involutive variant of Meyer’s theorem [11, Corollary 2.1.15], is useful in treating involutions on operator algebras with no identity or approximate identity.

**Lemma 2.2.** Let $A$ be a nonunital operator algebra with an involution of one of the types (1)–(4) at the start of Section 1.2. Then the involution on $A$ has a
unique extension to an involution of the same type on the unitization $A^1$, with the involution of 1 being 1.

**Proof.** For operator $\ast$-algebras this is [9, Lemma 1.15]. If $\theta : A \to A$ is a linear involution (type (2) in the list at the start of Section 1.2), then by Meyer’s theorem $a \mapsto \theta(a)^\circ$ extends to a unital completely isometric homomorphism $A^1 \to (A^\circ)^1$. Composing this with $\circ$, and using the fact that $(A^1)^\circ = (A^\circ)^1$ from Proposition 2.1, we obtain our result. The other cases are similar or easier. □

**Proposition 2.3.** Let $A$ and $B$ be operator algebras with $A$ nonunital. Also suppose that there exists involutions on $A$ and $B$ of one of the types (1)–(4) at the start of Section 1.2. Let $\pi : A \to B$ be a completely contractive (resp. completely isometric) involution preserving homomorphism, then there is a unital completely contractive (resp. completely isometric) involution preserving homomorphism extending $\pi : A^1 \to B^1$ (for the completely isometric case we also need $B$ nonunital).

**Proof.** By Lemma 2.2 we know that both $A^1$ and $B^1$ are operator algebras with the same type of involution. The unital extension of $\pi$ to $A^1$ is completely contractive (resp. completely isometric) by [11, Theorem 2.1.13 and Corollary 2.1.15]. It is easy to check that it is also involution preserving. □

**Remark.** One may replace completely contractive (resp. completely isometric) by contractive (resp. isometric) in the last result. Thus the unitization $A^1$ of an involutive operator algebra is unique up to (completely) isometric involutive isomorphism.

We now characterize class (2) from the start of Section 1.2. A **conjugation** on a complex Hilbert space $H$ is a conjugate linear period 2 isometry $u : H \to H$. If $j : H \to \bar{H}$ is the canonical conjugate linear map into the conjugate Hilbert space, then $ju$ is unitary, so $\langle jux, juy \rangle = \langle ux, uy \rangle = \overline{\langle x, y \rangle}$ for $x, y \in H$. An operator $T$ on $H$ is called $c$-symmetric if $cTc = T^*$, and is called complex symmetric if it is $c$-symmetric for a conjugation $c$ on $H$. The class of complex symmetric operators is very large and significant (see e.g. [20]).

**Theorem 2.4.** Let $A$ be an operator algebra. The following are equivalent:

(i) $A$ is an operator algebra with linear involution $\theta$.

(ii) There exists a $C^*$-algebra $B$ generated by $A$ (or by $A^1$), and a period 2 $*$-anti-isomorphism $\rho : B \to B$ with $\rho(A) = A$.

(iii) There exists a conjugation $c$ on a complex Hilbert space $H$ on which $A$ may be completely isometrically represented as an operator algebra such that $cA^*c \subset A$ (here we are identifying $A$ with its image in $B(H)$).

We may take $\theta$ in (i) to be the restriction to $A$ of the $\rho$ in (ii), or of the map $T \mapsto cT^*c$ in (iii). We may take $B$ in (iii) to be $C^*_c(A)$ (or $C^*_c(A^1)$), or $C^*_\max(A)$ (or $C^*_\max(A^1)$).

**Proof.** (i) $\Rightarrow$ (ii) We may assume that $A$ is unital by Proposition 2.3. Given (i), the map $A \to A^\circ : a \mapsto \theta(a)^\circ$ is a completely isometric isomorphism, so extends to $*$-isomorphism $B = C^*_c(A) \to C^*_c(A^\circ)$. The latter algebra equals $C^*_c(A^\circ)$ by Proposition 2.1. This gives a $*$-anti-isomorphism on $B$ taking $A$ onto $A$, which is easily checked to be period 2. Similarly with $B = C^*_\max(A)$ using the universal property of these $C^*$-algebras and the appropriate item in Proposition 2.1.
The algebra generated by any operator on a Hilbert space is isometrically isomorphic to the algebra generated by a complex symmetric operator on another Hilbert space.

**Corollary 2.5.** An operator algebra $A$ on a complex Hilbert space $H$ may be completely isometrically represented as an operator algebra, such that $cac = a^*$ for all $c$ in $A$ if and only if $A$ is symmetric (i.e. which are symmetric with respect to the transpose as matrices). By an operator algebra of matrices we mean a subalgebra of $M_n$ for a cardinal $n$, where $M_n = B(ℓ^2)$ thought of as $I × I$ matrices. Any operator algebra of matrices is the algebra generated by a set of commuting symmetric matrices in $M_n$.

**Corollary 2.6.** An operator algebra $A$ is commutative if and only if it is isometrically isomorphic to an operator algebra of matrices that equal their transpose.

**Proof.** If $A$ is commutative then it is isometrically isomorphic to $\{(a, a^*) ∈ A ⊕ A^* : a ∈ A\}$, which is a symmetric operator algebra. The rest follows from Corollary 2.5.

**Corollary 2.7.** The algebra generated by any operator on a Hilbert space is isometrically isomorphic to the algebra generated by a complex symmetric operator on another Hilbert space.

There are similar characterizations for the other three classes of ‘involutions’ considered at the start of Section 1.2. Indeed the result matching Theorem 2.4 for operator $*$-algebras is the following, mostly from [9] Section 1.]
Theorem 2.8. Let $A$ be an operator algebra. The following are equivalent:

(i) $A$ is an operator $\ast$-algebra.

(ii) There exists a $C^\ast$-algebra $B$ generated by $A$ (or of $A^1$), and a period 2 $\ast$-automorphism $\rho : B \to B$ with $\rho(A) = A^\ast$.

(iii) There exists a symmetry $u$ on a complex Hilbert space $H$ on which $A$ may be completely isometrically represented as an operator algebra such that $uA^\ast u \subset A$ (here we are identifying $A$ with its image in $B(H)$).

We may take $\theta$ in (i) to be the restriction to $A$ of $\rho(\cdot)^\ast$ for $\rho$ as in (ii), or to be the map $T \mapsto uT^\ast u$ in (iii). We may take $B$ in (ii) to be $C_e^\ast(A)$ (or $C_e^\ast(A^1)$), or $C_{\max}^\ast(A)$ (or $C_{\max}^\ast(A^1)$).

Proof. This is proved in [9, Section 1], except for the assertion about $C_{\max}^\ast(A)$. If $\rho : A \to C_{\max}^\ast(A)$ is the canonical ‘inclusion’ let $\pi : A \to C_{\max}^\ast(A)$ be the completely isometric homomorphism defined by $\pi(a) = \rho(a^1)^\ast$. By the universal property of $C_{\max}^\ast(A)$, there exists a unique $\ast$-homomorphism $\sigma : C_{\max}^\ast(A) \to C_{\max}^\ast(A)$ such that $\sigma(\rho(a)) = \pi(a) = \rho(a^1)^\ast$ for any $a \in A$. Moreover, $\sigma$ has order 2 since

$$\sigma^2(\rho(a)) = \sigma(\sigma(\rho(a))) = \sigma(\rho(a)^\ast) = \rho(a),$$

and since $\rho(A)$ generates $C_{\max}^\ast(A)$ as a $C^\ast$-algebra. The final assertion follows by extending to the unitization and using $C_{\max}^\ast(A^1) = C_{\max}^\ast(A^1)$ from Proposition 2.1.

It is natural to ask if in item (ii) in Theorems 2.4 or 2.8 (ii), or in matching results for the other types of involutions, one may use any $C^\ast$-algebra generated by a completely isometric copy of $A$. The answer is in the negative, as one sees in the following result and the example following it.

Definition 2.9. Suppose that an operator algebra $A$ has an involution $\nu$ of one of the types (1)–(4) at the start of Section 1.2. If a $C^\ast$-cover $(B, j)$ of $A$ has an involution $\omega$ of the same type, and if $j(a)^\omega = j(a^\nu)$, for any $a \in A$, we say that the involution on $B$ is compatible with $A$.

Lemma 2.10. Suppose that $A$ is an operator algebra (possibly not approximately unital) with involution $\nu$ of type (1) (resp. type (2)) at the start of Section 1.2 and $(B, j)$ is a $C^\ast$-cover of $A$. Then $B$ has an involution compatible with $A$ if and only if there exists an order 2 $\ast$-automorphism (resp. $\ast$-anti-automorphism) $\sigma : B \to B$ such that $\sigma(j(a^\omega)) = j(a)^\ast$ (resp. $\sigma(j(a^\nu)) = j(a^\nu)$) for any $a \in A$.

Proof. (⇒) If $B$ has an involution $\omega$ compatible with $A$, then $j(a)^\omega = j(a^\nu)$, for all $a \in A$. Define $\sigma : B \to B$ by $\sigma(b) = (b^\ast)^\omega$ (resp. $b^\nu$) for any $b \in B$. Then it is easy to see that $\sigma$ is an order 2 $\ast$-automorphism (resp. $\ast$-anti-automorphism).

(⇐) The involution on $B$ is defined by $b^\omega = \sigma(b)^\ast$ (resp. $b^\nu = \sigma(b)$) for any $b \in B$. Then $B$ is a $C^\ast$-algebra with involution which is compatible with $A$.

Example 2.11. The Toeplitz $C^\ast$-algebra is a well known $C^\ast$-cover of the disk algebra $A(D)$. We show that it is not compatible with the involution $f(\zeta)$ on $A(D)$. Let $S$ be the unilateral shift on $l^2(\mathbb{N}_0)$ and $oa(S)$ be the operator algebra generated by $S$. Then $oa(S)$ is an operator $\ast$-algebra with trivial involution induced by $S^\dagger = S$. Suppose that the Toeplitz $C^\ast$-algebra $C^\ast(S)$ has an involution compatible with $oa(S)$. Then there exists an order-2 $\ast$-isomorphism $C^\ast(S)$ such that $\sigma(S^\dagger) = S^\ast$. Moreover, we have

$$I = \sigma(I) = \sigma(S^\ast S) = \sigma(S)^\ast \sigma(S) = SS^\ast \neq I,$$
which is a contradiction.

It is similarly not hard to find (using \(\text{Lemma 3.2}\) below) commutative \(C^\ast\)-algebras \(C(K)\) generated by a function algebra \(A\) with linear involution \(\theta\) (such as \(A = A(\mathbb{D})\)), such that \(\theta\) does not extend to a linear involution on \(C(K)\).

**Lemma 2.12.** Let \(A\) be an operator algebra with an involution of one of the types (1)–(4) at the start of Section 1.2. Then the involution on \(A\) has a unique extension to a weak* continuous involution of the same type on the bidual \(A^{**}\).

**Proof.** We will just prove this in the case of a linear involution \(\theta\); the others are similar. The associated completely isometric homomorphism \(A \to A^\circ : a \mapsto \theta(a)^\circ\) extends to a weak* continuous completely isometric homomorphism \(A^{**} \to (A^\circ)^{**}\). However it is an easy exercise to see that \((A^\circ)^{**} \cong (A^{**})^\circ\). Composing with \(\circ\) we obtain a weak* continuous linear involution on \(A^{**}\) extending \(\theta\). \(\square\)

We mention that the Cayley transform \(\kappa\) and the \(\mathfrak{K}\) transform of [16 Section 2.2], important tools in the area of the later sections of our paper, do work well with respect to involutions. For example suppose that \(\dagger\) is the involution on an operator \(\ast\)-algebra, and \(\sigma\) the associated \(\ast\)-automorphism on a (compatible) \(C^\ast\)-cover. If \(x\) is real positive then \(\sigma(x^\dagger) = x^\dagger\) is real positive,

\[
\sigma\kappa(x) = \sigma((x^\dagger - 1)(x^\dagger + 1)^{-1}) = (x^\ast - 1)(x^\ast + 1)^{-1} = \kappa(x)^\ast.
\]

So \(\kappa(x^\dagger) = \kappa(x)^\dagger\). Similarly, if \(x\) is a contraction with \(1 - x\) invertible then the same is true for \(x^\dagger\) and the inverse Cayley transform \(\kappa^{-1}(x^\dagger) = (1 + x^\dagger)(1 - x^\dagger)^{-1}\) is real positive, and must equal \(\kappa^{-1}(x)^\dagger\). The \(\mathfrak{K}\)-map is \(\mathfrak{K}(x) = \frac{1}{\kappa^2}(1 + \kappa(x)) = x(1 + x)^{-1}\). Following the proof in Lemma 2.5 in [16], it is easy to see that for any operator \(\ast\)-algebra \(A\), the \(\mathfrak{K}\)-map maps the \(\dagger\)-selfadjoint elements in \(\mathfrak{r}_A\) bijectively onto the set of \(\dagger\)-selfadjoint elements in \(\frac{1}{\kappa}\mathfrak{K}_A\) of norm \(< 1\).

### 3. Examples

We give many examples of operator \(\ast\)-algebras and operator algebras with linear involution here. Of course any real operator algebra at all gives an example of the third type of involution mentioned at the start of Section 1.2, namely the complexification. We will not consider these here.

#### 3.1. Examples from noncommutative differential geometry.

Several examples of operator \(\ast\)-algebras were given in [9], most of them examples from noncommutative differential geometry (historically the first such example being due to Mesland). Other examples from noncommutative differential geometry may be found in other recent papers of Kaad, Mesland, and their coauthors.

#### 3.2. Function algebra examples.

Let \(A\) be a uniform algebra (with minimal operator space structure, see 1.2.21 in [11]). Then \(A \subset \mathcal{C}_\varepsilon(A) = \mathcal{C}(\partial A)\), where \(\partial A\) is the Shilov boundary of \(A\) (see e.g. [11 Section 4.1]). If \(A\) is an operator \(\ast\)-algebra (resp. has linear involution \(\theta\)), then there exists a period 2 homeomorphism \(\tau : \partial A \to \partial A\) such that \(f^\dagger(\omega) = f(\tau(\omega))\) (resp. \(f^\theta = f \circ \tau\)) for any \(f \in A\). From this formula it is easy to write down function algebra examples. For example, the disk algebra \(\mathcal{A}(\mathbb{D})\) is an operator \(\ast\)-algebra with \(f^\dagger(z) = \overline{f(z)}\), and so are its closed \(\dagger\)-ideals of functions e.g. vanishing at 0, or at 1. The latter ideal is interesting from the perspective of approximate identities: it is nonunital, is a \(\dagger\)-ideal, and has a real positive \(\dagger\)-selfadjoint cai (see Lemma 4.2 below, etc). Similarly \(H^\infty(\mathbb{D})\) is a dual
operator ∗-algebra with the same involution. This involution is weak* continuous, and extends to an involution on the von Neumann algebra $L^\infty(\mathbb{T})$.

These two algebras also have linear involution $f^\theta(z) = f(-z)$. This and the identity map are the only linear involutions on $A(\mathbb{D})$ and $H^\infty$, by the well known theory of automorphisms of these algebras.

We recall that a $Q$-algebra is an operator algebra quotient of a function algebra (with minimal operator space structure) by a closed ideal. Q-algebras are symmetric operator algebras, and in particular have a linear involution. If the function algebra has an involution making it an operator ∗-algebra, and the ideal is involutive, then we call the quotient an involutive $Q$-algebra. We will see later that for example the algebra generated by the Volterra operator is an involutive $Q$-algebra.

### 3.3. Examples from complex symmetric and ∗-exchangeable operators.

An operator $T$ in a $C^*$-algebra $B$ will be called ∗-exchangeable if $\|p(T, T^*)\| = \|p(T^*, T)\|$ for any polynomial $p$ in two free variables. One may use polynomials without constant term here if one wishes. Indeed if the equality holds for such polynomials $p$ then it follows that the map $p(T, T^*) \mapsto p(T^*, T)$ is well defined on a dense subset of $C^*(T)$, hence extends to a ∗-homomorphism $\sigma$ on $C^*(T)$ taking $T$ to $T^*$, and extends further to $C^*(1,T)$. This shows that the norm equality holds for polynomials with constant terms too. It is easy to see that $\sigma$ is a period 2 ∗-automorphism.

For a polynomial $p$ of one variable we write $p^*$ for the same polynomial but with coefficients replaced by their complex conjugate (that is, $p^*(z) = \overline{p(z)}$).

**Theorem 3.1.** Let $A$ be an operator algebra with a single generator $T$. The following are equivalent:

(i) For $n \in \mathbb{N}$ and polynomials $p_{ij}$ for $1 \leq i, j \leq n$ we have

$$\|p_{ji}^*(T)\| = \|p_{ij}(T)\|.$$  

(ii) $T$ is ∗-exchangeable in some $C^*$-algebra generated by $A$.

(iii) $A$ is an operator ∗-algebra with $T$ ∗-selfadjoint.

(iv) There exists a symmetry $u$ on a Hilbert space $H$ on which $A$ may be completely isometrically represented as an operator algebra such that $T^* = uT^*u$ (here we are identifying $T$ with its image in $B(H)$).

In (i) one may if one wishes use only polynomials with no constant term.

**Proof.** (iv) ⇒ (i) Given linear symmetry $u : H \to H$ with $T^* = uT^*u$ then $p(T)^* = p^*(T^*) = u p^*(T) u^*$, so that

$$\|p_{ji}^*(T)\| = \|u p_{ij}(T)^* u\| = \|p_{ij}(T)\|.$$  

(i) ⇒ (iii) If for polynomials $p_{ij}$ with no constant term we have $\|p_{ij}(T)\| = \|p_{ji}^*(T)\| = \|p_{ij}(T^*)\|$, then the map $p(T) \mapsto p(T^*)$ is well defined and completely isometric. Here ∗ is the involution on a $C^*$-algebra containing $A$. It extends to a completely isometric surjective homomorphism $oa(T) \to oa(T^*)$. Composing this with the involution ∗ we obtain an involution on $oa(T)$ making it an operator ∗-algebra with $T$ ∗-selfadjoint. (If one wishes then we can extend the involution to the unitization by Proposition 2.3 which implies the equality in (i) for polynomials with constant term.)

(iii) ⇒ (ii) If $oa(T)$ has such involution then by the characterization of operator ∗-algebras in Theorem 2.5 there exists a ∗-isomorphism $C^*_e(oa(T)) \to C^*_e(oa(T))$.
taking $T^\dagger = T$ to $T^*$. Equivalently (as in the discussion above the theorem), $p(T, T^*) \mapsto p(T^*, T)$ is a well-defined isometry.

(ii) $\Rightarrow$ (iii) If $T$ is $*$-exchangeable in some $C^*$-algebra $B$ generated by $A$, then as explained above the theorem we have a period 2 $*$-automorphism $\sigma : B \to B$ with $\sigma(T) = T^*$. The restriction of $\sigma$ to $A$ maps onto $A^*$. So $A$ is an operator $*$-algebra with $T$ $\dagger$-selfadjoint if we define $a^\dagger = \sigma(a)^*$.

(iii) $\Rightarrow$ (iv) By the characterization of operator $*$-algebras in Theorem 2.8 there exists a symmetry $u$ on a Hilbert space $H$ on which $A$ may be completely isometrically represented as an operator algebra such that $a^\dagger = ua^*u$ for all $a \in A$. Setting $a = T$ we obtain $T^* = uTu$. \hfill $\Box$

There is a similar result for operator algebras with linear involution $\theta$ with $\theta(T) = T$. The analogue of condition (ii) in Theorem 3.1 is the condition called $g$-normality in [22], namely that $\|p(T, T^*)\| = \|p^*(T^*, T)\|$ for any polynomial $p$ in two free variables. Here $p^\dagger$ is obtained from $p$ by conjugating each coefficient. The equivalence of (ii) and (iv) is known: after our paper was written we found this equivalence in [31] with a quite different proof. The paper [33] also contains some other very interesting related results.

**Theorem 3.2.** Let $A$ be an operator algebra with a single generator $T$. The following are equivalent:

(i) For $n \in \mathbb{N}$ and polynomials $p_{ij}$ for $1 \leq i, j \leq n$ we have

$$\|p_{ij}(T)\| = \|p_{ij}(T)^\dagger\|.$$

(ii) $T$ is $g$-normal in some $C^*$-algebra generated by $A$.

(iii) $A$ is a symmetric operator algebra (that is $I_A$ is a linear involution).

(iv) There exists a Hilbert space $H$ on which $A$ may be completely isometrically represented as an operator algebra such that $T$ becomes a complex symmetry on $H$ (in the sense defined above Theorem 2.4).

In (i) one may wish to use only polynomials with no constant term.

**Proof.** (iv) $\Rightarrow$ (i) Given conjugation $c : H \to H$ with $T^* = cTc$ then $p(T)^* = p^\dagger(T^*) = cp(T)c$, so that

$$\|p_{ij}(T)\| = \|p_{ij}(T)^\dagger\| = \|cp_{ij}(T)c\| = \|p_{ij}(T)^\dagger\|.$$

(i) $\Leftrightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (ii) If $oa(T)$ has such involution then by Theorem 2.8 there exists a $*$-antiautomorphism $C^*_c(\text{oa}(I, T)) \to C^*_c(\text{oa}(I, T))$ taking $T$ to $T$. Composing with $*$, we get a conjugate linear $*$-automorphism of $C^*_c(\text{oa}(I, T))$ taking $T$ to $T^*$. Equivalently, $p(T, T^*) \mapsto p^\dagger(T^*, T)$ is a well-defined isometry.

(ii) $\Rightarrow$ (iii) If $T$ is $g$-normal in some $C^*$-algebra $B$ generated by $A$, then we have a period 2 $*$-antiautomorphism $\sigma : B \to B$ with $\sigma(T) = T^*$. The restriction of $\sigma$ to $A$ maps onto $A$. So $A$ is an operator algebra with linear involution $\theta$ with $T^\theta = T$.

(iii) $\Rightarrow$ (iv) Immediate from Corollary 2.5. \hfill $\Box$

**Example 3.3.** One may ask if all operators $T$ satisfy the conditions in the last theorem, or in the one before it. However Halmos’ example $x = 2E_{12} + E_{23}$ in $M_3$ may be shown to be a counterexample. Since $x^\dagger = x^*$ (we write $\dagger$ for the transpose) the same example will work for both. Indeed one can show that $x$ generates $M_3$ as a $C^*$-algebra, and since this is simple we have $C^*_c(\text{oa}(x)) = M_3$. Any $*$-automorphism of $M_3$ is inner, and also $*$-antiautomorphisms of $M_3$ are of form $u^*a^\dagger u$ for a unitary
Thus $P\in M_n$. An easy matrix computation show that there are no unitary solutions to $u^*xu = x^T = x^*$. Thus $x$ is not $*$-exchangeable or $g$-normal in $M_n$, hence $oa(x)$ is not symmetric nor is an operator $*$-algebra with $x$ self-adjoint. On the other hand, the matrix $x \perp x^T$ in $M_6$ does satisfy the conditions in the last two theorems (this may be seen similarly to the idea in the proof of Corollary 2.26).

Many ‘truncated Toeplitz operators’ are complex symmetric, and some are $*$-exchangeable, giving by the theorems above examples of operator $*$-algebras, and operator algebras with linear involution. To see these assertions it is helpful to recall the Sz. Nagy-Foias model theory for contractions [26, 4]. For many contractions $T$ it is known that $T$ is unitarily isomorphic to a truncated Toeplitz operator, a so-called Jordan block [4, Chapter 3], namely the compression $S(u) = P_K S|_K$ of the unilateral shift $S$, viewed as multiplication by $z$ on $H^2$, to the subspace $K = H^2 \ominus uH^2$, for a (nonconstant) inner function $u$ on the disk. Thus the weak* closed algebra $A_T$ generated by $T$ (and $I$) is completely isometrically and weak* homeomorphically isomorphic to the weak* closed algebra generated by $S(u)$ (and $I$). On the other hand, the last weak* closed algebra is known to be equal to the commutant $(S(u))^\prime$, and is isometrically weak* homeomorphic to the quotient $H^\infty/uH^\infty$ (see [4, Corollary 1.20]).

**Lemma 3.4.** The weak* closed operator algebra generated by a Jordan block $S(u)$ is symmetric, indeed is a $Q$-algebra, for every inner function $u$. Thus $S(u)$ satisfies the conditions of the last theorem.

**Proof.** In [4, Corollary 1.20], it is shown that $A_{S(u)} = (S(u))^\prime$ is isometrically weak* homeomorphic to the quotient $H^\infty/uH^\infty$. Following the ideas in the proof of [4, Corollary 1.20] one can see that this isometry is a complete isometry. The functional calculus $H^\infty \to A_{S(u)}$ for $S(u)$ is a complete contraction since it has a positive unital, hence completely positive and completely contractive, extension to $L^\infty$. Thus we have an isometric complete contraction $H^\infty/uH^\infty \to A_T$. The unilateral shift $S$ is a minimal isometric dilation of $S(u)$. Suppose that $x = [x_{ij}] \in \text{Ball}(M_n(A_{S(u)}))$. Then $S^{(n)}$ is a minimal isometric dilation of $S^{(n)}$. By the commutant lifting theorem (e.g. [4, Theorem 1.10]) there exists $y = [y_{ij}] \in \text{Ball}(M_n(B(H^2)))$ such that $y \in \{S^{(n)}\}^\prime$, so that $y_{ij} \in \{S\}^\prime$, and $P_{K^{(n)}}y_{K^{(n)}} = x$. Thus $P_K(y_{ij})|_K = x_{ij}$. Since $y_{ij} \in \{S\}^\prime$, and the $H^\infty$ functional calculus is a complete isometry $H^\infty \to \{S\}^\prime$, we see that $y_{ij} = f_{ij}(S)$ for $[f_{ij}] \in \text{Ball}(M_n(H^\infty))$. Thus $[f_{ij} + uH^\infty] \in \text{Ball}(M_n(H^\infty/uH^\infty))$ is a preimage of $x$. It follows that $A_{S(u)} \cong H^\infty/uH^\infty$ completely isometrically (and weak* homeomorphically). Now $H^\infty/uH^\infty$ is a $Q$-algebra. Hence $A_{S(u)}$ is a $Q$-algebra and symmetric operator algebra. Its subalgebra $oa(S(u))$ is thus also symmetric, so $S(u)$ satisfies the conditions of the last theorem. □

As a consequence, the large class of contractions $T$ unitarily equivalent to a Jordan block $S(u)$ for some (nonconstant) inner function $u$ on the disk, all generate symmetric operator algebras, in particular operator algebras having linear involution.

Turning to more specific examples, the Volterra operator $Vf(x) = \int_0^x f(t) \, dt$ on $L^2((0,1))$ is both $*$-exchangeable and complex symmetric (the latter via the conjugation $cf(t) = f(1-t)$). Thus the operator algebra generated by $V$ is both an operator $*$-algebra and has linear involution. The same is true for the weak*
closed algebra generated by \( V \). These may be viewed as infinite dimensional versions of the upper triangular matrices. Indeed the Volterra operator \( V \) is unitarily equivalent to \( S(u) \) with \( u(z) = \exp((z + 1)(z - 1)^{-1}) \), by e.g. [4, Lemma 3.18 on p. 97], and this \( u \) is invariant under the involution \( f^\dagger(z) = f(z) \). Thus \( H^\infty/uH^\infty \) is an operator \(*\)-algebra, indeed is an involutive \( Q \)-algebra, and also is a dual operator \(*\)-algebra. This is because \( \dagger \) is a weak* continuous involution on \( H^\infty \). Hence \( A_V \) is a dual operator \(*\)-algebra. Similarly, the norm closed algebra \( \text{oa}(V) \) generated by the Volterra algebra is completely isometrically isomorphic to \( A_1(\mathbb{D})/uA_1(\mathbb{D}) \) where \( A_1(\mathbb{D}) \) are the disk algebra functions vanishing at 1 (the isomorphism \( H^\infty/uH^\infty \to A_V \) restricts to an isomorphism \( A_1(\mathbb{D})/uA_1(\mathbb{D}) \to \text{oa}(V) \), see [28]). The latter quotient again is an operator \(*\)-algebra (since \( A_1(\mathbb{D}) \) and its ideal \( uA_1(\mathbb{D}) \) are invariant under the involution \( \dagger \)). So \( \text{oa}(V) \) is an involutive \( Q \)-algebra, with \( \dagger \)-selfadjoint generator. The associated \( \dagger \)-selfadjoint contractive generator is \( 1 - (1 + V)^{-1} = V(I + V)^{-1} \) (see the discussion just above Section 3.4), which corresponds to the image of \((1 - z)/2 \in A_1(\mathbb{D})\). It is known to be a radical Banach algebra [28] so the spectrum of every element is \((0)\). Thus this is an example of an operator \(*\)-algebra such that every \( \dagger \)-selfadjoint element has real spectrum, but which is not a \( C^* \)-algebra. Indeed in this algebra for every \( a \in A \) we have \( \text{Sp}(a^*a) \subset [0, \infty) \).

Slightly more generally a contraction operator unitarily equivalent to Jordan block \( S(\theta) \) for an inner function \( \theta \), generates an operator \(*\)-algebra with \( T^\dagger = T \) if \( \theta^\dagger(z) = \overline{\theta(z)} \). Such inner functions include Blaschke products with real zeroes and the function \( u \) in the previous paragraph.

3.4. Examples based on upper triangular matrices. The upper triangular \( n \) by \( n \) matrix algebra is an example which has all four types of involutions mentioned at the start of Section 1.2. The \(*\)-algebra involution is given by \( x^\dagger = u_n x^* u_n \) where \( u_n \) is the order reversing \( n \times n \) permutation matrix. Similarly \( u_n x^\tau u_n \) is a linear involution, where \( \tau \) is the transpose. Similarly, the infinite dimensional version of the upper triangulars acting on \( l^2(\mathbb{Z}) \) is an operator \(*\)-algebra and operator algebra with linear involution. Here \( u((a_n)) = (a_{-n}) \) for \( (a_n) \in l^2(\mathbb{Z}) \), and \( x^\dagger = u x^* u \), etc. These algebras have as one ‘involutive ideal’ the strictly upper triangular subalgebra.

The following is an example of an operator \(*\)-algebra which is a maximal subdiagonal algebra in the sense of Arveson [2] within the hyperfinite \( II_1 \) factor \( R \). One could call this the hyperfinite upper triangulars. In \( M_{2^n} \) consider conjugation by the order reversing permutation matrix which for convenience we write as \( u_n \) (in the notation above it is \( u_{2^n} \)). Then we have \( u_{n+1}(x \oplus x)u_{n+1} = (u_n x u_n) \oplus (u_n x^* u_n) \) for \( x \in M_{2^n} \). It follows that \( (u_n x^* u_n) \) gives rise to a well defined period 2 \(*\)-automorphism on the copy of \( M_{2^n} \) . This extends by density to a period 2 \(*\)-automorphism \( \theta \) of the CAR algebra \( B \), and this gives an involution on the subalgebra \( A \) of the union of the upper triangular matrices in \( M_{2^n} \) for all \( n \in \mathbb{N} \), since \( \theta(A) \subset A^* \). So \( A \) is an operator \(*\)-algebra. A similar construction using the transpose in place of \(*\) gives a linear involution on \( A \). Note that for the normalized trace

\[ \tau_n(\theta(x)y^*) = \tau_n(u_n x u_n y^*) = \tau_n(x \theta(y)^*), \quad x, y \in M_{2^n}, \]

so that \( \theta \) extends to a symmetry \( U \) on the Hilbert space of the GNS representation of the trace of \( B \). Since \( UAU \subset A^* \), it is easy to argue that the weak* closure
$N$ of $A$ is a dual operator $*$-algebra inside $\mathcal{R}$, the hyperfinite II$_1$ factor. Similarly one has a linear involution on $N$. We claim that $N$ is a subdiagonal algebra in the sense of Arveson. If $\Phi_n : M_n \to D_n \subset M_n$ is the canonical projection onto the matrices supported on main diagonal in $M_n$, then $\Phi_{n+1}(x \oplus x) = \Phi_n(x) \oplus \Phi_n(x)$ for $x \in M_{2n}$. Thus we obtain a trace preserving projection $\Psi$ from the CAR algebra $B$ onto the ‘main diagonal’ part $D_0$ of $B$. Indeed $\tau(\Psi(x)y) = \tau(\Psi(xy)) = \tau(xy)$ for $x \in B, y \in D_0$. On the other hand the canonical trace preserving conditional expectation $\Phi$ from $R$ onto the ‘main diagonal’ part $D$ of $R$ restricts to a trace preserving normal conditional expectation from $B$ onto $D_0$, so by the unicity of the trace preserving normal conditional expectation we get that $\Phi$ extends $\Psi$. Since $\Psi$ is multiplicative on $B$, by density $\Phi$ is a homomorphism onto $D$. Also since $A + A^*$ is clearly dense in $B$, by density we have $N + N^*$ is weak* dense in $R$, so $N$ is a maximal subdiagonal algebra in the sense of Arveson.

3.5. The algebra of an involutive operator space. Let $X$ be an operator system or selfadjoint subspace of $B(H)$, and consider, as in 2.2.10 in [11],

$$U(X) = \left\{ \begin{bmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{bmatrix} : x \in X, \lambda_1, \lambda_2 \in \mathbb{C} \right\} \subset B(H \oplus H),$$

where $\lambda_1$ and $\lambda_2$ stand for the operators $\lambda_1 I_H$ and $\lambda_2 I_H$ respectively. By definition, $U(X)$ may be regarded as a subspace of the Paulsen system. Give $U(X)$ the involution that we gave the upper triangular matrices, namely $(u_2 \otimes I_H) a^* (u_2 \otimes I_H)$, where $u_2$ is the usual permutation matrix on $\mathbb{C}^2$. Then $U(X)$ is an operator $*$-algebra.

More abstractly we define an operator $*$-space to be an operator space $X$ with a period 2 conjugate linear bijection $\ast : X \to X$ satisfying $\|[a_1^*]\| = \|[a_2]\|$. As shown in the introduction to [10] if $u : X \to B(H)$ is a linear complete isometry, then

$$\Theta(x) = \begin{bmatrix} 0 & u(x)^* \\ u(x) & 0 \end{bmatrix} \in B(H \oplus H), \quad x \in X,$$

is a $*$-linear complete isometry. So $X$ ‘is’ a selfadjoint subspace of $B(K)$ for a Hilbert space $K$.

A special case that is sometimes used is when $X$ is an ‘involutive Banach space’. Then $X$ with its Min or Max operator space structure (see 1.2.21 and 1.2.22 in [11]) will be an operator $*$-space. There is a similar construction for the other types of ‘involution’ mentioned at the start of Section 1.2.2. Namely, if $X$ is an operator space with an involution satisfying the conditions at the start of Section 1.2.2 of one of these four types, but with no multiplicativity or anti-multiplicativity condition assumed, then the operator algebra $U(X)$ may be given an operator algebra involution of the matching type.

3.6. The algebra of an involutive bimodule. There is an operator module version of the last example. Since we plan to study operator modules in an involutive setting later we will be brief here. In [9] a kind of involutive operator module is studied that is quite different to, and much more interesting than, the ones below, although the representation in the next result has a superficial similarity inspired by the ‘standard forms’ considered there.

**Theorem 3.5.** Let $A$ be an approximately unital operator $*$-algebra and let $X$ be an operator $*$-space in the sense of Subsection 3.5 above. Suppose that $X$ is a nondegenerate operator $A$-$A$-bimodule in the sense of [11] Chapter 3] such that
Then there exist a Hilbert space $H$, a completely isometric linear map $\sigma : X \to B(H)$, and a nondegenerate completely isometric homomorphism $\pi$ of $A$ on $H$, and selfadjoint unitary $u$ on $H$, such that

$$\sigma(x^\dagger) = u\sigma(x)^*u \quad \text{and} \quad \pi(a^\dagger) = u\pi(a)^*u,$$

$$\pi(a)\sigma(x) = \sigma(ax) \quad \text{and} \quad \sigma(x)\pi(a) = \sigma(xa),$$

for all $a \in A$ and $x \in X$.

Proof. By [11, Theorem 3.3.1, Lemma 3.3.5], there exist a Hilbert space $H_0$, a completely isometric linear map $\phi : X \to B(H_0)$, and a nondegenerate completely isometric homomorphism $\Theta$ of $A$ such that

$$\Theta(a)\phi(x) = \phi(ax) \quad \text{and} \quad \phi(x)\Theta(b) = \phi(xb),$$

for all $a, b \in A$ and $x \in X$. We consider the Hilbert space $H := H_0 \oplus H_0$ and the completely isometric homomorphism $\pi : A \to B(H)$ given by

$$\pi(a) = \begin{pmatrix} \Theta(a) & 0 \\ 0 & \Theta(a^\dagger)^* \end{pmatrix},$$

and the complete isometry $\sigma : X \to B(H)$ given by

$$\sigma(x) = \begin{pmatrix} \phi(x) & 0 \\ 0 & \phi(x^\dagger)^* \end{pmatrix}.$$
4. Operator \(*\)-algebras

Henceforth in our paper for specificity our involutive algebras will be operator \(*\)-algebras. As said earlier, we leave the case of the remaining material for the other kinds of involutions to the reader. We remark that the \(C^*\)-algebras which are operator \(*\)-algebras are exactly the \(\mathbb{Z}_2\)-graded \(C^*\)-algebras.

An involutive ideal or \(\dagger\)-ideal in an operator algebra with involution \(\dagger\) is an ideal \(J\) with \(J^\dagger \subset J\).

**Proposition 4.1.** Let \(A\) be an operator \(*\)-algebra. Suppose \(J\) is a closed \(\dagger\)-ideal, then \(J\) and \(A/J\) are operator \(*\)-algebras.

**Proof.** This follows from the matching fact for operator algebras [11] Proposition 2.3.4], and the computation

\[
\|a_{ij}^\dagger + J\| \leq \|a_{ij} + x_{ij}\| \leq \|a_{ij} + J\|, \quad x_{ij} \in J,
\]

so that \(\|a_{ij}^\dagger + J\| \leq \|a_{ij} + J\|\) for \(a_{ij} \in A\). Similarly, we have \(\|a_{ij} + J\| \leq \|a_{ij}^\dagger + J\|\). \(\square\)

**Remark.** There are \(*\)-algebra variants of the usual consequences of the matching fact in operator algebra theory. For example one may deduce easily from Proposition 4.1 following the method in e.g. 1.2.30, 2.3.6, 2.3.7 in [11], that one may interpolate between operator \(*\)-algebras. Indeed suppose that \((A_0, A_1)\) is a compatible couple of Banach \(*\)-algebras which happen to be operator \(*\)-algebras. Just like in the general operator space case [11] 1.2.30], let \(S\) be the strip of all complex numbers \(z\) with \(0 \leq \text{Re} z \leq 1\) and let \(F = F(A_0, A_1)\) be the space of all bounded and continuous functions \(f : S \to A_0 + A_1\) such that the restriction of \(f\) to the interior of \(S\) is analytic, and such that the maps \(t \mapsto f(it)\) and \(t \mapsto f(1 + it)\) belong to \(C_0(\mathbb{R}; A_0)\) and \(C_0(\mathbb{R}, A_1)\) respectively. For any \(f \in F\), the function \(f^{\dagger}\) is defined by \(f^{\dagger}(z) = f(\overline{z}) \in F\). Then \(F(A_0, A_1)\) with the operator space considered in 1.2.30 in [11] is an operator \(*\)-algebra with the involution \(\dagger\). For any \(0 \leq \theta \leq 1\), let \(F_\theta(A_0, A_1)\) be the two-sided closed ideal of all \(f \in F\) for which \(f(\theta) = 0\). This is \(\dagger\)-selfadjoint. The interpolation space \(A_\theta = [A_0, A_1]_\theta\) is the subspace of \(A_0 + A_1\) formed by all \(x = f(\theta)\) for some \(f \in F\). As operator spaces, the interpolation space \(A_\theta \cong F(A_0, A_1)/F_\theta(A_0, A_1)\) through the map \(\pi : f \mapsto f(\theta)\). It is easy to see that \(\pi\) is \(\dagger\)-linear. By Proposition 4.1, the quotient \(A_\theta \cong F(A_0, A_1)/F_\theta(A_0, A_1)\) is an operator \(*\)-algebra.

4.1. Contractive approximate identities.

**Lemma 4.2.** Let \(A\) be an operator \(*\)-algebra. Then the following are equivalent:

(i) \(A\) has a cai.
(ii) \(A\) has a \(\dagger\)-selfadjoint cai.
(iii) \(A\) has a left cai.
(iv) \(A\) has a right cai.
(v) \(A^{**}\) has an identity of norm 1.

**Proof.** (i) \(\Rightarrow\) (ii) If \((e_i)\) is a cai for \(A\), then \((e_i^\dagger)\) is also a cai for \(A\). Let \(f_t = (e_t + e_t^\dagger)/2\), then \((f_t)\) is a \(\dagger\)-selfadjoint cai for \(A\).

(iii) \(\Rightarrow\) (iv) If \((e_i)\) is a left cai for \(A\), then \((e_i^\dagger)\) is a right cai. Analogously, it is easy to see that (iv) \(\Rightarrow\) (iii)
(iv) ⇒ (v) By a well-known fact in operator algebra that if \( A \) has a left cai and right cai, then \( A^{**} \) has an identity of norm 1 (see e.g. [11] Proposition 2.5.8).

That (ii) ⇒ (i), and (i) ⇒ (iii), are obvious. That (v) ⇒ (i) follows from Proposition 2.5.8 in [11]. \( \square \)

**Corollary 4.3.** If \( A \) is an operator \(*\)-algebra with a countable cai \((f_n)\), then \( A \) has a countable \( \dagger \)-selfadjoint cai in \( \frac{1}{2}A \).

**Proof.** By [14] Theorem 1.1, \( A \) has a cai \((e_t)\) in \( \frac{1}{2}A \). Denote \( e_t' = \frac{e_t + e_{1-t}}{2} \), then \((e_t')\) is also a cai in \( \frac{1}{2}A \). Choosing \( t_n \) with \( \|f_n e_{t_n} - f_n\| + \|e_{t_n} f_n - f_n\| < 2^{-n} \), it is easy to see that \((e_{t_n}')\) is a countable \( \dagger \)-selfadjoint cai in \( \frac{1}{2}A \). \( \square \)

**Corollary 4.4.** If \( J \) is a closed two-sided \( \dagger \)-ideal in an operator \(*\)-algebra \( A \) and if \( J \) has a cai, then \( J \) has a \( \dagger \)-selfadjoint cai \((e_t)\) with \( \|1 - 2e_t\| \leq 1 \) for all \( t \), which is also quasicentral in \( A \).

**Proof.** By the proof of Corollary 1.3 we know that \( J \) has a \( \dagger \)-selfadjoint cai, denoted \((e_t)\), in \( \frac{1}{2}A \). The rest is as in the proof of cite[Corollary 1.5][BRI]. \( \square \)

Let \( A \) be an operator algebra (possibly not unital). Then the left (resp. right) support projection of an element \( x \) in \( A \) is the smallest projection \( p \in A^{**} \) such that \( px = x \) (resp. \( xp = x \)), if such a projection exists (it always exists if \( A \) has a cai, see e.g. [14]). If the left and right support projection exist, and are equal, then we call it the support projection, written \( s(x) \).

**Theorem 4.5.** [15] Corollary 3.4] For any operator algebra \( A \), if \( x \in \tau_A \) and \( x \neq 0 \), then the left support projection of \( x \) equals the right support projection, and equals the weak* limit of \((a^{1/n})\). It also equals \( s(y) \), where \( y = x(1 + x)^{-1} \in \frac{1}{2}A \). Also, \( s(x) \) is open in \( A^{**} \).

**Proposition 4.6.** In an operator \(*\)-algebra \( A \), \( \mathfrak{F}(A) \) and \( \tau_A \) are \( \dagger \)-closed, and if \( x \in \tau(A) \) we have \( s(x) = x(1 + x)^{-1} \) and if \( x \in \mathfrak{F}(A) \) then \( s(x) \vee s(x)^* = s(x + x^*) \). In particular if \( x \) is \( \dagger \)-selfadjoint then so is \( s(x) \).

**Proof.** Indeed applying \( \dagger \) we see that \( \|1 - x\| \leq 1 \) implies \( \|1 - x^*\| \leq 1 \). For the \( \dagger \)-invariance of \( \tau_A \) note that this is easy to see for a \( C^* \)-cover \( B \) with compatible involution (Definition 2.9), and then one may use the fact that \( \tau_A = A \cap \tau_B \). Since \( x^{1/n} \) may be written as a power series in \( 1 - x \) with real coefficients, it follows that \((x^{1/n})^{1/n} = (x^{1/n})^1 \). Then \( s(x)^1 = (w^* \lim_n x^{1/n})^1 = s(x^1) \). The \( s(x + x^1) \) assertion follows from e.g. the proof of [14] Proposition 2.14. \( \square \)

Thus the theory of real positivity studied in many of the first authors recent papers will have good involutive variants.

**Corollary 4.7.** For any operator \(*\)-algebra \( A \), if \( x \in \tau_A \) is \( \dagger \)-selfadjoint, then \( a = \mathfrak{F}(x) = x(1 + x)^{-1} \in \frac{1}{2}A \) is \( \dagger \)-selfadjoint, and \( xA = aA = s(x)A^{**} \cap A \) is an \( r\dagger \)-ideal in \( A \). Also, \( x\bar{A} = a\bar{A}a \) is the \( \dagger \)-HSA matching \( \bar{A} \).

**Proof.** It is an exercise that \( a = x(1 + x)^{-1} \) is \( \dagger \)-selfadjoint, and is in \( \frac{1}{2}A \) by the previous result. Since \((a^{1/n})\) is \( \dagger \)-selfadjoint by a fact in the last proof, \((a^{1/n})\) serves as a \( \dagger \)-selfadjoint left cai for \( aA \). Besides, \( a\bar{A}a \) is \( \dagger \)-selfadjoint and the weak* limit of \((a^{1/n})\) is \( s(a) \). The rest follows from [15] Corollary 3.5. \( \square \)
Lemma 4.8. If \( x \in \mathfrak{F}_A \), with \( x \neq 0 \), then the operator \( * \)-algebra generated by \( x \), denoted \( \text{oa}^*(x) \), has a cai. Indeed, the operator \( * \)-algebra \( \text{oa}^*(x) \) has a \( \dagger \)-selfadjoint sequential cai belonging to \( \frac{1}{2} \mathfrak{F}_A \).

Proof. If \( x \in \mathfrak{F}_A \), then \( x^\dagger \in \mathfrak{F}_A \) as we proved above. Denote \( B = C^*_s(A) \), then \( p = s(x) \vee s(x^\dagger) = s(x + x^\dagger) \) in \( B^\ast \ast \) is in \( \text{oa}^*(x)^\ast \). Clearly \( px = xp = x \) and \( px^\dagger = x^\dagger p = x^\dagger \). Therefore, \( p \) is an identity in \( \text{oa}^*(x)^\ast \). By \([11] \) Theorem 2.5.8, \( \text{oa}^*(x) \) has a cai.

Moreover, since \( \text{oa}^*(x) \) is separable, by \([14] \) Corollary 2.17, there exists \( a \in \mathfrak{F}_A \) such that \( s(a) = 1_{\text{oa}^*(x)[\ast \ast]} \). Therefore \( \text{oa}^*(x) \) has a countable \( \dagger \)-selfadjoint cai by applying to \([14] \) Theorem 4.9 and Corollary 4.3.

We write \( x \preceq y \) if \( y - x \in \mathfrak{r}_A \). The ensuing ‘order theory’ in the involutive case is largely similar to the operator algebra case from \([16] \). For example:

Theorem 4.9. Let \( A \) be an operator \( * \)-algebra which generates a \( C^\ast \)-algebra \( B \) with compatible involution \( \dagger \), and let \( \mathcal{U}_A = \{ a \in A : \|a\| < 1 \} \). The following are equivalent:

1. \( A \) is approximately unital.
2. For any \( \dagger \)-selfadjoint positive \( b \in \mathcal{U}_B \) there exists \( \dagger \)-selfadjoint \( a \in \mathcal{c}_A \) with \( b \preceq a \).
3. Same as (2), but also \( a \in \frac{1}{2} \mathfrak{F}_A \) and ‘nearly positive’ in the sense of the introduction to \([16] \): we can make it as close in norm as we like to an actual positive element.
4. For any pair of \( \dagger \)-selfadjoint elements \( x, y \in \mathcal{U}_A \) there exist nearly positive \( \dagger \)-selfadjoint \( a \in \frac{1}{2} \mathfrak{F}_A \) with \( x \preceq a \) and \( y \preceq a \).
5. For any \( \dagger \)-selfadjoint \( b \in \mathcal{U}_A \) there exist nearly positive \( \dagger \)-selfadjoint \( a \in \frac{1}{2} \mathfrak{F}_A \) with \( -a \preceq b \preceq a \).
6. \( \mathfrak{r}_A \) is a generating cone, indeed any \( \dagger \)-selfadjoint element in \( A \) is a difference of two \( \dagger \)-selfadjoint elements in \( \mathfrak{r}_A \).
7. Same as (6) but with \( \mathfrak{r}_A \) replaced by \( \mathfrak{F}_A \).

Proof. (1) \( \Rightarrow \) (2') By the proof in \([16] \) Theorem 2.1] for any \( \dagger \)-selfadjoint positive \( b \in \mathcal{U}_B \) there exists \( c \in \frac{1}{2} \mathfrak{F}_A \) and nearly positive with \( b \leq \text{Re } c \). Hence it is easy to see that \( c \leq \text{Re } (c^\dagger) \) and \( b \leq \text{Re } a \) where \( a = (c + c^\dagger)/2 \).

(2') \( \Rightarrow \) (3) By \( C^\ast \)-algebra theory there exists positive \( b \in \mathcal{U}_B \) with \( \text{Re } x \) and \( \text{Re } y \) both \( \leq b \). It is easy to see that \( b^\dagger = \sigma(b) \geq 0 \). Then \( \text{Re } x \leq b^\dagger \), so that \( \text{Re } x \leq (b + b^\dagger)/2 \). Similarly for \( y \). Then apply (2') to obtain \( a \) from \( (b + b^\dagger)/2 \).

The remaining implications follow the proof in \([16] \) Theorem 2.1] but using tricks similar to the ones we have used so far in this proof. We leave the details to the reader.

Remark. Similarly as in Proposition 2.6 in \([16] \), but using our Theorem 4.9(3) in the proof in place of the matching result referenced there, one can show that for an approximately unital operator \( * \)-algebra \( A \), the \( \dagger \)-selfadjoint elements of norm \( < 1 \) in \( \frac{1}{2} \mathfrak{F}_A \) is a directed set in the \( \preceq \) ordering, and is a cai for \( A \) which is increasing in this ordering.

The following is a version of the Aarnes-Kadison Theorem for operator \( * \)-algebras.
Theorem 4.10 (Aarnes-Kadison type Theorem). If $A$ is an operator $\ast$-algebra then the following are equivalent:

(i) There exists a $\dagger$-selfadjoint $x \in \mathfrak{g}_A$ with $A = x^* A x$.

(ii) There exists a $\dagger$-selfadjoint $x \in \mathfrak{g}_A$ with $A = x A = A x$.

(iii) There exists a $\dagger$-selfadjoint $x \in \mathfrak{g}_A$ with $s(x) = 1_{A^{\ast\ast}}$.

(iv) $A$ has a countable $\dagger$-selfadjoint cai.

(v) $A$ has a $\dagger$-selfadjoint and strictly real positive element.

Indeed these are all equivalent to the same conditions with ‘$\dagger$-selfadjoint’ removed.

Proof. In (i)–(iii) we can assume that $x \in \mathfrak{f}_A$ by replacing it with the $\dagger$-selfadjoint element $x (1 + x)^{-1} \in \frac{1}{2} \mathfrak{g}_A$ (see [16, Section 2.2]). Then the equivalence of (i)–(iv) follows as in [14, Lemma 2.10 and Theorem 2.19], for (iv) using that $x^{1/n}$ is $\dagger$-selfadjoint as we said in the proof of Corollary 4.7. Similarly (v) follows from these by [14, Lemma 2.10], and the converse follows since strictly real positive elements have support projection 1 (see [16, Section 3]). The final assertion follows since if $A$ has a countable cai, then $A$ has a $\dagger$-selfadjoint countable cai (Lemma 4.2). □

4.2. Cohen factorization for operator $\ast$-algebras and their modules. The Cohen factorization theorem is a crucial tool for Banach algebras, operator algebras and their modules. In this section we will give a variant that works for operator $\ast$-algebras and their modules. Recall that if $X$ is a Banach space and $A$ is a Banach algebra then $X$ is called a Banach $A$-module if there is a module action $A \times X \to X$ which is a contractive linear map. If $A$ has a bounded approximate identity $(e_t)$ then we say that $X$ is nondegenerate if $e_t x \to x$ for $x \in X$. A Banach $A$-bimodule is both a left and a right Banach $A$-module such that $a (xb) = (ax)b$.

The following is an operator $\ast$-algebra version of the Cohen factorization theorem:

Theorem 4.11. If $A$ is approximately unital operator $\ast$-algebra, and if $X$ is a nondegenerate Banach $A$-module (resp. $A$-bimodule), if $b \in X$ then there exists an element $b_0 \in X$ and a $\dagger$-selfadjoint $a \in \mathfrak{g}_A$ with $b = ab_0$ (resp. $b = ab_0 a$). Moreover if $\|b\| < 1$ then $b_0$ and $a$ may be chosen of norm $< 1$.

Proof. In [27] Theorem 4.1], the $a$ is constructed from convex combinations of elements in a cai, and in our case the cai may be chosen $\dagger$-selfadjoint by Lemma 4.2. The details are left as an exercise to the reader. □

4.3. Multiplier algebras.

Theorem 4.12. Let $A$ be an approximately unital operator $\ast$-algebra. Then the following algebras are completely isometrically isomorphic:

(i) $LM(A) = \{ \eta \in A^{\ast\ast} : \eta A \subset A \}$.

(ii) $LM(\pi) = \{ T \in B(H) : T \pi(A) \subset \pi(A) \}$, where $\pi$ is a nondegenerate completely isometric representation of $A$ on a Hilbert space $H$ such that there exists an order 2 $\ast$-automorphism $\sigma : B(H) \to B(H)$ satisfying $\sigma(\pi(a))^* = \pi(\sigma(a))$ for any $a \in A$.

(iii) the set of completely bounded right $A$-module maps $CB_A(A)$.

Proof. See [11 Theorem 2.6.3]. □

Definition 4.13. Let $A$ be an approximately unital operator $\ast$-algebra. Then we define
(i) \( RM(A) = \{ \xi \in A^{**} : A \xi \subset A \} \);
(ii) \( RM(\pi) = \{ S \in B(H) : \pi(A)S \subset \pi(A) \} \), for any nondegenerate completely isometric representation \( \pi \) of \( A \) on a Hilbert space \( H \) and there exists order-
2 *-automorphism \( \sigma : B(H) \rightarrow B(H) \) satisfies \( \sigma(\pi(a))^* = \pi(a^\dagger) \) for any \( a \in A \);
(iii) the set of completely bounded left \( A \)-module maps, which we denote as \( \_ACB(A) \).

**Corollary 4.14.** Let \( A \) be an approximately unital operator *-algebra. Then
\begin{itemize}
  \item[(a)] \( \eta \in LM(A) \) if and only if \( \eta^\dagger \in RM(A) \), where \( \eta, \eta^\dagger \in A^{**} \) and \( \dagger \) is the involution in \( A^{**} \);
  \item[(b)] \( T \in LM(\pi) \) if and only if \( T^\dagger \in RM(\pi) \), where \( T^\dagger = \sigma(T)^* \);
  \item[(c)] \( L \in \_BCB(A) \) if and only if \( L^\dagger \in \_ACB(A) \), where the map \( L^\dagger \) is defined by
\[ L^\dagger(a) = L(a^\dagger)^\dagger. \]
\end{itemize}

**Proof.** We just give the proof of (b). Suppose that \( T \in LM(\pi) \), then
\[ \pi(a)T^\dagger = \sigma(\pi(a^\dagger))^* \sigma(T)^* = (\sigma(T\pi(a^\dagger)))^* \in \sigma(\pi(A))^* \subset \pi(A). \]
Thus, \( T^\dagger \in RM(\pi) \). Similarly, if \( T^\dagger \in RM(\pi) \) then \( T \in LM(\pi) \).

We consider pairs \( (D, \mu) \) consisting of a unital operator *-algebra \( D \) and a completely isometric \( \dagger \)-homomorphism \( \mu : A \rightarrow D \), such that \( D\mu(A) \subset \mu(A), \mu(A)D \subset \mu(A) \). We use the phrase **multiplier operator *-algebra** of \( A \), and write \( M(A) \), for any pair \( (D, \mu) \) which is completely \( \dagger \)-isometrically \( A \)-isomorphic to \( M(A) = \{ x \in A^{**} : xA \subset A \text{ and } Ax \subset A \} \). Note that by Lemma 2.12, the inclusion of \( A \) in \( A^{**} \) is a \( \dagger \)-homomorphism, hence the canonical map \( i : A \rightarrow M(A) \), is a \( \dagger \)-homomorphism. From this it follows that there is a unique involution on \( M(A) \) for which \( i \) is a \( \dagger \)-homomorphism.

**Proposition 4.15.** Suppose that \( A \) is an approximately unital operator *-algebra. If \( (D, \mu) \) is a left multiplier operator algebra of \( A \), then the closed subalgebra
\[ \{ d \in D : \mu(A)d \subset \mu(A) \} \]
of \( D \), together with the map \( \mu \), is a multiplier operator *-algebra of \( A \).

**Proof.** Let \( E \) denote the set \( \{ d \in D : \mu(A)d \subset \mu(A) \} \). By [11, Proposition 2.6.8], we know that \( E \) is a multiplier operator algebra of \( A \). Thus, there exists a completely isometric surjective homomorphism \( \theta : M(A) \rightarrow E \) such that \( \theta \circ i_A = \mu \). Now we may define an involution on \( E \) by \( d^\dagger = \theta(\eta^\dagger) \) if \( d = \theta(\eta) \). Then it is easy to check that \( E \) is an operator *-algebra which is completely \( \dagger \)-isometrically \( A \)-isomorphic to \( M(A) \). \( \square \)

**Example 4.16.** Let \( A = A_2(\mathbb{D}) \), the functions in the disk algebra vanishing at 1, which is the norm closure of \( (z - 1)A(\mathbb{D}) \), and let \( B = \{ f \in C(T) : f(1) = 0 \} \). By the nonunital variant of the Stone-Weierstrass theorem, \( B \) is generated as a \( C^* \)-algebra by \( A \). Indeed \( B = C_\circ^*(A) \), since any closed ideal of \( B \) is the set of functions that vanish on a closed set in the circle containing 1. Also for any \( z_0 \in \mathbb{T} \), \( z_0 \neq 1 \), there is a function in \( A \) that peaks at \( z_0 \), if necessary by the noncommutative Urysohn lemma for approximately unital operator algebras [12]. So the involution on \( A \) descends from the natural involution on \( B \). It is easy to see, for example by examining the bidual of \( B^{**} \) and noticing that \( A \) and \( B \) have a common cai, that \( M(A) = \{ T \in M(B) : TA \subset A \} = \{ g \in C_b(T \setminus \{1\}) : g(z - 1) \in A(\mathbb{D}) \} \). For such \( g \),
since the negative Fourier coefficients of \( k = g(1 - z) \) are zero, the negative Fourier coefficients of \( g \) are constant, hence zero by the Riemann-Lebesgue lemma. Thus \( g \) is in \( H^\infty \), and has an analytic extension to the open disk. Viewing \( g \) as a function \( h \) on \( \bar{\mathbb{D}} \setminus \{1\} \) we have \( h = k/(z - 1) \) for some \( k \in A(\mathbb{D}) \). So \( M(A) \) consists of the bounded continuous functions on \( \bar{\mathbb{D}} \setminus \{1\} \) that are analytic in the open disk, with involution \( f(\bar{z}) \).

Let \( A, B \) be approximately unital operator \(*\)-algebras. A completely contractive \( \dagger \)-homomorphism \( \pi : A \to M(B) \) will be called a multiplier-nondegenerate \( \dagger \)-homomorphism, if \( B \) is a nondegenerate bimodule with respect to the natural module action of \( A \) on \( B \) via \( \pi \). This is equivalent to saying that for any cai \( (e_i) \) of \( A \), we have \( \pi(e_i) b \to b \) and \( b \pi(e_i) \to b \) for \( b \in B \).

**Proposition 4.17.** If \( A, B \) are approximately operator \(*\)-algebras, and if \( \pi : A \to M(B) \) is a multiplier-nondegenerate \( \dagger \)-homomorphism then \( \pi \) extends uniquely to a unital completely contractive \( \dagger \)-homomorphism \( \hat{\pi} : M(A) \to M(B) \). Moreover \( \hat{\pi} \) is completely isometric if and only if \( \pi \) is completely isometric.

**Proof.** Regard \( M(A) \) and \( M(B) \) as \( \dagger \)-subalgebras of \( A^{**} \) and \( B^{**} \) respectively. Let \( \hat{\pi} : A^{**} \to B^{**} \) be the unique \( w^* \)-continuous \( \dagger \)-homomorphism extending \( \pi \). From [11, Proposition 2.6.12], we know that \( \hat{\pi} = \pi(\pi)(M(A)) \) is the unique bounded homomorphism on \( M(A) \) extending \( \pi \), and \( \hat{\pi}(M(A)) \subset M(B) \).

Let \( (e_i) \) be a \( \dagger \)-selfadjoint cai for \( A \). Then for any \( \eta \in M(A), \eta e_i \in A \) and \( \eta e_i w^* \to \eta \). Hence

\[
\hat{\pi}(\eta) = w^* - \lim_i \pi(\eta e_i).
\]

On the other hand, \( (\eta e_i)^\dagger \to \eta^\dagger \), which implies

\[
\hat{\pi}(\eta^\dagger) = w^* - \lim_i \pi((\eta e_i)^\dagger) = w^* - \lim_i \pi((\eta e_i))^\dagger.
\]

Since the involution on \( A^{**} \) is \( w^* \)-continuous, we get that

\[
\hat{\pi}(\eta^\dagger) = \hat{\pi}(\eta)^\dagger.
\]

The rest follows from [11, Proposition 2.6.12]. \( \square \)

### 4.4. Dual operator \(*\)-algebras.

**Definition 4.18.** Let \( M \) be a dual operator algebra and operator \(*\)-algebra such that the involution on \( M \) is weak* continuous. Then \( M \) is called a dual operator \(*\)-algebra.

We will identify any two dual operator \(*\)-algebras \( M \) and \( N \) which are \( w^* \)-homeomorphically and completely \( \dagger \)-isometrically isometric.

**Proposition 4.19.** Let \( M \) be a dual (possibly nonunital) operator \(*\)-algebra.

1. The \( w^* \)-closure of a \(*\)-subalgebra of \( M \) is a dual operator \(*\)-algebra.
2. The unitization of \( M \) is also a dual operator \(*\)-algebra.

**Proof.** For (1), the weak* -closure of \(*\)-subalgebra of \( M \) is a dual operator algebra by [11, Proposition 2.7.4 (4)].

For (2), suppose that \( M \) is a nonunital operator \(*\)-algebra and write \( I \) for the identity in \( M^\dagger \). Suppose that \( (x_t) \) and \( (\lambda_t) \) are nets in \( M \) and \( \mathbb{C} \) respectively, with \( (x_t + \lambda_t I) \) converging in \( w^* \)-topology. By Hahn-Banach theorem, it is easy to see that \( (\lambda_t) \) converges in \( \mathbb{C} \). It follows that \( (x_t) \) converges in \( M \) in the \( w^* \)-topology.
Thus, $(x_I + \lambda I)^1$ converges in $M^1$, in the $w^*$-topology. The rest follows immediately from [11] Proposition 2.7.4 (5).

**Proposition 4.20.** Let $A$ be an operator $*$-algebra, and $I$ any cardinal. Then $\mathbb{K}_{I}(A)^{**} \cong M_{I}(A^{**})$ as dual operator $*$-algebras.

**Proof.** The canonical embedding $A \subset A^{**}$ induces a completely isometric $\dagger$-homomorphism $\theta : \mathbb{K}_{I}(A) \to \mathbb{K}_{I}(A^{**}) \subset M_{I}(A^{**})$. Notice that the involutions on $\mathbb{K}_{I}(A)^{**}$, $M_{I}(A^{**})$ are $w^*$-continuous and $\mathbb{K}_{I}(A)^{**} \cong M_{I}(A^{**})$ as operator algebras. Thus, $\mathbb{K}_{I}(A)^{**} \cong M_{I}(A^{**})$ as dual operator $*$-algebras. □

**Lemma 4.21.** If $X$ is a weak* closed selfadjoint subspace of $B(H)$ for a Hilbert space $H$, then $\mathcal{U}(X)$ as defined as in Example 3.19 is a dual operator $*$-algebra.

**Proof.** We leave this as an exercise to the reader. □

In connection with the last result we note that any operator $*$-space $X$ in the sense of 3.3 which is a dual operator space, and whose involution is weak* continuous, may be embedded weak* homeomorphically, via a $*$-linear complete isometry, as a weak* closed selfadjoint subspace of $B(H)$ for a Hilbert space $H$. So $\mathcal{U}(X)$ is a dual operator $*$-algebra, again by Lemma 4.21 To see this simply use the proof in 3.3 taking $u$ there to be a weak* homeomorphic complete isometry from $X$ into $B(H)$.

The last result can be used to produce counterexamples concerning dual operator $*$-algebras, such as algebras with two distinct preduals, etc. Similarly one may use the $\mathcal{U}(X)$ construction to easily obtain an example of a dual operator algebra which is an operator $*$-algebra, but the involution is not weak*-continuous. We omit the details.

Recall that in [17] the maximal $W^*$-algebra $W^*_{max}(M)$ was defined for unital dual operator algebras $M$. If $M$ is a dual operator algebra but is not unital we define $W^*_{max}(M)$ to be the von Neumann subalgebra of $W^*_{max}(M^1)$ generated by the copy of $M$. Note that it has the desired universal property: if $\pi : M \to N$ is a weak* continuous completely contractive homomorphism into a von Neumann algebra $N$, then by the normal version of Meyer’s theorem we may extend to a weak* continuous completely contractive unital homomorphism $\pi^1 : M^1 \to N$. Hence by the universal property of $W^*_{max}(M^1)$, we may extend further to a normal unital $*$-homomorphism from $W^*_{max}(M^1)$ into $N$. Restricting to $W^*_{max}(M)$ we have shown that there exists a normal $*$-homomorphism $\tilde{\pi} : W^*_{max}(M) \to N$ extending $\pi$.

**Proposition 4.22.** Let $B = W^*_{max}(M)$. Then $M$ is a dual operator $*$-algebra if and only if there exists an order two $*$-automorphism $\sigma : B \to B$ such that $\sigma(M) = M^*$. In this case the involution on $M$ is $a^\dagger = \sigma(a)^*$.

**Proof.** This follows from a simple variant of the part of the proof of Theorem 2.8 that we did prove above, where one ensures that all maps there are weak* continuous. □

**Proposition 4.23.** For any dual operator $*$-algebra $M$, there is a Hilbert space $H$ (which may be taken to be $K \oplus K$ if $M \subset B(K)$ as a dual operator algebra completely isometrically), and a symmetry (that is, a selfadjoint unitary) $u$ on $H$, and a weak* continuous completely isometric homomorphism $\pi : M \to B(H)$ such that $\pi(a)^* = u\pi(a^\dagger)u$ for $a \in M$. 


Proposition 4.24. Let \( \pi \) be a weak* continuous completely isometric homomorphism \( \rho : M \to B(K) \) and checking that \( \pi : M \to B(H) \) produced in that proof is weak* continuous. □

Proof. From [11] Proposition 2.7.11], we know that \( M/I \) is a dual operator algebra. As dual operator spaces, \( M/I \cong (I_\perp)^\ast \), from which it is easy to see that the involution on \( M/I \) is \( w^\ast \)-continuous.

Lemma 4.25. If \( A \) is an operator \( \ast \)-algebra then \( \Delta(A) = A \cap A^\ast \) (adjoint taken in any containing \( C^\ast \)-algebra; see 2.1.2 in [11]), is a \( C^\ast \)-algebra and \( \Delta(A) = \Delta(A) \).

Proof. That \( \Delta(A) \) does not depend on the particular containing \( C^\ast \)-algebra may be found in e.g. 2.1.2 in [11]. as is the fact that it is spanned by its selfadjoint (with respect to the usual involution) elements. If \( A \) is also an operator \( \ast \)-algebra then \( \Delta(A) \) is invariant under \( \ast \). Indeed suppose that \( B \) is a \( C^\ast \)-cover of \( A \) with compatible involution coming from a \( \ast \)-automorphism \( \sigma \) as usual. If \( x = x^\ast \in \Delta(A) \) then \( \sigma(x) \) is also selfadjoint, so is in \( \Delta(A) \). This holds by linearity for any \( x \in \Delta(A) \). So \( \Delta(A)^\ast = \Delta(A) \).

If \( M \) is a dual operator algebra then \( \Delta(M) = M \cap M^\ast \), is a \( W^\ast \)-algebra (see e.g. 2.1.2 in [11]). If \( M \) is a dual operator \( \ast \)-algebra then \( \Delta(M) \) is a dual operator \( \ast \)-algebra, indeed it is a \( W^\ast \)-algebra with an extra involution \( \dagger \) inherited from \( M \).

Proposition 4.26. Suppose that \( M \) is a dual operator \( \ast \)-algebra. Suppose that \( (p_i)_{i \in I} \) is a collection of projections in \( M \). Then \( \left( \bigwedge_{i \in I} p_i \right)^\dagger = \bigwedge_{i \in I} p_i^\dagger \) and \( \left( \bigvee_{i \in I} p_i \right)^\dagger = \bigvee_{i \in I} p_i^\dagger \).

Proof. By the analysis above the proposition we may assume that \( M \) is a \( W^\ast \)-algebra with an extra involution \( \dagger \), which is weak* continuous and is of the form \( x^\dagger = \sigma(x)^\ast \) for a weak* continuous \( \ast \)-automorphism \( \sigma \) of \( M \). If \( p_i \) and \( p_j \) are two projections in \( M \), then \( p_i \wedge p_j = \lim_n (p_i p_j)^\ast = \lim_n (p_j p_i)^\ast \). By the weak*-continuity of involution on \( M \) we have
\[
(p_i \wedge p_j)^\dagger = \lim_n [(p_j p_i)^\dagger]^n = \lim_n (p_i^\dagger p_i^\dagger)^n = p_i^\dagger \wedge p_j^\dagger.
\]
Thus for any finite subset \( F \) of \( I \), we have \( \left( \bigwedge_{i \in F} p_i \right)^\dagger = \bigwedge_{i \in F} p_i^\dagger \). Note that the net \( (\bigwedge_{i \in F} p_i)_{F} \) indexed by the directed set of finite subsets \( F \) of \( I \), is a decreasing net with limit \( \bigwedge_{i \in F} p_i \). We have
\[
\left( \bigwedge_{i \in F} p_i \right)^\dagger = \lim_F \left( \bigwedge_{i \in F} p_i \right)^\dagger = \lim_F \left( \bigwedge_{i \in F} p_i^\dagger \right)^\dagger = \bigwedge_{i \in F} p_i^\dagger.
\]
by weak* continuity of involution. The statement about suprema of projections follows by taking orthocomplements. □

4.5. Involutive \( M \)-ideals. An \( M \)-projection \( P \) on a Banach \( \ast \)-space is called a \( \dagger \)-\( M \)-projection if \( P \) is \( \dagger \)-preserving. A subspace \( Y \) of a Banach \( \ast \)-space is called a \( \dagger \)-\( M \)-summand if \( Y \) is the range of a \( \dagger \)-\( M \)-projection. Such range is \( \dagger \)-closed. Indeed, if \( y \in Y \), then \( y = P(x) \) for some \( x \in X \). Thus, \( y^\dagger = P(x)^\dagger = P(x^\dagger) \in Y \).

A subspace \( Y \) of \( E \) is called an involutive \( M \)-ideal or a \( \dagger \)-\( M \)-ideal in \( E \) if \( Y^\perp \) is a \( \dagger \)-\( M \)-summand in \( E^\ast \). If \( X \) is an operator \( \ast \)-space, then an \( M \)-projection is called a complete \( \dagger \)-\( M \)-projection if the amplification \( P_n \) is a \( \dagger \)-\( M \)-projection on \( M_n(X) \)
for every $n \in \mathbb{N}$. Similarly, we could define complete $\dagger$-$M$-summand, complete $\dagger$-$M$-ideal, left $\dagger$-$M$-projection, right $\dagger$-$M$-summand and right $\dagger$-$M$-ideal.

**Proposition 4.27.** Let $X$ be an operator $\ast$-space.

1. A linear idempotent $\dagger$-linear map $P : X \to X$. $P$ is a left $\dagger$-$M$-projection if and only if it is a right $\dagger$-$M$-projection, and these imply $P$ is a complete $\dagger$-$M$-projection.
2. A subspace $Y$ of $X$ is a complete $\dagger$-$M$-summand if and only if it is a left $\dagger$-$M$-summand if and only if it is a right $\dagger$-$M$-summand.
3. A subspace $Y$ of $X$ is a complete $\dagger$-$M$-ideal if and only if it is a left $M$-$\dagger$-ideal if and only if it is a right $\dagger$-$M$-ideal.

**Proof.** (1) If $P$ is a left $\dagger$-$M$-projection, then the map

$$
\sigma_P(x) = \left( \begin{array}{c} P(x) \\ x - P(x) \end{array} \right)
$$

is a completely isometry from $X$ to $C_2(X)$. Also,

$$
\|x\| = \|\sigma_P(x)\| = \left\| \begin{array}{c} P(x) \\ x - P(x) \end{array} \right\| = \left\| \begin{array}{c} P(x) \\ x - P(x) \end{array} \right\| = \left\| \begin{array}{c} P(x) \\ x - P(x) \end{array} \right\| = \|x\|.
$$

One can easily generalize this to matrices, so that $P$ is a right $\dagger$-$M$-projection. Similarly, if $P$ is is a right $\dagger$-$M$-projection then $P$ is a left $\dagger$-$M$-projection. By Proposition 4.8.4 (1) in [11], we know that $P$ is a complete $\dagger$-$M$-projection.

(2) It follows from (1) and [11] Proposition 4.8.4 (2). Now (3) is also clear. □

**Theorem 4.28.** Let $A$ be an approximately unital operator $\ast$-algebra.

1. The right $\dagger$-$M$-ideals are the $\dagger$-$M$-ideals in $A$, which are also the complete $\dagger$-$M$-ideals. These are exactly the approximately unital $\dagger$-ideals in $A$.
2. The right $\dagger$-$M$-summands are the $\dagger$-$M$-summands in $A$, which are also the complete $\dagger$-$M$-summands. These are exactly the principal ideals $Ae$ for a $\dagger$-selfadjoint central projection $e \in M(A)$.

**Proof.** (ii) By Proposition 4.27 (2), the right $\dagger$-$M$-summands are exactly the complete $\dagger$-$M$-summands. Moreover, by [11] Theorem 4.8.5 (3), the $M$-summands in $A$ are exactly the complete $M$-summands. If $D$ is a $\dagger$-$M$-summand, then $D$ is a complete $\dagger$-$M$-summand and there exists a central projection $e \in M(A)$ such that $D = eA$. Then $D^\perp = eA^\ast$ and $e$ is an identity for $D^\perp$. Also, $e'$ serves as an identity in $D^\perp$, so that $e = e'$.

(i) By a routine argument, the results follow as in [11] Theorem 4.8.5 (1) and Proposition 4.27 (3). □

5. **Involutive hereditary subalgebras, ideals, and $\dagger$-projections**

5.1. **Involutive hereditary subalgebras.** Throughout this section $A$ is an operator $\ast$-algebra (possibly not approximately unital). Then $A^\ast$ is an operator
-algebra. Recall that a projection in $A^{**}$ is open in $A^{**}$, or $A$-open for short, if $p \in (pA^{**}p \cap A)^\perp\perp$. That is, if and only if there is a net $(x_t)$ in $A$ with

$$x_t = px_t = x_t p = px_t p \to p, \text{ weak}^*.$$ 

This is a derivative of Akemann’s notion of open projections for $C^*$-algebras. If $p$ is open in $A^{**}$ then clearly

$$D = pA^{**}p \cap A = \{a \in A : a = ap = pa = pap\}$$

is a closed subalgebra of $A$, and the subalgebra $D^{\perp\perp}$ of $A^{**}$ has identity $p$. By [11] Proposition 2.5.8 $D$ has a cai. If $A$ is also approximately unital then a projection $p$ in $A^{**}$ is closed if $p^{\perp}$ is open.

We call such a subalgebra $D$ is hereditary subalgebra of $A$ (or HSA) and we say that $p$ is the support projection of the HSA $pA^{**}p \cap A$. It follows from the above that the support projection of a HSA is the weak* limit of any cai from the HSA. If $p$ is $A$-open, then $p^{\perp}$ is also $A$-open. Indeed, if $x_t = px_t = x_t p = px_t p \to p$ weak*, then $x_t^\dagger = p^\dagger x_t^\dagger x_t^\dagger p^\dagger = p^\dagger x_t^\dagger p^\dagger \to p^\dagger$ weak*, which means $p^{\perp}$ is also open.

If $p$ is $\dagger$-selfadjoint and open, then we say $p$ is $\dagger$-open in $A^{**}$. That is, if and only if there exists a $\dagger$-selfadjoint net $(x_t)$ in $A$ with

$$x_t = px_t = x_t p = px_t p \to p \text{ weak}^*.$$ 

If also $A$ is approximately unital then we say that $p^{\dagger} = 1 - p$ is $\dagger$-closed. If $p$ is $\dagger$-open in $A^{**}$ then clearly

$$D = pA^{**}p \cap A = \{a \in A : a = ap = pa = pap\}$$

is a closed $\dagger$-subalgebra of $A$. We call such a $\dagger$-subalgebra $D$ is an involutive hereditary subalgebra or a $\dagger$-hereditary subalgebra of $A$ (or, $\dagger$-HSA).

In the following statements, we often omit the proof details where are similar to usual operator algebras case (see e.g. [11, 14, 15])

**Proposition 5.1.** A subalgebra $D$ of an operator $*$-algebra $A$ is a HSA and $D^{\dagger} \subset D$ if and only if $D$ is a $\dagger$-HSA.

**Proof.** One direction is trivial.

Conversely, if $D$ is a HSA, $D = pA^{**}p \cap A$, for some open projection $p \in A^{**}$. Here, $p \in D^{\perp\perp}$ and $p$ is an identity for $D^{\perp\perp}$. If also, $D$ is $\dagger$-selfadjoint, then $p^{\dagger} \in D^{\perp\perp}$ also serves as identity. By uniqueness of identity for $D^{\perp\perp}$, then $p = p^{\dagger}$. □

**Proposition 5.2.** A subspace of an operator $*$-algebra $A$ is a $\dagger$-HSA if and only if it is an approximately unital $\dagger$-selfadjoint inner ideal.

**Proof.** If $J$ is a $\dagger$-HSA, then $J$ is an approximately unital $\dagger$-selfadjoint inner ideal.

If $J$ is an approximately unital $\dagger$-selfadjoint inner ideal, then by Proposition 5.1, $J$ is a HSA and $\dagger$-selfadjoint which means that $J$ is a $\dagger$-HSA. □

**Remark.** If $J$ is an approximately unital ideal or inner ideal of operator $*$-algebra, we cannot necessarily expect $J$ to be $\dagger$-selfadjoint. For example, let $A(\mathbb{D})$ be the disk algebra and

$$A_1(\mathbb{D}) = \{f : f \in A(\mathbb{D}), f(i) = 0\}.$$ 

Then $A_1(\mathbb{D})$ is an approximately unital ideal but obviously it is not $\dagger$-selfadjoint.

The following is another characterization of $\dagger$-HSA’s.
Corollary 5.3. Let $A$ be an operator $*$-algebra and suppose that $(e_t)$ is a $\dagger$-selfadjoint net in $\text{Ball}(A)$ such that $e_t e_s \to e_s$ with $t$. Then
\[
\{ x \in A : x e_t \to x, e_t x \to x \}
\]
is a $\dagger$-HSA of $A$. Conversely, every $\dagger$-HSA of $A$ arises in this way.

Proof. Let $J = \{ x \in A : x e_t \to x, e_t x \to x \}$. Then it is easy to see that $J$ is an inner ideal and $J^\dagger \subset J$. By Proposition 5.2, $J$ is a $\dagger$-HSA. Conversely, if $D$ is a $\dagger$-HSA and $(e_t)$ is a $\dagger$-selfadjoint cai for $D$, then
\[
D = pA^{**}p \cap A = \{ x \in A : x e_t \to x, e_t x \to x \},
\]
where $p$ is the weak* limit of $(e_t)$. □

Closed right ideals $J$ of an operator $*$-algebra $A$ possessing a $\dagger$-selfadjoint left cai will be called $r$-$\dagger$-ideals. Similarly, closed left ideals $J$ of an operator $*$-algebra $A$ possessing a $\dagger$-selfadjoint right cai will be called $l$-$\dagger$-ideals. Note that there is a bijective correspondence between $r$-$\dagger$-ideals and $l$-$\dagger$-ideals, namely $J \to J^\dagger$. For $C^*$-algebras $r$-ideals are precisely the closed right ideals, and there is an obvious bijective correspondence between $r$-ideals and $l$-ideals, namely $J \to J^*$. 

Theorem 5.4. Suppose that $A$ is an operator $*$-algebra (possibly not approximately unital), and $p$ is a $\dagger$-projection in $A^{**}$. Then the following are equivalent:

(i) $p$ is $\dagger$-open in $A^{**}$.

(ii) $p$ is the left support projection of an $r$-$\dagger$-ideal of $A$.

(iii) $p$ is the right support projection of an $l$-$\dagger$-ideal of $A$.

(iv) $p$ is the support projection of a $\dagger$-hereditary algebra of $A$.

Proof. The equivalence of (i) and (iv) is just the definition of being $\dagger$-open in $A^{**}$.

Suppose (i), if $p$ is $\dagger$-open then $p$ is the support projection for some $\dagger$-HSA $D$. Let $(e_t)$ be a $\dagger$-selfadjoint cai for $D$, then $p = w^*\lim_t e_t$. Let
\[
J = \{ x \in A : e_t x \to x \},
\]
then $J$ is a right ideal of $A$ with $\dagger$-selfadjoint left cai $(e_t)$ and $p$ is the left support projection of $J$.

Suppose (ii), if $p$ is the left support projection of an $r$-$\dagger$-ideal $J$ of $A$ with $\dagger$-selfadjoint left cai $(e_t)$, then $J = pA^{**} \cap A$. Therefore $J^\dagger = A^{**}p \cap A$, which is an $l$-$\dagger$-ideal and $p$ is the right support projection of $J^\dagger$.

Suppose (iii), if $p$ is the right support projection of an $l$-$\dagger$-ideal $A$ with $\dagger$-selfadjoint right cai $(e_t)$, then $p = \text{weak}^*\lim_t e_t = p^\dagger$, which means that $p$ is $\dagger$-open.

Similarly we can get the equivalence between (i) and (iii). □

If $J$ is an operator $*$-algebra with an $\dagger$-selfadjoint left cai $(e_t)$, then we set
\[
\mathcal{L}(J) = \{ a \in J : ae_t \to a \}.
\]

Corollary 5.5. A subalgebra of an operator $*$-algebra $A$ is $\dagger$-hereditary if and only if it equals $\mathcal{L}(J)$ for an $r$-$\dagger$-ideal $J$. Moreover the correspondence $J \mapsto \mathcal{L}(J)$ is a bijection from the set of $r$-$\dagger$-ideals of $A$ onto the set of $\dagger$-HSA’s of $A$. The inverse of this bijection is the map $D \to DA$. Similar results hold for the $l$-$\dagger$-ideals of $A$.

Proof. If $D$ is a $\dagger$-HSA, then by Corollary 5.3 we have
\[
D = \{ x \in A : x e_t \to x, e_t x \to x \},
\]
where \((e_t)\) is a \(\dagger\)-selfadjoint cai for \(D\). Set \(J = \{x \in A : e_t x \to x\}\), then \(J\) is an r-\(\dagger\)-ideal with \(D = \mathcal{L}(J)\).

Conversely, if \(J\) is an r-\(\dagger\)-ideal and \((e_t)\) is a \(\dagger\)-selfadjoint left cai for \(J\), then
\[
D = \{x \in A : x e_t \to x, e_t x \to x\}
\]
is a \(\dagger\)-HSA by Corollary 5.3 and \(D = \mathcal{L}(J)\). The remainder is as in \cite{8} Corollary 2.7.

As in the operator algebra case \cite{8} Corollary 2.8, if \(D\) is a \(\dagger\)-hereditary subalgebra of an operator \(\ast\)-algebra \(A\), and if \(J = DA\) and \(K = AD\), then \(JK = J \cap K = D\).

Also as in the operator algebra case \cite{8} Theorem 2.10, any \(\dagger\)-linear functional on a HSA \(D\) of an approximately unital operator \(\ast\)-algebra \(A\) has a unique \(\dagger\)-linear Hahn-Banach extension to \(A\). This is because if \(\varphi\) is any Hahn-Banach extension to \(A\), then \(\varphi(x')\) is another, so these must be equal by \cite{8} Theorem 2.10.

**Proposition 5.6.** Let \(D\) be an approximately unital \(\dagger\)-subalgebra of an approximately unital operator \(\ast\)-algebra \(A\). The following are equivalent:

(i) \(D\) is a \(\dagger\)-hereditary subalgebra of \(A\).

(ii) Every completely contractive unital \(\dagger\)-linear map from \(D^{**}\) into a unital operator \(\ast\)-algebra \(B\), has a unique completely contractive unital \(\dagger\)-extension from \(A^{**}\) into \(B\).

(iii) Every completely contractive \(\dagger\)-linear map \(T\) from \(D\) into a unital weak* closed operator \(\ast\)-algebra \(B\) such that \(T(e_t) \to 1_B\) weak* for some cai \((e_t)\) for \(D\) has a unique completely contractive weakly \(\dagger\)-extension \(\tilde{T}\) from \(A\) into \(B\) with \(\tilde{T}(f_s) \to 1_B\) weak* for some(or all) cai \((f_s)\) for \(A\).

**Proof.** Let \(e\) be the identity of \(D^{**}\). Obviously, \(e\) is \(\dagger\)-selfadjoint. (iii)\(\Rightarrow\)(i) If (iii) holds, then the inclusion from \(D\) to \(D^{\perp\perp}\) extends to a unital complete \(\dagger\)-contraction \(T : A \to D^{**} \subset eA^{**}e\). The map \(x \to exe\) on \(A^{**}\) is also a completely contractive unital \(\dagger\)-extension of the inclusion map \(D^{**} \to eD^{**}e\). It follows from the hypothesis that these maps coincide, and so \(eA^{**}e = D^{**}\), which implies that \(D\) is a \(\dagger\)-HSA. The rest is left as an exercise to the reader, being very similar to the proof of \cite{8} Proposition 2.11.

5.2. **Support projections and \(\dagger\)-HSA’s.**

**Lemma 5.7.** If \((J_i)\) is a family of r-\(\dagger\)-ideals in an operator \(\ast\)-algebra \(A\), with matching family of \(\dagger\)-HSA’s \((D_i)\), and if \(J = \sum_i J_i\) then the \(\dagger\)-HSA matching \(J\) is the \(\dagger\)-HSA \(D\) generated by \((D_i)\).

**Proof.** This follows from the matching operator algebra result, since every r-\(\dagger\)-ideal is an r-ideal and any \(\dagger\)-HSA is a HSA.

**Proposition 5.8.** Let \(A\) be an operator \(\ast\)-algebra (not necessarily with an identity or approximate identity). Suppose that \((x_k)\) is a sequence of \(\dagger\)-selfadjoint elements in \(\mathcal{A}_A\), and \(\alpha_k \in (0, 1]\) add to 1. Then the closure of the sum of the r-\(\dagger\)-ideals \(x_k A\), is the r-\(\dagger\)-ideal \(z A\), where \(z = \sum_{k=1}^{\infty} \alpha_k x_k \in \mathcal{A}_A\). Similarly, the \(\dagger\)-HSA generated by all the \(x_k A x_k\) equals \(z A\).

**Proof.** As an r-ideal, \(z A\) is the closure of the sum of the r-ideals \(x_k A\). If \(z \in \mathcal{A}_A\) is \(\dagger\)-selfadjoint then \(z A\) is a r-\(\dagger\)-ideal.

If \(S \subset A\), define \(S_{\dagger}\) to be the set of \(\dagger\)-selfadjoint elements in \(S\).
Lemma 5.9. Let $A$ be an operator $*$-algebra, a subalgebra of a $C^*$-algebra $B$.

(i) The support projection of a $\dagger$-HSA $D$ in $A$ equals $\vee_{a \in (D)_1} s(a)$ (which equals $\vee_{a \in (D)_1} s(a)$).

(ii) The support projection of an $r$-$\dagger$-ideal $J$ in $A$ equals $\vee_{a \in (J)_1} s(a)$ (which equals $\vee_{a \in (J)_1} s(a)$).

Proof. (i) Suppose $p$ is the support projection of $D$, then $p = \vee_{b \in \mathfrak{A}_D} s(b)$ by the operator algebra variant of [18] Lemma 3.12. Thus,

$$p \geq \vee_{a \in (D)_1} s(a) \geq \vee_{a \in (D)_1} s(a).$$

For any $b \in \mathfrak{A}_D$, we have $b^\dagger \in \mathfrak{A}_D$ and notice that $s(b) \lor s(b^\dagger) = s(b + b^\dagger)$ (see e.g. Proposition 2.14 in [14]) and $(b + b^\dagger)/2 \in (\mathfrak{A}_D)_1$. Hence,

$$p = \vee_{b \in \mathfrak{A}_D} s(b) \leq \vee_{a \in (D)_1} s(a).$$

Therefore, $p \leq \vee_{a \in (D)_1} s(a) \leq \vee_{a \in (D)_1} s(a)$.

(ii) This is similar. \qed

Lemma 5.10. For any operator $*$-algebra $A$, if $E \subseteq (r_A)_1$, then the smallest $\dagger$-hereditary subalgebra of $A$ containing $E$ is $pA^{**}p \cap A$, where $p = \vee_{x \in E} s(x)$.

Proof. By Lemma 5.9, $pA^{**}p \cap A$ is a $\dagger$-hereditary subalgebra of $A$, and it contains $E$. Conversely, if $D$ is a $\dagger$-HSA of $A$ containing $E$ then $D^{\perp \perp}$ contains $p$ by a routine argument, so $pA^{\perp \perp}(D^{\perp \perp}) \subseteq D^{\perp \perp}$ and $pA^{\perp \perp}p \cap A \subseteq D^{\perp \perp} \cap A = D$. \qed

Corollary 5.11. For any operator $*$-algebra $A$, suppose that a convex set $E \subseteq r_A$ and $E^\perp \subseteq E$. Then the smallest hereditary subalgebra of $A$ containing $E$ is $pA^{**}p \cap A$, where $p = \vee_{x \in E} s(x)$. Indeed, this is the smallest $\dagger$-HSA of $A$ containing $E$.

Proof. The smallest HSA containing $E$ is $pA^{**}p \cap A$, where $p = \vee_{a \in E} s(a)$. For any $a \in E$, $\frac{a + a^\dagger}{2} \in E$ by convexity of $E$. Notice that $s(\frac{a + a^\dagger}{2}) \leq p$ and $s(\frac{a + a^\dagger}{2}) \geq s(a)$, then $p = \vee_{x \in E} s(x)$ and $pA^{**}p \cap A$ is a $\dagger$-HSA. \qed

Theorem 5.12. If $A$ is an operator $*$-algebra then $\dagger$-HSA’s (resp. $r$-$\dagger$-ideals) in $A$ are precisely the sets of form $E\Phi A E$ (resp. $E\Phi A$) for some $E \subseteq (r_A)_1$. The latter set is the smallest $\dagger$-HSA (resp. $r$-$\dagger$-ideal) of $A$ containing $E$.

Proof. If $D$ is a $\dagger$-HSA (resp. $r$-$\dagger$-ideal) and taking $E$ to be a $\dagger$-selfadjoint left cai for the $\dagger$-HSA $D$ (resp. a $\dagger$-selfadjoint left cai for the $r$-$\dagger$-ideal), then the results follows immediately.

Conversely for any $x \in (r_A)_1$, we have $x(1 + x)^{-1} \in (\frac{1}{2} \mathfrak{A}_A)_1$ as we said in Corollary 4.7. Then as in [18] Theorem 3.18 we may assume that $E \subseteq (\frac{1}{2} \mathfrak{A}_A)_1$. Note that $D = E\Phi A$ is the smallest HSA containing $E$ by [18] Theorem 3.18 and $D$ is $\dagger$-selfadjoint, so that $D$ is the smallest $\dagger$-HSA containing $E$. Similarly, $E\Phi A$ is the smallest right ideal with a $\dagger$-selfadjoint left contractive identity of $A$ containing $E$. Moreover, for any finite subset $F \subseteq E$ if $a_F$ is the average of the elements in $F$, then $(a_F^{1/n})$ will serve as a $\dagger$-selfadjoint left cai for $E\Phi A$. \qed

In particular, the largest $\dagger$-HSA in an operator $*$-algebra $A$ is the largest HSA in $A$, and the largest approximately unital subalgebra in $A$ (see [15] Section 4), namely $A_H = r_A A r_A = (r_A)_1 (A (r_A)_1)_1$. The latter equality follows because $A_H$ has a cai in $r_A$, hence has a cai in $(r_A)_1$. 

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Theorem 5.13. Let $A$ be an operator $*$-algebra (not necessarily with an identity or approximate identity.) The $\hat{\tau}$-HSA’s (resp. r-$\hat{\tau}$-ideals) in $A$ are precisely the closures of unions of an increasing net of $\hat{\tau}$-HSA’s (resp. r-$\hat{\tau}$-ideals) of the form $x\overline{Ax}$ (resp. $\overline{xA}$) for $x \in (\tau_A)^\dagger$.

Proof. Suppose that $D$ is a $\hat{\tau}$-HSA (resp. an r-$\hat{\tau}$-ideal). The set of $\hat{\tau}$-HSA’s (resp. r-$\hat{\tau}$-ideals) $\overline{a_FAx_F}$ (resp. $\overline{a_FA}$) as in the last proof, indexed by finite subsets $F$ of $(\mathfrak{F}_{D})^\dagger$, is an increasing net. Lemma 3.9 can be used to show, as in [18], that the closure of the union of these $\hat{\tau}$-HSA’s (resp. r-$\hat{\tau}$-ideals) is $D$. \hfill $\Box$

As in the theory we are following, it follows that $\hat{\tau}$-open projections are just the sup’s of a collection (an increasing net if desired) of $\hat{\tau}$-selfadjoint support projections $s(x)$ for $\hat{\tau}$-selfadjoint $x \in \tau_A$.

Theorem 5.14. Let $A$ be any operator $*$-algebra (not necessarily with an identity or approximate identity). Every separable $\hat{\tau}$-HSA or $\hat{\tau}$-HSA with a countable cai (resp. separable r-$\hat{\tau}$-ideal or r-$\hat{\tau}$-ideal with a countable cai) is equal to $x\overline{Ax}$ (resp. $\overline{xA}$) for some $x \in (\mathfrak{F}_{A})^\dagger$.

Proof. If $D$ is a $\hat{\tau}$-HSA with a countable cai, then $D$ has a countable $\hat{\tau}$-selfadjoint cai $(e_n)$ in $\frac{1}{2}\mathfrak{F}_{D}$. Also, $D$ is generated by the $\hat{\tau}$-HSA’s $e_n A e_n$ so $D = x\overline{Ax}$, where $x = \sum_{n=1}^{\infty} \frac{e_n}{n^2}$. For the separable case, note that any separable approximately unital operator $*$-algebra has a countable cai. For r-$\hat{\tau}$-ideals, the result follows from the same argument. \hfill $\Box$

Corollary 5.15. If $A$ is a separable operator $*$-algebra, then the $\hat{\tau}$-open projections in $A^{**}$ are precisely the $s(x)$ for $x \in (\tau_A)^\dagger$.

Proof. If $A$ is separable, then so is any $\hat{\tau}$-HSA. So the result follows from Theorem 5.13 \hfill $\Box$

Corollary 5.16. If $A$ is a separable operator $*$-algebra with cai, then there exists an $x \in (\mathfrak{F}_{A})^\dagger$ with $A = x\overline{A} = \overline{Ax} = x\overline{Ax}$.

5.3. Involutive compact projections. Throughout this section, $A$ is an operator $*$-algebra. We will say that a projection $q \in A^{**}$ is compact relative to $A$ if it is closed and $q = qx$ for some $x \in \text{Ball}(A)$. Furthermore, if $q$ is $\hat{\tau}$-selfadjoint, we say that such $q$ is an involutive compact projection, or is $\hat{\tau}$-compact in $A^{**}$.

Proposition 5.17. A $\hat{\tau}$-projection $q$ is compact if only if there exists a $\hat{\tau}$-selfadjoint element $a \in \text{Ball}(A)$ such that $q = qa$.

Proof. One direction is trivial. Conversely if $q$ is compact, then there exists $a \in \text{Ball}(A)$ such that $q = qa$. It is easy to argue from elementary operator theory that we have $aq = q$. Thus, $q = q\left(\frac{a + a^*}{2}\right)$. \hfill $\Box$

Theorem 5.18. Let $A$ be an approximately unital operator $*$-algebra. If $q$ is a projection in $A^{**}$ then the following are equivalent:

(i) $q$ is a $\hat{\tau}$-closed projection in $(A^1)^{**}$,

(ii) $q$ is $\hat{\tau}$-compact in $A^{**}$

(iii) $q$ is closed such that there exists a $\hat{\tau}$-selfadjoint element $x \in \frac{1}{2}\mathfrak{F}_{A}$ such that $q = qx$. 

Proof. This follows from a variant of the proof of [12, Theorem 2.2]: one just needs to go carefully through the proof noting that all elements may be chosen to be $\dagger$-selfadjoint. □

Corollary 5.19. Let $A$ be an approximately unital operator $*$-algebra. Then the infimum of any family of $\dagger$-compact projections in $A^{**}$ is a $\dagger$-compact projection in $A^{**}$. Also, the supremum of two commuting $\dagger$-compact projections in $A^{**}$ is a $\dagger$-compact projection in $A^{**}$.

Proof. Note that the infimum and supremum of $\dagger$-projections are still $\dagger$-projections. Then the results follow immediately from [12, Corollary 2.3]. □

Corollary 5.20. Let $A$ be an approximately unital operator $*$-algebra, with an approximately unital closed $\dagger$-subalgebra $D$. A projection $q \in D^{1\dagger}$ is $\dagger$-compact in $D^{**}$ if and only if $q$ is $\dagger$-compact in $A^{**}$.

Corollary 5.21. Let $A$ be an approximately unital operator $*$-algebra. If a $\dagger$-projection $q$ in $A^{**}$ is dominated by an open projection $p$ in $A^{**}$, then $q$ is $\dagger$-compact in $pA^{**}$. In much of what follows we use the peak projections $u(a)$ defined and studied in e.g. [12, 15]. These may be defined to be projections in $A^{**}$ which are the weak* limits of $a^n$ for some $a \in \text{Ball}(A)$, in the case such weak* limit exists. We will not take the time to review the properties of $u(a)$ here. We will however several times below use silently the following fact:

Lemma 5.22. If $a \in \text{Ball}(A)$ for an operator $*$-algebra $A$, and if $u(a)$ is a peak projection, with $a^n \to u(a)$ weak*, then $u((a + a^\dagger)/2) = u(a) \wedge u(a)^\dagger$ in $A^{**}$ and this is a peak projection. Indeed $((a + a^\dagger)/2)^n \to u((a + a^\dagger)/2)$ weak*.

Proof. Clearly $(a^\dagger)^n \to u(a)^\dagger$ weak*, so that $u(a^\dagger) = u(a)^\dagger$ is a peak projection. Then $u((a + a^\dagger)/2) = u(a) \wedge u(a)^\dagger$ by [12, Proposition 1.1], and since this is a projection it is by [12, Section 3] a peak projection, is $\dagger$-selfadjoint, and $(a + a^\dagger)/2)^n \to u((a + a^\dagger)/2)$ weak*. □

The following is the involutive variant of the version of the Urysohn lemma for approximately unital operator $*$-algebras in [12, Theorem 2.6].

Theorem 5.23. Let $A$ be an approximately unital operator $*$-algebra. If a $\dagger$-compact projection $q$ in $A^{**}$ is dominated by a $\dagger$-open projection $p$ in $A^{**}$, then there exists $b \in (\frac{1}{2}\mathbb{S}_A)^\dagger$ with $q = qb, b = pb$. Moreover, $q \leq u(b) \leq s(b) \leq p$, and $b$ may also be chosen to be ‘nearly positive’ in the sense of the introduction to [16]: we can make it as close in norm as we like to an actual positive element.

Proof. If $q \leq p$ as stated, then by the last corollary we know $q$ is $\dagger$-compact in $D^{**} = pA^{**}p$, where $D$ is a $\dagger$-HSA supported by $p$. By Theorem 5.18 there exists a $\dagger$-selfadjoint $b \in \frac{1}{2}\mathbb{S}_D$ such that $q = qb$ and $b = pb$. The rest follows as in [12, Theorem 2.6]. □

Theorem 5.24. Suppose that $A$ is an operator $*$-algebra (not necessarily approximately unital), and that $q \in A^{**}$ is a projection. The following are equivalent:

1. $q$ is $\dagger$-compact with respect to $A$.
2. $q$ is $\dagger$-closed with respect to $A^1$ and there exists $a \in \text{Ball}(A)^\dagger$ with $aq = qa = q$.
(3) \( q \) is a decreasing weak* limit of \( u(a) \) for \( \dagger \)-selfadjoint element \( a \in \Ball(A) \).

**Proof.** (2) \( \Rightarrow \) (3) Given (2) we certainly have \( q \) compact with respect to \( A \) by [15 Theorem 6.2]. By [12 Theorem 3.4], \( q = \lim_t u(z_t) \), where \( z_t \in \Ball(A) \) and \( u(z_t) \) is decreasing. We have \( q = q^\dagger = \lim_t u(z_t^\dagger) \). Moreover, \( u(z_t) \wedge u(z_t^\dagger) = u(\frac{z_t + z_t^\dagger}{2}) \).

Hence, \( q \) is a decreasing weak* limit of \( u(\frac{z_t + z_t^\dagger}{2}) \) since the involution preserves order.

The rest follows from [15, Theorem 6.2]. \( \square \)

**Corollary 5.25.** Let \( A \) be a (not necessarily approximately unital) operator \(*\)-algebra. If \( q \) is \( \dagger \)-compact then \( q \) is a weak* limit of a net of \( \dagger \)-selfadjoint elements \((a_i)\) in \( \Ball(A) \) with \( a_i q = q \) for all \( t \).

**5.4. Involutive peak projections.** Let \( A \) be an operator \(*\)-algebra. A \( \dagger \)-projection \( q \in A^{**} \) is called an *involutive peak projection* or a *\( \dagger \)-peak projection* if it is a peak projection.

**Proposition 5.26.** Suppose \( A \) is a separable operator \(*\)-algebra (not necessarily approximately unital), then the \( \dagger \)-compact projections in \( A^{**} \) are precisely the peak projections \( u(a) \), for some \( \dagger \)-selfadjoint \( a \in \Ball(A) \).

**Proof.** If \( A \) is separable then a projection in \( A^{**} \) is compact if and only if \( q = u(a) \), for some \( a \in \Ball(A) \) (see [15 Proposition 6.4]). If \( q \) is \( \dagger \)-selfadjoint, then

\[
q = u(a^\dagger) = u(a) \wedge u(a^\dagger) = u((a^\dagger + a)/2),
\]

using e.g. Lemma 5.22. \( \square \)

**Proposition 5.27.** If \( a \in \frac{1}{2}\mathbb{F}_A \) with \( a^\dagger = a \), then \( u(a) \) is a \( \dagger \)-peak projection and it is a peak for \( a \).

**Proof.** Since \( u(a) = \lim a^n \) weak* in this case, we see that \( u(a) \) is \( \dagger \)-selfadjoint. From [12, Lemma 3.1, Corollary 3.3], we know that \( u(a) \) is a peak projection and is a peak for \( a \). \( \square \)

**Theorem 5.28.** If \( A \) is an approximately unital operator \(*\)-algebra, then

(i) A projection \( q \in A^{**} \) is \( \dagger \)-compact if only if it is a decreasing limit of \( \dagger \)-peak projections.

(ii) If \( A \) is a separable approximately unital operator \(*\)-algebra, then the \( \dagger \)-compact projections in \( A^{**} \) are precisely the \( \dagger \)-peak projections.

(iii) A projection in \( A^{**} \) is a \( \dagger \)-peak projection in \( A^{**} \) if and only if it is of form \( u(a) \) for some \( a \in \left(\frac{1}{2}\mathbb{F}_A\right)^\dagger \).

**Proof.** (ii) Follows from Proposition 5.26 and Proposition 5.27.

(i) One direction is obvious. Conversely, let \( q \in A^{**} \) be a \( \dagger \)-compact projection with \( q = q x \) for some \( \dagger \)-selfadjoint element \( x \in \Ball(A) \). Then \( q \leq u(x) \). Now \( 1 - q \) is an increasing limit of \( s(x_t) \) for \( \dagger \)-selfadjoint elements \( x_t \in \frac{1}{2}\mathbb{F}_A \), by Theorem 5.13 and the remark after it. Therefore, \( q \) is a decreasing weak* limit of the \( q_t = s(x_t)^\dagger = u(1 - x_t) \). Let \( z_t = \frac{1 - x_t + x}{2} \), then \( u(z_t) \) is a projection. Since \( q \leq q_t \) and \( q \leq u(x) \), then \( q \leq u(z_t) \). Note that \( z_t \) is \( \dagger \)-selfadjoint, so \( u(z_t) = u(z_t^\dagger) \). Let \( a_t = z_t x \in \Ball(A) \), then \( u(a_t) = u(z_t) \) by the argument in [12, Lemma 3.1]. Thus, \( u(a_t) = u(z_t) \wedge q \) as in that proof. Moreover, \( u(a_t^\dagger) = u(a_t)^\dagger \wedge q \), which implies by an argument above that \( u(\frac{a_t + a_t^\dagger}{2}) \wedge q \).
(iii) One direction is trivial. For the other, if \( q \) is a \( \dagger \)-peak projection, then by the operator algebra case there exists \( a \in \mathcal{F}A \) such that \( q = u(a) \). Let \( b = (a + a^\dagger)/2 \), then \( q = u(b) \) by e.g. Lemma 5.22.

\[ \square \]

**Corollary 5.29.** Let \( A \) be an operator \(*\)-algebra. The supremum of two commuting \( \dagger \)-peak projections in \( A^{**} \) is a \( \dagger \)-peak projection in \( A^{**} \).

**Lemma 5.30.** For any operator \(*\)-algebra \( A \), the \( \dagger \)-peak projections for \( A \) are exactly the weak* limits of \( a^n \) for \( \dagger \)-selfadjoint element \( a \in \text{Ball}(A) \) if such limit exists.

**Proof.** If \( q \) is a \( \dagger \)-peak projection, then there exists \( a \in \text{Ball}(A) \) such that \( q = u(a) \) which is also the weak* limit of \( a^n \). Since \( q \) is \( \dagger \)-selfadjoint, by Lemma 5.22 we have \( q = u(a^\dagger) = u(\frac{a + a^\dagger}{2}) \), which is the weak* limit of \( ((a + a^\dagger)/2)^n \). The converse follows from \([15\text{, Lemma }1.3]\). \( \square \)

**Remark.** Similarly the theory of peak projections for operator \(*\)-algebras \( A \) which are not necessarily approximately unital follows the development in \([15\text{, Section }6]\), with appropriate tweaks in the proofs. Thus a projection is called a \( \dagger \)-\( \mathcal{F} \)-peak projection for \( A \) if it is \( \dagger \)-selfadjoint and \( \mathcal{F} \)-peak. A projection in \( A^{**} \) is \( \dagger \)-\( \mathcal{F} \)-compact if it is a decreasing limit of \( \dagger \)-\( \mathcal{F} \)-peak projections. We recall that \( A_H \) was discussed above Theorem 5.13

One may then prove:

(i) A projection \( q \) in \( A^{**} \) is \( \dagger \)-\( \mathcal{F} \)-compact if only if \( q \) is a \( \dagger \)-compact projection for \( A_H \).

(ii) A projection in \( A^{**} \) is a \( \dagger \)-\( \mathcal{F} \)-peak projection if and only if it is a \( \dagger \)-peak projection for \( A_H \).

(iii) If \( A \) is separable then every \( \dagger \)-\( \mathcal{F} \)-compact projection in \( A^{**} \) is a \( \dagger \)-\( \mathcal{F} \)-peak projection.

5.5. **Some interpolation results.** Item (ii) in the following should be compared with Theorem 5.23 which gets a slightly better result in the case that \( A \) is approximately unital.

**Theorem 5.31** (Noncommutative Urysohn lemma for operator \(*\)-algebras). Let \( A \) be a (not necessarily approximately unital) operator \( \dagger \)-subalgebra of \( C^* \)-algebra \( B \) with a second involution \( \dagger \). Let \( q \) be a \( \dagger \)-compact projection in \( A^{**} \).

(i) For any \( \dagger \)-open projection \( p \in B^{**} \) with \( p \geq q \) and any \( \varepsilon > 0 \), there exists an \( a \in \text{Ball}(A) \) with \( aq = q \) and \( \|a(1 - p)\| < \varepsilon \).

(ii) For any \( \dagger \)-open projection \( p \in A^{**} \) with \( p \geq q \), there exists a \( \dagger \)-selfadjoint element \( a \in \text{Ball}(A) \) with \( qa = q \) and \( a = pa \).

**Proof.** (i) By \([15\text{, Theorem }6.6]\) there exists \( b \in \text{Ball}(A) \) such that

\[ bq = q, \quad \|b(1 - p)\| < \varepsilon \quad \text{and} \quad \|(1 - p)b\| < \varepsilon. \]

Then \( a = \frac{b + b^\dagger}{2} \in \text{Ball}(A) \) does the trick, since

\[ \|\left(\frac{b + b^\dagger}{2}\right)(1 - p)\| \leq \frac{1}{2}\|b(1 - p)\| + \frac{1}{2}\|(1 - p)b^\dagger\| < \varepsilon. \]

(ii) Apply Theorem 5.23 in \( A^\dagger \) to obtain a \( \dagger \)-selfadjoint element \( a \in \text{Ball}(A^\dagger) \), \( p \in A^{\perp\perp} \) and \( ap = q \). Then \( a \in A^{\perp\perp} \cap A^\dagger = A \). \( \square \)

The following is an involutive variant of the noncommutative peak interpolation type result in \([15\text{, Theorem }5.1]\).
Theorem 5.32. Suppose that $A$ is an operator $*$-algebra and that $q$ is a $\dagger$-closed projection in $(A^\dagger)^{**}$. If $b = b^\dagger \in A$ with $bq = bq$, then $b$ achieves its distance to the right ideal $J = \{ a \in A : qa = 0 \}$ (this is a $r\dagger$-ideal if $1 - q \in A^{**}$), and also achieves its distance to $\{ x \in A : qx = qx = 0 \}$ (this is a $\dagger$-HSA if $1 - q \in A^{**}$). If further $\|bq\| \leq 1$, then there exists a $\dagger$-selfadjoint element $g \in \text{Ball}(A)$ with $gq = qg = bq$.

Proof. Proceed as in the proof of [15, Theorem 5.1]. The algebra $\hat{D}$ is a $\dagger$-HSA in $A^\dagger$. Thus if $C$ is as in that proof, $C$ is $\dagger$-selfadjoint and $\hat{D}$ is a $\dagger$-ideal in $C$. Also $I = C \cap A$ and $D = I \cap \hat{D}$ are $\dagger$-selfadjoint in $C$. Note that if $x \in A$ with $qx = qx = 0$ then $x \in \hat{D} \cap A \subset C \cap A = I$, so $x \in \hat{D} \cap A \subset \hat{D} \cap I = D$. So $D = \{ x \in A : qx = qx = 0 \}$. By the proof we are following, there exists $y \in D \subset J$ such that

$$\|b - y\| = \|b - y^\dagger\| = d(b, D) = \|bq\| = d(b, J) \geq \|b - z\|,$$

where $z = (y + y^\dagger)/2$. Set $g = b - z$, then $g$ is $\dagger$-selfadjoint with $gq = qg = bq$ (since $D$ is $\dagger$-selfadjoint), and $\|g\| = \|bq\|$.\hfill $\square$

Theorem 4.10 in [16] is the (noninvolutive) operator algebra version of the last result (and [15, Theorem 5.1]), but with the additional feature that the ‘interpolating element’ $g$ in the last result is also in $\frac{1}{2}\mathfrak{S}_A$. Whence after replacing $g$ by $g^{\dagger}$, it is ‘nearly positive’ in the sense of the introduction to [16]: we can make it as close in norm as we like to an actual positive element. As we have pointed out elsewhere, there seems to be a mistake in Theorem 4.10 in [16]. It is claimed there (and used at the end of the proof) that $D$ is approximately unital. However this error disappears in what is perhaps the most important case, namely that $q$ is the ‘perp’ of a (open) projection in $A^{**}$. Then $D$ is certainly a HSA in $A$, and is approximately unital. Hence we have, also in the involutive case:

Corollary 5.33. Suppose that $A$ is an operator $*$-algebra $p$ is a $\dagger$-open projection in $A^{**}$, and $b = b^\dagger \in A$ with $bp = pb$ and $\|b(1 - p)\| \leq 1$ (where $1$ is the identity of the unitization of $A$ if $A$ is nonunital). Suppose also that $\|b(1 - b)(1 - p)\| \leq 1$. Then there exists a $\dagger$-selfadjoint element $g \in \frac{1}{2}\mathfrak{S}_A \subset \text{Ball}(A)$ with $g(1 - p) = (1 - p)g = b(1 - p)$. Indeed such $g$ may be chosen ‘nearly positive’ in the sense of the introduction to [16], indeed it may be chosen to be as close as we like to an actual positive element.

The following is the ‘nearly positive’ case of a simple noncommutative peak interpolation result which has implications for the unitization of an operator $*$-algebra.

Proposition 5.34. Suppose that $A$ is an approximately unital operator $*$-algebra, and $B$ is a $C^*$-algebra generated by $A$ with compatible involution $\dagger$. If $c = c^\dagger \in B_+$ with $\|c\| < 1$ then there exists a $\dagger$-selfadjoint $a \in \frac{1}{2}\mathfrak{S}_A$ with $|1 - a|^2 \leq 1 - c$. Indeed such $a$ can be chosen to also be nearly positive.

Proof. As in [16, Proposition 4.9], but using our Theorem [4.9] (2), there exists nearly positive $\dagger$-selfadjoint $a \in \frac{1}{2}\mathfrak{S}_A$ with

$$c \leq \text{Re}(a) \leq 2\text{Re}(a) - a^*a,$$

and $|1 - a|^2 \leq 1 - c$.\hfill $\square$
We end by verifying the involutive case of the best noncommutative peak interpolation result (from [7]), a noncommutative generalization of a famous interpolation result of Bishop. See [7] for more context and an explanation of the classical variant, and the significance of the noncommutative variant.

**Theorem 5.35.** Suppose that $A$ is an operator $*$-algebra, a subalgebra of a unital $C^*$-algebra $B$ with compatible involution $\dagger$. Suppose that $q$ is a $\dagger$-closed projection in $B^{**}$ which lies in $(A^\dagger)^{\perp\perp}$. If $h$ is a $\dagger$-selfadjoint element in $A$ with $bq = qh$, and if $qh^*bq \leq qd$ for an invertible positive $d \in B$ with $d = d^q$ which commutes with $q$, then there exists a $\dagger$-selfadjoint $g \in \text{Ball}(A)$ with $gq = qg = bq$, and $g^*g \leq d$.

**Proof.** By the proof of [7, Theorem 3.4], there exists $h \in A$ with $qh = hq = bq$, and $h^*h \leq d$. (We remark that $f = d^{-\frac{1}{2}}$ in that proof.) Thus also $(h^*h)^\dagger = \sigma(h^*h) \leq \sigma(d) = d$. Let $g = \frac{h + h^\dagger}{2}$. Then $g$ is $\dagger$-selfadjoint and $qg = gq = bq$. Also

$$g^*g \leq \left(\frac{h + h^\dagger}{2}\right)^\dagger\left(\frac{h + h^\dagger}{2}\right) + \left(\frac{h - h^\dagger}{2}\right)^\dagger\left(\frac{h - h^\dagger}{2}\right).$$

Thus

$$g^*g \leq \frac{h^*h}{2} + \frac{(h^\dagger)^*h^\dagger}{2} = \frac{h^*h}{2} + \frac{(h^\dagger)^*h^\dagger}{2} \leq d,$$

as desired. $\square$

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