On the spectral radius of bi-block graphs with given independence number $\alpha$

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Abstract

A connected graph is called a bi-block graph if each of its blocks is a complete bipartite graph. Let $B(k, \alpha)$ be the class of bi-block graph on $k$ vertices with given independence number $\alpha$. It is easy to see that every bi-block graph is a bipartite graph and for a bipartite graph $G$ on $k$ vertices, the independence number $\alpha(G)$, satisfies $\left\lceil \frac{k}{2} \right\rceil \leq \alpha(G) \leq k - 1$. In this article, we prove that the maximum spectral radius $\rho(G)$ among all graphs $G$ in $B(k, \alpha)$, is uniquely attained for the complete bipartite graph $K_{\alpha,k-\alpha}$.

Keywords: complete bipartite graphs, bi-block graphs, independence number, spectral radius.

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1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, connected graph with $V(G)$ as the set of vertices and $E(G)$ as the set of edges in $G$. We simply write $G = (V, E)$ if there is no scope of confusion. We write $u \sim v$ to indicate that the vertices $u, v \in V$ are adjacent in $G$. The degree of the vertex $v$, denoted by $d_G(v)$ (or simply $d(v)$), equals the number of vertices in $V$ that are adjacent to $v$. A graph $H$ is said to be a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any subset $S \subseteq V(G)$, a subgraph $H$ of $G$ is said to be an induced subgraph with vertex set $S$, if $H$ is a maximal subgraph of $G$ with vertex set $V(H) = S$. We write $|S|$ to denote the cardinality of the set $S$. A graph $G = (V, E)$ is said to be bipartite if the vertex set can be partitioned into two subsets $M$ and $N$ such that $E \subseteq M \times N$. A complete bipartite graph is a bipartite graph with partition $V = M \cup N$, in which every vertex of $M$ is adjacent to every vertex of $N$. A complete bipartite graph with $|M| = m$ and $|N| = n$ is denoted by $K_{m,n}$. To emphasize the vertex partition $M$ and $N$, we use the notation $K(M, N)$ to represent $K_{m,n}$ whenever $|M| = n$ and $|N| = n$.

A vertex $v$ of a connected graph $G = (V, E)$ is a cut-vertex of $G$ if $G - v$ is disconnected. A block of the graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. A block is said to be a leaf block if its deletion does not disconnects the graph. Given two blocks $F$ and $H$ of graph $G$ are said to be neighbours, if they are connected via a cut-vertex. We denote $F \odot H$, to represent the induced subgraph on the vertex set of two neighbouring blocks $F$ and $H$. A connected graph is called a bi-block graph if each of its blocks is a complete bipartite graph (see Figure 1). Given a vertex $v \in V$, the block index of $v$ is denoted by $bi_G(v)$, equals the number of blocks in $G$ contain the vertex $v$. It is easy to see that if $v$ is not a cut vertex, then $bi_G(v) = 1$. Also note that, the star $K_{1,n}$, is bi-block graph with a central cut vertex $v$(say), where $bi_G(v) = d_G(v) = n$, where each of its blocks are edges. In this article we consider the star $K_{1,n}$ as a complete bipartite graph instead of a bi-block graph.

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A set \( I \) of vertices in a graph \( G \) is an independent set if no pair of vertices of \( I \) are adjacent. The independence number of \( G \), denote by \( \alpha(G) \), is the maximum cardinality of an independent set in \( G \). An independent set of cardinality \( \alpha(G) \) is called an \( \alpha(G) \)-set.

Let \( G = (V,E) \) be a graph. For \( x, y \in V \), the adjacency matrix of the graph \( G \) is, \( A(G) = [a_{xy}] \), where \( a_{xy} = 1 \) if \( x \sim y \) and 0 otherwise. For any column vector \( X \), if \( x_u \) represent the entry of \( X \) corresponding to vertex \( u \in V \), then

\[
X^t A(G) X = 2 \sum_{u \sim w} x_u x_w,
\]

where \( X^t \) represent the transpose of \( X \). For a connected graph \( G \) on \( k \geq 2 \) vertices, by Perron-Frobenius theorem, the spectral radius \( \rho(G) \) of \( A(G) \) is a simple positive eigenvalue and the associated eigenvector is entry-wise positive (for details see [1]). We will refer to such an eigenvector as the Perron vector of \( G \). We simply write \( A \) and \( \rho \) to represent the adjacency matrix and spectral radius, if there is no scope of confusion. Now we state a few known results on spectral radius useful in our subsequent calculations. By Min-max theorem, we have

\[
\rho(G) = \max_{X \neq 0} \frac{X^t A(G) X}{X^t X} = \max_{X \neq 0} \frac{2 \sum_{u \sim w} x_u x_w}{\sum_{u \in V} x_u^2}.
\]

The lemma below gives a relation between spectral radius and degree of vertices.

**Lemma 1.1.** [1] For a graph \( G \), if \( \Delta(G) \) and \( \delta(G) \) denote the maximum and the minimum of the vertex degrees of \( G \), then \( \delta(G) \leq \rho(G) \leq \Delta(G) \).

Given a graph \( G = (V,E) \), for \( x, y \in V(G) \) we will use \( G + xy \) to denote the graphs obtained by from \( G \) by adding an edge \( xy \notin E(G) \) and we have the following result.

**Lemma 1.2.** [2] If \( G \) is a graph such that for \( x, y \in V(G), xy \notin E(G) \), then \( \rho(G) < \rho(G + xy) \).

In literature, problems related to maximal and minimal spectral radius of graphs for a given class is an active area in spectral graph theory and has been extensively studied (for example see [2]-[9]). In particular, few interesting article related to maximal and minimal spectral radius with given independence number has been studied for different class of graphs (for details see [2, 3, 5, 9]). We are interested in maximizing spectral radius for bi-block graphs with given independence number. Let \( \mathcal{B}(k,\alpha) \) be the class of bi-block graph on \( k \) vertices with a given independence number \( \alpha \). In this article, we prove that the maximum spectral radius \( \rho(G) \), among all graphs \( G \) in \( \mathcal{B}(k,\alpha) \) is uniquely attained for the complete bipartite graph \( K_{\alpha,k-\alpha} \).

![Figure 1: bi-block graph with blocks \( K_{1,5} - K_{3,3} - K_{4,3} - K_{3,2} \)](image)
2 Main Results

We begin with a few results that gives us some insight to dependency of the independence number of a bi-block graph and its leaf blocks. It is easy to see, if \( G \) is a bipartite graph with vertex partition \( M \) and \( N \), then \( \alpha(G) = \max\{|M|, |N|\} \). Since every bi-block graph is a bipartite graph, so given a bi-block graph \( G \) on \( k \) vertices, the independence number \( \alpha(G) \), satisfies \( \left\lceil \frac{k}{2} \right\rceil \leq \alpha(G) \leq k - 1 \).

Let \( G \) be a bi-block graph. Let \( H \) be any leaf block connected to the graph \( G \) at a cut vertex \( v \in V(G) \) and \( G - H \) be the graph obtained from \( G \) by removing \( H \) – \( \{v\} \). Given a \( \alpha(G) \)-set \( I \), we denote

\[
I|_{G-H} = \{ u \in I \mid u \in V(G - H) \}.
\]

Note that, \( I|_{G-H} \) is an independent set of the graph \( G - H \) which need not be an \( \alpha(G - H) \)-set and hence \( |I|_{G-B} \leq \alpha(G - B) \). The result below gives us a relation between \( I|_{G-H} \) and a \( \alpha(G - H) \)-set.

**Lemma 2.1.** Let \( G \) be a bi-block graph and \( H = K_{m,n} \) be any leaf block connected to the graph \( G \) at a cut vertex \( v \in V(G) \) and \( G - H \) be the graph obtained from \( G \) by removing \( H - \{v\} \). Let \( I \) be an \( \alpha(G) \)-set and \( I|_{G-H} \) be defined as above. Then

\[
\alpha(G - H) = \begin{cases} 
|I|_{G-H} + 1 & \text{if } v \notin I \text{ and } I|_{G-H} \cup \{v\} \text{ is an } \alpha(G - H) \text{-set,} \\
|I|_{G-H} & \text{otherwise.}
\end{cases}
\]

**Proof.** Suppose \( I|_{G-H} \) is not an \( \alpha(G - H) \)-set. Then there is a vertex \( u \in V(G - H) \) and \( u \notin I|_{G-H} \) such that \( I|_{G-H} \cup \{u\} \) is an independent set. If \( u \neq v \), then \( L = I|_{G-H} \cup \{v\} \cup (I \setminus I|_{G-H}) \) is an independent set of \( G \), which is a contradiction as \( |L| > \alpha(G) \). Now if \( u = v \), then \( I|_{G-H} \cup \{v\} \) is an independent set and \( v \notin I \). Thus \( I|_{G-H} \cup \{v\} \) is an \( \alpha(G - H) \)-set and hence \( \alpha(G - H) = |I|_{G-H} + 1 \).

Next, if \( I|_{G-H} \) is an \( \alpha(G - H) \)-set, then we are done. \( \square \)

It is clear from the argument in the above proof, if \( I \) be an \( \alpha(G) \)-set and \( I|_{G-H} \) is not an \( \alpha(G - H) \)-set, then \( I|_{G-H} \cup \{v\} \) is an \( \alpha(G - H) \)-set. The result below relates the independence number of a bi-block of a graph with its leaf block.

**Proposition 2.2.** Let \( G \) be a bi-block graph and \( H = K_{m,n} \) be any leaf block connected to the graph \( G \) at a cut vertex \( v \) and \( G - H \) be the graph obtained from \( G \) by removing \( H - \{v\} \). If \( m \geq n \), then \( \alpha(G) \) equals to either \( \alpha(G - H) + m \) or \( \alpha(G - H) + m - 1 \).

**Proof.** Let \( H = K(M, N) \) where \( |M| = m, |N| = n \) and \( m \geq n \). Let \( I \) be an \( \alpha(G) \)-set and \( I|_{G-H} \) defined as before. We consider the following cases to complete the proof. First consider the case whenever \( v \notin I \) and \( I|_{G-H} \) is an \( \alpha(G - H) \)-set. If \( v \in M \) and \( m > n \), then \( \alpha(G) = |I|_{G-B} + m - 1 \).

Otherwise \( \alpha(G) = |I|_{G-B} + m \). Thus by Lemma 2.1 the result holds true.

Next, consider \( v \in I \) and \( I|_{G-H} \) is an \( \alpha(G - H) \)-set. If \( m > n \), then the cut vertex \( v \) necessarily belongs to \( M \), else \( L = \left(I|_{G-B} \setminus \{v\}\right) \cup M \) is an independent set of \( G \) with \( |L| > \alpha(G) \), which leads to a contradiction. Thus \( \alpha(G) = |I|_{G-B} + m - 1 \) and by Lemma 2.1, we get \( \alpha(G) = \alpha(G - B) + m - 1 \).

If \( m = n \), then either \( I|_{G-H} \cup (M \setminus \{v\}) \), whenever \( v \in M \) or \( I|_{G-H} \cup (N \setminus \{v\}) \), whenever \( v \in N \) is an \( \alpha(G) \)-set. Thus, by Lemma 2.1 we have \( \alpha(G) = \alpha(G - H) + m - 1 \).

Finally, consider the case \( v \notin I \) and \( I|_{G-H} \cup \{v\} \) is an \( \alpha(G - H) \)-set. If \( m > n \), then \( v \) necessarily belongs to \( N \), else \( L = I|_{G-H} \cup M \) is an independent set of \( G \) with \( |L| > \alpha(G) \) which is
a contradiction. Therefore \( \alpha(G) = |T_{G-B}| + m \) and by Lemma 2.1 we have \( \alpha(G) = \alpha(G-H) + m - 1 \). If \( m = n \), then both \( |T_{G-H} \cup M| \) and \( |T_{G-H} \cup N| \) are \( \alpha(G) \)-sets. Thus, by Lemma 2.1 we have \( \alpha(G) = \alpha(G-H) + m - 1 \). This completes the proof. \( \blacksquare \)

Now we consider our main goal, to maximize the spectral radius for the class of bi-block graphs \( B(k, \alpha) \). We begin with the result for bi-block graphs consisting of two-blocks. Before proceeding for the result, we list a few identities as an observation below.

**Observation 2.3.** Let \( G = (V, E) \) be a bi-block graph consisting of two blocks \( F \) and \( H \) connected by cut vertex \( v \), i.e. \( G = F \otimes H \). Let \( F = K(P, Q) \), where \( |P| = p \), \( |Q| = q \) and \( H = K(M, N) \), where \( |M| = m \), \( |N| = n \) such that \( Q \cap M = \{v\} \). Let \( A \) be the adjacency matrix of \( G \) and \( \rho, X \) be the eigen-pair corresponding to the spectral radius of \( A \). Let \( x_a \) denote the entry of \( X \) corresponding to the vertex \( u \in V \).

Let \( q, m \geq 2 \). Using \( AX = \rho X \), we have \( px_u = \sum_{w \sim u} x_w = \sum_{w \in M} x_w \), for all \( u \in N \). Thus \( x_u \) is a constant, whenever \( u \in N \) and we denote it by \( a_n \). Using similar arguments, let us denote

\[
x_u = \begin{cases} 
  a_n & \text{if } u \in N, \\
  a_m & \text{if } u \in M, u \neq v, \\
  a_p & \text{if } u \in P, \\
  a_q & \text{if } u \in Q, u \neq v.
\end{cases} \tag{2.1}
\]

Now using \( AX = \rho X \), we have the following identities:

\( (I1) \ (q - 1)a_q + x_v = \rho a_p. \)
\( (I2) \ pa_p = \rho a_q. \)
\( (I3) \ pa_p + n a_n = \rho x_v. \)
\( (I4) \ na_n = \rho a_m. \)
\( (I5) \ x_v + (m - 1)a_m = \rho a_n. \)

Using the identities (I2),(I3) and (I4), we have \( x_v = a_q + a_m \). Substituting of \( x_v = a_q + a_m \) in (I1) and (I5), we have

\( (I1^*) \ qa_q + a_m = \rho a_p. \)
\( (I5^*) \ a_q + ma_m = \rho a_n. \)

Without loss of generality if we assume that \( a_p = 1 \), then

\( (I6) \ a_q = \frac{p}{\rho}, \ a_m = \frac{\rho^2 - pq}{p} \) and \( a_n = \frac{\rho^2 - pq}{n} \).

Similarly, if we assume that \( a_n = 1 \), then

\( (I7) \ a_m = \frac{n}{p}, \ a_q = \frac{\rho^2 - mn}{\rho} \) and \( a_p = \frac{\rho^2 - mn}{p} \).

Moreover, since the ratio \( \frac{a_p}{a_n} \) is constant for the Perron vector \( X \), so using (I6) and (I7), we have

\( (I8) \ pn = (\rho^2 - pq)(\rho^2 - mn). \)

If \( m = 1 \) and \( q > 1 \), then by choosing \( a_m = x_v - a_q \), all the above identities is true. Similarly, for \( q = 1 \) and \( m > 1 \), we choose \( a_q = x_v - a_m \).
Remark 2.4. Under the assumption of Observation 2.3, using identity (I8) the spectral radius $\rho$ of adjacency matrix $A$ is given by

$$\rho = \sqrt{\frac{(pq + mn) + \sqrt{(pq - mn)^2 + 4mn}}{2}}.$$ 

The next lemma gives us a result on maximal spectral radius among bi-block graphs having two blocks with fixed number of vertices and independence number.

**Lemma 2.5.** Let $G \in \mathcal{B}(k, \alpha)$ consists of two blocks. Then $\rho(G) < \rho(K_{n,k-\alpha}).$

**Proof.** Let $G$ be a bi-block graph consisting of two blocks $F$ and $H$ connected by cut vertex $v$. Let $F = K(P, Q)$, where $|P| = p$, $|Q| = q$ and $H = K(M, N)$, where $|M| = p$, $|N| = q$ such that $Q \cap M = \{v\}$. Then $k = p + q + m + n - 1$.

If $m = 1$ and $q = 1$, then $k = p + n + 1$. and $G = K_{1,p+n}$ with independence number $\alpha(G) = p + n$. Thus, for $\alpha = p + n$ the class $\mathcal{B}(k, \alpha)$ consists of only the star $G = K_{1,p+n}$ and hence result is vacuously true. We complete the proof by considering the following cases.

**Case 1:** If $p < q$ and $n < m$, then $I = P \cup N$ is the $\alpha(G)$-set. We consider the complete bipartite graph $G^* = K(\tilde{P}, \tilde{Q})$, where $\tilde{P} = P \cup N$ and $\tilde{Q} = Q \cup M$. Thus $\alpha(G) = \alpha(G^*) = p + n$. Since $G^*$ is obtained from $G$ by adding extra edges, so by Lemma 1.2 we have $\rho(G) < \rho(G^*)$.

**Case 2:** If $q \geq p$ and $m \geq n$, then $I = Q \cup M$ is an $\alpha(G)$-set. We consider the complete bipartite graph $G^* = K(\tilde{P}, \tilde{Q})$, where $\tilde{P} = P \cup N$ and $\tilde{Q} = Q \cup M$. Thus $\alpha(G) = \alpha(G^*) = q + m - 1$. Since $G^*$ is obtained from $G$ by adding extra edges, so by Lemma 1.2 we have $\rho(G) < \rho(G^*)$.

**Case 3:** If $q > p$ and $n > m$, then $I = (Q \setminus \{v\}) \cup N$ is an $\alpha(G)$-set and hence $\alpha(G) = q + n - 1$. Now consider the complete bipartite graph $G^* = K(\tilde{P}, \tilde{Q})$, where $\tilde{P} = P \cup M$ and $\tilde{Q} = (Q \setminus \{v\}) \cup N$. So $\alpha(G) = \alpha(G^*) = q + n - 1$. Observe that, we can obtain the graph $G^*$ from $G$ using the following operations:

1. Delete the edges between vertex $v$ and the vertices of $P$.
2. Add edges between vertices of $M$ and $Q \setminus \{v\}$.
3. Add edges between vertices of $P$ and $N$.

Let $A$ be the adjacency matrix of $G$ and $(\rho, X)$ be the eigen-pair corresponding to the spectral radius of $A$. Let $A^*$ be the adjacency matrix of $G^*$. Using the notations and identities in Observation 2.3, we have

$$\frac{1}{2}X^T(A^* - A)X = -x_v \sum_{w \in P} x_w + \sum_{u \in M, w \in Q \setminus \{v\}} x_u x_w + \sum_{u \in P, w \in N} x_u x_w$$

By Eqn.(2.1)]

$$= -pa_p(a_q + a_m) + (q - 1)a_q(a_q + ma_m) + pma_pa_n$$

[Using (I5*)]

$$= -pa_p(a_q + a_m) + (q - 1)pa_qa_n + pma_pa_n$$

[Using (I2)]

$$= p((q - 1)a_n + p\rho a_m - (a_q + a_m))$$

[Using $a_p = 1$]

$$= \frac{p}{\rho n} [\rho(q - 1)(\rho^2 - pq) + \rho n(\rho^2 - pq) - n(p + \rho^2 - pq)]$$

[Using (I6)]

$$= \frac{p}{\rho n} [\rho(q - 1)(\rho^2 - pq) + \rho n(\rho^2 - pq) - n(\rho^2 - pq) - (\rho^2 - pq)(\rho^2 - mn)]$$

[Using (I8)]

$$= \frac{p(\rho^2 - pq)}{\rho n} [\rho(q - 1) + \rho n - n - (\rho^2 - mn)]$$

$$= \frac{p(\rho^2 - pq)}{\rho n} [\rho(q + n - 1) - \rho^2 + n(m - 1)].$$
By Lemma 1.1 we have \( \rho \leq \max\{p + n, q\} \). And using the assumption \( p < q \), we always have \( q + n - 1 \geq \rho \). Since \( X \) is Perron vector of \( G \), so \( X'(A^* - A)X > 0 \). Hence by Min-max theorem we have \( \rho(G) < \rho(G^*) \).

**Case 4:** If \( p > q \) and \( m > n \), then \( \mathcal{I} = P \cup (M \setminus \{v\}) \) is a \( \alpha(G) \)-set. We consider the complete bipartite graph \( G^* = K(\tilde{P}, \tilde{Q}) \), where \( \tilde{P} = P \cup (M \setminus \{v\}) \) and \( \tilde{Q} = Q \cup N \). This case is analogous to Case 3, hence proceeding similarly we have \( \rho(G) < \rho(G^*) \). 

In the above lemma we have considered a bi-block graph \( G \) with two blocks and hence the cut-vertex \( v \) have the block index \( b_{\mathcal{I}}(v) = 2 \). In the next lemma, we will consider bi-block graphs such that the block index of each of the cut-vertex is exactly 2.

**Lemma 2.6.** Let \( G \in \mathcal{B}(k, \alpha) \). If \( b_{\mathcal{I}}(u) = 2 \), for all cut-vertex \( u \) in \( G \), then \( \rho(G) \leq \rho(K_{\alpha,k-\alpha}) \) and equality holds if and only if \( G = K_{\alpha,k-\alpha} \).

**Proof.** We will use induction on the number of blocks to prove the lemma. Let \( G \in \mathcal{B}(k, \alpha) \) consists of \( b \) blocks and \( b_{\mathcal{I}}(c) = 2 \) for every cut vertex \( c \) in \( G \). By Lemma 2.5, the result is true for \( b = 2 \). Suppose the result is true for all bi-block graphs \( \mathcal{B}(k, \alpha) \) consisting of \( b - 1 \) blocks. Let \( H = K(M, N) \) with \( |M| = m \) and \( |N| = n \) be any leaf block connected to the graph \( G \) at a cut vertex \( v \). Since \( b_{\mathcal{I}}(v) = 2 \), there exist a unique block \( F = K(P, Q) \), with \( |P| = p \) and \( |Q| = q \) which is a neighbour of \( H \) connected via the cut vertex \( v \). Without loss of generality we assume that \( M \cap Q = \{v\} \). Let \( \mathcal{I} \) be an \( \alpha(G) \)-set of \( G \), i.e. \( |\mathcal{I}| = \alpha \).

**Case 1:** \( I \cap P = \emptyset \) and \( I \cap Q = \emptyset \). For this case either \( M \setminus \{v\} \subset I \) or \( N \subset I \). We consider the complete bipartite graph \( K(\tilde{P}, \tilde{Q}) \), where \( \tilde{P} = P \cup N \) and \( \tilde{Q} = Q \cup M \). Let \( G^* \) be the graph obtained from \( G \) by replacing the induced subgraph \( F \odot H \) with \( K(\tilde{P}, \tilde{Q}) \). Then \( G^* \) consists of \( b - 1 \) blocks and \( \mathcal{I} \) is an \( \alpha(G^*) \)-set, i.e. \( G^* \in \mathcal{B}(k, \alpha) \). Since \( G^* \) obtained from \( G \) by adding additional edges, so by Lemma 1.2 we have \( \rho(G) < \rho(G^*) \). Hence the induction hypothesis yields the result.

**Case 2:** \( I \cap P = \emptyset \) and \( I \cap Q \neq \emptyset \). For \( m \geq n \), we can assume \( M \subset I \). We consider graph \( G^* \) be the graph obtained from \( G \) by replacing the induced subgraph \( F \odot H \) with \( K(\tilde{P}, \tilde{Q}) \), where \( \tilde{P} = P \cup N \) and \( \tilde{Q} = Q \cup M \), which implies that \( \mathcal{I} \) is an \( \alpha(G^*) \)-set. Thus arguing similar to the Case 1 yields the result.

**Case 3:** \( I \cap P = \emptyset \) and \( I \cap Q \neq \emptyset \). For \( n > m \), if \( v \in \mathcal{I} \), then \( \mathcal{L} = (I \setminus \{v\}) \cup N \) is an independent set of \( G \) and \( |\mathcal{L}| > |\mathcal{I}| \), which leads to a contradiction. Thus \( v \notin \mathcal{I} \) and we have the following:

\[
\begin{aligned}
\{ v \notin \mathcal{I} \text{ and } &I = I |_{G-H} \cup N, \\
\alpha(G) = &I |_{G-H} + n. \\
\end{aligned}
\]

Next, we subdivide the case \( I \cap P = \emptyset \) and \( I \cap Q \neq \emptyset \) with \( n > m \), into the following sub cases.

**Subcase 1:** Suppose all the vertices of \( Q \) are cut-vertices. Let \( u \in Q \setminus \{v\} \) be a cut-vertex and \( u \in \mathcal{I} \). Since \( b_{\mathcal{I}}(u) = 2 \), so let \( B = K(R, S) \) be the neighbour of the block \( F \) via the cut-vertex \( u \), where \( R \cap Q = \{u\} \). Thus \( u \in \mathcal{I} \) and \( u \in R \) implies that \( I \cap S = \emptyset \). Consider the bi-block graph \( G^* \) obtained from \( G \) by replacing the induced subgraph \( F \odot B \) with the complete bipartite graph \( K(\tilde{P}, \tilde{Q}) \), where \( \tilde{P} = P \cup S \) and \( \tilde{Q} = Q \cup R \). It is easy to see, \( \mathcal{I} \) is an \( \alpha(G^*) \)-set and \( G^* \in \mathcal{B}(k, \alpha) \) consists of \( b - 1 \) blocks. Hence the result follows from the Lemma 1.2 and the induction hypothesis.

**Subcase 2:** Let \( c \in Q \) and \( c \) is not a cut vertex. Since \( I \cap P = \emptyset \), so \( c \in \mathcal{I} \). Let \( A \) be the adjacency matrix of \( G \) and \( (\rho, X) \) be the eigen-pair corresponding to the spectral radius of \( A \). Let \( x_u \) denote the entry of \( X \) corresponding to the vertex \( u \in V \). Using \( AX = \rho X \) and arguing similar to the Observation 2.3, we find a few identities as follows. For \( m \geq 2 \), let us denote

\[
x_u = \begin{cases} 
  b_n & \text{if } u \in N, \\
  b_m & \text{if } u \in M, u \neq v.
\end{cases}
\]

Using \( c \in Q \), \( c \) is not a cut vertex and \( AX = \rho X \), we have the following identities:
(J1) \( \rho x_c = \sum_{w \in P} x_w \).
(J2) \( \rho x_v = \sum_{w \in P} x_w + nb_n \).
(J3) \( \rho b_n = (m - 1)b_m + x_v \).
(J4) \( \rho b_m = nb_n \).

Using identities (J1), (J2) and (J4), we have \( x_v = x_c + b_m \). Thus the identity (J3) reduces to:

(J3*) \( \rho b_n = mb_m + x_c \).

Next, if \( m = 1 \), then by choosing \( b_m = x_v - x_c \), all the above identities holds true. Now we further subdivide the Subcase 2 as follows:

**Subcase 2.1:** Whenever \( b_m \geq b_n \).

Let \( G^* \) be a bi-block graph obtained from \( G \) by replacing the induced subgraph \( F \otimes H \) with the complete bipartite graph \( K(P, Q) \), where \( P = P \cup M \) and \( Q = (Q \setminus \{v\}) \cup N \). Thus, \( \mathcal{I} \) is an \( \alpha(G^*) \)-set and \( G^* \in \mathcal{B}(k, \alpha) \) consists of \( b - 1 \) blocks. Note that, we can obtain the graph \( G^* \) from \( G \) using the following operations:

1. Delete the edges between vertex \( v \) and the vertices of \( P \).
2. Add edges between vertices of \( M \) and \( Q \setminus \{v\} \).
3. Add edges between vertices of \( P \) and \( N \).

Let \( A^* \) be the adjacency matrix of \( G^* \). Using the above identities, we have

\[
\frac{1}{2} X^t (A^* - A) X = -x_v \sum_{w \in P} x_w + \sum_{u \sim w, w \in M, w \in Q \setminus \{v\}} x_u x_w + \sum_{u \sim w, w \in N} x_u x_w \\
= -(x_c + b_m) \sum_{w \in P} x_w + (mb_m + x_c) \sum_{w \in Q \setminus \{v\}} x_w + nb_n \sum_{w \in P} x_w \quad \text{[By Eqn. (2.3)]}
\]

\[
= -(x_c + b_m) \rho x_c + (mb_m + x_c) \sum_{w \in Q \setminus \{v\}} x_w + nb_n \rho x_c \quad \text{[Using (J1)]}
\]

\[
= -(x_c + b_m) \rho x_c + (mb_m + x_c) \sum_{w \in Q \setminus \{v\}} x_w + \rho^2 b_m x_c \quad \text{[Using (J4)]}
\]

\[
\geq -(x_c + mb_m) \rho x_c + (mb_m + x_c) \sum_{w \in Q \setminus \{v\}} x_w + \rho^2 b_m x_c \quad \text{[Using (J3*)]}
\]

\[
= -\rho^2 b_n x_c + (mb_m + x_c) \sum_{w \in Q \setminus \{v\}} x_w + \rho^2 b_m x_c
\]

Since \( b_m \geq b_n \), and \( X \) is Perron vector of \( G \), so \( X^t (A^* - A) X \geq 0 \). Hence by Min-max theorem we have \( \rho(G) \leq \rho(G^*) \) and the induction hypothesis yields the result.

**Subcase 2.2:** Whenever \( b_m < b_n \).

For this case we partition the set \( N \subset \mathcal{I} \) as \( N = N_1 \cup N_2 \) and \( N_1 \cap N_2 = \emptyset \) such that \( |N_1| = m \) and \( |N_2| = n - m \). We consider the complete bipartite graph \( K(P, Q) \), where \( P = P \cup N_1 \) and \( Q = Q \cup M \cup N_2 \). Let \( G^* \) be a bi-block graph obtained from \( G \) by replacing the induced subgraph \( F \otimes H \) with \( K(P, Q) \). Thus, by Eqn. (2.1) we get \( \mathcal{I}^* = \mathcal{I}\big|_{G-H} \cup M \cup N_2 \) is an \( \alpha(G^*) \)-set and \( \alpha(G^*) = \alpha(G) = |\mathcal{I}\big|_{G-H} | + n \), which implies that \( G^* \in \mathcal{B}(k, \alpha) \) consists of \( b - 1 \) blocks. Further note that, we can obtain the graph \( G^* \) from \( G \) using the following operations:

1. Delete the edges between vertices of \( M \) and \( N_2 \).
Let $A^*$ be the adjacency matrix of $G^*$. Then,
\[
\frac{1}{2}X^t(A^* - A)X = -\sum_{w\sim u} x_u x_w + \sum_{u\in M, w\in N_2} x_u x_w + \sum_{u\in N_1, w\in Q\setminus \{v\}} x_u x_w + \sum_{u\in N_2, w\in P} x_u x_w
\]
\[
= -(n - m)(mb_m + x_c)b_n + mb_n \sum_{w\in Q\setminus \{v\}} x_w + (n - m)b_n \sum_{w\in P} x_w + (n - m)mb_n^2 + b_m(m - 1) \sum_{w\in P} x_w \quad \text{[By Eqn.(2.3)]}
\]
\[
= -(n - m)mb_n b_n - (n - m)x_c b_n + mb_n \sum_{w\in Q\setminus \{v\}} x_w + \rho(n - m)b_n x_c + (n - m)mb_n^2 + \rho(m - 1)b_m x_c \quad \text{[Using (J1)]}
\]
\[
= (n - m)[mb_n(b_n - b_m) + (\rho - 1)x_c b_n] + mb_n \sum_{w\in Q\setminus \{v\}} x_w + \rho(m - 1)b_m x_c.
\]

Since $b_m < b_n$ and $\rho \geq 1$ (by Lemma 1.1), so using the fact that $X$ is Perron vector of $G$ we have $X^t(A^* - A)X \geq 0$. Hence by Min-max theorem we have $\rho(G) \leq \rho(G^*)$ and the induction hypothesis yields the result.

**Case 4:** $\mathcal{I} \cap P \neq \emptyset$ and $\mathcal{I} \cap Q = \emptyset$. For $n \geq m$ or $m = n + 1$, we have $\mathcal{N} \subset \mathcal{I}$. We consider graph $G^*$ obtained from $G$ by replacing the induced subgraph $F \otimes H$ with $K(\tilde{P}, \tilde{Q})$, where $\tilde{P} = P \cup N$ and $\tilde{Q} = Q \cup M$, which implies that $\mathcal{I}$ is an $\alpha(G^*)$-set. Thus arguments similar to the Case 1 yields the result.

**Case 5:** $\mathcal{I} \cap P \neq \emptyset$ and $\mathcal{I} \cap Q = \emptyset$. For $m > n + 1$, we have $(M \setminus \{v\}) \subset \mathcal{I}$. We consider all neighbouring blocks of $F = K(P, Q)$, say $B_i = K(R_i, S_i); 1 \leq i \leq j$, connected via cut-vertices to partition $P$. Without loss of generality we assume $S_j \cap P \neq \emptyset$. For any one of the such neighbour, if $\mathcal{I} \cap R_i = \emptyset$, then we consider graph $G^*$ be obtained from $G$ by replacing the induced subgraph $F \otimes B_i$ with $K(\tilde{P}, \tilde{Q})$, where $\tilde{P} = P \cup S_i$ and $\tilde{Q} = Q \cup R_i$. Since $\mathcal{I} \cap P \neq \emptyset$, so $\mathcal{I}$ is an $\alpha(G^*)$-set and argument similar to the Case 1 leads to desired the result. If no such neighbours exists, then proceeding inductively we need to look for $B_i$’s neighbours with similar properties. Since $G$ is a finite graph either we will reach a neighbour with suitable properties or reach a leaf block does not satisfies requisite properties. For the later case, we find a finite chain of blocks $C_i = K(M_i, N_i)$; $i = 1 \leq i \leq t$ satisfies the following:

1. $C_1 = H$ and $C_t$ are leaf blocks.
2. For $i = 1, 2, \ldots, t - 1$, the blocks $C_i$ and $C_{i+1}$ are neighbours such that $M_i \cap N_{i+1} \neq \emptyset$.
3. $\mathcal{I} \cap N_t = \emptyset$, for all $i = 1, \ldots, t$.

Since $C_t$ is a leaf block and is connected to $C_{t-1}$ via a cut vertex $u$(say) with $bi_G(u) = 2$, so it can be seen $\mathcal{I} \cap N_{t-1} = \emptyset$ and $\mathcal{I} \cap N_t = \emptyset$ implies that $|M_i| > |N_i|$. Now if we begin with the leaf block $C_t$, then this case is analogous to the Case 3. Hence the desired result follows.

Moreover, by Lemma 2.5 and combining all the above cases, the maximum spectral radius $\rho(G)$, among all graphs $G$ in $\mathcal{B}(k, \alpha)$ with $bi_G(u) = 2$, for all cut-vertex $u$ in $G$, is uniquely attained for the complete bipartite graph $K_{\alpha,k-\alpha}$.
Lemma 2.7. If $G \in \mathcal{B}(k, \alpha)$, then there exists a bi-block graph $G^* \in \mathcal{B}(k, \alpha)$ with $\rho(G^*) = 2$, for every cut-vertex $u$ such that $\rho(G) \leq \rho(G^*)$.

Proof. Let $v$ be a cut-vertex of $G$ with $b_{iG}(v) = t$, where $t \geq 3$. Let $B_i = K(M_i, N_i); i = 1, 2, 3$ be any three neighbours connected via the cut-vertex $v$ such that $v \in N_1 \cap N_2 \cap N_3$. Let $\mathcal{I}$ be an $\alpha(G)$-set. If $V(B_i) \cap \mathcal{I} \neq \emptyset$ for all $i = 1, 2, 3$, then either $M_i \cap \mathcal{I} \neq \emptyset$ or $N_i \cap \mathcal{I} \neq \emptyset$. Thus by pigeonhole principle, there exists $i, j \in \{1, 2, 3\}$ such that either $\mathcal{I} \cap N_i = \emptyset$ and $\mathcal{I} \cap N_j = \emptyset$ or $\mathcal{I} \cap M_i = \emptyset$ and $\mathcal{I} \cap M_j = \emptyset$. Let us consider a bi-block graph $G^*$ obtained from $G$ by replacing the induced subgraph $B_i \oplus B_j$ with $K(\tilde{M}, \tilde{N})$, where $\tilde{M} = M_i \cup M_j$ and $\tilde{N} = N_i \cup N_j$. It is easy to see that, $\mathcal{I}$ is an $\alpha(G^*)$-set and $b_{iG^*}(v) = t - 1$. By Lemma 1.2 we have $\rho(G) \leq \rho(G^*)$. Hence proceeding inductively the result follows. If $V(B_{i_0}) \cap \mathcal{I} = \emptyset$ (i.e. $M_{i_0} \cap \mathcal{I} = \emptyset$ and $N_{i_0} \cap \mathcal{I} = \emptyset$) for some $i_0 \in \{1, 2, 3\}$, then for $j \neq i_0$ and choosing $K(\tilde{M}, \tilde{N})$, where $\tilde{M} = M_{i_0} \cup M_j$ and $\tilde{N} = N_{i_0} \cup N_j$, similar argument yields the desired result. \hfill \Box

Next we state the main result of the article (without proof) which maximizes the spectral radius for the class $\mathcal{B}(k, \alpha)$ and the proof follows from Lemmas 2.6 and 2.7.

Theorem 2.8. If $G \in \mathcal{B}(k, \alpha)$, then $\rho(G) \leq \rho(K_{\alpha, k-\alpha})$ and equality holds if and only if $G = K_{\alpha, k-\alpha}$.

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