I. INTRODUCTION

Quantum correlations, like non-locality [1], steering [2], and entanglement [3], are very often used as the key resources in quantum information tasks, such as quantum state discrimination [4] and key distribution [5]. Unfortunately, quantum systems are never perfectly isolated from the external influences, which leads to a harmful dissipation and decoherence that ultimately destroy quantum correlations [4]. These dynamical processes for the open systems can be described using quantum channels, which are completely positive, trace-preserving maps from quantum states to quantum states [7]. Exceptionally detrimental kinds of open quantum system dynamics are described by the so-called entanglement breaking (EB) [8] and incompatibility breaking (ICB) [9] quantum channels. The former maps any entangled input state to a separable output. The latter, on the other hand, has the corresponding dual map that, when applied to a pair of incompatible (not jointly measurable) observables, maps them to a pair of compatible (jointly measurable) observables [10]. Therefore, it is of general interest to determine the likelihood to encounter such channels in the space of all channels, measured in terms of ratios of their corresponding volumes.

In this article, we focus on continuous variable (CV) systems; i.e., the systems described with the help of the canonical position and momentum operators. Laboratories equipped with linear optical elements and photodetectors can routinely prepare, manipulate, and perform quantum measurements on the states of such systems [11][12]. An important special class of states for continuous variable systems is the set of Gaussian states, characterized by the Gaussian Wigner function [12][14]. Closely associated to them is the set of Gaussian quantum channels, mapping any Gaussian state to a Gaussian state. These sets of states and channels form basic building blocks for current experiments on photonic systems in the field of quantum information [15][16]. The main reasons for their appeal is the fact that the Gaussianity-preserving unitary operations can be implemented in linear optics, and Gaussian systems are relatively easy to handle mathematically.

In this article, we study the geometry of the manifold of Gaussian quantum channels. We provide a rigorous route to investigate how likely it is, among all Gaussian quantum channels, to encounter a channel that is either entanglement breaking [17] or incompatibility breaking [18]. So far, the investigations on the information geometry in the Gaussian domain have been focused on the geometry of the state space [19][20] and the typical properties of quantum correlations [21][22]. In [23], first steps are taken to study the geometry of the Gaussian quantum channels. The main hindrance for further development have been the technical difficulties that are encountered when one tries to formulate the Choi-Jamiołkowski (CJ) isomorphism [25][26] for continuous variable systems [17][27].

Recently, new results have shed some light on how to formulate the Choi-Jamiołkowski correspondence between Gaussian states and channels in such a way that divergence problems do not occur [28][29]. In this article, we use the approach developed in [28] for the Choi-Jamiołkowski isomorphism in combination with the results on the geometry of Gaussian states in [23] to investigate the geometry of Gaussian quantum channels. In particular, we report the likelihood of encountering a one-mode entanglement or incompatibility breaking channel among all the one-mode Gaussian channels. It should be noted that such results, in general, depend on the choice of the metric. Here, as the metric on the space of channels is defined with the help of the Choi-Jamiołkowski isomorphism, it will also depend on the reference state of that isomorphism.

The rest of the article is organized as follows. In Section II, we provide a quick review on the main properties of the Gaussian states. Then, in Section III, we introduce the notion of the Gaussian channels. We present the generalization of the Choi-Jamiołkowski correspondence.
In the above equations, \( p \) canonical operators acting on the corresponding Hilbert space \( \mathcal{H} \). The canonical operators acting on \( \mathcal{H} \) can be arranged to create a vector \( R := (q_1, p_1, \ldots, q_n, p_n)^T \) with \( q_k := a_k^\dagger + a_k \) and \( p_k := i(a_k^\dagger - a_k) \). The creation and annihilation operators satisfy the bosonic commutation relations \([a_i, a_j^\dagger] = \delta_{ij}\), \([a_i, a_j] = 0\), which induce the following relation for the vector components,

\[
[R_i, R_j^\dagger] = 2i\Omega_{ij}, \quad \Omega := \sum_{k=1}^n \omega_k, \quad \omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

In the above equations, \( \Omega \) is the symplectic form. Now, for every quantum state \( \rho \), let us introduce the characteristic function \( \chi(\xi) := \text{Tr}[D(\xi)\rho] \), where \( \xi := (\xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_n^{(1)}, \xi_n^{(2)})^T \in \mathbb{R}^{2n} \) are the phase space coordinates and

\[
D(\xi) := e^{iR^T \Omega \xi}, \quad D(\xi)D(\xi') = e^{-i\xi'^T \Omega \xi} D(\xi')D(\xi),
\]

are the displacement (or Weyl) operators \( \chi(\xi) \). By definition, a Gaussian state is a quantum state whose characteristic function \( \chi(\xi) \) is a Gaussian function \( \mathcal{N} \). We write it as

\[
\chi(\xi) = \exp \left[ -\frac{1}{2} \xi^T \Sigma \Omega T \xi + i\ell^T \Omega \xi \right],
\]

where \( \ell_k := \text{Tr}(\rho R_k) \) is the displacement vector and \( \Sigma_{ij} := \frac{1}{2}(\text{Tr}[\rho (R_i R_j + R_j R_i)] - \ell_i \ell_j) \) is the covariance matrix of the underlying Gaussian quantum state \( \rho = \rho(\Sigma, \ell) \). Then, the state can be expressed as

\[
\rho(\Sigma, \ell) = \int_{\mathbb{R}^{2n}} d^{2n} \xi \frac{1}{\pi^n} \chi(\xi)D(-\xi).
\]

Note that \( \Sigma \) is a covariance matrix of a Gaussian state if and only if

\[
\Sigma + i\Omega \geq 0, \quad (1)
\]

due to the canonical commutation relations. We would like to stress that, despite the apparent similarities between the Gaussian states in the classical and quantum domain, the quantum case is fundamentally different due to eq. (1). The classical Gaussian states (probability densities) can become arbitrarily narrow approaching the Dirac \( \delta \) function in a limiting sense, whereas eq. (1) sets the minimal admissible width for the Gaussian quantum states compatible with Heisenberg’s uncertainty relation.

## III. GAUSSIAN CHANNELS AND THE CHOJI-JAMIOLKOWSKI ISOMORPHISM

The Gaussian quantum channels \( \Lambda : \mathcal{H}_A \to \mathcal{H}_B \) are completely positive, trace-preserving maps that transform Gaussian quantum states into Gaussian quantum states. The action of a Gaussian channel leads to a dual map on the displacement operators \( [13, 31, 32] \)

\[
\Lambda^*[D(\xi)] = D(M\xi) \exp \left[ -\frac{1}{2} \xi^T N \xi + ic^T \xi \right],
\]

with the matrices \( M \in \mathbb{M}_{2n_A \times 2n_B}(\mathbb{R}), \ N = N^T \in \mathbb{M}_{2n_B \times 2n_B}(\mathbb{R}), \) and the \( 2n_A \)-dimensional vector \( c \). Therefore, each channel is completely characterized by a triple \( (M, N, c) \). The action of a Gaussian channel on a Gaussian state \( \rho(\Sigma, \ell) \) is then efficiently expressed in terms of the covariance matrix and the displacement vector,

\[
\Sigma \mapsto M^T \Sigma M + N, \quad \ell \mapsto M^T \ell + c.
\]

The complete positivity condition

\[
N - iM^T \Omega M + i\Omega \geq 0
\]

follows directly from eq. (1).

In order to leverage the known results on the geometry of Gaussian states \( [19, 23] \), we use the Choi-Jamiolkowski isomorphism to express the Gaussian channels in terms of the state parameters. Let us recall Lemma 4 from Kiukas et. al. [28].

**Lemma 1.** There exists a one-to-one correspondence between bipartite Gaussian states \( \rho_{AB} \) with a common marginal \( \sigma = \text{Tr}_A \rho_{AB} \), a covariance matrix \( \Sigma_\sigma \) of full symplectic rank, and a displacement \( \ell_\sigma \), and Gaussian channels \( \Lambda : \mathcal{H}_B \to \mathcal{H}_A \), such that

\[
\rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho_\Omega).
\]

The Gaussian state \( \rho_\Omega \) is characterized by the following covariance matrix and displacement,

\[
\Sigma_\Omega := \begin{pmatrix} \Sigma_\sigma & S_\sigma^T Z_\sigma S_\sigma \\
S_\sigma^T Z_\sigma S_\sigma & \Sigma_\sigma \end{pmatrix}, \quad \ell_\Omega := \ell_\sigma \oplus \ell_\sigma.
\]

In the above definition,

\[
\Sigma_\sigma := S_\sigma^T D_\sigma S_\sigma, \quad D_\sigma := \text{diag}(\nu_{\sigma,1}, \nu_{\sigma,1}, \ldots, \nu_{\sigma,N}, \nu_{\sigma,N})
\]

\[
Z_\sigma := \bigoplus_{k=1}^N \sigma_3 \sqrt{\nu_{\sigma,k}^2 - 1},
\]

with \( S_\sigma \) being the symplectic matrix \( (S_\sigma^T \Omega S_\sigma = \Omega) \) diagonalizing \( \Sigma_\sigma \).
The correspondence between the Gaussian channel $\Lambda(M,N,c)$ and the Gaussian Choi-Jamiolkowski (CJ) state $\rho_{AB}(\Sigma,\ell)$ is given as follows [25],

$$
\begin{align*}
\Sigma &= \left( \begin{array}{cc} \Sigma_A & \Gamma^T \\ \Gamma & \Sigma \end{array} \right), \\
\ell &= \ell_A \oplus \ell_\sigma,
\end{align*}
$$

(2)

Hence, the line element and therefore a non-zero displacement produces the multivaluedness in [23]. Note that $d\Sigma = 0$ implies $\ell$, and so will the metric properties of the channel space.

## IV. GEOMETRY OF GAUSSIAN STATES AND CHANNELS

In order to discuss the geometric properties in the space of quantum channels, we need to define a metric in terms of a line element. For finite-dimensional systems, there exists the unique unitarily invariant line element induced by the Fubini-Study metric [33]. In the infinite-dimensional case, however, there are many possible non-equivalent choices for the metric [25]. We base our calculations on the Hilbert-Schmidt distance defined by $ds^2 = \text{Tr}(d\rho^2)$. On the manifold of the Gaussian states, it takes the following form,

$$
ds^2 = \text{Tr}[\rho(\Sigma + d\Sigma, \ell + d\ell)]^2 + \text{Tr}[\rho(\Sigma, \ell)]^2 - 2\text{Tr}[\rho(\Sigma, \ell)\rho(\Sigma + d\Sigma, \ell + d\ell)].
$$

For more details considering the computation of the line element, see Appendix A. The final result is

$$
ds^2 = \frac{1}{16\sqrt{\det \Sigma}} \left\{ 2\text{Tr}[\Sigma^{-1}d\Sigma]^2 + [\text{Tr}(\Sigma^{-1}d\Sigma)]^2 - 8d\ell^T\Sigma^{-1}d\ell \right\}. 
$$

(3)

Hence, the line element $ds^2$ and the volume element $dV$ can be written as

$$
ds^2 = (d\Sigma^T \ d\ell^T) \left( \begin{array}{cc} G & 0 \\ 0 & g \end{array} \right) \left( d\Sigma \ d\ell \right),
$$

$$
dV = \sqrt{\det G} \sqrt{\det g} \prod_{i=1}^{4n^2} d\Sigma_i \prod_{j=1}^{2n} d\ell_j,
$$

(4)

where $d\Sigma = \text{vec}(d\Sigma)$ is the matrix vectorization. For $\ell = 0$, eq. (3) is in correspondence with the results obtained in [25]. Note that $d\Sigma$ and $d\ell$ are not coupled, and therefore a non-zero displacement produces the multiplicative factor

$$
\sqrt{\det g} = \left[ \text{det} \left( \frac{\Sigma^{-1}}{2\sqrt{\det \Sigma}} \right) \right]^{-1} = 2^{-n}(\det \Sigma)^{-\frac{n+1}{2}}
$$

in the volume element.

Let us consider the one-mode Gaussian channel $\Lambda(M,N,c)$. According to eqs. (2), the corresponding two-mode Gaussian CJ state is given by

$$
\Sigma = \left( \begin{array}{cc} N + M^T\Sigma_M M & M^T \Sigma_M S \Sigma_S \\ S^T \Sigma_S S M & \Sigma \end{array} \right),
$$

(5)

$\ell = (c + M^T\ell_\sigma) \oplus \ell_\sigma$.

A non-zero displacement vector $\ell$ corresponds to local unitary contributions of the channel. When considering the effect of the channel on the non-local correlations, we can – without the loss of generality – set $\ell = 0$, as we do for the rest of the article.

Note that any two-mode Gaussian covariance matrix can be expressed in the standard form $\Sigma = S\gamma S^T$, where $S$ is a local symplectic transformation [25] and

$$
W = \left( \begin{array}{cc} \nu_A & 0 & \gamma_+ & 0 \\ 0 & \nu_A & 0 & \gamma_- \\ \gamma_+ & 0 & \nu_\sigma & 0 \\ 0 & \gamma_- & 0 & \nu_\sigma \end{array} \right).
$$

Here, $\nu_A$ and $\nu_\sigma$ are the symplectic eigenvalues of the marginal states, and $\gamma_{\pm}$ describe the correlations between the two modes. Following the method presented in [25], we compute the Hilbert-Schmidt volume element for the Gaussian states with the covariance matrix given by eq. (5) (for more details, see Appendix B). This way, we obtain

$$
dV = \sqrt{\det G} d\nu_A d\gamma_+ d\gamma_- d\theta dm(S_A),
$$

where $dm(S_A)$ is the measure of the non-compact symplectic group $Sp(2)$ and

$$
\sqrt{\det G} = \frac{\nu_A^3 \nu_\sigma^3 (\gamma_+^2 - \gamma_-^2)}{32\sqrt{2}(\gamma_+^2 - \nu_A \nu_\sigma)^{17/4}(\gamma_-^2 - \nu_A \nu_\sigma)^{17/4}}.
$$

It turns out that quantum correlations in the two-mode Gaussian states are most conveniently analyzed in the purity-seralian coordinates [34]. For the readers’ convenience, we recall the definitions,

$$
\mu_A/\sigma := \frac{1}{\sqrt{\det \Sigma_{A/\sigma}}},
$$

$$
\mu := \frac{1}{\sqrt{\det \Sigma}} = \frac{1}{\sqrt{(\gamma_+^2 - \nu_A \nu_\sigma)(\gamma_-^2 - \nu_A \nu_\sigma)}},
$$

$$
\Delta := \det \Sigma_A + \det \Sigma_\sigma + 2\det \Gamma = \nu_A^2 + \nu_\sigma^2 + 2\gamma_+ \gamma_-.
$$

(6)

As it is apparent from their definitions, these four new coordinates are local symplectic invariants. The inverse relations read

$$
\nu_A/\sigma = \frac{1}{\mu_A/\sigma},
$$

$$
\gamma_{\pm} = \frac{\sqrt{\mu_A \mu_\sigma}}{2} (\epsilon_{+} \pm \epsilon_{-}),
$$

where

$$
\epsilon_{\pm} := \sqrt{\left( \Delta - \frac{(\mu_A \pm \mu_\sigma)^2}{\mu_A^2 \mu_\sigma^2} \right)^2 - \frac{4}{\mu^2}}.
$$
Finally, we obtain the formula
\[ dV = \frac{\mu^{1/2}}{64\sqrt{2\mu_\Delta \mu_\sigma^2}} d\mu d\mu d\Delta d\vartheta d\mu(S_A). \]

The range of coordinates is determined by \( 0 \leq \mu_A / \sigma \leq 1, \quad e_1^2 \geq 0, \) and by the complete positivity condition \( [1] \), which is equivalent to
\[ 1 + \frac{1}{\mu^2} - \Delta \geq 0. \]

Combining all these requirements results in the following conditions for the two-mode Gaussian CK states that correspond to legitimate one-mode Gaussian channels \([34]\),
\[ 0 \leq \mu_A / \sigma \leq 1, \quad \mu_A \mu_\sigma \leq \mu \leq \frac{\mu_A \mu_\sigma}{\mu_A \mu_\sigma + |\mu_A - \mu_\sigma|}, \]
\[ \frac{2}{\mu} + \frac{(\mu_A - \mu_\sigma)^2}{\mu^2 \mu_\sigma} \leq \Delta \leq \min \left\{ \frac{2}{\mu} + \frac{(\mu_A + \mu_\sigma)^2}{\mu^2 \mu_\sigma}, 1 + \frac{1}{\mu^2} \right\}. \]

Now, we want to express conditions \([7]\) for the complete positivity of the channel in terms of the channel parameters directly. Interestingly, it turns out that these conditions depend only on the determinants of \( M \) and \( N \).

**Proposition 1.** Any one-mode Gaussian map \( \Lambda \) characterized by \((M,N,c)\) is completely positive if and only if
\[ \det N \geq (\det M - 1)^2. \]

**Proof.** For one-mode Gaussian channels, the complete positivity condition in eq. \([1]\) is equivalent to \([35]\)
\[ \det(\Sigma + i\Omega) \geq 0. \]

Note that \( \Sigma + i\Omega \) is a \( 4 \times 4 \) matrix, so its determinant can be calculated using the property
\[ \det(\Sigma + i\Omega) = \det D \det F. \]

In the above formula, \( D := \Sigma_\sigma + i\omega \) and \( F := \Sigma_A + i\omega - \Gamma D^{-1}\Gamma^T \) for \( \Sigma \) being a block matrix from eq. \([2]\). Simple calculations on \( 2 \times 2 \) matrices show that \( \det D = \nu_\sigma^2 - 1 \) and
\[ F = N + i\omega(1 - \det M), \]
where we implemented the formulas for \( \Sigma_A \) and \( \Gamma \) given on the r.h.s. of eq. \([2]\). As \( \det D \geq 0 \), condition \([6]\) can be rewritten into
\[ \det F = \det N - (1 - \det M)^2 \geq 0. \]

\[ \Box \]

**V. ENTANGLEMENT AND INCOMPATIBILITY BREAKING CHANNELS**

Let us consider a special class of quantum channels, for which \( \rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho) \) is separable for any (even entangled) state \( \rho \). These are known as the entanglement breaking channels and can always be written in the Holevo form \([8]\)
\[ \Lambda[\rho] = \sum_k \omega_k \Tr(F_k \rho), \]
where \( \omega_k \) are quantum states, and \( F_k \) form a POVM (positive operator-valued measure). For finite-dimensional quantum systems, it is straightforward to show that \( \Lambda \) is entanglement breaking if and only if \( \rho_{AB} \) is separable for \( \rho \), being a maximally entangled state. This notion can be extended to infinite-dimensional systems if one replaces the maximally entangled state with \( \rho_\Omega \) from Lemma \([4]\).

**Lemma 2.** A Gaussian channel \( \Lambda \) is entanglement breaking if and only if
\[ \rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho_\Omega), \]
with a marginal \( \sigma = \Tr_{A}\rho_{AB} \), is separable.

**Proof.** Note that if \( \Lambda \) is entanglement breaking, then trivially \( \rho_{AB} \) is separable. Now, assume that \( \rho_{AB} \) is separable; i.e., \( \rho_{AB} = \sum_k p_k \omega_k \otimes \beta_k \) with density operators \( \omega_k, \beta_k \) and a probability distribution \( p_k \). Then, we show that
\[ \Tr_A[\rho_{AB}(A \otimes \mathbb{I}_B)] = \sum_k p_k \beta_k \Tr(\omega_k A). \]

Recall that for an arbitrary \( \rho_{AB} \) with \( \sigma = \Tr_{A}\rho_{AB} \), one has \([23]\)
\[ \sqrt{\sigma} \Lambda^*[A] \sqrt{\sigma} = \Tr_A[\rho_{AB}(A \otimes \mathbb{I}_B)]^T, \]
where \( \Lambda^* \) is a map dual to the Gaussian channel \( \Lambda \). Therefore, eq. \([11]\) is equivalent to
\[ \sqrt{\sigma} \Lambda^*[A] \sqrt{\sigma} = \sum_k p_k \beta_k \Tr(\omega_k A). \]

One cannot simply invert \( \sqrt{\sigma} \), as the inverse of a full-rank state is unbounded. However, \( \sigma^{-1/2} p_k \beta_k \sigma^{-1/2} \) extends to a bounded operator \( F_k \) for which \( \sqrt{\sigma} F_k \sqrt{\sigma} = p_k \beta_k \), because \( \|\sqrt{\sigma} F_k \sqrt{\sigma} \|^2 \leq \sum_k \langle \psi | F_k \psi \rangle = \|\psi\|^2 \) and \( \|F_k\| = \|\sqrt{F_k}\|^2 \). Now, we can see that \( \Lambda^* \) is dual to the entanglement breaking channel of the form \([10]\). Indeed, the \( F_k \) define a POVM, as
\[ \sum_k \sqrt{\sigma} F_k \sqrt{\sigma} = \sum_k \sqrt{p_k \beta_k} = \Tr_A \rho_{AB} = \sigma. \]

\[ \Box \]
It was shown [35] that for two-mode Gaussian states, the Peres-Horodecki criterion [36, 37] is necessary and sufficient for separability. Namely, $\rho_{AB}$ is separable if and only if

$$\det(\Sigma_{\rho_{PT}} + i\Omega) \geq 0,$$

(12)

where $\Sigma_{\rho_{PT}} = \Theta \Sigma \Theta$ is the covariance matrix of the partially transposed state, and $\Theta = \text{diag}(-1, 1, 1, 1)$. In the seralian-purity coordinates, condition (12) reads

$$1 + \frac{1}{\mu^2} + \Delta - \frac{2}{\mu_A^2} - \frac{2}{\mu_S^2} \geq 0.$$

(13)

**Proposition 2.** Any one-mode Gaussian channel $\Lambda$ characterized by $(M, N, c)$ is entanglement breaking if and only if

$$\det N \geq (\det M + 1)^2.$$

(14)

The proof is analogous to the proof of Proposition 1.

Now, consider another class of quantum channels, for which $\rho_{AB} = (\Lambda \otimes I_B)(\rho_A)$ is non-steerable for any choice of $\rho_A$. These channels are the so-called incompatibility breaking channels [18]. It is known that a one-mode Gaussian channel is incompatibility breaking if and only if

$$\Sigma + i(0 \oplus \omega) \geq 0,$$

and, in terms of the purities,

$$\mu \leq \mu_A,$$

(15)

which are, in fact, just the conditions for the steerability of the CJ state.

**Proposition 3.** Any one-mode Gaussian channel $\Lambda$ characterized by $(M, N, c)$ is incompatibility breaking if and only if

$$\det N \geq \det M^2.$$

(16)

In Fig. 1, one can see the graphical representation of the conditions from Propositions 1–3 for the one-mode Gaussian channels. The complete positivity domain from ineq. (13) is gray, the entanglement breaking domain from ineq. (14) is double-hatched, and the incompatibility breaking domain from ineq. (16) is single-hatched. We see that the EB domain is contained within the ICB domain, and both of these domains are contained in the CP domain. Note that the inequalities presented in Propositions 1–3 are known [17, 18, 34]. Here, however, we were able to bring them to a unified concise form involving the simple channel parameters $\det M$ and $\det N$.

**VI. RELATIVE VOLUMES**

We analyze the geometry of the one-mode Gaussian channels by considering the manifold of the corresponding Gaussian CJ states. In order to do this, we make use of the local symplectic decomposition of the covariance matrix $\Sigma$. Recall that the local symplectic group $Sp(2)$ is non-compact [38], which means that the volume of two-mode Gaussian states, and hence the one-mode Gaussian channels, is not finite. The non-compactness emerges due to the possibility of unbounded squeezing. Regardless, we can compute the relative volumes of the quantities that are invariant with respect to the local symplectic transformations, such as entanglement. This quantity can be seen as the likelihood of encountering special classes of channels among all one-mode Gaussian channels.

To calculate the total volume of all one-mode Gaussian channels, we need to integrate the volume element in eq. (4) over the range of parameters determined by ineq. (7). Namely, one has

$$V_{GC} = C \int \int \int_{\mathcal{CP}} \frac{\mu_{11/2}}{4\sqrt{2}\mu_A^2\mu_S^2} d\mu_A d\mu_S d\Delta,$$

where $\mathcal{CP}$ is the region given by conditions (7).

We use the shorthand notation

$$C = \int_M dm(S_A) \int_0^{2\pi} d\theta$$

for the divergent part of the integral. Analogously, one obtains the volume of all entanglement breaking channels $V_{EBC}$ and incompatibility breaking channels $V_{ICBC}$:

$$V_{EBC} = C \int \int \int_{\mathcal{SEP}} \frac{\mu_{11/2}}{4\sqrt{2}\mu_A^2\mu_S^2} d\mu_A d\mu_S d\Delta,$$

$$V_{ICBC} = C \int \int \int_{\mathcal{NS}} \frac{\mu_{11/2}}{4\sqrt{2}\mu_A^2\mu_S^2} d\mu_A d\mu_S d\Delta.$$
The regions of integration $\mathcal{SEP}, \mathcal{NS}$ are given by conditions \((7, 13)\) and \((7, 15)\), respectively. Each of the above integrals can be solved analytically. The results are

\[ V_{GC} = C \frac{4 + \mu_\sigma^9/2(9\mu_\sigma^2 - 13)}{18018\sqrt{2}\mu_\sigma^3}, \]

\[ V_{EBC} = C \frac{\sqrt{\mu_\sigma}(1 - \mu_\sigma)^2(11 + 9\mu_\sigma)}{18018\sqrt{2}}, \]

\[ V_{ICBC} = C \frac{\sqrt{\mu_\sigma}(-13\mu_\sigma + 9\mu_\sigma^3 - 8\sqrt{2}(-11 + 7\mu_\sigma))}{18018\sqrt{2}}. \]

It is easy to see that the divergent part $C$ drops out when one considers a ratio of volumes. Interestingly, such ratio still depends on the choice of $\rho_0$ in the Choi-Jamiolkowski isomorphism through the marginal purity $\mu_\sigma$. The relative volumes of the entanglement and incompatibility breaking channels are presented in Fig. 2 as dashed and solid lines, respectively. Both curves grow monotonically as functions of $\mu_\sigma$. Two points, $\mu_\sigma = 0$ and $\mu_\sigma = 1$, have to be excluded from our considerations, even though the curves seem to behave well. The former point would correspond to the maximally entangled $\rho_0$, which is not a trace-class operator. The latter point does not satisfy the conditions in Lemma 1.

\[
\frac{\mu_A\mu_\sigma}{\sqrt{\mu_A^2 + \mu_\sigma^2 - \mu_A\mu_\sigma}} \leq \mu \leq \frac{\mu_A\mu_\sigma}{\mu_A + \mu_\sigma - \mu_A\mu_\sigma},
\]

then the associated two-mode Gaussian CJ states are separable or entangled, respectively. There also exists the so-called coexistence region, which corresponds to

\[
\frac{\mu_A\mu_\sigma}{\mu_A + \mu_\sigma - \mu_A\mu_\sigma} \leq \mu \leq \frac{\mu_A\mu_\sigma}{\sqrt{\mu_A^2 + \mu_\sigma^2 - \mu_A\mu_\sigma}}.
\]

For such values of $\mu_A/\mu_\sigma$ and $\mu$, it is impossible to distinguish between the separable and entangled states without the full knowledge about the system. Having the expressions for the volumes at hand, we can even compute the probability of finding an entangled CJ state in the coexistence region (see the density plots in Fig. 3).

In Fig. 3 we plot the separability (double-hatched), coexistence (single-hatched), and entanglement (shaded) regions for the Gaussian CJ states as functions of $\mu$ and $\mu_A$ for fixed $\mu_\sigma$. The unhatched white region is unphysical. Interestingly, knowing the value of seralian is not necessary to determine the steerability of Gaussian states. The shading indicates the relative conditional volume of entangled Gaussian CJ states. This is computed as a ratio of the volume of entangled states with respect to the volume of all states for fixed purities.

Discussing these figures in terms of the one-mode Gaussian quantum channels, we can simply read off the incompatibility and entanglement breaking regions in the parameter space. When $\mu \leq \mu_A$, the channel is incompatibility breaking. All the channels in the double-hatched region are entanglement breaking. For the single-hatched region, the color coding gives the probability for the channel not to be entanglement breaking.

There are no entanglement breaking channels in the dark blue non-hatched region, and there are not incompatibility breaking channels when $\mu > \mu_A$.

**VII. CONCLUSIONS AND OUTLOOK**

In this paper we calculate the relative volumes of entanglement and incompatibility breaking one-mode Gaussian quantum channels. We use the Choi-Jamiolkowski isomorphism to define the geometry in the space of Gaussian quantum channels. We explicitly determine the Hilbert-Schmidt line and volume elements for one-mode Gaussian channels, together with the regions corresponding to completely positive maps, as well as entanglement breaking and incompatibility breaking channels. Interestingly, these regions are completely characterized by inequalities involving only two channel parameters: $\det M$ and $\det N$. We find it useful to express the volume element in terms of symplectic invariants.

We base all of our calculations on the Hilbert-Schmidt distance. It would be interesting to compare our results with the volumes obtained from the Fisher-Rao and
channels. Having the general framework at hand, it is now straightforward to study the geometrical properties of subclasses of special interest, such as Weyl-covariant and quantum limited channels. Another open problem is finding relative volumes for canonical classes of channels, as introduced by Holevo in [31] and further explored in [39].

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**Appendix A: Hilbert-Schmidt line element**

Our computation follows the lines of [19, 23]. From our definition, the Hilbert-Schmidt line element reads

\[
\mathrm{ds}^2 = \text{Tr}[\rho(\Sigma + d\Sigma, \ell + d\ell)]^2 + \text{Tr}[\rho(\Sigma, \ell)]^2 - 2\text{Tr}[\rho(\Sigma, \ell)\rho(\Sigma + d\Sigma, \ell + d\ell)].
\]  
(A1)

Observe that

\[
\text{Tr}[\rho(\Sigma, \ell)\rho(\Sigma', \ell')] = \frac{1}{\sqrt{\det \frac{1}{2}(\Sigma + \Sigma')}} \times \exp \left[ -\frac{1}{2}(\ell - \ell')^T(\Sigma + \Sigma')^{-1}(\ell - \ell') \right],
\]  
(A2)

where we used the property of the trace

\[
\text{Tr}[D(-\xi)D(-\xi')] = \pi^n \delta^{2n}(\xi + \xi') \text{ and the } 2n\text{-dimensional Gaussian integral } [19]
\]

\[
\int_{\mathbb{R}^{2n}} \frac{d^{2n}\xi}{\pi^n} \exp \left[ -\frac{1}{2} \xi^T A \xi + B^T \xi \right] = \frac{2^n}{\sqrt{\det A}} \exp \left[ \frac{1}{2} B^T A^{-1} B \right].
\]  
(A3)

Hence, eq. (A1) simplifies to

\[
\mathrm{ds}^2 = \frac{1}{\sqrt{\det \Sigma}} + \frac{1}{\sqrt{\det(\Sigma + d\Sigma)}} - \frac{2}{\sqrt{\det \frac{1}{2}(2\Sigma + d\Sigma)}} \times \exp \left[ -\frac{1}{2} d\ell^T (2\Sigma + d\Sigma)^{-1} d\ell \right].
\]  
(A4)

By expanding the matrix \((2\Sigma + d\Sigma)^{-1} \approx \frac{1}{2}(1 - \frac{1}{2} d\ell^T \Sigma^{-1} d\ell)\) and the exponential

\[
\exp \left[ -\frac{1}{2} d\ell^T (2\Sigma + d\Sigma)^{-1} d\ell \right] \approx 1 - \frac{1}{2} d\ell^T (2\Sigma + d\Sigma)^{-1} d\ell \text{ and }
\]

\[
1 - \frac{1}{4} d\ell^T \Sigma^{-1} d\ell
\]  
(A5)
up to the quadratic terms in $d\Sigma$, $dl$, we find the final formula for the line element.

Appendix B: Hilbert-Schmidt volume element

The one-mode Gaussian channels correspond to the two-mode Gaussian CJ states with
\[
\Sigma = \left( \begin{array}{cc} N + M^T \Sigma_M M & M^T \Sigma_Z Z \Sigma_M \\ \Sigma_Z Z \Sigma_M M & \Sigma \end{array} \right).
\] (B1)

There always exists a symplectic transformation $S_A$ such that $N + M^T \Sigma_M M = \nu_A S_A^T S_A$. Hence, we can write
\[
\Sigma = (S_A \oplus S_M)^T \left( \begin{array}{cc} \nu_A \mathbb{I}_2 & S_A^{-1} M^T S_M^T Z \Sigma_M \\ Z \Sigma_M M S_A^{-1} \nu_M \mathbb{I}_2 \\ \end{array} \right) (S_A \oplus S_M).
\] (B2)

The off-diagonal block has the singular value decomposition $S_A^{-1} M^T S_M^T Z \Sigma_M = Q^T R$ with two orthogonal matrices $Q$, $R$ and $\Gamma := \text{diag}(\gamma_+, \gamma_-)$ [41]. Therefore, one has
\[
\Sigma = (S_A^T \oplus S_M^T R^T) W (S_A^T \oplus R S_M).
\] (B3)

with $S_A' := QS_A$ and
\[
W = \left( \begin{array}{cc} \nu_A \mathbb{I}_2 & \Gamma \\ \Gamma & \nu_M \mathbb{I}_2 \end{array} \right).
\] (B4)

The line element follows from eq. [3]. We use the fact that the covariance matrix has the structure $\Sigma = S^T W S$ with symplectic $S = \exp(H)$ that is generated by a traceless Hamiltonian matrix $H$. This gives
\[
d\Sigma = S^T (dW + dH^TW + WdH)S,
\] (B5)

and therefore
\[
ds^2 = \frac{1}{16\sqrt{\det W}} \left\{ 2\text{Tr}[W^{-1}(dW + dH^TW + RdW)]^2 + [\text{Tr}(W^{-1}dW)]^2 \right\}.
\] (B6)

In the last step, we perform the change of coordinates to the purity-seralian coordinates in eq. [6].

[1] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[2] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. G "ohne, Quantum Steering (2019), arXiv:1903.06663 [quant-ph].
[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[4] J. Bae and L.-C. Kwek, J. Phys. A: Math. Theor. 48, 083001 (2015).
[5] M. Razavi, A. Leverrier, X. Ma, B. Qi, and Z. Yuan, JOSA B 36, QKD1-QKD2 (2019).
[6] T. Yu and J. H. Eberly, Science 323, 598 (2009).
[7] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford 2003.
[8] M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. 15, 629–641 (2003).
[9] T. Heinosaari, J. Kiukas, D. Reitzner, and J. Schultz, J. Math. Phys. A: Math. Theor. 48, 435301 (2015).
[10] T. Heinosaari, T. Miyadera, and M. Ziman, J. Math. Phys. A: Math. Theor. 49, 123001 (2016).
[11] A. Ferrari, S. Olivares, and M. G. A. Paris, Gaussian states in continuous variable quantum information (2005), arXiv:quant-ph/0503237.
[12] S. Olivares, Eur. Phys. J. Spec. Top. 203, 3–24 (2012).
[13] A. S. Holevo, IEEE Trans. Info. Theor. 44, 269–273 (1998).
[14] G. Adesso, S. Ragy, and A. R. Lee, Open. Syst. Inf. Dyn. 21, 1440001 (2014).
[15] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
[16] C. Weedbrook, S. Pirandola, R. García-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621 (2012).
[37] P. Horodecki, Phys. Lett. A 232, 333–339 (1997).
[38] G. Adesso and F. Illuminati, J. Phys. A: Math. Theor. 40, 7821 (2007).
[39] F. Caruso, V. Giovannetti, and A. S. Holevo, New J. Phys. 8, 310 (2006).
[40] F. Byron and R. Fuller, Mathematics of Classical and Quantum Physics, Dover Publications, New York 2012.
[41] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).