Perturbed nonlinear systems control using extended robust right coprime factorisation

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ABSTRACT
In this paper, extended operator-based robust right coprime factorisation is investigated for dealing with a class of nonlinear systems with unknown bounded perturbations. First, a new kind of operator is introduced, by which operator-based right coprime factorisation approach is extended to consider the perturbed nonlinear systems. By regulating the exponent of the proposed operator, a broader class of nonlinear systems can be handled using the extended right coprime factorisation approach. Second, for guaranteeing robust stability of the perturbed nonlinear systems, feasible design schemes based on some sufficient conditions are discussed, which can reduce complicated calculation in control processing using a different unimodular operator. Finally, a simulation example is involved to illustrate the proposed design scheme for confirming the effectiveness of the proposed method.

1. Introduction

The original idea of coprime factorisation for systems can be traced to the work (Rosenbrock, 1970), which is proposed for dealing with finite-dimensional linear time-invariant systems. The approach of coprime factorisation provides a convenient framework for researching the input–output stability problem of nonlinear systems. Since 1980s, many insights and thoughts even for cases when linearity assumptions are relaxed are generated (Desoer & Kabuli, 1988; Hammer, 1984). The operator-based right coprime factorisation approach is extended form cases of linear systems to cases of nonlinear systems due to the usefulness in stabilisation for nonlinear systems, which has been consistently pursued with tremendous effort by many researchers in many fields (Chen & Han, 1998; Deng, Inoue, & Ishikawa, 2006; Paice & Moore, 1990).

In terms of the right coprime factorisation approach, roughly speaking, the main idea includes two steps: first, factorise a given plant operator $P$ as a composition of two different operators $N$ and $D$ such that $P = ND^{-1}$, where $N$ is a stable operator and $D$ is a stable and invertible operator; second, design other two stable operators $A$, $B$ satisfying the Bezout identity $AN + BD = M$, based on $N$ and $D$, where $M$ is an unimodular operator. Then, the given plant is said to have a right coprime factorisation (Chen & Han, 1998). After that, since 1998, the robust right coprime factorisation method has been attracting an increasing attention and many important results even for the real systems have been obtained (Bi & Deng, 2011; Bi, Deng, & Wen, 2011; Chen & Han, 1998; Deng, 2014; Deng & Bu, 2010; Deng, Wen, & Inoue, 2011; Deng et al., 2006). Meanwhile, aiming at the similar control objects, there are some other interesting and challenging methods (Chen, Zhai, & Fukuda, 2004; Deng & Kawashima, 2016; Katsurayama, Deng, & Jiang, 2016). Especially, there are some practical experiments which are nonlinear on robust stability respect, using the operator-based right coprime factorisation approach (Deng & Kawashima, 2016; Katsurayama et al., 2016). In these cases, the concrete design schemes are provided for dealing with the real systems, such as, in Deng and Kawashima (2016), combining passivity with isomorphism-factorisation, robust stability is obtained and for tracking the output, the adaptive control is proposed.

Although there are a vast amount of general results available in the literatures on existence, uniqueness, characterisation and construction of the right coprime factorisation approach for the nonlinear systems (Chen & Han, 1998; Deng, 2014; Deng & Bu, 2010; Deng et al., 2006; Hammer, 1984), which mainly focus on the in-depth direction, there is yet a promising direction which seems to be relatively ignored by researchers: the application range of the robust right coprime factorisation approach. As to the robust right coprime factorisation approach, the fundamental theory tool is the Lipschitz
operator due to the fact that a kind of nonlinear system can be regarded as Lipschitz operators, which provides a viewpoint to study the nonlinear system from operator theory. However, there are some nonlinear systems that cannot be satisfied with the Lipschitz requirement, leading to the right coprime factorisation approach unavailable. Thus, in this paper, a new kind of operator, $L_\alpha$ operator, is introduced to extend the application range of the right coprime factorisation approach for the nonlinear system. Proposed operator can contain a broader class of nonlinear systems by regulating the parameter. Combining with the proposed operator, a kind of nonlinear system with unknown bounded perturbations is investigated. After that, feasible design schemes are proposed based on a new unimodular operator in this paper, instead of the previous unimodular operator (Deng et al., 2006) for robust stability of the nonlinear system with unknown bounded perturbations. In the proposed design scheme, the complicated calculation of the previous unimodular operator and its inverse can be omitted, resulting in the design scheme more practical than the former method (Chen & Han, 1998; Deng et al., 2006).

The main concepts that motivate the present research are the observation that Lipschitz operator has a restriction on dealing with the nonlinear systems and the former methods in Chen and Han (1998) and Deng et al. (2006) for guaranteeing robust stability of the nonlinear system with unknown bounded perturbations. Specially, by introducing the $L_\alpha$ operator, robust right coprime factorisation is extended from the case where nonlinear systems are considered from the Lipschitz operator viewpoint to the case where nonlinear systems are considered from the $L_\alpha$ operator viewpoint. Then, based on the proposed operator, feasible design schemes are proposed for the nonlinear system with unknown bounded perturbations to guarantee robust stability. The main idea of the practical design scheme is to prove a stabilising operator to be an unimodular operator, so we can utilise the proposed unimodular operator to omit the complicated calculation in process of designing the nonlinear system with unknown bounded perturbations.

An outline on the rest of this paper is as follows. In Section 2, related knowledge on operator theory is reviewed and problem statements are given. In Section 3, main results are discussed. The definition of the $L_\alpha$ operator is proposed and discussed. Moreover, based on the proposed definition, fundamental theorems are obtained and the effectiveness on applying the proposed operator to the robust right coprime factorisation approach is verified. Then, the practical design scheme for the nonlinear system with unknown bounded perturbations is proposed. A simulation example is given in Section 4, illustrating the effectiveness of the proposed method. Finally, we draw conclusions for this paper in Section 5.

## 2. Mathematical preliminaries and problem statement

In this section, first, we review related definitions and notations on the robust right coprime factorisation approach (Deng, 2014) used in this paper. Then, problem statements are given.

### 2.1 Mathematical preliminaries

Let $\mathbf{U}$ and $\mathbf{Y}$ be two normed linear spaces, $\mathbf{U}_s$ and $\mathbf{Y}_s$ be subspaces of $\mathbf{U}$ and $\mathbf{Y}$, called the stable spaces of $\mathbf{U}$ and $\mathbf{Y}$, respectively, defined suitably under norm denoted as $\mathbf{U}_s = \{ u \in \mathbf{U} : \| u \| < \infty \}$ and $\mathbf{Y}_s = \{ y \in \mathbf{Y} : \| y \| < \infty \}$. Based on the spaces, an operator $Q : \mathbf{U} \rightarrow \mathbf{Y}$ is called to be stable if $Q(u_s) \subseteq Y_s$. Otherwise, if $Q$ maps someone input from $\mathbf{U}_s$ to the set $\mathbf{Y} \setminus \mathbf{Y}_s$, then $Q$ is said to be unstable.

Based on the stable definition mentioned above, the definition unimodular operators is recalled, which plays an important role in the operator theory.

**Definition 2.1:** Let $\mathcal{S}(\mathbf{U}, \mathbf{Y})$ be the set of stable operators from $\mathbf{U}$ to $\mathbf{Y}$. Then, $\mathcal{S}(\mathbf{U}, \mathbf{Y})$ contains a subset defined by

$$\mathcal{U}(\mathbf{U}, \mathbf{Y}) = \{ Q : Q \in \mathcal{S}(\mathbf{U}, \mathbf{Y}), Q \text{ is invertible with } Q^{-1} \in \mathcal{S}(\mathbf{U}, \mathbf{Y}) \}.$$  

Elements of $\mathcal{U}(\mathbf{U}, \mathbf{Y})$ are called unimodular operators.

A semi-norm on $\mathcal{F}(\mathbf{U}, \mathbf{Y})$, which is the family of operators from $\mathbf{U}$ to $\mathbf{Y}$ for the element $Q$, is defined as follows:

$$\| Q \| := \sup_{u_1, u_2 \in \mathbf{U}, u_1 \neq u_2} \frac{\| Q(u_1) - Q(u_2) \|_Y}{\| u_1 - u_2 \|_U},$$

if $\| Q \|$ is finite.

**Definition 2.2:** Let $\text{Lip}(\mathbf{U}, \mathbf{Y})$ be the subset of $\mathcal{F}(\mathbf{U}, \mathbf{Y})$ with each element $Q$ satisfying $\| Q \| < \infty$. Each $Q \in \text{Lip}(\mathbf{U}, \mathbf{Y})$ is said to be a Lipschitz operator mapping from $\mathbf{U}$ to $\mathbf{Y}$.

After that, since all of the control signals are time-limited in the real practice, the extended linear space defined below is more useful for considering the practical nonlinear system.

Let $F$ be the family of real-valued measurable functions defined on $[0, \infty)$, which is a linear space. As to each constant $T \in [0, \infty)$, assume $P_T$ be a projection mapping

from $F$ to another linear space, $F_T$, of measurable functions such that
\[ f_T(t) := P_T(f)(t) = \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases} \]
where $f_T(t) \in F_T$ is called the truncation of $f(t)$ with respect to $T$. Then, for any given Banach space (a complete vector space with a norm), $X_B$ of measurable functions, set
\[ X^c = \{ f \in X_B : \| f_T \| < \infty \text{ for all } T < \infty \}. \] (1)

Obviously, $X^c$ is a linear subspace of $X_B$. The space $X^c$ as defined (1) is called the extended linear space associated with the Banach space $X_B$.

**Definition 2.3:** In this paper, let $U$ and $Y$ denote the input and output spaces of a given plant operator $P$, respectively, i.e. $P : U \rightarrow Y$. Then, the given plant operator $P$ is said to have a right factorisation, if there exists a linear space $W$ and two stable operators $D : W \rightarrow U$ and $N : W \rightarrow Y$ such that $D$ is invertible and $P = ND^{-1}$ on $U$. And provided that there exist two stable operators $A : Y \rightarrow U$ and $B : U \rightarrow U$ and $B$ is invertible, satisfying the Bezout identity $AN + BD = M$, for $M \in U(W, U)$, then the right factorisation is said to be coprime, or $P$ is said to have a right coprime factorisation.

Note that $P$ is unstable and $(N, D, A, B)$ are to be determined in general. The right coprime factorisation for a nonlinear system can be described as in Figure 1, where $r$, $u$ and $y$ are reference input, control input and output, respectively, and $w$ belongs to the linear space $W$. Moreover, it is worth mentioning that the initial state is supposed to be considered, that is, $AN(w_0, t_0) + BD(w_0, t_0) = M(w_0, t_0)$ should be satisfied.

### 2.2 Problem statement

The nonlinear system with unknown bounded perturbations is shown in Figure 2, where the nominal system and the perturbed system are given $P$ and $\Delta P$, respectively. The overall system $\overline{P}$ is denoted as $\overline{P} = P + \Delta P$. Assume that the right factorisation of $P$ and $\overline{P}$ are denoted as $P = ND^{-1}$ and $\overline{P} = (N + \Delta N)D^{-1}$, respectively, where $N$ is stable; $D$ is stable and invertible, and $\Delta N$ is unknown bounded perturbations of the nonlinear system.

In Chen and Han (1998), robust stability of the nonlinear system with unknown bounded perturbations shown in Figure 2 can be guaranteed provided that
\[ A(N + \Delta N) - AN = 0. \] (2)

In Deng et al. (2006), authors proposed the following condition to guarantee robust stability of the perturbed nonlinear system:
\[ \| [A(N + \Delta N) - AN]M^{-1} \| < 1. \] (3)

Under Condition (3), the operator $A(N + \Delta N) + BD$ can be guaranteed to be unimodular.

First, when the right coprime factorisation approach is applied to consider a nonlinear system, the property of the nonlinear system would be explicit, verifying whether the nonlinear system is satisfied with the Lipschitz condition. If not, the right coprime factorisation approach cannot be used to establish the Bezout identity for guaranteeing stability of the nonlinear system.

Second, in Deng et al. (2006), the proposed sufficient condition is based on an inequality of Lipschitz norm, which extends the application range compared with the proposed condition of Chen and Han (1998). However, in some cases, there are some mathematical intractable difficulties resulting from calculating the unimodular operator $M$ and the inverse of $M$ in Deng et al. (2006), while we design controllers to satisfy the Inequation (3) proposed in Deng et al. (2006).

Thus, in this paper, in order to consider the issues mentioned above, a new kind of operator is proposed to extend application range of the right coprime factorisation approach. The merit of the proposed operator lies in that we can regulate the exponent of the operator to make the nonlinear system available for the right coprime factorisation approach. Moreover, aiming at issues of the
robust conditions mentioned above, a practical design scheme is proposed for stabilising the perturbed nonlinear systems using a new unimodular operator instead of $M$ in Deng et al. (2006), which relaxes the restriction of the previous methods, at least for the mathematical aspects of the perturbed nonlinear systems.

3. Main result

In this section, first, a new kind of operator is proposed. According to the proposed operator, fundamental theorems are discussed for applying the proposed operator to the right coprime factorisation approach. After that, sufficient conditions for robust stability of the nonlinear system with unknown bounded perturbations are given and discussed.

3.1 Extended right coprime factorisation

Definition 3.1: Let $U$ and $Y$ be two normed linear spaces over the field of the real number field $R$, endowed, respectively, with norms $\| \cdot \|_U$ and $\| \cdot \|_Y$. Let $T : U \rightarrow Y$ be an operator mapping from $U$ to $Y$. If the operator $T$ satisfies that

$$\| T \|_\alpha := \sup_{u_1, u_2 \in U, u_1 \neq u_2} \frac{\| T(u_1) - T(u_2) \|_Y}{\| u_1 - u_2 \|_U^\alpha},$$

(4)

is finite, for all $u_1, u_2 \in U$ and where the parameter $\alpha > 0$, then $T$ is said to be an $L_\alpha$ operator.

Note that, the proposed $L_\alpha$ has finite incremental gain to guarantee stability of nonlinear feedback systems (Chitour, Liu, & Sontag, 1995; Georgiou & Smith, 1990, 1997). A real plant is regarded as an $L_\alpha$ operator, whose domain and range are input space and output spaces of the real plant, respectively. The considered plant should be satisfied with the $L_\alpha$ condition such that the finite gain exists. In the proposed operators, the parameter $\alpha$ can be considered as one freedom for guaranteeing existence of finite gain issue by designing the suitable value in order to adjust input of the real plant.

The $\| \cdot \|_\alpha$ is a semi-norm for the $L_\alpha$ operator in the sense that $\| T \|_\alpha = 0$ does not necessarily imply $T = 0$. We can find a case where $T \neq 0$, but $\| T \|_\alpha$ is 0, for example, a constant operator which maps any input to a fixed output. And note that when $\alpha$ equals to 1, the $L_\alpha$ operator is reduced to the Lipschitz operator. The main merit of the $L_\alpha$ operator lies in that the $L_\alpha$ operator can include a broader class of nonlinear systems than Lipschitz operators. This fact provides a framework to deal with these nonlinear systems that cannot be coped with the Lipschitz operator based on the right coprime factorisation approach. In what follows, the necessary theoretical foundation is discussed for applying the $L_\alpha$ operator to the right coprime factorisation approach for the nonlinear system.

Corollary 3.1: The class $Lip_\alpha(U, Y)$ of the all $L_\alpha$ operators form $U$ to $Y$ defined in Definition 3.1 is a linear space over the real number field $R$. Moreover, if $T_1, T_2 \in Lip_\alpha(U, Y)$ and $a \in R$, then the following results are obtained:

1. $\| T_1 \|_\alpha = 0$ only if $T_1$ is a constant operator;
2. $\| T_1 + T_2 \|_\alpha \leq \| T_1 \|_\alpha + \| T_2 \|_\alpha$;
3. $\| aT_1 \|_\alpha \leq |a| \| T_1 \|_\alpha$.

Note that $Lip_\alpha(U, Y)$ be the subset of $N(U, Y)$ (the family of all nonlinear operators mapping from $U$ to $Y$) with each element $T$ with $\| T \|_\alpha < \infty$. It is clear that an element $T$ of $N(U, Y)$ is in $Lip_\alpha(U, Y)$ if and only if there is a number $L \geq 0$ such that $\| T(u_1) - T(u_2) \|_Y \leq L \| u_1 - u_2 \|_U$, for all $u_1, u_2 \in U$. The fact that $Lip_\alpha(U, Y)$ is broader than the set of the family of Lipschitz operators from $U$ to $Y$ can be shown in the following example.

An example to explain the proposed $L_\alpha$ operator

The nonlinear system $P(u)(t) = |u(t)|^\alpha$ defined on $[-1, 1]$. It is not a Lipschitz operator, because this nonlinear system becomes infinitely steep as $u$ approaches 0 as shown in Figure 3 based on the fact that the derivation of the nonlinear system tends to be infinite as $u$ approaches 0. That is, there is no constant finite number to satisfy the Lipschitz condition. However, it is a $L_\alpha$ operator by choosing the parameter such that $0 < \alpha < \frac{1}{2}$. Thus, the proposed $L_\alpha$ operator can deal with a broader class of nonlinear systems based on the proposed example than the Lipschitz operator.

Note that as to the design of proposed operator exponent, while considering the nonlinear system, there are
some points in many cases like the example shown, which leads to the norm tending to be infinite. For dealing with these systems, the exponent should be designed to satisfy the norm requirement according to the whole infinite points.

After that, the norm of $L_{\alpha}$ operator is considered, which is the fundamental premises using the definition of $L_{\alpha}$ operator for the robust right coprime factorisation approach. For any fixed $u_0 \in U$, define a norm for all $T \in Lip_\alpha(U, Y)$ as follows:

$$
\| T \|_{Lip}^\alpha := \| T(u_0) \| + \| T \|_{\alpha}
$$

where $\| T \|_{\alpha}$ is defined as (4). $\| T \|_{Lip}^\alpha$ is called the $L_{\alpha}$ norm of the $L_{\alpha}$ operator $T$.

A convenient choice for $u_0$ is $u_0 = 0$, where note that $T(0)$ is not zero in general if $T$ is nonlinear. To prove $\| T \|_{Lip}^\alpha$ to be a norm of the $L_{\alpha}$ operator, $T$, it amounts to showing that $\| T \|_{Lip}^\alpha = 0$ implies $T = 0$, where $0$ is the zero operator. This, however, is an immediate consequence of result (1) of Corollary 3.1.

**Lemma 3.1:** Let $U$, $Y$ be Banach spaces. The set $Lip_\alpha(U, Y)$ of all $L_{\alpha}$ operators from the normed space $U$ to $Y$ is a Banach space under the $L_{\alpha}$ norm.

**Proof:** Based on Corollary 3.1, $Lip_\alpha(U, Y)$ is a linear space since $Y$ is a linear space. Hence, it suffices to verify its completeness under the $L_{\alpha}$ norm.

Let $T_n$ be a Cauchy sequence in $Lip_\alpha(U, Y)$ such that $\| T_m - T_n \|_{Lip}^\alpha \to 0$ as $m, n \to \infty$. Then, for any $u \in U$,

$$
\begin{align*}
\| T_m(u) - T_n(u) \| &\leq \| (T_m - T_n)(u) - (T_m - T_n)(u_0) \| + \| T_m(u_0) - T_n(u_0) \| \\
&\leq \| (T_m - T_n) \|_{Lip}^\alpha \| u - u_0 \| + \| T_m(u_0) - T_n(u_0) \|
\end{align*}
$$

which shows that the sequence $T_n$ is, in fact, uniformly Cauchy on each bounded subset of $U$. Since $Y$ is complete, $T(u)$ exists and is unique. Moreover, since $T_n$ is a Cauchy sequence, $\lim_{n \to \infty} \| T_n \|_{Lip}^\alpha = c$, where $c$ is a constant number, so that

$$
\| T(u_1) - T(u_2) \| = \lim_{n \to \infty} \| T_n(u_1) - T_n(u_2) \| \\
\leq \lim_{n \to \infty} \| T_n \|_{Lip}^\alpha \| u_1 - u_2 \| \\
= c \| u_1 - u_2 \|
$$

for all $u_1, u_2 \in U$. This shows that $T \in Lip_\alpha(U, Y)$ with

$$
\| T \|_{Lip}^\alpha \leq c + \| T(u_0) \|.
$$

We finally verify that $\| T_n - T \|_{Lip}^\alpha \to 0$ as $n \to \infty$. Since the above equation also proves $\| T_n(u_0) - T(u_0) \| \to 0$ as $n \to \infty$, for $\epsilon > 0$, we can let $N$ be such that $\| T_m - T_n \|_{Lip}^\alpha \leq \frac{\epsilon}{2}$ and $\| (T_m - T_n)(u_0) \| \leq \frac{\epsilon}{2}$ for $m, n \geq N$. Then, for any $u_1, u_2 \in U$, it follows that

$$
\begin{align*}
\| (T - T_n)(u_1) - (T - T_n)(u_2) \| &= \lim_{n \to \infty} \| (T_m - T_n)(u_1) - (T_m - T_n)(u_2) \| \\
&\leq \lim_{n \to \infty} \| T_m - T_n \|_{Lip}^\alpha \| u_1 - u_2 \| \\
&\leq \frac{\epsilon}{2} \| u_1 - u_2 \|
\end{align*}
$$

So that $\| T_n - T \|_{Lip}^\alpha \leq \epsilon$ for $n \geq N$. This implies that $\| T_n - T \|_{Lip}^\alpha \leq \epsilon$ as $n \to \infty$, completing the proof of the theorem.

**Definition 3.2:** Let $U^c$ and $Y^c$ be two extended linear spaces, which are associated, respectively, with two given Banach spaces $U_B$ and $Y_B$ of measurable functions defined on the time domain $[0, \infty)$, where a Banach space is a complete vector space with a norm. A nonlinear operator $D : U^c \to Y^c$ is called a generalised $L_{\alpha}$ operator on $U^c$ if there exists a constant $L \geq 0$ such that

$$
\| [D(u_1)]_T - [D(u_2)]_T \|_{Y^c} \leq L \| [u_1]_T - [u_2]_T \|_{U^c}
$$

for all $u_1, u_2 \in U^c$ and for all $T \in [0, \infty)$.

Note that the least such constant $L$ is given by

$$
L := \sup_{T \in [0, \infty)} \sup_{u_1, u_2 \in U^c, u_1 \neq u_2} \frac{\| [D(u_1)]_T - [D(u_2)]_T \|_{Y^c}}{\| [u_1]_T - [u_2]_T \|_{U^c}} \quad (5)
$$

which is a semi-norm for the generalised $L_{\alpha}$ operator. The actual norm for a generalised $L_{\alpha}$ operator $D$ is given by

$$
\| D \|_{Glip} = \| D(u_0) \|_{Y^c} + \sup_{T \in [0, \infty)} \sup_{u_1, u_2 \in U^c, u_1 \neq u_2} \frac{\| [D(u_1)]_T - [D(u_2)]_T \|_{Y^c}}{\| [u_1]_T - [u_2]_T \|_{U^c, \alpha}}
$$

for any fixed $u_0 \in U^c$.

For simplicity, throughout the following section, by an $L_{\alpha}$ operator, we always mean one defined in this generalised sense. The reason for considering the extended linear space is that all control signals in real application are time-limited, but in the control processing, we sometimes do not know when the processing will stop. Hence, because of the finite time duration of practice, the function $f(t) = e^t + t^2$, $t \geq 0$ and the like should be considered under the underlying spaces.

Note that from the given example of the $L_{\alpha}$ operator, we can find the difference between the $L_{\alpha}$ operator and the Lipschitz operator.

After introducing the definition of the generalised $L_{\alpha}$ operator of the extended linear space, the robust right
The robust factorisation approach will be discussed under the framework of the proposed operator to guarantee the perturbed Bezout identity for stabilising the nonlinear system with unknown bounded perturbations. One of the main motivations of this paper is to extend the application range of the robust right coprime factorisation approach based on the proposed operator. In the following subsection, robustness of nonlinear systems with unknown bounded perturbations is considered. Based on the \( L_\alpha \) operator, practical design schemes for dealing with the unknown bounded perturbations are proposed to guarantee robust stability of the perturbed nonlinear system.

### 3.2 Robust stability design scheme

The following preparatory results are used throughout this subsection for developing the sufficient conditions for guaranteeing robust stability for the nonlinear system with unknown bounded perturbations. Throughout this section, the whole operators are considered in the context of the definition of the \( L_\alpha \) operator.

**Lemma 3.2:** Provided that a \( L_\alpha \) operator \( H \in \text{Lip}_\alpha(U_\alpha, U_\alpha) \), where \( U_\alpha \) is the stable space of \( U_B \) which is a Banach space, is satisfied with \( \| H \|_{Lip} < 1 \), then, \( I - H \) is invertible, in \( \text{Lip}_\alpha(U_\alpha, U_\alpha) \)

\[
\| (I - H)^{-1} \|_{Lip} \leq (1 - \| H \|_{Lip})^{-1}.
\]

**Proof:** In fact, for each \( u_1, u_2 \in U_\alpha \)

\[
\| (I - H)u_1 - (I - H)u_2 \| \geq \| u_1 - u_2 \| - \| Hu_1 - Hu_2 \| \\
\geq (1 - \| H \|_{Lip}) \| u_1 - u_2 \|.
\]

Thus, the operator \( I - H \) is injective. After that, the fact that \( I - H \) is surjective is verified as follows.

Define that \( K_0 := I \) and \( K_n := I + HK_{n-1} \), \( \forall n = 1, 2, \ldots \), for each \( u \in U_B \)

\[
\| K_{n+1}(u) - K_n(u) \| \leq \left( \| H \|_{Lip}^n \right) \| H(u) \|, \quad n = 1, 2, \ldots
\]

Then, for any positive integer \( m \), obtain

\[
\| K_{n+m}(u) - K_n(u) \| = \| \sum_{k=0}^{m-1} (K_{n+k+1}(u) - K_{n+k}(u)) \|
\leq \sum_{k=0}^{m-1} \left( \| H \|_{Lip}^{n+k} \right) \| H(u) \|
\leq \frac{\left( \| H \|_{Lip}^n \| H(u) \| \right)}{1 - \| H \|_{Lip}^n}.
\]

Since \( \| H \|_{Lip} < 1 \) and \( U_B \) is a Banach space, then,

\[
S(u) = \lim_{n \to \infty} K_n(u)
\]

exists and

\[
\| S(u) - K_n(u) \| = \lim_{n \to \infty} \| K_{n+m}(u) - K_n(u) \|
\leq \frac{\left( \| H \|_{Lip}^n \| H(u) \| \right)}{1 - \| H \|_{Lip}^n}.
\]

Since \( H \) is a \( L_\alpha \) operator and thus is continuous, we have

\[
S(u) = \lim_{n \to \infty} K_n(u) = \lim_{n \to \infty} (I + HK_{n-1})u = u + HSu
\]

that is,

\[
S = I + HS,
\]

namely, \((I - H)S = I\), which implies that \( I - J \) is surjective in \( \text{Lip}_\alpha(U_\alpha, U_\alpha) \). Then, for \( u_1, u_2 \in R(I - H) \),

\[
\| (I - H)^{-1}u_1 - (I - H)^{-1}u_2 \| \\
\leq (1 - \| H \|_{Lip}^n)^{-1} \| u_1 - u_2 \|.
\]

Thus, we can get the conclusion, \( I - H \) is invertible and

\[
\| (I - H)^{-1} \|_{Lip} \leq (1 - \| H \|_{Lip}^n)^{-1}.
\]

The proof of this lemma is completed.

**Lemma 3.3:** The nonlinear system shown in Figure 1 possesses a right coprime factorisation if and only if the composite operator \((I + AB^{-1}) : R(B) \to \text{U}^c\) is injective and its inverse is causal, and all the operators \( A, B, D, N, B^{-1}, \) and \((I + AB^{-1})\) are causally stable.

The proof of Lemma 3.3 is attached in the Appendix. Lemma 3.3 can show that the operator \( B \) is unimodular. In this paper, we will consider the unimodular \( B \) instead of operator \( M \) of Deng et al. (2006) to study the nonlinear system with unknown bound perturbations.

**Theorem 3.1:** Suppose that \( U_B \) and \( Y_B \) be Banach spaces. Let \( G, R \in \text{Lip}_\alpha(U_\alpha, Y_\alpha) \), where \( U_\alpha, Y_\alpha \) are the stable spaces of \( U_B, Y_B \), respectively, such that \( G \) is invertible in \( \text{Lip}(U_\alpha, Y_\alpha) \) and satisfied with

\[
\| G - R \|_{Lip} < 1
\]

Then, \( R \) is invertible in \( \text{Lip}_\alpha(U_\alpha, Y_\alpha) \) with

\[
\| R^{-1} \|_{Lip} \leq \left( \frac{\| G^{-1} \|_{Lip}}{\| G^{-1} \|_{Lip}^2} \right) \| G^{-1}u \| \\
+ \frac{\| G^{-1} \|_{Lip}^2}{1 - \| G - R \|_{Lip}^2} \| G^{-1} \|_{Lip}.
\]

(9)
for any \( u_0 \in U_B \).

**Proof:** By Lemma 3.2, show that \( G^{-1}R \) is invertible in \( L_{\alpha}(U, Y) \), considering
\[
\| I - RG^{-1}\|_{Lp} = \| G - R \|_{Lp} \alpha G^{-1} \|_{Lp} \alpha < 1.
\]

Also, from (6),
\[
\| (RG^{-1})^{-1}\|_{Lp} \leq \frac{1}{1 - \| I - RG^{-1}\|_{Lp} \alpha} \leq \frac{1}{1 - \| G - R \|_{Lp} \alpha G^{-1} \|_{Lp} \alpha}.
\]

Since \( R = (RG^{-1})G \), we see that \( R \) has an inverse in \( L_{\alpha}(U, Y) \), namely,
\[
R^{-1} = G^{-1}(RG^{-1})^{-1}.
\]

Hence,
\[
\| R^{-1}\|_{Lp} \leq G^{-1} \| G^{-1}\|_{Lp} \alpha (RG^{-1})^{-1} \|_{Lp} \alpha.
\]

The estimate (9) follows from the above inequation, (10) and the definition of norm for the generalised \( L_{\alpha} \) operator.

Note that Theorem 3.1 provides a condition on how to guarantee an operator be invertible. This condition in Theorem 3.1 is sufficient, not necessary. The reason for this result lies in that Theorem 3.1 is obtained based on Lemma 3.2, but in the proof of Lemma 3.2, for proving the surjective property of the operator \( I - H \), the condition \( H \|_{Lp} < 1 \) is sufficient, not necessary.

After the preparatory work, we give the design scheme for the nonlinear system with unknown bounded perturbations.

**Theorem 3.2:** In terms of the nonlinear feedback system with unknown bounded perturbations in Figure 2, provided that the following condition is guaranteed,
\[
\| (A(N + \Delta N) - B) \|_{Lp} \alpha B^{-1} \|_{Lp} < 1
\]
then, \( A(N + \Delta N) \) is an unimodular operator.

**Proof:** The fact that \( B \) is unimodular implies that \( B \) is invertible and \( B^{-1} \) is also stable.

Therefore, according to Theorem 3.1, we can obtain \( A(N + \Delta N) \) is invertible.
\[
\| A(N + \Delta N) \|_{Lp} \leq B^{-1} \|_{Lp} \| A(N + \Delta N) \|_{Lp} \alpha (x_0) \]
\[
+ \frac{\| B^{-1} \|_{Lp} \}}{1 - \| B^{-1} - A(N + \Delta N) \|_{Lp} \alpha B^{-1} \|_{Lp} \}
\]

Since
\[
A(N + \Delta N) = B - (B - A(N + \Delta N))
\]
\[
= [I - (B - A(N + \Delta N))B^{-1}]B.
\]

Thus, \( I - [B - A(N + \Delta N)]B^{-1} \) is proved to be invertible. Considering the inverse of \( I - (B - A(N + \Delta N))B^{-1} \) is stable, the obtained inverse of \( A(N + \Delta N) \) is stable. The proof of the theorem is completed.

Based on Theorem 3.2, we will prove the perturbed Bezout identity is unimodular as the following theorem. Therefore, the proposed design scheme can guarantee the nonlinear system with unknown bounded perturbations to be stable.

**Theorem 3.3:** For the perturbed nonlinear system as shown in Figure 2, if the following condition
\[
\| BD \|_{Lp} \| [A(N + \Delta N)]^{-1}\|_{Lp} < 1
\]

is satisfied, then the nonlinear system with unknown bounded perturbations is robust stable, i.e. \( A(N + \Delta N) + BD \) is unimodular.

**Proof:** Since
\[
\| BD \|_{Lp} \| [A(N + \Delta N)]^{-1}\|_{Lp} < 1
\]

Thus,
\[
\| [BD + A(N + \Delta N)] - A(N + \Delta N) \|_{Lp} \| [A(N + \Delta N)]^{-1}\|_{Lp} < 1
\]

It follows that \( A(N + \Delta N) + BD \) is unimodular by Theorem 3.2. Thus, the nonlinear system with unknown bounded perturbations is satisfied with robust right coprime factorisation, which results in that the overall system is stable.

Note that Theorems 3.2 and 3.3 are both sufficient conditions on guaranteeing robust stability of the nonlinear system with unknown bounded perturbations. From the viewpoint of guaranteeing the unimodular property, the proposed conditions can guarantee it, but the proposed conditions are not necessary because of conservativeness of Theorem 3.1 and the nonlinearity property of the considered operator.

The merits of this paper lie in two aspects: (1) based on the proposed \( L_{\alpha} \) operator, the application range of the robust right coprime factorisation approach is extended, which is verified by the proposed example; (2) for the broader class of nonlinear systems, sufficient conditions for robust stability is obtained by the proposed design scheme. From Theorems 3.2 and 3.3, the proposed design scheme do not employ the unimodular operator \( M^{−1} \).
in Deng et al. (2006), by which the complicated work of calculating the Bezout identity and the inverse of the unimodular operator \( M \) is omitted. Meanwhile, using the operator \( BD \) directly and evaluating the output signals of \( BD(w(t)) \) and \( [AN + \Delta N](w) \)^{-1}(t), we can verify robust stability by using signal \( w \) only. In real cases, the meaning is that the noised sensor signal from \( A(y)(t) \) is avoided.

4. A numerical example

In this section, a numerical example is given to show the effectiveness of the proposed method. Consider a nonlinear feedback system shown as in Figure 4, where reference input, control input and plant output are \( r, u \) and \( y \), respectively. We assume, in this nonlinear feedback system, that \( X_B = L_\infty \) is the standard Banach space of real-valued measurable functions defined on \([0, \infty)\), with the associated extended linear space \( X^* = L_{\infty}^* \).

Suppose that the plant operator \( P \) is given by the following unstable, \( L_\infty \) and time-varying nonlinear operator, where \( \alpha = \frac{1}{4} \):

\[
P(u(t)) = \int_0^t [u(\tau)]\frac{1}{4}d\tau + e^{\frac{3t}{4}}[u(t)]^{\frac{3}{4}}
\]

Based on the proposed plant, the operators \( D, N \) are given as follows:

\[
N(w(t)) = \int_0^t e^{-\frac{3t}{4}[w(\tau)]^\frac{3}{4}}d\tau + [w(t)]^\frac{3}{4}
\]

\[
D(w(t)) = e^{-\frac{3}{4}[w(t)]^\frac{3}{4}}
\]

The stability in terms of \( D, N \) is verified easily. And we can get the inverse operator of \( D \) is unstable from \( L_\infty \) to \( L_\infty \).

Next step for establishing a Bezout identity, we pick a stable controller \( A \) such that the \( I - AN \) is invertible as follows:

\[
A(y(t)) = (e^{-\frac{3t}{4}} - 1) \left[ \int_0^t [u(\tau)]\frac{1}{4}d\tau - y(t) \right]^\frac{3}{4}
\]

Then, we have

\[
AN(w(t)) = (e^{-\frac{3t}{4}} - 1) \int_0^t [u(\tau)]\frac{1}{4}d\tau - \left( e^{-\frac{3t}{4}} - 1 \right) \left( \int_0^t e^{-\frac{3t}{4}[w(\tau)]^\frac{3}{4}}d\tau - [w(t)]^\frac{3}{4} \right)
\]

Therefore, the controller \( B \) is provided based on the proposed controllers \( A \),

\[
B(u(t)) = (I - AN)D^{-1}(u(t)) = [u(t)]^{\frac{3}{4}}
\]

Finally, it can be verified that \( A \) and \( B \) satisfy the Bezout identity. Indeed, we have

\[
(AN + BD)(w(t)) = I(w(t))
\] (13)

According to the above analysis, the proposed unstable nonlinear system, \( P \), is stable by the proposed design scheme. Note that in order to realise the Bezout identity, the controllers are chosen from the simplicity viewpoint. The controller \( A \) is chosen from the following set: \( A = \{ A \in Lip_a(X_B, X_B) : (I - AN)D^{-1} \in Lip_a(X_B, X_B) \} \) in this case, the controller \( A \) will stabilise the unstable operator \( D^{-1} \). There is a cancellation of unstable factor between \( D^{-1} \) and \( I - AN \), hence, the stability of the controller \( B \) is guaranteed. Aside from that, there exist a great number of choices only if the designed controllers are satisfied with the Bezout identity condition.

After that, the case of the nonlinear system with unknown bounded perturbations is proposed to confirm the effectiveness of the robust design scheme.

After that, the perturbations and right factorisation of the overall plant are given as follows, where the perturbations \( \delta(t) \) is chosen as \( \delta(t) = 0.5e^{\frac{3t}{4}} \) for confirming the effectiveness of the proposed design scheme.

\[
(P + \Delta P)(u(t)) = \delta(t) \left( \int_0^t [u(\tau)]\frac{1}{4}d\tau + e^{-\frac{3t}{4}}[u(t)]^\frac{3}{4} \right)
\]

\[
(N + \Delta N)(w(t)) = \delta(t) \int_0^t e^{-\frac{3t}{4}[w(\tau)]^\frac{3}{4}}d\tau + \delta(t)[w(t)]^\frac{3}{4}
\]

\[
D(w(t)) = e^{-\frac{3}{4}[w(t)]^\frac{3}{4}}
\]

Based on the above perturbed system, Conditions (11) and (12) are verified as shown in Figures 6 and 7 to make the design scheme available, where the reference input is chosen as \( r(t) = 1.5(1 + e^{\frac{3t}{4}}) \) shown in Figure 5. As shown in Figures 6 and 7, the bounded perturbations is satisfied with the conditions for guaranteeing robust stability. In other words, calculation of the left item of Conditions (11) and (12) is less than 1. In this paper, under
the proposed condition for robust stability, the perturbed system does not alter the relationship of the Bezout identity, because $A(N + \Delta N) + BD$ is still unimodular for the same controllers $A$ and $B$ as shown in Theorems 3.2 and 3.3.

Next, in order to show the effectiveness of the proposed design scheme, the simulation result of the plant output of the nonlinear system with bounded perturbations is given in Figure 8. Thus, based on the simulation results, robust stability of the nonlinear system with unknown bounded perturbations is obtained by the proposed design scheme.

5. Conclusion

In this paper, extended right coprime factorisation is discussed based on the proposed $L_\alpha$ operator and robust control for the nonlinear system with unknown bounded perturbations is considered using a new unimodular operator for guaranteeing robust stability. The application range of the robust right coprime factorisation approach was extended by the proposed operator. Feasible design schemes were proposed for avoiding the difficulties in calculation based on proposed operators, which meant that robust stability of the perturbed nonlinear system can be guaranteed based on the proposed unimodular operator $B$. Finally, the effectiveness of the proposed design scheme was confirmed by a simulation example. It is difficult to find a systematic scheme for computing the appropriate coprime factorisation. Recently, Deng and Bu (2010) and Tao and Deng (2016) concerned the appropriate coprime factorisation. Therefore, we can combine the methods with the proposed method to obtain appropriate coprime factorisation. However, the detail research will be the future work.
Disclosure statement

No potential conflict of interest was reported by the authors.

References

Bi, S., & Deng, M. (2011). Operator-based robust control design for nonlinear plants with perturbation. International Journal of Control, 84, 815–821.

Bi, S., Deng, M., & Wen, S. (2011). Operator-based output tracking control for non-linear uncertain systems with unknown time-varying delays. IET Control Theory & Applications, 5, 693–699.

Chen, G., & Han, Z. (1998). Robust right factorization and robust stabilization of nonlinear feedback control systems. IEEE Transactions on Automatic Control, 43, 1505–1510.

Chen, X., Zhai, G., & Fukuda, T. (2004). An approximate inverse system for nonminimum-phase systems and its application to disturbance observer. Systems & Control Letters, 52, 193–207.

Chitour, Y., Liu, W., & Sontag, E. (1995). On the continuity and incremental-gain process of certain saturated linear feedback loops. International Journal of Robust and Nonlinear Control, 5, 413–440.

Deng, M. (2014). Operator-based nonlinear control systems design and applications. Hoboken, NJ: Wiley-IEEE Press.

Deng, M., & Bu, N. (2010). Isomorphism-based robust right coprime factorisation of non-linear unstable plants with perturbations. IET Control Theory & Applications, 4, 2381–2390.

Deng, M., Inoue, A., & Ishikawa, K. (2006). Operator-based nonlinear feedback control design using robust right coprime factorization. IEEE Transactions on Automatic Control, 51, 645–648.

Deng, M., & Kawashima, T. (2016). Adaptive nonlinear sensorless control for an uncertain miniature pneumatic curling rubber actuator using passivity and robust right coprime factorization. IEEE Transactions on Control Systems Technology, 24, 318–324.

Desoer, C.A., & Kabuli, M.G. (1988). Right factorization of a class of time-varying nonlinear systems. IEEE Transactions on Automatic Control, 33, 755–757.

Georgiou, T., & Smith, M. (1990). Optimal robustness in the gap metric. IEEE Transactions on Automatic Control, 35, 673–686.

Georgiou, T., & Smith, M. (1997). Robustness analysis of nonlinear feedback systems: An input–output approach. IEEE Transactions on Automatic Control, 42, 1200–1221.

Hammer, J. (1984). Nonlinear systems: Stability and rationality. International Journal of Control, 40, 1–35.

Katsurayama, Y., Deng, M., & Jiang, C. (2017). Operator based experimental studies on nonlinear vibration control for aircraft vertical tail with considering low order modes.

Transactions of the Institute of Measurement and Control. doi:10.1177/0142331215592063

Paice, A.D.B., & Moore, J.B. (1990). Robust stabilisation of nonlinear plants via left coprime factorizations. Systems & Control Letters, 15, 125–135.

Rosenbrock, H.H. (1970). State-space and multivariable theory. New York, NY: Wiley.

Tao, F., and Deng, M. (2016). Adjoint operator-based isomorphism realization and control design for nonlinear systems. IET Control Theory & Applications, 10, 919–925.

Appendix

Proof of Lemma 3.3: First, observe that if a nonlinear system has a right coprime factorisation, then there exist two causal operators \( N : U \to U^c \) and \( D : U \to U^c \) with a causal inverse \( D^{-1} : U^c \to U \) such that \( ND^{-1} = P \) and \( AN + BD = I \). Since \( B : U \to U \) is one-to-one, with domain \( X \), and is onto \( \mathcal{R}(B) \), we see that \( \mathcal{R}(B^{-1}) = X \subset \mathcal{D}(D^{-1}) \). On the other hand, observe that \( \mathcal{R}(D) = U^c \).

It, hence, follows that

\[
I + APB^{-1} = I + AND^{-1}B^{-1} = [BD + AN]D^{-1}B^{-1} = MD^{-1}B^{-1},
\]

which implies that the operator \((I + APB^{-1}) : \mathcal{R}(B) \to U\) is causal, one-to-one and onto, and hence is causally invertible, with the inverse equal to \( BD : U \to \mathcal{R}(B) \).

Conversely, if the operator \((I + APB^{-1}) : \mathcal{R}(B) \to U\) is causally invertible, then we can define

\[
D = B^{-1}(I + APB^{-1})^{-1}M.
\]

Since \((I + APB^{-1})^{-1} : U \to \mathcal{R}(B)\) is onto, we have \(\mathcal{R}((I + APB^{-1})^{-1}) = \mathcal{R}(B) = \mathcal{D}(B^{-1})\). On the other hand, observe that \(\mathcal{R}(B^{-1}) = U\). Hence, the operator \(D\) so defined is causal, one-to-one and onto. It follows that

\[
D = [M^{-1}(I + APB^{-1})B]^{-1} = [M^{-1}(B + AP)]^{-1}.
\]

Consequently, \(D^{-1} = M^{-1}(AP + B) : U^c \to U\) exists and is causal. Therefore, if we define \(N = PD\), then \(N\) is causal, and, moreover, we obtain both \(ND^{-1} = PDD^{-1} = P\) and

\[
AN + BD = (AND^{-1} + B)D = (AP + B)D = MD^{-1}D = M.
\]

This, along with the stability conditions imposing on the related operators, completes the proof of the lemma.