Area-preserving Structure and Anomalies in 1+1-dimensional Quantum Gravity

G. Amelino-Camelia, D. Bak, and D. Seminara

Center for Theoretical Physics
Laboratory for Nuclear Science and Department of Physics
Building 6, Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

ABSTRACT

We investigate the gauge-independent Hamiltonian formulation and the anomalous Ward identities of a matter-induced 1+1-dimensional gravity theory invariant under Weyl transformations and area-preserving diffeomorphisms, and compare the results to the ones for the conventional diffeomorphism-invariant theory. We find that, in spite of several technical differences encountered in the analysis, the two theories are essentially equivalent.

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I Introduction

It is well known[1] that in the quantization of the 1+1-dimensional matter-gravity field theory with action

\[ \mathcal{I}(X, g) = \frac{1}{2} \int d^2 \xi \sqrt{-g} g^{\mu \nu} \partial_\mu X_A \partial_\nu X^A , \]  

(1)

where \( g^{\mu \nu} \) is a metric tensor with signature (1,-1), \( A = 1, 2, ..., d \), and \( X \) is a \( d \)-component massless scalar field, one encounters anomalies that break part of the symmetry of the classical theory, which, as seen from \( \mathcal{I}(X, g) \), has Weyl and diffeomorphism invariance.

In the conventional quantization approach[1-8] diffeomorphism invariance is preserved, while renouncing Weyl invariance. In the path integral formulation this can be accomplished by choosing the following measures for the functional integrations[3, 4]

\[ \int \mathcal{D} \delta X \exp \left( i \int d^2 \xi \sqrt{-g} \delta X_A \delta X^A \right) = 1 , \]  

(2)

\[ \int \mathcal{D} \delta g \exp \left( i \int d^2 \xi \sqrt{-g} g^{\mu \sigma} g^{\nu \rho} \delta g_{\mu \nu} \delta g_{\sigma \rho} \right) = 1 , \]  

(3)

which are diffeomorphism invariant, but are not Weyl-invariant.

Integrating out the matter degrees of freedom using the measure (2) one obtains an effective pure gravity theory with action

\[ \Gamma_D(g) = -\Lambda_D \int d^2 \xi \sqrt{-g(\xi)} \right. + \frac{d}{96 \pi} \int d^2 \xi_1 d^2 \xi_2 \sqrt{-g(\xi_1)} R(g(\xi_1)) \Box^{-1}(\xi_1, \xi_2) \sqrt{-g(\xi_2)} R(g(\xi_2)) , \]  

(4)

where \( \Box^{-1} \) is the inverse of the Laplace-Beltrami operator, and the index \( D \) stands for “diffeomorphism invariant approach”. The cosmological term \( \Lambda_D \int d^2 \xi \sqrt{-g} \) is induced by renormalization[1].

As it should be expected, \( \Gamma_D(g) \) is diffeomorphism-invariant, but it is not Weyl-invariant; in fact, the energy-momentum tensor

\[ T^{\mu \nu}_D \equiv \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_D(g)}{\delta g^{\mu \nu}} . \]  

(5)

is covariantly conserved, but possesses non-vanishing trace

\[ \nabla_\mu (g^{\mu \nu} T^{\nu \alpha}_D) = 0 , \quad g^{\mu \nu} T^{\nu \alpha}_D = 2\Lambda_D + \frac{d}{24 \pi} R(g) . \]  

(6)

Recently, an alternative approach to the quantization of the classical theory of action (4) has been considered[9, 10], in which part of the diffeomorphism invariance is sacrificed in order to obtain a Weyl-invariant theory. This alternative approach is motivated by the observation that Eq.(4) depends on the metric only through the Weyl-invariant combination

\[ \gamma^{\mu \nu} \equiv \sqrt{-g} g^{\mu \nu} . \]  

(7)

In the conventional diffeomorphism-invariant quantization approach one is forced to introduce, through the path integral measures, a field describing the determinant of the metric \( g \), even though (4) is independent of this determinant. Instead, the Weyl-invariant approach does not lead to the introduction of this additional field and might therefore be a more natural[10] candidate as a quantized version of the original classical theory.
In the path integral formulation of this alternative Weyl-invariant quantization one can choose the following measures for the functional integrations

\[
\int \mathcal{D}\delta X \exp \left( i \int d^2 \xi \, \delta X_A \delta X^A \right) = 1 , \tag{8}
\]

\[
\int \mathcal{D}\delta \gamma \exp \left( i \int d^2 \xi \, \gamma^{\mu\sigma} \gamma^{\nu\rho} \delta \gamma_{\mu\nu} \delta \gamma_{\sigma\rho} \right) = 1 , \tag{9}
\]

which depend on \(g\) only through \(\gamma\), and can be obtained from (2) and (3) with the substitution \(g \to \gamma\) (also observing that \(\det \gamma = -1\)).

The measures (8) and (9) are evidently Weyl invariant, and are also invariant under the subgroup of the diffeomorphism group given by coordinate redefinitions with unit Jacobian, i.e. infinitesimally \(\xi^\mu \to \xi^\mu + f^\mu\) with \(\partial_\mu f^\mu = 0\), which are called area-preserving diffeomorphisms because they preserve local area \(\sqrt{-g} d^2 \xi\) on spaces where \(\sqrt{-g}\) is constant. However, (8) and (9) are not invariant under diffeomorphisms with non-unit Jacobian, and therefore, as already pointed out in [10], this Weyl-invariant approach has as many symmetries as the diffeomorphism-invariant approach.

The effective quantum gravity theory that follows from the action (1) upon integration of \(X\) using the measure (8) is described by the action

\[
\Gamma^W(\gamma) = -\Lambda^W \left( \int d^2 \xi \right) + \frac{d}{96\pi} \int d^2 \xi_1 d^2 \xi_2 \, R(\gamma(\xi_1)) \, \square^{-1}(\xi_1, \xi_2) \, R(\gamma(\xi_2)) , \tag{10}
\]

which is Weyl invariant, but is not diffeomorphism invariant; in fact

\[
T^W_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma^W(\gamma)}{\delta g^{\mu\nu}} = 2 \frac{\delta \Gamma^W(\gamma)}{\delta \gamma_{\mu\nu}} - \gamma_{\mu\nu} \gamma^{\alpha\beta} \frac{\delta \Gamma^W(\gamma)}{\delta \gamma^{\alpha\beta}} \tag{11}
\]

satisfies the following anomaly relations

\[
\nabla_\mu (g^{\mu\nu} T^W_{\nu\alpha}) = -\frac{d}{48\pi} \frac{1}{\sqrt{-g}} \partial_\alpha R(\gamma) , \quad g^{\mu\nu} T^W_{\mu\nu} = 0 . \tag{12}
\]

The term \(\Lambda^W \int d^2 \xi\) is induced by renormalization, and, even though it is \(\gamma\)-independent (and obviously does not contribute to the anomaly relations), can have a non-trivial role in the theory since it gives different weights to surfaces with different \(\int d^2 \xi\) in the evaluation of the partition function.

Also notice that the anomaly relations (12) can be put in the following \(\sqrt{-g}\)-independent form

\[
\hat{\nabla}_\mu (\gamma^{\mu\nu} T^W_{\nu\alpha}) = -\frac{d}{48\pi} \partial_\alpha R(\gamma) , \quad \gamma^{\mu\nu} T^W_{\mu\nu} = 0 , \tag{13}
\]

where \(\hat{\nabla}\) is the covariant derivative computed with the metric \(\gamma_{\mu\nu}\), and the invariance of \(\Gamma^W(\gamma)\) under area-preserving diffeomorphism is encoded in the fact that [10]

\[
\hat{\nabla}_\mu \hat{\nabla}_\nu (\gamma^\beta\sigma) e^{\mu\alpha} T^W_{\alpha\beta} = 0 . \tag{14}
\]

In this Letter, we investigate this alternative Weyl-invariant approach both in the gauge-independent Hamiltonian formulation [5] and using anomalous Ward identities [2], and compare the results to the ones for the conventional diffeomorphism invariant approach.

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Note that, 1+1-dimensional area-preserving diffeomorphisms are parametrized by one arbitrary function (in 1+1 dimensions \(\partial_\mu f^\mu = 0\) is locally solved by \(f^\mu = \epsilon^{\mu\nu} \partial_\nu \phi\)), and another arbitrary function is needed to parametrize Weyl transformations, whereas 1+1-dimensional diffeomorphisms are parametrized by two arbitrary functions.
II Classical Hamiltonian Formulation

In this section we discuss the classical Hamiltonian formulation of both approaches reviewed in the Introduction, and compare their respective simplectic structure and constraints. This type of Hamiltonian analysis should be affected very strongly by the different symmetry properties of the two approaches. In the diffeomorphism-invariant approach the symmetries impose that the Hamiltonian (which generates diffeomorphisms along the time direction) weakly vanish on the surface defined by the two diffeomorphism constraints. On the other hand, in the Weyl-invariant approach the fact that the Weyl symmetry is already enforced by working with $\gamma$ rather than $g$ suggests that the constraint surface be determined only by the constraint enforcing invariance under area-preserving diffeomorphisms. This should lead to a closed constraint algebra because the area-preserving diffeomorphisms form a group. Moreover, since diffeomorphisms in the time direction are not area-preserving, there might be room for a non-vanishing Hamiltonian.

We work with the localized versions\[5, 11\] of the nonlocal effective actions in (4) and (10), which are respectively given by

$$S^D = \frac{1}{2} \int dx \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \alpha R(g) \varphi - 2\Lambda^D \right), \quad (15)$$

$$S^W = \frac{1}{2} \int dx \left( \gamma^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \alpha R(\gamma) \varphi - 2\Lambda^W \right), \quad (16)$$

where $\varphi$ is an auxiliary scalar field, and the parameter $\alpha$ is defined in terms of $d$ by the relation $d = 1 + 12\pi \alpha^2$.

The diffeomorphism-invariant action $S^D$ depends on four independent fields\[3]: $g_{00}$, $g_{11}$, $g_{01}$, and $\varphi$. However, in the Weyl-invariant case, the fact that $\text{det} \gamma \equiv \gamma_{00} \gamma_{11} - \gamma_{01}^2 = -1$ gives a relation between $\gamma_{00}$, $\gamma_{11}$, and $\gamma_{01}$; therefore there are only 3 independent fields, and we choose to work with $\gamma_{11}$, $\gamma_{01}$, and $\varphi$.

From Eq. (15), by first using integration by parts to obtain an expression involving only first derivatives, and then following a standard procedure of Legendre transform, one finds the following first-order Lagrangian for the diffeomorphism-invariant case

$$L^D = \int dx \left( \pi^D_\varphi \dot{\varphi} + \pi^D_{11} \dot{g}_{11} - H^D \right)$$

$$H^D = - \frac{\sqrt{-g}}{g_{11}} \mathcal{E}^D + \frac{g_{01}}{g_{11}} \mathcal{P}^D,$$  

where

$$\pi^D_\varphi = \frac{1}{\sqrt{-g}} \left( g_{01} \varphi' - g_{11} \dot{\varphi} \right) + \frac{\alpha}{2\sqrt{-g}} \left( g_{11} - 2g_{01} + \frac{g_{01}}{g_{11}} g_{11}' \right),$$

$$\pi^D_{11} = \frac{\alpha}{2\sqrt{-g}} \left( \dot{\varphi} - \frac{g_{01}}{g_{11}} \varphi' \right) ,$$

Note that the Hamiltonian formulation of the conventional diffeomorphism-invariant approach was already discussed in Ref.\[3].

In analogy with the argument used in the proof\[11] of the equivalence of the diffeomorphism-invariant actions $S^D$ and $\Gamma^D$, we have verified the equivalence of the Weyl-invariant actions $S^W$ and $\Gamma^W$ by integrating out the scalar field $\varphi$, and evaluating the resulting determinant using a regularization which respects the invariance under Weyl transformations and area-preserving diffeomorphisms. (Obviously, in the diffeomorphism-invariant case a diffeomorphism-invariant regularization of the determinant is instead appropriate.)

$\pi^D_\varphi$
\[ \mathcal{E}^D = \frac{1}{2} (\phi'^2 - \frac{4}{\alpha^2} (g_{11} \pi_{11}^D)^2 - \frac{4}{\alpha} g_{11} \pi_{11}^D \pi_{\varphi}^D + 2 \alpha \varphi'' - \frac{g_{11}'}{g_{11}} \varphi' - 2 \Lambda^D g_{11}) \],
\[ \mathcal{P}^D = \pi_{\varphi}^D \varphi' - 2 \pi_{11}^D g_{11} - \pi_{11}^D g_{11}' . \]

\( g_{11} \) and \( \varphi \) are dynamical fields, whose canonical momenta are \( \pi_{\varphi}^D \) and \( \pi_{11}^D \), whereas \( \sqrt{-g}/g_{11} \) and \( g_{01}/g_{11} \) serve as Lagrange multipliers for the constraints \( \mathcal{E}^D \sim 0 \) and \( \mathcal{P}^D \sim 0 \). As it should be expected for this diffeomorphism-invariant theory, both the Poisson brackets of \( \mathcal{E}^D \) and \( \mathcal{P}^D \), which satisfy the diffeomorphism algebra \( [\mathcal{E}^D, \mathcal{P}^D] = 0 \), and the Hamiltonian \( \int dx \mathcal{H}^D \) vanish on the constraint surface.

The first order Lagrangian for the Weyl-invariant case is given by
\[ \mathcal{L}^W = \int dx (\pi_{\varphi}^W \dot{\varphi} + \pi_{11}^W \dot{\gamma}_{11} - \mathcal{H}^W) \]
\[ \mathcal{H}^W = -\frac{1}{\gamma_{11}} \mathcal{E}^W + \frac{\gamma_{01}}{\gamma_{11}} \mathcal{P}^W , \]

where
\[ \pi_{\varphi}^W = (\gamma_{01} \varphi' - \gamma_{11} \dot{\varphi}) + \frac{\alpha}{2} (\gamma_{11} - 2 \gamma_{01}' + \frac{\gamma_{01}}{\gamma_{11}} \gamma_{11}') , \]
\[ \pi_{11}^W = \frac{\alpha}{2} (\dot{\varphi} - \frac{\gamma_{01}}{\gamma_{11}} \varphi) , \]
\[ \mathcal{E}^W = \frac{1}{2} (\varphi'^2 - \frac{4}{\alpha^2} (\gamma_{11} \pi_{11}^W)^2 - \frac{4}{\alpha} \gamma_{11} \pi_{11}^W \pi_{\varphi}^W + 2 \alpha \varphi'' - \frac{\gamma_{11}'}{\gamma_{11}} \varphi' - 2 \Lambda^W \gamma_{11}) , \]
\[ \mathcal{P}^W = \pi_{\varphi}^W \varphi' - 2 \pi_{11}^W \gamma_{11} - \pi_{11}^W \gamma_{11}' . \]

Here again there are two dynamical variables, \( \gamma_{11} \) and \( \varphi \), whose canonical momenta are \( \pi_{\varphi}^W \) and \( \pi_{11}^W \); however, there is only one Lagrange multiplier, \( \gamma_{01}/\gamma_{11} \), which enforces the constraint \( \mathcal{P}^W \sim 0 \).

As expected based on the general arguments given at the beginning of this section, the Poisson brackets of \( \mathcal{P}^W \) close on \( \mathcal{P}^W \)
\[ \{ \mathcal{P}^W (x), \mathcal{P}^W (y) \} = [\mathcal{P}^W (x) + \mathcal{P}^W (y)] \delta' (x - y) \sim 0 , \]

and the Hamiltonian does not vanish on the constraint surface defined by \( \mathcal{P}^W \sim 0 \); in fact, \( \int dx \mathcal{H}^W \sim - \int dx (\mathcal{E}^W / \gamma_{11}) \). However, the Dirac Hamiltonian procedure for constrained systems \[12\] requires that the constraint surface be preserved by the time evolution, \( i.e. \) the commutator of the Hamiltonian with the constraints should weakly vanish, and instead one can show that
\[ \{ \mathcal{H}^W (x), \mathcal{P}^W (y) \} = \left\{ \frac{1 + \gamma_{01}(x)}{\gamma_{11}(x)}, \mathcal{P}^W (y) \right\} \mathcal{P}^W (x) + \frac{1 + \gamma_{01}(x)}{\gamma_{11}(x)} (\mathcal{P}^W (x) + \mathcal{P}^W (y)) \delta' (x - y) \]
\[ + \left[ \frac{1}{\gamma_{11}(x)} (\mathcal{P}^W (x) - \mathcal{E}^W (x)) \right]' \delta (x - y) , \]

which reduces to \( \{ \mathcal{H}^W (x), \mathcal{P}^W (y) \} \sim - \mathcal{E}^W (x)/\gamma_{11}(x) \delta (x - y) \) on the constraint surface defined by \( \mathcal{P}^W \sim 0 \). It is therefore necessary \[12\] to add a second constraint \( [- \mathcal{E}^W (x)/\gamma_{11}(x)]' \sim 0 , \)

\[ We use the standard notation \( A \sim 0 \) to indicate that \( A \) weakly vanishes, \( i.e. \) \( A \) vanishes on the constraint surface.
or equivalently $-\mathcal{E}^W(x)/\gamma_{11}(x) \sim \Lambda^0$, where $\Lambda^0$ is the constant mode of $-\mathcal{E}^W(x)/\gamma_{11}(x)$. Enforcing this additional constraint with a Lagrange multiplier $N(x)$, we obtain the Hamiltonian density

$$
\mathcal{H}^W = -\frac{1}{\gamma_{11}}\mathcal{E}^W + \frac{\gamma_{01}}{\gamma_{11}}\mathcal{P}^W - N\left(\frac{\mathcal{E}^W}{\gamma_{11}} + \Lambda^0\right) = -\frac{N+1}{\gamma_{11}}(\mathcal{E}^W + \Lambda^0\gamma_{11}) + \frac{\gamma_{01}}{\gamma_{11}}\mathcal{P}^W + \Lambda^0.
$$

(31)

In this version of $\mathcal{H}^W$, $\left(N+1\right)/\gamma_{11}$ serves as Lagrange multiplier for the constraint $\mathcal{E}^W + \Lambda^0\gamma_{11} \sim 0$, and the Lagrange multiplier $\gamma_{01}/\gamma_{11}$ enforces again $\mathcal{P}^W \sim 0$. Upon the identifications $\gamma_{11} \leftrightarrow g_{11}$, $\gamma_{01} \leftrightarrow g_{01}$, $N+1 \leftrightarrow \sqrt{-g}$, $\Lambda^D \leftrightarrow \Lambda^W - \Lambda^0/2$, one can immediately see that the theory with the Hamiltonian $\mathcal{H}^W$ in (31) is essentially equivalent to the theory with the Hamiltonian $\mathcal{H}^D$ in (18), the only difference being the additional contribution to $\mathcal{H}^W$ from the constant mode $\Lambda^0$. In particular, it is easy to verify that the constraints $\mathcal{E}^W + \Lambda^0\gamma_{11}$ and $\mathcal{P}^W$ generate general (i.e. not necessarily area-preserving) diffeomorphism transformations.

This result indicates that any quantization whose starting point is the Dirac Hamiltonian procedure would lead to equivalent theories for the two approaches.

### III Anomalies and Ward Identities

In Ref.[2] the 1+1-dimensional gravity theory defined by the diffeomorphism-invariant action (4) is investigated by using its anomalous Ward identities. The starting point is the functional integral

$$
Z^D = \int \frac{Dg}{\Omega_{diff}} \exp\left(i\Gamma^D(g)\right),
$$

(32)

where $\Omega_{diff}$ is the volume of the diffeomorphism group.

In order to factorize out the gauge volume one may choose to work in the light-cone gauge $g_-=0$, $g_+=1$, and introduce the corresponding action for the ghost fields

$$
Z^D = \int Dg_{++} Dc D\bar{b}^- D\chi_+ D\chi_- \exp\left[i\Gamma^D(g) + i\int d^2\xi \left(b^-\nabla_-\chi_- + c\nabla_+\chi_- + c\nabla_+\chi_+\right)\right]
$$

(33)

After integrating out the ghost fields, $Z^D$ takes the following form\[8\]

$$
Z^D[J] = \int Dg_{++} \exp \left(i\Gamma^D(g) + i\Gamma^D_{gh}(g)\right),
$$

(34)

where $\Gamma^D + \Gamma^D_{gh}$ is the gauge-fixed action for gravity.

In Ref.[2], in order to obtain the anomalous Ward identities, the following infinitesimal shift in the functional variable of integration is considered

$$
\delta f g_{++} = (2\partial_+ - g_{++}\partial_-)\delta f + \delta f \partial_- g_{++}.
$$

(35)

The variation of the gauge-fixed action $\Gamma^D + \Gamma^D_{gh}$ under this transformation can be computed by exploiting the fact that the corresponding energy momentum tensor

$$
\Theta^D_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta[\Gamma^D(g) + \Gamma^D_{gh}(g)]}{\delta g^{\mu\nu}}
$$

(36)
satisfies the anomaly relations.

\[ \nabla_\mu (g^{\mu \nu} \Theta_\mu^D) = 0 , \quad g^{\mu \nu} \Theta_\mu^D = 2\Lambda^D + \frac{d - 28}{24\pi} R(g) . \quad (37) \]

From these relations it follows that the variation of the gauge-fixed action under the transformation \( \delta g_+ + \delta g_+ \) is given by

\[ \int d\xi^2 \frac{\delta [\Gamma^D(g) + \Gamma^D(g)]}{\delta g_{++}} \delta f_{g++} = \int d\xi^2 [\nabla_\mu (g^{\mu \nu} \Theta_\nu^D) - \frac{1}{2} \nabla_\nu (g^{\mu \nu} \Theta_\mu^D)] \delta f = \int d\xi^2 \frac{28 - d}{48\pi \partial^3} g_{++} \delta f \quad (38) \]

which, following a standard procedure, leads to the anomalous Ward identities

\[ \sum_i \langle g_{++}(\xi_1) \cdots \delta g_{++}(\xi_i) \cdots g_{++}(\xi_n) \rangle + \frac{d - 28 + \lambda^D}{i48\pi} \int d\xi^2 \delta f(\xi) \left( \partial^3_{g_{++}(\xi)} g_{++}(\xi_1) \cdots g_{++}(\xi_n) \right) = 0 . \quad (39) \]

Here \( \lambda^D \) is the additional contribution to the anomaly which is due to the fact that \( \delta f g_{++} \) is a composition of a diffeomorphism and a Weyl transformation on \( g_{++} \), and therefore the diffeomorphism-invariant but not Weyl-invariant measure \( Dg_{++} \) is not invariant under \( g_{++} \rightarrow g_{++} + \delta g_{++} \). The direct evaluation of \( \lambda^D \) is not known, but a value of \( \lambda^D \) can be fixed by requiring that the theory be independent of the choice of gauge; specifically, in Ref.\,[2] the class of gauges \( g_{--} = g_{--}^B, \quad g_{++} = 1 \) is considered, and it is found that the independence of the partition function on the choice of \( g_{--}^B \) requires

\[ d - 28 + \lambda^D = \frac{d - 13 - \sqrt{(d - 1)(d - 25)}}{2} . \quad (40) \]

We now turn to the Weyl-invariant approach, and investigate its anomalous Ward identities. Let us start by considering the functional integral

\[ Z^W = \int \frac{D\gamma}{\Omega_{Sdiff}} \exp(i\Gamma^W(\gamma)) , \quad (41) \]

where \( \Omega_{Sdiff} \) is the volume of the area-preserving diffeomorphism group. The volume of the Weyl group does not appear because the functional integral is already written in terms of Weyl-invariant variables. Fixing the \( \gamma_{--} = 0 \) gauge, and introducing the corresponding ghost action, one can rewrite \( Z^W \) as

\[ Z^W = \int D\gamma_{++} Dc Db^- \exp \left[ i\Gamma^W(\gamma) + i \int d^2 \xi \ b^- (g_{--} \epsilon^{\alpha \beta} \partial_{\beta} c) \right] . \quad (42) \]

Our choice of gauge is motivated by the fact that in this gauge \( \Gamma^W \) takes the same form as the light-cone-gauge version of \( \Gamma^D \), and we intend to exploit this correspondence in the generalization of the Ward identities to the Weyl-invariant approach. Still, in the analysis we shall need to take into account the fact that the form of the ghost action in

|\footnotesize{The form of these anomaly relations can be derived from symmetry arguments, but the value of the coefficient in front of \( R(g) \) requires a calculation. The value indicated in (39) follows from the fact that, as shown in Ref.\,[2], \( \Gamma^D_{gh}(g) \sim -28\Gamma^D(g)/d \). Note that the \(-28\) is the ghost contribution to the anomaly in light-cone gauge, and is gotten by adding \(-26\) for the term \( b^- \nabla_- \chi_- \) and \(-2\) for the term \( c\nabla_+ \chi_- + c\nabla_- \chi_+ \).}
is very different \( \Gamma \) from the one in (43), and the measure \( \mathcal{D} \gamma \), which is Weyl-invariant but not diffeomorphism-invariant, is very different from the measure \( \mathcal{D} g \).

Integrating out the ghost fields in (42), \( Z^W \) takes the following form

\[
Z^W = \int \mathcal{D} \gamma^{++} \exp \left( i \Gamma^W (\gamma) + i \Gamma^W_{gh}(\gamma) \right).
\]  (43)

The gauge-fixed action \( \Gamma^W (\gamma) + \Gamma^W_{gh}(\gamma) \) that appears in (43) has energy-momentum tensor

\[
\Theta^{W}_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta [\Gamma^W (\gamma) + \Gamma^W_{gh}(\gamma)]}{\delta g^{\mu \nu}},
\]  (44)

which satisfies the following anomaly relations

\[
\nabla_{\mu} (g^{\mu \nu} \Theta^W_{\nu \alpha}) = - \frac{d - 28}{48 \pi} \frac{1}{\sqrt{-g}} \partial_{\alpha} R(\gamma), \quad g^{\mu \nu} \Theta^W_{\mu \nu} = 0,
\]  (45)

In order to obtain the anomalous Ward identities, in analogy with (35), we consider the following shift in the functional variable of integration

\[
\delta f \gamma^{++} = (2 \partial_{+} - \gamma^{++} \partial_{-}) \delta f + \delta f \partial_{-} \gamma^{++}.
\]  (46)

From the anomaly relations (43), one can show that the variation of the gauge-fixed action \( \Gamma^W + \Gamma^W_{gh} \) under the transformation (46) is given by

\[
\int d^2 \xi \frac{\delta [\Gamma^W (\gamma) + \Gamma^W_{gh}(\gamma)]}{\delta \gamma^{++}} = \int d^2 \xi [\nabla_{\mu} (g^{\mu \nu} \Theta^W_{\nu \alpha}) - \frac{1}{2} \nabla_{-} (g^{\mu \nu} \Theta^W_{\mu \nu})] \delta f = \int d^2 \xi \frac{28 - d}{48 \pi} \partial^{3} \gamma^{++} \delta f.
\]  (47)

The direct correspondence between (17) and (38) might be surprising considering that they were derived using very different anomaly relations [(43) and (37)] respectively; however, we notice that in (17) and (38) the anomaly relations only appear in the combination \( \nabla_{\mu} (g^{\mu \nu} \Theta^D_{\nu -}) - \nabla_{-} (g^{\mu \nu} \Theta^W_{\mu \nu})/2 \), and in the chosen gauges this combination is essentially insensitive to the difference between the anomaly relations (43) and (37), since

\[
\nabla_{\mu} (g^{\mu \nu} \Theta^D_{\nu -}) = 0, \quad - \frac{1}{2} \nabla_{-} (g^{\mu \nu} \Theta^D_{\mu \nu}) = \frac{28 - d}{48 \pi} \partial^{3} \gamma^{++},
\]  (48)

\[
\nabla_{\mu} (g^{\mu \nu} \Theta^W_{\nu -}) = \frac{28 - d}{48 \pi} \partial^{3} \gamma^{++}, \quad - \frac{1}{2} \nabla_{-} (g^{\mu \nu} \Theta^W_{\mu \nu}) = 0.
\]  (49)

Finally, from (47) it is straightforward to derive the following anomalous Ward identities

\[
\sum_{i}^{n} \langle \gamma^{++}(\xi_{1}) \cdots \delta \gamma^{++}(\xi_{i}) \cdots \gamma^{++}(\xi_{n}) \rangle + \frac{d - 28 + \lambda^{W}}{i48 \pi} \int d^2 \xi \delta f(\xi) \langle \partial^{3} \gamma^{++}(\xi) \gamma^{++}(\xi_{1}) \cdots \gamma^{++}(\xi_{n}) \rangle = 0.
\]  (50)

**One can understand the peculiar form of the ghost action encountered in the Weyl-invariant approach by noticing that the infinitesimal variation of \( \gamma_{\mu \nu} \) under an area-preserving diffeomorphism can be locally written as \( \delta g_{\mu \nu} = \nabla_{\mu} (g_{\nu \alpha} \epsilon^{\alpha \beta} \partial_{\beta} \phi) + \nabla_{\nu} (g_{\mu \alpha} \epsilon^{\alpha \beta} \partial_{\beta} \phi) \), where \( \phi \) is an arbitrary function.

**Also the structure of these anomaly relations is completely fixed by symmetry, and we determined the coefficient \( d - 28 \) by observing that \( \Gamma^W_{gh}(\gamma) \sim -28 \Gamma^W (\gamma) / d \) in \( \gamma_{- -} = 0 \) gauge.
The additional contribution $\lambda^W$ to the anomaly is due to the fact that, as one can easily verify using $\gamma^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$, $\delta f \gamma^{++}$ is an infinitesimal (not area-preserving) diffeomorphism transformation on $\gamma^{++}$, and therefore the measure $D\gamma^{++}$ is not invariant under $\gamma^{++} \rightarrow \gamma^{++} + \delta f \gamma^{++}$. Also the direct evaluation of $\lambda^W$ is not known, but we have verified that the value of $\lambda^W$ obtained by imposing the independence of this theory on the choice of gauge is equal to the value of $\lambda^D$ analogously obtained for the conventional diffeomorphism-invariant theory [see Eq.(40)]. This observation together with the results (39) and (50) indicates that the anomalous Ward identities satisfied by $\gamma^{++}$ in the Weyl-invariant approach are identical to the ones satisfied by $g^{++}$ in the diffeomorphism-invariant approach. Since these Ward identities completely determine the Green’s functions, also the Green’s functions are identical.

IV Conclusion

The results presented in the two preceding sections indicate that, in spite of several technical differences encountered along the way, the two approaches are ultimately equivalent. In the gauge-independent Hamiltonian formulation it appears that the Weyl-invariant approach leads to results that one would only expect in the diffeomorphism-invariant approach. In particular, by demanding the consistency of the Dirac analysis of the Weyl-invariant case one is led to the introduction of an additional independent field, the Lagrange multiplier $N$, which is related to the field $\sqrt{-g}$; so it appears that the action (1) describes a $\sqrt{-g}$-dependent theory, in spite of being $\sqrt{-g}$-independent (i.e. involving $\gamma^{\mu\nu}$ only).

In deriving the equivalence of the two approaches at the level of the anomalous Ward identities a key role is played by the combination $\nabla_{\mu}(g^{\mu\nu}\Theta_{\nu}) - \nabla_{\nu}(g^{\mu\nu}\Theta_{\mu})/2$, which (in the chosen gauges) takes the same form in both approaches. Clearly this combination encodes some essential feature of the theories, but its physical interpretation is not yet clear to us.

We believe that (at least part of) the results here found are a consequence of the fact that the group of the area-preserving diffeomorphisms is not invariant under arbitrary coordinate redefinitions along the time direction; for example, this leads to Eq.(30). It could be interesting to devise yet another quantization approach with symmetry under transformations of a subgroup (of the diffeomorphism group) that is invariant under coordinate redefinitions along the time direction.

Further insight might be gained by analyzing the one-parameter family of measures discussed in Ref.[13], which interpolates between the two limiting cases considered here: diffeomorphism invariance and Weyl invariance. In the chiral Schwinger model a one-parameter “$a$”-family of chiral symmetry breaking measures has also been identified[14] and the mass emergent in that theory depends on $a$; with two values of $a$ leading to the same mass. It is conceivable that the measures in the two approaches here considered are paired in a similar fashion within the one-parameter family of measures discussed in Ref.[13].

It would also be interesting to investigate some topological issues which have been ignored here. A differentiation between the outcome of the two approaches might be found if the topology of the space of metrics $g_{\mu\nu}$ modulo diffeomorphisms and the topology of the space of metrics $\gamma_{\mu\nu}$ modulo area-preserving diffeomorphisms do not coincide.

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