Collective excitations in an $F = 2$ Bose-Einstein condensate

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Abstract. We calculate the collective excitations in a homogeneous $F = 2$ spinor condensate in the absence of the magnetic field and also in a strong magnetic field. Almost all the excitations for a zero magnetic field are found to have either the Bogoliubov form or the free-particle form. We relate our results for a strong magnetic field to the concept of fragmented condensates and note that some observable quantities such as the speed of sound for the cyclic state depend on the existence of relative phase between the spinor amplitudes for the $M = \pm 2, 0$ atoms.

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1. Introduction

Due to experimental observations of Bose-Einstein condensation in dilute alkali gases [1, 2, 3] the weakly interacting quantum systems have become a highly relevant research topic. The spectrum of collective excitations for such a Bose condensate was predicted some time ago by Bogoliubov [4] and since then these studies have been extended to trapped systems in a number of papers, see reference [5]. Recently a sodium condensate of $^{23}\text{Na}$ atoms in the $F = 1$ hyperfine state was trapped in an optical dipole trap [6]. Such a trap confines simultaneously all the different Zeeman substates (quantum number $M$) of the atomic hyperfine ground state states (quantum number $F$), whether or not their energy degeneracy is removed by a magnetic field. Therefore the spin degree of freedom is not necessarily frozen, and the condensate must be described in terms of a multicomponent spinor wavefunction.

The possible ground states and excitations of the $F = 1$ spinor condensates have been studied theoretically by several authors [7, 8, 9]. So far only a few studies have been made for the $F = 2$ condensates, such as the calculation of the possible ground states, and their dependence on a weak magnetic field by Ciobanu et al. [10]. One can identify several possible ground states, and the relative magnitude of the scattering lengths for different $M, M'$ collisions ($M + M'$ is conserved) determines which of them is the true ground state. Our results for the ground states agree with those presented in reference [10], and we calculate further the collective excitations associated with each possible ground state.

Condensates have been produced on an $F = 2$ state experimentally in magnetic trap for $^{87}\text{Rb}$ [11, 12]. Also, one can achieve condensation on an $F = 1$ state, and then move the atoms with suitable microwave pulses to the $F = 2$ hyperfine ground state [13]. In that case the magnetic trap potentials are not affected by the change in the internal state of the atoms, so one could imagine of producing a condensed sample of atoms in a magnetic trap on the $F = 2$ state, and then moving it into an optical trap, after which one can remove the magnetic field and thus return to the case of degenerate Zeeman states. The main problem in this case are the inelastic atomic collisions, which may transfer atoms from the $F = 2$ manifold into the $F = 1$ manifold. However, for $^{87}\text{Rb}$ these rates are surprisingly small [14, 15]. Also, this problem is not present e.g. for $^{85}\text{Rb}$ as then the $F = 2$ hyperfine state is the lowest ground state. Very recently [16] a condensate of $^{87}\text{Rb}$ atoms on the $F = 1$ state was obtained in an all-optical trap via evaporative cooling, and this approach is expected to work also for atoms in the $F = 2$ state.

This paper proceeds as follows. In section 2 we introduce the physical model for the interacting spinor system, and in section 3 we discuss the ground states of the $F = 2$ condensate. We calculate the collective excitations in the absence of magnetic fields in section 4. We also study how the excitations change if a strong magnetic field is applied (section 5) and connect the results to the concept of fragmentation in spinor condensates (section 6). We also briefly note the superfluid properties of the different
2. Physical model

The weak interaction between condensed atoms is generally described very well with a contact potential whose strength is proportional to the s-wave scattering length \( a \). This has been thoroughly demonstrated with successful comparison of theory with experiments on single-component condensates and also for two-component condensates, see e.g. reference [17] and references therein. We generalize this interaction potential to apply for the \( F = 2 \) spinor condensates and use the interaction potential [18]

\[
V(r_2 - r_1) = \delta(r_2 - r_1) \sum_{f=0}^{4} g_f \sum_{m=-f}^{f} |f, m\rangle\langle f, m|,
\]

where \( |f, m\rangle \) is the total (molecular) hyperfine spin state formed by two atoms with spin \( F = 2 \), and \( g_f = 4\pi\hbar^2 a_f/m_a \) with \( a_f \) being the scattering length in the \( f \) channel and \( m_a \) the atomic mass.

Following the formalism of Pu et al. for the \( F = 1 \) condensate [18], we expand this interaction term in terms of the two-atom basis vectors \(|F_1 = 2, M_1\rangle \otimes |F_2 = 2, M_2\rangle = \hat{\Psi}_{M_1} \otimes \hat{\Psi}_{M_2} \). This procedure is straightforward, but the resulting equation is very long and thus we omit it here. To simplify things we limit our discussion to a homogeneous condensate and therefore the total Hamiltonian includes only the interaction potential and the kinetic energy \( K = -[\hbar^2/(2m_a)]\nabla^2 \). We also make the semiclassical approximation and replace the atomic creation and annihilation operators with complex numbers \( \Phi_M \).

The interaction term contains terms proportional to densities \( n_M = |\Phi_M|^2 \) of the different \( M \) components, but also terms that could result in complex spin-mixing dynamics, i.e. transient behavior of the populations in different \( M \) states. In the \( F = 2 \) case these processes are expected to be much more complicated than in the \( F = 1 \) condensate studied by Pu et al. [18]. To give an idea about the resulting Gross-Pitaevskii [19, 20] (GP) equations we present the equation for the \( M = 0 \) component:

\[
i\hbar \frac{\partial \Phi_0}{\partial t} = \left[ K + \lambda_s \sum_{M=-2}^{2} n_M + \alpha (n_1 + n_{-1}) + \beta (n_2 + n_{-2}) \right] \Phi_0 + 2\alpha \Phi_0^* \Phi_1 \Phi_{-1} - 2\beta \Phi_0^* \Phi_2 \Phi_{-2} + \gamma \left( \Phi_0^* \Phi_1 \Phi_{-1} + \Phi_1^* \Phi_2 \Phi_{-2} + \Phi_1^* \Phi_2 \Phi_{-1} + \Phi_2^* \Phi_1 \Phi_{-2} \right),
\]

where \( \lambda_s = (18g_1 + 10g_2 + 7g_0)/35, \alpha = (12g_1 - 5g_2 - 7g_0)/35, \beta = (-3g_1 + 10g_2 - 7g_0)/35, \) and \( \gamma = \sqrt{6}(g_4 - g_2)/7 \).

We mostly ignore mixing dynamics and focus on the condensate behavior around stationary states (ground states). At first we assume that the external magnetic field is absent, but in section 3 we discuss the behavior of the stationary states in a strong magnetic field.
Table 1. The possible ground states (spinor amplitudes $\Phi$ and energies $E$) for the $F = 2$ condensate in zero magnetic field. The angle variables $\phi, \alpha_{\pm 1}$ and $\alpha_{\pm 2}$ can be chosen freely.

| $F$ | $\Phi$ | $E$          |
|-----|--------|--------------|
| $F$ | $(1, 0, 0, 0, 0)$ | $ng_4$       |
| $F'$ | $(0, 1, 0, 0, 0)$ | $n(4g_4 + 3g_2)/7$ |
| $C$ | $\frac{1}{2} \left( e^{i\phi}, 0, \sqrt{2}, 0, -e^{-i\phi} \right)$ | $n(3g_4 + 4g_2)/7$ |
| $P$ | $\frac{1}{\sqrt{2}} \left( e^{i\alpha_2}, 0, 0, 0, e^{i\alpha_2} \right)$ | $n(18g_4 + 10g_2 + 7g_0)/35$ |
| $P_1$ | $(0, e^{i\alpha_1}, 0, e^{i\alpha_1-1}, 0)$ | $n(18g_4 + 10g_2 + 7g_0)/35$ |
| $P_0$ | $(0, 0, 1, 0, 0)$ | $n(18g_4 + 10g_2 + 7g_0)/35$ |

3. Possible ground states

Ground state structures for a $F = 2$ spinor condensate were calculated by Ciobanu et al. [10]. We follow their notation and terminology for the various ground states ($F, F', C, P, P_0$ and $P_1$). The ground states and their energies are given in table 1 for the case when the magnetic field is absent. When comparing these results with results in reference [10] it should be noted that in reference [10] the energy of the cyclic state was chosen to be zero. Taking this shift into account the results in table 1 coincide with those in reference [10].

Within the semiclassical approximation the true ground state is the state with the lowest energy in table 1. This approximation may ignore some subtle effects. For example, in the $F = 1$ spinor condensate with an antiferromagnetic interaction the true ground state will be a fragmented condensate [21]. In such a condensate the number of particles in each component is separately constant and there is no well defined phase relationship between different components. How much this matters in practice is still a matter of debate [22]. The semiclassically calculated results capture only states corresponding to a single condensate. In the $F = 1$ spinor condensate the energy difference between single and fragmented condensates vanishes at the thermodynamic limit so we expect that most energies in table 1 will be correct (in the thermodynamic limit) even if the true ground state would be fragmented. The cyclic state [23] is a possible exception. We will return to this issue in section 6.

As polar states are degenerate one might assume that a superposition of polar states would always give a state with same energy. This is clearly wrong. The cyclic state is a special superposition of $P_0$ and $P$ states, but it has nevertheless a different energy. Forming a superposition of polar states will change the systems energy non-linearly. In particular, the Hamiltonian has a term

$$H_{\pm 2, 0} = \frac{-3g_4 + 10g_2 - 7g_0}{35} \left[ \Phi^*_{-2} \Phi^*_0 \Phi_{-2} \Phi_0 + \Phi^*_2 \Phi^*_0 \Phi_2 \Phi_0 
- \Phi^*_0 \Phi^*_0 \Phi_{-2} \Phi_0 - \Phi^*_{-2} \Phi^*_2 \Phi_0 \Phi_0 \right]$$

which does not contribute to the $P_0$ and $P$ states separately, but will contribute to
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their superposition. For non-magnetic states and when $3g_4 - 10g_2 + 7g_0 > 0$ this term is minimized by having $1/4$ of the population at the $M = 2$ state and a relative phase $2\phi_0 - \phi_2 - \phi_{-2} = \pi$. This term is responsible for the peculiar phase-locking of the cyclic state.

4. Excitations in zero magnetic field

In our approach we simply linearize the Gross-Pitaevskii (GP) equations for each component around all stationary states. The resulting equations can be solved to get the energies of the collective excitations. Many of these equations are coupled, but all of them can be solved analytically. Some of the coupled equations have the form studied previously by Timmermans [24], and some lead to a $3 \times 3$ eigenvalue problem. In the following subsections and tables 2, 3, and 4 we present the results of our analysis. Our notation is such that index numbers refer to the $M$ states present in the excitation and index alphabets are used (when necessary) as labels to distinguish between excitations involving the same $M$ states. A negative excitation energy implies a thermodynamic (or energetic) instability since the system energy can be lowered by creating such excitations. On the other hand, an imaginary value of the excitation energy implies dynamical instability. This instability results in exponentially growing excitation amplitude and can occur even in the absence of dissipation.

4.1. Ferromagnetic states

The results given in table 2 show that the excitations of the ferromagnetic state $F$ are either of the Bogoliubov form (i.e. $E = \sqrt{K(K + 2\epsilon)}$) or the free-particle form $E = K + \Delta E_g$, where $\Delta E_g$ is the energy gap. The $F'$ state, on the other hand, seems more interesting. It is not stable energetically since when the magnetized ground state is favored the $F$ state has always a lower energy, but experimental preparation could leave the condensate in the $F'$ state, after which the relaxation to the $F$ state is slow and thus the $F'$ state could appear as a metastable one. We have the Bogoliubov branch and two free particle excitations but there are also two excitations which do not fall into these two basic categories. These excitations have the form

$$E = -\frac{\Delta E}{2} \pm \sqrt{\frac{(\Delta E + 2K)^2}{2} + 16K\Delta E},$$  \hspace{1cm} (4)

where $\Delta E = n(g_4 - g_2)/7$. At small kinetic energies the other one approaches 0 and the other one $-\Delta E$.

When $g_4 - g_2 > 0$ all excitations are real and $F'$ is thermodynamically unstable as excitations $E_{20-}$ and $E_{-2}$ both have negative values at small values of kinetic energy. On the other hand, if $g_4 - g_2 < 0$ the $F'$ state is also dynamically unstable as the $E_{20\pm}$ modes have imaginary energies. The imaginary value takes a maximum value when $K = 5n(g_2 - g_4)/14$. Presumably the related value of the wavenumber $k$ sets the scale for the spatial structures caused by the dynamical instability.
Table 2. Excitations in zero magnetic field for the ferromagnetic states $F$ and $F'$. The subscripts in the names of the branches refer to the $M$ state or states involved in the excitation superposition.

| State | Formulation |
|-------|-------------|
| $F$   | $E_2 = \sqrt{K(K + 2ng_4)}$ |
|       | $E_1 = K$ |
|       | $E_0 = K - \frac{4}{7}n(g_4 - g_2)$ |
|       | $E_{-1} = K - \frac{6}{7}n(g_4 - g_2)$ |
|       | $E_{-2} = K + \frac{2}{7}n(-17g_4 + 10g_2 + 7g_0)$ |
| $F'$  | $E_{2,0,a} = \frac{1}{2} \left[ -\frac{n(g_4 - g_2)}{7} + \sqrt{\left(\frac{n(g_4 - g_2)}{7} + 2K\right)^2 + \frac{16}{7}Kn(g_4 - g_2)} \right]$ |
|       | $E_{2,0,b} = \frac{1}{2} \left[ -\frac{n(g_4 - g_2)}{7} - \sqrt{\left(\frac{n(g_4 - g_2)}{7} + 2K\right)^2 + \frac{16}{7}Kn(g_4 - g_2)} \right]$ |
|       | $E_1 = \sqrt{K\left[K + \frac{2}{7}n(4g_4 + 3g_2)\right]}$ |
|       | $E_{-1} = K + \frac{2}{7}n(-2g_4 - 5g_2 + 7g_0)$ |
|       | $E_{-2} = K - \frac{3}{7}n(g_4 - g_2)$ |

4.2. Polar states

Table 3 shows that all excitations in the polar states have the Bogoliubov form. As the chemical potentials of these states are the same, the main branch is identical in all polar states. If the $P$ state is to be dynamically stable, then the terms $x = 17g_4 - 10g_2 - 7g_0$ and $y = -3g_4 + 10g_2 - 7g_0$ must be positive. This in turn implies that also the $P_0$ state would be dynamically stable. On the other hand, the $P_1$ state can be dynamically unstable if $y < \frac{4}{3}x$ or $y > 3x$. This instability is due to the imaginary energies in either of the modes $E_{\pm2,0,a}$ or $E_{\pm2,0,b}$ and will be reflected in the formation of domains (i.e. stripes) with $M = \pm2,0$ atoms.

4.3. Cyclic state

Most excitations in the cyclic state have the Bogoliubov form (table 4) with the exception the mode $E_{\pm2,0,b}$, which has the free particle form. This mode is a superposition of $M = \pm2,0$ atoms of the type $(-1,0,\alpha,0,1)$. When the cyclic state is the ground state, this excitation has a positive gap. When the polar state has a lower energy this mode has a negative gap, indicating thermodynamical instability.

5. Excitations in a strong magnetic field

Let us assume that the condensate is prepared in one of the possible ground states given in table 4. Next we turn on a strong longitudinal magnetic field $B = B\hat{z}$, for which the
quadratic Zeeman effect is at least on the order of the chemical potential. In a magnetic field the true ground state will be something different, but if dissipative processes are slow enough the condensate can still be metastable. Of course, the scattering lengths can be strongly affected by the magnetic field, and new inelastic two-body loss channels

**Table 3.** Excitations in zero magnetic field for polar states. The subscripts in the names of the branches refer to the $M$ state or states involved in the excitation superposition.

| $E_{\pm 2,a}$ | $\sqrt{K \left[ K + \frac{4n}{35} (18g_4 + 10g_2 + 7g_0) \right]}$ |
|----------------|------------------------------------------------------------------|
| $E_{\pm 2,b}$ | $\sqrt{K \left[ K + \frac{4n}{35} (17g_4 - 10g_2 - 7g_0) \right]}$ |
| $E_0$         | $\sqrt{K \left[ K + \frac{4n}{35} (-3g_4 + 10g_2 - 7g_0) \right]}$ |
| $E_{\pm 1}$   | $\sqrt{K \left[ K + \frac{4n}{35} (2g_4 + 5g_2 - 7g_0) \right]}$ |

| $E_{\pm 1,a}$ | $\sqrt{K \left[ K + \frac{4n}{35} (18g_4 + 10g_2 + 7g_0) \right]}$ |
|----------------|------------------------------------------------------------------|
| $E_{\pm 1,b}$ | $\sqrt{K \left[ K + \frac{4n}{35} (2g_4 + 5g_2 - 7g_0) \right]}$ |
| $E_{\pm 2,0,a}$ | $\sqrt{K \left[ K + \frac{4n}{35} (2g_4 + 5g_2 - 7g_0) \right]}$ |
| $E_{\pm 2,0,b}$ | $\sqrt{K \left[ K + \frac{4n}{35} (-13g_4 + 20g_2 - 7g_0) \right]}$ |
| $E_{\pm 2,0,c}$ | $\sqrt{K \left[ K + \frac{4n}{35} (27g_4 - 20g_2 - 7g_0) \right]}$ |

| $E_0$         | $\sqrt{K \left[ K + \frac{4n}{35} (18g_4 + 10g_2 + 7g_0) \right]}$ |
| $E_{\pm 2}$   | $\sqrt{K \left[ K + \frac{4n}{35} (-3g_4 + 10g_2 - 7g_0) \right]}$ |
| $E_{\pm 1}$   | $\sqrt{K \left[ K + \frac{4n}{35} (12g_4 - 5g_2 - 7g_0) \right]}$ |

**Table 4.** Excitations in zero magnetic field for the cyclic state. The subscripts in the names of the branches refer to the $M$ state or states involved in the excitation superposition.

| $E_{\pm 1}$ | $\sqrt{K \left[ K + \frac{4n}{7} (g_4 - g_2) \right]}$ |
|--------------|--------------------------------------------------|
| $E_{\pm 2,0,a}$ | $\sqrt{K \left[ K + \frac{4n}{7} (g_4 - g_2) \right]}$ |
| $E_{\pm 2,0,b}$ | $\sqrt{K \left[ K + \frac{4n}{35} (3g_4 - 10g_2 + 7g_0) \right]}$ |
| $E_{\pm 2,0,c}$ | $\sqrt{K \left[ K + \frac{4n}{7} (3g_4 + 4g_2) \right]}$ |
Collective excitations in an \( F = 2 \) Bose-Einstein condensate appear due to the Zeeman shifts, but we believe it still worthwhile to briefly examine what might happen at such a situation.

In the presence of a magnetic field the different \( M \) states are not degenerate, and for strong fields the energy separation of adjacent states are not equal, either. The Zeeman shifts for the \( M \) states are given by

\[
E_{Z,M} = -\mu_B \left( M g_F B + M^2 g_F^{(2)} B^2 \right),
\]

where \( \mu_B \) is the Bohr magneton, and \( g_F \) and \( g_F^{(2)} \) are the linear and quadratic Zeeman coefficients. The terms in the Gross-Pitaevskii equations that are responsible for spin-mixing dynamics begin to acquire time-dependent phase factors. As the magnetic field is strong these phase factors vary on a timescale much shorter than everything else in our system. This allows us to average over them, leaving the Gross-Pitaevskii equations without the spin-mixing terms.

The chemical potentials for different components are now different. We write the wavefunction as \( \Phi_M(t) = \Psi_M \exp(-i\mu_M t/\hbar) \) and solve for the chemical potential \( \mu_M \). For polar and ferromagnetic states they are given by \( \mu_M = \mu_0 + E_{Z,M} \), where \( \mu_0 \) is the chemical potential in the absence of magnetic field (see previous sections). For the cyclic state the the chemical potentials for \( M = \pm 2, 0 \) atoms are \( \mu_M = \frac{n}{70} (33g_4 + 30g_2 + 7g_0) + E_{Z,M} \), i.e. the constant in front of the Zeeman shift is different from the zero field chemical potential.

We have calculated the excitations without the spin-mixing terms in the same manner as before. The main branches of the collective excitations (i.e., the ones having a form \( \sqrt{K(K+2\mu_0)} \)) are unchanged for all the other states except for the cyclic state. For the cyclic state the main branch is now \( \sqrt{K \left( K + \frac{n}{35} (33g_4 + 30g_2 + 7g_0) \right)} \). In the absence of spin-mixing terms the components with very little population follow an ordinary free particle Schrödinger equation. In particular, the terms proportional to the complex conjugate of the wavefunction are missing. Therefore excitations corresponding to these components take the free particle form with different gaps.

In addition to the main branch the \( P \) and \( P1 \) states have also another Bogoliubov-type excitation with energies \( E_{\pm 2,b} \) and \( E_{\pm 1,b} \) respectively (here we follow the notation for the zero field results). These Bogoliubov type excitations result from two component GP equations of the type studied by Timmermans [24]. For the \( F' \) state all excitation, except the main branch, take the free particle form. This implies that possible dynamical instabilities of the \( F' \) state have been replaced with thermodynamical instabilities.

For the cyclic states we are left with three excitations of the Bogoliubov form, the main branch and two others (table [3]). These involve atoms from the components with large populations. The cyclic state can still be dynamically stable if \( (17g_4 - 10g_2 - 7g_0) > 0 \).

As for the thermodynamical stability of the cyclic state we have three magnetization conserving channels available. The first channel (a) is to take atoms from \( M = \pm 2 \) states and put them into \( M = \pm 1 \) states, the second channel (b) is to take two atoms from the \( M = 0 \) state and put them into \( M = \pm 1 \) states, and the last channel (c) is to take two atoms from the \( M = 0 \) state and put them into \( M = \pm 2 \) states. The first and second
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Table 5. Excitations in a strong magnetic field for the cyclic state. The subscripts in the names of the branches refer to the $M$ state or states involved in the excitation superposition. The energies for the $M = \pm 1$ atoms are scaled by $\frac{n}{70}(33g_4 + 30g_2 + 7g_0)$.

$$
\begin{align*}
E_{\pm 2,0,a} &= \sqrt{K \left[ K + \frac{1}{35} n (33g_4 + 30g_2 + 7g_0) \right]} \\
E_{\pm 2,0,b} &= \sqrt{K \left[ K + \frac{1}{35} n (17g_4 - 10g_2 - 7g_0) \right]} \\
E_{\pm 2,0,c} &= \sqrt{K \left[ K + \frac{1}{35} n (3g_4 - 10g_2 + 7g_0) \right]} \\
E_1 &= K + E_{Z,1} + \frac{n}{70} (17g_4 - 10g_2 - 7g_0) \\
E_{-1} &= K + E_{Z,-1} + \frac{n}{70} (17g_4 - 10g_2 - 7g_0)
\end{align*}
$$

processes tell us about relaxation towards the $P1$ state and the third process indicates relaxation to the $P$ (or $P0$) state. To ensure dynamical stability we will assume that $(17g_4 - 10g_2 - 7g_0) > 0$. The three processes come with energy costs

$$
\begin{align*}
\Delta E_a &= E_{\text{required}} - E_{\text{released}} = \\
&= 2K - 6\mu_B g_F^{(2)} B^2 + \frac{n}{35} (17g_4 - 10g_2 - 7g_0) \quad (5) \\
\Delta E_b &= E_{\text{required}} - E_{\text{released}} = \\
&= 2K + 2\mu_B g_F^{(2)} B^2 + \frac{n}{35} (17g_4 - 10g_2 - 7g_0) \quad (6) \\
\Delta E_c &= E_{\text{required}} - E_{\text{released}} = 2K + 8\mu_B g_F^{(2)} B^2 \quad (7)
\end{align*}
$$

If $g_F^{(2)}$ is positive and the term $\frac{n}{35} (17g_4 - 10g_2 - 7g_0)$ is small (which is very likely) one expects the process (a) to dominate since this does not necessarily cost energy. This is reflected in the increase of the number of $M = \pm 1$ atoms and in the reduction of the number of $M = \pm 2$ atoms. Finally, the spin-changing processes will drive the system towards the true ground state, the $F$ state. If, on the other hand, the term $\frac{n}{35} (17g_4 - 10g_2 - 7g_0)$ turns out to be larger than $6\mu_B g_F^{(2)} B^2$, all three processes cost energy, forming an energy barrier, and the cyclic state can be metastable.

If $g_F^{(2)}$ is negative the process (c) does not cost energy. Also, if $\frac{n}{35} (17g_4 - 10g_2 - 7g_0)$ is small the process (b) does not cost energy either, but as there is more energy to be gained in process (c) one expects a relaxation to the $P$ state, until spin-changing processes start to dominate.

If there is some other mechanism, beside the magnetic field, that randomizes the phase one could apply the strong field results in the limit $B \to 0$ and the metastability of the cyclic state might be a more relevant issue.

6. Fragmented state

In a fragmented condensate the numbers of particles in different $M$ states are separately constant \[17, 21\] and the terms in the Hamiltonian giving rise to the spin mixing
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dynamics average to zero. This suggests that we are describing the system in the
framework of fragmented states as soon as we have blocked the spin-mixing dynamics. 

Thus our previous results for the \( F = 2 \) spinor condensate in a strong magnetic
field could be equivalent to a study of a spinor condensate in a framework of fragmented
states. There is another argument which supports this interpretation. In a fragmented
state the relative phase between different \( M \) state wavefunctions is not a meaningful
concept. We can consider the relative phases as random variables between 0 and \( 2\pi \). As this randomness is present for all timescales we can replace all the terms which depend
on a relative phase of different \( M \) states, with zero, their average value.

Most of the chemical potentials do not change when spin-mixing terms are dropped.
As in the thermodynamic limit the energy of the \( F = 1 \) single condensate is the
same as the energy of the fragmented state, we find it quite likely that an unchanged
chemical potential reflects the connection between frameworks for single condensates and
fragmented condensates. Especially so since the only state whose chemical potential is
affected by ignoring spin-mixing dynamics is the cyclic state, the only fundamentally
new state not already present in the \( F = 1 \) condensate.

We conclude that many collective excitations for ferromagnetic and polar states are
different in a fragmented \( F = 2 \) condensate, compared to the zero field case. The main
branches are not changed, but especially excitations that are superpositions of different
\( M \) states are replaced with particle-like excitations. Also many possible dynamical
instabilities of the single condensate are replaced with thermodynamic instabilities
implying, perhaps, that the relaxation rates to the true ground states are different
for the fragmented condensates as opposed to the single condensates.

As the chemical potential of the cyclic state depends on whether we keep the spin-
mixing terms or not, it follows that the chemical potential depends on whether or not the
relative phase of the \( M = 2, 0, -2 \) components is well defined. Also, since some physical
properties such as sound velocity, for example, depend on the chemical potential we
conjecture that if the relative phase in a cyclic state is well defined (as it should be)
there should be a change in the sound velocity as we move from a zero magnetic field
to a strong one. The magnitude if this shift is given by

\[
\Delta c_s = c_{B \geq 0} - c_{B = 0} = \sqrt{\frac{n}{7m}} \left( \sqrt{\frac{33g_4 + 30g_2 + 7g_0}{10}} - \sqrt{3g_4 + 4g_2} \right). \tag{8}
\]

The relaxation of the non-magnetic cyclic state to a polar state could be faster
than the relaxation to the true ground state, the \( F \) state, since such relaxation does not
\( \dagger \) Often the precise nature of the condensate state is more or less irrelevant. The measurement
process can, for example, “induce” a relative phase even between number states \( \dagger \). Also the density
distribution is accurately given by the GP equation even though its theoretical justification in the case
of number states is dubious. Due to these complications the wave function in the absence of spin mixing
should be considered no more than the square root of the density. Whether the act of measurement
can induce a well defined phase to the different components (and thus destroy the fragmented state) is
an interesting problem that should be addressed.
require spin-changing collisions. But on the other hand, the before-mentioned energy barrier could make the cyclic state quite robust. A detailed study of the different relaxation rates in a spinor condensates is certainly warranted.

7. Superfluid properties

By applying a global rotation $U$ together with gauge transformation $e^{i\theta}$ to the spinor $\hat{\zeta}$ we get a spinor that is physically identical

$$\hat{\zeta}' = e^{i\theta} U \hat{\zeta}. \quad (9)$$

In terms of the Euler angles the rotation operator is given by

$$U(\alpha, \beta, \gamma) = e^{iF_\alpha} e^{iF_\beta} e^{iF_\gamma}. \quad (10)$$

By making the rotations and gauge transformation local we can induce velocity fields to the system. Superfluid velocity is defined as $v_s = -i(\hbar/m_a)\hat{\zeta}^\dagger \nabla \hat{\zeta}$ and using this definition it is straightforward to calculate the superfluid velocities of the polar, ferromagnetic, and cyclic states:

$$(v_s)_\text{polar} = \frac{\hbar}{m_a} \nabla \theta, \quad (11)$$

$$(v_s)_\text{ferro} = \frac{2\hbar}{m_a} \left[ \nabla (\gamma + \theta/2) + \cos \beta \nabla \alpha \right], \quad (12)$$

$$(v_s)_\text{cyclic} = \frac{\hbar}{m_a} \nabla \theta. \quad (13)$$

The superfluid velocity has the same form for all the polar states and the ferromagnetic superfluid velocity corresponds to the $F$ state. Inspecting these results it is clear that the polar and cyclic states support only ordinary vortices for which the density at the core is forced to zero. The ferromagnetic superfluid velocity has the same form as the corresponding result for the $F = 1$ spinor condensate [7] (there is a factor of 2 difference, though) and thus coreless vortices are possible although they are not expected to be topologically stable.

8. Discussion

In this paper we have calculated that almost all the excitations of a homogeneous $F = 2$ atomic condensate in zero magnetic field are either Bogoliubov-type excitations or free-particle excitations. The only exception is the ferromagnetic $F'$ state, which, however, is never a true ground state. In a strong magnetic field all the excitations have only the Bogoliubov form or the free-particle form. We have also noted that depending on the scattering lengths for collisions between atoms in different $M$ states and on the sign of the quadratic Zeeman coefficient $g_F^{(2)}$ there could be an energy barrier suppressing the decay of the cyclic state into the (possibly) lower lying polar states.

There are not many physical observables that depend on the relative phase of the different $M$ state wavefunctions in a spinor condensates. We have showed that the
value of the chemical potential in the cyclic state depends on the existence of relative phase of the $M = \pm 2, 0$ atoms. If there is a well-defined phase relationship between these different components (as there should be, for the cyclic state), then all physical quantities depending on the chemical potential will (in general) change in a strong magnetic field.

As discussed in the text, for simplicity we have ignored several important aspects of the real physical situation. We have not e.g. discussed dissipative processes or processes that do not conserve magnetization. Thus our results are indicative only, but we believe they reveal some basic characteristics and important a priori differences between the $F = 1$ and $F = 2$ spinor condensates. Regarding the true experimental situation, the trapping potential has also been absent. But since the main branch of the collective excitations for the cyclic state depends on the existence of well defined relative phase, it can be conjectured that similar dependence should also exist for the collective excitations of the trapped cyclic state.

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