CONVOLUTION ESTIMATES AND THE GROSS-PITAEVSKII HIERARCHY

WILLIAM BECKNER

Abstract. Extensions to higher-dimensions are given for a convolution estimate used by Klainerman and Machedon in their study of uniqueness of solutions for the Gross-Pitaevskii hierarchy. Such estimates determine more general forms of Stein-Weiss integrals involving restriction to smooth submanifolds.

Analysis of the Gross-Pitaevskii hierarchy has led to the development and application of functional analytic mappings for the rigorous description of many-body interactions in quantum dynamics. In their formative and influential paper on uniqueness of solutions for the Gross-Pitaevskii hierarchy, Klainerman and Machedon determine uniform bounds for a three-dimensional convolution integral. The idea of their argument rests on an extension of the classical convolution for Riesz potentials

$$\int_S \frac{1}{|w-g|} \frac{1}{|y|^\mu} d\sigma$$

where $S$ is a smooth submanifold in $\mathbb{R}^n$, $w \in \mathbb{R}^m$ and the objective is to bound the size of the integral by an inverse power of $|w|$ under suitable conditions on $\lambda$ and $\mu$. Such an estimate can be viewed as a step in the larger and dual program for understanding how smoothness controls restriction to a non-linear sub-variety (see [1]). Two natural extensions to higher dimensions are suggested here:

$$|w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ 1 + \sum' |x_k|^2 - |x_n|^2 \right] \delta \left( w - \sum x_k \right) \prod |x_k|^{-(n-1)} dx_1 \cdots dx_n$$  \hspace{1cm} (1)

$$|w|^{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \delta \left[ \tau + |z|^2 + |x|^2 - |y|^2 \right] \delta \left( w - x - y - z \right) \left[ |z| |x| |y| \right]^{-(n-1)} dx dy dz$$  \hspace{1cm} (2)

with the objective being to determine uniform bounds in terms of the variables $\tau > 0$ and $w \in \mathbb{R}^n$ with $n > 1$ (here the prime on the symbol for sum, product or sequence indicates that the last term should be dropped). From the dilation character of the expression, one can use “uniformity” to eliminate one variable so it suffices to consider $\tau = 1$:

$$\Lambda_n(w) = |w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ 1 + \sum' |x_k|^2 - |x_n|^2 \right] \delta \left( w - \sum x_k \right) \prod |x_k|^{-(n-1)} dx_1 \cdots dx_n$$

$$\Delta_n(w) = |w|^{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \delta \left[ 1 + |z|^2 + |x|^2 - |y|^2 \right] \delta \left( w - x - y - z \right) \left[ |x| |y| |z| \right]^{-(n-1)} dx dy dz$$

One observes that the first expression is an extension of the classical convolution form

$$(g * f_1 * \cdots * f_n)(w), \quad g \in L^1(\mathbb{R}^n), \quad f_k \in L^{n/(n-1)}(\mathbb{R}^n)$$

which is uniformly continuous and in the class $C_0(\mathbb{R}^n)$ using the Riemann-Lebesgue lemma. Here the convolution for Lebesgue classes is replaced by Riesz potentials, but the multivariable integration is constrained to be on a hyperbolic surface invariant under action by the indefinite orthogonal group.
Theorem 1. $\Lambda_n(w)$ is bounded for $n \geq 3$; $\Delta_n(w)$ is bounded for $n \geq 2$.

The argument for the proof of Theorem 1 will be developed in several steps and will be reduced to the second statement when the dimension is at least four. Note that $\Lambda_3 = \Delta_3$, and this is the case determined by Klainerman and Machedon.

Proof. Step 1: for $n = 2$, $\Lambda_2(w)$ is unbounded. This case is instructive and will identify the method used later in the proof of the second part.

$$\Lambda_2(w) = |w|^2 \int_{\mathbb{R}^2} \delta \left( 1 + |w - y|^2 - |y|^2 \right) \frac{1}{|w - y|} \frac{1}{|y|} \, dy$$

$$= |w|^2 \int_1^{\infty} \int_{-\pi/2}^{\pi/2} \delta \left[ 1 + |w|^2 - 2rw \cos \theta \right] \frac{1}{\sqrt{r^2 - 1}} \, dr \, d\theta$$

(since $\cos \theta$ must be positive and $|y| > 1$)

$$= 2|w|^2 \int_0^1 \frac{1}{\sqrt{1 - u}} \frac{1}{\sqrt{(1 + |w|^2)^2 - 4|w|^2 u^2}} \, du$$

$$= \frac{|w|^2}{1 + |w|^2} \int_0^1 \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1 - u}} \frac{1}{\sqrt{1 - \beta u}} \, du \quad \beta = \frac{4|w|^2}{(1 + |w|^2)^2} \leq 1$$

$$= \pi \frac{|w|^2}{1 + |w|^2} \, F \left( \frac{1}{2}, \frac{1}{2}; 1; \beta \right) = \frac{2|w|^2}{1 + |w|^2} \, K(\sqrt{\beta})$$

where $F$ denotes the hypergeometric function and $K$ the complete elliptic integral. $\Lambda_2(w) = \infty$ for any $w$ on the unit sphere $|w| = 1$. Observe that for $\beta \approx 1$ (e.g., $|w| \approx 1$)

$$\Lambda_2(w) \approx - \ln \left( \sqrt{1 - \beta}/2 \right)$$

A similar calculation will now give:

Lemma. For $n = 2$ and $\frac{1}{2} < \alpha < 1$

$$\Lambda_{2,\alpha}(w) = |w|^{2\alpha} \int_{\mathbb{R}^2} \delta \left( 1 + |w - y|^2 - |y|^2 \right) |w - y|^{-\alpha} |y|^{-\alpha} \, dy$$

is uniformly bounded in $w$.

Observe that by dilation symmetry this result is equivalent to uniform boundedness with $\tau > 0$, $w \in \mathbb{R}^2$ for

$$|w|^{2\alpha} \int_{\mathbb{R}^2} \delta \left( \tau + |w - y|^2 - |y|^2 \right) |w - y|^{-\alpha} |y|^{-\alpha} \, dy$$.
Step 2: let \( n \geq 4 \); then using the second delta function for the variable \( x_{n-2} \)

\[
\Lambda_n(w) = |w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ 1 + \sum_{k=1}^{n-3} |x_k|^2 + |w - \sum x_k - x_{n-1} - x_n|^2 + |x_{n-1}|^2 - |x_n|^2 \right] \times \\
\prod_{k=1}^{n-3} |x_k|^{-(n-1)} \left| w - \sum x_k \right|^{-(n-1)} \left( |x_{n-1}| \right)^{-(n-1)} dx_1 \cdots dx_{n-3} dx_{n-1} dx_n
\]

\[
= |w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \prod_{(n-3) \text{ copies}} |x_k|^{-(n-1)} \left| w - \sum x_k \right|^{-(n-1)} \times \\
\left[ |w - \sum x_k - x - y| |x| \right]^{-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta \left[ 1 + \sum_{k=1}^{n-3} |x_k|^2 + |w - \sum x_k - x| \right] \times \\
\left[ |w - \sum x_k - x - y| |x| \right]^{-1} \left[ |w - x - y| \right]^{-1} dx dy \right] dx_1 \cdots dx_{n-3}
\]

\[
\leq c_n |w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \prod_{(n-3) \text{ copies}} |x_k|^{-(n-1)} \left| w - \sum x_k \right|^{-(n-1)} dx_1 \cdots dx_{n-3}
\]

where

\[
c_n = \sup_{\tau, v} |v|^{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta \left[ \tau + |v - x - y|^2 + |x|^2 - |y|^2 \right] \left[ |v - x - y| \right]^{-1} dx dy = \sup_w \Delta_n(w)
\]

where in the earlier expression, \( \tau = 1 + \sum_{k=1}^{n-3} |x_k|^2 \) and \( v = w - \sum_{k=1}^{n-3} x_k \). Then

\[
\Lambda_n(w) \leq c_n |w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \prod_{(n-3) \text{ copies}} |x_k|^{-(n-1)} \left| w - \sum x_k \right|^{-(n-1)} dx_1 \cdots dx_{n-3}
\]

Using the following notation for the Fourier transform and its action on Riesz potentials

\[
(Ff)(x) = \int_{\mathbb{R}^n} e^{2\pi ixy} f(y) dy
\]

\[
\mathcal{F} \left[ |x|^{-\lambda} \right](\xi) = \pi^{-n/2 + \lambda} \frac{\Gamma \left( \frac{n-\lambda}{2} \right)}{\Gamma \left( \frac{\lambda}{2} \right)} |\xi|^{-n - \lambda}
\]

\[
|w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \prod_{(n-3) \text{ copies}} |x_k|^{-(n-1)} \left| w - \sum x_k \right|^{-(n-1)} dx_1 \cdots dx_{n-3}
\]

\[
= \pi^{[(n-1)^2-3]/2} \left[ \Gamma \left( \frac{n-1}{2} \right) \right]^{-(n-2)} \left[ \Gamma \left( \frac{n}{2} - 1 \right) \right]^{-1}
\]

Hence \( \Lambda_n(w) \) is bounded for \( n \geq 3 \) if \( \Delta_n(w) \) is bounded for \( n \geq 3 \).

An intriguing feature of this argument is that \( (n-1) \) is the unique uniform inverse power where one can preserve dilation invariance and obtain a reduction of this type that connects bounds for integrals of the form \( \Lambda_n, \Delta_n \). Perhaps this circumstance reflects a larger underlying symmetry in addition to the correspondence with the property that the convolution of \( n \) functions in \( L^{n/(n-1)}(\mathbb{R}^n) \) will be uniformly continuous.
STEP 3: consider \( n = 2 \)

\[
\Delta_2(w) = |w| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \delta \left[ 1 + |w-x|^2 + |x-y|^2 - |y|^2 \right] \frac{1}{|w-x| |x-y| |y|} \, dx \, dy
\]

\[
= |w| \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \, \delta \left[ 1 + |w-x|^2 + |x|^2 - 2|x| |y| \cos \theta \right] |w-x|^{-1} |y|^{-1} \left[ |y|^2 - 1 - |w-x|^2 \right]^{-1/2}
\]

\[
= 2|w| \int_{\mathbb{R}^2} dx |w-x|^{-1} \int_0^1 \frac{1}{\sqrt{1-u}} \left[ (1 + |w-x|^2 + |x|^2)^2 - 4|x|^2 u^2 (1 + |w-x|^2) \right]^{-1/2} du
\]

\[
= |w| \int_{\mathbb{R}^2} |w-x|^{-1} \left[ 1 + |w-x|^2 + |x|^2 \right]^{-1} \int_0^1 \frac{1}{u} \frac{1}{\sqrt{1-u}} \frac{1}{\sqrt{1 - \beta(x)u}} \, du \, dx
\]

where

\[
\beta(x) = \frac{4|x|^2 (1 + |w-x|^2)}{(1 + |w-x|^2 + |x|^2)^2} \leq 1
\]

Since for \( 0 < u < 1, \sqrt{1-u} \sqrt{1-\beta(x)} \leq 1 - u \beta(x) \)

\[
\Delta_2(w) \leq \left[ \int_0^1 u^{-1/2} (1-u)^{-3/4} \, du \right] |w| \int_{\mathbb{R}^2} |w-x|^{-1} \left[ 1 + |w-x|^2 + |x|^2 \right]^{-1/2} \left[ 1 + |w-x|^2 - |x|^2 \right]^{-1/2} \, dx
\]

Set \( \delta = \frac{1}{|w|} \), dilate by \( |w| \) and choose \( \xi \) as the \( x_1 \) direction

\[
\Delta_2(w) \leq c \int_{\mathbb{R}^2} \left[ |x_1 - 1|^2 + |x_2|^2 \right]^{-1/2} \left[ 1 + 4 \left( (x_1 - 1/2)^2 + |x_2|^2 \right) \right]^{-1/2} \left| x_1 - \frac{1}{2} \left( 1 + \frac{1}{|w|^2} \right) \right|^{-1/2} \, dx_1 \, dx_2
\]

Rearrange in the variable \( x_1 \) using

\[
\int_{\mathbb{R}^2} f(x, y) g(x, y) h(x, y) \, dx \, dy \leq \int_{\mathbb{R}^2} f_\#(x, y) g_\#(x, y) h_\#(x, y) \, dx \, dy
\]

where \( f_\#(x, y) \) is the equimeasurable symmetric decreasing rearrangement of \( |f(x, y)| \) in the variable \( x \in \mathbb{R} \). Then

\[
\Delta_2(w) \leq c \int_{\mathbb{R}^2} |x|^{-1} \left( 1 + 4 |x|^2 \right)^{-1/2} |x_1|^{-1/2} \, dx
\]

which is a convergent integral as one sees by using polar coordinates. Hence \( \Delta_2(w) \) is uniformly bounded. The option to directly use rearrangement depends on the choice of the inverse power, e.g., the value \( (n-1) \).

STEP 4: by using simple radial coordinate estimates, one can obtain for \( n > 2 \) (\( c \) denotes a generic constant)

\[
\Delta_n(w) \leq c \Delta_2(\bar{w})
\]

where \( \bar{w} \in \mathbb{R}^2 \) with \( |w| = |ar{w}| \).
Observe that the integrands for both expressions treated here, Δ_n(w) and Δ_n(\sigma), are functions only of lengths and polar angles so that facilitates the simplicity of the argument.

\[ \Delta_n(w) = |w|^{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta \left[ |x|^2 + |y|^2 - 2x \cdot y \right] |x| y^{-n-1} \times \left[ |y|^2 - 1 - |x|^2 \right]^{-(n-1)} \, dx \, dy \]

\[ = \frac{\sigma(S^{n-2})}{2} |w|^{n-1} \int_{\mathbb{R}^n} |w - x|^{-(n-1)} (2|x|)^{n-2} \left( 1 + |w - x| + |x| \right)^{-(n-1)} \times \int_0^1 u^{-1/2}(1 - u)^{(n-3)/2} (1 - \beta(x)u)^{-(n-1)/2} \, du \, dx \]

with \( \beta(x) \) as before. Since \( 0 \leq \beta(x) \leq 1 \) and \( 0 \leq u \leq 1 \)

\[ (1 - u)^{(n-3)/2}(1 - \beta(x)u)^{-(n-1)/2} \leq (1 - u)^{-1/2}(1 - \beta(x)u)^{-1/2} \]

In the integral over \( \mathbb{R}^n \), first make the change of variables \( z = w - x \), and then dilate \( z \) by \( |w| \) and integrate out the non-polar angle variables.

\[ \Delta_n(w) \leq 2^{n-4} \left[ \sigma(S^{n-2}) \right]^2 |w|^2 \int_0^\infty \int_0^\pi \left[ \frac{|w|^2 |z - \xi| |z|}{1 + |w|^2|z - \xi|^2 + |z|^2} \right]^{n-2} \times \left[ 1 + |w|^2 \left( |z - \xi|^2 + |z|^2 \right) \right]^{-1} d|z|(\sin \theta)^{n-2} \, d\theta \int_0^1 u^{-1/2}(1 - u)^{-1/2} (1 - \beta(|w|(z - \xi))u)^{-1/2} \, du \]

Using

\[ \left[ \frac{|w|^2 |z - \xi| |z|}{1 + |w|^2|z - \xi|^2 + |z|^2} \right] \leq \frac{1}{2} \]

\[ \Delta_n(w) \leq \frac{1}{8} \left[ \sigma(S^{n-2}) \right]^2 |w|^2 \int_0^\infty \int_{-\pi}^\pi \left[ 1 + |w|^2 \left( |z - \xi|^2 + |z|^2 \right) \right]^{-1} \int_0^1 u^{-1/2}(1 - u)^{-1/2} (1 - \beta(|w|(z - \xi))u)^{-1/2} \, du \, |z|d|z|d\theta \]

Now since we can take \( \xi \) as defining the polar angle for the coordinate system, and \( |z - \xi| \) only depends on this angle and the length \( |z| \), \( z \) and \( \xi \) can be repositioned as vectors in \( \mathbb{R}^2 \) so that

\[ \Delta_n(w) \leq \frac{1}{8} \left[ \sigma(S^{n-2}) \right]^2 |w|^2 \int_{\mathbb{R}^2} \left[ 1 + |w|^2 \left( |z - \xi|^2 + |z|^2 \right) \right]^{-1} \int_0^1 u^{-1/2}(1 - u)^{-1/2} (1 - \beta(|w|(z - \xi))u)^{-1/2} \, du \, dz \]

Reversing the previous coordinate changes of dilation and translation

\[ \Delta_n(w) \leq \frac{1}{8} \left[ \sigma(S^{n-2}) \right]^2 |w| \int_{\mathbb{R}^2} |w - x|^{-1} \left[ 1 + |x|^2 + |w - x|^2 \right]^{-1/2} \int_0^1 u^{-1/2}(1 - u)^{-1/2} (1 - \beta(x)u)^{-1/2} \, du \, dx \]

which then gives the required control

\[ \sup_{w \in \mathbb{R}^n} \Delta_n(w) \leq \frac{1}{8} \left[ \sigma(S^{n-2}) \right]^2 \sup_{w \in \mathbb{R}^2} \Delta_2(w) \]
and hence the uniform bound for $\Delta_2(w)$ gives a uniform bound for $\Delta_n(w)$, $n > 2$. This completes the proof of Theorem 1.

As noted above, the inverse power $|x|^{-(n-1)}$ has a special role for the convolution estimates discussed here; still the two-dimensional result from the Lemma is suggestive that useful bounds might be obtained for inverse powers close to $\alpha = n - 1$. Consider for $\tau > 0$ and $w \in \mathbb{R}^n$

$$|w|^{\rho} \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ \tau + \sum' |x_k|^2 - |x|^2 \right] \delta \left[ w - \sum x_k \right] \Pi|x_k|^{-\alpha} \, dx_1 \cdots dx_n \quad (3)$$

For dilation invariance, $p = 2 + n(\alpha - n + 1)$ with the further requirement of positivity for the possibility of boundedness; that means $(n - 1) \geq \alpha > (n - 1) - 2/n$ so that asymptotically $\alpha \simeq n - 1$ for large dimension. The upper bound is required by the nature of the proof for uniform boundedness. As with Theorem 1, the proof for uniform bounds will depend on a reduced integral form:

$$|w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta \left[ \tau + |x|^2 - |y|^2 \right] \delta(w - x - y)|x|^{-\alpha} |y|^{-\alpha} \, dx \, dy \quad (4)$$

Here $\sigma = 2\alpha + 2 - n$ for dilation invariance. For both forms, it suffices to show uniform bounds for $\tau = 1$, and in two dimensions they are the same and already proved in the argument for the Lemma.

**Theorem 2.** For $n \geq 2$, $\sigma = 2\alpha + 2 - n$ and $(n - 1)/2 < \alpha < (n - 1)$$$

$$\Theta_{n,\alpha}(w) = |w|^\sigma \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta \left[ 1 + |x|^2 - |y|^2 \right] \delta(w - x - y)|x|^{-\alpha} |y|^{-\alpha} \, dx \, dy$$

is uniformly bounded for $w \in \mathbb{R}^n$.

**Proof.** Let $n \geq 2$:

$$\Theta_{n,\alpha}(w) = |w|^\sigma \int_{\mathbb{R}^n} \delta \left( 1 + |w - y|^2 - |y|^2 \right) |w - y|^{-\alpha} |y|^{-\alpha} \, dy$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma((n - 1)/2)} |w|^\sigma \int_0^1 \int_0^1 \delta \left( 1 + |r|^2 - 2|wr| \right) (r^2 - 1)^{-\alpha/2} r^{n-\alpha-1} (1 - |u|^2)^{(n-3)/2} \, dr \, du$$

$$= \frac{\pi^{(n-1)/2} 2^{2\alpha-n}}{\Gamma((n - 1)/2)} \left[ \frac{|w|^2}{1 + |w|^2} \right]^{2\alpha-n+1} \int_0^1 (1 - u)^{(n-3)/2} u^{\alpha-(n-1)/2} - 1 (1 - \beta(2u))^{-\alpha/w} \, du$$

where $\beta(w) = \frac{4|w|^2}{(1 + |w|^2)^2} \leq 1$ with $(1 - \beta(w)u)^{-\alpha/2} \leq (1 - u)^{-\alpha/2}$. Then

$$\Theta_{n,\alpha}(w) \leq \frac{\pi^{(n-1)/2} 2^{2\alpha-n}}{\Gamma((n - 1)/2)} \left[ \frac{|w|^2}{1 + |w|^2} \right]^{2\alpha-n+1} \int_0^1 (1 - u)^{(n-1-\alpha)/2} u^{\alpha-(n-1)/2} \, du$$

$$= \frac{\pi^{(n-1)/2} 2^{2\alpha-n}}{\Gamma((n - 1)/2) \Gamma((2\alpha - n + 1)/2)} \left[ \frac{|w|^2}{1 + |w|^2} \right]^{2\alpha-n+1}$$

Hence $\Theta_{n,\alpha}(w)$ is bounded for $(n - 1)/2 < \alpha < n - 1$. Note that $\Lambda_{2,\alpha} = \Theta_{2,\alpha}$.

**Theorem 3.** For $n \geq 3$, $\rho = 2 + n(\alpha - n + 1)$ and $n - 1 > \alpha > n - 1 - 2/n$$$

$$\Lambda_{n,\alpha}(w) = |w|^{\rho} \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ 1 + \sum' |x_k|^2 - |x|^2 \right] \delta\left( w - \sum x_k \right) \Pi|x_k|^{-\alpha} \, dx_1 \cdots dx_n$$

is uniformly bounded for $w \in \mathbb{R}^n$. 


Proof. The argument here rests on the boundedness of $\Theta_{n,\alpha}$ following the method of Step 2 in the proof of Theorem 1. For $n \geq 3$ use the second delta function for the variable $x_{n-1}$ and write

$$
\Lambda_{n,\alpha}(w) = |w|^\rho \int_{\mathbb{R}^{n-1} \times \mathbb{R}^n} \prod_{k=n-2}^{n-1} |x_k|^{-\alpha} |w - \sum_{k=1}^{n-2} x_k|^{-\sigma} \times
$$

$$
\left[ |w - \sum_{k=1}^{n-2} x_k|^{\sigma} \int_{\mathbb{R}^n} \delta \left[ 1 + \sum_{k=1}^{n-2} |x_k|^2 + |w - \sum_{k=1}^{n-2} x_k - y|^2 - |y|^2 \right] \times
$$

$$
\left[ |w - \sum_{k=1}^{n-2} x_k - y| |y| \right]^{-\alpha} dy \right] dx_1 \ldots dx_{n-2}
$$

$$
\leq c_n |w|^\rho \int_{\mathbb{R}^{n-1} \times \mathbb{R}^n} \prod_{k=n-1}^{n-2} |x_k|^{-\alpha} |w - \sum_{k=1}^{n-2} x_k|^{-\sigma} dx_1 \ldots dx_{n-2}
$$

where

$$
c_n = \sup_{\tau,\nu} |v|^\sigma \int_{\mathbb{R}^n} \delta \left[ \tau + |v - y|^2 - |y|^2 \right] \left[ |v - g| |y| \right]^{-\alpha} dy = \sup_w \Theta_{n,\alpha}(w)
$$

and in the earlier expression, $\tau = 1 + \sum_{k=1}^{n-2} |x_k|^2$ and $v = w - \sum_{k=1}^{n-2} x_k$. Then

$$
\Lambda_{n,\alpha}(w) \leq c_n |w|^\rho \int_{\mathbb{R}^{n-1} \times \mathbb{R}^n} \prod_{k=n-1}^{n-2} |x_k|^{-\alpha} |w - \sum_{k=1}^{n-2} x_k|^{-\sigma} dx_1 \ldots dx_{n-2}
$$

with the integral

$$
|w|^\rho \int_{\mathbb{R}^{n-1} \times \mathbb{R}^n} \prod_{k=n-1}^{n-2} |x_k|^{-\alpha} |w - \sum_{k=1}^{n-2} x_k|^{-\sigma} dx_1 \ldots dx_{n-2}
$$

being constant in $w$ so that $\Lambda_{n,\alpha}(w)$ is bounded for $n \geq 3$ if $\Theta_{n,\alpha}(w)$ is bounded for $n \geq 3$.

In surveying the estimates outlined above, the critical computation would seem to be the surface integral

$$
\int_S \frac{1}{|w - y|^{\lambda}} \frac{1}{|y|^{\mu}} dy
$$

which then the convolution algebra for Riesz potentials allows an extended multilinear result. For completeness, an outline is given for non-uniform Riesz potentials.

**Theorem 4.** For $n \geq 2$, $\sigma = \alpha + \lambda + 2 - n$, $\alpha + \lambda > n - 1$ and $0 < \alpha < n - 1$

$$
\Theta_{n,\alpha,\lambda}(w) = |w|^\sigma \int_{\mathbb{R}^n} \delta \left[ 1 + |x|^2 - |y|^2 \right] \delta(w - x - y)|x|^{-\alpha} |y|^{-\lambda} dx dy
$$

is uniformly bounded for $w \in \mathbb{R}^n$.

**Proof.** Let $n \geq 2$; observe that since $|y| \geq 1$, there is no upper bound for $\lambda$ in this computation.

$$
\Theta_{n,\alpha,\lambda}(w) = |w|^\sigma \int_{\mathbb{R}} \delta \left[ 1 + |w - y|^2 - |y|^2 \right] |w - y|^{-\alpha} |y|^{-\lambda} dy
$$

$$
= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} |w|^\sigma \int_{1}^{\infty} \int_{0}^{1} \delta \left( 1 + |w|^2 - 2|w|ru \right) (r^2 - 1)^{-\alpha/2} r^{n-\lambda-1}(1 + u^2)^{(n-3)/2} dr du
$$

$$
= \frac{2^{\alpha+\lambda-n\pi(n-1)/2}}{\Gamma((n-1)/2)} \left[ \frac{|w|^2}{1 + |w|^2} \right]^{\alpha+\lambda-n+1} \int_{0}^{1} (1 - u)^{(n-3)/2} u^{(\alpha+\lambda-n+1)/2} - (1 - \beta(w)u)^{-\alpha/2} du
$$
where $\beta(w) = \frac{4|w|^2}{(1 + |w|)^2} \leq 1$ with $(1 - \beta(w)u)^{-\alpha/2} \leq (1 - u)^{-\alpha/2}$. Then

$$\Theta_{\alpha,\lambda}(w) = \frac{2^{\alpha+\lambda-n} \Gamma(n-1)/2}{\Gamma((n-1)/2)} \int_0^1 u^{(\alpha+\lambda-n+1)/2} (1 - u)^{(n-1)/2 - 1} du$$

$$= \frac{2^{\alpha+\lambda-n} \Gamma((\alpha + \lambda - n + 1)/2) \Gamma((n-1)/2)}{\Gamma((n-1)/2) \Gamma(1/2)} \frac{|w|^2}{1 + |w|^2}$$

Hence $\Theta_{\alpha,\lambda}(w)$ is bounded for $\lambda > 0$, $0 < \alpha < n - 1$ and $\alpha + \lambda > n - 1$. \qed

**Theorem 5.** For $n \geq 3$, consider real-valued exponents $0 < \alpha_k < n$, $k = 1, \ldots, n-1$ and $\lambda > 0$ with $\alpha = \sum \alpha_k$ and $\rho = 2 + \alpha + \lambda - n(n-1)$ so that $0 < \rho < n$. Further assume one exponent $\alpha_i$ together with $\lambda$ satisfies: $0 < \alpha_i < n - 1$ and $n - 1 < \alpha_i + \lambda < 2(n-1)$; relabel this $\alpha_i$ as $\alpha_{n-1}$. Then

$$\Lambda_{\alpha,\lambda} = \int_{\mathbb{R}^n} |w|^\sigma \int_{\mathbb{R}^n} \delta \left[ \tau + |x|^2 - |y|^2 \right] \delta(w - x - y)|x|^{-\alpha_{n-1}}|y|^{-\lambda} dx dy$$

for $\tau > 0$ and $w \in \mathbb{R}^n$ which is determined by Theorem 4. For $n \geq 3$ use the second delta function for the variable $x_{n-1}$ and write with $\sigma = \alpha_{n-1} + \lambda + 2 - n$

$$\Lambda_{\alpha,\lambda} = |w|^\rho \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-2} |x_k|^{-\alpha_k} \left| w - \sum_{k=1}^{n-2} x_k \right|^{-\sigma} \times$$

$$\left| w - \sum_{k=1}^{n-2} x_k \right|^{\alpha_{n-1}} \int_{\mathbb{R}^{n-1}} \left[ \left| w - \sum_{k=1}^{n-2} x_k - y \right|^{-\alpha_{n-1}}|y|^{-\lambda} \right] dx_{n-1} \ldots dx_{n-2}$$

$$\leq c_{\alpha,\lambda} |w|^\rho \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-2} |x_k|^{-\alpha_k} \left| w - \sum_{k=1}^{n-2} x_k \right|^{-\sigma} \times$$

$$\left| w - \sum_{k=1}^{n-2} x_k \right|^{\alpha_{n-1}} \int_{\mathbb{R}^{n-1}} \left[ \left| w - \sum_{k=1}^{n-2} x_k - y \right|^{-\alpha_{n-1}}|y|^{-\lambda} \right] dx_{n-1} \ldots dx_{n-2}$$

where

$$c_{\alpha,\lambda} = \sup_{\tau,\rho} |w|^\rho \int_{\mathbb{R}^n} \delta \left[ \tau + |v|^2 - |y|^2 \right] |v - y|^{-\alpha_{n-1}}|y|^{-\lambda}$$

and in the earlier expression $\tau = 1 + \sum_{k=1}^{n-2} |x_k|^2$ and $v = w - \sum_{k=1}^{n-2} x_k$. Then

$$\Lambda_{\alpha,\lambda}(w) \leq c_{\alpha,\lambda} |w|^\rho \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-2} |x_k|^{-\alpha_k} \left| w - \sum_{k=1}^{n-2} x_k \right|^{-\sigma} dx_{n-1} \ldots dx_{n-2}$$

with the integral

$$|w|^\rho \int_{\mathbb{R}^{n-1}} \prod_{k=1}^{n-2} |x_k|^{-\alpha_k} \left| w - \sum_{k=1}^{n-2} x_k \right|^{-\sigma} dx_{n-1} \ldots dx_{n-2}$$

being constant in $w$ so that $\Lambda_{\alpha,\lambda}(w)$ is bounded in $w$ for $n \geq 3$ if $\Theta_{\alpha,\lambda}(w)$ is bounded for $n \geq 3$ subject to the conditions on $\alpha_{n-1}$ and $\lambda$. \qed
Implicit in the formulation of the problems treated here is the continuing development of new forms that characterize control by smoothness for size. As an example and a consequence of the principal estimate obtained here, bounds for new Stein-Weiss integrals with a kernel determined by restriction to a smooth submanifold can be shown.

**Theorem 6.** Define

\[ K(w, v) = \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \Pi |x_k|^{-(n-1)} \left[ |w - \sum x_k| |v - \sum x_k| \right]^{-(n-1)} \times \delta \left[ 1 + \sum' |x_k|^2 - |x_n|^2 \right] dx_1 \ldots dx_n, \quad n \geq 3; \]

\[ K_{n, \alpha}(w, v) = \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \Pi |x_k|^{-\alpha} \left[ |w - \sum x_k| |v - \sum x_k| \right]^{-\lambda} \times \delta \left[ 1 + \sum' |x_k|^2 - |x_n|^2 \right] dx_1 \ldots dx_n, \quad n \geq 3 \]

\[ \lambda = n(n - \alpha + 1)/2 - 1, \quad n - 1 > \alpha > n - 2 - 2/n \]

\[ H_{n, \alpha}(w, v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x|^{-\alpha} |y|^{-\alpha} \left[ |w - x - y| |v - x - y| \right]^{-3n/2 - 1 - \alpha} \times \delta \left( 1 + |x|^2 - |y|^2 \right) dx dy, \quad n \geq 2, \quad (n - 1)/2 < \alpha < n - 1; \]

\[ J(w, v) = \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \left[ |x| |y| |z| \right]^{-(n-1)} |w - x - y - z| |v - x - y - z|^{-n+1/2} \times \delta \left[ 1 + |x|^2 + |y|^2 \right] dx dy dz, \quad n \geq 2; \]

then for non-negative \( f \in L^2(\mathbb{R}^n) \) and \( T \) representing the above kernels

\[ \int_{\mathbb{R}^n \times \mathbb{R}^n} f(w) T(w, v) f(v) dw dv \leq c \int_{\mathbb{R}^n} |f|^2 dx \quad (6) \]

**Proof.** Apply Pitt’s inequality and the uniform bounds obtained above for \( \Lambda_n, \Lambda_{n, \alpha}, \Theta_{n, \alpha} \) and \( \Delta_n \). Here \( c \) is a generic constant. \( \square \)

Practical application for such convolution-type estimates has proved to be efficient by replacing the Riesz potentials with the Fourier transform of Bessel potentials (\(2, 3\)); advantage is achieved by removing local singularities while gaining integrability on the potential side and improving the range of application as “smoothing operators”; still the lack of homogeneity limits determination of precise dependence on parameters in computing best size estimates. But as with exact model calculations, the role of Riesz potentials can result in “very elegant and useful formulae” that underline intrinsic geometric structure, capture essential features of symmetry and uncertainty, and provide insight to precise lower-order effects.

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Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin TX 78712-0257 USA

E-mail address: beckner@math.utexas.edu