Optimisation of Modelling of Finite Element Differential Equations with Modern Art Design Theory

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Abstract
A bridge structure is one of the most expressive forms of art design. The artistic expression of bridge structure combines different concepts of structural design and architectural art design. Finite element differential equations are widely used in bridge art design theory and based on these features, the paper adopts the bridge modal parameter recognition algorithm and uses the finite element model to modify and realise the bridge’s artistic design. The simulation results show the feasibility of the author’s attempt to use the finite element differential equation as the bridge structure art design carrier. After the finite element differential equation modelling, the bridge art structure correction is highly consistent with the experimental results.

Keywords: Bridge art design, finite element model modified differential equation modelling, constrained optimisation, modal parameters
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1 Introduction

The bridge is one of the most symbolic and expressive public buildings in the urban landscape. Because of its open and shared use attributes, it has attracted more attention from society. In recent years, with the continuous improvement of the quality of urban construction in China, higher requirements have been put forward on the
landscape and function of bridges. In the finite element model of bridge structure damage identification, health
diagnosis and bridge working condition assessment and prediction, a reliable and more accurate finite element
analysis model is the basis. However, most finite element models are established based on structural design
drawings, implying more idealised assumptions and simplifications [1]. Therefore, thalamic characteristics an
This phenomenon is a specific difference between the established finite element model and the actual struc-
ture. When the difference is significant, the calculated result of the model will be different from the actual
measurement result, even exceeding the accuracy allowed in the project. In this case, the finite element model
needs to be revised. Regarding the finite element model revision, scholars from various countries have con-
ducted extensive research [2]. The general approach is to minimise the residuals of various tests/calculations
of the structure. They proposed a series of structural model modification algorithms for different modification
objects according to different optimisation objectives and different optimisation constraints.

This paper proposes a parametric model correction algorithm based on optimisation design theory. This
algorithm takes the minimum weighted sum of the frequency residual and the measured degree of freedom
residuals as the optimisation of calculation objective. The correction objects are the geometrical, physical and
mechanical parameters of the components in the finite element model [3]. The algorithm imposes constraints
on the modification parameters and frequency changes based on engineering experience, and further reduces
the problem of finite element model modification to constrained optimisation. The problem is solved using an
optimised iterative algorithm based on gradient descent.

2 Theory and Method

2.1 Objective function

In this paper, frequency and mode shape are jointly used to construct the residual and objective functions.

\[ F(P) = W_\omega F_\omega(P) + W_\phi F_\phi(P) \]  

(1)

\( P \) is the vector of parameter set to be corrected \( P = [p_1, p_2, \cdots, p_n] \). There are a total of parameters to be cor-
rected. \( F_\omega(P) \) is the residual frequency term. \( F_\phi(P) \) is the mode shape residual term. \( W_\omega \)is the weight of the
frequency correction term. \( W_\phi \) is the weight of the mode shape correction term. The residual frequency term
\( F_\omega(P) \) is obtained by the weighted summation of the relative difference between the measured frequency and
the calculated frequency of each order

\[ F_\omega(P) = \sum_{i=1}^{m_1} W_\omega_i \left( \frac{\omega_{ei} - \omega_{ai}}{\omega_{ei}} \right)^2 \]  

(2)

where \( m_1 \) is the measured frequency order used. Note that \( \omega_{ei} \) and \( \omega_{ai} \) are the measured frequency and calculated
frequency of the i-th order of the structure, respectively.

Similarly, the modal residual term is defined as

\[ F_\phi(P) = \sum_{i=1}^{m_2} W_{\phi,i} r_i(\phi)^T r_i(\phi) \]  

(3)

where \( m_2 \) is the measured mode order used and \( r_i(\phi) \) is the residual vector of the i-th mode shape, which is
defined as follows:

\[ r_i(\phi) = \frac{\phi_{ie} - \phi_{ic}(P)}{\phi_{ic}(P)} \]  

(4)

Where \( \phi_{ie} \) and \( \phi_{ic}(P) \) are respectively the incomplete measured mode shape and the calculated mode shape of the
i order corresponding to the degree of freedom of the measuring point. \( \phi_{ie}^r \) and \( \phi_{ic}(P) \) are the components of the
measured mode shape and the calculated mode shape on the reference degrees of freedom, respectively [4]. The
reference degree of freedom is generally selected at the position of the mode amplitude, as shown in Figure 1.
2.2 Optimisation constraints

To avoid losing the physical meaning of modifying the parameters during the optimisation process, constraints to the optimisation iteration are added as follows:

\[
|\omega_{ei} - \omega_{ai}| \leq UL, \quad i = 1, 2, \cdots, m \tag{5}
\]

\[
p_L^k \leq p_k \leq p_U^k, \quad k = 1, 2, \cdots, n
\]

Equation (5) is the frequency constraint condition. UL is the maximum value of the frequency change of the state variable in each iteration [5]. Each parameter sets the upper limit and the lower limit \(p_L^k\). The value can be determined according to engineering experience and actual conditions.

2.3 Optimised solution

The general form of the constrained optimisation problem is

\[
\min : F = F(x) \tag{6}
\]

Restrictions:

\[
x^L_i \leq x_i \leq x^U_i, \quad (i = 1, 2, \cdots, n) \tag{7}
\]

\[
g_i \leq g^U_i, \quad (i = 1, 2, \cdots, m_1)
\]

\[
h_i^L \leq h_i(x), \quad (i = 1, 2, \cdots, m_2)
\]

\[
w_i^L \leq w_i(x) \leq w^U_i, \quad (i = 1, 2, \cdots, m_3)
\]

In the formula: \(x_i\) is the optimisation design variable, and \(g_i, h_i, w_i(x)\) is the state variable. In the optimisation calculation, we deal with the constrained optimisation problem by converting it into an unconstrained optimisation problem [6]. This paper uses the penalty function method to convert the above-constrained optimisation problem to unconstrained optimisation problem. The objective function after the penalty function is introduced in the experiment is recorded as

\[
Q(x, q) = F(x) + \sum_{i=1}^{n} P_x(x_i) + \left[ q \sum_{i=1}^{m_1} P_g(g_i) + \sum_{i=1}^{m_2} P_h(h_i) + \sum_{i=1}^{m_3} P_w(w_i) \right] \tag{8}
\]

\(P_x\) is the penalty function term of the optimisation design variable constraint condition. \(P_g, P_h, P_w\) are the penalty function terms of the constraint condition of the state variable. The more common and effective method to solve the unconstrained optimisation problem expressed by formula (8) is the iterative algorithm based on gradient descent. The iterative calculation starts from a particular initial point \(X^{(0)}\) of the design variable [7]. At each iteration step \(k\), the gradient of the objective function is calculated to determine the iterative calculation direction.

Fig. 1 The normalisation of the \(i\) mode shape and the reference degree of freedom \(r\).
of the design variables. Iterate repeatedly until the convergence condition is met. Whether the iteration converges or not can be judged from equations (9) and (10).

\[ |F^{(k+1)} - F^{(k)}| \leq tol \] (9)

\[ |F^{(k)} - F^b| \leq tol \] (10)

where \( tol \) is allowable error of objective function and \( F^b \) is the optimal value of the objective function. Equation (9) indicates that the difference between the value of the objective function at the iteration \( k + 1 \) and the previous iteration step \( k \) is less than the allowable error of the objective function [8]. Equation (10) indicates that the difference between the value of the objective function at step \( k \) and the optimal value (known) of the objective function is less than the allowable error of the objective function.

3 Calculation examples

The three-span continuous beam is shown in Figure 2. Its total length is 11 m. The span layout is 3 + 5 + 3 m. The rectangular cross-section is 0.2 m × 0.2 m. The corresponding moment of inertia is \( I = 1.33 \times 10^{-4} \text{ m}^4 \). The area is \( A = 0.04 \text{ m}^2 \). The material is \( E = 3.0 \times 10^4 \text{ MPa} \). \( \rho = 2500 \text{ kg/m}^3 \).

![Fig. 2 Three-span continuous beam/cm calculated by model modification simulation.](image)

The experiment uses plane beam elements to establish a finite element analysis model. It is divided into 22 units and 23 nodes at equal intervals [9]. We calculated the first eight modal frequencies and modal shapes using initial structural parameters that are mentioned above. At the same time, these values are used as the calculated values of the initial structural modal parameters. To simulate the unknown parameters, we reduce the moments of inertia of units 2, 9, 12 and 20 by 30%, 40%, 50% and 40%, respectively. And the experiment increases the density of all unit materials by 20% when other parameters remain unchanged. On this basis, the first eight modal frequencies and modal shapes after the parameter change are calculated and used as the simulation measured values [10]. The starting point of the parameter correction iteration is the initial structure parameter, and the ideal target of the correction is the changed parameter value. Among them, the parameter correction iteration start point is the initial structure parameter.

Frequency error: \( \omega \% = \left| \frac{\omega_e - \omega_a}{\omega_a} \right| \% \). Vibration shape error: \( \| \phi \|_1 \% = \left( \frac{\| \phi_e - \phi_a \|_1}{\| \phi_e \|_1} \right) \% \).

The calculated and measured values of the modal parameters before model modification and their errors are shown in Table 1. It can be seen from Table 1 that the calculated value of the modal parameters has a significant error with the actual measured value [11]. For example, the maximum error of frequency is 18.26%, and the maximum error of mode shape is 38.55%. Therefore, the experiment is divided into two cases.

Case 1. The target mode adopts the first eight orders. The trimming parameters only include the moments of inertia of elements 2, 9, 12 and 20 and the material parameter density \( \rho \). We set a total of 5 parameters.

Case 2. The target mode adopts the first eight orders. The inertia of all elements is selected as the design parameter, plus the material density \( \rho \), a total of 23 parameters.
Table 1  Comparison of calculated and measured modal parameters before correction

| Order | Frequency calculation value | Frequency measured value | Frequency error | Mode shape difference |
|-------|-----------------------------|--------------------------|-----------------|----------------------|
| 1     | 18.503                      | 15.674                   | 18.05           | 2.7                  |
| 2     | 39.549                      | 33.74                    | 17.22           | 12.69                |
| 3     | 46.472                      | 39.854                   | 16.61           | 10.2                 |
| 4     | 68.545                      | 58.921                   | 16.33           | 11.2                 |
| 5     | 78.746                      | 71.885                   | 9.54            | -                    |
| 6     | 122.81                      | 103.85                   | 18.26           | 17.78                |
| 7     | 153.04                      | 133.7                    | 14.47           | 38.55                |
| 8     | 167.12                      | 146.14                   | 14.36           | 36.81                |

Table 2  Comparison of calculated and measured values of modal parameters after correction

| Order | Frequency calculation value | Frequency measured value | Frequency error | Mode shape difference |
|-------|-----------------------------|--------------------------|-----------------|----------------------|
| 1     | 15.672                      | 15.674                   | 0.010           | 0.089                |
| 2     | 33.725                      | 33.740                   | 0.040           | 0.650                |
| 3     | 39.842                      | 39.854                   | 0.030           | 0.540                |
| 4     | 58.932                      | 58.921                   | 0.020           | 0.170                |
| 5     | 71.922                      | 71.885                   | 0.050           | -                    |
| 6     | 103.850                     | 103.850                  | 0                | 0.11                 |
| 7     | 133.750                     | 133.700                  | 0.04            | 0.15                 |
| 8     | 146.170                     | 146.140                  | 0.02            | 0.45                 |

Table 3  Parameter modification results

| Category | Before correction ×10^{-4} | After correction ×10^{-5} | True value ×10^{-5} | Error/% |
|----------|---------------------------|---------------------------|----------------------|---------|
| 1 Situation 1 | 1.33                       | 9.309                     | 9.31                 | 0.011   |
| Situation 2 | 9.204                     | 1.540                     | 1.54                 | 0.330   |
| 2 Situation 1 | 1.33                       | 7.974                     | 7.98                 | 0.075   |
| Situation 2 | 7.888                     | 0.040                     | 0.04                 | 2.4     |
| 3 Situation 1 | 1.33                       | 6.624                     | 6.65                 | 0.391   |
| Situation 2 | 6.61                      | 1.758                     | 1.76                 | 0.72    |
| 4 Situation 1 | 1.33                       | 7.899                     | 7.98                 | 1.015   |
| Situation 2 | 7.937                     | 0.02                      | 0.02                 | 0.53    |
| 5 Situation 1 | 2500                       | 2997.1                    | 3000                 | 0.097   |
| Situation 2 | 3031.6                     | 3000                      | 3000                 | 1.05    |

Modified calculation constraints: (1) Moment of inertia constraint: $5 \times 10^{-5} \leq I_i \leq 1.8 \times 10^{-4}$. (2) Density constraint: $2200 \leq \rho \leq 3200$. (3) Frequency constraint: $|\omega_i - \omega_{ai}| \leq 20, i = 1, 2, 3, 4|\omega_{ai} - \omega_{ai}| \leq$
50, $i = 5, 6, 7, 8$. The correction results are shown in Tables 2 and 3.

Tables 2 and 3 show that the calculated values of the modal parameters (frequency, mode shape) after the two cases of the proposed algorithm are very close to the measured values [12]. The maximum error of case 1 frequency is 0.05% and the maximum error of mode shape is 0.65%. The correction values of the five parameters are also very close to the actual values of the parameters. The maximum error of the case 2 frequency is 1.929%. Further, the maximum error of mode shape is 0.93%. The correction values of the five parameters are also relatively close to the actual values of the parameters. The maximum error is 2.4%. In case 1, the correction parameters are directly carried out on the five parameters with errors, and it only takes a few iterations to converge to the actual value. In case 2, there are 23 correction parameters and the algorithm can still effectively complete the model correction function [13]. However, the error of the parameter correction results has increased, and the iterative calculation workload has also increased. As shown in Figure 3, case 1 (Figure 3b) converges at the 9th iteration step, while Case 2 (Figure 3b) takes more than 15 iterations to converge.

![Fig. 3 Parameter ($I_2$, $I_9$, $I_{12}$, $I_{20}$) iterative process.](image)

The calculation examples prove that the parametric model correction algorithm proposed in this paper is feasible and effective. Satisfactory correction results can be obtained even when there are many correction parameters. However, when the parameters increase, the efficiency of iterative calculation decreases, and the amount of calculation is reduced. Larger.

4 Conclusion

The construction of the optimised objective function considers the frequency and mode shape errors and directly uses the measured mode shape components. Thus, there is no need for a complete mode vector, and the error introduced by mode expansion is avoided. Based on the experience of modern bridge art design engineering to impose constraints on the modification parameters and frequency changes, we attribute the model modification problem to the constraint optimisation problem. The problem is solved by an iterative optimisation algorithm based on gradient descent. The result of calculations of the example proves that the model correction method
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proposed in this paper is feasible and effective.

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