Managing Default Contagion in Inhomogeneous Financial Networks

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Abstract

The aim of this paper is to quantify and manage systemic risk in the interbank market. We model the market as a random directed network, where the vertices represent financial institutions and the weighted edges monetary exposures between them. Our model captures the strong degree of heterogeneity observed in empirical data and the parameters can easily be fitted to real data sets. One of our main results allows us to determine the impact of local shocks, where initially some banks default, to the entire system and the wider economy. Here the impact is measured by some index of total systemic importance of all eventually defaulted institutions. As a central application, we characterize resilient and non-resilient cases. In particular, for the prominent case where the network has a degree sequence without second moment, we show that a small number of initially defaulted banks can trigger a substantial default cascade. Our results complement and extend significantly earlier findings derived in the configuration model where the existence of a second moment of the degree distribution is assumed. Moreover, paralleling regulatory discussions, we determine minimal capital requirements for financial institutions sufficient to make the network resilient to small shocks. An appealing feature of these capital requirements is that they can be determined locally by each institution without knowing the complete network structure as they basically only depend on the institution’s exposures to its counterparties.

Keywords: systemic risk, financial contagion, capital requirements, inhomogeneous random graphs, weighted random graphs, directed random graphs

1 Introduction

State of the Art Systemic Risk was already listed in 2003 by Duffie and Singleton [17] as one of five types of risk financial institutions are exposed to, and after the financial crisis...
in 2007 it gained major importance. From today’s viewpoint, research is very diverse and parallels recent regulatory discussions that take systemic risk considerations into account, see [5, 9]. An overview of the different approaches to study systemic risk can be found e.g. in [20]. One important line of research addresses explicitly the network structure of the financial system, where institutions correspond to vertices in the network and edges represent dependencies among them, for example monetary exposures. Note that such exposures are more various than only usual Loans; they can result from Securities Cross-holdings, Derivatives and Foreign Exchange, see [28]. The recent monograph [23] serves as a good reference for methods relying on network models, which is also the focus of this paper. There, Hurd summarizes recent literature and identifies Systemic Risk as comprising of some triggering shock event and a propagation of it through the system that has a major impact on the macroeconomy. For the latter, he further lists four main channels via which the triggering shock event propagates: Asset Correlation, Default Contagion, Liquidity Contagion, and Market Illiquidity and Asset Fire Sales.

Of the above mentioned propagation mechanisms of systemic risk, Default Contagion probably is the one that has been studied the most in the existing literature. One of the most prominent models is the one by Eisenberg and Noe [18], where uniqueness of a clearing vector is investigated for liabilities when some institutions cannot fully pay off their debt. The model was extended in various works, see for example [29, 31], where several issues, as for instance the assumption that default happens without any additional costs or additional contagion channels, are addressed. Gai and Kapadia [21] even assumed a recovery rate of zero or, expressed differently, default costs of 100%. This might be reasonable for investigating the contagion mechanism, since processing defaults may take months or even years – time that institutions in financial distress do usually not have. Furthermore, directly after the default of an institution there is high uncertainty about its assets’ value and the mark-to-market recovery rate is hence likely to be very low. As a striking example, in a bond auction for the settlement of credit default swaps written on Lehman Brothers just three weeks after its default the realized recovery rate only amounted to 8.625% [1].

There are various possible approaches to building a network model for the financial system. The most direct one is to work with one concrete network structure that matches the observed financial network of interest. This is the approach used in [18] and its extensions. Similarly, [11] develops matrix majorization tools that allow to compare financial systems with different liability concentration in terms of the systemic loss generated. Financial networks also experience some change over time, however, and to make statements about the resilience of possible scenarios of the financial system in future it might not be advisable to simply consider today’s observed network. In [13] for example, the authors develop a structural default model and use a Bayesian network approach to derive formulas for the joint default and survival probability that can be computed explicitly. Although not directly related to a default cascade, another structural stochastic model that describes interbank lending was proposed in [19] and extended respectively modified in [27, 12].

An alternative approach, and the one being followed here, is to consider a random graph which is such that each typical sample resembles the important statistical characteristics such as the degree distribution of the real network. One of the biggest strengths of this approach is that it allows for the employment of the powerful tool of probabilistic limit theorems to asymptotically obtain analytic results for large networks in the analysis of systemic risk. Furthermore, these results are then robust with respect to local changes of the network over time as they are expressed in terms of statistical characteristics of financial networks which have been shown to stay relatively stable over time (see e.g. [14]). A popular choice for such a random model is the configuration model, as pursued by Amini et al. [3, 4] for example. Among other results, there it was shown that a financial network is resilient to small initial shocks if and only if
a specific measure depending only on the number of so-called contagious links is negative; a debt is referred to as contagious if the bank cannot sustain the default of the corresponding counterparty. Thus, in this approach resilience is a property that can be characterized in terms of local effects only.

The resilience criterion in [3] is rather strong, but that paper makes a crucial assumption on the structure of the networks: it requires that the underlying degree distribution has a finite second moment. The reason for this is a technical one: without this condition the probability that the configuration model generates a simple network (i.e., with no multiple edges or loops) is tiny, see also [24], and hence results that hold with high probability for the configuration model are not necessarily true when conditioning on simplicity of the graph. It is, however, the case that real systems are such inhomogeneous that their degree distribution does not necessarily have this property. Evidence of this in the context of financial networks is given in [14] for the Brazilian banking system and in [10] for the Austrian banking system. Apart from that, a key property of real world financial networks, see for example [15], is a distinct core-periphery structure, where a few large banks are connected to many other large or small banks, but small banks are connected only to few others. In models of random graphs such structures appear only when the underlying degree distribution has a second moment [30], (unless, of course, the structure is directly planted into the model) and so the model from [3] lacks this property.

**Contribution of this Work**

The first aim of this paper is to address the previously mentioned shortcomings. In particular, we will complement and extend the results in [3] so that they can be applied to broader and more prominent settings, where for example the degree distributions are allowed to have an unbounded second moment. As it turns out, deriving precise results in the more general setting with less assumptions is possible; however, we will also observe that the resilience criteria turn out to be not as simplistic as in [3] and comprise of many global effects and interactions. The second main contribution is therefore the determination of sufficient minimal capital requirements for controlling systemic risk in financial networks.

Our starting point is the paper [16], where the first three authors developed a directed random graph model that makes it possible to study random graphs where the underlying degree distribution is not necessarily required to possess a second moment, while simultaneously preserving the simplicity of the network. The model in [16], however, lacks the weighted edges that can represent values of exposures between different institutions. Instead, the contagion mechanism is described by integer-valued thresholds that represent the numbers of neighbors of each bank that need to default in order for the particular bank to default as well, rather than the actual monetary loss a bank can sustain. Moreover, first evidence that the absence of contagious links is not enough to ensure resilience of a system was given in that paper as an example.

In this paper we extend and develop further significantly the threshold-based model from [16] and the models from [3] so that it can be applied in the context of default contagion of financial networks. More specifically, the new model presented here is capable of describing monetary edge-weights and is hence, to our knowledge, the first simple, weighted and directed random graph model for financial systems that captures the heterogeneous structure of real world networks. In particular, the present paper makes the following contributions.

**A Random Graph Model for Financial Networks**

As already described briefly, we define a model that combines many properties observed for real financial networks such as simplicity and directness of the links between institutions, weighted connections and a strong degree of inhomogeneity. As a basic building block we use the model from [16], and we enhance it with weights on the edges and capitals on the vertices. This allows also to use some results from [16] to directly study the enhanced model. We are able to make asymptotic statements for large
networks about the impact that fundamental defaults have on the financial system and the wider economy due to their propagation through the network via a cascade mechanism. Compared to the existing literature where mostly only the number of defaulted banks is considered as systemic risk factor, our setup allows to account for more general systemic importance each bank has for the financial network or the real economy, for example by providing infrastructure for the payment system or a considerable share in the lending business to the economy. This is in line with latest regulatory methods where banks are placed into certain buckets depending on their systemic importance [5, 9]. While our model is intended primarily to model Default Contagion, it can also be reformulated to describe Illiquidity Contagion or other percolation processes on weighted networks.

Resilience Criteria and Systemic Risk Capital Requirements

We derive explicit criteria that determine whether a financial network is resilient or not to contagion with respect to small shocks caused by extraordinary events, such as for example some stock market crash, natural disasters or war, that trigger the default of a few institutions. Our model specification allows us to choose these shocks in a possibly highly correlated way, which is in line with the channel of Asset Correlation listed above.

We then employ the resilience criteria to derive a formula for the risk capital that is sufficient to make the system resilient to initial shocks. Due to the fact that we allow for infinite second moment for the degree sequences in our model, the derived risk capital turns out to be more restrictive than simply prohibiting contagious links as proposed in [5]. We derive a robust formula for a threshold that will make the system resilient and relies on the normalized expected in-degree $w^-$ of a certain bank by a sublinear form $\alpha(w^-)^\gamma$, where $\alpha > 0$ and $\gamma \in (0, 1)$. Here robustness refers to the dependence structure of in- and out-degrees. For the reasonable case of upper tail dependent degree sequences the threshold is sharp in the sense that barriers $\alpha_{\text{crit}}$ and $\gamma_{\text{crit}}$ for the values of $\alpha$ and $\gamma$ exist below which the system is non-resilient and above which it is always resilient. We then state how such threshold values can be transformed into monetary capital requirements. This contributes to the ongoing discussion about adequate risk capital that was for example discussed by the Basel Committee on Banking Supervision in [5] or by the Board of Governors of the Federal Reserve System in [9].

Most papers on systemic risk deal with the problem of assessing the total systemic risk present in a financial network. A second important aspect, however, that is dealt with much less in the existing literature is the allocation of total systemic risk to the individual institutions and the related question of deriving capital charges for them. In this context, complicated questions like ‘What is a fair allocation?’ or ‘Should one impose higher systemic risk charges on institutions that in some sense are prone to transmitting systemic risk or on institutions that connect to such systemic banks, i.e. on institutions that expose themselves to systemic risk?’ arise. Further, to the best of our knowledge all existing proposals of systemic risk allocations in the literature require the knowledge of the complete network to be determined. While a regulator might possess this knowledge, it seems complicated to communicate such systemic risk charges when the institutions cannot reconstruct them by themselves. Also in this setting the capital charge of one bank might depend on the action of the other banks, which gives rise to possible manipulation. A very appealing feature of our approach is that each institution can compute its own systemic risk charge by basically just knowing its local neighborhood, i.e. by knowing its counter parties. This is possible by averaging effects for large networks and inherently ensures fairness in the sense that the banks capital requirement only depends on its own business decisions. Furthermore, agents are prevented from manipulating capital charges of competitors. Further, a bank’s systemic risk capital requirement only depends on the default risk the bank itself is exposed to and not on risk that the bank raises to other banks, i.e. our systemic risk capital requirements are in line with the traditional risk management approach of
individual institutions. Also it has the advantage over models such as [18], which only consider the current state of a network, that also future configurations of the network are likely to be resilient using the same formula. A further insight is that it is in line with the analysis in [11] where it was found that the real network topology is unbalancing and as a consequence a deconcentration of exposures is desirable for stability.

Outline  The paper is structured as follows: In Section 2 we introduce our model for financial systems and relate it to the one from [16] in order to derive results about the final set of defaulted banks and its systemic importance. Our main result, Theorem 2.5, is proved there. In the third section, we first state sufficient criteria for resilience respectively non-resilience and then provide a rigorous proof of the formula describing the necessary risk capital of banks in a financial system. In Section 4 we pursue simulation studies to support our findings.

2 Default Contagion on a Weighted, Directed Random Graph

We shall present a stochastic model for a weighted, directed financial network. It will be based on the directed random graph model proposed in [16] for unweighted, directed financial networks but complemented by edge weights. We will then extend the results from [16] for an unweighted structure to our new model. In particular, the main objective will be to assess the damage caused by default contagion asymptotically when the network size grows to infinity.

2.1 Model Description

We want to describe a financial network consisting of financial institutions, such as banks, and monetary exposures, such as debt claims, between them. If \( n \in \mathbb{N} \) is the size of the network, we will label the institutions by indices \( i \in [n] \), where \([n]\) denotes the set \{1, \ldots, n\}, and interpret them as vertices in a graph. Exposures between institutions shall be represented by a set of weighted, directed edges. We do not allow for multiple edges between vertices or self loops, such that in total there are \( n^2 - n \) possible edges. We denote by \( \Omega \) the set of all possible configurations of weighted edges \( \Omega := \{ \omega \in \mathbb{R}^{n \times n}_+ | \omega_{i,i} = 0, i \in [n] \} \), represented by matrices \( \omega \) with non-negative entries and zero diagonal elements. For \( \omega \in \Omega \) a positive entry \( \omega_{i,j} > 0 \) denotes that in the network represented by \( \omega \), there is a directed edge from vertex \( i \) to \( j \) with weight \( \omega_{i,j} \).

We propose a way to define a probability measure \( P \) on \((\Omega, \mathcal{F})\) where \( \mathcal{F} \subseteq 2^{\Omega} \) is a \( \sigma \)-algebra to be specified later. The probability measure \( P \) is such that \( P(e_{i,j} \leq x | e_{i,j} > 0) = \tilde{P}(E_{i,j} \leq x) \) for \( x > 0 \), where the random matrix \( e : \Omega \rightarrow \mathbb{R}^{n \times n}_+ \), \( e(\omega) := \omega \), represents random draws of weighted edges and \( E_{i,j} \) is some additional random variable defined on an extended probability space with the property \( \tilde{P}(E_{i,j} > 0) = 1 \) for \( i \neq j \). That is, we want to uncouple the occurrence of an edge sent from \( i \) to \( j \) from the size of its possible edge-weight. In order to do so, we proceed by defining a probability measure for an unweighted, directed random network as proposed in [16].

2.1.1 Modeling Edges

For each \( n \in \mathbb{N} \), we consider the vertex set \([n]\) and the set \( D \) of unweighted, directed edges defined as \( D := \{ (i,j) | i,j \in [n], i \neq j \} \). Let \( \Omega_1 := \{0,1\}^{|D|} \) and \( \mathcal{F}_1 := 2^{\Omega_1} \), such that for \( \omega \in \Omega_1 \), a configuration of the links in the financial network, it is expressed by \( \omega_{(i,j)} = 1 \) if the directed edge \((i,j)\) is present in the network and by \( \omega_{(i,j)} = 0 \) otherwise. We define a probability measure \( P_1 \) on \((\Omega_1, \mathcal{F}_1)\) by the following procedure. To each vertex \( i \in [n] \) we assign
two deterministic vertex-weights $w_i^- = w_i^-(n) \in \mathbb{R}_+$ and $w_i^+ = w_i^+(n) \in \mathbb{R}_+$ and define the probability $p_{i,j} = p_{i,j}(n)$ of a directed edge from vertex $i$ to vertex $j$ being present by

$$p_{i,j} = \begin{cases} \min \left\{ 1, \frac{w_i^+ w_j^-}{n} \right\}, & \text{if } i \neq j, \\ 0, & \text{else.} \end{cases} \quad (2.1)$$

Further, we let $X_{i,j}$ be the indicator function for the event of edge $(i,j)$ sent from vertex $i$ to vertex $j$ being present and assume that these events are independent for all $(i,j) \in D$. We denote the weight sequences by $w^-(n) = (w_1^-(n), \ldots, w_n^-(n))$ respectively $w^+(n) = (w_1^+(n), \ldots, w_n^+(n))$. The role of $w_i^-$ respectively $w_i^+$ is to determine the tendency of vertex $i \in [n]$ to have incoming respectively outgoing edges. The vertex-weights are deterministic and purely used as a mean to specify the edge probabilities. They should not be confused with the edge-weights modeled in the next section, which are probabilistic and can only be read off from each sample individually. In the following, when we use the term "weighted" or "unweighted" networks, we always refer to the edge-weights $e_{(i,j)}$ and not the vertex-weights $(w_i^-, w_i^+)$. We shall assume the following regularity conditions of the sequences of weight sequences:

**Assumption 2.1.** Let the empirical distribution function of the vertex-weights be given by

$$F_n(x,y) = n^{-1} \sum_{i \in [n]} 1\{w_i^-(n) \leq x, w_i^+(n) \leq y\}, \quad (x,y) \in \mathbb{R}_+^2,$$

and let $(W_n^-, W_n^+)$ be a random vector distributed according to $F_n$. Then we make the following assumptions for the sequences of weight sequences:

1. **Convergence of weight distributions:** There exists a distribution function $F$ on $\mathbb{R}_+$ such that $F(x,y) = 0$ for all $x,y \leq x_0$ and $x_0 > 0$ small enough, and such that at all continuity points $(x,y)$ of $F$ it holds $\lim_{n \to \infty} F_n(x,y) = F(x,y)$.

2. **Convergence of average weights:** If we denote by $(W^-, W^+)$ a random vector distributed according to the limiting distribution $F$ from $[7]$, then both $W^-$ and $W^+$ are integrable and $\lim_{n \to \infty} E[W_n^-] = E[W^-]$ as well as $\lim_{n \to \infty} E[W_n^+] = E[W^+]$.

This assumption is of a technical nature and concerned with the behavior of the weight parameters as the size of the network tends to infinity. For practical purposes one can think of Assumption 2.1 ensuring that the limiting network keeps the observed parameter distribution of some real network we want to investigate. In particular, the expected weights are assumed to stay finite. Assumption 2.1 has already been required in [16] and will hence be necessary in order to extend the main result of [10] to our new model. In particular, it was derived in [16] that Assumption 2.1 implies $D_i^- \sim \text{Poi}(w_i^- E[W^+])$ respectively $D_i^+ \sim \text{Poi}(w_i^+ E[W^-])$, where $D_i^-$ and $D_i^+$ denote the random in- respectively out-degree of vertex $i$ with weights $(w_i^-, w_i^+)$. Conversely, we will show in Section 4.1 that for an observed network topology, that is, given in- and out-degrees, maximum likelihood estimators of the in- and out-weights are approximately given by the in- and out-degrees, simply normalized by some global factor. That is, morally one can always think of the in- respectively out-weight of a vertex as its in- respectively out-degree.

Further, note that we did not assume $W^-$ or $W^+$ to have finite second moment. By the result from [16] that the empirical degree distribution for the model above converges weakly to a random vector $(D^-, D^+)$ distributed as $\text{Poi}(W^- E[W^+], W^+ E[W^-])$, we see that hence our model is capable of modeling networks without a second moment condition on their degrees. In particular, choosing $W^-$ and $W^+$ power law distributed with parameters $\beta^-$ respectively $\beta^+$ results in power law distributions for the degrees $D^-$ and $D^+$ with the very same parameters. This
allows us to calibrate our model parameters to observed empirical in- and out-degree sequences. As we will see in Sections 3.2 and 3.3 these power law parameters carry the most important information about the network when it comes to determining resilient capital requirements.

2.1.2 Modeling Edge-Weights

We proceed with modeling the weight of the edges in a financial network. For this, we assume the existence of a second probability space \((\Omega_2, \mathcal{F}_2, \mathbb{P}_2)\) that captures for each pair of vertices \((i, j) \in [n]^2\) with \(i \neq j\) a random number \(E_{i,j} > 0\) that represents the possible exposure on \(j\) induced by \(i\). Further, we set \(E_{i,i} = 0\) for all \(i \in [n]\) and denote the resulting random matrix by

\[
E = \begin{pmatrix}
0 & E_{1,2} & \cdots & E_{1,n} \\
E_{2,1} & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
E_{n,1} & \cdots & E_{n,n-1} & 0
\end{pmatrix}
\]

To make the model analytically tractable, we assume that for each bank \(j\) the list of possible exposures \(E_{1,j}, \ldots, E_{j-1,j}, E_{j+1,j}, \ldots, E_{n,j}\) is an exchangeable sequence of random variables. That is, for each \(j \in [n]\) and each permutation \(\pi\) of \([n]\) \(\setminus \{j\}\)

\[
(E_{1,j}, \ldots, E_{j-1,j}, E_{j+1,j}, \ldots, E_{n,j}) \sim (E_{\pi(1),j}, \ldots, E_{\pi(j-1),j}, E_{\pi(j+1),j}, \ldots, E_{\pi(n),j}).
\]

This is equivalent to taking for each bank \(j \in [n]\) an arbitrary sequence of random variables \(\tilde{E}_{1,j}, \ldots, \tilde{E}_{j-1,j}, \tilde{E}_{j+1,j}, \ldots, \tilde{E}_{n,j}\) and transforming them into a list of exposures \(E_{1,j}, \ldots, E_{j-1,j}, E_{j+1,j}, \ldots, E_{n,j}\) by defining \(E_{i,j} = \tilde{E}_{\pi(i),j}\) for some random permutation \(\pi\) that is independent of the family \(\{\tilde{E}_{i,j}\}_{i \in [n]} \setminus \{j\}\) and uniformly drawn from the set of all permutations of \([n]\) \(\setminus \{j\}\). Note that the setting of \([3]\) implies exchangeability of the exposures as well and hence our assumption is not entirely new.

2.1.3 The Combined Model

Having modeled the occurrence of edges and their weights separately on the probability spaces \((\Omega_1, \mathcal{F}_1, \mathbb{P}_1)\) respectively \((\Omega_2, \mathcal{F}_2, \mathbb{P}_2)\), we can then define the product measure \(\tilde{\mathbb{P}} := \mathbb{P}_1 \times \mathbb{P}_2\) on the product space \((\tilde{\Omega}, \tilde{\mathcal{F}}) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\) that comprises all involved quantities and where the unweighted, directed random graph and the exposure lists are mutually independent.

Recall that \(\Omega = \{\omega \in \mathbb{R}^n_+^{\times n} \mid \omega_{i,i} = 0, i \in [n]\}\). We define the map \(F : \Omega \to \Omega\) by

\[
F(\omega_1, \omega_2) = (X_{i,j}(\omega_1)E_{i,j}(\omega_2))_{i,j},
\]

and the \(\sigma\)-algebra \(\mathcal{F} \subset 2^\Omega\) by

\[
\mathcal{F} := \{B \mid F^{-1}(B) \in \tilde{\mathcal{F}}\}.
\]

Clearly \(F\) is a measurable map from \((\tilde{\Omega}, \tilde{\mathcal{F}})\) to \((\Omega, \mathcal{F})\) and defines a probability measure \(\mathbb{P}\) on \((\Omega, \mathcal{F})\) by the pushforward measure \(\mathbb{P} := \tilde{\mathbb{P}} \circ F^{-1}\). The interpretation of the edge-weights \(X_{i,j}E_{i,j}\) is the following: it is positive if and only if \(X_{i,j} = 1\). That is, indeed the occurrence of an edge is fully covered by the unweighted random graph model on \((\Omega_1, \mathcal{F}_1, \mathbb{P}_1)\). The random variables \(E_{i,j}\) on \((\Omega_2, \mathcal{F}_2, \mathbb{P}_2)\) then model the actual edge-weight conditional on the event of their occurrence.
2.2 Default Contagion

In addition to the exposures, let \( c = c(n) = (c_1(n), c_2(n), \ldots, c_n(n)) \) be a list of non-negative valued (possibly deterministic) random variables that represent the banks’ capital/net worth. A bank \( i \in [n] \) is called insolvent if \( c_i = 0 \). The set of initially defaulted banks due to some shock to the network is hence given by \( D_0 = \{ i \in [n] \mid c_i = 0 \} \). They trigger a default cascade \( D_0 \subseteq D_1 \subseteq \ldots \) given by

\[
D_k = \{ i \in [n] \mid c_i \leq \sum_{j \in D_{k-1}} e_{j,i} \},
\]

(2.2)

where in each step \( k \) of the cascade process bank \( i \) has to write off its exposures to banks that defaulted in step \( k-1 \) and goes bankrupt as soon as its total write-offs exceed its initial capital. The chain of default sets clearly stabilizes after at most \( n-1 \) steps and we call \( D_n = D_{n-1} \) the final default cluster in the network induced by the initial defaults \( D_0 \). Note that we could easily introduce a constant recovery rate \( R \in [0,1) \) to our model simply by multiplying exposures \( e_{j,i} \) by a factor \( 1 - R \) in (2.2). In the following we spare this factor.

A special case of our model for contagion is when all exposures are equal to one, and the capitals are non-negative integers. In this setting the capital of each bank can be interpreted as a default threshold: as soon as \( c_i \) neighbors of bank \( i \) have defaulted, bank \( i \) will default as well. This model, which we denote in the rest of the paper as the threshold model, was studied in detail in [16] by the first three authors of this paper, and the distribution of size of the final default cluster could be determined in many cases with great accuracy. The setting studied here is more complex, since we cannot decide if a bank defaults only based on the number of its neighbors that default: (2.2) asserts that this also depends on the actual exposures between the banks. However, one crucial assumption that we made is that this exposures are exchangable, so intuitively it should actually make no difference which neighbors of a given bank default, but just their actual number.

To formalize this intuitive argument, define for each bank \( i \in [n] \) the random threshold value

\[
\tau_i(n) := \begin{cases} 0, & \text{if } c_i \leq 0, \\ \inf \{ s \in [n-1] \mid \sum_{\ell=1}^s E_{\rho_i(\ell),i} \geq c_i \}, & \text{else} \end{cases},
\]

(2.3)

where \( \rho_i(\ell) := \begin{cases} \ell, & \text{if } \ell < i \\ \ell + 1, & \text{if } \ell \geq i \end{cases} \) is the random threshold value of the number of neighbors that need to default in order to cause default of vertex \( i \), given that they go bankrupt in the order of their natural index given by \( \rho_i \) and assuming that all edges \((j,i), 1 \leq j \leq \rho_i(\tau_i), i \neq j, \) are present in the graph. The use of the enumeration \( \rho_i \) becomes necessary in (2.3) since we want to spare \( i \) in this natural ordering. Observe that \( \tau_i \) is also allowed to take the value \( \infty \) in the case that capital \( c_i \) is larger than the sum of all possible exposures. In this case, bank \( i \) can never default. We denote the (hypothetical) threshold sequence by

\[
\tau = \tau(n) = (\tau_1(n), \ldots, \tau_n(n)).
\]

(2.4)

The thresholds are only hypothetical, because not all of the first \( \rho_i(\tau_i) \) exposures must be present in the graph and also the natural order is usually not the one in which the vertices default. However, we know that the exposures are exchangable, so all these simplifications should have no effect; it will turn out in the proof of Theorem 2.5 that this is indeed the case and that the value \( \tau_i \) captures the actual dynamics: the qualitative characteristis of the contagion process in the financial network are the same as in the threshold model with thresholds given in (2.4).
2.3 Systemic Importance

In the previous section, we described a mechanism of how to determine the set $D_n$ of finally defaulted banks from a set of fundamental defaults $D_0$ caused by some external shock event. A first approach, that is often pursued in current literature, is to identify the damage caused to the financial network with the fraction $|D_n|/n$. That is, damage is bearable if only few banks default as a result of the external shock event and the hence started cascade process and it becomes the more threatening the larger the final fraction of defaulted banks $|D_n|/n$ gets. As already mentioned in [16], however, it might be more interesting to consider a more general index of systemic importance of defaulted banks, which measures the damage caused to the whole economical system, rather than only considering the count of defaults. This is in line with current regulator considerations. For instance, in its framework text from 2013, the Basel Committee on Banking Supervision states an indicator-based measurement approach in order to measure global systemic importance of banks. It takes into account Cross-jurisdictional activity, Size, Interconnectedness, Substitutability/financial institution infrastructure and the Complexity of the banks, all in equal shares. The focus is on the impact of a potential default on the real economy or the contribution to the global financial infrastructure. For example, another measure of such systemic importance that is very much in line with our formulation of default contagion in Section 2.2 is DebtRank as introduced in [7]. It focuses on the relative monetary impact of a bank in an interbank network.

We want to include such indicators in our model and assign systemic importance values $s_i(n) \in \mathbb{R}_+$ to each bank $i \in [n]$. Denote the importance sequence by $s(n) = (s_1(n), \ldots, s_n(n))$. We are then interested in the total importance $S_n := \sum_{i \in D_n} s_i$ of the set $D_n$ of finally defaulted banks respectively in the asymptotic normalized expression $\lim_{n \to \infty} S_n/n$ which further allows to compute the fraction of defaulted banks weighted by their systemic importance

$$\lim_{n \to \infty} \frac{\sum_{i \in D_n} s_i}{\sum_{i \in [n]} s_i} = \frac{1}{\mathbb{E}[S]} \lim_{n \to \infty} \frac{S_n}{n},$$

where $\mathbb{E}[S] = \lim_{n \to \infty} n^{-1} \sum_{i \in [n]} s_i$, as defined in Assumption 2.2 below. In Theorems 2.5 and 2.6 below, we will therefore derive an expression for $\lim_{n \to \infty} S_n/n$ in a shocked system. Note that in the case of $s_i = 1$ for all $i \in [n]$ the systemic importance $S_n$ reduces to the size of the final default cluster $|D_n|$ and our model hence covers this approach as a special case.

In order to make the model analytically tractable and to be able to reduce the model to the threshold model from [16], in addition to Assumption 2.1 we need to impose the following assumption:

**Assumption 2.2.** For $(x, y, v, l) \in \mathbb{R}_+^3 \times \mathbb{N}_0^\infty$ let $G_n(x, y, v, l)$ be the random empirical distribution function defined by

$$G_n(x, y, v, l) = n^{-1} \sum_{i \in [n]} 1\{w_i^-(n) \leq x, w_i^+(n) \leq y, s_i(n) \leq v, \tau_i(n) \leq l\}.$$

1. **Almost sure convergence in distribution:** We assume that there exists some distribution function $G : \mathbb{R}_+^3 \times \mathbb{N}_0^\infty \to [0, 1]$ such that

$$\lim_{n \to \infty} G_n(x, y, v, l) = G(x, y, v, l)$$
almost surely for all points \((x, y, v, l) \in \mathbb{R}_+^3 \times N_0^\infty\) for which \(G_i(x, y, v) := G(x, y, v, l)\) is continuous in \((x, y, v)\).

2. Convergence of average systemic importance: Let \(S_n\) be a random variable distributed according to the empirical distribution of \(s(n)\) and \((W^-, W^+, S, T)\) a random vector distributed according to \(G\). Then \(S\) is integrable and \(\lim_{n \to \infty} \mathbb{E}[S_n] = \mathbb{E}[S]\).

To ensure that Assumption 2.2 holds, a twofold regularity is needed. Firstly, for a vertex with given in- and out-weight, the distribution of the threshold value must stabilize, even though the number of exposures appearing in the sum in Assumption 2.3 increases. Secondly, a law of large numbers for the empirical distribution of the threshold values has to hold.

2.4 Examples

To get some intuition about systems satisfying Assumption 2.2, we give two examples where our assumptions are fulfilled.

Example 2.3. Let \((w^-, w^+, s)\) be a triple consisting of in-weight, out-weight and systemic importance sequences such that the empirical distribution

\[
\tilde{F}_n(x, y, v) = \frac{1}{n} \sum_{i \in [n]} 1 \{ w_i^-(n) \leq x, w_i^+(n) \leq y, s_i \leq v \}, \quad (x, y, v) \in \mathbb{R}_+^3,
\]

converges to some distribution function \(\tilde{F}(x, y, v)\). To be consistent with Assumption 2.1, we further require that \(\tilde{F}(x, y, v) = 0\) for all \(x, y \leq x_0\) and some \(x_0 > 0\). Further, assume that \(\lim_{n \to \infty} \mathbb{E}[(W^-, W^+, S)] = \mathbb{E}[(W^-, W^+, S)]\), where \((W^-, W^+, S)\) shall be distributed according to \(\tilde{F}_n\) and \((W^-, W^+, S)\) according to \(\tilde{F}\). Choose some partition of \([0, \infty) \times [0, \infty) \times [0, \infty)\) into countably many half open cubes \(D_{j, k, m} := [p_j, p_{j+1}] \times [p_k, p_{k+1}] \times [q_m, q_{m+1}]\), defined by sequences \(p_1, p_2, \ldots, p_j, p_{j+1}, \ldots, p_k, p_{k+1}, \ldots\), and \(q_1, q_2, \ldots\) determining the endpoints. Further, denote \(W_{j, k, m} := \{i \in [n] \mid (w_i^-, w_i^+, s_i) \in D_{j, k, m}\}\). Let the distributions of \(E_i\) and \(c_i\) equal across vertices \(i \in W_{j, k, m}\) and assume them all to be independent, also across different vertices.

We show that Assumption 2.2 is satisfied. For this let \((x, y, v) \in \mathbb{R}_+^3\) and define the deterministic set

\[
W^{(x, y, v)} := \{i \in [n] \mid w_i^- \leq x, w_i^+ \leq y, s_i \leq v\}.
\]

For each \(i \in W^{(x, y, v)}\) define the random variables \(Y_i^l := 1_{\{r < l\}}\). The set of random variables \(\{Y_i^l\}_{i \in W^{(x, y, v)}}\) is then mutually independent. Further note that for \(i_1, i_2 \in W^{(x, y, v)} \cap W_{j, k, m}\) the distributions of \(Y_{i_1}^l\) and \(Y_{i_2}^l\) equal. Furthermore, the distribution is stable for \(n \geq 1\), meaning it does not change as \(n\) increases. Altogether, this implies that the set of random variables \(\{Y_i^l\}_{i \in W^{(x, y, v)}}\) originates from a finite set of different distributions with the number of each category stabilizing as a fraction of \(n\) by the regularity of the weight sequence. Since \(Y_i^l\) clearly is in \(L^1\), this yields that

\[
\lim_{n \to \infty} n^{-1} \sum_{i \in W^{(x, y, v)}} \mathbb{E}[Y_i^l] = \lim_{n \to \infty} n^{-1} \sum_{i \in W^{(x, y, v)}} \mathbb{P}(Y_i = 1)
= \lim_{n \to \infty} n^{-1} \sum_{j, k, m \in N_0^\infty : \ p_j < x, p_k < y, q_m < v} \mathbb{P}(Y_{i, j, k, m} = 1)
= \lambda(x, y, v, l)
\]

10
converges, where \(i_{j,k,m} \in \mathcal{W}_{[x,y,v)} \cap \mathcal{W}_{k,m}\) are arbitrarily selected indices. By the strong law of large numbers applied to each category, it follows that
\[
\lim_{n \to \infty} G_n(x, y, v, l) = \lim_{n \to \infty} n^{-1} \sum_{i \in \mathcal{Y}(x,y,v)} Y^l_i = \lambda(x,y,v,l)
\]
almost surely.

Note that in the usual statement of the (strong) law of large numbers a single sequence \((X_i)_{i \in \mathbb{N}}\) is given and convergence of \(\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} X_i\), is then ensured. Here we have a slightly different situation since, although dropped from the notation, the random variables \(Y^l_i \in \mathcal{Y}(x,y,v)\) depend on \(n\) and in contrast to the standard situation where the \(n-1\) values \(X_1, \ldots, X_{n-1}\) involved in \((n-1)^{-1} \sum_{i=1}^{n-1} X_i\) are the same as the first \(n-1\) values in the sum of \(n^{-1} \sum_{i=1}^{n} X_i\). However, since the \(Y_i\) are in \(L^1\), the standard proofs using the generalized Chebyshov’s inequality and the Borel-Cantelli Lemma can be adjusted to show that the same result holds for this slightly modified situation.

It remains to show that the convergence holds almost surely for all choices of \((x, y, v, l)\). In fact one can show by partitioning the set \(\mathbb{R}_+^3 \times \mathbb{N}\) into pieces of small probability under the measure implied by \(G\) that
\[
\limsup_{n \to \infty} \sup_{(x,y,v,l) \in \mathbb{R}_+^3 \times \mathbb{N}} |G_n(x, y, v, l) - G(x, y, v, l)| = 0
\]
almost surely, which proves the claim.

It should be noted that the partitioning of the vertices into sets with equal exposure and capital distribution does not need to have the particular interval form as above.

As already observed in a similar setting in [3], both, the independence of the random variables \(\{E_{k,i,j}\}_{i \in [n-1]} \cup c_i\) for a fixed \(i \in [n]\) and the mutual independence of \(\{E_{k,i,j}\}_{k \in [n-1]} \cup c_i\) and \(\{E_{j,i,j}\}_{k \in [n-1]} \cup c_j\) for \(i \neq j\) can be weakened. This will be done in the following example.

**Example 2.4.** Similar as above, we assume that vertices are partitioned into the \(K\) classes \(\mathcal{W}_1, \ldots, \mathcal{W}_K\) with vertices with the same marginal distributions of the capitals and exposures. The sets may depend on \(n\) but we shall assume that
\[
\lim_{n \to \infty} n^{-1} |\mathcal{W}_i| =: \lambda(i),
\]
i.e. the fraction of vertices of a given class stabilizes. For each class \(i \in [K]\) we are given a generating sequence
\[
\{c_i^l, E_i^l\}_{l \in \mathbb{N}}
\]
with \(c_i^l\) and \(E_i^l\) random variables in \(\mathbb{R}_+\) respectively \(\mathbb{R}_+ \setminus \{0\}\). Further, we assume that the sequence \(\{c_i^l, E_i^l\}_{l \in \mathbb{N}}\) is an infinite exchangeable system for each \(i \in [K]\) (see for example [2] for the definition of exchangeability and infinite exchangeable systems). It will serve as the generating sequence for the capitals and exposures for vertices in \(\mathcal{W}_i\). To allocate them let \(j \in \mathcal{W}_i\) be a vertex. We assign to it the capital \(c_i^{k_n+(j-1)(n-1)+1}\), where \(k_n = \sum_{m=1}^{n-1} m(m-1) = n(n-1)(n-2)/3\), and the exposures \(E_i^{k_n+(j-1)(n-1)+1}, E_i^{k_n+(j-1)(n-1)+j-1}, \ldots, E_i^{k_n+(j-1)(n-1)+j-1}\). Observe that by this allocation many entries in the sequence are actually not being used but notation stays bearable. Define now the threshold value as in [2.3] and for a fixed \(m\) and vertex \(j\) in class \(i \in [K]\) indicator random variables \(Y_{i,j}^m := 1_{\{Y_{i,j} = m\}}\), that determine whether vertex \(j\) has threshold value \(m\). Observe that for \(n \geq m+1\) every vertex \(j \in [n]\) has more than \(m\) exposures and the distribution of \(Y_{i,j}^m\) is thus independent of \(n\). Let \(\beta_i(1), \ldots, \beta_i(|\mathcal{W}_i|)\) the indices of the
vertices in $W_i$. By construction then
\[ \mathcal{L}(Y_{i,j}(1), \ldots, Y_{i,j}(|W_i|)) = \mathcal{L}(Y_{i,j}(1), \ldots, Y_{i,j}(|W_i|)) \]
for all $\sigma_i \in \Sigma(W_i)$, that is, for each $i \in [K]$, the random variables $\{Y_{i,j}(m)\}$ build an exchangeable system. Since for fixed $n$ the sequence $\{Y_{i,j}(m)\}_{j \in W_i}$ is just the restriction to a finite subset of variables of an infinite exchangeable system for $|W_i| \to \infty$ it converges in law to an infinite exchangeable system. Let $\Delta_i$ be its directing measure. This implies that the system of random variables
\[ (Y_{i,j}(m))_{i \in [K], j \in W_i} \]
forms a multi-exchangeable system (see [22] for definition). Define the empirical measure by
\[ \Lambda_i^m := \frac{1}{|W_i|} \sum_{j=1}^{|W_i|} \delta_{Y_{i,j}(m)}(j) \]
for each $i \in [K]$. By [22, Thm. 2] convergence in distribution of $\{Y_{i,j}(m)\}_{j \in W_i}$ implies convergence in distribution of the empirical measure sequence $(\Lambda_i)_{i \in [K]}$, without any assumptions on the dependency structure across classes. Since the above considerations apply for all $m \in \mathbb{N}$, convergence in distribution of the empirical measure sequence
\[ \Lambda_i := \frac{1}{n} \sum_{j \in [n]} \delta_{Y_{i,j}(m)} \]
follows for all $i \in [K]$. By the Skorohod Coupling Theorem [22, Thm. 4.30], there exists a probability space with random elements $\{\tilde{\Lambda}_i\}_{i \in [K]}$ distributed as $\{\Lambda_i\}_{i \in [K]}$ such that $\{\tilde{\Lambda}_i\}_{i \in [K]}$ converges almost surely as required.

### 2.5 Asymptotic Results for Default Contagion

We assume in the following that we are given a shocked financial system $(w^-, w^+, s, E, c)$ satisfying Assumptions [2.1 and 2.2] and let $(W^-, W^+, S, T)$ a random vector distributed according to the limiting distribution $G$. By a shocked system we mean that the capitals $c_i$, $i \in [n]$, have already been adjusted for some loss due to an external shock event. In particular, we assume that $c_i = 0$ for some banks $i \in [n]$ and hence $P(T = 0) > 0$. Hence we are in a situation in which a default cascade is about to happen and we are interested in the final damage $S_n$. Since we are not given the exact network structure but only the weight sequences $w^-$ and $w^+$, the damage $S_n$ is a random number. When the network size gets large, however, we show that the damage value gets deterministic and determine its exact value. To this end, we denote
\[ f(z; (W^-, W^+, T)) := E[W^+ \psi_T(W^- z)] - z, \]
where
\[ \psi_r(x) := P(Poi(x) \geq r) = \begin{cases} \sum_{j \geq r} e^{-x} x^j / j!, & 0 \leq r < \infty, \\ 0, & r = \infty. \end{cases} \]
By Lebesgue’s dominated convergence theorem we know that the function $f(z; (W^-, W^+, T))$ is continuous in $z$. Further, $f(0; (W^-, W^+, T)) > 0$ and $\lim_{z \to -\infty} f(z; (W^-, W^+, T)) = -\infty$ and hence by the intermediate value theorem function $f$ must have a positive root. Also we can derive an integral representation of $f(z; (W^-, W^+, T))$ by Fubini’s theorem, using that $W^- W^+ P(Poi(W^- z) = T - 1) 1_{\{T \geq 1\}}$ is a non-negative, $B([0, \infty)) \otimes \sigma(W^-, W^+, T)$-measurable.
function:

\[
f(z; (W^-, W^+, T)) = \mathbb{E}[W^+ \mathbb{P}(\text{Poi}(W^- z) \geq T)] - z
\]

\[
= \mathbb{E}[W^+ 1_{\{T=0\}} + \int_0^z W^- W^+ \mathbb{P}(\text{Poi}(W^- \xi) = T - 1) 1_{\{T \geq 1\}} d\xi] - z
\]

\[
= \mathbb{E}[W^+ 1_{\{T=0\}} + \int_0^z (\mathbb{E}[W^- W^+ \mathbb{P}(\text{Poi}(W^- \xi) = T - 1) 1_{\{T \geq 1\}}] - 1) d\xi \quad (2.5)
\]

This representation identifies \( \mathbb{E}[W^- W^+ \mathbb{P}(\text{Poi}(W^- z) = T - 1) 1_{\{T \geq 1\}}] - 1 \) as the weak derivative of \( f(z; (W^-, W^+, T)) \). In particular, \( f(z; (W^-, W^+, T)) \) is continuously differentiable on some interval \( I \subset [0, \infty) \) with derivative \( \mathbb{E}[W^- W^+ \mathbb{P}(\text{Poi}(W^- z) = T - 1) 1_{\{T \geq 1\}}] - 1 \) if the latter is continuous on \( I \).

Then we derive the following result about \( S_n \), the damage caused to the system measured by the importance of the defaulted banks.

**Theorem 2.5.** Let \( (W^-, W^+, S, T) \) be a random vector distributed according to the limiting distribution of \( (w^-, w^+, s, r) \) and assume that \( \mathbb{P}(T = 0) > 0 \). Let further \( \hat{z} \) be the smallest positive root of \( f(z; (W^-, W^+, T)) \) and \( S_n = \sum_{i \in D_n} s_i \) be the total systemic importance of defaulted banks after the contagion process in the financial network \( (w^-, w^+, s, E, c) \). If the weak derivative \( \mathbb{E}[W^- W^+ \mathbb{P}(\text{Poi}(W^- z) = T - 1) 1_{\{T \geq 1\}}] \) of \( \mathbb{E}[W^+ \psi_T(W^- z)] \) is bounded from above by some constant \( \kappa < 1 \) on a neighborhood of \( \hat{z} \), then

\[
\frac{S_n}{n} \xrightarrow{p} \mathbb{E}[\text{Poi}(W^\hat{z}) \geq T], \quad n \to \infty.
\]

In particular, for \( f(z; (W^-, W^+, T)) \) continuously differentiable on a neighborhood of \( \hat{z} \) with \( f'(\hat{z}; (W^-, W^+, T)) < 0 \) (i.e. \( \hat{z} \) stable), Theorem 2.5 is applicable. Also note that in case the degree sequences have a finite second moment, it is straightforward to show that \( f(z; (W^-, W^+, T)) \) is always continuously differentiable. Without the assumption of stableness, it is in general not the case that the relative final damage \( S_n/n \) converges to a deterministic number (see [23] for a comparable result in a much simpler setting). However, in the following theorem we are still able to state asymptotic bounds rather than an exact limiting value. We believe that the derived bounds are sharp in the sense that they cannot be improved without further assumptions on the system. Proving this, however, is beyond the scope of this article.

In the following we say that a sequence of events \( (E_n)_{n \in \mathbb{N}} \) holds with high probability if \( \mathbb{P}(E_n) \to 1 \), as \( n \to \infty \).

**Theorem 2.6.** Let \( (W^-, W^+, S, T) \) a random vector distributed according to the limiting distribution of \( (w^-, w^+, s, r) \) and assume that \( \mathbb{P}(T = 0) > 0 \). Let \( \hat{z} \) be the smallest positive root of \( f(z; (W^-, W^+, T)) \) and further let \( \hat{z}^* \) be the smallest value of \( z > 0 \) at which \( f(z; (W^-, W^+, T)) \) crosses zero, that is

\[
\hat{z}^* := \inf \{ z > 0 : f(z; (W^-, W^+, T)) < 0 \}.
\]

Moreover, let \( S_n = \sum_{i \in D_n} s_i \) be the total systemic importance of defaulted banks after the contagion process in the financial network \( (w^-, w^+, s, E, c) \). Then the following holds:

1. For all \( \epsilon > 0 \) with high probability:

\[
\frac{S_n}{n} \geq \mathbb{E}[\text{Poi}(W^- \hat{z}) \geq T] - \epsilon.
\]
2. If further $\mathbb{E} \left[ W^-W^+ \mathbb{P}(\text{Poi}(W^-z) = T - 1) \mathbf{1}_{\{T \geq 1\}} \right]$ is continuous on some neighborhood of $\hat{z}^*$, then for all $\epsilon > 0$ with high probability:

$$\frac{S_n}{n} \leq \mathbb{E}[\mathbb{P}(\text{Poi}(W^\hat{z}^*) \geq T)] + \epsilon.$$

In particular, if $\hat{z} = \hat{z}^*$, then

$$\frac{S_n}{n} \xrightarrow{p} \mathbb{E}[\mathbb{P}(\text{Poi}(W^-) \geq T)], \text{ as } n \to \infty.$$ Note that Theorem 2.6 extends Theorem 2.5 to the case of $f'(\hat{z}; (W^-, W^+, T)) = 0$ under additional assumptions.

When introducing the threshold values, we remarked that they would depend on the particular choice of the procedure we apply in the cascade process. In the following, we will define a certain procedure for which, instead of exposing banks as described by (2.2), one exposes them sequentially, at each time step considering the contagion of one vertex only:

At the beginning we declare all initially defaulted vertices to be defaulted but yet unexposed. At each step, a single defaulted, unexposed vertex $i \in [n]$ is picked and exposed to its neighbors. That is, weighted edges to all of its neighbors are drawn. If a neighbor $j$ of $i$ goes bankrupt due to the new edge that is sent from $i$, it is added to the set of defaulted, unexposed vertices. Otherwise, the capital of $j$ is reduced by the amount $e_{i,j}$. Afterwards, vertex $i$ is removed from the set of unexposed vertices.

We keep track of the following sets and quantities at different steps $0 \leq t \leq n - 1$:

a. Set $U(t) \subset [n]$ which denotes unexposed vertices at step $t$. We set $U(0) := \{ i \in [n] \mid c_i = 0 \}$.

b. Set $N(t) \subset [n]$ of solvent vertices at step $t$. At $t = 0$, we set $N(0) := [n] \setminus U(0)$.

c. The updated capitals $\{ \tilde{c}_i(t) \}_{i \in [n]}$ with $\tilde{c}_i(0) = c_i$ for all $i \in [n]$.

At step $t \in [n - 1]$ the sets and quantities are updated according to the following scheme:

1. Choose a vertex $v \in U(t - 1)$ according to any rule.

2. Expose $v$ to all of its neighbors in $N(t - 1)$. That is, for all vertices $w \in N(t - 1)$ set $\tilde{c}_w(t) := \max(0, \tilde{c}_w(t - 1) - e_{v,w})$. Note that $\tilde{c}_w(t) = \tilde{c}_w(t - 1)$ if $e_{v,w} = 0$.

3. Fix $N(t) := \{ i \in N(t - 1) \mid \tilde{c}_w(t) > 0 \}$ and $U(t) := (U(t - 1) \setminus \{v\}) \cup \{ i \in N(t - 1) \mid \tilde{c}_w(t) = 0 \}$. Edges that are sent to already insolvent vertices are not exposed (but they could). Above steps are repeated until step $\hat{t}$, the first time set $U(t)$ becomes empty. Note that $\hat{t}$ is the final number of infected vertices independent of the rule chosen in Step 1. Further, we can complete the exposition of the entire graph by exposing also links to defaulted vertices and links sent from vertices in $N(\hat{t})$, that remain solvent until the end.

Now, observe that the rule chosen in 1 defines a permutation of the $\hat{t}$ elements of $[n]$ that go bankrupt. Further, for each $j \in [n]$ it defines an ordering of the set of insolvent vertices that send an edge to $j$, describing the order in which the edges are exposed. This ordering can be completed to a bijective map $\pi_j : [n - 1] \to [n] \setminus \{ j \}$ by adding vertices that either send no edge to $j$ or are still solvent in the end. To be precise, let $\pi_j$ denote the ordering for vertex $j$ and let this vertex (after the exposition) have $\hat{t}$ links sent from insolvent vertices. Then the entries $\pi_j(1), \ldots, \pi_j(\hat{t})$ list defaulted neighbors in $[n] \setminus \{ j \}$ in the order their edges are sent to vertex $j$. The entries $\pi_j(\hat{t} + 1), \ldots, \pi_j(n - 1)$ are, in their natural order, the remaining vertices in $[n] \setminus \{ j \}$.

Using this sequential procedure for the contagion process, we can then give the proof of Theorem 2.5.
Proof of Theorem 2.5. Since the exact rule chosen in Step 1 does not affect the final set \( D_n \), we assume that always the vertex with smallest global label is chosen. Further, the permutations \( \{ \pi_j \}_{j \in [n]} \) are completed as outlined above. We describe a sequential construction of a random graph that has the same distribution as the graph constructed in Section 2.4. We work on the same probability space \( (\Omega, \mathcal{F}) \) as constructed above but instead of assigning weight \( E_{i,j} \) to a potential edge sent from \( i \in [n] \) to \( j \), now the \( i \)-th \((i \in [n-1]) \) edge that is sent to vertex \( j \) during the sequential exposition shall receive weight \( E_{\rho_j(i), j} \), where as before \( p_j \) is the natural enumeration of \([n] \setminus \{ j \} \). That is, edge-weights are not linked to the natural indices of their knots anymore, but instead to the order of the exposition of the edges. One notes, however, that the random graph constructed that way has the same distribution as the random graph constructed before. To see this, observe that by the sequential procedure described by the orderings \( \{ \pi_j \}_{j \in [n]} \) and the assignment of exposures as described above, a potential edge sent from vertex \( i \) to vertex \( j \) is now assigned the edge-weight \( E_{\rho_j(i), j} \). By exchangeability of the lists \( \{E_{i,j}\}_{i \in [n] \setminus \{j\}} \) for \( j \in [n] \), the new random variables \( \{E_{\rho_j(\pi_j^{-1}(i)), j}\}_{j \in [n], i \in [n] \setminus \{j\}} \) have the same multivariate distribution as \( \{E_{i,j}\}_{j \in [n], i \in [n] \setminus \{j\}} \). Obviously, also the new exposures are independent of the edge-indicator functions \( \{X_{i,j}\}_{i,j \in [n]} \).

Hence, both constructions result in the same distribution for the random graph. Further, note that the assignment of edge-weights has been conducted in such a way that the threshold values in both versions of the network coincide. As before, they are given as

\[
\tau_i(n) := \begin{cases} 0, & \text{if } c_i(n) \leq 0, \\ \inf \{s \in [n-1] | \sum_{l=1}^{s} E_{\rho_l(i), i} \geq c_i(n) \}, & \text{else.} \end{cases}
\]

In the new random graph, however, the thresholds \( \tau_i \), \( i \in [n] \), have the interpretation of actual thresholds meaning that bank \( i \) goes bankrupt at the \( \tau_i \)-th of one of its neighbors’ default. The sequential description of the cascade process has then the advantage that we can reduce it to the threshold model as described in [16]. We can replace the capitals \( \hat{c}_i(t) \), which represent monetary thresholds, by integer values \( \hat{\tau}_i(t) \) (we set \( \hat{\tau}_i(0) := \tau_i \)), which count numbers of neighbors, and alter Steps 2 and 3 in the description of the sequential cascade process according to the rule that if there is an edge sent from \( v \) to \( w \) \((e_{v,w} > 0)\), then set \( \hat{\tau}_w(t) := \hat{\tau}_w(t-1) - 1. \) If there is no edge from \( v \) to \( w \) \((e_{v,w} = 0)\), set \( \hat{\tau}_w(t) := \hat{\tau}_w(t-1). \) Then the sets \( N(t) \) and \( U(t) \) are defined by \( N(t) := \{i \in N(t-1) | \hat{\tau}_i(t) > 0\} \) respectively \( U(t) := (U(t-1) \setminus \{v\}) \cup \{i \in N(t-1) | \hat{\tau}_i(t) = 0\} \).

Everything else stays unchanged. Note that the resulting threshold values \( \hat{\tau}_i(t) \) are only valid for an exposition in the order as specified above. In the threshold model from [16], however, we are free to choose exactly the same rule as we chose in Step 1 of our model since this does not affect the final set of defaulted vertices. Hence, we can replace our exposure model by the threshold model from [16], resulting in the same final set of defaulted vertices. In [16] Thm. 7.2., a regularity condition on the threshold distribution \( T \) is required. This is ensured after conditioning on the values of \( \{\tau_i\}_{i \in [n]} \) by Assumption 2.2 that \( G_n(x, y, v, l) \) converges to \( G(x, y, v, l) \) almost surely for all \((x, y, v, l). \)

Applying [16] Thm. 7.2. hence yields the desired statement.

Proof of Theorem 2.6. The first part follows from [16] Thm. 7.2.] by the same arguments as in the proof of Theorem 2.5.

In order to prove the second part, we will apply an additional small shock to the system such that each bank \( i \), regardless of its attributes \( w_i^- \), \( w_i^+ \) and \( \tau_i \), has its capital \( c_i \) and hence its threshold \( \tau_i \) set to 0 with probability \( p \), where \( p \) is some fixed small number. The new limiting distribution of the system is then given by \((W^-, W^+, TM_p)\), where \( M_p \) is a \( \{0,1\} \)-valued random variable independent of \( W^- \), \( W^+ \) and \( T \) and with \( \mathbb{P}(M_p = 0) = p. \) Instead of \( f(z; (W^-, W^+, T)) \)
we then have to consider the function
\[ f(z; (W^-, W^+, TM_p)) = p(\mathbb{E}[W^+] - z) + (1 - p)f(z; (W^-, W^+, T)). \]

Assuming \( P(T = 0) < 1 \) (\( P(T = 0) = 1 \) is a trivial case), this function is strictly larger than \( f(z; (W^-, W^+, T)) \) and hence we can conclude that the first positive root \( \hat{z}_p \) of the additionally shocked system is larger than \( \hat{z}^* \). By definition of \( \hat{z}^* \) we further derive that \( \hat{z}_p \to \hat{z}^* \) as \( p \to 0 \).

The idea will therefore be to choose \( p \) in such a way that \( \hat{z}_p \) satisfies the assumptions of Theorem 2.5 and then conclude by coupling the original system with the additionally shocked one to derive \( n^{-1} S_n \leq n^{-1} S_n^{(p)} \), where \( S_n^{(p)} := \sum_{i \in D_n^{(p)}} s_i \) and \( D_n^{(p)} \) denotes the set of finally defaulted vertices in the additionally shocked system. Since \( \hat{z}^* \) is a root of the continuously differentiable function \( f(z; (W^-, W^+, T)) \) it must hold \( f'(\hat{z}^*; (W^-, W^+, T)) \leq 0 \). We distinguish two cases:

In the first case, we assume that \( f'(\hat{z}^*; (W^-, W^+, T)) < 0 \) and hence
\[ \kappa := \mathbb{E} \left[ W^- W^+ P(\text{Poi}(W^-) = T - 1) 1_{\{T \geq 1\}} \right] < 1. \]

Then by continuity of \( \mathbb{E} \left[ W^- W^+ P(\text{Poi}(W^-) = T - 1) 1_{\{T \geq 1\}} \right] \) on a neighborhood of \( \hat{z}^* \) we conclude that also
\[ \mathbb{E} \left[ W^- W^+ P(\text{Poi}(W^-) = T M_p - 1) 1_{\{T M_p \geq 1\}} \right] \leq \mathbb{E} \left[ W^- W^+ P(\text{Poi}(W^-) = T - 1) 1_{\{T \geq 1\}} \right] \]
\[ < 1 + \frac{\kappa}{2} < 1 \]
on a neighborhood of \( \hat{z}_p \) for \( p \) small enough. As indicated above, an application of Theorem 2.5 together with a coupling argument then yields
\[ \frac{S_n}{n} \leq \frac{S_n^{(p)}}{n} = \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}_p) \geq T M_p) \right] + o_p(1) \leq \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}_p) \geq T) \right] + p + o_p(1) \]
and hence
\[ \frac{S_n}{n} \leq \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}^*) \geq T) \right] + \epsilon + o_p(1) \]
using continuity of \( \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}) \geq T) \right] \) (note that \( S \) is assumed integrable) and choosing \( p \) small enough such that
\[ \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}_p) \geq T) \right] - \mathbb{E} \left[ S P(\text{Poi}(W^- \hat{z}^*) \geq T) \right] + p \leq \epsilon. \]

In the second case, we have \( \mathbb{E} \left[ W^- W^+ P(\text{Poi}(W^- \hat{z}^*) = T - 1) 1_{\{T \geq 1\}} \right] = 1 \) respectively
\[ f'(\hat{z}^*; (W^-, W^+, T)) = \lim_{\epsilon \to 0} \frac{\mathbb{E} \left[ W^+ P(\text{Poi}(W^- (\hat{z}^* + \epsilon) \geq T) \right] - (\hat{z}^* + \epsilon)}{\epsilon} = 0. \]

Then let for each \( \epsilon > 0 \),
\[ \delta(\epsilon) := -\inf_{0 < \epsilon \leq \epsilon} \frac{\mathbb{E} \left[ W^+ P(\text{Poi}(W^- (\hat{z}^* + \epsilon) \geq T) \right] - (\hat{z}^* + \epsilon)}{\epsilon}, \]
which is positive for all \( \epsilon \) by definition of \( \hat{z}^* \). We can therefore find \( \tilde{\epsilon} > 0 \) arbitrarily small such that \( \delta(\epsilon) < \delta(\tilde{\epsilon}) \) for all \( \epsilon < \tilde{\epsilon} \). We then derive that
\[ 0 \leq f(\hat{z}^* + \epsilon; (W^-, W^+, T)) + \delta(\epsilon) \epsilon \leq f(\hat{z}^* + \epsilon; (W^-, W^+, T)) + \delta(\tilde{\epsilon}) \epsilon \]
for all \( \epsilon \leq \tilde{\epsilon} \) with equality only for \( \epsilon = \tilde{\epsilon} \). Hence at \( \epsilon = \tilde{\epsilon} \) the derivative of the last term must be
less or equal to zero or equivalently,

$$\mathbb{E}[W^+ \mathbb{P}(\text{Poi}(W^{-}(\hat{z}^* + \tilde{c})) = T - 1) \mathbf{1}_{\{T \geq 1\}}] \leq 1 - \delta(\tilde{c}) < 1.$$ 

By continuity, also $$\mathbb{E}[W^+ \mathbb{P}(\text{Poi}(W^{-} z) = T - 1) \mathbf{1}_{\{T \geq 1\}}] \leq 1 - \delta(\tilde{c})/2 < 1$$ on a neighborhood of $$\hat{z}^* + \tilde{c}$$. Hence $$\hat{z}^* + \tilde{c}$$ is a good candidate for the first positive root of the additionally shocked system. All that is left to show is that there indeed exists a certain value for the shock size $$p$$ such that $$\hat{z}^* + \tilde{c}$$ becomes the first positive root. To this end, let

$$p(\tilde{c}) := \frac{\tilde{c}\delta(\tilde{c})}{\mathbb{E}[W^+] - \hat{z}^* - \tilde{c}(1 - \delta(\tilde{c}))}.$$ 

Note that for $$\mathbb{P}(T = 0) < 1$$ the root $$\hat{z}^*$$ is always less than $$\mathbb{E}[W^+]$$ and hence for $$\tilde{c}$$ small enough $$p(\tilde{c})$$ becomes positive. As $$\tilde{c} \to 0$$, also $$p(\tilde{c})$$ tends to zero. Now note that for all $$0 < \epsilon \leq \tilde{c},$$

$$f(\hat{z}^* + \epsilon; (W^-, W^+, TM_{p(\tilde{c})})) = (1 - p(\tilde{c}))f(\hat{z}^* + \epsilon; (W^-, W^+, T)) + p(\tilde{c}) (\mathbb{E}[W^+] - (\hat{z}^* + \epsilon))$$

$$\geq (1 - p(\tilde{c}))(-\epsilon\delta(\tilde{c})) + p(\tilde{c}) (\mathbb{E}[W^+] - (\hat{z}^* + \epsilon))$$

$$= -\epsilon\delta(\tilde{c}) + \tilde{c}\delta(\tilde{c}) \frac{\mathbb{E}[W^+] - \hat{z}^* - \epsilon (1 - \delta(\tilde{c}))}{\mathbb{E}[W^+] - \hat{z}^* - \tilde{c}(1 - \delta(\tilde{c}))}$$

$$\geq (\epsilon - \tilde{c})\delta(\tilde{c})$$

$$\geq 0$$

with equality only for $$\epsilon = \tilde{c}$$. The additional shock strictly increases $$f(z; (W^-, W^+, T))$$ and hence there cannot be any root less or equal $$\hat{z}^*$$. In particular, $$\hat{z}^* + \tilde{c}$$ is the first positive root of the additionally shocked system. By letting $$\tilde{c} \to 0$$, we can as in the first case conclude that

$$\frac{S_n}{n} \leq \mathbb{E} \left[ S \mathbb{P}(\text{Poi}(W^{-}\hat{z}^*) \geq T) \right] + \epsilon + o_p(1)$$

for arbitrarily small $$\epsilon > 0$$.

For the case of $$\hat{z} = \hat{z}^*$$, we simply need to combine parts 1 and 2 of the theorem. 

\[\square\]

3 Resilient Financial Networks and Systemic Capital Requirements

In the previous section, we described the contagion mechanism in financial networks after some external shock to the banks’ capitals, i.e. in a network with some initially defaulted banks. It is of interest, however, to be able to determine how systemically risky a network is prior to some shock event. That is, for a financial system $$(w^-, w^+, s, E, c)$$ with $$\mathbb{P}(c_i > 0) = 1, i \in [n],$$ we want to observe today’s network topology and exposures and, keeping them unchanged, apply some small random shock to the capitals only ex post. A resilient and hence systemically unrisky network should only experience minor damage by this whereas in non-resilient and hence systemically risky networks even a small shock can cause huge harm to the whole system. An advantage over static models such as the Eisenberg-Noe model [18] is that we can assess stability already for an unshocked system. Further, this section will show that whether a financial network is judged resilient or non-resilient only depends on the distributions of $$W^-, W^+$$ and $$T$$. These have been shown to be relatively constant over time even if locally the network might change noticeably.
3.1 Resilience Criteria for Unshocked Networks

In order to incorporate such small random shocks into our model, we introduce a sequence \( m(n) = (m_1(n), \ldots, m_n(n)) \) of binary marks \( m_i \in \{0, 1\} \) to the system \((w^-, w^+, s, E, c)\), where \( m_i = 0 \) denotes that bank \( i \) defaults ex post due to some shock event and hence loses all its capital to start the cascade process. Otherwise, the capital distribution stays the same. In particular, \( c_i \) shall model post shock capitals conditional on them being positive. We extend Assumption 2.2 such that the new empirical distribution function

\[
\overline{G}_n(x, y, v, l, k) = \frac{1}{n} \sum_{i \in [n]} \mathbf{1}\{w_i^-(n) \leq x, w_i^+(n) \leq y, s_i \leq v, \tau_i(n) \leq l, m_i(n) \leq k\}
\]

converges almost surely at all continuity points \((x, y, v)\) of \( G_{l,k}(x, y, v) := G(x, y, v, l, k) \) and denote by \((w^-, w^+, S, T, M)\) a random vector distributed according to the limiting distribution \( \overline{G} \). We assume that \( P(T = 0) = 0 \), but \( P(M = 0) > 0 \) such that indeed \( M \) causes ex post defaults in an unshocked system. We can then derive a criterion for a system being non-resilient. Its proof uses part \([1]\) of Theorem 2.6 but besides this is analogue to the one of Theorem 7.3 in \([16]\).

**Theorem 3.1 (Non-resilience Criterion).** Assume that the random vector \((w^-, w^+, S, T)\) is such that \( P(T = 0) = 0 \) and there exists \( z_0 > 0 \) such that for any \( 0 < z < z_0 \),

\[
E\left[(w^+ P(\text{Poi}(W - z)) \geq T)\right] > z.
\]  
(3.1)

Then the system is non-resilient and in particular it holds that for all \( M \) with \( P(M = 0) > 0 \), with high probability

\[
\frac{S^M_n}{n} \geq E\left[SP(\text{Poi}(W - z_0) \geq T)\right],
\]

where \( S^M_n \) denotes the total systemic importance of the defaulted banks \( D^M_n \) at the end of the contagion process triggered by ex post default \( M \).

We can interpret Theorem 3.1 as follows: If a financial network satisfies condition (3.1), then no matter how small the fraction of banks which are driven to bankruptcy by an external shock event, after the cascade process of defaults always a damage larger than the constant \( E[SP(\text{Poi}(W - z_0) \geq T)] \) is caused to the system. In the reasonable case that \( E[S1_{\{T<\infty\}}] > 0 \), this lower bound for the damage is strictly positive. That is, even a tiny (vanishing) initial shock to the financial system will cause a large (non-vanishing) damage to the system and the economy. In particular, by choosing \( s_i = 1 \) for all \( i \in [n] \) and hence \( S \equiv 1 \), we derive that the final default fraction \( |D^M_n|/n \) is lower bounded by a positive constant. A network satisfying condition (3.1) will hence be called non-resilient in the following.

Condition (3.1) is an assumption on the function \( f(z; (W^-, W^+, T)) \) which is illustrated in Figure 1(a). Whereas for \( P(M = 0) = 0 \) the first non-negative root of the function is zero, any howsoever small increase in \( P(M = 0) \), and hence upwards shift of the function \( f(z; (W^-, W^+, T)) \), makes this first root jump above \( z_0 \) and hence causes default of a set of size larger than \( nE[SP(\text{Poi}(W - z_0) \geq T)] \) and with systemic importance larger than \( nE[SP(\text{Poi}(W - z_0) \geq T)] \).

If on the other hand function \( f(z; (W^-, W^+, T)) \) is such behaved that the first positive root \( z^M \) of \( f(z; (W^-, W^+, TM)) \) tends to zero as \( P(M = 0) \) becomes smaller, one can expect that also the final default cluster \( D^M_n \) and its systemic importance \( S^M_n \) vanish and the system can hence be regarded as resilient to small shocks. (See Figure 1(b)) for an exemplary illustration.) This intuition is formalized in the following Theorems and Proposition. Theorem 3.2 is the analogue of Theorem 7.4 in \([16]\) transferred to our exposure model.
Figure 1: Examples of functions $f(z) = E[W^+P(Poi(W^-z) \geq T)] - z$ (blue) satisfying conditions (3.1) (a) respectively (3.2) (b). In red: the function $g(z) = E[SP(Poi(W^-z) \geq T)]$. Dashed: the unshocked functions. Solid: the shocked functions.

Theorem 3.2 (Resilience Criterion). Assume that $(W^-, W^+, T)$ is such that there exists $z_0 > 0$ such that for any $0 < z < z_0$,

$$E[W^-W^+P(Poi(W^-z) = T - 1)] < 1. \quad (3.2)$$

Then the system is resilient and in particular the following holds: For any sequence of ex post defaults $\{M_i\}_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} P(M_i = 0) = 0$, it follows that for any $\epsilon > 0$, there exists $i_\epsilon$ such that for $i \geq i_\epsilon$ with high probability

$$\frac{S_n^{(i)}}{n} \leq \epsilon,$$

where $S_n^{(i)}$ denotes the total systemic importance of the set of defaulted banks $D_n^{(i)}$ at the end of the contagion process triggered by ex post default $M_i$.

Theorem 3.2 states that the systemic importance of all finally defaulted banks tends to zero as the initial default fraction tends to zero. However, it makes no statement about the rate of convergence. If we assume not only that $E[W^-W^+P(Poi(W^-z) = T - 1)] < 1$ for $z$ small enough but even $\limsup_{z \to 0^+} E[W^-W^+P(Poi(W^-z) = T - 1)] < 1$, then we derive the following result concerning convergence speed.

Proposition 3.3. Assume that $(W^-, W^+, S, T)$ is such that

$$\kappa := \limsup_{z \to 0^+} E\left[W^-W^+P(Poi(W^-z) = T - 1)\right] < 1$$

and

$$\kappa_S := \limsup_{z \to 0^+} E\left[W^-SP(Poi(W^-z) = T - 1)\right] < \infty.$$

Then for any sequence of ex post defaults $\{M_i\}_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} P(M_i = 0) = 0$, it follows that with high probability

$$\frac{S_n^{(i)}}{n} \leq \mathbb{E}[S1_{\{M_i=0\}}] + \mathbb{E}[W^+1_{\{M_i=0\}}] \frac{\kappa_S}{1 - \kappa} + o(\mathbb{E}[W^+1_{\{M_i=0\}}]).$$

If $\mathbb{E}[W^+P(Poi(W^-z) \geq T)]$ and $\mathbb{E}[SP(Poi(W^-z) \geq T)]$ are continuously differentiable from the
right at \( z = 0 \) with derivatives \( \kappa < 1 \) and \( \kappa_S < \infty \), then we derive

\[
\frac{S_{n}^{(i)}}{n} \xrightarrow{p} \mathbb{E}[S 1_{\{M_i=0\}}] + \mathbb{E}[W^+ 1_{\{M_i=0\}}] \frac{\kappa_S}{1 - \kappa} + o(\mathbb{E}[W^+ 1_{\{M_i=0\}}]).
\]

In particular, if \( \{M_i\}_{i \in \mathbb{N}} \) is independent of \( W^+ \) and \( S \), then

\[
\frac{S_{n}^{(i)}}{n} \leq \mathbb{P}(M_i = 0) \left( \mathbb{E}[S] + \mathbb{E}[W^+] \frac{\kappa_S}{1 - \kappa} \right) + o(\mathbb{P}(M_i = 0)) = \mathcal{O}(\mathbb{P}(M_i = 0))
\]

and

\[
1 + \frac{\mathbb{E}[W^+] \kappa_S}{\mathbb{E}[S]} \frac{1}{1 - \kappa}
\]

can be regarded as the maximal amplification factor of the systemic importance of initially defaulted banks \( \mathbb{E}[S 1_{\{M_i=0\}}] = \mathbb{P}(M_i = 0)\mathbb{E}[S] \). If further \( S \equiv 1 \) and \( W^- W^+ \) is integrable, above result reads as

\[
\frac{|P_{n}^{(i)}|}{n} \xrightarrow{p} \mathbb{P}(M_i = 0) \left( 1 + \frac{\mathbb{E}[W^+] \mathbb{E}[W^- 1_{\{T=1\}}]}{1 - \mathbb{E}[W^- W^+ 1_{\{T=1\}}]} \right) + o(\mathbb{P}(M_i = 0)).
\]

This is the analogon to Corollary 20 in [4].

**Proof.** By representation \((2.5)\), we derive

\[
\mathbb{E}[W^+ \mathbb{P}(\text{Poi}(W^- z) \geq TM_i)]
\]

\[
= \mathbb{E}[W^+ 1_{\{M_i=0\}}] + \int_0^z \mathbb{E} [W^- W^+ \mathbb{P}(\text{Poi}(W^- \xi) = T - 1) 1_{\{M_i=1\}}] \, d\xi
\]

\[
\leq \mathbb{E}[W^+ 1_{\{M_i=0\}}] + \kappa z + o(z).
\]

Similarly,

\[
\mathbb{E}[S \mathbb{P}(\text{Poi}(W^- z) \geq TM_i)] \leq \mathbb{E}[S 1_{\{M_i=0\}}] + \kappa_S z + o(z)
\]

and hence \( \hat{z}_i \), the first positive root of \( f(z; (W^-, W^+, TM_i)) \), satisfies

\[
\hat{z}_i \leq \frac{\mathbb{E}[W^+ 1_{\{M_i=0\}}]}{1 - \kappa} + o(\mathbb{E}[W^+ 1_{\{M_i=0\}}]).
\]

Together with Theorem \textbf{2.5} this shows

\[
\frac{S_{n}^{(i)}}{n} \leq \mathbb{E}[S 1_{\{M_i=0\}}] + \mathbb{E}[W^+ 1_{\{M_i=0\}}] \frac{\kappa_S}{1 - \kappa} + o(\mathbb{E}[W^+ 1_{\{M_i=0\}}])
\]

with high probability.

If \( \mathbb{E}[W^+ \mathbb{P}(\text{Poi}(W^- z) \geq T)] \) and \( \mathbb{E}[S \mathbb{P}(\text{Poi}(W^- z) \geq T)] \) are continuously differentiable from the right at \( z = 0 \) with derivatives \( \kappa < 1 \) and \( \kappa_S < \infty \), then above inequalities become equalities and hence

\[
\frac{S_{n}^{(i)}}{n} \xrightarrow{p} \mathbb{E}[S 1_{\{M_i=0\}}] + \mathbb{E}[W^+ 1_{\{M_i=0\}}] \frac{\kappa_S}{1 - \kappa} + o(\mathbb{E}[W^+ 1_{\{M_i=0\}}]).
\]

\( \Box \)

Both Theorem \textbf{3.3} and Proposition \textbf{3.3} are concerned with the behavior of the weak derivative \( \mathbb{E}[W^- W^+ \mathbb{P}(\text{Poi}(W^- z) = T - 1)] = \mathbb{E}[W^+ \psi_T(W^- z)] \) near \( z = 0 \). The following criterion that
assumes that also in the shocked systems $E_i$ that guarantee resilience. In the next section, we then discuss how to translate the threshold values into systemic capital requirements in the exposure model.

In this section we will first focus on identifying threshold requirements in the threshold model which concludes the proof.

**Theorem 3.4.** Assume that $(W^-, W^+, T)$ is such that $E[W^- W^+ P(\text{Poi}(W^- z) = T - 1)]$ is continuous on $(0, z_0)$ for some $z_0 > 0$ and

$$\inf \left\{ z > 0 : f(z; (W^-, W^+, T)) < 0 \right\} = 0. \quad (3.3)$$

Then for any sequence of ex post defaults $\{M_i\}_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} P(M_i = 0) = 0$, it follows that for any $\epsilon > 0$, there exists $i_\epsilon$ such that for $i \geq i_\epsilon$ with high probability

$$\frac{S_n(i)}{n} \leq \epsilon.$$ 

Assumption (3.3) describes that $f(z; (W^-, W^+, T))$ becomes negative immediately after $z = 0$. It is in some sense the opposite of assumption (3.1) and ensures that the roots $\hat{z}_i^*$ of the shocked systems tend to zero as the shock size $P(M_i = 0)$ shrinks to zero.

**Proof.** By assumption (3.3), we derive that $\hat{z}_i^* \to 0$ as $i \to \infty$, where

$$\hat{z}_i^* := \inf \left\{ z > 0 : f(z; (W^-, W^+, T M_i)) < 0 \right\}.$$ 

For $i$ large enough such that $\hat{z}_i^* < z_0$, we can then apply part 2 of Theorem 2.6 to derive

$$\frac{S_n(i)}{n} \leq E\left[ S P(\text{Poi}(W^- \hat{z}_i^* \geq T M_i)) \right] + \frac{\epsilon}{2} \leq E\left[ S P(\text{Poi}(W^- \hat{z}_i^* \geq T)) \right] + E\left[ S \mathbb{1}_{\{M_i = 0\}} \right] + \frac{\epsilon}{2}$$

with high probability. Note that from continuity of $E[W^- W^+ P(\text{Poi}(W^- z) = T - 1)]$ it follows that also in the shocked systems $E[W^- W^+ P(\text{Poi}(W^- z) = T - 1) \mathbb{1}_{\{M_i = 1\}}]$ is continuous by Lebesgue’s dominated convergence theorem. Since $S$ is integrable the first summand tends to zero as $\hat{z}_i^* \to 0$ and also the second summand vanishes as $P(M_i = 0) \to 0$. In particular, we can choose $i$ large enough such that

$$E\left[ S P(\text{Poi}(W^- \hat{z}_i^* \geq T)) \right] + E\left[ S \mathbb{1}_{\{M_i = 0\}} \right] \leq \frac{\epsilon}{2},$$

which concludes the proof. \hfill $\Box$

### 3.2 Systemic Threshold Requirements

A natural problem that is also of highest interest to regulators is to identify capital requirements for the individual banks which can be determined from observable quantities of the network and that are sufficient to make the network resilient to external shocks. Observable quantities are the in- and out-degrees $(d^-_i)_{i \in [n]}$ respectively $(d^+_i)_{i \in [n]}$, which function as estimators of the in- and out-weights $(w^-_i)_{i \in [n]}$ respectively $(w^+_i)_{i \in [n]}$ (see Section 4.1), and interbank exposures. In this section we will first focus on identifying threshold requirements in the threshold model (see Section 2.2) that guarantee resilience. In the next section, we then discuss how to translate the threshold values into systemic capital requirements in the exposure model.

More precisely, in this section we seek threshold requirements for bank $i$ of the form $\tau_i = \tau(w^-_i)$, where $\tau : \mathbb{R}_+ \to \mathbb{N}$. Such a functional form has the interpretation that the threshold (capital) requirement of a bank only depends on its risk of defaulting due to default of debtors (exposure risk). In contrast, if bank i’s threshold (capital) requirement $\tau(w^-_i, w^+_i)$ was also depending on the out-weight $w^+_i$, this would also take possible defaults caused by bank i into account.
account. This risk management policy would not be in line with traditional risk management techniques and would certainly be harder to communicate to the banks.

For the threshold model, Assumption 2.2 reduces to the one of an Extended Regular Vertex Sequence from [10] without the predicate almost surely since threshold values \( \{\tau_i(n)\}_{i \in [n]} \) are given deterministically. One would hence need to assume that

\[
\lim_{n \to \infty} n^{-1} \sum_{i \in [n]} 1\{w_i^-(n) \leq x, w_i^+(n) \leq y, s_i(n) \leq v, \tau(w_i^-) \leq l\} = G(x, y, v, l) \tag{3.4}
\]

for some distribution \( G : \mathbb{R}_+^3 \times \mathbb{N}_0^\infty \to [0, 1] \) and all points \((x, y, v, l) \in \mathbb{R}_+^3 \times \mathbb{N}_0^\infty \) for which \( G(x, y, v, l) := G(x, y, v, l) \) is continuous. For instance, this condition is satisfied if

\[
\lim_{n \to \infty} n^{-1} \sum_{i \in [n]} 1\{w_i^- \leq x, w_i^+ \leq y, s_i \leq v\} = \tilde{G}(x, y, v),
\]

for some distribution \( \tilde{G} : \mathbb{R}_+^3 \to [0, 1] \) at all points of continuity of \( \tilde{G} \), and \( \tau \) is a non-decreasing function which is either left-continuous or only admits left-discontinuities on \( \tilde{G} \)-null sets. The distribution \( G \) is then given by \( G(x, y, v, l) := \tilde{G}(x \wedge z_l, y, v) \), where \( \tau(w^-) \leq l \) if and only if \( w^- \leq z_l \) (for left-continuous jump discontinuities of \( \tau \)) respectively \( w^- < z_l \) (for right-continuous jump discontinuities of \( \tau \)). In the following, we will mostly work with absolutely continuous weights \( W^- \), \( W^+ \) and hence it is sufficient to choose \( \tau \) non-decreasing.

In empirical studies of financial networks such as [10] or [14], it was found that degrees follow extended regular vertex techniques and would certainly be harder to communicate to the banks. This risk management policy would not be in line with traditional risk management for some distribution \( \tilde{G} \).

Let weights \( W^- \) and \( W^+ \) be Pareto distributed and for each bank \( i \in [n] \) let the threshold value \( \tau_i \) depend on in-weight \( w_i^- \) by some functional form \( \tau_i = \tau(w_i^-) \), where \( \tau : \mathbb{R}_+ \to \mathbb{N}\{0, 1\} \) shall satisfy \( \{3.4\} \) (\( \tau \) non-decreasing for example). If we define

\[
\gamma_{\text{crit}} := 2 + \frac{\beta^- - 1}{\beta^+ - 1} - \beta^- \quad \text{and} \quad \alpha_{\text{crit}} := \frac{\beta^+ - 1}{\beta^+ - 2} \left( w_{\text{min}}^- \right)^{1 - \gamma_{\text{crit}}},
\]

then the system is resilient if one of the following holds:

1. \( \gamma_{\text{crit}} < 0 \)
2. \( \gamma_{\text{crit}} = 0 \) and \( \lim \inf_{w \to \infty} \tau(w) > \alpha_{\text{crit}} + 1 \).
3. \( \gamma_{\text{crit}} > 0 \) and \( \lim \inf_{w \to \infty} \tau(w) / w^\gamma_{\text{crit}} > \alpha_{\text{crit}} \).

The theorem identifies different criteria for \( \tau \) depending on the quantity \( \gamma_{\text{crit}} \) and hence the values of \( \beta^- \) and \( \beta^+ \). Since \( \beta^- > 2 \) and \( \beta^+ > 2 \), we note that always \( \gamma_{\text{crit}} < 1 \). That is, also in \[3\] it is possible to choose a sub-linear threshold function \( \tau \) that ensures resilience. On the other hand, even the constant threshold function \( \tau(w) = 2 \) for all \( w \in \mathbb{R}_+ \) ensures resilience by \[4\]
whenever $\gamma_{\text{crit}} < 0$. This is always the case if $\beta^- > 3$ and $\beta^+ > 3$, that is, if $W^-$ and $W^+$ both admit finite second moments. This is in line with the results from [3]. In addition, the theorem makes statements about cases when $\beta^- < 3$ and $\beta^+ > 3$ or vice versa. Such parameters were observed on real markets for example in [14]. In these cases, all $\gamma_{\text{crit}} < 0$, $\gamma_{\text{crit}} = 0$ or $\gamma_{\text{crit}} > 0$ are possible and only the exact values of $\beta^-$ and $\beta^+$ determine the condition for resilience.

**Remark 3.6.** In Theorem 3.5 we make the assumption of $\tau(w) \geq 2$. In other words, this means that each bank must at least be capable of sustaining the default of its largest debtor. This requirement has already been implemented in an even stricter form in the ‘Supervisory framework for measuring and controlling large exposures’ by the ‘Basel Committee on Banking Supervision’ from 2014 which will become applicable as from 2019 [6]. While being economically sensible, the assumption is actually not necessary in order to derive analytical results. For the case of $\gamma_{\text{crit}} < 0$ it is enough to postulate $\mathbb{E}[W^-W^+1_{\{\tau(W_\gamma)=1\}}] < 1$ in order to ensure resilience. Also in the case of $\gamma_{\text{crit}} \geq 0$, it suffices to adjust $\alpha_{\text{crit}}$ for a factor $(1 - \mathbb{E}[W^-W^+1_{\{\tau(W_\gamma)=1\}}])^{-1}$, whenever $\mathbb{E}[W^-W^+1_{\{\tau(W_\gamma)=1\}}] < 1$.

Note that Theorem 3.5 is formulated with assumptions on the marginal distributions of $W^-$ and $W^+$ only. That is, the result is robust with respect to the dependency structure of the weights. It will be clear from the proof, this is due to the fact that we can bound certain expected values by the comonotone dependency structure. However, we will further see in the proof from expression (3.5), that already $\liminf_{w \to \infty} \tau(w)/w^\gamma > 0$ will be sufficient for resilience of the system whenever $\mathbb{E}[W^+(W^-)^{1-\gamma}] < \infty$ for some $0 < \gamma \leq \gamma_{\text{crit}}$. This is an assumption on the dependency structure that is for example satisfied for independent weights but not for comonotone ones. On the other hand, our threshold requirement is also necessary for resilience in the sense that for comonotone weights $W^-$ and $W^+$, or more general for weights $W^-$ and $W^+$ that exhibit an upper tail dependence, the thresholds $\gamma_{\text{crit}}$ (and also $\alpha_{\text{crit}}$ for comonotone weights) are sharp. This will be the statement of Theorem 3.7. By $W^-$ and $W^+$ being upper tail dependent we mean that

$$\lambda := \liminf_{p \to 0} \mathbb{P}(F_{W^+}(W^+) > 1 - p | F_{W^-}(W^-) > 1 - p) > 0.$$ 

If even

$$\Lambda(x) := \lim_{p \to 0} \mathbb{P}(F_{W^+}(W^+) > 1 - xp | F_{W^-}(W^-) > 1 - p)$$

exists for all $x \geq 0$, Theorem 3.7 explicitly determines sharp thresholds $\alpha_{\text{crit}}(\Lambda)$ given by

$$\alpha_{\text{crit}}(\Lambda) := w^{\gamma_{\text{crit}}}_{\min}(w^{\gamma_{\text{crit}}}_{\min})^{\beta^-1-1/\beta^+ - 1} \int_0^\infty \Lambda \left( x^{1-\beta^+} \right) dx.$$ 

In the case of comonotone dependence, that is $\Lambda(x) = 1 \wedge x$, this threshold coincides with $\alpha_{\text{crit}}$ from Theorem 3.5.

**Theorem 3.7.** Let weights $W^-$ and $W^+$ be Pareto distributed and $\gamma_{\text{crit}} > 0$ (where $\gamma_{\text{crit}}$ as in Theorem 3.5). Let further the threshold value $\tau_i$ depend on in-weight $w^{\gamma_{\text{crit}}}_{\min}$ by some functional form $\tau_i = \tau(w^{\gamma_{\text{crit}}}_{\min})$, where $\tau : \mathbb{R}_+ \to \mathbb{N} \setminus \{0, 1\}$ shall satisfy (3.4). The following holds:

1. If $\limsup_{w \to \infty} \tau(w)/w^{\gamma_{\text{crit}} < w^{\gamma_{\text{crit}}}_{\min}(w^{\gamma_{\text{crit}}}_{\min})^{\beta^-1-1/\beta^+ - 1} \lambda$, then the system is non-resilient.
2. If $\Lambda(x)$ exists for each $x \geq 0$ and $\limsup_{w \to \infty} \tau(w)/w^{\gamma_{\text{crit}} < \alpha_{\text{crit}}(\Lambda)$, then the system is non-resilient. If $\liminf_{w \to \infty} \tau(w)/w^{\gamma_{\text{crit}} > \alpha_{\text{crit}}(\Lambda)$, then the system is resilient.
In part 2 of the theorem, we characterize threshold functions $\tau$ that are asymptotically smaller respectively larger than $\alpha_{\text{crit}}(w)w_{\text{crit}}^{\gamma_{\text{crit}}}$. In the proof we calculate the derivative of $f(z; (W^{-}, W^{+}, T))$ at $z = 0$ in order to show non-resilience ($f'(0; (W^{-}, W^{+}, T)) > 0$) respectively resilience ($f'(0; (W^{-}, W^{+}, T)) < 0$). If $\tau(w)$ asymptotically behaves like $\alpha_{\text{crit}}(w)w_{\text{crit}}^{\gamma_{\text{crit}}}$, we obtain $f'(0; (W^{-}, W^{+}, T)) = 0$ and hence both (3.1) and (3.3) are principally possible (surely not simultaneously). In this case the exact form of $\tau$ and not only its asymptotics are important to decide whether the system is resilient, non-resilient or neither of those.

Theorems 3.5 and 3.7 both describe financial systems whose weights are given by Pareto-type weight distributions. While such random variables model the tails of empirical degree distributions very well, it is possible that for small weights the distribution deviates from a Pareto-type distribution. In this case the exact form of $\tau$ not simultaneously. In this case the exact form of $\tau$ and not only its asymptotics are important to decide whether the system is resilient, non-resilient or neither of those.

Theorems 3.5 and 3.7 both describe financial systems whose weights are given by Pareto-type weight distributions. While such random variables model the tails of empirical degree distributions very well, it is possible that for small weights the distribution deviates from a Pareto distribution. Our results are robust with respect to such changes. To formalize this, we consider so-called Pareto-type distributions whose complementary distribution functions $F_{W^{-}}(w) = \mathbb{P}(W^{-} > w)$ and $F_{W^{+}}(w) = \mathbb{P}(W^{+} > w)$ are regularly varying at infinity with parameters $1 - \beta^{-}$ respectively $1 - \beta^{+}$. That is there shall exist continuously differentiable slowly varying functions $L_{W^{-}}$ and $L_{W^{+}}$ with $L_{W^{-}}(w_{\text{min}}^{+}) = L_{W^{+}}(w_{\text{min}}^{-}) = 1$ such that

$$F_{W^{-}}(w) = \begin{cases} 1 - L_{W^{-}} \left( \frac{w}{w_{\text{min}}} \right)^{1 - \beta^{-}}, & w \geq w_{\text{min}}^{-}, \\ 0, & w < w_{\text{min}}^{-}, \end{cases}$$

and

$$F_{W^{+}}(w) = \begin{cases} 1 - L_{W^{+}} \left( \frac{w}{w_{\text{min}}} \right)^{1 - \beta^{+}}, & w \geq w_{\text{min}}^{+}, \\ 0, & w < w_{\text{min}}^{+}. \end{cases}$$

By basic results for regularly varying functions, it then follows that the weight densities $f_{W^{-}}$ and $f_{W^{+}}$ are both regularly varying at infinity with parameters $-\beta^{-}$ respectively $-\beta^{+}$ and $F_{W^{+}}^{-1} \circ F_{W^{-}}$ is regularly varying at infinity with parameter $(\beta^{-} - 1)/(\beta^{+} - 1)$. We derive the following version of Theorem 3.5 for Pareto-type weight distributions.

**Theorem 3.8.** Let $W^{-}$ and $W^{+}$ be Pareto-type distributed random variables with parameters $\beta^{-} > 2$ respectively $\beta^{+} > 2$. Let further the threshold value $\tau_{i}$ depend on in-weight $w_{i}$ by some functional form $\tau_{i} = \tau(w_{i})$, where $\tau: \mathbb{R}_{+} \to \mathbb{R}_{+} \setminus \{0, 1\}$ shall satisfy (3.3). For $\gamma_{\text{crit}}$ defined as before, the system is resilient if one of following holds:

1. $\gamma_{\text{crit}} < 0$
2. $\gamma_{\text{crit}} \geq 0$ and $\lim\inf_{w \to \infty} \tau(w)/w^{\gamma} > 0$ for some $\gamma > \gamma_{\text{crit}}$.

We now turn to proofs of Theorems 3.5–3.8. To show resilience of the financial system in Theorem 3.5 we want to use Theorem 3.4. In order for this to work, we need to ensure that $\mathbb{E} [W^{-}W^{+}\mathbb{P}(\text{Poi}(W^{-}z) = T - 1)]$ is continuous for $z > 0$. This is done in the following lemma.

**Lemma 3.9.** Assume that for the limiting distribution $(W^{-}, W^{+}, T)$ we can find $Z \in (0, \infty)$ such that for each $\tilde{z} < Z$ there exist $\tilde{w} = \tilde{w}(\tilde{z})$, $\epsilon = \epsilon(\tilde{z})$ and $\delta = \delta(\tilde{z})$ such that

$$\mathbb{P} \left( \frac{T - 1}{W^{-}\tilde{z}} \leq \epsilon, W^{-} > \tilde{w} \right) = 0, \quad \forall z \in (\tilde{z} - \delta, \tilde{z} + \delta).$$

That is, $T$ shall be bounded away from some linear dependence on $W^{-}$ almost surely. Then the expression $\mathbb{E} [W^{-}W^{+}\mathbb{P}(\text{Poi}(W^{-}z) = T - 1)]$ is continuous on $(0, Z)$. 

24
Proof. We fix some $\tilde{z} < Z$ and aim to show that the family
\[
\{W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right)\}_{z \in [\tilde{z} - \delta, \tilde{z} + \delta]}
\]
is bounded by some integrable random variable almost surely. This will show continuity by Lebesgue’s dominated convergence theorem.

By definition of a Poisson random variable, we derive
\[
W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right) = W^{-}W^{+}e^{-W^{-}z}(W^{-}z)^{T-1}/\Gamma(T).
\]
Then, by applying Stirling’s approximation to the $\Gamma$-function,
\[
W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right) \leq W^{-}W^{+}\exp\left\{-W^{-}z\left(1 - \frac{T-1}{W^{-}z} + \frac{T-1}{W^{-}z}\log\left(\frac{T-1}{W^{-}z}\right)\right)\right\}.
\]
In the exponent, we identify the expression $g\left((T - 1)/(W^{-}z)\right)$, where $g(x) := 1 - x + x\log(x)$. The continuous function $g$ admits the unique minimum $g(x^*) = 0$ at $x^* = 1$. Since further $\lim_{x \to 0^+} g(x) = 1$ and $\lim_{x \to \infty} g(x) = \infty$, it holds that $g(x) \geq G$ for $|x - 1| > \epsilon$ and some suitable $G > 0$. Therefore,
\[
W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right) \leq W^{+}\left(W^{-}\exp\left\{-W^{-}(\tilde{z} - \delta)G\right\} + \tilde{w}\right) \leq W^{+}M(\tilde{z}) \in L^1,
\]
almost surely, where $M(\tilde{z})$ is a positive constant depending on $\tilde{z}$ only.

Remark 3.10. Lemma 3.9 is stated for general threshold distributions $T$ and hence needs a condition of almost sure boundedness. The case we are particularly interested in is $T = \tau(W^{-})$ for $\tau$ an integer-valued function satisfying $\tau(w) = o(w)$. We can then choose $Z = \infty$, $\epsilon = \frac{1}{2}$, $\delta < \tilde{z}$ arbitrary and $\tilde{w}$ such that $\tau(w) < (\tilde{z} - \delta)w/2$ for all $w > \tilde{w}$. The latter one is possible by $\tau(w) = o(w)$. It follows that $\mathbb{E}[W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right)]$ is continuous at all $\tilde{z} > 0$. In Theorem 3.5 we are in a situation where $\tau(w) = O(w^\gamma)$ for $0 \leq \gamma < 1$ and hence $\tau(w) = o(w)$.

Proof of Theorem 3.5. In order to ease notation, we will assume throughout all the proofs that $w_{\text{min}}^{-} = w_{\text{min}}^{+} = 1$. The arguments for general $w_{\text{min}}^{-}$ and $w_{\text{min}}^{+}$ are completely analogue.

First note that for $\gamma_\text{crit} < 0$,
\[
\mathbb{E}[W^{-}W^{+}] \leq \mathbb{E}\left[(W^{-})^{1 + \frac{\beta_{-} - 1}{\beta_{-}} - 1}\right] = (\beta_{-} - 1) \int_{1}^{\infty} w^\gamma w^{\frac{\beta_{-} - 1}{\beta_{-}} - 1} dw < \infty,
\]
where we used that $\mathbb{E}[W^{-}W^{+}]$ is maximized for comonotone dependence which is given by
\[
W^{+} = F_{W^{-}}^{w}(F_{W^{-}}(W^{-})) = (W^{-})^\frac{\beta_{-} - 1}{\beta_{-}}.
\]
By Lebesgue’s dominated convergence theorem we hence conclude that $f(z; (W^{-}, W^{+}, T))$ is continuously differentiable on $[0, \infty)$ with derivative $\mathbb{E}[W^{-}W^{+}\mathbb{P}\left(\text{Poi}(W^{-}z) = T - 1\right)] - 1$. By $T \geq 2$, in particular, $f'(0; (W^{-}, W^{+}, T)) = -1$ and hence by Theorem 3.2 or Theorem 3.4 the system is resilient to small shocks.

Now suppose that $\gamma_\text{crit} = 0$ and $\alpha := \liminf_{w \to \infty} \tau(w) > \alpha_\text{crit} + 1$. We assume that $\alpha < \infty$, otherwise we could simply truncate $\tau(w)$ at some $\mathbb{N} \ni \alpha > \alpha_\text{crit} + 1$. Since $\tau(w) \geq 2$ is an integer-valued function, we observe that $\alpha \in \mathbb{N}\setminus\{0, 1\}$ and $\tau(w) \geq \alpha$ for all $w > \tilde{w}$ and some
constant $\tilde{w} > 0$. By this we derive that
\[
E\left[ W^+ \psi_T(W^z) \right] \leq E\left[ W^+ \psi_\alpha(W^-z)1_{\{W^- > \tilde{w}\}} + W^+ \psi_2(W^-z)1_{\{W^- \leq \tilde{w}\}} \right]
\]
\[
\leq E\left[ W^+ \psi_\alpha(W^-z) \right] + \frac{w^2\tilde{z}^2}{2} E[W^+]
\]
\[
= E\left[ W^+ \psi_\alpha(W^-z) \right] + o(z).
\]
Now note that since $\psi_\alpha(x)$ is a strictly increasing function in $x$, this expression becomes maximized for comonotone dependence of $W^-$ and $W^+$. Hence,
\[
z^{-1}E\left[ W^+ \psi_T(W^-z) \right] \leq z^{-1} \sum_{k=\alpha}^\infty E\left[ (W^-)^{k-1} \frac{e^{-W^-z}(W^-z)^k}{k!} \right] + o(1)
\]
\[
= \alpha^{\text{crit}} \sum_{k=\alpha}^\infty \int_z^\infty \frac{\beta^{-1} - \beta^{-k} + x^{-x}}{k!} dx + o(1).
\]
Now,
\[
\lim_{z \to 0} \sum_{k=\alpha}^\infty \int_z^\infty \frac{(x)^{k-1} - \beta^{-k} + x^{-x}}{k!} dx = \sum_{k=\alpha}^\infty \frac{\Gamma(k-1)}{k!} = \sum_{k=\alpha}^\infty \frac{1}{k(k-1)} = \frac{1}{\alpha - 1} < \frac{1}{\alpha^{\text{crit}}},
\]
and therefore
\[
\limsup_{z \to 0^+} \frac{f(z; (W^-, W^+, T))}{z} = \limsup_{z \to 0^+} \frac{E[W^+ \psi_T(W^-z)]}{z} - 1 < 0.
\]
By Lemma 3.9 and Remark 3.10, we also know that $E[W^-W^+P(\text{Poi}(W^-z) = T - 1)]$ is continuous for $z > 0$ (we can simply cut $\tau(w)$ at $\alpha$) and hence by Theorem 3.4 we can conclude that the system must be resilient.

Finally, assume that $\gamma^{\text{crit}} > 0$ and $\alpha := \liminf_{w \to \infty} \tau(w)/w^{\gamma^{\text{crit}}} > \alpha^{\text{crit}}$. We can then choose some $\alpha^{\text{crit}} < \tilde{\alpha} < \alpha$ and $\tilde{w} < \infty$ such that $\tau(w) \geq \left( \tilde{\alpha}w^{\gamma^{\text{crit}}} \right)$ for all $w > \tilde{w}$. Hence we derive that
\[
E\left[ W^+ \psi_T(W^-z) \right] \leq E\left[ W^+ \psi_{\tilde{\alpha}w^{(\gamma^{\text{crit}})}}(W^-z)1_{\{W^- > \tilde{w}\}} \right] + o(z).
\]
For $\epsilon > 0$ small enough and $w \leq \left( \frac{(1+\epsilon)\alpha}{\tilde{\alpha}} \right)^{\frac{1}{\gamma^{\text{crit}} - 1}}$, we have $wz < \tilde{\alpha}w^{\gamma^{\text{crit}}}$ and hence a Chernoff bound yields
\[
\psi_{\tilde{\alpha}w^{\gamma^{\text{crit}}}}(wz) \leq \left( \frac{wz}{\tilde{\alpha}w^{\gamma^{\text{crit}}}} \right)^{\alpha^{\text{crit}}} \exp\left\{ -\tilde{\alpha}w^{\gamma^{\text{crit}}} - wz \right\} = \exp\left\{ -z^{\gamma^{\text{crit}}} g \left( wz^{x^{1-\gamma^{\text{crit}}}} \right) \right\},
\]
where
\[
g(x) := x - \tilde{\alpha}x^{\gamma^{\text{crit}}} \log \left( \frac{e^{x^{1-\gamma^{\text{crit}}}}}{\tilde{\alpha}} \right).
\]
Then for $\tilde{w} < w \leq \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}}$ and $\lambda > 0$, we derive

$$
\psi\left[\tilde{a}w^{-\gamma_{\text{crit}}} \right] (w^2) \leq \left(\frac{wze}{\tilde{a}w^{-\gamma_{\text{crit}}}}\right)^{1+\lambda} \leq \left(\frac{wze}{\tilde{a}w^{-\gamma_{\text{crit}}}}\right)^{1+\lambda} (1 + \epsilon - \tilde{a}w^{-\gamma_{\text{crit}}} + 1 + \lambda) \leq \left(\frac{ze}{\alpha}\right)^{1+\lambda},
$$

if $\tilde{w}$ large enough such that $\tilde{a}w^{-\gamma_{\text{crit}}} > 1 + \lambda$ and $w(1-\gamma_{\text{crit}})(1+\lambda)(1 + \epsilon - \tilde{a}w^{-\gamma_{\text{crit}}} + 1 + \lambda) \leq 1$ for all $w > \tilde{w}$. Hence

$$
E\left[W^+ \psi\left[\tilde{a}w^{-\gamma_{\text{crit}}} \right] (W^- z) 1\left\{ \tilde{w} < W^- \leq \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}} \right\} \right] \leq \left(\frac{ze}{\alpha}\right)^{1+\lambda} E[W^+] = o(z).
$$

Further, for $\left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}} < w \leq \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}}$, we have

$$
g\left(\frac{w}{1+\gamma_{\text{crit}}} \right) \geq \delta
$$

for some $\delta > 0$ and hence

$$
E\left[W^+ \psi\left[\tilde{a}w^{-\gamma_{\text{crit}}} \right] (W^- z) 1\left\{ \tilde{w} < W^- \leq \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}} \right\} \right] \leq \left(\frac{z}{\gamma_{\text{crit}}} \right) \exp \left\{ -\delta \frac{1}{\gamma_{\text{crit}}} \right\} = o(z).
$$

Hence, as $z \to 0$, only $W^- > \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}}$ contributes to $z^{-1}E[W^+\psi_T(W^- z)]$. On this set, by bounding with the comonotone dependence, we compute

$$
E\left[W^+ \psi_T(W^- z) 1\left\{ W^- > \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}} \right\} \right] \leq E\left[W^+ 1\left\{ W^- > \left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}} \right\} \right] \leq (\beta - 1) \int_{\left(\frac{(1+\epsilon)\epsilon z}{\alpha}\right)^{1/\gamma_{\text{crit}}}}^{\infty} \frac{\beta - 1}{w^{\beta - 1 + \beta - 1}} \, dw
$$

$$
= \alpha_{\text{crit}} \left(1 + \epsilon\right) \frac{z}{\alpha}
$$

Hence, choosing $\epsilon > 0$ small enough such that $\tilde{a}/(1 + \epsilon) > \alpha_{\text{crit}}$, we conclude that

$$
\limsup \frac{f(z; (W^-, W^+, T))}{z} = \limsup \frac{E[W^+ \psi_T(W^- z)]}{z} - 1 < 0
$$

which shows resilience by the same arguments as in part 2, noting that we can cut $\tau(w)$ at $w^\eta$ for some $\gamma_{\text{crit}} < \eta < 1$.

\textbf{Proof of Theorem 3.7.} Again, we simplify notation by setting $w^-_{\min} = w^+_{\min} = 1$.

We start by proving the second statement. Let $\alpha := \limsup_{w \to \infty} \tau(w) / w^{\gamma_{\text{crit}}}$ and choose $\alpha < \tilde{\alpha} < \int_0^\infty \Lambda \left(x^{1-\beta^+}\right) \, dx$ and $\tilde{w} < \infty$ such that $\tau(w) \leq \left[\tilde{a}w^{\gamma_{\text{crit}}} \right]$ for all $w > \tilde{w}$. Moreover, choose $\epsilon > 0$ and $\delta > 0$ such that $\tilde{a} < (1 - \epsilon)(1 - \delta) \int_0^\infty \Lambda \left(x^{1-\beta^+}\right) \, dx$ and let $z > 0$ small.
enough such that
\[ \tilde{w} \leq \left( \frac{(1 - \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1}{\gamma_{\text{crit}} - 1}} \]
as well as
\[ z < (\epsilon^2 \delta)^{1/\gamma_{\text{crit}}} \left( \frac{\tilde{\alpha}}{1 - \epsilon} \right)^{1/\gamma_{\text{crit}}}. \] (3.6)
For such \( z \) and \( w > ((1 - \epsilon)z/\tilde{\alpha})^{1/\gamma_{\text{crit}} - 1} \) it holds that
\[ P \left( \text{Poi}(wz) \geq \left\lceil \tilde{\alpha}w^{\gamma_{\text{crit}}} \right\rceil \right) \geq 1 - \text{P} (|\text{Poi}(wz) - wz| \geq \epsilon wz) \geq 1 - \frac{1}{\epsilon^2 wz} \geq 1 - \frac{1 - \epsilon}{\epsilon^2 \tilde{\alpha}w^{\gamma_{\text{crit}}}} \geq 1 - \delta, \]
by Chebyshev’s inequality and (3.6). Therefore,
\[
\mathbb{E} \left[ W^+ \psi_T(W^- z) \right] \\
\geq (1 - \delta) \mathbb{E} \left[ W^+ \mathbf{1} \left\{ W^- > \left( \frac{(1 - \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1}{\gamma_{\text{crit}} - 1}} \right\} \right] \\
= (1 - \delta) \int_0^\infty \mathbb{P} \left( W^+ > x, W^- > \left( \frac{(1 - \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1}{\gamma_{\text{crit}} - 1}} \right) dx \\
= (1 - \delta) \left( \frac{(1 - \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1}{\gamma_{\text{crit}} - 1} \beta_{\text{crit}} - 1} \\
x \int_0^\infty \mathbb{P} \left( F_{W^+}(W^+) > 1 - x^{1 - \beta^+} p(z), F_{W^-}(W^-) > 1 - p(z) \right) dx \\
= (1 - \epsilon)(1 - \delta) \frac{z}{\tilde{\alpha}} \int_0^\infty \mathbb{P} \left( F_{W^+}(W^+) > 1 - x^{1 - \beta^+} p(z) \mid F_{W^-}(W^-) > 1 - p(z) \right) dx, \] (3.7)
where we substituted
\[ p(z) := \left( \frac{(1 - \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1 - \beta^-}{\gamma_{\text{crit}} - 1}}. \]
Note that the conditional probability is bounded by \( 1 \wedge x^{1 - \beta^+} \). Hence, by Lebesgue’s dominated convergence theorem
\[ \frac{\mathbb{E} [W^+ \psi_T(W^- z)]}{z} \geq (1 - \epsilon)(1 - \delta) \int_0^\infty \Lambda \left( x^{1 - \beta^+} \right) dx + o(1) > 1 + o(1) \]
and thus \( \mathbb{E} [W^+ \psi_T(W^- z)] > z \) for \( z \) small enough which implies non-resilience by Theorem 3.1.
For resilience in part 2 we follow the same proof as for Theorem 3.5 until we arrive at expression (3.5) which we can now evaluate with the same means as above. That is,
\[
z^{-1} \mathbb{E} \left[ W^+ \mathbf{1} \left\{ W^- > \left( \frac{(1 + \epsilon)z}{\tilde{\alpha}} \right)^{\frac{1}{\gamma_{\text{crit}} - 1}} \right\} \right] \rightarrow \frac{1 + \epsilon}{\tilde{\alpha}} \int_0^\infty \Lambda \left( x^{1 - \beta^+} \right) dx < 1, \quad \text{as } z \to 0.
\]
For the first statement of the theorem, note that we can lower bound the integral in (3.7) by
\[ \mathbb{P} \left( F_{W^+}(W^+) > 1 - p(z) \mid F_{W^-}(W^-) > 1 - p(z) \right) \]
and thus
\[ \mathbb{E} [W^+ \psi_T(W^- z)] = z \int_0^\infty \mathbb{P} \left( F_{W^+}(W^+) > 1 - x^{1 - \beta^+} p(z), F_{W^-}(W^-) > 1 - p(z) \right) dx, \] (3.7)
and hence,
\[ \liminf_{z \to 0} \frac{\mathbb{E}[W^+ \psi_T(W^- z)]}{z} \geq \frac{(1 - \epsilon)(1 - \delta)}{\alpha} \lambda > 1, \]
which proves non-resilience as above.

**Proof of Theorem 3.8.** For \( \gamma_{\text{crit}} < 0 \), similar as in the proof of Theorem 3.5, we derive
\[
\mathbb{E}[W - W^+] \leq \mathbb{E}[L_{\bar{W}^+ \circ \bar{F}_W}(W^-)(W^-)^{1+\frac{\delta}{\beta-1}}] \leq K(\beta^- - 1) \int_1^\infty w^{\epsilon+\gamma_{\text{crit}}-1}dw < \infty,
\]
where we chose \( \epsilon < -\gamma_{\text{crit}} \) and used the slow variation of \( L_{\bar{W}^+ \circ \bar{F}_W}(w) \) to bound it by \( Kw^\epsilon \) for some constant \( K < \infty \). As in the proof of part 1 in Theorem 3.5 we conclude that the system must be resilient.

For the second part, we can follow the proof of part 2 in Theorem 3.5 substituting \( \gamma_{\text{crit}} \) by \( \gamma \) until we arrive at expression (3.5). This time, we bound \( L_{\bar{W}^+ \circ \bar{F}_W}(w) \leq Kw^\delta \) for some \( \delta < \gamma - \gamma_{\text{crit}} \), such that
\[
\mathbb{E}[W^+ 1\{W > \left(\frac{(1 + \epsilon)z}{\alpha}\right)^{\frac{1}{\gamma - 1}}\}] \leq \mathbb{E}[L_{\bar{W}^+ \circ \bar{F}_W}(W^-)(W^-)^{\frac{\delta}{\beta-1}} 1\{W > \left(\frac{(1 + \epsilon)z}{\alpha}\right)^{\frac{1}{\gamma - 1}}\}] = o(z).
\]
Resilience then follows as usual.

### 3.3 Systemic Capital Requirements

In this section, we shall translate the threshold requirements from Theorem 3.5 to capital requirements in the exposure model. That is, we state explicit capital amounts that the banks in the network have to be able to procure in stress scenarios in order for the system to be resilient to shock events. As for the threshold requirements, it is important to note that each bank can compute its capital requirements on its own by just knowing its local neighborhood in the network. Further, a bank’s capital requirement only depends on the default risk the bank exposes itself to and not on the default risk the bank poses to other banks. Proposition 3.11 states a straightforward robust way to translate threshold requirements into sufficient capital requirements. In general, it might lead to capital requirements that are too high and hence unnecessarily reduce interbank lending and liquidity, however. For this reason, we further provide Theorem 3.13 below, which accurately determines capital requirements under a certain regularity assumption on the exposure lists.

**Proposition 3.11.** Let \( (\bar{w}^- , \bar{w}^+, s, E, c) \) be a financial network such that \( (\bar{w}^- , \bar{w}^+) \) satisfies the assumptions in Theorem 3.5. Further let \( \tau : \mathbb{R} \to \mathbb{N} \setminus \{0, 1\} \) a function satisfying (3.4) and such that \( \liminf_{w \to \infty} \tau(w) > \alpha_{\text{crit}} + 1 \) if \( \gamma_{\text{crit}} = 0 \) respectively \( \liminf_{w \to \infty} \tau(w)/w_{\gamma_{\text{crit}}} > \alpha_{\text{crit}} \) if \( \gamma_{\text{crit}} > 0 \). Then the system is resilient if for each bank \( i \in [n] \),
\[
c_i > \max \left\{ \sum_{j \in J} E_{j,i} \right\} \quad J \subseteq [n], |J| = \tau(w_i^-) - 1
\]
almost surely, which is the sum of the \( \tau(w_i^-) - 1 \) largest exposures of \( i \).
Assumption 3.12. Motivated by above derivations, we want to assume that for each bank $i$ in (3.8) and (3.9) hold. Furthermore, we assume them to be uniform for $x > \tau$.

The capitals $\text{Proof.}$ The capitals $c_i$ are chosen such that the threshold values $\tau_i$ are at least $\tau(w_i^-)$. Hence coupling the weighted network to the corresponding threshold network yields the result. 

In the same spirit, also a robust translation of Theorems 3.7 and 3.8 to the exposure model is possible.

Proposition 3.11 requires each bank $i$ to be able to cope with default of its $\tau(w_i^-)$ largest exposures. But as we have seen in the proof of Theorem 3.5, only the thresholds and hence the capitals of large banks in the network matter for resilience. For large banks with many exposures on the other hand one can expect an averaging effect of the exposure sizes to occur if they are not too irregular. Hence, one can presume that in this case multiplying threshold exposures on the other hand one can expect an averaging effect of the exposure sizes to occur if $x > \tau$ is possible.

The capitals $\text{Proof.}$ Theorem 3.13 provides the banks with a formula that is easy to use and only requires the resilience characteristics. We formalize this in Theorem 3.13 under Assumption 3.12 on the exposure sequences. This assumption is motivated by the following reasoning:

For each bank $i$, let $\{E_{j,i}\}_{j \in \mathbb{N}\setminus \{i\}}$ be a sequence of i.i.d. positive random variables. Let $\mu_i := \mathbb{E}[E_{\sigma(1),i}] < \infty$ be their mutual expectation and denote $S^i_k := \sum_{j=1}^{k} E_{\sigma(j),i}$ the sum of the first $k$ exposures. If there is some $t > 1$ such that $\mathbb{E}\left[|E_{\sigma(1),i}|^t\right] < \infty$, then by the Baum-Katz-Theorem from $\mathbb{S}$ for all $\epsilon > 0$,

$$n^{t-1} \mathbb{P}(S^n_i \geq (1+\epsilon)n\mu_i) \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (3.8)$$

and for all $x > 1$,

$$n^{tx-1} \mathbb{P}(S^n_i \geq \epsilon \mu_i^x) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.9)$$

Assumption 3.12. Motivated by above derivations, we want to assume that for each bank $i \in [n]$ with exposure list $\{E_{j,i}\}_{j \in \mathbb{N}}$ of mutual mean $\mu_i$, we can find $t > 1$ such that the convergences in (3.8) and (3.9) hold. Furthermore, we assume them to be uniform for $i \in [n]$ (but not necessarily for $\epsilon$ or $x$).

Assumption 3.12 ensures a certain regularity of the exposures of different banks without bounding their mean.

Theorem 3.13. Let $(w^-, w^+, s, E, c)$ be a financial system satisfying Assumptions 2.1 and 2.2 such that the limiting weight distributions $W^-$ and $W^+$ are Pareto distributed as in Theorem 3.5 with parameters $w^-_{\min} > 0$ and $\beta^- > 2$ respectively $w^+_{\min} > 0$ and $\beta^+ > 2$. The quantities $\gamma^{\text{crit}}$ and $\alpha^{\text{crit}}$ shall be defined as in Theorem 3.5. Further, assume that $c_i > \max_{j \in [n]} E_{j,i}$ almost surely for all $i \in [n]$. Then the following holds:

1. If $\gamma^{\text{crit}} < 0$, then the system is always resilient.

Now further assume that the exposure lists $\{E_{j,i}\}_{j \in \mathbb{N}\setminus \{i\}}$, $i \in \mathbb{N}$, satisfy Assumption 3.12 for some $t > 1$. Then the system is resilient if one of the following holds:

2. $\gamma^{\text{crit}} = 0$ and there exist a function $\tau : \mathbb{R}^+ \rightarrow \mathbb{N}\setminus \{0,1\}$ and some $\gamma > 0$ such that $\lim_{w \rightarrow \infty} \tau(w)/w^\gamma > 0$ and for all $i \in [n]$, $c_i \geq \tau(w_i^-)\mu_i$ almost surely.

3. $\gamma^{\text{crit}} > 0$ and there exist a function $\tau : \mathbb{R}^+ \rightarrow \mathbb{N}\setminus \{0,1\}$ such that $\lim_{w \rightarrow \infty} \tau(w)/w^{\gamma^{\text{crit}}} > \alpha^{\text{crit}}$ and for all $i \in [n]$, $c_i \geq \tau(w_i^-)\mu_i$ almost surely.

Theorem 3.13 provides the banks with a formula that is easy to use and only requires the regulator to announce values $\alpha^{\text{crit}}$ and $\gamma^{\text{crit}}$. Resilient capital requirements are then determined according to average exposure size $\mu_i$ and the number of exposures $d_i^- \sim w_i^-$. Note that spreading a constant total exposure over more counterparties then reduces the capital requirements of a bank since average exposure size $\mu_i$ is proportional to $(d_i^-)^{-1}$ while the factor $\alpha^{\text{crit}}(d_i^-)^{\gamma^{\text{crit}}}$ is sublinear in $d_i^-$. 

30
Remark 3.14. Theorem 3.13 is an extension of Theorem 3.5 to the exposure model under Assumption 3.12 for the exposure sequences. By the same means, also Theorems 3.7 and 3.8 can be extended.

Proof. By the assumption of $c_i > \max_{j \in [n] \setminus \{i\}} E_{j,i}$ almost surely for all $i \in [n]$, we have that $\tau_i \geq 2$ almost surely. The proof of part 1 is hence completely analogue to the one of part 1 in Theorem 3.5.

We continue by proving part 2. By the means of the proof of Theorem 3.5 we derive that
\[
\limsup_{z \to 0^+} \mathbb{E} \left[ W^+ \psi_T(W^- z) 1 \left\{ T > (1 + \epsilon) \alpha^{\text{crit}}(W^-) \right\} \right] < 1,
\]
for each $\epsilon > 0$. It will hence suffice to prove
\[
\mathbb{E} \left[ W^+ \psi_T(W^- z) 1 \left\{ T \leq (1 + \delta) \alpha^{\text{crit}}(W^-) \right\} \right] = o(z),
\]
in order to show resilience. To this end, choose $0 < \delta < \gamma^{\text{crit}}(t - 1)$. Then
\[
\mathbb{E} \left[ W^+ \psi_T(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right] \leq \mathbb{E} \left[ W^+ \psi_2(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right]
= \lim_{M \to \infty} \mathbb{E} \left[ (W^+ \wedge M) \psi_2(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right]
\leq \liminf_{M \to \infty} \liminf_{n \to \infty} n^{-1} \sum_{i \in [n]} (w_i^+ \wedge M) \psi_2(w_i^- z) 1 \left\{ \tau_i \leq (w_i^-)^\delta \right\},
\]
where we approximated the non-continuous integrand by continuous ones and used almost sure weak convergence by Assumption 2.2. Now taking expectations with respect to the exposure lists, by Fatou’s lemma we derive
\[
\mathbb{E} \left[ W^+ \psi_T(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right] \leq \liminf_{M \to \infty} \liminf_{n \to \infty} n^{-1} \sum_{i \in [n]} (w_i^+ \wedge M) \psi_2(w_i^- z) \mathbb{P} \left( \tau_i \leq (w_i^-)^\delta \right)
\leq K_1 \liminf_{M \to \infty} \liminf_{n \to \infty} n^{-1} \sum_{i \in [n]} (w_i^+ \wedge M) \psi_2(w_i^- z)(w_i^-)^{\delta - \gamma^{\text{crit}}}
= K_1 \liminf_{M \to \infty} \mathbb{E} \left[ (W^+ \wedge M) \psi_2(W^- z)(W^-)^{\delta - \gamma^{\text{crit}}} \right]
= K_1 \mathbb{E} \left[ W^+ \psi_2(W^- z)(W^-)^{\delta - \gamma^{\text{crit}}} \right],
\]
where we used Assumption 3.12 to bound
\[
\mathbb{P} \left( \tau_i \leq (w_i^-)^\delta \right) = \mathbb{P} \left( \sum_{j=1}^{(w_i^-)^\delta} E_{\rho_i(j), i} \geq c_i \right) \leq \mathbb{P} \left( \sum_{j=1}^{(w_i^-)^\delta} E_{\rho_i(j), i} \geq \tau(w_i^-) \mu_i \right)
\leq \mathbb{P} \left( \sum_{j=1}^{(w_i^-)^\delta} E_{\rho_i(j), i} \geq \alpha^{\text{crit}}(w_i^-)^{\gamma^{\text{crit}}} \mu_i \right)
\leq K_1 (w_i^-)^{\delta - \gamma^{\text{crit}}},
\]
for $w_i^-$ large enough and some uniform constant $K_1 > \infty$. Note that for $W^- \leq \tilde{w}$, we have
\[ E \left[ W^+ \psi_T(W^- z) 1_{\{ W^- \leq \varrho \}} \right] = o(z) \] as in the proof of Theorem 3.5. Hence it holds

\[ \frac{E \left[ W^+ \psi_T(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right]}{z} \leq K_1 E \left[ \frac{W^+ \psi_2(W^- z)}{W^- z} (W^-)^{1+\delta-t\gamma_{\text{crit}}} \right] + o(1). \]

Since \( \psi_2(x) = o(x) \), by Lebesgue’s dominated convergence theorem it is sufficient to prove

\[ E \left[ W^+ (W^-)^{1+\delta-t\gamma_{\text{crit}}} \right] < \infty \] in order for \( E \left[ W^+ \psi_T(W^- z) 1 \left\{ T \leq (W^-)^\delta \right\} \right] = o(z) \). We can easily choose \( t > 1 \) and \( \delta > 0 \) in such a way that \( 1 + \delta - t\gamma_{\text{crit}} > 0 \) and can therefore estimate

\[ E \left[ W^+ (W^-)^{1+\delta-t\gamma_{\text{crit}}} \right] \] by the comonotone expectation \( E \left[ (W^-)^\beta \right] \) which is finite since

\[ \frac{\beta^+ - 1}{\beta^+ - 1} + \delta - t\gamma_{\text{crit}} - \beta^+ = \gamma_{\text{crit}} (1-t) + \delta - 1 < 0 \]

by our choice of \( \delta \).

Now let \( 2 \leq N < \gamma_{\text{crit}}/\delta \) and consider

\[ E \left[ W^+ \psi_T(W^- z) 1 \left\{ (W^-)^{(N-1)\delta} < T \leq (W^-)^{N\delta} \right\} \right]. \]

By the means of the proof of Theorem 3.5 it is enough to consider

\[ E \left[ W^+ 1 \left\{ W^- > z^{(N-1)\delta}, T \leq (W^-)^{N\delta} \right\} \right]. \]

Similar as above, we derive

\[ P \left( \tau_1 \leq (w^-_1)^{N\delta} \right) \leq K_N (w^-_1)^{N\delta-t\gamma_{\text{crit}}} \]

for some uniform \( K_N < \infty \) and

\[ E \left[ W^+ 1 \left\{ W^- > z^{(N-1)\delta - 1}, T \leq (W^-)^{N\delta} \right\} \right] \]

\[ \leq K_N E \left[ W^+ 1 \left\{ W^- > z^{(N-1)\delta - 1} \right\} (W^-)^{N\delta-t\gamma_{\text{crit}}} \right] \]

\[ \leq K_N E \left[ W^+ 1 \left\{ W^- > z^{(N-1)\delta - 1} \right\} \left( z^{N\delta-t\gamma_{\text{crit}}} \right)^{\frac{N\delta-t\gamma_{\text{crit}}}{(N-1)\delta-1}} \right] \]

\[ = K_N \frac{\beta^+ - 1}{\beta^+ - 1} \frac{z^{N\delta-t\gamma_{\text{crit}}}}{(N-1)\delta-1} \]

\[ = o(z) \]

since by the choice of \( \delta \) and \( N \),

\[ \gamma_{\text{crit}} - 1 + N\delta - t\gamma_{\text{crit}} < (N-1)\delta - 1 < 0. \]

Finally, we have to consider the part

\[ E \left[ W^+ \psi_T(W^- z) 1 \left\{ (W^-)^{\gamma_{\text{crit}}-\delta} < T \leq (1+\epsilon)\alpha_{\text{crit}}(W^-)^{\gamma_{\text{crit}}} \right\} \right]. \]

If we choose \( \epsilon > 0 \) small enough such that \( (1+2\epsilon)\alpha_{\text{crit}} < \liminf_{w \to \infty} \tau(w)/w^{\gamma_{\text{crit}}} \) and denote
\[ \tilde{\tau}(w) := (1 + \epsilon)\alpha_{\text{crit}} w^{\gamma_{\text{crit}}}, \]
then we observe that by Assumption 3.12 for \( w_i \) large enough
\[
P \left( \tilde{\tau}_i \leq \tilde{\tau}(w_i) \right) \leq P \left( \sum_{j=1}^{w_i} E_{\rho(j), i} \geq (1 + 2\epsilon)\alpha_{\text{crit}} (w_i)^{\gamma_{\text{crit}}} \mu_i \right)
= P \left( \sum_{j=1}^{\frac{1}{\gamma_{\text{crit}}+\delta-1}} E_{\rho(j), i} \geq \frac{1 + 2\epsilon}{1 + \epsilon} \tilde{\tau}(w_i) \mu_i \right)
\leq K \left( (1 + \epsilon)\alpha_{\text{crit}} (w_i)^{\gamma_{\text{crit}}} \right)^{-1}
\text{for some } K < \infty. \]
Similarly as above, we then have to consider
\[
E \left[ W^+ \mathbf{1} \left\{ W^- > z^{\frac{1}{\gamma_{\text{crit}}+\delta-1}}, T \leq (1 + \epsilon)\alpha_{\text{crit}} (W^-)^{\gamma_{\text{crit}}} \right\} \right]
\leq KE \left[ W^+ \mathbf{1} \left\{ W^- > z^{\frac{1}{\gamma_{\text{crit}}+\delta-1}} \right\} \left( 1 + \epsilon \right)\alpha_{\text{crit}} (W^-)^{\gamma_{\text{crit}}} \right]^{-(t-1)}
\leq K \left( 1 + \epsilon \right)\alpha_{\text{crit}}^{-(t-1)} E \left[ W^+ \mathbf{1} \left\{ W^- > z^{\frac{1}{\gamma_{\text{crit}}+\delta-1}} \right\} z^{\frac{\gamma_{\text{crit}}-(t-1)\gamma_{\text{crit}}}{\gamma_{\text{crit}}+\delta-1}} \right]
\leq K \left( 1 + \epsilon \right)\alpha_{\text{crit}}^{-(t-1)} \beta - 1 \frac{\gamma_{\text{crit}}-1-(t-1)\gamma_{\text{crit}}}{\gamma_{\text{crit}}+\delta-1}
= o(z),
\]
since
\[ \gamma_{\text{crit}} - 1 - (t - 1)\gamma_{\text{crit}} < \gamma_{\text{crit}} + \delta - 1 < 0. \]
Altogether, we have hence shown that
\[
E \left[ W^+ \psi_T (W^-) \mathbf{1} \left\{ T \leq (1 + \epsilon)\alpha_{\text{crit}} (W^-)^{\gamma_{\text{crit}}} \right\} \right] = o(z)
\]
(note that we decomposed the expectation in only finitely many summands) and hence
\[
\limsup_{z \to 0^+} \frac{f(z; (W^-, W^+, T))}{z} = \limsup_{z \to 0^+} \frac{E \left[ W^+ \psi_T (W^-) z \right]}{z} = 1 < 0,
\]
which shows resilience by Theorem 3.4. Note that we can cut off \( T \) at the value \( (W^-)^{\eta} \) for some \( \gamma_{\text{crit}} < \eta < 1 \) in order to ensure continuity of \( E \left[ W^- W^+ \mathbb{P} (\text{Poi}(W^- z) = T - 1) \right] \) by Lemma 3.9 together with Remark 3.10.

Part 2 follows by the same calculations as for part 3 but replacing \( \gamma_{\text{crit}} \) by \( \gamma \) and using \( \gamma > \gamma_{\text{crit}} \).

4 Simulation Study

All previous chapters have been formulated in the limit as the number of banks \( n \) tends to infinity and the fraction of initially defaulted banks \( p \) tends to zero. It is hence reasonable to investigate whether the results are good approximations also for real networks which are finite with only a few thousand institutions and experience a shock of a finite fraction of banks.

Since our model is based on the non-observable weight-parameters, one would have to estimate them from the degree-sequences which are observable for real network configurations at least by
regulating institutions. Hence, we will start this section by a short note on weight-estimation. Since specific transactions between banks are not disclosed to the public there is no data basis for us to investigate real networks, however. Instead, we will subsequently discuss our findings by simulating networks.

4.1 Estimation of Weights

Since the weight sequences are not directly observable from real networks, we give a few lines here on how to estimate them from data that is observable. First note that for a network $G$ of size $n$ with edge set $E(G)$ the likelihood of weight sequences $w^- = (w^-_1, \ldots, w^-_n)$ and $w^+ = (w^+_1, \ldots, w^+_n)$ is given by

$$L(w^-, w^+_1, \ldots, w^-_n, w^+_n | E(G)) = \prod_{(i,j) \in E(G)} \left( \frac{w^+_i w^-_j}{n} \right) \prod_{i \neq j \notin E(G)} \left( 1 - \frac{w^+_i w^-_j}{n} \right).$$

One can always derive the maximum-likelihood estimators $\hat{w}^-, \ldots, \hat{w}^-_n, \hat{w}^+_1, \ldots, \hat{w}^+_n$ by numerically maximizing $L$. In order to obtain some intuition about them, we further want to derive an approximation of the estimators. For this, we assume that $w^+_i w^-_j \ll n$ for all $i, j \in [n]$ which is a reasonable assumption at least when $W^+W^- \ll n$ is integrable. We can hence approximate

$$L(w^-, w^+_1, \ldots, w^-_n, w^+_n | E(G)) \approx \frac{1}{n^s} \prod_{i \in [n]} \left( w^-_i (w^+_i)^{d^-_i} (w^+_i)^{d^+_i} \exp \left( -w^+_i \frac{\sum_{j \in [n]} w^-_j}{n} \right) \right),$$

where $s := \sum_{i \in [n]} d^-_i = \sum_{i \in [n]} d^+_i$. By the product form $w^+_i w^-_j$ in (2.1), we are free to multiply all out-weights $w^+_i$ by some constant $\eta$ if, at the same time, we multiply all in-weights by its inverse $\eta^{-1}$. Motivated by the fact that $\sum_{i \in [n]} d^-_i = \sum_{i \in [n]} d^+_i$, we use this degree of freedom to set $\sum_{i \in [n]} w^-_i = \sum_{i \in [n]} w^+_i$ and want to maximize the approximated likelihood function under this constraint. (Other constraints, such as $\sum_{i \in [n]} w^-_i = \text{const.}$, are also possible and lead to the same result in the end.) By Lagrange’s multiplier method this leads to a maximization of

$$\prod_{i \in [n]} \left( w^-_i \right)^{d^-_i} \left( w^+_i \right)^{d^+_i} \exp \left( -w^+_i \frac{\sum_{j \in [n]} w^-_j}{n} \right) + \lambda \left( \sum_{k \in [n]} w^-_k - w^+_k \right).$$

Differentiating with respect to $w^-_l$ resp. $w^+_l$ for all $l \in [n]$, we are left with solving the equations

$$0 = \prod_{i \in [n]} \left( w^-_i \right)^{d^-_i} \left( w^+_i \right)^{d^+_i} \exp \left( -w^+_i \frac{\sum_{j \in [n]} w^-_j}{n} \right) \left( \frac{d^-_i}{w^-_i} - \frac{\sum_{k \in [n]} w^+_k}{n} \right) + \lambda$$

respectively

$$0 = \prod_{i \in [n]} \left( w^-_i \right)^{d^-_i} \left( w^+_i \right)^{d^+_i} \exp \left( -w^+_i \frac{\sum_{j \in [n]} w^-_j}{n} \right) \left( \frac{d^+_i}{w^+_i} - \frac{\sum_{k \in [n]} w^-_k}{n} \right) - \lambda,$$

which can be rewritten as $w^-_i = \lambda^- d^-_i$ and $w^+_i = \lambda^+ d^+_i$ for some constants $\lambda^-$ and $\lambda^+$ independent of $l$. Using the constraints $\sum_{i \in [n]} w^-_i = \sum_{i \in [n]} w^+_i$ and $\sum_{i \in [n]} d^-_i = \sum_{i \in [n]} d^+_i$, we obtain that $\lambda = 0$ and $\lambda^+ = \lambda^- = \sqrt{n/\sum_{i \in [n]} d^-_i}$ such that the approximated likelihood function is
maximized by
\[ w_i^- = d_i^- \sqrt{\frac{n}{\sum_{j \in [n]} d_j^-}}, \quad w_i^+ = d_i^+ \sqrt{\frac{n}{\sum_{j \in [n]} d_j^+}}. \]

That is, the approximated weight estimators are proportional to the observed degrees and only normalized in a certain sense. The normalization is necessary due to our choice of \( p_{i,j} \) in (2.1). If we had chosen \( p_{i,j} = 1 \wedge w_i^+ w_j^- / \sum_{k \in [n]} w_k^+ \) instead for example, then \( w_i^- \) and \( w_i^+ \) could be interpreted directly as expected degrees and estimated by \( d_i^- \) respectively \( d_i^+ \). All previous calculations would need to be adjusted by a factor \( n/\sum_{k \in [n]} w_k^+ \approx 1/E[W^+] \) but analogous results would still hold.

The smaller the observed fraction \( \max_{i,j \in [n]} d_i^+ d_j^- / \sum_{k \in [n]} d_k^- \), the better is above approximation of \( w_i^- w_j^- = d_i^- d_j^- n/\sum_{k \in [n]} d_k^- \ll n \). On networks where \( \max_{i,j \in [n]} d_i^+ d_j^- / \sum_{k \in [n]} d_k^- \) is large, \( w^- \) and \( w^+ \) have to be estimated numerically.

4.2 Simulations for the Threshold Model

For our simulations, we make use of the findings in [14] that the empirical in- and out-degrees as well as the exposure sizes in the Brazilian banking network are power law distributed. For November 2008, the authors of [14] estimated the power law exponents \( \beta^- = 2.132 \) and \( \beta^+ = 2.8861 \) for the degree sequences and \( \xi = 2.5277 \) for the exposures. In our weight-based model, these degree distributions are obtained by choosing in- and out-weights power law distributed with exponents \( \beta^- \) and \( \beta^+ \) as well. In addition to this, we assume them to be comonotone and Pareto distributed with minimal weights \( w_{\min}^- = w_{\min}^+ = 1 \).

In a first simulation, we consider a threshold model with above weight parameters and assume absence of contagious links but nothing more. That is, we set \( \tau_i = 2 \) for all \( i \in [n] \). In order to start the cascade process, we assume initial default of \( p = 1\% \) uniformly chosen banks in the network. We then simulate the default process for \( n \in \{100k \mid k \in [100]\} \) and 100 different configurations of the random network for each \( n \). The results for the final fraction of defaulted banks are plotted in Figure 2(a). As can be seen from Figure 2(b), the theoretical value of the final default fraction as \( n \) tends to infinity can be determined to be approximately 84.54\%. This value is drawn as a red line in Figure 2(a). Already for small \( n \), most of the simulations yield results that are close to this theoretical value and the networks can hence be understood as being non-resilient. As \( n \) grows to \( 10^4 \) the final fractions become even more precise. In particular, there is not a single resilient sample anymore for \( n \geq 500 \).

Instead of the absence of contagious links, Theorem 3.5 predicts certain threshold requirements to make our network model resilient to small initial shocks. Keeping above network parameters unchanged, we compute \( \alpha_{\text{crit}} \approx 2.13 \) and \( \gamma_{\text{crit}} \approx 0.468 \). A natural choice for the threshold of bank \( i \in [n] \) is then \( \tau_i = \min\{2, [\alpha(w_i^-)^\gamma]\} \), where \( \alpha = \alpha_{\text{crit}}(1 + \delta) \), \( \gamma = \gamma_{\text{crit}}(1 + \delta) \) and \( \delta \in [-1, \infty) \) denotes a (possibly negative) buffer. By Theorems 3.5 and 3.7 networks are resilient to initial shocks for \( \delta > 0 \) and non-resilient for \( \delta < 0 \). The influence of \( \delta \) on the function \( f(z; (W^-, W^+, T)) = E[W^+ \psi_T(W^- z)] - z \) can be seen in Figure 3(a). In particular, one notes that resilience for positive \( \delta \) stems from the negative hump of \( f(z; (W^-, W^+, T)) \) subsequent to zero. Further note, however, that resilience is only guaranteed to shocks whose size tends to zero. Even networks, where the number of banks tends to infinity but which are shocked by a strictly positive initial default fraction \( p \), will only be resilient for \( \delta > \delta_p \) for a certain \( \delta_p > 0 \). This is because \( f(z; (W^-, W^+, T)) \) depends on \( p \) by \( f(z; (W^-, W^+, T)) = (1 - p)E[W^+ \psi_T(W^- z)] + pE[W^+] - z \) if a uniformly chosen fraction \( p \) of all banks in the network defaults at the beginning. The influence of \( p \) on the function \( f(z; (W^-, W^+, T)) \) can be seen in Figure 3(b). In order for a network to be resilient to an
Figure 2: (a) Convergence of the final fraction of defaulted banks in the threshold model for networks of finite size. (b) Determination of the theoretical final default fraction in the threshold model for networks whose size grows to infinity and with $p = 1\%$ initial defaults and constant threshold 2. In blue: $f(z) = (1-p)E[W^+P(Poi(W-z) \geq 2)] + pE[W^+] - z$ with root $\hat{z} \approx 1.94433$. In red: $g(z) = (1-p)\mathbb{E}[Poi(W^-z)] + p$ with $g(\hat{z}) \approx 0.845434$.

Figure 3: (a) Influence of $\delta$ on the shape of $f(z) = \mathbb{E}[W^+P(Poi(W-z) \geq T)] - z$ with capital requirements $\tau_i = \min\{2, [(\alpha^{\text{crit}}(1 + \delta)(w_i^-)^{\gamma^{\text{crit}}(1+\delta)})\}$}. (b) Influence of $p$ on the shape of $f(z) = (1-p)\mathbb{E}[Poi(W^-z) \geq 2)] + pE[W^+] - z$ for the example of $\delta = 0.0839$.

Initial shock of $p$ the hump subsequent to 0 needs to become negative in Figure 3(b). By this, it is always possible to determine the least necessary buffer $\delta$ to make a system resilient to a shock of initial default fraction $p$ numerically. Whereas one needs a buffer of $\delta$ larger than 0.0839 to stop an initial default fraction of $p = 0.01$ from infecting a large part of the system, it only requires $\delta > 0.0235$ for $p = 0.001$ for example. All corresponding values of $\delta$ for $p = 0.001k$, $k \in [10]$, are listed in Table 1. Note that a buffer of $\delta = 0.0839$ yields $\alpha = 2.31$ and $\gamma = 0.507$ and hence the thresholds required to make the system resilient to shocks of 1% is still strongly sublinear.

We want to verify above results by simulations. For this, we simulate a very large network consisting of $n = 10^6$ banks and keeping the network topology constant we let $\delta$ vary between $-1$ and 1 in steps of $10^{-3}$. For each simulated network, we then find that it becomes resilient for $\delta$ large enough. This becomes visible by a jump of the final fraction of defaulted banks at this particular $\delta$ as illustrated in Figure 4(a) for a sample network. The jump shows that in the end it is only one bank whose default lets the whole system crash.
Table 1: List of values for buffer $\delta$ corresponding to initial default of $|pn|$ banks

| $p$ [%] | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\delta$ [%] | 2.35 | 3.44 | 4.30 | 5.04 | 5.71 | 6.36 | 6.89 | 7.42 | 7.91 | 8.39 |

Figure 4: (a) A typical result for the final fraction in a network of $10^6$ banks with initial default fraction of $p = 1\%$ as $\delta$ varies between $-1$ and $1$ in steps of $10^{-3}$. (b) The distribution of jump points for $10^4$ networks of size $n = 10^6$ with initial default fraction $p = 1\%$.

Keeping track of the values of $\delta$ at which the final fraction drops to near $p = 1\%$ for $10^4$ simulated networks yields the distribution shown in Figure 4(b). It shows a peak at about $\delta = 0.076$ and hence supports our theoretical findings from above. Deviations from the theoretical value of $\delta_{0.01} \approx 0.0839$ are small and can be explained by the finite (albeit very large) network size.

Having looked at the theoretical capital requirements for very large networks, it is now sensible to turn our attention to networks of a few thousand banks as they arise in the real world. Figure 5 shows the final fraction in networks of size $n \leq 10^4$ with initial default fraction $p = 0.01$ for $\delta$ between $-0.3$ and $0$. For each network size $n$, we have taken the average over $10^5$ simulations. The figure shows that finite networks of size $n \leq 10^4$ are already resilient for $\delta = 0$. Even for $\delta = -0.2$ the network is rather resilient if $n \leq 2,000$ respectively for $\delta = -0.1$ if $n \leq 6,000$. That is, our result is robust in the sense that already lower threshold requirements are sufficient to make the systems resilient to small shocks. The deviations stem from the relatively small network sizes of only a few thousand banks. Here, rare extreme values of vertex weights fail to appear despite the missing second moment condition or those large banks are not infected by the uniform initial infection.

For managing systemic risk in real networks it might, however, be of interest not only how some uniform initial default influences the system but also how the default of the largest banks does. In a further simulation, we hence choose the $|pn|$ largest (by weights) banks in the network to default at the beginning. The function $f(z; (W^-, W^+, T))$ then qualitatively keeps its shape as in Figure 3 but is shifted upwards. Again, we can compute corresponding values of $\delta$ and $p$ numerically. We list our results in Table 2. As one expects, the values of $\delta$ are larger in this case than the ones we obtained for uniform infection in Table 1 but only by a factor of about 2 and as before the resulting capital requirements are strongly sublinear.
Figure 5: Average final fraction of defaulted banks in finite networks

Table 2: List of values for buffer $\delta$ corresponding to initial default of the $|pn|$ largest banks

| $p$ [%] | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\delta$ [%] | 4.09 | 6.05 | 7.61 | 8.90 | 10.0 | 11.0 | 11.9 | 12.7 | 13.4 | 14.1 |

4.3 Simulations for the Exposure Model

We can now turn to the simulation of a weighted network as in the exposure model. In addition to the network parameters of the threshold model in the previous section, we assume that for $i \neq j \in [n]$, exposures $E_{j,i}$ are given by $E_{j,i} \sim E_i$ for some Pareto distributed random variables $E_i$ with exponent $\xi = 2.5277$ as in [14] and minimal value $E_{\min,i}$. The exposures are assumed independent of each other and independent of the network topology. The minimal exposures $E_{\min,i}$ can be chosen arbitrarily since they act as a constant factor for all exposures $E_{j,i}$ and capital $c_i$.

In a first simulation, again we assume absence of contagious links but nothing more. That is, we first simulate the network skeleton and the edge-weights independently and then determine the banks’ capitals as their largest exposure value plus some small buffer $\epsilon > 0$. For our simulation, we choose $\epsilon = 10^{-3}E[E_i] = 10^{-3}E_{\min,i}(\xi - 1)/(\xi - 2)$. As before, we assume initial default of $p = 1\%$ uniformly chosen banks in the network and simulate the default process for $n \in \{100k \mid k \in [100]\}$ and 100 different configurations of the random network for each $n$. The results for the final fraction of defaulted banks are plotted in Figure 6(a). We notice that already for small network sizes there are some non-resilient network samples with final default fraction of about 80%. As the number of banks $n$ grows, also the probability that the networks are non-resilient significantly increases. This can be seen from the red curve in Figure 6(a) which shows the average final default fraction taken over all 100 configurations. The simulation supports our analytical result that for networks without a second moment condition on their degree sequences, simply the absence of contagious links does not ensure resilience.

In a second simulation, we keep the network topology and the exposure sizes from the first simulation unchanged and choose capitals according to the formula in Proposition 3.11 with $\tau(w) = \max\{[\alpha w^\gamma], 2\}$ for $\alpha = \alpha^{\text{crit}}(1 + \delta)$, $\gamma = \gamma^{\text{crit}}(1 + \delta)$ and $\delta = 8.39\%$ as in Table 1. As can be seen from Figure 6(b), already for typical network sizes of less than $10^4$, these capital allocations make the system resilient (note the axis scale). The maximal final fraction we observed was given by 1.2%. As mentioned before, the capital requirements in Proposition 3.11 are too robust in general, however. In another simulation, we hence choose capitals as determined in Theorem 3.13 again for $\tau(w) = \max\{[\alpha w^7], 2\}$. Figure 6(b) shows that under these requirements the fundamental defaults still do not spread through the network. All observed
Figure 6: (a) Scatter plot of the final fraction of defaulted banks for weighted networks of finite size without contagious links. In red: the average value over all 100 configurations for each size. (b) Scatter plot of the final fraction of defaulted banks for weighted networks of finite size. In blue: Capitals are determined by Theorem 3.13. In red: Capitals are determined by Proposition 3.11.

Final fractions were less or equal 2.33%. However, keeping track of the total capitalization of the system further reveals that the capital requirements from Theorem 3.13 only amount to about 61% of the ones from Proposition 3.11 for our chosen network parameters.

References

[1] www.creditfixings.com/information/affiliations/fixings/auctions/2008/lehbro-res.shtml, 2017. Online; accessed April 2017.

[2] D. J. Aldous. École d’Été de Probabilités de Saint-Flour XIII — 1983, chapter Exchangeability and related topics, pages 1–198. Springer Berlin Heidelberg, Berlin, Heidelberg, 1985.

[3] H. Amini, R. Cont, and A. Minca. Resilience to contagion in financial networks. Mathematical Finance, 26(2):329–365, 2016.

[4] H. Amini and A. Minca. Inhomogeneous Financial Networks and Contagious Links. Operations Research, 5(64):1109–1120, 2016.

[5] Basel Committee on Banking Supervision. Global systemically important banks: updated assessment methodology and the higher loss absorbency requirement, 2013.

[6] Basel Committee on Banking Supervision. Standards: Supervisory Framework for Measuring and Controlling Large Exposures, 2014.

[7] S. Battiston, M. Puliga, R. Kaushik, P. Tasca, and G. Caldarelli. DebtRank: Too Central to Fail? Financial Networks, the FED and Systemic Risk. Scientific Reports, 2:541, 2012.

[8] L. E. Baum and M. Katz. Convergence rates in the law of large numbers. Transactions of the American Mathematical Society, 120(1):108–123, 1965.

[9] Board of Governors of the Federal Reserve System. Calibrating the GSIB Surcharge, 2015.
[10] M. Boss, H. Elsinger, M. Summer, and S. Thurner. Network topology of the interbank market. *Quantitative Finance*, 4(6):677–684, 2004.

[11] A. Capponi, P.-C. Chen, and D. D. Yao. Liability Concentration and Systemic Losses in Financial Networks. *Operations Research*, forthcoming, 2016.

[12] R. Carmona, J.-P. Fouque, and L.-H. Sun. Mean field games and systemic risk. *Communications in Mathematical Sciences*, 2014.

[13] C. Chong and C. Klüppelberg. Contagion in financial systems: A Bayesian network approach. *Preprint*, 2016.

[14] R. Cont, A. Moussa, and E. Santos. Network structure and systemic risk in banking systems. In J.-P. Fouque and J. Langsam, editors, *Handbook on Systemic Risk*. Cambridge University Press, Cambridge, 2013.

[15] B. Craig and G. von Peter. Interbank tiering and money center banks. *Journal of Financial Intermediation*, 23(3):322–347, 2014.

[16] N. Detering, T. Meyer-Brandis, and K. Panagiotou. Bootstrap percolation in directed and inhomogeneous random graphs. *ArXiv:1511.07993*, 2015.

[17] D. Duffie and K. J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. Princeton University Press, 2003.

[18] L. Eisenberg and T. H. Noe. Systemic Risk in Financial Systems. *Management Science*, 47(2):236–249, 2001.

[19] J. Fouque and T. Ichiba. Stability in a Model of Interbank Lending. *SIAM Journal on Financial Mathematics*, 4(1):784–803, 2013.

[20] J.-P. Fouque and J. A. Langsam, editors. *Handbook on systemic risk*. Cambridge University Press, Cambridge, 2013.

[21] P. Gai and S. Kapadia. Contagion in Financial Networks. *Proceedings of the Royal Society A*, 466:2401–2423, 2010.

[22] C. Graham. Chaoticity for Multiclass Systems and Exchangeability within Classes. *Journal of Applied Probability*, 45(4):1196–1203, 2008.

[23] T. R. Hurd. *Contagion! Systemic Risk in Financial Networks*. Springer, 2016.

[24] S. Janson. The Probability That a Random Multigraph Is Simple. *Comb. Prob. Comput.*, 18(1-2):205–225, 2009.

[25] S. Janson, T. Łuczak, T. Turova, and T. Vallier. Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.*, 22(5):1989–2047, 10 2012.

[26] O. Kallenberg. *Foundations of Modern Probability*. Springer, New York, 2nd edition, 2001.

[27] O. Kley, C. Klüppelberg, and L. Reichel. Systemic risk through contagion in a core-periphery structured banking network. 2014.

[28] S. Poledna, J. L. Molina-Borboa, S. Martínez-Jaramillo, M. van der Leij, and S. Thurner. The multi-layer network nature of systemic risk and its implications for the costs of financial crises. *Journal of Financial Stability*, 20:70–81, 2015.
[29] L. C. G. Rogers and L. A. M. Veraart. Failure and Rescue in an Interbank Network. *Management Science*, 59(4):882–898, 2013.

[30] R. van der Hofstad. Random Graphs and Complex Networks. Vol. II. [http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf](http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf) 2017. Online; accessed April 2017.

[31] S. Weber and K. Awiszus. The Joint Impact of Bankruptcy Costs, Fire Sales and Cross-Holdings on Systemic Risk in Financial Networks. *Working Paper, Leibniz Universität Hannover and Center for Risk and Insurance*, 2016.