CONNECTIVITY THROUGH BOUNDS FOR THE 
CASTELNUOVO-MUMFORD REGULARITY

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Abstract. We present a simple method to obtain information regarding the connectivity of the 1-skeleta of a wide family of simplicial complexes through bounds for the Castelnuovo-Mumford regularity of their Stanley-Reisner rings. In this way we generalize and unify two results on connectivity: one by Balinsky and Barnette, one by Athanasiadis. In particular, if $\Delta$ is a simplicial $d$-pseudomanifold, and $s$ is the highest integer such that there is an $s$-dimensional simplex not contained in $\Delta$, but such that its boundary is, then the 1-skeleton of $\Delta$ is $\lceil \frac{(s+1)d}{s} \rceil$-connected.

We also show that this bound on the connectivity is tight.

1. Introduction

We say that a graph $G$ having more than $m$ vertices is $m$-connected whenever it is impossible to disconnect it by removing less than $m$ vertices together with their incident edges. Interesting results on the connectivity of $G$ can be found if $G$ is the 1-skeleton of a pure polyhedral complex. The first step in this direction has been taken in 1922 by Steinz [12] who characterized the 1-skeleta of 3-dimensional polytopes as the planar 3-connected graphs. Later Balinsky [3] proved that the 1-skeleton of a $(d+1)$-dimensional convex polytope is $(d+1)$-connected. This result has been extended to polyhedral $d$-pseudomanifolds by Barnette [4].

For what concerns the simplicial case, Athanasiadis [2] proved that the 1-skeleton of a flag (i.e. coinciding with the clique complex of its 1-skeleton) simplicial $d$-pseudomanifold is $2d$-connected. This case has recently attracted interest among commutative algebraists. Björner and Vorwerk [6] extended Athanasiadis’ result interpolating it with Barnette’s one, thanks to a generalization of flag complexes. Recently, Adiprasito, Goodarzi and Varbaro [1] provided an algebraic method which allow them to obtain a more general result.

In this note we also present an algebraic approach, which allows us to unify the results by Athanasiadis and Barnette in a different direction from the one chosen by [6] and [1]. This can be done under even weaker hypotheses.

The strategy involves finding upper bounds for the Castelnuovo-Mumford regularity of the Stanley-Reisner ring of a simplicial complex $\Delta$ which are expressed in terms of the number of vertices of $\Delta$. Such bounds are common in literature, as the Castelnuovo-Mumford regularity itself acts as an upper bound for a wide variety of
algebraic invariants (maximum degree for which the local cohomology modules vanish, the integer from which the Hilbert function behaves polynomially, the maximum degree of the syzygies).

Furthermore, we require these bounds to well-behave up to restricting the simplicial complex (we call such bounds suitable). Then we focus on a set $T$ of vertices of $\Delta$ which have to be removed from the 1-skeleton $G$ of $\Delta$ in order to disconnected it. By setting some hypotheses on $\Delta$, we can bound from below the regularity of the Stanley-Reisner ring of the restriction $\Delta|_T$ (those hypotheses are weaker than requiring $\Delta$ to be a pseudomanifold). By reversing and applying a suitable bound for the regularity, we find a lower bound for the cardinality of the subset, therefore we can estimate the connectivity of $G$.

2. Preliminaries

Let $\Delta$ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. We call vertices and edges the faces of dimension 0 and 1 respectively. We denote by $G(\Delta)$ the 1-skeleton of a simplicial complex $\Delta$, i.e. the set of all the faces of $\Delta$ of dimension lower than or equal to 1. For our purposes we will deal with undirected simple graphs, therefore we define a graph to be the 1-skeleton $G(\Delta)$ of some simplicial complex $\Delta$. Given a subset $T \subseteq [n]$ we denote by $\Delta|_T$ the restriction of $\Delta$ to $T$, i.e. all the faces $\sigma$ of $\Delta$ such that $\sigma \subseteq T$. A subcomplex of $\Delta$ is called induced if it is a restriction of $\Delta$ to some set $T \subseteq [n]$.

Let $\mathbb{k}$ be an arbitrary field and $S = \mathbb{k}[x_1, \ldots, x_n]$ the polynomial ring on $n$ variables. The Stanley-Reisner ring of the complex $\Delta$ (with respect to the field $\mathbb{k}$) is the graded ring $\mathbb{k}[\Delta] = S/I_\Delta$ where the Stanley-Reisner ideal $I_\Delta$ is the ideal generated by all the squarefree monomials $x_{i_1} \cdots x_{i_s} \in S$ such that $\{i_1, \ldots, i_s\} \notin \Delta$.

Let $M$ be a finitely generated graded $S$-module and

$$0 \rightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{i,j}} \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_2} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{0,j}} \xrightarrow{\phi_1} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{0,j}} \rightarrow 0$$

a minimal graded free resolution of $M$, where the shifting numbers are chosen in order to let the maps $\phi_i$ be degree preserving. We call graded Betti numbers the exponents $\beta_{i,j} = \beta_{i,j}(M)$. Furthermore we define the Castelnuovo-Mumford regularity of $M$ as $\text{reg}(M) = \max \{j - i | \beta_{i,j}(M) \neq 0\}$.

We denote by $\tilde{H}_i(\Delta; \mathbb{k})$ the $i^{th}$ reduced (simplicial) homology of $\Delta$ over the field $\mathbb{k}$. Hochster’s formula [9] relates the Betti numbers of the Stanley-Reisner ring $\mathbb{k}[\Delta]$ to the reduced homology of restrictions of $\Delta$ as follows

$$\beta_{i,j}(\mathbb{k}[\Delta]) = \sum_{T \subseteq [n]} \dim_{\mathbb{k}} \tilde{H}_{j-i-1}(\Delta|_T; \mathbb{k}).$$

It immediately follows that if some restriction $\Delta|_T$ of $\Delta$ has nonzero homology in homological degree $h$ then

$$\text{reg}(\mathbb{k}[\Delta]) \geq h + 1;$$
note that the equality holds whenever $h$ is the maximum integer such that some restriction of $\Delta$ has nonzero homology in homological degree $h$.

3. Suitable bounds

We now introduce a new family of simplicial complexes to which we will extend the results on connectivity.

Definition 1. We say that the simplicial complex $\Delta$ is a vertex minimal $h$-cycle if, for some field $\mathbb{k}$, $H_h(\Delta|_T; \mathbb{k}) \neq 0$ if and only if $T = [n]$.

A simplicial $d$-pseudo-manifold is a simplicial complex which is strongly connected and whose $(d-1)$-dimensional faces are contained in exactly two facets. As a consequence of being strongly connected, a pseudo-manifold is also pure. Note that a $d$-dimensional simplicial pseudo-manifold is a vertex minimal $d$-cycle, while a vertex minimal cycle neither has to be strongly connected nor its $(d-1)$-dimensional faces must be contained in exactly two facets (they can actually be contained in any number of facets).

Example 2. Let $d \geq 2$, and let $V$ be the set whose elements are the $2d+2$ vertices of a $(d+1)$-prism, i.e. a prism whose basis is a $(d+1)$-gon. We define the $d$-dimensional simplicial complex $\Delta$ on the vertex set $V$ as the simplicial complex whose faces are all the subsets of $V$ whose elements lay on a common face of (the boundary of) the prism.

The following theorem relates the connectivity of a vertex minimal cycle $\Delta$ to the regularity of the Stanley-Reisner ring of specific restrictions of $\Delta$. It allows us to prove the main results of this note in Section 4.

Theorem 3. Let $\Delta$ be a vertex minimal $h$-cycle and let $T$ be a set of vertices of $\Delta$ such that $\Delta|_T$ is disconnected. Then $\text{reg}(k[\Delta|_T]) \geq h$.

Proof. Let $U_1$ be a subset of vertices of $T$ such that $\Delta|_{U_1}$ is one of the connected components of $\Delta|_T$ and let $U_2 = T \setminus U_1$. Let $\Gamma_1 = \Delta|_{[n] \setminus U_2}$ and $\Gamma_2 = \Delta|_{[n] \setminus U_1}$. Note that $\Gamma_1 \cup \Gamma_2 = \Delta$ and $\Gamma_1 \cap \Gamma_2 = \Delta|_{[n] \setminus T}$. By applying the Mayer-Vietoris sequence for the reduced homology of simplicial complexes to $\Gamma_1$ and $\Gamma_2$ we obtain the following exact sequence

$$0 \to \tilde{H}_h(\Delta; \mathbb{k}) \to \tilde{H}_{h-1}(\Delta|_{[n] \setminus T}; \mathbb{k}) \to \cdots,$$

where the first zero comes from the hypotheses, being $[n] \setminus U_1$ and $[n] \setminus U_2$ proper subsets of $[n]$ (and therefore $\tilde{H}_h(\Gamma_1; \mathbb{k}) = \tilde{H}_h(\Gamma_2; \mathbb{k}) = 0$). Then, since $\tilde{H}_h(\Delta; \mathbb{k}) \neq 0$, $\tilde{H}_{h-1}(\Delta|_{[n] \setminus T}; \mathbb{k}) \neq 0$.

We conclude by inequality (1).
Note that an upper bound for the regularity of the Stanley-Reisner ring $\mathbb{k}[\Delta]$ of a simplicial complex $\Delta$ given in terms of the number of vertices of $\Delta$ can be reversed and applied to Theorem 3. In this way it is possible to give a lower bound for the number of vertices one needs to remove from $\Delta$ in order to disconnect it, provided that the bound can be applied to restrictions of $\Delta$.

More specifically, we will say that an upper bound for $\text{reg}(\mathbb{k}[\Delta])$ in terms of $n$ is suitable for the purposes of this note, if the same bound holds true for $\text{reg}(\mathbb{k}[\Delta|_T])$ for each proper and nonempty subset $T \subset [n]$ by substituting $n$ by $|T|$ in the bound itself.

A family of suitable bounds can be achieved thanks to the Taylor resolution ([13], see also [5]).

Let $m_1, \ldots, m_r$ the minimal monomial generators of a monomial ideal $I$ in $S$. For each subset $X \subseteq [r]$, let $m_X = \text{lcm}(m_i | i \in X)$, $a_X$ its multidegree in $\mathbb{N}^n$, and $S(-a_X)$ the free $S$-module whose generator $e_X$ has degree $a_X$. The Taylor resolution of $S/I$ is the $S$-module $F = \bigoplus_{X \subseteq [r]} S(-a_X)$ graded on $\mathbb{Z}^n$ and equipped with the differential maps

$$\partial(e_X) = \sum_{i \in I} \text{sign}(i, X) \frac{m_X}{m_{X \setminus i}} e_{X \setminus i},$$

where sign$(i, X)$ is $(-1)^{j+1}$ if $i$ is the $j$th element in the ordering of $X$. The Taylor resolution is a free resolution of $S/I$. The degree $j$ part of $F$ in homological degree $i$ must have rank at least $\beta_{i,j}(S/I)$. If moreover $I$ is squarefree, let $s$ be the maximum degree of one of its minimal generators $m_1, \ldots, m_r$. In this case, the multigrade vector of each generator of the degree $j$ part of $F$ in homological degree $i$ is an element of $\{0,1\}^n$. Since the number of nonzero entries of this vector cannot exceed $si$, we conclude that $\beta_{i,j}(S/I) \neq 0$ only when $j \leq si$.

We can now obtain a bound for the Castelnuovo-Mumford regularity of $S/I$ in terms of $s$ and the number of indeterminates $n$. Indeed there must be a nonzero Betti number $\beta_{i,j}(S/I)$ such that $j - i = \text{reg}(S/I)$. As observed before $j \leq si = s(j - \text{reg}(S/I))$, therefore, since $I$ squarefree also implies that $j \leq n$, we obtain

$$(2) \quad \text{reg}(S/I) \leq \frac{n(s - 1)}{s}.$$ 

This bound is suitable as the maximum degree $s$ does not increase up to restrictions of $\Delta$.

Improved bounds can be obtained by strengthening the hypotheses. We report a result proved by Dao, Huneke, Schweig [7, Theorem 4.9], where the hypotheses has been rewritten thanks to the characterization for the $k$-step linearity given by Eisenbud, Green, Hulek and Popescu [8, Theorem 2.1].

**Theorem 4** ([7]). Let $\Delta$ be a flag simplicial complex and $k$ a positive integer such that $\Delta$ contains no induced $m$-cycles for $4 \leq m \leq k + 3$. Then

$$\text{reg}(\mathbb{k}[\Delta]) \leq \min \left\{ \log_{k+4} \left( \frac{n - 1}{k + 1} \right) + 2, \log_{k+4} \left( \frac{(n - 1) \ln \left( \frac{k+4}{k} \right)}{k + 1} + \frac{2}{k + 4} \right) + 2 \right\}.$$
The first term of the right hand side is tighter than the second one whenever \( k \geq 2 \).

Since no new induced subcycles are formed through restriction, also this bound is suitable.

4. Generalization of results on connectivity

Recall that a graph \( G \) is said to be \( m \)-connected, if it has more than \( m \) vertices and any subgraph obtained from \( G \) by deleting less than \( m \) vertices and their incident edges is connected (necessarily with at least one edge).

The following corollary generalizes and interpolates the result of Balinsky \([3]\) and Barnette \([4]\) with the one by Athanasiadis \([2]\).

**Corollary 5.** Let \( G \) be the 1-skeleton of a vertex minimal \( h \)-cycle \( \Delta \), and let \( s \) be the maximum degree of the minimal generators of the Stanley-Reisner ideal \( I_\Delta \). Then \( G \) is \( \left\lceil \frac{sh}{s-1} \right\rceil \)-connected.

**Proof.** Apply the suitable bound (2) to Theorem 3. \( \square \)

Note that Balinsky-Barnette’s result is obtained by looking at \( \left\lceil \frac{sh}{s-1} \right\rceil \) for \( s \gg 0 \), while Athanasiadis’ one is obtained by setting \( s = 2 \). Furthermore the class of vertex minimal cycles is ampler than the one of minimal cycles.

We now present a family of simplicial complexes for which the previous bound on the connectivity is tight. We thanks Eran Nevo for suggesting to build the following example in the same manner as the family of homology spheres he introduce in \([11]\) for different purposes. In our case, for each \( s \geq 2 \) and each \( h \geq s - 1 \) we build a vertex minimal \( h \)-cycle \( \Delta \) whose Stanley-Reisner ideal \( I_\Delta \) is generated by monomials of degree not exceeding \( s \) such that it is possible to disconnect the 1-skeleton of \( \Delta \) by removing exactly \( \left\lceil \frac{sh}{s-1} \right\rceil \) vertices. Note that the condition \( h \geq s - 1 \) is not restrictive as \( h = s - 2 \) holds true only if the vertex minimal \( h \)-cycle \( \Delta \) is the boundary of an \((s - 1)\)-simplex.

**Example 6.** Let \( s \geq 2 \) and \( h \geq s - 1 \) be two integers. Let \( sh = (s - 1)q' + r' \) the euclidean division of \( sh \) by \( s - 1 \), for a proper \( q' \geq 0 \) and a remainder \( 0 \leq r' \leq s - 2 \). Let moreover \( \left\lceil \frac{sh}{s-1} \right\rceil = sq + r \) the euclidean division of \( \left\lfloor \frac{sh}{s-1} \right\rfloor \) by \( s \), for a proper \( q \geq 0 \) and a remainder \( 0 \leq r \leq s - 1 \). Note that \( r' = 0 \) if and only if \( r = 0 \), as the second euclidean division has no reminder if and only if \( \left\lceil \frac{sh}{s-1} \right\rceil = \frac{sh}{s-1} \).

We first note that \( r \neq 1 \). Suppose otherwise, then \( r' \neq 0 \) and \( \left\lfloor \frac{sh}{s-1} \right\rfloor = q' + 1 \). The second euclidean division can be rewritten as

\[
q' + 1 = sq + 1
\]

and therefore, rewriting \( q' \) as \( \frac{sh - r'}{s-1} \), we get

\[
sh = (s - 1)sq + r'.
\]

So

\[
s(h - sq + q) = r',
\]

which is impossible, as \( s \geq 2 \) and \( 0 \leq r' \leq s - 2 \).
So the remainder $r$ must either equal 0 or satisfy $2 \leq r \leq s - 1$. In both cases we build $\Delta$ explicitly.

Remind that the simplicial join $\Delta_1 \ast \Delta_2$ of two simplicial complexes $\Delta_1$ and $\Delta_2$ on two disjoint vertex sets is the simplicial complex whose faces can be written as the union of a face of $\Delta_1$ with a face of $\Delta_2$. Moreover, remind that the simplicial join of two spheres of dimension $d_1$ and $d_2$ is a $(d_1 + d_2 + 1)$-sphere.

If $r = r' = 0$ we define $\Delta = \partial \sigma_1 \ast \partial \sigma^{s-1} \ast \cdots \ast \partial \sigma^{r-1}$, where $\partial \sigma_i$, the boundary of the $i$-simplex, appear $q$ times in the join. In this way, $\Delta$ is a sphere of dimension $q(s - 1)$. We now prove that $q(s - 1) = h$; by definition of $q$ we get

$$q(s - 1) = \frac{sh}{s(s - 1)}(s - 1) = h.$$ 

Conversely, let $2 \leq r \leq s - 1$. As observed before, $r' \neq 0$. In this case we define $\Delta$ as the join $\partial \sigma_1 \ast \partial \sigma^{s-1} \ast \cdots \ast \partial \sigma^{s-1} \ast \partial \sigma^{r-1}$, where $\partial \sigma^{s-1}$ appears $q$ times. In this way, $\Delta$ results a $(q(s - 1) + r - 1)$-sphere. We now prove that $q(s - 1) + r - 1 = h$. Indeed,

$$q(s - 1) + r - 1 = \frac{q' + 1 - r}{s}(s - 1) + r - 1 = h - \frac{r - r' - 1}{s}.$$

The quantity $\frac{r - r' - 1}{s}$ has to be an integer, and since $0 \leq r \leq s - 1$ and $0 \leq r' \leq s - 2$, the only integers value it can equal is 0.

In both the cases $\Delta$ is an $h$-sphere on $\lceil \frac{sh}{s-1} \rceil + 2$ vertices, indeed the second euclidean division counts exactly the number of vertices of the join except for the two vertices of $\partial \sigma^1$. Moreover, in both the cases, the largest $i$ such that $\partial \sigma_i$ is in $\Delta$ is $s - 1$, and therefore the Stanley-Reisner ideal $I_\Delta$ is generated by monomials of degree $s$ or lower. So $\Delta$ satisfies the hypothesis of Corollary 5.

If we remove from the 1-skeleton of $\Delta$ all the vertices but the two belonging to $\partial \sigma_1$, we disconnect it. Note that we are removing $n - 2 = \lceil \frac{sh}{s-1} \rceil$ vertices, therefore the 1-skeleton of $\Delta$ can not be more than $\lceil \frac{sh}{s-1} \rceil$-connected.

If we have sufficient hypotheses to apply the bound for the regularity given by Dao, Huneke, Schweig (see Theorem 4) we can obtain the following result in which the connectivity of the simplicial complex grows exponentially on $h$.

**Corollary 7.** Let $G$ be the 1-skeleton of a vertex minimal $h$-cycle $\Delta$ which is flag and without induced $k$-cycles for $k \geq 4$. Then $G$ is $M$-connected, where

$$M = \max \left\{ \left\lceil \left( k + 1 \right) \left( \frac{k + 4}{2} \right)^{\frac{h-2}{2}} + 1 \right\rceil, \left\lceil \frac{k + 1}{\ln \left( \frac{k + 4}{2} \right)} \left( \left( \frac{k + 4}{2} \right)^{\frac{h-2}{2}} - \frac{2}{k + 4} \right) + 1 \right\rceil \right\}.$$ 

**Proof.** Apply Theorem 4 to Theorem 3. \qed

For the sake of readability, we emphasize that from the corollary it follows that $G$ results at least $\left\lceil \left( \frac{k}{2} \right)^{\frac{h-1}{2}} \right\rceil$-connected.

A family of simplicial pseudomanifolds of arbitrary dimension which satisfy the hypotheses of the previous corollary has been built by Januszkwiecz and Świątkowski in [10].
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