Some Gabidulin Codes cannot be List Decoded Efficiently at any Radius

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Abstract—Gabidulin codes can be seen as the rank-metric equivalent of Reed-Solomon codes. It was recently proven, using subspace polynomials, that Gabidulin codes cannot be list decoded beyond the so-called Johnson radius. In another result, cyclic subspace codes were constructed by inspecting the connection between subspaces and their subspace polynomials. In this paper, these subspace codes are used to prove two bounds on the minimum possible size of a list in decoding certain Gabidulin codes. The first bound is an existential one, showing that exponentially-sized lists exist for codes with specific parameters. The second bound presents exponentially-sized lists explicitly, for a different set of parameters. Both bounds rule out the possibility of efficiently list decoding their respective families of codes for any radius beyond half the minimum distance. Such a result was known so far only for non-linear rank-metric codes, and not for Gabidulin codes.

Index Terms—Rank-metric codes, Gabidulin codes, list decoding, subspace polynomials.

I. INTRODUCTION

For a prime power \( q \), let \( \mathbb{F}_q \) be the field with \( q \) elements. For an integer \( n \), let \( \mathbb{F}_q^n \) be the extension field of degree \( n \) of \( \mathbb{F}_q \) (which may be seen as the vector space of dimension \( n \) over \( \mathbb{F}_q \), denoted \( \mathbb{F}_q^n \)), and \( \mathbb{F}_q^{n^*} = \mathbb{F}_q^n \setminus \{0\} \). The set \( G_q(n,k) \), called the Grassmannian, is the set of all subspaces of dimension \( k \) (\( k \)-subspaces, in short) of \( \mathbb{F}_q^n \). The size of \( G_q(n,k) \) is given by the Gaussian coefficient \( \binom{n}{k} \), which satisfies \( q^{k(n-k)} \leq \binom{n}{k} \leq 4q^{k(n-k)} \). A constant dimension subspace code \( \mathcal{C} \) is a subset of \( G_q(n,k) \) under the subspace metric \( d_S(U,V) = \dim U + \dim V - 2 \dim(U \cap V) \).

Rank-metric codes have recently attracted increasing interest due to their application in error correction in network coding [15]. A rank-metric code is a set of \( n \times n \) matrices over \( \mathbb{F}_q \), or alternatively, vectors of length \( n \) over the extension field \( \mathbb{F}_q^n \), where the distance between two matrices is the rank of their difference. Gabidulin codes, introduced by [13, Chapter 4, p. 108], may be seen as the rank-metric equivalent of Reed-Solomon codes. These codes are defined as evaluations of linearized polynomials (see below) of bounded degree at a given set of linearly independent evaluation points. We note that Gabidulin codes, and rank-metric codes in general, may be defined similarly as vectors of length \( n \) over \( \mathbb{F}_q^m \) for any \( m \geq n \). However, since our results only apply for the case \( m = n \), we use this more restricted definition in this paper.

Given a word \( w \in \mathbb{F}_q^n \) (or alternatively, \( w \in \mathbb{F}_q^{n \times n} \)), a list decoding algorithm outputs all Gabidulin codewords that are inside a ball of radius \( \tau \), centred at \( w \), where \( \tau \) is possibly larger than the unique decoding radius of the code. For a given code, a natural question to ask is: for which values of \( \tau \) can list decoding be done efficiently? List-decoding of rank-metric codes and Gabidulin codes was recently studied in [6], [17]. In [17], it was shown that Gabidulin codes cannot be list-decoded beyond the Johnson bound. This result was generalized to any rank-metric code by [6], which also showed that with high probability, a random rank-metric code can be efficiently list decoded. In [11], an explicit subcode of a Gabidulin code was shown to be efficiently list-decodable. In addition, [6], [11], and [17] have noted that it is not known if Gabidulin codes themselves can be efficiently list decoded beyond the unique decoding radius. In this paper, it is shown that the answer to this question is negative.

Clearly, if there exists a word \( w \in \mathbb{F}_q^n \) with exponentially many Gabidulin codewords in a radius \( \tau \) around it, then efficient list decoding is not possible for this radius. This combinatorial technique was used in [3] to show the limits of list decoding of Reed-Solomon codes, and in [17] to show the limits of list decoding of Gabidulin codes.

The main tool in [3], [17] is subspace polynomials, which are a special type of linearized polynomials. Linearized polynomials, defined by Ore [14], are polynomials of the form

\[
P(x) = a_k \cdot x^k + \cdots + a_1 \cdot x^1 + a_0 \cdot x,
\]

where \( [i] \triangleq q^i \) and the coefficients are in the finite field \( \mathbb{F}_q \) for some given \( n \). For a linearized polynomial \( P \), define the \( q \)-degree of \( P \) as \( \deg_q P \triangleq \log_q \deg P \). Using the isomorphism between \( \mathbb{F}_q^n \) and \( \mathbb{F}_q^q \), every linearized polynomial may be seen as an \( \mathbb{F}_q \)-linear function from \( \mathbb{F}_q^n \) to itself [13] Chapter 4, p. 108], that is, for every \( \alpha, \beta \in \mathbb{F}_q \) and \( u, v \in \mathbb{F}_q^n \), each linearized polynomial \( P \) satisfies \( \alpha P(u) + \beta P(v) = \alpha P(v) + \beta P(u) \). A subspace polynomial is defined as follows.

Definition 1. [11], [2], [3], [4], [17]

A monic linearized polynomial \( P \) is called a subspace polynomial with respect to \( \mathbb{F}_q^n \) if it satisfies the following equivalent conditions:

A1. \( P \) divides \( x^n - x \).

A2. \( P \) splits completely over \( \mathbb{F}_q^n \) and all its roots have multiplicity one.

A3. There exists a subspace \( V \in G_q(n,k) \) such that \( P(x) = \prod_{v \in V} (x - v) \).

By A3, each subspace \( V \) corresponds to a unique subspace polynomial, denoted \( P_V \).

Subspace polynomials are an efficient method of representing subspaces, from which one can directly deduce certain properties of the subspace which are not evident in some other representations. These objects were studied in the past for various other purposes, e.g., construction of affine dispersers [2], finding an element of high multiplicative order in a finite field [4], and construction of cyclic subspace codes [1]. Albeit this wide range of applications, not much is known about the coefficients of...
subspace polynomials and their connection to the properties of the subspace.

It is known that all roots of every linearized polynomial have the same multiplicity, which is an integer power of \( q \), and these roots form a subspace in the extension field \([13\) Theorem 3.50, p. 108]. Therefore, any monic linearized polynomial is a power of a subspace polynomial in its splitting field. However, the structure of the coefficients of subspace polynomials, comparing to other linearized polynomials of the same degree, is generally not known. A partial answer to this question was given by \([1\), and we use a similar technique to give an explicit large set of subspace polynomials in \( \mathbb{F}_q^n \), for infinitely many values of \( n \).

Some of the results of \([1\) will be used to show lists of limit decoding of Gabidulin codes.

Ben-Sasson et al. \([5\) proved that a given set of subspace polynomials with mutual top coefficients provides an upper bound on list decodability of Reed-Solomon codes. A counting argument was later applied in order to show that such large sets of subspace polynomials do exist. A similar technique was used in \([7\) to show the limits of list decoding of Gabidulin codes. In the sequel, the existence of a set of subspaces whose polynomials have a larger agreement is proved (Theorem 3). This set is a subset of a subspace code by \([1\). Furthermore, explicit dense sets of words in a Gabidulin code are provided (Theorem 4). Both bounds are used to show that the respective families of Gabidulin codes cannot be list decoded efficiently at all. That is, there exist received words that have exponentially many codewords around them, already for a radius which is only larger than the unique decoding radius by one (Example \([1\) and Theorem 4). Due to a technical limitation of our techniques, both presented families have rate at least \( \frac{1}{2} \). Our techniques may also be used for showing limits of list decoding or Reed-Solomon codes, but the resulting bounds are too weak to provide any useful insight.

The rest of the paper is organized as follows. The subspace code from \([1\) will be described in Section II together with the required background on cyclic shifts of subspaces and \( q \)-associates of polynomials. In Section III the code from Section II is used to prove the existence of a certain set of subspace polynomials, and the notion of \( q \)-associates is used to show an explicit set of another type of subspace polynomials. The improved bounds on list decodability of Gabidulin codes are discussed in Section IV and conclusions are given in Section V.

Detailed discussions about the applicability of our techniques to list decodability of subspace codes and Reed-Solomon codes will appear in the full version of this paper.

II. Preliminaries

An extensively used concept in this paper is cyclic shifts of subspaces, defined as follows.

**Definition 2.** For \( V \in \mathcal{G}_q(n,k) \) and \( \alpha \in \mathbb{F}_q^* \), let \( \alpha V \triangleq \{ \alpha v | v \in V \} \).

The set \( \alpha V \), which is clearly a subspace of the same dimension as \( V \), is called a cyclic shift of \( V \). Cyclic shifts were shown to be useful for construction of subspace codes \([1\), \([7\) . The set of all cyclic shifts of \( V \in \mathcal{G}_q(n,k) \) is called the orbit of \( V \), and its size is \( \frac{q^n-1}{q-1} \) for some integer \( t \) which divides \( n \). The size of the orbit and the structure of its subspace polynomials can be derived by inspecting the subspace polynomial of \( V \), as shown in the following lemmas.

**Lemma 1.** \([1\) If \( V \in \mathcal{G}_q(n,k) \) and \( \alpha \in \mathbb{F}_q^* \) then \( \alpha V \triangleq \{ \alpha v | v \in V \} \).

That is, if \( P_V(x) = x^{[k]} + \sum_{j=0}^{t} \alpha_j x^{[j]} \) then \( P_{\alpha V}(x) = x^{[k]} + \sum_{j=0}^{t} \alpha^{[j]}(\alpha x^{[j]}) \).

**Lemma 2.** \([1\) Let \( V \in \mathcal{G}_q(n,k) \) and \( P_V(x) = x^{[k]} + \sum_{j=0}^{t} \alpha_j x^{[j]} \). If \( \alpha_s \neq 0 \) for some \( s \in \{1, \ldots , t\} \) and \( \gcd(s,n) = t \), then \( V \) has at least \( \frac{q^n-1}{q^t-1} \) distinct cyclic shifts.

In \([1\) it shown that subspaces in \( \mathcal{G}_q(n,k) \) that may be considered as subspaces over a subfield of \( \mathbb{F}_q \), which is generally larger than \( \mathbb{F}_q \), admit a unique subspace polynomial structure. In what follows we cite the essentials from \([1\).

**Construction 1.** \([1\) For integers \( g, n, k \) such that \( 0 < k < n \) and \( g | \gcd(n,k) \), let

\[ C_g \triangleq \{ G(V) | V \in \mathcal{G}_q(n,g,k/g) \} , \]

where \( G \) is the \( \mathbb{F}_q \)-homomorphism from of Lemma \([\] .

The code \( C_g \) has minimum subspace distance \( 2g \), and it may alternatively be defined as a direct sum of cyclic shifts of \( \mathbb{F}_q \) or as the set of all subspace of \( \mathcal{G}_q(n,k) \) that are subspaces over \( \mathbb{F}_q \), as well \([1\). Since \( C_g \) is the image of an injective function from \( \mathcal{G}_q(n,g,k/g) \) to \( \mathcal{G}_q(n,k) \), we have the following.

**Corollary 1.** \([1\) \( |C_g| = \frac{n/g}{k/g+1} \).

The subspaces in \( C_g \) admit a unique subspace polynomial structure, from which the results in this paper follow.

**Lemma 4.** \([1\) If \( V \in \mathcal{G}_q(n,k) \) then \( V \in C_g \) if and only if 

\[ P_V(x) = \sum_{i=0}^{k/g} c_i x^{[gi]} \text{ for some } c_i's \in \mathbb{F}_q. \]

Another concept used in our constructions is the notion of \( q \)-associates. Two polynomials over \( \mathbb{F}_q \) of the form \( \ell(x) = \sum_{i=0}^{k} \alpha_i x^{i} \) and \( L(x) = \sum_{i=0}^{k} \alpha_i x^{qi} \), are called \( q \)-associates of each other. For any \( g \in \mathbb{N} \), one can similarly define \( q^g \)-associativity, which \( \ell(x) = \sum_{i=0}^{k} \alpha_i x^{i} \), and \( L(x) = \sum_{i=0}^{k} \alpha_i x^{qi} \) are \( q^g \)-associates of each other. Linearized polynomials over \( \mathbb{F}_q \) are deeply connected to their \( q \)-associates as follows.

**Lemma 5.** \([\] Theorem 3.62, p. 116] If \( L_1(x) \) and \( L(x) \) are linearized polynomials over \( \mathbb{F}_q \) with \( q \)-associates \( \ell_1(x) \) and \( \ell(x) \), then \( L_1(x) \) divides \( L(x) \) if and only if \( \ell_1(x) \) divides \( \ell(x) \).

III. Sets of Subspaces Polynomials with Mutual Top Coefficients

In \([3\) (resp. \([7\) ) it was shown that sets of subspace polynomials that agree on many of their top coefficients provide a bound on list decodability of Reed-Solomon (resp. Gabidulin) codes. By Lemma 4 it is evident that all subspace polynomials of subspaces in \( C_g \) agree on their topmost \( q \) coefficients \( (1,0, \ldots , 0) \). Using a counting argument we may prove the existence of a subset of \( C_g \) whose corresponding subspace polynomials agree on a larger number of top coefficients.
Theorem 1. If $g, n,$ and $k$ are integers such that $0 < k < n$, $g \mid \gcd(k, n)$, and $\ell$ is the unique non-negative integer such that $k = n - g(\ell + 1)$, then there exists a subset of $C_g$ of size at least
\[\frac{\binom{n}{g}}{k/g \cdot q^n}q^{\ell d},\]
whose subspace polynomials agree on their topmost $g(\ell + 1)$ coefficients.

Proof: Consider the set of all subspace polynomials of subspaces in $C_g$ (Construction 1). By Lemma 4 and since all of these polynomials are monic, they may be partitioned into $q^{\ell d}$ subsets according their $\ell + 1$ top coefficients which correspond to monomials whose $g$-degree is divisible by $g$. By the pigeonhole principle, there exists a subset of size at least $\frac{\binom{n}{g}}{k/g \cdot q^n}q^{\ell d}$ whose polynomials agree on their top $g(\ell + 1)$ coefficients.

Notice that for $g = 1$, Theorem 1 reduces to the ordinary counting argument employed by [3] and [17]. In addition, the case where $n - k = g(\ell + 1) > k$, in which the polynomials in the set agree on all coefficients, is also trivial, since it merely implies the existence of a set of size one. Therefore, all our bounds apply only for codes with $k \geq \frac{n}{2}$, implying that the rate of the code $\frac{k}{n}$ is at least $\frac{1}{2}$.

The notion of $q^g$-associativity, together with Lemma 1 allows us to construct an explicit large set of subspace polynomials. It will also be noted that in certain cases, this set of polynomials corresponds to the entire set $C_g$. The construction is based on the following lemma.

Lemma 6. If $g, s,$ and $k$ are integers such that $gs|k$ and $n \equiv k + sg$, then the polynomial $P(x) \equiv \sum x^{i[s]g} = s$ is a subspace polynomial in $F_{q^n}$.

Proof: Since $gs|k$, there exists an integer $\alpha$ such that $gs\alpha = k$, thus $n = gs(\alpha + 1)$ and $s | g$. It follows that
\[x^{n/g} - 1 = x^{\hat{\alpha} + s} + x^{\hat{\alpha} + 2s} + \ldots + 1,
\]
and hence $(x^{n/g} + x^{n/g - 2s} + \ldots + 1)\{x^{n/g} - 1\}$. According to Lemma 5 the $q^g$-associates of these polynomials satisfy $\sum_{i=0}^{n/g-1} x^{[i]g} = (x^n - 1)$, and thus $P$ is a subspace polynomial of a $k$-subspace in $F_{q^n}$ by Definition 1.

By Lemma 1 and Lemma 6 we have a large set of subspace polynomials whose coefficients may be given explicitly.

Construction 2. If $g, s,$ and $k$ are integers such that $gs|k$ and $n \equiv k + sg$, then
\[Z \equiv \left\{ \frac{n}{g} \sum_{i=0}^{n/g-1} \beta[i][s]g \mid \beta \in B \right\}
\]
consists of $\frac{n}{g}q^n$ subspaces polynomials of subspaces in $C_g(n, k)$, where $B$ is any set of nonzero representatives of the orbit of $F_{q^n}$.

Proof: Since $n = k + sg$ and $sg|k$, it follows that $sg|n$, and thus $F_{q^n}$ is a subfield of $F_{q^n}$. By Lemma 6 the polynomial $P_V(x) = \sum x^{i[s]g}$ is a subspace polynomial of some $V \in G_q(n, k)$. Let $B$ be any set of representatives of the orbit of $F_{q^n}$, that is, a set consisting of a single nonzero element from each subspace in $\{\alpha F_{q^n} \mid \alpha \in F_{q^n} \}$. Since the size of the orbit of $F_{q^n}$ is $q^{n-1}$, and since all subspaces in it intersect trivially [7 Section III], it follows that $|V| = q^{n-1}$. By Lemma 1 for all $\beta \in B$ we have that $P_{\beta V}(x) \in Z$. We are left to show that if $\beta_1, \beta_2 \in B$, then $\beta_1 B \neq \beta_2 B$.

Assume for contradiction that there exists $\beta_1, \beta_2 \in B$ such that $\beta_1 V = \beta_2 V$. It follows that $P_{\beta_1, \beta_2 V}(x) = P_{\beta_2 V}(x)$, and Lemma 1 implies that the coefficients of $x$ are equal, that is, $\beta_1[n - sg] - 1 = \beta_2[n - sg] - 1$. Therefore, since every $\alpha \in F_{q^n}$ satisfies $\alpha^q = \alpha$, we have that
\[\left( \beta_1[n - sg] - 1 \right)^g = \left( \beta_2[n - sg] - 1 \right)^g
\]

\[\beta_1[n - sg] - 1 = \beta_2[n - sg] - 1
\]

\[\left( \beta_1 \beta_2 \right)^{q^g} = 1.
\]

It is widely known (e.g., [13] Theorem 3.20, p. 91) that the subspace polynomial of $F_{q^n}$ is $x^{q^n} - x$, which implies that $\beta_1[\beta_2 - 1] \in F_{q^n}$, and thus $\beta_1 \in \beta_2 F_{q^n}$. Since $\beta_2 \in \beta_2 F_{q^n}$, it follows that $\beta_1$ and $\beta_2$ belong to the same cyclic shift $\beta_2 F_{q^n}$, a contradiction.

Remark 1. For $s = 1$, the set $Z$ from Construction 2 consists of all subspace polynomials of subspaces in $C_g$ (see Construction 7). This is since the number of cyclic shifts of $F_{q^n}$ is $\frac{q^n - 1}{q^s - 1}$ and the size of $C_g$ is $\frac{n}{g}q^n = \frac{n}{g}q^n - 1 - 1$.

IV. IMPROVED BOUNDS ON LIST DECODABILITY OF GABIDULIN CODES

We begin by formally defining Gabidulin codes, which are rank-metric codes that attain a Singleton-like bound. Any square rank-metric code over $F_{q^n}$ of length $n$, minimum distance $d$, and size $M$ satisfies $M \leq q^n(n - d + 1)$ [5]. [13]. For a linear rank-metric code of dimension $k$, this bound implies that $d \leq n - k + 1$. Codes which attain this bound are called maximum rank distance (MRD) codes. It can be shown that Gabidulin codes, defined below, are linear MRD codes, attaining $d = n - k + 1$.

Definition 3. [8] A linear Gabidulin code $G[n, k]$ over $F_{q^n}$, length $n$, and dimension $k \leq n$ is the set
\[G[n, k] \equiv \{(P(\alpha_1), \ldots, P(\alpha_n)) \mid \deg P < k\},
\]
where $P$ traverses all linearized polynomials, and $\alpha_1, \ldots, \alpha_n$ are some fixed elements of $F_{q^n}$ which are linearly independent over $F_{q^n}$.

In [17] it was shown that large sets of subspace polynomials that agree on many top coefficients may be used to show the limits of list decoding of Gabidulin codes. For the lack of knowledge about the structure of the coefficients of subspace polynomials, a counting argument was later applied to show the existence of such a set. The resulting bound on list decoding of Gabidulin codes is cited below. In what follows, for $w \in F_{q^n}$
For a linearized polynomial $T$, $c_T \triangleq (T(\alpha_1), \ldots, T(\alpha_n))$,
\[ \mathcal{P} = \{P_1, \ldots, P_r\}, \quad t = \left[\frac{n/g}{q^{n/\ell}}\right] q^{n/\ell} \]
c$_R$ \notin \text{Gab}[n, k] \text{, and for all } P_i \in \mathcal{P}, c_{R-P_i} \in \text{Gab}[n, k]

Fig. 1. An illustration of the proof of Theorem 3. The proof of Theorem 4 is similar.

and $\tau \in \mathbb{N}$, let $S_\tau(w) \triangleq \{c \mid \text{rank}(w-c) = \tau\}$, that is, a sphere of radius $\tau$ centered at $w$.

**Theorem 2.** ([17] Theorem 1) Consider the code Gab[n, k] over $\mathbb{F}_{q^n}$, with $d = n - k + 1$. If $\tau < d$, then there exists a word $w \in \mathbb{F}_{q^n}$ such that

\[ |\text{Gab}[n, k] \cap S_\tau(w)| \geq \left[\frac{n-\tau \cdot d}{q^{n/\ell}}\right]. \]

As a result, the following bound is achieved. This bound may be seen as the rank-metric equivalent of the Johnson radius [10], and for $\varepsilon = 0$ it is equal to the Hamming metric Johnson radius.

**Corollary 2.** ([17] Section III) The code Gab[n, k] over $\mathbb{F}_{q^n}$, with $d = n - k + 1$ cannot be list decoded efficiently for any list decoding radius $\tau \geq n - \sqrt{n(n-d+\varepsilon)}$, for any fixed $0 \leq \varepsilon < 1$.

By Lemma 4 in certain cases there exists a large set of subspace polynomials with a unique coefficient structure. Restricting the counting argument used in the proof of Theorem 2 to the set $C_n$ (Theorem 1) provides a bound which may outperform Corollary 2 in some cases.

The proof of the following theorem is illustrated in Fig. 1.

**Theorem 3.** Let Gab[n, k] be a linear Gabidulin code over $\mathbb{F}_{q^n}$ with $d = n - k + 1$, evaluation points $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_{q^n}$, and let $\tau, g$ be integers such that $\left[\frac{d-1}{2}\right] + 1 \leq \tau \leq d - 1$, $g \geq 2$, and $g \mid \gcd(n-\tau, n)$. If $\ell$ is the unique integer such that $n = n + g + \ell (\ell + 1)$ (and thus, $\tau = g(\ell + 1)$), then there exists a word $c_R \in \mathbb{F}_{q^n} \setminus \text{Gab}[n, k]$ such that

\[ |\text{Gab}[n, k] \cap S_\tau(c_R)| \geq \left[\frac{n/\ell}{q^{n/\ell}}\right]. \]

**Proof:** According to Theorem 1 there exists a set $\mathcal{P}$ of size $\left[\frac{n/g}{q^{n/\ell}}\right] / q^{n/\ell}$ of subspaces in $G_q(n, n - \tau)$ whose subspace polynomials agree on their topmost $\tau = g(\ell + 1)$ coefficients.

Let $R$ be any linearized polynomial of degree $n - \tau$ that has these top coefficients, and let $c_R \in \mathbb{F}_{q^n}$ be the word resulting from the evaluation of $R$ at $\alpha_1, \ldots, \alpha_n$. Similarly, for $P \in \mathcal{P}$ let $c_{R-P} \in \mathbb{F}_{q^n}$ be the word corresponding to the evaluation of $R - P$ at $\alpha_1, \ldots, \alpha_n$.

Since $\deg_q(R-P) \leq n - \tau - g(\ell + 1)$ and $\tau = g(\ell + 1) > \frac{d-1}{2} = \frac{n-k}{2}$ it follows that $2\tau = \tau + g(\ell + 1) > n - k$, and hence

\[ k > n - \tau - g(\ell + 1) \geq \deg_q(R-P). \]

Therefore, the word $c_{R-P}$ is a codeword of Gab[n, k] for all $P \in \mathcal{P}$. In addition, since $\tau \leq d - 1$ it follows that $\deg_q R = n - \tau \geq n - d + 1 = k$, and hence $c_R \notin \text{Gab}[n, k]$.

Since every linearized polynomial can be viewed as an $\mathbb{F}_q$-linear mapping (see Section 4), and since $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $\mathbb{F}_{q^n}$, it follows that for every $P \in \mathcal{P}$,

\[ \text{rank}(c_R - c_{R-P}) = \text{rank}((P(\alpha_1), \ldots, P(\alpha_n))) = \dim \text{Im}(P) = n - \dim \ker(P) = n - (n - \tau) = \tau. \]

Thus, the set $\{c_{R-P}\}_{P \in \mathcal{P}} \subseteq \text{Gab}[n, k]$ is a set of size $\left[\frac{n/\ell}{q^{n/\ell}}\right] / q^{n/\ell}$, which is contained in a ball of radius $\tau$ around the word $c_R$.

A simple analysis of (1) shows that

\[ |\text{Gab}[n, k] \cap S_\tau(c_R)| \geq \left[\frac{n/\ell}{q^{n/\ell}}\right] \geq \left(\frac{q^n}{\ell(q^{n/\ell} - q^{(n-\tau)/\ell})}\right) q^{n/\ell} = q^{n(\tau+1) - g(\tau+1)^2 - n\ell} = q^{g - n(\tau+1)^2} = q^{n - (\tau+1)}, \]

and hence, this bound results in a list of exponential size whenever $g(\ell + 1)^2 < c \cdot n$ for $c \in (0, 1)$, or alternatively, when $\tau < \frac{2\varepsilon}{\ell}$.

The following example provides an infinite set of Gabidulin codes, with rates from $\frac{1}{2}$ to 1, that cannot be list decoded efficiently at all according to the bound from Theorem 3. This result strictly outperforms the bound from Corollary 2 and provides an answer to an open problem by [6, 11] Section 6), and [17] Section V), that is, there exist Gabidulin codes that cannot be efficiently list decoded beyond the unique decoding radius.

**Example 1.** Let $n$ be an integer power of 2 and let $1 \leq i \leq \log n - 2$. Consider the code Gab[n, (1 - $\frac{1}{2}$)n + 2], and let $\tau$ be the smallest possible list decoding radius, that is,

\[ \tau = \left\lfloor \frac{d-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \frac{n}{2^{2i+1}}. \]

Let $g = \frac{n}{2^{2i+1}} = \tau$, and notice that $g \geq 2$. To see that $g \mid \gcd(n, n-\tau)$, notice that since $n$ is an integer power of 2, it follows that $\tau | n$, and thus $g | n$. In addition, we have that $\tau(2^{i+1} - 1) = n - \tau$, thus $\tau | (n-\tau)$ and $g | (n-\tau)$. Therefore, in Theorem 2 we may choose $g = \frac{n}{2^{\tau}}, \ell = 0$, and get that there
exists a word \( c_R \in \mathbb{F}_q^n \) with \( q^{(1-2^{-1})n} \) codewords in a ball of radius \( \tau \) around it. Since \( \tau \) is larger than the unique decoding radius by one, this code cannot be efficiently list decoded at all. By applying the bound of [12] (see Corollary [2]), we get that there exists a word with exponentially many codewords around it for approximately \( \tau \geq n(1 - \sqrt{2^{1/2} - 1})/2 \), which is strictly larger than \( \tau = \frac{n}{2} \) for all \( i \).

In the following we present a simple algorithmic way of constructing many dense sets of Gabidulin codewords. These sets also show that the corresponding Gabidulin codes cannot be efficiently list decoded beyond the unique decoding radius. In addition, we have that for certain Gabidulin codes, dense sets of codewords are abound and may easily be computed explicitly.

**Theorem 4.** Let \( g, s, n \) be integers such that \( g \geq 2 \) and \( sg/n \), and let \( \text{Gab}[n, n-2sg+1] \) be a linear Gabidulin code over \( \mathbb{F}_q^n \) with \( d = 2sg \) and evaluation points \( \alpha_1, \ldots, \alpha_n \in \mathbb{F}_q^\ast \). If \( \tau \triangleq \lfloor \frac{d-1}{2} \rfloor + 1 = sg \), then there exists an (explicitly defined) word \( c_R \in \mathbb{F}_q^n \setminus \text{Gab}[n, n-2sg+1] \) such that

\[
|\{ \text{Gab}[n, n-2sg+1] \cap S_\tau(c_R) \}| \geq q^n - 1 \quad \text{for all }\tau.
\]

In particular, if \( R \) is the polynomial whose evaluation in \( \alpha_1, \ldots, \alpha_n \) yields \( c_R \), then \( \frac{q^n-1}{q^sg-1} \) of the codewords in \( S_\tau(c_R) \) are given by the evaluations of \( \{ R - P \} \) \( P \in \mathbb{Z} \) (see Construction [2]) in \( \alpha_1, \ldots, \alpha_n \).

**Proof:** Since \( sg/n - sg \), by setting \( k = n - sg \) it follows from Construction [2] that the set \( Z \) is a set of subspace polynomials of subspaces in \( G_q(n, n - sg) \), whose size is \( q^{n-1}/q^{sg-1} - 1 \). Let \( R \) be any linearized polynomial of \( q \)-degree \( n - sg \) whose top \( sg \) coefficients are \( (1, 0, \ldots, 0) \), and let \( c_R \in \mathbb{F}_q^n \) be the codeword resulting from the evaluation of \( R \) at \( \alpha_1, \ldots, \alpha_n \). For each \( P \in Z \) let \( c_{R-P} \in \mathbb{F}_q^n \) be the word corresponding to the evaluation of \( R - P \) at \( \alpha_1, \ldots, \alpha_n \). For all \( P \in Z \) we have that \( \deg_\tau(R - P) \leq n - sg < n - 2sg + 1 \), and thus \( c_{R-P} \in \text{Gab}[n, n-2sg+1] \). In addition, \( \deg_\tau(R - P) \leq n - sg \), and thus \( c_R \notin \text{Gab}[n, n-2sg+1] \).

As in the proof of Theorem [3] for all \( P \in Z \) we have that \( \text{rank}(c_{R-P}) = n - \dim \ker P = sg \). Therefore, the set \( \{ c_{R-P} \} \) \( P \in Z \) is a set of \( \frac{q^n-1}{q^sg-1} \) codewords in \( \text{Gab}[n, n-2sg+1] \), all of which are of distance exactly \( \tau = sg \) from \( c_R \).

Notice that each code in the family of codes mentioned in Theorem [3] satisfies \( d = 2sg \), and hence the unique decoding radius is \( \lfloor \frac{d-1}{2} \rfloor = sg - 1 \). Furthermore, since \( sg/n \), it follows that \( sg \leq \frac{n}{2} \), and thus the word \( c_R \) has \( \Omega(q^{n/2}) \) codewords in a ball of radius \( \tau = \lfloor \frac{d-1}{2} \rfloor + 1 \) around it. Hence, this family of Gabidulin codes cannot be list decoded efficiently at all.

**V. Conclusions and Future Work**

We have shown that the worst case bound on list decodability of Gabidulin codes may be improved in some cases. This was shown by using the structure of the subspace polynomials of a subset of \( G_q(n, k) \) for \( n \) and \( k \) that have a non-trivial gcd. In addition, we have presented such subspace polynomials explicitly, using the notion of cyclic shifts and \( q \)-associativity. Both of these results outperform the counting argument applied in [17], and are the first example of infinite families of Gabidulin codes that cannot be list decoded efficiently beyond the unique decoding radius. This resolves an open question by [6], [11], and [12].

The work of [17] ruled out the existence of an efficient algorithm for list decoding of Gabidulin codes beyond the Johnson radius. Our work rules out the existence of an efficient list decoding algorithm that applies for any Gabidulin code and any radius beyond half the minimum distance. However, this certainly does not rule out the existence of list decoding very large subcodes of Gabidulin codes or Gabidulin codes with low rates, since our work requires the code parameters to satisfy some strict number-theoretic constraints, and have rate at least \( \frac{1}{2} \). E.g., [11] provides a subcode of a Gabidulin code which can be list-decoded efficiently.

Additional discussion about list decoding of subspace codes and Reed-Solomon codes will appear in the full version of this paper. For future research, we would like to have similar bounds for non-square Gabidulin codes, and for rates less than \( \frac{1}{2} \).

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