Thin-disk models in an Integrable Weyl-Dirac theory

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Abstract

We construct a class of static, axially symmetric solutions representing thin disks of matter in an Integrable Weyl-Dirac theory proposed by Mark Israelit. The main differences between these solutions and the corresponding general relativistic one are analyzed, focusing on the behavior of physical observables (rotation curves of test particles, density and pressure profiles). We consider the case in which test particles move on Weyl geodesics. The same rotation curve can be obtained from many different solutions of the Weyl-Dirac theory, although some of these solutions present strong qualitative differences with respect to the usual general relativistic model (such as a ring-like density profile). In particular, for typical galactic parameters all rotation curves of the Weyl-Dirac model present Keplerian fall-off. As a consequence, we conclude that a more thorough analysis of the problem requires the determination of the gauge function $\beta$ on galactic scales, as well as restrictions on the test-particle behavior under the action of the additional fields introduced by this theory.

1 Introduction

Axially symmetric solutions of a theory of gravitation are of great interest for its astrophysical applications, e.g. they can model spiral galaxies [1] and accretion disks around compact structures [2, 3]. There is a vast literature on analytical general relativistic self-gravitating disk models and extensions to theories of modified gravity. The first general relativistic solutions were obtained by Bonnor and Sackfield [4], representing a thin disk of matter without stress. Morgan and Morgan studied thin disks with transverse stress [5] and with radial stress [6]. Kuzmin-like thin disks in general relativity were constructed by Bičák, Lynden-Bell and Katz [7] and Vogt and Letelier [8], among others. An extension to six-dimensional gravity was proposed in [9]. For a survey on analytical thin-disk models in both general relativity and Newtonian gravitation, see [2, 3]. Relativistic thick disks extending the Miyamoto-Nagai model [10] were proposed in [11] and [12], with a six-dimensional counterpart presented in [13].

Spiral galaxies present a myriad of rotation curve profiles [14, 15, 16, 17, 18]. The Newtonian models, as well as their general relativistic counterparts, are not able to reproduce these profiles by considering only the corresponding density profiles inferred by photometry. The current explanation for these and other astrophysical phenomena is the presence of dark matter halos ([19, 20], see [21].

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for a comprehensive review of the dark matter paradigm and \cite{22} for a detailed historical overview of the subject), although recent work on quantum corrections to general relativity has given results as good as the dark matter framework (pseudothermal halo), at least in spiral galaxies \cite{23}. Furthermore, this discrepancy led many scientists to investigate whether the effect of modifications of the law of gravity in large scales could solve the “missing mass” problem (see \cite{24} for a modification of gravity which deals with rotation curves of spiral galaxies and \cite{22,25} for a thorough review of the MOND framework). However, so far no modification of the law of gravitation was able to explain these “dark” phenomena in all scales.

Concerning modified theories of gravity based on generalizations of the geometrical concept of spacetime, Hermann Weyl presented in 1918 a generalization of Riemannian geometry in an attempt to give a unified theory of gravitation and electromagnetism \cite{26}. After some criticisms concerning the non-integrability of length in his theory the idea was abandoned \cite{26}, reappearing later (in a cosmological context) in a work of Dirac \cite{27}. Integrable versions of Dirac’s theory were also proposed \cite{28,29}. A recent review on the applications of Weyl geometry to gravitation and quantum mechanics is presented in \cite{30}.

The aim of the present work is to construct analytical thin-disk models in the integrable Weyl-Dirac theory proposed by Israelit \cite{29} and to compare the observables with the corresponding GR ones obtained by the same procedure. Apart from matter creation itself \cite{29,31,32,33}, it is of extreme importance to analyze the motion of astrophysical bodies under the influence of the background fields generated by gravity and geometry, such as the motions of stars in a spiral galaxy. Our main goal in this paper is to compare the rotation curves and density profiles predicted by the present model to the corresponding general relativistic model, and see whether they are consistent with observations without the introduction of dark matter halos. Applications of Israelit’s Weyl-Dirac (W-D) theory to cosmology are presented in \cite{29,31,32,33}, and to spherically symmetric spacetimes in \cite{29}. As far as we know, the present paper is the first attempt to obtain exact solutions of an Integrable W-D theory which model galactic disks, as well as to analyze the behavior of test particles in a given background of \cite{29} (applications of the original non-integrable W-D theory \cite{27} to galactic rotation curves are presented in \cite{34}).

The paper is organized as follows: in section 2 we present a summary of Weyl geometry, mainly to fix our notation. In section 3 we reproduce the basics of the integrable version of the W-D theory proposed by Israelit \cite{29}. We review in some detail the equations of motion for massive test particles in the theory, focusing on the case when these equations describe Weyl geodesics. A thin-disk model is constructed in section 4. We obtain its density profile, pressure, total mass and the circular velocity of test particles. The radial stability of these circular motions is studied via Rayleigh’s angular momentum criterion \cite{35}. The results are discussed in section 5 and conclusions are presented in section 6.

\section{Weyl geometry}

In this section we present a summary of Weyl geometry in the context of tensor calculus, in order to fix our notation. We follow mainly the presentation by Folland \cite{36}, where a modern treatment of Weyl geometry is developed. Other presentations of Weyl geometry appear in \cite{26,29,30,33} and references therein.

Weyl geometry is constructed in a conformal manifold. There is an equivalence class of Lorentzian metrics, written in a coordinate basis as $[g_{\mu\nu}]$, where two elements of this class are related by $\tilde{g}_{\mu\nu} = e^{2\lambda(x^\alpha)} g_{\mu\nu}$ for some well-behaved function $\lambda$ \cite{36}. This is the most general transformation
which preserves the causal structure of spacetime \[37\]. In order to construct a connection which describes this conformal invariance we also associate for each metric \(g_{\mu\nu}\) a 1-form \(\omega_\mu\), in such a way that for a conformal transformation
\[
\tilde{g}_{\mu\nu} = e^{2\lambda(x^\alpha)} g_{\mu\nu}
\]
the respective 1-forms are related by
\[
\tilde{\omega}_\mu = \omega_\mu + \lambda_\mu,
\]
where we write \(\lambda_\mu \equiv \partial_\mu \lambda\). The equations (1)–(2), taken together, are called gauge transformations. Each pair \((g_{\mu\nu}, \omega_\mu)\) is called a gauge. We say that a family of tensor fields \([T]\) (or, for brevity, a tensor field \(T\)) indexed by the metric is gauge-covariant with Weyl power \(\Pi(T) = m\) if \(\tilde{T} = e^{m\lambda} T\) (indices suppressed) under the transformations (1)–(2). If \(m = 0\) the tensor field is called gauge-invariant \[27, 29\].

Weyl geometry is characterized by a symmetric connection satisfying, for each metric of the conformal class,
\[
\nabla_\sigma g_{\mu\nu} = 2 \omega_\sigma g_{\mu\nu},
\]
where \(\nabla\) is called the Weyl connection \[36\]. If (3) is valid in one gauge, it will also be valid in any gauge related to the former by the gauge transformations (1)–(2).

The coefficients of the Weyl connection are given by
\[
\Gamma^\sigma_{\mu\nu} = \{^\sigma_{\mu\nu}\} + g_{\mu\nu} \omega^\sigma - \delta^\sigma_\mu \omega_\nu - \delta^\sigma_\nu \omega_\mu,
\]
where
\[
\{^\alpha_{\mu\nu}\} = \frac{1}{2} g^{\alpha\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})
\]
are the Christoffel symbols of the metric \(g_{\mu\nu}\). These connection coefficients are invariant under gauge transformations. Thus, the geodesics of a Weyl manifold are gauge-invariant and can be written as
\[
\frac{d^2 x^\mu}{d\xi^2} + \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{d\xi} \frac{dx^\nu}{d\xi} = 0,
\]
where \(\xi\) is an affine parameter.

Since the 1-forms \(\omega_\mu\) are not necessarily closed and they are related in different gauges by an additive exact form, they all have the same exterior derivative, denoted by
\[
W_{\mu\nu} = \partial_\nu \omega_\mu - \partial_\mu \omega_\nu.
\]
\(W_{\mu\nu}\) is called length curvature (or distance curvature) \[33, 36\], since it is associated with the non-integrability of the length of a vector parallel transported along a closed curve. We say that the geometry is integrable if \(\omega_\mu = \partial_\mu \omega\) for some scalar field \(\omega\). In this case, \(W_{\mu\nu} = 0\).

The curvature tensor of a Weyl manifold is defined as the (3,1)-tensor field given by
\[
R_{\mu\nu\sigma} = \partial_\mu \Gamma^\sigma_{\nu\alpha} - \partial_\nu \Gamma^\sigma_{\mu\alpha} + \Gamma^\rho_{\mu\alpha} \Gamma^\sigma_{\nu\rho} - \Gamma^\rho_{\nu\alpha} \Gamma^\sigma_{\mu\rho}.
\]
The Ricci tensor and the Ricci scalar are defined, respectively, by
\[
R_{\mu\nu} = R_{\alpha\mu\nu}^\alpha,
\]
\[
R = R_{\mu}^\mu = g^{\mu\nu} R_{\mu\nu}.
\]
The curvature and Ricci tensors are gauge-invariant, since they depend only on the connection. The Ricci scalar is gauge-covariant with \( \Pi(R) = -2 \). In terms of the Riemannian curvature tensor \( K_{\mu\nu\sigma} \) for each metric, we have for a n-dimensional manifold that the Ricci scalar is

\[
R = K + 2(n - 1)\omega_{\sigma;\sigma} - (n - 2)(n - 1)\omega^{\sigma}\omega_{\sigma}. \tag{11}
\]

The semicolon denotes differentiation with respect to the Riemannian connection. In this paper we work in dimension 4 with a Lorentz metric of signature \((+ - - -)\).

3 Israelit’s version of the Integrable Weyl-Dirac theory

This section is based on the original paper [29], where the Integrable W-D theory studied here is fully developed. We present below the basics of this theory, following [29], in order to properly construct the thin-disk models in section 4. The equations reproduced below are the same equations which appear in [29]. However, some sign changes appear due to the conventions adopted in the former section.

Israelit’s version of Dirac’s theory is constructed on an integrable 4-dimensional Weyl manifold \((\omega_{\mu} = \partial_{\mu}\omega, \text{ with } \omega \text{ an scalar field})\). The action of the theory, as in the Dirac case, depends also on a gauge-covariant scalar field \( \beta \) with \( \Pi(\beta) = -1 \), and it is given by [29]

\[
S = \int_{\Omega} \left[ \beta^2 R + k(\beta^{\mu} + \beta\omega^{\mu})(\beta_{,\mu} + \omega_{\mu}) + 2\Lambda \beta^4 + 8\pi L_M \right] \sqrt{-g} d^4x, \tag{12}
\]

with \( k \) and \( \Lambda \) constants and \( L_M \) the Lagrangian of matter. Here, \( \beta_{,\mu} = g_{\mu\nu} \beta_{,\nu} \) and \( \omega_{\mu} = g_{\mu\nu} \omega_{,\nu} \). In terms of the associated Riemannian tensors we can write, discarding surface terms:

\[
S = \int_{\Omega} \left[ \beta^2 K + 2(k - 6)\beta\beta_{,\sigma}\omega^{\sigma} + (k - 6)\beta^2 \omega^{\sigma}\omega_{\sigma} + k\beta^2 \omega_{,\sigma}\omega^{\sigma} + 2\Lambda \beta^4 + 8\pi L_M \right] \sqrt{-g} d^4x. \tag{13}
\]

The action (12) is invariant under gauge transformations, as any action of a Weyl-type theory must be to preserve the form of the field equations under gauge transformations. Defining \( \alpha = k - 6 \), the field equations obtained from this action are (see [29]):

\[
G_{\mu\nu} = 8\pi \frac{T_{\mu\nu}}{\beta^2} - \alpha(Z_{\mu}Z_{\nu} - \frac{1}{2}g_{\mu
u}Z^{\sigma}Z_{\sigma}) - 2(g_{\mu\nu}\omega^{\sigma}_{,\sigma} - b_{\mu\sigma}b^{\sigma} - g_{\mu\nu}b^{\sigma}b_{,\sigma} + \Lambda \beta^2 g_{\mu\nu}, \tag{14}
\]

\[
2\alpha(\beta^{2}Z^{\nu})_{,\nu} = 16\pi S, \tag{15}
\]

\[
K + k(b^{\sigma}_{,\sigma} + b_{\sigma}b^{\sigma}) = -\alpha(\omega^{\sigma}\omega_{\sigma} - \omega^{\sigma}_{,\sigma}) - 4\Lambda \beta^2 + \frac{8\pi}{\beta} B, \tag{16}
\]

where \( G_{\mu\nu} := K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}, b_{\mu} := \ln(\beta)_{,\mu} = \frac{\beta_{,\mu}}{\beta}, Z_{\mu} := \omega_{\mu} + b_{\mu} \) and

\[
T_{\mu\nu} := -\frac{1}{\sqrt{-g}} \frac{\delta(L_M \sqrt{-g})}{\delta g^{\mu\nu}}, \tag{17}
\]

\[
S := -\frac{1}{2} \frac{\delta L_M}{\delta \omega}, \tag{18}
\]

\[
B := -\frac{1}{2} \frac{\delta L_M}{\delta \beta}. \tag{19}
\]
These equations were obtained varying \( g_{\mu\nu}, \omega \) and \( \beta \) in (12), respectively. They are also subjected to the conservation laws \[ T_{\mu\nu} - S\omega_{\mu} - B\beta_{\mu} = 0, \]

(20)

\[ T + S - \beta B = 0, \]

(21)

where \( T = T^\mu_\mu \), which come from the invariance of the action under diffeomorphisms and gauge transformations, respectively. With these conservation laws, Eq. (16) is automatically satisfied, and thus \( \beta \) has no dynamics (this is consistent with the gauge covariance of the field equations – see [29] for a more thorough discussion).

### 3.1 Equations of motion for massive test particles

Considering a cloud of noninteracting particles with the same mass \( m \), Israelit obtained the equations of motion for massive (timelike) test particles with the additional assumption that the curves are parametrized in each gauge in such a way that the four-velocity of the particles is rescaled to unity, \( u^\mu u_\mu = 1 \). The energy-momentum tensor for the cloud of particles is \( T_{\mu\nu} = \rho u^\mu u^\nu \), with \( \rho = m\rho_N \), where the particle density \( \rho_N \) satisfies in each gauge a continuity equation

\[ \rho_N u^\nu \]

(22)

With \( S = q_s\rho_N \) and \( B = q_b\rho_N \), and in virtue of the equations (20) and (21), we get [29]

\[ \frac{d^2 x^\mu}{ds^2} + \{\sigma_{\nu}\} \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} = (b_\nu + \frac{q_s}{m} Z_\nu \left( g^{\mu\nu} - \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right)), \]

(23)

where for each gauge \( s \) is the arc length of the curve.

We note that these equations can also be obtained from the variational principle

\[ \delta \int \beta \exp \left( \frac{q_s}{m} Z \right) ds = 0, \]

(24)

where the integral is gauge-invariant. This procedure is useful to obtain conserved quantities along a trajectory, and will be used in the next section.

When the Lagrangian for the cloud of test particles does not depend on \( \beta \), i.e., when \( q_b = 0 \), Eq. (21) gives \( \frac{q_s}{m} = -1 \) for all gauges. From (23), we find

\[ \frac{d^2 x^\mu}{ds^2} + \{\sigma_{\nu}\} \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} + \omega_{\nu} \left( g^{\mu\nu} - \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right) = 0, \]

(25)

which are the equations for Weyl geodesics reparametrized in such a way that \( u^\mu u_\mu = 1 \). The reparametrization \( d\xi = e^{-\omega} ds \) gives back the Weyl geodesics (6). From now on we will consider this case \( \frac{q_s}{m} = -1 \), in such a way that the particle’s orbit will be a Weyl geodesic.

#### 3.1.1 Circular timelike Weyl geodesics

We assume that an observer follows a timelike curve such that, in each gauge, its parametrization obeys \( u^\mu u_\mu = 1 \), in consistency with the above equations of motion. It is a consequence of this assumption that the norm of the 3-velocity measured by this observer is gauge-invariant. This fact
will be important in the study of disk models in Weyl-type theories, since it implies that the rotation curves measured by static local observers are gauge-invariant.

Let us now consider a diagonal static, axially symmetric metric

\[ ds^2 = g_{tt} dt^2 + g_{RR} dR^2 + g_{zz} dz^2 + g_{\varphi \varphi} d\varphi^2 \]  

(26)

for some gauge, with \( g_{tt}, g_{RR}, g_{zz}, g_{\varphi \varphi} \) functions of \( R \) and \( z \) only. Let \( \omega(R, z) \) be the corresponding Weyl field. We can rewrite Eq. (25) as

\[
\frac{d^2 x^\mu}{ds^2} + \left( \{\mu, \rho\} + g_{\sigma \nu} \omega^\nu \right) \frac{dx^\sigma}{ds} \frac{dx^\rho}{ds} = -\omega^\nu \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} = 0, \tag{27}
\]

from which we obtain the radial equation

\[
\ddot{R} + \left( \{R, \varphi \} + g_{\varphi \varphi} \omega^R \right) \dot{\varphi}^2 + \left( \{R, tt\} + g_{tt} \omega^R \right) (\dot{t})^2 = 0. \tag{28}
\]

In the following we consider the circular Weyl geodesics given by \( z = 0 \) and constant \( R \). We get from Eq. (28)

\[
\left( \{R, \varphi \} + g_{\varphi \varphi} \omega^R \right) (\dot{\varphi})^2 + \left( \{R, tt\} + g_{tt} \omega^R \right) (\dot{t})^2 = 0. \tag{29}
\]

The circular velocity of a particle, measured by a local observer at rest with respect to the coordinate system, is defined as the norm of the corresponding 3-velocity: \( (v_c)^2 := \frac{g_{\varphi \varphi}}{g_{tt}} (\dot{\varphi})^2 \). We have

\[
(v_c)^2 = \frac{g_{\varphi \varphi}}{g_{tt}} \left( \{R, \varphi \} + g_{tt} \omega^R \right) \tag{30}
\]

This expression is gauge-invariant, as it can be seen explicitly by using the gauge transformations \( (11-2) \).

For a static, axially symmetric metric in isotropic cylindrical coordinates

\[
ds^2 = e^{\nu(R, z)} dt^2 - e^{\lambda(R, z)} (dR^2 + dz^2 + R^2 d\varphi^2), \tag{31}
\]

we have \( \{R, tt\} = \frac{1}{2} \{e^{\nu}, \nu\}, \{R, \varphi \} = \frac{1}{2} R(2 + R \lambda, R) \). Then

\[
(v_c)^2 = \frac{R^2 e^\lambda}{e^{\nu}} \frac{(e^{\nu})_R}{(2 e^\lambda)_R} \frac{e^{\nu} (e^{\nu})_R - 2 e^\nu \omega_R}{(2 R^2 e^\lambda)_R - 2 R^2 e^\lambda \omega_R} \tag{32}
\]

The \( z \)-component of the specific angular momentum of a test particle, measured with respect to infinity, is defined as the conserved quantity associated with the coordinate \( \varphi \). By considering \( q_\rho = 0 \) and extremizing \( \int e^{-\omega} ds \) (see Eq. (23)), the Lagrangian \( L = e^{-\omega} \sqrt{g_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}} \) gives us

\[
h := - \frac{\partial L}{\partial \dot{\varphi}} = -e^{-\omega} g_{\varphi \varphi} \dot{\varphi} \frac{h}{\dot{\varphi}}.
\]

If the particle is moving on a circular orbit in the plane \( z = 0 \), we obtain

\[
h = \pm e^{-\omega} R^2 e^\lambda \sqrt{\frac{(e^{\nu})_R}{e^{\nu} (2 R^2 e^\lambda)_R} \frac{e^{\nu} (e^{\nu})_R - 2 e^\nu \omega_R}{(2 R^2 e^\lambda)_R - 2 R^2 e^\lambda \omega_R}}. \tag{33}
\]

The sign of \( h \) depends on the sign of \( \dot{\varphi} \).
4 Thin-disk models

In order to construct disk models we work in the gauge $\beta = 1$ (Einstein gauge, see [27, 29]). Moreover, the term proportional to $\Lambda$ in the action (12) is a cosmological factor and therefore can be neglected in galactic scales. Setting $\beta = 1$ and $\Lambda = 0$ in the field equations (14) and (15), we obtain

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + \Theta_{\mu\nu})$$  \hspace{1cm} (34)

and

$$2\alpha \omega^\nu_{\mu
u} = 16\pi S,$$  \hspace{1cm} (35)

where (see [29])

$$8\pi \Theta_{\mu\nu} = \alpha \left( \frac{1}{2} g_{\mu\nu} \omega^\lambda_{\lambda} - \omega_{\mu\nu} \right).$$  \hspace{1cm} (36)

In vacuum ($T_{\mu\nu} = 0$), the field equations reduce to

$$G_{\mu\nu} = \alpha \left( \frac{1}{2} g_{\mu\nu} \omega^\lambda_{\lambda} - \omega_{\mu\nu} \right),$$  \hspace{1cm} (37)

$$\omega^\nu_{\mu
u} = 0,$$  \hspace{1cm} (38)

which are the equations for a massless scalar field in general relativity. Buchdahl [38] found a family of spherically symmetric solutions of (37) and (38) in isotropic coordinates, given by

$$ds^2 = \left( \frac{1 - f}{1 + f} \right)^{2\gamma} dt^2 - (1 - f)^{2 - 2\gamma}(1 + f)^{2 + 2\gamma}(dR^2 + dz^2 + R^2 d\phi^2),$$  \hspace{1cm} (39)

$$\omega = 2\lambda \ln \left( \frac{1 - f}{1 + f} \right),$$  \hspace{1cm} (40)

where

$$f = \frac{m}{2\sqrt{R^2 + z^2}},$$  \hspace{1cm} (41)

and $\gamma$ and $\lambda$ are constants satisfying

$$\gamma^2 = 1 + 2\alpha \lambda^2.$$  \hspace{1cm} (42)

For the metric (39) and a function $\omega$ given by (40), with $f = f(R, z)$ an arbitrary function of $R$ and $z$, the nonvanishing components of the energy-momentum tensor (54) are given by

$$T^t_t = -\frac{\gamma - f}{2\pi(1 - f)^{3 - 2\gamma}(1 + f)^{3 + 2\gamma}} \left( f_{,RR} + f_{,zz} + \frac{1}{R} f_{,R} \right),$$

$$T^R_R = \frac{1}{4\pi(1 - f)^{3 - 2\gamma}(1 + f)^{3 + 2\gamma}} \left( f \left( f_{,zz} + \frac{1}{R} f_{,R} \right) + 2(f_{,R})^2 - (f_{,z})^2 \right),$$

$$T^z_z = -\frac{1}{4\pi(1 - f)^{3 - 2\gamma}(1 + f)^{3 + 2\gamma}} \left( f f_{,Rz} - 3f_{,R} f_{,z} \right),$$

$$T^R_z = T^z_R,$$

$$T^z_z = \frac{1}{4\pi(1 - f)^{3 - 2\gamma}(1 + f)^{3 + 2\gamma}} \left( f \left( f_{,RR} + \frac{1}{R} f_{,R} \right) + 2(f_{,z})^2 - (f_{,R})^2 \right),$$

$$T^\phi_\phi = \frac{1}{4\pi(1 - f)^{3 - 2\gamma}(1 + f)^{3 + 2\gamma}} \left( f \left( f_{,RR} + f_{,zz} \right) - ((f_{,R})^2 + (f_{,z})^2) \right).$$
These expressions are similar to the ones of GR [12], which are recovered setting $\gamma = 1$.

We now apply the “displace, cut and reflect” method (image method) to generate a family of disks from the above family of spherically symmetric vacuum solutions (39)–(42). This method consists in applying the transformation $z \mapsto |z| + a$ to the function $f$, with $a > 0$ (see [8]). The corresponding Newtonian solution is known as the Kuzmin disk (see [1]). A solution of the field equations is therefore mapped by this transformation to a solution which has a nonvanishing distributional energy-momentum tensor on the surface $z = 0$, which corresponds to a razor-thin disk of matter, as we see below.

The components of the metric now depend on $|z|$, which gives terms involving $\delta(z)$ in $f_{,zz}$, where $\delta(\cdot)$ is the Dirac delta distribution. This can be seen as follows: If $f$ depends on $|z|$, then

$$
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial |z|} \frac{|z|}{\partial z} = \frac{\partial f}{\partial |z|} \left( 1 - 2\theta(z) \right),
$$

(44)

where $\theta(z)$ is the Heaviside step function. Thus,

$$
\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial |z|^2} \left( 1 - 2\theta(z) \right)^2 - 2 \frac{\partial f}{\partial |z|} \delta(z).
$$

(45)

Substituting expressions (44–45) in (43), we see that the energy-momentum tensor now contains distributions (see [39]). From equations (43), we have that $T_{\mu}^{\nu}$ can be written as

$$
T_{\mu}^{\nu} = H_{\mu}^{\nu} \delta(z) + D_{\mu}^{\nu},
$$

(46)

where $H_{\mu}^{\nu}$ is defined for $z = 0$ and $D_{\mu}^{\nu}$ does not contain delta-like terms. The nonzero components of $H_{\mu}^{\nu}$ and $D_{\mu}^{\nu}$ are

$$
H_{t}^{t} = -\frac{\gamma - f}{2\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left( -2 \frac{\partial f}{\partial |z|} \right),
$$

$$
H_{R}^{R} = \frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left\{ -2 f \frac{\partial f}{\partial |z|} \right\},
$$

$$
H_{\varphi}^{\varphi} = \frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left\{ -2 f \frac{\partial f}{\partial |z|} \right\},
$$

(47)

$$
D_{t}^{t} = -\frac{\gamma - f}{2\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left( f_{,RR} + \frac{\partial^2 f}{\partial |z|^2} \left[ 1 - 2\theta(z) \right]^2 + \frac{1}{R} f_{,R} \right),
$$

$$
D_{R}^{R} = \frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left\{ f \left( \frac{\partial^2 f}{\partial |z|^2} \left[ 1 - 2\theta(z) \right]^2 + \frac{1}{R} f_{,R} \right) + 2(f_{,R})^2 - (f_{,z})^2 \right\},
$$

$$
D_{\varphi}^{\varphi} = -\frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left( f_{,RR} + \frac{1}{R} f_{,R} \right),
$$

$$
D_{z}^{R} = D_{R}^{z} = \frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left\{ f \left( f_{,RR} + \frac{1}{R} f_{,R} \right) + 2(f_{,z})^2 - (f_{,R})^2 \right\},
$$

$$
D_{z}^{z} = \frac{1}{4\pi(1 - f)^{3-2\gamma}(1 + f)^{3+2\gamma}} \left\{ f \left( f_{,RR} + \frac{\partial^2 f}{\partial |z|^2} \left[ 1 - 2\theta(z) \right]^2 \right) - ((f_{,R})^2 + (f_{,z})^2) \right\}.
$$

(48)
Note that $H_\mu^\nu$ is diagonal in these coordinates. For the specific case treated here

$$f(R, z) = \frac{m}{2\sqrt{R^2 + (|z| + a)^2}}.$$  \hspace{1cm} (49)$$

Thus we have, as in [8, 12], $D_\mu^\nu = 0$, and therefore $T_\mu^\nu = H_\mu^\nu \delta(z)$. In order to avoid horizons in the metric we must have $\frac{m}{2a} < 1$. We interpret this solution as a thin disk in the plane $z = 0$, without halos of matter ($D_\mu^\nu = 0$).

By analogy with the delta distribution in curvilinear coordinates [40, 41], the distribution which takes the role of the Dirac delta in this case is

$$\hat{\delta}(z) = \frac{1}{\sqrt{-g_{zz}}} \delta(z),$$

and thus

$$T_\mu^\nu = H_\mu^\nu \sqrt{-g_{zz}} \hat{\delta}(z)$$

$$\equiv Y_\mu^\nu \delta(z),$$  \hspace{1cm} (50)$$

where

$$Y_\mu^\nu := \sqrt{-g_{zz}} H_\mu^\nu$$  \hspace{1cm} (51)$$

is the physical energy-momentum tensor of the disk (see also [7, 8, 39]).

5 Rotation curves and density profiles

We find that the matter content $Y_\mu^\nu$ of the disk given by (49) is a perfect fluid with surface density $\sigma$ and isotropic pressure $P$ given by

$$\sigma = \frac{\tilde{a}}{m} \left( \gamma - \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right) \frac{2\pi \left( 1 - \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right)^{2-\gamma}}{\left( 1 + \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right)^{2+\gamma} (\tilde{R}^2 + \tilde{a}^2)^{3/2}},$$  \hspace{1cm} (52)$$

$$P = \frac{\tilde{a} / m}{8\pi \left( 1 - \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right)^{2-\gamma}} \frac{\left( 1 + \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right)^{2+\gamma}}{(\tilde{R}^2 + \tilde{a}^2)^2},$$  \hspace{1cm} (53)$$

where

$$\tilde{R} = R / m, \quad \tilde{a} = a / m,$$  \hspace{1cm} (54)$$

In order to the density be nonnegative at every point, we must have $\gamma > \frac{m}{2a}$. Integrating the surface density over the whole plane of the disk, we obtain its total mass

$$M_{DISK} = \int \sigma \sqrt{g_{R\varphi} g_{\varphi\varphi}} dR d\varphi = -a + \frac{1}{4a} \frac{(2a + m)^{\gamma + 1}}{(2a - m)^{\gamma - 1}}.$$  \hspace{1cm} (55)$$

Setting $\gamma = 1$ (and for consistency $\alpha = \lambda = 0$), we recover the GR Kuzmin model presented in [8]. The circular velocity of particles orbiting the plane of the disk is calculated from (32), giving

$$v_c = \tilde{R} \sqrt{\frac{\left( \gamma - 2\lambda \right)}{\tilde{R}^2 \left( 1 - \frac{1}{2\sqrt{R^2 + \tilde{a}^2}} \right) - (\gamma + 2\lambda) + (\tilde{R}^2 + \tilde{a}^2)^{3/2} \left( 1 - \frac{1}{4(\tilde{R}^2 + \tilde{a}^2)} \right)}}.$$  \hspace{1cm} (56)$$
For typical galactic scales \((a \sim 1 \text{ kpc}, \frac{\Omega}{\sigma} \sim 10^{-5})\), by fixing the parameters \(m, a, \gamma\), and \(\lambda\) it is possible to find a value for the parameter \(m\) of the general relativistic Kuzmin model \([5]\) in such a way that the rotation curves of both models are practically indistinguishable. Also, fixing \(a\) and given the parameter \(m\) of the usual relativistic model, it is possible to find a family of parameters \(\gamma, \lambda, m\) of the W-D model with practically the same rotation curve as the general relativistic one (see Fig. (1)). This occurs because for these ranges of the parameters expression \(\text{(56)}\) differs from the general relativistic one basically by a multiplicative constant and a constant additive term in the denominator.

![Figure 1: Rotation curves of the Kuzmin model in the Einstein gauge: (a) Density profiles; (b) Pressure profiles; (c) Rotation curves predicted by the model. The values for the parameters of each curve are given in Table 1.](image)

In this way, the qualitative behavior of the rotation curves of the W-D Kuzmin model in the Einstein gauge and of the general relativistic model are the same. In particular, both of them present Keplerian fall-off. The difference between the two configurations occurs in the density profile: there are many different density profiles generating the same rotation curve. In particular, the presence of the term \(\left(\gamma - \frac{1}{2\sqrt{R^2 + a^2}}\right)\) in the numerator of the expression for \(\sigma\), Eq. \(\text{(52)}\), allows us to construct density configurations very close to rings of matter, which also give practically the same rotation curve. This is illustrated in Fig. 2 with the corresponding rotation curve in Fig. 1(a), where we present pressure and density profiles for specific values of the parameters (see Table 1).

The parameters presented in Table 1 were chosen in such a way that the rotation curves – Fig. 1(c) – are very close to each other, but with density profiles – Fig. 1(a) and Fig. 2(a) – which vary from disks more massive than the GR disk to disks with relative low density compared to...
the GR disk (including a ring-like configuration – Fig. 2(a)). The pressure profiles also vary in magnitude, but maintaining the same qualitative shape (as expected from (53), since there is no term \((\gamma - \frac{1}{2})\) in the numerator of the expression for \(P\)). From Table 2 we see that the values for the total masses of the disks vary significantly, in a range of about five orders of magnitude. The GR parameters were chosen to give a mass around \(10^{11} M_\odot\) and a maximum circular velocity of the order of the typical circular velocities of spiral galaxies (\(\sim 100 \text{ km/s}\)). However, the values of the total mass of the Weyl-Dirac disks do not bear any relationship with the rotation curves: even though they vary by five orders of magnitude, the resulting rotation curves are almost identical (see Fig. 1(c)).

![Figure 2: Physical observables of the Kuzmin model in the Einstein gauge: (a) Ring-like density profile; (b) Corresponding pressure profile. The corresponding rotation curve is presented in Fig. 1(c). The values for the parameters are given in Table 1.](image)

Since the circular velocity is gauge-invariant, any Kuzmin model with \(\beta \neq 1\) related to the present model via a gauge transformation will present the same rotation curves. However, from equation (11), we obtain \(\Pi(T^{\mu \nu}) = -4\), and therefore the density and pressure profiles of the disk are gauge-covariant but not gauge-invariant. This means that by a gauge transformation we can obtain a Kuzmin model with the same rotation curves but with different profiles for density and pressure, which will depend on the explicit form of the gauge transformations.

Concerning stability, the circular orbits calculated are all stable under small radial perturbations, according to the Rayleigh stability criterion for circular orbits \[ hh, R > 0, \] where the specific angular momentum \(h\) is given by Eq. (33). This is shown in Fig. 3.

6 Conclusions

We presented a Kuzmin-like model in the context of Israelit’s version of the Integrable Weyl-Dirac theory [29]. In particular, we considered the case in which the equations of motion for massive test particles reduce to Weyl (reparametrized) geodesics. In the Einstein gauge, the field equations in vacuum reduce to “GR + massless scalar field”. The rotation curves obtained are qualitatively identical to the general relativistic case for galactic scales, but there is a great variety of density profiles for the present model which produce the same rotation curve. In all cases analyzed, the
Figure 3: Specific angular momentum of the rotation curves of Fig. 1 as a function of coordinate radius $R$. All orbits satisfy Rayleigh stability criterion and therefore are stable under small radial perturbations. The values for the parameters of each curve are given in Table 1.

| Curve         | $a$ (kpc) | $m$ (pc) | $\gamma$ | $\lambda$ | $\alpha$      |
|---------------|-----------|----------|----------|------------|----------------|
| Solid (GR)    | 6.9       | 4.7x10^{-3} | 1        | 0          | 0              |
| Long dashed   | 6.9       | 3.0x10^{-3} | 0.5      | -0.5       | -9.375x10^{-2} |
| Dot-dashed    | 6.9       | 2.2x10^{-3} | 0.1      | -1         | -0.495         |
| Dotted        | 6.9       | 4.68x10^{-3} | 1.2      | 0.132      | 3.83x10^{-3}   |
| Short dashed  | 7.2       | 0.1      | 7.0x10^{-6} | -0.0255   | -3.25x10^{-4}  |
Table 2: Total masses of the disks for the parameters of Figs. 1 and 2

| Curve            | Total mass of the disk ($M_\odot$) |
|------------------|-----------------------------------|
| Solid (GR)       | $9.8 \times 10^{10}$             |
| Long dashed      | $3.13 \times 10^{10}$            |
| Dot-dashed       | $4.6 \times 10^{9}$              |
| Dotted           | $1.174 \times 10^{11}$           |
| Short dashed     | $7.38 \times 10^{6}$             |

Rayleigh stability criterion for the stability of circular orbits under small radial perturbations is satisfied. Furthermore, all rotation curves present Keplerian fall-off.

The main difference between the present W-D model and the general relativistic one is the existence of ring-like configurations (or almost ring-like configurations) with the same rotation curve as the GR model. We stress that, even though the density profile may nearly vanish near the galactic center if $\gamma \gtrsim m/2a$, the corresponding pressure profile has a maximum at $R = 0$ and decreases with galactocentric radius for every value of $\gamma$. The physical origin of this behavior is still unclear. Also, the total mass of the disk in this case is about four orders of magnitude lower than the GR one.

We conclude that the extension of the general relativistic Kuzmin-like model to Israelit’s theory does not introduce significant changes in the form of the rotation curves when compared to the GR model [8]. Moreover, a change of gauge can modify the density and pressure profiles ($T_\mu^\nu$ is gauge-covariant but not gauge-invariant in Israelit’s theory), but not the rotation curves: the circular velocity is gauge-invariant. In view of this fact, the construction of models of disk galaxies in this theory, as well as the obtention of the appropriate gauge function $\beta$, deserve further investigation (see [33] for a discussion about the apparent arbitrariness of this gauge function). It may also be the case that the equations of motion for masssive test particles in Israelit’s theory are in fact more general than Weyl geodesics, as originally proposed in [29]. In this case, the particles will have a $\beta$-charge $q_\beta \neq 0$, and therefore $q_\mu/m \neq -1$. As far as we know, the possible physical origins of these new charges $q_\mu$ and $q_\beta$ for ordinary matter remain unclear. Nevertheless, we consider the results presented here a first step towards a definitive answer for the above issues.

However, it is valuable to remark that the family of solutions obtained from the spherically symmetric solution in vacuum [38] has the property that for a given rotation curve there are many possible density and pressure profiles, a situation that does not arise neither in the Newtonian Kuzmin model [1] nor in its general relativistic extension [8]. Since in Newtonian gravitation the density profile of a razor-thin disk and its corresponding circular velocity profile are in a one-to-one correspondence (see [1], section 2.6.2), we see that the existence of a variety of disks with virtually the same rotation curve and qualitatively different density profiles, as presented in section 5, is a particular feature of the Weyl-Dirac theory studied here. Also, this theory allows the existence of an exact matching between rotation curves whose corresponding density profiles are very different.
These configurations may be obtained, for instance, by a suitable gauge transformation applied to the GR Kuzmin model \[8\]. In this way, the obtention of an appropriate gauge function $\beta$ for galactic scales is mandatory, as well as physical constraints on the charge $q_a$ – and, as a consequence, on $q_b$ – which determines the equations of motion for test particles $^{23}$.

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