Upper bounds for the length of non-associative algebras*

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Abstract

We obtain a sharp upper bound for the length of arbitrary non-associative algebra and present an example demonstrating the sharpness of our bound. To show this we introduce a new method of characteristic sequences based on linear algebra technique. This method provides an efficient tool for computing the length function in non-associative case. Then we apply the introduced method to obtain an upper bound for the length of an arbitrary locally complex algebra. We also show that the obtained bound is sharp. In the last case the length is bounded in terms of Fibonacci sequence.

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1 Introduction

In the present paper $\mathcal{A}$ is a unital finite dimensional not necessarily associative algebra over a field $F$. We refer the reader to [11, 16] for the background on the topic. Let $\mathcal{S} = \{a_1, \ldots, a_k\}$ be a finite subset of elements of the algebra $\mathcal{A}$. We define the length function of $\mathcal{S}$ as follows.

Any product of a finite number of elements of $\mathcal{S}$ is a word in letters from $\mathcal{S}$, or simply a word in $\mathcal{S}$. The length of the word equals to the number of letters in the corresponding product. We consider 1 as a word in $\mathcal{S}$ of the length 0.

It is worth noting that different choices of brackets provide different words of the same length due to the non-associativity of $\mathcal{A}$.

The set of all words in $\mathcal{S}$ with lengths less than or equal to $i$ is denoted by $S_i$, here $i \geq 0$.

Note that similar to the associative case, $m < n$ implies that $S_m \subseteq S_n$.

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The set $\mathcal{L}_i(S) = \langle S_i \rangle$ is the linear span of the set $S_i$ (the set of all finite linear combinations with coefficients belonging to $F$). We write $\mathcal{L}_i$ instead of $\mathcal{L}_i(S)$ if $S$ is determined from the context. It should be noted that $\mathcal{L}_0(S) = \langle 1 \rangle = F$ for any $S$. The set $\mathcal{L}(S)$ stands for $\bigcup_{i=0}^{\infty} \mathcal{L}_i(S)$.

**Remark 1.1.** $S$ is a generating set of $\mathcal{A}$ if and only if $\mathcal{A} = \mathcal{L}(S)$.

**Definition 1.2.** The length of a generating set $S$ of a finite-dimensional algebra $\mathcal{A}$ is defined as follows $l(S) = \min \{ k \in \mathbb{Z}_+ : \mathcal{L}_k(S) = \mathcal{A} \}$.

**Definition 1.3.** The length of an algebra $\mathcal{A}$ is $l(\mathcal{A}) = \max \{ l(S) : \mathcal{L}(S) = \mathcal{A} \}$.

The problem of the associative algebra length computation was first discussed in [14], [15] for the algebra of $3 \times 3$ matrices in the context of the mechanics of isotropic continua. The problem of computing the length of the full matrix algebra $M_n(F)$ as a function of the matrix size $n$ was stated in the work [13] and is still an open problem. The known upper bounds for the length of the matrix algebra are in general nonlinear in $n$.

The first upper bound on the length function was established in 1984 by Paz, see [13].

**Theorem 1.4 ([13, Theorem 1, Remark 1]).** Let $F$ be an arbitrary field. Then

$$ l(M_n(F)) \leq \left\lceil \frac{n^2 + 2}{3} \right\rceil, $$

where $\lceil \cdot \rceil$ denotes the least integer function.

An (asymptotic) improvement of this bound was obtained in [12]. More precisely, Pappacena in 1997 provided an upper bound for the length of any finite dimensional associative algebra $\mathcal{R}$ as a function of two its invariants: the dimension and $m(\mathcal{R})$, which is the maximal degree of the minimal polynomials for the elements of the algebra.

**Theorem 1.5 ([12, Theorem 3.1]).** Let $F$ be an arbitrary field, $\mathcal{R}$ be an associative $F$-algebra, and let

$$ f(d, m) = m \sqrt{\frac{2d}{m-1} + \frac{1}{4} + \frac{m}{2} - 2}. $$

Then $l(\mathcal{R}) < f(\dim \mathcal{R}, m(\mathcal{R}))$.

For the matrix algebra the theorem above provides a bound with asymptotic behavior $O(n^{3/2})$.

These bounds are not sharp. However, there are sharp bounds on the length function, which are established for certain classes of associative algebras. For example, the length of commutative matrix subalgebras of size $n$ is bounded by $(n-1)$, see [7, 10], where this bound was proved and in particular it was shown that the bound $(n-1)$ can be achieved on the algebra of diagonal matrices.
over an infinite field. In the recent paper [3] we evaluate the length function for quaternion and octonion algebras.

In this paper we obtain an upper bound for the length of arbitrary non-associative algebra and provide an example demonstrating that our bound on the length is sharp. Namely, let \( \mathcal{A} \) be an \( F \)-algebra, \( \dim \mathcal{A} = n > 2 \). We show that \( l(\mathcal{A}) \leq 2^{n-2} \) and provide an example of an \( n \)-dimensional algebra of the length exactly \( 2^{n-2} \).

To show this we introduce a new method to compute the length. We call it the method of characteristic sequences. This method is based on linear algebraic technique and provides an efficient tool for computing the length function in non-associative case. Then in addition we apply our method to obtain a sharp upper bound for the lengths of locally-complex algebras. Here Fibonacci sequence \( F_n = (0, 1, 1, 2, 3, 5, 8, \ldots) \) appears. Namely, we show that if \( \mathcal{B} \) is an \( n \)-dimensional locally complex \( F \)-algebra, then \( l(\mathcal{B}) \leq F_{n-1} \). Moreover, we demonstrate that for each \( n \) there exists \( n \)-dimensional locally complex \( F \)-algebra of the length exactly \( F_{n-1} \). This method as well as most of the results in the present paper is based on basic concepts from linear algebra such as linear span and properties of linear subspaces or their dimensions.

Observe that in associative case there are different attempts to find general methods to compute lengths of algebras and generating sets. These methods depend on the structure of algebra. Namely different methods are developed for matrix algebras containing some matrices of a special structure, group algebras of abelian or non-abelian groups, incidence algebras, see respectively [4, 5, 6, 9]. However, nothing similar to the method of characteristic sequences was known.

We note that characteristic sequences belong to the class of integer sequences named additive chains. These sequences are known since ancient times and had several reincarnations. The detailed and self-contained survey of this theory still containing lots of open problems can be found in [8, Chapter 4.6.3].

In the subsequent paper [2] we characterize all integer sequences that may serve as characteristic sequences for some non-associative algebras as well as characteristic sequences for locally complex algebras.

Our paper is organized as follows. In Section 2 we discuss some very general properties of the length function and differences between associative and non-associative cases. In Section 3 we introduce the characteristic sequence of a generating set of an algebra and investigate its general properties. Section 4 is devoted to establishing the upper bounds for the lengths of non-associative algebras. Section 5 reminds some basic properties of locally-complex algebras and adopts for them general results from Section 2. In Section 6 we use characteristic sequence to find the upper bounds for the lengths of locally-complex algebras and to prove its sharpness.

2 Properties of the length in non-associative case

Lemma 2.1. Suppose \( m, n \in \mathbb{N} \) are given such that \( m < n \). Then the following statements are equivalent:
1. \( L_n(S) = L_m(S) \)

2. \( \dim L_n(S) = \dim L_m(S) \).

Proof. The statement follows directly from the fact that \( L_k(S) \) is a linear subspace of \( \mathcal{A} \) and \( L_m(S) \subseteq L_n(S) \) for \( m < n \).

Lemma 2.2. Let \( \mathcal{A} \) be an algebra and \( S_0 \) and \( S_1 \) be its finite subsets such that \( L_1(S_0) \subseteq L_1(S_1) \). Then \( L_k(S_0) \subseteq L_k(S_1) \) for every positive integer \( k \).

Proof. We will prove this statement by induction on \( k \).

The base: for \( k = 1 \) the statement is given.

The step: Directly from definitions we get \( L_k(S) = \langle \bigcup_{i=1}^{k-1} L_i(S) \cdot L_{k-i}(S) \rangle \) for a generating set. Let us assume that for every \( k = 1, \ldots, n-1 \), \( L_k(S_0) \subseteq L_k(S_1) \).

Then \( L_n(S_0) = \langle \bigcup_{i=1}^{n-1} L_i(S_0) \cdot L_{n-i}(S_0) \rangle \subseteq \langle \bigcup_{i=1}^{n-1} L_i(S_1) \cdot L_{n-i}(S_1) \rangle = L_n(S_1) \).

Corollary 2.3. If \( \mathcal{A} \) is an algebra, \( S_0 \) and \( S_1 \) are generating sets such that \( L_1(S_0) \subseteq L_1(S_1) \), then \( \dim L(S_0) \geq \dim L(S_1) \).

Proof. By using the result above, \( L_i(S_i) \subseteq L_i(S_1) \) for every positive integer \( i \), hence \( \dim L(S_0) \geq \dim L(S_1) \).

Lemma 2.4. If \( \mathcal{A} \) is an algebra and \( S_0 \) and \( S_1 \) are its finite subsets such that \( L_1(S_0) = L_1(S_1) \), then \( L_k(S_0) = L_k(S_1) \) for every natural \( k \).

Proof. Follows from Lemma 2.2 by applying it twice.

Corollary 2.5. If \( \mathcal{A} \) is an algebra, \( S_0 \) is its generating set and \( S_1 \) is a finite subset such that \( L_1(S_0) = L_1(S_1) \), then \( S_1 \) generates \( \mathcal{A} \) and \( \dim L(S_0) = \dim L(S_1) \).

Proof. By using the result above, we get: \( L_i(S_0) \subseteq L_i(S_1) \) for every positive integer \( i \), while \( L_i(S_0) = L_i(S_1) \) for \( i \geq 3 \), which means that \( S_1 \) is a generating set of \( \mathcal{A} \) and its length is equal to \( \dim L(S_0) \).

Remark 2.6. Note that unlike the associative case the equality \( \dim L_n(S) = \dim L_{n+1}(S) \) for some \( n \in \mathbb{N} \) may not imply that \( \dim L_n(S) = \dim L_m(S) \) for all \( m \geq n \) as the following example shows.

Example 2.7. Let \( \mathcal{A} \) be generated by \( 1, e_1, e_2, e_3 \) with the multiplication rules \( e_i^2 = e_2, e_2^3 = 0 \) for all pairs \( i, j \in \{ 1, 2, 3 \} \). Then \( S = \{ e_1 \} \) is a generating system and we have \( L_1 = \langle 1 \rangle, L_2 = \langle 1, e_1 \rangle, L_3 = \langle 1, e_1, e_2 \rangle \), and \( L_4 = \langle 1, e_1, e_2, e_3 \rangle = \mathcal{A} \), so \( \dim L_4 = \dim L_3 = \dim L_2 \).

However, the following proposition, which belongs to folklore, is true for any non-associative algebra.

Proposition 2.8. Let us consider a finite subset \( S \) of \( \mathcal{A} \) and integer \( n \geq 1 \). If \( \dim L_n(S) = \dim L_{n+1}(S) = \ldots = \dim L_{2n}(S) \), then for all \( t \in \mathbb{N} \) it holds that \( \dim L_n(S) = \dim L_{n+t}(S) \)).
Proof. We prove this statement using the induction on $t$. The base is $t \leq n$. In this case the assertion holds by the conditions.

Suppose we have proven the statement for all $t \leq n + k$, $k \geq 0$. Let us show that it is satisfied for $t = n + k + 1$.

If $s$ is a word of the length $n + k + 1$, then it can be represented as a product of two words of smaller non-zero lengths, $s = (s_1)(s_2)$. Both these words are elements of $L_n(S)$. Indeed, if the length of a word is less than or equal to $n$, then it is an element of $L_n(S)$ by the definition of $L_n(S)$. If the length of a word is greater than $n$, but strictly less than $n + k + 1$, then the inclusion follows from the induction hypothesis. In any case $s_1, s_2 \in L_n(S)$. Hence, $s \in L_n(S) \cdot L_n(S) \subseteq L_{2n}(S) = L_n(S)$.

This implies $L_{n+k+1}(S) = L_n(S)$, because the linear space $L_{n+k+1}(S)$ is generated by all words of length less than or equal to $n + k + 1$, which concludes the induction proof.

Remark 2.9. Example 2.7 above shows also that the assumption $\dim L_n(S) = \dim L_{n+1}(S) = \ldots = \dim L_{2n}(S)$ is indispensable in the case $n = 2$. For bigger $n$ we can extend this example in the following way in order to show that it is impossible to shorten the chain of equalities in the conditions of Proposition 2.8 even by 1.

Example 2.10. Let $A$ be generated by $1, e_1, e_2, \ldots, e_n, e_{n+1}$ with the multiplication rules

$$e_1^2 = e_2, e_1 e_i = e_{i+1}, e_i e_1 = 0, i = 2, \ldots, n-1, e_n^2 = e_{n+1},$$

$$e_{n+1}^2 = 0, e_i e_j = 0 \text{ for all pairs } i, j \in \{2, \ldots, n+1\}, i \neq j.$$ 

Then $S = \{e_1\}$ is a generating system and we have

$$L_1(S) = \langle 1, e_1 \rangle, \quad L_2(S) = \langle 1, e_1, e_2 \rangle, \quad \ldots, L_n(S) = \langle 1, e_1, \ldots, e_n \rangle,$$

further, $L_{2n-1}(S) = L_{2n-2}(S) = \ldots = L_n(S)$, but

$$L_{2n}(S) = \langle 1, e_1, \ldots, e_{n+1} \rangle \neq L_n(S).$$

For an arbitrary not necessarily associative algebra of dimension $n$ over $F$ we can achieve a sharp bound of length, as will be proven below. The key element of the proof of this bound is the concept of a fresh word.

Definition 2.11. A word $w$ of the length $n$ from generating set $S$ of algebra $A$ is a fresh word, if for all integer $m$, $0 \leq m < n$, it holds that $w \notin L_m(S)$.

Lemma 2.12. A fresh word of the length greater than 1 is a product of two fresh words of non-zero lengths.
Proof. Let us consider a word \( w \) of the length greater than 1. It can be represented as a product of two words \( s \) and \( t \) of the lengths \( a, b > 0 \), respectively. Let us assume that \( s \) is not fresh.

Then there exists \( a' \in \mathbb{N} \cup \{0\} \), which satisfies \( a' < a \) and \( s \in L_{a'}(S) \). Hence, \( w \) which is a word of the length \( a + b \), belongs to \( L_{a'+b}(S) \) and, by the definition, it is not fresh. It can be shown in the same way that if \( t \) is not fresh, then \( w \) is not fresh as well, which proves the statement of the lemma.

3 Characteristic sequences and their basic properties

In this section we introduce our main tool actual for all further considerations.

Definition 3.1. Consider a unital \( \mathbb{F} \)-algebra \( A \) of the dimension \( \dim A = n \), and its generating set \( S \). By the characteristic sequence of \( S \) in \( A \) we understand a monotonically non-decreasing sequence of natural numbers \( (m_0, m_1, \ldots, m_N) \), constructed by the following rules:

1. \( m_0 = 0 \).
2. Denoting \( s_1 = \dim L_1(S) - 1 \), we define \( m_1 = \ldots = m_{s_1} = 1 \).
3. If \( m_0, \ldots, m_r \) are already constructed and the sets \( L_1(S), \ldots, L_{k-1}(S) \) are considered, then we inductively continue the process in the following way. Denote \( s_k = \dim L_k(S) - \dim L_{k-1}(S) \). Then \( m_{r+1} = \ldots = m_{r+s_k} = k \).

Remark 3.2. In other words, to construct a characteristic sequence we start with \( m_0 = 0 \) and for each \( k = 0, \ldots, l(A) \) we add \( (\dim L_k(S) - \dim L_{k-1}(S)) \) elements equal to \( k \).

Remark 3.3. It is worth noting that in the associative case subsequent elements of a characteristic sequence are either equal or differ by 1, since for a generating set \( S \), \( \dim L_k(S) - \dim L_{k-1}(S) = 0 \) implies that for every integer \( h > k \), \( \dim L_h(S) - \dim L_{h-1}(S) = 0 \).

Lemma 3.4. Let \( A \) be an \( \mathbb{F} \)-algebra, \( \dim A = n > 2 \), and \( S \) be a generating set for \( A \). Then

1. Positive integer \( k \) appears in the characteristic sequence as many times as many there are linearly independent fresh words of the length \( k \).
2. For any term \( m_h \) of the characteristic sequence of \( S \) there is a fresh word in \( L(S) \) of the length \( m_h \).
3. If there is a fresh word in \( L(S) \) of the length \( k \), then \( k \) is included into the characteristic sequence of \( S \).

Proof. 1. Fresh words of lengths less than or equal to \( k \) form a basis of \( L_k(S) \), therefore the number of fresh words of the length exactly \( k \) is equal to \( \dim L_k(S) - \dim L_{k-1}(S) \).
2. Follows directly from 1.
3. Follows from the proof of 1.

\[ \square \]
Lemma 3.5. Let $A$ be an $\mathcal{F}$-algebra, $\dim A = n > 2$, and $S$ be a generating set for $A$. Then the characteristic sequence of $S$ contains $n$ terms, i.e., $N = n - 1$. Moreover, $m_N = l(S)$.

Proof. By the definition for each $k = 1, \ldots, l(S)$ on $k$th step we add $(\dim L_k(S) - \dim L_{k-1}(S))$ terms to the characteristic sequence. Hence, the total number of terms is

$$1 + (\dim L_1(S) - 1) + \ldots + (\dim L_k(S) - \dim L_{k-1}(S)) + \ldots$$

$$+ (\dim L_{l(S)}(S) - \dim L_{l(S)-1}(S)) = \dim L(S) = n$$

since $S$ is a generating set. Also, by Definition 2.2, the maximal $k$ such that $\dim L_k(S) - \dim L_{k-1}(S) > 0$ is $l(S)$, hence, by Definition 3.1, we obtain $m_N = l(S)$.

Lemma 3.6. Let $A$ be an $\mathcal{F}$-algebra, $\dim A = n > 2$. Assume, $S$ is a generating set of $A$, and $(m_0, m_1, \ldots, m_{n-1})$ is a characteristic sequence of $S$. Then for any integer $k \geq 0$ it holds that $dim L_k(S) = \max\{t|m_t \leq k\} + 1$.

Proof. We use the induction on $k$.

Induction base. For $k = 0$ the statement is trivial.

Induction step. Let us assume that the statement is true for $k = q$. Then for $k = q + 1$ one has $\dim L_{q+1}(S) = (\dim L_{q+1}(S) - \dim L_q(S)) + \dim L_q(S)$. By Definition 3.1 the summand $(\dim L_{q+1}(S) - \dim L_q(S))$ equals to the number $N_0$ of terms $(q + 1)$ in the characteristic sequence. By the induction hypothesis $N_1 = \dim L_q(S) = \max\{t|m_t \leq q\} + 1$, i.e., the increased by 1 index of the last position in which $m_t \leq q$. By Definition 3.1 the sum $N_0 + N_1$ equals to the increased by 1 index of the last position in which $m_t \leq q + 1$, or $\max\{t|m_t \leq q + 1\} + 1$.

Proposition 3.7. Let $A$ be an $\mathcal{F}$-algebra, $\dim A = n > 2$. Assume, $S$ is a generating set for $A$ and $(m_0, m_1, \ldots, m_{n-1})$ is the characteristic sequence of $S$. Then for each $h$ satisfying $m_h \geq 2$ it holds that there are indices $0 < t_1 \leq t_2 < h$ such that $m_h = m_{t_1} + m_{t_2}$.

Proof. By Lemma 3.4 Item 1 each term $m_h$ of the characteristic sequence corresponds to a fresh word of the length $m_h$, denote it by $w_{m_h}$. By Lemma 2.12 each fresh word of the length $m_h \geq 2$ can be represented as a product of two fresh words, possibly equal, of lesser lengths. Thus, $w_{m_h} = w_{k_1} \cdot w_{k_2}$ for some fresh words $w_{k_1}, w_{k_2}$ of the lengths $k_1, k_2$, correspondingly. Assume $k_1 \leq k_2$. Then by Lemma 3.4 Item 3 there are indices $0 < t_1 \leq t_2 < h$ such that $m_{t_1} = k_1$ and $m_{t_2} = k_2$. Assume $k_1 > k_2$. Then by Lemma 3.4 Item 3 there are indices $0 < t_1 \leq t_2 < h$ such that $m_{t_1} = k_2$ and $m_{t_2} = k_1$. In both cases, the additivity of word length concludes the proof.
4 Upper bound for the lengths of non-associative algebras

Theorem 4.1. Let \( A \) be an \( F \)-algebra of the dimension \( \dim A = n, n > 2 \), \( S \) be a generating set for \( A \), \((m_0, m_1, \ldots, m_{n-1})\) be the characteristic sequence of \( S \). Then for each positive integer \( h \leq n - 1 \) it holds that \( m_h \leq 2^{h-1} \).

Proof. We prove this statement using the induction on \( h \).

The base. Case \( h = 1 \) is trivial since \( m_1 = 1 \leq 2^0 \).

The step. Let us assume that for all positive integers \( k \) such that \( h \leq k < n - 1 \) the statement holds. We have to prove it now for \( h = k + 1 \leq n - 1 \).

By Proposition 3.7 we have \( m_{k+1} = m_{t_1} + m_{t_2} \), where \( 0 < t_1 \leq t_2 < k + 1 \). According to the induction hypothesis,

\[
m_{t_1} + m_{t_2} \leq 2^{t_1-1} + 2^{t_2-1} \leq 2^{k-1} + 2^{k-1} = 2^k,
\]

which concludes the proof.

Proposition 4.2. Let \( A \) be an \( F \)-algebra, \( \dim A = n > 2 \). Then \( l(A) \leq 2^{n-2} \).

Proof. Let \( S \) be an arbitrary generating set of \( A \). By Lemma 3.5 the length \( l(S) \) is equal to the last element of characteristic sequence of \( S \). The index of this element is \( \dim A - 1 = n - 1 \). Hence by Theorem 4.1 we get \( l(S) \leq 2^{(n-1)-1} = 2^{n-2} \).

The example below demonstrates that the obtained bound is sharp.

Example 4.3. Let us consider an arbitrary field \( F \) and non-associative \( F \)-algebra \( A \) of the dimension \( n > 2 \) with the basis \( \{e_0 = 1, e_1, \ldots, e_{n-1}\} \) and following multiplication rules: for every \( k \) such that \( 1 \leq k \leq n - 2 \) we take \( e_k^2 = e_{k+1}, e_{n-1}^2 = 0 \), and for all \( p, q, p \neq q, 1 \leq p, q \leq n - 1 \), \( e_p e_q = 0 \).

Then the set \( S = \{e_1\} \) generates the algebra \( A \). Its characteristic sequence is exactly \((0, 1, 2, \ldots, 2^{n-3}, 2^{n-2})\), since each new fresh word, except the first one, is the square of the previous one. Then by Lemma 3.5 we have \( l(S) = 2^{n-2} \).

Since any generating set of \( A \) should contain \( e_1 \), we get \( l(A) = l(S) = 2^{n-2} \). Since by the previous proposition \( l(A) \leq 2^{n-2} \) and in general \( l(A) \geq l(S) \) we get \( l(A) = l(S) = 2^{n-2} \).

5 Basic properties of locally-complex algebras and their length

The class of locally-complex algebras provides a natural generalization of the field of complex numbers \( \mathbb{C} \), namely

Definition 5.1. \( A \) is a locally-complex algebra, if it is finitely generated non-associative algebra over the field \( \mathbb{R} \), such that any 1-generated subalgebra of \( A \), which is generated by an element of \( A \setminus \mathbb{R} \), is isomorphic to the field of complex numbers, \( \mathbb{C} \).
These algebras were introduced and investigated in [1]. In this work we deal with the following equivalent definitions of locally-complex algebras, established in [1].

**Lemma 5.2** ([1] Lemma 4.1). The following conditions are equivalent for a real unital algebra $A$:

1. $A$ is locally-complex;
2. every $0 \neq a \in A$ has a multiplicative inverse lying in $\mathbb{R}a + \mathbb{R}$;
3. $A$ is quadratic and $A$ has no nontrivial idempotents or square-zero elements;
4. $A$ is quadratic and $n(a) > 0$ for every $0 \neq a \in A$.

Moreover, if $2 \leq \dim A = n < \infty$, then (1)-(4) are equivalent to

5. $A$ has a basis $\{1, e_1, \ldots, e_{n-1}\}$ such that $e_i^2 = -1$ for all $i$ and $e_ie_j = -e_je_i$ for all $i \neq j$.

Here by a quadratic algebra we understand such a unital $\mathbb{R}$-algebra $A$ that for every $a \in A$ the elements $1, a$ and $a^2$ are linearly dependent. By $n(a)$ we understand $a^2$ if $a \in \mathbb{R}$ and a real number $n$ such that $a^2 - t(a)a + n = 0$ ($t(a) \in \mathbb{R}$) if $a \notin \mathbb{R}$. Since we mainly use multiplication tables as a way of defining algebras, the property (v) is the one which is the most relevant for the present paper.

For locally-complex algebras we can improve the bound obtained in Proposition 5.3.

**Proposition 5.3.** Let us consider a finite subset $S$ of a locally-complex algebra $A$ and an integer $n \geq 2$. Let

(i) $\dim L_{n-1}(S) + 1 = \dim L_n(S)$, and  
(ii) $\dim L_n(S) = \dim L_{n+1}(S) = \ldots = \dim L_{2n-1}(S)$.

Then for all $t \in \mathbb{N}$ it holds that $\dim L_n(S) = \dim L_{n+t}(S)$.

**Proof.** By Proposition 2.8 it is sufficient to show that in these conditions $\dim L_n(S) = \dim L_{2n}(S)$.

Let us consider $w$, a word of the length $2n$. It can be represented as a product $w = st$ of two words $s,t$ such that $l(s) = k > 0$ and $l(t) = 2n - k > 0$. Without loss of generality $k \leq n$. If $k < n$, then, since $t$ is an element of $L_{2n-k}(S) = L_n(S)$, $w$ belongs to $L_{n+k}(S) = L_n(S)$. If $k = n$, then $s$ and $t$ can be represented as $s = s_0 + r_sc$ and $t = t_0 + r_tc$ respectively, where $s_0, t_0 \in L_{n-1}(S)$, $r_s, r_t \in \mathbb{R}$ and $c$ is an element of $L_n(S) \setminus L_{n-1}(S)$. This representation is correct since $\dim L_{n-1}(S)+1 = \dim L_n(S)$, and, hence, $c$ with the basis of $L_{n-1}(S)$ composes the basis of $L_n(S)$.

It follows that

$$w = st = (s_0 + r_sc)(t_0 + r_tc) = s_0t_0 + (r_sc)t_0 + s_0(r_tc) + (r_sc)(r_tc)$$
The coefficients $r_i$ and $r_s$ are real numbers, which allows us to omit the brackets and rearrange the order of multiplication. Thus, we get

$$w = s_0 t_0 + r_s t_0 + r_t s_0 c + r_s r_t c^2.$$ 

Let us consider each term separately.

$s_0 t_0 \in L_{2n-2}(S)$ as a product of two elements of $L_{n-1}(S)$.

1. $r_s t_0$ and $r_t s_0$ are elements of $L_{2n-1}(S)$ as products of a real number, an element of $L_{n-1}(S)$ and an element of $L_n(S)$.

2. $r_s r_t c^2$ is an element of $L_n(S)$, since $c^2 = u + v$, where $u, v \in \mathbb{R}$, as $A$ is locally-complex.

Since both $L_{2n-2}(S)$ and $L_{2n-1}(S)$ are equal to $L_n(S)$ by the conditions, it follows that $w$ is an element of $L_n(S)$. This implies $L_{2n}(S) = L_n(S)$. \qed

Next example shows that to improve Proposition 5.3 both additional conditions, that $A$ is locally-complex and condition (i), are necessary.

**Example 5.4.** Let $A$ be generated by $1, e_1, e_2, \ldots, e_6$ with the multiplication given by

$$e_1 e_2 = e_4 = -e_2 e_1, e_1 e_3 = e_5 = -e_3 e_1, e_4 e_5 = e_6 = -e_5 e_4, e_i^2 = -1, i = 1, \ldots, 6,$$

and all other products are zero. Then for $S = \{e_1, e_2, e_3\}$ we have that $\dim L_1(S) = 4, \dim L_2(S) = 6 = \dim L_3(S), \text{ but } \dim L_4(S) = 7$.

For $n > 2$ we consider the algebra $A$ generated by $1, e_1, e_2, \ldots, e_{n+3}$ with the multiplication rules $e_1 e_2 = e_3 = -e_2 e_1, e_1 e_3 = e_4 = -e_3 e_1, \ldots,$

$$e_1 e_{n-1} = e_n = -e_{n-1} e_1, e_1 e_n = e_{n+1} = -e_n e_1, e_2 e_n = e_{n+2} = -e_n e_2,$$

$e_{n+1} e_{n+2} = e_{n+3} = -e_{n+2} e_{n+1}, e_i^2 = -1, i = 1, \ldots, n + 3, \text{ and all other products are zero.}$ Then for $S = \{e_1, e_2\}$ we have $\dim L_{n-1}(S) = n + 1, \dim L_n(S) = \ldots = \dim L_{2n-1}(S) = n + 3, \text{ but } \dim L_{2n}(S) = n + 4$. So, comparing with Proposition 5.3 we see that (i) is not satisfied, (ii) is satisfied, and the result does not hold. It is straightforward to see that the introduced algebra $A$ is locally-complex.

Example 2.10 shows that conditions (i) and (ii) are not sufficient if $A$ is not locally complex.

**Corollary 5.5.** Let $S$ be a finite generating set of a locally-complex algebra $A$ and $n \geq 2$ be an integer. Assume that $\dim L_{n-1}(S) < \dim L_n(S)$ and $\dim L_{n-1}(S) \leq \dim A - 2$. Then $\dim L_{n-1}(S) \leq \dim L_{2n-1}(S) - 2$.

**Proof.** Assume the opposite that $\dim L_{n-1}(S) > \dim L_{2n-1}(S) - 2$. Then there are the following possibilities for $\dim L_{n-1}(S)$:

1. $\dim L_{n-1}(S) < \dim L_n(S) - 1$, or, in other words, $\dim L_{n-1}(S) \leq \dim L_n(S) - 2$. Since $\dim L_n(S) \leq \dim L_{2n-1}(S)$, we get $\dim L_{n-1}(S) \leq \dim L_{2n-1}(S) - 2$.  

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2. \( \dim \mathcal{L}_{n-1}(S) = \dim \mathcal{L}_n(S) - 1 \). By Proposition \( \ref{prop:dim_difference} \) if \( \dim \mathcal{L}_{2n+1}(S) = \dim \mathcal{L}_n(S) \), then \( \mathcal{L}(S) = \mathcal{L}_n(S) \). On the other hand,

\[
\dim \mathcal{L}_n(S) = \dim \mathcal{L}_{n-1}(S) + 1 \leq \dim \mathcal{A} - 2 + 1 < \dim \mathcal{A}.
\]

However, this contradicts to Remark \( \ref{rem:dim_contradiction} \) namely, \( \mathcal{L}(S) = \mathcal{A} \). Thus our assumption is wrong and \( \dim \mathcal{L}_{2n+1}(S) \geq \dim \mathcal{L}_n(S) + 1 = \dim \mathcal{L}_{n-1}(S) + 2 \).

**Proposition 5.6.** Let \( \mathcal{A} \) be a locally-complex algebra of the dimension \( \dim \mathcal{A} = n, n > 2 \). Assume \( \mathcal{S} \) is a generating set for \( \mathcal{A} \) and \( (m_0, m_1, \ldots, m_{n-1}) \) is the characteristic sequence of \( \mathcal{S} \). Then for each \( h \) satisfying \( m_h \geq 2 \) it holds that there are indices \( 0 < t_1 < t_2 < h \) such that \( m_h = m_{t_1} + m_{t_2} \).

**Proof.** By Lemma \( \ref{lem:char_sequence} \) Item 1 each term \( m_h \) of the characteristic sequence corresponds to a fresh word of the length \( m_h \), denote it by \( w_{m_h} \). By Lemma \( \ref{lem:word_representation} \) each fresh word of the length \( m_h \geq 2 \) can be represented as a product of two fresh words, possibly equal, of lesser lengths. Thus, \( w_{m_h} = w_{k_1} \cdot w_{k_2} \) for some fresh words \( w_{k_1}, w_{k_2} \) of the lengths \( k_1, k_2 \), correspondingly. We consider three cases separately.

1. Assume \( k_1 < k_2 \). Then by Lemma \( \ref{lem:char_sequence} \) Item 3 there are indices \( 0 < t_1 < t_2 < h \) such that \( m_{t_1} = k_1 \) and \( m_{t_2} = k_2 \).

2. Assume \( k_1 > k_2 \). Then by Lemma \( \ref{lem:char_sequence} \) Item 3 there are indices \( 0 < t_1 < t_2 < h \) such that \( m_{t_1} = k_2 \) and \( m_{t_2} = k_1 \).

3. Assume \( k_1 = k_2 \). If \( w_{k_1} \) and \( w_{k_2} \) are not linearly independent modulo \( \mathbb{R} \), then there exist real numbers \( r_0, r_1, r_2 \) such that \( r_0 + r_1 w_{k_1} + r_2 w_{k_2} = 0 \) and \( r_0^2 + r_1^2 + r_2^2 \neq 0 \). Obviously, at least one of \( r_1 \) and \( r_2 \) is non-zero. Let us assume that it is \( r_2 \), the second case is similar. It follows that \( w_{k_2} = r_0^* + r_1^* w_{k_1} \), where \( r_0^* = -r_0/r_2, r_1^* = -r_1/r_2 \). Hence,

\[
w_{m_h} = w_{k_1} w_{k_2} = w_{k_1}(r_0^* + r_1^* w_{k_1}) = r_0^* w_{k_1} + r_1^* w_{k_1}^2.
\]

Note that since \( \mathcal{A} \) is locally-complex, \( w_{k_1}^2 \in \langle 1, w_{k_1} \rangle \), thus \( w_{m_h} \in \langle 1, w_{k_1} \rangle \). However, this contradicts to the fact that \( w_{m_h} \) is fresh. Thus, by Lemma \( \ref{lem:char_sequence} \) Items 1 and 3, there are at least two distinct indices \( t_1 \) and \( t_2 \) such that \( m_{t_1} = m_{t_2} = k_1 = k_2 \).

In all cases, the additivity of word length concludes the proof.

**Remark 5.7.** Observe that we proved that characteristic sequences of locally complex algebras belong to the class of additive chain without doubling, see \( \cite{8} \), i.e., each term is a sum of different previous terms.

This small difference with Proposition \( \ref{prop:chain_difference} \) allows us to improve the upper bound on the length function established in Theorem \( \ref{thm:length_bound} \) for general algebras in the case of locally-complex algebras.
6 Upper bound for the lengths of locally-complex algebras

Definition 6.1. Let \( \mathcal{F}_n = (F_1, \ldots, F_n, \ldots) \) denote the Fibonacci sequence, i.e. the sequence of positive integers satisfying the recurrent relations \( F_1 = F_2 = 1, \ F_i = F_{i-1} + F_{i-2} \) for all \( i \geq 3 \).

Theorem 6.2. Let \( \mathcal{A} \) be a locally-complex algebra of the dimension \( \dim \mathcal{A} = n, n > 2 \). Assume \( S \) is a generating set of \( \mathcal{A} \), and \( (m_0, m_1, \ldots, m_{n-1}) \) is the characteristic sequence of \( S \). Then for each positive integer \( h \leq n-1 \) it holds that \( m_h \leq F_h \).

Proof. We prove this statement using the induction on \( h \).

The base. If \( h = 1 \), then \( m_1 = 1 \leq F_1 \).

The step. Assume that for all positive integers \( h = 1, \ldots, k \) the statement holds. We have to prove it now for \( h = k + 1 \leq n - 1 \). By Proposition 5.6 \( m_{k+1} = m_{t_1} + m_{t_2} \), where \( 0 < t_1 < t_2 < k + 1 \). According to the induction hypothesis,

\[
m_{t_1} + m_{t_2} \leq F_{t_1} + F_{t_2} \leq F_{k-1} + F_k = F_{k+1},
\]

which concludes the proof.

Now we can prove our main result.

Theorem 6.3. Let \( \mathcal{A} \) be a locally-complex algebra of the dimension \( \dim \mathcal{A} = n \), \( n > 2 \). Then the length of \( \mathcal{A} \) is less than or equal to the \( (n-1) \)-th Fibonacci number \( F_{n-1} \).

Proof. Let \( S \) be an arbitrary generating set of \( \mathcal{A} \), and \( (m_0, m_1, \ldots, m_{n-1}) \) be its characteristic sequence. By Lemma 3.5 and Theorem 6.2 \( l(S) = m_{n-1} \leq F_{n-1} \). Hence, length of \( \mathcal{A} \) is less than or equal to \( F_{n-1} \).

Proposition 6.4. If \( \mathcal{A} \) is a locally-complex algebra of the dimension \( \dim \mathcal{A} = n \) and \( S \) is its generating set, containing \( k \) linearly independent modulo \( \mathbb{R} \) elements, then \( l(S) \leq F_{n-k+1} \).

Proof. Let \( (m_0, \ldots, m_{n-1}) \) be the characteristic sequence of \( S \). It should be noted that \( m_1 = \ldots = m_k = 1 \), since \( \dim \mathcal{L}_1(S) - \dim \mathcal{L}_0(S) = k \). We use the induction to prove that \( m_{k+h} \leq F_{h+2} \) for all integer \( h, -1 \leq h \leq n - k - 1 \).

The base. For \( h = -1 \) and \( h = 0 \) one has \( m_{k-1} = m_k = 1 = F_1 = F_2 \).

The step. Let us assume that for \( h = -1, 0, \ldots, d \) the statement holds. We have to prove now that it holds for \( h = d + 1 \leq n - k - 1 \). By Proposition 5.6 we have \( m_{k+d+1} = m_{t_1} + m_{t_2} \), where \( 0 < t_1 < t_2 < k + d + 1 \). According to the induction hypothesis,

\[
m_{t_1} + m_{t_2} \leq F_{t_1-k+2} + F_{t_2-k+2} \leq F_{d+1} + F_{d+2} = F_{d+3},
\]

which concludes the proof.
The example below demonstrates that the obtained bound is sharp in the class of locally-complex algebras.

**Example 6.5.** Let us consider locally-complex algebra \( \mathcal{A} \) over real numbers with basis \( \{e_0 = 1, \ldots, e_{n-1}\} (n > 2) \) and following multiplication rule: for every \( k, \) such that \( 1 \leq k \leq n-3 \)

\[
e_k e_{k+1} = e_{k+2}
\]

\[
e_{k+1} e_k = -e_{k+2},
\]

for every \( m, \) such that \( 1 \leq m \leq n-1 \)

\[
e_m e_m = -1,
\]

and for other combinations of \( p, q: \) \( 1 \leq p, q \leq n-1 \)

\[
e_p e_q = 0.
\]

The set \( \mathcal{S} = \{e_1, e_2\} \) generates \( \mathcal{A}, \) and its characteristic sequence is exactly \((0, 1, 1, 2, \ldots, F_{n-1})\), since every fresh word is obtained as a product of two previous fresh words. We get \( F_{n-1} = l(\mathcal{S}) \leq l(\mathcal{A}) \leq F_{n-1}, \) which means \( l(\mathcal{A}) = F_{n-1}. \)

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