Complete objects in categories

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Abstract

We introduce the notions of proto-complete, complete, complete* and strong-complete objects in pointed categories. We show under mild conditions on a pointed exact protomodular category that every proto-complete (respectively complete) object is the product of an abelian proto-complete (respectively complete) object and a strong-complete object. This together with the observation that the trivial group is the only abelian complete group recovers a theorem of Baer classifying complete groups. In addition we generalize several theorems about groups (subgroups) with trivial center (respectively, centralizer), and provide a categorical explanation behind why the derivation algebra of a perfect Lie algebra with trivial center and the automorphism group of a non-abelian (characteristically) simple group are strong-complete.

1 Introduction

Recall that Carmichael [19] called a group $G$ complete if it has trivial center and each automorphism is inner. For each group $G$ there is a canonical homomorphism $c_G$ from $G$ to Aut($G$), the automorphism group of $G$. This homomorphism assigns to each $g$ in $G$ the inner automorphism which sends each $x$ in $G$ to $gxg^{-1}$. It can be readily seen that a group $G$ is complete if and only if $c_G$ is an isomorphism. Baer [11] showed that a group $G$ is complete if and only if every normal monomorphism with domain $G$ is a split monomorphism. We call an object in a pointed category complete if it satisfies this latter condition. Completeness, which corresponds to being injective in abelian categories, has been studied in other contexts (although not always under that name) and as explained by B. J. Gardner in [23] the following is known. Completeness corresponds to:

1. having trivial center (annihilator) and each derivation being inner in each category of Lie algebras;

2. having multiplicative identity in each category of associative algebras;

and follows from:
3. the existence of a multiplicative identity in each category of alternative algebras and each category of autodistributive algebras.

Note that complete objects need not be injective nor absolute retracts, in fact Baer showed in [1] that there are no non-trivial absolute retracts of groups.

One of the main purposes of this paper is to introduce and study four (three main) notions of completeness, and to give a categorical explanation of Baer’s result mentioned above. In addition to completeness which we have already defined above we call an object \( X \) **proto-complete** if every normal monomorphism with domain \( X \) which is the kernel of a split epimorphism is a split monomorphism, and **strong-complete** if it satisfies the same condition except with the additional requirement that the normal monomorphisms in question are required to have a unique splitting. The forth notion is obtain by replacing normal by **Bourn-normal** in the definition of completeness. Let us immediately mention that pointed protomodular categories [10] in which every object is strong-complete turn out to be precisely what F. Borceux and D. Bourn in [4] called coarsely action representable. In the pointed protomodular context we show that strong-completeness implies completeness (Proposition 4.17), and that every proto-complete (respectively complete) object, satisfying certain additional conditions which automatically hold in every such variety of universal algebras, is the product of an abelian proto-complete (respectively complete) object and a strong-complete object (Theorem 4.10). We show that a partial converse to the previous fact holds (Proposition 4.17). We give classifications of proto-completeness and strong-completeness (see Theorems 4.19 and 4.25) relating to the existence of generic split extensions in the sense of [7], which are closely related to Problem 6 of the open problems of [8]. For a group \( G \) these theorems imply: (a) \( G \) is proto-complete if and only if \( c_G : G \to \text{Aut}(G) \) is a normal monomorphism with trivial centralizer (Proposition 3.6); (b) \( G \) is strong-complete (= complete) if and only if \( c_G \) is an isomorphism.

Other aims include a brief study of objects with trivial center and of subobjects with trivial centralizer, in Section 3 and to study characteristic monomorphism and their interaction with completeness in Section 5. The main results of Section 3 applied to the category of groups, recover the following known facts about groups:

(i) If \( G \) has trivial center, then \( c_G : G \to \text{Aut}(G) \) is a normal monomorphism with trivial centralizer (Proposition 3.6);

(ii) If \( n : N \to G \) is a normal monomorphism with trivial centralizer, then each automorphism of \( N \) admits at most one extension to \( G \) (Proposition 3.7).

In Section 5 we provide a common categorical explanation behind why the derivation algebra of a perfect Lie algebra with trivial center, and the automorphism group of a (characteristically) simple group are (strong) complete. This explanation depends on several facts including: (a) two new characterizations of characteristic monomorphisms with domain satisfying certain conditions (see Theorems 5.2 and 5.3); (b) Theorem 5.3 which generalizes the following fact for
a group $G$: the homomorphism $c_G : G \to \text{Aut}(G)$ is a characteristic monomorphism if and only if $G$ has trivial center and $\text{Aut}(G)$ is (strong) complete.

2 Preliminaries

In this section we recall preliminary definitions and introduce some notation.

Let $C$ be a pointed category with finite limits. We will write $0$ for both the zero object and for a zero morphism between objects. For objects $A$ and $B$ we will write $A \times B$ for the product of $A$ and $B$, and write $\pi_1$ and $\pi_2$ for the first and second product projections, respectively. For a pair of morphisms $f : W \to A$ and $g : W \to B$ we will write $x_{f,g} : W \to A \times B$ for the unique morphism with $\pi_1 x_{f,g} = f$ and $\pi_2 x_{f,g} = g$. For objects $A$ and $B$ we write $[u,v] : A + B \to Z$ for the unique morphism with $[u,v] \pi_1 = u$ and $[u,v] \pi_2 = v$.

The category $C$ is called unital [3] if for each pair of objects $A$ and $B$ the morphisms $(1,0) : A \to A \times B$ and $(0,1) : B \to A \times B$ are jointly strongly epimorphic. When these morphism are only jointly epimorphic $C$ is called weakly unital [32]. A pair of morphisms $f : A \to X$ and $g : B \to X$ in a (weakly) unital category are said to (Huh)-commute if there exists a (unique) morphism $\varphi : A \times B \to X$ making the diagram commute. Let us recall some well-known facts and definitions related to the commutes relation (see e.g. [3], and the references there for the unital context, and [24, 26] for the weakly unital context.)

Lemma 2.1. Let $C$ be a weakly unital category and let $e : S \to A$, $f : A \to X$, $g : B \to X$, $f' : A' \to X'$, $g' : B' \to X'$ and $h : X \to Y$ be morphisms in $C$, then

(i) $f$ and $g$ commute if and only if $g$ and $f$ commute;

(ii) if $f$ and $g$ commute, then $fe$ and $g$ commute;

(iii) if $f$ and $g$ commute, then $hf$ and $hg$ commute;

(iv) $f \times f'$ and $g \times g'$ commute if and only if $f$ and $g$, and $f'$ and $g'$ commute.

Moreover, the converse of (ii) holds when $e$ is a pullback stable regular epimorphism, while the converse of (iii) holds when $C$ is unital and $h$ is a monomorphism.
Definition 2.2. Let $\mathcal{C}$ be a weakly unital category. The centralizer of a morphism $f : A \to B$ is the terminal object in the category of morphisms commuting with $f$.

We will write $z_f : Z_X(A, f) \to X$ for the centralizer of $f$ when it exits. Note that $z_f$ is always a monomorphism. When $f = 1_X$ the centralizer of $f$ is called the center of $X$ and will be denoted $z_X : Z(X) \to X$.

A split extension in $\mathcal{C}$ is a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\kappa} & A \\
\downarrow & & \downarrow \alpha \\
A' & \xrightarrow{\alpha'} & B'
\end{array}
$$

in $\mathcal{C}$ where $\kappa$ is the kernel of $\alpha$ and $\alpha \beta = 1_B$. A morphism of split extensions in $\mathcal{C}$ is a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\kappa} & A \\
\downarrow & & \downarrow \alpha \\
A' & \xrightarrow{\alpha'} & B'
\end{array}
$$

in $\mathcal{C}$ where the top and bottom rows are split extensions (the domain and codomain, respectively), such that $\kappa' u = u \kappa$, $\beta' w = v \beta$ and $\alpha' v = w \alpha$. Let us denote by $\text{SplExt}(\mathcal{C})$ the category of split extensions, and by $K$ the functor sending (1) and (2) to $X$ and $u$, respectively. Let us write $\text{KGpd}(\mathcal{C})$ for the category with objects 8-tuples $(X, G_0, G_1, d, c, e, m, k)$ consisting of objects and morphisms such that the diagram on the left

$$
\begin{array}{ccc}
G_1 \times G_1 & \xrightarrow{m} & G_1 \\
\downarrow & & \downarrow \beta \\
G_1 \times G_0 & \xrightarrow{\alpha} & G_0
\end{array}
$$

is a groupoid and the diagram on the right is a split extension. A morphism $(X, G_0, G_1, d, c, e, m, k) \to (X', G_0', G_1', d', c', e', m', k')$ is a triple $(u, v, w)$ where $u : X \to X'$, $v : G_1 \to G_1'$ and $w : G_0 \to G_0'$ are morphisms in $\mathcal{C}$ such that the diagram on the left is a functor

$$
\begin{array}{ccc}
G_1 \times G_1 & \xrightarrow{m} & G_1 \\
\downarrow & & \downarrow \beta \\
G_1 \times G_0 & \xrightarrow{\alpha} & G_0
\end{array}
$$

and the diagram on the right is a morphism of split extensions. Note that the forgetful functor $U : \text{KGpd}(\mathcal{C}) \to \text{SplExt}(\mathcal{C})$ is monadic since it is essentially the same as the forgetful functor from the category of internal groupoids in $\mathcal{C}$ to the category of split epimorphisms in $\mathcal{C}$ shown to be monadic by D. Bourn in [9] (see also the discussion above Theorem 4.2 of [28]).
A pointed category $\mathcal{C}$ can be equivalently defined to be (Bourn)-protomodular if the split short five lemma holds, that is, for each morphism of split extensions (2) if $u$ and $w$ are isomorphisms, then $v$ is an isomorphism. A category $\mathcal{C}$ is semi-abelian in the sense of G. Janelidze, L. Marki and W. Tholen if it is pointed, (Barr)-exact, protomodular and has binary coproducts. Following, F. Borceux, G. Janelidze, G. M. Kelly, in we define a generic split extension with kernel $X$ to be a terminal object in a fiber $K^{-1}(X)$ of $K : \text{SplExt}(\mathcal{C}) \to \mathcal{C}$. We denote such a generic split extension as follows:

$$X \xrightarrow{k} [X] \times X \xrightarrow{\pi_1} [X].$$

A semi-abelian category is called action representable if each object admints a generic split extension with kernel $X$. Examples of action representable categories such include the category of groups where $[X] = \text{Aut}(X)$ is the automorphism group of $X$, and the category of Lie algebras over a commutative ring $R$ where $[X] = \text{Der}(X)$ is the Lie algebra of derivations of $X$ (see [7]). Other examples can be found in [8], [5], [6], [25], [27]. For a pointed protomodular category $\mathcal{C}$ and for an object $X$ in $\mathcal{C}$ such that the generic split extension with kernel $X$ exists, the object $[X]$ is called the split extension classifier for $X$ and has the following universal property: For each split extension (1) there exist a unique morphism $v : B \to X$ such that there exist $u : A \to [X] \times X$ making the diagram a morphism of split extensions

$$X \xrightarrow{k} [X] \times X \xrightarrow{\pi_1} [X].$$

(see the last theorem of Section 6 of [7].)

D. Bourn and G. Janelidze call a split extension with kernel $X$ faithful when there is at most one morphism to it from any split extension in $K^{-1}(X)$. A semi-abelian category is called action accessible if for each $X$ in $\mathcal{C}$ there is a morphism from each split extension to a faithful one in $K^{-1}(X)$. Examples of action accessible categories include the category of not-necessarily unital rings, associative algebras and more generally categories of interest in the sense of G. Orzech (see [16] and [34].)

Throughout the paper we denote by $2$ the category with objects 0 and 1, and with one non-identity morphism $0 \to 1$. We will identify the functor category $\mathcal{C}^2$ with the category of morphism of $\mathcal{C}$, and its objects will be written as triples $(X, Z, f)$ where $X$ and $Z$ are objects and $f : X \to Z$ is a morphism in $\mathcal{C}$. A morphism $(X, Z, f) \to (X', Z', f')$ with be written as a pair $(u, v)$ where $u : X \to X'$ and $v : Z' \to Z'$ are morphisms in $\mathcal{C}$ with $fu = vf'$. Recalling from [27], in the semi-abelian context, or from [28] for the general pointed context, that each generic split extension in $\mathcal{C}^2$ has codomain a generic
split extension in $C$, we will denote a generic split extensions of a morphism $f : X \to Z$ in $C^2$ as follows:

$$
\begin{array}{c}
X \xrightarrow{k} [X, Z, f] \times X \xrightarrow{p_1} [X, Z, f] \\
\downarrow f \quad \downarrow q_2 \circ f \\
Z \xrightarrow{k} [Z] \times Z \xrightarrow{p_1} [Z].
\end{array}
$$

(6)

Note that according to the universal property of generic split extensions there is also a unique morphism

$$
\begin{array}{c}
X \xrightarrow{k} [X, Z, f] \times X \xrightarrow{p_1} [X, Z, f] \\
\downarrow q_1 \circ 1 \quad \downarrow q_1 \\
X \xrightarrow{k} [X] \times X \xrightarrow{p_1} [X].
\end{array}
$$

When $X$ admits a generic split extension, writing $(R, r_1, r_2)$ for the kernel pair of $p_1$, it turns out that the unique morphism

$$
\begin{array}{c}
X \xrightarrow{\langle 0, k \rangle} R \xrightarrow{r_1} [X] \times X \\
\downarrow s \quad \downarrow p_2 \\
X \xrightarrow{k} [X] \times X \xrightarrow{p_1} [X]
\end{array}
$$

is an algebra structure for the monad induced by the adjunction where $U : KGpd(C) \to SplExt(C)$ is a right adjoint, and hence determines a groupoid structure on $X$. Putting all the structure together we obtain an object $(X, [X] \times X, [X], p_1, p_2, i, m, k)$ in $KGpd(C)$ which turns out to be terminal in the category of $X$-groupoids, that is, the fiber $(KU)^{-1}(X)$ (see Proposition 5.1 of [4] where this is proved directly in the pointed protomodular context, or combine Proposition 2.26 and Theorem 4.1 via the remarks before Theorem 4.2 of [28], for the pointed finitely complete context). Writing $\nabla(X)$ for the object in $KGpd(C)$ with underlying groupoid the indiscrete groupoid on $X$ and with $\langle 0, 1 \rangle : X \to X \times X$ as kernel of its domain morphism $\pi_1 : X \times X \to X$, we call the morphism $c_X : X \to [X]$ which forms part of the unique morphism of $X$-groupoids $\nabla(X) \to (X, [X] \times X, [X], p_1, p_2, i, m, k)$ the conjugation morphism of $X$ (see Theorem 3.1 of [4]). In particular this means it forms part of the unique morphism

$$
\begin{array}{c}
X \xrightarrow{\langle 0, 1 \rangle} X \times X \xrightarrow{\pi_1} X \\
\downarrow \varphi \quad \downarrow c_X \\
X \xrightarrow{k} [X] \times X \xrightarrow{p_1} [X].
\end{array}
$$

(7)
and that \( p_2 k = c_X \). The kernel of \( c_X \) turns out to be \( z_X : Z(X) \to X \) the center of \( X \) (see Proposition 5.2 of [10] and Proposition 4.1 of [4] which is closely related).

For a semi-abelian category \( \mathcal{C} \) let us briefly recall the equivalence between \( \text{SplExt}(\mathcal{C}) \) and \( \text{Act}(\mathcal{C}) \) the category of internal object actions from [7]. The objects of \( \text{Act}(\mathcal{C}) \) are triples \((B, X, \zeta)\) where \( B \) and \( X \) and are objects in \( \mathcal{C} \) and \( \zeta : B \circ X \to X \) is a morphism making \((X, \zeta)\) an algebra over a certain (monad whose object functor sends \( X \) to \( B \circ X \) where \( \kappa_{B,X} : B \circ X \to B + X \) is the kernel of \([1, 0] : B + X \to B \). Given morphisms \( g : B \to B' \) and \( f : X \to X' \) the unique morphism \( g \circ f : B \circ X \to B' \circ X' \) making the diagram

\[
\begin{array}{ccc}
B \circ X & \xrightarrow{\kappa_{B,X}} & B + X \\
\downarrow{g \circ f} & & \downarrow{g + f} \\
B' \circ X' & \xrightarrow{\kappa_{B',X'}} & B' + X'
\end{array}
\]

a morphism of split extensions, makes \( \dashv \circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) a functor. A morphism \((B, X, \zeta) \to (B', X', \zeta') \) in \( \text{Act}(\mathcal{C}) \) is a pair \((g, f)\), where \( g : B \to B' \) and \( f : X \to X' \) are morphisms in \( \mathcal{C} \) such that \( f \zeta = \zeta'(g \circ f) \).

Given a split extension \([1] \) the unique morphism \( \zeta : B \circ X \to X \) making the diagram

\[
\begin{array}{ccc}
B \circ X & \xrightarrow{\kappa_{B,X}} & B + X \\
\downarrow{\zeta} & & \downarrow{[\beta, \kappa]} \\
X & \xrightarrow{\kappa} & A \\
\end{array}
\]

a morphism of split extensions, makes \((B, X, \zeta)\) an object in \( \text{Act}(\mathcal{C}) \) and this assignment determines the object map of a functor \( \text{SplExt}(\mathcal{C}) \to \text{Act}(\mathcal{C}) \) which is an equivalence of categories. Using this equivalence of categories \( G \). Janelidze produced an equivalence of categories between internal categories in \( \mathcal{C} \) and internal crossed modules in \( \mathcal{C} \) [30]. Let us write \( \gamma_X : X \circ X \to X \) for the action corresponding to the domain of morphism of split extensions \([1] \). When star multiplicative graphs coincide with multiplicative graphs (which N. Martins-Ferreira and T. Van der Linden showed, in [33], happens exactly when the Huq [29] and Smith-Pedicchio [30] commutators coincide) an internal crossed module can be equivalently defined as a triple \((B, X, \zeta, f)\) where \((B, X, \zeta)\) is an object in \( \text{Act}(\mathcal{C}) \) and \( f : X \to B \) is a morphism in \( \mathcal{C} \) such that \((f, 1_X) : (X, X, \gamma_X) \to (B, X, \zeta)\) and \((1_B, f) : (B, X, \zeta) \to (B, B, \gamma_B)\) are morphisms in \( \text{Act}(\mathcal{C}) \) (see [30] noting that \( \gamma_X = [1, 1] \kappa_{X,X} \) and that \([1, f] \kappa_{B,X} = \gamma_B(1_B \circ f) \)). When \( \mathcal{C} \) is a semi-abelian category and \( \tau \) is the action corresponding to split extension \([1] \) then, via the equivalence between internal categories and internal crossed modules, it follows that the quadruple \(([X], X, \tau, c_X)\) is the terminal object in the category internal crossed modules with domain of the underlying morphism the object \( X \).
A monomorphism \( m : S \to X \) in a finitely complete category \( C \) is Bourn-normal \([12]\) to an equivalence relation \( r_1, r_2 : R \to X \) if there exists a morphism \( \tilde{m} : S \times S \to R \) such that (either and hence both of) the squares of the diagram on left, or equivalently the diagram on the right

\[
\begin{array}{c}
\xymatrix{ S \ar[r]^\pi_1 \ar[d]_m & S \times S \ar[d]_{m \times 1} \ar[r]^\pi_2 & S \ar[d]_m & R \ar[d]_{\langle r_1, r_2 \rangle} \\
X \ar[r]_\tilde{m} & R & S \times X \ar[r]_{m \times 1} \ar[d]_{1 \times m} & X \times X
\end{array}
\]

are pullbacks. When \( C \) is pointed, such a morphism \( \tilde{m} \) exists as soon as there is a morphism \( k : S \to R \) making the lower left hand square in the diagram

\[
\begin{array}{c}
\xymatrix{ S \ar[r]^m & S \times S \ar[d]_{m \times m} \ar[dr]^{\langle 0,1 \rangle} & S \ar[d]_{\langle 1,1 \rangle} \\
X \ar[r]_{\langle r_1, r_2 \rangle} & X \times X \ar[r]_\pi_1 & X
\end{array}
\]

a pullback. Note that, writing \( s \) for the unique morphism such that \( r_1 s = 1_X = r_2 s \), this means that the entire diagram consists of morphisms of split extensions. Note also that this means that for an equivalence relation \( (R, r_1, r_2) \) if \( k \) is the kernel of \( r_1 \), then the composite \( r_2 k \) is Bourn-normal to \( (R, r_1, r_2) \). We call a morphism \( m \) a Bourn-normal monomorphism as soon as there exists an equivalence relation to which it is Bourn-normal.

### 3 Objects with trivial centers and morphisms with trivial centralizers

We will say that:

(i) An object \( X \) in a weakly unital category has trivial center if \( Z(X) = 0 \);

(ii) A morphism \( f : A \to X \) in a weakly unital category has trivial centralizer if \( Z_X(A, f) = 0 \).

In this section we study objects and morphisms with these properties. The following lemma is closely related to Proposition 3.18 of \([13]\).

**Lemma 3.1.** Let \( C \) be a (weakly) unital category. An object \( X \) in \( C \) has trivial center if and only if for each object \( Y \) the morphism \( \langle 1, 0 \rangle : X \to X \times Y \) has a unique section.
Proof. The claim follows by noting that for each commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \times Y \\
\downarrow_{1_X} & & \downarrow_{\varphi} \\
X & \xleftarrow{f} & X
\end{array}
\]

\(f\) is a zero morphism if and only if \(\varphi = \pi_2\). \(\square\)

Since for any morphisms \(e : S \to A, f : A \to X\) and \(g : B \to X\) in a weakly unital category, if \(f\) and \(g\) commute, then so do \(fe\) and \(g\), we obtain:

**Proposition 3.2.** Let \(e : S \to A\) and \(f : A \to X\) be morphisms in a weakly unital category \(\mathbb{C}\). If \(fe\) has trivial centralizer, then so does \(f\). In particular when \(A = X\) and \(f = 1_X\) this means that if \(e\) has trivial centralizer, then \(X\) has trivial center.

**Proposition 3.3.** Let \(\mathbb{C}\) be a weakly unital category and let \((X, (\pi_i : X \to X_i)_{i \in I})\) be the product of a family of objects \((X_i)_{i \in I}\) in \(\mathbb{C}\).

(i) A morphism \(f : A \to X\) commutes with \(1_X\) if and only if \(\pi_i f : A \to X_i\) commutes with \(1_{X_i}\).

(ii) If the center \(z_{X_i} : Z(X_i) \to X_i\) of each \(X_i\) exists and the product of family \((Z(X_i))_{i \in I}\) exists, then the center of \(X\) exists and is the product of the family of morphisms \((z_{X_i} : Z(X_i) \to X_i)_{i \in I}\);

(iii) if the center of \(X\) exists, then the center of each \(X_i\) exist and their product exists and is the center of \(X\).

Proof. Since (ii) is a straight forward consequence of (i) we prove (i) and (iii). To prove (i) let \(f : A \to X\) be a morphism in \(\mathbb{C}\). Since each \(\pi_i\) being a split epimorphism is a pullback stable regular epimorphism it follows by Proposition 3.14 in [26] that \(\pi_i f\) commutes with \(\pi_i\) if and only if \(\pi_i f\) commutes with \(1_{X_i}\). Therefore we need only show that \(f\) commutes with \(1_X\) if and only if \(\pi_i f\) commutes with \(\pi_i\). However, this follows from the fact that the existence of a morphism \(\varphi : A \times X \to X\) such that \(\varphi(1,0) = f\) and \(\varphi(0,1) = 1_X\) is equivalent to the existence of a family of morphisms \((\varphi_i : A \times X \to X_i)_{i \in I}\) such that for each \(i\) in \(I\), \(\varphi_i(1,0) = \pi_i f\) and \(\varphi_i(0,1) = \pi_i\). To prove (iii) let \(z : Z \to X\) be the center of \(X\). For each \(i\) in \(I\) let \(\lambda_i : X_i \to X\) be the unique morphism with \(\pi_i \lambda_i\) identity if \(i = j\) and zero otherwise, and let \(z_i : Z \to X\) be the morphism obtained by pullback as displayed in the square of the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\rho_i} & A \\
\downarrow^{\pi_i} & & \downarrow^{\eta_i} \\
Z_i & \xrightarrow{\zeta_i} & Z \\
\downarrow^{z_i} & & \downarrow^{z} \\
X_i & \xrightarrow{\lambda_i} & X
\end{array}
\]
We will show that $z_i$ is the center of $X_i$. Suppose $f : A \to X_i$ a central morphism. Since as easily follows from (i) the morphism $\lambda_i f : A \to X$ is central, it follows that there exists a unique morphism $\bar{f} : A \to Z_i$ such that $zf = \lambda_i f$ and hence a unique morphism $\bar{f} : A \to Z_i$ such that $z_i \bar{f} = f$ and $\eta_i \bar{f} = \bar{f}$. Since $z_i$ is a monomorphism this is sufficient to show that it is the center of $X_i$. Since $z$ commutes with $1_X$ it follows that $\pi_i z$ commutes with $\pi_i$ and hence, by Proposition 3.14 in [20], that $\pi_i z$ is central. This means that there is a unique morphism $\rho_i : Z \to Z_i$ such that $z_i \rho_i = \pi_i z$. We will show that the family $(Z, (\rho_i : Z \to Z_i)_{i \in I})$ is a product. Suppose $(\alpha_i : A \to Z_i)_{i \in I}$ is family of morphisms. By composition we obtain a family $(z_i \alpha_i : A \to X_i)_{i \in I}$ and hence a unique morphism $\alpha : A \to X$ such that $\pi_i \alpha = \pi_i \alpha_i$ for each $i \in I$. Since $(i)$ we see that $\alpha$ is central it follows that there exists a unique morphism $\bar{\alpha} : A \to Z$ such that $z \bar{\alpha} = \alpha$. However, since for each $i$ in $I$ the morphism $z_i$ is a monomorphism and $z_i \rho_i \bar{\alpha} = \pi_i z \bar{\alpha} = \pi_i \alpha = z_i \alpha_i$ it follows that $\rho_i \bar{\alpha} = \alpha_i$. The proof of claim is completed by noting that the family $(\rho_i)_{i \in I}$ is jointly monomorphic. To see why note that $(\pi_i z)_{i \in I}$ is jointly monomorphic (since $z$ is a monomorphism and $(\pi_i)_{i \in I}$ is jointly monomorphic) and for each $i$ in $I$, $z_i$ is a monomorphism and $z_i \rho_i = \pi_i z$.

As a corollary we obtain:

**Corollary 3.4.** Let $C$ be a weakly unital category and let $(X, (\pi_i : X \to X_i)_{i \in I})$ be the product of a family of objects $(X_i)_{i \in I}$ in $C$. The object $X$ has trivial center if and only if for each $i$ in $I$ the object $X_i$ has trivial center.

Recall that an object in the category of $X$-groupoids is called faithful [10] if it admits at most one morphism from each object in the category of $X$-groupoids. Recall also that a faithful $X$-groupoid has underlying split extension faithful (see Lemma 3.2 [10]), and that an internal reflexive graph in a protomodular (more generally Mal’tsev) category admits at most one groupoid structure. We will therefore, in the protomodular (more generally Mal’tsev) context, consider groupoids as special kinds of reflexive graphs.

**Lemma 3.5.** Let $C$ be a pointed protomodular category. If $X$ has trivial center and

\begin{equation}
\begin{array}{c}
X \\
\xrightarrow{\kappa} A \\
\xrightarrow{\alpha_1} B
\end{array}
\begin{array}{c}
\xrightarrow{\pi_1} X \times X \\
\xrightarrow{\pi_2} X
\end{array}
\end{equation}

is a morphism of $X$-groupoids, then $(A, \alpha_1, \alpha_2)$ is an equivalence relation and $c$ is a Bourn-normal monomorphism normal to $(A, \alpha_1, \alpha_2)$. When, in addition, the $X$-groupoid at the bottom of (9) is faithful, then $c$ has trivial centralizer.

**Proof.** Since according to Lemma 2.4 of [20] (see also Proposition 5.2 of [10]) the morphism $\ker(c) : \text{Ker}(c) \to X$ is a subobject of $X$ such that $\langle 0, 1 \rangle$ and
\( \langle 1, 1 \rangle \ker(c) \) commute, it easily follows that \( \ker(c) \) and \( 1_X \) commute (see Observation 5.3 of [16] or Corollary 2.6 of [20]). This means that \( \ker(c) = 0 \) and hence, by protomodularity, \( c \) is a monomorphism [11]. Since \( c = c\pi_2 \langle 0, 1 \rangle = \alpha_2 \varphi \langle 0, 1 \rangle = \alpha_2 \kappa \) it follows that the diagram

\[
\begin{array}{c}
X \xrightarrow{\kappa} A \xrightarrow{\alpha_1} B \\
\downarrow c \quad \downarrow \beta \\
B \xrightarrow{\langle 0, 1 \rangle} B \times B \xrightarrow{\pi_1} B
\end{array}
\]

commutes. Therefore, since \( \ker(\langle \alpha_1, \alpha_2 \rangle) = \ker(c) = 0 \) it follows by protomodularity that \( \langle \alpha_1, \alpha_2 \rangle \) is a monomorphism and \( c \) is Bourn-normal to the equivalence relation \( (A, \alpha_1, \alpha_2) \). For a morphism \( u : S \to B \) it follows from Lemma [24] that the conditions:

(i) \( u \) and \( c \) commute;

(ii) \( \langle \alpha_1, \alpha_2 \rangle \kappa = \langle 0, 1 \rangle c = (c \times c) \langle 0, 1 \rangle \) and \( \langle \alpha_1, \alpha_2 \rangle \beta u = \langle 1, 1 \rangle u = (u \times u) \langle 1, 1 \rangle \) commute;

(iii) \( \kappa \) and \( \beta u \) commute

are all equivalent. Therefore, if the upper split extension in (10) is faithful it follows that \( u \) and \( c \) commute if and only if \( u = 0 \) (as follows immediately from e.g. Proposition 2.5 of [20]).

Using these facts we obtain the following proposition which should be compared to Theorem 7.1 of [4]:

**Proposition 3.6.** Let \( C \) be a pointed protomodular category and let \( X \) be an object with trivial center, such that the generic split extension with kernel \( X \) exists.

(i) The morphism \( c_X \) is a Bourn-normal monomorphism with trivial centralizer, and the object \([X]\) has trivial center.

(ii) Every \( X \)-groupoid is an equivalence relation.

(iii) The morphism \( c_X \) is terminal in the category of Bourn-normal monomorphisms with domain \( X \).

(iv) When \( C \) is in addition semi-abelian, for each internal crossed module \( (B, X, \zeta, f) \) the morphism \( f \) is a normal monomorphism.

**Proof.** The claims (i) and (ii) are direct corollaries of the previous lemmas, while the (iv) is obtained from (ii) via the equivalence of categories between internal groupoids and internal crossed modules. The final claim follows by noting that there is an equivalence of categories between \( X \)-groupoids which are equivalence relations and Bourn-normal monomorphisms with domain \( X \), and under this equivalence the terminal \( X \)-groupoid is sent to \( c_X \).
Recalling that for a group homomorphism \( f : X \to Y \) the group \([X, Y, f]\) is the subgroup of \( \text{Aut}(X) \times \text{Aut}(Y) \) consisting of those pairs of automorphisms \((\theta, \phi)\) such that \( f \theta = \phi f \), and \( q_1 \) and \( q_2 \) are the first and second projections, one sees that the following proposition applied to the category of groups explains why for a normal subgroup \( S \) of a group \( X \) if the centralizer of \( S \) in \( X \) is trivial, then each automorphism of \( S \) has at most one extension to \( X \).

**Proposition 3.7.** Let \( C \) be a pointed protomodular category, and let \( m : S \to X \) be a normal monomorphism in \( C \) such that the generic split extensions with kernel \( S \) and \((S, X, m)\) exist in \( C \) and \( C^2 \), respectively. If \( m \) has trivial centralizer, then \( q_1 : [S, X, m] \to [S] \) is a monomorphism.

**Proof.** By considering the diagram (5), via the universal property of the split extension classifiers of \((S, X, m)\) and \( S \), there is a (unique) morphism \( u : X \to [S, X, m] \) making the diagram commute. Let \( \kappa : K \to [S, X, m] \) be the kernel of \( q_1 \). We will show that \( K = 0 \).

Consider the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{m} & X \\
\downarrow{c_X} & & \downarrow{u} \\
[S] & \xrightarrow{q_1} & [S, X, m] \\
\end{array}
\]

in which all squares are pullbacks. Since \( S \) has trivial center and \( c_S = q_1 u m \) it follows that \( I = \text{Ker}(c_S) = Z(S) = 0 \). This means that \( m \) and \( \lambda \) commute (see e.g. Proposition 3.3.2 of [3]) and hence \( J = 0 \). However, since \( X \) has trivial center it follows, by Proposition 3.6 that \( c_X \) is a normal monomorphism, and hence since \( q_2 \) is a monomorphism (see Proposition 4.5 of [27]) it follows that \( u \) is a normal monomorphism too. This means that \( u \) and \( \kappa \) commute and therefore so do \( c_X \) and \( q_2 \kappa \). But, according to Proposition 3.6 \( c_X \) has trivial centralizer, and hence \( K = 0 \) as desired.

The following proposition, applied to the category of groups, implies that each automorphism \( \theta \) of a group \( X \) with trivial center admits a unique extension \( \varphi \) to the automorphism group \( \text{Aut}(X) \) in such a way that \( c_X \theta = \varphi c_X \).

**Proposition 3.8.** Let \( C \) be a pointed protomodular category and let \( X \) be an object in \( C \) such that the generic split extension with kernel \( X \) exists. If the generic split extension with kernel \((X, [X], c_X)\) exists in \( C^2 \), and \( X \) has trivial
center, then there is a unique morphism \( \varphi : [X] \times X \to [[X]] \times [X] \) making the diagram

\[
\begin{align*}
X & \xrightarrow{k} [X] \times X \xrightarrow{p_1} [X] \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
[X] & \xrightarrow{\varphi} [[X]] \times [X] \xrightarrow{p_1} [[X]]
\end{align*}
\]

a generic split extension in \( C^2 \).

**Proof.** Let \( X \) be an object with trivial center. Consider the diagram

\[
\begin{align*}
X & \xrightarrow{k} [X] \times X \xrightarrow{p_1} [X] \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
[X] & \xrightarrow{\varphi} [[X]] \times [X] \xrightarrow{p_1} [[X]]
\end{align*}
\]

where the morphism in \( C^2 \) displayed on the right is the unique morphism obtained from the split extension on the left via the universal property of the split extension classifier of \( (X, [X], c_X) \) and which has lower morphism \( c_{[X]} \) according to Lemma 4.2 in [27]. The universal property of the split extension classifier \([X]\) then shows that \( q_1 \theta = 1_{[X]} \). However, since by Proposition 3.6 the morphism \( c_X \) is a Bourn-normal monomorphism with trivial centralizer, it follows, by the previous proposition, that \( q_1 : [X, [X], c_X] \to [X] \) is a monomorphism and hence an isomorphism. This means that \( \theta \) is an isomorphism which completes the claim. \( \square \)

## 4 Complete objects

In this section we study four notions of completeness and explain how Baer’s result can be recovered categorically. For a pointed category \( C \) we call a morphism a protosplit monomorphism [8] if it is the kernel of a split epimorphism. As mentioned above we define:

**Definition 4.1.** Let \( C \) be a pointed category. An object \( X \) is called

(i) proto-complete if every protosplit monomorphism with domain \( X \) is a split monomorphism;

(ii) complete if every normal monomorphism with domain \( X \) is a split monomorphism;

(iii) complete* if every Bourn-normal monomorphism with domain \( X \) is a split monomorphism.
(iv) strong-complete if every protosplit monomorphism with domain $X$ is a split monomorphism with a unique section.

**Remark 4.2.** Since in a pointed Barr-exact category every Bourn-normal monomorphism is a normal monomorphism it follows that completeness and completeness* coincide in the pointed Barr-exact context.

Most of the content of the following lemma follows from Corollary 3.3.3 of [3].

**Lemma 4.3.** Let $X$ be a proto-complete object in a pointed protomodular category $C$ with finite limits. For each split extension

$$
\begin{array}{ccc}
X & \overset{\kappa}{\rightarrow} & A \\
\downarrow & & \downarrow \alpha \\
B & \overset{\beta}{\leftarrow} & B
\end{array}
$$

there exists a morphism $\theta : B \to X$ and an isomorphism $\psi : A \to B \times X$ making the diagram

$$
\begin{array}{ccc}
X & \overset{\kappa}{\rightarrow} & A \\
\downarrow & & \downarrow \alpha \\
\langle 0,1 \rangle \ B \times X & \overset{\pi_1}{\rightarrow} & B
\end{array}
$$

an isomorphism of split extensions.

**Proof.** Suppose that $X$ is a proto-complete object in a pointed protomodular category $C$ and let

$$
\begin{array}{ccc}
X & \overset{\kappa}{\rightarrow} & A \\
\downarrow & & \downarrow \beta \\
B & \overset{\psi}{\leftarrow} & B
\end{array}
$$

be a split extension. By assumption there exists a morphism $\lambda : A \to X$ such that $\lambda \kappa = 1_X$. It easily follows that the diagram

$$
\begin{array}{ccc}
X & \overset{\kappa}{\rightarrow} & A \\
\downarrow & & \downarrow \langle \alpha, \lambda \rangle \\
\langle 0,1 \rangle \ B \times X & \overset{\pi_1}{\rightarrow} & B
\end{array}
$$

is a morphism of split extensions and hence by protomodularity $\langle \alpha, \lambda \rangle$ is an isomorphism. \qed

We will also need the following easy lemma (see e.g. Corollary 3.3.3 of [3]).

**Lemma 4.4.** Let $C$ be a pointed protomodular category and let $m : S \to X$ be a Bourn normal monomorphism. If $m$ is a split monomorphism, then $m$ is a product inclusion, that is, there exists an object $T$ and an isomorphism $\varphi : S \times T \to X$ such that $\varphi(1,0) = m$. 

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Proposition 4.5. Let $\mathbb{C}$ be a pointed protomodular category. An object $X$ in $\mathbb{C}$ is strong-complete if and only if $X$ is proto-complete and has trivial center.

Proof. Let $X$ be a proto-complete object in $\mathbb{C}$. Since every protosplit normal monomorphism with domain $X$ is, by Lemma 4.3 up to isomorphism, of the form $\langle 1, 0 \rangle : X \to X \times Y$ for some $Y$, the claim follows from Lemma 5.1.

Remark 4.6. As mentioned above categories in which every object is strongly-complete are called coarsely action representable. These include the opposite category of pointed sets.

Proposition 4.7. In a pointed protomodular category the implications hold: strong-completeness $\Rightarrow$ completeness $\Rightarrow$ proto-completeness.

Proof. Trivially completeness $\Rightarrow$ proto-completeness. Suppose $X$ is strong-complete, we need to show that it is complete. Let $n : X \to B$ be a Bourn-normal monomorphism and let $r_1, r_2 : R \to B$ be the (projections of the) relation of which it is the zero class. This means that there are morphisms $k : X \to R$ and $\tilde{n} : X \times X \to R$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\langle 0,1 \rangle} & X \times X & \xrightarrow{\pi_1} & X \\
\downarrow n & & \downarrow \tilde{n} & & \downarrow n \\
X & \xrightarrow{\langle r_1, r_2 \rangle} & R & \xrightarrow{r_1} & B \\
\downarrow n & & \downarrow s & & \downarrow n \\
B & \xrightarrow{\langle 0,1 \rangle} & B \times B & \xrightarrow{\pi_1} & B
\end{array}
\]

in which $s$ is the unique morphism $r_1s = r_2s = 1_B$, is a morphism of split extensions. According to Lemma 4.3 the middle split extension is isomorphic to a split extension of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\langle 0,1 \rangle} & B \times X & \xrightarrow{\pi_1} & B \\
\downarrow \nu & & \downarrow \pi_2 & & \downarrow \nu \\
\langle 0,1 \rangle & \xrightarrow{\phi} & X \times X & \xrightarrow{\pi_1} & X
\end{array}
\]

and hence composing the upper morphism of diagram (11) with this isomorphism we obtain a morphism of split extensions of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\langle 0,1 \rangle} & X \times X & \xrightarrow{\pi_1} & X \\
\downarrow \nu & & \downarrow \pi_2 & & \downarrow \nu \\
\langle 0,1 \rangle & \xrightarrow{\phi} & X \times X & \xrightarrow{\pi_1} & X
\end{array}
\]

for some morphism $\phi : X \times X \to X$. This implies that $\phi$ is a splitting of $\langle 0,1 \rangle$ and hence must be $\pi_2$ (since $X$ is strong-complete). It now follows that $\theta n = \pi_2 \langle 1, \theta \rangle n = \pi_2 \langle n \pi_1, \pi_2 \rangle \langle 1, 1 \rangle = 1_X$ as desired.
Since in an abelian category every monomorphism is a normal monomorphism and every protosplit monomorphism is a split monomorphism we see that:

**Proposition 4.8.** Let \( \mathcal{C} \) be an abelian category.

(i) Every object \( X \) satisfies the condition: if \( \kappa : X \to A \) is the kernel of a split epimorphism which is also a normal epimorphism, then \( \kappa \) is a split monomorphism;

(ii) Every object is proto-complete;

(iii) An object is complete if and only if it is (regular) injective;

(iv) An object is strong-complete if and only if it is a zero object.

More generally (i), (iii) and (iv) hold if \( \mathcal{C}^{\text{op}} \) is semi-abelian.

**Proof.** Noting that (i) and (ii) are equivalent for an abelian category. We prove the dual of (i), (iii) and (iv) for \( \mathcal{C} \) a semi-abelian category. To prove the dual of (i) suppose that \( \alpha : A \to B \) is the cokernel of a normal split monomorphism \( \kappa : X \to A \). If \( \lambda \) is a splitting of \( \kappa \), then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa} & A & \xrightarrow{\alpha} & B \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\langle 0,1 \rangle} & B \times X & \xrightarrow{\pi_1} & B
\end{array}
\]

is a morphism of short exact sequences and hence by the short five lemma \( \langle \alpha, \lambda \rangle \) is an isomorphism and \( \alpha \) a split epimorphism. The dual of (iii) is a standard fact about regular projective objects in regular categories. The dual of (iv) follows easily from the fact that if \( X \) is an object in \( \mathcal{C} \), then \( \langle 1,1 \rangle \) and \( \langle 1,0 \rangle \) are both sections of the normal epimorphism \( \pi_1 \), and \( \langle 1,1 \rangle = \langle 1,0 \rangle \) if and only if \( 1_X = 0 \) if and only if \( X \) is a zero object. \( \square \)

On the other hand in the category \( \text{Rng} \) of not necessarily unitary rings, proto-completeness, completeness and strong-completeness coincide and are equivalent to being unitary. Recall that as mentioned in the introduction it is known that completeness is equivalent to being unitary.

**Proposition 4.9.** Let \( X \) be an object in \( \text{Rng} \). The object \( X \) is a unitary if and only if it satisfies any of Definition 4.1 (i)-(iv).

**Proof.** Let \( X \) be a ring. If \( X \) is proto-complete, then we see that the morphism \( k \), which forms part of the split extension

\[
\begin{array}{ccc}
X & \xrightarrow{k} & \mathbb{Z} \times X & \xrightarrow{p_1} & \mathbb{Z} \\
\end{array}
\]
obtained by “adding a multiplicative identity to $X$”, splits. Since a surjective ring homomorphism sends a multiplicative identity to a multiplicative identity it follows that $X$ is unitary. Conversely if $X$ is unitary and $X$ is an ideal of $Y$, then the map $l : Y \to X$ defined $y \mapsto ey$ (where $e$ is the multiplicative identity in $X$) is a morphism which splits the inclusion. Indeed, using that $e$ is a multiplicative identity it immediately follows that it splits the inclusion, and since multiplication by an element is always an abelian group homomorphism one only needs to check that $l$ preserves multiplication. However since for any $y \in Y$, $ey$ is in $X$ and hence $eye = ey$ it follows that $l(y_1y_2) = ey_1y_2 = ey_1ey_2 = l(y_1)l(y_2)$. Finally note that $X$ has trivial center since $ex = x$ for all $x \in X$.

**Remark 4.10.** The previous proof can be easily adapted to prove that proto-completeness, completeness and strong-completeness coincide and are equivalent to being unitary for the category of algebras over an arbitrary ring. It is also easy to show that a morphism $f : X \to Y$ in $\text{Rng}$ is a unitary ring morphism if and only if it is strong-complete in the category of morphisms of $\text{Rng}$. In this way one can recover the category of unitary rings from the category of rings.

**Remark 4.11.** The Propositions 4.7 and 4.8 show that in general for $\text{SC}$, $\text{C}$ and $\text{PC}$ the classes of strong-complete, complete and proto-complete objects in a semi-abelian category the inclusions $\text{SC} \subset \text{C} \subset \text{PC}$ are strict. However we will recall the above mentioned fact that a group is strong-complete if and only if it is what we called complete, and we will show that a Lie algebra over a commutative ring is strong-complete if and only if it is complete if and only if it is proto-complete. We will also show that there are groups, in particular the group $\mathbb{Z}/2\mathbb{Z}$ is such an example, that are proto-complete but not complete.

**Proposition 4.12.** Let $\mathbf{C}$ be a pointed protomodular category, let $S$ and $T$ be objects in $\mathbf{C}$, and let $X = S \times T$. If $X$ is proto-complete, complete, complete* or strong-complete, then so are both $S$ and $T$.

**Proof.** For the cases where $X$ is proto-complete, complete or complete* note that if $n : S \to Y$ is a protosplit/normal/Bourn-normal monomorphism, then $n \times 1_T$ is too. This means that in each case $n \times 1_T$ is a split monomorphism. Since the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{n} & Y \\
\downarrow{(1,0)} & & \downarrow{(1,0)} \\
S \times T & \xrightarrow{n \times 1_T} & Y \times T
\end{array}
$$

commutes and the composite of two split monomorphisms is a split monomorphism it follows that $(1,0)n = (n \times 1_T)(1,0)$ is a split monomorphism and hence $n$ is too. The claim now follows from Corollary 3.4 and Proposition 4.9.

The converse of the implications of the previous proposition don’t hold in general, but do hold in certain categories.
Example 4.13. The product of strong-complete objects need not be proto-complete. If $X$ is a non-zero strong-complete group, then $X \times X$ has trivial center and hence is strong-complete if and only if it is proto-complete. However the automorphism $X \times X \to X \times X$ defined by $(x, y) \mapsto (y, x)$ is clearly not inner (which is sufficient via Baer’s theorem). On the other hand since the product of unitary rings is unitary we see that product of strong-complete objects in the category of rings is always strong-complete. It is also easy to check that the converse of the implications of the previous proposition do hold in each abelian category.

Recall that every unital variety of universal algebras has centers of objects. Recall also, that in a unital category $C$, central subobjects are normal whenever $C$ satisfies the condition that for each composite $f = \alpha n$ where $\alpha$ is a split epimorphism and $n$ is a normal monomorphism $f$ is a normal monomorphism as soon as it is a monomorphism. In particular every semi-abelian variety of universal algebras and every semi-abelian algebraically cartesian closed category satisfies these properties. Furthermore, we have (the probably known fact which we couldn’t find a reference for):

Proposition 4.14. Let $C$ be a pointed protomodular category. If $m : S \to X$ is a central monomorphism in $C$, then $m$ is Bourn-normal.

Proof. Suppose $m : S \to X$ is a central monomorphism with cooperator $\varphi : S \times X \to X$. It follows that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{(1,0)} & S \times X \\
\downarrow{m} & & \downarrow{\langle \varphi, \pi_2 \rangle} \\
X & \xrightarrow{(1,0)} & X \times X \\
\downarrow{\langle \varphi, \pi_2 \rangle} & & \downarrow{\langle 1, 1 \rangle}
\end{array}
\]

is a morphism of split extensions and hence by protomodularity $\langle \varphi, \pi_2 \rangle$ is a monomorphism and $m$ a Bourn-normal monomorphism. 

For an object $X$ in a pointed weakly unital category consider the condition:

Condition 4.15. The center $z_X : Z(X) \to X$ exists, $\langle z_X, z_X \rangle : Z(X) \to X \times X$ is a normal monomorphism, and the cokernels of $z_X$ and $\langle z_X, z_X \rangle$ exist.

Note that the above condition holds for each object in a semi-abelian category admitting centers and hence in each semi-abelian variety of universal algebras. On the other hand it holds for each abelian object in a unital category.

Theorem 4.16. Let $C$ be a pointed protomodular category and let $X$ be an object in $C$. If $X$ is proto-complete (respectively complete or complete*) and satisfies Condition 4.15, then $X$ is the product of its center $Z(X)$ which is an abelian proto-complete (respectively complete or complete*) object and the quotient object $X/Z(X)$ which is a strong-complete object.
Proof. Let \( X \) be a proto-complete object in \( C \) and let \( (Z, z) \) be the center of \( X \). Consider the diagram

\[
\begin{array}{ccc}
Z & \overset{z}{\longrightarrow} & Z \\
\downarrow^{\langle z, z \rangle} & & \downarrow^{z} \\
X & \overset{\varphi}{\longrightarrow} & X \\
\downarrow^{c} & & \downarrow^{c} \\
X & \overset{\langle 0, 1 \rangle}{\longrightarrow} & B \\
\end{array}
\]

in which \( c \) and \( \varphi \) are cokernels of \( z \) and \( \langle z, z \rangle \), respectively, and the lower induced split extension is of the form presented by Lemma 4.3. This means that the diagram

\[
\begin{array}{ccc}
X & \overset{\langle 1, 0 \rangle}{\longrightarrow} & X \\
\downarrow^{\pi_2 \varphi} & & \downarrow^{\pi_2 \varphi} \\
X & \overset{1_X}{\longrightarrow} & X \\
\end{array}
\]

commutes and so the morphism \( \pi_2 \varphi \langle 1, 0 \rangle \) is central. It follows there exists \( \tilde{f} : X \to Z \) such that \( z \tilde{f} = \pi_2 \varphi \langle 1, 0 \rangle \) and hence \( c \pi_2 \varphi \langle 1, 0 \rangle = 0 \). Therefore, since the morphisms \( \langle 1, 0 \rangle \) and \( \langle 0, 1 \rangle \) are jointly epimorphic it follows that \( c \pi_2 \varphi = c \pi_2 \) and hence that \( c b c = c \pi_2 \langle 1, \theta \rangle c = c \pi_2 \varphi \langle 1, 1 \rangle = c \pi_2 \langle 1, 1 \rangle = c \). This means that \( c \) is a split epimorphism which means that its kernel being central is a product inclusion. Indeed, if \( \gamma : B \times Z \to X \) is the cooperator of \( z \) and \( \theta \), then since \( \langle 1, 0 \rangle \) and \( \langle 0, 1 \rangle \) are jointly epimorphic it follows that the diagram

\[
\begin{array}{ccc}
Z & \overset{\langle 0, 1 \rangle}{\longrightarrow} & B \\
\downarrow^{\gamma} & & \downarrow^{\gamma} \\
X & \overset{c}{\longrightarrow} & B \\
\end{array}
\]

is a morphism of split extensions and \( \gamma \) is an isomorphism. Since \( \langle 0, 1 \rangle : Z \to B \times Z \) is the center of \( B \times Z \) it follows by Proposition 3.3 that \( B \) has trivial center. The claim now follows from Propositions 3.3, 4.7 and 4.12. 

The previous theorem raises the question of whether the product of a proto-complete abelian object and a strong-complete object is necessarily proto-complete. This turns out to be false in general. Let \( S_3 \) be the symmetric group on a three element set. It is well-known (and easy to check) that \( S_3 \) is complete=strong-complete. If the group \( X = \mathbb{Z}/2\mathbb{Z} \times S_3 \) is proto-complete, then by the previous theorem since \( \langle 1, 0 \rangle : \mathbb{Z}/2\mathbb{Z} \to X \) is the center of \( X \) we must have the Aut(\( X \)) \cong S_3 \cong \text{Aut}(S_3) \) and so every automorphism of \( X \) must be of the form \( 1 \times \phi \) where \( \phi \in \text{Aut}(S_3) \). However, recalling that the group \( S_3 \) is isomorphic to the semi-direct product \( \mathbb{Z}/3\mathbb{Z} \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z} \) where \( \theta : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/3\mathbb{Z}) \) is...
(the isomorphism) defined by \(1 \mapsto (x \mapsto 2 \cdot x)\) and hence forms part of a split extension
\[
\mathbb{Z}/3\mathbb{Z} \xrightarrow{k} S_3 \xrightarrow{p} \mathbb{Z}/2\mathbb{Z}
\]
and writing \(a : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}\) for the addition morphism of \(\mathbb{Z}/2\mathbb{Z}\), we see that the morphism \(\langle a(1_{\mathbb{Z}/2\mathbb{Z}} \times p), \pi_2 \rangle : X \to X\) is an automorphism which is not of this form. On the other hand we do have:

**Proposition 4.17.** Let \(\mathcal{C}\) be a pointed protomodular category in which centralizers of Bourn-normal monomorphisms exists and are Bourn-normal. Suppose \(A\) is an abelian and complete* object of \(\mathcal{C}\), and \(B\) is a strong-complete object of \(\mathcal{C}\). If \(\text{hom}(B, A) = \{0\}\), then \(B \times A\) is complete*.

**Proof.** Suppose \(n : B \times A \to Y\) is a Bourn-normal monomorphism. Since the pullback of the centralizer of a monomorphism \(f\) along \(f\) is the center of its domain it follows that there is a unique morphism \(m\) making the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{m} & Z_Y(B \times A, n) \\
\langle 0,1 \rangle & \downarrow & \downarrow z_n \\
B \times A & \xrightarrow{n} & Y
\end{array}
\]
a pullback. This means that \(n\langle 0,1 \rangle\) (being the binary meet of Bourn-normal monomorphisms) is Bourn-normal and hence by Lemma 4.14 there is an isomorphism \(\theta : Y \to C \times A\) such that \(\theta n\langle 0,1 \rangle = \langle 0,1 \rangle\). Therefore, since \(\text{hom}(B, A) = \{0\}\) it follows that \(\pi_2 \theta n(1, 0) = 0\) and hence that \(\theta n(1, 0) = \langle \hat{n}, 0 \rangle\) where \(\hat{n} = \pi_1 \theta n(1, 0)\). Since \(\langle 1,0 \rangle\) and \(\langle 0,1 \rangle\) are jointly epimorphic, it follows that \(\theta n = \hat{n} \times 1\). Noting that the diagram
\[
\begin{array}{ccc}
B & \xrightarrow{\hat{n}} & C \\
\langle 1,0 \rangle & \downarrow & \downarrow \langle 1,0 \rangle \\
B \times A & \xrightarrow{\hat{n} \times 1} & C \times A
\end{array}
\]
is a pullback, we see that \(\hat{n}\) is a Bourn-normal monomorphism and hence it and \(n\) are split monomorphisms. \(\square\)

**Lemma 4.18.** Let \(\mathcal{C}\) be a pointed protomodular category and let \(X\) be an object admitting a generic split extension with kernel \(X\). The following are equivalent:

(a) the generic split extension with kernel \(X\) is of the form
\[
X \xrightarrow{\langle 0,1 \rangle} [X] \times X \xrightarrow{\pi_1} [X] \tag{a}
\]
for some morphism \(\theta\);
(b) \( c_X \) is a split epimorphism.

Proof. (a) \(\Rightarrow\) (b): Consider the diagram

\[
\begin{array}{c}
\xymatrix{
X^{(0,1)} \ar[r]^\theta \ar[d]_{\theta \times 1} & X \times X \ar[r]^{\pi_1} \ar[d]_\theta & [X] \ar[d]^\theta \\
X \ar[r]^\varphi & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X] \\
X^{(0,1)} \ar[r] & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X] \\
X \ar[r]^k & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X]
}\end{array}
\]

where the lower part is the unique morphism into the generic split extension with kernel \( X \) which defines \( c_X \). The universal property of the generic split extension implies that the composite \( c_X \theta = 1_{[X]} \).

(b) \(\Rightarrow\) (a): Suppose that \( c_X \) is a split epimorphism with splitting \( \theta : [X] \to X \) and consider the diagram

\[
\begin{array}{c}
\xymatrix{
X^{(0,1)} \ar[r]^\theta \ar[d]_{\theta \times 1} & X \times X \ar[r]^{\pi_1} \ar[d]_\theta & [X] \ar[d]^\theta \\
X \ar[r]^\varphi & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X] \\
X^{(0,1)} \ar[r] & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X] \\
X \ar[r]^k & X \times X \ar[r]^{\pi_1} \ar[d]^\epsilon_X & [X]
}\end{array}
\]

in which the lower part is the unique morphism into the generic split extension with kernel \( X \) which defines \( c_X \). The claim now follows from the fact that since \( c_X \theta = 1_{[X]} \), the split short five lemma implies that \( \varphi(\theta \times 1) \) is an isomorphism.

\[\square\]

Theorem 4.19. Let \( \mathbb{C} \) be a pointed protomodular category. For an object \( X \) in \( \mathbb{C} \) the following are equivalent:

(a) \( X \) is proto-complete and satisfies Condition 4.15

(b) the generic split extension with kernel \( X \) exists and is of the form

\[
X^{(0,1)} \ar[r]^{\theta} & [X] \times X \ar[r]^{\pi_1} & [X]
\]

for some morphism \( \theta \).

(c) the generic split extension with kernel \( X \) exists and \( c_X \) is a split epimorphism.
Proof. The equivalence of (b) and (c) follow from the previous lemma. Suppose the assumptions in (a) hold. According to Theorem 4.16 there exists objects $A$ and $[X]$ in $C$ such that $A$ is abelian and proto-complete, $[X]$ is strong-complete, and $X = [X] \times A$. We will show that

$$X \xrightarrow{(0,1)} [X] \times X \xrightarrow{\pi_1} [X]$$

(12)

is a generic split extension with kernel $X$. For any split extension with kernel $X$, which by Lemma 4.3 is (up to isomorphism) of the form displayed at the top of the following diagram, the composite of the morphisms

$$X \xrightarrow{(0,1)} B \times X \xrightarrow{\pi_1 \langle 1, (f,g) \rangle} B$$

$$X \xrightarrow{\langle 1, (f,0) \rangle} B$$

$$X \langle (0,1) \rangle [X] \times X \xrightarrow{\pi_1 \langle 1, (1,0) \rangle} [X],$$

gives a morphism to (12). It remains only to show that this is the unique such morphism. By protomodularity it is sufficient to show that for a morphism of split extensions

$$X \xrightarrow{(0,1)} B \times X \xrightarrow{\pi_1 \langle 1, (f,g) \rangle} B$$

$$X \langle (0,1) \rangle [X] \times X \xrightarrow{\pi_1 \langle 1, (1,0) \rangle} [X],$$

we must have $v = f$. To do so note that the diagram

$$[X] \xrightarrow{(0,1)} X \xrightarrow{\langle 1,0,1 \rangle} B \times X$$

$$[X] \xrightarrow{1_{[X]}} X \xrightarrow{(0,1)} [X] \times X$$

commutes, and so since $[X]$ is strong-complete, and $\langle 0, (1,0) \rangle$ is normal with splitting $\pi_1 \pi_2$ it follows that $\pi_1 \pi_2 u = \pi_1 \pi_2$. This means that

$$v = \pi_1 \pi_2 (1, (1,0)) v = \pi_1 \pi_2 u (1, (f,g)) = \pi_1 \pi_2 (1, \langle f, g \rangle) = f$$

as desired. The claim now follows by noting that: (b) implies that every protosplit monomorphism with domain $X$ factors through $\langle 0,1 \rangle : X \rightarrow [X] \times X$.
and hence splits; in the protomodular context (c) implies that the morphism $c_X$ and $\varphi$ in (7), whose kernels are $z_X$ and $\langle z_X, z_X \rangle$ respectively, are normal epimorphisms.

**Corollary 4.20.** Let $\mathbb{C}$ be a pointed Barr-exact protomodular category admitting centers. If $X$ is complete (respectively proto-complete), then, $[X]$, the split extension classifier for $X$ exists and is strong-complete, $Z(X)$ is an abelian complete (respectively proto-complete) object, and $X \cong Z(X) \times [X]$.

**Corollary 4.21.** Let $\mathbb{C}$ be a pointed Barr-exact protomodular category admitting centers. An object $X$ in $\mathbb{C}$ is complete and abelian, if and only if the split extension classifier for $X$ is the zero object.

**Proof.** Let $X$ be an object. Since every complete object has a split extension classifier, we may assume that the split extension classifier for $X$ exists, and hence so does $c_X$. Recall that $z_X$ is the kernel of $c_X$, $X$ is abelian if and only if $z_X$ is an isomorphism, and according to Theorem 4.19, $X$ is proto-complete if and only if $c_X$ splits. Now, if $[X] \cong 0$ then $c_X = 0$ and hence is a split epimorphism and its kernel $z_X$ is an isomorphism. Conversely, if $z_X$ is an isomorphism and $c_X$ is a split epimorphism, then $c_X$ being the cokernel of $z_X$ is zero and $[X] \cong 0$.

Let us recall the following two facts (rephrased with our terminology):

**Proposition 4.22.** If $X$ is a group such that $\text{Aut}(X) \cong 0$, then $X$ is isomorphic to $0$ or $\mathbb{Z}/2\mathbb{Z}$. The only (up to isomorphism) non-zero abelian proto-complete group is $\mathbb{Z}/2\mathbb{Z}$.

**Proof.** Since $X$ is abelian it follows that the map sending $x$ to $-x$ is an automorphism and hence must be equal to $1_X$. This means that for each $x$ in $X$, $0 = x + -x = x + x = 2x$ and so $X$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$. Since any vector space has a non-trivial automorphism corresponding to any non-trivial permutation of elements in its basis it follows that $X$ has dimension at most one and hence is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or 0.

**Proposition 4.23.** If $X$ is a Lie algebra over a commutative ring $R$ such that $\text{Der}(X) \cong 0$, then $X \cong 0$. There are no non-zero abelian proto-complete Lie algebras.

**Proof.** Since $X$ is abelian it follows that any $R$-linear map is a derivation so that in particular the identity map is a derivation and hence must be equal to 0.

**Proposition 4.24.** Let $\mathbb{C}$ be an anti-additive action representable semi-abelian category. If $[X] \cong 0$, then $X \cong 0$.

**Proof.** The claim is immediate since $X$ is necessarily abelian and hence isomorphic to 0.
Recall that for an object $X$, in a pointed protomodular category admitting a generic split extension with kernel $X$, the object $\text{Out}(X)$ is codomain of the cokernel $q : [X] \to \text{Out}(X)$ of $c_X$. The following theorem should be compared to Theorems 9.1 and 9.2 of [4].

**Theorem 4.25.** Let $\mathcal{C}$ be a pointed protomodular category. For an object $X$ the following are equivalent:

(a) $X$ is strong-complete;

(b) the generic split extension with kernel $X$ exists and the conjugation morphism $c_X : X \to [X]$ is an isomorphism;

(c) the split extension

$$X \xrightarrow{(0,1)} X \times X \xrightarrow{\pi_1} X \tag{13}$$

is a generic split extension;

(d) the generic split extension with kernel $X$ exists, $X$ has trivial center, and $\text{Out}(X)$ is trivial.

If in addition every abelian complete object in $\mathcal{C}$ is a zero object, then these conditions are further equivalent to:

(e) $X$ is complete and satisfies Condition 4.15.

**Proof.** Considering the definition of $c_X$ and recalling that the kernel of $c_X$ is center of $X$ and cokernel of $c_X$ is $\text{Out}(X)$ it easily follows that (b), (c) and (d) are equivalent. Using again the fact that the center of $X$ is kernel of $c_X$ the equivalence of (a) and (b) follows from Theorem 4.19. The final claim follows from Theorem 4.16 since it implies that $Z(X)$ is abelian and complete and hence by assumption a zero object.

**Remark 4.26.** Note that Conditions 4.25 (a) and (c) are equivalent for an object $X$ in a pointed finitely complete category.

The content of the following proposition is well-known, we include a proof to keep the paper more self contained.

**Proposition 4.27.** There are no (non-zero) abelian complete groups. The group $\mathbb{Z}/2\mathbb{Z}$ is proto-complete but not complete.

**Proof.** Since every complete group is proto-complete and the only non-zero abelian proto-complete group is $\mathbb{Z}/2\mathbb{Z}$ it suffices to prove the final claim. However, this follows trivially from fact that the canonical monomorphism from $\mathbb{Z}/2\mathbb{Z}$ into $\mathbb{Z}/4\mathbb{Z}$ is a normal monomorphism which isn’t a split monomorphism.

**Remark 4.28.** Note that Baer’s theorem which can stated using our terminology, as: every complete group is strong-complete, is now a corollary of Theorem 4.25 via the previous proposition. Furthermore, we have that if a group $G$ is proto-complete, then $G$ is complete or $G \cong \mathbb{Z}/2\mathbb{Z} \times H$ where $H$ is complete.
5 Characteristic monomorphisms

The main purpose of this section is show that there is a common categorical explanation behind why the derivation algebra of a perfect Lie algebra with trivial center and the automorphism group of a (characteristically) simple nonabelian group are complete.

Recall that a subgroup $S$ of a group $X$ is called characteristic if every automorphism of $X$ restricts to an automorphism of $S$. Recall also that a subgroup $S$ of $X$ is characteristic if and only if whenever $X$ is normal in $Y$, then $S$ is normal in $Y$. Generalizing this latter condition A. S. Cigoli and A. Montoli introduced and studied the notion of a characteristic subobjects in semi-abelian categories [22]. Later D. Bourn gave a different definition in a more general context [14], which coincides with the previously mentioned one in the semi-abelian context. Here we say that a morphism $u : S \to X$ in a category $C$ is characteristic monomorphism if for each Bourn-normal monomorphism $n : X \to Y$ the composite $un$ is a Bourn-normal monomorphism. In [21] it was shown that for a semi-abelian category, a morphism $u : S \to X$ is a characteristic monomorphism if and only if for each protosplit monomorphism $\kappa : X \to A$ the composite $\kappa u$ is a normal monomorphism. This fact essentially remains true in any pointed Mal’tsev category using the above definition of characteristic monomorphism. Recall that a category is Mal’tsev if it is finitely complete and each (internal) reflexive relation is an (internal) equivalence relation. Note that Mal’tsev categories were first introduced and studied in [17] with exactness as part of the definition, exactness was removed in [18].

**Proposition 5.1.** Let $\mathcal{C}$ be a pointed finitely complete Mal’tsev category and let $u : S \to X$ be morphism in $\mathcal{C}$. The following are equivalent:

(a) $u$ is a characteristic monomorphism;

(b) for each protosplit normal monomorphism $\kappa : X \to A$ the composite $\kappa u$ is a Bourn-normal monomorphism;

(c) for each reflexive relation $(R, r_1, r_2)$ on an object $Y$ with $k : X \to R$ as kernel of $r_1$, there exists a monomorphism of reflexive relations $v : (T, t_1, t_2) \to (R, r_1, r_2)$ with $l : S \to T$ as kernel of $t_1$ such that $vl = ku$;

(d) the same as (c) but replace “reflexive relation” by “equivalence relation”.

**Proof.** Trivially (a) implies (b) and (c) is equivalent to (d). Now suppose that (b) holds and let $(R, r_1, r_2)$ be a reflexive relation on $Y$ with $k : X \to R$ as kernel of $r_1$. Since $r_1$ is a split epimorphism we know that $k$ is a protosplit monomorphism and hence $ku$ is a Bourn-normal monomorphism by assumption. Let us write $(\bar{T}, \bar{t}_1, \bar{t}_2)$ for the equivalence relation on $R$ that $k$ is zero-class of and $\bar{l} : S \to \bar{T}$ for the kernel of $\bar{t}_1$ so that $\bar{t}_2\bar{l} = ku$. Now, suppose that $e : Y \to R$ and $f : R \to T$ are the unique morphisms such that $r_1e = 1_Y$ and $t_1f = 1_R$ and
consider the pullback of split extensions

\[ S \rightarrow l \rightarrow T \rightarrow t_1 \rightarrow Y \]
\[ u \]
\[ (0,1) \]
\[ X \]
\[ k \]
\[ R \leftrightarrow e \rightarrow T \rightarrow t_1 \rightarrow Y \]
\[ f \]
\[ \langle (r_1,1) \rangle \]
\[ (1,1) \]
\[ \pi_1 \]
\[ R \]
\[ \delta \]
\[ B \]
\[ g \]
\[ 0 \]
\[ \gamma \]
\[ C \]
\[ \sigma \]
\[ Z \]
\[ \tau \]
\[ k \]
\[ X \rightarrow \kappa \rightarrow A \rightarrow \alpha \rightarrow B \]
\[ f \]
\[ g \]
\[ \beta \]
\[ \gamma \]

Setting \( t_2 = r_2 v \) we obtain the desired monomorphism of reflexive relations \( v : (T, t_1, t_2) \rightarrow (R, r_2, r_2) \) proving (b) implies (c). To complete the claim we will show that (d) implies (a). Suppose that (d) holds and \( n : X \rightarrow Y \) is a Bourn-normal monomorphism. By assumption there is an equivalence relation \((R, r_1, r_2)\) on \( Y \) with \( k : X \rightarrow R \) as kernel of \( r_1 \) such that \( n = r_2 k \). According to (d) there is a monomorphism of equivalence relations \( v : (T, t_1, t_2) \rightarrow (R, r_1, r_2) \) with \( l : S \rightarrow T \) as kernel of \( t_1 \) such that \( vl = ku \). This means that \( t_2 l = r_2 vl = r_2 ku = nu \) proving that \( nu \) is Bourn-normal.

Recall that a pointed category with finite limits can be equivalently defined to be strongly protomodular in the sense of D. Bourn [11] if it is protomodular and for each morphism of split extensions

\[ X \rightarrow \kappa \rightarrow A \rightarrow \alpha \rightarrow B \]
\[ f \]
\[ g \]
\[ \beta \]
\[ \gamma \]
\[ \delta \]
\[ B \]
\[ Z \rightarrow \sigma \rightarrow C \rightarrow \gamma \rightarrow B \]
\[ \delta \]

the morphism \( f \) is Bourn-normal if and only if \( \sigma f \) is Bourn-normal. Note that part (i) of the following theorem is essentially known, see Proposition 3.1 of [14].

**Theorem 5.2.** Let \( C \) be a pointed protomodular category, let \( X \) be an object in \( C \) such that the generic split extension with kernel \( X \) exists, and let \( u : S \rightarrow X \) be a morphism.

(i) The morphism \( u : S \rightarrow X \) is a characteristic monomorphism if and only if the composite \( ku \) of \( u \) and \( k : X \rightarrow [X] \times X \) is a Bourn-normal monomorphism;

(ii) If \( u : S \rightarrow X \) is a characteristic monomorphism, then the generic split extension with kernel \((S, X, u)\) exists in \( C^2 \) and \( q_2 : [S, X, u] \rightarrow [X] \) is an isomorphism.

Furthermore, when \( C \) is strongly protomodular the converse of (ii) holds.
Proof. Suppose \( u : S \to X \) is a morphism in \( C \). By definition, if \( u \) is a characteristic monomorphism, then \( ku : S \to [X] \times X \) is Bourn-normal. Conversely, suppose \( ku : S \to [X] \times X \) is Bourn-normal and \( \kappa : X \to A \) is the kernel of a split epimorphism \( \alpha : A \to B \) with splitting \( \beta : B \to A \). Accordingly there is a unique morphism of split extensions

\[
\begin{array}{c}
X \xrightarrow{k} S \xrightarrow{\kappa} A \xrightarrow{\alpha} B \\
X \xrightarrow{k} [X] \xrightarrow{i} X \xrightarrow{p_1} [X].
\end{array}
\]  

(14)

By protomodularity the right and downward directed arrows of the right hand square of (14) form a pullback and hence the morphism \( \langle f, \alpha \rangle : A \to ([X] \times X) \times B \) is a monomorphism. Therefore, since \( \langle f, \alpha \rangle ku = \langle ku, 0 \rangle \) which is a Bourn-normal monomorphism (being the intersection of the Bourn-normal monomorphisms \( 1 \times ku \) and \( \langle 1, 0 \rangle \) - see e.g. Proposition 3.2.6 of [4]) it follows that \( ku \) is a Bourn-normal monomorphism as desired. Now suppose \( u \) is a characteristic monomorphism, \( H = [X] \times X \) and \( (T, t_1, t_2) \) is the equivalence relation that \( ku \) is the zero class of. Since \( C \) is protomodular there is a unique (up to isomorphism) equivalence relation to which \( ku \) is normal. This means that \( (ku, (T, t_1, t_2)) \) is the normalizer of \( ku \) (in the sense of [15]) and hence by Proposition 2.4 of [15] the front face of the pullback of split extensions

is a \( K \)-precartesian. However, by Lemma 2.7 of [16] this means that the back face is also \( K \)-precartesian and hence by Lemma 2.6 of [28] is the generic split extension with kernel \( S, X, u \) in \( C^2 \).

To see that the final claim follows from strong protomodularity. Just note that in the diagram (16) with \( f = u \) and \( g \) an isomorphism, strong protomodularity implies that the composite \( kf \) is Bourn-normal.

Theorem 5.3. Let \( C \) be a pointed protomodular category, and let \( X \) be an object in \( C \) with trivial center such that the generic split extension with kernel \( X \) exists. A morphism \( u : S \to X \) is a characteristic monomorphism if and only if the composite \( c_X u : S \to [X] \) is Bourn-normal.
Proof. Since $X$ has trivial center, the morphism $c_X$ is monomorphism and hence (by protomodularity) so is the middle morphism $\langle p_1, p_2 \rangle$ in the morphism

$$
\begin{array}{c}
X \xrightarrow{k} [X] \times X \xrightarrow{p_1, p_2} [X] \\
\downarrow c_X \quad \quad \downarrow \langle p_1, p_2 \rangle \\
[X] \xrightarrow{(0,1)} [X] \times [X] \xrightarrow{\pi_1, \pi_2} [X]
\end{array}
$$

of split extensions. Since $\langle 0, c_X u \rangle = \langle p_1, p_2 \rangle ku$ is a Bourn-normal monomorphism and $\langle p_1, p_2 \rangle$ is a monomorphism, it follows that $ku$ is a Bourn-normal monomorphism. The claim now follows by the previous proposition.

**Proposition 5.4.** Let $C$ be a pointed protomodular category. Every normal subobject of a proto-complete object is characteristic.

Proof. Let $X$ be a proto-complete object and let $n : S \to X$ be a Bourn-normal monomorphism. If $\kappa : X \to A$ is a protosplit monomorphism, then according to Lemma 4.3 there is an isomorphism $\psi : B \times X \to A$, where $B$ is the (object part of the) cokernel of $\kappa$, such that $\kappa = \psi(0,1)$. Since $\langle 0, n \rangle$ is Bourn-normal it follows that $\kappa n = \psi(0,1)n$ is Bourn-normal.

**Theorem 5.5.** Let $C$ be a pointed protomodular category and let $X$ be an object in $C$ such that the generic split extension with kernel $X$ exists. There is a generic split extension with kernel $[X]$ and the conjugation morphism $c_X : X \to [X]$ is a characteristic monomorphism if and only if $X$ has trivial center and $[X]$ is strong-complete.

Proof. Since (i) the morphism $c_X$ is a Bourn-normal monomorphism if and only if $X$ has trivial center (for the “if” part use Proposition 3.6, and for the converse just recall that the center is the kernel of $c_X$); (ii) by Theorem 4.25 if $[X]$ is strong-complete, then there is a generic split extension with kernel $[X]$; it follows that in addition to the assumptions above we may assume that $X$ has trivial center and that there is a generic split extension with kernel $[X]$, and prove that $c_X$ is a characteristic monomorphism if and only if $[X]$ is strong-complete. Suppose that $c_X$ is a characteristic monomorphism. Then according to Theorem 5.2 the generic split extension with kernel $(X, [X], c_X)$ exists in $C^2$. By Proposition 6.8 this means that the split extension classifier of $(X, [X], c_X)$ is $([X], [[X]], c_{[[X]]})$ and so by Theorem 6.2 (again) the morphism $c_{[[X]]}$ is an isomorphism. Therefore, $[X]$ is strong-complete by Theorem 4.25. The converse follows from Proposition 6.4.

We obtain the following known fact as a corollary:

**Proposition 5.6.** If a group $X$ is characteristically simple (i.e. it has no proper characteristic subobjects) and non-abelian, then $\text{Aut}(X)$ is complete (=strong-complete).
Proof. Suppose that $X$ is a characteristically simple non-abelian group and suppose that $\theta$ is an automorphism of $\text{Aut}(X)$. Since the center is always characteristic it follows that $Z(X) = 0$ and hence $c_X$ is a normal monomorphism. Forming the pullback

$$
\begin{array}{c}
K \\
\downarrow^{u}
\end{array}
\begin{array}{c}
X \\
\downarrow^{c_X}
\end{array}

\begin{array}{c}
\downarrow^{X}
\end{array}
\begin{array}{c}
\theta c_X \\
\downarrow^{c_X}
\end{array}
\begin{array}{c}
\downarrow^{\text{Aut}(X)}
\end{array}

we see that $c_X u$ is normal and hence by Theorem 5.3 that $u$ is characteristic. By assumption this means that either $K = 0$ or $u$ is an isomorphism. However, since by Proposition 3.6 $c_X$ has trivial centralizer it follows that $K$ is not trivial and hence $u$ must be an isomorphism. Essentially the same argument implies that $v$ is an isomorphism which proves that $c_X$ is characteristic and hence by the previous theorem implies that $\text{Aut}(X)$ is complete.

Proposition 5.7. Let $\mathbb{C}$ be a category of interest in the sense of G. Orzech [33] such that the group operation (required to exist) is commutative, and let $X$ be an object in $\mathbb{C}$.

(i) If $X$ is perfect, then every normal monomorphism with domain $X$ is a characteristic monomorphism;

(ii) if $X$ is perfect, has trivial center, and the generic split extension with kernel $X$ exists, then $[X]$ is strong-complete.

Proof. Recall that for such a variety of universal algebras a subobject $S \subseteq Y$ is normal if and only if for each $s$ in $S$, each $y$ in $Y$ and each binary operation $*$ (excluding addition), $s * y$ and $y * s$ are in $S$. Recall also that $[X,X]$ is the subalgebra of $X$ generated by elements of the form $x_1 * x_2$ where $x_1$ and $x_2$ are elements of $X$ and $*$ is a binary operation (excluding addition). Now suppose $X$ is perfect (i.e. $[X,X] = X$), $X$ is a normal subobject of $Y$, and $Y$ is a normal subobject of $Z$. Since $\mathbb{C}$ is category of interest for binary operations $*$ and $\cdot$ we known that for some positive integer $n$ there are binary operations $*_1, \ldots, *_n$ and $\cdot_1, \ldots, \cdot_n$ and a term $w$ such that (in particular) for all $x_1, x_2$ in $X$ and $z$ in $Z$

$$(x_1 * x_2) \cdot z = w(x_1 *_1 (x_2 \cdot_1 z), \ldots, x_1 *_m (x_2 *_{m+1} z), x_2 *_{m+1} (x_1 \cdot_{m+1} z), \ldots, x_2 *_n (x_1 \cdot_n z)),$$

where $m$ is an integer between 0 and $n$. Therefore, since each $x_j \cdot_i z$ is in $Y$ it follows that $x_k *_i (x_j \cdot_i z)$ is in $X$ and hence $(x_1 * x_2) \cdot z$ is in $X$. A similar calculation shows that $z \cdot_0 (x_1 * x_2)$ is in $X$. Since $X$ is perfect we know that $X$ is generated by products and hence via the previous calculations is normal in $Z$. This proves (i). Combining (i) with Proposition 3.6 and Theorem 5.5 we obtain (ii).

Remark 5.8. Applying the previous proposition to the category of Lie algebras shows that a perfect Lie algebra with trivial center has derivation algebra complete.

Proof. Suppose that $X$ is a characteristically simple non-abelian group and suppose that $\theta$ is an automorphism of $\text{Aut}(X)$. Since the center is always characteristic it follows that $Z(X) = 0$ and hence $c_X$ is a normal monomorphism. Forming the pullback

$$
\begin{array}{c}
K \\
\downarrow^{u}
\end{array}
\begin{array}{c}
X \\
\downarrow^{c_X}
\end{array}

\begin{array}{c}
\downarrow^{X}
\end{array}
\begin{array}{c}
\downarrow^{\theta c_X}
\end{array}
\begin{array}{c}
\downarrow^{\text{Aut}(X)}
\end{array}

we see that $c_X u$ is normal and hence by Theorem 5.3 that $u$ is characteristic. By assumption this means that either $K = 0$ or $u$ is an isomorphism. However, since by Proposition 3.6 $c_X$ has trivial centralizer it follows that $K$ is not trivial and hence $u$ must be an isomorphism. Essentially the same argument implies that $v$ is an isomorphism which proves that $c_X$ is characteristic and hence by the previous theorem implies that $\text{Aut}(X)$ is complete.

Proposition 5.7. Let $\mathbb{C}$ be a category of interest in the sense of G. Orzech [33] such that the group operation (required to exist) is commutative, and let $X$ be an object in $\mathbb{C}$.

(i) If $X$ is perfect, then every normal monomorphism with domain $X$ is a characteristic monomorphism;

(ii) if $X$ is perfect, has trivial center, and the generic split extension with kernel $X$ exists, then $[X]$ is strong-complete.

Proof. Recall that for such a variety of universal algebras a subobject $S \subseteq Y$ is normal if and only if for each $s$ in $S$, each $y$ in $Y$ and each binary operation $*$ (excluding addition), $s * y$ and $y * s$ are in $S$. Recall also that $[X,X]$ is the subalgebra of $X$ generated by elements of the form $x_1 * x_2$ where $x_1$ and $x_2$ are elements of $X$ and $*$ is a binary operation (excluding addition). Now suppose $X$ is perfect (i.e. $[X,X] = X$), $X$ is a normal subobject of $Y$, and $Y$ is a normal subobject of $Z$. Since $\mathbb{C}$ is category of interest for binary operations $*$ and $\cdot$ we known that for some positive integer $n$ there are binary operations $*_1, \ldots, *_n$ and $\cdot_1, \ldots, \cdot_n$ and a term $w$ such that (in particular) for all $x_1, x_2$ in $X$ and $z$ in $Z$

$$(x_1 * x_2) \cdot z = w(x_1 *_1 (x_2 \cdot_1 z), \ldots, x_1 *_m (x_2 *_{m+1} z), x_2 *_{m+1} (x_1 \cdot_{m+1} z), \ldots, x_2 *_n (x_1 \cdot_n z)),$$

where $m$ is an integer between 0 and $n$. Therefore, since each $x_j \cdot_i z$ is in $Y$ it follows that $x_k *_i (x_j \cdot_i z)$ is in $X$ and hence $(x_1 * x_2) \cdot z$ is in $X$. A similar calculation shows that $z \cdot_0 (x_1 * x_2)$ is in $X$. Since $X$ is perfect we know that $X$ is generated by products and hence via the previous calculations is normal in $Z$. This proves (i). Combining (i) with Proposition 3.6 and Theorem 5.5 we obtain (ii).

Remark 5.8. Applying the previous proposition to the category of Lie algebras shows that a perfect Lie algebra with trivial center has derivation algebra complete.
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