ARCS, HYPERCUBES, AND GRAPHS AS QUOTIENTS OF PROJECTIVE FRAÎSSI LIMITS

GIANLUCA BASSO* AND RICCARDO CAMERLO**

Abstract. We establish some basic properties of quotients of projective Fraïssé limits and exhibit some classes of compact metric spaces that are the quotient of a projective Fraïssé limit of a projective Fraïssé family in a finite language. We prove the result for the arcs directly, and by applying some closure properties we obtain all hypercubes and graphs as well.

1. Introduction

Projective Fraïssé families of topological structures and their limits — called projective Fraïssé limits — for a given language $\mathcal{L}$ have been introduced by T. Irwin and S. Solecki in [3]. In that paper, the authors focused on a particular example, where $\mathcal{L} = \{R\}$ contained a unique binary relation symbol and the limit $P = (P, R^P)$ turned out to be endowed with an equivalence relation $R^P$, with the quotient $P/R^P$ being a pseudo-arc. The characterisation of all spaces that can be obtained, up to homeomorphism, as quotients $P/R^P$, where $(P, R^P)$ is the projective Fraïssé limit of a projective Fraïssé family of finite topological $\mathcal{L}$-structures, for $\mathcal{L}$ as above and $R^P$ an equivalence relation, has been settled in [2] and consists of the following list:

- Cantor space;
- disjoint sums of $m$ singletons and $n$ pseudo-arcs, with $m + n > 0$;
- disjoint sums of $n$ spaces each of the form $X = P \cup \bigcup_{j \in \mathbb{N}} Q_j$, where $P$ is a pseudo-arc, each $Q_j$ is a Cantor space which is clopen in $X$ and $\bigcup_{j \in \mathbb{N}} Q_j$ is dense in $X$.

The purpose of this note is to begin a systematic study of which spaces can be obtained as quotients of projective Fraïssé limits of projective Fraïssé families for more general languages. In section 2 we recall some basic definitions and fix some terminology and notations to be used in the rest of the paper. In section 3 we show some simple facts to be used later and note that, if we admit infinite languages, then every compact metric space can be obtained as a quotient of a projective Fraïssé limit. We therefore restrict our attention to finite languages and prove, in section 4, that the class of spaces that can be

2010 Mathematics Subject Classification. Primary 03E75; Secondary 54F15.

Key words and phrases. Topological structures, projective Fraïssé limits, hypercubes, graphs.

The research presented in this paper has been done while the second author was visiting the Department of information systems of the University of Lausanne. He wishes to thank the Équipe de logique, and in particular its director prof. Jacques Duparc, for providing such a friendly environment for research.
obtained as quotients of projective Fraïssé limits is closed under finite disjoint unions, finite products, and particular quotients satisfying some extra technical conditions. Section 5 presents some examples: after showing that arcs can be obtained as quotients of projective Fraïssé limits, the results of section 4 allow to extend this property to hypercubes and graphs. Finally, in section 6 we discuss some questions that appear naturally and that could lead to further research in the subject.

We note here that the construction of the projective Fraïssé limit has already been successfully employed in the literature, leading to interesting results on compact metric spaces and groups of homeomorphisms (see for example [4], [5], [1]).

2. Basic terminology and definitions

We recall here some basic definitions, mainly from [3], [2].

Let a first order language \( \mathcal{L} \) be given. A topological \( \mathcal{L} \)-structure is a zero-dimensional, (Hausdorff) compact, second countable space that is also an \( \mathcal{L} \)-structure such that:

- the interpretations of the relation symbols are closed sets;
- the interpretations of the function symbols are continuous functions.

An epimorphism between topological \( \mathcal{L} \)-structures \( A, B \) is a continuous surjection \( \varphi : A \to B \) such that:

- \( r^B = \varphi \times \ldots \times \varphi (r^A) \) for every \( n \)-ary relation symbol \( r \);
- \( f^B (\varphi (a_1), \ldots, \varphi (a_n)) = \varphi f^A (a_1, \ldots, a_n) \) for every \( n \)-ary function symbol \( f \) and \( a_1, \ldots, a_n \in A \);
- \( \varphi (c^A) = c^B \) for every constant symbol \( c \).

An isomorphism is a bijective epimorphism, so in particular it is an homeomorphism between the supports. An epimorphism \( \varphi : A \to B \) refines a covering \( \mathcal{U} \) of \( A \) if the preimage of any element of \( B \) is included in some element of \( \mathcal{U} \).

A family \( \mathcal{F} \) of topological \( \mathcal{L} \)-structures is a projective Fraïssé family if the following properties hold:

(JPP): (joint projection property) for every \( D, E \in \mathcal{F} \) there are \( F \in \mathcal{F} \) and epimorphisms \( F \to D, F \to E \);

(AP): (amalgamation property) for every \( C, D, E \in \mathcal{F} \) and epimorphisms \( \varphi_1 : D \to C, \varphi_2 : E \to C \) there are \( F \in \mathcal{F} \) and epimorphisms \( \psi_1 : F \to D, \psi_2 : F \to E \) such that \( \varphi_1 \psi_1 = \varphi_2 \psi_2 \).

Given a family \( \mathcal{F} \) of topological \( \mathcal{L} \)-structures, a topological \( \mathcal{L} \)-structure \( \mathbb{F} \) is a projective Fraïssé limit of \( \mathcal{F} \) if the following hold:

(L1): (projective universality) for every \( D \in \mathcal{F} \) there is some epimorphism \( \mathbb{F} \to D \);

(L2): for every finite discrete topological space \( A \) and continuous function \( f : \mathbb{F} \to A \), there are \( D \in \mathcal{F} \), an epimorphism \( \varphi : \mathbb{F} \to D \) and a function \( f' : D \to A \) such that \( f = f' \varphi \).
(L3): (projective ultrahomogeneity) for every $D \in \mathcal{F}$ and epimorphisms $\varphi_1, \varphi_2 : \mathcal{F} \to D$ there exists an isomorphism $\psi : \mathcal{F} \to \mathcal{F}$ such that $\varphi_2 = \varphi_1 \psi$.

Property (L2) is equivalent to

(L2'): for any clopen covering $\mathcal{U}$ of $\mathcal{F}$ there are $D \in \mathcal{F}$ and an epimorphism $\mathcal{F} \to D$ refining $\mathcal{U}$.

In [3] it is proved that every non-empty, at most countable, projective Fraïssé family of finite topological $\mathcal{L}$-structures has a projective Fraïssé limit, which is unique up to isomorphism.

If $\mathcal{F}$ is a class of topological $\mathcal{L}$-structures, a fundamental sequence $(D_n, \pi_n)$ is a sequence of elements of $\mathcal{F}$ together with epimorphisms $\pi_n : D_{n+1} \to D_n$ such that, denoting $\pi^m_n = \pi_n \cdot \ldots \cdot \pi_{m-1} : D_m \to D_n$ for $n < m$ and letting $\pi_n^n : D_n \to D_n$ be the identity, the following properties hold:

- for every $D \in \mathcal{F}$ there are $n$ and an epimorphism $D_n \to D$;
- for any $n$, any $E, F \in \mathcal{F}$ and any epimorphisms $\varphi_1 : F \to E$, $\varphi_2 : D_n \to E$, there exist $m \geq n$ and an epimorphism $\psi : D_m \to F$ such that $\varphi_1 \psi = \varphi_2 \pi^m_n$.

To study projective Fraïssé limits it is enough to consider fundamental sequences, due to the following fact whose details can be found in [2].

**Proposition 1.** Let $\mathcal{F}$ be a non-empty, at most countable projective Fraïssé family of finite topological $\mathcal{L}$-structures. Then the following are equivalent.

1. $\mathcal{F}$ is a projective Fraïssé family;
2. $\mathcal{F}$ has a projective Fraïssé limit;
3. $\mathcal{F}$ has a fundamental sequence.

Moreover, in this case the projective Fraïssé limits of $\mathcal{F}$ and of its fundamental sequence coincide. A limit for both is the inverse limit of the fundamental sequence.

In the sequel, whenever we denote a language with a subscript, like in $\mathcal{L}_R$, we mean that the language contains a distinguished binary relation symbol represented in the subscript (in this case the symbol $R$). The following definition is central.

**Definition 1.** A compact metric space $X$ is $\mathcal{L}_R$-representable if there exists an at most countable projective Fraïssé family of finite topological $\mathcal{L}_R$-structures such that, denoting $\mathcal{F}$ its projective Fraïssé limit, $R^\mathcal{F}$ is an equivalence relation and $\mathcal{F}/R^\mathcal{F}$ is homeomorphic to $X$.

Space $X$ is finitely representable if it is $\mathcal{L}_R$-representable for some finite $\mathcal{L}_R$.

In this terminology, when $\mathcal{L}_R = \{R\}$, the $\mathcal{L}_R$-representable spaces have been characterised in [2].

3. **Some preliminary facts**

In this section we collect some basic properties of projective Fraïssé families and their limits.
Proposition 2. Let $\mathcal{L}_R = \{R, \ldots\}$ and suppose $\mathcal{F}$ is a projective Fraïssé family in the language $\mathcal{L}_R$. Let $\mathbb{F} = (\mathbb{F}, R^2, \ldots)$ be the projective Fraïssé limit of $\mathcal{F}$.

1. If $R$ is interpreted by all structures in $\mathcal{F}$ as a reflexive relation, then $R^2$ is reflexive as well.
2. If $R$ is interpreted by all structures in $\mathcal{F}$ as a symmetric relation, then $R^2$ is symmetric as well.
3. If $R$ is interpreted by all structures in $\mathcal{F}$ as an anti-symmetric relation, then $R^2$ is anti-symmetric as well.
4. If $R$ is interpreted by all structures in $\mathcal{F}$ as a transitive relation, then $R^2$ is transitive as well.
5. If $R$ is interpreted by all structures in $\mathcal{F}$ as a total relation, then $R^2$ is total as well.
6. If $R$ is interpreted by all structures in $\mathcal{F}$ as having a first (respectively, last) element, then $R^2$ has a first (respectively, last) element as well.
7. If $R$ is interpreted by all structures in $\mathcal{F}$ as a connected relation, then for any partition $\{U, V\}$ of $\mathbb{F}$ into clopen sets there are $x \in U, y \in V$ with $xR^2y$.

Proof. In this proof we will use property (L2') extensively.

1. and (2) The proof is similar to the argument carried out in the proof of [3, lemma 4.1].

3. Let $x, y \in \mathbb{F}$ be distinct elements such that $xR^2y$. Pick a clopen subset $U$ of $\mathbb{F}$ such that $x \in U, y \notin U$ and find $A \in \mathcal{F}$ with an epimorphism $\varphi : \mathbb{F} \to A$ refining $\{U, \mathbb{F} \setminus U\}$. Then $\varphi(x), \varphi(y)$ are distinct and $\varphi(x)R^A \varphi(y)R^A \varphi(x)$.

4. Let $x, y, z \in \mathbb{F}$, with $xR^2yR^2z$. Since $R^2$ is closed, it is enough to show that for any neighbourhoods $U$ of $x$ and $V$ of $z$ there are $x' \in U, z' \in V$ with $x'R^2z'$. Let $U' \subseteq U, V' \subseteq V$ be clopen neighbourhoods of $x, z$, respectively, with $U' = V'$ if $x = z$ and $U' \cap V' = \emptyset$ otherwise. Let $A \in \mathcal{F}$ and $\varphi : \mathbb{F} \to A$ be an epimorphism refining the clopen covering $\{U', V', \mathbb{F} \setminus (U' \cup V')\}$. Since $\varphi(x)R^A \varphi(z)$, there are $x' \in U', z' \in V'$ with $\varphi(x) = \varphi(x'), \varphi(z) = \varphi(z')$, $x'R^2z'$.

5. It is enough to show that, given $x, y \in \mathbb{F}$, whenever $U, V$ are clopen neighbourhoods of $x, y$, respectively, there are $x' \in U, y' \in V$ such that either $x'R^2y'$ or $y'R^2x'$. Moreover, if $x = y$ it can be assumed that $U = V$, while for $x \neq y$ one can take $U \cap V = \emptyset$. Let $A \in \mathcal{F}$ with an epimorphism $\varphi : \mathbb{F} \to A$ refining the clopen covering $\{U, V, \mathbb{F} \setminus (U \cup V)\}$. Since $\varphi(x)R^A \varphi(y)$ or $\varphi(y)R^A \varphi(x)$, there are $x' \in U, y' \in V$ such that $\varphi(x') = \varphi(x), \varphi(y') = \varphi(y)$ and either $x'R^2y'$ or $y'R^2x'$.

6. Argue for the first element, the situation for the last being similar. Fix a compatible complete metric on $\mathbb{F}$ and, for each positive integer $n$, let $U_n$ be a partition of $\mathbb{F}$ with clopen sets of diameter less than $\frac{1}{n}$ such that $U_{n+1}$ refines $U_n$. Let $\varphi_n : \mathbb{F} \to A_n$ be an epimorphism refining $U_n$ onto some $A_n \in \mathcal{F}$. Let $x_n \in \mathbb{F}$ be such that $\varphi_n(x_n)$ is the first element of $R^{A_n}$ and fix a limit point $x$ of the sequence $x_n$, in order to show that $\forall y \in \mathbb{F} xR^2y$. For this it is enough to prove that given clopen neighbourhoods $U, V$ of $x, y$, respectively, where it can
be assumed that \( U = V \) if \( x = y \) and that \( U \cap V = \emptyset \) if \( x \neq y \), there are \( x', y' \in U \), \( y' \in V \) with \( x'R^f y' \). Take \( n \) such that if \( x \in W \in \mathbb{U}_n \) and \( y \in W' \in \mathbb{U}_n \), then \( W \subseteq U, W' \subseteq V \). Let \( n' \geq n \) be such that \( x_{n'} \in W \). Notice that \( \varphi_{n'} \) refines \( \{ W, W', \mathbb{F} \setminus (W \cup W') \} \). Since \( \varphi_{n'}(x_{n'}) = \varphi(y) \), there are \( x' \in W, y' \in W' \) such that \( \varphi_{n'}(x') = \varphi_{n'}(x_{n'}), \varphi_{n'}(y') = \varphi_{n'}(y), x'R^f y' \).

(7) As for the argument in the proof of [3, lemma 4.3].

Notice that for (1),(2),(5),(6),(7) the converse holds as well.

**Proposition 3.** Let \( \mathcal{L} \) be a language and let \( \mathcal{L}' \) be obtained from \( \mathcal{L} \) by replacing each constant symbol \( c \) with a unary relation symbol \( R_c \) and each function symbol \( f \), say with \( m \) arguments, with a new relation symbol \( R_{f} \), with \( m + 1 \) arguments. For every \( \mathcal{L} \)-structure \( A \), let \( A' \) be the \( \mathcal{L}' \)-structure defined as follows.

- \( A' \) has the same universe as \( A \);
- the interpretations in \( A' \) of the relation symbols of \( \mathcal{L} \) are the same as in \( A \);
- if \( c \) is a constant symbol of \( \mathcal{L} \), then \( R^A_c(a) \iff c^A = a \);
- if \( f \) is an \( m \)-ary function symbol of \( \mathcal{L} \), then \( R^A_f \) is the graph of \( f^A \).

Then

1. \( \varphi : A \to B \) is an \( \mathcal{L} \)-epimorphism if and only if \( \varphi : A' \to B' \) is an \( \mathcal{L}' \)-epimorphism.
2. If \( \mathcal{F} \) is a non-empty, at most countable, projective Fraïssé family of \( \mathcal{L} \)-structures and \( \mathcal{F}' \) is the class obtained from \( \mathcal{F} \) by replacing each \( A \in \mathcal{F} \) with the \( \mathcal{L}' \)-structure \( A' \), then \( \mathcal{F}' \) is a projective Fraïssé family.

Moreover, if \( \mathcal{F} \) is a projective Fraïssé limit of \( \mathcal{F} \), then a projective Fraïssé limit \( \mathcal{F}' \) of \( \mathcal{F}' \) can be constructed as follows:

- the universes of \( \mathcal{F}, \mathcal{F}' \) are the same;
- the interpretations of all relation symbols of \( \mathcal{L} \) are the same;
- for every constant symbol \( c \) of \( \mathcal{L} \), \( R^F_c(x) \iff c^F = x \);
- for every function symbol \( f \) of \( \mathcal{L} \), \( R^F_f \) is the graph of \( f^F \).

**Proof.** (1) Suppose \( \varphi : A \to B \) is an epimorphism. Take an \( m \)-ary function symbol \( f \) in \( \mathcal{L} \). Then \( R^B_f(b_1, \ldots, b_m, b_{m+1}) \iff f^B(b_1, \ldots, b_m) = b_{m+1} \iff \exists a_1, a_2, a_{m+1} (\varphi(a_1) = b_1 \land \ldots \land \varphi(a_m) = b_m \land \varphi(a_{m+1}) = b_{m+1} \land f^A(a_1, \ldots, a_m) = a_{m+1} \iff \exists a_1, \ldots, a_{m+1} (\varphi(a_1) = b_1 \land \ldots \land \varphi(a_{m+1}) = b_{m+1} \land R^A_f(a_1, \ldots, a_{m+1})) \). Similarly, if \( c \in \mathcal{L} \) is a constant symbol, \( R^B_c(b) \iff c^B = b \iff c = \varphi(c^A) \iff \exists a(b = \varphi(a) \land R^A_c(a)) \). So \( \varphi : A' \to B' \) is an epimorphism.

Conversely, assume \( \varphi : A' \to B' \) is an epimorphism and consider again an \( m \)-ary function symbol \( f \in \mathcal{L} \). Then \( f^A(a_1, \ldots, a_m) = a_{m+1} \iff R^A_f(a_1, \ldots, a_m, a_{m+1}) \iff \exists a^B(\varphi(a_1), \ldots, \varphi(a_m), \varphi(a_{m+1})) \iff f^B(\varphi(a_1), \ldots, \varphi(a_m)) = \varphi(a_{m+1}) \). Similarly, for a constant symbol \( c \in \mathcal{L} \), one has \( a = c^A \iff R^A_c(a) \iff R^B_c(\varphi(a)) \iff \varphi(a) = c^B \). Thus \( \varphi : A \to B \) is an epimorphism too.
(2) From (1), if \((D_n)\) is a fundamental sequence for \(\mathcal{F}\), then \((D'_n)\) is a fundamental sequence for \(\mathcal{F}'\) endowed with the same family of epimorphisms. Since the projective Fraïssé limits can be computed as inverse limits, the result about the universe of the limits and the interpretation of the relation symbols of \(\mathcal{L}\) follows. Finally, denote by \(\pi_n^\infty\) the projections of the limits onto the members of the fundamental sequence. Let \(c\) be a constant symbol of \(\mathcal{L}\); then \(R_c^\mathcal{L}(x) \iff \forall n \in \mathbb{N} \ R_{c}^{D_n}(\pi_n^\infty(x)) \iff \forall n \in \mathbb{N} \ c^{D_n} = \pi_n^\infty(x) \iff c^\mathcal{L} = x\). Similarly, for an \(m\)-ary function symbol \(f\) of \(\mathcal{L}\), one has \(R_f^\mathcal{L}(x_1, \ldots, x_m, x_{m+1}) \iff \forall n \in \mathbb{N} \ R_{f}^{D_n}(\pi_n^\infty(x_1), \ldots, \pi_n^\infty(x_m), \pi_n^\infty(x_{m+1})) \iff \forall n \in \mathbb{N} \ f^{D_n}(\pi_n^\infty(x_1), \ldots, \pi_n^\infty(x_m)) = \pi_n^\infty(x_{m+1}) \iff f^\mathcal{L}(x_1, \ldots, x_m) = x_{m+1}\). □

The import of proposition 3 is that for our purposes we are always allowed to consider only relational languages. For later use, note that it follows that the groups of isomorphisms of \(\mathcal{F}\) and of \(\mathcal{F}'\) coincide.

Now we show that if one admits infinite languages, then every compact metric space is homeomorphic to the quotient of a projective Fraïssé limit. Consequently, in the sequel we will be interested in studying what kind of spaces can be obtained with finite languages.

**Lemma 4.** Let \(\mathcal{L}\) be any language and let \(D_0, D_1, \ldots\) be a family of finite topological \(\mathcal{L}\)-structures. If for every \(n \leq m\) there is exactly one epimorphism \(\pi_n^m : D_m \to D_n\), then \((D_n, \pi_n^{n+1})\) is a fundamental sequence.

**Proof.** First notice that from the hypothesis it follows that for any \(n, m\) there is at most one epimorphism \(D_m \to D_n\). If \(n \leq m\) this is in the hypothesis; on the other hand, if \(n > m\), the existence of an epimorphism \(\varphi : D_m \to D_n\) implies that \(\varphi\) and \(\pi_n^m\) are actually isomorphisms; if there were two different isomorphisms \(D_m \to D_n\), their compositions with \(\pi_n^m\) would yield two different isomorphisms \(D_n \to D_n\).

Consequently, any two epimorphisms \(\varphi_1 : D_h \to D_k, \varphi_2 : D_p \to D_k\) and letting \(m = \max(h, p)\), one has \(\varphi_1\pi_n^m = \varphi_2\pi_n^m\). □

**Proposition 5.** Let \(\mathcal{L}_R = \{R, \rho_s\}_{s \in 2^{<\omega}}\), where the \(\rho_s\) are unary relation symbols for all \(s \in 2^{<\omega}\). Then every compact metric space is \(\mathcal{L}_R\)-representable.

**Proof.** Let \(X\) be a compact metric space and let \(\equiv\) be a closed equivalence relation on \(2^\mathbb{N}\) such that \(X \cong 2^\mathbb{N} / \equiv\). Define \(\mathcal{L}_R\)-structures \(D_n = (2^n, R_{D_n}, \rho_{D_n}^s)_{s \in 2^{<\omega}}\) by letting

\[
\begin{align*}
&u R_{D_n} u' \iff \exists x, x' \in 2^n \ (u \subseteq x \wedge u' \subseteq x' \wedge x \equiv x') \\
&\rho_{D_n}^s(u) \iff s \subseteq u \lor u \subseteq s
\end{align*}
\]

Now notice that, given \(n \leq m\), the only epimorphism \(D_m \to D_n\) is the restriction map \(\pi_n^m\) defined by \(\pi_n^m(w) = w_{1n}\). Indeed, if \(w, w' \in 2^m\) are such that \(w R_{D_n} w'\), let \(x, x' \in 2^\mathbb{N}\) with \(w \subseteq x, w' \subseteq x', x \equiv x'\); since \(x_{1n} = \pi_n^m(w), x'_{1m} = \pi_n^m(w')\), it follows that \(\pi_n^m(w) R_{D_n} \pi_n^m(w')\). Moreover, if \(w \in 2^m\) satisfies \(\rho_{D_n}^s(w)\) for some \(s \in 2^{<\omega}\), so that \(w\) is compatible with \(s\), its restriction \(w_{1n}\) is compatible with \(s\) as well, so \(\rho_{D_n}^s(\pi_n^m(w))\) holds. Conversely, assume first that \(u, u' \in 2^m\) fulfill \(u R_{D_n} u'\) and let \(x, x' \in 2^\mathbb{N}\) such that \(u \subseteq x, u' \subseteq x', x \equiv x'\); then \(x_{1n} R_{D_n} x'_{1m}, \pi_n^m(x_{1m}) = u, \pi_n^m(x'_{1m}) = u'\). Finally, suppose
that $s \in 2^{<\omega}$, $u \in 2^n$ are such that $\rho^{D_n}_s(u)$; then there is at least an element $w \in 2^m$ such that $\pi^m(w) = w^1_n = u$ and $w$ is compatible with $s$, so that $\rho^{D_m}_s(w)$. To see that $\pi^m$ is the unique epimorphism $D_m \to D_n$, notice that for any $w \in 2^m$, the unique element $u \in 2^n$ such that $\rho^{D_n}_w(u)$ is $w^1_n$.

Consequently, by lemma 4, $(D_n, \pi^{n+1})$ is a fundamental sequence. Let $F = (2^m, R^F, \rho^F_n)_{n \in 2^{<\omega}}$ be its inverse limit. It is now enough to prove $R^F = \equiv$, so let $x, x' \in 2^m$. If $x = x'$, then $\forall n \in \mathbb{N} x^1_n R^D_n x^1_n$, so that $x R^F x'$. Conversely, if $x R^F x'$, so that $\forall n \in \mathbb{N} x^1_n R^D_n x'^1_n$, for every $n \in \mathbb{N}$ there are $x_n, x'_n \in 2^S$ such that $x^1_n \subseteq x_n, x'^1_n \subseteq x'_n$, $x_n \equiv x'_n$, so that $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} x'_n = x'$, $x \equiv x'$, since $\equiv$ is closed. \qed

4. Closure under topological operations

This section collects some closure properties of finitely representable spaces. We will need the following notion.

Definition 2. Let $F$ be a projective Fraïssé limit of a projective Fraïssé family of topological structures in some language $L_R$ and suppose that $R^F$ is an equivalence relation. A point $x \in F$ is almost stable if, for all isomorphisms $\varphi$ of $F$, one has that $\varphi(x) R^F x$.

Notice that the set of almost stable points is invariant under the equivalence relation $R^F$.

Theorem 6. The finite disjoint sum of finitely representable spaces is finitely representable.

Proof. It is enough to prove the result for the disjoint sum of two spaces\footnote{Notice that for the sum of $n$ spaces, a direct proof would provide a smaller language than the one resulting by iterating the construction in the proof.}. So, for $i \in \{1, 2\}$ let $X_i$ be $L^i_R$-representable for some finite $L^i_R$, as witnessed by a projective Fraïssé family $F_i$ with limit $F_i$. By proposition 3 one can assume that each $L^i_R$ is a relational language, and moreover $L^1_R \cap L^2_R = \{ R \}$. Let $L_R = L^1_R \cup L^2_R \cup \{ P_1, P_2 \}$, where $P_1, P_2$ are new unary relation symbols.

Given topological $L^i_R$-structures $A_i$, for $i \in \{1, 2\}$, define an $L_R$-structure $A = A_1 \oplus A_2$ as follows:

- $A$ is a disjoint union $A_1 \cup A_2$, with each $A_i$ clopen in $A$;
- $R^A = R^{A_1} \cup R^{A_2}$;
- $P^A_i = A_i$;
- if $S \in L^i_R$ is a relation symbol different from $R$, then $S^A = S^{A_i}$.

Notice that if $\varphi_i : A_i \to B_i$ are $L_R$-epimorphisms for $i \in \{1, 2\}$, then $\varphi_1 \cup \varphi_2 : A_1 \oplus A_2 \to B_1 \oplus B_2$ is an $L_R$-epimorphism. Conversely, if $\varphi : A_1 \oplus A_2 \to B_1 \oplus B_2$ is an $L_R$-epimorphism, then by the interpretations of symbols $P_1, P_2$, the restriction $\varphi_i$ of $\varphi$ to $A_i$ has range included in — in fact, equal to — $B_i$; moreover $\varphi_i : A_i \to B_i$ is an $L^i_R$-epimorphism.

Define $\mathcal{F}$ as the class of $L_R$-structures $A = (A, R^A, \ldots, P_1^A, P_2^A)$ of the form $A = A_1 \oplus A_2$, where $A_i \in \mathcal{F}_i$.

Claim 6.1. $\mathcal{F}$ is a projective Fraïssé family.
Proof of claim. JPP: Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2 \in \mathcal{F}$. By (JPP) of $\mathcal{F}$, let $C_i \in \mathcal{F}$, with epimorphisms $\varphi_i : C_i \rightarrow A_i$, $\psi_i : C_i \rightarrow B_i$. Set $C = C_1 \oplus C_2 \in \mathcal{F}$, $\varphi = \varphi_1 \cup \varphi_2 : C \rightarrow A$, $\psi = \psi_1 \cup \psi_2 : C \rightarrow B$. Then $\varphi, \psi$ are epimorphisms.

AP: Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2 \in \mathcal{F}$, with epimorphisms $\varphi : B \rightarrow A$, $\psi : C \rightarrow A$. So let $\varphi_i = \varphi_1 B_i$, $\psi_i = \psi_1 c_i$, then $\varphi_i : B_i \rightarrow A_i$, $\psi_i : C_i \rightarrow A_i$ are epimorphisms. By (AP) for $\mathcal{F}$, let $D_i \in \mathcal{F}$, $\varphi'_i : D_i \rightarrow B_i$, $\psi'_i : D_i \rightarrow C_i$ be epimorphisms such that $\varphi_i \varphi'_i = \psi_i \psi'_i$. Let $D = D_1 \oplus D_2 \in \mathcal{F}$. So $\varphi' = \varphi'_1 \cup \varphi'_2 : D \rightarrow B$, $\psi' = \psi'_1 \cup \psi'_2 : D \rightarrow C$ are epimorphisms such that $\varphi \varphi' = \psi \psi'$.

Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$.

Claim 6.2. $\mathcal{F}$ is the projective Fraïssé limit of $\mathcal{F}$.

Proof of claim. It is enough to carry out the following three verifications:

- (L1) Let $A = A_1 \oplus A_2 \in \mathcal{F}$. By projective universality of $\mathcal{F}$, let $\varphi_i : \mathcal{F}_i \rightarrow A_i$ be an epimorphism. Then $\varphi = \varphi_1 \cup \varphi_2 : \mathcal{F} \rightarrow A$ is an epimorphism.

- (L2') Let $\mathcal{U}$ be a partition of $\mathcal{F}$ into clopen sets, which can be assumed to refine $\{\mathcal{F}_1, \mathcal{F}_2\}$. So $\mathcal{U} \cap \mathcal{P}(\mathcal{F}_i)$ is a partition of $\mathcal{F}_i$ into clopen sets. Let $A_i \in \mathcal{F}_i$ with an epimorphism $\varphi_i : \mathcal{F}_i \rightarrow D_i$ refining $\mathcal{U} \cap \mathcal{P}(\mathcal{F}_i)$. So $\varphi = \varphi_1 \cup \varphi_2 : \mathcal{F} \rightarrow A_1 \oplus A_2$ is an epimorphism refining $\mathcal{U}$.

- (L3) Let $A = A_1 \oplus A_2 \in \mathcal{F}$, with epimorphisms $\varphi_1, \varphi_2 : \mathcal{F} \rightarrow A$. So $\varphi_1, \varphi_2$ are epimorphisms $\mathcal{F}_i \rightarrow A_i$. By projective ultrahomogeneity of $\mathcal{F}$, let $\psi_i : \mathcal{F}_i \rightarrow \mathcal{F}_i$ be an isomorphism such that $\varphi_1 \psi : \mathcal{F}_i \rightarrow \mathcal{F}_i$. So $\psi_i : \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism such that $\varphi_1 \psi = \varphi_2$.

Notice that $R^2$ is an equivalence relation on $\mathcal{F}$ and $\mathcal{F} / R^2$ is a disjoint sum of $\mathcal{F}_1 / R_{\mathcal{F}_1}, \mathcal{F}_2 / R_{\mathcal{F}_2}$, completing the proof.

For later use we remark that in the proof of theorem 6, if $x$ is an almost stable point in one of the $\mathcal{F}_i$, then $x$ is almost stable also in the resulting $\mathcal{F}$.

Theorem 7. The finite product of finitely representable spaces is finitely representable.

Proof. It is enough to prove the assertion for products of two factors$^2$. So, for $i \in \{ 1, 2 \}$ let $X_i$ be $\mathcal{L}_R^1$-representable, for some finite $\mathcal{L}_R^0$, as witnessed by a projective Fraïssé family $\mathcal{F}_i$ with limit $\mathcal{F}_i$. By proposition 3 it can be assumed that $\mathcal{L}_R^1, \mathcal{L}_R^2$ are relational languages, and moreover $\mathcal{L}_R \cap \mathcal{L}_R^0 = \{ R \}$. Let $\mathcal{L}_R = \mathcal{L}_R^1 \cup \mathcal{L}_R^2 \cup \{ r_1, r_2 \}$, where $r_1, r_2$ are two new binary relation symbols.

Let $\mathcal{F} = \{ A \times B \mid A \in \mathcal{F}_1, B \in \mathcal{F}_2 \}$ where:

- $(a, b) R^A \times B(a', b') \iff a R^A a' \land b R^B b'$;
- $S^A \times B((a_1, b_1), \ldots, (a_m, b_m)) \iff S^A(a_1, \ldots, a_m)$ for any $m$-ary relation symbol $S \in \mathcal{L}_R^1 \setminus \{ R \}$;
- $S^A \times B((a_1, b_1), \ldots, (a_m, b_m)) \iff S^B(b_1, \ldots, b_m)$ for any $m$-ary relation symbol $S \in \mathcal{L}_R^2 \setminus \{ R \}$.

$^2$Remarks about the language similar to those in theorem 6 apply here.
\[ r_1^{A \times B}((a_1, b_1), (a_2, b_2)) \iff a_1 = a_2; \]
\[ r_2^{A \times B}((a_1, b_1), (a_2, b_2)) \iff b_1 = b_2. \]

**Claim 7.1.** \( \varphi : A \times B \to C \times D \) is an epimorphism if and only if \( \varphi = \psi \times \theta \) for some epimorphisms \( \psi : A \to C, \theta : B \to D \).

**Proof of claim.**
Let \( \varphi : A \times B \to C \times D \) be an epimorphism. Since \( r_1^{A \times B}(a, b_1), (a, b_2)) \), from \( r_1^{C \times D}(\varphi(a, b_1), \varphi(a, b_2)) \) it follows that \( \varphi(a, b_1), \varphi(a, b_2) \) have the same first component; similarly for \( \varphi(a_1, b), \varphi(a_2, b) \). This means that \( \psi = \varphi \times \theta \) for some surjective \( \psi : A \to C, \theta : B \to D \). It remains to prove that \( \psi \), and similarly \( \theta \), are epimorphisms.

Suppose \( cR^C \). Since \( R^D \) is reflexive, by reflexivity of \( R^F \) and proposition 2(1), it follows that for any \( d \in D \) one has \( (c, d)R^{C \times D}(c', d) \). So there are \( (a, b), (a', b') \in A \times B \) such that \( \varphi(a, b) = (c, d), \varphi(a', b') = (c', d), (a, b)R^{A \times B}(a', b') \). Consequently, \( \psi(a) = c, \psi(a') = c', \varphi^{A \times B}(a, b) \). Conversely, if \( aR^{A \times B}d' \), for any \( b \in B \) one has \( (a, b)R^{A \times B}(a', b) \), hence \( (\psi(a), \theta(b))R^{C \times D}(\psi(a'), \theta(b)) \), so \( \psi(a)R^C \psi(a') \).

Let \( S \in L_2^R \setminus \{R\} \) be an \( m \)-ary relation symbol. If \( S^C(c_1, \ldots, c_m) \), for any \( d \in D \) one has \( S^{C \times D}((c_1, d), \ldots, (c_m, d)) \). Let \( a_1, \ldots, a_m \in A, b_1, \ldots, b_m \in B \) with \( \varphi(a_1, b_1) = (c_1, d), \varphi(a_m, b_m) = (c_m, d) \), \( S^{A \times B}((a_1, b_1), \ldots, (a_m, b_m)) \). This implies \( \psi(a_1) = c_1, \ldots, \psi(a_m) = c_m \), \( S^A(a_1, \ldots, a_m) \). Conversely, whenever \( S^A(a_1, \ldots, a_m) \), picking any \( b \in B \), one has \( S^{A \times B}((a_1, b), \ldots, (a_m, b)) \), whence \( S^{C \times D}(\varphi(a_1, b), \ldots, \varphi(a_m, b)) \), which allows to conclude that \( S^C(\psi(a_1), \ldots, \psi(a_m)) \).

Assume now \( \psi : A \to C, \theta : B \to D \) are epimorphisms, and set \( \varphi = \psi \times \theta \).

Then, for any \( (c, d), (c', d') \in C \times D \),
\[
(c, d)R^{C \times D}(c', d') \iff cR^C c' \land dR^D d' \iff \\
\iff \exists a, a' \in A \exists b, b' \in B \\
(\psi(a) = c \land \psi(a') = c' \land \theta(b) = d \land \theta(b') = d' \land aR^A a' \land bR^B b') \iff \\
\iff \exists a, a' \in A \exists b, b' \in B \\
(\varphi(a, b) = (c, d) \land \varphi(a', b') = (c', d') \land (a, b)R^{A \times B}(a', b')).
\]

Moreover, if \( S \in L_2^R \setminus \{R\} \) is an \( m \)-ary relation symbol and \( S^{A \times B}((a_1, b_1), \ldots, (a_m, b_m)) \), then \( S^A(a_1, \ldots, a_m) \), whence \( S^C(\psi(a_1), \ldots, \psi(a_m)) \) and finally \( S^{C \times D}(\varphi(a_1, b_1), \ldots, \varphi(a_m, b_m)) \). Conversely, suppose \( S^{C \times D}((c_1, d_1), \ldots, (c_m, d_m)) \), which is equivalent to \( S^C(c_1, \ldots, c_m) \). So there are \( a_1, \ldots, a_m \in A \) such that \( \psi(a_1) = c_1, \ldots, \psi(a_m) = c_m \), \( S^A(a_1, \ldots, a_m) \). Taking any \( b_1, \ldots, b_m \in B \) such that \( \theta(b_1) = d_1, \ldots, \theta(b_m) = d_m \) one has \( \varphi(a_1, b_1) = (c_1, d_1), \ldots, \varphi(a_m, b_m) = (c_m, d_m) \), \( S^{A \times B}((a_1, b_1), \ldots, (a_m, b_m)) \). Similarly for symbols in \( L_2^R \).

**Claim 7.2.** \( \mathcal{F} \) is a projective Fraïssé family.

**Proof of claim.** JPP: Let \( A \times B, C \times D \in \mathcal{F} \). By (JPP) of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), let \( E \in \mathcal{F}_1, F \in \mathcal{F}_2 \) with epimorphisms \( \varphi_1 : E \to A, \varphi_2 : E \to C, \psi_1 : F \to B \),
$\psi_2 : F \to D$. Then $\varphi_1 \times \psi_1 : E \times F \to A \times B$, $\varphi_2 \times \psi_2 : E \times F \to C \times D$ are epimorphisms.

**Claim 8.1.** Assume that $\varphi_1, \varphi_2, \psi_1, \psi_2$ such that $\varphi = \varphi_1 \times \varphi_2$, $\psi = \psi_1 \times \psi_2$. Using (AP) of $\mathcal{F}_1, \mathcal{F}_2$, let $D_1 \in \mathcal{F}_1$, $D_2 \in \mathcal{F}_2$ with epimorphisms $\theta_1 : D_1 \to B_1$, $\rho_1 : D_1 \to C_1$, $\theta_2 : D_2 \to B_2$, $\rho_2 : D_2 \to C_2$ be such that $\varphi \theta_1 = \psi \rho_1$, $\varphi_2 \theta_2 = \psi_2 \rho_2$. Thus $\theta_1 \times \theta_2 : D_1 \times D_2 \to B_1 \times B_2$, $\rho_1 \times \rho_2 : D_1 \times D_2 \to C_1 \times C_2$ are epimorphisms such that $\varphi(\theta_1 \times \theta_2) = \psi(\rho_1 \times \rho_2)$. □

Let now $(A_n, \pi_n), (B_n, \rho_n)$ be fundamental sequences for $\mathcal{F}_1, \mathcal{F}_2$, respectively.

**Claim 7.3.** $(A \times B_n, \pi_n \times \rho_n)$ is a fundamental sequence for $\mathcal{F}$.

**Proof of claim.** Let $A \times B \in \mathcal{F}$. There are $n, m \in \mathbb{N}$ and epimorphisms $\varphi : A_n \to A$, $\psi : B_m \to B$. If $n \leq m$, then $(\varphi \pi_n) \times \psi : A_m \times B_m \to A \times B$ is an epimorphism; otherwise, $\varphi \times (\psi \rho_n) : A_n \times B_n \to A \times B$ is.

Let now $E_1 \times E_2, F_1 \times F_2 \in \mathcal{F}$, $n \in \mathbb{N}$, with epimorphisms $\varphi_1 \times \varphi_2 : F_1 \times F_2 \to E_1 \times E_2$, $\psi_1 \times \psi_2 : A_n \times B_n \to E_1 \times E_2$. Let $m, m' \geq n$ with epimorphisms $\theta_1 : A_m \to F_1$, $\theta_2 : B_{m'} \to F_2$ be such that $\varphi_1 \theta_1 = \psi_1 \pi_m$, $\varphi_2 \theta_2 = \psi_2 \rho_{m'}$. Suppose for instance that $m \leq m'$. Then $(\varphi_1 \times \varphi_2)((\theta_1 \pi_{m}) \times \theta_2) = (\psi_1 \times \psi_2)(\pi_{m'} \times \rho_{m'}^m) : A_{m'} \times B_{m'} \to E_1 \times E_2$. □

So $\mathcal{F} = F_1 \times F_2$ is the support of the projective Fraïssé limit of $\mathcal{F}$. Moreover, denoting $\pi_n^\infty : F_1 \to A_n$, $\rho_n^\infty : F_2 \to B_n$ the projections of the limits onto the members of the fundamental sequences, and given $(a, b), (a', b') \in \mathcal{F}$,

$$(a, b)R^F(a', b') \iff \forall n \in \mathbb{N} (\pi_n^\infty(a), \rho_n^\infty(b))R^{A_n \times B_n}(\pi_n^\infty(a'), \rho_n^\infty(b')) \iff$$

$$\forall n \in \mathbb{N} (\pi_n^\infty(a)R^{A_n \pi_n^\infty(a')} \land \rho_n^\infty(b)R^{B_n \rho_n^\infty(b')} \iff aR^F_1 a' \land bR^F_2 b'$$

So $\mathcal{F}/R^F$ is homeomorphic to $\mathcal{F}_1/R^F_1 \times \mathcal{F}_2/R^F_2$.

**Theorem 8.** Let $X$ be a finitely representable metric space; say this is witnessed by a language $L_R$ and a homeomorphism $\Phi : X \to \mathcal{F}/R^F$. Let $G$ be the set of all almost stable points of $\mathcal{F}$. Let $\equiv$ be a closed equivalence relation on $\mathcal{F}$ such that $R^F \subseteq \equiv \subseteq \equiv \setminus G^2 = R^F \setminus G^2$. Let $\equiv$ be the equivalence relation defined on $X$ by letting

$$x \equiv y \iff \exists u, v \in \mathcal{F} (u \equiv v \land \pi(u) = \Phi(x) \land \pi(v) = \Phi(y))$$

where $\pi : \mathcal{F} \to \mathcal{F}/R^F$ is the quotient map.

Then $X' = X/\equiv$ is finitely representable.

**Proof.** By proposition 3 and the remark following it, we can assume that $L_R$ is a relational language. Let $\mathcal{F}$ be a projective Fraïssé family of finite topological $L_R$-structures of which $\mathcal{F}$ is a projective Fraïssé limit.

Notice that since $G$ is $R^F$-invariant, it is also $\equiv$-invariant.

**Claim 8.1.** Assume that $A \in \mathcal{F}$ and let $\varphi, \psi : F \to A$ be $L_R$-epimorphisms. Then $\varphi \times \varphi (\equiv) = \psi \times \psi (\equiv)$, that is, if $a, b \in A$ then there are $u, v \in \mathcal{F}$ such that $\varphi(u) = a$, $\varphi(v) = b$, $u \equiv v$ if and only if there are $u', v' \in \mathcal{F}$ such that $\psi(u') = a$, $\psi(v') = b$, $u' \equiv v'$.
Proof of claim. Let $\alpha : F \to F$ be an isomorphism such that $\varphi = \psi \alpha$. Assume that $a, b \in A$, $a, v \in F$ are such that $\varphi(u) = a$, $\varphi(v) = b$, $u \equiv v$. Denote $u' = \alpha(u)$, $v' = \alpha(v)$, so that $\psi(u') = a$, $\psi(v') = b$. By the $\equiv$-invariance of $G$, we have that either $u, v$ are both in $G$ or they are both outside $G$. If $u, v \notin G$, then $uR^F v$, so that $u'R^F v'$ and consequently $u' \equiv v'$. If instead $u, v \in G$, then $u'R^F u \equiv vR^F v'$ and again $u' \equiv v'$ and we are done. \hfill \Box

Set $\mathcal{L}_S = \mathcal{L}_R \cup \{S\}$, where $S$ is a new binary relation symbol. For every $A \in \mathcal{F}$ let $A'$ be the expansion of $A$ to $\mathcal{L}_S'$ defined by letting $S^{A'} = \varphi \times \varphi(\equiv)$ for any arbitrary $\mathcal{L}_R$-epimorphism $\varphi : F \to A$. Let $\mathcal{F}' = \{A'\}_{A \in \mathcal{F}}$.

Claim 8.2. Given $A, B \in \mathcal{F}$, a function $\varphi : A \to B$ is an $\mathcal{L}_R$-epimorphism if and only if it is an $\mathcal{L}_S'$-epimorphism from $A'$ to $B'$.

Proof of claim. The backward implication holds that as $A', B'$ are expansions of $A, B$, respectively.

For the forward direction, it is enough to show that $\varphi$ respects $S$. So let $a, b \in A$ be such that $aS^{A'} b$; pick any $\mathcal{L}_R$-epimorphism $\psi : F \to A$ and let $u, v \in F$ be such that $\psi(u) = a$, $\psi(v) = b$, $u \equiv v$. So, by claim 8.1, $u, v$, together with the $\mathcal{L}_R$-epimorphism $\varphi\psi : F \to B$, witness that $\varphi(u)S^{B'} \varphi(b)$. Conversely, let $a, b \in B$ be such that $aS^{B'} b$ and fix an arbitrary $\mathcal{L}_R$-epimorphism $\psi : F \to B$; then there are $u, v \in F$ such that $a = \psi(u)$, $b = \psi(v)$, $u \equiv v$. Let $\theta : F \to A$ be an $\mathcal{L}_R$-epimorphism such that $\varphi\theta = \psi$; such an epimorphism exists by combining (L1) and (L3). Then, again by claim 8.1, $\theta(u)S^A \theta(v)$, $\varphi\theta(u) = a$, $\varphi\theta(v) = b$ and we are done. \hfill \Box

By the claim, $\mathcal{F}'$ is a projective Fraïssé family and a projective Fraïssé limit $F'$ of $\mathcal{F}'$ is an expansion of $F$ to $\mathcal{L}'_S$. As for the interpretation of $S$ in $F'$, we have the following.

Claim 8.3. $S^{F'} = \equiv$.

Proof of claim. Let $u, v \in F'$ and assume first $uS^{F'} v$. By the closure of $\equiv$, to show $u \equiv v$ it is enough to prove that for any clopen neighbourhoods $U, V$ of $u, v$ respectively, there are $u' \in U$, $v' \in V$ with $u' \equiv v'$, where we can take $U = V = \emptyset$ if $u = v$, and $U \cap V = \emptyset$ otherwise. So let $A' \in \mathcal{F}'$ with an epimorphism $\varphi : F' \to A'$ refining the clopen covering $\{U, V, F' \setminus (U \cup V)\}$. Since $\varphi(u)S^A \varphi(v)$, there are $u', v' \in F'$ with $\varphi(u') = \varphi(u)$, $\varphi(v') = \varphi(v)$, $u' \equiv v'$. Since it follows that $u' \equiv v' \in V$, we are done.

Conversely, suppose $u \equiv v$. Again, fix any clopen neighbourhoods $U, V$ of $u, v$, respectively, such that $U = V = \emptyset$ if $u = v$, and $U, V$ disjoint otherwise. Pick $A' \in \mathcal{F}'$ and an epimorphism $\varphi : F' \to A'$ refining the clopen covering $\{U, V, F' \setminus (U \cup V)\}$. Since $\varphi(u)S^A \varphi(v)$, there are $u', v' \in F'$ (actually $u' \in U$, $v' \in V$) with $u'S^F v'$, and we are done again. \hfill \Box

To finish the proof, notice that $X'$ is homeomorphic to $F/\equiv$. \hfill \Box
5. ARCS, HYPERCUBES, GRAPHS

We now apply the results of the preceding sections to demonstrate the finite representability of some classes of continua. We begin by establishing the following.

**Theorem 9.** Arcs are finitely representable.

We prove theorem 9 through a sequence of lemmas.

Let $\mathcal{L}_R = \{R, \leq\}$, where $\leq$ is a binary relation symbol. Let $\mathcal{X}$ be the class of those finite topological $\mathcal{L}_R$-structures $A$ such that:

- $\leq^A$ is a total order;
- $aR^Ab$ if and only if $a = b$ or $a, b$ are $\leq^A$-consecutive.

**Lemma 10.** Class $\mathcal{X}$ is a projective Fraïssé family.

**Proof.** If $A = \{1\} \in \mathcal{X}$ is defined by letting $R^A = \leq^A = \{(1, 1)\}$, then for any $B \in \mathcal{X}$ the constant map $\varphi : B \to A$ is an epimorphism. So it is enough to verify (AP).

Let $A, B, C \in \mathcal{X}$ with epimorphisms $\varphi : B \to A$, $\psi : C \to A$. Let $a_1 \leq^A \ldots \leq^A a_{\text{card}(A)}$ be an enumeration of $A$. Let $N_j = \max(\text{card}(\varphi^{-1}(\{a_j\})), \text{card}(\psi^{-1}(\{a_j\})))$, for each $j \in \{1, \ldots, \text{card}(A)\}$, and define $D \in \mathcal{X}$ such that

$$\text{card}(D) = \sum_{j=1}^{\text{card}(A)} N_j$$

and enumerate it as $D = \{d_{jl} \mid j \in \{1, \ldots, \text{card}(A)\}, l \in \{1, \ldots, N_j\}\}$. Let $\leq^D$ be the total order on $D$ determined by the lexicographic order on the pairs of indices $(j, l)$. This determines relation $R^D$ too.

Now define $\theta : D \to B$ by mapping $\{d_{j1}, \ldots, d_{jN_j}\}$ onto $\varphi^{-1}(\{a_j\})$ in an increasing way, and similarly define $\rho : D \to C$. So $\theta, \rho$ are epimorphisms and $\varphi\theta = \psi\rho$. □

Let $\mathbb{X}$ be the projective Fraïssé limit of $\mathcal{X}$.

**Lemma 11.** Relation $\leq^\mathbb{X}$ is a total order on $\mathbb{X}$ having a least and a last element.

**Proof.** By proposition 2, parts (1)(3)(4)(5)(6). □

**Lemma 12.** Relation $R^\mathbb{X}$ is an equivalence relation.

**Proof.** By proposition 2, parts (1)(2), $R^\mathbb{X}$ is reflexive and symmetric. To complete the proof, it will be shown that every $x \in \mathbb{X}$ is $R^\mathbb{X}$-related to at most one element different from itself.

So suppose, towards a contradiction, that $x, y_1, y_2$ are distinct elements in $\mathbb{X}$ such that $y_1R^\mathbb{X}xR^\mathbb{X}y_2$. Let $U, V_1, V_2$ be disjoint clopen neighbourhoods of $x, y_1, y_2$, respectively. If $\varphi : \mathbb{X} \to A$ is any epimorphism onto an element of $\mathcal{X}$ refining $\{U, V_1, V_2, \mathbb{X} \setminus (U \cup V_1 \cup V_2)\}$, since $\varphi(x), \varphi(y_1), \varphi(y_2)$ are distinct and $\varphi(y_1)R^A\varphi(x)R^A\varphi(y_2)$, it follows that $\varphi(y_1), \varphi(x), \varphi(y_2)$ are $\leq^A$-consecutive,
with \( \varphi(x) \) being the midpoint. Say, for instance, \( \varphi(y_1) \leq A \varphi(x) \leq A \varphi(y_2) \).

Then let \( B = A \cup \{ z \} \), where \( z \notin A \), with the symbols of \( L_R \) interpreted as follows:

- \( \leq^B \) is obtained from \( \leq^A \) by inserting \( z \) between \( \varphi(x) \), \( \varphi(y_2) \);
- \( R^B \) is the only extension of \( R^A \) compatible with the definition of \( \leq^B \) that turns \( B \) in an element of \( A \).

Define \( \psi : B \to A \) as the identity on the elements of \( A \) and by letting \( \psi(z) = \varphi(x) \). Then there cannot be any epimorphism \( \theta : X \to B \) such that \( \varphi = \psi \theta \), since \( \theta(x) \) could not be \( R^B \)-related to both \( \varphi(y_1), \varphi(y_2) \).

**Lemma 13.** If \( x \in X \) then \( x \) has a basis of clopen neighbourhoods that are convex sets with respect to \( \leq^X \).

**Proof.** Let \( U \) be a clopen subset of \( X \) containing \( x \). Let \( \varphi : X \to A \) be an epimorphism onto some \( A \in X \) refining the clopen covering \( \{ U, X \setminus U \} \). Let \( V = \varphi^{-1}(\{ \varphi(x) \}) \), so that \( V \) is clopen. If \( y, z \in V \) with \( y \leq^X z \), then for any \( w \in X \) with \( y \leq^X w \leq^X z \) one has \( \varphi(w) = \varphi(x) \), whence \( w \in V \).

**Lemma 14.** If \( x, y \in X \), then \( x, y \) are \( \leq^X \)-consecutive if and only if they are distinct and \( R^X \)-related.

**Proof.** Suppose \( x \leq^X y \), so that in particular \( \varphi(x) \leq^A \varphi(y) \) for any epimorphism \( \varphi \) from \( X \) onto some \( A \in X \).

Assume first they are consecutive (in particular, \( x \neq y \)). First, notice that for any \( A \in X \) and epimorphism \( \varphi : X \to A \) either \( \varphi(x) = \varphi(y) \) or \( \varphi(x), \varphi(y) \) are \( \leq^A \)-consecutive, since \( \varphi \) is monotone with respect to the orders. So it follows that \( \varphi(x)R^A\varphi(y) \). By the arbitrarity of \( A \) and \( \varphi \), this implies \( xR^Xy \).

Conversely, assume \( x \neq y \), \( xR^Xy \) and suppose there is \( z \in X \) with \( x <^X z <^X y \). Let \( U, V, W \) be disjoint clopen neighbourhoods of \( x, y, z \), respectively. Let \( A \in X \) with an epimorphism \( \varphi : X \to A \) refining \( \{ U, V, W, X \setminus (U \cup V \cup W) \} \). Then \( \varphi(x) <^A \varphi(z) <^A \varphi(y) \), so \( \varphi(x), \varphi(y) \) are not \( R^A \)-related, a contradiction.

**Lemma 15.** A closed total order \( \leq \) on a compact metric space \( X \) is complete.

**Proof.** Let \( A \) be a bounded non-empty subset of \( X \). Let \( A' = \{ x \in X \mid \forall y \in A \ y \leq x \} \), the set of upper bounds of \( A \), which is a closed non-empty subset of \( X \). It is then enough to establish the existence of \( \min A' \). Let \( \{ x_\alpha \}_{\alpha \in \beta} \) be a maximal decreasing sequence in \( A' \). Since every \( \leq \)-open interval is an open subset of \( X \), by separability of \( X \) ordinal \( \beta \) must be countable. If \( \beta = \gamma + 1 \) is a successor ordinal, then \( x_\gamma = \min A' \). Otherwise, by compactness, \( \inf \{ x_\alpha \}_{\alpha \in \beta} \) exists and it equals \( \min A' \).

Let \( Q = X/R^X \) and let \( \pi : X \to Q \) be the quotient map. On \( Q \) define \( [x] \leq' [y] \) if and only if \( x \leq^X y \). By lemma 14 this is well defined. Moreover, by lemmas 14, 15 and 11, this is a dense, complete total order with a first and a last element.

**Lemma 16.** The quotient topology on \( Q \) is the order topology induced by \( \leq' \).
Proof. We first show that sets of the form $I_{[a]} = \{ [x] \in Q \mid [a] \prec' [x] \}$, $I_{[b]} = \{ [x] \in Q \mid [x] \prec' [b] \}$ are open in $Q$. For the first kind, since $[a]$ contains at most two elements, let $a^*$ be its maximum with respect to $\leq_X$. Then $I_{[a]}$ is the image under $\pi$ of $\{ x \in X \mid a^* \prec_X x \}$, which is open (since $\leq_X$ is closed and total) and $R^X$-invariant. The same argument works for the second type of intervals.

Conversely, let $U$ be open in $Q$ and fix $[x] \in U$. By lemma 13 for each point in $[x]$ there is a $\leq_X$-convex, clopen subset of $X$ containing that point and included in $\pi^{-1}(U)$. Since $[x]$ is either a singleton or a doubleton consisting of two $\leq_X$-consecutive points, the union of these clopen sets, call it $I$, is $\leq_X$-convex. It is then enough to show that, if $\min Q \neq [x]$, then $I$ contains some element that strictly precedes all elements of $[x]$, and similarly that if $\max Q \neq [x]$ then $I$ contains some element strictly bigger than the elements of $[x]$. So suppose $\min Q \neq [x]$. If, towards a contradiction, $[x]$ contained the least element of $I$, let $J$ be the set of all strict predecessors of $\min I$. Since $I$ is clopen and $\leq_X$ is closed, $J$ is a clopen, non-empty, bounded subset of $X$. By lemma 15, $J$ has a maximum $z$. So $z$ is an immediate predecessor of $\min I$, but $z$ and $\min I$ are not $R^X$-related, since $\min I \in [x] \subseteq I$. This contradicts lemma 14.

Lemma 17. $\leq'$ has order type $1 + \lambda + 1$, where $\lambda$ is the order type of the real line.

Proof. We already noted that $\leq'$ is bounded and complete. We remark that it is also a separable order: indeed, it is a dense order, so every open interval is non-empty and, by lemma 16, open in the Polish space $Q$, thus every interval contains a point of a fixed countable dense subset of $Q$. Now apply [7, theorem 2.30].

Since the topology of $Q$ is induced by an order of type $1 + \lambda + 1$, it follows that $Q$ is an arc, concluding the proof of theorem 9.

An immediate consequence is now the following. Recall that a hypercube is a space homeomorphic to $[0, 1]^n$, for some $n$.

Corollary 18. Every hypercube is finitely representable.

Proof. By theorems 9 and 7.

For the next consequence recall that, in continuum theory, a graph is defined as a finite union of arcs any two of them meeting at most in one or both of their endpoints (see for example [6]).

Corollary 19. Every graph is finitely representable.

Proof. Notice that in the proof of theorem 9 each endpoint of arc $Q$ is the image under the quotient map of an almost stable point, since the extrema of a total order — in this case $\leq_X$ — are preserved under isomorphism. So we can use theorem 6 to obtain a disjoint union of arcs; the remark following that theorem allows us to apply theorem 8 to glue endpoints and thus obtain any possible graph.
6. Questions

In the previous sections we exhibited some simple classes of finitely representable spaces, enlarging the examples given in [2]. This suggests the following general question.

**Question 1.** What spaces are finitely representable?

In our examples, due to the application of the constructions of section 4, the languages and the structures associated to the spaces were in some sense always related to the obvious structural characteristics of the spaces, starting from an order representing the arc. The following rather vague question comes to mind.

**Question 2.** Given a finitely representable space, what are the minimal, or most natural, language and structures representing it? Can some specific features of the space be derived directly from the language? What are the obstructions that forbid a space to be represented with a given language?

**References**

[1] D. Bartošová, A. Kwiatkowska, *Lelek fan from a projective Fraïssé limit*, Fundamenta Mathematicae 231 (2015), 57–79.
[2] R. Camerlo, *Characterising quotients of projective Fraïssé limits*, Topology and its Applications 157 (2010), 1980–1989.
[3] T. Irwin, S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Transactions of the American Mathematical Society 358 (2006), 3077–3096.
[4] A. Kwiatkowska, *The group of homeomorphisms of the Cantor set has ample generics*, Bulletin of the London Mathematical Society 44 (2012), 1132–1146.
[5] A. Kwiatkowska, *Large conjugacy classes, projective Fraïssé limits and the pseudo-arc*, Israel Journal of Mathematics 201 (2014), 85–97.
[6] S.B. Nadler, Jr., *Continuum theory*, Marcel Dekker 1992.
[7] J.G. Rosenstein, *Linear orderings*, Academic Press 1982.

* Département des systèmes d’information
Université de Lausanne
Quartier UNIL-Dorigny Bâtiment Internef
1015 Lausanne
SWITZERLAND
E-mail address: gianluca.basso@unil.ch

** Dipartimento di scienze matematiche «Joseph-Louis Lagrange»
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino
ITALY
E-mail address: riccardo.camerlo@polito.it