A NEW ANALYSIS OF THE TIPPE TOP:
ASYMPTOTIC STATES AND LIAPUNOV STABILITY

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Abstract

We study the asymptotic behaviour of a spinning top whose shape is spherical, while its mass distribution has axial symmetry only, and which is subject to sliding friction on the plane of support (so-called tippe top). By a suitable choice of variables the equations of motion make explicit the conservation of Jelett’s integral (rediscovered by Leutwyler) and allow to construct explicitly all solutions of constant energy. The latter are the possible asymptotic states of the solutions with arbitrary initial conditions. Their stability or instability in the sense of Liapunov is determined for all possible choices of the moments of inertia. We conclude with some numerical examples which illustrate our general analysis of Liapunov stability.
1. Introduction

The so-called tippe top may be modeled by a sphere whose mass distribution is axially symmetric, but not spherically symmetric, so that its center-of-mass does not coincide with its geometrical center. As described in the historical introduction to the article by R.J. Cohen on the subject [1], this top’s astonishing behaviour has puzzled several generations of physicists: Provided certain conditions on the moments of inertia $I_1 = I_2$ and $I_3$ are fulfilled, the rapidly spinning top will quickly tip over to a completely inverted position where the center-of-mass sits vertically above the geometric center. Thus, in the initial phase of the motion the center-of-mass is raised, with the rotational and the total energy decreasing to a constant value. In the same process the direction of rotation, with respect to a body-fixed frame, is reversed.

A detailed analysis of the tippe top and a numerical study of its equations of motion was presented by Cohen [1]. In particular, this analysis established definitely an earlier contention by Pliskin, Braams, and Hugenholtz [2, 3, 4]: It is the sliding frictional force acting at the point of contact between the top and the plane of support which is responsible for the inversion.

Recently, Leutwyler showed that if that frictional force is the dominant one and if it is proportional to the sliding velocity then, while the individual projections $L_3$ and $\overline{L}_3$ of the angular momentum onto the spatial vertical and onto the body’s symmetry axis, respectively, decrease by dissipation, the specific linear combination $\lambda = L_3 - \alpha \overline{L}_3$, (with $\alpha$ denoting the distance of the center-of-mass from the sphere’s center, in units of its radius), remains constant in time [5]. This conservation law is of a purely geometric nature and does not depend on the specific form of the frictional force, as a function of velocity, (for a proof using the geometry of the system only, see [6]). Using this constraint, Leutwyler showed, by a simple energy consideration, that the inverted state of the spinning top, indeed, has rotational energy lower than the non-inverted state.

In fact, that conservation law, called Jelett’s integral [7], was known to the experts in the field much earlier [8]. In these references as well as in [9] the dynamics of gyroscopes was studied. Synge, in particular, studied the stability of an asymmetric tippe top spinning in inverted position [10].

To the best of our knowledge none of these articles addressed the long-term orbital stability of this system, in the sense of a Liapunov analysis. The questions to be asked being: Which of the possible inverted positions (whose nature depends on $I_1$ and $I_3$) is asymptotically stable? Which initial configurations are driven into these asymptotic states, and how do they move towards them? In this work we present a rigorous and complete analysis of the (symmetric) tippe top. We study its long-term behaviour by means of a Liapunov function, making use of the conservation law, and thereby give a complete description of the asymptotic solutions and of their stability or instability. The conservation law, clearly, allows to reduce the number of variables, as compared to Cohen’s analysis. Furthermore, by a suitable choice of these variables the equations of motion are simplified further. Repeating then the numerical study, the pattern of
explicit solutions is rendered considerably more transparent.

In sec. 2 we define the system that we study, in more precise terms, list our assumptions, and state the conservation law. In sect. 3 we write down the equations of motion and derive all solutions with constant energy, i.e. the asymptotic solutions towards which the spinning top will tend if they are found to be stable. Sect. 4 addresses this stability analysis for any initial condition. This section contains our main results. In sect. 5 we give some examples of our own numerical study and summarize our results.

2. Definitions, assumptions, and conservation law

The top is taken to be a sphere with unit radius, \( r = 1 \), and an axially symmetric mass distribution. Its symmetry axis is taken to be the body-fixed \( \bar{3} \)-axis so that for the moments of inertia \( I_1 = I_2 \neq I_3 \). The center-of-mass \( S \) has the distance \( \alpha \) from the sphere’s center \( M \), with \( 0 < \alpha < 1 \). \( A \) is the point of contact with the plane of support as sketched in fig. 1.

Generally speaking, there are three different types of motion which are possible: (i) the top rotates about a vertical axis through a fixed point on the plane. In this state of motion that we call rotational below, only rotational friction is active; (ii) the spinning top rolls over the plane without sliding. For this motion that we shall call tumbling below, only rolling friction (and, possibly, rotational friction) is active; (iii) more complicated spinning whereby the top slides over the plane of support and, hence, is subject to sliding friction. Among the type (iii) solutions we consider only those for which the top is in permanent contact with the plane, that is, we do not consider hopping states of motion.

To the extent that rolling friction and rotational friction can be neglected as compared to sliding friction, solutions of type (i) or (ii) are asymptotic solutions, with constant energy, which may or may not be stable. We shall assume, indeed, that rolling friction as well as rotational friction are absent. The problem to be solved then is twofold: to classify all solutions with constant energy and, by means of a Liapunov analysis, to study the long-term behaviour of dissipative solutions with general initial conditions.

Sliding friction slows down the projection \( L_3 \) of the angular momentum onto the vertical direction, i.e. the laboratory 3-axis, as well as its projection \( \bar{L}_3 \) onto the top’s, body-fixed, 3-axis, through torques \( R \) and \( \bar{R} \), respectively, due to the frictional force that acts at the point of contact \( A \), viz.

\[
\frac{d}{dt} L_3 = -R, \quad \frac{d}{dt} \bar{L}_3 = -\bar{R}. \tag{1}
\]

The (sliding) velocity components of the instantaneous point of support \( A \) which are due to infinitesimal rotations about the 3-axis and about the \( \bar{3} \)-axis, respectively, have the same direction in the plane of support and are perpendicular to the plane spanned by the 3- and \( \bar{3} \)-axes. The velocity component of \( A \) which is due to a change \( d\theta \) of the angle between the 3- and the \( \bar{3} \)-axis, on the other hand, lies in that plane and, hence, is perpendicular to the former two components. Obviously, analogous statements hold
true for the components of the frictional force acting on \( A \), and are independent of its explicit functional dependence. As a consequence, the torques \( R \) and \( R' \) depend on the same component of that force and differ only by the moment arms whose lengths are seen to be \( \alpha \sin \theta \) and \( \sin \theta \), respectively, from fig. 1. Thus, \( R = \alpha R' \) and, from eq. (1), the linear combination

\[
\lambda := L_3 - \alpha L_3
\]

is a constant of the motion.

If the rotational kinetic energy is large as compared to the potential energy in the gravitational field, the energy of the spinning top can be rewritten in terms of \( \lambda \) as follows

\[
E \approx T_{\text{rot}} = \lambda^2 \left\{ I_1(1 - z^2) + I_3(z - \alpha)^2 \right\}^{-1}
\]

where \( z := \cos \theta \). This formula can be used for deriving criteria for partial or complete inversion \( [5] \). For instance, if the moments of inertia obey the inequalities

\[
(1 - \alpha)I_3 < I_1 < (1 + \alpha)I_3
\]

the expression (3) assumes its smallest value in the completely inverted position \( z = -1 \).

In our analysis below we will recover the conservation law (2) from the equations of motion. The criterion (4), as well as analogous criteria for other types of inverted motion, will appear in the stability analysis, though modified in the presence of gravity.

3. Equations of motion and solutions with constant energy

3.1 The equations of motion

As customary in the theory of rigid bodies the motion of the top is described most conveniently using three systems of reference \( [6] \): An inertial, laboratory, system \( K_0 \) whose 3-axis is taken to be the vertical; a (non-inertial) system \( K \) which is attached to the center-of-gravity and whose axes are parallel, at all times, to the axes of \( K_0 \); and a body fixed, principal-axes-system \( \tilde{K} \) whose 3-axis is the symmetry axis of the top. If \( \hat{e}_3 \) denotes the unit vector along the top’s symmetry axis with respect to \( \tilde{K} \), and \( \mathbf{R}(t) \) the SO(3) rotation matrix which connects \( \tilde{K} \) and \( K \), that same unit vector is expressed with respect to \( K \) as follows

\[
\hat{\eta} = \mathbf{R}(t)\hat{e}_3 .
\]

Again with respect to \( K \) the inertia tensor is given by

\[
\mathbf{I}(t) = I_1 \left\{ \mathbb{1} + \frac{I_3 - I_1}{I_1} \left| \hat{\eta}(t) \right\rangle \left\langle \hat{\eta}(t) \right| \right\},
\]

in an obvious notation for the dyadic constructed from the vector \( \hat{\eta}, (\left| \left. \right\rangle \langle \left. \right\rangle \right) \), and its transposed (\( \langle \left| \left. \right\rangle \langle \left. \right\rangle \right) \)). The inverse of the inertia tensor is seen to be

\[
\Gamma^{-1}(t) = \frac{1}{I_1} \left\{ \mathbb{1} - \frac{I_3 - I_1}{I_3} \left| \hat{\eta}(t) \right\rangle \left\langle \hat{\eta}(t) \right| \right\} ,
\]
The angular velocity $\omega(t)$ is defined by the formula
\[
ad \omega(t) \equiv \omega(t) \times = \dot{R}(t)R^T(t) = -R(t)\dot{R}^T(t),
\]
while the angular momentum is $L(t) = I(t)\omega(t)$. By eq. (7) $\omega$ is expressed in terms of $L$ as follows:
\[
\omega(t) = \frac{1}{I_1} \left\{ L(t) - \frac{I_3 - I_1}{I_3} \langle \hat{\eta} \mid L \rangle \hat{\eta} \right\}. \tag{9}
\]
The variables $\hat{\eta}$ and $L$ are shown in fig. 2, for the same position of the top as in fig. 1.

Let $s(t)$ be the coordinate vector of $S$, $v$ the velocity of the momentaneous point of support $A$, $F$ the external force in the laboratory system $K_0$, and let $N$ denote the external torque in the system $K$. By the axial symmetry of the top $F$ and $N$ depend only on $(\hat{\eta}, L, \dot{s})$. The equations of motion read
\[
\begin{align*}
\frac{d}{dt} \dot{\eta} &= \omega \times \dot{\eta} = \frac{1}{I_1} L \times \dot{\eta} \\
\frac{d}{dt} L &= N(\dot{\eta}, L, \dot{s}) \\
m \ddot{s} &= F(\dot{\eta}, L, \dot{s}), \tag{10}
\end{align*}
\]
with $m$ denoting the total mass of the top. In the first of these equations we have made use of eq. (9). As we require the top to be in contact with the plane of support at all times, the component $s_3$ of $s$ is not an independent coordinate. Clearly, the requirement is that the $3$-component of the coordinate vector of the momentaneous point of contact $A$ must be zero at all times. With $a$ denoting the the vector that joins the center-of-mass $S$ to the point of contact $A$ (cf. fig. 2)
\[
a = \alpha \hat{\eta} - \hat{e}_3 \tag{11}
\]
this implies
\[
\langle \hat{e}_3 \mid s + a \rangle = s_3 + \alpha \langle \hat{e} \mid \hat{\eta} \rangle - 1 = 0. \tag{12}
\]

With $v$ and $v^{(s)}$ defined by
\[
\begin{align*}
v := \dot{s} + \omega \times a, & \quad v^{(s)} := \omega \times a,
\end{align*}
\]
and making use of eqs. (9) and (11), there follows from eq. (12)
\[
v_3 = \langle \hat{e}_3 \mid \dot{s} + \omega \times a \rangle = \dot{s}_3 + \frac{\alpha}{I_1} \langle \hat{e}_3 \mid L \times \dot{\eta} \rangle = 0.
\]
This result shows that $\dot{s}_3$, the $3$-component of the center-of-mass’ velocity, is not an independent variable but is a function of $L$ and $\dot{\eta}$, i.e. $\dot{s}_3 = \dot{s}_3(\dot{\eta}, L)$. Therefore, the third equation of the system (10) must be replaced by $m \ddot{s}_{1,2} = \text{Proj}_{1,2} F$, where $\text{Proj}_{1,2}$ denotes the projection onto the plane of support.
It remains to derive explicit expressions for the external force \( F \) and the external torque \( N \). The external force is the sum of the gravitational force \( F_g = -mg\hat{e}_3 \), the normal force \( F_n \) describing the action of the plane of support at \( A \), \( F_n = g_n\hat{e}_3 \), and the frictional force \( F_f \) whose direction is opposite to the velocity of \( A \) in the plane, \( F_f = -g_f\hat{v} \), with \( g_n \) and \( g_f \) positive semi-definite functions, \( F = F_g + F_n + F_f \). The force exerted in the point \( A \), in turn, is the sum of \( F_n \) and \( F_f \), \( F_A = F_n + F_f \), so that the torque \( N \) is given by

\[
N = \mathbf{a} \times F_A = (\alpha\hat{\eta} - \hat{e}_3) \times (g_n\hat{e}_3 - g_f\hat{v}).
\]  

(13)

The final form of the equations of motion is then

\[
\begin{align*}
\frac{d}{dt}\hat{\eta} &= \frac{1}{I_1} \mathbf{L} \times \hat{\eta} \\
\frac{d}{dt}\mathbf{L} &= (\alpha\hat{\eta} - \hat{e}_3) \times (g_n\hat{e}_3 - g_f\hat{v}) \\
m\ddot{s}_{1,2} &= -g_f\hat{v}.
\end{align*}
\]  

(14)

Before we move on we note that the conservation of the quantity \( \lambda \), eq. (2), follows from the equations of motion (14). We have

\[
\lambda(\hat{\eta}, \mathbf{L}) = L_3 - \alpha L_3 = -\langle \mathbf{a} | \mathbf{L} \rangle
\]  

and

\[
-\dot{\lambda} = \langle \dot{\mathbf{a}} | \mathbf{L} \rangle + \langle \mathbf{a} | \mathbf{N} \rangle = 0.
\]  

(15)

The first term is independent of the force of friction and, hence, vanishes because for a force-free, axially symmetric top both \( L_3 \) and \( \mathbf{L}_3 \) are conserved. Alternatively, this may also be seen from \( d\mathbf{a}/dt = \alpha d\hat{\eta}/dt \) and the first of eqs. (14). The second term vanishes because the torque \( \mathbf{N} = \mathbf{a} \times F_A \) is perpendicular to \( \mathbf{a} \). As stated in the introduction this result is independent of the explicit functional dependence of the frictional force. The way we have chosen the independent variables the conservation law is already encoded in the equations of motion (14)\(^1\). Therefore, the seven variables that appear in the system of equations (14) form an optimal set of independent variables. It will become clear, furthermore, that this set is optimal in the sense of being well adapted to the problem we are studying.

In our numerical analysis below we shall assume the frictional force to be proportional to the normal force, i.e.

\[
g_f = \mu g_n,
\]

with \( \mu \) a constant, positive coefficient of friction \([\text{I}]\). The coefficient \( g_n \), that is the magnitude of the normal force, is calculated from the orbital derivative of \( \dot{s}_3 \) and from Newton’s law. It is found to be a function of the seven independent variables, \( g_n = g_n(\hat{\eta}, \mathbf{L}, \dot{s}_{1,2}) \). The result is given in equation (A.1) of the appendix \([\text{I}]\).

\(^1\)Indeed, Cohen’s analysis \([\text{I}]\) leads to 10 first-order differential equations while the system (14) involves only 9 equations of first order in time.
Finally, we note that the frictional force, if it is taken to be proportional to \( v/\|v\| \), is undefined for \( v = 0 \). As the asymptotic states of the top involve configurations where \( v \) vanishes, we replace the expression for the frictional force by a functional form that is continuous in \( v = 0 \) and vanishes at that point. This is achieved most easily by replacing the unit vector \( v/\|v\| \) by 
\[
\hat{v} = h(\|v\|)\frac{v}{\|v\|},
\]
where the function \( h(x) \) is chosen such that it fulfills the conditions
\[
h \geq 0, \quad h(x) = 0 \iff x = 0, \quad |h(x) - 1| \leq \delta \text{ for all } x \geq \varepsilon \text{ for given } \varepsilon, \delta.
\]

An example for such a function is \( h(x) = \tanh(Nx) \) with \( N \) a sufficiently large positive integer.

Any solution \( \{\hat{\eta}, L, \hat{s}_{1,2}\} \) of the system (14) for which the coefficient \( g_n(\hat{\eta}, L, \hat{s}_{1,2}) \) is positive, is physically admissible. Therefore, the domain of definition for the equations of motion is
\[
\Omega = g_n^{-1}([0, \infty]) \subseteq S^2 \times \mathbb{R}^3 \times \mathbb{R}^2.
\]
On this domain the equations of motion are real analytic.

### 3.2 Solutions of constant energy

Recall that we assume sliding friction to be the only frictional force present, rotational and rolling friction being neglected in our analysis of the tippe top. Clearly, any solution of the system (14) for which the sliding velocity \( v \) of the point of contact vanishes at all times, must have constant energy. The converse is also true: any solution with constant total energy has the property \( v = 0 \). Indeed, as we shall confirm in sect. 4, \( \dot{E} = -\mu g_n h(\|v\|)\|v\| \). Therefore, all solutions with constant energy satisfy a system of differential equations which follows from (14) by setting \( \dot{v} = 0 \), viz.
\[
\begin{align*}
\frac{d}{dt}\hat{\eta} &= \frac{1}{I_1} L \times \hat{\eta} \\
\frac{d}{dt}L &= \alpha g_n \hat{\eta} \times \hat{e}_3 \\
m\ddot{s}_{1,2} &= 0,
\end{align*}
\]
where the coefficient \( g_n \) is given by eq. (A.1) with \( \dot{v} = 0 \),
\[
g_n \equiv g_n(\hat{\eta}, L) = mg \frac{1 + \alpha(\eta_3 L^2 - L_3 T_3)/(gI_1^2)}{1 + m\alpha^2(1 - \eta_3^2)/I_1}.
\]
Note that the velocity \( v \) refers to the laboratory system \( K_0 \), i.e. \( v \equiv v_{1,2} = \hat{s}_{1,2} + v_{1,2}^{(s)} = 0 \), and, from the third equation of the system (16), \( v_{1,2}^{(s)} = \text{const.} \), \( v^{(s)} \) referring to the system \( K \) which is attached to the center-of-mass. Therefore, the third equation (16)
is equivalent to the condition \( v_{1,2}^{(s)} = \text{const.} \) or, for its orbital derivative with respect to eqs. (16), \( \dot{v}_{1,2}^{(s)} = 0 \), so that we may as well study solutions of the first two equations to which we add that subsidiary condition

\[
\dot{v}_{1,2}^{(s)}(\hat{\eta}, L) = 0.
\] (18)

Before giving the explicit form of the solutions with constant energy we collect their properties in the following

**Proposition 1:** If rotational and rolling friction are absent all spinning solutions of the tippe top with constant total energy are characterized by the properties

(i) the projections of the angular momentum \( L \) onto the vertical and onto the top’s symmetry axis are conserved, \( L_3 \equiv \langle \hat{e}_3 \mid L \rangle = \text{const.}, \quad \overline{L}_3 \equiv \langle \hat{\eta} \mid L \rangle = \text{const.} \),

(ii) the square of the angular momentum is conserved, \( L^2 = \text{const.} \), and so is the projection of \( \hat{\eta} \) onto the vertical, \( \eta_3 \equiv \langle \hat{e}_3 \mid \hat{\eta} \rangle = \text{const.} \),

(iii) at all times \( \hat{e}_3, \hat{\eta}, \) and \( L \) lie in a plane,

(iv) the center-of-mass stays fixed in space, \( \dot{s} = 0 \).

Proof: (i) is a well-known result for the children’s top in a gravitational field \([3]\) and follows from the first two equations of motion (16). To prove (ii) and (iii) we must calculate \( v^{(s)} \), the velocity of the point \( A \) with respect to the system of reference \( K \), and make use of the condition (18).

With \( \omega \) as given by eq. (9) and \( a \) as given by eq. (11), one has

\[
v^{(s)} = \omega \times a = \frac{1}{I_1} \left\{ \alpha L \times \hat{\eta} - L \times \hat{e}_3 + \frac{I_3 - I_1}{I_3} \overline{L}_3 \hat{\eta} \times \hat{e}_3 \right\}. \quad (19)
\]

From this expression one calculates the orbital derivative of \( v^{(s)} \), by means of the equations of motion (16). As the vectors \( \hat{e}_3 \times \hat{\eta} \) and \( \hat{e}_3 \times (\hat{e}_3 \times \hat{\eta}) = \eta_3 \hat{e}_3 - \hat{\eta} \) both are in the (1,2)-plane, the condition (18) requires their scalar products with \( \dot{v}^{(s)} \) to vanish. This leads to the conditions

\[
\overline{L}_3 \left( \alpha - \frac{I_3 - I_1}{I_3} \eta_3 \right) P = 0,
\]

\[
\alpha (1 - \eta_3^2) \left( L^2 - g_n I_1 (1 - \alpha \eta_3) \right) + \overline{L}_3 \left\{ \alpha (\eta_3 L_3 - \overline{L}_3) - \frac{I_3 - I_1}{I_3} (L_3 - \eta_3 \overline{L}_3) \right\} = 0,
\quad (20)
\]

where \( P \) stands for the scalar product \( P := \langle \hat{e}_3 \times L \mid \hat{\eta} \rangle \) (and its cyclic permutations). By the equations of motion the orbital derivatives of \( \eta_3 \) and of \( L^2 \) are also proportional to \( P \), \( \frac{d\eta_3}{dt} = \frac{P}{I_1}, \quad \frac{dL^2}{dt} = 2\alpha g_n P \). Therefore, upon multiplication of the second equation by \( P \), the above conditions can be rewritten as follows

\[
\overline{L}_3 \left( \alpha - \frac{I_3 - I_1}{I_3} \eta_3 \right) \frac{d}{dt} \eta_3 = 0,
\]
\begin{align}
(1 - \eta_3^2) \left\{ \frac{1}{2} (1 - \alpha \eta_3) \frac{dL^2}{dt} - \alpha L^2 \frac{d\eta_3}{dt} \right\} - \\
T_3 \left\{ \alpha (\eta_3 L_3 - L_3) - \frac{I_3 - I_1}{I_3} (L_3 - \eta_3 L_3) \right\} \frac{d\eta_3}{dt} = 0. \tag{21}
\end{align}

As long as $T_3 \neq 0$ the first eq. (21) implies $\eta_3 = \text{const.}$, hence $\dot{\eta}_3 = 0$ and $P = 0$. Then we have also $d/dt(L^2) = 0$, thus proving (ii) and (iii). The case $T_3 = 0$ is a little more complicated. If $T_3 = 0$ and if $\eta_3^2 \neq 1$, the second equation (21) implies that the product

$$L^2(1 - \alpha \eta_3)^2 = \text{ const.}$$

is a constant. The second equation (20), in turn, reduces to $L^2 = g_n I_1 (1 - \alpha \eta_3)$. Inserting here the expression (17) for $g_n$ gives the relation

$$mg I_1 (1 - \alpha \eta_3) - L^2 (1 - \frac{m\alpha}{I_1} \eta_3 + \frac{m\alpha^2}{I_1}) = 0.$$

Finally, replacing $L^2$ by means of eq. (22) we obtain a cubic equation for $\eta_3$ with constant coefficients. This equation has at least one real solution which is a constant.

This proves that $\eta_3$ is constant in all cases and, thus, that the quantity $P$ vanishes, i.e. that $\dot{\hat{e}}_3, L, \text{ and } \hat{\eta}$ lie indeed in a plane. The velocity $\mathbf{v}^{(s)}$, eq. (19), lies in the (1,2)-plane and, by eq. (12), $\dot{s}_3 = 0$.

The last part (iv) follows from the explicit solutions that we give next. These solutions pertain to the following classes:

(A): $\eta_3 = 1$, (B): $\eta_3 = -1$, (rotating solutions). As $\hat{\eta} = \pm \hat{e}_3$ the first eq. (16) implies $L = \Lambda_0 \hat{e}_3$, with $\Lambda_0 = \pm \sqrt{T_3}$, $L_3 = \Lambda_0$, $\bar{T}_3 = \pm \Lambda_0$, and, according to eq. (2), $\lambda = \Lambda_0 (1 \mp \alpha)$. Eq. (17) reduces to $g_n = mg = \text{ const.}$. One verifies that the second condition (20) is fulfilled and that eq. (19) yields $\mathbf{v}^{(s)} = 0$.

(C): $-1 < \eta_3 < +1$ (tumbling solutions). As the projection $\eta_3$ of the unit vector $\hat{\eta}(t)$ on the vertical is constant, its time dependence must be of the form

$$\hat{\eta}(t) = R_3(\phi(t)) \hat{\eta}^{(0)} \bigg|_{t=0}, \text{ with } R_3(\phi) = \exp\{\phi \hat{e}_3 \times \}.$$

Furthermore, according to (iii), the angular momentum has the decomposition $L = \Lambda \hat{e}_3 + \Lambda \bar{\eta}(t)$, where $\Lambda$ and $\bar{\eta}$ are constants. Then $L_3 = \Lambda + \bar{\eta} \eta_3$, $\bar{T}_3 = \bar{\eta} + \Lambda \eta_3$, and, from eq. (19), $\mathbf{v}^{(s)}$ is again seen to vanish, $\mathbf{v}^{(s)} = 0$.

The first equation of motion (16) gives $\dot{\phi} = \Lambda/I_1 = \text{ const.}$, the second equation of motion, together with eq. (17) yields the relations

$$\Lambda \bar{\eta} = -\alpha mg I_1, \quad g_n = mg. \tag{23}$$

\footnote{Thus, an oscillatory or rolling motion whereby the top swings or rolls about an axis perpendicular to the symmetry axis and parallel to the plane of support, is subject to sliding friction and does not have constant energy.}
If these are inserted into the second equation (20) one obtains

$$\Lambda^2 = \frac{mg\alpha I_1^2}{I_1\eta_3 + I_3(\alpha - \eta_3)}.$$  \hspace{1cm} (24)

For a given value $\lambda$ of the conserved quantity (2) the parameter $\eta_3$ can be expressed in terms of $\Lambda$ in two different ways, viz.

$$\eta_3 = \frac{\alpha(mgI_1^2 - I_3\Lambda^2)}{\Lambda^2(I_1 - I_3)} = \frac{\Lambda^2 - \Lambda\lambda + \alpha^2mgI_1}{\alpha(\Lambda^2 + mgI_1)}.$$  \hspace{1cm} (25)

The first of these follows from eq. (24), the second form follows from the expression $\lambda = L_3 - \alpha T_3 = \Lambda(1 - \alpha\eta_3) + \overline{T}(\eta_3 - \alpha)$ and from eq. (23). From eq. (25), finally, one obtains the fourth-order equation for the quantity $\Lambda$

$$\left(\frac{I_3 - I_1}{I_3} - \alpha^2\right)\Lambda^4 - \lambda\frac{I_3 - I_1}{I_3}\Lambda^3 + (\alpha mgI_1)^2\frac{I_1}{I_3} = 0.$$  \hspace{1cm} (26)

Thus, in all cases the velocity $v(s)$ vanishes and, from the condition $v = \dot{s} + v^{(s)} = 0$ also the 1- and 2-components of the center-of-mass’ velocity vanish. This proves (iv). \(\blacksquare\)

3.3 More about the tumbling motions

As a preparation for the stability analysis we need to know how many tumbling solutions there are, given the two moments of inertia, for a given value of the constant of the motion $\lambda$, eq. (2). The answer to this question is provided by the following statements and results.

**Tumbling solutions:** (T1) Equation (26) which is of degree $\leq 4$ in the unknown $\Lambda$, has at most two real solutions. Indeed, the function $y(x) = a_4x^4 + a_3x^3 + a_0$ whose derivative has a double zero at $x = 0$, has at most one extremum. As two zeroes are separated by at least one extremum, $y$ can have no more than two real zeroes.

A solution of eq. (26) is admissible only if $\Lambda$ is real and if $\eta_3$, as calculated from the equations (25), lies in the interval $(-1, 1)$. The following special cases are particularly easy:

(T2) For $I_1 = I_3$ the solutions of eq. (26) are $\Lambda_{\pm} = \pm\sqrt{mgI_1}$. For $\eta_3$ to be in the admissible interval, the constant $\lambda$ must obey the inequalities

$$(1 - \alpha)^2 < \frac{\lambda}{\Lambda_{\pm}} < (1 + \alpha)^2.$$  \hspace{1cm} (27)

As these can hold at most for one of the two solutions, there is at most one tumbling solution. The derivative $y'(x = \Lambda_{\pm}) = -4\alpha^2mgI_1$ being different from zero there is then a neighbourhood of the value $\lambda^{(0)} = \Lambda_{\pm}$, and of $I_3 = I_1$ for which there is precisely one real zero of eq. (26) which guarantees the condition $-1 < \eta_3 < +1$.

(T3) Let $I_1 = I_3(1 - \alpha^2)$. In this special case eq. (26) has exactly one real solution which
is \( \Lambda_0 = ((mgI_1)^2(1 - \alpha^2)/\lambda)^{1/3} \). There is at most one type of tumbling motion.

(T4) In the limit \( |\lambda| \gg \sqrt{mgI_1} \), i.e. in the limit where the rotational kinetic energy is large as compared to the gravitational energy, counting the tumbling solutions becomes particularly simple:

(i) If \( I_1 > I_3(1 + \alpha) \), or if \( I_1 < I_3(1 - \alpha) \), there exists a positive number \( c_0 \) such that for \( |\lambda| > c_0 \) eq. (26) has one and only one solution which yields \( \eta_3 \) with \(-1 < \eta_3 < +1\). In other words, there is exactly one tumbling solution.

(ii) For \( I_3(1 - \alpha) < I_1 < I_3(1 + \alpha) \) there exists a positive \( c_1 \) such that for all \( |\lambda| > c_1 \) there is no real solution of eq. (26) satisfying the subsidiary condition \(-1 < \eta_3 < +1\). There are no tumbling solutions.

The proof goes as follows: We consider first the special cases (T2) and (T3) both of which belong to case (ii). If \( I_1 = I_3 \) the inequalities (27) are violated for all \( |\lambda| \geq 4\sqrt{mgI_1} \). If \( I_1 = I_3(1 - \alpha^2) \), \( \Lambda_0 \) is as given above, and \( \eta_3 \) (as calculated from the first equation (25)) is in the right interval only if \( \lambda \) lies in the interval

\[
\frac{1 - \alpha}{\sqrt{1 + \alpha}} < \frac{\lambda}{\sqrt{mgI_1}} < \frac{1 + \alpha}{\sqrt{1 - \alpha}}.
\]

Clearly, this is not the case whenever \( |\lambda| \gg \sqrt{mgI_1} \).

We then consider the general situation where \( I_1 \) neither is equal to \( I_3 \) nor to \( I_3(1 - \alpha^2) \). Eq. (26) which we rewrite as follows

\[
\left(1 - \frac{I_3}{I_3 - I_1}\alpha^2\right)\Lambda + \frac{(mgI_1\alpha)^2}{I_3 - I_1} \frac{1}{\lambda^3} = \lambda,
\]

for sufficiently large \( |\lambda| \), has two real solutions whose asymptotics is

\[
\Lambda_1 \sim \frac{\lambda}{1 - \frac{I_3}{I_3 - I_1}\alpha^2}, \quad \Lambda_2 \sim (mgI_1\alpha)^{2/3}\left(\frac{I_1}{I_3 - I_1}\right)^{1/3}\lambda^{-1/3}.
\]

The first of these, \( \Lambda_1 \) tends to infinity as \( \lambda \to \infty \) and, from eq. (25),

\[
\eta_3(\Lambda_1) \to \frac{I_3}{I_3 - I_1}\alpha \quad (28)
\]

which is indeed in the right interval if the inequalities (i) hold true. The second solution tends to zero, for large values of \( \lambda \), while \( \eta_3(\Lambda_2) \) tends to infinity and, hence is not in the right interval. This proves (ii).

4. Long term behaviour and stability

In this section we show that the total energy of the top is a suitable Liapunov function and, in case of asymptotic stability, that the solutions tend towards some solution with constant energy. This analysis makes essential use of the conservation law (2) and of the
solutions with constant energy that we studied in secs. 3.2 and 3.3 above.

4.1 The energy is a Liapunov function

With our choice of variables the total energy of the spinning top is given by

\[
E(\hat{\eta}, L, \dot{s}_{1,2}) = \frac{m}{2} \left[ \dot{s}_{1,2}^2 + \dot{s}_3^2(\hat{\eta}, L) \right] + \frac{1}{2I_1} \left( L^2 - \frac{I_3 - I_1}{I_3} \dot{L}_3^2 \right) + mg s_3(\hat{\eta}).
\]

The first term is the kinetic energy of the center-of-mass motion, the second is the rotational energy \( T_{rot} = \omega \cdot \frac{L}{2} \), with \( \omega \) as given in eq. (9), the third term is the potential energy. The orbital derivative of \( E(\hat{\eta}, L, \dot{s}_{1,2}) \) is calculated from the equations of motion (10) or (14), and from eq. (12) for \( \dot{s}_3 \). The result is

\[
\frac{d}{dt}E(\hat{\eta}, L, \dot{s}_{1,2}) = v(\hat{\eta}, L, \dot{s}_{1,2}) \cdot F_f(\hat{\eta}, L, \dot{s}_{1,2}) = -\mu g_n h(\|v\|) \|v\|. \tag{30}
\]

We recall that \( v \) is the velocity of \( A \) with respect to \( K_0 \) (cf. sec. 3.1), and that \( g_n \) and \( v = \dot{s}_{1,2} + v^{(s)}_{1,2} \) depend on \( \hat{\eta}, L \), and \( \dot{s}_{1,2} \), cf. eqs. (A.1) and (19). The coefficient of friction \( \mu \) being positive, the orbital derivative of \( E(t) \) is negative semi-definite. It vanishes if and only if \( v(\hat{\eta}, L, \dot{s}_{1,2}) \) vanishes. The function \( E(t) \) decreases monotonically and, hence, is a Liapunov function. Furthermore, the equations of motion and the function \( E \) are real analytic. Thus \( E(t) \) is either strictly monotonous or is a constant. As expected, \( E \) becomes constant whenever the sliding velocity \( v \) vanishes. As \( E(t) \) is a Liapunov function the asymptotic states, for \( t \to \infty \), are solutions of constant energy.

4.2 Extrema of the Liapunov function

In determining the extrema of the Liapunov function \( E(t) = E(\hat{\eta}(t), L(t), \dot{s}_{1,2}(t)) \) on the hypersurfaces which are defined by the condition

\[
\lambda(\hat{\eta}, L) \equiv \lambda^{(0)} = \text{const.} \tag{31}
\]

we make use of the following idea: In case of asymptotic stability the system will tend towards one of the solutions of constant energy. In these asymptotic states \( v^{(s)} \) vanishes and, from eq. (19), the angular velocity \( \omega \) is proportional to the vector \( a \), eq. (11). Therefore, \( L \) is proportional to \( Ia \), with \( I \) as given in eq. (6). Furthermore, \( L \) is in the same plane as \( \hat{\eta} \) and \( \hat{e}_3 \). For an arbitrary spinning state of the top (in which sliding friction is still active) we decompose the angular momentum \( L \) into a component

\[
L_\parallel = \frac{1}{I_1} L_\parallel I(\hat{\eta})a = \frac{1}{I_1} L_\parallel I(\hat{\eta})(\alpha \hat{\eta} - \hat{e}_3)
\]

parallel to what its direction would be if \( E \) were already constant, and a component \( L_\perp \) perpendicular to \( a, L_\perp \cdot a = 0 \). With \( a \) extracted from eq. (32) this means, in fact, that \( L_\parallel \) and \( L_\perp \) are orthogonal with respect to the scalar product defined by \( I^{-1}(\hat{\eta}) \), viz.

\[
\langle L_\perp | I^{-1}(\hat{\eta}) | L_\parallel \rangle = \frac{L_\parallel}{I_1} L_\perp \cdot a = 0.
\]
When the spinning state tends towards a solution of constant energy, \( L_\perp \) will tend to zero, \( L_\parallel \) will tend to its asymptotics \( L_\parallel \).

With \( L = L_\parallel + L_\perp \), \( L_\parallel \) being defined by eq. (32), we have

\[
\langle \omega \mid L \rangle = \langle \mathbf{I}^{-1} L \mid L \rangle = \langle \mathbf{I}^{-1} L_\parallel \mid L_\parallel \rangle + \langle \mathbf{I}^{-1} L_\perp \mid L_\perp \rangle
\]

and \( \langle \mathbf{e}_3 \mid L \times \hat{n} \rangle = \langle \mathbf{e}_3 \mid L_\perp \times \hat{n} \rangle \). The total energy can be written as the sum of two terms

\[
E = E^{(1)}(\eta_3, L_\parallel) + E^{(2)}(\hat{n}, L_\perp, \dot{s}_{1,2}),
\]

the second of which contains all terms that will vanish asymptotically

\[
E^{(2)} = \frac{1}{2} \langle \mathbf{I}^{-1} L_\perp \mid L_\perp \rangle + \frac{m}{2} \left[ \langle \dot{s}_{1,2} \mid \dot{s}_{1,2} \rangle + \alpha^2 \langle \mathbf{e}_3 \times \hat{n} \mid L_\perp \rangle^2 / I_1^2 \right],
\]

while the first depends on \( \eta_3 \) and \( L_\parallel \) only, and is given by

\[
E^{(1)} = \frac{L_\parallel^2}{2I_1} G(\eta_3) + mg(1 - \alpha\eta_3)
\]

with

\[
G(\eta_3) = \frac{1}{I_1} \langle \mathbf{a} \mid \mathbf{I} \mid \mathbf{a} \rangle = \frac{1}{I_1} \langle \alpha \hat{n} - \hat{e}_3 \mid \mathbf{I} \mid \alpha \hat{n} - \hat{e}_3 \rangle = 1 - \eta_3^2 + \frac{I_3}{I_1}(\eta_3 - \alpha)^2.
\]

The constant of the motion (2) is given by

\[
\lambda = -\langle \mathbf{a} \mid L \rangle = -\frac{L_\parallel}{I_1} \langle \mathbf{a} \mid \mathbf{I} \mid \mathbf{a} \rangle = -L_\parallel G(\eta_3)
\]

so that \( L_\parallel \) can be calculated from \( \eta_3 \) and \( \lambda \), viz.

\[
L_\parallel(\eta_3, \lambda) = -\frac{\lambda}{G(\eta_3)},
\]

which, by proposition 1 (ii) becomes a constant for \( t \to \infty \).

Clearly, both \( E(\hat{n}, L, \dot{s}_{1,2}) \) and \( \lambda(\hat{n}, L) \) are invariant under rotations \( R_3(\phi) \) about the vertical. Therefore, \( E \) will be extremal under the subsidiary condition \( \lambda = \lambda^{(0)} = \text{const.} \) at most on the sets

\[
\Gamma(\hat{n}, L, \dot{s}_{1,2}) := \{(R_3(\phi)\hat{n}, R_3(\phi)L, R_3(\phi)\dot{s}_{1,2}) \mid \phi \in [0, 2\pi]\}
\]

The following proposition shows that determining the extrema of the function \( E \), eq. (33), on the hypersurface defined by the condition \( \lambda = \lambda^{(0)} \), in fact, is equivalent to finding
the extrema of $E^{(1)}$ as a function of $\eta_3$ and, thus, is a one-dimensional problem. We set $\eta_3 = \cos \theta$ and write $E^{(1)}(\theta)$ for $E^{(1)}(\eta_3, L_\parallel(\eta_3, \lambda(0)))$. Then,

Proposition 2: The following assertions are equivalent:

(i) the function $E(\hat{\eta}, L, \hat{s}_{1,2})$ assumes an extremum under the condition $\lambda(\hat{\eta}, L) = \lambda(0)$ for $(\hat{\eta}, L, \hat{s}_{1,2}) \in \Gamma(\hat{\eta}(0), L(0), \hat{s}_{1,2}(0))$;
(ii) $L(0)$ and $\hat{s}_{1,2}(0)$ have the values

\[ L(0) = L_\parallel(0) = \frac{1}{I_1}L_\parallel(\eta_3(0), \lambda(0))I(\hat{\eta}(0))(\alpha \hat{\eta}(0) - \hat{e}_3), \quad \hat{s}_{1,2}(0) = 0 \]  \hspace{5cm} (38)

and the function $E^{(1)}(\theta)$ has an extremum for $\cos \theta = \eta_3(0)$. In particular, the minima of $E^{(1)}(\theta)$ correspond to minima of $E$ on the hypersurface $\lambda(\hat{\eta}, L) = \lambda(0)$, while maxima or saddle points of $E^{(1)}$ yield saddle points of $E$ with $\lambda = \lambda(0)$.

The proof is easy and we do not write it down here. It makes use of eq. (34) which shows that $E^{(2)}$ is a strictly monotonous function of $\|L_\perp\|$ and of $\|\hat{s}_{1,2}\|$, and vanishes when these vectors vanish.

Finding the extrema of the function $E^{(1)}(\theta)$ as defined by eq. (35), with $\lambda = \lambda(0)$ fixed, with $G(\eta_3 = \cos \theta)$ as given by eq. (36) and with $L_\parallel = -\lambda(0)/G(\eta_3)$, eq. (37), is tedious but straightforward. We merely give the result in

Proposition 3: Let $\theta = \arccos \eta_3(0)$ be such that $dE^{(1)}(\theta)/d\theta = 0$.

(i) If $\eta_3(0) = \pm 1$, $E^{(1)}(\theta)$ assumes a minimum (maximum) iff

\[ I_3(1 \mp \alpha) - I_1 \pm \frac{mg\alpha I_3^2}{\lambda(0)^2}(1 \mp \alpha)^4 \]  \hspace{5cm} (39)

is positive (negative).

(ii) If $-1 < \eta_3(0) < 1$, $E^{(1)}(\theta)$ assumes a minimum (maximum) iff

\[ I_3(1 - \alpha^2)/\left[1 + 3((I_3 - I_1)\eta_3(0) - \alpha I_3)^2/I_1^2\right] - I_1 \]  \hspace{5cm} (40)

is negative (positive).

On the basis of proposition 3, and taking into account the condition (38) as well as eq. (37) for $L_\parallel$, it is plausible that the set $\Gamma \subset \Omega$ on which $E$ is extremal, with $\lambda = \lambda(0)$ fixed, coincides with the equivalence classes of solutions of constant energy. That this is indeed so is the content of proposition 4 whose proof we again omit.\[ \]  

Proposition 4: The sets $\Gamma(\hat{\eta}(0), L(0), \hat{s}_{1,2})$ for which $L(0)$ and $\hat{s}_{1,2}(0)$ are given by eq. (38) and for which $E^{(1)} = 0$ at $\theta = \arccos \eta_3(0)$, are precisely the trajectories with $E(t) = \text{const.}$ and $\lambda(t) = \lambda(0)$. In particular, the cases $\eta_3(0) = \pm 1$ and $-1$ correspond to the classes $(A)$ and $(B)$ above (rotating solutions), respectively. The case $-1 < \eta_3(0) < 1$ corresponds

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3This generalizes Leutwyler’s result who neglected gravity.

4Two solutions are called equivalent if they have the same trajectory.
to the tumbling solutions (C). Since for a given value \( \lambda^{(0)} \) of \( \lambda \) there exist at most two tumbling solutions, \( E^{(1)}(\theta) \) can have at most two extrema, that is, can have at most one minimum on the open interval \( 0 < \theta < \pi \).

In the first two cases, \( \eta^{(i)}_3 = \pm 1 \), we have from sec. 3.2

\[
\Gamma(\eta^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) = \{ (\pm \dot{e}_3, \Lambda_0 \dot{e}_3, 0) \}.
\]

In the second case \( \dot{s}^{(0)}_{1,2} = 0 \), and \( \eta^{(i)}_3 \) and \( L^{(0)} \) are obtained from eqs. (23) - (26).

4.3 Asymptotics of solutions and Liapunov stability

Let \( \Phi_t \) denote the flux of the equations of motion (14), or, for \( E = \text{const.} \), (16), \( (\eta^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \) the initial conditions. We have shown above that if the energy is constant, \( E = \text{const.} \), and if \( \lambda = \lambda^{(0)} \) is given, there are at most four different trajectories. We denote the time-positive trajectories of constant energy, if they exist, by

\[
\gamma_{\pm} = \{ \Phi_t(\pm \dot{e}_3, \Lambda_0 \dot{e}_3, 0) \mid t \geq 0 \}; \quad \gamma_i = \{ \Phi_t(\Gamma(\eta^{(i)}, L^{(i)}, 0)) \mid t \geq 0 \}.
\]

The first of these are the rotating solutions, the second are the tumbling solutions with \(-1 < \eta^{(i)}_3 < +1 \) that we studied in sec. 3.3.

Consider now the solution of the full equations of motion (14) pertaining to the arbitrary initial condition \( (\eta^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \) and let \( \omega \) denote its limit set\(^5\). The asymptotic behaviour of the solution is fixed by the following proposition.

**Proposition 5**: Let \( (\dot{\eta}(t), L(t), \dot{s}_{1,2}(t)) \) be the solution defined on the interval \( I_{\text{max}} \subset \mathbb{R}_t \), with initial condition \( (\dot{\eta}(0) = \eta^{(0)}, L(0) = L^{(0)}, \dot{s}_{1,2}(0) = \dot{s}^{(0)}_{1,2}) \). Assume further that there is a positive constant \( g^{(0)}_n \) such that

\[
g^{(0)}_n(\dot{\eta}(t), L(t), \dot{s}_{1,2}(t)) \geq g^{(0)}_n > 0, \quad \text{for all } t \in I_{\text{max}}.
\]

Then \( I_{\text{max}} \supset [0, \infty[ \) and there exists exactly one trajectory \( \gamma \in \{ \gamma_{\pm}, \gamma_{-}, \gamma_1, \gamma_2 \} \) such that, as \( t \to \infty \), the solution tends to \( \gamma \). The \( \omega \) limit set of \( (\dot{\eta}^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \) is \( \gamma \).

We sketch the proof: For all \( t \geq 0 \) the energy is bounded \( E(t) \leq E(t = 0) \). Therefore, \( \gamma^{(+)} = \{ \Phi_t(\dot{\eta}^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \mid t \geq 0 \} \) is bounded. By assumption the solution is contained in the set \( \mathcal{M} = \{ (\dot{\eta}, L, \dot{s}_{1,2}) \in \Omega \mid g_n \geq g_n^{(0)} \} \) which is a closed subset of \( \Omega \). Therefore \( I_{\text{max}} \subset [0, \infty[ \) and \( \gamma^{(+)} \) is relatively compact. Following standard theory of ordinary differential equations \( [11] \) we conclude that (i) \( \omega(\dot{\eta}^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \) is compact, connected and positive invariant by \( \Phi_t \); (ii) there exists a real value \( E^{(1)} \) such that \( \omega(\dot{\eta}^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) = E^{-1}(E^{(1)}) \); (iii) for \( t \to \infty \) the solution tends to \( \omega(\dot{\eta}^{(0)}, L^{(0)}, \dot{s}^{(0)}_{1,2}) \).

The statements (i) and (ii) imply that this limit set is the connected closure of trajectories with \( E = E^{(1)} = \text{const.} \), and \( \lambda = \lambda^{(0)} = \lambda(\dot{\eta}^{(0)}, L^{(0)}) \). Knowing that any two

\(^5\)Recall that the \( \omega \) limit set of \( x^{(0)} \) for the flux \( \Phi_t \) is the set of all accumulation points of \( \Phi_t(x^{(0)}) \) as \( t \to \infty \), cf. e.g. \( [11] \).
of these trajectories have a finite distance we see that there is exactly one trajectory \( \gamma \in \{ \gamma_+, \gamma_-, \gamma_1, \gamma_2 \} \) such that \( \omega(\vec{\eta}^{(0)}, \vec{L}^{(0)}, \dot{s}_{1,2}^{(0)}) = \gamma \neq \emptyset \).

We now turn to the central topic of our investigation: the question of orbital stability of the spinning motions of the tippe top. The answer is contained in the following

**Theorem:**

(i) If the quantity (39) with the upper sign is positive then \( \gamma_+ \), the non-inverted, rotating motion, is Liapunov stable. If it is negative, \( \gamma_+ \) is unstable.

(ii) If the quantity (39) with the lower sign is positive then \( \gamma_- \), the completely inverted, rotating motion, is Liapunov stable. If it is negative \( \gamma_- \) is unstable.

(iii) Let the quantity (40) be negative (positive). Then in as much as the tumbling motion corresponding to \( \eta_3^{(0)} \) exists, \( \gamma_i \) is Liapunov stable (unstable).

To prove this theorem let \( \gamma \) be \( \gamma = \gamma_+ \) and \( \gamma = \gamma_- \), in the cases (i) and (ii), respectively, and let \( \gamma = \gamma_i \) in the case (iii). According to proposition 3 the conditions given in the theorem are sufficient for \( E^{(i)} \) to assume a minimum (maximum) for \( \cos \theta = 1 \) and \(-1\), for (i) and (ii), respectively, or \( \cos \theta = \eta_3^{(i)} \) for (iii). By proposition 2 this means that the total energy \( E \) has a minimum (saddle point), with \( \lambda = \lambda^{(0)} \), for \( (\vec{\eta}, \vec{L}, \dot{s}_{1,2}) \in \gamma \).

Now, if \( E \) is minimal the stability follows by the Liapunov stability theorem. If \( E \) has a saddle point for \( (\vec{\eta}, \vec{L}, \dot{s}_{1,2}) \in \gamma \), and with \( \lambda = \lambda^{(0)} \), then one argues as follows. In any neighbourhood of \( \gamma \) there exists a solution with \( E(\vec{\eta}, \vec{L}, \dot{s}_{1,2}) < E(\gamma) \) and \( \lambda = \lambda^{(0)} \). There are two possibilities: (a) there is a \( \gamma^{(0)} \in \{ \gamma_+, \gamma_-, \gamma_1, \gamma_2 \} - \{ \gamma \} \) such that \( \Phi_t(\vec{\eta}, \vec{L}, \dot{s}_{1,2}) \to \gamma^{(0)} \). As any two trajectories have a finite distance, \( \gamma \) is Liapunov unstable; (b) there exists a series \( t_k \to t_+ \) for which \( g_n(\Phi_t(\vec{\eta}, \vec{L}, \dot{s}_{1,2})) \to 0 \). As \( g_n(\gamma) = mg \), by proposition 1, this implies instability. \( \square \)

4.4 Stability and instability for \( \lambda \gg \sqrt{mgI_1} \)

The theorem of the preceding section, together with the propositions 2 – 5, completely solves the stability problem. However, the answers and criteria are somewhat intricate and not very transparent at first sight. Furthermore, in practice, the top will usually be launched with an initial rotational energy large as compared to the potential energy, i.e. with a value of the conserved quantity \( \lambda \) large as compared to \( \sqrt{mgI_1} \). In this limit the stability criteria simplify considerably. We summarize them, for all possible choices of the moments of inertia, as they follow from the theorem above.

**Criteria for stability for large \( \lambda \):** We distinguish three cases corresponding to three possible choices of the moments of inertia:

(I) \( I_1 > I_3(1+\alpha) \): In this case there exists a positive number \( c_0 \) such that for all \( |\lambda| > c_0 \) the rotating solutions \( \gamma_+ \) and \( \gamma_- \) are Liapunov unstable, and there is exactly one equivalence class of tumbling motions which is Liapunov stable.

(II) \( I_1 < I_3(1-\alpha) \): In this case there exists a positive number \( c_0 \) such that for all \( |\lambda| > c_0 \) both \( \gamma_+ \) and \( \gamma_- \) are stable. There is exactly one equivalence class of tumbling motions which, however, is unstable.

(III) \( I_3(1-\alpha) < I_1 < I_3(1+\alpha) \): In this case there exists a positive number \( c_1 \) such
that for all $|\lambda| > c_1$ the non-inverted, rotating solution $\gamma_+$ is Liapunov unstable, while the completely inverted rotating solution $\gamma_-$ is Liapunov stable. There are no tumbling motions.

These statements are easily verified: Case (I) follows from parts (i) and (ii) of the theorem. Indeed, for large $|\lambda|$ the criterion (i) of proposition 3, eq. (39), simplifies to

$$I_3(1 \mp \alpha) - I_1 > 0 \text{ (stability), or } < 0 \text{ (instability) for } \gamma_{\pm}.$$  

With $0 < \alpha < 1$ we have $I_1 > I_3(1 + \alpha) > I_3(1 - \alpha)$. Thus, by (i) $\gamma_+$ is unstable, and, by (ii), $\gamma_-$ is unstable too. Furthermore, using eq. (28) for $\eta_3$ in the limit of large $|\lambda|$ the expression (40) simplifies to $I_3(1 - \alpha^2) - I_1$. With $I_3(1 + \alpha) < I_1$ as assumed we also have the inequality $I_3(1 - \alpha^2) < I_1$. By (T4) (i) of sec. 3.3 there exists one tumbling solution $\gamma_1$ which, by part (iii) of the theorem, is Liapunov stable.

Case (II) is completely analogous to case (I) but this time $\gamma_+$ is stable, and so is $\gamma_-$. The tumbling solution $\gamma_1$ (the only one that exists) is unstable.

In case (III), finally, part (i) of the theorem implies that $\gamma_+$ is unstable, while part (ii) implies that $\gamma_-$ is stable. Furthermore, the result (T4) (ii) of sec. 3.3 tells us that there is no tumbling solution. This completes the proof of the criteria in the three cases. Clearly, case (III) is the genuine tippe top whose strange behaviour we described in the introduction and which triggered this analysis.

5. Numerical results and summary

In order to illustrate our results we have studied numerical solutions of the equations of motion (14) for the three characteristic situations described in sec. 4.4 above. It is useful to write the variables $L$ and $\dot{s}_{1,2}$ in dimensionless form. Having chosen the radius $r$ of the sphere to be unity this is achieved by expressing $L$ in units of $mg r^{3/2}$, $\dot{s}_{1,2}$ as well as any other velocity in units of $\sqrt{g}$, time in units of $g^{-1/2}$. In what follows all variables are given in these rational units, viz.

$$L \equiv \frac{1}{mg^{1/2}r^{3/2}} L, \quad u \equiv \frac{1}{r^{1/2}g^{1/2}} \dot{s}_{1,2}, \quad t \equiv \frac{g^{1/2}}{r^{1/2}t}.$$  

The values of the original physical quantities are recovered by multiplying angular momenta by $mg^{1/2}r^{3/2}$, velocities by $(gr)^{1/2}$, times by $r^{1/2}g^{-1/2}$. The relevant constants which determine the behaviour of the top are the distance $\alpha$ of the center-of-mass from the geometric center, the asymmetry $\varepsilon := (I_3 - I_1)/I_3$ of the moments of inertia, and $c := 1/I_1$, the inverse of the transversal moment of inertia $I_1 (= I_2)$, expressed in units of $mr^2$. In all examples we have chosen the coefficient to be $\mu = 0.75$, in order to have the solutions approach their asymptotics rapidly.

Figs. 3 and 4 pertain to case (III) of the criteria in sec. 4.4, with $\alpha = 0.1$, $\varepsilon = 0$, $c = 2.5$. Fig. 3 shows the position of the top’s symmetry axis as a function of time, for
an initial condition close to rotation about the (positive) vertical. The rapid oscillations which are superimposed to the inversion of the top are easy to understand: they are a remnant of the nutational nodding that the top would perform if it were force-free [3]. Indeed, for $\alpha = 0$ and $\mu = 0$ the equations of motion (14) simplify to

$$\frac{d\hat{\eta}}{dt} = \frac{1}{I_1} L \times \hat{\eta}, \quad \frac{dL}{dt} = 0, \quad m\ddot{s}_{1,2} = 0,$$

whose solution describes uniform rotation of the symmetry axis about the constant angular momentum. As $\alpha$ and $\mu$ are small in the chosen example, by a theorem of Poincaré [12] which guarantees a smooth transition of the solution to the force-free solution, as $\alpha$ and $\mu$ tend to zero, the behaviour of the top still reflects that nutation. Fig. 4 shows the time evolution of $\hat{\eta}(t)$, the motion starting at the top of the figure and ending in the completely inverted position.

Figs. 5 and 6 illustrate case (I) of the criteria, i.e. $I_1 > I_3(1 + \alpha)$, or $\varepsilon < -\alpha$ for an initial rotation close to the positive vertical (fig. 5), and an initial rotation close to the negative vertical (fig. 6). The constants are chosen to be $\alpha = 0.1$ and $\varepsilon = -0.3$. Both solutions move quickly towards the tumbling motion $\gamma_1$ which is asymptotically stable.

The case (II) of the criteria, sec. 4.4, describes an "indifferent" top for which both $\gamma_+$ and $\gamma_-$ are asymptotically stable. This is illustrated by figs. 7 and 8 where we have chosen $\alpha = 0.1$ and $\varepsilon = 0.2$. In fig. 7 the top starts at the bottom of the figure and moves towards the upright position, the initial value being chosen in the attractive basin of $\gamma_+$. In fig. 8 the top starts at the top and quickly tends to the inverted position $\gamma_-$. 

Fig. 9, finally, also belongs to the case (II) which also admits one equivalence class of tumbling motions. Unlike $\gamma_+$ and $\gamma_-$, however, these are Liapunov unstable.

In summary, the equations of motion for the tippe top, subject to gravitation and to sliding friction, can be formulated in terms of an optimally adapted, minimal set of coordinates. The conservation law (2), called "Jelett’s integral" in the early literature on this topic, and which can be derived by a purely geometric argument [3], follows from these equations in an elementary and transparent manner. The total energy is found to be a suitable Liapunov function for the stability analysis of the spinning tippe top. Its extrema on the hypersurfaces defined by the conservation law $\lambda = \lambda^{(0)} = \text{const.}$ are studied. The solutions of constant energy which are the asymptotic states of the top, in case of stability, are obtained explicitly from the equations of motion in the limit of vanishing sliding friction. Our main result is contained in the theorem of sec. 4.3 which answers the question of asymptotic Liapunov stability for all choices of the constants. The criteria provided by the stability theorem simplify somewhat in case the rotational kinetic energy is large. The results are given in sec. 4.4. Finally, numerical sample calculations, illustrated by figs. 3 – 9, confirm the salient features of our analysis.

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Appendix

The coefficient $g_n$ describing the normal force $F_n = g_n \hat{e}_3$ is calculated as follows. One calculates first the acceleration $\ddot{s}_3$ of the center-of-mass in the vertical direction by taking the orbital derivative of eq. (12). The result is

$$\ddot{s}_3 = -\frac{\alpha}{I_1} g_n(\eta, L, \dot{s}_{1,2}) \left\langle \hat{\eta} \times \hat{e}_3 \right| \left[ \alpha \hat{\eta} \times \hat{e}_3 - \mu (\alpha \hat{\eta} - \hat{e}_3) \times \hat{v} \right]\right\rangle - \frac{\alpha}{I_2} \left\langle |L \times [L \times \hat{\eta}]| \right\rangle.$$  

This must be equal, by Newton’s law, to $g_n/m - g$. Working this out yields the desired formula for $g_n$,

$$g_n(\eta, L, \dot{s}_{1,2}) = \frac{mgI_1 \left\{ 1 + \alpha(\eta L^2 - L_3 L_3)/(gI_1^2) \right\}}{I_1 + m\alpha^2(1 - \eta_3^2) + m\alpha \mu \{ (\eta_3 - \alpha)\hat{e}_3 - (1 - \alpha \eta_3)\hat{\eta} \} \cdot \hat{v}} \quad (A.1)$$
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Figure captions

Fig. 1 Axially symmetric top of spherical shape (radius $r = 1$). The center-of-mass $S$ is at distance $\alpha$ from the center $M$. Sliding friction related to rotation about the $\hat{3}$- or the $\hat{3}$-axis, active in the point of support $A$, is perpendicular to the plane of the drawing. The moment arms of the corresponding torques $R$ and $\overline{R}$ are as indicated.

Fig. 2 The symmetry axis of the top is described by $\hat{\eta}$, $L$ is the angular momentum, both with respect to the system $K$ attached to the center-of-mass whose axes are parallel to the ones of the inertial system $K_0$ (not shown). $L$ need not be in the plane of $\hat{\eta}$ and $\hat{e}_3$. Also not shown are the velocities of $S$ and $A$ w. r. t. $K_0$.

Fig. 3 Type (III) top: time evolution of $\eta_3$ for initial condition $\hat{\eta} = (0, 0.4, 0, \sqrt{0.84})$, $L = (-1, 0, 5)$, $u(0) = 0$. The oscillations show the nutational nodding of the top.

Fig. 4 Inversion of the tippe top (type (III)) with initial conditions $\hat{\eta} = (0, 0.2, 0, \sqrt{0.96})$, $L = (0, 0, 5)$, $u(0) = 0$. The top starts near $\hat{\eta} = \hat{e}_3$ and moves quickly towards $\hat{\eta} = -\hat{e}_3$.

Fig. 5 Motion of a type (I) top towards a tumbling solution, the initial conditions being $\hat{\eta} = (0.2, 0, \sqrt{0.96})$, $L = (0, 0, 5)$, $u(0) = 0$. The top starts near $\hat{\eta} = \hat{e}_3$ but stabilizes in a tumbling motion.

Fig. 6 Same case as in fig. 5, except that the top is launched near $\hat{\eta} = -\hat{e}_3$, i.e. with initial conditions $\hat{\eta} = (0.2, 0, -\sqrt{0.96})$, $L = (0, 0, 5)$, $u(0) = 0$.

Fig. 7 Indifferent top (case (II)) for which both the upright position and the inverted position are Liapunov stable. The initial conditions are $\hat{\eta} = (0.8, 0, 0.6)$, $L = (0, 0, 5)$, $u(0) = 0$. It moves towards the upright position.

Fig. 8 Same top as in fig. 7 but launched with initial conditions $\hat{\eta} = (\sqrt{0.84}, 0, 0.4)$, $L = (0, 0, 5)$, $u(0) = 0$, i.e. in the basin of attraction of the inverted asymptotic state.

Fig. 9 Same top as in figs. 7 and 8, launched with initial conditions $\hat{\eta} = (\sqrt{1 - 0.495^2}, 0, 0.495)$, $L = (0, 0, 5)$, $u(0) = 0$. This state remains in the tumbling regime which, however, is Liapunov unstable.