IMPROVED BOUNDS FOR ACYCLIC COLORING PARAMETERS

LEFTERIS KIROUSIS © AND JOHN LIVIERATOS ©

ABSTRACT. The acyclic chromatic number of a graph is the least number of colors needed to properly color its vertices so that none of its cycles has only two colors. The acyclic chromatic index is the analogous graph parameter for edge colorings. We first show that the acyclic chromatic index is at most $\Delta - 1$, where $\Delta$ is the maximum degree of the graph. We then show that for all $\epsilon > 0$ and for $\Delta$ large enough (depending on $\epsilon$), the acyclic chromatic number of the graph is at most $\left(\frac{4^{1/3} + \epsilon}{3^{4/3}}\right)\Delta + 1$. Both results improve long chains of previous successive advances. Both are algorithmic, in the sense that the colorings are generated by randomized algorithms. Previous randomized algorithms assume the availability of enough colors to guarantee properness deterministically and use additional colors in dealing with the bichromatic cycles in a randomized fashion. In contrast, our algorithm initially generates colorings that are not necessarily proper; it only aims at avoiding cycles where all pairs of edges, or vertices, that are one edge, or vertex, apart in a traversal of the cycle are homochromatic (of the same color). When this goal is reached, the algorithm checks for properness and if necessary it repeats until properness is attained. Thus savings in the number of colors is attained.

1. Introduction

Let $\chi(G)$ denote the chromatic number of a graph, i.e. the least number of colors needed to color the vertices of $G$ in a way that no adjacent vertices are homochromatic. The acyclic chromatic number of a graph $G$, a notion introduced back in 1973 by Grünbaum [10] and denoted here by $\chi_a(G)$, is the least number of colors needed to properly color the vertices of $G$ so that no cycle of even length is bichromatic (has only two colors). Notice that in any properly colored graph, no cycle of odd length can be bichromatic.

The literature on the acyclic chromatic number for general graphs with arbitrary maximum degree $\Delta$ includes:

- Alon et al. [1] proved that $\chi_a(G) \leq \lceil 50\Delta^{4/3} \rceil$. They also showed that there are graphs for which $\chi_a(G) = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)\Delta} \right)$.
- Ndreca et al. [14] proved that for $\chi_a(G) \leq \lceil 6.59\Delta^{4/3} + 3.3\Delta \rceil$.
- Sereni and Volec [15] proved that $\chi_a(G) \leq \left(\frac{9}{5^{5/3}}\right)\Delta^{4/3} + \Delta < 2.835\Delta^{4/3} + \Delta$.

National and Kapodistrian University of Athens, Greece

E-mail addresses: lkirousis@math.uoa.gr, jlivier89@math.uoa.gr.

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Corresponding author: L. Kiousis; part of his research was carried out while visiting NYU Abu Dhabi.
Finally, Gonçalves et al. [9] provided the best previous bound, namely that for \( \Delta \geq 24 \),

\[
\chi_a(G) \leq \left( \frac{3}{2} \Delta^{4/3} + \min \left( 5\Delta - 14, \Delta + \frac{8\Delta^{4/3}}{\Delta^{2/3} - 4} + 1 \right) \right).
\]

There is also extensive literature on special cases of graphs. Here we show that for all \( \epsilon > 0 \) and for \( \Delta \) large enough (depending on \( \epsilon \)),

\[
\chi_a(G) \leq \left\lceil \left(4 - \frac{1}{3} + \epsilon\right)\Delta^{4/3}\right\rceil + \Delta + 1.
\]

With respect to edge coloring, the chromatic index of \( G \), often denoted by \( \chi'(G) \), is the least number of colors needed to properly color the edges of \( G \), i.e., to color them so that no edges coincident on a common vertex get the same color. It is known that the chromatic index of any graph is either \( \Delta \) or \( \Delta + 1 \) (Vizing [18]). Nevertheless, observe that to generate a proper edge coloring by successively coloring its edges in a way that at each step the color is arbitrarily chosen so that properness is not destroyed, and no color is ever changed, necessitates a palette of at least \( 2\Delta - 1 \) colors, because \( 2\Delta - 2 \) edges might be coincident with any given edge. The acyclic chromatic index of \( G \), often denoted by \( \chi'_a(G) \), is the least number of colors needed to properly color the edges of \( G \) so that no cycle of even length is bichromatic. Notice again that in any properly edge-colored graph, no cycle of odd length can be bichromatic. It has been conjectured (J. Fiamčík [5] and Alon et al. [2]) that the acyclic chromatic index of any graph with maximum degree \( \Delta \) is at most \( \Delta + 2 \).

Besides the numerous publications for special cases of graphs, the literature on the acyclic chromatic index for general graphs with max degree \( \Delta \) includes:

- Alon et al. [1] proved \( \chi'_a(G) \leq 64\Delta \), Molloy and Reed improved this to \( \chi'_a(G) \leq 16\Delta \), and then Ndreca et al. [14] showed \( \chi'_a(G) \leq [9.62(\Delta - 1)] \).
  Subsequently,
- Esperet and Parreau [3] proved that \( \chi'_a(G) \leq 4(\Delta - 1) \).
- The latter bound was improved to \([3.74(\Delta - 1)] + 1\) by Giotis et al. [8].
  Also, an improvement of the \( 4(\Delta - 1) \) bound was announced by Gutowski et al. [11] (the specific coefficient for \( \Delta \) is not given in the abstract of the announcement).
- Finally, the best bound until now was given by Fialho et al. [4], who proved that \( \chi'_a(G) \leq [3.569(\Delta - 1)] + 1 \).

Here we show that

\[
\chi'_a(G) \leq 2\Delta - 1.
\]

The most recent results from both groups above are based on the algorithmic proofs of Lovász Local Lemma (LLL) by Moser [12] and Moser and Tardos [13], which use an approach that has been known as the entropy compression method. The main difficulty in this approach is to prove the eventual halting of a randomized algorithm that successively and randomly assigns colors to the vertices, or edges, unassigning some colors when a violation of the desired properties arises. Towards proving the eventual halting (actually proving that the expected time of duration of the process is constant), a structure called witness forest is associated with the process so that at every step, the history of the random choices made can be reconstructed from the current witness forest and the current coloring. The key observation is that the number of such forests (entropy) would not have been compatible, probabilistically, with the number of random choices made if the process had lasted too long. For
nice expositions, see Tao [17] and Spencer [16]. It should be kept in mind that as the algorithm develops, dependencies are introduced between the colors of vertices or edges (see Remark 1 in Subsection 2.3 below).

Very roughly and in qualitative terms, to get our improvements on acyclic colorings, we again design a Moser-type algorithm, but not caring for properness. We only aim at avoiding badly colored cycles, i.e. cycles where all pairs of vertices, or edges, that are an odd number of vertices, or edges, apart in a traversal of the cycle are homochromatic. Parenthetically, observe that in the case of properly colored graphs, the notions of being badly colored and bichromatic for a cycle are equivalent, whereas if non-properness is considered, being badly colored is the natural formulation of being at most bichromatic. With our approach, we avoid the need for a number of colors that guarantee that choices are made in a way that does not destroy properness. If the coloring generated by the Moser-type algorithm of our approach is not proper, we just repeat the process, until properness is achieved.

Of course, this rough sketch does not explain many things: for example, we need to show that the probability that the graph generated by the Moser-type algorithm is not proper is bounded away from 1. To prove this, we approach the correctness proof of the Moser-type algorithms not via the entropy compression method, but via a direct probabilistic argument, first introduced in Giotis et al. [7]. Also, actually, our algorithm during its Moser-type phases not only ignores properness but a stronger property that has to do with the colorings of 4-cycles. This is because the Moser-type algorithm does not work well for 4-cycles. This stronger properness notion is defined differently for the cases of vertex and edge colorings. It turns out that with the number of colors that we assume to have, the graph generated during a Moser-type phase has a positive probability of having the strong properness property.

1.1. Notation and terminology. In the sequel, we give some general notions and introduce the notation and terminology we use.

Throughout this paper, $G$ is a simple graph with $l$ vertices and $m$ edges, and these parameters are considered constant. On the other hand, we denote by $n$ the number of steps an algorithm takes, and it is only with reference to $n$ that we make asymptotic considerations.

The maximum degree of $G$ is denoted by $\Delta$ and we assume, to avoid trivialities, that it is greater than 1. A (simple) $k$-path is a succession $u_1, \ldots, u_k, u_{k+1}$ of $k+1 \geq 2$ distinct vertices any two consecutive of which are connected by an edge. A $k$-cycle is a succession of $k \geq 3$ distinct vertices $u_1, \ldots, u_k$ any two consecutive of which, as well as the pair $u_1, u_k$, are connected by an edge. A path (respectively, cycle) is a $k$-path (respectively, $k$-cycle) for some $k$. Vertices of a cycle or a path separated by an odd number of other vertices are said to have equal parity. Analogously, we define equal parity edges of a cycle.

A vertex coloring of $G$ is an assignment of colors to its vertices selected from a given palette of colors. A vertex coloring is proper if no neighboring vertices have the same color. We define in an analogous way edge coloring and proper edge coloring (no coincident on a common vertex pair of edges is homochromatic).

A path or a cycle of a properly vertex-colored graph is called bichromatic if the vertices of the path or the cycle are colored by only two colors. Analogously for edge colorings. A proper coloring is $k$-acyclic if there are no bichromatic $k$-cycles. A proper coloring is called acyclic if there are no bichromatic cycles of any length.
Note that for a cycle to be bichromatic in a proper coloring, its length must be even. The acyclic chromatic number of $G$, denoted by $\chi_a(G)$, is the least number of colors needed to produce a proper, acyclic vertex coloring of $G$. Analogously, we define the acyclic chromatic index of $G$, denoted by $\chi'_a(G)$, for edge colorings.

In the algorithms of this paper, not necessarily proper colorings are constructed by independently selecting one color from a palette of $K$ colors, for suitable values of $K$, uniformly at random (u.a.r.). Thus, for any vertex $v \in V$, or edge $e \in E$ for the case of edge coloring, and any color $i \in \{1, \ldots, K\}$,

\[(1) \quad \Pr[v \ (\text{or } e) \ \text{receives color } i] = \frac{1}{K}.\]

We call such colorings random colorings (they are not necessarily proper).

In all that follows, we assume the existence of some arbitrary (total, strict) ordering among all vertices, paths, and cycles of the given graph to be denoted by $\prec$.

Among the two possible traversals of a cycle, we arbitrarily select one and call it positive. Given a vertex $v$ and a $2k$-cycle $C$ containing it, we define $C(v) := \{v = v_1^C, \ldots, v_{2k}^C\}$ to be the set of vertices of $C$ in the positive traversal starting from $v$. The two disjoint and equal cardinality subsets of $C(v)$ comprised of vertices of the same parity that are at even (odd, respectively) distance from $v$ are to be denoted by $C^0(v)$ ($C^1(v)$, respectively). We define analogously $C(e) := \{e = e_1^C, \ldots, e_{2k}^C\}$, $\mathcal{C}^0(e)$ and $\mathcal{C}^1(e)$, for the case of edge colorings and an edge $e$ of $C$.

We call badly colored cycles whose sets of equal parity vertices (or edges for edge colorings) are monochromatic (have a single color). Notice that in the case of not proper colorings, a badly colored cycle might have all its vertices (edges) of the same color. Also, again for non-proper colorings, the bichromaticity of a cycle does not imply that its coloring is bad.

We define the scope of $C(v)$, $\mathcal{S}(C(v))$, to be the set $\{v = v_1^C, \ldots, v_{2k-2}^C\}$, i.e. all but the last two of the vertices of $C$ in the positive traversal starting from $v$. Analogously, we define the scope of $C(e)$ to be the set of all but the last two edges in a traversal starting from $e$. Roughly, the reason we introduce this notion is that if we recolor the scope of a badly colored cycle, all information about it being badly colored is lost, and thus the uniform randomness for the colors before discovering that the cycle is badly colored is re-established.

In the following sections, edge and vertex colorings will be investigated separately. We start with edge coloring, because we consider, perhaps quite subjectively, that the corresponding result is more interesting.

Probabilistic considerations. In all algorithms of this work, except where we explicitly take another probabilistic approach, we assume that we are given a sequence $\rho$ of color choices (irrespective of the edges, or vertices, those colors will be assigned to) that are selected independently and u.a.r. from the palette; then each examined algorithm assigns colors to those edges, or vertices, that are selected by its commands, following successively the color choices of $\rho$. We always assume that $\rho$ is long enough to carry out the execution of the algorithm until it halts, assuming it ever halts. Probabilistic considerations are made in relation to the space of such sequences (except when another space is explicitly considered).
2. ACYCLIC EDGE COLORINGS

In this section, the term “coloring” refers to edge coloring, even in the absence of the specification “edge”.

We assume that we have $2\Delta - 1$ colors at our disposal, and we show that this number of colors suffice to algorithmically construct, with positive probability, a proper, acyclic edge coloring for $G$, in other words, we prove that $\chi'_a(G) \leq 2\Delta - 1$.

We now give a cornerstone result proven by Esperet and Parreau [3]:

**Lemma 1** (Esperet and Parreau [3]). At any step of any successive coloring of the edges of a graph (possibly with recolorings), there are at most $2\Delta - 2$ colors that should be avoided to produce a proper 4-acyclic coloring.

**Proof Sketch.** Notice that for each edge $e$, it suffices to avoid (i) the colors of all edges adjacent to $e$, and (ii) for each pair of currently homochromatic edges $e_1, e_2$ coincident with $e$ at its two endpoints, respectively (these edges contribute one to the count of colors to be avoided), the color of the edge $e_3$ that together with $e_1, e_2$ define a cycle of length 4. Thus, easily, the total count of colors to be avoided does not exceed the number of adjacent edges of $e$, which is at most $2(\Delta - 1) = 2\Delta - 2$. □

We now give the following definition:

**Definition 1.** We call a coloring strongly proper if it is proper and 4-acyclic. The notion naturally extends to partial colorings if the graph obtained by deleting its uncolored edges is strongly proper.

So by the above Lemma, $2\Delta - 1$ colors are sufficient to produce a strongly proper coloring, by arbitrarily choosing colors successively for all edges from the colors that retain strong properness.

2.1. **EdgeColor.** We start by presenting below the algorithm EdgeColor.

Each iteration of a call of the Recolor procedure from line 5 of the algorithm EdgeColor, which entails coloring all but two edges of a cycle, is called a phase. Phases are nested. Also the loop of lines 1–3 is called the initial phase. The number of steps of a phase is at most $m - 2$ (recall that $m$ denotes the number of edges). In the sequel, we count phases rather than color assignments. Because the number $m$ of the edges of the graph is constant, this does not affect the form of the asymptotics of the number of steps.

Notice also that EdgeColor may not halt, and perhaps worse, even if it stops, it may generate a coloring that is not strongly proper. However, it is obvious, because of the while-loops in the main part of EdgeColor and in the procedure Recolor, that if the algorithm halts, then it outputs a coloring with no badly colored cycles of even length $\geq 6$. So in the MainAlgorithm-Edges that follows, we repeat EdgeColor until the desired coloring is obtained.

Obviously MainAlgorithm-Edges, if and when it stops, outputs a proper acyclic coloring. The rest of the paper is devoted to computing the probability distribution of the number of steps it takes.

We prove the following progression lemma, which shows that at every time a Recolor($e, C$) procedure terminates, some progress has indeed been made, which is then preserved in subsequent phases.

**Lemma 2.** Consider an arbitrary call of Recolor($e, C$) and let $\mathcal{E}$ be the set of edges that at the beginning of the call are not contained in a cycle of even length
Algorithm 1 EdgeColor

1: for each $e \in E$ do
2:   Choose a color for $e$ from the palette, independently for each $e$, and u.a.r. (not caring for properness)
3: end for
4: while there is an edge contained in a badly colored cycle of even length $\geq 6$, let $e$ be the least such edge and $C$ be the least badly colored cycle among those that contain $e$, and do
5:   Recolor($e, C$)
6: end while
7: return the current coloring

Recolor($e, C$), where $C = C(e) = \{e = e_1^C, \ldots, e_{2k}^C\}$, $k \geq 3$.

1: for $i = 1, \ldots, 2k - 2$ do
2:   Choose a color for $e_i^C$ independently and u.a.r. (not caring for properness)
3: end for
4: while there is an edge in $\text{sc}(C(e)) = \{e_1^C, \ldots, e_{2k-2}^C\}$ contained in a badly colored cycle of even length $\geq 6$, let $e'$ be the least such edge and $C'$ the least badly colored cycle among those that contain $e'$ and do
5:   Recolor($e', C'$)
6: end while

Algorithm 2 MainAlgorithm-Edges

1: Execute EdgeColor and if it stops, let $c$ be the coloring it generates.
2: while $c$ is not strongly proper do
3:   Execute EdgeColor anew and if it halts, set $c$ to be the newly generated coloring
4: end while

$\geq 6$ with homochromatic edges of the same parity. Then, if and when that call terminates, no edge in $E \cup \{e\}$ is contained in a cycle of even length $\geq 6$ with homochromatic edges of the same parity.

Proof. Suppose that Recolor($e, C$) terminates and there is an edge $e' \in E \cup \{e\}$ contained in a cycle of even length $\geq 6$ whose edges of the same parity are homochromatic.

If $e' = e$, then by line 4, Recolor($e, C$) could not have terminated. Thus, $e' \in E$.

Since at the beginning of Recolor($e, C$), $e'$ is not contained in a cycle with homochromatic edges of the same parity, it must be the case that at some point during this call, some cycle, with $e'$ among its edges, turned into one whose edges of the same parity are homochromatic, because of some call of Recolor. Let Recolor($e^*, C^*$) the last call that leads to the appearance of some cycle $C'$ that contains $e'$ and the recoloring of the edges of $C^*$ resulted in $C'$ having all edges of the same parity homochromatic and staying such until the end of the Recolor($e, C$) call. Then there is at least one edge $e''$ contained in both $C^*$ and $C'$ that was recolored by Recolor($e^*, C^*$). By line 4 of Recolor($e^*, C^*$), this procedure could not terminate, and thus neither could Recolor($e, C$), a contradiction. □
By Lemma 2, we get:

**Lemma 3.** There are at most $m$, the number of edges of $G$, i.e. a constant, repetitions of the **while**-loop of the main part of **EdgeColor**.

However, a **while**-loop of **Recolor** or **MainAlgorithm-Edges** could last infinitely long. In the next subsection, we analyze the distribution of the number of steps they take.

### 2.2. Analysis of the algorithms

In this subsection we will prove the following two facts:

**Fact 1.** The probability that **EdgeColor** lasts at least $n$ phases is inverse exponential in $n$, i.e. there exists a constant $c \in (0, 1)$ such that this probability is at most $c^n$.

**Fact 2.** The probability that the **while**-loop of **MainAlgorithm-Edges** is repeated at least $n$ times is inverse exponential in $n$.

From the above two facts, *yet to be proved*, we get Theorem 1 below and its corollary, Corollary 1, the main results of this section.

**Theorem 1.** Assume that there are $2\Delta - 1$ colors available, where $\Delta$ is the maximum degree of the graph. Then the probability that **MainAlgorithm-Edges** lasts at least $n^2$ steps is inverse exponential in $n$.

**Proof.** By Fact 1, the probability that one of the first $n$ repetitions of the **while**-loop of **MainAlgorithm-Edges** entails an execution of **EdgeColor** with $n$ or more phases is inverse exponential in $n$. The result now follows by Fact 2. $\square$

Therefore:

**Corollary 1.** $2\Delta - 1$ colors suffice to properly and acyclically color a graph.

The possible successions of edges that are colored by **EdgeColor** are depicted by graph structures called **feasible forests**. These structures are described next.

### 2.3. Feasible forests

We will depict an execution of **EdgeColor** organized in phases with a rooted forest, that is an acyclic graph whose connected components (trees) all have a designated vertex as their root. We label the vertices of such forests with pairs $(e, C)$, where $e$ is an edge and $C$ a $2k$-cycle containing $e$, for some $k \geq 3$. If a vertex $u$ of $F$ is labeled by $(e, C)$, we will sometimes say that $e$ is the edge-label and $C$ the cycle-label of $u$. The number of nodes of a forest is denoted by $|F|$.

**Definition 2.** A labeled rooted forest $F$ is called **feasible**, if the following two conditions hold:

1. Let $e$ and $e'$ be the edge-labels of two distinct vertices $u$ and $v$ of $F$. Then, if $u, v$ are both either roots of $F$ or siblings (i.e. they have a common parent) in $F$, then $e$ and $e'$ are distinct.
2. If $(e, C)$ is the label of a vertex $u$ that is not a leaf, where $C$ has half-length $k \geq 3$, and $e'$ is the edge-label of a child $v$ of $u$, then $e' \in \text{sc}(C(e)) = \{e_{1}^C, \ldots, e_{2k-2}^C\}$.
Notice that because the edge-labels of the roots of the trees are distinct, a feasible forest has at most as many trees as the number \( m \) of edges.

Given an execution of \( \text{EdgeColor} \) with at least \( n \) phases, we construct a feasible forest with \( n \) nodes by creating one node \( u \) labeled by \( (e, C) \) for each phase. We structure these nodes according to the order their labels appear in the recursive stack implementing \( \text{EdgeColor} \): the children of a node \( u \) labeled by \( (e, C) \) correspond to the recursive calls of \( \text{Recolor} \) made by line 5 of \( \text{Recolor}(e, C) \), with the leftmost child corresponding to the first such call and so on. We order the roots and the siblings of the witness forest according to the order they were examined by \( \text{EdgeColor} \). By traversing \( F \) in a depth-first fashion, respecting the ordering of roots and siblings, we obtain the label sequence \( \mathcal{L}(F) = (e_1, C_1), \ldots, (e_{|F|}, C_{|F|}) \) of \( F \).

**Definition 3.** Given a finite sequence \( \rho \) of mutually independent and u.a.r. color-choices, let \( F_\rho \) be the uniquely defined feasible forest generated by \( \text{EdgeColor} \) if it follows the random choices \( \rho \), assuming it halts. \( F_\rho \) is a random variable, defined for sequences of random choices \( \rho \) for which \( \text{EdgeColor} \) halts. \( F_\rho \) is called the witness structure for \( \rho \).

Let \( P_n \) be the probability that \( \text{EdgeColor} \) lasts at least \( n \) phases, and \( Q \) be the probability that \( \text{EdgeColor} \) halts and the coloring generated is not strongly proper. In the next subsection, we will compute upper bounds for these probabilities. To bound \( P_n \), we introduce an algorithm that given the label sequence of a feasible forest as input, checks whether this forest can be obtained as the forest of an execution of \( \text{EdgeColor} \). We call this algorithm the validation algorithm. The reason for introducing the validation algorithm is to circumvent the fact that the coloring generated by \( \text{EdgeColor} \) is not uniformly random (see the following Remark):

**Remark 1.** A uniformly random coloring can be generated by choosing a color from the palette u.a.r. for each edge. However, if we first color all edges of a graph with colors chosen u.a.r. and independently and then we recolor some of its edges in the same manner, but with the choice of the edges to be recolored depending on the first coloring, then the resulting coloring is not uniform, because the choice of the edges to be recolored may introduce a bias, i.e. reveal information about existing colors. However, conditional that an edge is assigned a given color, the probability that an edge colored later will be assigned another given color is \( 1/K \) where \( K \) is the number of colors. Formally, given two positive integers \( i < j \), and two colors \( c, c' \) (possibly the same), at the end of every step of Algorithm 2.1 the probability that the chronologically \( j \)-th edge gets color \( c' \) conditional that the \( i \)-th edge gets color \( c \) is \( 1/K \), whereas this is not necessarily true if this conditional probability is computed with \( i, j \) swapped (see also Remark 2). Notice that the chronological order is itself a random variable, i.e. depends on the sequence \( \rho \) of color choices.

### 2.4. Validation Algorithm

In this subsection, we first give the validation algorithm (see Algorithm 3).

We call each iteration of the \textbf{for}-loop of lines 2–8 a \textbf{phase} of \( \text{EdgeValidation}(F) \). Also, the initial color assignment of line 1 is called the initial phase.

Given a (finite) sequence \( \rho \) of color choices (chosen u.a.r. and mutually independently) and a feasible \( F \), we say that \( \text{EdgeValidation}(F) \) if executed following \( \rho \) is successful if it goes through all cycles of \( \mathcal{L}(F) \) without reporting failure. Let
Algorithm 3 \texttt{EdgeValidation}(\mathcal{F})

Input: $\mathcal{L}(\mathcal{F}) = (e_1, C_1), \ldots, (e_{|\mathcal{F}|}, C_{|\mathcal{F}|})$: $C_i(e_i) = \{e_{1}^{C_i}, \ldots, e_{2k_i}^{C_i}\}$.

1: Color the edges of $G$, independently and selecting for each a color u.a.r. from $\{1, \ldots, K\}$.
2: for $i = 1, \ldots, |\mathcal{F}|$ do
3:  if $C_i$ is badly colored then
4:    Recolor the edges in $\text{sc}(C(e)) = \{e_{1}^{C_i}, \ldots, e_{2k_i-2}^{C_i}\}$ by selecting colors independently and u.a.r.
5:  else
6:    return failure and exit
7:  end if
8: end for
9: return success

$V_\mathcal{F}$ be the event comprised of sequences of color choices $\rho$ such that \texttt{EdgeValidation}(\mathcal{F}), if executed following $\rho$, is successful. A phase that is executed during an execution of \texttt{EdgeValidation}(\mathcal{F}) following $\rho$ and for which the check of line 3 is true, i.e., no failure is reported, is also called successful, independently of whether \texttt{EdgeValidation}(\mathcal{F}) is finally successful or not.
Lemma 4. The following hold:

(a) For every (fixed) feasible $F$, the coloring generated at the end of a successful phase of the execution of $\text{EdgeValidation}(F)$ (independently of whether this execution of $\text{EdgeValidation}(F)$ is finally successful or not) is random, i.e. every coloring can be equiprobably generated. Formally, if $c$ denotes a coloring of the graph, then for any phase of the execution of $\text{EdgeValidation}(F)$,

the probability of the event $E(c; F)$ that this phase is successful and the coloring $c$ is generated is the same for all colorings $c$.

(b) For every (fixed) feasible $F$, if $\mathcal{L}(F) = (e_1, C_1), \ldots, (e_{|F|}, C_{|F|})$ and if $C_i$ has half-length $k_i \geq 3$, $i = 1, \ldots, n$, then

$$\Pr[V_F] = \prod_{i=1}^{|F|} \left( \frac{1}{(2\Delta - 1)^{2k_i - 2}} \right).$$

Proof. For the first statement, observe that at each successful phase of $\text{EdgeValidation}(F)$, all colors in the scope of the corresponding cycle are replaced by random choices of colors.

The second statement is an immediate corollary of the first and the fact that for a cycle of half-length $k$, the probability that is badly colored for a random coloring is $\frac{1}{(2\Delta - 1)^{2k-2}}$.

Lemma 5. We have that:

$$Q \leq 1 - \left(1 - \frac{2\Delta - 2}{2\Delta - 1}\right)^m = 1 - \left(\frac{1}{2\Delta - 1}\right)^m.$$

Proof. Let us call a coloring step of an edge $e$ of an execution of $\text{EdgeColor}$ good if (i) the color assigned to $e$ is different from the colors that are presently on edges coincident with $e$ on a vertex and (ii) if $e$ belongs to a 4-cycle $e = e_1, e_2, e_3, e_4$ and $e_2$ and $e_4$ are presently homochromatic, then the color assigned to $e$ is different from the current color of $e_3$ (given that $e_3$ is already colored). From the cornerstone result of Esperet and Parreau given in Lemma 1, for a coloring step to be good, it suffices to avoid at most $2\Delta - 2$ colors. Therefore by considering the colors on the graph in chronological order and by Remark 1, we conclude that at every step of $\text{EdgeColor}$ the probability that the coloring obtained is strongly proper is at least $\left(\frac{1}{2\Delta - 1}\right)^m$. By Lemma 6 that follows, for any $\epsilon > 0$ there is an $n$ large enough so that the probability that $\text{EdgeColor}$ has not halted by step $n$ is at most $\epsilon$. The result follows because $\epsilon$ is arbitrary and because

(2) $Q \leq \Pr[\text{the colors of all edges are finalized by step n of EdgeColor} \text{ to a strongly proper coloring}] + \epsilon \leq 1 - \left(\frac{1}{2\Delta - 1}\right)^m + \epsilon.$

□

Now, Fact 2 follows from Lemma 5, since $Q$ is bounded away from 1. To prove Fact 1, we first let $\tilde{P}_n$ be the probability that $\text{EdgeValidation}(F)$ succeeds for at least one $F$ with $n$ nodes.
**Remark 2.** Perhaps interestingly, given an arbitrary coloring of all edges (as a mapping from edges to colors), it is not necessarily true that at the end of a phase (except the original one), the probability of getting this coloring is \((1/K)^m\). The argument with considering in chronological order the probability of each edge taking the color specified by the given coloring is not correct, because to find e.g. the probability of the \(i\)-th edge taking the specified color, we have first to determine which edge is the \(i\)-th one, which may introduce dependencies.

**Lemma 6.** We have that
\[
P_n \leq \hat{P}_n \leq \sum_{|F|=n} \Pr[V_F].
\]

**Proof.** For the first inequality, consider an execution of EdgeColor whose sequence of random choices is \(\rho\) and let \(F_{\rho}\) be the corresponding feasible forest. Execute now EdgeValidation\((F_{\rho})\) making the random choices of \(\rho\). The second one is obvious from union-bound. \(\square\)

So all that remains to be proved to complete the proof of Fact 1, is to show that \(\sum_{|F|=n} \Pr[V_F]\) is inverse exponential in \(n\). We do this in the next subsection, expressing the sum as a recurrence.

### 2.5. The recurrence.

We will estimate \(\sum_{|F|=n} \Pr[V_F]\) by purely combinatorial arguments. Towards this end, we first define the weight of a forest, denoted by \(\|F\|\), to be the number
\[
\prod_{i=1}^{|F|} \left( \frac{1}{(2\Delta - 1)^{2k_i - 2}} \right),
\]
(recall that \(|F|\) denotes the number of nodes of \(F\)), and observe that by Lemma 4
\[
\sum_{|F|=n} \Pr[V_F] = \sum_{|F|=n} \|F\|.
\]

From the definition of a feasible forest, we have that such a forest is comprised of some trees at most as many as the number of edges. For \(j = 1, \ldots, m\), let \(T_j\) be the set of all possible feasible trees whose root has as edge-label the edge \(e_j\) together with the empty tree. Assume that the weight of the empty tree is one, i.e. \(\|\emptyset\| = 1\). (Of course, the number of nodes of the empty tree is 0, i.e. \(|\emptyset| = 0\). Let also \(T\) be the collection of all \(m\)-ary sequences \((T_1, \ldots, T_m)\) with \(T_j \in T_j\).

Now, obviously:
\[
\sum_{|F|=n} \|F\| = \sum_{(T_1, \ldots, T_m) \in T \mid |T_1| + \cdots + |T_m| = n} \|T_1\| \cdots \|T_m\|
\]
\[
= \sum_{n_1 + \cdots + n_m = n \mid n_1, \ldots, n_m \geq 0} \left( \sum_{T_1 \in T_1; |T_1| = n_1 \mid T_1 \in T_1} \|T_1\| \right) \cdots \left( \sum_{T_m \in T_m; |T_m| = n_m \mid T_m \in T_m} \|T_m\| \right).
\]

We will now obtain a recurrence for each factor of the right-hand side of (4). Let:
\[
q = \frac{\Delta - 1}{2\Delta - 1} \quad \text{(obviously } q < 1/2\text{)}.
\]
Lemma 7. Let $T^c$ be any one of the $T_j$. Then:

\begin{equation}
\sum_{T \in T^c} \|T\| \leq R_n,
\end{equation}

where $R_n$ is defined as follows:

\begin{equation}
R_n := \sum_{k \geq 3} q^{2k-2} \left( \sum_{n_1 + \cdots + n_{2k-2} = n-1} R_{n_1} \cdots R_{n_{2k-2}} \right)
\end{equation}

and $R_0 = 1$.

Proof. Indeed, the result is obvious if $n = 0$, because the only possible $T$ is the empty tree, which has weight 1. Now if $n > 0$, observe that there are at most $(\Delta - 1)^{2k-2}$ possible cycles with $2k$ edges, for some $k \geq 3$, that can be the cycle-edge of the root of a tree $T \in T^c$ with $|T| > 0$. Since the probability of each such cycle having homochromatic equal parity sets is $\left( \frac{1}{2\Delta - 1} \right)^{2k-2}$, the lemma follows. \hfill \Box

2.6. The solution of the recurrence. For the solution we will follow the technique presented by Flajolet and Sedgewick in [6, Proposition IV.5, p. 278]. Towards this, we will first find the Ordinary Generating Function (OGF) $\sum_{n=1}^{\infty} R_n z^n$ (notice that the sum starts from $n = 1$).

Multiply both sides of (7) by $z^n$ and sum for $n = 1, \ldots, \infty$ to get

\begin{equation}
R(z) = \sum_{k \geq 3} q^{2k-2} z \left( 1 + R(z) \right)^{2k-2}.
\end{equation}

Letting $R := R(z)$ we get:

\begin{equation}
R = z \left( \sum_{k \geq 2} q^{2k}(R + 1)^{2k} \right) = z \left( \frac{q^4(R + 1)^4}{1 - q^2(R + 1)^2} \right).
\end{equation}

Now, set:

\begin{equation}
\phi(x) = \frac{q^4(x + 1)^4}{1 - q^2(x + 1)^2},
\end{equation}

to get from (9):

\begin{equation}
R = z \phi(R).
\end{equation}

Observe now that:

- $\phi$ is a function analytic at 0 with nonnegative Taylor coefficients (with respect to $x$),
- $\phi(0) \neq 0$,
- the radius of convergence $r$ of the series representing $\phi$ at 0 is $\frac{1}{q} - 1$ and $\lim_{x \to r^-} \frac{x \phi'(x)}{\phi(x)} = +\infty$,

so all the hypotheses to apply [6, Proposition IV.5, p. 278] are satisfied. Therefore, $[z^n]R \sim \rho^n$, i.e. $\limsup ([z^n]R)^{1/n} = \rho$, where $\rho = \phi'(\tau)$, and $\tau$ is the (necessarily unique) solution of the characteristic equation:

\begin{equation}
\frac{x \phi'(x)}{\phi(x)} = 1
\end{equation}
within \((0,r) = (0, \frac{1}{q} - 1)\) (for the asymptotic notation “\(\asymp\)" see [6, IV.3.2, p. 243]).

Using a computational tool like Maple, we can easily see that \(\rho < 1\), however, let us work this out without the use of computational tools.

**Lemma 8.** We have that \(\rho < 1\).

**Proof.** Indeed, using (12) and (10) and after simple algebraic manipulations, the characteristic equation in (12) reduces to

\[
2x = (1 - x)(1 - q^2(x + 1)^2) \iff q^2x^3 + q^2x^2 - (3 + q^2)x + 1 - q^2 = 0.
\]

We call the polynomial \(q^2x^3 + q^2x^2 - (3 + q^2)x + 1 - q^2\) the characteristic polynomial.

To show that for the unique root \(\tau \in (0, \frac{1}{q} - 1)\) of the characteristic polynomial, the value \(\rho = \phi'(\tau) = \frac{\phi(\tau)}{\tau}\) is < 1, we first claim that \(\tau \in (0, 1/2)\). Indeed, since \(\tau\) lies within \((0, \frac{1}{q} - 1)\), we get \(0 < 1 - q^2(\tau + 1)^2\); also since \(q = \frac{\Delta - 1}{\Delta}\) (recall (3)), we deduce by elementary Calculus methods that \(q\) starts from \(q = 1/3\) for \(\Delta = 2\) and increasingly approaches 1/2 as \(\Delta \to \infty\), and therefore \(1 - q^2(\tau + 1)^2 < 1 - \frac{1}{q} = \frac{2}{q}\).

So, from the left-hand side of (13), we get

\[
2\tau < \frac{8}{9}(1 - \tau) \iff \tau < 4/13 < 1/2,
\]

our first claim.

Now, again from the left-hand side of (13) we get that

\[
1 - q^2(\tau + 1)^2 = \frac{2\tau}{1 - \tau}
\]

and also

\[
q^2 = \frac{1 - 3\tau}{(1 - \tau)(\tau + 1)^2}.
\]

Therefore, using (10), we have that

\[
\rho = \frac{\phi(\tau)}{\tau} = \phi'(\tau) = \frac{(1 - 3\tau)^2}{2\tau^2(1 - \tau)}.
\]

We will now find for what values of positive \(x\) the function \(\frac{(1 - 3x)^2}{2x^2(1 - x)}\), which by (15) gives the value of \(\rho\) for \(x = \tau\), is < 1. Equivalently, we will find for which values of positive \(x\), the polynomial \(p(x) = (1 - 3x)^2 - 2x^2(1 - x) = 2x^3 + 7x^2 - 6x + 1\) is negative. It can be easily verified that all three roots of \(p(x)\) are \(-\sqrt{5} - 2, \sqrt{5} - 2, 1/2\), and that \(p(x)\) approaches \(-\infty\) as \(x \to -\infty\), therefore the only positive \(x\)'s for which \(p(x)\) is negative are in the interval \((\sqrt{5} - 2, 1/2)\). Therefore, because we already know that \(\tau < 1/2\), all that remains to be shown to prove that \(\rho < 1\) is to show that \(\tau > \sqrt{5} - 2\). We know that \(\tau\) is the unique solution in \((0, 1/2)\) of the characteristic equation \(q^2x^3 + q^2x^2 - (3 + q^2)x + 1 - q^2 = 0\). Notice now that the polynomial \(q^2x^3 + q^2x^2 - (3 + q^2)x + 1 - q^2\) is negative for \(x = 1/2\) and positive for \(x = \sqrt{5} - 2\) and \(q < 1/2\) (for \(x = \sqrt{5} - 2\) this polynomial is equal to \(3(\sqrt{5} - 7)\)(4\(q^2 - 1))\), so its unique solution in \((0, \frac{1}{q} - 1)\) actually lies in the interval \((\sqrt{5} - 2, 1/2)\), so \(\rho < 1\). \(\square\)

By the above, and since there are at most \(n^m\) sequences \(n_1, \ldots, n_m\) of integers that add up to \(n\) and since \(m\) is considered to be constant, we get by the second inequality of Lemma 6, then using Equations (3) and (4) and finally using Lemma 7 that:

\[
\hat{P}_n \asymp \rho^n,
\]
where $\rho$ is a positive constant $< 1$. By Equation (16) and the first inequality of Lemma 6, we get Fact 1, and thus the proofs of Theorem 1 and its corollary, Corollary 1, our main results of this section, are completed.

3. ACYCLIC VERTEX COLORING

In this section, since we only deal with vertex colorings, we often avoid the specification “vertex” for colorings (unless the context is ambiguous). It is known that the acyclic chromatic number is $O(\Delta^{4/3})$ and that this result is optimal within a logarithmic factor (see Introduction). Intuitively, the reason we need asymptotically more colors for acyclic vertex coloring than acyclic edge coloring is essentially the fact that given a vertex, we have a choice of $k-1$ other vertices to form a $k$-cycle, whereas given an edge we have $k-2$ choices of other edges (the last edge is uniquely determined), so the probability of selecting a vertex should be smaller to offset the larger number of choices. We will prove that (the main result of this section):

**Theorem 2.** For all $\alpha > 4^{-1/3}$ and for $\Delta$ large enough (depending on $\alpha$), the acyclic chromatic number of the graph is at most $K := \lceil \frac{\alpha \Delta^{4/3}}{3} \rceil + \Delta + 1$.

Again the main idea is to design a Moser-type algorithm that ignores properness (actually a stronger notion, as we will see immediately below). Since several theorems, and their proofs, are very analogous to the edge coloring case, we will often give only an outline of, or even omit, proofs when they are fully analogous to their edge coloring counterpart. As in the case of edge coloring, the stronger properness condition entails 4-cycles. However, the stronger properness condition will not force all 4-cycles not to be bichromatic, as in edge coloring (recall Definition 1). Rather besides properness, it will require that any two vertices $u$ and $v$ such that many 4-cycles exist having $u$ and $v$ as opposite vertices are differently colored. In the next subsection, we formalize this notion.

3.1. Special pairs. We define the notion of special pairs originally introduced by Alon et al. [1]. Gonçalves et al. [9] generalized this notion and proved results of which we make strong use. The reason that the notion of special pairs is useful for us is on one hand that 4-cycles through a given vertex that forms a special pair with its opposing vertex can be handled with respect to bichromaticity directly and on the other that 4-cycles through a given vertex that does not form a special pair with its opposing vertex, although commoner, their number (see Lemma 9) allows them to be handled with respect to bichromaticity with a Moser-type algorithm.

Below, we follow the notation and terminology of Gonçalves et al. slightly adjusted to our needs. We give in detail the relevant definitions and proofs.

Given a vertex $u$, let $N(u)$ and $N^2(u)$, respectively, denote the set of vertices at distance one and two, respectively, from $u$. Among the vertices in $N^2(u)$ define a strict total order $\prec_u$ as follows: $v_1 \prec_u v_2$ if either $|N(u) \cap N(v_1)| < |N(u) \cap N(v_2)|$ or alternatively $|N(u) \cap N(v_1)| = |N(u) \cap N(v_2)|$ and $v_1$ precedes $v_2$ in the ordering $\prec$ between vertices we assumed to exist.

**Definition 4** (Gonçalves et al. [9]). A pair $(u, v)$ of vertices such that $v \in N^2(u)$ is called an $\alpha$-special pair if it belongs to the at most $\lceil \frac{\alpha \Delta^{4/3}}{3} \rceil$ highest, in the sense of $\prec_u$, elements of $N^2(u)$. The set of vertices $v$ for which $(u, v)$ form a special pair is denoted by $S_\alpha(u)$. Also $N^2(u) \setminus S_\alpha(u)$ is denoted by $\overline{S_\alpha(u)}$. 
It is possible that \( v \in S_\alpha(u) \) but \( u \not\in S_\alpha(v) \). Also by definition,
\[
|S_\alpha(u)| = \min(|\alpha \Delta^{4/3}|, |N^2(u)|).
\]
\[(17)\]

We now give the proof of the following, that is essentially the proof presented by Gonçalves et al. [9].

**Lemma 9** (Gonçalves et al. [9, Claim 11]). For all vertices \( u \), there are at most
\[
\frac{\Delta^{8/3}}{8\alpha}
\]
4-cycles that contain \( u \) but contain no vertex \( v \in S_\alpha(u) \).

**Proof.** Let \( d \) be an integer such that
\[
\begin{align*}
\text{if } v &\in S_\alpha(u) \text{ then } |N(u) \cap N(v)| \geq d \text{ and} \\
\text{if } v &\not\in S_\alpha(u) \text{ then } |N(u) \cap N(v)| \leq d.
\end{align*}
\]
(18) (19)

Now, because cycles that contain \( u \) and a given \( v \not\in S_\alpha(u) \) are in one to one correspondence with a subset of the at most \( \left(\begin{array}{c}N(u)\cap N(v)\end{array}\right) \) pairs of distinct edges from \( u \) to \( N(u) \cap N(v) \), and because of Equation (19), we conclude that the 4-cycles through \( u \) whose opposing vertex is not in \( S_\alpha(u) \) are at most
\[
\sum_{v \in S_\alpha(u)} \left(\begin{array}{c}N(u) \cap N(v)\end{array}\right) \leq \frac{1}{2}d \sum_{u} |N(u) \cap N(v)|.
\]
Assume now that \( |\alpha \Delta^{4/3}| \leq |N^2(u)| \), and therefore by Equation (17) that \( |S_\alpha(u)| = |\alpha \Delta^{4/3}| \) (otherwise all vertices in \( N^2(u) \) are special and so there is nothing to prove). Observe that because there at most \( \Delta^2 \) edges between \( N(u) \cap N(v) \) and \( N^2(u) \), and because of Equation (18) above,
\[
\sum_{v \in S_\alpha(u)} |N(u) \cap N(v)| \leq \Delta^2 - dS_\alpha(u) \leq \Delta^2 - d\alpha \Delta^{4/3}
\]
and therefore the number of 4-cycles through \( u \) whose opposing vertex \( v \not\in S_\alpha(u) \) is at most \((1/2)d(\Delta^2 - d\alpha \Delta^{4/3})\), a binomial in \( d \) whose maximum is \( \frac{\Delta^{8/3}}{8\alpha} \).

We now give the following definition.

**Definition 5.** We call a coloring of a graph \( \alpha \)-specially proper if for any two vertices \( u, v \) such that \( v \) is a neighbor of \( u \) or \( v \in S_\alpha(u) \), and \( u \) and \( v \) are differently colored.

The notion naturally extends to partial colorings by allowing at least one of the \( u, v \) to be uncolored.

We have that:

**Lemma 10.** For any positive \( \alpha \), at any step of any successive coloring, possibly with recolorings, of the vertices of a graph with maximum degree \( \Delta \), there are at most \( |\alpha \Delta^{4/3}| + \Delta \) colors that should be avoided to finally produce a \( \alpha \)-specially proper coloring.

**Proof.** Given a vertex \( u \), there are at most \( |\alpha \Delta^{4/3}| \) vertices forming an \( \alpha \)-special pair with \( u \) and also at most \( \Delta \) neighbors of \( u \).

3.2. The Moser part of the proof. In this section we will show that, for any \( \alpha > 2^{-1/3} \), \( |\alpha \Delta^{4/3}| + \Delta + 1 \) colors suffice to color the vertices of a graph in a way that although may produce a not \( \alpha \)-specially proper (or not even proper) coloring, it succeeds nevertheless in producing a coloring where for every vertex \( u \), for all 4-cycles that contain \( u \) and the opposing to \( u \) vertex does not form a \( \alpha \)-special pair with \( u \) (these are the commoner 4-cycles), as well as for all cycles of length at least
6 that contain \( u \), not both equal parity sets are monochromatic, i.e. not all equal parity pairs of vertices are homochromatic. For this we will need to assume that the maximum degree of the graph is at least as small as an integer depending on \( \alpha \) (but not depending on the graph).

In what follows, assume we have a palette of \([a\Delta^{4/3}] + \Delta + 1\) colors, \( \alpha > 2^{-1/3}\). Let also \( B \) be the set comprised (i) of all 4-cycles whose opposing vertices do not form \( \alpha \)-special pairs and (ii) of all 5-paths, that is paths containing five edges and six vertices. Recall that the elements of \( B \) are ordered according to \( \prec \). Given a set \( B \in B \), a pivot vertex \( u \) of \( B \) is any vertex in \( B \) if \( B \) is a 4-cycle, or any of \( B \)'s endpoints if \( B \) is a 5-path. In the former case, let \( B(u) := \{ u = u_1^B, \ldots, u_4^B \} \) be the set of consecutive vertices of \( B \) in its positive traversal beginning from \( u \), while in the latter case let \( B(u) := \{ u = u_1^B, \ldots, u_6^B \} \) be the set of consecutive vertices of \( B \) starting from \( u \). Also let \( |B| = 4 \) or \( 6 \) be the number of vertices of \( B \). Given a pivot vertex \( u \) of a set \( B \in B \), we define the scope of \( B(u) \) to be the set \( sc(B(u)) := \{ u_1^B, \ldots, u_{k-2}^B \} \), where \( k = 4 \) or \( 6 \). In the sequel, we call badly colored the sets in \( B \) whose both equal parity sets are monochromatic. Consider now \( \text{VertexColor} \), Algorithm 4 defined below.

**Algorithm 4 VertexColor**

1: for each \( u \in V \) do
2: 
3:  Choose a color from the palette, independently for each \( u \), and u.a.r.
4: end for
5: while there is a pivot vertex of a badly colored set in \( B \), let \( u \) be the least such vertex and \( B \) be the least such set and do
6: 
7:  Recolor\((u, B)\)
8: end while
9: return the current coloring

**Recolor\((u, B)\)**

1: Choose a color independently for each \( v \in sc(B(u)) \), and u.a.r.
2: while there is vertex in \( sc(B(u)) \) which is a pivot vertex of a badly colored set in \( B \), let \( u' \) be the least such vertex and \( B' \) be the least such set and do
3: 
4:  Recolor\((u', B')\)
5: end while

Because of lack of \( \alpha \)-special properness of colorings generated by \( \text{VertexColor} \) we introduce:

**Algorithm 5 MainAlgorithm-Vertices**

1: Execute \( \text{VertexColor} \) and if it stops, let \( c \) be the coloring it generates.
2: while \( c \) is not \( \alpha \)-specially proper do
3:  Execute \( \text{VertexColor} \) anew and if it halts, set \( c \) to be the newly generated coloring
4: end while

The following two lemmas are analogous to Lemmas 2 and 3, respectively. We do not prove them, neither do we give any comment, as both their proofs and their role should be clear by analogy.
**Lemma 11.** Let $V$ be the set of vertices that at the beginning of some procedure \textsc{Recolor}(u, B) are not pivot vertices in a badly colored set $B$. Then, if and when that call terminates, no such vertex in $V \cup \{u\}$ exists.

**Lemma 12.** There are at most $l$, the number of vertices of $G$, i.e. a constant, repetitions of the \texttt{while}-loop of line 4 of the main part of \textsc{VertexColor}.

Let $P_n$ be the probability that \textsc{VertexColor} lasts at least $n$ phases, and $Q$ be the probability that \textsc{VertexColor} halts and the coloring generated is not $\alpha$-specially proper. We will now show that for any $\alpha > 2^{-1/3}$, there is an integer $\Delta_\alpha$ such that for any graph whose maximum degree is at least $\Delta_\alpha$, the following two facts hold:

**Fact 3.** The probability $P_n$ is inverse exponential in $n$.

**Fact 4.** The probability that the \texttt{while}-loop of \textsc{MainAlgorithm-Vertices} is repeated at least $n$ times is inverse exponential in $n$.

From the above two facts, \textit{yet to be proved}, the proof of Theorem 2 follows.

As in edge coloring, we depict the phases of an execution of the algorithm \textsc{VertexColor} with a labeled rooted forest, the witness structure. The labels are pairs $(u, B)$, where $u$ is a pivot vertex of $B \in B$. We call $u$ the vertex-label and $B$ the set-label of the node of the tree.

**Definition 6.** A labeled rooted forest $F$ is called feasible, if the following conditions hold:

i. Let $u$ and $v$ be the vertex-labels of two distinct nodes $x$ and $y$ of $F$. If $x$ and $y$ are both either roots of $F$ or siblings in $F$, then $u$ and $v$ are distinct.

ii. If $(u, B)$ is the label of an internal node $x$ of the forest, the vertex-labels of the children of $x$ comprise the set $\text{sc}(B(u))$.

As in edge coloring, we order the vertices of a feasible forest, and we set:

$$L(F) := ((u_1, B_1), \ldots, (u_{|F|}, B_{|F|})).$$

We also associate a feasible forest with every halting execution of \textsc{VertexColor} (the forest may differ for different executions). If \textsc{VertexColor} halts when executed following a sequence of color choices $\rho$, then the associated feasible forest is denoted by $F_\rho$.

Again as in the case of edge-coloring, we introduce below a validation algorithm, \textsc{VertexValidation}. 
Algorithm 6 VertexValidation(\mathcal{F})

\begin{algorithmic}
\State Input: Feasible forest \mathcal{F}, where \mathcal{L}(\mathcal{F}) = (u_1, B_1), \ldots, (u_{|\mathcal{F}|}, B_{|\mathcal{F}|}).
\State Color the vertices of \mathcal{G}, independently and selecting for each a color u.a.r. from the palette.
\For {i = 1, \ldots, |\mathcal{F}|} \Do
\If {\text{\text{scape}}} \Then
\State recolor the vertices in sc(B_i(u_i)) independently by selecting for each a color u.a.r. from the palette.
\Else
\State return failure and exit
\EndIf
\EndFor
\State return success
\end{algorithmic}

Let \mathcal{V}_\mathcal{F} be the event comprised of sequences of color choices \rho such that VertexValidation(\mathcal{F}) if executed following \rho is successful. Also a successful phase is defined to be one that is executed and the check of line 3 is true, independently of whether VertexValidation(\mathcal{F}) is finally successful. The following two Lemmas, namely Lemmas 13 and 14 are analogous to the corresponding ones for edge coloring, namely Lemmas 4 and 5, and so we do not give any proofs.

Lemma 13. The following hold:

(a) For every (fixed) feasible \mathcal{F}, the coloring generated at the end of every successful phase of the execution of VertexValidation(\mathcal{F}) is uniformly random.

(b) For every feasible \mathcal{F}, if \mathcal{L}(\mathcal{F}) = (u_1, B_1), \ldots, (u_{|\mathcal{F}|}, B_{|\mathcal{F}|}), then

\[
\Pr[V_{\mathcal{F}}] = \prod_{i=1}^{|\mathcal{F}|} \left( \frac{1}{K(|B_i|-2)} \right).
\]

Analogously to Lemma 5, and taking into account the proof of Lemma 10, we have that

Lemma 14. Q is bounded away from 1 by a constant.

From Lemma 14 we get Fact 4.

Let now \hat{P}_n be the probability that VertexValidation(\mathcal{F}) succeeds for at least one \mathcal{F} with exactly \(n\) nodes.

Lemma 15. We have that \(P_n \leq \hat{P}_n \leq \sum_{|\mathcal{F}|=n} \Pr[V_{\mathcal{F}}].\)

To show Fact 3, we will bound \(\sum_{|\mathcal{F}|=n} \Pr[V_{\mathcal{F}}].\) Let:

\[
q := \frac{1}{\alpha \Delta^{4/3}} > \frac{1}{\left[\alpha \Delta^{4/3}\right] + \Delta + 1}.
\]

We define the weight \(\|\mathcal{F}\|\) of a feasible forest \mathcal{F} by taking the product of weights assigned to its nodes as follows: for each node with set-label \(B\), if \(B\) is a 4-cycle,
assign weight $q^2$; if $B$ is a 5-path, assign weight $q^4$. We get by the second item of Lemma 13:

\begin{equation}
\sum_{|F| = n} \Pr[V_F] \leq \sum_{|F| = n} ||F||.
\end{equation}

We will bound $\sum_{|F| = n} ||F||$ by purely combinatorial arguments.

Let $T_j$ be the set of all possible feasible trees, whose root has as a vertex-label the vertex $u_j$, together with the empty tree (for the latter, we assume $|\emptyset| = 0$ and $||\emptyset|| = 1$). Let also $T$ be the collection of all $l$-ary sequences $(T_1, \ldots, T_l)$ with $T_j \in T_j$.

Now, obviously:

\begin{equation}
\sum_{|F| = n} ||F|| = \sum_{(T_1, \ldots, T_l) \in T} ||T_1|| \cdots ||T_l|| = \sum_{n_1 + \cdots + n_l = n, n_1, \ldots, n_l \geq 0} \left( \left( \sum_{T_1 \in T_j, |T_1| = n_1} ||T_1|| \right) \cdots \left( \sum_{T_l \in T_j, |T_l| = n_l} ||T_l|| \right) \right).
\end{equation}

We now obtain a recurrence for each factor of (22).

**Lemma 16.** Let $T^u$ be anyone of the $T_j$. Then:

\begin{equation}
\sum_{T \in T^u, |T| = n} ||T|| \leq R_n,
\end{equation}

where $R_n$ is defined as follows:

for $n \geq 1$,

\begin{equation}
R_n := \frac{\Delta^{8/3}}{8\alpha} q^2 \sum_{n_1 + n_2 = n-1, n_1, n_2 \geq 0} \left( R_{n_1} R_{n_2} \right)
+ \Delta^5 q^4 \sum_{n_1 + n_2 + n_3 + n_4 = n-1, n_1, n_2, n_3, n_4 \geq 0} \left( R_{n_1} R_{n_2} R_{n_3} R_{n_4} \right)
= \frac{1}{8\alpha^3} \sum_{n_1 + n_2 = n-1, n_1, n_2 \geq 0} \left( R_{n_1} R_{n_2} \right)
+ \frac{1}{\Delta^{1/3} \alpha^4} \sum_{n_1 + n_2 + n_3 + n_4 = n-1, n_1, n_2, n_3, n_4 \geq 0} \left( R_{n_1} R_{n_2} R_{n_3} R_{n_4} \right),
\end{equation}

$R_0 = 1$.

**Proof.** The result is obvious for $n = 0$, because the empty tree has weight 1. For $n > 0$, we have two cases for the set-label $B$ of the root of $T \in T^u$. If it is one of the, by Lemma 9, $\frac{\Delta^{8/3}}{8\alpha}$ 4-cycles whose opposing to $u$ vertices do not form special pairs, then the root $u$ has weight $q^2$ and two children. Otherwise, observe that there are at most $\Delta^5$ 5-paths beginning from $u$. In this case, $u$ has weight $q^4$ and four children. \qed
To estimate the asymptotic behavior of the sequence $R_n$, we will find the Ordinary Generating Function $\sum_{n=1}^{\infty} R_n z^n$ (note that the sum starts from $n = 1$) and apply [6, Proposition IV.5, p. 278].

Multiply the first and third sides of Eq. (24) by $z^n$ and sum for $n = 1, \ldots, +\infty$, to get:

$$R(z) = \frac{1}{8\alpha^3} z (1 + R(z))^2 + \frac{1}{\Delta^{1/3} \alpha^4} z (1 + z R(z))^4 \Rightarrow$$

$$R(z) = z \left( \frac{1}{8\alpha^3} (1 + R(z))^2 + \frac{1}{\Delta^{1/3} \alpha^4} (1 + R(z))^4 \right)$$

Set $R := R(z)$ and observe that for:

$$\phi(x) = \frac{(1 + x)^4}{\Delta^{1/3} \alpha^4} + \frac{(1 + x)^2}{8\alpha^3},$$

we have that $R = z \phi(R)$.

Therefore following [6, Proposition IV.5, p. 278], we consider the characteristic equation:

$$x\phi'(x) - \phi(x) = 0 \iff \frac{3x(1 + x)^3 - (1 + x)^4}{\alpha^4 \Delta^{1/3}} + \frac{2x(1 + x) - (1 + x)^2}{8\alpha^3} = 0,$$

and we let $\tau$ be its unique positive solution. As $\Delta$ approaches $\infty$, $\tau$ gets arbitrarily close to 1. Indeed the polynomial (in $x$) $x\phi'(x) - \phi(x)$ is negative at 0 and positive at 1 and since the interval $[0, 1]$ is compact, the function $x\phi'(x) - \phi(x)$ approaches the function $\frac{2x(1 + x) - (1 + x)^2}{8\alpha^3}$ as $\Delta$ approaches $+\infty$ uniformly in $x \in [0, 1]$; but the latter function’s unique positive solution is 1. It only remains to find the range $\alpha$ for which $\phi'(\tau) < 1$.

Again because $\Delta$ approaches $\infty$, the range of $\alpha$ for which $\phi'(\tau) < 1$ is found by observing:

$$2 \frac{1}{8\alpha^3} < 1 \iff \frac{1}{4} < \alpha^3$$

It follows that, for every $\alpha > 4^{-1/3}$, there is a $\Delta_\alpha$ (depending on $\alpha$), such that if the maximum degree $\Delta$ of the graph is at least $\Delta_\alpha$, then with $\lceil \alpha \Delta^{4/3} \rceil + \Delta + 1$ colors, $P_n$ is exponentially small, so the proof of Fact 3 and therefore Theorem 2, our main result in this section, is complete.

**Remark 3.** *Our technique does not lead to the conclusion that for large enough maximum degree $\Delta$, the chromatic number is at most $\lceil 4^{-1/3} \Delta^{4/3} \rceil + \Delta + 1$, because we cannot exclude that $\Delta_\alpha$ approaches $+\infty$ as $\alpha$ approaches $2^{-1/3}$.*

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