Nonlocality and fluctuations near the optical analog of a sonic horizon

S. Bar-Ad, R. Schilling, and V. Fleurov

1 Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel
2 Johannes Gutenberg University, Mainz, Germany

We consider the behavior of fluctuations near the sonic horizon and the role of the nonlocality of interaction (nonlinearity) on their regularization. The nonlocality dominates if its characteristic length scale is larger than the regularization length. The influence of nonlocality may be important in the current experiments on the transonic flow in Kerr nonlinear media. Experimental conditions, under which the observation of straddled fluctuations can be observed, are discussed.

PACS numbers: 42.65.-k, 47.40.Hg, 47.60.Kz, 04.70.-s

I. INTRODUCTION.

A possibility of creating a sonic horizon in a transonically accelerating fluid, mimicking either black or white hole horizons, inspired proposals and attempts to realize them. Quite a number of theoretical models realizing such artificial black (white) holes has been put forward. Experimentally a white-hole horizon was observed in optical fibers, where the probe light was back-reflected from a moving soliton, and a black-hole horizon was observed in a Bose-Einstein condensate (BEC) system. A radiation in optical fibers was observed, which is a promising contender to being analogous to Hawking radiation in a tabletop experiment. A "horizon physics" is currently intensively studied also in the surface water waves and nematic liquid crystals. The whole field of analog gravity has been recently reviewed.

Propagation of the coherent laser beam in Kerr nonlinear media is in its many aspects analogous to a flow of fluid. Properties of such "luminous fluid", in particular formation of the so called "dispersive shock waves" were studied theoretically and experimentally. Similar approach appeared to be very successful in studying temporal dynamics of tunneling. This analogy prompted a proposal to create an optical analog of the Laval nozzle, in which the luminous fluid is accelerated to a supersonic velocity and a sonic horizon is created. Our recent experiment, realizing the proposal, demonstrated a possibility of transonic acceleration of a luminous fluid in an all-optical analog of the Laval nozzle. A laser beam is propagating in a specially profiled cavity filled with ethanol with an addition of iodine. The beam disperses and also deviates from the straight line (along the z axis) propagation so that the beam angle with respect to the axis mimics the flow velocity and its variation mimics acceleration. The defocusing nonlinearity in this experiment appears due to a local temperature variation under the action of the laser beam. Such mechanism directly leads to nonlocality of the Kerr nonlinearity (see review). It was demonstrated both theoretically and experimentally in Ref. that the nonlocality can be described by a nonlocal response function with a finite radius. The nonlocality in Kerr nonlinear media allows one to control stability and mobility of solitons (bright or dark, depending on the sign of the nonlinearity) as shown theoretically and confirmed experimentally. In Bose-Einstein condensates the nonlocality may become of importance at high enough densities. It is also typical of the dipolar bosonic quantum gases and of nematic liquids.

However the issue of nonlocal nonlinearity has not yet been addressed in the context of the horizon physics. A number of papers present detailed analysis of fluctuations and Hawking radiation in transonically accelerated fluid accounting for local nonlinearity only, which is a quite reasonable approximation in BEC at least for very short-range interactions. The same approximation in Kerr nonlinear media may be less harmless and an analysis of the role of nonlocal nonlinearity in dynamics of fluctuations and formation of Hawking radiation near the Mach horizon is necessary. The nonlocality introduces a new length \( \sigma \) (see Ref.) so that we may expect that fluctuations with shorter lengths will be smeared out. The interplay of the nonlocality length \( \sigma \) with other lengths characterizing the problem will be discussed in what follows and conditions for observation of analog Hawking radiation in nonlocal Kerr nonlinear media will be formulated.

II. NONLOCAL NONLINEARITY.

We consider here the Nonlinear Schrödinger (NLS) equation in 1 + 1 dimensions,
\[ -\frac{1}{2\beta} \partial^2_x \Psi(x, z) + g(R|\Psi|^2)(x, z)\Psi(x, z) = 0 \tag{1} \]

where \( \Psi \) is a paraxial amplitude of light propagating along the \( z \) axis, which plays the part of time. \( \beta \) is the wave vector of the laser beam, which plays now the role of mass. \( R \) is a linear integral nonlocal operator,

\[ \tilde{R}(|\Psi|^2) = \int dx' dz' R(x' - x, z' - z)|\Psi(x', z')|^2 \tag{2} \]

The kernel of this operator is normalized to one and is assumed to be not singular and characterized by a finite length scale \( \sigma \). For example it may be chosen in the form

\[ R(x, z) = \frac{1}{2\pi \sigma^2 \sqrt{\sigma^2 + \beta^2}} \tag{3} \]

In Kerr nonlinear optics the nonlocality depends obviously both on \( x \) ("coordinate") and \( z \) ("time"). The dependence is expected to be symmetric. The Fourier transform of \( \tilde{R} \) reads

\[ \tilde{R}(k_x, k_z) = \frac{1}{(1 + \sigma^2 k_x^2 + \sigma^2 k_z^2)^{3/2}}. \]

It tends to one in the long wave limit \( k_x, k_z \to 0 \), and to zero in the short wave limit \( k_x, k_z \to \infty \). These generic features (see, e.g. discussion in Ref. 41) will be important for the analysis to be presented in the following.

The Madelung transformation \( \Psi = f e^{-i\varphi} \) allows one to represent the NLS equation for a complex function as two hydrodynamic equations

\[ \partial_t \rho + \partial_x (\rho v) = 0, \tag{4} \]

\[ \partial_t v + \frac{1}{\beta} \partial_x v^2 = \frac{1}{\beta} \partial_x \left[ U_{qu} + g \tilde{R}_0 \right] \tag{5} \]

for two real functions. Here

\[ U_{qu} = \frac{1}{2\beta} \frac{\partial^2 f}{f}. \]

These two equations describe an equivalent luminous fluid, where the light intensity \( \rho = f^2 \) plays the role of the density and \( \beta v = -\partial_x \varphi \) defines the flow velocity \( v \) in the \( x \) direction.

Our aim here will be to analyze the behavior of fluctuations near the Mach horizon of the transomically accelerating luminous fluid. This analysis will be quite analogous to the one described in detail in Ref. 38 (references to some earlier papers can also be found there). The only difference is due to the nonlocality term in the Euler equation \( \tilde{R} \). That is why only principal steps will be outlined below, which are necessary to introduce the notations and arrive at the result.

We consider fluctuations on the background of a given stationary flow profile described by a function \( \Psi_0 = f_0 e^{-i\varphi_0} \). They can appear spontaneously or induced artificially by experimental means. Linearized equations for these fluctuations are deduced from \( \Psi_0 \) and \( \tilde{R} \),

\[ \partial_z + v_0\partial_x)(\chi - \frac{1}{\beta \rho_0})\partial_x (\rho_0 \partial_x \xi) = 0, \]

\[ \partial_z + v_0\partial_x) = \frac{1}{4\beta \rho_0} \partial_x (\rho_0 \partial_z \chi) - g(\tilde{R}_0 \chi) = 0. \tag{6} \]

where the density \( \delta \rho(x, z) \) and velocity \( \delta v(x, z) \) fluctuations are defined by the relations

\[ \delta v(x, z) = -\frac{1}{\beta} \partial_x \chi(x, z), \]

\[ \delta \rho(x, z) = \rho_0(x) \chi(x, z), \tag{7} \]

and \( \rho_0 = f_0^2 \), \( v_0 = -1/(\beta \chi) \partial_x \varphi_0 \).

Equations resulting from linearization around a known solution are usually called modulation equations, which may or may not lead to a modulation instability. Such an instability appears typically in the case of negative \( g < 0 \), i.e. focusing nonlinearity, whereas we deal here with positive, \( g > 0 \), i.e. defocusing nonlinearity. Refs. 41, 42 present a general discussion in the context of the Kerr nonlinear media. In the simple case of a constant background density \( \rho_0 \) and velocity \( v_0 \) we readily obtain the spectrum of the fluctuations in the form

\[ (k_z - v_0 k_x)^2 = \frac{g \rho_0 k_x^2}{\beta} \left[ \frac{k_x^2 \rho_0}{2} + \tilde{R}(k_x, k_z) \right], \tag{8} \]

which corresponds to the Bogoliubov excitation spectrum in the case of nonlocal interaction. Here \( l_n^2 = 1/(2\beta g \rho_0) \) is the nonlinearity length (healing length in BEC). If the nonlocality kernel is chosen in the form \( \tilde{R} \) then the long wave limit in \( \tilde{R} \) holds at \( k_x, k_z \ll 1 \) and \( k_x, k_z \ll 1 \) and the sound velocity keeps its standard form \( c^2 = g \rho_0 / \beta \). We assume the nontrivial situation when \( \sigma > l_n \). Then the quartic dependence

\[ (k_z - v_0 k_x)^2 = \frac{k_x^4}{4\beta^2} \]

becomes dominant under the condition that \( k_x, k_z > (l_n^2 \sigma^2)^{-1/5} \), which may be fulfilled even at \( l_n k_x, k_z < 1 \).

### III. Regularization due to the Nonlocality

In order to consider the fluctuations close to the Mach horizon we will use the relations

\[ v = \tilde{v}(1 + \alpha x), \quad \frac{g \rho_0(x)}{\beta} = \tilde{v}^2(1 - \alpha x). \]
describing a spatially accelerating flow. Here $\bar{v}$ is the sound velocity at $x = 0$, i.e., at the throat of the Laval nozzle. Then Eqs. (9) take the form

$$
\partial_z \chi + \bar{v}(1 + \alpha x) \partial_x \chi + \frac{1}{\beta} \left[ \alpha \partial_x - \partial_x^2 \right] \xi = 0,
$$

$$
\frac{1}{4\beta} \left\{ -\alpha \partial_x + \partial_x^2 \right\} \chi - \bar{v}^2 \bar{R} \chi + \beta \bar{v}^2 \alpha \bar{R}(x\chi) + \left[ \partial_z + \bar{v}(1 + \alpha x) \partial_x \right] \xi = 0.
$$

where the terms $O(\alpha^2 x^2)$ are omitted. These equations can be solved in the Fourier space to within the terms $O(\alpha/k_x)$ (see Ref. 38). The Fourier transformed first equation (10) becomes

$$
\partial_{k_x} \ln \chi_k \approx \frac{1}{2i} \left\{ \bar{r}^2(2i\alpha k_x^3 + k_x^4) + (i\alpha k_x + k_x^2)\bar{R}(k_x, \nu) - (\nu - k_x^2)^2 - i\alpha k_x \right\}
$$

$$
\alpha k_x \{2\nu - [2 + \bar{R}(k_x, \nu)]k_x - \bar{R}(k_x, \nu)\}
$$

where $\nu = k_x/\bar{v}$.

The integration of the r.h.s. of Eq. (10) although possible, may result in very cumbersome expression. That is why we consider here two limits. If $\sigma k_{x,\nu} \ll 1$, then $\bar{R}(k_x, \nu) \approx 1$ and we get

$$
\partial_{k_x} \ln \chi_k(k_x, \nu) = \frac{\bar{r}^2}{2i} \left\{ (2i\alpha k_x^3 + k_x^4) - (\nu^2 + 2\nu k_x) \right\}
$$

$$
\alpha k_x (2\nu - 3k_x - \nu) \right\}
$$

(11)

It means that we return to the situation of the local nonlinearity. As shown in Ref. 38 there is a singular real space solution of Eq. (11)

$$
\chi_s(x, \nu) \propto x^{-\gamma},
$$

where

$$
\gamma = \frac{2\nu}{3\alpha} + \frac{4i}{81} \frac{\bar{r}^2}{\alpha} - \frac{2}{27n^2} \nu^2.
$$

(13)

Finally it results in the $\nu$-dependent Hawking temperature

$$
T_H(\nu) = T_H(0)/(1 + \frac{1}{2\nu^2} \nu^2)
$$

where

$$
T_H(0) = \frac{3\hbar \bar{v}}{4\pi k_B}
$$

This solution holds for the distances from the Mach horizon satisfying the condition $\min\{1/\nu, 1/\alpha\} \gg |x|$. In the absence of nonlocality the other condition is $|x| \gg l_r$ where $l_r = l_n/(\nu^2/\alpha)^{1/3}$ is the regularization length (38) (see also 27). As we will see below the nonlocality also leads to a regularization which means that the final condition reads $|x| \gg \max\{\sigma, l_r\}$.

We also have to consider the limit $k_x, \sigma \gg 1$. It produces nontrivial results only if $\sigma > l_n$. Otherwise the problem can be reduced to the local one. In the limit $k_x, \sigma \gg 1$ when $\bar{R}(k_x, \nu) \ll 1$ equation (10) becomes

$$
\partial_{k_x} \ln \chi_k(k_x, \nu) =
$$

$$
\frac{\bar{r}^2}{2i} \left\{ (2i\alpha k_x^3 + k_x^4) - \nu^2 + 2\nu k_x - k_x^2 \right\}
$$

$$
2\alpha k_x (\nu - k_x) \right\}
$$

(14)

Carrying out the procedure, as outlined in Ref. 38, the real space density fluctuations for a given $\nu$ are described by the function

$$
\chi_{\nu}(x, z) = e^{-ik_x z} \int dk k^-i(k - \nu)\gamma'_{\nu} e^{ikx + \Lambda}
$$

(15)

where

$$
\gamma'_1 = \frac{i\nu}{2\alpha},
$$

$$
\gamma'_2 = \frac{1}{2} + \frac{4i}{27n^2} \nu^2 - \frac{i}{4\alpha} \nu^3
$$
ularization is not violated at even smaller distances \( \alpha \) becomes regular at small enough \( \chi \) used. It means that the function \( \sigma > l \) should be taken into account if its characteris-
tic length scale exceeds the regularization length,

\[ \Lambda = \frac{i}{2\alpha} k + \frac{1}{2} k^2 \nu - \frac{i}{4\alpha} k^2 \nu^2 k + \frac{1}{4} k^2 \nu - \frac{i}{8\alpha} k^2 \nu - \frac{i}{12\alpha} k^2 \nu^3. \]

The behavior of the density fluctuations can be readily deduced from (15). At \( \nu \ll l^{-1} \) and \( \sigma \gg |x| \) we get

\[ \chi_\nu(x, z) = e^{-ikz} \int dk k \gamma_1 (k - \nu) \gamma_2 e^{ikx +k} k \approx e^{-ikz} \left( x + \frac{1}{2\alpha} \right)^{-\gamma_1 + \gamma_2 + 1} \int dy y \gamma_1 + \gamma_2 e^{iy} \]

where integration variable \( y = |x + 1/(2\alpha)|k \) is used. It means that the function \( \chi \) tends to a con-
tant (since \( \alpha |x| \ll 1 \)) and the singular solution (12) becomes regular at small enough \( x \). This reg-
ularization is not violated at even smaller distances \( |x| \ll l_n \) and may be obtained in the way similar to that used in Ref. \[ 28 \].

IV. DISCUSSION AND SUMMARY.

As discussed in Introduction, the non-locality of the nonlinearity may play an important part in experi-
diments designed to observe Hawking-type classical and quantum fluctuations near the Mach hori-
zon formed by a transonic flow of luminous fluid. The above derivation shows that the non-locality
should be taken into account if its characteris-
tic length scale exceeds the regularization length,

\( \sigma > l_r \). The largest of these determines the point, below which the singular fluctuation eigenfunc-
tion (12) is regularized. Consequently, if \( \sigma > l_r \) it is the length scale \( \sigma \), characterizing the nonlocality, which sets the lower bound for the distance \( |x| \) such that Hawking radiation or straddled fluctuations appear.

At the same time it should be taken into account that the derivation of (12) holds for distances from the horizon that are not too large, and in particular shorter than \( 1/\alpha \) (the inverse acceleration rate) and \( 1/\nu \) (the length scale of a fluctuation in the \( z \) direction). The latter sets an upper limit for the distance from the horizon where a fluctu-
ation may develop (see e.g. Ref. \[ 24 \] for a dis-
cussion of a Kerr nonlinear medium). A window \[ \min\{1/\nu, 1/\alpha\} > |x| > \max\{l_r, \sigma\} \] in real space must therefore exist in order for fluctuations to be observed experimentally. In a typical experiment, such as the one described in \[ 25 \], the nonlinear length is on the order of several tens of microns, while the non-locality length is on the order of 1 mm. \[ 26 \]. This means that \( 1/\nu \) and \( 1/\alpha \) must both be at least on the order of a few millimeters. \( 1/\nu \) can

be as large as the size of the experimental apparatus (a few centimeters), and is not expected to be the limiting constraint. On the other hand, the experimental setup must be designed in such a way that the acceleration of the luminous fluid should occur over a distance of several millimeters in order that the required window for the fluctuations exists. Changing the stationary profile allows one to change the spatial acceleration \( \alpha \) and therefore \( l_r \). This offers the possibility by increasing \( \alpha \) to turn to the regime where \( \sigma \) is the length scale separ-
ating the regular from the singular behavior for all \( \alpha > \alpha_c = (l_n/\sigma)^2 / \sigma \). In case when the quantum potential \( U_{qu} \) can be neglected, i.e. we can assume that \( l_n = 0 \), it is the nonlocality regulariz-
ing the fluctuations for \( |x| < \sigma \). The relationships between these different length scales are schematically shown in Fig. \[ 1 \].

FIG. 1. (Color online) A sketch of an all-optical Laval
nozzle demonstrating the relative length scales which
determine the window in real space where straddled
fluctuations may be generated in an experiment. \( \sigma \) is the
non-locality length (typically 1 mm); \( l_r \) is the reg-
ularization length, determined by the nonlinear length
and the acceleration rate (typically a few tenths of a
mm); \( \alpha \) is the acceleration rate (typically \( \geq 1 \) mm\(^{-3}\)); \( \nu \) is the characteristic "frequency" of the fluctuations,
and should be such that \( 1/\nu \) be of the order of the size of
the experimental test cell (a few centimeters).

To summarize, we considered the effect of the
nonlocality of the nonlinear response on the feasibility of observing, in an actual experiment, fluctuations near a “sonic” horizon of an accelerating luminous fluid that are analogs of Hawking radiation. We find that the non-locality sets a lower bound on the distance from the horizon where the fluctuations may develop, and this, in turn, sets an upper bound on the acceleration rate of the fluid. With proper design of the experimental apparatus, there appears to be a large enough window for observing such fluctuations.

Acknowledgment SB and VF acknowledge the financial support of the Israel Science Foundation, grant N 1309/11. VF acknowledges fruitful discussions with G. Shlyapnikov and E. Bogomolny during his visit to LPTMS, Univ. Paris-Sud, Orsay.

1 W. G. Unruh, Phys. Rev. Lett., 46, 1351 (1981).
2 B. Reznik, Phys. Rev. D, 62, 044044 (2000).
3 S. Giovanazzi, Phys. Rev. Lett., 94, 061302 (2005).
4 C. Barcelo, S. Liberati, and M. Visser, Phys. Rev. A, 68, 053613 (2003).
5 I. Carusotto, S. Fagnocchi, A. Recati, R. Balbinot, and A. Fabbri, New J. Phys., 10, 103001 (2008).
6 A. Recati, N. Pavloff and I. Carusotto, Phys. Rev., A 80, 043603 (2009).
7 P. D. Nation, M. P. Blencowe, A. J. Rimberg, and E. Buks, Phys. Rev. Lett., 103, 087004 (2009).
8 G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. König, and U. Leonhardt, Science, 319, 1367 (2008).
9 O. Lahav, A. Itah, A. Blumkin, C. Gordon, S. Rinott, A. Zayats, J. Steinhauer, Phys. Rev. Lett., 105, 240401 (2010).
10 F. Belgiorno, S. L. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V. G. Sala, and D. Faccio, Phys. Rev. Lett., 105, 203901 (2010).
11 G. Rousseaux, C. Mathis, P. Maissa, T. G. Philbin, and U. Leonhardt, New J. Phys., 10, 053015 (2008).
12 G. Rousseaux, P. Maissa, C. Mathis, P. Coullet, T. G. Philbin, and U. Leonhardt, New J. Phys., 12, 095018 (2010).
13 S. Weinfurtner, E. W. Tedford, M. C. J. Penrice, W. G. Unruh, G. A. Lawrence, Phys. Rev. Lett., 106, 021302 (2011).
14 J. S. Robertson, J. Phys. B 45, 163001 (2012).
15 G. A. El, V. V. Geogjaev, A. V. Gurevich, A. L. Krylov, Physica D 87, 186 (1995).
16 G. A. El, A. M. Kamchatnov, Phys. Lett. A 350, 192 (2006).
17 G. A. El, A. M. Kamchatnov, V. V. Khodorovskii, E. S. Annibale, and A. Gammal, Phys. Rev. E 80, 046317 (2009).
18 M. A. Hoffer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard, Phys. Rev. A 74, 023623 (2006).
19 W. Wan, S. Jia and J. W. Fleischer, Nature Physics, 3, 46 (2007).
20 S. Jia, W. Wan, J. W. Fleischer, Phys. Rev. Lett., 99, 223901 (2007).
21 G. Dekel, V. Fleurov, A. Soffer, C. Stucchio, Physical Review A 75, 043617 (2007).
22 G. Dekel, O. V. Farberovich, A. Soffer, V. Fleurov, Physica D: Nonlinear Phenomena 238, 1475 (2009).
23 G. Dekel, V. Farberovich, V. Fleurov, A. Soffer, Phys. Rev. A 81, 063638 (2010).
24 I. Fouzou, O.V. Farberovich, S. Bar-Ad and V. Fleurov, Europhys.Lett., 92, 14002 (2010).
25 M. Elazar, V. Fleurov, and S. Bar-Ad, Phys. Rev. A 86, 063821 (2012).
26 W. Krolikowski, O. Bang, N.I. Nikolov, D. Neshov, J. Wyller, J.J. Rasmussen, and D. Edmundson, Modulational instability, solitons and beam propagation in spatially nonlocal nonlinear media. J. of Optics B: Quantum and Semiclassical Optics 6, S288 (2004).
27 A. Minovich, D. N. Neshov, A. Dreischuh, W. Krolikowski, and Y. S. Kivshar, Optics Letters, 32, 1599 (2007).
28 Y. V. Kartashov, V. A. Vysloukh, and L. Torner Physical Review Letters, 93, 153903 (2004).
29 F. Ye, Y. V. Kartashov, and L. Torner, Physical Review A 77, 043821 (2008).
30 N. I. Nikolov, D. Neshov and W. Krolikowski, O. Bang, J. J. Rasmussen, P. L. Christiansen, Optics Letters 29, 286 (2004).
31 A. Dreischuh, D.N. Neshov, D.E. Petersen, O. Bang, W. Krolikowski, Phys. Rev. Lett. 96, 043901 (2006).
32 A. Parola, L. Salasnich, and L. Reatto, Physical Review, A 57, R3180 (1998).
33 T. Lahaye, C. Menotti, L. Santos, M. Lewenstein and T. Pfau, Rep. Prog. Phys. 72 126401 (2009).
34 P.D. Rasmussen, O. Bang, W.Z. Krolikowski, Phys. Rev. E 72, 066611 (2005).
35 J. Macher and R. Parentani, Phys. Rev. D 79, 124008 (2009).
36 J. Macher and R. Parentani, Phys. Rev. A 80, 043601 (2009).
37 S. Finazzi and R. Parentani, Phys. Rev. D 83, 084010 (2011).
38 V. Fleurov and R. Schilling, Phys. Rev. A85, 045602 (2012); arXiv:1105.0799 (2011).
39 P.-E. Larré, A. Recati, I. Carusotto and N. Pavloff, Phys. Rev. A 85, 013621 (2012).
40 A. M. Kamchatnov and N. Pavloff, Phys. Rev. A 85, 033603 (2012).
41 J. Wyller, W. Krolikowski, O. Bang, J.J. Rasmussen, Phys. Rev. E 66, 066615 (2002).
42 W. Krolikowski, O. Bang, J.J. Rasmussen, J. Wyller, Phys. Rev. E 64, 016612 (2001).