On the Profile of Multiplicities of Complete Subgraphs

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March 29, 2017

Abstract

Previous work on Ramsey multiplicity focused primarily on the multiplicity of complete graphs of constant size. We study the question for larger complete graphs, specifically showing that in every 2-coloring of a complete graph on \( n \) vertices, there are at least \( \left( \frac{n}{\log n} \right) \frac{\log n}{2} \) monochromatic complete subgraphs of size between \( \frac{33}{128} \log n \) and \( \log n \).

We also study bounds on the ratio between a maximum and a random red clique in a graph, and the ratio between a maximum and a random monochromatic clique in a graph.

1 Introduction

The classic diagonal Ramsey number question asks the following: what is the minimum \( n \) such that all 2-colored (edge colored) complete graphs of size \( n \) contain a monochromatic clique of size \( t \)? Letting \( \alpha \) be the size of the largest monochromatic clique, the diagonal Ramsey number question can be rephrased as: what is the minimum \( \alpha \) over all possible 2-colorings of the complete graph on \( n \) vertices?

These questions are the basis for the field of Ramsey Theory, the study of which dates back to the 1920’s. Known bounds say that all graphs of size \( n \) contain a monochromatic clique of size at least \( \frac{\log n}{2} \), and that there exists a graph with maximum monochromatic clique of size at most \( 2 \log n \). These results date back to 1935 (Erdős and Szekeres \textsuperscript{9}) and 1947 (Erdős \textsuperscript{6}), respectively. There have since been improvements to these bounds, but only to the lower-order terms.
In other words, this gap has proved difficult to narrow. In this paper, we do not seek to directly improve results to the Ramsey bounds, but rather turn our attention to a related question that is usually phrased “RamseyMultiplicity”: what is the minimum number of monochromatic cliques of size \( t \) in a 2-colored graph?

At first glance this may seem like a more challenging question, and it is, since a full answer to this question would also give us an answer to the classic Ramsey problem. However, the Ramsey Multiplicity question may be easier to solve for different values of \( t \), and may thus give insight into solving it for \( t = \alpha \). In fact, for constant values of \( t \) it has previously been studied somewhat extensively (see the Related Work section for more details).

In this paper, we study the Ramsey Multiplicity question for values of \( t \) close to \( \log n \). In other words, we already know from Ramsey bounds that in every graph there exists at least one monochromatic clique of size \( \log \frac{n}{2} \), but how many are there? We restate previously known upper bounds in Section 2 and give some new lower bounds in Section 3 specifically proving the following:

**Theorem 1.** Every 2-colored complete graph on \( n \) vertices contains at least

\[
\left( \frac{n - t}{\log n} \right)^{\log n} \text{ monochromatic cliques of size } \frac{33}{128} \log n \leq t \leq \log n.
\]

Our study of Ramsey Multiplicity then led us to a related question: what is the ratio between the maximum size of a monochromatic clique and the expected size of a random monochromatic clique? One might see why we became interested in this question – having proved that there exist many monochromatic cliques of a size \( t \), leading to a possible lower bound on the expected size of a random monochromatic clique, we hoped it could lead to a lower bound on the maximum size monochromatic clique as well. This ratio question ultimately became interesting to us for its own sake, and our preliminary results can be found in Section 4.

1.1 Some Notes, Definitions, and Background

Some of the related and cited work talk about cliques and independent sets, which are equivalent to a monochromatic clique in a 2-colored graph, if we let one color be edges and the other color be non-edges. Therefore, when we discuss or use their results we sometimes re-word them to coloring terminologies, without further comment.

Let the Ramsey Number \( R(s, t) \) be the minimum size such that all 2-colored graphs of this size have either a blue clique of size \( s \) or a red clique of size \( t \). Ramsey’s Theorem states that there exists a positive integer \( R(s, t) \) such that this holds. Let \( R(s) \) be the diagonal Ramsey Number, when \( s = t \). Simple known bounds for the diagonal \( R(s) \) are of the form:

\[
2^s \leq R(s) \leq 4^s - 1
\]  \hspace{1cm} (1)
There exist improvements on these bounds in lower order terms, but we do not use them in this paper so do not include them here.

The following inequality is used to prove Ramsey’s Theorem, and is related to the present work as well:

\[ R(s, t) \leq R(s, t - 1) + R(s - 1, t) \]  \hspace{1cm} (2)

A proof sketch for this inequality goes as follows: We prove the statement by induction. Assume we have a graph \( G \) of size \( R(s, t - 1) + R(s - 1, t) \). Pick a vertex \( v \) in \( G \), let \( A(v) \) be the set of vertices connected to \( v \) by a red edge and \( B(v) \) be the set of vertices connected to \( v \) by a blue edge. So, we have that \( R(s, t - 1) + R(s - 1, t) = A(v) + B(v) + 1 \). Therefore, either \( A(v) \geq R(s, t - 1) \) or \( B(v) \geq R(s - 1, t) \). If \( A(v) \geq R(s, t - 1) \), then either \( A(v) \) contains a blue clique of size \( s \) in which case \( G \) contains a blue clique size \( s \), or \( A(v) \) contains a red clique of size \( t - 1 \), in which case \( G \) contains a red clique of size \( t \) by adding \( v \) to it. The analogous arguments can be used if \( B(v) \geq R(s - 1, t) \).

Let \( k_t(G) \) be the number of monochromatic cliques of size \( t \) in a graph \( G \) of size \( n \). Let

\[ k_t(n) = \min\{k_t(G) : |G| = n\} \]

Let \( c_t(n) = \frac{k_t(n)}{\binom{n}{t}} \), and lastly let \( c_t = \lim_{n \to \infty} c_t(n) \) so that \( c_t \) gives the minimum fraction of all subsets of size \( t \) that are monochromatic cliques. This notation is consistent with the (most closely) related work.

Let \( K_i \) denote a clique of size \( i \).

1.2 Related Work

The study of the multiplicity of monochromatic cliques was introduced by Erdös in 1962 in [7], where Erdös proves that for all graphs,

\[ c_t \geq \left( \frac{R(t)}{t} \right)^{-1} \]  \hspace{1cm} (3)

This lower bound is proved by inductively applying Inequality (2); a similar result and proof can be found in Section 3.1.

In the same paper, Erdös proves that there exists a graph for which

\[ c_t \leq 2^{1 - \binom{t}{2}} \]  \hspace{1cm} (4)

This upper bound is proved using the probabilistic method; a similar result and proof are shown in Section 2.

Erdös conjectured that the upper bound in Inequality (4) was tight, or in other words, that an Erdös-Rényi \( G(n, \frac{1}{2}) \) random graph is the graph with the
smallest number of monochromatic cliques of every size. In 1959, this conjecture
was proved true for the case \( t = 3 \) by Goodman [13].

A survey on Ramsey Multiplicity results was published in 1980 by Burr and
Rosta [4], in which they extend Erdős’s conjecture to the multiplicity of any
subgraph, not just monochromatic cliques.

The conjecture was later disproved by counterexamples in 1989 by Thomason
[17], who showed that it does not hold for \( t \geq 4 \). Subsequently, several
others worked on upper bounds for \( c_t \) for small \( t \). Soon after Thomason’s work,
Franek and Rödl [10] also gave some different counterexamples based on Cayley
graphs for \( t = 4 \). Then in 1994, Jagger, Štovíček, and Thomason [14] studied for
which subgraphs the Burr-Rosta conjecture holds, and found that it does not
hold for any graph with \( K_4 \) as a subgraph, which is consistent with the \( t \geq 4 \)
found by Thomason.

For larger values of \( t \) dependent on \( n \), there appears to be little previous
work improving Erdős’s bound. However, a closely related line of research in
Frankl, Rödl, and Wilson [11] has found some classes of graphs (for example,
graphs with a Hadamard matrix as their adjacency matrix) for which the con-
jecture holds for \( t \leq (\frac{1}{2} - \delta) \log n \), where \( \delta \) is an arbitrary positive real. Similar
results were found in Thomason [16]. In other words, there exist graphs that
contain the same proportion of monochromatic cliques of size \( \leq (\frac{1}{2} - \delta) \log n \) as
a random \( G(n, \frac{1}{2}) \) graph.

On the flip side, with regards to the lower bound in Inequality (3), in 1979
Giraud [12] proved that \( c_4 > \frac{1}{46} \). The lower bound was not further improved
until more recently in 2012, in a work of Conlon [5]. Conlon proved that there
must exist at least \( \frac{n'}{C(1+\alpha(1))r^2} \) monochromatic \( K_t \)'s in any two-colouring of the
edges of \( K_n \), where \( C \approx 2.18 \) and \( t \) is a constant independent of \( n \). In the
present work, we also improve the lower bound, but for larger \( t \) dependent on
\( n \).

In the second part of this paper we study the ratio between the size of a
random and the size of a maximum monochromatic clique in a graph \( G \). There
appears to be no previous work directly on this topic. However, one of the
primary motivations for our work on this topic is from the study of the minimum
of the maximum independent set size over all \( K_r \)-free graphs of size \( n \).

In 1995, Shearer [15] used the probabilistic method to prove that \( \alpha \geq c'(r)\frac{n\log d}{d \log \log d} \), where \( \alpha \) is the size of the maximum independent set and \( d \)
is the average degree in the graph. Following his technique, Alon [1] proved
that for a graph in which the neighborhood of every vertex is \( r \)-colorable,
\( \alpha \geq \frac{c}{\log(r+1)} \frac{n\log d}{d \log \log d} \) for some constant \( c \). Note that an \( r \)-colorable graph is
\( K_{r+1} \)-free, since a clique can contain at most one vertex of each color.

The latest improvement for \( K_r \)-free graphs is due to Bansal, Gupta, and
Guruganesh [3], proving that \( \alpha \geq \frac{n}{d} \cdot \max\{\frac{\log d}{r \log \log \log d}, \left(\frac{\log d}{\log r}\right)^\frac{1}{2}\} \). There is still
a gap in this question, the upper bound being \( \frac{n\log d}{d \log \log r} \) for \( K_r \)-free graphs (also
given in Bansal et al \cite{3}). All three of these papers actually prove that a random independent set in $G$ is of the given size, and then conclude that therefore the maximum independent set must be at least that size as well.

Thus, knowing the relationship between a random independent set and a maximum one could be useful in improving these bounds. Our study of the Ramsey Multiplicity problem also stems from these results, since these proofs begin by showing that there exist many independent sets within certain subgraphs, and therefore that a random independent set is at least of a certain size.

2 Upper Bound

In this section we show that there exist graphs with a small number of monochromatic cliques by calculating the expectation of the number of monochromatic cliques in a random graph. This is a restatement of previous results, mostly for educational benefit and completeness.

**Theorem 2.** There exists a graph with $\leq \binom{n^\frac{1}{2}}{\log n} \cdot 2n^\frac{1}{2}$ monochromatic cliques.

**Proof.** Let $G$ be a random 2-coloring of a complete graph on $n$ vertices where each edge is blue with probability $\frac{1}{2}$ and red otherwise. In other words, this is equivalent to a $G(n, \frac{1}{2})$ Erdős-Rényi random graph.

For every subset $S \subseteq G$ of size $i$, let $A_S$ be the event that $S$ is a blue monochromatic clique, and let $X_S$ be its indicator variable. Then the expectation $E(X_S) = \Pr(A_S) = 2^{-\binom{i}{2}}$. Now set $X_i = \sum_{S| |S|=i} X_S$. Thus $X_i$ is the number of blue monochromatic cliques of size $i$. Using linearity of expectation, we get

$$E(X_i) = \binom{n}{i} 2^{-\binom{i}{2}} \leq \left(\frac{ne}{\log n}\right)^i 2^{-\binom{i}{2}}$$

Over all $0 \leq i \leq n$, this expression is maximized at $i = \log n$, yielding

$$E(X_{\log n}) \leq \left(\frac{ne}{\log n}\right)^{\log n} 2^{-\left(\log n\right)^{\frac{3}{2}}}$$

$$= \left(\frac{n^\frac{1}{2}}{\log n}\right)^{\log n} \cdot n^\frac{3}{2}$$

Therefore, the expected total number of blue monochromatic cliques in the graph is at most

$$\sum_{1 \leq i \leq n} E(X_i) \leq n \cdot E(X_{\log n}) = \left(\frac{n^\frac{1}{2}}{\log n}\right)^{\log n} \cdot n^\frac{3}{2} \quad (5)$$

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Notice that since \( p = \frac{1}{2} \), these calculations are the same for blue and red monochromatic cliques, and thus \( G \) also contains the same number of red monochromatic cliques in expectation. So in total \( G \) contains at most 
\[
2n^{\frac{3}{4}} \log \frac{n}{\log n}
\]
monochromatic cliques in expectation.

Thus, since the expectation is 
\[
\left( \frac{n^{\frac{3}{4}}}{\log n} \right) \log n \cdot 2n^{\frac{3}{2}},
\]
then there exists at least one graph with at most this many monochromatic cliques.

Note that the first part of this proof, calculating the expectation of \( X_i \), is the same as Erdős’s proof for Equation 4.

### 3 Lower Bound

#### 3.1 Preliminary Lower Bound

In this section we present a lower bound on the total number of monochromatic cliques in every 2-colored graph. This preliminary result is not as strong as the subsequent bound presented in Section 3.2 but it is much simpler.

**Theorem 3.** For every 2-coloring of a graph \( G \) on \( n \) vertices, \( G \) contains at least \( n^{\frac{1}{8}} \log n \) monochromatic cliques.

**Proof.** There are \( \binom{n}{n^{\frac{3}{4}}} \) subsets size \( n^{\frac{3}{4}} \). Each one of these contains, by Ramsey bounds, a monochromatic clique of size \( \frac{1}{2} \log(n^{\frac{3}{4}}) = \frac{1}{4} \log n \). However, one size \( \frac{1}{4} \log n \) monochromatic clique is a part of \( \binom{n^{\frac{3}{4}}}{n^{\frac{3}{4}} - \frac{1}{4} \log n} \) size \( n^{\frac{3}{4}} \) subsets. Therefore, total, we have:

\[
\frac{n!}{(n^{\frac{3}{4}})!(n - n^{\frac{3}{4}})!} \frac{(n - n^{\frac{3}{4}})!(n^{\frac{3}{4}} - \frac{1}{4} \log n)!}{(n - \frac{1}{4} \log n)!(n^{\frac{3}{4}})!}(n^{\frac{3}{4}} - \frac{1}{4} \log n)! \leq \frac{n!}{(n - \frac{1}{4} \log n)!(n^{\frac{3}{4}})!} \frac{(n^{\frac{3}{4}})!}{(n^{\frac{3}{4}} - \frac{1}{4} \log n)!(n^{\frac{3}{4}})!} \]

monochromatic cliques of size \( \frac{1}{4} \log n \) in the graph.

Note that with Erdős’ result given in Equation (3), which is found in a similar fashion to this proof, we can achieve the same result by plugging the Ramsey lower bound in to Equation (3) and maximizing the resulting expression over \( 0 < t \leq \frac{\log n}{2} \). We find the maximum at \( t = \frac{\log n}{4} \), and thus that the total number of monochromatic cliques of size \( \frac{\log n}{4} \) is at least \( n^{\frac{1}{8}} \log n \).

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3.2 Tighter Lower Bound

In this section we take a different combinatorial approach to our question, and find a tighter lower bound on the minimum number of monochromatic cliques in a 2-colored graph.

3.2.1 Construction

Recall the Ramsey number inequality \( R(s, t) \leq R(s - 1, t) + R(s, t - 1) \). This expression is recursive, and when applied inductively proves an upper bound for diagonal Ramsey numbers, in particular proving that a complete 2-colored graph on \( n \) vertices contains a monochromatic clique of size \( \log n \). We consider a construction below that is an unwrapping of this simple recursive expression and allows us to find not just one large monochromatic clique, but many.

Let \( V = \{v_1, v_2, ..., v_n\} \) be the set of vertices of a given graph \( G \). Let red and blue be the two colors of the edges in \( G \). Let \( A(v) \) be the set of vertices connected to vertex \( v \) by a red edge, and \( B(v) \) be the set of vertices connected to \( v \) by a blue edge.

For the sake of clarity, hereon in this section we will use the term vertex to refer to vertices of a graph \( G \), and the term node to refer to nodes of our tree construction below. With each node \( t \) in the tree, we will associate a vertex \( v(t) \) in the graph \( G \), and a bag \( H(t) \), where a bag is simply a set of vertices that will be explained in a few lines. There will be many nodes in the tree construction associated with a given vertex \( v \).

We will build \( n \) trees \( T_1, T_2, ..., T_n \). Each tree \( T_r \) is rooted at node \( r \), and \( v(r) = v_r \), so that we have one tree per vertex in \( G \). Each root node \( r \) has bag \( H(r) = V \setminus v(r) \), for all \( r \in [n] \).

Now we will explain how to build the trees. Each tree is built recursively, as follows. There is one child of node \( t \) for every vertex in the bag \( H(t) \), and the children of node \( t \) are split into “left” and “right” children, their sets denoted \( L(t) \) and \( R(t) \), respectively. Thus \( |L(t)| + |R(t)| = |H(t)| \). So, for node \( t \):

- \( L(t) \) contains all the vertices in \( H(t) \) that are connected to \( v(t) \) in \( G \) by a red edge, or in other words: \( L(t) = A(v(t)) \cap H(t) \). For each left child \( w \in L(t) \), we let its bag \( H(w) = L(t) \setminus v(w) \).

- \( R(t) \) contains all the vertices in \( H(t) \) that are connected to \( v(t) \) in \( G \) by a blue edge, or in other words: \( R(t) = B(v(t)) \cap H(t) \). For each right child \( w \in R(t) \), we let its bag \( H(w) = R(t) \setminus v(w) \).

We apply this recursively, beginning at the root of the tree, then the new children nodes of the root, then their children, etcetera. The recursions end when the bags are all empty, and then we have our completed tree.
3.2.2 Proofs

Lemma 1. In every 2-colored complete graph $G$ on $n$ vertices, there exists a monochromatic clique of size at least $\log_2 n$.

Proof. Given $G$, we build the $n$ trees as in the construction above. In any one of these trees, there must exist at least one path of length $\log n$: starting at the root node $r$, choose the larger of $L(r)$ and $R(r)$. The larger is at least half of the size of $H(r)$, so at least size $\frac{n-1}{2}$. Repeat on one of the nodes on the larger side, and continue repeating recursively, choosing the larger side at each step, until there are no more children.

Let $n_i$ be the number of vertices in the $i$th node’s bag, where the root node $r$ is the 0th node and $n_0 = |H(r)| = n - 1$. The process above gives us the recursion $n_i = n_i - 1 - 1$, which yields $n_i = n_i = \frac{n}{2^i} - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^i}\right) \geq \frac{n}{2^i} - 1$. Then, solving for when $n_i < 1$, we get $i > \log n - 1$. In other words, there exists at least one path of length at least $i + 1$ (including also the root node), so length at least $\log_2 n$.

Now consider one of these paths of length at least $\log n$ nodes, and so at least $\log n - 1$ edges. Every edge in the path is either “going left” or “going right”. In other words, each edge either limits the new bag to a red-connected neighborhood or a blue-connected neighborhood of the parent node’s corresponding vertex in $G$. Without loss of generality, at least half of the edges in the path are “going left”, limiting to red-connected neighborhoods. Now take the set of every parent node of each edge “going left” on this path, and also the last vertex on the path. These vertices form a monochromatic red clique, and thus we have a monochromatic clique of size $\frac{1}{2}(\log n - 1) + 1 \geq \log n$. \qed

Lemma 2. For every 2-colored complete graph, the corresponding forest built with the construction above contains $n \frac{\log n}{4} - 1$ paths of length $\log n$.

Proof. Like in the previous proof, we start at the root node $r$ and choose the larger of $L(r)$ and $R(r)$, without loss of generality $L(r)$. This side is at least size $|L(r)| \geq \frac{n-1}{2}$. For each $t \in L(r)$, we can choose the larger side $L(t)$ or $R(t)$, which will be at least size $\frac{n}{4} - \left(\frac{1}{2} + \frac{1}{4}\right)$. Continuing by induction on all the nodes in the larger side at each step until reaching length $\log n$ from the root, we get at least

\[
\left(\frac{n}{2} - \frac{1}{2}\right) \cdot \left(\frac{n}{4} - \frac{1}{2} + \frac{1}{4}\right) \cdot \left(\frac{n}{8} - \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \cdot \cdots \cdot \left(\frac{n}{2\log n} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2\log n}\right)\right) \geq
\]

\[
\geq \prod_{i=1}^{\log n} \frac{n}{2^i} - 1 \geq \prod_{i=1}^{\log n} \frac{n}{2^i+1} = n \frac{\log n}{4} - 2
\]

different paths, each of which is length $\log n$. Since we have $n$ such trees, we get $n \frac{\log n}{4} - 1$ paths of length $\log n$ for the whole forest.

Note that, as seen in the proof of Lemma 1, it is always possible when going through only the larger sides to reach depth $\log n$. \qed
Notice that in a perfectly balanced forest – that is, one in which at each node the child nodes are split half to the left and half to the right – we get instead $n^{\log_2 n}$ paths of length $\log n$. Does there exist a graph with an exactly balanced forest construction? The answer is no, because such a graph would have an adjacency matrix that is a Hadamard matrix, since every vertex has degree $\frac{n}{2}$ and every pair of vertices have half of their neighborhood’s in common. Furthermore, we know that a Hadamard matrix contains a submatrix size $\sqrt{n}$ that is all 1’s (for a proof of this fact, see for example [2], Chapter 1.1). Therefore there would exist a monochromatic clique of size $\sqrt{n}$ in the graph, and therefore there would be some path of length at least $\sqrt{n}$ somewhere in the forest, which is a contradiction with the forest being balanced.

On the other hand, an example of a graph with a fairly balanced forest is, again, a $G(n, \frac{1}{2})$ random graph. In the forest construction of such a random graph, the expected number of paths of length $t$ is $\frac{n!}{(n-t)!}2^t-1\cdot \binom{t}{1}$. Here $\frac{n!}{(n-t)!}$ is the number of ways of naming the vertices on the path, $2^t-1$ is the number of ways of choosing where the path goes left and where the path goes right, and $\binom{t}{1}$ is the probability that these turns are consistent with the random graph. At length around $t = 2 \log n$, the expected number of paths of length $t$ or longer drops below 1, and hence there is a graph whose forest construction does not contain any such path.

Now, note that every path in a forest construction defines some monochromatic cliques, but each one is not represented uniquely – far from it. This is the subject of the next proof.

**Theorem 4.** Every 2-colored complete graph $G$ contains at least $\left(\frac{n^{\frac{1}{2}}}{\log n}\right)^{\frac{1}{2}} \log n$ $n^{-\frac{1}{2}}$ monochromatic cliques.

*Proof.* Given the forest construction corresponding to $G$, we begin by truncating all paths of length longer than $\log n$ to length exactly $\log n$. From Lemma 2, we know that we have at least $n^{\log_2 n - 1}$ distinct such paths.

Since each path contains $\log n$ vertices, the set of vertices in given path $p$ can appear in at most $(\log n)!$ different paths – they can appear in every possible order. Therefore, there are at least $\frac{\log n - 1}{(\log n)!} \geq \frac{\log n - 1}{(\log n)^{\log n}} = \left(\frac{n^{\frac{1}{2}}}{\log n}\right)^{\log n} \cdot n^{-1}$ paths with distinct sets of vertices. As in the proof of Lemma 1, each one of these paths defines a blue clique and a red clique. If two paths have different sets of vertices, they may define either the same red clique or the blue clique, but not both. Assume that there are the same number of red and blue cliques in the graph, each $k$. Then by pairing them in all combinations we get at most $k^2$ paths. If the number of red and blue cliques are different, we get less than $k^2$ paths, because $(x - d) \cdot (x + d) < x^2$.

Therefore if we have $x$ paths, we have at least $x^{\frac{1}{2}}$ different monochromatic
cliques. Thus, we have at least \( \left( \frac{n^{1/2}}{\log n} \right)^{1/2} = n^{-1/2} \) different monochromatic cliques in every graph.

The next question is: how large are these monochromatic cliques? Since there are \( \frac{n^{\log n}}{2} \) paths with distinct sets of vertices of length \( \log n \), this suggests that there should actually be \( \frac{n^{\log n}}{2} \) monochromatic cliques, which is the case when the tree is balanced. This is the subject of the next section, where we prove a weaker result.

### 3.3 Lower Bound with Size Bound

We restate Theorem 1 here for convenience:

**Theorem 5.** Every 2-colored complete graph on \( n \) vertices contains at least \( n^{\log n / 2} \) monochromatic cliques of size \( \frac{3}{128} \log n \leq t \leq \log n \).

**Proof.** First note the following fact: every monochromatic clique of size \( t \) appears \( t! \) times as a prefix to paths starting from the root. Therefore when we upper bound the number of prefixes of length \( t \), we are also upper bounding the number of size \( t \) monochromatic cliques.

Let \( \alpha, \beta, \gamma > 0 \). We will first prove that every graph contains at least \( n^{\log n / 4} \) monochromatic cliques of size \( \geq \left( \frac{1}{4} + \gamma \right) \log n \), then choose \( \gamma \) according to the constraints from the proof.

In the forest construction corresponding to \( G \), consider all the paths starting from the root. Call a prefix of a path “good” if its length is \( \log n / 8 \), and the size of the bag of the node at the end of this prefix is at least \( n^{1-\alpha} \). Call a prefix “bad” if its length is \( \log n / 8 \), and the size of the bag of the node at the end of this prefix is less than \( n^{1-\alpha} \). We will now give some facts about these good and bad prefixes.

**Fact 1.** A good prefix is the prefix to at least \( n^{1-\alpha} \cdot n^{1-\alpha} \cdot \cdots \cdot n^{1-\alpha} \cdot n^{(1-\alpha)^2} \) different paths.

**Proof.** We use the same logic as in the proof of Lemma 2 except that now we are starting with a node with bag size \( n^{1-\alpha} \). So we get:

\[
\frac{n^{1-\alpha}}{2^1} \cdot \frac{n^{1-\alpha}}{2^2} \cdots \cdot \frac{n^{1-\alpha}}{2^{(1-\alpha)n}} = \frac{n^{(1-\alpha)^2} \log n}{2^{(1-\alpha)^2} \log n} = n^{1-\alpha} \cdot n^{(1-\alpha)^2}.
\]

**Fact 2.** A good prefix is the prefix to at most \( n^{(1-\alpha) \cdot \log n} \) different paths of length \( (\frac{1}{4} + \gamma) \log n \), including the length of the prefix.
Proof. In this case, we use $n$ as an upper bound on the starting bag size. Then, at each step at least one vertex is removed in the next bag, so we get:

$$n(n - 1)(n - 2)\ldots(n - (\frac{1}{8} + \gamma) \log n) \leq n(\frac{1}{8}+\gamma)^{\log n}$$

Fact 3. A bad prefix is the prefix to at most $n(1-\alpha)(\frac{1}{8}+\gamma)^{\log n}$ paths of length $(\frac{1}{8}+\gamma)^{\log n}$, including the length of the prefix.

Proof. With the same logic as for Fact 2, but starting with a node with bag size less than $n^{1-\alpha}$, we find:

$$n^{1-\alpha}(n^{1-\alpha} - 1)(n^{1-\alpha} - 2)\ldots(n^{1-\alpha} - (\frac{1}{8} + \gamma) \log n) \leq n(1-\alpha)(\frac{1}{8}+\gamma)^{\log n}$$

Fact 4. There exist at most $n^{\frac{\log n}{8}}$ bad prefixes.

Proof. With again the same logic as for Fact 2, we find:

$$n(n - 1)(n - 2)\ldots(n - \frac{\log n}{8}) \leq n^{\frac{\log n}{8}}$$

Now equipped to prove Theorem 5, consider the following two cases:

1. There are fewer than $n(\frac{1}{8} - \beta)^{\log n}$ good prefixes. Using Fact 2, 3, and 4, we add the good and bad prefixes times the number of length $(\frac{1}{8} + \gamma)^{\log n}$ prefixes they can precede, to find that there are at most

$$n^{(\gamma + \frac{1}{8})^{\log n}} \cdot n(\frac{1}{8} - \beta)^{\log n} \cdot n^{\frac{\log n}{8}} \cdot n(1-\alpha)(\frac{1}{8}+\gamma)^{\log n}$$

$$= n(\frac{1}{8}+\gamma-\beta)^{\log n} + n(\frac{1}{8}+(1-\alpha)(\frac{1}{8}+\gamma))^{\log n}$$

prefixes of length $(\frac{1}{8} + \gamma)^{\log n}$. By adding the following constraints,

$$\gamma < \beta \quad \text{and} \quad (1-\alpha)\left(\frac{1}{8} + \gamma\right) < \frac{1}{8}$$

we get that Expression 6 is less than $n^{\frac{\log n}{8}}$, which is the total number of monochromatic cliques. Thus we get that the number of length $(\frac{1}{8} + \gamma)^{\log n}$ prefixes is less than the total number of monochromatic cliques, so almost all monochromatic cliques are larger than $(\frac{1}{8} + \gamma)^{\log n}$, and thus so is their average size.
2. There are at least \( n^{\left(\frac{1}{2} - \beta\right) \log n} \) good prefixes. Using Fact 1, the number of root to leaf paths is at least

\[
n \left(\frac{1}{2} - \beta\right) \log n \leq n^{\left(\frac{1}{2} - \beta\right) \log n} = n^{\left(\frac{1}{2} (1 - \alpha) + \frac{1}{8} - \beta\right) \log n}.
\]

By adding the constraint \( \frac{1}{2} (1 - \alpha)^2 + \frac{1}{8} - \beta \geq \frac{1}{2} + 2\gamma \), we thus get that the number of root to leaf paths is at least \( n^{\left(\frac{1}{2} + \gamma\right) \log n} \), and hence the number of monochromatic cliques is significantly larger than \( n^{\left(\frac{1}{2} + \gamma\right) \log n} \), and hence the average size of a monochromatic clique is at least \( \left(\frac{1}{4} + \gamma\right) \log n \).

It remains to choose \( \alpha, \beta, \gamma \) to satisfy the constraints

\[
\alpha, \beta, \gamma > 0 \\
\gamma < \beta \\
(1 - \alpha)\left(\frac{1}{8} + \gamma\right) < \frac{1}{8} \\
\frac{1}{2} (1 - \alpha)^2 + \frac{1}{8} - \beta \geq \frac{1}{2} + 2\gamma
\]

For example, \( \beta = \frac{1}{64} \), \( \gamma = \frac{1}{128} \), and \( \alpha = \frac{1}{16} \) satisfy these constraints. Thus, using these numbers we have proved that every graph contains at least \( n^{\log n} \) distinct monochromatic cliques of size \( \geq \left(\frac{1}{4} + \frac{1}{128}\right) \log n = \frac{33}{128} \log n \).

The upper bound of size \( \log n \) on these monochromatic cliques comes directly from the fact that we did not consider paths of length longer than \( \log n \) in our counting.

\[\square\]

4 Ratio of Average to Maximum Size Monochromatic Clique

In this section, we are interested in the ratio between maximum and average monochromatic cliques.

Let \( \alpha \) denote the size of the largest monochromatic clique in a graph \( G \). Let \( \beta \) denote the average size over all monochromatic cliques in \( G \). We find bounds on the ratio \( \frac{\alpha}{\beta} \).

We begin by considering the analogous question, but for one color only. Let \( \alpha^\ast \) denote the size of the largest red clique in a graph \( G \), and let \( \beta^\ast \) denote the average size over all red cliques in \( G \).

4.1 Upper Bounds on the Ratios

4.1.1 For One Color Only

**Theorem 6.** There exists a family of graphs for which the ratio \( \frac{\alpha^\ast}{\beta^\ast} \to \log n \) as \( n \to \infty \).
Proof. Consider the following construction: \( \alpha^* = c \log n \) vertices form a red clique and connected to the remainder of the graph by only blue edges, and the rest of the graph is a pair of blue cliques each of size \( \frac{n}{2} \). This gives us \( \frac{n^2}{4} \) red cliques of size 2 and \( n \) red cliques of size 1. Now here is our ratio:

\[
\frac{\alpha^*}{\beta^*} = \frac{c \log n(n^c + \frac{n^2}{2} + n)}{2c \log n \cdot \frac{n^c \cdot \log n}{2} + \frac{n^2}{2} + n \cdot 1} = \frac{c \log n(n^c) + c \log n(\frac{n^2}{4}) + c \log n(n)}{n^c \cdot \log n \cdot \frac{n^c}{2} + \frac{n^2}{2} + n} = \frac{2cn^c \log n + \frac{n^2}{4}n^2 \log n + 2cn \log n}{n^c \log n + n^2 + 2n}
\]

Now notice that if \( c = 1 \), we get a ratio of \( \log n \) as \( n \to \infty \), and when \( c > 2 \), we get a ratio of 2 as \( n \to \infty \). However, if we take the limit as \( c \) approaches 2 from below, we get:

\[
\lim_{c \to 2} \frac{\alpha^*}{\beta^*} = \log n
\]

This is because as long as \( c < 2 \), we get that the \( n^2 \) terms dominate the expression, which gives a ratio of \( \frac{c \log n}{c^2 \log n} \), which goes to \( \log n \) as \( c \) goes to 2.

Theorem 7. For every graph \( G \) of large enough \( n \), the ratio \( \frac{\alpha^*}{\beta^*} < \log n \).

Proof. Assume, for the sake of contradiction, that there exists a graph with ratio \( a \log n \) where \( a \geq 1 \). This graph must contain a red clique of size \( \alpha = c \log n \), where \( c \geq a \), and have average red clique size of at most \( \beta \leq \frac{\alpha^*}{c \log n} = \frac{c}{a} \).

There are at most \( \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) \leq n^{\frac{\log n}{2}} \) subsets of size less than or equal to \( \frac{c}{a} \). On the other hand, there are \( 2^{\alpha^*} = 2^{c \log n} = n^c \) subsets of \( \alpha^* \), which are of average size \( \frac{c \log n}{2} \). Thus the average will be at least

\[
\beta^* \geq \frac{c \log n(n^c + \frac{n^2}{2})}{n^c + \frac{n^2}{2}}
\]

Recalling that \( a > 1 \) and \( c > a \), we can now take the limit of \( \frac{\alpha^*}{\beta^*} \) to get

\[
\lim_{n \to \infty} \frac{\alpha^*}{\beta^*} \leq \lim_{n \to \infty} \frac{c \log n(n^c + \frac{n^2}{2})}{c \log n \cdot n^c + \frac{n^2}{2a}} = 2
\]

Here is the contradiction, since we assumed that the graph would have ratio \( a \log n \). Therefore, the ratio must be smaller than \( \log n \). \( \square \)

Corollary 1. For every graph, the ratio \( \frac{\alpha^*}{\beta^*} < \log n \).

The proof of Theorem 7 works for this result as well, replacing “red clique” with “monochromatic clique”, \( \alpha^* \) with \( \alpha \), and \( \beta^* \) with \( \beta \).
4.1.2 For Both Colors

Theorem 8. There exists a graph $G$ on $n$ vertices for which the ratio $\frac{\alpha}{\beta} = (\frac{1}{2} - \varepsilon) \log n$ for some small positive $\varepsilon$.

Proof. Consider the following construction: $\alpha = (\frac{1}{2} - \varepsilon) \sqrt{n} \log n$ vertices form a red clique and are connected to the remainder of the graph by only blue edges. This gives us $n(\frac{1}{2} - \varepsilon)n^\frac{1}{2}$ red cliques of average size $(\frac{1}{4} - \varepsilon^2) \sqrt{n} \log n$.

The rest of the graph is composed of $\sqrt{n}$ disjoint red cliques of size $\sqrt{n}$, all interconnected by only blue edges. This gives us $\geq n^{\frac{1}{2}}n^{\frac{1}{2}}$ cliques of size $\sqrt{n}$, and smaller ones that only increase the ratio.

Thus since there are many more cliques of size $\leq \sqrt{n}$, we have a ratio of $(\frac{1}{2} - \varepsilon) \log n$, which as $\varepsilon \to 0$, goes to $\frac{\alpha}{\beta} \to \frac{1}{2} \log n$. \qed

Note that we can achieve the same ratio with a graph having $\alpha = \frac{1}{2} \log^2 n$ and the rest of the graph be a $G(n, \frac{1}{2})$ graph.

There is still a gap between Theorem 8 and Corollary 1, and thus the remaining question is whether there exists a graph with ratio close to $\log n$ for monochromatic cliques, or whether there is none with ratio greater than $\frac{1}{2} \log n$.

4.2 Lower Bounds on the Ratios

First of all, trivially, a graph with only blue edges has $\alpha^* = \beta^* = 1$, giving us a ratio of 1. However, the question becomes more interesting and difficult when we talk about the ratio for both colors.

Given $G$, recall that $k_t(G)$ is the number of monochromatic cliques of size $t$ in $G$. When discussing a specific $G$, let $k_t = k_t(G)$. Let the mode of $G$ be the value of $t$ for which $k_t$ is largest. Some graphs may have multiple modes.

Theorem 9. Let $K$ be the size of a maximum monochromatic clique in a graph $G$. If $G$ has only one mode $M$, then $k_K < k_M$.

Proof. Each monochromatic clique of size $K$ contains $K$ subgraphs of size $K-1$, each monochromatic cliques as well. Thus, there exists at least one monochromatic clique $C$ of size $K-1$ that is a subgraph of $\frac{k_K}{k_{K-1}}$ different size $K$ monochromatic cliques. Without loss of generality, let $C$ be a red monochromatic clique.

Taking all the vertices from the size $K$ monochromatic cliques that are supersets of $C$, excluding the vertices of $C$, we have a subgraph size $\frac{k_K}{k_{K-1}}$. This subgraph cannot contain a red edge, or else the edge’s two endpoint vertices and $C$ would form a size $K + 1$ red clique. This subgraph also cannot contain a size $K + 1$ blue clique. Therefore, we have the following inequality:

$$\frac{k_K \cdot K}{k_{K-1}} \leq K$$

$$k_K \leq k_{K-1}$$
If these two values are equal, then they cannot be modes since we are discussing
the case where $G$ has only one mode. Therefore if there is only one mode we
actually have $k_K < k_{K-1}$, and this implies $k_K < k_M$. □

We now show in the theorems below that these different extreme cases exist,
at least for some values of $K$.

**Theorem 10.** There exist graphs in which one of the modes equals the size
of the maximum monochromatic clique.

**Proof.** Take the 5-cycle graph. This graph has 10 monochromatic cliques of size
2, 10 monochromatic cliques of size 1 (each single vertex can be a blue or red
monochromatic clique), and maximum monochromatic clique size 2, thus 2 is
the mode as well as the maximum.

A second example is the 17-Paley graph (for an explanation of Paley graphs,
see for example [8]), for which $K = 3$. It has 136 monochromatic cliques size 3
and 136 edges, and only 17 vertices. Thus 3 is the mode as well as the maximum.

**Theorem 11.** Let $K$ be the size of a maximum monochromatic clique. There
exists a graph in which the single mode $M = K - 1$.

**Proof.** Consider a Turán graph composed of $n^k$ blue cliques of size $n^k$, com-
pletely interconnected by red edges.

In this graph, there are $n^k (n^k) + (n^k) (n^k - t)$ monochromatic cliques (blues plus reds). This expression has a unique maximum at $t = 1$, and
thus here the mode is one less than the maximum. □

### 4.3 A Thought on the Lower Bounds

The technique of the proof of Theorem 10 can theoretically be extended to $k_{K-2}$
and then to $k_{K-i}$ in the following way. As before, there exists a size $K - 2$
red (w.l.o.g.) clique $C$ that is the subgraph of at least $\frac{k_{K-i}(K-1)}{k_{K-2}}$ size $K - 1$
monochromatic cliques. In the subgraph of the monochromatic supersets
excluding the vertices of $C$ itself, we know there cannot be a red triangle, or
else we would have a size $K + 1$ red clique. Thus we can now use the Ramsey
bounds on $R(3, K + 1)$ to lower bound $k_{K-2}$.

Our original hope was to combine this technique with our lower bounds
from Section 3 to then find even more interesting bounds. Unfortunately Ram-
sey bounds are too weak to give very interesting bounds for large values of $i$.

### 5 Concluding Thoughts

Our work in this paper asks how many monochromatic cliques there can or must
be of different sizes close to the maximum size. We have shown that there must
be at least \( n^{\log n} \) monochromatic cliques of size larger than \( \frac{33}{128} \log n \), and our proof of Theorem 4 implies that if there exists a graph with significantly fewer than \( n^{\log n} \) monochromatic cliques that are significantly larger than \( \frac{\log n}{2} \), then it must have a monochromatic clique significantly larger than \( \frac{\log n}{2} \) (beating the current bounds on Ramsey numbers). Our results lead to two further questions.

The first question is whether there can exist a graph containing \( \binom{n}{K}^{\delta} \) monochromatic cliques of size \( K \), where for example \( \delta = \frac{1}{2} \), and this graph contains no monochromatic cliques of size \( (1 + \varepsilon)K \)? We have shown in Theorems 10 and 11 that this is possible when \( K \) is a constant or when \( K \) is of size \( n^{\frac{1}{2}} \).

There are different conjectures one can formulate. For instance, since in a \( G(n, \frac{1}{2}) \) random graph the ratio between the maximum and the mode is 2, one might conjecture that this is the minimum ratio for values of \( K \) around \( \log n \). If this were true, combined with Theorem 5 this improves Ramsey bounds.

The second question is to close the gap in the minimum total number of monochromatic cliques in a graph, which is currently roughly \( n^{\frac{\log n}{2}} \leq \ ? \leq n^{\frac{\log n}{2}} \).

This gap lies in the same realm as the gaps of the bounds for \( c_t \) in our related works. If the true bound is close to the upper bound, this fact combined with a proof like that in Section 3.3 would improve Ramsey bounds.

Acknowledgements

The work of the authors was supported in part by the Israel Science Foundation (grant No. 1388/16).
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