Quantitative Estimates for Homogenization of Nonlinear Elliptic Operators in Perforated Domains

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Abstract

This paper is devoted to study the quantitative homogenization problems for nonlinear elliptic operators in perforated domains. In terms of $L^2$-norm, we obtain $O(\varepsilon^{1/2})$ convergence rates on a $C^{1,1}$ region intersected a “regular” perforated domains. The extension arguments developed in [9, Theorem 2.1] and [29, Theorem 4.3] have been applied in a subtle way to weaken the regularity assumption on given data. In this regard, the result is new even for a linear model. Equipped with the error estimates, we may further develop an interior Lipschitz estimate at large scales, and the extension technique still plays a key role there.

Key words. homogenization; perforated domains; nonlinear elliptic operators; convergence rates; large-scale Lipschitz estimates.

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1 Introduction and main results

The aim of the present paper is to establish some error estimates and large-scale Lipschitz estimates for a class of monotone operators in periodically perforated domains, arising in the homogenization theory. More precisely, let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \geq 2$, and $\omega \subset \mathbb{R}^d$ is an unbounded Lipschitz domain with 1-periodic structure, i.e., if $l^+(y)$ denotes the characteristic function of $\omega$, then $l^+$ is a 1-periodic function. We denote $\varepsilon$-homothetic set $\{x \in \mathbb{R}^d : x/\varepsilon \in \omega\}$ by $\varepsilon \omega$. 

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The function $l_\varepsilon^+(x) = l^+(x/\varepsilon)$ is the characteristic function of $\varepsilon \omega$. Consider the following elliptic equations in the divergence form depending on a parameter $0 < \varepsilon \ll 1$.

\[
\begin{cases}
  L_\varepsilon u_\varepsilon \equiv -\text{div} A(x/\varepsilon, \nabla u_\varepsilon) = F & \text{in } \Omega_\varepsilon, \\
  \sigma_\varepsilon(u_\varepsilon) = 0 & \text{on } S_\varepsilon, \\
  u_\varepsilon = g & \text{on } \Gamma_\varepsilon, 
\end{cases}
\]  

where $\Omega_\varepsilon := \Omega \cap \varepsilon \omega$, $\Gamma_\varepsilon := \partial \Omega_\varepsilon \cap \partial \Omega$, $S_\varepsilon := \partial \Omega_\varepsilon \cap \Omega$ and $\sigma_\varepsilon(u_\varepsilon) = \vec{n} \cdot A(x/\varepsilon, \nabla u_\varepsilon)$ being known as the conormal derivative of $u_\varepsilon$ on related boundaries. Given three constants $\mu_0, \mu_1, \mu_2 > 0$, the function $A(y, \xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous with $A(y, \cdot) \in C^1(\mathbb{R}^d)$ for any $y \in \mathbb{R}^d$, which additionally satisfies the following structure conditions.

1. For any $y, \xi, \xi' \in \mathbb{R}^d$, there hold the coerciveness and growth conditions

\[
\begin{align*}
  &\langle A(y, \xi) - A(y, \xi'), \xi - \xi' \rangle \geq \mu_0 |\xi - \xi'|^2; \\
  &|A(y, \xi) - A(y, \xi')| \leq \mu_1 |\xi - \xi'|.
\end{align*}
\]  

2. For every $\xi \in \mathbb{R}^d$, $A(\cdot, \xi)$ is $1$-periodic and

\[ A(y, 0) = 0 \quad \text{for } y \in \mathbb{R}^d. \]  

3. The smoothness assumption is also imposed, i.e.,

\[ |A(y, \xi) - A(y', \xi)| \leq \mu_2 |y - y'|^\tau |\xi|, \]

where $\tau \in (0, 1)$.

We say $u_\varepsilon$ is a weak solution to (1) if

\[
\int_{\Omega_\varepsilon} A(x/\varepsilon, \nabla u_\varepsilon) \nabla w dx = \int_{\Omega_\varepsilon} F w dx
\]  

for any $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$, and $u_\varepsilon - g \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$, where $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ denotes the closure in $H^1(\Omega_\varepsilon)$ of $C^\infty(\mathbb{R}^d)$ with functions vanishing on $\Gamma_\varepsilon$. Under our assumptions of $A(y, \xi)$, the existence and uniqueness of a weak solution to (1) follows from a strong monotone, hemicontinuous and coercive property (see for example [42, Theorem 26.A]). Moreover, the following qualitative homogenization result has been included in V. Zhikov and M. Rychago's work [43, 46], i.e., there hold that $l_\varepsilon^+ u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$, and $l_\varepsilon^+ \nabla u_\varepsilon \rightharpoonup \nabla u_0$ with $l_\varepsilon^+ A(x/\varepsilon, \nabla u_\varepsilon) \rightharpoonup \hat{A}(\nabla u_0)$ weakly in $L^2(\Omega; \mathbb{R}^d)$. Here $u_0$ is the solution to the effective (homogenized) equation

\[
\begin{cases}
  \mathcal{L}_0 u_0 \equiv -\text{div} \hat{A}(\nabla u_0) = F & \text{in } \Omega, \\
  u_0 = g & \text{on } \partial \Omega.
\end{cases}
\]

The function $\hat{A} : \mathbb{R}^d \to \mathbb{R}^d$ is defined for every $\xi \in \mathbb{R}^d$ by

\[ \hat{A}(\xi) = \int_{Y \cap \omega} A(y, \xi + \nabla y N(y, \xi)) dy, \]
and \( N(y, \xi) \) is the so-called corrector, associated with the following cell problem

\[
\begin{align*}
\begin{dcases}
\text{div} A(y, \xi + \nabla_y N(y, \xi)) &= 0 \quad \text{in } Y \cap \omega, \\
\vec{n} \cdot A(y, \xi + \nabla_y N(y, \xi)) &= 0 \quad \text{on } Y \cap \partial \omega, \\
N(\cdot, \xi) &\in H^1_{\text{per}}(Y \cap \omega), \\
\int_{Y \cap \omega} N(\cdot, \xi) &= 0,
\end{dcases}
\end{align*}
\]

(8)

where the notation \( \frac{1}{|\Omega|} \int_{\Omega} \) represents the average of integral, and \( |\Omega| \) is the volume of \( \Omega \). The existence and uniqueness of the weak solution to (6) and (8) also follows from the properties of \( A(y, \xi), \tilde{A}(\xi) \) (see Lemma 2.3).

In order to make the statement clear, we introduce some notation and terminology used throughout the paper. Let \( r_0 > 1 \) denote the diameter of \( \Omega \). Let \( B(x, r) \subset \mathbb{R}^d \) represent a ball centered at \( x \) with radius \( r > 0 \), and \( B_\varepsilon(x, r) := B(x, r) \cap (\varepsilon \Omega) \), \( S_\varepsilon(r) := B(x, r) \cap \partial(\varepsilon \Omega) \). We call \( \varepsilon \) a “regular” domain, if it satisfies the following two conditions: (I) a separated property. It’s assumed that \( (\mathbb{R}^d \setminus \omega) \cap Y \subset Y \) in which \( Y \) is the unit cube, and any two connected components of \( \mathbb{R}^d \setminus \omega \) are separated by some positive distance. Specifically, if \( \mathbb{R}^d \setminus \omega = \bigcup_{k=1}^{\infty} H_k \) in which \( H_k \) is connected and bounded for each \( k \), then there exists a constant \( g^\omega \) such that

\[
0 < g^\omega \leq \inf_{i \neq j} \left\{ \text{dist}(H_i, H_j) \right\}. \tag{9}
\]

(II) regular boundaries. For each of the components \( \{H_k\} \), the boundary of \( H_k \) is assumed to be \( C^{1,\alpha} \) with \( \alpha \in (0, 1) \), where the component \( H_k \) is usually referred to as a “hole” in the context. Without additional notes, the constant \( C \) may change from line to line and may depend on \( \mu_0, \mu_1, \mu_2, \tau, d, g^\omega \) and \( r_0 \), as well as, the boundary character of \( \omega \) and \( \Omega \), but never relies on \( \varepsilon \). We write \( A \lesssim B \) by meaning that there exists the constant \( C \) which is independent of \( \varepsilon \) such that \( A \leq CB \).

The main results of the paper are stated as follows.

**Theorem 1.1** (convergence rates). Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^{1,1} \) domain and \( \omega \) be a regular one. Suppose that \( \mathcal{L}_\varepsilon \) satisfies the conditions (2), (3) and (4). Given \( F \in L^2(\Omega) \) and \( g \in H^{3/2}(\partial \Omega) \), let \( u_\varepsilon \in H^1(\Omega_\varepsilon), u_0 \in H^1(\Omega) \) be the weak solution to (1) and (6), respectively. Then there holds

\[
\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{1/2} \left\{ \|F\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial \Omega)} \right\}, \tag{10}
\]

in which \( C \) depends on \( \mu_0, \mu_1, \mu_2, \tau, d, r_0, g^\omega \) and the boundary character of \( \omega \) and \( \Omega \).

**Remark 1.2.** The loss of the power in the error estimate (10) caused by a boundary layer. Our methods could be extended to the whole space without any real difficulty, and therefore one may derive a sharp error estimate (see for example [16, 30]). As an application of this development, we employ the duality argument to derive the optimal error estimate for elasticity systems on perforated domains in a separated work.

**Theorem 1.3** (interior Lipschitz estimates at large scales). Let \( B(0, 2) \subset \Omega \). Suppose that \( \mathcal{L}_\varepsilon \) satisfies the same conditions as in Theorem 1.1. Let \( u_\varepsilon \) be a weak solution of \( \mathcal{L}_\varepsilon u_\varepsilon = F \) in \( B_\varepsilon(0, 2) \) and \( \sigma_\varepsilon(u_\varepsilon) = 0 \) on \( S_\varepsilon(2) \), where \( F \in L^p(B(0, 2)) \) with \( p > d \). Then one may derive that

\[
\left( \int_{B_\varepsilon(0, r)} |\nabla u_\varepsilon|^2 \, dx \right)^{1\over 2} \lesssim \left\{ \left( \int_{B_\varepsilon(0, 2)} |\nabla u_\varepsilon|^2 \, dx \right)^{1\over 2} + \left( \int_{B(0, 2)} |F|^p \, dx \right)^{1\over p} \right\}, \tag{11}
\]

for any \( \sqrt[3]{\varepsilon} \leq r < (1/4) \).
The stated theorems above parallel with our previous results in \[39, \text{Theorem 1.1 and Theorem 1.2}], which seems to be not surprising at first glance. However, the difficulties arose from perforated domains are essential. For example, let \( A(y, \xi) = \xi \) and then the related corrector is given by \( N(y, \xi) = \phi \cdot \xi \), while it is not hard to check that the equation (8) may be reduced to
\[
- \Delta \phi_k = 0 \quad \text{in } Y \cap \omega, \quad \text{and} \quad \vec{n} \cdot \nabla \phi_k = -n_k \quad \text{on } Y \cap \partial \omega, \tag{12}
\]
where \( n_k \) is the \( k \)-th component of \( \vec{n} \), and \( k = 1, \ldots, d \). If we impose the condition \( \int_{Y \cap \omega} \phi_k = 0 \), then there exists a unique nontrivial weak solution to (12), compared to the case \( \omega = \mathbb{R}^d \). Clearly, one may observe that \( \int_{Y \cap \omega} \nabla \phi_k \cdot \nu \neq 0 \), which will bring in some new influence to the process of homogenization, mostly coming from the geometry of \( \omega \). If going back to macroscopic scales, one of the dangerous operations would be eager to employ the trace theorem on \( S_\varepsilon \), because of its dependence on the size of the “holes”. Meanwhile, as \( \varepsilon \) approaches to zero, too many “holes” also lead to a severe problem especially when repeating operations around them however fail to have a cancelation. A fortunate remedy seems to be a careful extension argument, which just relies on the character of the boundary \( \partial \omega \). In general, it is proved to be the fundamental idea for homogenization on perforated domains, such as the extension theorem developed by E. Acervi, V. Piat, G. Maso and D. Percivale [1, Theorem 2.1] and by O. Oleinik, A. Shamaev, G. Yosifian [29, Theorem 4.3], which ultimately makes our previous framework in \[39\] workable to the present model. In this connection, we would like to address some specific difficulties encountered in the proof of Theorem 1.1, as well as, the related ideas and comments.

(i). The fact that \( \int_{Y \cap \omega} \nabla N(\cdot, \xi) \, dy \neq 0 \) prevents us from simply repeating the proof used in \[39, \text{Lemma 2.3}] or \[30, \text{Lemma 1}] to prove the coercive property of \( \hat{A} \) (see Lemma 2.3), while this property plays a crucial role in the quantitative homogenization theory as we have explained in \[38, 39\] with details. Given its importance, we employ the extension theorem developed in \[1, \text{Theorem 2.1}] to show a clear proof for this property, inspired by a similar result stated in \[32, 46\]. We remark that this difficulty can not be observed from the linear models, such as the example mentioned above.

(ii). For later two-scale expansions, we will impose an composite function \( N(x/\varepsilon, \varphi) \) with \( \varphi \in H^1_0(\Omega; \mathbb{R}^d) \), which may wreck the periodicity of \( N(\cdot, \xi) \) for any fixed \( \xi \). This loss causes that we can not use the so-called periodic cancellation (which is quite useful to error estimates), i.e.,
\[
\| \varpi(\cdot/\varepsilon) f \|_{L^2(\Omega)} \leq C \| \varpi \|_{L^2(Y)} \| f \|_{L^2(Y)} + o(1), \quad \text{as } \varepsilon \to 0, \tag{13}
\]
where \( \varpi \in L^2_{\text{per}}(Y) \) and \( f \in C(\tilde{\Omega}) \). Because of this, we have to show \( \nabla \xi N(y, \xi) \in L^\infty((Y \cap \omega) \times \mathbb{R}^d) \). Our argument relies on the local boundedness estimate coupled with the weak Harnack inequality, originally developed by L. Caffarelli [11] in unperforated settings. Since the behavior of \( N(y, \xi) \) near \( \partial \omega \) is also involved, we provide the boundary estimates in Lemma 5.1 for the completeness. Until now, we do not require any smoothness assumption on \( A \) with respective to the first variable. However, the imposed flux corrector \( E \) (see Lemma 2.7) will meet the same problem when \( E \) and \( \varphi \) are composed to be the form of \( E(\cdot/\varepsilon, \varphi) \). This in turn requires a uniform \( L^p \)-bound of the quantity \( \frac{\nabla N(\cdot, \xi) - \nabla N(\cdot, \xi')}{|\xi' - \xi|} \) on the region \( Y \cap \omega \) with \( p \geq 2 \) for any \( \xi, \xi' \in \mathbb{R}^d \) (see Lemma 2.5), and therefore the smoothness assumption (4) is necessary here. In this respect, the influence of the regularity of \( \partial \omega \) should be considered too, and it is the reason why we focus ourselves on the so-called “regular” domain in the present work. Finally, we mention that to weaken the assumption on \( f \) in the inequality (13), the Steklov averaging operator was originally introduced by V. Zhikov, S. Pastukhova [44], and the smoothing operator by Z. Shen [35] (see Definition 2.8), and here we prefer the latter.
Another difficulty caused by perforated domains is that the boundary part $\Gamma_\varepsilon \subset \partial \Omega_\varepsilon$ would be very complicated and irregular. Hence, we can not employ Poincaré’s inequality in perforated domains as freely as in unperforated cases. The main ideas to overcome this obstacle is to extend the function from the perforated domain $\Omega_\varepsilon$ to a little larger region $\Omega_0$ (see Lemma 2.12), which has already been established in [29, Theorem 4.3]. Besides, as stated in Theorem 1.1 we merely ask $F$ to be square integrable, compared with the assumption $F \in H^1(\Omega)$ which is necessary in [9, 13, 14]. To the authors’ best known, it is a new result even for linear equations in perforated domains, and we will explain the related tricks later on.

In this paragraph, we outline the main idea on error estimates concerning the perforated domains. As developed in the previous framework [38, 39] for unperforated domains, consider the two-scale expansion $w_\varepsilon = u_\varepsilon - v$ with $v = u_0 + \varepsilon N(y, \varphi)$, and
\[
\nabla w_\varepsilon = \nabla u_\varepsilon - \nabla v, \tag{14}
\]
where $\varphi \in H^1_0(\Omega; \mathbb{R}^d)$ will be fixed later. Then, the variational inequality below reveals an important information of convergence rates for nonlinear elliptic equations in the divergence form,
\[
\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq \left| \int_{\Omega} (l_\varepsilon^+ - \theta \psi_\varepsilon') F \tilde{w}_\varepsilon \, dx + \int_{\Omega} \left\{ \tilde{A}(\nabla u_0) - \tilde{A}(\varphi) \right\} \right|_{T_2} + \tilde{A}(\varphi) - A(x/\varepsilon, \varphi + \nabla_y N(x/\varepsilon, \varphi)) \right|_{T_3} + A(x/\varepsilon, \varphi + \nabla_y N(x/\varepsilon, \varphi)) - A(x/\varepsilon, \nabla v) \right| \cdot \nabla \tilde{w}_\varepsilon \, dx \right|_{T_4}
\]
where $\tilde{w}_\varepsilon$ is the extension of $w_\varepsilon$ given by Lemma 2.12. Compared to the previous work in [38, 39], some new tricks have been developed for the term $T_1$ in the present paper, which do not require any higher regularity assumption on $F$. In fact, the integrand function in $T_1$ may be divided into two parts:
\[
\int_{\Omega} (l_\varepsilon^+ - \theta) F \tilde{w}_\varepsilon \psi_\varepsilon' \, dx \quad \text{and} \quad \int_{\Omega} (1 - \psi_\varepsilon') l_\varepsilon^+ F \tilde{w}_\varepsilon \, dx,
\]
where $\psi_\varepsilon'$ is a cut-off function satisfying (46). In the “co-layer part”, observing $\int_Y (l^+ - \theta) \, dy = 0$, one may construct the auxiliary equation (48) to produce the term like $\varepsilon \nabla \Phi \cdot \nabla (F \tilde{w}_\varepsilon \psi_\varepsilon')$. The providential term $\tilde{w}_\varepsilon \psi_\varepsilon' \in H^1_0(\Omega)$ is such that the quantity $\nabla F$ belongs to $H^{-1}(\Omega)$, which is the dual space of $H^1_0(\Omega)$. This together with the fact that $\|\nabla F\|_{H^{-1}(\Omega)} \lesssim \|F\|_{L^2(\Omega)}$ completes the argument. In terms of “layer part”, we actually employ Poincaré’s inequality down to the scale $\varepsilon$ for $\tilde{w}_\varepsilon = 0$ on $\Gamma_\varepsilon$ (see Lemma 2.14).

Regarding the remainder terms $T_2$, $T_3$ and $T_4$, roughly speaking, one may follow from a similar philosophy summarized in [38, 39], and some related challenges have been stated previously. Here the ultimate aim of the computations is reducing the right-hand side of (15) to the so-called “layer” and “co-layer” type estimates
\[
\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \quad \text{and} \quad \|\nabla^2 u_0\|_{L^2(\Omega \setminus O_{4\varepsilon})},
\]
where $O_{4\varepsilon} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < 4\varepsilon \}$. We need to mention that the approach still heavily relies on the antisymmetric property of the flux correctors, which is closer to B. Russell’s argument in linear elasticity systems [31] than to A. Belyaev, A. Pyatnitskiǐ and G. Chechkin’s [9].
Recently, error estimates have been studied extensively, and without attempting to exhaustive we refer the reader to [3, 4, 5, 6, 8, 16, 18, 22, 24, 27, 30, 31, 33, 34, 36, 37, 40, 41, 45] and their references therein for more results. Based upon the estimate 11, it is possible to develop some regular estimates at large-scales, and we outline some background and ideas in the following.

The uniform Lipschitz estimate was first obtained by M. Avellaneda, F. Lin [7] for the linear case $A(y, \xi) = A(y)\xi$, in which a compactness method had been well developed. Recently, S. Armstrong, J. Mourrat and C. Smart [4, 6] obtained the large-scale Lipschitz estimate for stochastic homogenization of convex integral functionals and their new idea is based upon a convergence rate coupled with the so-called Campanato’s iteration. In terms of perforated domains, to the authors’ best knowledge, the only known large-scale regularity appeared in B. Russell’s work for a linear elasticity system \[\text{\cite{31}}. \]

Since iteration is a kind of nonlinear methods, it is not very surprising to be extended from the linear case, but we still need a nontrivial effort to the estimate (10) regarding a homogenization problem in perforated domains.

Concretely speaking, the ideas developed in [31] and [38] are combined to make a proof for Theorem 1.3, and we outline the main ingredients as follows.

1. Thanks to the error estimate (11), we can derive an approximating lemma

\[
\left( \int_{B_\varepsilon(0,r)} |u_\varepsilon - w|^2 \right)^{1/2} \leq C \left( \frac{\varepsilon}{r} \right)^{1/4} \left( \int_{B_\varepsilon(0,2r)} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left( \int_{B(0,2r)} |F|^2 \right)^{1/2}
\]

for $\sqrt{\varepsilon} \leq r < (1/4)$. Here $w \in H^1(B(0,\bar{r}))$ satisfies $\mathcal{L}_0w = F$ in $B(0,\bar{r})$ with $w = (\tilde{u}_\varepsilon)_\delta$ on $\partial B(0,\bar{r})$, in which $r < \bar{r} < 3r/2$, and $(\tilde{u}_\varepsilon)_\delta \in H^{3/2}(\partial B(0,\bar{r}))$ satisfies the estimate (38). Note that the extension argument still plays a very important role. However, $u_\varepsilon \in H^1(B_\varepsilon(0,r))$ doesn’t satisfy the preconditions in Lemma 2.12, we overcome this by multiplying $u_\varepsilon$ with a cut-off function.

2. To carry out the iteration program, we define the following quantities:

\[
G_\varepsilon(r, v) = \frac{1}{r} \inf_{M \in \mathbb{R}} \inf_{c \in \mathbb{R}} \left\{ \left( \int_{B_\varepsilon(0,r)} |v - Mx - c|^2 dx \right)^{1/2} + r^2 \left( \int_{B(0,r)} |F|^p \right)^{1/p} \right\};
\]

\[
G(r, v) = \frac{1}{r} \inf_{M \in \mathbb{R}} \inf_{c \in \mathbb{R}} \left\{ \left( \int_{B(0,r)} |v - Mx - c|^2 dx \right)^{1/2} + r^2 \left( \int_{B(0,r)} |F|^p \right)^{1/p} \right\},
\]

in which the requirement $p > d$ is a natural assumption. Although the quantity $w - Mx - c$ is not a solution of $\mathcal{L}_0w = F$ in $B(0,\bar{r})$ in general, the key observation is that it verified the same linearized equation as $w$ did. Moreover, combining the property of $\hat{\Lambda}(\xi)$ with the De Giorgi-Nash-Moser theorem of the linearized homogenization equation, we may obtain that

\[G_\varepsilon(r, w) \leq CG(2r, w)\]

(see Lemma 4.2), where the constant $C$ depends only on $\mu_0, \mu_1$ and $d$.

3. Then we use the iteration argument (see Lemma 4.5) to prove our result, which was proved by Z. Shen in [34], originally shown in [4, 6]. Then the remainder parts of the proof is standard.

Finally, we point out that due to the irregularity of the boundary of perforated domains intersected with any ball, it is not easy to verify the small-scale Lipschitz regularity even though $\omega$ is assumed
to be a “regular” domain here. In the end, we refer the reader to [2, 3, 19, 20, 23, 28] and their references therein for the recent developments regarding the regularity estimates in homogenization theory.

The paper is organized in five sections. In Section 2, we establish some properties on correctors and flux correctors, and some properties on smoothing operators and extension operators are also introduced there. Section 3 is devoted to show a proof of Theorem 1.1, and then we present a proof for Theorem 1.3 in Section 4. Finally, some local boundary estimates used for correctors have been arranged in Section 5.

2 Preliminaries

2.1 Properties of correctors and flux correctors

Most of the properties of correctors and flux correctors associated with perforated domains are similar to those established in unperforated ones. Roughly speaking, ideas here mainly inspired by [17, 25, 32, 39, 46].

Lemma 2.1. Suppose that $A$ satisfies the conditions (2) and (3). Let $N(\cdot, \xi) \in H^1_{\text{per}}(Y \cap \omega)$ be the weak solution to the equation (8), and then for any $\xi \in \mathbb{R}^d$, we have the following estimates

$$
\int_{Y \cap \omega} |N(\cdot, \xi)|^2 + \int_{Y \cap \omega} |\nabla N(\cdot, \xi)|^2 \leq C|\xi|^2 \tag{17}
$$

and

$$
\int_{Y \cap \omega} |\nabla \xi N(\cdot, \xi)|^2 + \int_{Y \cap \omega} |\nabla \xi \nabla N(\cdot, \xi)|^2 \leq C, \tag{18}
$$

where $C$ depends only on $\mu_0, \mu_1, \omega$ and $d$. Moreover,

$$
|N(y, \xi) - N(y, \xi')| \lesssim |\xi - \xi'| \quad \text{for} \; y \in \omega, \; \xi, \xi' \in \mathbb{R}^d, \tag{19}
$$

i.e. $|\nabla \xi N(y, \xi)| \leq C$, for any $y \in \omega$, and $\xi \in \mathbb{R}^d$.

Proof. Some of these results have already been given in [39, 38] in the case of unperforated domain, and also included in [46] in perforated domains. We provide a proof for the sake of the completeness. Multiplying both sides of (8) by $N(y, \xi)$ and then integrating by parts, we have

$$
0 = \int_{Y \cap \omega} A(y, \xi + \nabla_y N(y, \xi)) \cdot \nabla_y N(y, \xi) dy
$$

$$
= \int_{Y \cap \omega} A(y, \xi + \nabla_y N(y, \xi)) \cdot (\xi + \nabla_y N(y, \xi)) dy - \int_{Y \cap \omega} A(y, \xi + \nabla_y N(y, \xi)) dy \cdot \xi
$$

$$
\geq \mu_0 \int_{Y \cap \omega} |\xi + \nabla_y N(y, \xi)|^2 dy - \mu_1 |\xi| \int_{Y \cap \omega} |\xi + \nabla_y N(y, \xi)| dy,
$$

where we use the assumptions (3) in the last inequality. By Young’s inequality,

$$
\int_{Y \cap \omega} |\xi + \nabla_y N(y, \xi)|^2 dy \leq C(\mu_0, \mu_1)|\xi|^2.
$$

Thus this together with Poincaré’s inequality will give the stated estimate (17).
To show the estimate (18), we start with the following identity
\[ \int_{Y \cap \omega} \left[ A(y, \xi + \nabla_{y} N(y, \xi)) - A(y, \xi' + \nabla_{y} N(y, \xi')) \right] \cdot \left[ \xi - \xi' + \nabla_{y} N(y, \xi) - \nabla_{y} N(y, \xi') \right] dy \]
\[ = \int_{Y \cap \omega} \left[ A(y, \xi + \nabla_{y} N(y, \xi)) - A(y, \xi' + \nabla_{y} N(y, \xi')) \right] dy \cdot (\xi - \xi'), \tag{20} \]
where we use the fact that \( N(\cdot, \xi), N(\cdot, \xi') \in H^{1}_{\text{per}}(Y \cap \omega) \) satisfy the equation (8) for \( \xi, \xi' \in \mathbb{R}^{d} \), respectively. By the assumption (3), the left-hand side above is greater than
\[ \mu_{0} \int_{Y \cap \omega} |\xi - \xi' + \nabla_{y} N(y, \xi) - \nabla_{y} N(y, \xi')|^{2} dy, \]
while it follows from Young’s inequality that its right-hand side is less than
\[ \frac{\mu_{0}}{2} \int_{Y \cap \omega} |\xi - \xi' + \nabla_{y} N(y, \xi) - \nabla_{y} N(y, \xi')|^{2} dy + C(\mu_{0}, \mu_{1})|\xi - \xi'|^{2}. \]
Thus it is not hard to derive that
\[ \left( \int_{Y \cap \omega} |\nabla_{y} N(y, \xi) - \nabla_{y} N(y, \xi')|^{2} dy \right)^{1/2} \leq C|\xi - \xi'|, \tag{21} \]
and this will give the estimate (18) in a similar way.

Then we proceed to show (19). Let \( u(y, \xi) = N(y, \xi) + y : \xi \) and \( \tilde{u}(y, \xi) = u(y, \xi) + \tilde{M} \), in which we choose \( \tilde{M} \) such that \( \tilde{u} \) is positive in \( Y \cap \omega \). Due to the estimate in Lemma 5.1, if \( \xi \) is bounded, the existence of \( \tilde{M} \) is not hard to see, and such the boundedness of \( \xi \) will be given by \( S_{\varepsilon}(\psi_{\varepsilon} \nabla u_{0}) \) later. Note that \( \tilde{u} \) still satisfies the equation
\[ \text{div} A(y, \nabla u(y, \xi)) = 0, \quad \text{in} \ Y \cap \omega. \]
Thus, it follows from the local boundedness estimate and the weak Harnack inequality (see Lemma 5.1 for the case \( B_{r} \cap \partial \omega \neq \emptyset \), and [26, Corollary 3.10, Theorem 3.13] for the case \( B_{r} \subset 2Y \cap \omega \)) that
\[ \sup_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi) \lesssim \int_{Y \cap \omega \cap B_{r}} \tilde{u}(\cdot, \xi) \quad \text{and} \quad \inf_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi) \gtrsim \int_{Y \cap \omega \cap B_{r}} \tilde{u}(\cdot, \xi), \tag{22} \]
in which \( B_{r} \) and \( B_{\tilde{r}} \) satisfy that \( B_{r}, B_{\tilde{r}} \subset 2Y \) and they are centered at \( Y \cap \omega \) with \( r < \tilde{r} \). Then for any \( y \in Y \cap \omega \) such that \( \tilde{u}(y, \xi) - \tilde{u}(y, \xi') > 0 \), according to (22), there holds
\[ \tilde{u}(y, \xi) - \tilde{u}(y, \xi') \lesssim \sup_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi) - \inf_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi') \lesssim \int_{Y \cap \omega \cap B_{r}} |\tilde{u}(\cdot, \xi) - \tilde{u}(\cdot, \xi')|. \]
Similarly, for any \( y \in Y \cap \omega \) such that \( \tilde{u}(y, \xi') - \tilde{u}(y, \xi) > 0 \), we may have
\[ \tilde{u}(y, \xi') - \tilde{u}(y, \xi) \lesssim \sup_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi') - \inf_{y \in Y \cap \omega \cap B_{r}} \tilde{u}(y, \xi) \lesssim \int_{Y \cap \omega \cap B_{r}} |\tilde{u}(\cdot, \xi') - \tilde{u}(\cdot, \xi)|. \]
Therefore, for any \( y \in Y \cap \omega \), we obtain that
\[ |\tilde{u}(y, \xi) - \tilde{u}(y, \xi')| \lesssim \int_{Y \cap \omega \cap B_{r}} |\tilde{u}(\cdot, \xi) - \tilde{u}(\cdot, \xi')| \lesssim \int_{Y \cap \omega} |N(\cdot, \xi) - N(\cdot, \xi')| + |\xi - \xi'|. \]
This together with (18) implies (19), and we have completed the proof. \( \square \)
Remark 2.2. In view of the estimate (17), one may conclude that \( N(y, 0) = 0 \) for \( y \in Y \cap \omega \).

Lemma 2.3. Suppose \( \mathcal{L}_\varepsilon \) satisfies the assumptions (2), (3). Let \( \hat{A} \) be given in (7). Then the effective operator \( \mathcal{L}_0 \) is still strongly monotone, coercive, i.e.,
\[
\begin{align*}
\langle \hat{A}(\xi) - \hat{A}(\xi'), \xi - \xi' \rangle &\geq C_1 |\xi - \xi'|^2; \\
|\hat{A}(\xi) - \hat{A}(\xi')| &\leq C|\xi - \xi'|; \\
\hat{A}(0) &= 0,
\end{align*}
\]
where \( C, C_1 \) depend on \( \mu_0, \mu_1, \omega \) and \( d \).

Proof. The proof relies on the extension theorem heavily, and the idea is inspired by [32, 46]. Due to the formula (20), we have that
\[
\langle \hat{A}(\xi) - \hat{A}(\xi'), \xi - \xi' \rangle \geq \mu_0 \int_{Y \cap \omega} |\xi - \xi' + \nabla_y (N(y, \xi) - N(y, \xi'))|^2 dy.
\]
It follows from [1, Lemma 2.6] that there is a linear extension operator from \( H^1(Y \cap \omega) \) to \( H^1(Y) \) such that the extended function (still denoted by \( N(y, \xi) \)) satisfies the inequality
\[
\int_{Y \cap \omega} |\nabla_y N(y, \xi)|^2 dy \geq C \int_Y |\nabla_y N(y, \xi)|^2 dy,
\]
where \( C \) is independent of \( N \) and \( \xi \). Thus, we may have
\[
\int_{Y \cap \omega} |\nabla_y N(y, \xi) + \xi|^2 dy \geq C \int_Y |\nabla_y N(y, \xi) + \xi|^2 dy
\]
for any \( \xi \in \mathbb{R}^d \), where we also employ the facts that the extension operator is linear and the extension of a linear function is itself. Combining with the periodic property of \( N(y, \xi) \), one may derive the well-known property of correctors:
\[
\int_{Y \cap \omega} |\nabla_y N(y, \xi) + \xi|^2 dy \geq C|\xi|^2.
\]
According to (24), (25) and the fact that \( \int_{\partial Y} [N(y, \xi) - N(y, \xi')] dS = 0 \) for \( N(\cdot, \xi), N(\cdot, \xi') \in H^1_{per}(Y \cap \omega) \) have the same periodicity, we may derive
\[
\int_{Y \cap \omega} |\xi - \xi' + \nabla_y (N(y, \xi) - N(y, \xi'))|^2 dy \geq C \int_Y |\xi - \xi' + \nabla_y (N(y, \xi) - N(y, \xi'))|^2 dy
\geq C|\xi - \xi'|^2,
\]
in which the constant \( C \) depends only on \( \omega, d \). Therefore, we derive the first inequality in (23). Note that
\[
|\hat{A}(\xi) - \hat{A}(\xi')| \leq \mu_1 \int_{Y \cap \omega} |A(y, \xi + \nabla N(y, \xi)) - A(y, \xi' + \nabla N(y, \xi'))| dy
\leq C|\xi - \xi'|,
\]
where the last step is due to the estimate (21). In view of Remark 2.2, we may have the third line of (23) and the proof is complete. \( \Box \)
Remark 2.4. Due to the second line of (23), it is known that $\nabla \hat{A}(z)$ exists for a.e. $z \in \mathbb{R}^d$. Moreover, there holds
\begin{equation}
\sum_{i,j=1}^{d} \nabla_j \hat{A}_i(z) \xi_j \xi_i = \lim_{t \to 0} \frac{\langle \hat{A}(z + t \xi) - \hat{A}(z), \xi \rangle}{t} \geq C_1 |\xi|^2
\end{equation}
for any $\xi \in \mathbb{R}^d$ and for a.e. $z \in \mathbb{R}^d$, and this property will guarantee that the $H^2$ theory is still valid for the effective operator $L_0$.

Lemma 2.5. Suppose that $A$ satisfies (2), (3) and (4). Assume that $N(y, \xi)$ is the corrector satisfying (8), then for any $p \geq 2$, there holds
\begin{equation}
\left( \int_{Y \cap \omega} |\nabla (N(y, \xi) - N(y, \xi'))|^p dy \right)^{\frac{1}{p}} \lesssim |\xi - \xi'|
\end{equation}
for any $\xi, \xi' \in \mathbb{R}^d$, where the up-to-constant depends on $\mu_0, \mu_1, \mu_2, \tau, d$ and the character of $\omega$.

Proof. For any fixed $\xi, \xi' \in \mathbb{R}^d$, setting $P_1 = \xi + \nabla_y N(y, \xi)$ and $P_2 = \xi' + \nabla_y N(y, \xi')$, we have
\[ \text{div}[A(y, P_1) - A(y, P_2)] = 0 \quad \text{in } Y \cap \omega. \]

Under the assumptions (2), (3) and (4) it is well-known that $P_1, P_2$ are Hölder continuous (see for example [15, Theorems 1.1, 1.3]). In view of the Newton-Leibniz formula,
\[ \frac{\partial}{\partial y_i} \left[ \int_0^1 \partial_{\xi_j} A(y, tP_1 + (1 - t)P_2) dt \cdot (P_1^j - P_2^j) \right] = 0. \]
We write $a_{ij}(y) = \int_0^1 \partial_{\xi_j} A(y, tP_1 + (1 - t)P_2) dt$. Thus, this together with $A \in C(\mathbb{R}^d \times \mathbb{R}^d)$ and $A(y, \cdot) \in C^1(\mathbb{R}^d)$ further implies that $a_{ij}$ is continuous on $\mathbb{R}^d$. Setting $\pi = N(y, \xi) - N(y, \xi')$ we have
\[ -\text{div}[a(y)\nabla \pi] = \text{div}[a(y)](\xi - \xi') \quad \text{in } Y \cap \omega \]
with a natural boundary condition $\bar{n} \cdot a(P_1 - P_2) = 0$ on $\partial \omega$ and $\pi$ being periodic on $\partial Y$. It follows from the $L^p$ estimate (see for example [23, Theorem 1.1]) that for any $p \geq 2$, there holds
\[ \left( \int_{Y \cap \omega} |\nabla \pi|^p dy \right)^{\frac{1}{p}} \lesssim |\xi - \xi'| + \left( \int_{Y \cap \omega} |\nabla \pi|^2 dy \right)^{\frac{1}{2}} \lesssim |\xi - \xi'|, \]
and the proof is complete. \hfill \Box

Remark 2.6. In fact, the range of $p$ relies on the regularity of the boundary of $\omega$. There are at least two types of Lipschitz domains which may guarantee the range $2 \leq p < \infty$. The one is the so-called Reifenberg-flat domains, whose boundary is even permitted to be a fractal structure but merely owns a “small” Lipschitz constant. The other one is a class of Lipschitz domains with convex properties. Boundary estimates involving non-smooth domains have been extensively studied in the past decades, and we refer the reader to [10, 12, 34] and the references therein for more details. Besides, assuming the same conditions as in Lemma 2.5, on account of the Sobolev’s embedding theorem, the desired estimate (19) may derive from (18) and (27) straightforwardly by setting $p > d$. However, this argument inevitably relies on the additional smoothness assumption both on $A$ and boundary of $\omega$. 

Lemma 2.7 (flux correctors). Suppose $A$ satisfies (2) and (3). Let $b(y, \xi) = \theta \hat{A}(\xi) - l^+(y) A(y, \xi + \nabla N(y, \xi))$, where $y \in Y$ and $\xi \in \mathbb{R}^d$. Then we have two properties: (i) $\int_Y b(\cdot, \xi) = 0$; (ii) $\operatorname{div} b(\cdot, \xi) = 0$ in $Y$. Moreover, there exists the so-called flux correctors $E_{ji}(\cdot, \xi) \in H^1_{per}(Y)$ such that

$$b_i(y, \xi) = \frac{\partial}{\partial y_j} \left\{ E_{ji}(y, \xi) \right\} \quad \text{and} \quad E_{ji} = -E_{ij}, \quad (28)$$

and

$$\int_Y |\nabla_{\xi} E_{ji} (\cdot, \xi)|^2 + \int_Y |\nabla_{\xi} \nabla E_{ji} (\cdot, \xi)|^2 \leq C, \quad (29)$$

where $C$ depends only on $\mu_0, \mu_1$ and $d$. Moreover if we additional assume (4), then there holds

$$|E(y, \xi) - E(y, \xi')| \lesssim |\xi - \xi'| \quad \text{for any } y, \xi, \xi' \in \mathbb{R}^d, \quad (30)$$

i.e., $|\nabla\xi E(y, \xi)| \leq C$ for any $y, \xi \in \mathbb{R}^d$.

Proof. The proof is quite similar to the linear case (see for example [33, 45]) and surprisingly depends on a linear structure of an auxiliary equation. It is clear to see that (i) and (ii) follow from the formula (7) and the equation (8), respectively. By (i), there exists $f_i(\cdot, \xi) \in H^2_{per}(Y)$ such that $\Delta f_i(\cdot, \xi) = b_i(\cdot, \xi)$ in $Y$. Let $E_{ji}(y, \xi) = \frac{\partial}{\partial y_j} \left\{ f_i(y, \xi) \right\} - \frac{\partial}{\partial y_n} \{ f_j(y, \xi) \}$. Thus $E_{ji} = -E_{ij}$, and one may derive the first expression in (28) from the fact (ii). Then, the rest thing is to show the estimate (29). For any $\xi, \xi' \in \mathbb{R}^d$, note that

$$\int_Y |\nabla E_{ji}(y, \xi) - \nabla E_{ji}(y, \xi')|^2 \, dy \leq 2 \int_Y |\nabla^2 (f(y, \xi) - f(y, \xi'))|^2 \, dy \leq C \int_{2Y} |b_i(y, \xi) - b_i(y, \xi')|^2 \, dy \leq C|\xi - \xi'|^2,$$

where we employ $H^2$ theory in the second step, (3) and (21) in the last one. This together with Poincaré’s inequality finally leads to the desired estimate (29). To show (30), we claim that

$$\|\nabla f(\cdot, \xi) - \nabla f(\cdot, \xi')\|_{L^\infty(Y)} \lesssim |\xi - \xi'|.$$

According to $\Delta f(y, \xi) = b(y, \xi)$ in $Y$, we have

$$\Delta [f(y, \xi) - f(y, \xi')] = b(y, \xi) - b(y, \xi') \text{ in } Y,$$

and it follows from the assumption (2) and the estimate (23) that

$$|b(y, \xi) - b(y, \xi')| \lesssim |\xi - \xi'| + l^+|\nabla(N(y, \xi) - N(y, \xi'))|.$$}

Due to Lemma 2.5, it is known that $\|\nabla(N(\cdot, \xi) - N(\cdot, \xi'))\|_{L^p(Y \cap \omega)} \lesssim |\xi - \xi'|$ for any $p \geq 2$, and this coupled with the above estimate gives

$$\|b(y, \xi) - b(y, \xi')\|_{L^p(Y)} \lesssim |\xi - \xi'| \quad (31)$$

for any $p \geq 2$. By interior Lipschitz’s estimates one may derive that

$$\|\nabla f(\cdot, \xi) - \nabla f(\cdot, \xi')\|_{L^\infty(Y)} \lesssim \|\nabla f(\cdot, \xi) - \nabla f(\cdot, \xi')\|_{L^2(Y)} + \|b(y, \xi) - b(y, \xi')\|_{L^q(Y)} \lesssim |\xi - \xi'|$$

with $q > d$, where we employ the energy estimate and Hölder’s inequality in the second inequality, and the estimate (31) in the last one. Hence, by the definition of $E_{ij}$, we obtain that

$$|E(y, \xi) - E(y, \xi')| \lesssim |\nabla f(\cdot, \xi) - \nabla f(\cdot, \xi')| \lesssim |\xi - \xi'| \quad \text{for any } y, \xi, \xi' \in \mathbb{R}^d,$$

which means that $|\nabla\xi E(y, \xi)| \leq C$ for any $y, \xi \in \mathbb{R}^d$, and we have completed the proof. \qed
2.2 Smoothing and extension operators

Definition 2.8. Fix a nonnegative function \( \zeta \in C_0^\infty(B(0,1/2)) \), and \( \int_{\mathbb{R}^d} \zeta(x)dx = 1 \). Define the smoothing operator
\[
S_\varepsilon(f)(x) = f \ast \zeta_\varepsilon(x) = \int_{\mathbb{R}^d} f(x-y)\zeta_\varepsilon(y)dy,
\]
where \( \zeta_\varepsilon = \varepsilon^{-d}\zeta(x/\varepsilon) \). Let \( \tilde{B}(0,1/2) \subset \mathbb{R}^{d-1} \) be a ball, and \( \eta \in C_0^\infty(\tilde{B}(0,1/2)) \) be a nonnegative function such that \( \int_{\mathbb{R}^{d-1}} \eta(x)dx = 1 \). Then one may similarly define
\[
K_\delta(g)(x) = g \ast \eta_\delta(x) = \int_{\mathbb{R}^{d-1}} g(x-y)\eta_\delta(y)dy,
\]
where \( \eta_\delta = \delta^{-1-d}\zeta(x/\delta) \).

Lemma 2.9. Let \( f \in L^p(\mathbb{R}^d) \) for some \( 1 \leq p < \infty \). Then for any \( \varpi \in L^p_{\text{per}}(\mathbb{R}^d) \),
\[
\| \varpi(\cdot/\varepsilon)S_\varepsilon(f) \|_{L^p(\mathbb{R}^d)} \leq C \| \varpi \|_{L^p(Y)} \| f \|_{L^p(\mathbb{R}^d)},
\]
where \( C \) depends on \( d \). Moreover, if \( f \in W^{1,p}(\mathbb{R}^d) \) for some \( 1 < p < \infty \), we have
\[
\| S_\varepsilon(f) - f \|_{L^p(\mathbb{R}^d)} \leq C \varepsilon \| \nabla f \|_{L^p(\mathbb{R}^d)},
\]
where \( C \) depends only on \( d \).

Proof. See [34, Lemmas 2.1 and 2.2]. \( \square \)

Lemma 2.10. Let \( g \in H^1(\mathbb{R}^{d-1}) \), and then we have
\[
\| K_\delta(g) \|_{H^{3/2}(\mathbb{R}^{d-1})} \leq C\delta^{-1/2}\| g \|_{H^1(\mathbb{R}^{d-1})}
\]
and
\[
\| K_\delta(g) - g \|_{H^{1/2}(\mathbb{R}^{d-1})} \leq C\delta^{1/2}\| g \|_{H^1(\mathbb{R}^{d-1})},
\]
where \( C \) depends on \( d \) and \( \eta \).

Proof. The main idea has been in [34, Lemma 2.2], see also [39, Lemma 2.9]. \( \square \)

Remark 2.11. If \( g \in H^1(\partial B(0,r)) \) for any \( r > 0 \), then there exists \( (g)_\delta \in H^{3/2}(\partial B(0,r)) \) such that
\[
\| (g)_\delta \|_{H^{3/2}(\partial B(0,r))} \leq C\delta^{-1/2}\| g \|_{H^1(\partial B(0,r))},
\]
\[
\| (g)_\delta - g \|_{H^{1/2}(\partial B(0,r))} \leq C\delta^{1/2}\| g \|_{H^1(\partial B(0,r))},
\]
in which the constant \( C \) is independent of \( r \). This estimate is based upon the above results (36) and (37), respectively. We mention that it has already been given in [36] without a proof. Similarly, inspired by the estimate (35) we may have
\[
\| (g)_\delta - g \|_{L^2(\partial B(0,r))} \leq C\delta\| g \|_{H^1(\partial B(0,r))}. \tag{39}
\]

Lemma 2.12 (extension property). Let \( \Omega \) and \( \Omega_0 \) be a bounded Lipschitz domains with \( \bar{\Omega} \subset \Omega_0 \) and \( \text{dist}(\partial \Omega_0, \Omega) > 1 \). For \( 0 < \varepsilon < 1 \), there exists a linear extension operator \( P_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \rightarrow H^1_0(\Omega_0) \) such that
\[
\| P_\varepsilon w \|_{H^1_0(\Omega_0)} \leq C_1 \| w \|_{H^1(\Omega_\varepsilon)} \tag{40},
\]
\[
\| \nabla P_\varepsilon w \|_{L^2(\Omega_0)} \leq C_2 \| \nabla w \|_{L^2(\Omega_\varepsilon)} \tag{41}
\]
for some constants \( C_1, C_2 \) depending on \( \Omega \) and \( \omega \).
Proof. See [29, Theorem 4.3].

Remark 2.13. The extension property is very important during this paper. Due to this lemma, we can extend the error term $w_\varepsilon$ to $H^1(\Omega)$ and calculate error estimate on a domain without holes. We mention that it requires that function $w$ should equal 0 on the boundary $\Gamma_\varepsilon$.

Lemma 2.14. For $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$, let $\tilde{w}$ be the extension of $w$ given by Lemma 2.12. Then we have
\[
\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla \tilde{w}\|_{L^2(\Omega)},
\]
where $C$ depends on $d, \Omega$ and $\omega$.

Proof. See [32, Lemma 3.4].

2.3 Fundamental regularities

Lemma 2.15 (interior Caccioppoli’inequality). Assume that $\mathcal{L}_\varepsilon$ satisfies the conditions (2), (3). Let $u_\varepsilon \in H^1(B_\varepsilon(0,2r))$ be a weak solution of
\[
\begin{align*}
\mathcal{L}_\varepsilon u_\varepsilon &= F \quad \text{in } B_\varepsilon(0,2r), \\
\sigma_\varepsilon(u_\varepsilon) &= 0 \quad \text{on } S_\varepsilon(2r).
\end{align*}
\]
Then for any $c \in \mathbb{R}$ we have
\[
\int_{B_\varepsilon(0,\frac{7}{4}r)} |\nabla u_\varepsilon|^2 \, dx \leq \frac{C}{r^2} \inf_{c \in \mathbb{R}} \int_{B_\varepsilon(0,2r)} |u_\varepsilon - c|^2 \, dx + C r^2 \int_{B(0,2r)} |F|^2 \, dx,
\]
where $C$ depends on $\mu_0, \mu_1$ and $d$.

Proof. It’s a classical result and we provide the proof for the sake of completeness. By the definition of the weak solution,
\[
\int_{B_\varepsilon(0,2r)} A(x/\varepsilon, \nabla u_\varepsilon) \cdot \nabla \phi \, dx = \int_{B_\varepsilon(0,2r)} F \phi \, dx
\]
holds for any $\phi \in H^1(B_\varepsilon(0,2r), \Gamma_\varepsilon(2r))$. Set $\phi = \psi_r^2(u_\varepsilon - c)$ for any $c \in \mathbb{R}$, where $\psi_r \in C_0^1(B(0,2r))$ is a cut-off function, satisfying $\psi_r = 1$ in $B(0,7r/4)$ and $\psi_r = 0$ outside $B(0,15r/8)$ with $|\nabla \psi_r| \leq C/r$. The stated estimate (40) follows from the assumptions (2), (3) coupled with Young’s inequality.

Theorem 2.16 ($H^1$ theory). Let $\Omega$ be a bounded Lipschitz domain. Assume that $\mathcal{L}_\varepsilon$ satisfies the conditions (2), (3). Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the solution of (1). Then we have
\[
\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \left\{ r_0 \|F\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial \Omega)} \right\},
\]
where $C$ depends on $\mu_0, \mu_1, d$ and the character of $\Omega$.

Proof. The proof is standard, and we provide a proof for the sake of the completeness. By the definition of the weak solution, one may have
\[
\int_{\Omega_\varepsilon} A(x/\varepsilon, \nabla u_\varepsilon) \cdot \nabla (u_\varepsilon - z) \, dx = \int_{\Omega_\varepsilon} F(u_\varepsilon - z) \, dx,
\]
where \( z \in H^1(\Omega) \) satisfies
\[
\Delta z = 0 \quad \text{in} \quad \Omega, \quad z = g \quad \text{on} \quad \partial \Omega.
\]
It is well known that
\[
\| \nabla z \|_{L^2(\Omega)} \leq C \| g \|_{H^{3/2}(\partial \Omega)},
\]
where \( C \) depends on \( d \) and the character of \( \Omega \). Then by the assumptions (2) and (3) we have
\[
\int_{\Omega_\epsilon} A(x/\epsilon, \nabla u_\epsilon) \cdot (\nabla u_\epsilon - \nabla z) \, dx \\
\geq \mu_0 \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx - \frac{\mu_0}{2} \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx - C(\mu_0, \mu_1) \int_{\Omega} |\nabla z|^2 \, dx,
\]
in which we use Young’s inequality, and
\[
\left| \int_{\Omega_\epsilon} F(u_\epsilon - z) \, dx \right| \leq C r_0 \| F \|_{L^2(\Omega)} \| \nabla (u_\epsilon - z) \|_{L^2(\Omega_\epsilon)} \\
\leq \frac{\mu_0}{4} \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx + C r_0^2 \int_{\Omega} |F|^2 \, dx + C \int_{\Omega} |\nabla z|^2 \, dx,
\]
where we employ Poincaré’s inequality in the first step. Thus we have
\[
\int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx \leq C r_0^2 \int_{\Omega} |F|^2 \, dx + C \int_{\Omega} |\nabla z|^2 \, dx.
\]
(44)
Consequently, this together with the estimate (43) leads to the desired estimate (42), and we are done.

\[\square\]

**Theorem 2.17 (H² theory).** Let \( \Omega \) be a bounded \( C^{1,1} \) domain. Given \( g \in H^{3/2}(\partial \Omega) \) and \( F \in L^2(\Omega) \), assume that \( u_0 \in H^1(\Omega) \) is the weak solution of \( L_0 u_0 = F \) in \( \Omega \) with \( u_0 = g \) on \( \partial \Omega \). Then we have \( u_0 \in H^2(\Omega) \) satisfying
\[
\| \nabla^2 u_0 \|_{L^2(\Omega)} \leq C \left\{ \| F \|_{L^2(\Omega)} + \| g \|_{H^{3/2}(\partial \Omega)} \right\},
\]
where \( C \) depends on \( \mu_0, \mu_1, d \) and the character of \( \Omega \).

**Proof.** The proof is standard, see for example [39, Theorem 2.17].

\[\square\]

### 3 Convergence rates

In this section, we derive the convergence rates of (1) by calculating the integral \( \int_{\Omega_\epsilon} A(x/\epsilon, \nabla u_\epsilon) \nabla w_\epsilon \, dx \).

As a start, we introduce some cut-off functions. Let \( O_\epsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \epsilon \} \), \( \psi_\epsilon, \psi'_\epsilon \in C_0^\infty(\Omega) \) satisfy
\[
\begin{align*}
0 \leq \psi_\epsilon, \psi'_\epsilon &\leq 1 \quad \text{for} \quad x \in \Omega, \\
\text{supp}(\psi_\epsilon) &\subset \Omega \setminus O_{3\epsilon}, \quad \text{supp}(\psi'_\epsilon) \subset \Omega \setminus O_{\epsilon}, \\
\psi_\epsilon &= 1 \quad \text{in} \quad \Omega \setminus O_{4\epsilon}, \quad \psi'_\epsilon = 1 \quad \text{in} \quad \Omega \setminus O_{2\epsilon}, \\
|\nabla \psi_\epsilon| &\leq C \epsilon^{-1}, \quad |\nabla \psi'_\epsilon| \leq C \epsilon^{-1}.
\end{align*}
\]
(46)

By the definition of \( \psi'_\epsilon, \psi_\epsilon \), it’s known that \( \psi_\epsilon(1 - \psi'_\epsilon) = 0 \) in \( \Omega \).
Lemma 3.1. Let $\Omega \subset \mathbb{R}^d$ and $\omega$ be Lipschitz domains. Assume that $A$ satisfies (3) and (2). Suppose that $u_\varepsilon \in H^1(\Omega_\varepsilon)$ and $u_0 \in H^1(\Omega)$ satisfy equations (1) and (6), respectively. Let $w_\varepsilon = u_\varepsilon - v_\varepsilon$, $v_\varepsilon = u_0 + \varepsilon N(x/\varepsilon, \varphi)$ in which $\varphi = S_\varepsilon(\psi_\varepsilon \nabla u_0)$. Then we have

$$
\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \left\{ \varepsilon \left( \|F\|_{L^2(\Omega)} + \|\nabla \varepsilon E(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega)} + \|\nabla \varepsilon N(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega)} \right) + \|\nabla u_0 - \varphi\|_{L^2(\Omega)} \right\},
$$

(47)
in which the constant $C$ depends only on $\mu_0, \mu_1, d$.

Proof. By the definition of $\varphi$, it's known that $\varphi \in H^1_0(\Omega)$. In view of $u_\varepsilon$ and $u_0$ are solutions to (1) and (6), respectively, we have

$$
\int_{\Omega_\varepsilon} A(x/\varepsilon, \nabla u_\varepsilon) \nabla w_\varepsilon dx = \int_{\Omega_\varepsilon} Fw_\varepsilon dx = \int_{\Omega} l_+^+ F\bar{w}_\varepsilon dx,
$$

$$
\int_{\Omega} \hat{A}(\nabla u_0) \nabla (\tilde{w}_\varepsilon \psi'_\varepsilon) dx = \int_{\Omega} F(\tilde{w}_\varepsilon \psi'_\varepsilon) dx,
$$

where we use the fact that $w_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and $\bar{w}_\varepsilon$ is the extension of $w_\varepsilon$ given by Lemma 2.12. In fact, because of $\varphi = S_\varepsilon(\psi_\varepsilon \nabla u_0)$, we have $\varphi = 0$ on $O_{2\varepsilon}$ and in view of Remark 2.2, we see that $N(x/\varepsilon, \varphi) = 0, \nabla_y N(x/\varepsilon, \varphi) = 0$ for any $x \in O_{2\varepsilon} \cap \Omega_\varepsilon$. This coupled with $u_\varepsilon = u_0$ on $\Gamma_\varepsilon$ leads to the fact $w_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$.

It follows from the above two equalities that

$$
\int_{\Omega_\varepsilon} \left( A(x/\varepsilon, \nabla u_\varepsilon) - A(x/\varepsilon, \nabla v_\varepsilon) \right) \cdot \nabla w_\varepsilon dx
$$

$$
= \int_{\Omega} l_+^+ F\bar{w}_\varepsilon - \theta F\bar{w}_\varepsilon \psi'_\varepsilon dx + \theta \int_{\Omega} \hat{A}(\nabla u_0) \nabla (\tilde{w}_\varepsilon \psi'_\varepsilon) dx - \int_{\Omega} l_+^+ A(x/\varepsilon, \nabla v_\varepsilon) \nabla \bar{w}_\varepsilon dx
$$

$$
= \int_{\Omega} l_+^+ F\bar{w}_\varepsilon - \theta F\bar{w}_\varepsilon \psi'_\varepsilon dx + \theta \int_{\Omega} [\hat{A}(\nabla u_0) - \hat{A}(\varphi)] \nabla (\tilde{w}_\varepsilon \psi'_\varepsilon) dx
$$

$$
+ \int_{\Omega} [\theta \hat{A}(\varphi) - l_+^+ A(x/\varepsilon, \varphi + \nabla_y N(x/\varepsilon, \varphi))] \nabla (\tilde{w}_\varepsilon \psi'_\varepsilon) dx
$$

$$
+ \int_{\Omega} l_+^+ A(x/\varepsilon, \varphi + \nabla_y N(x/\varepsilon, \varphi)) \nabla (\tilde{w}_\varepsilon \psi'_\varepsilon) - l_+^+ A(x/\varepsilon, \nabla v_\varepsilon) \nabla \bar{w}_\varepsilon dx
$$

$$
:= I_1 + I_2 + I_3 + I_4.
$$

Next, we will estimate every $I_i, i = 1, 2, 3, 4$. As for $I_1$,

$$
I_1 = \int_{\Omega} (l_+^+ - \theta) F\bar{w}_\varepsilon \psi'_\varepsilon dx + \int_{\Omega} (1 - \psi'_\varepsilon) l_+^+ F\bar{w}_\varepsilon dx := I_{11} + I_{12}.
$$

Since supp$(1 - \psi'_\varepsilon) = O_{2\varepsilon}$ and Lemma 2.14, we have

$$
|I_{12}| \leq \int_{O_{2\varepsilon}} |F\bar{w}_\varepsilon| dx \lesssim \|F\|_{L^2(O_{2\varepsilon})} \|\bar{w}_\varepsilon\|_{L^2(O_{2\varepsilon})}
$$

$$
\lesssim \varepsilon \|F\|_{L^2(\Omega)} \|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)}.
$$
To deal with the first term $I_{11}$, we consider the auxiliary equation
\begin{equation}
\begin{aligned}
- \Delta \Psi(y) &= l^+(y) - \theta \quad \text{in } Y, \\
\int_Y \Psi \, dy &= 0, \ \Psi \in H^1_{\text{per}}(Y).
\end{aligned}
\end{equation}
 According to $\int_Y l^+(y) - \theta \, dy = 0$, it’s known that (48) has a solution $\Psi \in H^1_{\text{per}}(Y)$, and by Schauder’s estimates we have $[\nabla \Psi]_{C^0(Y)} \lesssim \|l^+ - \theta\|_{L^\infty(Y)}$. Therefore,

\[ |I_{11}| = | - \int_{\Omega} \Delta_y \Psi(F \tilde{w}_x \psi_x') \, dx | = | - \varepsilon \int_{\Omega} \nabla_x \cdot (\nabla_y \Psi)(F \tilde{w}_x \psi_x') \, dx | \]
\[ = | \varepsilon \int_{\Omega} \nabla_y \Psi \cdot \nabla(F \tilde{w}_x \psi_x') \, dx | \lesssim \varepsilon \| \nabla F \|_{H^{-1}(\Omega)} \| \tilde{w}_x \psi_x' \|_{H^1_0(\Omega)} + \varepsilon \| F \|_{L^2(\Omega)} \| \nabla(\tilde{w}_x \psi_x') \|_{L^2(\Omega)} \]
\[ \lesssim \varepsilon \| F \|_{L^2(\Omega)} \| \nabla(\tilde{w}_x \psi_x') \|_{L^2(\Omega)} \lesssim \varepsilon \| F \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)}, \]

where $y = x/\varepsilon$, we use the fact that $\| \nabla F \|_{H^{-1}(\Omega)} \leq C \| F \|_{L^2(\Omega)}$ and Lemma 2.14 in the last two inequalities. Hence, $|I_1| \lesssim \varepsilon \| F \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)}$.

By the properties of $\hat{A}(\xi)$, see (23), we have

\[ |I_2| = | \theta \int_{\Omega} (\hat{A}(\nabla u_0) - \hat{A}(\varphi)) \nabla(\tilde{w}_x \psi_x') \, dx | \]
\[ \lesssim \int_{\Omega} | \nabla u_0 - \varphi | | \nabla(\tilde{w}_x \psi_x') | \, dx \leq \| \nabla u_0 - \varphi \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)}, \]

where we use Hölder’s inequality and Lemma 2.14 in the last inequality. Recalling that $b(y, \xi) = \theta \hat{A}(\xi) - l^+(y)A(y, \xi + \nabla N(y, \xi))$, it follows from Lemmas 2.7 and 2.14 that

\[ |I_3| = | \int_{\Omega} b(x/\varepsilon, \varphi) \nabla(\tilde{w}_x \psi_x') \, dx | \]
\[ = | \varepsilon \int_{\Omega} \frac{\partial}{\partial x_j} \{ E_{ji}(x/\varepsilon, \varphi) \} \frac{\partial}{\partial x_i}(\tilde{w}_x \psi_x') \, dx + \varepsilon \int_{\Omega} \frac{\partial}{\partial \xi_k} \{ E_{ji}(x/\varepsilon, \varphi) \} \frac{\partial}{\partial x_i}(\tilde{w}_x \psi_x') \, dx | \]
\[ = | - \varepsilon \int_{\Omega} E_{ji}(x/\varepsilon, \varphi) \frac{\partial^2}{\partial x_j \partial x_i}(\tilde{w}_x \psi_x') \, dx + \varepsilon \int_{\Omega} \frac{\partial}{\partial \xi_k} \{ E_{ji}(x/\varepsilon, \varphi) \} \frac{\partial}{\partial x_i}(\tilde{w}_x \psi_x') \, dx | \]
\[ = | \varepsilon \int_{\Omega} \frac{\partial}{\partial \xi_k} \{ E_{ji}(x/\varepsilon, \varphi) \} \frac{\partial}{\partial x_j \partial x_i}(\tilde{w}_x \psi_x') \, dx | \lesssim \varepsilon \| \nabla E_{ji} \cdot \psi_x' \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)}, \]

where we use the fact that $E_{ji}(x/\varepsilon, \varphi) = 0$ on $\partial \Omega$ according to $\varphi \in H^1_0(\Omega)$ in the third equality, and we employ the anti-symmetric property of $E$ in the fourth step.

For the last term $I_4$,

\[ I_4 = \int_{\Omega} \int_{\varepsilon} [A(x/\varepsilon, \varphi + \nabla y N(y, \varphi)) - A(x/\varepsilon, \nabla y)] \nabla w_x \, dx + \int_{\Omega} \int_{\varepsilon} A(x/\varepsilon, \varphi + \nabla y N(y, \varphi)) \nabla(\tilde{w}_x \psi_x' - \tilde{w}_x) \, dx \]
\[ := I_{41} + I_{42}. \]

We have

\begin{align}
|I_{41}| &\leq \int_{\Omega} \int_{\varepsilon} | \varphi - \nabla u_0 - \varepsilon \nabla \xi N(x/\varepsilon, \varphi) \nabla \varphi | \cdot | \nabla \tilde{w}_x | \, dx \\
&\leq \| \varphi - \nabla u_0 \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)} + \varepsilon \| \nabla \xi N(\cdot/\varepsilon, \varphi) \nabla \varphi \|_{L^2(\Omega)} \| \nabla \tilde{w}_x \|_{L^2(\Omega)}, \quad (49) \\
|I_{42}| &\leq \int_{\Omega} \int_{\varepsilon} | \varphi + \nabla y N(y, \varphi) | \cdot | \nabla(\tilde{w}_x (1 - \psi_x')) | \, dx.
\end{align}
According to $\text{supp}(1 - \psi'_\varepsilon) = O_{2\varepsilon}$ and $\nabla_y N(x/\varepsilon, \varphi) = 0$ for any $x \in O_{2\varepsilon} \cap \Omega_\varepsilon$, we see that $I_{42} = 0$. Hence, it follows that

$$|I_4| \leq \left( \|\varphi - \nabla u_0\|_{L^2(\Omega)} + \varepsilon \|\nabla N(\cdot/\varepsilon, \varphi)\|_{L^2(\Omega, \Omega_{3\varepsilon})} \right) \|\nabla \bar{w}_\varepsilon\|_{L^2(\Omega)}.$$  

Combining the above estimates for $I_i$ with $i = 1, 2, 3, 4$ and the condition (3), we derive the estimate (47).

**Lemma 3.2.** Assume the same conditions as in Theorem 1.1. Then we have the following estimate

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \left\{ \|\nabla u_0\|_{L^2(O_{4\varepsilon})} + \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega, O_{3\varepsilon})} + \varepsilon \|F\|_{L^2(\Omega)} \right\} \quad (50)$$

and

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \leq Cr_0 \left\{ \|\nabla u_0\|_{L^2(\Omega, O_{3\varepsilon})} + \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega, O_{3\varepsilon})} + \varepsilon \|F\|_{L^2(\Omega)} \right\}, \quad (51)$$

where $C$ depends on $\mu_0, \mu_1, \mu_2, \tau, d$ and the boundary character of $\omega$.

**Proof.** According to Lemma 3.1, to show the estimate (50), we need to estimate $\|\nabla \xi E(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega)}$, $\|\nabla \xi N(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega_\varepsilon)}$, and $\|\nabla u_0 - \varphi\|_{L^2(\Omega)}$. By Lemmas 2.1 and 2.7, and the inequalities (34) and (35), we can easily derive that

$$\|\nabla \xi E(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega)} + \|\nabla \xi N(\cdot/\varepsilon, \varphi) \cdot \nabla \varphi\|_{L^2(\Omega_\varepsilon)} \lesssim \|\nabla^2 u_0\|_{L^2(\Omega, O_{3\varepsilon})} + \varepsilon^{-1} \|\nabla u_0\|_{L^2(\Omega)}.$$  

According to the properties of $\psi_\varepsilon$ and (34) and (35), we have the following inequalities,

$$\|\nabla u_0 - \varphi\|_{L^2(\Omega)} \lesssim \|\nabla u_0 - S_\varepsilon (\psi_\varepsilon \nabla u_0)\|_{L^2(\Omega)} + \|\nabla u_0 - S_\varepsilon (1 - \psi_\varepsilon) \nabla u_0\|_{L^2(\Omega)} \lesssim \varepsilon \|\nabla (\psi_\varepsilon \nabla u_0)\|_{L^2(\Omega)} + \|\nabla u_0 - S_\varepsilon (1 - \psi_\varepsilon) \nabla u_0\|_{L^2(\Omega)} \lesssim \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega, O_{3\varepsilon})} + \|\nabla u_0\|_{L^2(O_{3\varepsilon})}.$$  

Thus, we obtain the estimate (50). As for estimate (51), it suffices to show

$$\int_{\Omega_\varepsilon} |N(x/\varepsilon, \varphi)|^2 dx \lesssim \int_{\Omega} |\psi_\varepsilon \nabla u_0|^2 dx, \quad (52)$$

and we recall that $\varphi = S_\varepsilon (\psi_\varepsilon \nabla u_0)$. To do so, we collect a family of small cubes by $Y^i_\varepsilon = \varepsilon (i + Y)$ for $i \in \mathbb{Z}^d$ with an index set $I_\varepsilon$, such that $\Omega_\varepsilon \setminus O_{2\varepsilon} \subset \bigcup_{i \in I_\varepsilon} Y^i_\varepsilon \subset \Omega$ and $Y^i_\varepsilon \cap Y^j_\varepsilon = \emptyset$ if $i \neq j$. Thus

$$\int_{\Omega_\varepsilon} |N(x/\varepsilon, \varphi)|^2 dx \lesssim \sum_{i \in I_\varepsilon} \int_{Y^i_\varepsilon \cap \omega} |N(x/\varepsilon, \varphi)|^2 dx + \int_{O_{2\varepsilon} \cap \Omega_\varepsilon} |N(x/\varepsilon, \varphi)|^2 dx \lesssim \sum_{i \in I_\varepsilon} |Y^i_\varepsilon| ||\varphi||^2 \lesssim \int_{\Omega} ||\varphi||^2 dx,$$

where we employ the estimate (17) and the fact that

$$N(x/\varepsilon, \varphi) = 0 \quad \forall x \in O_{2\varepsilon} \cap \Omega_\varepsilon.$$  

We take $\varphi^i = \inf_{x \in Y^i_\varepsilon} |S_\varepsilon (\psi_\varepsilon \nabla u_0)(x)|$, the second step is according to the fact that $N(y, \xi)$ is continuous about the second variable and (19), while the last step is due to Chebyshev’s inequality. Therefore, the estimate (52) consequently follows from

$$\|S_\varepsilon (\psi_\varepsilon \nabla u_0)\|_{L^2(\Omega)} \lesssim \|\psi_\varepsilon \nabla u_0\|_{L^2(\Omega)},$$
in which we use (34). Due to the properties of $\psi_\varepsilon$ and Poincaré’s inequality, one can derive that
\[
\|\psi_\varepsilon \nabla u_0\|_{L^2(\Omega)} \leq C \varepsilon R \left\{ \varepsilon^{-1} \|\nabla u_0\|_{L^2(\Omega \setminus \delta B)} + \|\nabla^2 u_0\|_{L^2(\Omega \setminus \delta B)} \right\}
\]
and we have completed the proof.

**Theorem 3.3.** Let $B = B(0, R) \subset \mathbb{R}^d$ be a ball with $R \in (16\varepsilon, 1]$. Assume $\mathcal{L}_\varepsilon$ satisfies the conditions (3), (2) and (4). Given $F \in H^1(B)$ and $g \in H^{3/2}(\partial B)$, let $u_\varepsilon \in H^1(B_\varepsilon)$ (noting that $B_\varepsilon = B \cap \varepsilon \omega$), $u_0 \in H^1(B)$ be the weak solutions of (1) and (6), respectively. Then we have
\[
\|u_\varepsilon - u_0\|_{L^2(B_\varepsilon)} \leq C \left( \frac{\varepsilon}{R} \right)^{1/2} \left\{ R^{3/2} \|F\|_{L^2(B)} + R \|g\|_{H^{3/2}(\partial B)} \right\},
\]
where $C$ depends on $\mu_0, \mu_1, \mu_2, \tau, d$ and the character of $\omega$, but independent of $R$.

**Proof.** In view of the estimates (51) and (45), one may have
\[
\|u_\varepsilon - u_0\|_{L^2(B_\varepsilon)} \leq CR \|\nabla u_0\|_{L^2(B(0, R-\varepsilon \cdot 4\varepsilon))} + CR^2 \left( \frac{\varepsilon}{R} \right) \left\{ \|F\|_{L^2(B)} + \|g\|_{H^{3/2}(\partial B)} \right\}. \tag{54}
\]
To complete the proof, it suffices to show
\[
\left\| \nabla u_0 \right\|_{L^2(B(0, R-\varepsilon \cdot 4\varepsilon))}^2 = \int_{R-\varepsilon \cdot 4\varepsilon}^R \int_{\partial B(0, r)} |\nabla u_0|^2 dS_r \, dr \leq 4\varepsilon \sup_{(3R/4) \leq r \leq R} \int_{\partial B(0, r)} |\nabla u_0|^2 dS_r \leq C\varepsilon/R \int_{B(0, R)} |\nabla u_0|^2 dx + C\varepsilon R \int_{B(0, R)} |\nabla^2 u_0|^2 dx,
\]
where we use the trace theorem
\[
\int_{\partial B(0, r)} |\nabla u_0|^2 dS_r \leq \frac{C}{R} \int_{B(0, R)} |\nabla u_0|^2 dx + CR \int_{B(0, R)} |\nabla^2 u_0|^2 dx
\]
for any $(3R/4) \leq r \leq R$. Thus,
\[
\left\| \nabla u_0 \right\|_{L^2(B(0, R-\varepsilon \cdot 4\varepsilon))} \leq C \left( \frac{\varepsilon}{R} \right)^{1/2} \left\{ \|\nabla u_0\|_{L^2(B(0, R))} + R \|\nabla^2 u_0\|_{L^2(B(0, R))} \right\} \tag{55}
\]
in which we employ the estimates (45) and (42), and the fact that $\|g\|_{H^{3/2}(\partial B)} \leq C \|g\|_{H^{3/2}(\partial B)}$. Consequently, the desired estimate (53) follows from (54) and (55) by noting that $\varepsilon/R < 1$, and we have completed the proof.

**Proof of Theorem 1.1.** If replacing $B(0, R)$ in Theorem 3.3 by a bounded $C^{1,1}$ domain, then we can derive that
\[
\|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)} \leq C \Omega \varepsilon^{1/2} \left\{ \|F\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial \Omega)} \right\},
\]
where $C_\Omega$ depends on $\mu_0, \mu_1, \mu_2, \tau, d, r_0$ and the character of $\omega$ and $\Omega$. This in fact proved the estimate (10), and we have completed the proof.
4 Interior estimates

Lemma 4.1 (approximating lemma). Let $\sqrt{\varepsilon} \leq r < (1/2)$. Assume the same conditions as in Theorem 1.3. Let $u_\varepsilon \in H^1(B_\varepsilon(0, 2r))$ be a weak solution of

\[
\begin{aligned}
\mathcal{L}_\varepsilon u_\varepsilon &= F &\text{in } B_\varepsilon(0, 2r), \\
\sigma_\varepsilon(u_\varepsilon) &= 0 &\text{on } S_\varepsilon(2r).
\end{aligned}
\]

Then there exists $w \in H^1(B(0, r))$ such that $\mathcal{L}_0w = F$, and there holds

\[
\left( \iint_{B_\varepsilon(0, r)} |u_\varepsilon - w|^2 \, dx \right)^{1/2} \leq C \left( \frac{\varepsilon}{r} \right)^{1/4} \left\{ \left( \int_{B_\varepsilon(0, 2r)} |u_\varepsilon|^2 \, dx \right)^{1/2} + r^2 \left( \iint_{B(0, 2r)} |F|^2 \, dx \right)^{1/2} \right\},
\]

(56)

where $C$ depends on $\mu_0, \mu_1$ and $d$.

Proof. The main idea may be found in [36, Lemma 11.2]. However, this result can not be obtained by rescaling arguments due to the nonlinearity of $\mathcal{L}_\varepsilon$. Before proceeding our proof, we need to extend $u_\varepsilon \in H^1(B_\varepsilon(0, 2r))$ to $H^1(B(0, 3r))$. The problem is that $u_\varepsilon$ does not equal to 0 on $\Gamma_\varepsilon(2r) (= \partial B(0, 2r) \cap \varepsilon \omega)$, we can not apply the extension operator in Lemma 2.12 directly. Suppose that

\[
\rho \in C^1_0(\bar{B}(0, 7r/4)), \; \rho(x) = 1 \text{ on } B(0, 3r/2), \; \text{ and } |\nabla \rho| \leq \frac{C}{r}.
\]

Then we have $\rho u_\varepsilon \in H^1(B_\varepsilon(2r), \Gamma_\varepsilon(2r))$. By Lemma 2.12, we have the extension function $\tilde{u}_\varepsilon$ satisfying that $\tilde{u}_\varepsilon(x) = \rho(x) u_\varepsilon(x) = u_\varepsilon(x)$ for $x \in B_\varepsilon(0, 3r/2)$, $\tilde{u}_\varepsilon \in H^1_0(B(0, 3r))$ and

\[
\|\tilde{u}_\varepsilon\|_{H^1_0(B(0, 3r))} \leq C \|\rho u_\varepsilon\|_{H^1(B_\varepsilon(0, 2r))} \leq C \left\{ \frac{1}{r} \|u_\varepsilon\|_{L^2(B_\varepsilon(0, 2r))} + \|\nabla u_\varepsilon\|_{L^2(B_\varepsilon(7r/4))} \right\}
\]

\[
\leq C \left\{ \frac{1}{r} \|u_\varepsilon\|_{L^2(B_\varepsilon(0, 2r))} + 2 \|F\|_{L^2(B(0, 2r))} \right\},
\]

(57)

in which the last step follows from Caccioppoli’s inequality (40). On account of (57) and co-area formula, it is true that there exists $\tilde{r} \in [r, 3r/2]$ such that

\[
\|\tilde{u}_\varepsilon\|_{H^1(\partial B(0, \tilde{r}))} \leq C \left\{ \frac{1}{r} \|u_\varepsilon\|_{L^2(B_\varepsilon(0, 2r))} + 2 \|F\|_{L^2(B(0, 2r))} \right\}.
\]

(58)

Then for some $0 < \delta \leq \tilde{r}$, we consider

\[
\begin{aligned}
\mathcal{L}_\varepsilon v_\varepsilon &= F, &\text{in } B_\varepsilon(0, \tilde{r}), \\
\sigma_\varepsilon(v_\varepsilon) &= 0, &\text{on } S_\varepsilon(\tilde{r}), \\
v_\varepsilon &= (\tilde{u}_\varepsilon)_\delta, &\text{on } \Gamma_\varepsilon(\tilde{r}),
\end{aligned}
\]

and

\[
\begin{aligned}
\mathcal{L}_0w &= F &\text{in } B(0, \tilde{r}), \\
w &= (\tilde{u}_\varepsilon)_\delta &\text{on } \partial B(0, \tilde{r}),
\end{aligned}
\]

in which $(\tilde{u}_\varepsilon)_\delta \in H^{3/2}(\partial B(0, \tilde{r}))$ satisfies the estimate (38). Consider

\[
\int_{B_\varepsilon(0, r)} |u_\varepsilon - w|^2 \, dx \leq \int_{B_\varepsilon(0, r)} |u_\varepsilon - v_\varepsilon - z_\varepsilon|^2 \, dx + \int_{B_\varepsilon(0, r)} |v_\varepsilon - w|^2 \, dx + \int_{B_\varepsilon(0, r)} |z_\varepsilon|^2 \, dx
\]

\[
=: I_1 + I_2 + I_3,
\]

(59)
where \( z_\varepsilon \in H^1(B(0, \bar{r})) \) satisfies
\[
\Delta z_\varepsilon = 0 \quad \text{in} \quad B(0, \bar{r}), \quad z_\varepsilon = \tilde{u}_\varepsilon - (\tilde{u}_\varepsilon)_\delta \quad \text{on} \quad \partial B(0, \bar{r}).
\]

We first handle \( I_2 \), and it follows from the estimates (53) and (38) that
\[
\sqrt{I_2} \leq \| v_\varepsilon - w \|_{L^2(B_\varepsilon(0, \bar{r}))} \\
\leq C \left( \frac{\varepsilon}{r} \right)^{1/2} \left\{ r^{3/2} \| F \|_{L^2(B(0, \bar{r}))} + r \| \tilde{u}_\varepsilon \|_{H^{3/2}(\partial B(0, \bar{r}))} \right\} \\
\leq C \left( \frac{\varepsilon}{\delta r} \right)^{1/2} \left\{ r^2 \| F \|_{L^2(B(0, 2\bar{r}))} + r \| \tilde{u}_\varepsilon \|_{H^1(\partial B(0, \bar{r}))} \right\} \\
\leq C \left( \frac{\varepsilon}{\delta r} \right)^{1/2} \left\{ \| u_\varepsilon \|_{L^2(B_\varepsilon(0, 2\bar{r}))} + r^2 \| F \|_{L^2(B(0, 2\bar{r}))} \right\},
\]
in which the last step follows from the estimate (58).

Before estimating \( I_1 \), we claim that
\[
\| \nabla u_\varepsilon - \nabla v_\varepsilon \|_{L^2(B_\varepsilon(0, \bar{r}))} \leq C \| \nabla z_\varepsilon \|_{L^2(B(0, \bar{r}))}
\]
where \( C \) depends only \( \mu_0 \) and \( \mu_1 \). In fact,
\[
\int_{B_\varepsilon(0, \bar{r})} \left[ A(x/\varepsilon, \nabla u_\varepsilon) - A(x/\varepsilon, \nabla v_\varepsilon) \right] \cdot \nabla \phi dx = 0
\]
for any \( \phi \in H^1(B_\varepsilon(0, \bar{r}), \Gamma_\varepsilon(\bar{r})) \). Set \( \phi = u_\varepsilon - v_\varepsilon - z_\varepsilon \), because of \( \tilde{u}_\varepsilon(x) = u_\varepsilon(x) \) for \( x \in \Gamma_\varepsilon(\bar{r}) \), we have \( \phi \in H^1(B_\varepsilon(0, \bar{r}), \Gamma_\varepsilon(\bar{r})) \) and then by employing the assumptions (3) to the above equation, we can arrive at the claim (61) immediately. Hence, from Poincaré’s inequality and (61), it follows that
\[
\sqrt{I_1} \leq Cr \| (u_\varepsilon - v_\varepsilon - z_\varepsilon) \|_{L^2(B_\varepsilon(0, \bar{r}))} \leq Cr \| \nabla z_\varepsilon \|_{L^2(B(0, \bar{r}))} \\
\leq Cr \| u_\varepsilon - (\tilde{u}_\varepsilon)_\delta \|_{H^{1/2}(\partial B(0, \bar{r}))} \leq Cr \delta^{1/2} \| \tilde{u}_\varepsilon \|_{H^1(\partial B(0, \bar{r}))} \\
\leq C \delta^{1/2} \left\{ \| u_\varepsilon \|_{L^2(B_\varepsilon(0, 2\bar{r}))} + r^2 \| F \|_{L^2(B(0, 2\bar{r}))} \right\},
\]
where we also use the estimates (38), (58) and the fact \( r \leq \bar{r} \leq \frac{3}{2} \bar{r} \), while the third step is due to the fact \( \| z_\varepsilon \|_{H^{1/2}(\partial B(0, \bar{r}))} \leq C \| z_\varepsilon \|_{H^{1/2}(\partial B(0, \bar{r}))} \).

The computation for \( I_3 \) relies on some properties of harmonic functions, by Hölder’s inequality, we have
\[
\sqrt{I_3} \leq Cr^{1/2} \| z_\varepsilon \|_{L^2(B_\varepsilon(0, \bar{r}))} \leq Cr^{1/2} \| (\tilde{z}_\varepsilon)^* \|_{L^2(\partial B(0, \bar{r}))} \\
\leq Cr^{1/2} \| z_\varepsilon \|_{L^2(\partial B(0, \bar{r}))} \leq Cr^{1/2} \delta \| \tilde{u}_\varepsilon \|_{H^1(\partial B(0, \bar{r}))} \\
\leq C r \delta^{1/2} \| \tilde{u}_\varepsilon \|_{H^1(\partial B(0, \bar{r}))} \leq C \delta^{1/2} \left\{ \| u_\varepsilon \|_{L^2(B_\varepsilon(0, 2\bar{r}))} + r^2 \| F \|_{L^2(B(0, 2\bar{r}))} \right\},
\]
in which the notation \( (z_\varepsilon)^* \) represents the nontangential maximal function of \( z_\varepsilon \) (see for example [41, Definition 2.19]). Here the second inequality follows from [23, Remark 9.3], and the third one is the so-called nontangential maximal function estimate (see for example [33, Theorem 7.5.14]). We employ the estimate (39) in the fourth inequality and the estimate (58) in the last step.
Consequently, combining the estimates (60), (62) and (63) with (59), we have
\[ \|u_\varepsilon - w\|_{L^2(B_\varepsilon(0,r))} \leq C\delta^{1/2} + \delta^{-1/2}(\varepsilon/r)^{1/2}\left\{ \|u_\varepsilon\|_{L^2(B_\varepsilon(0,2r))} + r^2\|F\|_{L^2(B(0,2r))} \right\} \]
\[ \leq C(\varepsilon/r)^{1/4}\left\{ \|u_\varepsilon\|_{L^2(B_\varepsilon(0,2r))} + r^2\|F\|_{L^2(B(0,2r))} \right\}, \]
in which the second line asks for \( \delta = (\varepsilon/r)^{1/2} \), and the assumption \( \sqrt{\varepsilon} \leq r < (1/2) \) meets this requirement. By multiplying \( r^{-d/2} \) in both sides of the above inequality, the desired estimate (56) follows, and we have completed the proof.

Before we proceed further, we recall the definition of \( G(r, v) \) and \( G_\varepsilon(r, v) \) in (16).

**Lemma 4.2.** Given \( F \in L^p(\Omega) \) for some \( p > d \), let \( u_0 \in H^1(B(0,2r)) \) be a solution of \( \mathcal{L}_0u_0 = F \) in \( B(0,2r) \). Then there exists \( \alpha \in (0,1) \), and a constant \( C > 0 \) depending on \( \mu_0, \mu_1, p, d \), such that
\[ [\nabla u_0]_{C^{0,\alpha}(B(0,r/2))} \leq Cr^{-\alpha}\left\{ \frac{1}{r}\left( \int_{B(0,r)} |u_0|^2 \right)^{1/2} + r\left( \int_{B(0,r)} |F|^p \right)^{1/p} \right\}. \] (64)

**Proof.** It is fine to assume \( u_0 \in H^2(B(0,r)) \) and we have the following equation
\[ \int_{B(0,r)} \nabla \xi_j \hat{A}^i(\nabla u_0)\nabla_{jk} u_0 \nabla_i \phi dx = -\int_{B(0,r)} F \nabla_k \phi dx \] (65)
for any \( \phi \in H^1_0(B(0,r)) \), and \( k = 1, \ldots, d \). Let \( \hat{a}_{ij}(x) = \nabla \xi_j \hat{A}^i(\nabla u_0) \), which will give a linear operator with the uniform ellipticity on account of (23) and (26). Hence, the De Giorgi-Nash-Moser theorem tells us that for any \( p > d \), there exists \( \alpha \in (0,1) \) and \( C > 1 \), depending only on \( \mu_0, \mu_2, d \) and \( p \), such that
\[ [\nabla u_0]_{C^{0,\alpha}(B(0,r/2))} \leq C r^{-\alpha}\left\{ \frac{1}{r}\left( \int_{B(0,r)} |u_0|^2 \right)^{1/2} + r\left( \int_{B(0,r)} |F|^p \right)^{1/p} \right\} \] (66)
(see for example [21, Theorem 8.13]).

**Lemma 4.3.** Suppose that \( \mathcal{L}_0(v) = F \) for \( x \in B(0,2r) \), \( r \geq \varepsilon \), there exists a constant \( C \) depending on \( \mu_0, \mu_1, d \) and \( \omega \) such that
\[ G(r, v) \leq CG_\varepsilon(2r, v). \] (67)

**Proof.** First, we decompose domain \( B(0, r) \) as in [31]. Let
\[ T_\varepsilon = \{ z \in \mathbb{Z}^d : \varepsilon(Y + z) \cap B(0, r) \neq \emptyset \}, \]
and fix \( z \in T_\varepsilon \). We denote the bounded, connected components of \( \mathbb{R}^d \setminus \omega \) by \( \{ H_k \}_{k=1}^N \) with \( H_k \cap (Y + z) \neq \emptyset \). Define cut-off function \( \varphi_k \in C_0^\infty(Y^*(z)) \) as
\[ \begin{cases} \varphi_k(x) = 1, & \text{if } x \in H_k, \\ \varphi_k(x) = 0, & \text{if } \text{dist}(x, H_k) > \frac{1}{4}g^w, \\ |\nabla \varphi_k| \leq C, \end{cases} \]
where \( C \) depends on \( \omega, g^w \) is defined in (9), and
\[ Y^*(z) = \bigcup_{j=1}^{3^d} (Y + z_j), z_j \in \mathbb{Z}^d \text{ and } |z - z_j| \leq \sqrt{d}. \]
Set $\varphi = \sum_{k=1}^{N} \varphi_k \in C_0^\infty(Y^*)$, where $Y^* = Y^*(z)$. We note that

$$\varphi(1 - \varphi) = 0 \text{ in } Y^* \setminus \omega, \text{ hence } \nabla \varphi = 0 \text{ in } Y^* \setminus \omega.$$ 

In the case of $\mathcal{L}_0(v) = F$ in $Y^*$, for any $M \in \mathbb{R}^d$, $c \in \mathbb{R}$, set $\tilde{v}(x) = v(x) - Mx - c$ and then we claim that

$$\int_{Y^*} |\tilde{v}|^2 dx \leq C \left\{ \int_{Y^* \cap \omega} |\tilde{v}|^2 dx + \int_{Y^*} F^2 dx \right\}, \quad (68)$$

where $C$ depends only on $\mu_0, \mu_1, d, \omega$ and is independent of $z$. By Poincaré’s inequality,

$$\int_{(Y^* \setminus \omega)} |\tilde{v}(x)|^2 dx \leq \sum_{k=1}^{N} \int_{H_k} |\tilde{v}(x)|^2 dx \leq C \int_{Y^*} |\nabla (\tilde{v} \varphi)|^2 dx. \quad (69)$$

By a routine calculation,

$$\int_{Y^*} |\nabla (\tilde{v} \varphi)|^2 dx \leq \int_{Y^*} |\nabla \tilde{v}|^2 |\varphi|^2 dx + \int_{Y^*} |\nabla \varphi|^2 |\tilde{v}|^2 dx, \quad (70)$$

and the second term in the right-hand side of the above inequality is good, and we just need to deal with the first term. According to $\mathcal{L}_0(v) = F$ in $Y^*$, there holds

$$\int_{Y^*} |\nabla \tilde{v}|^2 |\varphi|^2 dx = \int_{Y^*} |\nabla v - M|^2 |\varphi|^2 dx \leq C \int_{Y^*} |\varphi|^2 [\tilde{A}(\nabla v) - \tilde{A}(M)] \nabla \tilde{v} dx$$

$$= -2C \int_{Y^*} \varphi \tilde{v} [\tilde{A}(\nabla v) - \tilde{A}(M)] \nabla \varphi dx - C \int_{Y^*} \text{div}[\tilde{A}(\nabla v) - \tilde{A}(M)] |\varphi|^2 \tilde{v} dx,$$

in which we use (23) in the second step, and we employ divergence theorem in the last step. It follows from (23) and Young’s inequality that

$$\int_{Y^*} |\nabla |\varphi|^2 dx \leq C \int_{Y^*} |\varphi \nabla \tilde{v}| \cdot |\tilde{v} \nabla \varphi| dx + C \int_{Y^*} |F \varphi|^2 dx$$

$$\leq \delta \int_{Y^*} |\varphi \nabla \tilde{v}|^2 dx + C_\delta \int_{Y^*} |\tilde{v} \nabla \varphi|^2 dx + \delta \int_{Y^*} |\varphi \tilde{v}|^2 dx + C_\delta \int_{Y^*} |F \varphi|^2 dx$$

$$\leq \delta \int_{Y^*} |\varphi \nabla \tilde{v}|^2 dx + C_\delta \int_{Y^*} |\tilde{v} \nabla \varphi|^2 dx + C \delta \int_{Y^*} |\nabla (\varphi \tilde{v})|^2 dx + C_\delta \int_{Y^*} |F \varphi|^2 dx.$$
where the last inequality comes from the fact that $\nabla \varphi = 0$ on $Y^* \setminus \omega$. Hence, we have the desired claim (68).

In the following, we proceed to show (67). Regarding to $L_0(v) = F$ in $\varepsilon Y^*$ and recalling that $L_0 = -\text{div}_x[A(\nabla x)]$, set $x = \varepsilon y$ with $y \in Y^*$ and $\pi(y) = \frac{1}{\varepsilon} v(\varepsilon y) = \frac{1}{\varepsilon} v(x)$, $F(y) = \varepsilon (\varepsilon y) = \varepsilon F(x)$, and then $\pi$ satisfies the equation

$$L_0(\pi) = F \quad \text{in } Y^*. $$

(71)

Due to the claim (68), one may obtain

$$\int_{Y^* + z} |\pi(y) - My - c|^2 dy \lesssim \int_{Y^* \cap \omega} |\pi(y) - My - c|^2 dy + \int_{Y^*} |F(y)|^2 dy$$

for any $M \in \mathbb{R}^d$, $c \in \mathbb{R}$, which means

$$\int_{\varepsilon(Y^* + z)} \frac{1}{\varepsilon} v(x) - \frac{M}{\varepsilon} x - c|^2 dx \lesssim \int_{\varepsilon(Y^* \cap \omega)} \frac{1}{\varepsilon} v(x) - \frac{M}{\varepsilon} x - c|^2 dx + \int_{\varepsilon Y^*} \varepsilon F(x)^2 dx.$$

This further gives that

$$\int_{\varepsilon(Y^* + z)} |v(x) - Mx - c\varepsilon|^2 dx \lesssim \int_{\varepsilon(Y^* \cap \omega)} |v(x) - Mx - c\varepsilon|^2 dx + \varepsilon^4 \int_{\varepsilon Y^*} |F(x)|^2 dx.$$

Because of the arbitrariness of $c$, we may derive that

$$\|v - Mx - c\|_{L^2(\varepsilon(Y^* + z))} \leq C\|v - Mx - c\|_{L^2(\varepsilon(Y^* \cap \omega))} + C\varepsilon^2 \|F\|_{L^2(\varepsilon Y^*)}.$$  

According to the fact that there is a constant $N' < \infty$ depending only on $d$ such that $Y^*(z_1) \cap Y^*(z_2) \neq \emptyset$ for at most $N'$ coordinates if $z_1 \neq z_2$, and then summing over all $z \in T_\varepsilon$ gives

$$\|v - Mx - c\|_{L^2(B(0,r))} \leq C\|v - Mx - c\|_{L^2(B(0,2r))} + \varepsilon^2 C\|F\|_{L^2(B(0,2r))} \leq C\|v - Mx - c\|_{L^2(B(0,2r))} + r^2 C\|F\|_{L^2(B(0,2r))}, $$

(72)

in which we use the fact $r \geq \varepsilon$ in the last inequality. By recalling the definition of $G(r, v)$ and $G_\varepsilon(2r, v)$, we have completed the whole proof.

For the ease of the statement, we denote $\Phi(r)$ by

$$\Phi(r) = \frac{1}{r} \inf_{c \in \mathbb{R}} \left\{ \left( \int_{B(0,r)} |u_\varepsilon - c|^2 \right)^{1/2} + r^2 \left( \int_{B(0,r)} |F|^p \right)^{1/p} \right\}.$$

**Lemma 4.4.** Assume the same conditions as in Theorem 1.3. Let $u_\varepsilon$ be the solution of $L_\varepsilon(u_\varepsilon) = F$ in $B_\varepsilon(0,2r)$. Then there exists $\theta \in (0, 1/4)$ such that

$$G_\varepsilon(\theta r, u_\varepsilon) \leq \frac{1}{2} G_\varepsilon(r, u_\varepsilon) + C \left( \frac{\varepsilon}{r} \right)^{1/4} \Phi(2r) $$

(73)

for any $\sqrt{\varepsilon} \leq r < (1/4)$. 
Proof. Fixed \( r \in [\sqrt[3]{\varepsilon}, 1/4] \), let \( w \) be a solution to \( \mathcal{L}_\theta w = F \) in \( B(0, r) \) as in Lemma 4.1. For any \( \theta \in (0, 1) \) (which will be fixed later), we have

\[
G_\varepsilon(\theta r, u_\varepsilon) = \frac{1}{\theta r} \inf_{M \in \mathbb{R}^d} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - M x - c|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, \theta r)} |F|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{\theta r} \inf_{M \in \mathbb{R}^d} \left( \int_{B_r(0, \theta r)} |w - M x - c|^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{\theta r} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, \theta r)} |F|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{\theta r} \left[ \nabla w \right]_{C^{0, \alpha}(B_r(\theta r))} (\theta r)^{1+\alpha} + \frac{1}{\theta r} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, \theta r)} |F|^p \, dx \right)^{\frac{1}{p}}
\]

where we take \( M = \nabla w(0), c = w(0) \) and employ the mean value theorem for \( w(x) \) in the last step. It’s easy to see that the right-hand side above is less than

\[
\theta^\sigma r^{\alpha} [\nabla w]_{C^{0, \alpha}(B(\theta r))} + r^{-1} \theta^{-\frac{d - \frac{4}{p}}{2}} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}} + \theta^{1 - \frac{d}{p}} r^{\frac{1}{2}} \left( \int_{B(0, \theta r)} |F|^p \, dx \right)^{\frac{1}{p}}.
\]

Let \( \tilde{w}(x) = w(x) - M x - c \), and we take \( \tilde{a}(x) = (\tilde{a}_{ij}(x)) = \nabla \tilde{w}, \tilde{A}(\nabla w) \) as we did in Lemma 4.2, for \( 1 \leq k \leq d, \tilde{w}(x) \) satisfies

\[
- \text{div}[\tilde{a}(x) \nabla (\nabla_k \tilde{w})] = \nabla_k F \quad \text{in} \quad B(0, r).
\]

Hence, by Lemma 4.2 we have

\[
[\nabla \tilde{w}]_{C^{0, \alpha}(B(0, r/2))} \leq C r^{-\alpha} \left\{ \frac{1}{r} \left( \int_{B(0, r/2)} |\tilde{w}|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, r/2)} |F|^p \, dx \right)^{\frac{1}{p}} \right\}.
\]

According to the fact that \( \nabla \tilde{w} = \nabla w - M \), we see that

\[
[\nabla w]_{C^{0, \alpha}(B(0, \frac{r}{2}))} = [\nabla \tilde{w}]_{C^{0, \alpha}(B(0, \frac{r}{2}))} \leq C r^{-\alpha} \left\{ \frac{1}{r} \left( \int_{B(0, r/2)} |w - M x - c|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, r/2)} |F|^p \, dx \right)^{\frac{1}{p}} \right\}.
\]

Then we have

\[
G_\varepsilon(\theta r, u_\varepsilon) \leq \theta^\sigma \left[ r^{-1} \left( \int_{B(0, r/2)} |w - M x - c|^2 \, dx \right)^{\frac{1}{2}} + \theta r \left( \int_{B(0, r/2)} |F|^p \, dx \right)^{\frac{1}{p}} \right]
\]

\[+ r^{-1} \theta^{-\frac{d - \frac{4}{p}}{2}} \left( \int_{B(0, r/2)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}} + \theta^{1 - \frac{d}{p}} r \left( \int_{B(0, r/2)} |F|^p \, dx \right)^{\frac{1}{p}} \]

\[\leq \theta^\sigma G(r, w) + \theta^\sigma G_\varepsilon(r, w) + r^{-1} \theta^{-\frac{d}{2}} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}},\]

in which we take \( \sigma = \min\{\alpha, 1 - \frac{d}{p}\} \). And then, by Lemma 4.3, the right hand side above is less than

\[
C \theta^\sigma G(r, w) + r^{-1} \theta^{-\frac{d}{2}} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}}.
\]

By the definition of \( G_\varepsilon(r, w) \), it follows that

\[
G_\varepsilon(\theta r, u_\varepsilon) \leq C \theta^\sigma G_\varepsilon(r, u_\varepsilon) + C r^{-1} \theta^{-\frac{d}{2}} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} G_\varepsilon(r, u_\varepsilon) + C r^{-1} \left( \int_{B_r(0, \theta r)} |u_\varepsilon - w|^2 \, dx \right)^{\frac{1}{2}},
\]
where we choose $\theta$ small enough such that $C \theta^\sigma = \frac{1}{2}$. By Lemma 4.1, we arrive at

$$G_\varepsilon(\theta r, u_\varepsilon) \leq \frac{1}{2} G_\varepsilon(r, u_\varepsilon) + C(\frac{\varepsilon}{r})^{1/4}\left\{ \frac{1}{r} \left( \int_{B_r(0,2r)} |u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + r \left( \int_{B(0,2r)} |F|^p dx \right)^{\frac{1}{p}} \right\}.$$ 

Note that for any $c \in \mathbb{R}$, $u_\varepsilon - c$ is still a solution of $L_\varepsilon u_\varepsilon = F$ in $B(0, 2r)$, and the proof is complete. \(\square\)

**Lemma 4.5** (iteration lemma). Let $\Psi(r)$ and $\psi(r)$ be two nonnegative continuous functions on the integral $(0, 1]$. Let $0 < \varepsilon < \frac{1}{4}$. Suppose that there exists a constant $C_0$ such that

$$\begin{cases} \max_{r \leq t \leq 2r} \Psi(t) \leq C_0 \Psi(2r), \\ \max_{r \leq s, t \leq 2r} |\psi(t) - \psi(s)| \leq C_0 \Psi(2r). \end{cases} \quad (74)$$

We further assume that

$$\Psi(\theta r) \leq \frac{1}{2} \Psi(r) + C_0w(\varepsilon/r)\left\{ \Psi(2r) + \psi(2r) \right\} \quad (75)$$

holds for any $\varepsilon \leq r < (1/4)$, where $\theta \in (0, 1/4)$ and $w$ is a nonnegative increasing function in $[0, 1]$ such that $w(0) = 0$ and

$$\int_0^1 \frac{w(t)}{t} dt < \infty.$$ 

Then, we have

$$\max_{\varepsilon \leq r \leq 1} \left\{ \Psi(r) + \psi(r) \right\} \leq C \left\{ \Psi(1) + \psi(1) \right\}, \quad (76)$$

where $C$ depends only on $C_0, \theta$ and $\omega$.

**Proof.** The proof may be found in [34, Lemma 8.5]. \(\square\)

**Proof of Theorem 1.3.** It is fine to assume $0 < \varepsilon < 1/4$, otherwise it follows from the classical theory. In view of Lemma 4.5, we set $\Psi(r) = G_\varepsilon(r, u_\varepsilon)$, $w(t) = t^{1/4}$. To prove the desired estimate (11), it is sufficient to verify (74) and (75). Let $\psi(r) = |M_r|$, where $M_r$ is the matrix associated with $\Psi(r)$ such that

$$\Psi(r) = \frac{1}{2} \inf_{r \in \mathbb{R}} \left\{ \left( \int_{B_r(0, r)} |u_\varepsilon - M_r x - c|^2 \right)^{\frac{1}{2}} + r^2 \left( \int_{B(0, r)} |F|^p \right)^{\frac{1}{p}} \right\}.$$ 

Then it follows that

$$\begin{cases} \max_{r \leq t \leq 2r} \Psi(t) \leq C_0 \Psi(2r), \\ \Psi(2r) \leq C \left\{ \Psi(2r) + \psi(2r) \right\}, \\ \psi(r) \leq \Psi(r) + \frac{1}{2} \inf_{r \in \mathbb{R}} \left( \int_{B_r(0, 0)} |u_\varepsilon - c|^2 dx \right)^{\frac{1}{2}}. \end{cases}$$

According to Lemma 4.4, we have

$$\Psi(\theta r) \leq \frac{1}{2} \Psi(r) + C_0 w(\varepsilon/r)\left\{ \Psi(2r) + \psi(2r) \right\}.$$
for $\sqrt{c} \leq r < (1/4)$, so condition (75) in Lemma 4.5 holds. Let $t, s \in [r, 2r]$, and $v(x) = (M_t - M_s)x$. It is clear to see $v$ is harmonic in $\mathbb{R}^d$, $\mathcal{L}_0(v) = 0$ in $\mathbb{R}^d$ and combining with (72), we have

\[
|M_t - M_s| \leq \frac{C}{r} \left( \int_{B(0, r/2)} |(M_t - M_s)x - c|^2 \right)^{\frac{1}{2}} \leq \frac{C}{r} \left( \int_{B_c(0, r)} |(M_t - M_s)x - c|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{t} \left( \int_{B_c(0, t)} |u_x - M_t x|^2 \right)^{\frac{1}{2}} + \frac{C}{s} \left( \int_{B_c(0, s)} |u_x - M_s x|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \left\{ \Psi(t) + \Psi(s) \right\} \leq C \Psi(2r),
\]

where the third and the last steps are based on the fact that $s, t \in [r, 2r]$. Due to estimate (77), it’s known that $\psi(r)$ satisfies the second condition in (74). Before we get the estimate (11), we claim that

\[
\|u_x - c\|_{L^2(B_c(0, r))} \leq \|u_x - c\|_{L^2(\Sigma)} \leq C \|\nabla u_x\|_{L^2(B_c(0, 2r))},
\]

in which $c = \int_\Sigma u_x dx$, and $\Sigma$ is a “good” domain such that $B_c(0, r) \subset \Sigma \subset B_c(0, 2r)$ and Poincaré’s inequality holds on $\Sigma$. We can get this $\Sigma$ by avoiding the cusps on the boundary of $B_c(0, \frac{3}{2}r)$, and then the estimate (78) follows from Poincaré’s inequality. Hence, according to Lemma 4.5, for any $r \in [\sqrt{c}, 1)$, we have the following estimate

\[
\frac{1}{r} \inf_{c \in \mathbb{R}} \left( \int_{B_c(0, r)} |u_x - c|^2 \right)^{\frac{1}{2}} \leq \left\{ \Psi(r) + \psi(r) \right\} \leq C \left\{ \Psi(1) + \psi(1) \right\}
\]

\[
= C \left\{ G_\epsilon(1, u_x) + \psi(1) \right\} \leq G_\epsilon(1, u_x) + \inf_{c \in \mathbb{R}} \left( \int_{B_c(0, 1)} |u_x - c|^2 dx \right)^{\frac{1}{2}}.
\]

If we take $M = 0$ and the constant $c = \int_\Sigma u_x dx$ in the right-hand side above, then it follows from (78) that

\[
\frac{1}{r} \inf_{c \in \mathbb{R}} \left( \int_{B_c(0, r)} |u_x - c|^2 \right)^{\frac{1}{2}} \leq C \left\{ \left( \int_{B_c(0, 1)} |u_x - c|^2 \right)^{\frac{1}{2}} + \|F\|_{L^p(B(0, 2))} \right\}
\]

\[
\leq C \left\{ \left( \int_{B_c(0, 2)} |\nabla u_x|^2 \right)^{\frac{1}{2}} + \|F\|_{L^p(B(0, 2))} \right\}.
\]

Therefore, the desired estimate (11) is consequently obtained by the above estimate coupled with Caccioppoli’s inequality (40), and we have completed the whole arguments. \qed

5 Appendix

Lemma 5.1 (local boundary estimates). Suppose that $A$ satisfy the condition (3). Let $u \in H^1_{loc}(Y \cap \omega)$ be a nonnegative solution of $\text{div}A(y, \nabla u) = 0$ in $Y \cap \omega$ with $n \cdot A(y, \nabla u) = 0$ on $\partial \omega$. Then for any $B_r \subset B_R \subset Y$ centered at $\partial \omega$ with $0 < r < R/4$, there hold the local boundedness estimate

\[
\sup_{y \in Y \cap \omega \cap B_r} u(y, \xi) \leq \left( \int_{Y \cap \omega \cap B_R} |u(\cdot, \xi)|^p \right)^{1/p}
\]

for any $p > 0$, and the weak Harnack inequality

\[
\inf_{y \in Y \cap \omega \cap B_r} u(y, \xi) \geq \left( \int_{Y \cap \omega \cap B_R} |u(\cdot, \xi)|^q \right)^{1/q}
\]

is true for $1 < q < \frac{2d}{d-2}$, where the up to constant depends only on $\lambda, d, p$. 

Proof. Step 1. We claim that if \( u \in H^1_{\text{loc}}(Y \cap \omega) \) is a solution satisfying
\[
\int_{Y \cap \omega} A(y, \nabla u) \nabla v dx = 0
\]
for any \( v \in C^1_0(B_R) \) with \( B_R \subset Y \). Then \( u^+ = \max\{u, 0\} \) is a sub-solution, which means that
\[
\int_{Y \cap \omega} A(y, \nabla u^+) \nabla v dx = \int_{Y \cap \omega \cap \{u > 0\}} A(y, \nabla u) \nabla v dx \leq 0
\]
for any \( v \geq 0 \) and \( v \in C^1_0(B_R) \). To see this, let \( v_k = \min\{ku^+, 1\} \). Then for \( \varphi \geq 0, \varphi \in C^1_0(B_R) \) we have
\[
0 = \int_{Y \cap \omega} A(y, \nabla u) \nabla (\varphi v_k) dx = \int_{Y \cap \omega} A(y, \nabla u) \nabla \varphi v_k dx + \int_{Y \cap \omega} A(y, \nabla u) \nabla v_k \varphi dx,
\]
and this together with (3) implies that
\[
\int_{Y \cap \omega} A(y, \nabla u) \nabla \varphi v_k dx = -k \int_{Y \cap \omega \cap \{0 < u^+ \leq \frac{1}{k}\}} A(y, \nabla u) \nabla u^+ \varphi dx \leq -k \mu_0 \int_{Y \cap \omega \cap \{0 < u^+ \leq \frac{1}{k}\}} |\nabla u^+|^2 \varphi \leq 0.
\]
Hence, Let \( k \to \infty \) one may obtain
\[
\int_{Y \cap \omega} A(y, \nabla u^+) \nabla \varphi dx \leq 0.
\]
Step 2. Let \( B_R = B_R(x_0) \) with \( x_0 \in \partial \omega \) such that \( D_R = B_R \cap Y \). Let \( \eta \in C^1_0(B_R) \) be a cutoff function such that \( \eta = 1 \) on \( D_{R/2} \) and \( \eta = 0 \) on \( \mathbb{R}^d \setminus D_R \) with \( |\nabla \eta| \leq C/R \). For any \( \beta \geq 0 \), one may establish that
\[
\int_{D_R} \eta^2 |\nabla u|^2 u^\beta dx \leq C(\mu_0, \mu_1, d, \beta) \int_{D_R} |\nabla \eta|^2 u^{\beta+2} dx.
\]
To do so, it is firstly known by the assumption that \( u = u^+ \), and then we set \( v = \eta^2 u^\beta u > 0 \), where
\[
u_M = \begin{cases} u, & \text{if } 0 < u < M; \\ M, & \text{if } u \geq M. \end{cases}
\]
Then plugging \( v \) back into (82) one may obtain
\[
0 \geq \int_{D_R} A(y, \nabla u) \nabla (\eta^2 u^\beta u) dx
\]
\[
= \int_{D_R} A(y, \nabla u) \eta^2 (\beta u^\beta u u_M + u^\beta u_M \nabla u) dx + 2 \int_{D_R} A(y, \nabla u) \eta \nabla \eta u^\beta u_M dx := I_1 + I_2.
\]
It follows from the condition (3) that
\[
I_1 \geq \beta \mu_0 \int_{D_R} \eta^2 |\nabla u_M|^2 u^\beta dx + \mu_0 \int_{D_R} \eta^2 |\nabla u|^2 u^\beta dx
\]
\[
I_2 \geq -2 \mu_1 \int_{D_R} \eta |\nabla u| |\nabla \eta| u^\beta u_M dx \geq -\frac{\mu_0}{2} \int_{D_R} \eta^2 |\nabla u|^2 u_M^\beta dx - C(\mu_0, \mu_1) \int_{D_R} |\nabla \eta|^2 u^\beta u^2 dx,
\]
where we use Young’s inequality in the last step. Thus, on account of \( I_1 + I_2 \leq 0 \), we arrive at
\[
\frac{\mu_0}{2} \int_{D_R} \eta^2 |\nabla u|^2 u_M^\beta dx + \beta \mu_0 \int_{D_R \cap \{0 < u < M\}} \eta^2 |\nabla u_M|^2 u_M^\beta dx \leq C(\mu_0, \mu_1) \int_{D_R} |\nabla \eta|^2 u^\beta u^{\beta+2} dx,
\]
and letting $M \to \infty$ leads to the stated estimate (83), which is in fact a good formula for the later iteration.

**Step 3.** In this part, we plan to derive the same formula like (83) for the non-negative supersolution which is defined as follows:

$$\int_{D_R} A(y, \nabla u) \nabla v dx \geq 0$$

for any $v \in C^1_0(B_R)$ with $v \geq 0$. To achieve our goal, we set $v = \eta^2 u_k$, where $u_k = u + \frac{1}{k}$ and $\beta < 0$. Hence,

$$2 \int_{D_R} A(y, \nabla u) \eta \nabla \eta u_k^\beta dx + \beta \int_{D_R} \eta^2 A(y, \nabla u) u_k^{\beta-1} \nabla u dx \geq 0.$$

In terms of the condition (3), we obtain

$$-\beta \mu_0 \int_{D_R} \eta^2 |\nabla u|^2 u_k^{\beta-1} \leq 2 \mu_1 \int_{D_R} |\nabla u| |\nabla \eta| u_k^\beta dx$$

$$\leq -\frac{\beta \mu_0}{2} \int_{D_R} |\nabla u|^2 \eta^2 u_k^{\beta-1} dx + C(\mu_0, \mu_1, |\beta|, d) \int_{D_R} |\nabla \eta|^2 u_k^{\beta+1} dx,$$

where we employ Young’s inequality again, and it implies

$$\int_{D_R} \eta^2 |\nabla u|^2 u_k^{\beta-1} dx \leq C(\mu_0, \mu_1, |\beta|, d) \int_{D_R} |\nabla \eta|^2 u_k^{\beta+1} dx.$$

Let $k \to \infty$ and $\tilde{\beta} = \beta - 1$, we have

$$\int_{D_R} \eta^2 |\nabla u|^2 \tilde{u}^{\beta} dx \leq C(\mu_0, \mu_1, |\beta|, d) \int_{D_R} |\nabla \eta|^2 \tilde{u}^{\tilde{\beta}+2} dx. \quad (84)$$

**Step 4.** We claim that (83) implies the local boundedness estimate (80). We first prove the case $p \geq 2$. Let $w = u^{\frac{\tilde{\beta}+1}{2}}$, and then the estimate (83) may be rewrite as

$$\int_{D_R} \eta^2 |\nabla w|^2 dx \lesssim \int_{D_R} |\nabla \eta|^2 w^2 dx,$$

which together with Sobolev’s inequality gives

$$\left( \int_{D_R} |\eta w|^{2\chi} dx \right)^{1/\chi} \lesssim \int_{D_R} |\nabla \eta|^2 w^2 dx,$$

where $\chi = \frac{d}{d-2}$ if $d \geq 3$, and we prefer some $\chi > 2$ in the case of $d = 2$. Recalling $w = u^{\frac{\tilde{\beta}+1}{2}}$, there holds

$$\left( \int_{D_r} (u^{\tilde{\beta}+2})^\chi dx \right)^{1/\chi} \lesssim \frac{1}{(R-r)^2} \int_{D_R} u^{\tilde{\beta}+2} dx.$$

By setting $\gamma = \beta + 2 \geq 2$, the above inequality becomes

$$\left( \int_{D_r} u^{\gamma} dx \right)^{\frac{1}{\gamma}} \lesssim \frac{1}{(R-r)^{\frac{\gamma}{2}}} \left( \int_{D_R} u^\gamma dx \right)^{1/\gamma}.$$
In order to realize the iteration, we prefer \( R_i = \frac{R}{2} + \frac{R}{2^{i+1}} \), \( \rho_i = 2^i \) and \( \rho_i = \chi\rho_{i-1}, i = 0, 1, 2, \ldots \). Hence, one may have the formula
\[
\left( \int_{D_{R_{i+1}}} u^{\rho_{i+1}} dx \right)^{\frac{1}{\rho_{i+1}}} \leq C \frac{1}{\rho_i} \left( \int_{D_{R_i}} u^{\rho_i} dx \right)^{\frac{1}{\rho_i}} \leq C \sum_{i=1}^\infty \left( \int_{D_R} u^2 dx \right)^{\frac{1}{2}},
\]
in which the constant \( C \) is independent of \( R \). Consequently, letting \( i \to \infty \), we have proved the desired estimate (80) for \( p \geq 2 \). The case \( 0 < p < 2 \) easily follows from another iteration argument and we left it to the reader.

**Step 5.** We turn to show the estimate (81) for some \( p_0 > 0 \). In terms of the estimate (84), it is clear to see that \( u^{-1} \) in fact satisfies the estimate (83), which means \( u^{-1} \) plays a role as subsolution. Thus, there holds
\[
\sup_{D_{\frac{R}{2}}} u^{-1} \leq C \left( \int_{D_R} u^{-p} dx \right)^{\frac{1}{p}},
\]
for any \( p > 0 \), and this implies
\[
\inf_{D_{\frac{R}{2}}} u \geq C \left( \int_{D_R} u^{-p} dx \right)^{-\frac{1}{p}} = C \left( \int_{D_R} u^{-p} dx \int_{D_R} u^p dx \right)^{-\frac{1}{p}} \left( \int_{D_R} u^p dx \right)^{-\frac{1}{p}}.
\]
It’s reduced to show for some \( p_0 > 0 \), there holds
\[
\int_{D_R} u^{-p_0} dx \int_{D_R} u^{p_0} dx \leq C,
\]
and it would be done if we proved the following estimate
\[
\int_{D_R} e^{p_0 |w|} dx \leq C, \tag{85}
\]
where \( w = \ln u - \int_{B_R} \ln u \). To see so, we have the following computation,
\[
\int_{D_R} e^{p_0 \ln u - p_0 \int_{D_R} \ln u} dx = \int_{D_R} u^{p_0} e^{-p_0 \int_{D_R} \ln u} \int_{D_R} e^{p_0 \ln u} dx \\
\geq \int_{D_R} u^{p_0} \int_{D_R} e^{-p_0 \ln u} dx = \int_{D_R} u^{p_0} \int_{D_R} u^{-p_0} dx,
\]
where the third step follows from Jensen’s inequality. Now we just need to check (85). In fact, due to John-Nirenberg’s inequality it suffices to verify \( w = \ln u - \int_{B_R} \ln u \in \text{BMO} \). To do so, Recalling the estimate (84), we choose \( \beta = -2 \) and then
\[
\int_{D_R} \eta^2 |\nabla u|^2 u^{-2} dx \leq C \int_{D_R} |\nabla \eta|^2 dx.
\]
Noting that \( \nabla w = \frac{\nabla u}{u} \), the above estimate gives
\[
\int_{D_r} |\nabla w|^2 dx \preceq r^{d-2}.
\]
Thus, it’s clear to see
\[
\int_{D_r} |w - \int_{D_r} w| dx \leq \left( \int_{D_r} |w - \int_{D_r} w|^2 dx \right)^{1/2} \preceq r \left( \int_{D_r} |\nabla w|^2 dx \right)^{1/2} \preceq C.
\]
Hence, \( w \in \text{BMO} \), and the estimate (85) follows, and this leads to the desired estimate (81). We have completed the whole proof. \( \square \)
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