The Background Field Approximation in (quantum) cosmology

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Abstract
We analyze the Hamilton-Jacobi action of gravity and matter in the limit where gravity is treated at the background field approximation. The motivation is to clarify when and how the solutions of the Wheeler-DeWitt equation lead to the Schrödinger equation in a given background. To this end, we determine when and how the total action, solution of the constraint equations of General Relativity, leads to the HJ action for matter in a given background. This is achieved by comparing two neighboring solutions differing slightly in their matter energy content. To first order in the change of the 3-geometries, the change of the gravitational action equals the integral of the matter energy evaluated in the background geometry. Higher order terms are governed by the “susceptibility” of the geometry. These classical properties also apply to quantum cosmology since the conditions which legitimize the use of WKB gravitational waves are concomitant with those governing the validity of the background field approximation.
1 Introduction

There is a question that haunts all works in which both quantum matter effects and the dynamics of gravity play a crucial role, see e.g. [1, 2, 3]: What is the validity of the “semi-classical” Einstein’s equations

\[ R_{\mu\nu} - g_{\mu\nu}R/2 = 8\pi G \langle \Psi|\hat{T}_{\mu\nu}|\Psi\rangle \quad ? \]  

This question concerns the nature and the validity of the approximations that deliver eq. (1) starting from the solutions of the Wheeler-DeWitt equation. There are two different (but related) approximations involved in this reduction process. First, the gravitational wave functions must be described by WKB solutions so as to have their phases governed by the gravitational Hamilton-Jacobi action. Secondly, the spread in energy-momentum of the state \( |\Psi\rangle \) should be small enough so as to legitimate the description of its evolution by the Schrödinger equation

\[ i\partial_t |\Psi\rangle = \int d^3x \, N^\mu \hat{H}_m^\mu (g_{ij}) |\Psi\rangle \]  

in the gravitational background solution of eq. (1). The 4-D metric \( g_{\mu\nu} \) has been split in the 3+1 decomposition\[4\]: \( N^\mu \) is the lapse-shift 4-vector and \( g_{ij} \) the metric of the spacelike hypersurfaces. \( \hat{H}_m^\mu (g_{ij}) = \sqrt{g} \hat{T}_{\mu}^0 \) is the energy-momentum density operator acting on matter states.

To address the validity of this reduction to a single geometry, one should in principle first construct matrix elements of \( \hat{T}_{\mu\nu} \) in an enlarged framework in which each matter state is entangled with its own gravitational wave. Indeed, the Wheeler-DeWitt equation implies that each matter state determines its own gravitational wave (up to the specification of the quantum state of the linearized gravitons state that might be considered as part of the matter states in the present discussion). Only then one can ask in which circumstances it is legitimate to replace these entangled waves by a single one valid for all matter states.

At this point it should be stressed that this procedure is not the one which has been generally adopted in the literature\[5\]-\[12\]. Indeed, in the “conventional” treatment, it is a priori assumed that the solutions of the WDW equation can be factorized into a single WKB wave function describing the gravitational background and something which is identified, to the lowest order in the inverse Planck mass, as the matter wave function evolving in this background according to the Schrödinger equation\[1\]. Higher order terms determine the gravitational corrections to the matter propagation\[10, 12\]. The main problem of this approach is that a part of these corrections is of purely classical character and directly follows from the choice of

\[ ^1 \text{See in particular the recent article } [11]: \text{“In this way the complete solution of the Wheeler-DeWitt equation is split into a product of a purely gravitational piece and a mixed piece representing quantum field theory on that background.”} \]
the gravitational wave which has been factorized. In view of the present difficulties in the interpretation of the solutions of the WDW equation\cite{7, 13, 14}, one should carefully determine the origin of the corrections: Are they intrinsic to the WDW equation or are they induced by the approximations that have been applied to its solutions? This question is particularly relevant upon considering violations of unitarity\cite{9, 10}.

To clarify these issues, another procedure has been adopted in \cite{15, 16, 17}. The solutions of the WDW equation have been expressed as entangled superpositions of matter and gravitational waves and the evolution of the coefficients of this decomposition has been analyzed without factorizing a gravitational wave common to all matter states. After having obtained the dynamical equation governing this evolution, a gravitational background can be introduced by performing a first order expansion in the matter energy change. Then, the new expression of the dynamical equation reduces to the Schrödinger equation. Moreover, in order to minimize the corrections with respect to the exact background-free evolution, the background must be driven by the energy of the matter states under investigation. Finally, no violation of unitarity is produced by the passage to the new description based on a background.

In the present paper, we pursue this analysis which was restricted to mini-superspace. Our aim is to explicitize the specific aspects introduced by the implementation of a Background Field Approximation (BFA) to gravity. In order to separate these aspects from the problems associated with the quantization of gravity, we shall work in a classical framework. However, we shall lead our investigation so as to shed light on the quantum problem which consists in obtaining eq. (1) starting from the solutions of the WDW equation. To this end, we shall analyze its classical counterpart: Starting from the Hamilton-Jacobi action which describes the entangled evolution of matter and gravity, what is the nature and the validity of the approximations that lead to the matter action satisfying the usual Hamilton-Jacobi equation in a given background whose evolution is fixed? Specifically, we shall phrase this question in path integral terms by comparing neighboring Hamilton-Jacobi actions at fixed gravitational end points but differing in matter energy content. Having chosen this comparison, to apply a BFA to gravity is a well defined mathematical procedure.

To first order in the change of the 3-geometries interpolating from the initial to the final condition, the linear change of the gravitational action equals the matter energy change in the background geometry. This is how one recovers the usual HJ action governing matter propagation in a given background starting from the full HJ action, solution of the constraint equations \( \mathcal{H}_\text{gravity} + \mathcal{H}_m = 0 \): The matter energy term \( \int d^3x N^\mu \mathcal{H}_m^\mu \) (c.f. eq. (2)) is delivered by the linear “recoil” of the gravitational \( \int p\dot{q} dt \) term. We recall that the full HJ action contains only the sum of the \( \int p\dot{q} dt \) terms for gravity and matter. This recovery of the matter (light) HJ
action is not specific to gravity+matter systems. It occurs whenever one applies a BFA to the heavy sector of an enlarged entangled system composed of “heavy” and “light” degrees of freedom, see [18].

This classical result is just what we need in quantum cosmology. Indeed, the matter part of the full kernel, solution of the WDW equation, will obey a Schrödinger equation when the gravitational waves are WKB and when a first order expansion in the gravitational change makes sense.

The validity of this first order expansion is controlled by higher order terms. These determine the gravitational corrections to the usual HJ matter action and their importance establishes a posteriori whether it was legitimate to treat gravitational degrees of freedom as “heavy”. They may be analyzed in classical settings. This crucial point deserves more explanations: The second order term has a structure of the form $\Delta \epsilon^2 \times \partial^2 S_G$ where $S_G$ is the gravitational HJ action evaluated on the background. However, in quantum cosmology, the origin of the spread in energy $\Delta \epsilon$ may be purely quantum. Therefore, the structure of these corrections is fully determined by the classical analysis whereas its normalization may not be. This explains why we shall “recover” expressions that have been obtained in the quantum analysis[9, 12]. Moreover we shall show that the corrections to the WKB gravitational wave functions are governed by expressions similar to those governing the BFA. This legitimizes the implementation of a BFA to WKB variables.

Finally, the analysis of this classical problem has its own interest. It makes bridges between the techniques used when studying growth of local fluctuations (in particular as phrased by Steward and Salopek[19]) and those used in quantum cosmology. Moreover it also sheds light on the “issue of time” in (quantum) cosmology[13, 20]. Our analysis indeed does not coincides with the similar investigation performed by Barbour[21].

In this article, we proceed in three steps. In Section 2, we analyze the action in the simple case of an homogeneous matter field in minisuperspace. Furthermore, we restrict ourselves to matter hamiltonians such that the amplitude of the field stays constant. Then the implementation of the BFA can explicitly and easily be performed in gauge independent terms.

In Section 3, we work in a minisuperspace model in which the background is characterized by many degrees of freedom and we no longer put restrictions on the matter hamiltonian. Moreover we make use of techniques very similar to those required by the analysis of the solutions of the WDW equation.

In Section 4, we generalize these results to arbitrary 3-geometries. We shall also comment on the treatment of second order corrections to the BFA which has been recently presented in [12].
2 The background field approximation when matter is characterized by a constant of motion

As a warming up, we first show how a background field contribution is chosen and extracted from the HJ action describing the entangled evolution of matter and gravity. Then, we show how this choice determines the nature of the corrections to the description of matter evolution. To explicitize these corrections, we begin this Section by the analysis of the HJ action of matter and gravity without making any approximation, i.e. in a background free description. In order to simplify this analysis, we shall limit ourselves to cases in which the matter energy is determined by a constant of motion.

We recall that in this Section and the next one, we work in minisuperspace. For these geometries, the metric element can be written as

\[ ds^2 = -N^2(\xi)d\xi^2 + a^2(\xi)d^2\Omega_3 \]  

where \( N(\xi) \) is the lapse function, \( a(\xi) \) the scale factor and \( d^2\Omega_3 \) the constant line element of the homogeneous three surfaces.

The simplest matter system whose evolution is characterized by a constant of motion consists of an homogeneous distribution of comoving oscillators. Classically, these oscillators can be represented be the zero-component momentum of a massive scalar field\[15\]. Their action is

\[ S_M = \int d\xi \left\{ \frac{\left( \frac{\partial_\xi \phi}{2N(\xi)} \right)^2}{2} - \frac{M^2\phi^2}{2} \right\} \]

where \( M \) is the frequency of the oscillations. On the equation of motion, their energy with respect to \( \xi \) is

\[ N(\xi)e = N(\xi)M|A_M|^2 \]

where \( A_M \) is the constant amplitude of the field.

Another example consists of conformally coupled scalar photons of fixed (conformal) momentum \( k \). On the equation of motion, their energy is \( Ne = Nk|A_\gamma|^2/a \), see [13]. These examples can be generalized by considering non quadratic hamiltonians in the field amplitude or by considering oscillations with an arbitrary \( a \)-dependent frequency \( \omega(a) \) treated in the adiabatic approximation so as to guarantee that their energy is \( N(\xi)e_m(a) = N(\xi)\omega(a)|A_m|^2 \). One can of course also consider a sum of these systems.
The background-free description

The residual Bianchi identity (see Chap. 27 in [4]) requires that matter satisfies

\[ \partial_a \epsilon_m(a) = -4\pi P(a)a^2 \]  \hspace{1cm} (6)

where \( P \) is the pressure. In all the above examples, this equation is satisfied because \( \epsilon_m(a) \) characterizes on-shell matter propagation. Of crucial importance is the fact that the matter energy \( \epsilon_m(a(\xi)) \) depends on \( \xi \) through \( a \) only, thereby allowing eq. (6) to have this intrinsic lapse-free writing.

In minisuperspace, the gravitational hamiltonian is given by

\[ N(\xi)H_G = N(\xi) \frac{1}{2aG} \left( -G^2 \pi_a^2 + \kappa a^2 + \Lambda a^4 \right) \]  \hspace{1cm} (7)

where \( G \) is Newton’s constant, \( \pi_a \) the momentum of \( a \), \( \kappa \) is equal to 0, ± for flat, open or closed three surfaces and \( \Lambda \) is the cosmological constant. Thus when the matter energy \( \epsilon_m(a) \) is given, the HJ constraint equation, \( H_G + H_m = 0 \), reads

\[ -G^2(\partial_a S_G(a, \epsilon_m))^2 + \kappa a^2 + \Lambda a^4 + 2Ga \epsilon_m(a) = 0 \]  \hspace{1cm} (8)

The lapse \( N \) has also disappeared from this equation. In this Section, it will not be re-introduced thereby guaranteeing a gauge independent description.

The HJ action \( S_G \), solution of eq. (8), is

\[ S_G(a_2, \epsilon_m) = \int_{a_1}^{a_2} da \pi(a, \epsilon_m) \]  \hspace{1cm} (9)

where \( \pi(a, \epsilon_m) \) is the on-shell momentum driven by \( \epsilon_m(a) \)

\[ \pi(a, \epsilon_m) = -G^{-1}\sqrt{\kappa a^2 + \Lambda a^4 + 2Ga\epsilon_m(a)} \]  \hspace{1cm} (10)

Notice that the minus sign characterizes expanding universes. It arises from the unusual sign of the kinetic gravitational energy in eq. (8).

The total HJ action (gravity + matter), solution of the constraint equation, is thus

\[ S_T = S_G(a_2, \epsilon_m) + S_m(\phi_2, \epsilon_m) = \int_{a_1}^{a_2} da \pi(a, \epsilon_m) + \int_{\phi_1}^{\phi_2} d\phi p_\phi(\phi, \epsilon_m) \]  \hspace{1cm} (11)

where \( p_\phi(\phi, \epsilon_m) \) is the matter momentum, determined by the energy condition \( H_m(p_\phi, \phi, a) = \epsilon_m(a) \). For definiteness and simplicity, from now on we consider only the case of harmonic oscillators. In this case, one has \( p_\phi(\phi, \epsilon) = \sqrt{2\epsilon - M^2 \phi^2} \), see eqs. (4, 5) and compare it with eq. (10). Conformal photons or more general oscillators are treated along similar lines.
At this point we have imposed that the total energy vanishes and that both gravitational and matter propagations occur along classical orbits. However, when working with fixed the end point conditions \((a_1, \phi_1; a_2, \phi_2)\), \(S_T\) is not fully extremal: It should still be extremized with respect to variations of \(\epsilon\). This stationary condition determines the “saddle point” value \(\bar{\epsilon} = \bar{\epsilon}(a_1, \phi_1; a_2, \phi_2)\) and reads
\[
- \partial_\epsilon \int_{a_1}^{a_2} da \pi(a, \epsilon)|_{\epsilon=\bar{\epsilon}} = \partial_\phi \int_{\phi_1}^{\phi_2} d\phi \frac{p_\phi(\phi, \epsilon)}{G\pi(a, \epsilon)}|_{\epsilon=\bar{\epsilon}}
- \int_{a_1}^{a_2} da \frac{a}{G\pi(a, \epsilon)} = \int_{\phi_1}^{\phi_2} d\phi \frac{1}{p_\phi(\phi, \bar{\epsilon})}
\]
(12)

On the left hand side, the gravitational integral has defined \(t(a_2, \bar{\epsilon})\), the proper time lapse from \(a_1\) calculated from the propagation of \(a\) driven by the “saddle point” energy \(\bar{\epsilon}\).

On the r.h.s. the matter integral has defined the “cesium” time \(t_{\text{cesium}}(\phi_2, \bar{\epsilon})\).

In the case of massive harmonic oscillators at rest, it equals to the period \(2\pi/M\) times the number of periods. (In the case of conformal photons, one would have obtained respectively the conformal time on the l.h.s. and the number of conformal periods on the r.h.s., c.f. [15].)

Is it through the equality of these times that one obtains the orbit \(\phi_2 = \phi(a_2, \bar{\epsilon})\) characterized by \(\bar{\epsilon}\) and which passes by \(\phi_1\) at \(a_1\). Once this correlation is obtained, it can be recast in the usual way parametrized by \(t\): \(\phi_t(t), a_t(t)\). In this writing, the dependence in \(\bar{\epsilon}\) is “blamed” on the dynamical variables.

The unusual aspect in this derivation of the trajectory is that \(t\) has arisen \(a\) posteriori through the dynamics of both matter and gravity. This is because the usual linear term in \(E_t\) external was not present in the total action \(S_T\), eq. (11). Rather both \(S_G\) and \(S_m\) depend non-linearly on \(\epsilon\). Up to now indeed \(S_G\) and \(S_m\) have been treated on the same footings. By the implementation of a BFA, this symmetry will be broken since the BFA will concern only \(S_G\). In other words, only \(a\) will be treated as an “heavy” degree of freedom. Then, the notion of an “external” time to matter (light) dynamics will be recovered from the heavy character of \(a\). Notice that the division into light and heavy variables is not a question of taste: The physical circumstances decide whether it makes mathematical sense. Notice also that this procedure based on a division between light and heavy variables does not coincide with Barbour’s approach [21] to time in cosmology. In his approach, time is a redundant variables which depends on all degrees of freedom.

\[\text{There might be a discrete number of solutions to eq. (12) since we are dealing with periodic matter systems. However this multiplicity does not affect what follows since we can always choose to work with the lowest value.}\]
The background field approximation

We now apply a BFA to the gravitational part of the full HJ action $S_G + S_m$. To this end, one should first choose the reference background. To choose it, one needs to focus on a given “physical” problem. Indeed the choice of the background should be tied up to the problem under investigation and made so as to minimize the errors induced by the approximation process. This procedure has been applied in quantum cosmology [16]: Upon computing a given transition amplitude from an initial matter state to a final one, it was shown that the “best” background is determined symmetrically by the energy contents of the initial and final states. Any other choice leads to greater systematic deviations with respect to the exact (background free) description.

We use the same philosophy in the classical framework: We want to determine the simplified description of matter propagation in a single background geometry when the final values of $\phi_2$ are centered around a given $\phi_2^B$ and when the three other conditions ($a_1$, $\phi_1$; $a_2$) are held fixed. This imposes to work with the background determined by the end point values ($a_1$, $\phi_1$; $a_2$, $\phi_2^B$) since any other choice leads to systematic deviations. To obtain the simplified matter description, we expand only the gravitational part of the HJ action in power of $\Delta \epsilon = \epsilon - \bar{\epsilon}^B = \epsilon - \bar{\epsilon}(a_1, \phi_1; a_2, \phi_2^B)$.

(From now on, we use the subscript $\bar{B}$ to identify quantities evaluated on their background value.) Notice that we have kept $\epsilon$ free since the new description of matter evolution still involves the extremization of $S_T$ with respect to it.

By developing $S_G$ to first order in $\Delta \epsilon = \epsilon - \bar{\epsilon}^B$, one obtains

$$S_G(a_2, \epsilon) + S_M(\phi_2, \epsilon) = S_G(a_2, \bar{\epsilon}^B) + \int_{a_1}^{a_2} da \left( (\epsilon - \bar{\epsilon}^B) \partial_\epsilon \pi(a, \epsilon) \right)_{\epsilon = \bar{\epsilon}^B} + S_M(\phi_2, \epsilon) \quad (13)$$

As in the absence of approximation, $S_G + S_m$ must be stationary with respect to changes in $\epsilon$. Then, as before, $\partial_\epsilon S_T = 0$ determines the saddle point energy $\bar{\epsilon}$. However, the new expression of this condition now reads

$$t(a_2, \bar{\epsilon}^B) = \int_{\phi_1}^{\phi_2} d\phi \frac{1}{p_\phi(\phi, \bar{\epsilon})} \quad (14)$$

in the place of eq. [12]. The only difference with that equation is that $\bar{\epsilon}$ on the l.h.s. has been replaced by $\bar{\epsilon}^B$ for all values of $\phi_2$. Thus $t(a_2, \bar{\epsilon}^B)$, the background time, is external to matter dynamics and the orbit directly expressed as $\phi_2 = \phi_2(t)$.

The errors induced by this BFA are of two kinds: First $\bar{\epsilon}$, solution of eq. [14], differs (slightly when $a$ is heavy) from the solution of eq. [12] and secondly, $t(a_2, \bar{\epsilon}^B)$ is not the time in the actual geometry. This is the price to pay when one has chosen to work with a given background common for different values of $\phi_2$. It should be noticed that neither error affects the orbit $\phi_2 = \phi_2(t)$ since both are concerned with the choice of the orbit through $\bar{\epsilon}$ and the parametrization of $a_2$ by $t(a_2, \bar{\epsilon}^B)$. This absence of deformations of the orbit follows from the existence of the constant $\epsilon$.
which allows to separate the action $S_T$ into two unconnected pieces. This will no longer be the case in the next Sections.

Eq. (14) has been obtained from the HJ action, eq. (11), in three steps. One has first chosen the reference background driven by $\bar{\epsilon}^B$. One has then expand the gravitational part of the action to first order in $\epsilon - \bar{\epsilon}^B$ and finally requires extremization with respect to $\epsilon$. This is how the stationary condition determining matter propagation in the background $a^B(t) = a_{\bar{\epsilon}^B}(t)$ is obtained from the total action $S_G + S_M$ governing the entangled dynamics of matter and gravity.

This procedure generalizes the usual test-particle (or test-field) approximation for two reasons. First, in that approximation, there is no influence of the matter energy on the determination of the background evolution. Here instead, the “mean” matter energy $\bar{\epsilon}^B$ does determine $a^B(t)$, see eq. (10). Secondly, because of this partial backreaction effect, the corrections to the BFA differ from those to the test-particle approximation, see below.

**The validity of the background field approximation**

The validity of the approximate description based on eq. (14) is governed by the influence of the higher order terms in $\Delta \epsilon$. The quadratic term of the expansion of $S_G(a_2, \epsilon)$ around the background contribution $S_G(a_2, \bar{\epsilon}^B)$ is

$$\frac{(\Delta \epsilon)^2}{2} \partial^2 \epsilon S_G(a_2, \epsilon)_{\epsilon=\bar{\epsilon}^B} = \frac{(\Delta \epsilon)^2}{2} \partial_t t(a_2, \epsilon)_{\epsilon=\bar{\epsilon}^B} = \frac{(\Delta \epsilon)^2}{2} \int_{a_1}^{a_2} da a \frac{\partial \pi(a, \epsilon)}{G \pi^2(a, \epsilon)}_{\epsilon=\bar{\epsilon}^B}$$

$$= \int_{0}^{t^B(a_2)} dt (\Delta \epsilon) \left( \frac{G \Delta \epsilon}{2a^B(t) + \Lambda[a^B(t)]^3 + 2G \bar{\epsilon}^B} \right)$$  (15)

The second expression makes clear that this correction is governed by the “susceptibility” of the background geometry. It is indeed controlled by the amount of change in the background time lapse $dt = d\epsilon \partial_t t$ induced by the change in the matter energy $d\epsilon$.

In the last expression, we have factorized the contribution of the linear term $(dt \Delta \epsilon)$ so as to identify the relative correction to it. In order to give an estimate to this correction factor, it is appropriate to consider matter dominated universes for which $G \bar{\epsilon}^B \gg \kappa a + \Lambda a^3$. Then the correction factor is simply $\Delta \epsilon/4 \bar{\epsilon}^B$: the change in the matter energy divided by the matter energy $\bar{\epsilon}^B$ which drives the background. Notice that this ratio is independent of $G$. Thus the BFA is not an expansion in powers of $G$. Similarly, in quantum cosmology, we shall see that the BFA is not an expansion in the inverse Planck mass. On the contrary, the test-particle approximation can be viewed as the linearized theory in $(G \epsilon)$ since $\epsilon$ does not enter into the determination of the background. Eq. (15) also shows that the

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\[3\] In the language of MTW[4], it is called the “condition of constructive interference” in reference to the machinery of quantum mechanics. Upon working in quantum cosmology, we shall see indeed that both eqs. (12, 14) appear with this status.
best background geometry to describe matter propagation when the values of $\phi_2$ are centered around $\phi_2^B$ is determined by $\bar{\epsilon} = \bar{\epsilon}(a_2, \phi_2^B)$.

To determine the influence of the quadratic term, one should add it to the l.h.s. of eq. (14) and compute the new value of the saddle energy $\bar{\epsilon}$. One finds that its effect is to shift the saddle point energy $\bar{\epsilon}$ by $(\bar{\epsilon} - \bar{\epsilon}^B)\partial_t t^B / \partial t_{\text{cesium}}$. Thus, when $\phi$ is a light and rapidly changing variable when compared to $a$, this shift is negligible since $\partial_t t^B \ll \partial_t t_{\text{cesium}}$. This inequality shows that the value of the correction term does not control alone the validity of the BFA for describing matter evolution. Indeed this validity is governed by a ratio in which the lightness of $\phi$ also intervenes.

The fact that the change of $\bar{\epsilon}$ is controlled by a ratio of “specific heats” is reminiscent to statistical mechanics: Consider a microcanonical ensemble of energy $E$ composed of two systems 1 and 2 with densities of states given by $\Omega(E_1) = e^{S(E_1)}$ and $\omega(E_2) = e^{s(E_2)}$. When evaluating the density of states of the total system given by $\int d\epsilon \Omega(E - \epsilon) \omega(\epsilon)$ by a saddle point approximation, one obtains the equality of the temperatures: $\partial_\epsilon S(E - \epsilon) + \partial_\epsilon s(\epsilon) = 0$ which is the equivalent of eq. (12), see App. B in [15] for more details. Moreover, close to equilibrium and when $E - \bar{\epsilon} \gg \bar{\epsilon}$, one can treat the big system 1 at the BFA, i.e. develop $S(E)$ to first order in $\Delta \epsilon$. Then the errors induced by this canonical approximation are governed by the ratio of the specific heats $\partial^2 S / \partial^2 s$. For homogeneous systems this ratio scales like the ratio of the volumes. Therefore the BFA can be considered as a large reservoir limit.

We conclude this subsection by mentioning an interesting question [24]: What happens close to a turning point? In this case the quadratic correction term, eq. (15), becomes unbounded since the gravitational momentum $\pi(a, \bar{\epsilon}^B)$ vanishes. This signals the “instability” of the parametrization of matter evolution by $t(a, \bar{\epsilon}^B)$. Thus, close to a turning point, $t(a, \bar{\epsilon}^B)$ is not a good parameter. A very good parameter is $t(\pi_a, \bar{\epsilon}^B)$ obtained by performing a Legendre transformation to $S_G$ so as to fix the end value of the momentum rather than $a_2$. A similar situation occurs upon considering the validity of the WKB approximation: Close to a turning point, this approximation becomes also completely inappropriate. However the physics is not directly affected since, it is the state (the ket) and not its expression in the position representation which governs physical processes through matrix elements.

**Application to quantum cosmology**

The application of these results to quantum cosmology is straightforward if one postulates that gravitational waves are WKB. In this case indeed, their phases and norms are determined by the gravitational HJ action, eq. (9). At first sight it might seem that the validity of the WKB approximation introduce new restrictions. However, as pointed out in [8] and further clarified in [13], the conditions legitimizing the use of WKB gravitational waves are concomitant with those gov-
erning the validity of the BFA: Both approximations are valid when the relative change of the kinetical energy of the scale factor is negligible.

Upon quantizing matter systems governed classically by constants of motion, one obtains stationary eigenstates labeled by a discrete parameter \(n\) (e.g. the occupation number) and characterized by an eigenvalue:

\[
\hat{H}_m(a)|n\rangle = \epsilon(n)|n\rangle
\]  

In the time-free description, stationarity means that \(\partial_{\phi} |n\rangle = 0\). This implies that the kernel, solution of the WDW equation

\[
(\hat{H}_G + \hat{H}_m) K(a, \phi; a_1, \phi_1) = 0
\]

can be decomposed as

\[
K(a, \phi; a_1, \phi_1) = \sum_n K_n(a; a_1) K_n(\phi; \phi_1)
\]

wherein the matter kernel \(K_n(\phi; \phi_1)\) is equal to \(\langle \phi|n\rangle\langle n|\phi_1\rangle\) as in Schrödinger settings. Moreover since the structure of \(K\) is still a sum of products, the BFA will concern only the gravitational kernels \(K_n(a; a_1)\). This is true only for the simple mini-superspace cases we are dealing with.

The gravitational kernel \(K_n(a; a_1)\) is a solution the quantized version of eq. \(\Phi\), i.e.

\[
\left(G^2 \partial_a^2 + \kappa a^2 + \Lambda a^4 + 2Ga \epsilon(n)\right) K_n(a; a_1) = 0
\]

In the WKB approximation, for \(a > a_1\), it is equal to

\[
K_n(a; a_1) = \frac{\exp \left[ i \int_{a_1}^a da' \frac{\pi(a', \epsilon(n))}{\pi(a_1, \epsilon(n))} \right]}{\sqrt{\pi(a_1, \epsilon(n)) \pi(a_1, \epsilon(n)) \pi(a_2, \epsilon(n))}}
\]

We now determine the behavior of kernel \(K(a, \phi; a_1, \phi_1)\) when \(a\) is equal to \(a_2\) and \(\phi\) close to \(\phi_2^B\). To make contact with classical mechanics, we use a saddle point approximation (i.e. a constructive interference condition) to estimate the summation over \(n\) in eq. \(\Phi\). When using the WKB expressions eq. \(\Phi\), the equation which fixes the saddle point value \(\bar{n}^B\) is equal to that one would have obtained by treating gravity at BFA if the background was the solution driven by the saddle point energy \(\epsilon^B = \epsilon(\bar{n}^B)\). This is clearly exhibited by factorizing \(K_{\bar{n}^B}(a_2; a_1)\) and developing the gravitational kernels in \(\Delta n = n - \bar{n}^B\)

\[
K(a_2, \phi_2^B; a_1, \phi_1) = K_{\bar{n}^B}(a_2; a_1) \times \sum_n K_n(\phi_2^B; \phi_1) \sqrt{\frac{\pi(a_2, \epsilon(\bar{n}^B)) \pi(a_1, \epsilon(\bar{n}^B))}{\pi(a_2, \epsilon(n)) \pi(a_1, \epsilon(n))}} \exp \left[ i \int_{a_1}^{a_2} da \left\{ \pi(a, \epsilon(n)) - \pi(a, \epsilon(\bar{n}^B)) \right\} \right]
\]
\[ K_\text{nB}(a_2; a_1) \times \sum_n K_n(\phi_2^B; \phi_1) \left\{ 1 + O \left( \frac{\Delta_n \epsilon}{\pi^2(a, \epsilon(\bar{n}_B))} \right) \right\} \]

\[ \exp \left[ -i \int_0^{t_{nB}(a_2)} dt \{ \epsilon(n) - \epsilon(\bar{n}) \} \left\{ 1 + O \left( \frac{\Delta \epsilon_n \pi^2}{\pi^2} \right) \right\} \right] \]  

where \( \Delta_n \epsilon = \epsilon(n) - \epsilon(\bar{n}_B) \). The stationary phase conditions are respectively

\[ \frac{\partial_n}{\partial n} \left\{ \int_{a_1}^{a_2} da \pi(a, \epsilon(n)) + i \ln K_n(\phi_2^B; \phi_1) \right\} = 0 \]  

(22)

\[ \left( \frac{d\epsilon(n)}{dn} \right) t_{nB}(a_2) + i \partial_n \ln K_n(\phi_2^B; \phi_1) = 0 \]  

(23)

Eq. (22) is obtained from the first expression of eq. (21) and does no rely on any BFA. It can be viewed as the quantum version of eq. (12). Instead eq. (23) follows from the second expression in which a BFA has been already performed, c.f. eq. (14). This later condition is the usual “constructive interference condition” obtained from QFT in the background characterized by \( t_{nB}(a) \). By construction, these conditions coincide since the expansion in \( \Delta n \) was performed precisely around the saddle point value \( \bar{n}_B \).

From this agreement and the former classical analysis leading to eq. (14), one immediately deduces that the saddle point phase of the kernel considered as a function of \( \phi_2 \) in a neighborhood of \( \phi_2^B \) is equal to the BFA phase when non linear terms in \( \bar{n} - \bar{n}_B \) are discarded. (Notice that these correction terms are governed by \( \Delta_n \epsilon / \pi^2 \) as in the classical case.) Moreover, when they are discarded, the spread in \( n - \bar{n} \) agrees with the BFA spread since in both cases (with or without BFA) it is determined by the (second derivative of the log of the) matter kernel \( K_n(\phi; \phi_1) \) only. Notice that this spread is intrinsic, i.e. determined by the physical circumstances and \( \hbar \). This is to be opposed to the classical spread \( \Delta \epsilon \) in the former subsection which was fixed by the “external” choice of \( \phi_2 \). However, the mathematics involved are the same.

The above three agreements imply that \( K(a_2, \phi_2; a_1, \phi_1) \) satisfies the Schrödinger equation in the background geometry \( a^B(t) \) for \( \phi_2 \) sufficiently close to the background value \( \phi_2^B \) so as to legitimize the neglection of higher order terms in \( \Delta n \).

Notice however the unconventional property of this Schrödinger equation. Since the background geometry is dynamically determined by \( \bar{n} \), c.f. eq. (22), it changes when \( a_2 \) varies except if one tunes \( \phi_2^B \) so as to stay on the orbit \( \phi_2 = \phi(a_2, \bar{n}_B) \). Because of this, it is appropriate to work with fixed \( n \) rather than with fixed end point values on the matter field. This is what has been adopted in \([14, 17]\) to study transition amplitudes in quantum cosmology. This is also what we shall adopt in the next Sections in the definition of the background.

As in classical mechanics, higher order terms in \( \Delta n \) (or \( \Delta \epsilon \)) induces corrections to the determination of the saddle point value \( \bar{n} \). To determine their physical
consequences it is mandatory to introduce interactions leading to transitions among states, see [16, 17]. The main lesson of these works is that it is not necessary to first factorize the mean gravitational kernel and then search for the description of matter evolution\(^4\). Indeed, the WDW equation determines their evolution directly in terms of \(a\), i.e. in a background free description, see eq. (18) for a simple example. However, as in eq. (21), one can always \textit{a posteriori} perform a BFA (an expansion in \(n - n^B\)) in order to obtain a more conventional description formulated in a single background[20]. Doing so one induces non linear terms in \(n - n^B\) whose role among other things is to introduce phase shifts governed by \(\partial^2 S_G\).

In resume, eq. (21) shows that the corrections to the kernel with respect to its Schrödinger BFA expression are of two kinds: those due to quadratic and higher order terms in \(\Delta n\) and those due to the WKB approximation, eq. (20). In a next paper, we shall analyze the consequences of both types of corrections on possible violations of unitarity.

3 The background field approximation for general mini-superspace models

In this Section we no longer impose that matter dynamics are governed by a constant of motion. Therefore we are obliged to apply the BFA to extremized actions since we have lost the possibility of applying a BFA before having extremized the total action with respect to \(\epsilon\), see eq. (13). There are two ways to implement this approximation.

The first one consists in keeping the lapse function \(N(\xi)\) and working with trajectories parametrized by \(\xi\). The main advantage of this approach is that the whole space-time structure of the background emerges upon applying the BFA to gravity. Indeed since both \(a^B(\xi)\) and \(N^B(\xi)\) are determined, one obtains matter propagation in a curved background which possesses a given time parametrization.

A second advantage is that it offers the possibility of considering “off-shell” matter configurations. At first sight it might seem completely absurd to consider the gravitational reaction to off-shell matter configurations since r.h.s of Einstein’s equation will be incompatible with the Bianchi identity, i.e. eq. (3) will not be satisfied. However, in the path integral, most of the matter configurations from \(\phi_1\) to \(\phi_2\) do violate this equation. Therefore it is a legitimate question to

\(^4\) It is even a nuisance! The reason is that transition amplitudes are independent of the mean energy: they only depend of the energy of the two states characterizing the transition. However, when searching for the description of these transitions in the background geometry driven by the mean energy, one obtains series in which all terms depend parametrically on this energy. This is nothing but the consequence of having first factorized the mean gravitational kernel. By no means it is an intrinsic properties of the solutions of the WDW equation.
ask whether the action governing these configurations can be conceived as arising from an enlarged action governing the evolution of both gravity and matter. This analysis is performed in Appendix B.

The second approach is more intrinsic since it is based on the HJ action rather than on trajectories. It does not make use of the lapse function and furthermore proceeds along lines very similar to those used in quantum cosmology. Therefore, we shall present here this second approach. The first one is developed in Appendix A. The reader unfamiliar with HJ techniques is invited to first read this Appendix which proceeds along more conventional mechanical lines.

In order to prepare the analysis of the next Section which deals with general 3-surfaces, we work with backgrounds described by many degrees of freedom, $X^i$. These consist on the scale factor $a$, on the homogeneous part of some massive inflaton field and possibly also on gravitational Fourier modes\[5\] highly excited. The multidimensional character of the background trajectories will introduce new difficulties since there are now “transversal” directions to the background orbits\[11,12\].

The total hamiltonian we are considering is of the form

$$N(\xi)H_T = N(\xi) \left[ H_X(\pi_i, X^i) + H_m(\phi, p_\phi, X^i) \right]$$

(24)

We only impose that $H_X$ and $H_m$ be quadratic in the momenta $\pi_i$ and $p_\phi$. Notice also that we have not included in $H_m$ a “non-minimal” coupling to $\pi_i$. The inclusion of this additional coupling presents no difficulty however.

The total action satisfies the HJ constraint equation

$$\frac{1}{2} C^{ij}(X^k) \partial_{X^i} S_T \partial_{X^j} S_T + V(X^k) + H_m(\phi, \partial_\phi S_T, X^i) = 0$$

(25)

As in eq. (11), $S_T$ can be written as a sum of terms

$$S_T = S_X + S_m = \int_{X^i_1}^{X^i_2} dX^i \pi_i(X^i, \{X^j_2, \phi_2\}) + \int_{\phi_1}^{\phi_2} d\phi p_\phi(\phi, \{X^i_2, \phi_2\})$$

(26)

where each on-shell momentum is parametrized by its conjugate position. In brackets we have indicated that they depend parametrically of the end point values of all variables. It is on this dependence that the BFA will act.

\[5\] It might seem a little bit antinomic to consider a plurality of backgrounds. Moreover, there is a simple way to reject this plurality: It consists in treating all but one coordinates $X^i$ (say e.g. $a$) as “matter” degrees of freedom. One would then recover a situation similar to that of the former Section in which the “matter” energy univocally specifies the orbits of the remaining background coordinate. This is effectively what has been adopted by Halliwell and Hawking in \[5\]. In what follows however, we shall proceed with background orbits characterized by many coordinates $X^i$ in order to confront the new difficulties.
Before proceeding to the implementation of the BFA, we should carefully determine the background(s). In the former Section, this was easily achieved by the specification of the constant of motion $\epsilon^B$. As we discussed in the “quantum cosmology” subsection, this amounted to work with the end value of $\phi$ evaluated along the orbit characterized by $\epsilon^B$. In the present situation without constant of motion, we define the background by fixing the initial value of the matter energy $H_m(a_1, \phi_1, p_\phi)$, i.e. we fix the initial momentum $p_{\phi,1} = p^B_\phi$. Then the end value of the $\phi$ field is evaluated along the orbit specified by $p^B_\phi$ as well as the end-point values $X^i_2$. Indeed the specification of the matter energy no longer suffices to specify univocally the orbit. Therefore the set of $X^i_2$ will both determine the time lapse along the trajectory (as $a$ did in the former Section) but also determine the orbit, i.e. the location in the “transversal” directions.

With these aspects in mind, we now search for the approximate description of matter evolution when the change in $\delta \pi^i = \pi^i(X^i, \{X^j_2, \phi_2\}) - \pi^i(X^i, \{X^j_2, p^B_\phi\})$ induced by the change in the matter configurations is treated to first order. To this end, we write

$$S_T(X^i_2, \phi_2) = S^B_X(X^i_2) + \Delta S(X^i_2, \phi_2)$$  \hspace{1cm} (27)

where $S^B_X(X^i_2)$ is the background action given by $\int_{X^i_1}^{X^i_2} dX^i \pi^i(X^i, \{X^j_2, p^B_\phi\})$. By inserting eq. (27) into the constraint eq. (25) one obtains

$$C^{ij}(X^k) \partial^i_X S^B_X \partial^j_X \Delta S + H_m(\phi, \partial^i_\phi \Delta S, X^i) - H^B_m(X^i)$$

$$+ \frac{1}{2} C^{ij}(X^k) \partial^i_X \Delta S \partial^j_X \Delta S = 0$$  \hspace{1cm} (28)

where we have used the fact that $S^B_X$ is the solution of eq. (25) driven by the background matter energy $H^B_m(X^i)$. Notice that $H^B_m(X^i)$ is orbit dependent but nevertheless acts as a potential energy along each orbit. Moreover, along each orbit, the proper time lapse to go from $X^i_1, \phi_1, p^B_\phi$ to $X^i_2$ satisfies (by definition)

$$C^{ij}(X^k) \partial^i_X S^B_X \partial^j_X t^B(X^i) = 1$$  \hspace{1cm} (29)

This is just another way to express the relationship between velocities and momenta which is usually written as $dX^i / dt = C^{ij}_t \pi^j_\phi$ when one has $t$ at our disposal from the outset. Here instead, $t^B(X^i)$ is re-introduced and defined by $-\partial^i E S^B_X$ where $dE$ is an infinitesimal constant change of the matter energy $H^B_m$. Then by varying eq. (28) applied to $S^B_X$ with respect to $dH^B_m = dE$, one obtains eq. (29).

In the present formalism, to apply a BFA consists in neglecting the last term of eq. (28). This amounts to assume that $\partial^i_X \Delta S \ll \partial^i_X S^B_X$, i.e. $\delta \pi^i \ll \pi^i_t$. Then, using eq. (29), the first line of eq. (28) reads

$$\partial_t \Delta S(\phi, X^i \phi(t)) + H_m(\phi, \partial^i_\phi \Delta S, X^i \phi(t)) - H^B_m(t) = 0$$  \hspace{1cm} (30)
Thus, to first order in $\delta \pi_i$, $\Delta S$ satisfies the HJ equation for the matter field in the background determined by $X_i^2, p^B_\phi$. Then, as usual, it can be decomposed as

$$\Delta S(\phi_2, t) = \int_{\phi_1}^{\phi_2} d\phi p_\phi(\phi, \{\phi_2\}) - \int_0^t dt' \left\{ H_m(\phi, p_\phi, X^{iB}(t')) - H_m^B(t') \right\}$$  \hspace{1cm} (31)

Together with eqs. (26, 27), this expression shows that the energy term $-\Delta H_m dt$ originates from the linear recoil of gravity, $\delta \pi_i dX^i$, induced by the change in the matter trajectory.

Notice that all heavy degrees of freedom $X^i$ contribute to the determination of the background time. In this we agree with what has been advocated in [21] safe for the fact that the light variables only intervene in the definition of time through their “mean” energy $\tilde{H}_m^B$. To our opinion however, the important lesson of eq. (30) is not that $t^B(X^i)$ is a “redundant” parameter but that it is precisely the one which does the job: to parametrize the dynamical evolution of the light variables that did not directly participate in its determination. The reasons for which $t^B(X^i)$ is the right parameter are the following. First, the change of the light trajectory that we want to parametrize induces through the constraint equation a change in the energy available to the heavy variables. Secondly $t^B(X^i)$ is defined by $-\partial E_S^B X_i$, i.e. the conjugate to an infinitesimal change of that available energy. Then to first order in the finite matter energy change, these two reasons imply eq. (30).

Notice also that in spite of the fact that the new matter configuration defines a new background orbit, the “transversal” part of the change does not show up to first order in $\delta \pi_i$ since it is projected out by the contraction with the gradient $C^{ij}_k \partial_i S^B \partial_j$, which is perpendicular to the surfaces $S^B_X = \text{Const}$. This is a well known feature of first order perturbation theory in classical mechanics, see e.g. [25].

We now verify that the neglected term in eq. (30) governs back-reaction effects, i.e. the modifications of the matter dynamics induced by its own energy through the “susceptibility” of the background. To estimate this term perturbatively we use eq. (31) and we obtain

$$\frac{1}{2} C^{ij}_k (X^k) \partial_i \Delta S \partial_j \Delta S = \frac{1}{2} C^{ij}_k (X^k) \partial_i t^B \partial_j t^B (\Delta H_m)^2$$  \hspace{1cm} (32)

---

6 As pointed out to me by Serge Massar, eq. (32) is exact only in the limit of slowly varying $\Delta H_m(X^i)$, i.e. when $\Delta H_m \gg (X^2_2 - X^1_2) \partial X, \Delta H_m$. This is the usual condition appearing in adiabatic treatments[22]. Thus, for rapidly varying $\Delta H_m$ or extremely long trajectories, one should exactly take into account the transversal dependence of $\Delta H_m(X^i)$ associated with the change in the end point value $X^2$. This amounts to compute the Jacobi fields of $S^B_X$. The reader interested by the exact calculation will consult Section 5 of [12]. Let us mention here that it is necessary to perform the exact calculation only when the variables $X^i$ are not sufficiently “heavy”. Notice also that our estimate, eq. (32), does not coincide with that of [12]. In that article, the longitudinal is defined by the surfaces $S^B_X = \text{Const}$, whereas here it is defined by $t^B = \text{Const}$. A detailed comparison of these estimates goes beyond the scope of the present paper. For further discussion see next Section.
This term generalizes what we obtained in eq. (13). Indeed for backgrounds characterized by a single variable, one has $C(X)\partial_X t^B \partial_X t^B = 1/C(X)\pi^2_X$ which is the integrand of the fourth expression in eq. (15). Moreover, in multi dimensional cases, when the change in energy $\Delta H_m$ is constant, eq. (32) still contributes to $\Delta S$ given in eq. (31) by the addition of $(\Delta H_m)^2\partial_E t^B/2$. By $\partial_E t^B$ we designate $\partial^2 E_S/B X$. To evaluate it, one should vary eq. (29) with respect to $dH_B = dE$. One finds,

$$
\partial_E \left\{ C^{ij}(X^k)\partial_X t^B \partial_X t^B \right\} = \frac{1}{C(X)\pi^2_X} \pi^2_X X^i t^B - C^{ij}(X^k)\partial_X t^B \partial_X t^B = 0
$$

(33)

Using once more eq. (29) to rewrite the second term of eq. (33), one gets

$$
\partial_E t^B = \int_0^t dt' C^{ij}(X^k)\partial_X t^B \partial_X t^B \partial_X t^B
$$

(34)

From this equation one verifies that eq. (32) introduces a source term to eq. (30) which integrated over $t$ gives the announced result in the cases of constant $\Delta H_m$.

When the last term of eq. (28) is replaced by the r.h.s. of eq. (32), one obtains an equation for the matter propagation since all $X^i$ are evaluated on their background values and parametrized by $t$. Thus one can view the last three terms of eq. (28) as defining the “effective” matter hamiltonian. This hamiltonian gives rise to unusual Euler-Lagrange equation since it is quartic in $p_\phi = \partial_\phi \Delta S$. This results from the “elimination” of the degrees of freedom $X^i$. A simple procedure to handle this hamiltonian consists in treating the correction term perturbatively, in a manner similar to what is done in a Hartree approximation and also in agreement to what we have learned in the former Section. Namely, the quadratic correction term only shifts the saddle point energy $\bar{\epsilon}$ by a term proportional to $\partial^2 S_G/\partial^2 S_m$.

In this procedure, one first uses the unperturbed action eq. (31) to obtain the energy difference $\Delta H_m(t)$ evaluated along the unperturbed equations of motion. Having obtained this function of $t$ (a c-number in the quantum mechanical language) one linearizes eq. (28) with respect to $H_m(N, \partial_\phi \Delta S, X^i(t)) - H^B_m(t)$ so as to keep the quadracity of the HJ equation in $\partial_\phi \Delta S$. One gets

$$
\partial_t \Delta S = \left[ H_m(N, \partial_\phi \Delta S, X^i(t)) - H^B_m(t) \right] \left[ 1 + \frac{1}{2} \Delta H_m(t) C^{ij} \partial_\phi \partial_i t^B \partial_\phi \partial_j t^B \right]
$$

(35)

By construction, the correction term is now expressed as a factor. Moreover, since it depends only on $t$, it can be absorbed in the l.h.s. by a redefinition of $t$. Thus its sole effect is to modify the time-parametrization of the background orbit $X^B(t)$ and therefore the lapse of time between the end points $X^i_1$ and $X^i_2$. Notice that this procedure can be refined by determining self-consistently $\Delta H_m(t)$.

**Application to quantum cosmology**
The aim of this brief subsection is to show how similar are the quantized versions of the equations we just obtained.

As in the former Section, in order to extract the background contribution, we write the total kernel, solution of the WDW equation (the quantized version of eq. (25)), as a product of the background gravitational propagator times the rest:

$$K(X^i_2, \phi^2; X^i_1, \phi_1) = K^B(X^i_2; X^i_1) \times \Delta K(X^i_2, \phi^2; X^i_1, \phi_1)$$  \hspace{1cm} (36)

The background gravitational kernel is the solution of

$$\left[ -C^{ij}(X^k)\partial_{X^i}\partial_{X^j} + V(X^k) + H^B_m(X^k) \right] K^B(X^i_2, X^i_1) = 0$$ \hspace{1cm} (37)

Since the variables $X^i$ are heavy, it is legitimate to use the WKB approximation:

$$K^B_{WKB}(X^i_2; X^i_1) = \sqrt{Det.\partial_{X^i_2}\partial_{X^i_1}} S^B_X \exp \{ iS^B_X(X^i_2) \}$$ \hspace{1cm} (38)

where the normalization factor is given by the enlarged Van Vleck determinant which occurs when one works at fixed energy, see [26]. By $X^i_2$ and $X^i_1$, we designate the sets of variables $(X^i_2, dE)$ and $(X^i_1, dE)$. (Notice that we should not worry about the normal ordering ambiguities of the operator $C^{ij}(X^k)\partial_{X^i}\partial_{X^j}$ since we are working in the WKB approximation.)

Using the fact that the background kernel is the WKB solution of eq. (37), one immediately obtains the equation for $\Delta K$

$$\left[ C^{ij}(X^k)\partial_{X^i}S^B_X i\partial_{X^j} + \hat{H}_m(\phi, i\partial_{\phi}, X^i) - H^B_m(X^i) \right] \Delta K(X^i, \phi; X^i_1, \phi_1)$$

$$+ \frac{1}{2} \left[ -C^{ij}(X^k)\partial_{X^i}\partial_{X^j} \right] \sqrt{Det.\partial_{X^i_2}\partial_{X^i_1}} S^B_X \Delta K(X^i, \phi; X^i_1, \phi_1) = 0$$ \hspace{1cm} (39)

The parallelism between the classical version eq. (28) and this equation is manifest. In the quantum version, upon neglecting the second line of eq. (39), i.e. to first order in small gradients $\partial_{X^i}\Delta K$ and $\partial_{X^i}Det. \Delta K$ is the usual matter kernel and it satisfies the Schrödinger equation in the background geometry driven by $H^B_m(X^i)$.

Higher order terms govern the corrections to the WKB and the BFA approximations. Their structure results from the \textit{a priori} factorization of $K^B_{WKB}$. Thus one should be careful in interpreting these terms as providing \textit{intrinsic} corrections to the WDW equation. It should be indeed recalled that for unidimensional minisuperspace models, the intrinsic corrections have been determined from the exact background-free equation governing matter evolution [17] which was obtained without factorizing a given gravitational kernel nor postulating that the WKB expressions constitute a good approximation. It was show that the intrinsic corrections do not reproduce those resulting from a \textit{a priori} factorization. The challenging question is thus to extend this analysis to the present multidimensional case. We hope to be able to report on this question.
4 The background field approximation in cosmology

In this Section we indicate how the results of the former one can be generalize to arbitrary three-geometries whose metric is given by $g^{ij}(x)$. The total Hamiltonian which governs the entangled dynamics of $g^{ij}(x)$ and some local matter field $\phi(x)$ has the following structure

$$H_T = \int d^3 x N^\mu(\xi, x) \mathcal{H}_T^\mu(\xi, x) = \int d^3 x N^\mu(\xi, x) \left\{ \mathcal{H}^G_{\mu}(\pi_{ij}, g^{ij}) + \mathcal{H}^m_{\mu}(p_\phi, \phi, g^{ij}) \right\}$$

The four-vector field $N^\mu(\xi, x)$ describes how successive 3-surfaces are pasted to form a 4-dimensional space-time, see [4]. $\mathcal{H}^G_{\mu}(\xi, x)$ is the gravitational energy-momentum density. It generalizes eq. (7), it is quadratic in the gradients $\partial_k g^{ij}(x)$ and in the conjugate momenta $\pi_{ij}(x)$ but is a non-linear function of $g^{ij}(x)$. The matter energy-momentum vector density $\mathcal{H}^m_{\mu}(\xi, x)$ depends on $\phi(x), \partial_k \phi(x), p_\phi(x)$ and $g^{ij}(x)$.

The main difference with Section 3 is that there are now four local equations of constraint in the place of eq. (25) which result from the four Einstein’s equations $\partial_{\nu}S_T = H^T_\nu(\xi, x) = 0$. In the HJ guise, they read

$$\frac{1}{2} C^{ij'j}(g^{kl}(x)) \partial_{g^{ij'}} S_T \partial_{g^{ij'}} S_T + V(g^{kl}(x)) + \mathcal{H}^0_{\mu}(\phi(x), \partial_\phi S_T, g^{ij}(x)) = 0$$

$$-2(\partial_{g^{ij'}} S_T)^{ij} + \mathcal{H}^m_{\mu}(\phi(x), \partial_\phi S_T, g^{ij}(x)) = 0$$

For further details concerning the kinetical matrix $C$ or the potential $V$ see [4]. The important point is that these constraints do not decouple since the potential term $V(g^{kl}(x))$ contains gradients connecting the fields at different points. Upon neglecting this term one obtains an action which is just a sum of decoupled actions evaluated at each point. As shown in [19], one can then perform a perturbative treatment in those gradients. What follows can be conceived as a generalization of these works in that the small parameter which governs the expansion is the change of the matter Hamiltonian itself. Therefore, our (less explicit) treatment also applies to arbitrary large gradients.

In spite of these gradients, the total action from the initial configuration $g_1^{ij}(x), \phi_1(x)$ to the final one $g_2^{ij}(x), \phi_2(x)$ which is a solution of the constraints can be decomposed, as in eq. (26), as

$$S_T = \int_{g_1^{ij}(x)}^{g_2^{ij}(x)} dg^{ij}(x) \pi_{ij}(g^{ij}; \{g_2^{kl}, \phi_2\}) + \int_{\phi_1(x)}^{\phi_2(x)} d\phi(x) p_\phi(\phi; \{g_2^{kl}, \phi_2\})$$

where the local on-shell momenta are parametrized by their conjugate field. In this paper, we shall not discuss the difficulties that might prevent to actually compute
$S_T$ when the initial and final configurations are given, i.e. to solve the “thick” sandwich problem. Rather we shall only use the structure of $S_T$ and the fact that it is a solution of eqs. \cite{14,12} since it is all what we need in our implementation of a BFA to gravity.

To conform ourselves to the “standard procedure” \cite{10,12,13} (later on we shall comment on its appropriate character), we treat all $g^{ij}(x)$ in the same footing as “heavy” degrees of freedom. Thus, as in the former Section, we choose an initial matter configuration $(p_{\phi_1}(x), \phi_1(x))$ which determines both the background matter density $H^B_\mu$ and the background trajectories. Then, as in eq. (27), we decompose $S_T$ as $S^B_G(g^{ij}_2) + \Delta S(\phi_2, g^{ij}_2)$ and we perform an expansion in the changes $\delta \pi_{ij}(x)$ which result from the change in the matter field configuration induced by the replacement of the background value $\phi^B(x)$ by the new field configuration specified by the final value $\phi_2(x)$. By inserting this decomposition in eqs. \cite{11,12}, we obtain respectively

$$C^{ijij'}(g^{kl}(x))\partial_{g^{ij}}S^B_X \partial_{g^{ij'}}\Delta S + H_0(\phi(x), \partial_\phi \Delta S, g^{ij}(x)) - H^B_0(x, g^{ij}(x)) = 0 \quad (44)$$

$$- 2(\partial_{g^{ij}}\Delta S)^{ij} + \left(H^m_i(\phi(x), \partial_\phi S_T, g^{ij}) - H^{mB}_i(x, g^{ij}(x))\right) = 0 \quad (45)$$

In order to relate these local equations to a single HJ equation governing matter propagation in a given background metric, we must introduce by hand the lapse-shift vector field $N^\mu(x, t)$. These four functions then define univocally a single time parameter. It is given by $t = -\partial E S^B_X$ where $dE$ is an infinitesimal constant change in the total matter energy $\int d^3 x N^\mu(x) H^B_\mu(x)$. More precisely, this “redundant” parameter satisfies

$$\int d^3 x \left\{ N^0(x)C^{ijij}(g^{kl}(x))\partial_{g^{ij}}S^B_X \partial_{g^{ij'}}\Delta S + 2N^i(x)^{ij} \partial_{g^{ij}}\right\} t^B(g^{ij}(x), N^\mu(x)) = 1 \quad (46)$$

This equation is obtained by varying eq. \cite{10} with respect to $dE$ and by imposing $H_T = 0$. It allows to recover the usual relation between the momentum and the velocity: $d\phi^{ij}(x)/dt = N^0 C^{ijij'}\pi_{ij'} + 2N^{ij}$. Using this time parameter and eq. \cite{15}, and upon neglecting the second line of eq. \cite{14} which is quadratic in the small gradients $\partial_{g^{ij}}\Delta S$, the first line of that equation becomes the HJ equation

$$\partial_t \Delta S = \int d^3 x N^\mu(t, x) \left\{ H^m_\mu(\phi(t, x), \partial_\phi(t, x) \Delta S, g^{ijB}(t, x)) - H^{mB}_\mu(t, x)\right\} \quad (47)$$

evaluated in the curved geometry solution of the Einstein’s equation driven by the “mean” matter energy density $H^{mB}_\mu(x)$.

In brief, the novelty introduced by considering arbitrary 3-geometries relies on the necessity of introducing from the outset the four functions $N^\mu(x)$ in order to
rigidify the propagation in super-space from the initial 3-geometry to the next ones. We emphasize that the dynamical justification arises from the presence of gradients \( \partial_x g^{ij}, \partial_x \phi \) in the potential terms of \( H_T \).

In what follows, we shall not present the quantum version of this derivation which would lead to the Schrödinger equation in a curved geometry. It is straightforwardly obtained by generalizing eqs. (36-39). (The interested reader will find this quantum treatment in [12] which is however restricted to empty solutions of Einstein equations, i.e. to \( \mathcal{H}_{m \mu} = 0 \).) Once more, the reader will verify that the structure of the quantum version is fully determined by the implementation of a BFA in classical settings.

We shall neither present the computation of the quadratic corrections which arise through the second line of eq. (44). They can also be found in [12]. What we shall do instead is to discuss on physical grounds the appropriate character of applying, as we just did, a BFA to all gravitational degrees of freedom.

We shall base our discussion on the fact that the quadratic corrections to \( \Delta S \) are given by a term of the form \( \int \int T_{\mu \nu} D^{\mu \nu \rho \sigma} T^{\mu \nu \rho \sigma} \) where \( D \) is a gravitational Green function. These interactions are in strict analogy with the current-current interactions in electro-magnetism obtained by integrating over the photon field. However, as discussed in [12], the Green function which is obtained by treating all gravitational degrees of freedom on the same footing is rather unconventional in that it vanishes both on the initial and final 3-surfaces, in contradistinction with the usual retarded and Feynman functions in classical and quantum settings. This unconventional feature results from the fact that the \( g^{ij} \) have been completely specified on both the initial and final 3-surfaces.

This unpleasant feature reveals that the “heavy” character that has been a priori attributed to \( g^{ij} \) must be called into question. Indeed, so far, \( g^{ij} \) has been treated like the \( W \) gauge boson in the derivation of the four fermions weak interactions model. In that case, it is the rest mass of the \( W \) which legitimizes its elimination for low energy processes since no on-shell \( W \) is ever produced. For gravity, the Planck mass does not act like the \( W \) mass and no mass threshold prevents the production of on-shell gravitons. For this reason, it is inappropriate to functionally integrate out the gravitons. Indeed, in quantum gravity, these gravitons are correlated in phase and amplitude to the quantized matter sources. This quantum entanglement prevents the possibility of describing the final gravitational configurations by \( c \)-number functions giving rise to a classical background action. More intuitively, on which physical basis could one justify an unsymmetrical treatment in which rare (i.e. not coherent) on-mass gravitons are considered “heavy” and thus described by WKB waves on which one applies a BFA and photons with the same frequency are “light” and thus fully quantized?

In order to treat gravitons and photons on an equal footing, one must therefore abandon the idealized procedure in which all gravitational degrees of freedom are
treated on the same footing. By treating the light part of the gravitational degrees of freedom like matter degrees of freedom, one “recovers” what has been adopted in cosmology to study the growth of perturbations, see also [5] for a discussion on the dynamical role of light and fluctuating gravitational degrees of freedom in explaining why the heavy coordinates are driven by mean values.

In conclusion, it is physically unjustified to search for a treatment of the solutions of the WDW in which all gravitational degrees of freedom are treated on the same footing. However what is lost from a geometrical point of view is compensate by a dynamical justification: in each situation, it will be convenient and appropriate to treat some gravitational degrees of freedom as heavy but it will also be mandatory to keep the other ones fully quantized. The same division also applies to matter degrees of freedom. The implementation of this program requires further work which will hopefully be presented somewhere else.

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5 Appendix A. The background field approximation based on trajectories.

Given the hamiltonian eq. (24), the total “off-shell” action we are considering is

\[ S_T = \int_0^1 d\xi \left\{ \pi_i \dot{X}^i + p_\phi \dot{\phi} - N(\xi) \left[ H_X(\pi_i, X^i) + H_m(\phi, p_\phi, a) \right] \right\} \quad (48) \]

In order to have univocally specified expressions, we must choose a “representative” for \( N(\xi) \). Indeed, at fixed end point values \( X_1^i, \phi_1 \) to \( X_2^i, \phi_2 \), the action \( S_T \) is invariant under reparametrizations of \( \xi \) which leave the end point values \( \xi = 0,1 \) fixed. The reparametrizations which satisfy this restriction leave the integral \( \int_0^1 d\xi N(\xi) = N \) invariant. Thus one can only (but always) choose as a representative the constant lapse \( N(\xi) = N \). This choice is the one usually made when describing the path integral of a relativistic particle. In that case, \( ds = Nd\xi \) defines the Schwinger fifth-time. Adopting the same choice, our “representative” off-shell action is

\[ S_T = \int_0^N dt \left\{ \pi_i \dot{X}^i + p_\phi \dot{\phi} - \left[ H_X(\pi_i, X^i) + H_m(\phi, p_\phi, X^i) \right] \right\} \quad (49) \]

We have used the symbol \( t \) defined by \( dt = Nd\xi \), since in this “gauge” of constant lapse, it indeed corresponds to the proper time. In this gauge, the total hamiltonian
is a constant along the classical trajectory which relates \( X^i_1, \phi_1 \) to \( X^i_2, \phi_2 \) in a proper time \( N \) since it is not explicitly time dependent.

Since physical propagation must occur with vanishing total energy, one must impose \( \partial_N S_T = H_X + H_m = 0 \). This extremization condition determines the "saddle value" \( \bar{N} = N(X^i_2, \phi_2; X^i_1, \phi_1) \) which, in our gauge, equals \( t(X^i_1, \phi_2) \), the proper time lapse from \( X^i_1 \) to \( X^i_2 \) given \( \phi_1 \) and \( \phi_2 \). \( \partial_N S_T = 0 \) plays thus the role of \( \partial_i S_T = 0 \) in Section 2. Both fix the proper time lapse on the classical trajectory.

Thus the fully extremized action is

\[
S_T = S_X + S_m = \int_0^{\bar{N}} dt \left\{ \pi_i \dot{X}^i + p_{\phi} \dot{\phi} \right\}
\]  

(50)

In this action, \( X^i(t) \) and \( \phi(t) \) are evaluated along the equation on motion, they thus depend parametrically on \( \bar{N} \).

We are now in position to implement a BFA. As in Section 2, we shall linearize only the gravitational part of the action in order to obtain the simplified description of the matter evolution for values of \( \phi_2 \) centered around the background value \( \phi_2^B \). The difference with that Section is that we shall consider the difference of extremized actions since no longer can work with \( \epsilon \) off shell.

Thus, we want to expand \( S_T(X^i_2, \phi_2, \bar{N}(X^i_2, \phi_2)) \) around the background action \( S_T(X^i_2, \phi_2^B, \bar{N}^B = \bar{N}(X^i_2, \phi_2^B)) \) to a first order change of the background coordinates \( \delta X^i(t) \) and the lapse \( \delta N \). We proceed in three steps.

In the first step, we replace \( \bar{N}(X^i_2, \phi_2) \) by its background value \( \bar{N}^B = \bar{N}(X^i_2, \phi_2^B) \) in the first action. To first order in \( \delta N \), the action is left unchanged since we are working with vanishing total energy and we are not modifying the end point values of \( \phi \) and \( a \). To consider only the first order change in \( \delta N \) is a valid restriction because, when \( X^i(t) \) are heavy, i.e. when second order change in \( \delta X^i(t) \) are negligible with respect to first order change, the same in true for the changes due to \( \delta N \). In other words when \( X^i(t) \) are heavy, \( N \) is also heavy. (This is exactly like in statistical mechanics. The change of the temperature induced by a fluctuation of the energy repartition scale like the inverse volume of the heavy system.)

In a second step, we consider the first order change in \( \delta X^i(t) \) and \( \delta \pi_i(t) \). We have

\[
S_X(X^i_2, \phi_2, \bar{N}^B) = S_X(X^i_2, \phi_2^B, \bar{N}^B) + \int_0^{\bar{N}^B} dt \left\{ \delta \pi_i \dot{X}^i - \delta X^i \tilde{\pi}_i \right\} + \delta X^i \pi_i |_{X^i_1} (X^i_2)
\]  

(51)

Since we are working with fixed end points \( X^i_1, X^i_2 \), the boundary term vanishes. Moreover since we are on-shell,

\[
\delta \pi_i \dot{X}^i - \delta X^i \tilde{\pi}_i = \delta X^i_{\pi_i} (H_X + H_m)
\]  

(52)

where \( \delta X^i_{\pi_i} \) means the variation of the quantity induced by the sole variations of \( X^i \) and \( \pi_i \), i.e. with \( \phi(t) \) and \( p_{\phi}(t) \) evaluated along their background classical values.
In the third step, we re-use the vanishing of the total energy for both classical trajectories. Thus

\[
\delta X^i, \pi_i (H_X + H_m) = -\delta \phi, p_\phi (H_X + H_m) = -\delta \phi, p_\phi H_m = -H_m (p_\phi (t), \phi (t), X^{iB} (t)) + H_m (p_\phi^B (t), \phi^B (t), X^{iB} (t)) \tag{53}
\]

The second equality follows from the fact that \(H_X\) is independent of \(\phi\) and \(p_\phi\). The last one means that the variations with respect to \(\phi\) and \(p_\phi\) should be taken to all orders and not to first order only in the case of \(\delta X^i\) and \(\delta \pi_i\). We emphasize that both terms in this last expression are evaluated in the background geometry \(X^{iB} (t)\).

By collecting the results, to first order in \(\delta N, \delta X^i, \delta \pi_i\), we have

\[
S_T (X^2_2, \phi_2, \tilde{N} (X^2_2, \phi_2)) = S_X (X^2_2, \phi^B_2, \tilde{N}^B) + \int_0^{\tilde{N}^B} dt H^B_m (t) + \int_0^{\tilde{N}^B} dt \left\{ p_\phi (t) \dot{\phi} (t) - H_m (p_\phi (t), \phi (t), X^{iB} (t)) \right\} \tag{54}
\]

The \(\phi_2\) dependent part of this action is the usual HJ matter action evaluated in the time dependent background \(X^{iB} (t)\). The simplest way to verify it is to consider first order variation of the end point values \(X^i_2\). At this point indeed, we cannot vary the time lapse \(\tilde{N}^B\) since it is fixed by \(a_2\). The first order variation of \(S_T\) with respect to an arbitrary variation of the end points \(X^i_2\) gives

\[
\delta S_T = \delta X^i_2 \partial_{X^2_2} S_T = \delta X^i_2 \left\{ g_i (X^2_2) + \partial_{X^2_2} \tilde{N}^B (X^2_2, \phi_2) H_m (\partial_{\phi^B} S_T, \phi, X^i_2) \right\} \tag{55}
\]

where the dependence in \(X^2_2\) which is irrelevant for the matter dynamics has been put in the functions \(g_i\). Using \(\tilde{N}^B = t (X^2_2)\), we can now determine the change of \(S_T\) when the \(\delta X^i_2\) are evaluated along the classical trajectory, i.e. when they are of the form \(\delta X^i_2 = C^{ij} \partial_j S_X^B \delta t\). By definition of \(t\) one obtains

\[
\delta S_T = \delta t \partial_t S_T = \delta t \left\{ H_m (\partial_{\phi} S_T, \phi, a^B (t)) + g_i (t) \partial_t X^{iB} (t) \right\} \tag{56}
\]

It is thus through changes of \(X^i\) evaluated along the background orbit that one recuperates the time dependent HJ equation for the matter degrees of freedom.

6 Appendix B. The gravitational response to off-shell matter configurations

In the former Appendix we saw that the linear change of the background action \(S_X\) induced by the change of the matter configurations is equal to \(\int dt \Delta H_m\) evaluated along the background trajectory \(X^{iB} (t)\), see eq. \(\text{[4]}\). However it should
be noticed that this result has been obtained by comparing neighboring on-shell matter configurations. It would therefore be very useful to extend this result to off-shell matter configurations in order to be able to use this relationship in a path integral formulation of quantum mechanical matter propagation which does involve off-shell configurations.

Thus the problem one should analyze in classical settings is to determine the gravitational response to a given arbitrary matter configuration \( \phi(\xi), p_{\phi}(\xi) \). Strictly speaking, this cannot be achieved since one cannot find a stationary gravitational action whose source would be this particular matter configuration. Indeed, there is no reason that this configuration satisfies the conservation equation eq. (\text{[3]}). Therefore, the Bianchi identity tells us that no (on-shell) background could be driven by this source.

However this does not prevent us to characterize the departure from stationarity when the configuration \( \phi(\xi), p_{\phi}(\xi) \) is close to a classical orbit \( \phi^B(\xi), p_{\phi}^B(\xi) \). As we shall see, this departure is just what we need to recover the usual off-shell matter action in a given background that one uses in a path integral formulation. To prove this result, we shall simply adapt the procedure we used in the former Appendix so as to take into account the off-shell character of the configuration \( \phi(\xi), p_{\phi}(\xi) \).

The total unextremized action is given by eq. (\text{[19]}), since one can still work with constant lapses \( N \) as representatives. (The off-shellness of \( \phi(\xi) \) does not spoil reparametrization invariance). Therefore, at fixed \( N \), one can still extremize \( S_T \) with respect to arbitrary variations of \( X^i(t) \) and \( \pi_i(t) \) with fixed end points \((X_{i1}, X_{i2})\). Doing so one obtains an “almost” extremized action since it is stationary for all variations but those of the constant lapse \( N \).

At this point the “proximity” of \( \phi(t), p_{\phi}(t) \) with some classical trajectory \( \phi^B(t), p_{\phi}^B(t) \) must be called upon. Indeed, to first order in the change in the lapse around its background value \( \bar{N}^B \), one can replace \( N \) in the “almost” extremized action by the background value \( \bar{N}^B \) and obtain

\[
S_T = \int_0^{\bar{N}^B} dt \left\{ \pi_i \dot{X}^i + p_{\phi} \dot{\phi} - \left[ H_G(X^i(t), \pi_i(t)) + H_m(\phi, p_{\phi}, X^i(t)) \right] \right\} \tag{57}
\]

where \( X^i(t) \) is evaluated along the stationary orbit from \((X_{i1}, X_{i2})\) in a time \( \bar{N}^B \) which is driven by the off-shell “pressures” \( \partial_X H_m(\phi, p_{\phi}, X^i) \). The reason why this replacement can be performed is the following. Upon searching \( \bar{N} \), the solution of \( \partial_N S_T = 0 \) where \( S_T(N) \) is the “almost” extremized action defined above, one finds that \( \bar{N} \) is complex function since no classical (i.e. real) solution can be found. However, when \( \phi(t), p_{\phi}(t) \) is close to \( \phi^B(t), p_{\phi}^B(t) \), i.e. when \( H_m(\phi, p_{\phi}) - H_m(\phi^B, p_{\phi}^B) \ll H_m(\phi^B, p_{\phi}^B) \) for all \( t \), the real part of \( \bar{N} \) is close to \( \bar{N}^B \) and that its imaginary part is small. Remember that the location of the saddle is mainly determined by the heavy coordinates \( X^i \). Thus, one can correctly replace the complex \( \bar{N} \) by the background value \( \bar{N}^B \). Then, the off-shellness of the matter
configuration is made manifest by the fact the $H_G + H_m$ in eq. (57) does not vanish. Not surprisingly, it is through this term that the usual matter action will be recovered.

Having performed this first step, one can indeed proceed to the second and third step of Appendix A. The simplest way to proceed consists in "adding" $H_G(X_i^B, \pi_i^B) + H_m(\phi^B, p^B_B, X^B) = 0$ to the integrand of eq. (57) and implementing directly eqs. (51) and (52) in the resulting action. (Both equations rely only on the stationary character of the orbits $X^i(t)$ and $X^i(t).$) After some elementary algebra, one recovers eq. (54).

What we have learned from this generalization is that the determination of the linear gravitational response does not require mass-shell matter sources. Thus the matter action which determines the phase of off-shell configurations in a path integral formulation of quantum matter propagation in a given background can be considered as the action emerging, through a first order variation, from a more quantum framework in which gravity responded to these off-shell configurations. Even though the imaginary part of $\bar{N}$ is negligible for heavy coordinates $X^i$, strictly speaking this imaginary part is present. So what does it mean? Simply that in the path integral of quantum gravity, these gravito-matter configurations will be exponentially reduced by this imaginary contribution with respect to the BFA formalism in which one has discarded this contribution.

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