FACING NON-STATIONARY CONDITIONS WITH A NEW INDICATOR OF ENTROPY INCREASE: THE CASSANDRA ALGORITHM

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We address the problem of detecting non-stationary effects in time series (in particular fractal time series) by means of the Diffusion Entropy Method (DEM). This means that the experimental sequence under study, of size $N$, is explored with a window of size $L << N$. The DEM makes a wise use of the statistical information available and, consequently, in spite of the modest size of the window used, does succeed in revealing local statistical properties, and it shows how they change upon moving the windows along the experimental sequence. The method is expected to work also to predict catastrophic events before their occurrence.

1 Introduction

The main aim of this paper is to illustrate a promising strategy to study non-stationary processes. We prove that the method is efficient by means of the joint study of real and artificial sequences, and we reach a conclusion that makes it plausible to imagine the method at work to successfully predict the time of occurrence of catastrophic events.

The method here illustrated is a suitable extension of the Diffusion Entropy Method (DEM). The DEM is discussed in details in other publications. Here we limit ourselves to give a concise illustration of this technique so as to allow the reader to understand the spirit of the method of this paper, at least through a first reading, without consulting these earlier publications. The first step of this technique is the same as that of the pioneering work of Refs. This means that the experimental sequence is converted into a kind of Brownian-like trajectory. The second step aims at deriving many distinct diffusion trajectories with the technique of moving windows of size $l$. The reader should not confuse the mobile window of size $l$ with the mobile window of value $L$ that will be used later on in this paper to detect non-stationary properties. For this reason we shall refer to the mobile windows of...
size $L$ as large windows, even if the size of $L$ is relatively small, whereas the mobile windows of size $l$ will be called small windows. The large mobile window has to be interpreted as a sequence with statistical properties to reveal, and will be analyzed by means of small windows of size $l$, with $l < L$. The success of the method depends on the fact that the DE makes a wise use of the statistical information available. In fact, the small windows overlap among themselves and are obtained by locating their left border on the first site of the sequence, on the second second site, and so on. The adoption of overlapping windows is dictated by the wish to establish a connection with the Kolomogorov-Sinai (KS) method. This yields also, as a further beneficial effect, many more trajectories than the widely used method of Detrended Fluctuation Analysis (DFA).

In conclusion, we create a conveniently large number of trajectories by gradually moving the small window from the first position, with the left border of the small window coinciding with the first size of the sequence, to the last position, with the right border of the small window coinciding with the last site of the sequence. After this stage, we utilize the resulting trajectories, all of them with the initial position located at $x = 0$, to produce a probability distribution at “time” $l$. We evaluate the Shannon entropy of this distribution, $S_d(l)$, and with easy mathematical arguments we prove that, if the diffusion process undergoes a scaling of intensity $\delta$, then

$$S_d(l) = A + \delta ln(l).$$

Thus the parameter $\delta$ of the scaling condition, if this condition applies, can be measured without recourse to any form of detrending.

This is the original motivation for the DEM. In this paper we want to prove that the DEM does much more than detecting scaling. In a stationary condition the DEM not only detects with accuracy the final scaling but it also affords a way to monitor the regime of transition to the final thermodynamic condition. If the sequence under study is affected by biases and non-stationary perturbations, the attainment of the final regime of steady scaling can be cancelled, and replaced by an out of equilibrium regime changing in time under the influence of time dependent biases. We want to prove that the DEM can be suitably extended to face this challenging non-stationary condition.

The outline of the paper is as follows. In Section 2 we shortly review the main tenets of the DEM so as to make this paper, as earlier mentioned, as self-contained as possible. In Section 3 we illustrate the extension of the DEM and we discuss the fundamental problem of assessing which is the shortest portion, of size $L$, of the sequence under study, which is still large enough as to make the DEM work. In Section 3 we express our hope that this new method might serve prediction purposes.

## 2 Diffusion Entropy

Let us consider a sequence of $M$ numbers $\xi_i$, with $i = 1, \ldots, M$. The purpose of the DEM algorithm is to establish the possible existence of a scaling, either normal or anomalous, in the most efficient way as possible, without altering the data with any form of detrending. Let us select first of all an integer number $l$, fitting the
condition $1 \leq l \leq M$. As earlier mentioned, we shall refer ourselves to $l$ as "time". For any given time $l$ we can find $M - l + 1$ sub-sequences defined by

$$\xi^{(s)}_i \equiv \xi_{i+s}, \quad s = 0, \ldots, M - l.$$  \hspace{1cm} (2)

For any of these sub-sequences we build up a diffusion trajectory, labelled with the index $s$, defined by the position

$$x^{(s)}(l) = \sum_{i=1}^{l} \xi^{(s)}_i = \sum_{i=1}^{l} \xi_{i+s}.$$  \hspace{1cm} (3)

Let us imagine this position as referring to a Brownian particle that at regular intervals of time has been jumping forward or backward according to the prescription of the corresponding sub-sequence of Eq.(2). This means that the particle before reaching the position that it holds at time $l$ has been making $l$ jumps. The jump made at the $i$-th step has the intensity $|\xi^{(s)}_i|$ and is forward or backward according to whether the number $\xi^{(s)}_i$ is positive or negative.

We are now ready to evaluate the entropy of this diffusion process. To do that we have to partition the $x$-axis into cells of size $\epsilon(l)$. When this partition is made we have to label the cells. We count how many particles are found in the same cell at a given time $l$. We denote this number by $N_i(l)$. Then we use this number to determine the probability that a particle can be found in the $i$-th cell at time $l$, $p_i(l)$, by means of

$$p_i(l) \equiv \frac{N_i(l)}{M - l + 1}. \hspace{1cm} (4)$$

At this stage the entropy of the diffusion process at the time $l$ is determined and reads

$$S_d(l) = - \sum_i p_i(l) \ln[p_i(l)]. \hspace{1cm} (5)$$

The easiest way to proceed with the choice of the cell size, $\epsilon(l)$, is to assume it independent of $l$ and determined by a suitable fraction of the square root of the variance of the fluctuation $\xi(i)$.

Before proceeding with the illustration of how the DEM method works, it is worth making a comment on the way we use to define the trajectories. The method we are adopting is based on the idea of a moving window of size $l$ that makes the $s$–th trajectory closely correlated to the next, the $(s+1)$–th trajectory. The two trajectories have $l - 1$ values in common. A motivation for our choice is given by our wish to establish a connection with the Kolmogorov Sinai (KS) entropy. The KS entropy of a symbolic sequence is evaluated by moving a window of size $l$ along the sequence. Any window position corresponds to a given combination of symbols, and, from the frequency of each combination, it is possible to derive the Shannon entropy $S(l)$. The KS entropy is given by the asymptotic limit \(\lim_{l \to \infty} S(l)/l\). We believe that the same sequence, analyzed with the DEM method, at the large values of $l$ where $S(l)/l$ approaches the KS value, must yield a well defined scaling $\delta$. To realize this correspondence we carry out the determination of the Diffusion Entropy using the same criterion of overlapping windows as that behind the KS entropy.
Details on how to deal with the transition from the short-time regime, sensitive to the discrete nature of the process under study, to the long-time limit where both space an time can be perceived as continuous, are given in Ref. [1]. Here we make the simplifying assumption of considering so large times as to make the continuous assumption valid. In this case the trajectories, built up with the above illustrated procedure, correspond to the following equation of motion:

\[
\frac{dx}{dt} = \xi(t),
\]

where \( \xi(t) \) denotes the value that the time series under study gets at the \( l \)-th site. This means that the function \( \xi(l) \) is depicted as a function of \( t \), thought of as a continuous time \( t = l \). In this case the Shannon entropy reads

\[
S_d(t) = -\int_{-\infty}^{\infty} dx \, p(x,t) \ln[p(x,t)].
\]

We can derive with a simple treatment an analytical solution for Diffusion Entropy when the process is characterized by scaling, namely when

\[
p(x,t) = \frac{1}{t^{\delta}} F\left(\frac{x}{t^{\mu}}\right).
\]

Let us plug Eq.(8) into Eq.(7). After a simple algebra, we get:

\[
S_d(\tau) = A + \delta(\tau)\tau,
\]

where

\[
A \equiv -\int_{-\infty}^{\infty} dy \, F(y) \ln[F(y)]
\]

and

\[
\tau \equiv \ln(t).
\]

It is evident that this kind of technique to detect scaling does not imply any form of detrending, and this is one of the reasons why some attention should be devoted to it. It is also worth mentioning, as we prove now, that it yields the correct scaling values even for the so-called Lévy walks, where the time dependence of the second moment with respect to time has an exponent which is different from the scaling exponent of the Lévy process.

We therefore check the efficiency of this technique by the studying the artificial sequence of Refs. [8,9]. This sequence is built up in such a way as to realize long sequences of either +’s or −’s. The probability of finding a sequence of only +’s or only −’s of length \( t \) is given by

\[
\psi(t) = (\mu - 1) \frac{T^{t\mu - 1}}{(t + T)^{\mu - 1}}.
\]

Here we focus our attention on the condition \( \mu < 3 \) and we raise the reader’s attention on the interval \( 2 \leq \mu \leq 3 \). In fact, this kind of sequence is the same as that adopted in earlier work for a dynamic derivation of Lévy diffusion, which
shows up when the condition $2 < \mu < 3$ applies. It corresponds to a particle travelling with constant velocity throughout the whole time interval corresponding to either only +’s or only -’s, and changing direction with no rest, at the end of any string with the same symbols.

We will refer to this model as **Symmetric Velocity Model (SVM)**. We know from the theory of Ref. 9 that the scaling of the resulting diffusion process when $2 < \mu < 3$ is

$$\delta = \frac{1}{\mu - 1}. \quad (13)$$

Note, however, that this diffusion process has a finite propagation front, with ballistic peaks showing up at both $x = t$ and $x = -t$. The intensity of these peaks is proportional to the correlation function

$$\Phi_\xi(t) = \left( \frac{T}{T + t} \right)^{\mu - 2}. \quad (14)$$

As a consequence of this fact, the whole distribution does not have a single rescaling. In fact, the distribution enclosed between the two peaks rescales with $\delta$ of Eq. (13) while the peaks are associated to $\delta = 1$. Furthermore, it is well known that the scaling of the second moment is given by

$$\delta_H = \frac{4 - \mu}{2}. \quad (15)$$

Thus, it is expected that the scaling detected by the DE method might not coincide with the prediction of Eq. (13) for the whole period of time corresponding to the presence of peaks of significant intensity. We think that the Lévy scaling of Eq. (13) will show up at long times, when the peak intensity is significantly reduced. This conjecture seems to be supported by the numerical results illustrated in Fig. 1. We see in fact that the scaling predicted by Eq. (13) is reached after an extended transient, of the order of about 20,000 in the scale of Fig. 1. This time interval is about 2000 larger than the value assigned to the parameter $T$, of Eq. (12), which is, in fact, in the case of Fig. 1, $T = 10$.

In conclusion, this section proves that the DE method applied to the SVM yields, for the scaling parameter $\delta$, the correct value of Eq. (13), rather than the value that would be obtained measuring the variance of the diffusion process, Eq. (15). However, the time necessary to make this correct value emerge is very large. Furthermore, as shown in Fig. 2, the adoption of SVM would make the scaling parameter $\delta$ insensitive to $\mu$ in the whole interval $1 \leq \mu \leq 2$. This means that the adoption of the DE method would not allow us to distinguish a process with $\mu$ very close to 1 from one with $\mu$ very close to 2. This problem can be solved using different rules for the diffusion process: the random walker can, for instance, walk always in the same direction, and at the “time” when there is a passage from a laminar region of +’s to one of -’s and vice-versa. If this latter rule is adopted, then it is easy to prove that the resulting $p(x, t)$ is an asymmetric Lévy distribution, with a scaling $\delta = \mu - 1$ for $1 \leq \mu \leq 2$.

Throughout this paper, however, apply the DEM with the rule corresponding to the symmetric velocity model, with $\delta$ depending on $\mu$ as in Fig. 2. In the regime
of ordinary statistical mechanics (µ ≫ 3) the ordinary scaling is quickly attained, while the condition of anomalous statistical mechanics µ < 3 is characterized by a long transient regime, which is carefully recorded by the DEM. In this paper we want to use the DEM to monitor the time dependence of the “rules” generating the sequences under study.

3 The new method at work with nonstationary sequences

To illustrate the ideas that led us to propose the method of analysis with two moving window, let us begin with discussing the artificial sequence given by

$$\xi_b(t) = \kappa \xi(t) + A \cos(\omega t).$$

The second term on the right hand side of this equation is a deterministic contribution that might mimic, for instance, the season periodicity of Ref. [1]. The first term on the right hand side is a fluctuation with no correlation that can be correlated or not to the harmonic bias.

Fig. 3 refers to the case when the random fluctuation has no correlation with the harmonic bias. It is convenient to illustrate what happens when κ = 0. This is the case where the signal is totally deterministic. It would be nice if the entropy in this case did not increase upon increasing l. However, we must notice that the method of mobile windows implies that many trajectories are selected, the difference among them being, in the determinist case where $$\xi_b(t) = A \cos(\omega t)$$, a difference on initial conditions. Entropy initially increases. This is due to the fact that the statistical
average on the initial conditions is perceived as a source of uncertainty. However, after a time of the order of the period of the deterministic process a regression to the condition of vanishing entropy occurs, and it keeps repeatedly occurring for the multiple times. Another very remarkable fact is that the maximum entropy value is constant, thereby signalling correctly that we are in the presence of a periodic signal, where the initial entropy increase, due to the uncertainty on the initial conditions, is balanced by the recurrences. Let us now consider the effect of a non-vanishing $\kappa$. We see that the presence of an even very weak random component makes an abrupt transition to occur from the condition where the diffusion entropy is bounded from above, to a new condition where the recurrences are limited from below by an entropy increase proportional to $0.5 \ln l$. In the asymptotic time regime the DEM yields, as required, the proper scaling $\delta = 0.5$. However, we notice that it might be of some interest for a method of statistical analysis to give information on the extended regime of transition to the final thermodynamic condition. We notice that if the DEM is interpreted as a method of scaling detection, it might also give the impression that a scaling faster than the ballistic $\delta$ is possible. This would be misleading. However, this aspect of the DEM, if conveniently used, can become an efficient method to monitor the non-stationary nature of the sequence under study.

In the special case where the fluctuation $\xi(t)$ is correlated or anticorrelated to the bias, the numerical results illustrated in Fig. 4 show that the time evolution of the diffusion entropy is qualitatively similar to that of Fig. 3. The correlation between the first and the second term on the right hand side of Eq. (16) is established...
by assuming

$$\xi(t) = \xi_0(t) \cos(\omega t),$$

(17)

where $\xi_0(t)$ is the genuine independent fluctuation, without memory, whose intensity is modulated to establish a correlation with the second term. It is of some interest to mention what happens when $A = 0, \kappa = 1$, and consequently $\xi_0(t)$ coincides with $\xi(t)$ of Eq. (17). In this case we get the straight (solid) line of Fig. 3. This means that the adoption of the assumption that the process is stationary yields a result that is independent of the modulation.

We use this interesting case to illustrate the extension of the DEM, which is the main purpose of this paper. As earlier mentioned, this extension is based on the use of two mobile windows, one of length $L$ and the traditional ones of length $l \ll L$. This means that a large window of size $L$, with $L \ll T = 2\pi/\omega$, is located in a given position $i$ of the sequence under study, with $i \leq N - L$, and the portion of the sequence contained within the window is thought of as being the sequence under study. We record the resulting $\delta$ (obtained with a linear regression method) and then we plot it as a function of the position $i$. We show in Fig. 5 that this way of proceeding has the nice effect of making the periodic modulation emerge.

Let us now improve the method to face non-stationary condition even further. As we have seen, the presence of time dependent condition tends to postpone or to cancel the attainment of a scaling condition. Therefore, let us renounce to using Eq. (9) and let us proceed as follows. For any large mobile window of size $L$ let us call $l_{\text{max}}$ the maximum size of the small windows. Let us call $n$ the position of the

Figure 3: The diffusion entropy $S_d(l)$ as a function of time $l$ for different sequences of the type of Eq. (12).
Figure 4: The diffusion entropy $S_d(l)$ as a function of time $l$ for different sequences of the type of Eq. (12) with the prescription of Eq. (13) for the random component.

Figure 5: The method of the two mobile windows applied to a sequence given by Eq. (12) with $A = 0$ and $\xi(t)$ given by Eq. (13). The dashed line represents the sinus' amplitude (not in scale) corresponding to the position $i$ of the left border of the large moving window.
Figure 6: The method of the two moving windows with $l_{\text{max}} = 30$ applied to the analysis of an artificial CMM sequence with periodic parameter $\epsilon$. The period of the variation of $\epsilon$ is 5000 bps and the analysis is carried out with moving windows of size 2000 bps. Inset: Fourier spectral analysis of $I(n)$.

left border of the large window, and and let us evaluate the following property

$$I(n) \equiv \sum_{l=2}^{l_{\text{max}}} \frac{S_d(l) - \left[S_d(1) + 0.5 \ln l \right]}{l}.$$  \hspace{1cm} (18)

The quantity $I(n)$ detects the deviation from the condition of increase that the diffusion entropy would have in the random case. Since in the regime of transition the entropy increase can be much slower than in the corresponding random case, the quantity $I(n)$ can also bear negative values. This indicator affords a satisfactory way to detect local properties. As an example, Fig. 6 shows a case based on the DNA model of Ref. [8], called Copying Mistake Map (CMM). This is a sequence of symbols 0 and 1 obtained from the joint action of two independent sequences, one equivalent to tossing a coin and the other equivalent to establishing randomly a sequence of patches whose length is distributed as an inverse power law with index $\mu$ fitting the condition $2 < \mu < 3$. The probability of using the former sequence is $1 - \epsilon$ and that of using the latter is $\epsilon$. We choose a time dependent value of $\epsilon$

$$\epsilon = \epsilon_0 [1 - \cos(\omega t)].$$  \hspace{1cm} (19)

In Fig. 6 we show how this periodicity is perceived by using the two-windows generalization, proposed in this paper, of the DEM.

As a final example to show the efficiency of the new method of analysis, let us address the problem of the search of hidden periodicities in DNA sequences.
Fig. 7 shows a distinct periodic behavior for the human T-cell receptor alpha/delta locus. A period of about 990 base pairs is very neat in the first part of the sequence (promoter region), while several periodicities of the order of 1000 base pairs are distributed along the whole sequence.

These periodicities, probably due to DNA-proteins interactions in active eukaryotic genes, are expected by biologists, but the current technology is not yet adequate to deal with this issue, neither from the experimental nor from the computational point of view: such a behavior cannot be analyzed by means of crystallographic or structural NMR methods, nor would the current (or of the near future) computing facilities allow molecular dynamics studies of systems of the order of $10^6$ atoms or more.

4 Conclusions

The research work illustrated in this paper shows that the DEM is a very efficient way to detect the departure from ordinary Brownian motion with the shortest sequence as possible. On the basis of these results we are confident that it will be possible to predict the occurrence of catastrophic events, heart-quakes, heart attacks, stock-market crashes, and so on. We think that if all these misfortune events are anticipated by a correlation change, lasting for a fairly extended time period, then the DEM, within the double window procedure here illustrated, will signal in time their later occurrence. We refer to this method of analysis as Complex
Analysis of Sequences via Scaling AND Randomness Assessment (CASSANDRA), and we hope to prove by means of future research work that its prophetic power is worth of consideration. We wish that the CASSANDRA algorithm will have more fortune and will receive more credit than the daughter of Priam and Hecuba.

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