Neutrino dispersion relations at finite temperature and density in the Left-Right Symmetric Model

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Abstract

On this work we calculate the most general left-handed neutrino self-energy at one-loop order in perturbation theory using the Mellin summation technique. We perform this calculation in the real-time formalism of quantum field theory at finite temperature and density assuming that there exists an excess of leptons over antileptons in the medium. Thus we obtain for the first time a general expression for the fermionic effective thermal mass which depends on fermion mass, gauge boson mass, fermionic chemical potential and temperature. As an application of these results into the context of the Left-Right Symmetric Model we calculate the left-handed neutrino dispersion relations and we obtain the corresponding effective thermal masses from the unbroken, parity-broken and fully-broken phases.

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I. INTRODUCTION

The observational evidence obtained from neutrino oscillation experiments is well understood in terms of massive mixed neutrinos [1]. Due to neutrinos are massless in the Electroweak Standard Model (ESM), then neutrino oscillations are the first phenomenological evidence of physics beyond ESM. With the existence of neutrino oscillations, it is necessary to approach to ESM extensions in order to understand, in some way, the origin of neutrino masses and mixing.

Non-vanishing masses and mixing for neutrinos have, in principle, consequences in different scenarios in which these particles participate. One of these scenarios is the early universe in which neutrinos can influence processes such as the primordial generation of light nuclei (primordial nucleosynthesis) and possibly in other processes such as the generation of baryonic asymmetry via leptogenesis and large scale structure [2]. In the astrophysical context we know that neutrinos can influence strongly to the dynamics of supernova and neutron stars. One of the main factors that must be taken into account in analyzing this kind of effects is that neutrinos are propagating through a medium which interacts with them. As it has been shown in the literature [3, 4], the medium can alter the properties of propagation of particles. Therefore thermal and density effects can alter the oscillation patterns [5] tending to either suppress or enhance them depending on the way in which neutrinos interact with the medium [6]-[7]. Due to this the description of the neutrino propagation in a medium using thermal dispersion relations can be relevant for this purpose.

The main goal of this work is to calculate the most general left-handed neutrino self-energy at one-loop order in perturbation theory using the Mellin summation technique [8]. This calculation is performed in the real-time formalism of quantum field theory a finite temperature and density assuming that there exists an excess of leptons over antileptons in the thermal medium. From this calculation we obtain for the first time a general expression for a fermionic effective thermal mass which depends on the fermion mass, gauge boson mass, fermionic chemical potential and temperature. To illustrate the applicability of these results, we calculate the left-handed neutrino dispersion relations at finite temperature and density in the framework of the Left-Right symmetric model (LRSM). Starting from these neutrino dispersion relations, for the unbroken, parity-broken and fully-broken phases we obtain the neutrino effective masses at finite temperature and density.

In contrast with results obtained previously [9, 10], on this work we calculate the most general neutrino thermal self-energy analytically for the case where fermion and gauge are massive and with the presence of non-vanishing leptonic chemical potentials. We show how the general
expression for the neutrino effective thermal mass obtained from the most general neutrino thermal self-energy can be reduced for specific cases to the thermal effective masses known in the literature.

A former work about fermion dispersion relations at finite temperature has proceeded along the general remarks made by [11] and considering the possible extensions concerning parity and chirality violation. Later on reference [9], it was calculated the neutrino dispersion relations at finite temperature in the context of the ESM for an electroweak plasma with vanishing leptonic chemical potentials. On this reference the neutrino dispersion relations from the electroweak unbroken phase were calculated using an analytical approach, while these dispersion relations from the electroweak broken phase were calculated using a numerical approach for the difficulty to develop an analytical calculation. Additionally in [10], the neutrino dispersion relations from the electroweak unbroken phase of the ESM were obtained for a plasma with non-vanishing leptonic chemical potentials. On this last reference, the neutrino effective thermal masses were obtained on the basis of the leptonic chemical potentials.

Many other works approaching to the problem of fermionic dispersion relations at finite temperature and density do exist in the literature (for instance, to see the references into [9, 10]). Particularly, in [12] the finite temperature corrections to the density of neutralinos which could be relevant to the dark matter problem were calculated. Another relevant work on this field is presented in [13], where the propagation of Majorana fermions at finite temperature and density was studied and the results were applied to describe the thermodynamical properties of a system with neutralinos on the framework of a minimal supersymmetric extension of the ESM. In context of neutrino physics, related calculations have been performed in order to obtain the refraction index and other related thermodynamical quantities for these particles [14, 15].

This paper is organized as follows. First in section II we show some features of the LRSM which are relevant to our analysis. Next in section III the left-handed neutrino dispersion relations are determined analytically in a model-independent generalized way using the Mellin summation technique. Furthermore in section IV a general expression for a fermionic effective thermal mass is calculated and as an application of this result, we obtain the left-handed neutrino effective thermal masses from the unbroken, parity-broken and fully-broken phases in context of the LRSM. Finally, in section V we discuss some conclusions.
II. RELEVANT ASPECTS OF THE LRSM

The LRSM is an extension of the ESM motivated by the fact that parity should be restored at high energies [2]. On its simplest form, the model is based on the gauge group (ignoring strong interactions) $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. From this gauge group we expect to have right-handed neutrinos and three additional gauge bosons associated to weak right interactions. For this model just two gauge coupling constants are required from the left-right symmetry. As in the ESM, these coupling constants are parametrized in terms of the Weinberg angle and the positron charge. Thus the interaction Lagrangian for neutrinos is written as

$$L = -\frac{g}{\sqrt{2}}\left(\bar{\nu}_L\gamma^\mu l_L W^\mu_{L\mu} + \bar{l}_L\gamma^\mu \nu_L W^-_{L\mu}\right) + \frac{g}{\sqrt{2}}\left(\frac{g_2}{2}\left(c_w + s_w t_w\right)Z_\mu + \frac{g'}{2}t_w Z'_\mu\right) - \frac{g}{\sqrt{2}}\left(\bar{N}_{L\mu} W^\mu_{R\mu} + \bar{l}_R\gamma^\mu N_{L\mu} W^-_{R\mu}\right) + N_{L\mu} N_{R\mu} \frac{1}{2}\left(g \frac{\sqrt{c_{2w}}}{c_w} + g' t_w\right) Z'_\mu,$$

where $l$ runs over $e, \mu, \tau$, $\nu_L$ represents the left-handed neutrino fields, $N_{L\mu}$ the right-handed neutrino fields, $l_L$ the left-handed charged lepton fields, $l_R$ the right-handed charged lepton fields, $W^\pm_{L\mu}$ the left charged gauge boson fields, $W^\pm_{R\mu}$ the right charged gauge boson fields, $Z$ the left neutral gauge boson field, $Z'$ the right neutral gauge boson field, $c_w = \cos \theta_w$ and $c_{2w} = \cos(2\theta_w)$. Scalar multiplets are introduced on this model in such a way that parity is spontaneously broken at high energy scale. A second spontaneous symmetry breaking occurs at the electroweak scale and recovers the ESM phenomenology. The simplest way to obtain such a pattern of symmetry breaking and allow the existence of Majorana neutrinos is to include the following scalar multiplets: A bidoublet $\Phi : (1/2, 1/2, 0)$ and two triplets $\Delta_R : (0, 1, 2)$, $\Delta_L : (1, 0, 2)$. The presence of these fields allows the following Yukawa Lagrangian for leptons

$$-L_Y = \sum_{i,j} \left[ h_{ij} \bar{\Psi}_L \Phi \Psi_{Rj} + \tilde{h}_{ij} \bar{\Psi}_L \Phi \tilde{\Psi}_{Rj} + f_{ij} \left( \bar{\Delta}_L \Psi_{Lj} + \bar{\Psi}_{Li} \Delta_R \Psi_{Rj} \right) \right] + h.c.$$

where the sum runs over the three lepton flavors, $\Psi^T_L = (\nu_L, l_L)$ is the left-handed lepton doublet and $\Psi^T_R = (N_R, l_R)$ is the right-handed lepton doublet. After spontaneous symmetry breaking the scalar fields gain a non-vanishing vacuum expectation value

$$\langle \Phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} k_1 e^{i\alpha} & 0 \\ 0 & k_2 \end{pmatrix},$$
\[
\langle \Delta_L \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ v_L & 0 \end{pmatrix}, \\
\langle \Delta_R \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ v_R e^{i\theta} & 0 \end{pmatrix},
\]

(4)

(5)

with the phases \(\alpha\) and \(\theta\) being the sources of CP-violation on this model, which means that the violation of this symmetry is spontaneous. Because of the mentioned symmetry breaking, neutrinos gain mass. The resulting mass matrix can be block-diagonalized and one of such blocks is the mass matrix for heavy neutrinos \(M_{\text{heavy}} \approx f_{\text{diag}} v_R\), with \(f_{\text{diag}}\) representing the diagonalized form of the Majorana-Yukawa coupling matrix \(f_{ij}\). The other block corresponds to the mass matrix for the light neutrinos and can be written, in the limit \(v_R \gg k_1, k_2\), as

\[
M_\nu = f v_L + (k_1 e^{i\alpha} h + k_2 g)(f_{\text{diag}} v_R e^{i\theta})^{-1}(k_1 e^{i\alpha} h^t + k_2 g^t).
\]

(6)

Then a see-saw pattern of masses for neutrinos is obtained on this model. Given the present experimental bounds on neutrino masses \([1]\) (\(\sum \nu m_\nu < 1\) eV) and supposing the Majorana-Yukawa couplings are of order unity, we have that the mass of the heavy right neutrinos must be at least of order \(10^{15}\) GeV, thus the energy scale associated to the breaking of the left-right symmetry is of this order. For this reason we expect that the right gauge bosons also gain masses on this energy scale.

III. DISPERSION RELATIONS FOR LEFT-HANDED NEUTRINOS

As it has been discussed in the literature \([3]\), the emergence of thermal effective masses for particles in a medium can be seen as a consequence of the existence of a privileged frame of reference for the system, the rest frame of plasma. This lack of invariance makes that the poles of the propagators can no longer be located at physical vacuum mass but at a shifted value. Therefore, we must consider the corrections for the poles of the propagator of these fermions due to the medium at finite temperature and density in order to find the effective masses for the neutrinos. To compute this self-energy we use the real-time formalism of quantum field theory at finite temperature and density. Using this formalism it is possible to separate directly the contributions to the self-energy coming from the vacuum and the medium. The relevant diagrams for the calculation at one-loop order have the generic form shown on figure \([4]\). The corresponding Feynman rules on this formal-
ism are essentially the same as those in the conventional case and the only modification arises in the propagators. The propagator for fermions is

$$S(q) = (q + M_f) \left[ \frac{1}{q^2 - M_f^2} + i\Gamma_f(q) \right],$$  \hspace{1cm} (7)$$

whereas the propagators for gauge bosons (in a generalized $\xi$-gauge) and scalars are

$$D_{\mu\nu}(p) = \left(-g_{\mu\nu} + \frac{p_\mu p_\nu(1 - \xi)}{p^2 - \xi M_B^2}\right) \left[ \frac{1}{p^2 - M_B^2} - i\Gamma_b(p) \right],$$  \hspace{1cm} (8)$$

$$D(p) = \frac{1}{p^2 - M_B^2} - i\Gamma_b(p),$$  \hspace{1cm} (9)$$

respectively. The functions $\Gamma_{b,f}$ are depending on temperature $T$ and leptonic chemical potential $\mu$. These functions are given by

$$\Gamma_{b,f}(p) = 2\pi\delta(p^2 - M_{B,I}^2)n_{b,f}(p),$$  \hspace{1cm} (10)$$

where $n_b$ represents the Bose-Einstein distribution and $n_f$ the Fermi-Dirac distribution. These distributions are given by

$$n_b(p) = \frac{1}{\exp\left(\frac{p^0}{T}\right) - 1},$$  \hspace{1cm} (11)$$

$$n_f(p) = \theta(p \cdot u)n_f^-(p) + \theta(-p \cdot u)n_f^+(p),$$  \hspace{1cm} (12)$$

$$n_f^\pm(p) = \frac{1}{\exp\left(\frac{p^0 \pm u \cdot p}{T}\right) + 1}.\hspace{1cm} (13)$$

As it was discussed in [4], it is enough to consider the real part of the finite temperature contribution of the diagrams of figure 1 to find a dispersion relation and its corresponding effective thermal mass. At this point it is necessary to specify the phase of symmetry breaking of the LRSM in which we are working. For the sake of generalization we will approach to the case in which left-right symmetry is fully broken in such a way that all the fermions and bosons of this model
gain mass. Given the chiral nature of the interactions on this case, we expect that the real part of
the self-energy for neutrinos of momentum \( K^\mu \) propagating through a medium with four-velocity
\( u^\mu \) can be parametrized by virtue of the Lorentz covariance as \([11]\)

\[
Re \Sigma' = -\frac{K}{a_L + a_R} - \frac{\hat{\psi}(b_L + b_R)}{b_L + b_R} + m_\nu (c_L + c_R),
\]

(14)

where the Lorentz invariant functions \( a_{L,R}, b_{L,R} \) and \( c_{L,R} \) depend on the scalars \( K^2 \) and \( K \cdot u \).

These functions are separating the contributions associated to interactions of each chirality. In \([9]\)
it was shown that the presence of a term proportional to the mass in (14) makes impossible to
associate a specific chirality to the dispersion relations. However as will be discussed later, the
smallness of the left-handed neutrino masses causes the deviation from the chiral behavior of the
dispersion relations to be very small. Therefore we can neglect safely the third term in (14) and
we can assume that the real part of the left-handed neutrino self-energy has the general form

\[
Re \Sigma' = -\frac{K}{a_L + a_R} - \hat{\psi}(b_L + b_R).
\]

(15)

As we are interested in the dynamics of left-handed neutrinos, we will focus only on the contribu-
tions to the left-handed invariant functions. Forthcoming we omit the sub-index for the Lorentz
invariant functions \( a_L \) and \( b_L \). With this parametrization there is no coupling between left and right
handed contributions on the dispersion relation \([4]\).

The invariant functions can be calculated from the expressions

\[
a = \frac{1}{4\kappa^2} \left[ Tr \{ KRe \Sigma' \} - \omega Tr \{ \hat{\psi}Re \Sigma' \} \right],
\]

(16)

\[
b = \frac{1}{4\kappa^2} \left[ (\omega^2 - \kappa^2)Tr \{ \hat{\psi}Re \Sigma' \} - \omega Tr \{ KRe \Sigma' \} \right],
\]

(17)

where we have defined the Lorentz-scalar variables \( \omega \) and \( \kappa \) as \( \omega = K \cdot u \) and \( \kappa = ((K \cdot u)^2 - K^2)^{1/2} \). These variables are known in the literature as the invariants of energy and momentum,
respectively. With these definitions the problem of finding the thermal self-energy is reduced
to computing the traces \( Tr \{ KRe \Sigma' \} \) and \( Tr \{ \hat{\psi}Re \Sigma' \} \). The calculation of these traces using the
diagrams of figure \([4]\) is relatively long and proceeds in a similar way to the calculations shown in
\([9, 10]\). Thus we restrict ourselves to show the results for the traces in terms of integrals of the
thermal distribution functions. For the diagrams that involve gauge bosons, we have that the direct
calculation of these traces in the Feynman gauge (\( \xi = 1 \)) yields to

\[
Tr \{ KRe \Sigma' \} = \frac{8}{(2\pi)^2} \sum_{B,I} C_{B,I} \int_0^\infty d\rho \rho^2 \left[ \frac{n_B(\epsilon_B)}{\epsilon_B} + \frac{n_I^-(\epsilon_I)}{2\epsilon_I} + \frac{n_I^+(\epsilon_I)}{2\epsilon_I} \right]
\]
$$Tr\{\not{d}Re\Sigma\}' = \frac{2}{(2\pi)^2} \sum_{B,I} C'_{B,I} \int d\rho \rho^2 [n_f^-(\epsilon_I) - n_f^+(\epsilon_I)],$$  

(18)

$$\text{where } \epsilon_{I,B} = \sqrt{\rho^2 + M_{I,B}^2} \text{ and the sum runs over all the possible diagrams. The coefficients } C_{B,I} \text{ and } C'_{B,I} \text{ are coming from the respective vertex factors of Feynman rules. These coefficients in terms of the group generators are written as}$$

$$C_{B,I} = 2g^2 t^B_{ii} t^B_{Ii},$$  

(20)

$$C'_{B,I} = g^2 t^B_{ii} t^B_{II},$$  

(21)

$$\text{where } i \text{ labels the neutrino species whose thermal self-energy is calculated. Up to this point, we have not used the explicit constants for the LRSM. So these expressions can be evaluated straightforward for any non-abelian gauge theory. We must note that in the unbroken phase of non-abelian gauge theories with vanishing chemical potentials, the integrals can be extracted from those sums and these simplify to quadratic Casimir invariants. The explicit results for this situation in the LRSM will be shown later. All the calculations presented here have been performed in the Feynman gauge, however the case of a generalized } \xi \text{-gauge will be discussed briefly later on. It is necessary to point out that the last term of expression (18) corresponds to the contribution of the tadpole diagram represented on Fig.}$$

[1]  

This tadpole diagram is different from zero only for non-vanishing leptonic chemical potentials and massive gauge bosons. The contribution corresponding to the diagrams involving scalars has the same form but the constants } C \text{ are given now in terms of the Yukawa-Majorana constants that couple the neutrinos and the scalars divided by 2, while the constants } C' \text{ vanish due to the scalar tadpole on Figure}$$

[1]  

is identically zero.

The analytic evaluation of the integrals appearing in (18) and (19) is complicated and generally one needs to resort to numerical techniques or diverse approximations using series. For the last case, the series are convergent only in a limited range of temperatures and/or densities. Nevertheless for the purpose of this study we will use the method of Mellin transforms [8, 16] to arrive to series that converge for any value of the parameters. To outline the method we will evaluate the integral

$$I_1 = \int_0^\infty d\rho \rho^2 \frac{n(\rho)}{2\epsilon},$$  

(22)

where } $$\epsilon = \sqrt{\rho^2 + M^2} \text{ and}$$

$$n(\rho) = \frac{1}{e^\beta(\epsilon - \mu) - \eta} + \frac{1}{e^\beta(\epsilon + \mu) - \eta}.$$

(23)
The parameter $\eta$ tells us the kind of distribution we are using. Specifically $\eta = -1$ for the fermion distribution and $\eta = +1$ for the boson distribution. By using the geometric series, it is possible to write this integral as

$$I_1 = \sum_{n=1}^{\infty} \eta^{n-1} \frac{1}{\beta^2} \int_{x-y}^{\infty} du \sqrt{(u+y)^2 - x^2 e^{-nu} + \{y \to -y\}}, \quad (24)$$

where we have performed the change of variable $x = \beta M$ and $y = \beta \mu$. The inner integral is evaluated in terms of Laplace transforms. Substituting its value and simplifying it, we find that

$$I_1 = \sum_{n=1}^{\infty} \eta^{n-1} \frac{n^{-1}}{\beta^2} x \cosh(ny) K_1(nx), \quad (25)$$

where $K_1$ represents the modified Bessel function of order 1. Essentially, the expression (25) corresponds to the integral that we are looking for. However, this expression is not convenient at approaching our problem because it involves a double series which converges slowly. To solve this problem, we rewrite the series in terms of a simple series by using the Mellin resummation method [8]. This method is based on the use of the Mellin integral transform [16, 17]. Essentially the method is based on the use of the following identity

$$\sum_{n=1}^{\infty} \eta^{n-1} n^{-\nu} f(nx) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} (1 - \eta)_{1-s} \zeta(s) f^*(s-\nu) x^{-s+\nu}, \quad (26)$$

where $\zeta$ is the Riemann zeta function and the integration is performed on a line in the complex plane with $Re[z] = c$. The function $f^*$ is a Mellin transform of $f$ and is defined as

$$f^*(s) = \int_{0}^{\infty} f(x) x^{s-1} dx. \quad (27)$$

The identity (26) together with (25) allow us to write the integral $I_1$ as an integral in the complex plane. If certain conditions of asymptotic behavior are fulfilled, we can consider this complex integral as a contour integral which is evaluated by using the residue theorem [16, 17]. This evaluation is in general a sum over the residues which is the series that we are looking for. The Mellin transformation of the function $f(x) = \cosh(bx) K_1(x) (b = \mu/M)$ can be written as [18]

$$f^*(s) = \frac{2^s}{4} \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(s - \frac{1}{2}\right)} \times _2 \text{F}_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; b^2\right). \quad (28)$$

On this paper we limit ourselves to show the final result for $I_1$. Given that the series depends on the poles of the integrand in (26), the expansion of the integral depends strongly on $\eta$. Thus the integral $I_1$ can be written as

$$I_1 = \sum_{l=0}^{1} \sum_{n=0}^{l} (-1)^n (4^{n-l} - (1-\eta)4^{n-l})(2\pi)^{2-2l} \frac{B_{2-2l}(n-l)!}{(2-2l)!(l-n)!(2n)!} x^{2l-2n} y^{2n} \quad (29)$$

$$I_1 = \sum_{l=0}^{1} \sum_{n=0}^{l} (-1)^n (4^{n-l} - (1-\eta)4^{n-l})(2\pi)^{2-2l} \frac{B_{2-2l}(n-l)!}{(2-2l)!(l-n)!(2n)!} x^{2l-2n} y^{2n} \quad (29)$$
\[-(1 + \eta)\frac{\pi}{2}(x^2 - y^2)^{1/2} - \eta x^2 \left[ \log \frac{x}{4\pi} + (1 - \eta) \log 2 + \gamma - \frac{1}{2} \right] \]
\[+ 2 \sum_{l=1}^{\infty} \sum_{n=0}^{l} (-1)^{i+1}(4^{n-l-1} - (1 - \eta)4^{n-1}) \frac{(2\pi)^{-2l}(2l)!\zeta(2l + 1)}{(2n)!((l - n)!(l - n + 1)!)} x^{2+2l-2n} y^{2n}, \quad (29)\]

where \(B_{2-2l}\) are the Bernoulli numbers. This series is valid for any value of temperature, masses and leptonic chemical potential.

It is possible to obtain series of the form (29) for all the integrals appearing in (18) and (19). The convergence of these series is faster at high temperatures, so the partial sums approximate better the full expression when \(\beta \to 0\). In high temperature limit, we can approximate (18) and (19) as

\[Tr\{KRe\Sigma'\} = \frac{4}{(2\pi)^22\beta^2} \sum_{B,I} C_{B,I} \left\{ \frac{\pi^2}{2} - \pi x_B + \frac{y_I^2}{2} + \frac{2\gamma - 1}{4} (x_B^2 - x_B^2) \right\} \]
\[\quad - \frac{2\omega}{(2\pi)^2\beta} \sum_{B,I} C_{B,I}^\prime \left\{ \frac{\pi^2}{6} y_I + \frac{1}{64} y_I \left( \frac{2}{3} y_I^2 - x_B^2 \right) \right\} , \quad (30)\]

\[Tr\{\bar{d}Re\Sigma'\} = \frac{4}{(2\pi)^22\beta^2} \log \left( \frac{\omega_+}{\omega_-} \right) \sum_{B,I} C_{B,I} \left\{ \frac{\pi^2}{4} - \pi x_B + \frac{y_I^2}{4} + \frac{2\gamma - 1}{8} (x_B^2 - x_B^2) \right\} , \quad (31)\]

where we have defined \(\omega_\pm = \omega \pm \kappa\). From these expressions and (16) and (17), it is found that the Lorentz invariant functions \(a\) and \(b\) are given by

\[a = -\frac{M'^2}{\kappa^2} \left[ \frac{\omega}{2\kappa} \log \left( \frac{\omega_+}{\omega_-} \right) - 1 \right] , \quad (32)\]
\[b = -\frac{M'^2}{\kappa^2} \left[ \omega - \frac{\omega^2 - \kappa^2}{2\kappa} \log \left( \frac{\omega_+}{\omega_-} \right) \right] , \quad (33)\]

where the constant \(M'^2\) is written as

\[M'^2(\beta) = \frac{1}{(2\pi)^22\beta^2} \sum_{B,I} C_{B,I} \left[ \frac{\pi^2}{2} - \pi \beta M_B + \frac{\beta^2 \mu_I^2}{2} + \frac{2\gamma - 1}{4} \beta^2 (M_I^2 - M_B^2) \right] \]
\[- \frac{2\omega}{(2\pi)^2\beta} \sum_{B,I} C_{B,I}^\prime \left[ \frac{\pi^2}{6} \mu_I + \frac{\beta^2 \mu_I^2}{64} \left( \frac{2}{3} \mu_I^2 - M_I^2 \right) \right] . \quad (34)\]

This constant contains all the information about the interactions that neutrinos have with the medium. However, it can not be regarded as an effective thermal mass because of the dependency on the invariant energy \(\omega\) induced by the tadpole contribution. In spite of this, \(M'\) reduces to reported results for thermal effective masses in the limit of vanishing masses and chemical potentials [4, 9, 10].

It is well known that the dispersion relation can be found by determining the poles of the propagator corrected with the fermionic self-energy. For the massless fermions, it has been shown
that the dispersion relation for each chirality is given by the solution of
\[ \omega (1 + a_{L,R}) + b_{L,R} = \pm \kappa (1 + a_{L,R}), \] (35)
where the positive and negative signs refer to quasifermions associated to neutrinos and quasiantifermions associated with the antineutrinos in the plasma, respectively. In the literature the dispersion relations for each case are known as the normal and abnormal branches. Restricting to the left chirality, it is possible to arrive to an expression that involves both dispersion relations
\[ \omega \pm \kappa = \frac{M'}{2\kappa} \log \left( \frac{\omega_+}{\omega_-} \right) \left( 1 \mp \frac{\omega}{\kappa} \right) \pm \frac{M'^2}{\kappa}. \] (36)
On this last expression, we have neglected infrared-divergent terms that arise from the fact that the traces \(18\) and \(19\) are no longer proportional to each other as in the massless case. As the resummation technique has shown that the calculation is infrared-safe \([19]\), we ignore those divergent terms and define the finite part by demanding that the results reduce to the known expressions for the massless case in such a limit.

IV. NEUTRINO EFFECTIVE THERMAL MASSES

The quantity that appears in the right hand side of \((36)\) can be defined as an effective potential that acts on the (anti)neutrinos. For the neutrino case and following a similar procedure as shown in \([4]\), it is possible to find that the dispersion relation in the small momenta limit can be approximated by
\[ \omega = (M^2 + m' \omega) \left( 1 + \frac{2}{3} \frac{\kappa}{\omega} + \frac{\kappa^2}{\omega^2} + O\left( \frac{\kappa^3}{\omega^3} \right) \right), \] (37)
where
\[ M^2 = \frac{1}{(2\pi^2)^2 \beta^2} \sum_{B,I} C_{B,I} \left[ \frac{\pi^2}{2} - \pi \beta M_B + \frac{\beta^2 \mu_I^2}{2} - \frac{4}{3} \beta^2 (M_{I}^2 - M_B^2) \right], \] (38)
\[ m' = -\frac{2}{(2\pi^2)^2 \beta^2} \sum_{B,I} C_{B,I} \mu_I \left[ \frac{\pi^2}{6} + \frac{\beta^2}{64} \left( \frac{2}{3} \mu_I^2 - M_I^2 \right) \right], \] (39)
From the zero momentum limit of \((37)\), we obtain that the effective thermal masses for quasifermions \(M_+\) and quasiantifermions \(M_-\) are
\[ M_\pm = \frac{1}{2} (m' \pm \sqrt{m'^2 + 4M^2}). \] (40)
We can observe that these effective thermal masses are different from each other. This fact is reflecting the asymmetry between number of fermions and antifermions which is described by the
fermionic chemical potential. We remarkably notice that the quantity $M^2$ provides the effective mass for both particles and antiparticles in the vanishing chemical potential case, since $m'$ also vanishes.

We notice that the expression given by (40) is the most general fermionic effective thermal mass that is possible to obtain. This effective thermal mass depends on the fermion mass ($M_I$), the gauge boson mass ($M_B$), the leptonic chemical potential ($\mu_I$) and the temperature ($T = 1/\beta$).

We observe that if we fix $M_F = M_B = \mu_I = 0$, the expression (40) leads us to

$$M_\pm = M = \frac{g^2 C(R)}{8} T^2.$$  \hfill (41)

This result corresponds to the effective mass for the case of a massless fermion interacting through a massless gauge boson mass, being $C(R)$ the quadratic Casimir invariant of the representation defined by $(L^A L^A)_{mn} = C(R) \delta_{mn}$, in agreement with [4].

Now if we fix $M_F = M_B = 0$ into the expression (40), for this case the tadpole does not exist and the expression (40) implies $m' = 0$. For this reason the expression (40) leads us to

$$M_\pm = M = \frac{g^2 C(R)}{8} \left( T^2 + \frac{\mu_I^2}{\pi^2} \right),$$  \hfill (42)

in agreement with [10].

At this point, it is important to discuss further the validity of neglecting the term proportional to the neutrino mass in (14). For simplicity we will ignore the contribution due to the tadpole diagram, in such a way that the analysis is accurate in the vanishing chemical potential regime. The properties of the dispersion relation for massive neutrinos on this regime have been discussed in [20], therein it was found that for small momenta the normal and abnormal branches for neutrinos of mass $m_\nu$ can be written as

$$\omega = M'_+ + \frac{1}{M^2 + M'_+^2} \left[ M^2_{\nu} + \frac{1}{m_\nu} \left( \frac{3M_+^2 - M^2_{\nu}}{3M'_+} \right)^2 \right] K^2 + O(K^4),$$  \hfill (43)

$$\omega = M'_- + \frac{1}{M^2 + M'_-^2} \left[ M^2_{\nu} + \frac{1}{m_\nu} \left( \frac{3M'_- - M^2_{\nu}}{3M'_-} \right)^2 \right] K^2 + O(K^4),$$  \hfill (44)

where $M$ is the effective thermal mass at zero chemical potential defined as above and

$$M'_+ = \frac{1}{2} \left[ (m_\nu^2 + 4M^2)^{1/2} + m_\nu \right],$$  \hfill (45)

$$M'_- = \frac{1}{2} \left[ (m_\nu^2 + 4M^2)^{1/2} - m_\nu \right].$$  \hfill (46)
Considering the present bounds on neutrino masses \( \sum m_\nu \sim 0.1 \text{ eV} \) and that the temperatures of the systems of interest are at least of order \( 10^9 \text{ K} \sim 10^5 \text{ eV} \), we can expect safely that \( M \gg m_\nu \). Using the binomial series in expressions (45) and (46), we can find that

\[
M'_\pm \approx M \left( 1 + \frac{1}{8} \frac{m_\nu^2}{M^2} \pm \frac{m_\nu}{M} \right).
\]  

(47)

Thus, using (43) and (44), we see that for vanishing momenta the normal and abnormal branches both tend to the thermal effective mass (34) because the terms of order \( m_\nu \) can be neglected. So this argument validates our approximation. On the following section we will apply the formula developed here for the specific case of the left-handed neutrinos in the LRSM context by expressing the explicit couplings and diagrams corresponding to each symmetry breaking phase.

A. Effective thermal masses for the unbroken phase

On the unbroken phase all the fermions and gauge bosons described by LRSM are massless. For the special case in which all leptonic chemical potentials vanish, the neutrino effective thermal masses simplify considerably. The explicit form of the neutrino thermal self-energy depends on the diagrams in which the neutrino is involved. The expressions (32) and (33) specify the final form of the effective thermal masses for the left-handed neutrinos. The diagrams with an exchange of charged scalar and \( W_{L, R}^\pm \) charged electroweak gauge bosons induce a flavour change in the incoming neutrino \( I \) to a different outgoing neutrino \( F \). In the latter contributions, the flavour \( i \) of the internal charged lepton (inside the loop) runs over the three lepton flavors. The one-loop contribution to the real part of the self-energy lead us to write the squared value of the effective thermal masses as

\[
M_{2iF}^2 = \frac{T^2}{8} \left[ \left( \frac{3}{2} g^2 + g'^2 \right) \delta_{IF} + \frac{3}{2} \left[ (f f')_i + (\tilde{h} \tilde{h}')_i + (\tilde{g} \tilde{g}')_i \right]_{IF} \right].
\]  

(48)

After the see-saw mechanism is developed, the effective thermal masses can be written in terms of the left-handed neutrino masses \( m_{\nu_i} \), the charged lepton masses \( m_i \), the Maki-Nakagawa-Sakata (MNS) matrix \( U \) and the charged electroweak gauge boson mass \( M_{W_L} \) as

\[
M_{2iF}^2 = \frac{T^2}{8} \frac{g^2}{4} \left[ \left( 2 + \frac{1}{c_w} + \frac{m_{\nu_i}^2}{M_{W_L}^2} \right) \delta_{IF} + U_{iF}^\dagger \frac{m_i^2}{M_{W_L}^2} U_{iF} \right].
\]  

(49)

The different Yukawa-Majorana coupling constants for the distinct neutrino flavors introduce the flavour non-degeneracy of the branches that describe the quasi-particles. The neutrino effective masses are non-degenerate since the Yukawa-Majorana coupling constants are different for the
different flavors. The differences between the neutrino effective masses can not be unworthy and they can affect the left-handed neutrino oscillations.

In order to get a deeper insight about these results on the LRSM context, it is appropriate to compare the thermal effective masses given by (48) with their counterpart of the ESM without chemical potentials. After some simplification we obtain that the neutrino effective masses in the ESM are given by

\[ M_{i}^{2} = \frac{T^{2}}{8} \left( \frac{3}{4} g^{2} + g'^{2} + \frac{3}{2} |f|_{i}^{2} \right). \]  

Thus, it is possible to reproduce the results of [4] but with an effective thermal mass for each neutrino species. These effective masses can be written in terms of the charged lepton masses \( m_{i} \) and the charged electroweak gauge boson mass \( M_{W_{L}} \) as

\[ M_{i}^{2} = \frac{T^{2}}{8} \frac{g^{2}}{4} \left( 2 + \frac{1}{c_{w}} \frac{m_{i}^{2}}{M_{W_{L}}^{2}} \right). \]  

in agreement to a previous computation performed in [9]. On this reference the neutrino dispersion relations for the electroweak unbroken phase of the ESM were explicitly computed.

B. Effective thermal masses for the parity-broken phase

On the phase where the temperature is low enough so that the original gauge symmetry is broken down to the electroweak group \( SU(2)_{L} \times U(1)_{Y} \), the left-handed neutrinos do not possess mass but the contributions to the thermal effective mass that involves the right-handed particles begin to be thermally suppressed. Taking into account (25), we see that in the limit \( x = \beta M \rightarrow \infty \) the modified Bessel function \( K_{1}(nx) \) behaves asymptotically as \( \exp(-nx) \). So the contribution due to this integral tends to zero and the same property is satisfied by the remaining integrals in (18) and (19). Therefore it can be concluded that at temperatures \( T \ll v_{R} \) we can neglect the \( T \neq 0 \) contribution of the diagrams involving the right-handed particles. Entering to the unbroken electroweak symmetry regime but using the couplings given in the Lagrangians (1) and (2), from now on we will work in this regime.

Because in this case, the left gauge symmetry is respected the left-handed gauge bosons do not have mass. Furthermore, none of the fermions with which neutrinos couple neither have mass so the contribution of the gauge diagrams involving those particles is nearly the same as in the unbroken electroweak case but with the addition of terms depending on the fermionic chemical
potentials. Thus the thermal effective mass is

\[ M_i^{(g)^2} \frac{T^2}{8} \left( \frac{3}{2} g^2 + g'^2 \right) + \frac{1}{8\pi^2} \left[ g^2(\mu_c^2 + \mu_\mu^2 + \mu_\tau^2) + \frac{1}{2} g^2(c_w + s_w t_w)^2 \mu_i^2 \right], \tag{52} \]

where we have also gotten that the tadpole contribution mediated by \( Z \) vanishes. The only non vanishing tadpole term involves the \( Z' \) boson but this term is negligible in the regime in which we are working at \( (T \ll v_R) \). Hence, we obtain that in this regime \( m' \approx 0 \) and quasiparticles and quasiantiparticles gain nearly the same effective mass.

With respect to the scalar contribution, the structure of the effective thermal mass is heavily influenced by the pattern in which the different scalar fields gain masses and mixings. By counting the degrees of freedom of the scalar multiplets introduced to break the gauge symmetry of the LRSM, we obtain that there are 20 scalar fields involved on this dynamics. In order to give mass to scalar fields, a scalar potential is introduced and includes all possible terms respecting the gauge and Lorentz invariance. The complete form of this potential is complicated [21], so we will not enter into details here. Given the size of the matrices involved the analytic computation of scalar masses and mixings is very difficult in general. We will use the results of a numerical analysis presented in [22], so we can get an expression for effective thermal masses in this context.

An interesting feature is that, in general, the scalar spectrum depends on the CP-violating phases. For instance, with \( \alpha = 0 \) and \( \theta = \pi/2 \), we have 3 neutral bosons with masses of order \( v_R \) while the remaining neutrals are massless. For the situation in which both phases vanish, the number of massive neutral scalars increases to 5. In this work we will limit ourselves to work with \( \alpha = 0 \) and \( \theta = \pi/2 \) scenario, in which there are more Higgs bosons at low energies than in the ESM. With this, and using the values for scalar masses and mixing given by tables 7, 8 and 9 of [22], we can write the scalar contribution to the effective thermal masses as

\[ M_i^{(s)^2} = \frac{T^2}{16} \left( 3.00(\bar{h}h^i)_{ii} + (\bar{g}\bar{g}^i)_{ii} + 4.14(ff^i)_{ii} \right) + \frac{1}{2} \sum_j \left[ \bar{h}_{ij}\bar{h}^*_{ji}(2.00\mu_{\nu_j}^2 + \mu_{\tau_j}^2) + \bar{g}_{ij}\bar{g}^*_{ji}\mu_i^2 + f_{ij}f^*_{ji}(1.23\mu_{\nu_j}^2 + 3.09\mu_{\tau_j}^2) \right]. \tag{53} \]

We see that the additional scalar bosons have a decisive effect in the thermal mass. In contrast with the former case, we have that this result can be compared with the ESM predictions for contexts where the electroweak symmetry is restored as, for example, in the early universe.
C. Effective thermal masses in the fully-broken phase

Here we will present the thermal effective masses for the completely broken phase of the LRSM. It is remarkable that the formula, which we have presented before for the case of massive particles, allow us to obtain fully analytic results for this regime. Direct evaluation of equations (38) and (39), with the diagrams involving gauge bosons and neutrino species $i$, yield to

$$M_i^{(g)^2} = \frac{T^2}{8} \left( \frac{3}{2} g^2 + g'^2 \right) + \frac{1}{2\pi^2} \left[ -\pi T \left( \frac{g^2}{2} M_W + \frac{g^2}{4 c_w^2} M_Z \right) + \frac{1}{2} \left( \frac{g^2}{2} \sum_j U_{ij} U_{ij}^* \mu_{ij}^2 \right) \right] + \frac{g^2}{4 c_w^2} \sum_j U_{ij} U_{ij}^* \mu_{ij}^2 + \frac{2\gamma - 1}{4} \left( \frac{g^2}{2} \sum_j U_{ij} U_{ij}^* M_{ij}^2 + \frac{g^2}{4 c_w^2} \sum_j U_{ij} U_{ij}^* M_{ij}^2 \right) \left( -\frac{g^2}{2} M_W^2 - \frac{g^2}{4 c_w^2} M_Z^2 \right),$$

where we have taken into account the possibility of neutrino mixing through the MNS mixing matrix $U$. For the case of the ESM, it is relevant that this constant has the same form up to some couplings owning to the particle spectrum in which we are working with is the same. The $Z$-mediated tadpole diagram contribution takes the form

$$m_i = -\frac{T^2 g^2}{2\pi^2 M_Z^2} \left[ \frac{\pi^2}{6} \sum_j \left( \mu_{ij}^2 + \frac{1}{2}(c_w + s_w t_w)^2 \mu_{ij} \right) \right] + \frac{1}{64 T^2} \sum_j \left( \mu_{ij} \left( \frac{2}{3} \mu_{ij}^2 - M_{ij}^2 \right) + \frac{2}{6} (c_w + s_w t_w)^2 \mu_{ij}^3 \right).$$

(55)

It is interesting to note that this quantity takes the same value for all neutrino species. As in the parity-broken case, we expect that the main differences with the ESM arise from the larger Yukawa sector of the LRSM. Restricting ourselves to the same regime as the preceding section, we can write the scalar contribution as

$$M_i^{(s)^2} = \frac{T^2}{2(2\pi)^2} \left[ \frac{\pi^2}{2} \left( 3.00 (\tilde{h} h^\dagger)_{ii} + (\tilde{g} g^\dagger)_{ii} + 4.14 (f f^\dagger)_{ii} \right) \right] - \frac{\pi}{T} \left( (\tilde{h} h^\dagger)_{ii} (0.50 M_{\phi_D^0} + M_{\phi_E^0}) + (f f^\dagger)_{ii} (0.25 M_{\phi_L^+} + M_{\phi_l^+} + 0.50 M_{\phi_l^0} + M_{\phi_E^0}) \right) + \frac{1}{2 T^2} \sum_j \left( \tilde{h}_{ij} \tilde{h}_{ji}^* (2.00 \mu_{ij}^2 + \mu_{ij}^2) + \tilde{g}_{ij} \tilde{g}_{ji}^* \mu_{ij}^2 + f_{ij} f_{ji}^* (1.23 \mu_{ij}^2 + 3.09 \mu_{ij}^2) \right) + \frac{2\gamma - 1}{4 T^2} \left( \sum_j (\tilde{h}_{ij} \tilde{h}_{ji}^* + \tilde{g}_{ij} \tilde{g}_{ji}^* + 3.09 f_{ij} f_{ji}^* ) M_{ij}^2 + \sum_j (2 \tilde{h}_{ij} \tilde{h}_{ji}^* + 1.24 f_{ij} f_{ji}^*) M_{ij}^2 \right) - (\tilde{h} h^\dagger)_{ii} (0.5 M_{\phi_D^0}^2 + M_{\phi_E^0}^2) - (f f^\dagger) (0.25 M_{\phi_L^+}^2 + M_{\phi_l^+}^2 + 0.50 M_{\phi_l^0}^2 + M_{\phi_E^0}^2),$$

(56)
where we have used the nomenclature used in [22] for the scalar fields with definite mass. We note that these results are accurate in the case in which the temperatures are bigger than the masses of the particles. Even if the general series obtained before are convergent for any value of temperature, in the low $T$ regime the contribution of the imaginary part of the self energy must be taken into account. For this case (18) and (19) are not enough to describe the propagation of quasiparticles.

All these results have been calculated at Feynman gauge, however it is necessary to know if the results hold in other gauges. In the case of unbroken symmetry, former studies [9, 10] indicate that the self-energy calculated at one loop order is gauge invariant if just leading terms in temperature are kept. For the general case of broken symmetry, this property generally does not hold. But it has been shown [15] that although the neutrino self-energy can be gauge-dependent. In that case, the dispersion relation only receives gauge-dependent contributions at higher order on the coupling constants. So for the approximation regime that we have implemented here, we can state that the dispersion relation (36) is gauge invariant.

With these results it is possible to calculate a refraction index of plasma for neutrinos and antineutrinos. This index is defined as

$$n \equiv \frac{\kappa}{\omega_\kappa}.$$  \hspace{1cm} (57)

Using the dispersion relation (36), we have that the index of refraction for neutrinos can be written as

$$n_\nu = 1 - \frac{M'^2}{\kappa} \left[ 1 + \frac{1}{2} \log \frac{\omega_+}{\omega_-} \left( 1 - \frac{\omega}{\kappa} \right) \right].$$  \hspace{1cm} (58)

V. CONCLUSIONS

We have calculated the left-handed neutrino dispersion relations at finite temperature and density on the framework of the LRSM considering that there exists an excess of leptons over antileptons in the medium. On this context, we have obtained the neutrino effective thermal masses from the unbroken, parity-broken and fully-broken phases. These mentioned results were possible to be obtained due to our first calculations for the most general neutrino thermal self-energy using the Mellin summation technique. This last calculation has been performed using the real-time formalism of thermal field theory at finite temperature and density. For the first time we have found a general expression for the neutrino effective mass which depends on the fermion mass, the gauge boson mass, the leptonic chemical potential and the temperature. Different effective thermal
masses known in the literature can be obtained as specific cases from this general effective thermal mass.

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