A SURFACE OF GENERAL TYPE WITH \( p_g = q = 0, K^2 = 1 \).

CARYN WERNER

1. Introduction

In this paper we construct a minimal surface \( X \) of general type with \( p_g = q = 0, K^2 = 1 \), and \( \text{Tors} X \cong \mathbb{Z}/\mathcal{P} \). In [3], Campedelli noted that if a degree ten plane curve could be found having certain singularities, a double plane construction would yield a surface with \( p_g = q = 0 \). In [4], Oort and Peters construct such a double plane and compute the torsion group of their surface to be \( \mathbb{Z}/\mathcal{P} \); however we will show that the torsion group is actually \( \mathbb{Z}/\mathcal{P} \). (Weng Lin has also constructed surfaces with \( p_g = q = 0, K^2 = 1 \) using double covers, but I do not believe his results are published.)

For minimal surfaces of general type with \( p_g = q = 0 \) and \( K^2 = 1 \) it is known that \( |\text{Tors} X| \leq 5 \). (See for example [7].) By writing down generators for the pluricanonical rings, Reid [10] has described surfaces with torsion of order three, four, and five. Barlow [1, 2] has constructed surfaces with torsion of order two and four, as well as a simply connected surface.

The double plane constructions we call Campedelli surfaces, while numerical Godeaux surfaces are minimal surfaces of general type with the invariants \( p_g = q = 0, K^2 = 1 \). Here \( p_g = \dim H^0(X, \mathcal{O}_X(K)) = \dim H^2(X, \mathcal{O}_X), q = \dim H^1(X, \mathcal{O}_X), \) and \( K^2 = K \cdot K \) is the self-intersection number of the canonical class \( K \). Write \( h^i(D) = \dim \mathbb{C} H^i(X, \mathcal{O}_X(D)) \) for \( D \) a divisor on \( X \), \( \text{Tors} X \) for the torsion subgroup of the Picard group of \( X \), \( \equiv \) to represent linear equivalence of divisors, and \( |D| \) for the complete linear system of a divisor class \( D \).

2. The double plane construction.

Let \( D \) be a degree ten plane curve with an ordinary order four point at \( p \), five infinitely near triple points at \( p_1, \ldots, p_5 \), and no other singularities. An infinitely near triple point refers to a triple point which remains of order three after the plane is blown up at this point, so that all three tangent directions of \( D \) coincide. We assume that each triple point becomes ordinary after one blow up. Assume further that the six singular points do not lie on a conic, and that the system of plane quartics with double point at \( p \) and through each \( p_i \) with the same tangent direction as \( D \) is exactly a pencil.

Let \( \sigma_1 : Y_1 \to \mathbb{P}^5 \) be the blowup of \( \mathbb{P}^5 \) at \( p \), and let \( E = \sigma_1^{-1}(p) \) be the exceptional curve on \( Y_1 \).
The total transform of $D$ is $\sigma_1^*(D) = \tilde{D} + 4E$, where $\tilde{D}$ is the proper transform of $D$. Set $D_1 = \tilde{D} = \sigma_1^*(D) - 4E$.

Now let $\sigma_2 : Y_2 \to Y_1$ be the blowup of $Y_1$ at $p_1, \ldots, p_5$. With $E_i = \sigma_2^{-1}(p_i)$, the total transform of $D_1$ is $D_1 + \sum_{1}^{5}E_i$, where $D_1$ is the proper transform of $D_1$. Set

$$D_2 = \tilde{D}_1 + \sum_{1}^{5} E_i \equiv \sigma_2^*(D_1) - 2\sum_{1}^{5} E_i,$$

which is the reduced divisor consisting of the proper transform of the degree ten curve, together with the five exceptional curves $E_i$. As each $p_i$ is an infinitely near triple point of the original branch curve, $D_2$ has an order four point on each $E_i$.

Let $\sigma_3 : Y_3 \to Y_2$ be the blowup of each of these quadruple points, and let $F_1, \ldots, F_5$ be the corresponding exceptional divisors. We will write $E_i$ to denote both the exceptional curve on $Y_2$ and its proper transform on $Y_3$ (and similarly for $E$) so that $\sigma_3^*(E_i) = E_i + F_i$.

The total transform of $D_2$ is $\tilde{D}_2 + 4\sum_{1}^{5} F_i$, where $\tilde{D}_2$ is the proper transform of $D_2$; set

$$B = \tilde{D}_2 \equiv \sigma_3^*(D_2) - 4\sum_{1}^{5} F_i$$

$$= \sigma_3^*(\sigma_2^*(D) - 4E) - 2\sum_{1}^{5} E_i - 4\sum_{1}^{5} F_i$$

$$= \sigma^*(\tilde{D} - 4E - 2\sum_{1}^{5} E_i - 6\sum_{1}^{5} F_i)$$

where $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3$. If $H$ represents the pullback of the hyperplane class in $\mathbb{P}^2$, then

$$B \equiv 10H - 4E - 2\sum_{1}^{5} E_i - 6\sum_{1}^{5} F_i = 2\mathcal{L}$$

where

$$\mathcal{L} \equiv \nabla H - \epsilon E - \sum_{1}^{5} E_i - \sum_{1}^{5} F_i;$$

$B$ is now a non-singular even curve on the surface $Y_3$. The canonical divisor on $Y_3$ is

$$K_{Y_3} \equiv \sigma^*(K_{\mathbb{P}^2}) + E + \sum_{1}^{5} E_i + 2\sum_{1}^{5} F_i \equiv -3H + E + \sum_{1}^{5} E_i + 2\sum_{1}^{5} F_i.$$

Let $\pi : X \to Y_3$ be the double cover of $Y_3$ branched at $B$. Then

$$K_X \equiv \pi^*(K_{Y_3} + \mathcal{L})$$

$$\equiv \pi^*(2H - E - \sum_{1}^{5} F_i)$$

and $K_X^2 = 2[2H - E - \sum_{1}^{5} F_i]^2 = 2(4 - 1 + 5(-1)) = -4$. Since each $E_i$ is part of the branch locus and $E_i^2 = -2$ on $Y_3$, $[\pi^{-1}(E_i)]^2 = -1$. Let $X \to \tilde{X}$
be the map contracting these five \((-1)\) curves. Then
\[(K_X)^2 = (K_X)^2 + 5 = 1.\]

**Proposition 1.** \(\tilde{X}\) is a minimal surface of general type with \(p_g = 0, q = 0,\) and \(K^2 = \chi = 1.\)

Since \(\tilde{X}\) is obtained from \(X\) by blowing down five exceptional curves, we can compute \(p_g\) and \(q\) for the surface \(X.\) To compute these invariants, we will use the following,

**Projection formula.** Let \(\pi : X \to Y\) be a double cover branched along a smooth curve \(B \equiv 2L.\) For any divisor \(A\) on \(Y,\)
\[\pi_*O_X (\pi^* A) \cong O_Y (A) \oplus O_Y (A - L).\]

In particular,
\[\pi_*O_X (\mathcal{K}_X) \cong O_Y (\mathcal{K}_Y + \mathcal{L}) \oplus O_Y (\mathcal{K}_Y + (-\infty) \mathcal{L}).\]

Therefore in our example,
\[H^0 (O_X (\mathcal{K}_X)) \cong H^0 (O_{Y^3} (\mathcal{K}_{Y^3} + \mathcal{L})) \oplus H^0 (O_{Y^3} (\mathcal{K}_{Y^3})) ,\]
so that
\[p_g (X) = h^0 (O_X (\mathcal{K}_X)) = h^0 (K_{Y^3} + \mathcal{L}) + p_g (Y^3).\]

Since \(p_g (Y^3) = p_g (\mathbb{P}^3) = 0, p_g (X) = h^0 (K_{Y^3} + \mathcal{L}).\) The space \(H^0 (K_{Y^3} + \mathcal{L})\) corresponds to the the linear system \(|2H - E - \sum F_i|\) of conics through \(p, p_1, \ldots, p_5,\) so \(p_g (X) = 0.\) Also
\[H^0 (O_X (\mathcal{K}_X)) = H^0 (O_{Y^3} (\mathcal{K}_{Y^3} + \mathcal{L})) \oplus H^0 (O_{Y^3} (\mathcal{K}_{Y^3} + \mathcal{L})).\]

Since \(2K_{Y^3} + \mathcal{L} \equiv -\mathcal{H} + \sum \left(\mathcal{E}_j + \mathcal{F}_j\right), H^0 (2K_{Y^3} + \mathcal{L}) = 0.\) The divisor \(\sum E_i\)
is a fixed part of the linear system
\[|2K_{Y^3} + 2\mathcal{L}| = \left|4H - 2E - 2 \sum F_i\right|\]
since \(E_i \cdot F_i = 1\) and \(E_i^2 = -2;\) the difference \(|4H - 2E - 2 \sum F_i - \sum E_i|\) corresponds to quartics in \(\mathbb{P}^3\) with a double point at \(p,\) through each \(p_i,\) with the same tangent direction as the branch curve. By assumption this system is a pencil, thus
\[P_2 = h^0 (2K_X) = \dim H^0 (O_X (\mathcal{K}_X)) = 2.\]

Suppose \(S\) is the minimal model of \(\tilde{X};\) then \(P_2 (S) = 2\) and \(K_S^2 \geq K_X^2 = 1,\) so \(S\) is of general type (see for example [3]). But 2 = \(P_2 = \chi + K_S^2 = 1 + K_S^2,\) so \(K_S^2 = K_X^2\) and \(S : X.\) Thus \(X\) is minimal of general type with \(K^2 = 1;\) since \(q \leq p_g [3, 3.1, lemma 3], p_g = q = 0.\)
To construct a plane curve of degree ten with the necessary singularities, we will find an octic and a conic as follows.

We wish to find an octic $C$ with one order four point, one infinitely near triple point, and four tacnodes, where a tacnode refers to a double point which remains double after one blowup. Furthermore we want these tacnodes to lie on a conic $Q$ with the same tangent direction, so that the octic and conic will still intersect after the plane is blown up at these points.

Let $F$ be a homogeneous polynomial of degree eight in three variables defining an octic $C$ in $\mathbb{P}^2$. After imposing an order four point at $p = [1 : 0 : 0]$ and an infinitely near triple point at $p_1 = [0 : 1 : 0]$, $F$ has 23 free coefficients. Let $\gamma : \mathbb{P}^2 \to Q$, 

\[
\begin{align*}
[s : t] & \to [as^2 + bst + ct^2 : ds^2 + est + ft^2 : gs^2 + hst + it^2]
\end{align*}
\]

be a parametrization of a conic $Q$ in $\mathbb{P}^2$, where $a, b, c, d, e, f, g, h, i$ are variables over $\mathbb{C}$.

Set

\[
\begin{align*}
p_2 &= \gamma([0 : 1]) \\
p_3 &= \gamma([1 : 0]) \\
p_4 &= \gamma([1 : 1]) \\
p_5 &= \gamma([-1 : 1]).
\end{align*}
\]

The condition that $F$ have a double point at $p_i$ can be expressed by requiring the three partial derivatives of $F$ at $p_i$ to vanish, thus a double point is three linear conditions on the coefficients of $F$; a tacnode at a given point with a designated tangent direction puts six conditions on $F$, while a cusp at a given point with a given tangent direction is five linear conditions on the coefficients. If we impose tacnodes tangent to $Q$ on the octic at $p_2$ and $p_3$, this gives twelve linear relations on the coefficients of $F$. Imposing cusps tangent to $Q$ at $p_4$ and $p_5$ gives ten more relations; solving these gives an octic whose coefficients are polynomials in $a, b, c, d, e, f, g, h, i$. Imposing the conditions that $p_4$ and $p_5$ be tacnodes of $C$ gives two more linear relations in the coefficients of $F$, and therefore two higher degree polynomials in $a, b, c, d, e, f, g, h, i$. In solving these two relations for $a, b, c, d, e, f, g, h, i$ we hope to obtain an irreducible polynomial $F$ over $\mathbb{C}$, and thus an octic plane curve as desired.

We use Maple to compute the equations for these conditions on $F$, and to find the coefficients. Let $\{A_j\}_{1}^{22}$ be the equations corresponding to these conditions on $F$; the $A_j$ are linear in the coefficients of $F$.

Form the matrix $M$ generated by the $A_j$ where $M_{i,j}$ is the coefficient in $A_j$ of the $i$th coefficient of $F$. Then $M$ is a $22 \times 23$ matrix, and if we set $D_j$ to be the determinant of the matrix obtained from $M$ be deleting the $j$th column, we have $M((-1)^jD_j) = 0$, so that setting the $j$th coefficient of $F$ to be $(-1)^jD_j$ gives the desired octic.
In order for Maple to compute these determinants quickly enough, we first set $a = e = g = i = 1$ and $b = d = h = 0$ in the parametrization of $Q$. This reduces the number of free parameters in this problem to two, namely $c$ and $f$; since we will end up imposing two non-linear conditions on the remaining parameters, there is still the possibility of a non-degenerate solution. After finding the determinants $D_j$, the coefficients of $F$ become polynomials in $c$ and $f$.

Imposing the final two conditions on the octic, Maple finds several degenerate solutions, where the octic splits into several curves of smaller degree, and thus has more singularities, and also a solution for $c$ and $f$ giving an octic which we will show has the desired properties.

The branch curve $D$ is defined by the equations for the conic and the octic, which are both polynomials in three variables over $\mathbb{Z}[\alpha, \beta, \delta]$ where

\[
\begin{align*}
\alpha &= \sqrt{17} \\
\beta &= \sqrt{21 + 5\sqrt{17}} \\
\delta &= \sqrt[3]{5 + \sqrt{17}}.
\end{align*}
\]

The polynomial defining the conic $Q$, which is given parametrically by $\gamma$, is

\[
(9 \alpha \beta + 90 \alpha + 81 \beta + 234) x^2 + (+176 \alpha \beta + 1568 \alpha + 1200 \beta + 5920) y^2 \\
+ (57 \alpha \beta + 258 \alpha + 129 \beta + 1170) z^2 + (48 \alpha \beta \delta - 168 \alpha \delta - 48 \beta \delta - 936 \delta) xy \\
+ (-66 \alpha \beta - 348 \alpha - 210 \beta - 1404) xz + (48 \alpha \beta \delta + 168 \alpha \delta + 48 \beta \delta + 936 \delta) yz
\]

The octic $C$ is defined by $F = 0$ where $F$ is

\[
\begin{align*}
24 (14408408592 x^4 y^2 z + 50076004923 x^4 z^3 + 14182182144 x^3 y^4 &+ 219953469600 x^3 y^2 z^2 - 363210576777 x^3 z^4 - 109333732608 x^2 y^2 z^3 \\
+ 831133690121 x^2 z^5 + 858975454416 xy^2 z^4 - 772939669603 xz^6 &+ 254940551336 z^7) y \alpha \beta \delta \\
+ (-72389196288 x^4 y^4 - 333539379632 x^4 y^2 z^2 - 1065820046526 x^4 z^4 &- 8342111361024 x^3 y^4 z + 394542871808 x^3 y^2 z^3 + 10184161263912 x^3 z^5 \\
+ 20168534212608 x^2 y^4 z^2 + 53110587008192 x^2 y^2 z^4 - 32270723397636 x^2 z^6 &- 100932292129536 xy^2 z^5 + 38252243189640 xz^7 + 47211381447168 y^2 z^6 \\
- 15099861009390 z^8) \alpha \beta +
\end{align*}
\]
144 \left( 15490159728 x^4 y^2 z + 53840161671 x^4 z^3 + 15251365120 x^3 y^4 + 23648832416 x^3 y^2 z^2 - 39051259133 x^3 z^4 - 1175521621376 x^2 y^2 z^3 + 89360894925 x^2 z^5 + 923543229232 xy^2 z^4 - 831040262535 x z^6 + 274103997272 z^7 \right) y \alpha \delta + 24 \left( 59398585488 x^4 y^2 z + 2064680787 x^4 z^4 + 58496365824 x^3 y^4 + 90689935584 x^3 y^2 z^2 - 1497555836337 x^3 z^4 - 4507944789888 x^2 y^2 z^3 + 342652351793 x^2 z^5 + 3541646868816 xy^2 z^4 - 3186912029723 x z^6 + 1051146805480 z^7 \right) y \beta \delta + \left( -466877917440 x^4 y^4 - 21516582641184 x^4 y^2 z^2 - 6875617032333 x^4 z^4 - 53815549731840 x^3 y^2 z^3 + 65698123816692 x^3 z^5 + 130107481479168 x^2 y^4 z^2 + 342618718898784 x^2 y^2 z^4 - 208178748305934 x^2 z^6 - 651115201689280 xy^2 z^5 + 246765593291124 x z^7 + 304561087975168 y^2 z^6 - 97409351769549 z^8 \right) \alpha + \left( -2983444909312 x^4 y^4 - 13752106145280 x^4 y^2 z^2 - 4394491299054 x^4 z^4 - 34395989170176 x^3 y^2 z^3 + 41990375439720 x^3 z^5 + 8315076186176 x^2 y^4 z^2 + 218981688444672 x^2 y^2 z^4 - 133055599121316 x^2 z^6 - 41615449902208 xy^2 z^5 + 157718037119688 x z^7 + 194657513400832 y^2 z^6 - 62258322139038 z^8 \right) \beta + 48 \left( 191605550544 x^4 y^2 z + 665965983645 x^4 y^4 + 188641373952 x^3 y^4 + 391437865031 x^3 z^4 - 15450399432832 x^2 y^2 z^2 - 1105332893807 x^2 z^5 + 14123598740496 xy^2 z^4 - 10279400307101 x z^6 + 3390479204680 z^7 \right) y \delta - 1925078503680 x^4 y^4 - 88715185482528 x^4 y^2 z^2 + 28348893570645 x^4 z^4 - 221886790124544 x^3 y^2 z^3 + 104941198124928 x^3 y^4 z^2 + 1298803031999124 x^3 z^5 + 53644684151808 x^2 y^4 z^2 + 1412653204386144 x^2 y^2 z^4 - 858342969385662 x^2 z^6 - 2684616751584448 xy^2 z^5 + 1017440607056532 x z^7 + 1255737534555904 y^2 z^6 - 401629046099349 z^8.

The resulting singular points of the branch curve are

\[
p = [1 : 0 : 0]
\]
\[
p_1 = [0 : 1 : 0]
\]
\[
p_2 = [10 + 4 \alpha + 4 \beta : 3 \delta : 6]
\]
\[
p_3 = [1 : 0 : 1]
\]
\[
p_4 = [16 + 4 \alpha + 4 \beta : 3 \delta + 6 : 12]
\]
\[
p_5 = [16 + 4 \alpha + 4 \beta : 3 \delta - 6 : 12].
\]

We need to check that the branch curve \( D \) has no singularities outside the set \( \{p, p_1, \ldots, p_5\} \). Since \( F \) is a polynomial over the complex numbers, Maple is unable to quickly check that the cotic has no other singularities, so we use Macaulay to check the smoothness of \( C \) outside of the set \( \{p, p_1, \ldots, p_5\} \). As Macaulay only makes computations over finite fields, we first find a prime number \( P \) where \( \alpha, \beta, \) and \( \delta \) exist mod \( P \), so that we can map \( F \) to a polynomial over \( \mathbb{Z}/P \).
To check that $C$ has no singularities other than at the points $p, p_1, \ldots, p_5$, consider the map $\phi : \mathbb{Z}[\alpha, \beta, \delta] \to \mathbb{Z}/\mathfrak{p}[\mathfrak{p}]$ given by sending

$$
\begin{align*}
\alpha & \to 20452 \\
\beta & \to 6941 \\
\delta & \to 27962;
\end{align*}
$$

mapping $F$ to $F_\phi$ we obtain

$$
\begin{align*}
24082 x^4 y^4 + 3438 x^4 y^3 z + 4775 x^4 y^2 z^2 & + 29499 x^4 y z^3 + 12698 x^4 z^4 \\
+ 29927 x^3 y^5 & + 14121 x^3 y^4 z + 17243 x^3 y^3 z^2 + 3139 x^3 y^2 z^3 + 8704 x^3 y z^4 + 80 x^3 z^5 \\
+ 28712 x^2 y^4 z^2 & + 10654 x^2 y^3 z^3 + 12817 x^2 y^2 z^4 + 8239 x^2 y z^5 + 5515 x^2 z^6 \\
+ 28759 x y^3 z^4 & + 7372 x y^2 z^5 + 19696 x y z^6 + 28079 x z^7 \\
+ 1944 y^2 z^6 & + 24003 y z^7 + 13722 z^8.
\end{align*}
$$

**Claim 1.** $F_\phi$ has no singularities other than at $p, p_1, \ldots, p_5$.

First we have Macaulay compute $\text{Jac} F_\phi$, the Jacobian ideal of the octic generated by $\frac{\partial F_\phi}{\partial x}, \frac{\partial F_\phi}{\partial y}, \text{ and } \frac{\partial F_\phi}{\partial z}$, and the ideal $I$ associated to the points $p, p_1, p_2, p_3, p_4, p_5$. Since the zeros of $\text{Jac} (F_\phi)$ are precisely the singular points of the octic, the zeros of the saturation of $\text{Jac} (F_\phi)$ by $I$ are any singularities other than at the zeros of $I$. Macaulay computes

$$(\text{Jac} (F_\phi) : I^\infty) = \{ g : g I \subseteq \text{Jac} F_\phi \text{ for some } n \} = (1),$$

thus there are no zeros of $\text{Jac} F_\phi$ other than at the points $p, p_1, \ldots, p_5$ and therefore no other singularities of $C_\phi$.

**Claim 2.** To check that $F$ has no singularities outside the set $\{ p, p_1, \ldots, p_5 \}$, it suffices to check this for the polynomial $F_\phi$ over $\mathbb{Z}/\mathfrak{p}[\mathfrak{p}]$.

It is easy to check, using Maple, that $C_\phi$ has an ordinary quadruple point at $p$, and after one blow up, the triple point at $p_1$ and the double points at $p_2, \ldots, p_5$ become ordinary. Since $C$ maps to $C_\phi$, the same is true for the singularities on $C$.

Maple is not reliable about completely factoring polynomials in many variables; hence Maple cannot check directly that $C$ is irreducible. Hence we fall back on a more case-by-case analysis.

Maple can check that a given polynomial divides another; and so one can use Maple to conclude that $Q$ is not a component of $C$.

Next note that since $\deg Q = 2$ and $\deg C = 8$, $Q \cdot C = 16$. We know that $C$ and $Q$ meet four times at each $p_i$, $i = 2, \ldots, 5$; thus $Q$ cannot meet any component of $C$ at any other point.

We check that none of the tangent lines to $C$ at any $p_i$ are contained in $C$. If any other line was a component of $C$, say $C = \ell G$, then $(G \cdot Q) = 14$; however $G$ must meet $Q$ four times each at $p_2, \ldots, p_5$, so no line can be contained in $C$. 7
Suppose a conic $G$ is a component of $C$. Then $G$ must meet $Q$ at two of the four points, say $p_i$ and $p_j$, with the proper tangent directions, to multiplicity two.

Case 1. If $C$ breaks up into $G$ and an irreducible sextic $S$, then $S$ must have at least a triple point at $p$ and tacnodes at $p_k, p_l$ for $l, k \neq i, j$; since there is no conic through $p_1, p_i, p_j$ with the required tangent directions, $S$ must have an infinitely near triple point at $p_1$. But these conditions would drop the genus of $S$ by 13, while an irreducible degree six curve has genus 10, so no such sextic exists.

Case 2. If $C$ breaks up into two conics $G$ and $H$ and a degree four part, then $G$ meets $Q$ at $p_i, p_j$, $H$ meets $Q$ at $p_k, p_l$, so neither conic can pass through $p_1$. Therefore the degree four part of $C$ would have to have an infinitely near triple point, which is impossible (even for a reducible quartic).

Case 3. If $C$ is composed of a conic $G$ and two cubics $S_1, S_2$, then one of the cubics must have a tacnode at $p_1$, which is impossible.

Thus the octic $C$ cannot contain either a line or a conic as a component. We can conclude therefore that if $C$ does split, it splits into at most two components (of degrees 3 and 5 or 4 and 4).

Suppose $C$ is composed of a cubic $G$ and a quintic $S$, both of which are irreducible. The arithmetic genus of $S$ is six, and $S$ must have at least a double point at $p$ and a tacnode at $p_1$, which together drop the genus by three. Since $Q \cdot G = 6$, $G$ must meet $Q$ at three of the $p_i$, thus $S$ must have a tacnode along $Q$ (at the fourth point) which drops the genus by two more. Thus $S$ can have exactly a double point at $p$ and a tacnode at $p_1$, and $G$ must have a double point at $p$ and pass through $p_1$ and three of the $p_i$ with the necessary tangent directions. But no such cubics exist, as can be checked using Maple; (this gives 11 linear conditions on the cubic, and Maple checks that this linear system has no solutions).

Next, suppose $C$ is composed of two irreducible quartics $G$ and $S$. Then one of the quartics, say $G$, must have a tacnode at $p_1$ and pass through $p$. Also $G$ must meet $Q$ along $p_2, \ldots, p_5$. But these are all linear conditions on the quartic, and again Maple can be used to check that there are no such quartics.

Thus the octic $C$ is irreducible.

Since $C$ is irreducible, we can compute the arithmetic genus to be $\binom{7}{2} - 21$; after blowing up a point of multiplicity $n$, the genus of the proper transform goes down by $\binom{n}{2}$. After resolving the singularities of $C$ at $p, p_1, \ldots, p_5$, the resulting curve has genus equal to

$$21 - \binom{4}{2} - 2\binom{3}{2} - 8\binom{2}{2} = 1,$$

thus $C$ can have at most one more singularity of multiplicity two.

We will now prove that $C$ has no other singularities than the known ones at $p$ and $p_1, \ldots, p_5$. The curve $C$ is defined over the field $K = \mathbb{Q}(\alpha, \beta, \delta)$,
as is its strict transform $\tilde{C}$ after resolving the singularities at $p, p_1, \ldots, p_5$. Suppose that $\tilde{C}$ is singular; since it can have at most one singularity, the coordinates of this singular point are then invariant by the action of the Galois group of the algebraic closure of $K$ over $K$, hence lie in $K$. Thus the normalization $\tilde{C}$ of the curve $\tilde{C}$ is defined over $K$. Since the genus of $\tilde{C}$ is 0, its anti-canonical map induces an isomorphism, defined over $K$, onto a smooth conic in $\mathbf{P}^2_K$. Since the curve $\tilde{C}$ has a rational point over $K$ (namely the eighth point of intersection of the line $z = 0$ with the curve $C$; this line meets $C$ four times at $p$, three times at $p_1$, and then once at a point with coordinates in $K$), the projection from this point yields an isomorphism defined over $K$ between $\tilde{C}$ and $\mathbf{P}^1_K$. By composing with the map $\tilde{C} \to C$ (also defined over $K$), we obtain a parametrization $\psi: \mathbf{P}^1_K \to C$ defined over $K$; by clearing denominators we can take $\psi$ to be defined over $\mathbb{Z}[\alpha, \beta, \delta]$. Since $\mathbb{Z}[\alpha, \beta, \delta]$ maps to $\mathbb{Z}/30047$, we get a map $\mathbf{P}^1_{\mathbb{Z}/30047} \to C$. Thus $C$ is rational over $\mathbb{Z}/30047$, so the genus is zero and the genus of $C$ over the algebraic closure of $\mathbb{Z}/30047$ is also zero.

But Macaulay can and does check that $C$ has no other singularities in the algebraic closure of the finite field; so $C$ is smooth and its genus must be one (using the genus formula, which is essentially adjunction). Therefore the genus of $C$ must be one as well, which gives a contradiction. Hence $C$ can have no other singularities.

We can also use Maple to check that the system of quartics with a double point at $p$, through each $p_i$ with the necessary tangent direction is a pencil; thus $C$ and $Q$ give a degree ten curve as needed.

4. The torsion group of $\tilde{X}$

The following lemma will show that the torsion group is non-trivial.

**Lemma 1.** (Beauville [4]) Let $Y$ be a smooth surface with $\text{Tors}(\text{Pic}(Y)) = 0$, $\{C_i\}_{i \in I}$ a collection of smooth disjoint curves on $Y$, and $\pi: X \to Y$ a connected double cover branched along $\cup_{i \in I} C_i$. Define a map

$$\varphi: \mathbb{Z}/\mathbb{Z}\not\equiv I \to \text{Pic} Y \otimes \mathbb{Z}/\not\equiv$$

by sending $\sum n_i C_i$ to its class in $\text{Pic} Y$. If $e = \sum_{i \in I} C_i$, then the group $\text{Pic}_2X$ of $2-$ torsion elements in $\text{Pic} X$ is isomorphic to $\ker (\varphi) / (\mathbb{Z}/\not\equiv)e$.

If $\sum_{i \in J} C_i \equiv 2A$ for some divisor $A$, where $J$ is a subset of $I$, then the map from $\ker (\varphi)$ to the $2-$torsion elements in $\text{Pic} X$ sends $\sum_{i \in J} C_i$ to $\sum_{i \in J} \pi^{-1}(C_i) - \pi^*(A)$; for components $C_i$ of the branch locus

$$2\pi^{-1}(C_i) \equiv \pi^*(C_i),$$

so that $\sum_{i \in J} \pi^{-1}(C_i) - \pi^*(A)$ is in $\text{Pic}_2(X)$.

Let $\bar{Q}$ be the strict transform of $Q$ on $Y_3$. Since $\bar{Q} + \sum E_i$ is a sum of components of the branch locus and $\bar{Q} + \sum E_i \equiv 2(H - \sum F_i)$, the lemma
Thus if  shows that the divisor
\[ \pi^{-1}\left(Q + \sum_{i=1}^{5} E_i\right) - \pi^\ast\left(H - \sum_{i=1}^{5} F_i\right) \]
has order two in Pic (X). Thus Tors (X) is non-trivial.

For numerical Godeaux surfaces, the torsion group has order less than or equal to five, and it is known that \(\mathbb{Z}/\xi \oplus \mathbb{Z}/\xi\) does not occur. (See \([8]\).) To determine whether Tors X is \(\mathbb{Z}/\xi\) or \(\mathbb{Z}/\xi^2\), we use a base point lemma due to Miyaoka \([8]\): for a minimal Godeaux surface, the number of base points of \(|3K|\) is equal to
\[ \# \{ T \in \text{Pic} \cdot X : T \neq -T \} / 2. \]

Thus if \(|3K|\) has no base points, the torsion group is \(\mathbb{Z}/\xi\).

Write \(\epsilon : X \to \tilde{X}\) for the map contracting the \(\pi^{-1}(E_i)\). Then \(3K_X \equiv \epsilon^\ast(3K_{\tilde{X}}) + 3 \sum \pi^{-1}(E_i)\). To compute \(|3K_X|\), first consider the system \(|3K_{Y_1} + 3C|\). The divisor \(2 \sum E_i\) is fixed in this system; the difference \(6H - 3E - 2 \sum E_i - 3 \sum F_i\) is the pencil of sextics with a triple point at \(p\) and double points at each \(p_i\) with one tangent direction coinciding with the branch curve. Set \(M = \pi^\ast(6H - 3E - 2 \sum E_i - 3 \sum F_i)\); we have \(\epsilon^\ast(3K_{\tilde{X}}) \equiv M + \sum \pi^{-1}(E_i)\), so any base point must either lie on \(\sum \pi^{-1}(E_i)\) or be a base point of \(|M|\).

We use Maple to find two sextics in \(M\) and their two points of intersection. These two points do not lie on \(Q\) or \(C\), so there is no base point of \(|M|\) on the branch curve.

Since \(3K_X \equiv \pi^{-1}(B) + \pi^\ast(H - E) + 2 \sum \pi^{-1}(E_i)\), we also have \(\epsilon^\ast(3K_{\tilde{X}}) \equiv \pi^{-1}(B) + \pi^\ast(H - E) - \sum \pi^{-1}(E_i)\), so any base point must lie on the branch curve, away from the divisor \(\sum \pi^{-1}(E_i)\).

Therefore there are no base points of the tricanonical system. From the Miyaoka lemma, this shows that Tors X \(\cong \mathbb{Z}/\xi\).

5. The Oort and Peters Example

In \([9]\), Oort and Peters construct a branch curve \(B\) from two conics \(Q_1, Q_2\) and two cubics \(C_1, C_2\) where
\[
\begin{align*}
Q_1 &= y^2 + 2x^2 - 2xy - 5xz + 2yz + 3z^2 \\
Q_2 &= y^2 + 2x^2 + 2xy - 5xz - 2yz + 3z^2 \\
C_1 &= y^2z + x^3 - 4x^2z + 3xz^2 \\
C_2 &= 2y^2z - xy^2 + 4x^2z - 12xz^2 + 9z^3.
\end{align*}
\]

We have
\[
\begin{align*}
(Q_1 \cdot Q_2) &= P + 3P_1 \\
(Q_1 \cdot C_1) &= 2(P_1 + P_2 + P_3) \\
(Q_1 \cdot C_2) &= 2(P + P_2 + P_3) \\
(Q_2 \cdot C_1) &= 2(P_1 + P_4 + P_3) \\
(Q_2 \cdot C_2) &= 2(P_4 + P_5 + P) \\
(C_1 \cdot C_2) &= 2(P_2 + P_3 + P_4 + P_5) + \infty
\end{align*}
\]
where

\[
\begin{align*}
P &= \left[ \frac{3}{2} : 0 : 1 \right] \\
P_1 &= \left[ 1 : 0 : 1 \right] \\
P_2 &= \left[ \frac{3+i\sqrt{3}}{2} : \frac{3+i\sqrt{3}}{2} : 1 \right] \\
P_3 &= \left[ \frac{3-i\sqrt{3}}{2} : \frac{3-i\sqrt{3}}{2} : 1 \right] \\
P_4 &= \left[ \frac{3+i\sqrt{3}}{2} : -\frac{3-i\sqrt{3}}{2} : 1 \right] \\
P_5 &= \left[ \frac{3-i\sqrt{3}}{2} : -\frac{3+i\sqrt{3}}{2} : 1 \right] \\
\infty &= \left[ 0 : 1 : 0 \right].
\end{align*}
\]

In this case the branch curve has two extra ordinary double points, one at \(\infty\), and the other which occurs on the second blowup above \(P_1\), since \(Q_1\) and \(Q_2\) intersect with multiplicity three at this point. However these double points do not affect the invariants of the double plane \(\tilde{Z}\) constructed.

Write \(\pi : Z \to Y\) for the double cover, where \(Y\) is the blowup of the plane resolving the singularities of the branch curve; although the double points do not affect the computations, we will blow them up to obtain a smooth branch divisor \(B\) with

\[
B = 2\mathcal{L} \equiv \infty\mathcal{H} - \triangle\mathcal{E} - \sum_{\infty} \mathcal{E}_i - \sum_{\infty} \mathcal{F}_j - \forall\mathcal{G}_{\infty} - \epsilon\mathcal{E}/
\]

where we use the notation for the exceptional curves as above, with \(G_1\) being the divisor lying above the extra double point on \(F_1\) and \(E_6\) the exceptional divisor above \(\infty\). Let \(\tilde{Z}\) be the minimal surface obtained from \(Z\) by blowing down the \(E_i\).

Note that we have the following equivalences of divisors:

\[
\begin{align*}
Q_1 &= 2H - E - E_1 - E_2 - E_3 - 2F_1 - 2F_2 - 2F_3 - 3G_1 \\
Q_2 &= 2H - E - E_1 - E_4 - E_5 - 2F_1 - 2F_4 - 2F_5 - 3G_1 \\
C_1 &= 3H - \sum_{1}^{5} E_i - 2 \sum_{1}^{5} F_i - 2G_1 - E_6 \\
C_2 &= 3H - 2E - \sum_{2}^{5} E_i - 2 \sum_{2}^{5} F_i - E_6 \\
K_Y &= -3H + E + \sum_{1}^{5} E_i + 2 \sum_{1}^{5} F_i + 3G_1 + E_6 \\
\mathcal{L} &= 5H - 2E - \sum_{i} E_i - 3 \sum_{i} F_i - 4G_1 - E_6.
\end{align*}
\]

Let \(\mathcal{L}_1 = 2H - E - E_1 - 2F_1 - \sum_{2}^{3} F_i - 3G_1\) and \(B_1 = 2\mathcal{L}_1 \equiv Q_1 + Q_2 + \sum_{2}^{5} E_i\); set \(\mathcal{L}_2 = \mathcal{L} - \mathcal{L}_1\) and \(B_2 = 2\mathcal{L}_2 \equiv C_1 + C_2 + E_1\).

It follows from Beauville’s lemma that

\[
T = \pi^{-1}(B_1) - \pi^*(\mathcal{L}_1) \equiv -\pi^{-1}(B_2) + \pi^*(\mathcal{L}_2)
\]

is of order two, thus \(\text{Tors} \tilde{Z}\) is either \(\mathbb{Z}/\mathcal{E}\) or \(\mathbb{Z}/\bar{\mathcal{E}}\).

We will show that \(\text{Tors} \tilde{Z} \cong \mathbb{Z}/\bar{\mathcal{E}}\). Note that it was previously believed that \(\text{Tors} \tilde{Z} \cong \mathbb{Z}/\bar{\mathcal{E}}\) (\cite{6, 9}).

Oort and Peters use the base point lemma of Miyaoka to argue that \(\text{Tors} \tilde{Z}\) is \(\mathbb{Z}/\bar{\mathcal{E}}\); however they miss a base point of the system \(3K_{\tilde{Z}}\) in their computation. Again if \(\epsilon : Z \to \tilde{Z}\) is the map from \(Z\) to its minimal model, we have

\[
\epsilon^*(3K_{\tilde{Z}}) \equiv M + \sum \pi^{-1}(E_i) \equiv \pi^{-1}(B) + \pi^*(H - E) - \sum \pi^{-1}(E_i),
\]

where

B = 2L \equiv \infty H - \triangle E - \sum_{\infty} E_i - \sum_{\infty} F_j - \forall G_{\infty} - \epsilon E/
where $M = \pi^* (3K_Y + 3\mathcal{L} - 2\sum E_i)$. Thus any base point must lie on $\pi^{-1} (B) - \sum \pi^{-1} (E_i)$ and be a base point of $|M|$. The divisors $\bar{Q}_1 + \bar{Q}_2 + \bar{Q} + F_1$ and $\bar{\ell} + \bar{C}_2 + \bar{Q}$ are in $|M|$, where $\bar{Q} = 2H - E - \sum_2^3 (E_i + F_i)$ is the proper transform of the conic $\bar{Q} = 2x^2 - 9xz + y^2 + 9z^2$ through $P, P_2, \ldots, P_3$, $\bar{\ell} = H - E - E_1 - F_1 - G_1$ is the proper transform of the line $y = 0$, and $\bar{Q} = 2H - \sum_2^3 E_i - 2F_1 - \sum_2^3 F_i - 2G_1$ is the proper transform of the conic $\bar{Q} = 3xz - 3z^2 - y^2$ through each $P_i$ where the tangent direction at $P_1$ coincides with that of the branch curve.

The point $[3 : 0 : 1]$ lies on the curves $Q, \ell, C_1$, and therefore is a base point of $|\epsilon^* (3K_Z)|$. It follows from Miyaoka’s result that $\text{Tors} Z$ is $\mathbb{Z}/2\mathbb{Z}$.

In [4], Dolgachev assumes that there exists an order four divisor on $Z$ and gets a contradiction after finding a fixed part of the pencil $|2K_Z|$. However his computation of generators for $|2K_Z|$ is incorrect. We find divisors in the system

$$|\epsilon^* (2K_Z)| = 2K_Z - 2\sum \pi^{-1} (E_i) |$$

$$= |\pi^* (2K_Y + 2\mathcal{L} - \sum \mathcal{E}_i)|$$

$$= |\pi^* (4H - 2E - \sum E_i - 2\sum F_i - 2G_1)|.$$  

This pencil has generators $y_0 = \bar{Q}_1 + \bar{Q}_2 + 2F_1 + 4G_1 + E_1$ and $y_1 = C_2 + \bar{\ell} + 2E_6$, where $\bar{\ell}$ is the line tangent to the branch curve at $P_1$; thus there is no fixed part to this system.

We can also check that $\bar{Z}$ has order four torsion by calculating the bicanonical system of a double cover of $\bar{Z}$. Form the double cover $S$ of $\bar{Z}$ branched over $2T \equiv 0$, $\rho : S \to \bar{Z}$. Since there is no ramification, $\rho$ is étale over $\bar{Z}$. Also $K_S \equiv \rho^* (K_Z + T)$ and $K_S^2 = 2$. We have already found two sections $y_0$ and $y_1$ in $H^0 (2K_Z)$, and hence two sections $\rho^* (y_0)$ and $\rho^* (y_1)$ in $H^0 (2K_S)$. Since $\rho^* (T) \equiv 0$, we also have

$$2K_S \equiv \rho^* (\pi^{-1} (B_1) + \pi^* (2H - E - \sum_2^5 (E_i + F_i) + G_1))$$

and

$$2K_S \equiv \rho^* \left( \pi^{-1} (B_2) + \pi^* (H - E - E_1 - F_1 - G_1 + E_6) \right).$$

We have seen that the proper transform $\bar{Q}$ of the conic $Q$ is in the linear system $|\pi^* (2H - E - \sum_2^3 (E_i + F_i))|$ and the proper transform of the line $\bar{\ell}$ is in $|\pi^* (H - E - E_1 - G_1)|$; set $y_2 = \pi^{-1} (B_1) + \pi^* (\bar{Q} + G_1)$ and $y_3 = \pi^{-1} (B_2) + \pi^* (\bar{\ell} + E_6)$. We have $(y_0 - 2y_1)^2 - y_2^2 + 4y_3^2 = 0$. This gives a quadratic relation among the four elements of $H^0 (2K_S)$; in fact we obtain a quadric cone as the bicanonical image of $S$. By [4], if the bicanonical image is a cone then $\text{Tors} S \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1 (S) \cong \mathbb{Z}/2\mathbb{Z}$. Since $S$ is a covering space of $\bar{Z}$ of degree two, $\left[ \pi_1 (\bar{Z}) : \pi_1 (S) \right] = 2$. Thus $\pi_1 (\bar{Z})$ is abelian of order four and $\pi_1 (\bar{Z}) \cong \text{Tors} (\bar{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
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Department of Mathematics, Colorado State University, Fort Collins, CO 80523
E-mail address: werner@lagrange.math.colostate.edu