A nonlinear deformed su(2) algebra with a two-colour quasitriangular Hopf structure

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Abstract. Nonlinear deformations of the enveloping algebra of $\text{su}(2)$, involving two arbitrary functions of $J_0$ and generalizing the Witten algebra, were introduced some time ago by Delbecq and Quesne. In the present paper, the problem of endowing some of them with a Hopf algebraic structure is addressed by studying in detail a specific example, referred to as $A_q^+(1)$. This algebra is shown to possess two series of $(N + 1)$-dimensional unitary irreducible representations, where $N = 0, 1, 2, \ldots$. To allow the coupling of any two such representations, a generalization of the standard Hopf axioms is proposed by proceeding in two steps. In the first one, a variant and extension of the deforming functional technique is introduced: variant because a map between two deformed algebras, $\text{su}_q(2)$ and $A_q^+(1)$, is considered instead of a map between a Lie algebra and a deformed one, and extension because use is made of a two-valued functional, whose inverse is singular. As a result, the Hopf structure of $\text{su}_q(2)$ is carried over to $A_q^+(1)$, thereby endowing the latter with a double Hopf structure. In the second step, the definition of the coproduct, counit, antipode, and $R$-matrix is extended so that the double Hopf algebra is enlarged into a new algebraic structure. The latter is referred to as a two-colour quasitriangular Hopf algebra because the corresponding $R$-matrix is a solution of the coloured Yang-Baxter equation, where the ‘colour’ parameters take two discrete values associated with the two series of finite-dimensional representations.

Running title: Two-colour quasitriangular Hopf algebra

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I INTRODUCTION

Quantized universal enveloping algebras, also called $q$-algebras, refer to some specific deformations of (the universal enveloping algebra of) Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to one [1]. The simplest example of $q$-algebra, $\text{su}_q(2) \equiv \text{U}_q(\text{su}(2))$, was first introduced by Sklyanin [2], and independently by Kulish and Reshetikhin [3] in their work on Yang-Baxter equations. A Jordan-Schwinger realization of $\text{su}_q(2)$ in terms of $q$-bosonic operators was then derived by Biedenharn [4] and Macfarlane [5]. Since then, $\text{su}_q(2)$ has been applied in various branches of physics. It has been found suitable, for instance, for the solution of deformed spin-chain models [6], as well as for the approximate description of rotational spectra of deformed nuclei [7], superdeformed nuclei [8], and diatomic molecules [9].

In addition to the usual version of the deformed $\text{su}(2)$ algebra, namely $\text{su}_q(2)$, several generalized forms of the algebra have been introduced. Deformations involving one arbitrary function of $J_0$ were independently proposed by Polychronakos [10] and Roček [11]. Their representation theory is characterized by a rich variety of phenomena, which might be of interest in applications to particle physics. Bonatsos et al [12] considered a subclass of these algebras having a representation theory as close as possible to the usual $\text{su}(2)$, so that they might prove useful in applications to nuclear and molecular physics similar to those above-mentioned in connection with $\text{su}_q(2)$.

Deformations of $\text{su}(2)$ involving two arbitrary functions of $J_0$ were introduced by Delbecq and Quesne [13]. Contrary to the former deformations, for which the spectrum of $J_0$ is linear as for $\text{su}(2)$, the latter give rise to exponential spectra. Such spectra did recently arouse much interest in various contexts, for instance in connection with alternative Hamiltonian quantization [14], exactly solvable potentials [15], $q$-deformed supersymmetric quantum mechanics [16], and $q$-deformed interacting boson models [17].

From a mathematical viewpoint, $q$-algebras are just special classes of Hopf algebras [1, 18]. One of the basic data defining a Hopf algebra is the so-called comultiplication rule. The latter plays an important role in representation theory since it allows one to define a product of two independent representations that is still a representation. Hence, for $\text{su}_q(2)$,
for instance, Wigner-Racah calculus can be developed in much the same way as for the standard \( \text{su}(2) \) Lie algebra (see e.g. Ref. \[19\] and references quoted therein).

The existence of a comultiplication rule for more general deformations of \( \text{su}(2) \) is an important problem, which remains largely unsolved. In principle, a coproduct can be induced from the ordinary coupling rule for \( \text{su}(2) \) \[10, 20\] by the deforming functional technique \[21\]. However such a procedure leads in general to complicated and untractable coproducts. More direct methods were recently used to construct a coproduct for some Polychronakos-Roček algebras (PRA’s) \[22\] and for another deformation of \( \text{su}(2) \) involving a single function of \( J_0 \) \[23\].

In the present paper, we shall address the problem of endowing some Delbecq-Quesne algebras (DQA’s) with a Hopf algebraic structure. For such purpose, we shall use a variant and extension of the deforming functional technique, wherein functionals mapping PRA generators to those of some DQA’s will be determined. By considering the special case where the PRA reduces to \( \text{su}_{q}(2) \), we shall obtain DQA’s whose Hopf algebraic structure can be inferred from that of \( \text{su}_{q}(2) \).

This procedure will be carried out in detail for a specific example of DQA, referred to as \( \mathcal{A}_{q}^{+}(1) \), which will be shown to have two sets of \((N + 1)\)-dimensional unitary irreducible representations (unirreps), where \( N = 0, 1, 2, \ldots \). For such an algebra, the functional to be considered will be two-valued, so the same will be true for the resulting Hopf algebraic structure. The meaning of this property will be clarified as far as representation theory is concerned.

To allow the coupling of any two representations of \( \mathcal{A}_{q}^{+}(1) \), we shall be led to enlarge the double Hopf structure of the algebra into a generalized Hopf structure, obeying extended coassociativity, counit, and antipode axioms, which we propose to call a two-colour Hopf algebra. We shall indeed prove that the generalized coproduct non-cocommutativity is controlled by a generalized \( \mathcal{R} \)-matrix, satisfying the coloured Yang-Baxter equation \[24\]–\[31\], where the ‘colour’ parameters take two discrete values associated with the two sets of unirreps with the same dimension that the algebra possesses.

This paper is organized as follows. In Sec. \[1\], deforming functionals mapping PRA’s to DQA’s are introduced with special emphasis on the case where the PRA is \( \text{su}_{q}(2) \). The
algebra $A_q^+(1)$ is introduced in Sec. III. In Sec. IV, the theory developed in Sec. III is applied to endow $A_q^+(1)$ with a double Hopf algebraic structure. In Sec. V, the latter is enlarged into a generalized Hopf structure, and the corresponding $R$-matrix is studied in Sec. VI. Sec. VII contains some concluding remarks.

II DEFORMING FUNCTIONALS MAPPING PRA’S TO DQA’S

PRA’s are associative algebras over $\mathbb{C}$ generated by three operators $j_0 = (j_0)^\dagger$, $j_+$, and $j_- = (j_+)^\dagger$, satisfying the commutation relations

$$[j_0, j_+] = j_+, \quad [j_0, j_-] = -j_-, \quad [j_+, j_-] = f(j_0),$$

(2.1)

where $f(z)$ is a real, parameter-dependent function of $z$, holomorphic in the neighbourhood of zero, and going to $2z$ for some values of the parameters. These algebras have a Casimir operator given by

$$c = j_- j_+ + h(j_0) = j_+ j_- + h(j_0) - f(j_0),$$

(2.2)

in terms of another real function $h(z)$, related to $f(z)$ through the equation $h(z) - h(z - 1) = f(z)$. In the case where $f(z)$ is a $\lambda$-degree polynomial, an explicit expression for $h(z)$ has been found by Delbecq and Quesne in terms of Bernoulli polynomials and Bernoulli numbers.

In the case of polynomial functions $h(z), f(z)$, the PRA can be identified with the linear space spanned by the monomials

$$j^m_0 j^n_+ j^p_- \quad \text{where} \quad m, n, p \in \mathbb{N}.$$ 

(2.3)

The above basis is not unique, because we can consider other bases corresponding to different normal orderings, as the following ones:

$$j^m_- j^n_0 j^p_+ \quad \text{where} \quad m, n, p \in \mathbb{N};$$

(2.4)

or

$$j^m_0 j^n_+ j^p_- \quad \text{where} \quad m, n, p \in \mathbb{N}.$$ 

(2.5)
Any normal ordering is actually permitted.

DQA’s differ from PRA’s by the replacement of (2.1) by

\[ [J_0, J_+] = G(J_0)J_+, \quad [J_0, J_-] = -J_-G(J_0), \quad [J_+, J_-] = F(J_0), \quad (2.6) \]

where the generators \( J_0 = (J_0)\dagger, J_+, J_- = (J_+)\dagger \) are now denoted by capital letters to distinguish them from those of PRA’s, and their commutators involve two real, parameter-dependent functions of \( z \), \( F(z) \) and \( G(z) \), holomorphic in the neighbourhood of zero, and going to \( 2z \) and \( 1 \) for some values of the parameters, respectively. These functions are further restricted by the assumption that the algebras have a Casimir operator given by

\[ C = J_-J_+ + H(J_0) = J_+J_- + H(J_0) - F(J_0), \quad (2.7) \]

in terms of some real function \( H(z) \), holomorphic in the neighbourhood of zero. The latter restriction implies that \( F(z) \), \( G(z) \), and \( H(z) \) satisfy the consistency condition \( H(z) - H(z-G(z)) = F(z) \).

As in the case of the PRA, the existence of polynomial functions \( G(z) \), \( F(z) \), implies that the DQA can be identified with the linear space spanned by the monomials

\[ J_-^mJ_0^nJ_+^p \quad \text{where} \quad m, n, p \in \mathbb{N}. \quad (2.8) \]

We must notice here that in general, if the function \( G(z) \) is not invertible, only the normal ordering (2.8) is permitted and different orderings, as in the cases (2.4) and (2.5), are not allowed.

Let us now try to find a deforming functional that converts the PRA generators into operators satisfying the commutation relations of a DQA. For such purpose, we first remark that the first equation in (2.1) can be rewritten as

\( (j_0 - 1)j_+ = j_+j_0. \quad (2.9) \)

Then, for every entire function \( p(z) \), one can prove

\[ p(j_0 - 1)j_+ = j_+p(j_0). \quad (2.10) \]
Let us consider the equation

\[ p(z) - p(z - 1) = G(p(z)) \]  (2.11)

for a given function \( G(z) \). If this equation has a solution \( p(z) \) that is an entire function, then Eq. (2.11) can be written as \( [p(j_0), j_+] = G(p(j_0)) j_+ \). Similarly, one finds \( [p(j_0), j_-] = -j_- G(p(j_0)) \).

The above equations suggest that a correspondence between the PRA’s and the DQA’s might exist. In terms of the function \( p(z) \), the first two relations in (2.1) can indeed be reduced to the first two relations in (2.6) with the identification

\[ J_0 = p(j_0), \quad J_+ = j_+, \quad J_- = j_. \]  (2.12)

Assuming \( p \) is invertible, the third defining equation (2.1) of a PRA can then be reduced to the third defining equation (2.6) of a DQA with the identification

\[ F \circ p = f \quad \text{or} \quad F = f \circ g, \quad \text{where} \quad g = p^{-1}, \]  (2.13)

and \( f \circ g \) means the composition of the two functions, i.e., \( (f \circ g)(z) = f(g(z)) \). The correspondence between the relevant Casimir operators, given in (2.2) and (2.7), similarly implies

\[ H \circ p = h \quad \text{or} \quad H = h \circ g. \]  (2.14)

The existence of the map transferring the commutation relations (2.1) to the commutation relations (2.6), using the function \( p(z) \), does not mean that the algebra generated by Eqs. (2.12) is the same as the DQA. This fact is evident in the case where the function \( g(z) \) is a polynomial, because the map (2.12) transfers the PRA basis given by (2.3) to the basis

\[ J_-^m g(J_0)^n J_+^p \quad \text{where} \quad m, n, p \in \mathbb{N}, \]  (2.15)

and the latter constitutes a subspace of the linear space (2.8) defining the DQA. Hence, we have shown the following proposition:

Proposition 1: If there exists an invertible entire function \( p(z) \) satisfying the equations

\[ p(z) - p(z - 1) = G(p(z)) \quad \text{and} \quad F \circ p = f, \]
then the PRA, defined by the commutation relations
\[ [j_0, j_+] = j_+, \quad [j_0, j_-] = -j_-, \quad [j_+, j_-] = f(j_0), \]
is mapped through the generator mapping \( P \):
\[ J_0 = p(j_0), \quad J_+ = j_+, \quad J_- = j_-; \]
into the DQA, defined by the commutation relations
\[ [J_0, J_+] = G(J_0)J_+, \quad [J_0, J_-] = -J_-G(J_0), \quad [J_+, J_-] = F(J_0). \]

For a given DQA, the existence of a solution \( p(z) \) of Eq. (2.11) implies that the algebraic (and eventually the coalgebraic) structure of the root algebra PRA can be mapped to the DQA generated by \( P \).

Let consider the special case where the PRA is \( su_q(2) \), i.e, the function \( f(j_0) \) in Eq. (2.1) is given by
\[ f(j_0) = [2j_0]_q, \tag{2.16} \]
where \([x]_q \equiv (q^x - q^{-x})/(q - q^{-1}) \) denotes a \( q \)-number and \( q \) is either a real number or a phase \([4, 5]\). Throughout this paper, we shall assume that \( q \in \mathbb{R}^+ \). The function \( h(j_0) \) of Eq. (2.2) can be taken as \( h(j_0) = [j_0][j_0 + 1]_q \).

The Hopf algebraic structure of \( su_q(2) \) \([4, 18]\) is defined in terms of a comultiplication map \( \Delta : su_q(2) \rightarrow su_q(2) \otimes su_q(2) \), a counit map \( \epsilon : su_q(2) \rightarrow \mathbb{C} \), and an antipode map \( S : su_q(2) \rightarrow su_q(2) \), given by
\[
\Delta(j_0) = j_0 \otimes 1 + 1 \otimes j_0, \quad \Delta(j_\pm) = j_\pm \otimes q^{j_0} + q^{-j_0} \otimes j_\pm, \\
\epsilon(j_0) = \epsilon(j_\pm) = 0, \\
S(j_0) = -j_0, \quad S(j_\pm) = -q^{\pm 1}j_\pm, \tag{2.17}
\]
respectively. These maps satisfy the following relations:
\[
(\Delta \otimes \text{id}) \circ \Delta(a) = (\text{id} \otimes \Delta) \circ \Delta(a), \\
(\epsilon \otimes \text{id}) \circ \Delta(a) = (\text{id} \otimes \epsilon) \circ \Delta(a) = a, \\
m \circ (S \otimes \text{id}) \circ \Delta(a) = m \circ (\text{id} \otimes S) \circ \Delta(a) = \iota \circ \epsilon(a). \tag{2.18}
\]
Here $a$ denotes any element of $\text{su}_q(2)$, $m$ its multiplication map, $m : \text{su}_q(2) \otimes \text{su}_q(2) \to \text{su}_q(2)$, and $\iota$ its unit map, $\iota : \mathbb{C} \to \text{su}_q(2)$, defined by $m(a \otimes b) = ab$ and $\iota(\lambda) = \lambda 1$ respectively, where $1$ is the unit element of $\text{su}_q(2)$. In addition, $\Delta$ and $\epsilon$ are algebra homomorphisms,

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1, \quad \epsilon(ab) = \epsilon(a)\epsilon(b), \quad \epsilon(1) = 1.$$  \hspace{1cm} (2.19)

The functions $F(J_0)$ and $H(J_0)$ of Eqs. (2.13) and (2.14) become

$$F(J_0) = -\frac{(\phi(J_0))^2 - (\phi(J_0))^{-2}}{q - q^{-1}},$$

$$H(J_0) = \frac{q^{-1}(\phi(J_0))^2 + q(\phi(J_0))^{-2} - q - q^{-1}}{(q - q^{-1})^2},$$ \hspace{1cm} (2.20)

where

$$\phi(z) \equiv q^{-g(z)}.$$

The Hopf algebraic structure of $\text{su}_q(2)$ can then be transferred to that member of the DQA set whose function $F(J_0)$ is given by (2.20), provided the function $g(z)$ can be determined. It should be noticed that the solution of Eq. (2.11) is not a trivial task in general. In the next sections, we shall proceed to study a special case, which proves tractable.

III THE LINEAR $G(J_0)$ CASE

Let consider the case where the function $G(J_0)$ is linear, i.e.,

$$G(J_0) = 1 + (1 - q)J_0,$$ \hspace{1cm} (3.1)

and the $q$ values are restricted to the interval $0 < q < 1$ (note that the algebras with $q > 1$ are related to those with $0 < q < 1$ by an automorphism). The DQA’s, corresponding to (3.1) and to functions $F(J_0)$ that are polynomials of degree $\lambda$ in $J_0$, have been studied in Ref. [13], where they are denoted by $A^+_q(\alpha_2, \alpha_3, \ldots, \alpha_{\lambda-1}; \lambda, 1)$, in terms of some extra real parameters $\alpha_2, \alpha_3, \ldots, \alpha_{\lambda-1}$, entering the definition of $F(J_0)$. In particular, the representation theory of the algebras $A^+_q(2, 1)$ (equivalent to Witten’s first deformation [32] of su(2)), and $A^+_{p,q}(3, 1)$ has been studied in detail. In this respect, the point $z = (q - 1)^{-1}$, where $G(z)$
vanishes, appears as a singular point. The unirreps indeed separate into two classes according to whether the eigenvalues of $J_0$ are contained in the interval $(-\infty, (q-1)^{-1})$, or in the interval $((q-1)^{-1}, +\infty)$. In addition, there is a one-dimensional unirrep corresponding to the eigenvalue $(q-1)^{-1}$ of $J_0$.

With a linear choice like Eq. (3.1) for $G(z)$, Eq. (2.11) can be easily solved. One finds a family of solutions

$$p_\delta(z) = \frac{1 - \delta q^{-z}}{q - 1};$$

where any real, nonvanishing value of the parameter $\delta$ is acceptable. The functions $p_\delta$ are entire functions, which are invertible as

$$p_\delta^{-1}(z) = g_{|\delta|}(z) = \frac{\ln \left((1 + (1 - q)z)^2/|\delta|^2\right)}{\ln (1/q^2)} = \frac{\ln (G^2(z))/|\delta|^2}{\ln (1/q^2)}.$$  \hfill (3.3)

If $z$ is real, the range of $p_\delta$ (and consequently the domain of $p_\delta^{-1}$) is the interval $(-\infty, (q-1)^{-1})$ or $((q-1)^{-1}, +\infty)$ according to whether $\delta < 0$ or $\delta > 0$. In the case of the linear $G(J_0)$ model, Eq. (2.20) gives

$$F(J_0) = -\frac{(G(J_0)/|\delta|)^2 - (G(J_0)/|\delta|)^{-2}}{q - q^{-1}},$$

$$H(J_0) = \frac{q^{-1}(G(J_0)/|\delta|)^2 + q(G(J_0)/|\delta|)^{-2} - q - q^{-1}}{(q - q^{-1})^2}.$$ \hfill (3.4)

Without loss of generality, we can set $\delta = \pm 1$, because any $|\delta| \neq 0$ would lead to similar results. Therefore, we shall henceforth use $p_\pm(z)$ and its inverse $p_\pm^{-1} = g(z)$. In this case,

$$p_\pm^{-1}(z) = g(z) = \frac{\ln \left((1 + (1 - q)z)^2\right)}{\ln (1/q^2)} = \frac{\ln (G^2(z))}{\ln (1/q^2)}.$$ \hfill (3.5)

As for any PRA, $j_+$ is the raising generator, while for the DQA with linear $G(J_0)$, $J_+$ (resp. $J_-$) is the raising generator in the interval $((q-1)^{-1}, +\infty)$ corresponding to $\delta = +1$ (resp. $(-\infty, (q-1)^{-1})$, corresponding to $\delta = -1$), one has to use there the map $p_+(z)$ (resp. $p_-(z)$). The function $g(z)$ is well-behaved everywhere on $\mathbb{R}$, except in the neighbourhood of the point $z = (q-1)^{-1}$.

Let denote by $\mathcal{A}_q^+(1)$ the DQA generated from $\text{su}_q(2)$ by the mapping $p_\delta$, with $\delta = \pm 1$:

$$P_\delta : \text{su}_q(2) \rightarrow \mathcal{A}_q^+(1).$$ \hfill (3.6)
By a procedure similar to that used in Ref. [13], it can be easily shown that $A_q^+(1)$ has no infinite-dimensional unirrep, but has two $(N + 1)$-dimensional unirreps for any $N = 0, 1, 2, \ldots$. The corresponding spectrum of $J_0$ is given by

$$m^\delta = \frac{1 - \delta q^{-(N-2n)/2}}{q - 1}, \quad n = 0, 1, \ldots, N, \quad \delta = \pm 1,$$

with maximum and minimum eigenvalues

$$J^\delta = \frac{1 - \delta q^{-\delta N/2}}{q - 1}, \quad -J^\delta = \frac{1 - \delta q^{\delta N/2}}{q - 1},$$

respectively. The unirrep specified by $J^+$ (resp. $J^-$) is entirely contained in the interval $((q - 1)^{-1}, +\infty)$ (resp. $(-\infty, (q - 1)^{-1})$). For both unirreps, the eigenvalue of the Casimir operator is given by

$$\langle C \rangle = H(\gamma^\delta),$$

where $\gamma^\delta = \frac{1 - \delta q^{-N/2}}{q - 1}$.

In the carrier space $V_{J^\delta}$ of the unirrep characterized by $J^\delta$, whose basis vectors are specified by the values of $J^\delta$ and $m^\delta$, the $A_q^+(1)$ generators are represented by some linear operators $\Phi^\delta(A), A \in A_q^+(1)$, defined by

$$\Phi^\delta (J_0) \left| J^\delta, m^\delta \right\rangle = m^\delta \left| J^\delta, m^\delta \right\rangle,$$
$$\Phi^\delta (J_-) \left| J^\delta, m^\delta \right\rangle = \sqrt{H(\gamma^\delta) - H(qm^\delta - 1)} \left| J^\delta, qm^\delta - 1 \right\rangle,$$
$$\Phi^\delta (J_+) \left| J^\delta, m^\delta \right\rangle = \sqrt{H(\gamma^\delta) - H(m^\delta)} \left| J^\delta, q^{-1}(m^\delta + 1) \right\rangle.$$

IV DOUBLE HOPF STRUCTURE OF THE ALGEBRA $A_q^+(1)$

The theory developed in Sec. [1] can be applied provided one replaces the entire function $p(z)$ by the functions $p_+(z)$ and $p_-(z)$ given by Eq. (3.2), with $\delta = \pm 1$.

The functionals $p_\delta$ can be used to transfer the coalgebra structure and the antipode map from $su_q(2)$ to $A_q^+(1)$. The existence of two functionals however leads to similar properties.
for the latter. According to whether $p_{+}$ or $p_{-}$ is employed, one obtains a comultiplication map $\Delta_{+}$ or $\Delta_{-}$, defined by

$$
\Delta_{\delta}(J_{0}) = \Delta(p_{\delta}(j_{0})) = \frac{1}{q-1} \left( 1 \otimes 1 - \delta q^{-j_{0}} \otimes q^{-j_{0}} \right),
$$

$$
\Delta_{\delta}(J_{\pm}) = \Delta(j_{\pm}) = j_{\pm} \otimes q^{j_{0}} + q^{-j_{0}} \otimes j_{\pm}.
$$

The operators $\Delta_{\delta}(J_{0})$, $\Delta_{\delta}(J_{\pm})$ satisfy the commutation relations (2.6) and from the properties of the comultiplication (2.18) and (2.19), they satisfy the equation

$$
\Delta_{\delta}(J_{0}) = p_{\delta}(\Delta(j_{0})),
$$

(4.2)

hence correspond to a representation of the algebra $A_{q}^{+}(1)$ characterized by $\delta$.

The above relations define the comultiplication rules as morphisms $\Delta_{\delta}$ from the algebra $A_{q}^{+}(1)$ to the algebra $\text{su}_{q}(2) \otimes \text{su}_{q}(2)$.

By using the generator mapping $P_{\delta}$ associated with the functional $p_{\delta}$, as defined in Proposition 1 and Eq. (3.6), we can go from the algebra $\text{su}_{q}(2) \otimes \text{su}_{q}(2)$ to the algebra $A_{q}^{+}(1) \otimes A_{q}^{+}(1)$,

$$
P_{\delta} \otimes P_{\delta} : \text{su}_{q}(2) \otimes \text{su}_{q}(2) \longrightarrow A_{q}^{+}(1) \otimes A_{q}^{+}(1).
$$

(4.3)

The above map transfers the Hopf structure of the algebra $\text{su}_{q}(2)$ to a Hopf-like structure and we obtain the following proposition:

**Proposition 2:** The algebra $A_{q}^{+}(1)$ is equipped with comultiplication, counit and antipode maps, given by

$$
\Delta_{\delta}(J_{0}) = \frac{1}{q-1} \left( 1 \otimes 1 - \delta G(j_{0}) \otimes G(j_{0}) \right),
$$

$$
\Delta_{\delta}(J_{\pm}) = \delta \left( J_{\pm} \otimes (G(j_{0}))^{-1} + G(j_{0}) \otimes J_{\pm} \right),
$$

$$
\epsilon_{\delta}(j_{0}) = \frac{1-\delta}{q-1}, \quad \epsilon_{\delta}(j_{\pm}) = 0,
$$

$$
S_{\delta}(j_{0}) = -j_{0}(G(j_{0}))^{-1}, \quad S_{\delta}(j_{+}) = -qJ_{+}, \quad S_{\delta}(j_{-}) = -q^{-1}J_{-},
$$

(4.4)

respectively. Both $\Delta_{+}$, $\epsilon_{+}$, $S_{+}$, and $\Delta_{-}$, $\epsilon_{-}$, $S_{-}$ satisfy the Hopf algebra axioms (2.18) and (2.19), but the former are only valid for the representations of $A_{q}^{+}(1) \otimes A_{q}^{+}(1)$ with
eigenvalues of $\Delta_\delta(J_0)$ in the interval $((q-1)^{-1}, +\infty)$, whereas the latter act in $(-\infty, (q-1)^{-1})$. The algebra $A^+_q(1)$ is therefore endowed with a double Hopf algebraic structure.

Remark. Contrary to the comultiplication and counit maps, the antipode one does not depend explicitly upon $\delta$.

As a consequence of Eq. (4.4), from the operators $\varphi^{N/2}(a), a \in su_q(2)$, representing the $su_q(2)$ generators in the $(N + 1)$-dimensional unirrep carrier space $v^{N/2}$, spanned by the vectors $|\frac{N}{2}, \frac{N}{2} - n\rangle$, $n = 0, 1, \ldots, N$, one obtains the operators $\Phi^{J_\delta}(A), A \in A^+_q(1)$, representing the $A^+_q(1)$ generators in $V^{J_\delta}$ as follows:

\begin{align*}
\varphi^{N/2}(j_0) |\frac{N}{2}, \frac{N}{2} - n\rangle &= (\frac{N}{2} - n) |\frac{N}{2}, \frac{N}{2} - n\rangle, \\
\varphi^{N/2}(j_-) |\frac{N}{2}, \frac{N}{2} - n\rangle &= \sqrt{[n+1]_q[N-n]_q} |\frac{N}{2}, \frac{N}{2} - n - 1\rangle, \\
\varphi^{N/2}(j_+) |\frac{N}{2}, \frac{N}{2} - n\rangle &= \sqrt{[n]_q[N-n+1]_q} |\frac{N}{2}, \frac{N}{2} - n + 1\rangle, 
\end{align*}

(4.5)

On the right-hand side of (4.6), $N$ and $n$ have to be replaced by their expression in terms of $J^\delta$ and $m^\delta$, obtained by inverting (3.7) and (3.8). The results can be written as in Eq. (3.10), which was constructed by a direct procedure. They also confirm that $A^+_q(1)$ has no infinite-dimensional unirrep.

Considering now an $(N_1 + 1)$-dimensional unirrep of $A^+_q(1)$, characterized by $J^\delta_1$, in a carrier space $V^{J^\delta_1}$, and another $(N_2 + 1)$-dimensional unirrep of the same, specified by $J^\delta_2$, in a carrier space $V^{J^\delta_2}$, one can couple them by using the coproduct $\Delta_\delta$ to obtain a reducible representation of $A^+_q(1)$ in $V^{J^\delta_1} \otimes V^{J^\delta_2}$. The corresponding operators $\Phi^{J_\delta^1, J_\delta^2}(A), A \in A^+_q(1)$, are given by
where the quantity on the right-hand side is a $\text{su}_q$ from the first formula in Eq. (4.4) that the Wigner coefficient (4.9) can be different from the comultiplication rule definition can be extended so as to allow the coupling of any two unirreps $\text{A}_q^+(1)$ of $\text{su}_q(2)$.

Such a reducible representation can be decomposed into a direct sum of $(N + 1)$-dimensional unirreps, characterized by $J^\delta$, and whose basis states $|J^\delta_1 J^\delta_2 J^\delta m^\delta\rangle$ can be written as

$$|J^\delta_1 J^\delta_2 J^\delta m^\delta\rangle = \sum_{m^\delta_1, m^\delta_2} \langle J^\delta_1 m^\delta_1, J^\delta_2 m^\delta_2 | J^\delta m^\delta \rangle_{DQ} |J^\delta_1 m^\delta_1\rangle \otimes |J^\delta_2 m^\delta_2\rangle,$$

in terms of some Wigner coefficients $\langle J^\delta_1 m^\delta_1, J^\delta_2 m^\delta_2 | J^\delta m^\delta \rangle_{DQ}$. From the relation between the representations of $\text{A}_q^+(1)$ and those of $\text{su}_q(2)$, it follows that

$$\langle J^\delta_1 m^\delta_1, J^\delta_2 m^\delta_2 | J^\delta m^\delta \rangle_{DQ} = \langle \frac{N_1}{2}, \frac{N_1}{2} - n_1, \frac{N_2}{2}, \frac{N_2}{2} - n_2 | \frac{N}{2}, \frac{N}{2} - n \rangle_q,$$

where the quantity on the right-hand side is an $\text{su}_q(2)$Wigner coefficient [13]. It follows from the first formula in Eq. (4.4) that the Wigner coefficient (4.3) can be different from zero only if

$$m^\delta = \frac{1}{q - 1} \left(1 - \delta G(m^\delta_1)G(m^\delta_2)\right).$$

It is worth noting that up to now no comultiplication rule is available for coupling two unirreps $J^+_1$ and $J^+_2$, or $J^-_1$ and $J^-_2$. The purpose of the next section will be to show that the comultiplication rule definition can be extended so as to allow the coupling of any two $\text{A}_q^+(1)$ unirreps. This extension will lead us to enlarge the double Hopf structure of the algebra into a generalized one.
V GENERALIZED HOPF STRUCTURE OF THE ALGEBRA $A_q^+(1)$

To connect the two types of unirreps specified by $\delta = +1$ and $\delta = -1$ respectively, let us introduce some linear operators $T^{J^\delta} : V^{J^\delta} \to V^{J^{-\delta}}$, where $J^\delta$ may be any unirrep label, as given in Eq. (3.8). They are defined by their action on the $V^{J^\delta}$ basis vectors $|J^\delta, m^\delta\rangle$ as follows:

$$T^{J^\delta} |J^\delta, m^\delta\rangle = |J^{-\delta}, m^{-\delta}\rangle. \quad (5.1)$$

Such operators will be referred to as transmutation operators as they change the basis states of an $(N + 1)$-dimensional unirrep into those of its companion with the same dimension. They obviously satisfy the relation

$$T^{J^{-\delta}} T^{J^\delta} = I^{J^\delta}, \quad (5.2)$$

where $I^{J^\delta}$ denotes the unit operator in $V^{J^\delta}$.

By applying $T^{J^\delta}$ on both sides of Eq. (4.6) and using Eqs. (5.1), (5.2), and (3.7), it can be easily proved that for any $A_q^+(1)$ generator $A$,

$$T^{J^\delta} \Phi^{J^\delta}(A) T^{J^{-\delta}} = \Phi^{J^{-\delta}}(\sigma(A)), \quad (5.3)$$

where $\sigma : A_q^+(1) \to A_q^+(1)$, defined by

$$\sigma(J_0) = \frac{2}{q-1} - J_0, \quad \sigma(J_\pm) = J_\pm, \quad (5.4)$$

is an involutive automorphism of the algebra $A_q^+(1)$. This clearly shows that at the algebra level, the operator $\sigma$ is responsible for the transmutation.

Let us define $\sigma_\delta : A_q^+(1) \to A_q^+(1)$ by

$$\sigma_\delta = \left\{ \begin{array}{ll} \text{id} & \text{if } \delta = +1 \\ \sigma & \text{if } \delta = -1 \end{array} \right.. \quad (5.5)$$

The basic mapping $P_\delta$ of Eq. (3.6) transforms under (5.5) as follows:

$$\sigma_\zeta \circ P_\eta = P_\zeta, \quad (5.6)$$
where $\zeta, \eta = \pm 1$. This equation is equivalent to the commuting diagram

\[
\begin{array}{c}
\text{su}_q(2) \xrightarrow{P_\eta} \mathcal{A}_q^+ (1) \\
P_\zeta \downarrow \quad \downarrow \sigma_{\zeta \eta} \\
\mathcal{A}_q^+ (1) \xleftrightarrow{\text{id}} \mathcal{A}_q^+ (1)
\end{array}
\]

(5.7)

We can now extend the comultiplication and antipode maps, $\Delta_\delta$ and $S_\delta$ ($\delta = \pm 1$), of Eq. (4.4) by setting

\[
\Delta_\delta^{\zeta, \eta} (A) = (\sigma_{\zeta \delta} \otimes \sigma_{\eta \delta}) \circ \Delta_\delta (A), \quad S_\delta^{\zeta} (A) = \sigma_{\zeta \delta} \circ S_\delta (A),
\]

(5.8)

where $\zeta, \eta, \delta = \pm 1$, while leaving unchanged the counit map $\epsilon_\delta$, defined in the same equation. We note that in particular,

\[
\Delta_\delta^{\delta, \delta} = \Delta_\delta, \quad S_\delta^{\delta} = S_\delta.
\]

(5.9)

By using Eqs. (4.4), (5.4), and (5.5), we obtain

\[
\begin{align*}
\Delta_\delta^{\zeta, \eta} (J_0) &= \frac{1}{q-1} (1 \otimes 1 - \delta \zeta \eta G(J_0) \otimes G(J_0)), \\
\Delta_\delta^{\zeta, \eta} (J_\pm) &= \eta J_\pm \otimes (G(J_0))^{-1} + \zeta G(J_0) \otimes J_\pm, \\
S_\delta^{\zeta} (J_0) &= \frac{1}{q-1} \left(1 - \zeta \delta (G(J_0))^{-1}\right), \\
S_\delta^{\zeta} (J_\pm) &= -q^{\pm 1} J_\pm.
\end{align*}
\]

(5.10)

Alternatively, $\Delta_\delta^{\zeta, \eta}$, $\epsilon_\delta$, and $S_\delta^{\zeta}$ can be defined directly in terms of the comultiplication, counit, and antipode maps $\Delta$, $\epsilon$, $S$ of $\text{su}_q(2)$, as well as the map $P_\delta$, by the commuting diagrams

\[
\begin{array}{cccc}
\text{su}_q(2) & \xrightarrow{\Delta} & \text{su}_q(2) \otimes \text{su}_q(2) & \text{su}_q(2) \xrightarrow{\epsilon} \mathbb{C} \\
p_\delta \downarrow & & p_\delta \otimes p_\eta & p_\delta \downarrow \\
\mathcal{A}_q^+ (1) & \xrightarrow{\Delta_\delta^{\zeta, \eta}} & \mathcal{A}_q^+ (1) \otimes \mathcal{A}_q^+ (1) & \mathcal{A}_q^+ (1) \xrightarrow{\epsilon_\delta} \mathbb{C} \\
& & p_\delta \downarrow & p_\delta \downarrow \\
& & \mathcal{A}_q^+ (1) & \xrightarrow{S_\delta^{\zeta}} \mathcal{A}_q^+ (1)
\end{array}
\]

(5.11)

As shown in the Appendix, $\Delta_\delta^{\zeta, \eta}$, $\epsilon_\delta$, and $S_\delta^{\zeta}$ transform under $\sigma_\delta$ as follows:

\[
(\sigma_{\mu \zeta} \otimes \sigma_{\nu \eta}) \circ \Delta_\delta^{\zeta, \eta} = \Delta_\rho^{\mu, \nu} \circ \sigma_{\rho \delta},
\]

(5.12)

\[
\epsilon_\delta \circ \sigma_{\delta \zeta} = \epsilon_\zeta,
\]

(5.13)

\[
\sigma_{\zeta \eta} \circ S_\delta^{\eta} = S_\mu^{\zeta} \circ \sigma_{\mu \delta}.
\]

(5.14)
By using Eqs. (5.12)–(5.14) and the Hopf algebra axioms (2.18), (2.19), satisfied by \( \Delta, \epsilon, \) and \( S \), or alternatively the diagrammatic method of the Appendix, we obtain

**Proposition 3:** The algebra \( A_q^+ (1) \) is endowed with a generalized Hopf algebraic structure, whose comultiplication, counit, and antipode maps, \( \Delta^\zeta, \epsilon, S^\zeta, \) defined in Eqs. (5.10) and (5.11), satisfy the following generalized coassociativity, counit, and antipode axioms:

\[
\begin{align*}
(\Delta^\zeta \otimes \text{id}) \circ \Delta^\mu = (\text{id} \otimes \Delta^\nu) \circ \Delta^\rho(A), \\
(\epsilon \otimes \sigma) \circ \Delta^\zeta = (\sigma \otimes \epsilon_q) \circ \Delta^{\zeta_q}(A) = A, \\
m \circ \left( S^\mu \otimes \sigma_q \right) \circ \Delta^\zeta = m \circ \left( \sigma_{\mu_q} \otimes S^\eta_q \right) \circ \Delta^{\zeta_q}(A) = \iota \circ \epsilon_q(A),
\end{align*}
\]

where \( A \) denotes any element of \( A_q^+ (1) \), \( m \) and \( \iota \) are the multiplication and unit maps of \( A_q^+ (1) \), \( \delta, \zeta, \eta, \mu, \nu, \rho \) take any values in the set \( \{-1, +1\} \), and no summation over repeated indices is implied. Moreover, \( \Delta^{\zeta_q} \) and \( \epsilon_q \) are algebra homomorphisms, while \( S^\zeta_q \) is both an algebra and a coalgebra antihomomorphism.

**Remark.** In principle, the multiplication and unit maps of \( A_q^+ (1) \) should be distinguished from those of \( \text{su}_q(2) \), but as no confusion can arise, for simplicity’s sake we use the same notation.

By using the generalized coproduct \( \Delta^{\zeta_q} \), any \((N_1 + 1)\)- and \((N_2 + 1)\)-dimensional unirreps of \( A_q^+ (1) \), specified by \( J^\zeta_1 \) and \( J^\zeta_2 \) respectively, can now be coupled to provide two reducible representations in \( V^{J^\zeta_1} \otimes V^{J^\zeta_2} \), which are characterized by \( \delta = +1 \) and \( \delta = -1 \), respectively. From Eq. (5.11), we obtain for the corresponding operators \( \Phi_{\delta}^{J^\zeta_1, J^\zeta_2}(A), \ A \in A_q^+ (1), \)

\[
\begin{align*}
\Phi_{\delta}^{J^\zeta_1, J^\zeta_2} (J_0) \left( |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle \right) &= \frac{1}{q-1} \left( 1 - \delta q^{-\left( N_1 + N_2 - 2n_1 - 2n_2 \right)/2} \right) |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle, \\
\Phi_{\delta}^{J^\zeta_1, J^\zeta_2} (J_-) \left( |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle \right) &= \sqrt{\left[ N_1 + 1 \right]_q \left[ N_1 - n_1 \right]_q} q^{\left( N_2 - 2n_2 \right)/2} |J^\zeta_1, qm^\zeta_1 - 1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle \\
& \quad + q^{-\left( N_1 - 2n_1 \right)/2} \sqrt{\left[ N_2 + 1 \right]_q \left[ N_2 - n_2 \right]_q} |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, qm^\zeta_2 - 1 \rangle, \\
\Phi_{\delta}^{J^\zeta_1, J^\zeta_2} (J_+) \left( |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle \right) &= \sqrt{\left[ N_1 + 1 \right]_q \left[ N_1 - n_1 \right]_q} q^{\left( N_2 - 2n_2 \right)/2} |J^\zeta_1, qm^\zeta_1 - 1 \rangle \otimes |J^\zeta_2, m^\zeta_2 \rangle \\
& \quad + q^{-\left( N_1 - 2n_1 \right)/2} \sqrt{\left[ N_2 + 1 \right]_q \left[ N_2 - n_2 \right]_q} |J^\zeta_1, m^\zeta_1 \rangle \otimes |J^\zeta_2, qm^\zeta_2 - 1 \rangle,
\end{align*}
\]
By comparing Eq. (5.18) with Eq. (4.7), we note that the \( \delta \)-type reducible representation in \( V^{J_1^+} \otimes V^{J_2^+} \) coincides with the reducible representation in \( V^{J_1^+} \otimes V^{J_2^-} \), previously considered. Therefore the same transformation decomposes both representations into a direct sum of \((N+1)\)-dimensional unirreps characterized by \( J^\delta \). Hence, the states

\[
\ket{J_1^\delta J_2^\delta m^\delta} = \sum_{m_1^\delta, m_2^\delta} \braket{J_1^\delta m_1^\delta, J_2^\delta m_2^\delta | J^\delta m^\delta}_{DQ} \ket{J_1^\delta m_1^\delta} \otimes \ket{J_2^\delta m_2^\delta},
\]

with

\[
\braket{J_1^\delta m_1^\delta, J_2^\delta m_2^\delta | J^\delta m^\delta}_{DQ} = \left( \frac{N_1}{2} \frac{N_2}{2} - n_1, \frac{N_1}{2} \frac{N_2}{2} - n_2 \right)_q \left( \frac{N_1}{2} \frac{N_2}{2} - n \right)_q,
\]

span the carrier space of the unirrep \( J^\delta \) in \( V^{J_1^+} \otimes V^{J_2^+} \).

It should be stressed that the space \( V^{J_1^+} \otimes V^{J_2^+} \) does not contain two \((N+1)\)-dimensional unirreps, characterized by \( J^+ \) and \( J^- \) respectively, for \( N \) in the range \( |N_1 - N_2|, |N_1 - N_2| + 2, \ldots, N_1 + N_2 \), but a single one, which may be considered as that specified by \( J^+ \) or that specified by \( J^- \), according to whether the coproduct \( \Delta^\zeta_+ \eta \) or \( \Delta^\zeta_- \eta \) is used. In other words, a given linear combination of states \( \ket{J_1^\delta m_1^\delta} \otimes \ket{J_2^\delta m_2^\delta} \), as that contained in Eq. (3.13), may be regarded as a basis state of a unirrep \( J^+ \) or \( J^- \), where \( J^+ \) and \( J^- \) are determined from \( N \) by using Eq. (3.8) (and in the same way, the corresponding \( m^+ \) or \( m^- \) is determined from \( N \) and \( n \) by using Eq. (3.7)).

For instance, the state \( \ket{J_1^+, J_1^+} \otimes \ket{J_2^+, J_2^+} \), where \( J_1^+ = J_2^+ = \left( \frac{\sqrt{q}}{2} \right)^{-1} \left( \frac{\sqrt{q}}{2} + 1 \right) \), corresponding to \( N_1 = N_2 = 1, n_1 = n_2 = 0 \), may be considered as the highest-weight state of the three-dimensional unirrep characterized by \( J^+ = q^{-1} \), or the lowest-weight state of the three-dimensional unirrep specified by \( J^- = (q + 1)/(q - 1) \), both of these states corresponding to \( N = 2 \) and \( n = 0 \) in Eqs. (3.7) and (3.8),

\[
\ket{J_1^+, J_1^+} \otimes \ket{J_2^+, J_2^+} = \ket{J_1^+ J_2^+ J^+ = q^{-1}, m^+ = q^{-1}} = \ket{J_1^+ J_2^- J^- = \frac{q+1}{q-1}, m^- = q\frac{q+1}{(q-1)}}.
\]
By direct use of Eq. (5.10), we indeed obtain for instance
\[
\Delta_{J_+}^{\pm,+} (J_0) \left( |J_1^+, J_1^+\rangle \otimes |J_2^+, J_2^+\rangle \right) = q^{-1} \left( |J_1^+, J_1^+\rangle \otimes |J_2^+, J_2^+\rangle \right),
\]
\[
\Delta_{J_+}^{\pm,+} (J_0) \left( |J_1^+, J_1^+\rangle \otimes |J_2^+, J_2^+\rangle \right) = \frac{q + 1}{q(q - 1)} \left( |J_1^+, J_1^+\rangle \otimes |J_2^+, J_2^+\rangle \right),
\]
\[
\Delta_{J_+}^{\pm,+} (J_0) \left( |J_1^+, J_1^+\rangle \otimes |J_2^+, J_2^+\rangle \right) = 0.
\]

(5.22)

In the next section, we shall examine how the universal \( R \)-matrix definition valid for \( su_q(2) \), hence for the double Hopf structure of \( A_q^+ (1) \), can be extended to the generalized Hopf structure of the latter.

**VI GENERALIZED \( R \)-MATRIX OF THE ALGEBRA \( A_q^+ (1) \)**

It is well known [18] that \( su_q(2) \) is a quasitriangular Hopf algebra, which means that there exists an invertible element \( R \in su_q(2) \otimes su_q(2) \) (completed tensor product), called the \( su_q(2) \) universal \( R \)-matrix,

\[
R = q^{2j_0 \otimes j_0} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n-1)/2} \left( q^{j_0} j_+ \otimes q^{-j_0} j_- \right)^n,
\]

such that its comultiplication \( \Delta \) and its opposite comultiplication \( \Delta^{\text{op}} \equiv \tau \circ \Delta \) (where \( \tau \) is the twist map, defined by \( \tau (a \otimes b) = b \otimes a \)) only differ by a conjugation by \( R \),

\[
\Delta^{\text{op}} (a) = R \Delta (a) R^{-1}, \quad a \in su_q(2),
\]

and in addition

\[
(\Delta \otimes \text{id})(R) = R_{13} R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13} R_{12}.
\]

(6.3)

By applying the map \( P_\delta \otimes P_\delta \), considered in Eq. (4.3), to the \( su_q(2) \) \( R \)-matrix, given in Eq. (5.1), we obtain an invertible element \( R^\delta \) of \( A_q^+ (1) \otimes A_q^+ (1) \) (completed tensor product),

\[
R^\delta = q^{2 \log_q (\delta G(J_0)) \otimes \log_q (\delta G(J_0))} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n-1)/2} \left( (G(J_0))^{-1} J_+ \otimes G(J_0) J_- \right)^n.
\]

(6.4)
It can be easily checked that it satisfies with respect to the comultiplication \( \Delta_\delta \), defined in Eq. (4.4), and the opposite comultiplication \( \Delta^\text{op}_\delta \equiv \tau \circ \Delta_\delta \), relations similar to Eqs. (6.2) and (13), namely

\[
\Delta^\text{op}_\delta(A) = R_\delta \Delta_\delta(A) (R_\delta)^{-1}, \quad A \in A^+_q(1),
\]

(6.5)

\[
(\Delta_\delta \otimes \text{id}) (R_\delta) = R_{13}^\delta R_{23}^\delta, \quad (\text{id} \otimes \Delta_\delta) (R_\delta) = R_{13}^\delta R_{12}^\delta.
\]

(6.6)

Hence, \( A^+_q(1) \) has a double quasitriangular Hopf structure with a double universal \( R \)-matrix \( R_\delta \), \( \delta = \pm 1 \), where \( R^+ \) (resp. \( R^- \)) corresponds to the coalgebra structure and antipode \( (\Delta^+, \epsilon^+, S^+) \) (resp. \( (\Delta^-, \epsilon^-, S^-) \)), and therefore acts in the interval \( ((q - 1)^{-1}, +\infty) \) (resp. \( (-\infty, (q - 1)^{-1}) \)).

It is worth noting that as direct consequences of Eqs. (6.5), (6.6), and of the Hopf algebra axioms (2.18), (2.19), satisfied by \( \Delta_\delta, \epsilon_\delta, S_\delta \), the double \( R \)-matrix \( R_\delta \), \( \delta = \pm 1 \), satisfies the relations

\[
R_{12}^\delta R_{13}^\delta R_{23}^\delta = R_{23}^\delta R_{13}^\delta R_{12}^\delta,
\]

(6.7)

\[
(\epsilon_\delta \otimes \text{id}) (R_\delta) = (\text{id} \otimes \epsilon_\delta) (R_\delta) = 1,
\]

(6.8)

\[
(S_\delta \otimes \text{id}) (R_\delta) = (\text{id} \otimes S^{-1}_\delta) (R_\delta) = (R_\delta)^{-1},
\]

(6.9)

which are similar to well-known properties of the \( \text{su}_q(2) \) \( R \)-matrix [18]. In particular, Eq. (6.7) shows that both \( R^+ \) and \( R^- \) are solutions of the (ordinary) Yang-Baxter equation (YBE).

Turning now to the generalized Hopf structure introduced in the previous section, let us consider in \( A^+_q(1) \otimes A^+_q(1) \) (completed tensor product) the four elements

\[
R^\zeta,\eta = (\sigma_\zeta \otimes \sigma_\eta) (R_\delta), \quad \zeta, \eta = \pm 1,
\]

(6.10)

where, on the right-hand side, \( \delta \) takes any value in the set \( \{-1, +1\} \) and no summation over it is implied. From Eqs. (5.4), (5.5), and (6.4), it follows that the explicit form of \( R^\zeta,\eta \) is given by

\[
R^\zeta,\eta = q^{2\log_q(\zeta G(J_0)) \otimes \log_q(\eta G(J_0))} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n-1)/2} \left( (\zeta G(J_0))^{-1} J_+ \otimes \eta G(J_0) J_- \right)^n.
\]

(6.11)
We also note that $R^{\delta,\delta} = R^\delta$.

By using now Eqs. (5.8), (5.12)–(5.14), (6.5), (6.6), and (6.10), we easily obtain

Proposition 4: The generalized Hopf algebra, defined in Proposition 3 of the previous section, has a generalized universal $R$-matrix made of four invertible pieces $R^{\zeta,\eta}, \zeta, \eta = \pm 1$, as defined in Eq. (6.10) or (6.11), which satisfy the following properties:

\[
\left(\sigma_{\mu\zeta} \otimes \sigma_{\nu\eta}\right) \left(R^{\zeta,\eta}\right) = R^{\mu,\nu},
\]

\[
\tau \circ \Delta_\delta^\zeta(A) = R^{\zeta,\eta} \Delta_\delta^\zeta(A) \left(R^{\zeta,\eta}\right)^{-1}, \quad A \in A^+_q(1),
\]

\[
\left(\Delta_\zeta^{\lambda,\mu} \otimes \sigma_{\nu\eta}\right) \left(R^{\zeta,\eta}\right) = R^{\lambda,\nu}_{13} R^{\mu,\nu}_{23}, \quad \left(\sigma_{\lambda\zeta} \otimes \Delta_\eta^{\mu,\nu}\right) \left(R^{\zeta,\eta}\right) = R^{\lambda,\nu}_{13} R^{\lambda,\mu}_{12}.
\]

From the results of Proposition 4 or, more simply, by combining definitions (5.8), (5.10), and properties (5.12)–(5.14), (6.12), with Eqs. (6.7), (6.8), and (6.9), we find that the latter can be generalized as follows:

Corollary 1: The generalized universal $R$-matrix of $A^+_q(1)$ satisfies the relations

\[
R_{12}^{\zeta,\eta} R_{13}^{\zeta,\mu} R_{23}^{\eta,\mu} = R_{23}^{\eta,\nu} R_{13}^{\zeta,\mu} R_{12}^{\zeta,\eta},
\]

\[
\left(\epsilon_\zeta \otimes \text{id}\right) \left(R^{\zeta,\eta}\right) = \left(\text{id} \otimes \epsilon_\eta\right) \left(R^{\zeta,\eta}\right) = 1,
\]

\[
\left(S_\zeta^{\lambda} \otimes \sigma_{\mu\eta}\right) \left(R^{\zeta,\eta}\right) = \left(\sigma_{\lambda\zeta} \otimes (S_\eta^\mu)^{-1}\right) \left(R^{\zeta,\eta}\right) = \left(R^{\lambda,\mu}\right)^{-1}.
\]

In particular, Eq. (6.13) shows that the generalized $R$-matrix (6.11) is a solution of the coloured YBE [24]–[31], where the ‘colour’ parameters $\zeta, \eta, \mu$ take discrete values in the set \{-1, +1\}. We therefore propose to call $(A^+_q(1), +, m, t, \Delta_\delta^\zeta, \epsilon_\delta, S_\delta^\zeta, R^{\zeta,\eta}; \mathbb{C})$ a two-colour quasitriangular Hopf algebra over $\mathbb{C}$.

VII CONCLUDING REMARKS

In the present paper, we did construct a DQA, denoted as $A^+_q(1)$, which has two series of (N+1)-dimensional unirreps, where $N = 0, 1, 2, \ldots$, and we did show that it can be
endowed with a generalized quasitriangular Hopf structure, providing us with composition laws for all couples of unirreps. This new algebraic structure was termed a two-colour quasitriangular Hopf algebra because the corresponding generalized $R$-matrix satisfies the coloured YBE, where the colour parameters take two discrete values.

It should be noted that various approaches have been previously used to construct solutions of the coloured YBE \[24\]–\[30\]. In the works of Akutsu and Deguchi \[25\], and Ge et al. \[29\], an infinite-dimensional representation of $sl_q(2)$ was considered and the colour parameter was introduced as the value of the corresponding Casimir operator. To get finite-dimensional matrix solutions of the coloured YBE, $q$ had to be restricted to a root of unity. In the approach pioneered by Burdík and Hellinger \[27, 28, 29\], deformations of a non-semisimple Lie algebra, such as $gl_q(2)$, were considered, then the colour parameter was taken as the eigenvalue of the extra Casimir operator, related with the invariant $u(1)$ subalgebra. Another method, proposed by Kundu and Basu-Mallick \[30\], used a symmetry transformation of the YBE (for $gl_q(N)$) to derive solutions of the coloured one.

Another alternative approach was used in the present work. The colour parameter now turns out to be related with an involutive automorphism of the algebra considered. Its two-valuedness is a direct consequence of this property and contrasts with its continuous character in previous approaches. However, as in the work of Ge et al. \[25\], this parameter serves to distinguish between the representations of the algebra with the same dimension.

It is also worth pointing out that here the colour parameter does not make any appearance in the algebra defining relations, as it is only needed in the generalized coalgebraic structure and antipode. This again contrasts with both the coloured Fadeev-Reshetikhin-Takhtajan algebra and coloured quantum group (generalizing both $GL_q(2)$ and $GL_{p,q}(2)$), recently constructed by Basu-Mallick \[29\], where the colour parameter enters both the algebraic structure definition and the generator realization, while the coalgebraic structure remains free from such dependence.

The two-colour quasitriangular Hopf algebra considered here bears some similarity to the coloured quasitriangular Hopf algebras previously introduced by Ohtsuki \[31\], which are also characterized by the existence of a coloured universal $R$-matrix. There are however
some differences between both algebraic structures, the most striking one being the fact that
the generalized comultiplication depends upon two colour parameters in Ref. [31], instead
of three in the present work.

Construction of other DQA’s with a generalized Hopf structure similar to that considered
in the present paper, as well as the investigation of possible relationships with other coloured
algebraic structures, might be some interesting problems for future study.

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APPENDIX: DIAGRAMMATIC PROOFS OF EQUATIONS (5.12)–(5.14) AND (5.15)–(5.17)

In this Appendix, we prove Eqs. (5.12)–(5.14) and (5.15)–(5.17) by combining the diagrammatic definitions (5.11) of the generalized coproduct, counit, and antipode with the action (5.7) of \( \sigma_{\pm} \) on the basic mapping \( P_\delta \), and the diagrammatic representation of standard Hopf algebra axioms [18]. In the following diagrams, the shorthand notations \( \mathcal{H} \) and \( \mathcal{A} \) are used for \( su_q(2) \) and \( A^+_q(1) \), respectively.

Eqs. (5.12)–(5.14) are given by the inner low rectangular, the outer square, and the outer rectangular diagrams hereunder, respectively:

\[
\begin{align*}
\mathcal{H} \otimes \mathcal{H} & \xleftarrow{\Delta} \mathcal{H} \xrightarrow{id} \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \\
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\mathcal{H} \otimes \mathcal{H}} \mathcal{D} \xrightarrow{\mathcal{D} \otimes \mathcal{D}} \mathcal{H} \otimes \mathcal{H}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A} \otimes \mathcal{A} & \xleftarrow{id} \mathcal{A} \xrightarrow{\mathcal{A} \otimes \mathcal{A}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{id} \mathcal{A} \otimes \mathcal{A}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A} & \xrightarrow{id} \mathcal{A} \xrightarrow{\mathcal{A} \otimes \mathcal{A}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{id} \mathcal{A} \otimes \mathcal{A}
\end{align*}
\]

Eq. (5.15) corresponds to the outer rectangular diagram:
The two parts of Eq. (5.16) are given by the outer rectangular diagrams:

\[
\begin{array}{c}
\text{A} \xrightarrow{\Delta^\zeta,\rho} \text{A} \otimes \text{A} & \xleftarrow{\text{id} \otimes \text{id}} & \text{A} \otimes \text{A} & \xrightarrow{\text{id} \otimes \text{id}} & \text{A} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{H} \otimes \text{H} & \xrightarrow{\Delta} & \text{H} \otimes \text{H} \otimes \text{H} & \xrightarrow{P_\zeta \otimes P_\rho \otimes P_\sigma} & \text{A} \otimes \text{A} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{A} \xrightarrow{P_\delta} & \text{H} \xrightarrow{\Delta} & \text{H} \otimes \text{H} \xrightarrow{P_\zeta \otimes P_\eta} & \text{A} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{A} \xrightarrow{\text{id}} & \text{A} \xrightarrow{\Delta_{\zeta,\eta}^\delta} & \text{A} \otimes \text{A} \xrightarrow{\text{id} \otimes \text{id}} & \text{A} \otimes \text{A}
\end{array}
\]

and

\[
\begin{array}{c}
\text{A} \xrightarrow{\text{id}} & \text{A} \xrightarrow{\zeta} & \text{C} \otimes \text{A} \xrightarrow{\text{id}} & \text{C} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{H} \xrightarrow{\Delta} & \text{H} \otimes \text{H} \xrightarrow{P_\zeta \otimes P_\eta} & \text{C} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{A} \xrightarrow{P_\delta} & \text{H} \xrightarrow{\Delta} & \text{H} \otimes \text{H} \xrightarrow{P_\zeta \otimes P_\eta} & \text{A} \otimes \text{A} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{id} & \downarrow & \text{id} & \downarrow & \text{id} \\
\text{A} \xrightarrow{\text{id}} & \text{A} \xrightarrow{\Delta_{\zeta,\eta}^\delta} & \text{A} \otimes \text{A} \xrightarrow{\text{id} \otimes \text{id}} & \text{A} \otimes \text{A}
\end{array}
\]

Finally, the two parts of Eq. (5.17) correspond to the right-hand lower rectangular diagrams:
\[ \mathcal{H} \xleftrightarrow{\text{id}} \mathcal{H} \xrightarrow{P_\mu} \mathcal{A} \xleftrightarrow{\text{id}} \mathcal{A} \]

\[ \mathcal{H} \xleftarrow{\epsilon} \mathcal{C} \xleftrightarrow{\text{id}} \mathcal{C} \xrightarrow{\epsilon} \mathcal{A} \]

\[ \mathcal{H} \xrightarrow{m} \mathcal{H} \xrightarrow{\Delta} \mathcal{A} \xrightarrow{\Delta_{\mathcal{S}}^{\mathcal{C},\mathcal{A}}} \mathcal{A} \]

\[ \mathcal{H} \otimes \mathcal{H} \xleftrightarrow{\text{id} \otimes \text{id}} \mathcal{H} \otimes \mathcal{H} \xrightarrow{P_{\mathcal{H} \otimes \mathcal{H}}} \mathcal{A} \otimes \mathcal{A} \xleftrightarrow{\text{id} \otimes \text{id}} \mathcal{A} \otimes \mathcal{A} \]

and

\[ \mathcal{H} \xleftrightarrow{\text{id}} \mathcal{H} \xrightarrow{P_\mu} \mathcal{A} \xleftrightarrow{\text{id}} \mathcal{A} \]

\[ \mathcal{H} \xleftarrow{\epsilon} \mathcal{C} \xleftrightarrow{\text{id}} \mathcal{C} \xrightarrow{\epsilon} \mathcal{A} \]

\[ \mathcal{H} \xrightarrow{m} \mathcal{H} \xrightarrow{\Delta} \mathcal{A} \xrightarrow{\Delta_{\mathcal{S}}^{\mathcal{C},\mathcal{A}}} \mathcal{A} \]

\[ \mathcal{H} \otimes \mathcal{H} \xleftrightarrow{\text{id} \otimes \text{id}} \mathcal{H} \otimes \mathcal{H} \xrightarrow{P_{\mathcal{H} \otimes \mathcal{H}}} \mathcal{A} \otimes \mathcal{A} \xleftrightarrow{\text{id} \otimes \text{id}} \mathcal{A} \otimes \mathcal{A} \]

\[ (A7) \]

\[ (A8) \]
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