Dynamic risk measures on variable exponent
Bochner–Lebesgue spaces

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Abstract Risk measures are a critical factor not only in risk analysis, but also in insurance and financial applications. However, as any given risk assessment may change over time, traditional static risk measures may be inappropriate. In this paper, we consider dynamic risk measures on a special space $L^{p(\cdot)}$, where the variable exponent $p(\cdot)$ is no longer a given real number, as in the space $L^p$, but a random variable. By further developing the axioms related to this class of risk measures, we are able to derive dual representation for them.

Keywords risk measure · Banach · time consistency

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1 Introduction

Research on risk is a hot topic in both quantitative and theoretical research, and risk models have attracted considerable attention. The quantitative calculation of risk involves two problems: choosing an appropriate risk model and allocating the risk to individual institutions. This has led to further research on risk measures.

If the time required to quantify the risk is unknown, it is important not to contradict oneself over time in one’s risk assessments. This idea has encouraged the study of dynamic risk measures. In a seminal paper, [19] first introduced the class of dynamic coherent risk measures. Further, [9] introduced dynamic convex risk measures and studied the related time consistency properties. Other studies on dynamic risk measures include those of [1], [5], [11], [17], and the references therein.

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In the abovementioned research on risk measures, the space of financial positions is described by the linear space of bounded random variables, which can be regarded as the space or subspace of $L^p$ with $p \in [1, \infty)$. However, as financial markets become more complicated, the usual risk measures may failed to capture the complexity of markets. This has raised awareness of the urgent need for more appropriate risk measures under a financial systems with greater uncertainty and volatility. Taking this into consideration, we would like to emphasize that our study of risk measures will not focus on the common space of financial positions, but on a special space: the variable exponent Bochner–Lebesgue space, which is denoted by $L^{p(.)}$. Under this space, the order $p(.)$ is no longer a fixed positive number like $L^p$, but a measurable function.

Variable exponent Lebesgue spaces were first described by [18]. More recent studies on variable exponent Lebesgue spaces include those of [2], [8], [13], [14], [16], [22], [23], and the references therein.

The main focus of this paper is dynamic risk measures on the variable exponent Bochner–Lebesgue space $L^{p(.)}$. To this end, we first introduce convex risk measures on $L^{p(.)}$. With the help of these convex risk measures on $L^{p(.)}$, we consider dynamic and cash sub-additive risk measures on $L^{p(.)}$. By further developing the axioms related to these classes of risk measures, we are able to derive their dual representations. Moreover, the optimized certainty equivalent on variable exponent Bochner–Lebesgue spaces is investigated as an example.

The remainder of this paper is organized as follows. In Sect. 2, we briefly review the definition and main properties of variable exponent Bochner–Lebesgue spaces and some preliminaries that are used throughout this paper.

From now on, let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite complete measurable space, and $E$ be a given reflexive Banach space with zero element $\theta$ and dual space $E^*$ having the Radon-Nikodým property. Throughout this paper, we assume that $E^*$ is partially ordered by a given cone $K_0$ and $E$ is partially ordered by $K$, where $K := \{ f \in E : \langle X, f \rangle \geq 0 \text{ for any } X \in K_0 \}$ is the positive dual cone of $K_0$. We suppose that the numeraire asset $z$ is some interior point of $K$. The asset $z$ is actually either a ‘reference cash stream’ ([20]) or a ‘relatively secure cash stream’ ([15]).
Remark 21 The partial order relation $\geq_K$ is defined as follows, for any $X, Y \in E$,

$$X \geq_K Y \iff X - Y \in K.$$ 

The cone $K$ consists of the ‘admissible’ price functionals. $K$ also plays the role of the solvency set of financial positions, which denotes the way that a set of investors jointly interprets the common notion of the cost of financial positions.

The Banach-space-valued Bochner–Lebesgue spaces with variable exponents were first introduced by [6]. We now recall the definition and related properties of this special space. We denote the set of all $\mathcal{F}$-measurable functions $p(\cdot) : \Omega \to [1, \infty]$ by $S(\Omega, \mu)$; these functions are said to be variable exponent on $\Omega$. For a function $p(\cdot) \in S(\Omega, \mu)$, we define $p'(\cdot) \in S(\Omega, \mu)$ by $1/p(y) + 1/p'(y) = 1$. The following definitions and properties come from [6].

Definition 21 A function $f : \Omega \to E$ is strongly $\mathcal{F}$-measurable if there exists a sequence $\{f_n\}_{n \geq 1}$ converging to $f \mu$-almost everywhere.

Definition 22 The Bochner–Lebesgue space with variable exponent, which is denoted by $L^{p(\cdot)}(\Omega, E)$, is the collection of all strongly $\mathcal{F}$-measurable functions $f : \Omega \to E$ endowed with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega, E)} := \inf \{\lambda > 0, \rho_{p(\cdot)}(f/\lambda) \leq 1\}$$

where

$$\rho_{p(\cdot)}(f) := \int_{\Omega} \|f(y)\|^{p(y)}d\mu(y) \quad \text{and} \quad p(\cdot) \in S(\Omega, \mu).$$

Remark 22 If $E$ is a reflexive Banach space, then the dual of $L^{p(\cdot)}(\Omega, E)$ is characterized by the mapping $g \mapsto V_g$ with $g \in L^{p'(\cdot)}(\Omega, E^*)$ and $V_g \in (L^{p(\cdot)}(\Omega, E))^*$ as follows:

$$\langle V_g, f \rangle = \int_{\Omega} \langle g, f \rangle d\mu, \quad \text{for any} \quad f \in L^{p(\cdot)}(\Omega, E).$$

See [6].

3 Convex risk measures on $L^{p(\cdot)}$

The main aim of this paper is to study the dual representation of dynamic risk measures on variable exponent Bochner–Lebesgue spaces. To this end, this section first considers convex risk measures, which will be used later for the dynamic risk measures. The convex risk measure was first introduced by [11] and [12]. Furthermore, [7] and [21] extend convex risk measures to loss-based cases.

In the absence of ambiguity, we denote the variable exponent Bochner–Lebesgue space by $L^{p(\cdot)} := L^{p(\cdot)}(\Omega, E)$. Let $T$ be a discrete time horizon which can reach infinity and consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$.
with \( \{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T = \mathcal{F} \). Let \( L^p(\mathcal{F}_t) \) be the space of all strongly \( \mathcal{F}_t \)-measurable functions \( f_t \) which satisfy Definition \([22]\). Note that \( L^p(\mathcal{F}_0) = L^p(\mathcal{F}_T) \). We denote \( L^p(\mathcal{F}_0) := \{ f \in L^p(\mathcal{F}_0) : \Omega \to K \} \) and \( L^p(\mathcal{F}_0) := \{ g \in L^p(\mathcal{F}_0) : \Omega \to K_0 \} \). We denote the space of all essentially bounded \( \mathcal{F}_t \)-measurable random variables by \( L^\infty := L^\infty(\Omega, \mathcal{F}_t, \mu) \). Next, the definition of a convex risk measure on \( L^p(\cdot) \) is introduced using an axiomatic approach.

**Definition 31** Let \( E \) be a Banach space ordered by the partial ordering relation induced by a cone \( K \) with interior point \( z \), and \( L^p(\cdot) \) be a variable exponent Bochner–Lebesgue space. A function \( \varrho : L^p(\cdot) \to \mathbb{R} \) is said to be a \( p(\cdot) \)-convex risk measure if it satisfies the following:

A1 Monotonicity: for any \( f_1, f_2 \in L^p(\cdot), f_1 \leq_K f_2 \) a.s. implies \( \varrho(f_1) \geq \varrho(f_2) \);
A2 Translation invariance: for any \( m \in \mathbb{R} \) and \( f \in L^p(\cdot), \varrho(f + mz) = \varrho(f) - m \);
A3 Convexity: for any \( f_1, f_2 \in L^p(\cdot) \) and \( \lambda \in (0, 1), \varrho(\lambda f_1 + (1 - \lambda)f_2) \leq \lambda \varrho(f_1) + (1 - \lambda) \varrho(f_2) \).

**Remark 31** The order in A1 is the partial order under a cone \( K \) which is defined by Remark \([21]\). The interior point \( z \) of \( K \) in A2 is considered to be the numeraire asset, which means that \( mz \in E \) for any \( m \in \mathbb{R} \). Before we study the dual representation of the \( p(\cdot) \)-convex risk measures, the acceptance sets should be defined.

**Definition 32** The acceptance set of the \( p(\cdot) \)-risk measure \( \varrho \) is defined as

\[
\mathcal{A}_\varrho := \{ f \in L^p(\cdot) : \varrho(f) \leq 0 \}
\]

and we denote \( \mathcal{A}_\varrho^0 \) by

\[
\mathcal{A}_\varrho^0 := \{ g \in (L^p(\cdot))^* : \langle g, f \rangle \geq 0 \text{ for any } f \in \mathcal{A}_\varrho \}.
\]

**Remark 32** It is relatively easy to check that \( \mathcal{A}_\varrho \) is a convex set if \( \varrho \) satisfies the convexity property. \( \mathcal{A}_\varrho^0 \) can be considered as the positive polar cone of \( \mathcal{A}_\varrho \).

Now, we provide the dual representation of \( p(\cdot) \)-convex risk measures which will be used in the proof of \( p(\cdot) \)-dynamic risk measures in Sect. \([5]\).

**Theorem 31** If \( \varrho : L^p(\cdot) \to \mathbb{R} \) is a \( p(\cdot) \)-convex risk measure, then for any \( f \in L^p(\cdot) \),

\[
\varrho(f) = \sup_{g \in Q_{\varrho(\cdot)}} \{ \langle g, -f \rangle - \alpha(g) \}
\]

where

\[
Q_{\varrho(\cdot)} := \{ g \in (L^p(\cdot))^* : \langle \frac{dg}{d\mu}, z \rangle = 1, \frac{dg}{d\mu} \in L^p(\cdot)(K_0) \}
\]

and \( \alpha : Q_{\varrho(\cdot)} \to \mathbb{R} \) is the penalty function while the minimal penalty function \( \alpha_{\text{min}} \) is denoted by

\[
\alpha_{\text{min}}(g) := \sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho(f) \} = \sup_{f \in \mathcal{A}_\varrho} \{ \langle g, -f \rangle \}. \]
Proof. For any $g \in Q_{p(\cdot)}$, we denote
\[ \alpha(g) = \sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho(f) \} \]
and
\[ \alpha_{\min}(g) = \sup_{f \in \mathcal{A}_p} \{ \langle g, -f \rangle \}. \]
We now show that $\alpha(g) = \alpha_{\min}(g)$ for any $g \in Q_{p(\cdot)}$. Note that $\alpha(g) \geq \alpha_{\min}(g)$.
Indeed, for any $f \in \mathcal{A}_p$, $\langle g, -f \rangle - \varrho(f) \geq \langle g, -f \rangle$. Hence,
\[ \sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho(f) \} \geq \sup_{f \in \mathcal{A}_p} \{ \langle g, -f \rangle \}. \]
We now prove that $\alpha(g) \leq \alpha_{\min}(g)$. For any $f \in L^p(\cdot)$, consider $f_1 = f + \varrho(f)z \in \mathcal{A}_p$. Thus,
\[ \alpha_{\min}(g) \geq \langle g, -f_1 \rangle = \langle g, -f \rangle - \varrho(f)g(z) \]
\[ = \langle g, -f \rangle - \varrho(f) \int_{\Omega} \langle \frac{dg}{d\mu}, z \rangle d\mu \]
\[ = \langle g, -f \rangle - \varrho(f). \]
Hence, we have $\alpha(g) = \alpha_{\min}(g)$, and it is easy to check that
\[ \varrho(f) \geq \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle - \alpha(g) \}. \]
Next, we show that the above inequality only holds in the case of equality.
Suppose there is some $f_0 \in L^p(\cdot)$ such that
\[ \varrho(f_0) > \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f_0 \rangle - \alpha(g) \}. \]
Hence, there exists some $m \in \mathbb{R}$ such that
\[ \varrho(f_0) > m > \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f_0 \rangle - \alpha(g) \}. \]
Thus, we have
\[ \varrho(f_0 + mz) = \varrho(f_0) - m > 0, \]
which means that $f_0 + mz \notin \mathcal{A}_p$. As $\{f_0 + mz\}$ is a singleton set, it is also a convex set. Meanwhile, $\mathcal{A}_p$ is also a closed convex set because $\varrho$ is a $p(\cdot)$-convex risk measure. Then, by the Strong Separation Theorem for convex sets, there exists some $\pi \in (L^p(\cdot))^*$ such that
\[ \langle \pi, f_0 + mz \rangle > \sup_{f \in \mathcal{A}_p} \langle \pi, f \rangle. \] (3.1)
By Remark 22, $\langle \pi, f \rangle = \int_{\Omega} \langle h, f \rangle d\mu$, where $h \in L^p(\cdot)(\Omega, E^*)$. It is easy to check that $\langle \pi, f \rangle = \int_{\Omega} \langle h, f \rangle d\mu \leq 0$ for any $f \in L^p(\cdot)(K)$. Then, we have that
−h ∈ L^{p(·)}(K_0). For any −π ∈ Q_{p(·)}, \langle h, z \rangle = −1. Thus, by (3.1), we can conclude that

\[
\langle π, f_0 + mz \rangle > \sup_{f \in A_α} \langle π, f \rangle \Rightarrow \int_Ω ⟨h, f_0 + mz⟩dμ > \sup_{f \in A_α} \int_Ω ⟨h, f⟩dμ \\
\Rightarrow \int_Ω (⟨h, f_0⟩ - m)dμ > \sup_{f \in A_α} \int_Ω ⟨h, f⟩dμ \\
\Rightarrow \int_Ω ⟨h, f_0⟩dμ - m > \sup_{f \in A_α} \int_Ω ⟨h, f⟩dμ \\
\Rightarrow ⟨π, f_0⟩ - \sup_{f \in A_α} ⟨π, f⟩ > m \\
\Rightarrow ⟨π, f_0⟩ - α(-π) > m.
\]

Replacing −π by g_0, we have

\[
⟨g_0, -f_0⟩ - α(g_0) > m.
\]

This is a contradiction, because in this case

\[
m > \sup_{g \in Q_{p(·)}} \{ ⟨g, -f_0⟩ - α(g) \} ≥ (g_0, -f_0) - α(g_0) > m.
\]

The contradiction arises from the assumption that some f_0 ∈ L^{p(·)} exists such that

\[
φ(f_0) > \sup_{g \in Q_{p(·)}} \{ ⟨g, -f_0⟩ - α(g) \}.
\]

Hence, we have

\[
φ(f) = \sup_{g \in Q_{p(·)}} \{ ⟨g, -f⟩ - α(g) \}.
\]

For the opposite direction, it is relatively simple to check that φ satisfies the properties of a p(·)-convex risk measure. This completes the proof of Theorem 3.1.

The p(·)-convex risk measures can be considered as extensions of the convex risk measures studied by [12]. A special example of p(·)-convex risk measures, the so-called OCE, is discussed in the next section. Finally, in Sect. 5 the p(·)-convex risk measures are used to study the dual representation of the p(·)-dynamic risk measures.

4 Optimized Certainty Equivalent on L^{p(·)}

In this section, we study a special class of p(·)-convex risk measures: the Optimized Certainty Equivalent (OCE), which will be used as an example of dynamic risk measures in Sect. 5. The OCE was first introduced by [3] and later developed by the same researchers ([4]). In this section, we define the OCE on variable exponent Bochner–Lebesgue spaces L^{p(·)}. Further, we establish its
main properties, and show how it can be used to generate $p(\cdot)$-convex risk measures. Note that the OCE can be used as an application of convex risk measures.

**Definition 41** Let $u : E \to [-\infty, +\infty]$ be a closed, concave, and non-decreasing (partially ordered by $K$) function. Suppose that $u(\theta) = 0$, where $\theta$ is the zero element of $E$. We denote the set of such $u$ by $U$.

**Remark 41** For any $u \in U$ and $f \in L^p(\Omega)$, we denote by $u(f) : \Omega \to \mathbb{R}$ and $\mathbb{E}u(f)$ the expectation of $u(f)$ with respect to a probability measure $\mu$.

**Definition 42** For any $u \in U$ and $f \in L^p(\Omega)$, the OCE of some uncertain outcome $f$ is defined by the map $S_u : L^p(\Omega) \to \mathbb{R}$,

$$S_u(f) = \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(f - \eta z) \}$$

where the domain of $S_u$ is defined as $\text{dom}S_u = \{ f \in L^p(\Omega) | S_u(f) > -\infty \} \neq \emptyset$ and $S_u$ is finite on $\text{dom}S_u$.

**Theorem 41** For any $u \in U$, the following properties hold for $S_u$:

(a) For any $f \in L^p(\Omega)$ and $m \in \mathbb{R}$, $S_u(f + mz) = S_u(f) + m$;
(b) For any $f_1, f_2 \in L^p(\Omega)$, $f_1 \leq_K f_2$ a.s. implies that $S_u(f_1) \leq S_u(f_2)$;
(c) For any $f_1, f_2 \in L^p(\Omega)$ and $\lambda \in (0, 1)$, $S_u(\lambda f_1 + (1 - \lambda)f_2) \geq \lambda S_u(f_1) + (1 - \lambda)S_u(f_2)$.

**Proof.**

(a) For any $f \in L^p(\Omega)$, $m \in \mathbb{R}$,

$$S_u(f + mz) = \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(f + mz - \eta z) \}$$

$$= \sup_{\eta \in \mathbb{R}} \{ \eta - m + \mathbb{E}u(f - (\eta - m)z) \}$$

$$= m + S_u(f).$$

(b) For any $f_1, f_2 \in L^p(\Omega)$ with $f_1 \leq_K f_2$, we have $f_1 - \eta z \leq_K f_2 - \eta z$. As $u$ is non-decreasing, we have

$$S_u(f_1) = \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(f_1 - \eta z) \} \leq \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(f_2 - \eta z) \} = S_u(f_2).$$

(c) For any $f_1, f_2 \in L^p(\Omega)$ and $\lambda \in (0, 1)$,

$$S_u(\lambda f_1 + (1 - \lambda)f_2) = \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(\lambda f_1 + (1 - \lambda)f_2 - \eta z) \}.$$
We take \( \eta = \lambda \eta_1 + (1 - \lambda) \eta_2 \). Then,

\[
S_u(\lambda f_1 + (1 - \lambda) f_2) = \sup_{\eta_1, \eta_2 \in \mathbb{R}} \left\{ \lambda \eta_1 + (1 - \lambda) \eta_2 + \mathbb{E} u \left( \lambda (f_1 - \eta_1 z) + (1 - \lambda)(f_2 - \eta_2 z) \right) \right\} \\
\geq \sup_{\eta_1, \eta_2 \in \mathbb{R}} \left\{ \lambda \eta_1 + (1 - \lambda) \eta_2 + \lambda \mathbb{E} u (f_1 - \eta_1 z) + (1 - \lambda) \mathbb{E} u (f_2 - \eta_2 z) \right\} \\
= \sup_{\eta_1, \eta_2 \in \mathbb{R}} \left\{ \lambda (\eta_1 + \mathbb{E} u (f_1 - \eta_1 z)) + (1 - \lambda) (\eta_2 + \mathbb{E} u (f_2 - \eta_2 z)) \right\} \\
= \lambda S_u(f_1) + (1 - \lambda) S_u(f_2).
\]

This completes the proof of Theorem 4.1.

**Theorem 42**  The function \( \gamma \), defined as \( \gamma(f) := -S_u(f) \) for any \( f \in L^p(\cdot) \), is a \( p(\cdot) \)-convex risk measure.

**Proof:** The proof of Theorem 4.2 is straightforward from Theorem 4.1.

**Proposition 41**  For any \( u \in U \), \( \alpha \in \mathbb{R}^+ \), and \( f \in L^p(\cdot) \), the OCE \( S_u(f) \) is sub-homogeneous, i.e.

(a) \( S_u(\alpha f) \leq \alpha S_u(f) \), \quad \forall \alpha > 1; \\
(b) \( S_u(\alpha f) \geq \alpha S_u(f) \), \quad \forall 0 \leq \alpha \leq 1.

**Proof.** Denote \( S(\alpha) := \frac{1}{\alpha} S_u(\alpha f) \). Then,

\[
S(\alpha) = \frac{1}{\alpha} S_u(\alpha f) = \sup_{\eta \in \mathbb{R}} \left\{ \eta + \mathbb{E} u \left( \frac{\alpha}{\alpha - \gamma} (f - \eta z) \right) \right\}.  \tag{4.1}
\]

Next, we show that \( S(\alpha) \) is non-increasing in \( \alpha > 0 \) for any \( f \in L^p(\cdot) \). For \( \alpha_2 \geq \alpha_1 \geq 0 \), we have

\[
\frac{u(\alpha_2 t) - u(\alpha_1 t)}{\alpha_2 - \alpha_1} \leq \frac{u(\alpha_1 t) - u(\theta)}{\alpha_1 - 0}, \quad \text{for any } t \in E
\]

by the concavity of \( u \). As \( u(\theta) = 0 \), we have

\[
\frac{1}{\alpha_2} u(\alpha_2 t) \leq \frac{1}{\alpha_1} u(\alpha_1 t).
\]

Then, from (4.1), we have \( S(\alpha_1) \geq S(\alpha_2) \), which clearly implies (a) and (b).

**Proposition 42**  (Second-order stochastic dominance) We denote \( C_u(f) \) by \( uC_u(f) := \mathbb{E} u(f) \) for any \( u \in U \) and \( f \in L^p(\cdot) \). We also assume that the supremum in the definition of \( S_u \) is attained. Then, for any \( f_1, f_2 \in L^p(\cdot) \),

\[
S_u(f_1) \geq S_u(f_2) \quad \text{if and only if} \quad C_u(f_1) \geq C_u(f_2).
\]
Proof. We first show the “if” part. If \( C_u(f_1) \geq C_u(f_2) \), we have \( \mathbb{E}u(f_1) \geq \mathbb{E}u(f_2) \) by the fact that \( u \) is non-decreasing. Then, from the definition of \( S_u \), it follows that \( S_u(f_1) \geq S_u(f_2) \). We now show the “only if” part. Let \( \ell_{f_1}, \ell_{f_2} \) be the points where the suprema of \( S_u(f_1) \) and \( S_u(f_2) \) are attained, respectively. Then, for any \( u \in U \),

\[
S_u(f_1) = \ell_{f_1} + \mathbb{E}u(f_1 - \ell_{f_1}z) \geq \ell_{f_2} + \mathbb{E}u(f_2 - \ell_{f_2}z) \geq \ell_{f_1} + \mathbb{E}u(f_2 - \ell_{f_1}z),
\]

where the first inequality comes from \( S_u(f_1) \geq S_u(f_2) \). Therefore, for any \( u \in U \),

\[
\mathbb{E}u(f_1 - \ell_{f_1}z) \geq \mathbb{E}u(f_2 - \ell_{f_1}z),
\]

which implies \( \mathbb{E}u(f_1) \geq \mathbb{E}u(f_2) \). Then, \( C_u(f_1) \geq C_u(f_2) \).

5 Dynamic risk measures on \( L^p(\cdot) \)

A person’s risk assessments may change over time. This observation motivated us to study the dynamic \( p(\cdot) \)-convex risk measures on the variable exponent Bochner–Lebesgue spaces.

In fact, in dynamic cases, the risk measures can be regarded as both the minimum capital requirement of some real number and the hedging of some financial positions denoted by bounded random variables.

Conditional \( p(\cdot) \)-convex risk measures are now introduced using an axiomatic approach.

Definition 51 A map \( g_t : L^p(\cdot) \rightarrow L^\infty(\cdot) \) is called a conditional \( p(\cdot) \)-convex risk measure if it satisfies the following properties for all \( f, f_1, f_2 \in L^p(\cdot) \):

i. Monotonicity: \( f_1 \leq_K f_2 \) a.s. implies \( g_t(f_1) \geq g_t(f_2) \);

ii. Conditional cash invariance: for any \( m_t \in L^\infty(\cdot) \), \( g_t(f + m_tz) = g_t(f) - m_t \);

iii. Conditional convexity: for any \( \lambda \in L^\infty(\cdot) \) with \( \lambda \in [0, 1] \), \( g_t(\lambda f_1 + (1 - \lambda)f_2) \leq \lambda g_t(f_1) + (1 - \lambda)g_t(f_2) \);

iv. Normalization: \( g_t(\theta) = 0, g_t(f) < \infty \).

Remark 51 Note that any element in \( L^\infty(\cdot) := L^\infty(\Omega, F_t, \mu) \) is a random variable, where \( F_t \) is a sub-\( \sigma \)-algebra of \( F \). As stated by [3], if the additional information is described by a sub-\( \sigma \)-algebra \( F_t \) of the total information \( F_T \), then a conditional risk measure is a map assigning an \( F_T \)-measurable random variable \( g_t(f) \), representing the conditional riskiness of \( f \), to every \( F_T \)-measurable function \( f \), representing a final payoff.

The acceptance set of a conditional \( p(\cdot) \)-convex risk measure \( g_t \) is defined as

\[
\mathcal{A}_t := \{ f \in L^p(\cdot) : g_t(f) \leq 0 \} \text{ for any } 0 \leq t \leq T.
\]

The corresponding stepped acceptance set is defined as

\[
\mathcal{A}_{t,t+s} := \{ f \in L^p(\cdot)(F_{t+s}) : g_t(f) \leq 0 \} \text{ for any } 0 \leq t < t+s \leq T.
\]
**Proposition 51** The acceptance set $\mathcal{A}_t$ of a conditional $p(\cdot)$-convex risk measure $\mathcal{g}_t$ has the following properties:

1. Conditional convexity: for any $f_1, f_2 \in \mathcal{A}_t$, and an $\mathcal{F}_t$-measurable function $\alpha$ with $0 \leq \alpha \leq 1$, we have $\alpha f_1 + (1 - \alpha) f_2 \in \mathcal{A}_t$;
2. Solidity: for any $f_1 \in \mathcal{A}_t$ with $f_1 \leq_K f_2$ a.s. implies $f_2 \in \mathcal{A}_t$;
3. Normalization: $\theta \in \mathcal{A}_t$.

**Proof.** It is easy to check properties 1–3 using Definition 51.

**Definition 52** A sequence $(\mathcal{g}_t)_{t=0}^T$ is called a dynamic $p(\cdot)$-convex risk measure if each $\mathcal{g}_t$ is a conditional $p(\cdot)$-convex risk measure for any $0 \leq t \leq T$.

We now study the dual representation of a conditional $p(\cdot)$-convex risk measure. First, the notion of the $\mathcal{F}_t$-conditional inner product related to $L^p(\cdot)$ should be defined.

**Definition 53** For any $f \in L^p(\cdot)$ and $g \in (L^p(\cdot))^*$, we define the $\mathcal{F}_t$-conditional inner product $\langle g, - f \rangle_t$ by

$$
\int_A \langle g, - f \rangle_t d\mu = \langle g, - f \rangle 
$$

for any $A \subseteq \mathcal{F}_t$. (5.3)

We also define the minimal penalty function $\alpha_t^\min$ as

$$
\alpha_t^\min(g) := \text{ess sup}_{f \in \mathcal{A}_t} \langle g, - f \rangle_t.
$$

(5.4)

**Lemma 51** For any $g \in Q_{p(\cdot)}$, $0 \leq t \leq T$, and $A \subseteq \mathcal{F}_t$,

$$
\int_A \alpha_t^\min(g) d\mu = \sup_{f \in \mathcal{A}_t} \langle g, - f \rangle.
$$

(5.5)

**Proof.** We first show that there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{A}_t$ such that

$$
\text{ess sup}_{f \in \mathcal{A}_t} \langle g, - f \rangle_t = \lim_{n \to \infty} \langle g, - f_n \rangle_t.
$$

(5.6)

Indeed, for any $f_1, f_2 \in \mathcal{A}_t$, we define $\hat{f} := f_1 I_B + f_2 I_{B^c}$ where $B := \{ \langle g, - f_1 \rangle_t \geq \langle g, - f_2 \rangle_t \}$. By property 1 of Proposition 51 we know that $\hat{f} \in \mathcal{A}_t$. Hence, by the definition of $\hat{f}$,

$$
\langle g, - \hat{f} \rangle_t = \max\{ \langle g, - f_1 \rangle_t, \langle g, - f_2 \rangle_t \}.
$$
Thus, (5.5) holds. We now have

$$
\int_A \alpha_t^{\text{min}}(g) d\mu = \int_A \esssup_{f \in A} \langle g, -f \rangle_t d\mu
$$

= \int_A \lim_{n \to \infty} \langle g, -f_n \rangle_t d\mu

= \lim_{n \to \infty} \int_A \langle g, -f_n \rangle_t d\mu

\leq \sup_{f \in A} \langle g, -f \rangle.t

The converse inequality is easy to check.

The following theorem gives the dual representation of conditional $p(\cdot)$-convex risk measures.

**Theorem 51** Suppose $\varrho_t$ is a conditional $p(\cdot)$-convex risk measure. Then, the following statements are equivalent.

1. $\varrho_t$ has the robust representation

$$
\varrho_t(f) = \esssup_{g \in Q_p(\cdot)} \{ \langle g, -f \rangle_t - \alpha_t(g) \}
$$

for any $f \in L^p(\cdot)$, (5.7)

where

$$
Q_p(\cdot) := \left\{ g \in (L^p(\cdot))^* : \left( \frac{dg}{d\mu}, z \right) = 1, \frac{dg}{d\mu} \in L^{p^*(\cdot)}(K_0) \right\},
$$

and $\alpha_t$ is the penalty function from $Q_p(\cdot)$ to the set of $\mathcal{F}_t$-measurable random variables such that $\esssup_{g \in Q_p(\cdot)} \{-\alpha_t(g)\} = 0$;

2. $\varrho_t$ has a robust representation in terms of the minimal function, i.e.

$$
\varrho_t(f) = \esssup_{g \in Q_p(\cdot)} \{ \langle g, -f \rangle_t - \alpha_t^{\text{min}}(g) \}
$$

for any $f \in L^p(\cdot)$; (5.8)

3. $\varrho_t$ is continuous from above under $K$, i.e.

$$
f_n \searrow f \Rightarrow \varrho_t(f_n) \nearrow \varrho_t(f).
$$

**Proof.** (2) $\Rightarrow$ (1) is obvious. We first prove (1) $\Rightarrow$ (3). Using Lemma 5 of [6], suppose that $f_n \searrow f$. Then, by the monotonicity of $\varrho_t$, we have $\varrho_t(f_n) \nearrow \varrho_t(f)$. Next, we show (3) $\Rightarrow$ (2). The inequality

$$
\varrho_t(f) \geq \esssup_{g \in Q_p(\cdot)} \{ \langle g, -f \rangle_t - \alpha_t^{\text{min}}(g) \}
$$

is a direct consequence of the definition of $\alpha_t^{\text{min}}$. Now, we need only show the inverse inequality. To this end, we define a map $\tilde{\varrho} : L^p(\cdot) \to \mathbb{R}$ as $\tilde{\varrho}(f) = \int_A \varrho_t(f) d\mu$. It is easy to check that $\tilde{\varrho}$ is a $p(\cdot)$-convex risk measure as defined
in Sect. 3 which is continuous from above. Hence, by Theorem 3, we know that $\tilde{\varrho}$ has the dual representation

$$\tilde{\varrho}(f) = \sup_{g \in Q_{p(\cdot)}} \{(g, -f) - \alpha(g)\}, \quad f \in L^{p(\cdot)},$$

where the minimum penalty function $\alpha_{\text{min}}$ is given by $\alpha_{\text{min}}(g) := \sup_{f \in A_t} \{(g, -f)\}$. By Lemma 3, we have

$$\int_A \alpha_{\text{min}}(g) d\mu = \sup_{f \in A_t} \langle g, -f \rangle \leq \alpha(g)$$

for any $g \in Q_{p(\cdot)}$. Thus, we have

$$\int_A \varrho_t(f) d\mu = \tilde{\varrho}(f)$$

for any $f \in Q_{p(\cdot)}$. As $\tilde{\varrho}(f) \leq 0$ for all $f \in A_t$,

$$\int_A \alpha_{\text{min}}(g) d\mu = \sup_{f \in A_t} \langle g, -f \rangle \leq \alpha(g)$$

for any $g \in Q_{p(\cdot)}$. Thus, we have

$$\int_A \varrho_t(f) d\mu = \tilde{\varrho}(f)$$

$$= \sup_{g \in Q_{p(\cdot)}} \{(g, -f) - \alpha(g)\}$$

$$\leq \sup_{g \in Q_{p(\cdot)}} \left\{ \int_A \langle g, -f \rangle d\mu - \int_A \alpha_{\text{min}}(g) d\mu \right\}$$

$$= \sup_{g \in Q_{p(\cdot)}} \left\{ \int_A \langle g, -f \rangle - \alpha_{\text{min}}(g) d\mu \right\}$$

$$\leq \int_A \max \sup_{g \in Q_{p(\cdot)}} \{(g, -f) - \alpha_{\text{min}}(g)\} d\mu.$$

Thus, (5.8) holds.

Now, with the definition and dual representation, we consider the time consistency of dynamic $p(\cdot)$-convex risk measures.

**Definition 54** A dynamic $p(\cdot)$-convex risk measure $(\varrho_t)^T_{t=0}$ is said to be time consistent if, for all $f_1, f_2 \in L^{p(\cdot)}$ and $0 \leq t < t + s \leq T$,

$$\varrho_{t+s}(f_1) \leq \varrho_{t+s}(f_2) \Rightarrow \varrho_t(f_1) \leq \varrho_t(f_2). \quad (5.10)$$

**Remark 52** Time consistency means that if two payoffs will have the same riskiness tomorrow in every state of nature, then the same conclusion should be drawn today.

**Theorem 52** Let $(\varrho_t)^T_{t=0}$ be a dynamic $p(\cdot)$-convex risk measure such that each $\varrho_t$ is continuous from above. Then, the following conditions are equivalent for any $0 \leq t < t + s \leq T$:

1. $(\varrho_t)^T_{t=0}$ is time consistent;
2). $\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$;
3). $\varrho_t(-\varrho_{t+s}(f)z) = \varrho_t(f)$ for any $f \in L^p(t)$.

**Proof.** We first show the equivalence between 1) and 3). Suppose that 3) holds and $\varrho_{t+s}(f_1) \leq \varrho_{t+s}(f_2)$ for any $f_1, f_2 \in L^p(t)$. Then, by the monotonicity of $\varrho_t$,

$$\varrho_t(f_1) = \varrho_t(-\varrho_{t+s}(f_1)z) \leq \varrho_t(-\varrho_{t+s}(f_1)z) = \varrho_t(f_2).$$

Next, suppose that $(\varrho_t^r)_{t=0}^T$ is time consistent, and set $f_2 := -\varrho_{t+s}(f_1)z$ and then by Definition $5\!1$ we get that $-\varrho_{t+s}(f_1)z = -\varrho_{t+s}(f_2)z$ for any $f_1 \in L^p(t)$. Thus,

$$\varrho_t(f_1) = \varrho_t(f_2) = \varrho_t(-\varrho_{t+s}(f_1)z).$$

We now show the equivalence between 2) and 3). To this end, suppose that 3) holds and let $f_1 \in \mathcal{A}_{t,t+s}$, $f_2 \in \mathcal{A}_{t+s}$. Then, setting $f := f_1 + f_2$, we have

$$\varrho_{t+s}(f) = \varrho_{t+s}(f_1 + f_2) = \varrho_{t+s}(f_2) - \frac{f_1}{z} \leq -\frac{f_1}{z}.$$

Thus, by the monotonicity of $\varrho_t$, we know that

$$\varrho_t(f) = \varrho_t(-\varrho_{t+s}(f)z) \leq \varrho_t(f_1) \leq 0,$$

which implies

$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}.$$

For the inverse relation, let $f \in \mathcal{A}_t$ and define $f_2 := f + \varrho_{t+s}(f)z$, $f_1 := f - f_2 = -\varrho_{t+s}(f)z$. Then, by the conditional cash invariance of $\varrho_t$, it is easy to check that $f_1 \in \mathcal{A}_{t,t+s}$, $f_2 \in \mathcal{A}_{t+s}$, which implies

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}.$$

Let us now suppose that 2) holds and $f \in \mathcal{A}_t$. It is easy to check that $f + \varrho_{t+s}(f)z \in \mathcal{A}_{t+s}$. Then, with $\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$, we have $-\varrho_{t+s}(f)z \in \mathcal{A}_{t,t+s}$. Hence, we know that $\varrho_t(-\varrho_{t+s}(f)z) \leq 0$, which implies

$$\varrho_t(-\varrho_{t+s}(f)z) \leq \varrho_t(f).$$

Now, we need only show the inverse inequality. Indeed, for any $f \in L^p(t)$ such that $-\varrho_{t+s}(f)z \in \mathcal{A}_{t,s}$, we have $\varrho_t(-\varrho_{t+s}(f)z) \leq 0$. It is easy to check that $f + \varrho_{t+s}(f)z \in \mathcal{A}_{t,s}$. Thus, by $\mathcal{A}_t \supseteq \mathcal{A}_{t,s} + \mathcal{A}_{s,s}$, we have $f \in \mathcal{A}_t$, which implies

$$\varrho_t(-\varrho_{t+s}(f)z) \geq \varrho_t(f).$$

The case for the recursive property strongly relies on the validity of conditional cash invariance for $\varrho_t$, and hence on the interpretation as conditional capital requirements. In fact, if $\varrho_{t+s}(f)$ is the conditional capital requirement that has to be set aside at date $t + s$ in view of the final payoff $f$, then the risky position is equivalently described, at date $t$, by the payoff $\varrho_t(-\varrho_{t+s}(f)z)$.
occurring in $t + s$.

We end this section with a special example of conditional $p(\cdot)$-convex risk measures.

**Example 51 (Conditional OCE)** Let $u : E \to \mathbb{R}$ be a closed, concave, and non-decreasing (partially ordered by $K$) function and suppose that $u(\theta) = 0$, where $\theta$ is the zero element of $E$. Then, for any $f \in L^{p(\cdot)}$, the conditional OCE of some uncertain outcome $f$ is defined by the map $S_u : L^{p(\cdot)} \to L^\infty$:

$$S_u(f) = \text{ess sup}_{\eta \in L^\infty} \left\{ \eta + \mathbb{E}\left[(u(f - \eta z))|F_t]\right\}.$$ 

Thus, by Definition 51, it is easy to check that the function $\varphi_t$ defined as $\varphi_t(f) := -S_u(f)$ for any $f \in L^{p(\cdot)}$ is a conditional $p(\cdot)$-convex risk measure.

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