Ratios of the Gauss hypergeometric functions with parameters shifted by integers: part I

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Abstract. Given real parameters $a, b, c$ and integer shifts $n_1, n_2, m$, we consider the ratio $R(z) = \frac{2F_1(a + n_1, b + n_2; c + m; z)}{2F_1(a, b; c; z)}$ of the Gauss hypergeometric functions. We find a formula for $\text{Im} R(x \pm i0)$ with $x > 1$ in terms of real hypergeometric polynomial $P$, beta density and the absolute value of the Gauss hypergeometric function. This allows us to construct explicit integral representations for $R$ when the asymptotic behaviour at unity is mild and the denominator does not vanish. Moreover, for arbitrary $a, b, c$ and $\omega \leq 1$ the product $P(z + \omega)R(z + \omega)$ is proved to belong to the generalized Nevanlinna class $\mathcal{N}_{\lambda}^{\kappa}$. We give an in-depth analysis of the case $n_1 = 0, n_2 = m = 1$ known as the Gauss ratio. Furthermore, we establish a few general facts relating generalized Nevanlinna classes to Jacobi and Stieltjes continued fractions, as well as to factorization formulae for these classes. The results are illustrated with a large number of examples.

Keywords: Gauss hypergeometric function, Gauss continued fraction, Integral representation, Stieltjes class, Generalized Nevanlinna classes

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1 Introduction

The Gauss hypergeometric functions ([25], [6, Chapter II], [34, Chapter 15])

\begin{equation}
2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n
\end{equation}

and $2F_1(a + n_1, b + n_2; c + m; z)$, $n_1, n_2, m \in \mathbb{Z}$, are called contiguous in a wide sense [23]. Any three functions of this type satisfy a linear relation with coefficients rational in $a, b, c, z$. If $n_1, n_2, m \in \{-1, 0, 1\}$ the coefficients of this relation are linear in $z$ and the functions are called contiguous in a narrow sense. Such a contiguous relation was used by Euler to derive a continued fraction (much later termed T-fraction) for the ratio $2F_1(a + b + 1; c + 1; z)/2F_1(a, b; c; z)$. Gauss described all three-term relations among the functions contiguous in the narrow sense and found another

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continued fraction for the above ratio which has the form [25, p. 134] (see also [41, (89.9)] or [35, p. 123])

\[ G(z) = \frac{F(a, b + 1; c + 1; z)}{F(a, b; c; z)} = \frac{\alpha_0}{1 - \frac{\alpha_1 z}{1 - \cdots}}, \]

where \( \alpha_0 = 1 \), and for \( n \geq 0 \),

\[ \alpha_{2n + 1} = \frac{(a + n)(c - b + n)}{(c + 2n)(c + 2n + 1)}, \quad \alpha_{2n + 2} = \frac{(b + n + 1)(c - a + n + 1)}{(c + 2n + 1)(c + 2n + 2)}. \]

The main protagonist of this paper is the following generalization of the Gauss ratio (1.2)

\[ R_{n_1, n_2, m}(z) = \frac{2F_1(a + n_1, b + n_2; c + m; z)}{2F_1(a, b; c; z)}, \]

where \( n_1, n_2, m \in \mathbb{Z} \) are arbitrary. Our main goal is to derive an integral representation for this ratio under relatively general conditions, and to employ the continued fraction (1.2)–(1.3) to show that a similar representation exists when those conditions fail. In order to explain in more detail which specific representations and properties of \( R_{n_1, n_2, m}(z) \) will be established in this paper, we need to introduce some preliminaries.

Any continued fraction of the form given by the right hand side of (1.2) corresponds to (it may be turned into) a unique power series \( c_0 + c_1 z + c_2 z^2 + \ldots \) with \( \alpha_0 = c_0 \). The coefficients of this formal power series then satisfy the conditions \( D_n^{(0)}, D_n^{(1)} \neq 0 \) for all \( n \), where

\[ D_n^{(p)} := \det(c_{i+j+p})_{i,j=0}^{n-1}. \]

These conditions are also sufficient for turning the power series back into a continued fraction as in (1.2) via:

\[ \alpha_{2n - 1} = \frac{D_n^{(0)} D_{n-1}^{(1)}}{D_n^{(0)} D_{n-1}^{(1)}}, \quad \alpha_{2n} = \frac{D_{n+1}^{(0)} D_{n-1}^{(1)}}{D_n^{(0)} D_{n-1}^{(1)}}, \quad \text{where} \quad D_0^{(0)} = D_0^{(1)} = 1 \quad \text{and} \quad n = 1, 2, \ldots, \]

see [35, §§21,23] for the details. If a fraction as in (1.2) is terminating, then it corresponds to a rational function, and there is an index \( n_0 \) such that \( D_n^{(0)}, D_{n-1}^{(1)} \neq 0 \) for all \( n \leq n_0 \) and \( D_n^{(p)} = 0 \) for all \( n > n_0 \) and \( p \geq 0 \).

We need the following digest of certain classical results by Stieltjes [40], Van Vleck [35, pp. 148–151] or [41, Theorem 54.2 with footnote 19], Blumenthal [12, pp. 122–124] and others adapted to our situation:

**Theorem 1.1.** Let \( \varphi(z) \) be a continued fraction of the form (1.2) with arbitrary coefficients \( \alpha_0, \alpha_1, \ldots \in \mathbb{C} \setminus \{0\} \). Then the following assertions are true:

(a) If \( \sup \alpha_n = l_0 \), then for any positive \( r < \frac{1}{l_0} \) the continued fraction \( \varphi(z) \) and the power series corresponding via (1.5)–(1.6) uniformly converge in the disc \( |z| \leq r \) to an analytic function, say \( f(z) \).

(b) If \( \exists \lim_{n \to \infty} \alpha_n = l \in [0, +\infty) \), then \( f(z) \) may be analytically continued to a function meromorphic in whole \( \mathbb{C} \setminus \{\frac{1}{l}, +\infty\} \), or in \( \mathbb{C} \) when \( l = 0 \). Moreover, \( \varphi(z) \) uniformly converges to \( f(z) \) on compact subsets of \( \mathbb{C} \setminus \{\frac{1}{l}, +\infty\} \) excluding neighbourhoods of its poles. In such cases we identify \( \varphi(z) \) with \( f(z) \) and write \( \varphi(z) = f(z) \).
(c) If $0 < 4\alpha_n \leq \gamma < +\infty$ for all $n$, then there exists a unique function $\mu(x)$ non-decreasing on $[0, \gamma]$ such that

$$c_n = \int_{[0,\gamma]} x^n d\mu(x) < \infty, \quad n = 0, 1, \ldots.$$ 

In this case, $\varphi(z)$ uniformly converges on compact subsets of $\mathbb{C} \setminus \left[\frac{1}{\gamma}, +\infty\right)$ and may be expressed as

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{for} \ |z| < \frac{1}{\gamma}, \quad \text{and} \quad \varphi(z) = \int_{[0,\gamma]} \frac{d\mu(x)}{1-xz} \quad \text{for} \ z \in \mathbb{C} \setminus \left[\frac{1}{\gamma}, +\infty\right). \quad (1.7)$$

(d) If both (b) and (c) hold, and $l > 0$, then the points of growth of $\mu(x)$ are dense in $[0, 4l]$.

The coefficients $(c_n)_{n=0}^{\infty}$ in Theorem 1.1 (c) are called moments. The restriction $\gamma < +\infty$ separates out the (scaled) Hausdorff moment problem (the problem consists in finding $\mu(x)$ from $(c_n)_{n=0}^{\infty}$). On lifting this restriction, one obtains the more general Stieltjes moment problem, where $\gamma = +\infty$, the power series in (1.7) turns to be only asymptotic, and an analogous integral representation persists (namely (1.9) with $C = 0$ under a suitable summability condition on $\mu(s)$). However, the continued fraction $\varphi(z)$ may then diverge (and oscillate for $z < 0$), in which case $\mu(x)$ is determined non-unique (the indeterminate moment problem). More details on moment problems may be found in [2, 12, 35, 40, 41].

In the case of the Gauss ratio we have $\lim_{n \to \infty} \alpha_n = 1/4$, while $\sup_n |\alpha_n| =: \gamma/4 \geq 1/4$. So, if $\alpha_n > 0, \ n = 1, 2, \ldots$, then there is a unique positive measure $d\mu(s)$ on $[0, \gamma]$ whose support is dense in $[0, 1]$ and has at most finitely\footnote{Theorem 1.1 only implies that the measure $d\mu(s)$ is discrete in $(1, \gamma]$. The fact that $d\mu(s)$ has at most finitely many atoms in this interval directly follows from that $\gamma F_1(a, b; c; z)$ has finitely many zeros in $[0, 1)$. The latter is given by Theorem 2.1, a corollary of [37].} many points in $(1, \gamma)$, such that

$$G(z) = \int_{[0,\gamma]} \frac{d\mu(s)}{1-sz}. \quad (1.8)$$

Moreover, [41, Theorem 69.2] asserts that one may take $\gamma = 1$ in (1.8) if $\alpha_n = (1 - g_{n-1})g_n$ for all $n \geq 1$ with some numbers $g_n \in [0, 1]$ (the equality cases correspond to rational $G(z)$). It is immediate to see that the condition $\alpha_n > 0$ is satisfied for the Gauss fraction for all $n$ when $-1 < b < c$ and $0 < a < c + 1$. The more restrictive condition $g_n \in [0, 1]$ holds true if $0 \leq a \leq c + 1$, $0 \leq b \leq c$, see [30, Proof of Theorem 1.1] for details. Surprisingly enough, the representing measure $d\mu$ for the Gauss fraction has only been computed in 1984 by Vitold Belevitch [10]. Around the same time Jet Wimp [42] constructed explicit formulae for the convergents of the Gauss continued fraction in terms of hypergeometric polynomials.

The class of all functions possessing the integral representation

$$f(z) = -\frac{C}{z} + \int_{[0,\infty)} \frac{d\mu(s)}{1-sz} \quad (1.9)$$

for some $C \geq 0$ and a positive measure $\mu$ supported on $[0, \infty)$ making the integral convergent is known as the Stieltjes cone $S$ [8]. It is a much studied and important class in many areas from analysis and operator theory to combinatorics and probability. The Stieltjes cone is a subclass of another famous class in function theory commonly known as the Pick-Nevanlinna class $N_0$ comprising functions $f$ holomorphic in $\mathbb{C} \setminus \mathbb{R}$ satisfying $\text{Im}(f(z)) \geq 0$ for all $\text{Im}(z) > 0$ and possessing the symmetry property

$$f(\zeta) = \overline{f(z)} \quad \text{for all} \ z \in \mathbb{C}, \text{where} \ f(z) \text{is defined.} \quad (1.10)$$
Any function $f(z)$ meromorphic in $\mathbb{C} \setminus \mathbb{R}$ possessing the above symmetry will be called real throughout the paper. It is well-known that $f \in \mathcal{N}_0$ in fact satisfies $\text{Im}(f(z)) > 0$ for all $\text{Im}(z) > 0$ unless it is identically equal to a real constant. The elements $f$ of $\mathcal{N}_0$ are characterized by the canonical integral representation

$$f(z) = \nu_1 z + \nu_2 + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t),$$

(1.11)

where $\nu_1 \geq 0$, $\nu_2 \in \mathbb{R}$ and $d\sigma(t)$ is nonnegative locally finite Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} d\sigma(t)/(1+t^2) < \infty$. If $f \in \mathcal{N}_0$ is holomorphic in $I \cup (\mathbb{C} \setminus \mathbb{R})$ for an open interval $I \subset \mathbb{R}$ then it is a consequence of the Schwarz reflection principle that $\text{supp}(\sigma) \subset \mathbb{R} \setminus I$. In particular, it can be shown [2, pp. 127–128] that $f \in \mathcal{S}$ iff both $f \in \mathcal{N}_0$ and $zf \in \mathcal{N}_0$. Another way to characterize the elements of $\mathcal{N}_0$ is in terms of the Hermitian form [2, Chapter 3]

$$\left[ \xi_1, \ldots, \xi_n \right] \cdot H_f \cdot \begin{bmatrix} \overline{\xi}_1 \\ \vdots \\ \overline{\xi}_n \end{bmatrix}, \quad \text{where} \quad H_f = H_f(z_1, \ldots, z_n) := \left[ \frac{f(z_k) - f(z_j)}{z_k - z_j} \right]_{k,j=1}^n.$$

(1.12)

Namely, $f \in \mathcal{N}_0$ if and only if for every choice of the points $z_1, \ldots, z_n$ this Hermitian form has no negative squares, that is $H_f$ has no negative eigenvalues. We call $H_f$ the Pick matrix of $f(z)$. Based on this characterization Krein and Langer [28] defined a generalized class known as $\mathcal{N}_\kappa$:

**Definition 1.** $f \in \mathcal{N}_\kappa$ whenever it is meromorphic in $\mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} z > 0 \}$, the Hermitian form (1.12) has at most $\kappa \in \mathbb{N}_0$ negative squares for arbitrarily chosen $n \in \mathbb{N}$ and $(z_k)_{k=1}^n \subset \mathbb{C}_+$, and there are precisely $\kappa$ negative squares for a certain choice of $n$ and $(z_k)_{k=1}^n$. Here $\mathbb{N}_0 := \{0, 1, \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$.

Functions in generalized Nevanlinna classes may be represented in a form more general than (1.11). One of such representations was given by Krein and Langer [28, 29], the other representation [18] is cited here in Theorem 3.18 and may be combined with the formula (1.11).

**Definition 2.** We will say that $f \in \mathcal{N}_\kappa^\lambda$ whenever the function $f(z)$ lies in $\mathcal{N}_\kappa$ and simultaneously $zf(z)$ lies in $\mathcal{N}_\lambda$, cf. [15, 17].

In this new notation the Stieltjes class is $\mathcal{S} = \mathcal{N}_0^0$. Krein and Langer also considered the case $\mathcal{N}_\kappa^+ := \mathcal{N}_\kappa^0$ with an integral representation remarkably simpler than for $\mathcal{N}_\kappa$, see [21] for the details. There is a moment problem corresponding to the class $\mathcal{N}_\kappa^\lambda$ in the same way as the Stieltjes moment problem corresponds to $\mathcal{S}$. Analogously to the Stieltjes case, its solutions may be studied via continued fractions, see e.g. [29, 17].

It is convenient to introduce the unions of the generalized Nevanlinna classes as follows

$$\mathfrak{N} = \bigcup_{\kappa \geq 0} \mathcal{N}_\kappa \quad \text{and} \quad \mathfrak{S} = \bigcup_{\kappa, \lambda \geq 0} \mathcal{N}_\kappa^\lambda.$$

(1.13)

This paper gives a necessary and sufficient condition for $\pm R_{n_1,n_2,m}(z)$ to belong to the class $\mathfrak{S}$. In particular, we prove that the Gauss ratio $R_{0,1,1} \in \mathfrak{S}$ or $-R_{0,1,1} \in \mathfrak{S}$ for all real values of parameters. For general shifts, our condition is given in terms of positivity of a certain explicitly given polynomial $P_r(x)$ of degree

$$r = (n_1 + n_2 - m)_+ + (m)_+ - \min(n_1, n_2) - 1, \quad \text{where} \quad (x)_+ := \max\{0, x\}.$$

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\footnote{Another reasoning may be found in the beginning of the proof of Lemma 3.20 on page 30.}
We also determine a constant $B_{n_1,n_2,m}$ such that $B_{n_1,n_2,m}P_r(1/(z + \omega))R_{n_1,n_2,m}(z + \omega)$ belongs to $\mathcal{S}$ for each $\omega \leq 1$. In fact, an explicit factor times $B_{n_1,n_2,m}P_r(1/z)$ turns out to be equal to $\text{Im}(R_{n_1,n_2,m}(x \pm i0))$ for the values of $x > 1$. This further enabled us to derive an integral representation for $R_{n_1,n_2,m}(z)$ under certain asymptotic restrictions and the assumption that $R_{n_1,n_2,m}(z)$ is analytic in the cut plane $\mathbb{C} \setminus [1, +\infty)$ and on the banks of the branch cut. The latter amounts to non-vanishing of $\alpha F_1(a, b; c; z)$ there, which can be rendered in terms of the parameters $a, b, c$ by a remarkable theorem due to Runckel [37].

Given values of $n_1, n_2, m$, our integral representation provides a criterion of whether $R_{n_1,n_2,m}(z)$ belongs to the Stieltjes class $\mathcal{S}$ and its representing measure. Finally, to exemplify our results, for 15 triples $(n_1, n_2, m)$ we furnish explicit integral representations of the ratios $R_{n_1,n_2,m}(z)$ and conditions of their applicability, as well as conditions for $\pm R_{n_1,n_2,m}(z)$ to belong to $\mathcal{S}$.

In the course of the proof we also establish a number of general facts regarding the classes $\mathcal{N}_k^\lambda$ which are of their own merit. A part of these facts is scattered over the literature and often presented from a different viewpoint; another part may be considered folklore, so we were unable to locate corresponding proofs elsewhere. In particular, we give conditions for Jacobi and Stieltjes continued fractions to belong to $\mathcal{S}$ and an algorithm for computing their indices $\kappa, \lambda$. Moreover, we study integral representations for the class $\mathcal{S}$ and deduce conditions for the combination $r_1f + r_2$ to be in $\mathcal{S}$ once $f \in \mathcal{S}$ and $r_1, r_2$ are rational. To make our exposition more or less self-contained, certain basic properties of generalized Nevanlinna classes are also established.

The ratios of Gauss hypergeometric functions is a recurring theme in literature. Their relation to certain generalized Nevanlinna classes is probably best studied for the case of the Jacobi polynomials

$$P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(n + 1){\prod}_{k=0}^{n}{(\alpha + k; z)\Gamma(\beta + n + k)}}{\Gamma(\beta + 1){\prod}_{k=0}^{n}{(\alpha + \beta + n + k; z)\Gamma(1-z)\Gamma(2-z)}},$$

where it reflects the interlacing or partial interlacing of their zeros (see e.g. [4, 19]). Some researchers also go beyond the polynomials case, see for instance [38]. A general result connected to our Theorem 3.4 was obtained in [14].

There are many intriguing open questions related to our work. For instance, the case when the shifts are no longer integer is also of interest for applications, but requires additional tools. For the Jacobi polynomials, certain relevant results are presented in [20]. For the non-polynomial case, there are only very fragmentary results of this type, such as [32, Lemma 4.5].

Another interesting topic is rational approximations to the ratios $R_{n_1,n_2,m}(z)$, which are connected to orthogonal or multiple orthogonal polynomials. As we mentioned above, Wimp gave explicit expressions for the diagonal Padé approximants to the Gauss ratio in [42]. These approximants are related to the so-called associated Jacobi polynomials defined as solution of the same recurrence relation but with the index shifted by a positive number (for integer shifts but more general weights see [33]). The orthogonality measure computed by Wimp bears noticeable resemblance to the measures appearing in our work. Further results in this direction are in [43]. We would also like to mention the recent work [31] investigating multiple orthogonal polynomials with respect to a pair of particular measures, that corresponds to $R_{0,1,0}(z)$. A compelling problem is to obtain counterparts of our results for the ratios of the generalized hypergeometric functions $\alpha F_q$ which, for certain integer shifts, have explicitly known branched continued fractions generalizing the Gauss ratio [39, Sections 13–14]. A similar problem may be posed, mutatis mutandis, for basic hypergeometric functions, cf. [39, Section 15]. The basic analogue of the Gauss continued fraction has been considered in detail in [1, 7].

This paper is organized as follows. In Section 2, we obtain the values of $\text{Im}(R_{n_1,n_2,m}(x \pm i0))$ for $x > 1$ and exploit them to write the corresponding integral representation. The basic ingredients are Theorem 2.1 presenting a corollary of [37] and Lemma 2.4 connecting $\text{Im}(R_{n_1,n_2,m}(x \pm i0))$ with a Cauchy-type integral. Then Subsection 2.1 deals with the asymptotic behaviour of $R_{n_1,n_2,m}(z)$ near the point $z = 1$ and at infinity. Theorems 2.8 and 2.11 from Subsection 2.2 are at the heart of our
work: they derive a formula for $\text{Im}(R_{n_1,n_2,m}(x \pm i0))$. The corresponding integral representations are established in Theorem 2.12. Section 3 is concerned with the generalized Nevanlinna classes: Theorem 3.1 answers when $\pm R_{n_1,n_2,m} \in \mathcal{G}$, Theorem 3.3 exhibit expressions that belong to $\mathcal{G}$ for all real $a, b, c$ such that $-c \not\in \mathbb{N}_0$, and Theorem 3.4 settles the case of the Gauss ratio $R_{0,1,1}(z)$. Subsection 3.1 briefly introduces properties of generalized Nevanlinna classes and their relation to continued fractions, which imply then the proof of Theorem 3.4. Subsection 3.1 uses the multiplicative representation [18] to obtain the proofs of Theorems 3.1 and 3.3. The last section of this paper – Section 4 – illustrates our study with examples.

In Part II, we plan to give more general integral representations that handle singularity at $z = 1$ and possible zeros of the denominator. We hope it will also provide us with a method for calculating $\varepsilon \in \{-1, 1\}$ and $\kappa, \lambda$ such that $\varepsilon R_{n_1,n_2,m} \in N^{\lambda}_{\kappa}$ whenever $\pm R_{n_1,n_2,m} \in \mathcal{G}$. Part II will also give deeper and more detailed examination of the particular examples.

2 Boundary values and integral representation

Given $\xi \in \mathbb{R}$, let $[\xi]$ be the maximal integer number $\leq \xi$. Note that if $\xi$ is non-integer, then $[-\xi] = -[\xi] - 1$. We need the following corollary of an important result due to Runckel [37].

**Theorem 2.1.** Suppose $c \neq 0$ and any of the following conditions is true:

(I) $-1 < \min(a, b) \leq c \leq \max(a, b) \leq 0$

(II) $-1 < \min(a, b) \leq 0 \leq \max(a, b) \leq c$

(III) $-1 < c \leq \min(a, b) \leq 0 \leq \max(a, b) < c + 1$

(IV) $0 \leq \min(a, b) \leq c$ and $0 \leq \max(a, b) < c + 1$

(V) $a, b, c-a, c-b$ are non-integer negative numbers, such that $[\xi_1] + 1 = [\xi_4]$ and $[\xi_2] = [\xi_3]$, where $\xi_1, \ldots, \xi_4$ are the numbers $a, b, c-a, c-b$ taken in non-decreasing order:

$$\min(a, b, c-a, c-b) = \xi_1 \leq \xi_2 \leq \xi_3 \leq \xi_4 = \max(a, b, c-a, c-b).$$

Then $z \to \mathcal{F}_1(a, b; c; z)$ does not vanish in $\mathbb{C} \setminus [1, \infty]$ as well as on the banks of the branch cut.

For any real numbers $a, b, c$ with $-c \not\in \mathbb{N}_0$, the number of zeros of the function $z \to \mathcal{F}_1(a, b; c; z)$ in $\mathbb{C} \setminus [1, \infty)$ and on the banks of the branch cut is finite.

**Remark 2.2.** The conditions of the theorem are not necessary as is seen from the example $\mathcal{F}_1(-n, c, c, z) = (1 - z)^n$, $n \in \mathbb{N}$, $-c \not\in \mathbb{N}_0$, non-vanishing in $\mathbb{C} \setminus [1, \infty)$. One can show that, apart from this example, the conditions are in fact necessary.

**Remark 2.3.** Under condition (V), one necessarily has $c - \xi_4 = \xi_1 < \xi_2$ and $c - \xi_2 = \xi_3 < \xi_4$. Indeed: $\xi_1 + \xi_4 = c = \xi_2 + \xi_3$ in view of $a + (c-a) = c = b + (c-b)$. So, if we had one of the equalities $\xi_1 = \xi_2$ and $\xi_3 = \xi_4$, we automatically had the other. However, assuming the last two equalities together will contradict to $[\xi_1] + 1 = [\xi_4]$ on account of $[\xi_2] = [\xi_3]$.

In fact, (V) is generated by the following two basic cases

$$-k - 1 < a < \min(b, c-b) \leq \max(b, c-b) < -k < c-a < -k + 1, \quad k \in \mathbb{N}, \quad \text{and}$$

$$-k - 1 < a < -k < \min(b, c-b) \leq \max(b, c-b) < c-a < -k + 1, \quad k \in \mathbb{N},$$

further extended through the symmetry $a \leftrightarrow b$ and Euler’s transformation (2.1) exchanging $(a, b) \leftrightarrow (c-a, c-b)$. 

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The right-hand side is equal to $-\lfloor \xi \rfloor$ we may apply the last option in (2.1) and similarly for $b = c$, so that again the conclusion is true. Next, according to [37, Theorem, p. 56]

$$\#\{\text{zeros of } 2F_1(a, b; c; z) \text{ in } \mathbb{C} \setminus [1, \infty)\} = 0$$

if $c \geq a + b$ and $a \geq b > 0$. Exchanging the roles of $a, b$ we get the condition $a, b > 0$ and $c \geq a + b$ which implies $a < c$, $b < c$. Note next, that the number of zeros of $2F_1(a, b; c; z)$ in $\mathbb{C} \setminus [1, \infty)$ is equal to the number of zeros of $2F_1(c - a, c - b; c; z)$ by the celebrated Euler’s identity

$$2F_1(a, b; c; z) = (1 - z)^{-a - b}2F_1(c - a, c - b; c; z).$$

Hence, we conclude that $c - a, c - b > 0$ and $c \geq a + b$ ($\Rightarrow a + b \geq c$) are also sufficient conditions. Union of these two sets of conditions in view of the aforementioned degenerate case yields $0 \leq a, b \leq c$.

Next, again by [37, Theorem, p. 56] if $a < 0$, $b \geq a$, $c > a$ and $c \geq a + b$, then

$$\#\{\text{zeros of } 2F_1(a, b; c; z) \text{ in } \mathbb{C} \setminus [1, \infty)\} = [-a] + (1 + S)/2, \quad S := \text{sign}((\Gamma(a)\Gamma(b)\Gamma(c-b)\Gamma(c-a)))$$

This implies that for $-1 < a < 0$ and $\Gamma(b)\Gamma(c-b) > 0$ the quantity on the right hand side is equal to zero. Condition $\Gamma(b)\Gamma(c-b) > 0$ is satisfied if $c > b > 0$ or if $-1 < c < b < 0$. Combining these conditions and exchanging the roles of $a, b$ (in view of the remark about $ab = 0$), we get the conditions (I) and (II). Finally, applying conditions (I) and (II) to the function $2F_1(c - a, c - b; c; z)$ we arrive at (III) and $0 \leq \min(a, b) \leq c \leq \max(a, b) < c + 1$. Combined with $0 \leq a, b \leq c$ this leads to (IV).

To obtain the condition (V), let us start with the case $\xi_1 = a$. Then $a \leq b$, and hence $c - b \leq c - a$. Moreover, $a \leq c - b$ implies $b \leq c - a$ and further $b + a \leq c$. Consequently, we have $\xi_4 = c - a$ and the assumptions of (V) yield

$$[a] + 1 = [c - a], \quad [b] = [c - b], \quad \text{and} \quad a < \min(b, c - b) \leq \max(b, c - b) < c - a < 0.$$

None of the numbers $a, b, c - a, c - b$ is integer, so the above relations give $\Gamma(a)\Gamma(c - a) < 0$ and $\Gamma(b)\Gamma(c - b) > 0$, which imply that $S$ defined as above equals $-1$. Due to $a + b \leq c < a \leq b < 0$, we may apply the last option in [37, Theorem, p. 56] asserting

$$\#\{\text{zeros of } 2F_1(a, b; c; z) \text{ in } \mathbb{C} \setminus [1, \infty)\} = [-a] + (1 + S)/2 + S \cdot [a - c + 1].$$

The right-hand side is equal to $-a - 1 + (1 - 1)/2 + [c - a] = 0$, so the case $\xi_1 = a$ is proved.\(^3\)

The case $\xi_1 = b$ follows by exchanging the roles of $a$ and $b$. Now, if $\xi_1 = c - a$ or $\xi_1 = c - b$, we apply Euler’s identity (2.1) to reduce the proof of (V) to, respectively, the cases $\xi_1 = a$ or $\xi_1 = b$. \(\square\)

Integral representations of the forms (1.8) or (1.11) may already be too restrictive for the Gauss ratio $G(z) = R_{0,1,1}(z)$ when none of the conditions (I)–(V) of Theorem 2.1 is satisfied. In particular, (1.8) and (1.11) are analytic in $\mathbb{C} \setminus \mathbb{R}$, while $R_{0,1,1}(z)$ may have complex poles for certain values of $a, b, c$. Moreover, the measure $d\sigma(t)$ related to $R_{0,1,1}(z)$ via the Stieltjes-Perron formula (3.17) may grow too fast for the integral in (1.11) to be convergent. Introduction of the arbitrary integer shifts $n_1, n_2, m$ makes the problem even more complicated, since the prospective

\(^3\)In fact, our reasoning is reversible: if $a + b \leq c < a \leq b < 0$, $\{-a, -b, a - c, b - c\} \cap \mathbb{N}_0 = \emptyset$ and the right-hand side of (2.2) is zero, then we necessarily arrive at (V) with $\xi_1 = a$. One only needs to note that (2.2) implies $S = -1$ in this case.
measure determined via the Stieltjes-Perron formula may easily be signed, see Section 4 for examples (e.g. Example 5).

To account the complex poles is not an easy task, unless we exactly know the zeros of the denominator of \( R_{n_1,n_2,m}(z) \). As a certain compromise, Section 3 shows that the ratio \( R_{n_1,n_2,m}(z) \) may always be expressed as the product \( h(z)g(z) \), where \( h(z) \) is a real rational function and \( g \in \mathcal{N}_0 \) has the form (1.11) (in fact even the form (1.8) with \( \gamma = 1 \) up to addition of another real rational function, see Theorems 3.3 and 3.18).

This section aims at constructing explicit integral representations for \( R_{n_1,n_2,m}(z) \) in the case where there are no poles in \( \mathbb{C} \setminus [1,\infty) \) as well as on the banks of the branch cut. If so, the corresponding signed measure (or charge) turns to be supported on \([1,\infty)\) and have an analytic density, thus, to obtain the integral representation we only need to deal with the asymptotic behaviour of \( R_{n_1,n_2,m}(z) \) near the points \( z = 1 \) and \( z = \infty \). Such representations up to rational correction terms may be expressed via the boundary values using the standard Schwarz formula; more specifically, the following fact is true:

**Lemma 2.4.** Let \( f(z) \) be a real analytic function defined in the cut plane \( \mathbb{C} \setminus [1,\infty) \) and suppose that \( u(x) := \frac{1}{\pi} \text{Im} f(x + \imath 0) \) is continuous on \((1,\infty)\). Suppose that there exists \( n \in \mathbb{N}_0 \) such that

\[
\lim_{|z-1| \to 1} |f(z)(1-z)| = \lim_{|z| \to \infty} |f(z)z^{-n}| = 0 \tag{2.3}
\]

and \( u(x)x^{-n-1} \) is absolutely integrable over \((1,\infty)\). Then

\[
f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k + z^n \int_1^{\infty} \frac{u(x) \, dx}{(x-z)x^n}. \tag{2.4}
\]

**Proof.** Let \( C \) be the closed contour consisting of a small circle around the point \( z = 1 \) of radius \( \epsilon < 1/2 \), then (the upper bank of) the interval \((1+\epsilon + \imath 0, 1/\epsilon + \imath 0)\) followed by a large circle \(|z| = 1/\epsilon\) and (the lower bank of) the interval \((1+\epsilon - \imath 0, 1/\epsilon - \imath 0)\). The contour is traversed in the direction that leaves the bounded domain inside it on the left (so that the large circle is traversed counterclockwise). The Cauchy formula for the Taylor coefficients of \( f(z) \) implies

\[
2\pi i \frac{f^{(n+k)}(0)}{(n+k)!} = \oint_C f(z) z^{-(n+k+1)} \, dz = \oint_{|z| = 1/\epsilon} f(z) \frac{dz}{z^{n+k+1}} + \int_{1+\epsilon}^{\infty} \frac{f(x + \imath 0) - f(x - \imath 0)}{x^{n+k+1}} \, dx + \oint_{|z| = 1/\epsilon} f(z) \frac{dz}{z^{n+k+1}}.
\]

Here \( k \in \mathbb{N}_0 \); note also that \( f(x + \imath 0) - f(x - \imath 0) = 2i \text{Im} f(x + \imath 0) \). On letting \( \epsilon \to 0 \), the first and the last integrals on the right-hand side vanish due to (2.3), and hence

\[
\frac{f^{(n+k)}(0)}{(n+k)!} = \frac{1}{\pi} \int_1^{\infty} \frac{\text{Im} f(x + \imath 0) \, dx}{x^{n+k+1}} = \int_1^{\infty} \frac{u(x) \, dx}{x^{n+k+1}} =: c_k, \quad k \in \mathbb{N}_0.
\]

The Taylor series uniformly converges on compact subsets of the unit disc (in fact, \( c_k \to 0 \) as \( k \to \infty \) since for each \( x > 1 \) the integrands monotonically tend to zero). Therefore, if \(|z| < 1\) we have

\[
\frac{1}{z^n} \left( f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right) = \sum_{k=0}^{\infty} c_k z^k = \int_1^{\infty} \left( \sum_{k=0}^{\infty} \frac{z^k}{x^{k+1}} \right) \frac{u(x) \, dx}{x^n} = \int_1^{\infty} \frac{u(x) \, dx}{(x-z)x^n}.
\]

The ratio \( x/(x-z) \) is bounded in \( z \) on compact subsets of \( \mathbb{C} \setminus [1,\infty) \) uniformly in \( x > 1 \), so the integral on the right-hand side uniformly converges there to an analytic function. This analytic function coincides with \( f(z) \) in the unit disc, and hence in \( \mathbb{C} \setminus [1,\infty) \).
2.1 Asymptotic behaviour

In this section, we will record the behavior of $R_{n_1,n_2,m}(z)$ in the neighborhood of the singular points $z = 1$ and $z = \infty$. We begin with some remarks. Define

$$A(x_1, x_2, x_3) = \frac{\Gamma(x_3)\Gamma(x_2 - x_1)}{\Gamma(x_3 - x_1)\Gamma(x_2)}, \quad A_1 = A(a + n_1, b + n_2, c + m),$$

$$A_2 = A(b + n_2, a + n_1, c + m), \quad A_3 = A(a, b, c), \quad A_4 = A(b, a, c).$$

(2.5)

Is it clear that $A(x_1, x_2, x_3) = 0$ iff $x_1 - x_3 \in \mathbb{N}_0$ or $x_2 \in \mathbb{R}_0$. In this case $2F_1(x_1, x_2; x_3; z) = 2F_1(x_2, x_1; x_3; z)$ reduces to a polynomial possibly times a power of $1 - z$. Hence, the condition $A_3A_4 = 0$ is equivalent to the fact that $2F_1(a, b; c; z)$ reduces to a polynomial, possibly times a power of $1 - z$; similar claim holds for $A_1A_2 = 0$ and $2F_1(a + n_1, b + n_2; c + m; z)$. Note, finally, that the condition $A_1A_2A_3A_4 \neq 0$ is equivalent to

$$\{a, a + n_1, b, b + n_2, c - a, c + m - a - n_1, c - b, c + m - b - n_2\} \cap -\mathbb{N}_0 = \emptyset. \quad (2.6)$$

Denote, as customary, $(x)_- = \min(x, 0)$. Our first lemma deals with the singular point $z = 1$.

**Lemma 2.5.** Suppose that condition (2.6) holds true for some $a, b, c \in \mathbb{C}$ and $n_1, n_2, m \in \mathbb{Z}$. Denote $\eta = c - a - b, q = m - n_1 - n_2$ and write $\delta_{x,y}$ for the Kronecker delta. Then

$$R_{n_1,n_2,m}(z) = M \frac{(1 - z)^{(n+q)} - [1 - \delta_{\eta+q,0} + \delta_{\eta+q,0} \log(1 - z)]}{(1 - z)^{(n)} - [1 - \delta_{\eta,0} + \delta_{\eta,0} \log(1 - z)]} (1 + o(1)) \quad (2.7)$$

as $z \to 1$ with some constant $M \neq 0$ independent of $z$. If (2.6) is violated, formula (2.7) should be modified as follows:

(a) If $-a \in \mathbb{N}_0$ and/or $-b \in \mathbb{N}_0$, then the denominator should be replaced by 1.
(b) If $-(a + n_1) \in \mathbb{N}_0$ and/or $-(b + n_2) \in \mathbb{N}_0$, then the numerator should be replaced by 1.
(c) If $-a, -b \notin \mathbb{N}_0$ but $a - c \in \mathbb{N}_0$ and/or $b - c \in \mathbb{N}_0$, then the denominator should be replaced by $(1 - z)^\eta$.
(d) If $-a - n_1, -b - n_2 \notin \mathbb{N}_0$, but $a + n_1 - c - m \in \mathbb{N}_0$ and/or $b + n_2 - c - m \in \mathbb{N}_0$, then the numerator should be replaced by $(1 - z)^{n+q}$.

**Proof.** Suppose first that (2.6) is satisfied. Then, if $\eta = c - a - b \notin \mathbb{Z}$, according to [34, (15.8.4)] we have

$$2F_1(a, b; c; 1 - z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} 2F_1(a, b; 1 + \eta; z) + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} z^\eta 2F_1(c - a - c - b; 1 - \eta; z).$$

If $\eta = c - a - b = s \in \mathbb{N}_0$, then according to [6, 2.10(12-13)] or [34, (15.8.10)] we have

$$2F_1(a, b; c; 1 - z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \sum_{n=0}^{s-1} (a)_n(b)_n \frac{1}{(1 - \eta)_n n!} z^n + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b) s!} \sum_{n=0}^{\infty} \frac{(c - a)n(c - b)n}{(1 + \eta)_n n!} H_n z^n$$

$$- \frac{\Gamma(c)}{\Gamma(a)\Gamma(b) s!} z^\eta \log(z) 2F_1(c - a - c - b; 1 + \eta; z),$$

where the sum over the empty index set equals zero, and

$$H_n = \psi(n + 1) + \psi(n + s + 1) - \psi(a + n + s) - \psi(b + n + s), \quad \psi(z) = \Gamma'(z)/\Gamma(z).$$
If \( \eta = c - a - b = -s, s \in \mathbb{N}_0 \), according to [6, 2.10(14-15)] we have

\[
2F_1(a, b; c; 1-z) = \frac{\Gamma(c)\Gamma(a+b-c)z^\eta}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(c-a)\eta(n-c-b)_n}{(1+\eta)_n n!} z^n + \frac{(-1)^n\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)s!} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1-\eta)_n n!} \hat{H}_n z^n
\]

and

\[
\hat{H}_n = \psi(n+1) + \psi(n+s+1) - \psi(a+n) - \psi(b+n).
\]

These formulae imply that

\[
2F_1(a, b; c; 1-z) = \begin{cases}
A(1 + \alpha_1 z + \alpha_2 z^2 + \cdots) + B z^\eta(1 + \beta_1 z + \beta_2 z^2 + \cdots), & \eta \notin \mathbb{Z}; \\
\hat{A}(1 + \hat{\alpha}_1 z + \hat{\alpha}_2 z^2 + \cdots) + \hat{B} z^\eta log(z)(1 + \hat{\beta}_1 z + \hat{\beta}_2 z^2 + \cdots), & \eta \in \mathbb{N}_0; \\
Cz^\eta(1 + \gamma_1 z + \gamma_2 z^2 + \cdots) + \hat{D} log(z)(1 + \hat{\gamma}_1 z + \hat{\gamma}_2 z^2 + \cdots), & -\eta \notin \mathbb{N},
\end{cases}
\]

where the constants \( A, \hat{A}, \hat{B}, \hat{B}, \hat{D} \) do not vanish due to condition (2.6). In a similar fashion,

\[
2F_1(a + n_1, b + n_2; c + m; 1-z)
\]

\[
= \begin{cases}
C(1 + \delta_1 z + \delta_2 z^2 + \cdots) + D z^q \log(z)(1 + \gamma_1 z + \gamma_2 z^2 + \cdots), & \eta + q \notin \mathbb{Z}; \\
\hat{C}(1 + \hat{\delta}_1 z + \hat{\delta}_2 z^2 + \cdots) + \hat{D} z^q \log(z)(1 + \hat{\gamma}_1 z + \hat{\gamma}_2 z^2 + \cdots), & \eta + q \in \mathbb{N}_0; \\
\hat{C}z^q(1 + \hat{\delta}_1 z + \hat{\delta}_2 z^2 + \cdots) + \hat{D} \log(z)(1 + \hat{\gamma}_1 z + \hat{\gamma}_2 z^2 + \cdots), & -\eta - q \notin \mathbb{N},
\end{cases}
\]

where the constants \( C, \hat{C}, \hat{C}, \hat{D}, \hat{D}, \hat{D} \) do not vanish due to condition (2.6). Substituting these formulae into definition (1.4) of the function \( R_{n_1,n_2,m}(z) \) and analyzing the principal asymptotic term in each of the five possible cases (1) \( \eta \notin \mathbb{Z} \); (2) \( \eta \in \mathbb{N}_0 \) and \( \eta + q \notin \mathbb{N}_0 \); (3) \( \eta \in \mathbb{N}_0 \) and \( -\eta - q \notin \mathbb{N} \); (4) \( -\eta \in \mathbb{N} \) and \( \eta + q \in \mathbb{N}_0 \); (5) \( -\eta \in \mathbb{N} \) and \( -\eta - q \in \mathbb{N} \), we arrive at formula (2.7).

If condition (2.6) is violated, then the claims (a)-(d) of the lemma follow from the following two facts: (1) if \( -a \in \mathbb{N}_0 \) and/or \( -b \in \mathbb{N}_0 \), then \( 2F_1(a, b; c; z) \) reduces to a polynomial; (2) If \( -a, -b \notin \mathbb{N}_0 \), but \( a-c \in \mathbb{N}_0 \) and/or \( b-c \notin \mathbb{N}_0 \), then Euler’s transformation

\[
2F_1(a, b; c; z) = (1-z)^n 2F_1(c - a, c - b; c; z)
\]

implies that \( 2F_1(a, b; c; z) = (1-z)^n \times \) polynomial. In view of a similar statement for \( 2F_1(a + n_1, b + n_2; c + m; z) \) we arrive at the conclusions contained in claims (a)-(d) of the lemma on the basis of case-by-case analysis.

The above lemma implies that in all possible cases the asymptotics takes the form

\[
R_{n_1,n_2,m}(z) = M(1-z)^\nu [\log(1-z)]^\epsilon (1 + o(1)) \quad \text{as} \quad z \to 1,
\]

where \( \epsilon \in \{ -1, 0, 1 \} \) and \( \nu = (\eta + q)_- - (\eta)_- \) if condition (2.6) is satisfied. Otherwise, the formula for \( \nu \) is modified according to statements (a)-(d) of Lemma 2.5. Suppose \( \nu > -1 \). Take \( \theta \in (-1, \nu) \). Clearly,

\[
R_{n_1,n_2,m}(z) = M(1-z)^\nu [\log(1-z)]^\epsilon (1 + o(1)) = M(1-z)^\theta o(1)(1 + o(1)),
\]

so that in a neighborhood of \( z = 1 \) we get the estimate

\[
|R_{n_1,n_2,m}(z)| \leq M|1-z|^\theta.
\]

For the neighborhood of infinity we will break the result in two sub-cases. First, define

\[
A_{1,2} = \begin{cases}
A_1, \text{ if } \min(a + n_1, b + n_2) = a + n_1, \\
A_2, \text{ if } \min(a + n_1, b + n_2) = b + n_2
\end{cases}, \quad A_{3,4} = \begin{cases}
A_3, \text{ if } \min(a, b) = a, \\
A_4, \text{ if } \min(a, b) = b
\end{cases}.
\]
Lemma 2.6. Suppose \( a - b \notin \mathbb{Z}, -c \notin \mathbb{N}_0, A_1, \ldots, A_4, A_{1,2}, A_{3,4} \) are defined in (2.5) and (2.10), respectively. Then the asymptotic expansion of \( R_{n_1,n_2,m}(z) \) has the form

\[
R_{n_1,n_2,m}(z) \sim \frac{A_{1,2}}{A_{3,4}} z^{\alpha - \gamma} \left( 1 + \sum_{k=1}^{\infty} \frac{a_k}{z^{\sigma_k}} \right) \quad \text{as} \quad z \to \infty,
\]

where \( a_k \neq 0 \) and

1. \( \alpha = -\min(a + n_1, b + n_2) \) if \( A_1 A_2 \neq 0 \), \( \alpha = -a - n_1 \) if \( A_2 = 0 \), \( \alpha = -b - n_2 \) if \( A_1 = 0 \) (the case \( A_1 = A_2 = 0 \) is impossible under conditions of the lemma);

2. \( \gamma = -\min(a, b) \) if \( A_3 A_4 \neq 0 \), \( \gamma = -a \) if \( A_4 = 0 \), \( \gamma = -b \) if \( A_3 = 0 \) (the case \( A_3 = A_4 = 0 \) is impossible under conditions of the lemma);

3. the term \( \sigma_k \) of the positive increasing sequence \( \{\sigma_k\}_{k \in \mathbb{N}} \) equals the \( k \)-th smallest element of the multi-set

\[
\{k_1 \varepsilon + k_2 \delta + k_3 : \ k_1 \in \{0,1\}, \ k_2 \in \mathbb{N}_0, k_3 \in \mathbb{N}_0, k_1 k_2 k_3 \neq 0\},
\]

where \( \varepsilon = |a + n_1 - b - n_2| \) if \( A_1 A_2 \neq 0 \) or \( \varepsilon = 0 \) otherwise and \( \delta = |a - b| \) if \( A_3 A_4 \neq 0 \) or \( \delta = 0 \) otherwise.

Proof. According to [3, (2.3.12)] or [34, (15.8.2)] as long as \( a - b \notin \mathbb{Z} \) we have

\[
\mathcal{2}_1 F_1(a,b;c; -z) = A(a,b,c) z^{-a}(1 + \alpha z^{-1} + \beta_2 z^{-2} + \cdots) + B(b,a,c) z^{-b}(1 + \gamma z^{-1} + \beta_2 z^{-2} + \cdots)
\]

\[
= A_{3,4} z^\gamma \left( 1 + \alpha_1 z^{-\delta} + \alpha_2 z^{-\delta-1} + \cdots + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots \right),
\]

where \( \gamma = -\min(a,b) \), \( \delta = |a - b| \),

\[
\alpha_j = \frac{(a)_j (a - c + 1)_j}{(a - b + 1)_j j!}, \quad \alpha_j = (b)_j (b - c + 1)_j \quad \frac{(b - a + 1)_j j!}{(b - a + 1)_j j!}
\]

and \( A_{3,4} \) is defined in (2.10). Hence,

\[
\left( \mathcal{2}_1 F_1(a,b;c; -z) \right)^{-1} = (A_{3,4})^{-1} z^{-\gamma} \left( 1 + \sum_{k=1}^{\infty} \frac{\alpha_k}{z^{\sigma_k}} \right),
\]

where \( \sigma_k \) is the \( k \)-th smallest number in the (multi)-set \( \{k_1 \delta + k_2 : k_1, k_2 \in \mathbb{N}_0, k_1 k_2 \neq 0\} \)

In a similar fashion,

\[
\mathcal{2}_1 F_1(a + n_1, b + n_1; c + m; -z) = A_{1,2} z^\alpha \left( 1 + \beta_1 z^{-\varepsilon} + \beta_2 z^{-\varepsilon-1} + \cdots + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots \right),
\]

where \( \alpha = -\min(a + n_1, b + n_2) \) if \( A_1 A_2 \neq 0 \), \( \alpha = -a - n_1 \) if \( A_2 = 0 \), \( \alpha = -b - n_2 \) if \( A_1 = 0 \); and \( A_{1,2} \) is defined in (2.10). Multiplying these two expansions we arrive at (2.11).

The condition \( a - b \notin \mathbb{Z} \) in Lemma 2.6 ensures that no logarithms appear in the asymptotics. If, on the contrary \( a - b \in \mathbb{Z} \) so that also \( a + n_1 - b - n_2 \in \mathbb{Z} \) the asymptotic expansions of the hypergeometric functions in both numerator and denominator of \( R_{n_1,n_2,m}(z) \) will contain logarithmic terms if (2.6) holds true (i.e. \( A_1 A_2 A_3 A_4 \neq 0 \)). We will treat this situation in the lemma below. If (2.6) is violated, however, then either numerator (if \( A_1 A_2 = 0 \)) or denominator (if \( A_3 A_4 = 0 \)) or both reduce to a polynomial possibly time a power of \((1 - z)\) in which case logarithmic terms are missing. More precisely by Euler’s transformation if \(-x_1, -x_2 \notin \mathbb{N}_0, n \in \mathbb{N}_0, \) then

\[
\mathcal{2}_1 F_1(x_1, x_2; x_1 - n; -z) = \frac{\mathcal{2}_1 F_1(-n, x_1 - x_2 - n; x_1 - n; -z)}{(1 + z)^{x_1 + x_2}} = \frac{(1 + x_2 - x_1)_n}{z^{x_2}(1 - x_1)_n} \left( 1 + \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots \right).
\]

If \(-x_1 \in \mathbb{N}_0 \) and/or \(-x_2 \in \mathbb{N}_0 \) then \( \mathcal{2}_1 F_1(x_1, x_2; x_3; -z) \) is a polynomial of degree determined by the minimal among the integer values of \(-x_j, j = 1, 2\).
Lemma 2.7. Suppose \( n_2 - n_1 \neq a - b \in \mathbb{Z} \setminus \{0\} \), \( A_1A_2A_3A_4 \neq 0 \) and \( A_{1,2}, A_{3,4} \) is defined \((2.10)\). Then the asymptotic expansion of \( R_{n_1,n_2,m}(-z) \) as \( z \to \infty \) has the form

\[
R_{n_1,n_2,m}(-z) \sim \frac{A_{1,2}}{A_{3,4}}z^{\alpha - \gamma} \left( 1 + \sum_{k=1}^{\min(\delta, \varepsilon)-1} \frac{a_k}{z^k} + \sum_{k=\min(\delta, \varepsilon)}^{\infty} \frac{a_k}{z^k} \left[ 1 + b_{1,k} \log(z) + \cdots + b_{k,k} \log^k(z) \right] \right),
\]

where the sum over the empty index set is zero, \( a_k \) and \( b_k \) are real numbers (possibly vanishing), \( \alpha = -\min(a + n_1, b + n_2) \); \( \gamma = -\min(a, b) \); and \( \varepsilon = |a + n_1 - b - n_2| \) and \( \delta = |a - b| \) are positive integers.

Proof. Indeed if \( |a - b| \geq 1 \) we apply \([34, (15.8.8)]\) which can be written in the form:

\[
2F_1(a, b; c; -z) = A_{3,4}z^\gamma \left( 1 + \sum_{j=1}^{\infty} \frac{f_j}{z^j} + \log(z) \sum_{k=\delta}^{\infty} \frac{e_k}{z^k} \right).
\]

where as before \( \gamma = -\min(a, b), \delta = |a - b| \in \mathbb{N} \), and \( A_{3,4} \) is defined in \((2.10)\). Hence,

\[
[2F_1(a, b; c; -z)]^{-1} = (A_{3,4})^{-1}z^{-\gamma} \left( 1 + \sum_{j=1}^{\infty} \frac{\hat{f}_j}{z^j} \left[ 1 + \hat{e}_{j,1} \frac{\log(z)}{z^{\delta-1}} + \hat{e}_{j,2} \frac{\log^2(z)}{z^{2(\delta-1)}} + \cdots + \hat{e}_{j,j} \frac{\log^j(z)}{z^{j(\delta-1)}} \right] \right).
\]

In a similar fashion,

\[
2F_1(a + n_1, b + n_2; c + m; -z) = A_{1,2}z^\alpha \left( 1 + \sum_{j=1}^{\infty} \frac{g_j}{z^j} + \log(z) \sum_{k=\varepsilon}^{\infty} \frac{q_k}{z^k} \right),
\]

where as before \( \alpha = -\min(a + n_1, b + n_2), \varepsilon = |a + n_1 - b - n_2| \in \mathbb{N} \) and \( A_{1,2} \) is defined in \((2.10)\).

Note that in the above lemma \( \alpha - \gamma \in \mathbb{Z} \). The remaining cases not covered by Lemmas 2.6 and 2.7 are the following. If \( a = b \), but \(-a, a - c \notin \mathbb{N}_0\) according to \([34, (15.8.8)]\) we have

\[
2F_1(a, a; c; -z) = \frac{\log(z)\Gamma(c)}{\Gamma(a)\Gamma(c-a)}z^{\alpha} \left( 1 + \frac{f_0}{\log(z)} + \sum_{k=1}^{\infty} \frac{e_k}{z^k} \left[ 1 + \frac{f_k}{\log(z)} \right] \right),
\]

so that

\[
[2F_1(a, a; c; -z)]^{-1} = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)\log(z)}z^{\alpha} \left( 1 + \sum_{k=1}^{\infty} \frac{f_k}{\log(z)^k} + \sum_{j=1}^{\infty} \frac{\hat{f}_j}{z^j} \left[ 1 + \frac{\hat{e}_{j,1}}{\log(z)} + \cdots + \frac{\hat{e}_{j,j}}{\log(z)^j} \right] \right).
\]

In a similar fashion if \( a + n_1 = b + n_2 \), but \(-a - n_1, a + n_1 - c - m \notin \mathbb{N}_0\) we will have

\[
2F_1(a + n_1, a + n_1; c + m; -z) = \frac{z^{-a-n_1}\log(z)\Gamma(c+m)}{\Gamma(a+n_1)\Gamma(c-a+m-n_1)} \left( 1 + \frac{g_0}{\log(z)} + \sum_{k=1}^{\infty} \frac{q_k}{z^k} \left[ 1 + \frac{g_k}{\log(z)} \right] \right).
\]

Hence, when both \( a = b \) and \( a + n_1 = b + n_2 \), but there no nonnegative integers among the numbers \(-a, a - c, -a - n_1, a + n_1 - c - m \) the asymptotic expansion of \( R_{n_1,n_2,m}(-z) \) is obtained by multiplication of \((2.16)\) and \((2.17)\). If \( a = b \) but \( a + n_1 \neq b + n_2 \) we have to multiply \((2.16)\) by \((2.15)\) or, if \( a + n_1 = b + n_2 \) but \( a \neq b \), then multiply \((2.17)\) by \((2.14)\). If \( a - c \in \mathbb{N}_0 \) or/and \( a + n_1 - c - m \in \mathbb{N}_0 \) formula \((2.12)\) should be used for the corresponding asymptotic expansion. Finally, if \(-a \in \mathbb{N}_0 \) and/or \(-b \in \mathbb{N}_0 \), then \( 2F_1(a, b; c; z) \) reduces to a polynomial; a similar claim applies to \( 2F_1(a + n_1, b + n_2; c + m; z) \) if \(-a - n_1 \in \mathbb{N}_0 \) and/or \(-b - n_2 \in \mathbb{N}_0 \).
2.2 Main theorems

For any integer \(r\) define the Pochhammer symbol by \((z)_r = \Gamma(z + r)/\Gamma(z)\). Given three integers \(n_1, n_2, m \in \mathbb{Z}\) define the following related quantities:

\[
\underline{n} = \min(n_1, n_2), \quad \overline{n} = \max(n_1, n_2), \quad p = (m - n_1 - n_2)_+, \quad l = (n_1 + n_2 - m)_+
\]

\[
r = l + (m)_+ - \underline{n} - 1 = \begin{cases} 
\max(m - \underline{n}, \overline{n}) - 1, & m \geq 0 \\
\max(-\underline{n}, \overline{n} - m) - 1, & m \leq 0.
\end{cases}
\]

(2.18)

Note that \(p - l = m - n_1 - n_2\) and \(r\) may only be negative when \(n_1 = n_2 = m = 0\) in which case \(r = -1\). The key fact that we will need is a more precise version of a particular case of [26, Theorem 1] which (after some change of notation) reads:

**Theorem 2.8.** Assume that \(n_1, n_2, m \in \mathbb{Z}\). Then

\[
\frac{(\gamma - \alpha)_{n_2}(\gamma - \beta)_{m-n_2}t^{n_1}}{(\gamma - 1)_{n_1-n_2+1}} {}_2F_1 \left( \frac{1 - \gamma + \alpha, 1 - \gamma + \beta}{2 - \gamma} \right) {}_2F_1 \left( \frac{\gamma - \alpha - n_2, \gamma - \beta + m - n_2}{\gamma + n_1 - n_2} \right) t + \frac{(1 - \alpha)_{n_1}(1 - \beta)_{m-n_2}t^{n_2}}{(1 - \gamma)_{n_2-n_1+1}} {}_2F_1 \left( \frac{\alpha, \beta}{\gamma} \right) {}_2F_1 \left( \frac{1 - \alpha - n_1, 1 - \beta + m - n_1}{2 - \gamma + n_2 - n_1} \right) t = \frac{pmP_r(t)}{(1-t)^p},
\]

(2.19)

where \(P_r(t)\) is a polynomial of degree \(r\) (\(P_{-1} \equiv 0\)) given by

\[
P_r(t) = (-1)^{\overline{n}} \sum_{k=0}^{r} (-t)^k \sum_{j=(k-p)+-\overline{n}}^{k-\overline{n}} (-1)^j \binom{p}{k-\overline{n}-j} K_j.
\]

(2.20a)

The coefficients \(K_j\) are given by

\[
K_j = \frac{(1 - \alpha)_{j}(1 - \beta)_{m+j}}{(1 - \gamma)_{n_2+j+1}(j + n_1)!} {}_4F_3 \left( \begin{array}{c} -j - n_1, \alpha, \beta, \gamma - 1 - n_2 - j \\ \alpha - j, \beta - m - j, \gamma \end{array} ; 1 \right) + \frac{(\gamma - \alpha)_{j}(\gamma - \beta)_{m+j}}{(\gamma - 1)_{n_1+j+1}(j + n_2)!} {}_4F_3 \left( \begin{array}{c} -j - n_2, 1 - \gamma + \alpha, 1 - \gamma + \beta, 1 - \gamma - n_1 - j \\ 1 - \gamma + \alpha - j, 1 - \gamma + \beta - m - j, 2 - \gamma \end{array} ; 1 \right),
\]

(2.20b)

where we use the convention \(1/(-i)! = 0\) for \(i \in \mathbb{N}\). This polynomial can also be computed by multiplying the left hand side of (2.19) by \(t^{-\underline{n}}(1-t)^p\) and calculating the first \(r+1\) Taylor coefficients on the left hand side.

**Remark 2.9.** The particular \(2F_1\) case of our general identity [26, Theorem 1] used above has been essentially discovered by Ebisu in [22]. Formula (2.19) can be derived by combining Theorem 3.7 with Proposition 3.4 from [22].

**Remark 2.10.** Our identity from [26, Theorem 1] does not contain explicit expression (2.20) for the polynomial \(P_r\). This expression is found in [11]. It can also be computed by taking the limit \(q \to 1\) in [44, Theorem 2]. For specific values of \(n_1, n_2, m\), the second method of computing \(P_r(t)\) indicated in the above theorem is more practical.

In the following theorem we give an explicit formula for the imaginary part of \(R_{n_1,n_2,m}(z)\) on the banks of the branch cut \([1, \infty)\). Note that for \(x > 1\) the function \(2F_1(a, b; c; x \pm i0)\) may vanish at finitely many points in the degenerate case \(\{-a, -b, c - a, c - b\} \cap \mathbb{N}_0 \neq \emptyset\), and does not vanish otherwise, see resp. Theorem 2.1 and [37, Lemma 2, p. 54].
Theorem 2.11. Assume that \( n_1, n_2, m \in \mathbb{Z} \). In terms of notation (2.18), on the branch cut \( x > 1 \) we have

\[
\text{Im}[R_{n_1, n_2, m}(x \pm i0)] = \pm \pi B_{n_1, n_2, m}(a, b, c) \frac{x^{l-n-c}(x-1)^{c-a-b-l}P_r(1/x)}{|2F_1(a, b; c; x)|^2}, \tag{2.21a}
\]

where

\[
B_{n_1, n_2, m}(a, b, c) = - \frac{\Gamma(c)\Gamma(c+m)}{\Gamma(a)\Gamma(b)\Gamma(c-a+m-n_1)\Gamma(c-b+m-n_2)} \tag{2.21b}
\]

and \( P_r(t) \) is the polynomial (2.20) with \( \alpha = a, \beta = 1-c+a, \gamma = 1-b+a \). Note that \(|2F_1(a, b; c; x)|^2\) takes the same values on both banks of the branch cut.

Proof. The boundary values of the generalized hypergeometric function on the cut \([1, \infty)\) have been found in [27, Theorem 3]. For \( 2F_1 \) this theorem takes the form \((x > 1)\):

\[
2F_1(a, b; c; x \pm i0) = - \frac{\pi\Gamma(c)}{\Gamma(a)\Gamma(b)} G_{2,1}^{2,1} \left( \frac{1}{x}, 1, 3/2, c \right) \pm \pi i \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G_{2,2}^{2,0} \left( \frac{1}{x}, 1, c, a, b \right),
\]

where \( G_{p,q}^{m,n} \) is Meijer’s \( G \) function [34, section 16.17]. As

\[
\text{Im} \left( \frac{\alpha + i\beta}{\gamma + i\delta} \right) = \frac{\beta \gamma - \alpha \delta}{|\gamma + i\delta|^2},
\]

denoting \( \phi_{\pm}(x) = \text{Im}[R_{n_1, n_2, m}(x \pm i0)] \), we get

\[
\phi_{\pm}(x) = \text{Im} \left[ \frac{2F_1(a + n_1, b + n_2; c + m; x \pm i0)}{2F_1(a, b; c; x \pm i0)} \right] = \pm \frac{\pi^2\Gamma(c)\Gamma(c+m)}{|2F_1(a, b; c; x)|^2\Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)} \left\{ G_{3,3}^{2,1} \left( \frac{1}{x}, 1, 3/2, c \right) = G_{3,3}^{2,0} \left( \frac{1}{x}, 1, c \right) - G_{3,3}^{2,1} \left( \frac{1}{x}, 1, 3/2, c \right) \right\}.
\]

Meijer’s \( G \) function here can be expanded as follows [27, Proof of Theorem 3]:

\[
- G_{3,3}^{2,1} \left( \frac{1}{x}, 1, 3/2, c \right) = \frac{\Gamma(b-a)\Gamma(a)\Gamma(c-a)}{\pi\Gamma(c-a)} 2F_1 \left( a, 1-c+a \right) \frac{t \cos(\pi a)}{1-b+a} + \frac{\Gamma(a-b)\Gamma(b)\Gamma(b-c)}{\pi\Gamma(c-b)} 2F_1 \left( b, 1-c+b \right) \frac{t \cos(\pi b)}{1-a+b}
\]

and

\[
G_{3,3}^{2,0} \left( \frac{1}{x}, 1, c \right) = \frac{\Gamma(b-a)\Gamma(a)\Gamma(c-a)}{\pi\Gamma(c-a)} 2F_1 \left( a, 1-c+a \right) \frac{t \sin(\pi a)}{1-b+a} + \frac{\Gamma(a-b)\Gamma(b)\Gamma(b-c)}{\pi\Gamma(c-b)} 2F_1 \left( b, 1-c+b \right) \frac{t \sin(\pi b)}{1-a+b}.
\]

Substituting these expansion into the above formula for \( \phi_{\pm}(x) \) and collecting terms, the expression in braces becomes:

\[
\frac{\Gamma(b-a)\Gamma(a+n_1-b-n_2)\Gamma(b+n_2)\Gamma(a)\Gamma(c-a+m-b-n_2)}{\pi^2\Gamma(c-a)\Gamma(c+m-b-n_2)} 2F_1 \left( a, 1-c+a \right) \frac{1}{1-b+a} \times 2F_1 \left( b, 1-c+b \right) \frac{1}{1-a+b}
\]

\[
\times 2F_1 \left( b+n_2, 1-c-m+b+n_2 \right) \frac{1}{1-a+n_1+b+n_2} \sin(\pi(b+n_2-a))
\]

\[
+ \frac{\Gamma(a-b)\Gamma(b+n_2-a-n)\Gamma(a+n_1)\Gamma(b)\Gamma(c-a-b-n_1)}{\pi^2\Gamma(c-b)\Gamma(c+m-a-n_1)} 2F_1 \left( b, 1-c+b \right) \frac{1}{1-a+b} \times 2F_1 \left( a, 1-c+m+a+n_1 \right) \frac{1}{1-b-n_2+a+n_1} \sin(\pi(a+n_1-b)).
\]
Then, writing $t = 1/x$, applying Euler’s transformation and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/(\sin(\pi z))$, we get

$$
|\text{cF}_1(a, b; c; 1/t)|^2 \frac{\Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)}{\Gamma(c)\Gamma(c+m)} \phi_+(1/t) = 
$$

\[
= \frac{\Gamma(a-b)\Gamma(b+n_2-a-n_1)\Gamma(a+n_1)\Gamma(b)^{a+b+n_1}}{\Gamma(c-b)\Gamma(c+m-a-n_1)} \text{cF}_1\left(\begin{array}{c} b, 1-c+b \\ 1-a+b \\ \end{array} \right) t 
\]

\[
+ \frac{2\text{F}_1\left(\begin{array}{c} a+n_1, 1-c-m+a+n_1 \\ 1-b-n_2+a+n_1 \\ \end{array} \right) \sin(\pi(a+n_1-b))}{\Gamma(b-a)\Gamma(a+n_1-b-n_2)\Gamma(b+n_2)\Gamma(a)^{a+b+n_2}} \text{cF}_1\left(\begin{array}{c} a, 1-c+a \\ 1-b+a \\ \end{array} \right) t
\]

\[
+ \frac{\pi\Gamma(b-a)\Gamma(a+b+n_1)\Gamma(b)\Gamma(b+n_2)\Gamma(a)^{a+b+n_2}}{\Gamma(c-a)\Gamma(c+m-b-n_2)\Gamma(b-a+n_2)} \text{cF}_1\left(\begin{array}{c} a, 1-c+a \\ 1-b+a \\ \end{array} \right) t
\]

Further, writing $a = \alpha, b = 1 - \gamma + \alpha, c = 1 - \beta + \alpha$ after tedious but elementary transformations using the relations

$$(1 - z)_k = \frac{(-1)^k}{(z)_k} \text{ and } (z - r)_k = \frac{(z)_{k-r}}{(z)_r} = (-1)^k \frac{(1 - z)_r}{(z)_r - k}$$

the above expression reduces to:

$$
|\text{cF}_1(a, b; c; 1/t)|^2 \frac{\Gamma(a)\Gamma(b)\Gamma(a+n_1)\Gamma(b+n_2)}{\Gamma(c)\Gamma(c+m)} \phi_+(1/t) = 
$$

\[
= -\pi^{2\alpha-\gamma+1}(1-t)^{\gamma-a-\beta+m-n_1-n_2}\Gamma(\alpha+n_1)\Gamma(1+\alpha-\gamma+n_2) \times 
\]

\[
\left\{ \frac{(\gamma-a-n_2)(\gamma-\beta)m-n_2}{(\gamma-1)n_1-n_2+1} \text{cF}_1\left(\begin{array}{c} 1-\gamma+a, 1-\gamma+b \\ 2-\gamma \\ \end{array} \right) t \right\} \text{cF}_1\left(\begin{array}{c} \gamma-a-n_2, \gamma-\beta+m-n_2 \\ \gamma+n_1-n_2 \\ \end{array} \right) t 
\]

\[
+ (1-\alpha)_n(1-\beta)_n \frac{\text{cF}_1\left(\begin{array}{c} \alpha, \beta \\ \gamma \\ \end{array} \right) t \right\} \text{cF}_1\left(\begin{array}{c} 1-\alpha-n_1, 1-\beta+m-n_1 \\ 2-\gamma+n_2-n_1 \\ \end{array} \right) t \right\} 
\]

\[
= -\pi^{2\alpha-\gamma+1+\frac{m}{2}}(1-t)^{\gamma-a-\beta+m} \Gamma(\alpha+n_1)\Gamma(1+\alpha-\gamma+n_2) P_r(t), 
\]

where the ultimate equality is an application of Theorem 2.8 with the notation introduced in (2.18).

Now substituting back $\alpha = a, \beta = 1 - c + a, \gamma = 1 - b + a$ we get

$$
|\text{cF}_1(a, b; c; 1/t)|^2 \phi_+(1/t) = -\pi^{2\alpha-\gamma+1}(1-t)^{\gamma-a-\beta+m-n_1-n_2} \text{cF}_1\left(\begin{array}{c} \alpha, \beta \\ \gamma \\ \end{array} \right) t \right\} \text{cF}_1\left(\begin{array}{c} 1-\alpha-n_1, 1-\beta+m-n_1 \\ 2-\gamma+n_2-n_1 \\ \end{array} \right) t \right\} 
\]

\[
= -\pi^{2\alpha-\gamma+1+\frac{m}{2}}(1-t)^{\gamma-a-\beta+m} \Gamma(\alpha+n_1)\Gamma(1+\alpha-\gamma+n_2) P_r(t), 
\]

It remains to plug back $x = 1/t$ to arrive at (2.21a).
Lemmas 2.6 and 2.7 and the subsequent remarks show that the asymptotic expansion of \( R_{n_1,n_2,m}(z) \) at infinity is a combination of terms of the form \( A z^\mu |\log(z)|^k \), where \( A \) and \( \mu \) are real numbers while \( k \) is an integer. Condition (2.22) in the theorem below requires that each exponent \( \mu \) in such terms satisfying \( \mu \geq N \), \( N \in \mathbb{N}_0 \), is, in fact, integer and contains no logarithm (i.e. \( k = 0 \)). The following theorem is the main result of this section.

**Theorem 2.12.** Suppose conditions of Theorem 2.1 are satisfied, so that \( _2F_1(a,b;c;z) \neq 0 \) for \( z \in \mathbb{C} \setminus [1,\infty) \). Assume further that \( \nu > -1 \) in (2.8) and for some \( N \in \mathbb{N}_0 \) the asymptotic expansion of \( R_{n_1,n_2,m}(z) \) at infinity has the form

\[
R_{n_1,n_2,m}(z) - Q_{a,b,c}(z) = o(z^N) \quad \text{as} \quad z \to \infty,
\]

where \( Q_{a,b,c}(z) \) is a (possibly vanishing) polynomial with real coefficients and the lowest degree \( N \). Then the following representation holds true

\[
R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + \sum_{k=0}^{N-1} \frac{R_{n_1,n_2,m}^{(k)}(0)}{k!} z^k + z^N B_{n_1,n_2,m}(a,b,c) \int_1^\infty \frac{x^{l-2-c-N} (x-1)^{c-a-b-l} P_r(1/x)}{|_2F_1(a,b;c;x)|^2 (x-z)} dx,
\]

where \( r, l \) and \( B_{n_1,n_2,m}(a,b,c) \) retain their meanings from Theorem 2.11 and \( P_r \) is defined in (2.20). In particular, if (2.22) holds with \( N = 0 \) we obtain

\[
R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + B_{n_1,n_2,m}(a,b,c) \int_1^\infty \frac{x^{l-2-c}(x-1)^{c-a-b-l} P_r(1/x)}{|_2F_1(a,b;c;x)|^2 (x-z)} dx.
\]

If \( n_1, n_2 \geq 0 \) and (2.6) holds, then (2.22) is true with \( N = 0 \) and \( Q_{a,b,c}(z) = Q_{a,b,c} \) being a constant.

**Remark 2.13.** Note that the choice of \( N \) and \( Q_{a,b,c}(z) \) in (2.22) is not unique. In particular, it follows from Lemmas 2.6 and 2.7 that we can always take \( Q_{a,b,c}(z) = 0 \) by choosing large enough \( N \).

**Remark 2.14.** The first two terms of the Taylor expansion of \( R_{n_1,n_2,m}(z) \) are given by

\[
R_{n_1,n_2,m}(z) = 1 + \frac{(an_2 + bn_1 + n_1 n_2) c - abm}{c(c + m)} z + O(z^2)
\]

**Remark 2.15.** Substitution \( x = 1/t \) brings formula (2.24) to the form (we write \( B = B_{n_1,n_2,m}(a,b,c) \) for brevity):

\[
R_{n_1,n_2,m}(z) = Q_{a,b,c}(z) + \sum_{k=0}^{N-1} \frac{R_{n_1,n_2,m}^{(k)}(0)}{k!} z^k + z^N B \int_0^1 \frac{t^{a+b+n+N-1}(1-t)^{c-a-b-l} P_r(t)}{|_2F_1(a,b;c;1/t)|^2 (1-zt)} dt.
\]

This form turns out to be more convenient in most applications. Moreover, taking \( z = 0 \) or \( z = 1 \) we get the following curious integral evaluations:

\[
\int_0^1 \frac{t^{a+b+n+N-1}(1-t)^{c-a-b-l-1} P_r(t)}{|_2F_1(a,b;c;1/t)|^2} dt = \frac{R_{n_1,n_2,m}^{(N)}(0) - Q_N N!}{N! B},
\]

where \( Q_N \) denotes the coefficient at \( z^N \) in \( Q_{a,b,c}(z) \), and

\[
\int_0^1 \frac{t^{a+b+n+N-1}(1-t)^{c-a-b-l-1} P_r(t)}{|_2F_1(a,b;c;1/t)|^2} dt = \frac{R_{n_1,n_2,m}(1) - Q_{a,b,c}(1)}{B} - \frac{1}{B} \sum_{k=0}^{N-1} \frac{R_{n_1,n_2,m}^{(k)}(0)}{k!},
\]

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where, in view of the Gauss summation formula,

\[ R_{n_1, n_2, m}(1) = \frac{(c)_n (c - a - b)_{m - n_1} (c - b)_{m - n_2}}{(c - a)_{m - n_1} (c - b)_{m - n_2}}. \]

Multiplying the integrand in (2.27) by \((1 - t)\), splitting the result in two summands, and using both formulae (2.26) and (2.27), we also obtain

\[
\int_0^1 \frac{a + b + 2N(1 - t)c - a - b - t - 1}{|2F_1(a, b; c; 1/t)|^2} P_r(t) dt = \frac{R_{n_1, n_2, m}(1) - Q_{a,b,c}(1) + Q_N}{B} - \frac{1}{B} \sum_{k=0}^{N} \frac{R_{n_1, n_2, m}(0)}{k!}.
\]

**Remark 2.16.** The absolute value of \(2F_1\) on the branch cut in the integrands in (2.23) and (2.24) can be computed as follows \((x > 1)\):

\[
|2F_1(a, b; c; x)|^2 = \frac{\pi^2 \Gamma(c)^2}{\Gamma(a)^2 \Gamma(b)^2} \left\{ \frac{(x - 1)^{2(c - a - b)}}{\Gamma(c - a - b)^2} \left[ 2F_1\left( \frac{c - a, c - b}{c - a - b}; \frac{1 - x}{1}\right) \right]^2 + \frac{\Gamma(b - a) \Gamma(a)x^a}{\Gamma(c - a) \Gamma(a) \Gamma(1/2 - a) \Gamma(1/2 + a)} 2F_1\left( \frac{a, 1 - c + a}{1 - b + a}; \frac{1/x}{1}\right) + \frac{\Gamma(a - b) \Gamma(b)x^{-b}}{\Gamma(c - b) \Gamma(1/2 - b) \Gamma(1/2 + b)} 2F_1\left( \frac{b, 1 - c + b}{1 - a + b}; \frac{1/x}{1}\right) \right\}.
\]

**Proof of Theorem 2.12.** Define \(f(z) = R_{n_1, n_2, m}(z) - Q_{a,b,c}(z)\). As lowest degree term in \(Q_{a,b,c}(z)\) is \(z^N\), in view of Theorem 2.1 we see that the function

\[
\hat{f}(z) = R_{n_1, n_2, m}(z) - Q_{a,b,c}(z) - \sum_{k=0}^{N-1} \frac{f^{(k)}(0)}{k!} z^k = R_{n_1, n_2, m}(z) - Q_{a,b,c}(z) - \sum_{k=0}^{N-1} \frac{R_{n_1, n_2, m}(0)}{k!} z^k
\]

is holomorphic in \(z \in \mathbb{C} \setminus [1, \infty)\) and has no singularities on the banks of the branch cut other than \(z = 1\) and \(z = \infty\). We aim to apply Lemma 2.4 to the function \(\hat{f}(z)\). Condition \(\nu > -1\) implies that the first term in (2.3) vanishes for \(f = \hat{f}(z)\), while (2.22) leads to the second equality in (2.3) with \(n = N\). Denote \(u(x) = \text{Im}(\hat{f}(x + i0))\). As \(Q_{a,b,c}(z)\) has real coefficients we conclude that \(u(x) = \text{Im}[R_{n_1, n_2, m}(x + i0)]\). Then Lemmas 2.6 and 2.7 imply that the asymptotics must have one of the forms

\[ z^{-N} \hat{f}(z) = \frac{C}{\log(z)} \left( 1 + O\left( |\log(z)|^{-1}\right) \right) \quad \text{as} \quad z \to \infty \]

or for some \(\tau > 0\)

\[ z^{-N} \hat{f}(z) = \frac{C}{z^\tau} \left( 1 + o(1) \right) \quad \text{as} \quad z \to \infty. \]

Then, in view of

\[
\left| \frac{1}{\log(x + i0)} \right| \leq \frac{\pi}{\log^2 |x| + \pi^2},
\]

we obtain

\[
\frac{u(x)}{x^{N+1}} = O\left( \frac{1}{x \log^2(x)} \right) \quad \text{or} \quad \frac{u(x)}{x^{N+1}} = O\left( \frac{1}{x^{1+\tau}} \right) \quad \text{as} \quad x \to \infty.
\]

Together with the condition \(\nu > -1\) this leads to absolute integrability of \(x^{-N-1}u(x)\) on \((0, +\infty)\). Hence, we are in the position to apply Lemma 2.4 leading to (2.23) in view of Theorem 2.11. The ultimate claim follows directly from Lemmas 2.6 and 2.7. \(\square\)
3 Generalized Nevanlinna classes

Integral representation (2.24) shows that the ratio \( R_{n_1, n_2, m}(z) \) belongs, possibly up to an additive polynomial term, to the Stieltjes class \( S = \mathcal{N}_0^\infty \) under the condition of positivity of the measure. The natural question is then to determine classes that extend \( S \) and contain \( R_{n_1, n_2, m}(z) \) if some of the conditions required for (2.24) are violated. To answer this question we adopt the machinery of the generalized Nevanlinna classes. Functions of these classes may be expressed in a very convenient multiplicative form due to Dijksma, Langer, Luger and Shondin, see Theorem 3.18 below. Another option is to use the representation by Krein and Langer [28, Satz 3.1] based on the standard ‘additive’ regularization approach.

For the general ratio \( \pm R_{n_1, n_2, m}(z) \) we give a simple and explicit criterion for being a generalized Nevanlinna function: our Theorem 3.1 provides conditions under which \( \pm R_{n_1, n_2, m}(z) \) belongs to the class \( \mathcal{G} \) or (see Remark 3.2) to \( \mathcal{R} \). Theorem 3.3 shows how to modify \( R_{n_1, n_2, m}(z) \) in order to obtain expressions that lie in \( \mathcal{G} \) for the case of arbitrary integer shifts \( n_1, n_2, m \) and parameters \( a, b, c \in \mathbb{R} \). Theorem 3.4 not only proves that \( \varepsilon R_{0,1,1} \in \mathcal{N}_\kappa^\lambda \) for some indices \( \kappa, \lambda \) and \( \varepsilon = \pm 1 \), which is a particular case for Theorems 3.1 and 3.3 – it also gives a method for calculating these indices in terms of the parameters \( a, b, c \). For some other shifts there is a similar way to determine \( \kappa, \lambda \) and \( \varepsilon \), see Examples 2–4 and 8–9 in Section 4.

Recall that each rational function \( f \) has a unique (up to multiplication by a common constant) representation as a ratio of two polynomials with no common zeros. The maximal degree of these polynomials is called the degree of \( f \) and denoted by \( \deg f \). Together with a possible zero or pole at infinity, \( f \) has exactly \( \deg f \) zeros and \( \deg f \) poles (counting with their multiplicities and, resp., orders).

**Theorem 3.1.** Suppose that \( a, b, c \in \mathbb{R} \) and \( -c \notin \mathbb{N}_0 \). The function \( R_{n_1, n_2, m} \in \mathcal{G} \) if and only if
\[
B_{n_1, n_2, m}(a, b, c)P_r(t) \geq 0, \quad \text{for all} \quad t \in (0, 1),
\]
where \( P_r(t) \) is the polynomial from (2.20) and \( B_{n_1, n_2, m} \) is given in (2.21b). In turn, if the reverse inequality holds above for all \( t \in (0, 1) \), then (3.1) is equivalent to \( -R_{n_1, n_2, m} \in \mathcal{G} \).

In particular, \( R_{n_1, n_2, m}(z) \) is a rational function if and only if \( B_{n_1, n_2, m}P_r(t) = 0 \) for all \( t \). If so, \( R_{n_1, n_2, m} \in \mathcal{N}_\kappa^\lambda \) and \( -R_{n_1, n_2, m} \in \mathcal{N}_{K-\kappa}^{\lambda-1} \) for some \( \kappa \leq K \) and \( \lambda \leq \Lambda \), where \( K \) and \( \Lambda \) are the degrees of \( R_{n_1, n_2, m} \) and \( zR_{n_1, n_2, m} \), respectively.

**Remark 3.2.** It is seen from the formulae (2.21) and (3.14) that the first part of Theorem 3.1 has the following equivalent form: the condition
\[
\varepsilon \lim_{y \to +0} \text{Im} \, R_{n_1, n_2, m}(x + iy) \geq 0
\]
for all but finitely many points \( x \in \mathbb{R} \) and \( \varepsilon \in \{-1, 1\} \) is necessary for \( \varepsilon R_{n_1, n_2, m} \in \mathcal{G} \) and sufficient for \( \varepsilon R_{n_1, n_2, m} \in \mathcal{G} \subseteq \mathcal{R} \). The next theorem shows that this condition may be removed through multiplication by an appropriate rational factor fixing the sign of \( \text{Im} \, R_{n_1, n_2, m}(x + i0) \).

**Theorem 3.3.** Suppose that \( a, b, c \in \mathbb{R} \), \( -c \notin \mathbb{N}_0 \), and \( n_1, n_2, m \in \mathbb{Z} \). If \( P_r(z) \) is the corresponding polynomial from (2.20), \( B_{n_1, n_2, m} \) is defined by (2.21b) and \( B_{n_1, n_2, m}P_r(z) \neq 0 \), then both functions
\[
\frac{R_{n_1, n_2, m}(z + \omega)}{B_{n_1, n_2, m}P_r(1/(z + \omega))} \quad \text{and} \quad B_{n_1, n_2, m}P_r\left(\frac{1}{z + \omega}\right)R_{n_1, n_2, m}(z + \omega)
\]
belong to \( \mathcal{G} \) for any \( \omega \leq 1 \).
Expressing the indices $\kappa, \lambda$ of the particular classes $\mathcal{N}^\lambda_\kappa \subset \mathfrak{S}$ emerging in Theorems 3.1 and 3.3 through the shifts $n_1, n_2, m$ and the parameters and $a, b, c$ does not look as an easy task. It is nevertheless doable in various specific cases. In particular, if a function has a regular C-fraction, then the coefficients of this continued fraction make it possible to determine the corresponding indices $\kappa$ and $\lambda$ uniquely (see Corollary 3.15 and Theorem 3.16). The following theorem extending [14, Theorem 3.4] applies this idea to the Gauss ratio $R_{0,1,1}(z)$, whose continued fraction is given in (1.2)–(1.3). We will also employ this result as an intermediary step in our proofs of Theorems 3.1 and 3.3 (alternative to, e.g., direct derivation of the multiplicative representation presented in Theorem 3.18).

**Theorem 3.4.** For any real values of parameters $a, b, c$ with $-c \notin \mathbb{N}_0$, there are $\kappa, \lambda \in \mathbb{N}_0$ such that $\varepsilon R_{0,1,1}(z) \in \mathcal{N}^\lambda_\kappa$ for a certain $\varepsilon \in \{-1, 1\}$.

More specifically, if $\{-a, -b - 1, a - c - 1, b - c\} \cap \mathbb{N}_0 = \emptyset$, put

$$\theta_j := \text{sign} \left[ (c + 2j) \Gamma(a + j) \Gamma(b + j + 1) \Gamma(c - a + j + 1) \right] \quad \text{and} \quad \eta_j := \text{sign} \left[ (c + 2j - 1) \Gamma(a + j) \Gamma(c - b + j) \Gamma(b + j) \Gamma(c - a + j) \right]$$

for $j \in \mathbb{N}_0$. Then $R_{0,1,1}(z)$ is not rational, $\varepsilon = \theta_0$, and $\lambda$ and $\kappa$ are, respectively, the number of negative entries in the sequences $(\theta_j)_{j=0}^{\infty}$ and $(\eta_j)_{j=0}^{\infty}$.

If $\{-a, -b - 1, a - c - 1, b - c\} \cap \mathbb{N}_0 \neq \emptyset$, then $R_{0,1,1}(z)$ is a rational function of degree $K = \left\lfloor \frac{s + 1}{2} \right\rfloor$ satisfying $\varepsilon R_{0,1,1} \in \mathcal{N}^\lambda_\kappa$ and $-\varepsilon R_{0,1,1} \in \mathcal{N}^{\lambda - \kappa}$, where

$$s_1 := \min \left( \{-a, b - c\} \cap \mathbb{N}_0 \right), \quad s_2 := \min \left( \{-b, a - c\} \cap \mathbb{N} \right), \quad s := \min \{2s_1, 2s_2 - 1\}$$

and $\Lambda = \left\lfloor \frac{s + 2}{2} \right\rfloor$ equals the degree of $zR_{0,1,1}(z)$; here we assume that $\min(\emptyset) = +\infty$. If $s = 2s_1 > 0$, put

$$\varepsilon_{2j+1+\delta} := \text{sign} \left[ (a + j + \delta)_{s_1-j-\delta}(c - b + j + \delta)_{s_1-j}(b + j + 1)_{s_1-j}(c - a + j + 1)_{s_1-j} \right] \frac{(c + 2j + \delta)(c + 2s_1)}{(c + 2j + \delta)(c + 2s_1)}$$

for $j = 0, \ldots, s_1 - 1$ and $\delta \in \{0, 1\}$. Then $\varepsilon = \varepsilon_1$ and $\lambda$ (resp. $\kappa$) is the number of negative entries in the sequence $\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2s_1-1}$ (resp. $\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{2s_1}$), and $\lambda = \kappa = 0$, $\varepsilon = 1$ if $s_1 = 0$.

If $s = 2s_2 - 1$, put

$$\varepsilon_{2j+\delta} := \text{sign} \left[ (a + j)_{s_2-j}(c - b + j)_{s_2-j}(b + j + \delta)_{s_2-j}(c - a + j + \delta)_{s_2-j-\delta} \right] \frac{(c + 2j + \delta)(c + 2s_2 - 1)}{(c + 2j + \delta)(c + 2s_2 - 1)}$$

and $\varepsilon = \varepsilon_1$, where $j = 0, \ldots, s_2 - 1$, $\delta \in \{0, 1\}$ and $j + \delta \neq 0$. Then $\lambda$ (resp. $\kappa$) is the number of negative entries in the sequence $\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2s_2-1}$ (resp. $\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{2s_2-2}$) if $s_2 > 1$, and $\lambda = 1/2 - \varepsilon/2$, $\kappa = 0$ if $s_2 = 1$.

The proofs of Theorems 3.1 and 3.3 will hinge on Theorem 3.4. The next subsection presents some background material and closes with the proof of Theorem 3.4. Several results in this subsection represent rather general facts connecting generalized Nevanlinna classes with continued fractions. The main result in this direction is formulated in Theorem 3.16 which gives explicit formulae for the Nevanlinna indices in terms of the coefficients of continued fractions. Our starting point is a continued fraction – it simplifies our task so we do not need to deal with degenerate cases of the corresponding moment problem, cf. [17].
3.1 Nevanlinna indices of continued fractions and proof of Theorem 3.4

Let us recall some basic properties of Hermitian matrices, i.e. matrices satisfying $H = H^*$, where $H^*$ denotes conjugate-transpose of $H$. These matrices have real spectrum, they possess a complete system of eigenvectors that may be chosen to be orthogonal. The inertia of a Hermitian matrix $H$ is defined as the triplet comprising the number of positive, vanishing, and negative eigenvalues of $H$, respectively. Sylvester’s law of inertia asserts that the inertia of a Hermitian matrix $H$ equals the inertia of $PHP^*$, where $P$ is an arbitrary non-degenerate matrix. Moreover, for each Hermitian matrix $H$ there is a unitary transformation $U = (U^*)^{-1}$, such that $UHU^*$ is diagonal (the so-called reduction to principal axes). Jacobi’s algorithm for reduction of Hermitian matrices (which relies on the LDU-decomposition, see [24, p. 38 and p. 300]) shows that the signs of their eigenvalues equal the signs of the ratios of consecutive leading principal minors (if these minors do not vanish). As a result, we therefore have:

**Lemma 3.5.** Given two $n \times n$ Hermitian matrices $H_1$ and $H_2$, let the number of their negative eigenvalues be $\kappa_1$ and $\kappa_2$, respectively. Then $H_1 + H_2$ has at most $\kappa_1 + \kappa_2$ negative eigenvalues. If $H_1$ has $\lambda_1$ positive eigenvalues, then $H_1 + H_2$ cannot have less than $\kappa_2 - \lambda_1$ negative eigenvalues.

**Proof.** Let $U$ diagonalize $H_1$:

$$UH_1U^* = \text{diag}\{d_1, \ldots, d_n\} := \sum_{j=1}^{n} d_j e_j \cdot e_j^T,$$

where $e_1, \ldots, e_n$ is the standard column basis in $\mathbb{R}^n$, and $(e_j)^T$ is the transpose of $e_j$. It is enough to check the case $UH_1U^* = d_je_j \cdot (e_j)^T$, then the general case will follow by consecutive application of that particular result for every $j$. Moreover, we may also assume $j = n$, which can always be achieved through permuting the rows of $U$.

For each $\varepsilon > 0$ small enough, no leading principal minor of the matrix $G(\varepsilon) := UH_2U^* + \varepsilon I$ vanishes (as nontrivial polynomial in $\varepsilon$), and this matrix has precisely $\kappa_2$ negative eigenvalues (by continuity: the spectrum of $G(\varepsilon)$ is just the spectrum of $G(0)$ shifted to the right by $\varepsilon$).

Denote the $j$th-order leading principal minor of $G(\varepsilon)$ by $\Delta_j$, $\Delta_0 := 1$, and observe that the matrix $\tilde{G}(\varepsilon) := d_ne_n \cdot (e_n)^T + G(\varepsilon)$ has the same leading principal minors as $G(\varepsilon)$ with the only exception for

$$\det \tilde{G}(\varepsilon) = \Delta_n + d_n \Delta_{n-1}.$$

According to Jacobi’s algorithm, the sequence $\Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_n}{\Delta_{n-1}}$ has exactly $\kappa_2$ negative entries. The analogous sequence for $\tilde{G}(\varepsilon)$ comprises the same entries, except for the last one equal to $\frac{\Delta_n}{\Delta_{n-1}} + d_n$.

Thus, $\tilde{G}(\varepsilon)$ has either $\kappa_2$ or $\kappa_2 - \text{sign} d_n$ negative eigenvalues for all small enough $\varepsilon > 0$. The same is true for number $\kappa$ of negative eigenvalues of $\tilde{G}(0)$, since the spectrum of $\tilde{G}(\varepsilon)$ is the right-shifted by $\varepsilon$ spectrum of $\tilde{G}(0)$. In other words, due to

$$\text{sign} d_n = \begin{cases} \lambda_1, & \text{if } d_n \geq 0, \\ -\kappa_1, & \text{if } d_n \leq 0, \end{cases}$$

the number $\kappa$ satisfies $\kappa_2 - \lambda_1 \leq \kappa \leq \kappa_2 + \kappa_1$ (the positive eigenvalues of $\tilde{G}(\varepsilon)$ remain nonnegative as $\varepsilon \to +0$). Since the inertia of $H_1 + H_2$ coincides with the inertia of $\tilde{G}(0)$, the matrix $H_1 + H_2$ has between $\kappa_2 - \lambda_1$ and $\kappa_2 + \kappa_1$ negative eigenvalues. The lemma is thereby proved for our rank-one matrix $H_1$. As is noted above, then we also obtain the lemma in the general case. \[\square\]

Next we obtain certain basic properties of the generalized Nevanlinna classes.
Lemma 3.6. Given $a, c \in \mathbb{R}$ and $b, d > 0$, let either $h(z) = a + \frac{b}{c - zd}$ or $h(z) = a + bz$. Then a function $f(z)$ belongs to the class $\mathcal{N}_\kappa$ precisely when $h \circ f(z) := h(f(z))$ belongs to $\mathcal{N}_\kappa$.

Proof. Take any finite sequence of numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}_+$ such that none of them is a pole of $f(z)$. If $h(z) = a + bz$, then the Pick matrix $H_{ho f}$ defined in (1.12) satisfies

$$H_{ho f}(z_1, \ldots, z_n) = b H_f(z_1, \ldots, z_n),$$

so the lemma is trivial. If $h(z)$ is not a polynomial, we have

$$h(f(z_i)) - h(f(z_j)) = a + \frac{b}{c - f(z_i)d} - a - \frac{b}{c - f(z_j)d} = \frac{\sqrt{bd} \cdot (f(z_i) - f(z_j)) \cdot \sqrt{bd}}{(c - f(z_i)d) \cdot (c - f(z_j)d)}$$

and, therefore,

$$H_{ho f}(z_1, \ldots, z_n) = D \cdot H_f(z_1, \ldots, z_n) \cdot D^*, \quad \text{where} \quad D := \text{diag} \left[ \frac{\sqrt{bd}}{c - f(z_i)d} \right]_{i=1}^n \quad (3.2)$$

and $D^*$ stands for the conjugate transpose of $D$. So, the lemma follows by Sylvester’s law of inertia. \hfill \square

Lemma 3.7. Let $f(z)$ be a real function, and let

$$g(z) := -f(-z), \quad h(z) := -f \left( \frac{1}{z} \right) \quad \text{and} \quad \tilde{h}(z) := \frac{1}{f \left( \frac{1}{z} \right)}.$$

Then $f \in \mathcal{N}_\kappa \iff g \in \mathcal{N}_\kappa \iff h \in \mathcal{N}_\kappa \iff \tilde{h} \in \mathcal{N}_\kappa$.

Proof. Indeed, $H_g(z_1, \ldots, z_n)$ is precisely the transpose of $H_f(w_1, \ldots, w_n)$ with $w_i := -\overline{z_i}$:

$$\frac{g(z_i) - g(z_j)}{z_i - z_j} = \frac{f(-z_j) - f(-z_i)}{z_i - z_j} = \frac{f(\overline{w_j}) - f(\overline{w_i})}{-\overline{w_i} + w_j} = \frac{f(\overline{w_i}) - f(\overline{w_i})}{w_j - w_i},$$

and $h(z_1, \ldots, z_n)$ is diagonally similar to $H_g(\zeta_1, \ldots, \zeta_n)$ with $\zeta_i := -\frac{1}{z_i}$:

$$\frac{h(z_i) - h(z_j)}{z_i - z_j} = \frac{1}{z_i} \frac{g(-1/z_i) - g(-1/z_j)}{1/z_i + 1/z_j} = \zeta_i \cdot \frac{g(\zeta_i) - g(\zeta_j)}{\zeta_i - \zeta_j} \cdot \frac{1}{z_j}.$$

Here $i, j$ run over $1, \ldots, n$. Therefore, $f \in \mathcal{N}_\kappa \iff g \in \mathcal{N}_\kappa \iff h \in \mathcal{N}_\kappa$.

Now, $\tilde{h}(z) = -\frac{1}{m(\overline{z})}$, and hence $\tilde{h} \in \mathcal{N}_\kappa \iff h \in \mathcal{N}_\kappa$ by the case $a = c = 0$, $b = d = 1$ of Lemma 3.6. \hfill \square

Lemma 3.8 (cf. [36, p. 17]). If $f(z)$ is a real rational function and $n \geq m := \deg f$, then the rank of $H_f(z_1, \ldots, z_n)$ equals $m$ provided that the points $z_1, \ldots, z_n$ are distinct.

Proof. Let $m > 0$, otherwise the lemma is trivial. Take any

$$\delta \in \mathbb{R} \setminus \{f(\xi_1), \ldots, f(\xi_r), f(\infty)\},$$

where $\xi_1, \ldots, \xi_r$ are the real solutions to the equation $f'(z) = 0$. Put

$$g_\delta(z) := \frac{1}{\delta - f(z)},$$

When $m > 0$, by (3.2) we have

$$H_{ho f}(z_1, \ldots, z_n) = D \cdot H_f(z_1, \ldots, z_n) \cdot D^*, \quad \text{where} \quad D := \text{diag} \left[ \frac{\sqrt{bd}}{c - f(z_i)d} \right]_{i=1}^n \quad (3.2)$$

and $D^*$ stands for the conjugate transpose of $D$. So, the lemma follows by Sylvester’s law of inertia. \hfill \square
so that \( \deg g_\delta = m \) and, according to (3.2) with \( h(z) = 1/(\delta - z) \), the matrices \( H_f(z_1, \ldots, z_n) \) and \( H_{gs}(z_1, \ldots, z_n) \) have the same inertia. Moreover, \( g_\delta(z) \) only has simple poles, and \( g_\delta(\infty) \) is finite. As a result, there exists \( l \), such that \( 0 \leq 2l \leq m \) and for some numbers \( A_0 \in \mathbb{R}, \ a_{2k-1} = \overline{a_{2k}} \) and \( A_{2k-1} = \overline{A_{2k}} \neq 0 \) when \( 0 < k \leq l \),

\[
g_\delta(z) = A_0 + \sum_{k=1}^{m} \frac{A_k}{z - a_k} = A_0 + \sum_{k=1}^{l} \left( \frac{\overline{A_{2k}}}{z - \overline{a_{2k}}} + \frac{A_{2k}}{z - a_{2k}} \right) + \sum_{k=2l+1}^{m} \frac{A_k}{z - a_k}.
\]

Then

\[
\frac{g_\delta(z_i) - g_\delta(z_j)}{z_i - z_j} = \frac{1}{z_i - z_j} \sum_{k=1}^{m} \left( \frac{A_k}{z_i - a_k} - \frac{A_k}{z_j - a_k} \right) = - \sum_{k=1}^{m} \frac{A_k}{z_i - a_k(z_j - a_k)}.
\]

Now, choose arbitrary \( a_{m+1}, \ldots, a_n \in \mathbb{R} \) so that the numbers \( a_1, \ldots, a_n \) are distinct. Observe that \( H_{gs}(z_1, \ldots, z_n) = P_n \cdot M \cdot P_n^* \), where

\[
M := - \text{diag} \left( \begin{bmatrix} 0 & \overline{A_2} \\ A_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \overline{A_2} \\ A_2 & 0 \end{bmatrix}, A_{2l+1}, \ldots, A_m, 0, \ldots, 0 \right), \quad P_n := \begin{bmatrix} 1 \\ \overline{z_j - a_k} \end{bmatrix}_{j,k}
\]

and \( P_n^* \) denotes the conjugate transpose of \( P_n \). Since \( P_n \) is the Cauchy matrix, the well-known formula for its determinant yields \( \det P_n \neq 0 \). By Sylvester’s law of inertia, the inertia of the Pick matrix \( H_{gs}(z_1, \ldots, z_n) \) is the same as that of \( M \). The latter, however, equals \( m \), since all the numbers \( A_1, \ldots, A_m \) are distinct from zero.

**Remark 3.9.** The conditions of Lemma 3.8 exclude the case \( n < \deg f \), in which the rank of \( H_f(z_1, \ldots, z_n) \) may be less than \( n \): for instance, if \( f(z) = \prod_{j=1}^{n} (z - z_j)(z - \overline{z_j}) \), then \( \deg f = 2n \) and \( \text{rank } H_f(z_1, \ldots, z_n) = 0 \).

**Remark 3.10.** As is seen from the above identity \( H_{gs}(z_1, \ldots, z_n) = P_n \cdot M \cdot P_n^* \), one can relate the inertia of \( H_{gs}(z_1, \ldots, z_n) \) to the location of poles of \( g_\delta(z) \), and (for real poles) to the signs of the corresponding residues. Moreover, this relation may be extended to multiple poles. The works [16, 18] present a far-reaching generalization of this idea, which is cited here as Theorem 3.18.

**Remark 3.11.** On letting \( z_1, \ldots, z_n \to \infty \) and \( f(z) = \sum_{k=0}^{\infty} c_k z^{-k-1} \), the (scaled) matrix \( H_f(z_1, \ldots, z_n) \) tends to the Hankel matrix \( D_n^{(0)} \) as in (1.5). Lemma 3.8 is therefore related to Kronecker’s theorem: \( D_n^{(0)} \neq 0 = D_{n+1}^{(0)} = D_{n+2}^{(0)} = \cdots \) if and only if \( f(z) \) represents a rational function of degree \( n \).

Rational functions may be seen as building blocks for the generalized Nevanlinna classes. From this viewpoint, it looks natural to study these classes via continued fractions.

**Lemma 3.12** (e.g. [13] or [14, Prop. 3.3]). Given \( \epsilon, a, b > 0 \) and \( c \in \mathbb{R} \), let \( f(z) \) be a function meromorphic in \( \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im } z > 0 \} \) such that

\[
f(\epsilon y) = o(y) \quad \text{as} \quad y \to +\infty.
\]

Define

\[
g_\epsilon(z) := -\frac{\epsilon b}{az - c + \epsilon f(z)} = -\frac{b}{\frac{\epsilon}{a} z - \frac{c}{a} + f(z)}.
\]

Then \( g_{-\epsilon}(z) \) does not belong to \( \mathcal{N}_0 \). Moreover, the following three conditions are equivalent:

\[
f \in \mathcal{N}_n, \quad g_\epsilon \in \mathcal{N}_n \quad \text{and} \quad g_{-\epsilon} \in \mathcal{N}_{n+1}.
\]

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Remark 3.13. The choice \( f(z) = 2z \) and \( \varepsilon = a \) yields \( f, g_{-\varepsilon} \in N_0 \), so Lemma 3.12 fails to be generally true without the condition on the growth of \( f(z) \) at infinity. (It is missed out in [14], but fortunately does not affect other results of [14]).

Proof. Lemma 3.6 implies that \( g_{\pm \varepsilon} \in N_\kappa \) for some \( \kappa \) if and only if \( f + h_{\pm \varepsilon} \in N_\kappa \), where

\[
h_{\varepsilon}(z) := \frac{a}{\varepsilon} z, \quad \text{and hence } f(z) + h_{\pm \varepsilon}(z) = \pm \frac{c}{\varepsilon} - \frac{b}{g_{\pm \varepsilon}(z)}.
\]

Observe that as \( y \to +\infty \)

\[
H_{f + h_{\varepsilon}}[z_1, \ldots, z_n, iy] = \begin{bmatrix}
H_f[z_1, \ldots, z_n] + \left[ \frac{a}{\varepsilon} \right]_{k, m=1}^{n} & o(1) + \frac{a}{\varepsilon} \\
o(1) + \frac{a}{\varepsilon} & \vdots & o(1) + \frac{a}{\varepsilon} & o(1) + \frac{a}{\varepsilon}
\end{bmatrix} \to P \cdot \begin{bmatrix}
H_f[z_1, \ldots, z_n] & 0 \\
0 & 0
\end{bmatrix} \cdot P^*,
\]

where

\[
P := \begin{bmatrix} I_n \end{bmatrix}.
\]

If \( y > 0 \) is large enough, the \( 1 \times 1 \) matrix \( H_{f + h_{\varepsilon}}[iy] \) turns out to be negative definite, so \( g_{-\varepsilon} \notin N_0 \).

For arbitrary \( n \in \mathbb{N} \) and \( z_1, \ldots, z_n \in \mathbb{C}_+ \), let \( \lambda \) be the number of negative eigenvalues of \( H_f[z_1, \ldots, z_n] \). The matrix \( H_{h_{\varepsilon}}[z_1, \ldots, z_n] \) is positive semidefinite of rank one, thus

\[
H_{f + h_{\varepsilon}}[z_1, \ldots, z_n] = H_f[z_1, \ldots, z_n] + H_{h_{\varepsilon}}[z_1, \ldots, z_n]
\]

has \( \lambda \) or \( \lambda - 1 \) negative eigenvalues by Lemma 3.5. Therefore, the inclusion \( f \in N_{\kappa_1} \) for some \( \kappa_1 \) implies \( g_{\varepsilon} \in N_{\kappa_2} \) with \( \kappa_2 \leq \kappa_1 \), while \( g_{-\varepsilon} \in N_{\kappa_2} \) for some \( \kappa_2 \) implies \( f \in N_{\kappa_1} \) with \( \kappa_1 \leq \kappa_2 + 1 \), in view of the equivalence \( g_{\varepsilon} \in N_\kappa \iff f + h_{\varepsilon} \in N_\kappa \).

Let \( f \in N_{\kappa_1} \). Then there exist numbers \( z_1, \ldots, z_n \in \mathbb{C}_+ \) such that \( H_f[z_1, \ldots, z_n] \) has precisely \( \kappa_1 \) negative eigenvalues. Then \( H_{f + h_{\varepsilon}}[z_1, \ldots, z_n, iy] \) has \( \kappa_1 \) negative eigenvalues due to (3.3) whenever \( y > 0 \) is large enough. As a result, \( g_{\varepsilon} \in N_{\kappa_2} \) with \( \kappa_2 \geq \kappa_1 \), and hence \( \kappa_2 = \kappa_1 \).

Analogously, for any \( z_1, \ldots, z_n \in \mathbb{C}_+ \) and \( y > 0 \) the matrix \( H_{h_{-\varepsilon}}[z_1, \ldots, z_n] \) is negative semidefinite of rank one, by Lemma 3.5 the number of negative eigenvalues of \( H_{f + h_{-\varepsilon}}[z_1, \ldots, z_n] \) may only equal \( \lambda \) or \( \lambda + 1 \). In particular, \( f \in N_{\kappa_1} \) for some \( \kappa_1 \) implies \( g_{-\varepsilon} \in N_{\kappa_3} \) with \( \kappa_3 \leq \kappa_1 + 1 \), while \( g_{-\varepsilon} \in N_{\kappa_3} \) for some \( \kappa_3 \) implies \( f \in N_{\kappa_1} \) with \( \kappa_1 \leq \kappa_3 \).

If \( f \in N_{\kappa_1} \), the formula (3.3) for the above choice of \( z_1, \ldots, z_n \in \mathbb{C}_+ \), \( \varepsilon \mapsto -\varepsilon \), and sufficiently large \( y > 0 \) yields that \( H_{f + h_{-\varepsilon}}[z_1, \ldots, z_n, iy] \) has \( \kappa_1 + 1 \) negative eigenvalues. So, \( g_{-\varepsilon} \in N_{\kappa_3} \) with \( \kappa_3 \geq \kappa_1 + 1 \), and hence \( \kappa_3 = \kappa_1 + 1 \). \qed

Lemma 3.14. Given numbers \( m, \lambda \in \mathbb{N}_0 \), \( \varepsilon_{2m+1} \in \{-1, 1\} \) and a function \( \psi_{2m+1} \in N_\lambda \) satisfying \( \psi_{2m+1}(iy) = o(y) \) as \( y \to +\infty \), let

\[
\psi(z) := -\frac{\alpha_0}{z - \beta_0} - \frac{\alpha_1 \alpha_2}{z - \beta_1 - \cdots - \frac{\alpha_2 m + 1}{z - \beta_m + \varepsilon_{2m+1} \psi_{2m+1}(z)}},
\]

(3.4)
where the coefficients \( \beta_0, \ldots, \beta_m \) and \( \alpha_0, \ldots, \alpha_{2m} \neq 0 \) are real. Put

\[
\varepsilon_j := \varepsilon_{2m+1} \text{sign} \prod_{k=j}^{2m} \alpha_k \quad \text{for} \quad j = 0, 1, \ldots, 2m,
\]

and let \( \kappa \) be the number of negative entries in the sequence \( \varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2m+1} \), that is

\[
\kappa := \frac{m+1}{2} - \frac{1}{2} \sum_{j=0}^{m} \varepsilon_{2j+1}.
\]

Then \( \varepsilon_0 \psi \in \mathcal{N}_{\kappa+\lambda} \).

The particular choice \( \psi_{2m+1}(z) \equiv 0 \) and \( \varepsilon_{2m+1} = 1 \) yields \( \varepsilon_0 \psi \in \mathcal{N}_\kappa \) and \( -\varepsilon_0 \psi \in \mathcal{N}_{m+1-\kappa} = \mathcal{N}_{\deg \psi - \kappa} \) for \( \kappa = m/2 - \sum_{j=1}^{m} (\varepsilon_{2j-1}/2) \).

**Proof.** It is enough only to consider the case \( m \geq 1 \): for \( m = 0 \) the proof immediately follows from Lemma 3.12. In view of \( \varepsilon_0 = \varepsilon_1 \text{sign} \alpha_0 \) and \( \varepsilon_{2j-1} = \varepsilon_{2j+1} \text{sign}(\alpha_{2j-1} - \alpha_{2j}) \) we see that

\[
\varepsilon_0 \varepsilon_1 \alpha_0 = \varepsilon_1^2 |\alpha_0| > 0 \quad \text{and} \quad \varepsilon_{2j-1} \varepsilon_{2j+1} \alpha_{2j-1} - \alpha_{2j} = \varepsilon_{2j+1}^2 |\alpha_{2j-1} - \alpha_{2j}| > 0,
\]

for all \( j = 1, 2, \ldots, m \). In other words, \( \varepsilon_0 \alpha_0 = \varepsilon_1 |\alpha_0| \) and \( \alpha_{2j-1} \alpha_{2j} = \varepsilon_{2j-1} \varepsilon_{2j+1} |\alpha_{2j-1} - \alpha_{2j}| \), so

\[
\varepsilon_0 \psi(z) = -\frac{\varepsilon_1 |\alpha_0|}{z - \beta_0 - \frac{\varepsilon_1 \varepsilon_3 |\alpha_1 \alpha_2|}{z - \beta_1 - \cdots - \frac{\varepsilon_{2m-1} \varepsilon_{2m+1} |\alpha_{2m-1} \alpha_{2m}|}{z - \beta_m + \varepsilon_{2m+1} \psi_{2m+1}(z)}}.
\]

Define

\[
\psi_{2j-1}(z) := -\frac{\varepsilon_{2j+1} |\alpha_{2j-1} \alpha_{2j}|}{z - \beta_j + \varepsilon_{2j+1} \psi_{2j+1}(z)}, \quad j = m, m-1, \ldots, 1.
\]

In particular, for each \( j \) the estimate \( \psi_{2j+1} = o(z) \) as \( z \to \infty \) implies \( \psi_{2j-1} = O(z^{-1}) \). By Lemma 3.12, \( \psi_{2m+1} \in \mathcal{N}_0 \iff \psi_{2m-1} \in \mathcal{N}_{\kappa_m} \), where \( \kappa_m := (1 - \varepsilon_{2m+1})/2 \). Analogously, Lemma 3.12 yields that \( \psi_{2j+1} \in \mathcal{N}_{\kappa_{j+1}} \iff \psi_{2j-1} \in \mathcal{N}_{\kappa_j} \) with \( \kappa_j := \kappa_{j+1} + (1 - \varepsilon_{2j+1})/2 \) as \( j \) runs over \( m - 1, m - 2, \ldots, 1 \). As

\[
\varepsilon_0 \psi(z) = -\frac{\varepsilon_1 |\alpha_0|}{z - \beta_0 + \varepsilon_1 \psi_1(z)}.
\]

Lemma 3.12 also implies \( \varepsilon_0 \psi \in \mathcal{N}_\kappa \), where the index \( \kappa \) is determined by the signs of \( \varepsilon_1, \ldots, \varepsilon_{2m-1} \):

\[
\kappa = \kappa_1 + \frac{1 - \varepsilon_1}{2} = \kappa_2 + \frac{2 - \varepsilon_1 - \varepsilon_3}{2} = \cdots = \frac{m + 1}{2} - \sum_{j=0}^{m} \frac{\varepsilon_{2j+1}}{2}.
\]

If \( \psi_{2m+1}(z) \equiv 0 \), the corresponding assertion of the lemma follows by noting that \( \varepsilon_{2m+1} \psi_{2m+1} \in \mathcal{N}_0 \) for both \( \varepsilon_{2m+1} = 1 \) and \( \varepsilon_{2m+1} = -1 \). With \( \varepsilon_{2m+1} = 1 \), we obtain \( \varepsilon_0 \psi \in \mathcal{N}_\kappa \), where

\[
\kappa = \frac{m}{2} - \sum_{j=0}^{m-1} \frac{\varepsilon_{2j+1}}{2} \quad \text{due to} \quad \frac{1 - \varepsilon_{2j+m}}{2} = 0.
\]

On the other hand, there are \( m + 1 - \kappa \) positive numbers among \( \varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2m+1} \). So, the inclusion \( -\varepsilon_0 \psi \in \mathcal{N}_{m+1-\kappa} \) follows by an application of the previous part of the proof, where \( \varepsilon_{2m+1} \) and, hence, \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2m-1} \) are chosen to have the opposite signs. (Alternatively, one can use Lemma 3.38.)
Consider a terminating or non-terminating continued fraction

\[
\varphi_n(z) := \frac{\alpha_n}{1 - \frac{\alpha_{n+1}z}{1 - \frac{\alpha_{n+2}z}{1 - \cdots}}} = \frac{\alpha_n}{1 - z\varphi_{n+1}(z)}, \quad n = 0, 1, \ldots. \quad (3.7)
\]

Fractions of this form are called C-fractions. Here \(\varphi_n(z)\) is terminating when \(\alpha_j = 0\) for some \(j \geq n\), in that case we assume \(\varphi_j(z) \equiv 0\). For every \(n\),

\[
\varphi_n(z) = \frac{\alpha_n}{1 - \frac{\alpha_{n+1}z}{1 - \frac{\alpha_{n+2}z}{1 - \cdots}}} \sim \frac{\alpha_n(1 - z\varphi_{n+2}(z))}{1 - z\varphi_{n+2}(z) - \alpha_{n+1}z}\sim \frac{\alpha_n + \alpha_n\alpha_{n+1}z}{1 - \alpha_{n+1}z - z\varphi_{n+2}(z)}
\]

where \(\sim\) denotes the ‘correspondence’, i.e. that the corresponding (formal) power series coincide. This is the so-called contraction of continued fractions, cf. [35, p. 135]; it connects regular C-fractions and J-fractions as in (3.4). For our goals, it is enough to observe that

\[
\psi_n(z) := \alpha_n - \varphi_n \left(\frac{1}{z}\right) \sim -\frac{\alpha_n\alpha_{n+1}z}{1 - \alpha_{n+1}z - z\varphi_{n+2}(z)} \sim -\frac{\alpha_n\alpha_{n+1}}{z - \alpha_{n+1} - \alpha_{n+2} + \psi_{n+2}(z)}. \quad (3.8)
\]

This formula implies that defining

\[
\psi(z) := -\frac{\alpha_0}{z - \alpha_1 + \psi_1(z)} \quad (3.9)
\]

we obtain (3.4) with \(\beta_0 = \alpha_1\) and \(\beta_j = \alpha_{2j} + \alpha_{2j+1}\) for \(j = 1, \ldots, m\). On the other hand, in view of (3.8) and (3.7), we have

\[
\psi(z) \sim -\frac{\alpha_0}{z - \varphi_1(1/z)} \equiv -\frac{\alpha_0/z}{1 - (1/z)\varphi_1(1/z)} = -\frac{1}{z}\varphi_0 \left(\frac{1}{z}\right). \quad (3.10)
\]

**Corollary 3.15.** Consider the terminating continued fraction

\[
\varphi(z) = \frac{\alpha_0}{1 - \frac{\alpha_1z}{1 - \cdots - \frac{\alpha_{k-1}z}{1 - \alpha_kz}}},
\]

whose coefficients \(\alpha_0, \ldots, \alpha_k\) are real and non-zero. Denote \(\varepsilon_j := \text{sign} \prod_{l=j}^{k} \alpha_l\) and put \(\alpha_{k+1} := 0\). Then

\[
\varepsilon_0\varphi \in N^\Lambda_\kappa \quad \text{and} \quad -\varepsilon_0\varphi \in N^{\Lambda-\kappa}_K,
\]

where \(\Lambda := \left\lfloor \frac{k+2}{2} \right\rfloor\) and \(K := \left\lfloor \frac{k+1}{2} \right\rfloor\), while \(\lambda\) and \(\kappa\) are the numbers of negative entries in the sequences \(\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2\Lambda-3}, \alpha_{2\Lambda-1}\) and \(\varepsilon_4, \varepsilon_4, \ldots, \varepsilon_{2K-2}, \alpha_{2K}\), respectively. (Here, \(K\) and \(\Lambda\) are resp. equal to the degrees of the rational functions \(\varphi(z)\) and \(z\varphi(z)\).

---

\(^4\)This is possible since we start with a continued fraction \(\varphi_n(z)\) with finite coefficients (in our application to \(R_{0,1,1}(z)\), the finiteness follows from the assumption \(-c \notin \mathbb{N}_0\)). If \(\varphi_n(z)\) would be introduced as a (formal) power series, and some coefficient \(\alpha_j\) calculated via (1.6) would vanish, then a neighbouring coefficient might need to be assumed infinite. In such a case, the series cannot be expressed as a regular C-fraction (although one can use the general C-fraction [35, § 21], or follow [17, 29] an use the P-fractions).
Proof. Suppose that \( \varphi(z) \neq 0 \), otherwise the corollary is trivial. Then \( \varphi_{2m+1}(z) = \alpha_{2m+1} \) for \( m := \Lambda - 1 \) in view of (3.7); hence, (3.8) implies \( \psi_{2m+1}(z) \equiv 0 \) and, on account of (3.9)–(3.10),

\[
\psi(z) := \frac{1}{z} \varphi \left( \frac{1}{z} \right) = - \frac{\alpha_0 \beta}{1 - \frac{1}{z} \varphi_1 \left( \frac{1}{z} \right)} = - \frac{\alpha_0}{z - \alpha_1 + \psi_1(z)},
\]

which is (3.4) with \( \beta_0 = \alpha_1 \) and \( \beta_j = \alpha_{2j} + \alpha_{2j+1} \) for \( j = 1, \ldots, m \). By Lemma 3.7, if one of the functions \( \varepsilon_0 z \varphi_0(z) = \varepsilon_0 z \varphi_0(z) \) or \( \varepsilon_0 \psi_0(z) \) lies in \( N_\lambda \), then the other is also in \( N_\lambda \). Analogously, \( -\varepsilon_0 z \varphi(z) \) lies in \( N_{\lambda - \lambda} \) if and only if \( -\varepsilon_0 \psi \in N_{\lambda - \lambda} = N_{m+1-\lambda} \). For \( k = 2m, \) Lemma 3.14 with \( \varepsilon_{2m+1} = 1 \) and \( \psi_{2m+1}(z) \equiv 0 \) applied to \( \psi(z) \) immediately yields \( \varepsilon_0 \psi \in N_\lambda \) and \( -\varepsilon_0 \psi \in N_{m+1-\lambda} \), where \( \lambda \) is as required.

For \( k = 2m + 1 \), we also put \( \psi_{2m+1}(z) \equiv 0 \) and apply Lemma 3.14: to get \( \varepsilon_0 \psi \in N_\lambda \) it is enough to take \( \varepsilon_{2m+1} = \text{sign} \alpha_{2m+1} \), which makes the present values of \( \varepsilon_j \) equal to those of Lemma 3.14. In turn, the inclusion \( -\varepsilon_0 \psi \in N_{m+1-\lambda} \) follows by taking the opposite signs of \( \varepsilon_{2m+1} \) and, hence, of \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2m} \) as well.

Observe that \( \psi_0(z) \) defined by (3.8) may be expressed as

\[
\psi_0(z) = - \frac{\alpha_0 \alpha_1}{z - (\alpha_1 + \alpha_2) - \frac{\alpha_2 \alpha_3}{z - (\alpha_3 + \alpha_4) - \cdots - \frac{\alpha_{2K-2} \alpha_{2K-1}}{z - (\alpha_{2K-1} + \alpha_{2K})}}},
\]

By Lemmas 3.6 and 3.7, the required inclusions \( \varepsilon_0 \varphi \in N_\kappa \) and \( -\varepsilon_0 \varphi \in N_{K - \kappa} \) are equivalent to \( \varepsilon_0 \psi_0 \in N_\kappa \) and \( -\varepsilon_0 \psi_0 \in N_{K - \kappa} \). If \( k = 2K - 1 \), the latter pair of inclusions readily follows from Lemma 3.14 (in which we take \( \varepsilon_{2m+1} = 1 \) and \( \psi_{2m+1}(z) \equiv 0 \) applied to \( \psi_0(z) \)).

If \( k = 2K \), letting \( \varepsilon_{2K} = \text{sign} \alpha_{2K} \) and \( \psi_{2K}(z) \equiv 0 \) yields \( \varepsilon_0 \psi_0 \in N_\kappa \). The inclusion \( -\varepsilon_0 \psi_0 \in N_{K - \kappa} \) follows in a similar way by taking the opposite signs of \( \varepsilon_{2K} \) and, hence, \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2K-2} \).

**Theorem 3.16.** Let the continued fraction \( \varphi(z) := \varphi_0(z) \) defined in (3.7) be non-terminating and have real coefficients satisfying \( \sup_n |\alpha_n| < \infty \). If there exists \( m \in \mathbb{N} \) such that \( \alpha_n > 0 \) for all \( n > 2m \), then \( \varepsilon_0 \varphi \in N_\kappa^0 \) and \( -\varepsilon_0 \varphi \notin \frak{S} \), where \( \varepsilon_0 \) is found from

\[
\varepsilon_j := \text{sign} \prod_{k=j}^{2m} \alpha_k = \prod_{k=j}^{\infty} \text{sign} \alpha_k,
\]

and \( \lambda \) and \( \kappa \) are the number of negative entries in the sequences \( \varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2m-1} \) and \( \varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{2m} \), respectively. Conversely,\(^5\) \( \varphi \notin \frak{S} \) for both \( \varepsilon \in \{-1, 1\} \) if and only if there are infinitely large \( n \) such that \( \alpha_n < 0 \).

**Proof.** Theorem 1.1 (a) says that, for each \( n = 0, 1, \ldots \) and any positive \( \gamma < \gamma_0 := 1/(4 \sup_n |\alpha_n|) \), the continued fraction \( \varphi_n(z) \) defined in (3.7) uniformly converges in the disc \( D_\gamma := \{ z \in \mathbb{C} : |z| \leq \gamma \} \) to a function analytic in \( D_\gamma \).

If \( m \) is a number satisfying the assumptions of the theorem, then the functions determined by the continued fractions \( \varphi_{2m+1}(z) \) and \( \varphi_{2m+2}(z) \) allow analytic continuations to \( \mathbb{C} \setminus [\gamma_0, +\infty) \) by Theorem 1.1 (c); moreover, on denoting these continuations also by \( \varphi_{2m+1}(z) \) and \( \varphi_{2m+2}(z) \), respectively, we have \( \varphi_{2m+1}, \varphi_{2m+2} \in N_0^0 \). Then, according to (3.8)

\[
\varphi_{2m+1}(z) = \alpha_{2m+1} - \psi_{2m+1} \left( \frac{1}{z} \right) \quad \text{and} \quad \varphi_{2m+2}(z) = \alpha_{2m+2} - \psi_{2m+2} \left( \frac{1}{z} \right),
\]

\(^5\)We still may have \( \varepsilon \varphi \in \frak{S} \) when there are infinitely large \( n \) such that \( \alpha_n < 0 \); for instance, if \( \alpha_j = (-1)^j \) for all \( j \).
Lemmas 3.6 and 3.7 yields $\psi_{2m+1} \in \mathcal{N}_0$ and $\psi_{2m+2} \in \mathcal{N}_0$. So, if we use each of the last two functions and $\varepsilon_{2m+1} = \varepsilon_{2m+2} = 1$ as the ‘initial data’ for Lemma 3.14, we obtain $\varepsilon_0 \psi_0 \in \mathcal{N}_\kappa$ and $\varepsilon_0 \psi \in \mathcal{N}_\lambda$ with $\kappa$ and $\lambda$ as defined in the theorem. Since

$$z \varphi(z) = -\psi \left( \frac{1}{z} \right)$$

and

$$\varphi(z) = \varphi_0(z) = \alpha_0 - \psi_0 \left( \frac{1}{z} \right),$$

Lemmas 3.6 and 3.7 imply that the pair of conditions $\varepsilon_0 \psi \in \mathcal{N}_\lambda$ and $\varepsilon_0 \psi_0 \in \mathcal{N}_\kappa$ is equivalent to $\varepsilon_0 \varphi \in \mathcal{N}^\lambda_\kappa$.

Now let us prove the converse: assuming $\varepsilon \varphi \in \mathcal{N}^\lambda_\kappa$ for a certain $\varepsilon = \pm 1$ and $\kappa, \lambda \in \mathbb{N}_0$, and hence $\varepsilon \psi \in \mathcal{N}_\lambda$ and $\varepsilon \psi_0 \in \mathcal{N}_\kappa$, we will show that there is some $m > 0$ such that $\alpha_n > 0$ for all $n > 2m$, and that the signs of $\alpha_1, \ldots, \alpha_{2m}$ determine $\kappa, \lambda$ via (3.11). Put $\varepsilon_0 := \varepsilon$, $\varepsilon_1 := \varepsilon \text{sign} \alpha_0$ and $\varepsilon_2 := \varepsilon_1 \text{sign} \alpha_1$ so that in view of (3.8) and (3.9)

$$\varepsilon \psi(z) = -\frac{\varepsilon_1 |\alpha_0|}{z - \alpha_1 + \varepsilon_1^2 \psi_1(z)} \quad \text{and} \quad \varepsilon \psi_0(z) = -\frac{\varepsilon_2 |\alpha_0 \alpha_1|}{z - (\alpha_1 + \alpha_2) + \varepsilon_2^2 \psi_2(z)}.$$ 

According to Lemma 3.12, we have $\varepsilon_1 \psi_1 \in \mathcal{N}_{\kappa_1}$ and $\varepsilon_2 \psi_2 \in \mathcal{N}_{\kappa_2}$, where $\kappa_1 = \lambda - \frac{1-\varepsilon_2}{2}$ and $\kappa_2 = \kappa - \frac{1-\varepsilon_2}{2}$. Analogously, given $\varepsilon_{n-2} \psi_{n-2} \in \mathcal{N}_{\kappa_{n-2}}$, where some known $\varepsilon_{n-2} = \pm 1$, and $\kappa_{n-2}$, we put $\varepsilon_n := \varepsilon_{n-2} \text{sign} (\alpha_{n-2} \alpha_{n-1})$ and $\kappa_n := \kappa_{n-2} - \frac{1-\varepsilon_{n-2}}{2}$, so that $\psi_n(z)$ satisfies

$$\varepsilon_{n-2} \psi_{n-2}(z) = -\frac{\varepsilon_{n-2} \alpha_{n-2} \alpha_{n-1}}{z - (\alpha_{n-1} + \alpha_n) + \psi_n(z)} = -\frac{\varepsilon_n |\alpha_{n-2} \alpha_{n-1}|}{z - (\alpha_{n-1} + \alpha_n) + \varepsilon_n^2 \psi_n(z)}$$

$$\implies \varepsilon_n \psi_n(z) \in \mathcal{N}_{\kappa_n}$$

due to Lemma 3.12. The sequence of indices $\kappa_1, \kappa_3, \ldots$ is nonnegative and non-increasing, as well as $\kappa_2, \kappa_4, \ldots$. So, proceeding further this way, one obtains

$$\kappa_* = \lim_{n \to \infty} \kappa_{2n} \geq 0 \quad \text{and} \quad \lambda_* = \lim_{n \to \infty} \kappa_{2n+1} \geq 0.$$ 

As these sequences are built from integers, there exists $m > 0$ such that

$$\kappa_{2m} = \kappa_{2m+2} = \cdots = \kappa_*; \quad \kappa_{2m+1} = \kappa_{2m+3} = \cdots = \lambda_*; \quad \text{and thus} \quad \varepsilon_{2m+1} = \varepsilon_{2m+2} = \cdots = 1.$$ 

For $n > 1$ the formulae $\varepsilon_0 = \varepsilon$, $\varepsilon_1 = \varepsilon \text{sign} \alpha_0$ and $\varepsilon_n = \varepsilon_n \text{sign} (\alpha_{n-2} \alpha_{n-1})$ give

$$\varepsilon_n = \varepsilon_{n-2} \prod_{j=n-2}^{n-1} \text{sign} \alpha_j = \varepsilon_{n-4} \prod_{j=n-4}^{n-2} \text{sign} \alpha_j = \cdots = \varepsilon_0 \prod_{j=0}^{n-1} \text{sign} \alpha_j$$

and, therefore, $\text{sign} \alpha_n = \varepsilon_n \text{sign} = 1$ as soon as $n > 2m$. In particular, our new definition of $\varepsilon_j$ coincides with the above formula (3.11).

Now, to complete the proof it is enough to show that $\kappa_* = \lambda_* = 0$. By virtue of Theorem 1.1 (c), both continued fractions $\varphi_{2m+1}(z)$ and $\varphi_{2m+2}(z)$ converge in $\mathbb{C} \setminus \{\gamma_0, +\infty\}$ to analytic functions; moreover, on keeping the same labelling for these functions, we also have $\varphi_{2m+1}, \varphi_{2m+2} \in \mathcal{N}_0^\theta$. By Lemma 3.7, the expressions (3.12) yield $\psi_{2m+1} \in \mathcal{N}_0$ and $\psi_{2m+2} \in \mathcal{N}_0$. The latter, however, contradicts to $\psi_{2m+1} \in \mathcal{N}_{\lambda_{2m+1}} = \mathcal{N}_{\lambda_*}$ and $\psi_{2m+2} \in \mathcal{N}_{\kappa_{2m+2}} = \mathcal{N}_{\kappa_*}$ unless $\kappa_* = \lambda_* = 0$. \hfill \Box

**Remark 3.17.** The condition $\sup_n |\alpha_n| < \infty$ may be relaxed. Nevertheless, Theorem 3.16 fails if one drops this condition completely: roughly speaking, $\lambda_*$ and $\kappa_*$ in the proof could remain positive. This depends on whether the corresponding moment problem determinate or not, see [15, Theorem 5] for the details.
Proof of Theorem 3.4. Let \(-a, -b - 1, a - c - 1, b - c\) \(\cap \mathbb{N}_0 \neq \emptyset\). Then the coefficients (1.3) of the C-fraction (1.2) corresponding to \(R_{0,1,1}(z)\) satisfy \(\alpha_0, \ldots, \alpha_s \neq 0\) and \(\alpha_{s+1} = 0\), where

\[
s_1 := \min \left\{ \{ -a, b - c \} \cap \mathbb{N}_0 \right\}, \quad s_2 := \min \left\{ \{ -b, a - c \} \cap \mathbb{N} \right\} \quad \text{and} \quad s := \min \{2s_1, 2s_2 - 1\};
\]

here we assume that \(\min(\mathbb{S}) = +\infty\). In order to apply Corollary 3.15, let us calculate the numbers \(\varepsilon_j = \text{sign} \prod_{l=j}^s \alpha_l\) for \(j = 0, \ldots, s\).

Due to \(\alpha_0 = 1\), we have \(\varepsilon := \varepsilon_0 = 1\) if \(s = 0\), and \(\varepsilon := \varepsilon_0 = \text{sign}(\alpha_0)\varepsilon_1 = \varepsilon_1\) otherwise. Moreover, given any integer numbers \(m > j \geq 0\)

\[
\xi_j(m) := \text{sign} \prod_{l=2j+1}^{2m} \alpha_l = \text{sign} \prod_{n=j}^{m-1} \alpha_{2n+1}\alpha_{2n+2}
\]

\[
= \text{sign} \prod_{n=j}^{m-1} \frac{(a + n)(b - n + 1)(c - n + 1)}{(c + 2n)(c + 2n + 1)}
\]

\[
= \text{sign} \frac{(a + j)_m}{(c + 2j)(c + 2j + 1)}
\]

\[
= \text{sign} \frac{(a + j)_m(c - b + j)_m}{(c + 2j)(c + 2j + 1)}
\]

\[
= \text{sign} \frac{(a + j)_m(c - b + j)_m}{(c + 2j)(c + 2j + 1)}
\]

\[
(3.13)
\]

In the case \(s = 2s_1\) we have \(\varepsilon_{2j+1} = \xi_j(s_1)\) and

\[
\varepsilon_{2j+2} = \frac{\varepsilon_{2j+1}}{\text{sign} \alpha_{2j+1}} = \text{sign} \frac{(a + j + 1)_{s_1-j-1}(c - b + j + 1)_{s_1-j}}{(c + 2j + 1)(c + 2s_1)}
\]

for \(j = 0, \ldots, s_1 - 1\). Corollary 3.15 yields

\[
\varepsilon_1 R_{0,1,1} \in \mathcal{N}_\kappa^\lambda \quad \text{and} \quad -\varepsilon_1 R_{0,1,1} \in \mathcal{N}_\kappa^{\lambda - \lambda},
\]

where \(\Lambda = \left\lfloor \frac{s_1 + 2}{2} \right\rfloor = s_1 + 1\), \(K = \left\lfloor \frac{s_1 + 1}{2} \right\rfloor = s_1\), and \(\lambda\) (resp. \(\kappa\)) is the number of negative entries in the sequence \(\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2s_1-1}\) (resp. \(\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{2s_2-1}\)). In particular, for \(s_1 = 0\) we have \(\lambda = \kappa = 0\).

In the case \(s = 2s_2 - 1\), if \(s_2 = 1\) we obtain \(\varepsilon_1 = \text{sign} \alpha_1\), otherwise \(s_2 > 1\) and for \(j = 0, \ldots, s_2 - 1\) the expression (3.13) implies

\[
\varepsilon_{2j+1} = \xi_j(s_2-1) \text{sign} \alpha_{2s_2-1} = \text{sign} \frac{(a + j)_{s_2-j}(c - b + j)_{s_2-j}}{(c + 2j)(c + 2s_2 - 1)}
\]

and hence for \(j = 1, \ldots, s_2 - 1\)

\[
\varepsilon_{2j} = \text{sign} \alpha_{2j} \varepsilon_{2j+1} = \text{sign} \frac{(a + j)_{s_2-j}(c - b + j)_{s_2-j}}{(c + 2j - 1)(c + 2s_2 - 1)}
\]

Moreover, \(K = \left\lfloor \frac{s_2 + 1}{2} \right\rfloor = s_2\) and \(\Lambda = \left\lfloor \frac{s_2 + 2}{2} \right\rfloor = s_2\). By Corollary 3.15,

\[
\varepsilon_1 R_{0,1,1} \in \mathcal{N}_\kappa^\lambda \quad \text{and} \quad -\varepsilon_1 R_{0,1,1} \in \mathcal{N}_\kappa^{\lambda - \lambda},
\]

where \(\lambda\) (resp. \(\kappa\)) is the number of negative entries in the sequence \(\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2s_2-1}\) (resp. \(\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_{2s_2-2}\)). Here we obtain \(\kappa = 0\) when \(s_2 = 1\).

In both cases \(s = 2s_1\) and \(s = 2s_2 - 1\), the ratio \(R_{0,1,1}(z)\) is a rational function of degree \(K\), and \(\Lambda\) is the degree of \(z R_{0,1,1}(z)\), see Corollary 3.15.

Now, let \(\{ -a, -b - 1, a - c - 1, b - c\} \cap \mathbb{N}_0 = \emptyset\). None of the coefficients (1.3) of the C-fraction (1.2) vanishes, that is the function \(R_{0,1,1}(z)\) is not rational. Furthermore, \(\alpha_n > 0\) for
all \( n > m \) provided that \( m \) is large enough. So, Theorem 3.16 implies that \( \varepsilon R_{0,1} \in \mathcal{N}_\kappa' \) for \( \varepsilon = \varepsilon_0 \) and \( \kappa, \lambda \in \mathbb{N}_0 \) determined by the numbers \( \varepsilon_j := \prod_{n=j}^{\infty} \sin \alpha_n \). These numbers may be calculated via the identity (3.13):
\[
\varepsilon_{2j+1} = \lim_{m \to \infty} \frac{\varepsilon_j(m)}{\varepsilon_{2j+1}} = \frac{\lim_{m \to \infty} \text{sign} \left[ (c + 2m) \Gamma(a + m) \Gamma(c - b + m) \Gamma(b + m + 1) \Gamma(c - a + m + 1) \right]}{\lim_{m \to \infty} \text{sign} \left[ (c + 2j) \Gamma(a + j) \Gamma(c - b + j) \Gamma(b + j + 1) \Gamma(c - a + j + 1) \right]}
\]

and
\[
\varepsilon_{2j+2} = \frac{\varepsilon_{2j+1}}{\text{sign} \alpha_{2j+1}} = \text{sign}(c + 2j + 1) \Gamma(a + j + 1) \Gamma(c - b + j + 1) \Gamma(b + j + 1) \Gamma(c - a + j + 1)
\]

where \( j \in \mathbb{N}_0 \). Furthermore, \( \varepsilon_0 = \text{sign}(\alpha_0) \varepsilon_1 = \varepsilon_1 \). □

3.2 Multiplicative factorizations

Recall the striking description of the generalized Nevanlinna classes given in [16, 18]:

**Theorem 3.18.** A function \( f(z) \) belongs to the class \( \mathcal{N}_\kappa \) if and only if it may be written in the form \( f(z) = Q(z)g(z) \), where \( g \in \mathcal{N}_0 \) and \( Q(z) \) is a real rational function of degree \( 2\kappa \) nonnegative on the real line (except for its poles).

Under the assumptions of this theorem, for \( x \in \mathbb{R} \) excluding the real poles and zeros of \( Q(z) \) we immediately obtain
\[
\liminf_{y \to +0} \text{Im} f(x + iy) = Q(x) \liminf_{y \to +0} \text{Im} g(x + iy) \in [0, +\infty],
\]
where the right-hand side is due to \( \text{Im} g(x + iy) \geq 0 \) for all \( y > 0 \).

The representation offered by Theorem 3.18 is unique up to multiplication of \( Q(z) \) by a positive constant: the poles and zeros of \( Q(z) \) in \( \mathbb{C} \cup \{\infty\} \) determine the local analytic properties of \( f(z) \) uniquely. If \( f \in \mathcal{N}_\kappa \subset \mathcal{R} \) with \( \mathcal{R} \) defined in (1.13) is written as \( f(z) = Q(z)g(z) \) according to Theorem 3.18 and \( T(z) \) is a rational function nonnegative on the real axis, then also \( Tg = TQg \in \mathcal{N}_{\frac{1}{2}\deg(TQ)} \subset \mathcal{R} \). The following theorem states the conditions, under which a similar fact holds without the assumption \( T(z) \geq 0 \) on \( \mathbb{R} \).

**Theorem 3.19.** Let \( g \in \mathcal{R} \), and let \( T(z) \) be a real rational function. Denote
\[
\mathcal{I} := \{ t \in \mathbb{R} : -\infty < T(t) < 0 \}.
\]

Then \( f := Tg \in \mathcal{R} \) if and only if \( g(z) \) is analytic and real on \( \mathcal{I} \) excluding at most a finite number of poles.

**Proof.** By Theorem 3.18, we can write \( g(z) = Q(z)\bar{g}(z) \) for some \( \bar{g} \in \mathcal{N}_0 \) and some real rational function \( Q(z) \) nonnegative on the real line. The set of poles of \( g(z) \) is finite if and only if the same is true for \( \bar{g}(z) \). Thus, the proof reduces to an application of a particular case of Theorem 3.19 stated here as Lemma 3.20 to the function \( f(z) = Q(z)\bar{g}(z) \), where \( \bar{Q}(z) := T(z)Q(z) \). □

**Lemma 3.20.** Let \( g \in \mathcal{N}_0 \), and let \( T(z) \) be a real rational function. Define \( \mathcal{I} \) via (3.15). Then in order for \( f(z) := T(z)g(z) \) to lie in \( \mathcal{R} \) it is necessary and sufficient that \( g(z) \) is analytic and real on \( \mathcal{I} \) excluding at most a finite number of poles.

Moreover, if \( f \in \mathcal{N}_\kappa \) for some \( \kappa \), then \( \mathcal{I} \) may contain at most \( \kappa \) of its poles.
Proof. Write \( g(z) \) in the form (1.11)
\[
g(z) = \nu_1 z + \nu_2 + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t) = \nu_1 z + \nu_2 + \int_{\mathbb{R}} \frac{1+tz}{t-z} \cdot \frac{d\sigma(t)}{t^2+1},
\]
(3.16)
where \( \nu_1 \geq 0, \ \nu_2 \in \mathbb{R} \) and \( \sigma(t) \) is a non-decreasing function satisfying \( \int_{-\infty}^{\infty} (1+t^2)^{-1} d\sigma(t) < \infty \). According to the Stieltjes-Perron inversion formula (see e.g. [2, pp. 124–126]), if \((a,b)\) is a real interval, then
\[
\frac{\sigma(b+) - \sigma(b-)}{2} - \frac{\sigma(a+) - \sigma(a-)}{2} = \lim_{y \to +0} \frac{1}{\pi} \int_a^b \text{Im} g(x+iy) \, dx.
\]
(3.17)
In particular, \( g(z) \) is analytic and real on \((a,b)\) precisely when \( \sigma(a+) = \sigma(b-) \), which is equivalent to that \( \sigma'(t) \) is defined and equals zero for all \( t \in (a,b) \).

We prove the necessity part of the lemma by contradiction. Suppose that \( f \in \mathcal{N}_\kappa \) and yet there are points \( t_0, \ldots, t_\kappa \in \mathcal{I} \) such that
\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon>0} \frac{\sigma(t_j + \varepsilon) - \sigma(t_j - \varepsilon)}{2\varepsilon} \in (0, +\infty], \quad j = 0, \ldots, \kappa,
\]
(3.18)
(this ratio is non-negative since \( \sigma(t) \) is non-decreasing). Then all poles of \( g(z) \) lying in \( \mathcal{I} \) are among the numbers \( t_0, \ldots, t_\kappa \). For \( y \in (0,1) \),
\[
yg(t_j + iy) = y\nu_1(t_j + iy) + y\nu_2 + y \int_{\mathbb{R}} \frac{1+t(t_j+iy)}{t-(t_j+iy)} \frac{d\sigma(t)}{t^2+1}
\]
\[= y\nu_1(t_j + iy) + y\nu_2 + y \int_{\mathbb{R}\setminus\{t_j\}} \frac{1+t(t_j+iy)}{t-(t_j+iy)} \frac{d\sigma(t)}{t^2+1} + y \int_{\mathbb{R}\setminus\{t_j\}} \frac{d\sigma(t)}{t^2+1},
\]
so that
\[
|yg(t_j + iy) - iy(\sigma(t_j+) - \sigma(t_j-))|
\]
\[\leq y \left| \nu_1(t_j + iy) + \nu_2 - (\sigma(t_j+) - \sigma(t_j-)) \right| \frac{t_j}{t_j^2+1} + \left| y \int_{\mathbb{R}\setminus\{t_j\}} \frac{1+t(t_j+iy)}{t-(t_j+iy)} \frac{d\sigma(t)}{t^2+1} \right|
\]
\[\leq Ay + y \int_{\mathbb{R}\setminus\{t_j\}} \frac{d\sigma(t)}{\sqrt{(t-t_j)^2+y^2+1}},
\]
where \( A \geq 0 \) does not depend on \( y \). On the right-hand side, the second factor in the integrand satisfies \( \int_{-\infty}^{\infty} (1+t^2)^{-1} d\sigma(t) < \infty \), while the first factor is bounded uniformly in \( y \in (0,1) \) and \( t \in \mathbb{R} \) and vanishes for each \( t \neq t_j \) as \( y \to 0 \). Thus, the right-hand side tends to zero by the dominated convergence theorem. This leads to the asymptotic formula
\[
yg(t_j + iy) = iy(\sigma(t_j+) - \sigma(t_j-)) + o(1), \quad \text{where } y \to 0 \text{ and } j = 0, \ldots, \kappa.
\]
(3.19)
According to definition (3.15) the rational function \( T(z) \) is bounded on \( \mathcal{I} \), and hence for \( y \to +0 \) and \( j = 0, \ldots, \kappa \) we have \( T(t_j + iy) = T(t_j) + O(y) \) and \( f(t_j + iy) = (T(t_j) + O(y)) g(t_j + iy) \). So, the formula (3.19) gives
\[
yf(t_j + iy) = iy \text{Im} f(t_j + iy) + \text{Re} \left[ (T(t_j) + O(y)) yg(t_j + iy) \right]
\]
\[= iy \text{Im} f(t_j + iy) + o(1) \quad \text{as} \quad y \to +0.
\]
(3.20)
Taking the imaginary part of (3.16) we get the Poisson representation

$$\text{Im } g(t_j + iy) = \nu_1 y + \int_{\mathbb{R}} \frac{yd\sigma(t)}{(t - t_j)^2 + y^2}.$$ 

In view of $\nu_1 \geq 0$ and $y > 0$, the above equality leads to the estimate

$$\text{Im } g(t_j + iy) \geq \int_{(t_j - y,t_j + y)} \frac{yd\sigma(t)}{(t - t_j)^2 + y^2} \geq \int_{(t_j - y,t_j + y)} \frac{yd\sigma(t)}{2y} = \frac{\sigma(t_j + y) - \sigma(t_j - y)}{2y}.$$ 

Therefore, from $T(t_j) < 0$ and $T(t_j + iy) = T(t_j) + T'(t_j)iy + O(y^2)$ we obtain

$$\text{Im } f(t_j + iy) = (T(t_j) + O(y^2)) \text{Im } g(t_j + iy) + (T'(t_j) + O(y)) \text{Re } (yg(t_j + iy)) \leq (T(t_j) + O(y^2)) \frac{\sigma(t_j + y) - \sigma(t_j - y)}{2y} + o(1) \quad \text{as } y \to +0,$$

where the last inequality is due to (3.19). Now, the last estimate along with (3.18) implies that there exist a number $C > 0$ and a sequence $y_1, y_2, \ldots$ of positive reals such that

$$\lim_{t \to +\infty} y_i = 0 \quad \text{and} \quad \text{Im } f(t_j + iy_i) \leq -C \quad (3.21)$$

for all $l = 1, 2, \ldots$ and $j = 0, \ldots, \kappa$. Denote $z_{j,l} := t_j + iy_l$ and observe that the matrix

$$M_l = (m_{j,k})_{j,k=0}^\kappa, \quad m_{j,k} := \frac{y_i}{\sqrt{\text{Im } f(z_{j,l}) \text{Im } f(z_{k,l})}} \frac{f(z_{j,l}) - f(z_{k,l})}{z_{j,l} - z_{k,l}},$$

has the diagonal entries $m_{j,j} = -1$ for all $j$. Due to (3.20) and (3.21), the off-diagonal entries of $M_l$ as $l \to \infty$ are

$$m_{j,k} = \frac{y_i}{z_{j,l} - z_{k,l}} \frac{f(z_{j,l}) - f(z_{k,l})}{\sqrt{\text{Im } f(z_{j,l}) \text{Im } f(z_{k,l})}} = \frac{i}{z_{j,l} - z_{k,l}} \sqrt{\frac{y_i \text{Im } f(z_{j,l})}{\text{Im } f(z_{j,l})}} + o(1) \to 0.$$ 

The ultimate $o(1)$ is due to the estimate

$$\frac{1}{\sqrt{\text{Im } f(z_{j,l}) \text{Im } f(z_{k,l})}} \leq \frac{1}{C}$$

following from (3.21). For $l \to \infty$, the matrix $M_l$ therefore satisfies

$$D \cdot H_f(t_0 + iy_l, \ldots, t_\kappa + iy_l) \cdot D = M_l \to -I_{\kappa+1}, \quad \text{where} \quad D := \text{diag} \left( \sqrt{-\text{Im } f(t_j + iy_l)} \right)_{j=0}^\kappa$$

and $I_{\kappa+1}$ is the $(\kappa + 1) \times (\kappa + 1)$ identity matrix. For $l$ large enough, the left-hand side is negative definite, which contradicts our assumption $f \in \mathcal{N}_\kappa$.

Now, let us prove the sufficiency part. Since $g(z)$ may only have a certain finite number of poles in $\mathcal{I}$ (say $m$), it may be written in the form $g(z) = g_1(z) + g_2(z)$, where

$$g_1(z) := \sum_{k=1}^m \frac{\sigma(t_k^+) - \sigma(t_k^-)}{t_k - z} \quad \text{and} \quad g_2(z) := \nu_1 z + \nu_2 + \int_{\mathbb{R} \setminus \mathcal{I}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma(t),$$

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and \( t_1, \ldots, t_m \in \mathcal{I} \). Let the co-prime polynomials \( p(z) \) and \( q(z) \) denote, respectively, the numerator and the denominator of \( T(z) \), so that \( T(z) = \frac{p(z)}{q(z)} \). Choose arbitrary \( t_* \in \mathcal{I} \), then the function \( T_1(z) := p(z)q(z)(z - t_*)^{-2 \deg T} \) is nonnegative, bounded and analytic on \( \mathbb{R} \setminus \mathcal{I} \). Therefore, the new positive measure \( dt(t) := T_1(t) \, d\sigma(t) \) on \( \mathbb{R} \setminus \mathcal{I} \) is properly defined and satisfies \( \int_{\mathbb{R} \setminus \mathcal{I}} (1 + t^2)^{-1} \, dt(t) < \infty \), so the integral

\[
g_3(z) := \int_{\mathbb{R} \setminus \mathcal{I}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) \, dt(t)
\]
defines a function \( g_3 \in \mathcal{N}_0 \). Therefore,

\[
S_1(z) := T_1(z)g_2(z) - g_3(z) = T_1(z)(\nu_1 z + \nu_2) + \int_{\mathbb{R} \setminus \mathcal{I}} \frac{(1 + tz)(T_1(z) - T_1(t))}{t - z} \, dt(t),
\]

where the integrand is analytic and bounded in \( z \) outside each small disc centred at \( t_* \). So, \( S_1(z) \) is manifestly analytic in \( \mathbb{C} \setminus \{t_*\} \) and real on \( \mathbb{R} \setminus \{t_*\} \). We want to show that it is actually rational. On writing the Laurent expansion of \( T_1(z) \) around \( t_* \) as

\[
T_1(z) = \sum_{j=0}^{2 \deg T} \frac{c_{-j}}{(z - t_*)^j}
\]

with some coefficients \( c_{-2 \deg T}, \ldots, c_0 \in \mathbb{R} \), we see that

\[
\frac{T_1(z) - T_1(t)}{t - z} = \sum_{j=1}^{2 \deg T} \frac{c_{-j} (z - t_*)^{-j} - (t - t_*)^{-j}}{(t - t_*) - (z - t_*)} = \sum_{j=1}^{2 \deg T} j \, |c_{-j}| \frac{1}{|t - t_*|^j - \delta^j} \frac{1}{|t - t_*|}
\]

Put \( \delta := \text{dist}\{t_*, \mathbb{R} \setminus \mathcal{I}\} \), then for \( |t - t_*| \geq \delta \) and \( |z - t_*| \geq \delta \)

\[
\left| \frac{T_1(z) - T_1(t)}{t - z} \right| \leq \sum_{j=1}^{2 \deg T} j \, |c_{-j}| \frac{1}{|t - t_*|^j} = \frac{C}{|t - t_*|^j}, \quad \text{where} \quad C := \sum_{j=1}^{2 \deg T} j \, |c_{-j}| \delta^j.
\]

Therefore,

\[
|S_1(z)| \leq |T_1(z)(\nu_1 z + \nu_2)| + \int_{\mathbb{R} \setminus \mathcal{I}} \frac{C + C|zt|}{|t - t_*|^j} \, dt(t) = O(z) \quad \text{as} \quad z \to \infty.
\]

By definition of \( S_1(z) \), its only finite singularity at the point \( t_* \) is a pole of order \( 2 \deg T \). So, Liouville’s theorem implies that \( S_1(z) \) may only be a real rational function of degree at most \( 2 \deg T + 1 \). Now, for the real rational functions

\[
Q(z) := \frac{(z - t_*)^{\deg T}}{q(z)} \quad \text{and} \quad S(z) := T(z)g_1(z) + Q^2(z)S_1(z)
\]
satisfying \( \deg S \leq (\deg T + m) + (2 \deg T + 1) = 3 \deg T + m + 1 \), we obtain

\[
f(z) = T(z)g(z) = T(z)g_1(z) + Q^2(z)T_1(z)g_2(z) = S(z) + Q^2(z)g_3(z).
\]

It remains to employ a reasoning analogous to [16, Section 3] to show that \( f \in \mathfrak{N} \): given some numbers \( z_1, \ldots, z_n \in \mathbb{C} \), write the entries of the matrix \( H_f(z_1, \ldots, z_n) \) in the following form:

\[
\frac{f(z_i) - f(z_j)}{z_i - z_j} = \frac{S(z_i) - S(z_j)}{z_i - z_j} + \frac{Q(z_i)g_3(z_i)Q(z_j)}{z_i - z_j} - \frac{Q(z_j)g_3(z_j)Q(z_i)}{z_i - z_j} + g_3(z_i)Q(z_j) - Q(z_j)g_3(z_j)
\]

\[
+ g_3(z_i)Q(z_j) - Q(z_i)g_3(z_j) + \frac{Q(z_i) - Q(z_j)}{z_i - z_j} + \frac{Q(z_j) - Q(z_i)}{z_j - z_i}.
\]
The right-hand side is the \((i,j)\)-th entry of a sum of three Hermitian matrices:

\[
H_f(z_1, \ldots, z_n) = H_S(z_1, \ldots, z_n) + D_1 \cdot H_{g_1}(z_1, \ldots, z_n) \cdot D_1^* + \left( D_2 \cdot H_Q(z_1, \ldots, z_n) + (D_2 \cdot H_Q(z_1, \ldots, z_n))^* \right),
\]

where \(D_1, D_2\) are given by \(D_1 = D_1^* = \text{diag} \left( Q(z_j) \right)_{j=1}^n\) and \(D_2 = \text{diag} \left( g_3(z_j)Q(z_j) \right)_{j=1}^n\). The first matrix \(H_S(z_1, \ldots, z_n)\) comes from a real rational function \(S\) and hence its rank is at most \(\deg S\). The second is positive semidefinite, as it is diagonally similar to \(H_{g_1}(z_1, \ldots, z_n)\) and \(g_3 \in \mathcal{N}_0\). The rank of the third matrix is bounded from above by \(2 \deg Q\): it is the Hermitian part of the product \(D_2 \cdot H_Q(z_1, \ldots, z_n)\) whose rank does not exceed \(\deg Q\). By Lemma 3.5, the number of negative eigenvalues of \(H_f(z_1, \ldots, z_n)\) cannot exceed \(\deg S + 2 \deg Q\).

**Corollary 3.21.** Let \(f \in \mathcal{N}\). Then \(f \in \mathcal{S}\) if and only if \(f(z)\) is real and analytic for \(z < 0\) excluding at most finitely many poles.

**Proof.** The corollary follows from Theorem 3.19 by choosing \(T(z) = z\), since \(f \in \mathcal{N}_\lambda\) precisely when \(f(z)\) and \(zf(z)\) belong to \(\mathcal{N}_k\) and \(\mathcal{N}_\lambda\), respectively.

**Corollary 3.22.** Let \(f \in \mathcal{S}\), and let \(h(z)\) and \(g(z)\) be two real rational functions. Then \(hf + g \in \mathcal{S}\) if and only if \(f(z)\) is real and analytic in \(I := \{ t \in \mathbb{R} : -\infty < h(t) < 0 \}\) excluding at most a finite number of poles.

**Proof.** Recall that \(f \in \mathcal{S}\) if and only if \(f, \tilde{f} \in \mathcal{N}\), where \(\tilde{f}(z) := zf(z)\). The function \(\tilde{f}(z)\) has the same singular points in \(I\) as \(f(z)\), possibly excluding the origin. So, by Theorem 3.19, one has both \(hf \in \mathcal{N}\) and \(hf \in \mathcal{N}\) if and only if \(f(z)\) is real and analytic in \(I\) except for at most a finite number of poles.

Now, for any points \(z_1, \ldots, z_n \in \mathbb{C}_+\)

\[
H_{hf + g}(z_1, \ldots, z_n) = H_{hf}(z_1, \ldots, z_n) + H_g(z_1, \ldots, z_n)
\]

and

\[
H_{h\tilde{f} + \tilde{g}}(z_1, \ldots, z_n) = H_{hf}(z_1, \ldots, z_n) + H_{\tilde{g}}(z_1, \ldots, z_n),
\]

where \(\tilde{g}(z) := zg(z)\). Since both matrices \(H_g(z_1, \ldots, z_n)\) and \(H_{\tilde{g}}(z_1, \ldots, z_n)\) have rank \(\leq \deg g + 1\) according to Lemma 3.8, the conditions \(hf + g \in \mathcal{S}\) and \(hf \in \mathcal{S}\) are equivalent.

**Proof of Theorem 3.1.** If \(R_{n_1,n_2,m}(z)\) is rational, then we immediately have both: the equality in (3.1) in view of (2.21a) and \(\pm R_{n_1,n_2,m} \in \mathcal{S}\) by Lemma 3.8. Thus, it is enough only to consider the case when \(R_{n_1,n_2,m}(z)\) is not rational. Under this assumption we will prove both directions of the theorem simultaneously.

Choose the integer \(\delta \geq 0\) to be large enough so that

\[
0 \leq a + \delta \leq c + 2\delta \quad \text{and} \quad 0 \leq b + \delta + 1 \leq c + 2\delta.
\]

Then the coefficients (1.3) of the continued fraction (1.2) corresponding to

\[
\tilde{R}_{0,1,1}(z) := \frac{\beta F_1(a + \delta, b + \delta + 1; c + 2\delta + 1; z)}{\beta F_1(a + \delta, b + \delta; c + 2\delta; z)}
\]

are all strictly positive, and hence \(\tilde{R}_{0,1,1}(z)\) is not rational. Moreover, Theorem 1.1 implies \(\tilde{R}_{0,1,1} \in \mathcal{S} \subset \mathcal{S}\). The condition (IV) of Theorem 2.1 is satisfied, so \(\tilde{R}_{0,1,1}(x + i0)\) is continuous for all \(x > 1\) and, according to Theorem 2.11 and Example 1 below, has strictly positive imaginary part.\(^6\)

\(^6\)One can avoid Theorem 2.1 here by invoking Theorem 1.1 (d).
Given shifts \( n_1, n_2, m \in \mathbb{Z} \) and \( \delta \in \mathbb{N}_0 \), there exist (see [25, pp. 130–133] or [22, eq. (1.1)]) nontrivial real coprime polynomials \( p_\delta(z) \), \( q_\delta(z) \) and \( r_\delta(z) \) in \( z \) (as well as in \( a, b, c \)) such that

\[
p_\delta(z) \frac{2F_1(a + n_1, b + n_2; c + m; z)}{2F_1(a + \delta, b + \delta; c + 2\delta; z)} = q_\delta(z) \tilde{R}_{0,1,1}(z) + r_\delta(z).
\]

(3.22)

In our case, the numbers \( a, b, c \) are fixed. Since \( \tilde{R}_{0,1,1}(z) \) is non-rational for our choice of \( \delta \), the polynomial \( p_\delta(z) \) cannot vanish identically.

The ratio of two version of the equality (3.22) – the second one with \( n_1 = n_2 = m = 0 \) and polynomials \( \tilde{p}_\delta(z) \equiv 0 \), \( \tilde{q}_\delta(z) \) and \( \tilde{r}_\delta(z) \) – may be written as

\[
\frac{p_\delta(z)}{p_\delta(z)} R_{n_1, n_2, m}(z) = \frac{q_\delta(z) \tilde{R}_{0,1,1}(z) + r_\delta(z)}{q_\delta(z) \tilde{R}_{0,1,1}(z) + \tilde{r}_\delta(z)}.
\]

(3.23)

If \( \tilde{q}_\delta(z) \equiv 0 \), then we obtain

\[
R_{n_1, n_2, m}(z) = \tilde{h}(z) \tilde{R}_{0,1,1}(z) + \tilde{g}(z)
\]

(3.24)

with real rational functions \( \tilde{h}(z) \) and \( \tilde{g}(z) \). Here \( \tilde{h}(z) \not\equiv 0 \), since \( R_{n_1, n_2, m}(z) \) is non-rational. Then for all \( x > 1 \) excluding zeros and poles of \( \tilde{h}(x) \),

\[
0 < \text{Im}[\tilde{R}_{0,1,1}(x + i0)] = \text{Im} \frac{R_{n_1, n_2, m}(x + i0)}{\tilde{h}(x)} = \frac{\pi B_{n_1, n_2, m} P_r(1/x)x^{\frac{c-a-b-l}{2}}}{\tilde{h}(x) \cdot |2F_1(a, b; c; x)|^2},
\]

(3.25)

where the last equality is (2.21a). The inequality (3.1) holds if and only if \( \tilde{h}(x) > 0 \) for all \( x > 1 \) excluding possible poles and zeros. The latter is equivalent to \( R_{n_1, n_2, m} \in \mathcal{S} \) due to (3.24) and Corollary 3.22.

If \( \tilde{q}_\delta(z) \not\equiv 0 \), then (3.23) yields

\[
R_{n_1, n_2, m}(z) = \frac{\tilde{p}_\delta(z)}{p_\delta(z)} \left( \frac{r_\delta(z) - \tilde{r}_\delta(z) q_\delta(z) / \tilde{q}_\delta(z)}{\tilde{q}_\delta(z) \tilde{R}_{0,1,1}(z) + \tilde{r}_\delta(z)} \right) = \frac{-h(z)}{\tilde{R}_{0,1,1}(z) + g(z)} + g_1(z)
\]

(3.26)

for certain real rational functions \( h(z), g(z) \) and \( g_1(z) \): note that \( h(z) \not\equiv 0 \), otherwise \( R_{n_1, n_2, m}(z) \) would be rational. Analogously to (3.25) we have

\[
0 < \left| \frac{\text{Im} \tilde{R}_{0,1,1}(x + i0)}{\tilde{R}_{0,1,1}(x + i0) + g(x)} \right|^2 = \text{Im} \frac{-1}{\tilde{R}_{0,1,1}(x + i0) + g(x)} = \frac{\pi B_{n_1, n_2, m} P_r(1/x)x^{\frac{c-a-b-l}{2}}}{\tilde{h}(x) \cdot |2F_1(a, b; c; x)|^2}
\]

for all \( x > 1 \) excluding possible poles and zeros of \( \tilde{h}(x) \). Consequently, \( h(x) > 0 \) if and only if \( B_{n_1, n_2, m} P_r(1/x) > 0 \); the latter inequality is equivalent to (3.1) combined with \( B_{n_1, n_2, m} P_r(x) \not\equiv 0 \).

From (3.26) we obtain

\[
\tilde{R}_{0,1,1} \in \mathcal{S} \iff \tilde{R}_{0,1,1} + g \in \mathcal{S} \iff R_{n_1, n_2, m} - g_1 \in \mathcal{S} \iff R_{n_1, n_2, m} \in \mathcal{S}
\]

by applying Lemma 3.5, then Lemma 3.7 with Corollary 3.22, and finally Lemma 3.5 again.

---

\(^7\)Under ‘nontrivial’ polynomials we understand that one cannot simultaneously have \( p_\delta(z) \), \( q_\delta(z) \) and \( r_\delta(z) \) identically equal to zero (as polynomials in \( z \)). The universal choice \( \delta = 0 \) would shorten the proof, but then it would be possible that \( \tilde{R}_{0,1,1}(z) = R_{0,1,1}(z) \) is rational. The latter would lead to the undesired case \( p_\delta(z) \equiv 0 \) and \( q_\delta(z), r_\delta(z) \not\equiv 0 \), since \( R_{n_1, n_2, m}(z) \) is not rational.
Proof of Theorem 3.3. Suppose that \( R_{n_1,n_2,m}(z) \) is not rational, since the case when it is rational reduces to Theorem 3.1. In the proof of Theorem 3.1, for the case \( \tilde{q}_d(z) \equiv 0 \) identity (3.24) yields

\[
B_{n_1,n_2,m}P_r(1/z)R_{n_1,n_2,m}(z) = \tilde{h}(z)B_{n_1,n_2,m}P_r(1/z)\tilde{R}_{0,1,1}(z) + B_{n_1,n_2,m}P_r(1/z)\tilde{g}(z).
\]

The right-hand side is in \( \mathfrak{S} \subset \mathfrak{N} \) due to \( \tilde{h}(z)B_{n_1,n_2,m}P_r(1/z) > 0 \) for all \( z > 1 \) except for poles and zeros. In particular, the product

\[
f(z) := B_{n_1,n_2,m}P_r\left(1/(z + \omega)\right)R_{n_1,n_2,m}(z + \omega)
\]

lies in the class \( \mathfrak{N} \) whose definition does not depend on real shifts. According to the ultimate claim of Theorem 2.1, the function \( f(z) \) is real and has at most finitely many zeros and poles in \( z < 1 - \omega \). So, Corollary 3.22 implies that in fact \( f \in \mathfrak{S} \). The same conclusion \( f \in \mathfrak{S} \) holds for \( \tilde{q}_d(z) \neq 0 \): it is enough to use (3.26) and Lemma 3.6 instead of (3.24).

Now, note that

\[
g(z) := \frac{R_{n_1,n_2,m}(z + \omega)}{B_{n_1,n_2,m}P_r(1/(z + \omega))} = f(z)\left[\frac{B_{n_1,n_2,m}P_r(1/z)}{B_{n_1,n_2,m}P_r(1/(z + \omega))}\right]^{-2},
\]

and therefore \( f \in \mathfrak{S} \) implies \( g \in \mathfrak{S} \) by Theorem 3.19. \( \square \)

4 Examples

Recall that \( R_{n_1,n_2,m}(z) \) is defined in (1.4). Below we apply our main theorems to 15 specific triples \( n_1, n_2, m \). The resulting integral representations are only valid if \( R_{n_1,n_2,m}(z) \) is well-behaved near \( z = 1 \) and its denominator \( {}_2F_1(a, b; c; z) \neq 0 \) in the cut plane \( \mathbb{C} \setminus [1, +\infty) \) and on the banks of the branch cut. Conditions for the latter are given in Theorem 2.1, while the former in ensured by the inequality \( \nu > -1 \) with \( \nu \) defined in (2.8). To relax these restrictions, one needs a kind of regularization near the point \( z = 1 \), as well as near all zeros of the denominator. We plan to tackle these issues in part II of our work.

Example 1. For the Gauss ratio \( R_{0,1,1}(z) \) according to (2.18) we obtain \( p = l = r = 0 \). Theorem 2.8 and definition (2.21b) yield:

\[
B_{0,1,1}P_0(t) \equiv \frac{\Gamma(c)\Gamma(c + 1)}{\Gamma(a)\Gamma(b + 1)\Gamma(c - a + 1)\Gamma(c - b)}.
\]

Next, using (2.11) and (2.13) or directly it is easy to verify that

\[
Q_{a,b,c} = \lim_{z \to \infty} R_{0,1,1}(z) = \begin{cases} 0, \quad b \leq a \\ \frac{c(b - a)}{|b(c - a)|}, \quad b > a. \end{cases}
\]

Then Theorem 2.12 with \( N = 0 \) yields:

\[
R_{0,1,1}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c + 1)}{\Gamma(a)\Gamma(b + 1)\Gamma(c - b)\Gamma(c - a + 1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}}{(1-zt)^2} {}_2F_1(a, b; c; t^{-1}) dt.
\]

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction the condition \( \nu > -1 \) from Theorem 2.12 holds automatically since the parameter \( q = m - n_1 - n_2 \) in Lemma 2.5 vanishes, so that \( R_{0,1,1}(z) \) is integrable in the neighborhood of 1. We remark that the integrand is symmetric with respect to
the interchange of $a$ and $b$ and the asymmetry of $R_{0,1,1}(z)$ is only reflected in the constants $Q_{a,b,c}$ and $B_{0,1,1}$.

The above integral representation was first found by V. Belevitch in [10, formula (72)] under the restrictions $0 \leq a, b, c$, $c \geq 1$ (there is a small mistake in Belevitch’s paper - a superfluous 2 in the denominator of the constant $Q_{a,b,c}$). Independently, using the Gauss continued fraction (1.2) and Wall’s theorem Küstner [30, Theorem 1.5] proved that $R_{0,1,1}(z)$ is a generating function of a Hausdorff moment sequence if $0 < a \leq c + 1$, $0 < b \leq c$. As we mentioned in introduction, the coefficients of the Gauss continued fraction (1.2) for $R_{0,1,1}(z)$ are all positive if $(a) -1 < a < 0$ and either $-1 < b < c < 0$ or $0 < c < b < c + 1$ or $(b) 0 < a < c + 1$, $c > 0$ and $-1 < b < c$. If these conditions hold while conditions of Runckel’s theorem 2.1 are violated, then Theorem 1.1(c) implies that representation (1.7) is true while the above integral representation is not. Hence, in this situation $R_{0,1,1}(z)$ has pole(s) in the interval $(0,1)$ which are reflected by the atoms of the representing measure in (1.7). This is the case, for instance, if $0 < c < a < c + 1$ and $-1 < b < 0$.

For all $\omega \leq 1$ and $a, b, c \in \mathbb{R}$ such that $-\omega \notin \mathbb{N}_0$, Theorem 3.3 implies that $B_{0,1,1}P_0 R_{0,1,1}(z + \omega)$ belongs to $\mathcal{S}$; the case $\omega = 0$ is considered in detail in Theorem 3.4.

**Example 2.** For the ratio $R_{0,1,0}(z)$ according to (2.18) we obtain $l = 1$, $p = r = 0$. Theorem 2.8 and definition (2.21b) yield:

$$B_{0,1,0} P_0(t) = \frac{[\Gamma(c)]^2}{\Gamma(a) \Gamma(b+1) \Gamma(c-a) \Gamma(c-b)}.$$  

Next, using (2.11) and (2.13) or directly we can verify that

$$Q_{a,b} = \lim_{z \to \infty} R_{0,1,0}(z) = \begin{cases} 
0, & b \leq a \\
(b-a)/b, & b > a.
\end{cases}$$

Then Theorem 2.12 with $N = 0$ yields:

$$R_{0,1,0}(z) = Q_{a,b} + \frac{[\Gamma(c)]^2}{\Gamma(a) \Gamma(b+1) \Gamma(c-a) \Gamma(c-b)} \int_0^1 \frac{t^{a-b-1}(1-t)^{c-a-b-1} dt}{(1-zt)^2 F_1(a,b;c;1/t)^2}.$$  

Note that similarly to Example 1, the integrand is symmetric with respect to the interchange of $a$ and $b$ and the asymmetry of the left hand side is only reflected in the constants. In order for this representation to hold, we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases $ab = 0$, $(c-a)(c-b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$(c-a-b-1)_- - (c-a-b)_- > -1,$$

which is easily seen to be equivalent to $c > a + b$. The above set of conditions holds, for example, if $-1 < a < 0$ and $0 < b < c$ or $a > 0$ and $-1 < b < c - a$.

Using continued fractions Küstner [30, Theorem 1.5] proved that $R_{0,1,0}(z)$ is a generating function of a Hausdorff moment sequence if $-1 \leq b \leq c$ and $0 < a < c$. Askitis [5, Lemma 6.2.2] found another proof for the this claim (without use of continued fractions). We also remark that the continued fraction for $R_{0,1,0}$ was also found by Gauss, see [25, eq. 26] or [30, eq. (2.7)], in the form

$$\frac{1}{1 - \frac{\alpha_1 z}{1 - \frac{\alpha_2 z}{1 - \ldots}}},$$
Furthermore, shows how yields we obtain:

\[ \alpha_{2k} = \frac{(b + k)(c - a + k - 1)}{(c + 2k - 2)(c + 2k - 1)}, \quad \alpha_{2k+1} = \frac{(a + k)(c - b + k - 1)}{(c + 2k - 1)(c + 2k)}. \]

From these formulae, it is not difficult to formulate sufficient conditions for \( \alpha_n \geq 0 \) ensuring that \( R_{0,1,0} \in \mathcal{S} \) (the Stieltjes class). For general values of \( a, b, c \in \mathbb{R}, -c \notin \mathbb{N}_0 \), Theorem 3.3 yields that \( B_{0,1,0}P_0R_{0,1,0}(z + \omega) \) belongs to \( \mathcal{G} \) whenever \( \omega \leq 1 \). In particular, \( B_{0,1,0}P_0R_{0,1,0} \in \mathcal{N}_{\alpha}^\kappa \) for non-rational \( R_{0,1,0}(z) \) where the indices \( \kappa, \lambda \) are computed in Theorem 3.16; the case of rational \( \pm R_{0,1,0} \) is treated in Corollary 3.15.

**Example 3.** For the ratio \( R_{1,1,1}(z) \) according to (2.18) we obtain \( l = 1, p = r = 0 \). Theorem 2.8 and definition (2.21b) yield:

\[ B_{1,1,1}P_0(t) = \frac{\Gamma(c)\Gamma(c + 1)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a)\Gamma(c - b)} \]

Next, it is easy to verify using (2.11) and (2.13) or directly that

\[ Q_{a,b,c} = \lim_{z \to \infty} R_{1,1,1}(z) = 0. \]

Then according to the case \( N = 0 \) of Theorem 2.12 we obtain:

\[ R_{1,1,1}(z) = \frac{\Gamma(c)\Gamma(c + 1)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a)\Gamma(c - b)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b-1}dt}{(1-zt)|2F_1(a, b; c; 1/t)|^2}. \]

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases \( ab = 0 \), \( (c - a)(c - b) = 0 \) the condition \( \nu > -1 \) from (2.8) reads

\[ (c - a - b - 1)_+ - (c - a - b)_- > -1, \]

which is easily seen to be equivalent to \( c > a + b \). All these conditions are satisfied, for example, if \( a \) \(-1 < a < 0 \) and \( 0 < b < c \) or \( b \) \( 0 < a < c \) and \(-1 < b < c - a \).

In this case, the corresponding C-fraction is not as simple as in the previous two examples. Instead of trying to write it directly, let us employ the following contiguous relation ([25, eq. 17] or [30, eq. (2.2)]):

\[ R_{1,1,1}(z) = \frac{2F_1(a + 1, b + 1; c + 1; z)}{2F_1(a, b; c; z)} = \frac{c}{az} \left( \frac{2F_1(a, b + 1; c; z)}{2F_1(a, b; c; z)} - 1 \right) = \frac{c}{az} (R_{0,1,0}(z) - 1). \]

Since \( R_{1,1,1}(z) \) is analytic near the origin and only has finitely many poles for \( z < 1 \), we obtain

\[ \frac{a}{c} R_{1,1,1} \in \mathcal{N}_\kappa^\lambda \iff R_{0,1,0} \in \mathcal{N}_{\kappa+\delta}^\lambda \quad \text{and} \quad -\frac{a}{c} R_{1,1,1} \in \mathcal{N}_\kappa^\lambda \iff -R_{0,1,0} \in \mathcal{N}_{\kappa+\delta}^\lambda, \]

where \( \delta \in \{-1, 0, 1\} \) is a certain number depending on the asymptotics at infinity.\(^8\) Furthermore, Theorem 3.3 yields that \( B_{1,1,1}P_0R_{1,1,1}(z + \omega) \) belongs to \( \mathcal{G} \) whenever \( \omega \leq 1 \).

---

\(^8\)By Theorem 3.18, \( \frac{a}{c} R_{1,1,1} \in \mathcal{N}_\kappa^\lambda \) implies that \( \frac{a}{c} z^2 R_{1,1,1}(z) \) lies in \( \mathcal{N}_{\kappa-1} \cup \mathcal{N}_\kappa \cup \mathcal{N}_{\kappa+1} \). Lemma 3.5 shows how the number of negative eigenvalues of the Pick matrix changes when a function is increased by \( z \) or by \( 1/z \). In particular, \( z R_{0,1,0}(z) = \frac{a}{c} z^2 R_{1,1,1}(z) + z \) belongs to \( \mathcal{N}_{\kappa-1} \cup \mathcal{N}_\kappa \cup \mathcal{N}_{\kappa+1} \). To obtain the exact value of \( \delta \) and the converse implication, one can check what happens with the \textit{generalized pole of nonpositive type} of \( R_{1,1,1}(z) \) at infinity (if any), since its generalized poles at finite points remain unaffected, see the definition and further details in [16, 18].
In particular, for \(-1 \leq a \leq c\) and \(0 < b \leq c\) Küstner proved [30, Theorem 1.5] that the ratio \(R_{1,1,1}(z)\) is the generating function of a Hausdorff moment sequence, which is equivalent to \(z \mapsto R_{1,1,1}(z + \omega) \in S = N_0^\ast\) for each \(\omega \leq 1\).

**Example 4.** For the ratio \(R_{1,1,2}(z)\) according to (2.18) we obtain \(l = p = r = 0\). Theorem 2.8 and definition (2.21b) yield:

\[
B_{1,1,2}P_0(t) = B_{1,1,2}P_0 = \frac{\Gamma(c + 1)\Gamma(c + 2)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a + 1)\Gamma(c - b + 1)}.
\]

Next, it is easy to verify using (2.11) and (2.13) or directly that

\[
Q_{a,b,c} = \lim_{z \to \infty} R_{1,1,2}(z) = 0.
\]

Then according to the case \(N = 0\) of Theorem 2.12 we obtain:

\[
R_{1,1,2}(z) = \frac{\Gamma(c + 1)\Gamma(c + 2)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a + 1)\Gamma(c - b + 1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{(1-zt)^2} dt.
\]

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases \(ab = 0, (c - a)(c - b) = 0\) the condition \(\nu > -1\) from (2.8) reads

\[(c - a - b)_+ - (c - a - b)_- > -1\]

and is trivially satisfied.

For general values of \(a, b, c \in \mathbb{R}\) such that \(-c \notin \mathbb{N}_0\), we again employ a contiguous relation (see [25, eq. 19] or [35, p. 122 eq. (2)]) instead of looking for the corresponding C-fraction:

\[
R_{1,1,2}(z) = \frac{\binom{2}{2}F_1(a + 1, b + 1; c + 2; z)}{\binom{2}{2}F_1(a, b; c, z)} = \frac{c(c + 1)}{a(c - b)z} \left(\frac{2F_1(a + 1, b + 1; c + 1; z)}{2F_1(a, b; c, z)} - 1\right).
\]

Since \(R_{1,1,2}(z)\) is analytic near the origin and only has finitely many poles for \(z < 1\), we obtain

\[
\frac{a(c - b)}{c(c + 1)}R_{1,1,2} \in \mathcal{N}_\kappa^\lambda \iff R_{0,1,1} \in \mathcal{N}_\kappa^\lambda + \delta \quad \text{and} \quad -\frac{a(c - b)}{c(c + 1)}R_{1,1,2} \in \mathcal{N}_\kappa^\lambda \iff -R_{0,1,1} \in \mathcal{N}_\kappa^\lambda + \delta,
\]

where \(\delta \in \{-1, 0, 1\}\) is a certain number depending on the asymptotics at infinity, cf. Example 3. This relation for \(a(c - b) \neq 0\) refines what Theorem 3.1 states:

\[
B_{1,1,2}P_0 \geq 0 \implies R_{1,1,2} \in \mathcal{G}, \quad B_{1,1,2}P_0 \leq 0 \implies -R_{1,1,2} \in \mathcal{G}.
\]

However, the latter also holds for \(R_{1,1,2}(z + \omega)\) for all \(\omega \leq 1\) due to Theorem 3.3.

**Example 5.** For the ratio \(R_{0,2,2}(z)\) according to (2.18) we obtain \(l = p = 0, r = 1\). Theorem 2.8 and definition (2.21b) yield:

\[
B_{0,2,2}P_1(t) = \frac{\Gamma(c)\Gamma(c + 2)(ct + b - a + 1)}{\Gamma(a)\Gamma(b + 2)\Gamma(c - a + 2)\Gamma(c - b + 2)}.
\]

Next, it is easy to verify using (2.11) and (2.13) or directly that

\[
Q_{a,b,c} = \lim_{z \to \infty} R_{0,2,2}(z) = \begin{cases} 0, & b \leq a \\ \frac{c(c + 1)(b - a)(b - a + 1)/[b(b + 1)(c - a)(c - a + 1)],} & b > a. \end{cases}
\]
is satisfied. Under this restriction and except for the degenerate cases \( ab = 0 \), \((c - a)(c - b) = 0\) the condition \( \nu > -1 \) from (2.8) reads
\[(c - a - b)_- - (c - a - b)_- > -1\]
and is trivially satisfied.

Further, \( B_{0,2,2}P_1(t) \) does not change sign on \((0, 1)\) if the zero \( t_* = (a - b - 1)/c \) of the polynomial \( P_1(t) = ct + b - a + 1 \) does not lie in \((0, 1)\), which is the case if \( a \leq b + 1 \) or \( a \geq c + b + 1 \). If this case, according to Theorem 3.1 \( R_{0,2,2} \in \mathcal{O} \) if \( B_{0,2,2}P_1(t) \geq 0 \) on \((0, 1)\) or \(-R_{0,2,2} \in \mathcal{O}\) if \( B_{0,2,2}P_1(t) \leq 0 \) on \((0, 1)\). Furthermore, according to Theorem 3.3 the functions
\[
B_{0,2,2}P_1 \left( \frac{1}{z + \omega} \right) R_{0,2,2}(z + \omega) \quad \text{and} \quad \frac{R_{0,2,2}(z + \omega)}{B_{0,2,2}P_1(1/(z + \omega))}
\]
lie in \( \mathcal{O} \) for all real values of parameters \( a, b, c, \omega \) such that \(-c \not\in \mathbb{N}_0 \) and \( \omega \leq 1 \).

**Example 6.** For the ratio \( R_{0,2,0}(z) \) according to (2.18) we obtain \( p = 0 \), \( l = 2 \), \( r = 1 \). Theorem 2.8 and definition (2.21b) yield:
\[
B_{0,2,0}P_1(t) = \frac{[\Gamma(c)]^2 (t(c - 2b - 2) + b + 1 - a)}{\Gamma(a)\Gamma(b + 2)\Gamma(c - a)\Gamma(c - b)}.\]

Next, it is easy to verify using (2.11) and (2.13) or directly that
\[
Q_{a,b} = \lim_{z \to \infty} R_{0,2,0}(z) = \begin{cases} 0, & b \leq a \\ (b-a)(b-a+1)/[b(b+1)], & b > a. \end{cases}
\]

Then according to the case \( N = 0 \) of Theorem 2.12 we obtain:
\[
R_{0,2,0}(z) = Q_{a,b} + \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b + 2)\Gamma(c - a)\Gamma(c - b)} \left[ 1_{\Gamma}^{a+b-1}(b - a + 1 + t(c - 2b - 2))(1 - t)^{c - a - b - 2} dt \right]_{0}^{1 - z} 2F_1(a, b; c; 1/t)^2.
\]

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases \( ab = 0 \), \((c - a)(c - b) = 0\) the condition \( \nu > -1 \) from (2.8) reads
\[(c - a - b)_- - (c - a - b)_- > -1\]
which is easily seen to be equivalent to \( c > a + b + 1 \).

Further, \( B_{0,2,0}P_1(t) \) does not change sign on \((0, 1)\) if the zero \( t_* = (a - b - 1)/(c - 2b - 2) \) of the polynomial \( P_1(t) = t(2b + 2 - c) + a - b - 1 \) does not lie in \((0, 1)\), which is the case if \( a \leq b + 1 \) or \( a + b + 1 \geq c \). If this case, according to Theorem 3.1 \( R_{0,2,0} \in \mathcal{O} \) if \( B_{0,2,0}P_1(t) \geq 0 \) on \((0, 1)\) or \(-R_{0,2,0} \in \mathcal{O}\) if \( B_{0,2,0}P_1(t) \leq 0 \) on \((0, 1)\). Furthermore, according to Theorem 3.3 the functions
\[
B_{0,2,0}P_1 \left( \frac{1}{z + \omega} \right) R_{0,2,0}(z + \omega) \quad \text{and} \quad \frac{R_{0,2,0}(z + \omega)}{B_{0,2,0}P_1(1/(z + \omega))}
\]
lie in $\mathcal{S}$ for all real values of parameters $a, b, c, \omega$ such that $-c \notin \mathbb{N}_0$ and $\omega \leq 1$.

**Example 7.** For the ratio $R_{1,1,0}(z)$ according to (2.18) we obtain $p = 0$, $l = 2$, $r = 0$. Theorem 2.8 and definition (2.21b) yield:

$$B_{1,1,0}\rho_0(t) = \frac{[\Gamma(c)]^2(c - a - b - 1)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a)\Gamma(c - b)}.$$

Next, it is easy to verify using (2.11) and (2.13) or directly that

$$Q_{a,b,c} = \lim_{z \to \infty} R_{1,1,0}(z) = 0.$$

Then according to the case $N = 0$ of Theorem 2.12 we obtain:

$$R_{1,1,0}(z) = -\frac{[\Gamma(c)]^2(c - a - b - 1)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a)\Gamma(c - b)} \int_0^1 \frac{t^{a+b}(1 - t)^{c-a-b-2}dt}{(1 - zt)^2 F_1(a, b; c; 1/t)}.$$

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases $ab = 0$, $(c - a)(c - b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$(c - a - b - 2) - (c - a - b) > -1$$

which is easily seen to be equivalent to $c > a + b + 1$.

Further, $B_{1,1,0}\rho_0(t) = B_{1,1,0}\rho_0$ does not depend on $t$, and thus does not change sign on $(0, 1)$. Hence, according to Theorem 3.1 $R_{1,1,0} \in \mathcal{S}$ if $B_{1,1,0}\rho_0(t) \geq 0$ on $(0, 1)$ or $-R_{1,1,0} \in \mathcal{S}$ if $B_{1,1,0}\rho_0(t) \leq 0$ on $(0, 1)$. Furthermore, according to Theorem 3.3 the function $B_{1,1,0}\rho_0 R_{1,1,0}(z + \omega)$ lies in $\mathcal{S}$ for all real values of parameters $a, b, c, \omega$ such that $-c \notin \mathbb{N}_0$ and $\omega \leq 1$.

**Example 8.** For the ratio $R_{0,0,1}(z)$ according to (2.18) we obtain $p = 1$, $l = r = 0$. Theorem 2.8 and definition (2.21b) yield:

$$B_{0,0,1}\rho_0(t) = \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)}.$$

Next, it is easy to verify using (2.11) and (2.13) or directly that

$$Q_{a,b,c} = \lim_{z \to \infty} R_{0,0,1}(z) = \begin{cases} c/(c-b), & b \leq a \\ c/(c-a), & b > a. \end{cases}$$

Then the case $N = 0$ of Theorem 2.12 leads to the representation:

$$R_{0,0,1}(z) = Q_{a,b,c} - \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(1 - t)^{c-a-b}dt}{(1 - zt)^2 F_1(a, b; c; 1/t)}.$$

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases $ab = 0$, $(c - a)(c - b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$(c - a - b + 1) - (c - a - b) > -1$$

which is easily seen to be satisfied for all real $a, b, c$.
For general values of $a, b, c \in \mathbb{R}$ such that $-c \notin \mathbb{N}_0$ and $b \neq 0$, we can use the contiguous relation [25, eq. 12] instead of calculating the corresponding C-fraction directly:

$$\frac{c-b}{b} R_{0,0,1}(z) = \frac{c-b}{b} 2F_1(a,b;c+1;z) = \frac{c}{b} - \frac{2F_1(a,b+1;c+1;z)}{2F_1(a,b;c;z)} = \frac{c}{b} - R_{0,1,1}(z).$$

Since $R_{0,1,1}(z)$ is analytic near the origin and only has finitely many poles for $z < 1$, we obtain

$$-\frac{c-b}{b} R_{0,0,1} \in \mathcal{N}_{\kappa}^\lambda \iff R_{0,1,1} \in \mathcal{N}_{\kappa}^{\lambda+\delta} \quad \text{and} \quad \frac{c-b}{b} R_{0,0,1} \in \mathcal{N}_{\kappa}^\lambda \iff -R_{0,1,1} \in \mathcal{N}_{\kappa}^{\lambda+\delta}$$

for certain $\delta \in \{-1, 0, 1\}$ depending on $c/b$ and the asymptotics of $R_{0,0,1}(z)$ at infinity, cf. Example 3.

This relation for $b \neq 0$ refines what Theorem 3.1 states:

$$B_{0,0,1} P_0 \geq 0 \iff R_{0,0,1} \in \mathcal{G}, \quad B_{0,0,1} P_0 \leq 0 \iff -R_{0,0,1} \in \mathcal{G}. $$

However, the latter also holds for $R_{0,0,1}(z + \omega)$ due to Theorem 3.3.

**Example 9.** For the ratio $R_{0,0,-1}(z)$ according to (2.18) we obtain $l = 1, p = r = 0$. Theorem 2.8 and definition (2.21b) then yield:

$$B_{0,0,-1} P_0(t) = \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}$$

Next, it is easy to verify using (2.11) and (2.13) or directly that

$$Q_{a,b,c} = \lim_{z \to \infty} R_{0,0,-1}(z) = \left\{ \begin{array}{ll} (c-b-1)/(c-1), & b \leq a \\ (c-a-1)/(c-1), & b > a. \end{array} \right.$$ 

Then the case $N = 0$ of Theorem 2.12 leads to the representation:

$$R_{0,0,-1}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b-1}dt}{(1-zt)^{2F_1(a,b;c;1/t)}}.$$ 

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases $ab = 0, (c-a)(c-b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$(c-a-b-1)_- - (c-a-b)_+ > -1$$

which is easily seen to be equivalent to $c > a + b$. All these conditions are satisfied, for example, if (a) $-1 < a < 0$ and $0 < b < c-a$ or (b) $0 < a < c$ and $-1 < b < c-a$.

For general values of $a, b, c \in \mathbb{R}$ such that $-c \notin \mathbb{N}_0$, one may express $R_{0,0,-1}(z)$ via $R_{0,1,0}(z)$ using the contiguous relation [25, eq. 12]:

$$\frac{c-1}{b} R_{0,0,-1}(z) = \frac{c-b-1}{b} + R_{0,1,0}(z)$$

and further apply Example 2. Another option is to relate to the case considered in Example 8:

$$\frac{1}{R_{0,0,-1}(z)} = \frac{2F_1(a,b;c;z)}{2F_1(a,b;c-1;z)} =: \tilde{R}_{0,0,1}(z),$$

here the right-hand side is $R_{0,0,1}(z)$ with $c$ decreased by 1. Therefore, from Lemma 3.7 and Theorem 3.18 we get

$$-R_{0,0,-1} \in \mathcal{N}_{\kappa}^\lambda \iff \tilde{R}_{0,0,1} \in \mathcal{N}_{\kappa}^{\lambda+\delta} \quad \text{and} \quad R_{0,0,-1} \in \mathcal{N}_{\kappa}^\lambda \iff -\tilde{R}_{0,0,1} \in \mathcal{N}_{\kappa}^{\lambda+\delta}$$

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for certain $\delta \in \{-1, 0, 1\}$ depending on the asymptotics at infinity, cf. Example 3. By Theorem 3.3, $B_{0,0,-1}P_0 \geq 0$ implies that $R_{0,0,-1}(z+\omega)$ is in $\mathcal{S}$, and the opposite inequality $B_{0,0,-1}P_0 \leq 0$ implies that $-R_{0,0,-1}(z+\omega)$ is in $\mathcal{S}$ for all $\omega \leq 1$.

**Example 10.** For the ratio $R_{0,0.2}(z)$ according to (2.18) we obtain $p = 2$, $l = 0$, $r = 1$. Application of Theorem 2.8 and definition (2.21b) yield:

$$B_{0,0.2}P_1(t) = \frac{\Gamma(c)\Gamma(c+2)[ct+a+b-2c-1]}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)},$$

Next, it is easy to verify using (2.11) and (2.13) or directly that

$$Q_{a,b,c} = \lim_{z \to \infty} R_{0,0.2}(z) = \begin{cases} c(c+1)/[(c-b)(c-b+1)], & b \leq a \\ c(c+1)/[(c-a)(c-a+1)], & b > a. \end{cases}$$

Then the case $N = 0$ of Theorem 2.12 leads to the representation:

$$R_{0,0.2}(z) = Q_{a,b,c} + \frac{\Gamma(c)\Gamma(c+2)}{\Gamma(a)\Gamma(b)\Gamma(c-a+2)\Gamma(c-b+2)} \int_0^1 t^{a+b-1}(ct+a+b-2c-1)(1-t)^{c-a-b}dt.$$ 

In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases $ab = 0$, $(c-a)(c-b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$(c-a+b-2) - (c-a-b) > -1$$

which is true for all real $a, b, c$.

Further, $B_{0,0.2}P_1(t)$ does not change sign on $(0,1)$ if the zero $t_* = (2c+1-a-b)/c$ of the polynomial $P_1(t) = 2c+1-a-b-ct$ does not lie in $(0,1)$, which is the case if $c > 0$ and either $2c \leq a+b-1$ or $c \geq a+b-1$; or $c < 0$ and either $2c \geq a+b-1$ or $c \geq a+b-1$. If this case, according to Theorem 3.1 $R_{0,0.2} \in \mathcal{S}$ if $B_{0,0.2}P_1(t) \geq 0$ on $(0,1)$ or $-R_{0,0.2} \in \mathcal{S}$ if $B_{0,0.2}P_1(t) \leq 0$ on $(0,1)$. Furthermore, according to Theorem 3.3 the functions

$$B_{0,0.2}P_1\left(\frac{1}{z+\omega}\right)R_{0,0.2}(z+\omega) \quad \text{and} \quad \frac{R_{0,0.2}(z+\omega)}{B_{0,0.2}P_1(1/(z+\omega))}$$

lie in $\mathcal{S}$ for all real values of parameters $a, b, c, \omega$ such that $-c \notin \mathbb{N}_0$ and $\omega \leq 1$.

**Example 11.** For the ratio $R_{0,1.2}(z)$ according to (2.18) we obtain $p = 1$, $l = 0$, $r = 1$. Theorem 2.8 and definition (2.21b) yield:

$$B_{0,1.2}P_1(t) = -\frac{\Gamma(c)\Gamma(c+2)(ct+b-c)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+2)\Gamma(c+b+1)}.$$

Next, it is easy to verify using (2.11) and (2.13) or directly that

$$Q_{a,b,c} = \lim_{z \to \infty} R_{0,1.2}(z) = \begin{cases} 0, & b \leq a \\ c(c+1)(b-a)/[b(c-a)(c-a+1)], & b > a. \end{cases}$$

Then the case $N = 0$ of Theorem 2.12 leads to the representation:

$$R_{0,1.2}(z) = Q_{a,b,c} - \frac{\Gamma(c)\Gamma(c+2)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+2)\Gamma(c-b+1)} \int_0^1 t^{a+b-1}(ct+b-c)(1-t)^{c-a-b}dt.$$
In order for this representation to hold we need to assume that any of the conditions (I)-(V) of Theorem 2.1 is satisfied. Under this restriction and except for the degenerate cases \( ab = 0, (c - a)(c - b) = 0 \) the condition \( \nu > -1 \) from (2.8) reads

\[
(c - a - b + 1)_- - (c - a - b)_+ > -1
\]

which is true for all real \( a, b, c \).

Further, \( B_{0,1,2}P_1(t) \) does not change sign on \((0, 1)\) if the zero \( t_\ast = (c - b)/c \) of the polynomial \( P_1(t) = (ct + b - c)/b \) does not lie in \((0, 1)\), i.e. \((c - b)/c \in (-\infty, 0] \cup [1, \infty)\). If this case, according to Theorem 3.1 \( R_{0,1,2} \in \mathcal{S} \) if \( B_{0,1,2}P_1(t) \geq 0 \) on \((0, 1)\) or \( -R_{0,1,2} \in \mathcal{S} \) if \( B_{0,1,2}P_1(t) \leq 0 \) on \((0, 1)\). Furthermore, according to Theorem 3.3 the functions

\[
B_{0,1,2}P_1\left(\frac{1}{z + \omega}\right)R_{0,1,2}(z + \omega) \quad \text{and} \quad \frac{R_{0,1,2}(z + \omega)}{B_{0,1,2}P_1(1/(z + \omega))}
\]

lie in \( \mathcal{S} \) for all real values of parameters \( a, b, c, \omega \) such that \(-c \notin \mathbb{N}_0 \) and \( \omega \leq 1 \).

**Example 12.** For the ratio \( R_{0,-1,0}(z) \) according to (2.18) we obtain \( l = 1, \ p = r = 0 \). Theorem 2.8 and definition (2.21b) yield:

\[
B_{0,-1,0}P_0(t) = -\frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b + 1)}.
\]

Using Lemmas 2.6 and 2.7 or by direct, albeit tedious calculation, we get the asymptotic approximations

1. if \( b + 1 < a \), then \( R_{0,-1,0}(z) = Az + B + o(1) \) as \( z \to \infty \);
2. if \( b < a \leq b + 1 \), then \( R_{0,-1,0}(z) = Az + o(z) \) as \( z \to \infty \);
3. if \( b - 1 \leq a \leq b \), then \( R_{0,-1,0}(z) = o(z) \) as \( z \to \infty \);
4. if \( a < b - 1 \), then \( R_{0,-1,0}(z) = C + o(1) \) as \( z \to \infty \),

where

\[
A = \frac{b - a}{c - b}, \quad B = \frac{b(b + 1) - 2ab + c(a - 1)}{(c - b)(a - b - 1)}, \quad C = \frac{b - 1}{b - a - 1}.
\]

Hence, if \( |a - b| > 1 \), we have \( R_{0,-1,0}(z) = \beta z + \alpha + o(1) \) as \( z \to \infty \), with \( (\beta, \alpha) = (A, B) \) if \( a > b + 1 \) and \( (\beta, \alpha) = (0, C) \) if \( a < b - 1 \). Then for \( |a - b| > 1 \) we can choose \( N = 0 \) in Theorem 2.12 leading to the representation:

\[
R_{0,-1,0}(z) = \alpha + \beta z - \frac{[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b + 1)} \int_0^1 \frac{t^{a+b-2}z^c-1}{(1-zt)^2}F_1(a, b; c; 1/t)^2 dt. \tag{4.2}
\]

In addition to the condition \( |a - b| > 1 \) we need to assume any of the conditions (I)-(V) of Theorem 2.1. Under these restrictions and except for the degenerate cases \( ab = 0, (c - a)(c - b) = 0 \) the condition \( \nu > -1 \) from (2.8) reads

\[
(c - a - b + 1)_- - (c - a - b)_+ > -1
\]

which is true for all real \( a, b, c \).

For arbitrary \( a, b \) we obtain \( R_{0,-1,0}(z) = \beta z + o(z) \) as \( z \to \infty \), with \( \beta = A \) if \( b < a \) and \( \beta = 0 \) if \( a \leq b \). Hence, we can remove the restriction \( |a - b| > 1 \) by taking \( N = 1 \) in Theorem 2.12 which leads to

\[
R_{0,-1,0}(z) = 1 + \beta z - \frac{z[\Gamma(c)]^2}{\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b + 1)} \int_0^1 \frac{t^{a+b-1}z^c-1}{(1-zt)^2}F_1(a, b; c; 1/t)^2 dt, \tag{4.3}
\]
or, taking \( N = 2 \) to get

\[
R_{0,-1,0}(z) = 1 - \frac{ac}{c^2} z - \frac{z^2 \Gamma(c)^2}{\Gamma(a) \Gamma(b) \Gamma(c-a) \Gamma(c-b+1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{(1-zt)^2 F_1(a, b; c; 1/t)} dt.  \tag{4.4}
\]

In particular, for \( a = b = 1, \, c = 2 \) we arrive at the curious identity

\[
\frac{z}{\log(1+z)} = 1 + z \int_1^\infty \frac{dx}{(\log^2(x-1) + \pi^2)(x+z)}.
\]

If \( z \) is replaced by 1 in the numerator on the left hand side, a similar representation can be found in [9, (34)].

Further, \( B_{0,-1,0}P_0(t) = B_{0,-1,0}P_0 \) does not depend on \( t \), and thus does not change sign on \((0, 1)\). Hence, according to Theorem 3.1 \( R_{0,-1,0} \in \mathcal{S} \) when \( B_{0,-1,0}P_0 \geq 0 \) or \( -R_{0,-1,0} \in \mathcal{S} \) when \( B_{0,-1,0}P_0 \leq 0 \). Furthermore, according to Theorem 3.3 the function \( B_{0,-1,0}P_0R_{0,-1,0}(z+\omega) \) lies in \( \mathcal{S} \) for all real values of parameters \( a, b, c, \omega \) such that \(-c \notin \mathbb{N}_0\) and \( \omega \leq 1 \).

**Example 13.** For the ratio \( R_{1,-1,0}(z) \) according to (2.18) we obtain \( p = 2, \, l = r = 0 \). Theorem 2.8 and definition (2.21b) yields:

\[
B_{1,-1,0}P_0(t) = -\frac{\Gamma(c)^2(c-a-b+1)}{\Gamma(a) \Gamma(b) \Gamma(c-a+1) \Gamma(c-b+1)}.
\]

Using Lemmas 2.6 and Lemma 2.7 or by direct, albeit tedious calculation we get the asymptotic approximations

1. If \( a > b + 1 \), then \( R_{1,-1,0}(z) = B(a, b)z + A(a, b) + o(1) \) as \( z \to \infty \);
2. If \( b \leq a \leq b + 1 \), then \( R_{1,-1,0}(z) = B(a, b)z + o(z) \) as \( z \to \infty \);
3. If \( b - 1 \leq a \leq b \), then \( R_{1,-1,0}(z) = B(b, a)z + o(z) \) as \( z \to \infty \);
4. If \( a < b - 1 \), then \( R_{1,-1,0}(z) = B(b, a)z + A(b, a) + o(1) \) as \( z \to \infty \),

where

\[
B(a, b) = \frac{a-1}{b-c}, \quad A(a, b) = \frac{(a-1)(2b-c)}{(c-b)(1+b-a)}.
\]

Hence, if \( |a-b| > 1 \), then \( R_{1,-1,0}(z) = \beta z + o(z) \) as \( z \to \infty \), where \((\beta, \alpha) = (B(a, b), A(a, b))\) if \( a > b + 1 \) and \((\beta, \alpha) = (B(b, a), A(b, a))\) if \( a < b - 1 \). Hence, for \( |a-b| > 1 \) the \( N = 1 \) case of Theorem 2.12 leads to the representation:

\[
R_{1,-1,0}(z) = \alpha + \beta z - \frac{\Gamma(c)^2(c-a-b+1)}{\Gamma(a) \Gamma(b) \Gamma(c-a+1) \Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-2}(1-t)^{c-a-b}}{|2 F_1(a, b; c; 1/t)|^2(1-zt)} dt. \tag{4.5}
\]

In addition to the condition \( |a-b| > 1 \) we need to assume any of the conditions (I)-(V) of Theorem 2.1. Under these restrictions and except for the degenerate cases \( ab = 0, \, (c-a)(c-b) = 0 \) the condition \( \mu > -1 \) from (2.8) reads

\[
(c-a-b) - (c-a-b) > -1
\]

which is true for all real \( a, b, c \). As \( R_{1,-1,0}(z) = \beta z + o(z) \) as \( z \to \infty \), where \( \beta = B(a, b) \) if \( a \geq b \) and \( \beta = B(b, a) \) if \( a \geq b \), we can lift the restriction \( |a-b| > 1 \) by taking \( N = 1 \) in Theorem 2.12 which leads to

\[
R_{1,-1,0}(z) = 1 + \beta z - \frac{z \Gamma(c)^2(c-a-b+1)}{\Gamma(a) \Gamma(b) \Gamma(c-a+1) \Gamma(c-b+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}}{|2 F_1(a, b; c; 1/t)|^2(1-zt)} dt. \tag{4.6}
\]
or, taking \( N = 2 \), we get

\[
R_{-1,1,0}(z) = 1 + \frac{(a + b - 1)c}{c^2} - \frac{z^2[\Gamma(c)]^2(c - a - b + 1)}{\Gamma(a)\Gamma(b)\Gamma(c - a + 1)\Gamma(c - b + 1)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{|2F_1(a, b; c; 1/t)|^2(1-zt)} dt.
\]

Further, \( B_{-1,1,0}P_0(t) = B_{-1,1,0}P_0 \) does not depend on \( t \), and thus does not change sign on \((0, 1)\). Hence, according to Theorem 3.1 \( R_{-1,1,0} \in \mathcal{S} \) when \( B_{-1,1,0}P_0 \geq 0 \) or \(-R_{-1,1,0} \in \mathcal{S} \) when \( B_{-1,1,0}P_0 \leq 0 \). Furthermore, according to Theorem 3.3 the function \( B_{-1,1,0}P_0R_{-1,1,0}(z + \omega) \) lies in \( \mathcal{S} \) for all real values of parameters \( a, b, c, \omega \) such that \(-c \notin \mathbb{N}_0 \) and \( \omega \leq 1 \).

**Example 14.** For the ratio \( R_{-1,1,0}(z) \) according to (2.18) we obtain \( p = l = r = 0 \). Theorem 2.8 and definition (2.21b) yield:

\[
B_{-1,1,0}P_0 = \frac{[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}.
\]

The asymptotic behavior of \( R_{-1,1,0}(z) \) as \( z \to \infty \) is rather complicated and depends on the relation between \( a \) and \( b \). Application of Lemmas 2.6 and 2.7 yield:

1. If \( b + 1 < a \), then \( R_{-1,1,0}(z) = o(1) \) as \( z \to \infty \);
2. If \( b \leq a \leq b + 1 \), then \( R_{-1,1,0}(z) = o(z) \) as \( z \to \infty \);
3. If \( b - 1 \leq a < b \), then \( R_{-1,1,0}(z) = Bz + o(z) \) as \( z \to \infty \);
4. If \( a < b - 1 \), then \( R_{-1,1,0}(z) = Bz + C + o(1) \) as \( z \to \infty \),

where

\[
B = \frac{(b-a)(b-a+1)}{b(a-c)}, \quad C = \frac{(b-a)(b-a+1)(c(a+b-1)-2ab)}{b(c-a)(a-b+1)(a-b+1)}.
\]

Hence, if \( |a-b| > 1 \) we have \( R_{-1,1,0}(z) = \beta z + o(1) \) as \( z \to \infty \), where \( (\beta, a) = (0, 0) \) when \( a > b + 1 \) and \( (\beta, a) = (B, C) \) when \( a < b - 1 \). Then for \( |a-b| > 1 \) the \( N = 0 \) case of Theorem 2.12 leads to the representation:

\[
R_{-1,1,0}(z) = \beta z + o + \frac{[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-2}(1-t)^{c-a-b}}{|2F_1(a, b; c; 1/t)|^2(1-zt)} dt. \quad (4.8)
\]

In addition to the condition \( |a-b| > 1 \) we need to assume any of the conditions (I)-(V) of Theorem 2.1. Under these restrictions and except for the degenerate cases \( ab = 0 \), \( (c-a)(c-b) = 0 \) the condition \( \nu > -1 \) from (2.8) reads

\[
(c - a - b)_- - (c - a - b)_- > -1
\]

which is true for all real \( a, b, c \).

For arbitrary values of \( a, b \) we have \( R_{-1,1,0}(z) = \beta z + o(z) \) as \( z \to \infty \), where \( \beta = 0 \) when \( a \geq b \) and \( \beta = B \) when \( a < b \). Hence, we can use representation (2.23) with \( N = 1 \) yielding

\[
R_{-1,1,0}(z) = 1 + \beta z + \frac{z[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}}{|2F_1(a, b; c; 1/t)|^2(1-zt)} dt. \quad (4.9)
\]

or with \( N = 2 \) yielding

\[
R_{-1,1,0}(z) = 1 + \frac{(a - b - 1)c}{c^2} z + \frac{z^2[\Gamma(c)]^2(a-b-1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)} \int_0^1 \frac{t^{a+b}(1-t)^{c-a-b}}{|2F_1(a, b; c; 1/t)|^2(1-zt)} dt. \quad (4.10)
\]

Further, \( B_{-1,1,0}P_0(t) = B_{-1,1,0}P_0 \) does not depend on \( t \), and thus does not change sign on \((0, 1)\). Hence, according to Theorem 3.1 \( R_{-1,1,0} \in \mathcal{S} \) when \( B_{-1,1,0}P_0 \geq 0 \) or \(-R_{-1,1,0} \in \mathcal{S} \) when
$B_{-1,1,0} P_0 \leq 0$. Furthermore, according to Theorem 3.3 the function $B_{-1,1,0} P_0 R_{-1,1,0}(z + \omega)$ lies in $\mathcal{S}$ for all real values of parameters $a, b, c, \omega$ such that $-c \notin \mathbb{N}_0$ and $\omega \leq 1$.

**Example 15.** For the ratio $R_{-2,-2,0}(z)$ according to (2.18) we obtain $p = 4, l = 0, r = 1$. Theorem 2.8 and definition (2.21b) yield:

$$
B_{-2,-2,0} P_1(t) = -\frac{[\Gamma(c)](c-a-b+2)(\rho_0 + \rho_1 t)}{\Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)},
$$

where $\rho_0 = a^2 + b^2 - (c+2)(a+b) + 3c + 1$, $\rho_1 = c(c-a-b+1) + 2(ab-a-b+1)$. Using Lemmas 2.6 and 2.7 or by direct, albeit tedious calculation we obtain the asymptotic approximations

1. if $a > b + 2$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + \beta a,b,c z + \alpha a,b,c + o(1)$ as $z \to \infty$;
2. if $b + 1 < a \leq b + 2$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + \beta a,b,c z + o(1)$ as $z \to \infty$;
3. if $b \leq a \leq b + 1$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + o(z^2)$ as $z \to \infty$;
4. if $b - 1 \leq a \leq b$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + o(z^2)$ as $z \to \infty$;
5. if $b - 2 \leq a < b - 1$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + \beta a,b,c z + o(1)$ as $z \to \infty$;
6. if $a < b - 2$, then $R_{-2,-2,0}(z) = \gamma a,b,c z^2 + \beta a,b,c z + \alpha a,b,c + o(1)$ as $z \to \infty$,

where

$$
\gamma a,b,c = \frac{(a-2)(a-1)}{(c-b)(c-b+1)}, \quad \beta a,b,c = \frac{2(a-2)(a-1)(c+1-2b)}{(c-b)(c-b+1)(b-a+1)},
$$

$$
\alpha a,b,c = \gamma a,b,c \frac{c(c+1)(a-1)+2b^2(a+4c-3b)-2ab(c+2)-b(3c^2-c-6)}{(a-b-2)(a-b-1)^2}.
$$

Hence, for $|a - b| > 2$ we have $R_{-2,-2,0}(z) = \gamma z^2 + \beta z + o(1)$ as $z \to \infty$, where $(\gamma, \beta, \alpha) = (\gamma a,b,c, \beta a,b,c, \alpha a,b,c)$ when $a > b + 2$ and $(\gamma, \beta, \alpha) = (\gamma b,a,c, \beta b,a,c, \alpha b,a,c)$ when $a < b - 2$. Then for $|a - b| > 2$ the case $N = 0$ of Theorem 2.12 leads to the representation:

$$
R_{-2,-2,0}(z) = \gamma z^2 + \beta z + o(1) = \frac{\int \left(\rho_0 + \rho_1 t\right) t^{a+b-3} (1-t)^{c-a-b}}{\Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)} dt.
$$

In addition to the condition $|a - b| > 2$ we need to assume any of the conditions (I)-(V) of Theorem 2.1. Under these restrictions and except for the degenerate cases $ab = 0, (c-a)(c-b) = 0$ the condition $\nu > -1$ from (2.8) reads

$$
(c-a-b+4)_- - (c-a-b)_+ > -1
$$

which is true for all real $a, b, c$. If $1 < |a - b| \leq 2$, we see that the asymptotics takes the form $R_{-2,-2,0}(z) = \gamma z^2 + \beta z + o(1)$ as $z \to \infty$, where $(\gamma, \beta) = (\gamma a,b,c, \beta a,b,c)$ when $a > b + 1$ and $(\gamma, \beta) = (\gamma b,a,c, \beta b,a,c)$ when $a < b - 1$. Hence, for $1 < |a - b|$ according to (2.23) with $N = 1$ we get

$$
R_{-2,-2,0}(z) = \gamma z^2 + \beta z + 1 - \frac{z \Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)}{\Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)} \int_0^1 \left(\rho_0 + \rho_1 t\right) t^{a+b-2} (1-t)^{c-a-b} \frac{dt}{\Gamma(1-a,b,c;1/t)^2 (1-z)}.
$$

Similarly, for $|a - b| \leq 1$ the asymptotics takes the form $R_{-2,-2,0}(z) = \gamma z^2 + o(z^2)$, where $\gamma = \gamma a,b,c$ when $a \geq b$ and $\gamma = \gamma b,a,c$ when $a \leq b$. Hence, without additional restrictions according to (2.23) with $N = 2$ we get

$$
R_{-2,-2,0}(z) = 1 + \frac{2(a-b)}{c^2} z + \gamma z^2
$$

$$
- \frac{z \Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)}{\Gamma(a) \Gamma(b) \Gamma(c-a+2) \Gamma(c-b+2)} \int_0^1 \left(\rho_0 + \rho_1 t\right) t^{a+b-1} (1-t)^{c-a-b} \frac{dt}{\Gamma(1-a,b,c;1/t)^2 (1-z)}.
$$
under any of the conditions (I)-(V) of Theorem 2.1, but without any other restrictions.

Further, $B_{-2,-2,0}P_1(t)$ does not change sign on $(0,1)$ if the zero $t_\ast = -\rho_0/\rho_1$ of the polynomial $P_1(t) = \rho_0 + \rho_1 t$ does not lie in $(0,1)$, i.e. $-\rho_0/\rho_1 \in (-\infty,0) \cup [1,\infty)$. If this case, according to Theorem 3.1 $R_{-2,-2,0} \in \mathcal{S}$ if $B_{-2,-2,0}P_1(t) \geq 0$ on $(0,1)$ or $-R_{-2,-2,0} \in \mathcal{S}$ if $B_{-2,-2,0}P_1(t) \leq 0$ on $(0,1)$. Furthermore, according to Theorem 3.3 the functions

$$B_{-2,-2,0}P_1\left(\frac{1}{z + \omega}\right)R_{-2,-2,0}(z + \omega) \quad \text{and} \quad \frac{R_{-2,-2,0}(z + \omega)}{B_{-2,-2,0}P_1(1/(z + \omega))}$$

lie in $\mathcal{S}$ for all real values of parameters $a,b,c,\omega$ such that $-c \notin \mathbb{N}_0$ and $\omega \leq 1$.

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References

[1] S. Agrawal and S.K. Sahoo, Geometric properties of basic hypergeometric functions, J. Difference Equ. Appl. 20 (2014), no. 11, 1502–1522, doi:10.1080/10236198.2014.946501.

[2] N.I. Akhiezer, The classical moment problem and some related questions in analysis, Oliver & Boyd, Edinburgh-London, 1965.

[3] G.E. Andrews, R. Askey and R. Roy, Special functions, Cambridge University Press, 1999.

[4] J. Arvesú, K. Driver, L. Littlejohn, Zeros of Jacobi and Ultraspherical polynomials. Preprint (2020), available online at https://arxiv.org/abs/2009.10196.

[5] D. Askitis, Geometric function theory, completely monotone sequences and applications in special functions. Thesis for the Master Degree in Mathematics, Copenhagen, 2015.

[6] A. Erdélyi, Higher Transcendental Functions, Volume I. Bateman Manuscript Project, Mc Graw-Hill Book Company, Inc, 1953.

[7] Á. Baricz and A. Swaminathan, Mapping properties of basic hypergeometric functions, J. Class. Anal. 5 (2014), no. 2, 115–128, doi:10.7153/jca-05-10.

[8] C. Berg, Quelques remarques sur le cône de Stieltjes, in Séminaire de Théorie du Potentiel, Paris, No.5, in: F. Hirsch, G. Mokobodzki (Eds.), Lecture Notes in Mathematics, vol.814, Springer, Berlin, Heidelberg, New York, 1980, 70–79.

[9] C. Berg and H.L. Pedersen, A one-parameter family of Pick functions defined by the gamma function and related to the volume of the unit ball in n-space, Proc. Amer. Math. Soc. 139 (2011), no. 6, 2121–2132.

[10] V. Belevitch, The Gauss hypergeometric ratio as a positive real function, SIAM J. Math. Anal., 13 (1982), no. 6, 1024–1040.

[11] A. Çetinkaya, D. Karp, E. Prilepkina, Generalized hypergeometric function at unity: simple derivation of old and new identities, in preparation.

[12] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York-London-Paris, 1978.
[13] M. Derevyagin, *On the Schur algorithm for indefinite moment problem*, Methods Funct. Anal. Topol. 9 (2003), no. 2, 133–145.

[14] M. Derevyagin, *Jacobi matrices generated by ratios of hypergeometric functions*, J. Difference Equ. Appl. 24 (2018), no. 2, 267–276.

[15] V.A. Derkach, *Generalized resolvents of a class of Hermitian operators in a Kreîn space*, Dokl. Akad. Nauk SSSR 317 (1991), no. 4, 807–812; translation in Soviet Math. Dokl. 43 (1991), no. 2, 519–524.

[16] V. Derkach, S. Hassi, H. de Snoo, *Operator models associated with Kac subclasses of generalized Nevanlinna functions*, Methods Funct. Anal. Topology, 5 (1999), no. 1, 65–87.

[17] V. Derkach, I. Kovalyov, *The Schur algorithm for an indefinite Stieltjes moment problem*, Math. Nachr. 290 (2017), no. 11–12, 1637–1662, doi:10.1002/mana.201600189.

[18] A. Dijksma, H. Langer A. Luger and Yu. Shondin, *A factorization result for generalized Nevanlinna functions of the class Nκ*, Integral Equations Oper. Theory 36 (2000), no.1, 121–125.

[19] K. Driver, K.H. Jordaan, *Zeros of quasi-orthogonal Jacobi polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016), 042, 13 pp., doi:10.3842/SIGMA.2016.042.

[20] K. Driver, K.H. Jordaan, N. Mbuyi, *Interlacing of the zeros of Jacobi polynomials with different parameters*. Num. Alg. 49 (2008), 143–152.

[21] A. Dyachenko, *Functions of classes Nκ+. Preprint (2016), available online at https://arxiv.org/abs/1503.05894v2.*

[22] A. Ebisu, *Three Term Relations for the Hypergeometric Series*, Funkcial. Ekvac., 55 (2012), 255–283.

[23] A. Ebisu, K. Iwasaki, *Three-term relations for 3F2(1)*, J. Math. Anal. Appl. 463 (2018), 593–610.

[24] F. R. Gantmacher, *The theory of matrices*. Vol.1, AMS Chelsea Publ., Providence (R.I.), 1959.

[25] C.F. Gauss, *Disquisitiones generales circa seriem infinitam. . . ,* Commentationes Societatis Regiae Scientiarum Gottingensis Recentiore 2 (1812), 1–46; reprint in C.F. Gauß, *Werke, Band III*, Königliche Gesellschaft der Wissenschaften zu Göttingen, Göttingen, 1876, 123–162.

[26] D. Karp and A. Kuznetsov, *A new identity for a sum of products of the generalized hypergeometric functions*, Proc. Amer. Math. Soc., Articles in Press, 2019, doi:10.1090/proc/14803.

[27] D.B. Karp and E.G. Prilepkina, *Applications of the Stieltjes and Laplace transform representations of the hypergeometric functions*, Integral Transforms Spec. Funct. 28 (2017), no.10, 710–731.

[28] M. Krein and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. 77 (1977), 187–236.

[29] M.G. Krein and H. Langer, *On some extension problems which are closely connected with the theory of Hermitian operators in a space IIκ, III: Indefinite analogues of the Hamburger and Stieltjes moment problems*, Beiträge Anal., 14 (1979), 25–40. and 15 (1981), 27–45.
[30] R. Küstner, Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order $\alpha$, Comput. Methods Funct. Theor. 2 (2002), no. 2, 597–610.

[31] H. Lima, A. Loureiro, Multiple orthogonal polynomials with respect to Gauss’ hypergeometric function. Preprint (2020), available online at http://arxiv.org/abs/2012.13913.

[32] B.-Y. Long, T. Sugawa, Q.-H. Wang, Completely monotone sequences and harmonic mappings. Preprint (2021), available online at https://arxiv.org/abs/2102.00138v1.

[33] P. Nevai, A new class of orthogonal polynomials, Proc. Amer. Math. Soc. 91 (1984), no. 3, 409–415.

[34] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.19 of 2018-06-22. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds.

[35] O. Perron, Die Lehre von den Kettenbrüchen. Band II. Dritte, verbesserte und erweiterte Aufl., B.G. Teubner Verlagsgesellschaft, Stuttgart, 1957.

[36] G. Pick, Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, Math. Ann. 77 (1916), 7–23.

[37] H.-J. Runckel, On the zeros of the hypergeometric function, Math. Ann. 191 (1971), 53–58.

[38] J. Segura, Interlacing of the zeros of contiguous hypergeometric functions. Numer. Algorithms 49 (2008), 387–407.

[39] M. Pétréolle, A.D. Sokal, B.-X. Zhu, Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes–Rogers and Thron–Rogers polynomials, with coefficientwise Hankel-total positivity. Preprint (2018/2020), to appear in Mem. Amer. Math. Soc., available online at http://arxiv.org/abs/1807.03271.

[40] T.J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse 8 (1894), J1–122, 9 (1895), A1–47.

[41] H.S. Wall, Analytic Theory of continued fractions, Chelsea Publishing Company, 1948.

[42] J. Wimp, Explicit Formulas for the Associated Jacobi Polynomials and Some Applications, Can. J. Math., Vol. XXXIX, No. 4, 1987, 983–1000.

[43] J. Wimp and B. Beckermann, Some explicit formulas for Padé approximants of ratios of hypergeometric functions. Contributions in Numerical Mathematics, WSSIAA 2 (1993), 427–434.

[44] Y. Yamaguchi, Three-term relations for basic hypergeometric series, J. Math. Anal. Appl. 464 (2018), no. 1, 662–678.