SMALL COHERENCE IMPLIES THE WEAK NULL SPACE PROPERTY

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Abstract. In the Compressed Sensing community, it is well known that given a matrix $X \in \mathbb{R}^{n \times p}$ with $\ell_2$ normalized columns, the Restricted Isometry Property (RIP) implies the Null Space Property (NSP). It is also well known that a small Coherence $\mu$ implies a weak RIP, i.e. the singular values of $X_T$ lie between $1 - \delta$ and $1 + \delta$ for "most" index subsets $T \subset \{1, \ldots, p\}$ with size governed by $\mu$ and $\delta$. In this short note, we show that a small Coherence implies a weak Null Space Property, i.e. $\|h_T\|_2 \leq C \|h_T\|_1/\sqrt{s}$ for most $T \subset \{1, \ldots, p\}$ with cardinality $|T| \leq s$. We moreover prove some singular value perturbation bounds that may also prove useful for other applications.

Keywords: Restricted Invertibility, Coherence, Null Space Property.

1. Introduction

1.1. Motivations. Compressed Sensing is a new paradigm for data acquisition which was discovered in [6] and [12] and has had a paramount impact on modern Signal Processing, Statistics, Applied Harmonic Analysis, Machine Learning, to name just a few. The whole field started after it was discovered that if $\beta$ is sufficiently sparse, one could recover the support and sign pattern of a high dimensional vector $\beta \in \mathbb{R}^p$ from just a few linear measurements $y = X\beta + \varepsilon$, where $X \in \mathbb{R}^{n \times p}$, with $n \ll p$, by solving a simple convex programming problem of the form

$$\min_{b \in \mathbb{R}^p} \frac{1}{2}\|y - Xb\|_2^2 + \lambda\|b\|_1.$$ 

In the remainder of this paper, we will assume that the columns of $X$ are $\ell_2$ normalized.

One condition implying that both support and sign pattern can be recovered is called the Restricted Isometry Property (RIP) [4]. More precisely, RIP is the property that for all index subset $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$, all the singular values of the submatrix $X_{T_0}$ whose columns are the columns of $X$ indexed by $T_0$, lie in the interval $(1 - \delta, 1 + \delta)$. 

One key result relating RIP and recovery of the basic features of a sparse vector is the fact that RIP implies the so-called Null Space Property, which says that the kernel of $X$ does not contain any sparse vector. More precisely, the NSP is the property that for all $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$, and for all $h \in \text{Ker}(X)$,

$$\|h_{T_0}\|_2 \leq C \|h_{T_0}\|_1/\sqrt{s_0}$$

with $C \in (0, 1)$. It is well known that the NSP is the key property behind sparse recovery using Basis Pursuit type of methods, whereas RIP is not. The main reason for introducing the RIP is that it provides a pedagogical step for proving the NSP in the case of random matrices. It was recently shown that the NSP can also be derived without the RIP for random design [1]. Thus, understanding more precisely what are the conditions on the design matrix for which we can obtain a kind of NSP is quite an important question in this field.

Some very interesting work has been published recently in order to test if the NSP or weaker version of this property hold for a given matrix using convex programming; see e.g. [11]. On the other hand, one of the main drawbacks of the Restricted Isometry Property is that one cannot in general check if

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a given matrix $X$ satisfies it in polynomial time. Therefore, RIP is usually not considered of practical interest. Another property often used in many sparse recovery problems is the property of small coherence.

The coherence of a matrix is an important quantity in the study of designs for sparse recovery is the coherence. It will be denoted by $\mu$, will be defined as

$$\mu = \max_{1 \leq k < i \leq p} |\langle X_k, X_i \rangle |.$$  

(1.2)

If the columns are almost orthogonal, then, one usually expects that the performance of Basis Pursuit should be almost as good as in the orthogonal case. This have been rigorously studied in e.g. [5]. The main motivation for using the coherence is that it is conceptually intuitive and also very easy to compute.

On the other hand, it was also proved in [13], [5, Theorem 3.2 and following comments] that if a matrix $X$ has small coherence, then, for most index subsets $T_0 \subset \{1, \ldots, p\}$ with cardinal $|T_0| = s_0$, and for all $h \in \text{Ker}(X)$, (1.1) holds for some positive $\mu$. In other words, small coherence implies a kind of weak Null Space Property which holds for most, instead of all, $T_0$ with $|T_0| = s_0$.

1.2. Goal of the paper. Our aim in the present paper is to understand better the role of the coherence for Compressed Sensing by understanding how a small coherence implies a weaker version of the Null Space Property. The main result of the present work is the following. We prove that if a matrix $X$ has small coherence, then, for most index subsets $T_0 \subset \{1, \ldots, p\}$ with cardinal $|T_0| = s_0$, and for all $h \in \text{Ker}(X)$, (1.1) holds for some positive $\mu$. In other words, small coherence implies a kind of weak Null Space Property which holds for most, instead of all, $T_0$ with $|T_0| = s_0$.

1.3. Additional notation. For $T \subset \{1, \ldots, p\}$, we denote by $|T|$ the cardinal of $T$. Given a vector $x \in \mathbb{R}^p$, we set $x_T = (x_j)_{j \in T} \in \mathbb{R}^{|T|}$. The canonical scalar product in $\mathbb{R}^p$ is denoted by $\langle \cdot, \cdot \rangle$.

For any matrix $A \in \mathbb{R}^{d_1 \times d_2}$, we denote by $A^T$ its transpose. The set of symmetric real matrices is denoted by $\mathbb{S}_n$. We denote by $\|A\|$ the operator norm of $A$. We use the Loewner ordering on symmetric real matrices: if $A \in \mathbb{S}_n$, $0 \preceq A$ denotes positive semi-definiteness of $A$, and $A \preceq B$ stands for $0 \preceq B - A$. The singular values of $A$ will be denoted by $\sigma_{\max}(A) = \sigma_1(A) \geq \cdots \geq \sigma_{\min(d_1,d_2)} = \sigma_{\min}(A)$.

2. Background

In this section, we recall some well known previous results relating coherence, singular value concentration, RIP and NSP. We begin with some definitions.

2.1. Weak NSP and weak RIP.

2.1.1. Weak Null Space Property. First, the weak-Null Space Property.

**Definition 2.1.** A matrix $X \in \mathbb{R}^{n \times p}$ satisfies the Weak Null Space Property (weak-NSP$(s,C,\pi)$) if for at least a proportion $\pi$ of all index subsets $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$, and for all $h \in \text{Ker}(X)$,

$$\|h_{T_0}\|_2 \leq C \|h_{T_0^c}\|_1 / \sqrt{s_0}.$$  

(2.3)

Notice that when $\pi = 1$, we recover the definition of the standard Restricted Isometry Property.

The main consequence of the weak Null Space Property is that exact recovery holds for the basis pursuit problem. Since the work [10], this can be proved swiftly as follows. Let us first recall the framework: we assume that $y = X\beta$, i.e. we are in the noise free setting and $\beta$ has support $T_0$ with $|T_0| \leq s_0$. Then, we solve

$$\min_{b \in \mathbb{R}^p} \|b\|_1 \text{ s.t. } y = Xb.$$  

Let $\hat{\beta}$ denote a minimizer. Then, we have

$$\|\hat{\beta}\|_1 \leq \|\beta\|_1,$$

*the precise result underpinning this statement will be recalled in Section 2.3 below*
which gives
\[(2.4) \quad \|\hat{\beta}_T - \beta_T\|_1 \leq \|\hat{\beta}_{T_0} - \beta_{T_0}\|_1 + 2\|\beta_{T_0}\|\]
and thus, by the Cauchy-Schwartz inequality
\[(2.5) \quad \|\hat{\beta}_T - \beta_T\|_1 \leq \sqrt{s_0} \|\hat{\beta}_{T_0} - \beta_{T_0}\|_2 + 2\|\beta_{T_0}\|\]
Since \(\beta\) has support \(T_0\), we obtain that \(\beta_{T_0} = 0\). Using the fact that \(\hat{\beta} - \beta\) lies in the kernel of \(X\) and using \((2.6)\), we obtain from \((2.24)\) that \(\|\hat{\beta}_{T_0} - \beta_{T_0}\|_1 = 0\). Using \((2.6)\) again, we conclude that \(\|\hat{\beta} - \beta\|_1 = 0\), i.e. exact recovery holds. More results of this type can be found in \([3] \) and \([13] \).

2.1.2. Weak Restricted Isometry Property. The weak-Restricted Isometry Property is the subject of the next definition.

**Definition 2.2.** A matrix \(X \in \mathbb{R}^{n \times p}\) satisfies the Weak Restricted Isometry Property weak-RIP\((s, \rho, \pi)\) if for at least a proportion \(\pi\) of all index subsets \(T_0 \subset \{1, \ldots, n\}\) with \(|T_0| = s_0\),
\[(2.6) \quad (1 - \rho) \leq \sigma_{\min}(X_{T_0}) \leq \cdots \leq \sigma_{\max}(X_{T_0}) \leq (1 + \rho).\]
Notice that when \(\pi = 1\), we recover the definition of the standard Restricted Isometry Property.

2.2. On the relationship between RIP and NSP. One of the cornerstones of Compressed Sensing is the Null Space Property. It is well known that RIP implies NSP as stated in the next theorem. We will use the standard notations RIP\((s_0, \rho)\) for RIP\((s_0, \rho, 1)\) and NSP\((s_0, C)\) for NSP\((s_0, C, 1)\).

**Theorem 2.3.** \([1]\) Any matrix \(X \in \mathbb{R}^{n \times p}\) satisfying RIP\((2s_0, \delta)\) satisfies NSP\((s_0, C)\) with \(C \leq \sqrt{2}(1 + \delta)/(1 - \delta)\).

2.3. On the relationship between the Coherence and weak-RIP. The first result relating small coherence with weak-RIP was established by \([5]\) based on a result about column selection due to Tropp \([18]\). A refinement of this result is recalled in the next theorem.

**Theorem 2.4. Chrétien and Darses** \([7]\) Let \(r \in (0, 1), \alpha \geq 1\). Let us be given a full rank matrix \(X \in \mathbb{R}^{n \times p}\) and a positive integer \(s_0\), such that
\[(2.7) \quad \mu \leq \frac{r}{(1 + \alpha)\log p}\]
\[(2.8) \quad s_0 \leq \frac{r^2}{(1 + \alpha)\log p}.\]
Let \(T_0 \subset \{1, \ldots, p\}\) be a random support with uniform distribution on index sets satisfying \(|T_0| = s_0\). Then the following bound holds:
\[(2.9) \quad \mathbb{P}(\|X_{T_0}^T X_{T_0} - I\| \geq r) \leq \frac{1944}{p^\alpha}.\]

This theorem was used in, e.g. \([9]\) for a study of the LASSO when the variance is unknown. It has also been used in remote sensing \([15]\), in the study of Gaussian erasure channels \([17]\), Kaczmarcz type methods for least squares \([16]\), extentions of RIP \([3]\); see also \([14]\).

2.4. The Gershgorin bound. The Gershgorin theorem gives a bound on the operator norm as a function of the coherence. More precisely, as discussed e.g. in \([2]\), for each index subset \(T \subset \{1, \ldots, p\}\) with cardinal \(|T_0| = s_0\),
\[(2.10) \quad \|X_{T_0}^T X_{T_0} - I\| \leq \mu(s_0 - 1).\]
Clearly, this result starts being useful when \(\mu\) is much smaller than \(s_0\). In the application for the LASSO, it is often assumed that this indeed the case as in e.g. \([5]\).
3. Main results: small coherence implies weak-NSP

In this section, we state and prove the main result of this paper, namely that small coherence implies weak-NSP. Our main theorem is the following.

**Theorem 3.1.** Let \( X \in \mathbb{R}^{n \times p}, s_0 \leq n \) and \( \alpha > 0 \). Assume that

\[
s_0 \leq \frac{1}{16(1 + \alpha)e^2} \frac{p}{\|X\|^2 \log p}.
\]

Let \( \mu \) denote the coherence of \( X \). Let

\[
\varepsilon_{\text{min}} = \frac{1}{2} s_0^3 \mu^2 + \frac{s_0^3}{2} \mu \quad (3.11)
\]

\[
\varepsilon_{\text{max}} = 144 s_0^3 \mu^2 + 72 s_0^{3/2} \mu.
\]

Assume that

\[
\mu \leq \min \left\{ \frac{1}{\sqrt{288 s_0^{5/2} (2 s_0^{3/2} + 1)}}, \frac{1}{\sqrt{4 s_0^{5/2} + 6 s_0^{5/2} + 2 s_0}}, \frac{1}{4(1 + \alpha) \log p} \right\}.
\]

Then, the matrix \( X \) verifies the weak-NSP(\( s_0, C, \pi \)) with \( \pi = 1 - 1944/p^\alpha \) and

\[
C = 1 + \frac{1}{\lambda_1} \frac{s_0 (\varepsilon_{\text{max}} + \varepsilon_{\text{min}})}{s_0 \varepsilon_{\text{min}}}.
\]

In particular, if

\[
\mu \leq \min \left\{ c_0 \frac{s_0}{s_0^{5/2}}, \frac{1}{4(1 + \alpha) \log p} \right\}
\]

for some positive constant \( c_0 \), then the matrix \( X \) verifies the weak-NSP(\( s_0, C, \pi \)) with \( \pi = 1 - 1944/p^\alpha \) and

\[
\varepsilon_{\text{min}} = \frac{1}{4} \frac{c_0^2 s_0^{-2/4} + c_0 s_0^{-1}}{1/2 - c_0^2 s_0^{4}}
\]

\[
\varepsilon_{\text{max}} = \frac{1}{4} \frac{144 s_0^{-1/2} c_0^2 + 72 c_0 s_0^{-2}}{\lambda_1 - 1}
\]

Then, the matrix \( X \) verifies the weak-NSP(\( s_0, C, \pi \)) with \( \pi = 1 - 1944/p^\alpha \) and

\[
C = \frac{1 + \frac{3}{4} \left( \frac{c_0^2 s_0^{-1/4} + c_0}{1/2 - c_0^2 s_0^{-4}} + \frac{144 c_0^{4} + 72 c_0 s_0^{-1}}{\lambda_1 - 1} \right)}{1 - \frac{3}{4} \frac{c_0^2 s_0^{-1/4} + c_0}{1/2 - c_0^2 s_0^{-4}}}.
\]

**Proof.** Using Theorem 2.1 for

\[
\mu \leq \frac{1}{4(1 + \alpha) \log p}
\]

with probability larger that \( \pi \), an index subset \( T_0 \) with cardinality \( s_0 \)

\[
s_0 \leq \frac{1}{16(1 + \alpha)e^2} \frac{p}{\|X\|^2 \log p}.
\]

satisfies

\[
\frac{5}{4} \geq \lambda_1 \geq \lambda_{s_0} \geq \frac{3}{4},
\]

where

\[
\lambda_1 := \lambda_1(X_{T_0} X_{T_0}^t)
\]
and
\begin{equation}
\lambda_{s_0} := \lambda_{s_0}(X_{T_0}X_{T_0}^T).
\end{equation}

Let \( h \in \text{Ker}(X) \) and let \( T_0 \) be a subset of \( \{1, \ldots, p\} \) with cardinality \( |T_0| = s_0 \) verifying (3.16), (3.17) and (3.18). Define
\begin{enumerate}
  \item \( T_1 \) as the index set of the \( s_0 \) largest entries of \( h_{T_0} \) in absolute value,
  \item \( T_2 \) as the index set of the \( s_0 \) largest entries of \( h_{(T_0 \cup T_1) c} \) in absolute value,
  \item etc . . .
\end{enumerate}

Let \( J \) denote the number of subsets obtained in this process. Let \( T = T_0 \cup T_1 \). By (A.29) in Corollary A.5, we have that
\begin{equation}
(\lambda_{s_0} - 3 s_0 \varepsilon_{\text{min}}) \| h_{T} \|_2^2 \leq \| X_{T} h_{T} \|_2^2.
\end{equation}

Moreover, since \( h \) belongs to the kernel of \( X \),
\begin{align*}
\| X_{T} h_{T} \|_2^2 &= |\langle X_{T} h_{T}, X h \rangle - \langle X_{T} h_{T}, X_{T}^c h_{T}^c \rangle|,
\end{align*}
\begin{align*}
&= \left| \sum_{j=2,\ldots,J} \langle X_{T} h_{T}, X_{T}^c h_{T}^c \rangle \right|.
\end{align*}

On the other hand, by Lemma A.6 we have for \( j = 2, \ldots, J \),
\begin{equation}
\langle X_{T} h_{T}, X_{T}^c h_{T}^c \rangle \leq (\lambda_1 + 3 s_0 \varepsilon_{\text{max}}) \| h_{T} \|_2 \| h_{T}^c \|_2.
\end{equation}

Therefore,
\begin{align*}
\| X_{T} h_{T} \|_2^2 &= \sum_{j=2,\ldots,J} \langle X_{T} h_{T}, X_{T}^c h_{T}^c \rangle \\
&\leq \sum_{j=2,\ldots,J} |\langle X_{T} h_{T}, X_{T}^c h_{T}^c \rangle| \\
&\leq (\lambda_1 - \lambda_{s_0} + 3 s_0 (\varepsilon_{\text{max}} + \varepsilon_{\text{min}})) \| h_{T} \|_2 \sum_{j=2,\ldots,J} \| h_{T}^c \|_2.
\end{align*}

By Lemma [13, Lemma A.4], we get
\begin{equation}
\sum_{j=2,\ldots,J} \| h_{T}^c \|_2 \leq \frac{\| h_{T_0} \|_1}{\sqrt{s_0}}
\end{equation}

and we can deduce that
\begin{equation}
\| X_{T} h_{T} \|_2^2 \leq (\lambda_1 - \lambda_{s_0} + 3 s_0 (\varepsilon_{\text{max}} + \varepsilon_{\text{min}})) \| h_{T} \|_2 \frac{\| h_{T_0} \|_1}{\sqrt{s_0}}.
\end{equation}

Combined (3.21) with (3.19) gives
\begin{equation}
\| h_{T} \|_2 \leq \frac{\lambda_1 - \lambda_{s_0} + 3 s_0 (\varepsilon_{\text{max}} + \varepsilon_{\text{min}})}{\lambda_{s_0} - 3 s_0 \varepsilon_{\text{min}}} \frac{\| h_{T_0} \|_1}{\sqrt{s_0}}.
\end{equation}

The bound (3.13) on \( C \) in this Theorem can be made arbitrarily small by taking \( c_0 \) accordingly sufficiently small. 

\[\text{\textsuperscript{†}}\text{The last set contains the remaining smallest terms in absolute value and may not contain } s \text{ terms}\]
4. Conclusion

In this paper, we established a relationship between the coherence and a weak version of the Null Space Property for design matrices in Compressed Sensing. Our approach is based on perturbation theory and no randomness assumption on the design matrix is used to establish this property. We expect that this result will be helpful to study a larger class of designs than usually done in the literature. In a future paper, we will show that such bounds can be fruitfully applied to simplify the analysis of Robust PCA.

Appendix A. Technical lemmæ

A.1. Some perturbation results. Perturbation after appending a column to a given matrix is a special type of perturbation. A survey on this topic is [3].

A.1.1. Background. Recall that for a matrix $A$ in $\mathbb{R}^{n \times n}$, $p_A$ denotes the characteristic polynomial of $A$.

Lemma A.1. Cauchy’s Interlacing theorem. If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and associated eigenvectors $v_1, \ldots, v_n$, and $v \in \mathbb{R}^n$, then

$$
p_{A+vv^t}(x) = p_A(x) \left(1 - \sum_{i=1}^n \frac{(v, u_i)^2}{x - \lambda_i}\right).
$$

The previous lemma states in particular that the eigenvalues of $A$ interlace those of $A + vv^t$.

A.1.2. Appending one vector: perturbation of the smallest non zero eigenvalue. If we consider a subset $T_0$ of $\{1, \ldots, p\}$ and a submatrix $X_{T_0}$ of $X$, the problem of studying the eigenvalue perturbations resulting from appending a column $X_j$ to $X_{T_0}$, with $j \notin T_0$ can be studied using Cauchy’s Interlacing Lemma as in the following result.

Lemma A.2. Let $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$ and $X_{T_0}$ a submatrix of $X$. Let $\lambda_1 \geq \cdots \geq \lambda_{s_0}$ be the eigenvalues of $X_{T_0}X_{T_0}^t$. Let $\tilde{\lambda}_{s_0} \leq \lambda_{s_0}$. Assume that $\tilde{\lambda}_{s_0} < 1 - s_0\mu^2$, we have

$$
\lambda_{s_0+1} (X_{T_0}X_{T_0}^t + X_jX_j^t) \geq \tilde{\lambda}_{s_0} - \varepsilon_{s_0,\text{min}}
$$

with

$$
\varepsilon_{s_0,\text{min}} = \frac{1}{2} \left( \frac{s_0^2 \mu^2 \|X_{T_0}\|^2 + 4 s_0^3 \mu \|X_{T_0}\| \tilde{\lambda}_{s_0}}{1 - s_0\mu^2 - \tilde{\lambda}_{s_0}} \right).
$$

Proof. Setting $v = X_j$

$$
A = X_{T_0}X_{T_0}^t
$$

we obtain that the smallest nonzero eigenvalue of $X_{T_0}X_{T_0}^t + X_jX_j^t$ is the smallest root $\rho_{\text{min}}$ of

$$
f(x) = 1 - \sum_{i=1}^n \frac{(v, u_i)^2}{x - \lambda_i}.
$$

Therefore, $\rho_{\text{min}}$ is larger than the smallest positive root of

$$
\tilde{f}(x) = 1 - \frac{s_0 \gamma}{x - \tilde{\lambda}_{s_0}} - \frac{1 - s_0\mu^2}{x}
$$

for any upper bound $\gamma$ to $(v, u_i)^2$ for $i = 1, \ldots, s_0$. Thus, we find that

$$
\rho_{\text{min}} \geq \frac{1}{2} \left( s_0(\gamma - \mu^2) + \tilde{\lambda}_{s_0} + 1 - \sqrt{\frac{2 s_0 \gamma + 2 s_0 \gamma (\tilde{\lambda}_{s_0} + 1 - s_0\mu^2)}{1 - s_0\mu^2 - \tilde{\lambda}_{s_0}}} \right).
$$
As long as $1 - s_0\mu^2 > \lambda_{s_0}$, we have
\[
\rho_{\min} \geq \frac{1}{2} \left( s_0(\gamma - \mu^2) + \tilde{\lambda}_{s_0} + 1 - \left( 1 - s_0\mu^2 - \tilde{\lambda}_{s_0} \right) \sqrt{1 + \frac{s_0^2\gamma^2 + 2s_0\gamma(\tilde{\lambda}_{s_0} + 1 - s_0\mu^2)}{(1 - s_0\mu^2 - \tilde{\lambda}_{s_0})^2}} \right).
\]
Moreover, since $\sqrt{1 + a} \leq 1 + \frac{1}{2}a$, we get
\[
\rho_{\min} \geq \frac{1}{2} \left( s_0(\gamma - \mu^2) + \tilde{\lambda}_{s_0} + 1 - \left( 1 - s_0\mu^2 - \tilde{\lambda}_{s_0} \right) \left( 1 + \frac{s_0^2\gamma^2 + 2s_0\gamma(\tilde{\lambda}_{s_0} + 1 - s_0\mu^2)}{2 \left( 1 - s_0\mu^2 - \tilde{\lambda}_{s_0} \right)^2} \right) \right)
\]
which gives
(A.24) \[\rho_{\min} \geq \tilde{\lambda}_{s_0} - \varepsilon_{s_0,\min}\] with
\[
\varepsilon_{s_0,\min} = \frac{1}{2} \left( \frac{s_0^2\gamma^2 + 4s_0\gamma\tilde{\lambda}_{s_0}}{2 \left( 1 - s_0\mu^2 - \tilde{\lambda}_{s_0} \right)} \right).
\]
Let us now find out a reasonable value of $\gamma$. Let $X_{T_0} = U_0\Sigma_0V_0^T$ denote the singular value decomposition of $X_{T_0}$. We have
\[
|\langle X_j, u_{j_0} \rangle| = |\langle X_j, X_{T_0}V_0\Sigma_0e_{j_0} \rangle| = \|X_{T_0}^TX_j\|_2\|V_0\Sigma_0e_{j_0}\|_2 \leq \sqrt{s_0} \mu \|X_{T_0}\|.
\]
Therefore we can take
\[
\gamma = \sqrt{s_0} \mu \|X_{T_0}\|.
\]
Combining this result with (A.24), we get the desired result. \(\square\)

A.1.3. Appending one vector: perturbation of the largest eigenvalue. For the largest eigenvalue, we obtain

Lemma A.3. Let $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$ and $X_{T_0}$ a submatrix of $X$. Let $\lambda_1 \geq \ldots \geq \lambda_{s_0}$ be the eigenvalues of $X_{T_0}X_{T_0}^T$. Let $\tilde{\lambda}_1 \geq \lambda_1$, with $\tilde{\lambda}_1 > 1$. Then, we have
\[
\lambda_1 \left( X_{T_0}X_{T_0}^T + X_jX_j^T \right) \leq \tilde{\lambda}_1 + \varepsilon_{s_0,\max}.
\]
with
\[
\varepsilon_{s_0,\max} = \frac{1}{2} \left( \frac{s_0^3\mu^2\|X_{T_0}\|_2^2 + 4s_0^{3/2} \mu \|X_{T_0}\| \tilde{\lambda}_1}{2(\lambda_1 - 1)} \right).
\]

Proof. Setting $v = X_j$
\[
A = X_{T_0}X_{T_0}^T
\]
we obtain that the largest nonzero eigenvalue of $X_{T_0}X_{T_0}^T + X_jX_j^T$ is the largest root $\rho_{\max}$ of
\[
f(x) = 1 - \sum_{i=1}^n \frac{|\langle v, u_i \rangle|^2}{x - \lambda_i}.
\]
Therefore, $\rho_{\max}$ is smaller than the largest positive root of
\[
\tilde{f}(x) = 1 - \frac{s_0 \gamma}{x - \lambda_1} - \frac{1}{x}
\]
for any upper bound $\gamma$ to $|\langle v, u_i \rangle|^2$ for $i = 1, \ldots, s_0$. Hence, we find that
(A.25) \[\rho_{\max} \leq \frac{1}{2} \left( s_0 \gamma + \tilde{\lambda}_1 + 1 + \sqrt{s_0^2\gamma^2 + 2s_0\gamma(\tilde{\lambda}_1 + 1) + \left( 1 - \tilde{\lambda}_1 \right)^2} \right).
\]
Since the columns of $X$ have unit $\ell_2$-norm, we have $1 < \lambda_1$, and thus one obtains from (A.25) that

$$\rho_{\text{max}} \leq \frac{1}{2} \left( s_0 \gamma + \tilde{\lambda}_1 + 1 + \left( \tilde{\lambda}_1 - 1 \right) \sqrt{1 + \frac{s_0^2 \gamma^2 + 2 s_0 \gamma (\tilde{\lambda}_1 + 1)}{(\tilde{\lambda}_1 - 1)^2}} \right)$$

which gives

$$\rho_{\text{max}} \leq \tilde{\lambda}_1 + \varepsilon_{s_0,\text{max}}$$

with

$$\varepsilon_{s_0,\text{max}} = \frac{1}{2} \left( \frac{s_0^2 \gamma^2 + 4 s_0 \gamma \tilde{\lambda}_1}{2(\tilde{\lambda}_1 - 1)} \right).$$

We finally plug in the value of $\gamma$ found earlier in the proof of Lemma A.2 to get the desired result. □

A.1.4. Successive perturbations. If we append $s_0$ columns successively, we obtain the following result.

**Lemma A.4.** Let $T_0 \subset \{1, \ldots, p\}$ with $|T_0| = s_0$ and $X_{T_0}$ a submatrix of $X$. Let $\lambda_1 \geq \ldots \geq \lambda_{s_0}$ be the eigenvalues of $X_{T_0}^T X_{T_0}$. Let $\lambda_1 \geq \tilde{\lambda}_1$ and $\lambda_{s_0} \leq \tilde{\lambda}_{s_0}$. Let $T_1 \subset \{1, \ldots, p\}$ with $|T_1| = s_1$ and $T_0 \cap T_1 = \emptyset$. Assume

1. $1 - (s_0 + s_1) \mu > \tilde{\lambda}_{s_0} > \eta$;
2. $1 < \tilde{\lambda}_1 < 2 - \eta$;
3. $s_1 < \min \left( \frac{\tilde{\lambda}_1 - \eta}{\varepsilon_{\text{min}}}, \frac{2 - \eta - \tilde{\lambda}_1}{\varepsilon_{\text{max}}} \right)$;

with

$$\varepsilon_{\text{min}} = \frac{1}{4} \left( \frac{s_0^3 \mu^2 \eta^2 + 4 s_0^3 / 2 \mu \eta^2}{1 - s_0 \mu^2 - \eta} \right)$$

and

$$\varepsilon_{\text{max}} = \frac{1}{4} \left( \frac{(s_0 + s_1)^3 \mu^2 (2 - \eta)^2 + 4(s_0 + s_1)^3 / 2 \mu (2 - \eta)^2}{\lambda_1 - 1} \right)$$

Then

(A.26) \hspace{1cm} \lambda_1 \left( X_{T_0 \cup T_1}^T X_{T_0 \cup T_1} \right) \leq \tilde{\lambda}_{s_0} - s_1 \varepsilon_{\text{min}}

and

(A.27) \hspace{1cm} \lambda_{s_0+s_1} \left( X_{T_0 \cup T_1}^T X_{T_0 \cup T_1} \right) \geq \tilde{\lambda}_1 + s_1 \varepsilon_{\text{max}}

**Proof.** The proof relies on induction. First of all, note that from assumption (3)

(i) $\tilde{\lambda}_{s_0} - s_1 \varepsilon_{\text{min}} > \eta$;
(ii) $\tilde{\lambda}_1 + s_1 \varepsilon_{\text{max}} < 2 - \eta$.

We apply lemma A.2 to $X_{T_0}^T X_{T_0} + X_{j_1}^T X_{j_1}$ with $j_1 \in T_1$. We have

$$\lambda_{s_0+1} \left(X_{T_0}^T X_{T_0} + X_{j_1}^T X_{j_1}\right) \geq \tilde{\lambda}_{s_0} - \varepsilon_{s_0,\text{min}}$$

with $\varepsilon$ defined in A.2. Since $\varepsilon_{s_0,\text{min}} \leq \varepsilon_{\text{min}}$, we get

$$\lambda_{s_0} \left(X_{T_0}^T X_{T_0}\right) \geq \lambda_{s_0+1} \left(X_{T_0}^T X_{T_0} + X_{j_1}^T X_{j_1}\right) \geq \tilde{\lambda}_{s_0} - \varepsilon_{\text{min}}.$$

It implies by (i) that

$$1 - (s_0 + s_1) \mu \geq \lambda_{s_0+1} \left(X_{T_0}^T X_{T_0} + X_{j_1}^T X_{j_1}\right) > \eta.$$

Thus, the induction hypothesis is verified and we can apply Lemma A.2 for the next step of the induction. This leads to (A.26).

For the lower bound (A.27), we have from lemma A.1.3

$$\lambda_1 \left(X_{T_0}^T X_{T_0} + X_{j_1}^T X_{j_1}\right) \leq \tilde{\lambda}_1 + \varepsilon_{s_0,\text{max}}$$
Proof. \[ \lambda_1(X_{T_0}^T X_{T_0}^* + X_{j_1}^T X_{j_1}^*) < 2 - \eta. \]

We can then apply lemma A.1.3 to the next step. The result follows by induction. \( \square \)

**Corollary A.5.** Let \( T_0 \subset \{1, \ldots, p\} \) with \( |T_0| = s_0 \) and \( X_{T_0} \) a submatrix of \( X \). Let \( \lambda_1 \geq \ldots \geq \lambda_{s_0} \) be the eigenvalues of \( X_{T_0}^T X_{T_0}^* \). Let \( \lambda_1 \geq \lambda_0 \) and \( \lambda_{s_0} \leq \lambda_0 \). Let \( T_1 \subset \{1, \ldots, p\} \) with \( |T_1| = s_1 \) and \( T_0 \cap T_1 = \emptyset \). Set \( \eta = \frac{1}{2} \) and \( s_1 = 3s_0 \). Assume

1. \( 1 - (s_0 + s_1)\mu > \tilde{\lambda}_{s_0} > \eta; \)
2. \( 1 < \lambda_1 < 2 - \eta; \)
3. \( s_1 < \min \left( \frac{\lambda_{s_0} - \eta}{\lambda_{s_0}}, \frac{2\lambda_{s_0} - \eta}{\lambda_1} \right); \)

with

\[
\varepsilon_{\min} = \frac{1}{4} \left( \frac{\mu^2}{s_0^2} \right) \left( 1 - s_0 \mu^2 - \frac{1}{2} \right)
\]

\[
\varepsilon_{\max} = \frac{1}{4} \left( \frac{144 s_0^4 \mu^2 + 32 s_0^{3/2} \mu (2 - \eta)^2}{(\lambda_1 - 1)} \right)
\]

Assume also

\[
\mu \leq \min \left\{ \frac{1}{\sqrt{288s_0^{5/2} (2s_0^{3/2} + 1)}}, \frac{1}{\sqrt{\frac{3}{2} s_0^4 + 6s_0^{5/2} + 2s_0^3}} \right\}.
\]

Then,

\[(A.28) \quad \lambda_1 \left( X_{T_0 \cup T_1}^T X_{T_0 \cup T_1} \right) \leq \tilde{\lambda}_1 + 3s_0 \varepsilon_{\max} \]

and

\[(A.29) \quad \lambda_{s_0 + s_1} \left( X_{T_0 \cup T_1}^T X_{T_0 \cup T_1} \right) \geq \tilde{\lambda}_{s_0} - 3s_0 \varepsilon_{\min}. \]

**Proof.** Set \( \eta = \frac{1}{2} \), assumption \( \Box \) writes

\[
s_1 < \frac{4(\tilde{\lambda}_{s_0} - \frac{1}{2}) (\frac{1}{2} - s_0 \mu^2)}{s_0^3 \mu^2 \frac{1}{4} + s_0^{3/2} \mu}
\]

and

\[
s_1 < \frac{4(\frac{3}{2} - \tilde{\lambda}_1) (\tilde{\lambda}_1 - 1)}{(s_0 + s_1)^3 \mu^2 \frac{9}{4} + 9(s_0 + s_1)^{3/2} \mu}
\]

which leads to

\[
s_1 \left( s_0^3 \mu^2 \frac{1}{4} + s_0^{3/2} \mu \right) + 4 \left( \tilde{\lambda}_{s_0} - \frac{1}{2} \right) s_0 \mu^2 < 2 \left( \tilde{\lambda}_{s_0} - \frac{1}{2} \right)
\]

and

\[
s_1 \left( (s_0 + s_1)^3 \mu^2 \frac{9}{4} + 9(s_0 + s_1)^{3/2} \mu \right) < 4 \left( \frac{3}{2} - \tilde{\lambda}_1 \right) (\tilde{\lambda}_1 - 1).
\]

Since \( \mu < 1 \)

\[
s_1 \left( s_0^3 \mu^2 \frac{1}{4} + s_0^{3/2} \mu^2 \right) + 4 \left( \tilde{\lambda}_{s_0} - \frac{1}{2} \right) s_0 \mu^2 < 2 \left( \tilde{\lambda}_{s_0} - \frac{1}{2} \right)
\]

and

\[
s_1 \left( (s_0 + s_1)^3 \mu^2 \frac{9}{4} + 9(s_0 + s_1)^{3/2} \mu^2 \right) < 4 \left( \frac{3}{2} - \tilde{\lambda}_1 \right) (\tilde{\lambda}_1 - 1)
\]

The result follows by factoring out \( \mu^2 \) and setting \( s_1 = 3s_0 \). \( \square \)
A.2. Bounding scalar products.

Lemma A.6. Let $|T_0| = s_0$ and $|T_1| = s_0$, $T_0$, $T_1$ disjoint. Let $T = T_0 \cup T_1$ and $T'$ be two disjoint subsets of $\{1, \ldots, p\}$ with $|T'| = 2s_0$. Let $g$ and $h$ be vectors in $\mathbb{R}^p$. Assume that

$$
\mu \leq \min \left\{ \frac{1}{\sqrt{288s_0^{5/2}}} \left( \frac{(2s_0^{3/2} + 1)}{2s_0^4 + 6s_0^{5/2} + 2s_0} \right) \right\}.
$$

Then,

$$
(A.30)

\langle X_T g_T, X_T^T h_{T'} \rangle \leq (\lambda_1 + 3 \ s_0 \ \varepsilon_{\max}) \ \| g_T \|_2 \ \| h_{T'} \|_2.
$$

Proof. Assume first that $\| g_T \|_2 = \| h_{T'} \|_2 = 1$. The parallelogram law now gives

$$
\langle X_T g_T, X_T^T h_{T'} \rangle \leq \frac{1}{4} \| X_T g_T + X_T^T h_{T'} \|_2^2 - \| X_T g_T - X_T^T h_{T'} \|_2^2
$$

\begin{align*}
&\leq \frac{1}{4} \| X_T g_T + X_T^T h_{T'} \|_2^2 - \| X_T g_T - X_T^T h_{T'} \|_2^2 \\
&= \frac{1}{4} \| (X_T g_T) \pm (X_T^T h_{T'}) \|_2^2 = \| (X_{T,T'}) (g_T \pm h_{T'}) \|_2^2.
\end{align*}

Notice that

$$
\| X_T g_T \pm X_T^T h_{T'} \|_2^2 = \| X_{T,T'} (g_T \pm h_{T'}) \|_2^2
$$

By Corollary A.5, we have

$$
(\lambda_{s_0} - 3 \ s_0 \ \varepsilon_{\min}) \ \| g_T + h_{T'} \|_2^2 \leq \| g_T + h_{T'} \|_2^2 \| X_T g_T \pm X_T^T h_{T'} \|_2^2 \leq (\lambda_1 + 3 \ s_0 \ \varepsilon_{\max}) \ \| g_T + h_{T'} \|_2^2.
$$

From this, and the fact that $g_T$ and $h_T$ are unit norm, we deduce that

$$
\langle X_T g_T, X_T^T h_{T'} \rangle \leq \lambda_1 - \lambda_{s_0} + 3 \ s_0 \ (\varepsilon_{\max} + \varepsilon_{\min}).
$$

The proof is completed using homogeneity. □

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