Short Distance Asymptotics of Ising Correlations

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Abstract

We prove that the short distance asymptotics for the even Ising model scaling functions from below $T_c$ is given by the Luther-Peskel formula. Generalizations to the odd scaling functions and Holonomic Fields are given.

1 Introduction

In this paper we will use the Sato, Miwa, Jimbo characterization of the scaling functions for the two dimensional Ising model to show that the short distance asymptotics of the even scaling functions below the critical point are given by the Luther-Peskel formula (see Theorem(1) below). We will then present results for the odd correlations below $T_c$ and also for holonomic quantum fields which are a consequence of the same technique used to prove Theorem(1).

This paper is a sequel to [10] and the reader is referred to that paper for a more detailed explanation of the Ising model scaling limits than we will give here. Continuum limits for the two dimensional Ising correlations on a lattice were first considered in [12], where, in addition, a connection with Painlevé transcendents was discovered. In a series of papers Sato, Miwa, and Jimbo showed that the continuum correlations (the scaling functions) were associated with monodromy preserving deformations of the Euclidean Dirac equation and that this connection sufficed to account for the appearance of the Painlevé transcendents, [12]-[16]. Here we exploit the fact that the SMJ formula for the log derivative of the scaling function (a $\tau$ function in their terminology) can be expressed in terms of the Fourier coefficients of a solution to the linear Dirac equation. We analyse the linear problem in order to control the short distance asymptotics. This analysis was suggested by the success of Riemann-Hilbert techniques in obtaining asymptotics for non linear integrable systems [3], where a similar connection with a linear problem is a central feature.

We would like to point out that the two point function both for the Ising model and for Holonomic Fields in general has been analysed in more detail than
the result we obtain here, \[17, 1, 18\]. In particular, the constant term in the short distance asymptotics is obtained–our result for the log derivative has nothing to say about this.

We will begin by recalling some of the results of \[10\] where a sketch of the the proof was presented. The SMJ characterization involves certain solutions to the Dirac equation in two dimensions so we will start with a description of the situation of interest to us. The Euclidean Dirac operator in \(\mathbb{R}^2\) (with a mass perturbation) is given by

\[
mI - \partial = \begin{bmatrix} m & -2\partial \\ -2\bar{\partial} & m \end{bmatrix},
\]

where,

\[
\partial := \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right),
\]

\[
\bar{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).
\]

Although we will not be working exclusively with holomorphic functions, the presence of \(\partial\) and \(\bar{\partial}\) in the Dirac operator makes it very convenient to introduce the complex variable \(z = x_1 + ix_2\) with \(\bar{z} = x_1 - ix_2\); we thus identify \(\mathbb{R}^2\) with \(\mathbb{C}\) in the usual fashion. For brevity we will write \(f(z)\) for a function of two real variables even though it is customary to use a notation like \(f(z, \bar{z})\) to avoid the temptation to regard \(f(z)\) as a holomorphic function of \(z\).

Let \(a = \{a_1, a_2, \ldots, a_N\}\) denote a collection of \(N\) distinct points in \(\mathbb{C}\). The solutions of the Dirac equation that we are interested in are smooth sections of a rank 2 vector bundle over the punctured plane \(\mathbb{C}\setminus a\).

For the purpose of allowing some later remarks we will begin by defining a slightly more general family of line bundles, \(E_\lambda\), than is relevant for the Ising model. For \(j = 1, 2, \ldots, N\) suppose real numbers \(\lambda_j\) are given with \(|\lambda_j| \leq \frac{1}{2}\). Define

\[
\Lambda_j = e^{2\pi i \lambda_j},
\]

and write,

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N).
\]

Roughly speaking the smooth sections of the bundle \(E_\lambda \rightarrow \mathbb{C}\setminus a\) will be multivalued functions on \(\mathbb{C}\setminus a\) with values in \(\mathbb{C}\) which have multiplier \(\Lambda_j\) when continued about a loop that circles \(a_j\) counterclockwise. This can be made precise in an elegant fashion by working on the simply connected covering space of \(\mathbb{C}\setminus a\) and then restricting attention to smooth sections that transform appropriately under the action of \(\pi_1(\mathbb{C}\setminus a)\) by deck transformations. However, some later developments will be clearer for us if we can use functions with specific branching behavior as multipliers taking sections of \(E_\lambda\) to sections of the trivial bundle over \(\mathbb{C}\setminus a\). It will be easiest to be precise about this multiplier action if we define the bundles \(\mathcal{E}_\lambda\) by giving transition functions, in spite of the fact that this is a little clumsy.
To begin, note that there are only a finite number of lines each of which consists of all multiples of \( a_i - a_j \) for \( i \) and \( j \) distinct. Thus it is possible to choose a vector \( r \neq 0 \) which is not contained in any of these lines. Then the rays, \( r_j \), defined by
\[
r_j = \{ z : z = a_j + tr, t > 0 \},
\]
do not intersect. Choose an argument \( \theta_r \) for \( r \) so that \( r = |r|e^{i\theta_r} \) with \( |\theta_r| \leq \pi \) and let \( \theta(z) \) denote the polar angle with
\[
\theta_r - \pi < \theta(z) < \theta_r + \pi
\]
\[
z = |z|e^{i\theta(z)},
\]
This angle is branched along the ray \( -r \). For \( \epsilon > 0 \) define a tubular neighborhood, \( t_j(\epsilon) \), of \( r_j \) by,
\[
n_j(\epsilon) = \{ z : \text{dist}(z, r_j) < \epsilon \} \cap \{ z : |\theta(z) - \theta_r| < \frac{\pi}{4} \}.
\]
Now choose \( \epsilon > 0 \) small enough so that the tubular neighborhoods \( t_j(\epsilon) \) are mutually disjoint and so that the disks,
\[
D_j(2\epsilon) := \{ z : |z - a_j| < 2\epsilon \},
\]
are also mutually disjoint (this will be useful later on).

We now introduce a covering of \( C \setminus \mathfrak{a} \) over each element of which the bundle \( E_\lambda \) is trivial. Let
\[
U_0 := \{ z \in C \setminus \mathfrak{a} : z \notin r_j \text{ for } j = 1, 2, \ldots, N \},
\]
Let
\[
U_j := t_j(\epsilon) \text{ for } j = 1, 2, \ldots, N.
\]
Now we glue together the trivial bundles,
\[
U_k \times C \to U_k \text{ for } k = 0, 1, \ldots, N,
\]
by giving the transition functions \( s_j \) that define the bundle \( E_\lambda \). For \( j = 1, \ldots, N \) define,
\[
s_j(z) = \begin{cases} 
\Lambda_j \text{ for } \theta(z_j) < 0 \\
1 \text{ for } \theta(z_j) > 0.
\end{cases}
\]
Then the bundle \( E_\lambda \) is defined by the following transition maps between vectors \((z, v)_0 \in U_0 \times C\) in the trivial bundle over \( U_0 \) and vectors \((z, v)_j \in U_j \times C\) in the trivial bundle over \( U_j \) for \( k = 1, 2, \ldots, N \),
\[
(z, v)_0 = (z, s_j(z)v)_j \text{ for } z \in U_0 \cap U_j.
\]
The function \( s_j(z) \) is smooth since it is constant on each of the two components of \( U_0 \cap U_j \). The bundle that is relevant for the Ising model is the one with the choice \( \Lambda_j = -1 \) for all \( j = 1, 2, \ldots, N \). For simplicity we will denote this bundle by \( \mathcal{E} \) with no subscript.
The rank 2 vector bundles that are more directly of interest to us are $E_\lambda \otimes \mathbb{C}^2$ and $E \otimes \mathbb{C}^2$, the direct sum of two copies of $E_\lambda$ and $E$ respectively. For simplicity we will use the same notation, $E_\lambda$ and $E$, to denote these vector bundles and when necessary make distinctions by referring to the line bundles $E_\lambda$ and $E$.

The differential operator $mI - \varphi$ acts on $C^\infty(E_\lambda)$, the space of smooth sections of the vector bundle $E_\lambda$, since it commutes with multiplication by constants. We will now define a family of local smooth sections of $C^\infty(E_\lambda)$ which are simultaneously solutions of the Dirac equation, $(mI - \varphi)w = 0$ and eigenfunctions for the infinitesimal rotation about $a_j$, $R_j = z_j \partial_j - \bar{z}_j \bar{\partial}_j + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which commutes with $mI - \varphi$. We write $z_j = z - a_j$ and $\partial_j = \partial_{z_j}$. Note that this infinitesimal rotation has eigenvalues which are translated by $\pm \frac{i}{2}$ compared to the infinitesimal monodromy. Following SMJ we will parametrize our local wave functions by the $R_j$ eigenvalue rather than the infinitesimal monodromy.

Let $\Theta(z)$ denote the angular coordinate at 0 defined so that for $z \neq \{tr : t > 0\}$ we have,

$$z = |z|e^{i\Theta(z)}, \text{ with } \theta_\pi < \Theta(z) < \theta_\pi + 2\pi.$$

For $\ell$ a real number we define a function $w_\ell(z)$ for $z \in \mathbb{C}\setminus\{r\}$ by,

$$w_\ell^0(z) = \begin{pmatrix} e^{i(\ell - \frac{j}{2})\Theta(z)} I_{\ell - \frac{1}{2}}(m|z|) \\ e^{i(\ell + \frac{j}{2})\Theta(z)} I_{\ell + \frac{1}{2}}(m|z|) \end{pmatrix},$$

where $I_k$ is the modified Bessel function of order $k$. For $z \in \mathbb{C}\setminus\{-r\}$ we define,

$$w_\ell^r(z) = \begin{pmatrix} e^{i(\ell - \frac{j}{2})\Theta(z)} I_{\ell - \frac{1}{2}}(m|z|) \\ e^{i(\ell + \frac{j}{2})\Theta(z)} I_{\ell + \frac{1}{2}}(m|z|) \end{pmatrix}.$$

The only difference being, of course, the choice of angle. Where defined these are solutions to the Dirac equation $(mI - \varphi)w = 0$ and are eigenfunctions of the infinitesimal rotation $Rw_\ell = \ell w_\ell$ about 0 $[\mathbb{4}]$. Now let $\ell$ denote a real number and define (for $|z_j| < 2\varepsilon$ say),

$$w_\ell(z_j) = w_\ell^0(z_j) \text{ in the } U_0 \text{ trivialization}$$
$$w_\ell(z_j) = w_\ell^r(z_j) \text{ in the } U_j \text{ trivialization}$$

Then it is easy to check that $w_\ell(z_j)$ is a local section of $C^\infty(E_\lambda)$ provided $\ell \equiv \frac{1}{2} + \lambda_j \bmod \mathbb{Z}$. Now define a conjugation on $\mathbb{C}^2$ by,

$$\begin{pmatrix} a \\ b \end{pmatrix}^* = \begin{pmatrix} b \\ a \end{pmatrix}.$$

This conjugation commutes with the Dirac operator $\varphi$ and we define,

$$w_\ell^*(z) = \begin{pmatrix} \bar{w}_\ell^0(z) \\ \bar{w}_\ell^r(z) \end{pmatrix}.$$
One can check that \( w^\ast_j(z) \) is a local smooth section of \( C^\infty(\mathcal{E}_\lambda) \) if and only if \( \ell \equiv \frac{1}{2} - \lambda_j \mod \mathbb{Z} \). It is a result of SMJ that every solution to \((mI - \partial)w = 0\) in \( C^\infty(\mathcal{E}_\lambda) \) has local expansions,

\[
 w(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} a_k(w)w_{k+\lambda_j}(z) + b_k(w)w^\ast_{k-\lambda_j}(z),
\]

valid for \( 0 < |z_j| < 2\epsilon \) [14], [9]. As the reader may check the coefficients \( a_j(w) \) and \( b_j(w) \) are simply related to Fourier coefficients in the expansion of the restriction of \( w \) to say the circle \( |z_j| = \epsilon \). We will refer to these coefficients as local expansion coefficients.

For the Ising case \( \lambda_j = \pm \frac{1}{2} \) and it is better not to use this form of the expansion (which would require a somewhat arbitrary choice of \( \pm \frac{1}{2} \) at each \( a_j \)); instead we will just write,

\[
 w(z) = \sum_{n \in \mathbb{Z}} c_n(w)w_n(z_j) + c_n^\ast(w)w_n^\ast(z_j).
\]

Note that we have changed the names of the local expansion coefficients in \( [2] \) to \( c_j(w) \) and \( c_j^\ast(w) \) so that it coincides with the terminology in [14]. Our way of writing \( [2] \) is different than the corresponding local expansions in [14] and so we have given different names to the local expansion coefficients.

Before we move on we will make one further observation about local expansions in a neighborhood of \( \infty \). Suppose that \( R > 0 \) is big enough so that all the points \( a_j \) for \( j = 1, 2, \ldots, N \) are inside the circle of radius \( R \). Then \( \{z : |z| > R\} \setminus \bigcup_{j} r_j \) splits into \( N \) distinct components and the \( U_0 \) trivialization is not very convenient for the description of sections of \( \mathcal{E} \) over this set. In particular suppose that \( N \) is even. Then we can alternately flip the signs of sections supported in adjacent components of the \( U_0 \) trivialization to produce a trivialization \( U_\infty \) for \( \mathcal{E} \) over \( \{z : |z| > R\} \). Actually the \( U_0 \) trivialization is not defined over the rays \( r_j \) but because of the sign flips on adjacent components it is easy to see that \( U_\infty \) extends to a trivialization of \( \mathcal{E} \) over the exterior of the disk of radius \( R \). It is also clear that \( U_\infty \) is only determined itself up to an overall sign which we fix by declaring the \( U_\infty \) trivialization of the \( U_0 \) section \( \prod_j (z - a_j)^{\epsilon_j} \) for \( |\epsilon| = 0 \) to be,

\[
 \prod_j \left(1 - \frac{a_j}{z}\right)^{\epsilon_j} \text{ for } |z| > R,
\]

where the fractional powers in this last product are the holomorphic functions of \( z \) normalized to be \( 1 \) at \( z = \infty \).

It can be shown (\cite{[13], [14]}) that sections \( w \in L^2(\mathcal{E}) \) which are solutions to the Dirac equation in the exterior of the disk of radius \( R \) have convergent expansions (in the \( U_\infty \) trivialization),

\[
 w(z) = \sum_{n \in \mathbb{Z}} c_n^\infty(w)\tilde{w}_n(z), \tag{3}
\]
where,
\[ \hat{w}_n(z) := \begin{bmatrix} -e^{-in\theta}K_n(m|z|) \\ e^{-i(n-1)\theta}K_{n-1}(m|z|) \end{bmatrix}. \]

The functions \( K_n \) are the modified Bessel functions that tend to zero at \( \infty \). The reader should note that there is more than one definition of these functions (differing by a factor \( e^{in\pi} \)). We are using the version defined in [8]. Also note that because \( n \) is an integer the choice of angle \( \theta \) is irrelevant.

Now write \( x \cdot y = x_1 y_1 + x_2 y_2 \) for the standard bilinear form on \( \mathbb{C}^2 \), so that \( \bar{x} \cdot y \) is the standard Hermitian form. For \( w, v \in C^\infty_0(\mathcal{E}_\lambda) \) define an inner product,
\[ (w, v) = \frac{i}{2} \int_{\mathbb{C}} \bar{w} \cdot v \, dz d\bar{z}, \]

which is well defined since \( \bar{w}(z) \cdot v(z) \) decends to a compactly supported function on \( \mathbb{C} \setminus a \). We will write \( L^2(\mathcal{E}_\lambda) \) for the Hilbert space completion of \( C^\infty_0(\mathcal{E}_\lambda) \) with respect to the norm induced by this inner product.

For the rest of this introduction we will specialize our considerations to the situation relevant to the Ising model. For \( n \) an integer we write,
\[ w_n^R = \frac{1}{2} (w_n + w_n^*), \]
\[ w_n^I = \frac{1}{2i} (w_n - w_n^*), \]

for the real and imaginary parts of \( w_n \) with respect to the conjugation *. Since \( \Lambda_j = -1 \) is real for all \( j \) it follows that \( w_n^R(z_j) \) and \( w_n^I(z_j) \) are local sections of \( C^\infty(\mathcal{E}) \). In [10] it is shown that for \( j = 1, 2, \ldots, N \) there exists a real solution \( W_j \) (\( W_j^* = W_j \)) to the Dirac equation,
\[ (m I - \partial) W_j = 0, \]

which is in \( L^2(\mathcal{E}) \) and which has leading order local expansions given by,
\[ W_j(z) = \delta_{ij} w_0^I(z_i) + T_{ij} w_0^R(z_i) + \cdots \text{ for } i = 1, 2, \ldots, N \] (4)

Note that the coefficients \( w_n(z) \) are less and less locally singular at \( z = 0 \) as \( n \) increases. The \( + \cdots \) in (4) refer to terms with \( w_n \) and \( w_n^* \) for \( n > 0 \). Also note that in [4] it is not necessary to specify what the coefficients \( T_{ij} \) are—they are already uniquely determined by the other conditions on \( W_j \) [4].

We are now ready to present the SMJ characterization of the Ising model scaling function from below \( T_c \), \( \tau_-(ma) = \tau_-(ma_1, ma_2, \ldots, ma_N) \). It is,
\[ d_a \log \tau_-(ma) = \frac{m}{2\bar{t}} \sum_j c_j^1(\mathcal{W}_j) da_j - \overline{c_j^1(\mathcal{W}_j) \bar{d}_j}. \] (5)

The reader might want to consult [13] or [16] for an explanation of what exactly \( \tau_- \) is and how it is related to two dimensional Ising correlations. Most of the rest of this paper will be devoted to understanding the solution \( W_j \) well enough in the limit \( m \to 0 \) so that we can compute the limiting values of the coefficients \( mc_j^1(\mathcal{W}_j) \) which appear in [18]. Our principal result is,
Theorem 1 (Luther-Peschel Asymptotics) Suppose that \( N \) is even. Then
\[
\lim_{m \to 0} d_a \log \tau_-(ma) = \frac{1}{2} d_a \log \sum_{|\epsilon| = 0} \prod_{i < j} |a_i - a_j|^{2\epsilon_i \epsilon_j}
\]
where the sum is over all choices of \( \epsilon_k = \pm \frac{1}{2} \) with,
\[
|\epsilon| := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_N = 0.
\]
After the proof of this result we will indicate the changes that are needed to adapt the proof to the case where \( N \) is odd. We find for \( N \) odd,
\[
\lim_{m \to 0} d_a \log \tau_-(ma) = \frac{1}{2} d_a \log \sum_{|\epsilon| = \pm \frac{1}{2}} \prod_{i < j} |a_i - a_j|^{2\epsilon_i \epsilon_j}.
\]
We will also indicate how to derive the short distance behavior of the correlations for Holonomic Fields.

Very briefly the rest of the paper is organized as follows. In section 2 we characterize \( W_j \) as the solution to an inhomogeneous boundary value problem on a finite domain. In the third section we introduce the Green function for the \( m \to 0 \) limit of this boundary value problem. In the fourth section we introduce the associated boundary value projection. In the fifth section we discuss the inversion of a suitable restriction of this projection. In the sixth section we discuss how to put these results together to give the perturbation scheme which we use to approximate \( W_j \) in the limit \( m \to 0 \). In the seventh section we examine the application of the same technique to other problems.

2 An Equivalent Boundary Value Problem

The tool we will use in dealing with the \( m \to 0 \) limit of \( W_j \) is a characterization of \( W_j \) as the solution to an inhomogeneous boundary value problem. We will now describe this characterization. It is a result of SMJ\[14\] that the space of solutions \( w \in C^\infty(E) \) to the Dirac equation,
\[
(mI - \phi)w = 0,
\]
which are also in \( L^2(E) \) is \( N \) dimensional. We write \( N \) for this space of solutions. For \( w \in N \) define,
\[
c_0(w) = (c_0^1(w), c_0^2(w), \ldots, c_0^N(w)) \in C^N,
\]
with a similar definition for \( c_0^0(w) \). Now let \( \mathcal{N} \) denote the image of \( N \) in \( C^N \oplus C^N \) under the map,
\[
N \ni w \to (c_0(w), c_0^0(w)).
\]
Suppose now that \( \mathcal{I} \) is any subspace of \( C^N \oplus C^N \) which is transverse to \( \mathcal{N} \). If \( f \in C_0^\infty(E) \) then in \[10\] it was proved that there exists a unique solution \( w \in L^2(E) \) to
\[
(mI - \phi)w = f
\]
which satisfies the boundary condition \((c_0(w), c_0^\star(w)) \in I\). It was also shown there that the subspace \(I\) given by the set of vectors \((v, v)\) for \(v \in C^N\) (the diagonal subspace) is transverse to \(N\). Henceforth we will work with the subspace \(I\) which corresponds to the boundary condition,

\[
c_0(w) = c_0^\star(w). \tag{7}
\]

Now we will make a subtraction from \(W_j\) which will put the result in the subspace of sections of \(E\) satisfying (7). Let \(\varphi(z)\) denote a non-negative function in \(C_0^\infty(\mathbb{R}^2)\) which is identically 1 for \(|z| < 1\) and identically 0 outside the ball of radius 2. Define,

\[
\varphi_j,\epsilon(z) = \varphi\left(\frac{z - a_j}{\epsilon}\right).
\]

Then since \(\epsilon\) has been chosen small enough we know that \(\varphi_j,\epsilon\) is one in a neighborhood of \(a_j\) and vanishes near \(a_i\) for all \(i \neq j\). Now define,

\[
\delta W_j(z) = m^\frac{1}{2}(W_j(z) - \varphi_j,\epsilon(z)w_0^I(z - a_j)). \tag{8}
\]

Then consulting (7) we see that if we look at the local expansion for \(\delta W_j\) in an \(\epsilon\) neighborhood of \(a_j\) then the local expansion coefficients satisfy the condition (7). The scale factor \(m^\frac{1}{2}\) has been introduced so that the following limit exists,

\[
\lim_{m \to 0} m^\frac{1}{2}w_0^I(z_j) = \frac{1}{\sqrt{2\pi i}} \begin{pmatrix} z_j^{-\frac{1}{2}} \\ -\bar{z}_j^{-\frac{1}{2}} \end{pmatrix}.
\]

Here we used \(\Gamma(\frac{1}{2}) = \sqrt{\pi}\), the fractional powers of \(z_j\) and \(\bar{z}_j\) that occur are branched along \(z = r_j\) and we employ the convention that the section \(w_0^I\) can be identified with its \(U_0\) trivialization (which appears on the right hand side). Using the fact that both \(W_j\) and \(w_0^I(z_j)\) satisfy the massive Dirac equation we find that,

\[
(m - \tilde{\varphi})\delta W_j = \begin{pmatrix} 0 & -2\partial \varphi_j \\ -2\partial \bar{\varphi}_j & 0 \end{pmatrix} m^\frac{1}{2}w_0^I(z_j) := f_j. \tag{9}
\]

We are now prepared to give an alternative characterization of \(\delta W_j\). Choose \(R > 0\) big enough so that \(D_i(2\epsilon)\) is contained inside \(|z| = R\) for \(i = 1, 2, \ldots, N\). Let \(D_\infty = \{z : |z| \geq R\}\) and define the bounded domain, \(\Omega\), by,

\[
\Omega = \mathbb{C} \setminus \{\cup_{i=1}^n D_i(\epsilon) \cup D_\infty\}.
\]

We write \(H^k(\mathcal{E}_\Omega)\) for the Sobolev space of sections of \(\mathcal{E}\) over \(\Omega\) which are in \(L^2(\Omega)\) together with all their weak derivatives up to and including those of order \(k\).

**Lemma 1** The smooth section \(\delta W_j\) of \(\mathcal{E}_\Omega\) (the restriction of \(\mathcal{E}\) to \(\Omega\)) is uniquely characterized by the following three properties,

1. \(\delta W_j \in H^1(\mathcal{E}_\Omega)\) satisfies the inhomogeneous Dirac equation (9) in \(\Omega\).
2. For \( i = 1, 2, \ldots, N \) the local Fourier expansions \((\ref{eq:local_fourier})\) for \( \delta \mathcal{W}_j \) restricted to \( C_\epsilon(a_i) \) have coefficients \( c_1^j(\delta \mathcal{W}_j) \) and \( c_k^j(\delta \mathcal{W}_j) \) that vanish for \( k < 0 \) and are equal for \( k = 0 \).

3. The section \( \delta \mathcal{W}_j \) has a Fourier expansion \( \delta \mathcal{W}_j = \sum_{n \in \mathbb{Z}} c_n^\infty(\delta \mathcal{W}_j) \hat{w}_n(z) \), on the circle of radius \( R \).

Remark: Henceforth we interpret the local expansions \((\ref{eq:local_fourier})\) as the Fourier expansions of the restrictions of the \( \mathcal{U}_0 \) trivialization of \( \delta \mathcal{W}_j \) to \( |z_i| = \epsilon \) in powers \( e^{i(n+\frac{3}{2})\Theta} \), for \( n \in \mathbb{Z} \). In a similar fashion we interpret \((\ref{eq:local_fourier})\) as the Fourier expansions of the \( \mathcal{U}_\infty \) trivialization of \( \delta \mathcal{W}_j \) restricted to the circle of radius \( R \).

Proof of Lemma \((\ref{eq:lemma})\). Because the solution, \( \delta \mathcal{W}_j \), of condition (1) of the Lemma is assumed to be in \( H^1(\mathcal{E}_\Omega) \) it follows from local elliptic regularity that the solution is actually in \( C^\infty(\mathcal{E}_\Omega) \). The support properties of the inhomogeneous term \( f_j \) makes it possible to enlarge each circle \( C_\epsilon(a_i) \) to an annular region in which \( \delta \mathcal{W}_j \) satisfies the homogeneous Dirac equation. In this region it will have a convergent local expansion of type \((\ref{eq:local_fourier})\). Since the Fourier coefficients \( c_1^j(\delta \mathcal{W}_j) \) and \( c_k^j(\delta \mathcal{W}_j) \) (for the restriction of \( \delta \mathcal{W}_j \) to \( C_\epsilon(a_i) \)) vanish for \( k < 0 \) and are equal for \( k = 0 \) it follows (by the uniqueness of Fourier expansions) that the same is true for the local expansion coefficients in the annulus. Since the Bessel functions \( I_\ell(r) \) are monotone increasing functions of \( r \) for \( \ell \geq 0 \) this restriction on the local expansions implies that they converge in a domain \( 0 < |z_j| < \epsilon' \) where \( \epsilon' \) is slightly bigger than \( \epsilon \) (only a finite number of Fourier coefficients will get larger for smaller values of \( |z_j| \)). This shows that a solution, \( \delta \mathcal{W}_j \), to (1) and (2) of the Lemma extends to a solution of the Dirac equation which is in \( L^2 \) near \( a_i \) and has appropriate restrictions on its local expansion coefficients. The same sort of argument shows that the restriction (3) allows one to extend \( \delta \mathcal{W}_j \) to an \( L^2 \) solution to the Dirac equation in a neighborhood of \( \infty \). QED

Without much difficulty the reader should be able to verify the following formula for the local expansion coefficients \( mc_1^j(\mathcal{W}_j) \) that appear in the SMJ formula for the log derivative of the \( \tau \) function,

\[
mc_1^j(\mathcal{W}_j) = \frac{\sqrt{m}}{2\pi I_\frac{1}{2}(me)} \int_{\theta_\epsilon}^{\theta_\epsilon+2\pi} (\delta \mathcal{W}_j)_1(\epsilon e^{i\theta}) e^{-i\frac{\epsilon}{2}} d\theta.
\]

(10)

In this formula \((\delta \mathcal{W}_j)_1\) is the first component of \( \delta \mathcal{W}_j \) in the \( \mathcal{U}_0 \) trivialization. The formula follows easily from the standard formula for Fourier coefficients and the fact that the subtraction of \( \varphi_j w_0(z_j) \) does not alter the local expansion coefficients at level 1, so that \( c_1^j(\delta \mathcal{W}_j) = \sqrt{mc_1^j(\mathcal{W}_j)} \).

Our strategy in controlling the \( m \to 0 \) limit of the coefficients \( mc_1^j(\mathcal{W}_j) \) will be to use the characterization of Lemma \((\ref{eq:lemma})\) in conjunction with the formula \((\ref{eq:local_fourier})\). Since,

\[
\lim_{m \to 0} \frac{\sqrt{m}}{I_\frac{1}{2}(me)} = \sqrt{2} \Gamma \left( \frac{3}{2} \right) = \sqrt{\frac{\pi}{2\epsilon}},
\]

it will suffice for our purposes to control the \( m \to 0 \) convergence of \( \delta \mathcal{W}_j \) in \( L^p(C_\epsilon(a_i)) \) for any \( p \geq 1 \) and all \( i \). Here \( C_\epsilon(a_i) \) is the circle of radius \( \epsilon \) about \( a_i \).
Next we introduce convenient orthonormal bases for the subspaces that are of interest to us. Define,

\[ e_n^m(r, \Theta) = \begin{bmatrix} e^{i(n-\frac{1}{2})\Theta} \alpha_n^m(mr) \\ e^{i(n+\frac{1}{2})\Theta} \beta_n^m(mr) \end{bmatrix}, \]

where,

\[ \alpha_n^m(mr) = \frac{I_n - \frac{1}{2}(mr)}{\sqrt{I_n^2 - \frac{1}{4}(mr) + I_{n+\frac{1}{2}}^2(mr)}}, \]
\[ \beta_n^m(mr) = \frac{I_n + \frac{1}{2}(mr)}{\sqrt{I_n^2 - \frac{1}{4}(mr) + I_{n+\frac{1}{2}}^2(mr)}}. \]

Also define \( e_n^m(r, \Theta) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{e}_n^m(r, \Theta). \) The collection,

\[ \{e_n^m(\epsilon, \Theta_j), e_n^m(\epsilon, \Theta_j)^*\}, \]

as \( n \) ranges over the integers is an orthonormal basis for \( L^2(C_\epsilon(a_j)) \) (with values in \( C^2 \)). Here we write,

\[ \Theta_j(z) := \Theta(z - a_j). \]

**Definition 1** Let \( W_j^m \) denote the subspace of \( L^2(C_\epsilon(a_j)) \) which is the \( L^2 \) closure of the span of \( e_n^m(\epsilon, \Theta_j) \), and \( e_n^m(\epsilon, \Theta_j)^* \) for \( n > 0 \) and the vector \( e_0^m(\epsilon, \Theta_j) + e_0^m(\epsilon, \Theta_j)^* \).

Define

\[ \tilde{e}_n^m(r, \theta) = \begin{bmatrix} e^{-i\theta} \alpha_n^\infty(mr) \\ e^{-i(n-1)\theta} \beta_n^\infty(mr) \end{bmatrix}, \]

with

\[ \alpha_n^\infty(mr) = \frac{-K_n(mr)}{\sqrt{K_{n-1}^2(mr) + K_n^2(mr)}}, \]
\[ \beta_n^\infty(mr) = \frac{K_{n-1}(mr)}{\sqrt{K_{n-1}^2(mr) + K_n^2(mr)}}. \]

Then \( \{\tilde{e}_n^m(R, \theta)\} \), where \( n \) ranges over the integers, is an orthonormal set in \( L^2(C_R) \).

**Definition 2** Let \( W_n^m \) be the \( L^2 \) closure of the span of \( \tilde{e}_n^m(R, \theta) \) for \( n \in \mathbb{Z} \).
The boundary conditions (2) and (3) in Lemma(1) become,

\[ \delta W_j|_{C_{(a_i)}} \in W^{(m)}_i, \]
\[ \delta W_j|_{C_R} \in W^{(m)}_\infty. \]

As a first step towards controlling the \( m \to 0 \) limit of the solution of the boundary value problem described in Lemma(1) we will now record some elementary estimates for the convergence of \( e_n^{(m)}(r, \Theta) \) and \( \hat{e}_n^{(m)}(r, \theta) \) to their limits as \( m \to 0 \). For \( \ell > 0 \) the Bessel function \( I_\ell(r) \) behaves like a constant times \( r^\ell \) as \( r \to 0 \). It immediately follows that (for \( n \geq 1 \)),

\[ \lim_{m \to 0} a_n^{(m)}(mr) = 1, \]
\[ \lim_{m \to 0} \beta_n^{(m)}(mr) = 0. \]

and hence that,

\[ \lim_{m \to 0} e_n^{(m)}(r, \Theta) = e_n(\Theta) := \begin{bmatrix} e^{i(n-\frac{1}{2})\Theta} \\ 0 \end{bmatrix} \text{ for } n \geq 1. \]

For similar reasons we find that,

\[ \lim_{m \to 0} \hat{e}_n^{(m)}(r, \theta) = \hat{e}_n(\theta) := \begin{bmatrix} 0 \\ e^{-i(n-\frac{1}{2})\theta} \end{bmatrix} \text{ for } n \leq 0. \]

For \( \ell > 0 \), \( I_\ell(r) \) is an increasing function of \( r \) and since \( I_\ell'(r) = -\frac{\ell}{r}I_\ell(r) + I_{\ell-1}(r) \) the right hand side is non negative and hence,

\[ \frac{I_\ell(r)}{I_{\ell-1}(r)} \leq \frac{r}{\ell} \text{ for } \ell > 0. \]  \hspace{1cm} (11)

For \( n > 0 \), \( K_n(r) \) is a decreasing function of \( r \) and since \( K'_n(r) = \frac{\ell}{r}K_n(r) - K_{n+1}(r) \) the right hand side is non positive and hence,

\[ \frac{K_n(r)}{K_{n+1}(r)} \leq \frac{r}{n} \text{ for } n > 0. \]  \hspace{1cm} (12)

Now suppose that \( 0 < a < b \), then we have,

\[ \frac{a}{\sqrt{a^2 + b^2}} \leq \frac{a}{b}, \]  \hspace{1cm} (13)

and

\[ 1 \geq \frac{b}{\sqrt{b^2 + a^2}} = \frac{1}{\sqrt{1 + (\frac{a}{b})^2}} \geq 1 - \frac{1}{2} \left( \frac{a}{b} \right)^2. \]  \hspace{1cm} (14)
Using equations (11), (12), (13), and (14), we obtain the following estimates,

\[ |\alpha_n^{(m)}(mr) - 1| \leq \frac{1}{2} \left( \frac{I_{n+\frac{1}{2}}(mr)}{I_{n-\frac{1}{2}}(mr)} \right)^2 \leq \frac{1}{2} \left( \frac{mr}{n+\frac{1}{2}} \right)^2 \text{ for } n \geq 1, \]  

(15)

\[ |\beta_n^{(m)}(mr)| \leq \frac{I_{n+\frac{1}{2}}(mr)}{I_{n-\frac{1}{2}}(mr)} \leq \frac{mr}{n+\frac{1}{2}} \text{ for } n \geq 1, \]  

(16)

and

\[ |\alpha_n^{\infty}(mr) + 1| \leq \frac{1}{2} \left( \frac{K_{n-1}(mr)}{K_{n}(mr)} \right)^2 \leq \frac{1}{2} \left( \frac{mr}{n-1} \right)^2 \text{ for } n \geq 2, \]  

(17)

\[ |\beta_n^{\infty}(mr)| \leq \frac{K_{n-1}(mr)}{K_{n}(mr)} \leq \frac{mr}{n-1} \text{ for } n \geq 2. \]  

(18)

For \( n \leq -1 \) we also have,

\[ |\alpha_n^{\infty}(mr)| \leq \frac{K_{|n|}(mr)}{K_{|n|+1}(mr)} \leq \frac{mr}{|n|}, \]  

(19)

and

\[ |\beta_n^{\infty}(mr) - 1| \leq \frac{1}{2} \left( \frac{K_{|n|}(mr)}{K_{|n|+1}(mr)} \right)^2 \leq \frac{1}{2} \left( \frac{mr}{|n|} \right)^2. \]  

(20)

We will use these bounds in the fifth section to estimate the norm of a graph operator for the inversion of a certain projection. The \( n \) dependence in these bounds will be of use to us there. Note that estimates (17) and (18) obviously fail for \( n = 1 \).

For the same reason even though (19) and (20) are not valid for \( n = 0 \) we still find that \( |\alpha_0^{\infty}(mr)| \) and \( |\beta_0^{\infty}(mr) - 1| \) both tend to 0 as \( m \to 0 \).

We write

\[ W^{(m)} := W^{(m)}_{\infty} \oplus W^{(m)}_1 \oplus \cdots \oplus W^{(m)}_N, \]

and \( W^{(0)} \) for the \( m \to 0 \) limit of \( W^{(m)} \). The estimates we’ve just given allow us to calculate the limiting behavior of the basis elements of \( W^{(m)} \). This makes it natural to define,

\[ W_j^{(0)} := \text{span}_{n \geq 1} \left\{ \begin{bmatrix} e^{i(n-\frac{1}{2})\theta_j} & 0 \\ 0 & e^{-i(n-\frac{1}{2})\theta_j} \end{bmatrix} \right\} \oplus \mathbb{C} \left[ \begin{bmatrix} e^{i\theta_j} \\ e^{-i\theta_j} \end{bmatrix} \right], \]  

(21)

for \( j = 1, \ldots, N \) and

\[ W_{\infty}^{(0)} := \text{span}_{n \geq 1} \left\{ \begin{bmatrix} e^{-i\theta} \\ 0 \end{bmatrix} \right\} \oplus \text{span}_{n \geq 1} \left\{ \begin{bmatrix} 0 \\ e^{i\theta} \end{bmatrix} \right\}, \]  

(22)
with

\[ W^{(0)} := W^{(0)}_{\infty} \oplus W^{(0)}_{1} \oplus W^{(0)}_{2} \oplus \cdots \oplus W^{(0)}_{N}. \]

To be a little more precise we will write \( W^{(0)} \) for the \( L^2 \) closure of the span of the basis vectors given above (recall that \( W^{(m)} \) was also a closed subspace of \( L^2 \)).

We will always regard,

\[ W^{(0)}_{j} \subset L^2(\mathcal{E}_{C_r(a_j)}), \]
\[ W^{(0)}_{\infty} \subset L^2(\mathcal{E}_{C_R}), \]

so that \( W^{(0)} \subset L^2(\mathcal{E}_{\partial}) \). Notice that the basis elements of the subspaces \( W^{(0)}_{j} \) can be regarded as smooth sections of \( \mathcal{E}_{C_r(a_j)} \) in the \( U_0 \) trivialization. Similarly the basis elements of \( W^{(0)}_{\infty} \) can be regarded as smooth sections of \( \mathcal{E}_{C_R} \) in the \( U_\infty \) trivialization. We will do this henceforth and it will make a difference for us when we look at the subspaces of \( W^{(0)} \) obtained by taking the closure of the span of the same basis elements in the Sobolev spaces \( W^{1,p}_{\infty}(\mathcal{E}_{\partial}) \).

3 The \( m = 0 \) Green Function

The Green function we want to understand has the following matrix kernel,

\[ G_0(z, z') = -\frac{1}{4\pi i} \left[ \sum_{j} \frac{u_j(z)v_j(z')g(z, z')}{g(z, z')} \sum_{j} \frac{u_j(z)v_j(z')}{g(z, z')} \right], \tag{23} \]

where,

\[ u_j(z) := (z - a_j)^{-\frac{1}{2}} \prod_{k \neq j} \frac{(z - a_k)^{\frac{1}{2}}}{(a_j - a_k)^{\frac{1}{2}}}, \tag{24} \]
\[ g(z, z') := \sum_{|\epsilon|=0} c(\epsilon) \prod_{j} (z - a_j)^{\epsilon_j} (z' - a_j)^{-\epsilon_j} |z' - z|, \tag{25} \]

with \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) and each \( \epsilon_j = \pm \frac{1}{2} \). Also

\[ |\epsilon| := \sum_{j=1}^{N} \epsilon_j, \]
\[ c(\epsilon) := \frac{\prod_{j<k} |a_j - a_k|^{2\epsilon_j \epsilon_k}}{\sum_{|\epsilon|=0} \prod_{j<k} |a_j - a_k|^{2\epsilon_j \epsilon_k}}, \tag{26} \]

and

\[ v_j(z) = (z - a_j)^{-\frac{1}{2}} \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \prod_{k \neq j} \frac{(z - a_k)^{\epsilon_k}}{(a_j - a_k)^{\epsilon_k}}, \tag{27} \]
The multivalued functions \((z - a_j)^{\epsilon_j}\) are all defined using the argument \(\Theta_j\) and are consequently branched along \(z \in r_j\). We regard \(G_0(z, z')\) as defining an operator, \(G_0\), acting on sections of \(E_{\Omega}\) in the following manner,

\[
G_0 f(z) := \int_{\Omega} G_0(z, z') f(z') dz' d\bar{z}',
\]

where the section \(f(z')\) is identified with its \(U_0\) trivialization. We also regard \(G_0 f\) as a section of \(E_{\Omega}\) given in the \(U_0\) trivialization.

When working with \(G_0 f(z)\) for \(|z| > R\) it is useful to rewrite \(v_j(z)\) and \(g(z, z')\) in terms appropriate for the \(U_\infty\) trivialization. The conversion from the \(U_0\) to the \(U_\infty\) trivialization is given as follows,

\[
\prod_k (z - a_k)^{\epsilon_k} \rightarrow \prod_k \left(1 - \frac{a_k}{z}\right)^{\epsilon_k} \text{ for } |\epsilon| = 0 \text{ and } |z| > R,
\]

where the fractional powers on the right are holomorphic functions functions of \(z\) normalized to be 1 at \(z = \infty\). In a similar fashion,

\[
v_j(z) \rightarrow z^{-1} \left(1 - \frac{a_j}{z}\right)^{-\frac{1}{2}} \sum_{|\epsilon| = 0, \epsilon_j = \frac{1}{2}} c(\epsilon) \prod_{k \neq j} \left(1 - \frac{a_k}{a_j - a_k}\right)^{\epsilon_k} \text{ for } |z| > R.
\]

Our goal in this section is to show that \(G_0\) inverts \(-\partial/\partial z\) with \(W^{(0)}\) boundary conditions. It is precisely this property that determines our interest in \(G_0\). We will at the same time establish some elementary but useful estimates.

It is helpful to recall some well known results for the kernel \(1/(z - z')\). Let \(f \in C^1(\bar{\Omega})\) (the continuously differentiable functions on \(\Omega\) which are continuous together with their derivative on the closure of \(\Omega\)) and define,

\[
T f(z) := \frac{1}{2\pi i} \int_{\Omega} \frac{f(z')}{z' - z} dz' d\bar{z}'.
\]

Then

**Theorem 2** The distribution derivative of \(T f\) is

\[
dT f = T(\partial_z f) dz + f d\bar{z}.
\]

For \(p > 2\) one has the estimate,

\[
||T f||_{W^{1,p}(\Omega)} \leq C_p ||f||_{W^{1,p}(\Omega)},
\]

for a constant \(C_p\) that depends only on \(p\) and \(\Omega\), and \(W^{k,p}(\Omega)\) is the subspace of \(L^p(\Omega)\) which consists of functions whose first \(k\) weak derivatives are in \(L^p(\Omega)\).

**Proof.** Suppose \(f \in C^1(\Omega)\) and \(\phi \in C_0^\infty(\Omega)\). To compute the distribution derivative \(\partial_z T f(z)\) we wish to “integrate by parts” in

\[
- \int_{\Omega} T f(z) \partial_z \phi(z) dz d\bar{z}.
\]
Hence, derivative, which is the first part of (31). To obtain the second part consider the exterior \( L \) set of \( z \) \( f \) Since both \( \phi \) and \( f \) are \( C^1 \) functions, \( \partial f \) is chosen small enough so that the distance from the support of \( \phi \) to the boundary of \( \Omega \) is greater than \( \epsilon \). For \( \epsilon \) this small it follows that for all \( z \) in the support of \( \phi \) the set of \( z' \) with \( |z - z'| = \epsilon \) is completely contained in \( \Omega \).

In order to do the “integration by parts” efficiently we calculate the exterior derivative of a particular 3 form on the domain \( \Omega \times \Omega \setminus \{(z, z') : |z' - z| < \epsilon \} \),

\[
d\left( \frac{f(z')\phi(z) - f(z)\phi(z')}{z' - z} \right) dzd\bar{z}'d\bar{z} = \\
\left( \frac{f(z')\partial\phi(z) - \partial f(z)\phi(z')}{z' - z} + \frac{f(z')\phi(z) - f(z)\phi(z')}{(z' - z)^2} \right) dzd\bar{z}'d\bar{z}'.
\]

Now integrate this last identity over,

\[
(\Omega \times \Omega)_\epsilon := \Omega \times \Omega \setminus \{(z, z') : |z' - z| < \epsilon \},
\]

and use Stokes’ theorem. Then make use of,

\[
\int_{(\Omega \times \Omega)_\epsilon} \frac{f(z')\phi(z) - f(z)\phi(z')}{(z' - z)^2} dzd\bar{z}'d\bar{z} = 0,
\]

which follows from the fact that the integrand is odd under the transformation \((z, z') \rightarrow (z', z)\) and the domain \((\Omega \times \Omega)_\epsilon\) is invariant under this map. After multiplication by \( \frac{1}{2\pi i} \) the resulting identity simplifies directly to,

\[
-\int_\Omega T_{\epsilon}f(z)\partial_\epsilon\phi(z) dzd\bar{z} = \int_\Omega T_{\epsilon}(\partial f)(z)\phi(z) dzd\bar{z}
\]

\[
+ \frac{1}{2\pi i} \int_\Omega dz'd\bar{z}' \int_{|z' - z| = \epsilon} \frac{f(z')\phi(z) - f(z)\phi(z')}{z' - z} d\bar{z}.
\]

Since both \( f \) and \( \phi \) are \( C^1 \) functions, it is easy to see that the second term on the right hand side of this last equation tends to 0 in the limit \( \epsilon \rightarrow 0 \). Because \( \frac{1}{z' - z} \) is in \( L^1_{\text{loc}}(\mathbb{C}^2) \) it follows that \( T_{\epsilon}f \rightarrow Tf \) in the sense of distributions as \( \epsilon \rightarrow 0 \). Hence, \( \partial T f = T \partial f \), which is the first part of (31). To obtain the second part consider the exterior derivative,

\[
d\left( \frac{f(z')\phi(z)}{z' - z} \right) dzd\bar{z}'d\bar{z} = -\bar{z}' \left( \frac{f(z')\phi(z)}{z' - z} \right) dzd\bar{z}'d\bar{z}'
\]

\[
= -\left( \frac{f(z')\partial\phi(z)}{z' - z} \right) dzd\bar{z}'d\bar{z}'.
\]
Integrate this over \((\Omega \times \Omega)\_\epsilon\) and multiply the result by \(\frac{1}{2\pi i}\). Using Stokes’ theorem again, one finds that,

\[-\int_{\Omega} T_{\epsilon} f(z) \partial \phi(z) dz d\xi = -\frac{1}{2\pi i} \int_{\Omega} d\xi' d\xi \int_{|z-z'|=\epsilon} \frac{f(z') \phi(z)}{z'-z} dz.\]

As \(\epsilon \to 0\) the right hand side tends to,

\[\int_{\Omega} d\xi' d\xi f(z') \phi(z').\]

Hence,

\[\bar{\partial} T f = f,\]

which is the second part of (31).

To prove the second part of the theorem we first observe that since \(\Omega\) is a bounded domain it is straightforward to show that \(T\) defines a bounded operator on \(L^p(\Omega)\) for \(p > 2\). Hölder’s inequality implies that,

\[\left| \int_{\Omega} \frac{f(z')}{z' - z} idz' d\xi' \right| \leq \|f\|_{L^p(\Omega)} \left( \int_{\Omega} \left| z' - z \right|^{-q} idz' d\xi' \right)^{\frac{1}{q}},\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). However since \(p > 2\) it follows that \(1 < q < 2\) and hence that,

\[z \to \int_{\Omega} \left| z' - z \right|^{-q} idz' d\xi',\]

is a bounded function on \(\Omega\). It follows at once that \(T\) is bounded on \(L^p(\Omega)\) since \(\Omega\) is a finite domain and bounded functions are in \(L^p(\Omega)\) (note: the analogue of \(T\) on \(\Omega = \mathbb{C}\) is also bounded on \(L^p(\mathbb{C})\) for \(p > 2\), (see [5]).

To see that it defines a bounded operator on \(W^{1,p}(\Omega)\) for \(p > 2\) it is enough to use (31) which implies that for \(f \in C^1(\Omega)\),

\[||dT f||_{L^p(\Omega)} \leq ||T \partial f||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \leq C||f||_{W^{1,p}(\Omega)}\]

for \(p > 2\) where we used the fact that \(T\) is bounded on \(L^p\) \((p > 2)\). Since the boundary of \(\Omega\) is smooth, \(C^1(\Omega)\) is dense in \(W^{1,p}(\Omega)\), and the second part of the theorem is proved. QED

We will now use theorem (2) to establish that \(G_0\) is a Green function for \(-\partial\) on \(W^{1,p}(E_\Omega)\) for \(p > 2\). Incidentally, we work in the space \(W^{1,p}(E_\Omega)\) only in order to simplify some boundary estimates by using the Sobolev trace theorems; we could work in \(L^p(E_\Omega)\) at the cost of using more complicated global ellipticity estimates (see [4]).

**Theorem 3** Suppose \(p > 2\) and suppose that \(f \in W^{1,p}(E_\Omega)\) and that \(f\) is compactly supported in \(\Omega\). Then \(G_0 f \in W^{1,p}(E_\Omega)\) and

1. \(\|G_0 f\|_{W^{1,p}} \leq C_p \|f\|_{W^{1,p}}\)
2. \(-\partial G_0 f = f\)
3. $G_0 f|_{\partial \Omega} \in W^{(0)}$

Proof. Let $z_j' = (z - a_j)'$ be defined using the argument $\Theta_j$ so that these functions are branched along $r_j$. For any choice $\epsilon_j = \pm \frac{1}{2}$ the function,

$$z^\epsilon := \prod_{j=1}^N z_j'^{\epsilon_j},$$

defines a map,

$$C^\infty(\mathcal{E}_\Omega) \ni f(z) \rightarrow z^\epsilon f(z) \in C^\infty(\Omega),$$

which has an inverse,

$$C^\infty(\Omega) \ni f(z) \rightarrow z^{-\epsilon} f(z) \in C^\infty(\mathcal{E}_\Omega),$$

where in each case sections of $C^\infty(\mathcal{E}_\Omega)$ are identified with their $U_0$ trivializations. Since the derivatives of $z^\pm \epsilon$ are bounded on $\Omega$ it follows that these maps induce bounded maps,

$$W^{1,p}(\mathcal{E}_\Omega) \ni f(z) \rightarrow z^\epsilon f(z) \in W^{1,p}(\Omega)$$

and

$$W^{1,p}(\Omega) \ni f(z) \rightarrow z^{-\epsilon} f(z) \in W^{1,p}(\mathcal{E}_\Omega).$$

The upper right matrix element of the kernel $G_0(z', z)$ is a linear combination of terms,

$$\frac{z^\epsilon(z')^{-\epsilon}}{z' - z},$$

each of which is the kernel of an operator we can interpret as a composition,

$$W^{1,p}(\mathcal{E}_\Omega) \xrightarrow{z^{-\epsilon}} W^{1,p}(\Omega) \xrightarrow{z^\epsilon} W^{1,p}(\Omega) \xrightarrow{\cdot} W^{1,p}(\mathcal{E}_\Omega),$$

which is bounded for $p > 2$ as a consequence of Theorem (3). To be more precise we note that it is the line bundle $\mathcal{E}_\Omega$ which appears in this composition. The same argument for $\bar{z}^{\pm \epsilon}$ and $\frac{1}{\bar{z}}$ coupled with the complex conjugate version of Theorem (3) shows that the lower left kernel in $G_0(z, z')$ determines a bounded linear transformation on $W^{1,p}(\mathcal{E}_\Omega)$. The diagonal elements of $G_0(z, z')$ are finite rank $L^2$ kernels with a range that consists of smooth sections of $\mathcal{E}_\Omega$. Consequently, they determine bounded operators on $W^{1,p}(\mathcal{E}_\Omega)$ as well and this finishes the proof of part (1) of the theorem.

The proof of part (2) of the theorem is a straightforward computation using (1) of Theorem (3) (and its complex conjugate), the fact that $\bar{\partial}_z z^\epsilon = 0$, $\partial \bar{z}^\epsilon = 0$, $\partial_z u_j(z) = 0$, $\partial_{\bar{z}} \bar{u}_j(z) = 0$, and finally,

$$\sum_{|\epsilon| = 0} c(\epsilon) = 1.$$  

Note that both $\bar{\partial}$ and $\partial$ act on $C^\infty(\mathcal{E}_\Omega)$ since the transition functions that define the bundle are piecewise constant.
To establish part (3) of the theorem it is useful to observe that the subspace $W_j^{(0)}$ consists of $L^2$ boundary values on $C_\epsilon(a_j)$, of functions,

$$\left[ \frac{z^\frac{1}{2}}{\bar{z}^\frac{1}{2}} h_1(z) \right] + c_0 \left[ \frac{z^\frac{1}{2}}{\bar{z}^\frac{1}{2}} h_2(z) \right],$$

where $h_1$ and $h_2$ are holomorphic functions on the disk $D_\epsilon(a_j)$ and $c_0$ is a complex constant (technically, $h_1$ and $h_2$ should be in the appropriate Hardy space). We wish to show that

$$\int_{\Omega} G_0(z, z') f(z') dz' d\bar{z}'$$

restricted to $z \in C_\epsilon(a_j)$ lies in $W_j^{(0)}$. Because we have assumed that the support of $f$ is contained inside $\Omega$ it follows that for $z' \in$ the support of $f$ we have $|z' - a_j| > |z - a_j|$ and so,

$$\frac{1}{z' - z} = \frac{1}{z' - z_j} = \sum_{n=0}^\infty \frac{1}{z_j} \left( \frac{z_j}{z_j} \right)^n,$$

will converge uniformly for $z \in D_\epsilon(a_j)$ and $z'$ in the support of $f$. Substituting this expansion in the formula for $G_0 f(z)$ (and the analogue obtained by taking complex conjugates) one sees easily that the boundary value of $G_0 f(z)$ has the form,

$$\left[ \frac{z^\frac{1}{2}}{\bar{z}^\frac{1}{2}} h_1(z) \right] + c_0 \left[ \frac{z^\frac{1}{2}}{\bar{z}^\frac{1}{2}} h_2(z) \right],$$

where $h_1$ and $h_2$ are holomorphic in $D_\epsilon(a_j)$. The only issue is whether the coefficient of $z^\frac{1}{2}$ in the first component is the same as the coefficient of $\bar{z}^\frac{1}{2}$ in the second component. A computation shows that the coefficient of $z^\frac{1}{2}$ in the Fourier expansion on $C_\epsilon(a_j)$ of the first component of $G_0 f$ is,

$$-\frac{1}{4\pi i} \int_\Omega dz' d\bar{z}' \left\{ \bar{v}_j(z') f_1(z') + \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) z_j^\epsilon \prod_{k \neq j} (z_j - a_k)^{\epsilon_k} f_2(z') \right\}$$

where we used the fact that

$$\sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} S(\epsilon) = \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} S(-\epsilon).$$

Computing the coefficient of $\bar{z}^\frac{1}{2}$ in the Fourier expansion of the second component of $G_0 f$ we find,

$$-\frac{1}{4\pi i} \int_\Omega dz' d\bar{z}' \left\{ v_j(z') f_2(z') + \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \bar{z}_j^\epsilon \prod_{k \neq j} \frac{(z_j - a_k)^{\epsilon_k}}{(a_j - a_k)^{\epsilon_k}} f_1(z') \right\}.$$

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Comparing these two coefficients using the definition of \( v_j(z) \) we see that they are the same. Thus \( G_0 f|_{C_\epsilon(a_j)} \in W_0^{(0)}. \)

To finish the proof we need to show that,

\[ G_0 f|_{C_R} \in W_\infty^{(0)}. \]

Recalling the definition of \( W_\infty^{(0)} \) we see that it consists of boundary values on \( C_R \) of functions,

\[ \left[ \begin{array}{c} h_1(z) \\ h_2(z) \end{array} \right], \]

where \( h_1(z) \) and \( h_2(z) \) are holomorphic functions on \( D_\infty \) which vanish at \( z = \infty \).

The condition \( |\epsilon| = 0 \) in the sum that defines \( g(z, z') \) makes it easy to see that,

\[ \int_{\Omega} g(z, z') f_2(z') dz' d\bar{z}', \]

is holomorphic for \( z \in D_\infty \) and tends to 0 at \( \infty \). For precisely the same reason,

\[ \int_{\Omega} \bar{g}(z, z') f_1(z') dz' d\bar{z}', \]

is anti-holomorphic in \( D_\infty \) and vanishes at \( \infty \). The diagonal contributions,

\[ -\frac{1}{4\pi i} \int_{\Omega} \sum_j u_j(z) \bar{v}_j(z') f_1(z') dz' d\bar{z}', \]

and

\[ -\frac{1}{4\pi i} \int_{\Omega} \sum_j \bar{u}_j(z) v_j(z') f_2(z') dz' d\bar{z}', \]

do not at first appear to vanish at infinity since \( u_j(z) \) does not tend to 0 at \( \infty \). However in the lemma which follows this theorem we will prove the homogeneous function identity,

\[ \sum_j u_j(z) \bar{v}_j(z') = \sum_j v_j(z) \bar{u}_j(z'). \]

(33)

Since \( v_j(z) \) is holomorphic for \( |z| > R \) and does tend to 0 at \( \infty \) (see (29)) this identity finishes the proof of the theorem. QED

We turn to the proof of the identity (33).

**Lemma 2** The following identity is true for the functions \( u_j \) and \( v_j \) defined in (24) and (27) above,

\[ \sum_j u_j(z) \bar{v}_j(z') = \sum_j v_j(z) \bar{u}_j(z'). \]
Proof. Suppose that \( v(z) \) is a holomorphic function branched along the rays \( r_j \) such that,

\[
V(z) = v(z) \prod_k z_k^{-\frac{1}{2}},
\]

is holomorphic in the punctured plane, \( \mathbb{C}\backslash \mathbf{a} \), with simple poles at each \( a_j \) and which tends to 0 as \( z \to \infty \) (this will be the case for each of the functions \( v_j \)). Then \( V(z) \) has the partial fraction decomposition,

\[
V(z) = \sum_k \frac{V_k}{z - a_k},
\]

where \( V_k = \text{Res}_{z=a_k} V(z) \). Rewriting this in terms of \( v(z) \) one finds,

\[
v(z) = \sum_k V_k z_k^{-\frac{1}{2}} \prod_{\ell \neq k} z_\ell^{\frac{1}{2}} = \sum_k V_k \prod_{\ell \neq k} (a_k - a_\ell)^{\frac{1}{2}} u_k(z).
\]

Thus we have,

\[
v_j(z) = \sum_k v_{kj} u_k(z),
\]

where the coefficients \( v_{kj} \) are found by residue calculation to be,

\[
v_{jj} = \sum_{|\epsilon|=0, \epsilon_j = \frac{1}{2}} \epsilon_j c(\epsilon),
\]

and for \( k \neq j \),

\[
\begin{align*}
    v_{kj} &= \frac{(a_k - a_j)^{\frac{1}{2}}}{(a_j - a_k)^{\frac{1}{2}}} \sum_{|\epsilon|=0} \epsilon_{j \neq k} = \sum_{|\epsilon|=0} \epsilon_{j \neq k} \prod_{\ell \neq j, k} \frac{(a_k - a_\ell)^{\epsilon_\ell}}{(a_j - a_\ell)^{\epsilon_\ell}}, \quad (34)
\end{align*}
\]

We will now show that \( v_{kj} = \bar{v}_{kj} \), which will have (33) as a simple consequence. Since \( v_{jj} \) is real it is clear that \( v_{jj} = \bar{v}_{jj} \). Now suppose that \( k \neq j \) and note that,

\[
\frac{(a_k - a_j)^{\frac{1}{2}}}{(a_j - a_k)^{\frac{1}{2}}} = \frac{(a_j - a_k)^{\frac{1}{2}}}{(a_k - a_j)^{\frac{1}{2}}},
\]

since cross multiplication produces the identity,

\[
|a_k - a_j| = |a_j - a_k|.
\]

Thus the first factor in (34) is Hermitian symmetric and it remains only to check that the second factor is also. First we rewrite \( c(\epsilon) \) for \( \epsilon_j = \frac{1}{2} \) and \( \epsilon_k = -\frac{1}{2} \) in the following manner,

\[
c(\epsilon)_{\epsilon_j = \frac{1}{2}, \epsilon_k = -\frac{1}{2}} = \prod_{\alpha < \beta} |a_\alpha - a_\beta|^{2\epsilon_\alpha \epsilon_\beta} \prod_{\ell \neq j, k} |a_\ell - a_k|^\epsilon \prod_{\ell \neq j, k} |a_\ell - a_j|^\epsilon.
\]
where \( c := \sum_{\alpha < \beta} |a_\alpha - a_\beta|^{2\varepsilon_\alpha \varepsilon_\beta} \). Now define,

\[
s_{kj}(\epsilon) = \frac{|a_j - a_k|^{-\frac{1}{2}}}{c} \prod_{\alpha < \beta \neq (j,k)} |a_\alpha - a_\beta|^{2\varepsilon_\alpha \varepsilon_\beta}.
\]

The second factor in (34) becomes,

\[
\sum_{s_{kj}(\epsilon)} s_{kj}(\epsilon) \prod_{\ell \neq j,k} |a_j - a_\ell|^{\varepsilon_\ell} (a_k - a_\ell)^{\varepsilon_\ell} = \prod_{\ell \neq j,k} |a_j - a_\ell|^{\varepsilon_\ell} (a_k - a_\ell)^{\varepsilon_\ell}.
\]

Since \( s_{kj}(\epsilon) \) is real and obviously equal to \( s_{jk}(\epsilon) \) this last expression will be Hermitian symmetric provided that,

\[
\prod_{\ell \neq j,k} |a_j - a_\ell|^{\varepsilon_\ell} (a_k - a_\ell)^{\varepsilon_\ell} = \prod_{\ell \neq j,k} |a_k - a_\ell|^{\varepsilon_\ell} (a_j - a_\ell)^{\varepsilon_\ell}.
\]

But this follows directly from cross multiplication as before. We’ve shown that \( v_{kj} = \bar{v}_{jk} \) and the following simple calculation now proves the identity (33),

\[
\sum_j u_j(z)\bar{v}_j(z') = \sum_{j,k} u_j(z)\bar{v}_{kj}\bar{u}_k(z')
\]

\[
= \sum_{j,k} v_{jk}u_j(z)\bar{u}_k(z') = \sum_k v_k(z)\bar{u}_k(z').
\]

QED

4 The Projection on \( W^{(0)} \)

In this section we will introduce the projection \( P_0 \) from \( L^2(\mathcal{E}_{\partial\Omega}) \) onto \( W^{(0)} \) which is naturally associated with the Green function \( G_0 \). Another goal is a description of the complementary subspace for \( P_0 \) acting on \( H^\frac{1}{2}(\mathcal{E}_{\partial\Omega}) \). We will show that the complementary projection \( I - P_0 \) projects \( H^\frac{1}{2}(\mathcal{E}_{\partial\Omega}) \) onto the boundary values of sections \( \Psi \in H^1(\mathcal{E}_{\Omega}) \) which are solutions to the Dirac equation, \( \Phi \Psi = 0 \) in \( \Omega \).

It is useful to start with a calculation. Write,

\[
G_0(z, z') = \begin{bmatrix}
G_{11}(z, z') & G_{12}(z, z') \\
G_{21}(z, z') & G_{22}(z, z')
\end{bmatrix},
\]

for the matrix elements of \( G_0 \).

Now suppose that \( f \in C^1(\mathcal{E}_{\Omega}) \) and choose \( z \in \mathbb{C} \setminus \bar{\Omega} \). Then,

\[
-G_0(\bar{\Phi}f)(z) = -2 \int_{\Omega} \begin{bmatrix}
G_{11}(z, z')\partial_z f_2(z') + G_{12}(z, z')\bar{\partial}_z f_1(z') \\
G_{21}(z, z')\partial_z f_2(z') + G_{22}(z, z')\bar{\partial}_z f_1(z')
\end{bmatrix} dz'd\bar{z}'
\]

\[
= -2 \int_{\Omega} \begin{bmatrix}
G_{11}(z, z')f_2(z')dz' - G_{12}(z, z')f_1(z')dz' \\
G_{21}(z, z')f_2(z')dz' - G_{22}(z, z')f_1(z')dz'
\end{bmatrix}.
\]
which Stokes’ theorem transforms into,

\[ -G_0(\partial f)(z) = -2 \int_{\partial \Omega} \left[ G_{11}(z, z')f_2(z')dz' - G_{12}(z, z')f_1(z')dz' \right]. \]

The first equality follows from the fact that \( G_{k2}(z, z') \) is holomorphic for \( z' \in \Omega \) and \( G_{k1}(z, z') \) is anti-holomorphic in \( z' \in \Omega \) (remember \( z \) is outside of \( \Omega \)). Of course, this is not precisely accurate since these functions are branched along the rays \( r_j \). However, it is not hard to argue that the application of Stokes’ theorem is still correct using the fact that \( G_{k2}(z, z')f_1(z') \) and \( G_{k1}(z, z')f_2(z') \) are continuous for \( z' \) on the rays \( r_j \). Also note that the orientation of \( \partial \Omega \) appropriate for this Stokes’ calculation is that \( C_R \) has the usual counterclockwise orientation and the circles \( C_c(a_j) \) are all clockwise oriented. With this as our motivation, we define, for \( f \in L^2(E_{\partial \Omega}) \),

\[ P_0f(z) := -2 \int_{\partial \Omega} \left[ G_{11}(z, z')f_2(z')dz' - G_{12}(z, z')f_1(z')dz' \right]. \]

where we understand this as a section of \( E_{\partial \Omega} \) by letting \( z \to \partial \Omega \) from outside of \( \Omega \). As usual sections of \( E \) are identified with their \( U_0 \) or \( U_\infty \) trivializations. If \( -\partial f \) is compactly supported in \( \Omega \) we saw in the last section that the restriction of \( -G_0\partial f \) to \( \partial \Omega \) is in \( W^{(0)} \). Thus (35) suggests that if \( f|_{\partial \Omega} \in W^{(0)} \) we should have \( P_0f = f \). Our first result in this section is,

**Theorem 4** The map \( P_0 \) defined by (35) is a projection from \( L^2(E_{\partial \Omega}) \) onto \( W^{(0)} \). \( P_0 \) restricts to a continuous map,

\[ P_0 : H^\tau(E_{\partial \Omega}) \to H^\tau(E_{\partial \Omega}). \]

**Proof.** We will show that \( P_0 \) maps \( L^2(E_{\partial \Omega}) \) into \( W^{(0)} \). Observe first that the functions, \( f \), in \( L^2(E_{C_c(a_j)}) \) which are restrictions to \( z \in C_c(a_j) \) of the type

\[ f(z) = \sum_{n=-L}^{L} \left[ f_{1n}z_j^{n+\frac{1}{2}} + f_{2n}z_j^{n+\frac{1}{2}} \right], \]

for \( L \) finite, are dense in \( L^2(E_{C_c(a_j)}) \), and have extensions to \( C\setminus r_j \) which are solutions to \( \partial f = 0 \). If \( \varphi \) is a \( C^\infty_0 \) function which is 1 for \( |z| \leq 1.5\epsilon \) and 0 for \( |z| > 2\epsilon \) and \( f \) is a function of type (35) then it is easy to see that \( \varphi_j f(z) := \varphi(z_j)f(z) \) is a section of \( E_{\Omega} \) and \( \partial(\varphi_j f) \) is compactly supported inside \( \Omega \). For such a function the calculation that we began this section with shows that,

\[ -G_0\partial(\varphi_j f)|_{\partial \Omega} = P_0f, \]

and it follows from Theorem (3) that \( P_0f \in W^{(0)} \). The first part of the theorem now follows from the fact that \( W^{(0)} \) is closed in \( L^2(E_{\partial \Omega}) \) and \( P_0 \) is continuous on \( L^2 \). We won’t bother to give the proof that \( P_0 \) is continuous on \( L^2 \) since the argument we now present to show that \( P_0 \) is continuous on \( H^\tau(E_{\partial \Omega}) \) adapts directly to show \( L^2 \) continuity.
Observe first that the finite rank part of $P_0$ associated with the kernels $G_{11}$ and $G_{22}$ has a range which is a subset of $C^\infty(\mathcal{E}_{\partial\Omega}) \subset H^\frac{1}{2}(\mathcal{E}_{\partial\Omega})$ and is clearly continuous in $L^2$ and hence also in $H^\frac{1}{2}$. Next consider the part of $P_0$ associated with $G_{12}$. The component of this operator which maps $H^\frac{1}{2}(\mathcal{E}_{C_e(a_j)})$ into $H^\frac{1}{2}(\mathcal{E}_{C_e(a_j)})$ can be written as a sum of operators each of which has a factorization of the following sort,

\[ H^\frac{1}{2}(\mathcal{E}_{C_e(a_j)}) \xrightarrow{\varphi(z)} H^\frac{1}{2}(C_e(a_j)) \xrightarrow{\phi(z)} H^\frac{1}{2}(C_e(a_j)) \]

\[ \xrightarrow{\psi(z)} H^\frac{1}{2}(C_e(a_k)) \xrightarrow{\phi(z)} H^\frac{1}{2}(\mathcal{E}_{C_e(a_k)}), \]

where the first, second, fourth and fifth maps are multiplication operators and the third map is,

\[ f(z) \rightarrow \frac{1}{2\pi i} \int_{C_e(a_j)} \frac{f(z')}{z'-z} \, dz', \quad (37) \]

which must be interpreted as a suitable boundary value when $j = k$. In this factorization $\phi(z)$ and $\psi(z)$ are smooth functions and hence determine bounded maps on $H^\frac{1}{2}$. The Cauchy projection $(37)$ is easily seen to be continuous from $H^\frac{1}{2}(C_e(a_j))$ to $H^\frac{1}{2}(C_e(a_k))$ even when $j = k$. Nothing changes if $C_R$ is one or both of the two components of $\partial\Omega$ that are involved and it follows that the part of $P_0$ associated with $G_{12}$ is bounded on $H^\frac{1}{2}(\mathcal{E}_{\partial\Omega})$. The kernel $G_{21}$ is just the complex conjugate of $G_{12}$ and so it too defines a bounded operator on $H^\frac{1}{2}(\mathcal{E}_{\partial\Omega})$. This completes the proof that $P_0$ is continuous on $H^\frac{1}{2}(\mathcal{E}_{\partial\Omega})$.

To finish the proof of the theorem we need to show that if $f \in W^{(0)}$ then $P_0 f = f$. Clearly it is enough to show this for the basis elements $(21)$ and $(22)$. For the calculation on $W^{(0)}$ it is preferable to use the alternate forms for $G_{11}$ and $G_{22}$ found in Lemma $(4)$. Thus $(33)$ becomes,

\[ P_0 f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \left[ \sum_k u_k(z) u_k(z') f_2(z') d\bar{z}' - g(z, z') f_1(z') d\bar{z}' - \sum_k \bar{v}_k(z) u_k(z') f_1(z') d\bar{z}' \right]. \quad (38) \]

The following residue calculations suffice to evaluate $P_0$ on the basis elements of $W^{(0)}$ (note that in these results $C_e(a_j)$ is counterclockwise oriented, as usual),

\[ \frac{1}{2\pi i} \int_{C_e(a_j)} u_k(z) z^{n-\frac{1}{2}} d\bar{z} = \delta_{jk} \delta_{n0} \text{ for } n = 0, 1, 2, \ldots \]

And for $n = 0, 1, 2, \ldots$,

\[ \frac{1}{2\pi i} \int_{C_e(a_j)} \sum_{|z|=0} c(z) \prod_{k=1} z_{\epsilon_k}^{e_k} (z')^{-e_k} \frac{(z')^{n-\frac{1}{2}}}{z'-z} d\bar{z}' \]

\[ = \begin{cases} z_j^{n-\frac{1}{2}} - \delta_{n0} v_j(z) & \text{for } z \in C_e(a_j) \\ -\delta_{n0} v_j(z) & \text{for } z \in \partial\Omega \setminus C_e(a_j) \end{cases} \]

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One finds (being careful to recall the orientation of the $C_i(a_j)$ component of $\partial\Omega$ is clockwise) that $P_0$ fixes the elements of the basis for $W_j^{(0)}$. The reader might find the cancelation of the $v_j(z)$ terms that appear in the calculation of the action of $P_0$ on \[
abla_j \begin{bmatrix} z_j^{-\frac{1}{2}} \\ \bar{z}_j^{-\frac{1}{2}} \end{bmatrix}\] instructive.

To compute the action of $P_0$ on the basis elements for $W_j^{(0)}$ the original form for the kernel of the Green function $G_0$ is preferable and one can do the needed calculation with the following residues,

$$
\frac{1}{2\pi i} \int_{C_R} v_k(z)z^{-n}dz = 0 \text{ for } n = 1, 2, 3, \ldots
$$

which follows from the fact that $v_k(z)$ is holomorphic in the exterior of $C_R$ and vanishes at $\infty$ in the $U_\infty$ trivialization. And for $n = 1, 2, 3, \ldots$,

$$
\frac{1}{2\pi i} \int_{C_R} \sum c(\epsilon) \prod \frac{z_k^{(\epsilon)}(z_k')^{-\epsilon_k}}{z_k' - z} \,(z')^{-n}dz' = \begin{cases} 
\text{for } z \in C_R \\
0 \text{ for } z \in \partial\Omega \setminus C_R
\end{cases}
$$

Again one finds that $P_0$ fixes the basis (22) and this finishes the proof of the theorem. QED

Next we turn to a characterization of the complementary projection $I - P_0$.

**Theorem 5** The projection $I - P_0$ maps $H^\perp(E_{\partial\Omega})$ into the subspace of $H^\perp(E_{\partial\Omega})$ which consists of boundary values of functions $\Psi \in H^1(E_{\Omega})$ which satisfy the Dirac equation

$$
\partial\Psi = 0
$$
in $\Omega$. Furthermore, there exists a constant $C$ so that for all $f \in H^\perp(E_{\partial\Omega})$ we have,

$$
|| (I - P_0)f ||_{H^1(E_{\Omega})} \leq C || f ||_{H^\perp(E_{\partial\Omega})}
$$

**Proof.** Using (28) and the well known identity,

$$(z' - z_{int})^{-1} - (z' - z_{ext})^{-1} = 2\pi i \delta(z' - z),$$

for the difference of the interior and exterior boundary values of the Cauchy kernel on a circle we find the following formula for $P_0^c := I - P_0$,

$$
P_0^c f(z) = -\frac{1}{2\pi i} \int_{\partial\Omega} \left[ \sum_k \frac{\bar{v}_k(z)\bar{g}(z,z')f_2(z')dz'}{g(z,z')f_2(z')dz'} - \sum_k \bar{v}_k(z)u_k(z')f_1(z')dz' \right], \quad (40)
$$

with the difference compared to (28) that in this formula $z$ is to tend to $\partial\Omega$ from the interior of $\Omega$. It is clear from this formula that $(I - P_0)f(z)$ extends to a section of $E_{\Omega}$ which is in the null space of $\Phi$. We need only establish the estimate (39) to finish the proof. The finite rank operator,

$$
f \rightarrow -\frac{1}{2\pi i} \int_{\partial\Omega} \sum_k v_k(z)\bar{u}_k(z')f(z')dz',
$$
is obviously continuous on $H^\gamma(E_{\partial\Omega})$ since each $u_k$ is in $L^2$ and the functions $v_k \in C^\infty(E_{\Omega}) \subset H^1(E_{\Omega})$. The other finite rank operator that occurs in (40) is continuous from $H^\gamma(E_{\partial\Omega})$ into $H^1(E_{\Omega})$ for the same reason. Next we turn to the operator,

$$f \rightarrow \frac{1}{2\pi i} \int_{\partial\Omega} g(z, z') f(z)dz'.$$

This operator is linear combination of operators each of which we wish to interpret as a composition,

$$H^\gamma(E_{\partial\Omega}) \xrightarrow{z'} H^\gamma(\partial\Omega) \xrightarrow{\partial} H^1(\Omega) \xrightarrow{x'} H^1(E_{\Omega}).$$

The first and third maps are multiplication operators which are obviously continuous. The middle map is shorthand for the operator,

$$f \rightarrow \frac{1}{2\pi i} \int_{\partial\Omega} f(z') z' - z dz',$$

which is well known to be continuous from $H^\gamma(\partial\Omega)$ to $H^1(\Omega)$. For the reader’s convenience we recall a simple argument for this continuity.

Write $z_j = z - a_j$ and consider a function, $f$, defined on $\partial\Omega$ by,

$$f(z) = \begin{cases} \sum_{n=-L}^L f_n z_j^n & \text{for } z \in C_\epsilon(a_j) \\ 0 & \text{for } z \in \partial\Omega \setminus C_\epsilon(a_j) \end{cases}$$

Define $P_{int}f(z)$ for $z$ in $\Omega$ by,

$$P_{int}f(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{f(z')}{z' - z} dz'.$$

Then

$$P_{int}f(z) = \sum_{n=-L}^{-1} f_n z_j^n.$$

Since $\Omega$ is bounded the Poincare’ inequality [6] implies that the $H^1(\Omega)$ norm of $P_{int}f$ is bounded by the $L^2(\Omega)$ norm of,$

$$\partial P_{int}f(z) = \sum_{n=-L}^{-1} n f_n z_j^{n-1}.$$
We see then that,

$$||P_{int}f||_{H^1(\Omega)}^2 \leq C \sum_{n=-L}^{-1} e^{2n} |n| f_n \tilde{f}_n \leq C ||f||_{H^2(C, (a_j))}^2.$$ 

Now suppose that,

$$f(z) = \left\{ \begin{array}{ll} \sum_{n=-L}^{L} f_n z^n & \text{for } z \in C_R \\ 0 & \text{for } z \in \partial \Omega \setminus C_R. \end{array} \right.$$ 

(42)

Then the argument we just gave is easily modified to show that,

$$||P_{int}f||_{H^1(\Omega)} \leq C ||f||_{H^2(C_R)}.$$ 

Sums of functions of type (41) for \( j = 1, 2, \ldots, N \) and type (42) are dense in \( H^1_1(\partial \Omega) \) and it follows that \( P_{int} \) extends to a continuous map from \( H^1_1(\partial \Omega) \) to \( H^1(\Omega) \).

Taking complex conjugates the result we just proved also shows that the map,

$$f \to -\frac{1}{2\pi i} \int_{\partial \Omega} g(\bar{z}; z') f(z') d\bar{z'},$$

is bounded from \( H^1_1(\partial \Omega) \) to \( H^1(\Omega) \). This finishes the proof of the theorem.

QED

5 Inverting the Projection \( P_0 : W^r(m) \to W^r(0) \)

Remark. In this section we will write \( W^r(m) \) for \( W^r(m) \cap H^1_0(\partial \Omega) \) and \( W^r(0) \) for \( W^r(0) \cap H^1_0(\partial \Omega) \). We will prove the following theorem.

Theorem 6 For all sufficiently small values of \( m \) the projection

$$P_0 : W^r(m) \to W^r(0),$$

is an isomorphism. Furthermore, there is a linear map \( \delta \) from \( W^r(0) \) into \( (I - P_0)H^1_0(\partial \Omega) \) and a constant \( C \) which is independent of \( f \) and \( m \) so that for all \( f \in W^r(0) \),

1. \( f + \delta f \in W^r(m) \)
2. \( ||\delta f||_{H^1_0(\partial \Omega)} \leq C m ||f||_{H^1_0(\partial \Omega)} \)

Remark. The map \( \delta \) gives \( W^r(m) \) as a graph over \( W^r(0) \).

Proof. Suppose first that \( f \in W^r(m) \) and let \( f_j \) denote the restriction of \( f \) to \( C_r(a_j) \) and let \( f_\infty \) denote the restriction of \( f \) to \( C_R \). Write \( e_{n,j}^m \) for \( e_{n,j}^m(\epsilon, \Theta_j) \)
and \( \hat{e}_{n,\infty}^{(m)} \) for \( \hat{e}_{n}^{(m)}(R, \theta) \). Then the Fourier expansions of \( f \) on \( C_{r}(a_{j}) \) and \( C_{R} \) can be written,

\[
f_{j} = a_{0,j}(e_{0,j}^{(m)} + e_{0,j}^{(m)*}) + \sum_{n=1}^{\infty} \left\{ a_{n,j}e_{n,j}^{(m)} + b_{n,j}e_{n,j}^{(m)*} \right\}
\]

(43)

and

\[
f_{\infty} = \sum_{n=-\infty}^{\infty} a_{n,\infty}\hat{e}_{n,\infty}^{(m)}
\]

(44)

Now let \( p_{0}f \) denote the element of \( W^{(0)} \) which has the same “Fourier coefficients” in the \( m \to 0 \) limiting basis. That is,

\[
p_{0}f_{j} = a_{0,j}(e_{0,j} + e_{0,j}^{*}) + \sum_{n=1}^{\infty} \left\{ a_{n,j}e_{n,j} + b_{n,j}e_{n,j}^{*} \right\}
\]

(45)

and

\[
p_{0}f_{\infty} = \sum_{n=-\infty}^{\infty} a_{n,\infty}\hat{e}_{n,\infty},
\]

(46)

where \( e_{n,j} \) denotes the basis vector \( e_{n}(\Theta_{j}) \) and \( \hat{e}_{n,\infty} \) denotes the vector \( \hat{e}_{n}(\theta) \).

It is easy to check that,

\[
\langle e_{k,j}^{(m)} - e_{k,j}^{(m)*} - e_{\ell,j}^{(m)} \rangle_{H^{1}(\mathcal{E}_{\partial \Omega})} = \delta_{k\ell}||e_{k,j}^{(m)} - e_{k,j}^{(m)*}||^{2}_{H^{1}(\mathcal{E}_{\partial \Omega})},
\]

and from (13) and (16) it follows that,

\[
||e_{k,j}^{(m)} - e_{k,j}^{(m)*}||_{H^{1}(\mathcal{E}_{\partial \Omega})} \leq Cm
\]

Analogous results for \( e_{k,j}^{(m)*} \) and for \( \hat{e}_{k,\infty}^{(m)} \) (which follow from (17) and (18)) imply the inequality,

\[
||p_{0}f - f||_{H^{1}(\mathcal{E}_{\partial \Omega})} \leq Cm||f||_{H^{1}(\mathcal{E}_{\partial \Omega})},
\]

(47)

for some constant \( C \) and all \( f \in W^{(m)} \). Now write \( \Delta p_{0} = p_{0} - I \). Then for \( f \in W^{(m)} \) we have,

\[
f + \Delta p_{0}f \in W^{(0)}.
\]

It follows from this that,

\[
(I - P_{0})(f + \Delta p_{0}f) = 0,
\]

or

\[
(I - P_{0})f = -(I - P_{0})\Delta p_{0}f.
\]

From this (17) and the fact that \( P_{0} \) is bounded on \( H^{1}(\mathcal{E}_{\partial \Omega}) \) it follows that,

\[
||f - P_{0}f||_{H^{1}(\mathcal{E}_{\partial \Omega})} \leq Cm||f||_{H^{1}(\mathcal{E}_{\partial \Omega})},
\]
for some constant $C$ and all $f \in W^{(m)}$. When $m$ is small enough so that $Cm < 1$ this last inequality implies that $P_0 : W^{(m)} \to W^{(0)}$ is injective since $P_0 f = 0$ gives $\|f\| \leq Cm\|f\|$ with $Cm < 1$ which in turn forces $\|f\| = 0$.

Now start with $f_0 \in W^{(0)}$ with Fourier expansion given by (45) and (46) and define $p_m f_0 \in W^{(m)}$ to be the element of $W^{(m)}$ with the Fourier expansion (43) and (44). Then the same estimates we gave above imply that for $f_0 \in W^{(0)}$ we have,

$$\|\Delta p_m f_0\|_{H^1(E_{\partial \Omega})} \leq Cm \|f_0\|_{H^\frac{1}{2}(E_{\partial \Omega})},$$

for $\Delta p_m := p_m - I$ and some constant $C$. Now choose $m$ small enough so that the map,

$$P_0 + P_0 \Delta p_m : W^{(0)} \to W^{(0)},$$

is invertible. Define,

$$g_0 := (P_0 + P_0 \Delta p_m)^{-1} f_0.$$

Then one can easily check that,

$$g_0 + \Delta p_m g_0 \in W^{(m)}$$

and,

$$P_0(g_0 + \Delta p_m g_0) = f_0.$$

This shows that $P_0 : W^{(m)} \to W^{(0)}$ is surjective. Furthermore if we define,

$$\delta = (I - P_0) \Delta p_m (P_0 + P_0 \Delta p_m)^{-1},$$

as a map from $W^{(0)}$ to $(I - P_0) H^\frac{1}{2}(E_{\partial \Omega})$ then it is easy to check that $f_0 + \delta f_0 \in W^{(m)}$ and $\delta$ satisfies the estimate (2) of the theorem. QED

6 Convergence Results

In this section we provide the details for the approximation scheme for $\delta W_j$ that was outlined in (10). Let $f_j$ denote the section of $E_\Omega$ defined in (9) above. As a first approximation to $\delta W_j$ we define,

$$\delta_1 W_j = G_0 (1 + mG_0)^{-1} f_j$$

We will show that for all sufficiently small $m$, $\delta_1 W_j$ is well defined and

$$(m - \ddot{\theta}) \delta_1 W_j = f_j,$$

with $\delta_1 W_j |_{\partial \Omega} \in W^{(0)}$. Thus $\delta_1 W_j$ satisfies the same differential equation as $\delta W_j$ but has boundary values in $W^{(0)}$ instead of $W^{(m)}$. Next we define,

$$\delta_2 W_j = \delta (\delta_1 W_j).$$

Then the boundary values of $\delta_1 W_j + \delta_2 W_j$ are in $W^{(m)}$ but since $\ddot{\theta} \delta_2 W_j = 0$ on $\Omega$ we find that,

$$(m - \ddot{\theta}) (\delta_1 W_j + \delta_2 W_j) = f_j + m \delta_2 W_j.$$
Now let $\delta_3 W_j$ denote the solution of,

$$(m - \partial)\delta_3 W_j = -m\delta_2 W_j,$$

such that $\delta_3 W_j|_{\partial\Omega} \in W^{(m)}$. The solution we are interested in is then,

$$\delta W_j = \delta_1 W_j + \delta_2 W_j + \delta_3 W_j.$$ 

The following theorem gives convergence results that will allow us to calculate \((10)\) in the limit $m \to 0$. We write $f_{0j}$ for the $m \to 0$ limit of $f_j$,

$$f_{0j}(z) = i\sqrt{2\pi} \left[ -\frac{\text{sgn}_j}{z_j^2}\partial\varphi_j(z) \right].$$

**Theorem 7** For $\delta_k W_j$ defined above ($k = 1, 2, 3$) we have,

1. As $m \to 0$, $\delta_1 W_j$ converges to $G_0 f_{0j}$ in $W^{1,p}(\Omega)$ for all $p > 2$.
2. $||\delta_2 W_j||_{H^1(\Omega)} \leq C m$ for some constant $C$ independent of $m$.
3. The Fourier coefficient,

$$\int_{\partial\Omega} (\delta_3 W_j) (e^{i\theta}) e^{-\frac{m}{2}r^2} d\theta_j,$$

**tends to 0 as $m \to 0$.**

**Remark.** As we shall see below, the upshot of these estimates is that we can compute the $m \to 0$ limit of $mc_i^1(W_j)$ by calculating the appropriate Fourier coefficient of $G_0 f_{0j}$. Also in the course of proving 1-3 of theorem (7) we will confirm the properties asserted for $\delta_k W_j$, $k = 1, 2, 3$, when they were introduced above.

**Proof.** Estimate 1 of theorem (3) shows that for $p > 2$, $G_0$ is bounded on $W^{1,p}(\Omega)$. Thus for small enough $m$, the map $I + mG_0$ is invertible on $W^{1,p}(\Omega)$. Since it is clear that $f_j \in C_0^\infty(\Omega) \subset W^{1,p}(\Omega)$ it follows that $\delta_1 W_j \in W^{1,p}(\Omega)$ for $p > 2$. Since $\delta_1 W_j$ is in the image of $G_0$, part 3 of theorem (3) implies that the boundary value of $\delta_1 W_j$ on $\partial\Omega$ is in $W^{(0)}$. We use part 2 of theorem (3) to do the following calculation,

$$(m - \partial) G_0(I + mG_0)^{-1} f_j = mG_0(I + mG_0)^{-1} f_j + (I + mG_0)^{-1} f_j = (mG_0 + I)(I + mG_0)^{-1} f_j = f_j,$$

which confirms (3). Since $G_0$ is bounded on $W^{1,p}(\Omega)$ the operator $(I + mG_0)^{-1}$ converges uniformly to $I$ on $W^{1,p}(\Omega)$ as $m \to 0$. Thus to finish the proof of 1 we need only show that for all $p > 2$ the section $f_j$ converges in $W^{1,p}(\Omega)$ to $f_{0j}$. Using (3) one finds,

$$f_j = i\sqrt{m} \left[ e^{i\frac{m}{2}r^2} (I_{\frac{1}{2}}(mr) - I_{-\frac{1}{2}}(mr)) \partial\varphi_j(z) \right].$$
The simple estimate,
\[ I_{\pm \frac{1}{2}}(mr) = \left( \frac{mr}{2} \right)^{\pm \frac{1}{2}} \sum_{n=0}^{\infty} \frac{(mr)^{2n}}{2^{2n}n! \Gamma(n + \frac{1}{2})} \leq \left( \frac{mr}{2} \right)^{\pm \frac{1}{2}} \exp \left( \frac{2mr^2}{\Gamma(\frac{3}{2})} \right), \]
which is valid for \( r < 2\epsilon \) (which contains the support of \( \partial \varphi_j \) and \( \bar{\partial} \varphi_j \)) shows that dominated convergence applies to,
\[ \lim_{m \to 0} \int_{\Omega} |f_j - f_{0j}|^p idzd\bar{z} = 0, \]
for all \( p \geq 1 \). The same estimate shows that dominated convergence applies to the \( m \to 0 \) limit of the integral,
\[ \int_{\Omega} |df_j - df_{0j}|^p idzd\bar{z} \]
and this proves that \( f_j \) converges to \( f_{0j} \) in \( W^{1,p}(E_\Omega) \). Now fix \( p > 2 \). Since \( f_j \) converges in \( W^{1,p}(E_\Omega) \) as \( m \to 0 \) it follows that its norm in this space is uniformly bounded. Hence the \( W^{1,p} \) norm of \( \delta_1W_j \) is also uniformly bounded. However since the domain \( \Omega \) is bounded the \( W^{1,p}(E_\Omega) \) norm for \( p > 2 \) dominates (a constant times) the \( H^1(E_\Omega) \) norm. This shows that \( \delta_1W_j \) is uniformly bounded in \( H^1(E_\Omega) \) as \( m \to 0 \). The Sobolev trace theorem implies that the boundary value of \( \delta_1W_j \) is uniformly bounded in \( H^{\frac{1}{2}}(E_{\partial\Omega}) \) and estimate 2 of theorem (6) then shows that \( \delta_2W_j = \delta(\delta_1W_j) \) has norm in \( H^{\frac{1}{2}}(E_\Omega) \) bounded by \( Cm \). Finally estimate (3) shows that the extension of \( \delta_2W_j \) to \( \Omega \) has \( H^1(E_\Omega) \) norm dominated by \( Cm \), which is estimate 2 of theorem (7).

Before we turn to the proof of part 3 of theorem (7) it will be useful to establish the following estimates for solutions to the massive Dirac equation.

**Theorem 8** Suppose that \( \Psi \in L^2(E_\Omega) \) is a weak solution to the Dirac equation,
\[ (m - \partial)\Psi = f, \]
in \( \Omega \), where \( f \in C_0^\infty(E_\Omega) \). Suppose that \( \Psi|_{\partial\Omega} \in W^{(m)} \) so that for \( z \in C_r(a_j) \) the section \( \Psi \) has the local expansion,
\[ \Psi(z) = \sum_{n \geq 0} c_n^0(\Psi)w_n(z_j) + c_n^0(\Psi)w_n^*(z_j), \]
with \( c_0^0(\Psi) = c_{0s}^0(\Psi) \), and for \( z \in C_R \) the section \( \Psi \) has the local expansion,
\[ \Psi(z) = \sum_{n \in \mathbb{Z}} c_n^\infty(\Psi)\hat{w}_n(z). \]

Then,
\[ \|\Psi\|_{L^2(E_\Omega)} \leq \frac{1}{m} \|f\|_{L^2(E_\Omega)}, \]  
(55)
\[\sum_j |c_j^0(\Psi)|^2 \leq \frac{1}{8} \|f\|_{L^2(\Omega)}^2 \]  
(56)

\[4\pi \sum_{n \geq 0} \left\{ |c_n^0(\Psi)|^2 + |c_n^{*0}(\Psi)|^2 \right\} I_{n+\frac{1}{2}}(me) I_{n-\frac{1}{2}}(me) \epsilon \leq \frac{1}{m} \|f\|_{L^2(\Omega)}^2 \]  
(57)

\[4\pi \sum_{n \in \mathbb{Z}} |c_n^\infty(\Psi)|^2 K_n(mR) K_{n-1}(mR) R \leq \frac{1}{m} \|f\|_{L^2(\Omega)}^2. \]  
(58)

**Proof.** Since \( f \in C_0^\infty(\Omega) \), the existence theorem in [10] then gives us a weak solution \( \Psi \in L^2(\Omega) \) of \((m - \partial)\Psi = f\) which is smooth as a consequence of local elliptic regularity. Next we calculate the exterior derivative of \( 2i\bar{\Psi}_1\Psi_2 d\bar{z} \) using the fact that \( \Psi \) satisfies the Dirac equation,
\[
d(2i\bar{\Psi}_1\Psi_2 d\bar{z}) = m|\Psi|^2 idzd\bar{z} - (\bar{\Psi}_1 f_1 + \Psi_2 \bar{f}_2) idzd\bar{z}.
\]
Integrating this equality over \( \Omega \) and using Stokes' theorem we find,
\[
m\|\Psi\|^2_{L^2(\Omega)} - 2i \int_{\partial\Omega} \bar{\Psi}_1\Psi_2 d\bar{z} = \int_\Omega (\bar{\Psi}_1 f_1 + \Psi_2 \bar{f}_2) idzd\bar{z}. \]  
(59)

From this we deduce the inequality,
\[
m\|\Psi\|^2_{L^2(\Omega)} - 2i \int_{\partial\Omega} \bar{\Psi}_1\Psi_2 d\bar{z} \leq \|\Psi\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \]  
(60)

Next we compute the boundary term in (60) using the local expansions for \( \Psi \). We find,
\[
-2i \int_{\partial\Omega} \bar{\Psi}_1\Psi_2 d\bar{z} = 4\pi \sum_{n \in \mathbb{Z}} |c_n^\infty(\Psi)|^2 K_n(mR) K_{n-1}(mR) R \]  
(61)

and recalling the appropriate orientation of \( \partial\Omega \),
\[
2i \int_{\partial\Omega} \bar{\Psi}_1\Psi_2 d\bar{z} = 4\pi \sum_{n \geq 0} \left( |c_n^0(\Psi)|^2 + |c_n^{*0}(\Psi)|^2 \right) I_{n+\frac{1}{2}}(me) I_{n-\frac{1}{2}}(me) \epsilon \\
+ 4\pi c_0^2(\Psi) I_{\frac{1}{2}}^2(\epsilon) + 4\pi \bar{c}_0^2(\Psi) I_{\frac{1}{2}}^2(\epsilon) \]  
(62)

The boundary condition \( c_0^0(\Psi) = c_0^{*0}(\Psi) \) implies that the right hand side of this last equation is positive definite. Thus the boundary term on the left hand side of (60) is positive and we immediately deduce,
\[
m\|\Psi\|^2_{L^2(\Omega)} \leq \|\Psi\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}, \]  
which is (55). Now (55) and (60) coupled with the positivity of all the boundary contributions and (55) together imply (57) and (58). For the same reasons we
can pick out just one term from each of the $C_\epsilon(a_j)$ boundary terms to find the inequality,

$$4\pi \sum_j |c_0^j(\Psi)|^2 I^2_{\frac{1}{2}}(m \epsilon) \epsilon \leq \frac{1}{m} ||f||^2_{L^2(\mathcal{E}_\Omega)}.$$ 

This must be true for all $\epsilon$ and since,

$$\lim_{\epsilon \to 0} I^2_{\frac{1}{2}}(m \epsilon) \epsilon = 2 \frac{m \pi}{m \epsilon},$$

we have proved (56). QED (Theorem(8))

Next suppose as in the preceding theorem that $\Psi$ is an $L^2(\mathcal{E}_\Omega)$ solution to,

$$(m - \phi) \Psi = f,$$

where $f \in C_0^\infty(\mathcal{E}_\Omega)$. To finish the proof of 3 of Theorem(7) we want to estimate $c_j^1(\Psi)$ in terms of $f$. To do this we first introduce the function,

$$V_j(z) = (z - a_j)^{-\frac{3}{2}} \prod_{k \neq j} (a_j - a_k)^{\frac{1}{2}} (z - a_k)^{-\frac{1}{2}},$$

which we take to be branched along the rays $r_j$. Next observe that if $\Psi$ is identified with its $\mathcal{U}_0$ trivialization then $V_j \Psi$ is differentiable on $\Omega$ and,

$$d(2V_j \Psi_1 dz) = -\bar{\partial}(2V_j \Psi_1) dz d\bar{z} = -2V_j \bar{\partial} \Psi_1 dz d\bar{z} = mV_j \Psi_2 dz d\bar{z} - V_j f_2 dz d\bar{z}.$$

Stokes' theorem implies,

$$\int_{\partial \Omega} 2V_j \Psi_1 dz = \int_{\Omega} mV_j \Psi_2 dz d\bar{z} - \int_{\Omega} V_j f_2 dz d\bar{z},$$

from which, together with (55), we deduce the inequality,

$$\left| \int_{\partial \Omega} V_j \Psi_1 dz \right| \leq ||V_j||_{L^2(\Omega)} ||f||_{L^2(\mathcal{E}_\Omega)}. \quad (62)$$

Next we wish to estimate the boundary integrals over $C_\epsilon(a_k)$. First observe that in a neighborhood of $a_k$ the function $V_j$ has a “Laurent” expansion in powers of $z_k = z - a_k$,

$$V_j(z) = \sum_{n \geq -1} c_n^k(V_j) z_k^{n-\frac{1}{2}},$$

and $z$ is restricted to an annulus about $|z_k| = \epsilon$. One finds,

$$\frac{1}{2\pi i} \int_{C_\epsilon(a_k)} V_j \Psi_1 dz = c_0^k(\Psi) c_{-1}^k(V_j) I_{\frac{3}{2}}(m \epsilon) \epsilon^{-\frac{3}{2}} + c_0^k(\Psi) c_0^k(V_j) I_{-\frac{1}{2}}(m \epsilon) \epsilon^{-\frac{1}{2}} + \sum_{n \geq 0} c_n^k(\Psi) c_n^k(V_j) I_{n+\frac{1}{2}}(m \epsilon) \epsilon^{n-\frac{1}{2}}.$$
Observe that the term with \( c_k^j(\Psi) \) is present only for \( k = j \) since one can easily check that,

\[
c_{k-1}^j(V_j) = \delta_{jk}.
\]

Next we use the fact that the Taylor expansion of \( V_j \) for \( |z| \geq R \) has the form,

\[
V_j(z) = \sum_{n=-1}^\infty c_n^\infty(V_j)z^n,
\]

to calculate,

\[
\frac{1}{2\pi Ri} \int_{C_R} V_j \Psi_1(z)dz = -\sum_{n=-1}^\infty c_n^\infty(V_j)R^n c_{n+1}^\infty(\Psi)K_{n+1}(mR).
\]

Note: the coefficients \( c_n^\infty(V_j) \) are zero for \( n > -\frac{N}{2} - 1 \), where \( N \) is the number of branch points, but we will not need this.

Next we use (57) and Cauchy’s inequality for the \( \ell^2 \) norm with weight,

\[
I_{n+\frac{1}{2}}(me)I_{n-\frac{1}{2}}(me)e,
\]

to estimate,

\[
\sum_{n \geq 0} |e_n^{k*}(\Psi)||c_n^k(V_j)||I_{n+\frac{1}{2}}(me)ee^{n-\frac{1}{2}} \leq A_k B_k,
\]

where

\[
A_k^2 = \sum_{n \geq 0} |e_n^{k*}(\Psi)|^2 I_{n+\frac{1}{2}}(me)I_{n-\frac{1}{2}}(me)e
\]

and

\[
B_k^2 = \sum_{n \geq 0} \left| \frac{e_n^k(V_j)e^{n-\frac{1}{2}}}{I_{n-\frac{1}{2}}(me)} \right|^2 I_{n+\frac{1}{2}}(me)I_{n-\frac{1}{2}}(me)e.
\]

Combining this with (11) we see that,

\[
B_k^2 \leq \sum_{n \geq 0} |e_n^k(V_j)e^{n-\frac{1}{2}}|^2 \frac{me^2}{n+\frac{1}{2}} \leq em||V_j||_{L^2(C_\epsilon)}^2,
\]

since \( e_n^k(V_j)e^{n-\frac{1}{2}} \) are Fourier coefficients for \( V_j \) on the circle \( C_\epsilon(a_k) \). The norm of \( V_j \) that appears here is actually the \( H^{-\frac{1}{2}} \) norm, but this won’t matter for us. This last estimate for \( B_k \) combined with (57) for \( A_k \) give us,

\[
A_k B_k \leq \sqrt{\frac{\epsilon}{4\pi}} ||V_j||_{L^2(C_\epsilon(a_k))} ||f||_{L^2(\epsilon_0)}.
\]

In a similar fashion we estimate,

\[
\sum_{n \leq -1} |e_n^\infty(V_j)R^n||e_{n+1}^\infty(\Psi)||K_{n+1}(mR)R \leq A_\infty B_\infty,
\]

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where,

\[ A_\infty^2 = \sum_{n < -1} |c_{n+1}^\infty(\Psi)|^2 K_{n+1}(mR) K_n(mR) R \leq \frac{1}{4\pi m} ||f||_{L^2(\varepsilon_0)}^2, \]

and

\[ B_\infty^2 = \sum_{n < -1} |c_n^\infty(V_j) R^n|^2 \frac{K_{n+1}(mR)}{K_n(mR)} R. \]

In this equation it is important for us that \( n < -1 \). For \( n < -1 \) we have,

\[ \frac{K_{n+1}(mR)}{K_n(mR)} \leq \frac{mR}{|n| - 1}, \]

so that

\[ B_\infty^2 \leq mR ||V_j||_{L^2(C_R)}, \]

and

\[ A_\infty B_\infty \leq \sqrt{\frac{R}{4\pi} ||V_j||_{L^2(C_R)} ||f||_{L^2(\varepsilon_0)}}. \] (66)

Combining the expressions we found for the boundary integrals with the estimates that follow from (63)–(66) we find the following lower bound,

\[
\frac{1}{2\pi} \left| \int_{\partial \Omega} V_j \Psi_1 dz \right| \geq |c_1^1(\Psi)| I_{\frac{3}{2}}(me) e^{-\frac{1}{2}} - \sum_k |c_k^1(V_j)| |c_k^0(\Psi)| I_{-\frac{3}{2}}(me) e^{\frac{1}{2}} - C ||V_j||_{L^2(\varepsilon_0)} ||f||_{L^2(\varepsilon_0)},
\]

for a constant \( C \) which is independent of \( m \). Now we put together this lower bound with (62) to find,

\[ I_{\frac{3}{2}}(me) |c_1^1(\Psi)| \leq C(I_{-\frac{3}{2}}(me) \sum_k |c_k^0(\Psi)| + ||f||_{L^2(\varepsilon_0)}), \] (67)

where the constant \( C \) is independent of \( m \) but incorporates all the dependence on \( V_j \). The form of this estimate that we will use is now obtained by combining (66) with (67). We find,

\[ I_{\frac{3}{2}}(me) |c_1^1(\Psi)| \leq C(I_{-\frac{3}{2}}(me) + 1) ||f||_{L^2(\varepsilon_0)}, \] (68)

for a different constant \( C \).

We are now prepared to finish the proof of part 3 of Theorem (7). Recall that \( \delta_3 W_j \) is defined as the solution to,

\[ (m - \vartheta)\delta_3 W_j = -m\delta_2 W_j, \] (69)

with boundary values in \( W^{(m)} \). We could make this description technically precise and prove the existence of such a solution using \( H^1 \) estimates along the
lines of the $L^2$ estimate (55). The relevant estimates can be obtained via a
Stokes’ theorem calculation involving the exterior derivative,
$$2id \left( (\bar{\Psi}_1 \bar{\partial} \Psi_1 + \bar{\Psi}_2 \bar{\partial} \Psi_2) d\bar{z} - (\Psi_1 \partial \Psi_1 + \bar{\Psi}_2 \partial \Psi_2) dz \right),$$
for a solution $\Psi$ to
$$(m - \bar{\partial}) \Psi = f.$$  
However it will be simpler to proceed differently. The solution of (69) which
is relevant to us is the one constructed via functional analysis in \[10\]. This
solution is a weak $L^2$ solution to (69) inside $\Omega$ which extends to a solution of
the homogeneous equation $(m - \partial) \delta_3 W_j = 0$ outside $\Omega$ and is globally in $L^2(\mathcal{E})$.
We can obtain such a solution by approximating the right hand side $\delta_2 W_j$
in $L^2(\mathcal{E}_\Omega)$ by a sequence of functions $f_n \in C_0^\infty(\mathcal{E}_\Omega)$. Theorem (8) shows that the
resulting sequence of solutions to the approximate equations tends strongly in
$L^2(\mathcal{E}_\Omega)$ to a weak solution to (69). The norms on the left hand sides of (57)
and (58) are equivalent to the $L^2$ norms of limiting solution in the components
of the exterior of $\Omega$ and so the resulting solution is globally in $L^2(\mathcal{E})$ (this is a
consequence of the same Stokes’ theorem calculation that went into the proof
of Theorem (5) but done in the components of the exterior of $\Omega$). We may thus
identify the limiting solution with $\delta_3 W_j$ and by obtaining the solution $\delta_3 W_j$
in this fashion we see that estimate (68) remains valid, so that,
$$I_1^{1/2} \left( m_\epsilon \right) \leq C (I_2^{1/2} + 1) m \| \delta_2 W_j \|_{L^2(\Omega)}.$$ 
The right hand side of this last inequality is the Fourier coefficient (58) and the
left hand side tends to 0 using estimate 2 of Theorem (7). This finishes the
proof of Theorem (7). QED

Remark. The norms on the left hand side of (57) and (58) which are the
appropriate norms for boundary values of $L^2$ solutions also appear to be equivalent
to the $H^{-1/2}$ norms on the corresponding circles. The loss of one half derivative
for boundary values of solutions to $(m = 0)$ Dirac equations is a general property \[5\]. In our case, this would follow from the following Bessel function estimate,
$$\frac{K_n(r)}{K_{n-1}(r)} \leq 2(n - 1) \left( 1 + \frac{1}{r} \right),$$
for $r > 0$ and $n \geq 2$ which seems to be true.

We now substitute $\delta W_j = \delta_1 W_j + \delta_2 W_j + \delta_3 W_j$ into (10) and use Theorem (7) to determine the limit as $m \to 0$. According to part 1 of Theorem (7), $\delta_1 W_j$
converges to $G_0 f_{0,j}$ in $W^{1,p}(\mathcal{E}_\Omega)$ for $p > 2$. The Sobolev trace theorem implies
that it converges in $W^{1/2,p}(\mathcal{E}_{\partial \Omega})$ and this is enough to show that the Fourier
coefficient of $\delta_1 W_j$ which appears in (10) converges to,
$$\sqrt{\frac{\pi}{2e}} \frac{1}{2\pi} \int_{\theta_j}^{\theta_{j+2\pi}} (G_0 f_{0,j})_1 (ee^{i\Theta}) e^{-\frac{m}{2}} d\Theta_j.$$ 
(70)
Estimate 2 of Theorem (7) implies that the $H^1(\mathcal{E}_\Omega)$ norm of $\delta_2 W_j$ tends to 0 as
$m \to 0$ and again the Sobolev trace theorem implies that the boundary value
tends to 0 in $H^2(\mathcal{E}_{\partial\Omega})$ which is enough to show that the contribution made by $\delta_2 W_j$ to (10) is 0 in this limit. Finally part 3 of Theorem (11) shows that $\delta_3 W_j$ makes no contribution to the $m \to 0$ limit of (10). Thus to finish the proof of Theorem (1) we need only calculate (70).

We turn now to the calculation of (70). Using the definition of $f_{0,j}$ found in (52) above we see that,

$$\left( G_0 f_{0,j} \right)_1(z) = i \sqrt{\frac{2}{\pi}} \int_{\Omega} \left( -G_{11}(z, z') \bar{z}' \frac{1}{2} \partial z' \bar{\varphi}(z') + G_{12}(z, z') \bar{z}' \frac{1}{2} \partial \varphi(z') \right) dz' \bar{dz}' .$$

Using the fact that $G_{11}(z, z')$ is anti-holomorphic in $z'$ and $G_{12}(z, z')$ is holomorphic in $z$ we can rewrite this last integral as the integral of an exact form,

$$i \sqrt{\frac{2}{\pi}} \int_{\Omega} d \left( -G_{11}(z, z') \bar{z}' \frac{1}{2} \varphi(z') dz' - G_{12}(z, z') \bar{z}' \frac{1}{2} \bar{\varphi}(z') dz' \right) .$$

Since $\varphi(z) = 1$ on $C_\epsilon(a_j)$ and vanishes on the rest of $\partial \Omega$, Stokes’ theorem implies that the last integral is,

$$i \sqrt{\frac{2}{\pi}} \int_{C_\epsilon(a_j)} G_{11}(z, z') \bar{z}' \frac{1}{2} \varphi(z') dz' + G_{12}(z, z') \bar{z}' \frac{1}{2} \bar{\varphi}(z') dz' .$$

Now substitute,

$$G_{11}(z, z') = - \frac{1}{4 \pi i} \sum_k v_k(z) \bar{u}_k(z) ,$$

and (25) for $G_{12}(z, z')$ in this last integral and use the residue calculations that are found in the results that follow (38) to find,

$$\left( G_0 f_{0,j} \right)_1(z) = - i \frac{z - a_j}{2} + i \sqrt{\frac{2}{\pi}} v_j(z) .$$

Using this result it is now a simple matter to convert the Fourier integral (70) into the following residue calculation,

$$\frac{1}{2 \pi} \int_{C_\epsilon(a_j)} \frac{z - a_j}{2} \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \prod_{k \neq j} \frac{1}{(a_j - a_k)^{\epsilon_k}} dz ,$$

which in turn gives,

$$i \frac{\partial}{\partial z} \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \prod_{k \neq j} \left( \frac{z - a_k}{a_j - a_k} \right)^{\epsilon_k} \bigg|_{z=a_j} = i \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \sum_{k \neq j} \frac{\epsilon_k}{a_j - a_k} .$$

Dividing by $2i$ to get the limiting value of the coefficient that appears in the $m \to 0$ of the log derivative of the tau function we find,

$$\lim_{m \to 0} \frac{mc_j^2(W_j)}{2i} = \frac{1}{2} \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} c(\epsilon) \sum_{k \neq j} \frac{\epsilon_k}{a_j - a_k} .$$
To finish the proof of Theorem (1) we compare this result with,

\[ \frac{\partial}{\partial a_j} \sum_{|\epsilon|=0} \prod_{\alpha<\beta} |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta} = \sum_{|\epsilon|=0} \sum_{k \neq j} \frac{\epsilon_j \epsilon_k}{a_j - a_k} \prod_{\alpha<\beta} |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta} \]

which we obtained using,

\[ \frac{\partial}{\partial a_j} |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta} = \frac{\epsilon_\alpha \epsilon_\beta}{a_{\alpha} - a_{\beta}} (\delta_{\alpha j} - \delta_{\beta j}) |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta}. \]

In (71) observe that the \(|\epsilon|=0\) sum has two different possible values for \(\epsilon_j\), either \(\epsilon_j = \frac{1}{2}\) or \(\epsilon_j = -\frac{1}{2}\). However since the summand on the right hand side of (71) is clearly invariant under the complete sign reversal \(\epsilon_{\alpha} \rightarrow -\epsilon_{\alpha}\) it follows that the whole sum is just twice the result for \(\epsilon_j = \frac{1}{2}\). That is,

\[ \frac{\partial}{\partial a_j} \sum_{|\epsilon|=0} \prod_{\alpha<\beta} |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta} = \sum_{|\epsilon|=0, \epsilon_j=\frac{1}{2}} \sum_{k \neq j} \frac{\epsilon_k}{a_j - a_k} \prod_{\alpha<\beta} |a_{\alpha} - a_{\beta}|^{2\epsilon_\alpha \epsilon_\beta}. \]

Comparing this with (71) and recalling the definition of \(c(\epsilon)\) we have finished the proof of Theorem (1).

7 Odd Correlations and Holonomic Fields

In this section we make some observations about the application of the technique used to prove Theorem (1) to work out the asymptotics of the odd Ising scaling functions from below \(T_c\) and also the short distance behavior of the correlations for Holonomic Quantum Fields.

First we treat the case where \(n\) is odd. The one difference in the analogue of Lemma (1) for \(n\) odd is that the subspace \(W^{(m)}(\infty)\) is now the \(L^2\) closure of the span of,

\[ \hat{w}_n(z) = \begin{bmatrix} -e^{-i(n+\frac{1}{2})\theta} K_{n+\frac{1}{2}}(m|z|) \\ e^{-i(n-\frac{1}{2})\theta} K_{n-\frac{1}{2}}(m|z|) \end{bmatrix}, \]

for \(n \in \mathbb{Z}\). For definiteness we make the choice \(0 < \theta < 2\pi\) and choose the \(U_\infty\) trivialization (in the complement of \(\theta = 0\)) so that finite linear combinations of the \(\hat{w}_n(z)\) are smooth sections of \(\mathcal{E}\) in the \(U_\infty\) trivialization. Without difficulty one can compute the \(m \rightarrow 0\) limit of the normalized versions of these vectors and as a consequence we define \(W^{(0)}(\infty)\) as the \(L^2\) closure of the span of

\[ \left\{ \begin{bmatrix} -e^{-i(n+\frac{1}{2})\theta} \\ 0 \end{bmatrix} \right\}_{n \geq 1}, \left\{ \begin{bmatrix} e^{-i\theta n} \\ e^{i\theta n} \end{bmatrix} \right\}_{n \geq 1}. \]

Next we introduce a Green function \(-\hat{\theta}\) with \(W^{(0)}(\infty)\) boundary conditions in the following manner.

\[ G_0(z, z') = -\frac{1}{4\pi i} \left[ \sum_j u_j(z)v_j(z') \frac{g(z, z')}{g(z, z')} \sum_j u_j(z)v_j(z') \right], \quad (73) \]
where,

\[ u_j(z) := (z - a_j)^{-\frac{1}{2}} \prod_{k \neq j} \frac{(z - a_k)^{\frac{1}{2}}}{(a_j - a_k)^{\frac{1}{2}}}, \]  

and

\[ g(z, z') := \sum_{|\epsilon| = \pm \frac{1}{2}} c(\epsilon) \prod_{j \neq j} \frac{(z - a_j)^{\epsilon_j}}{(z' - a_j)^{-\epsilon_j}}, \]  

with \( \epsilon = (\epsilon_1, \ldots, \epsilon_N) \) and each \( \epsilon_j = \pm \frac{1}{2} \). Also

\[ |\epsilon| := \sum_{j=1}^N \epsilon_j, \]

\[ c(\epsilon) := \frac{\prod_{j<k} |a_j - a_k|^{2\epsilon_j \epsilon_k}}{\prod_{|\epsilon| = \pm \frac{1}{2}} \prod_{j<k} |a_j - a_k|^{2\epsilon_j \epsilon_k}}, \]

and

\[ v_j(z) = (z - a_j)^{-\frac{1}{2}} \sum_{|\epsilon| = \pm \frac{1}{2}, \epsilon_j = \frac{1}{2}} c(\epsilon) \prod_{k \neq j} \frac{(z - a_k)^{\epsilon_k}}{(a_j - a_k)^{\epsilon_k}}, \]

The multivalued functions \((z - a_j)^{\epsilon_j}\) are all defined using the argument \(\Theta_j\) and are consequently branched along \(z \in \mathfrak{r}_j\). We regard \(G_0(z, z')\) as defining an operator, \(G_0\), acting on sections of \(\mathcal{E}_\Omega\) in the following manner,

\[ G_0 f(z) := \int_{\Omega} G_0(z, z') f(z') d\bar{z}'dz', \]

where the section \(f(z')\) is identified with its \(\mathcal{U}_0\) trivialization. We also regard \(G_0 f\) as a section of \(\mathcal{E}_\Omega\) given in the \(\mathcal{U}_0\) trivialization.

The homogeneous function identity,

\[ \sum_k \bar{a}_k(z) v_k(z') = \sum_k \bar{v}_k(z) u_k(z'), \]

can be proved along the lines of Lemma 3 and this makes it possible to establish the desired results concerning the Green function and the projection \(P_0\). One matter that requires a little further analysis is the proof that that \(G_0 f\) has boundary values on \(C_R\) which are in \(W_0^{1,0}\). For this purpose it is useful to introduce a \(\mathcal{U}_\infty\) trivialization for \(\mathcal{E}\) over \(\{z : |z| > R\} \setminus \{t \in \mathbb{R} : t > 0\}\) by introducing square root \(z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\theta}\) for \(0 < \theta < 2\pi\), which is branched along the positive real axis. Smooth sections of \(\mathcal{E}\) over \(\{z : |z| > R\} \setminus \{t \in \mathbb{R} : t > 0\}\) can then be represented in the \(\mathcal{U}_\infty\) trivialization as products \(z^{\frac{1}{2}} \phi(z)\) for a smooth map \(\phi\) from \(D_\infty := \{z : |z| > R\}\) into \(\mathbb{C}^2\). For the purpose of analysing the behavior of the Green function \(G_0(z, z')\) for \(|z| > R\) it is useful to note that \(v_j(z)\) has a representation in this domain given by,

\[ v_j(z) = z^{-\frac{1}{2}} \left(1 - \frac{a_j}{z}\right)^{-\frac{1}{2}} \sum_{|\epsilon| = \pm \frac{1}{2}, \epsilon_j = \frac{1}{2}} c(\epsilon) \prod_{k \neq j} \frac{(1 - a_k)^{\epsilon_k}}{(a_j - a_k)^{\epsilon_k}} \]
\[ z^{-\frac{3}{2}} \left( 1 - \frac{a_j}{z} \right)^{-\frac{1}{2}} \sum_{|\epsilon| = -\frac{1}{2}, \epsilon_j = \frac{1}{2}} c(\epsilon) \prod_{k \neq j} \frac{(1 - \frac{a_k}{a_j})^{\epsilon_k}}{(a_j - a_k)^{\epsilon_k}}. \]

Using this one can check that for \( \varphi \in C_0^\infty(E_{\Omega}) \) we have \( G_0 \varphi | c_R \in W^{(0)} \) provided the following reality conditions are satisfied,

\[ \sum_{|\epsilon| = \frac{1}{2}, \epsilon_j = \frac{1}{2}} c(\epsilon) \prod_{k \neq j} |a_j - a_k|^{-\epsilon_k} = \sum_{|\epsilon| = \frac{1}{2}, \epsilon_j = \frac{1}{2}} \bar{c}(\epsilon) \prod_{k \neq j} (a_j - a_k)^{-\epsilon_k}. \]

This will be true for our choice of \( c(\epsilon) \) provided that,

\[ \sum_{|\epsilon| = \frac{1}{2}, \epsilon_j = \frac{1}{2}} \prod_{\alpha < \beta \neq j} |a_\alpha - a_\beta|^{2\epsilon_\alpha \epsilon_\beta} \prod_{k \neq j} |a_j - a_k|^{\epsilon_k} (a_j - a_k)^{-\epsilon_k}, \]

is real. However, under the transformation \( \epsilon_k \to -\epsilon_k \) for \( k \neq j \) the product,

\[ \prod_{k \neq j} |a_j - a_k|^{\epsilon_k} (a_j - a_k)^{-\epsilon_k}, \]

maps into its complex conjugate while in the preceding sum the coefficient of this product is real and invariant. This implies reality for the sum.

The rest of the analysis closely follows that in the even case and so we will only quote the final result. For \( N \) odd we have,

\[
\lim_{m \to 0} d_n \log \tau_-(ma) = \frac{1}{2} d_n \log \sum_{|\epsilon| = \frac{1}{2}, \epsilon_j = \frac{1}{2}} \prod_{\alpha < \beta \neq j} |a_\alpha - a_\beta|^{2\epsilon_\alpha \epsilon_\beta}.
\]

Finally we describe the situation for the tau functions for holonomic fields in the formalism of [9]. Suppose that for \( j = 1, \ldots, N \) we have,

\[ -\frac{1}{2} < \lambda_j < \frac{1}{2}, \]

and for simplicity we also suppose that,

\[ \sum_j \lambda_j = 0. \]

The restricted local expansion that determines the subspace \( W^{(m)}_j \) is,

\[ w(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}, k > 0} a_k^j (w) w_{k - \lambda_j} (z_j) + b_k^j (w) w_{k + \lambda_j}^* (z_j). \]

At infinity the restricted expansion that determines \( W^{(m)}_\infty \) is,

\[ w(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} c_k (w) \hat{w}_k (z). \]
Without difficulty one can check that the limiting subspaces, $W_j^{(0)}$, are spanned by,
\[ \begin{bmatrix} z_j^{k - \lambda_j} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z_j^{k + \lambda_j} \end{bmatrix} \text{ for } k = \frac{1}{2}, \frac{3}{2}, \ldots \]
and $W_\infty^{(0)}$ is spanned by,
\[ \begin{bmatrix} z^{-n} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z^{-n} \end{bmatrix} \text{ for } n = 1, 2, 3 \ldots \]

The mass 0 Green function for the Dirac operator of interest is clearly (see Proposition 1.1 in [9]),
\[ G_0(z, z') = -\frac{1}{4\pi} \begin{bmatrix} 0 \\ g(z, z') \end{bmatrix}, \begin{bmatrix} g(z, z') \\ 0 \end{bmatrix}, \]
where
\[ g(z, z') = \frac{\prod_j z_j^{-\lambda_j}(z'_j)^{\lambda_j}}{z' - z}. \]

There are no “chiral symmetry breaking” terms. In the notation of (4.3) of [9] we have,
\[ da \log \tau(ma, \lambda) = \frac{m}{2} \sum_j \left\{ a_j^i (\lambda) da_j + \bar{a}_j^i (\lambda) d\bar{a}_j \right\}, \]
and we find for the $m \to 0$ limit,
\[ \lim_{m \to 0} da \log \tau(ma, \lambda) = \sum_{j=1}^{N} \left\{ \sum_{k \neq j} \frac{\lambda_j \lambda_k}{a_j - a_k} da_j + \sum_{k \neq j} \frac{\lambda_j \lambda_k}{\bar{a}_j - \bar{a}_k} d\bar{a}_j \right\}, \]
which is also just,
\[ da \log \prod_{j < k} |a_j - a_k|^{\lambda_j \lambda_k}. \]

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