Phase structure of Abelian Chern-Simons gauge theories

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Abstract. – We study the effect of a Chern-Simons (CS) term in the phase structure of two different Abelian gauge theories. For the compact Maxwell-Chern-Simons theory, with the CS-term properly defined, we obtain that for values $g = n/2\pi$ of the CS coupling with $n = \pm 1, \pm 2$, the theory is equivalent to a gas of closed loops with contact interaction, exhibiting a phase transition in the $3dXY$ universality class. We also employ Monte Carlo simulations to study the noncompact $U(1)$ Abelian Higgs model with a CS term. Finite size scaling of the third moment of the action yields critical exponents $\alpha$ and $\nu$ that vary continuously with the strength of the CS term, and a comparison with available analytical results is made.

Gauge theories in $2 + 1$ dimensions are frequently proposed as effective theories of strongly correlated electron systems in two spatial dimensions and zero temperature. Strong local constraints on the dynamics of lattice fermion systems are enforced by fluctuating compact gauge fields which exhibit topological defects in the form of space-time instantons [1]. It is conceivable that such effective gauge theories may feature phase transitions from a phase of bound instantons (deconfined phase) to a phase of unbound free instantons (confined phase). Such theories have been proposed as effective theories for high-temperature cuprate superconductors, chiral spin liquids, and Mott insulators [2–4]. There is some hope that confinement-deconfinement transitions in $2 + 1$ dimensions may shed light on quantum phenomena such as spin-charge separation and the breakdown of Fermi liquid theory in two spatial dimensions, and quantum phase transitions in Mott insulators. One central issue is how to describe exotic physical phenomena which do not comply with the Fermi-liquid paradigm such as the fractional quantum Hall effect, high-$T_c$ superconductivity, and heavy fermion physics.

In this paper we will consider two different $(2 + 1)$-dimensional Abelian lattice gauge theories containing a Chern-Simons (CS) term [5]. The first of these theories is defined by a kind of lattice Maxwell-Chern-Simons Lagrangian. Polyakov has demonstrated that the compact Maxwell theory is permanently confined in $2 + 1$ dimensions and does not exhibit any phase transition [1]. When this theory is coupled to bosonic matter fields with an integer charge $q > 1$, a deconfinement transition occurs [6,7]. The case where the bosonic matter fields are coupled to the fundamental charge is more controversial and at present it is not yet known if a deconfinement transition indeed occurs [8,9]. In this paper we will study the effect...
of adding a CS-term to Polyakov theory. Such a theory was studied previously by a number of authors [10,11]. In this paper we will use duality techniques to obtain new exact results on a lattice Abelian compact theory with a special type of CS-term. We will show that a phase transition does occur as a function of the gauge coupling for a fixed value of the CS-coupling, while confinement of electric charges is suppressed.

The second theory considered in this paper is a non-compact lattice Abelian Higgs model with a CS-term. This theory has also been much studied in the past by many authors [12–15], often in connection with condensed matter systems. In this paper, we will compute the critical exponents of this theory as a function of the CS-coupling using Monte Carlo simulations. To our knowledge, this is the first time Monte Carlo simulations are employed on a lattice CS-theory.

The lattice Maxwell-Chern-Simons theory with a compact gauge field used in this paper is defined by the partition function

$$Z = \int_{\mathcal{P}} \left[ \prod_{\mu} \exp \left( \frac{DA_{\mu}}{2\pi} \right) \right] \sum_{\{n,N\}} e^{-\sum \mathcal{L}}$$

$$\mathcal{L}_i = \frac{1}{2\pi} (\Delta \times A_i - 2\pi n_i)^2 + \frac{i\lambda}{2} (A_i - 2\pi N_i) \cdot (\Delta \times A_i - 2\pi n_i),$$

where $A_i$ is a periodic gauge field, $n_i$ and $N_i$ are integer fields taking care of the compactness of the theory and $\Delta$ is the lattice difference operator. The above Lagrangian is written in Villain-like form in order to allow the use of standard duality transformations. Note that it differs from the usual compact Maxwell-Chern-Simons Lagrangian [11], where the constraint $\Delta \cdot n_i = 0$ is expected to hold. Physically, this constraint leads to a complete suppression of magnetic monopoles (which are instantons in $2+1$ dimensions), since it implies $\Delta \cdot n_i = 0$.

Relaxation of this constraint incorporates the magnetic monopoles in the theory, since then $\Delta \cdot n_i = Q_i$, where $Q_i$ is an integer, just like in Polyakov's $(2+1)$-dimensional compact QED [1]. As we will see, removing the constraint $\Delta \cdot n_i = n_i$ implies new interesting physics.

A Lagrangian similar to the one in Eq. (1) arises in effective descriptions of chiral spin states [16]. Due to compactness, Eq. (1) is gauge invariant only if $g = n/2\pi$, with $n$ integer [10]. This is in contrast with the non-compact Abelian theory where $g$ can be any real number. Furthermore, Eq. (1) is also invariant under the integer gauge transformation $A_i \rightarrow A_i + 2\pi M_i$, $n_i \rightarrow n_i + \Delta \times M_i$, and $N_i \rightarrow N_i + M_i$.

Let us associate a vector field $\mathbf{a}$ with $\Delta \times \mathbf{A} - 2\pi \mathbf{n}$ and $\mathbf{b}$ with $\mathbf{A} - 2\pi \mathbf{N}$ by introducing the Lagrange multiplier vector fields $\mathbf{\lambda}$ and $\mathbf{\sigma}$ (in the following we omit sometimes lattice subindices to simplify the notation)

$$\tilde{\mathcal{L}} = \frac{1}{2\pi^2} a^2 + \frac{i\lambda}{2} a \cdot b + i\sigma \cdot (\Delta \times A - 2\pi n - \mathbf{a}) + i\mathbf{\sigma} \cdot (A - 2\pi N - b).$$

Next, we use the Poisson summation formula to replace $\mathbf{\lambda}$ and $\mathbf{\sigma}$ by a new set of integer valued fields denoted $\mathbf{L}$ and $\mathbf{S}$

$$\tilde{\mathcal{L}} = \frac{1}{2\pi^2} a^2 + \frac{i\lambda}{2} a \cdot b + i\mathbf{L} \cdot (\Delta \times A - \mathbf{a}) + i\mathbf{S} \cdot (A - \mathbf{b}).$$

Straightforward integration of $A$ leads to the constraint $\Delta \times \mathbf{L} = \mathbf{S}$ in the partition function. Summation over the field $\mathbf{S}$ followed by integration of the fields $\mathbf{a}$ and $\mathbf{b}$ yields

$$\tilde{\mathcal{L}} = \frac{8\pi^2}{n^2 f^2} (\Delta \times \mathbf{L})^2 - \frac{i4\pi}{n} \mathbf{L} \cdot (\Delta \times \mathbf{L}),$$

where we have written explicitly $g = n/2\pi$. The crucial point to note is that when $n = \pm 1, \pm 2$, the integer CS-term in Eq. (1) does not contribute to the partition function. In such a case,
the theory can be written in terms of a new integer field $R = \Delta \times L$ to obtain

$$Z = \sum_{\{R\}'} \exp \left( - \sum_i \frac{8\pi^2}{n^2 f^2 R_i^2} \right),$$

(5)

where $\sum_{\{R\}'}$ denotes a constrained sum over closed vortex loops, i.e., with the constraint $\Delta \cdot R = 0$ implied. The theory defined by Eq. (5) is a theory of closed loops interacting through contact repulsion. This is the well known loop gas representation of the Abelian Higgs model with zero screening length which exhibits a loop-proliferation phase transition in the $3dXY$ universality class. There is a family of critical points given by $f_c^2 \approx 0.33 \times 8\pi^2/n^2$, with the values $n = \pm 1, \pm 2$. When $n = \pm 2$, the loop gas partition function (5) is equivalent to a non-compact abelian Higgs model in the so called “frozen” limit [17], which is the basis for the “inverted” $3dXY$ universality class in superconductors [18]. For $n = \pm 1$, on the other hand, Eq. (5) corresponds to a frozen superconductor where the charge of the Cooper pair $f = 2e$ is fractionalized. Therefore, in this case the theory in (5) can be thought as corresponding to a “frozen” superconductor with charge $f/2 = e$.

The constraint $\Delta \cdot R = 0$ implies that there are no monopoles in the spectrum of the theory with $n = \pm 1, \pm 2$. Therefore, there is no confinement of electric charges in the corresponding $3dXY$ phase transition. This is entirely complementary to the theory without a CS-term: Here, one has permanent confinement and no phase transition.

In Ref. [11] a continuum version of a similar model is studied using a Hamiltonian approach via a variational analysis. There a $XY$-like phase transition is also found, but only as a function of $n$. In contrast with our model, the compact Maxwell-Chern-Simons theory studied in Ref. [11] does not undergo any phase transition for fixed $n$.

We next consider the Abelian Higgs model with a noncompact CS gauge field given in the Villain approximation by

$$S = \sum_i \left[ \frac{\beta}{2} (\Delta \theta_i - f A_i - 2\pi n_i)^2 + ig A_i \cdot \Delta A_i + \lambda (\Delta \times A_i)^2 \right],$$

(6)

where $A_i$ is a noncompact gauge field and $\theta_i$ is a scalar phase field. Note that here the CS coupling $g$ can be any real number, since now we are dealing with a non-compact Abelian gauge field. The topological defects of this model are vortex loops. An Abelian Higgs theory with a CS term has been proposed as an effective theory for the Laughlin state of fractional quantum Hall systems [3, 12].

By introducing an auxiliary continuous field $v_i$, we can rewrite the action in Eq. (6) as

$$S = \sum_i \left[ \frac{1}{2\beta} v_i^2 - iv_i \cdot (\Delta \theta_i - f A_i - 2\pi n_i) + ig A_i \cdot \Delta A_i + \lambda (\Delta \times A_i)^2 \right].$$

(7)

Besides the usual gauge invariance, the action (6) is also invariant under the integer gauge transformation $\theta_i \to \theta_i + 2\pi l_i$, $n_i \to n_i + \Delta l_i$, where $l_i$ is an integer. This suggests that a gauge fixing on $n_i$ would allow us to extend the limit of integration for $\theta_i$ over the whole real line. However, since we are dealing with integer fields, not all gauge fixings work consistently. The widely employed gauge fixing $\Delta \cdot n_i = 0$ does not work because it would lead to a decoupling of $\theta_i$ from $n_i$ and consequently to a wrong loop-gas representation in the $3dXY$ limit. Instead, we follow Ref. [19] and fix the axial gauge $n_3 = 0$. Within this gauge fixing we can integrate out $\theta_i \in (-\infty, \infty)$ in Eq. (7) to obtain the constraint $\Delta \cdot v_i = 0$. This constraint is solved by introducing a new vector field $h_i$ such that $v_i = \Delta \times h_i$. Next we perform a partial summation
in the term $2\pi i \mathbf{n}_i \cdot (\Delta \times \mathbf{h}_i)$ to give $2\pi i \mathbf{h}_i \cdot (\Delta \times \mathbf{n}_i)$ and identify the integer field $\mathbf{m}_i = \Delta \times \mathbf{n}_i$ as the vortex field. The end result is

$$S = \sum_i \left[ \frac{1}{2\beta} (\Delta \times \mathbf{h}_i)^2 - i f(\Delta \times \mathbf{h}_i) \cdot \mathbf{A}_i + 2\pi i \mathbf{h}_i \cdot \mathbf{m}_i + ig A_i \cdot \mathbf{A}_i + \lambda (\Delta \times \mathbf{A}_i)^2 \right]. \quad (8)$$

where the constraint $\Delta \cdot \mathbf{m}_i = 0$ is understood. Setting $\lambda = 0$ and integrating out $\mathbf{A}_i$ yields a well known self-duality [15] at $\beta \to \infty$.

After integrating out $\mathbf{A}_i$ and $\mathbf{h}_i$ and performing a lattice Fourier transform, we obtain

$$V^{\mu\nu}_q = \sum_q m^q_i V^{\mu\nu}_q m^q_i,$$

$$S = \sum_q m^q_i V^{\mu\nu}_q m^q_i.$$

where $Q_q Q_q \equiv Q^q Q^q$, with $Q^q = e^{iq^\mu}/2 - e^{-iq^\mu}/2$ being the Fourier representation of the symmetrized difference operator, and we have defined $\varphi = \lambda/g$ and $\chi = \beta f^2/(2g)$. $V^{\mu\nu}_q$ is the Fourier transform of the vortex-vortex interaction tensor. In the two limits $g \to \infty$ and $f = 0$, it reduces to the interaction potential of vortices in the 3dXY model $V^{\mu\nu}_{3dXY} = (2\pi^2 \beta/Q_q Q_q)\delta^{\mu\nu}$.

We next study the model Eq. (9) using Monte Carlo (MC) simulations. Having real-valued vortex variables requires working in real-space. However, the action defined by Eq. (9) is complex in real-space, $S = S_R + i S_I$, leading to a complex transition-probability between various vortex-configurations in the Metropolis algorithm. This is analogous to the "sign problem" in quantum Monte Carlo simulations, which is often treated by using the absolute value of the probability $\rho$ in the Metropolis update and sampling expectation values by $\langle \mathcal{O} \rangle = \langle \mathcal{O} s | \rho | s \rangle / \langle s | \rho | s \rangle$, where $s$ is the sign of the probability [20]. The expectation value of an operator $\mathcal{O}$ is in our case defined schematically by

$$\langle \mathcal{O} \rangle_S = \frac{\int_D \mathcal{D} \Psi \mathcal{O} e^{i S_R + i S_I}}{\int_D \mathcal{D} \Psi e^{i S_R + i S_I}} \quad (10)$$

where $\Psi$ denotes the complete set of eigenstates given by $S$. Since the real part of the action $S_R$ defines the same set of eigenstates, i.e. closed vortex loops, we may express $\langle \mathcal{O} \rangle_S$ in terms of expectation values $\langle \mathcal{O} \rangle_{S_R}$

$$\langle \mathcal{O} \rangle_S = \frac{\int_D \mathcal{D} \Psi (\mathcal{O} e^{i S_I}) e^{i S_R} / Z_{S_R}}{\int_D \mathcal{D} \Psi (e^{i S_I}) e^{i S_R} / Z_{S_R}} = \frac{\langle \mathcal{O} e^{i S_I} \rangle_{S_R}}{\langle e^{i S_I} \rangle_{S_R}}, \quad (11)$$

where $Z_{S_R}$ is the partition function for the system defined by the real part of the action. The MC simulations are performed using $e^{S_R}$ as the Boltzmann weight and expectation values are calculated from Eq. (11).

The system size $L$ and the lattice constant define the only length scales of the system. Critical properties are governed by the long range physics i.e. the $q \to 0$ limit. In this limit the real part of the potential (9) has an effective screening length $\lambda_{\text{eff}}^{-1} \sim \lambda^2/(2\varphi + 1)$. When this screening length is of the order of $L/2$, the critical behavior will probably be a crossover to the 3dXY model with an infinite screening length. On the other hand, when $\lambda_{\text{eff}} < a$ the critical behavior will experience a crossover towards a system with steric repulsion. Hence, MC simulations on finite lattices can only provide true critical exponents in a limited region of coupling space where $a \ll \lambda_{\text{eff}} \ll L/2$. 

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Fig. 1 – a) $(1 + \alpha)/\nu$ from finite size scaling of $M_3$. The solid line denotes the well known 3dXY values and is included here for the sake of comparison. b) Critical exponent $\nu$ computed directly from the width of $M_3$. c) $\alpha$ from combining $M_3$ results for $(1 + \alpha)/\nu$ and $1/\nu$.

We have performed MC simulations on the model defined by

$$Z = \sum_{\{m_\mu(r_i)\}} \exp \left[ - \sum_{i,j} m_\mu(r_i) V^{\mu\nu}(r_i - r_j) m_\nu(r_j) \right], \quad (12)$$

where the potential $V^{\mu\nu}(r)$ is the inverse Fourier transform of Eq. (9). A MC move is an attempt to insert a unitary closed vortex loop of random orientation, and the move is accepted or rejected according to the standard Metropolis algorithm using the real part of the action.
One sweep consists of traversing the lattice performing one MC move for each lattice site. The system with size $L \times L \times L$ has periodic boundary conditions. Simulations with up to $10^6$ sweeps per coupling constant combined with Ferrenberg-Swendsen multihistogram reweighting are used to produce finite size scaling (FSS) plots. System sizes $L \in [4, 6, 8, 10, 12, 14, 16, 20, 24]$ have been used.

To investigate the critical properties of the model, we compute critical exponents $\alpha$ and $\nu$. Recently we have proposed a FSS method based on the third moment $M_3$ of the action $S$ [7],

$$M_3 = \langle (S - \langle S \rangle)^3 \rangle_S,$$

for which the asymptotically correct behavior is reached for accessibly small system sizes. The peak to peak value of this quantity scales with system size as $L^{(1+\alpha)/\nu}$ and the width between the peaks scales as $L^{-1/\nu}$. In this way one can resolve $\alpha$ and $\nu$ independently from one measurement without invoking hyperscaling [7].

The denominator $|\langle e^{iS_I} \rangle_{S_R}|$ decreases with system size and coupling strength, contaminating expectation values Eq. (11). Hence, as the strength of the CS term $g$ increases, smaller system sizes are accessible for the FSS. Together with the limitations on the effective screening length $\lambda_{\text{eff}}$, this restricts the range in coupling space for which critical exponents can be calculated. We compute exponents along a line in coupling space corresponding to fixing $f = 1$, $\lambda = \frac{1}{4}$, and tuning $g$ so that Eq. (11) is meaningful and crossover effects are absent. Along this line exponents can only be extracted from $M_3$-analysis for $g \leq 0.5$. We simulate system sizes $L = 4, 6, 8, \ldots$ where the maximum system size varies from $L = 12$ for $g = 0.5$ to at least $L = 24$ for $g = 0.01$. The results of the FSS analysis are shown in Fig. 1. We find non-universal exponents $\alpha$ and $\nu$ approaching 3dXY values for large $g$ and when $g \to 0$. Continuously varying critical exponents are a consequence of the vanishing of the renormalization group (RG) $\beta$-function of the CS-coupling [13, 21]. They are associated with a marginal operator, which in this case is just the CS-term. Critical exponents for the Abelian Higgs model with a CS-term have been obtained previously using RG methods [14, 21]. Here we provide for the first time a MC calculation of these exponents. Due to the technical difficulties explained in Ref. [14], the RG calculations are more reliable for a generalized model with $N/2$ complex field components where $N$ is large enough. Although our results are for $N = 1$, it is useful to compare at least qualitatively the results obtained here with those obtained from the $1/N$-expansion. In this case the critical exponent $\nu$ is given at order $1/N$ by [21]

$$\nu = 1 - \frac{96}{\pi^2 N} \left[ 1 - \frac{8 \bar{g}^2 (\bar{g}^2 + 4)}{9 (\bar{g}^2 + 1)^2} \right],$$

(13)

where we have defined $\bar{g} = 4g/\pi$. The critical exponent $\nu$ as given by Eq. (13) exhibits a similar qualitative behavior in comparison with panel (b) in Fig. 1 having a maximum at some value of $g$. For smaller values of $N$, on the other hand, the RG treatment is in poor agreement with our numerical results, both from qualitative and quantitative points of view [14].

In summary, we have shown in two different gauge theories that the presence of a CS term changes dramatically their behavior. In both cases the analysis was entirely non-perturbative and based on duality arguments.

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