POINCARÉ DUALITY IN P.A. SMITH THEORY

CHRISTOPHER ALLDAY, BERNHARD HANKE, AND VOLKER PUPPE

Abstract. Let $G = S^1$, $G = \mathbb{Z}/p$ or more generally $G$ be a finite $p$-group, where $p$ is an odd prime. If $G$ acts on a space whose cohomology ring fulfills Poincaré duality (with appropriate coefficients $k$), we prove a mod 4 congruence between the total Betti number of $X^G$ and a number which depends only on the $k[G]$-module structure of $H^*(X;k)$. This improves the well known mod 2 congruences that hold for actions on general spaces.

1. Introduction and statement of results

Let $X$ be a finite dimensional connected CW complex and let $k$ be a commutative ring with unit. We say that $X$ is a Poincaré duality space over $k$ (we will often write $k$-PD space instead) of formal dimension $n$, if $H^*(X;k)$ is a finitely generated $k$-module and if there is a class $\nu \in H^*_n(X;k)$ such that

$$H^*(X;k) \to H_{n-*}(X;k), \quad c \mapsto c \cap \nu$$

is an isomorphism. Note that if $k$ is a field, this is equivalent to requiring that

$$H^*(X;k) \times H^*(X;k) \xrightarrow{\cup} H^*(X;k) \xrightarrow{\nu} k$$

is a nonsingular pairing (viewing $\nu$ as an element in $\text{Hom}(H^*(X;k),k)$).

Let $G$ denote the group $S^1$ (with its usual topology) or $\mathbb{Z}/p$, where $p$ is an odd prime number. Let $k = \mathbb{Q}$, if $G = S^1$, and $k = \mathbb{F}_p$, if $k = \mathbb{Z}/p$. By a well known result proven independently by Chang-Skjelbred in [2] and Bredon in [3], each component of the fixed point set of a finite dimensional $G$-CW complex $X$ fulfills Poincaré duality over $k$, if this is property holds for $X$. By now there are several further versions and variants of proofs of this result ([4, 5, 6]). In this paper, we will use certain consequences of Poincaré duality to deduce relations between the total Betti number of $X^G$ and the $k[G]$-module $H^*(X;k)$.

Theorem 1. Let $G = S^1$ and let $X$ be a finite dimensional connected $G$-CW complex such that $X$ is a $\mathbb{Q}$-PD space of formal dimension $n$. If

- $n$ is even or

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\[ n = 2m + 1, \ X^G \neq \emptyset, \ H^i(X; \mathbb{Q}) = 0 \text{ for } 0 < i \leq m, \ i \text{ even}, \]

the following congruence holds.

\[ \dim_{\mathbb{Q}} H^*(X^G; \mathbb{Q}) \equiv \dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) \mod 4. \]

If \( G = S^1 \times \ldots \times S^1 \) and if \( X \) is a finite dimensional connected \( G \)-CW complex with finitely many orbit types that fulfills Poincaré duality over \( \mathbb{Q} \), then an analogue of Theorem 1 holds. This is true, because in this case we can choose a subcircle \( S^1 \subset G \), such that \( X^{S^1} = X^G \).

A version of Theorem 1 for actions of \( G = \mathbb{Z}/p \), where \( p \) is an odd prime, can be proven under additional assumptions on the space \( X \). It turns out that one also has to take into account the fact that now the induced \( G \)-action on \( H^*(X; F_p) \) might be nontrivial. Using a spectral sequence argument together with results from [6], we show:

**Theorem 2.** Let \( G = \mathbb{Z}/p \) (where \( p \) is an odd prime) and let \( X \) be a finite dimensional connected \( G \)-CW complex such that \( X \) is an \( F_p \)-PD space of formal dimension \( n \). Furthermore, assume that \( H^*(X; \mathbb{Z}(p)) \) does not contain \( \mathbb{Z}/p \) as a direct summand. Then, we get a decomposition as graded \( F_p[G] \)-modules

\[ H^*(X; F_p) = F^* \oplus T^* \oplus R^*, \]

where \( F^* \) is a free \( F_p[G] \)-module, \( T^* \) is a trivial \( F_p[G] \)-module and \( R^* \) is a direct sum of \( F_p[G] \)-modules of the form \( \ker \epsilon \), where \( \epsilon : F_p[G] \to F_p \) is the augmentation map. If

- \( n \) is even or
- \( n = 2m + 1, \ X^G \neq \emptyset, \ T^i = 0 \text{ for } 0 < i \leq m, \ i \text{ even}, \ R^i = 0 \text{ for } 0 < i \leq m, \ i \text{ odd}, \)

the following congruence holds.

\[ \dim_{F_p} H^*(X^G; F_p) \equiv \dim_{F_p} T^* + \frac{1}{p-1} \dim_{F_p} R^* \mod 4. \]

Theorem 2 was first proved by A. Sikora in his PhD thesis. This and Theorem 3 for the case that \( R^* = 0 \) and \( H^*(X; \mathbb{Z}) \) does not contain any \( p \)-torsion are also contained in his paper [9], where a somewhat different line of argument is used.

It is well known that the cited result by Bredon, Chang and Skjelbred immediately generalizes to actions of finite \( p \)-groups \( G \). This is achieved by applying induction on the order of \( G \) and using the fact that every finite (nontrivial) \( p \)-group contains a normal subgroup of order \( p \). This procedure can be applied in our situation as well and shows the following result.

**Theorem 3.** Let \( G \) be a finite \( p \)-group (where \( p \) is an odd prime) and let \( X \) be a finite dimensional connected \( G \)-CW complex that is an \( F_p \)-PD space of even formal dimension. If \( p > \dim H^*(X; F_p) \), then

\[ \dim_{F_p} H^*(X^G; F_p) \equiv \dim_{F_p} H^*(X; F_p) \mod 4. \]
If we impose more restrictions on $X$ and on the group operation, then using a result from [7], we can use a similar induction argument for comparing rational Betti numbers:

**Theorem 4.** Let $G$ be a finite $p$-group (where $p$ is an odd prime) and let $X$ be a finite connected simplicial complex with $G$ acting simplicially such that $X$ is an even dimensional orientable $\mathbb{Z}_p$-homology manifold. Furthermore, assume that $p > \dim H^*(X; \mathbb{F}_p)$. Then

$$\dim_{\mathbb{Q}} H^*(X^G; \mathbb{Q}) \equiv \dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) \mod 4.$$ 

In the case of smooth actions, this result can be proven without making use of [7].

In Section 5, we will show by examples that none of the additional assumptions in the second part of Theorem 1 and Theorem 2 can be dropped.

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2. **Algebraic preliminaries**

For our later applications to spectral sequences, it will be useful to provide a notion of Poincaré duality for bigraded algebras.

**Definition 1.** Let $k$ be a field and let $A^{*,*}$ be a $(\mathbb{Z}/2 \times \mathbb{N})$-bigraded associative and graded commutative $k$-algebra with unit that is finitely generated as $k$-vector space. (The graded commutativity is required with respect to the total $\mathbb{Z}/2$-grading, where $A^{i,j}$ has total degree $i + j \mod 2$.)

i. We call $A^{*,*}$ connected, if $A^{0,0} \cong k$.

ii. $A^{*,*}$ is a $k$-Poincaré duality algebra of formal dimension $n$, if there is a surjective linear map $\phi : A^{*,*} \to k$ (called orientation) with $\phi(A^{i,j}) = 0$ for $(i,j) \neq (0,n)$ such that the bilinear form $\zeta : A^{*,*} \times A^{*,*} \to k$ is nondegenerate.

Note that in a connected Poincaré duality algebra of formal dimension $n$, we have $A^{0,n} \cong k$. The following elementary fact provides a connection between the Euler characteristic and the total Betti number of Poincaré duality algebras of even formal dimension.

**Lemma 1.** Let $A^{*,*}$ be a Poincaré duality algebra of even formal dimension over a field $k$. If char $k \neq 2$, then

$$\dim_k A^{*,*} \equiv \chi(A^{*,*}) \mod 4,$$

where the Euler characteristic is calculated using the total $\mathbb{Z}/2$-grading of $A$.

**Proof.** Let $2m$ be the formal dimension of $A^{*,*}$. Using the induced total $\mathbb{Z}/2$ grading of $A^{*,*}$, we claim that the dimension of $A^{\text{odd}}$ is even, which obviously
proves the lemma. For all $i$, we get isomorphisms
\[
A^{0,2i+1} \cong \text{Hom}(A^{0,2m-2i-1}, k) \\
A^{1,2i} \cong \text{Hom}(A^{1,2m-2i}, k)
\]
by Poincaré duality. Hence, $A^{0,2i+1}$ and $A^{0,2m-2i-1}$, respectively $A^{1,2i}$ and $A^{1,2m-2i}$ have the same dimension. Furthermore, if $m$ is even, the module $A^{1,m}$, carries a skew nonsingular form by Poincaré duality and therefore has even dimension, as $\text{char } k \neq 2$. The same is true for $A^{0,m}$, if $m$ is odd.

**Proposition 1.** Let $(A^{*,*}, \delta)$ be a connected $(\mathbb{Z}/2 \times \mathbb{N})$-graded differential algebra over the field $k$ with a differential $\delta$ of arbitrary bidegree, but lowering the second grading parameter and acting as a derivation. Furthermore, assume that $A^{*,*}$ is a Poincaré duality algebra of formal dimension $n$. Then, after taking homology, we get either $H(A^{*,*}, \delta) = 0$ or $H(A^{*,*}, \delta)$ is again a connected Poincaré duality algebra of formal dimension $n$.

**Proof.** Assume that $H(A^{*,*}) \neq 0$. This implies that the unit of $A$ (that generates $A^{0,0}$) is not in the image of $\delta$. Let $c \in A^{0,n} \cong k$ be a generator. Assume that $\delta(c) = x \neq 0$. By Poincaré duality, we then find an element $y \in A^{*,*}$ with $c = yx \in A^{0,n}$. We therefore get
\[
0 \neq x = \delta(c) = \delta(yx) = \delta(y)x,
\]
which implies that $\delta(y)$ is a generator of $A^{0,0}$, contrary to what we said before. Hence, because $c$ is not hit by $\delta$, any orientation $\phi$ of $A$ induces a surjective linear map $H^{*,*}(A) \to k$. To complete the proof, observe that by the derivation property of $\delta$, we have
\[
\zeta(\ker \delta, \text{im } \delta) = 0,
\]
using the bilinear form $\zeta$ from Definition I. As
\[
\dim_k \ker \delta + \dim_k \text{im } \delta = \dim_k A^{*,*},
\]
we may conclude that $\ker \delta$ is exactly the orthogonal complement of $\text{im } \delta$ with respect to $\zeta$. This proves that the induced bilinear form on $H^{*,*}(A)$ is nonsingular. \qed

**Proposition 2.** Let $(A^{*,*}, \delta)$ be as in the last Proposition. Additionally, we assume that $\text{char } k \neq 2$, the formal dimension of $A^{*,*}$ is an odd number $2m+1$, the differential $\delta$ has odd (total) degree, lowering the second grading parameter in $A^{*,*}$ and $A^{0,i} = 0$ for $0 < i \leq m$, $i$ even, $A^{1,i} = 0$ for $0 < i \leq m$, $i$ odd. If $H(A^{*,*}) \neq 0$. Then
\[
\dim_k A^{*,*} \equiv \dim_k H(A^{*,*}, \delta) \mod 4.
\]

**Proof.** Let $Z^* = \ker \delta \subset A^*$ denote the cycles in $A^*$ with respect to $\delta$ (here we use the total $\mathbb{Z}/2$ grading, again). By our assumption on $A^*$, the even dimensional part $H^{even}(A^*, \delta)$ of the homology of $A^*$ with respect to $\delta$,
coincides with \( Z^{ev} \), cf. the proof of Proposition \( \text{[1]} \). Now we define a bilinear form
\[
\gamma: A^{ev} \times A^{ev} \rightarrow k; \quad (x, y) \mapsto \phi(x \cdot \delta(y)),
\]
using an orientation \( \phi \) of \( A \). It is easy to check, that \( \gamma \) induces a well defined, nonsingular and skew symmetric form on \( A^{ev}/Z^{ev} \), hence this vector space has even dimension over \( k \) (using the fact that \( \text{char} \ k \neq 2 \)). As the Euler characteristic of \( A^* \) and of \( H(A^*) \) are equal, we get the equation
\[
A^{ev} - H^{ev}(A^*) = A^{odd} - H^{odd}(A^*).
\]
Therefore, the number \( \dim A^* - \dim H^*(A^*) \) is divisible by 4.

3. Proof of Theorems \( \text{[1]}, \text{[3]} \) and \( \text{[4]} \)

Here, when applying the results from the last section, we usually forget about the first grading parameter of \( A^{*+} \) and use the \( N \)-grading by the second parameter. Part of the argument is based on the following well known fact (cf. \( \text{[1]}, \text{Exercise (3.29)} \)):

**Lemma 2.** Let \( G = S^1 \) or \( G = \mathbb{Z}/p \) and let \( X \) be a finite dimensional \( G \)-CW complex such that \( H^*(X; \mathbb{Z}) \) is a finitely generated \( \mathbb{Z} \)-module. Then
i. \( \chi(X^G) = \chi(X) \), if \( G = S^1 \).
ii. \( \chi(X^G) = \Lambda(g) \), if \( G = \mathbb{Z}/p \) and \( g \in G, g \neq 1 \).

Here, \( \Lambda(g) \) denotes the Lefschetz number
\[
\Lambda(g) = \sum_i (-1)^i \text{trace}(g_*: H_i(X; \mathbb{Q}) \rightarrow H_i(X; \mathbb{Q})),
\]
regarding \( g \) as a map \( X \rightarrow X \).

If we set \( A^{0,1} = 0 \) and \( A^{*,0} = H^*(X; \mathbb{Q}) \), respectively \( A^{*,0} = H^*(X^G; \mathbb{Q}) \), then Lemma \( \text{[1]} \) and Lemma \( \text{[3]} \) give the following sequence of equations:
\[
\dim H^*(X^G; \mathbb{Q}) \equiv \chi(X^G) = \chi(X) \equiv \dim H^*(X; \mathbb{Q}) \mod 4.
\]
This shows the first part of Theorem \( \text{[1]} \).

The proof of Theorem \( \text{[3]} \) proceeds by induction on the order of \( G \). Assume \( |G| \neq 1 \) and choose a normal subgroup \( H \subset G, H \cong \mathbb{Z}/p \) that exists by group theory. Each component of \( X^H \) is an \( \mathbb{F}_p \)-PD space of even formal dimension by the Theorem of Bredon-Chang-Skjelbred. As \( \dim H^*(X; \mathbb{Q}) < p \) by our assumption on \( p \), the induced action of \( H \) on \( H^*(X; \mathbb{Q}) \) is trivial: Let \( V \) be a rational vector space of dimension smaller than \( p - 1 \) and let \( f \) be a linear endomorphism of \( V \) with \( f^p = \text{id} \). Then the minimal polynomial of \( f \) in \( \mathbb{Q}[x] \) must divide \( x^p - 1 \). The last polynomial splits over \( \mathbb{Q}[x] \) into \( (x - 1) \) and an irreducible factor of degree \( p - 1 \). Because the minimal polynomial of \( f \) has degree at most \( \dim V \), it must therefore be equal to \( x - 1 \). This covers all cases, where \( H^*(X; \mathbb{Q}) \) is not concentrated in degree 0. In the remaining case, the assertion follows from the fact that \( H^0(X; \mathbb{Q}) \) is a permutation module.
Altogether, for \( 1 \neq h \in H \), we obtain
\[
\Lambda(h) = \chi(X).
\]
By Lemma 1 and Lemma 2 above, we get
\[
\dim H^\ast(X^H; \mathbb{F}_p) \equiv \chi(X) \equiv \dim H^\ast(X; \mathbb{F}_p) \mod 4.
\]
(Note that the Euler characteristic does not depend on the coefficient field used). For the induction step, observe that by Smith theory
\[
\dim H^\ast(F; \mathbb{F}_p) \leq \dim H^\ast(X; \mathbb{F}_p) < p
\]
for each component \( F \) of \( X^G \). Furthermore, the group \( G' = G/H \) has order less than the order of \( G \) and each component of \( X^H \) is invariant under the induced \( G' \)-action by our assumption on \( p \). Using the Theorem of Bredon-Chang-Skjelbred, each component of \( X^H \) is again an \( \mathbb{F}_p \)-PD space of even formal dimension. Hence the induction hypothesis applies to each component of \( X^H \). The proof of Theorem 3 is now complete.

For the proof of Theorem 4, we recall the following fact.

**Proposition 3.** (\cite{7}, Proposition 13) Let \( G = \mathbb{Z}/p \) act simplicially on a finite simplicial complex \( X \) that is an orientable \( \mathbb{Z}(p) \)-homology manifold. If \( F \subset X \) is a component of the fixed point set \( X^G \), then \( F \) is an orientable \( \mathbb{Z}(p) \)-homology manifold of even codimension in \( X \).

Using this fact, Theorem 4 follows from Theorem 3 by observing that the total Betti numbers with either \( \mathbb{F}_p \) or \( \mathbb{Q} \) coefficients of a space are congruent modulo 4, if this space fulfills Poincaré duality both over \( \mathbb{Q} \) and over \( \mathbb{F}_p \) and has even formal dimension for both fields of coefficients. This follows by the independence of the Euler characteristic of the coefficient field and by using Lemma 1 twice.

For the proof of the second part of Theorem 1 we consider the cohomological Leray-Serre spectral sequence \((E_r, \delta_r)\) for the Borel fibration
\[
X \to X_G = EG \times_G X \to BG
\]
with coefficients in \( \mathbb{Q} \). For the \( E_2 \)-term of this spectral sequence as a module over \( H^\ast(BG; \mathbb{Q}) \cong \mathbb{Q}[t] \), where \( t \in H^2(BG; \mathbb{Q}) \) is a generator, we get
\[
E_2^{s,\mu} \cong H^\mu(X; \mathbb{Q}) \otimes \mathbb{Q}[t].
\]
Because all the differentials in the spectral sequence are \( \mathbb{Q}[t] \)-linear, we can evaluate the terms \( E_r, r \geq 2 \) at \( t = 1 \) and because evaluation at \( t = 1 \) commutes with taking homology with respect to each \( \delta_r \) (see \cite{1}, Lemma (A.7.2)), we get a converging spectral sequence \((\overline{E}_r, \overline{\delta}_r)\), where
\[
\overline{E}_r = (E_r)_{t=1}
\]
and \( \overline{\delta}_r \) is induced by \( \delta_r \). This spectral sequence is concentrated in the first column and \( \overline{E}_\infty^{s,\ast} \) is (noncanonically and not preserving the grading) isomorphic to \( H^\ast(X_G; \mathbb{Q})_{t=1} \) as a \( \mathbb{Q} \)-vector space. By the evaluation theorem
(cf. [P], Theorem (3.5.1)), we have a canonical isomorphism (of ungraded modules)
\[ H(X_G; \mathbb{Q})_{t=1} \cong H(X^G; \mathbb{Q}) . \]
Because \( X \) fulfills Poincaré duality over \( \mathbb{Q} \), the term \( E_2^{0,*} = E_2^{0,0} \) is an \( \mathbb{N} \)-graded \( \mathbb{Q} \)-Poincaré duality algebra of formal dimension \( 2m + 1 \), where we use the grading induced by \( E_2 \). As \( X^G \neq \emptyset \), we have \( E_r \neq 0 \) for all \( r \). Now the claim follows from Propositions [1] and [2] in the second section of this paper.

4. Proof of Theorem 2

Let \( p \) be an odd prime number, let \( G = \mathbb{Z}/p \) and let \( X \) be a finite dimensional connected \( G \)-CW complex that is a \( \mathbb{F}_p \)-PD space of formal dimension \( n \). Furthermore, we assume that \( \beta(H^*(X; \mathbb{F}_p)) = 0 \), where \( \beta : H^*(X; \mathbb{F}_p) \to H^{*+1}(X; \mathbb{F}_p) \) is the Bockstein operator associated to the exact sequence of coefficients
\[
0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0 .
\]
This condition is equivalent to the requirement, that each cohomology class in \( H^*(X; \mathbb{F}_p) \) can be lifted to a cohomology class in \( H^*(X; \mathbb{Z}/p^2) \), which is the case, if and only if \( H^*(X; \mathbb{Z}/p) \) does not contain a direct summand of the form \( \mathbb{Z}/p \). Note that the \( G \)-action on \( X \) induces a \( \mathbb{Z}/p^2 \)-module structure on \( H^*(X; \mathbb{Z}/p^2) \). By our assumption on \( \beta \), \( H^*(X; \mathbb{Z}/p^2) \) is a free \( \mathbb{Z}/p^2 \)-module, so the following proposition is an immediate consequence of [P], Proposition 6, which is shown using a little representation theory over \( \mathbb{Z}/p^2 \).

**Proposition 4.** For each \( i \) there is a \( \mathbb{F}_p[G] \)-linear decomposition
\[ H^i(X; \mathbb{F}_p) = H^i(X; \mathbb{Z}/p^2) \otimes \mathbb{F}_p \cong T^i \oplus F^i \oplus R^i , \]
where \( T^i \) is a trivial \( \mathbb{F}_p[G] \)-module, \( F^i \) is a direct sum of free \( \mathbb{F}_p[G] \)-modules and \( R^i \) is a direct sum of modules of the form \( \ker \epsilon \), where \( \epsilon : \mathbb{F}_p[G] \to \mathbb{F}_p \) is the augmentation map.

Let \( (E_r, \delta_r) \) denote the cohomological Leray-Serre spectral sequence for the Borel fibration
\[ X = X_G = EG \times_G X \to BG \]
with coefficients in \( \mathbb{F}_p \). For \( r \geq 2 \), each \( E_r \) is a differential bigraded algebra over \( H^*(BG; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda(s) \), where \( t \in H^2(BG; \mathbb{F}_p) \) and \( s \in H^1(BG; \mathbb{F}_p) \) are generators and \( \beta(s) = t \). By the evaluation theorem (cf. [P], Theorem (1.4.5)), the inclusion \( X^G \hookrightarrow X \) induces an isomorphism of (ungraded) \( \Lambda(s) \)-algebras
\[ H(X_G; \mathbb{F}_p)_{t=1} \cong H(X^G; \mathbb{F}_p) \otimes \Lambda(s) , \]
hence, after evaluating at \( s = 0 \), we get an induced isomorphism
\[
H(X_G; \mathbb{F}_p)_{t=1, s=0} \cong H(X^G; \mathbb{F}_p).
\]

Now we encounter the following difficulty that did not arise in the consideration of \( S^1 \)-actions above: In order to calculate the dimension of the left hand side of the second isomorphism one has to get around the difficulty that evaluation at \( s = 0 \) does not commute with taking homology in general, and so a spectral sequence argument as in Section 3 does not seem to be feasible in this case. However, under the additional assumption \( \beta = 0 \), we can get around this difficulty by using the following fact (cf. [6], Proposition 9, and the last part of the proof of Proposition 10).

**Proposition 5.** For all \( r \geq 2 \), the localized terms \( E_r[t^{-1}] \) are finitely generated free \((\mathbb{Z} \times \mathbb{N})\)-bigraded differential \( \Lambda(s) \otimes \mathbb{F}_p[t, t^{-1}] \)-algebras. Furthermore, evaluation at \( t = 1 \) and \( s = 0 \) on \( E_r[t^{-1}] \) commutes with taking homology with respect to the differentials induced by \( \delta_r \).

Now, we set
\[
\overline{E}_r = (E_r)_{t=1, s=0}
\]
and use the induced differentials \( \overline{\delta}_r \). Each term \( \overline{E}_r \) has an induced \((\mathbb{Z}/2 \times \mathbb{N})\)-grading and for \( r = 2 \) we obtain
\[
\overline{E}_2^\gamma,\mu = H^\gamma(G; H^\mu(X; \mathbb{F}_p))_{t=1, s=0} \cong \begin{cases} 
\mathbb{F}_p^{\dim T^\mu}, & \text{if } \gamma \text{ is even,} \\
\mathbb{F}_p^{\dim R^\mu}, & \text{if } \gamma \text{ is odd.}
\end{cases}
\]
Here, we used Proposition 4 and the fact (which follows from usual dimension shifting) that \( H^i(\mathbb{F}_p; \ker e)[t^{-1}] \) is isomorphic to \( \mathbb{F}_p \) in odd degrees and is equal to zero in even degrees. Further, we get from Proposition 5 and the evaluation theorem
\[
\dim_{\mathbb{F}_p} \overline{E}_\infty = \frac{1}{2} \dim_{\mathbb{F}_p}(E_\infty)_{t=1} = \dim_{\mathbb{F}_p} H^*(X^G; \mathbb{F}_p).
\]

Now, the proof of Theorem 3 is completed by using induction in the spectral sequence. Notice that \( \overline{E}_2^2 \) is a \((\mathbb{Z}/2 \times \mathbb{N})\)-bigraded connected Poincaré duality algebra over \( \mathbb{F}_p \) of formal dimension \( n \) in the sense of Definition 1 (cf. [3], proof of Proposition 9). For the induction step, if \( n \) is even, we use Lemma 1 and the fact that the Euler characteristic of a \( \mathbb{Z}/2 \)-graded differential complex does not change after taking homology with respect to a differential of odd degree. If \( n \) is odd, we use Proposition 1 and Proposition 2.

5. Examples, applications and concluding remarks

The following examples illustrate the significance of the conditions on the Betti numbers in the second part of Theorem 1 and 2. The example of free \( S^1 \) or \( \mathbb{Z}/p \)-actions on spheres of odd dimension shows that the requirement \( X^G \neq \emptyset \) in the second part of Theorem 1 and in the second part of Theorem 2 is necessary. In [3], p. 425, an example of an \( S^1 \)-action on \( X = S^3 \times S^5 \times S^9 \) is
constructed whose fixed point set is an $S^7$-bundle over $S^3 \times S^5$ and has total Betti number 6 (with coefficients in $\mathbb{Q}$). As the total Betti number of $X$ is 8, one sees that even if fixed points exist, the assumption on the vanishing of certain Betti numbers in the second part of Theorem 1 is needed. Restricting this $S^1$-action to $\mathbb{Z}/p \subset S^1$, we similarly may conclude that all additional assumptions in the second part of Theorem 2 are necessary.

Now, we will construct examples of Poincaré duality spaces of odd formal dimension that fulfill the additional requirement on Betti numbers in the second part of Theorem 1 and 2. Let $X$ be an arbitrary connected finite simplicial complex with the property that its even dimensional integral cohomology is concentrated in degree 0. Now embed $X$ in a Euclidean space $\mathbb{R}^{2m+1}$, where $2m + 1 \geq 2 \dim X + 1$, and take a regular neighbourhood $R$ of $X$ inside $\mathbb{R}^{2m+1}$ which can be assumed to be a compact oriented smooth manifold with boundary. Gluing two copies of this manifold (the orientation of one of which had been reversed) along their boundaries yields a $(2m+1)$-dimensional connected oriented smooth manifold $Y$, whose even dimensional integral cohomology below dimension $m+1$ is concentrated in degree 0. This follows from the Mayer-Vietoris sequence and the fact that by general position, the inclusion $\partial R \hookrightarrow R$ induces isomorphisms of homotopy groups up to degree $2m + 1 - \dim X - 2 \geq m - 1$ and a surjection in degree $2m + 1 - \dim X - 1 \geq m$. In particular, the connecting homomorphism $H^i(\partial R; \mathbb{Z}) \to H^{i+1}(Y; \mathbb{Z})$ in the Mayer-Vietoris sequence is 0, if $i \leq m - 1$. Hence, for actions on $Y$, the second part of Theorem 1 and 2 can be applied (assuming that the induced action on $H^*(Y; \mathbb{F}_p)$ is trivial in the case of $G = \mathbb{Z}/p$).

Another example illustrating the second part of Theorem 1 and Theorem 2 can be constructed as follows. Consider the space $X = S^1 \times S^{2m}$ equipped with the $S^1$-action that acts trivially on the first factor and is the usual rotation action (fixing north and south pole) on the second factor. The fixed point set of this $S^1$-action is the union of two circles. Next we choose one point in each fixed point component and remove small $S^1$-invariant neighborhoods equivariantly diffeomorphic to $D^{2m+1}$ (with a linear $S^1$-action) around each of these two fixed points which gives an $S^1$-manifold $Z$ with two boundary components each of which is equivariantly diffeomorphic to $S^{2m}$ with the rotation action by $S^1$. Now we form the equivariant connected sum of $Z$ and $[0,1] \times S^{2m}$, where on the last space $S^1$ acts trivially on the first factor and by a rotation action on the second. In this way, we obtain a $(2m+1)$-dimensional oriented $S^1$-manifold which fulfills the requirement of the second part of Theorem 1 and 2 (using the induced action by $\mathbb{Z}/p \subset S^1$). It is easy to check that the integral cohomology of this space has rank 1 in degrees 0 and $2m + 1$, rank 2 in degrees 1 and $2m$ and is 0 in all other degrees. The total Betti number of the fixed set (which is just a single copy of $S^1$) is two. So the Leray-Serre spectral sequence for the Borel construction does not collapse at the $E_2$-level in this case.
In this paper we have been working in the category of $G$-CW complexes. But using Čech cohomology and the usual somewhat more technical machinery (see, e.g. [1]), one can extend all the results to general $G$-spaces which fulfill the hypothesis (LT) for the localization theorem (see [1], p. 208). As this generalization is straightforward, we leave it to the interested reader.

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C. ALLDAY, UNIVERSITY OF HAWAII, DEPARTMENT OF MATHEMATICS, 2565 MCCARTHY MALL, HONOLULU, HI  E-mail address: chris@math.hawaii.edu

B. HANKE, UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY  E-mail address: hanke@rz.mathematik.uni-muenchen.de

V. PUPPE, UNIVERSITÄT KONSTANZ, 78457 KONSTANZ, GERMANY  E-mail address: Volker.Puppe@uni-konstanz.de