The embedding of the Teichmüller space in geodesic currents

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Abstract

We prove that the embedding of the Teichmüller space in the space of geodesic currents is totally linearly independent.

1 Introduction

Let $S$ be a closed connected oriented surface of genus $g \geq 2$. By the uniformization theorem, the Teichmüller space of $S$, $\text{Teich}(S)$, identifies with the moduli space of marked hyperbolic structures. That is $\text{Teich}(S) = \{(X, f_X)\}/\sim$ where $X$ is an hyperbolic surface and $f_X : S \to X$ is an orientation preserving homeomorphism. The equivalence relation is given by $(X, f_X) \sim (Y, f_Y)$ if and only if there exists an isometry $g : X \to Y$ such that $g \circ f_X$ is isotopic to $f_Y$.

The universal cover $\tilde{X}$ of a hyperbolic surface identifies with the hyperbolic plane $\mathbb{H}^2$, and the marking induces a representation of $(f_X)_* : \pi_1(S) \to \pi_1(X) < \text{Isom}^+(\mathbb{H}^2) \simeq \text{PSL}_2(\mathbb{R})$, well defined up to a conjugacy by an isometry of $\mathbb{H}^2$. This give an embedding of the Teichmüller space into the space of representations: $\text{Rep} := \text{Hom}(\pi_1(S) \to \text{PSL}_2(\mathbb{R}))/\text{PSL}_2(\mathbb{R})$. The Teichmüller space actually is one of the two connected components of $\text{Rep}$ which consists of only discrete and faithful representation. The other such component is $\text{Teich}(\bar{S})$ the Teichmüller space of the surface with the opposite orientation.

Our work concerns another embedding of the Teichmüller space in a very large vector space, called the space of geodesic currents and denoted by $\mathcal{C}(S)$. This embedding, say $L : \text{Teich}(S) \to \mathcal{C}(S)$, has been introduced by F. Bonahon in [Bon88], and this article aims to understand the linear properties of this embedding. We prove that this embedding is as linearly independent as possible, namely:

**Theorem 1.1.** $\{L(S) | S \in \text{Teich}(S)\} \subset \mathcal{C}(S)$ is a linearly independent set of vectors.

One gets the following corollary:

**Corollary 1.2.** Quasi-Fuchsian AdS$_3$-manifolds are determined by their length spectrum.
In the next section, we present the space of geodesic currents, Bonahon’s embedding of the Teichmüller space and the intersection function.

Then the proof of Theorem 1.1 decomposes into two parts. In Section 3, we show that $n$ different surfaces induce via the direct sum representations of the holonomies, a Zariski dense subgroup of $\text{PSL}_2(\mathbb{R})^n$ and in Section 4 we use a famous theorem of Benoist to show the total linear independence of $L$.

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2 Geodesic currents

Recall that $S$ is a closed connected oriented surface of genus $g \geq 2$. We denote by $\Gamma := \pi_1(S)$ its fundamental group and $\tilde{S}$ the universal cover of $S$. The group $\Gamma$ is a hyperbolic group and we denote by $\partial \Gamma$ its Gromov boundary. If $S$ is endowed with a negatively curved metric, then one can identify the boundary of $\tilde{S}$ with $S^1$, homeomorphic to the boundary of $\Gamma$. In this case, the geodesics of $\tilde{S}$ identify, via their endpoints to $\partial(2)S^1 := (\partial S^1 \times \partial S^1 \setminus \Delta) / \mathbb{Z}/2\mathbb{Z}$, where $\Delta$ is the diagonal of the product and the action of $\mathbb{Z}/2\mathbb{Z}$ is given by the exchange of the two factors.

Definition 2.1. The space of geodesic currents is the set of $\Gamma$ invariant measures on $\partial(2)\Gamma \simeq \partial(2)S^1$. It will be denoted by $\mathcal{C}(S)$.

This does not depend on the identification of $\partial \Gamma$ with $\partial S^1$.

For the purpose of this article, we will not need a precise description of geodesic currents and will instead focus on two major examples.

For the first example it will be more convenient to endow $S$ with a negatively curved metric. Then for any closed curve $c$ on $S$, there is a unique geodesic representative. It lifts to a $\Gamma$ invariant subset of geodesic on $\tilde{S}$, that we see as a subset of $\partial(2)S^1$. The Dirac measure on each of this lift give a $\Gamma$ invariant measure, that is a geodesic current. In the same manner, we can also see a (positive) linear combination of different closed curve as a geodesic current. We will follow the usual slight abuse of notations, and will not make the difference between the curve $c$ and its associated current.

Theorem 2.2. [Bon88] The set of linear combinations of closed curved is dense in $\mathcal{C}(S)$.

The second major example is the Liouville current. The Liouville measure $L$ is a $\text{PSL}_2(\mathbb{R})$ invariant measure on $\partial(2)S^1$, absolutely continuous to the Lebesgue measure $d\alpha d\beta$. 

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given by
\[ L = \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}. \]

Now let \((S, m)\) be the surface \(S\) endowed with a hyperbolic metric \(m\). This gives an isometric diffeomorphism \(\phi_m : \tilde{S} \to \mathbb{H}^2\), well defined up to composition by an isometry of \(\mathbb{H}^2\). This defines a boundary homeomorphism \(\partial \phi_m : \partial \tilde{S} \to \partial \mathbb{H}^2\), well defined up composition by an element of \(\text{PSL}_2(\mathbb{R})\). Since the Liouville measure on \(\mathbb{H}^2\) is invariant by \(\text{PSL}_2(\mathbb{R})\) we can pull back \(L\) on \(\partial(\tilde{S}) \simeq \partial(S^1)\). We denote it by \(L_m\). We have the following:

**Theorem 2.3.** [Bon88] The map \(L : \text{Teich}(S) \to \mathcal{C}(S)\) that sends \((S, m)\) to \(L_m\) is a proper, continuous, embedding.

Bonahon also introduced a bilinear form on \(\mathcal{C}(S)\), the so called intersection function, denoted by \(i(\cdot, \cdot) : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}\). It generalizes the notion of geometric intersection between two closed curves. Although the definition is a bit technical, the only property we will need will be:

**Theorem 2.4.** Let \(L_m\) be a Liouville current associated to \((S, m)\), let \(c\) be a closed curve on \(S\). Then
\[ i(L_m, c) = \ell_m(c), \]
where \(\ell_m(c)\) is the length on \((S, m)\) of the unique geodesic representative of \(c\).

### 3 Zariski closure of diagonal representation

We will need the following result, known as Goursat’s Lemma:

**Theorem 3.1** (Goursat’s Lemma). Let \(G, G'\) be two groups. Let \(p_1 : H \to G, p_2 : H \to G'\) be two surjective homomorphisms. Identify \(N'\) the kernel of \(p_1\) with a normal subgroup of \(G'\) and \(N\), the kernel of \(p_2\) with a normal subgroup of \(G\). Then the image of \(H\) in \(G/N \times G'/N'\) is the graph of an isomorphism \(G/N \simeq G'/N'\).

Recall that to a hyperbolic surface \((S, m)\), the holonomy map gives a discrete and faithful representation \(\rho_m : \Gamma \to \text{PSL}_2(\mathbb{R})\), well defined up to conjugacy by \(\text{PSL}_2(\mathbb{R})\). The outer automorphism \(\tau\) of \(\text{PSL}_2(\mathbb{R})\) is given by the conjugation with \(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) and gives the corresponding hyperbolic structure on the surface with the opposite orientation.

Finally recall that any discrete and faithful representation of a compact surface fundamental group into \(\text{PSL}_2(\mathbb{R})\) is Zariski dense.

**Theorem 3.2.** Let \(S\) be a closed connected oriented surface of genus \(g \geq 2\). Let \(X_1, \ldots, X_n\) be \(n\) hyperbolic surfaces and let \(\rho_i\) be the corresponding holonomies. Let \(\rho = (\rho_1, \ldots, \rho_n) : \Gamma \to \text{PSL}_2(\mathbb{R})^n\) be the direct sum representation. Then \(\rho(\Gamma)\) is Zariski dense if and only if all surfaces are different in \(\text{Teich}(S)\).
Proof. If two surfaces, say $X_1, X_2$, correspond to the same point in the Teichmüller space, then there exists $h \in \text{PSL}_2(\mathbb{R})$ such that their holonomies satisfy $\rho_1 = h \rho_2 h^{-1}$. Therefore the Zariski closure of $\rho$ is contained in $H \times \text{PSL}_2(\mathbb{R})^{n-2}$, where $H$ is the diagonal copy of $\text{PSL}_2(\mathbb{R})$ given by $H = \{(g, ghg^{-1}) \mid g \in \text{PSL}_2(\mathbb{R})\}$.

We will prove the converse by induction.

For $n = 1$ the result is trivial. Suppose that the result is true for $n$ and consider $(n+1)$ non isometric hyperbolic structures on $S : X_1, \ldots, X_{n+1}$. Let $\rho = (\rho_1, \ldots, \rho_{n+1})$ be the diagonal representation corresponding to their holonomies. Let $H$ be the Zariski closure of $\rho(\Gamma)$. Consider the projection on the last factor $\pi_{n+1} : H \to \{\text{Id}\}^n \times \text{PSL}_2(\mathbb{R})$ and the one on the first $n$ factors: $p : H \to \text{PSL}_2(\mathbb{R})^n \times \{\text{Id}\}$.

By induction the projection $p$ is surjective and by hypothesis $\pi_{n+1}$ also is. Since $\text{PSL}_2(\mathbb{R})$ is simple, the kernel of $p$ can either be $\text{PSL}_2(\mathbb{R})$ or $\{\text{Id}\}$. In the first case, by Goursat’s lemma, the kernel of $\pi_{n+1}$, $N$, has to verify $\text{PSL}_2(\mathbb{R})^n/N \simeq \text{PSL}_2(\mathbb{R})/\text{PSL}_2(\mathbb{R})$ and therefore $H = \text{PSL}_2(\mathbb{R})^{n+1}$.

In the second case, the kernel of $\pi_{n+1}$, $N$, has to verify $\text{PSL}_2(\mathbb{R})^n/N \simeq \text{PSL}_2(\mathbb{R})$. Since $\text{PSL}_2(\mathbb{R})$ is simple, we claim that the only possibility for $N = \text{PSL}_2(\mathbb{R})^k \times \{\text{Id}\} \times \text{PSL}_2(\mathbb{R})^{n-k-1}$ for some $k \in \{0, \ldots, n-1\}$, see Lemma 3.3. Then by Goursat’s Lemma, the projection of $H$ in $\text{PSL}_2(\mathbb{R}) \times \left(\text{PSL}_2(\mathbb{R})^n/N\right)$ identifies with the graph of an isomorphism from $\text{PSL}_2(\mathbb{R})$ to $\left(\{\text{Id}\}^k \times \text{PSL}_2(\mathbb{R}) \times \{\text{Id}\}^{n-k-1}\right) \simeq \text{PSL}_2(\mathbb{R})$, and finally implies that $\theta(\rho_1) = \rho_k$ for an automorphism of $\text{PSL}_2(\mathbb{R})$.

Since the surface is oriented, $\theta$ has to be a inner automorphism, and therefore $\rho_1$ is conjugated to $\rho_k$. This is a contradiction.

We prove the claim from the second part of the proof

Lemma 3.3. Let $N$ be a normal subgroup of $\text{PSL}_2(\mathbb{R})^n$, such that $\text{PSL}_2(\mathbb{R})^n/N \simeq \text{PSL}_2(\mathbb{R})$. Then there exists $k \in \{0, \ldots, n-1\}$ such that $N = \text{PSL}_2(\mathbb{R})_k \times \{\text{Id}\} \times \text{PSL}_2(\mathbb{R})^{n-k-1}$.

Proof. For the sake of clarity we will denote $G_i$ the $i$-th $\text{PSL}_2(\mathbb{R})$ factor in $\text{PSL}_2(\mathbb{R})^n$ and $G$ will designated any abstract $\text{PSL}_2(\mathbb{R})$. Consider $p_i : G^n \to G_i$. Then $p_i(N)$ is a normal subgroup of $G_i$. By simplicity $p_i(N)$ is either trivial or the whole group $G_i$. Suppose that there exists $i \neq j$ such that $p_i(N) = p_j(N) = \{\text{Id}\}$, and by symmetry with suppose that $(i, j) = (1, 2)$ then $N \subset \{\text{Id}\} \times \{\text{Id}\} \times G^{n-2}$ and therefore $G^n/N \supset G^2$ which is absurd. Consequently there is at most one projection such that the image of $N$ is trivial.

Consider two projections $p_i, p_j$ with non trivial image. Denote by $N_{ij} := N \cap (G_i \times G_j)$. Remark that $N_{ij}$ is normal in $G_i \times G_j$. Then applying Goursat lemma, one see that either $N_{ij} = G_i \times G_j$ or $N_{ij}$ is the graph of a homomorphism between $G_i$ and $G_j$. The latter cannot happen since it would not be a normal subgroup. Therefore $N_{ij} = G_i \times G_j$.

Applying this argument for all pairs with non trivial image, we see that $N = \Pi_k G_k$ where the product if taken over all $k$ such that $p_k(G) \neq \text{Id}$. Since $G^n/N \simeq G$, there is at least one factor such that $p_k(G) \neq \text{Id}$, and finally exactly one by the previous argument. 

\[\square\]
Remark: This lemma generalizes to any non-abelian product of simple groups.

4 Total independance of Liouville’s current

We are going to give Benoist’s theorem [Ben97] in the context of $\text{PSL}_2(\mathbb{R})^n$. Any hyperbolic element $g \in \text{PSL}_2(\mathbb{R})$ is conjugated to an element of the form \[
\begin{pmatrix}
es^{\lambda(g)/2} & 0 \\
0 & e^{-\lambda(g)/2}
\end{pmatrix}.
\] The number $\lambda(g) \in \mathbb{R}^+$ is the translation length of $g$.

Let $g = (g_1, ..., g_n) \in \text{PSL}_2(\mathbb{R})^n$ be a loxodromic element, that is all $g_i$ are hyperbolic elements. The vector $\lambda(g) := (\lambda(g_1), ..., \lambda(g_n)) \in (\mathbb{R}^+)^n$ is called the Jordan projection of $g$.

Definition 4.1. Let $H$ be a subgroup of $\text{PSL}_2(\mathbb{R})^n$. The limit cone of $H$ is defined by
\[
C(H) := \bigcup_{h \in H_{\text{lox}}} \lambda(h) \mathbb{R}^+ \subset (\mathbb{R}^+)^n.
\]

One of the striking properties of the limit cone is given by the following theorem of Y. Benoist:

Theorem 4.2. [Ben97] If $H$ is Zariski dense in $\text{PSL}_2(\mathbb{R})^n$, then its limit cone has non-empty interior.

We are now going to show the main result of this paper.

Theorem 4.3. \{ $L_m \mid (S, m) \in \text{Teich}(S) \}$ is a linearly independent family of geodesic currents.

Proof. Let $(S_1, ..., S_n)$ a finite family of distinct hyperbolic surfaces, and denote by $L_k$ the corresponding Liouville currents. Suppose that there exists $(a_1, ..., a_n) \in \mathbb{R}^n$ a family of real number such that
\[
\sum_{k=1}^n a_k L_k = 0.
\]
We have therefore, for all closed curve $c \in \mathcal{C}$:
\[
\sum_{k=1}^n a_k i(L_k, c) = \sum_{k=1}^n a_k \ell_{S_k}(c) = 0. \tag{1}
\]

Consider the holonomy representations $\rho_k : \pi_1(S) \to \text{PSL}_2(\mathbb{R})$ and let $\rho = (\rho_1, ..., \rho_n)$ be the diagonal representation of $\Gamma \to \text{PSL}_2(\mathbb{R})^n$.

By Theorem 3.2 since $(S_k)$ are $n$ distinct hyperbolic surfaces $\rho(\Gamma)$ is Zariski dense. Therefore by Benoist’s Theorem 4.2 the limit cone of $\rho(\Gamma)$, $C(\rho(\Gamma)) \subset \mathfrak{a}^+$ is of non-empty interior.

However, by Equation (1), it is also contained in the kernel of the linear form $\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(x_1, ..., x_n) = \sum_k a_k x_k$. A proper vector subspace of $\mathbb{R}^n$ has empty interior. Therefore, the linear form $\ell$ is the zero map, and $(a_1, ..., a_n) = (0, ..., 0)$. \qed
4.1 Length spectrum rigidity for AdS$^3$-quasi-Fuchsian manifolds

An AdS$^3$-quasi-Fuchsian manifold is a manifold locally isometric to the anti-de Sitter space of dimension 3, AdS$^3$, admitting a compact Cauchy surface $S$. For more details, we refer to [Mes90, BM12, Glo17]. We denote by $QF^{AdS}(S)$ the space of AdS$^3$-quasi-Fuchsian manifolds whose Cauchy surface is homeomorphic to $S$. These manifolds are homeomorphic to $S \times (0, 1)$ and for any curve on $S$ there is a unique geodesic representative inside the Lorentzian manifold.

Mess proved in [Mes90] that one can parametrize $QF^{AdS}(S)$ by two points in the Teichmüller space. We call this the Mess parametrization of $QF^{AdS}(S)$. In [Glo17], we proved that if a AdS$^3$-quasi-Fuchsian manifold $M$ is given by Mess parametrization $((S, m_1), (S, m_2))$, then for any closed curve $c$ on $S$, the unique closed geodesic in $M$ corresponding to $c$ has length $\ell_M(c) = \frac{\ell_1(c) + \ell_2(c)}{2}$, where $\ell_i(c)$ is the hyperbolic length of the curve $c$ on $(S, m_i)$, $i \in \{1, 2\}$.

We can now give the proof of the corollary:

**Corollary 4.4.** AdS$^3$-Quasi-Fuchsian manifolds are determined by their marked length spectrum.

**Proof.** Let $M, N$ be two AdS$^3$ quasi-Fuchsian manifolds, parametrized through Mess coordinates by $M = ((S, m_1), (S, m_2))$ and $N = ((S, m_3), (S, m_4))$.

Suppose that the two marked spectrum are equal. Then for all closed curve $c$ on $S$ we have

$$2\ell_M(c) = \ell_1(c) + \ell_2(c) = \ell_3(c) + \ell_4(c) = 2\ell_N(c).$$

Let $L_i$ be the Liouville currents associated to $(S, m_i)$ for $i \in \{1, ..., 4\}$. The previous relation gives $i(L_1 + L_2, c) = i(L_2 + L_3, c)$ for all closed curve $c$. By density of current associated to closed curves, Theorem 2.2 one has $i(L_1 + L_2, \eta) = i(L_2 + L_3, \eta)$ for all $\eta \in C(S)$. Otal showed in [Ota90], that geodesic currents are determined by their intersection function, therefore,

$$L_1 + L_2 = L_3 + L_4.$$

Finally Theorem 1.1 implies that $(L_1, L_2) = (L_3, L_4)$ and therefore $M = N$. □

**References**

[Ben97] Yves Benoist. Propriétés asymptotiques des groupes linéaires. *Geometric and functional analysis*, 7(1):1–47, 1997.

[BM12] Thierry Barbot and Quentin Mérigot. Anosov ads representations are quasi-fuchsian. *Groups, Geometry, and Dynamics*, 6(3):441–484, 2012.

[Bon88] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Inventiones mathematicae*, 92(1):139–162, 1988.
[Glo17] Olivier Glorieux. Counting closed geodesics in globally hyperbolic maximal compact AdS 3-manifolds. *Geometriae Dedicata*, 188(1):63–101, 2017.

[Mes90] G Mess. Lorentz spacetimes of constant curvature,” institut des hautes etudes scientifiques preprint ihes. Technical report, M/90/28, 1990.

[Ota90] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Annals of Mathematics*, 131(1):151–162, 1990.