POLYNOMIALS IN ONE VARIABLE AND RANKS OF CERTAIN TANGENT MAPS

YURI G. ZARHIN

Abstract. We study a map that sends a monic degree \( n \) complex polynomial \( f(x) \) without multiple roots to the collection of \( n \) values of its derivative at the roots of \( f(x) \). We give an answer to a question posed by Yu.S. Ilyashenko.

MSC: 14A25; 37F10

1. Definitions, Notation, Statements

We write \( \mathbb{C} \) for the field of complex numbers. Our aim is to compute the rank of the following map.

Let us consider the \( n \)-dimensional complex manifold \( P_n \subset \mathbb{C}^n \) of all monic complex polynomials of degree \( n \geq 2 \)

\[
f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i
\]

with coefficients \( a = (a_0, \ldots, a_{n-1}) \) and without multiple roots. We denote the roots of \( f(x) \) by

\[
\alpha = \{\alpha_1, \ldots, \alpha_n\},
\]

somehow ordering them; locally (with respect to \( a \)), one may choose each \( \alpha_i \) (using Implicit Function Theorem) as a smooth (univalued) function in \( a \). Further, we will try to differentiate these functions with respect to coordinates, without computation of the roots. And here is our map

\[
M : a = (a_0, \ldots, a_{n-1}) \mapsto f'(\alpha) = (f'(\alpha_1), \ldots, f'(\alpha_n)) \in \mathbb{C}^n.
\]

Abusing notation, we may assume that \( M \) is defined locally on \( P_n \) and write \( M(f) \) instead of \( M(a_0, \ldots, a_{n-1}) \). Let

\[
dM : \mathbb{C}^n \to \mathbb{C}^n
\]

be the corresponding tangent map (at the point \( f(x) \)). It is convenient to identify the tangent space \( \mathbb{C}^n \) with the space of all polynomials \( p(x) \) of degree \( \leq n - 1 \). Namely, to a polynomial \( p(x) = \sum_{i=0}^{n-1} c_i x^i \) one assigns the tangent vector \( (c_0, \ldots, c_{n-1}) \in \mathbb{C}^n \). So, to the derivative \( f'(x) \) corresponds the tangent vector \( (a_1, \ldots, (n-1)a_{n-1}, n) \in \mathbb{C}^n \). We denote by \( W = W(f) \) the complex vector space of all polynomials \( p(x) \) of degree \( \leq n - 2 \) such that the polynomial

\[
R(x) := f''(x)p(x) - f'(x)p'(x)
\]

is divisible by \( f(x) \).

**Theorem 1.1.** The rank of the tangent map \( dM : \mathbb{C}^n \to \mathbb{C}^n \) is \( n - 1 \) at all points of \( P_n \). The kernel of \( dM \) always contains \( f'(x) \) and coincides with the one-dimensional subspace \( \mathbb{C} \cdot f'(x) \).
Corollary 1.3. The tangent map $dM$ is related to the fact that for each (small) complex number $\epsilon$ the map $M$ sends $f(x)$ and $f(x + \epsilon)$ to the same vector in $\mathbb{C}^n$.

Denote by $H_n$ the space of all monic complex polynomials $g(x)$ of degree $n \geq 2$ such that the map $G : \mathbb{C} \to \mathbb{C}$, $z \mapsto g(z)$ has exactly $n$ fixed points. Clearly, $g \in H_n$ if and only if $g(x) - x$ has no multiple roots, i.e.,

$$g(x) - x \in P_n.$$ 

It is also clear that the roots $\beta_1, \ldots, \beta_n$ of $f(x) = g(x) - x$ are exactly the fixed points of $G$ and the corresponding multiplier $g'(\beta_i)$ of $\beta_i$ coincides with $f(\beta_i) + 1$.

Let us consider the locally defined map

$$\text{Mult} : H_n \to \mathbb{C}^n, \ g(x) \mapsto (g'(\beta_1), \ldots, g'(\beta_n)) = M(g(x) - x) + (1, \ldots, 1),$$

which assigns to $G$ the collection of its multipliers.

Corollary 1.3. The tangent map $d\text{Mult} : \mathbb{C}^n \to \mathbb{C}^n$ to the multiplier map $\text{Mult}$ has rank $n - 1$ at all points of $H_n$.

Remark 1.4. The non-triviality of the kernel of $d\text{Mult}$ is related to the fact that for each (small) complex number $\epsilon$ the maps $z \mapsto g(z)$ and $z \mapsto g(z + \epsilon) - \epsilon$ are conjugate and have the same collection of multipliers.

In the course of the proof of Theorem 1.5 we use the following purely algebraic assertion that has a certain independent interest.

Theorem 1.5. Let $n \geq 2$ be an integer and $f(x)$ a complex degree $n$ polynomial. Suppose that there exists a nonzero complex polynomial $p(x)$ of degree $\leq n - 2$ such that $f''(x)p(x) - f'(x)p'(x)$ is divisible by $f(x)$. Then:

(i) $n \geq 4$, $\deg(p) \geq 2$.

(ii) There exists a quadratic polynomial $\bar{p}(x)$ such that $f(x)$ is divisible by $\bar{p}(x)^2$.

In particular, $f(x)$ has a multiple root.

(iii) If $\deg(p) = 2$ then $p(x)$ divides both $f(x)$ and $f'(x)$; in particular, all the roots of $p(x)$ are multiple roots of $f(x)$.

(iv) If $n = 4$ then $\deg(p) = 2$ and there exists a nonzero complex number $c$ such that $f(x) = c \cdot p(x)^2$.

2. Differentiation

The first question that naturally arises is how to deal with $M$? We interpret the ordering of the roots as a choice of an isomorphism of commutative semisimple $\mathbb{C}$-algebras

$$\psi : \Lambda = \mathbb{C}[x]/f(x)\mathbb{C}[x] \cong \mathbb{C}^n, \ u(x) + f(x) \cdot \mathbb{C}[x] \mapsto u(\alpha) := (u(\alpha_1), \ldots, u(\alpha_n))$$

and carry out all the computations, including the differentiation with respect to $\alpha$, of functions that take values in the algebra $\Lambda$, despite of the fact that this algebra does depend on the coefficients $\alpha$! (However, its isomorphism class does not depend on the coefficients.) Of course, while differentiating, we will use Leibnitz’s rule and that $f(x) = 0$ in $\Lambda$. In what follows we will often mean under polynomials their images in $\Lambda$ (i.e., the collection of their values at the roots of $f(x)$, while we try not refer to the roots explicitly). Notice that the absence of multiple roots means...
that \( f'(x) \) is an invertible element of \( \Lambda \). Notice also that \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \) is the image (under \( \psi \)) of the independent variable \( x \).

The first thing that we want to compute is the derivatives \( da_i/da_i \). Since \( f(\alpha) = 0 \), \( df(\alpha)/da_i = 0 \). We have

\[
df(\alpha)/da_i = \frac{\partial f}{\partial a_i}(\alpha) + f'(\alpha) \cdot da_i/da_i.
\]

Since \( \partial f/\partial a_i = x^i \), we obtain that

\[
0 = \alpha^i + f'(\alpha) \cdot da_i/da_i,
\]

which gives us

\[
da_i/da_i = -[f'(\alpha)]^{-1} \alpha^i.
\]

It follows that for any polynomial \( u(x) \) (whose coefficients may depend on \( a \))

\[
du(\alpha)/da_i = \frac{\partial u}{\partial a_i}(\alpha) + u'(\alpha) \times \{-[f'(\alpha)]^{-1} \alpha^i\}.
\]

We are interested in the case when

\[
u(x) = f'(x) = nx^{n-1} + \sum_{i=1}^{n-1} i\alpha_i x^{i-1}.
\]

We obtain that

\[
df'(\alpha)/da_i = i\alpha^{i-1} - [f'(\alpha)]^{-1} \alpha^i f''(\alpha)
\]

(of course, if \( i = 0 \) then the first term disappears).

Actually, the rank of \( dM \) at \( f(x) \) is the dimension of the subspace of \( \Lambda \) generated by \( n \) elements \( df'/da_i(\alpha) \). Suppose that a collection of \( n \) complex numbers \( c_0, \ldots, c_{n-1} \) satisfies \( \sum_{i=0}^{n-1} c_i df'/da_i(\alpha) = 0 \) in \( \Lambda \). If we put \( p(x) = \sum_{i=0}^{n-1} c_i x^i \) then one may easily observe that \( p'(x) = \sum_{i=1}^{n-1} ic_i x^{i-1} \) and in \( \Lambda \) the equality

\[
0 = \sum_{i=0}^{n-1} c_i \frac{df'}{da_i}(\alpha) = p'(\alpha) - [f'(\alpha)]^{-1} p(\alpha) f''(\alpha)
\]

holds. Multiplying (without loss of generality) this equality by the invertible element \( f'(\alpha) \), we obtain the equivalent condition: \( f'(\alpha)p'(\alpha) - p(\alpha)f''(\alpha) = 0 \) in \( \Lambda \).

In other words, the polynomial \( f'(x)p'(x) - p(x)f''(x) \) is divisible by \( f(x) \). Now it is clear that the rank of \( dM \) at \( f(x) \) equals \( n - \{ \text{dimension of the space of polynomials } p(x) \text{ of degree } \leq n - 1 \text{ such that } f'(x)p'(x) - p(x)f''(x) \text{ is divisible by } f(x) \} \).

Obviously, this space contains nonzero \( f'(x) \), which implies that the rank of \( dM \) always does not exceed \( n - 1 \). Since the degree of \( f'(x) \) is \( n - 1 \), it is easy to observe that the kernel of \( dM \) at \( f(x) \) coincides with the direct sum \( \mathbb{C} \cdot f'(x) \oplus W(f) \).

It follows readily that the rank of \( dM \) at \( f(x) \) equals

\[
(n - 1) - \dim_{\mathbb{C}}(W).
\]

**Proof of Theorem 1.3** (modulo Theorem 1.3). Since \( f(x) \) has no multiple roots, it follows from Theorem 1.5 that

\[
W = W(f) = \{0\}
\]

and therefore the rank of \( dM \) at \( f(x) \) equals

\[
(n - 1) - \dim_{\mathbb{C}}(W) = n - 1 - 0 = n - 1.
\]
Remark 2.1. A priori, it is clear that the set of all \( f(x) \) at which the rank of \( dM \) reaches its maximum value is a non-empty Zariski-open subset of \( P_n \). That is why if we are interested only in the general position case then it suffices to check that the rank is \( n-1 \) at least for one \( f(x) \): this would imply that the rank is \( n-1 \) for typical polynomials, i.e., for all polynomials that belong to a certain non-empty Zariski-open subset of \( P_n \).

3. POLYNOMIAL ALGEBRA

**Proof of Theorem 1.5** So, let \( f(x) \) be a complex polynomial of degree \( n \geq 2 \) and \( p(x) \) a nonzero polynomial of degree \( \leq n-2 \) such that

\[
R(x) := f''(x)p(x) - f'(x)p'(x)
\]

is divisible by \( f(x) \). Without loss of generality, we may and will assume that both \( f(x) \) and \( p(x) \) are monic, i.e., their leading coefficients are equal to 1. Notice that there is no “cancelation of degrees” in the expression \( f''(x)p(x) - f'(x)p'(x) \) (since \( p(x) \) and \( f'(x) \) have distinct degrees). Indeed \( \deg(R) \leq n + \deg(p) - 2 \) and the coefficient at \( x^{n+\deg(p)-2} \) of \( R(x) \) equals

\[
n(n-1) - n \deg(p) = n[(n-1) - \deg(p)] \neq 0,
\]

because \( \deg(p) \leq n - 2 < n - 1 \). It follows that \( R(x) \neq 0 \) and

\[
\deg(R) = n - 2 + \deg(p) \leq n - 2 + n - 2 = 2n - 4.
\]

(In addition, \( n[(n-1) - \deg(p)] \) is the leading coefficient of \( R(x) \).) Since \( R(x) \) is a nonzero polynomial divisible by \( f(x) \),

\[
n = \deg(f) \leq \deg(R) = n - 2 + \deg(p),
\]

that is why \( \deg(p) \geq 2 \) and therefore

\[
n - 2 \geq \deg(p) \geq 2,
\]

which implies that \( n \geq 4 \). This proves (i). It is also clear that if \( n = 4 \) then \( \deg(p) = 2 \).

Let us assume now that \( p(x) \) is a monic quadratic polynomial. Then

\[
\deg(R) = n - 2 + \deg(p) = n - 2 + 2 = n = \deg(f)
\]

and therefore there exists a nonzero constant \( h_0 \) such that

\[
R(x) = f''(x)p(x) - f'(x)p'(x) = h_0 f(x).
\]

Since \( f(x) \) is monic while the leading coefficient of \( R(x) \) is

\[
n[(n-1) - \deg(p)] = n(n-1 - 2) = n(n-3),
\]

we obtain that \( h_0 = n(n-3) \) and

\[
R(x) = f''(x)p(x) - f'(x)p'(x) = n(n-3)f(x).
\]

Differentiating, we obtain that

\[
R'(x) = [f'''(x)p(x) + f''(x)p'(x)] - [f''(x)p'(x) + f'(x)p''(x)] = n(n-3)f'(x).
\]

Since \( p(x) \) is a monic quadratic polynomial, \( p''(x) = 2 \). Taking this into account, opening the parentheses and grouping together like terms in the formula for \( R'(x) \), we obtain that

\[
R'(x) = f'''(x) \cdot p(x) - 2f'(x) = n(n-3)f'(x),
\]

which gives us

\[ f''(x) \cdot p(x) = (n^2 - 3n + 2)f'(x). \]

We obtain that \( p(x) \) divides \( f'(x) \). (Recall that \( n \geq 4 \) and therefore \( n^2 - 3n + 2 = (n - 1)(n - 2) \neq 0 \).) Since

\[ f''(x) \cdot p(x) - f'(x)p'(x) = n(n - 3)f(x), \]

we obtain that \( p(x) \) divides \( f(x) \). (Recall that \( n \geq 4 \) and therefore \( n(n - 3) \neq 0 \).) This proves (iii). It is also clear that if (a quadratic polynomial) \( p(x) \) has no multiple roots then \( f(x) \) is divisible by \( p(x)^2 \). On the other hand, if \( p(x) \) has a double root \( \beta \), i.e., \( p(x) = (x - \beta)^2 \) then \( f(x) \) is divisible by \( (x - \beta)^3 \).

Let us prove assertions (ii) and (iv) of Theorem 1.5. If all the roots of \( f(x) \) are multiple then (ii) holds true. Let us assume that \( f(x) \) has a simple root and denote it by \( \alpha \). Then \( f'(:) \neq 0 \) and there exists a complex polynomial \( f_1(x) \) of degree \( n - 1 \) such that

\[ f(x) = (x - \alpha)f_1(x). \]

The simplicity of \( \alpha \) means that

\[ f_1(\alpha) \neq 0, \quad f'(\alpha) \neq 0. \]

Let \( V \) be the two-dimensional vector (sub)space of polynomials generated by \( f'(x) \) and \( p(x) \). (It is two-dimensional, because nonzero \( f'(x) \) and \( p(x) \) have different degrees.) Clearly, the degree of any polynomial from \( V \) does not exceed \( n - 1 \). It is also clear that for each \( q(x) \in V \) the polynomial \( f''(x)q(x) - f'(x)q'(x) \) is divisible by \( f(x) \), since this is true for both \( q(x) = p(x) \) and \( q(x) = f'(x) \). Choose in the two-dimensional \( V \) a nonzero polynomial \( q(x) \) such that

\[ q(\alpha) = 0. \]

Clearly,

\[ q(x) \notin \mathbb{C} \cdot f'(x), \quad \deg(q) \leq n - 1. \]

We have

\[ f''(x)q(x) - f'(x)q'(x) = h(x)f(x) \quad (*) \]

for some complex polynomial \( h(x) \). Since

\[ f(\alpha) = 0, \quad f'(\alpha) \neq 0, \quad q(\alpha) = 0, \]

we conclude that \( q'(\alpha) = 0 \), i.e., \( \alpha \) is a multiple root of nonzero \( q(x) \), hence, there exists a nonzero complex polynomial \( q_1(x) \) such that

\[ q(x) = (x - \alpha)^2q_1(x), \quad \deg(q_1) = \deg(q) - 2 \leq (n - 1) - 2 = \deg(f_1) - 2. \]

I am going to prove that \( f_1''(x)q_1(x) - f_1'(x)q_1'(x) \) is divisible by \( f_1(x) \) (and apply induction by \( n \)).

We have

\[ f(x) = (x - \alpha)f_1(x), \quad f'(x) = f_1'(x) + (x - \alpha)f_1(x), \quad f''(x) = 2f_1''(x) + (x - \alpha)f_1'(x), \quad q(x) = (x - \alpha)^2q_1(x), \quad q'(x) = (x - \alpha)^2q_1'(x) + 2(x - \alpha)q_1(x). \]

Plugging in these expressions in \((*)\), we obtain

\[ [2f_1'(x) + (x - \alpha)f_1''(x)](x - \alpha)^2q_1(x) - [f_1(x) + (x - \alpha)f_1'(x)][(x - \alpha)^2q_1'(x) + 2(x - \alpha)q_1(x)] = h(x)(x - \alpha)f_1(x). \]

Dividing both sides by \( (x - \alpha) \), we obtain

\[ [2f_1'(x) + (x - \alpha)f_1''(x)]q_1(x)(x - \alpha) - [f_1(x) + (x - \alpha)f_1'(x)][(x - \alpha)q_1'(x) + 2q_1(x)] = h(x)f_1(x). \]
Moving to the right hand side all the terms containing \( f_1(x) \), we obtain that 
\[
[2f_1'(x) + (x - \alpha)f_1''(x)]q_1(x)(x - \alpha) - (x - \alpha)f_1'(x)[(x - \alpha)q_1'(x) + 2q_1(x)] = \{h(x) + [(x - \alpha)q_1'(x) + 2q_1(x)]\}f_1(x).
\]
If we put \( h_1(x) := h(x) + [(x - \alpha)q_1'(x) + 2q_1(x)] \) then we get 
\[
[2f_1'(x) + (x - \alpha)f_1''(x)]q_1(x)(x - \alpha) - (x - \alpha)f_1'(x)[(x - \alpha)q_1'(x) + 2q_1(x)] = h_1(x)f_1(x).
\]

The left hand side is divisible by \( (x - \alpha) \), while \( f_1(x) \) (in the right hand side) is not divisible by \( (x - \alpha) \). This means that there exists a complex polynomial \( h_2(x) \) such that \( h_1(x) = (x - \alpha)h_2(x) \) and 
\[
[2f_1'(x) + (x - \alpha)f_1''(x)]q_1(x) - f_1'(x)[(x - \alpha)q_1'(x) + 2q_1(x)] = h_2(x)f_1(x).
\]
Opening the parentheses in the left hand side and grouping together like terms, we obtain 
\[
(x - \alpha)f_1''(x)q_1(x) - f_1'(x)(x - \alpha)q_1'(x) = h_2(x)f_1(x).
\]
Again the left hand side is divisible by \( (x - \alpha) \), while \( f_1(x) \) (in the right hand side) is not divisible by \( (x - \alpha) \). This means that there exists a complex polynomial \( h_3(x) \) such that \( h_2(x) = (x - \alpha)h_3(x) \) and 
\[
f_1''(x)q_1(x) - f_1'(x)q_1'(x) = h_3(x)f_1(x).
\]

Let us treat separately the cases \( n = 4 \) and \( n > 4 \).

Suppose that \( n = 4 \). Then \( \deg(f_1) = 4 - 1 = 3 \) and we get a contradiction to the already proven (i). Therefore all the roots of \( f(x) \) are multiple, which implies that \( f(x) \) has either two double roots or one root of multiplicity 4. In particular \( f(x) \) has no roots of multiplicity 3. We also know that \( \deg(p) = 2 \) while \( p(x) \) divides both \( f(x) \) and \( f'(x) \). We have seen that if quadratic \( p(x) \) has no multiple roots then \( f(x) \) is divisible by \( p(x)^2 \) and the comparison of degrees and leading coefficients implies that \( f(x) = p(x)^2 \). On the other hand, if \( p(x) \) has a double root \( \beta \) then we have seen that \( p(x) = (x - \beta)^2 \) and \( f(x) \) is divisible by \( (x - \beta)^3 \). Since \( f(x) \) has no roots of multiplicity 3, we conclude that monic \( f(x) = (x - \beta)^4 \). Since \( p(x) = (x - \beta)^2 \) we conclude that \( f(x) = p(x)^2 \). This proves (iv). In addition, it also proves (ii) for \( n = 4 \).

Suppose that \( n > 4 \). Then \( \deg(f_1) = n - 1 \). Using induction by \( n \), we may apply the assertion (ii) of Theorem 1.5 to \( n - 1 \) (instead of \( n \)) and \( f_1 \) and \( q_1 \) (instead of \( f \) and \( p \) respectively) and obtain that there exists a quadratic polynomial \( \tilde{p}(x) \) such that \( f_1(x) \) is divisible by \( \tilde{p}(x)^2 \). It follows that \( f(x) = (x - \alpha)f_1(x) \) is also divisible by \( \tilde{p}(x)^2 \). This ends the proof.

**Remark 3.1.** Inspecting the proof of Theorem 1.5, we observe that it works (not only over arbitrary fields of characteristic zero but also) over fields of characteristic \( > n \). In particular, the assertion of Theorem 1.5 remains true over fields of positive characteristic \( > n \).

**Example 3.2.** Let \( p(x) \) be a monic quadratic polynomial. Let us put \( n = 4 \) and \( f(x) = p(x)^2 \). Then
\[
p''(x) = 2, \quad f'(x) = 2p(x)p'(x), \]
\[
f''(x) = 2p'(x)^2 + 2p(x)p''(x) = 2p'(x)^2 + 4p(x).
\]
Then
\[
f''(x)p(x) - f'(x)p'(x) = [2p'(x)^2 + 4p(x)]p(x) - 2p(x)p'(x)p'(x) = 4p(x)^2 = 4f(x).
\]
Example 3.3. Let \( d \) be an arbitrary complex number. Let us put \( n = 5 \),

\[
f(x) := (x + d) \left( x^2 - \frac{d}{2} x + \frac{9}{4} \right)^2 = x^5 + \frac{15}{4} d^2 x^3 + \frac{5}{2} d^3 x^2 + \frac{45}{16} d^4 x + \frac{81}{16} d^5,
\]

\[
p(x) := x^3 + 2d^2 x + \frac{9}{8} d^3.
\]

Then

\[
f''(x)p(x) - f'(x)p'(x) = 5x \cdot f(x).
\]

Remark 3.4. If \( f(x) \) is the polynomial \( x^n - 1 \), which has no multiple roots, then one may directly check that there does not exist a nonzero polynomial \( p(x) \) of degree \( \leq n - 2 \) such that \( R(x) = f''(x)p(x) - f'(x)p'(x) \) is divisible by \( f(x) \), i.e., \( R(x) = f(x)h(x) \) for some polynomial \( h(x) \). Indeed, let us assume that such a \( p(x) \) does exist; we may assume that \( p(x) \) is monic. Obviously (see the very beginning of the proof of theorem 1.5)

\[
R(x) \neq 0, \ deg(R) \leq 2n - 4.
\]

This implies that \( h(x) \neq 0 \) and

\[
\deg(h) = \deg(R) - n \leq n - 4.
\]

Plugging in into the formula for \( R(x) \) explicit formulas for \( f(x) \) and its first and second derivatives, we obtain that

\[
n(n-1)x^{n-2}p(x) - nx^{n-1}p'(x) = h(x)(x^n - 1).
\]

This implies that nonzero \( h(x) \) is divisible by \( x^{n-2} \) and therefore \( \deg(h) \geq n - 2 \), which is not true. The obtained contradiction proves that such a \( p(x) \) does not exist and therefore \( W(f) = \{0\} \), i.e., the rank of \( dM \) at \( f(x) = x^n - 1 \) has rank \( n - 1 \). According to Remark 2.1 it follows immediately that the rank of \( dM \) equals \( n - 1 \) for typical polynomials.

Remark 3.5. After having read previous versions of this paper, Elmer Rees [1] and (independently and simultaneously) Victor S. Kulikov outlined a more direct way of approaching the proof of Theorem 1.1. Namely, they consider the Jacobi matrix of the map

\[
\tilde{M} : (\alpha_1, \ldots, \alpha_n) \mapsto (f'(\alpha_1), \ldots, f'(\alpha_n))
\]

and prove that its first principal \((n-1) \times (n-1)\) minor equals

\[
c \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2,
\]

where \( c \) is a constant that does not depend on \( f(x) \). (In order to prove Theorem 1.1 it suffices to check that \( c \neq 0 \).) Rees [1] proved that

\[
c = (-1)^{(n-1)(n-2)/2}(n-1)!.\]

Kulikov’s approach is presented in the Appendix to this paper that he has kindly agreed to contribute.
Let us consider a polynomial that is of independent interest and whose proof requires minimal computations. This observation allows us to deduce Theorem 1.1 from the following statement
\[
\alpha
\]
where
Let
\[
j
\]
v
\[
i,i
\]
and two rows of a matrix does not change its determinant, \(M\). The rank \(\text{rg } D\) of the matrix \(D\) is an eigenvector of \(D\). Therefore the curve \(f(x + t)\) lies in a fiber of the map \(M\) and the tangent vector to this curve at the point \(f(x)\) is \(f'(x)\).

**Proof of Theorem 4.1.** The map \(\tilde{M}\) with respect to the coordinates \((\alpha_1, \ldots, \alpha_n)\) is defined by polynomials \(y_j = f_j'(\alpha_1, \alpha) = (\alpha_i - \alpha_1) \ldots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \ldots (\alpha_i - \alpha_n), 1 \leq i \leq n\). Computing the partial derivatives, one may easily observe that the entry \(d_{i,j}\) of \(D\) equals \(d_{i,j} = -f_j'(\alpha_i, \alpha)\) if \(j \neq i\) and \(d_{i,i} = \sum_{j \neq i} f_j'(\alpha_i, \alpha)\). Therefore the sum of all columns of \(D\) equals zero. That is why \(\text{det } D = 0\) and the vector \(v = (1, \ldots, 1)\) is an eigenvector of \(D\) with eigenvalue 0.

Let us prove that the minor \(M_{i,i}\) of \(D\) equals \(c \prod_{j<k,j \neq i,k \neq i} (\alpha_k - \alpha_j)^2\). When \(n = 2\) this is obvious. Suppose that \(n > 2\). First notice that \(M_{i,i}\) is a polynomial in variables \(\alpha_j\) of degree \((n - 1)(n - 2)\). Besides, one may easily observe that the minors \(M_{i,i}\) are invariant with respect to transformations \(\sigma_{j,k}\) defined as follows: an exchange of \(j\)th and \(k\)th rows followed by the exchange of columns with the same numbers and a change of variables \(\alpha_j \leftrightarrow \alpha_k\). Since the operation of exchanging two columns and two rows of a matrix does not change its determinant, \(M_{i,i}\) is a symmetric

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4. Appendix by Victor S. Kulikov

Recall (see Remarks 2.4 and 3.4) that it is much more easier to prove that the differential \(dM\) has rank \(n - 1\) at a generic point rather than Theorem 1.1 in full. This observation allows us to deduce Theorem 1.1 from the following statement that is of independent interest and whose proof requires minimal computations. Let us consider a polynomial
\[
f(x, \alpha) = (x - \alpha_1) \ldots (x - \alpha_n) = x^n + \sum_{i=0}^{n-1} a_i x_i,
\]
where \(\alpha = (\alpha_1, \ldots, \alpha_n)\). Let us define the map \(\tilde{M} : \mathbb{C}^n \to \mathbb{C}^n\) by
\[
\tilde{M}(\alpha) = (f'_1(\alpha_1, \alpha), \ldots, f'_n(\alpha_n, \alpha)).
\]
Let \(D\) be the Jacobi matrix of the map \(\tilde{M}\).

**Theorem 4.1.** The rank \(\text{rg } D\) of the matrix \(D\) is \(n - 1\). The vector \(v = (1, \ldots, 1)\) is an eigenvector of \(D\) with eigenvalue 0. If \(n > 1\) then for each \(i, 1 \leq i \leq n\) the minor \(M_{i,i}\) of \(D\) equals \(c \prod_{j<k,j \neq i,k \neq i} (\alpha_k - \alpha_j)^2\), where \(c\) is a certain nonzero constant.

Clearly, Theorem 1.1 is a corollary of Theorem 4.1. Indeed, first \(M_{i,i} \neq 0\) at every point with pairwise distinct coordinates. Second, for each \(\alpha\) the line \(l_\alpha(t) = \alpha - t(1, \ldots, 1)\) lies in a fiber of the map \(\tilde{M}\). Therefore the curve \(f(x + t)\) lies in a fiber of the map \(M\) and the tangent vector to this curve at the point \(f(x)\) is \(f'(x)\).

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1Steklov Mathematical Institute of the Russian Academy of Sciences, email: kulikov@mi.ras.ru
function in variables $\alpha_j$, $j \neq i$. Further, let us restrict $\widetilde{M}$ to the hyperplane $\alpha_i = \text{const}$ and consider the composition of this restriction with the projection map to the hyperplane $y_i = 0$: $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$. Denote the obtained map by $\widetilde{M}_i : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$. Clearly, the Jacobi matrix of $\widetilde{M}_i$ coincides with the matrix $D_\alpha$ obtained from $D$ by deleting its $i$th row and $i$th column. It is easy to observe that for each triple $i,j,k$ of pairwise distinct indices the image of the hyperplane $\alpha_j = \alpha_k$ under $\widetilde{M}_i$ has codimension $\geq 2$, because it lies in the intersection of the hyperplanes $y_j = 0$ and $y_k = 0$. Therefore the jacobian of $\widetilde{M}_i$ vanishes at $\alpha_j = \alpha_k$ and any value of $\alpha_i$. On the other hand, the jacobian of $\widetilde{M}_i$ coincides with $M_{i,i}$ and therefore $M_{i,i}$ is divisible by $\alpha_j - \alpha_k$. Since $M_{i,i}$ is a symmetric polynomial in variables $\alpha_j$, $j \neq i$, we obtain that $M_{i,i}$ is divisible by $(\alpha_j - \alpha_k)^2$ for each triple $i, j, k$ of pairwise distinct indices. Taking into account that $\deg(M_{i,i}) = (n-1)(n-2)$, we obtain that $M_{i,i} = c \prod_{j<k, j \neq i, k \neq i}(\alpha_k - \alpha_j)^2$ for a certain constant $c$. Since the vector $(1, \ldots, 1)$ is not tangent to the hyperplane $\alpha_i = \text{const}$, one may easily observe that $c \neq 0$ because $dM$ (and therefore $d\widetilde{M}$) has rank $n - 1$ at a generic point. \hfill $\square$

**Theorem 4.2.** The image of the map $\widetilde{M} : \mathbb{C}^n \to \mathbb{C}^n$ lies in the irreducible hypersurface $S_{n-1}$ that is defined in $\mathbb{C}^n$ by the equation

$$\sum_{i=1}^{n} y_1 \cdots y_{i-1}y_{i+1}\cdots y_n = 0.$$  

The image $\widetilde{M}(\mathbb{C}^n)$ is everywhere dense in $S_{n-1}$ with respect to the complex topology.

**Proof.** The irreducibility of $S_{n-1}$ follows from the irreducibility of the polynomial

$$s_{n-1} = \sum_{i=1}^{n} y_1 \cdots y_{i-1}y_{i+1}\cdots y_n,$$

which can be easily checked. The map $\widetilde{M}$ is defined by polynomials

$$y_1 = f_1'(\alpha_1, \alpha), \ldots, y_n = f_n'(\alpha_n, \alpha).$$

Let us view $s_{n-1}$ as a polynomial in $\alpha_1, \ldots, \alpha_n$, plugging in $y_i = f_i'(\alpha_i, \alpha)$. We observe that $s_{n-1}$ is a homogeneous symmetric polynomial in $\alpha_1, \ldots, \alpha_n$ of degree $(n-1)^2$.

Clearly, if $\alpha_i = \alpha_j$, then $y_i = y_j = 0$. Therefore the polynomial $s_{n-1}$ vanishes if $\alpha_i = \alpha_j$ and therefore it is divisible by $\alpha_i - \alpha_j$ for each $i$ and $j$, $i \neq j$. Since $s_{n-1}$ is symmetric, it is divisible by $(\alpha_i - \alpha_j)^2$. Therefore $s_{n-1}$ is divisible by the polynomial $\prod_{i \neq j}(\alpha_i - \alpha_j)$, whose degree is $n(n-1)$. Since $n(n-1) > (n-1)^2$, we have $s_{n-1}(\alpha) \equiv 0$ and therefore $\widetilde{M}(\mathbb{C}^n)$ lies in $S_{n-1}$. Let $X$ be the open algebraic subvariety of all points in $\mathbb{C}^n$ with pairwise distinct coordinates and let $Y \subset S_{n-1}$ be the Zariski closure of the (sub)set $\widetilde{M}(X)$. Clearly, $Y$ is an irreducible closed algebraic subvariety in $S_{n-1}$; in particular, $\dim(Y) \leq \dim(S_{n-1}) = n - 1$. It follows from [4] Ch. 1, Sect. 5, Th. 6 that $\widetilde{M}(X)$ contains a Zariski-open non-empty subset of $Y$; in particular, it contains a nonsingular point of $Y$. Theorem 4.1 combined with irreducibility of $S_{n-1}$ implies that $\dim(Y) \geq n - 1$. It follows that $\dim(Y) = n - 1 = \dim(S_{n-1})$ and therefore $Y = S_{n-1}$. Since the image $\widetilde{M}(\mathbb{C}^n)$ contains $\widetilde{M}(X)$, we conclude that $\widetilde{M}(\mathbb{C}^n)$ contains a Zariski-open non-empty subset.
of $S_{n-1}$; in particular, $\tilde{M}(\mathbb{C}^n)$ is everywhere dense in $S_{n-1}$ with respect to the complex topology.

\[\square\]

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