EMERGENCE OF AGGREGATION IN THE SWARM SPHERE MODEL WITH ADAPTIVE COUPLING LAWS

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Abstract. We present aggregation estimates for the swarm sphere model equipped with the adaptive coupling laws on a sphere. The temporal evolution of coupling strength is determined by a feedback rule incorporating the balance between relative spatial variations and linear damping. For the analytical treatment, we employ two adaptive feedback laws, namely anti-Hebbian and Hebbian laws. For the anti-Hebbian law, we provide a sufficient framework leading to the complete aggregation in which all particles aggregate to the same position and behave like one big point cluster asymptotically. Our frameworks are given in terms of the initial positions and the coupling strengths. For the Hebbian law, we provide proper subsets of the basin of attractions for the complete aggregation and bi-polar aggregation where particles aggregate to the north pole and south pole simultaneously. We also present a uniform $\ell_p$-stability of the swarm sphere model with an adaptive coupling with respect to the initial data when the complete aggregation occurs exponentially fast.

1. Introduction. Collective coherent motions are ubiquitous in classical and quantum many-body systems such as cardiac pacemaker cells, biological clocks in the brain, Josephson junction arrays and a group of fireflies [1, 10, 26, 30, 34, 37]. Recently, the collective behaviors of many-body systems have received lots of attention in diverse scientific and engineering disciplines such as applied mathematics, biology, control theory and statistical physics due to the emerging applications in coordinated motions of drones, unmanned vehicles, robot systems and sensor networks, etc. Among phenomenological and mechanical models for synchronization, the Kuramoto model serves a prototype one for phase synchronization. In relation with the flocking realization problem [5], one natural question will be whether we...
can design a particle system exhibiting aggregation behaviors on the unit $d$-sphere $\mathbb{S}^d$ embedded in $\mathbb{R}^{d+1}$ as a multi-dimensional generalization of the Kuramoto model. In this paper, we are interested in the swarm sphere model proposed in [24, 25, 28] generalizing the Kuramoto model as a special case. Let $x_i$ and $\kappa_{ij}$ be the position of the $i$-th particle on a sphere and the coupling strength between $i$ and $j$-th particles, respectively. Then, the dynamics of $(x_i, \kappa_{ij})$ is governed by a Cauchy problem for the swarm sphere model [28]:

$$
\begin{cases}
\dot{x}_i = \Omega_i \frac{x_i}{|x_i|^2} + \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \frac{\langle x_i, x_j \rangle}{\langle x_i, x_i \rangle} x_i \right), & t > 0, \quad 1 \leq i \leq N, \\
\dot{\kappa}_{ij} = \mu \Gamma(x_i, x_j) - \gamma \kappa_{ij}, \\
(x_i(0), \kappa_{ij}(0)) = (x_i^0, \kappa_{ij}^0).
\end{cases}
$$

(1)

Here $\Omega_i$ and $\langle \cdot, \cdot \rangle$ are real $(d+1) \times (d+1)$ skew-symmetric matrix and the standard inner product in $\mathbb{R}^{d+1}$, respectively. The nonnegative constants $\mu$ and $\gamma$ are proportional to learning enhancement rate and friction, respectively, and $|x|$ denotes the standard $\ell_2$-norm of $x$ in $\mathbb{R}^{d+1}$. It is easy to see that as long as particles stay on a sphere initially, particles stay on the same sphere. Thus, system (1) is well-posed as long as all particles stays away from zero.

From the modeling viewpoint, there might be many different choices for the adaptive coupling law which reflects physical reality. In neuroscience, Hebbian rule [20] roughly says that the growth rate for the coupling strength is maximized when interacting neurons are in phase and is decreasing in the phase differences, e.g., $\Gamma(\theta) = \cos \theta$. In contrast, anti-Hebbian rule refers to the opposite situation. In analogy with this, we consider the following two adaptive coupling laws (anti-Hebbian and Hebbian laws):

- **Anti-Hebbian**: $\Gamma_{ah}(x, y) = |x - y|^2,$
- **Hebbian**: $\Gamma_{h,g}(x, y) = 1 - \frac{|x - y|^2}{2}$ and $\Gamma_{h,p}$.

(2)

System (1) can be reduced to the Kuramoto model with adaptive couplings for $d = 1$, which has been extensively studied in [2, 10, 14, 18, 19, 27, 29, 31, 32, 33, 35]. For the uniform all-to-all coupling in which $\kappa_{ij}$ is a uniform constant, i.e., $\kappa_{ij} = \kappa > 0$, emergent dynamics of (1) has been extensively studied in [5, 7, 17] in which several analytic frameworks for the complete aggregation and practical aggregation (see Definition 2.1) have been investigated in terms of initial data and parameters.

The purpose of this paper is to study the emergent dynamics of (1) under the feedback laws (2) and $\Omega_i = \Omega$, $i = 1, \cdots, N$, and we present sufficient frameworks leading to the complete aggregation and bi-polar aggregation. More precisely, our main results in this paper are three-fold. First, we consider anti-Hebian rule $\Gamma_{ah}(s)$ in (2). In this case, $\Gamma(0) = 0$, and this degeneracy causes lots of difficulty in the aggregation analysis. For this, we use the Lyapunov functional approach together with the Barbalat’s lemma. For a given solution $(X, K)$ to (1) and $i, j$, we introduce Lyapunov functionals $L_{ij} = L_{ij}(X, K)$:

$$
L_{ij}(t) = \frac{1}{2} |x_i(t) - x_j(t)|^2 + \frac{1}{4 \mu N} \sum_{k=1}^{N} |\kappa_{ik}(t) - \kappa_{jk}(t)|^2, \quad t \geq 0.
$$
Then, as long as the initial data \((X^0, K^0)\) satisfy
\[
\max_{1 \leq i,j \leq N} L_{ij}(0) < 1, \quad \min_{1 \leq i,j \leq N} \kappa_{ij}(0) > 0,
\]
\(L_{ij}\) is non-increasing and satisfies aggregation estimates (see Proposition 1 and Theorem 3.4):
\[
\lim_{t \to \infty} \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)| = 0 \quad \text{and} \quad \lim_{t \to \infty} \max_{1 \leq i,j \leq N} \kappa_{ij}(t) = 0.
\]

Second, we consider the Hebbian feedback law (2). In this case, \(\Gamma_{h,p}\) has a positive value at the origin and is decreasing until its first positive zero. Unlike the anti-Hebbian case, \(\Gamma_{ah}\) can take positive and negative values so that diverse asymptotic patterns can emerge from the initial configurations. In particular, we are interested in the basin of attractions for the complete aggregation and bi-polar aggregation. For the former situation, if the initial position of the particles is on the same hemisphere and the ratio between the maximum and minimum of coupling strength is sufficiently small, the complete aggregation can occur exponentially fast (Theorem 4.2):
\[
\max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)| \leq C e^{-\lambda t}, \quad \text{as} \quad t \to \infty,
\]
where \(\lambda\) is a positive constant independent of \(N\) and \(t\). On the other hand, the bi-polar aggregation can also emerge from the well-prepared initial configurations. More precisely, consider a configuration consisting of two approximate clusters scattered around the north and south poles, respectively. Furthermore, the intra coupling strengths between the particles belonging to the same subgroup are assumed to be positive(attraction), whereas the inter coupling strengths between the particles belonging to different groups are assumed to be negative(repulsion). Under this well-prepared situation, the particles belonging to the same group are likely to aggregate each other. In contrast, the particles belonging to different clusters are likely to depart each other. In fact, this plausible scenario can be made rigorous (see Theorem 6.3). Third, as a direct application of an exponential aggregation estimate, we present a uniform stability estimate of (1) with respect to the initial data in the sense of the Definition 6.1: For two pair of solutions \((X, K)\) and \((\tilde{X}, \tilde{K})\) to (1), we have
\[
\sup_{0 \leq t < \infty} \|X(t) - \tilde{X}(t)\|_p \leq G \|X^0 - \tilde{X}^0\|_p,
\]
where \(\|\cdot\|_p\) is the \(\ell_p\)-norm in \(\mathbb{R}^{N(d+1)}\) (see Theorem 6.3).

The rest of this paper is organized as follows. In Section 2, we review basic properties of (1) and discuss its relation with the Kuramoto model, and recall previous results on the emergent dynamics of the Kuramoto model with the adaptive coupling laws. In Section 3, we consider the anti-Hebbian adaptive law and present the formation of complete aggregation from some admissible class of initial data using the Lyapunov functional approach. In Section 4, we consider the two Hebbian adaptive laws: one is a positive and non-increasing law which generalizes a cosine-type function and the other is also decreasing but it can be both positive and negative in their domain. For these Hebbian adaptive laws, we show that the complete aggregation can emerge from some class of initial configurations. In Section 5, we consider the second adaptive law and show that the bi-polar aggregation can emerge from the well-prepared initial data. In Section 6, we present the uniform \(\ell_p\)-stability estimate with respect to the initial data once we have an exponential
aggregation estimate. Finally, Section 7 is devoted to a brief summary of our main
result and future directions.

Notation: For \((d + 1) \times (d + 1)\) matrix \(\Omega\) and \(y \in \mathbb{R}^{d+1}\), we denote
\[\|\Omega\| := \max_{1 \leq l, m \leq d+1} |\Omega_{l,m}| \text{ and } |y| := \sqrt{y_1^2 + \cdots + y_{d+1}^2}.\]
We set
\[X := (x_1, \cdots, x_N) \in \mathbb{R}^{(d+1)N}, \quad D(X) := \max_{1 \leq i,j \leq N} |x_i - x_j|,\]
\[K = (\kappa_{ij}) \in \mathbb{M}_N(\mathbb{R}), \quad h_{ij} := \langle x_i, x_j \rangle,\]
where \(\mathbb{M}_N(\mathbb{R}_+)\) is the set of all \(N \times N\) real matrices, and define a \(\ell_p\)-norm of the vector \(X\):
\[\|X\|_p := \left( \sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty).\]

2. Preliminaries. In this section, we present elementary estimates for the swarm
sphere model to be used in later sections, and briefly recall previous results on the
emergent dynamics of the model with the adaptive coupling laws.

2.1. A swarm sphere model. First, we recall two concepts of emergent phe-
nomena such as the complete aggregation and bi-polar aggregation in the following
definition.

Definition 2.1. Let \((X, K)\) be a solution to system (1).
1. System (1) exhibits the complete aggregation if the relative positions tend to
zero asymptotically:
\[\lim_{t \to \infty} D(X(t)) = 0.\]
2. System (1) exhibits the bi-polar aggregation if there exists a non-empty subset
\(N_0\) of \(N := \{1, \cdots, N\}\) such that
\[\lim_{t \to \infty} \frac{x_i}{|x_i|} \frac{x_j}{|x_j|} = \begin{cases} 1 & \text{for } (i, j) \in (N_0 \times N_0) \cup (N_0^c \times N_0^c), \\ -1 & \text{for } (i, j) \in (N_0 \times N_0^c) \cup (N_0^c \times N_0). \end{cases}\]

Next, we present two elementary properties of the swarm sphere model (1) with
adaptive couplings.

Lemma 2.2. Let \((X, K)\) be a solution to system (1) with initial data \((X^0, K^0)\)
satisfying the symmetry relations:
\[\kappa_{ij}^0 = \kappa_{ji}^0 \geq 0, \quad 1 \leq i, j \leq N.\]  \(\text{(3)}\)
Then, the coupling matrix \(K = [\kappa_{ij}]\) is symmetric and its entries are nonnegative:
\[\kappa_{ij}(t) = \kappa_{ji}(t) \geq 0, \quad 1 \leq i, j \leq N, \quad t > 0.\]  \(\text{(4)}\)
Proof. We can obtain the following integro-differential equation from (1)_2:
\[\kappa_{ij}(t) = e^{-\gamma t} \left( \kappa_{ij}^0 + \int_0^t e^{\gamma s} \Gamma(x_i, x_j)ds \right).\]
Then, conditions (3) and (4) yield desired estimates. \(\square\)

Below, we show that the unit sphere \(S^d\) is a positively invariant set for (1).
Lemma 2.3. Let \((X, K)\) be a solution to system (1) with initial data \((X^0, K^0)\) satisfying the relations:

\[
|x^0_i| = r > 0, \quad \kappa^0_{ij} = \kappa^0_{ji}, \quad i, j = 1, \cdots, N.
\]

Then, \(x_i\) stays on the sphere \(\partial B_r(0)\) along the flow (1):

\[
|x_i(t)| = r, \quad t > 0, \quad i = 1, \cdots, N.
\]

Proof. Note that the pairwise coupling strengths satisfy the symmetry condition (4) due to Lemma 2.2. Now, we take the standard inner product of \(x_i\) and (1) and use the symmetry of \(\kappa_{ij}\) to get

\[
\frac{1}{2} \frac{d}{dt} |x_i|^2 = \left< x_i, \Omega_i \frac{x_i}{|x_i|^2} \right> + \frac{1}{N} \sum_{j=1}^N \kappa_{ij} \left< \frac{x_i}{|x_i|^2}, -\frac{x_j}{|x_j|^2} \right> = \left< x_i, \Omega_i \frac{x_i}{|x_i|^2} \right> = 0
\]

where the last equality is due to skew-symmetry of \(\Omega_i\):

\[
\left< x_i, \Omega_i \frac{x_i}{|x_i|^2} \right> = \left< \Omega_i^T x_i, x_i \right> = \left< \Omega_i x_i, x_i \right> = -\left< x_i, \Omega_i \frac{x_i}{|x_i|^2} \right>, \quad \text{i.e.,}
\]

\[
\left< x_i, \Omega_i \frac{x_i}{|x_i|^2} \right> = 0.
\]

For identical particles with \(\Omega_i = \Omega\), \(1 \leq i \leq N\), system (1) restricted on the set \((\partial B_r(0))^N\) with \(r > 0\) has a solution splitting property as a composition of the rotation and Lohe coupling. More precisely, we consider the following two sub-systems on \(|x_i| = r\):

\[
\dot{x}_i = \Omega \frac{x_i}{r^2} \quad \text{and} \quad \dot{x}_i = \frac{\kappa}{N} \sum_{k=1}^N \kappa_{ki} \left( x_i - \frac{1}{r^2} \langle x_i, x_k \rangle x_i \right). \tag{5}
\]

First, we introduce two solution operators \(R_\Omega(t)\) and \(L(t)\) for subsystems in (5).

For \(u \in \mathbb{R}^{d+1}\) and \((d + 1) \times (d + 1)\) skew-symmetric matrix \(\Omega\), \(R_\Omega(t)\) is a solution operator corresponding to (5) \(_1\): for \(u \in \mathbb{R}^{d+1}\),

\[
R_\Omega(t)u := e^{\Omega t}u,
\]

and \(L(t)\) is a solution operator corresponding to (5) \(_2\): for a solution \(x_i(t)\) to (5) \(_2\) with initial data \(u \in \mathbb{R}^d\),

\[
x_i(t) := L(t)x^0_i.
\]

Any solution to (1) can be written as a composite of two solution operators \(R_\Omega(t)\) and \(L(t)\) as follows.

Lemma 2.4. Suppose that the initial positions and \(\Omega_i\) satisfy

\[
|x_i^0| = r, \quad \Omega_i \equiv \Omega, \quad \text{for all} \quad i = 1, \cdots, N.
\]

Then, for any solution \((X, K)\) to (1), we have

\[
x_i(t) = R_\Omega(t) \circ L(t)x^0_i, \quad t > 0, \quad 1 \leq i \leq N.
\]

Proof. Note several elementary relations:

\[
(e^{-\frac{\Omega t}{r}})^T = e^{-\frac{(\Omega t)^T}{r}} = e^{\frac{\Omega t}{r}} = (e^{-\frac{\Omega t}{r}})^{-1},
\]

\[
(e^{-\frac{\Omega t}{r}})x_i, e^{-\frac{\Omega t}{r}}x_j = (x_i, x_j), \quad \langle e^{-\frac{\Omega t}{r}}x_i, e^{-\frac{\Omega t}{r}}x_i \rangle = (x_i, x_i).
\]
Then, we multiply \( e^{-\Omega t} \) to (5) and use the above relations to obtain
\[
e^{-\Omega t} \dot{x}_i - \frac{\Omega^2}{r^2} e^{-\Omega t} x_i = e^{-\Omega t} \frac{\kappa}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \frac{\langle x_i, x_j \rangle}{\langle x_i, x_i \rangle} x_i \right)
\]
\[
= \frac{\kappa}{N} \sum_{j=1}^{N} \kappa_{ij} \left( e^{-\Omega t} x_j - \frac{\langle e^{-\Omega t} x_i, e^{-\Omega t} x_j \rangle}{\langle e^{-\Omega t} x_i, e^{-\Omega t} x_i \rangle} e^{-\Omega t} x_i \right).
\]

Thus, we have
\[
\frac{d}{dt} (e^{-\Omega t} x_i) = \frac{\kappa}{N} \sum_{j=1}^{N} \kappa_{ij} \left( e^{-\Omega t} x_j - \frac{\langle e^{-\Omega t} x_i, e^{-\Omega t} x_j \rangle}{\langle e^{-\Omega t} x_i, e^{-\Omega t} x_i \rangle} e^{-\Omega t} x_i \right).
\] (6)

We use the change of variables \( y_i := e^{-\Omega t} x_i = R_{-\Omega}(t)x_i \) to see that (6) equals to
\[
\frac{dy_i}{dt} = \frac{\kappa}{N} \sum_{j=1}^{N} \kappa_{ij} \left( y_j - \frac{\langle y_i, y_j \rangle}{\langle y_i, y_i \rangle} y_i \right).
\]

Thus, we have
\[
x_i(t) = R_{\Omega}(t) \circ L(t)x_i^0.
\]

\[\square\]

**Remark 1.** 1. By Lemma 2.4, it suffices to consider the following Lohe system with \( \Omega_i = 0, \quad 1 \leq i \leq N, \quad r = 1: \)
\[
\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \langle x_i, x_j \rangle x_i \right), \quad t > 0,
\]
\[
\kappa_{ij} = \mu \Gamma(x_j - x_i) - \gamma \kappa_{ij},
\]
for the complete aggregation analysis.

2. For \( d = 1 \), system (1) can be reduced to the Kuramoto model under the following setting: for \( i, j = 1, \cdots, N, \)
\[
x_i := \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix}, \quad \kappa_{ij} = \kappa \quad \text{and} \quad \Omega_i := \begin{pmatrix} 0 & -\nu_i \\ \nu_i & 0 \end{pmatrix}.
\] (7)

Then, we use the relations \( \dot{x}_i = \dot{\theta}_i x_i^+ \), \( |x_i| = 1 \) to see that system (7) becomes
\[
\dot{\theta}_i x_i^+ = \Omega_i x_i + \frac{\kappa}{N} \sum_{j=1}^{N} \left( x_j - \langle x_i, x_j \rangle x_i \right).
\] (8)

We take the inner product of (8) and \( x_i^+ \) to find
\[
\langle \dot{\theta}_i x_i^+, x_i^+ \rangle = \langle \Omega_i x_i, x_i^+ \rangle + \frac{\kappa}{N} \sum_{j=1}^{N} \left( \langle x_j, x_i^+ \rangle - \langle x_i, x_j \rangle \langle x_i, x_i^+ \rangle \right).
\] (9)

Now, we use (9) and the relations:
\[
\langle x_i^+, x_i^+ \rangle = 1, \quad \langle \Omega_i x_i, x_i^+ \rangle = \nu_i, \quad ||x_i||^2 = 1, \quad \langle x_j, x_i^+ \rangle = \sin(\theta_j - \theta_i), \quad \langle x_i, x_i^+ \rangle = 0
\]
to get the Kuramoto model:
\[
\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i).
\] (10)
2.2. **Previous results.** As noticed in Remark 1, the Kuramoto model (10) can be understood as a special case of the swarm sphere model on the unit sphere. Next, we briefly review previous results on the Kuramoto model with adaptive couplings.

Consider the generalized Kuramoto model with the adaptive couplings [19]:

\[
\dot{\theta}_i = \nu_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \\
\dot{k}_{ij} = \mu \Gamma(\theta_j - \theta_i) - \gamma k_{ij},
\]

(11)

where \(\mu\) and \(\gamma\) are nonnegative constants proportional to the learning enhancement rate and friction, respectively and \(\Gamma\) is a feedback law satisfying the parity and periodicity conditions:

\[
\Gamma(-\theta) = \Gamma(\theta), \quad \Gamma(\theta + 2\pi) = \Gamma(\theta), \quad \theta \in \mathbb{R}.
\]

Note that for \(\mu = \gamma = 0\) and \(k_{0ij}^0 = \frac{k}{N}\), system (11) reduces to the Kuramoto model [22, 23]:

\[
\dot{\theta}_i = \nu_i + \frac{k}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0.
\]

In this case, the emergent dynamics of the Kuramoto model [22, 23] has been extensively studied in [1, 4, 8, 9, 11, 12, 13, 17, 15, 21, 35, 36] and we refer the reader to recent survey articles [12, 16] in relation with the complete synchronization.

2.2.1. **A Hebbian model.** Consider the Kuramoto model with a Hebbian adaptive rule:

\[
\dot{\theta}_i = \nu_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \\
\dot{k}_{ij} = \mu \cos(\theta_j - \theta_i) - \gamma k_{ij}.
\]

(12)

Then, the emergent dynamics for (12) can be summarized as follows.

**Theorem 2.5.** [19] (Identical particles) Suppose that the natural frequency vector and the initial data \(\Theta^0\) satisfy

\[
\nu_i = 0, \quad 1 \leq i \leq N, \quad D(\Theta^0) < \frac{\pi}{2}, \quad \mu > 0, \quad \gamma > 0.
\]

Then, for any solution \(\Theta = \Theta(t)\) to (12), there exists a positive number \(t_1 > 0\) such that

\[
(i) \|\Theta(t)\| \leq \|\Theta(t_1)\| \exp \left[ -\frac{\mu \cos D(\Theta^0) \sin D(\Theta^0)}{2\gamma D(\Theta^0)} (t - t_1) \right], \quad t \geq t_1.
\]

\[
(ii) \|\Theta(t)\| \geq \|\Theta(t_1)\| \exp \left[ -\frac{2\mu}{\gamma} (t - t_1) \right].
\]

**Theorem 2.6.** [19] (Nonidentical particles) Suppose that the parameters \(\mu, \gamma\), the initial phase configuration and the coupling strength satisfy the following conditions:

\[
\frac{\mu}{\gamma} > k_{m}^0 := \min_{i,j} k_{ij}^0 > 0, \quad D^\infty = \arccos \left( \frac{\gamma k_{m}^0}{\mu} \right) \in \left( 0, \frac{\pi}{2} \right), \quad D(\Theta^0) < D^\infty,
\]

\[
\sum_{i=1}^{N} \theta_i^0 = 0, \quad \sum_{i=1}^{N} \nu_i = 0, \quad 0 < D(\nu) < \frac{k_{m}^0 N D(\Theta^0) \sin D^\infty}{D^\infty}.
\]
Then, for any solution $\Theta = \Theta(t)$ to (12), the following ‘practical synchronization’ is achieved:

$$\limsup_{t \to \infty} D(\Theta(t)) \leq \sqrt{2} \|\Omega\| D^{\infty} N \kappa_0 \sin D^{\infty}$$

2.2.2. An anti-Hebbian rule. Consider an adaptive Kuramoto model with an anti-Hebbian coupling:

$$\dot{\theta}_i = \nu_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0,$$

$$\dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}.$$ (13)

Next, we recall two emergent dynamics on (13) as follows.

**Theorem 2.7.** [19] (Identical oscillators) Let $\Theta = \Theta(t)$ be a global smooth solution to (13) satisfying

$$\nu_i = 0, \quad 1 \leq i \leq N \quad \text{and} \quad D(\Theta^0) < \frac{\pi}{2}.$$ Then, we have an asymptotic complete synchronization:

$$\lim_{t \to \infty} D(\Theta(t)) = 0, \quad \lim_{t \to \infty} D(\dot{\Theta}(t)) = 0, \quad \lim_{t \to \infty} \max_{1 \leq i, j \leq N} |k_{ij}(t)| = 0.$$

**Theorem 2.8.** [19] (Nonidentical oscillators) Let $\Theta = \Theta(t)$ be a solution to (13) with the a priori assumption:

$$\sup_{0 \leq t < \infty} D(\Theta(t)) < \frac{\pi}{2}.$$ Then, we have an asymptotic complete synchronization:

$$\lim_{t \to \infty} |D(\dot{\Theta})(t)| = 0, \quad 1 \leq i, j \leq N.$$ 3. Emergence of complete aggregation: Anti-Hebbian rule. In this section, we study an emergent dynamics of (1) with an anti-Hebbian rule $\Gamma_{ab}(x, y) = |x - y|^2$ from well-prepared initial data. As long as there is no confusion, from now on we assume that $x_i$ stays on the unit sphere:

$$|x_i| = 1, \quad i = 1, \ldots, N.$$ Consider the swarm sphere model on the unit sphere for identical particles:

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \langle x_i, x_j \rangle x_i \right), \quad t > 0,$$

$$\dot{\kappa}_{ij} = \mu |x_j - x_i|^2 - \gamma \kappa_{ij}, \quad |x_i| = 1.$$ (14)

Note that (14) can also be rewritten as

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - h_{ij} x_i \right), \quad t > 0.$$ (15)

For $i, j = 1, \ldots, N$ and $(X, K)$, we introduce the Lyapunov functionals $L_{ij}$:

$$L_{ij}(t) := \frac{1}{2} |x_i - x_j|^2 + \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2.$$ (16)
Note that the functional $\mathcal{L}_{ij}$ is always nonnegative. Using the angle functional $h_{ij} := \langle x_i, x_j \rangle$, $\mathcal{L}_{ij}(t)$ can be rewritten as follows.

$$\mathcal{L}_{ij}(t) = (1 - h_{ij}) + \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 \geq 0. \quad (17)$$

Next, we study the rate of changes for $\mathcal{L}_{ij}(t)$.

**Lemma 3.1.** Let $(X, K)$ be a solution to (14). Then, $\mathcal{L}_{ij}$ satisfies

$$\frac{d\mathcal{L}_{ij}}{dt} = -\frac{1}{2N} \sum_{k=1}^{N} (\kappa_{ik}h_{ik} + \kappa_{jk}h_{jk})|x_i - x_j|^2 - \frac{\gamma}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2, \quad t > 0. \quad (18)$$

**Proof.** Note that the unit modulus of $x_i$ and $x_j$ yields

$$|x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2\langle x_i, x_j \rangle = 2(1 - \langle x_i, x_j \rangle).$$

We use the above relations and (15) to get

$$\frac{d}{dt} \frac{1}{2} |x_i - x_j|^2 = -\frac{d}{dt} \langle x_i, x_j \rangle$$

$$= -\frac{1}{N} \sum_{k=1}^{N} \left[ \kappa_{ik}h_{jk} + \kappa_{jk}h_{ik} - (h_{ik}\kappa_{ik} + h_{jk}\kappa_{jk})h_{ij} \right]$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left[ (\kappa_{ik} - \kappa_{jk})(h_{jk} - h_{ik}) + (h_{ik}\kappa_{ik} + h_{jk}\kappa_{jk}) \frac{|x_i - x_j|^2}{2} \right], \quad (19)$$

and

$$\frac{d}{dt} \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 = \frac{1}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(\dot{\kappa}_{ik} - \dot{\kappa}_{jk})$$

$$= \frac{1}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(-\gamma\kappa_{ik} - 2\mu h_{ik} + \gamma\kappa_{jk} + 2\mu h_{jk}) \quad (20)$$

$$= -\frac{\gamma}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 - \frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(h_{ik} - h_{jk}).$$

Now, we use (16) and combine (19) and (20) to yield the desired estimate. \hfill \Box

**Remark 2.** Note that $\dot{\mathcal{L}}_{ij}$ does not have definite sign due to the indefinite sign of $h_{ij}$.

Next, we show that $\mathcal{L}_{ij}$ satisfies the non-increasing property under some restricted initial data.

**Proposition 1.** Suppose that the initial data $(X^0, K^0)$ satisfy the following relations: for $i, j = 1, \cdots, N$,

$$\max_{1 \leq i,j \leq N} \mathcal{L}_{ij}^0 = \max_{1 \leq i,j \leq N} \left[ (1 - h_{ij}^0) + \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik}^0 - \kappa_{jk}^0)^2 \right] < 1, \quad \min_{1 \leq i,j \leq N} \kappa_{ij}^0 > 0. \quad (21)$$

Then, for any solution $(X, K)$ to (14) and $i, j \in \{1, \cdots, N\}$, we have the following two assertions:
1. $h_{ij}$ and $\kappa_{ij}$ are strictly positive for all $t$: 
   $$h_{ij} > 0, \quad \kappa_{ij} > 0, \quad i, j = 1, \cdots, N.$$ 

2. $\mathcal{L}_{ij}$ is nonincreasing along the flow (14).

Proof. We will use the continuity argument. First, we define a set $\mathcal{T}$: 
   $$\mathcal{T} := \{T \in [0, \infty) : h_{ij}(t) > 0, \quad t \in (0, T)\}.$$ 
Due to the assumption (21), 
   $$h_{ij}^0 > \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik}^0 - \kappa_{jk}^0)^2 \geq 0,$$ 
and the continuity of the solution, we can see that the set $\mathcal{T}$ is nonempty, and we set 
   $$T_* := \sup T.$$ 
We claim: 
   $$T_* = \infty.$$ 
Suppose not, i.e., $T_* < \infty$. Then we have 
   $$\lim_{t \to T_*} h_{ij}(t) = 0.$$ 
Since $h_{ij} > 0$, $t \in (0, T_*)$, it follows from Lemma 3.1 that we have 
   $$1 - h_{ij}(t) \leq \mathcal{L}_{ij}(t) < \mathcal{L}_{ij}^0, \quad \text{so} \quad h_{ij}(t) > 1 - \mathcal{L}_{ij}^0, \quad t \in [0, T_*).$$ 
This implies 
   $$\lim_{t \to T_*} h_{ij}(t) \geq 1 - \mathcal{L}_{ij}^0 > 0,$$ 
which is contradictory to (22), i.e., 
   $$h_{ij}(t) > 0, \quad \text{for all } i, j = 1, \cdots, N \text{ and } t > 0.$$ 
On the other hand, since $\kappa_{ij}^0 > 0$ and $1 - h_{ij} \geq 0$, we have 
   $$\kappa_{ij}(t) = \kappa_{ij}^0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}(1 - h_{ij}(s))ds > 0.$$ 
Therefore, we have 
   $$h_{ij}(t) > 0 \quad \text{and} \quad \kappa_{ij}(t) > 0, \quad t > 0, \quad i, j = 1, \cdots, N,$$ 
which also yield the nonpositivity of the R.H.S. of (18). Hence, we can conclude that $\mathcal{L}_{ij}(t)$ is non-increasing. 

Next, we recall Barbalat’s lemma to be used throughout the paper.

Lemma 3.2. [3] Suppose that a real-valued function $f : [0, \infty) \to \mathbb{R}_+$ is uniformly continuous and satisfies 
   $$\lim_{t \to \infty} \int_0^t f(s)ds \quad \text{exists.}$$ 
Then, $f$ tends to zero as $t \to \infty$: 
   $$\lim_{t \to \infty} f(t) = 0.$$ 

As a direct corollary of Proposition 1 and Lemma 3.2, we next show that the coupling matrix $K$ becomes a constant multiple of the matrix $J_N$ with entries 1: 
   $$J_N = (a_{ij}), \quad a_{ij} = 1, \quad i, j = 1, \cdots, N.$$
Corollary 1. Let \((X, K)\) be a solution to (14) with the initial data satisfying 
\[
\max_{i,j} L^0_{ij} < 1 \quad \text{and} \quad \min_{i,j} \kappa^0_{ij} > 0.
\]
Then, we have 
\[
\lim_{t \to \infty} \max_{i,j,k} |\kappa_{ik}(t) - \kappa_{jk}(t)| = 0. \tag{23}
\]

Proof. Basically, we will use Barbalat’s lemma in (3.2) and estimate (18).

• Step A. We first show that 
\[
\int_0^\infty (\kappa_{ik} - \kappa_{jk})^2 \, ds < \infty, \quad \forall \, i, j, k = 1, \ldots, N. \tag{24}
\]
For given \(i, j\), it follows from Lemma 3.1 that we have 
\[
\mathcal{L}_{ij}(t) + \frac{1}{2N} \sum_{k=1}^N \int_0^t (\kappa_{ik} h_{ik} + \kappa_{jk} h_{jk}) |x_i - x_j|^2 \, ds + \frac{\gamma}{2\mu N} \sum_{k=1}^N \int_0^t (\kappa_{ik} - \kappa_{jk})^2 \, ds = \mathcal{L}^0_{ij}.
\]
This implies the desired estimate (24).

• Step B. In order to derive (23) from the integrability estimate (24), we use Barbalat’s lemma. For this, we set 
\[
f_{ij,k}(t) := (\kappa_{ik}(s) - \kappa_{jk}(s))^2 \geq 0.
\]
It suffices to check that \(f_{ij,k}\) is uniformly continuous. For the uniform continuity of \(f_{ij,k}(t)\), we need to check the uniform boundedness of \(\dot{f}_{ij,k} = 2(\kappa_{ik}(t) - \kappa_{jk}(t))(\dot{\kappa}_{ik}(t) - \dot{\kappa}_{jk}(t))\) which is implied by the uniform boundedness of \(\kappa_{ij}(t)\) and (14)_2. For this, we use (14)_2 to get 
\[
\frac{d}{dt} \kappa_{ij} = \mu |x_i - x_j|^2 - \gamma \kappa_{ij} \leq 2\mu - \gamma \kappa_{ij}.
\]
By Gronwall’s lemma, we have the uniform boundedness of \(\kappa_{ij}\).
\[
\kappa_{ij}(t) \leq \kappa^0_{ij} e^{-\gamma t} + 2\mu (1 - e^{-\gamma t}) \leq \kappa^0_{ij} + 2\mu. \tag{25}
\]
Thus, \(\kappa_{ij}\) is uniformly bounded and so is \(\dot{\kappa}_{ij}\). Hence, \(f_{ij,k}\) is uniformly bounded. Now, we apply Lemma 3.2 to see 
\[
\lim_{t \to \infty} (\kappa_{ik} - \kappa_{jk})^2 = 0.
\]

Next, we recall the Gronwall type lemma to be used later.

Lemma 3.3. [6] Let \(y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}\) be a \(C^1\)-function satisfying the following differential inequality: 
\[
y' \leq -\alpha y + f, \quad t > 0, \quad y(0) = y_0,
\]
where \(\alpha\) is a positive constant and \(f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}\) is a continuous function decaying to zero as its argument goes to infinity. Then \(y\) satisfies 
\[
y(t) \leq \frac{1}{\alpha} \max_{s \in [t/2, t]} |f(s)| + y_0 e^{-\alpha t} + \|f\|_{L^\infty} e^{-\frac{\alpha}{2} t}, \quad t \geq 0.
\]

Proof. The proof can be found in Appendix A of [6].

Finally, we arrive at our first main result on the emergence of the complete aggregation.
Theorem 3.4. Let \((X, K)\) be a solution to (14) with initial data \((X^0, K^0)\) satisfying
\[
\max_{1 \leq i, j \leq N} L_{ij}^0 < 1 \quad \text{and} \quad \min_{1 \leq i, j \leq N} \kappa_{ij}^0 > 0.
\]
Then for every \(i, j = 1, \ldots, N\), we have
\[
\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \quad \text{and} \quad \lim_{t \to \infty} \kappa_{ij}(t) = 0.
\]

Proof. • Step A (Derivation of Gronwall’s inequality for \(L_{ij}\)): It follows from Proposition 1 that we have
\[
1 - h_{ij}(t) < L_{ij}^0 \quad \text{or equivalently} \quad h_{ij}(t) > 1 - L_{ij}^0, \quad t \in [0, \infty).
\]
Then, the relation (18) and \(\kappa_{ij} > 0\) imply
\[
\frac{dL_{ij}}{dt} < - \frac{1 - L_{ij}^0}{2N} \sum_{k=1}^{N} (\kappa_{ik} + \kappa_{jk}) |x_i - x_j|^2 - \frac{\gamma}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 - \frac{1}{4\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 \cdot 2\gamma.
\]
This yields
\[
L_{ij}(t) \leq L_{ij}^0 \exp \left( - \int_0^t \Lambda_m(s) ds \right), \quad t \geq 0.
\]

• Step B (Derivation of estimates (26)): Depending on the integrability of \(\Lambda_m\), consider the following two cases:

○ Case B.1: Suppose that \(\Lambda_m\) is not integrable:
\[
\lim_{t \to \infty} \int_0^t \Lambda_m(s) ds = \infty.
\]
In this case, the relation (29) yields
\[
\lim_{t \to \infty} L_{ij}(t) = 0.
\]
By definition of \(L_{ij}\) and (30), we have
\[
\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0.
\]
Next, we also show that \(\kappa_{ij}\) tends to zero asymptotically. For this, it follows from the relation (14) that we have
\[
\dot{\kappa}_{ij} = -\gamma \kappa_{ij} + \mu |x_i - x_j|^2, \quad t > 0.
\]
Then, we apply Lemma 3.3 using the relations (31) and (32) to get
\[
\lim_{t \to \infty} \kappa_{ij}(t) = 0.
\]
Case B.2: Suppose that $\Lambda_m$ is integrable. Then, the relation (28) yields
$$\int_0^\infty \Lambda_m(t)dt < \infty,$$
or, equivalently,$$
\int_0^\infty (\kappa_{ik}(t) + \kappa_{jk}(t)) dt < \infty.$$
In this case, we can apply Barbalat’s lemma to get
$$\lim_{t \to \infty} \kappa_{ik}(t) = \lim_{t \to \infty} \kappa_{jk}(t) = 0.$$We use (25) to see that $\dot{h}_{ij}$ and $\dot{\kappa}_{ij}$ are uniformly bounded:
$$\dot{h}_{ij} \leq \frac{2}{N} \sum_{k=1}^{N} (\kappa_{ik}^0 + 2\mu)(\kappa_{jk}^0 + 2\mu),$$
$$\dot{\kappa}_{ij} \leq 2\mu + \gamma(\kappa_{ij}^0 + 2\mu).$$
(33)
Then, we use (33) and differentiate the dynamics of $\kappa_{ij}$ in (14) to yield that $\ddot{\kappa}_{ij}$ is also uniformly bounded:
$$|\ddot{\kappa}_{ij}| = | - \gamma \dot{\kappa}_{ij} - 2\mu \dot{h}_{ij}| \leq \gamma(2\mu + \gamma \kappa_{ij}^0 + 2\mu \gamma) + \frac{4\mu}{N} \sum_{k=1}^{N} (\kappa_{ik}^0 + 2\mu)(\kappa_{jk}^0 + 2\mu).$$Hence we apply Barbalat’s lemma to conclude that
$$\lim_{t \to \infty} \kappa_{ik}(t) = \lim_{t \to \infty} \kappa_{jk}(t) = 0.$$Finally we use the dynamics of $\kappa_{ij}$ in (14) to conclude that
$$\lim_{t \to \infty} |x_i - x_j| = 0.$$\[\square\]

Remark 3. Note that our convergence estimates does not yield explicit convergence rate due to the use of Barbalat’s lemma.

4. Emergence of complete aggregation: Hebbian rule. In this section, we consider two Hebbian rules denoted by $\Gamma_{h,p}$ and $\Gamma_{h,g}$. The former case is called the “positive Hebbian rule” in the sense that $\Gamma_{h,p}$ takes positive values in its domain, and the latter case $\Gamma_{h,g}$ is called the “general Hebbian rule” where adaptive law can attain both positive and negative values in its domain. For a positive Hebbian rule, we motivate its structure based on the cosine function, and for a general Hebbain rule, we take the following explicit ansatz:
$$\Gamma_{h,g}(x,y) = \langle x, y \rangle,$$or $$\Gamma_{h,g}(x,y) = 1 - \frac{|x - y|^2}{2}.$$In the following two subsections, we will show that the complete aggregation can occur for both positive and general Hebbian rules.

4.1. A positive Hebbian rule. In this subsection, we present aggregation estimate for positive Hebbian rule $\Gamma_{h,p}$ satisfying the following structural conditions:
$$\Gamma_{h,p}(x,y) = \bar{\Gamma}(|x - y|), \quad \bar{\Gamma}(0) \neq 0, \quad r_* := \inf\{r \geq 0 : \bar{\Gamma}(r) = 0\} > 2,$$
$$\bar{\Gamma}(\cdot) \text{ is non-increasing on the interval } [0, r_*].$$
(34)
We first present basic estimates for the coupling matrix.

Lemma 4.1. Suppose that the feedback law $\Gamma_{h,p}$ satisfies the structural relations (34), and let $(X, K)$ be a solution to (1). Then, there exist a positive time $t_*$ such that
$$\kappa_m := \frac{1}{2} \frac{\mu \bar{\Gamma}(2)}{\gamma} \leq \kappa_{ij}(t) \leq \frac{3}{2} \frac{\mu \bar{\Gamma}(0)}{\gamma} =: \kappa_M \quad \forall \ t \geq t_*.$$
Proof. Since \(0 \leq |x_i - x_j| \leq 2\), we have
\[
\bar{\Gamma}(2) \leq \bar{\Gamma}(|x_i - x_j|) \leq \bar{\Gamma}(0).
\]
We use the above relations and (1) to obtain
\[
-\gamma \kappa_{ij} + \mu \bar{\Gamma}(2) \leq \dot{\kappa}_{ij} \leq -\gamma \kappa_{ij} + \mu \bar{\Gamma}(0).
\]
Then, Gronwall’s lemma implies
\[
\left| \kappa_{0ij} - \frac{\mu \bar{\Gamma}(0)}{\gamma} \right| e^{-\gamma t} + \frac{\mu \bar{\Gamma}(0)}{\gamma}, \quad \forall \, t \geq 0.
\]
Finally, we set
\[
\kappa_m := \frac{1}{2} \frac{\mu \bar{\Gamma}(2)}{\gamma}, \quad \kappa_M := \frac{3}{2} \frac{\mu \bar{\Gamma}(0)}{\gamma}
\]
to get the desired estimate.  

Thanks to Lemma 4.1, we have positive lower and upper bounds for coupling strengths.

Now, we are ready to present our second main result on the complete aggregation.

**Theorem 4.2.** Suppose that the initial data and the feedback law \(\bar{\Gamma}\) in (34) satisfy
\[
\rho := \frac{\kappa_M}{\kappa_m} = \frac{3 \bar{\Gamma}(0)}{\bar{\Gamma}(2)} \in (1, 2), \quad \min_{1 \leq i,j \leq N} h_{ij}^0 > \frac{2 - \rho}{\rho}, \quad (35)
\]
and let \((X, K)\) be a solution to (1) with (34). Then, there exist positive constants \(C\) and \(\lambda\) such that
\[
\max_{i,j} |x_i(t) - x_j(t)| \leq C e^{-\lambda t}, \quad \text{as } t \to \infty.
\]

Proof. First, note that
\[
|\dot{h}_{ij}| = |x_i - x_j|^2 = 2(1 - h_{ij}), \quad 1 \leq i, j \leq N.
\]
Thus, it suffices to show the exponential zero convergence of \(1 - h_{ij}\) as \(t \to \infty\). For this, we set
\[
\Delta(H) := \max_{1 \leq i,j \leq N} (1 - h_{ij}).
\]
Since \(h_{ij} \leq 1\), the functional \(\Delta(H)\) is always nonnegative, and we will show that \(\Delta(H)\) decays to zero exponentially fast. Since it is Lipschitz continuous, for a given \(t \geq 0\), we set indices \(i, j\) depending on \(t\) such that
\[
h_{ij} = \max_{1 \leq k,l \leq N} (1 - h_{kl}).
\]
Then, for such \(i, j\), we use (15) to get
\[
\frac{dh_{ij}}{dt} = \frac{1}{N} \sum_{k=1}^{N} \left[ h_{jk} \kappa_{ik} + h_{ik} \kappa_{jk} - (h_{ik} \kappa_{ik} + h_{jk} \kappa_{jk}) h_{ij} \right].
\]
After tedious algebraic manipulation, we have
\[
\frac{d}{dt}(1 - h_{ij}) = -(\kappa_i^c + \kappa_j^c)(1 - h_{ij})
\]
\[
+ \frac{1}{N} \sum_{k=1}^{N} \left( \kappa_{ik}(1 - h_{ik}) + \kappa_{jk}(1 - h_{jk}) \right)(1 - h_{ij})
\]
\[
+ \frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(1 - h_{jk}) - (1 - h_{ik})
\]
\[
=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13},
\]
where
\[
\kappa_i^c := \frac{1}{N} \sum_{\ell=1}^{N} \kappa_{i\ell} \quad \text{and} \quad \kappa_j^c := \frac{1}{N} \sum_{\ell=1}^{N} \kappa_{j\ell}.
\]
Below, we estimate the terms \(\mathcal{I}_{1i}\) separately.

- (Estimate of \(\mathcal{I}_{11}\)): We use Lemma 4.1 to find
  \[
  \kappa_i^c + \kappa_j^c \geq 2 \kappa_m.
  \]
  This yields
  \[
  \mathcal{I}_{11} \leq -2 \kappa_m \Delta(H).
  \] (36)

- (Estimate of \(\mathcal{I}_{12}\)): We again use Lemma 4.1 to get
  \[
  \left( \kappa_{ik}(1 - h_{ik}) + \kappa_{jk}(1 - h_{jk}) \right)(1 - h_{ij}) \leq 2 \kappa_M (\Delta(H))^2.
  \]
  This yields
  \[
  \mathcal{I}_{12} \leq 2 \kappa_M (\Delta(H))^2.
  \] (37)

- (Estimate of \(\mathcal{I}_{13}\)): Since
  \[
  (\kappa_{ik} - \kappa_{jk})(1 - h_{jk}) - (1 - h_{ik}) \leq 2(\kappa_M - \kappa_m)\Delta(H),
  \]
  we have
  \[
  \mathcal{I}_{13} \leq 2(\kappa_M - \kappa_m)\Delta(H).
  \] (38)

Finally, we combine all estimates (36), (37) and (38) to obtain a Riccati differential inequality:
\[
\frac{d}{dt} \Delta(H) \leq -2 \kappa_m \Delta(H) + 2 \kappa_M \Delta(H)^2 + 2(\kappa_M - \kappa_m)\Delta(H)
\]
\[
= -2(2 \kappa_m - \kappa_M)\Delta(H) + 2 \kappa_M (\Delta(H))^2.
\]
Then, by the comparison principle of ODEs and explicit representation of solutions to the Riccati equation, we have
\[
\Delta(H(t)) \leq \frac{1}{\left( \frac{\kappa_M}{2 \kappa_m - \kappa_M} + e^{2(2 \kappa_m - \kappa_M)t} \left( \frac{1}{2 \kappa_M} - \frac{\kappa_M}{2 \kappa_m - \kappa_M} \right) \right)},
\] (39)

where
\[
\Delta(H^0) = 1 - \min_{1 \leq i, j \leq N} h_{ij}^0,
\]
and the two terms inside the parentheses in the denominator are positive due to the assumptions (35). Finally, we take the limit \(t \to \infty\) to the both sides of (39) to obtain the desired result.
Remark 4. We recall the dynamics of $\kappa_{ij}$ with (34):

$$\dot{\kappa}_{ij} = -\gamma \kappa_{ij} + \bar{\Gamma}(|x_i - x_j|),$$

and this can be written as

$$\left(\kappa_{ij} - \frac{\mu}{\gamma} \bar{\Gamma}(0)\right)' = -\gamma \left(\kappa_{ij} - \frac{\mu}{\gamma} \bar{\Gamma}(0)\right) + \mu \left(\bar{\Gamma}(|x - y|) - \bar{\Gamma}(0)\right).$$

It follows from Theorem 4.2 that the second term in the right-hand side converges to zero. Then, Lemma 3.3 yields

$$\lim_{t \to \infty} \kappa_{ij}(t) = \frac{\mu}{\gamma} \bar{\Gamma}(0), \text{ for all } i, j = 1, \cdots, N.$$

4.2. A general Hebbian rule. In this subsection, we present the complete aggregation estimate for an adaptive rule:

$$\Gamma_{h,g}(x_i - x_j) = (x_i, x_j) \quad \text{or equivalently} \quad \Gamma_{h,g}(x_i - x_j) = 1 - \frac{|x_i - x_j|^2}{2}. \quad (40)$$

Note that compared to the adaptive rule (34) which always attain positive values, this type (40) can be both positive and negative. Then, the system reads as

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left(x_j - (x_i, x_j)x_i\right), \quad t > 0, \quad (41)$$

$$\dot{\kappa}_{ij} = \mu (x_i, x_j) - \gamma \kappa_{ij}, \quad |x_i| = 1, \ 1 \leq i \leq N.$$ 

We define the functionals for (41) as

$$\tilde{L}_{ij}(t) := \frac{1}{2} |x_i - x_j|^2 - \frac{1}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2. \quad (42)$$

Since $\tilde{L}_{ij}(t)$ is defined as the difference of two positive functionals, the sign of $\tilde{L}_{ij}(t)$ is indefinite unlike that of $L_{ij}$. Next, we study the rate of change of $\tilde{L}_{ij}$.

Lemma 4.3. Let $(X, K)$ be a solution to (41). Then $\tilde{L}_{ij}$ satisfies

$$\frac{d}{dt} \tilde{L}_{ij} = -\frac{1}{2N} \sum_{k=1}^{N} (\kappa_{ik} h_{ik} + \kappa_{jk} h_{jk}) |x_i - x_j|^2 + \frac{\gamma}{\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2, \quad t > 0.$$ 

Proof. We estimate the terms in the R.H.S. of (42) separately below. Note that

$$\frac{d}{dt} \frac{1}{2} |x_i - x_j|^2 = \frac{d}{dt} (x_i, x_j)$$

$$= -\frac{1}{N} \sum_{k=1}^{N} \left(\kappa_{ik} h_{jk} + \kappa_{jk} h_{ik} - (h_{ik} \kappa_{ik} + h_{jk} \kappa_{jk}) h_{ij}\right)$$

$$= -\frac{1}{N} \sum_{k=1}^{N} \left((\kappa_{ik} - \kappa_{jk}) (h_{jk} - h_{ik}) + (h_{ik} \kappa_{ik} + h_{jk} \kappa_{jk}) \frac{|x_i - x_j|^2}{2}\right) \quad (43)$$

and

$$\frac{1}{2\mu N} \frac{d}{dt} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 = \frac{1}{\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk}) (\dot{\kappa}_{ik} - \dot{\kappa}_{jk})$$
\begin{align*}
&= \frac{1}{\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(-\gamma \kappa_{ik} + \mu h_{ik} + \gamma \kappa_{jk} - \mu h_{jk}) \\
&= \frac{1}{\mu N} \sum_{k=1}^{N} \frac{-\gamma (\kappa_{ik} - \kappa_{jk})^2 + \mu (\kappa_{ik} - \kappa_{jk})(h_{ik} - h_{jk})}{N} \\
&= -\frac{\gamma}{\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 + \frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(h_{ik} - h_{jk}).
\end{align*}

We use (43) - (44) to obtain

\[
\frac{d}{dt} \left( \frac{1}{2} |x_i - x_j|^2 - \frac{1}{2\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2 \right) = -\frac{1}{2N} \sum_{k=1}^{N} (\kappa_{ik} h_{ik} + \kappa_{jk} h_{jk}) |x_i - x_j|^2 + \frac{\gamma}{\mu N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})^2.
\]

For the adaptive rule \( \Gamma_{ah}(x, y) = |x - y|^2 \), we can associate the corresponding Lyapunov functionals as the sum of two positive functionals (see (18).) However, for the general Hebbian law (40), the sign of the functionals in (42) is indefinite. Hence, it is difficult to apply the Lyapunov functional method as it is.

Next, we introduce two frameworks (\( \mathcal{F}_A \)) and (\( \mathcal{F}_B \)) in which the complete aggregation will be verified.

(Framework (\( \mathcal{F}_A \))): Let \( \mu \) and \( \gamma \) be positive constants.

- (\( \mathcal{F}_A1 \)): The initial pairwise coupling strengths are moderately strong in the sense that
  \[
  \gamma > \frac{3}{2}, \quad \frac{2\mu \kappa_{ij}}{\gamma + 2\mu} < \kappa_{ij}^0 < \frac{2\mu (3 - 1)}{\gamma},
  \]
  \[
  \max_{i,j} \kappa_{ij}^0 > \frac{2\mu (\gamma - 1)}{\gamma}, \quad 1 \leq i, j \leq N.
  \]

- (\( \mathcal{F}_A2 \)): The initial configuration \( X^0 \) satisfies
  \[
  \Delta(H^0) < \frac{4\mu \gamma}{\gamma \kappa_{ij}^0 + 2\mu} - 1.
  \]

Next, we introduce another framework for the complete aggregation.

(Framework (\( \mathcal{F}_B \))): Let \( \mu \) and \( \gamma \) be positive constants.

- (\( \mathcal{F}_B1 \)): The initial pairwise coupling strengths are moderately strong in the sense that
  \[
  \frac{3}{4} < \gamma < \frac{3}{2}, \quad \frac{2\mu}{3} \kappa_{ij}^0 < \frac{\mu}{\gamma}, \quad 1 \leq i, j \leq N.
  \]

- (\( \mathcal{F}_B2 \)): The initial configuration \( X^0 \) satisfy
  \[
  \Delta(H^0) < \frac{4}{3} \left( \gamma - \frac{3}{4} \right).
  \]
Next, we introduce the extremal critical coupling strengths:

\[
\kappa_M := \max \left\{ \max_{i,j} (\kappa_{ij}^0), \frac{\mu}{\gamma} \right\}, \quad \kappa_m := \frac{2\mu \gamma \kappa_M}{\gamma \kappa_M + 2\mu}.
\] (45)

**Lemma 4.4.** Suppose that one of frameworks \((\mathcal{F}_A)\) and \((\mathcal{F}_B)\) hold, and let \((X, K)\) be a solution to (41). Then, we have

\[
\frac{1}{2} \kappa_M < \kappa_m < \kappa_M \quad \text{and} \quad \kappa_{ij}^0 > \kappa_m, \quad 1 \leq i, j \leq N.
\] (46)

**Proof.** We use the relation (45) to see the equivalence:

\[
\frac{1}{2} \kappa_M < \kappa_m < \kappa_M \iff \frac{2\mu (\gamma - 1)}{\gamma} < \kappa_M < \frac{2\mu (2\gamma - 1)}{\gamma}.
\] (47)

(i) (Framework \((\mathcal{F}_A))\): Suppose that the framework \((\mathcal{F}_A)\) holds.

\(\diamond\) (Derivation of (46)_1): Since \(\gamma > \frac{3}{2}\), we have

\[
\frac{2\mu (\gamma - 1)}{\gamma} > \frac{\mu}{\gamma}
\] (48)

and from \((\mathcal{F}_A)_1\), we see that

\[
\max_{i,j} \kappa_{ij}^0 > \frac{2\mu (\gamma - 1)}{\gamma}
\] (49)

We combine (48) with (49) to conclude that

\[
\max_{i,j} \kappa_{ij}^0 > \frac{\mu}{\gamma}, \quad 1 \leq i, j \leq N
\]

which implies

\[
\kappa_M = \max_{i,j} \kappa_{ij}^0.
\] (50)

From the assumption \((\mathcal{F}_A)_1\) which holds for all \(i, j\) and (50), we show that \(\kappa_M\) satisfies relation (47).

\(\diamond\) (Derivation of (46)_2): We use the first relation of the framework \((\mathcal{F}_A)_2\), (50) and the definition of \(\kappa_m\) to see that (46)_2 holds.

(ii) (Framework \((\mathcal{F}_B))\) Suppose that the framework \((\mathcal{F}_B)\) holds.

\(\diamond\) (Derivation of (46)_1) From \((\mathcal{F}_B)_1\), we have \(\kappa_M = \frac{\mu}{\gamma}\). Since (51) is equivalent to

\[
\frac{2\mu (\gamma - 1)}{\gamma} < \frac{\mu}{\gamma} < \frac{2\mu (2\gamma - 1)}{\gamma} \iff 2(\gamma - 1) < 1 < 2(2\gamma - 1) \iff \frac{3}{4} < \gamma < \frac{3}{2},
\]

(46)_1 is verified.

\(\diamond\) (Derivation of (46)_2) Since (46)_2 is equivalent to

\[
\kappa_{ij}^0 > \kappa_m = \frac{2\mu \gamma \cdot \frac{\mu}{\gamma}}{\gamma \cdot \frac{\mu}{\gamma} + 2\mu} = \frac{2\mu}{3},
\]

(46)_2 is verified from \((\mathcal{F}_B)_2\). □

In the following theorem, we show that the exponential aggregation can occur under the above two frameworks.
Theorem 4.5. Suppose that exactly one of frameworks \((F_A)\) and \((F_B)\) hold, and let \((X,K)\) be a solution to (41). Then, the complete aggregation occurs exponentially, i.e., there exist positive constants \(C > 0\) and \(\lambda > 0\) such that
\[
\Delta(H(t)) \leq Ce^{-\lambda t} \quad \text{as } t \to \infty.
\]

Proof. Since the proof is rather lengthy, we split its proof into several steps below.

- **Step A** (Uniform upper and lower bounds for the coupling strength): We use (41) and a rough estimate \(h_{ij} \leq 1\) to see
  \[
  \dot{\kappa}_{ij} \leq -\gamma \kappa_{ij} + \mu, \quad t > 0.
  \]
  This implies
  \[
  \kappa_{ij}(t) \leq \left(\kappa_{ij}^0 - \frac{\mu}{\gamma}\right)e^{-\gamma t} + \frac{\mu}{\gamma}, \quad t \geq 0.
  \]
  Thus, we have a uniform upper bound for \(\kappa_{ij}\):
  \[
  \sup_{0 \leq t < \infty} \kappa_{ij}(t) \leq \kappa_M.
  \]
  For a uniform lower bound for the coupling strengths, we define a set \(T\) and its supremum as follows:
  \[
  T := \{T \in (0, \infty) : \kappa_{ij}(t) > \kappa_m, \quad \forall t \in (0, T)\}, \quad T_* := \sup T.
  \]

- **Step B** \((T_* = \infty)\): Next, we claim:
  \[
  \lim_{t \to T_*^-} \kappa_{ij}(t) = \kappa_m. \quad (51)
  \]
  For a given \(t > 0\), we choose indices \(i, j\) depending on \(t\) such that
  \[
  1 - h_{ij} = \Delta(H) := \max_{1 \leq k,l \leq N}(1 - h_{kl}).
  \]
  Then, for such \(i\) and \(j\), \(1 - h_{ij}\) satisfies
  \[
  \frac{d}{dt}(1 - h_{ij}) = -\frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik} + \kappa_{jk})(1 - h_{ij})
  \]
  \[
  + \frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik}(1 - h_{ik}) + \kappa_{jk}(1 - h_{jk}))(1 - h_{ij})
  \]
  \[
  + \frac{1}{N} \sum_{k=1}^{N} (\kappa_{ik} - \kappa_{jk})(h_{ik} - h_{jk}). \quad (52)
  \]
  In (52), we use the arguments similar to that in the proof of Theorem 4.2 using the relation:
  \[
  \kappa_{ij} \geq \kappa_m, \quad |\kappa_{ik} - \kappa_{jk}| < \kappa_M - \kappa_m, \quad |h_{ik} - h_{jk}| < 2\Delta(H), \quad t \in (0, T_*),
  \]
  to obtain
  \[
  \frac{d}{dt}\Delta(H) \leq -2\kappa_m\Delta(H) + 2\kappa_M(\Delta(H))^2 + 2(\kappa_M - \kappa_m)\Delta(H)
  \]
  \[
  = -2(2\kappa_m - \kappa_M)\Delta(H) + 2\kappa_M(\Delta(H))^2, \quad t \in [0, T_*].
  \]
  \[
  (53)
  \]
Note that the coefficient $2(2\kappa_m - \kappa_M)$ of $\Delta(H)$ in the R.H.S. of (53) is strictly negative under our two frameworks (see (46)). Now we use the comparison principle and the initial condition
\[
\Delta(H^0) < \frac{2\kappa_m - \kappa_M}{\kappa_M} = \frac{2\kappa_m}{\kappa_M} - 1 \tag{54}
\]
to get
\[
\Delta(H(t)) \leq \frac{2\kappa_m - \kappa_M}{\kappa_M}, \quad t \in (0, T_*). \tag{55}
\]
Next, we use the dynamics of $\kappa_{ij}(t)$ and (55) to obtain
\[
\dot{\kappa}_{ij} = -\kappa_{ij} + \mu - \mu(1 - h_{ij}) \geq -\gamma \kappa_{ij} + \mu - \mu \Delta(H) \\
\geq -\gamma \kappa_{ij} + \frac{2\mu(\kappa_M - \kappa_m)}{\kappa_M}.
\]
We integrate the above relation using the integrating factor to get
\[
\kappa_{ij} \geq \left(\kappa_{ij}^0 - \frac{2\mu \kappa_M - \kappa_m}{\gamma \kappa_M}\right) e^{-\gamma t} + \frac{2\mu}{\gamma} \left(\frac{\kappa_M - \kappa_m}{\kappa_M}\right), \quad t \in [0, T_*). \tag{56}
\]
• **Step C** (Exponential decay estimate of $\Delta(H)$): Note that the defining condition (45) yields
\[
\frac{2\mu}{\gamma} \left(\frac{\kappa_M - \kappa_m}{\kappa_M}\right) = \kappa_m.
\]
Then, relation (56) becomes
\[
\kappa_{ij}(t) \geq (\kappa_{ij}^0 - \kappa_m) e^{-\gamma t} + \kappa_m, \quad t \in [0, T_*).
\]
In particular, we have
\[
\lim_{t \to T_*} \kappa_{ij}(t) \geq (\kappa_{ij}^0 - \kappa_m) e^{-\gamma T_*} + \kappa_m > \kappa_m,
\]
which is contradictory to (51). Thus $T_* = \infty$ and the following Riccati type differential inequality holds for $\Delta(H)$:
\[
\frac{d}{dt} \Delta(H) \leq -2(2\kappa_m - \kappa_M) \Delta(H) + 2\kappa_M (\Delta(H))^2, \quad t \in (0, \infty).
\]
Now, consider the Riccati differential equation:
\[
\frac{dZ}{dt} = -2(2\kappa_m - \kappa_M)Z + 2\kappa_M Z^2, \quad t > 0, \quad Z(0) = \Delta(H^0),
\]
which can be integrated to give
\[
Z(t) = \frac{1}{\left(\frac{1}{\Delta(H^0)} - \frac{2\kappa_m - \kappa_M}{\kappa_M}\right) e^{(2\kappa_m - \kappa_M)t} + \frac{2\kappa_m - \kappa_M}{\kappa_M}}, \quad t > 0. \tag{57}
\]
On the other hand, it follows from the comparison principle that
\[
\Delta(H(t)) \leq Z(t), \quad t \in [0, \infty). \tag{58}
\]
Finally, we combine (57) and (58) to obtain
\[
\Delta(H) \leq \frac{1}{\left(\frac{1}{\Delta(H^0)} - \frac{2\kappa_m - \kappa_M}{\kappa_M}\right) e^{(2\kappa_m - \kappa_M)t} + \frac{2\kappa_m - \kappa_M}{\kappa_M}}, \quad t > 0,
\]
which yields the exponential decay of $\Delta(H)$. Furthermore, we check that the initial condition in (54)
\[
\Delta(H^0) < \frac{2\kappa_m}{\kappa_M} - 1 = \frac{4\mu \gamma}{\gamma \kappa_M + 2\mu} - 1
\]
is equivalent to \((F_A^2)\) when \(\kappa_M = \max_{i,j} \kappa_{ij}^0\) and to \((F_B^2)\) when \(\kappa_M = \frac{\mu}{\gamma}\).

**Remark 5.** Note that in Theorem 3.4, we employed the adaptive law \(\Gamma(s) = |s|^2\). Thus, we do not know the exact decay rate of \(\Delta(H)\), since we use Barbalat’s lemma to obtain the desired limit. However in Theorems 4.2 and 4.5, we can guarantee that the decay rate is exponential due to the uniform positiveness of \(\kappa_{ij}(t)\) for all time. This explicit exponential decay of \(\Delta(H)\) will be crucially used in the uniform stability analysis in Section 6.

5. **Emergence of bi-polar aggregation.** In this section, we study the emergence of bi-polar aggregation of the swarm sphere model with the Hebbian adaptive rule in Section 4.2:

\[ \Gamma_{h,g}(x_i - x_j) = \langle x_i, x_j \rangle. \]

Note that \(\Gamma_{h,g}\) does not have a definite sign, for example, when \(x_i\) and \(x_j\) are located on the different hemisphere, \(\Gamma_{h,g}\) takes a negative value. Thus, in the following two subsections, we will show that under some well-controlled situations, the bi-polar aggregation can emerge from the well-prepared initial configuration.

5.1. **A two-particle system.** Consider the two-particle system:

\[
\begin{align*}
\dot{x}_1 &= \kappa_{12}(x_2 - \langle x_2, x_1 \rangle x_1), \\
\dot{x}_2 &= \kappa_{21}(x_1 - \langle x_1, x_2 \rangle x_2), \\
\dot{\kappa}_{12} &= \mu \langle x_1, x_2 \rangle - \gamma \kappa_{12}, \\
\dot{\kappa}_{21} &= \mu \langle x_2, x_1 \rangle - \gamma \kappa_{21}.
\end{align*}
\]

We assume \(\kappa_{21} = \kappa_{12} = \kappa\) and set \(h := \langle x_1, x_2 \rangle\). Then, it follows from (59) that

\[
\dot{h} = \kappa(1 - h^2), \quad \dot{\kappa} = \mu h - \gamma \kappa.
\]

Note that system (60) has three equilibria \((\kappa_\infty, h_\infty)\):

\[
(\kappa_\infty, h_\infty) = \left( \frac{\mu}{\gamma}, 1 \right), \left( -\frac{\mu}{\gamma}, -1 \right), (0, 0).
\]

From the linear stability analysis, the first two equilibria are (asymptotically) stable and the last equilibrium is unstable. For the stability of the first two equilibria, see the vector field in Figure 1 below.

![Figure 1. Vector field of (60) for \(\mu = \gamma = 1\)](image)
5.2. A many-body system. In this subsection, we study a bi-polar aggregation estimate for the many-body system (41) with \( N \geq 3 \):

\[
\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \langle x_i, x_j \rangle x_i \right), \quad t > 0,
\]

\[
\kappa_{ij} = \mu(x_i, x_j) - \gamma \kappa_{ij}, \quad |x_i| = 1, \ 1 \leq i \leq N.
\]

As noticed in the two-particle system in the previous subsection, the bi-polar configuration can emerge from the well-prepared initial configuration. Throughout this subsection, the index letter \( i \) denotes an element of \( G_1 \), i.e., \( i \) runs from 1 to \( N_1 \) and the index letter \( j \) denotes an element of \( G_2 \), i.e., \( j \) runs from 1 to \( N_2 \). For the coupling strength \( \kappa^1_{ij} \) and \( \kappa^2_{ij} \), they are concerned with the same subgroups \( G_1 \) and \( G_2 \), respectively and for the \( \kappa^d_{ij} \), it is concerned with the other groups. For example, for the notation \( \kappa^1_{1j} \) and for \( j \in G_1 \), then we automatically realize that \( i \) also belongs to \( G_1 \). By the same way, for \( \kappa^d_{ij} \) and \( i \in G_1 \), then \( j \) must belong to \( G_2 \).

In order to describe a sufficient framework leading to the bi-polar aggregation, we assume that the system is composed of two sub-ensemble \( G_1 \) and \( G_2 \):

\[
|G_1| = N_1, \quad |G_2| = N_2 \quad \text{and} \quad N_1 + N_2 = N.
\]

We relabel the position of particles as \( \{x_{1i}\}_{i=1}^{N_1} \) and \( \{x_{2j}\}_{j=1}^{N_2} \), respectively, and the coupling matrix \( K = (\kappa_{\ell m}) \) is also represented as follows:

\[
\kappa^1_{\ell m} = -\gamma \kappa^1_{\ell m} + \mu(x_{1i}, x_{1\ell}), \quad i, \ell = 1, \ldots, N_1,
\]

\[
\kappa^2_{\ell m} = -\gamma \kappa^2_{\ell m} + \mu(x_{2j}, x_{2\ell}), \quad j, \ell = 1, \ldots, N_2,
\]

\[
\kappa^d_{ij} = -\gamma \kappa^d_{ij} + \mu(x_{1i}, x_{2j}), \quad i = 1, \ldots, N_1, \quad j = 1, \ldots, N_2.
\]

Note that system (61) can be rewritten as follows. For \( i = 1, \ldots, N_1 \) and \( j = 1, \ldots, N_2 \),

\[
\dot{x}_{1i} = \frac{1}{N} \sum_{\ell=1}^{N_1} \kappa^1_{\ell i} \left( x_{1\ell} - \langle x_{1i}, x_{1\ell} \rangle x_{1i} \right) + \frac{1}{N} \sum_{\ell=1}^{N_2} \kappa^d_{i\ell} \left( x_{2\ell} - \langle x_{1i}, x_{2\ell} \rangle x_{1i} \right),
\]

\[
\dot{x}_{2j} = \frac{1}{N} \sum_{\ell=1}^{N_2} \kappa^2_{\ell j} \left( x_{2\ell} - \langle x_{2j}, x_{2\ell} \rangle x_{2j} \right) + \frac{1}{N} \sum_{\ell=1}^{N_1} \kappa^d_{j\ell} \left( x_{1\ell} - \langle x_{2j}, x_{1\ell} \rangle x_{2j} \right).
\]

Since we are interested in the bi-polar aggregation, we look for a framework in which the particles in the same sub-ensemble \( G_i \), \( i = 1, 2 \) aggregate to the same position:

\[
\lim_{t \to \infty} \langle x_k(t), x_l(t) \rangle = 1, \quad (k, l) \in (G_1 \times G_1) \cup (G_2 \times G_2),
\]

whereas for the particles belonging to different sub-ensembles,

\[
\lim_{t \to \infty} \langle x_k(t), x_l(t) \rangle = -1, \quad (k, l) \in G_1 \times G_2.
\]

To quantify (63) and (64), we introduce the angle functionals and maximal functionals:

\[
h^d_{\ell m} := \langle x_{\alpha k}, x_{\alpha l} \rangle, \quad \alpha = 1, 2, \ 1 \leq k, l \leq N_\alpha,
\]

\[
h^d_{\ell m} := \langle x_{1k}, x_{2l} \rangle, \quad 1 \leq k \leq N_1, \ 1 \leq l \leq N_2,
\]

\[
\Delta(H^1) := \max_{1 \leq i, j \leq N} \left( 1 - \langle x_{1i}, x_{1j} \rangle \right), \quad \Delta(H^2) := \max_{1 \leq i, j \leq N} \left( 1 - \langle x_{2i}, x_{2j} \rangle \right),
\]

\[
\Delta(H^d) := \max_{1 \leq i, j \leq N} \left( 1 + \langle x_{1i}, x_{2j} \rangle \right).
\]
Note that $\Delta(H^1)$ and $\Delta(H^2)$ measure the maximal distances between the particles in the sub-ensembles $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, and $\Delta(H^d)$ measures the maximal distance between the particles in $\mathcal{G}_1$ and $\mathcal{G}_2$. In this regard, we will design our framework leading to the estimate:

$$\lim_{t \to \infty} \left( \Delta(H^1) + \Delta(H^2) + \Delta(H^d) \right) = 0.$$ 

In the following lemma, we provide the rate of changes for the above functionals.

**Lemma 5.1.** Let $(X, K)$ be a solution to (61). Then, we have

$$\frac{d}{dt} (1 - h^2_{ik})$$

$$= -\frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{i\ell}^1 + \kappa_{k\ell}^1)(1 - h^2_{ik}) + \frac{1}{N} \sum_{\ell=1}^{N_1} \left( \kappa_{i\ell}^1(1 - h^1_{i\ell}) + \kappa_{k\ell}^1(1 - h^1_{k\ell}) \right) (1 - h^1_{ik})$$

$$+ \frac{1}{N} \sum_{\ell=1}^{N_1} (\kappa_{i\ell}^1 - \kappa_{k\ell}^1) \left( (1 - h^1_{i\ell}) - (1 - h^1_{k\ell}) \right)$$

$$+ \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{i\ell}^d + \kappa_{k\ell}^d)(1 - h^1_{ik}) - \frac{1}{N} \sum_{\ell=1}^{N_2} \left( \kappa_{i\ell}^d(1 + h^d_{i\ell}) + \kappa_{k\ell}^d(1 + h^d_{k\ell}) \right) (1 - h^1_{ik})$$

$$- \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{i\ell}^d - \kappa_{k\ell}^d) \left( (1 + h^d_{i\ell}) - (1 + h^d_{k\ell}) \right),$$

(65)

$$\frac{d}{dt} (1 - h^2_{jm})$$

$$= -\frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{j\ell}^2 + \kappa_{m\ell}^2)(1 - h^2_{jm}) + \frac{1}{N} \sum_{\ell=1}^{N_2} \left( \kappa_{j\ell}^2(1 - h^2_{j\ell}) + \kappa_{m\ell}^2(1 - h^2_{m\ell}) \right) (1 - h^2_{jm})$$

$$+ \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{j\ell}^2 - \kappa_{m\ell}^2) \left( (1 - h^2_{j\ell}) - (1 - h^2_{m\ell}) \right)$$

$$+ \frac{1}{N} \sum_{\ell=1}^{N_1} (\kappa_{j\ell}^d + \kappa_{m\ell}^d)(1 - h^2_{jm}) - \frac{1}{N} \sum_{\ell=1}^{N_1} \left( \kappa_{j\ell}^d(1 + h^d_{j\ell}) + \kappa_{m\ell}^d(1 + h^d_{m\ell}) \right) (1 - h^2_{jm})$$

$$- \frac{1}{N} \sum_{\ell=1}^{N_1} (\kappa_{j\ell}^d - \kappa_{m\ell}^d) \left( (1 + h^d_{j\ell}) - (1 + h^d_{m\ell}) \right),$$

(66)

and

$$\frac{d}{dt} (1 + h^d_{ij})$$

$$= \frac{1}{N} \sum_{\ell=1}^{N_1} (-\kappa_{i\ell}^d + \kappa_{j\ell}^d)(1 + h^d_{ij}) + \kappa_{i\ell}^d(1 + h^d_{i\ell})(1 - h^1_{i\ell}) - \kappa_{j\ell}^d(1 + h^d_{j\ell})(1 + h^d_{j\ell})$$

$$+ \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{i\ell}^d + \kappa_{j\ell}^d) \left( (1 + h^d_{i\ell}) - (1 - h^1_{i\ell}) \right)$$

(67)
\[ + \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{j\ell}^2 + \kappa_{i\ell}^d)(1 + h_{ij}^d) + \kappa_{j\ell}^2(1 + h_{ij}^d)(1 - h_{ij}^2) - \kappa_{i\ell}^d(1 + h_{ij}^d)(1 + h_{ij}^d) \]
\[ + \frac{1}{N} \sum_{\ell=1}^{N_2} (\kappa_{j\ell}^d + \kappa_{i\ell}^d)(1 + h_{ij}^d) \left( (1 + h_{ij}^d) - (1 - h_{ij}^2) \right). \]

**Proof.** The relations (65), (66) and (67) follow from direct algebraic manipulation. \qed

Now, we introduce a class of well-prepared initial framework which leads to the bi-polar aggregation state.

(The framework \( F_C \)) Let \( \mu \) and \( \gamma \) be positive constants.

- (\( F_C 1 \)): The initial pairwise coupling strengths are moderately strong in the sense that
  \[ \frac{\mu}{\gamma} < \frac{4}{9}, \quad \frac{9\mu^2 + 6\mu\gamma}{10\gamma^2 + 9\mu^2} < (\kappa_{ij}^a)^0, \quad \frac{\mu}{\gamma} < (\kappa_{ij}^d)^0 < -\frac{9\mu^2 + 6\mu\gamma}{10\gamma^2 + 9\mu^2}, \quad a = 1, 2. \]

- (\( F_C 2 \)): The initial position configuration satisfies
  \[ \Delta(H_1) + \Delta(H_0^2) + \Delta(H_0^d) < \frac{10\delta_1 - 6\rho}{9\rho^2}, \quad \rho := \frac{\mu}{\gamma}, \quad \delta_1 := \frac{9\mu^2 + 6\mu\gamma}{10\gamma^2 + 9\mu^2} = \frac{9\rho^2 + 6\rho}{9\rho + 10}. \]

(Framework \( F_D \)) Let \( \mu \) and \( \gamma \) be positive constants.

- (\( F_D 1 \)): The initial pairwise coupling strengths are moderately strong in the sense that
  \[ \frac{\mu}{\gamma} > \frac{4}{9}, \quad \frac{9\mu^2 + 16\mu\gamma}{24\gamma^2 + 9\mu^2} < (\kappa_{ij}^a)^0, \quad \frac{\mu}{\gamma} < (\kappa_{ij}^d)^0 < -\frac{9\mu^2 + 16\mu\gamma}{24\gamma^2 + 9\mu^2}, \quad a = 1, 2. \]

- (\( F_D 2 \)): The initial position configuration satisfies
  \[ \Delta(H_1) + \Delta(H_0^2) + \Delta(H_0^d) < \frac{20\delta_2 - 12\rho}{9\rho^2 + 4\rho}, \quad \rho := \frac{\mu}{\gamma}, \quad \delta_2 := \frac{9\mu^2 + 16\mu\gamma}{24\gamma^2 + 9\mu^2} = \frac{9\rho^2 + 16\rho}{9\rho + 24}. \]

**Lemma 5.2.** Suppose that one of frameworks (\( F_C \)) and (\( F_D \)) holds and let \( (X, K) \) be a solution to (61). Then, we have
\[ \frac{3\mu}{5\gamma} < \delta_a < \frac{\mu}{\gamma}, \quad a = 1, 2. \] \hspace{1cm} (68)

**Proof.** (i) (Framework (\( F_C \))): Suppose that framework (\( F_C \)) holds. Then, we use previous notation \( \rho \) to see the equivalence of (68)
\[ \frac{3\mu}{5\gamma} < \delta_a < \frac{\mu}{\gamma} \iff \frac{3}{5}\rho < \frac{9\rho^2 + 6\rho}{9\rho + 10} < \rho \iff \frac{3}{5} < \frac{9\rho + 6}{9\rho + 10} < 1. \]

Since \( \rho > 0 \), we can see that
\[ 27\rho + 10 < 45\rho + 30, \quad 9\rho + 6 < 9\rho + 10. \]

Hence, (68) is verified.

(ii) (Framework (\( F_D \))): Suppose that framework (\( F_D \)) holds. Then, we see the equivalence of (68)
\[ \frac{3\mu}{5\gamma} < \delta_a < \frac{\mu}{\gamma} \iff \frac{3}{5}\rho < \frac{9\rho^2 + 16\rho}{9\rho + 24} < \rho \iff \frac{3}{5} < \frac{9\rho + 16}{9\rho + 24} < 1. \]

Since \( \rho > 0 \), we see that
\[ 27\rho + 72 < 45\rho + 80, \quad 9\rho + 16 < 9\rho + 24. \]
Hence, (68) is verified.

**Theorem 5.3.** Suppose that exactly one of frameworks (FC) and (FD) hold, and let (X, K) be a solution to (61). Then, the complete aggregation occurs exponentially, i.e., there exist positive constants $\bar{C} > 0$ and $\lambda > 0$ such that

$$\Delta(H^1(t)) + \Delta(H^2(t)) + \Delta(H^d(t)) \leq \bar{C} e^{-\lambda t} \quad \text{as} \quad t \to \infty.$$  

**Proof.** We basically follow the strategy similar to that in the proof of Theorem 4.5.

- **Step A** (One-sided boundedness of coupling strengths): We define a set $\mathcal{T}$

  $$\tilde{T} := \{ T \in (0, \infty) : \kappa_{i\ell}^1(t) > \delta_a, \quad \kappa_{j\ell}^2(t) > \delta_a, \quad \kappa_{ij}^d(t) < -\delta_a, \quad t \in (0, T) \},$$

  where $\delta_a$, $a = 1, 2$ is defined in the frameworks (FC) and (FD), respectively. We use either (FC-1) or (FD-1) and the continuity of the coupling strengths to see that the set $\mathcal{T}$ should contain a small time interval $[0, \varepsilon)$, i.e., $\varepsilon \in \mathcal{T}$. Hence, we have $\mathcal{T} \neq \emptyset$, and $\tilde{T}_* = \sup \mathcal{T}$.

  It follows from Lemma 5.1 that for $t \in [0, T_*]$,

  $$\frac{d}{dt} \Delta(H^1) \leq -2 \left( 2\delta_a + \delta_a - \frac{\mu}{\gamma} \right) \Delta(H^1) + 2\frac{\mu}{\gamma} \Delta(H^1)^2 + 2 \left( \frac{\mu}{\gamma} - \delta_a \right) \Delta(H^d) + \frac{\mu}{\gamma} \Delta(H^d) \Delta(H^1),$$

  $$\frac{d}{dt} \Delta(H^2) \leq -2 \left( 2\delta_a + \delta_a - \frac{\mu}{\gamma} \right) \Delta(H^2) + 2\frac{\mu}{\gamma} \Delta(H^2)^2 + 2 \left( \frac{\mu}{\gamma} - \delta_a \right) \Delta(H^d) + \frac{\mu}{\gamma} \Delta(H^d) \Delta(H^2),$$

  $$\frac{d}{dt} \Delta(H^d) \leq \left( -\delta_a - \delta_a - 4\delta_a + \frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^d) + 2\frac{\mu}{\gamma} \Delta(H^d)^2 + \Delta(H^1) \left( \frac{\mu}{\gamma} \Delta(H^d) + \frac{\mu}{\gamma} - \delta_a \right) + \Delta(H^2) \left( \frac{\mu}{\gamma} \Delta(H^d) + \frac{\mu}{\gamma} - \delta_a \right).$$

  We add the above relations to obtain the differential inequality for $\Delta(H^1) + \Delta(H^2) + \Delta(H^d)$: for $t \in [0, T_*]$

  $$\frac{d}{dt} \left( \Delta(H^1) + \Delta(H^2) + \Delta(H^d) \right) \leq \left( -5\delta_a - 2\delta_a + 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^1) + \left( -5\delta_a - 2\delta_a + 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^2)$$

  $$+ \left( -6\delta_a - 4\delta_a + 2\frac{\mu}{\gamma} + 2\frac{\mu}{\gamma} \right) \Delta(H^d) + 2\frac{\mu}{\gamma} \Delta(H^d)^2 + \Delta(H^1) \left( \frac{\mu}{\gamma} \Delta(H^d) + \frac{\mu}{\gamma} - \delta_a \right) + \Delta(H^2) \left( \frac{\mu}{\gamma} \Delta(H^d) + \frac{\mu}{\gamma} - \delta_a \right).$$

  Then, at least one of the following relations hold:

  $$\lim_{t \to T_* -} \kappa_{i\ell}^1(t) = \delta_a, \quad \lim_{t \to T_* -} \kappa_{j\ell}^2(t) = \delta_a, \quad \lim_{t \to T_* -} \kappa_{ij}^d(t) = -\delta_a, \quad a = 1, 2. \quad (69)$$

  Suppose not, i.e., $\tilde{T}_* < \infty$. Then, let $\tilde{T}_* \to \infty$. It follows from Lemma 5.1 that for $t \in (0, \tilde{T}_*)$,
\[ \begin{align*}
+ \left( -\delta_a - \delta_a - 8\delta_a + \frac{\mu}{\gamma} + \frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^d) + 2\frac{\mu}{\gamma} \Delta(H^1)^2 + 2\frac{\mu}{\gamma} \Delta(H^2)^2 \\
+ \left( 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^d) \Delta(H^1) + \left( 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^d) \Delta(H^2) \\
\leq \left( -5\delta_a - 2\delta_a + 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^1) + \left( -5\delta_a - 2\delta_a + 2\frac{\mu}{\gamma} + \frac{\mu}{\gamma} \right) \Delta(H^2) \\
+ \left( -\delta_a - \delta_a - 8\delta_a + \frac{\mu}{\gamma} + \frac{\mu}{\gamma} + 4\frac{\mu}{\gamma} \right) \Delta(H^d) \\
+ \left( 2\frac{\mu}{\gamma} + \frac{(2\mu + \frac{\mu}{\gamma})^2}{2} \right) \Delta(H^1)^2 + \left( 2\frac{\mu}{\gamma} + \frac{(2\mu + \frac{\mu}{\gamma})^2}{2} \right) \Delta(H^2)^2 \\
+ \left( \frac{(2\mu + \frac{\mu}{\gamma})^2}{2} + \frac{(2\mu + \frac{\mu}{\gamma})^2}{2} \right) \Delta(H^d)^2 \\
= \left( -7\delta_a + 3\frac{\mu}{\gamma} \right) \Delta(H^1) + \left( -7\delta_a + 3\frac{\mu}{\gamma} \right) \Delta(H^2) + \left( -10\delta_a + 6\frac{\mu}{\gamma} \right) \Delta(H^d) \\
+ \left( 2\frac{\mu}{\gamma} + \frac{9\mu^2}{2\gamma^2} \right) \Delta(H^1)^2 + \left( 2\frac{\mu}{\gamma} + \frac{9\mu^2}{2\gamma^2} \right) \Delta(H^2)^2 + \left( \frac{9\mu^2}{\gamma^2} \right) \Delta(H^d)^2 \\
\end{align*} \]

(70)

where we used the following inequalities: for \( \alpha = 1, 2, \)
\[
\Delta(H^d) \Delta(H^\alpha) \leq \frac{1}{2} \Delta(H^d)^2 + \frac{1}{2} \Delta(H^\alpha)^2.
\]

We set
\[
\Lambda_m := \min \left\{ 7\delta_a - 3\frac{\mu}{\gamma}, 10\delta_a - 6\frac{\mu}{\gamma} \right\}, \quad \kappa_M := \max \left\{ \frac{2\mu}{\gamma} \frac{\mu}{\gamma}, \frac{9\mu^2}{2\gamma^2}, \frac{9\mu^2}{\gamma^2} \right\}.
\]

Then, (70) can be rewritten as follows: for \( t \in [0, T_*], \)
\[
\frac{d}{dt} \left( \Delta(H^1) + \Delta(H^2) + \Delta(H^d) \right) \\
\leq -\Lambda_m (\Delta(H^1) + \Delta(H^2) + \Delta(H^d)) + \kappa_M \left( \Delta(H^1)^2 + \Delta(H^2)^2 + \Delta(H^d)^2 \right) \\
\leq -\Lambda_m (\Delta(H^1) + \Delta(H^2) + \Delta(H^d)) + \kappa_M \left( \Delta(H^1) + \Delta(H^2) + \Delta(H^d) \right)^2.
\]

(72)

For notational simplicity, we set
\[
\mathcal{X} := \Delta(H^1) + \Delta(H^2) + \Delta(H^d).
\]

Then, relation (72) can be rewritten as
\[
\dot{\mathcal{X}} \leq -\Lambda_m \mathcal{X} + \kappa_M \mathcal{X}^2, \quad t \in [0, T_*].
\]

(73)

Then, it is easy to see that if the initial configuration satisfies
\[
\mathcal{X}(0) < \frac{\Lambda_m}{\kappa_M},
\]

then we have
\[
\mathcal{X}(t) < \frac{\Lambda_m}{\kappa_M}, \quad t \in (0, T_*).
\]

This implies
\[
\Delta(H^1) < \frac{\Lambda_m}{\kappa_M}, \quad \Delta(H^2) < \frac{\Lambda_m}{\kappa_M}, \quad \Delta(H^d) < \frac{\Lambda_m}{\kappa_M}.
\]

(75)
So, the definition of $\Delta(H^1)$ and (75) yield
\[
\kappa_{i\ell}^1 = -\gamma \kappa_{i\ell}^1 + \mu - \mu(1 - \bar{h}_{i\ell}^1) \geq -\gamma \kappa_{i\ell}^1 + \mu - \mu \Delta(H^1) \geq -\gamma \kappa_{i\ell}^1 + \mu - \mu \frac{\Lambda_m}{\kappa_M}.
\]
This gives
\[
\kappa_{i\ell}^1 > \left( (\kappa_{i\ell}^1) - \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} \right) \right) e^{-\gamma t} + \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} \right), \quad t \in [0, T_*). \quad (76)
\]
Similarly, we have
\[
\kappa_{ij}^d < \left( (\kappa_{ij}^d) - \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} - 1 \right) \right) e^{-\gamma t} + \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} - 1 \right). \quad (77)
\]
\begin{itemize}
  \item **Step C** (Exponential decay estimate of $\Delta(H)$): Our defining relation (68) yields
    \[
    \delta_a < \frac{\mu}{\gamma} \iff 10\delta_a - 6\frac{\mu}{\gamma} < 7\delta_a - 3\frac{\mu}{\gamma}
    \]
    and hence
    \[
    \Lambda_m = 10\delta_a - 6\frac{\mu}{\gamma} > 0, \quad a = 1, 2.
    \]
\end{itemize}

We use Lemma 5.2 to see that the positivity of $\Lambda_m$ follows from the framework $(F_C)$ and $(F_D)$. From the definition of $\kappa_m$ in (71), we have the following two cases:

\begin{itemize}
  \item **Case (i)** In this case, we use $(F_C1)$ to verify that
    \[
    \delta_1 = \frac{9\mu^2 + 6\mu\gamma}{10\gamma^2 + 9\mu\gamma} \iff \delta_1 = \frac{\mu}{\gamma} \left( 1 - \frac{10\delta_1 - 6\frac{\mu}{\gamma}}{9 \left( \frac{\mu}{\gamma} \right)^2} \right) = \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} \right).
    \]
  \item **Case (ii)** In this case, we use $(F_D1)$ to verify that
    \[
    \delta_2 = \frac{9\mu^2 + 16\mu\gamma}{24\gamma^2 + 9\mu\gamma} \iff \delta_2 = \frac{\mu}{\gamma} \left( 1 - \frac{10\delta_2 - 6\frac{\mu}{\gamma}}{2\gamma + 9 \left( \frac{\mu}{\gamma} \right)^2} \right) = \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} \right).
    \]
\end{itemize}

Hence, we conclude that
\[
\delta_a := \frac{\mu}{\gamma} \left( 1 - \frac{\Lambda_m}{\kappa_M} \right), \quad a = 1, 2.
\]

Then, (76) and (62) yield
\[
\kappa_{i\ell}(T_*) \geq ((\kappa_{i\ell}^1) - \delta_a)e^{-\gamma T_*} + \delta_a > \delta_a \quad (78)
\]
We use the same argument to get
\[
\kappa_{ij}^2(T_*) \geq ((\kappa_{ij}^2) - \delta_a)e^{-\gamma T_*} + \delta_a > \delta_a. \quad (79)
\]
Similarly, (77) and (62) imply
\[
\kappa_{ij}^d(T_*) < -\delta_3. \quad (80)
\]
However, (78), (79) and (80) contradict (69):
\[
\lim_{t \to T_*} \kappa_{i\ell}(t) = \delta_1, \quad \lim_{t \to T_*} \kappa_{ij}^2(t) = \delta_2, \quad \lim_{t \to T_*} \kappa_{ij}^d(t) = -\delta_3.
\]
Hence, we have
\[ T_\ast = \infty. \]
Finally, the differential inequality \((73)\) holds for whole time \( t \in (0, \infty) \):
\[ \dot{X} \leq -\Lambda_m X + \kappa M X^2, \quad t \in (0, \infty), \quad X(0) < \frac{\Lambda_m}{\kappa M}. \]
Now, consider the Riccati differential equation:
\[ \dot{Z} = -\Lambda_m Z + \kappa M Z^2, \quad t > 0, \quad Z(0) = X(0). \]
which can be integrated to give
\[ Z(t) = \frac{1}{\left(\frac{1}{X(0)} - \frac{\kappa M}{\Lambda_m}\right)e^{\Lambda_m t} + \frac{\kappa M}{\Lambda_m}}, \quad t > 0. \]  
Then, it follows from the comparison principle that
\[ X(t) \leq Z(t), \quad t \in [0, \infty). \]
Finally, we combine \((81)\) with \((82)\) to obtain that
\[ X(t) \leq \frac{1}{\left(\frac{1}{X(0)} - \frac{\kappa M}{\Lambda_m}\right)e^{\Lambda_m t} + \frac{\kappa M}{\Lambda_m}}, \quad t > 0. \]
which yields the exponential decay of \(\Delta(H_1(t)), \Delta(H_2(t))\) and \(\Delta(H_d(t))\). Furthermore, we can check that the initial condition in \((74)\)
\[ \Delta(H_0^1) + \Delta(H_0^2) + \Delta(H_0^d) = X(0) < \frac{\Lambda_m}{\kappa M} = \begin{cases} \frac{10\delta_1 - 6\rho}{9\rho^2}, & \text{for the framework } \mathcal{F}_C, \\ \frac{20\delta_2 - 12\rho}{9\rho^2 + 4\rho}, & \text{for the framework } \mathcal{F}_D. \end{cases} \]

6. Uniform \(\ell_p\)-stability. In this section, we provide the uniform \(\ell_p\)-stability estimate of an adapative swarm sphere model \((1)\). First, we recall definition of the stability with respect to the initial data as follows.

**Definition 6.1.** For \( p \in [1, \infty] \), system \((1)\) is uniformly \(\ell_p\)-stable with respect to the initial data, if for two solutions \(X\) and \(\tilde{X}\) to \((1)\) with initial data \(X_0, \tilde{X}_0\), respectively, there exists a uniform positive constant \(G\) independent of \(N\) and \(t\) such that
\[ \sup_{0 \leq t < \infty} \|X(t) - \tilde{X}(t)\|_p \leq G\|X^0 - \tilde{X}^0\|_p. \]

**Remark 6.**
1. Note that Definition 6.1 does not take into account the coupling strength function. In fact, in Proposition 2, we will see that \(|\kappa_{ij} - \tilde{\kappa}_{ij}|\) tends to 0 exponentially fast.
2. Since our stability mainly relies on the exponential aggregation estimate, we use the following result proved in Theorem 4.2 and Theorem 4.5: there exist uniform positive constants \(C_0\) and \(D_0\) such that
\[ \max_{1 \leq i, j \leq N} |x_i - x_j| < C_0e^{-D_0t}. \]  
Before we discuss the uniform stability, we first present several basic Gronwall-type lemmas.
Lemma 6.2. Let \( y : \mathbb{R}_+ \to \mathbb{R}_+ \) be a \( C^1 \)-function satisfying a differential inequality:
\[
y' \leq \alpha_1 e^{-\beta_1 t} y + \alpha_2 e^{-\beta_2 t}, \quad t > 0.
\] (84)

Then we have the following assertions:
1. If \( \alpha_i \) and \( \beta_i \) satisfy
\[
\alpha_1 < 0, \quad \beta = 0, \quad \alpha_2 > 0, \quad \beta_2 > 0,
\]
there exist uniform positive constants \( C_0 \) and \( D_0 > 0 \) such that
\[
y(t) \leq C_0 e^{-D_0 t}, \quad t \geq 0.
\]
2. If \( \alpha_i \) and \( \beta_i \) satisfy
\[
\alpha_1 > 0, \quad \beta_1 > 0, \quad \alpha_2 = 0,
\]
there exists a uniform constant \( C_1 \) such that
\[
y(t) \leq C_1 y_0, \quad t \geq 0.
\]
3. If \( \alpha_i \) and \( \beta_i \) satisfy
\[
\alpha_1 > 0, \quad \beta_1 > 0, \quad \alpha_2 > 0, \quad \beta_2 > 0,
\]
there exists a uniform constant \( C_2 \) such that
\[
y(t) \leq C_2 y_0, \quad t \geq 0.
\]

Proof. (i) By the comparison principle of ODE and method of integrating factor, we have
\[
y(t) \leq \left( y_0 + \frac{\alpha_2}{\alpha_1 + \beta_2} \right) e^{-\beta_1 t} - \frac{\alpha_2}{\alpha_1 + \beta_2} e^{-\beta_2 t}, \quad t \geq 0.
\]
Hence, there exist uniform positive constants \( C_0 \) and \( D_0 > 0 \) such that
\[
y(t) \leq C_0 e^{-D_0 t}, \quad t \geq 0.
\]
(ii) We multiply (84) with the integrating factor
\[
\exp \left( -\int_0^t \alpha_1 e^{-\beta_1 s} ds \right) = \exp \left( -\frac{\alpha_1}{\beta_1} (1 - e^{-\beta_1 t}) \right)
\]
(85)
to find
\[
y(t) \leq y_0 e^{\frac{\alpha_1}{\beta_1} (1 - e^{-\beta_1 t})} \leq y_0 e^{\frac{\alpha_1}{\beta_1} y_0} =: C_1 y_0, \quad t \geq 0.
\]
(iii) We also multiply the same integrating factor (85) to obtain
\[
y(t) \exp \left( -\frac{\alpha_1}{\beta_1} (1 - e^{-\beta_1 t}) \right) - y_0 \leq \alpha_2 e^{-\frac{\alpha_1}{\beta_1} t} \int_0^t e^{-\beta_2 s} e^{\frac{\alpha_1}{\beta_1} e^{-\beta_2 t} s} ds \leq \frac{\alpha_2}{\beta_2}.
\]
This implies
\[
y(t) \leq \left( y_0 + \frac{\alpha_2}{\beta_2} \right) e^{\frac{\alpha_1}{\beta_1} (1 - e^{-\beta_1 t})} \leq \left( y_0 + \frac{\alpha_2}{\beta_2} \right) e^{\frac{\alpha_1}{\beta_1} y_0}.
\]
Thus we can find a uniform constant such that
\[
y(t) \leq e^{\frac{\alpha_1}{\beta_1} \left( 1 + \frac{\alpha_2}{\beta_2 y_0} \right) y_0} =: C_2 y_0.
\]

\[\square\]

In next proposition, we show that \(|\kappa_{ij} - \tilde{\kappa}_{ij}|\) tends to zero exponentially fast and \(\kappa_{ij}\) tends to the positive value exponentially fast.
Proposition 2. Let $(X, K)$ and $(\tilde{X}, \tilde{K})$ be two solutions to (1) with either (34) or (40). Then, there exist positive uniform constants $(C_i, D_i)$, $i = 1, 2$ such that

$$|\kappa_{ij}(t) - \tilde{\kappa}_{ij}(t)| \leq C_1 e^{-D_1 t}, \quad |\kappa_{ij}(t) - \kappa_\infty| \leq C_2 e^{-D_2 t}, \quad t \geq 0. \quad (86)$$

Proof. • (Derivation of the first estimate): It follows from (1) that

$$\frac{d}{dt}(\kappa_{ij} - \tilde{\kappa}_{ij}) = -\gamma(\kappa_{ij} - \tilde{\kappa}_{ij}) + \mu(\Gamma(x_i - x_j) - \Gamma(\tilde{x}_i - \tilde{x}_j)). \quad (87)$$

We multiply $\kappa_{ij} - \tilde{\kappa}_{ij}$ to the both sides of (87) to obtain

$$\frac{1}{2} \frac{d}{dt}|\kappa_{ij} - \tilde{\kappa}_{ij}|^2 = -\gamma|\kappa_{ij} - \tilde{\kappa}_{ij}|^2 + \mu|\kappa_{ij} - \tilde{\kappa}_{ij}|\Gamma(x_i - x_j) - \Gamma(\tilde{x}_i - \tilde{x}_j)| \leq -\gamma|\kappa_{ij} - \tilde{\kappa}_{ij}|^2 + \mu|\kappa_{ij} - \tilde{\kappa}_{ij}|\Gamma_{Lip}\left(|x_i - x_j| + |\tilde{x}_i - \tilde{x}_j|\right).$$

Then, we apply (83) to obtain that

$$\frac{d}{dt}|\kappa_{ij} - \tilde{\kappa}_{ij}| \leq -\gamma|\kappa_{ij} - \tilde{\kappa}_{ij}| + 2\mu\Gamma_{Lip}C_0 e^{-D_0 t}. \quad (88)$$

We use Lemma 3.3 to conclude that there exist uniform positive constants $C_1, D_1$ such that

$$|\kappa_{ij}(t) - \tilde{\kappa}_{ij}(t)| \leq C_1 e^{-D_1 t}, \quad t \geq 0. \quad \square$$

• (Derivation of the second estimate): We use (83), dynamics of $\kappa_{ij}$ and Lemma 6.2(i) to conclude that there exist uniform positive constants $C_2, D_2$ such that

$$|\kappa_{ij}(t) - \kappa_\infty| \leq C_2 e^{-D_2 t}, \quad \kappa_\infty := \frac{\mu}{\gamma}\Gamma(0).$$

Now, we are ready to present a uniform $\ell_p$-stability of (1).

Theorem 6.3. Let $(X, K)$ and $(\tilde{X}, \tilde{K})$ be two solutions to (1) with either (34) or (40). Suppose a priori that exponential aggregation occurs. Then for $p \in [1, \infty)$, there exists a uniform positive constant $C > 0$ such that

$$\|X(t) - \tilde{X}(t)\|_p \leq C\|X^0 - \tilde{X}^0\|_p, \quad t \geq 0. \quad (89)$$

Proof. It follows from (1) that

$$\frac{d}{dt}(x_i - \tilde{x}_i) = \frac{1}{N} \sum_{j=1}^{N} \tilde{\kappa}_{ij}\left((x_j - \tilde{x}_j) - (x_i - \tilde{x}_i)\right) + \frac{1}{N} \sum_{j=1}^{N} (\kappa_{ij} - \tilde{\kappa}_{ij})(x_j - x_i)$$

$$+ \frac{1}{2N} \sum_{j=1}^{N} \tilde{\kappa}_{ij}|\tilde{x}_i - \tilde{x}_j|^2(x_i - \tilde{x}_i) + \frac{1}{2N} \sum_{j=1}^{N} (\kappa_{ij} - \tilde{\kappa}_{ij})|\tilde{x}_i - \tilde{x}_j|^2 x_i$$

$$+ \frac{1}{2N} \sum_{j=1}^{N} \kappa_{ij}\left(|x_i - x_j|^2 - |\tilde{x}_i - \tilde{x}_j|^2\right) x_i. \quad (88)$$
We now take an inner product (88) with \((x_i - \tilde{x}_i)\) to obtain
\[
\frac{1}{2} \frac{d}{dt} |x_i - \tilde{x}_i|^2 \leq -\frac{1}{N} \sum_{j=1}^{N} \left( \tilde{\kappa}_{ij} |x_i - \tilde{x}_i|^2 - |\tilde{\kappa}_{ij}||x_j - \tilde{x}_j||x_i - \tilde{x}_i| \right) \\
+ \frac{1}{N} \sum_{j=1}^{N} |\kappa_{ij} - \tilde{\kappa}_{ij}||x_j - x_i||x_i - \tilde{x}_i| \\
+ \frac{1}{2N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |\tilde{x}_i - \tilde{x}_j|^2 |x_i - \tilde{x}_i|^2 + \frac{1}{2N} \sum_{j=1}^{N} |\kappa_{ij} - \tilde{\kappa}_{ij}||\tilde{x}_i - \tilde{x}_j|^2 |x_i - \tilde{x}_i|^2 \\
+ \frac{1}{2N} \sum_{j=1}^{N} |\kappa_{ij}(|x_i - x_j|^2 - |\tilde{x}_i - \tilde{x}_j|^2)||x_i - \tilde{x}_i|.
\]

We divide the above inequality by \(|x_i - \tilde{x}_i|\) to find a differential inequality for \(|x_i - \tilde{x}_i|\):
\[
\frac{d}{dt} |x_i - \tilde{x}_i| \leq -\frac{1}{N} \sum_{j=1}^{N} \left( \tilde{\kappa}_{ij} |x_i - \tilde{x}_i| - \tilde{\kappa}_{ij} |x_j - \tilde{x}_j| \right) + \frac{1}{N} \sum_{j=1}^{N} |\kappa_{ij} - \tilde{\kappa}_{ij}||x_j - x_i| \\
+ \frac{1}{2N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |\tilde{x}_i - \tilde{x}_j|^2 |x_i - \tilde{x}_i| + \frac{1}{2N} \sum_{j=1}^{N} |\kappa_{ij} - \tilde{\kappa}_{ij}||\tilde{x}_i - \tilde{x}_j|^2 \\
+ \frac{1}{2N} \sum_{j=1}^{N} |\kappa_{ij}(|x_i - x_j|^2 - |\tilde{x}_i - \tilde{x}_j|^2)|.
\]

Next, we multiply the above equation with \(p|x_i - \tilde{x}_i|^{p-1}\) and use the estimates (83) and (86) to obtain
\[
\frac{d}{dt} |x_i - \tilde{x}_i|^p \leq -\frac{p}{N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |x_i - \tilde{x}_i|^p + \frac{p}{N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1} \\
+ pC_0 C_1 e^{-(D_0 + D_1)t} |x_i - \tilde{x}_i|^{p-1} + \frac{p}{2N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} C_0^2 e^{-2D_0 t} |x_i - \tilde{x}_i|^p \\
+ \frac{p}{2} C_0^2 C_1 e^{-(2D_0 + D_1)t} |x_i - \tilde{x}_i|^{p-1} + pC_0^2 \sum_{j=1}^{N} \kappa_{ij} e^{-2D_0 t} |x_i - \tilde{x}_i|^{p-1} \\
\leq -\frac{p}{N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |x_i - \tilde{x}_i|^p + \frac{p}{N} \sum_{j=1}^{N} \tilde{\kappa}_{ij} |x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1} \\
+ p2^{p-1} C_0 C_1 e^{-(D_0 + D_1)t} + \frac{p}{2} \kappa_M C_0^2 e^{-2D_0 t} |x_i - \tilde{x}_i|^p \\
+ p2^{p-2} C_0^2 C_1 e^{-(2D_0 + D_1)t} + p2^{p-1} \kappa_M C_0^2 e^{-2D_0 t},
\]

where \(\kappa_M\) is an upper bound for \(\kappa_{ij}\). Again, we use (86) to find
\[
-\frac{p}{N} \sum_{i,j=1}^{N} \tilde{\kappa}_{ij} |x_i - \tilde{x}_i|^p + \frac{p}{N} \sum_{i,j=1}^{N} \tilde{\kappa}_{ij} |x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1}
\]
\[
\begin{align*}
&\leq -\frac{p\kappa_{\infty}}{N} \sum_{i,j=1}^{N} |x_i - \tilde{x}_i|^p - \sum_{i,j=1}^{N} |x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1} \\
&+ \frac{p}{N} \sum_{i,j=1}^{N} |x_i - \tilde{x}_i|^p + \sum_{i,j=1}^{N} |x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1} \\
\leq 2pC_2e^{-D_2t} \|X - \tilde{X}\|_p,
\end{align*}
\]
where we used H"{o}lder's inequality. We sum up (89) over the index \(i\) to have
\[
\frac{d}{dt} \|X - \tilde{X}\|_p = \frac{d}{dt} \sum_{i=1}^{N} |x_i - \tilde{x}_i|^p
\leq -\frac{p}{N} \sum_{i,j=1}^{N} \tilde{\kappa}_{ij}|x_i - \tilde{x}_i|^p + \frac{p}{N} \sum_{i,j=1}^{N} \tilde{\kappa}_{ij}|x_j - \tilde{x}_j||x_i - \tilde{x}_i|^{p-1}
\]
\[
+ \frac{p}{2} \kappa_M C_0^2 e^{-2D_0t} \sum_{i=1}^{N} |x_i - \tilde{x}_i|^p + \Lambda_2 e^{-\Gamma_2 t}
\]
\[
\leq \left(2pC_2e^{-D_2t} + \frac{p}{2} \kappa_M C_0^2 e^{-2D_0t}\right) \|X - \tilde{X}\|_p + \Lambda_2 e^{-\Gamma_2 t}
\]
\[
\leq \Lambda_1 e^{-\Gamma_1 t} \|X - \tilde{X}\|_p + \Lambda_2 e^{-\Gamma_2 t}.
\]

Finally, we apply Lemma 6.2 (iii) to find a uniform constant \(G > 0\) such that
\[
\|X(t) - \tilde{X}(t)\|_p \leq G\|X - \tilde{X}_0\|_p, \quad t \geq 0.
\]

7. Conclusions. In this paper, we have studied the emergent dynamics of the swarm sphere model equipped with adaptive couplings whose adaptiveness is influenced by the closeness between the particles. The coupling strength in the swarm sphere model is designed to evolve via the competing mechanism between a linear damping and spatial variations. To make a realistic modeling for the real world phenomena, we basically employ the two adaptive feedback rules characterized by the function \(\Gamma = \Gamma(s)\). The first law is called Hebbian, and the second law is called anti-Hebbian. For the Hebbian case, we just determine that the adaptive function has a special form, say, \(\Gamma(s) = s^2\) so that we can apply the Lyapunov functional method. By the decay estimate for the Lyapunov functional, we show that complete aggregation can occur under some restricted initial data. In contrast, for the anti-Hebbian case, we also specify the form of adaptive function, say \(\Gamma(s) = 1 - \frac{s^2}{2}\). In this case, this anti-Hebbian function can attain both positive and negative values in their domain. Thus, several interesting dynamic patterns other than the complete aggregation can emerge depending on the initial configuration and parameters. Finally, as a direct application of the (complete) aggregation estimate, we also show that swarm sphere model is uniformly \(\ell_p\)-stable with respect to the initial data. This generalizes the earlier result for the swarm sphere model with a uniform coupling constant.

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