Abstract—In a recent paper by V. Barra et al. [3] it was shown that the consistency of the quantum theory of a sterile scalar coupled to massive fermions requires inclusion of odd-power terms in the potential of scalar self-interaction. One of the most important examples of a sterile scalar is the inflaton, which is typically a real scalar field that does not belong to representations of particle physics gauge groups, such as $SU(2)$. Here we explore the effects of odd-power terms in the inflation potential on main observables, such as the scalar spectral index $n_s$ and the tensor-to-scalar ratio $r$, in the case of the inflaton’s strong non-minimal coupling to gravity.

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1. INTRODUCTION

Inflaton-based models play an important role in the understanding of inflation. There is an extensive variety of inflaton models [1], however, typically the inflaton is a specially designed real scalar field with the potential providing a special dynamics of the vacuum, such that the Universe can inflate in a proportion required by the existing observational data. Since the inflaton is a scalar which is not a representation of a gauge group of the Standard Model, it can be called a sterile scalar. On the other hand, this scalar has to be coupled to ordinary matter in order for the reheating phase to take place at the end of the inflationary period [2].

Recently, it was shown that a sterile scalar coupled to massive fermions has to satisfy certain consistency conditions related to quantum corrections [3]. In particular, the renormalizability of such a theory can be achieved only if the inflaton potential is supplemented by three terms which have odd powers of the sterile scalar field. A relevant detail is that these odd-power terms are not necessary in other models, e.g., in the Higgs inflation [4]. In this model the loop corrections are also important as they determine the value of the nonminimality parameter $\xi$ and even impose constraints on the Higgs mass (see, e.g., [5, 6] and [7–9]), but there is no need to include odd terms, since the Higgs field is not a sterile field.

The situation described above opens the following interesting possibility. Up to some extent, the inflaton-based models can be mapped to an $f(R)$-type modified theory of gravity. In general, this requires a conformal transformation (or even two of them in the case of a non-minimal coupling of the inflaton to gravity), but in the case of strong non-minimal coupling, this can be achieved for inflationary trajectories in phase space even without using it. This was done, for example, in [10] for the case of the $\alpha$-attractors class of inflationary models [11], and in [12] for the mixed Higgs-$R^2$ inflationary model. This feature certainly remains valid for inflaton models with odd potential. But if the inflaton is a real field, after such a mapping we shall meet very specific additions to the function $f(R)$ that may produce observables...
which are different from the ones of other, most frequently used, functions. As far as odd terms in the potential are typical only for inflaton-based models, one can use the observable consequences of these terms to learn whether the inflation is caused by the inflaton or by some form of modified gravity theory.

In the present paper we will explore this possibility. To do so, we use the following strategy. We map the inflaton potential with odd terms to an $f(R)$ gravity model without a conformal transformation, i.e., remaining in the same Jordan frame, which can be done approximately in the case of a strong non-minimal inflaton coupling to gravity and a sufficiently smooth behavior of the inflaton like that during slow-roll inflation. Thus, the particle masses, the Hubble parameter and the space-time curvature keep their original physical values during this mapping. The corresponding function $f(R)$ has additional terms due to odd powers in the inflaton potential, and the further work will concern these additional terms. Since the odd terms are assumed to be numerically small, the resulting $f(R)$ will be a sum of the usual $R^2$ term, typical of the Starobinsky model of inflation [13, 14], plus extra terms which produce the effect of our interest.

The paper is organized as follows. In Section 2 we use the perturbation theory and a conformal transformation to perform mapping of the original potential with odd terms [3] to the Einstein frame, which is the most standard way of the analysis of inflationary parameters. An alternative mapping without using a conformal transformation is considered in Section 3. It leads to an $f(R)$ model having the same inflationary stage with the same predictions for primordial perturbation spectra as the original model. In Section 4 we derive and analyze the inflationary slow-roll parameters. Finally, in Section 5 we draw our conclusions.

2. SCALAR FIELD WITH ODD TERMS AND TRANSFORMATION TO THE EINSTEIN FRAME

Assuming that the inflaton (or another kind of sterile scalar) $\varphi$ couples to fermions $\psi_k$ by means of a Yukawa-type interaction, i.e., $h_k \psi_k \varphi \psi_k$, the picture is qualitatively different from the one for the fermion-Higgs interactions. The Higgs scalar belongs to the fundamental representation of the $SU(2)$ gauge group and therefore has the corresponding group in the fundamental representation of the Higgs interactions. The Higgs scalar belongs to the fermions, with odd terms [3] to the Einstein frame, which is the most standard way of the analysis of inflationary parameters. An alternative mapping without using a conformal transformation is considered in Section 3. It leads to an $f(R)$ model having the same inflationary stage with the same predictions for primordial perturbation spectra as the original model. In Section 4 we derive and analyze the inflationary slow-roll parameters. Finally, in Section 5 we draw our conclusions.

According to standard arguments, this means that odd terms should be included already at the classical level in order for the theory to be renormalizable. If we do not follow this standard procedure, the odd terms will emerge anyway, being proportional to the leading logarithms in momenta or the scalar field, and they will be more difficult to control.

Taking into account a nonminimal interaction between the sterile scalar field and the scalar curvature, the potential of the sterile scalar reads

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 - \frac{1}{2} \xi \varphi^2 R + \frac{g}{3!} \varphi^3 - \tau \varphi + \hat{g} R \varphi,$$

where $\lambda$ is the usual dimensionless scalar self-coupling parameter, and $m$ is the scalar mass.

We are using the notation $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. As a consequence, $R > 0$ during inflation, and so $\xi > 0$ is a necessary condition for a nonminimal coupling. In fact, we assume a strong nonminimal coupling, $\xi \gg 1$, similar to what is required for the Higgs inflationary model. On the top of this, we need $\lambda > 0$ for the stability of the vacuum state. Furthermore, $g$, $\tau$, and $\lambda$ are nonminimal parameters corresponding to the odd powers of the scalar. Unlike $\lambda$ and $\xi$, these parameters are dimensional, $[g] = \text{[mass]}$, $[\hat{g}] = \text{[mass]}$, and $[\tau] = \text{[mass$^3$]}$.

An analysis of the renormalization group equations for $g$, $\tau$ and $\hat{g}$ shows that the minimum possible magnitudes of these parameters are determined by the masses of heaviest fermions, i.e., the top quark in the Standard Model. Compared to the value of the Hubble parameter, even at the end of inflation, these values are small. However, even relatively small parameters can produce measurable effects if the corresponding terms are qualitatively different. Thus in what follows we will try to explore such traces for the odd terms in the potential (1).

As a first step, let us transform the action of gravity and the nonminimal scalar with the potential (1),

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_0^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\}$$

(2)
to the Einstein frame. We assume that the terms $V_0(\varphi) = \frac{1}{3} \varphi^3 - \frac{1}{2} \xi \varphi^2 R$ are dominating and treat the rest of the potential, $V_1(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 - \tau \varphi + \hat{g} R \varphi$, as a small perturbation, taking only first-order effects into account. Also, during the inflationary epoch, the kinetic term can be neglected, so we will not take it into account, even as a perturbation. We denote the solution of the corresponding equation of...
motion with $V_0(\varphi)$ as $\varphi_0$, and the solution of the full equation, with $V(\varphi) = V_0(\varphi) + V_1(\varphi)$, as $\varphi_0 + \varphi_1$.

First we consider the theory with the basic potential $V_0(\varphi)$. It proves useful to perform the following change of variables in the zero-order action:

$$M_p^2 B = \xi \varphi_0^2,$$

where $B$ is a new scalar field. Then the reduced (without the kinetic term) form of the action (2) is

$$S_0 = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{M_p^2}{2} (B + 1) R - \frac{M_p^4}{8\alpha} B^2 \right\},$$

where $\alpha$ is the first of the useful new parameters

$$\alpha = \frac{3\kappa^2}{\lambda}, \quad \beta = \tilde{g} + \frac{g \xi}{\lambda}, \quad \gamma = \sqrt{\frac{2\alpha}{\xi}}.$$

Note that the parameter $\beta$ is invariant under an arbitrary shift in $\varphi$: $\varphi \rightarrow \varphi + \delta \varphi$.

Making the conformal transformation of the metric (not of the scalar field),

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{2\rho}, \quad e^{2\rho} = 1 + B,$$

after some algebra we arrive at the action

$$S_0 = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{M_p^2}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U_0(\chi) \right\},$$

and the minimal potential term is

$$U_0(\chi) = \frac{M_p^4}{8\alpha} \sigma^2 B^2,$$

written in terms of the variables

$$\sigma = \sigma(\chi) = e^{-\sqrt{\frac{\xi}{M_p}} \tilde{\chi}},$$

$$B = B(\chi) = e^{\sqrt{\frac{\xi}{M_p}} \tilde{\chi}} - 1.$$

This potential is the Einstein-frame mapping of the $R + R^2$ action of the Starobinsky inflationary model.

As the next step, consider the first order in perturbations. Starting from the modified version of the change of variables (3), we get

$$M_p^2 B = \xi \varphi^2 - 2\tilde{g} \varphi.$$

Replacing $\varphi = \varphi_0 + \varphi_1$ in this equation, in the first order in $\tilde{g}$ we get

$$\varphi^2 = \frac{M_p^2 B}{\xi} + \frac{2\tilde{g} M_p B^{-1/2}}{\xi^{3/2}}.$$

The action in terms of the field $B$ has the form

$$S_1 = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{M_p^2}{2} R(B + 1)$$

$$- \frac{\lambda M_p^4 B^2}{24\xi^2} - V_1(B) \right\},$$

where

$$V_1(B) = \frac{m^2 M_p^2 B}{2\xi} + \frac{g M_p^3 B^{3/2}}{\xi^{3/2}}$$

$$+ \frac{\tilde{g} \lambda M_p^3 B^{3/2}}{6\xi^{5/2}} - \frac{\tau M_p B^{1/2}}{\xi^{1/2}}.$$

Before making the conformal transformation, let us write the action (12) in terms of a more useful notation. From now on, we will express the parameters in units of the (reduced) Planck mass $M_p = (8\pi G)^{-1/2}$, that is,

$$m' = \frac{m}{M_p}, \quad \tau' = \frac{\tau}{M_p^2},$$

$$\tilde{g}' = \frac{\tilde{g}}{M_p}, \quad \gamma' = \frac{\gamma}{M_p}.$$

In these units, according to what we have discussed above, $m', |\tau'|, |\gamma'|, |\gamma'| \ll 1$.

It is important that the conformal transformation is not affected by the small perturbation terms and should have the same form (13) as in the unperturbed version of the theory. The reason is that the curvature $R$ enters into Eq. (12) in exactly the same way as in the action (4), so it is the unique form of conformal transformation providing the canonical kinetic term for the scalar $\chi$.

Dropping the primes, the new action has the form (7) with the full potential

$$U(\chi) = \frac{M_p^4}{8\alpha} \sigma^2 B^2 + \frac{M_p^2 m^2}{2\xi} \sigma^2 B$$

$$+ \frac{\sqrt{2}\gamma M_p^3}{4\alpha^{3/2}} \sigma B^{3/2} - \frac{\tau M_p^3}{\sqrt{2\alpha}} \sigma^2 B^{1/2}.$$

As could be expected, Eq. (40) includes the potential corresponding to the $R + R^2$ model plus a perturbation.

It is remarkable that the small parameters $g$, $\lambda$, $\tilde{g}$, and $\tau$ and the parameters $\lambda$ and $\xi$, enter into the expression (15) only in the combinations $\alpha$, $\beta$, $\gamma$ and $\tau$, the first three defined in (5). The scalar field $\chi$ combines into the quantities defined in (9).

The derivative of the above potential (15) is

$$U'(\chi) = \frac{M_p^4}{2\sqrt{6\alpha}} \sigma^2 B + \frac{M_p m^2}{\sqrt{6\xi}} \sigma(2\sigma - 1)$$

$$+ \frac{\gamma M_p^3}{4\sqrt{3\alpha^{3/2}}} \sigma B^{1/2}(3 - 4\sigma B)$$

$$- \frac{\tau M_p}{2\sqrt{3\alpha}} \sigma B^{1/2}(1 - 4\sigma B).$$
In this expression, the first term is the usual one in the Starobinsky model. Now, for $\chi \to \infty$, we have $\sigma \to 0$. Noting that $\sigma B = 1 - \sigma$ and keeping the leading orders in $\sigma$, in this limit we have then:

$$U'(\chi) \sim \frac{M_B^2}{2\sqrt{6}\alpha} \sigma + \frac{M_P m^2}{\sqrt{6}\beta} \sigma(-1)$$

$$+ \frac{\gamma \beta M_B^3}{4\sqrt{3}\alpha^{3/2}} \chi^{1/2}(-1) - \frac{\tau \gamma M_B^3}{2\sqrt{3}\alpha^{3/2}} \chi^{3/2}(-3). \quad (17)$$

It is easy to see that $U' \to 0$ in the limit $\chi \to 0$. Indeed, there are various contributions, the second term being of the same order $\sigma$ as for the Starobinsky model, plus the third one, which is of order $\sigma^{1/2}$, and therefore dominant (of course, not taking into account the smallness of the parameters in the coefficients), and the last one which is subdominant. Note that $U'$ must be positive in order to allow for the slow-roll phase towards small values of $\chi$, so we must have, roughly speaking, that

$$m^2/\xi \ll M_B^2/\alpha, \quad \gamma \beta \ll \alpha, \quad \tau \gamma \ll \sqrt{\alpha}. \quad (18)$$

3. INDUCED ACTION OF GRAVITY WITH ODD TERMS

Another way of obtaining the results of the previous section, which is even simpler in fact, is to use the possibility of an approximate representation of the theory (2) with $\xi \gg 1$ as $f(R)$ gravity in the same Jordan frame (i.e., without a conformal transformation) up to small terms $\propto \xi^{-1}$. This possibility follows already from the fact that the effective Brans-Dicke parameter $\omega_{BH}$ is very small ($\approx \frac{1}{30}$) for this theory, while it is exactly zero for $f(R)$ gravity. The alternative derivation presented below demonstrates the possibility to avoid a conformal transformation and to work directly in the Jordan frame all the time. It is useful in the case of large fermion masses since neither particle rest masses nor the physical values of the Hubble function $H(t)$ are invariant under the conformal transformation.

Our strategy will be as follows. We perform mapping of the scalar theory with the potential (1) strongly coupled to the Ricci scalar $R$ to the form of modified $f(R)$ gravity

$$S = \frac{M_B^2}{2} \int d^4x \sqrt{-g}f(R). \quad (19)$$

We shall describe it here in more detail than in [3]. After that, the analysis of consequences for inflation, for the odd terms in the action of original scalar, becomes trivial and can be done either directly in the physical (Jordan) frame or, after the conformal transformation, in the Einstein frame, see, e.g., [18], where this procedure is used for a wide class of models.

Let us start with the potential (1) and, as in the previous section, assume that the main nonminimal term $\frac{\lambda}{4} R \varphi^2$ and the interaction term $\frac{g}{3} \varphi^3$ are dominating over other terms, which are regarded as small corrections, their effects being perturbational. The kinetic term in the classical action of the scalar field $\varphi$ will be simply neglected. This approximation corresponds to the part of the inflationary epoch when the potential term dominates. As we will see in what follows, this approximation provides the mapping of the scalar potential to the $R + c R^2$ action with a sufficiently large coefficient $c$. A known fact is that this theory fits well with observations, justifying the approximation.

Without the kinetic term, the equation for the scalar field follow from Eq. (1),

$$V'(\varphi) = m^2 \varphi + \lambda \varphi^3 + g \varphi^2 - \tau + \tilde{g} R - \xi \varphi R = 0. \quad (20)$$

Let us solve Eq. (20) perturbatively, in the first order in the small parameters $\tau$, $\tilde{g}$ and $g$. It is useful to divide the potential in two parts, $V(\varphi) = V_0(\varphi) + V_1(\varphi)$, where

$$V_0(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 - \frac{1}{2} \xi R \varphi^2,$$

$$V_1(\varphi) = \frac{g}{3!} \varphi^3 - \tau \varphi + \tilde{g} R \varphi. \quad (21)$$

For the sake of generality, we keep the mass dependence exact till the end of the consideration. The zero-order reduction of (20) has the form

$$V_0'(\varphi) = m^2 \varphi_0 + \frac{\lambda}{4!} \varphi_0^3 - \xi \varphi_0 R = 0$$

$$\implies \varphi_0^2 = \frac{6}{\lambda} (\xi R - m^2). \quad (22)$$

Substituting this result into the first-order equation, with $\varphi = \varphi_0 + \varphi_1$, after small algebra we obtain from Eq. (20)

$$\varphi_1 = -\frac{3\tilde{g}}{2\lambda} + \frac{\tau - \tilde{g} R}{2(\xi R - m^2)}. \quad (23)$$

According to the simplified version of the mapping (see, e.g., [21]), the function $f(R)$ in the first order of perturbation theory has the form

$$f(R) = R - \frac{2}{M_P^2} [V_0(\varphi_0) + V_1(\varphi_0)$$

$$+ \varphi_1 V_0'(\varphi_0)], \quad (24)$$

\footnote{Taking into account leads to nonlocalities, which were discussed in [19, 20].}
where the last term in square brackets obviously van-
ishes. In this way, substituting (22) into the potential,
we arrive at the expression
\[
\frac{M_P^2}{2} f(R) = \frac{3}{2\lambda} m^4 + \left( \frac{M_P^2}{2} - \frac{3\xi}{\lambda} m^2 \right) R
+ \frac{3\xi^2}{2\lambda} R^2 + \sqrt{\frac{6}{\lambda}(\xi R - m^2)} \left[ -\frac{g}{\lambda}(\xi R - m^2) \right]
+ \tau - \tilde{g}R
\]  
(25)

The first term in this expression is the induced cos-
ological constant, the second one is the Einstein-
ential constant, cf. Eq. (27), which is irrelevant
under the condition \(\xi m^2 \ll \lambda M_P^2\), mentioned above.
The third term is \(R^2\), which is an important element
of the inflationary model of [13]. According to the
standard evaluation [14], the magnitude of the coef-
ficient \(\frac{3\xi^2}{2\lambda}\) should be approximately \(5 \times 10^8\); hence,
the natural value of the main nonminimal parameter
is \(\xi \sim 10^4\). In the inflationary regime, \(\xi R \gg \frac{\lambda M_P^2}{\xi} \gg m^2\).
Since \(\lambda \ll 1\), this coefficient of the \(R^2\) term in the effective
\(f(R)\) representation of the original model (1) (valid during inflation only) much exceeds
the quantum-gravitational loop correction to it due to
scalar fields strongly nonminimally coupled to gravity,
which is of the order of \(\xi^2\) up to a logarithmic factor
[15, 16]. The imaginary part of this correction just
akens the decay rate of the scalaron into pairs of
such scalar particles and antiparticles after the end
of inflation [17], that is the most effective channel of
reheating in the Starobinsky model.\(^2\)

All odd parameters and \(m^2\) are small, and their ef-
ct on the cosmic perturbations should be considered
in the linear approximation. As we are interested in
the odd terms, we can safely set \(m^2\) to zero in Eq. (25).
In this way, we arrive at the action
\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M_P^2}{2} R - \frac{3\xi m^2}{\lambda} R + \frac{3\xi^2}{2\lambda} R^2
+ M_P \sqrt{\frac{6\xi R}{\lambda}} \left[ \tau M_P^2 - \left( \tilde{g} + \frac{g\xi}{\lambda} \right) R \right] \right\}
\]  
(26)

As far as \(\xi m^2 \ll \lambda M_P^2\), the second term in the in-
tegrand produces only a small shift in the inverse

\(^2\) Note that we do not agree with the statement in [16] on the appear-
ance of a new scalar degree of freedom in this system,
different from that existing in any scalar-tensor gravity or
in \(f(R)\) gravity. In fact, the effective scalar particle in the
Starobinsky model (scalaron) in our model (1) and its \(f(R)\)
representation during inflation (25) is the same. Its mass
at the end of inflation is \(M \sim M_P \sqrt{\lambda\xi}^{-1}\).

Newton constant,
\[
M_P^2 \rightarrow M_P^2 - \frac{6\xi m^2}{\lambda},
\]
so it can be omitted, and we arrive at
\[
S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} \left[ R + \frac{\alpha}{M_P^2} R^2
+ \frac{2}{M_P} \sqrt{\frac{6\xi R}{\lambda}} (M_P^2 \tau - \beta R) \right]
\]  
(28)

We note that, due to the odd terms in the potential
(1), the resulting function \(f(R)\) has an unusual form
with the noninteger powers \(1/2\) and \(3/2\) of the scalar curvature.

The following observation is in order. It is clear
that the term proportional to the root of scalar curva-
ture in the gravitational action leads to an inconsis-
tency, since in the presence of this term there is no
flat metric solution to the equations for the metric.
In the present case, this does not mean that the theory
which we are dealing with is inconsistent. Let us
remember that (26) is not the fundamental action
of gravity, but only the intermediate form of a mapping
of the scalar theory with the potential (1), which is
valid in the inflationary epoch only, more precisely,
in the slow-role phase.

Indeed, the above effective action is valid only
when the \(R^2\) term dominates, and therefore the new
contributions \(R^{1/2}\) and \(R^{3/2}\) can indeed be treated as
perturbations. At low energies, one has \(\xi \rightarrow 0\), and so
the mapping from the potential (1) to the action (26)
cannot be performed. This allows us to avoid a pos-
sible disruption of the graceful exit or effects such as
strong particle production or tachyonic instabilities,
see, e.g., [22].

If one tries to consider (26) as a fundamental the-
ory, valid at all \(R\), then many conditions and require-
ments apply for its viability, as discussed extensively,
e.g., in [23]. For example, one must have \(f'(R) > 0\)
and \(f''(R) > 0\) in order to guarantee that gravity is
an attractive force and to avoid ghosts, and these
requirements put constraints on the parameter space
\((\xi, \lambda, \tau, \gamma, g)\). In the present case, these constraints
do not apply because (26) is not regarded as a fun-
damental action, but only as an intermediate stage of
mapping of the scalar theory with the potential (1) to
the minimal scalar model. Furthermore, according
to the analysis of [23], one must extend an \(f(R)\)
theory to negative values of \(R\) in order to guarantee
a graceful exit from the inflationary era. As it stands,
the \(f(R)\) theory of Eq. (26) has no such extension
because of the \(\sqrt{R}\) term and thus can only be regarded
as an effective theory for large \(R\).
Our further analysis will be based on the action (28) that can be regarded as a particular case of the $f(R)$ theory (19). This action can be mapped to the usual scalar-metric action (see, e.g., [21] and further references therein) in the Jordan frame:

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} [\varphi R - V(\phi)],$$

(29)

where

$$\phi = f'(R) = 1 + \frac{2\alpha}{M_P^2} R + \chi \left( \frac{\tau M_P}{\sqrt{R}} - \frac{3\beta}{M_P} \sqrt{R} \right),$$

$$V(\phi) = \phi R - f(R).$$

(30)

Here the prime denotes a derivative with respect to $R$, and we used the notation (5). As before, we assume all “odd” parameters to be small and perform all calculations perturbatively, in the first order in these parameters. This approach simplifies the general procedure of [21]. Let us call $R_0$ the solution without odd terms and $\Delta R_1$ the first-order correction to it. Writing $R$ as

$$R = R_0 + \Delta R_1$$

(31)

where $|\Delta R_1 = O(1) (\tau, \beta)| \ll |R_0|$, we arrive at

$$R^{1/2} \approx R_0^{1/2} \left( 1 + \frac{\Delta R_1}{2R_0} \right),$$

$$R^{-1/2} \approx R_0^{-1/2} \left( 1 - \frac{\Delta R_1}{2R_0} \right).$$

(32)

Solving Eq. (30), in the zero order we get

$$\phi = 1 + \frac{2\alpha}{M_P^2} R_0 \Rightarrow R_0 = \frac{M_P^2}{2\alpha} (\phi - 1),$$

(33)

and in the first order

$$\Delta R_1 \approx \frac{3M_P \beta \gamma R_0^{1/2} - M_P^{3/2} \tau \gamma R_0^{-1/2}}{2\alpha}. $$

(34)

The explicit form of the solution is

$$R(\phi) = \frac{dV(\phi)}{d\phi} = \frac{M_P^2}{2\alpha} (\phi - 1)$$

$$+ \frac{3\sqrt{2}M_P^2 \beta \gamma}{4} \frac{1}{\alpha^{3/2}} (\phi - 1)^{1/2} - \frac{\tau \gamma M_P^2}{\sqrt{2\alpha}} (\phi - 1)^{-1/2}.$$ 

(35)

Finally, after integration, we obtain the potential

$$V(\phi) = \frac{M_P^2}{4\alpha} (\phi - 1)^2 + \frac{\sqrt{2}M_P^2}{2} \frac{\beta \gamma}{\alpha^{3/2}} (\phi - 1)^{3/2}$$

$$- \frac{2\tau \gamma M_P^2}{\sqrt{2\alpha}} (\phi - 1)^{1/2}. $$

(36)

It is useful to work with the action of the standard form, in the Einstein frame:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M^2}{2} R - g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right\}.$$

(37)

Since the terms originating from the odd terms in Eq. (1) contribute only to the potential part of the action, after the Weyl transformation we get the usual relation between the field $\chi$ with a canonically normalized kinetic term and the scalar $\phi$,

$$U(\chi) = \frac{M_P^2}{2\phi^2} V(\phi(\chi)),$$

(38)

where

$$\phi(\chi) = e^{\sqrt{\frac{3}{4} M_P \chi}}.$$ 

(39)

After small algebra, we find for the potential the expression (15). In the $m^2 = 0$ approximation, it boils down to

$$U(\chi) = \frac{M_P^4}{8\alpha \sigma^2} B^2 + \frac{\sqrt{2} M_P^4 \beta \gamma}{4} \frac{1}{\alpha^{3/2}} \sigma^2 B^{3/2}$$

$$- \frac{M_P^4 \tau \gamma}{\sqrt{2\alpha}} \sigma^2 B^{1/2},$$

(40)

The two new terms $R^{1/2}$ and $R^{3/2}$, with the corresponding two corrections in the above potential, therefore modify the dynamics of inflation. Which of the two dominates, depends on the parameters, but if these are of the same order, then the $R^{3/2}$ term dominates. Gravitational vacuum polarization from massive fermions (i.e., with masses $m \gg H$) during inflation provides a contribution $\sim \frac{H^3}{M_P^2}$ [3] (see also [24] and [25] for more examples), but, owing to the Planck suppression, this is small compared to the new terms due to fermions. See also [26] in connection with RG corrections to $\xi$ resulting in its running which transforms to the running of the $R^2$ coefficient in the $f(R)$ representation.

4. DERIVATION OF THE SLOW-ROLL PARAMETERS

In the previous sections we saw that the potential (40) in the Einstein frame can be derived from the original potential (1) in two different (albeit ultimately equivalent) ways. Now we are in a position to use this expression to obtain the parameters characterizing inflation.

It is useful to derive the first and second derivatives of the potential, which have the form

$$U'(\chi) = \frac{\beta \gamma M_P^2 \sigma B^{1/2}}{4\sqrt{3\alpha^{3/2}}} (3 - 4\sigma B)$$
As in the general case (see, e.g., [18]), in the slow-roll approximation the slow-roll parameters are related to the scalar field potential as follows (see, e.g., [27]):

\[
\epsilon = \frac{M_P^2}{2} \left( \frac{U'(\chi)}{U(\chi)} \right)^2, \quad \eta = \frac{M_P^2}{2} \frac{U''(\chi)}{U(\chi)}. \tag{42}
\]

Keeping only the \(O^{(1)}(\tau, \beta)\) terms, we get

\[
\epsilon = \frac{4}{3} \beta - \frac{4}{\sqrt{3}} \sqrt{2} \beta M_P^2 \left( 3 - 4 \sigma B \right) + \frac{8\sigma \sqrt{2} \beta}{\sqrt{3} \alpha B^2}, \quad \eta = \frac{4(2\sigma - 1)}{3} + \frac{\sqrt{2} \beta \gamma (16B^2 \sigma^2 - 18sb + 3)}{3\alpha B^2},
\]

or

\[
\epsilon = \frac{4}{3} \sigma^2 - \frac{4\beta \gamma \sqrt{2} \sigma^3}{3\sqrt{\alpha}} + 8\sigma \sqrt{2} \beta \gamma \sqrt{\alpha} e^{-\frac{3}{\sqrt{2}} \frac{\sqrt{2} \sigma}{\sqrt{\sigma}}}.
\]

Expanding the square root in the integrand, we get

\[
N(\chi) = -\frac{3}{4} \int_{\chi_{\text{end}}}^{\chi} \frac{d\chi'}{\sigma^2} \left( 1 + \frac{\beta \gamma}{\sqrt{2} \alpha^2} - 3 \tau \gamma \sqrt{\frac{2\alpha}{\sigma^2}} \right),
\]

where

\[
d\sigma' = -\sqrt{\frac{2\sigma'}{3} \frac{d\chi'}{M_P}}. \tag{49}
\]

After integration and using the condition \(\chi_{\text{end}} \ll \chi\) (i.e., assuming that \(\chi\) is deep in the inflationary era), we find

\[
N(\chi) = \frac{3}{4} \left[ \frac{\sigma^{-1}}{\alpha^2} + \frac{\sqrt{2} \beta \gamma}{3\sqrt{\alpha}} - 4\sqrt{2} \alpha \tau \gamma \sigma^{-1/2} \right]. \tag{50}
\]

We can expand \(\sigma\) in Eq. (50) to find the field \(\chi\) in terms of the number of e-folds:

\[
\sigma = \sigma_0 + \delta \sigma_1 = \frac{3}{4N} + \frac{\sqrt{2} \beta \gamma}{\sqrt{\alpha}} \left( \frac{3}{4N} \right)^{1/2} - 4\sqrt{2} \alpha \tau \gamma \left( \frac{3}{4N} \right)^{3/2}.
\]

Then, by plugging it in Eqs. (45), (46), and keeping only terms up to \(O^{(1)}(\tau, \beta)\), we get

\[
\epsilon = \frac{3}{4N^2} - \frac{\sqrt{6} \beta \gamma}{6\sqrt{\alpha} N^{3/2}} + \frac{9\sqrt{6}}{4} \tau \gamma \sqrt{\alpha} \frac{1}{N^{5/2}},
\]

\[
\eta = -\frac{1}{N} - \frac{\sqrt{6} \beta \gamma}{18\sqrt{\alpha} N^{1/2}} + \frac{3\sqrt{6}}{4} \tau \gamma \sqrt{\alpha} \frac{1}{N^{3/2}}.
\]

The main inflationary observables are the scalar spectral index \(n_s\) and the tensor-to-scalar ratio \(r\) (see, e.g., [27]), whose expressions in terms of the slow-roll parameters are

\[
n_s - 1 = -6\epsilon + 2\eta, \quad r = 16\epsilon. \tag{54}
\]

These observables are constrained by the Planck mission [28] to \(n_s = 0.9649 \pm 0.0042\) at 68% CL and \(r_{0.002} < 0.056\) at 95% CL.

Therefore, keeping the leading order of the slow-roll approximation only, we arrive at

\[
n_s - 1 = -\frac{2}{\sqrt{3}} - \frac{\sqrt{6} \beta \gamma}{9\sqrt{\alpha} N} + \frac{3\sqrt{3}\alpha}{2N^3} \tau \gamma,
\]

\[
r = \frac{12}{N^2} - \frac{8\sqrt{6} \beta \gamma}{\sqrt{3} \alpha N^{3/2}} - \frac{36\sqrt{6} \alpha \tau \gamma}{N^{5/2}},
\]

and in terms of the original parameters:

\[
n_s - 1 = -\frac{2}{N} - \frac{2\sqrt{3}}{9\sqrt{\xi N}} (\tilde{g} + g \xi) \frac{1}{N^3/2} + \frac{9\sqrt{3} \xi^{3/2}}{N^{3/2} \lambda}, \tag{56}
\]
We have derived the above results in the Einstein frame, but as long as we retain up to first-order corrections, they are valid also in the Jordan one, although the number of e-folds $N$ becomes a different function of the present wave vector modulus $k$. Moreover, the results (56) can be derived directly in the Jordan frame, without a conformal transformation to the Einstein frame, by using the formulas presented in [29] where a generic $f(R) = A(R)R^2$ model, with $A(R)$ slowly varying, is explored. Note that for the case studied in the present paper, cf. Eq. (28), the corresponding $A(R)$ function is

\[ A(R) = \frac{M_p^2}{R} + \alpha + 2M_P \sqrt{\frac{6\xi}{\lambda^3}} (M_p^2 \tau - \beta R). \]  

(58)

So, it is slowly varying at large values of $R$ at which the slow-roll inflationary regime is realized in this model.

To estimate the value of the inflationary observables coming from the new terms, we need the values of the parameters $\xi, \lambda, \tilde{g}, g, \tau$. If $\lambda = 1$, we can set $\xi = 1 \times 10^4$, so that we recover Starobinsky inflation if $\tilde{g} = g = \tau = 0$. The values of $\tilde{g}, g$ and $\tau$ depend on the masses of fermions to which the scalar is coupled, via the renormalization group equations, as explained in [3]. Without repeating the corresponding arguments, we just mention that quantum contributions to the dimensional parameters $\tilde{g}, g$ and $\tau$ from a fermion loop are proportional to the respective power of the fermion mass.

For the first evaluation let us assume the fermion masses at the upper bound of the Standard Model, $m_f \sim 1$ TeV. Then, according to the previous considerations, $g = \tilde{g} \approx 1$ TeV and $\tau \approx 10^4$ TeV. For these values of the parameters and $N = 60$ we find

\[ n_s = 0.965417, \quad r = 0.003333, \]  

(59)

and for $N = 50$:

\[ n_s = 0.9582, \quad r = 0.0048, \]  

(60)

which are the same, to this precision, to the $R^2$ case. If we increase the $m_f$ values to the GUT scale, for $m_f \approx 10^{14}$ GeV we find $n_s = 0.965374$ and $r = 0.003313$. For supersymmetric GUT models, with $m_f \approx 10^{16}$ GeV, there is $n_s = 0.961231$ (a difference of 0.43% from the $R + R^2$ model) and $r = 0.001332$ (a difference of 60% from $R + R^2$).

We can compare the contribution to the inflationary observables coming from the induced action (28) with second-order corrections coming from pure Starobinsky inflation. The next-to-leading order contributions to $\epsilon$ and $\eta$ are, when $N = 60$,

\[ \epsilon_{NL} = \frac{9}{8N^3} \approx 5.21 \times 10^{-6}, \]  

(61)

\[ \eta_{NL} = -\frac{3}{2N^2} \approx -4.17 \times 10^{-4}. \]  

(62)

For $m_f \approx 1$ TeV we have that the corrections for $\epsilon$ and $\eta$ from the $\sqrt{R}$ and $R^{3/2}$ terms, at $N = 60$, are $\epsilon_{\text{odd}} = -1.24 \times 10^{-17}$ and $\eta_{\text{odd}} = -2.48 \times 10^{-16}$, and are indeed smaller than the next-to-leading order corrections of the $R + R^2$ model. However, if $m_f$ is as large as $m_f \approx 10^{16}$ GeV, the new contributions to $\epsilon$ and $\eta$ are $\epsilon_{\text{odd}} = -1.25 \times 10^{-4}$ and $\eta_{\text{odd}} = -2.47 \times 10^{-3}$, becoming more relevant than the second-order corrections to the Starobinsky model.
Figure 1 shows the potential (40) for a few different values of $m_f$, together with the pure Starobinsky model. For negative values of the field, our potential becomes imaginary because of the square root. But the inflation corresponds to $R > 0$ and the plateau, so in principle this does not make a problem.

To better understand the results, at this point we have to return to the problem statement in [3] and describe the physical situation in which the sterile scalar can be coupled to the Standard Model fermions.

The Standard Model left-handed fermions are doublets under the $SU(2)$ gauge group of the form

$$\begin{pmatrix} u^L_i \\ d^L_i \end{pmatrix}$$

in the case of left-handed quarks. We have $u^L_i = \{u_L, t_L, c_L\}$ and $d^L_i = \{d_L, b_L, s_L\}$. On the other hand, the right-handed fermions are singlets under the $SU(2)$ group, namely $u^R_i$ and $d^R_i$. So, in order that the Yukawa interaction in the Lagrangian be invariant under the $SU(2)$ gauge group, the scalar must be at least a doublet under $SU(2)$, so it can multiply the $SU(2)$ index of left-handed fermions. This is exactly why we cannot directly introduce the fermion mass terms into the Lagrangian, and we have to do so using the Higgs mechanism.

The only possibility to couple a sterile scalar to Standard Model fermions is that the $SU(2) \times U(1)$ symmetry is broken by the Higgs mechanism at energies below 125 GeV. At this regime, terms in the effective low-energy Lagrangian do not need to be invariant under $SU(2)$, as this is no longer a manifest symmetry of the theory. In particular, if the sterile...
scalar is mixed with Higgs at high energies, in the process of symmetry breaking the Yukawa interactions with the sterile scalar emerge in a natural way.

On the other hand, as we have seen above, the coupling of a sterile scalar (inflaton) with fermions at the Standard Model energy and mass scale does not produce essential changes in the inflationary observables. Assuming physics beyond the Standard Model, there may be new heavy fermion singlet fields that could couple to a sterile scalar. Alternatively, coupling of the inflaton with a Higgs-like scalar of GUT model can give the effect of mixing similar to the one described above for the Standard Model. This possibility gives a chance to detect traces of GUTs in cosmological observations.

5. CONCLUSION

We have explored basic consequences of odd terms in the inflaton potential in the case where the inflaton is strongly non-minimally coupled to gravity. The presence of these odd terms is motivated by the structure of renormalization of a generic sterile scalar coupled to fermions by means of the Yukawa interaction. The analysis has been performed both by a direct transformation to the Einstein frame and by means of mapping to the $f(R)$ inflationary model in the original Jordan frame, which has the same predictions for primordial perturbation spectra. Along with an additional control, this second approach provides more intuitive understanding of the role of the odd terms.

An advantage of our approach is that the physical analysis can be performed in terms of the underlying particle physics model to which the inflaton is coupled. The values of the constants of the odd terms in the potential satisfy the lower bounds related to running of the corresponding parameters. In practice, it means that these dimensional constants should be at least of the same order of magnitude as the heaviest fermions of the model. The main result obtained here is that the effect of the odd terms is negligible if the typical mass $m_f$ of the heaviest fermions is smaller than the GUT scale $\sim 10^{16}$ GeV. Thus the odd terms in the inflaton potential become relevant only in the presence of GUT with the corresponding fermions. If these conditions are satisfied, there is, in principle, a chance to distinguish the inflaton models from legitimate $f(R)$ models by measuring such quantities as the spectral index $n_s$ and the tensor-to-scalar ratio $r$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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