

Zero-bias conductance quantization in a normal / superconducting junction of nano wire

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Abstract. We discuss a strong relationship between Majorana fermions and odd-frequency Cooper pairs which appear at a disordered normal nano wire attached to a topologically nontrivial superconducting one. The zero-bias differential conductance in a normal / superconducting nano wire junctions is quantized at $2e^2/h$ irrespective of degree of disorder, length of disordered segment, and random realization of disordered potential. Such behaviors are exactly the same as those in the anomalous proximity effect of $p_x$-wave spin-triplet superconductors. We show that odd-frequency Cooper pairs assist the unusual transport properties.

1. Introduction

Finding of Majorana fermions (MFs) and controlling of Majorana bound states (MBSs) are hot research issues in condensed matter physics [1] from the view of potential application of MBS to the topological quantum computation [2, 3]. Although we have already known a number of promising systems hosting MFs, the semiconductor nano wire fabricated on top of a superconductor has attracted much attention [4, 5, 6, 7, 8, 9]. Actually, a plenty of theoretical studies have discussed MFs or MBSs in such nano wires [10, 11, 12, 13, 14, 15, 16]. These researches have stimulated a number of theoretical investigation on unusual charge transport phenomena through the MBS in normal-metal / superconductor (NS) and superconductor /normal-metal/ superconductor (SNS) junctions on nano wires [17, 18, 19, 20, 21, 22, 23]. The zero-bias conductance peak reported very recently would be considered as an experimental evidence of MFs (MBS) [24, 25].

Odd-frequency Cooper pairing was originally proposed to understand nature of unconventional superfluidity and superconductivity[26]. Ubiquitous appearance of the odd-frequency pairs at the surface of superconductors and near the interface of superconducting junctions has been established and widely accepted in recent years [27, 28, 29, 30, 31]. In particular, the odd-frequency Cooper pairs make the background of the anomalous proximity effect in a diffusive normal metal attached to a spin-triplet superconductor: (i) the large zero-energy quasiparticle density of states in a normal metal [32], (ii) the quantized zero-bias conductance at twice of the Sharvin’s value in diffusive NS junctions [33], (iii) the fractional current($J$)-phase($\phi$) relationship of $J \propto \sin (\phi/2)$ in diffusive SNS junctions [34], (iv) the zero-bias anomaly in nonlocal conductance spectra [35] and (v) the anomalous surface impedance in NS bilayers [36].
In a previous paper [37], we have shown that disordered NS and SNS junctions of nano wire indicate the properties of (i)-(iii) when the superconducting nano wire is topologically nontrivial by tuning the Zeeman field and the spin-orbit coupling. We have also suggested a strong relationship between the Majorana fermions and the odd-frequency Cooper pairs. In this paper, we theoretically study details of the differential conductance, the local density of states, and the pairing function in disordered NS nano wires.

2. Model

Figure 1. A Schematic figure of a normal / superconducting junction of a nano wire.

Let us consider a nano wire with strong spin-orbit coupling fabricated on a junction of an insulator and a metallic superconductor as shown in Fig. 1. A segment on the insulator and that on the superconductor are in the normal and the superconducting states, respectively. The diameter of the nano wire is sufficiently small so that the number of propagating channel is unity for each spin degree of freedom. We describe the present nano wire by using the tight-binding model in one-dimension,

\[
H_0 = -t \sum_{j,\alpha} \left( c_{j+1,\alpha}^\dagger c_{j,\alpha} + c_{j,\alpha}^\dagger c_{j+1,\alpha} \right) + i \lambda \sum_{j,\alpha,\beta} \left( c_{j+1,\alpha}^\dagger (\hat{\sigma}_2)_{\alpha,\beta} c_{j,\beta} - c_{j,\alpha}^\dagger (\hat{\sigma}_2)_{\alpha,\beta} c_{j+1,\beta} \right) + \mu \sum_{j,\alpha} c_{j,\alpha}^\dagger c_{j,\alpha},
\]

\[
H_d = \sum_{1 \leq j,\alpha,\beta \leq L} V_{j,\alpha} c_{j,\alpha}^\dagger c_{j,\alpha},
\]

\[
H_s = \sum_{j \geq L+1} \left[ \Delta e^{i\varphi} c_{j,\alpha}^\dagger c_{j,\alpha} + \Delta e^{-i\varphi} c_{j+1,\alpha}^\dagger c_{j+1,\alpha} \right],
\]

where \( c_{j,\alpha}^\dagger (c_{j,\alpha}) \) is the creation (annihilation) operator of an electron at the lattice site \( j \) with spin \( \alpha = (\uparrow \text{ or } \downarrow) \), \( t \) denotes the hopping integral, \( \mu \) is the chemical potential, and \( \Delta \) is the pair potential in the superconducting segment. The Pauli matrices in spin space are denoted by \( \hat{\sigma} \) for \( j = 1 - 3 \) and the unit matrix of \( 2 \times 2 \) is \( \hat{\sigma}_0 \). The on-site potential in the normal segment is given randomly in the range of \( -W/2 \leq V_j \leq W/2 \) for \( 1 \leq j \leq L \). We measure the energy and the length in units of \( t \) and the lattice constant, respectively. Throughout this paper, we fix several parameters as \( \mu = t, \Delta = 2t, \) the superconducting macroscopic phase \( \varphi = 0 \), and the pair potential at the zero temperature \( \Delta_0 = 0.01t \). The number of samples used for the random ensemble averaging is typically 1000. By tuning the external magnetic field \( B \) as shown in Fig. 1, it is possible to introduce the exchange potential \( V_{ex} \). For \( V_{ex} > V_c \equiv \sqrt{\Delta_0^2 + \mu^2} \), the number of propagating channels in the nano wire is unity and the superconducting segment becomes topologically nontrivial. Therefore MBS is expected at the boundary between the normal and the superconducting segments.

In the following, we study the transport properties of two nano wires. One is the nano wire with \( V_{ex} = 1.5t \) and \( \lambda = 0.5t \), which we call topological nano wire hosting MF. The other is the
non-topological nano wire with $V_{ex} = \lambda = 0$ in which MF is absent. The latter one is studied as a reference. By comparing the results of the two nano wires, we discuss anomalous charge transport properties of the topological nano wire.

3. Results

At first, we study the normal conductance $g_N$ of disordered nanowires based on the linear response theory with using the recursive Green function method [38]. The Hamiltonian is given by $H_0 + H_d$. In Fig. 2, the normal conductance $g_N$ in units of $2e^2/h$ is plotted as a function of the length of the disordered segment $L$. The results show $\ln(g_N) \propto -L/\xi_{AL}$ with $\xi_{AL}$ being the localization length because one-dimensional disordered wires are in the localization regime. From Fig. 2, we estimate that $\xi_{AL}$ for the topological and the non-topological nano wires are about 11 and 8.5 lattice constants, respectively. The localization length of the topological nano wire is slightly larger than that of the reference nanowire. This may be due to the breaking down the time-reversal and the spin-rotation symmetries. The normal nano wires are in the Anderson insulators. Namely there is no propagating channels at the fermi level.

![Figure 2. The normal conductance in a disordered nano wires.](image)

Secondly, we study the differential conductance $g_{NS}$ of NS junctions based on the Blonder-Tinkham-Klapwijk formula [39],

$$g_{NS} = \frac{e^2}{h} \sum_{\alpha,\beta} \left[ \delta_{\alpha,\beta} - |r_{\alpha,\beta}^n|^2 + |r_{\alpha,\beta}^h|^2 \right]_{E = eV}$$

(4)

where we consider the Hamiltonian $H_0 + H_d + H_s$. In Eq. (4), $r_{\alpha,\beta}^n$ and $r_{\alpha,\beta}^h$ are the normal and Andreev reflection coefficients of the junction at energy $E$. By solving the Bogoliubov-de Gennes equation numerically, we obtain the Green function

$$\tilde{G}_E(j, j') = \begin{bmatrix} \tilde{G}_E(j, j') & \tilde{F}_E(j, j') \\ -\tilde{F}^*_E(j, j') & \tilde{G}^*_E(j, j') \end{bmatrix},$$

(5)

where the symbols $\cdot \cdot$ and $\cdot \cdot$ indicate $2 \times 2$ matrices in spin space and $4 \times 4$ matrices in spin plus Nambu space. The Andreev reflection coefficients can be calculated from the asymptotic behavior of the Green function [40]. We show the differential conductance $g_{NS}$ in units of $2e^2/h$ as a function of the bias-voltage $eV$ for the non-topological nano wire in Fig. 3(a) and the topological nano wires in Fig. 3(b). The conductance for the non-topological nano wires in (a) decreases with increasing the length of disordered segment $L$ for all $eV$. The similar tendency can be seen also in the conductance in the topological nano wires in (b) for finite
$eV$. However the zero-bias conductance of the topological nano wires is quantized at $2e^2/h$ irrespective of $L$, which is an intrinsic phenomenon of Majorana fermion. The results suggest a perfect transmission channel in the disordered nanowire even in the localization regime.

![Figure 3](image1.png)

**Figure 3.** The differential conductance of NS nano wires for several choices of length of disordered segment $L$. The results for $V_{ex} = \lambda = 0$ and those for $V_{ex} = 1.5t$ and $\lambda = 0.5t$ are shown in (a) and (b), respectively.

![Figure 4](image2.png)

**Figure 4.** The local density of states at the center of the disordered segment. The results for $V_{ex} = \lambda = 0$ and those for $V_{ex} = 1.5t$ and $\lambda = 0.5t$ are shown with broken and solid lines, respectively.

Next we discuss the local density of states (LDOS) defined by

$$N(E, j) = -\frac{\text{Im}}{\pi} \text{Tr} \left[ \hat{G}(j, j) \right].$$

The LDOS in the disordered nano wire is shown in Fig. 4, where we plot the LDOS at $j = 10$ as a function of $E$ for $L = 20$. The LDOS for the non-topological nano wire (broken line) is almost flat around the zero energy and is slightly smaller than the density of states at the fermi level in clean normal nano wire $N_0$. On the other hand, the LDOS for the topological nano wire (solid line) has large zero-energy peak, which reflects the presence of the resonant state in the normal disordered wire. Strictly speaking, the resonant state is no longer a bound state because a quasiparticle can escape from the disordered normal segment to a semi-infinitely long clean normal lead wire ($j \leq 0$). This is responsible for the finite width of the zero-energy peak. The LDOS also shows a peak around $E = 0.3\Delta_0$, where the number of the propagating channels in the superconducting segment changes from 0 to 2. The dip structures in the differential conductance around $eV = 0.3\Delta_0$ in Fig. 3(b) reflect this fact. The LDOS shows another peak structures around $E = 0.6\Delta_0$ and $E = 0.85\Delta_0$ as shown in Fig. 4. At present, we do not know well the reason of the peaks. In the conductance, there are neither peak nor dip structures in the conductance at corresponding bias voltage as shown in Fig. 3(b).
Finally we analyze the pairing function in the normal segment. The anomalous Green function can be decomposed into
\[ \tilde{F}_E(j, j) = i [f_0(j, E) + f(j, E) \cdot \sigma] \tilde{\sigma}_2. \]
(7)
The spin-singlet pairing function is denoted by \( f_0 \) and the spin-triplet pairing functions are \( f_i \) for \( i = 1 - 3 \). According to the definition, the spin configuration of \( f_3 \), \( f_{↑↑} = -f_1 + i f_2 \) and \( f_{↓↓} = f_1 + i f_2 \) are \((| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle)/\sqrt{2}, | \uparrow \uparrow \rangle \) and \(| \downarrow \downarrow \rangle \), respectively. In Fig. 5, we show the pairing function in non-topological and topological nano wire in (a) and (b), respectively. In the non-topological nano wire, the spin-triplet components are zero. Only the spin-singlet component has finite amplitude. The real part of \( f_0 \) is the even function of \( E \), whereas the imaginary part of \( f_0 \) is the odd-function of \( E \), (i.e., \( f_0(-E) = f_0^*(E) \)). This behavior means the even-frequency spin-singlet Cooper pairs penetrate into the normal segment. On the other hand in the topological nano wire, \( f_{↑↑} \) is the most dominant component. In fact, \( \text{Re}(f_0) \) takes its maximum value -0.14 at \( E = 0 \). Thus the amplitudes of the even-frequency spin-singlet component is much smaller than \( f_{↑↑} \). The results in Fig. 5(b) show that the real part of \( f_{↑↑} \) is the odd function of \( E \), whereas the imaginary part of \( f_{↑↑} \) is the even function of \( E \), (i.e., \( f_{↑↑}(-E) = -f_{↑↑}^*(E) \)). This fact means the penetration of odd-frequency Cooper pairs into the normal segment of the topological nano wire [31, 36]. The odd-frequency Cooper pairs assist the conductance quantization at the zero-bias in Fig. 3(b). We note that the pair density in Eq. (8) is obtained average over a non-topological wire for \( V_{ex} < V_c \approx 1 \). For \( V_{ex} > V_c \), the amplitude of \( n_{↑↑} \) suddenly grows. In the topological wire \( V_{ex} > V_c \), the fraction of the odd-frequency pairs becomes almost unity. Thus we conclude that the odd-frequency Cooper pairs assists the conductance quantization at the zero-bias in Fig. 3(b). We note that the pair density in Eq. (8) is obtained average over a number of samples with different random potential configurations. In our simulation, we find very large sample-to-sample fluctuations of \( n_{↑↑} \) for \( V_{ex} > V_c \). In fact, the results of \( n_{↑↑} \) for \( V_{ex} > V_c \) seems to be not a smooth function of \( V_{ex} \). This may be a characteristic feature in the localization regime.

**Figure 5.** The pairing function at the center of the disordered segment. The spin-singlet component \( f_0 \) for \( V_{ex} = \lambda = 0 \) are shown in (a). The equal-spin triplet component \( f_{↑↑} \) for \( V_{ex} = 1.5t \) and \( \lambda = 0.5t \) are shown in (b). We choose \( L = 20 \). The results calculated at \( j = L/2 = 10 \).
4. Conclusion
We have studied transport properties of disordered normal / superconducting (NS) junctions of nano wires which have strong spin-orbit coupling. When the superconducting pair potential is absent, the normal nano wires are in the Anderson localization regime. By introducing the exchange field $V_{ex}$, superconducting nano wires become topologically nontrivial when the exchange potential is larger than a critical value of $V_c$. For $V_{ex} > V_c$, the superconducting nano wires host the Majorana fermions at their edges. In such NS junction, the zero-bias conductance is quantized at the universal value of $2e^2/h$ irrespective of the length of disordered normal segment, degree of disorder, and the random potential configurations. The penetration of the Majorana fermions into the normal segment is responsible for the universal behavior. We have also studied the pairing function in normal segment and found that the amplitude of the odd-frequency Cooper pairs suddenly grows for $V_{ex} > V_c$. We conclude that the odd-frequency Cooper pairs and the Majorana fermions always coexist with each other in solids.

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