On \(n\)-tuple principal bundles

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Abstract

We develop the concept of a double (more generally \(n\)-tuple) principal bundle departing from a compatibility condition for a principal action of a Lie group on a groupoid.

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1 Introduction

The concept of double structures, i.e. two structures of the same type satisfying some compatibility condition, became recently very popular. Let us mention double vector bundles \([4, 11, 22, 23, 24]\), double affine bundles \([8]\), double Lie algebroids and groupoids \([1, 16, 17]\), double graded bundles \([7]\), etc. Of course, one can consider also \(n\)-tuple instead of double structures. It is a particular case of mathematical objects consisting with a pair (or \(n\) pieces) of compatible (not necessary of the same type) structures, which is crucial for the whole mathematics.

In \([12]\) the authors introduced double principal bundles (DPBs) as a diagram of four principal bundles and two exact sequences of their structure groups. The definition is \textit{ad hoc} with no real motivation and explanation. In this paper we introduce DPBs as two principal bundle structures which are naturally compatible in the sense that one of the structures, with the structure group \(G_1\), is compatible in a natural sense with the groupoid determined by the second structure, so that we have a \(G_1\)-groupoid (see \([3]\)). It turns out that the concept of a DPB in this sense is equivalent to the concept of a \(G\)-DPB in \([12]\), i.e. the principal bundle structure of a \textit{double principal group} \(G\) – a Lie group generated by two its normal subgroups \(G_1, G_2\).

The advantage of this approach is that it can be easily generalized to the \(n\)-tuple case with natural examples of \(n\)-tuple principal groups associated with \(n\)-tuple vector (or graded) bundles.

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2 \( G \)\text{-}groupoids

2.1 The compatibility condition

Our general reference to the theory of Lie groupoids and Lie algebroids will be Mackenzie’s book [19].

Let \( \mathcal{G} \rightrightarrows M \) be an arbitrary Lie groupoid with source map \( s: \mathcal{G} \to M \) and target map \( t: \mathcal{G} \to M \). There is also the inclusion map \( \iota_M : M \to \mathcal{G} \), \( \iota_M(x) = 1_x \), and a partial multiplication \( (g,h) \mapsto gh \) which is defined on \( \mathcal{G}^{(2)} = \{(g,h) \in \mathcal{G} \times \mathcal{G} : s(g) = t(h)\} \). Moreover, the manifold \( \mathcal{G} \) is foliated by \( s \)-fibres \( \mathcal{G}_s = \{ g \in \mathcal{G} \mid s(g) = x \} \), where \( x \in M \). As by definition the source and target maps are submersions, the \( s \)-fibres are themselves smooth manifolds.

Let \( \mathcal{G}_i \rightrightarrows M_i \) \( (i=1,2) \) be a pair of Lie groupoids. Then a Lie groupoid morphisms is a pair of maps \((\Phi, \phi)\) such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{G}_1 & \xrightarrow{\Phi} & \mathcal{G}_2 \\
\downarrow{s_1} & & \downarrow{t_1} \\
M_1 & \xrightarrow{t_2} & M_2
\end{array}
\]

in the sense that

\[
s_2 \circ \Phi = \phi \circ s_1, \quad \text{and} \quad t_2 \circ \Phi = \phi \circ t_1
\]

subject to the further condition that \( \Phi \) respects the (partial) multiplication; if \( g, h \in \mathcal{G}_1 \) are composable, then \( \Phi(gh) = \Phi(g)\Phi(h) \). It then follows that for \( x \in M_1 \) we have \( \Phi(1_x) = 1_{\phi(x)} \) and \( \Phi(g^{-1}) = \Phi(g)^{-1} \).

In our study of Jacobi and contact groupoids [3] we deal with Lie groupoids that have a compatible action of \( \mathbb{R}^\times \) upon them; compatibility to be defined shortly. However, as the basic theory of compatible group actions on Lie groupoids is independent of the actual Lie group, we recall the general setting from [3]. We will use the characterization of a proper action \( P \times G \ni (p,g) \mapsto pg \in P \) of a Lie group \( G \) on a manifold \( P \) by the condition that for each compact \( K \subset P \) the set \( K(G) = \{ g \in G \mid Kg \cap K \neq \emptyset \} \) is relatively compact in \( G \) (cf. [21]).

**Definition 2.1.** An action \( \rho : G \times \mathcal{G} \to \mathcal{G} \) of a Lie group \( G \) on a Lie groupoid \( \mathcal{G} \rightrightarrows M \) is called compatible with the groupoid structure if \( h_g : \mathcal{G} \to \mathcal{G} \) are groupoid isomorphisms for all \( g \in G \), and principal if the action is free and proper. A groupoid equipped with a compatible principal action we call a principal bundle \( G \)-groupoid, \((G\text{-}groupoid \text{ in short})\).

**Remark 2.2.** The reader should immediately be reminded of Mackenzie’s notion of a PBG-groupoid [14, 15] which is close to ours but not exactly the same.

**Proposition 2.3.** (cf. [3]) A compatible action \( \rho : G \times \mathcal{G} \to \mathcal{G} \) of a Lie groupoid \( \mathcal{G} \rightrightarrows M \) induces a canonical action \( \rho_M : G \times M \to M \) of units. If \( \rho \) is principal, then \( \rho_M \) is also principal. In particular, in this case the set of orbits \( M_0 = M/G \) has a canonical manifold structure.

**Proof.** The action of \( G \) on \( \mathcal{G} \) commutes with the source and target maps, thus projects onto a \( G \)-action on the manifold \( M \). Moreover, \( M \) as an immersed submanifold of \( \mathcal{G} \) is invariant with respect to the \( G \)-action, and the projected and restricted actions coincide. As the action of \( G \) on \( \mathcal{G} \) is principal, it is also principal on the immersed submanifold \( M \), so \( M \) inherits a structure of a principal \( G \)-bundle. \( \Box \)

In some cases we have to deal with actions which are originally not free but induce a principal action \([\rho]\) of \( G/\ker \rho \), where \( \ker(\rho) = \{ g \in G \mid \rho_g = \text{id} \} \) is the kernel of the action. In this case we will speak about a pre-principal action.
Proposition 2.4. A compatible action \( \rho : G \times \mathcal{G} \to \mathcal{G} \) of a Lie group \( G \) on a Lie groupoid \( \mathcal{G} \to M \) is pre-principal if and only if the induced action \([\rho]_M \) of \( G/\ker(\rho) \) on the manifold \( M \) of units is principal.

**Proof.** We already know that if \([\rho] \) is principal, then \([\rho]_M \) is principal, so assume \([\rho]_M \) is principal. Since \([\rho]_M \) is free, \( \rho \) is clearly free, so it remains to show that \( \rho \) is proper. Let \( K \) be a compact subset of \( P \) and \( K_M = \pi(K) \). As \( K_M \) is clearly compact, the set

\[
(G/\ker(\rho), K_M) = \{ [g] \in G/\ker(\rho) \mid (K_M[g]) \cap K_M \neq \emptyset \}
\]

is compact. Since \((K_M[g]) \cap K \neq \emptyset \) implies \((K_M[g]) \cap K_M \neq \emptyset \),

\[
(G/\ker(\rho), K) = \{ [g] \in G/\ker(\rho) \mid (K[g]) \cap K \neq \emptyset \}
\]

is compact. \( \square \)

2.2 Structure of \( G \)-groupoids

For any principal \( G \) groupoid the reduced manifold \( \mathcal{G}/G = \mathcal{G}_0 \) is canonically a Lie groupoid \( \mathcal{G}/G = \mathcal{G}_0 \Rightarrow M/G = M_0 \), with the set of units \( M_0 \), defined by the following structure:

\[
\begin{array}{ccc}
G & \overset{\pi}{\longrightarrow} & \mathcal{G}_0 \\
\downarrow{s} & & \downarrow{\tau} \\
M & \overset{\sigma}{\longrightarrow} & M_0 \\
\downarrow{p} & & \\
\end{array}
\]

\[
\sigma \circ \pi = p \circ s, \\
\tau \circ \pi = p \circ t, \\
1_p(x) = \pi(1_x) \text{ for all } x \in M, \\
\pi(y)^{-1} = \pi(y^{-1}) \text{ for all } y \in \mathcal{G}, \\
\pi(y y') = \pi(y) \pi(y') \text{ for all } (y, y') \in \mathcal{G}^{(2)},
\]

where \( \pi : \mathcal{G} \to \mathcal{G}_0 \) is the canonical projection. In fact, the above construction implies, tautologically, that \( (\pi, p) : \mathcal{G} \Rightarrow M \Rightarrow \mathcal{G}_0 \Rightarrow M_0 \) is a morphism of Lie groupoids with the above structures.

**Theorem 2.1.** [3] The map

\[
S : \mathcal{G} \to \mathcal{G}_0 \times M_0 \quad M := \{ (y_0, x) \in \mathcal{G}_0 \times M \mid p(x) = \sigma(y_0) \}, \quad S(y) = (\pi(y), s(y)),
\]

is a diffeomorphism. With respect to the above identification, the \( G \)-action is \( (y_0, x)g = (y_0, xg) \), the embedding of units is \( \iota_M(x) = (1_x, x) \), and the source map reads \( s(y_0, x) = x \). As the projection \((y_0, x) \mapsto y_0 \) is a groupoid morphism, the groupoid structure is uniquely determined by its target map \( t : \mathcal{G}_0 \times M_0 \Rightarrow M, t(y_0, x) =: y_0 x \). On the other hand, such a map \( t \) is a target map if and only if it has the following properties (holding for all \( x \in M \)):

1. \( p(y_0, x) = \tau(y_0) \) for all \( y_0 \in \mathcal{G}_0 \),
2. \( y_0, (y_0', x) = (y_0 y_0', x) \) for all \( (y_0, y_0') \in \mathcal{G}_0^2 \),
3. \( 1_p(x) \cdot x = x \),
4. \( y_0, (x g) = (y_0, x) g \) for all \( y_0 \in \mathcal{G}_0 \) and all \( g \in G \).

Note that (i) - (iii) mean that \( t \) is an action of \( \mathcal{G}_0 \) on \( \mathcal{G} \) as \( p : M \to M_0 \) (c.f. [19] Definition 1.6.1), and (iv) means that the action is \( G \)-equivariant. The \( G \)-groupoid determined by \( t \) as above we will denote \( \mathcal{G}_0 \times^t M_0 \) and called \( t \)-split \( G \)-groupoid. Thus, any \( G \)-groupoid (??) is \( t \)-split for some \( t(y_0, x) = y_0 x \) satisfying (i) - (iv).

An important particular case of the above theorem is the case of a trivial principal bundle, \( M = M_0 \times G \) which is always a local form of any \( G \)-groupoid. In this case we can use the identification \( \mathcal{G}_0 \times M_0 \Rightarrow \mathcal{G}_0 \times G \) and replace the map \( t \) satisfying (i) with a map \( b : \mathcal{G}_0 \to G \).
Indeed, any map on a Lie group commuting with all the right-translations is a left-translation, so can we write $t(y_0, \sigma(y_0), g) = (\tau(y_0), b(y_0)g)$. Now, the properties (i) – (iv) can be reduced to
\[
b(y_0)b(y'_0) = b(y_0y'_0)
\]
for all $(y_0, y'_0) \in G_0^2$, i.e. to the fact that $b : G_0 \to G$ is a groupoid morphism. This is of course always the local form of any $G$-groupoid. The corresponding $G$-groupoid structure, denoted with $G_0 \times^b G$, is an obvious generalisation of the groupoid extension by the additive $\mathbb{R}$ with a help of a multiplicative function considered in the literature (cf. [5, Definition 2.3]), and we have shown that this construction is in a sense universal. Thus we get the following.

**Theorem 2.2.** [3] For any $G$-groupoid structure on the trivial $G$-bundle $G = G_0 \times G$ there is a Lie groupoid structure on $G_0$ with the source and target maps $\sigma, \tau : G_0 \to M_0$ and a groupoid morphism $b : G_0 \to G$ such that the source map $s$, the target map $t$ and the partial multiplication in $G$ read
\[
s(y_0, g) = (\sigma(y_0), g), \quad t(y_0, g) = (\tau(y_0), b(y_0)g), \quad (y_0, g_1)(y_0, g_2) = (y_0y'_0, g_2).
\]

### 3 Double principal bundles

It is well known that every principal bundle $\pi : \mathbb{P} \to M$ with the structure group $G$ acting on $\mathbb{P}$ by (from the right)
\[
\rho : \mathbb{P} \times G \to \mathbb{P}, \quad (p, g) \mapsto pg,
\]
induces a canonical Lie groupoid structure on $G = (\mathbb{P} \times \mathbb{P})/G$ being the space of orbits $\langle p, q \rangle_G$ of the canonical action of $G$ on $\mathbb{P} \times \mathbb{P}$, $\langle (p, q), g \rangle \mapsto \langle pg, qg \rangle$. The set of units is identified with $M$ embedded by $\iota_M(x) = \langle p_x, p_x \rangle_G$, where $p_x$ is any element of $\mathbb{P}$ satisfying $\pi(p_x) = x$, and the partial multiplication reads
\[
\langle p, q \rangle_G \bullet \langle q, r \rangle_G = \langle p, r \rangle_G.
\]

If one looks for a compatibility condition of two principal group actions, it is natural to expect that each action should induce a compatible action on the groupoid associated with the other action. Of course, as one should assume that each principal action is compatible with itself, it is impossible to assume that the other group action on the groupoid $\langle \mathbb{P} \times \mathbb{P} \rangle/G$ is principal, since it need not to be free. We will thus use a weaker condition and assume that the action is pre-principal.

**Definition 3.1.** Let
\[
\rho' : \mathbb{P} \times G' \to \mathbb{P}, \quad (p, g') \mapsto pg',
\]
be a principal action of another Lie group, $G'$, on $\mathbb{P}$. We say that the action $\rho'$ is $(\mathbb{P}, G')$-principal if $\rho'$ induces a compatible pre-principal action of $G'$ on the groupoid $\langle \mathbb{P} \times \mathbb{P} \rangle/G$ by
\[
\langle p, q \rangle g' = \langle pg', qg' \rangle.
\]

We say that a two principal actions $\rho : \mathbb{P} \times G \to \mathbb{P}$ and $\rho' : \mathbb{P} \times G' \to \mathbb{P}$ on the same total space $\mathbb{P}$ are compatible if the action $\rho'$ is $(\mathbb{P}, G')$-principal and vice versa, the action $\rho$ is $(\mathbb{P}, G')$-principal. A manifold $\mathbb{P}$ equipped with two compatible principal actions we call a double principal bundle.

**Remark 3.2.** As we will see later, our definition of a double principal bundle is stronger than the definition introduced in [12] and corresponds to the definition of a $G$-DPB therein. We find this definition better motivated and better suited to the concept of associated bundles.
Theorem 3.1. If \( \mathbb{P} \) is a double principal bundle with respect to two principal actions \( \rho, \rho' \) of Lie groups \( G, G' \), respectively, then there is a canonical structure of a Lie group on \( G = \{ \rho_g g' : g \in G, \ g' \in G' \} \) such that \( G \cong \{ \rho_g : g \in G \} \) and \( G' \cong \{ \rho'_g : g' \in G' \} \) are closed normal Lie subgroups of \( G \).

Moreover, the obvious action of \( G \) on \( \mathbb{P} \) is principal and induces principal actions of \( [G] = G/G_0 \) and \( [G'] = G'/G_0 \), with \( G_0 = G \cap G' \), on \( M' = \mathbb{P}/G' \) and \( M = \mathbb{P}/G \), respectively, such that the commutative diagram of canonical maps,

\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\pi'} & M' \\
\downarrow{\pi} & & \downarrow{[\pi]} \\
M & \xrightarrow{[\pi']^*} & M_0 ,
\end{array}
\]

with \( M_0 = \mathbb{P}/G = M/[G'] = M'/[G] \), consists of principal bundle morphisms: \( G \)-principal bundles for the horizontal maps, and \( G' \)-principal bundles for the vertical ones.

The group \( G \) in the above proposition we will call the structure group of the double principal bundle \( \mathbb{P} \).

Lemma 3.3. If \( \pi : \mathbb{P} \to M \) is a \( G \)-principal bundle, then there is a Borel cross section \( \sigma : M \to \mathbb{P} \) of \( \pi \) such that \( \sigma(K) \) is relatively compact if \( K \subset M \) is compact.

Proof of Lemma. Let \( \{ U_i \}_{i \in \mathbb{N}} \) be a locally finite open covering of \( M \) such that there is a local smooth cross section \( \sigma_i : U_i \to \mathbb{P}, i \in \mathbb{N} \). Define \( \sigma \) so that

\[
\sigma(m) = \sigma_i(m) \quad \text{for} \quad m \in U_i \setminus \bigcup_{j=0}^{i-1} U_j .
\]

The section \( \sigma \) is clearly a Borel section. Any compact \( K \subset M \) intersects with only a finite number of \( U_i \), say, \( U_{i_1}, \ldots, U_{i_k} \). Then \( \sigma(K) \) is contained in \( \sigma_{i_1}(K \cap U_{i_1}) \cup \cdots \cup \sigma_{i_k}(K \cap U_{i_k}) \), thus is relatively compact.

Remark 3.4. One of the instances of the above result is the existence of the corresponding cross section for the canonical projection of a Lie group \( G \) onto the space \( G/H \) of its cosets modulo a closed subgroup \( H \). Extensions to more general cases of groups can be found in \([20], [21]\) and \([10]\).

Proof of Theorem 3.1. Let \( (p, q) \in \mathbb{P} \times \mathbb{P} \) and \( g' \in G' \). According to \([2]\), for each \( g \in G \) we have

\[
(pg', qg')_G = (pg, qg)_{G'} = (pg, qg)_G g' = (pgg', qgg')_G .
\]

Hence, there is a uniquely determined element \( g_{g'} \) such that \( pgg' g_{g'} = pgg' \) and \( qg' g_{g'} \). In consequence, for each \( g' \in G' \) and each \( g \in G \) there is \( g_{g'} \in G \) such that, for each \( p \in \mathbb{P} \), we have

\[
pgg' = pgg' g_{g'} ,
\]

or equivalently

\[
q(g')^{-1} gg' = qg' .
\]

It is now easy to see from \([11]\) that \( g' \mapsto (g \mapsto g_{g'}) \) is an action of \( G' \) on \( G \) by group homomorphisms:

\[
g_{g_1' g_2'} = (g_{g_1'} g_{g_2'}) \quad \text{and} \quad (g_1 g_2)_{g'} = (g_1)_{g'} (g_2)_{g'} .
\]

It is also clear from \([5]\) that the map \( G \times G' \ni (g, g') \mapsto g_{g'} \in G \) is smooth.
Let now $G' \ltimes G = G' \times G$ with the group multiplication
\[
(g', g)(g'_1, g_1) = (g'g'_1, g_1g_1).
\]

The neutral element is the pair of neutral elements $(e', e)$ and the inverse reads
\[
(g', g)^{-1} = ((g')^{-1}, g^{-1}_{(g')^{-1}}).
\]

It is easy to see that $G$ and $G'$ are closed subgroups generating $G' \ltimes G$ and a direct inspection shows that $G$ is a normal subgroup. Actually, if $\text{Ad}$ is the adjoint representation of $G' \ltimes G$, then $\text{Ad}_g g = g(g')^{-1}$ for $g \in G$ and $g' \in G'$.

Of course, changing the roles of $G$ and $G'$, we get an action $g \mapsto (g' \mapsto g'_g)$ of $G'$ on $G$ by group homomorphisms and we can construct the corresponding group $\hat{G}'$. Moreover, as
\[
pg'g^{-1}(g')^{-1} = p(g')g^{-1}(g')^{-1} = pg^{-1}_{(g')^{-1}},
\]
we have the identity
\[
(g')_{g^{-1}}(g')^{-1} = g^{-1}_{(g')^{-1}}.
\]

We have a canonical action of $G' \ltimes G$ on $\mathbb{P}$ given by $(g', g) \mapsto \rho_{g'}^\prime \circ \rho_g$. Indeed, according to (9), $(g', g)(g'_1, g_1) = (g'g'_1, g_1g_1)$ acts as
\[
\rho_{g'_1}^\prime \circ \rho_{g_1} = \rho_{g'_1}^\prime \rho_{g_1} = \rho_{g'_1}^\prime \rho_g \rho_{g_1} = \rho_{g'_1}^\prime \rho_g \rho_{g_1}.
\]

The kernel the $G' \ltimes G$-action is the closed normal subgroup $G_0 = \{(g', g) \in G' \times G | \rho_{g'}^\prime \circ \rho_g = \text{id}\}$. Put $\mathbb{G} = (G' \ltimes G)/G_0$ and denote the coset of $(g', g)$ with $[g', g]$. Clearly, $\mathbb{G}$ is a Lie group acting on $\mathbb{P}$ in the obvious way, and $G, G'$ can be identified canonically with Lie subgroups of $\mathbb{G}$. Moreover, $G$ and $G'$ are normal subgroups of $\mathbb{G}$ and
\[
\text{Ad}_{g'} g = g_{(g')^{-1}}, \quad \text{and} \quad \text{Ad}_g g' = g'_{g^{-1}}
\]
for $g \in G$ and $g' \in G'$.

Let us see that the action of $\mathbb{G}$ is free. For, let $p_0 [g', g] = p_0$ for some $p_0 \in \mathbb{P}$. This means that $p_0 g' = p_0 g^{-1}$. This implies that $(p_0, p_0) = (p_0 g', p_0 g^{-1}) = (p_0, p_0)$ so that $(p_0, p_0)$ is a fixed point of the action induced on $(\mathbb{P} \times \mathbb{P})/G$ by $G'$. As this action is $(\mathbb{P} \times \mathbb{P})/G$-principal, $g' \in \ker([\rho'])$, i.e. $\langle pg', gg' \rangle = \langle p, q \rangle$ for all $p, q \in \mathbb{P}$, so that $\rho_{g'}^\prime$ acts in fibers of $\pi : \mathbb{P} \to M$. In particular,
\[
\langle p_0, q \rangle = \langle p_0, q \rangle g' = \langle p_0 g^{-1}, q g' \rangle,
\]
so that $q g' = qg^{-1}$ for all $q \in \mathbb{P}$, thus $\rho_{g'}^\prime \circ \rho_g = \text{id}$.

Actually, what we have proved is also
\[
G_0 = \ker([\rho']) = G' \cap G
\]
where $G' \cap G$ is the intersection of $G'$ with $G$ as subgroups of $\mathbb{G}$. Note that, since $G, G'$ generate $\mathbb{G}$, the subgroup $G_0$ normal in $\mathbb{G}$ and $\mathbb{G}/G \simeq [G']$. Note also that $G$ and $G'$ are closed normal subgroup of $\mathbb{G}$. Indeed, $G$ can be characterized as the set of elements of $\mathbb{G}$ which preserve (closed) fibers of $\pi$. For, if $g' \in G'$ preserves the fibers of $\pi$, then $g' \in \ker(\rho')$, so according to (8) $g' \in G$.

We will prove now that the action of $\mathbb{G}$ on $\mathbb{P}$ is principal. Let $K$ be a compact subset of $\mathbb{P}$ and $K_M = \pi(K)$ and $K_{M'} = \pi'(K)$ be its projection onto $M$ and $M'$, respectively. As $K_M$ is compact and the $[G']$-action on $M'$,
\[
\pi(p)[g'] = \pi(pg'),
\]
is principal,
\[
K_M([G']) = \{[g'] \in [G'] | (K_M)[g'] \cap K_M \neq \emptyset\}
\]
is compact. According to Lemma, there is a Borel cross section $\sigma$ of $\text{pr} : G' \to [G'] = G'/G_0$ such that $K_M^{\sigma}([G']) = \sigma(K_M([G']))$ is relatively compact. We can also assume that $\sigma([e']) = e'$. Hence, the closure $K'$ of $KK_M^{\sigma}([G'])$ is compact in $\mathbb{P}$. Consequently, as the action of $G$ is principal,

$$(K \cdot K_M^{\sigma})([G']) = \{ [g] \in [G] \mid (K \cdot K_M^{\sigma})([G'])[g] \cap (K \cdot K_M^{\sigma})([G']) \neq \emptyset \}$$

is compact, thus

$$K_M^{\sigma}([G']) \cdot (K \cdot K_M^{\sigma})([G']) \subset \mathbb{G}$$

is compact.

Suppose that $K[g', g] \cap K \neq \emptyset$. Since $\pi(pg'g) = \pi(pg) = \pi(p)[g']$, we have

$$K_M[g'] \cap K_M \neq \emptyset,$$

so that $[g'] \in K_M([G'])$. In consequence, there is $g_0 \in G_0$ such that $g' = \sigma([g'])g_0$ and

$$Kg'g = K\sigma([g'])(g_0g) \subset KK_M^{\sigma}([G'])\cdot (g_0g)$$

intersects $K$, thus $KK_M^{\sigma}([G'])$. Hence, $g_0g \in (K \cdot K_M^{\sigma})([G'])$ and

$$g'g = \sigma([g'])(g_0g) \in K_M^{\sigma}([G']) \cdot (K \cdot K_M^{\sigma})([G']).$$

This shows that

$$(\mathbb{G}, K) = \{ [g', g] \in \mathbb{G} \mid K[g', g] \cap K \neq \emptyset \}$$

is a subset of $K_M^{\sigma}([G']) \cdot (K \cdot K_M^{\sigma})([G'])$, thus compact, and proves that the $\mathbb{G}$-action on $\mathbb{P}$ is proper.

Finally, as $\pi(p[g', g]) = \pi(pg'g) = \pi(p)[g']$ and $\pi'(p[g', g]) = \pi'(pg'g) = \pi'(pg')[g] = \pi'(p)[g]$, the projections $\pi, \pi'$ are morphisms of the corresponding principal bundles.

\[\square\]

## 4 Double principal groups and their actions

The group $\mathbb{G}$ which appears in the formulation of Theorem 3.1 is a Lie group with two distinguished closed normal subgroups (which are automatically Lie subgroups) $G, G'$ such that $G \cup G'$ generates $\mathbb{G}$. The triples $\mathbb{G} = (\mathbb{G}; G, G')$ we will call double principal groups. This concept coincides with the concept of a double Lie group in [12]. Indeed, the triple of Lie groups $\mathbb{G} = (G; G, G')$ as above induces the exact sequence of group homomorphisms

$$1 \to G_0 \to \mathbb{G} \xrightarrow{\phi} [G] \times [G'],$$

(10)

where $G_0 = G \cap G'$, $[G] = G/G_0$, and $[G'] = G/G_0$. Conversely, the exact sequence (10) gives rise to the triple $(\mathbb{G}, \phi^{-1}([e]) \times [G']), \phi^{-1}([G] \times \{e\}))$. We prefer the term “double principal group” to prevent any confusion with double Lie groups of Lu and Weinstein [13] (cf. also [16]) or Drinfel’d doubles.

By a morphism of double principal groups $\hat{\mathbb{G}} = (\mathbb{G}_i, G_i, G'_i)$, $i = 1, 2$, we understand a group homomorphism $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ such that $\phi(G_1) \subset G_2$ and $\phi(G'_1) \subset G'_2$. A double principal group is called vacant if the normal subgroups $G$ and $G'$ intersect trivially, i.e. $G \cap G' = \{ e \}$.

**Proposition 4.1.** A double principal group $\hat{\mathbb{G}} = (\mathbb{G}; G, G')$ is vacant if and only if the map

$$m : G \times G' \ni (g, g') \mapsto gg' \in \mathbb{G}$$

is a diffeomorphism.
Proof. If $G \cap G' = \{e\}$, then the map $m$ is clearly smooth and injective. As it is also surjective, it is a smooth bijection. It remains to show that $m$ is a local diffeomorphism. Let $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}'$ be the decomposition of the Lie algebra $\mathfrak{g}$ of $G$ into a direct sum of the Lie algebras of $G$ and $G'$. Let $X \in \mathfrak{g}$, $X' \in \mathfrak{g}'$ and $\hat{X}$ and $\hat{X}'$ be the corresponding left-invariant vector fields on $G$. We have

$$D_{(x,g')}m \left( \hat{X}(g), \hat{X}'(g') \right) = \frac{d}{dt} \bigg|_{t=0} \left( (g \exp(tX)g')^{\prime} \exp(tX') \right) = \frac{d}{dt} \bigg|_{t=0} \left( g' \exp(t\text{Ad}_{(g')^{-1}}X) \exp(tX') \right) = (g' \ast \text{Ad}_{(g')^{-1}}X + X') .$$

As $X + X' \mapsto \text{Ad}_{(g')^{-1}}X + X'$ is an isomorphism of $\mathfrak{g}$ for any $g' \in G'$, this shows that the derivative $D_{(x,g')}m$ of $m$ is an isomorphism, thus $m$ is a diffeomorphism. The converse is trivial.

**Theorem 4.1.** Let $\hat{G} = (\hat{G}; G, G')$ be a double principal group. Then, any principal action $r : \mathbb{P} \times G \to \mathbb{P}$ induces on $\mathbb{P}$ a double principal structure relative to the actions $\rho = r|_{G}$, $\rho' = r|_{G'}$.

**Proof.** Of course, the principal action of $G$ on $\mathbb{P}$ induces principal actions of closed subgroups, so that $\rho, \rho'$ are principal. Since $G, G'$ are normal subgroup, the adjoint action of $G$ on itself induces the ‘dressing actions’ of $G'$ on $G$ and vice versa:

$$G \times G' \ni (g, g') \mapsto g'g \in G' \quad G' \times G \ni (g', g) \mapsto g'g^{-1}g \in G ,$$

so that

$$g'g = g'g'g = g'g = g'g^{-1}g' .$$

(11)

In particular, as $G$ is generated by $G \cup G'$,

$$G = GG' = \{gg' \mid g \in G, g' \in G'\} \quad G = \{g'g \mid g \in G, g' \in G'\} = G'G .$$

Identities (11) imply also that $\rho$ and $\rho'$ induce canonical actions of $G$ and $G'$ on $(\mathbb{P} \times \mathbb{P})/G'$ and $(\mathbb{P} \times \mathbb{P})/G$, respectively. The kernels of these actions coincide with $G_0 = G \cap G'$, so that we get actions of $[G] = G/G_0 = G/G'$ and $[G'] = G'/G_0 = G/G$ by

$$\langle p, q \rangle G'_{\mathbb{P}} [g] = \langle pg, qq \rangle G' , \quad \langle p, q \rangle G'_{\mathbb{P}} [g'] = \langle pg', qq' \rangle G .$$

These actions are free. Indeed, if $\langle p, q \rangle G'_{\mathbb{P}} [g'] = \langle p, g \rangle G$, then there is $g \in G$ such that $pg' = pg$, $qq' = gg$. In particular, $p$ is a fixed point of $g'g^{-1} \in G$, thus $g'g \in G_0$. Similarly we prove that the action of $[G']$ is free.

The action is clearly compatible with the groupoid structure: if $\langle p, q \rangle G'_{\mathbb{P}}$ and $\langle p', q' \rangle G'_{\mathbb{P}}$ are composable, then there is $g' \in G'$ such that $q = p'g'q$, so $qq'g = p'g'g = p'gg'g$ and $\langle p, q \rangle G'_{\mathbb{P}} [g]$ and $\langle p', q' \rangle G'_{\mathbb{P}} [g]$ are composable. Moreover,

$$\langle p, q \rangle G'_{\mathbb{P}} [g] \bullet \langle p', q' \rangle G'_{\mathbb{P}} [g] = \langle pg, qq'gg'g \rangle G = \left( \langle p, q \rangle G' \bullet \langle p', q' \rangle G' \right) [g] .$$

It remains to show that the actions are proper. Due to symmetry of conditions, it is enough to prove that the action of $[G]$ on $(\mathbb{P} \times \mathbb{P})/G'$ is proper. Note that $(\mathbb{P} \times \mathbb{P})/G'$ is a principal bundle over $M' = \mathbb{P}/G'$ and the induced action of $[G]$ on $M'$ is the same as induced from the action of $G$ on $\mathbb{P}$:

$$\pi'(p)[g] = \pi'(pg) .$$

Clearly, it is enough to prove that the latter action is proper. Let $K'$ be a compact subset of $M'$ and $\sigma : M' \to \mathbb{P}$ be a section from Lemma 3.3. The set $K = \sigma(K')$ is relatively closed on $\mathbb{P}$, so $K(\mathbb{P}) = \{gg' \in \mathbb{P} \mid Kg'g \cap K' \neq \emptyset\}$ is relatively compact in $G$. If $[g] \in K\prime([G]) = \{[g] \mid [g] \in [G] \mid K'([g]) \cap K' \neq \emptyset\}$, then there is $g' \in G'$ such that $gg' \in K(\mathbb{P})$. Hence, $K\prime([G])$ is contained in $\text{pr}(K(\mathbb{P}))$, where $\text{pr} : G \to [G] = G/G'$ is the canonical projection. As $\text{pr}(K(\mathbb{P}))$ is relatively compact, $K\prime([G])$ is relatively compact.
**Definition 4.2.** Double principal bundles described by the above theorem we will call \( \hat{G} \)-principal bundles. A *morphism of \( \hat{G}_i \)-principal bundles* \( \Phi : \mathbb{P}_1 \to \mathbb{P}_2 \) which is equivariant with respect to a morphism of double principal groups \( \phi : \hat{G}_1 \to \hat{G}_2 \), i.e. \( \Phi(pg) = \Phi(p)\phi(g) \).

In general, any \( G \)-manifold, i.e. a manifold \( \mathbb{P} \) with an action of \( G \), gives rise to the commutative diagram of maps between the spaces of orbits:

\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\pi'} & \mathbb{P}/G' \\
\downarrow{\pi} & & \downarrow{[\pi]} \\
\mathbb{P}/G & \xrightarrow{[\pi']} & \mathbb{P}/G \\
\end{array}
\]

Conversely, if we have a commutative diagram \( \hat{\mathbb{P}} \) of (locally trivial) smooth fibrations

\[
\hat{\mathbb{P}} = \begin{array}{ccc}
\mathbb{P} & \xrightarrow{\pi'} & P' \\
\downarrow{\pi} & & \downarrow{[\pi']} \\
P & \xrightarrow{[\pi]} & P_0 \\
\end{array}
\]

on which \( G \) acts (from the left) in such a way that \( G \) acts in fibers of \( \pi \) and \( G' \) acts in fibers of \( \pi' \) (thus \( G' \) acts on \( P \) and \( G \) acts on \( P' \)), then we call \( \hat{\mathbb{P}} \) (or simply \( \mathbb{P} \)) a \( \hat{G} \)-fibered manifold. If \( P_0 \) is just one point, we speak about a \( \hat{G} \)-fibered space.

**Example 4.3.** Consider a trivial \( \hat{G} \)-fibered space \( \mathbb{P} = P \times P' \times P^{(1,1)} \) with coordinates \((y, y', z)\) and fibrations

\[
\pi(y, y', z) = (y') \quad \pi'(y, y', z) = (y)
\]

on which the double principal group \( \hat{G} = (\mathbb{G}; G, G') \) acts by

\[
\hat{g}(y, y', z) = (A(\hat{g})(y), A'(\hat{g})(y'), B(\hat{g})(y, y', z))
\]

such that \( A(\hat{g}) \) is id for \( \hat{g} \in G \) and \( A'(\hat{g}) \) is id for \( \hat{g} \in G' \).

**Example 4.4.** More specifically, define for \( \overline{d} = (d, d', d^0) \in \mathbb{N}^3 \) the trivial double space \( \mathbb{A} = \mathbb{R}_{aff}^{\overline{d}} = \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R}^{d^0} \) with canonical projections \( \rho : \mathbb{A} \to \mathbb{R}^d \) and \( \rho' : \mathbb{A} \to \mathbb{R}^{d'} \), which is viewed as a trivial double affine space (cf. [8]), i.e. we view \( \mathbb{R}^d, \mathbb{R}^{d'}, \mathbb{R}^{d^0} \) as affine spaces with affine coordinates \((y, y', z)\) and \( G = \text{Aut}(\mathbb{R}_{aff}^{d}) \) of double affine space automorphisms, i.e. diffeomorphisms of the form

\[
\hat{g}(y, y', z) = (\alpha^0_j(\hat{g})y_i, \alpha^0_j(\hat{g})y_i + (\alpha')^0_j(\hat{g})y_i', \beta^0_{ij}(\hat{g})y_i + \beta^0_{ij}(\hat{g})y_i' + \sigma^0_{ij}(\hat{g})z_0).
\]

Here \( G \) and \( G' \) act by automorphisms of the affine fibrations \((y, y', z) \to (y')\) and \((y, y', z) \to (y)\) which act trivially on \( y \) and \( y' \), respectively.

**Example 4.5.** Similarly we can consider the case of a double vector space \( \mathbb{V} \) of dimension \( \overline{d} \), \( \mathbb{V} = \mathbb{R}_{vect}^{\overline{d}} = \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R}^{d^0} \), and its group of automorphisms \( G = \text{Aut}(\mathbb{R}_{vect}^{\overline{d}}) \) which acts by

\[
\hat{g}(y, y', z) = (\alpha_j^0(\hat{g})y_i, \alpha_j^0(\hat{g})y_i + (\alpha')^0_j(\hat{g})y_i', \beta^0_{ij}(\hat{g})y_i + \beta^0_{ij}(\hat{g})y_i' + \sigma^0_{ij}(\hat{g})z_0),
\]

with the action of the subgroups \( G \) and \( G' \) given by

\[
g(y, y', z) = (y, \alpha_j^0(y)y_i', \beta^0_{ij}(y)y_i' + \sigma^0_{ij}(y)z_0),
\]
and
\[ g'(y, y', z) = (\alpha'_j(g')y_i, \ y', \beta'^u_v(g')y_i y'_a + \sigma'_v(g')z_u). \]  
(17)

Note that in this case the normal subgroup \( G_0 = G \cap G' \) of \( \text{Aut}(\mathbb{V}) = \text{Aut}(\mathbb{R}^d_{\text{vec}}) \) contains another normal subgroup consisting of statomorphism, i.e. the automorphism with all linear part vanishing:
\[ g_0(y, y', z) = (y, y', \beta'^u_v(g_0)y_i y'_a + z_v). \]  
(18)

For the theory of double vector bundles as vector bundles in the category of vector bundles we refer to the original papers by Pradines [22, 23, 24] (see also [11, 16]). A substantially simplified approach via homogeneity structures can be found in [7].

5 \( n \)-tuple principal groups and bundles

Our definition of double principal group and double principal bundle can be generalized to an \( n \)-tuple case, \( n \geq 2 \). This definition is inductive.

**Definition 5.1.** An \( n \)-tuple principal group is a system \( \hat{G} = (G; G^1 \ldots, G^n) \), where \( G \) is a Lie group and \( G^j \), \( i = 1, \ldots, n \) are its closed normal subgroups such \( \bigcup_{i=1}^n G^i \) generates \( G \) and any system
\[ \hat{G}_i = (G^i; G^1 \cap G^i, \ldots, G^{i-1} \cap G^i, G^{i+1} \cap G^i, \ldots, G^n \cap G^i) \]
is an \( (n-1) \)-tuple principal group. An \( n \)-tuple principal bundle with the structure \( n \)-tuple principal group \( \hat{G} \) is a manifold \( P \) equipped with a principal action of \( G \).

Suppose \( n \geq 2 \). From the above definition we obtain easily the following.

**Proposition 5.2.** If \( \hat{G} = (G; G^1 \ldots, G^n) \) is an \( n \)-tuple principal group, then any triple \( \hat{G}^{ij} = (G; G^i, G^j) \), \( i \neq j \), is a double principal group. Any \( n \)-tuple principal bundle \( P \) with the structure group \( \hat{G} \) is a double principal bundle with respect to the action of each double principal group \( \hat{G}^{ij} \), \( i \neq j \). Moreover, the bundle \( P \) is a principal fibration with respect to all projections \( \pi_i: P \to P_i = P/G^i \), where each \( P_i \) is itself an \( (n-1) \)-tuple principal bundle with the structure group \( \hat{G}_i \).

**Proof.** It is enough to prove that, for each \( i, j = 1, \ldots, n, i \neq j \), the set \( G^i \cup G^j \) generate \( G \). For \( n = 2 \) this is true by definition, so by the inductive assumption, each \( G^j \) is generated by \( G^i \cup G^j \). Because \( \bigcup_{i=1}^n G^i \) generate \( G \), the proposition follows. \( \Box \)

**Example 5.3.** Example [4.5] can be generalized to the \( k \)-tuple case if we start with a \( k \)-tuple vector bundle bundle, i.e. a manifold \( E \) equipped with coordinate systems \((y^a_\sigma, x_\sigma)_{(a, \sigma)} \) which yield diffeomorphisms of members of an open cover of \( E \) onto \( U \times \prod_{\sigma \in \{0,1\}^k} \mathbb{R}^{d_\sigma} \), where \( U \subset \mathbb{R}^{d_0} \), and transformation rules
\[ x^0_a = \phi_a(x^0_0), \quad x^\sigma_b = \sum_{\sigma = \sigma_1 + \ldots + \sigma_m} g_{b, \sigma_1 \ldots, \sigma_m}^b(x^0_0) x^\sigma_{b_1} \ldots x^\sigma_{b_m}. \]

Here, \( 0 = (0, \ldots, 0) \in \{0,1\}^k \) and \( \{0,1\}^k = \{0,1\}^k \setminus \{0\} \) (cf. [6]). It is easy to see that \( E \) is a polynomial fibration over some manifold \( M \) (with local coordinates \((x^0_a)\)) and fibers \( E_\sigma \) of \( k \)-tuple vector bundle spaces (with local coordinates \((x^\sigma_a)\)). All fibers are isomorphic \( k \)-tuple vector spaces,
\[ E_0 = E = \prod_{\sigma \in \{0,1\}^k} \mathbb{R}^{d_\sigma} \]
with the automorphism group \( G = \text{Aut}(E) \) acting by
\[ x^\sigma_b = \sum_{\sigma = \sigma_1 + \ldots + \sigma_m} g_{b, \sigma_1 \ldots, \sigma_m}^b(x^\sigma_{b_1} \ldots x^\sigma_{b_m}), \quad \sigma \neq 0. \]

It is easy to see that \( G \) acts linearly in factors \( \mathbb{R}^{d_{\varepsilon_i}} \) in \( \prod_{\sigma \in \{0,1\}^k} \mathbb{R}^{d_\sigma}, \varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0,1\}^k \). The following observation is trivial.

**Proposition 5.4.** The collection \( \hat{G} = (G; G^1 \ldots, G^k) \), where \( G^i \) is the subgroup of \( G \) acting identically on the factor \( \mathbb{R}^{d_{\varepsilon_i}} \) in \( \prod_{\sigma \in \{0,1\}^k} \mathbb{R}^{d_\sigma} \), is a \( k \)-tuple principal group.
6 Associated bundles and principal connections

6.1 Associated double bundles

Let now $\hat{G} = (G; G, G')$ be a double principal group, $\hat{P}$ be a double principal bundle with the structure group $\hat{G}$:

$$\hat{P} = \begin{array}{ccc}
P & \xrightarrow{\pi} & P' \\
\downarrow \pi' & & \downarrow \pi \\
\hat{P} & \xrightarrow{\hat{\pi}} & P_0
\end{array},$$

and

$$\hat{M} = \begin{array}{ccc}
M & \xrightarrow{\rho'} & M' \\
\downarrow \rho & & \downarrow \rho \\
\hat{M} & \xrightarrow{\hat{\rho}} & \{\ast\}
\end{array},$$

be a $\hat{G}$-fibered space. Then, it is easy to see that the associated bundle $\hat{P} \times G M$ is canonically a $\hat{G}$-fibered manifold, namely

$$\hat{P} \times G \hat{M} = \begin{array}{ccc}
P \times G M & \xrightarrow{\Pi'} & P \times G' M \\
\downarrow \Pi & & \downarrow \Pi \\
P' \times G M' & \xrightarrow{[\Pi']} & P_0
\end{array},$$

where

$$(p, u) \in \hat{P} \times G \hat{M} = \left\{(p, u, \bar{p}, g^{-1} u) \mid \bar{p} \in \hat{P}, u \in M, g \in G\right\},$$

and $\Pi'(p, u) = [\pi'(p), \rho'(u)]$, $\Pi(p, u) = [\pi(p), \rho(u)]$. Note that we can make this construction starting from a $G_1$-fibered space and a faithful morphism $\tau : G \to G_1$ of double principal bundles to produce the $G$-action on $M$ and the associated bundle $\hat{P} \times G \hat{M} = \hat{P} \tau \hat{M}$.

**Example 6.1.** If $\hat{M}$ is a $\hat{G}$-fibered space, fibered over $M$ and $M'$ and $\hat{P}$ is a $\hat{G}$-principal bundle, then the associated bundle $\hat{P} \times G \hat{M}$ is a $\hat{G}$-fibered manifold which is a locally trivial fibration over $P_0$, locally diffeomorphic to $U \times \hat{M}$ with the same transition functions as the $G$-principal bundle $\hat{P}$ by maps $\sigma : (U \cap U') \to \hat{G}$. Indeed, we apply the classical result on the associated bundle construction to the $G$-principal bundle $\hat{P}$. The added value is only the double fibration (22) on $\hat{P} \times G \hat{M}$.

6.2 Frame bundle of a double vector bundle

An inverse construction is the following.

**Example 6.2** (Frame bundle construction). Let

$$\hat{V} = \begin{array}{ccc}
V & \xrightarrow{\rho'} & V' \\
\downarrow \rho & & \downarrow \rho \\
V & \xrightarrow{[\rho']} & V_0
\end{array},$$
be a double vector bundle modelled on the double vector space $\mathbb{R}^d_{\text{vect}}$ and let $G = \text{Aut}(\mathbb{R}^d_{\text{vect}})$ which is canonically a double principal group (see Example 4.5). Then, for any $x \in V_0$ the space $V_x = (\rho' \circ \rho)^{-1}(\{x\})$ is a double vector space of dimension $d$ and the space $\text{Iso}(V_x, \mathbb{R}^d_{\text{vect}})$ of isomorphisms of double vector spaces is a $G$-principal space with the obvious (right) action $\psi \circ \tilde{g} = \psi \circ \tilde{g}$. The corresponding $G$-bundle $P = \text{Iso}(V, \mathbb{R}^d_{\text{vect}})$ over $V_0$ with fibers $\text{Iso}(V_x, \mathbb{R}^d_{\text{vect}})$ is canonically a double principal bundle with the structure group $G$. We call it the frame bundle of the double vector bundle $V$. Like in the classical case, one can show that the $V$ is canonically isomorphic with the associated bundle $P \times_G \mathbb{R}^d_{\text{vect}}$. This construction can be generalized to any case when the double fibration $(23)$ is locally trivial modelled on a fibre with a geometric structure admitting a Lie group as the automorphism group.

The mutually converse constructions, that of an associated bundle and the frame bundle, lead to the following ‘double analog’ of a classical result.

**Theorem 6.1.** Given a double vector space $\mathbb{R}^d_{\text{vect}}$, we have the double principal group $G = \text{Aut}(\mathbb{R}^d_{\text{vect}})$ of its automorphisms. Then the associated bundle construction gives rise to an equivalence from the category of principal $G$-bundles on $P_0$ to the category of double vector bundles of dimension $d$ on $P_0$.

Actually, the above theorem and constructions have full analogues in the $k$-tuple case.

### 6.3 Higher double graded bundles

All the above constructions for double vector bundles can be generalized to higher graded bundles.

Consider now a smooth action $h : \mathbb{R} \times F \to F$ of the monoid $(\mathbb{R}, \cdot)$ on a manifold $F$ and assume that $h_0(F) = 0^F$ for some element $0^F \in F$. Such an action we will call a homogeneity structure. The set $F$ with a homogeneity structure will be called a graded space. The reason for the name is the following theorem.

**Theorem 6.2** (Grabowski-Rotkiewicz [7]). Any graded space $(F, h)$ is diffeomorphically equivalent to a dilation structure, i.e. to a certain $(\mathbb{R}^d, h^d)$, where $d = (d_1, \ldots, d_k)$, with positive integers $d_i$, and $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ is equipped with the dilation action $h^d$ of multiplicative reals given by

$$h^d_t(y_1, \ldots, y_k) = (t \cdot y_1, \ldots, t^k \cdot y_k), \quad y_i \in \mathbb{R}^{d_i}.$$  

In other words, $F$ can be equipped with a system of (global) coordinates $(y^i_j)$, $i = 1, \ldots, k$, $j = 1, \ldots, d_i$, such that $y^i_j$ is homogeneous of degree $i$ with respect to the homogeneity structure $h$:

$$y^i_j \circ h_t = t^i \cdot y^i_j.$$  

Of course, in these coordinates $0^F = (0, \ldots, 0)$.

It is natural to call a morphism between graded spaces $(F_a, h^a)$, $a = 1, 2$, a smooth map $\Phi : F_1 \to F_2$ which intertwines the homogeneity structures:

$$\Phi \circ h^a_t = h^2_t \circ \Phi \quad (24)$$

**Theorem 6.3** (Grabowski-Rotkiewicz [7]). Any morphism of graded spaces is polynomial in homogeneous fiber coordinates $y$’s. In particular the group $\text{Aut}(\mathbb{R}^d)$ of automorphisms of $\mathbb{R}^d$ is a Lie group.

Note that automorphisms of $(\mathbb{R}^d, h^d)$ need not to be linear, so the category of graded spaces is different from that of vector spaces. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees 1, 2, respectively, then the map

$$\mathbb{R}^2 \ni (y, z) \mapsto (y, z + y^2) \in \mathbb{R}^2$$

is an automorphism of $(\mathbb{R}^d, h^d)$.
is an automorphism of the homogeneity structure, but is nonlinear.

A straightforward generalisation is the concept of a graded bundle $\tau : F \to M$ of rank $d$, with a local trivialization by $U \to \mathbb{R}^d$, and with the difference that the transition functions of local trivialisations:

$$U \cap V \times \mathbb{R}^d \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^d,$$

respect the weights of coordinates $(y^1, \ldots, y^d)$ in the fibres, i.e. $A(x, \cdot)$ are automorphisms of the graded space $(\mathbb{R}^d, h^d)$. In other words, a graded bundle of rank $d$ is a locally trivial fibration with fibers modelled on the graded space $\mathbb{R}^d$. As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. If all $w_i \leq r$, we say that the graded bundle is of degree $r$.

**Example 6.3.** Consider the second-order tangent bundle $T^2M$, i.e. the bundle of second jets of smooth maps $(\mathbb{R}, 0) \to M$. Writing Taylor expansions of curves in local coordinates $(x^A)$ on $M$:

$$x^A(t) = x^A(0) + \dot{x}^A(0)t + \ddot{x}^A(0)\frac{t^2}{2} + o(t^2),$$

we get local coordinates $(x^A, \dot{x}^B, \ddot{x}^C)$ on $T^2M$, which transform

$$x'^A = x'^A(x),$$

$$\dot{x}'^A = \frac{\partial x'^A}{\partial x^B}(x) \dot{x}^B,$$

$$\ddot{x}'^A = \frac{\partial x'^A}{\partial x^B}(x) \ddot{x}^B + \frac{\partial^2 x'^A}{\partial x^B \partial x^C}(x) \dddot{x}^B \dddot{x}^C.$$

This shows that associating with $(x^A, \dot{x}^B, \ddot{x}^C)$ the weights $0, 1, 2$, respectively, will give us a graded bundle structure of degree $2$ on $T^2M$. Note that, due to the quadratic terms above, this is not a vector bundle. All this can be generalised to higher tangent bundles $T^kM$.

One can pick an atlas of the graded bundle $F$ consisting of charts for which the degrees of homogeneous local coordinates $(x^A, y^a_w)$ are $\deg(x^A) = 0$ and $\deg(y^a_w) = w, \ 1 \leq w \leq k$, where $k$ is the degree of the graded bundle. The local changes of coordinates are of the form

$$x'^A = x'^A(x),$$

$$y'^a_w = y^b_w T^a_{b^w}(x) + \sum_{1 \leq a \leq k} \frac{1}{n!} b_1^{w_1} \cdots b_n^{w_n} T^a_{b_1 \cdots b_n}(x),$$

where $T^a_{b^w}$ are invertible and $T^a_{b_1 \cdots b_n}$ are symmetric in indices $b_1, \ldots, b_n$.

Note that the homogeneity structure in the typical fiber of a graded bundle $F$, i.e. the action $h : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, is preserved under the transition functions, that defines a globally defined homogeneity structure $h : \mathbb{R} \times F \to F$. In local homogeneous coordinates it reads

$$h_t(x^A, y^a_w) = (x^A, t^w y^a_w).$$

We call a function $f : F \to \mathbb{R}$ homogeneous of degree (weight) $w$ if

$$f \circ h_t = t^w \cdot f.$$

The whole information about the degree of homogeneity is contained in the weight vector field (for vector bundles called the Euler vector field)

$$\nabla_F = \sum_i w y^a_i \partial y^a_i,$$

so $f : F \to \mathbb{R}$ is homogeneous of degree $w$ if and only if $\nabla_F(f) = w \cdot f$. Clearly, the fiber bundle morphism $\Phi$ is a smooth map which relates the weight vector fields $\nabla_{F^1}$ and $\nabla_{F^2}$.

The fundamental fact is that graded bundles and homogeneity structures are actually equivalent concepts.
Theorem 6.4 (Grabowski-Rotkiewicz [7]). For any homogeneity structure $h$ on a manifold $F$, there is a smooth submanifold $M$ of $F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi^k : F \to T^k F|_M$ which identifies $F$ with a graded submanifold of the graded bundle $T^k F$. In particular, there is an atlas on $F$ consisting of local homogeneous coordinates.

The theory of vector bundles is a part of the theory of graded bundles thanks to the following.

Theorem 6.5 (Grabowski-Rotkiewicz [7]). In the above terminology, vector bundles are just graded bundles of degree 1. The corresponding homogeneity structure is determined by the $\mathbb{R}$-action by homotheties. The corresponding weight vector field is the Euler vector field.

Since morphisms of two homogeneity structures are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_s = h^2_s \circ \Phi$, this describes also morphism of graded bundles. Consequently, a graded subbundle of a graded bundle $F$ is a smooth submanifold $S$ of $F$ which is invariant with respect to homotheties, $h^t(S) \subset S$ for all $t \in \mathbb{R}$.

Definition 6.4. A double graded bundle is a manifold equipped with two homogeneity structures $h^1, h^2$ which are compatible in the sense that

$$h^1_s \circ h^2_t = h^2_s \circ h^1_t$$

for all $s, t \in \mathbb{R}$.

The above condition can also be formulated as commutation of the corresponding weight vector fields, $[\nabla^1, \nabla^2] = 0$. For vector bundles this is equivalent to the concept of a double vector bundle in the sense of Pradines.

Theorem 6.6 (Grabowski-Rotkiewicz [6]). The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

All this can be extended to $n$-fold graded bundles in the obvious way:

Definition 6.5. An $n$-fold vector bundle (n-fold graded bundle) is a manifold $F$ equipped with $n$ homogeneity structures $h^1, \ldots, h^n$ of vector (graded) bundle structures which are compatible in the sense that

$$h^i_s \circ h^j_t = h^j_t \circ h^i_s$$

for all $s, t \in \mathbb{R}$ and $i, j = 1, \ldots, n$.

If $h^1 \circ \cdots \circ h^n(F)$ is just a single point, we speak about an $n$-fold vector (graded space).

Example 6.6. If $\tau : F \to M$ is a graded bundle of degree $k$, then there are canonical lifts of the graded structure to the tangent and to the cotangent bundle. In this way $TF$ and $T^*F$ carry canonical double graded bundle structure: one is the obvious vector bundle, the other is the lifted one (of degree $k$). There are also lifts of graded structures on $F$ to $T^r F$.

Like in the $n$-tuple vector space case, the group $\mathcal{G}$ of automorphisms of an $n$-tuple graded space, i.e. diffeomorphisms respecting each of homogeneity structures, are polynomials of a fixed degree in $n$-homogeneous coordinates. This means that the automorphism group of an $n$-tuple graded space is a Lie group. The subgroups $G^i$ acting by identity on the homogeneous functions of $n$-degree $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0, 1\}^n$ are normal subgroups. By a full analogy to the case of $n$-tuple vector bundles we have the following (cf. Proposition 5.4 and Theorem 6.1).

Theorem 6.7. If $(F, h^1, \ldots, h^n)$ is an $n$-tuple graded space and $\mathcal{G}$ is the Lie group of its automorphism, then the collection $\hat{\mathcal{G}} = (\mathcal{G}, G^1, \ldots, G^n)$, where $G^i$ is the subgroup of $\mathcal{G}$ acting identically on homogeneous functions of $n$ degree $\varepsilon_i$, $i = 1, \ldots, n$, is an $n$-tuple principal group. The associated bundle construction gives rise to an equivalence from the category of principal $\mathcal{G}$-bundles on $\mathbb{P}_0$ to the category of $n$-tuple graded bundles with the model fiber $(F, h^1, \ldots, h^n)$.
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