Let $\mathbf{R}$ be a commutative ring with unity $1$ and zero $0$.

Let $x = (x_1, \ldots, x_n)$ be variables, $\mathbf{R}[x]$ is the ring of all polynomials in variables $x$ with coefficients in the ring $\mathbf{R}$.

The degree of a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$. The degree of a polynomial $F(x)$ is called the maximal degree of a monom with a nonzero coefficient, and such a degree is denoted by $\deg(F)$; if $F(x) = 0$, then put $\deg(F) = -\infty$.

**Definition 1.** Let $x = (x_1, \ldots, x_n)$ be variables; we denote by $\mathbf{R}[x]_{\leq d}$ the set of all polynomials of degree $\leq d$. Note that $\mathbf{R}[x]_{\leq \infty} = \mathbf{R}[x]$ and if $d < 0$, then $\mathbf{R}[x]_{\leq d} = \{0\}$.

**Definition 2.** Let $x = (x_1, \ldots, x_n)$ be variables; we denote by $\mathbf{R}[x]_*$ the set of all maps from $\mathbf{R}[x]$ to $\mathbf{R}$ that are linear over $\mathbf{R}$, write such maps as $l(x_*)$, where $x_* = (x_1^*, \ldots, x_n^*)$, and call such maps linear functionals or simply functionals. We denote the action of $l(x_*)$ on $F(x) \in \mathbf{R}[x]$ by $l(x_*)F(x)$.

**Definition 3.** Let $x = (x_1, \ldots, x_n)$ be variables, and let $f(x) = (f_1(x), \ldots, f_s(x))$ be polynomials.

For a covector of polynomials $g(x) = (g^1(x), \ldots, g^s(x))^\top$, we denote $f(x)g(x) = \sum_{i=1}^{s} f_i(x)g^i(x)$.

Denote $(f(x))_{\leq d} = \{\sum_{i=1}^{s} f_i(x)g^i(x) | \forall i = 1, s : g^i(x) \in \mathbf{R}[x] \text{ and } \deg(f_i) + \deg(g^i) \leq d\}$.

Denote $(f(x))_* = \{\sum_{i=1}^{s} f_i(x)g^i(x) | \forall i = 1, s : g^i(x) \in \mathbf{R}[x]\}$.

Note that $(f(x))_{\leq \infty} = (f(x))_*$, and if $d < 0$, then $(f(x))_{\leq d} = \{0\}$.

We call a functional in $\mathbf{R}[x]_*$ that annihilates $(f(x))_*$ a root functional, and a functional in $\mathbf{R}[x]_*$ that annihilates $(f(x))_{\leq d}$ a bounded root functional.

**Definition 4.** Let $x = (x_1, \ldots, x_n)$ be variables, and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$; we denote by $1_x(\lambda) = 1_{(x_1, \ldots, x_n)}(\lambda_1, \ldots, \lambda_n)$ the map such that $1_x(\lambda).F(x) = F(\lambda)$ for any $F(x) \in \mathbf{R}[x]$.

**Definition 5.** Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ be variables. We call a difference derivative of a polynomial $F(x) \in \mathbf{R}[x]$ a covector $\hat{u} \nabla F(x, y) = \sum_{k=1}^{n} \hat{u}_k \nabla^k F(x, y)$, such that

$$
\hat{u} \nabla F(x, y) = \sum_{k=1}^{n} \hat{u}_k \nabla^k F(x, y) \mapsto (x - y) \nabla F(x, y) = \sum_{k=1}^{n} (x_k - y_k) \nabla^k F(x, y) = F(x) - F(y),
$$

where $\forall k = 1, n : \nabla^k F(x, y) \in \mathbf{R}[x, y]$. 

We call a difference derivative monotonous if the degree of $\nabla F(x,y)$ in $(x,y)$ is $\leq \deg(F) - 1$.

We call a mapping that linear over $\mathbb{R}$ and assign, to a polynomial $F(x) \in \mathbb{R}[x]$, a covector of a difference derivative $\nabla F(x,y)$, an operator of difference derivative and denote it by $\nabla_x(x,y)$; then, we have $\nabla_x(x,y)F(x) = \nabla F(x,y)$. Moreover,

$$\hat{u}\nabla x(x,y) = \sum_{k=1}^{n} \hat{u}_k \nabla^k x(x,y) \mapsto (x-y)\nabla x = \sum_{k=1}^{n} (x_k - y_k)\nabla^k x(x,y) = \mathbf{1}_x(x) - \mathbf{1}_x(y).$$

We call an operator of difference derivative monotonous if, for any polynomial $F(x) \in \mathbb{R}[x]$, the degree of $\nabla_x(x,y)F(x)$ is $\leq \deg(F) - 1$.

**Lemma 1.** Let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ be variables. A difference derivative of a polynomial $F(x)$ exists, for example, $\forall k = 1, n$:

$$\nabla^k F(x,y) = \frac{F(y_1,\ldots,y_{k-1},x_k,x_{k+1},\ldots,x_n) - F(y_1,\ldots,y_{k-1},y_k,x_{k+1},\ldots,x_n)}{x_k - y_k},$$

its degree is $\leq \deg(F) - 1$. A mapping that assigns, to any polynomial $F(x)$, a covector $\nabla F(x,y)$ is linear over $\mathbb{R}$. Thus, there exists a monotonous difference derivative and a monotonous operator of difference derivative.

**Lemma 2.** Let $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$, and $\hat{u} = (\hat{u}_1,\ldots,\hat{u}_n)$ be variables.

1. For any polynomial $F(x)$, a covector $\hat{u}\nabla F(x,y) = \hat{u}\nabla F(y,x)$ is a difference derivative of the polynomial $F(x)$, and $\hat{u}\nabla^k_x(x,y) = \hat{u}\nabla^k_y(x,y)$ is a operator of difference derivative.

2. Let $V(x) = F(x) \cdot G(x)$, then $\hat{u}\nabla F(x,y) \cdot G(y) + F(x) \cdot \hat{u}\nabla G(x,y)$ is a difference derivation of the polynomial $V(x)$.

3. Let $F(x) \in \mathbb{R}[x]$ and let $\nabla^l F(x,y)$ and $\nabla^{kl} F(x,y)$ be two difference derivatives of the polynomial $F(x)$ of degrees $\leq d - 1$; then

$$\hat{u}\nabla^l F(x,y) = \hat{u}\nabla^{kl} F(x,y) + \sum_{k,l} ((x_k - y_k) \cdot \hat{u}_l - (x_l - y_l) \cdot \hat{u}_k) \cdot T^{kl}(x,y),$$

where $k < l$ and $\deg(T^{kl}) \leq d - 2$.

**Proof 1.**

$$\begin{align*}
(x-y) \cdot \nabla F(x,y) &= (x-y) \cdot \nabla F(y,x) \\
&= -(y-x) \cdot \nabla F(y,x) = -(F(y) - F(x)) = F(x) - F(y)
\end{align*}$$

It follows from the first part of Statement 1 that $\hat{u}\nabla^l_x(x,y) = \hat{u}\nabla^l_y(x,y)$ assigns, to any polynomial $F(x) \in \mathbb{R}[x]$, its a difference derivative. The linearity of the map $\hat{u}\nabla^l_x(x,y)$ over $\mathbb{R}$ implies the linearity of the map $\hat{u}\nabla^l_x(x,y) = \hat{u}\nabla^l_y(x,y)$ over $\mathbb{R}$. We finally obtain that $\hat{u}\nabla^l_x(x,y)$ is an operator of difference derivative.

**Proof 2.**

$$\begin{align*}
((x-y) \cdot \nabla F(x,y)) \cdot G(y) + F(x) \cdot ((x-y) \cdot \nabla G(x,y)) &= \\
&= (F(x) - F(y)) \cdot G(y) + F(x) \cdot (G(x) - G(y)) = \\
&= F(x) \cdot G(x) - F(y) \cdot G(y) = V(x) - V(y).
\end{align*}$$

**Proof 3.** Set $W^k(x,y) = \nabla^k F(x,y) - \nabla^{mk} F(x,y)$, and set

$$T^{kl}(x,y) = \nabla^k x(x,y)W^l(x,y) = \frac{1}{x_k - y_k} \cdot (W^l(y_{<k},x_k,x_{>k},y) - W^l(y_{<k},y_k,x_{>k},y)).$$

36  ISSN 1025-6415  Reports of the National Academy of Sciences of Ukraine, 2002, no. 7
It is directly verified that the equality in the statement is true. Further, since the degrees of difference derivatives $\nabla F(x, y)$ and $\nabla^n F(x, y)$ are $\leq d - 1$, then we have $\deg(W^i) \leq d - 1$, hence, $\deg(T^{kl}) \leq \deg(W^i) - 1 \leq d - 2$.

**Assumption 1.** In the sequel, unless otherwise stated, we will consider only monotonous difference derivatives of polynomials and only monotonous operators of difference derivative.

If $x = (x_1, \ldots, x_n)$ are variables, then by $y \simeq x$ we mean $y = (y_1, \ldots, y_n)$.

**Theorem 1.** Let $x = (x_1, \ldots, x_n)$ and $y \simeq x$ be variables, let $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, let $\delta_f = \sum_{i=1}^n (\deg(f_i) - 1)$, and let $F(x) \in \mathbb{R}[x^{\leq d}]$; then we have

$$\det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{bmatrix} = \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(y) \end{bmatrix}.$$  

Denote this polynomial by $R(x, y)$; then we have the following:

$R(x, y)$ have a degree $\leq \delta_f + d$,

$R(x, y)$ is uniquely determined up to an addend of the form

$$\sum_{i,j} (f_i(x) \cdot f_j(y) - f_i(y) \cdot f_j(x)) \cdot \omega^{ij}(x, y)$$

independently of the choice of $\nabla F(x, y)$, where $i < j$ and $\deg(f_i) + \deg(f_j) + \deg(\omega^{ij}) \leq \delta_f + d$,

$R(x, y)$ is uniquely determined up to an addend of the form

$$\sum_{i,j} (f_i(x) \cdot f_j(y) - f_i(y) \cdot f_j(x)) \cdot \omega^{ij}(x, y) + \sum_i (f_i(x) \cdot F(y) - f_i(y) \cdot F(x)) \cdot \Omega^i(x, y)$$

independently of the choice of $\nabla f(x, y)$, where $i < j$ and $\deg(f_i) + \deg(f_j) + \deg(\omega^{ij}) \leq \delta_f + d$ for the first summand, and $\deg(f_i) + \deg(\Omega^i) \leq \delta_f$ for the second summand.

**Proof.** Since $f_i(x) - \sum_{k=1}^n (x_k - y_k) \cdot \nabla^k f_i(x, y) = f_i(x) - (f_i(x) - f_i(y)) = f_i(y)$ and $F(x) - \sum_{k=1}^n (x_k - y_k) \cdot \nabla^k F(x, y) = F(x) - (F(x) - F(y)) = F(y)$, by adding, to the last row, the linear combination of the rest rows of the first determinant matrix, we obtain the second determinant matrix. It implies the equality of determinants.

It follows from the monotony of a difference derivative that the degree of $\nabla F(x, y)$ is $\leq d - 1$; and the degree of $\nabla f_i(x, y)$ is $\leq \deg(f_i) - 1$ for any $i$, then the degree of the polynomial $R(x, y)$ is

$$\sum_{i=1}^n (\deg(f_i) - 1) + (\deg(F) - 1) + 1 \leq \delta_f + d.$$

Since the degree of $\nabla F(x, y)$ is $\leq d - 1$, by Statement 3 of Lemma 2, variation of $\nabla F(x, y)$ is of the form

$$\hat{u} \cdot \nabla^l F(x, y) = \hat{u} \cdot \nabla F(x, y) + \sum_{k,l} ((x_k - y_k) \cdot \hat{u}_l - (x_l - y_l) \cdot \hat{u}_k) \cdot T^{kl}(x, y),$$

where $k < l$, and $\deg(T^{kl}) \leq d - 2$. Then $R(x, y)$ is uniquely determined up to the addend

$$\sum_{k,l} \pm \det \begin{bmatrix} \nabla^{\neq k,l} f(x, y) & 0 \\ \nabla^k f(x, y) & -(x_l - y_l) \\ \nabla^l f(x, y) & (x_k - y_k) \\ f(x) & 0 \end{bmatrix} \cdot T^{kl}(x, y) =$$
where \( i < j \). The second equality is true since \( \forall i = 1, n : \)

\[
-f_i(y) = (x_k - y_k) \cdot \nabla^k f_i(x, y) + (x_l - y_l) \cdot \nabla^l f_i(x, y) +
\]

\[
+ \sum_{m \neq k, l} (x_m - y_m) \cdot \nabla^m f_i(x, y) - f_i(x),
\]

i. e., the last but one row of the third determinant matrix is the sum of the last but one row and the linear combination of the rest row of the second determinant matrix. The last equality is obtained by decomposition of the determinant into minors of the two last rows. Moreover, we have \( \deg(f_i) + \deg(f_j) + \deg(\omega^{ij}) \leq \delta_f + d \).

Permuting \( f_i(x) \) and \( F(x) \) in the statement proved above, we obtain that \( R(x, y) \) is uniquely determined up to an addend

\[
\sum_{i,j \neq t} (f_i(x) \cdot f_j(y) - f_i(y) \cdot f_j(x)) \cdot \omega^{ij}(x, y) + \sum_{i \neq t} (f_i(x) \cdot F(y) - f_i(y) \cdot F(x)) \cdot \Omega^t(x, y)
\]

under lack of uniqueness of \( \nabla f_i(x, y) \), where \( i < j \) and \( \deg(f_i) + \deg(f_j) + \deg(\omega^{ij}) \leq \delta_f + d \) for the first summand, and \( \deg(f_i) + \deg(F) + \deg(\Omega^t) \leq \delta_f + \deg(F) \), hence, \( \deg(f_i) + \deg(\Omega^t) \leq \delta_f \), for the second summand. Summing the additional addends appearing on changing \( \nabla f_i(x, y) \) for all \( t = 1, n \), we obtain that \( R(x, y) \) is uniquely determined up to an addend of the form

\[
\sum_{i,j} (f_i(x) \cdot f_j(y) - f_i(y) \cdot f_j(x)) \cdot \omega^{ij}(x, y) + \sum_i (f_i(x) \cdot F(y) - f_i(y) \cdot F(x)) \cdot \Omega^t(x, y)
\]

under lack of uniqueness of \( \nabla f_i(x, y) \), where \( i < j \) and \( \deg(f_i) + \deg(f_j) + \deg(\omega^{ij}) \leq \delta_f + d \) for the first summand, and \( \deg(f_i) + \deg(\Omega^t) \leq \delta_f \) for the second summand.

**Theorem 2.** Let \( x = (x_1, \ldots, x_n) \) and \( y \simeq x \) be variables, \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials, let \( \delta_f = \sum_{i=1}^n (\deg(f_i) - 1) \), let a functional \( L(x) \) annuls \( (f(x))_{x}^{\leq \delta_f + \delta} \), where \( \delta \geq 0 \), and let \( F(x) \in \mathbb{R}[x^{\leq d}] \). We set

\[
H(x) = L(y_*), \det \left| \begin{array}{cc} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{array} \right| = L(y_*), \det \left| \begin{array}{cc} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(y) \end{array} \right|.
\]

Then we have the following:

1. \( H(x) \in \mathbb{R}[x^{\leq \max(\delta_f, d-\delta-1)}] \).

2. \( H(x) \) is uniquely determined up to an addend in \( (f(x))_{x}^{\leq \max(\delta_f, d-\delta-1)} \), independently of the choice of \( \nabla f(x, y) \), and uniquely determined up to an addend in \( (f(x))_{x}^{\leq d-\delta-1} \), independently of the choice of \( \nabla F(x, y) \).
3. If $F(x) \in (f(x))_{x \leq d}^{\leq d}$, then $H(x) \in (f(x))_{x \leq d}^{\leq d-\delta}$.

4. $H(x)$ is uniquely determined up to an addend in $(f(x))_{x \leq d}^{\leq d-\delta}$, independently of the determination of $L(x)$ outside $R[x^{\leq \delta}]$.

**Proof 1.** We have

$$H(x) = L(y_*) \cdot \det \begin{vmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(y) \end{vmatrix} = L(y_*) \cdot \det \| \nabla f(x, y) \| + L(y_*) \cdot \det \begin{vmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & 0 \end{vmatrix}.$$  

The first summand $\in R[x^{\leq \delta}]$, and the second summand $\in L(y_*) \sum_{\alpha, \beta} (f(y))^{\leq \alpha} \cdot R[x^{\leq \beta}]$, where $\alpha + \beta \leq \delta_d + d$. Since $L(y_*)$ annuls $(f(y))^{\leq \delta_d + \delta}$, without changing of the sum we can retain only these terms for which $\alpha \geq \delta_d + \delta + 1$, this means that $-\alpha \leq -\delta_d + \delta + 1$, and, hence, for the remaining terms, we have $\beta = (\alpha + \beta) - \alpha \leq (\delta_d + d) - (\delta_d + \delta + 1) = d - \delta - 1$. Hence, the second summand $\in \sum_{\beta} R[x^{\leq \delta}] \subseteq R[x^{\leq d-\delta-1}]$, where $\beta \leq d - \delta - 1$. Then the sum of the both summands $\in R[x^{\leq \delta}] + R[x^{\leq d-\delta-1}] \subseteq R[x^{\leq \max(\delta_d, d-\delta-1)}]$. Hence, we have $H(x) \in R[x^{\leq \max(\delta_d, d-\delta-1)}]$.

**Proof 2.** Under lack of uniqueness of $\nabla F(x, y)$, by Theorem 1, $H(x)$ is uniquely determined up to an addend

$$L(y_*) \cdot \sum_{i, j} f_i(x) \cdot f_j(y) \cdot \omega^j(x, y) \in L(y_*) \cdot \sum_{\alpha, \beta} (f(y))^{\leq \alpha} \cdot (f(x))^{\leq \beta},$$

where $\alpha + \beta \leq \delta_d + d$. The last inclusion is true since $\forall i : \deg(f_i) + \deg(f_j) + \deg(\omega^j) \leq \delta_d + d$. Since $L(y_*)$ annuls $(f(y))^{\leq \delta_d + \delta}$, without changing of the sum we can retain only these terms for which $\alpha \geq \delta_d + \delta + 1$; this means that $-\alpha \leq -\delta_d + \delta + 1$, and, hence, for the remaining terms, we have $\beta = (\alpha + \beta) - \alpha \leq (\delta_d + d) - (\delta_d + \delta + 1) = d - \delta - 1$. Hence, this addend $\in \sum_{\beta} R[x^{\leq \delta}] \subseteq (f(x))^{\leq d-\delta-1}$, where $\beta \leq d - \delta - 1$. We finally obtain that, under lack of uniqueness of $\nabla F(x, y)$, $H(x)$ is uniquely determined up to an addend in $(f(x))^{\leq d-\delta-1}$.

Under lack of uniqueness of $\nabla f(x, y)$, by Theorem 1, $H(x)$ is uniquely determined up to an addend of the form

$$L(y_*) \cdot \sum_{i, j} f_i(x) \cdot f_j(y) \cdot \omega^j(x, y) + L(y_*) \cdot \sum_i (f_i(x) \cdot F(y) - f_i(y) \cdot F(x)) \cdot \Omega^i(x, y),$$

where $\forall i, j : \deg(f_i) + \deg(f_j) + \deg(\omega^j) \leq \delta_d + d$; $\forall i : \deg(f_i) + \deg(\Omega^j) \leq \delta_d$. As shown above, the first summand $\in (f(x))^{\leq d-\delta-1}$. Since $\sum_i (-f_i(y) \cdot F(x) \cdot \Omega^i(x, y)) \in (f(y))^{\leq d-\delta-1} \cdot R[x]$, it is annulled by $L(y_*)$. The polynomial $\sum_i (f_i(x) \cdot F(y) \cdot \Omega^i(x, y)) \in R[y] \cdot (f(x))^{\leq d-\delta-1}$. Acting by $L(y_*)$ on this polynomial, we obtain a polynomial $\in (f(x))^{\leq d-\delta-1}$. We finally obtain that this sum $\in (f(x))^{\leq d-\delta-1} + (f(x))^{\leq d-\delta-1} \subseteq (f(x))^{\leq \max(\delta_d, d-\delta-1)}$. Hence, under lack of uniqueness of $\nabla f(x, y)$, $H(x)$ is uniquely determined up to an addend in $(f(x))^{\leq \max(\delta_d, d-\delta-1)}$. 

ISSN 1025-6415 Dopovidi Natsionalnoi Akademiyi Nauk Ukraini, 2002, no. 7
Proof 3. In the proofs of Theorems 1 and 2, we use a weaker condition than the condition under which a difference derivative of the polynomial \( F(x) \) is monotonous, namely, the condition under which its degree is \( \leq d - 1 \). Hence, these theorems are true if the last condition is satisfied instead of the first condition.

Let \( F(x) = f(x)g(x) \in (f(x))^{\leq d} \). By Statement 2 of Lemma 2, \( F(x) \) have two difference derivatives \( \nabla F(x, y) \) and \( \nabla f(x, y)g(y) + f(x)\nabla g(x, y) \), and their degrees are \( \leq d - 1 \), although the second difference derivative may be not monotonous when \( \deg(F) < d \). We have

\[
H(x) = L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(x) \end{bmatrix} = (f(x))^{\leq d - 1}
\]

(by Statement 2 on the uniqueness of \( H(x) \) under lack of uniqueness of \( \nabla F(x, y) \)

\[
\equiv L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla f(x, y)g(y) + f(x)\nabla g(x, y) \\ f(y) & f(y)g(y) \end{bmatrix} = 
\equiv L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla g(x, y) \\ f(y) & 0 \end{bmatrix} \in L(y_\ast) \cdot \sum_{\alpha, \beta}(f(y))^{\leq \alpha} \cdot (f(x))^{\leq \beta},
\]

where \( \alpha + \beta \leq \delta_f + d \). Since the functional \( L(y_\ast) \) annuls \( (f(y))^{\leq \delta_f + \delta} \), without changing the sum, we can retain only these terms for which \( \alpha \geq \delta_f + \delta + 1 \), this means that \( -\alpha \leq -(\delta_f + \delta + 1) \), and, hence, for the remaining terms, we have \( \beta = (\alpha + \beta) - \alpha \leq (\delta_f + d) - (\delta_f + \delta + 1) = d - \delta - 1 \). Hence, the obtained polynomial \( \in \sum_{\beta}(f(x))^{\leq \beta} \subseteq (f(x))^{\leq d - \delta - 1} \), where \( \beta \leq d - \delta - 1 \). Since difference of \( H(x) \) and the obtained polynomial \( \in (f(x))^{\leq d - \delta - 1} \), we have \( H(x) \in (f(x))^{\leq d - \delta - 1} \).

Proof 4. Let \( L'(x_\ast) = L(x_\ast) \in R[x^{\leq \delta_f + \delta}] \), then it, as well as the functional \( L(x_\ast) \), annuls \( (f(x))^{\leq \delta_f + \delta} \subseteq R[x^{\leq \delta_f + \delta}] \), and \( l(x_\ast) = L'(x_\ast) - L(x_\ast) \) annuls \( R[x^{\leq \delta_f + \delta}] \). We have

\[
L'(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{bmatrix} = L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(x) \end{bmatrix} = 
L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) & \nabla F(x, y) \\ f(x) & F(x) & 0 \end{bmatrix} = 
L(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & F(x) \end{bmatrix} + l(y_\ast) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(x) & 0 \end{bmatrix}.
\]

Since \( l(y_\ast) \) annuls \( R[y^{\leq \delta_f + \delta}] \subseteq R[y^{\leq \delta_f}] \) and \( \det \| \nabla f(x, y) \| \in R[y^{\leq \delta_f}] : R[x] \), the first summand is equal to 0. The second summand \( \in l(y_\ast) \cdot \sum_{\alpha, \beta} R[y^{\leq \alpha}] \cdot (f(x))^{\leq \beta}, \) where \( \alpha + \beta \leq \delta_f + d \). Since \( l(y_\ast) \) annuls \( R[y^{\leq \delta_f + \delta}] \), without changing the sum we can retain only these terms for which \( \alpha \geq \delta_f + \delta + 1 \), this means that \( -\alpha \leq -(\delta_f + \delta + 1) \), and, hence, for the remaining terms \( \beta = (\alpha + \beta) - \alpha \leq (\delta_f + d) - (\delta_f + \delta + 1) = d - \delta - 1 \). Hence, the obtained polynomial \( \in \sum_{\beta}(f(x))^{\leq \beta} \subseteq (f(x))^{\leq d - \delta - 1} \), where \( \beta \leq d - \delta - 1 \). We finally obtain that \( H(x) \) is uniquely determined up to an addend in \( (f(x))^{\leq d - \delta - 1} \), independently of the determination of \( L(x_\ast) \) outside \( R[x^{\leq \delta_f + \delta}] \).
Theorem 3. Let \( x = (x_1, \ldots, x_n) \) and \( y \simeq x \) be variables, let \( f(x) = (f_1(x), \ldots, f_n(x)) \) be polynomials, and let \( \delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1) \). Let \( \forall i = 1, 2 : L_i(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_i} \), where \( \delta_i \geq 0 \). We set

\[
L(x_*) = L_1(x_*)L_2(y_*), \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(x) & 1_x(x) \end{bmatrix} =
= L_1(x_*)L_2(y_*), \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(y) & 1_y(y) \end{bmatrix}.
\]

Then we have the following:

1. \( L(x_*) \) is uniquely determined in \( \mathbb{R}^{x_{\leq \delta_f+\delta_1+\delta_2+1}} \), independently of the choice of \( \nabla f(x, y) \) and the choice of \( \nabla x(x, y) \).

2. \( L(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1+\delta_2+1} \).

3. \( L(x_*) \) is uniquely determined in \( \mathbb{R}^{x_{\leq \delta_f+\delta_1+\delta_2+1}} \), independently of the determination of \( L_1(x_*) \) outside \( \mathbb{R}^{x_{\leq \delta_f+\delta_1}} \), and the determination of \( L_2(x_*) \) outside \( \mathbb{R}^{x_{\leq \delta_f+\delta_2}} \).

Proof. Since \( \nabla x(x, y) \) is an operator linear over \( \mathbb{R} \), \( L(x_*) \) is a map that linear over \( \mathbb{R} \), i.e., it is a linear functional. Let a polynomial \( F(x) \in \mathbb{R}^{x_{\leq \delta_f+\delta_1+\delta_2}} \). Set \( d = \delta_f + \delta_1 + \delta_2 + 1 \) and \( \delta = \delta_2 \). Then \( L_2(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta} \) and \( F(x) \in \mathbb{R}^{x_{\leq d}} \). Also, we have \( d - \delta - 1 = (\delta_f + \delta_1 + \delta_2 + 1) - \delta_2 - 1 = \delta_f + \delta_1 \), \( \max(\delta_f, d - \delta - 1) = \max(\delta_f, \delta_f + \delta_1) = \delta_f + \delta_1 \), since \( \delta \geq 0 \). Set

\[
H(x) = L_2(y_*), \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(y) \end{bmatrix},
\]

then

\[
L(x_*)F(x) = L_1(x_*)L_2(y_*), \det \begin{bmatrix} \nabla f(x, y) & \nabla F(x, y) \\ f(y) & F(y) \end{bmatrix} = L_1(x_*)H(x).
\]

By Statement 1 of Theorem 2, \( H(x) \in \mathbb{R}^{x_{\leq \max(\delta_f, d - \delta - 1)}} = \mathbb{R}^{x_{\leq \delta_f+\delta}} \).

Proof 1. By Statement 2 of Theorem 2, the polynomial \( H(x) \) is uniquely determined up to an addend in \( (f(x))^\ell_{d - \delta - 1} = (f(x))^\ell_{\delta_f+\delta_1} \), independently of the choice of \( \nabla F(x, y) \), and is uniquely determined up to an addend in \( (f(x))^\ell_{\max(\delta_f, d - \delta - 1)} = (f(x))^\ell_{\delta_f+\delta_1} \), independently of the choice of \( \nabla f(x, y) \). Since \( L_1(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1} \), \( L_2(x_*)F(x) = L_1(x_*) \) is uniquely determined, independently of the choice of \( \nabla x(x, y) \). \( F(x) = \nabla F(x, y) \), and the choice of \( \nabla f(x, y) \).

From the arbitrariness of \( F(x) \in \mathbb{R}^{x_{\leq \delta_f+\delta_1+\delta_2+1}} \), we obtain that \( L_1(x_*) \) is uniquely determined in \( \mathbb{R}^{x_{\leq \delta_f+\delta_1+\delta_2+1}} \), independently of the choice of \( \nabla f(x, y) \) and the choice of \( \nabla x(x, y) \).

Proof 2. Let \( F(x) \in (f(x))^\ell_{\delta_f+\delta_1+\delta_2+1} = (f(x))^\ell_{\delta_f+\delta_1} \), then, by Statement 3 of Theorem 2, the polynomial \( H(x) \in (f(x))^\ell_{d - \delta - 1} = (f(x))^\ell_{\delta_f+\delta_1} \), and since \( L_1(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1} \), we have \( L(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1+\delta_2+1} \), and since \( L(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1} \), \( L_2(x_*)F(x) = L_1(x_*)H(x) = 0 \). From the arbitrariness of \( F(x) \in (f(x))^\ell_{\delta_f+\delta_1+\delta_2+1} \), we obtain that \( L(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1+\delta_2+1} \).

Proof 3. By Statement 4 of Theorem 2, \( H(x) \) is uniquely determined up to an addend in \( (f(x))^\ell_{d - \delta - 1} = (f(x))^\ell_{\delta_f+\delta_1} \), independently of the determination of \( L_2(x_*) \) outside \( \mathbb{R}^{x_{\leq \delta_f+\delta}} \). Since \( L(x_*) \) annuls \( (f(x))^\ell_{\delta_f+\delta_1} \), \( L_2(x_*)F(x) = L_1(x_*)H(x) \) is uniquely determined independently of the determination of \( L_2(x_*) \) outside \( \mathbb{R}^{x_{\leq \delta_f+\delta_2}} \).
Since the polynomial $H(x) \in \mathbb{R}[x^{\leq \delta_f + \delta_i}]$, $L(x_*)F(x) = L_1(x_*)H(x)$ is uniquely determined independently of the determination of $L_1(x_*)$ outside $\mathbb{R}[x^{\leq \delta_f + \delta_i}]$.

Hence, it follows from the arbitrariness of $F(x) \in \mathbb{R}[x^{\leq \delta_f + \delta_i + \delta_2 + 1}]$ that the functional $L(x_*)$ is uniquely determined in $\mathbb{R}[x^{\leq \delta_f + \delta_i + \delta_2 + 1}]$, independently of the determination of $L_1(x_*)$ outside $\mathbb{R}[x^{\leq \delta_f + \delta_i}]$, and the determination of $L_2(x_*)$ outside $\mathbb{R}[x^{\leq \delta_f + \delta_2}]$.

**Theorem 4.** Let $x = (x_1, \ldots, x_n)$ and $y \preceq x$ be variables, let $f(x) = (f_1(x_1, \ldots, f_n(x))$ be polynomials, and let $\delta_f = \sum_{i=1}^n (\deg(f_i) - 1)$. Let $\forall i = 1, 2: L_i(x_*)$ annuls $(f(x))_{x^{\leq \delta_f + \delta_i}}$, where $\delta_i \geq 0$; then we have

$$L_1(x_*)L_2(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(x) & 1_x(x) \end{bmatrix} = L_1(x_*) \cdot L_2(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(y) & 1_x(y) \end{bmatrix}$$

in $\mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + 1}]$.

**Proof.**

$$L_1(x_*)L_2(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(x) & 1_x(x) \end{bmatrix} = L_1(x_*) \cdot L_2(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(y) & 1_x(y) \end{bmatrix} =$$

(permuting $L_1(x_*)$ and $L_2(y_*)$ and substituting $x \mapsto y, y \mapsto x$)

$$= L_2(x_*) \cdot L_1(y_*) \cdot \det \begin{bmatrix} \nabla f(y, x) & \nabla x(y, x) \\ f(x) & 1_x(x) \end{bmatrix} =$$

$$= L_2(x_*) \cdot L_1(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(x, y) \\ f(x) & 1_x(x) \end{bmatrix} =$$

$$= L_2(x_*) \cdot L_1(y_*) \cdot \det \begin{bmatrix} \nabla f(x, y) & \nabla x(y, x) \\ f(x) & 1_x(x) \end{bmatrix} .$$

By Statement 1 of Lemma 2, $\nabla_x(x, y) = \nabla_y(y, x)$ is operator of a difference derivative, $\forall i = 1, n: \nabla^i f_i(x, y) = \nabla f_i(y, x)$ is a difference derivative of the polynomial $f_i(x)$. The last equality is true in $\mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + 1}]$ by Statement 1 of Theorem 3, since this functional is uniquely determined in $\mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + 1}]$, independently of the choice of $\nabla_x(x, y)$ and the choice of $\nabla f(x, y)$. Hence, the both functionals coincide in $\mathbb{R}[x^{\leq \delta_f + \delta_1 + \delta_2 + 1}]$.

1. Seifullin, T. R. Root functionals and root polynomials of a system of polynomials. (Russian) Dopov. Nats. Akad. Nauk Ukraini – 1995, – no. 5, 5–8.
2. Seifullin, T. R. Root functionals and root relations of a system of polynomials. (Russian) Dopov. Nats. Akad. Nauk Ukraini – 1995, – no. 6, 7–10.
3. Seifullin, T. R. Homology of the Koszul complex of a system of polynomial equations. (Russian) Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 1997, no. 9, 43–49.
4. Seifullin, T. R. Koszul complexes of embedded systems of polynomials and duality. (Russian) Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 2000, no. 6, 26–34.
5. Seifullin, T. R. Koszul complexes of systems of polynomials connected by linear dependence. (Russian) Some problems in contemporary mathematics (Russian), 326–349, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 25, Natsional. Akad. Nauk Ukraini, Inst. Mat., Kiev, 1998.

V. M. Glushkov Institute of Cybernetics of the NAS of Ukraine, Kiev

Received 06.07.2001

ISSN 1025-6415 Reports of the National Akademy of Sciences of Ukraine, 2002, no. 7