Uncertainty of data obtained in SRF cavity vertical test

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Abstract: Vertical test is a commonly used experimental method to qualify Superconducting Radio Frequency (SRF) cavities. Taking the experiences at Jefferson Lab (JLab) in US for example, over thousand of vertical tests have been performed on over 500 different cavities up to now \cite{1}. Most of the tests at JLab followed the method as described in \cite{1}, but all the uncertainties of the calculated quality factors as well as the gradients were in-accurate due to the wrong algorithm used. In this paper, a first-principle method was applied to analyze the uncertainty of the data, and the results were compared with those in \cite{1} under typical experiment conditions.

Key words: SRF cavity, vertical test, uncertainty
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1 Introduction and motivation

Over years more than 1000 of vertical tests on over 500 SRF cavities at JLab have been following the algorithm as described in \cite{1}. Though, during the author’s visiting at JLab, the algorithm to calculate the uncertainty of intrinsic quality factor (Q0) and gradient (Eacc) in \cite{1} was found to be wrong, since the correlations between variables were not properly treated. In general, it makes the calculated uncertainties in-accurate, e.g. the uncertainty of coupling factor $\beta$ was under-estimated by more than 25% when it is critically coupled. Another significant consequence was that the uncertainty of Eacc for decay measurement under strongly over-coupled condition was severely over-estimated. In this paper, a first-principle method was applied to the uncertainties of external quality factor of input port (Qe1) and pickup port (Qe2), Q0, and Eacc for decay measurement, as well as to the uncertainty of Q0 and Eacc for continuous wave (CW) measurement.

2 Algorithm and assumptions

A first-principle method to calculate random error with any distribution is used:

\[
\begin{align*}
\Delta f &= \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right) \Delta x_i^2, \\
\Delta f &= \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right) f(x_1, x_2, ..., x_n). 
\end{align*}
\]

Note Eq. (1) is valid only if all variables are independent to each other (i.e. uncorrelated). The $\Delta$ means standard deviation which is defined as $\Delta y^2 = ((y - \bar{y})^2)$.

In case the power meter readings are well above the noise floor level, all the directly measured data, i.e. decay time constant $\tau$ and power meter readings, are uncorrelated. The only figure of merit to judge whether two calculated values are correlated is that whether they both use $\tau$ or at least one same power meter reading during the calculation.

Since the uncertainties of measured data are in the form of relative values, it is convenient to rewrite the differential of function $f$ in Eq. (1) as:

\[
\frac{df}{f} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\Delta x_i}{x_i}.
\]

Applying Eq. (2) to (1), one gets:

\[
\frac{\Delta f}{f} = \frac{1}{f} \sqrt{\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 \Delta x_i^2} = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2}. 
\]

So, Qe1, Qe2, Eacc and Q0 will be presented as function of independent variables as in Eq. (2), and the relative error will be calculated with Eq. (3).

Note one more frequently used trick is that:

\[
F(x_i) = \prod_i G_i(x_i) \Rightarrow \frac{dF}{F} = \sum_i \frac{dG_i}{G_i}. 
\]

Eq. (4) is generally valid regardless of the correlations between $G_i$.

3 Define the variable names

In a vertical test, first of all the cables need to be calibrated. Typically the scaling factor $C_i$, $C_r$, and $C_t$ need to be determined, which calibrate the power meter readings to the real power at the entrance and exit of the cavity, as shown in Figure 1.

![Diagram](image-url)

Figure 1: scaling factors for cable calibration

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The detailed procedure for cable calibration is well defined in [1]. Here the variable names are defined in Figure 2 for the uncertainty analysis later.

There are two typical measurement techniques: decay measurement which is also used as lower power calibration to determine $Q_{e2}$; and CW measurement using a known $Q_{e2}$. For the later one, the $Q_{e2}$ is obtained by low power measurement; whether the forward power cable damping $\alpha_1$ is re-calibrated at high power level or not will make difference to the calculation of uncertainties. The variable names for $Q$-E measurement are defined as below:

For decay measurement (which also determine the $Q_{e2}$), the power meter readings in the control room of incident, reflected, and transmitted power are set as $P_{ir}$, $P_{rr}$, and $P_{tr}$, respectively. The decay time constant is set as $t$. For high power measurement with known $Q_{e2}$, the power meter readings in the control room of incident, reflected, and transmitted power are set as $P_{ih}$, $P_{rh}$, and $P_{th}$, respectively. In case the $Q_{e2}$ obtained by a lower power measurement is used but the driving cable damping $\alpha_1$ is re-calibrated, then use $P_{rd}$ and $P_{rd}$ to replace the $P_{ic}$ and $P_{rc}$ for the new cable calibration.

\[
\begin{align*}
Q_{e1} &= Q_{load} \times \frac{1 + \beta}{\beta^2} = \frac{4\pi f_0 t}{1 + |\Gamma|}, \\
Q_{e2} &= \frac{\omega U}{P_t} = Q_{load} \frac{2P_i}{P_t} \left(1 + |\Gamma|\right), \\
Q_{0} &= Q_{e2} \frac{P_t}{P_c} = Q_{load} \frac{2P_i}{P_i - P_r - P_t} \left(1 + |\Gamma|\right), \\
E_{acc} &= \sqrt{R/Q/L} \times \sqrt{Q_{load} \cdot 2P_i \cdot \left(1 + |\Gamma|\right)}
\end{align*}
\]

\[
\begin{align*}
\frac{\Delta Q_{e1}}{Q_{e1}} &= \left(\frac{\Delta t}{t}\right)^2 + \left(1 - \beta_1 + \beta_2\right) \left(1 + |\Gamma|\right) \left(\frac{\Delta P_{rr}}{P_{tr}}\right)^2 + \left(1 + \frac{1 - \beta_1 + \beta_2}{2\beta_1}\right) \left(\frac{\Delta P_{rr}}{P_{tr}}\right)^2 + \left(\frac{\Delta P_{rr}}{P_{tr}}\right)^2 + \left(\frac{\Delta P_{rr}}{P_{tr}}\right)^2 \\
\frac{\Delta Q_{e2}}{Q_{e2}} &= \left(\frac{\Delta t}{t}\right)^2 + \frac{\Delta \theta_1}{\theta_1}^2 + \left(1 + \frac{1 - \beta_1 + \beta_2}{2\beta_1}\right) \left(\frac{\Delta \theta_2}{\theta_2}\right)^2 + \left(\frac{\Delta \theta_3}{\theta_3}\right)^2 + \left(\frac{\Delta \beta_1}{\beta_1}\right)^2 + \left(\frac{\Delta \beta_2}{\beta_2}\right)^2 + \left(\frac{\Delta \beta_3}{\beta_3}\right)^2 + \left(\frac{\Delta \beta_4}{\beta_4}\right)^2 + \left(\frac{\Delta \beta_5}{\beta_5}\right)^2 + \left(\frac{\Delta \beta_6}{\beta_6}\right)^2 \left(\frac{\Delta P_{rr}}{P_{tr}}\right)^2
\end{align*}
\]

4 Cable calibration

As defined in the last section, the scaling factors for incident, reflected, and transmitted power are called $C_i$, $C_r$, and $C_t$, respectively. The three scaling factors are correlated, but they could be presented as function of four independent variables as shown in Eq. (5). More details could be found in appendix A.

\[
\begin{align*}
\theta_1 &= \sqrt{B_i / P_{rB}}, & \theta_2 &= \sqrt{P_{rC} / P_{iC}}, & \theta_3 &= a_2 / P_{rA}, & \theta_4 &= \sqrt{P_V}
\end{align*}
\]

\[
C_i = \theta_1 \times \theta_2 \times \theta_4, \quad C_r = \theta_1 \times \theta_4 / \theta_2, \quad C_t = \theta_3 \times \theta_4^2.
\]

5 Decay measurement

There are in all eight independent variables in a decay measurement: $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, $t$, $P_{ir}$, $P_{rr}$, and $P_{tr}$. The four results obtained from decay measurement, i.e. $Q_{e1}$, $Q_{e2}$, $Q_{0}$, and $E_{acc}$, are presented in Eq. (6), and their uncertainties are presented in Eq. (7), (8), (9), and (10). Note the uncertainty of frequency $f_0$ is ignored, and upper sign is for under-coupled. Detailed definition and derivation could be found in appendix B.

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6 CW measurement with known Qe2

If the cable calibrations are not changed after the decay measurement which determines the Qe2, then there are in all eleven independent variables, i.e. eight same as in the decay measurement, and PiH, PrH and PH from the high power measurement readings. The uncertainty of Q0 and Eacc are shown in Eq. (11) and (13).

But in case the cable loss on the RF driving cable is changed by heating effect, the cable loss factor α1 will usually be re-calibrated. In this case, there are two more independent variables PiD and PrD as defined in section 3. Accordingly, define $\theta_5 = \sqrt{PrD/PiD}$. The uncertainty of Q0 is different, as shown in Eq. (12), and Eacc is the same. Detailed definitions and derivation could be found in appendix D.

\[
\frac{\Delta Q_{\text{re-cal}}}{Q_{\text{re-cal}}} = \frac{1}{2} \left[ \frac{(\Delta \tau)^2}{\tau^2} + \frac{(\Delta \theta_1)^2}{\theta_1^2} + \frac{(\Delta c)^2}{c^2} + \frac{(\Delta \theta_2)^2}{\theta_2^2} + \frac{(\Delta \theta_3)^2}{\theta_3^2} + \frac{(\Delta \theta_4)^2}{\theta_4^2} + \frac{(\Delta \theta_0)^2}{\theta_0^2} + \frac{(\Delta \gamma)^2}{\gamma^2} \right]
\]

\[
\frac{\Delta \theta_1}{\theta_1} = 1 \left[ \frac{(\Delta \tau)^2}{\tau^2} + \frac{(\Delta \theta_1)^2}{\theta_1^2} + \frac{(\Delta \theta_2)^2}{\theta_2^2} + \frac{(\Delta \theta_3)^2}{\theta_3^2} + \frac{(\Delta \theta_4)^2}{\theta_4^2} + \frac{(\Delta \theta_0)^2}{\theta_0^2} + \frac{(\Delta \gamma)^2}{\gamma^2} \right]
\]

\[
\frac{\Delta \theta_2}{\theta_2} = 1 \left[ \frac{(\Delta \tau)^2}{\tau^2} + \frac{(\Delta \theta_1)^2}{\theta_1^2} + \frac{(\Delta \theta_2)^2}{\theta_2^2} + \frac{(\Delta \theta_3)^2}{\theta_3^2} + \frac{(\Delta \theta_4)^2}{\theta_4^2} + \frac{(\Delta \theta_0)^2}{\theta_0^2} + \frac{(\Delta \gamma)^2}{\gamma^2} \right]
\]

7 Calculated typical uncertainties

The uncertainty of Q0 and Eacc for decay measurement are obtained in Eq. (9) and (10), while for CW measurement with known Qe2 they are obtained in Eq. (11), (12), and (13).

For a typical vertical test, the tolerance of power meter readings in control room during the test (i.e. Pir, Pr, Pr, PiH, PrH, PH) is within 5% when attenuators are well distributed in the low level control system. For cable calibration, uncertainty of Py, PtA, and PiB are 5% too. Additional error induced by the standing wave in circulator makes the tolerance of Ps and PtB 6-7%. The standing wave introduces about 7% extra error to the sampling of directional coupler, which makes a tolerance of about 10% to PrA, PyC, PrD, PiA, PiC, and PiD.

The trend of calculated uncertainties are illustrated in Figure 3, following the assumptions above, and assuming $\beta_2$ is 0.02, $\beta_1$ is 2.5 when calibrating Qe2 for CW measurement, and accuracy of decay time constant is 3%.

Note a significant source of uncertainty when it is way off critical coupling is that the difference between Pi and Pr have big error bar. Though, from Eq. (6) it is noticed that uncertainties of Q0 and Eacc should not diverge when it is under-coupled and over-coupled, respectively. It agrees very well with the curve shown in Figure 3.

Note: in case the coefficient Bi, Br, Bt, and a2 are measured at multiple power levels, and the averages are taken for each of them, then the correlations between them become much weaker, and it is reasonable to treat Bi, Br, a1, and Bt/ a2 as four independent variables instead of $\theta_1$-$\theta_4$. Eq. (9)-(13) could be re-derived accordingly.
If the wrong algorithm in [1] is used, which doesn’t consider the correlations in between variables, then the differences of Q0 and Eacc for decay measurement are illustrated in Figure 4. Note same assumptions are used as in Figure 3, together with the fact that tolerance of 5% is typically assumed for all the power meter readings when using algorithm in [1]. Note detailed calculation could be found in [1], and it is recalled in appendix D.

Reference:
[1] Tom Powers, “Theory and Practice of Cavity Test Systems”, in SRF2005. Cornell, USA, 2005.

Appendix A: present cable calibration coefficients as function of independent variables

During the cable calibration, there are in all nine independent power meter readings: Pv, Ps, PrA, PtA, PiA, PiB, PtB, PiC, and PrC. Usually Bi, Br, Bt, α1, α2, Ci, Cr, and Ct are calculated from those measured values as below:

\[
\begin{align*}
\alpha_1 & = \frac{\text{PrC} \times \text{Br}}{\text{PiC} \times \text{Bi}}, \quad \alpha_2 = \frac{\sqrt{\text{PtB} / \text{Ps}}}{\sqrt{\text{Pv}}} \\
\text{Ci} & = \text{Bi} \cdot \alpha_1 = \frac{\sqrt{\text{PrC} / \text{PiC}}}{\sqrt{\text{PtA} / \text{PrA}}} \\
\text{Cr} & = \frac{\sqrt{\text{PrC} / \text{PiC}}}{\sqrt{\text{PtA} / \text{PrA}}} \times \frac{\text{Bt}}{\alpha_2} \times \frac{\sqrt{\text{Pv}}}{\sqrt{\text{PtB}}}
\end{align*}
\] (A1)

Apparently, Ci, Cr, and Ct are correlated. But they could be presented as functions of four independent variables defined as θ1, θ2, θ3 and θ4 in Eq. (5), which is recalled as below.

\[
\begin{align*}
\theta_1 & \triangleq \sqrt{\frac{\text{Bi}}{\text{PrA}}} \quad \theta_2 \triangleq \sqrt{\frac{\text{PrC}}{\text{PiC}}} \\
\theta_3 & \triangleq \theta_2 / \theta_1, \quad \theta_4 \triangleq \sqrt{\frac{\text{Pv}}{\text{PrC}}}
\end{align*}
\]

The uncertainties of θ1 to θ4 are calculated in Eq. (A2)

\[
\begin{align*}
\Delta \theta_1 \quad & = \frac{1}{2} \sqrt{\left(\frac{\Delta \text{Bi}}{\text{PiB}}\right)^2 + \left(\frac{\Delta \text{PrA}}{\text{PiA}}\right)^2 + \left(\frac{\Delta \text{PtA}}{\text{PrA}}\right)^2} \\
\Delta \theta_2 \quad & = \frac{1}{2} \sqrt{\left(\frac{\Delta \text{PrC}}{\text{PiC}}\right)^2 + \left(\Delta \text{PiC}\right)^2} \\
\Delta \theta_3 \quad & = \frac{1}{2} \sqrt{\left(\frac{\Delta \text{PtB}}{\text{PtB}}\right)^2 + \left(\frac{\Delta \text{Ps}}{\text{Ps}}\right)^2 + \left(\Delta \text{PrA}\right)^2} \\
\Delta \theta_4 \quad & = \frac{1}{2} \sqrt{\left(\Delta \text{Pv}\right)^2}
\end{align*}
\] (A2)

Appendix B: uncertainty analysis for decay measurement

As defined in section 3, there are in all eight independent variables in a decay measurement: θ1, θ2, θ3, θ4, τ, Pir, Prr, and Ptr. Some variables that help the calculation are listed as below:

8 Conclusion

The wrong uncertainty calculation method for processing SRF cavity vertical test data at JLab is corrected in this paper. The formula of uncertainty calculation for decay and CW measurement is provided in Eq. (9) to (13). The trend of uncertainty is illustrated as the input coupling changes, and the accuracy with decay measurement is found better than CW measurement. The suggestion in [1] that CW measurement should be performed with 0.5<β<2 is still valid.

9 Acknowledgement

I would like to appreciate Tom Powers at JLab for providing the information on how standing wave affects the accuracy of power meter readings. I would also like to appreciate Gianluigi Ciovati at JLab for the discussion on how averaging affects the calculation. Moreover, I would like to appreciate Haipeng Wang at JLab for the suggestions on making this paper more readable.
The error on $Q_e$ is derived as below. It gives the Eq. (7):

$$
\frac{dQ_e}{Q_e} = \frac{dr}{\tau} \pm \frac{d|\Gamma|}{1 + |\Gamma|} = \frac{dr}{\tau} \pm \frac{|\Gamma|}{1 + |\Gamma|} \left( \frac{1}{2} \frac{dP_{rr}}{P_{rr}} - \frac{1}{2} \frac{dP_{ir}}{P_{ir}} - \frac{d\theta_2}{\theta_2} \right)
$$

The error on $Q_e$ is derived as below. It gives the Eq. (8):

$$
\frac{\Delta Q_e}{Q_e} = \sqrt{\left( \frac{\Delta r}{r} \right)^2 + \left( \frac{\Delta \theta_1}{\theta_1} \right)^2 + \left( \frac{\Delta \theta_2}{\theta_2} \right)^2 + \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta \theta_5}{\theta_5} \right)^2}
$$

The error on $Q_o$ is derived as below. It gives the Eq. (9):

$$
\frac{dQ_o}{Q_o} = \frac{dr}{\tau} + (1 - f_2 + f_3) \frac{d\theta_1}{\theta_1} + (f_1 + f_2 + f_3 - f_4 - f_5) \frac{d\theta_2}{\theta_2} + f_6 \left( \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} + \frac{dP_{rr}}{P_{rr}} \right)
$$
\[
\Delta Q_0 = \frac{\Delta \tau}{\tau} + \beta^2 \left( \frac{\Delta \theta_1}{\theta_1} \right)^2 + \left( \beta_1 - \beta_1^2 - \beta_2 - \beta_2^2 \right)^2 \cdot \left( \frac{\Delta \theta_2}{\theta_2} \right) + \beta^2 \cdot \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta P_{tr}}{P_{tr}} \right)^2
\]

The error on \(E_{\text{acc}}\) is derived as below. It gives the Eq. (10):

\[
\begin{align*}
\Delta E_{\text{acc}} &= 1 \cdot \left( \frac{\Delta \tau}{\tau} \right) + \left( \beta_1 - \beta_1^2 - \beta_2 - \beta_2^2 \right)^2 \cdot \left( \frac{\Delta \theta_2}{\theta_2} \right) + \beta^2 \cdot \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta P_{tr}}{P_{tr}} \right)^2 \\
&= \frac{1}{2} \cdot \left( \frac{\Delta \tau}{\tau} \right) + \left( \beta_1 - \beta_1^2 - \beta_2 - \beta_2^2 \right)^2 \cdot \left( \frac{\Delta \theta_2}{\theta_2} \right) + \beta^2 \cdot \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta P_{tr}}{P_{tr}} \right)^2
\end{align*}
\]

**Appendix C: uncertainty analysis for CW measurement with known \(Q_2\)**

C.1: the cable calibrations are not changed after the decay measurement which determines the \(Q_2\).

In this case there are in all eleven independent variables, i.e. eight same as in the decay measurement, and \(P_{H}, P_{H}\) and \(P_{F}\) from the high power measurement readings. The \(Q_0\) and \(E_{\text{acc}}\) are calculated using Eq. (C.1).

\[
\begin{align*}
f_7 \triangleq P_{\text{ccw}} &= \frac{\theta_1 \cdot \theta_2 \cdot P_{\text{ih}}}{\theta_1 / \theta_2 - \theta_3 / \theta_4 - 0.0304 \text{Pth}} = \frac{1 + \beta_1 \text{c}_{w} + \beta_2 \text{c}_{w}}{4 \beta_1 \text{c}_{w}} \\
f_8 \triangleq P_{\text{cpr}} &= \frac{\theta_1 \cdot \theta_2 \cdot P_{\text{ph}}}{\theta_1 / \theta_2 - \theta_3 / \theta_4 - 0.0304 \text{Pth}} = \frac{1 - \beta_1 \text{c}_{w} + \beta_2 \text{c}_{w}}{4 \beta_1 \text{c}_{w}} \\
f_9 \triangleq P_{\text{ptr}} &= \frac{\theta_1 \cdot \theta_2 \cdot P_{\text{ptr}}}{\theta_1 / \theta_2 - \theta_3 / \theta_4 - 0.0304 \text{Pth}} = \beta_2 \text{c}_{w}
\end{align*}
\]

The error on \(Q_0\) is derived as below. It gives the Eq. (11)

\[
\begin{align*}
\Delta Q_0 &= \frac{1}{2} \cdot \left( \frac{\Delta \tau}{\tau} \right) + \left( \beta_1 \text{c}_{w} + \beta_2 \text{c}_{w} \right) \cdot \left( \frac{\Delta \theta_1}{\theta_1} \right) + \left( \beta_1 \text{c}_{w} \beta_2 \text{c}_{w} - \beta_1 \left( \beta_1 \text{c}_{w} \beta_1 \text{c}_{w} + \beta_1 \left( \beta_2 \text{c}_{w} \right)^2 \right) \cdot \left( \frac{\Delta \theta_2}{\theta_2} \right) + \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta P_{tr}}{P_{tr}} \right)^2
\end{align*}
\]

The error on \(E_{\text{acc}}\) is derived as below. It gives the Eq. (13)

\[
\begin{align*}
\Delta E_{\text{acc}} &= 1 \cdot \left( \frac{\Delta \tau}{\tau} \right) + \left( \beta_1 \text{c}_{w} + \beta_2 \text{c}_{w} \right) \cdot \left( \frac{\Delta \theta_1}{\theta_1} \right) + \left( \beta_1 \text{c}_{w} \beta_2 \text{c}_{w} - \beta_1 \left( \beta_1 \text{c}_{w} \beta_1 \text{c}_{w} + \beta_1 \left( \beta_2 \text{c}_{w} \right)^2 \right) \cdot \left( \frac{\Delta \theta_2}{\theta_2} \right) + \left( \frac{\Delta \theta_3}{\theta_3} \right)^2 + \left( \frac{\Delta \theta_4}{\theta_4} \right)^2 + \left( \frac{\Delta P_{tr}}{P_{tr}} \right)^2
\end{align*}
\]
Case 2: the cable loss on the RF driving cable is changed by heating effect, and the cable loss factor $\alpha_1$ is re-calibrated. But the $Q_e$ obtained before re-calibrating the cable is used to calculate $Q_0$ and $E_{acc}$.

In this case, there are two more independent variables $\Pi_D$ and $\Pi_R$ defined as in section 3. By replacing $\theta_2$ with $\theta_5 = \sqrt{\Pi_D/\Pi_H}$ when calculating $Q_0$, and keeping $\theta_2$ unchanged inside $Q_e$, the uncertainty of $Q_0$ could be calculated as in Eq.(C2).

$$Q_0 = Q_e \times \frac{0.03 \cdot 0.04 \cdot \Pi_H}{0.05 \cdot \Pi_H - \Pi_H / 0.05} - 0.03 \cdot 0.04 \cdot \Pi_H$$ (C2)

Note the accuracy of $Q_0$ is different from that in case 1, but $E_{acc}$ is the same. By replacing $\theta_2$ with $\theta_5$, define $f_{10}$, $f_{11}$ and $f_{12}$ as following:

$$f_{10} \triangleq \frac{P_{pcw}}{1 - 0.05 \cdot \Pi_H} = \frac{\theta_1 \cdot 0.05 \cdot \Pi_H}{1 - 0.05 \cdot \Pi_H} - 0.03 \cdot 0.04 \cdot \Pi_H$$

$$f_{11} \triangleq \frac{P_{pcw}}{1 - 0.05 \cdot \Pi_H} = \frac{\theta_1 / 0.05 \cdot \Pi_H - 0.03 \cdot 0.04 \cdot \Pi_H}{\Pi_H}$$

$$f_{12} \triangleq \frac{P_{pcw}}{1 - 0.05 \cdot \Pi_H} = \frac{\theta_1 / 0.05 \cdot \Pi_H - 0.03 \cdot 0.04 \cdot \Pi_H}{\Pi_H}$$

The error on $Q_0$ is then derived as below:

$$\frac{dQ_0}{Q_0} = \frac{d\theta_1}{\theta_1} + (1 - f_{10} + f_{11}) \cdot \frac{d\theta_1}{\theta_1} + \frac{d\theta_2}{\theta_2} + f_{12} \cdot \left( \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} - f_{10} + f_{11} \right) \frac{d\theta_5}{\theta_5}$$

$$= \frac{d\theta_1}{\theta_1} + \frac{d\theta_2}{\theta_2} + \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} - \frac{f_{10} + f_{11}}{\theta_5} \frac{d\theta_5}{\theta_5}$$

$$= \frac{d\theta_1}{\theta_1} + \frac{d\theta_2}{\theta_2} + \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} - \frac{f_{10} + f_{11}}{\theta_5} \frac{d\theta_5}{\theta_5}$$

$$= \frac{d\theta_1}{\theta_1} + \frac{d\theta_2}{\theta_2} + \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} - \frac{f_{10} + f_{11}}{\theta_5} \frac{d\theta_5}{\theta_5}$$

$$= \frac{d\theta_1}{\theta_1} + \frac{d\theta_2}{\theta_2} + \frac{d\theta_3}{\theta_3} + \frac{d\theta_4}{\theta_4} - \frac{f_{10} + f_{11}}{\theta_5} \frac{d\theta_5}{\theta_5}$$

Appendix D: recall the wrong algorithm used in [1] to calculate error of $Q_0$ and $E_{acc}$ for decay measurement

In the “Derivation of measurement errors – decay measurement” section in the Appendix A of [1], the error of $Q_0$ and gradient is calculated as following:

$$\frac{\Delta Q_0}{Q_0} = \sqrt{\left( \frac{\Delta Q_L}{Q_L} \right)^2 + \left( \frac{\Delta \beta_1}{\beta_1} \right)^2 + \left( \frac{\Delta \beta_2}{\beta_2} \right)^2 + \left( \frac{\Delta E_{acc}}{E_{acc}} \right)^2}$$

All the relative errors could be presented in the form of $\sqrt{\left( \frac{\Delta L}{L} \right)^2 + F(\beta_1, \beta_2) \cdot \left( \frac{\Delta \beta}{\beta} \right)^2}$, by adopting the same assumptions as described in the section 2 and 7, except that identical deviation of each power meter reading is assumed. Note all the calculations below follow the method in appendix A in [1]:

$$\Delta Q_{PC} = \frac{\Delta L}{L} \sqrt{\frac{\Delta \Pi_i^2 + \Delta \Pi_r^2 + \Delta \Pi_t^2}{P_{pc}}} = \sqrt{f_4^2 \left( \frac{\Delta \Pi_i}{\Pi_i} \right)^2 + f_5^2 \left( \frac{\Delta \Pi_r}{\Pi_r} \right)^2 + f_6^2 \left( \frac{\Delta \Pi_t}{\Pi_t} \right)^2}$$

Where $f_4$, $f_5$, and $f_6$ are defined in Appendix B.

$$\frac{\Delta \beta_2}{\beta_2} = \sqrt{\left( \frac{\Delta \Pi_i}{\Pi_i} \right)^2 + \left( \frac{\Delta \Pi_r}{\Pi_r} \right)^2} = \sqrt{1 + f_4^2 + f_5^2 + f_6^2} \frac{\Delta P}{P}$$

$$= \frac{\beta_2^4 + (1 + \beta_2)^4 + 2 \beta_2(7 + \beta_2(6 + 7 \beta_2)) \Delta P}{8 \cdot \beta_1^2}$$
Thus, the error of $Q_0$ and $E_{acc}$ are shown in

$$\Delta Q_0 = \sqrt{\frac{1}{\beta} \left( \frac{d\beta}{\tau} \right)^2 + \frac{1}{2} \left( \frac{d\beta_1}{P} \right)^2 + \frac{1}{2} \left( \frac{d\beta_2}{P^2} \right)^2}$$

$$\Delta E_{acc} = \sqrt{\frac{1}{\beta} \left( \frac{d\beta}{\tau} \right)^2 + \frac{1}{2} \left( \frac{d\beta_1}{P} \right)^2 + \frac{1}{2} \left( \frac{d\beta_2}{P^2} \right)^2}$$

Thus, the error of $Q_0$ and $E_{acc}$ are shown in Eq.