Large deviations for stochastic PDE with Lévy noise *

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Abstract

We prove a large deviation principle result for solutions of abstract stochastic evolution equations perturbed by small Lévy noise. We use general large deviations theorems of Varadhan and Bryc, viscosity solutions of integro-partial differential equations in Hilbert spaces, and deterministic optimal control methods. The Laplace limit is identified as a viscosity solution of a Hamilton-Jacobi-Bellman equation of an associated control problem. We also establish exponential moment estimates for solutions of stochastic evolution equations driven by Lévy noise. General results are applied to stochastic hyperbolic equations perturbed by subordinated Wiener process.

Keywords: Large deviation principle, Lévy process, viscosity solutions, integro-PDE, Hamilton-Jacobi-Bellman equation, stochastic PDE.

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1 Introduction

Let $L(t)$ be a square integrable Lévy martingale on a Hilbert space $H$, starting from 0, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $\mathcal{F}_t$. It is well

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known, see e.g. [21], that

\[ L(t) = \int_0^t \int_H z\hat{\pi}(ds, dz) + W(t) \]  \hspace{1cm} (1.1)

where \( W \) is an \( H \) valued Wiener process, independent of the compensated random measure \( \hat{\pi}(ds, dz) = \pi(ds, dz) - ds\nu(dz) \) with the intensity measure \( \nu \), satisfying

\[ \int_H \|z\|^2 \nu(dz) < +\infty. \]

Here

\[ \pi([0, t], \Gamma) = \#\{s \in [0, t]; L(t) - L(t-) \in \Gamma\}, \]

is the random measure of jumps of the process \( L \), see e.g. [25], [3] and [21]. Define

\[ L_n(t) = \frac{1}{n} L(nt), \]

and note that

\[ \mathbb{E}\|L_n(t)\|^2 = \frac{t}{n} \int_H \|z\|^2 \nu(dz). \]

We study large deviation principle for the family of processes \( \{X_n\} \) satisfying

\[ dX_n(s) = (-AX_n(s) + F(X_n(s)))ds + G(X_n(s-))dL_n(s), \quad X_n(0) = x \in H, \]  \hspace{1cm} (1.2)

where \( A \) is a linear, densely defined, maximal monotone operator in \( H \) and \( F, G \) are certain continuous functions. These abstract stochastic differential equations may be for instance semilinear stochastic PDE with small Lévy noise. For the theory of such equations we refer to [21] and the references therein. We excluded from our considerations the Gaussian part of the noise. If \( L \) is a Wiener process, large deviation results are well known, see e.g. [4, 5, 6, 9, 13, 14, 16, 22, 26, 27, 29] and the references therein. We think that our methods, combined with the techniques of [29], should apply to the general case, however we do not attempt to do it here. Thus, we will always assume that

\[ L_n(t) = \frac{1}{n} L(nt), \text{ where } L(t) = \int_0^t \int_H z\hat{\pi}(ds, dz). \]  \hspace{1cm} (1.3)

There are two types of large deviation results; at a single time, i.e. for \( X_n(T) \) with \( T \) fixed, and in the path space, i.e. for \( X_n(\cdot) \). Our goal is to show the large deviation principle and identify the rate function for the single time case since this is where the PDE theory is used. Once this is done a general strategy to pass to the path space case can be found in [13]. Such a strategy was employed in [29] when \( L \) was a Wiener process. We don’t know if it can be successful here.
The problem of large deviations for infinite dimensional processes with jumps seems to be wide open although for the finite dimensional spaces basic results are presented in [30]. We are only aware of three papers that specifically address it in the path space. In [1] the large deviation principle is proved for a family of Banach space valued Lévy processes and in [28] for solutions of linear evolution equations of type (1.2) with additive Lévy noise and the operator $A$ with a discrete spectrum. Paper [31] deals with the case of two-dimensional stochastic Navier-Stokes equations driven by additive Lévy noise. We also refer to [2, 13] for related results.

Our approach uses the classical theorems of Varadhan and Bryc [10]. According to them the processes $X_n$ satisfy the large deviation principle in a metric space $E$ if and only if the family $\{X_n\}$ is exponentially tight and the Laplace limit

$$
\Lambda(g) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} e^{ng(X_n)}
$$

exists for all $g \in C_b(E)$. We will choose $E$ to be any Hilbert space $V$ such that $H \subset V$ and $H \hookrightarrow V$ is compact. Our main result, the existence of the Laplace limit and its identification, will be a consequence of a much more general result about convergence of viscosity solutions of certain integro-PDE in $H$ to the viscosity solution of the limiting first order Hamilton-Jacobi-Bellman (HJB) equation.

After recalling basic definitions and introducing main hypotheses in Section 2, exponential estimates and continuous dependence estimates for solutions of (1.2) are established in Section 3 see Proposition 3.1 and Proposition 3.3. In the proofs we use a new result on convergence of solutions of the equation (1.2) with $A$ replaced by Yosida approximations of $A$. Associated nonlinear PDE in Hilbert spaces are investigated in Section 4. The fact that functions

$$
v_n(t, x) = \frac{1}{n} \log \mathbb{E} e^{ng(X_n(T))},
$$

where $X_n$ solves (3.1), are viscosity solutions of proper nonlinear PDE, is the content of Theorem 4.1. Moreover Theorem 4.4 establishes existence of a viscosity solution to the limiting HJB equation. The main results on the Laplace limits are subjects of Theorem 5.1, Theorem 5.4, and Corollary 5.3 of Section 5. Finally Theorem 6.1 states conditions under which the large deviation principle holds for solutions of (1.2). Various examples are discussed in Sections 7 and 8. In the Appendix we give a proof of the convergence result used in Section 3.
2 Preliminaries

2.1 Basic definitions and assumptions

Throughout this paper $H$ will be a real separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We recall that $A$ is a linear, densely defined, maximal monotone operator in $H$.

Let $B$ be a bounded, linear, positive, self-adjoint operator on $H$ such that $A^* B$ is bounded on $H$ and

$$\langle (A^* B + c_0 B)x, x \rangle \geq 0 \quad \text{for all } x \in H$$

for some $c_0 \geq 0$. Such an operator always exists, for instance $B = ((A + I)(A^* + I))^{-1/2}$ (see [24]). We refer to [7] for various examples of $B$. Using the operator $B$ we define for $\gamma > 0$ the space $H_{-\gamma}$ to be the completion of $H$ under the norm

$$\| x \|_{-\gamma} = \| B^{\frac{\gamma}{2}} x \|.$$

Let $\Omega \subset [0, T] \times H$. We say that $u : \Omega \to \mathbb{R}$ is $B$-upper-semicontinuous (respectively, $B$-lower-semicontinuous) on $\Omega$ if whenever $t_n \to t$, $x_n \rightharpoonup x$, $Bx_n \to Bx$, $(t, x) \in \Omega$, then $\limsup_{n \to +\infty} u(t_n, x_n) \leq u(t, x)$ (respectively, $\liminf_{n \to +\infty} u(t_n, x_n) \geq u(t, x)$). The function $u$ is $B$-continuous on $\Omega$ if it is $B$-upper-semicontinuous and $B$-lower-semicontinuous on $\Omega$.

The following assumptions will be made about the functions $F : H \to H$ and $G : H \to L(H)$, where $L(H)$ is the space of bounded linear operators on $H$:

$$\| F(0) \| \leq M, \quad \| F(x) - F(y) \| \leq M \| x - y \|_{-1} \quad \text{for all } x, y \in H,$$

(2.2)

$$\| G(x) - G(y) \| \leq M \| x - y \|_{-1} \quad \text{for all } x, y \in H,$$

(2.3)

$$\| G(x) \| \leq M \quad \text{for all } x \in H$$

(2.4)

for some $M \geq 0$, and

$$\int_H \| z \|^2 e^{K \| z \|} \nu(dz) < +\infty \quad \text{for every } K > 0.$$  

(2.5)

Condition (2.5) is equivalent to the requirement that the noise process has exponential moments:

$$\mathbb{E} e^{K \| L(t) \|} < +\infty, \quad \text{for all } t, K > 0.$$

If (2.5) holds then the Laplace transform of the process $L$ is well defined. Namely if $L$ is given by (1.1) and $Q_W$ is the covariance of $W$, then

$$\mathbb{E} e^{\langle p, L(t) \rangle} = e^{H(p)} \quad \text{where } H(p) = 1/2 \langle Q_W p, p \rangle + \int_H \left[ e^{\langle p, z \rangle} - 1 - \langle p, z \rangle \right] \nu(dz), \ p \in H.$$  

(2.6)
We set
\[ H_0(p) = \int_H \left[ e^{(p,z)} - 1 - \langle p, z \rangle \right] \nu(dz), \ p \in H, \]
if \( L \) is without the Gaussian part as in (1.3).

**Remark 2.1.** If instead of (2.1) we suppose that
\[ \langle (A^*B + c_0B)x, x \rangle \geq \|x\|^2 \quad \text{for all } x \in H \] (2.7)
then (2.2) can be replaced by a weaker condition
\[ \|F(0)\| \leq M, \quad \|F(x) - F(y)\| \leq M\|x - y\| \quad \text{for all } x, y \in H. \] (2.8)
We refer the reader to [7] for examples of operators satisfying (2.7) and to [24] for conditions guaranteeing the existence of \( B \) for which (2.7) holds.

We will need the following simple fact which we record for future use.

**Lemma 2.2.** If \( f \in C^2(H) \) then for every \( x, y \in H \)
\[ f(x + y) = f(x) + \langle Df(x), y \rangle + \int_0^1 \int_0^1 \langle D^2f(x + s\sigma y), y \rangle \sigma dsd\sigma. \]

For a square integrable martingale \( M \) we will denote by \( \langle M, M \rangle_t \) its angle bracket and by \([M, M]_t\) its quadratic variation (see [23], p. 57, or [19], p. 150). It is easy to see that \( \langle L(nt), L(nt) \rangle_t = cnt \) for some \( c > 0 \).

For a Hilbert space \( Z \) we will be using the following function spaces.
\[ C_b(Z) = \{ u : Z \to \mathbb{R} : u \text{ is continuous and bounded} \}, \]
\[ \text{Lip}_b(Z) = \{ u \in C_b(Z) : u \text{ is Lipschitz continuous} \}, \]
\[ C^2(Z) = \{ u : Z \to \mathbb{R} : Du, D^2u \text{ are continuous} \}, \]
\[ C^{1,2}((0, T) \times Z) = \{ u : (0, T) \times Z \to \mathbb{R} : u_t, Du, D^2u \text{ are continuous} \}, \]
\[ C^2_{uc}(Z) = \{ u : Z \to \mathbb{R} : u, Du, D^2u \text{ are uniformly continuous} \}, \]
where \( Du, D^2u \) denote the Fréchet derivatives of \( u \) with respect to the spatial variable.

We will denote by \( S(\cdot) \) the \( C_0 \)-semigroup generated by \(-A\). For \( \lambda > 0 \) we denote by \( A_\lambda \) the Yosida approximation of \( A \), \( A_\lambda = \lambda A R_\lambda \), where \( R_\lambda = (\lambda I + A)^{-1} \). The \( C_0 \)-semigroup generated by \(-A_\lambda \) will be denoted by \( S_\lambda(\cdot) \). Both \( S(\cdot) \) and \( S_\lambda(\cdot) \) are semigroups of contractions. It is well known (see for instance [20]) that
\[ \|R_\lambda\| \leq \frac{1}{\lambda}, \quad \text{and } \lim_{\lambda \to +\infty} \lambda R_\lambda x = x \quad \text{for } x \in H. \] (2.9)
For \( C \in L(H) \) we will denote by \( \|C\|_{HS} \) its Hilbert-Schmidt norm.
\section{2.2 Viscosity solutions}

To minimize the technicalities we will be using a slightly simplified definition of viscosity solution. This simplified definition will be enough since in this paper we only deal with bounded solutions. We also point out that Definition 2.4 applies to terminal value problems.

\textbf{Definition 2.3.} A function $\psi$ is a test function if $\psi = \varphi + h(\|x\|)$, where:

(i) $\varphi \in C^{1,2}((0,T) \times H)$, is $B$-lower semicontinuous, $\varphi, \varphi_t, D\varphi, D^2\varphi, A^*D\varphi$ are uniformly continuous on $[\epsilon, T - \epsilon] \times H$ for every $\epsilon > 0$, and $\varphi$ is bounded on every set $[\epsilon, T - \epsilon] \times \{\|x\| - 1 \leq r\}$.

(ii) $h \in C^2([0, +\infty))$ is such that $h'(0) = 0, h'(r) \geq 0$ for $r \in (0, +\infty)$, and $h, h', h''$ are uniformly continuous on $[0, +\infty)$.

We will be concerned with terminal value problems for integro-PDE of the form

$$v_t - \langle Ax, Dv \rangle + F(t, x, Dv, v(t, \cdot)) = 0 \quad \text{in } (0, T) \times H, \quad (2.1)$$

where $F : (0, T) \times H \times H \times C^{2,uc}_\infty(H) \to \mathbb{R}$.

\textbf{Definition 2.4.} A locally bounded $B$-upper semicontinuous function $u : (0, T) \times H \to \mathbb{R}$ is a viscosity subsolution of (2.1) if whenever $u - \varphi - h(\|\cdot\|)$ has a maximum over $(0, T) \times H$ at a point $(t, x)$ for some test functions $\varphi, h(\|y\|)$ then

$$\psi_t(t, x) - \langle x, A^*D\varphi(t, x) \rangle + F(t, x, D\psi(t, x), \psi(t, \cdot)) \geq 0,$$

where $\psi(s, y) = \varphi(s, y) + h(\|y\|)$.

A locally bounded $B$-lower semicontinuous function $u : (0, T) \times H \to \mathbb{R}$ is a viscosity supersolution of (2.1) if whenever $u + \varphi + h(\|\cdot\|)$ has a minimum over $(0, T) \times H$ at a point $(t, x)$ for some test functions $\varphi, h(\|y\|)$ then

$$\psi_t(t, x) + \langle x, A^*D\varphi(t, x) \rangle + F(t, x, D\psi(t, x), \psi(t, \cdot)) \leq 0,$$

where $\psi(s, y) = -\varphi(s, y) - h(\|y\|)$.

A viscosity solution of (2.1) is a function which is both a viscosity subsolution and a viscosity supersolution.
3 Estimates for solutions of stochastic PDE with Lévy noise

In this section we recall basic facts and show various estimates about mild solutions of the equations,

\[ dX_n(s) = (-AX_n(s) + F(X_n(s)))ds + G(X_n(s-))dL_n(s), \quad X_n(t) = x \in H, \quad (3.1) \]
on a fixed time interval \([0, T]\), where \(L_n\) are the processes defined in (1.3).

Let us recall that if (1.3) holds then

\[
\mathbb{E}e^{\langle p, L_n(t) \rangle} = e^{ntH_0(p)} = e^{nt \int_H \left[ e^{\frac{1}{n}(p,z)} - 1 - \frac{1}{n}(p,z) \right] \nu(dz)}, \quad p \in H. \quad (3.2)
\]

The covariance operator of the process \(L\) will be denoted by \(Q\) and then the covariance operator of \(L_n\) is \(\frac{1}{n}Q\).

We refer the readers to Chapter 9 of [21] for the definition of a mild solution. We will also need solutions \(X^m_n\) of the equations

\[ dX^m_n(s) = (-A_mX^m_n(s) + F(X^m_n(s)))ds + G(X^m_n(s-))dL_n(s), \quad X^m_n(t) = x \in H, \quad (3.3) \]
where the operators \(A_m\) are Yosida approximations of \(A\) for \(\lambda = m = 1, 2, \ldots\).

**Proposition 3.1.** Let \(0 \leq t \leq T\). Let (2.5) be satisfied and let

\[ \|G(x) - G(y)\|, \|F(x) - F(y)\| \leq C\|x - y\| \quad \text{for all } x, y \in H, \quad (3.4) \]

for some \(C \geq 0\). Then:

(i) There exists a unique mild solution \(X_n\) of (3.1). The solution \(X_n\) has a càdlàg modification.

(ii) If \(X^m_n\) is the solution of (3.3) then

\[
\lim_{m \to +\infty} \mathbb{E} \left( \sup_{t \leq s \leq T} \|X^m_n(s) - X_n(s)\|^2 \right) = 0. \quad (3.5)
\]

(iii) If in addition (2.4) holds then there exist constants \(c_1 > 0, c_2 > 0\) (depending only on \(T, M, \) with \(c_2\) depending also on \(\|x\|\)) such that

\[
\mathbb{E} \left( \sup_{t \leq s \leq T} e^{nc_1\|X_n(s)\|} \right) \leq e^{nc_2}. \quad (3.6)
\]

**Remark 3.2.** It follows from the proof that (3.6) is also satisfied for the processes \(X^m_n\) with the same constants \(c_1, c_2\). In particular this implies that there exists a constant \(C(\|x\|, T)\) such that for every \(n, m\)

\[
\mathbb{E} \left( \sup_{t \leq s \leq T} e^{nc_1\|X^m_n(s)\|} \right) \leq C(\|x\|, T) \quad (3.7)
\]

with the same estimate being also true for the processes \(X_n\).
Proof. (i) This is a standard result, see Theorem 9.29 in [21].

(ii) We will need two general results on convergence of stochastic and deterministic convolutions, Propositions 3.3 and 3.4. The proof of Proposition 3.3 will be postponed to the Appendix and the classical proof of Proposition 3.4 will be omitted.

Denote by $\mathcal{L}$ the space of all predictable processes $\psi(\cdot)$ whose values are linear operators from the space $Q^{1/2}(H)$ into $H$, equipped with the scalar product

$$<\psi_1, \psi_2>_{\mathcal{L}} = \sum_{n=1}^{+\infty} <\psi_1 Q^{1/2}e_n, \psi_2 Q^{1/2}e_n>_H, \quad \psi_1, \psi_2 \in \mathcal{L}.$$ 

Here $(e_n)$ is any orthonormal basis in $H$. Moreover two operators on $H$, even unbounded, identical on $Q^{1/2}(H)$, are identified. The norm on $\mathcal{L}$ is given by the formula.

$$|\psi|_1 = \left( \mathbb{E} \int_0^T \|\psi(s)Q^{1/2}\|^2_{HS} ds \right)^{1/2} < +\infty.$$ 

Proposition 3.3. Let $L(t)$ be a square integrable Lévy martingale in $H$ with the covariance operator $Q$, and $\psi \in \mathcal{L}$. Then the processes

$$\int_0^t S(t-s)\psi(s) dL(s), \quad \int_0^t S_{\lambda}(t-s)\psi(s) dL(s), \quad t \in [0,T], \quad \lambda > 0, \quad (3.8)$$

have càdlàg modifications and

$$\lim_{\lambda \to +\infty} \mathbb{E} \sup_{0 \leq t \leq T} \| \int_0^t S(t-s)\psi(s) dL(s) - \int_0^t S_{\lambda}(t-s)\psi(s) dL(s) \|^2 = 0. \quad (3.9)$$

Proposition 3.4. Assume that $\psi$ is an $H$-valued predictable process such that

$$\mathbb{E} \int_0^T \|\psi(s)\|^2 ds < +\infty$$

Then the processes

$$\int_0^t S(t-s)\psi(s) ds, \quad \int_0^t S_{\lambda}(t-s)\psi(s) ds, \quad t \in [0,T], \quad \lambda > 0,$$

have continuous modifications and

$$\lim_{\lambda \to +\infty} \mathbb{E} \sup_{0 \leq t \leq T} \| \int_0^t S(t-s)\psi(s) ds - \int_0^t S_{\lambda}(t-s)\psi(s) ds \|^2 = 0.$$

We can now proceed with the proof of (ii). Let $\mathcal{X}$ denote the space of all càdlàg, adapted to the filtration $\mathcal{F}_t$, $H$-valued processes $X$, equipped with the norm $| \cdot |_0$:

$$|X|_0 = \left( \mathbb{E} \sup_{t \leq T} \|X(t)\|^2 \right)^{1/2}.$$
Define transformations $K_n$, $K_{nm}$, $n, m = 1, 2, \ldots$ by the formulae,

$$
K_n(X)(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)G(X(s-))dL_n(s),
$$

$$
K_{nm}(X)(t) = S_m(t)X_0 + \int_0^t S_m(t-s)F(X(s))ds + \int_0^t S_m(t-s)G(X(s-))dL_n(s).
$$

It will follow from the first part of the proof of Proposition 3.3 that the processes $K_n(X)$, $K_{nm}(X)$ have càdlàg modifications. Moreover, as in the proof of existence of mild solutions, see e.g. [21] and using arguments similar to the proof of (9.1) one can show that for arbitrary $\alpha \in (0, 1)$ there exists $T_\alpha$ such that all transformations $K_n$, $K_{nm}$ satisfy Lipschitz conditions on $X$ with a constant smaller than $\alpha$. Moreover processes $X_n, X_{nm}^m$, are unique solutions in $X$ of the following fixed point problems

$$
X = K_n(X), \quad X = K_{nm}(X).
$$

Therefore, it is easy to see, that to prove the results it is enough to show that for each $X \in \mathcal{X}$,

$$
\lim_{m} K_{nm}(X) = K_n(X),
$$

and this follows from Proposition 3.3 [4] The case of arbitrary $T > 0$ follows by repeating the same argument on intervals $[0, T_\alpha], [T_\alpha, 2T_\alpha], \ldots, [(k - 1)T_\alpha, kT_\alpha]$, where $kT_\alpha > T$.

(iii) Without loss of generality we will assume that $t = 0$. We will denote by $\pi_n(dt, dz)$, respectively $\pi_{k}^n(dt, dz), k \geq 1$, the Poisson random measure for the process $L(nt)$, respectively $L^k(nt)$, where $L^k(nt)$ is the process $L(nt)$ with jumps restricted to size $k$. It is easy to see that the intensity measure of $L(nt)$ is equal to $n\nu(dz)$ and the intensity measure of $L^k(nt)$ is equal to $n\nu^k(dz)$, where $\nu^k(dz) = \chi_{\{|z| \leq k\}}\nu(dz)$.

Denote by $X_{mk}^m, m, k = 1, 2, \ldots$ the solution of (3.1) with $A$ replaced by $A_m$ and $L_n$ replaced by $L_n^k$, where $L_n^k = \frac{1}{n}L^k(nt)$. We will show (3.6) for the processes $X_{mk}^m$ and then pass to the limit as $k \to +\infty$ and $m \to +\infty$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth even function such that $h(0) = 1$, $h$ is increasing on $(0, +\infty)$, $h'(0) = 0$, $|h'(r)| \leq 1, h(r) \geq (1 + r)/2$ for $r > 0$. (We can take for instance $h(r) = \sqrt{1 + r^2}$.) For $l > 0$ denote by $\tau_l$ the exit time of $X_n^{mk}$ from $\{\|y\| \leq l\}$. Let $\alpha > 0$ be a number which will be specified later. By Ito’s formula, see [19], Theorem 27.2, p.
190, we have
\[ e^{-\alpha(s\wedge \tau_t)} h(||X_{n+1}(s\wedge \tau_t)||) = e^{nh(||x||)} \]
\[ - \int_0^{s\wedge \tau_t} \alpha n e^{-\alpha r} h(||X_n(r)||) e^{nh(||X_n(r)||)} dr \]
\[ + \int_0^{s\wedge \tau_t} n e^{-\alpha r} e^{nh(||X_n(r)||)} h'(X_n(r)) \left< -A_nX_n(r) + F(X_n(r), \frac{X_n(r)}{||X_n(r)||}) \right> dr \]
\[ + \int_0^s n e^{-\alpha r} e^{nh(||X_n(r)||)} h'(X_n(r)) \left< \frac{X_n(r)}{||X_n(r)||}, G(X_n(r)) \right> dL_k \]
\[ + \int_0^s \int_M 1_{[0,\tau_t]} \left[ \frac{h'(X_n(r))}{h(||X_n(r)||)} \right] \pi^k \]
\[ - e^{-\alpha r} e^{nh(||X_n(r)||)} h'(X_n(r)) \left< \frac{X_n(r)}{||X_n(r)||}, G(X_n(r)) \right> \nu(dz) dr. \]

(3.10)

To proceed further we compensate the measure \( \pi \) and recall that stochastic integrals with respect to the compensated random measures form martingales. Thus taking expectation in (3.10), using \( (2.4) \), \( (3.4) \), martingale property, the fact that \( \left< -A_ny, y \right> \leq 0 \) for \( y \in H \) and \( 1 + r \leq 2h(r) \), we therefore obtain
\[ \mathbb{E} e^{-\alpha(s\wedge \tau_t)} h(||X_{n+1}(s\wedge \tau_t)||) \leq e^{nh(||x||)} \]
\[ + \mathbb{E} \int_0^{s\wedge \tau_t} n e^{-\alpha r} e^{nh(||X_n(r)||)} \left[ C(1 + ||X_n(r)||) - \alpha h(||X_n(r)||) \right] dr \]
\[ + \mathbb{E} \int_0^s \int_M n h'(X_n(r)) \left< \frac{X_n(r)}{||X_n(r)||}, G(X_n(r)) \right> \nu(dz) dr \]
\[ - e^{-\alpha r} e^{nh(||X_n(r)||)} h'(X_n(r)) \left< \frac{X_n(r)}{||X_n(r)||}, G(X_n(r)) \right> \nu(dz) dr. \]

(3.11)

where \( I(r) \) is the integrand of the last term in the middle line of (3.11). Applying Lemma **2.3** to the function \( f(x) = e^{-\alpha h(||x||)} \) we have
\[ I(r) = \int_H \int_0^1 \int_0^1 n D^2 f \left( X_n(r) + \frac{t\sigma}{n} G(X_n(r)) z \right) \frac{1}{n} G(X_n(r)) z \sigma dt d\sigma \nu(dz). \]

(3.12)

Elementary calculation gives us
\[ D^2 f(x) = ne^{-\alpha r} e^{nh(||x||)} (n\psi_1(x) + \psi_2(x)), \]
where
\[
\psi_1(x) = e^{-\alpha r}(h'(|x|))^2 \frac{x}{|x|} \otimes \frac{x}{|x|},
\]
\[
\psi_2(x) = \left(h''(|x|) - \frac{h'(|x|)}{|x|}\right) \frac{x}{|x|} \otimes \frac{x}{|x|} + \frac{h'(|x|)}{|x|}I.
\]
We observe that both \(\psi_1, \psi_2\) are bounded as functions from \(H\) to \(L(H)\). Therefore

\[
I(r) \leq e^{ne^{-\alpha r}h(||X_n^{mk}(r)||)} \int_H \int_0^1 \int_0^1 M^2 e^{-\alpha r} e^{ne^{-\alpha r}h(||X_n^{mk}(r)||)} \left(n||\psi_1||_{\infty} + ||\psi_2||_{\infty}\right) ||z||^2 dt \, d\nu(dz)
\]

\[
\leq nM_1 e^{-\alpha r} e^{ne^{-\alpha r}h(||X_n^{mk}(r)||)} \int_H ||z||^2 e^{M||z||} \nu(dz) \leq nM_2 e^{-\alpha r} e^{ne^{-\alpha r}h(||X_n^{mk}(r)||)}
\]

(3.13)

for some \(M_1, M_2 > 0\). Plugging (3.13) into (3.11), choosing \(\alpha = 2C + M_2 + 1\) and recalling that \(h(r) \geq 1\) we thus obtain

\[
\mathbb{E}e^{ne^{-\alpha(s\wedge \tau_1)}h(||X_n^{mk}(s\wedge \tau_1)||)} + \mathbb{E} \int_0^{s\wedge \tau_1} ne^{-\alpha r} e^{ne^{-\alpha r}h(||X_n^{mk}(r)||)} dr \leq e^{nh(|x|)}
\]

which in particular implies that

\[
\mathbb{E}e^{ne^{-\alpha h(||X_n^{mk}(s\wedge \tau_1)||)} } \leq e^{nh(|x|)}.
\]

Since \(\lim_{l \to +\infty} (T \wedge \tau_1) = T\) a.s., letting \(l \to +\infty\) and using Fatou’s lemma we obtain

\[
\mathbb{E}e^{ne^{-\alpha h(||X_n^{mk}(s)||)} } \leq e^{nh(|x|)}.
\]

We can now send \(k \to +\infty\), employ once again Fatou’s lemma and the fact that \(X_n^{mk}(s) \to X_n^m(s)\) a.s. (at least along a subsequence). This can be shown using the arguments from the proof of \((ii)\). This way we arrive at

\[
\mathbb{E}e^{ne^{-\alpha h(||X_n^m(s)||)} } \leq e^{nh(|x|)}.
\]

(3.14)

We can now go back to Ito’s formula (3.10) but apply it to the function \(e^{\frac{r}{2}e^{-\alpha r}h(|z|)}\),
the process $X_n^m$ and without stopping time. It yields

$$e^{\frac{n}{2} e^{-\alpha h(\|X_n^m(r)\|)}} = e^{\frac{n}{2} h(\|x\|)} - \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m(s)\|)} e^{\frac{n}{2} e^{-\alpha h(\|X_n^m\|)}} dr \quad \times \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m\|)} h'(X_n^m(s)) \langle A_m X_n^m(s) + F(X_n^m(s)), \frac{X_n^m(s)}{\|X_n^m(s)\|} \rangle dr$$

$$+ \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m\|)} h'(X_n^m(s)) \langle \frac{X_n^m(s)}{\|X_n^m(s)\|}, G(X_n^m(s)) dL_n(s) \rangle + \int_0^s \int_0^s \left[ e^{\frac{n}{2} e^{-\alpha h(\|X_n^m\|)}} \langle \frac{X_n^m(s)}{\|X_n^m(s)\|}, G(X_n^m(s)) z \rangle \right] \pi_n(dr, dz).$$

Arguing like in (3.11) and (3.13), applying $\sup_{0 \leq s \leq T}$ to both sides and taking expectation give us

$$\mathbb{E} \sup_{0 \leq s \leq T} e^{\frac{n}{2} e^{-\alpha h(\|X_n^m(s)\|)}} \leq e^{\frac{n}{2} h(\|x\|)} + \mathbb{E} \sup_{0 \leq s \leq T} \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m\|)} (2C + M_2 - \alpha) h(\|X_n^m\|) dr$$

$$+ \mathbb{E} \sup_{0 \leq s \leq T} \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m\|)} h'(X_n^m(s)) \langle \frac{X_n^m(s)}{\|X_n^m(s)\|}, G(X_n^m(s)) dL_n(s) \rangle$$

$$+ \mathbb{E} \sup_{0 \leq s \leq T} \int_0^s \int_0^s \left[ e^{\frac{n}{2} e^{-\alpha h(\|X_n^m\|)}} \langle \frac{X_n^m(s)}{\|X_n^m(s)\|}, G(X_n^m(s)) z \rangle \right] \pi_n(dr, dz).$$

Denote

$$N(s) = \int_0^s \frac{n}{2} e^{-\alpha h(\|X_n^m\|)} h'(X_n^m(s)) \langle \frac{X_n^m(s)}{\|X_n^m(s)\|}, G(X_n^m(s)) dL_n(s) \rangle.$$

Then $N$ is a square integrable martingale. From the definition of the quadratic variation process, see [23],

$$\mathbb{E}[N, N]_T = \mathbb{E}N^2(T).$$

Therefore, from the Burkholder-Davis-Gundy inequality [23], [21],

$$\mathbb{E} \sup_{0 \leq s \leq T} |N(s)| \leq C_1 \mathbb{E}[N, N]_T^{\frac{1}{2}} \leq C_1 (\mathbb{E}[N, N]_T)^{\frac{1}{2}} = C_1 (\mathbb{E}N^2(T))^{\frac{1}{2}}$$

$$\leq C_2 \left[ \mathbb{E} \int_0^T n^2 e^{n e^{-\alpha h(\|X_n^m\|)}} M_n^2 n^2 ndr \right]^{\frac{1}{2}} \leq M_3 n^{\frac{1}{2}} e^{\frac{n}{2} h(\|x\|)}.$$
for some constant $M_3 > 0$, where we used (3.14) to get the last inequality. As regards the last term of (3.15), by Theorem 8.23 of [21],

$$
\mathbb{E} \sup_{0 \leq s \leq T} \left| \int_0^s \int_H \left( e^{\frac{1}{2}e^{-\alpha h(t)}} X^n_m(r) - e^{\frac{1}{2}e^{-\alpha h(t)}} G(X^n_m(r-)) z \right) \nu(dz) dt \right|
$$

if we once again argue like in (5.13) and then use (3.14).

Therefore, plugging (3.16) and (3.17) in (3.15) we finally obtain

$$
\mathbb{E} \sup_{0 \leq s \leq T} e^{\frac{1}{2}e^{-\alpha h(t)}} X^n_m(s) \leq M_5 n e^{\frac{1}{2}e^{-\alpha h(t)}} \leq e^{M_6 h(t)}
$$

for some $M_6 > 0$. We can now pass to the limit as $m \to +\infty$ using (3.5) and use that $(1 + r)/2 \leq h(r)$ to complete the proof.

**Proposition 3.5.** Let $0 \leq t \leq T$ and let (2.3)-(2.5) be satisfied. Let $X_n(s)$, and $Y_n(s)$ are solutions of (3.4) with initial conditions $x$ and $y$ respectively. Then

$$
\mathbb{E} \| X_n(s) - Y_n(s) \|_1^2 \leq C_1(T) \|x - y\|_1^2,
$$

(3.18)

$$
\mathbb{E} \| X_n(s) - x \|_1^2 \leq C_2(\|x\|, T)(s - t),
$$

(3.19)

and

$$
\mathbb{E} \| X_n(s) - x \|_1^2 \leq \omega_x(s - t)
$$

(3.20)

for some modulus $\omega_x$.

**Proof.** The proofs are rather typical for these kinds of estimates. We first show (3.18). By Itô’s formula we have

$$
\mathbb{E} \| X^n_m(s) - Y^n_m(s) \|_1^2 = \| x - y \|_1^2 + 2 \mathbb{E} \int_t^s \langle (X^n_m - Y^n_m, A^n_m B(X^n_m - Y^n_m) \rangle d\tau
$$

$$
+ \langle F(X^n_m) - F(Y^n_m), B(X^n_m - Y^n_m) \rangle d\tau
$$

$$
+ \frac{1}{n} \mathbb{E} \int_t^s \int_H \| G(X^n_m) - G(Y^n_m) \|_1^2 \nu(dz) d\tau.
$$

(3.21)
Using (3.5) and moment estimates (3.7) for $X^m_n$ and $Y^m_n$ we can pass to the limit above to obtain that (3.21) is still true if $X^m_n$ and $Y^m_n$ are replaced by $X_n$ and $Y_n$ respectively and $A^m$ is replaced by $A$. We then use (2.1), (2.2) and (2.3) to get

$$
\mathbb{E}\|X_n(s) - Y_n(s)\|_{-1}^2 \leq \|x - y\|_{-1}^2 \\
+ (2c_0 + M\|B^{1/2}\|) \mathbb{E} \int_t^s \|X_n(\tau) - Y_n(\tau)\|_{-1}^2 d\tau \\
+ \frac{M\|B^{1/2}\|}{n} \mathbb{E} \int_t^s \int_H \|X_n(\tau) - Y_n(\tau)\|_{-1}^2 \|z\|_{-1}^2 \nu(dz) d\tau \\
\leq \|x - y\|_{-1}^2 + C \int_t^s \mathbb{E}\|X_n(\tau) - Y_n(\tau)\|_{-1}^2 d\tau
$$

and the claim follows from Gronwall’s inequality.

To show (3.19) we again employ Ito’s formula and (2.2), (2.4) to find that

$$
\mathbb{E}\|X^m_n(s) - x\|_{-1}^2 = 2\mathbb{E} \int_t^s \left[ -\langle X^m_n(\tau), A^* B(X^m_n(\tau) - x) \rangle \\
+ \langle F(X^m_n(\tau)), B(X^m_n(\tau) - x) \rangle d\tau + \frac{1}{n} \mathbb{E} \int_t^s \int_H \|G(X^m_n(\tau)) z\|_{-1}^2 \nu(dz) d\tau \right] \text{(3.22)}
$$

$$
\leq C(\|x\|) \mathbb{E} \int_t^s (1 + \|X^m_n(\tau)\|_{-1}^2) d\tau \leq C_2(\|x\|, T)(s - t).
$$

As regards (3.20) it follows from the definition of mild solution that

$$
X_n(s) = S(s - t)x + \int_t^s S(s - \tau) F(X_n(\tau)) d\tau + \int_t^s S(s - \tau) G(X_n(\tau)) dL_n(\tau).
$$

Therefore

$$
\mathbb{E}\|X_n(s) - x\|^2 \leq 4 \left[ \|S(s - t)x - x\|^2 + \mathbb{E} \left| \int_t^s M(1 + \|X_n(\tau)\|) d\tau \right|^2 \right] \\
+ \mathbb{E} \left| \int_t^s S(s - \tau) G(X_n(\tau)) dL_n(\tau) \right|^2 \text{(3.23)}
$$

$$
\leq C \left( \|S(s - t)x - x\|^2 + (s - t)^2 + \mathbb{E} \int_t^s \frac{1}{n} d\tau \right),
$$

where we have used the isometric formula to obtain the last inequality. \hfill \blacksquare

Finally we state for future use the following lemma which can be shown rather easily using again Ito’s formula applied first to the process $X^m_n$ and then letting $m \to +\infty$. Its proof will thus be omitted.
Lemma 3.6. Let the assumptions of Proposition 3.1 be satisfied. Let \( t \leq s \leq T \). Let \( \psi = \varphi + h(\| \cdot \|) \) be a bounded test function. Then

\[
\mathbb{E} e^{\psi(s,X_n(s))} \leq e^{\psi(t,x)} + \mathbb{E} \int_t^s e^{\psi(\tau,X_n(\tau))} [\psi_t(\tau,X_n(\tau)) + \langle F(X_n(\tau)), D\psi(\tau,X_n(\tau)) \rangle + \langle A^* D\varphi(\tau,X_n(\tau)), z \rangle]
\]

for all \( \tau \in [t,s] \).

4 Associated nonlinear integro-PDE

For \( g \in C_b(H) \) we define the function

\[
v_n(t,x) = \frac{1}{n} \log \mathbb{E} \left( e^{ng(X_n(T))} \right), \tag{4.1}
\]

where \( X_n \) solves (3.1). As we have stated earlier one of our main aims is to establish convergence of the sequence \( (v_n) \) and to identify its limit as a solution of a Hamilton-Jacobi-Bellman equation. In the present section we investigate the approximating and the limiting equations.

4.1 Approximating equations

We first show that for each \( n \) the function \( v_n \) is a viscosity solution of an integro-PDE.

Theorem 4.1. Let (2.2)-(2.5) be satisfied and let \( g \in \text{Lip}_b(H_{-1}) \). Then there exist a constant \( C_1 \) and, for every \( R > 0 \), a constant \( C_2 = C_2(R) \) (both possibly depending on \( n \)) such that

\[
|v_n(t,x) - v_n(s,y)| \leq C_1 \| x - y \|_{-1} + C_2 (\max \{ \| x \|, \| y \| \}) |t - s|^\frac{1}{2}
\]

for \( x, y \in H, t, s \in [0,T] \) \( \tag{4.2} \)

and \( v_n \) is a viscosity solution of an integro-PDE

\[
\begin{cases}
(v_n)_t + \langle -Ax + F(x), Dv_n \rangle \\
\quad + \int_H \left[ e^{n(v_n(t,x+\frac{1}{n}G(x)z)-v_n(t,x))} - 1 - \langle Dv_n, G(x)z \rangle \right] \nu(dz) = 0,
\end{cases} \tag{4.3}
\]

and \( v_n(T,x) = g(x) \) in \( (0,T) \times H \).
Proof. Estimate (4.2) is a direct consequence of (3.18), (3.19), and the Markov property of the process $X_n$. The proof that $v_n$ is a viscosity solution of (4.3) is similar to the proof of Theorem 7.1 in [29]. We will only show that $v_n$ is a viscosity subsolution since the supersolution part is similar.

Suppose that $v_n - h(\| \cdot \|) - \varphi$ has a global maximum at $(t, x)$. Since $v_n$ is bounded by Remark 4.3 of [29] without loss of generality we can also assume that $h, h', h''$ and $\varphi$ are bounded. Denote $\psi(s, y) = h(\| y \|) + \varphi(s, y)$. Then for small $\epsilon > 0$

$$v_n(t + \epsilon, X_n(t + \epsilon)) - \psi(t + \epsilon, X_n(t + \epsilon)) \leq v_n(t, x) - \psi(t, x).$$

Therefore, setting $u_n = e^{nv_n}$ we have

$$\frac{u_n(t + \epsilon, X_n(t + \epsilon))}{u_n(t, x)} \leq e^{n\psi(t + \epsilon, X_n(t + \epsilon))} e^{-n\psi(t, x)}.$$

which, upon taking the expectation of both sides of the above inequality and using the Markov property of $X_n(s)$, produces

$$e^{n\psi(t, x)} \leq \mathbb{E} e^{n\psi(t + \epsilon, X_n(t + \epsilon))}.$$

Therefore, applying Lemma 3.6, we obtain

$$0 \leq \mathbb{E} \frac{1}{\epsilon} \left\{ e^{n\psi(t + \epsilon, X_n(t + \epsilon))} - e^{n\psi(t, x)} \right\}$$

$$\leq \mathbb{E} \frac{1}{\epsilon} \int_t^{t+\epsilon} ne^{n\psi(\tau, X_n(\tau))} \left[ \psi_t(\tau, X_n(\tau)) + \langle F(X_n(\tau)), D\psi(\tau, X_n(\tau)) \rangle d\tau - \langle X_n(\tau), A^* D\varphi(\tau, X_n(\tau)) \rangle \right] d\tau$$

$$+ \mathbb{E} \frac{n}{\epsilon} \int_t^{t+\epsilon} \int_H \left[ e^{n\psi(\tau, X_n(\tau)) + \frac{1}{\epsilon} G(X_n(\tau))z} - e^{n\psi(\tau, X_n(\tau))} \right] \nu(dz) d\tau. \quad (4.4)$$

Using (3.20), (2.2), boundedness of $\psi, \psi_t, D\psi, A^* \varphi$, and moment estimates (in particular (3.6)) it is easy to see that

$$\mathbb{E} \frac{1}{\epsilon} \int_t^{t+\epsilon} ne^{n\psi(\tau, X_n(\tau))} \left[ \psi_t(\tau, X_n(\tau)) + \langle F(X_n(\tau)), D\psi(\tau, X_n(\tau)) \rangle d\tau - \langle X_n(\tau), A^* D\varphi(\tau, X_n(\tau)) \rangle \right] d\tau$$

$$= \frac{1}{\epsilon} \left[ \int_t^{t+\epsilon} ne^{n\psi(t, x)} \left[ \psi_t(t, x) + \langle F(x), D\psi(t, x) \rangle d\tau - \langle x, A^* D\varphi(t, x) \rangle \right] d\tau + o(\epsilon) \right]. \quad (4.5)$$
As regards the other term, by Lemma \[2.2\] (2.3) (2.4), (2.5), (3.6), (3.20), boundedness of \(\psi\) and uniform continuity of \(\psi, D\psi, D^2\psi\), we have

\[
\mathbb{E} \frac{\nu}{\epsilon} \int_t^{t+\epsilon} \int_H \left[ e^{n\psi(t, x_n(\tau)) + \frac{1}{n} G(x_n(\tau)) z} - e^{n\psi(t, x_n(\tau))} \right] \nu(dz) d\tau
\]

\[
-e^{n\psi(t, x_n(\tau))} \langle D\psi(t, x_n(\tau)), G(x_n(\tau)) z \rangle \nu(dz) d\tau
\]

\[
= \mathbb{E} \frac{\nu}{\epsilon} \int_t^{t+\epsilon} \int_H \left[ \int_0^1 \int_0^1 \langle D^2 e^{n\psi(t, x_n(\tau))} + s \sigma \frac{1}{n} G(x_n(\tau)) z \rangle \frac{1}{n} G(x_n(\tau)) z, \frac{1}{n} G(x) z \rangle \sigma ds d\sigma d\tau
\]

\[
+ C_1 (1 + \|X_n(\tau)\|^2 + ||z||^2) \|z\|^2 \omega(||X_n(\tau) - x|| (1 + ||z||)) \nu(dz) d\tau
\]

\[
= \frac{n}{\epsilon} \int_t^{t+\epsilon} \left[ \int_H \left[ e^{n\psi(t, x + \frac{1}{n} G(x) z)} - e^{n\psi(t, x)} \right] (D\psi(t, x), G(x) z) \nu(dz) + \omega_1(\epsilon) \right] d\tau.
\] (4.6)

(Above \(\omega, \omega_1\) are some moduli and \(C_1, C_2\) are constants, all depending on \(\psi\).) Therefore plugging (4.5) and (4.6) into (4.4) and sending \(\epsilon \to 0\) we obtain

\[
0 \leq ne^{n\psi(t, x)} \left( \psi(t, x) - \langle x, A^* D\varphi(t, x) \rangle + \langle F(x), D\psi(t, x) \rangle
\]

\[
+ \int_H \left[ e^{n\psi(t, x + \frac{1}{n} G(x) z) - \psi(t, x))} - 1 - \langle D\psi(t, x), G(x) z \rangle \nu(dz) \right)
\]

which completes the proof after we divide both sides by \(ne^{n\psi(t, x)}\).

\[\Box\]

### 4.2 Limiting Hamilton-Jacobi-Bellman equation

The limiting equation (obtained by letting \(n \to +\infty\) in (4.3)) can be formally identified as

\[
\begin{cases}
    v_t + \langle -Ax + F(x), Dv \rangle + H_0(G^*(x) Dv) = 0 \\
    v(T, x) = g(x) \quad \text{in} \quad (0, T) \times H,
\end{cases}
\] (4.7)

where

\[
H_0(p) = \int_H \left[ e^{\langle p, z \rangle} - 1 - \langle p, z \rangle \right] \nu(dz).
\]

It is the Bellman equation corresponding to a deterministic control problem. For \(0 \leq t \leq T, x \in H\), and \(u(\cdot) \in M_t = \{u : [t, T] \to H : u \text{ is strongly measurable}\}\) we consider the
The state equation

\[ X'(s) = -AX(s) + F(X(s)) + G(X(s))u(s), \quad X(t) = x, \quad (4.8) \]

and we want to maximize the cost functional

\[ J(t, x; u(\cdot)) = \int_t^T -L_0(u(s))ds + g(X(T)) \]

over all controls \( u(\cdot) \in M_t \), where \( L_0 \) is the Legendre transform of \( H_0 \), i.e.

\[ L_0(z) = \sup_{y \in H} \{ \langle z, y \rangle - H_0(y) \}. \quad (4.9) \]

The value function for the problem is

\[ v(t, x) = \sup_{u(\cdot) \in M_t} J(t, x; u(\cdot)). \quad (4.10) \]

The Hamiltonian \( H_0 \) and Lagrangian \( L_0 \) are both convex. By (2.5) and the definition of \( H_0 \) we see that \( 0 \leq H_0(y) < +\infty \) for every \( y \in H \), \( H_0(0) = 0 \), and \( H_0 \) is locally Lipschitz continuous on \( H \). Therefore \( L_0(0) = 0 \), \( L_0(z) \geq 0 \) for every \( z \in H \), and moreover

\[ L_0(z) \geq \|z\| - H_0\left(\frac{z}{\|z\|}\right) \to +\infty \quad \text{as} \quad \|z\| \to +\infty \quad (4.11) \]

(but \( L_0 \) can possibly take infinite values). Since \( g \) is bounded it is then obvious that

\[ v(t, x) = \sup_{u(\cdot) \in \tilde{M}_t} J(t, x; u(\cdot)), \]

where

\[ \tilde{M}_t = \{ u(\cdot) \in M_t : \int_t^T L_0(u(s))ds \leq K = 2\|g\|_\infty \}. \quad (4.12) \]

We will need the following simple lemma.

**Lemma 4.2.** For every \( \epsilon > 0 \) there exists a constant \( N_\epsilon = N_\epsilon(\nu) \) such that for every \( z \in H \)

\[ \|z\| \leq \epsilon L_0(z) + N_\epsilon. \]

**Proof.** It follows from (1.9), (2.5), and \( L_0(0) = 0 \) that

\[ \|z\| = \langle \epsilon z, \frac{z}{\epsilon \|z\|} \rangle \leq L_0(\epsilon z) + H_0\left(\frac{z}{\epsilon \|z\|}\right) \leq \epsilon L_0(z) + N_\epsilon. \]

\[ \blacksquare \]
Lemma 4.3. Let (2.2)-(2.4) be satisfied. Let $0 \leq t \leq T$ and $u(\cdot) \in \tilde{M}_t$. Then:

(i) There exists a unique mild solution $X \in C([t,T];H)$ of (4.8). Moreover there exists a constant $C_1 = C_1(T, K, M)$ such that

$$\sup_{t \leq s \leq T} \|X(s)\| \leq C_1(1 + \|x\|). \quad (4.13)$$

(ii) There exists a constant $C_2 = C_2(T, K, M, c_0, \|B^\sharp\|)$, such that if $X$, and $Y$ are solutions of (4.8) with initial conditions $x$ and $y$ respectively then

$$\|X(s) - Y(s)\|_{-1} \leq C_2\|x - y\|_{-1} \quad \text{for } t \leq s \leq T, \quad (4.14)$$

(iii) For every $R > 0$ there exists a modulus $\omega_R$, depending on $R, K, T, \|A^*B\|$, such that if $\|x\| \leq R$ then

$$\|X(s) - x\|_{-1} \leq \omega_R(s - t) \quad \text{for } t \leq s \leq T, \quad (4.15)$$

and for every $x \in H$ there exists a modulus $\omega_x$, independent of $u(\cdot)$, such that

$$\|X(s) - x\| \leq \omega_x(s - t) \quad \text{for } t \leq s \leq T. \quad (4.16)$$

Proof. We first notice that by Lemma 4.2 (applied with $\epsilon = 1$)

$$\int_t^T \|u(\tau)\| d\tau \leq K + N_1 \quad (4.17)$$

for every $u(\cdot) \in \tilde{M}_t$. Therefore the existence and uniqueness of a mild solution of (4.8) and estimate (4.13) are well known. We refer for instance to [18], Chapter 2, Proposition 5.3.

To show (4.14) we notice that

$$\|X(s) - Y(s)\|_{-1}^2 = \|x - y\|_{-1}^2 - 2 \int_t^s \langle A^*B(X(\tau) - Y(\tau)), X(\tau) - Y(\tau) \rangle d\tau$$

$$+ 2 \int_t^s \langle B(X(\tau) - Y(\tau)), F(X(\tau)) - F(Y(\tau)) + (G(X(\tau)) - G(Y(\tau)))u(\tau) \rangle d\tau$$

and therefore using (2.1), (2.2) and (2.3) we have

$$\|X(s) - Y(s)\|_{-1}^2 \leq \|x - y\|_{-1}^2 + C \int_t^s \|X(\tau) - Y(\tau)\|_{-1}^2(1 + \|u(\tau)\|) d\tau.$$  

Therefore (4.14) follows from (4.17) and Gronwall’s inequality.

To prove (4.13) we write

$$\|X(s) - x\|_{-1}^2 = -2 \int_t^s \langle A^*B(X(\tau) - x), X(\tau) \rangle d\tau$$

$$+ 2 \int_t^s \langle B(X(\tau) - x), F(X(\tau)) + G(X(\tau))u(\tau) \rangle d\tau$$
and thus using (2.2)-(2.4), (4.13) and Lemma 4.2 we obtain
\[ \|X(s) - x\|_{-1}^2 \leq \int_t^s C_R(1 + \|u(\tau)\|)d\tau \leq \epsilon C_R \int_t^s L(u(\tau))d\tau + C_R N(\epsilon(s - t)) \leq \epsilon C_R K + C_R N(\epsilon(s - t)). \]
Therefore we obtain (4.15) with
\[ \omega_R(\tau) = \inf_{\epsilon > 0} (\epsilon C_R K + C_R N(\epsilon\tau))^{\frac{1}{2}}. \]
Estimate (4.16) is proved similarly noticing that
\[ \|X(s) - x\| \leq \|S(s-t)x - x\| + \int_t^s C_R(1 + \|u(\tau)\|)d\tau. \]

The definition of viscosity solution of (4.7) is the same as Definition 2.4 after we disregard the nonlocal part and of course it is enough to have test functions which are only once continuously differentiable. For more on viscosity solutions of first order PDE in Hilbert spaces we refer to [7, 8, 18].

**Theorem 4.4.** Let (2.2)-(2.4) be satisfied and let \( g \in \text{Lip}_b(H_{-1}) \). There exist a constant \( D_1 \) and, for every \( R > 0 \), a modulus \( \omega_R \) such that the value function \( v \) satisfies
\[ |v(t, x) - v(s, y)| \leq D_1 \|x - y\|_{-1} + \omega_R(\|t - s\|) \quad \text{for } x, y \in H, \|x\|, \|y\| \leq R, t, s \in [0, T]. \]
Moreover \( v \) is a viscosity solution of the HJB equation (4.7).

**Proof.** The proof is very similar to the proof of Theorem 7.3 in [29]. We include it here for completeness.

The Lipschitz continuity in \( x \) follows from (4.14) and the fact that \( g \in \text{Lip}_b(H_{-1}) \). To show the continuity in time let \( x \in H \) and \( s < t \) and let \( \epsilon > 0 \). Let \( u_\epsilon(\cdot) \in M_t \) be such that
\[ v(t, x) \leq J(t, x; u_\epsilon(\cdot)) + \epsilon. \]
Extending \( u_\epsilon(\cdot) \) by 0 to \([s, T]\) we can assume that \( u_\epsilon(\cdot) \in M_s \). Therefore
\[ v(s, x) - v(t, x) \geq J(s, x; u_\epsilon(\cdot)) - J(t, x; u_\epsilon(\cdot)) - \epsilon \geq g(X(T; s, x)) - g(X(T; t, x)) + \epsilon \geq -C_2 D_2 \omega_R(\|s - t\|) - \epsilon, \]
where we have used (4.14), (4.15), and \( D_2 \) is the Lipschitz constant of \( g \). For the opposite inequality if \( u_\epsilon(\cdot) \in M_s \) is such that
\[ v(s, x) \leq J(s, x; u_\epsilon(\cdot)) + \epsilon \]
then \( u_\epsilon(\cdot) \in M_t \) and by (4.14), (4.15) we again have

\[
v(s, x) - v(t, x) \leq J(s, x; u_\epsilon(\cdot)) + \epsilon - J(t, x; u_\epsilon(\cdot))
\]

\[
\leq g(X(T; s, x)) - g(X(T; t, x)) - \int_t^s L_0(u_\epsilon(\tau))d\tau + \epsilon
\]

\[
\leq C_2 D_2 \omega_R(|s - t|) + \epsilon.
\]

Therefore since \( \epsilon \) was arbitrary we have obtained

\[
|v(s, x) - v(t, x)| \leq C_2 D_2 \omega_R(|s - t|).
\]

We will only show that \( v \) is a viscosity subsolution as the proof of the supersolution property is similar but easier. We will use the dynamic programming principle. It asserts that if \( 0 \leq t < t + \epsilon \leq T, x \in H \) then

\[
v(t, x) = \sup_{u(\cdot) \in M_t} \left\{ \int_t^{t+\epsilon} -L_0(u(s))ds + v(t + \epsilon, X(t + \epsilon)) \right\}.
\]

Let now \( v - \varphi - h(\| \cdot \|) \) have a local maximum at \((t, x)\). By the dynamic programming principle for every \( 0 < \epsilon < T - t \) there exists a control \( u_\epsilon(\cdot) \) such that

\[
v(t, x) \leq \int_t^{t+\epsilon} -L_0(u_\epsilon(s))ds + v(t + \epsilon, X_\epsilon(t + \epsilon)) + \epsilon^2
\]

We recall that in particular this implies that \( u_\epsilon(\cdot) \) is integrable.

Denote \( \psi(s, y) = -\varphi(s, y) - h(\| y \|) \). For simplicity we will write \( h(y) := h(\| y \|) \).

We have

\[
\varphi(t + \epsilon, X_\epsilon(t + \epsilon)) = \varphi(t, X_\epsilon(t)) + \int_t^{t+\epsilon} \left[ -\langle X_\epsilon(s), A^* D\varphi(X_\epsilon(s)) \rangle + \langle F(X_\epsilon(s)) + G(X_\epsilon(s))u_\epsilon(s), D\varphi(X_\epsilon(s)) \rangle \right] ds
\]

and

\[
h(X_\epsilon(t + \epsilon)) \leq h(x) + \int_t^{t+\epsilon} \langle F(X_\epsilon(s)) + G(X_\epsilon(s))u_\epsilon(s), Dh(X_\epsilon(s)) \rangle ds.
\]

The first equality above is proved for instance in [18], Chapter 2, Proposition 5.5 and the inequality is also standard and can be shown using Yosida approximations similarly to what we have done in the stochastic case.
Using this we therefore have

\[ -\epsilon \leq \frac{1}{\epsilon} \left( v(t + \epsilon, X_\epsilon(t + \epsilon)) - v(t, x) - \int_t^{t+\epsilon} L_0(u_\epsilon(s))ds \right) \]

\[ \leq \frac{1}{\epsilon} \left( \varphi(t + \epsilon, X_\epsilon(t + \epsilon)) - \varphi(t, x) + h(X_\epsilon(t + \epsilon)) - h(x) - \int_t^{t+\epsilon} L(u_\epsilon(s))ds \right) \]

\[ \leq \frac{1}{\epsilon} \left\{ \int_t^{t+\epsilon} \left[ \varphi_t(s, X_\epsilon(s)) - (X_\epsilon(s), A^*D\varphi(s, X_\epsilon(s))) \right. \]
\[ \left. + \langle F(X_\epsilon(s)) + G(X_\epsilon(s))u_\epsilon(s), D\psi(s, X_\epsilon(s)) \rangle - L_0(u_\epsilon(s)) \right] ds \right\} \]

\[ \leq \frac{1}{\epsilon} \left\{ \int_t^{t+\epsilon} \left[ \varphi_t(s, X_\epsilon(s)) - (X_\epsilon(s), A^*D\varphi(s, X_\epsilon(s))) \right. \]
\[ \left. + \langle F(X_\epsilon(s)), D\psi(s, X_\epsilon(s)) \rangle + H_0(G^*(X_\epsilon(s))D\psi(s, X_\epsilon(s))) \right] ds \right\}. \quad (4.19) \]

Therefore, using (4.16), we can pass to the limit as \( \epsilon \to 0 \) in (4.19) to obtain

\[ 0 \leq \psi_t(t, x) - \langle x, A^*D\varphi(t, x) \rangle + \langle F(x), D\psi(t, x) \rangle + H_0(G^*(x)D\psi(t, x)). \]

\[ \blacksquare \]

5 Existence of Laplace limit

Define

\[ H(x, p) = H_0(G^*(x)p). \]

By (2.3), (2.4) and local Lipschitz continuity of \( H_0 \) we have that for every \( R > 0 \) there exists a constant \( K_R \) such that

\[ |H(x, p) - H(y, q)| \leq K_R(\|x - y\|_1 + \|p - q\|) \quad \text{for all } x, y, p, q \in H, \|p\|, \|q\| \leq R. \quad (5.1) \]

The theorems below are our key results on the existence of the Laplace limit.

**Theorem 5.1.** Let (2.2)-(2.5) hold. Let \( g \in \text{Lip}_b(H_{-1}) \). Let \( v_n \) be bounded viscosity solutions of (4.3), and \( v \) be a bounded viscosity solution of (4.7) such that

\[ \lim_{t \to T} \{ |v_n(t, x) - g(x)| + |v(t, x) - g(x)| \} = 0, \text{ uniformly on bounded sets} \quad (5.2) \]

for every \( n \) and

\[ |v(t, x) - v(t, y)| \leq D_1 \|x - y\|_1 \quad (5.3) \]

for some \( D_1 \geq 0 \) and all \( t \in (0, T], x, y \in H \). Let \( K := \|v\|_\infty + \sup_n \|v_n\|_\infty < +\infty. \) Then

\[ \|v_n - v\|_\infty \to 0 \quad \text{as } n \to +\infty. \quad (5.4) \]
The proof of this theorem is postponed until the end of the section.

**Remark 5.2.** We point out that Theorem 5.1 implies that if (2.2)-(2.5) hold and $g \in \text{Lip}_b(H_{-1})$ then the value function (4.10) of the control problem of Section 4.2 is the unique bounded viscosity solution of (4.7) satisfying (5.2) and (5.3).

Let $X_n(T)$ be the solution of (1.2) (i.e. the solution of (3.1) with $t = 0$). Theorems 4.1, 4.4, and 5.1 yield the following corollary.

**Corollary 5.3.** Let (2.2)-(2.5) hold and let $g \in \text{Lip}_b(H_{-1})$. Then

$$\Lambda(g) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} e^{ng(X_n(T))} = v(0, x),$$

where $v$ is the value function defined by (4.10).

This result can now be easily extended to larger class of functions $g$.

**Theorem 5.4.** Let (2.2)-(2.5) hold and let $g$ be bounded and weakly sequentially continuous on $H$. Then $\Lambda(g)$ exists and

$$\Lambda(g) = v(0, x),$$

(5.5)

where $v$ is the value function defined by (4.10).

**Proof.** We use exponential moment estimate (3.6) and the fact that $g$ can be approximated uniformly on balls in $H$ by functions in $\text{Lip}_b(H_{-1})$. Since (5.5) is true for every $g \in \text{Lip}_b(H_{-1})$, it will be preserved in the limit. Since the argument is rather standard it will not be repeated here. Instead we refer to the proofs of Lemma 7.6 and Proposition 7.7 of [29].

We now pass to the proof of Theorem 5.1.

**Proof of Theorem 5.1.** If (5.4) is not satisfied then without loss of generality we can assume that there exists $\epsilon > 0$ and a subsequence $n_k$ such that

$$\sup(v_{n_k} - v) \geq 4\epsilon.$$  

(5.6)

Let $a > 0$ be such that $aT \leq \epsilon$ and let $m > 0$ be such that

$$m \geq K + \frac{D_1^2}{\epsilon}, \quad \text{and} \quad \frac{2D_1^2}{m}(c_0 + M\|B^2\|) + \frac{D_1}{m}K_2\|B^4\| + 1 \leq \frac{a}{2}.$$  

Let $\psi : [0, +\infty) \to [0, +\infty)$ be a smooth and nondecreasing function such that $\psi(r) = r^2$ for $0 \leq r \leq 1$ and $\psi(r) = 2$ for $r \geq 2$. For each $k$ we choose $\mu_k > 0$ such that

$$\sup(v_{n_k} - v - \frac{\mu_k}{t} - \frac{\mu_k}{s}) \geq 3\epsilon.$$  

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For $\delta, \beta > 0$ we now consider the function

$$
\Phi(t, s, x, y) = v_{nk}(t, x) - v(s, y) - a(T-t) - \frac{\mu_k}{t} - \frac{\mu_k}{s} - m\psi(\|x - y\|_1^2) - \frac{(t-s)^2}{2\beta} - \delta \sqrt{1 + \|x\|^2} - \delta \sqrt{1 + \|y\|^2}.
$$

(5.7)

Since $\Phi$ is $B$-upper semicontinuous,

By a perturbed optimization technique of [8] (see page 424 there or [18], Chapter 6.4), we obtain for every sufficiently big $i > 0$ elements $p_i, q_i \in H$ and $a_i, b_i \in \mathbb{R}$ such that $\|p_i\| + \|q_i\| + |a_i| + |b_i| \leq 1/i$ and such that

$$
\Phi(t, s, x, y) + a_i t + b_i s + \langle Bp_i, x \rangle + \langle Bq_i, r \rangle
$$

has a global maximum over $[0, T] \times H$ at some points $\bar{t}, \bar{s}, \bar{x}, \bar{y}$, where $0 < \bar{t}, \bar{s}$. Following standard arguments (see for instance [15]) is is easy to see that

$$
\lim_{\delta \to 0} \lim_{\beta \to 0} \lim_{i \to +\infty} \delta(\sqrt{1 + \|\bar{x}\|^2} + \sqrt{1 + \|\bar{y}\|^2} = 0 \mbox{ for fixed } k,
$$

(5.8)

$$
\lim_{\beta \to 0} \lim_{i \to +\infty} \frac{(\bar{t} - \bar{s})^2}{2\beta} = 0 \mbox{ for fixed } k, \delta.
$$

(5.9)

Moreover it is clear that $\psi(\|\bar{x} - \bar{y}\|_1^2) = \|\bar{x} - \bar{y}\|_1^2$ and, since $\Phi(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \leq \Phi(\bar{t}, \bar{s}, \bar{x}, \bar{y})$, we obtain

$$
m\|\bar{x} - \bar{y}\|_1^2 \leq D_1 \|\bar{x} - \bar{y}\|_1 + \delta \sqrt{1 + \|\bar{x}\|^2} + \langle q_i, \bar{y} - \bar{x} \rangle
$$

which, in light of (5.8) and the fact that $\|\bar{x}\|, \|\bar{y}\| \leq c_\delta$ for every $i$ for some constant $c_\delta$, implies

$$
\lim_{\delta \to 0} \lim_{\beta \to 0} \lim_{i \to +\infty} m\|\bar{x} - \bar{y}\|_1 \leq D_1.
$$

(5.10)

Therefore, by (5.6), (5.8), (5.9), (5.10) and the definition of $m$, for small $\delta, \beta$, and big $i$ we have $0 < \bar{t}, \bar{s} < T$.

We now use (5.7) and the definition of viscosity solution to obtain

$$
- a - a_i - \frac{\mu_k}{t^2} + \frac{\bar{t} - \bar{s}}{\beta} - \langle \bar{x}, A^* B(2m(\bar{x} - \bar{y}) - p_i) \rangle
$$

$$
+ \left\langle F(\bar{x}), 2mB(\bar{x} - \bar{y}) + \frac{\delta \bar{x}}{\sqrt{1 + \|\bar{x}\|^2}} - Bp_i \right\rangle
$$

$$
+ \int_H \left[ e^{\eta_k m(\psi(\|\bar{x} + \frac{1}{\eta_k} G(\bar{x}) z + \|\bar{y}\|_1^2) - \psi(\|\bar{x} - \bar{y}\|_1^2)) + \eta_k (\sqrt{1 + \|\bar{x} + \frac{1}{\eta_k} G(\bar{x}) z\|^2 - \sqrt{1 + \|\bar{x}\|^2}) - \langle Bp_i, G(\bar{x}) z \rangle}
$$

$$
- 1 - \left( 2mB(\bar{x} - \bar{y}) + \frac{\delta \bar{x}}{\sqrt{1 + \|\bar{x}\|^2}} - Bp_i, G(\bar{x}) z \right) \nu(dz) \right] \geq 0
$$

(5.11)
and
\[ b_i + \frac{\mu_k}{\beta \bar{s}} + \frac{\bar{t} - \bar{s}}{\beta} - \langle \bar{y}, A^*(2mB(\bar{x} - \bar{y} + q_i)) \rangle + \left\langle F(\bar{y}), 2mB(\bar{x} - \bar{y}) - \frac{\delta \bar{y}}{\sqrt{1 + \|\bar{y}\|^2}} + Bq_i \right\rangle \]
\[ + H(\bar{y}, 2mB(\bar{x} - \bar{y}) - \frac{\delta \bar{y}}{\sqrt{1 + \|\bar{y}\|^2}} + Bq_i) \leq 0. \]  
(5.12)

But
\[ e^{n_k}\psi(\|\bar{x} + \frac{\bar{t}}{\beta}G(\bar{x} - \bar{y})z - \bar{y}\|^2_1) - \psi(\|\bar{x} - \bar{y}\|^2_1) + \delta n_k(\sqrt{1 + \|\bar{x} + \frac{\bar{t}}{\beta}G(\bar{x} - \bar{y})z\|^2} - \sqrt{1 + \|\bar{x}\|^2}) - (Bp, G(\bar{x})z) \]
\[ = e^{(2mB(\bar{x} - \bar{y}) + \frac{\delta \bar{x}}{\sqrt{1 + \|\bar{x}\|^2}}) - Bp, G(\bar{x})z + \sigma_k(z)}, \]  
(5.13)

where for small \( \delta, \beta \) and big \( i \)
\[ |
\sigma_k(z) \leq C_m \min(\|z\|, \frac{\|z\|^2}{n_k}) \]
for some constant \( C_m \) independent of \( k \). Using this in (5.11) we therefore obtain that for small \( \delta, \beta \) and big \( i \)
\[ - a - a_i - \frac{\mu_k}{\beta^2} + \frac{\bar{t} - \bar{s}}{\beta} - \langle \bar{x}, A^*B(2m(\bar{x} - \bar{y}) - p_i) \rangle \]
\[ + \left\langle F(\bar{x}), 2mB(\bar{x} - \bar{y}) + \frac{\delta \bar{x}}{\sqrt{1 + \|\bar{x}\|^2}} - Bp_i \right\rangle \]
\[ + H(\bar{x}, 2mB(\bar{x} - \bar{y}) + \frac{\delta \bar{x}}{\sqrt{1 + \|\bar{x}\|^2}} - Bp_i) \geq - \int_{\{\|z\| \leq 1\}} \tilde{C}_m \|z\|^2 \nu(dz) \]
\[ + \int_{\{\|z\| > 1\}} e^{(2D_1\|B\|^{\frac{1}{2}} + 1)\|z\|}(e^{\sigma_k(z)} - 1)\nu(dz) \geq -\omega(k, \delta, \beta, \bar{s}), \]
where \( \lim_{k \to +\infty} \lim \sup_{\delta \to 0} \lim \sup_{\beta \to 0} \lim \sup_{i \to +\infty} \omega(k, \delta, \beta, \bar{s}) = 0 \) by (2.5) and the Lebesgue dominated convergence theorem.

Combining (5.12) and (5.14) and using (5.8), (5.10), (2.2), (2.3), (2.4) we thus obtain
\[ a \leq -2\frac{\mu_k}{T^2} + 2m(c_0 + M\|B\|^{\frac{3}{2}})\|\bar{x} - \bar{y}\|^2_1 + K\frac{D_1}{2D_1\|B\|^{\frac{3}{2}} + 1}\|\bar{x} - \bar{y}\|_1 + \omega_1(k, \delta, \beta, i) \]
\[ \leq \frac{2D_1^2}{m}(c_0 + M\|B\|^{\frac{3}{2}}) + \frac{D_1}{m}K\frac{D_1}{2D_1\|B\|^{\frac{3}{2}} + 1} + \omega_2(k, \delta, \beta, i) \]
\[ \leq \frac{a}{2} + \omega_2(k, \beta, \delta, i), \]  
(5.15)

where \( \lim \sup_{k \to +\infty} \lim \sup_{\delta \to 0} \lim \sup_{\beta \to 0} \lim \sup_{i \to +\infty} \omega_j(k, \beta, \delta, i) = 0 \) for fixed \( j = 1, 2, i \). This yields a contradiction after we send \( i \to +\infty, \beta \to 0, \delta \to 0 \) and then \( k \to +\infty \).

Similar argument gives us that \( \lim_{n \to +\infty} \sup(v - v_n) = 0 \) and therefore (5.4) follows for some modulus \( \omega \).
6 Large deviation principle

Let $V$ be a Hilbert space such that $H \subset V$ and $H \hookrightarrow V$ is compact. We remark that on every closed ball in $H$, the topology of $V$ is equivalent to the weak topology in $H$. We have the following large deviation result.

**Theorem 6.1.** Let (2.2)-(2.5) hold. Let $T > 0, x \in H$, and let $X_n$ be the solutions of (1.2). Then the random variables $X_n(T)$ satisfy large deviation principle in $V$ with the rate function

$$I(y) = \liminf_{z \to y} \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s))ds : X \text{ satisfies (4.8), } X(0) = x, X(T) = z \right\},$$

(6.1)

(where the liminf above is taken in the topology of $V$).

**Proof.** By Bryc’s theorem (see for instance [10], Theorem 1.3.8) to show that $X_n(T)$ satisfy large deviation principle in $V$ it is enough to prove that $X_n(T)$ are exponentially tight in $V$ and that for every $g \in C_b(V)$ the Laplace limit $\Lambda(g)$ exists. Since closed balls in $H$ are compact in $V$, exponential tightness of $X_n(T)$ follows from the exponential moment estimates (3.6). Since every $g \in C_b(V)$ is weakly sequentially continuous on $H$, the Laplace limit $\Lambda(g)$ exists by Theorem 5.4. It remains to prove the representation formula for the rate function. We recall that

$$\Lambda(g) = \sup_{u(\cdot) \in M_0} \left\{ \int_0^T -L_0(u(s))ds + g(X(T)) \right\},$$

where $X(0) = x$.

We have (see [10], page 27 or [13], page 47)

$$I(y) = \sup_{g \in C_b(V), g(y) = 0} \{-\Lambda(g)\}
= \sup_{g \in C_b(V), g(y) = 0, g \geq 0} \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s))ds + g(X(T)) \right\}.$$

Denote the right-hand side of (6.1) by $I_1(y)$ and for $m > 0$ define the function

$$g_m(z) = m\|z - y\|_V,$$

where $\| \cdot \|_V$ is the norm in $V$. Then for $m, n \geq 1$

$$I(y) \geq \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s))ds + g(X(T)) \right\}
\geq \min \left\{ \frac{m}{n}, \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s))ds : \|X(T) - y\|_V \leq \frac{1}{n} \right\} \right\}.$$
Therefore, letting \( m \to +\infty \) we obtain

\[
I(y) \geq \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s)) ds : \|X(T) - y\|_V \leq \frac{1}{n} \right\},
\]

which implies \( I(y) \geq I_1(y) \). To show the reverse inequality, for \( g \in C_b(V) \) let \( \omega_y \) be a modulus of continuity of \( g \) at \( y \). Then for \( n \geq 1 \) we have

\[
\inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s)) ds + g(X(T)) \right\} \\
\leq \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s)) ds : \|X(T) - y\|_V \leq \frac{1}{n} \right\} + \omega_y \left( \frac{1}{n} \right).
\]

Taking the \( \lim \inf_{n \to +\infty} \) in the above inequality and then supremum over \( g \) gives us \( I(y) \geq I_1(y) \).

**Remark 6.2.** Since if \( \int_0^T L_0(u(s)) ds \leq n \) the solution of (4.8) with \( X(0) = x \) satisfies \( \|X(T)\| \leq C_n \) for some absolute constant \( C_n \) it is clear that \( I(y) = +\infty \) if \( y \in V \setminus H \).

In some cases \( \lim \inf_{z \to y} \) can be removed from (6.1). We present below one such case.

**Proposition 6.3.** Suppose that, in addition to the assumptions of Theorem 6.1, there exists \( p > 1 \) such that

\[ \|z\|^p \leq C(1 + L_0(z)) \quad \text{for all} \quad z \in H, \quad (6.2) \]

and that for every \( x \in H \) and \( K > 0 \) there exists a modulus \( \omega_{x,K} \) such that if \( X \) satisfies (4.8), \( X(0) = x, \int_0^T \|u(s)\|^p ds \leq K \), then

\[ \|X(s_1) - X(s_2)\|_V \leq \omega_{x,K}(|s_1 - s_2|) \quad \text{for all} \quad s_1, s_2 \in [0,T]. \quad (6.3) \]

Then

\[ I(y) = \inf_{u(\cdot) \in M_0} \left\{ \int_0^T L_0(u(s)) ds : X \text{ satisfies (4.8), } X(0) = x, X(T) = y \right\}. \quad (6.4) \]

**Proof.** To show (6.4), suppose that \( X_m \) satisfies (4.8) with \( u_m(\cdot) \in M_0, X_m(0) = x, X_m(T) = z_m \), where \( z_m \to y \) in \( V \), and

\[ \int_0^T L_0(u(s)) ds \to \alpha \in \mathbb{R} \quad \text{as} \quad m \to +\infty. \]

Then by (1.13), (6.2) and (6.3) the family \( \{X_m\} \) is equibounded in \( H \) and equicontinuous in \( V \) and since balls in \( H \) are compact in \( V \), by the Arzela-Ascoli theorem a subsequence, still denoted by \( X_m \), converges uniformly in \( C[0,T];V \) to \( Y : [0,T] \to H \) which also
satisfies (6.3). Moreover we can assume that \( u_m \rightharpoonup u \) in \( L^p(0, T; H) \) for some \( u \). By the definition of mild solution for \( 0 \leq s \leq T \)

\[
X_m(s) = S(s)x + \int_0^s S(s - \tau)(F(X_m(\tau)) + G(X_m(\tau))u_m(\tau))d\tau.
\]

Since the topology of \( V \) on closed balls of \( H \) is equivalent to the weak topology in \( H \), we have that

\[
sup_{0 \leq \tau \leq T} \|X_m(\tau) - Y(\tau)\| \xrightarrow{m \to +\infty} 0.
\]

Therefore (6.5), combined with \( u_m \rightharpoonup u \) in \( L^p(0, T; H) \), yields that for every \( p \in H \)

\[
\langle Y(s), p \rangle = \lim_{m \to +\infty} \langle X_m(s), p \rangle = \langle S(s)x + \int_0^s S(s - \tau)(F(Y(\tau)) + G(Y(\tau))u(\tau))d\tau, p \rangle.
\]

This means that \( Y \) is the mild solution of (4.8) with \( u(\cdot) \in M_0, Y(0) = x, Y(T) = y \).

Since \( u_m \rightharpoonup u \) in \( L^p(0, T; H) \)

\[
\sum_{i=1}^k \lambda_i^k u_{m_i}^k \to u \quad \text{in} \quad L^p(0, T; H)
\]

where for every \( k \geq 1, \sum_{i=1}^k \lambda_i^k = 1 \) and \( \inf_{1 \leq i \leq k} m_i^k \geq k \). Moreover, upon taking another subsequence, we can assume that we have pointwise convergence in (6.6) a.e. on \([0, T] \). It now follows from Fatou’s lemma that

\[
\int_0^T L_0(u(s))ds = \int_0^T \lim_{k \to +\infty} L_0(\sum_{i=1}^k \lambda_i^k u_{m_i}^k(s))ds
\]

\[
\leq \liminf_{k \to +\infty} \int_0^T L_0(\sum_{i=1}^k \lambda_i^k u_{m_i}^k(s))ds \leq \liminf_{k \to +\infty} \sum_{i=1}^k \lambda_i^k \int_0^T L_0(u_{m_i}^k(s))ds = \alpha
\]

which completes the proof.

**Remark 6.4.** Condition (6.3) is satisfied for instance if \( S(\cdot) \) is a compact semigroup. We also remark that in the above proof, (2.2) cannot be replaced by (2.8) even if (2.7) is satisfied.

### 7 Examples of noise processes

We will consider two specific cases of small perturbations: compound Poisson processes and subordinated Wiener processes. We will try to calculate the functions

\[
H_0(p) = \int_H \left[ e^{\langle p, z \rangle} - 1 - \langle p, z \rangle \right] \nu(dz),
\]

(7.1)

\[
L_0(z) = \sup_{y \in H} \{ \langle z, y \rangle - H_0(y) \}.
\]

(7.2)

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7.1 Compound Poisson noise

Let \( L \) be a compound Poisson process with the Gaussian jump measure \( \nu = N(0, Q) \) with the trace class covariance operator \( Q \geq 0, \text{Tr} \ Q < +\infty \). It is easy to see, compare also Proposition 4.18 in [21], that the operator \( Q \) is identical with the covariance of \( L \). It is well known, see e.g. [9], that in this specific case for each \( k > 0 \)

\[
\int_H \|z\|^2 e^{k\|z\|} \nu(dz) < +\infty. \tag{7.3}
\]

To calculate the function \( H_0(\cdot) \) remark that for a random variable \( \xi \) such that \( L(\xi) = \nu \),

\[
\int \langle p, z \rangle^2 \nu(dz) = \mathbb{E} \langle p, \xi \rangle^2 = \langle Qp, p \rangle = \|Q^{1/2}p\|^2.
\]

Moreover, for a real valued random variable \( \eta \) such that \( L(\eta) = N(0, 1) \),

\[
\mathbb{E} e^{\lambda \eta} = e^{1/2 \lambda^2}, \quad \lambda \in \mathbb{R}.
\]

Consequently,

\[
\int_H e^{\langle p, z \rangle} \nu(dz) = \mathbb{E} e^{\|Q^{1/2}p\|} = e^{1/2 \langle Qp, p \rangle}. \tag{7.4}
\]

Thus, in the present situation

\[
H_0(p) = e^{1/2 \langle Qp, p \rangle} - 1 = e^{1/2 \|Q^{1/2}p\|^2} - 1 \tag{7.5}
\]

We denote by \( Q^{-1/2} \) the pseudo inverse of \( Q^{1/2} \). Since \( Q^{1/2} \) is self-adjoint we have an orthogonal decomposition \( H = \text{Im} Q^{1/2} \times \text{Ker} Q^{1/2} \) and we notice that \( Q^{-1/2}z \) is the unique element \( p_0 \in \text{Im} Q^{1/2} \) such that \( Q^{-1/2}p_0 = z \). For \( x \in H \) will write \( x = x_0 + x^\perp \) to indicate the orthogonal decomposition of \( x \). We have the following general result.

**Proposition 7.1.** Assume that

\[
H_0(p) = h(\|Q^{1/2}p\|), \quad p \in H,
\]

where \( Q \) is a trace class nonnegative operator and \( h \) is a convex, even function with the Legendre transform \( l \). Then the Legendre transform \( L_0 \) of \( H_0 \) is of the form:

\[
L_0(z) = \begin{cases} 
\|Q^{-1/2}z\|, & \text{if } z \in \text{Im} Q^{1/2}, \\
+\infty, & \text{if } z \notin \text{Im} Q^{1/2}.
\end{cases}
\]

**Proof.** Let \( z = z_0 + z^\perp \). If \( z^\perp \neq 0 \) then

\[
L_0(z) = \sup_p \left[ \langle z, p \rangle - h(\|Q^{1/2}p\|) \right] \geq \sup_{p^\perp \in \text{Ker} Q^{1/2}} \langle z^\perp, p^\perp \rangle - h(0) = +\infty.
\]
If \( z = Q^{1/2} \bar{p}, \bar{p} \in \overline{\text{Im} Q^{1/2}} = H_1, \) then

\[
L_0(z) = \sup_p \left( \langle z, p \rangle - h(\|Q^{1/2} p\|) \right) = \sup_p \left[ \langle \bar{p}, Q^{1/2} p \rangle - h(\|Q^{1/2} p\|) \right]
\]

\[
= \sup_{v \in H_1} \left[ \langle \bar{p}, v \rangle - h(\|v\|) \right] = \sup_{t \geq 0} \left[ \sup_{\|v\|=t} \left( \langle \bar{p}, v \rangle - h(t) \right) \right]
\]

\[
= \sup_{t \geq 0} \left[ \sup_{\|v\|=t} \left( \langle \bar{p}, v \rangle \right) - h(t) \right] = \sup_{t \geq 0} (\|\bar{p}\| - h(t)) = l(\|\bar{p}\|) = l(\|Q^{-1/2} z\|),
\]
as required.

Let now \( z \in \overline{\text{Im} Q^{1/2}} \setminus \text{Im} Q^{1/2}. \) When restricted to \( \overline{\text{Im} Q^{1/2}}, Q^{1/2} \) is a positive, self-adjoint, compact operator and \( Q^{-1/2} \) exists in the usual sense. Let \( \{e_1, e_2, \ldots\} \) be an orthonormal basis of \( \overline{\text{Im} Q^{1/2}} \) composed of eigenvectors of \( Q^{1/2}. \) Then \( z_n = \sum_{i=1}^n \langle z, e_i \rangle e_i \in \text{Im} Q^{1/2}. \) Let \( H_n \) be the linear subspace of \( H \) spanned by the vectors \( \{e_1, \ldots, e_n\} \) and \( p = p_n + p_n^\perp, \ z = z_n + z_n^\perp, \) be the orthogonal decompositions of \( p \) and \( z \) with respect to \( H_n \) and \( H_n^\perp. \) Thus

\[
L_0(z) = \sup_{p_n + p_n^\perp} \left[ \langle z, p_n + p_n^\perp \rangle - h(\|Q^{1/2} (p_n + p_n^\perp)\|) \right] \geq \sup_{p_n} \left[ \langle z, p_n \rangle - h(\|Q^{1/2} p_n\|) \right]
\]

\[
\geq \sup_{p_n} \left[ \langle z_n + z_n^\perp, p_n \rangle - h(\|Q^{1/2} p_n\|) \right] \geq \sup_{p_n} \left[ \langle z_n, p_n \rangle - h(\|Q^{1/2} p_n\|) \right]
\]

\[
= \sup_{p_n} \left[ \langle z_n, p \rangle - h(\|Q^{1/2} p\|) \right] = l(\|Q^{-1/2} z_n\|).
\]

But the sequence \( (\|Q^{-1/2} z_n\|) \) tends to \( +\infty \) and since \( l(+\infty) = +\infty, \ L(z) = +\infty, \) as required.

As a corollary we get the following proposition

**Proposition 7.2.** Assume that \( H_0 \) is given by (7.3). Let \( f : \mathbb{R}_+^1 \to \mathbb{R}_+^1 \) be the inverse function to \( g(\sigma) = \sigma e^{2\sigma^2/2}, \ \sigma \geq 0. \) Then

\[
L_0(z) = \begin{cases} 
\left( \left[ f(\|Q^{-1/2} z\|) \right]^2 - 1 \right) e^{\frac{1}{2} f^2(\|Q^{-1/2} z\|)} + 1, & \text{if } z \in \text{Im} Q^{1/2}, \\
+\infty, & \text{if } z \notin \text{Im} Q^{1/2}.
\end{cases}
\]

**Remark 7.3.** It is immediate that \( f \) is a concave function and for every \( 0 < a < 2 \) we have

\[
\sqrt{a \ln x} \leq f(x) \leq \sqrt{2 \ln x}, \text{ for large } x.
\]

### 7.2 Subordinated Wiener process

Take \( L(t) = W(Z(t)), t \geq 0, \) where \( W \) is a Wiener process on \( H, \) say \( L(W(1)) = N(0, Q_W) \) and \( Z \) is a subordinator with the jump measure \( \rho \) on \( [0, +\infty). \) Thus \( Z \) is an increasing
process starting from 0 and such that
\[
\mathbb{E} e^{-\lambda Z(t)} = e^{-t \psi(\lambda)}, \quad \lambda \geq 0,
\]
\[
\psi(\lambda) = \gamma \lambda + \int_0^{+\infty} (1 - e^{-\lambda \sigma}) \rho(d\sigma), \quad \lambda \geq 0,
\]
(7.6)
where \( \gamma \geq 0 \) and \( \int_0^1 \sigma \rho(d\sigma) < +\infty, \int_1^{+\infty} \rho(d\sigma) < +\infty \). If \( \gamma = 1, \rho \equiv 0 \), then \( Z(t) = t, t \geq 0 \) and we have \( L \) identical to the Wiener process \( W \).

We will assume that \( \gamma = 0 \), find the function \( H_0 \) and check under what assumptions on \( \rho \) the crucial condition (7.3) is satisfied.

It is well known, see e.g. [25], [21], that for the Lévy process \( L \), the measure \( \nu \) is of the form
\[
\nu = \int_0^{+\infty} N(0, tQ_W) \rho(dt).
\]
(7.7)
By direct calculations we get that the covariance operator \( Q \) of \( L \) is equal to,
\[
Q = \left[ \int_0^{+\infty} t \rho(dt) \right] Q_W = \left[ \mathbb{E}Z(1) \right] Q_W.
\]
(7.8)
To simplify notation we will assume that
\[
\mathbb{E}Z(1) = 1, \text{ and then, } Q_W = Q.
\]
(7.9)
Therefore,
\[
H_0(p) = \int_H (e^{\langle \rho, z \rangle} - 1) \nu(dz) = \int_0^{+\infty} \left( \int_H (e^{\langle \rho, z \rangle} - 1) N(0, tQ_W)(dz) \right) \rho(dt)
\]
\[
= \int_0^{+\infty} \left( e^{\frac{1}{2} Q_{\rho, p}} - 1 \right) \rho(dt).
\]
Thus
\[
H_0(p) = h(\|Q^{1/2} p\|), \text{ where } h(u) = \int_0^{+\infty} (e^{\frac{1}{2} t u^2} - 1) \rho(dt), \quad u \geq 0,
\]
(7.10)
and Proposition 7.1 applies. An explicit formula for \( L_0 \) can be easily derived.

Note that
\[
I = \int_H \|z\|^2 e^{\kappa \|z\|^2} \nu(dz) = \int_0^{+\infty} \rho(dt) \left[ \int_H \|z\|^2 e^{\kappa \|z\|^2} N(0, tQ_W)(dz) \right]
\]
\[
= \int_0^{+\infty} \rho(dt) \mathbb{E} \left[ \|W(t)\|^2 e^{\kappa \|W(t)\|^2} \right].
\]
But \( L(W(t)) = L(\sqrt{t}W(t)) \). Therefore
\[
I = \int_0^{+\infty} t \rho(dt) \left[ \mathbb{E} \|W(1)\|^2 e^{\kappa \sqrt{t} \|W(1)\|^2} \right].
\]
We will need the following lemma.
Lemma 7.4. There exists $a > 0$ such that for all $s \geq 0$,
\[ \mathbb{E} e^{s\|W(1)\|} \leq e^{as^2}. \]

Proof. By [17], page 55, there exists $\delta > 0$ such that
\[ P(\|W(1)\| > u) \leq e^{-\delta u^2}, \quad u > 0. \]

Therefore
\[ \mathbb{E}(s\|W(1)\|) = \int_0^{+\infty} P(s\|W(1)\| \geq u) \, du = 1 + \int_1^{+\infty} P(\|W(1)\| > \frac{\ln u}{s}) \, du. \]

Note that
\[ \int_1^{+\infty} P(\|W(1)\| > \frac{\ln u}{s}) \, du \leq \int_1^{+\infty} e^{-\delta(\ln u/s)^2} \, du. \]

Substituting $v = \frac{\ln u}{s}$, $du = us \, dv = se^{sv} \, dv$,
\[ \int_1^{+\infty} e^{-\delta(\ln u/s)^2} \, du = s \int_0^{+\infty} e^{-\delta v^2} e^{sv} \, dv = s \left( \int_0^{+\infty} e^{-\delta(v-s/(2\delta))^2} \, dv \right) e^{s^2/(4\delta)} \leq s \left( \int_{-\infty}^{+\infty} e^{-\delta v^2} \, dv \right) e^{s^2/(4\delta)}. \]

The required result now follows. \hfill \blacksquare

Proposition 7.5. If
\[ \int_0^{+\infty} t \rho(dt) = 1 \quad \text{and} \quad \int_1^{+\infty} e^{\lambda t} \rho(dt) < +\infty, \quad \lambda \geq 0, \]
then the measure $\nu$ given by (7.7) satisfies (7.3) and $H_0$ is given by (7.10).

Proof. It is enough to remark that,
\[ \mathbb{E}[\|W(1)\|^2 e^{\kappa \sqrt{t} \|W(1)\|}] \leq \left( \mathbb{E}[\|W(1)\|^2] \right)^{1/2} \left( \mathbb{E}[e^{2\kappa \sqrt{t} \|W(1)\|}] \right)^{1/2} \leq ce^{\frac{2}{\kappa^2}t}. \]

Example 7.6. The assumptions of the above proposition are satisfied if, for instance,
\[ \rho(dt) = \frac{1}{t^{1+\alpha}} e^{-t^2} dt \quad \text{for} \ \alpha < 1. \]

In some cases asymptotic behavior of the function $\psi$ can be determined.
Example 7.7.

\[ \rho(dt) = 1_{[0,1]}(t) \frac{1}{t^{1+\alpha}} dt, \quad \alpha < 1, \]

\[ -\psi(-\lambda) = \int_0^1 (e^{\lambda\sigma} - 1) \frac{1}{\sigma^{1+\alpha}} d\sigma. \]

After substitution, \( \lambda \sigma = u \), for \( \lambda > 1 \),

\[ \int_0^1 (e^{\lambda\sigma} - 1) \frac{1}{\sigma^{1+\alpha}} d\sigma = \frac{1}{\lambda} \int_0^\lambda (e^{u} - 1) \frac{1}{(u^{1+\alpha})} du = \lambda^\alpha \int_0^\lambda (e^{u} - 1) \frac{1}{u^{1+\alpha}} du \]

\[ \leq \lambda^\alpha \left[ \int_0^1 e^{u} - 1 \frac{1}{u} \cdot \frac{1}{u^{\alpha}} du + \int_1^\lambda e^{u} du \right]. \]

Thus, for large \( \lambda \),

\[ \int_0^1 (e^{\lambda\sigma} - 1) \frac{1}{\sigma^{1+\alpha}} d\sigma \sim c \lambda^\alpha e^{\lambda} \]

Remark 7.8. In the considered examples, the Legendre transforms \( L_0 \) of \( H_0 \) were of the form \( l(||Q^{-\frac{1}{2}}z||) \), \( z \in H \). Thus the control system, which defines the rate function, can be written in a more convenient way,

\[ X'(s) = -AX(s) + F(X(s)) + G(X(s))Q^{1/2}u(s), \quad X(t) = x, \quad (7.11) \]

and to find the rate function one has to look for the infimum of the cost functional

\[ J(x; u(\cdot)) = \int_0^T l(u(s)) ds + g(X(T)) \]

over all controls \( u(\cdot) \in M_0 \).

8 Stochastic PDE of hyperbolic type

We present an example of a class of stochastic PDE which can be handled by the developed theory. To begin consider a nonlinear stochastic wave equation which can be formally written as

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, \xi) = \Delta u(t, \xi) + f(u(t, \xi)) + \frac{\partial}{\partial t} \tilde{L}_n(t, \xi), & t > 0, \ \xi \in \mathcal{O}, \\
u(t, \xi) = 0, & t > 0, \ \xi \in \partial \mathcal{O}, \\
u(0, \xi) = u_0(\xi), & \xi \in \mathcal{O}, \\
\frac{\partial u}{\partial t}(0, \xi) = v_0(\xi), & \xi \in \mathcal{O}.
\end{cases}
\]

with \( \tilde{L}_n \), \( L^2(\mathcal{O}) \) valued Lévy process (properly normalized), \( \mathcal{O} \) a bounded regular domain in \( \mathbb{R}^d \), \( f : \mathbb{R} \to \mathbb{R} \) is a Lipschitz function and \( u_0 \in H^1_0(\mathcal{O}), v_0 \in L^2(\mathcal{O}) \).
Setting
\[ X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad t \geq 0, \]
we can rewrite (8.1) in an abstract way:
\[ dX(t) = \left( \begin{array}{cc} 0 & I \\ -A & 0 \end{array} \right) X(t) + F(X(t)) \, dt + dL_n(t), \quad (8.2) \]
where
\[ F\left( \begin{array}{c} u \\ v \end{array} \right) = \begin{pmatrix} 0 \\ F_1(u) \end{pmatrix}, \quad L_n(t) = \begin{pmatrix} 0 \\ \tilde{L}_n(t) \end{pmatrix} \quad (8.3) \]
and \( A = -\Delta \) in \( H = L^2(\mathcal{O}) \) with \( D(A) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}) \). Moreover the same setup applies to other equations of hyperbolic type.

Therefore let us assume that \( A \) in (8.2) is a strictly positive, self-adjoint operator in a Hilbert space \( H \) with a bounded inverse. It is then well known that the operator
\[ A = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad D(A) = \begin{pmatrix} D(A^{1/2}) \\ \times \end{pmatrix} \]
is maximal monotone in the Hilbert space \( H = \begin{pmatrix} D(A^{1/2}) \\ \times \end{pmatrix} \), equipped with the following “energy” type inner product
\[ \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle_H = \left\langle A^{1/2}u, A^{1/2}\bar{u} \right\rangle _H + \left\langle v, \bar{v} \right\rangle _H, \quad \left( \begin{array}{c} u \\ v \end{array} \right), \left( \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right) \in H. \]
Moreover, \( A^* = -A \).

It is easy to check that the operator
\[ B = \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \]
is bounded, positive, self-adjoint on \( H \), and such that \( A^*B \) is bounded. Moreover (2.1) holds with constant \( c_0 = 1 \). In fact
\[ \left\langle (A^* + I)B \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle _H = \left\langle B \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle _H = \|A^{1/4}u\|^2 + \|A^{-1/4}v\|^2. \]
In particular we see that
\[ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{-1} = \left( \|A^{1/4}u\|^2 + \|A^{-1/4}v\|^2 \right)^{1/2}. \]
Thus \( F = \begin{pmatrix} 0 & F_1 \end{pmatrix} \) is Lipschitz from \( \mathcal{H}_{-1} \) into \( \mathcal{H} \) (condition (2.2)) if and only if

\[
\| A^{-1/4}(F_1(u) - F_1(\bar{u}) \|_H \leq c\| A^{1/2}(u - \bar{u}) \|, \quad u, \bar{u} \in D(A^{1/2}).
\] (8.4)

It is easy to see that if \( F_1(u)(\xi) = f(u(\xi)), \quad \xi \in \mathcal{O}, \) and \( f \) is a Lipschitz function, then (8.4) is satisfied.

9 Appendix: Proof of Proposition 3.3

Let us recall that the spaces \( X, L \) were introduced in Section 4. Define, for each \( \psi \in L \), processes

\[
K(\psi)(t) = \int_0^t S(t-s)\psi(s) dL(s), \quad t \in [0, T],
\]

\[
K_\lambda(\psi)(t) = \int_0^t S_\lambda(t-s)\psi(s) dL(s), \quad \lambda > 0, \quad t \in [0, T].
\]

We can treat \( K \) and \( K_\lambda \) as linear transformations from the space \( L \) into \( X \). We prove this now and establish that there exists a constant \( C_1 > 0 \) such that

\[
\|K_\lambda\| \leq C_1 \quad \text{for} \quad \lambda > 1.
\] (9.1)

In the proof we omit the subscript \( \lambda \). Let \( \widehat{H} \), and the unitary semigroup \( \widehat{S} \), be the extensions, respectively of \( H \) and of the semigroup \( S \), given by the delation theorem, see e.g. [21 Theorem 9.24]. Thus \( H \hookrightarrow \widehat{H} \) is an isometry and the semigroup \( S \) is the restriction of \( P \widehat{S} \) to \( H \), where \( P \) is the orthogonal projection of \( \widehat{H} \) onto \( H \). Therefore we have:

\[
\int_0^t S(t-s)\psi(s) dL(s) = \int_0^t P\widehat{S}(t-s)\psi(s) dL(s) = P\widehat{S}(t) \int_0^t \widehat{S}(-s)\psi(s) dL(s), \quad t \in [0, T].
\]

Moreover the process

\[
\widehat{Y}(t) = \int_0^t \widehat{S}(-s)\psi(s) dL(s), \quad t \geq 0,
\]

is a \( \widehat{H} \) martingale and therefore has càdlàg modification. This implies that the stochastic convolution has \( H \)-valued, càdlàg modifications and

\[
\left\| \int_0^t S(t-s)\psi(s) dL(s) \right\| \leq \|\widehat{Y}(t)\|_{\widehat{H}}, \quad t \in [0, T].
\]

However, \( \|\widehat{Y}(t)\|_{\widehat{H}}, t \in [0, T], \) is a submartingale and by the classical Doob inequality for all \( p > 1 \)

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \|\widehat{Y}(t)\|_{\widehat{H}}^p \right) \leq \left( \frac{p}{p-1} \right)^p \mathbb{E}\|\widehat{Y}(T)\|_{\widehat{H}}^p.
\]
In particular

\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)\psi(s)\,dL(s) \right\|_H^2 \right) \leq 4\mathbb{E}\left( \int_0^T \left\| \hat{S}(-s)\psi(s)\,d\hat{L}\right\|_{\hat{H}}^2 \right) \]

\[ \leq 4\mathbb{E}\int_0^T \left\| \hat{S}(-s)\psi(s)Q^{1/2}\right\|_{L_{\text{HS}}(H,\hat{H})}^2\,ds \leq 4\mathbb{E}\int_0^T \|\psi(s)Q^{1/2}\|_{\text{HS}}^2\,ds. \]

Thus the existence of the constant $C_1$ follows, and by the Banach-Steinhaus theorem it is enough to establish (3.9) for a dense set of $\psi$.

**Lemma 9.1.** For each $k = 1, 2, \ldots$ the set

\[ \mathcal{L}_k = \left\{ \psi \in \mathcal{L} : \mathbb{E}\int_0^T \| A^k\psi(u)Q^{1/2}\|_{\text{HS}}^2\,du < +\infty \right\} \]

is dense in $\mathcal{L}$.

**Proof.** Let $\psi \in \mathcal{L}$. Since for $\mu > 0$ the operator $\mu AR_\mu$ is bounded we have

\[ \mathbb{E}\int_0^T \| A^k(\mu R_\mu)^k\psi(u)Q^{1/2}\|_{\text{HS}}^2\,du = \mathbb{E}\int_0^T \| (\mu AR_\mu)^k\psi(u)Q^{1/2}\|_{\text{HS}}^2\,du < +\infty, \]

and thus $(\mu R_\mu)^k\psi \in \mathcal{L}_k$. Moreover it follows from (2.9) that

\[ \| ( (\mu R_\mu)^k - I )\psi(u)Q^{1/2}\|_{\text{HS}}^2 \leq C\|\psi(u)Q^{1/2}\|_{\text{HS}}^2, \]

and $\lim_{\mu \to +\infty}(\mu R_\mu)^kx = x$ for every $x \in H$. Therefore the dominated convergence theorem yields

\[ \lim_{\mu \to +\infty} \mathbb{E}\int_0^T \| ( (\mu R_\mu)^k - I )\psi(u)Q^{1/2}\|_{\text{HS}}^2\,du = 0. \]

**Lemma 9.2.** Assume that $M(t)$, $t \geq 0$, is a $D(A)$-valued process with locally bounded trajectories, $H$-square integrable martingale, and $M(0) = 0$. Then

\[ \int_0^t S(t-s)\,dM(s) = M(t) - \int_0^t S(t-s)AM(s)\,ds. \quad (9.2) \]

**Proof.** Let $e \in D((A^*)^2)$ and

\[ \varphi(s,x) = \langle S(t-s)x,e \rangle = \langle x,S^*(t-s)e \rangle. \]

Then $\varphi \in C^2((-\infty,t) \times H)$ and has uniformly continuous derivatives. In fact it can be extended to a function in $C^2(\mathbb{R} \times H)$ in an obvious way. Therefore, applying Ito’s formula
for Hilbert space valued semimartingales (see [19, Theorem 27.2] or [21, Theorem D2]) we obtain
\[
\langle M(t), e \rangle = \int_0^t \langle S(t-s)AM(s), e \rangle ds + \int_0^t \langle S(t-s)dM(s), e \rangle ds
\]
which proves the claim since \(D((A^*)^2)\) is dense in \(H\).

Applying Lemma 9.2 to the martingale \(M(t) = \int_0^t \psi(u) dL(u), t \in [0, T]\) we arrive at the following lemma.

**Lemma 9.3.** If \(\mathbb{E} \int_0^T \|A\psi(u)Q^{1/2}\|_{HS}^2 du < +\infty\) then for all \(t \in [0, T], \lambda > 0,
\[
\begin{align*}
\int_0^t S(t-s)\psi(s) dL(s) &= \int_0^t \psi(s) dL(s) - \int_0^t S(t-s) \left( \int_0^s A\psi(u) dL(u) \right) ds, \\
\int_0^t S_\lambda(t-s)\psi(s) dL(s) &= \int_0^t \psi(s) dL(s) - \int_0^t S_\lambda(t-s) \left( \int_0^s A_\lambda\psi(u) dL(u) \right) ds.
\end{align*}
\]

We can now continue the proof of the theorem. We will show that (3.9) holds for every \(\psi \in \mathcal{L}_2\). Note that
\[
\begin{align*}
\mathcal{K}\psi(t) - \mathcal{K}_\lambda\psi(t) &= \int_0^t S(t-s) \left[ \int_0^s -A\psi(u) dL(u) + \int_0^s A_\lambda\psi(u) dL(u) \right] ds \\
&+ \int_0^t \left[ S(t-s) - S_\lambda(t-s) \right] \left( \int_0^s -A_\lambda\psi(u) dL(u) \right) ds = I_\lambda^1\psi(t) + I_\lambda^2\psi(t).
\end{align*}
\]
Now
\[
I_\lambda^1\psi(t) = \int_0^t S(t-s)(A_\lambda - A) \int_0^s \psi(u) dL(u) ds,
\]
and
\[
\sup_{0 \leq t \leq T} \left\| I_\lambda^1\psi(t) \right\| \leq \int_0^T \left\| (A - A_\lambda) \int_0^s \psi(u) dL(u) \right\| ds.
\]
But \(\|(A - A_\lambda)x\| = \|R_\lambda A^2x\| \leq \frac{1}{\lambda}\|A^2x\|, x \in D(A^2)\). Therefore, since
\[
\mathbb{E} \int_0^T \|A^2\psi(u)Q^{1/2}\|_{HS}^2 du < +\infty,
\]
we have, by isometric identity,
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} \left| I_\lambda^1\psi(t) \right|^2 &\leq \mathbb{E} \left( \int_0^T \left\| (A - A_\lambda) \int_0^s \psi(u) dL(u) \right\| ds \right)^2 \\
&\leq T \int_0^T \mathbb{E} \int_0^s \left\| (A - A_\lambda)\psi(u)Q^{1/2} \right\|_{HS}^2 du ds \\
&\leq \frac{1}{\lambda^2} T \int_0^T \mathbb{E} \int_0^s \left\| A^2\psi(u)Q^{1/2} \right\|_{HS}^2 du ds \\
&\leq \frac{1}{\lambda^2} T^2 \int_0^T \mathbb{E} \left\| A^2\psi(u)Q^{1/2} \right\|_{HS}^2 du.
\end{align*}
\]
Therefore, if (9.3) holds,
\[
\lim_{\lambda \to +\infty} \mathbb{E} \|I_\lambda^2 \psi(t)\|^2 = 0.
\]
Since for every \(x \in D(A), \lambda > 0,\)
\[
\|S_\lambda(t)x - S(t)x\| \leq t\|A_\lambda x - Ax\|
\]
(see for instance [20, page 10), we have

Thus
\[
\sup_{0 \leq t \leq T} \|I_\lambda^2 \psi(t)\|^2 \leq \sup_{0 \leq t \leq T} \left( \int_0^t \left\| \left( S(t-s) - S_\lambda(t-s) \right) A_\lambda \int_0^s \psi(u) dL(u) \right\| ds \right)^2
\]
\[
\leq \sup_{0 \leq t \leq T} \left[ \int_0^t \| (A - A_\lambda) A_\lambda \int_0^s \psi(u) dL(u) \| ds \right]^2
\]
\[
\leq T^2 \sup_{0 \leq t \leq T} \left[ \int_0^t \| (A - A_\lambda) A_\lambda \int_0^s \psi(u) dL(u) \| ds \right]^2.
\]
Moreover,
\[
(A - A_\lambda) A_\lambda = (A - \lambda R_\lambda A) \lambda R_\lambda A = \lambda R_\lambda (I - \lambda R_\lambda) A^2.
\]
Therefore
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|I_\lambda^2 \psi(t)\|^2 \leq T^2 \mathbb{E} \int_0^T \left\| (I - \lambda R_\lambda) A^2 \int_0^s \psi(u) dL(u) \right\|^2 ds
\]
\[
\leq T^2 \mathbb{E} \int_0^T \int_0^s \| (I - \lambda R_\lambda) A^2 \psi(u) Q^{1/2} \|_{HS}^2 du ds
\]
\[
\leq T^3 \mathbb{E} \int_0^T \| (I - \lambda R_\lambda) A^2 \psi(u) Q^{1/2} \|_{HS}^2 du.
\]
Thus, if (9.3) holds, we can conclude by the dominated convergence theorem that
\[
\lim_{\lambda \to +\infty} \mathbb{E} \sup_{0 \leq t \leq T} \|I_\lambda^2 \psi(t)\|^2 = 0.
\]
This finishes the proof of the proposition.

\[\square\]

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