This paper is a follow-up to the author’s article [1], where the functors $P_{\tau}$ and $\hat{P}$ of spaces of probability $\tau$-smooth and Radon measures were defined. We continue with the same numbering of chapters and propositions as in [1].

In this part, for spaces of Radon probability measures, the barycenter map is studied, this map will be used to show that the functor $\hat{P}$ is monadic in the category of metrizable spaces.

Another question addressed in this article concerns the metrizability of the functors $P_{\tau}$ and $\hat{P}$. In this direction, we shall show that the functors $P_{\tau}$ and $\hat{P}$ can be lifted to the category $\mathcal{B}\mathcal{M}etr$ of bounded metric spaces and also to the category $Unif$ of uniform spaces, and investigate properties of those liftings. We should note that similar problems concerning the metrizability of the functor $P_{\beta}$ were explored in [2]–[6].

### 3 Barycenters and monadicity of the functor $\hat{P}$

Let $E$ be a locally convex vector space. By $E^*$ we shall denote the vector space of all continuous linear functionals on $E$, equipped with the topology of uniform convergence on weakly bounded subsets of $E$. (Let us recall that a subset $A \subset E$ is weakly bounded if each functional $f \in E^*$ is bounded on $A$). A subset $K \subset E^*$ is be called equicontinuous if there exists a neighborhood $U \subset E$ of zero such that $\|f(x)\| < 1$ for any $x \in U$ and $f \in K$. By $E^{**}$ we denote the space of continuous linear functionals on $E^*$ endowed with the natural topology, i.e. the topology of uniform convergence on equicontinuous subsets of $E^*$. It is well-known [7, c.182] that the canonical map $E \to E^{**}$ is an embedding with respect to the natural topology on $E^{**}$, so from now on, we will treat the space $E$ as a subspace of $E^{**}$.

Let $X$ be a weakly bounded subset of a locally convex space $E$. Then for any measure $\mu \in \hat{P}(X)$, the formula $b_X(\mu)(f) = \mu(f\chi_X)$, $f \in E^*$, defines a linear functional $b_X(\mu)$ on $E^*$, which is called the barycenter of the measure $\mu$. The functional $b_X(\mu)$ is continuous, since the set $X$ is weakly bounded and $E^*$ carries the topology of uniform convergence on weakly bounded sets. Thus, we have defined the barycenter map $b_X : \hat{P}(X) \to E^{**}$.

A map $f : C \to C'$ between convex sets is called affine if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \text{ for any } x, y \in C \text{ and } t \in [0, 1].$$

The following lemma is immediate:

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Lemma 3.1. The barycenter map \( b_X : \hat{P}(X) \to E^{**} \) is affine and \( b_X(\delta_x) = x \) for all \( x \in X \) (where \( \delta_x \) denotes the Dirac measure, concentrated at the point \( x \)).

Theorem 3.2. The map \( b_X : \hat{P}(X) \to E^{**} \) is continuous if and only if the set \( X \) is bounded.

Proof. Let us first assume that the map \( b_X : \hat{P}(X) \to E^{**} \) is continuous. Fix \( x \in X \). In order to prove that the set \( X \) is bounded, it is sufficient to show that for any open set \( U \ni 0 \) in \( E^{**} \) there exists \( n \in \mathbb{N} \) such that \( X \subset n \cdot U + x \). Since the map \( b_X : \hat{P}(X) \to E^{**} \) is continuous, the set \( b_X^{-1}(U + x) \) is open in \( \hat{P}(X) \).

Therefore, there exists a *-weakly open set \( W \) in the space \( C_b^*(X) \ni \hat{P}(X) \) such that \( W \cap \hat{P}(X) = b_X^{-1}(U + x) \). Since the subset \( \hat{P}(X) \subset C_b^*(X) \) is bounded, there exists a number \( n \in \mathbb{N} \) such that \( \frac{1}{n} \hat{P}(X) + \frac{n-1}{n} \delta_x \subset W \). Since the set \( \hat{P}(X) \) is convex, \( \frac{1}{n} \hat{P}(X) + \frac{n-1}{n} \delta_x \subset W \cap \hat{P}(X) = b_X^{-1}(U + x) \). Then for every \( y \in X \), \( \frac{1}{n}y \in U + \frac{1}{n}x \) and \( y \in nU + x \). This implies that \( \frac{1}{n}y \in U + \frac{1}{n}x \) and \( y \in nU + x \). Thus, \( X \subset nU + x \), i.e. the set \( X \) is bounded.

Now we will prove that if \( X \) is a bounded subset of \( E \), then the map \( b_X : \hat{P}(X) \to E^{**} \) is continuous.

Fix a measure \( \mu_0 \in \hat{P}(X) \) and a neighborhood

\[ U(b_X(\mu_0)) = \{ F \in E^{**} : |(F - b_X(\mu_0))(f)| < 1 \text{ for all } f \text{ from some equicontinuous set } K \subset E^* \} \]

of its barycenter. Since the set \( X \) is bounded and \( K \) is an equicontinuous family in \( E^* \), there exists a constant \( C \geq 1 \) such that \( |f(x)| < C \) for all \( x \in X \) and \( f \in K \).

Now let us recall that the measure \( \mu_0 \) on \( X \) is Radon. Therefore, there exists a compact \( K \subset X \) such that \( \mu_0(K) > 1 - \frac{1}{12C} \). Since the family \( K \) is equicontinuous, there exists a neighborhood of zero \( U \subset E \) such that \( |f(x)| < \frac{1}{12} \) for all \( f \in K \) and \( x \in U \). Let \( V = (U + K) \cap X \).

By the Ascoli theorem [8, 3.4.20], the family of maps \( \{ f|K : f \in K \} \subset C(K) \) is precompact in the topology of uniform convergence. Therefore, there exists a finite subset \( K' \subset K \) such that for any functional \( f \in K \) there exists a functional \( g \in I \) with \( |f(x) - g(x)| < \frac{1}{12} \) for any \( x \in K \). Moreover, \( |f(x) - g(x)| < \frac{1}{6} \) for all \( x \in V \).

By Lemma 1, 1.19, the set \( \{ \mu \in \hat{P}(X) : |\mu(V)| > 1 - \frac{1}{12C} \} \) is an open neighborhood of the measure \( \mu_0 \).

Then the set

\[ W = \{ \mu \in \hat{P}(X) : |\mu(V)| > 1 - \frac{1}{12C} \} \]

is an open neighborhood of the measure \( \mu_0 \) in \( \hat{P}(X) \). We claim that \( b_X(W) \subset U(b_X(\mu_0)) = \{ F \in E^{**} : |(F - b_X(\mu_0))(f)| < 1 \} \). Indeed, let \( \mu \in W \). Then for any \( f \in K \) there exists a functional \( g \in I \) such that \( |f(x) - g(x)| < \frac{1}{12} \) for all \( x \in K \). Then

\[ |(b_X(\mu) - b_X(\mu_0))(f)| = |\mu(f)X - \mu_0(f)| \leq \mu(|f(x)|)X + |g(x) - \mu_0(f)| \leq \int_V |f - g|d\mu + \int_K |f - g|d\mu + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{2C}{12C} < 1. \]

Thus, \( b_X(\mu) \in U(b_X(\mu_0)) \), which means that the map \( b_X : \hat{P}(X) \to E^{**} \) is continuous.

Remark 3.3. If \( E \) is a closed subset of \( E^{**} \), then \( b_X(\hat{P}(X)) \subset E \) for any bounded set \( X \subset E \). This implies from Lemma 3.1 and Theorem 3.2.

Definition 3.4. A convex subset \( X \) of a locally convex space \( E \) is called

- barycentric if \( X \) is bounded in \( E \) and \( b_X(\hat{P}(X)) \subset X \);
- \( \infty \)-convex if for any bounded sequence of points \( (x_n)_{n \in \mathbb{N}} \) of \( X \) and any sequence \( (\lambda_n)_{n \in \mathbb{N}} \) of non-negative real numbers with \( \sum_{n=0}^{\infty} \lambda_n = 1 \) the series \( \sum_{n=0}^{\infty} \lambda_n x_n \) converges to some point \( x \in X \);
- compactly convex if each compact subset \( K \subset X \) its closed convex hull \( \text{cl}_X(\text{conv}(K)) \subset X \) is compact.

Proposition 3.5. Each compactly convex subset \( X \) of a locally convex space is \( \infty \)-convex.
Proof. To show that $X$ is $\infty$-convex, fix a bounded sequence $(x_n)_{n \in \omega}$ in $X$ and a sequence $(\lambda_n)_{n \in \omega}$ of non-negative real numbers with $\sum_{n \in \omega} \lambda_n = 1$. Choose any number $m \in \omega$ with $\lambda_m > 0$. Then $\sum_{n \neq m} \lambda_n = 1 - \lambda_m < 1$ and we can choose a sequence of positive real numbers $\varepsilon_n < 1$, $n \in \omega$, such that $\lim_{n \to \infty} \varepsilon_n = 0$ and $\sum_{n \neq m} \lambda_n / \varepsilon_n \leq 1$. Let $\lambda_n = \lambda_n / \varepsilon_n$ for $n \neq m$ and $\lambda_m = 1 - \sum_{n \neq m} \lambda_n'$.

For every $n \in \omega$ consider the point $x'_n = (1 - \varepsilon_n)x_m + \varepsilon_nx_n \in X$ and observe that the sequence $(x'_n)_{n \in \omega}$ tends to $x_m = x_n$. Then the set $K = \{x'_n\}_{n \in \omega} \subset X$ is compact and so is its closed convex hull $cl_X(\text{conv}(K))$ in $X$. Now observe that

$$\sum_{n \in \omega} \lambda_n x_n = \lambda_m x_m + \sum_{n \neq m} \lambda'_n \varepsilon_n x_n =$$

$$= \lambda_m x_m - \sum_{n \neq m} \lambda'_n (1 - \varepsilon'_n)x_m + \sum_{n \neq m} \lambda'_n ((1 - \varepsilon_n)x_m + \varepsilon_n x_n) =$$

$$= (\lambda_m - \sum_{n \neq m} \lambda'_n + \sum_{n \neq m} \lambda'_n \varepsilon'_n)x_m + \sum_{n \neq m} \lambda'_n x'_n =$$

$$= (\lambda_m - \sum_{n \neq m} \lambda'_n + \sum_{n \neq m} \lambda_n)x_m + \sum_{n \neq m} \lambda'_n x'_n =$$

$$= \sum_{n \in \omega} \lambda_n - \sum_{n \neq m} \lambda'_n)x_m + \sum_{n \neq m} \lambda'_n x'_n =$$

$$= (1 - (1 - \lambda'_m))x_m + \sum_{n \neq m} \lambda'_n x'_n = \sum_{n \in \omega} \lambda_n x'_n \in cl_X(\text{conv}(K)) \subset X,$$

witnessing that the set $X$ is $\infty$-convex. \hfill \Box

Theorem 3.6. A bounded convex subset $X$ of a locally convex space $E$ is barycentric if and only if it is compactly convex.

Proof. To prove the “only if” part, assume that the set $X$ is barycentric. Then for each compact subset $K \subset X$ its probability measure space $P(K) \subset \hat{P}(X)$ is compact [9, VII.3.5]. Since the map $b_X : \hat{P}(X) \to X$ is continuous, $b_X(P(K))$ is compact in $X$ and so is the closed convex hull $cl_X(\text{conv}(K)) = b_X(P(K))$ of $K$ in $X$. This means that the set $X$ is compactly convex.

To prove the “if part”, assume that the convex set $X \subset E$ is compactly convex. By Proposition 3.5, the set $X$ is $\infty$-convex. To prove that $X$ is barycentric, fix a Radon measure $\mu \in \hat{P}(X)$ on $X$. If the support $\text{supp}(\mu) = \{x \in X : \text{every neighborhood } U \ni x \text{ has a non-zero } \mu\text{-measure}\}$ of the measure $\mu$ is compact, then, by the compact convexity of $X$ its closed convex hull $cl_X(\text{conv}(\text{supp}(\mu)))$ is compact. In this case, $b_X(\mu) \in cl_X(\text{conv}(\text{supp}(\mu))) \subset X$. If the support of the measure $\mu$ is not compact, then there exists a sequence if compacta $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset X$, such that $0 = \mu(K_0) < \mu(K_1) < \cdots < \mu(K_n) < \cdots < 1$ and $\mu(K_n) > 1 - 2^{-n}$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $\varepsilon_n = \mu(K_n \setminus K_{n-1})$ and let $\mu_n = P(K_n) \subset \hat{P}(X)$ be the measure, defined by the formula $\mu_n(A) = \mu(K_n \setminus K_{n-1}) / \varepsilon_n$, where $A$ is a Borel subset of $X$. One can easily observe that $\sum_{n=1}^{\infty} \varepsilon_n = 1$ and $\mu = \sum_{n=1}^{\infty} \varepsilon_n \mu_n$. Since the map $b_X : \hat{P}(X) \to E^{**}$ is affine and continuous, $b_X(\sum_{n=1}^{\infty} \varepsilon_n \mu_n) = \sum_{n=1}^{\infty} \varepsilon_n b_X(\mu_n) \in E^{**}$. By the compact convexity of $X$, $b_X(\mu_n) \in X$, $n \in \mathbb{N}$, and by the $\infty$-convexity of $X$, $b_X(\mu) = \sum_{n=1}^{\infty} \varepsilon_n b_X(\mu_n) \in X$. So, the set $X$ is barycentric. \hfill \Box

Let us recall that a map $f : A \to B$ between topological spaces is called quotient if a set $U \subset B$ is open if and only if the set $f^{-1}(U) \subset A$ is open.

Proposition 3.7. For any barycentric set $X$ an a locally convex space, the barycenter map $b_X : \hat{P}(X) \to X$ is surjective and quotient.

The proof follows the fact that for each point $x \in X$ the Dirac measure $\delta_x$ concentrated in $x$ has barycenter $b_X(\delta_x) = x$.

Now let us prove several results on the preservation of compactly or $\infty$-convex sets by some operations. One can easily prove the following propositions.

Proposition 3.8. If a convex subset $X$ of a locally convex space is compactly ($\infty$-) convex, then each closed convex subset of $X$ also has that property.

Proposition 3.9. Let $X_i$, $i \in I$, be convex subsets of locally convex spaces $E_i$, $i \in I$, respectively. If all $X_i$, $i \in I$, are compactly ($\infty$-) convex, then their product $\prod_{i \in I} X_i \subset \prod_{i \in I} E_i$ also has that property.
Proposition 3.10. Let $X_i$, $i \in I$, be convex subsets of a locally convex space $E$. If all $X_i$, $i \in I$, are compactly ($\infty$-) convex, then their intersection $\bigcap_{i \in I} X_i$ also has this property.

Proposition 3.11. If a convex bounded subset $X$ of a locally convex space is $\infty$-convex and $X$ a countable union of compactly convex Borel subsets, then $X$ is compactly convex.

Proof. By Theorem 3.6, it suffices to prove that $X$ is barycentric. So, let $\mu \in \hat{P}(X)$ be a Radon measure on $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \subset X$ are compactly convex Borel subsets of $X$. Since the measure $\mu$ is Radon, for every $n, m \in \mathbb{N}$ there exists a compact subset $K^m_n \subset X_n \setminus (\bigcup_{i<n} X_i)$ such that $\mu(K^m_n) \geq (1 - 2^{-m})\mu(X_n \setminus (\bigcup_{i<n} X_i))$. For any $n, m \in \mathbb{N}$ let $\varepsilon^m_n = \mu(K^m_n \setminus K^{m-1}_n)$ (we assume that $\mu(K^m_n) = 2\varepsilon^m_n$) and, if $\varepsilon^m_n > 0$, then we can define a measure $\mu^m_n \in \hat{P}(X)$ by the following formula: $\mu^m_n(A) = \mu((K^m_n \setminus K^{m-1}_n) \cap A)/\varepsilon_n^m$, where $A$ is a Borel subset of $X$. One can easily see that $\sum_{n,m=1}^{\infty} \varepsilon^m_n = 1$ and $\sum_{n,m=1}^{\infty} \varepsilon^m_n \mu^m_n = \mu$. Since the support of every measure $\mu^m_n$ is contained in the compact set $K^m_n \subset X_n$, $b_X(\mu^m_n) \in X_n$, $n, m \in \mathbb{N}$ (let us recall that the sets $X_n$, $n \in \mathbb{N}$, are compactly convex). Let us show that $b_X(\mu) \in \hat{X}$. Indeed, since the map $b_X : \hat{P}(X) \to E^{\ast \ast}$ is affine and continuous, $b_X(\mu) = b_X(\sum_{n,m=1}^{\infty} \varepsilon^m_n \mu^m_n) = \sum_{n,m=1}^{\infty} \varepsilon^m_n b_X(\mu^m_n) \in X$ by the $\infty$-convexity of $X$.

Proposition 3.12. For any $\infty$-convex subset $X$ of a locally convex space $E$ and any affine continuous map $T : E \to E'$ to a locally convex space $E'$, the image $T(X)$ is $\infty$-convex.

Krein’s Theorem [7, IV.11.5] implies:

Proposition 3.13. Every complete bounded convex subset of a locally convex set is barycentric.

Corollary 3.14. Any convex compact subset of a locally convex space is barycentric.

Propositions 3.8 and 3.11 imply:

Corollary 3.15. Any open bounded convex subset of a Banach space is barycentric.

Now we will consider some specific examples of barycentric sets. Let us recall that $Q = [-1,1]^{\omega}$ is a Hilbert cube, $s = (-1,1)^{\omega}$ is its pseudo-interior, $\Sigma = \{(x_i)_{i=1}^{\omega} \in Q : \sup_{i \in \mathbb{N}} |x_i| < 1\}$ is its radial-interior and $\sigma = \{(x_i)_{i=1}^{\omega} \in s : x_i \neq 0 \text{ for a finite number of indices } i\}$. All these spaces are assumed to be subsets of the locally convex space $\mathbb{R}^{\omega}$. By Corollary 3.14, Hilbert cube $Q$ is barycentric, the pseudo-interior $s$ is barycentric by Proposition 3.10 and Corollary 3.15. Then, let us note that the radial-interior $\Sigma$ is an image of the unit open ball $\{x \in l_\infty : ||x|| < 1\}$ of the Banach space $l_\infty$ under the “identity” linear operator $T : l_\infty \to \mathbb{R}^{\omega}$. Corollaries 3.12 and 3.15 imply that the set $\Sigma$ is $\infty$-convex. Since $\Sigma$ is a union of convex compacta, Propositions 3.8 and Corollary 3.11 imply that the radial-interior $\sigma$ is a convex barycentric set. Since $\Sigma^\omega = \bigcap_{n=1}^{\infty} (\Sigma^\omega \times Q^{-n})$, Proposition 3.9 implies that the convex set $\Sigma^\omega$ is also barycentric. Further in the text we will see that there exist barycentric convex sets of any Borel complexity. On the other hand, the convex set $\sigma$ is not barycentric since it is not $\infty$-convex. For the same reason, the set $\sigma \times Q$ is also barycentric neither. It is worth noting that the spaces $\sigma \times Q$ and $\Sigma$ are homeomorphic.

In connection with barycentric sets the following problem appears naturally:

Problem 3.16. Is the set $\hat{P}(X)$ barycentric for every Tychonoff space $X$?

We do not have the answer to this question. We will show, however, that the set $\hat{P}(X)$ is barycentric provided $\hat{P}(X) = P_\tau(X)$, or, if $X$ is a metrizable space. But let us first clarify a few points concerning the barycenter map $b_{P_\tau(X)}$. Let $X$ be a Tychonoff space. In this case the barycenter map $b_{P_\tau(X)} : P^2(\beta X) \to P(\beta X)$ is well-defined and coincides with the component $\psi_{P_\tau(X)}$ of the monad multiplication $\mathbb{P} = (P, \delta, \psi)$ (see [10]). Since the space $\hat{P}(X)$ can be naturally identified with a subspace of $P(\beta X)$, and $P^2(\beta X)$ with a subspace of $P^2(\beta X)$, by the definition we have that the barycenter map $b_{P_\tau(X)} : \hat{P}(\beta X) \to (C^*_\tau(X))^{\ast \ast}$ is a restriction of the map $b_{P(\beta X)}$. According to [1, 1.26], $b_{P(\beta X)}(P^2(\beta X)) \subset P_\tau(X)$. If $\hat{P}(X) = P_\tau(X)$, then $b_{P_\tau(X)}(\hat{P}(\beta X)) = b_{P(\beta X)}(\hat{P}(\beta X)) \subset b_{P(\beta X)}(P^2(\beta X)) \subset P_\tau(X) = P(X)$, i.e. the set $\hat{P}(X)$ is barycentric.

Proposition 3.17. For every Tychonoff space $X$, the set $\hat{P}(X)$ is $\infty$-convex.

Proof. Given sequences $\{\mu_i\}_{i=1}^{\infty} \subset \hat{P}(X)$ and $\{t_n\}_{n=1}^{\infty} \subset [0, 1]$, $\sum_{n=1}^{\infty} t_n = 1$, we should prove that $\sum_{n=1}^{\infty} t_n \mu_n \in \hat{P}(X)$. Given any $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} t_n > 1 - \frac{\varepsilon}{2}$. For every $i \leq m$
the Radon property of the measure $\mu_i$ yields a compact subset $K_i \subset X$ with $\mu_i(K_i) > 1 - \frac{\varepsilon}{2}$. Then $K = \bigcup_{n=1}^{\infty} K_n$ is a compact subset of measure
\[ (\sum_{n=1}^{\infty} t_n \mu_n)(K) = \sum_{n=1}^{\infty} t_n \mu_n(K) \geq \sum_{n=1}^{m} t_n \mu_n(K_n) > \left(1 - \frac{\varepsilon}{2}\right) \sum_{n=1}^{m} t_n > \left(1 - \frac{\varepsilon}{2}\right)^2 > 1 - \varepsilon, \]
which means that the measure $\sum_{n=1}^{\infty} t_n \mu_n$ is Radon. \qed

**Theorem 3.18.** For every metrizable space $X$, the set $\hat{P}(X)$ is barycentric.

**Proof.** Let $X$ be a metrizable space and $cX$ be any compactification of $X$. By Theorem 3.6, it suffices to prove that $\hat{P}(X)$ is compactly convex. For this purpose, take any compact subset $K \subset \hat{P}(X) \subset P(cX)$. By Theorem 2 [11, c.96], for every $n \in \mathbb{N}$ there exists a compact subset $K_n \subset X$ such that $\mu(K_n) > 1 - 2^{-n}$ for any measure $\mu \in K$. By Lemma [1, 1.19], the set $K_n = \{ \mu \in P(cX): \mu(K_n) \geq 1 - 2^{-n}\}$ is closed in $P(cX)$ for all $n \in \mathbb{N}$. Then $\hat{K} = \bigcap_{n=1}^{\infty} K_n$ is a convex compact in $P(cX)$. Moreover, $\hat{K} \subset \hat{P}(X)$. Obviously, $\text{cl}_{P(X)}(\text{conv}(K)) = \text{cl}_{\hat{K}}(\text{conv}(K))$ is a compact subset of $\hat{P}(X)$. \qed

The question whether the set $\hat{P}(X)$ is barycentric is closely connected with the question whether the functor $\hat{P}$ is monadic (for the definition of the latter see [1, §1]).

**Theorem 3.19.** The restriction $\hat{P} : \mathcal{Metr} \to \mathcal{Metr}$ of the functor $\hat{P}$ to the category $\mathcal{Metr}$ of metrizable spaces and their continuous maps is a monad.

**Proof.** The functor $\hat{P}$ can be included into the triple $\hat{P} = (\hat{P}, \delta, \psi)$, where $\delta$ is the Dirac’s transform, and the component $\psi_X : P^2(X) \to \hat{P}(X)$ of the multiplication $\psi$ coincides with the barycenter map $b_{\hat{P}(X)} : P^2(X) \to \hat{P}(X)$. Here we use Theorems 3.15 and [1, 2.27], which says that for any metrizable space $X$ the set $\hat{P}(X)$ is both barycentric and metrizable. The fact that equalities $\psi \circ T\delta = \psi \circ \delta T = \text{id}_T$ and $\psi \circ \psi T = \psi \circ T\psi$ hold follows from the corresponding equalities that hold for the monad $\mathbb{P}$. \qed

Closely connected with the notion of a monad is the notion of an algebra. Let us recall the definition. By a $\mathbb{T}$-algebra of a monad $\mathbb{T} = (T, \delta, \psi)$ on the category $\mathcal{C}$ we understand a pair $(X, \xi)$ consisting of an object $X$ of the category $\mathcal{C}$ and a morphism $\xi : T(X) \to X$ such that $\xi \circ \delta_X = \text{id}_X$ and $\xi \circ \psi_X = \xi \circ T(\xi)$. A morphism of $\mathbb{T}$-algebras $(X, \xi)$ and $(X', \xi')$ is a morphism $f : X \to X'$ such that $\xi' \circ T(f) = f \circ \xi$.

**Theorem 3.20.** If $X$ is a convex bounded barycentric metrizable subset of a locally convex space, then the pair $(X, b_X)$ is an algebra of the monad $\hat{P} : \mathcal{Metr} \to \mathcal{Metr}$. In this case the barycentric map is a morphism of $\mathbb{P}$-algebras.

**Proof.** The equality $b_X \circ \delta_X = \text{id}_X$ follows form Lemma 3.1. Let us show that $b_X \circ b_{\hat{P}(X)} = b_X \circ \hat{P}(b_X)$. Let us note that the maps $b_X \circ b_{\hat{P}(X)}$ and $b_X \circ \hat{P}(b_X)$ are affine and continuous. Also, if $\delta_\mu \in P^2(X)$ is a Dirac measure on $\hat{P}(X)$, then $b_X \circ b_{\hat{P}(X)}(\delta_\mu) = b_X(\mu)$ and $b_X \circ \hat{P}(b_X)(\delta(\mu)) = b_X(\delta(b_X(\mu))) = b_X(\mu)$. As the convex hull of all Dirac measures is dense in $P^2(X)$, the sets $b_X \circ b_{\hat{P}(X)}$ and $b_X \circ \hat{P}(b_X)$ coincide. Thus, $(X, b_X)$ is a $\hat{P}$-algebra.

In order to prove the second statement of the theorem, observe that for any affine continuous map $f : X \to Y$ between barycentric subsets of locally convex spaces, the maps $b_Y \circ P(f)$ and $f \circ b_X$ are continuous and affine. Taking into account that $b_Y \circ P(f)(\delta_x) = b_Y(\delta_{f(x)}) = f(x) = f \circ \delta_X(\delta_x)$ for any $x \in X$ and the convex hull of all Dirac measures is dense in $\hat{P}(X)$, we conclude that $b_Y \circ P(f) = f \circ b_X$, i.e. $f$ is a morphism of $\mathbb{P}$-algebras $(X, b_X)$ and $(Y, b_Y)$. The theorem is proved. \qed

It is known [10], [12] that the category of $\mathbb{P}$-algebras is isomorphic to the category $\mathcal{Conv}$ of convex compacta that lie in locally convex spaces, and their continuous affine maps.

**Problem 3.21.** Is the category of algebras of the functor $\hat{P}$ isomorphic to the category whose objects are convex bounded barycentric sets in locally convex spaces, and whose morphisms are their affine continuous maps.

In light of the monadicity of the functor $P_\tau : \mathcal{Ych} \to \mathcal{Ych}$, the following problem arises naturally:

**Problem 3.22.** Describe the category of algebras of the monad $P_\tau : \mathcal{Ych} \to \mathcal{Ych}$.

**Remark 3.23.** Let us note that Theorem 3.20 implies that an algebra of the monad $P_\tau$ is any pair $(X, b_X)$, where $X$ is a convex bounded barycentric subset of a locally convex space such that $\hat{P}(X) = P_\tau(X)$. In addition, affine continuous maps between such sets are morphisms of $P_\tau$-algebras.
4 LIFTING THE FUNCTORS $P_\tau, \hat{P}$ TO CATEGORIES OF METRIC AND UNIFORM SPACES

In this section, for every bounded (pseudo)metric $d$ on a Tychonoff space $X$, we will construct a (pseudo)metric $d_\tau$ on $P_\tau(X)$ that extends the (pseudo)metric $d$. This will allow us to lift the functors $P_\tau$ and $P$ to the category of bounded metric spaces, and also to the category of uniform spaces. All constructions will be made first for the functor $\hat{\tau}$. Then, using the facts that the functor $P_\tau$ preserves embeddings, and that $P_\tau(X) = \hat{P}(X)$ for any complete metric space, we will generalize the obtained results to the functor $P_\tau$.

Let $d$ be a bounded pseudometric on a Tychonoff space $X$. For Radon measures $\mu, \eta \in \hat{P}(X)$ let

$$\hat{d}(\mu, \eta) = \inf\{\lambda(d) : \lambda \in \hat{P}(X \times X), \ P(pr_1)(\lambda) = \mu, \ P(pr_2)(\lambda) = \eta\},$$

where $pr_i : X \times X \to X$, $i = 1, 2$, is the projection onto the $i$-th factor. Let us note that for any measures $\mu, \eta \in \hat{P}(X)$ a measure $\lambda \in \hat{P}(X \times X)$ with $P(pr_1)(\lambda) = \mu$ and $P(pr_2)(\lambda) = \eta$ always exists. Indeed, for $\lambda$ we can take the tensor product $\mu \otimes \eta$ of the measures $\mu$ and $\eta$ (see [1, 2.21 and 2.22]).

Formally, the metric $\hat{d}$ was introduced by L.V. Kantorovitch in [13] and for compact spaces was studied by V.V. Fedorchuk [14]. Very recently, Yu.V. Sadovnichyi has shown [3] that for a metric space $(X, d)$, the metric $\hat{d}$ induces the subspace topology of the subspace $P_\tau(X) \subset \hat{P}(X)$, consisting of measures with compact supports.

The following proposition is analogous to [14, §4, Lemma 5].

**Lemma 4.1.** For any Radon measures $\mu, \eta \in \hat{P}(X)$, there exists a measure $\lambda \in \hat{P}(X \times X)$ with $P(pr_1)(\lambda) = \mu$, $P(pr_2)(\lambda) = \eta$ and $\hat{d}(\mu, \eta) = \lambda(d)$.

**Proof.** Let $\{\lambda_n\}_{n=1}^\infty \subset \hat{P}(X \times X)$ be a sequence of Radon measures such that $P(pr_1)(\lambda_n) = \mu$, $P(pr_2)(\lambda_n) = \eta$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} \lambda_n(d) = \hat{d}(\mu, \eta)$. Let $cX$ be a compactification of the space $X$. Since the functor $\hat{\tau}$ preserves embeddings, we can assume that $\hat{P}(X) \subset \hat{P}(cX)$ and $\hat{P}(X \times X) \subset \hat{P}(cX \times cX)$. In addition, $P(pr_1)(\lambda_n) = \mu$ and $P(pr_2)(\lambda_n) = \eta$, $n \in \mathbb{N}$ (here $pr_i : cX \times cX \to cX$, $i = 1, 2$, is the projection onto the $i$-th factor). As $\hat{P}(cX \times cX)$ is a compact, there exists an accumulation point $\lambda \in \hat{P}(cX \times cX)$ of the sequence $\{\lambda_n\}_{n=1}^\infty$. Since the maps $P(pr_i) : P(cX \times cX) \to P(cX)$, $i = 1, 2$, are continuous, $P(pr_1)(\lambda) = \mu$ and $P(pr_2)(\lambda) = \eta$. Let us show that $\lambda \in \hat{P}(X \times X) \subset \hat{P}(cX \times cX)$. For this purpose, fix $\varepsilon > 0$ and find two compacts $K_1 \subset X$ and $K_2 \subset X$ such that $\mu(K_1) > 1 - \frac{\varepsilon}{2}$ and $\eta(K_2) > 1 - \frac{\varepsilon}{2}$. Then

$$\lambda((cX \times cX) \setminus (K_1 \times K_2)) = \lambda((cX \setminus K_1) \times (cX \times cX)) \leq \lambda((cX \setminus K_1) \times cX) + \lambda(cX \times (cX \setminus K_2)) = \mu(cX \setminus K_1) + \eta(cX \setminus K_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\lambda(K_1 \times K_2) > 1 - \varepsilon$, i.e., $\lambda \in \hat{P}(X \times X)$. By definition of the topology on $\hat{P}(X \times X)$, the function $D : \hat{P}(X \times X) \to \mathbb{R}$, $D : \nu \mapsto \nu(d)$, is continuous. Since $\lambda \in \hat{P}(X \times X)$ is a limit point of the sequence $\{\lambda_n\}_{n=1}^\infty \subset \hat{P}(X \times X)$ and $\lim_{n \to \infty} \lambda_n(d) = \lim_{n \to \infty} D(\lambda_n) = \hat{d}(\mu, \eta)$, $D(\lambda) = \lambda(d) = \hat{d}(\mu, \eta)$.

**Lemma 4.2.** For any bounded continuous (pseudo)metric $d$ on a Tychonoff space $X$, the function $\hat{d} : \hat{P}(X) \times \hat{P}(X) \to \mathbb{R}$ is a continuous (pseudo)metric on $\hat{P}(X)$. Moreover, diam$(X, d) = \text{diam}(\hat{P}(X), \hat{d})$.

**Proof.** Similarly to Lemma 6 from [14, §4], it can be proved that the function $\hat{d} : \hat{P}(X) \times \hat{P}(X) \to \mathbb{R}$ is a bounded (pseudo)metric on $\hat{P}(X)$ for any bounded continuous (pseudo)metric $d$ on $X$.

Let us check that the pseudometric $\hat{d}$ on $\hat{P}(X)$ is continuous. For that purpose, fix a measure $\mu_0 \in \hat{P}(X)$ and $\varepsilon > 0$, and let $D = \text{diam}(X, d) + 1 = \sup\{d(x, x') : x, x' \in X\} + 1$. Let us fix a compact subset $K \subset X$ such that $\mu_0(K) > 1 - \frac{\varepsilon}{2D}$. Let $U = \{U_1, \ldots, U_n\}$ be a finite cover of the compact $K$ by open $\frac{\varepsilon}{2D}$-balls (with respect to the pseudometric $d$). Since the measure $\mu_0$ is countable additive, there exists an open set $V_1 \subset V_1 = d_X(V_1) \subset U_1$ such that $\mu_0(V_1) > \mu_0(U_1) - \frac{\varepsilon}{2D}$. By induction, construct open sets $V_k \subset X$, $1 < k \leq n$, such that $V_k \subset V_k \subset U_k \setminus \bigcup_{\ell < k} V_\ell$ and $\mu(V_k) > \mu(U_k) - \mu(U_1) - \ldots - \mu(U_{k-1}) - \frac{\varepsilon}{2D}$. Let us note that the sets $V_1, \ldots, V_n$ are disjoint. Apart from that, $\mu_0(V_1), \mu_0(V_2), \ldots, \mu_0(V_n), \mu_0(U_1), \mu_0(U_2 \setminus V_1), \ldots, \mu_0(U_n \setminus \bigcup_{i \leq n} U_i), \mu_0(U_1 \setminus V_1), \ldots, \mu_0(U_n \setminus \bigcup_{i \leq n} U_i) > 1 - \frac{\varepsilon}{D}$. By [1, 1.19], the set $U = \{\mu \in \hat{P}(X) : \mu(V_i) > \mu(U_i) - \frac{\varepsilon}{2D}, 1 \leq i \leq n\}$ is an open neighborhood of the measure $\mu_0$.

We will show that $\hat{d}(\mu, \mu_0) < \varepsilon$ for any measure $\mu \in U$. Let $V_0 = X \setminus \bigcup_{i=1}^n V_i$. For a Borel subset $B \subset X$ let $\mu[B]$ be the measure on $X$ defined by $\mu[B](A) = \mu(B \cap A)$ for any Borel set $A \subset X$. Let us consider the measure $\lambda = \sum_{i,j=0}^n \alpha_{ij} \cdot (\mu_0(V_i) \times \mu(V_j))$ on $X \times X$, where coefficients $\alpha_{ij} \geq 0$ are chosen to satisfy the following conditions for every $0 \leq i, j \leq n$:

$$\sum_{j=0}^n \alpha_{ij} \mu(V_j) = 1, \sum_{i=0}^n \alpha_{ij} \mu_0(V_i) = 1 \quad \text{and} \quad \alpha_{ij} \mu_0(V_i) \mu(V_j) = \min\{\mu_0(V_i), \mu(V_j)\}.$$
One can easily check that \( \lambda \) is a Radon measure on \( X \times X \), \( \hat{P}(pr_1)(\lambda) = \mu_0 \) \( \cup \) \( \hat{P}(pr_2)(\lambda) = \mu \). Let us show that \( \lambda(d) = \int_{X \times X \setminus \bigcup_{i=1}^{n} V_i \times V_i} d\lambda + \int_{\bigcup_{i=1}^{n} V_i \times V_i} d\lambda < \varepsilon \). Let us note that

\[
\lambda\left(\bigcup_{i=1}^{n} V_i \times V_i\right) = \sum_{i=1}^{n} \min\{\mu_0(V_i), \mu(V_i)\} > \sum_{i=1}^{n} \left(\mu_0(V_i) - \frac{\varepsilon}{4D}\right) > 1 - \frac{\varepsilon}{4D} - \frac{\varepsilon}{4D} = 1 - \frac{\varepsilon}{2D}.
\]

Since every set \( V_i \) is contained in some \( \frac{\xi}{4} \)-ball, for any \((x, y) \in \bigcup_{i=1}^{n} V_i \times V_i \) we get \( d(x, y) < \frac{\xi}{4} \). Then \( \lambda(d) < \frac{\varepsilon}{2D} + (1 - \frac{\varepsilon}{2D}) \frac{\xi}{4} < \varepsilon \). This implies that \( d(\mu, \mu_0) < \varepsilon \). Thus, \( d \) is a continuous pseudometric on \( \hat{P}(X) \).

The fact that \( \text{diam}(X, d) = \text{diam}(\hat{P}(X), \hat{d}) \) can be easily deduced from the definition of the pseudometric \( \hat{d} \). \( \square \)

Let us recall that a metric on a topological space \( X \) is called \textit{compatible} if it generates the initial topology on \( X \).

**Lemma 4.3.** If \( d \) is a compatible bounded metric on \( X \) then \( \hat{d} \) is a compatible metric on \( \hat{P}(X) \).

**Proof.** Let \((X, d)\) be a bounded metric space. Lemma 4.2 implies that \( \hat{d} \) is a continuous metric on \( \hat{P}(X) \).

Let us show that the metric \( \hat{d} \) induces the topology on \( \hat{P}(X) \). By Theorem 4 [11,III], the sets of the form \( \{\mu \in \hat{P}(X) : |\mu(f) - \mu_0(f)| < \varepsilon\} \), where \( f \) runs over uniformly continuous bounded functions on \((X, d)\), constitute a subbase of open neighborhoods of the measure \( \mu_0 \). Let \( f : (X, d) \to \mathbb{R} \) be a uniformly continuous bounded function on \( X \). Let us show that there exists an \( \varepsilon > 0 \), such that for any measure \( \mu \in \hat{P}(X) \) if \( d(\mu, \mu_0) < \varepsilon \), then \( |\mu(f) - \mu_0(f)| < 1 \). Let \( M = \sup\{f(x) | x \in X\} + 1 \). By the uniform continuity of \( f \), there exists a \( 0 \) \( < \delta < \frac{1}{2M} \) such that \( |f(x) - f(y)| < \frac{1}{2} \) for any for any \( x, y \in X \) with \( d(x, y) < \delta \). Let \( \varepsilon = \delta^2 \). We claim that \( |\mu(f) - \mu_0(f)| < 1 \) for any measure \( \mu \in \hat{P}(X) \) with \( d(\mu, \mu_0) < \varepsilon \). Lemma 4.1 implies the existence of a measure \( \lambda \in \hat{P}(X \times X) \) such that \( \hat{P}(pr_1)(\lambda) = \mu_0 \), \( \hat{P}(pr_2)(\lambda) = \mu \) and \( \lambda(d) < \varepsilon \). This implies that \( \lambda(A) \leq \delta \) for the set \( A = \{(x, y) \in X \times X | d(x, y) \geq \delta \} \). In this case \( |\mu(f) - \mu_0(f)| = |\int_{X \times X} f(x)d\lambda - \int_{X \times X} f(y)d\lambda| \leq \int_{X \times X} |f(x) - f(y)|d\lambda \leq \int_A |f(x) - f(y)|d\lambda + \int_{(X \times X) \setminus A} |f(x) - f(y)|d\lambda \leq \frac{1}{2} \varepsilon + 2M \delta < 1 \). Thus, the topology on \( \hat{P}(X) \), induced by the metric \( \hat{d} \) coincides with the initial topology and the lemma is proved. \( \square \)

Let \( C \) be a category and \( U : C \to \mathcal{T}_{\text{ych}} \) be a "forgetful" functor. We say that a functor \( F : \mathcal{T}_{\text{ych}} \to \mathcal{T}_{\text{ych}} \) can be lifted to the category \( C \) if there exists a functor \( \tilde{F} : C \to C \) such that \( U \circ \tilde{F} = F \circ U \). Lemmas 4.2, 4.3 imply

**Theorem 4.4.** The functor \( \hat{P} \) can be lifted to the category \( \mathbf{BMetr} \) of bounded metric spaces and their continuous maps.

**Lemma 4.5.** Let \( K \) be a closed subset of a bounded metric space \((X, d), \varepsilon > 0 \) and \( \delta > 0 \). Then, for any measures \( \mu, \eta \in \hat{P}(X) \) with \( d(\mu, \eta) \leq \frac{\xi}{4} \delta \) and \( \mu(K) \geq 1 - \frac{\xi}{4} \) we get \( \eta(O_\delta(K)) \geq 1 - \varepsilon \), where \( O_\delta(K) = \{x \in X : d(x, K) \leq \delta\} \) is a \( \delta \)-neighborhood of the set \( K \).

**Proof.** Let \( \mu, \eta \in \hat{P}(X) \) be measures such that \( d(\mu, \eta) \leq \frac{\xi}{4} \delta \) \( \cup \) \( \mu(K) \geq 1 - \frac{\xi}{4} \). By Lemma 4.1, there exists a measure \( \lambda \in \hat{P}(X \times X) \) such that \( \hat{P}(pr_1)(\lambda) = \mu \), \( \hat{P}(pr_2)(\lambda) = \eta \) and \( \lambda(d) = d(\mu, \eta) \). Let \( A = \{(x, y) \in X \times X : d(x, y) \leq \delta, x \in K\} \). One can easily see that \( \eta(O_\delta(K)) \geq \lambda(A) \). Let us note that \( (X \times X) \setminus A = \{(x, y) \in X \times X | d(x, y) > \delta\} \cup \{pr_1^{-1}(X \setminus K)\} \). As \( \lambda(d) \leq \frac{\xi}{4} \delta \), \( \lambda((x, y) \in X \times X | d(x, y) > \delta) \leq \frac{\xi}{4} \). The fact that \( \mu(K) \geq 1 - \frac{\xi}{4} \) \( \cup \) \( \mu(\lambda) = \lambda \) implies \( \lambda(pr_1^{-1}(X \setminus K)) \leq \frac{\xi}{4} \). In this case \( \lambda((X \times X) \setminus A) \leq \frac{\xi}{4} + \frac{\xi}{4} \). Therefore, \( \eta(O_\delta(K)) \geq \lambda(A) \geq 1 - \varepsilon \). \( \square \)

**Theorem 4.6.** If \( d \) is a complete bounded metric on \( X \), then \( \hat{d} \) is a complete bounded metric on \( \hat{P}(X) \).

**Proof.** Let \( cX \) be any compactification of \( X \) and \( \{\mu_n\}_{n=1}^{\infty} \subset \hat{P}(X) \) be a \( \hat{d} \)-Cauchy sequence. Then there exists a measure \( \mu \in P(cX) \) which is a limit point of the sequence \( \{\mu_n\}_{n=1}^{\infty} \subset \hat{P}(X) \subset P(cX) \). Let us show that \( \mu \in \hat{P}(X) \). Fix \( \varepsilon > 0 \). We will construct inductively a sequence \( \{k(n)\}_{n=1}^{\infty} \subset \mathbb{N} \) of numbers and an increasing sequence \( \{K_n\}_{n \in \mathbb{N}} \subset X \) of compact subsets of \( X \) such that for any \( n \in \mathbb{N} \) and any \( k, m \geq k(n) \) we get \( d(\mu_k, \mu_m) \leq \frac{\xi}{4} 2^{-n} \) and \( \mu(\lambda) \geq 1 - \frac{\xi}{4} \). Since the space \((X, d)\) is complete, the intersection \( K = \cap_{n=1}^{\infty} O_{2^{-n}}(K_n) \subset X \) is compact [8, 4.3.29]. For every \( n \in \mathbb{N} \) let \( C_n = cl_{cX}(O_{2^{-n}}(K_n)) \). One can easily check that \( K = \bigcap_{n=1}^{\infty} C_n \). By [1, 1.19], the set \( \mathcal{K}_n = \{\mu \in P(cX) : \mu(C_n) \geq 1 - \varepsilon\} \) is closed in \( P(cX) \).
Lemma 4.5 implies that for any \( n \in \mathbb{N} \) and \( k \geq k(n) \) we get \( \mu_k \in \mathcal{K}_n \). Consequently, \( \mu \in \bigcap_{n=1}^{\infty} \mathcal{K}_n \) and hence \( \mu(C_n) \geq 1 - \varepsilon \) for every \( n \in \mathbb{N} \). Taking into account that \( K = \bigcap_{n=1}^{\infty} C_n \), we conclude that \( \mu(K) \geq 1 - \varepsilon \) and hence \( \mu \in \hat{P}(X) \). Being Cauchy, the the sequence \( \{\lambda_n\}_{n=1}^{\infty} \subset \hat{P}(X) \) converges to its accumulation point \( \mu \in \hat{P}(X) \).

Now we will investigate the action of the functor \( \hat{P} \) on various classes of maps connected with the metric structure.

**Proposition 4.7.** The functor \( \hat{P} \) preserves the class of isometric embeddings.

**Proposition 4.8.** The functor \( \hat{P} \) preserves non-expanding mappings.

**Proof.** Let \( f : (X, d) \to (Y, g) \) be a non-expanding map and \( \mu, \eta \in \hat{P}(X) \). Let us show that \( \hat{d}(\hat{P}(f)(\mu), \hat{P}(f)(\eta)) \leq \hat{d}(\mu, \eta) \). By Lemma 4.1, there exists a measure \( \lambda \in \hat{P}(X \times X) \) such that \( \hat{P}(\pi_1)(\lambda) = \mu, \hat{P}(\pi_2)(\lambda) = \eta \) and \( \hat{d}(\mu, \eta) = \hat{d}(\lambda) \). Then the measure \( \hat{P}(f \times f)(\lambda) \in \hat{P}(Y \times Y) \) satisfies the following conditions:

\[
\hat{P}(\pi_1)(\hat{P}(f \times f)(\lambda)) = \hat{P}(f)(\mu) \quad \text{and} \quad \hat{P}(\pi_2)(\hat{P}(f \times f)(\lambda)) = \hat{P}(f)(\eta).
\]

Therefore, \( \hat{d}(\hat{P}(f)(\mu), \hat{P}(f)(\eta)) \leq \hat{P}(f \times f)(\lambda)(g) = \lambda(g \circ (f \times f)) \). Since the map \( f \) is non-expanding, \( \lambda(g \circ (f \times f)) \leq d(x, y) \) for every \( (x, y) \in X \times X \). Thus, \( \hat{d}(\hat{P}(f)(\mu), \hat{P}(f)(\eta)) \leq \lambda(g \circ (f \times f)) \leq \hat{d}(\mu, \eta) \).

Now we will deal with the question of extension of pseudometrics from a Tychonoff space \( X \) to the space \( P_\tau(X) \) of \( \tau \)-smooth probability measures on \( X \).

Let \( p \) be a continuous pseudometric on a Tychonoff space \( X \). By \((X_p, d_p)\) we denote the metric space induced by the pseudometric \( p \) and by \( \pi : X \to X_p \) the corresponding projection. Let \((X'_p, d'_p)\) be the completion of the metric space \((X_p, d_p)\). As \( X'_p \) is a complete metric space, \( P_\tau(X'_p) = \hat{P}(X'_p) \). Let us consider a map \( P_\tau(\pi) : P_\tau(X) \to P_\tau(X_p) \subset P_\tau(X'_p) = \hat{P}(X'_p) \) and define a pseudometric \( p_\tau \) on \( P_\tau(X) \) by the formula \( p_\tau(\mu, \eta) = d'_p(P_\tau(\pi)(\mu), P_\tau(\pi)(\eta)) \), \( \mu, \eta \in P_\tau(X) \), where \( d_p \) is a metric on the space \( \hat{P}(X'_p) = P_\tau(X'_p) \) induced by the metric \( d'_p \) on \( X'_p \).

Since the functors \( P_\tau \) and \( \hat{P} \) preserve the class of embeddings, Lemmas 4.2, 4.3, and the definition of the pseudometric \( p_\tau \) imply

**Proposition 4.9.** If \( p \) is a continuous bounded pseudometric on a Tychonoff space \( X \), then \( p_\tau \) is a continuous pseudometric on \( P_\tau(X) \). Moreover, \( \text{diam}(X, p) = \text{diam}(P_\tau(X), p_\tau) \).

**Proposition 4.10.** If \( d \) is a compatible bounded metric on the space \( X \), then \( d_\tau \) is a compatible bounded metric on \( P_\tau(X) \).

**Theorem 4.11.** The functor \( P_\tau \) can be lifted to the category \( B\text{Metr} \) of bounded metric spaces and their continuous maps.

**Remark 4.12.** For any bounded pseudometric \( d \) on \( X \) and any Radon measures \( \mu, \eta \in \hat{P}(X) \subset P_\tau(X) \) the distance \( d_\tau(\mu, \eta) \) coincides with the distance \( d(\mu, \eta) \), defined at the beginning of this section. This follows from the fact that the pseudometrics \( d_\tau \) and \( d \) are continuous on \( \hat{P}(X) \) and the equality \( d(\mu, \eta) = d_\tau(\mu, \eta) \) holds for measures \( \mu, \eta \in \hat{P}(X) \) with finite supports.

Since for complete metric spaces the spaces of Radon and \( \tau \)-smooth measures coincide, Theorem 4.6 implies:

**Theorem 4.13.** If \( d \) is a complete bounded metric on \( X \), then \( d_\tau \) is a complete bounded metric on \( P_\tau(X) \).

Using Propositions 4.7, 4.8, we can prove:

**Proposition 4.14.** The functor \( P_\tau \) preserves the class of isometric embeddings.

**Proposition 4.15.** The functor \( P_\tau \) preserves non-expanding mappings.

For maps \( f, g : Y \to X \) and a bounded pseudometric \( d \) on \( X \) let \( d(f, g) = \sup\{d(f(y), g(y)) : y \in Y\} \).

**Proposition 4.16.** For any bounded pseudometric \( d \) on a Tychonoff space \( X \) and any maps \( f, g : Y \to X \) we have that \( d(f, g) = d_\tau(P_\tau(f), P_\tau(g)) \).
Indeed, let $\mu, \eta \in P_r(Y)$.

One can easily observe that balls are convex with respect to a convex pseudometric. Moreover, if $\tau(x,y) = \inf \{ d'(x,x') : d'(x,x') \leq d(f,g) \}$ then $d_r(P_r(f)(\mu), P_r(g)(\mu)) = \tau(\mu,\eta)$. The proposition is proved.

Now let us look into the connection between the metric and convex structures on $P_r(X)$.

A pseudometric $\gamma$ on a convex subset $Y$ of a vector space will be called convex if for any $x, x', y, y' \in Y$ and $t \in [0,1]$ we get

$$\gamma(tx + (1-t)y, tx' + (1-t)y') \leq t\gamma(x, x') + (1-t)\gamma(y, y').$$

One can easily observe that balls are convex with respect to a convex pseudometric. Moreover, if $d$ is a convex pseudometric on $Y$, then for any $x, y \in Y$ and $t \in [0,1]$ we get $d(tx + (1-t)y, t\gamma(x, y) \leq td(x, y)$.

One can easily prove the following

**Proposition 4.17.** For any bounded pseudometric $d$ on a Tychonoff space $X$, the pseudometric $d_r$ on $P_r(X)$ is convex.

**Proposition 4.18.** Let $X$ be a convex barycentric subset of a locally convex space, and let $d$ be a convex bounded metric on $X$. Then, the barycenter map $b_X : \hat{P}(X), \hat{d} \to (X, d)$ is non-expanding.

**Proof.** For Dirac measures $\delta_x, \delta_y \in \hat{P}(X)$ we have that $d(b_X(\delta_y), b_X(\delta_x)) = d(y, x)$. From the fact that $d$ is convex it follows that for any $n \in \mathbb{N}$, any points $x_i, y_i \in X$, $1 \leq i \leq n$, and numbers $t_i \in [0,1], 1 \leq i \leq n$, with $\sum_{i=1}^{n} t_i = 1$ we get

$$d\left(\sum_{i=1}^{n} t_i x_i, \sum_{i=1}^{n} t_i y_i\right) \leq \sum_{i=1}^{n} t_i d(x_i, y_i).$$

Let us show that for any measures $\mu, \eta \in P_\omega(X) \subset \hat{P}(X)$ with finite supports, $d(b_X(\mu), b_X(\eta)) \leq \hat{d}(\mu, \eta)$.

Indeed, let $\lambda \in \hat{P}(X \times X)$ be a measure such that $\hat{P}(pr_1)(\lambda) = \mu$, $\hat{P}(pr_2)(\lambda) = \eta$ and $\lambda(d) = \hat{d}(\mu, \eta)$. The measures $\mu$ and $\eta$ can be written as the convex combinations $\mu = \sum_{i=1}^{n} \alpha_i \delta_x$, and $\eta = \sum_{j=1}^{m} \beta_j \delta_y$, of Dirac measures. In this case the measure $\lambda$ can be written as $\lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \delta_{x_i, y_j}$, where $\sum_{j=1}^{m} \gamma_{ij} = \alpha_i$ and $\sum_{i=1}^{n} \gamma_{ij} = \beta_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Then

$$d(b_X\left(\sum_{i=1}^{n} \alpha_i \delta_x\right), b_X\left(\sum_{j=1}^{m} \beta_j \delta_y\right)) = d\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j\right) =
$$

$$d\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} x_i, \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} y_j\right) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} d(x_i, y_j) = \lambda(d) = \hat{d}(\mu, \eta).$$

Thus, the inequality $d(b_X(\mu), b_X(\eta)) \leq \hat{d}(\mu, \eta)$ holds for all measures $\mu, \eta$ from the dense subset $P_\omega(X) \subset \hat{P}(X)$. By the continuity of the map $b_X : \hat{P}(X) \to X$, the inequality $d(b_X(\mu), b_X(\eta)) \leq \hat{d}(\mu, \eta)$ holds for all measures $\mu, \eta \in \hat{P}(X)$.

A connection between the metric and monadic structures of the function $P_r$ is described in the following:

**Theorem 4.19.** For any bounded metric space $(X, d)$ the identity $\delta_X : X \to P_r(X)$ of the monad $P_r$ is a closed isometric embedding, and the multiplication $\psi_X = b_{P_r(X)} : P^2_r(X) \to P_r(X)$ is a non-expanding map. Moreover, for every point $x \in X$ and a measure $\mu \in P^2_r(X)$, we get $d_{P^2_r(X)}(\delta_x, \mu) = d_{P_r(X)}(\delta_x, \psi_X(\mu))$. 
Proof. The first statement of the theorem follows from Propositions 4.17, 4.18 and the fact that the set \(P^2(X)\) is dense in \(P^2(X)\). The last statement of the theorem will follow as soon as we prove the equality 
\[d_{P^2(X)}(\delta_{x,y}, \mu) = d_{P^2(X)}(\delta_{x}, \psi_X(\mu))\]
for any point \(x \in X\) and any measure \(\mu\) from the dense subset \(P_n(P_n(X)) \subset P_n(P_n(X))\). In this case the support of \(\mu\) is equal to the set \(K = \{x\} \cup \{\supp(\eta) \mid \eta \in \supp(\mu)\} \subset X\). Observe that \(\mu \in P(P(K)) \subset P_n(P_n(X))\). For the compact metric space \((K, d(K))\) the equality 
\[d_{P^2(K)}(\delta_{x,y}, \mu) = d_{P^2(K)}(\delta_{x}, \psi_X(\mu))\]
is proved in [22, §4, лемма 11]. Since the functor \(P_n\) preserves isometric embeddings, the last equality implies that 
\[d_{P^2(X)}(\delta_{x,y}, \mu) = d_{P_n(X)}(\delta_{x}, \psi_X(\mu))\].
The theorem is proved.

Now we will deal with the problem of lifting the functors \(\hat{P}_n, P_n : \mathcal{T}_{ych} \to \mathcal{T}_{ych}\) to the category \(\mathcal{U}_{uni} f\) of uniform spaces and their continuous maps (see [8, chapter 8] for the basics of the theory of uniform spaces).

Let us recall that a map \(f : (X, U) \to (Y, V)\) between two uniform spaces is a \emph{uniform homeomorphism} if \(f\) is bijective and the maps \(f\) and \(f^{-1}\) are uniformly continuous. A map \(f : (X, U) \to (Y, V)\) is called a \emph{uniform embedding} if \(f\) is a uniform homeomorphism of the space \((X, U)\) onto its image \(f(X) \subset (Y, V)\).

We shall say that \(U\) is a \emph{uniformity on a topological space} \(X\) it generates the topology of \(X\). A pseudometric \(d : X \times X \to \mathbb{R}\) is called \emph{\(U\)-uniform} if for any \(\varepsilon > 0\) there exists a neighborhood of the diagonal \(U \in U\) such that \(d(U) \subset \{0, \varepsilon\}\).

For a uniform space \((X, U)\) a family \(P\) of all \(U\)-uniform pseudometrics on \(X\) has the following properties:

(UP1) if \(\rho_1, \rho_2 \in P\), then \(\max\{\rho_1, \rho_2\} \in P\);

(UP2) for every pair \(x, y\) of distinct points from \(X\) there exists a pseudometric \(\rho \in P\) such that \(\rho(x, y) > 0\).

Conversely, for every family \(P\) of pseudometrics on the set \(X\) that satisfies (UP1)–(UP2), the family \(B\) of all neighborhoods of the diagonal of the form \(\{(x, y) \mid \rho(x, y) < 2^{-n}\}\), where \(\rho \in P\) and \(n \in \mathbb{N}\), forms a base for some uniformity on \(X\).

Now, we will show how to construct a uniformity on the space \(P_n(X)\) given a uniformity \(U\) on the space \(X\).

For every pseudometric \(p\) on \(X\) by \((X_p, d_p)\) we will denote the metric space induced by \(p\), and by \(\pi_p : X \to X_p\) the corresponding quotient map. Every uniformity \(U\) on the space \(X\) induces a uniform embedding \(\pi_U : (X, U) \to \prod_{p \in P(U)}(X_p, d_p)\), defined by the formula \(\pi_U(x) = (\pi_p(x))_{p \in P(U)}\), where \(P(U)\) is the family of all \(U\)-uniform bounded pseudometrics on \(X\) (see [6, 8.2.2]). The set \(P(U)\) is equipped with the natural partial order \(\leq\). One can easily observe that any two pseudometrics \(\rho, \rho' \in P(U)\), \(\rho \leq \rho'\), induce a non-expanding map \(\pi^\rho_p : (X_p, d_p) \to (X_p, d_p)\). Evidently, the product \(\prod_{p \in P(U)}X_p\) is homeomorphic (uniformly homeomorphic even) to the limit \(\varprojlim X_p\) of the inverse system \(\{X_p, \pi^\rho_p, P(U)\}\). Since the set \(P(U)\) is directed (i.e. for any \(\rho_1, \rho_2 \in P(U)\) there exists \(\rho \in P(U)\) such that \(\rho \geq \rho_1\) and \(\rho \geq \rho_2\)), Proposition 1, 1.11] guarantees that the map \(R : P_n(\prod_{p \in P(U)}X_p) = P_n(\varprojlim X_p) \to \varprojlim P_n(X_p)\) is a topological embedding. Therefore, the map \(R \circ \pi_U : (X, U) \to \varprojlim P_n(X_p)\) is also a topological embedding. On every space \(P_n(X_p)\) we consider the uniformity induced by the metric \((d_p)\), which, by Proposition 4.10, is compatible with the topology of \(P_n(X_p)\). Finally, let us equip the space \(P_n(X)\) with the uniformity \(U_r\) of the subspace of the inverse limit \(\varprojlim P_n(X_p)\) of uniform spaces \(P_n(X_p)\). The space \(\hat{P}(X)\) will be equipped with the uniformity \(U\) of the subspace of the uniform space \((P_n(X), U_{r})\).

Thus for every uniformity \(U\) on the space \(X\) we have defined a uniformity \(U_r\) on the space \(P_n(X)\). One can easily observe that the uniformity \(U_r\) can be defined in a direct fashion. In particular, the base for that uniformity consists of the entourages \(U = \{(\mu, \eta) \in P_n(X) \times P_n(X) \mid p_\tau(\mu, \eta) < 2^{-n}\}\), where \(p \in P(U)\) and \(n \in \mathbb{N}\). Having chosen the roundabout way (using inverse limits), we didn’t need to check that the family \(B\), indeed, satisfies the axioms of a base for a uniformity. (see [8, p.624]). Moreover, we have also proved that if the uniformity \(U\) induces the topology of \(X\), the uniformity \(U_r\) induces the topology of \(P_n(X)\).

Proposition 4.20. Let a family \(P\) of bounded pseudometrics on a uniform space \((X, U)\) satisfy the conditions (UP1)–(UP2). If the family of entourages \(B = \{(x, y) \in X^2 : p(x, y) < 2^{-n}\} : p \in P, \ n \in \mathbb{N}\}\) is a base of the uniformity \(U\), then the family \(B_r = \{(\mu, \eta) \in P_n(X) \times P_n(X) : p_r(\mu, \eta) < 2^{-n}\} : p \in P, \ n \in \mathbb{N}\}\), is a base for the uniformity \(U_r\) on \(P_n(X)\).

Proof. First, let us note that since very pseudometric \(\rho \in P\) is \(U\)-uniform, every entourage \(V \in B_r\) belongs to \(U_r\) according to the definition of the uniformity \(U_r\).

Now fix a neighborhood of the diagonal \(V \in U_r\). The definition of the uniformity \(U_r\) implies that there exists a bounded pseudometric \(\rho \in P(U)\) such that \(\{(\mu, \eta) \in P_n(X)^2 : p_r(\mu, \eta) < 1\} \subset V\). Let
\[ D = \sup\{\rho(x,y) \mid x, y \in X\} + 1 \] and \( U = \{(x,y) \mid \rho(x,y) < \frac{1}{4}\}. \) As \( \mathcal{B} \) is a base for the uniformity \( \mathcal{U} \), there exist \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \) such that \( W = \{(x,y) : x^2 + p(x,y) < 2^{-n}\} \subset U. \)

We are going to show that \( \mathcal{W} = \{(\mu, \eta) \in P_r(X)^2 : \rho_r(\mu, \eta) < 2^{-(n+1)/D}\} \subset \mathcal{A} = \{(\mu, \eta) \in P_r(X)^2 : \rho_r(\mu, \eta) \leq \frac{2}{5}\}. \) Since the set \( \mathcal{A} \) is closed in \( P_r(X) \times P_r(X) \), and the set \( \mathcal{W} \) is open, the inclusion \( \mathcal{W} \subset \mathcal{A} \) follows from the inclusion \( (\mu, \eta) \in \mathcal{A} \) for any pair \( (\mu, \eta) \in \mathcal{W} \) of measures with finite supports. Fix a pair \( (\mu, \eta) \in \mathcal{W} \) of measures with finite supports. By Lemma 4.1 and Remark 4.12, there exists a measure \( \lambda \in P_r(X \times X) \) such that \( P_r(pr_1)(\lambda) = \mu, P_r(pr_2)(\lambda) = \eta \) and \( \lambda(p) = \rho_r(\mu, \eta) = \int_{X \times X} \rho d\lambda < 2^{-(n+1)/D}. \)

Taking into account that the value of the function \( p \) on the set \( (X \times X) \setminus W \) is not less than \( 2^{-n} \), we conclude that \( 2^{-(n+1)/D} > \int_{X \times X} \rho d\lambda \geq \int_{(X \times X) \setminus W} \rho d\lambda \geq 2^{-n} \lambda((X \times X) \setminus W) \), which implies that \( \lambda((X \times X) \setminus U) \leq \lambda((X \times X) \setminus W) < 2^n \cdot 2^{-(n+1)/D} = \frac{2}{5}. \) Then, \( \rho_r(\mu, \eta) \leq \lambda(p) = \int_{X \times X} \rho d\lambda = \int_{(X \times X) \setminus U} \rho d\lambda + \int_{U} \rho d\lambda \leq D \cdot \lambda((X \times X) \setminus U) + \frac{1}{2} \lambda(U) \leq \frac{1}{2} + \frac{1}{2} = \frac{2}{5}. \) The proposition is proved. \( \square \)

**Corollary 4.21.** Let \( (X,d) \) be a bounded metric space and \( \mathcal{U} \) be a uniformity on \( X \) induced by the metric \( d. \) Then the uniformity on \( P_r(X) \) induced by the metric \( d_r \) coincides with the uniformity \( \mathcal{U}_r. \)

The idea of the proof of the following proposition belongs to Yu. Sadovnichiy.

**Proposition 4.22.** If \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniformly continuous map, then the map \( P_r(f) : (P_r(X), \mathcal{U}_r) \to (P_r(Y), \mathcal{V}_r) \) is also uniformly continuous.

**Proof.** Let \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) be a uniformly continuous map. As we have mentioned before, the basis for the uniformity \( (P_r(Y), \mathcal{V}_r) \) is the family of sets of the form \( U^\rho_p = \{(\mu, \eta) \in P_r(Y) \times P_r(Y) \mid \rho_r(\mu, \eta) < \varepsilon\}, \) where \( \varepsilon > 0 \) and \( p \in P(\mathcal{U}). \) Fix an \( \varepsilon > 0 \) and \( p \in P(\mathcal{U}). \) Since the map \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is uniformly continuous, the pseudometric \( \rho = \rho (f \times f) \) is uniform with regard to \( \mathcal{U}. \) Let us note that the definition of the pseudometric \( \rho_r \) implies that for any \( \mu, \eta \in P_r(X), \rho_r(\mu, \eta) = \rho_r(f(\mu), f(\eta)). \) Therefore, \( (P_r(f) \times P_r(f))^{-1}(U^\rho_p) = U^\rho_p = \{(\mu, \eta) \in P_r(X) \times P_r(X) \mid \rho_r(\mu, \eta) < \varepsilon\} \subset \mathcal{U}_r, \) i.e. the map \( P_r(f) : (P_r(X), \mathcal{U}_r) \to (P_r(Y), \mathcal{V}_r) \) is uniformly continuous. The Proposition is proved. \( \square \)

Using Proposition 4.22 we can prove:

**Theorem 4.23.** The functors \( P_r : \text{Tych} \to \text{Tych} \) and \( \hat{P} : \text{Tych} \to \text{Tych} \) can be lifted to the category \( \text{Unif} \) of uniform spaces and their uniformly continuous maps.

One can immediately prove

**Proposition 4.24.** For every uniform space \( (X, \mathcal{U}) \) the map \( \delta : (X, \mathcal{U}) \to (\hat{P}(X), \hat{U}) \subset (P_r(X), \mathcal{U}_r), \delta : x \mapsto \delta_x, x \in X, \) is a uniform embedding.

Since every bounded uniformly continuous pseudometric on a subspace \( Y \) of a uniform space \( (X, \mathcal{U}) \) can be extended to a \( \mathcal{U} \)-uniform bounded pseudometric on \( X \) (cm. \cite{15}, or \cite[8.5.6]{8}), we have the following

**Proposition 4.25.** For every uniform embedding \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) the map \( P_r(f) : (P_r(X), \mathcal{U}_r) \to (P_r(Y), \mathcal{V}_r) \) is also a uniform embedding.

It is well-known (see \cite[8.1.8]{8}) that a complete uniform space is compact if and only if it is totally bounded.

**Proposition 4.26.** A uniform space \( (X, \mathcal{U}) \) is totally bounded if and only if the space \( (P_r(X), \mathcal{U}_r) \) is totally bounded too.

**Proof.** Since the space \( (X, \mathcal{U}) \) can be uniformly embedded in \( (P_r(X), \mathcal{U}_r), \) the fact that \( (P_r(X), \mathcal{U}_r) \) is totally bounded implies that the space \( (X, \mathcal{U}) \) is also totally bounded.

Now, if \( (X, \mathcal{U}) \) is a totally bounded uniform space, its completion \( (\hat{X}, \hat{U}) \) is compact. By Proposition 4.26, the space \( (P_r(X), \mathcal{U}_r) \) can be uniformly embedded in the compact space \( P_r(\hat{X}) = \hat{P}(X) \) and, therefore, is a totally bounded uniform space. The Proposition is proved. \( \square \)

**Remark 4.27.** Despite the fact that the functor \( \hat{P} \) preserves complete metric spaces, it does not, generally speaking, preserve complete uniform spaces. This follows form the fact that for an uncountable set \( A \) the uniform space \( (P(\mathbb{R}^A), \mathcal{U}) \) is not complete (here \( \mathcal{U} \) is the product uniformity on \( \mathbb{R}^A \)). Indeed, let \( \mu \in \hat{P}(\mathbb{R}) \) be
an arbitrary measure on $\mathbb{R}$ with a non-compact support. For every finite $B \subset A$ let $\mu_B = \otimes_{\alpha \in A} \mu_{\alpha} \in \hat{P}(\mathbb{R}^A)$, where

$$\mu_{\alpha} = \begin{cases} \mu, & \text{for } \alpha \in B; \\ \delta_0, & \text{for } \alpha \notin B. \end{cases}$$

It can be shown that $\{\mu_{\alpha}\}_{B \subset A}$ is a Cauchy net in $\hat{P}(\mathbb{R}^A)$, which does not have an accumulation point. The author is unaware of whether the functor $P_\tau$ preserves complete uniform spaces$^1$.

Let us recall that a topological space $X$ is called (Hewitt) Dieudonne complete if $X$ is homeomorphic to a closed subset of the Tychonov product of (separable) complete metric spaces. It is known [8] that every Lindelof space is Hewitt complete, and every metrizable space is Dieudonne complete.

**Problem 4.28.** Do the functors $P_\tau$ and $\hat{P}$ preserve Hewitt or Dieudonne complete spaces?$^1$

We will finish this section with the following simple statement, which follows from Proposition 4.17.

**Proposition 4.29.** For every uniform space $(X, \mathcal{U})$ the map

$$\alpha : (P_\tau(X), \mathcal{U}_\tau) \times (P_\tau(X), \mathcal{U}_\tau) \times [0, 1] \rightarrow (P_\tau(X), \mathcal{U}_\tau), \quad \alpha(\mu, \eta, t) = t\mu + (1-t)\eta,$$

is uniformly continuous.

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The answer to this question is given in the following articles:

- V.V.Fedorchuk On completeness-type properties of spaces of $\tau$-additive probability measures (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1998, no. 5, 19–22, 71; transl. in Moscow Univ. Math. Bull. 53 (1998), no. 5, 20–23 (1999).

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