The cone construction via intersection theory

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Abstract
Using intersection theory, we give a construction in the category of smooth projective varieties over $\mathbb{C}$. It proves the Lefschetz standard conjecture over $\mathbb{C}$.

Contents

1 Introduction 1
1.1 Lefschetz standard conjecture 1
1.2 Outline of the proof 2

1.2.1 Cone construction on $\mathbb{Z}$ groups 3
1.2.2 Cohomological descend 5

2 Cone construction 6
2.1 Decomposition of the diagonal 6
2.2 The infinite end-cycle of the rational equivalence 11
2.3 The finite end-cycle of the rational equivalence 13
2.4 The final proof 15

1 Introduction

1.1 Lefschetz standard conjecture
The Lefschetz standard conjecture were proposed by Grothendieck ([2]) in formulating a solution to Weil’s conjectures. The conjecture addresses a smooth projective variety $X$ of dimension $n \geq 3$ over an algebraically closed field of arbitrary characteristic. In this paper, we step back to assume the ground field is $\mathbb{C}$, and the cohomology is Betti cohomology with rational coefficients. Let $u$ be the hyperplane section class in the rational cohomology $H^2(X; \mathbb{Q})$. Let $p, q$ be
whole numbers satisfying $p + q = n, q \geq p$. Let $u^{q-p}$ denote the homomorphism on the cohomology

$$u^{q-p} : H^{2p}(X; \mathbb{Q}) \to H^{2q}(X; \mathbb{Q})$$

$$\alpha \to \alpha \cdot u^{q-p}.$$  \hspace{1cm} (1.1)

The hard Lefschetz theorem says $u^{q-p}$ is an isomorphism.

**Conjecture 1.1. (Lefschetz)**

Let $A^i(X) \subset H^{2i}(X; \mathbb{Q})$ be the subspace spanned by algebraic cycles. Then the restriction $u^{q-p}_a$ of $u^{q-p}$ to $A^p(X)$,

$$A^p(X) \to A^q(X)$$

$$\alpha \to \alpha \cdot u^{q-p}.$$  \hspace{1cm} (1.2)

is also an isomorphism.

Conjecture 1.1 is known as the $A$-conjecture or the Lefschetz standard conjecture.

In this paper, we prove that

**Theorem 1.2. (Main theorem)**

Over the complex numbers $\mathbb{C}$, the Lefschetz standard conjecture is correct.

### 1.2 Outline of the proof

Our tool is the intersection theory [1] from where the most of notations are adopted. The key notion among them is the spread of a family of algebraic cycles, which is somewhat non standard, hence precisely defined in definition 1.3 of the following.

**Notation:**

1. We use $Z$ to denote the total Abelian groups of algebraic cycles with rational coefficients, called $Z$ groups, $CH$ to denote the total Chow groups with rational coefficients, and $H$ to denote the total Betti cohomology groups with rational coefficients. Also we sometimes add the superscript index to denote the codimension of homogeneous cycles and subscript index to denote the dimension of homogeneous cycles.

2. $a^*$ denotes a pull-back in various situation depending on the context.

3. $a_*$ denotes a push-forward in various situation depending on the context.
The cycle associated to a variety $\alpha$ is still denoted by $\alpha$.

(5) $\langle \alpha \rangle$ denotes the cohomology class represented by a scheme, a variety or a cycle $\alpha$.

(6) Continuing from (2), (3) and (5), we denote the correspondences on the cohomology by angle brackets $\langle \bullet \rangle_*, \langle \bullet \rangle^*$ and on $\mathbb{Z}$ groups and Chow groups by parentheses $(\bullet)_*, (\bullet)^*$.

(7) The polarization of $X$ is fixed.

(8) A smooth projective variety is connected.

**Definition 1.3.** Let $\Upsilon, X$ be two smooth projective varieties over $\mathbb{C}$. Let $C_z$ for generic $z \in \Upsilon$ be a family of algebraic cycles of $X$, i.e. there is a non empty open set $U \subset \Upsilon$ such that $C_z, z \in U$ on $X$ is a family of algebraic cycles. Let $C$ be the cycle of its universal family on $U \times X$. We define the spread of $C_z$ for generic $z$ to be the closure in $X$, of the push-forward $(P_X)_*(C)$, where $P_X : U \times X \to X$ is the projection. The spread of $C_z$ is denoted by

$$\bigcup_z C_z, \quad \text{and} \quad C \text{ in the context.}$$

The spread is independent of the choice of the open set $U$. It only depends on the family $C_z$ for generic $z$. It plays a crucial role in our arguments.

Notations try to distinguish the same object in three different categories: I) cohomology groups, II) Chow groups, III) the Abelian groups of algebraic cycles called the $\mathbb{Z}$ groups. Going from III) to I) is called descending. Being able to descend to I) is called “cohomological”.

**1.2.1 Cone construction on $\mathbb{Z}$ groups**

To show the $A$-conjecture, we construct a linear map

$$\langle \text{Con}_{q-p} \rangle : A^q(X) \to A^p(X)$$

which will be proved to be the inverse of the map (1.2), i.e.

$$\langle \text{Con}_{q-p} \rangle \circ u_{q-p} = \text{identity}$$

$$u_{q-p} \circ \langle \text{Con}_{q-p} \rangle = \text{identity.}$$

---

1 The name “spread” has been used in other literature, but in a slightly different way. Our description is standard. However its application to correspondences is non-standard. It needs an attention.

2 The extension of the construction to the entire cohomology requires the real ($\mathbb{R}$) intersection theory which is discussed elsewhere. Our extension is a hybrid that, on one hand, coincides with the existing, topological inverse of $u_{q-p}$, and, on the other, carries the algebro-geometric structure.
The construction occurs in the $\mathbb{Z}$ groups from where the cohomological descent is not expected. However when it is restricted to one particular homogeneous part of the cohomology, the cohomological descent does exist. Let’s describe these two seemingly conflicting steps. Let $h$ be a natural number $< n = \dim(X)$. Let

$$\mu : X \to \mathbb{P}^{n+1}$$

be a birational morphism to a hypersurface of $\mathbb{P}^{n+1}$. We decompose $\mathbb{P}^{n+1}$ linearly as

$$\mathbb{P}^{n+1} = \mathbb{P}^{n-h+1} \# \mathbb{P}^{h-1},$$

where $\mathbb{P}^{n-h+1}$ is a fixed subspace of dimension $n - h + 1$, and $\mathbb{P}^{h-1}$ is a varied but generic subspace of dimension $h - 1$ parametrized by a smooth curve $\mathcal{Y}$ in the Grassmannian. We’ll use $z$ to represent a point in $U$, which represents the varied $\mathbb{P}^{h-1}$.

Let $f_h, f_{n-h+1}$ be the rational projections,

$$\mathbb{P}^{n+1} \supset \mathbb{P}^{n-h+1} \xrightarrow{f_h} \mathbb{P}^{n+1} \xrightarrow{f_{n-h+1}} \mathbb{P}^{h-1} \subset \mathbb{P}^{n+1}$$

respectively. Let $L_h^z$ be the transpose of the graph of $f_h$ in $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$. Let $L_{n-h+1}$ be the graph of $f_{n-h+1}$ in $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$. Then the diagonal $\Delta_{\mathbb{P}^{n+1}}$ of $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ has a “linear decomposition”

$$\Delta_{\mathbb{P}^{n+1}} \text{ rational equi } L_h^z + L_{n-h+1}. \quad (1.5)$$

Restricting (1.5) to $X \times X$, we obtain a decomposition of a hypersurface of the diagonal.

$$(\Delta_X)^z \text{ rational equi } \Phi_1^z + \Phi_2^z, \quad (1.6)$$

where $\Delta_X$ is the diagonal in $X \times X$, and $(\Delta_X)^z$ is a hypersurface (obtained through a specialization) of the diagonal varied with $z$. The both sides of the formula (1.6) has lower dimension. But next taking the union of (1.6) over $z \in U$, then taking the closure, we obtain the decomposition of the diagonal in $X \times X$,

$$m\Delta_X \text{ rational equi } \Phi_1 + \Phi_2, \quad (1.7)$$

where the natural number $m$ could be 1 for a suitable choice of $\mathcal{Y}$. This step of taking union is the spreading of a generic cycle as in definition 1.3. So far the construction still has Chow descend and even cohomological descend. But what follows changes all. Applying the notion of the spread once again to both sides of (1.7), we construct two linear operators $(\Phi_i)\circ$, $i = 1, 2$, from $Z$ groups of $X$ to itself. The main purpose of using $(\Phi_i)\circ$ instead of the conventional correspondences $(\Phi_i)_*$ is to remove the “bad” components in the intersection. The operator $(\bullet)_*$ does not have any descends. But it leads to a decomposition of the identity on $Z$ groups,

$$(m\Delta_X)\circ \text{ rational equi } (\Phi_1)\circ + (\Phi_2)\circ$$

$$\downarrow$$

$$m \cdot id_X \text{ rational equi } (\Phi_1)\circ + (\Phi_2)\circ, \quad (1.8)$$
1 INTRODUCTION

where \( id_X \) is the identity and \((\Phi_2)_{\oplus}\) vanishes if it acts on the cycles of the dimension \( q > \frac{n}{2} \). By observing the construction (especially the subvariety \( L_h \)), we are able to factorize \((\Phi_1)_{\oplus}\),

\[
(\Phi_1)_{\oplus} = \text{Con}_h \circ v^h
\]

where \( v^h \) is the intersection with an \( h \)-power of the hyperplane section, and \( \text{Con}_h \) called the cone operator is constructed using the notion of the spread. Thus the cone construction gives the sequence of maps,

\[
\sum_{q > \frac{n}{2}} Z_q^s(X) \xrightarrow{v^h} \sum_{q > \frac{n}{2}} Z_{q-h}(V^h) \xrightarrow{\text{Con}_h} \sum_{q > \frac{n}{2}} Z_q^s(X),
\]

where \( V^h \) is a generic \( h \)-codimensional plane section of \( X \), and the superscript \( s \) means the subgroup of cycles meeting \( V^h \) properly. The maps satisfy the formula (1.9). In this initial step we proved that the identity on the \( Z \) groups of larger dimensions can be deformed (in a rational equivalence) to a factorization which is a plane section followed by a linear operator on algebraic cycles called cone operator, denoted by \( \text{Con}_h \) for any natural number \(< n \).

1.2.2 Cohomological descend

It is unfortunate that the factorization

\[
(\Phi_1)_{\oplus} = \text{Con}_h \circ v^h
\]

on \( Z \) groups does not have a cohomological descend. However it does when it is restricted to one particular homogeneous part. In the following step we would like to show that when (1.11) is restricted to the homogeneous part of cycles of dimension \( q = \frac{n+h}{2} \), it has the cohomological descend. This is due to the hard Lefschetz theorem. The following is the description. This amounts to show that if \( \sigma \in Z^s_{n-q}(V^h) \) is cohomologous to zero, so is \( \text{Con}_h(\sigma) \). In this homogeneous part, \( q + (q-h) = n \). Then we use the hard Lefschetz theorem to obtain that \( v^h \) is reduced to an isomorphism on the cohomology \( H^{2p}(X; \mathbb{Q}) \), where \( p = n - q \) (this is the key in this approach). Thus it suffices to prove that \( v^h \circ \text{Con}_h \) sends cohomologically trivial \( p \)-cycles of \( V^h \) to cohomologically trivial \( p \)-cycles of \( X \). By displaying the construction of \( \text{Con}_h \), we conclude that indeed there is a formula

\[
v^h \circ \text{Con}_h(\sigma) = \sigma \quad (1.12)
\]

on \( Z^s_p(V^h) \). Therefore (1.12) does not only prove \( \text{Con}_h \) is cohomological on this particular homogeneous part but also continues with (1.9) to yield that \( \langle \text{Con}_h \rangle \) is the inverse of \( u^h \) when restricted to \( A_p(V^h) \). To connect the cohomology of \( V^h \) with that of \( X \), we use the same deformation in (1.5), where the parameter is in \( \mathbb{P}^1 \). But this time we use the real deformation or homotopy for the parameter \( t \) in the real axis \([0, 1]\). It yields the isomorphism

\[
A^p(V^h) \simeq A^q(X).
\]

(1.13)
So we conclude that the composition of maps in the following is the inverse of $u^h_0$,

$$A^q(X) \xrightarrow{\text{homotopy}} A^p(V^h) \xrightarrow{\langle\text{Con}_h\rangle} A^p(X) \xrightarrow{\text{rational equi}} A^p(X) \quad (1.14)$$

where the two ends are isomorphisms. The composition will also be denoted by $\langle\text{Con}_h\rangle$.

In the following section we give the details. In subsection 2.1, we decompose the diagonal $\Delta_X$ to have the initial construction (1.7). It breaks down to 3 parts: (1) the finite end-cycle of the rational equivalence ($m \cdot id_X$ in (1.8)); (2) the other end-cycles at $\infty$ of the rational equivalence ($\Phi_1 \otimes$ in (1.8)); (3) the cycle $I^h_\infty$ in $V^h \times X$, that gives the factorization of $\Phi_1 \otimes$ in (1.8). In subsection 2.2, using intersection theory, we calculate infinite end-cycles which involves non-correspondence operator ($\Phi_1 \otimes$). In subsection 2.3, using intersection theory, we calculate finite end-cycle. In subsection 2.4, we combine the calculations to conclude the proof of Main theorem.

# Cone construction

## 2.1 Decomposition of the diagonal

Let $X$ be a smooth projective variety over $\mathbb{C}$ with dimension $n$. Let $P^{n+1}$ be a projective space such that there is a birational morphism to a hypersurface

$$\mu : X \to P^{n+1}. \quad (2.1)$$

Let $0 < h < n$ be an integer. Let

$$P^{n+1-h} \subset P^{n+1}$$

be a fixed, $h$-codimensional subspace of $P^{n+1}$. Let $P^{h-1}$ be a generic summand of $P^{n+1-h}$ in $P^{n+1}$, i.e.

$$P^{n+1} = P^{n+1-h} \# P^{h-1}. \quad (2.2)$$

where $\#$ is the join operator in the projective space. Use $z$ to denote the point in the Grassmannian representing the subspace $P^{h-1}$. Let $Y \subset P G(h, n+1)$ be a generic curve of the Grassmannian. Let $U \subset Y$ be an open set consisting of those $z$ whose represented $P^{h-1}$ is disjoint with the fixed $P^{n+1-h}$. Let

$$\pi : P^1 \times X \times X \to P^1 \times P^{n+1} \times P^{n+1}$$

\[(t, x, y) \to (t, \mu(x), \mu(y)). \quad (2.3)\]
Let \( t = [t_0, t_1] = \mathbb{P}^1 \) be the homogeneous coordinates, for which we denote \([0, 1]\) by \( \infty \) and \([1, 1]\) by 1. Next we define a subvariety depending on \( z \in \mathcal{U} \),

\[
\Omega^z = \{(t_0, t_1), [t_1 \beta_1, t_0 \beta_0], [\beta_1, \beta_0], ) : [\beta_0] \in \mathbb{P}^{h-1}, [\beta_1] \in \mathbb{P}^{n+1-h}\}
\]

where \( \beta_0, \beta_1 \) are affine coordinates for \( \mathbb{C}^h, \mathbb{C}^{n+1-h} \). Let

\[
\Omega = \bigcup_{z \in \mathcal{U}} \Omega^z
\]

be the subvariety of the union. Then the fibre of \( \Omega \) over any \((z, t) \in \mathcal{U} \times \mathbb{P}^1\) is denoted by \( \Omega^z \), the fibre over \( z \in \mathcal{U} \) is denoted by \( \Omega^z \), etc.

**Lemma 2.1.**

1. Let \( z \in \mathcal{U} \). Then \( \Omega^z \) is a variety of dimension \( n+2 \), and \( \Omega^z_\infty \), the fibre over \( \infty \) has two components \( L_h, L_{n-h+1} \) of dimension \( n+1 \).

2. The spread \( \Omega = \bigcup_z \Omega^z \) is a variety of dimension \( n+3 \) and \( \Omega_\infty = \bigcup_z \Omega^z_\infty \) is a variety of dimension \( n+2 \).

**Proof.** (1) Let \([x_0, \ldots, x_{n+1}]\) be the homogeneous coordinates of \( \mathbb{P}^{n+1} \). Let \([t_0, t_1]\) be the homogeneous coordinates for \( \mathbb{P}^1 \). Let

\[
[x_0, \cdots, x_{n+1-h}], [x_{n-h+2}, \cdots, x_{n+1}]
\]

be the homogeneous coordinates for \( \mathbb{P}^{n+1-h} \) and \( \mathbb{P}^{h-1} \) in \( \mathbb{P}^{n+1} \). Use \( x, y \) for the homogeneous coordinates of the first and the second copies of \( \mathbb{P}^{n+1} \) in the product

\[
\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}.
\]

Use \( z \in \mathcal{U} \) to denote the point in the Grassmanian corresponding to \( \mathbb{P}^{h-1} \). Then the subvariety

\[
\Omega^z \subset \mathbb{P}^1 \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}
\]

can be explicitly defined by the following equations.

\[
x_i y_j = x_j y_i, 0 \leq i, j \leq n + 1 - h \\
x_i y_j = x_j y_i, n - h + 2 \leq i, j \leq n + 1 \\
x_i y_j t_1 = y_i x_j t_0, 0 \leq j \leq n + 1 - h, n - h + 2 \leq i \leq n + 1.
\]

Let \( \Omega^z_t \) be the fibre over \( t \in \mathbb{P}^1 \) which is isomorphic to \( \mathbb{P}^{n+1} \). Hence \( \Omega^z_t \) is the diagonal, but the scheme

\[
\Omega^z_\infty
\]

is split into two components. It can be explicitly defined by
\[ x_iy_j = x_jy_i, 0 \leq i, j \leq n + 1 - h \]
\[ x_iy_j = x_jy_i, n - h + 2 \leq i, j \leq n + 1 \]
\[ x_iy_j t_1 = 0, 0 \leq j \leq n + 1 - h, n - h + 2 \leq i \leq n + 1. \] (2.7)

Any component with at least one of
\[ x_i \neq 0, n - h + 2 \leq i \leq n + 1 \]
will have all \( y_j = 0, 0 \leq j \leq n - h + 1. \) Thus there are two types of components of \( \Omega_\infty \) in \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \)
(a) \( L_h, \) a subvariety defined by
\[ x_iy_j = x_jy_i, 0 \leq i, j \leq n + 1 - h \]
\[ x_k = 0, n - h + 2 \leq k \leq n + 1. \] (2.8)

(b) \( L_{n-h+1}, \) a subvariety defined by
\[ x_iy_j = x_jy_i, n - h + 2 \leq i, j \leq n + 1 \]
\[ y_k = 0, 0 \leq k \leq n + 1 - h. \] (2.9)

It is easy to see the dimensions of \( L_h, L_{n-h+1} \) are both \( n + 1, \) and \( L_{n-h+1} \) depends on \( z \in U. \)
(2) We observe the projective curve \( \Upsilon. \) There is a universal family
\[ \Omega' \subset U \times \mathbb{P}^1 \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}. \] (2.10)
Notice that \( \Omega' \) is surjective to \( \Upsilon \times \mathbb{P}^1 \) and generically 1-to-1 to its image in \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}. \) Then all the assertions on the dimensions follow.

\[ \square \]

Remark There is no need to describe more complicated \( \Omega', z \notin U, \) which requires details of the boundary scheme \( \Upsilon - U \) and is the source of the “bad” components.

To transform the cycles, we use the following type of intersection (or equivalently the correspondence).

**Definition 2.2.**
(1) Let
\[ \phi : W_1 \rightarrow W_2 \]
be a regular map between two smooth projective varieties over \( \mathbb{C}. \) Let \( G_\phi \) be the graph of \( \phi \) in
\[ W_1 \times W_2. \]

Let
\[ P_{W_2} : W_1 \times W_2 \rightarrow W_1 \]
be the projection. Let $A, B$ be algebraic cycles on $W_1$ and $W_2$ respectively. Assume the intersection

$$(A \times B) \cap G_{\phi}$$

is proper. We denote the algebraic cycle

$$(P_{W_1})_*((A \times B) \cdot G_{\phi})$$

in $Z(W_1)$ by $A \cdot_{\phi} B$, called intersection of $A$ and $B$.

(2) If $\phi$ is the identity, we simplify $A \cdot_{\phi} B$ to $A \cdot B$, and leave the notations for the ambient spaces $W_1, W_2$ in the context.

The definition coincides with that in §6, [1], where arguments on multiplicities imply that all formulas there on Chow groups should work the same way in $Z$ groups, as long as there is no excessive intersection.

**Lemma 2.3.** $\{1\} \times \Delta_X$ is a distinguished variety of the intersection

$$(P^1 \times X \times X) \cdot_{\pi} \Omega^z,$$  \hspace{1cm} (2.11)

for each $z$, where $\Delta_X$ is the diagonal of $X \times X$.

**Proof.** We may assume $z \in U$. Notice the $\mu(X)$ is a hypersurface of $P^{n+1}$. Thus $\text{image}(\pi)$ is a complete intersection of

$$P^1 \times P^{n+1} \times P^{n+1}$$

of codimension 2. The intersection

$$\pi(P^1 \times X \times X) \cap \Omega^z$$

is proper for a generic choice of birational map $\mu$. Since the dimension of $\Omega^z$ is $n + 2$, the cycle

$$(P^1 \times X \times X) \cdot_{\pi} \Omega^z$$

(2.12)

has dimension $n$. It is straightforward that $\{1\} \times \Delta_X$ is contained in

$$(P^1 \times X \times X) \cdot_{\pi} \Omega^z.$$  \hspace{1cm} (2.13)

Due to its dimension it must be a component of the cycle

$$(P^1 \times X \times X) \cdot_{\pi} \Omega^z.$$  \hspace{1cm} (2.14)
Definition 2.4.

(1) Let $z$ be generic in $U$. Let $a$ be the multiplicity of $\{1\} \times \Delta_X$ in

$$
(P^1 \times X \times X) \cdot \pi \Omega^2.
$$

(2.15)

Defined $\Theta^z$ to be the cycle

$$
(P^1 \times X \times X) \cdot \pi \Omega^2 - a(\{1\} \times \Delta_X)
$$

(2.16)

in $P^1 \times X \times X$.

(2) For each such $a$, we define $\Theta^z_t, t \in P^1$ to be the fibre over the point $t$ in the projection

$$
\Theta^z \rightarrow P^1.
$$

In particular,

$$
\Theta^z_\infty = (X \times X) \cdot \mu^2 (L_h + L_{n-h+1}),
$$

(2.17)

where $\mu^2$ is the map

$$
\mu^2 : X \times X \rightarrow P^{n+1} \times P^{n+1}
$$

$$(x,y) \rightarrow (\mu(x), \mu(y)).
$$

(2.18)

Furthermore we defined $\Phi_1^z, \Phi_2^z$ to be the algebraic cycle

$$
(X \times X) \cdot \mu^2 L_h
$$

$$
(X \times X) \cdot \mu^2 L_{n-h+1},
$$

(2.19)

and $\Phi_1, \Phi_2$ be the spreads of $\Phi_1^z, \Phi_2^z$.

Definition 2.5. (the operator $(\cdot)_\sigma$)

Let $Y, X, Y$ be three smooth projective varieties. Let $J^z$ for generic $z \in Y$ be a family of algebraic cycles in $Y \times X$. Let $\sigma$ be an algebraic cycle in $Y$ such that

$$
J^z \cap (\sigma \times X)
$$

(2.20)

is proper for generic $z \in Y$. Define

$$
J_\sigma(\sigma)
$$

(2.21)

to be the spread of the family

$$(P_X)_* (J^z \cdot (\sigma \times X))
$$

(2.22)

over $z$, where $P_X : Y \times X \rightarrow X$ is the projection. Hence the family $J^z$ for generic $z$ gives a homomorphism,

$$
J_\sigma : Z^*(Y) \rightarrow Z(X)
$$

$$
\sigma \rightarrow J_\sigma(\sigma)
$$

where the superscript $s$ means the subgroup of cycles meeting $P_X)_* (J^z)$ properly for generic $z$. 

Remark We should note that $\mathcal{J}_\circ$ is obtained from the usual correspondence $\mathcal{J}$, where $\mathcal{J}$ is the spread $\bigcup_z \mathcal{J}$. So it is not expected that it'll always respect adequate equivalence relations on cycles.

Let $\sigma$ be an algebraic cycle in $X$ such that $\sigma$ meeting $V^h$ properly. We call $(\Theta_\infty)_\circ(\sigma)$ the infinite end-cycle and $(\Theta_1)_\circ(\sigma)$ the finite end-cycle.

**Proposition 2.6.** (Rational equivalence). For $z \in U$

\[
(\Theta_\infty)_*(\sigma) \quad (\Theta_1)_*(\sigma)
\]

are rationally equivalent, and furthermore two end-cycles are rationally equivalent.

**Proof.** Let $\sigma$ be a cycle meeting $V^h$ properly. Then $\Phi_1^z$ meets $\sigma \times X$ properly for $z \in U$. The first statement is the definition of the intersection. For the second statement, we let $\Psi$ be the spread of the family

\[
\Theta^z \cdot (P^1 \times \sigma \times X)
\]

over $z$. Let $r(1)$ be its fibre of

\[
\Psi \rightarrow P^1
\]

over $1 \in P^1$. A direct formula for $\Theta^z$ shows that the boundary

\[
|\Psi| - \left( \left( \bigcup_{z \in U} \Theta^z \right) \cap (P^1 \times \sigma \times X) \right)
\]

is not contained in $r(1)$. Similarly the boundary is not contained in the fibre $r(\infty)$ over $\infty$. Therefore the spreads of

\[
(\Theta_\infty)_*(\sigma) \quad (\Theta_1)_*(\sigma)
\]

are rationally equivalent. \hfill \square

### 2.2 The infinite end-cycle of the rational equivalence

**The sliced cone**

Let $V^h = X \cdot P^{n+1-h} \subset X$ be a generic $h$-codimensional plane section of $X$. Recall

\[
f_h : P^{n+1} \rightarrow P^{n+1-h}
\]
is the rational projection with infinity $\mathbb{P}^{h-1}$.

Define $I_z^h$ for $z \in U$ to be the closure

$$
\{(x_1, x_2) : \mu(x_1) = f_h \circ \mu(x_2), \mu(x_2) \notin \mathbb{P}^{h-1}, \} \tag{2.28}
$$

in $V^h \times X$. By the definition, for each $z \in U$, $I_z^h$ is embedded into $X \times X$ as the cycle $\Phi_z^2$. We should note that $z$ varies the map $f_h$.

**Proposition 2.7.** Let $\sigma$ be an algebraic cycle in $X$ that meets $V^h$ properly. Then

$$(I_z^h)_*(V^h \cdot \sigma) = (\Phi_z^2)_*(\sigma) \tag{2.29}$$

**Proof.** Let

$$V^h \times X \xleftarrow{i} V^h \times X \xrightarrow{j} X \times X \tag{2.30}$$

be the embeddings. By the associativity in 8.1.1 (a), [1],

$$
(I_z^h \cdot (V^h \times X)) \cdot_j (\sigma \times X) = I_z^h \cdot_i \left( (V^h \cdot (\sigma \times X)) \right) \tag{2.31}
$$

in $Z(V^h \times X) \times_j (X \times X)$. After the projection to $X \times X$, the left hand side is the intersection

$$\Phi_z^2 \cdot (\sigma \times X). \tag{2.32}$$

After the projection to $V^h \times X$, the right hand side is

$$I_z^h \cdot ((V^h \cdot \sigma) \times X). \tag{2.33}$$

Therefore their projections to the second factor $X$ are the same. This completes the proof.

The vanishing cone $(\Phi_2)_\oplus(\sigma)$

**Proposition 2.8.** Let $\sigma$ be a homogeneous algebraic cycle on $X$ such that the intersection in operation $|\sigma| \cap V^h$ is proper. If $\text{dim}(\sigma) > h$,

$$(\Phi_2)_\oplus(\sigma) = 0.$$
Proof. By the definition, if it is non-zero, \((\Phi_2)_{\otimes}(\sigma)\) is a cycle supported on an algebraic set \([P_2_*(X \times L_{n-h+1})]\) of dimension \(h\), where \(P_2 : X \times X \to X\) is the projection to the second factor. Hence \(\dim((\Phi_2)_{\otimes}(\sigma)) \leq h\). On the other hand, for the operations of intersection and projection, the dimension is determined unless the resulting cycle is zero. Notice that
\[
(\Phi_2)_{\otimes}(\sigma) = \cup_z (P_2)_*(\Phi_z \otimes (\sigma \times X)).
\] (2.34)
Since \(\dim(\Phi_z) = n - 1\), if \((\Phi_2)_{\otimes}(\sigma) \neq 0\), since the intersection in (2.34) is proper,
\[
\dim((\Phi_2)_{\otimes}(\sigma)) = \dim(\sigma) > h.
\] (2.35)
This contradiction says \((\Phi_2)_{\otimes}(\sigma)\) must be zero.

We complete the proof. \(\Box\)

**Definition 2.9.**

We define a linear operator
\[
\text{Con}_h : \sum_{q > \frac{h}{2}} Z_q^{n-h}(V^h) \to \sum_{q > \frac{h}{2}} Z_q^n(X)
\]
\[
\sigma \to (I_h)_{\otimes}(\sigma)
\] (2.36)
to be the cone operator.

Using Propositions 2.7 and 2.8, we obtain

**Corollary 2.10.** Let \(\sigma\) be a cycle meeting \(V^h\) properly. The infinite end-cycle
\[
(\Theta_\infty)_{\otimes}(\sigma) = (\Phi_1)_{\otimes}(\sigma)
\]
which is the spread of the family \((I_z^\infty)_*(V^h \cdot \sigma)\) over the \(z\) in \(U\). Furthermore the infinite end-cycle can be factorized as
\[
\text{Con}_h \circ v^h(\sigma).
\] (2.37)

### 2.3 The finite end-cycle of the rational equivalence

**Proposition 2.11.** Let \(\sigma\) be an algebraic cycle in \(X\) meeting \(V^h\) properly. For a generic curve \(Y\) in the Grassmannian of subspaces \(\mathbb{P}^{h-1}\), the spread
\[
(\Theta_1)_{\otimes}(\sigma) = m\sigma
\] (2.38)
where \(m\) is a natural number.
\section{Cone Construction}

\begin{proof}
First we observe the spread of $\Theta^1_z$. Because the map $\mu^2$ is generically 1-to-1, the intersection of algebraic cycles (not classes) in $\Theta^1_z$ can be pushed forward to the projective spaces
\[ P^{n+1} \times P^{n+1}, \]
where the intersection exists as an algebraic cycle in the $Z$ group. Notice $\mu(X)$ is a hypersurface of $P^{n+1}$. Assume $\mu(X)$ is defined by a polynomial $f$. Then $(\mu^2)_*(X \times X)$ is a complete intersection defined by two polynomials $f(x), f(y)$ in
\[ P^{n+1} \times P^{n+1}. \]
Inside of
\[ \mathbb{P}^1 \times P^{n+1} \times P^{n+1} \]
there is the closure of the scheme
\[ \Sigma = \{ \{ t \} \times \cup_z (\Omega^1_t \cap \mu^2(X \times X)) : t \neq 1 \} \]
where $\cup$ is defined as the spread of a family of algebraic cycles. Then we have the projection
\[ \Sigma \to \mathbb{P}^1. \]

The push-forward
\[ \pi_*(\cup_z \Theta^1_z) \]
is the specialization $\Sigma_1$ of a generic fibre $\Sigma_t, t \neq 1$ at 1. Let's use the homogeneous coordinates $[\beta_1, \beta_0]$ for $P^{n+1}$ as before in (2.4). For $t \neq 1$, $\Theta^1_z$ is a smooth projective variety isomorphic to $P^{n+1}$ expressed in coordinates as
\[ \{(t[\beta_1, \beta_0]) \times [\beta_1, \beta_0]\} \subset P^{n+1} \times P^{n+1}. \]
Then $\Sigma^1_z$ is explicitly defined by
\[ f(t\beta_1, \beta_0) = f(\beta_1, \beta_0) \]
inside of $\Omega^1_z \simeq P^{n+1}$. As $t \to 1$, we observe the expansion
\[ f(t\beta_1, \beta_0) - f(\beta_1, \beta_0) = (t - 1)^r g^z_r(\beta_1, \beta_0) + (t - 1)^{r+1} g^z_{r+2}(\beta_1, \beta_0) + \cdots. \]
Then the specialization $\Sigma^1_z$ as $t \to 1$ in $\Delta_{P^{n+1}}$ of coordinates $[\beta_1, \beta_0]$ is defined by two polynomials
\[ f(\beta_1, \beta_0) = g^z_r(\beta_1, \beta_0) = 0. \]
Therefore $\Sigma_1$ is the spread over $z \in U$, of a family of hypersurfaces $\{g^z_r = 0\}$ of the divisor
\[ \{ f = 0 \} \simeq X \]
in the diagonal
\[ \Delta_{P^{n+1}} \simeq P^{n+1}. \]
\end{proof}
Hence $\Sigma_1$ is an integral multiple of the divisor $\{f = 0\} \subset \Delta_{\mathbb{P}^{n+1}}$. Now we consider the cycle $\sigma$. By the argument above,

$$\mu_* ((\Theta^1_1)_*(\sigma))$$

is just a Cartier divisor of $\mu(\sigma)$ depending on $z$. Taking the spread, we obtain the assertion of the proposition, i.e.

$$(\Theta^1_1)_*(\sigma) = m\sigma,$$ \hspace{1cm} (2.43)

where $m$ is a natural number.

$\square$

### 2.4 The final proof

**Proof.** of Main theorem: The calculations in previous sections showed that there is a sequence,

$$\sum_{q > \frac{n}{2}} Z^h_q(X) \xrightarrow{\nu^h} \sum_{q > \frac{n}{2}} Z_{q-h}(V^h) \xrightarrow{\text{Con}_h} \sum_{q > \frac{n}{2}} Z^h_q(X),$$ \hspace{1cm} (2.44)

where the superscript $s$ means the subspace of cycles meeting $V^h$ properly. The maps satisfy

$$\frac{1}{m} \text{Con}_h \circ \nu^h \xrightarrow{\text{rational equi}} \text{id}_X.$$ \hspace{1cm} (2.45)

To reduce the sequence (2.44) to the cohomology, we have a couple of steps where the main idea is to have the restriction to a particular homogeneous part of the total cohomology. For this we let

$$q = \frac{n + h}{2}.$$ \hspace{1cm} (2.46)

The first step is to show the map

$$Z_{q-h}(V^h) \xrightarrow{\text{Con}_h} Z^s_q(X)$$

is cohomological, i.e sends a cohomologically trivial cycle to a cohomologically trivial cycle. So we let $\delta \in Z^s_p(V^h)$ be a cycle cohomologous to zero in $V^h$, where $p = n - q$. The assumption (2.46), which is the assumption of Main theorem, allows us to apply the hard Lefschetz theorem. It yields that the map $\nu^h$ is reduced to an isomorphism on the homogeneous part $H^p(X; \mathbb{Q})$ of the cohomology. Then it is sufficient to prove

$$\nu^h \circ \text{Con}_h(\delta)$$
is cohomologous to zero in $X$. To continue, we calculate the intersection of varieties in $V^h \times X$.

$$I_h^\alpha \cap \left( (V^h \cap |\delta|) \times X \cap (V^h \times V^h) \right)$$

$$= I_h^\alpha \cap \left( (V^h \cap |\delta|) \times V^h \right) \cap \Delta_{V^h \cap |\delta|}$$

where $\Delta_{V^h \cap |\delta|}$ is the diagonal of $V^h \cap |\delta|$. Hence $v^h \circ Con_h(\delta)$ is supported on the cycle $|\delta|$. If $\delta$ is prime (i.e. an irreducible subvariety), so is $v^h \circ Con_h(\delta)$. Then because $\dim(v^h \circ Con_h(\delta)) = \dim(\delta)$, we must have

$$v^h \circ Con_h(\delta) = l\delta.$$  \hspace{1cm} (2.48)

for some integer $l$. Next we would like to show the multiplicity $l$ only depends on the choice of $\Upsilon$. In general each curve $\Upsilon$ gives subvariety

$$\Gamma \subset \Upsilon \times \mathbb{P}^{n+1-h} \times \mathbb{P}^{h-1} \times \mathbb{P}^{n+1}$$  \hspace{1cm} (2.49)

Let

$$Proj : \Gamma \rightarrow \mathbb{P}^{n+1-h} \times \mathbb{P}^{h-1} \times \mathbb{P}^{n+1}$$

be the projection. By the construction, $\dim(\Gamma) = n + 1$ and

$$\dim(\text{Image}(Proj)) = n + 1.$$  

Hence the map $Proj : \Gamma \rightarrow \text{Image}(Proj)$ is a covering map. Then the degree of $Proj$ is the multiplicity $l$. Let’s give this a detailed proof for $l = 1$. To see this, we choose a special $\Upsilon$. Let $\Upsilon$ give the $Proj$ of degree 1. Then the corresponding map

$$\eta : \Upsilon \times \mathbb{P}^{n+1-h} \times \mathbb{P}^{h-1} \rightarrow \mathbb{P}^{n+1}$$

must be generically 1-to-1, i.e an isomorphism on an open set. Then the intersection above (the spread of (2.47))

$$(\cup_{z} I_h^\alpha) \cap \left( (\delta \cap V^h) \times V^h \right)$$

can be adjusted to be generically transversal (for instance by varying $\delta$ in the same cohomology class). Therefore the multiplicity $l = 1$. From now on we use such a $\Upsilon$ with $l = 1$. Therefore $m$ from (2.43) is also 1. Next we let $\delta$ be non prime. Then by the linearity,

$$v^h(Con_h(\delta)) = \delta.$$  \hspace{1cm} (2.52)

Hence if $\delta$ is cohomologous to zero in $V^h$, so is $Con_h(\delta)$ in $X$. This shows that $Con_h$ is cohomological. Then (2.44) is reduced to the cohomology. Therefore we have

$$A^p(X) \xrightarrow{v^h} A^p(V^h) \xrightarrow{\langle Con_h \rangle} A^p(X)$$  \hspace{1cm} (2.53)
such that
\[ \langle \text{Con}_h \rangle \circ u_h^m = \text{id}, \text{ by } (2.45) \] (2.54)
where \( m = 1 \), and
\[ u_h^m \circ \langle \text{Con}_h \rangle = \text{id}, \text{ by } (2.52). \] (2.55)

So we have chosen a particular \( \Upsilon \) such that \( \langle \text{Con}_h \rangle \) is the inverse of \( u_a^{q-p} \) when restricted to \( V^h \), i.e.
\[ A^p(V^h) \simeq A^p(X). \] (2.56)

At last (the 2nd step) we consider any \( p \) dimensional algebraic cycle \( \sigma \) in \( X \). We may choose all \( h - 1 \) subspaces from \( \Upsilon \) are disjoint with \( \sigma \) (this requires \( p \neq 0 \)). Using the same deformation \( \Theta_t \) for the real parameter \( t \in [0,1] \), we obtain a homotopy for the spread
\[ (\Theta_t)_\ast(\sigma), \text{ for } t \in [0,1]. \] (2.57)

At \( t = 1 \), we have the identity as before. At \( t = 0 \), for each generic \( z \), the cycle \( (\Theta_0)_\ast(\sigma) \) lies in \( V^h \). Therefore
\[ (\Theta_0)_\ast(\sigma) = \bigcup_z \left( (\Theta_0^z)_\ast(\sigma) \right), \] (2.58)
must lie in \( V^h \). This shows that \( \sigma \) is homotopic to an algebraic cycle lying in \( V^h \). Therefore
\[ A^p(V^h) \simeq A^p(X). \] (2.59)

We complete the proof of Main theorem.

References

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