VERSAL DEFORMATION THEORY OF ALGEBRAS OVER A QUADRATIC OPERAD

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Abstract. We develop versal deformation theory of algebras over quadratic operads where the parameter space is a complete local algebra and give a construction of a distinguished deformation of this type—the so-called ‘versal deformation’—which induces all other deformations of a type of algebras.

1. Introduction

Formal one-parameter deformation theory for algebras was originally introduced for associative algebras by M. Gerstenhaber in the 60’s (see [9, 10]). Since then it has been applied to many other algebraic categories. Most of these cases turned out to be algebras over a suitable quadratic operad. Gerstenhaber’s theory was generalized to an algebra over a quadratic operad by D. Balavoine in 1997 (see [2]), and further developed by M. Kontchevich and Soibelman (see [14]).

Classical deformation theory is not general enough to describe all nonequivalent deformations of a given object. To take care of this, one needs to enlarge the base of deformations from one parameter power series ring to a local commutative algebra, or more generally, to a complete local algebra. It is known that under certain cohomology restrictions in this general setup there exists a "characteristic" versal deformation with complete local algebra base, which induces all nonequivalent deformations and is universal at the infinitesimal level (see e.g. [22, 6]). Versal deformation has been constructed for Lie [4, 6], associative, infinity [7] and Leibniz algebras [8].

The aim of this paper is to give a construction of versal deformation for algebras over a quadratic operad. Our approach gives a unified treatment for algebras over quadratic operads like Comm, Ass, Lie, Leib, Zinb, Dend, Dias, see [15]. The structure of the paper is as follows. In Section 2 we review the necessary definitions of operads, algebras over an operad and cohomology of an algebra over an operad (operadic cohomology) with coefficients in itself. In Section 3 we develop deformation theory of algebras over an operad with local commutative algebra base. In Section 4 we study infinitesimal deformations and their properties. In Section 5 we develop -
using cohomology - the necessary obstruction theory for extending a deformation with a given base to a larger base. We prove that obstructions are 2-cocycles, and that a given deformation can be extended if the obstruction cocycles vanish. In Section 6 we introduce the notion of formal deformation over a complete local algebra base and define the notion of versal deformation. We also give a construction of versal deformation of an algebra over a quadratic algebra.

2. Preliminaries on operads and operadic cohomology

In this section we recall some basic definitions and results about (algebraic) operads [18, 19, 20, 21] and the operadic cohomology of an algebra over a quadratic operad with coefficients in itself [1, 2].

The symbol \( \mathbb{N} \) denotes the set of positive integers. Throughout this paper, we work over a fixed field \( k \) of characteristic zero. \( \text{Vect} \) denotes the category of vector spaces over \( k \). The tensor product of vector spaces over \( k \) is denoted by \( \otimes \). For any positive integer \( n \), \( S_n \) denotes the group of permutations on \( n \) elements. For \( \sigma \in S_n \), \( \epsilon(\sigma) \in \{-1,1\} \) stands for the sign of \( \sigma \), and \( \text{sgn}_n \) denotes the sign representation of \( S_n \). For any map \( f : \otimes^n i = 1 E_i \rightarrow F \), \( f(x_1, \cdots, x_n) \) will stand for \( f(x_1 \otimes \cdots \otimes x_n) \).

2.1. \( S \)-module.

**Definition 2.2.** An \( S \)-module over \( k \) is a family

\[
M = \{M(0), M(1), \ldots, M(n), \ldots\}
\]

of right \( k[S_n] \) modules \( M(n) \). An \( S \)-module \( M \) is finite dimensional if \( M(n) \) is finite dimensional for all \( n \). A morphism \( f : M \rightarrow N \) between two \( S \)-modules \( M \) and \( N \) is a family of maps \( f_n : M(n) \rightarrow N(n) \) which are \( S_n \) equivariant for all \( n \).

Note that to every \( S \)-module \( M \) is associated a functor, called the *Schur functor* \( \widetilde{M} : \text{Vect} \rightarrow \text{Vect} \) such that \( \widetilde{M}(V) = \oplus_{n \geq 0} M(n) \otimes_{S_n} V^\otimes n \), where the left action of \( S_n \) on \( V^\otimes n \) is given by \( \sigma(a_1, a_2, \ldots, a_n) = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots, v_{\sigma^{-1}(n)}) \).

**Tensor product** of two \( S \)-modules is the \( S \)-module \( M \otimes N \) defined by

\[
(M \otimes N)(n) = \oplus_{i+j=n} \text{Ind}_{S_i \times S_j}^{S_n} M(i) \otimes N(j).
\]

**Composite** of two \( S \)-modules \( M \) and \( N \) is the \( S \)-module

\[
M \circ N = \oplus_{n \geq 0} M(n) \otimes N^\otimes n,
\]

where \( N^\otimes n \) is the tensor product of \( n \) copies of the \( S \)-module \( N \).
With respect to this composite product the category $(\mathcal{S}\text{-Mod}, \circ, I)$ is a monoidal category, where $I$ is the $\mathcal{S}$-module $(0, k, 0, \ldots)$.

2.3. Operads and cohomology.

Definition 2.4. A symmetric operad $\mathcal{P} = (\mathcal{P}, \gamma, \eta)$ is a monoid in the monoidal category $(\mathcal{S}\text{-Mod}, \circ, I)$. In other words, a symmetric operad is a $\mathcal{S}$-module $\mathcal{P} = \{\mathcal{P}(n), n \geq 0\}$ with morphism of $\mathcal{S}$-modules

$$\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$$

called the composition map and

$$\eta : I \longrightarrow \mathcal{P}$$

called the unit map such that the Schur functor $\mathcal{P}$ becomes a monoid.

More explicitly, a $k$-linear operad is a sequence $\mathcal{P} = \{\mathcal{P}(n), n \in \mathbb{N}\}$ of vector spaces over $k$ such that

1. Each $\mathcal{P}(n)$ is a right $k[\Sigma_n]$-module.
2. For all $n \geq 1$ and $1 \leq i \leq n$, there exist linear maps

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \rightarrow \mathcal{P}(n + m - 1),$$

such that the following identities hold: for $l, m, n \in \mathbb{N}$, $\lambda \in \mathcal{P}(l)$, $\mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$,

$$(\lambda \circ_i \mu) \circ_j \nu = \begin{cases} 
(\lambda \circ_j \nu) \circ_{i+n-1} \mu & \text{if } 1 \leq j \leq i - 1, \\
\lambda \circ_i (\mu \circ_{j-i+1} \nu) & \text{if } i \leq j \leq m + i - 1, \\
(\lambda \circ_{j-m+1} \nu) \circ_i \mu & \text{if } i + m \leq j.
\end{cases} \quad (2.4.1)$$

Furthermore, the following equivariant conditions hold: for all $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$ and $i \in \{1, \ldots, m\}$,

$$(\mu \circ_i \nu)(\sigma \circ_i \tau) = \mu \sigma \circ_{\sigma^{-1}(i)} \nu \tau;$$

where $\sigma \circ_i \tau = \hat{\sigma} \circ (1 \times \cdots \times \tau \times \cdots \times 1)$ ($\tau$ is in the $i$th place), and $\hat{\sigma}$ permutes the $m$ blocks corresponding to $1 \times \cdots \times \tau \times \cdots 1$ in the same way as $\sigma$ permutes $\{1, \ldots, m\}$.

3. There exists a unit element $1 \in \mathcal{P}(1)$ such that for all $n \in \mathbb{N}$, $\mu \in \mathcal{P}(n)$, and $1 \leq i \leq n$, $\mu \circ_i 1 = \mu$ and $1 \circ_1 \mu = \mu$. 
Definition 2.5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. A morphism of operads from $\mathcal{P}$ to $\mathcal{Q}$ is a sequence $a = \{a(n), n \in \mathbb{N}^*\}$ of $k[\Sigma_n]$-linear maps $a(n): \mathcal{P}(n) \to \mathcal{Q}(n)$ satisfying the conditions, $a(1)(1) = 1$ and

$$a(n + m - 1)(\mu \circ_i \nu) = a(n)(\mu) \circ_i a(m)(\nu)$$

for $n, m \in \mathbb{N}$, $1 \leq i \leq n$, $\mu \in \mathcal{P}(n)$, and $\nu \in \mathcal{P}(m)$.

Example 2.6. Let $V$ be a vector space over $k$ and for every $n \in \mathbb{N}$ let $\text{End}(V)(n) = \text{Hom}_k(V^\otimes n, V)$. Then $\text{End}(V) = \{\text{End}(V)(n), \ n \in \mathbb{N}\}$ is naturally an operad, called the endomorphism operad of $V$. The right action of $\sigma \in \Sigma_n$ on $f \in \text{End}(V)(n)$ is given by

$$(f\sigma)(a_1, \cdots, a_n) = f(v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(n)}), \ a_1, a_2, \cdots, a_n \in V.$$ 

For $1 \leq i \leq n$, $f \in \text{End}(V)(n)$ and $g \in \text{End}(V)(m)$, the $\circ_i$ operation in $\text{End}(V)$ is given by

$$f \circ_i g(a_1, \cdots, v_{n+m-1}) = f(a_1, \cdots, g(v_i, \cdots, v_{i+m-1}), v_{i+m}, \cdots, v_{n+m-1}).$$

The unit is given by $\text{id}: V \longrightarrow V$.

Definition 2.7. Let $\mathcal{P}$ be an operad. A $\mathcal{P}$-algebra or an algebra over $\mathcal{P}$, denoted by $(V, a)$, is a vector space $V$ over $k$ equipped with a morphism of operads $a: \mathcal{P} \to \text{End}(V)$.

A morphism of $\mathcal{P}$-algebras $\phi: (V, a) \longrightarrow (W, b)$ is a $k$-linear map $\phi: V \longrightarrow W$ such that for any $a_1, \cdots, a_n \in V$ and $\mu \in \mathcal{P}(n)$,

$$\phi(a(\mu)(a_1, \cdots, a_n)) = b(\mu)(\phi(a_1), \cdots, \phi(a_n)).$$

The existence of free operad is given by the following proposition [11].

Proposition 2.8. Let $E$ be a right $k[\Sigma_2]$-module. Then there exists an operad $\mathcal{F}(E)$ with $\mathcal{F}(E)(1) = k$ and $\mathcal{F}(E)(2) = E$ such that the following property holds: For any operad $Q$ and for any morphism of right $k[\Sigma_2]$-modules $a: E \rightarrow Q(2)$, there exists a unique morphism of operads, $\check{a}: \mathcal{F}(E) \rightarrow Q$, such that $\check{a}(2) = a$.

The operad $\mathcal{F}(E)$ is called the free operad generated by $E$ and is unique up to operad isomorphisms.

Definition 2.9. Let $\mathcal{P}$ be an operad. An ideal of $\mathcal{P}$ is a sequence $I = \{I(n), n \in \mathbb{N}\}$ of $k[\Sigma_n]$-submodules of $\mathcal{P}(n)$ such that, for $\mu \in \mathcal{P}(n)$, $\nu \in \mathcal{P}(m)$, $x \in I(m)$, $y \in I(n)$, $1 \leq i, j \leq n$, we have $\mu \circ_i x \in I(n + m - 1)$ and $y \circ_j \nu \in I(n + m - 1)$.

For deformation theory, we will be concerned with algebras over a special type of operads, called quadratic operads, for which Koszul duality makes sense (see [11]).
Definition 2.10. Let \( E \) be a right \( k[\Sigma_2] \)-module and \( R \) be a right \( k[\Sigma_3] \)-submodule of \( \mathcal{F}(E)(3) \). Let \( (R) \) be the ideal generated by \( R \). Then the quotient operad \( \mathcal{F}(E)/(R) \) is called the \textit{quadratic operad generated by} \( E \) with relations \( R \), denoted by \( \mathcal{P}(k, E, R) \). A quadratic operad \( \mathcal{P}(k, E, R) \) is said to be \textit{finitely generated} if \( E \) is a finite dimensional vector space.

Remark 2.11. (1) For a quadratic operad \( \mathcal{P} = \mathcal{F}(E) \) we have \( \mathcal{P}(2) = E, \mathcal{P}(3) = \mathcal{F}(E)(3)/R \); moreover, \( \mathcal{F}(E)(3) = (E \otimes E) \otimes_{\Sigma_2} k[\Sigma_3] \) where \( \Sigma_2 \) acts trivially on the first factor \( E \).

(2) The operads \( \text{Com}, \text{Ass}, \text{Lie}, \text{Poiss}, \text{Leib}, \text{Zinb}, \text{Dend}, \text{Dias} \) are examples of quadratic operads.

We shall use the following proposition (see [2]).

Proposition 2.12. Let \( \mathcal{P} = \mathcal{P}(k, E, R) \) be a quadratic operad, and let \( V \) be a vector space. A \( \mathcal{P} \)-algebra structure on \( V \) is determined by a morphism of right \( k[\Sigma_2] \)-modules \( a: \mathcal{P}(2) = E \to \text{End}(V)(2) \) such that \( \hat{a}(3)(r) = 0 \) for any \( r \in R \), where \( \hat{a} \) is the unique morphism of operads determined by \( a \).

The morphism \( a \), or equivalently its adjoint
\[
a: \mathcal{P}(2) \otimes_{\Sigma_2} V^\otimes 2 \to V,
\]
is called the \textit{structural morphism} of the \( \mathcal{P} \)-algebra \( V \).

Let us recall some facts about the Koszul duality for quadratic operads. Let \( F \) be a right \( k[\Sigma_n] \)-module. By \( F^\vee \) we mean the right \( k[\Sigma_n] \)-module
\[
F^\vee = \text{Hom}_k(F, k) \otimes \text{sgn}_n,
\]
where the right \( \Sigma_n \)-action is given by
\[
(\phi \cdot \sigma)(x) = \epsilon(\sigma)\phi(x \cdot \sigma^{-1})
\]
for \( \phi \in \text{Hom}_k(F, k) \) and \( x \in F \).

Let \( E \) be a right \( k[\Sigma_2] \)-module, and \( \mathcal{F}(E) \) denote the free operad generated by \( E \). Then as a right \( k[\Sigma_3] \)-module, we have
\[
\mathcal{F}(E^\vee)(3) \cong (\mathcal{F}(E)(3))^\vee.
\]

Let \( R \subset \mathcal{F}(E)(3) \) be a right \( k[\Sigma_3] \)-submodule, and let \( R^\perp \subset \mathcal{F}(E^\vee)(3) \) be the annihilator of \( R \) in \( (\mathcal{F}(E))(3)^\vee \equiv \mathcal{F}(E^\vee)(3) \). The \textit{Koszul dual} of the quadratic operad \( \mathcal{P} = \mathcal{P}(k, E, R) \) is defined as the quadratic operad \( \mathcal{P}^! = \mathcal{P}(k, E^\vee, R^\perp) \).

Remark 2.13. We have \( \mathcal{P}^!(2) = E^\vee \) and \( (\mathcal{P}^!)(3)^\vee \equiv R \). Furthermore, as a \( k[\Sigma_3] \)-module, \( R \) is generated by \( \mu \circ_i \nu \), where \( \mu, \nu \in \mathcal{P}(2) \) and \( i \in \{1, 2\} \).
We briefly recall the cohomology of an algebra over a finitely generated quadratic operad, due to Balavoine [1, 2]. For deformation theory we need to use the cohomology with coefficients in the algebra itself, hence we restrict our attention to this particular case.

Let $\mathcal{P} = \mathcal{P}(k, E, R)$ be a finitely generated quadratic operad. Let $V$ be a vector space. Consider the Lie algebra $L^\mathcal{P}(V)$ defined as follows. For $n \geq 0$, let $L^n\mathcal{P}(V)$ denote the vector space $L^n\mathcal{P}(V) = \mathcal{P}^! (n+1) \otimes \Sigma_{n+1} \operatorname{End}(V)(n+1)$, where $\operatorname{End}(V)(n) = \operatorname{End}(V)(n) \otimes \operatorname{sgn}_n$. The module $\operatorname{End}(V)(n)$ is the same as $\operatorname{End}(V)(n)$ equipped with the left $\Sigma_n$-action given by

$$\sigma f(a_1, \cdots, a_n) = f(\sigma^{-1}(a_1, \cdots, a_n))$$

and $\mathcal{P}^!(n)$ has the obvious right $\Sigma_n$-action. Let $L^\mathcal{P}(V)$ denote the graded vector space $L^\mathcal{P}(V) = \bigoplus_{n \geq 0} L^n\mathcal{P}(V)$.

Let $\mu^* \otimes f \in L^n\mathcal{P}(V)$ and $\nu^* \otimes g \in L^m\mathcal{P}(V)$. For $1 \leq i \leq n+1$, we have linear maps

$$\circ_i : L^n\mathcal{P}(V) \otimes L^m\mathcal{P}(V) \to L^{n+m}\mathcal{P}(V)$$

(2.13.1) defined by

$$(\mu^* \otimes f) \circ_i (\nu^* \otimes g) = (\mu^* \circ_i \nu^*) \otimes (f \circ_i g),$$

where $\mu^* \circ_i \nu^*$ on the right are the elementary operations on the operad $\mathcal{P}^!(n)$ and $f \circ_i g$ is basically the $\circ_i$ operation in the operad $\operatorname{End}(V)$. We use these linear maps $\circ_i$ in $L^\mathcal{P}(V)$ to define

$$(\mu^* \otimes f) \circ (\nu^* \otimes g) = \sum_{i=1}^{n+1} (-1)^{m(i-1)} (\mu^* \circ_i \nu^*) \otimes (f \circ_i g),$$

$$[\mu^* \otimes f, \nu^* \otimes g] = (\mu^* \otimes f) \circ (\nu^* \otimes g) + (-1)^{nm+1} (\nu^* \otimes g) \circ (\mu^* \otimes f)$$

for $\mu^* \otimes f \in L^n\mathcal{P}(V)$ and $\nu^* \otimes g \in L^m\mathcal{P}(V)$. Then we have

**Proposition 2.14.** The graded vector space $L^\mathcal{P}(V)$ equipped with the bracket $[-,-]$ is a graded Lie algebra.

Let $V$ be a $\mathcal{P}$-algebra determined by the structural morphism

$$\pi : \mathcal{P}(2) \otimes \Sigma_2 \to V \otimes^2 \to V.$$ 

To define cohomology modules of an algebra over a quadratic operad we need the following proposition.
**Proposition 2.15.** Let \((\mathcal{P}^1)^\vee(n)\) stand for \((\mathcal{P}^1(n))^\vee\). There is an isomorphism of vector spaces

\[
\Gamma : L^n_\mathcal{P}(V) \cong \text{Hom}_k((\mathcal{P}^1)^\vee(n+1) \otimes_{\Sigma_{n+1}} V^\otimes n+1, V)
\]
defined by

\[
\Gamma(\mu^* \otimes f)(\mu \otimes x) = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \epsilon(\sigma) \langle \mu, \mu^* \sigma \rangle f(\sigma x)
\]

\[
= \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \langle \mu \sigma^{-1}, \mu^* \rangle f(\sigma x)
\]

for \(\mu^* \otimes f \in L^n_\mathcal{P}(V)\) and \(\mu \otimes x \in (\mathcal{P}^1)^\vee(n+1) \otimes_{\Sigma_{n+1}} V^\otimes n+1\).

For \(n \geq 1\), define the vector space

\[
C^n_\mathcal{P}(V) = \text{Hom}_k((\mathcal{P}^1)^\vee(n) \otimes_{\Sigma_n} V^\otimes n, V) \cong L^{n-1}_\mathcal{P}(V)
\]
and the map \(\delta^n_\pi : C^n_\mathcal{P}(V) \to C^{n+1}_\mathcal{P}(V)\) by the formula

\[
\delta^n_\pi(f) = -\frac{1}{2}(n+1)\Gamma([\Gamma^{-1}(f), \Gamma^{-1}(\pi)]).
\]

Then the map \(\delta^n_\pi\) is a differential. The homology of the cochain complex \((C^*_\pi(V), \delta^n_\pi)\), denoted by \(H^*_\mathcal{P}(V)\), is called the cohomology of the \(\mathcal{P}\)-algebra \(V\) with coefficients in itself, or simply the operadic cohomology of \(V\). More generally, if \(M\) is any finite dimensional vector space, then we may consider the complex \((M \otimes C^*_\mathcal{P}(V), id \otimes \delta^n_\pi)\) and the corresponding cohomology is \(M \otimes H^*_\mathcal{P}(V)\). This version of cohomology with more general coefficients will be needed later.

**Remark 2.16.** For future reference, we recall the following special cases.

**The case \(n=1\):** Let \(f \in C^1_\mathcal{P}(V)\). For \(\mu \in \mathcal{P}(2)\), \(a_1, a_2 \in V\), the differential \(\delta^1_\pi\) is given by

\[
\delta^1_\pi(f)(\mu, a_1, a_2) = \pi(\mu, f(a_1), a_2) + \pi(\mu, a_1, f(a_2)) - f(\pi(\mu, a_1, a_2)). \quad (2.16.1)
\]

**The case \(n=2\):** Let \(f \in C^2_\mathcal{P}(V)\) and \(\overline{f} : \mathcal{F}(E)(3) \otimes_{\sigma_3} V^\otimes 3 \to V\) be the map defined by

\[
\overline{f}(\mu \circ_1 \nu, a_1, a_2, a_3) = f(\mu, \nu(a_1, a_2), a_3) + \pi(\mu, f(\nu, a_1, a_2), a_3)
\]
\[
\overline{f}(\mu \circ_2 \nu, a_1, a_2, a_3) = f(\mu, a_1, \nu(a_2, a_3)) + \pi(\mu, a_1, f(\nu, a_2, a_3));
\]
where \(\mu, \nu \in C^2_\mathcal{P}(V)\), and \(v_i \in V, i = 1, 2, 3\). Then the differential \(\delta^2_\pi(f) \in C^3_\mathcal{P}(V)\) is given by:

\[
\delta^2_\pi(f) = -\overline{f}|_{R \otimes_{\Sigma_3} V^\otimes 3}. \quad (2.16.2)
\]
3. Deformations

Let $\mathcal{R}$ be a commutative local algebra with $1_\mathcal{R}$ over $k$. Let $\epsilon : \mathcal{R} \rightarrow k, \epsilon(1) = 1$ be the canonical augmentation map, and $\mathfrak{M} = \ker(\epsilon)$ be the unique maximal ideal in $\mathcal{R}$. In this section, we introduce the notion of deformation of an algebra over a quadratic algebra with base $\mathcal{R}$ and study its properties.

Let $\mathcal{P} = \mathcal{P}(k,E,R)$ be a quadratic operad. We will denote by $\mathcal{P}_\mathcal{R}$ the operad which is obtained by extension of $\mathcal{P}$ to the category of modules over $\mathcal{R}$, in other words, $\mathcal{P}_\mathcal{R}(n) = \mathcal{R} \otimes \mathcal{P}(n)$ for all $n \in \mathbb{N}$. Let $(A,\pi)$ be a $\mathcal{P}$-algebra. Let $A_\mathcal{R} = \mathcal{R} \otimes A$ denote the extension of $A$. Then $A_\mathcal{R}$ can be viewed as a $\mathcal{P}_\mathcal{R}$-algebra by extending $\pi : \mathcal{P} \rightarrow \text{End}(A)$ to $\pi_\mathcal{R} : \mathcal{P}_\mathcal{R} \rightarrow \text{End}(A_\mathcal{R})$, since

$$\text{Hom}_\mathcal{R}(A^{\otimes \mathcal{R} n}, A_\mathcal{R}) \cong \mathcal{R} \otimes \text{Hom}_k(A^\otimes n, A).$$

Moreover, $A = k \otimes A$ can be viewed as a $\mathcal{P}_\mathcal{R}$-algebra by considering $A$ as a module over $\mathcal{R}$ via $\epsilon : r \cdot a = \epsilon(r)a$, for $r \in \mathcal{R}$ and $a \in A$.

**Definition 3.1.** A deformation $\lambda$ of a $\mathcal{P}$-algebra $(A,\pi)$ with base $(\mathcal{R},\mathfrak{M})$ is a morphism of operads $\lambda : \mathcal{P}_\mathcal{R} \rightarrow \text{End}(A_\mathcal{R})$ such that $(\epsilon \otimes \text{Id}) : A_\mathcal{R} \rightarrow k \otimes A$ is a $\mathcal{P}_\mathcal{R}$-algebra morphism.

**Remark 3.2.**

1. Since we are working with algebras over quadratic operads, a deformation $\lambda$ is determined by $\lambda(2)$ and observe that as $\lambda(2)(\mu) : A^{\otimes \mathcal{R} 2} \rightarrow A_\mathcal{R}$ is $\mathcal{R}$-linear, for all $\mu \in \mathcal{P}_\mathcal{R}(2)$,

$$\lambda(2)(\mu)\{r_1 \otimes a_1, r_2 \otimes a_2\} = r_1 r_2 \lambda(2)(\mu)\{1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2\}.$$

Thus $\lambda(2)(\mu)$ is determined by $\lambda(2)(\mu)\{1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2\}$.

2. Moreover, by $\mathcal{R}$-linearity of $\lambda(n)$, it is determined by its values on $1_\mathcal{R} \otimes \mu \in \mathcal{R} \otimes \mathcal{P}(n)$ which is identified with $\mu \in \mathcal{P}(n)$.

**Remark 3.3.** Since $\epsilon \otimes \text{Id} : A_\mathcal{R} \rightarrow k \otimes A$ is a $\mathcal{P}_\mathcal{R}$-algebra morphism, we have

$$\langle \epsilon \otimes \text{Id} \rangle \lambda(2)(\mu)\{1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2\} = \pi(2)(\mu)\{(\epsilon \otimes \text{Id})(1_\mathcal{R} \otimes a_1), (\epsilon \otimes \text{Id})(1_\mathcal{R} \otimes a_2)\},$$

$$\pi(2)(\mu)\{1 \otimes a_1, 1 \otimes a_2\},$$

$$\pi(2)(\mu)\{a_1, a_2\},$$

$$\langle \epsilon \otimes \text{Id} \rangle\{1 \otimes \pi(2)(\mu)\{a_1, a_2\}\}.$$ 

Thus, $\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2) - 1 \otimes \pi(2)(\mu)\{a_1, a_2\} \in \ker(\epsilon \otimes \text{Id})$ for all $\mu \in \mathcal{P}_\mathcal{R}(2)$.

So if $\mathcal{R}$ is finite dimensional with dimension $r + 1$ with $\{m_i\}_{i=1}^r$ a basis of $\mathfrak{M}$, then

$$\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2)$$

$$= 1 \otimes \pi(2)(\mu)\{a_1, a_2\} + \sum_{i=1}^r m_i \otimes v_i, v_i \in A.$$
Definition 3.4. Suppose \( \lambda_1 \) and \( \lambda_2 \) are two deformations of a \( \mathcal{P} \)-algebra \((A, \pi)\) with the same base \((R, \mathfrak{M})\). They are said to be equivalent if there exists a \( \mathcal{P}_R \)-algebra isomorphism \( \phi: (A_R, \lambda_1) \to (A_R, \lambda_2) \) such that
\[
\epsilon \otimes \text{Id} \两名箭头 \to \epsilon \otimes \text{Id}
\]
\[k \otimes A\]
commutes.

Definition 3.5. Let \( R \) be a complete local algebra, that is, \( R = \varprojlim_{n \to \infty} (R/\mathfrak{m}^n) \), \( \mathfrak{M} \) denoting the maximal ideal in \( R \). A formal deformation of a \( \mathcal{P} \)-algebra \((A, \pi)\) with base \((R, \mathfrak{M})\) is a \( \mathcal{P}_R \)-algebra structure on the completed tensor product \( \hat{R} \otimes A = \varprojlim_{n \to \infty} ((R/\mathfrak{m}^n) \otimes A) \), such that \( \epsilon \hat{\otimes} \text{Id} : \hat{R} \otimes A \to k \otimes A = A \) is a \( \mathcal{P}_R \)-algebra morphism.

Example 3.6. Let \( R = k[[t]] \) be the ring of formal power series with coefficients in \( k \). Then a formal deformation of a \( \mathcal{P} \)-algebra \((A, \pi)\) over \( R \) is precisely the formal ‘1-parameter’ deformation of \((A, \pi)\) as defined in [2].

Definition 3.7. Let \( \lambda \) be a deformation of the \( \mathcal{P} \)-algebra \((A, \pi)\) with base \((R, \mathfrak{M})\) and augmentation \( \epsilon: R \to k \). Let \( R' \) be another commutative local algebra with identity, and augmentation \( \epsilon': R' \to k \) with \( \ker(\epsilon') = \mathfrak{M}' \). Let \( \phi: R \to R' \) be an algebra homomorphism with \( \phi(1) = 1 \). Then \( \epsilon' \circ \phi = \epsilon \).

Consider \( R' \) as an \( R \)-module by the map \( r' \cdot r = r' \phi(r) \) so that
\[
R' \otimes A = (R' \otimes_R R) \otimes A = R' \otimes_R (R \otimes A).
\]

Then the push-out of the deformation \( \lambda \) is the deformation \( \phi_\ast \lambda \) of \((A, \pi)\) with base \((R', \mathfrak{M}')\), defined by
\[
\phi_\ast \lambda(n)(\mu) \{ r'_1 \otimes_R (r_1 \otimes a_1), r'_2 \otimes_R (r_2 \otimes a_2), \ldots, r'_n \otimes_R (r_n \otimes a_n) \}
= r'_1 r'_2 \ldots r'_n \otimes_R \lambda(n)(\mu)(r_1 \otimes a_1, r_2 \otimes a_2, \ldots, r_n \otimes a_n), \mu \in \mathcal{P}(n).
\]

It is straightforward to see that \( \phi_\ast \lambda \) is a deformation of \((A, \pi)\) with base \((R', \mathfrak{M}')\).

Remark 3.8. If the deformation \( \lambda \) of the \( \mathcal{P} \)-algebra \((A, \pi)\) over a finite dimensional base \((R, \mathfrak{M})\) is given by:
\[
\lambda(2)(\mu)(1 \otimes a_1, 1 \otimes a_2) = 1 \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{i=1}^{r} m_i \otimes v_i
\]
then the deformation \( \phi_\ast \lambda \) is given by:
\[
(\phi_\ast \lambda(2))(\mu)(1 \otimes a_1, 1 \otimes a_2) = 1 \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{i=1}^{r} \phi(m_i) \otimes v_i.
\]
Definition 3.9. A deformation $\lambda$ of $(A, \pi)$ with base $(\mathcal{R}, \mathcal{M})$ is called infinitesimal if, in addition, $\mathcal{M}^2 = 0$.

To consider the equivalence of infinitesimal deformations the cohomology comes into play naturally.

Let $\lambda$ be an infinitesimal deformation of $(A, \pi)$, over a finite dimensional local base $\mathcal{R}$, with maximal ideal $\mathcal{M}$. Let $\xi \in \mathcal{M}' = \text{Hom}_k(\mathcal{M}, k)$. Clearly, $\xi$ can be viewed as an element of $\text{Hom}_k(\mathcal{R}, k)$ with $\xi(1_\mathcal{R}) = 0$. For every such $\xi$ define a 2-cochain
\[
\alpha_{\lambda, \xi} \in C_2^2(A, A) = \text{Hom}_k((\mathcal{P}^1(2)^\vee \otimes A^2), A)
\]
by
\[
\alpha_{\lambda, \xi}(\mu; a_1, a_2) = (\xi \otimes \text{Id})\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2)
\]
for all $\mu \in \mathcal{P}(2)$.

Theorem 3.10. For any infinitesimal deformation $\lambda$ of a $\mathcal{P}$-algebra $(A, \pi)$, $\alpha_{\lambda, \xi}$ is a 2-cocycle.

Proof. We denote $\delta_\pi$ by $\delta$, for the sake of simplicity of notation. Need to show $\delta \alpha_{\lambda, \xi} = 0$. Since, $(\mathcal{P}^1)^\vee(3) \cong R$, and $R$ is generated by the elements $\mu \circ_i \nu$'s ($\mu, \nu \in \mathcal{P}(2)$, $i \in \{1, 2\}$), it is enough to verify
\[
\delta \alpha_{\lambda, \xi}(\mu \circ_i \nu; a_1, a_2, a_3) = 0
\]
for $i = \{1, 2\}, \mu, \nu \in \mathcal{P}(2)$ and $a_1, a_2, a_3 \in A$ (cf. \[2.13\]). Consider the case $i = 1$. We claim
\[
\delta \alpha_{\lambda, \xi}(\mu \circ_1 \nu; a_1, a_2, a_3) = -((\xi \otimes \text{Id})\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2, 1_\mathcal{R} \otimes a_3))
\]
\[
= -((\xi \otimes \text{Id})\lambda(3)(\mu \circ_1 \nu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2, 1_\mathcal{R} \otimes a_3))
\]
\[
= 0.
\]
The last equality follows from the fact that $\lambda$ is a $\mathcal{P}_R$-algebra structure on $A_\mathcal{R}$.

Now,
\[
\lambda(2)(\mu)(\lambda(2)(\nu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2, 1_\mathcal{R} \otimes a_3))
\]
\[
= (\xi \otimes \text{Id})\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, a_2, 1_\mathcal{R} \otimes a_3)
\]
\[
= (\xi \otimes \text{Id})\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, a_2, 1_\mathcal{R} \otimes a_3)
\]
\[
+ (\xi \otimes \text{Id})\lambda(2)(\mu)(\sum_{i=1}^{r} m_i \otimes w_i, 1_\mathcal{R} \otimes a_3)
\]
\[
= (\xi \otimes \text{Id})\lambda(2)(\mu)(1_\mathcal{R} \otimes a_1, a_2, 1_\mathcal{R} \otimes a_3)
\]
\[
+ (\xi \otimes \text{Id})\lambda(2)(\mu)(\sum_{i=1}^{r} m_i (1_\mathcal{R} \otimes w_i), 1_\mathcal{R} \otimes a_3)
\]
\[
= \alpha_{\lambda, \xi}(\mu; \pi(2)(\nu)(a_1, a_2), a_3) + (\xi \otimes \text{Id})\sum_{i=1}^{r} m_i \lambda(2)(\mu)(1_\mathcal{R} \otimes w_i, 1_\mathcal{R} \otimes a_3).
\]
Now we look at the second term. As

\[ \lambda(2)(\mu)(1_R \otimes w_i, 1_R \otimes a_3) - 1_R \otimes \pi(2)(\mu)(w_i, a_3) \in \mathcal{M} \otimes A, \]

so,

\[ \lambda(2)(\mu)(1_R \otimes w_i, 1_R \otimes a_3) = 1_R \otimes \pi(2)(\mu)(w_i, a_3) + h; \ h \in \mathcal{M} \otimes A. \]

Hence,

\[
m_i(\lambda(2)(\mu)(1_R \otimes w_i, 1_R \otimes a_3)) = m_i(1_R \otimes \pi(2)(\mu)(w_i, a_3) + h) = m_i(1_R \otimes \pi(2)(\mu)(w_i, a_3)) \text{ because } m_ih = 0.
\]

Thus,

\[
(\xi \otimes \text{Id}) \sum_{i=1}^{r} m_i \lambda(2)(\mu)(1_R \otimes w_i, 1_R \otimes a_3)
= \sum_{i=1}^{r} (\xi \otimes \text{Id})(m_i \otimes \pi(2)(\mu)(w_i, a_3))
= \sum_{i=1}^{r} \xi(m_i)(\pi(2)(\mu)(w_i, a_3))
= \sum_{i=1}^{r} \pi(2)(\mu)(\xi(m_i)w_i, a_3)
= \pi(2)(\mu)((\xi \otimes \text{Id})(\sum_{i=1}^{r} m_i \otimes w_i), a_3)
= \pi(2)(\mu)\{(\xi \otimes \text{Id})(\lambda(2)(\nu)(1_R \otimes a_1, 1_R \otimes a_2) - 1_R \otimes \pi(2)(\nu)(a_1, a_2)), a_3\}
= \pi(2)(\mu)\{(\xi \otimes \text{Id})(\lambda(2)(\nu)(1_R \otimes a_1, 1_R \otimes a_2)), a_3\}
= \pi(2)(\mu)(\alpha_{\lambda, \xi}(\nu; a_1, a_2), a_3).
\]

Therefore,

\[
(\xi \otimes \text{Id})\{\lambda(2)(\mu)(\lambda(2)(\nu)(1_R \otimes a_1, 1_R \otimes a_2), 1_R \otimes a_3)\}
= \alpha_{\lambda, \xi}(\mu; \pi(2)(\nu)(a_1, a_2), a_3) + \pi(2)(\mu)(\alpha_{\lambda, \xi}(\nu; a_1, a_2), a_3).
\]

Now, by [2.16.2]

\[
\delta \alpha_{\lambda, \xi}(\mu \circ_2 \nu; a_1, a_2, a_3)
= -\alpha_{\lambda, \xi}(\mu; \pi(2)(\nu)(a_1, a_2), a_3) - \pi(2)(\mu)(\alpha_{\lambda, \xi}(\nu; a_1, a_2), a_3).
\]

Hence the claim follows.

Similarly, \( \delta \alpha_{\lambda, \xi}(\mu \circ_2 \nu; a_1, a_2, a_3) = 0 \). It follows that \( \delta \alpha_{\lambda, \xi} = 0 \). \( \square \)

Let us define for \( \xi \in \mathcal{M}' \) the cohomology class of the cocycle \( \alpha_{\lambda, \xi} \) by \( a_{\lambda, \xi} \). The correspondence \( \xi \mapsto a_{\lambda, \xi} \) defines a map

\[ a_{\lambda}: \mathcal{M}' \rightarrow H^2_P((A, \pi)). \]

**Theorem 3.11.** Let \( \lambda_1 \) and \( \lambda_2 \) be two infinitesimal deformations of \((A, \pi)\) with a finite dimensional base \((R, \mathcal{M})\). Then the deformations \( \lambda_1, \lambda_2 \) are equivalent iff \( \alpha_{\lambda_1, \xi} \) and \( \alpha_{\lambda_2, \xi} \) represent the same cohomology class, that is \( a_{\lambda_1, \xi} = a_{\lambda_2, \xi} \) for \( \xi \in \mathcal{M}' \).
Proof. By definition $\lambda_1$ and $\lambda_2$ are equivalent if and only if there exists a $\mathcal{P}_R$-algebra isomorphism

$$\rho : A_R \rightarrow A_R,$$ such that $(\epsilon \otimes \text{Id}) \circ \rho = \epsilon \otimes \text{Id}. \quad (3.11.1)$$

Since $A_R = R \otimes A = (k \oplus \mathfrak{M}) \otimes A \cong A \oplus (\mathfrak{M} \otimes A)$, the isomorphism $\rho$ can be written as $\rho = \rho_1 + \rho_2$ where $\rho_1 : A \rightarrow A$ and $\rho_2 : A \rightarrow \mathfrak{M} \otimes A$.

By compatibility $\rho_1 \rho = \rho$, we get $\rho_1 = \text{Id}$. Using the adjunction property of tensor products, we have

$$\text{Hom}_k(A; \mathfrak{M} \otimes A) \cong \mathfrak{M} \otimes \text{Hom}_k(A, A) \cong \text{Hom}_k(\mathfrak{M'}; \text{Hom}_k(A, A)),$$

where the isomorphisms are given by

$$\rho_2 \mapsto \sum_i m_i \otimes \phi_i \mapsto \sum_i \chi_i. \quad (3.11.2)$$

Here $\phi_i = (\xi_i \otimes \text{Id}) \circ \rho_2$, $\chi_i(\xi_j) = \delta_{i,j} \phi_i$ and as before, $\{m_i\}_{1 \leq i \leq r}$ denotes a basis of $\mathfrak{M}$, $\{\xi_i\}_{1 \leq i \leq r}$ the dual basis of $\mathfrak{M}'$.

Thus we can write,

$$\rho(1_R \otimes x) = \rho_1(1_R \otimes x) + \rho_2(1_R \otimes x) = 1_R \otimes x + \sum_i m_i \otimes \phi_i(x).$$

Recall that, $\rho$ is a $\mathcal{P}_R$-algebra morphism iff

$$\rho(\lambda_1(2)(\mu)(1_R \otimes x_1, 1_R \otimes x_2)) = \lambda_2(\mu)(\rho(1_R \otimes x_1), \rho(1_R \otimes x_2)), \quad \mu \in \mathcal{P}(2). \quad (3.11.3)$$

Let us set $\psi_k^i = \alpha_k \xi_i$, $i = 1, 2, \ldots, r$ and $k = 1, 2$. Then we have

$$\lambda_2(2)(\mu)(1_R \otimes x_1, 1_R \otimes x_2) = 1_R \otimes \pi(2)(\mu)(x_1, x_2) + \sum_i m_i \otimes \psi_i^k(\mu; x_1, x_2), \quad (3.11.4)$$

for all $\mu \in \mathcal{P}(2)$.

So,

$$\begin{align*}
\lambda_2(2)(\mu)(\rho(1_R \otimes x_1), \rho(1_R \otimes x_2)) &= \lambda_2(2)(\mu)(1_R \otimes x_1 + \sum_i m_i \otimes \phi_i(x_1), 1_R \otimes x_2 + \sum_i m_i \otimes \phi_i(x_2)) \\
&= \lambda_2(2)(\mu)(1_R \otimes x_1, 1_R \otimes x_2) + \sum_i m_i \lambda_2(2)(\mu)(1_R \otimes x_1, 1_R \otimes \phi_i(x_2)) \\
&\quad + \sum_i m_i \lambda_2(2)(\mu)(1_R \otimes \phi_i(x_1), 1_R \otimes x_2) \quad \text{(since } m_i m_j = 0) \\
&= 1_R \otimes \pi(2)(\mu)(x_1, x_2) + \sum_i m_i \otimes \psi_i^2(\mu; x_1, x_2) \\
&\quad + \sum_i m_i (1_R \otimes \pi(2)(\mu)(x_1, \phi_i(x_2))) + \sum_i m_i (1_R \otimes \pi(2)(\mu)(\phi_i(x_1), x_2)).
\end{align*}$$
On the other hand,
\[
\rho(\lambda_1(2)(\mu)(1_R \otimes x_1, 1_R \otimes x_2))
\]
\[= \rho(1_R \otimes \pi(2)(\mu)(x_1, x_2)) + \sum_{i=1}^r m_i \otimes \psi_i^1(\mu; x_1, x_2))
\]
\[= \rho(1_R \otimes \pi(2)(\mu)(x_1, x_2)) + \sum_{i=1}^r m_i \rho(1_R \otimes \psi_i^1(\mu; x_1, x_2))
\]
\[= 1_R \otimes \pi(2)(\mu)(x_1, x_2) + \sum_{i=1}^r m_i \otimes \phi_i(\pi(2)(\mu)(x_1, x_2))
\]
\[+ \sum_{i=1}^r m_i \{1_R \otimes \psi_i^1(\mu; x_1, x_2)\} \text{ (since } m_i m_j = 0).\]

It follows from 3.11.3,
\[
\sum_{i=1}^r m_i \otimes (\psi_i^2(\mu; x_1, x_2) - \psi_i^1(\mu; x_1, x_2)) + \sum_{i=1}^r m_i \otimes \delta \phi_i(\mu; x_1, x_2) = 0.
\]

Hence,
\[
\delta \phi_i = \psi_i^2 - \psi_i^1 = \alpha_{\lambda_2, \xi} - \alpha_{\lambda_1, \xi} \text{ for all } i = 1, \ldots, r.
\]

So, \(a_{\lambda_1, \xi} = a_{\lambda_2, \xi}\), for all \(\xi \in \mathfrak{M}'\). \(\square\)

Let \(\mathcal{R}\) be a finite dimensional local algebra with maximal ideal \(\mathfrak{M}\). Then \(\mathcal{R}/\mathfrak{M}^2\) is local with maximal ideal \(\mathfrak{M}/\mathfrak{M}^2\) with \((\mathfrak{M}/\mathfrak{M}^2)^2 = 0\). Let \(p_2: \mathcal{R} \rightarrow \mathcal{R}/\mathfrak{M}^2\) be the projection map.

**Definition 3.12.** Let \(\lambda\) be a deformation of \((A, \pi)\) with base \((\mathcal{R}, \mathfrak{M})\). The mapping
\[
a_p \lambda : (\mathfrak{M}/\mathfrak{M}^2)' \rightarrow H^2_{\mathcal{P}}(A),
\]
is called the **differential** of \(\lambda\) and is denoted by \(d\lambda\). In particular, if \(\mathfrak{M}^2 = 0\), then the differential \(d\lambda\) of the infinitesimal deformation \(\lambda\) is the map \(a_\lambda\).

**Corollary 3.13.** If two deformations \(\lambda_1\) and \(\lambda_2\) of a \(\mathcal{P}_\mathcal{R}\)-algebra \((A, \pi)\) are equivalent, then their differentials are equal.

### 4. Universal Infinitesimal deformation

In this section we construct a specific example of an infinitesimal deformation of a \(\mathcal{P}\)-algebra satisfying finite dimensionality of the second cohomology module. We shall also prove a fundamental property of this deformation. In the last section we will see that this infinitesimal deformation is the first step of our main construction.
Let \((A, \pi)\) be a given \(\mathcal{P}\)-algebra satisfying the condition \(\dim H_P^2(A) < \infty\). Let us denote \(H_P^2(A)\) by \(H\). Consider the \(k\)-algebra \(C_1 = k \oplus \mathbb{H}'\) with the following structure:

\[
(k_1, h_1) \cdot (k_2, h_2) = (k_1k_2, k_1h_2 + k_2h_1).
\]

Clearly, \(C_1\) is local with maximal ideal \(\mathfrak{M} = \mathbb{H}'\) and \(\mathfrak{M}^2 = 0\).

Fix a homomorphism

\[
\sigma : H \rightarrow \text{C}^2_P(A) = \text{Hom}_k((P^1) \otimes \Sigma_2 A^{\otimes 2}, A),
\]

which takes a cohomology class into a representative cocycle. We note that

\[
\text{Proposition 4.1. For any homomorphism } \sigma : H \rightarrow \text{C}^2_P(A), (A_{C_1}, \eta_1) \text{ is a } \mathcal{P}_{C_1} \text{-algebra.}
\]

**Proof.** In view of \([2.12]\) we need to check that \(\eta_1(3)(r) = 0\), for all \(r \in R\). Recall that \([2.13]\) as a module over \(k[\Sigma_3]\), \(R\) is generated by the \(\mu \circ_i \nu\)'s (\(\mu, \nu \in \mathcal{P}(2), i \in \{1, 2\}\)). Thus,

\[
\eta_1(3)(\mu \circ_1 \nu)((a_1, \phi_1), (a_2, \phi_2), (a_3, \phi_3))
\]

\[
\eta_1(2)(\mu) \circ_1 \left[ \eta_1(2)(\nu)(\eta_1(2)(\nu))(a_1, \phi_1), (a_2, \phi_2), (a_3, \phi_3) \right] \]

\[
\eta_1(2)(\mu) \circ_1 \left[ \eta_1(2)(\nu)(a_1, \phi_1), (a_2, \phi_2), (a_3, \phi_3) \right] \]

\[
\text{where } \psi_{\nu} : H \rightarrow A \text{ is given by}
\]

\[
\psi_{\nu}(\alpha) = \sigma(\alpha)(\mu; a_1, a_2) + \pi(\nu)(\phi_1(\alpha), a_2) + \pi(\nu)(a_1, \phi_2(\alpha))
\]

\[
= \sigma(\alpha)(\nu; a_1, a_2) + \pi_{\nu}(a_1, a_2, \psi_{\mu})
\]

\[
\eta_1(3)(\mu \circ_1 \nu)(a_1, a_2, a_3, \psi_{\mu})
\]
By definition of $\eta_1$, $\psi_\mu : \mathbb{H} \rightarrow A$ is given by

$$
\psi_\mu(\alpha) = \sigma(\alpha)(\mu; \pi(2)(\nu)(a_1, a_2), a_3) + \pi(2)(\mu)\{\psi_\nu(\alpha), a_3\}
+ \pi(2)(\mu)\{\pi(2)(\nu)(a_1, a_2), \phi_3(\alpha)\}
= \sigma(\alpha)(\mu; \pi(2)(\nu)(a_1, a_2), a_3)
+ \pi(2)(\mu)\{\sigma(\alpha)(\nu; a_1, a_2) + \pi(2)(\nu)(\phi_1(\alpha), a_2) + \pi(2)(\nu)(a_1, \phi_2(\alpha)), a_3\}
+ \pi(2)(\mu)\{\pi(2)(\nu)(a_1, a_2), \phi_3(\alpha)\}
= \sigma(\alpha)(\mu; \pi(2)(\nu)(a_1, a_2), a_3) + \pi(2)(\mu)\{\sigma(\alpha)(\nu; a_1, a_2), a_3\}
+ \pi(2)(\mu)\{\pi(2)(\nu)(\phi_1(\alpha), a_2), a_3\} + \pi(2)(\mu)\{\pi(2)(\nu)(a_1, \phi_2(\alpha)), a_3\}
+ \pi(2)(\mu)\{\pi(2)(\nu)(a_1, a_2), \phi_3(\alpha)\}
= \delta(\sigma(\alpha))(\mu \circ_1 \nu; a_1, a_2, a_3) + \pi(3)(\mu \circ_1 \nu)(\phi_1(\alpha), a_2, a_3)
+ \pi(3)(\mu \circ_1 \nu)(a_1, \phi_2(\alpha), a_3) + \pi(3)(\mu \circ_1 \nu)(a_1, a_2, \phi_3(\alpha))].
$$

Note that the first term in the last equality is obtained from the first two terms of the previous step using $[2,1,2]$ and is zero as $\sigma(\alpha)$ is a cocycle. Moreover the last three terms in the last step are zero as $(A, \pi)$ is an algebra over the quadratic operad $\mathcal{P}$ (cf. Proposition $2.12$). Therefore $\psi_\mu$ is zero. Thus $\eta_1(3)(\mu \circ_1 \nu) = 0$. Similarly, one can show that $\eta_1(3)(\mu \circ_2 \nu) = 0$. It follows that $\eta_1(3)$ vanishes on $R$ and hence $\eta_1$ is a $\mathcal{P}_{C_1}$-algebra structure on $A_{C_1}$. \qed

**Proposition 4.2.** Upto isomorphism, the $\mathcal{P}_{C_1}$-algebra structure of $A_{C_1}$ does not depend on the choice of $\sigma$.

**Proof.** Let $\sigma' : \mathbb{H} \rightarrow C^2_{\mathcal{P}}(A)$ be another choice of $\sigma$ and denote the corresponding $\mathcal{P}_{C_1}$-algebra structure on $C_1 \otimes A$ by $\eta'$. Then for $\alpha \in \mathbb{H}$, $\sigma(\alpha)$ and $\sigma'(\alpha)$ are two 2-cocycles of $A$, representing the same cohomology class. So, $\sigma(\alpha) - \sigma'(\alpha)$ is a 2-coboundary. Let $\sigma(\alpha) - \sigma'(\alpha) = \delta(\gamma(\alpha))$ where $\gamma : \mathbb{H} \rightarrow C^1_{\mathcal{P}}(A) \cong \text{Hom}_k(A, A)$. Using the identification $C_1 \otimes A \cong A \oplus \text{Hom}_k(\mathbb{H}, A)$, define an $C_1$-linear automorphism $\rho : C_1 \otimes A \rightarrow C_1 \otimes A$ by $\rho(\nu, \phi) = (\nu, \tilde{\phi})$ where $\tilde{\phi}(\alpha) = \phi(\alpha) + \gamma(\alpha)(\nu)$. Need to show that $\rho$ preserves the $\mathcal{P}_{C_1}$-algebra structure, that is,

$$
\rho(\eta_1(2)(\mu)((a_1, \phi_1), (a_2, \phi_2))) = \eta'(2)(\mu)(\rho(a_1, \phi_1), \rho(a_2, \phi_2)).
$$

Now observe that

$$
\rho(\eta_1(2)(\mu)((a_1, \phi_1), (a_2, \phi_2))) = \rho(\pi(2)(\mu)(a_1, a_2), \psi_\mu) = (\pi(\mu)(a_1, a_2), \overline{\psi_\mu})
$$
by the definitions of $\eta_1$ and $\rho$, respectively, where

$$\overline{\psi}_\mu(\alpha) = \psi_\mu(\alpha) + \gamma(\alpha)(\pi(2)(\mu)(a_1, a_2))$$

$$= \sigma(\alpha)(\mu; a_1, a_2) + \pi(2)(\mu)(\phi_1(\alpha), a_2)$$

$$+ \pi(2)(\mu)(a_1, \phi_2(\alpha)) + \gamma(\alpha)(\pi(2)(\mu)(a_1, a_2)).$$

On the other hand,

$$\eta'(2)(\mu)(\rho(a_1, \phi_1), \rho(a_2, \phi_2))$$

$$= \eta'(2)(\mu)((a_1, \bar{\phi}_1), (a_2, \bar{\phi}_2))$$

$$= (\pi(2)(\mu)(a_1, a_2), \psi'_\mu)$$

where

$$\psi'_\mu(\alpha) = \sigma'(\alpha)(\mu; a_1, a_2) + \pi(2)(\mu)(\bar{\phi}_1(\alpha), a_2) + \pi(2)(\mu)(a_1, \bar{\phi}_2(\alpha))$$

$$= \sigma'(\alpha)(\mu; a_1, a_2) + \pi(2)(\mu)(\phi_1(\alpha) + \gamma(\alpha)(a_1), a_2)$$

$$+ \pi(2)(\mu)(a_1, \phi_2(\alpha) + \gamma(\alpha)(a_2))$$

$$= \sigma'(\alpha)(\mu; a_1, a_2) + \pi(2)(\mu)(\phi_1(\alpha), a_2) + \pi(2)(\mu)(\gamma(\alpha)(a_1), a_2)$$

$$+ \pi(2)(\mu)(a_1, \phi_2(\alpha)) + \pi(2)(\mu)(a_1, \gamma(\alpha)(a_2)).$$

Therefore,

$$\overline{\psi}_\mu(\alpha) - \psi'_\mu(\alpha)$$

$$= \sigma(\alpha)(\mu; a_1, a_2) - \sigma'(\alpha)(\mu; a_1, a_2) + \gamma(\alpha)(\pi(2)(\mu)(a_1, a_2)$$

$$- \pi(2)(\mu)(\gamma(\alpha)(a_1), a_2) - \pi(2)(\mu)(a_1, \gamma(\alpha)(a_2)).$$

Now, by the definition of differential (2.16.1),

$$\delta(\gamma(\alpha))(\mu; a_1, a_2) = \pi(2)(\mu)(\gamma(\alpha)(a_1), a_2) + \pi(2)(\mu)(a_1, \gamma(\alpha)(a_2)) - \gamma(\alpha)(\pi(2)(\mu)(a_1, a_2)).$$

It follows that, $\overline{\psi}_\mu(\alpha) - \psi'_\mu(\alpha) = 0$ for all $\alpha$. Hence, $\overline{\psi}_\mu = \psi'_\mu$. Thus, $\rho$ is a $\mathcal{P}_{C_1}$-algebra isomorphism. \qed

**Remark 4.3.** Suppose $\{h_i\}_{1 \leq i \leq n}$ is a basis of $\mathbb{H}$ and $\{g_i\}_{1 \leq i \leq n}$ is the dual basis. Let $\sigma(h_i) = \sigma_i \in C^2_P(A)$. Under the identification $C_1 \otimes A = A \oplus \text{Hom}_K(\mathbb{H}, A)$ an element $(v, \phi) \in A \oplus \text{Hom}_K(\mathbb{H}, A)$ corresponds to $1_{C_1} \otimes v + \sum_{i=1}^n g_i \otimes \phi(h_i)$. Then for $(a_1, \phi_1), (a_2, \phi_2) \in A \oplus \text{Hom}_K(\mathbb{H}, A)$, $\eta_1(2)(\mu)\{(a_1, \phi_1), (a_2, \phi_2)\}$ corresponds to

$$1_{C_1} \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{i=1}^n g_i \otimes \{\sigma_i(\mu; a_1, a_2) + \pi(2)(\mu)(\phi_1(h_i), a_2) + \pi(2)(\mu)(a_1, \phi_2(h_i))\}. $$
In particular, for \(a_1, a_2 \in A\), we have
\[
1_{C_1} \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{i=1}^{n} g_i \otimes \sigma_i(\mu; a_1, a_2).
\]

The main property of the infinitesimal deformation \(\eta_1\) is its universality in the class of infinitesimal deformations with finite dimensional base.

**Theorem 4.4.** For any infinitesimal deformation \(\lambda\) of the \(P\)-algebra \(A\) with a finite dimensional local base \((R, \mathfrak{M})\), there exists a unique homomorphism \(\phi : C_1 \to R\) such that \(\lambda\) is equivalent to the push-out \(\phi \ast \eta_1\).

**Proof.** Let \(a_{\lambda} : \mathfrak{M}' \to H_{\mathcal{P}}^2(A) = \mathbb{H}\) denote the differential of \(\lambda\),
\[
a_{\lambda} : \xi \mapsto a_{\lambda, \xi} = [a_{\lambda}, \xi], \quad \xi \in \mathfrak{M}'.
\]
Consider the map \(\phi = \text{Id} \oplus a'_{\lambda} : k \oplus \mathbb{H}' \to k \oplus \mathfrak{M} = R\). As before, let \(\{m_i\}_{i=1}^{r}\) be a basis of \(\mathfrak{M}\) and \(\{\xi_i\}\) be the dual basis.

It is enough to show \(\alpha_{\phi \ast \eta_1} = \sigma \circ a_{\lambda}\) (Theorem 3.11). Let \(\{h_1, \ldots, h_n\}\) be a basis of \(\mathbb{H}\), and \(\{g_1, \ldots, g_n\}\) be the corresponding dual basis of \(\mathbb{H}'\).

Then by Remarks 4.3, 3.8 we have
\[
\phi \ast \eta_1(2)(\mu)(1_R \otimes a_1, 1_R \otimes a_2)
= 1_R \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{j=1}^{n} \phi(g_j) \otimes \sigma(h_j)(\mu; a_1, a_2).
\]

Now,
\[
a'_{\lambda}(g_j) = \sum_{i=1}^{r} \xi_i(a'_{\lambda}(g_j))m_i, \quad \text{and}
\]
\[
a_{\lambda}(\xi_i) = \sum_{j=1}^{n} g_j(a_{\lambda}(\xi_i))h_j.
\]
Thus

\[ \alpha_{\phi, \eta_1}(\xi_i)(\mu; a_1, a_2) = (\xi_i \otimes \text{Id})\phi_* \eta_1(2)(\mu)(1_\mathcal{R} \otimes a_1, 1_\mathcal{R} \otimes a_2) \]

\[ = (\xi_i \otimes \text{Id})\{1_\mathcal{R} \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{j=1}^{n} \phi(g_j) \otimes \sigma(h_j)(\mu; a_1, a_2) \} \]

\[ = (\xi_i \otimes \text{Id})\{\sum_{j=1}^{n} a_{\lambda}^j(g_j) \otimes \sigma(h_j)(\mu; a_1, a_2) \} \]

\[ = \sum_{j=1}^{n} \xi_i(a_{\lambda}^j(g_j)) \otimes \sigma(h_j)(\mu; a_1, a_2) \]

\[ = \sum_{j=1}^{k} g_j(a_{\lambda}(\xi_i)) \otimes \sigma(h_j)(\mu; a_1, a_2) \]

\[ = \sigma\{\sum_{j=1}^{n} g_j(a_{\lambda}(\xi_i))h_j\}(\mu; a_1, a_2) \]

\[ = \sigma \circ a_{\lambda}(\xi_i)(\mu; a_1, a_2) \]

Therefore, \( \alpha_{\phi, \eta_1} = \sigma \circ a_{\lambda} \). The uniqueness part follows from the definition of \( \phi \). \( \Box \)

5. Deformation Extensions

Let us recall some definitions and results from [12]. Let \( \mathcal{R} \) be a commutative algebra with 1 over \( k \). Let \( (C_q(A), \delta) \) denote the standard Hochschild complex, where \( C_q(A) \) is the \( \mathcal{R} \)-module \( \mathcal{R}^{\otimes (q+1)} \) with \( \mathcal{R} \) acting on the first factor by multiplication of \( \mathcal{R} \). Let \( Sh_q(\mathcal{R}) \) be the \( \mathcal{R} \)-submodule of \( C_q(\mathcal{R}) \) generated by chains

\[ s_p(r_1, r_2, \ldots, r_q) = \sum_{(i_1, i_2, \ldots, i_q) \in Sh(p,q-p)} sgn(i_1, i_2, \ldots, i_q)(a_{i_1}, a_{i_2}, \ldots, a_{i_q}) \in C_q(\mathcal{R}) \]

for \( r_1, r_2, \ldots, r_q \in \mathcal{R} ; 0 < p < q \).

Then \( Sh_\ast \) is a subcomplex of \( C_\ast(\mathcal{R}) \) and hence we have a complex called the Harrison complex

\[ Ch(A\mathcal{R}) = \{Ch_q(\mathcal{R}), \delta \} ; Ch_q(\mathcal{R}) = C_q(\mathcal{R})/Sh_q(\mathcal{R}) \]

For an \( \mathcal{R} \)-module \( M \), the Harrison cochain complex defining the Harrison cohomology with coefficients in \( M \) is given by \( Ch^q(\mathcal{R} ; M) = \text{Hom}(Ch_q(\mathcal{R}), M) \).

**Definition 5.1.** For an \( \mathcal{R} \)-module \( M \) we define

\[ H^q_{\text{Harr}}(\mathcal{R} ; M) = H^q(\text{Hom}(Ch(\mathcal{R}), M)) \].
Proposition 5.2. Let $R$ be a commutative local algebra with the maximal ideal $M$, and let $M$ be an $R$-module with $MM = 0$. Then we have the canonical isomorphism
\[ H^q_{\text{Harr}}(R; M) \cong H^q_{\text{Harr}}(R; \mathbb{K}) \otimes M. \]

Definition 5.3. An extension $R'$ of $R$ by an $R$-module $M$ is a $k$-algebra $R'$ together with an exact sequence of $k$-modules
\[ 0 \to M \xrightarrow{i} R' \xrightarrow{p} R \to 0, \]
where $p$ is an algebra homomorphism so that $N = i(M)$ is $R'$-module and we require that this $R'$-module structure is given by the $R$-module structure on $M$ as $r'i(m) = i(p(r')m)$. In particular, $N$ is an ideal in $R'$ satisfying $N^2 = 0$.

We will be concerned with those extensions of $R$ by $R$-modules $M$ which satisfy $MM = 0$.

Remark 5.4. Note that as $R$ is local, $R'$ is also local with $M_{R'} = p^{-1}(M)$ as its maximal ideal. Moreover, the condition $MM = 0$ clearly implies that for any $x \in (M)_{R'}$ and $n \in N$, $xn = 0$.

We will use the following results relating Harrison cohomology and extensions of the algebra $R$ by means of $M$. [12].

Proposition 5.5. (i) The space $H^1_{\text{Harr}}(R; M)$ is isomorphic to the space of derivations $R \rightarrow M$.

(ii) Elements of $H^2_{\text{Harr}}(R; M)$ correspond bijectively to isomorphism classes of extensions
\[ 0 \to M \to R' \to R \to 0 \]

of the algebra $R$ by means of $M$.

(iii) The space $H^1_{\text{Harr}}(R; M)$ can also be interpreted as the group of automorphisms of any given extension of $R$ by $M$.

Corollary 5.6. If $R$ is a local algebra with the maximal ideal $M$, then
\[ H^1_{\text{Harr}}(R; k) \cong \left( \frac{M}{M^2} \right)' = TR. \]

Let $R$ be a finite dimensional commutative, unital, local algebra with augmentation $\epsilon$, and maximal ideal $M$. Let $\lambda$ be a deformation of a $P = P(k, E, R)$-algebra $(A, \pi)$ over the base $(R, M)$. Let $(R', M_{R'})$ be an extension of $(R, M)$ by an $R$-module $M$ with $MM = 0$. In this section we consider the problem of extending the given deformation $\lambda$ to a deformation with base $(R', M_{R'})$. 
First let us consider the case of 1-dimensional extension. Let
\[ 0 \to k \xrightarrow{i} \mathcal{R}' \xrightarrow{p} \mathcal{R} \to 0 \]
be any 1-dimensional extension of \( \mathcal{R} \). By the above proposition the isomorphism classes of 1-dimensional extensions of \( \mathcal{R} \) are in one-one correspondence with the Harrison cohomology \( H^2_{\text{Harr}}(\mathcal{R}; k) \) of \( \mathcal{R} \) with coefficients in \( k \) where the \( \mathcal{R} \) module structure on \( k \) is given by \( r k = \epsilon(r) k \).

Let \( [f] \in H^2_{\text{Harr}}(\mathcal{R}; k) \). Suppose \( 0 \to k \xrightarrow{i} \mathcal{R}' \xrightarrow{p} \mathcal{R} \to 0 \) is a representative of the class of 1-dimensional extensions of \( \mathcal{R} \), corresponding to the cohomology class \([f]\). Let us recall how the algebra structure on \( \mathcal{R}' \) is related to \( f \). Fix a splitting \( q : \mathcal{R} \to \mathcal{R}' \). Let \( \hat{\epsilon} = \epsilon \circ p : \mathcal{R}' \to k \) denote the augmentation of \( \mathcal{R}' \). Set
\[
I = (i \otimes \text{Id}) : A \cong k \otimes A \to \mathcal{R}' \otimes A, \quad P = (p \otimes \text{Id}) : \mathcal{R}' \otimes A \to \mathcal{R} \otimes A, \\
E = (\hat{\epsilon} \otimes \text{Id}) : \mathcal{R}' \otimes A \to k \otimes A \cong A \quad \text{and} \quad Q = q \otimes \text{Id} : \mathcal{R} \otimes A \to \mathcal{R}' \otimes A.
\]
Then \( b \mapsto (p(b), i^{-1}(b - q \circ p(b))) \) is a \( k \)-module isomorphism \( \mathcal{R}' \cong \mathcal{R} \oplus k \). Let \((a, k)_{q} \in \mathcal{R}' \) denote the inverse of \((a, k) \in \mathcal{R} \oplus k \) under the above isomorphism. The cocycle \( f \) representing the extension is determined by \( f(a_1, a_2) = i^{-1}((a_1, 0)_{q}(a_2, 0)_{q} - (a_1 a_2, 0)_{q}) \). On the other hand, \( f \) determines the algebra structure on \( \mathcal{R}' \) by
\[
(a_1, k_1)_{q} \cdot (a_2, k_2)_{q} := (a_1 a_2, a_1 \cdot k_1 + a_2 \cdot k_1 + f(a_1, a_2))_{q}.
\]

As in Section 3, let \( \{m_i\}_{i=1}^{r} \) be a fixed basis of the maximal ideal \( \mathfrak{m} \) of \( \mathcal{R} \) with the dual basis \( \{\xi_i\} \). Let \( \psi_i = \alpha_{\lambda, \xi_i} \in C^2_{\mathcal{P}}(A) \) for \( 1 \leq i \leq r \). Then by Remark 3.3, the deformation \( \lambda \) can be written as
\[
\lambda(2)(\mu)(1_{\mathcal{R}} \otimes a_1, 1_{\mathcal{R}} \otimes a_2) = 1_{\mathcal{R}} \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{i=1}^{r} m_i \otimes \psi_i(\mu; a_1, a_2), \quad a_1, a_2 \in A.
\]

Let \( \{n_j\}_{1 \leq j \leq r+1} \) be defined by \( n_j = (m_j, 0)_{q} \) for \( 1 \leq j \leq r \) and \( n_{r+1} = (0, 1)_{q} \). Then \( \{n_j\}_{1 \leq j \leq r+1} \) is a basis of the maximal ideal \( \mathfrak{m}_{\mathcal{R}'} \) of \( \mathcal{R}' \). A deformation \( \Gamma \) with base \( \mathcal{R}' \) extending \( \lambda \) is entirely determined by the following two facts:
\begin{itemize}
  \item \( \Gamma(2) \) defined on \( \mathcal{P}(2) \) can be extended to the category of \( \mathcal{R}' \)-modules, and
  \item if \( \hat{\Gamma} \) is the unique extension of \( \Gamma(2) \) then \( \hat{\Gamma}(3)(r) = 0 \) for every \( r \in R \).
\end{itemize}

Let \( \psi \in C^2_{\mathcal{P}}(A) \) be any cochain. Define \( \Gamma(2)(\mu)(1_{\mathcal{R}'} \otimes a_1, 1_{\mathcal{R}'} \otimes a_2) \)
\[
= 1_{\mathcal{R}'} \otimes \pi(2)(\mu)(a_1, a_2) + \sum_{j=1}^{r} n_j \otimes \psi_j(\mu; a_1, a_2) + n_{r+1} \otimes \psi(\mu; a_1, a_2) \quad \text{for} \quad \mu \in \mathcal{P}(2).
\]

Extending to the category of \( \mathcal{R}' \)-modules this defines a \( \mathcal{R}' \)-linear map
\[ \Gamma(2) : \mathcal{P}_{\mathcal{R}'}(2) \rightarrow \text{Hom}_{\mathcal{R}'}((A_{\mathcal{R}'}^2)^2, A_{\mathcal{R}'}) \]

It is straightforward to check that the map \( \Gamma(2) \) satisfies the following properties:

1. \[ P[\Gamma(2)(\mu)(x_1, x_2)] = \lambda(2)(\mu)(P(x_1), P(x_2)), \quad x_1, x_2 \in A_{\mathcal{R}'} \]
2. \[ \Gamma(2)(\mu)(I(x), x_1) = I[\pi(2)(\mu)(x, \mathcal{E}(x_1))], \quad x \in A, \quad x_1 \in A_{\mathcal{R}'}, \mu \in \mathcal{P}(2). \]

Moreover, we have

\[ \mathcal{E}(\Gamma(\mu)(x_1, x_2)) = \pi(\mu)(\mathcal{E}(x_1), \mathcal{E}(x_2)). \quad (5.6.2) \]

By Proposition 2.13, we extend \( \Gamma(2) \) to a morphism of operads \( \mathcal{F}(E_{\mathcal{R}'}) \rightarrow \text{End}(A_{\mathcal{R}'}) \).

In view of the Remark 2.13, \( \Gamma \) induces a \( \mathcal{P}_{\mathcal{R}'} \)-algebra structure on \( A_{\mathcal{R}'} \) if and only if \( \mathcal{F}(E_{\mathcal{R}'})/(3)(\mu \circ \nu) = 0 \) for \( i \in \{1, 2\} \) and \( \mu, \nu \in \mathcal{P}(2) \) and it is clear from our construction of \( \Gamma \) that it extends the given deformation \( \lambda \).

Define a 3-cochain \( \phi_{\mathcal{R}'} \in C^3_{\mathcal{P}_{\mathcal{R}'}}(A_{\mathcal{R}'}) = \text{Hom}_{\mathcal{R}'}((\mathcal{P}_{\mathcal{R}'})^3 \otimes_{\Sigma_3} (A_{\mathcal{R}'}^3)^2, A_{\mathcal{R}'}) \) by

\[
\begin{align*}
\phi_{\mathcal{R}'}(\mu \circ \nu; a_1, a_2, a_3) &= \Gamma(2)(\mu)(\Gamma(2)(\nu)(a_1, a_2), a_3) \\
\phi_{\mathcal{R}'}(\mu \circ \nu; a_1, a_2, a_3) &= \Gamma(2)(\mu)(\Gamma(2)(\nu)(a_1, a_2), a_3), \quad v_1 \in A_{\mathcal{R}'}.
\end{align*}
\]

**Proposition 5.7.** The morphism of operads \( \mathcal{F}(E_{\mathcal{R}'}) \rightarrow \text{End}(A_{\mathcal{R}'}) \) defines a \( \mathcal{P}_{\mathcal{R}'} \)-algebra structure on \( A_{\mathcal{R}'} \) if and only if the 3-cochain \( \phi_{\mathcal{R}'} \) is zero.

**Proof.** By Proposition 2.12, \( \Gamma \) defines a \( \mathcal{P}_{\mathcal{R}'} \)-algebra structure on \( A_{\mathcal{R}'} \) if and only if \( \Gamma(3) : \mathcal{F}(E)(3) \rightarrow \text{End}(A_{\mathcal{R}'})/(3) \) is zero on \( R \).

Observe that

\[
\begin{align*}
\Gamma(3)(\mu \circ \nu)(a_1, a_2, a_3) &= \Gamma(2)(\mu)(\Gamma(2)(\nu)(a_1, a_2), a_3) \\
\Gamma(3)(\mu \circ \nu)(a_1, a_2, a_3) &= \Gamma(2)(\mu)(\Gamma(2)(\nu)(a_1, a_2), a_3), \quad v_1 \in A_{\mathcal{R}'}.
\end{align*}
\]

for \( \mu, \nu \in \mathcal{P}(2) \) and \( a_1, a_2, a_3 \in A_{\mathcal{R}'} \). Therefore by Remark 2.13, \( \phi_{\mathcal{R}'} = 0 \) if and only if \( \Gamma \) defines a \( \mathcal{P}_{\mathcal{R}'} \)-algebra structure on \( A_{\mathcal{R}'} \). \( \Box \)

Proof of the following remark is straightforward.

**Remark 5.8.** The 3-cochain \( \phi_{\mathcal{R}'} \) takes values in \( \text{Ker}(P) \) and \( \phi_{\mathcal{R}'}(\mu; a_1, a_2, a_3) = 0, \mu \in (\mathcal{P}^i)^3 = R \), whenever any of the arguments \( a_i \in \text{Ker}(\mathcal{E}) \), \( i \in \{1, 2, 3\} \).

We recall here the convolution Lie algebra \((\mathcal{P}_{\mathcal{R}'}, \mathfrak{g}_{\mathcal{P}_{\mathcal{R}'}, A} = (\text{Hom}_{\mathcal{E}}(\mathcal{P}^i, \text{End}_A)), [\cdot, \cdot]) \), where \( \mathcal{P}^i \) is the Koszul dual co-operad of \( \mathcal{P} \), [3]. We note that the set of \( \mathcal{P} \)-algebra structures on a space \( A \) is in one-to-one correspondence with the set of Maurer Cartan elements of \( \mathfrak{g}_{\mathcal{P}_{\mathcal{R}'}, A} \), i.e. solutions to the equation \( \sigma * \sigma = 0 \), where * denotes the pre-Lie product in \( \mathfrak{g}_{\mathcal{P}_{\mathcal{R}'}, A} \). Proposition 6.4.5, [16]. Given such an element \( \sigma \) one can define a differential \( \partial_{\sigma} \), called the twisted differential, on this Lie algebra in order to make it into a
differential graded Lie algebra, called the twisted differential graded Lie algebra. The
twisted differential $\partial_\sigma$ is defined using the Lie bracket as follows: $\partial_\sigma(f) = [\sigma, f]$. The
underlying cochain complex of this twisted dg Lie algebra is the deformation complex
that we intend to work with. Given any $\mathcal{P}$-algebra structure $\pi = \pi(2)$ on a space $A$, $\pi$
is a Maurer Cartan element in $\mathfrak{g}_{\mathcal{P}, A}$, i.e., $\pi * \pi = 0$. We also note that $\mathcal{P}(n)^\vee \cong \mathcal{P}(n)$.
So
$$C^n_{\mathcal{P}}(A) = \text{Hom}(\mathcal{P}(n)^\vee \otimes S^n A^n, A)$$
$$\cong \text{Hom}_{S_n}(\mathcal{P}(n), \text{End}(A^n, A))$$
$$= \mathfrak{g}_{\mathcal{P}, A}(n - 1)$$
as vector spaces. Moreover, these two cochain complexes are isomorphic, hence a
cocycle in $C^n_{\mathcal{P}}(A)$ can be thought of as a cocycle in $\mathfrak{g}_{\mathcal{P}, A}$. The fact that $\lambda(2)$ defines
an algebra structure on $\mathcal{R} \otimes A$ implies that $\lambda(2) * \lambda(2) = 0$, where $*$ denotes the
pre-Lie product in $\mathfrak{g}_{\mathcal{P}, A}$, Proposition 6.4.5, [16]. This implies
$$\sum_{i=1}^r m_i \otimes \partial_\pi(\psi_i) + \sum_{i,j=1}^r m_i m_j \otimes \psi_i * \psi_j = 0. \quad (5.8.1)$$
In order that $\Gamma(2)$ defines an algebra structure on $\mathcal{R}' \otimes A$ is equivalent to saying
that as an element of $\mathfrak{g}_{\mathcal{R}' \otimes \mathcal{P}, \mathcal{R}' \otimes A} \cong \mathcal{R}' \otimes \mathfrak{g}_{\mathcal{P}, A}$, $\Gamma(2) * \Gamma(2)$ vanishes. Now, from the
expression of $\Gamma(2)$ we get
$$\Gamma(2) * \Gamma(2) = \{ 1 \otimes \pi(2) + \sum_{i=1}^r n_i \otimes \psi_i + n_{r+1} \otimes \psi \} * \{ 1 \otimes \pi(2) + \sum_{i=1}^r n_i \otimes \psi_i + n_{r+1} \otimes \psi \}$$
$$= 1 \otimes [\pi(2), \pi(2)] + \sum_{i=1}^r n_i \otimes \partial_\pi \psi_i + n_{r+1} \otimes \partial_\pi \psi + \sum_{i,j=1}^r n_i n_j \otimes \psi_i * \psi_j$$
$$+ \sum_{i=1}^r n_i n_{r+1} \otimes \psi_i * \psi + \sum_{i=1}^r n_i n_{r+1} \otimes \psi * \psi_i + n_{r+1}^2 \otimes \psi * \psi$$
using the fact that $\partial_\pi(f) = [\pi(2), f] = \pi(2) * f + f * \pi(2)$.
We note that $[\pi(2), \pi(2)] = 0$. As $n_i n_{r+1} = 0$ for $1 \leq i \leq r$ ($\mathfrak{mM} = 0$) and $n_{r+1}^2 = 0$
the above expression is equal to
$$\sum_{i=1}^r n_i \otimes \partial_\pi \psi_i + n_{r+1} \otimes \partial_\pi \psi + \sum_{i,j=1}^r n_i n_j \otimes \psi_i * \psi_j.$$  
Now $\Gamma(2)$ defines a $\mathcal{P} \otimes \mathcal{R}'$-algebra structure on $A \otimes \mathcal{R}'$ extending $\lambda(2)$, the $\mathcal{P} \otimes \mathcal{R}$-
algebra structure on $A \otimes \mathcal{R}$, if and only if
$$\Gamma(2) * \Gamma(2) = 0$$
$$\Leftrightarrow n_{r+1} \otimes \partial_\pi \psi = -\sum_{i=1}^r n_i \otimes \partial_\pi \psi_i - \sum_{i,j=1}^r n_i n_j \otimes \psi_i * \psi_j$$
$$\Leftrightarrow (0, 1) \otimes \partial_\pi(\psi) = -\sum_{i=1}^r (m_i, 0) \otimes \partial_\pi(\psi_i) - \sum_{i,j=1}^r (m_i m_j, f(m_i, m_j)) \otimes \psi_i * \psi_j$$
(using the isomorphism between $\mathcal{R}' \cong \mathcal{R} \oplus \mathfrak{M}$)
$$\Leftrightarrow \partial_\pi(\psi) = -\sum_{i,j=1}^r f(m_i, m_j) \psi_i * \psi_j.$$
Let us define a 2 cochain $\Phi$ on $g_{P,A}$ as follows:

$$\Phi = \sum_{i,j=1}^{r} f(m_i, m_j)(\psi_i \ast \psi_j).$$

This cochain is called the obstruction cochain.

**Proposition 5.9.** The obstruction cochain is a 2 cocycle in $g_{P,A}$.

**Proof.** $R' \otimes g_{P,A}$ can be thought of as a cochain complex by defining a coboundary map $\delta' = I d \otimes \partial_\pi$. Also, $R' \otimes g_{P,A}$ can be thought of as a pre-Lie algebra by defining the pre-Lie product as

$$(r \otimes f)(s \otimes g) = rs \otimes f \ast g.$$  

One can easily check that a product so defined makes $R' \otimes g_{P,A}$ into a pre-Lie algebra. Now

$$\partial'(\sum_{i,j=1}^{r} m_i m_j \otimes \psi_i \ast \psi_j)$$

$$= \sum_{i,j=1}^{r} m_i m_j \otimes \partial_\pi(\psi_i \ast \psi_j)$$

$$= \sum_{i,j=1}^{r} m_i m_j \otimes \{\psi_i \ast \partial_\pi \psi_j - \partial_\pi \psi_i \ast \psi_j\}$$

$$= \sum_{i}(m_i \otimes \psi_i)\pi \sum_{j}(n_j \otimes \partial_\pi \psi_j) - \sum_{i}(m_i \otimes \partial_\pi \psi_i)\pi \sum_{j}(n_j \otimes \psi_j)$$

$$= \sum_{i}(m_i, 0)q \otimes \psi_i \pi \sum_{j}(m_j, 0)q \otimes \partial_\pi \psi_j - \sum_{i}(m_i, 0)q \otimes \partial_\pi \psi_i \pi \sum_{j}(m_j, 0)q \otimes \psi_j$$

$$= \sum_{i} m_i \otimes \psi_i \pi \sum_{j} m_j \otimes \partial_\pi \psi_j - \sum_{i} m_i \otimes \partial_\pi \psi_i \pi \sum_{j} m_j \otimes \psi_j \ast 0$$

using the isomorphism $R' \otimes g_{P,A} \cong R \otimes g_{P,A} \otimes g_{P,A}$

$$= \sum_{i} m_i \otimes \psi_i \pi \sum_{k,l=1}^{r} m_k m_l \otimes \psi_k \ast \psi_l - \sum_{k,l=1}^{r} m_k m_l \otimes (\psi_k \ast \psi_l)\pi \sum_{j} m_j \otimes \psi_j$$

$$= \sum_{i,j,k} m_i m_j m_k \otimes \{\psi_i \ast (\psi_k \ast \psi_l - (\psi_i \ast \psi_k) \ast \psi_l)\}.$$  

By Lemma 1 of [10], we assume $k \neq l$ in the expression $\psi_i \ast (\psi_k \ast \psi_l - (\psi_i \ast \psi_k) \ast \psi_l)$.

Now as in Proposition 3 of [10], the above sum can be written as a sum of terms of the form

$$\psi_i \ast (\psi_k \ast \psi_l + \psi_l \ast \psi_k) - ((\psi_i \ast \psi_k) \ast \psi_l + (\psi_i \ast \psi_l) \ast \psi_k),$$

where $i,j,k = 1 \cdots , r$ and each of these term vanishes by the following property [9]: If $\{V_m,*\}$ be a pre-Lie algebra and $f,g,h$ be elements of $V_m, V_n, V_p$ respectively, then $(f \ast g) \ast h - f \ast (g \ast h) = (-1)^{np}\{(f \ast h) \ast g - f \ast (h \ast g)\}$. Hence

$$\sum_{i,j} m_i m_j \otimes \partial_\pi(\psi_i \ast \psi_j) = 0$$

$$\Leftrightarrow \sum_{i,j} m_i m_j \otimes \partial_\pi(\psi_i \ast \psi_j) + f(m_i, m_j) \otimes \partial_\pi(\psi_i \ast \psi_j) = 0$$

$$\Leftrightarrow \sum_{i,j} f(m_i, m_j) \otimes \partial_\pi(\psi_i \ast \psi_j) = 0.$$  

The above consideration defines a map $\theta_\lambda : H^2_{arr}(R; \mathbb{K}) \rightarrow H^3_{P}(A, A)$ by $\theta_\lambda([f]) = [\Phi]$, where $[\Phi]$ is the cohomology class of $\Phi$. The map $\theta_\lambda$ is called the obstruction map. Proof of the following proposition is straightforward.
Proposition 5.10. Let $\lambda$ be a deformation of a $P$-algebra $A$ with base $R$ and let $R'$ be a 1-dimensional extension of $R$ corresponding to the cohomology class $[f] \in H^2_{Harr}(R; \mathbb{K})$. Then $\lambda$ can be extended to a deformation of $A$ with base $R'$ if and only if the obstruction $\theta_\lambda([f]) = 0$.

We state the following proposition, proof of which is similar to the proof of corollary 5.8 in [8].

Proposition 5.11. Suppose that for a deformation $\lambda$ of a $P$-algebra $A$ with base $R$, the differential $d\lambda : T^1 R \to \mathbb{H}$ is onto. Then the group of automorphisms $A$ of the extension

$$0 \to k \xrightarrow{i} R' \xrightarrow{p} R \to 0$$

(5.11.1)

operates transitively on the set of equivalence classes of deformations $\mu$ of $A$ with base $R$ such that $p_* \mu = \lambda$. In other words, if $\mu$ exists, it is unique up to an isomorphism and an automorphism of this extension.

Suppose now that $M$ is a finite dimensional $R$-module satisfying the condition $\mathfrak{M}M = 0$, where $\mathfrak{M}$ is the maximal ideal in $R$. The previous results can be generalized from the 1-dimensional extension (5.11.1) to a more general extension

$$0 \to M \xrightarrow{i} R' \xrightarrow{p} R \to 0.$$

The obstruction map for this extension is

$$\theta_\lambda : H^2_{Harr}(R; M) \to M \otimes H^2_P(A, A)$$

defined by $\theta_\lambda([f]) = [\Phi]$.

Then, as in the case of 1-dimensional extension, we have the following.

Proposition 5.12. Let $\lambda$ be a deformation of a $P$-algebra $A$ with base $(R, \mathfrak{M})$ and let $M$ be a finite dimensional $R$-module with $\mathfrak{M}M = 0$. Consider an extension $R'$ of $R$

$$0 \to M \xrightarrow{i} B \xrightarrow{p} R \to 0$$

corresponding to some $[f] \in H^2_{Harr}(R; M)$. A deformation $\mu$ of $A$ with base $R$ such that $p_* \mu = \lambda$ exists if and only if the obstruction $\theta_\lambda([f]) = 0$. If $d\lambda : T^1 R \to \mathbb{H}$ is onto, then the deformation $\mu$, if it exists, is unique up to an isomorphism and an automorphism of the above extension.

We end this section with the following naturality property of the obstruction map, proof of which is similar to Proposition 5.10 in [8].
Proposition 5.13. Suppose $R_1$ and $R_2$ are finite dimensional unital local algebras with augmentations $\varepsilon_1$ and $\varepsilon_2$, respectively. Let $\phi : R_2 \rightarrow R_1$ be an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon_1 \circ \phi = \varepsilon_2$. Suppose $\lambda_2$ is a deformation of a Leibniz algebra $L$ with base $R_2$ and $\lambda_1 = \phi_* \lambda_2$ is the push-out via $\phi$. Then the following diagram commutes.

\[ \xymatrix{ H^2_{Harr}(R_1; k) \ar[r]^{\theta_{\lambda_1}} & H^3_P(A; A) \ar[l]_{\phi^*} \ar[d]_{\theta_{\lambda_2}} \ar[r] & \cr H^2_{Harr}(R_2; k) \ar[r]_{\theta_{\lambda_2}} & H^3_P(A; A) \ar[l]_{\phi^*} } \]

Figure 1.

6. Construction of a Versal Deformation

In this section we give an explicit construction of versal deformation of an algebra over a quadratic operad following [6].

Consider the $P$-algebra $A$ with $\dim(\mathbb{H}) < \infty$. Set $S_0 = k$ and $S_1 = k \oplus \mathbb{H}'$. Consider the extension

\[ 0 \rightarrow \mathbb{H}' \xrightarrow{i} S_1 \xrightarrow{p} S_0 \rightarrow 0, \]

where the multiplication in $S_1$ is defined by

\[ (k_1, h_1) \cdot (k_2, h_2) = (k_1 k_2, k_1 h_2 + k_2 h_1) \text{ for } (k_1, h_1), (k_2, h_2) \in S_1. \]

Let $\eta_1$ be the universal infinitesimal deformation with base $S_1$ as constructed in Section 4. We proceed by induction. Suppose for some $k \geq 1$ we have constructed a finite dimensional local algebra $S_k$ and a deformation $\eta_k$ of $A$ with base $S_k$. Let

\[ \mu : H^2_{Harr}(S_k; k) \rightarrow (\text{Ch}_2(S_k))^' \]

be a homomorphism sending a cohomology class to a cocycle representing the class. Let

\[ f_{S_k} : \text{Ch}_2(S_k) \rightarrow H^2_{Harr}(S_k; k)' \]

be the dual of $\mu$. By Proposition 5.5 (ii) we have the following extension of $S_k$:

\[ 0 \rightarrow H^2_{Harr}(S_k; k)' \xrightarrow{\bar{\eta}_{k+1}} S_{k+1} \xrightarrow{\bar{p}_{k+1}} S_k \rightarrow 0. \quad (6.0.1) \]
The corresponding obstruction \( \theta([f_{\mathcal{S}}]) \in H^2_{\text{Harr}}(\mathcal{S}_k; k) \otimes H^3_{\mathcal{P}}(A; A) \) gives a linear map
\[
\omega_k : H^2_{\text{Harr}}(\mathcal{S}_k; k) \to H^3_{\mathcal{P}}(A; A)
\]
with the dual map
\[
\omega_k^\prime : H^3_{\mathcal{P}}(A; A) \to H^2_{\text{Harr}}(\mathcal{S}_k; k).
\]
We have an induced extension
\[
0 \to \text{coker}(\omega_k^\prime) \to \bar{\mathcal{S}}_{k+1}/\bar{i}_{k+1} \circ \omega_k^\prime(H^3_{\mathcal{P}}(A; A)) \to \mathcal{S}_k \to 0.
\]
Since \( \text{coker}(\omega_k^\prime) \cong (\ker(\omega_k))^\prime \), it yields an extension
\[
0 \to (\ker(\omega_k))^\prime \overset{i_{k+1}}{\longrightarrow} S_{k+1} \overset{p_{k+1}}{\longrightarrow} S_k \to 0
\]
where \( S_{k+1} = \bar{\mathcal{S}}_{k+1}/\bar{i}_{k+1} \circ \omega_k^\prime(H^3_{\mathcal{P}}(A; A)) \) and \( i_{k+1}, p_{k+1} \) are the mappings induced by \( \bar{i}_{k+1} \) and \( \bar{p}_{k+1} \), respectively. Observe that the algebra \( S_k \) is also local. Since \( S_k \) is finite dimensional, the cohomology group \( H^2_{\text{Harr}}(\mathcal{S}_k; k) \) is also finite dimensional and hence \( S_{k+1} \) is finite dimensional as well.

Remark 6.1. It follows from Proposition 5.2 that the specific extension (6.0.1) has the following “universality property”. For any \( \mathcal{S}_k \)-module \( M \) with \( \mathfrak{M}M = 0 \), (6.0.1) admits a unique morphism into an arbitrary extension of \( S_k \):
\[
0 \to M \to \mathcal{R}' \to S_k \to 0.
\]

Proof of the following proposition is along the same lines as Proposition 6.2 in [8].

Proposition 6.2. The deformation \( \eta_k \) with base \( S_k \) of a \( \mathcal{P} \)-algebra \( A \) admits an extension to a deformation with base \( S_{k+1} \), which is unique up to an isomorphism and an automorphism of the extension
\[
0 \to (\ker(\omega_k))^\prime \overset{i_{k+1}}{\longrightarrow} S_{k+1} \overset{p_{k+1}}{\longrightarrow} S_k \to 0
\]
Next, we give an algebraic description of the base \( S \) of the versal deformation. For that we need the following Proposition from [12].

Proposition 6.3. Let \( \mathcal{R} = k[x_1, x_2, \ldots, x_n] \) be the polynomial algebra, and let \( \mathfrak{M} \) be the ideal of polynomials without constant terms.
(a) If an ideal \( I \) of \( \mathcal{R} \) is contained in \( \mathfrak{M}^2 \), then \( H^2_{\text{Harr}}(\mathcal{R}/I; k) \cong (I/\mathfrak{M})^\prime \).
(b) There is an extension for \( \mathcal{R}' = \mathcal{R}/I \):
\[
0 \to I/\mathfrak{M} \overset{i}{\longrightarrow} \mathcal{R}/\mathfrak{M} \overset{p}{\longrightarrow} \mathcal{R}/I \to 0
\]
where \( i \) and \( p \) are induced by the inclusions \( I \hookrightarrow \mathcal{R} \) and \( \mathfrak{M}I \hookrightarrow I \).
Suppose $\text{dim}(\mathbb{H}) = n$. Let $\{h_i\}_{1 \leq i \leq n}$ be a basis of $\mathbb{H}$ and $\{g_i\}_{1 \leq i \leq n}$ be the corresponding dual basis. Let $k[[\mathbb{H}']]$ denote the formal power series ring $k[[g_1, \ldots, g_n]]$ in $n$ variables $g_1, \ldots, g_n$ over $k$. Now a typical element in $k[[\mathbb{H}']]$ is of the form

$$\sum_{i=0}^{\infty} a_i f_i(g_1, \ldots, g_n) = a_0 + a_1 f_1(g_1, \ldots, g_n) + a_2 f_2(g_1, \ldots, g_n) + \ldots$$

where $a_i \in k$ and $f_i$ is a monomial of degree $i$ in $n$-variables $g_1, \ldots, g_n$ for $i = 0, 1, 2, \ldots$. Let $M$ denote the unique maximal ideal in $k[[\mathbb{H}']]$, consisting of all elements in $k[[\mathbb{H}']]$ with constant term being equal to zero.

**Proposition 6.4.** For the local algebra $S_k$ we have $S_k \cong k[[\mathbb{H}']]^\prime/I_k$ for some ideal $I_k$, satisfying $M^2 = I_1 \supset I_2 \supset \ldots \supset I_k \supset M^{k+1}$.

**Proof.** By construction, $S_1 = k \oplus \mathbb{H}' \cong k[[\mathbb{H}']]^\prime/I^2$. Suppose we already know that $S_k \cong k[[\mathbb{H}']]^\prime/I_k$ where $M^2 \supset I_k \supset M^{k+1}$. Then by specifying $\mathcal{R} = k[[\mathbb{H}']]$ and $I = I_k$ in Proposition 6.3, we get $S_{k+1} \cong k[[\mathbb{H}']]^\prime/MI_k$. In the previous construction, $S_{k+1}$ is the quotient of $S_{k+1}$ by an ideal contained in $I_k/MI_k \subset M^2/MI_k$. Hence $S_{k+1} \cong k[[\mathbb{H}']]^\prime/I_{k+1}$ where $M^2 \supset I_{k+1} \supset MI_k \supset M^{k+2}$. The proof is now complete by induction. \hfill $\square$

**Corollary 6.5.** For $k \geq 2$ the projection $p_k : S_k \to S_{k-1}$ induces an isomorphism $TS_k \to TS_{k-1}$. In particular, for every $k \geq 1$, $TS_k \cong TS_1 = \mathbb{H}$. Moreover, under the above identification of $TS_k$ with $\mathbb{H}$, the differential $d\eta : TS_k \to \mathbb{H}$ is the identity map.

**Proof.** We have $S_0 = k$; $S_1 = k \oplus \mathbb{H}' \cong k[[\mathbb{H}']]^\prime/M^2$ and for $k \geq 2$, $S_k = k[[\mathbb{H}']]^\prime/I_k$ where $M^2 = I_1 \supset I_2 \supset \ldots \supset I_k \supset M^{k+1}$. The projection $p_k : S_k \to S_{k-1}$ is given by $p_k(f + I_k) = f + I_{k-1}$ for $f \in S_k$ and $k \geq 1$. The map $p_k$ gives rise to a surjective linear map $M/I_k \to M/I_{k-1}$. Taking the quotient map $M/I_{k-1} \to \mathbb{M}/I_{k-1}$, we get an epimorphism $M/I_k \to \mathbb{M}/I_{k-1}$ with kernel $M^2/I_k$ which corresponds to an isomorphism

$$\frac{M/I_k}{M^2/I_k} \to \frac{M/I_{k-1}}{M^2/I_{k-1}}.$$ 

As a result we get an isomorphism

$$\left(\frac{M/I_k}{M^2/I_k}\right)' = TS_k \to TS_{k-1} = \left(\frac{M/I_{k-1}}{M^2/I_{k-1}}\right)'.$$

Observe that for any $k \geq 1$, $TS_k = \frac{M/I_k}{M^2/I_k}' = \left(\frac{M}{M^2}\right)' \cong TS_1$. On the other hand, since $S_1 = k \oplus \mathbb{H}'$ with maximal ideal $\mathbb{H}'$ and $\left(\mathbb{H}'\right)' = 0$. Hence $TS_1 = (\mathbb{H})' = \mathbb{H}$. The last assertion follows from the definition of the differential. \hfill $\square$
Proposition 6.6. The complete local algebra \( S = \lim_{k \to \infty} S_k \) can be described as \( S \cong k[[H']] / I \), where \( I \) is an ideal contained in \( M^2 \).

Proof. Consider the map
\[
\phi : k[[H']] \to S_k = k[[H']]/I_k \text{ defined by } \phi(f) = f + I_k \text{ for } f \in k[[H']].
\]
Since \( I_k \supset M^k + 1 \), the map \( \phi \) induces an epimorphism
\[
\phi_k : k[[H']]/M^k + 1 \to S_k \text{ for each } k \geq 1.
\]
In the limit we get an epimorphism
\[
k[[H']] = \lim_{k \to \infty} k[[H']]/M^k + 1 \to \lim_{k \to \infty} S_k.
\]
Therefore \( S \cong k[[H']]/I \) where \( I = \bigcap_k I_k \) is the kernel of the epimorphism. \( \square \)

Finally we prove the versality property of the constructed deformation \( \eta \) with base \( S \). For this we use the following standard lemma.

Lemma 6.7. Suppose \( 0 \to M_s \xrightarrow{i} S'_s \xrightarrow{p} S \to 0 \) is an \( s \)-dimensional extension of \( S \). Then there exists an \((s-1)\)-dimensional extension
\[
0 \to M_{s-1} \xrightarrow{i} R'_{s-1} \xrightarrow{p} R \to 0
\]
of \( S \) and a \( 1 \)-dimensional extension
\[
0 \to k \xrightarrow{i'} R'_s \xrightarrow{p'} R'_s \to 0.
\]

Theorem 6.8. Let \( A \) be a \( P \)-algebra with \( \dim(H) < \infty \). Then the formal deformation \( \eta \) with base \( S \) constructed above is a versal deformation of \( A \).

Proof. Suppose \( \dim(H) = n \). Let \( \{h_i\}_{1 \leq i \leq n} \) be a basis of \( H \) and \( \{g_i\}_{1 \leq i \leq n} \) the corresponding dual basis of \( H' \). Let \( R \) be a complete local algebra with maximal ideal \( M \) and let \( \lambda \) be a formal deformation of \( A \) with base \( R \). We want to find a \( k \)-algebra homomorphism \( \phi : S \to R \) such that \( \phi_* \eta = \lambda \). Denote \( R_0 = R/M \cong k \); \( R_1 = R/M^2 \cong k \oplus (T'R)' \). Since \( R \) is complete, we have \( R = \lim_{k \to \infty} R/M^k \). Moreover, for each \( k \) we have the following finite dimensional extension
\[
0 \to \frac{M^k}{M^{k+1}} \xrightarrow{i} \frac{R}{M^{k+1}} \xrightarrow{p} \frac{R}{M^k} \to 0,
\]
because \( \dim(\frac{M^k}{M^{k+1}}) < \infty \).

Let \( \dim(\frac{M^k}{M^{k+1}}) = n_{k-1} \). A repeated application of Lemma 6.7 to the extension
\[
0 \to \frac{M^2}{M^3} \xrightarrow{i} \frac{R}{M^3} \xrightarrow{p} \frac{R}{M^2} = R_1 \to 0
\]
yields \( n_1 \) number of 1-dimensional extensions as follows.

\[
\begin{array}{c}
0 \rightarrow k \rightarrow R_2 \rightarrow R_1 \rightarrow 0 \\
0 \rightarrow k \rightarrow R_3 \rightarrow R_2 \rightarrow 0 \\
\vdots \\
0 \rightarrow k \rightarrow R_{n_1+1} \rightarrow 0.
\end{array}
\]

Similarly, the extension

\[
0 \rightarrow \mathcal{M}^3 \rightarrow R \rightarrow R = R_{n_1+1} \rightarrow 0
\]

splits into \( n_2 \) number of 1-dimensional extensions and so on. Thus we get a sequence of 1-dimensional extensions

\[
0 \rightarrow k \rightarrow R_{k+1} \rightarrow R_k \rightarrow 0; \quad k \geq 1.
\]

Since \( R = \lim_{k \rightarrow \infty} R/M_k \), it is clear that \( R = \lim_{k \rightarrow \infty} R_k \). Let \( Q_k : R \rightarrow R_k \) be the projection map for the inverse system \( \{R_k, q_k\}_{k \geq 1} \) with the limit \( R \), where \( Q_1 : R \rightarrow R_1 = R/\mathcal{M}^2 \) is the natural projection. Let \( Q_{k*} \lambda = \lambda_k \), then \( \lambda_k \) is a deformation of \( A \) with base \( R_k \). Thus \( \lambda_k = Q_{k*} \lambda = (q_{k+1} \circ Q_{k+1})_* \lambda = q_{k+1} \lambda_{k+1} \).

Now we will construct inductively homomorphisms \( \phi_j : S_j \rightarrow R_j \) for \( j = 1, 2, \ldots \), compatible with the corresponding projections \( S_{j+1} \rightarrow S_j \) and \( R_{j+1} \rightarrow R_j \), along with the conditions \( \phi_j \circ \eta_j \cong \lambda_j \). Define

\[
\phi_1 : S_1 \rightarrow R_1 \text{ as } id \oplus (d\lambda)' : k \oplus \mathbb{H}' \rightarrow k \oplus (TR)'.
\]

From Proposition 4.4 we have \( \phi_1, \eta_1 \cong \lambda_1 \).

Suppose we have constructed a \( k \)-algebra homomorphism \( \phi_k : S_k \rightarrow R_k \) with \( \phi_{k*} \eta_k \cong \lambda_k \). Consider the homomorphism \( \phi_k^* : H^2_{Harr}(R_k; k) \rightarrow H^2_{Harr}(S_k; k) \) induced by \( \phi_k \). Let

\[
0 \rightarrow k \xrightarrow{i_{k+1}} R' \xrightarrow{p_{k+1}} S_k \rightarrow 0
\]

represent the image under \( \phi_k^* \) of the isomorphism class of extension

\[
0 \rightarrow k \xrightarrow{j_{k+1}} R_{k+1} \xrightarrow{q_{k+1}} R_k \rightarrow 0
\]

(see Proposition 5.5). Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & k \xrightarrow{i_{k+1}} R' \xrightarrow{p_{k+1}} S_k \rightarrow 0 \\
\| & & \downarrow \psi \\
0 & \rightarrow & k \xrightarrow{j_{k+1}} R_{k+1} \xrightarrow{q_{k+1}} R_k \rightarrow 0
\end{array}
\]

Figure 2.
where $\psi$ is given by $\psi((x, k), q) = (\phi_k(x), k, q')$ for some fixed sections $q$ and $q'$ of $p_{k+1}$ and $q_{k+1}$ respectively. Observe that by Proposition 5.13 the obstructions in extending $\lambda_k$ to the base $R_{k+1}$ and that of $\eta_k$ to the base $R'$ coincide. Since $\lambda_k$ has an extension $\lambda_{k+1}$, the corresponding obstruction is zero. Hence there exists a deformation $\xi$ of $A$ with base $R'$ which extends $\eta_k$ with base $S_k$ such that $\psi_\ast \xi = \lambda_{k+1}$. By Remark 6.1 we get the following unique morphism of extensions.

$$0 \to H^2_{Harr}(S_k; k) \to \tilde{S}_{k+1} \to S_k \to 0$$

Figure 3.

Since the deformation $\eta_k$ has been extended to $R'$, the obstruction map

$$\omega_k : H^2_{Harr}(S_k; k) \to H^3_P(A; A)$$

is zero. Therefore the composition $\tau' \circ \omega_k' : H^3_P(A; A)' \to k$ is zero. So $\tau'$ will induce a linear map $\tau : H^2_{Harr}(S_k; k)' / \omega_k'(H^3_P(A; A)') \to k$. Also the map $\bar{\chi} : \bar{S}_{k+1} \to R'$ will induce a linear map $\chi : S_{k+1} = \bar{S}_{k+1} / \bar{i}_{k+1} \circ \omega_k'(H^3_P(A; A)') \to S'$. Since $\text{coker}(\omega_k') \cong (\text{ker}(\omega_k))'$, the last diagram yields the following commutative diagram.

$$0 \to (\text{Ker}(\omega_k))' \to S_{k+1} \to S_k \to 0$$

$$0 \to k \to R' \to S_k \to 0$$

Figure 4.

By Corollary 6.5 the differential

$$d\eta_k : T S_k \to \mathbb{H}$$

is onto, so by Corollary 5.11 the deformations $\chi_\ast \eta_{k+1}$ and $\xi$ are related by some automorphism $u : R' \to R'$ of the extension

$$0 \to k \to R' \to S_k \to 0$$

with $u_\ast (\chi_\ast \eta_{k+1}) = \xi$. Now set $\phi_{k+1} = (\psi \circ u \circ \chi) : S_{k+1} \to R_{k+1}$, where $\psi$ is as in Figure 3. Then

$$\phi_{k+1} = \psi \circ u \circ \chi_\ast \eta_{k+1} = \psi_\ast \xi = \lambda_{k+1}.$$ 

Thus by induction we get a sequence of homomorphisms $\phi_k : S_k \to R_k$ with $\phi_k \ast \eta_k = \lambda_k$. Consequently, taking the limit, we find a homomorphism $\phi : S \to R$ such that $\phi \ast \eta = \lambda$. If $\mathfrak{M}^2 = 0$, then the uniqueness of $\phi$ follows from the corresponding property in Theorem 4.4. \hfill $\square$
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