Spectral gap and definability

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1. Background on von Neumann algebras

2. Motivation

3. Definability

4. Spectral gap and unitary group representations

5. Spectral gap and subfactors
Defining von Neumann algebras

- $\mathcal{H}$ a complex Hilbert space, $B(H)$ the set of bounded operators on $H$.
- The **weak operator topology** on $B(H)$ is induced by the family of semi-norms given by, for every $\zeta, \eta \in H$,
  \[ a \mapsto |\langle a\zeta, \eta \rangle|. \]
- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital $*$-algebra closed in the weak operator topology.
- Equivalently, any unital $*$-algebra $M \subseteq B(H)$ which satisfies $M'' = M$ is a von Neumann algebra, where
  \[ M' = \{ a \in B(H) : ab = ba \text{ for all } b \in M \}. \]
Definition

A linear functional $\tau$ on a von Neumann-algebra $M$ is a (finite, normalized) trace if

- it is positive ($\tau(a^*a) \geq 0$ for all $a \in M$),
- $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$, and
- $\tau(1) = 1$.

We say it is faithful if $\tau(a^*a) = 0$ implies $a = 0$.

A tracial von Neumann algebra is a pair $(M, \tau)$ consisting of a von Neumann algebra and a faithful trace $\tau$ on $M$. $\tau$ induces a norm on $M$

$$\|a\|_2 = \sqrt{\tau(a^*a)}.$$
Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial vNa; $B(H)$ for infinite-dimensional $H$ is not.

- Inductive limits of tracial von Neumann algebras are tracial von Neumann algebras. In particular, $\mathcal{R}$, the inductive limit of the $M_n(\mathbb{C})$’s, is a tracial von Neumann algebra called the hyperfinite II$_1$ factor.

- $L(G)$ - Suppose $G$ is a group and $\mathcal{H}$ has an orthonormal generating set $\zeta_h$ for $h \in G$. Let $u_g$ for $g \in G$ be the operator determined by

  $$u_g(\zeta_h) = \zeta_{gh}.$$ 

$L(G)$ is the von Neumann algebra generated by the $u_g$’s. It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. 

Tracial ultraproducts

Suppose $M_i$ are von Neumann algebras with faithful traces $\tau_i$ for all $i \in I$ and $U$ is an ultrafilter on $I$. The bounded product is

$$\prod^b M_i := \{ \bar{a} \in \prod M_i : \lim_{i \to U} \| a_i \| < \infty \}$$

and we have a two-sided ideal

$$c_U = \{ \bar{a} \in \prod^b M_i : \lim_{i \to U} \tau_i(a_i^* a_i) = 0 \}.$$ 

The ultraproduct, $\prod_U M_i$, is defined as $\prod^b M_i / c_U$. It is a tracial von Neumann algebra with the trace given by $\tau(\bar{x}) = \lim_{i \to U} \tau_i(x_i)$.

The class of tracial von Neumann algebras forms an elementary class where the model-theoretic ultraproduct construction coincides with the tracial ultraproduct construction.
A von Neumann algebra whose center is $\mathbb{C}$ is called a factor.

A tracial factor is type I if all its projections have rational trace and is type II$_1$ if the range of the trace on projections is $[0,1]$.

$\mathcal{R}$, $\mathcal{R}^U$, $\prod_U M_n(\mathbb{C})$ and $L(\Gamma)$ (for $\Gamma$ a discrete ICC group) are all II$_1$ factors.

The class of II$_1$ factors is an elementary class.
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Continuum many theories of $\mathbb{II}_1$ factors

Theorem (BCI)

If $(M_\alpha)_{\alpha \in 2^\omega}$ is McDuff’s family of pairwise nonisomorphic separable $\mathbb{II}_1$ factors, then for any ultrafilters $\mathcal{U}, \mathcal{V}$ on any index sets and distinct $\alpha, \beta$, we have that $M^\mathcal{U}_\alpha \not\cong M^\mathcal{V}_\beta$.

Corollary

For distinct $\alpha, \beta$, we have that $M_\alpha \not\equiv M_\beta$.

Theorem (G.-Hart-Towsner)

There are concrete sentences distinguishing the McDuff factors. Moreover, if $d(\alpha, \beta) = 2^{-k}$, then the sentence distinguishing $M_\alpha$ and $M_\beta$ has “complexity” $5k + 3$. 
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A motivating conversation

During the 2017 NCGOA, Chifan and Hart had a conversation about the above results. Part of that conversation was the following quote of Chifan:

“It’s all spectral gap.”

Hart relayed this quote to me. My initial response:

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In some sense, the point of this talk is me understanding this comment model-theoretically.
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Definability in a structure

Definition

Suppose that $M$ is a structure, $A \subseteq M$, and $(\varphi_n)$ a sequence of formulae with parameters from $A$. If $\varphi\big|_n$ converges uniformly, we call the sequence a formula in $M$ over $A$.

Definition

Suppose that $\varphi$ is a formula in $M$ over $A$. We will say that $Z(\varphi\big|_M)$ is $\varphi$-definable if for every $\epsilon > 0$, there is $\delta > 0$ such that, for all $a \in M^\vec{x}$, if $\varphi\big|_M(a) < \delta$, then $d(a, Z(\varphi\big|_M)) \leq \epsilon$. $Z(\varphi)$ is definable in $M$ over $A$ if it is $\psi$-definable for some $\psi$.

Theorem

Suppose that $\varphi$ is a formula in $M$ over $A$. Then $Z(\varphi)$ is $\varphi$-definable if and only if $Z(\varphi\big|_M)^U = Z(\varphi\big|_MU)$ for every ultrafilter $U$. 

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Definability in a structure

Definition

Suppose that $\mathbf{M}$ is a structure, $A \subseteq \mathbf{M}$, and $(\varphi_n)$ a sequence of formulae with parameters from $A$. If $\varphi_n^\mathbf{M}$ converges uniformly, we call the sequence a **formula in $\mathbf{M}$ over $A$**.

Definition

Suppose that $\varphi$ is a formula in $\mathbf{M}$ over $A$. We will say that $Z(\varphi^\mathbf{M})$ is $\varphi$-**definable** if for every $\epsilon > 0$, there is $\delta > 0$ such that, for all $a \in \mathbf{M}^\mathcal{X}$, if $\varphi(\bar{a})^\mathbf{M} < \delta$, then $d(\bar{a}, Z(\varphi^\mathbf{M})) \leq \epsilon$. $Z(\varphi)$ is **definable in $\mathbf{M}$ over $A$** if it is $\psi$-definable for some $\psi$.

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Suppose that $\varphi$ is a formula in $\mathbf{M}$ over $A$. Then $Z(\varphi)$ is $\varphi$-definable if and only if $Z(\varphi^\mathbf{M})^\mathcal{U} = Z(\varphi^\mathbf{M}^\mathcal{U})$ for every ultrafilter $\mathcal{U}$. 
Definability in a structure

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Suppose that \( M \) is a structure, \( A \subseteq M \), and \((\varphi_n)\) a sequence of formulae with parameters from \( A \). If \( \varphi_n^M \) converges uniformly, we call the sequence a \textit{formula in } M \textit{ over } A.

Definition

Suppose that \( \varphi \) is a formula in \( M \) over \( A \). We will say that \( Z(\varphi^M) \) is \textit{\( \varphi \)-definable} if for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that, for all \( a \in M^\bar{x} \), if \( \varphi(\bar{a})^M < \delta \), then \( d(\bar{a}, Z(\varphi^M)) \leq \epsilon \). \( Z(\varphi) \) is \textit{definable in } M \textit{ over } A \textit{ if it is } \psi\textit{-definable for some } \psi.

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Suppose that \( \varphi \) is a formula in \( M \) over \( A \). Then \( Z(\varphi) \) is \textit{\( \varphi \)-definable} if and only if \( Z(\varphi^M)^U = Z(\varphi^{M^U}) \) for every ultrafilter \( U \).
An application: definability of relative commutants

Proposition

Suppose that $M$ is a tracial von Neumann algebra and $N$ is a definable subalgebra. Then $N' \cap M$ is also definable.

Proof.

This follows from the fact that, for $x \in M$, we have

$$\|x - \mathbb{E}_{N' \cap M}(x)\|_2 \leq \sup_{y \in N_1} \|[x, y]\|_2.$$ 

The right hand side is a formula if $N$ is definable.
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Introducing spectral gap

Proposition

Let $\pi : G \to U(H_\pi)$ be a unitary representation. The following are equivalent:

1. There exists finite $F \subseteq G$ and $c > 0$ such that, for all $\zeta \in H_\pi$, we have
   \[
   \max_{g \in F} \| \pi(g)\zeta - \zeta \| \geq c\|\zeta\|.
   \]

2. For any nonprincipal ultrafilter $\mathcal{U}$, $\pi^\mathcal{U}$ is ergodic.

3. For all $\epsilon > 0$, there is a finite $F \subseteq G$ and $\delta > 0$ such that, for all $\zeta \in H_\pi$, we have
   \[
   \max_{g \in F} \| \pi(g)\zeta - \zeta \| \leq \delta \Rightarrow \|\zeta\| \leq \epsilon.
   \]
Definition

A unitary representation $\pi$ has **spectral gap** if $\pi|_{\text{Erg}(\pi)}$ satisfies the equivalent properties above.

Fact (Hulanicki-Reiter)

$\lambda_\Gamma$ has spectral gap if and only if $\Gamma$ is non-amenable.

Lemma

$\pi$ has spectral gap if and only if, for any nonprincipal ultrafilter $\mathcal{U}$, we have $\text{Fix}(\pi^\mathcal{U}) = \text{Fix}(\pi)^\mathcal{U}$. 

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Introducing spectral gap (cont’d)

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$\pi$ has spectral gap if and only if, for any nonprincipal ultrafilter $\mathcal{U}$, we have $\text{Fix}(\pi^\mathcal{U}) = \text{Fix}(\pi)^\mathcal{U}$.
Let $T_{\Gamma}$ denote the theory of unitary representations of $\Gamma$.

Let $\varphi_{\Gamma}$ be the $T_{\Gamma}$-formula $\sum_m 2^{-m}||\gamma_m \cdot x - x||$.

For any unitary representation $\pi$ of $\Gamma$, we have $Z(\varphi_{\Gamma}^{H_{\pi}}) = \text{Fix}(\pi)$.

**Theorem**

For a given representation $\pi$ of $\Gamma$, $\pi$ has spectral gap if and only if $\text{Fix}(\pi)$ is a $\varphi_{\Gamma}$-definable subset of $H_{\pi}$.

**Remark**

One cannot replace “$\varphi_{\Gamma}$-definable” in the previous theorem with “definable.” For example, suppose that $\Gamma$ is an infinite amenable group. Then $\text{Fix}(\lambda_{\Gamma}) = \{0\}$ (which is clearly a definable subset of $\ell^2\Gamma$) but $\lambda_{\Gamma}$ does not have spectral gap.
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Property (T)

Definition

We say that $\Gamma$ has **property (T)** if every unitary representation of $\Gamma$ has spectral gap.

Lemma

$\Gamma$ has property (T) if and only if there is a finite $F \subseteq \Gamma$ and $\delta > 0$ such that: for every unitary representation $\pi$, if $\pi$ has a $(F, \delta)$-almost invariant vector, then $\text{Fix}(\pi) \neq \{0\}$. Such a pair $(F, \delta)$ is called a **Kazhdan pair** for $\Gamma$.

Proposition

Suppose that $(F, \delta)$ is a Kazhdan pair for $\Gamma$. Then for any unitary representation $\pi$ of $\Gamma$ and any $\epsilon > 0$, if $\xi \in \mathcal{H}_\pi$ is $(F, \delta \epsilon)$-invariant, then there is $\eta \in \text{Fix}(\pi)$ such that $\|\xi - \eta\| < \epsilon \|\xi\|$.
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The following are equivalent:

1. $\Gamma$ has property (T).
2. The $T$-functor $\text{Fix}$ is a $T_\Gamma$-definable set.

In this case, a simple $T_\Gamma$-formula witnesses the definability.

Let $T_{\Gamma\bowtie}$ denote the theory of pmp actions of $\Gamma$. Then the following are equivalent:

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The hard part uses the Connes-Shmidt-Weiss characterization of (T).
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Throughout, $M$ is a separable II$_1$ factor and $N$ is a von Neumann subalgebra.

**Definition**

We say that $N$ has **spectral gap in** $M$ if the unitary representation $U(N) \to L^2(M)$ given by $u \mapsto uxu^*$ has spectral gap.

**Example**

If $\Gamma$ has property (T), then $L(\Gamma)$ has spectral gap in any II$_1$ factor extension. This holds more generally for property (T) II$_1$ tracial vNas.
Introducing spectral gap for subfactors

Throughout, $M$ is a separable $\text{II}_1$ factor and $N$ is a von Neumann subalgebra.

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If $\Gamma$ has property (T), then $L(\Gamma)$ has spectral gap in any $\text{II}_1$ factor extension. This holds more generally for property (T) $\text{II}_1$ tracial vNas.
Observation

$N$ has spectral gap in $M$ if, for all $\epsilon > 0$, there are $u_1, \ldots, u_n \in U(M)$ and $\delta > 0$ such that, for all $x \in M$,

$$\|[x, u_i]\|_2 \leq \delta \|x\|_2 \Rightarrow \|x - E_{N' \cap M}(x)\|_2 \leq \epsilon \|x\|_2.$$ 

Definition

$N$ has weak spectral gap in $M$ (or w-spectral gap in) if for all $\epsilon > 0$, there are $u_1, \ldots, u_n \in U(M)$ and $\delta > 0$ such that, for all $x \in M_1$,

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$$\| [x, u_i] \|_2 \leq \delta \| x \|_2 \Rightarrow \| x - E_{N' \cap M}(x) \|_2 \leq \epsilon \| x \|_2.$$
Comparing the notions

**Lemma**

1. \( N \) has spectral gap in \( M \) if and only if \( N' \cap L^2(M)^U = L^2(N' \cap M)^U \).
2. \( N \) has w-spectral gap in \( M \) if and only if \( N' \cap M^U = (N' \cap M)^U \).

**Fact (Connes)**

Suppose that \( N \) is a \( \text{II}_1 \) factor. Then the following are equivalent:

1. \( N \) has spectral gap in \( N \);
2. \( N \) has w-spectral gap in \( N \) (i.e. \( N' \cap N^U = \mathbb{C} \cdot 1 \));
3. \( N \) does not have property Gamma.

Moreover, if these equivalent conditions hold, then \( N \) has spectral gap in \( N \otimes S \) for any tracial von Neumann algebra \( S \).
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Moreover, if these equivalent conditions hold, then \(N\) has spectral gap in \(N \otimes S\) for any tracial von Neumann algebra \(S\).
Let \( \{u_m\} \) be an enumeration of a countable dense subset \( U(N) \).

Let \( \varphi_N(x) := \sum_m 2^{-m} \|[x, u_m]\|_2 \), a formula in \( M \) over \( N \).

Note that \( Z(\varphi_N) = N' \cap M \).

**Theorem**

\( N \) has w-spectral gap in \( M \) if and only if \( N' \cap M \) is a \( \varphi_N \)-definable subset of \( M \). In this case, \( (N' \cap M)' \cap M \) is also definable.

**Remark**

Once again we cannot replace “\( \varphi_N \)-definable” with “definable” in the previous theorem. For instance, if \( N = M \), then \( M' \cap M = \mathbb{C} \), which is a definable subset of \( M \), but \( M \) has w-spectral in itself if and only if \( M \) does not have property Gamma.
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Proposition

Suppose that $M$ is an existentially closed II$_1$ factor and $N$ is a subalgebra of $M$ with w-spectral gap. Then $N$ satisfies the bicommutant condition $(N' \cap M)' \cap M = N$.

Proof.

Suppose, towards a contradiction, that $b \in (N' \cap M)' \cap M$ but $b \notin N$. Let $Q := M \ast_N (N \bar{\otimes} L(\mathbb{Z}))$. Since $M \subseteq Q$ and $M$ is e.c., there is $i : Q \to M^\mathcal{U}$ such that $i$ is the diagonal embedding on $M$. Let $c \in Q$ be the canonical unitary in $L(\mathbb{Z})$. Then $i(c) \in N' \cap M^\mathcal{U} = (N' \cap M)^\mathcal{U}$, so we can write $i(c) = (c_n) \ast$ with each $c_n \in N' \cap M$. By choice of $b$, we have $[b, c_n] = 0$ for all $n$, whence $[i(b), i(c)] = 0$ and hence $[b, c] = 0$, contradicting the fact that $b \notin N$.  

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Suppose, towards a contradiction, that $b \in (N' \cap M)' \cap M$ but $b \notin N$. Let $Q := M \ast_N (N \otimes L(\mathbb{Z}))$. Since $M \subseteq Q$ and $M$ is e.c., there is $i : Q \to M^U$ such that $i$ is the diagonal embedding on $M$. Let $c \in Q$ be the canonical unitary in $L(\mathbb{Z})$. Then $i(c) \in N' \cap M^U = (N' \cap M)^U$, so we can write $i(c) = (c_n)^\cdot$ with each $c_n \in N' \cap M$. By choice of $b$, we have $[b, c_n] = 0$ for all $n$, whence $[i(b), i(c)] = 0$ and hence $[b, c] = 0$, contradicting the fact that $b \notin N$.\qed
**II$_1$ factors do not have a model companion**

**Corollary (G.-Hart-Sinclair)**

There is an e.c. II$_1$ factor $M$ and an elementary extension $\tilde{M}$ of $M$ such that $\tilde{M}$ is not e.c.

**Proof (G.).**

Let $N$ be a property (T) factor and let $M$ be an e.c. factor containing $N$. We show that $M^U$ is not e.c. This follows from the previous slide and the computation:

$$N^U \subseteq ((N' \cap M)^U)' \cap M^U = (N' \cap M^U)' \cap M^U,$$

whence it follows that $N \neq (N' \cap M^U)' \cap M^U$ and thus $M^U$ is not e.c. □
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Questions

**Question**

Are any two e.c. $\text{II}_1$ factors elementarily equivalent?

Assuming CEP, $\mathcal{R}$ is e.c. It seems that if $M$ is an e.c. $\text{II}_1$ factor containing a property (T) factor, then since that subfactor is definable, we should not have $\mathcal{R} \equiv M$. Similar reasoning applies as well to:

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If $N$ is non-Gamma, then is it possible that $\mathcal{R} \equiv N \boxtimes \mathcal{R}$?

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Can a strongly McDuff II$_1$ factor ever be e.c.?

Call a non-Gamma factor $N$ bc-good if it has a proper subalgebra $\tilde{N}$ with w-spectral gap such that $(\tilde{N}' \cap N)' \cap N \neq \tilde{N}$.

Corollary

If $N$ is bc-good, then $N \otimes \mathbb{R}$ is not e.c.

If $N$ is not bc-good, then every w-spectral gap subfactor is definable.

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There are self-functors $T_0$ and $T_1$ on the category of countable groups.

Iterating gives us functors $T_\alpha$ for any $\alpha \in 2^{\leq \omega}$.

We set $M_\alpha(\Gamma) := L(T_\alpha(\Gamma))$.

**Theorem (G.-Hart-Towsner)**

For each non-amenable ICC group $\Gamma$, there is an integer $m(\Gamma)$ and a sequence $(c_n(\Gamma))$ of positive real numbers such that, for any $n, t \in \mathbb{N}$ with $t \geq 1$ and any $\alpha \in 2^n$, we have:

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\theta_{m,n}^{M_\alpha(\Gamma) \boxtimes t} = 0 \text{ for all } m \geq 1 \quad \text{if } \alpha(n-1) = 1;
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The role of spectral gap

- The base case uses the spectral gap characterization of nonamenability, which gives $m(\Gamma)$ and $c_0(\Gamma)$.

- A generalized McDuff ultraprocess for $\Gamma$ and $\alpha$ is one of the form $\prod_\mathcal{U} M_\alpha^{\otimes t}$.

- One defines pairs of good unitaries $(u, v)$ to be pairs of unitaries that generate a $w$-spectral gap subalgebra in a precise numerical way. Such pairs (in ultraproducts) are the zero set of a formula.

- Given two such pairs $(u_1, v_1)$ and $(u_2, v_2)$ in $M_\alpha(\Gamma)^\mathcal{U}$ with $C(u_2, v_2) \subseteq C(u_1, v_1)$, one has that $C(u_2, v_2)' \cap C(u_1, v_1)$ is a generalized McDuff ultraproduct with respect to $\Gamma$ and a string that is one digit shorter, hinting at an inductive procedure.

- One then needs to be able to relativize previously constructed sentences; this heavily uses the uniform definability of such relative commutants.
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