Benjamin-Ono equation: Rogue waves, generalized breathers, soliton bending, fission, and fusion

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Abstract

In this work, we construct various interesting localized wave structures of Benjamin-Ono equation describing the dynamics of deep water waves. Particularly, we extract the rogue wave and generalized breather solutions with the aid of bilinear form and by applying two appropriate test functions. Our analyses reveal the control mechanism of rogue wave with arbitrary parameters to obtain both bright and dark type first and second order rogue waves. Additionally, a generalization of homoclinic breather method also known as the three-wave method is used for extracting the generalized bright-dark breathers and solitons. Interestingly, we have observed the manipulation of breathers as well as soliton bending, fission and fusion. Our results are discussed categorically with the aid of clear graphical demonstrations.

Keywords: Benjamin-Ono equation; Rogue Waves; Breather; Soliton.

1. Introduction

The study of water waves gain tremendous interest over more than two centuries; theoretical as well as experimental investigations are performed to understand the dynamics of these waves [1]. These wave equations are modeled using prototype ordinary/partial/delay/fractional differential equations [2, 3]. The nonlinear dynamics of waves associated with such model equations explore several exciting phenomena including wave-mixing/breaking and interaction of several localized structures like solitons, breathers, lump and rogue waves, which have several applications in fluid dynamics, plasma, Bose-Einstein condensate, fibre optics and even in finance [3, 4]. These wave models also studied in the sense of wick type fractional stochastic systems, where the analysis of these models are more general [5]. One of the most agreeable localized structure is soliton. It can be viewed as a classical solution structure in the integrable models. Also, it is well known that the stability or identity-preserving nature of these solitons even after the collisions enable them to have phenomenal applications in diverse areas of science and technology. Especially, these nonlinear wave properties help in tackling the behaviour of real wave structures, including DNA, plasma,
wave transmissions in optical fibre and many more. Mathematically, these solitons are nothing but the solution prototype of integrable nonlinear partial differential equations. Different types of analytical solutions for such models can also be obtained to unearth the dynamics of various other localized structures.

One natural question arises here, why other localized solution structures apart from solitons, such as rogue waves and breathers, multi-shock waves and lumps, are required? It can be explained as, because of several highly unstable practical phenomena such nonlinear wave structures are necessary to explain them completely. Rogue waves are the comparatively new kind of localized structures and they are also known as “monster waves, killer waves, extreme waves” and “freak waves” [6]. Their behaviour is mysterious and it can also be explained with chaotic phenomenon. A famous saying for rogue waves are ‘coming from nowhere and disappear with no trace’ [7], because of the reason that the rogue waves are temporally and spatially localized disturbance and amplitude is increasing on the background by a few order of magnitude [7, 8], which can also be considered as a possible explanation for its chaotic behaviour. Further, these rogue waves appear due to several reasons including a universal route of modulation instability [8]. These waves appear as substantial, large localized structures compared to other localized waves. Their height is approximately two to three times height of the surrounding waves. In ocean their occurrence damages ships and oil drilling platforms. Although, the appearance of rogue waves are not limited to ocean but in finance [9], plasma [10], superfluid [11], Bose-Einstein condensate [12] and well known optical rogue waves [13]. A rigorous treatment for rogue waves including modelling and experimental observations are being done continuously with different models [14, 15, 8]. Various rational solution for different evolution equations were constructed using different analytical methods including the famous Inverse spectral transform, Darboux transformation, Hirota method, dressing method and several ansatz approaches. It is clear that the exact solutions of the nonlinear evolution equations help significantly to understand the dynamics of the waves and these solutions with different physical structures have phenomenal applications in a broader range of science and engineering [16, 17].

Being motivated by the increasing interest on the rogue waves, we devote our investigation in understanding them in the following familiar Benjamin-Ono (BO) equation [18]:

$$u_{tt} + 2\beta(u_x)^2 + 2\beta uu_{xx} + \gamma u_{xxxx} = 0,$$

where $\beta$ and $\gamma$ are arbitrary nonlinearity and dispersion coefficients, respectively. The Benjamin-Ono equation describes one-dimensional internal waves in deep water [19, 20] and it is mathematically an important nonlinear partial integro-differential equation possessing integrability and solvable by inverse scattering transform as well as Bäcklund transformation and singularity structure analysis admitting $N$-soliton solutions [21, 22, 23, 24] and have infinite conserved quantities apart from special rational and periodic solutions [25, 26]. Its various solutions including nonlocal symmetries [27] and rogue wave solutions [28] are available in the literature. Our aim of the present work is to explore various physically interesting solutions such as rogue waves and breathers with a newly developed method and to study their control mechanism along with bending, fusion and fission of nonlinear waves especially solitons.

Now by using a bilinearizing transformation we write the BO equation into a compact bilinear form and then solve it by introducing a polynomial test function of required order. For this purpose,
Substituting the above equation (1) is transformed into the following Bilinear form:

\[ (D^2_t + 2u_0\beta D_x^2 + \gamma D_y^4)f \cdot f = 0. \]

Here \( D \) represents the standard Hirota differential operator \[28, 29, 30\] and it can be defined as

\[ D^a D^b_x f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^a \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^b f(x, y, t) \cdot g(x', y', t')|_{(x, y, t) = (x', y', t)}. \]

In recent years, localized structures such as lump, breather and rogue wave arising in various nonlinear systems/models are highly appreciated and they become interesting both in mathematical and physical perspective due to their occurrence in diversified areas like plasmas, optics, Bose-Einstein condensate and financial systems \[6, 7, 8\].

The present work is organized as follows. In Sec. 2, the first and second order rogue wave solutions are constructed using the Hirota bilinear form and polynomial test functions \[34\]. In Sec. 3, a generalized breather solutions of different wave structures are obtained by using the three-wave method \[35\] along with their dynamics. Conclusions are provided in the final section.

2. Rogue Waves

In this section, we will construct first and second order rogue wave solutions by using the bilinear form \[3\] and appropriate polynomial test functions and investigate their evolution.

2.1. Rogue wave solution of order-one

To extract the rogue wave solution of order-one we choose the form of \( f(x, t) \) as \[34\]:

\[ f(x, t) = k_0 + (\alpha_1 x + \beta_1 t)^2 + (\alpha_2 x + \beta_2 t)^2. \]

Substituting the above \( f \) form \[4\] into bilinear equation \[3\] and collecting the coefficients of different powers of \( x^i y^j, i, j = 0, 1, 2 \), we get the following set of equations:

Coefficient of \( x^2 \) : 
\[-4u_0\alpha_1^2\beta - 8u_0\alpha_1^2\alpha_2\beta - 4u_0\alpha_2^4\beta - 2\alpha_1^2 + \beta_1^2 + 2\alpha_2\beta_1 - 16\alpha_1\alpha_2\beta_1\beta_2 + 8\alpha_1\alpha_2\beta_1^2 = 0, \]

Coefficient of \( t^2 \) : 
\[-4u_0\alpha_1^2\beta^2_1 + 4u_0\alpha_2^2\beta_1 - 2\beta_1^2 - 16u_0\alpha_1\alpha_2\beta_1\beta_2 + 4u_0\alpha_2^2\beta_2^2 - 4\alpha_2\beta_2 = 0. \]

Coefficient of \( tx \) : 
\[-8u_0\alpha_1\beta_1 - 8u_0\alpha_1\alpha_2\beta_1 - 4\alpha_1\beta_1^2 - 8u_0\alpha_1\alpha_2\beta_2 - 8u_0\alpha_2\beta_2^2 - 4\alpha_2\beta_2 = 0. \]

Constants : 
\[(4k_0u_0\alpha_1\beta + 4k_0u_0\alpha_2\beta + 2k_0\beta_2 + 2k_0\beta_2^2 + 12\alpha_1\gamma + 24\alpha_1^2\beta + 12\alpha_2\gamma) = 0. \]
Solving the above system of equations (5a)-(5d), we obtain the following relations among the parameters resulting to the first order rogue wave solution as

\[ u_0 \beta > 0, \quad \beta_1 = \pm \sqrt{2u_0 \beta \alpha_2}, \quad \beta_2 = \mp \sqrt{2u_0 \beta \alpha_1}, \quad k_0 = \frac{-3(\alpha_1^2 + \alpha_2^2) \gamma}{2u_0 \beta}. \]  

(6)

From Eqs. (6) and (4), we get the explicit form of \( f \) as

\[ f(x, t) = (\alpha_1 x + \sqrt{2u_0 \beta \alpha_2} t)^2 + (\alpha_2 x - \sqrt{2u_0 \beta \alpha_1} t)^2 - \frac{3(\alpha_1^2 + \alpha_2^2) \gamma}{2u_0 \beta}. \]  

(7)

Thus we can obtain the first order rogue wave solution from (2) and (7) as

\[ u(x, t) = u_0 + \frac{24u_0 \gamma(-2u_0 x^2 \beta + 4t^2 u_0^2 \beta^2 - 3\gamma)}{(2u_0 x^2 \beta + 4t^2 u_0^2 \beta^2 - 3\gamma)^2}, \]  

(8)

under the constraint condition \( u_0 \beta > 0 \). The above solutions carries three arbitrary parameters \( u_0, \beta \) and \( \gamma \). It is of natural interest to understand the importance and roles of these arbitrary parameters in defining the dynamics of the rogue wave (8). We can obtain two types of rogue waves namely bright and dark by tuning these parameters as shown in Fig. 1 by retaining the necessary condition \( u_0 \beta > 0 \). Mainly, we obtain bright single peak doubly localized excitation with the choice \( u_0 = -1.05, \beta = -1.01 \) and \( \gamma = 1.05 \). On the other hand, a dark type first order rogue wave structure is depicted for the choice \( u_0 = 1.05, \beta = 0.3 \) and \( \gamma = -0.05 \).

![Figure 1: Bright (left panel) and dark (right panel) type first order rogue waves through solution (8). The bottom panel shows the corresponding contour plots.](image)

For a much clear inference of the arbitrary parameters, we have demonstrated their role graphically for bright rogue wave with three different set of values in Fig. 2. Our analysis shows that the increase in the magnitude of \( \beta \) decreases the width of the rogue waves, while the \( \gamma \) parameter is altering its width in direct proportion with an appreciable change in their tail and without affecting the amplitude. However, the parameter \( u_0 \) is much simpler which increases the amplitude of the rogue wave along with a shifts from its constant background. Importantly, there occurs appreciable change in the significant wave height (amplitude) of the amplitude. Similar
effects can also be observed in the case of dark rogue wave, where the depth/darkness, background, width and tail of the dark rogue waves are controlled by tuning $\beta$, $\gamma$ and $u_0$ parameters.

![Image](65x554 to 577x672)

**Figure 2:** Impact of $\beta$, $\gamma$ and $u_0$ parameters in the first order bright rogue wave \[\text{(9)}\] for fixed values of other parameters.

### 2.2. Rogue wave solution of order-two

Next natural step is to construct and look for the dynamical behaviour of higher order rogue waves in the present BO system. Here we construct a simplest higher order rogue wave, that is the rogue wave of order two. For this purpose, we have adopted the following polynomial test function:

$$f(x, t) = \beta_1 + (\beta_2 x^2 + \beta_3 t^2)^3 + \beta_4 x^4 + \beta_5 x^2 t^2 + \beta_6 t^4 + \beta_7 x^2 + \beta_8 t^2,$$

where $\beta_j$, $(j = 1, 2, \ldots, 8)$ are parameters of second order rogue wave solution, will be determined later. Substituting \[\text{(9)}\] into the bilinear form \[\text{(3)}\] and equating to zero the coefficients of $\{x^m y^n, \ m, n = 0, 1, 2, 3, 4, 5, 6, 7, 8\}$, we obtained the following algebraic nonlinear system of equations:

| Coefficient of $x^{10}$ | $-12u_0\beta_2^6 + 6\beta_2^3\beta_3 = 0$, \[\text{(10a)}\] |
| Coefficient of $t^2 x^8$ | $-36u_0\beta_2^6\beta_3 + 18\beta_2^3\beta_3^2 = 0$, \[\text{(10b)}\] |
| Coefficient of $t^4 x^6$ | $-24u_0\beta_2^6\beta_3 + 12\beta_2^3 + \beta_3^2 = 0$, \[\text{(10c)}\] |
| Coefficient of $t^6 x^4$ | $24u_0\beta_2^6\beta_3 - 12\beta_2^3\beta_3^2 = 0$, \[\text{(10d)}\] |
| Coefficient of $t^8 x^2$ | $36u_0\beta_2^6\beta_3^2 - 18\beta_2^3\beta_3^2 = 0$, \[\text{(10e)}\] |
| Coefficient of $t^{10}$ | $12u_0\beta_2^6\beta_3 - 6\beta_3^2 = 0$, \[\text{(10f)}\] |
| Coefficient of $x^8$ | $-12u_0\beta_2^3\beta_4 + 6\beta_2^3\beta_3^2 + 2\beta_2^3\beta_5 + 180\beta_2^2\gamma = 0$, \[\text{(10g)}\] |
| Coefficient of $t^2 x^6$ | $-48u_0\beta_2^3\beta_3\beta_4 + 36\beta_2^3\beta_3^2\beta_4 + 16u_0\beta_2^3\beta_5 - 12\beta_2^3\beta_3\beta_5 + 12\beta_2^3\beta_6$ $+ 144\beta_2^2\beta_3\gamma = 0$, \[\text{(10h)}\] |
| Coefficient of $t^4 x^4$ | $-12u_0\beta_2^3\beta_3^2\beta_4 - 12u_0\beta_2^3\beta_2\beta_3 - 6\beta_2^3\beta_3^2\beta_5 + 60u_0\beta_2^3\beta_6 - 6\beta_2^3\beta_3\beta_6$ $- 72\beta_2^2\beta_3\gamma = 0$, \[\text{(10i)}\] |
Coefficient of \( t^6x^2 : 24u_0\beta_1^6\beta_4 - 24u_0\beta_1\beta_2^3\beta_5 + 8\beta_1^3\beta_5 + 72u_0\beta_1^3\beta_3\beta_6 - 24\beta_2^3\beta_6 + 144\beta_2^3\gamma = 0 \) (10j)

Coefficient of \( t^8 : 4u_0\beta_1^8\beta_3 + 12u_0\beta_1^2\beta_2^3\beta_5 - 6\beta_1\beta_4 + 180\beta_2^3\beta_3\gamma = 0, \) (10k)

Coefficient of \( x^6 : -8u_0\beta_1^6\beta_4 + 2\beta_5 + 16u_0\beta_1^4\beta_3\beta_6 + 6\beta_1^2\beta_6 + 2\beta_2\beta_8 + 48\beta_2^3\beta_4\gamma = 0, \) (10l)

Coefficient of \( t^4x^2 : -4u_0\beta_1^4\beta_3\beta_5 + 2\beta_3\beta_6 - 12u_0\beta_1^3\beta_3\beta_7 + 36\beta_2\beta_3\beta_7 + 60u_0\beta_1^3\beta_8 + 12\beta_2^2\beta_3\beta_4\gamma - 240\beta_2^2\beta_5\gamma = 0, \) (10m)

Coefficient of \( t^4x^2 : -4u_0\beta_1^4\beta_3\beta_5 - 2\beta_3\beta_6 - 24u_0\beta_1^3\beta_3\beta_7 + 30\beta_2^3\beta_7 + 72u_0\beta_1^3\beta_3\beta_8 - 6\beta_2^2\beta_3\beta_4\gamma - 72\beta_2^2\beta_3\beta_5\gamma + 360\beta_2^2\beta_6\gamma = 0, \) (10n)

Coefficient of \( t^6 : 4u_0\beta_1^6\beta_3 - 4\beta_5^3 + 4u_0\beta_1^3\beta_7 + 12u_0\beta_1^2\beta_3\beta_8 + 8\beta_2\beta_8 + 24\beta_3\beta_4\gamma + 72\beta_2^2\beta_3\beta_5\gamma \) \( + 72\beta_2^2\beta_3\beta_6\gamma = 0, \) (10o)

Coefficient of \( x^4 : 60u_0\beta_1^6\beta_3 + 6\beta_1^3\beta_3 - 4u_0\beta_1\beta_3 + 2\beta_3\beta_3 + 2\beta_2\beta_8 + 72\beta_2^3\gamma - 240\beta_2^3\beta_7\gamma = 0 \) (10p)

Coefficient of \( x^4 : 24u_0\beta_1\beta_4 + 2\beta_3\beta_3 - 4u_0\beta_1\beta_3 + 2\beta_2\beta_3 + 36\beta_2\beta_3\gamma - 24\beta_2\beta_7\gamma = 0, \) (10q)

Coefficient of \( t^4x^2 : 72u_0\beta_1^2\beta_2\beta_3 + 36\beta_1\beta_2\beta_3 - 8u_0\beta_1^2\beta_3 - 12u_0\beta_1\beta_4 - 4\beta_8 - 24\beta_3\gamma \) \( - 72\beta_2^2\beta_3\gamma + 360\beta_2^2\beta_5\gamma = 0, \) (10r)

Coefficient of \( t^4 : 12u_0\beta_1\beta_2\beta_3^2 + 30\beta_1\beta_3^2 + 4u_0\beta_1\beta_3 + 4u_0\beta_1\beta_3\beta_8 - 2\beta_2\beta_8 + 12\beta_2\gamma + 24\beta_3\beta_6\gamma + 72\beta_2^2\beta_7\gamma + 72\beta_2^2\beta_6\gamma = 0, \) (10s)

Coefficient of \( t^2 : 4u_0\beta_1\beta_3 + 12\beta_1\beta_3 + 4u_0\beta_1\beta_7 - 2\beta_2\beta_6 + 72\beta_2\beta_3\gamma + 24\beta_3\gamma + 24\beta_4\gamma = 0 \) (10t)

Constants:
\[ \begin{align*}
\beta_1 &= \frac{-1875\beta_3^3\gamma^3}{64u_0^6\beta^6}; \\
\beta_2 &= \frac{\beta_3}{2u_0\beta}; \\
\beta_4 &= \frac{-25\beta_3\gamma}{16u_0^4\beta^4}; \\
\beta_5 &= \frac{-45\beta_3^3\gamma}{4u_0^6\beta^6}; \\
\beta_6 &= \frac{-17\beta_3^3\gamma^2}{4u_0^6\beta^2}; \\
\beta_7 &= \frac{-125\beta_3^3\gamma^2}{32u_0^6\beta^5}; \\
\beta_8 &= \frac{475\beta_3^3\gamma^2}{16u_0^6\beta^4}.
\end{align*} \] (11)

From Eqs. (2), (9) and (11), we obtained the two-rogue wave solution of Benjamin-Ono equation [1] as follows:

\[ u(x,t) = u_0 + \frac{6y}{\beta} \left( \frac{F_1}{F_2^2} \right), \] (12a)

where

\[ \begin{align*}
F_1 &= 4u_0\beta(4608^6u_0^6x^2\beta^6 + 3072^10u_0^10\beta^10 + 531250u_0^2x^2\beta^2y^2 + 234375y^5 + 62500u_0^5\beta^5\gamma(-2x^4 + 7\gamma^2) + 768u_0^1\beta_3^3\gamma(-x^4 + 14\gamma^2) - 768u_0^1\beta_3^3\gamma(-2x^4 + 47\gamma^2) - 2000u_0^1\beta_3^3\gamma(7x^4 + 510\gamma^2) + 400u_0^1\beta_3^3\gamma(3x^8 + 210\gamma^2x^4 - 1850\gamma^2) - 96u_0^1\beta_3^3\gamma(x^8 + 20\gamma^2x^4 - 1250\gamma^2) + 64u_0^1\beta_3^3\gamma(-9x^8 + 90\gamma^2x^4 + 2830\gamma^2)), \\
F_2 &= 96t^4u_0^5x^2\beta^5 + 64t^4u_0^1\beta_3^3\gamma - 250u_0^1\beta_3^3\gamma - 100u_0^1\beta_3^3\gamma(x^4 - 19\gamma^2) + 16t^2u_0^1\beta_3^3\gamma(3x^4 - 17\gamma^2),
\end{align*} \] (12b)
Our categorical analysis on the above second order rogue wave solution reveal that there exist two types of rogue waves, namely bright and dark rogue waves, as appeared in the case of first order rogue wave solutions. For completeness, we have shown such a bright and dark type second order rogue wave structures in Fig. 3 and the choices of parameters are given in the caption. Further, the impact of arbitrary parameters $\beta$, $\gamma$, and $u_0$ give additional freedom in manipulating the amplitude or depth, width, background and tail of the second order rogue waves too which is shown in Fig. 4 for the second order bright rogue wave. Similarly, such effects can also be observed in the dark rogue wave which is not given here by considering the length of the article.

3. Generalized Breathers

As given in the introduction, the second part of this work is to investigate the dynamics of generalized breathers of Benjamin-Ono equation (1), which we carryout in this section. For this purpose, first we obtain the breather solutions of BO model (1) by adopting a generalized three wave test function approach suggested in Ref. [31], which is also referred as three-wave method.
Here the following form of homoclinic breather (two-wave) ansatz [32] is generalized to obtained the homoclinic breathers as well as traveling wave solutions:

\[ f(x, t) = m_1 e^{(p_1 x + k_1 t)} + m_2 \cos(q_1 x + a_1 t) + m_1 e^{-(p_1 x + k_1 t)} \]  

were \( m_1, m_2, p_1, k_1, q_1, a_1 \) are arbitrary constants. The above one is a combination of two waves, a periodic wave \( \cos(q_1 x + a_1 t) \) and a solitary wave \( \cos(p_1 x + k_1 t) \), which gives rise to homoclinic breathers. For more generalized breather wave solutions, we consider a combination of three waves as initial test function as given below [31].

\[ f(x, t) = b_1 e^{(p x + k t)} + b_2 \cos(q x - at) + b_3 e^{-(p x + k t)} + b_4 \cosh(mx + ct) \]  

where \( b_1, b_2, b_3, p, k, q, a, m \) and \( c \) are unknown parameters to be determined for generalized breathers. Here the choice \( b_4 = 0 \) in (14) correspond to the homoclinic breathers of two wave interaction method (13). Thus, without loss of generality, here we construct homoclinic breathers solutions which exhibit different nonlinear wave structures ranging from bright/dark/gray solitons, breathers, etc. by following the above three-wave interaction method. It is also clear that the number of arbitrary parameters in the three-wave method are higher than two-wave homoclinic breathers.

Further, one can also generalize the three-wave method as suggested in Ref. [33] for a KdV type system

\[ f(x, t) = b_1 \cosh(\Xi_1) + b_2 \cosh(\Xi_2) + b_3 \cosh(\Xi_3), \]  

where \( \Xi_i = k_i(n_i x + r_i y + s_i t + \alpha_i), \) \( i = 1, 2, 3, \) and \( n_i, r_i, s_i, \alpha_i \) are the arbitrary parameters. This newly introduced test function is a combination of homoclinic breather test function and three wave solution test function. Also, in the similar way a generalized N-wave method can also be used to find various solutions of nonlinear wave equations, which is beyond the scope of the current work and can be investigated separately to look for other types of possible nonlinear wave entities.

**Breather Solution using Three-Wave method**

Considering the three wave test function [14] and collecting the coefficients of \( e^{(p x + k t)}, \cos(q x - at), \sin(q x - at), \cosh(mx + ct), \sinh(mx + ct), (i = -1, 0, 1) \), we get the following system of algebraic equations from Eq. [3]:

\[
\begin{align*}
(8b_1b_3k^2 + 16b_1b_3p^2u_0\beta + 32b_1b_3p^4\gamma) &+ (-2a^2b_2^2 - 4b_2q^2u_0\beta + 8b_2q^4) \\
+ (2b_4c^2 + 4b_4m^2u_0\beta + 8b_4m^4) &\gamma = 0, \\
(2b_1b_4c^2 + 2b_1b_4k^2 + 4b_1b_4m^2u_0\beta + 4b_1b_4p^2u_0\beta + 2b_1b_4p^2\gamma + 12b_1b_4p^4\gamma + 2b_1b_4q^2\gamma + 2b_1b_4q^4) &\gamma = 0, \\
(2b_2b_4c^2 + 2b_2b_4k^2 + 4b_2b_4m^2u_0\beta + 4b_2b_4p^2u_0\beta + 2b_2b_4p^2\gamma + 12b_2b_4p^4\gamma + 2b_2b_4q^2\gamma + 2b_2b_4q^4) &\gamma = 0, \\
(2b_1b_4c^2 + 2b_1b_4k^2 + 4b_1b_4m^2u_0\beta + 4b_1b_4p^2u_0\beta + 2b_1b_4p^2\gamma + 12b_1b_4p^4\gamma + 2b_1b_4q^2\gamma + 2b_1b_4q^4) &\gamma = 0, \\
(2b_2b_4c^2 + 2b_2b_4k^2 + 4b_2b_4m^2u_0\beta + 4b_2b_4p^2u_0\beta + 2b_2b_4p^2\gamma + 12b_2b_4p^4\gamma + 2b_2b_4q^2\gamma + 2b_2b_4q^4) &\gamma = 0, \\
(4ab_1b_2k - 8b_1b_2pqu_0\beta - 8b_1b_2p^2\gamma + 8b_1b_2p^3q^3) &\gamma = 0, \\
(-4ab_1b_3k + 8b_2b_3pqu_0\beta + 8b_2b_3p^3\gamma - 8b_2b_3p^4q^3) &\gamma = 0,
\end{align*}
\]
\begin{align}
-4b_1b_4ck - 8b_1b_3mpu_0\beta - 8b_1b_3m^3py - 8b_1b_3mp^3\gamma &= 0, \\
4b_3b_4ck + 8b_3b_4mpu_0\beta + 8b_3b_4m^3py + 8b_3b_4mp^3\gamma &= 0, \\
4ab_2b_4c - 8b_2b_3mqu_0\beta - 8b_2b_3m^3qy + 8b_2b_3mq^3\gamma &= 0.
\end{align}

After solving the above nonlinear algebraic system of equations, different classes of solutions can be obtained for the arbitrary parameters which we discuss one by one in the following part.

**Case 1:** When $b_1 = b_3 = b_4 = 0$ and $b_2 \neq 0$, Eqs. (16) give $a = \sqrt{2(2q^4\gamma - q^2u_0\beta)}$ with $b_2$ as an arbitrary free parameter. This results into the following form of $f$:

$$f = b_2 \cos \left( qx - \sqrt{2(2q^4\gamma - q^2u_0\beta)}t \right).$$

Thus we get a singular solution of the Benjamin Ono equation [3] as

$$u(x, t) = u_0 - \frac{6\gamma}{\beta} \left( q^2 \sec^2(qx - t \sqrt{2(2q^4\gamma - q^2u_0\beta)}) \right).$$

The above solution always result into unbounded singluar form without much advantage or applications. So, here we do not discuss any further details of this solution (18).

**Case 2:** When $b_1 = b_3 = 0$ and $b_2, b_4 \neq 0$, the explicit form of $f$ is obtained from Eqs. (16) as

$$f = b_2 \cos \left( \frac{u_0\beta + 2m^2\gamma}{\sqrt{2\gamma}} \right) \left( \sqrt{3}x - 2m\sqrt{\gamma}t \right) + \frac{2\sqrt{-2b_2^2(u_0\beta + 2m^2\gamma)}}{u_0\beta + 8m^2\gamma} \times \cosh \left[ mx - \frac{3}{4\gamma} (u_0\beta + 4m^2\gamma) t \right],$$

where $b_4 = 2\sqrt{-2b_2^2(u_0\beta + 2m^2\gamma)}$, $q = \sqrt{\frac{3(u_0\beta + 2m^2\gamma)}{2\gamma}}$, $c = -\frac{3}{4\gamma}(u_0\beta + 4m^2\gamma)$, and $a = \sqrt{2m^2(u_0\beta + 2m^2\gamma)}$ while the other parameters ($u_0$, $\beta$, $\gamma$, $b_2$ and $m$) are arbitrary. From the above $f$ and Eqn. (2), we can obtain the exact solution as below.

$$u(x, t) = u_0 + \frac{6\gamma}{\beta} (\ln f)_{xx}.$$
Figure 5: Breathing soliton with periodical oscillation in amplitude and space for (a) \(m = 0.75\), (b) \(m = 0.0\) and (c) \(m = -0.75\). Other arbitrary parameters are fixed as \(u_0 = 1\), \(\gamma = 0.5\), \(\beta = 1.50\), and \(b_2 = 1.0\).

Fig. 5 traveling with positive, zero and negative velocity having different amplitudes.

**Case 3:** For the choice \(b_2 = b_4 = 0\) and \(b_1, b_3 \neq 0\), Eqs. (16) reduces to an explicit form of \(f\) as

\[
f = b_1 e^{(p x - \sqrt{2(-p^2 u_0 \beta - 2 p^2 \gamma)} t)} + b_3 e^{(-p x - \sqrt{2(-p^2 u_0 \beta - 2 p^2 \gamma)} t)},
\]

along with a condition \(k = -\sqrt{2p^2 (u_0 \beta + 2 p^2 \gamma)}\). Further, by setting \(b_1 = b_3 > 0\), the function \(f(x, t)\) becomes

\[
f(x, t) = 2b_1 \cosh \left( px - \sqrt{-2p^2 (u_0 \beta + 2 p^2 \gamma)} t \right),
\]

Hence the solution of BO equation (1) is obtained as:

\[
u(x, t) = u_0 + \frac{6 \gamma}{\beta} p^2 \text{sech}^2 \left( px - t \sqrt{-2p^2 (u_0 \beta + 2 p^2 \gamma)} \right) \rightarrow \text{Bright Soliton.}
\]

On the other hand, when \(b_1 = d_1, b_3 = -d_1\) with \(d_1 > 0\) the function \(f(x, t)\) reduces to

\[
f(x, t) = 2d_1 \sinh \left( px - \sqrt{-2p^2 (u_0 \beta + 2 p^2 \gamma)} t \right),
\]

Now, the corresponding solution of BO equation (1) can be derived as below.

\[
u(x, t) = u_0 - \frac{6 \gamma}{\beta} p^2 \text{cosech}^2 \left( px - t \sqrt{-2p^2 (u_0 \beta + 2 p^2 \gamma)} \right) \rightarrow \text{Singular Solution.}
\]

From the above solutions Eqs. (21b) and (22b) one can understand that they correspond to bright/dark soliton and unbounded singular structures, respectively. Though the singular solutions are of no further interest, solitons found multifaceted applications due to their stable propagation with variety of localized profiles of salient features. In the present case, the solution (21b) further divided into two categories namely bright and dark soliton based on the choice of arbitrary parameter \(\beta\) and \(\gamma\) apart from \(u_0\) parameter which controls the background and amplitude/depth of this bright/dark soliton. Particularly when \(\beta \gamma > 0\), we obtain a standard bright soliton for
Figure 6: Bright soliton, anti-dark soliton (bright soliton on a constant background $u_0 = 0$) and W-shaped (double well) dark soliton of BO equation for the choice $u_0 = 0 \& \gamma = -0.5$, $u_0 = 1 \& \gamma = 0.5$, and $u_0 = 1 \& \gamma = 0.5$, respectively with other arbitrary parameters as $\beta = -1.50$, $b_1 = 1.0$, and $p = -1.0$.

$u_0 = 0$ defined with velocity $\sqrt{-2p^2(u_0\beta + 2p^2\gamma)}$ and amplitude $\frac{6\gamma}{\beta} p^2$, while for the choice $u_0 \neq 0$ results into an anti-dark soliton (soliton on a constant non-zero background) having an amplitude $u_0 + \frac{6\gamma}{\beta} p^2$ and traveling with same velocity. In contrary, for the choice $\beta \gamma < 0$ and $u_0 \neq 0$, Eq. (21b) yields dark soliton solution with different shapes ranging from a single well or double well (W-shaped) profiles. Also, by tuning these $\beta$ and $\gamma$ parameters one can manipulate their width and amplitude along with the direction of propagation appropriately. For completeness and better understanding, we have shown such bright, anti-dark and dark solitons in Fig. 6.

Case 4: When $b_1 = 0$ and $b_2, b_3, b_4 \neq 0$, we can obtain the explicit form of $f$ from the set of equations (16) as below.

$$f(x, t) = b_2 \cos(qx - at) + b_3 e^{-(px + ki)} + b_4 \cosh(mx + ct), \quad (23a)$$

where the parameters $m$, $p$, $k$, $a$ and $q$ take the following form:

$$m = \frac{1}{2} \sqrt{\frac{2}{\gamma}} \sqrt{-u_0\beta - \sqrt{u_0^2\beta^2 - 4c^2\gamma}}, \quad p = \frac{3}{4\gamma} \sqrt{-u_0\beta + \sqrt{u_0^2\beta^2 - 4c^2\gamma}}, \quad (23b)$$

$$k = -\frac{1}{4c\gamma} \sqrt{3(u_0^2\beta^2 - 4c^2\gamma)} \sqrt{-\left(u_0\beta + \sqrt{u_0^2\beta^2 - 4c^2\gamma}\right) \left(u_0\beta - \sqrt{u_0^2\beta^2 - 4c^2\gamma}\right)}, \quad (23c)$$

$$a = \frac{1}{4c\gamma} \left(u_0\beta - \sqrt{u_0^2\beta^2 - 4c^2\gamma}\right) \sqrt{-\left(u_0\beta + \sqrt{u_0^2\beta^2 - 4c^2\gamma}\right)^2}, \quad (23d)$$

$$q = \frac{1}{2} \sqrt{\frac{2}{\gamma}} \sqrt{u_0\beta + \sqrt{u_0^2\beta^2 - 4c^2\gamma}}. \quad (23e)$$

In the above solution (23), $u_0$, $\beta$, $\gamma$, $c$, $b_2$, $b_3$ and $b_4$ are arbitrary parameters through which we can manipulate the resultant nonlinear wave pattern of soliton breather. Compared to the previous
Figure 7: Soliton breathers appearing on a constant background through (23a). The parameter choices are (left panel) \( u_0 = 1, \beta = -1.5, \gamma = -0.5, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5, \) and \( c = -1.0; \) (middle panel) \( u_0 = 1, \beta = -1.5, \gamma = -0.5, b_2 = -1.0, b_3 = -0.5, b_4 = 1.5, \) and \( c = 1.0; \) and (right panel) \( u_0 = 1, \beta = 1.5, \gamma = -0.75, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5, \) and \( c = 1.0. \)

Figure 8: Bending of W-shaped dark soliton with decrease/increase in the intensity due to increased/decreased width of the soliton given by Eqn. (23a) for \( c = -0.5 \) (left panel) and \( c = 0.5 \) (right panel) with other arbitrary parameters as \( u_0 = 1, \beta = -1.5, \gamma = 0.25, b_2 = 1.0, b_3 = 0.5, \) and \( b_4 = 1.5. \)

solutions, the above solution reveal interesting patterns of soliton breather explaining various phenomena. It starts from the breathing of solitons with periodic oscillations in its amplitude and also solitons with single or double hump/well structure undergoing fusion and fission processes in addition to their bending characteristics. Such type of soliton breather is shown in Fig. 7, while the bending, fission and fusion nature of solitons are demonstrated in Figs. 8–10 with appropriate choice of arbitrary parameters for illustrative purpose. Further, dynamics on these solitonic bending, fission and fusion properties require a separate dedicated investigation which shall unravel different features. Here also one can control the amplitude, period of oscillations/breathing, width, and velocity of solitons/breathers by tuning the arbitrary parameters.
Figure 9: Fission of a W-shaped dark soliton into two single well solitons (for $u_0 = 1, \beta = -1.5, \gamma = 0.5, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5$, and $c = 1.0$) and a single well soliton and double well gray solitons (for $u_0 = 1, \beta = -1.5, \gamma = 0.25, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5$, and $c = -1.0$) through Eqn. (23a).

Figure 10: Fusion of two single well solitons (for $u_0 = 1, \beta = -1.5, \gamma = 0.5, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5$, and $c = 1.0$) and a single well soliton with double well soliton (for $u_0 = 1, \beta = -1.5, \gamma = 0.25, b_2 = 1.0, b_3 = 0.5, b_4 = 1.5$, and $c = 1.0$) to form amplified W-shaped dark soliton through Eqn. (23a).

4. Conclusions

We have considered the familiar Benjamin-Ono equation and constructed various localized wave solutions starting from the rogue waves to breathers and solitons by employing a polynomial test functions and a recently proposed three-wave method with the aid of bilinear form. Through the obtained solutions we are able to control and manipulate the constructed localized waves with the available arbitrary parameters to realize their multifaceted nature such as tailoring of their amplitude, width, velocity and tail of both bright and dark type first as well as second order rogue waves, developing bright and dark solitons, exploring their fusion, fission and bending properties with clear demonstrations using graphical examples. The reported results will be helpful for a complete understanding on the dynamics of the considered system and further the analysis can be extended to other related nonlinear models.
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