A note on higher-charge configurations for the Faddeev-Hopf model

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Abstract. We identify higher-charge configurations that satisfy Euler-Lagrange equations for the (strong coupling limit of) Faddeev-Hopf model, by means of adequate changes of the domain metric and a reduction technique based on $\alpha$-Hopf construction. In the last case it is proved that the solutions are local minima for the reduced $\sigma_2$-energy and we identify among them those who are global minima for the unreduced energy.

1. Introduction

1.1. Motivation from hadrons physics. Skyrme model, stated in the early 60ties \cite{18}, as well as Faddeev model proposed about ten years later \cite{10} are attempts to apply the soliton mechanism for particle-like excitations. Explicitly, for the first case the idea was to model baryons as smooth stable finite energy solutions (solitons) of a modified nonlinear $\sigma$-model with pion fields, while in the second case, it was suggested that gluon flux tubes in hadrons are modelled by solitons in a similar $\sigma$-model, the main difference being that the former ones were point-like (localized around a point) while the latter are knotted (localized around a loop).

To be more specific, let us present the original version of Faddeev’s model, also known as Faddeev-Hopf or Faddeev-Skyrme model. The fields in this model are maps $\vec{n}$ from $\mathbb{R}^3$ to the two-sphere $S^2$, asymptotically constant at infinity. In the static limit the energy of the system is:

$$E_{\text{Faddeev}}(\vec{n}) = \frac{1}{2} \int_{\mathbb{R}^3} \left\{ ||d\vec{n}||^2 + K(d\vec{n} \times d\vec{n})^2 \right\} d^3x.$$

where $K$ is a positive coupling constant. The second (fourth power) term give the possibility of field configurations that are stable under a spatial rescaling. Moreover the following topological lower bound holds: $E_{\text{Faddeev}}(\vec{n}) \geq c \cdot |Q(\vec{n})|^{3/4}$, where $c \neq 0$ is a numerical constant and $Q(\vec{n}) \in \pi_3(S^2) \cong \mathbb{Z}$ denotes the Hopf invariant ("charge") of $\vec{n}$ seen as map on $S^3$. The position of a field configuration is defined as the preimage of the point $(0, 0, -1)$ (antipodal to the vacuum $\vec{n}_\infty$), so it forms a closed loop.

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Let us mention that solutions $\mathbb{S}^4 \to \mathbb{S}^2$ for the strongly coupled model play also a role, in the quantized version of the theory. For more details on physical models that allow topological solitons, see [15].

1.2. Differential geometric background. The static Hamiltonian of both Skyrme and Faddeev-Hopf models is interpreted as $\sigma_{1,2}$-energy of mappings $\varphi: (M, g) \to (N, h)$ between Riemannian manifolds (see [19] following [14]):

$$E_{\sigma_1,2}(\varphi) = E_{\sigma_1}(\varphi) + K \cdot E_{\sigma_2}(\varphi) = \frac{1}{2} \int_M \left[ |d\varphi|^2 + K \cdot |\nabla^2 d\varphi|^2 \right] \nu_g.$$  

The first term is the standard Dirichlet (quadratic) energy of $\varphi$ and the second (quartic) term is the $\sigma_2$-energy introduced in 1964 by Eells and Sampson [9]. Their critical points are the well-known harmonic maps and the less studied $\sigma_2$-critical maps, respectively. We shall refer to $E_{\sigma_2}$ as strongly coupled energy. The critical points for the full energy, or $\sigma_{1,2}$-critical maps, are characterized by the equations:

$$(1.2) \quad \tau(\varphi) + K \tau_{\sigma_2}(\varphi) = 0,$$

where

- $\tau(\varphi) = \text{trace}(\nabla d\varphi)$ is the tension field of $\varphi$,
- $\tau_{\sigma_2}(\varphi) = 2[\psi(\varphi) \tau(\varphi) + d\varphi(\text{grade}(\varphi))] - \text{trace}(\nabla d\varphi) \circ \mathcal{C}_\varphi - d\varphi(\text{div}(\mathcal{C}_\varphi))$ is the $\sigma_2$-tension field of $\varphi$, cf. [4, 19, 24], with $\mathcal{C}_\varphi = d\varphi^* \circ d\varphi \in \text{End}(TM)$ denoting the Cauchy-Green tensor of $\varphi$.

Obvious solutions for (1.2) are those maps that are both harmonic and $\sigma_2$-critical. This is the case for the standard Hopf map $S^3 \to S^2$, and the only exact solution between (round) spheres known until now. But this situation seems very rare and a heuristic reason for this, given in [19], is that while the prototype for harmonic maps (from a Riemann surface to $\mathbb{C}$) is provided by a holomorphic/conformal map, the prototype of $\sigma_2$-critical maps is an area-preserving map. In this 2-dimensional context, a map encompasses both conditions if and only if it is homothetic. For a further analysis of transversally (to the Reeb foliation) area-preserving maps between 3-dimensional contact manifolds and various stability results, see [19].

1.3. Sketch of the paper. The main idea of the paper is looking for mappings $\mathbb{S}^3 \to \mathbb{S}^2$ of higher Hopf invariant, which are either harmonic or $\sigma_2$-critical and ask if they produce full higher-charge solutions by paying the price of a (bi)conformal change of the domain metric. In section 2 we point out that a harmonic horizontally conformal submersion becomes $\sigma_2$ or $\sigma_{1,2}$-critical if we replace the domain metric with a (bi)conformally related one. Some known examples are revisited. In section 3 we find (non-conformal) $\sigma_2$-critical maps of arbitrary Hopf invariant on $(\mathbb{S}^3, \text{can})$ using a general reduction technique known as $\alpha$-Hopf construction [3, 5, 6, 8] and we study their stability. We mention that the integrability of the strongly coupled Faddeev-Hopf model on $\mathbb{S}^3 \times \mathbb{R}$ (endowed with a Lorentzian metric of warped product type) has already been proved in [7].

2. Horizontally conformal configurations and related metrics

Let us recall the following

Definition 2.1. ([5]) A smooth map $\varphi: (M^m, g) \to (N^n, h)$ between Riemannian manifolds is a horizontally conformal map if, at any point $x \in M$, $d\varphi_x$ maps the horizontal space $\mathcal{H}_x = (\ker d\varphi_x)^\perp$ conformally onto $T_{\varphi(x)}N$, i.e. $d\varphi_x$ is surjective
and there exists a number \( \lambda(x) \neq 0 \) such that \( (\varphi^*h)_x \bigg|_{\mathcal{H}_x} = \lambda^2(x)g_x \bigg|_{\mathcal{H}_x} \), or equivalently \( (\zeta \varphi)_x \big|_{\mathcal{H}_x} = \lambda^2(x)Id_{TM} \big|_{\mathcal{H}_x} \). The function \( \lambda \) is the \textit{dilation} of \( \varphi \); if \( \lambda \equiv 1 \), then \( \varphi \) is a \textit{Riemannian submersion}. If a horizontally conformal map is moreover harmonic, then it is a \textit{harmonic morphism}. The mean curvatures of the distributions \( \mathcal{H} \) and \( \mathcal{V} = \ker d\varphi \) are denoted \( \mu^H \) and \( \mu^V \).

In this section we look for horizontally conformal \( \sigma_{1,2} \)-critical mappings between two Riemannian manifolds. We mention that horizontally conformal condition for complex valued maps has been analyzed in physics literature under the name \textit{eikonal equation} (see [1] and references therein).

**Remark 2.2** (\( \sigma_2 \)-tension field for horizontally conformal maps, [19]). If \( \varphi \) is horizontally conformal of dilation \( \lambda \), then

\[
\tau(\varphi) = -d\varphi \left( (n - 2)\text{grad} \ln \lambda + (m - n)\mu^V \right),
\]

\[
\tau_{\sigma_2}(\varphi) = (n - 1)\lambda^2 \left[ \tau(\varphi) + 2d\varphi(\text{grad} \ln \lambda) \right] = \frac{n - 1}{n} \tau_4(\varphi),
\]

where \( \tau_4(\varphi) \) is the Euler-Lagrange operator for the 4-energy, \( (1/4) \int_M |d\varphi|^4 \nu_g \). In particular, a submersive harmonic morphism is \( \sigma_{1,2} \)-critical if and only if it is horizontally homothetic (with minimal fibres).

So the harmonicity and \( \sigma_2 \)-criticality of a horizontally conformal map are related as follows.

**Lemma 2.3.** Let \( \varphi : (M^m, g) \rightarrow (N^n, h) \) with \( m \neq 2 \) be a horizontally conformal map of dilation \( \lambda \). Then \( \varphi \) is \( \sigma_{1,2} \)-critical if and only if it is harmonic with respect to the conformally related metric \( \tilde{g} \) on \( M \), given by

\[
(2.1) \quad \tilde{g} = \left[ 1 + K(n - 1)\lambda^2 \right] a^{m-2} \cdot g.
\]

In particular, \( \varphi \) is \( \sigma_2 \)-critical if and only if it is harmonic with respect to the conformally related metric \( \tilde{g} = \lambda a^{m-2} \cdot g \).

**Proof.** Under an arbitrary conformal change of metric \( \tilde{g} = a^2 \cdot g \), the tension field of a map becomes:

\[
\tilde{\tau}(\varphi) = \frac{1}{a^2} \left\{ \tau(\varphi) + d\varphi(\text{grad} \ln a^{m-2}) \right\}
\]

But, according to the above remark we also have:

\[
\tau_{\sigma_{1,2}}(\varphi) = \left[ 1 + K(n - 1)\lambda^2 \right] \left\{ \tau(\varphi) + d\varphi(\text{grad} \ln \left[ 1 + K(n - 1)\lambda^2 \right]) \right\}
\]

\( \square \)

Now let us recall another important type of related metrics.

**Definition 2.4.** ([5]) Let \( (M^m, g) \) be a Riemannian manifold endowed with a distribution \( \mathcal{V} \) of codimension \( n \). Denote \( \mathcal{H} = \mathcal{V}^\perp \). Two metrics are \textit{biconformally related with respect to} \( \mathcal{V} \) if it exists a smooth function \( \rho : M \rightarrow (0, \infty) \) such that:

\[
(2.2) \quad g_\rho = \rho^{-2}g^\mathcal{H} + \rho^{\frac{m-n}{m-2}}g^\mathcal{V}.
\]

The harmonicity of almost submersive maps is invariant under biconformal changes of metric \( (2.2) \) with respect to \( \mathcal{V} = \ker d\varphi \), cf. [13] [17]. In particular, for any submersive harmonic morphism \( \varphi : (M^m, g) \rightarrow (N^n, h) \) with dilation \( \lambda \) and
$m > n$, if we take on $M$ the biconformally related metric $g^\lambda$, then it becomes a Riemannian submersion with minimal fibres (and in particular, $\sigma_2$-critical).

Therefore we got two ways to obtain $\sigma_{1,2}$-critical maps from harmonic morphisms, that we now resume in the following

**Proposition 2.1.** Let $\varphi : (M^m, g) \to (N^n, h)$ be a submersive harmonic morphism with $m > n$ and dilation $\lambda$. Then:

(i) $\varphi$ is $\sigma_{1,2}$-critical with respect to the biconformally related metric $g^\lambda$ on $M$;

(ii) $\varphi$ is $\sigma_{1,2}$-critical with respect to the conformally related metric $\tilde{g} = b^2 \cdot g$ on $M$ if and only if

\[
\nabla^H \left[ b^{m-4}(b^2 + K(n-1)\lambda^2) \right] = 0.
\]

In particular, if $m \neq 4$, then $\varphi$ is $\sigma_2$-critical with respect to the conformally related metric $\tilde{g} = \lambda^{\frac{m-4}{2}} \cdot g$.

**Remark 2.5.**

(a) If $n = 2$, then biconformally related metric needed above has a simpler form: $g^\lambda = \lambda^2 g^H + g^V$.

(b) Using the $\alpha$-Hopf construction [8], for each pair of positive integers $k, \ell$, one can construct a smooth harmonic morphism $\varphi_{k,\ell} : (S^3, e^{2\gamma} \cdot \text{can}) \to (S^2, \text{can})$ with Hopf invariant $kl$, cf. [5] Example 13.5.3 (some details will be also given in the next section). So, by applying Proposition 2.1 we can obtain a $\sigma_{1,2}$-critical (or a $\sigma_2$-critical) configuration in every nontrivial class of $\pi_3(S^2) = \mathbb{Z}$ with respect to a metric (bi)conformally related to the canonical one.

(c) By composing a semiconformal map from $S^4$ to $S^3$ (used in [3]) with the above mentioned map $\varphi_{k,\ell}$, Burel [6] has obtained a family of non-constant harmonic morphisms $\Phi_{k,\ell} : (S^4, g_{k,\ell}) \to (S^2, \text{can})$ which represents the (non)trivial class of $\pi_4(S^2) = \mathbb{Z}_2$ whenever $k\ell$ is even (respectively odd). In this case too, $g_{k,\ell}$ is in the conformal class of the canonical metric (on $S^4$).

Again applying Proposition 2.1 we can obtain a $\sigma_{1,2}$-critical configuration in the nontrivial class of $\pi_4(S^2) = \mathbb{Z}_2$ with respect to a metric (bi)conformally related to the canonical one. Indeed we have only to choose a suitable function $\vartheta$ constant along the horizontal curves and to take $\tilde{g}_{k,\ell} = (\vartheta - K\lambda^2) \cdot g_{k,\ell}$.

On the other hand, to obtain a $\sigma_2$-critical point $(S^4, e^{2\nu} \cdot \text{can}) \to (S^2, \text{can})$ (i.e. an instanton for the strong coupling limit of the Faddeev-Hopf model on Minkowski space) is no more possible with the same procedure, due to conformal invariance in 4 dimensions. A $\sigma_2$-critical map defined on the same pattern as in [6] may still exist, but it might be not horizontally conformal.

### 3. Non-conformal higher-charge configurations for the strongly coupled model

In [23] Ward has proposed the investigation of the following maps

\[
(3.1) \quad \Psi_{k,\ell} : S^3_R \to \mathbb{CP}^1, \quad (z_0, z_1) \mapsto \left[ \frac{z_0^k}{|z_0|^{k-1}}, \frac{z_1^\ell}{|z_1|^{\ell-1}} \right], \quad k, \ell \in \mathbb{N}^*
\]
as higher-charge configurations for the Faddeev-Hopf model. He estimated their energy and then compared it to a conjectured topological lower bound.

It is easy to see that $\Psi_{k,\ell}$ are particular cases (via the composition with a version of stereographic projection) of the $\alpha$-Hopf construction (applied to $F : S^1 \times S^1 \to S^1$, $F(z, w) = zw^*\ell$) that provides us $\varphi^\alpha_{k,\ell} : S^3_R \to S^2$ defined by:

$$\varphi^\alpha_{k,\ell}(R \cos s \cdot e^{iz_1}, R \sin s \cdot e^{iz_2}) = \left(\cos \alpha(s), \sin \alpha(s) \cdot e^{i(kz_1 + \ell z_2)}\right),$$

where $k, \ell \in \mathbb{Z}^*$ and $\alpha : [0, \pi/2] \to [0, \pi]$ satisfies the boundary conditions $\alpha(0) = 0$, $\alpha(\pi/2) = \pi$. When $(k, \ell) = (\mp 1, 1)$ and $\alpha(s) = 2s$, this construction provides us the (conjugate) Hopf fibration.

The maps $\varphi^\alpha_{k,\ell}$ are equivariant with respect to some isoparametric functions (projections in the argument $s$) and their Hopf invariant is $Q(\varphi^\alpha_{k,\ell}) = k\ell$ (for more details see \[8\]). They have been considered in many places as the toroidal ansatz, see e.g. \[11\] \[17\] \[18\] \[22\] \[16\].

Let us work out explicitly the condition of being $\sigma_2$-critical for $\varphi^\alpha_{k,\ell}$.

Consider the open subset of the sphere $S^3_R$ parametrized by

$$\{p = (\cos s \cdot e^{iz_1}, \sin s \cdot e^{iz_2}) \mid (x_1, x_2, s) \in (0, 2\pi)^2 \times (0, \pi/2)\}.$$

The (standard) Riemannian metric of $S^3_R$ is $g_p = R^2 \left(\cos^2 s \, dx_1^2 + \sin^2 s \, dx_2^2 + ds^2\right)$.

We can immediately construct the orthonormal base for $T_p S^3_R$:

$$f_1 = \frac{1}{R \cos s} \frac{\partial}{\partial x_1}; \quad f_2 = \frac{1}{R \sin s} \frac{\partial}{\partial x_2}; \quad f_3 = \frac{1}{R} \frac{\partial}{\partial s}.$$

Analogously, if we consider $\{x = (\cos t, \sin t \cdot e^{iu}) \mid (t, u) \in (0, \pi) \times (0, 2\pi)\}$, an open subset of $S^2$, then the standard round metric reads $h = dt^2 + \sin^2 t \, du^2$.

The differential of the map $\varphi = \varphi^\alpha_{k,\ell}$ operates as follows:

$$\frac{d\varphi(f_1)}{dt} = \frac{k}{R \cos s} \frac{\partial}{\partial u}; \quad \frac{d\varphi(f_2)}{dt} = \frac{\ell}{R \sin s} \frac{\partial}{\partial u}; \quad \frac{d\varphi(f_3)}{dt} = \frac{\alpha'(s)}{R} \frac{\partial}{\partial t}.$$

As we can easily check, the vertical space $V = \mathrm{Ker} \, d\varphi$ is spanned by the unitary vector

$$E_3 = \frac{k\ell}{\sqrt{k^2 \sin^2 s + \ell^2 \cos^2 s}} \begin{pmatrix} \cos s \\ k f_1 - \ell f_2 \end{pmatrix}$$

and the horizontal distribution $H = (\mathrm{Ker} \, d\varphi)^\perp$, by the unitary vectors

$$E_1 = f_3, \quad E_2 = \frac{k\ell}{\sqrt{k^2 \sin^2 s + \ell^2 \cos^2 s}} \begin{pmatrix} \sin s \\ \ell f_1 + k f_2 \end{pmatrix}.$$

A key observation in what follows is that

- $E_1$ is an eigenvector for $\varphi^* h$, corresponding to the eigenvalue
  $$\lambda_1^2 = \left[\frac{\alpha'(s)}{R}\right]^2;$$

- $E_2$ is an eigenvector for $\varphi^* h$, corresponding to the eigenvalue
  $$\lambda_2^2 = \frac{\sin^2 \alpha(s)}{R^2} \cdot \frac{k^2 \sin^2 s + \ell^2 \cos^2 s}{\sin^2 s \cos^2 s}.$$
Remark 3.1. \textit{(a) (Horizontally conformal maps.)} Clearly $\varphi : (S^3,\text{can}) \to (S^2,\text{can})$ is horizontally conformal provided that $\lambda_1^2 = \lambda_2^2$. This is a (first order) ODE in $\alpha$:

\begin{equation}
\alpha' = \pm \sin \alpha \sqrt{\frac{k^2}{\cos^2 s} + \frac{\ell^2}{\sin^2 s}}.
\end{equation}

It has a solution for all $k$ and $\ell$, explicitly given in \cite[Example 13.5.3]{5}. Then taking $\alpha_0$ a solution and performing an appropriate conformal change of metric, we obtain a harmonic morphism $\varphi_{k,\ell}^{\alpha_0} : (S^3, e^{2\gamma\text{can}}) \to (S^2, \text{can})$.

\textit{(b) (Harmonic maps.)} For a submersion $\varphi : (M^n, g) \to (N^2, h)$, the equations of harmonicity can be translated in terms of eigenvalues of $\varphi^* h$ as follows, cf. \cite[2, 13]{Harmonic Maps}.

\begin{equation}
\begin{align*}
\frac{1}{2} E_1 (\lambda_1^2 - \lambda_2^2) &= \left( \lambda_1^2 - \lambda_2^2 \right) g(\nabla E_2, E_1) - \lambda_1^2 g(\mu^V, E_1) \\
\frac{1}{2} E_2 (\lambda_1^2 - \lambda_2^2) &= \left( \lambda_1^2 - \lambda_2^2 \right) g(\nabla E_1, E_2) - \lambda_2^2 g(\mu^V, E_2)
\end{align*}
\end{equation}

For our map given by \cite[(3.2)]{Harmonic Maps}, the second equation is satisfied trivially and the first one leads to a second order ODE in $\alpha$, cf. also \cite{Horizontally Conformal}.

According to \cite[Theorem (3.13)]{Horizontally Conformal} and \cite[(3.6)]{Horizontally Conformal} has a solution if and only if $\ell = \pm k$; moreover, it is explicitly given by $\alpha(s) = 2 \arctan \left( C \tan^k s \right)$, $C > 0$. The corresponding harmonic map $\varphi_{k,\ell}^0$ is the standard Hopf map followed by a weakly conformal map of degree $k$.

We will apply a strategy analogous to the harmonic case described above. Firstly, we need a general result, whose proof is to be find in \cite{Harmonic Maps}.

Lemma 3.2. Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersion. Then $\varphi$ is $\sigma_2$-critical if and only if the following equation is satisfied:

\begin{equation}
\text{grad}^H (\ln \lambda_1 \lambda_2) - (m-2) \mu^V = 0.
\end{equation}

Moreover, $\varphi$ remains $\sigma_2$-critical when we replace $g$ with $\overline{g} = \sigma^{-2} g^H + \rho^{-2} g^V$ if and only if $\text{grad}^H (\sigma^2 \rho^2 - m) = 0$, where $\sigma$ and $\rho$ are functions on $M$.

We can directly check that, if $\varphi$ is horizontally conformal, i.e. $\lambda_1^2 = \lambda_2^2$, then \text{(3.7)} is equivalent to the 4-harmonicity equation and that \text{(3.7)} is equivalent to:

\begin{equation}
\begin{align*}
\frac{1}{2} \lambda_1^2 E_1 (\lambda_1^2) + \frac{1}{2} \lambda_2^2 E_1 (\lambda_2^2) - (m-2) \lambda_1^2 \lambda_2^2 g(\mu^V, E_1) &= 0 \\
\frac{1}{2} \lambda_2^2 E_2 (\lambda_1^2) + \frac{1}{2} \lambda_1^2 E_2 (\lambda_2^2) - (m-2) \lambda_1^2 \lambda_2^2 g(\mu^V, E_2) &= 0
\end{align*}
\end{equation}

For our map given by \cite[(3.2)]{Horizontally Conformal}, the second equation in \text{(3.8)} is trivially satisfied and the first one leads to the following second order ODE in $\alpha$:

\begin{equation}
\alpha' \sin \alpha \left\{ \alpha'' \sin \alpha + (\alpha')^2 \cos \alpha \cdot \left( \frac{k^2}{\cos^2 s} + \frac{\ell^2}{\sin^2 s} \right) + \alpha' \sin \alpha \cdot \left( \frac{k^2}{\sin s \cos^3 s} - \frac{\ell^2}{\cos s \sin^3 s} \right) \right\} = 0.
\end{equation}

Contrary to the harmonic case, this equation always has a (unique) solution, for all $\ell$, $k$ which satisfies boundary conditions $\alpha(0) = 0$, $\alpha(\pi/2) = \pi$: 
\( \alpha(s) = \begin{cases} 
\arccos \left( 1 - 2 \frac{\ln \frac{k^2 \sin^2 s + \cos^2 s}{t^2}}{\ln \frac{k^2}{t^2}} \right), & \text{if } |k| > |\ell| \\
2s, & \text{if } |k| = |\ell| 
\end{cases} \)

Therefore, with \( \alpha \) given above, we have obtained a \( \sigma_2 \)-critical map \( \varphi_{k,\ell}^\alpha \) in every nontrivial homotopy class of \( \pi_3(\mathbb{S}^2) \). Moreover, they are local minima among equivariant maps of the same type.

**Proposition 3.1.** The equation (3.9) is the Euler-Lagrange equation for the reduced \( \sigma_2 \)-energy functional:

\[
(3.11) \quad \varepsilon_{\sigma_2}(\alpha) = \frac{2\pi^2}{R} \int_0^{\frac{\pi}{\ell}} (k^2 \tan s + \ell^2 \cot s) (\alpha')^2 \sin^2 \alpha \, ds.
\]

The solutions (3.10) are stable critical points for the energy functional \( \varepsilon_{\sigma_2} \).

**Proof.** Note that \( \mathcal{E}_{\sigma_2}(\varphi_{k,\ell}^\alpha) = \varepsilon_{\sigma_2}(\alpha) \). Consider a fixed endpoints variation \( \{\alpha_t\} \) of \( \alpha \). To prove the result, we only have to follow a direct computation using integration by parts. For the second variation we get:

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \varepsilon_{\sigma_2}(\alpha_t) = \frac{4\pi^2}{R} \int_0^{\frac{\pi}{\ell}} \left( \frac{d\alpha_t}{dt} \bigg|_{t=0} \right)^2 (\alpha')^2 (k^2 \tan s + \ell^2 \cot s) \, ds \geq 0.
\]

\( \Box \)

**Remark 3.3.**

(a) The \( \sigma_2 \)-critical maps \( \varphi_{k,\ell}^\alpha \) with \( \alpha \) given by (3.10) are also harmonic (so critical points for the full energy) only if \( |k| = |\ell| = 1 \), that corresponds to the (conjugate) Hopf fibration.

Recall that in [21, 22] it has been proved that Hopf map is a stable critical point for \( \mathcal{E}_{\sigma_1,2} \) (if \( K \geq 1 \)) and an absolute minimizer for \( \mathcal{E}_{\sigma_2} \) (the equivalent quartic energy term used in [21, 22] is \( \int_M \|\varphi^* \Omega\|^2 \nu_g \)). For further discussions about critical configurations for the Faddeev-Hopf model on \( \mathbb{S}^3 \) inside the same ansatz, see also [1].

(b) The \( \sigma_2 \)-energy of critical maps obtained from (3.10) is:

\[
(3.12) \quad \mathcal{E}_{\sigma_2}(\varphi_{k,\ell}^\alpha) = \frac{16\pi^2}{R} \cdot \frac{k^2 - \ell^2}{\ln (k^2/\ell^2)} \quad (|k| > |\ell|), \quad \mathcal{E}_{\sigma_2}(\varphi_{k,k}^\alpha) = \frac{16\pi^2 k^2}{R} \quad (|k| = |\ell|)
\]

where \( Q = k\ell \) is the Hopf charge of the solution (compare with [27 (29)]).

In particular, the \( \sigma_2 \)-energy of the Hopf map is \( \frac{16\pi^2}{R} \).

(c) The \( \sigma_2 \)-critical maps given by (3.10) becomes \( \sigma_{1,2} \)-critical with respect to an appropriately perturbed domain metric \( \mathcal{F} = \sigma^{-2} g^H + \sigma^{-4} g^V \). Indeed, being \( \sigma_2 \)-critical is invariant under these changes of metric, cf. Lemma 3.2 while the tension field becomes \( \mathcal{T}(\varphi_{k,\ell}^\alpha) = \sigma^2 \left[ \tau(\varphi_{k,\ell}^\alpha) + d\varphi_{k,\ell}^\alpha (\text{grad} \ln \sigma^{-2}) \right] \).

But, at least locally, it is possible to find \( \sigma \) such that \( \mathcal{T}(\varphi_{k,\ell}^\alpha) = 0 \), i.e. \( \varphi_{k,\ell}^\alpha \) is also harmonic.

Recall the topological lower bound found in [22]: \( \mathcal{E}_{\sigma_2}(\varphi) \geq \frac{16\pi^2}{R} Q(\varphi) \). In the family of solutions \( \varphi_{k,\ell}^\alpha \) this bound is attained if and only if \( k = \ell \). Therefore:

**Proposition 3.2.** The solution \( \varphi_{k,k}^\alpha \) is an absolute \( \sigma_2 \)-minima in its homotopy class.
Notice that, denoting by $\Omega$ the area-form on $S^2$ and by $\eta$ the standard contact form on $S^3$ (see [19] for details), we have $(\varphi_{\ell,k}^\alpha)^*\Omega = k \, d\eta$, i.e. $\varphi_{\ell,k}^\alpha$ is transversally area-preserving up to rescale, as expected for heuristic reasons presented in the introduction.

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