Products of radial derivative and integral-type operators from Zygmund spaces to Bloch spaces

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. In this paper we investigate the boundedness and compactness of the products of radial derivative operator and the following integral-type operator

$$I^g_f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \quad z \in B$$

where $g \in H(B), g(0) = 0, \varphi$ is a holomorphic self-map of $B$, between Zygmund spaces and Bloch spaces.

Keywords: radial derivative operator; integral-type operator; Zygmund space; Bloch space

1. Introduction

Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in the complex vector space $\mathbb{C}^n$ and $< z, w > = z_1\overline{w_1} + \ldots + z_n\overline{w_n}$. Let

$$\Re f(z) = \sum z_j \frac{\partial f}{\partial z_j}(z)$$

stand for the radial derivative of $f \in H(B)[1]$. It is easy to see that, if $f \in H(B), f(z) = \sum_\alpha a_\alpha z^\alpha$, where $\alpha$ is a multi-index, then $\Re f(z) = \sum_\alpha |\alpha|a_\alpha z^\alpha$. We write $\Re^m f = \Re(\Re^{m-1} f)$.

The Bloch space $\mathcal{B}(B)$ is the space of all $f \in H(B)$ such that

$$\|f\|_B := |f(0)| + \sup_{z \in B}(1 - |z|^2)|\nabla f(z)| < \infty,$$

The little Bloch space $\mathcal{B}_0(B)$ is the space of all $f \in H(B)$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)|\nabla f(z)| = 0.$$
It is well known that $f \in B(B)$ if and only if $b(f) := \sup_{z \in B}(1 - |z|^2)|\Re f(z)| < \infty$, and that $f \in B_0$ if and only if $\lim_{|z| \to 1}(1 - |z|^2)|\Re f(z)| = 0$. Moreover, the following asymptotic relation holds[2]:

$$\|f\|_B \asymp |f(0)| + b(f).$$

Let $Z$ denote the class of all $f \in H(B)$, such that

$$\sup_{z \in B}(1 - |z|^2)|\Re^2 f(z)| < \infty. \tag{1}$$

Therefore, $Z$ is called the Zygmund class. Under the natural norm

$$\|f\|_Z := |f(0)| + |f'(0)| + \sup_{z \in B}(1 - |z|^2)|\Re^2 f(z)| < \infty. \tag{2}$$

$Z$ becomes a Banach space. Zygmund class with this norm will be called the Zygmund space.

The little Zygmund space $Z_0$ denote the closure in $Z$ of the set of all polynomials. From Theorem 7.2 of[3], we see that

$$f \in Z_0 \iff \lim_{z \in B}(1 - |z|^2)|\Re^2 f(z)| = 0. \tag{3}$$

Suppose that $g \in H(B), g(0) = 0$ and $\varphi$ is a holomorphic self-map of $B$, then an integral-type operator, denote by $I^g_\varphi$ on $H(B)$, is defined as follows:

$$I^g_\varphi f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \quad g \in H(B), \ z \in B \tag{4}$$

Operator (4) is related to the following operators

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad I_g(f) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}. \tag{5}$$

acting on $H(B)$, introduced in [4] and studied in [5-10], as well as the operator $T_g$ introduced in [11]acting on holomorphic functions on the unit polydisc (see,also[12],[13],as well as [14] for a particular case of the operator). One of motivations for introducing operator $I^g_\varphi$ stems from the operator introduced in [15]. Some characterizations of the boundedness and compactness of these and some other integral-type operators mostly in $\mathbb{C}^n$, can be found, for example, in[4,6,7-9,15-31].
In this paper we study the boundedness and compactness of products of $\Re$ and $I_\varphi$ between Zygmund space and Bloch spaces on the unit Ball.

Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other.

2. Auxiliary results

Lemma 1. Let $\Re$ be the radial derivative operator. The product of $\Re$ and $I_\varphi$
\[ \Re[I_\varphi(f)](z) = \Re(f(\varphi(z)))g(z) \] (6)

Lemma 2. Suppose $f \in \mathcal{Z}$. The following statements are true.
(a). There is a positive constant $C$ independent of $f$ such that
\[ |\Re f(z)| \leq C\|f\|_\mathcal{Z} \ln \frac{e}{1-|z|^2}. \] (7)
(b). There is a positive constant $C$ independent of $f$ such that
\[ \|f\|_\infty = \sup_{|z|<1} |f(z)| \leq C\|f\|_\mathcal{Z}. \] (8)

For studying the compactness of the operator $\Re I_\varphi : \mathcal{Z} \to \mathcal{B}$, we need the following Lemma. The proof of the lemma is standard, hence we omit the details.

Lemma 3. Assume that $g \in H(\mathbb{B})$, $\varphi$ be a holomorphic self-map of $\mathbb{B}$. Let $T = \Re I_\varphi$, then $T : \mathcal{Z}(or \mathcal{Z}_0) \to \mathcal{B}$ is compact if and only if $T$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{Z}(or \mathcal{Z}_0)$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $T f_k \to 0$ as $k \to \infty$.

Lemma 4. A closed set $K$ in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies
\[ \lim_{|z|\to 1} \sup_{f \in K} (1 - |z|^2)|\Re f(z)| = 0. \] (9)

3. The boundedness and compactness of $\Re I_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to \mathcal{B}(\mathcal{B}_0)$

Theorem 1. Let $\varphi$ be a holomorphic self-map of $\mathbb{B}$. Then the following statements are equivalent.
(a) $\Re I_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded;
(b) $\Re I_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded;
(c) \[ \sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)|\Re \varphi(z)||g(z)|}{1 - |\varphi(z)|^2} < \infty, \] (10)
and
\[ \sup_{z \in \mathbb{B}} (1 - |z|^2)|\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \infty. \] (11)
Proof. (a)⇒(b) This implication is obvious.

(b)⇒(c) Assume that $\mathcal{R}^g_{\varphi}: \mathcal{Z}_0 \rightarrow \mathcal{B}$ is boundedness, i.e., there exists a constant $C$ such that

$$
\|\mathcal{R}^g_{\varphi}(f)\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{Z}},
$$

for all $f \in \mathcal{Z}_0$. Taking the functions $f_j(z) = z_j \in \mathcal{Z}_0$ and $f_j(z) = z_j - z_j^2 \in \mathcal{Z}_0$, $j = 1, 2, \cdots, n$, we get

$$
\sup_{z \in \mathcal{B}} (1 - |z|^2)|\varphi_j(z)||\mathcal{R}\varphi(z)g(z) + \Re g(z)| < \infty, \quad (12)
$$

and

$$
\sup_{z \in \mathcal{B}} (1 - |z|^2)((\varphi_j(z) - 4\varphi_j^2(z))(\Re\varphi(z)g(z) + \Re g(z)) + 2\varphi_j^2(z)\Re g(z)) < \infty. \quad (13)
$$

Using (12) and the boundedness of functions $\varphi_j$, we have that

$$
\sup_{z \in \mathcal{B}} (1 - |z|^2)|\Re\varphi(z)g(z) + \Re g(z)| < \infty. \quad (14)
$$

Then with (13), (14) and the boundedness of functions $\varphi_j$, we have that

$$
\sup_{z \in \mathcal{B}} (1 - |z|^2)|\Re g(z)| < \infty, \sup_{z \in \mathcal{B}} (1 - |z|^2)|\Re\varphi(z)g(z)| < \infty \quad (15)
$$

Set

$$
h(\zeta) = (\zeta - 1)((1 + \ln \frac{1}{1 - \zeta})^2 + 1), \zeta \in \mathbb{C},
$$

and

$$
h_a(z) = \frac{h(<z,a>)}{|a|^2}(\ln \frac{1}{1 - |a|^2})^{-1},
$$

for $a \in \mathbb{B} \setminus \{0\}$. It is known that $h_a(z) \in \mathcal{Z}_0$(see[17]). Since

$$
\Re h_a(z) = \frac{<z,a>}{|a|^2}(\ln \frac{1}{1 - <z,a>})^2(\ln \frac{1}{1 - |a|^2})^{-1},
$$

and

$$
\Re^2 h_a(z) = \Re h_a(z) + \frac{2 <z,a>^2}{|a|^2(1 - <z,a>)}(\ln \frac{1}{1 - <z,a>})(\ln \frac{1}{1 - |a|^2})^{-1},
$$

for $|\varphi(z)| > \sqrt{1 - 1/e}$ we have

$$
C\|\mathcal{R}^g_{\varphi}\|_{\mathcal{Z}}\|\mathcal{R}^g_{\varphi}(h\varphi(z))\|_{\mathcal{B}} \geq (1 - |z|^2)\ln \frac{1}{1 - |\varphi(z)|^2}\Re g(z) - \frac{2(1 - |z|^2)}{1 - |\varphi(z)|^2}|\varphi(z)|^2|\Re\varphi(z)||g(z)| - (1 - |z|^2)\ln \frac{1}{1 - |\varphi(z)|^2}|\Re\varphi(z)||g(z)|.
$$
Hence

\[
(1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re g(z)| \leq C + \frac{2(1 - |z|^2)}{1 - |\varphi(z)|^2} |\varphi(z)| |\Re \varphi(z)||g(z)| + (1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)|
\]

\[
\leq C + (2 + e) \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)|, 
\]

which using the fact of \((1 - |\varphi(z)|^2) \ln \frac{1}{1 - |\varphi(z)|^2} \leq e\).

For \(|a| > \sqrt{1 - 1/e}\), set

\[
f_a(z) = \frac{h(< z, a >)}{|a|^2}(\ln \frac{1}{1 - |a|^2})^{-1} - \frac{1}{\ln \frac{1}{1 - |a|^2}}.
\]

Then \(f_a \in Z_0\). It is easy to see that

\[
\Re f_a(z) = \frac{< z, a >}{|z|^2}(\ln \frac{1}{1 - |a|^2})^{-1}(\ln \frac{1}{1 - |a|^2})^{-1} - \frac{1}{\ln \frac{1}{1 - |a|^2}},
\]

\[
\Re^2 f_a(z) = \Re f_a(z) + \frac{2 < z, a >^2 |a|^2(1 - z, a >) (\ln \frac{1}{1 - |a|^2})(\ln \frac{1}{1 - |a|^2})^{-1} }{- < z, a > + \ln \frac{1}{1 - |a|^2}}.
\]

Therefore

\[
C\|\Re I_{\varphi}\| \geq \|\Re I_{\varphi}^g(f_{\varphi(z)})\|_B = \sup_{z \in B} (1 - |z|^2) |\Re^2 I_{\varphi}^g(f_{\varphi(z)}) (z)|
\]

\[
= (1 - |z|^2)(\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} + \ln \frac{1}{1 - |\varphi(z)|^2}) |\Re \varphi(z)||g(z)|
\]

\[
\geq (1 - |z|^2)(\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} + 1) |\Re \varphi(z)||g(z)|
\]

\[
= \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)|. 
\]

On the other hand, from (15) we have that

\[
\sup_{|\varphi(z)| \leq \sqrt{1 - 1/e}} (1 - |z|^2) |\Re g(z)| \ln \frac{1}{1 - |\varphi(z)|^2} \leq \sup_{|\varphi(z)| \leq \sqrt{1 - 1/e}} (1 - |z|^2) |\Re g(z)| < \infty. 
\]

Hence from (15),(16),(17)and (18), we obtain (11). Further, from (17), we have

\[
\sup_{|\varphi(z)| > \sqrt{1 - 1/e}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)| \leq C. 
\]
On the other hand, from (15) we have that
\[
\sup_{|\varphi(z)| \leq \sqrt{1-1/e}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)| \leq e, \quad \sup_{|\varphi(z)| \leq \sqrt{1-1/e}} (1 - |z|^2) |\Re \varphi(z)||g(z)| < \infty. \quad (20)
\]
Combining (19) and (20), (10) follows.

**Theorem 2.** Let \( \varphi \) be a holomorphic self-map of \( \mathbb{B} \). Then the following statements are equivalent.

(a) \( \Re I^g_\varphi : \mathbb{Z} \to \mathcal{B} \) is compact;
(b) \( \Re I^g_\varphi : \mathbb{Z}_0 \to \mathcal{B} \) is compact;
(c) \( \Re I^g_\varphi : \mathbb{Z} \to \mathcal{B} \) is bounded,
\[
\lim_{|\varphi(z)| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)| = 0, \quad (21)
\]
and
\[
\lim_{|\varphi(z)| \to 1} (1 - |z|^2)|\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0. \quad (22)
\]

**Proof.** (a) \( \Rightarrow \) (b) This is obvious.

(b) \( \Rightarrow \) (c) Assume that \( \Re I^g_\varphi : \mathbb{Z}_0 \to \mathcal{B} \) is compact, then it is clear that \( \Re I^g_\varphi : \mathbb{Z}_0 \to \mathcal{B} \) is bounded. By theorem 1, we know that \( \Re I^g_\varphi : \mathbb{Z} \to \mathcal{B} \) is bounded. Let \((z^k)_{k \in \mathbb{N}}\) be a sequence in \( \mathbb{B} \) such that \( |\varphi(z^k)| \to 1 \) as \( k \to \infty \) and \( \varphi(z^k) \neq 0, k \in \mathbb{N} \). Set
\[
h_k(z) = \frac{h(<z, \varphi(z^k)>)}{|\varphi(z^k)|^2} (\ln \frac{1}{1 - |\varphi(z^k)|^2})^{-1}, k \in \mathbb{N}.
\]
Then from the proof of theorem 1, we see that \( h_k \in \mathbb{Z}_0 \), for each \( k \in \mathbb{N} \). Moreover \( h_k \to 0 \) uniformly on compact subsets of \( \mathbb{B} \), as \( k \to \infty \).

Since \( \Re I^g_\varphi : \mathbb{Z}_0 \to \mathcal{B} \) is compact, by Lemma 3
\[
\lim_{k \to \infty} \|\Re[I^g_\varphi(h_k)]\|_{\mathcal{B}} = 0.
\]
On the other hand, similar to the proof of Theorem 1, we have
\[
0 \leq \|\Re I^g_\varphi(h_k)\|_{\mathcal{B}} \geq (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)|
- \frac{2(1 - |z^k|^2)}{1 - |\varphi(z^k)|^2} |\varphi(z^k)|^2 |\Re \varphi(z^k)||g(z^k)|
- (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|
= (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)|
- M_1 \frac{(1 - |z^k|^2)}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|,
\]
where \( M_1 := 2|\varphi(z^k)|^2 - (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} \).

From this we obtain

\[
\lim_{k \to \infty} (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)| \leq \lim_{k \to \infty} \frac{(1 - |z^k|^2)}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)| = 0, \quad (23)
\]

if one of these two limits exists, which use the case of

\[
\lim_{k \to \infty} [2|\varphi(z^k)|^2] + (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} = 2.
\]

Next, set

\[
f_k(z) = \frac{h(<z, \varphi(z^k)>)}{|\varphi(z^k)|^2} (\ln \frac{1}{1 - |\varphi(z^k)|^2})^{-1} - \int_0^1 \ln^2 t \frac{1}{1 - |t z, \varphi(z^k)|} \frac{dt}{t (\ln \frac{1}{1 - |\varphi(z^k)|^2})^{-2}}.
\]

Since \( \Re f : \mathcal{Z}_0 \to \mathcal{B} \) is compact, we have \( \| \Re f_k \| = 0 \) as \( k \to \infty \). Thus, similar to the proof of Theorem 1, when \( \sqrt{1 - \frac{1}{e}} < |\varphi(z^k)| < 1 \)

\[
0 \leq \| \Re f_k \| \leq (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} - \frac{|\varphi(z^k)|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|
\]

\[
\leq (1 - |z^k|^2) \frac{|\varphi(z^k)|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|
\]

\[
= (1 - |z^k|^2) \frac{1 - |\varphi(z^k)|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|
\]

\[
= \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|
\]

\[
= \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)|,
\]

where \( M_2 := 1 - \frac{1}{e} - (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} \).

Hence

\[
\lim_{k \to \infty} \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)||g(z^k)| = \lim_{k \to \infty} (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)| = 0. \quad (24)
\]

From (24) easily following that \( \lim_{k \to \infty} (1 - |z^k|^2)|\Re g(z^k)| = 0 \), which altogether imply (21) and (22).

(c) \( \Rightarrow \) (a)

\[
C_1 = (1 - |z|^2)|\Re \varphi(z)||g(z)| < \infty, \quad C_2 = (1 - |z|^2)|\Re g(z)| < \infty. \quad (25)
\]
For every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re\varphi(z)||g(z)| < \varepsilon, \quad (1 - |z|^2)|\Re g(z)|\ln \frac{e}{1 - |z|^2} < \varepsilon.$$ (26)

Assume that $(f_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k \in \mathbb{N}} \|f_k\|_Z \leq L$ and $f_k$ converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Let $K = \{z \in \mathbb{B} : |\varphi(z)| \leq \delta\}$. Then by (25) and (26), we have that

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |(\Re^2 I_{\varphi}^\delta (f_k))(z)| = \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f_k(\varphi(z))\Re \varphi(z)g(z) + \Re f_k(\varphi(z))\Re g(z)|$$

$$\leq \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f_k(\varphi(z))\Re \varphi(z)g(z)|$$

$$+ \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re f_k(\varphi(z))\Re g(z)|$$

$$\leq \sup_{z \in K} (1 - |z|^2) |\Re^2 f_k(\varphi(z))\Re \varphi(z)g(z)|$$

$$+ \sup_{z \in K} (1 - |z|^2) |\Re f_k(\varphi(z))\Re g(z)|$$

$$\leq \sup_{z \in K} (1 - |z|^2) |\Re^2 f_k(\varphi(z))\Re \varphi(z)g(z)|$$

$$+ \sup_{z \in K} (1 - |z|^2) |\Re f_k(\varphi(z))\Re g(z)|$$

$$+ \sup_{z \in \mathbb{B}\setminus K} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)g(z)| \|f_k\|_Z$$

$$+ C \sup_{z \in \mathbb{B}\setminus K} \ln \frac{e}{1 - |\varphi(z)|^2} |\Re g(z)| \|f_k\|_Z$$

$$\leq C_1 \sup_{z \in K} |\Re^2 f_k(\varphi(z))| + C_2 \sup_{z \in K} |\Re f_k(\varphi(z))| + (C + 1)\varepsilon \|f_k\|_Z$$

Hence

$$\|\Re^2 I_{\varphi}^\delta (f_k)\|_B \leq C_1 \sup_{z \in K} |\Re^2 f_k(\varphi(z))| + C_2 \sup_{z \in K} |\Re f_k(\varphi(z))|$$

$$+ (C + 1)\varepsilon \|f_k\|_Z + |f_k'(\varphi(0))| |\varphi'(0)|$$

Since $(f_k)_n$ converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$, Cauchy’s estimate gives that $\Re f_k \to 0$ and $\Re^2 f_k \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{B}$. Hence, letting $k \to \infty$, we obtain

$$\lim_{k \to \infty} \|\Re^2 I_{\varphi}^\delta (f_k)\|_B = 0.$$
**Theorem 4.** Let $\varphi$ be a holomorphic self-map of $\mathbb{B}$. Then $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded if and only if $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}$ is bounded

\[
\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}g(z)| = 0, \quad (27)
\]

\[
\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}\varphi(z)||g(z)| = 0. \quad (28)
\]

**Proof:** Assume that $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded. Then, it is clear that $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}$ is bounded. Taking the function $f_j(z) = z_j$ and $f_j(z) = z_j - z_j^2, j = 1, 2, \ldots, n$, we obtain (27),(28).

Assume that $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}$ is bounded and (27),(28) holds. Then for each polynomial $p$, we have that

\[
(1 - |z|^2)|\mathcal{R}^2(I^{g}_{\varphi}p)(z)|| \leq (1 - |z|^2)|\mathcal{R}^2p(z)||\mathcal{R}\varphi(z)||g(z)|
\]

\[
+ (1 - |z|^2)|\mathcal{R}p(\varphi(z))||\mathcal{R}g(z)|, \quad (29)
\]

from (27),(28) it follows that $\mathcal{R}I^{g}_{\varphi}p \in \mathcal{B}_0$. Since the set of all polynomials is dense in $\mathcal{Z}_0$, we have that for every $f \in \mathcal{Z}_0$, there is a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ such that $\|f - p_n\|_\mathcal{Z} \to 0$ as $n \to \infty$. Hence

\[
\|\mathcal{R}I^{g}_{\varphi}(f) - \mathcal{R}I^{g}_{\varphi}(p_n)\|_\mathcal{B} \leq \|\mathcal{R}I^{g}_{\varphi}\|_{\mathcal{Z}_0 \to \mathcal{B}}\|f - p_n\|_\mathcal{Z} \to 0
\]

as $n \to \infty$. Since the operator $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}$ is bounded, hence $\mathcal{R}I^{g}_{\varphi}(\mathcal{Z}_0) \subseteq \mathcal{B}_0$.

**Theorem 5.** Let $\varphi$ be a holomorphic self-map of $\mathbb{B}$. Then the following statements are equivalent.

(a) $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z} \to \mathcal{B}$ is compact;

(b) $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}_0$ is compact;

(c) 

\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\mathcal{R}\varphi(z)||g(z)| = 0, \quad (30)
\]

and

\[
\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0. \quad (31)
\]

**Proof:** (a)$\Rightarrow$(b). It is clear.

(b)$\Rightarrow$(c). Assume that $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}_0$ is compact, then $\mathcal{R}I^{g}_{\varphi} : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded. From the proof of Theorem 4, we known that

\[
\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}g(z)| = 0,
\]

\[
\lim_{|z| \to 1} (1 - |z|^2)|\mathcal{R}\varphi(z)||g(z)| = 0,
\]
Hence, if \( \| \varphi \| < 1 \),
\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)| \leq \frac{1}{1 - \|\varphi\|_\infty} \lim_{|z| \to 1} (1 - |z|^2)|\Re g(z)| = 0,
\]
\[
\lim_{|z| \to 1} (1 - |z|^2)|\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} \leq \ln \frac{e}{1 - \|\varphi\|_2^2} \lim_{|z| \to 1} (1 - |z|^2)|\Re g(z)| = 0.
\]
from which the result follows in this case.

Assume \( \| \varphi \| = 1 \). Let \((\varphi(z^k))_{k \in \mathbb{N}}\) be a sequence such that \( |\varphi(z^k)| \to 1 \) as \( k \to \infty \). Since \( \Re I_\varphi^g : \mathcal{Z}_0 \to \mathcal{B} \) is compact, by Theorem 2,
\[
\lim_{|\varphi(z)| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)||g(z)| = 0,
\] (32)
and
\[
\lim_{|\varphi(z)| \to 1} (1 - |z|^2)|\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0.
\] (33)
It is not difficult to see that (28),(32) implies (30). Similar, (27) and (33) imply (31).

(c)\(\Rightarrow\)(a). Let \( f \in \mathcal{Z} \), we have
\[
(1 - |z|^2)|\Re^2(I_\varphi^g(f))(z)| \leq (1 - |z|^2)|\Re \varphi(z)||g(z)| + (1 - |z|^2) \ln \frac{e}{1 - |\varphi(z)|^2} |\Re g(z)||f|_\mathcal{Z}.
\]
Taking the supremum in this inequality over all \( f \in \mathcal{Z} \) such that \( \|f\|_\mathcal{Z} \leq 1 \). Letting \( |z| \to 1 \) and using (30),(31)
\[
\lim_{\|z\| \to 1} \sup_{\|f\|_\mathcal{Z} \leq 1} (1 - |z|^2)|\Re^2(I_\varphi^g(f))(z)| = 0.
\]
Using Lemma 3, we obtain that the operator \( \Re I_\varphi^g : \mathcal{Z} \to \mathcal{B}_0 \) is compact.

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