On asymptotic behavior of solutions to non-uniformly elliptic equations with generalized Orlicz growth

O.V. Hadzhy, M.O. Savchenko, I.I. Skrypnik, M.V. Voitovych

August 12, 2022

Abstract

We study asymptotic behavior of sub-solutions to non-uniformly elliptic equations with nonstandard growth. In particular, Harnack type inequalities are proved. Our approach gives new results for the cases with \((p,q)\) nonlinearity and generalized Orlicz growth.

Keywords: non-uniformly elliptic equations, generalized Orlicz growth, Harnack type inequalities.

MSC (2010): 35B40, 35D30, 35J60.

1 Introduction and main results

To explain the point of view of this research consider the following equations

\[
div \left( H_i(x,|\nabla u|) \frac{\nabla u}{|\nabla u|^2} \right) = 0, \quad i = 1, 2,
\]

\[
H_1(x,v) = v^p + a_1(x)v^q, \quad a_1(x) \geq 0, \quad v > 0,
\]

and

\[
H_2(x,v) = v^p \left( 1 + a_2(x) \log(1 + v) \right), \quad a_2(x) \geq 0, \quad v > 0.
\]

It is well known (see [8]) that the correspondent non-negative bounded local weak solutions satisfy Harnack’s type inequality if

\[
\text{osc}_{B_r(x_0)} a_1(x) \leq A_1 r^{q-p} \quad \text{and} \quad \text{osc}_{B_r(x_0)} a_2(x) \leq A_2 \left( \log \frac{1}{r} \right)^{L_1}, \quad A_1, A_2 > 0,
\]

or more generally (see [23]), Harnack’s type inequality holds under the conditions

\[
\text{osc}_{B_r(x_0)} a_1(x) \leq A_1 \left( \log \frac{1}{r} \right)^{L_1} r^{q-p} \quad \text{and} \quad \text{osc}_{B_r(x_0)} a_2(x) \leq A_2 \left[ \log \log \frac{1}{r} \right]^{L_2} \left( \log \frac{1}{r} \right)^{-1}, \quad A_1, A_2 > 0,
\]

if \(L_1, L_2 > 0\) are sufficiently small.

Now let \(a_1(x) = a_2(x) = \left| \log \left| \log \frac{1}{|x-x_0|} \right| \right|^L, L \in \mathbb{R}^1\) and set \(G_1(v) = v^p + v^q\), \(G_2(v) = v^p \left( 1 + \log(1 + v) \right)\), then evidently we have

\[
\gamma^{-1} \left| \log \left| \log \frac{1}{|x-x_0|} \right| \right|^L G_i(v) \leq H_i(x,v) \leq \gamma G_i(v), \quad i = 1, 2, \quad \text{if} \quad L < 0,
\]
and
\[ \gamma^{-1} G_i(v) \leq H_i(x,v) \leq \gamma \left| \log \frac{1}{|x-x_0|} \right|^L G_i(v), \quad i = 1, 2, \quad \text{if } L > 0, \]
for \( v > 0 \) and for \( x \in B_R(x_0) \), if \( R \) is sufficiently small.

By our main Theorems (see results below) Harnack’s type inequality for non-negative bounded local weak solutions of the correspondent equation will be valid if \( L \) is sufficiently small. One can see that in this sense our results improve the previous ones. However our point of view is to take under consideration also non uniformly elliptic equations of the previous type. Of course, here it would be interesting to unify our approach.

More precisely, in this paper we are concerned with elliptic equations
\[ \text{div} \left( H(x,|\nabla u|) \frac{\nabla u}{|\nabla u|^2} \right) = 0, \quad H(x,v) := \int_0^v h(x,s) \, ds, \quad v > 0, \quad x \in \Omega, \tag{1.1} \]
where \( \Omega \) is bounded domain in \( \mathbb{R}^n, n \geq 2 \). We suppose that the function \( h(x,v) : \Omega \times \mathbb{R}^1_+ \to \mathbb{R}^1_+ \) is such that \( h(\cdot, v) \) is Lebesgue measurable for all \( v \in \mathbb{R}^1_+ \), and \( h(x, \cdot) \) is increasing and continuous for almost all \( x \in \Omega \), \( \lim_{v \to 0} h(x,v) = 0 \), \( \lim_{v \to \infty} h(x,v) = \infty \).

We assume also that the following structure conditions are satisfied
\[ K_1 a(x) g(x,v) \leq h(x,v) \leq K_2 b(x) g(x,v), \quad x \in \Omega, v \in \mathbb{R}^1_+, \tag{1.2} \]
where \( K_1, K_2 \) are positive constants.

This type of equations belongs to a wide class of non uniformly elliptic equations with generalized Orlicz growth. In terms of the function \( g \), this class can be characterized as follows.

\( (g) \) There exist \( 1 < p < q \) such that for \( w \geq v > 0 \) there holds
\[ \left( \frac{w}{v} \right)^{p-1} \leq \frac{g(x,w)}{g(x,v)} \leq \left( \frac{w}{v} \right)^{q-1}, \quad x \in \Omega. \]

We also assume that one of the following conditions holds:

\( (g_\mu) \) there exists \( R_0 > 0 \) and non-increasing function \( \mu(r) \geq 1 \) for \( r \in (0,R_0) \), \( \lim_{r \to 0} \mu(r) = +\infty \), \( \lim_{r \to 0} \mu(r)r = 0 \), such that for any \( K > 0 \) there holds
\[ g \left( x, \frac{v}{r} \right) \leq C(K) \mu(r)g \left( y, \frac{v}{r} \right), \quad \text{for any } x,y \in B_r(x_0) \subset B_{R_0}(x_0) \subset \Omega \text{ and } r \leq v \leq K, \]
with some \( C(K) > 0 \),
or

\( (g_\lambda) \) there exists \( R_0 > 0 \) and non-decreasing function \( 0 < \lambda(r) \leq 1 \) for \( r \in (0,R_0) \), \( \lim_{r \to 0} \lambda(r) = 0 \), \( \lim_{r \to 0} \frac{\lambda(r)}{r} = +\infty \), such that for any \( K > 0 \) there holds
\[ g \left( x, \frac{v}{r} \right) \leq C(K) g \left( y, \frac{v}{r} \right), \quad \text{for any } x,y \in B_r(x_0) \subset B_{R_0}(x_0) \subset \Omega \text{ and } r \leq v \leq \lambda(r) K, \]
with some \( C(K) > 0 \).
For the functions \( a(x), b(x) \geq 0 \) we assume that
\[
a^{-1}(x) \in L^t_{\text{loc}}(\Omega), \quad b(x) \in L^s_{\text{loc}}(\Omega), \quad \frac{1}{tp} + \frac{1}{sq} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}, \tag{1.3}
\]
\[
t \in \left( \max \left( 1, \frac{q-p+1}{p-1} \right), \infty \right], \quad s \in (1, \infty].
\]

**Remark 1.1.** We note that the assumptions on the functions \( a(x) \) and \( b(x) \) imply the local boundedness of sub-solutions to Eq. (1.1), for this we refer the reader to the paper by Cupini, Marcellini and Mascaro [16].

Here some typical examples of the functions \( g(x,v) \) and \( h(x,v) \), which satisfy the above conditions:

- \( g(x,v) = v^{p(x)-1}, \quad \text{osc}_{B_r(x_0)} p(x) \leq A \frac{\bar{\mu}(r)}{\log \frac{1}{r}}, \quad \lim_{r \to 0} \bar{\mu}(r) = \infty, \quad \lim_{r \to 0} \frac{\bar{\mu}(r)}{\log \frac{1}{r}} = 0 \)
  - satisfies condition \((g_a)\) with \( \mu(r) = \exp(A\bar{\mu}(r)) \),
- \( g(x,v) = v^{p-1} + a_1(x)v^{q-1}, \quad \text{osc}_{B_r(x_0)} a_1(x) \leq A_1 \bar{\mu}(r) r^{q-p}, \quad \lim_{r \to 0} \bar{\mu}(r) = \infty, \quad \lim_{r \to 0} \frac{\bar{\mu}(r)}{r^{q-p}} = 0 \)
  - satisfies conditions \((g_a), (g_\lambda)\) with \( \mu(r) = \bar{\mu}(r) \) and \( \lambda(r) = [\bar{\mu}(r)]^{-\frac{1}{q-p}} \),
- \( g(x,v) = v^{p-1}(1+\log(1+a_2(x)v)), \quad \text{osc}_{B_r(x_0)} a_2(x) \leq A_2 \bar{\mu}(r) r, \quad \lim_{r \to 0} \bar{\mu}(r) = \infty, \lim_{r \to 0} \frac{\bar{\mu}(r)}{r} = 0 \)
  - satisfies conditions \((g_a), (g_\lambda)\) with \( \mu(r) = \log(A_2\bar{\mu}(r)) \) and \( \lambda(r) = [\bar{\mu}(r)]^{-1} \),
- \( h(x,v) = a(x) v^{p(x)-1}, \quad h(x,v) = v^{p-1} + a(x)v^{q-1} \)
  - satisfies conditions \((1.2), (1.3)\) and \((g)\).

Before formulating the main results, let us recall the definition of a weak solution to Eq. (1.1). We write \( W(\Omega) \) for the class of functions \( u \in W^{1,1}(\Omega) \) with \( \int_{\Omega} H(x, |\nabla u|) \, dx < +\infty \). We also need a class of functions \( W_0(\Omega) \) which consists of functions \( u \in W^{1,1}_0(\Omega) \) such that \( \int_{\Omega} H(x, |\nabla u|) \, dx < +\infty \).

**Definition 1.1.** We say that a function \( u \) is a weak sub(super)-solution to Eq. (1.1) if \( u \in W(\Omega) \) and the integral identity
\[
\int_{\Omega} H(x, |\nabla u|) \frac{\nabla u}{|\nabla u|^2} \nabla \varphi \, dx \leq \geq 0, \tag{1.4}
\]
holds for all non-negative test functions \( \varphi \in W_0(\Omega) \).

Let \( a_1(x) = 1 + a^{-1}(x), \quad b_1(x) = 1 + b(x) \), for any \( x_0 \in \mathbb{R}^n \) and \( r > 0 \) set
\[
\Lambda(x_0,r) := \left( \int_{B_r(x_0)} a_1^s(x) \, dx \right)^{\frac{1}{s}} \left( \int_{B_r(x_0)} b_1^s(x) \, dx \right)^{\frac{1}{s}} +
\]
\[
+ \left( \int_{B_r(x_0)} a_1^s(x) \, dx \right)^{\frac{1}{s}} \left( \int_{B_r(x_0)} b_1^s(x) \, dx \right)^{\frac{1}{s}}.
\]
We refer to the parameters $n, p, q, t, s, K_1, K_2, M := \sup_{\Omega} u$ and $C(M)$ as our structural data, and we write $\gamma$ if it can be quantitatively determined as a priori in terms of the above quantities. The generic constant $\gamma$ may change from line to line.

Our first main result of this paper reads as follows.

**Theorem 1.1.** Let $u$ be a non-negative weak super-solution to Eq. (1.1), let conditions (1.2), (1.3), (g) and (g,IN) be fulfilled, then there exist positive constants $\gamma$ and $\beta_1$ depending only on the data, such that for any $m > 0$ and any $B_{\rho}(x_0) \subset \Omega$ there holds

$$m \lambda(\rho) \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{t+1}{t(p-1)}} \leq \gamma \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \right) \left( \inf_{B_{\rho}(x_0)} u(x) + \rho \right), \quad (1.5)$$

where $E(\rho, m) := B_{\rho}(x_0) \cap \{ u(x) > m \}$.

**Theorem 1.2.** Let $u$ be a non-negative weak super-solution to Eq. (1.1), let conditions (1.2), (1.3), (g) and (g,IN) be fulfilled, then there exist positive constants $\gamma$ and $\beta_1$ depending only on the data, such that for any $m > 0$ and any $B_{\rho}(x_0) \subset \Omega$ there holds

$$m \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{t+1}{t(p-1)}} \leq \gamma \exp \left( \gamma \left[ \mu \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \right) \left( \inf_{B_{\rho}(x_0)} u(x) + \rho \right). \quad (1.6)$$

The following two results are consequences of Theorems 1.1, 1.2. The first one is the weak Harnack type inequality.

**Theorem 1.3.** Let $u$ be a weak non-negative super-solution to Eq. (1.1), let conditions of Theorem 1.1 be fulfilled. Then for all $\theta \in \left( 0, \min \left( 1, \frac{t}{t+1} (p-1) \right) \right)$ there holds

$$\left( \rho^{-n} \int_{B_{\rho}(x_0)} u^{\theta}(x) \, dx \right)^{\frac{1}{\theta}} \leq (t(p-1) - \theta(t+1))^{-\frac{1}{p+1}} \gamma \lambda(\rho) \times$$

$$\times \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \right) \left( \inf_{B_{\rho}(x_0)} u(x) + \rho \right), \quad (1.7)$$

provided that $B_{\rho}(x_0) \subset \Omega$.

Likewise, let $u$ be a weak non-negative super-solution to Eq. (1.1) and let conditions of Theorem 1.2 be fulfilled. Then for all $\theta \in \left( 0, \min \left( 1, \frac{t}{t+1} (p-1) \right) \right)$ there holds

$$\left( \rho^{-n} \int_{B_{\rho}(x_0)} u^{\theta}(x) \, dx \right)^{\frac{1}{\theta}} \leq (t(p-1) - \theta(t+1))^{-\frac{1}{p+1}} \times$$

$$\times \gamma \exp \left( \gamma \left[ \mu \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \right) \left( \inf_{B_{\rho}(x_0)} u(x) + \rho \right), \quad (1.8)$$

provided that $B_{\rho}(x_0) \subset \Omega$. Here $\beta_1 > 0$ is the number defined in Theorems 1.1, 1.2 and $\gamma > 0$ is a constant, depending only on the data.

Next result is the Harnack inequality.
**Theorem 1.4.** Let $u$ be a non-negative weak solution to Eq. (1.1), let conditions (1.2), (1.3), (g), $(g_\lambda)$ and $(g_\mu)$ be fulfilled, then there exist positive constants $\gamma$, $\beta_2$, $\beta_3$ depending only on the data, such that

$$
\sup_{B_{\frac{1}{2}}(x_0)} u(x) \leq \frac{\gamma}{\lambda(\rho)} \left[ \mu \left( \frac{\rho}{4} \right) \right]^{\beta_2} \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_3} \right) \left( \inf_{B_{\frac{1}{2}}(x_0)} u(x) + \rho \right),
$$

provided that $B_{\frac{1}{2}}(x_0) \subset \Omega$.

Likewise, let $u$ be a non-negative weak solution to Eq. (1.1), let conditions (1.2), (1.3), $(g)$ and $(g_\mu)$ be fulfilled, then

$$
\sup_{B_{\frac{1}{2}}(x_0)} u(x) \leq \gamma \exp \left( \gamma \left[ \mu \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_4} \right) \left( \inf_{B_{\frac{1}{2}}(x_0)} u(x) + \rho \right),
$$

provided that $B_{\frac{1}{2}}(x_0) \subset \Omega$.

Before describing the method of proof, a few words about the history of the problem. Qualitative properties of solutions to the corresponding elliptic equations in the standard case, i.e. if $p = q$ are well known since the famous results by De Giorgi [19], Nash [38] and Moser [37] (we refer the reader to the well-known monograph of Ladyzhenskaya and Ural’tseva [30], and the seminal papers of Serrin [40], DiBenedetto and Trudinger [18]). Harnack’s inequality for non-uniformly elliptic equations has been known since the well-known paper of Trudinger [48].

The study of regularity of minima of functionals with non-standard growth has been initiated by Zhikov [49-52, 54], Marcellini [34, 35], and Lieberman [33], and in the last thirty years, the qualitative theory of second order elliptic equations with so-called log-condition (i.e. if $\lambda(r) \equiv 1$ and $\mu(r) \equiv 1$) has been actively developed. Moreover, many authors have established local boundedness, Harnack’s inequality and continuity of solutions to such equations, as well as local minimizers, $Q$-minimizers, and $\omega$-minimizers of the corresponding minimization problems (see, e.g. [1-15, 20-28, 39, 41, 45-47] and references therein).

The case when conditions $(g_\mu)$ or $(g_\lambda)$ hold differs substantially from the logarithmic case. To our knowledge, there are few results in this direction. Zhikov [53] obtained a generalization of the logarithmic condition which guarantees the density of smooth functions in Sobolev space $W^{1,p(x)}(\Omega)$. Particularly, this result holds if $1 < p < p(x)$ and

$$
|p(x) - p(y)| \leq \frac{|\log \mu(|x - y|)|}{|\log |x - y||}, \quad x, y \in \Omega, \quad \int_0^1 [\mu(r)]^{-\frac{n}{p}} \frac{dr}{r} = \infty.
$$

Particularly, the function $\mu(r) = [\log \frac{1}{r}]^L$ satisfies the above condition if $L \leq \frac{p}{n}$. Later Zhikov and Pastukhova [55] under the same condition proved higher integrability of the gradient of solutions to the $p(x)$-Laplace equation.

Continuity and Harnack’s inequality to the $p(x)$-Laplace equation were proved in [4, 7, 47] under the condition $(g_\mu)$, where $\mu(r)$ satisfies

$$
\int_0^1 \exp \left( - c[\mu(r)]^\beta \right) \frac{dr}{r} = \infty,
$$

(1.11)
with some positive constants $c, \beta$. Particularly, the function $\mu(r) = \left[ \log \log \frac{1}{r} \right]^L, L \beta < 1$ satisfies the above condition. These results were generalized in [41, 45, 46] for a wide class of elliptic equations with non-logarithmic Orlicz growth, furthermore, Harnack’s inequality was proved in [41] under similar conditions. In the proof, the authors used Trudinger’s ideas [48]. These results were refined in [23], in addition, the Harnack inequality was proved under the conditions $(g_\lambda), (g_\mu)$ and

$$\int_0^1 \lambda(r)[\mu(r)]^{-\beta} \frac{dr}{r} = \infty,$$

(1.12)

with some $\beta > 0$.

In this paper, as already mentioned, we improve previous results, in particular (1.11) and (1.12). The main difficulty arising in the proof of the main results is related to the so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the ”positivity set” of $u$ over the ball $B_r(\bar{x})$:

$$|\{x \in B_r(\bar{x}) : u(x) \geq m\}| \geq \alpha(r) |B_r(\bar{x})|,$$

with some $r, m > 0$ and $\alpha(r) \in (0, 1)$, we cannot use the covering argument of Krylov and Safonov [29], as was done in the logarithmic case, i.e. if $\alpha$ is independent of $r$ (see e.g. [8]). Moreover, we cannot use Moser’s method, as was done in [41, 48] (it seems that it can be done under condition $(g_\mu)$, but not under conditions $(g_\lambda)$ and $(g_\mu)$). We also cannot use the local clustering lemma [17], since in this case we will inevitably arrive at an estimate of the form

$$\inf_{B_\rho(\bar{x})} u \geq \gamma m \exp \left( -\gamma \int_{\bar{r}}^{\rho} [\Lambda(\bar{x}, s)]^\beta \frac{ds}{s} \right), \quad \bar{r} = \varepsilon r, \alpha^2(r) \mu(r),$$

with some positive $\gamma, \varepsilon, \alpha$ and $\beta$, from which the weak Harnack type inequality, Theorem 1.3 does not follows. At the end of the paper we will demonstrate the possibility of using the local clustering lemma, this can only be done under the conditions $(g_\lambda)$ and $\Lambda(x_0, \rho) \propto \text{const}$. Note that this proof is valid for the corresponding $DG_{\lambda, \Lambda}(\Omega)$ De Giorgi classes (see Section 2 below).

In the present paper we use the workaround that goes back to Mazya [36] and Landis papers [31, 32], due to which we obtain an analogue of the local clustering lemma, namely, Theorems 1.1, 1.2. The proof of Theorems 1.1 and 1.2 is based on the precise point-wise estimates of solutions of the nonlinear potential type, see Section 3 below.

The rest of the paper contains the proof of the above theorems. In Section 2, we collect some auxiliary propositions and required integral estimates of solutions. Sections 3 and 4 contain the upper and lower bounds for auxiliary solutions and proof of Theorems 1.1, 1.2. Finally, in Section 5 we give a proof of Harnack type inequalities, Theorems 1.3, 1.4.

2 Auxiliary material and integral estimates

2.1 Auxiliary Lemmas

The following lemma will be used in the sequel, it is the well-known De Giorgi-Poincaré Lemma (see [30, Chap. 2]).
Lemma 2.1. Let \( u \in W^{1,1}(B_r(y)) \) for some \( r > 0 \), and \( y \in \mathbb{R}^n \). Let \( k, l \) be real numbers such that \( k < l \). Then there exists a constant \( \gamma \) depending only on \( n \) such that
\[
(l - k)|A_{k,r}| \cdot |B_r(y) \setminus A_{l,r}| \leq \gamma r^{n+1} \int_{A_{l,r} \setminus A_{k,r}} |\nabla u| \, dx,
\]
where \( A_{k,r} = B_r(y) \cap \{ u < k \} \).

The following lemma can be also found in [30, Chap. 2].

Lemma 2.2. Let \( \{ y_j \}_{j \in \mathbb{N}} \) be a sequence of non-negative numbers such that for any \( j = 0, 1, 2, \ldots \) the inequality
\[
y_{j+1} \leq \gamma c^j y_j^{1+\varepsilon}
\]
holds with some \( \varepsilon, \gamma > 0, c \geq 1 \). Then the following estimate is true
\[
y_j \leq \frac{\gamma (1+\varepsilon)^j}{c^{(1+\varepsilon)^j-1} \varepsilon} y_0^{(1+\varepsilon)^j}.
\]
Particularly, if \( y_0 \leq \frac{\gamma^\frac{1}{2} c^{-\frac{1}{2}}}{2} \), then \( \lim_{j \to +\infty} y_j = 0 \).

The following lemma can be found in [33].

Lemma 2.3. Let \( Q \) be an increasing, convex, continuous function such that \( Q(0) = 0 \) and \( \frac{1}{s} Q(s) \leq \frac{1}{t} Q(t) \) for \( 0 < s \leq t \). If \( u \in W^{1,1}(\Omega) \) with \( u = 0 \) on \( \partial \Omega \) and \( \int_{\Omega} Q(|\nabla u|) \, dx \) finite, and \( R = \text{diam} \, \Omega \) then
\[
\int_{\Omega} Q\left(\frac{u}{R}\right) \, dx \leq \int_{\Omega} Q(|\nabla u|) \, dx.
\]

2.2 Local energy estimates

The following inequality is a simple analogue of Young’s inequality
\[
g(x, a)b \leq \varepsilon ag(x, a) + \max(\varepsilon^{1-p}, \varepsilon^{1-q})bg(x, b), \quad \varepsilon, a, b > 0, \quad x \in \Omega, \quad (2.1)
\]
indeed, if \( b \leq \varepsilon a \), then \( g(x, a)b \leq \varepsilon ag(x, a) \), and if \( b \geq \varepsilon a \), then by condition \((g)\) \( g(x, a)b \leq \max(\varepsilon^{1-p}, \varepsilon^{1-q})bg(x, b) \).

The following lemma, namely inequalities (2.2) and (2.3), can be used as a definition of the corresponding \( DG_{y, \Lambda}(\Omega) \) De Giorgi classes, we refer the reader to [45,46] for the case \( \Lambda(x_0, r) \propto \text{const}, r > 0 \).

Lemma 2.4. Let \( u \) be bounded local weak solution to (1.1), then for any \( B_{\theta r}(x_0) \subset \Omega \), any \( k < l \), any \( \sigma \in (0, 1) \) and any \( \zeta(x) \in C_0^\infty(B_r(x_0)), \zeta(x) = 1 \) in \( B_{(1-\sigma)r}(x_0) \), \( 0 \leq \zeta(x) \leq 1 \), \(|\nabla \zeta(x)| \leq \frac{1}{\sigma r} \) next inequalities hold
\[
\int_{A_{k,r}^+ \setminus A_{l,r}^+} |\nabla u|^q \zeta(x) \, dx \leq \gamma \sigma^{-q} \left( \frac{g_{B_r(x_0)}^\ast \left( \frac{M_{q,k,r}}{r} \right)}{g_{B_r(x_0)}^\ast \left( \frac{M_{q,k,r}}{r} \right)} \right)^{\frac{1}{p}} (\frac{1}{1-p}) \left( \frac{1}{1-p} \right) \Lambda(x_0, r) \left( \frac{|A_{k,r}^+ \setminus A_{l,r}^+|}{|B_r(x_0)|} \right)^{\frac{1}{n(1-p)}} \left( \frac{|A_{k,r}^+|}{|B_r(x_0)|} \right)^{\frac{1}{n(1-p)}} \Lambda(x_0, r), \quad (2.2)
\]
\[
\int_{A_{k,r}^+ \setminus A_{k,r}^-} |\nabla u| \zeta^q(x) \, dx \leq \gamma \sigma^{-q} \left( \frac{g^+_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)}{g^-_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)} \right)^{\frac{\beta}{p}} M_-(l,r) r^{n-1} \times \\
\times \Lambda(x_0, r) \left( \frac{|A_{l,r}^- \setminus A_{k,r}^-|}{|B_r(x_0)|} \right) \left( 1 - \frac{1}{\tau(p-1)} \right) \left( \frac{|A_{l,r}^-|}{|B_r(x_0)|} \right)^{\frac{1}{q}} \frac{1}{M}, \quad (2.3)
\]

\[
\int_{A_{k,r}^+} |\nabla u| \zeta^q(x) \, dx \leq \gamma \sigma^{-q} \left( \frac{g^+_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)}{g^-_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)} \right)^{\frac{\beta}{p}} \times \\
\times M_\pm(k,r) r^{n-1} \Lambda(x_0, r) \left( \frac{|A_{k,r}^\pm|}{|B_r(x_0)|} \right) \left( 1 - \frac{1}{\tau(p-1)} \right) \left( \frac{1}{M} - \frac{1}{2} \right), \quad (2.4)
\]

Here \((u-k)_\pm := \max\{ \pm(u-k), 0 \}, A_{k,r}^+ := B_r(x_0) \cap \{(u-k)_\pm > 0 \}, M_\pm(k,r) := \text{ess sup}(u-k)_\pm, \)

\(g^+_{B_r(x_0)}(v) := \max_{x \in B_r(x_0)} g(x,v), \quad g^-_{B_r(x_0)}(v) := \min_{x \in B_r(x_0)} g(x,v), \quad v > 0.\)

Proof. Test identity \((1.4)\) by \(\varphi = (u-k)_+ \zeta^q(x),\) by conditions \((1.2)\) we obtain

\[
\int_{A_{k,r}^+} a(x)G(x, |\nabla u|) \zeta^q(x) dx \leq \gamma \int_{A_{k,r}^+} b(x)G \left( x, \frac{(u-k)_+}{\sigma r} \right) dx \leq \\
\leq \gamma \sigma^{-q} M_+(k,r) r \frac{g^+_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)}{g^-_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)} \int_{A_{k,r}^+} b(x) dx, \quad G(x,v) := \int_0^v g(x,s) ds, \quad v > 0,
\]

and use inequality \((2.1)\) with \(a = \frac{M_+(k,r)}{r}, \quad b = |\nabla u|\) and \(\varepsilon\) replaced by \(\varepsilon^{-1} a_1(x), \quad \varepsilon > 0,\)

integrating over the set \(A_{k,r}^+ \setminus A_{k,r}^-\), using condition \((g)\) we arrive at

\[
\int_{A_{k,r}^+ \setminus A_{k,r}^-} |\nabla u| \zeta^q(x) \, dx = \int_{A_{k,r}^+ \setminus A_{k,r}^-} \frac{g^-_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)}{g^-_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right)} |\nabla u| \zeta^q(x) \, dx \leq \\
\leq \varepsilon^{-1} M_+(k,r) r \int_{A_{k,r}^+ \setminus A_{k,r}^-} a_1(x) \frac{1}{p-1} dx + \max(\varepsilon^{p-1}, \varepsilon^{q-1}) \int_{A_{k,r}^+ \setminus A_{k,r}^-} a(x)G(x, |\nabla u|) \zeta^q(x) dx \leq \\
\leq \varepsilon^{-1} M_+(k,r) r \int_{A_{k,r}^+ \setminus A_{k,r}^-} a_1(x) \frac{1}{p-1} dx + \\
+ \gamma \sigma^{-q} \max(\varepsilon^{p-1}, \varepsilon^{q-1}) \frac{M_+(k,r)}{r} g^+_{B_r(x_0)} \left( \frac{M_+(k,r)}{r} \right) \int_{A_{k,r}^+ \setminus A_{k,r}^-} b(x) dx \leq \\
\leq \gamma \sigma^{-q} M_+(k,r) r^{n-1} \left\{ \varepsilon^{-1} \left( \frac{1}{p-1} \int_{B_r(x_0)} a_1(x) dx \right)^{\frac{1}{n(p-1)}} + \right\}^{\frac{n(p-1)}{n(p-1)+1}}
\]
On asymptotic behavior ....

\[ + \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \max(\varepsilon^{p-1}, \varepsilon^{q-1}) \left( r^{-n} \int_{B_r(x_0)} b^n(x)dx \right)^{\frac{1}{p}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{q}} \].

Choose \( \varepsilon \) by the condition

\[ \varepsilon^p = \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \left( r^{-n} \int_{B_r(x_0)} a_1(x)dx \right)^{\frac{1}{(p-1)}} \left( \frac{|A_{k,r}^{+} \setminus A_{l,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{p(1-p)}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{-1+\frac{1}{q}}, \]

if

\[ \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \left( r^{-n} \int_{B_r(x_0)} a_1(x)dx \right)^{\frac{1}{(p-1)}} \left( \frac{|A_{k,r}^{+} \setminus A_{l,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{p(1-p)}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{-1+\frac{1}{q}} \leq 1, \]

and

\[ \varepsilon^q = \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \left( r^{-n} \int_{B_r(x_0)} b^n(x)dx \right)^{\frac{1}{q}} \left( \frac{|A_{k,r}^{+} \setminus A_{l,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{2}{p(1-p)}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{-1+\frac{1}{q}}, \]

in the opposite case, from the previous we arrive at

\[ \int_{A_{k,r}^{+} \setminus A_{l,r}^{+}} |\nabla u| \zeta^q(x) dx \leq \gamma \sigma^{-q} \left( \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \right)^{\frac{1}{p}} \left( \frac{|A_{k,r}^{+} \setminus A_{l,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{p(1-p)}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{q}} \times \]

\[ \left\{ \left( \frac{|A_{k,r}^{+} |}{|B_r(x_0)|} \right)^{1-\frac{1}{q}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{q}} \right\} \leq \gamma \sigma^{-q} \left( \frac{g_{B_r(x_0)}^{+}(M_{r,k,r})}{g_{B_r(x_0)}^{-}(M_{r,k,r})} \right)^{\frac{1}{q}} M_+(k,r)^{p-1} \times \]

\[ \Lambda(x_0,r) \left( \frac{|A_{k,r}^{+} \setminus A_{l,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{p(1-p)}} \left( \frac{|A_{k,r}^{+}|}{|B_r(x_0)|} \right)^{1-\frac{1}{q}} \],

which proves (2.2). The proof of (2.3) and (2.4) is completely similar.

\[ \square \]

2.3 De Giorgi Type Lemmas

The following lemmas are De Giorgi type lemmas and their formulation under condition \((g_0)\) or condition \((g_\lambda)\) is different. In the proof we closely follow [30, Chap. 2].

Lemma 2.5. (De Giorgi Type Lemma Under the Condition \((g_\lambda)\)) Let \( u \) be a local bounded weak solution to Eq. (1.1), \( B_{8r}(x_0) \subset B_r(x_0) \subset \Omega \), assume that condition \((g_\lambda)\) be
fulfilled and let
\[ \mu^+_r \geq \text{ess sup}_{B_r(x_0)} u, \quad \mu^-_r \leq \text{ess inf}_{B_r(x_0)} u, \quad \omega_r := \mu^+_r - \mu^-_r \]
and set \( v_+ := \mu^+_r - u, \ v_- := u - \mu^-_r \). Fix \( \xi \in (0, \lambda(r)) \) and \( \delta \in (0, 1) \), then there exists \( \nu_1 \in (0, 1) \) depending only on the data and \( \delta \) such that if
\[ |\{ x \in B_r(x_0) : v_+(x) \leq \xi \omega_r \}| \leq \nu_1 \frac{A(x_0, r)}{B_r(x_0)}, \quad \kappa = \frac{1}{n} - \frac{1}{tp} - \frac{1}{qs} + \frac{1}{p} - \frac{1}{q}. \] (2.5)
then either
\[ \delta(1 - \delta)\xi \omega_r \leq r, \] (2.6)
or
\[ v_\pm(x) \geq \delta \xi \omega_r \text{ for a.a. } x \in B_{r/2}(x_0). \] (2.7)

**Proof.** We provide the proof of (2.7) for \( v_+ \), while the proof for \( v_- \) is completely similar. For \( j = 0, 1, 2, \ldots \) we set \( r_j := r \left( 1 + 2^{-j} \right), \ r_j = \frac{r_j + r_{j+1}}{2} = k_j := \mu^+_r - \delta(2 - \delta)\xi \omega_r - \xi \omega_r \frac{2 - \delta^2}{2j+1} \), and let \( \zeta_j \in C^\infty_0(B_{r_j}(x_0)), \ 0 \leq \zeta_j \leq 1, \ \zeta_j = 1 \text{ in } B_{r_{j+1}}(x_0), \) and \( |\nabla \zeta_j| \leq \gamma \frac{2^j}{r} \).

Since \( M_+(k_j, r) \leq \xi \omega_r \leq 2M\lambda(r) \) for \( j = 0, 1, 2, \ldots \), so if (2.6) is violated then condition \((g_\lambda)\) is applicable and we obtain that
\[ g^+_r \left( \frac{M_+(k_j, r_j)}{r_j} \right) \leq g \left( \frac{M_+(k_j, r_j)}{r} \right). \]
Therefore inequality (2.4) can be rewritten as
\[ \int_{A^r_{k_j, r_j}} |\nabla u| \zeta_j^q \ dx \leq \gamma 2^j \xi \omega_r r^{n-1} \Lambda(x_0, r) \left( \frac{A^r_{k_j, r_j}}{|B_r(x_0)|} \right)^{\left(1 - \frac{1}{t(p-1)}\right)\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right)\frac{1}{q}}. \]
From this, using the Sobolev embedding theorem we obtain that
\[ (1 - \delta)^2 \xi \frac{\omega_r}{2^{j+1}} \left| A^r_{k_j+1, r_{j+1}} \right| \leq \int_{A^r_{k_j, r_j}} (u - k_j) \zeta_j^q \ dx \leq \gamma \int_{A^r_{k_j, r_j}} |\nabla (u - k_j) + \zeta_j^q(x)| \ dx \left| A^r_{k_j, r_j} \right|^\frac{1}{q} \leq \gamma 2^j \xi \omega_r r^{n} \Lambda(x_0, r) \left( \frac{A^r_{k_j, r_j}}{|B_r(x_0)|} \right)^{\left(1 - \frac{1}{t(p-1)}\right)\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right)\frac{1}{q}} \]
from which we arrive at
\[ y_j := \frac{|A^r_{k_j+1, r_{j+1}}|}{|B_r(x_0)|} \leq \gamma (1 - \delta)^{-2} 2^j \Lambda(x_0, r) y_j^{1 + \kappa}, \quad j = 0, 1, 2, \ldots \]
By our choices \( \kappa > 0 \), so choosing \( \nu_1 \) by the condition
\[ \nu_1(r) = \gamma^{-1} (1 - \delta)^{\frac{2}{\kappa}} \]
and using Lemma 2.2 from this we arrive at the required (2.7), which completes the proof of the lemma. \( \square \)
Lemma 2.6. (De Giorgi Type Lemma Under the Condition \((g_\mu)\)) Let \(u\) be a local bounded weak solution to Eq. \((1.1)\), \(B_{8r}(x_0) \subset B_R(x_0) \subset \Omega\), assume that condition \((g_\mu)\) be fulfilled. Fix \(\xi, \delta \in (0, 1)\), then there exists \(\nu_1 \in (0, 1)\) depending only on the data and \(\delta\) such that if
\[
\{x \in B_r(x_0) : v_{\pm}(x) \leq \xi \omega_r\} \leq \nu_1 \|u(r)\|^{-\frac{1}{\gamma}} |\Lambda(x_0, r)|^{-\frac{1}{\gamma}} |B_r(x_0)|, \tag{2.8}
\]
then either \((2.6)\) holds, or
\[
v_{\pm}(x) \geq \delta \xi \omega_r \quad \text{for a.a. } x \in B_{r/2}(x_0). \tag{2.9}
\]
Here \(\kappa > 0\) is the number, defined in Lemma 2.6.

Proof. For \(j = 0, 1, 2, \ldots\) we set \(r_j := \frac{r}{2}(1 + 2^{-j}), \bar{r}_j = \frac{r_j + r_{j+1}}{2}, k_j := \mu^r - \delta(2 - \delta)\xi \omega_r - \frac{(1 - \delta)^2}{2j} \xi \omega_r\), and let \(\zeta_j \in C_0^\infty(B_{\bar{r}_j}(x_0))\), \(0 \leq \zeta_j \leq 1\), \(\zeta_j = 1\) in \(B_{r_{j+1}}(x_0)\), and \(|\nabla \zeta_j| \leq \gamma \frac{2^j}{r} \). We assume that \(M_+(k_{\infty}, r/2) \geq \delta(1 - \delta)\omega_r\), because in the opposite case, the required \((2.9)\) is evident. If \((2.6)\) is violated then \(M_+(k_{\infty}, r/2) \geq r\). In addition, since \(M_+(k_j, r) \leq \xi \omega_r\) for \(j = 0, 1, 2, \ldots\), then condition \((g_\mu)\) is applicable and we obtain that
\[
g_{B_r(x_0)}^r \left( \frac{M_+(k_j, r_j)}{r_j} \right) \leq \gamma 2^j \mu(r) g_{B_r(x_0)}^r \left( \frac{M_+(k_j, r_j)}{r} \right) \tag{2.10}.
\]
Therefore inequality \((2.4)\) similarly to Lemma 2.5 can be rewritten as
\[
\int_{A_{k_j}^+} \partial u \zeta_j^q dx \leq \gamma 2^j \mu(r) \omega_r r^\alpha |\Lambda(x_0, r)| \left( \frac{|A_{k_j}^+|}{|B_r(x_0)|} \right)^{\left(\frac{1-q}{p\alpha-1}\right) \left(1 - \frac{\alpha}{2} \right)} \left(1 + \frac{\alpha}{2} \right)^\frac{\alpha}{2}.
\]
From this, using the Sobolev embedding theorem, similarly to Lemma 2.4 we obtain
\[
y_{j+1} = \frac{|A_{k_j}^+|}{|B_r(x_0)|} \leq \gamma 2^j \left(1 - \delta\right)^{-2} \mu(r) |\Lambda(x_0, r)| y_j^{1+\kappa}, \quad j = 0, 1, 2, \ldots
\]
choosing \(\nu_1\) from the condition
\[
\nu_1 = \gamma^{-1} \left(1 - \delta\right)^{\frac{\alpha}{2}}
\]
and using Lemma 2.2 from this we arrive at the required \((2.9)\), which completes the proof of the lemma.

2.4 Expansion of the Positivity Lemma

The following two \(\infty\) will be used in the sequel. In the proof we closely follow to [30, Chap. 2]. The first one is a variant of the expansion of positivity lemma under the \((g_\lambda)\) condition.

Lemma 2.7. Let \(u\) be a local bounded weak solution to Eq. \((1.1)\), let \(B_{8r}(x_0) \subset B_R(x_0) \subset \Omega\) and \(\xi \in (0, 1)\), in addition let condition \((g_\lambda)\) be fulfilled. Assume that with some \(\alpha_0 \in (0, 1)\) and \(\bar{\lambda}(r) \in (0, \lambda(\frac{r}{2}))\) there holds
\[
\{x \in B_{3r/4}(x_0) : v_{\pm}(x) \leq \xi \bar{\lambda}(r) \omega_r\} \leq \left(1 - \alpha_0\right) |B_{3r/4}(x_0)|. \tag{2.10}
\]
Then there exists number $C_*$ depending only on the known data, $\alpha_0$ and $\xi$, such that either
\[ \omega_r \leq \frac{r}{\lambda(r)} \exp \left( C_* [\Lambda(x_0, r)]^{3\bar{\alpha}} \right), \quad (2.11) \]
or
\[ v_+ (x) \geq \omega_r \tilde{\lambda}(r) \exp \left( -C_* [\Lambda(x_0, r)]^{3\bar{\alpha}} \right) \text{ for a.a. } x \in B_r/2(x_0). \quad (2.12) \]
Here $3\bar{\alpha} > 0$ is some fixed number depending only upon the data.

Proof. We provide the proof of $2.12$ for $v_+$, while the proof for $v_-$ is completely similar. We set $k_j := \mu_j^+ - \frac{\bar{\lambda}(r) \omega_r}{2^j}$, $j = \log 1/\xi + 1, 2, \ldots, j_*$, where $j_*$ is large enough to be chosen later.

We will assume that $M_+ (k_j, \frac{r}{2}) \geq \frac{M \lambda(r)}{r}$, because in the opposite case, the required $2.12$ is evident. If $(2.11)$ is violated, then $M_+ (k_j, \frac{r}{2}) \geq r$, and since $M_+ (k_j, r) \leq 2^{-j} \bar{\lambda}(r) \omega_r \leq 2M \lambda(r)$, $j = \log 1/\xi + 1, j_*$, then by $(g_1)$ we obtain that
\[ g^+_B(x_0) \left( \frac{M_+(k_j, r)}{r} \right) \leq g^-_B(x_0) \left( \frac{M_+(k_j, r)}{r} \right). \]
Therefore inequality $(2.2)$ can be rewritten as
\[ \int_{A^+_{k_j}} |\nabla u| \zeta^q \, dx \leq \gamma \bar{\lambda}(r) \frac{\omega_r}{2^j} r^{n-1} \Lambda(x_0, r) \left( \frac{|A^+_{k_j} \setminus A^+_{k_{j+1}}|}{|B_r(x_0)|} \right)^{(1-\frac{1}{p}) \frac{1}{2^j}}, \]

where $\zeta \in C^\infty_0 (B_r(x_0))$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{3r/4}(x_0)$, $|\nabla \zeta| \leq 4/r$. From this, using $(2.10)$, De Giorgi-Poincaré Lemma $2.1$, we obtain
\[ \bar{\lambda}(r) \frac{\omega_r}{2^j} |A^+_{k_j} \setminus A^+_{k_{j+1}}| \leq \frac{\gamma}{\alpha_0} \frac{r}{2^j} \int_{A^+_{k_j} \setminus A^+_{k_{j+1}}} |\nabla u| \zeta^q \, dx \leq \frac{\gamma}{\alpha_0} \bar{\lambda}(r) \frac{\omega_r}{2^j} r^{n-1} \Lambda(x_0, r) \left( \frac{|A^+_{k_j} \setminus A^+_{k_{j+1}}|}{|B_r(x_0)|} \right)^{\kappa_1}, \quad \kappa_1 = \left( 1 - \frac{1}{p} - \frac{1}{(p-1)} \right) \left( 1 - \frac{1}{p} \right), \]
raising the left and right hand-sides to the power $\frac{1}{\kappa_1}$ and summing up the resulting inequalities in $j, j = \log 1/\xi + 1, 2, \ldots, j_*$, we conclude that
\[ (j_* - \log 1/\xi - 1) \left| A^+_{k_j} \setminus A^+_{k_{j+1}} \right|^\frac{1}{\kappa_1} \leq \gamma \alpha_0^{-\frac{1}{\kappa_1}} [\Lambda(x_0, r)]^{\frac{1}{\kappa_1}} \left| B_{3r/4}(x_0) \right|^\frac{1}{\kappa_1}. \]
Choosing $j_*$ by the condition
\[ \gamma^{\kappa_1} \alpha_0^{-1} \left( j_* - \log 1/\xi - 1 \right)^{-\kappa_1} \leq \nu_1 [\Lambda(x_0, r)]^{-1 - \frac{1}{p}}, \]
where $\nu_1$ and $\kappa$ are the constants, defined in Lemma $2.5$ therefore by $(2.7)$ we obtain inequality $(2.12)$, which proves Lemma $2.7$ with $\bar{\beta}_1 = \frac{1}{\kappa_1} (1 + \frac{1}{p})$. \(\square\)

The following lemma is a variant of the expansion of positivity lemma under the $(g_\mu)$ condition.
Lemma 2.8. Let $u$ be a local bounded weak solution to Eq. (1.1), let $B_{3r}(x_0) \subset B_R(x_0) \subset \Omega$ and $\xi \in (0,1)$, in addition let condition (g$_\mu$) be fulfilled. Assume that with some $\alpha_0 \in (0,1)$ there holds

$$\left| \{ x \in B_{3r/4}(x_0) : v_{\pm}(x) \leq \xi \omega_r \} \right| \leq (1-\alpha_0) |B_{3r/4}(x_0)|.$$ \hfill (2.13)

Then there exists number $C_*$ depending only on the known data, $\alpha_0$ and $\xi$ such that either

$$ \omega_r \leq r \exp \left( C_* \left[ \mu(r) \right]^{\beta_1} \left[ \Lambda(x_0, r) \right]^{\beta_1} \right),$$ \hfill (2.14)

$$ v_{\pm}(x) \geq \xi \omega_r \exp \left( - C_* \left[ \mu(r) \right]^{\beta_1} \left[ \Lambda(x_0, r) \right]^{\beta_1} \right), \quad \text{for a.a.} \ x \in B_{r/2}(x_0),$$ \hfill (2.15)

where $\beta_1 > 0$ is the number, defined in Lemma 2.7.

Proof. We provide the proof of (2.15) for $v_+$, while the proof for $v_-$ is completely similar. We set $k_j := \mu^+_r - \frac{\omega_r}{2j}$, $j = [\log |\xi|] + 1, 2, \ldots, j_*$, where $j_*$ to be chosen. We will assume that $M_+ (k_j, \frac{\omega_r}{2j}) \geq \frac{\omega_r}{2j+1}$, because in the opposite case, the required inequality (2.15) is evident. If (2.14) is violated, then $M_+ (k_j, \frac{\omega_r}{2j}) \leq r$, and since $M_+ (k_j, r) \leq 2^{-j} \omega_r$, we get $\mu_0 \leq 2^{-j} \mu_0$, $j = \log |\xi| + 1, j_*$, then by (g$_\mu$) we obtain that

$$ g_+^{\alpha_0} (M_+ (k_j, r)) \leq \gamma r g_{B_r(x_0)} \left( \frac{M_+ (k_j, r)}{r} \right).$$

Therefore, completely similar to the previous lemma, inequality (2.2) can be rewritten as

$$ \int_{A_{k_j}^{\alpha_0} \setminus A_{k_{j+1}}^{\alpha_0}} |\nabla u| \zeta^q \, dx \leq \frac{\gamma}{\alpha_0} \mu(r) \frac{\omega_r}{2j} r^{n-1} \Lambda(x_0, r) \left( \frac{|A_{k_j}^{\alpha_0} \setminus A_{k_{j+1}}^{\alpha_0} |}{|B_r(x_0)|} \right)^{\kappa_1},$$

where $\kappa_1 = \left( 1 - x_0 \left[ \mu(r) \right]^{-\frac{1}{\alpha_0}} \right \left( 1 - \frac{1}{p} \right)$ and $\zeta \in C_0^\infty (B_r(x_0))$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{3r/4}(x_0)$, $|\nabla \zeta| \leq 4/r$. From this, using (2.13) and De Giorgi-Poincaré Lemma 2.1 we obtain

$$ \frac{\omega_r}{2j} \left| A_{k_j}^{\alpha_0} \right| \leq \gamma r \int_{A_{k_{j+1}}^{\alpha_0} \setminus A_{k_{j+1}}^{\alpha_0}} |\nabla u| \zeta^q \, dx \leq \frac{\gamma}{\alpha_0} \mu(r) \frac{\omega_r}{2j} r^{n} \Lambda(x_0, r) \left( \frac{|A_{k_j}^{\alpha_0} \setminus A_{k_{j+1}}^{\alpha_0} |}{|B_r(x_0)|} \right)^{\kappa_1},$$

raising the left and right hand-sides to the power $\frac{1}{\kappa_1}$ and summing up the resulting inequalities in $j$, $j = \log |\xi| + 1, 2, \ldots, j_*$, we conclude that

$$ (j_* - \log |\xi| - 1) \left| A_{k_j}^{\alpha_0} \right|^{\frac{1}{\kappa_1}} \leq \gamma \alpha_0 \left[ \mu(r) \right]^{\frac{1}{\kappa_1}} \left[ \Lambda(x_0, r) \right]^{\frac{1}{\kappa_1}} \left| B_{3r/4}(x_0) \right|^{\frac{1}{\kappa_1}}.$$

Choosing $j_*$ by the condition

$$ \gamma^{\kappa_1} \alpha_0^{-1} \left( j_* - \log |\xi| - 1 \right)^{-\kappa_1} \leq \nu_1 \left[ \mu(r) \right]^{-1-\frac{\delta}{4}} \left[ \Lambda(x_0, r) \right]^{-1-\frac{\delta}{4}},$$

by Lemma 2.6 we obtain inequality (2.15), which proves Lemma 2.8. \qed
3 Upper and lower estimates of auxiliary solutions under \((g_\lambda)\) condition. Proof of Theorem 1.1

Fix \(x_0 \in \Omega\), let \(E \subset B_r(x_0) \subset B_\rho(x_0) \subset B_R(x_0) \subset \Omega\) and consider the solution \(v\) of the following problem

\[
\text{div} \left( H(x, |\nabla v|) \frac{\nabla v}{|\nabla v|^2} \right) = 0, \quad x \in \mathcal{D} = B_{\rho}(x_0) \setminus E, \quad v - m\psi \in W_0(\mathcal{D}),
\]

where \(m \in [\rho, \lambda(\rho)]\) is some fixed number and \(\psi \in W_0(\mathcal{D}), \psi = 1\) on \(E\).

We note that our assumptions on the function \(h(x, \cdot)\) imply the monotonicity condition

\[
\left( H(x, |\xi|) \frac{\xi}{|\xi|^2} - H(x, |\eta|) \frac{\eta}{|\eta|^2}, \xi - \eta \right) > 0, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \neq \eta,
\]

indeed, by the Cauchy inequality and since \(h(x, \cdot)\) increases

\[
\left( H(x, |\xi|) \frac{\xi}{|\xi|^2} - H(x, |\eta|) \frac{\eta}{|\eta|^2}, \xi - \eta \right) =
\]

\[
= \left( \int_0^1 \frac{d}{dt} \left[ H(x, |t\xi + (1-t)\eta|) \right] dt, \xi - \eta \right) =
\]

\[
= |\xi - \eta|^2 \int_0^1 \frac{H(x, |t\xi + (1-t)\eta|)}{|t\xi + (1-t)\eta|^2} dt + \int_0^1 \frac{h(x, |t\xi + (1-t)\eta|)}{|t\xi + (1-t)\eta|^3} (t\xi + (1-t)\eta, \xi - \eta)^2 dt -
\]

\[
- 2 \int_0^1 \frac{H(x, |t\xi + (1-t)\eta|)}{|t\xi + (1-t)\eta|^4} (t\xi + (1-t)\eta, \xi - \eta)^2 dt \geq
\]

\[
\geq \int_0^1 \left[ \frac{h(x, |t\xi + (1-t)\eta|)}{|t\xi + (1-t)\eta|^3} - \frac{H(x, |t\xi + (1-t)\eta|)}{|t\xi + (1-t)\eta|^4} \right] (t\xi + (1-t)\eta, \xi - \eta)^2 dt > 0.
\]

So, the existence of the solutions \(v\) follows from the general theory of monotone operators. We will assume that the following integral identity holds:

\[
\int_\mathcal{D} H(x, |\nabla v|) \frac{\nabla v}{|\nabla v|^2} \nabla \varphi \, dx = 0 \quad \text{for any} \quad \varphi \in W_0(\mathcal{D}). \tag{3.1}
\]

Testing (3.1) by \(\varphi = (v - m)_+\) and by \(\varphi = v_-\) and using condition (1.4), we obtain that

\(0 \leq v \leq m\).

To formulate our next result, we need the notion of the capacity. For this set

\[
C_H(E, B_{\rho}(x_0); m) := \frac{1}{m} \inf_{\varphi \in \mathfrak{M}(E)} \int_{B_{\rho}(x_0)} H(x, m|\nabla \varphi|) \, dx,
\]

where the infimum is taken over the set \(\mathfrak{M}(E)\) of all functions \(\varphi \in W_0(B_{\rho}(x_0))\) with \(\varphi \geq 1\) on \(E\). If \(m = 1\), this definition leads to the standard definition of \(C_H(E, B_{\rho}(x_0))\) capacity (see, e.g., [23]).

Further we will assume that

\[
g_{x_0}^{-1} \left( C_H(E, B_{\rho}(x_0); m) \right) \leq \bar{c}, \tag{3.2}
\]

where \(\bar{c}, \bar{c}_1 > 0\) to be chosen later depending only on the data.
3.1 Upper bound for the function \( v \)

We note that in the standard case (i.e. if \( p = q \)) the upper bound for the function \( v \) was proved in \cite{42} (see also \cite{43} Chap. 8, Sec. 3, \cite{44}).

**Lemma 3.1.** There exists \( \bar{\beta} > 0 \) depending only on the data such that

\[
v(x) \leq \gamma \left[ A \left( x_0, \frac{\rho}{4} \right) \right]^{\beta} \rho g_{x_0}^{-1} \left( \frac{C_H(E, B_{8\rho}(x_0) ; m)}{\rho^{n-1}} \right), \quad x \in B_{\rho}(x_0) \setminus B_{\frac{\rho}{2}}(x_0).
\]

For \( i, j = 0, 1, 2, \ldots \) set \( k_j := k(1 - 2^{-j}), k > 0 \) to be chosen later, \( \rho_{i,j} := 2^{-i-j-3} \rho \),

\[
M_i := \text{ess sup}_v, \quad F_i := \left\{ x \in \mathcal{D} : \frac{\rho}{4}(1 + 2^{-i}) \leq |x - x_0| \leq \frac{\rho}{2}(3 - 2^{-i}) \right\}.
\]

Fix \( \bar{\tau} \in F_i \) and suppose that \((v(\bar{\tau}) - k)_+ \geq \rho \), then \( M_{i,j}(k_j) := \text{ess sup}_v(v - k_j) \geq (v(\bar{\tau}) - k)_+ \geq \rho \geq \rho_{i,j} \). And let \( \zeta_{i,j} \in C_0^\infty(B_{\rho_{i,j}}(\bar{\tau})) \), \( 0 \leq \zeta_{i,j} \leq 1 \), \( \zeta_{i,j} = 1 \) in \( B_{\rho_{i,j+1}}(\bar{\tau}) \), \(|\nabla \zeta_{i,j}| \leq 2^{i+j+4}/\rho \).

By our choice we have \( v(x) \leq m \leq 2M \lambda(\rho) \leq \frac{2^{i+j+1}}{\rho_{i,j}} \lambda(\rho_{i,j}) \), \( x \in B_{\rho_{i,j}}(\bar{\tau}) \). Therefore condition \((g_\lambda)\) is applicable in \( B_{\rho_{i,j}}(\bar{\tau}) \) and we have by \((g_\lambda)\) with \( K = 2^{i+j} \gamma \) that

\[
g_{B_{\rho_{i,j}}(\bar{\tau})}^+(M_{i,j}(k_j+1)/\rho_{i,j}) \leq 2^{i+j} \gamma g_{B_{\rho_{i,j}}(\bar{\tau})}^-(M_{i,j}(k_j+1)/\rho_{i,j}).
\]

So, by \cite{2} we obtain

\[
\int_{B_{\rho_{i,j}}(\bar{\tau})} |\nabla(v - k_{j+1})_+| \zeta_{i,j}^q \, dx \leq \gamma 2^\gamma(i+j) M_{i+1} \Lambda \left( x_0, \frac{\rho}{4} \right) \times 
\]

\[
\rho^{-1-n} \left( \frac{|A_{\rho_{i,j},k_{j+1}}|}{|B_{\rho_{i,j}}(\bar{\tau})|} \right) \left( 1 - \frac{1}{n} \right) q \left( 1 - \frac{1}{q} \right) \frac{1}{q} \leq 2^\gamma(i+j) M_{i+1} \Lambda \left( x_0, \frac{\rho}{4} \right) \rho^{-n} k^{-1-\frac{\kappa}{n}} \left( \int_{B_{\rho_{i,j}}(\bar{\tau})} (v - k_j)_+ \, dx \right)^{1+\kappa+n},
\]

here \( A_{\rho_{i,j},k_{j+1}}^+ := B_{\rho_{i,j}}(\bar{\tau}) \cap \{ v \leq k_{j+1} \} \) and \( \kappa = \frac{1}{n} - \frac{1}{tp} - \frac{1}{sq} - \frac{1}{p} + \frac{1}{q} > 0 \).

From this, similarly to that of Section \cite{2}, and choosing \( k \) from the condition

\[
k^{1+\frac{1}{\kappa}} = 2^{i+j} M_{i+1} \left[ A \left( x_0, \frac{\rho}{4} \right) \right]^{\frac{1}{n}} \rho^{-n} \int_{B_{\rho/2^{i+3}}(\bar{\tau})} v \, dx,
\]

since \( \bar{\tau} \in F_i \) is an arbitrary point, using the Young inequality and keeping in mind our assumption that \( v(\bar{\tau}) \geq k + \rho \), from the previous we obtain for any \( \varepsilon \in (0, 1) \)

\[
M_i \leq \varepsilon M_{i+1} + \frac{2^{i+j}}{\varepsilon^{\frac{1}{n}}} \rho^{-n} \left[ A \left( x_0, \frac{\rho}{4} \right) \right]^\frac{1}{n} \int_{F_{i+1}} v \, dx + \gamma \rho, \quad i = 0, 1, 2, \ldots \quad (3.3)
\]

Let us estimate the second term on the right-hand side of \((3.3)\). For this we assume that \( M_0 \geq \rho \), because otherwise, by \cite{32} the upper estimate is evident. Since \( \rho \leq M_0 \leq M_i \leq \)
\( \leq 2 M \lambda(\rho), i = 0, 1, 2, \ldots \), condition \((g_\lambda)\) is applicable. Set \( v_{M_{i+1}} := \min\{v, M_{i+1}\} \), by \((2.1)\) and \((g_\lambda)\) with \( K = 2M \) we have with arbitrary \( \varepsilon_1 \in (0, 1) \)

\[
\int_{F_i+1} v \ dx = \int_{F_i+1} v_{M_{i+1}} \ dx \leq \gamma \rho \int_D |\nabla v_{M_{i+1}}| \ dx
\]

\[
= \gamma \rho \int_D |\nabla v_{M_{i+1}}| \frac{g_{B_{\rho_0}(x_0)}(M_i/\rho)}{g_{B_{\rho_0}(x_0)}(M_i/\rho)} \ dx \leq \varepsilon_1 M_i \rho^n + \gamma \varepsilon_1^{-q} \Lambda(x_0, \frac{\rho}{4}) \int_D a(x) G(x, |\nabla v_{M_{i+1}}|) \ dx \leq \varepsilon_1 M_i \rho^n + \gamma \varepsilon_1^{-q} \Lambda(x_0, \frac{\rho}{4}) \int_D H(x, |\nabla v_{M_{i+1}}|) \ dx.
\]

Collecting the last two inequalities and choosing \( \varepsilon_1 \) from the condition

\[
\gamma = 2^{r_\gamma} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\frac{1}{\gamma}} \varepsilon^{-\frac{1}{\gamma} \varepsilon_1} = \varepsilon,
\]

we obtain

\[
M_i \leq 2\varepsilon M_{i+1} + \gamma 2^{r_\gamma} \varepsilon^{-\gamma} \frac{\left[ \Lambda(x_0, \frac{\rho}{4}) \right]^{1+\frac{2\gamma}{\gamma}} \rho^{1-n}}{\mu(x, M_i/\rho)} \int_D H(x, |\nabla v_{M_{i+1}}|) \ dx + \gamma \rho,
\]

which by \((2.1)\) implies

\[
M_i g(x_0, M_i/\rho) \leq 3\varepsilon M_{i+1} g(x_0, M_{i+1}/\rho)
\]

\[
+ \gamma 2^{r_\gamma} \varepsilon^{-\gamma} \frac{\left[ \Lambda(x_0, \frac{\rho}{4}) \right]^{1+\frac{2\gamma}{\gamma}} \rho^{1-n}}{\mu(x, M_i/\rho)} \int_D H(x, |\nabla v_{M_{i+1}}|) \ dx + \gamma \rho^n.
\]

Let \( \psi \in W_0(D) \) be such that \( \frac{1}{m} \int_D H(x, m|\nabla \psi|) \ dx \leq C_H(E, B_{\rho_0}(x_0), m) + \gamma \rho^n \). Testing identity \((3.1)\) by \( \varphi = v - m \psi \), by the Young inequality \((2.1)\) and using condition \((g_\lambda)\) we obtain

\[
\int_D H(x, m|\nabla \psi|) \ dx \leq \gamma m \int_{B_{2r}(x_0)} H(x, m|\nabla \psi|) \ dx \leq \gamma m (C_H(E, B_{\rho_0}(x_0), m) + \rho^n).
\]

Testing \((3.1)\) by \( \varphi = v_{M_{i+1}} - M_{i+1} v/m \) and using the Young inequality \((2.1)\) and \((3.5)\), we have

\[
\int_D H(x, |\nabla v_{M_{i+1}}|) \ dx \leq \gamma M_{i+1} \int_D H(x, |\nabla v|) \ dx \leq \gamma M_{i+1} (C_H(E, B_{\rho_0}(x_0), m) + \rho^n).
\]

This inequality and \((3.4)\) imply that

\[
M_i g(x_0, M_i/\rho) \leq 3\varepsilon M_{i+1} g(x_0, M_{i+1}/\rho)
\]

\[
+ \gamma 2^{r_\gamma} \varepsilon^{-\gamma} M_{i+1} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+\frac{2\gamma}{\gamma}} \rho^{1-n} C_H(E, B_{\rho_0}(x_0), m) + \rho^n, \ i = 0, 1, 2, \ldots
\]

which yields for any \( \varepsilon_2 \in (0, 1) \)

\[
g(x_0, M_i/\rho) \leq \frac{1}{\varepsilon_2} \frac{M_i}{M_{i+1}} g(x_0, M_i/\rho) + \varepsilon_2^{-1} g(x_0, M_{i+1}/\rho) \leq \left( \frac{3\varepsilon}{\varepsilon_2} + \varepsilon_2^{-1} \right) g(x_0, M_{i+1}/\rho) + \gamma 2^{r_\gamma} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+\frac{2\gamma}{\gamma}} \rho^{1-n} C_H(E, B_{\rho_0}(x_0), m) + \rho^n \leq \left( \frac{3\varepsilon}{\varepsilon_2} + \varepsilon_2^{-1} \right) g(x_0, M_{i+1}/\rho) + \gamma 2^{r_\gamma} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+\frac{2\gamma}{\gamma}} \rho^{1-n} C_H(E, B_{\rho_0}(x_0), m), \ i = 0, 1, 2, \ldots
\]
provided that \( \bar{c} \) in \((3.2)\) is chosen to satisfy \( \bar{c} \geq 1 \). Iterating the last inequality and choosing \( \varepsilon_2 \) and \( \varepsilon \) small enough, we arrive at

\[
g(x_0, M_0/\rho) \leq \gamma \left( \Lambda \left( x_0, \frac{\rho}{4} \right) \right)^{1+\frac{2\varepsilon_1}{\rho}} \rho^{1-n} C_H(E, B_{8\rho}(x_0), m),
\]

which proves the upper bound of the function \( v \) with

\[
\beta = \frac{1}{p-1} \left( 1 + \frac{q-1}{\kappa} \right).
\]

### 3.2 Lower bound for the function \( v \)

The main step in the proof of the lower bound is the following lemma.

**Lemma 3.2.** There exist numbers \( \varepsilon, \vartheta \in (0, 1) \), \( \tilde{\beta}_2, \tilde{\beta}_3 > 0 \) depending only on the data such that

\[
\left\{ K_{\rho/4, 2\rho} : v(x) \leq \varepsilon m \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\tilde{\beta}_2} \rho g_{x_1}^{-1} \left( C_H(E, B_{8\rho}(x_0); m) \right) \right\} \subseteq \left\{ \left( 1 - \vartheta \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\tilde{\beta}_3} \right) |K_{\rho/4, 2\rho}| \right\}, \tag{3.6}
\]

where \( K_{\rho_1, \rho_2} := B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0), \ 0 < \rho_1 < \rho_2. \)

**Proof.** Let \( \zeta_1 \in C_0^\infty(B_{\rho}(x_0)), 0 \leq \zeta_1 \leq 1, \ \zeta_1 = 1 \) in \( B_{\rho/2}(x_0), \ |\nabla \zeta_1| \leq 2/\rho. \) Testing \((3.1)\) by \( \varphi = v - m \zeta_1^q \) and using condition \((g_\lambda)\) with \( K = 2M \) and the Young inequality \((2.1)\), we obtain for any \( \varepsilon_1 \in (\rho, 2M \lambda(\rho)) \)

\[
\int_{\mathcal{D}} H(x, |\nabla v|) \, dx \leq \frac{\gamma m}{\rho} \int_{K_{\rho/2, \rho}} h(x, |\nabla v|) \zeta_1^{q-1} \, dx \leq \frac{\gamma m}{\rho} \sum_{\varepsilon_1} \int_{K_{\rho/2, \rho}} h(x, \varepsilon_1/\rho) \, dx \leq \frac{\gamma m}{\rho} \sum_{\varepsilon_1} \int_{K_{\rho/2, \rho}} b(x) g(x, \varepsilon_1/\rho) \, dx \leq \frac{\gamma m}{\rho} \int_{K_{\rho/2, \rho}} b(x) G(x, v/\rho) \, dx \leq \frac{\gamma m}{\rho} \int_{K_{\rho/2, \rho}} b(x) G^+(x, v/\rho) \, dx,
\]

Let \( \zeta_2 \in C_0^\infty(K_{\rho/4, 2\rho}), 0 \leq \zeta_2 \leq 1, \ \zeta_2 = 1 \) in \( K_{\rho/2, \rho}, \ |\nabla \zeta_2| \leq 4/\rho. \) Testing \((3.1)\) by \( \varphi = v \zeta_2^q \) and using the Young inequality \((2.1)\), we estimate the first term on the right-hand side of the previous inequality as follows:

\[
\int_{K_{\rho/4, 2\rho}} H(x, |\nabla v|) \, dx \leq \int_{K_{\rho/4, 2\rho}} H(x, |\nabla v|) \zeta_2^q \, dx \leq \gamma \int_{K_{\rho/4, 2\rho}} H(x, v/\rho) \zeta_2^q \, dx \leq \gamma \int_{K_{\rho/4, 2\rho}} b(x) G(x, v/\rho) \, dx \leq \gamma \int_{K_{\rho/4, 2\rho}} b(x) G^+(x, v/\rho) \, dx,
\]
here we used the notation $G_{K_{p/4,2p}}^+(v/\rho) = \sup_{x \in K_{p/4,2p}} G(x,v/\rho)$. Combining the last two inequalities and using the definition of capacity, we obtain

$$C_H(E,B_{8\rho}(x_0);m) \leq \frac{1}{m} \int_D H(x,\lfloor \nabla v \rfloor) \, dx \leq \frac{\gamma}{\varepsilon_1} \int_{K_{p/4,2p}} b(x)G_{K_{p/4,2p}}^+(v/\rho) \, dx + \gamma \Lambda \left( x_0, \frac{\rho}{4} \right) g(x_0,\varepsilon_1/\rho) \rho^{n-1}.$$  

Choose $\varepsilon_1$ by the condition

$$\Lambda \left( x_0, \frac{\rho}{4} \right) g(x_0,\varepsilon_1/\rho) = \tau_1 \rho^{1-n} C_H(E,B_{8\rho}(x_0);m), \quad \tau_1 \in (0,1).$$

By (3.2) $\varepsilon_1 \geq 8\rho$ if $\bar{c} = \sigma(\tau_1)$ is large enough and if $\bar{c}_1 \geq 1$. To apply condition $(g_{\lambda})$, we still need to check the inequality $\varepsilon_1 \leq 2M \lambda(\rho)$. Indeed, let $\psi \in C_0^\infty(B_{2\rho}(x_0)), 0 \leq \psi \leq 1, \psi = 1$ in $B_{\rho}(x_0)$ and $|\nabla \psi| \leq 2/\rho$, then by $(g_1)$ and $(g_\lambda)$ with $K = 2M$ we have

$$C_H(E,B_{8\rho}(x_0);m) \leq C_H(E,B_{\rho}(x_0),B_{8\rho}(x_0);m) \leq \frac{1}{m} \int_{B_{8\rho}(x_0)} H(x,m|\nabla \psi|) \, dx \leq \frac{\gamma}{\rho} \int_{B_{2\rho}(x_0)} b(x) g(x,m/\rho) \, dx \leq \gamma \Lambda \left( x_0, \frac{\rho}{4} \right) g(x_0,\varepsilon_1/\rho) \rho^{n-1},$$

and therefore, if $\tau_1 = 1/2\gamma$, then $\varepsilon_1 \leq \rho g_{x_0}^{-1}(\tau_1 \gamma g(x_0,m/\rho)) \leq m \leq 2M \lambda(\rho)$. So, by our choices, from the previous, we obtain

$$C_H(E,B_{8\rho}(x_0);m) \leq \frac{\gamma}{\varepsilon_1} \int_{K_{p/4,2p}} b(x)G_{K_{p/4,2p}}^+(v/\rho) \, dx.$$  

Let us estimate the term on the right-hand side of (3.7). For this we decompose $K_{p/4,2p}$ as $K_{p/4,2p} = K_{p/4,2p}' \cup K_{p/4,2p}''$, where

$$K_{p/4,2p}':= K_{p/4,2p} \cap \left\{ v \leq \varepsilon g_{x_0}^{-1} \left( \frac{C_H(E,B_{8\rho}(x_0);m)}{\rho^{n-1}[\Lambda(x_0,\frac{\rho}{4})]^{\beta_4}} \right) \right\}, K_{p/4,2p}'':= K_{p/4,2p} \setminus K_{p/4,2p}'.$$

and $\varepsilon \in (0,1), \beta_4 > 0$ to be determined later. If $\bar{c}^{-1} \geq 1/\varepsilon$ and $\bar{c}_1 \geq \bar{c}_4$ then by (3.2)

$$\rho \leq \varepsilon g_{x_0}^{-1} \left( \frac{C_H(E,B_{8\rho}(x_0);m)}{\rho^{n-1}[\Lambda(x_0,\frac{\rho}{4})]^{\beta_4}} \right) \leq 1,$$

and by $(g_{\lambda})$ with $K = 1$ we have

$$\frac{\gamma}{\varepsilon_1} \int_{K_{p/4,2p}'} b(x)G_{K_{p/4,2p}}^+(v/\rho) \, dx \leq \frac{\gamma}{\varepsilon_1} \Lambda \left( x_0, \frac{\rho}{4} \right) G_{K_{p/4,2p}'} \left( \varepsilon g_{x_0}^{-1} \left( \frac{C_H(E,B_{8\rho}(x_0);m)}{\rho^{n-1}[\Lambda(x_0,\frac{\rho}{4})]^{\beta_4}} \right) \right) \rho^{\beta_4} \leq \frac{\gamma}{\varepsilon_1} \Lambda \left( x_0, \frac{\rho}{4} \right) G \left( x_0, \varepsilon g_{x_0}^{-1} \left( \frac{C_H(E,B_{8\rho}(x_0);m)}{\rho^{n-1}[\Lambda(x_0,\frac{\rho}{4})]^{\beta_4}} \right) \right) \leq \varepsilon \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\frac{m}{1-p}-p\beta_4} C_H(E,B_{8\rho}(x_0);m).$$  

(3.8)
By the upper bound for the function \( v \), \([5.2]\) and our choice of \( \varepsilon_1 \), we obtain
\[
\frac{\gamma}{\varepsilon_1} \int_{K''_{p/4,2p}} b(x) G_{K''_{p/4,2p}}(v/\rho) \, dx \leq \gamma(\varepsilon) \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+q\beta} \rho^{-n} C_H(E, B_{8\rho}(x_0); m) |K''_{p/4,2p}|. \tag{3.9}
\]
Collecting estimates \([3.7] – [3.9]\) we obtain
\[
C_H(E, B_{8\rho}(x_0); m) \leq \varepsilon \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+q\beta} \rho^{-n} C_H(E, B_{8\rho}(x_0); m) + \gamma(\varepsilon) \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{1+q\beta} \rho^{-n} CH(E, B_{8\rho}(x_0); m) |K''_{p/4,2p}|.
\]
choosing \( \varepsilon, \beta \) from the conditions \( \varepsilon \gamma = 1/2, \beta_4 = \frac{1}{p-1} \), we arrive at
\[
\left\{ K_{p/4,2p} : v \geq \varepsilon \rho \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\frac{(p-1)^2}{n}} g_{x_0}^{-1} \left( \frac{C_H(E, B_{8\rho}(x_0); m)}{\rho^{n-1}} \right) \right\} \geq \gamma^{-1} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-q\beta} \rho^n.
\]
This completes the proof of the lemma with \( \beta_2 = \frac{1}{(p-1)^2} \) and \( \beta_3 = 1 + q\beta \).

3.3 Proof of Theorem 1.1 under (g\( _A \)) conditions

To prove Theorem 1.1 we need to estimate the term on the left hand side of inequality \( [3.6] \). Let \( \delta = \frac{t}{t+1} \) and set \( \overline{g}(x, v) := \frac{1}{\rho} \int_{0}^{v} g(x, s) \, ds \), \( \overline{G}(x, v) := \frac{1}{\rho} \int_{0}^{v} \overline{g}(x, s) \, ds \) for \( v > 0 \). By our assumptions \( p - 1 \leq \overline{g}(x, v) \leq q - 1, \quad x \in \Omega \). We use Lemma 2.3 with \( Q(v) = [G_{B_{8\rho}(x_0)}^{-}(v)]^{\delta} \), for this we need to check that \( Q''(v) \geq 0 \), indeed by our choices
\[
\frac{1}{\delta} Q''(v) \overline{G}_{B_{8\rho}(x_0)}^{-}(v)^{2-\delta} = \overline{G}_{B_{8\rho}(x_0)}^{-}(v) \overline{g}_{B_{8\rho}(x_0)}^{-} - \frac{1}{t+1} \overline{g}_{B_{8\rho}(x_0)}^{-}(v)^{2} \geq \frac{1}{\delta} (p - 1) \overline{G}_{B_{8\rho}(x_0)}^{-}(v) - \frac{1}{t+1} v \overline{g}_{B_{8\rho}(x_0)}^{-}(v)^{2} \geq (p - 1) \overline{G}_{B_{8\rho}(x_0)}^{-}(v)^{2} \left( \frac{p - 1}{q} - \frac{1}{t+1} \right) > 0.
\]
Let \( \varphi \in W_0(B_{8\rho}(x_0)), \varphi \geq 1 \) on \( E \), then by Lemma 2.3 and by the Hölder inequality we have
\[
\left[ \overline{G}_{B_{8\rho}(x_0)}^{-} \left( \frac{m}{\rho} \right) \right]^{\delta} |E| \leq \int_{B_{8\rho}(x_0)} \left[ \overline{G}_{B_{8\rho}(x_0)}^{-} \left( \frac{m}{\rho} \varphi \right) \right]^{\delta} \, dx \leq \gamma \int_{B_{8\rho}(x_0)} \left[ G_{B_{8\rho}(x_0)}^{-} (m|\nabla \varphi|) \right]^{\delta} \, dx \leq \gamma \left( \int_{B_{8\rho}(x_0)} [a(x)]^{-1} \, dx \right)^{1-\delta} \left( \int_{B_{8\rho}(x_0)} a(x) G(x, m|\nabla \varphi|) \, dx \right)^{\delta} \leq \gamma \rho^{n(1-\delta)} [\Lambda(x_0, \rho)]^{p\delta} \left(m C_H(E, B_{8\rho}(x_0); m) \right)^{\delta},
\]
which yields by our choice of \( m \) and \( \delta \)
\[
\gamma^{-1} g \left( x_0, \frac{m}{\rho} \right) \left( \frac{|E|}{\rho^n} \right)^{1+1} \left( \frac{|E|}{\rho^n} \right) \leq \rho^{1-n} C_H(E, B_{8\rho}(x_0); m),
\]
and hence
\[
\rho \, g^{-1}_x \left( \rho^{1-n} C_H(E, B_{8\rho}(x_0); m) \right) \geq \gamma^{-1} m \left[ \Lambda(x_0, \rho) \right]^{-\beta_5} \left( \frac{|E|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}}, \quad \beta_5 = \frac{p}{p-1}. \tag{3.10}
\]

Note that by (3.10), inequality (3.24) holds if
\[
m \left( \frac{|E|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \geq \gamma \, \rho \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_5} \left( \frac{|E|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}}, \quad \beta_5 = \frac{p}{p-1} + \beta_5, \tag{3.11}
\]
with some \( \gamma = \gamma(\bar{c}) > 0. \)

In this case inequality (3.6) translates into
\[
\left\{ \begin{array}{l}
K_{\rho/4, 2\rho} : v(x) \leq \gamma^{-1} \varepsilon m \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_2 - \beta_5} \left( \frac{|E|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \\
\end{array} \right. \leq \leq \left( 1 - \dot{\vartheta} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_3} \right) |K_{\rho/4, 2\rho}|. \tag{3.12}
\]

To complete the proof of Theorem 1.1, construct the sets \( E(\rho, m) := B_\rho(x_0) \cap \{ u(x) \geq m \} \) and \( \bar{E}(\rho, m) := B_\rho(x_0) \cap \{ u(x) \geq m \lambda(\rho) \}, \quad 0 < m < M, \quad E(\rho, m) \subset \bar{E}(\rho, m). \) Let \( v \) be an auxiliary solution to the correspondent problem in \( D := B_{8\rho}(x_0) \setminus \bar{E}(\rho, m). \) Since \( u \geq v \) on \( \partial D \) then by the monotonicity to the correspondent problem in \( D \) and inequality (3.12) implies that
\[
\left\{ \begin{array}{l}
B_{2\rho}(x_0) : u(x) \leq \gamma^{-1} \varepsilon m \lambda(\rho) \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_2 - \beta_5} \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \\
\end{array} \right. \leq \leq \left( 1 - \frac{1}{8} \dot{\vartheta} \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_3} \right) |B_{2\rho}(x_0)|,
\]
provided that inequality (3.11) holds.

Now we apply Lemma 2.7 with \( r \) replaced by \( \rho, \) \( m \) replaced by \( m \lambda(\rho), \) \( \xi \lambda(r) \omega_r \) replaced by \( \gamma^{-1} \varepsilon m \lambda(\rho)\Lambda(x_0, \frac{\rho}{4})^{-\beta_2 - \beta_5} \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \) and \( a_0 \) replaced by \( \frac{1}{8} \dot{\vartheta} \left[ \Lambda(x_0, \frac{\rho}{4}) \right]^{-\beta_3}. \) Lemma 2.7 yields that if
\[
m \lambda(\rho) \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \geq \gamma(\varepsilon) \rho \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_2 + \beta_5} \exp \left( C_\ast \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\frac{\beta_2 + \beta_5}{\kappa_1}} \right), \tag{3.13}
\]
then
\[
\min_{x \in B_{2\rho}(x_0)} u(x) \geq \gamma^{-1} \varepsilon m \lambda(\rho) \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{\ell+1}{\ell(p-1)}} \times \times \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\beta_2 - \beta_5} \exp \left( -C_\ast \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\frac{\beta_2 + \beta_5}{\kappa_1}} \right), \tag{3.14}
\]
here \( \kappa_1 > 0 \) is the number fixed by Lemma 2.7. We can assume without loss that \( \left[ \Lambda(x_0, \frac{\rho}{4}) \right]^{\beta_2 + \beta_5} \leq \exp \left( \Lambda \left( x_0, \frac{\rho}{4} \right) \right), \) therefore since \( E(\rho, m) \subset \bar{E}(\rho, m), \) inequalities (3.13), (3.14) imply (1.5) with \( \beta_1 = 1 + \beta_2 + \frac{\beta_5}{\kappa_1}, \) which completes the proof of Theorem 1.1.
4 Proof of Theorem 1.2 under \((g_\mu)\) conditions

The proof is similar to that of Paragraphs 3.1 – 3.3, we give only the sketch of the proof.

Fix \(x_0 \in \Omega\), let \(E \subset B_r(x_0) \subset B_\rho(x_0) \subset B_R(x_0) \subset \Omega\) and consider the solution \(w\) of the following problem

\[
\text{div} \left( H(x, |\nabla w|) \frac{\nabla w}{|\nabla w|^2} \right) = 0, \quad x \in \mathcal{D} = B_{8\rho}(x_0) \setminus E, \quad w - m\psi \in W_0(\mathcal{D}),
\]

where \(m \in [\rho, 2M]\) is some fixed number and \(\psi \in W_0(\mathcal{D})\), \(\psi = 1\) on \(E\).

We will assume that the following integral identity holds:

\[
\int_{\mathcal{D}} H(x, |\nabla w|) \frac{\nabla \varphi}{|\nabla w|^2} \nabla \varphi \, dx = 0 \quad \text{for any } \varphi \in W_0(\mathcal{D}), \quad \mathcal{D} = B_{8\rho}(x_0) \setminus E. \tag{4.1}
\]

Testing (4.1) by \(\varphi = (w - m)_+\) and by \(\varphi = w_-\) and using condition (1.4), we obtain that \(0 \leq w \leq m\). For \(i, j = 0, 1, 2, \ldots\) set \(k_j := k(1 - 2^{-j})\), \(k > 0\) to be chosen later, \(\rho_{i,j} := 2^{-i-j-\frac{3}{2}}\rho\), \(\kappa \leq 0\).

Fix \(\overline{\xi} \in F_i\) and suppose that the following integral identity holds:

\[
\int_{B_{\rho_{i,j}}(\overline{\xi})} |\nabla (w - k_{j+1})_+| \, \zeta_{i,j} \, dx \leq 2^{(i+j)} M_{i+1} \mu \left( \frac{\rho}{4} \right) A \left( x_0, \frac{\rho}{4} \right) M_{i+1} \rho^{-n} \frac{\rho^2}{\rho_{i,j}} \left( \int_{B_{\rho_{i,j}}(\overline{\xi})} (w - k_{j+1})_+ \, dx \right)^{1+\kappa - \frac{1}{q}} \leq \gamma 2^{(i+j)} M_{i+1} \mu \left( \frac{\rho}{4} \right) A \left( x_0, \frac{\rho}{4} \right) M_{i+1} \rho^{-n} \frac{\rho^2}{\rho_{i,j}} \left( \int_{B_{\rho_{i,j}}(\overline{\xi})} (w - k_{j+1})_+ \, dx \right)^{1+\kappa - \frac{1}{q}},
\]

where \(\kappa = \frac{1}{n} - \frac{1}{lp} - \frac{1}{sq} - \frac{1}{p} + \frac{1}{q} > 0\).

From this, by standard arguments and choosing \(k\) from the condition

\[
k^{1+\frac{1}{q}} = \gamma 2^{(i+j)} M_{i+1} \mu \left( \frac{\rho}{4} \right) A \left( x_0, \frac{\rho}{4} \right) \frac{1}{\rho_{i,j}} M_{i+1} \rho^{-n} \int_{B_{\rho/2^{i+1}}(\overline{\xi})} w \, dx,
\]

since \(\overline{\xi} \in F_i\) is an arbitrary point, using the Young inequality and keeping in mind our assumption that \(w(\overline{\xi}) \geq k + \rho\), from the previous we obtain for any \(\varepsilon \in (0, 1)\)

\[
M_i \leq \varepsilon M_{i+1} + \frac{\gamma 2^{(i+j)} \mu \left( \frac{\rho}{4} \right) A \left( x_0, \frac{\rho}{4} \right) \frac{1}{\rho_{i,j}} M_{i+1} \rho^{-n} \int_{F_{i+1}} w \, dx + \gamma \rho}{\varepsilon \rho^3}, \quad i = 0, 1, 2, \ldots \tag{4.2}
\]
Lemma 4.1. There exist numbers \( \varepsilon, \vartheta, \bar{\beta}_2, \bar{\beta}_3 > 0 \) depending only on the data such that

\[
\left\{ K_{\rho/4, 2\rho} : w(x) \leq \varepsilon \right\} \leq \varepsilon \rho \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\bar{\beta}_2 - \bar{\beta}_3} \left( \frac{|E(r, m)|}{\rho^n} \right)^{\frac{4+4}{1(n-1)}} \left| K_{\rho/4, 2\rho} \right|
\]

Similarly to [12], inequality (4.3) translates into

\[
\left\{ K_{\rho/4, 2\rho} : w(x) \leq \gamma - 1 \varepsilon \right\} \leq \left( 1 - \vartheta \right) \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\bar{\beta}_3} \left| K_{\rho/4, 2\rho} \right|
\]

Set \( E = E(\rho, N) := B_\rho(x_0) \cap \{ u(x) \geq m \} \) with any \( 0 < m \leq M \), by Lemma 2.8 with \( \rho \) replaced by \( \rho, \xi, \omega_r \) replaced by \( \gamma^{-1} \varepsilon \rho \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\bar{\beta}_2 - \bar{\beta}_3} \left( \frac{|E(r, m)|}{\rho^n} \right)^{\frac{4+4}{1(n-1)}} \) and \( \alpha_0 \) replaced by \( \frac{1}{\varepsilon} \vartheta \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\bar{\beta}_3} \), from the previous inequality, using monotonicity condition we obtain that either

\[
m \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{4+4}{1(n-1)}} \leq \gamma \rho \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^\beta_3 \exp \left( C_\ast \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^\beta_3 \right)
\]

or

\[
\min_{x \in B_{\frac{\rho}{2}}(x_0)} u(x) \geq \gamma^{-1} \varepsilon m \left( \frac{|E(\rho, m)|}{\rho^n} \right)^{\frac{4+4}{1(n-1)}} \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{-\bar{\beta}_2 - \bar{\beta}_3} \exp \left( -C_\ast \left( x_0, \frac{\rho}{4} \right)^{\beta_3} \right)
\]

here \( \kappa_1 > 0 \) is the number fixed by Lemma 2.7. We assume without loss that \( \left[ \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^\beta_3 \leq \exp \left( \left( \frac{\rho}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right) \), therefore inequalities (4.3), (4.5) imply (4.6) with

\[
\beta_1 = 1 + \bar{\beta}_2 + \bar{\beta}_3, \quad \kappa_1, \text{ which completes the proof of Theorem 1.2}
\]

5 Harnack type inequalities, proof of Theorems 1.3, 1.4

5.1 Proof of Theorem 1.3

Theorem 1.3 follows immediately from Theorems 1.1, 1.2, indeed set

\[
\tilde{m}(\rho) = \frac{1}{\lambda(\rho)} \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \left( \min_{x \in B_{\frac{\rho}{2}}(x_0)} u(x) + \rho \right) \right)
\]
where $\gamma$ and $\beta_1$ are the positive numbers defined in Theorem 1.1. By (1.3) we obtain for $\theta \in (0, \frac{1}{t+1}(p-1))$

$$\rho^{-n} \int_{B_\rho(x_0)} u^\theta \, dx = \theta \rho^{-n} \int_{0}^{\infty} |\{B_\rho(x_0) : u(x) > m\}| m^{\theta-1} \, dm \leq \tilde{m}^\theta(\rho) +$$

$$+ \gamma \tilde{m}^{\frac{t}{t+1}(p-1)}(\rho) \int_{\tilde{m}(\rho)}^{\infty} m^{\theta-\frac{t}{t+1}(p-1)-1} \, dm \leq \gamma ((p-1) - \theta(t+1))^{-1} \tilde{m}^\theta(\rho), \quad (5.1)$$

which proves inequality (1.7).

To prove (1.8) we use inequality (1.6) with $\tilde{m}(\rho) = \exp \left( \left( \mu \left( \frac{\rho}{2} \right) \Lambda \left( x_0, \frac{\rho}{2} \right) \right)^{\beta_1} \right) \left( \min_{B_{\rho}^*} u(x) + \rho \right)$, completely similar to the previous

$$\rho^{-n} \int_{B_\rho(x_0)} u^\theta \, dx \leq \tilde{m}^\theta(\rho) + \gamma \tilde{m}^{\frac{t}{t+1}(p-1)}(\rho) \int_{\tilde{m}(\rho)}^{\infty} m^{\theta-\frac{t}{t+1}(p-1)-1} \, dm \leq \gamma ((p-1) - \theta(t+1))^{-1} \tilde{m}^\theta(\rho),$$

which completes the proof of Theorem 1.3.

### 5.2 Proof of Theorem 1.4

The proof of Theorem 1.4 is almost standard. Fix $\sigma \in (0, \frac{1}{2})$, $s \in (\frac{3}{4} \rho, \frac{5}{4} \rho)$ and for $j = 0, 1, 2, \ldots$ set $\rho_j := s(1 - \sigma + \sigma 2^{-j})$, $B_j := B_{\rho_j}(x_0)$ and let $M_0 := \max_{B_0} u$, $M_\sigma := \max_{B_{\infty}} u$.

Similarly to Section 4 we obtain

$$M_\sigma^{1+\frac{1}{n}} \leq \gamma \sigma^{-\gamma} M_0^{\frac{1}{n}} \left[ \mu \left( \frac{\rho}{2} \right) \Lambda \left( x_0, \frac{\rho}{2} \right) \right]^{\frac{1}{n}} \rho^{-n} \int_{B_0} u \, dx + \gamma \rho,$$

which implies for any $\varepsilon, \theta \in (0, 1)$ that

$$M_\sigma \leq \varepsilon M_0 + \gamma \sigma^{-\gamma} \left[ \mu \left( \frac{\rho}{2} \right) \Lambda \left( x_0, \frac{\rho}{2} \right) \right]^{\frac{1}{n}} \rho^{-n} \int_{B_0} u^\theta \, dx + \gamma \rho.$$

Iterating this inequality we arrive at

$$\sup_{B_{\rho}^*(x_0)} u \leq \gamma \left[ \mu \left( \frac{\rho}{2} \right) \Lambda \left( x_0, \frac{\rho}{2} \right) \right]^{\frac{1}{n}} \rho^{-n} \int_{B_0} u^\theta \, dx + \gamma \rho. \quad (5.2)$$

Collecting inequalities (1.7) and (5.2) we obtain with $\theta = \frac{1}{2} \min \left( 1, \frac{1}{t+1}(p-1) \right)$

$$\sup_{B_{\rho}^*(x_0)} u \leq \frac{\gamma}{\lambda(\rho)} \left[ \mu \left( \frac{\rho}{2} \right) \Lambda \left( x_0, \frac{\rho}{2} \right) \right]^{\frac{1}{n}} \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1} \right) \left( \inf_{B_{\rho}^*(x_0)} u(x) + \rho \right) \leq$$

$$\leq \frac{\gamma}{\lambda(\rho)} \left[ \mu \left( \frac{\rho}{4} \right) \right]^{\frac{1}{n}} \exp \left( \gamma \left[ \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1+1} \right) \left( \inf_{B_{\rho}^*(x_0)} u(x) + \rho \right),$$
which proves Theorem 1.4 under the \((g_\lambda)\) and \((g_\mu)\) conditions.

Similarly, by (1.8) and (5.2) we obtain for any \(\theta = \frac{1}{2}(1, \frac{1}{r+1}(p-1))\)

\[
\sup_{B_{\frac{3}{2}}(x_0)} u \leq \gamma \exp \left( \gamma \left[ \mu \left( \frac{r}{4} \right) \Lambda \left( x_0, \frac{\rho}{4} \right) \right]^{\beta_1 + \frac{1}{\beta_2}} \right) \left( \inf_{B_{\frac{3}{2}}(x_0)} u(x) + \rho \right),
\]

which completes the proof of Theorem 1.4 under \((g_\mu)\) condition.

5.3 Proof of Theorem 1.4 using the local clustering lemma under the conditions \((g_\lambda)\) and \(\Lambda(x_0, \rho) \approx\) const

In this Section, as it was mentioned, we give a simple proof of Theorem 1.3 for the function \(u\) which belongs to the corresponding \(DG_{g_1}(\Omega)\) De Giorgi classes, using the local clustering lemma \([17]\) under the conditions \((g_\lambda)\) and \(\Lambda(x_0, \rho) \approx\) const, \(\rho > 0\). For the function \(\lambda\) we additionally assume that there exists \(c > 0\) such that

\[
\lambda(\rho) = \left( \frac{\rho}{r} \right)^c \lambda(r), \quad 0 < r < \rho.
\]

For \(\rho > 0\) denote by \(K_\rho(y)\) a cube of edge \(\rho\) centered at \(y\).

**Lemma 5.1. (Local clustering lemma [17])** Let \(u \in W^{1,1}(K_\rho(x_0))\) and let

\[
\int_{K_\rho(x_0)} |\nabla(u - k)_-| \, dx \leq K k \rho^{n-1} \quad \text{and} \quad |\{ K_\rho(x_0) : u > k \}| \geq \alpha |K_\rho(x_0)|,
\]

for some \(k, K > 0\) and \(\alpha \in (0, 1)\). Then for any \(\eta \in (0, 1)\) and any \(\delta \in (0, 1)\) there exist \(\bar{x} \in K_\rho(x_0)\) and \(\bar{\varepsilon} = \bar{\varepsilon}(n) \in (0, 1)\) such that

\[
|\{ K_r(\bar{x}) : u \geq \delta k \}| \geq (1 - \eta) |K_r(\bar{x})|, \quad r = \bar{\varepsilon}(1 - \delta) \frac{\alpha^2}{K} \rho.
\]

Let \(\zeta \in C_0(2\rho(x_0))\), \(\zeta = 1\) on \(K_\rho(x_0), 0 \leq \zeta \leq 1\), \(|\nabla \zeta| \leq \frac{2}{\rho}\). For \(m \in (0, M)\) set \(k = m \lambda(r)\), where \(r \in (0, \rho)\) to be chosen. If \(m \lambda(r) \geq \rho\), then by condition \((g_\lambda)\)

\[
\frac{g_{K_\rho(x_0)}^{+}(m \lambda(r))}{m \lambda(r)} \leq C(M),
\]

therefore inequality (2.4) can be rewritten as

\[
\int_{K_\rho(x_0)} |\nabla(u - k)_-| \, dx \leq \gamma k \rho^{n-1}.
\]

Assume that with some \(\alpha \in (0, 1)\)

\[
|E(\rho, m)| := |\{ K_\rho(x_0) : u \geq m \}| \geq \alpha |K_\rho(x_0)|, \quad (5.3)
\]

then since

\[
|E(r, \rho, m)| := |\{ K_\rho(x_0) : u \geq \lambda(r) m \}| \geq |E(\rho, m)| \geq \alpha |K_\rho(x_0)|,
\]
by the local clustering lemma with \( k = \lambda(r) m, \delta = \frac{1}{7} \) and \( \eta = \nu_1 \) we obtain that
\[
|\{ K_r(\bar{x}) : u \geq \frac{1}{2} \lambda(r) m \}| \geq (1 - \nu_1) |K_r(\bar{x})|, \quad r = \varepsilon \alpha^2 \rho, \quad \varepsilon = \bar{\varepsilon} \frac{\nu_1}{2\gamma},
\]
here \( \nu_1 \in (0, 1) \) is the number fixed by Lemma 2.5. Inequality (5.4) together with Lemma 2.5 imply that
\[
\inf_{K_r(\bar{x})} u \geq \frac{1}{8} \lambda(r) m,
\]
provided that \( \lambda(r) m \geq 8 r \). By (5.5) and Lemma 2.7 there exists \( C_* > 0 \) depending only on the data such that
\[
\inf_{K_r(\bar{x})} u \geq 2^{-C_* - 3} \lambda(r) m,
\]
provided that \( \lambda(r) m \geq 2^{C_* + 3} r \). Repeating the previous arguments, on the \( j \)-th step we obtain
\[
\inf_{K_{2^j r}(\bar{x})} u \geq 2^{-jC_* - 3} \lambda(r) m,
\]
provided that \( \lambda(r) m \geq 2^{j(C_* + 3) 2^j r} \). Choosing \( j \) from the condition \( 2^j r = \rho \), from the previous we obtain
\[
\inf_{K_{2^j r}(x_0)} u \geq \inf_{K_{2^j r}(\bar{x})} u \geq 2^{-jC_* - 3} \lambda(r) m = \varepsilon \alpha^{2C_*} \lambda(r) m \geq \varepsilon \alpha^{2C_* + c} \lambda(r) m \geq \varepsilon \alpha^{2(C_* + c)} \lambda(\rho) m,
\]
provided that \( \lambda(\rho) m \geq 8 \varepsilon^{-C_* - c} \alpha^{-2(C_* + c)} \rho \), which yield
\[
m \lambda(\rho) \alpha^{2(C_* + c)} \leq \gamma \left( \inf_{K_{\rho}(x_0)} u + \rho \right),
\]
This is an analog of inequality (5.5) of Theorem 1.1. From this, using our choice of \( \alpha \), similarly to inequality (5.1) we obtain for any \( \theta \in \left( 0, \frac{1}{2(C_* + c)} \right) \)
\[
\left( \rho^{-n} \int_{B_{\rho}(x_0)} u^\theta \, dx \right)^{\frac{1}{\theta}} \leq \gamma (1 - 2(C_* + c)\theta)^{-\frac{1}{\theta}} [\lambda(\rho)]^{-1} \left( \inf_{B_{\rho}(x_0)} u + \rho \right),
\]
which completes the proof of Theorem 1.3 under the conditions \((g_\lambda)\) and \( \Lambda(x_0, \rho) \propto \text{const.} \).

Acknowledgements. This work is supported by grants of Ministry of Education and Science of Ukraine (project number is 0121U109525) and by the National Academy of Sciences of Ukraine (project numbers are 0121U111851, 0120U100178).

References

[1] Yu. A. Alkhutov, The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition (Russian), Differ. Uravn. 33 (1997), no. 12, 1651–1660; translation in Differential Equations 33 (1997), no. 12, 1653–1663 (1998).
[2] Yu. A. Alkhutov, On the Hölder continuity of $p(x)$-harmonic functions, Sb. Math. 196 (2005), no. 1-2, 147–171.

[3] Yu. A. Alkhutov, O. V. Krasheninnikova, Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition, Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 6, 3–60 (in Russian).

[4] Yu. A. Alkhutov, O. V. Krasheninnikova, On the continuity of solutions of elliptic equations with a variable order of nonlinearity (Russian), Tr. Mat. Inst. Steklova 261, (2008), Differ. Uravn. i Din. Sist., 7–15; translation in Proc. Steklov Inst. Math. 261 (2008), no. 1–10.

[5] Yu. A. Alkhutov, M. D. Surnachev, A Harnack inequality for a transmission problem with $p(x)$-Laplacian, Appl. Anal. 98 (2019), no. 1-2, 332–344.

[6] Yu. A. Alkhutov, M. D. Surnachev, Harnack’s inequality for the $p(x)$-Laplacian with a two-phase exponent $p(x)$, J. Math. Sci. (N.Y.) 244 (2020), no. 2, 116–147.

[7] Yu. A. Alkhutov, M. D. Surnachev, Behavior at a boundary point of solutions of the Dirichlet problem for the $p(x)$-Laplacian (Russian), Algebra i Analiz 31 (2019), no. 2, 88–117; translation in St. Petersburg Math. J. 31 (2020), no. 2, 251–271.

[8] P. Baroni, M. Colombo, G. Mingione, Harnack inequalities for double phase functionals, Nonlinear Anal. 121 (2015), 206–222.

[9] P. Baroni, M. Colombo, G. Mingione, Non-autonomous functionals, borderline cases and related function classes, St. Petersburg Math. J. 27 (2016), 347–379.

[10] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, Calc. Var. Partial Differential Equations 57, 62 (2018).

[11] A. Benyaiche, P. Harjulehto, P. Hästö, A. Karpinnen, The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth, arXiv:2006.06276v1 [math.AP].

[12] K. O. Buryachenko, I. I. Skrypnik, Local continuity and Harnack’s inequality for double-phase parabolic equations, Potential Analysis, 56 (2020), 137–164.

[13] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Rational Mech. Anal. 218 (2015), no. 1, 219–273.

[14] M. Colombo, G. Mingione, Regularity for double phase variational problems, Arch. Rational Mech. Anal. 215 (2015), no. 2, 443–496.

[15] M. Colombo, G. Mingione, Calderon-Zygmund estimates and non-uniformly elliptic operators, J. Funct. Anal. 270 (2016), 1416–1478.

[16] G. Cupini, P. Marcellini, E. Mascolo, Nonuniformly elliptic energy integrals with $p,q$-growth, Nonlinear Anal. 177 (2018), 312–324.
[17] E. DiBenedetto, U. Gianazza, V. Vespri, Local clustering of the non-zero set of functions in $W^{1,1}(E)$, Rend. Lincei Mat. Appl. 17 (2006), 223–225

[18] E. DiBenedetto, N. S. Trudinger, Harnack inequalities for quasi-minima of variational integrals, Ann. Inst. Henri Poincare, 1 (1984), no. 4, 295–308.

[19] E. De Giorgi, Sulla differenziabilita’ e l’analiticita’ delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 3 (1957), 25–43.

[20] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, in: Lecture Notes in Mathematics, 2017, Springer, Heidelberg, 2011, x+509 pp.

[21] X. Fan, A Class of De Giorgi Type and Hölder Continuity of Minimizers of Variational with $m(x)$-Growth Condition. China: Lanzhou Univ., 1995.

[22] X. Fan, D. Zhao, A class of De Giorgi type and Hölder continuity, Nonlinear Anal. 36 (1999) 295–318.

[23] O. V. Hadzhy, I. I. Skrypnik, M. V. Voitovych, Interior continuity, continuity up to the boundary and Harnack’s inequality for double-phase elliptic equations with non-logarithmic growth, Math. Nachrichten, in press.

[24] P. Harjulehto, P. Hästö, Orlicz Spaces and Generalized Orlicz Spaces, in: Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019, p. X+169

[25] P. Harjulehto, P. Hästö, Boundary regularity under generalized growth conditions, Z. Anal. Anwend. 38 (2019), no. 1, 73–96.

[26] P. Harjulehto, P. Hästö, M. Lee, Hölder continuity of quasiminimizers and $\omega$-minimizers of functionals with generalized Orlicz growth, Ann. Sc. Norm. Super Pisa Cl. Sci 5 XXII (2021), no. 2, 549–582.

[27] P. Harjulehto, P. Hästö, O. Toivanen, Hölder regularity of quasiminimizers under generalized growth conditions, Calc. Var. Partial Differential Equations 56 (2017), no. 2, Art. 22, 26 pp.

[28] O. V. Krasheninnikova, On the continuity at a point of solutions of elliptic equations with a nonstandard growth condition, Proc. Steklov Inst. Math. (2002), no. 1(236), 193–200.

[29] N. V. Krylov, M. V. Safonov, A property of the solutions of parabolic equations with measurable coefficients, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 1, 161–175 (in Russian).

[30] O. A. Ladyzhenskaya, N. N. Ural’tseva, Linear and quasilinear elliptic equations, Nauka, Moscow, 1973.
On asymptotic behavior ....

[31] E. M. Landis, Some questions in the qualitative theory of second-order elliptic equations (case of several independent variables), Uspehi Mat. Nauk 18 (1963), no. 1 (109), 3–62 (in Russian).

[32] E. M. Landis, Second Order Equations of Elliptic and Parabolic Type, in: Translations of Mathematical Monographs, vol. 171, American Math. Soc., Providence, RI, 1998.

[33] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations, Comm. Partial Differential Equations 16 (1991), no. 2-3, 311–361.

[34] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, Arch. Rational Mech. Anal. 105 (1989), no. 3, 267–284.

[35] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, J. Differential Equations 90 (1991), no. 1, 1–30.

[36] V. G. Maz’ya, Behavior, near the boundary, of solutions of the Dirichlet problem for a second-order elliptic equation in divergent form, Math. Notes of Ac. of Sciences of USSR, 2 (1967), 610–617.

[37] J. Moser, On Harnack’s theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577–591.

[38] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958), 931–954.

[39] J. Ok, Regularity for double phase problems under additional integrability assumptions, Nonlinear Anal. 194 (2020) 111408.

[40] J. Serrin, Local behavior of solutions of quasi-linear equation, Acta Math, 111 (1964), 302–347.

[41] M. A. Shan, I. I. Skrypnik, M. V. Voitovych, Harnack’s inequality for quasilinear elliptic equations with generalized Orlicz growth, Electr. J. of Diff. Equations, 2021 (2021), no. 27, 1–16.

[42] I. V. Skrypnik, Pointwise estimates of certain capacitative potentials, in: General theory of boundary value problems, pp. 198–206, Naukova Dumka, Kiev, 1983 (in Russian).

[43] I. V. Skrypnik, Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, in: Translations of Mathematical Monographs, vol. 139, American Math. Soc., Providence, RI, 1994.

[44] I. V. Skrypnik, Selected works, in: Problems and Methods. Mathematics. Mechanics. Cybernetics, vol. 1, Naukova Dumka, Kiev, 2008 (in Russian).
[45] I. I. Skrypnik, M. V. Voitovych, $B_1$ classes of De Giorgi, Ladyzhenskaya and Ural’tseva and their application to elliptic and parabolic equations with nonstandard growth, Ukr. Mat. Visn. 16 (2019), no. 3, 403–447.

[46] I. I. Skrypnik, M. V. Voitovych, $B_1$ classes of De Giorgi-Ladyzhenskaya-Ural’tseva and their applications to elliptic and parabolic equations with generalized Orlicz growth conditions, Nonlinear Anal. 202 (2021) 112135.

[47] M. D. Surnachev, On Harnack’s inequality for $p(x)$-Laplacian (Russian), Keldysh Institute Preprints 10.20948/prepr-2018-69, 69 (2018), 1–32.

[48] N. S. Trudinger, On the regularity of generalized solutions of linear nonuniformly elliptic equations, Arch. Rat. Mech. Anal. 42 (1971), 50–62.

[49] V. V. Zhikov, Questions of convergence, duality and averaging for functionals of the calculus of variations (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 5, 961–998.

[50] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 50, (1986), no. 4, 675–710, 877.

[51] V. V. Zhikov, On Lavrentiev’s phenomenon, Russian J. Math. Phys. 3 (1995), no. 2, 249–269.

[52] V. V. Zhikov, On some variational problems, Russian J. Math. Phys. 5 (1997), no. 1, 105–116 (1998).

[53] V. V. Zhikov, On the density of smooth functions in Sobolev-Orlicz spaces (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 310 (2004), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34], 67–81, 226; translation in J. Math. Sci. (N.Y.) 132 (2006), no. 3, 285–294.

[54] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, 1994.

[55] V. V. Zhikov, S. E. Pastukhova, On the improved integrability of the gradient of solutions of elliptic equations with a variable nonlinearity exponent (Russian), Mat. Sb. 199 (2008), no. 12, 19–52; translation in Sb. Math. 199 (2008), no. 11–12, 1751–1782.

CONTACT INFORMATION

Oleksandr V. Hadzhy
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Gen. Batiouk Str. 19, 84116 Sloviansk, Ukraine
aleksanderhadzhy@gmail.com

Maria O. Savchenko
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine,
On asymptotic behavior ....

Gen. Batiouk Str. 19, 84116 Sloviansk, Ukraine
Vasyl’ Stus Donetsk National University, 600-richcha Str. 21, 21021 Vinnytsia, Ukraine
shanunaria@ukr.net

Igor I. Skrypnik
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine,
Gen. Batiouk Str. 19, 84116 Sloviansk, Ukraine
Vasyl’ Stus Donetsk National University, 600-richcha Str. 21, 21021 Vinnytsia, Ukraine
ihor.skrypnik@gmail.com

Mykhailo V. Voitovych
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine,
Gen. Batiouk Str. 19, 84116 Sloviansk, Ukraine
voitovichmv76@gmail.com