Mass Superselection, Canonical Gauge Transformations, and Asymptotically Flat Variational Principles

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Abstract

The phase space reduction of Schwarzschild black holes by Thiemann and Kastrup and by Kuchař is reexamined from a different perspective on gauge freedom. This perspective introduces additional gauge transformations which correspond to asymptotically nontrivial diffeomorphisms. Various subtleties concerning variational principles for asymptotically flat systems are addressed which allow us to avoid the usual conclusion that treating such transformations as gauge implies the vanishing of corresponding total charges. Instead, superselection rules are found for the (nonvanishing) ADM mass at the asymptotic boundaries. The addition of phenomenological clocks at each asymptotic boundary is also studied and compared with the ‘parametrization clocks’ of Kuchař.

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I. INTRODUCTION

In the study of constrained canonical systems, the term ‘gauge transformation’ is typically taken to mean ‘transformation generated by a constraint’ (see [1]). However, when considering the equations of motion for a system, gauge transformations are usually identified through the appearance of arbitrary functions of time in the set of solutions. These notions of gauge may differ in several ways. In particular, when acting on the canonical data corresponding to a Cauchy slice $\Sigma$ of a spacetime $M$, the orbits of the canonical gauge transformations may be smaller than the orbits of the covariant gauge transformations. A much cited example (see, for example [1–6]) is the case of general relativity on asymptotically flat spacetimes, in which the covariant gauge group contains all diffeomorphisms of the manifold while the constraint-generated gauge group contains only those transformations that preserve the asymptotic reference frame.

In this work we use canonical methods to study asymptotically flat gravity but do not assume that the constraints define the gauge transformations. Instead, we treat the gauge transformations as an additional physical input which must be correctly incorporated into the canonical description of our system. The reasons for this are threefold, as we shall now describe.

The first aspect is philosophical. The mathematical definition of a physical system always requires physical input and, for the cases below, we wish to include in our system physics not captured by the usual treatment. As a result, we choose a somewhat different mathematical definition.

The second aspect is observational. For the case of so-called ‘large’ gauge transformations (those which are not continuously connected to the identity), it is in fact standard that their inclusion (or not) into the formulation of the system is taken as an additional physical input. For our study of asymptotically flat gravitating systems, the ‘additional gauge transformations’ are those that act nontrivially in the asymptotic regions and we observe that, for the appropriate choice of topology on the space of gauge transformations, they are in fact disconnected from the identity. As a result, the treatment below may be viewed as taking a set of large transformations to be gauge.

The third aspect is that Dirac canonical methods [7] can be regarded as a special example of a class of algebraic descriptions of gauge theories (see [8]) based on so-called ‘generalized Peierls algebra’ instead of the Poisson algebra. Quantization schemes based on these methods have ‘generalized constraints’ which act much like the constraints of Dirac quantization. Now, if our system is defined by an action principle, the action also defines a notion of gauge through its invariance under transformations that are compactly supported in time. In this case, the orbits generated by the generalized constraints are always contained in gauge orbits.

\[\text{1For example, we may take a topology in which a sequence of infinitesimal gauge transformations converges to the identity only if their gauge parameters } \epsilon_i(x) \text{ are such that } \int d^n x \, r^m |\epsilon_i(x)|^2 \text{ (for appropriate } n, m) \text{ vanishes in the limit of large } i.\]

\[\text{2All of these algebras induce the same (standard) algebra on gauge invariants, but differ on gauge dependent functions.}\]
defined by the action principle. However, the constraint generated suborbits may be much smaller than the full orbits by an amount that depends on the particular scheme employed. It follows that the generalized constraints do not always generate all of the gauge. We take this as an indication that constraints and gauge transformations are in fact two related but different inputs for the mathematical formulation of a physical system.

In the work below, we are interested in asymptotically flat general relativity and we take our ‘physical’ notion of gauge transformations to be the entire group of diffeomorphisms of some manifold \( M \). In particular, we consider spherically symmetric asymptotically flat gravity, but include gauge transformations that do not preserve the asymptotic structure. Our task, then, is to do this consistently.

The reduced phase space of this system has been thoroughly investigated by Thiemann and Kastrup \[3,4\] and by Kuchar \[5\] using only the constraint-generated gauge transformations. By making use of the results of Kuchar \[5\], it is straightforward to include the additional gauge transformations which lead immediately to superselection rules for the ADM masses in the asymptotic regions. This is because, while the ADM Hamiltonian does not vanish, it generates the new gauge transformations and so commutes with every gauge invariant. With this perspective on gauge, the ADM Hamiltonian is thus not helpful in addressing what is known as ‘the problem of time in quantum gravity’ \[9,10\].

We perform this analysis in section II and show the consistency of this approach with various subtle points involving variational principles for asymptotically flat gravitating systems. In particular, we show how to avoid the conclusion of \[6\] that our notion of gauge requires the asymptotic charges to vanish. The main point will be made in subsection II A, which treats a model of the asymptotic boundary without the interior spacetime. The new physical input enters only at this boundary and attaching the interior is largely a technical issue, which we postpone until subsection II B. In section III, we study the case of spherically symmetric gravity coupled to asymptotic phenomenological clocks. This is done both to provide an additional example of our approach and for comparison with the parametrization clocks of \[5\]. From these examples, the corresponding results for full asymptotically flat gravity (and indeed any field theory) should be clear.

Below, the reader will encounter a number of ‘Remarks’ which are separated from the main text. These clarify various technical points, but may be skipped without loss of continuity.

**II. THE ISOLATED SCHWARZSCHILD BLACK HOLE**

In this section, we address the isolated primordial Schwarzschild black hole, extending the analyses of Thiemann and Kastrup \[3,4\] and Kuchar \[5\] to treat asymptotically nontrivial diffeomorphisms as gauge. We find the language and notation of \[5\] convenient and use it below.

As stated in the introduction, we treat the gauge transformations of our system as a physical input and choose it to be the entire group of spacetime diffeomorphisms. We will take our system to be defined by a variational principle using the Einstein-Hilbert action exactly because this captures the desired notion of gauge. Recall then that the usual Einstein-Hilbert action
on a spacetime $M$ with boundary $\partial M$ is invariant under all diffeomorphisms of $M$. Such diffeomorphisms necessarily map the boundary $\partial M$ onto itself, but need not act trivially on any part of $\partial M$. Here, $R$ is the scalar curvature, $K$ is the trace of the extrinsic curvature on the boundary, and $dS$ is the proper volume element on the boundary.

As we are interested in Schwarzschild black holes, we consider the restriction of this action to spherically symmetric geometries. This also restricts the allowed gauge transformations to those that preserve spherical symmetry and amounts to a partial gauge fixing, but the details of this process will not concern us here.

A simple form of this restricted action was derived in [5], and we will take this as our starting point. By passing to spherically symmetric canonical variables $(g, \pi)$, performing a canonical transformation to the new variables $M, P_M, R, P_R$, and using the Lagrange multipliers $N_M, N_R$, Kuchař arrives at the action

$$S = \lim_{\rho \to +\infty} \left[ \int_{T_1}^{T_2} dt \int_{-\rho}^{\rho} dr (P_M(r) \dot{M}(r) + P_R(r) \dot{R}(r) - N_M(r) M'(r) - N_R(r) P_R(r)) ight.$$ 
$$+ \int_{T_1}^{T_2} dt (N^M(\rho) M(\rho) - N^M(-\rho) M(-\rho)), \right]$$

(2.2)

where a dot indicates a time derivative, a prime denotes $\partial/\partial r$, and the quantity $M(r)$ can be interpreted on the constraint surface as the black hole mass. Note that, because we consider primordial black holes, the ‘radial’ coordinate $r$ takes values on the entire real line and that our spacetime has two asymptotic regions. Here, we have written the action in a slightly different form than in [5] in order to make it clear this action can be well defined even when the asymptotic limits $\lim_{r \to \pm \infty} N^M(r)$ of the lapse do not exist.

### A. A model of a boundary

The viewpoint expressed in the introduction affects the analysis only at the asymptotic boundaries. The remaining task, to connect the boundaries to the interior spacetime, is largely a technical issue. As such, the main idea can be clearly illustrated for a finite dimensional system which models a component of the boundary. We analyze such a system here, and then return the technical complications produced by the interior in II B below.

We will in fact study our model system in two different ways, first through the a variational principle and then through a canonical description. Following the philosophy stated in the introduction, both formulations of the system will be designed to be compatible with our ‘physical’ idea that all diffeomorphisms should be treated as gauge. As a result, they will differ somewhat from the familiar treatments.

Our starting point will be the action

$$S = \int_{T_1}^{T_2} dt (P \dot{M} - NM).$$

(2.3)

This $M$ roughly corresponds to the total mass of an asymptotically flat system, $P$ is it’s conjugate momentum (as in [5]), and $N$ plays the role of the asymptotic value of the lapse.
Following this analogy, the standard treatment \[3–5\] of asymptotically flat systems specifies \(N(t)\) as part of the boundary data since this is a part of the complete 3-metric around the boundary of the 4-dimensional spacetime.

The action 2.3 is invariant under the transformation
\[
\begin{align*}
\delta P(t) &= -\epsilon(t) \\
\delta M(t) &= 0 \\
\delta N(t) &= \dot{\epsilon}(t)
\end{align*}
\]
for any \(\epsilon(t)\) such that \(\epsilon(T_1) = \epsilon(T_2) = 0\). We wish to consider such ‘time reparametrizations’ as gauge transformations. There is, however, a subtle point in considering such general gauge transformations when our system is formulated through a variational principle. Recall that a variational principle consists of an action functional together with a set of boundary conditions that the variations of the field must respect. These boundary conditions must be chosen so that the stationary points of the variational principle include all of the desired solutions. If the gauge transformations do not preserve the boundary conditions, they will not lead to multiple solutions of the variational principle and need not be interpreted as gauge. Thus, if \(N(t)\) is fixed (as in the usual treatment), there are no gauge transformations and \((M, P)\) forms a canonical pair of invariants. We will not follow this approach, but instead fix only boundary data compatible with interpreting 2.4 as gauge.

On the other hand, unrestricted variation of \(N(t)\) would impose \(M = 0\) as an equation of motion, a feature that we wish to avoid to discuss interesting black holes. Our solution is to impose only that \(\int_{T_1}^{T_2} N(t) dt\) be specified as boundary data. The transformations 2.4 preserve \(\int_{T_1}^{T_2} N(t) dt\) as is appropriate to their interpretation as gauge. We will be interested only in the correspondingly gauge invariant quantities 3.

The effects of varying \(N(t)\) subject to this constraint are most easily seen by expanding \(N(t)\) and \(M(t)\) in Fourier series on the interval \(t \in (T_1, T_2)\). The constant Fourier component of \(N(t)\) has been fixed, but the other components may be varied freely. As a result, these variations vanish exactly when \(M(t)\) is a constant. Varying \(N(t)\) subject to our constraint imposes only \(dM/dt = 0\), which is also the equation of motion obtained by varying \(P(t)\). Thus, our boundary condition imposes no new equations of motion. This completes our study of the variational principle for the model system.

We now turn to a canonical description of this system. Note that \(P, M\) are the only canonical degrees of freedom, and that they are conjugate. While the lapse \(N\) is not a canonical variable, its treatment can again be made clear by expanding \(N(t)\) in Fourier modes. The constant component is fixed by the boundary conditions, while the other components act as Lagrange multipliers in the usual way. Thus, \(N\) is in no way a coordinate on the phase space, but its values are related to those of the canonical coordinates through the equations of motion.

Let us consider the transformation induced by 2.4 on the phase space \(\Gamma\) by identifying \(\Gamma\) with the canonical data at time \(t\). This is just \(\delta M = 0, \delta P = -\epsilon\), which is clearly

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3As pointed out by a referee, one can obtain similar results by adding a term of the form \(k \left(\int_{T_1}^{T_2} N(t) dt - A\right)\) to the action, where \(k\) is a time independent Lagrange multiplier and \(A\) is a constant which is not to be varied. However, we will not pursue this approach here.
generated by taking Poisson bracket with $\epsilon M$. As a result, all (canonical) gauge invariants are functions of $M$.

**Remark 1:** Although we have found a generator for our gauge transformations, we see no a priori reason why such a generator need always exist. Indeed, in the setting of generalized Peierls algebras [8] or Poisson algebras [11], there are examples for which such a generator does not exist (c.f., axial ‘gauge-breaking’ for Maxwell fields).

It seems that the gauge invariant phase space is just one dimensional. As such, it is clearly not a symplectic space; that is, $M$ has no conjugate momentum. This means that there are no invariants that generate nontrivial canonical transformations. In a sense then, each different value of the mass parameter $M$ defines a separate phase space and we have a classical superselection rule [12] between different values of the mass. Similarly, the quantization of this reduced phase space leads to a collection of superselected one dimensional Hilbert spaces, one for each allowed value of the mass.

**B. The full Schwarzschild spacetime**

We now return to our study of the full spherically symmetric vacuum theory as described by 2.2. Note that this action is invariant under the transformations

$$
\begin{align*}
\delta M(r, t) &= 0, \quad \delta R(r, t) = \epsilon R(r, t) \\
\delta P_M(r, t) &= \epsilon'_M(r, t), \quad \delta P_R(r, t) = 0 \\
\delta N^M(r, t) &= \dot{\epsilon}_M(r, t), \quad \delta N^R(r, t) = \dot{\epsilon}_R(r, t)
\end{align*}
$$

(2.5)

for any (smooth) $\epsilon_M$ that vanishes at $t = T_1, T_2$ and any (smooth) $\epsilon_R$. The gauge transformations 2.5 correspond to diffeomorphisms of the spacetime volume enclosed between $t = T_1, T_2$ which map the boundaries of this region onto themselves. We will be interested only in objects that are invariant under all such transformations.

As in II A, we will choose boundary conditions in a manner appropriate to treating 2.5 as gauge. To do so, let us consider a general variation $\delta N_M$ of $N_M$. The corresponding variation of the action is

$$
\delta S = -\lim_{\rho \to \infty} \left[ \int_{T_1}^{T_2} \int_{\rho}^{\rho} dr \, \delta N^M(r) M'(r) - \int_{T_1}^{T_2} dt (\delta N^M(\rho) M(\rho) - \delta N^M(-\rho) M(-\rho)) \right]. \quad (2.6)
$$

Requiring $S$ to be stationary under such a general $\delta N$ would impose $M'(r) = 0$, $\lim_{r \to \pm \infty} M(r) = 0$ as equations of motion; i.e., $M(r) = 0$. To avoid this difficulty, the lapse $N_M$ is typically taken (as in 2.3) to be fixed at spatial infinity. Note that this $N_M$ is simply part of the 3-metric on the (timelike) boundaries whose specification as boundary data makes the usual variational principle well-defined. This leads to the usual equations of

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4Technically, each value of $M$ corresponds to a different symplectic leaf [12] of the reduced phase space.
motion, but removes the gauge transformations \( 2.5 \) with \( \lim_{r \to \pm \infty} \epsilon_M(r) \neq 0 \) from the class of allowed variations. As a result, they are not typically treated as gauge.

This is not, however, the only possible solution. Since the variation of the action vanishes exactly under \( 2.5 \) whether or not \( \lim_{r \to \pm \infty} \epsilon_M(r) = 0 \), there is no reason to specify the entire lapse on the boundary. Instead, it is only necessary to specify the lapse ‘up to a diffeomorphism of the boundary.’ In particular, we may choose to fix not \( N_\pm(t) = \mp \lim_{r \to \pm \infty} N^M(r) \), but only the total proper time \( \int_{T_1}^{T_2} dt N_\pm(t) \) that elapses along the boundaries of our space-time. Then the variation \( \delta N_\pm(t) \) would not be completely free; the freedom would only be that contained in the arbitrariness of \( \delta N_\pm \), and these additional variations enforce only \( dM_\pm/dt = 0 \), where \( M_\pm = \lim_{r \to \pm \infty} M(r) \). Since this follows from the original equations of motion, the variation of the action is still well defined (and vanishing) on the usual space of solutions.

**Remark 2:** One may wish to ask that the action in fact be differentiable with respect to the allowed class of variations. This is a stronger condition than that the vanishing of a variation be well defined and requires all boundary terms to disappear in the variation \( 2.6 \). Since, however, the variation of the action vanishes completely under \( 2.5 \), it is differentiable with respect to \( 2.5 \) and we may still add these to the class of variations allowed by Kuchař [5]. This results in a smaller class of variations than that allowed above. Note in particular that \( 2.5 \) requires the variations of \( N_M \) to be correlated with those of \( P_M \) so that this remark does not contradict the argument given below \( 2.6 \). Since the action is unchanged by these additional variations, this new variational principle leads to the same equations of motion as that of [5]. For the purposes below, the use of this variational principle or of the one stated above lead to precisely the same conclusions.

**Remark 3:** With either class of variations above, the boundary data need only be given up to a diffeomorphism of the boundary. Specifying only an equivalence class of boundary data may be unfamiliar, but leads to no problems. For example, Ref. [13] uses boundary data defined only up to an even larger equivalence relation.

We now turn to the (Dirac style) canonical description of this system. Note that \( P_M(r), P_R(r) \) are the canonical conjugates of \( M(r) \) and \( R(r) \) and that variation of the Lagrange multipliers \( N_M \) and \( N_R \) enforce the constraints

\[
M'(r) = 0, \quad P_R(r) = 0. \tag{2.7}
\]

However, the boundary term in the action \( 2.2 \) leads to the much publicized fact (see [3,14–16] and others) that the Hamiltonian is not just a sum of constraints. Instead, it is given by

\[
H(N^M, N^R) = \int dr (N^M(r)M'(r) + N^R(r)P_R(r)) + N_+ M_+ + N_- M_- \tag{2.8}
\]

where \( N^M(r) \) and \( N^R(r) \) are now treated as specified functions of \( r \) and are chosen so that \( N_\pm = \mp \lim_{r \to \pm \infty} N^M(r) \) exist. With the inclusion of the boundary term, the Hamiltonian is a differentiable function on the phase space. Taking Poisson Brackets with the Hamiltonian generates equations of motion which, together with the constraints \( 2.7 \), completely specifies the dynamics of our system.
Following the philosophy outlined in the introduction, we take as gauge all of the transformations \(2.5\). That is, we base our notion of gauge directly on the action and do not take it to be determined by the form of the constraints \(5\). Following the usual canonical procedure, we are particularly interested in the transformation under \(2.5\) of a function \(A(M, P_M, R, P_R)\) on the phase space. Many of these transformations are generated by a phase space function as follows.

Note that when the integral
\[
G(\epsilon_M, \epsilon_R) = \int dr (-\epsilon_M'(r)M(r) + \epsilon_R(r)P_R(r))
\] (2.9)
converges, it is differentiable on the phase space and generates the transformations
\[
\{G, A\} = -\int d^3x \frac{\delta A}{\delta P_M} \epsilon_M' + \frac{\delta A}{\delta R} \epsilon_R.
\] (2.10)

If we think of the phase space as the space of canonical data on some Cauchy surface \(\Sigma\) in \(M\), then this is just the transformation of \(A\) induced by the gauge transformations \(2.5\). Thus, when \(2.9\) converges, \(G(\epsilon_M, \epsilon_R)\) generates gauge transformations. When \(\epsilon_M\) and \(\epsilon_R\) are such that \(2.9\) does not converge, the corresponding transformation \(2.5\) has no generator. This is, however, consistent with our viewpoint as expressed in Remark 1.

- **Remark 4:** In studying asymptotically flat gravity, a number of conditions are typically imposed on the rate at which the canonical variables approach their asymptotic values. We shall not explicitly choose a set of fall-off conditions here, but our viewpoint on this is the same as for the other boundary data: in principle, the fall-off conditions need only be imposed *up to diffeomorphism (gauge) equivalence*, in which case our transformations preserve the allowed class of data. If the fall-off conditions are not imposed in a diffeomorphism invariant manner, then this is tantamount to gauge fixing and the system must be reinterpreted accordingly.

- **Remark 5:** If one insists that all gauge transformations be generated by phase space functions, one may impose fall-off conditions such as \(M(r) \to \text{const}\) and \(\int_0^r dr P(r) \to \text{const}\) (where neither constant is specified in the boundary data). The integral \(2.9\) then converges whenever \(\epsilon_M(r) \to \text{const}\) and \(\epsilon_R(r) \to \text{const}\). The corresponding class of gauge transformations is large enough to imply all of our results below.

Note that \(G\) is not just a combination of the constraints. In fact, \(G(\epsilon_M, \epsilon_R)\) is identical to \(H(\epsilon_M, \epsilon_R)\), the Hamiltonian evaluated at \(N^M = \epsilon_M, N^R = \epsilon_R\). Thus, any gauge invariant phase space function must Poisson commute with \(H(N^M, N^R)\) and, while it is true that the ADM Hamiltonian is nonvanishing, it generates only the trivial transformation on the algebra of gauge invariants. With our perspective toward gauge, we find that the ADM Hamiltonian is of little use in solving ‘the problem of time’ \(\textbf{[9,10]}\).

\(^{5}\)It is, however, guaranteed that the transformations generated by the constraints will always be gauge (see, for example, \(\textbf{[7,18]}\)).
What then are the physical degrees of freedom of this system? We answer this question by finding the complete set of gauge invariant functions on the constraint surface. From the form of $G(\epsilon_M, \epsilon_R)$, it is clear that any such invariant $A$ must be independent of the momenta $P_M(r)$ and the coordinates $R(r)$. Furthermore, on the constraint surface, the momenta $P_R(r)$ vanish and $M(r)$ is independent of $r$. Thus, the only gauge invariants are functions of the $r$-independent quantity $M$, which is just the total mass of the system. This coincides with the answer that would be expected from Birkhoff’s theorem: every spherically symmetric solution to the vacuum Einstein equations is a piece of a Schwarzschild spacetime, characterized only by its mass $M$. As in II A, we find a superselection rule for the only invariant degree of freedom, the total mass.

III. ASYMPTOTIC PHENOMENOLOGICAL CLOCKS

For comparison, we now wish to study the system in which two phenomenological clocks are added to the black hole, one in each asymptotic region. By phenomenological clocks, we mean subsystems whose mathematical description is concocted to ensure that their configuration variables $\tau$ increment in direct proportion to the passage of proper time along their worldlines and such that this constant of proportionality is independent of the state of the clock. Nevertheless, we take the clock to have a canonical description of the usual type, with both a coordinate and a conjugate momentum (so that its phase space is a symplectic space). We shall also take the clock to be fixed (at $r = \pm\infty$) and to ignore any degrees of freedom associated with its movement. We expect that such a description could be approximated by realistic systems in certain regimes, though we shall not discuss such approximations here.

In a fixed background $M$ foliated by slices $\Sigma_t$ orthogonal to the clock’s worldline, the standard description of a phenomenological clock is through the action principle $\int dt P_\tau \dot{\tau} - NP_\tau$ where $N$ is the lapse function defined at the clock’s position by the foliation. If we were to couple this clock to the finite dimensional model of II A, the resulting action would be

$$S = \int_{T_1}^{T_2} dt(\dot{P}M + P_\tau \dot{\tau} - N(M + P_\tau)).$$

(3.1)

Again imposing $\int_{T_1}^{T_2} dt N = \text{const}$, the gauge transformations are

$$\delta P(t) = -\epsilon(t), \delta \tau(t) = \epsilon(t)$$
$$\delta M(t) = 0, \delta P_\tau(t) = 0$$
$$\delta N(t) = \dot{\epsilon}(t)$$

(3.2)

for $\epsilon(T_1) = \epsilon(T_2) = 0$. There are now three independent gauge invariants: $M$, $P_\tau$, and $P + \tau$ and the invariant $M + P_\tau$ is superselected. Attaching an asymptotic clock to the full black hole is somewhat more subtle, as we shall see below.

To study the full Schwarzschild system coupled to asymptotic clocks, we use the action

$$S = \int_{T_1}^{T_2} dt \int_{-\infty}^{\infty} \left( P_M(r) \dot{M}(r) + P_R(r) \dot{R}(r) - N^M(r) M'(r) - N^R(r) P_R(r) \right)$$
$$+ \int_{T_1}^{T_2} dt \left( P_{\tau^-} \dot{\tau}^+ + P_{\tau^+} \dot{\tau}^- + N_- P_{\tau^-} - N_+ P_{\tau^+} - N_+ M_+ - N_- M_- \right).$$

(3.3)
where we have now assumed that the asymptotic values \(N_{\pm} = \lim_{r \to \pm\infty} \mp N(r)\) exist and we have implicitly fixed the foliation to be orthogonal to the worldlines of the (asymptotic) clocks. Thus, we have introduced a partial gauge fixing, of which we shall have to take proper account.

- **Remark 6:** While it would have been nice to keep the full gauge symmetry manifest, this is complicated by the asymptotic nature of the clocks. The angles between the clocks’ world lines and a \(t = \text{constant}\) hypersurface is a function of the clock velocity \(\dot{X}^\mu\), the spatial metric, and the Lagrange multipliers. Thus, to explicitly display this (gauge) degree of freedom would require the introduction of a clock at finite position or the use of a new formalism to describe the velocity of an asymptotic clock. We choose to follow the conceptually simpler gauge-fixed approach here and we shall see that no problems arise as long as we remember the physical setup behind this description.

Note that the action 3.3 is exactly the same as that used by Kuchař to describe ‘parametrization clocks.’ However, we will require this action to be stationary under a different class of variations; we would like the addition of the clocks to affect the gravitational degrees of freedom only through their explicit interactions (as governed by 3.3) and through the gauge fixing that we have already performed. Since we have assumed that the limits \(\lim_{r \to \pm\infty} N^M(r)\) exist and that each \(t = \text{constant}\) surface is orthogonal to the (asymptotic) clock worldlines, we require that the limits \(\lim_{r \to \pm\infty} \frac{\partial}{\partial r} N^M(r) = 0\). Otherwise, however, we allow the same class of variations for the gravitational field here as when the clocks were absent (as in the action 2.2). The clock fields \(\tau^\pm\) are to be varied keeping \(\tau^\pm(T_1), \tau^\pm(T_2)\) fixed while the clock momenta \(P_{\tau^\pm}\) may be varied freely.

As before, we proceed by identifying the gauge transformations associated with this action. They are given by

\[
\begin{align*}
\delta M(r, t) &= 0 & \delta R(r, t) &= \epsilon(r, t) \\
\delta P_M(r, t) &= \epsilon'_M(r, t) & \delta P_R(r, t) &= 0 \\
\delta N^M(r, t) &= \epsilon_M(r, t) & \delta N^R(r, t) &= \epsilon_R(t, r) \\
\delta P_{\tau^\pm}(t) &= 0 \\
\delta \tau^\pm(t) &= \mp \lim_{r \to \pm\infty} \epsilon_M(r, t).
\end{align*}
\]

(3.4)

where \(\epsilon_M(T_1) = \epsilon_M(T_2) = 0\) and, as opposed to the case of 2.3, we must assume that \(\lim_{r \to \pm\infty} \frac{\partial}{\partial r} \epsilon_M(r) = 0\) in order to take account of our gauge fixing. Note that the gravitational degrees of freedom transform just as they did in 2.3.

- **Remark 7:** The detailed relationship between this set of gauge transformations and the action 3.3 is somewhat subtle, again due to the asymptotic nature of the clocks. The action 3.3 is in fact invariant under all transformations of the form 3.4 for which \(\epsilon_M(t) = 0\) at \(t = T_1, T_2\) whether or not \(\lim_{r \to \pm\infty} \frac{\partial}{\partial r} \epsilon_M(r)\) vanishes or even exists. However, we know that such a transformation breaks our gauge fixing condition that the slices be orthogonal to the clock worldlines; in fact, a transformation of the form 3.4 for which \(\frac{\partial}{\partial r} \epsilon_M(r, t) \neq 0\) at the clocks position would have changed the factor in the action that describes the motion of the clocks. (An additional subtlety is that the
ADM energies and so contribute separately to the ADM energy. Indeed, we can now identify two distinct to the ADM energy. This is because the clocks were added on the ‘outside’ of the system a phase space function; in this case, the function is  for and that gauge invariant phase space functions are independent of As mentioned above, the Hamiltonian in fact defines two independent functions on the constraint surface, so this will lead to two superselection rules, one for each asymptotic region. We see that these energies need no longer agree.

Again, the gauge transformations can often be generated by taking Poisson brackets with a phase space function; in this case, the function is

for such that \( \lim_{r \to \pm \infty} N^M(r) \) exists and \( \lim_{r \to \pm \infty} \frac{\partial}{\partial r} N^M(r) = 0 \). Note that while \( M(r) \) is still \( r \)-independent on the constraint surface, the corresponding mass \( M \) is no longer equal to the ADM energy. This is because the clocks were added on the ‘outside’ of the system and so contribute separately to the ADM energy. Indeed, we can now identify two distinct ADM energies \( E^\pm = M \pm P_\tau \), one at each asymptotic region. We see that these energies need no longer agree.

Again, the gauge transformations can often be generated by taking Poisson brackets with a phase space function; in this case, the function is

for \( \epsilon_M(r) \) such that \( \lim_{r \to \pm \infty} \epsilon_M(r) \) exists and \( \lim_{r \to \pm \infty} \frac{\partial}{\partial r} \epsilon_M(r) = 0 \) and such that converges. Again, we find that all gauge invariants commute with the Hamiltonian. As mentioned above, the Hamiltonian in fact defines two independent functions \( E^\pm \) on the constraint surface, so this will lead to two superselection rules, one for each asymptotic region.

This is easily seen by explicitly carrying out the reduction. Again, we have that \( P_R = 0 \) and that gauge invariant phase space functions are independent of \( R \). However, because of the additional terms in the gauge generator, the analysis of the transformations parametrized by \( \epsilon_M \) differs from that of section II. A gauge invariant need not be completely independent of \( P_M(r) \), so long as it depends on this momentum only through the combination \( P = \int_{-\infty}^{+\infty} dr P_M(r) + \tau^+ - \tau^- \). The other gauge invariants are just the mass \( M \) (as before) and the two ADM energies \( E^\pm \). Since the \( E^\pm \) are superselected, all of their Poisson brackets vanish while \( \{M, P\} = 1 \). The observable \( P \) has just the interpretation given in as the difference between the two clock readings on any spacelike slice of constant killing time through the black hole, and forms the conjugate of \( M \) as in . The corresponding quantum theory is given by a set of superselected Hilbert spaces (labeled by the values of \( E^\pm \)), each of which is isomorphic to \( L^2(\mathbb{R}^+, dm) \) on which \( M \) acts by multiplication by \( m \) and \( P \) acts as \( -id/dm \). Here we have incorporated the classical positivity condition \( M > 0 \). Note that by coupling clocks we have introduced four canonical degrees of freedom (\( \tau^\pm, P_{\tau^\pm} \)) and found three additional observables and one new superselection rule.

- **Remark 8:** In fact, due to our choice of gauge fixing and boundary conditions (in particular, the restriction \( \lim_{r \to \pm \infty} \frac{\partial}{\partial r} \epsilon_M(r) = 0 \)), there are two other gauge invariants, \( \lim_{r \to \pm \infty} P_M(r) \) which describe the asymptotic rate of change of Killing time
along the hypersurfaces. Since our slices are asymptotically orthogonal to the clocks’
worldlines, \( \lim_{r \to \pm \infty} P_M(\infty) \) is a function of the angle between these worldlines and the
timelike Killing field of the Schwarzschild metric. As such, these invariants describe
the the velocities of the clocks with respect to the black hole. Since they lie at the
boundaries, the \( P_M(\pm \infty) \) commute with \( M(r) \) and thus with all gauge invariants; they
too are superselected. This is as it should be since we did not include the dynamics
of the clock velocities in \[5\]. Note also that the fall-off conditions of \[5\] would set
\( P_M(\pm \infty) \) to zero. Here, we have two options consistent with our philosophy:

- 1) Impose that the fall-off conditions of \[5\] hold on some Cauchy slice where the
clocks read finite values. This would set \( P_M(\pm \infty) = 0 \).

- 2) Impose that the fall-off conditions imposed by \[5\] on the metric and momenta
hold up to a diffeomorphism of the spacetime, but require no relation between
slices satisfying these fall-off conditions and the clock variables. In this case,
we may have nonzero \( P_M(\pm \infty) \). This changes the above counting of degrees
of freedom by adding two new invariants \( P_M(\pm \infty) \) which result from the two
restrictions \( \lim_{r \to \pm \infty} \frac{\partial}{\partial r} \epsilon_M(r) = 0 \) placed on the gauge transformations.

• **Remark 9:** Note that \( P \) can be defined whenever \( P_M(\pm \infty) = -P_M(\mp \infty) \) and \( P_M \to P_M(\pm \infty) \) sufficiently rapidly.

We now present a brief comparison of this system to the black hole with ‘parametrization
clocks’ of \[8\]. This type of clock is essentially defined by its appearance through the
coordinate \( \tau^\pm \) and the momenta \( P_{\tau^\pm} \) in the variational principle used in \[8\]. This variational
principle uses the same action \( (3.3) \) as our phenomenological clocks but allows a larger class
of variations. For the parametrization clocks, the lapse \( N^M \) is allowed to be varied freely
at spatial infinity. This has the effect of setting the ADM energies \( E^\pm \) to zero. Thus,
‘parametrization clocks’ can be seen to be phenomenological clocks at rest with respect to
the black hole (since \( P_M(\pm \infty) = 0 \)) and whose internal states are chosen to carry a negative
energy that exactly cancels the mass-energy of the black hole.

The ‘addition’ of parametrization clocks to the black hole system is somewhat different
from the usual notion of coupling an additional system. Their inclusion modifies the varia-
tional principle for the black hole not only by adding new parameters to vary, but also by
changing the class of allowed variations for the lapse, a part of the metric field. As a result,
the counting of ‘additional’ degrees of freedom is more complicated than usual. Using the
notion of gauge transformation of section \[4\], the counting is just as that for the phenomeno-
logical clocks, except that we are now also free to vary \( N^\pm \), imposing two new equations of
motion \( (E^\pm = 0) \). As a result, the parametrization clocks create only one new canonical
degree of freedom. Using the more limited notion of gauge generated by constraints (as in
\[8\]), they create no new degrees of freedom.

One additional remark is that the free variation of \( N^M \) is allowed only by the presence of
the clocks at the boundary. Thus, if the clocks were first included at a finite position (say,
a fixed value of \( R \)) and then moved outward toward the boundaries, we would subject the
lapse to the boundary condition \( \delta \int dt N^\pm(t) = 0 \) as in section \[4\]. There would then be no
reason for the total ADM mass to vanish and we would arrive at the variation principle for
the ‘phenomenological clocks.’
IV. DISCUSSION

As stated in the introduction, we consider the ‘proper’ set of boundary conditions and the definition of gauge transformations to be a matter of physical input. Clearly, in the case of a gravitational system, there are two possible interpretations for a system with boundaries (asymptotic or not). The first corresponds to the treatment presented here, in which the system is considered in isolation and no structure outside of these boundaries is utilized. Another interpretation naturally yields the treatment of [18, 19]. In this case, some observer is assumed to sit at or just outside the boundaries but is not explicitly included in the action functional. This observer merely supplies a coordinate chart on the boundaries (perhaps, through his ‘parametrization clocks’) which we may use to fix the gauge of our system at the boundaries; for example, to fix the lapse $N_\pm$. If one wishes this external observer to construct his clocks to yield zero ADM energy, we have seen that he may do this as well.

However, if the first interpretation is chosen, we may naturally take the full set of asymptotic transformations to be gauge. It is clear that the results found above carry over to much more general cases. Fixing the boundary data for an asymptotically flat variational principle only up to a diffeomorphism of the boundary allows the full set of diffeomorphisms of the spacetime volume to act as gauge symmetries of the action. While this will not force the ADM energies to vanish, it will introduce a superselection law for the ADM energy at each boundary. If the requirement of spherical symmetry is dropped, similar considerations apply to all of the generators of the asymptotic Poincaré group. As a result, none of these generators will be gauge invariant themselves, but the symplectic leaves of the reduced phase space (or the superselected sectors of the quantum theory) will be labeled by the representations of the Poincaré group; for example by the mass $P_\mu P^\mu$. In much the same way, consideration of these additional gauge transformations renders the quantities constructed in [19] gauge dependent and implies that they generate pure gauge.

The idea of a superselection law for the mass of a gravitating system is now new. In particular, it was discussed in [6] and [20], although the motivations and arguments in these works were rather different both from each other and from those presented here.

The existence of a superselection rule for the total mass is quite in accord with our understanding of other gauge theories, in which charge superselection is well known (see for example [21]). However, our derivation of this superselection rule (which could be applied to Yang-Mills theories as well) is somewhat different from the usual one given for Yang-Mills theories, where the local (or ‘quasilocal’) nature of the observables is stressed. Such a derivation is not applicable here because, for diffeomorphism invariant systems, gauge invariants can never be local. Given any function $f$ on the (unconstrained) phase space built from the fields in a compact region $R$ of a spacetime $M$, we may simply perform a diffeomorphism that moves $R$ into $M \setminus R$. Since the the fields may vanish outside $R$, only the field independent (constant) function on $R$ can be invariant under such a transformation. It is precisely due to the non-local nature of observables in general relativity that great care is needed in formulating the gauge transformations of the system.
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