Notes on the Super Nambu Bracket

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Abstract

We define a super Nambu-Poisson algebra over a super manifold. A super Nambu bracket does not satisfy the usual skew-symmetric property, and we propose another skew-symmetric property. We show that the divergence of super Nambu-Hamiltonian vector fields leads to a generalization of the Batalin-Vilkovisky algebra.

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§1. Introduction

The Nambu-Hamiltonian (NH) system is a generalization of the Hamilton system that was introduced by Nambu.\(^1\) Many authors\(^2\) have studied the fundamental properties of the NH system of bosons. In order to add fermions to the system, we must extend the Nambu bracket to a super Nambu bracket (SNB). A SNB does not satisfy the usual skew-symmetric property, and we propose another skew-symmetric property. In this paper, we demonstrate the three properties that a SNB satisfies, and define a super Nambu-Poisson algebra on a super manifold. Some authors have remarked on the relations\(^2,9\) between the Nambu-Poisson algebra and the \(L_\infty\) algebra, or the Batalin-Vilkovisky (BV) algebra,\(^5\) especially with regard to the Nambu bracket and higher brackets of the \(L_\infty\) algebra. In order to determine these relations, we extend the Nambu-Poisson algebra to a \(\mathbb{Z}_2\)-graded algebra.

We show that the divergence of the NH vector fields of a SNB leads to a generalization of the BV algebra. Throughout this paper, \(\partial_l/\partial x\) (resp., \(\partial_r/\partial x\)) denotes a left (resp., right) derivative, and \(|f| = 0,1\) is the degree (Grassmann parity) of \(f\), modulo 2.

§2. Super Nambu-Poisson algebra

An example of the SNB over \(\mathbb{R}^2\) with coordinates \((x_1, x_2, \theta)\) is

\[
\{f, g, h\} = (-)^{|g|} \left( \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} \frac{\partial h}{\partial \theta} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial \theta} \frac{\partial h}{\partial x_1} \right) + \left( \frac{\partial_r f}{\partial \theta} \frac{\partial g}{\partial x_2} \frac{\partial h}{\partial x_1} - \frac{\partial_r f}{\partial x_2} \frac{\partial g}{\partial \theta} \frac{\partial h}{\partial x_1} \right),
\]

with degree \(\epsilon = 1\). This bracket satisfies the \(\mathbb{Z}_2\)-graded Nambu-Poisson algebra. In this section, we define the even and odd super Nambu-Poisson algebra.

2.1. Super Nambu-Poisson manifold

The bosonic Nambu bracket is characterized by the following three properties:\(^2\) (i) skew-symmetry, (ii) the Leibniz rule, and (iii) the fundamental identity (FI). A manifold with a Nambu bracket is called a Nambu-Poisson manifold. We define a SNB \(\{\cdot, \cdot, \cdot\}\) as a bracket over a \(d\)-dimensional super manifold \(M\) that satisfies three properties with the \(\mathbb{Z}_2\)-grading. First, we assume there exist SNBs and demonstrate these properties as the necessary conditions for this existence. We define a Nambu-Poisson tensor\(^*\) \(\eta \in TM\otimes^n\) as \(\eta(df_1, \cdots, df_n) = \{f_1, \cdots, f_n\}\) for any \(f_i \in C^\infty(M)\) and the map \(\sharp : T^*M^{\otimes n-1} \rightarrow TM\) as

\[
\sharp(\alpha_1 \otimes \cdots \otimes \alpha_{n-1}) = \eta(\cdot, \alpha_1, \cdots, \alpha_{n-1}),
\]

\(^*\) We do not assume that \(\eta\) is an element of \(A^n TM\).
for $\alpha_i \in T^*M$. Owing to the action principle\(^2\) of the NH system, we assume that a NH vector field is given by

$$X_{f_1\cdots f_{n-1}} = \sharp(df_1 \otimes \cdots \otimes df_{n-1}) = \frac{1}{(n-1)!} \sharp(df_1 \wedge \cdots \wedge df_{n-1}).$$

(2.3)

The degree of a NH field is

$$|X_{f_1\cdots f_{n-1}}| = \epsilon + \sum_{i=1}^{n-1} |f_i|,$$

(2.4)

where $\epsilon = |\eta| = 0, 1$, and the degree of a SNB is

$$|\{f_1, \cdots, f_n\}| = |df_1(X_{f_2\cdots f_n})| = \epsilon + \sum_{i=1}^{n} |f_i|.$$  

(2.5)

Below we demonstrate three properties that a SNB satisfies under these assumptions.

(i) The skew-symmetric property

Owing to the skew-symmetry of $df_i \wedge df_{i+1}$, a SNB possesses the skew-symmetric property

$$\{f_1, \cdots, f_i, f_{i+1}, \cdots \} = -(-)^{|f_i||f_{i+1}|}\{f_1, \cdots, f_{i+1}, f_i, \cdots \},$$

(2.6)

for $i = 2, \cdots, n-1$. Note that we cannot impose the same condition for $i = 1$ and $\epsilon = 1$. For $n = 2$ and $i = 1$, this property differs from the skew-symmetry of an odd Poisson bracket.

Remark For a class of bosonic Nambu brackets $\{f_1, f_2, f_3\}$ over a “symplectic” manifold $(M, \omega^{(3)})$, where $\omega^{(3)}$ is a non-degenerate\(^*\) closed 3-form, the Nambu bracket also possesses the following skew-symmetric property. A usual Poisson bracket has the skew-symmetric property, owing to the equation $\{f, g\} = \omega(X_f, X_g)$. For a bosonic Nambu bracket of this class, we have the equation

$$\omega^{(3)}(X_{f_1f_2}, X_{g_1g_2}, X_{h_1h_2}) = (df_1 \wedge df_2)(X_{g_1g_2}, X_{h_1h_2})$$

$$= \{f_1, g_1, g_2\}\{f_2, h_1, h_2\} - \{f_2, g_1, g_2\}\{f_1, h_1, h_2\},$$

(2.7)

for any $f_i, g_i$ and $h_i$ that do not depend on the local coordinates. Since the LHS of (2.7) is skew-symmetric, we have

$$\omega^{(3)}(X_{f_1f_2}, X_{g_1g_2}, X_{h_1h_2}) = -\omega^{(3)}(X_{g_1g_2}, X_{f_1f_2}, X_{h_1h_2}).$$

(2.8)

\(^*\) If the equation $i(X)\omega^{(3)} = df_1 \wedge df_2$, where $i$ denotes an interior product, has a solution $X$, we say that $\omega^{(3)}$ is non-degenerate.
The RHS of (2.7) has the same skew-symmetric property,
\[
\{f_1, g_1, g_2\} \{f_2, h_1, h_2\} - \{f_2, g_1, g_2\} \{f_1, h_1, h_2\} = -\{g_1, f_1, f_2\} \{g_2, h_1, h_2\} + \{g_2, f_1, f_2\} \{g_1, h_1, h_2\},
\]
and so on. We can easily extend the property (2.9) to the case that \(n > 3\) and to a SNB. This fact implies that we should impose the above property instead of the usual skew-symmetric property on the general (bosonic and super) Nambu bracket.

(ii) The Leibniz rule
Owing to the Leibniz rule for \(X_{f_1, \ldots, f_{n-1}}\), a SNB satisfies the equation,
\[
\{gh, f_2, \ldots, f_n\} = g\{h, f_2, \ldots, f_n\} + (-)^{(e+\sum_{i=2}^n |f_i|)} \sum_{i=2}^n \{g_1, f_1, f_2, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n\} h.
\]
(2.10)

For \(i \geq 2\), from the Leibniz rule for an exterior derivative and the linearity of the map \(\sharp\), we obtain the Leibniz rule
\[
\{f_1, f_2, \ldots, f_i, gh, \ldots, f_{n-1}\} = (-)^{(|h|+\sum_{j=i+1}^n |f_j|)} \sum_{j=i+1}^n \{g_1, f_1, f_2, \ldots, f_{i-1}, g, f_i, f_{i+1}, \ldots, f_{n-1}\} g.
\]
(2.11)

Note that this Leibniz rule leads to a \(\mathbb{Z}_2\)-graded cyclic cocycle-like property of a SNB,
\[
\sum_{i=1}^{n-1} (-)^{i+1} \{f, f_1, \ldots, f_i, f_{i+1}, \ldots, f_n\} + (-)^{n+|f_n|(|f_1|+\sum_{j=1}^{n-1} |f_j|)} \{f, f_n f_1, \ldots, f_{n-1}\} = 0,
\]
(2.12)
for any fixed \(f\). This property is a necessary condition for (2.11) to hold.

(iii) The fundamental identity
The FI is generalized as
\[
\{\{g_1, \ldots, g_n\}, f_1, \ldots, f_{n-1}\}
\]
\[
= \sum_{i=2}^{n-1} (-)^{(e+\sum_{j=1}^{i-1} |f_j|)(e+\sum_{k=i+1}^n |g_k|)} \{g_1, \ldots, g_i, f_1, \ldots, f_{i-1}, g_{i+1}, \ldots, g_n\} + (-)^{(e+\sum_{j=1}^{n-1} |f_j|)(e+\sum_{k=1}^n |g_k|)} \{g_1, f_1, \ldots, f_{n-1}, g_2, \ldots, g_n\}.
\]
(2.13)

This property implies that the SNB satisfies the Leibniz rule for the SNB itself.
Here, we obtain the three properties of the SNB. We also define the SNB as the bracket that satisfies all three of these properties and define the super Nambu-Poisson manifold as the manifold that is equipped with a SNB.

**Definition** A super Nambu-Poisson manifold is a super manifold $M$ with a polylinear map, called a super Nambu bracket, $\{,\cdots,\} : A^\otimes n \to A$, where $A$ is $C^\infty(M)$, that satisfies (i) the skew-symmetric property given in (2.6), (ii) the Leibniz rule given in (2.10) and (2.11), and (iii) the fundamental identity given in (2.13). We call the algebra $(A, \{,\cdots,\})$ a super Nambu-Poisson algebra.

An example of a super Nambu-Poisson algebra is the algebra with the odd SNB (2.1) over $\mathbb{R}^{2|1}$ that satisfies the above three properties and (2.9) with the $\mathbb{Z}_2$-grading. Note that the super Jacobian in this case is not a SNB, unlike in the bosonic case. This results from the difference between the differential forms and the volume forms over a super manifold.

**Remarks** (i) A SNB satisfies all of our assumptions. (ii) The order $n$ of a SNB need not satisfy $n \leq \dim M$. (iii) For a bosonic Nambu bracket, the $n - p$ order bracket defined by $\{f_1, \cdots, f_p\}' := \{f_1, \cdots, f_p, g_1, \cdots, g_{n-p}\}$ for fixed $\{g_i\}$ is also a Nambu bracket. For a SNB, this is not true, because a bracket $\{,\cdots,\}'$ does not satisfy the FI, as $\{\cdots, f, f, \cdots\} \neq 0$ in general.

### 2.2. Decomposition of the super Nambu bracket

The fact that SNBs do not possess the usual skew-symmetric property suggests a decomposition of the SNB, as in the case of a bosonic canonical Nambu bracket.\(^3\) In fact, Theorem 1 in Ref. 3) can be generalized to the SNB as follows.

**Theorem 1** Let $\mathfrak{g}$ be a super Lie algebra with degree $\epsilon$. Let $\tau$ be a degree 0 polylinear map $\mathfrak{g}^\otimes n-1 \to \mathfrak{g}$ that is skew-symmetric:

$$\tau(\cdots, a, b, \cdots) = -(\epsilon)^{|a||b|} \tau(\cdots, b, a, \cdots). \quad (2.14)$$

We assume that this map satisfies the equation

$$[a, \tau(b_1, \cdots, b_{n-1})] = -(\epsilon)^{|[a|b]} \sum_{i=1}^{n-1} |b_i| \sum_{i=1}^{n-1} \tau(b_1, \cdots, [a, b_i], \cdots, b_{n-1})]. \quad (2.15)$$

Then, the degree $\epsilon$ bracket $\{a_1, a_2, \cdots, a_n\} := [a_1, \tau(a_2, \cdots, a_n)]$ is a polylinear map $\mathfrak{g}^\otimes n \to \mathfrak{g}$ that satisfies the skew-symmetry (2.6) and the fundamental identity (2.13).

We can easily prove this theorem using (2.15) and the Jacobi identity of $\mathfrak{g}$. Note that the set of all linear combinations of elements $\tau(a_1, \cdots, a_{n-1})$ is a Lie sub-algebra of $\mathfrak{g}$.
Remarks (i) The bracket \{, \cdots, \} is not skew-symmetric in the sense of (2.9). (ii) Equation (2.12) and the Leibniz rule (2.10) suggest that we extend \( g \) to a Poisson algebra and impose the equation
\[
\sum_{i=1}^{n-1} (-)^{i+1} \tau(a_1, \cdots, a_ia_{i+1}, \cdots, a_n) + (-)^{n+|a_n|} e^{\sum_{i=1}^{n-1} |a_i|} \tau(a_n, a_1, \cdots, a_{n-1}) = 0. \tag{2.16}
\]
In fact, for the bosonic canonical Nambu bracket,\(^3\) the Lie bracket corresponds to the Poisson (Dirac) bracket, and the map \( \tau \) corresponds to the cyclic cocycle over the algebra \( C^\infty(T^{n-2}) \).

§3. Generalization of the Batalin-Vilkovisky Algebra

Over an odd symplectic manifold \( M \) with the coordinates \( (z_1, \cdots, z_d) \), the Hamilton vector fields need not be divergenceless, and the divergence plays an important role in the geometric realization of the BV quantization.\(^5\) The volume element \( \mu \) on \( M \) is specified by the density function \( \rho(z) \). The divergence of the Hamilton vector field \( X_H = \frac{\partial}{\partial z_i} X^i_H \) is determined by\(^4\)
\[
\text{div}_\mu X_H = 2\Delta H = \sum_i (-)^{|z_i|} \rho \frac{\partial}{\partial z_i} (\rho X^i_H), \tag{3.1}
\]
and the anti-bracket satisfies\(^6, 7\)
\[
(-)^{|f|} \{ f, g \} = \Delta(fg) - \Delta(f)g - (-)^{|f|} f \Delta(g). \tag{3.2}
\]
The divergence \( \Delta \) satisfies the Leibniz rule for the anti-bracket:
\[
\Delta(\{ f, g \}) = \{ \Delta(f), g \} + (-)^{|f|+1} \{ f, \Delta(g) \}. \tag{3.3}
\]
In the BV quantization, we impose the nilpotency condition \( \Delta^2 = 0 \), which gives the condition on \( \rho \).

On the super Nambu-Poisson manifold with the volume element \( \mu \), the NH vector fields need not be divergenceless over both an even and an odd Nambu-Poisson manifold. We can define the divergence of a NH vector field in the same same way as (3.1):
\[
\text{div}_\mu X_{H_1, \cdots, H_{n-1}} =: 2\Delta(H_1, \cdots, H_{n-1}). \tag{3.4}
\]
This divergence has the skew-symmetric property
\[
\Delta(\cdots, f, g, \cdots) = -(-)^{|f||g|} \Delta(\cdots, g, f, \cdots). \tag{3.5}
\]
and its degree is $\epsilon$. For example, we consider an even SNB over $\mathbb{R}^{1|2}$ with coordinates $(x, \theta_1, \theta_2)$,

$$\{f, g, h\} = (-)^{|g|} \partial f \left( \frac{\partial g \partial h}{\partial \theta_1 \partial \theta_2} + \frac{\partial g \partial h}{\partial \theta_2 \partial \theta_1} \right) + (-)^{|h|+|f|} \partial f \left( \frac{\partial g \partial h}{\partial x \partial \theta_2} - \frac{\partial g \partial h}{\partial \theta_1 \partial \theta_2} \right).$$

In this case, the divergence is given by

$$\Delta(f, g) = (-)^{|g|+1} \left( \frac{\partial^2 f}{\partial x \partial \theta_1 \partial \theta_2} + \frac{\partial f}{\partial \theta_1} \frac{\partial^2 g}{\partial x \partial \theta_2} \right) + \frac{\partial^2 f}{\partial x \partial \theta_2 \partial \theta_1} + \frac{\partial f}{\partial \theta_1} \frac{\partial^2 g}{\partial \theta_2 \partial x \theta_1}. \quad (3.7)$$

Here, we give the properties of the divergence. By the Leibniz rule (2.11), the NH vector fields satisfy

$$X_{f_1, f_2, \ldots, f_{n-1}} = (-)^{|g||F|} X_{f_1, f_2, \ldots, f_{n-1}} g + (-)^{|f|(|g|+|F|)} X_{g, f_2, \ldots, f_{n-1}} f, \quad (3.8)$$

where $|F| = \sum_{i=2}^{n-1} |f_i|$, and $X_f$ represents $(Xf)(g) = (dg)(X) \cdot f$. Taking the divergence of both sides of (3.8), and using the equation $\sum \frac{\partial f_k}{\partial x_i} X^i_{f_2, \ldots, f_n} = \{f_1, \ldots, f_n\}$, we obtain

$$\frac{1}{2} \left( (-)^{|f|} \{f, g, f_2, \ldots, f_{n-1}\} - (-)^{|f|+|g|+1} \{g, f, f_2, \ldots, f_{n-1}\} \right)$$

$$= \Delta(f, g, f_2, \ldots, f_{n-1}) - (-)^{|g||F|} \Delta(f, f_2, \ldots, f_{n-1}) g$$

$$- (-)^{|f|} f \Delta(g, f_2, f_3, \ldots, f_{n-1}). \quad (3.9)$$

For $n = 2$, the above equation is reduced to (3.2). (A similar equation$^*$) appears in Refs. 7 and 8.) In contrast to the case of the anti-bracket, in the present case we cannot define the SNB in terms of $\Delta$, because the SNB does not have the usual skew-symmetric property. By the FI (2.13), the NH vector field satisfies

$$[X_{g_1, \ldots, g_{n-1}}, X_{f_1, \ldots, f_{n-1}}]$$

$$= \sum_{i=1}^{n-1} (-)^{(\epsilon+|F|)(\sum_{k=i+1}^{n-1} |g_k|)+1} X_{g_1, \ldots, (g_i, f_1, \ldots, f_{n-1}), \ldots, g_{n-1}}, \quad (3.10)$$

where $|F| = \sum_{i=1}^{n-1} |f_i|$, and $XY$ represents $XY(f) = d(df(X))(Y)$. Taking the divergence of both sides of (3.10), we obtain

$$\left\{ \Delta(f_1, \ldots, f_{n-1}), g_1, \ldots, g_{n-1} \right\}$$

$^*$ By the skew-symmetric property of $\Delta$ and the identification $(C^\infty(M), \epsilon, \Delta)$ with $(\mathcal{A}, |\Delta|, \Phi_{\Delta}^{n-1})$ given in Ref. 8, the RHS of (3.9) is shown to be the same as that of (7) in Ref. 8, up to a constant factor.
\[
\sum_{i=1}^{n-1} (-)^{i+|\mathcal{F}|} \Delta (g_1, \cdots, \{g_i, f_1, \cdots, f_{n-1}\}, \cdots, g_{n-1}) + (-)^{i+|\mathcal{F}|} \{\Delta (g_1, \cdots, g_{n-1}), f_1, \cdots, f_{n-1}\},
\]

where \(|G| = \sum_{i=1}^{n-1} |g_i|\). For \(n = 2\), the above equation reduces to (3.3). We want to generalize the nilpotency condition for \(\Delta\). However, for example, the divergence (3.7) is not nilpotent as a bi-differential operator nor as a co-derivation. Note that if the SNB is given by the Lie (or Poisson) bracket and the map \(\tau\) of Theorem 1, we obtain the relations between them and \(\Delta\). This generalized BV algebra should be useful for the quantization of the NH system using the BV method, and extended objects.

§4. Discussion

In this paper, we defined the super Nambu-Poisson algebra and demonstrated its connection with the generalized BV algebra. Here, we comment on some physical applications. The super Nambu-Hamilton system has the equation of motion

\[
\frac{df}{dt} = \{f, h_1, \cdots, h_{n-1}\},
\]

with the Hamiltonians \(h_1, \cdots, h_{n-1}\).

One trivial example is a system with one free boson \((x, p) = (x_1, x_2)\) and one free fermion \(\theta\) over the phase space \(\mathbb{R}^{2|1}\). In this case, the equation of motion is given by the SNB (2.1) and the Hamiltonians \(h_1 = \frac{1}{2}p^2\), \(h_2 = \theta\). Other non-trivial examples, including Euler’s top, \(^{1)}\) have not yet been elucidated and must be studied. We consider a Lagrangian system, \(^{(10)}\) such as that with the Polyakov action with the light-cone gauge, that is invariant under the \(n\)-dimensional volume preserving diffeomorphism

\[
\delta \epsilon \mathcal{X}_\mu (\sigma) = \{\mathcal{X}_\mu, \epsilon_1, \cdots, \epsilon_{n-1}\},
\]

where the bracket is the bosonic Nambu bracket. To apply the BRS or BV method to such a constrained Lagrangian system, the parameters \(\{\epsilon_j\}\) must be replaced by the “ghosts” \(\{c_j\}\), which are fermions, and the above Nambu bracket becomes the SNB with super coordinates. It would be interesting to study the supersymmetric extension of the above Lagrangian with the SNB. The quantization of the SNB involves the same difficulties as that of the bosonic Nambu bracket. \(^{1), 2)}\) The triple brackets \(^{1)}\) are meaningless for our Nambu bracket, which does not possess the usual skew-symmetric property. The difficulty \(^{1)}\) that the triple bracket for the bosonic Nambu bracket does not satisfy the Leibniz rule may arise from this fact. On the other hand, to quantize the Nambu bracket, we should construct the representation of the algebra \((\mathfrak{g}, \tau)\) in Theorem 1, instead of that of the Nambu-Poisson algebra itself.
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