Nonlocal Metric Realizations of MOND

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ABSTRACT

I discuss relativistic extensions of MOND in which the metric couples normally to matter. I argue that MOND might be a residual effect from the vacuum polarization of infrared gravitons produced during primordial inflation. If so, MOND corrections to the gravitational field equations would be nonlocal. Nonlocality also results when one constructs metric field equations which reproduce the Tully-Fisher relation, along with sufficient weak lensing. I give the full field equations for the simplest class of models, and I specialize these equations to the geometries relevant for cosmology. I conclude by sketching the direction of future studies.

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1 Introduction

Milgrom’s MOdified Newtonian Dynamics (MOND) [1, 2, 3] has been wonderfully successful in explaining galactic structure without the need for dark matter [4]. I work in quantum gravity, not astronomy, but I cannot forbear to present a list that Bob Sanders compiled of the regularities of rotationally supported systems which MOND explains and which would otherwise need to be accidents of galaxy formation [5]:

- The Baryonic Tully-Fisher Relation $v^4_\infty = a_0 GM$ between the asymptotic rotational velocity $v_\infty$, the total mass $M$ and the MOND acceleration $a_0$, which holds over five decades in mass [6];
- Milgrom’s Law, that the need for dark matter (or MOND) seems always to occur when the Newtonian acceleration drops below $a_0$, [7];
- Freeman’s Law $G\Sigma < a_0$ for the surface density $\Sigma$ of a rotationally supported system [8]; and
- Sancisi’s Law that for every feature in the surface brightness there is a corresponding feature in the rotation curve, and vice versa [9].

Bob has an equally impressive list of successes for pressure-supported systems [5] but I will let him make that case. The bottom line for a fundamental theorist like me is that MOND works better than anyone would expect if dark matter were the actual determinant of galactic structure.

Just as Newtonian dynamics is the nonrelativistic limit of general relativity, so MOND must be the nonrelativistic limit of some larger theory. We need that larger theory if we are to understand whether or not it can accomplish all the things that dark matter does for general relativity. The search for an extension of MOND began early with the development of an action for the MOND-corrected Newtonian potential [10]. Although not relativistic, this model demonstrates that MOND conserves energy, 3-momentum and angular momentum.

There are now two classes of fully relativistic models:

- Those in which the extra, MOND force is carried by some field other than the metric; and
- Those in which the MOND force is carried by the metric.
Examples of the former are Bekenstein’s TeVeS \[11\] and Moffat’s STVG \[12\]. TeVeS has been studied extensively by some of the world’s top cosmologists \[13, 14\] and, while not without problems \[15\], it agrees better with observation than critics of MOND thought possible before the first relativistic extension was available for study. STVG (now called MOG) has not received independent attention but interesting results have been claimed for it \[16, 17, 18\].

When the MOND force is carried by fields other than the metric it raises the philosophical question of whether or not these other fields are a form of dark matter in disguise. Another peculiar feature of this class of models is that gravitons and matter particles propagate along the geodesics of different metrics. This leads to an easily observable time lag between the arrival of the gravity wave pulse from some cosmic event (such as a supernova) and the corresponding optical and neutrino signal \[19, 20, 21, 22\]. A single coincident detection of gravitational and optical (or neutrino) radiation from such an event would rule these models out.

If the MOND force is carried by the normal metric then the gravitational field equations must take the form,

\[ G_{\mu \nu} + \Delta G_{\mu \nu} = 8\pi G T_{\mu \nu} , \]

where \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \) is the Einstein tensor, \( T_{\mu \nu} \) is the normal stress-energy tensor, and \( \Delta G_{\mu \nu} \) represents the MOND correction. It has been difficult to build models of this type because the Tully-Fisher relation precludes MOND corrections which derive from a local, invariant action. To see why, let us specialize to a static, spherically symmetric geometry in the weak field regime,

\[ ds^2 = -\left[1 + b(r)\right] dt^2 + \left[1 + a(r)\right] dr^2 + r^2 d\Omega^2 . \]  

(2)

Suppose the mass density \( \rho(r) \) is always within the MOND regime of \( \frac{GM(r)}{c^2} < a_0 \), where \( M(r) \) is the mass enclosed within radius \( r \),

\[ M(r) \equiv 4\pi \int_0^r dr' r'^2 \rho(r') . \]  

(3)

These conditions would pertain throughout a low surface brightness galaxy. For particles moving with \( v(r) \) around circular orbits of radius \( r \), the geodesic equation implies \( v^2 = rb'/2 \). Hence the Tully-Fisher relation becomes,

\[ \left( \frac{1}{2} rb'(r) \right)^2 = a_0 GM(r) . \]  

(4)
We can compare with the $\mu = 0 = \nu$ component of (1) by differentiating with respect to $r$ and then dividing by $\frac{1}{2}a_0 r^2$ \[23\].
\[
\frac{1}{2a_0 r^2} \frac{d}{dr} \left( r b'(r) \right)^2 = 8\pi G \rho(r). \tag{5}
\]
The factors of $r$ are simple to understand in terms the spherical coordinate measure but a striking property of the left hand — gravitational — side of equation (5) is the presence of three derivatives. Local curvature invariants, and invariant derivatives of local curvatures, contain even, nonzero, numbers of derivatives — 2, 4, 6 and so on — so variations of them cannot produce the three derivatives which MOND phenomenology clearly requires.

In addition to precluding a local action, expression (5) is peculiar because the left hand side is quadratic in the weak field $b(r)$ \[24\]. This means that the MOND correction $\Delta G_{\mu\nu}$ must cancel the linear term from the Einstein tensor $G_{\mu\nu}$ — at least for the $\mu = 0 = \nu$ component — and then substitute the quadratic term we see in (5). Abandoning locality allows us to accomplish both things. The basic idea is that acting the inverse of a second order differential operator on a curvature produces a weak field term with no derivatives, so the desired MOND correction to the gravitational Lagrangian can take the form,
\[
\Delta L = - \left[ \partial \times \frac{1}{\partial^2} (\text{Curvature}) \right]^2 + \frac{1}{a_0} \left[ \partial \times \frac{1}{\partial^2} (\text{Curvature}) \right]^3 + \ldots \tag{6}
\]

The first attempt to construct such an action was able to recover the Tully-Fisher relation but failed to produce the MOND enhancement of weak lensing which the data demand if dark matter is absent \[25\]. It was finally possible to accommodate both requirements by involving a more complicated curvature \[23\]. (See also \[26, 27, 28\].) The purpose of this article is to explain how models of this type might arise from fundamental theory, to give the field equations for a general metric, and to specialize these equations to cosmology.

Section 2 discusses how significant nonlocal corrections to the effective field equations might arise from the vacuum polarization of infrared gravitons created during an extended phase of primordial inflation. Section 3 describes the nonlocal structures from which a phenomenologically viable model can be constructed. Section 4 gives the full field equations for a general metric, and their specialization to cosmology. I also discuss the possibility of making the MOND acceleration $a_0$ dynamical so that it changes with the cosmological expansion rate. My conclusions comprise section 5.
2 Gravitational Vacuum Polarization

Abandoning locality is a profound step that requires justification. I believe that justification can be found in the phenomenon of gravitational vacuum polarization. To understand this I begin by reviewing electromagnetic polarization, both in a medium and in vacuum. I then explain why gravitational vacuum polarization is so small in flat space background, and why there should have been a much larger effect during primordial inflation. I believe that MOND might be a residual consequence of this.

Even undergraduate physics students are familiar with the phenomenon of polarization in a medium. The medium contains a vast number of bound charges. When an electric field is applied, the positive charges tend to move with the field and the negative changes move opposite. That charge separation polarizes the medium and tends to reduce the electric field strength.

One of the amazing predictions of quantum field theory is that virtual particles are continually emerging from the vacuum, existing for a brief period, and then disappearing. How long these virtual particles can exist is controlled by the energy-time uncertainty principle, which gives the minimum time $\Delta t$ needed to resolve and energy difference $\Delta E$,

$$\Delta t \Delta E \gtrsim \hbar.$$  \hspace{1cm} (7)

(I will henceforth work in units where Planck’s constant $\hbar$ and the speed of light $c$ are both unity.) If one imagines the emergence of a pair of positive and negatively charged particles of mass $m$ and momentum $\pm \vec{k}$ then the energy went from zero to $E = 2[m^2 + k^2]^{\frac{1}{2}}$. To not resolve a violation of energy conservation, the energy-time uncertainty principle requires the pair to disappear after a time $\Delta t$ given by,

$$\Delta t \sim \frac{1}{\sqrt{m^2 + k^2}}.$$  \hspace{1cm} (8)

The rest is an exercise is classical (that is, non-quantum) physics. If we ignore the change in the particles’ momentum then their positions obey,

$$\frac{d^2}{dt^2} \left( \sqrt{m^2 + k^2} \Delta \vec{x}_\pm \right) = \pm e\vec{E} \quad \implies \quad \Delta \vec{x}_\pm (\Delta t) = \pm e\vec{E} \frac{2[m^2 + k^2]^{\frac{1}{2}}}{2[m^2 + k^2]^{\frac{3}{2}}}.$$  \hspace{1cm} (9)

Hence the polarization induced by wave vector $\vec{k}$ is,

$$\vec{p} = +e\Delta \vec{x}_+ (\Delta t) - e\Delta \vec{x}_- (\Delta t) = \frac{e^2 \vec{E}}{[m^2 + k^2]^{\frac{3}{2}}}.$$  \hspace{1cm} (10)
The full vacuum polarization density comes from integrating $d^3k/(2\pi)^3$.

The simple analysis I have just sketched gives pretty nearly the prediction from one loop quantum electrodynamics, which is in quantitative agreement with experiment. It allows us to understand two features of vacuum polarization which would be otherwise obscure:

- That the largest effect derives from the lightest charged particles because they have the longest persistence times $\Delta t$ and therefore induce the greatest polarization; and

- That the electrodynamic interaction becomes stronger at short distances because the longest wavelength (hence smallest $k$) virtual particles could induce more polarization than is allowed by the travel time between two very close sources.

We can understand one more thing of great significance for the point I wish to make in this section: vacuum polarization makes nonlocal corrections to the field equations and it would be a macroscopic phenomenon if there were massless charged particles. In flat space background the quantum-corrected Maxwell equations take the form \[ \text{[29]} \]

\[
\partial_\nu F^{\nu\mu}(x) - \partial_\nu \int d^4x' \chi_e(x; x') F^{\nu\mu}(x') = J^\mu(x), \tag{11}
\]

where $\chi_e(x; x')$ is the vacuum susceptibility. The one loop result for massless, scalar quantum electrodynamics is \[ \text{[30]} \]

\[
\chi_e(x; x') = \frac{\alpha \partial^4}{96\pi^2} \left\{ \theta(\Delta t - \Delta x) \left[ \ln[\mu^2(\Delta t^2 - \Delta x^2)] - 1 \right] \right\} + O(\alpha^2), \tag{12}
\]

where $\alpha$ is the fine structure constant, $\Delta t \equiv t - t'$, $\Delta x \equiv ||x - x'||$ and $\mu$ is the renormalization scale. Note the causality which is manifest in the factor of $\theta(\Delta t - \Delta x)$. By solving (11) for a point charge it is easy to show that one loop corrections screen the charge on distances $r > 1/\mu$ by a factor which grows like $\ln(2\mu r)$ \[ \text{[31]} \]. In fact the effect becomes so large that perturbation theory breaks if the interaction remains on for too long.

The source of gravitation is stress-energy, and unlike electromagnetism it has only one sign. This means that the gravitational analogue of polarization strengthens gravity, rather than weakening it. The application of a gravitational field to a classical medium attracts the material of the medium
towards the gravitating source, which increases its gravitational field. Although gravitons are massless, and quantum gravitational vacuum polarization makes nonlocal corrections to the effective field equations, macroscopic effects are suppressed in flat space background because the “charge” of a graviton with wave vector $\vec{k}$ is $k$, which goes to zero for long wave length gravitons.

The situation can be very different in an expanding universe. The associated geometry is characterized by a time dependent scale factor $a(t)$,

$$ds^2 \equiv g_{\mu \nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \vec{d}x \cdot \vec{d}x.$$  \hspace{1cm} (13)

Two derivatives of $a(t)$ characterize the expansion,

$$H(t) \equiv \frac{\dot{a}}{a} \quad , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2}.$$  \hspace{1cm} (14)

The Hubble parameter $H(t)$ gives the rate of cosmological expansion, and the first slow roll parameter $\epsilon(t)$ provides a dimensionless measure of how fast that rate is changing. The crucial boundary for this discussion is $\epsilon = 1$. For larger values of $\epsilon$ the universe is said to be decelerating because $\ddot{a} < 0$, whereas $\epsilon < 1$ corresponds to accelerated expansion ($\ddot{a} > 0$) or inflation. We know inflation can happen because it is going on right now, as witness the current values of $H$ and $\epsilon$ [32],

$$H_0 \approx 2.2 \times 10^{-18} \text{ Hz} \quad , \quad \epsilon_0 \approx .47.$$  \hspace{1cm} (15)

However, the period of relevance to this discussion is the epoch of primordial inflation, which is thought to have occurred at about $10^{-32}$ seconds after the Big Bang. If the BICEP detection of primordial $B$-mode polarization is correct then the values of $H$ and $\epsilon$ at that time were [33],

$$H_i \approx 1.8 \times 10^{38} \text{ Hz} \quad , \quad \epsilon_i \approx 0.013.$$  \hspace{1cm} (16)

These values tell us two crucial things about primordial inflation:

- The quantum gravitational loop counting parameter of $(\hbar/c^5)GH_i^2 \approx 7.7 \times 10^{-9}$ is small enough that perturbation theory is valid, but not so small that quantum gravitational effects are negligible; and

- The slow roll parameter is close to the de Sitter value of $\epsilon = 0$, at which point the Hubble parameter becomes constant and the scale factor is $a(t) \approx a_i e^{H_i t}$. 
Cosmological expansion can strengthen quantum effects because it causes the virtual particles which drive them to persist longer. This is easy to see from the geometry \([13]\). Because spatial translation invariance is unbroken, particles still have conserved wave numbers \(\vec{k}\). However, because the physical distance is the coordinate distance scaled by \(a(t)\), the physical energy of a particle with mass \(m\) and wave number \(k = 2\pi/\lambda\) becomes time dependent,

\[
E(t, \vec{k}) = \sqrt{m^2 + \frac{k^2}{a^2(t)}}. \tag{17}
\]

Hence the relation for the persistence time \(\Delta t\) of a virtual pair which emerges at time \(t\) changes from \([8]\) to,

\[
\int_{t}^{t+\Delta t} dt' E(t', \vec{k}) \sim 1. \tag{18}
\]

Massless particles persist the longest, just as they do in flat space. However, for inflation it is the lower limit of \([18]\) which dominates, so that even taking \(\Delta t\) to infinity does not cause the integral to grow past a certain point. One can see this from the de Sitter limit,

\[
\int_{t}^{t+\Delta t} dt' \frac{k}{a(t')} \frac{a}{H_0 a(t)} \left[1 - e^{-H_0 \Delta t} \right]. \tag{19}
\]

A particle with \(k < H(t)a(t)\) is said to be super-horizon, and we have just shown that any massless virtual particle which emerges from the vacuum with a super-horizon wave number during inflation will persist forever.

It turns out that almost all massless particles possess a symmetry known as \textit{conformal invariance} which suppresses the rate at which they emerge from the vacuum. This keeps the density of virtual particles small, even though any that do emerge can persist forever. One can see the problem by specializing the Lagrangian of a massless, conformally coupled scalar \(\psi(t, \vec{x})\) to the cosmological geometry \([13]\),

\[
\mathcal{L} = -\frac{1}{2} \partial_\mu \psi \partial^\mu \psi g^{\mu\nu} \sqrt{-g} - \frac{R}{12} \psi^2 \sqrt{-g} \rightarrow \frac{a^3}{2} \left[\dot{\psi}^2 - \frac{\partial_\mu \psi \partial^\mu \psi}{a^2} - (\dot{H} + 2H^2)\psi^2\right]. \tag{20}
\]

The equation for a canonically normalized, spatial plane wave of the form \(\psi(t, \vec{x}) = v(t, k)e^{i\vec{k} \cdot \vec{x}}\) can be solved for a general scale factor \(a(t)\),

\[
\ddot{v} + 3H \dot{v} + \left[\frac{k^2}{a^2} + \dot{H} + 2H^2\right] v = 0 \quad \implies \quad v(t, k) = \frac{1}{a(t) \sqrt{2k}} \exp \left[-ik \int_{t, a(t')} dt'\right]. \tag{21}
\]
The factor of $1/a(t)$ in (21) suppresses the emergence rate, even though destructive interference from the phase dies off, just as the energy-time uncertainty principle (19) predicts. The stress-energy contributed by this field is,

$$T_{\mu\nu} = \left[\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}\right] \partial_\rho \psi \partial_\sigma \psi + \frac{1}{6} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Box - D_\mu D_\nu\right] \psi^2,$$

(22)

where $D_\mu$ is the covariant derivative and $\Box$ is the covariant d’Alembertian.

We can get the 0-point energy of a single wave vector $\vec{k}$ by specializing $T_{00}$ to the cosmological geometry (13) and multiplying by a factor of $a^3(t)$,

$$E(t, \vec{k}) = \frac{a^3(t)}{2} \left[|\dot{\psi}|^2 + \left(\frac{k^2}{a^2} + H^2\right) |v|^2 + H \left(\dot{v} - \dot{v}^* + \dot{\psi} - \dot{\psi}^*\right)\right] = \frac{k}{2a(t)}.$$  

(23)

This is just the usual $\frac{1}{2} \hbar \omega$ term which is not strengthened but rather weakened by the cosmological expansion.

Only gravitons and massless, minimally coupled scalars are both massless and not conformally invariant so that they can engender significant quantum effects during inflation. Because they obey the same mode equation [34, 35] it will suffice to specialize the scalar Lagrangian to the cosmological geometry (13),

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} \rightarrow \frac{1}{2} a^3 \left[\dot{\phi}^2 - \frac{1}{a^2} \partial_i \phi \partial_i \phi\right].$$

(24)

The equation for a canonically normalized, spatial plane wave of the form $\phi(t, \vec{x}) = u(t, k) e^{i\vec{k} \cdot \vec{x}}$ is simpler than that of its conformally coupled cousin (21) but more difficult to solve, so I will specialize the solution to de Sitter,

$$\ddot{u} + 3H \dot{u} + \frac{k^2}{a^2} u = 0 \implies u(t, k) = \frac{1}{a(t) \sqrt{2k}} \left[1 + \frac{iH t a(t)}{k} \right] \exp \left[-ik \int_t^{t'} dt' a(t')\right].$$

(25)

The minimally coupled mode function $u(t, k)$ has the same phase factor as the conformal mode function (21), and they both fall off like $1/a(t)$ in the far sub-horizon regime of $k \gg H_i a(t)$. However, they disagree strongly in the super-horizon regime during which $v(t, k)$ continues to fall off whereas $u(t, k)$ approaches a phase times $H_i / \sqrt{2k^3}$. One can see from the equation on the left of (25) that $u(t, k)$ approaches a constant for any inflating geometry.

The 0-point energy in wave vector $\vec{k}$ is,

$$\mathcal{E}(t, \vec{k}) = \frac{1}{2} a^3 \left[|\dot{u}|^2 + \frac{k^2}{a^2} |u|^2\right] = \frac{k}{a(t)^2} \frac{1}{2} + \left(\frac{H_i a(t)}{2k}\right)^2.$$  

(26)
Because each wave vector is an independent harmonic oscillator with mass proportional to $a^3(t)$ and frequency $k/a(t)$ we can read off the occupation number from expression (26),

$$N(t, \vec{k}) = \left[ \frac{H_i a(t)}{2k} \right]^2. \quad (27)$$

As one might expect, this number is small in the sub-horizon regime. It becomes of order one at the time $t_k$ of horizon crossing, $k = H(t_k)a(t_k)$, and $N(t, \vec{k})$ grows explosively afterwards. This is the basis of the power spectra of scalar [36] and tensor [37] perturbations which are predicted by primordial inflation.

We can use the occupation number (27) to compute the energy density of infrared ($k < H_i a(t)$) gravitons,

$$\rho_{\text{IR}} = \int \frac{d^3k}{(2\pi a)^3} \theta\left(H_i a - k\right) \times 2 \times N(t, \vec{k}) \times \frac{k}{a} = \frac{H_i^4}{8\pi^2}. \quad (28)$$

This is smaller than the energy density which caused inflation by a factor of $GH_i^2/3\pi \sim 10^{-9}$, but it is still a huge energy density by today’s standards, and one can easily see that the continual particle production needed to maintain $\rho_{\text{IR}}$ must limit the duration of inflation. To keep things finite, suppose that the spatial topology is that of a 3-torus, with radius $R(t) = a(t)/H_i$. If de Sitter expansion proceeds unimpeded then the total mass at time $t$ is,

$$M(t) \sim \rho_{\text{IR}} R^3(t) \sim H_i a^3(t) . \quad (29)$$

If this mass were in causal contact with itself then the universe would fall within its own Schwarzschild radius at time $t$ such that,

$$1 \sim \frac{G M(t)}{R(t)} = GH_i^2 a^2(t) . \quad (30)$$

That corresponds to only about 9 e-foldings, which is far less than the 50 to 60 e-foldings of inflation which must occur to solve the horizon problem.

My estimate (30) was based on assuming that all the mass is instantaneously in contact with itself. Of course causality imposes a time lag between when a particle first emerges from the vacuum and when its gravity can affect the particles which have already emerged. That time lag is the only thing holding the universe up against gravitational collapse. It seems obvious
that causality cannot extend the duration of inflation forever. A simple measure of how fast things grow causally is the volume of the past light-cone back to the beginning of time. For flat space this goes like \( t^4 \), but its growth during inflation is only like \( \ln\left(\frac{a(t)}{a_i}\right) \sim H_i t \). So one should expect the back-reaction from inflationary graviton production to become effective when \( \ln\left(\frac{a(t)}{a_i}\right) \sim 1/GH_i^2 \).

With Nick Tsamis I have proposed a model in which inflation begins due to a large, positive cosmological constant that is screened by the gradual build-up of self-gravitation between inflationary gravitons \(^{38,39}\). Direct computations of quantum gravity on de Sitter background are difficult, and plagued by problems of interpretation \(^{40,41}\), but they do show secular corrections of \( \ln\left(\frac{a(t)}{a_i}\right) \) \(^{42,43}\) according to well-defined rules \(^{44}\). These factors eventually grow so large that perturbation theory breaks down, at which point some sort of nonperturbative scheme must be employed to follow subsequent evolution. Developing such a technique is not hopeless because Starobinsky and Yokoyama were able to accomplish it for scalar potential models \(^{45,46}\), and the technique has been generalized to scalars which interact with photons \(^{47}\) and with fermions \(^{48}\). However, the generalization to quantum gravity has not been achieved yet \(^{49,50}\).

In the absence of a nonperturbative resummation technique there have been efforts to use perturbative results to make a plausible guess for the most cosmologically significant part of the effective field equations \(^{51,52,53}\). The point of relevance to MOND is that these models are necessarily nonlocal \(^{54}\), and that they inevitably change the force of gravity. I will close this section by commenting that this physical picture helps to justify two features phenomenological models of MOND which would be otherwise inexplicable:

- There is a beginning of time, corresponding roughly to the end of inflation when quantum gravitational corrections became nonperturbatively strong; and
- Because the mechanism derives from *cosmological* gravitons, we should expect the largest modifications of gravity on large scales, not on small scales.
3 Building Blocks for Nonlocal Models

It turns out that MOND phenomenology can be implemented using only two nonlocal structures [23]:

- The inverse (with retarded boundary conditions) of the scalar d'Alembertian [51],

\[ \Box \equiv \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \right) \; ; \text{ and} \]

- The normalized gradient of some nonlocal scalar \( \chi[g] \), such as the volume of the past lightcone [55], which grows in the timelike direction,

\[ u^\mu[g] \equiv \frac{-g^{\mu \nu} \partial_{\nu} \chi}{\sqrt{-g^{\alpha \beta} \partial_{\alpha} \partial_{\beta} \chi}}. \]

The apparent sufficiency of these quantities may result from adopting too narrow a focus; one should remain open to the possibility of higher spin operators [56] and more general forms of nonlocality. In this section I specialize these quantities and some relevant curvatures to the static, spherically symmetric geometry (2) and to the homogeneous, isotropic and spatially flat geometry (13) of cosmology. It turns out that this can be done without giving a precise specification for the scalar \( \chi[g] \).

Because I do not necessarily make the weak field approximation it is useful to define \( A(r) \equiv 1 + a(r) \) and \( B(r) \equiv 1 + b(r) \) for the static, spherically symmetric geometry. The scalar d'Alembertian and normalized, timelike 4-velocity are,

\[ \Box F(r) = \frac{1}{r^2 \sqrt{AB}} \frac{d}{dr} \left[ r^2 \sqrt{B(r)} F'(r) \right], \quad u^\mu = \frac{\delta^\mu_0}{\sqrt{B(r)}}. \]

The 00 component and trace of the Ricci tensor are,

\[ R_{00} = \frac{B''}{2A} - \frac{B'}{4A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{B'}{rA}, \]

\[ R = -\frac{B''}{AB} + \frac{B'}{2AB} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{2}{r} \left( \frac{A'}{A} - \frac{B'}{B} \right) + \frac{2(A-1)}{r^2 A} \]

Assuming regularity at \( r = 0 \) one can recover the potential \( B(r) \) from a nonlocal scalar,

\[ \frac{1}{\Box} (u^\alpha u^\beta R_{\alpha \beta}) = \frac{1}{2} \ln \left[ \frac{B(r)}{B(0)} \right]. \]
For the cosmological geometry \((13)\) the scalar d’Alembertian and the timelike 4-velocity take the forms,

\[
\Box F(t) = -\frac{1}{a^3(t)} \frac{d}{dt} \left[ a^3(t) \dot{F}(t) \right], \quad u^\mu = \delta_0^\mu .
\] (37)

The 00 component and the trace of the Ricci tensor are,

\[
R_{00} = -3(\dot{H} + H^2) ,
\]

\[
R = 6\dot{H} + 12H^2 .
\] (38) (39)

One can very largely reconstruct the scale factor from another nonlocal scalar,

\[
\frac{1}{\Box} \left( R + u^\alpha u^\beta R_{\alpha\beta} \right) = -3 \int_{t_i}^{t} \frac{dt'}{a^3(t')} \int_{t_i}^{t} \frac{dt''}{a^3(t'')} \left( H(t'') a^3(t'') \right) ,
\]

\[
= -3 \ln \left[ \frac{a(t)}{a_i} \right] + 3 \int_{t_i}^{t} \frac{dt'}{a^3(t')} H_i a^3 .
\] (40) (41)

Note the implementation of retarded boundary conditions.

## 4 MOND Field Equations and Cosmology

The static, spherically symmetric geometry \((2)\) has two gravitational potentials, \(B(r)\) and \(A(r)\). Requiring that the ultra-weak field limit of the \(B(r)\) equation should reproduce the Tully-Fisher relation, along with no change in the \(A(r)\) equation, restricts invariant, metric-based gravitational Lagrangians to take the form of general relativity plus \([23]\),

\[
\Delta \mathcal{L} = \frac{a_0^2}{16\pi G} f \left( \frac{Y[g]}{a_0^2} \right) \sqrt{-g} , \quad Y[g] \equiv g^{\mu\nu} \partial_\mu \frac{2}{\Box} \left( u^\alpha u^\beta R_{\alpha\beta} \right) \partial_\nu \frac{2}{\Box} \left( u^\rho u^\sigma R_{\rho\sigma} \right) .
\] (42)

MOND phenomenology requires \(f(Z) = \frac{1}{2}Z - \frac{1}{6}Z^{2} + O(Z^2)\), with major suppression for large, positive \(Z\). It does not fix the behavior of \(f(Z)\) for negative \(Z\). The purpose of this section is to give the general field equations for this class of models and then specialize them to the cosmological geometry \((13)\). For simplicity I will assume that the nonlocal scalar \(\chi[g]\) is,

\[
\chi[g] \equiv -\frac{1}{\Box} 1 .
\] (43)
An easy way of getting the field equations is through the auxiliary scalar formalism that Nojiri and Odintsov \[57\] introduced to localize a nonlocal model of dark energy \[58\]. The general MOND model \((42)\) requires four scalars: \(\chi[g] = -\frac{1}{10} \) and \(\phi[g] = \frac{\dot{g}}{2}(u^\alpha u^\beta R_{\alpha\beta})\), along with Lagrange multiplier fields \(\psi\) and \(\xi\) to enforce these conditions. The localized form is \[59\],

\[
\Delta \mathcal{L} = \frac{1}{16\pi G} \left\{ a^2 f\left(\frac{\mu^\mu{\phi}\nu{\phi}}{a^2} \right) - \left[ \partial_\mu \xi \partial_\nu \phi g^{\mu\nu} + 2\xi R_{\mu\nu} u^\mu u^\nu \right] - \left[ \partial_\mu \psi \partial_\nu \chi g^{\mu\nu} - \psi \right] \right\} \sqrt{-g} .
\] (44)

The MOND correction \(\Delta G_{\mu\nu}\) to the Einstein tensor is \[59\],

\[
\frac{16\pi G \delta \Delta S}{\sqrt{-g}} \delta g_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left[ -a^2 f + g^{\rho\sigma} \left( \partial_\rho \xi \partial_\sigma \phi + \partial_\rho \psi \partial_\sigma \chi \right) + 2\xi u^\rho u^\sigma R_{\rho\sigma} - \psi \right]
+ \partial_\mu \phi \partial_\nu \phi f' - \partial_\mu \xi \partial_\nu \phi - \partial_\mu \psi \partial_\nu \chi
+ 2\xi \left[ 2u_\mu u_\nu R_{\alpha\beta} + u_\mu u_\nu u^\alpha u^\beta R_{\alpha\beta} \right]
+ \left[ \partial_\mu \left( \xi u_\mu + \psi \right) + g_{\mu\nu} D_\alpha D_\beta (\xi u^\alpha u^\beta) - 2D_\alpha D_{\mu}(\xi u_\nu u^\alpha) \right] .
\] (45)

Unlike Nojiri and Odintsov, we do not take the various scalars to have independent initial value data (which would result in two linear combinations of them being ghosts \[60, 61\]) but instead define each scalar using \[1\] with retarded boundary conditions \[59\],

\[
\phi[g] = \frac{2}{10} (u^\alpha u^\beta R_{\alpha\beta}) , \quad \xi[g] = \frac{2}{10} \left( D_\mu \left[ D^\mu \phi f' \left( \frac{g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi}{a^2} \right) \right] \right) ,
\] (46)

\[
\chi[g] = -\frac{1}{10} (1) , \quad \psi[g] = \frac{4}{10} \left( D_\mu \left[ \frac{\xi g^\mu_\nu u^\rho R_{\rho\sigma}}{\sqrt{-g} a^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi} \right] \right) ,
\] (47)

where \(g^\mu_\nu \equiv g^{\mu\nu} + u^\mu u^\nu\) is the projected metric.

The scalars take simple forms in the cosmological geometry \[13, 59\],

\[
\phi(t) = 6 \int_{t_i}^{t} \frac{d\tau'}{a^2(\tau')} \int_{t_i}^{\tau'} dt'' a^3 \left[ H + H^2 \right] , \quad \xi(t) = 2 \int_{t_i}^{t} \frac{d\tau'}{a^2(\tau')} f' \left( -\frac{\dot{\phi}^2(t')}{a^2} \right) ,
\] (48)

\[
\chi(t) = \int_{t_i}^{t} \frac{d\tau'}{a^2(\tau')} \int_{t_i}^{\tau'} dt'' a^3 (t'') , \quad \psi(t) = 0 .
\] (49)

Homogeneity and isotropy imply that any second rank tensor such as \(\Delta G_{\mu\nu}\) has only two distinct components when specialized to cosmology \[13, 59\],

\[
\Delta G_{00}(t) = \frac{a^2}{2} f \left( \frac{\dot{\phi}^2}{a^2} \right) + 3H \dot{\xi} + 6H^2 \xi ,
\] (50)
\[
\Delta G_{ij}(t) = -\left[ \frac{a_0^2}{2} f\left(\frac{-\dot{\phi}^2}{a_0^2}\right) + \ddot{\xi} + \frac{\dot{\phi}}{2} + 4H \dot{\xi} + (4\dot{H} + 6H^2)\xi \right] g_{ij} .
\] (51)

As one can see from expressions (48) and (50-51), the MOND function \( f(Z) \) is evaluated at a negative definite argument \( Z = -\dot{\phi}^2(t)/a_0^2 \) for cosmology. MOND phenomenology only fixes the behavior of \( f(Z) \) for positive \( Z \) [23]. We are therefore free to adjust the negative \( Z \) branch so that the MOND additions to the Einstein tensor (50-51) make up for the absence of dark matter in determining the expansion history \( a(t) \). This is accomplished by regarding \( a(t) \) as known, with the matter density \( \rho \) also known in terms of \( a(t) \), and then treating the modified Friedmann equation \( 3H^2 + \Delta G_{00} = 8\pi G \rho \) as an integro-differential equation for \( f(Z) \) which can be solved numerically [59]. A very similar exercise was worked out in detail [62] to determine the negative branch of the free function of nonlocal cosmology [58] so as to make this model reproduce the ΛCDM expansion history without a cosmological constant. Because the nonlocal MOND equations (42) are fixed once the function \( f(Z) \) has been defined, one can subject the model to meaningful tests by studying its response to perturbations around the cosmological geometry. Dodelson and Park have recently done this for nonlocal cosmology [63, 64], and one can directly apply their treatment of the operator \( 1/\Box \).

Although there seems to be no obstacle to reconstructing \( f(Z) \) to make nonlocal MOND support the known expansion history without dark matter, a potential problem is the enormous range of \( Z = -\dot{\phi}^2(t)/a_0^2 \) that would be involved. This becomes apparent from the time derivative of (48),

\[
\dot{\phi}(t) = -\frac{6}{a^3(t)}\int_{t_i}^{t} dt' a^3(t')H^2(t') \epsilon(1) ,
\] (52)

\[
= -6H \frac{(\epsilon - 1)}{3 - \epsilon} + 6H_i \frac{(\epsilon_i - 1)}{3 - \epsilon_i} \left( \frac{a_i}{a} \right)^3 + \frac{6}{a^3} \int_{t_i}^{t} dt' a^3 H \frac{d(\epsilon - 1)}{dt'} .
\] (53)

Because the slow roll parameter \( \epsilon(t) \equiv -\dot{H}/H^2 \) has only small temporal variation, the integral in expression (53) is negligible and the first term dominates. It has been noted from the earliest papers [2] that the MOND acceleration \( a_0 \) is close to the current value of the Hubble parameter \( H_0 \) times the speed of light (which is one in my units). Hence we see that cosmological evolution results in staggering changes in \( Z(t) \equiv -\dot{\phi}^2(t)/a_0^2 \sim -H^2(t)/H_0^2 \). For example, it would be about \( Z \sim -10^{32} \) during nucleosynthesis, and about \( Z \sim -10^{10} \) at the time of recombination.
There may be no problem with such dramatic variation in \( Z(t) \), but it is surely worth considering the alternative of replacing the MOND acceleration \( a_0 \) by some dynamical quantity \( \alpha[g] \) which changes with the expansion of spacetime. Although this step would make major changes in cosmology, there is no strong evidence for or against it from observable cosmic structures \[65, 66, 67\]. One of the simplest choices for \( \alpha[g] \) is proportional to the local expansion derived from the divergence of the timelike 4-velocity \( u^\mu[g] \) \[59\],

\[
a_0 \rightarrow \alpha[g] \equiv \frac{D_\mu u^\mu[g]}{6\pi}. \tag{54}
\]

The replacement \((54)\) engenders only three changes in the MOND correction \( \Delta G_{\mu\nu}[g] \) \[45-47\] for a general metric. Of course the factors of \( a_0 \) in \((45)\) and \((46)\) must be replaced by \( \alpha[g] \). The second difference is that \( \Delta G_{\mu\nu} \) acquires an extra contribution from the metric dependence of \( \alpha[g] \) \[59\],

\[
(\Delta G_{\mu\nu})_{\text{new}} = (\Delta G_{\mu\nu})_{\text{old}} + g_{\mu\nu} \left[ \alpha^2 f - g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f' \right]
+ \frac{1}{6\pi} \left[ g_{\mu\nu} u^\gamma \partial_\gamma - 2u_\mu \partial_\nu - u_\mu u_\nu u^\gamma \partial_\gamma \right] \left[ \alpha f - \frac{1}{\alpha} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f' \right]. \tag{55}
\]

The final difference is that the auxiliary scalar \( \psi[g] \) picks up an extra term from the \( \chi \) dependence of \( \alpha \) \[59\],

\[
\psi[g] = \frac{1}{\Box} \left( D_\mu \left[ 4\xi g^{\mu\nu} u^\rho R_{\rho\nu} + \frac{1}{3\pi} g^{\mu\nu} \partial_\nu \left[ \alpha f - \frac{1}{\alpha} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f' \right] \right] \right). \tag{56}
\]

These slight changes in the functional form of the field equations conceal vast differences in their numerical values when specialized to the cosmological geometry \[13\]. For that case the functional \( \alpha[g] \) degenerates to \( H(t)/2\pi \). The auxiliary scalars \( \phi, \chi \) and \( \psi \) are unchanged from \((48-49)\) but \( \xi \) is \[59\],

\[
\xi(t) = 2\int_{t_i}^{t} dt' \dot{\phi}(t') f' \left( \frac{-4\pi^2 \dot{\phi}^2(t')}{H^2(t')} \right). \tag{57}
\]

The MOND addition to the Friedmann equation becomes \[59\],

\[
\Delta G_{00} = -\frac{H^2}{8\pi^2} f \left( \frac{-4\pi^2 \dot{\phi}^2}{H^2} \right) - \dot{\phi}^2 f' \left( \frac{-4\pi^2 \dot{\phi}^2}{H^2} \right) + 3H \dot{\xi} + 6H^2 \xi. \tag{58}
\]

Because the argument of the function \( f(Z) \) is now nearly constant it is no longer clear that the reconstruction problem can be solved.
Relation (54) is probably good enough to study the cosmological implications of dynamical $a_0$, but it suffers from the obvious problem of vanishing inside any static system. Of course that would make the MOND corrections to gravity vanish in precisely the low acceleration, gravitationally bound systems for which MOND was originally proposed! So a more nonlocal version of (54) must be devised to make the dynamical MOND acceleration inside a gravitationally bound system depend upon the cosmological acceleration around it.

5 Conclusions

There is no question that MOND phenomenology requires any relativistic, metric realization of MOND to be nonlocal [23]. Nonlocal modifications of gravity have been much studied for a variety of reasons [68]. Section 2 argues that this particular one might arise as a residual effect from the quantum gravitational vacuum polarization left over from the epoch of primordial inflation. This in no way changes the sorts of nonlocal Lagrangians (42) one must consider to implement MOND phenomenology, but it does justify some of their properties, in particular the appearance of an initial time and the fact that gravity is modified on large scales rather than small ones.

Section 3 describes the nonlocal building blocks from which successful models can be constructed. One result of great significance is the realization (36) that the Newtonian potential can be represented as a nonlocal scalar. This means that the general class of models (42) I developed with Cedric Deffayet and Gilles Esposito-Farese are a relativistic extension of the Newtonian model originally proposed by Bekenstein and Milgrom [10]. It follows that the implications for nonspherical galaxies and strong fields have already been worked out from studies of this model.

Section 4 gives the general field equations (45-47) and their specialization (48-51) to the geometry (13) of cosmology. That completes what might be termed the “0th order program” in developing a relativistic, metric-based extension of MOND. The “1st order program” of subsequent research is clear:

- Apply same techniques that were used for nonlocal cosmology [60, 61] to check that the nonlocal addition (42) introduces no extra degrees of freedom in addition to those present in general relativity and that none of the old degrees of freedom becomes a ghost; and
Solve the reconstruction problem to determine the function \( f(Z) \) for \( Z < 0 \) needed to support the \( \Lambda \)CDM expansion history without dark matter, both for constant \( a_0 \) and for dynamical extensions.

If these investigations have a successful outcome one can envisage a “2nd order program” of further studies:

- Apply the same techniques that Dodelson and Park developed for non-local cosmology [63, 64] to analyze cosmological perturbations to see if the extra MOND force can make up for the absence of dark matter in the cosmic microwave background and during structure formation;

- Develop a more nonlocal dynamical extension of the MOND acceleration which reduces to \( a_0 \) inside gravitationally bound systems; and

- See if dynamical extensions of \( a_0 \) can resolve the problems MOND has in reproducing the dynamics of galactic cluster cores [69] and in explaining recently disturbed systems such as the Bullet Cluster [70].

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References

[1] M. Milgrom, Astrophys. J. 270, 365 (1983).

[2] M. Milgrom, Astrophys. J. 270, 371 (1983).

[3] M. Milgrom, Astrophys. J. 270, 384 (1983).

[4] R. H. Sanders and S. S. McGaugh, Ann. Rev. Astron. Astrophys. 40, 263 (2002) [astro-ph/0204521].

[5] R. H. Sanders, arXiv:0806.2585 [astro-ph].
[6] S. S. McGaugh, Astrophys. J. 632, 859 (2005) [astro-ph/0506750].

[7] M. Kaplinghat and M. S. Turner, Astrophys. J. 569, L19 (2002) [astro-ph/0107284].

[8] K. C. Freeman, Astrophys. J. 160, 811 (1970).

[9] R. Sancisi, astro-ph/0311348.

[10] J. Bekenstein and M. Milgrom, Astrophys. J. 286, 7 (1984).

[11] J. D. Bekenstein, Phys. Rev. D 70, 083509 (2004) [Erratum-ibid. D 71, 069901 (2005)] [astro-ph/0403694].

[12] J. W. Moffat, JCAP 0603, 004 (2006) [gr-qc/0506021].

[13] C. Skordis, D. F. Mota, P. G. Ferreira and C. Boehm, Phys. Rev. Lett. 96, 011301 (2006) [astro-ph/0505519].

[14] S. Dodelson and M. Liguori, Phys. Rev. Lett. 97, 231301 (2006) [astro-ph/0608602].

[15] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]].

[16] J. W. Moffat, S. Rahvar and V. T. Toth, arXiv:1204.2985 [astro-ph.CO].

[17] J. W. Moffat and S. Rahvar, Mon. Not. Roy. Astron. Soc. 436, 1439 (2013) [arXiv:1306.6383 [astro-ph.GA]].

[18] J. W. Moffat and S. Rahvar, arXiv:1309.5077 [astro-ph.CO].

[19] E. O. Kahya and R. P. Woodard, Phys. Lett. B 652, 213 (2007) arXiv:0705.0153 [astro-ph]].

[20] S. Desai, E. O. Kahya and R. P. Woodard, Phys. Rev. D 77, 124041 (2008) [arXiv:0804.3804 [astro-ph]].

[21] E. O. Kahya, Class. Quant. Grav. 25, 184008 (2008) [arXiv:0801.1984 [gr-qc]].

[22] E. O. Kahya, Phys. Lett. B 701, 291 (2011) arXiv:1001.0725 [gr-qc]].
[23] C. Deffayet, G. Esposito-Farese and R. P. Woodard, Phys. Rev. D 84, 124054 (2011) [arXiv:1106.4984 [gr-qc]].

[24] M. E. Soussa and R. P. Woodard, Phys. Lett. B 578, 253 (2004) [astro-ph/0307358].

[25] M. E. Soussa and R. P. Woodard, Class. Quant. Grav. 20, 2737 (2003) [astro-ph/0302030].

[26] F. W. Hehl and B. Mashhoon, Phys. Lett. B 673, 279 (2009) [arXiv:0812.1059 [gr-qc]].

[27] F. W. Hehl and B. Mashhoon, Phys. Rev. D 79, 064028 (2009) [arXiv:0902.0560 [gr-qc]].

[28] H.-J. Blome, C. Chicone, F. W. Hehl and B. Mashhoon, Phys. Rev. D 81, 065020 (2010) [arXiv:1002.1425 [gr-qc]].

[29] T. Prokopec and R. P. Woodard, Am. J. Phys. 72, 60 (2004) [astro-ph/0303358].

[30] T. Prokopec, O. Tornkvist and R. P. Woodard, Annals Phys. 303, 251 (2003) [gr-qc/0205130].

[31] H. Degueldre and R. P. Woodard, Eur. Phys. J. C 73, 2457 (2013) [arXiv:1303.3042 [gr-qc]].

[32] P. A. R. Ade et al. [Planck Collaboration], arXiv:1303.5076 [astro-ph.CO].

[33] P. A. R. Ade et al. [BICEP2 Collaboration], arXiv:1403.3985 [astro-ph.CO].

[34] E. Lifshitz, J. Phys. (USSR) 10, 116 (1946).

[35] L. P. Grishchuk, Sov. Phys. JETP 40, 409 (1975) [Zh. Eksp. Teor. Fiz. 67, 825 (1974)].

[36] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. 33, 549 (1981)].

[37] A. A. Starobinsky, JETP Lett. 30, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. 30, 719 (1979)].
[38] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B 474 (1996) 235 [hep-ph/9602315].

[39] N. C. Tsamis and R. P. Woodard, Int. J. Mod. Phys. D 20, 2847 (2011) arXiv:1103.5134 [gr-qc].

[40] J. Garriga and T. Tanaka, Phys. Rev. D 77, 024021 (2008) arXiv:0706.0295 [hep-th].

[41] N. C. Tsamis and R. P. Woodard, Phys. Rev. D 78, 028501 (2008) arXiv:0708.2004 [hep-th].

[42] S. -P. Miao and R. P. Woodard, Class. Quant. Grav. 23, 1721 (2006) gr-qc/0511140.

[43] S. P. Miao and R. P. Woodard, Phys. Rev. D 74, 024021 (2006) gr-qc/0603135.

[44] T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. 323, 1324 (2008) arXiv:0707.0847 [gr-qc].

[45] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D 50, 6357 (1994) astro-ph/9407016.

[46] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B 724, 295 (2005) gr-qc/0505115.

[47] S. -P. Miao and R. P. Woodard, Phys. Rev. D 74, 044019 (2006) gr-qc/0602110.

[48] S. -P. Miao and R. P. Woodard, Class. Quant. Grav. 25, 145009 (2008) arXiv:0803.2377 [gr-qc].

[49] H. Kitamoto and Y. Kitazawa, Phys. Rev. D 83, 104043 (2011) arXiv:1012.5030 [hep-th].

[50] H. Kitamoto and Y. Kitazawa, Phys. Rev. D 85, 044062 (2012) arXiv:1109.4892 [hep-th].

[51] N. C. Tsamis and R. P. Woodard, Annals Phys. 267, 145 (1998) hep-ph/9712331.
[52] N. C. Tsamis and R. P. Woodard, Phys. Rev. D 80, 083512 (2009) [arXiv:0904.2368 [gr-qc]].

[53] N. C. Tsamis and R. P. Woodard, Phys. Rev. D 81, 103509 (2010) [arXiv:1001.4929 [gr-qc]].

[54] M. G. Romania, N. C. Tsamis and R. P. Woodard, Lect. Notes Phys. 863, 375 (2013) [arXiv:1204.6558 [gr-qc]].

[55] S. Park and R. P. Woodard, Gen. Rel. Grav. 42, 2765 (2010) [arXiv:0910.4756 [gr-qc]].

[56] P. G. Ferreira and A. L. Maroto, Phys. Rev. D 88, 123502 (2013) [arXiv:1310.1238 [astro-ph.CO]].

[57] S. ’i. Nojiri and S. D. Odintsov, Phys. Lett. B 659, 821 (2008) [arXiv:0708.0924 [hep-th]].

[58] S. Deser and R. P. Woodard, Phys. Rev. Lett. 99, 111301 (2007) [arXiv:0706.2151 [astro-ph]].

[59] C. Deffayet, G. Esposito-Farese and R. P. Woodard, “Field Equations and Cosmology for a Class of Nonlocal Metric Models of MOND,” in preparation.

[60] S. Deser and R. P. Woodard, JCAP 1311, 036 (2013) [arXiv:1307.6639 [astro-ph.CO]].

[61] R. P. Woodard, Found. Phys. 44, 213 (2014) [arXiv:1401.0254 [astro-ph.CO]].

[62] C. Deffayet and R. P. Woodard, JCAP 0908, 023 (2009) [arXiv:0904.0961 [gr-qc]].

[63] S. Park and S. Dodelson, Phys. Rev. D 87, 024003 (2013) [arXiv:1209.0836 [astro-ph.CO]].

[64] S. Dodelson and S. Park, [arXiv:1310.4329 [astro-ph.CO]].

[65] M. Milgrom, [arXiv:0801.3133 [astro-ph]].

[66] M. Milgrom, Astrophys. J. 698, 1630 (2009) [arXiv:0810.4065 [astro-ph]].

21
[67] J. D. Bekenstein and E. Sagi, Phys. Rev. D 77, 103512 (2008) [arXiv:0802.1526 [astro-ph]].

[68] L. Parker and D. J. Toms, Phys. Rev. D32 (1985) 1409; T. Banks, Nucl. Phys. B309 (1988) 493; C. Wetterich, Gen. Rel. Grav. 30 (1998) 159, gr-qc/9704052; A. O. Barvinsky, Phys. Lett. B572 (2003) 109, hep-th/0304229; D. Espriu, T. Multamaki and E. C. Vagenas, Phys. Lett. B628 (2005) 197, gr-qc/0503033; H. W. Hamber and R. M. Williams, Phys. Rev. D72 (2005), 044026, hep-th/0507017; T. Biswas, A. Mazumdar and W. Siegel, JCAP 0603 (2006) 009, hep-th/0508194; D. Lopez Nacir and F. D. Mazzitelli, Phys. Rev. D75 (2007) 024003, hep-th/0610031; J. Khoury, Phys. Rev. D76 (2007) 123513, hep-th/0612052; S. Capozziello, E. Elizalde, S. ’i. Nojiri and S. D. Odintsov, Phys. Lett. B671 (2009) 193, arXiv:0809.1535; F. W. Hehl and B. Mashhoun, Phys. Lett. B673 (2009) 279, arXiv:0812.1059; Phys. Rev. D79 (2009) 064028, arXiv:0902.0560; T. Biswas, T. Koivisto and A. Mazumdar, JCAP 1011 (2010) 008, arXiv:1005.0590; Y.-l. Zhang and M. Sasaki, Int. J. Mod. Phys. D21 (2012) 1250006, arXiv:1108.2112; A. O. Barvinsky, Phys. Lett. B710 (2012) 12, arXiv:1107.1463; Phys. Rev. D85 (2012) 104018, arXiv:1112.4340; E. Elizalde, E. O. Pozdeeva and S. Y. Vernov, Phys. Rev. D85 (2012) 044002, arXiv:1110.5806; A. O. Barvinsky and Y. V. Gusev, Phys. Part. Nucl. 44 (2013) 213, arXiv:1209.3062; T. Biswas, A. Conroy, A. S. Koshelev and A. Mazumdar, Class. Quant. Grav. 31 (2013) 015022 (2013), arXiv:1308.2319; P. G. Ferreira and A. L. Maroto, Phys. Rev. D88 (2013) 123502, arXiv:1310.1238; S. Foffa, M. Maggiore and E. Mitsou, arXiv:1311.3421; arXiv:1311.3435; E. O. Pozdeeva and S. Y. Vernov, arXiv:1401.7550; A. Kehagias and M. Maggiore, arXiv:1401.8289; M. Maggiore and M. Mancarella, arXiv:1402.0448; J. F. Donoghue and B. K. El-Menoufi, arXiv:1402.3252; Y. Dirian, S. Foffa, N. Khosravi, M. Kunz and M. Maggiore, arXiv:1403.6068.

[69] A. Aguirre, J. Schaye and E. Quataert, Astrophys. J. 561, 550 (2001) [astro-ph/0105184].

[70] D. Clowe, M. Bradac, A. H. Gonzalez, M. Markevitch, S. W. Randall, C. Jones and D. Zaritsky, Astrophys. J. 648, L109 (2006) [astro-ph/0608407].