Bifurcation Phenomena in Optimal Velocity Model for Traffic Flows

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In the optimal velocity model with a time lag, we show that there appear multiple exact solutions in some ranges of car density, describing a uniform flow, a stable and an unstable congested flows. This establishes the presence of subcritical Hopf bifurcations. Our analytic results have far-reaching implications for traffic flows such as hysteresis phenomena associated with discontinuous transitions between uniform and congested flows.

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The optimal velocity (OV) model [1] for traffic flows attracts considerable interest recently, since it describes congested flows as well as uniform flows. For low mean densities, the uniform flow is realized, while it becomes unstable above some critical density and the congested flow shows up, with high and low density regions appearing alternately. At the critical density, the model predicts the presence of a discontinuity in the correlation diagram between car density and traffic flux, called the fundamental diagram. This discontinuous behavior is consistent with observed data. (In Fig. 9 of [2] their results are compared to an observed data [3].)

Although it is an important consequence of the OV model, the precise origin of the discontinuity has not been identified. The purpose of this letter is to demonstrate the dynamical origin of the discontinuous transitions between uniform and congested flows. We consider in this letter an OV model with a time lag whose exact solutions have been found recently by three of the present authors [5] (see also ref. [6]). We observe that in some ranges of density three exact solutions coexist; a stable uniform flow, a stable and an unstable congested flows. In a numerical simulation one of them is realized depending on its initial condition. Any congested flow, where all cars have the same maximum \(v_{\text{max}}\) as well as the minimum value of velocities \(v_{\text{min}}\), may be characterized by the amplitude in velocity, \(\Delta v = v_{\text{max}} - v_{\text{min}}\). As for the coexisting solutions, \(\Delta v\) of the stable congested flow is larger than that of the unstable flow, while \(\Delta v = 0\) for the uniform flow. These values are functions of mean headway \(h\) and can be used to establish the bifurcation to be of subcritical type. Further the discontinuity in the traffic flux mentioned earlier is interpreted as the hysteresis phenomenon associated with the subcritical bifurcation. The coexistence of stable uniform and congested flows [2] and the hysteresis phenomena [8] were noticed earlier in simulations. However let us emphasize here that the subcritical bifurcation is the key concept to explain properties of the transition between the two flows.

The system of first-order differential-difference equations [8] [9] we consider is given by

\[
\dot{x}_n(t + \tau) = V[\Delta x_n(t)] = \xi + \eta \tanh \left( \frac{\Delta x_n(t) - \rho}{2\sigma} \right),
\]

where \(x_n\) and \(\Delta x_n = x_{n-1} - x_n\) correspond to the position of \(n\)th car and its headway, the distance between the car and the preceding \((n-1)\)th car. \(\tau\) is the time lag to reach the optimal velocity \(V(\Delta x)\) when the traffic flow changes. We impose the periodic boundary conditions, \(x_n+N = x_n-L\), where \(N\) is the total number of cars in a circuit of length \(L\). The OV function \(V(\Delta x)\) is described by a hyperbolic tangent with four parameters, \(\xi, \eta, \rho\) and \(\sigma\).

The trivial solution of the system [8] corresponding to a uniform flow is given by

\[
x_n^{(0)}(t) = V(h) t - n h,
\]

where \(h = L/N\) is the common headway. A linear stability analysis shows that the uniform flow is unstable for the region \(|h - \rho| < H_0\). The critical values \(\rho \pm H_0\) are determined by

\[
\frac{\tau \sin \pi/N}{\sigma} = \cosh^2 \left( \frac{H_0}{2\sigma} \right),
\]

where \(\sigma = \sqrt{\frac{\Delta v_{\text{min}}}{h}}\). The amplitude \(\Delta v_{\text{min}}\) is given by

\[
\Delta v_{\text{min}} = \frac{x_0}{h} \left( 1 - \sin \frac{\pi}{N} \right)
\]

with \(x_0 = L / h\) and \(x_0 \approx L / h = L / (L / N) = N\).

The optimal velocity (OV) model with a time lag is interpreted as the hysteresis phenomenon associated with the subcritical bifurcation. This establishes the presence of subcritical Hopf bifurcations. Our analytic results have far-reaching implications for traffic flows such as hysteresis phenomena associated with discontinuous transitions between uniform and congested flows. PACS numbers: 05.45.Yv, 04.20.Jb, 45.70.Vn, 47.54.+r

The optimal velocity (OV) model [1] for traffic flows attracts considerable interest recently, since it describes congested flows as well as uniform flows. For low mean densities, the uniform flow is realized, while it becomes unstable above some critical density and the congested flow shows up, with high and low density regions appearing alternately. At the critical density, the model predicts the presence of a discontinuity in the correlation diagram between car density and traffic flux, called the fundamental diagram. This discontinuous behavior is consistent with observed data. (In Fig. 9 of [2] their results are compared to an observed data [3].)
where \( \tau_c = \sigma/\eta \). Thus, for \(-H_0 + \rho < h < H_0 + \rho\), an initially prepared uniform flow becomes unstable, and a congested flow develops. As shown in ref. [10], the resulting congested flow is described by an analytic function of the form

\[
x_n(t) = Ct - nh + A \ln \frac{\vartheta_0(\nu t - (2n+1)\beta + \delta, q)}{\vartheta_0(\nu t - (2n+1)\beta - \delta, q)},
\]

where \( A, \beta, \nu, C, \delta \) are constants. \( \vartheta_0(v, q) \) is one of the theta functions with the modulus parameter \( q \). It follows from the periodicity that \( 2\beta = n_b/N \), where an integer \( n_b \) is the number of low density regions in the circuit. The width parameter \( 0 < 2\delta < 1 \) determines the ratio of the length of a low density region to \( L/n_b \). For each length \( L/n_b \), the length of the low density region is given by \( 2\delta L/n_b \), while that of the high density region by \( (1-2\delta)L/n_b \). In this sense, by replacing \( 2\delta \) by \( 1-2\delta \), we exchange the length of two regions.

A crucial point for observing bifurcation is the recognition of rich structure of the exact solution [10]. The stable uniform flows always appear in the region \( H_0 < h \). Unlike this, there must be critical values of headway \( \rho \pm H_1 \) (\( H_1 > H_0 \)) such that the periodic solutions do not exist for \( |h - \rho| > H_1 \): no congested flows are generated if the mean density is too high or too low. Thus, \( h \) can be divided into five regions: \( h > H_1 + \rho \) (I), \( H_0 + \rho < h < H_1 + \rho \) (II), \( -H_0 + \rho < h < H_0 + \rho \) (III), \( -H_1 + \rho < h < -H_0 + \rho \) (IV) and \( -H_1 + \rho > h \) (V).

We have uniform flow solutions over the whole region; They are stable except in the region (III). There are no congested flow solutions in (I) and (V); The stable ones are known to appear in (III). Since the stability of the uniform flows in (II) and (IV) is confirmed against small perturbations only, one cannot exclude the presence of other stable solutions than the uniform flows. We shall show that stable as well as unstable periodic solutions do coexist actually in these regions, and that both are described by [10].

To this end, we discuss the relations of the model parameters \( (\tau, \xi, \eta, \rho, \sigma, N, L) \) in [10] to those in the ansatz [10], \( (A, \nu, \beta, \xi, \eta, \rho, C, h) \). In addition to \( L = Nh \) the following \([10]\) should obey in order for \([10]\) to solve \([10]\):

\[
\begin{align*}
\hbar - \rho &= A \ln \frac{\vartheta_1(2\delta - \beta, q)}{\vartheta_1(2\delta + \beta, q)}, \\
\eta &= \frac{A\beta}{2\tau} \frac{d}{d\beta} \ln \frac{\vartheta_1^2(\beta, q)}{\vartheta_1(2\delta + \beta, q) \vartheta_1(2\delta - \beta, q)}, \\
C - \xi &= -\frac{A\beta}{2\tau} \frac{d}{d\beta} \ln \frac{\vartheta_1(2\delta + \beta, q)}{\vartheta_1(2\delta - \beta, q)}, \\
\sigma &= A, \\
\beta &= \tau \nu.
\end{align*}
\]

Inclusion of the relation \( 2\beta = n_b/N \) with a given \( n_b \) makes seven relations between the above two sets of seven parameters [10]: thus, we may construct exact solutions for a given set of the model parameters. In this paper we restrict ourselves to the case \( n_b = 1 \), for simplicity. (See ref. [10] for solutions with \( n_b \neq 1 \).) The eq.\([10]\) is the Whitham’s dispersion relation [10], an important characteristic of the delayed model. Note that \( h - \rho \) and \( C - \xi \) change their signs while \( \eta \) remains invariant when we replace \( 2\delta \) by \( 1-2\delta \).

We show the presence of multiple solutions for a given set, \( (\tau, \xi, \eta, \rho, \sigma, N, L) \). To obtain the parameters of the solutions, one first fixes \( (A, \nu, \beta, \xi, \eta, \rho, C, h) \) and \( 2\beta N = 1 \). As the modulus parameter, we use \( \kappa = -\pi/(\ln q) \) instead of \( q \). By solving \([10]\), \( \delta \) is computed as a function of \( \kappa \). (As a result of the property of \( \delta \) discussed earlier, it has two branches, \( \delta_1(\kappa) \) and \( \delta_2(\kappa) \) such that \( 2\delta_1(\kappa) = 1-2\delta_2(\kappa) \).) Then substituting \( \delta(\kappa) \) to \([10]\) gives us a function, \( h(\kappa) \). The result is shown in Fig.1.

![Fig. 1. The relation between the mean headway \( h \) and \( \kappa \). The solid and broken lines correspond to stable and unstable solutions, respectively. Here the parameters are \( \tau = 0.58228, \xi = \tanh 2, \eta = 1, \rho = 2, \sigma = 1/2 \) and \( N = 20 \), for which \( H_0 = 0.38978 \) and \( H_1 = 0.56290 \).](image-url)
We discuss now flux-density correlation diagram, the fundamental diagram. The flux \( Q \) may be defined by the number of cars passing through some reference point in a certain time interval. For a uniform flow, it is given by product of the density \( 1/h \) and the common velocity \( V(h) \):

\[
Q = V(h)/h.
\]  

We calculate the flux of a congested flow as follows. Since high density regions are moving backward with velocity \( v_B \), it will be convenient to take the rest frame of the density waves. The period \( T_0 \) defined for a car to make a round in the rest frame is given by

\[
T_0 = \frac{1}{\nu} = 2\tau N,
\]

where we have used the Whitham’s relation, \( \beta = \tau \nu = 1/(2N) \). Since the average velocity of cars in the original coordinate is to be \( L/T_0 - v_B \), the flux \( Q \) may be given by

\[
Q = \frac{1}{h} \left( \frac{L}{T_0} - v_B \right) = \frac{1}{2\tau} - v_B/h.
\]  

Let us calculate \( v_B \) by considering the repetitive spatio-temporal pattern in a congested flow:

\[
x_{n-1}(t) = x_n(t + 2\tau) + 2v_B \tau.
\]  

This implies that the \( x_n \) has the same time dependence as \( x_{n-1} \) apart from a time delay \( 2\tau \) and a distance \( 2v_B \tau \). It follows from (14) and (15) that

\[
v_B = \frac{h}{2\tau} - C.
\]

Therefore, one obtains a simple expression for \( Q \)

\[
Q = C/h.
\]

As discussed previously, one calculates the parameter \( C \) as a function of \( \kappa \) using (7). Together with \( h(\kappa) \), it gives \( C(h) \). Our analytic results are shown in Fig.3: Discontinuities in the traffic flow show up as a result of a hysteresis phenomenon, which is consistent with the observed data qualitatively. Since the size of the discontinuous jumps may be determined analytically, a quantitative comparison is also possible.

A few comments are in order. (1) The presence of the subcritical Hopf bifurcations naturally define the boundaries between (I) and (II), and between (IV) and (V). This should be compared with the Fig. 8 of [2]. (2) In Fig.3, the discontinuities in the low density side appears after passing through the uniform-flow’s peak. Since the position of the peak is determined by the OV function, we may take a set of model parameters so that the discontinuities appear before the peak.
FIG. 3. The fundamental diagram for traffic flow. The dotted line is for uniform flows, while the solid and broken lines are for stable and unstable congested flows, respectively.

In summary, the presence of subcritical Hopf bifurcations in a delayed optimal velocity model which admits exact solutions is established by showing coexistence of stable and unstable solutions. Although our results are obtained in a specific OV model, we believe that any OV type model will show up the subcritical bifurcations. Actually, we have confirmed by numerical simulations the presence of multiple stable solutions for the model given by \( \ddot{x}_n = a[V(\Delta x_n) - \dot{x}_n] \) (First studied in \[7\] in this context). Therefore, our results given here should be considered, at least qualitatively, to be a universal feature of the OV models describing spatio-temporal patterns. They may be of fundamental importance in the area of the traffic engineering. In order to confirm the subcritical bifurcation directly highly desirable are more precise data of the \( h-\Delta v \) relation.

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