SOLUTIONS OF THE KPI EQUATION WITH SMOOTH INITIAL DATA *

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June 23, 1993

Abstract

The solution $u(t, x, y)$ of the Kadomtsev–Petviashvili I (KPI) equation with given initial data $u(0, x, y)$ belonging to the Schwartz space is considered. No additional special constraints, usually considered in literature, as $\int dx u(0, x, y) = 0$ are required to be satisfied by the initial data. The problem is completely solved in the framework of the spectral transform theory and it is shown that $u(t, x, y)$ satisfies a special evolution version of the KPI equation and that, in general, $\partial_t u(t, x, y)$ has different left and right limits at the initial time $t = 0$. The conditions of the type $\int dx u(t, x, y) = 0$, $\int dx xu_y(t, x, y) = 0$ and so on (first, second, etc. ‘constraints’) are dynamically generated by the evolution equation for $t \neq 0$. On the other side $\int dx \int dy u(t, x, y)$ with prescribed order of integrations is not necessarily equal to zero and gives a nontrivial integral of motion.

1 Introduction

We consider the Kadomtsev–Petviashvili equation \[1, 2, 3\] in its version called KPI

$$ \left( u_t - 6uu_x + u_{xxx} \right) x = 3u_{yy}, \quad u = u(t, x, y), \quad (1.1) $$

for $u(t, x, y)$ real. Already in 1974 \[4\] it has been acknowledged to be integrable since it can be associated to a linear spectral problem and, precisely, to the non–stationary Schrödinger equation

$$ (-i\partial_y + \partial_x^2 - u(x, y))\Phi = 0. \quad (1.2) $$

However, to building a complete and coherent theory for the spectral transform of the potential $u(x, y)$ in \(1.2\) that could be used to linearize the initial value problem of \(1.1\)

\[\text{\* Work supported in part by Ministero delle Università e della Ricerca Scientifica e Tecnologica, Italy
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resulted to be unexpectedly difficult. The real breakthrough has been the discover that the problem is solvable via a non local Riemann–Hilbert formulation \[5, 6\].

Successively other progresses have been made. The characterization problem for the spectral data was solved in \[7\]. The extension of the spectral transform to the case of potential \(u(x, y)\) approaching to zero in every direction except a finite number has been given in \[8\]. The questions of the associated conditions (often called ‘constraints’) and how to choose properly \(\partial_{x}^{-1}\) in the evolution form of KPI

\[
 u_t(t, x, y) - 6u(t, x, y)u_x(t, x, y) + u_{xxx}(t, x, y) = 3\partial_{x}^{-1}u_{yy}(t, x, y) \quad (1.3)
\]

have been studied in \[9\]. Some additional relevant points on constraints are known, but not yet published \[10\].

In our opinion, the problem of the proper choice of \(\partial_{x}^{-1}\) in (1.3) requires an additional investigation paying special attention to the behavior of \(u\) and spectral data at the initial time, say at \(t = 0\). This study can now be done since we have at hands some additional theoretical tools in the theory of the spectral transform developed in \[8, 11, 12\].

In the following we will find convenient to consider the problem also in terms of the Fourier transform of \(u\)

\[
 v(t, p) \equiv v(t, p_1, p_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx dy e^{i(p_1 x - p_2 y)} u(t, x, y), \quad (1.4)
\]

where \(p\) denotes a 2–component vector

\[
 p \equiv \{p_1, p_2\}. \quad (1.5)
\]

Then, the evolution equation (1.3) can be rewritten as \((d^2q \equiv dq_1 dq_2)\)

\[
 \frac{\partial v(t, p)}{\partial t} = -3ip_1 \int d^2q v(t, q) v(t, p - q) - i p_1^4 + 3p_2^2 \frac{p_1}{p_1} v(t, p) \quad (1.6)
\]

and we are faced with the problem of properly defining the distribution \(p_1^{-1}\). The integrations here and in the following when it is not differently indicated are performed all along the real axis from \(-\infty\) to \(+\infty\).

According to the usual scheme of the spectral transform theory we expect that the spectral data evolve in time as the linear part of (1.6). This makes clear that the special behavior of the considered quantities at the initial time \(t = 0\) is just complicated by the nonlinearity of the evolution equation but it is, in fact, inherent to the singular character of its linear part.

The final answer is that, for initial data \(u(0, x, y)\) belonging to the Schwartz space which are arbitrarily chosen and, therefore, not necessarily subjected to the constraint

\[
 \int dx u(0, x, y) = 0, \quad \text{or} \quad v(0, 0, p_2) = 0, \quad (1.7)
\]

the function \(u(t, x, y)\) reconstructed by solving the inverse spectral problem for (1.2) evolves in time according to the equation

\[
 u_t(t, x, y) - 6u(t, x, y)u_x(t, x, y) + u_{xxx}(t, x, y) = 3\int_{-t\infty}^{t\infty} dx' u_{yy}(t, x', y). \quad (1.8)
\]
Therefore, for smooth initial data not satisfying constraint (1.7), \( u_t(t, x, y) \) has at \( t = 0 \) different left and right limits and the condition
\[
\int dx \ u(t, x, y) = 0,
\] (1.9)
is dynamically generated by the evolution equation at times \( t \neq 0 \), and thus for these times we recover the result obtained in [4]. On the other side, the initial time \( t = 0 \) requires a special investigation and the evolution form of the KPI equation we deduce in (1.8) is different from that proposed in [3].

In the same way higher conditions ('constraints') can be obtained. But some special care is needed as, e.g., for the next condition we have
\[
\int dx \ y \ u(t, x, y) = 0, \quad t \neq 0,
\] (1.10)
and the \( y \)-derivative cannot be extracted from the integral. In spite of all these facts there exists
\[
\int dx \int dy \ u(t, x, y) = \int \int dx dy u(0, x, y),
\] (1.11)
where the order of integrations in the l.h.s. is explicitly shown. It is clear that this quantity is not necessarily equal to zero and gives a nontrivial integral of motion.

In terms of the Fourier transform \( v(t, p) \) we have
\[
\frac{\partial v(t, p)}{\partial t} = -3ip1 \int d^2q v(t, q) v(t, p - q) - \frac{i p_1^4 + 3p_2^2}{p_1 + i0t} v(t, p)
\] (1.12)
and
\[
\lim_{p_1 \to 0} v(t, p_1, p_2) = 0, \quad \text{for} \quad t \neq 0,
\] (1.13)
where the limit is understood in the sense of distributions in the \( p_2 \) variable.

2 The Linearized Equation

According to the remark made in the introduction, insofar as initial value problem
\[
\partial_x (\partial_t U(t, x, y) + \partial^2_x U(t, x, y)) = 3\partial^2_y U(t, x, y),
\] (2.1)
\[
U(0, x, y) = U(x, y)
\] (2.2)
is concerned, we expect the linear equation and the nonlinear one (1.1) to be closely related. We consider here the case of initial data \( U(x, y) \) belonging to the test function Schwartz space \( S \) in the \( x \) and \( y \) variables.

The study of the existence and uniqueness of the solution of the initial value problem defined in (2.1), (2.2) requires to pay special attention to the behavior of the solution \( U(t, x, y) \) at \( t = 0 \). In fact if this function is continuously differentiable proper number of times in \( x, y \) and \( t \) variables and vanishing with its derivatives for \( (x, y) \to \infty \) at any time, then the initial value problem has no solution if the initial data does not satisfy the constraint (1.7). However we show in the following that if for \( t = 0 \) (and only for \( t = 0 \) \( U_t(t, x, y) \) is allowed to be discontinuous and not decreasing at large \( (x, y) \) then
the solution \( U(t, x, y) \) of the initial value problem exists and is uniquely determined for any initial data \( U(x, y) \in \mathcal{S} \). In addition an evolution form of (2.1) satisfied by \( U(t, x, y) \) at all times including \( t = 0 \) (in the sense of left/right limits) is uniquely determined.

First of all we see that if such \( U(t, x, y) \) exists then, by integrating both sides of (2.1) for any \( t \neq 0 \), we can rewrite (2.1) in the form

\[
\partial_t U(t, x, y) = -\partial_x^2 U(t, x, y) + 3 \int_{-\infty}^{x} dx'' U_{yy}(t, x'', y), \quad t \neq 0,
\]

and derive, again for any \( t \neq 0 \), the condition

\[
\int dx U_{yy}(t, x, y) = 0.
\]

(2.4)

Let \( V(t, p) \) and \( V(p) \) denote the Fourier transforms of \( U(t, x, y) \) and \( U(x, y) \) according to the definition (1.4). Then (2.3) gives

\[
\frac{\partial V(t, p)}{\partial t} = -i \frac{p_1^4 + 3p_2^2}{p_1} V(t, p), \quad t \neq 0
\]

and due to (2.2)

\[
V(0, p) = V(p) \in \mathcal{S}.
\]

(2.6)

Notice that the exact meaning of \( p_1^{-1} \) in (2.5) is irrelevant due to condition (2.4). For any \( p_1 \neq 0 \) the unique solution of the problem (2.3), (2.6) is given as

\[
V(t, p) = \exp \left( -i t \frac{p_1^4 + 3p_2^2}{p_1} \right) V(p).
\]

(2.7)

To proceed further we need the following

**Lemma 1** Function \( \exp(it/p_1) \) defines a distribution, depending continuously on parameter \( \tau \), in the Schwartz space of the variable \( p_1 \).

i) This distribution is continuously differentiable in \( \tau \) for any \( \tau \neq 0 \) :

\[
\partial_\tau \exp \left( \frac{it}{p_1} \right) = \frac{i}{p_1} \exp \left( \frac{it}{p_1} \right), \quad \tau \neq 0,
\]

where the r.h.s. is a well defined distribution in the same space.

ii) At \( \tau = 0 \) there exist right/left limits

\[
\lim_{\tau \to \pm 0} \frac{1}{p_1} \exp \left( \frac{it}{p_1} \right) = \frac{1}{p_1 \mp i0}.
\]

(2.8)

Thus due to (2.8) we can write for arbitrary \( \tau \)

\[
\partial_\tau \exp \left( \frac{it}{p_1} \right) = \frac{i}{p_1 - i0} \tau \exp \left( \frac{it}{p_1} \right).
\]

(2.9)
Proof

i) Let us consider

\[ f(\tau) = \int dp_1 e^{i\tau/p_1}\varphi(p_1), \]  

(2.11)

where \( \varphi(p_1) \in S \). Subtracting and adding terms we can write

\[ f(\tau) = \int dp_1 e^{i\tau/p_1}[\varphi(p_1) - \partial(1- | p_1 |)\varphi(0)] + \varphi(0) \int_{-1}^{1} dp_1 e^{i\tau/p_1}. \]

Changing \( p_1 \) to \( 1/p_1 \) in the second term we have

\[ f(\tau) = \int dp_1 e^{i\tau/p_1}[\varphi(p_1) - \partial(1- | p_1 |)\varphi(0)] + \varphi(0) \int_{|p_1| \geq 1} dp_1 e^{i\tau/p_1}. \]  

(2.12)

Now it is obvious that both terms are differentiable in \( \tau \) for \( \tau \neq 0 \) and

\[ \partial_\tau f(\tau) = i \int \frac{dp_1}{p_1} e^{i\tau/p_1} [\varphi(p_1) - \partial(1- | p_1 |)\varphi(0)] + \varphi(0) \int_{|p_1| \geq 1} \frac{dp_1}{p_1} e^{i\tau/p_1}. \]  

(2.13)

Substituting again \( p_1 \) for \( 1/p_1 \) in the second term we obtain that it cancels out with the second term in brackets. This proves \( i) \). Let us remark that (2.13) shows that for \( \tau \neq 0 \) it is not necessary to regularize the factor \( 1/p_1 \) in front of the exponent in (2.8). One can consider this factor indifferently as principal value or as \((p_1 \pm i0)^{-1}\).

ii) Again subtracting and adding terms we can write

\[ \int \frac{dp_1}{p_1} e^{i\tau/p_1} \varphi(p_1) = \int \frac{dp_1}{p_1} e^{i\tau/p_1} [\varphi(p_1) - \partial(1- | p_1 |)\varphi(0)] + sgn \tau \varphi(0) \int_{|p_1| \geq |\tau|} \frac{dp_1}{p_1} e^{ip_1}, \]

where now in the second term \( p_1 \) was substituted for \( \tau p_1 \). To compute the limit for \( \tau \to 0 \) notice that

\[ \lim_{\tau \to 0} \int_{|p_1| \geq |\tau|} \frac{dp_1}{p_1} e^{ip_1} = i\pi. \]  

(2.14)

Thus

\[ \lim_{\tau \to 0} \int \frac{dp_1}{p_1} e^{i\tau/p_1} \varphi(p_1) = \int \frac{dp_1}{p_1} [\varphi(p_1) - \partial(1- | p_1 |)\varphi(0)] + i\pi \varphi(0) \equiv \int \frac{dp_1}{p_1} \varphi(p_1) + i\pi \varphi(0), \]

where to get the second equality we chose \( p_1^{-1} \) in the integral as the principal value. Then (2.10) follows due to (2.8) and the remark made after the proof of i). □

Applying this Lemma we see that \( V(t, p) \) given in (2.7), for \( V(p) \) belonging to the test–function space \( S \) of Schwartz, is a distribution (in the \( p_1 \) and \( p_2 \) variables) belonging to the dual space \( S' \) and that \( V(t, p) \) obeys for any \( t \) the equation

\[ \frac{\partial V(t, p)}{\partial t} = -i \frac{p_1^2 + 3p_2^2}{p_1 + i0} V(t, p). \]  

(2.15)
Note that the $i0$-term in the denominator is relevant only for $t \to \pm 0$ since for any $t \neq 0$

due to rapid oscillations in (2.7)

\[
\lim_{p_1 \to 0} V(t, p_1, p_2) = 0
\]  

(2.16)

in the sense of distributions in $p_2$. Thus $U_t$ is discontinuous at $t = 0$.

These properties of the distribution

\[
\exp \left( -it \frac{p_1^4 + 3p_2^2}{p_1} \right)
\]  

(2.17)

can be received as well by noting that it can be obtained as a limit in the sense of $S'$

according to the following formula

\[
\exp \left( -it \frac{p_1^4 + 3p_2^2}{p_1} \right) = \lim_{\epsilon \to +0} \exp \left( -it \frac{p_1^4 + 3p_2^2}{p_1 + i\epsilon t} \right).
\]

(2.18)

Then, since in the r.h.s. we have functions belonging to $S$ for $t \neq 0$, we can use the

property that the derivation with respect to $t$ is a continuous operation in the space of

distributions in the variables $t, p_1$ and $p_2$ (cf. [13]).

Coming back to $U(t, x, y)$ by making the inverse Fourier transform

\[
U(t, x, y) = \int \int d^2q \exp \left( -i p_1 x + i p_2 y \right) V(q)
\]

(2.19)

we get that it exists and solves the initial value problem (2.1), (2.2). To study the

properties of this solution it is convenient to extract the nonsingular part of the dispersive

function in the exponent, i.e. the part proportional to $q_3^1$. We introduce

\[
U_0(t, x, y) = \int \int d^2q \exp \left( -i p_1 x + i p_2 y \right) V(q),
\]

(2.20)

which solves the equation

\[
\partial_t U_0(t, x, y) = -\partial_x^2 U_0(t, x, y).
\]

(2.21)

and belongs to $S$ in the $x$ and $y$ variables for any $t$ since $V(p) \in S$. Then from (2.13) we have

\[
U(t, x, y) = \int \int dx' dy' J(t, x - x', y - y') U_0(t, x', y'),
\]

(2.22)

where we have introduced the distribution

\[
J(t, x, y) = \int \int d^2p \exp \left( -i x p_1 + i y p_2 - 3it \frac{p_2^2}{p_1} \right).
\]

(2.23)

By computing the Fourier transform we get

\[
J(t, x, y) = \frac{\text{sgn} t}{\pi} \partial_x (12xt - y^2)^{-1/2},
\]

(2.24)

where the standard definition (cf. [13])

\[
x_+^{-1/2} = \vartheta(x) \frac{1}{\sqrt{x}}
\]

(2.25)
is used. The distribution $J(t, x, y)$ obeys the following properties

$$J(0, x, y) = \delta(x) \delta(y),$$  \hspace{1cm} (2.26)

$$\partial_t J(t, x, y) = 3\partial_y^2 \int_{-t\infty}^{t\infty} dx' J(t, x', y),$$  \hspace{1cm} (2.27)

$$\lim_{t \to \pm 0} \partial_t J(t, x, y) = \pm 3\partial(\pm x) \partial_y^2 \delta(y).$$  \hspace{1cm} (2.28)

Now by means of (2.22) and (2.24) it is easy to check that (2.19) gives the solution of (2.1) obeying all properties mentioned above. To prove that this solution is unique let us notice that any solution of the problem (2.5), (2.6) can differ from the one given in (2.7) only for a distribution localized at $p_1 = 0$, i.e. for terms proportional to $\delta(p_1)$ or its derivatives. Addition of such terms in the r.h.s. of the (2.19) violates the condition of decreasing of $U(t, x, y)$ for large $x$. Thus in the chosen class the solution (2.19) is unique.

Recall that we imposed to the solution to be continuously differentiable in $t$ only for $t \neq 0$. We have therefore to study separately the properties of (2.19) at $t = 0$. Due to (2.22), (2.21), (2.27) and (2.28) we derive that $U(t, x, y)$ satisfies the evolution equation

$$\partial_t U(t, x, y) = -\partial_x^3 U(t, x, y) + 3\partial_y^2 \int_{-t\infty}^{t\infty} dx' U(t, x', y),$$  \hspace{1cm} (2.29)

and that the condition

$$\int dx U(t, x, y) = 0, \quad t \neq 0,$$  \hspace{1cm} (2.30)

is generated dynamically. Of course (2.29) and (2.30) are just the Fourier transforms of (2.15) and (2.16).

In spite of (2.30) integrating (2.22) first in $y$ and then in $x$ we get due to (2.23) and (2.24) that

$$\int dx \int dy U(t, x, y) = V(0) = \int \int dx dy U(x, y)$$  \hspace{1cm} (2.31)

for any $t$, i.e. for this order of integrations the l.h.s. gives an integral of motion, which is not equal to zero in a generic situation.

By means of (2.22), (2.23) and (2.24) we derive that $U(t, x, y)$, for $t \neq 0$ and $y$ fixed, decreases rapidly for $x \to -t\infty$ while for $x \to t\infty$ it satisfies the following asymptotic behavior

$$U(t, x, y) = \frac{-1}{4\pi \sqrt{3tx|x|}} \int \int dx' dy' U(x', y'),$$  \hspace{1cm} (2.32)

$$-\frac{1}{32\pi |t| \sqrt{3txx^2}} \int \int dx' dy' [12tx' + (y - y')^2] U(x', y') + o(|x|^{-5/2}).$$

This asymptotic expansion is differentiable in $t$ and twice differentiable in $y$ and obeys (2.1). Therefore we can differentiate in $t$ the condition (2.30) and use (2.24). This procedure leads to the next condition

$$\int dx x U_{yy}(t, x, y) = 0, \quad t \neq 0.$$  \hspace{1cm} (2.33)
Thanks to the asymptotic behaviour of \( U(t,x,y) \) at large \( x \) derived in (2.32) one \( y \)-derivative can be extracted from the integral getting
\[
\int dx \, x \, U_y(t,x,y) = 0, \quad t \neq 0, \tag{2.34}
\]
but it is impossible to remove the second derivative since \( \int dx \, U(t,x,y) \) is divergent. This procedure can be continued to get an infinite set of dynamically generated conditions.

Equation (2.32) explains the role of constraints, i.e. conditions of the type (2.30) and (2.34) imposed on the initial data \( U(x,y) \), in the asymptotic behaviour of \( U(t,x,y) \) at large \( x \). Precisely, for each additional constraint that is satisfied we get an additional \( x^{-1} \) factor in the decreasing law.

Finally we notice that it is possible to weaken the conditions on the initial data. For example, it is enough for \( V(p_1, p_2) \) to be continuous at point \( p = 0 \) separately in \( p_1 \) and \( p_2 \).

In the nonlinear case (1.1) we will consider solutions \( u(t,x,y) \) that have the same mentioned smoothness properties in the \( x, y \) and \( t \) variables as \( U(t,x,y) \). The possible discontinuity of \( u_t(t,x,y) \) at \( t = 0 \) can be factorized by considering the following representation for the Fourier transform \( \tilde{v}(t,p) \) of \( u(t,x,y) \)
\[
\tilde{v}(t,p) = \exp \left( -it \frac{p_1^2}{p_1} + \frac{3}{2} p_2^2 \right) \tilde{v}(t,p), \tag{2.35}
\]
which can be considered a natural generalization of (2.7). Then \( \tilde{v}(t,p) \) is required to be continuously differentiable also at \( t = 0 \) and continuous at the point \( p = 0 \) separately in \( p_1 \) and \( p_2 \).

3 The Nonlinear Equation

3.1 The direct problem at \( t = 0 \)

The direct problem has been extensively studied in [5, 6, 8, 11, 12]. We report here the main definitions and formulae we need in the following emphasizing specific features of the considered problem.

The Jost solutions \( \Phi(x,y|k) \) are special eigenfunctions of the non–stationary Schrödinger equation (1.2), which are analytic in the upper and lower half plane of the complex spectral parameter \( k \) and whose values \( \Phi^\pm(x,y|k) \) on the two sides \( \pm \Im k > 0 \) of the real \( k \)–axis
\[
\Phi^\pm(x,y|k) = e^{-ikx + ik^2y} \mu^\pm(x,y|k) \tag{3.1}
\]
are determined by the integral equation
\[
\mu^\pm(x,y|k) = 1 + \int dx' \, dy' \, G^\pm(x - x', y - y'|k) \, u(x', y') \, \mu^\pm(x', y'|k), \tag{3.2}
\]
where
\[
G^\pm(x,y|k) = \frac{1}{(2\pi)^2} \int dp \, \frac{e^{-ip_1 x + ip_2 y}}{L(p) - 2p_1 k \mp ip_1} \tag{3.3}
\]
\[ L(p) = p_2 - p_1^2. \]  

The special role played by the spectral parameter \( k \) is stressed by separating it from other variables by a vertical bar. The Green function introduced in (3.3) obviously can be analytically continued in the upper and bottom half planes in \( k \):

\[ G^\pm(x, y|k) = \frac{1}{(2\pi)^2} \int d^2 p \frac{e^{-ip_1 x + ip_2 y}}{L(p) - 2p_1 k} \pm \Im k > 0. \]  

(3.5)

It decreases for \( k \to \infty \) and in what follows we need the first coefficient of its \( 1/k \) expansion. By means of (3.5) we have

\[ \lim_{k \to \infty} kG^\pm(x, y|k) = \lim_{k \to \infty} -\frac{1}{2(2\pi)^2} \int d^2 p \frac{e^{-ip_1 x + ip_2 y}}{p_1 \pm i0p_2} = \frac{1}{4\pi} \left( \mp \frac{1}{y} + i\pi \text{sgn} x \delta(y) \right), \quad \pm \Im k > 0. \]  

(3.6)

Thus we see that the term in the expansion which is nonlocal in \( y \) explicitly depends on the half plane of \( k \) in which the limit is performed. Correspondingly for the first two terms in the \( 1/k \) expansion of \( \mu \) we have

\[ \mu^\pm(x, y|k) = 1 + \frac{1}{4\pi k} \left( \mp \int \frac{dx' dy'}{y - y'} u(x', y') + i\pi \int dx' \text{sgn}(x - x') u(x', y) \right) + o\left( \frac{1}{k} \right), \quad k \to \infty, \quad \pm \Im k > 0, \]  

(3.7)

which also demonstrates the dependence on the half plane.

The spectral data are defined as

\[ r^\pm(\alpha, \beta) = \frac{1}{(2\pi)^2} \int dx dy e^{i\alpha(x - \beta) - iy(\alpha^2 - \beta^2)} u(x, y) \mu^\pm(x, y|\beta). \]  

(3.8)

Note that, due to specific dependence of the exponent in (3.8) on \( \alpha \) and \( \beta \), spectral data decrease for large values of the arguments only if \( \alpha - \beta \) or \( \alpha^2 - \beta^2 \) increase. It means that if we consider for some fixed parameter \( \gamma \)

\[ r^\pm \left( \beta + \frac{\gamma}{\beta}, \beta \right) = \frac{1}{(2\pi)^2} \int dx dy e^{ix/y} \left( 2\gamma^2 + \frac{x^2}{\beta^2} \right) u(x, y) \mu^\pm(x, y|\beta), \]  

then, since \( \mu(x, y|\beta) \to 1 \) for \( \beta \to \infty \),

\[ \lim_{\beta \to \infty} r^\pm \left( \beta + \frac{\gamma}{\beta}, \beta \right) = \frac{1}{(2\pi)^2} \int dx dy e^{-2iy\gamma} u(x, y), \]  

(3.10)

which is not necessarily equal to zero. This remark will be crucial in the following.

We need in the following to consider the counterpart of all these formulae in the Fourier transformed space. The Fourier transformed Jost solution is defined as

\[ \nu^\pm(p|k) = \int dx dy e^{ip_1 x - ip_2 y} \mu^\pm(x, y|k), \quad \pm \Im k > 0, \]  

(3.11)
and satisfies the integral equation

\[
\nu^\pm(p|k) = \delta^2(p) + \frac{1}{\mathcal{L}(p) - 2p_1 k \mp i0 p_1} \int d^2 p' v(p - p') \nu^\pm(p'|k),
\]

(3.12)

where notation (3.4) has been used and \(\delta^2(p) = \delta(p_1) \delta(p_2)\). From this equation the expansion in \(1/k\) can be easily derived. For following use we write it up to the third term:

\[
\nu^\pm(p|k) = \delta^2(p) - \frac{1}{2k} \frac{v(p)}{p_1 \pm i0 p_2} - \frac{1}{(2k)^2} \left\{ \frac{\mathcal{L}(p) v(p)}{(p_1 \pm i0 p_2)^2} - \frac{1}{p_1 \pm i0 p_2} \int d^2 p' \frac{v(p - p') v(p')}{p_1' \pm i0 p_2'} \right\} + o\left(\frac{1}{k^2}\right),
\]

\(\pm \Im k > 0, k \to \infty\).

Formula (3.10) reads

\[
\lim_{\beta \to \infty} r^\pm(\beta + \gamma, \beta) = v(0, 2\gamma).
\]

(3.14)

If we introduce the function

\[
\rho^\pm(p|k) = [\mathcal{L}(p) - 2p_1 k] \nu^\pm(p|k),
\]

(3.15)

the spectral data (3.8) can be written as

\[
r^\pm(\alpha, \beta) = \rho^\pm(\ell(\alpha) - \ell(\beta)|\beta),
\]

(3.16)

or

\[
r^\pm(\alpha, \beta) = \int d^2 q v(q) \nu^\pm(\ell(\alpha) - q|\beta),
\]

(3.17)

where the special two component vector

\[
\ell(k) = (k, k^2)
\]

(3.18)

has been introduced. Note that for \(u\) real (as we are considering) \(r^\pm(\alpha, \beta) = r^{\mp}(\beta, \alpha)\).

For more details the reader can refer to [12], where a slightly different notation has been used. Precisely one has to make the following identifications

\[
\nu^\pm(p|k) \equiv \nu^\pm(p, \ell(k))
\]

(3.19)

\[
\rho^\pm(\ell(\alpha) - \ell(\beta)|\beta) \equiv \rho^\pm(\ell(\alpha) - \ell(\beta), \ell(\beta)).
\]

(3.20)

### 3.2 The inverse problem at \(t = 0\)

The inverse problem can be solved equivalently in the \((x, y)\) space or in the Fourier transformed \((p_1, p_2)\) space. However, we prefer to work in the \(p\)-space because, as we will see in the following section, the time evolution can be more easily handled.
We start from the known fact that the advanced/retarded eigenfunctions $\nu_{\pm}(p|k)$ of the non–stationary Schrödinger equation can be indifferently related to the Jost solution $\nu^\sigma(p|k)$ for $\sigma = +$ or for $\sigma = -$ according to the formula

$$\nu_{\pm}(p|k) = \nu^\sigma(p|k) \mp 2i\pi \int d\beta \theta(\pm \sigma(\beta - k)) r^{-\sigma}(k, \beta) \nu^\sigma(p + \ell(k) - \ell(\beta)|\beta), \quad \sigma = +, -,$$

(3.21)

i.e. the l.h.s. is independent on the sign of $\sigma$ (for a proof see [8, 11, 12, 14]). From (3.21) we deduce that also the sum $\nu_+(p|k) + \nu_-(p|k)$ is $\sigma$ independent or that

$$\nu^\sigma(p|k) + i\pi \int d\beta \operatorname{sgn}(k - \beta) r^{-\sigma}(k, \beta) \nu^\sigma(p + \ell(k) - \ell(\beta)|\beta) = (\sigma \to -\sigma).$$

(3.22)

Due to this the non–local Riemann–Hilbert problem for the discontinuity of the Jost solution across the real $k$–axis (see also [12]) is given by the equation

$$\nu^+(p|k) - \nu^-(p|k) = -i\pi \sum_{\sigma = +, -} \int d\beta \operatorname{sgn}(k - \beta) r^{-\sigma}(k, \beta) \nu^\sigma(p + \ell(k) - \ell(\beta)|\beta), \quad (3.23)$$

that has to be compared with the usual one in literature which is quadratic in the spectral data.

From the analytic properties of $\nu^\pm$ and the Cauchy formula we have that $\nu^\pm$ can be reconstructed from the spectral data $\nu^\pm(k, \beta)$ by solving the singular linear integral equation

$$\nu^\pm(p|k) = \delta^2(p) - \frac{d\alpha}{\alpha - k \mp i0} \sum_{\sigma = +, -} \int d\beta \frac{\operatorname{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta)|\beta).$$

(3.24)

One can use the explicit $k$–dependence in the r.h.s. to get a $1/k$ expansion at large $k$ of $\nu^\pm(p, k)$. However, before expanding the denominator in the r.h.s. it is necessary to perform a subtraction in the integration over $\alpha$. In fact, due to (3.14) and (3.13), we have (for any sign of $\alpha$)

$$\lim_{\alpha \to \infty} \alpha \int d\beta \frac{\operatorname{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta), \beta) = \delta(p_1) \frac{\operatorname{sgn} p_2}{2} v(p),$$

(3.25)

where to get the second line the substitution $\beta \to \alpha - \beta/\alpha$ has been performed. This proves that a direct expansion would furnish a divergent result.

Therefore, we rewrite (3.24) as

$$\nu^\pm(p|k) = \delta^2(p) - \frac{d\alpha}{\alpha - k \mp i0} \left\{ \sum_{\sigma = +, -} \int d\beta \frac{\operatorname{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta)|\beta) + \frac{\delta(p_1)}{2(\alpha \mp i0) \operatorname{sgn} p_2 v(p)} \right\}.$$

(3.26)
The additional term corrects the bad behavior of the integrand at large $\alpha$ and does not modify the total value of the r.h.s. of (3.24) since, once integrated over $\alpha$, gives a zero contribution.

Now, we can compute the first term in the expansion at large $k$ and we get
\[
\lim_{k \to \infty} k [\nu^\pm(p|k) - \delta^2(p)] = \int d\alpha \left\{ \sum_{\sigma = +, -} \int d\beta \frac{\text{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta) | \beta) + \frac{\delta(p_1)}{2(\alpha \mp i0)} \text{sgn} p_2 v(p) \right\}, \quad \pm \Im k > 0.
\]

Then, by using (3.13), we obtain
\[
\frac{v(p)}{p_1 \pm i0 p_2} = \int d\alpha \left\{ \sum_{\sigma = +, -} \int d\beta \frac{\text{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta) | \beta) + \frac{\delta(p_1)}{\alpha \mp i0} \text{sgn} p_2 v(p) \right\},
\]
or, taking into account that the delta distributions coming from the $\pm i0$ terms cancel out, we get
\[
\frac{v(p)}{p_1} = -\int d\alpha \left\{ \sum_{\sigma = +, -} \int d\beta \frac{\text{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta) | \beta) + \frac{\delta(p_1)}{\alpha} \text{sgn} p_2 v(p) \right\},
\]
where $1/p_1$ and $1/\alpha$ have the sense of principal value.

Finally we solve the inverse problem by reconstructing the potential $v(p)$ in terms of the spectral data and Jost solutions according to the following formula
\[
v(p) = -\int d\alpha \frac{1}{p_1} \sum_{\sigma = +, -} \int d\beta \text{sgn}(\alpha - \beta) r^{-\sigma}(\alpha, \beta) \nu^\sigma(p + \ell(\alpha) - \ell(\beta) | \beta),
\]
where we used that $p_1 \delta(p_1) = 0$. Note that due to this fact $p_1$ cannot be extracted from the integral.

### 3.3 The time evolution

The time evolution in (3.24) and (3.30) is switched on by choosing spectral data that depend parametrically on the time as follows
\[
r^\sigma(t, \alpha, \beta) = e^{-4it(\alpha^3 - \beta^3)} r^\sigma(\alpha, \beta).
\]

We need, then, to compute directly from
\[
\nu^\pm(t, p|k) = \delta^2(p) - \int \frac{d\alpha}{\alpha - k \mp i0} \sum_{\sigma = +, -} \int d\beta e^{4it(\alpha^3 - \beta^3)} \frac{\text{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \nu^\sigma(t, p + \ell(\alpha) - \ell(\beta) | \beta)
\]
the evolution equation satisfied by the Jost solution $\nu^\pm(t, p|k)$, i.e. the second equation of the Lax pair. We expect it to have the form of the Fourier transform of the standard one

$$\frac{\partial}{\partial t} \nu^\sigma(t, p|k) = -2i[3(p_1 + k)\mathcal{L}(p) + 2p_1^3] \nu^\sigma(t, p|k) - 3i \int d^2p' \mathcal{L}(p') \frac{v(t, p')}{p_1' \nu^\sigma(t, p - p'|k)}, \quad t \not= 0,$$

where possibly the sense of $1/p_1'$ must be specified. Then we have to prove that

$$v(t, p) = -\int d\alpha \, p_1 \sum_{\sigma = +, -} \sigma \int d\beta \, e^{4it(\alpha^3 - \beta^3)} \text{sgn}(\alpha - \beta) \frac{r^{-\sigma}(\alpha, \beta)}{r^{-\sigma}(\alpha, \beta)} \nu^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta)$$

satisfies the Fourier transform of the KPI equation. Note that for $t \not= 0$ the oscillating term $\exp[4it(\alpha^3 - \beta^3)]$ improves the behavior at large $\alpha$ of the integrand. Consequently, the potential $v(t, p)$ satisfies the condition

$$\lim_{p_1 \to 0} v(t, p) = 0 \quad \text{for} \quad t \not= 0 \quad (3.35)$$

in the sense of distribution in $p_2$.

Taking into account the asymptotic behavior in (3.25) we see that differentiation of (3.34) under the sign of integral can give a divergent result at $t = 0$. The main advantage of the use of the Fourier transformed Jost solutions $\nu^\pm(t, p|k)$ is that they give an explicit possibility to remove the rapidly oscillating terms, given by the ‘free’ part of (3.33). Indeed, let us introduce the new functions

$$\tilde{\nu}^\pm(t, p|k) = e^{2it[3(k + p_1)|\mathcal{L}(p) + 2p_1^3]} \nu^\pm(t, p|k). \quad (3.36)$$

The kernel of the integral equation for $\tilde{\nu}^\pm(t, p|k)$ does not contain a term oscillating in time depending on $\beta$. In fact, by using the identity

$$3(k + p_1)\mathcal{L}(p) + 2p_1^3 + 2(\alpha^3 - \beta^3) = 3(p_1 + \alpha)\mathcal{L}(p + \ell(\alpha) - \ell(\beta)) + 2(p_1 + \alpha - \beta)^3 + 3\mathcal{L}(p)(k - \alpha), \quad (3.37)$$

we can rewrite (3.32) in the form

$$\tilde{\nu}^\pm(t, p|k) = \delta^2(p) - \int d\alpha \, \frac{\delta^{(k+1)\mathcal{L}(p)|(k-\alpha)}}{\alpha - k + i0} \sum_{\sigma = +, -} \int d\beta \, \frac{\text{sgn}(\alpha - \beta)}{2} r^{-\sigma}(\alpha, \beta) \tilde{\nu}^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta). \quad (3.38)$$

In order to get the evolution equation for $\tilde{\nu}^\pm(t, p|k)$ we have to derive this equation with respect to the time and exploit the exponential time dependence in the r.h.s. by exchanging the time derivative with the integration over $\alpha$. This can be done for $t \not= 0$ but at $t = 0$ it is forbidden because of the following behavior at large $\alpha$

$$\lim_{\alpha \to \infty} \frac{\delta(p_1)}{4} \text{sgn} p_2 v(0, p), \quad (3.39)$$
that can be easily proved in analogy with (3.23) taking into account that the property
\( \tilde{\nu}^\pm(t, p, k) \rightarrow \delta^2(p) \) for \( k \rightarrow \infty \) holds for any \( t \). Note that this limit is time independent
and determined by the initial data \( u(0, x, y) \).

In conclusion we rewrite (3.32) as
\[
\tilde{\nu}^\pm(t, p|k) = \delta^2(p) - \int \frac{d\alpha e^{\imath \theta_{(\mp tL(p))}(\alpha - k \mp i0)}}{2(\alpha - k \mp i0)} \left\{ \sum_{\sigma = +, -} \int d\beta \ \text{sgn}(\alpha - \beta) \frac{r^{-\sigma}(\alpha, \beta)}{r^{-\sigma}(\alpha, \beta)} \tilde{\nu}^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta) + \frac{1}{\alpha \mp i0} \delta(p_1) \ sgn \ p_2 \ v(0, p) \right\} \mp i\pi \delta(p_1) \ v(0, p) \ sgn \ p_2 \ \theta(\mp tL(p)) \frac{\ e^{\imath \theta(\mp tL(p))} - 1}{k}.
\]

The last term compensates the subtracted term since
\[
\int \frac{d\alpha e^{\imath \theta_{(\mp tL(p))}(\alpha - k \mp i0)}}{(\alpha \mp i0)(\alpha - k \mp i0)} = \mp 2i\pi \theta(\mp tL(p)) \frac{\ e^{\imath \theta_{(\mp tL(p))}} - 1}{k}.
\]

3.3.1 The first operator in the Lax pair

The non–stationary Schrödinger equation (1.2) in the Fourier transformed space (see (3.1) and (3.11)) reads
\[
\mathcal{L}(p) - 2p_1k \ v(t, p|k) = \int d^2p' \ v(t, p - p') \ v(t, p'|k).
\]

We want to prove, now, that the function \( \nu^\pm \) solution of (3.24) obeys (3.42) with the potential \( v(t, p) \) defined in (3.34). Let us rewrite (3.40) in terms of the function
\[
\tilde{\rho}^\pm(t, p|k) = \mathcal{L}(p) - 2p_1k \ \tilde{\nu}^\pm(t, p|k),
\]
that, due to (3.30), is related to the function \( \rho \) introduced in (3.15) by
\[
\tilde{\rho}^\pm(t, p|k) = e^{2\imath t[\mathcal{L} + 2p_1\mathcal{L}(p)]} \rho^\pm(t, p|k).
\]

By using the identity
\[
\mathcal{L}(p) - 2p_1k = \mathcal{L}(p + \ell(\alpha) - \ell(\beta)) - 2(p_1 + \alpha - \beta)\beta - 2p_1(k - \alpha)
\]
we get
\[
\tilde{\rho}^\pm(t, p|k) = - \int \frac{d\alpha e^{\imath \theta_{(\mp tL(p))}(\alpha - k \mp i0)}}{2(\alpha - k \mp i0)} \left\{ \sum_{\sigma = +, -} \int d\beta \ \text{sgn}(\alpha - \beta) \frac{r^{-\sigma}(\alpha, \beta)}{r^{-\sigma}(\alpha, \beta)} \tilde{\rho}^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta) + \frac{1}{\alpha \mp i0} \delta(p_1) \ |p_2| \ v(0, p) \right\} + i\pi \delta(p_1) \ v(0, p) \ |p_2| \ \theta(\mp tL(p)) \frac{\ e^{\imath \theta(\mp tL(p))} - 1}{k} - p_1 \int \frac{d\alpha e^{\imath \theta_{(\mp tL(p))}(\alpha - k \mp i0)}}{2(\alpha - k \mp i0)} \left\{ \sum_{\sigma = +, -} \int d\beta \ \text{sgn}(\alpha - \beta) \frac{r^{-\sigma}(\alpha, \beta)}{r^{-\sigma}(\alpha, \beta)} \tilde{\nu}^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta) + \frac{1}{\alpha \mp i0} \delta(p_1) \ sgn \ p_2 \ v(0, p) \right\},
\]

(3.46)
Using again (3.43) and (3.44) and the definition of the potential in (3.34) we have

\[ \rho^\pm(t, p|k) = v(t, p) - \int \frac{\, d\alpha}{\, 2(\alpha - k \mp i0)} \sum_{\sigma = +, -} \int d\beta \, e^{\, i\text{it}(\alpha^3 - \beta^3)} \, \text{sgn}(\alpha - \beta) \, r^{-\sigma}(\alpha, \beta) \rho^\sigma(t, p + \ell(\alpha) - \ell(\beta)|\beta). \]

Comparing this integral equation for \( \rho^\pm(t, p|k) \) with the integral equation (3.32) we deduce:

\[ \rho^\pm(t, p|k) = \int d^2p' \, v(t, p - p') \, \nu^\pm(t, p'|k) \quad (3.47) \]

that, due to (3.15), is just the Fourier transformed non–stationary Schrödinger equation (3.42).

### 3.3.2 The second operator in the Lax pair

As we already noted, for computing the time derivative at \( t = 0 \) of \( \tilde{\nu}^\sigma(t, p|k) \) it is necessary to use the subtracted equation (3.40). We get different left and right limits

\[
\frac{\partial \tilde{\nu}^\sigma}{\partial t}(\pm 0, p|k) + \sum_{\sigma' = +, -} \int \frac{\, d\alpha}{\, 2(\alpha - k - i0\sigma)} \int d\beta \, \text{sgn}(\alpha - \beta) \, r^{-\sigma}(\alpha, \beta) \frac{\partial \tilde{\nu}^{\sigma'}}{\partial t}(\pm 0, p + \ell(\alpha) - \ell(\beta)|\beta) = 3iL(p) \left\{ \int d\alpha \left[ \sum_{\sigma' = +, -} \int d\beta \, \text{sgn}(\alpha - \beta) \, r^{-\sigma}(\alpha, \beta) \nu^{\sigma'}(\pm 0, p + \ell(\alpha) - \ell(\beta)|\beta) + \frac{1}{\alpha - i0\sigma} \delta(p_1) \text{sgn}(p_2) \nu(0, p) \right] - 2i\pi \sigma \delta(p_1) \theta(\mp p_2\sigma) \nu(0, p) \right\}.
\]

The formula in curl brackets can be expressed in terms of \( \nu(0, p) \) by using (3.28) and we have

\[
\frac{\partial \tilde{\nu}^\sigma}{\partial t}(\pm 0, p|k) + \sum_{\sigma' = +, -} \int \frac{\, d\alpha}{\, 2(\alpha - k - i0\sigma)} \int d\beta \, \text{sgn}(\alpha - \beta) \, r^{-\sigma}(\alpha, \beta) \frac{\partial \tilde{\nu}^{\sigma'}}{\partial t}(\pm 0, p + \ell(\alpha) - \ell(\beta)|\beta) = -3iL(p) \frac{\nu(0, p)}{p_1 \mp i0} \quad (3.48)
\]

Comparing this integral equation with (3.24) we deduce that

\[
\frac{\partial}{\partial t} \nu^\sigma(\pm 0, p|k) = -2i[3(p_1 + k)L(p) + 2p_1^2] \nu^\sigma(\pm 0, p|k) - 3i \int d^2p' \, L(p') \frac{\nu(0, p')}{p_1' \mp i0} \nu^\sigma(\pm 0, p - p'|k).
\]

For \( t \neq 0 \) the time derivative of \( \nu^\sigma(t, p|k) \) can be directly computed by using (3.32). We get (3.33), where the distribution \( 1/p_1' \) does not need to be specified as for these \( t \) the
potential satisfies the condition (3.33). Thus we proved that the reconstructed \( \nu^\sigma(t, p|k) \) satisfies the evolution equation

\[
\frac{\partial v}{\partial t}(t, p|k) = -2i[3(p_1 + k)\mathcal{L}(p) + 2p_1^4]\nu^\sigma(t, p|k) - 3i\int d^2 p' \mathcal{L}(p') \frac{v(t, p')}{p_1^2 + i\nu} v^\sigma(t, p - p'|k).
\] (3.51)

3.3.3 The evolution version of KPI

To derive the evolution equation for the reconstructed \( v(t, p) \) we substitute in (3.51) the expansion (3.13) for large \( k \) and omit terms of order \( 1/k^2 \). Again it is necessary to consider separately the case \( t = 0 \). Noting in (3.13) \( \text{sgn}\, 3k = \sigma \) and multiplying the result by \( k \) we get

\[
\frac{1}{p_1 + i\sigma 0 p_2}\frac{\partial v}{\partial t}(\pm 0, p) =
\]

\[
6ik\mathcal{L}(p) v(0, p) \left\{ \frac{1}{p_1 \pm i0} - \frac{1}{p_1 + i\sigma 0 p_2} \right\} - \frac{3i\mathcal{L}(p)^2 + 6p_1^2\mathcal{L}(p) + 4p_1^4}{(p_1 + i\sigma 0 p_2)^2} v(0, p) +
\]

\[
\frac{3i\mathcal{L}(p)}{p_1 + i\sigma 0 p_2} \int d^2 p' \frac{v(0, p - p') v(0, p')}{p_1^2 \pm i0} - 3i \int d^2 p' \mathcal{L}(p') \frac{v(0, p - p') v(0, p')}{(p_1 \pm i0) (p_1 - p_1^1 + i\sigma 0 p_2)}.
\]

To compensate the terms of first order in \( k \) for \( k \to \infty \) we have to chose

\[
\sigma = \pm \text{sgn} p_2.
\] (3.53)

It means that the limits \( k \to \infty \) and \( t \to 0 \) commute only if this condition is satisfied. Notice that in this case the last term in (3.40) is absent. Thus

\[
\frac{\partial v}{\partial t}(\pm 0, p) = -i \frac{3p_2^2 + p_1^4}{p_1 \pm i0} v(0, p) +
\]

\[
3i \int d^2 p' [\mathcal{L}(p) - \mathcal{L}(p') - \mathcal{L}(p - p')] \frac{v(0, p - p') v(0, p')}{p_1^2 \pm i0}
\]

which due to the identity

\[
\mathcal{L}(p) - \mathcal{L}(p') - \mathcal{L}(p - p') = 2p_1^1 (p_1^1 - p_1)
\] (3.55)

takes the form

\[
\frac{\partial v}{\partial t}(\pm 0, p) = -i \frac{3p_2^2 + p_1^4}{p_1 \pm i0} v(0, p) - 6i \int d^2 p' (p_1 - p_1^1) v(0, p - p') v(0, p').
\] (3.56)

For \( t \neq 0 \) it is not necessary to use the subtracted form for the reconstructed \( v(t, p) \). Finally we deduce that \( v(t, p) \) satisfies the evolution equation

\[
\frac{\partial v}{\partial t}(t, p) = -i \frac{3p_2^2 + p_1^4}{p_1 + t i0} v(t, p) - 6i \int d^2 p' (p_1 - p_1^1) v(t, p - p') v(t, p').
\] (3.57)

which is the Fourier transform of the evolution equation

\[
u(t, x, y) - 6u(t, x, y)\partial_x u(t, x, y) + u_{xxx}(t, x, y) = 3 \int_{-\infty}^{\infty} dx' u_{yy}(t, x', y).
\] (3.58)
Acknowledgements  A. P. thanks A. Fokas and M. Ablowitz for fruitful discussions and his colleagues at Dipartimento di Fisica dell’Università di Lecce for hospitality.

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