SEMI-NEGATIVITY OF HODGE BUNDLES ASSOCIATED TO DU BOIS FAMILIES

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ABSTRACT. In this note we show that the sheaves $R^if_*\mathcal{O}_X$ are anti-nef vector bundles (i.e., their duals are nef) for $i < d$, where $f : X \to Y$ is a family of Du Bois schemes of pure dimension $n$ with $S_d$ fibers. Further, in the case of $d = n$ we may allow also $i$ to be equal to $d$.

1. INTRODUCTION

In this note we show that the sheaves $R^if_*\mathcal{O}_X$ are anti-nef vector bundles (i.e., their duals are nef) for $i < d$, where $f : X \to Y$ is a family of Du Bois schemes of pure dimension $n$ with $S_d$ fibers. Further, in the case of $d = n$ we may allow also $i$ to be equal to $d$. Note that by [KK10], $R^if_*\mathcal{O}_X$ is known to be a vector bundle, so our contribution is proving anti-nefness. This statement is the generalization of the classical result stating that if $f$ is smooth, then the Hodge metric on $R^if_*\mathcal{O}_X$ has semi-negative curvature [Gri70]. For the definition and properties of Du Bois singularities we refer to [KS11b], and here we only note that they can be viewed as the largest class of singularities where vanishing theorems hold [Kol95, 9.12, 12.7].

We should also point out that the semi-negativity of $R^if_*\mathcal{O}_X$ in this case is related but not equivalent to the widely investigated semi-positivity of $R^{n-i}f_*\omega_{X/Y}^n$ (e.g., [FF12, Theorem 1.4]). In fact, the semi-negativity of $R^if_*\mathcal{O}_X$ is equivalent in this case to the semi-positivity of $R^{-i}f_*\omega^*_{X/Y}$, which sheaf is not equal to $R^{n-i}f_*\omega_{X/Y}$. (Recall that $\omega_{X/Y}$ is the $-n$-th cohomology sheaf of $\omega^*_{X/Y}$, as defined in Section 1.1)

Theorem 1.1. If $f : X \to Y$ is a flat, projective family of connected, Du Bois, $S_d$ schemes of pure dimension $n$ over $\mathbb{C}$ for some $n \geq d \geq 2$, then $R^if_*\mathcal{O}_X$ is an anti-nef or equivalently $R^{-i}f_*\omega^*_{X/Y}$ is a nef vector bundle for every $i < d$. Furthermore, if $d = n$, then nefness holds also for $i = n$.

Remark 1.2. One would be tempted to use directly the available semipositivity results for reducible fiber spaces [FF12], [Kaw11] to prove Theorem 1.1. However, the author does not see a way of doing it, due to certain assumptions on the strata and monodromies in [FF12] and [Kaw11]. Instead, we use an injectivity theorem for Du Bois schemes.

The two main ingredients in proving Theorem 1.1 are the following. First, we show Theorem 1.3 and Corollary 1.4 in Section 3. Note that Theorem 1.3 was shown in [Kol95, Thm 9.12] for normal schemes. Though we believe the arguments of [Kol95, Thm 12.10] can be generalized to non-normal schemes, for the convenience of the reader we include a different proof here.

Theorem 1.3. If $X$ is a projective, Du Bois scheme, $N > 0$ an integer, $\mathcal{L}$ a line bundle on $X$, such that $\mathcal{L}^N$ is globally generated and $F$ a general effective divisor of $\mathcal{L}^N$, then the natural map

$$H^i(X, \omega^*_X \otimes \mathcal{L}) \to H^i(X, \omega^*_X \otimes \mathcal{L}(F))$$

is injective.

Corollary 1.4. Let $f : X \to Y$ be a flat, projective Du Bois family over a smooth projective curve, $y_0 \in Y$ and $N > 0$ such that $|N^X_{y_0}|$ is base-point free. Then for any $i$, $R^if_*(\omega^*_{X/Y} \otimes \omega_Y((N+1)y_0))$ is generically globally generated.
Second, in Section 4, we show the following decomposition result, in the spirit of the celebrated article of Kollár [Kol86].

**Theorem 1.5.** Let \( n \geq d \geq 2 \) be arbitrary integers and \( f : X \to Y \) a flat projective morphism with connected fibers, such that \( X \) is a reduced scheme of pure dimension \( n \) and \( Y \) a smooth curve. Furthermore, assume that \( X \) is \( S_d \). Then

\[
Rf_* \omega^*_X \cong R^{\leq -d} f_*(\omega^*_X) \oplus \bigoplus_{i > -d} R^i f_* \omega^*_X[-i].
\]

### 1.1. Notation

The base field is the field of complex numbers \( \mathbb{C} \). For a complex \( \mathcal{C}^\bullet \) of sheaves, \( h^i(\mathcal{C}^\bullet) \) is the \( i \)-th cohomology sheaf of \( \mathcal{C} \). For a morphism \( f : X \to Y \), \( \omega^*_X/Y := f^! \mathcal{O}_Y \), where \( f^! \) is the functor obtained in [Har66, Corollary VII.3.4.a]. If \( f \) has equidimensional fibers of dimension \( n \), then \( \omega^*_X/Y := h^{-n}(\omega^*_X/Y) \). Every complex and morphism of complexes is considered in the derived category \( D(qc/\mathbb{C}) \) of quasi-coherent sheaves up to the equivalences defined there.

## 2. The proof of semi-positivity

Since nefness is checked on curves, proving Theorem 1.1 for curve base turns out to be the main issue. This is proved in Proposition 2.1, assuming Corollary 1.4 and Theorem 1.5, which will be showed in Section 3 and 4, respectively. We conclude this section with the (short) proof of Theorem 1.1 using Proposition 2.1.

**Proposition 2.1.** If \( f : X \to Y \) is a flat, projective family of connected, Du Bois and \( S_d \) schemes of pure dimension \( n \) for some \( n \geq d \geq 2 \) over a smooth, projective curve, then \( R^if_* \mathcal{O}_X \) is an anti-nef or equivalently \( R^{-i} f_* \omega^*_X/Y \) is a nef vector bundle for every \( i < d \). Furthermore, if \( d = n \), then nefness holds also for \( i = n \).

We will prove Proposition 2.1 at the end of this section, after listing a few lemmas.

**Lemma 2.2.** If \( f : X \to Y \) is a flat, projective family with Du Bois fibers, then \((R^i f_* \mathcal{O}_X)^* \cong R^{-i} f_* \omega^*_X/Y^*\).

**Proof.** By [KK10, Theorem 7.8], \( R^i f_* \mathcal{O}_X \) is locally free. Hence the following computation concludes our proof.

\[
R^{-i} f_* \omega^*_X/Y \cong R^{-i} f_* R\mathcal{H}om_X(\mathcal{O}_X, \omega^*_X/Y) \cong R^{-i} \mathcal{H}om_Y(Rf_* \mathcal{O}_X, \mathcal{O}_Y) \cong (R^i f_* \mathcal{O}_X)^*.
\]

Since \( \omega^*_X/Y \) is the main object of Proposition 2.1 for fibrations \( X \to Y \) that are not necessarily Cohen-Macaulay, we need the following technical lemma. The most important consequence is stated in Lemma 2.4, a formula relating the relative and absolute dualizing complexes. It turns out that, at least over Gorenstein bases, nothing surprising happens.

**Lemma 2.3.** If \( f : X \to Y \) is a flat, projective morphism between projective schemes, then for every \( \mathcal{C}^\bullet \in D(X) \),

\[
f^!(\mathcal{C}^\bullet) \cong Lf^*(\mathcal{C}^\bullet) \otimes_L f^! \mathcal{O}_Y.
\]

**Proof.** For a projective morphism \( f \), Neeman’s [Nee96] and Hartshorne’s definition [Har66] of \( f^! \) agree, since both are right adjoint functors of \( Rf_* \). Hence we may use the results of [Nee96]
to prove the lemma. By [Nee96, Theorem 5.4], it is enough to show that \( f^1 \) commutes with coproducts. Fix an ample line bundle \( L \) on \( X \). By the discussion of [Nee96, Example 1.10] for every \( M \in \mathbb{Z} \), \( \{ L^m[n] | m, n \in \mathbb{Z}, m > M \} \) is a compact generating set for \( D(qc/X) \). Fix \( M \) such that 
\[ H^i(X, L^m) = 0 \] for all \( m > M \) and all \( y \in Y \). Then for every \( m > M \), \( Rf_* (L^m[n]) \) is supported only in cohomological degree \(-n\) and furthermore with locally free cohomology sheaf according to [Har77, Theorem 12.11]. In particular, it is a compact object of \( D(qc/X) \) [Nee96, Example 1.10] (to be precise in [Nee96, Example 1.10], it is only stated that line bundles are compact, but verbatim the same proof works for a locally free sheaves, by replacing inverse with dual). Hence \( Rf_* (\mathcal{F}) \) is compact for every element \( \mathcal{F} \) of the generating set \( \{ L^m[n] | m, n \in \mathbb{Z}, m > M \} \) of \( D(qc/X) \). Therefore, by [Nee96, Theorem 5.1] \( f^1 \) commutes with coproduct, which finishes our proof. 

\[ \square \]

**Lemma 2.4.** If \( f : X \to Y \) is a flat projective morphism between projective schemes with Gorenstein base of pure dimension \( d \), then 
\[
\omega^\bullet_{X/Y} \otimes f^* \omega_Y[d] \cong \omega^\bullet_X.
\]

**Proof.**
\[
\omega^\bullet_X \cong f^! \omega^\bullet_Y \cong f^! \omega_Y[d] \cong f^! O_Y \otimes f^* \omega^\bullet_Y[d] \cong \omega^\bullet_{X/Y} \otimes f^* \omega^\bullet_Y[d],
\]
Lemma 2.3 and flatness of \( f \) and \( \omega_Y \)

\[ \square \]

We need a third lemma as well about the behavior of relative dualizing complexes, for which we introduce first some notation.

**Notation 2.5.** For a morphism \( f : X \to Y \) of schemes, define 
\[
X^m_Y := X \times_Y X \times_Y \cdots \times_Y X.
\]
and \( f^m_Y : X^m_Y \to Y \) the base morphism. In most cases, when \( Y \) is obvious from the context, we omit \( Y \) from our notation. We denote then the \( i \)-th projection morphisms \( X^m \to X \) by \( p_i \).

**Lemma 2.6.** Using Notation 2.5, if \( f : X \to Y \) is a flat projective morphism of projective schemes, then 
\[
\omega^\bullet_{X^m/Y} \cong \bigotimes_{i=1}^m Lp^*_i \omega^\bullet_{X/Y}.
\]

**Proof.** The statement is vacuous for \( m = 1 \). For \( m > 1 \) we prove by induction. By the inductive hypothesis 
(2.6.c)
\[
\omega^\bullet_{X^{m-1}/Y} \cong \bigotimes_{i=1}^{m-1} Lp^*_i \omega^\bullet_{X/Y},
\]
where \( p_i \) is the \( i \)-th projection \( X^{m-1} \to X \). Let \( q : X^m \to X^{m-1} \) be the projection on the first \( m - 1 \) factors. Then the following computation concludes our proof.

\[
\omega^\bullet_{X^m/Y} \cong Lq^* \omega^\bullet_{X^{m-1}/Y} \otimes_L \omega^\bullet_{X^m/X^{m-1}} \cong Lq^* \left( \bigotimes_{i=1}^{m-1} Lp^*_i \omega^\bullet_{X/Y} \right) \otimes_L \omega^\bullet_{X^m/X^{m-1}} \cong \bigotimes_{i=1}^m Lp^*_i \omega^\bullet_{X/Y}.
\]

\[ \square \]
Having finished the lemmas about the relative dualizing complex, we need two other auxiliary lemmas used in the proof of Proposition 2.1.

**Lemma 2.7.** If $\mathcal{F}$ is a vector bundle on a smooth curve $Y$ and $\mathcal{L}$ is a line bundle such that for every $m > 0$, $S^m(\mathcal{F}) \otimes \mathcal{L}$ is generically globally generated, then $\mathcal{F}$ is nef.

**Proof.** Take a finite cover $\tau : Z \to Y$ by a smooth curve and a quotient line bundle $\mathcal{E}$ of $\tau^* \mathcal{F}$. Since $S^m(\mathcal{F}) \otimes \mathcal{L}$ is generically globally generated, so is $S^m(\tau^* \mathcal{F}) \otimes \tau^* \mathcal{L}$ and hence $\mathcal{E}^m \otimes \tau^* \mathcal{L}$ as well. Therefore $m \deg(\mathcal{E}) + \deg(\tau^* \mathcal{L}) \geq 0$ for all $m > 0$. In particular then $\deg(\mathcal{E}) \geq 0$. Since this is true for arbitrary $\tau$ and $\mathcal{E}$, $\mathcal{F}$ is nef indeed. \(\square\)

**Proof of Proposition 2.1.** According to Lemma 2.2, we only have to prove that $R^{-i} f_*(\omega_{X/Y}^\bullet)$ is nef for $i < d$. Since the fibers of $f$ are reduced, so is $X$. By flatness and [Har77, Corollary III.9.6], $X$ is also of pure dimension $n + 1$. Furthermore, by [PS13, Lemma 4.2], $X$ is $S_d$ and if $d = n$, then $X$ is $S_{n+1}$ (or equivalently Cohen-Macaulay). Therefore, Theorem 1.5 applies. Also, by Lemma 2.4 we may replace $\omega_X^\bullet$ in the statement of Theorem 1.5 by $\omega_{X/Y}^\bullet$ if we also shift the indices. That is, for all $i < d$ (or $i \leq n$ in the $d = n$ case) the following holds.

$$Rf_* \omega_{X/Y}^\bullet \cong R^{\leq -i} f_*(\omega_{X/Y}^\bullet) \oplus \bigoplus_{l > -i} R^l f_*(\omega_{X/Y}^\bullet[-l]) \quad (2.7.d)$$

Fix integers $m > 0$ and $i < d$ unless $d = n$, when $i = n$ is also allowed. Consider the following stream of isomorphisms and surjections, using Notation 2.5.

$$R^{-im}(f^m)_* (\omega_{X/Y}^\bullet) \cong R^{-im}(f^m)_* \left( \bigotimes_{L}^{m} \bigoplus_{i=1}^{L} Lp^*_1 (\omega_{X/Y}^\bullet) \right) \quad \text{Lemma 2.6}$$

$$\cong h^{-im} \left( \bigotimes_{L}^{m} \bigoplus_{j=1}^{L} R^{\leq -i} f_*(\omega_{X/Y}^\bullet) \right) \quad \text{K"unneth formula}$$

$$\cong h^{-im} \left( \bigotimes_{L}^{m} \left( R^{\leq -i} f_*(\omega_{X/Y}^\bullet) \oplus \bigoplus_{l > -i} R^l f_*(\omega_{X/Y}^\bullet[-l]) \right) \right) \quad (2.7.d)$$

$$\Rightarrow h^{-im} \left( \bigotimes_{L}^{m} R^{\leq -i} f_*(\omega_{X/Y}^\bullet) \right) \quad (2.7.e)$$

$$\Rightarrow \bigotimes_{L}^{m} R^{\leq -i} f_*(\omega_{X/Y}^\bullet)$$

**Fix any** $y_0 \in Y$ and $N \in \mathbb{Z}$, such that $|N y_0|$ is base-point free. By Corollary 1.4, $R^{-im}(f^m)_* (\omega_{X/Y}^\bullet) \otimes \omega_Y((N+1)y_0)$ is generically globally generated. Hence by (2.7.e), So is $S^m(R^{\leq -i} f_*(\omega_{X/Y}^\bullet)) \otimes \omega_Y((N+1)y_0)$. Therefore, by Lemma 2.7, $R^{\leq -i} f_*(\omega_{X/Y}^\bullet)$ is nef for every $i < d$ and also for $i = d$ when $d = n$, which concludes our proof. \(\square\)
Proof of Theorem 1.1. By Lemma 2.2, the statements on $R^i f_\ast O_X$ and $R^{-i} \omega_{X/Y}^\bullet$ are equivalent indeed. By [KK10, Theorem 7.8], $R^i f_\ast O_X$ is compatibale with arbitrary base-change. Furthermore, since nefness is decided on curves, we may assume that $Y$ is a smooth curve. However, then using Lemma 2.2 again, Proposition 2.1 concludes our proof.

\section{Injectivity and surjectivity for Du Bois schemes}

Here we prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Consider a closed embedding of $X$ into a smooth scheme $Y$, and let $\rho : Z \to Y$ be an embedded log-resolution of $(Y, X)$, which is isomorphism on $Y \setminus X$. Set $E := \rho^{-1}(X)_{\text{red}}$ and $\pi := \rho|_E$. By [Sch07, Theorem 4.6], the natural homomorphism $O_X \to R\pi_* O_E$ is quasi-isomorphism. This yields the following isomorphisms.

\begin{align}
(3.0.f) \quad R\pi_* \omega_E^\bullet \cong R\pi_* R\mathcal{H}\mathcal{O}\mathcal{M}_E(O_E, \omega_E^\bullet) \cong R\mathcal{H}\mathcal{O}\mathcal{M}_X(R\pi_* O_E, \omega_X^\bullet) \cong \omega_X^\bullet \quad \text{Grothendieck-duality} \quad \text{[Sch07, Theorem 4.6]}
\end{align}

\begin{align}
(3.0.g) \quad H^{i+\dim E}(E, \omega_E \otimes \pi^* \mathcal{L}) & \cong H^i(E, \omega_E^\bullet \otimes \pi^* \mathcal{L}) \\
& \cong H^i(Y, \rho_* (\omega_E^\bullet \otimes \pi^* \mathcal{L})) \quad \text{E Gorenstein, hence } \omega_E^\bullet \cong \omega_E[\dim E] \\
& \cong H^i(Y, R\pi_* (\omega_E^\bullet \otimes \pi^* \mathcal{L})) \quad \text{Grothendieck spectral sequence} \\
& \cong H^i \mathcal{H}\mathcal{O}\mathcal{M}(\omega_X^\bullet \otimes \mathcal{L}) \quad \text{projection formula} \quad \text{(3.0.f)}
\end{align}

Furthermore, by replacing $\mathcal{L}$ in (3.0.g) with $\mathcal{L}(F)$, one obtains that

\begin{align}
(3.0.h) \quad H^{i+\dim E}(E, \omega_E \otimes \pi^* \mathcal{L}(F)) \cong H^i(\omega_X^\bullet \otimes \mathcal{L}(F)),
\end{align}

and (3.0.g) and (3.0.h) are compatible with the natural maps induced by $\mathcal{L} \to \mathcal{L}(F)$. Hence, by setting $j = i + \dim E$, it is enough to prove that the natural homomorphisms

\begin{align}
(3.0.i) \quad H^j(E, \omega_E \otimes \pi^* \mathcal{L}) \to H^j(E, \omega_E \otimes \pi^* \mathcal{L}(F))
\end{align}

are injective for every $j$. Note at this point that since $\pi^* F$ is a general member of a base-point free linear system, it does not contain any strata of $E$. In particular then [Fuj09, Theorem 2.38] (setting $X := E, D' := 0, D := \pi^* F, H$ be any divisor such that $O_E(H) \cong \pi^* t, t := N, B := 0, S := 0$) implies the injectivity of (3.0.i).

Remark 3.1. Theorem 1.3 also follows from the arguments of [Kol95, Theorem 9.12] using [KK10, Corollary 7.7]. Unfortunately, [Kol95, Theorem 9.12] is stated for irreducible $X$, hence we included a full proof of Theorem 1.3.

To prove Corollary 1.4, we need two more lemmas.

Lemma 3.2. If $X$ is a quasi-projective scheme and $H$ an effective Cartier divisor on it, then there is an adjunction exact triangle as follows.

$$
\omega_X^\bullet \quad \longrightarrow \quad \omega_X^\bullet(H) \quad \longrightarrow \quad \omega_H^\bullet[1] \quad \longrightarrow
$$

Proof. If $t : H \to X$ is the embedding morphism, then

$$
\omega_H^\bullet \cong t^! \omega_X^\bullet = R\mathcal{H}\mathcal{O}\mathcal{M}_H(O_H, t^! \omega_X^\bullet) \cong R\mathcal{H}\mathcal{O}\mathcal{M}_X(O_H, \omega_X^\bullet) \quad \text{by Grothendieck duality}
$$

Consider then the exact sequence

$$
0 \longrightarrow O_X(-H) \longrightarrow O_X \longrightarrow O_H \longrightarrow 0,
$$
and apply $\mathcal{R}\text{Hom}_X(\omega_X^\bullet)$ to it:

\[(3.2.j) \quad \omega^*_H \cong \mathcal{R}\text{Hom}_X(\mathcal{O}_H, \omega_X^\bullet) \xrightarrow{\omega^*_X} \omega_X^\bullet(H) \xrightarrow{+1} \]

Rotating (3.2.j) yields the statement of the lemma. \hfill \Box

**Lemma 3.3.** Let $f : X \to Y$ be a flat, projective Du Bois family over a smooth projective curve, $y_0 \in Y$, $N > 0$ such that $|NX_{y_0}|$ is base-point free and $A \in |NX_{y_0}|$ a generic element. Then for any $i$ and any $y \in Y$ such that $X_y \subseteq A$, the natural map $\alpha$ in the following diagram is surjective.

\[(3.3.k) \quad H^i(X, \omega_X^{\bullet/Y} \otimes f^*\omega_Y((N + 1)X_{y_0})) \cong H^i(X, \omega_X^\bullet(A + X_{y_0})[-1]) \xrightarrow{\alpha} H^i(A, \omega_A^\bullet(X_{y_0})) \cong H^i(A, \omega_A^\bullet)
\]

Here the horizontal homomorphism is induced by the adjunction map $\omega_X^\bullet(A)[-1] \to \omega_A^\bullet$ of Lemma 3.2.

**Proof.** The vertical arrow of (3.3.k) is surjective because $X_y$ is a connected component of $A$. Therefore, it is enough to prove that the horizontal arrow of (3.3.k) is surjective. However, then equivalently we may also show that

\[(3.3.l) \quad H^i(X, \omega_X^{\bullet}(X_{y_0})[-1]) \to H^i(X, \omega_X^{\bullet}(X_{y_0} + A)[-1])
\]

is injective for all $i$. Note at this point that by [KS11a, Main Theorem], $X$ itself is Du Bois. Hence, (3.3.l) follows from Theorem 1.3. \hfill \Box

**Proof of Corollary 1.4.** For any $y \in Y$,

\[(3.3.m) \quad \dim_{k(y)} \left( \mathcal{R}^i f_* \omega_{X/Y}^\bullet \otimes k(y) \right) = \dim_{k(y)} \left( \mathcal{R}^{-i} f_* \mathcal{O}_X \otimes k(y) \right) \]

\[
\overset{\text{Lemma 2.2}}{=} \dim_{k(y)} H^{-i}(X_y, \mathcal{O}_{X_y}) \overset{\text{[KK10, Theorem 7.8]}}{=} \dim_{k(y)} H^i(X_y, \omega_{X_y}^\bullet). \]

Consider then the following diagram for a generic closed point $y \in Y$.

\[(3.3.n) \quad \begin{array}{c}
H^0(Y, \mathcal{R}^i f_* (\omega_{X/Y}^\bullet) \otimes \omega_Y((N + 1)y_0)) \\
\xrightarrow{\beta} \mathcal{R}^i f_* (\omega_{X/Y}^\bullet) \otimes \omega_Y((N + 1)y_0)_y \\
\xrightarrow{\gamma} H^i(X_y, \omega_{X_y}^\bullet)
\end{array}
\]

The arrow $\alpha$ is surjective by Lemma 3.3, and by (3.3.m) the two ends of $\gamma$ have the same dimensions over $\mathbb{C}$. Hence $\beta$ also has to be surjective. This finishes our proof. \hfill \Box

### 4. The proof of direct decomposition

Here we show Theorem 1.5. First, the following two lemmas state certain preservations of properties by passing to generic hypersurfaces. Since the first one is well known, we do not prove it here.

**Lemma 4.1.** If $X$ is a quasi-projective, $S_d$ scheme of pure dimension $n$, then a generic hyperplane section is also $S_d$. 

Lemma 4.2. If \( f : X \to Y \) is a flat quasi-projective morphism, such that \( X \) is a \( S_1 \) scheme of pure dimension \( n \geq 2 \), \( Y \) a smooth curve, and \( H \) is a general hyperplane section, then the induced morphism \( g : H \to Y \) is flat as well or equivalently \( H \) does not contain any component of any fiber of \( f \).

**Proof.** By Lemma 4.1, \( H \) is \( S_1 \), hence all its associated points are generic points. Therefore, by [Har77, Proposition III.9.7], \( H \) is flat if and only if all its components dominate \( Y \). However, since \( n \geq 2 \), the restriction of \( H \) to every component of \( X \) is irreducible [Har77, Exercise III.11.3]. Using genericity of \( H \) once more, we obtain that every component of \( H \) dominates \( Y \) indeed. \( \square \)

Having finished the preparatory lemmas, we prove in Proposition 4.3 the direct sum decomposition for \( R_f(\omega^\bullet_{X/Y}) \) when \( \dim(X/Y) = 1 \).

**Proposition 4.3.** If \( f : X \to Y \) a flat projective morphism with connected fibers, such that \( Y \) has rational singularities and pure dimension \( d \), then \( R_f(\omega^\bullet_X) \cong R^\leq -d - 1 f^*\omega^\bullet_Y \oplus R^{-d} f^*\omega^\bullet_Y \).

**Proof.** According to [Bha10, Theorem 4.1.3], the natural inclusion \( O_Y \to R_f O_X \) splits. Since \( f_* O_X \cong O_Y \) by the connectedness and flatness assumptions, this means that \( R_f O_X \cong O_Y \oplus R^{\geq 1} f_* O_X \).

\[
R_f(\omega^\bullet_X) \cong R_f R_{\text{Hom}}(O_X, \omega^\bullet_X) \cong R_{\text{Hom}}(R_f O_X, \omega^\bullet_Y) \cong R_{\text{Hom}}(O_Y \oplus R^{\geq 1} f_* O_X, \omega^\bullet_Y)
\]

\[
\cong R_{\text{Hom}}(O_Y \oplus R^{\geq 1} f_* O_X, \omega^\bullet_Y)[d] \oplus \omega^\bullet_Y[d]
\]

Our proof is concluded by noting that \( R^{\geq 1} f_* O_X \) is supported in cohomological degrees \([1, \ldots, d]\), and therefore \( \text{Hom}(R^{\geq 1} f_* O_X, \omega^\bullet_Y)[d] \) is supported in cohomological degrees less than \(-d\). \( \square \)

We show the direct sum decomposition for \( f_*(\omega^\bullet_{X/Y}) \) when \( \dim(X/Y) > 1 \) by induction on dimension. Some of the inductive arguments are isolated in the following lemmas.

**Lemma 4.4.** Let \( n > 2 \) and \( n \geq d \geq 2 \) be arbitrary integers and \( f : X \to Y \) a flat projective morphism with connected fibers, such that \( X \) is a reduced, \( S_d \) scheme of pure dimension \( n \) and \( Y \) a smooth curve. Let \( H \) be a generic hyperplane section. Then, \( g : H \to Y \) is a flat projective morphism with connected fibers, such that \( H \) is a reduced, \( S_d \) scheme of pure dimension \( n - 1 \).

**Proof.** We check the properties of \( H \) one by one.

- \( g \) is flat by Lemma 4.2.
- Since \( H \) is general, it does not contain any component of \( X \). Therefore, \( \dim(H \cap X') = n - 1 \) for every component \( X' \) of \( X \), and consequently \( H \) is of pure dimension \( n - 1 \).
- To prove, that \( H_y \) is connected, it is enough to prove it for a generic fiber, since then the Stein-factorization of \( g \) is a finite birational extension of \( Y \), which has to be \( Y \) itself by the normality of \( Y \). However, for generic \( y \), \( H_y \) is a generic hyperplane section of \( X_y \), which then is connected, because \( \dim X_y \geq 2 \) [Har77, Exercise III.11.3].
- \( H \) is \( S_d \) by Lemma 4.1. \( \square \)

**Lemma 4.5.** Let \( n > 2 \), \( n \geq d \geq 2 \) be arbitrary and \( f : X \to Y \) a flat projective morphism with connected fibers, such that \( X \) is a reduced, \( S_d \) scheme of pure dimension \( n \) and \( Y \) a smooth curve. Let \( H \) be a generic hypersurface of large enough degree and \( g : H \to Y \) the induced morphism. If

\[
R_g(\omega^\bullet_H) \cong R^\leq -d g_*(\omega^\bullet_H) \oplus \bigoplus_{i > -d} R^i g_*(\omega^\bullet_H)[-i]
\]
inducing identity on cohomology sheaves, then

\begin{equation}
R_f \omega_X^* \cong R_{\leq -d} f_* \omega_X^* \oplus \left( \bigoplus_{i > -d} R^i f_* \omega_X^*[-i] \right).
\end{equation}

also inducing identity on cohomology sheaves.

**Proof.** Since $H$ is of high enough degree, $H^j(X, h^i(\omega_X^*(H))) = 0$ for all $i$ and every $j > 0$. In particular, $R^i f_* \omega_X^*(H) \cong f_* h^i(\omega_X^*(H))$ for every $i$. Therefore, by [Pat13, Proposition 3.3.6],

\begin{equation}
R^i f_* \omega_X^*(H) \text{ is zero for } i > -d.
\end{equation}

Consider then the exact triangle of Lemma 3.2, rotate it, shift it, and push it forward.

\begin{equation}
R_f \omega_X^*(H)[-1] \longrightarrow Rg_* \omega_H^* \longrightarrow Rf_* \omega_X^*[-1] \longrightarrow.
\end{equation}

By (4.5.q), this induces the following isomorphisms.

\begin{equation}
R^i g_* \omega_H^* \cong R^i f_* \omega_X^* \text{ if } i > 1 - d, \text{ and } R^{1-d} f_* \omega_X^* \cong R^{1-d} g_* \omega_H^* / \text{im}(R^{-d} f_* \omega_X^*(H) \rightarrow R^{1-d} g_* \omega_H^*)
\end{equation}

To prove (4.5.p), it is enough to exhibit a homomorphism

\begin{equation}
R_{\leq -d} f_* \omega_X^* \oplus \left( \bigoplus_{i > -d} R^i f_* \omega_X^*[-i] \right) \rightarrow Rf_* \omega_X^*,
\end{equation}

which is identity on every cohomology sheaf. There is a natural homomorphism $R_{\leq -d} f_* \omega_X^* \rightarrow Rf_* \omega_X^*$ which is identity on the cohomology sheaves of degree at most $-d$. Hence, it is enough to exhibit a homomorphism

\begin{equation}
\bigoplus_{i > -d} R^i f_* \omega_X^*[-i] \rightarrow Rf_* \omega_X^*,
\end{equation}

which is identity on cohomology sheaves of degree greater than $-d$. For that consider the following composition: the inclusion

\begin{equation}
\bigoplus_{i > -d} R^i g_* \omega_H^*[i] \hookrightarrow R_{\leq -d} g_* \omega_H^* \oplus \left( \bigoplus_{i > -d} R^i g_* \omega_H^*[i] \right),
\end{equation}

the isomorphism (4.5.o), and finally the natural homomorphism $Rg_* \omega_H^* \rightarrow Rf_* \omega_X^*$ given by (4.5.r).

Call this composition $\phi$. According to (4.5.s),

\begin{equation}
\ker \phi = \text{im}(R^{-d} f_* \omega_X^*(H) \rightarrow R^{1-d} g_* \omega_H^*)[d-1].
\end{equation}

Hence, $\ker \phi$ yields a natural map

\begin{equation}
\bigoplus_{i > -d} R^i f_* \omega_X^*[i] \cong \text{im} \phi \rightarrow Rf_* \omega_X^*
\end{equation}

inducing identity on cohomology sheaves. This finishes our proof.

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**Proof of Theorem 1.5.** For $n = 2$, then necessarily $d = 2$. Hence $X$ is Cohen-Macaulay and hence $\omega_X^* = \omega_X[-2]$. Furthermore in this case $f$ is 1 relative dimensional. Therefore $Rf_* \omega_X^*$ is supported in $-2$ and $-1$ cohomological degrees. So, the statement for $n = 2$ is shown in Proposition 4.3.

We show the $n > 2$ cases by induction. Assume that the statement is known for $n$ replaced by $n' := n - 1$ and $d$ replaced by $d' := \min(d, n-1)$. Choose a general hypersurface $H$ of $X$ of large enough degree. By Lemma 4.4, all the conditions assumed for $X$ hold for $H$ (using $n'$ and $d'$ instead of $n$ and $d$). Hence, we may assume that (1.5.b) holds for all $X$ replaced by $H$. However then Lemma 4.5 concludes our proof.

\[ \Box \]
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