Diagram automorphisms and canonical bases for quantized enveloping algebras

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Abstract. Let $X$ be a Cartan datum of symmetric type, with an admissible automorphism $\sigma$ on $X$, and $\overline{X}$ the Cartan datum induced from $(X, \sigma)$. Let $U_q^-$ (resp. $\overline{U}_q^-$) be the negative part of the quantized enveloping algebra associated to $X$ (resp. $\overline{X}$). Lusztig constructed the canonical basis $B$ of $U_q^-$ and the canonical signed basis $\overline{B}$ of $U_q^-$ by making use of the geometric theory of quivers. By normalizing the sign of $\overline{B}$, he obtained the canonical basis $B$ of $U_q^-$, and a natural bijection $B \sim_{\sigma} \overline{B}$. In this paper, assuming the existence of $B$ for $U_q^-$, we construct the canonical basis $\overline{B}$ of $U_q^-$, and a bijection $\overline{B} \sim_{\sigma} \overline{B}$, by an elementary method, subject to the condition that the order of $\sigma$ is odd. In the case where the order is even, we obtain the corresponding result for the canonical signed basis.

Introduction

Let $X$ be the Cartan datum corresponding to a Kac-Moody algebra $\mathfrak{g}$, and $U_q^-$ the negative part of the quantized enveloping algebra $U_q = U_q(\mathfrak{g})$. Let $\sigma : X \to X$ be the diagram automorphism of $X$. Then $\sigma$ induces an algebra automorphism $\sigma : U_q^- \to U_q^-$. In the case where $\sigma$ is admissible (see 2.1 for the definition), the pair $(X, \sigma)$ induces a Cartan datum $\overline{X}$, which corresponds to the orbit algebra $\mathfrak{g}^\sigma$ of $\mathfrak{g}$. Let $U_q^-$ be the negative part of the quantized enveloping algebra $U_q^-$ associated to $\overline{X}$.

In the case where $X$ is a symmetric Cartan datum, the canonical basis $B$ of $U_q^-$ was constructed by Lusztig [L1] by making use of the geometric theory of quivers. In [L2], this result was generalized to an arbitrary Cartan datum $\overline{X}$. The outline of the proof is as follows; for a given Cartan datum $\overline{X}$, there exists a Cartan datum $X$ of symmetric type, and an admissible diagram automorphism $\sigma : X \to X$ such that the Cartan datum induced from $(X, \sigma)$ is isomorphic to $\overline{X}$. Let $B$ be the canonical basis of $U_q^-$, and $\overline{B}$ gives a permutation of $B$, and we denote by $B^\sigma$ the set of $\sigma$-fixed elements in $B$. In the first step, by making use of the geometric theory of quivers with automorphisms, he constructed the canonical signed basis $\overline{B}$ of $U_q^-$ (see 1.18), and proved that there exists a natural bijection $\overline{B}^\sigma \sim \overline{B}$, where $\overline{B} = B \sqcup -B$, and $\overline{B} = \overline{B} \sqcup -\overline{B}$ for some basis $\overline{B}$ of $U_q^-$. In the second step, he proved the existence of the canonical basis $B$ of $U_q^-$ such that $\overline{B} = B \sqcup -B$ and constructed a natural bijection $B^\sigma \sim \overline{B}$ ([L2, Thm. 14.4.3, 19.2.3]), by normalizing the sign of $\overline{B}$ in terms of the theory of crystals due to Kashiwara [K].

In this paper, we take up a similar problem, but from a different point of view. This is a generalization of [SZ1,2], where the case $X$ is finite or affine type was discussed. We consider $X$ of symmetric type, with an automorphism $\sigma$ on $X$, and $\overline{X}$ induced from $(X, \sigma)$ as before. Assuming the existence of the canonical basis
\(\mathbf{B}\) for \(U_q^-\), we shall construct the canonical signed basis \(\tilde{B}\) of \(U_q^-\) and a natural bijection \(\tilde{B}^\sigma \sim \tilde{B}\) in an elementary way, without appealing the geometric theory, nor the theory of crystal basis. In the case where the order of \(\sigma\) is odd, we obtain the canonical basis \(B\) of \(U_q^-\) and a natural bijection \(B^\sigma \sim B\).

The main ingredient for our approach is the isomorphism \(\Phi: A' U_q^- \sim V_q\) as discussed in [SZ1, SZ2], where \(X\) is assumed to be finite type or affine type. We generalize those results to the following situation. Let \(X\) be a Cartan datum of arbitrary type, with an admissible automorphism \(\sigma\) on \(X\). Then the pair \((X,\sigma)\) induces a Cartan datum \(\tilde{X}\). We consider the quantized enveloping algebra \(U_q^-\) (resp. \(U_q^\tilde{X}\)) associated to \(X\) (resp. \(\tilde{X}\)). Here we assume that the order of \(\sigma\) is a power of a prime number \(p\). Let \(A = \mathbb{Z}[q, q^{-1}]\), and \(A' = F[q, q^{-1}]\), where \(F = \mathbb{Z}/p\mathbb{Z}\) is the finite field of \(p\)-elements. Let \(A' U_q^-\) be Lusztig’s integral form of \(U_q^-\), and set \(A' U_q^- = A' \otimes_A U_q^-\). \(\sigma\) acts naturally on \(A' U_q^-\), and we denote by \(A' U_q^-\sigma\) the \(\sigma\)-fixed point subalgebra of \(A' U_q^-\). The \(A'\)-algebra \(V_q\) is defined as \(V_q = A' U_q^-/J\), where \(J\) is the two-sided ideal generated by the orbit sum \(O(x)\) such that \(\sigma(x) \neq x\) (see 3.2). The \(A'\)-algebra \(A' U_q^-\) is defined similarly to the case of \(U_q^-\). In turn, we give an axiomatic definition of the canonical basis for \(U_q^-\) (see 1.12) by focussing some properties of the canonical basis in the symmetric case. It is shown that the canonical basis is unique if it exists. We show, in Theorem 3.4, that if the canonical (signed) basis \(B\) exists for \(U_q^-\), then there exists an isomorphism \(\Phi: A' U_q^- \sim V_q\) of \(A'\)-algebras. Moreover, in Theorem 4.18, under the assumption that \(p\) is odd, we construct the canonical basis \(\tilde{B}\) of \(U_q^-\) from \(B\), by making use of this isomorphism, and show that there exists a natural bijection \(B^\sigma \sim B\). In the case where \(p = 2\), we only obtain the canonical signed basis (see Remark 4.19).

Returning to the original problem, we consider \(X\) of symmetric type, and \(\sigma: X \rightarrow X\) such that \(\tilde{X}\) is induced from \((X, \sigma)\). In this case, the order of \(\sigma\) is not necessarily a prime power, but one can find a sequence \(X = X_0, X_1, \ldots, X_k = \tilde{X}\) of Cartan data, and an automorphism \(\sigma_i: X_i \rightarrow X_i\) such that \(X_{i+1}\) is isomorphic to the Cartan datum induced from \((X_i, \sigma_i)\) and that the order of \(\sigma_i\) is a prime power, with \(\sigma = \sigma_k \cdots \sigma_1 \sigma_0\). In the case where the order of \(\sigma\) is odd, since the canonical basis \(B\) exists for \(X_0 = X\), by the repeated use of Theorem 4.18, one can find the canonical basis \(\tilde{B}\) of \(X_k = \tilde{X}\), and a bijection \(B^\sigma \sim B\) (Theorem 4.27). While in the case where the order of \(\sigma\) is even, we obtain the corresponding result concerning with the canonical signed basis.

In the course of the proof of Theorem 3.4, one needs to show that \(\Phi\) is a homomorphism. In [SZ1], where \(X\) is finite or affine type, this was proved by case by case verification, by making use of PBW-bases. In our case, we can not use PBW-bases. Instead, in this paper we prove this by a purely combinatorial argument, in a uniform way. The discussion here is in some sense simpler, and more transparent than the direct computation in [SZ1].

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1. Preliminaries

1.1. Let $X = (I, (\ , \ ))$ be a Cartan datum, where $I$ is a finite set and $(\ , \ )$ is a symmetric bilinear form on the vector space $\bigoplus_{i \in I} \mathbb{Q} \alpha_i$ with basis $\alpha_i$, such that $(\alpha_i, \alpha_i) \in \mathbb{Z}$ satisfies the property

(i) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ for any $i \in I$,
(ii) $2(\alpha_i, \alpha_i) \in \mathbb{Z}_{\leq 0}$ for any $i \neq j$ in $I$.

For $i, j \in I$, set $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbb{Z}$. The matrix $A = (a_{ij})$ is called the Cartan matrix associated to $X$. The Cartan datum is said to be symmetric if $(\alpha_i, \alpha_i) = 2$ for any $i \in I$, and simply-laced if it is symmetric and $(\alpha_i, \alpha_j) \in \{0, -1\}$ for any $i \neq j$. If $X$ is symmetric, then $A$ is a symmetric matrix. Let

$Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the root lattice, and set $Q_+ = \sum_{i \in I} N_{\alpha_i}, Q_- = -Q_+$. For a weight $\nu = \sum_{i \in I} n_i \alpha_i \in Q_+$, set $|\nu| = \sum_i |n_i|$. Let $\mathfrak{g}$ be the Kac-Moody algebra associated to $X$.

We fix a weight lattice $P$, a free abelian group of finite rank, such that $Q \subset P$, and set $P^* = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$. Let $(\ , \): P^* \times P \rightarrow \mathbb{Z}$ be the canonical pairing. We fix $h_i \in P^*$ for each $i \in I$ satisfying the property that $(h_i, \lambda) = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ for any $\lambda \in P$.

1.2. Let $q$ be an indeterminate, and for an integer $n$, a positive integer $m$, set

$$ [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]^! = [1][2] \cdots [m], \quad [0]^! = 1. $$

Also, for $n \in \mathbb{Z}, m \in \mathbb{N}$, put

$$ \left[ \frac{n}{m} \right] = \frac{[n][n-1] \cdots [n-m+1]}{[m]^!}, \quad (m \geq 1), \quad \left[ \frac{n}{0} \right] = 1. $$

In particular, if $0 \leq m \leq n$, then we have \[\left[ \frac{n}{m} \right] = \left[ \frac{[n]^!}{[m]^!} \right] = \left[ \frac{n}{n-m} \right].\]

For $d \in \mathbb{N}$, we denote by $[n]_d$ the element obtained from $[n]$ by replacing $q$ by $q^d$. For each $i \in I$, set $d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{N}$, and $q_i = q^{d_i}$.

Let $U_q = U_q(\mathfrak{g})$ be the quantized enveloping algebra associated to $X$ and $P$, namely, an associative algebra over $\mathbb{Q}(q)$ with generators $e_i, f_i$ $(i \in I)$ and $q^h$ $(h \in P^*)$, and relations

\begin{align*}
(1.2.1) \quad & q^0 = 1, \quad q^{h+h'} = q^h q^{h'} \quad \text{for } h, h' \in P^*, \\
(1.2.2) \quad & q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \quad \text{for } i \in I, h \in P^*, \\
(1.2.3) \quad & e_i f_j - f_j e_i = \delta_{ij} t_i - t_i^{-1} \frac{q_i - q_i^{-1}}{q_j - q_j^{-1}} \quad \text{for } i, j \in I, \\
(1.2.4) \quad & \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_i e_i^{(k)} = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_i f_i^{(k)} = 0 \quad \text{for } i \neq j \in I,
\end{align*}
where $t_i = q^{d_i h_i}$, and $e_i^{(n)} = e_i^n/[n]_{id}^1$, $f_i^{(n)} = f_i^n/[n]_{id}^1$. Let $U_q^-$ (resp. $U_q^+$) be the subalgebra of $U_q$ generated by $f_i$ ($i \in I$) (resp. by $e_i$ ($i \in I$)). Then $U_q^-$ (resp. $U_q^+$) is an associative algebra over $Q(q)$ with generators $f_i$ (resp. $e_i$) satisfying the fundamental relations in (1.2.4).

Set $A = Z[q, q^{-1}]$, and let $\mathcal{A} U_q^-$ be Lusztig’s integral form of $U_q^-$, namely, the $A$-subalgebra of $U_q^-$ generated by $f_i^{(n)} = f_i^n/[n]_{id}^1$ for $i \in I, n \in \mathbb{N}$.

We define a $Q$-algebra automorphism, called the bar-involution, $\bar{\cdot} : U_q^- \to U_q^-$ by $\bar{q} = q^{-1}, \bar{f}_i = f_i$ for $i \in I$. We define an anti-algebra automorphism $^* : U_q^- \to U_q^-$ by $f_i^* = f_i$ for any $i \in I$.

**1.3. $U_q^-$ has a weight space decomposition**

$U_q^- = \bigoplus_{\nu \in Q_-} (U_q^-)_\nu$, where $(U_q^-)_\nu$ is a subspace of $U_q^-$ spanned by elements $f_{i_1} \cdots f_{i_N}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_N} = -\nu$. $x \in U_q^-$ is called homogeneous with wt $x = \nu$ if $x \in (U_q^-)_\nu$. We define a multiplication on $U_q^- \otimes U_q^-$ by

$$
(1.3.1) \quad (x_1 \otimes x_2) \cdot (x'_1 \otimes x'_2) = q^{-(\text{wt} x_2, \text{wt} x'_1)} x_1 x'_1 \otimes x_2 x'_2
$$

where $x_1, x'_1, x_2, x'_2$ are homogeneous in $U_q^-$. Then $U_q^- \otimes U_q^-$ becomes an associative algebra with respect to this twisted product. We define a homomorphism $r : U_q^- \to U_q^- \otimes U_q^-$ by $r(f_i) = f_i \otimes 1 + 1 \otimes f_i$ for each $i \in I$. It is known (see [L2, 1.2]). Note that $v_i$ in [L2] coincides with $q_i^{-1}$ in our paper) that there exists a unique bilinear form $(\ , \ )$ on $U_q^-$ satisfying the following properties; $(1, 1) = 1$ and

$$
(1.3.2) \quad (f_i, f_j) = \delta_{ij}(1 - q_i^2)^{-1},
(x, y'y'') = (r(x), y' \otimes y''),
(x'x'', y) = (x' \otimes x'', r(y)),
$$

where the bilinear form on $U_q^- \otimes U_q^-$ is defined by $(x_1 \otimes x_2, x'_1 \otimes x'_2) = (x_1, x'_1)(x_2, x'_2)$. Thus defined bilinear form is symmetric and non-degenerate. Using the property $r((U_q^-)_\nu) \subset \bigoplus_{\nu' + \nu'' = \nu} (U_q^-)_{\nu'} \otimes (U_q^-)_{\nu''}$, we have

$$
(1.3.3) \quad ((U_q^-)_\nu, (U_q^-)_{\nu'}) = 0 \quad \text{for} \ \nu \neq \nu'.
$$

For any $i \in I$, we define $Q(q)$-linear maps $\iota, r_i : U_q^- \to U_q^-$ by

$$
(1.3.4) \quad r(x) = f_i \otimes \iota r(x) + \sum y \otimes z, \quad r(x) = r_i(x) \otimes f_i + \sum z \otimes y,
$$

where $y$ are homogenous such that wt $y \neq -\alpha_i$. From the definition, we have

$$
(1.3.5) \quad (f_i y, x) = (f_i, f_i)(y, \iota r(x)), \quad (y f_i, x) = (f_i, f_i)(y, r_i(x)).
$$
The following properties for $i, r, r_i$ are also immediate from the definition. Assume that $x, x'$ are homogeneous. Then

$$ i r(1) = 0, \quad i r(f_j) = \delta_{ij}, \quad r_i(1) = 0, \quad r_i(f_j) = \delta_{ij}. $$  

(1.3.6) \quad i r(x x') = q^{(\text{wt}, x, \alpha)}_{\alpha} x i r(x') + i r(x) x', \quad r_i(x x') = q^{(\text{wt}, x', \alpha)}_{\alpha} r_i(x) x' + x r_i(x').  

Moreover, we have

$$ r_i = * \circ i r \circ *. $$  

(1.3.7)

**Lemma 1.4.** Assume that $(\alpha_i, \alpha_j) = 0$. Then

(i) $i r$ commutes with the left action of $f_j$ on $U_q$.

(ii) $i r$ and $j r$ commute each other.

**Proof.** (i) is immediate from (1.3.6). We show (ii). Assume that $x$ is homogeneous. We prove (*) $i r_j r(x) = j r_i r(x)$ by induction on $|\text{wt} x|$. If $x = 1$, this is trivial. So assume that $x \neq 0$, and (*) holds for $x'$ such that $|\text{wt} x'| < |\text{wt} x|$. Write $x = yz$ with $y, z$ homogeneous, not equal to 1. Then by (1.3.6), we have $i r(yz) = q^{(\text{wt}, y, \alpha)}_{\alpha} y z r_i(z) + r(y) z$, and so

$$ j r_i r(y z) = q^{(\text{wt}, y, \alpha)}_{\alpha} q^{(\text{wt}, y, \alpha)}_{\alpha} (y j r_i r(z) + j r_i r(y) z) + (q^{(\text{wt}, y + \alpha, \alpha)}_{\alpha} j r_i r(y) z + j r_i r(y) z) $$

$$ = q^{(\text{wt}, y, \alpha)}_{\alpha} + (\text{wt}, y, \alpha) j r_i r(z) + (q^{(\text{wt}, y, \alpha)}_{\alpha} j r(y) z + q^{(\text{wt}, y, \alpha)}_{\alpha} j r(y) z) + j r_i r(y) z. $$

In the last formula, $j r_i r(y) = i r_j r(y), j r_i r(z) = i r_j r(z)$ by induction hypothesis, and the second term is symmetric with respect to $i$ and $j$. Thus we have $j r_i r(y z) = i r_j r(y z)$. Hence (ii) holds. \hfill $\square$

The following result is known (cf. [L2, Lemma 1.2.15]).

**Lemma 1.5.** $\bigcap_{i \in I} \text{Ker } r_i = Q(q)1, \quad \bigcap_{i \in I} \text{Ker } r_i = Q(q)1$.

**1.6.** The following formula is easily verified by induction on $n$ (see [L2, Lemma 1.4.2]).

$$ r(f^{(n)}_i) = \sum_{k + k' = n} q_{i}^{-kk'} f_{i}^{(k)} \otimes f_{i}^{(k')}.$$  

(1.6.1)

It follows from (1.6.1) that $r : U_q^- \to U_q^- \otimes U_q^-$ induces a homomorphism $r : A U_q^- \to A U_q^- \otimes A U_q^-$. This implies that the maps $i r, r_i : U_q^- \to U_q^-$ induce $A$-linear maps $i^r, r_i : A U_q^- \to A U_q^-$. 
1.7. For $i \in I$ and any $t \geq 0$, we consider the operator

\[(1.7.1) \quad \Pi_{i,t} = \sum_{s \geq 0} (-1)^s q_i^{-s(s-1)/2} f_i^{(s)} (r)^s t : U_q^{-} \to U_q^-.\]

The following result is known by [L2, Lemma 16.1.2] (originally due to Kashiwara [K, 3.2]). For the orthogonality relations, see (1.9.2) below. For $i \in I$, set $\text{Ker} r_i = U^-_q[i]$.

Then $\text{Ker} r_i = *(U_q^-[i])$ which we denote by $*U_q^-[i]$. Note that the statements for $*U_q^-[i]$ follows from that for $U_q^-[i]$ by applying the $*$-operation.

**Lemma 1.8.** For $i \in I$, the followings holds.

(i) $U_q^- = \bigoplus_{n \geq 0} f_i^{(n)} U_q^-[i] = \bigoplus_{n \geq 0} *U_q^-[i] f_i^{(n)}$ as vector spaces. The direct summands $f_i^{(n)} U_q^-[i]$ are mutually orthogonal with respect $(\ , \ )$. A similar result holds for $*U_q^-[i] f_i^{(n)}$.

(ii) The map $x \mapsto f_i^{(n)} x$ gives an isomorphism $U_q^-[i] \cong f_i^{(n)} U_q^-[i]$, and similarly, the map $x \mapsto x f_i^{(n)}$ gives an isomorphism $*U_q^-[i] \cong *U_q^-[i] f_i^{(n)}$.

(iii) By (i) and (ii), $x \in U_q^-$ is written uniquely as $x = \sum_{n \geq 0} f_i^{(n)} x_n$ with $x_n \in U_q^-[i]$. Then

\[(1.8.1) \quad x_n = q_i^{n(n-1)/2} \Pi_{i,n}(x).\]

In particular, the projection $U_q^- \to f_i^{(n)} U_q^-[i]$ preserves the weights, namely, if $x \in (U_q^-)_\nu$, then $f_i^{(n)} x_n \in (U_q^-)_\nu$ for any $n \geq 0$.

1.9. For any subspace $Z$ of $U_q^-$, set $A Z = Z \cap A U_q^-$. Take $x \in U_q^-$. then $x$ is written uniquely as $x = \sum_{n \geq 0} f_i^{(n)} x_n$ with $x_n \in U_q^-[i]$ by Lemma 1.8. Since $r$ gives a map $A U_q^- \to A U_q^-$ by 1.6, $\Pi_{i,t}$ induces a map $A U_q^- \to A U_q^-$. Assume that $x \in A U_q^-$. Then by (1.8.1), $x_n \in A U_q^-$, and so $f_i^{(n)} x_n \in A U_q^-$. It follows that the decomposition in Lemma 1.8 (i) induces a direct sum decomposition

\[(1.9.1) \quad A U_q^- = \bigoplus_{n \geq 0} A (f_i^{(n)} U_q^-[i]) = \bigoplus_{n \geq 0} f_i^{(n)} A (U_q^-[i]),\]

where $A (f_i^{(n)} U_q^-[i]) = f_i^{(n)} A U_q^-[i]$.

The following orthogonality relations hold for the decomposition in (1.9.1). For the proof, see [L2, Lemma 16.2.6], where the case $m = n$ is discussed. The case $m \neq n$ is also treated by a similar argument.

(1.9.2) Assume that $x, y \in A U_q^-[i]$. Then

\[(f_i^{(n)} x, f_i^{(m)} y) = \begin{cases} c(x, y) \text{ with } c \in 1 + (qZ[[q]] \cap Q(q)), & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}\]
1.10. Let $V$ be a $\mathbf{Q}(q)$-subspace of $\mathbf{U}_q^-$. A basis $\mathcal{B}$ of $V$ is said to be almost orthonormal if
\[
(b, b') \in \begin{cases} 
1 + q\mathbf{Z}[q] \cap \mathbf{Q}(q) & \text{if } b = b', \\
q\mathbf{Z}[q] \cap \mathbf{Q}(q) & \text{if } b \neq b'.
\end{cases}
\]

Recall that $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$. Let $\mathbf{A}_0 = \mathbf{Q}[q] \cap \mathbf{Q}(q)$. Set
\[
(1.10.1) \quad \mathcal{L}_\mathbf{Z}(\infty) = \{ x \in \mathbf{A} \mathbf{U}_q^- \mid (x, x) \in \mathbf{A}_0 \}.
\]

Then $\mathcal{L}_\mathbf{Z}(\infty)$ is a $\mathbf{Z}[q]$-submodule of $\mathbf{A} \mathbf{U}_q^-$. It is known that if $\mathcal{B}$ is an $\mathbf{A}$-basis of $\mathbf{A} \mathbf{U}_q^-$, which is almost orthonormal, then $\mathcal{B}$ gives a $\mathbf{Z}[q]$-basis of $\mathcal{L}_\mathbf{Z}(\infty)$ by [L2, Lemma 16.2.5].

For a fixed $i \in I$, we consider the decomposition $\mathbf{A} \mathbf{U}_q^- = \bigoplus_{n \geq 0} f_i^{(n)} \mathbf{A} \mathbf{U}_q^- [i]$ as in (1.9.1). Write $x = \sum_{n \geq 0} y_n = \sum_{n \geq 0} f_i^{(n)} x_n$, where $y_n = f_i^{(n)} x_n$ with $x_n \in \mathbf{A} \mathbf{U}_q^- [i]$. The following result is known.

Lemma 1.11 ([L2, Lemma 16.2.7]). Let $x = \sum_{n \geq 0} y_n$ be as above.

(i) Assume that $x \in \mathcal{L}_\mathbf{Z}(\infty)$. Then each $x_n$ and $y_n$ is in $\mathcal{L}_\mathbf{Z}(\infty)$. If, in addition, $(x, x) \in 1 + q\mathbf{A}_0$, then there exists $n_0 \geq 0$ such that $(y_{n_0}, y_{n_0}), (x_{n_0}, x_{n_0}) \in 1 + q\mathbf{A}_0$ and $(y_n, y_n), (x_n, x_n) \in q\mathbf{A}_0$ for all $n \neq n_0$.

(ii) Assume that $\mathcal{B}$ is an $\mathbf{A}$-basis of $\mathbf{A} \mathbf{U}_q^-$, which is almost orthonormal. Then in the setup of (i), $y_{n_0} \equiv \pm b \mod q \mathcal{L}_\mathbf{Z}(\infty)$ for some $b \in \mathcal{B}$, and $x_n \equiv 0 \mod q \mathcal{L}_\mathbf{Z}(\infty)$ for all $n \neq n_0$.

1.12. For a fixed $i \in I$, we consider the direct sum decomposition of $\mathbf{A} \mathbf{U}_q^-$ as in (1.9.1). For each $x \in \mathbf{U}_q^-$, let $\varepsilon_i(x)$ be the largest integer $n$ such that $x \in f_i^{(n)} \mathbf{U}_q^-$, and $x_{[i,a]}$ the projection of $x$ on $f_i^{(a)} \mathbf{U}_q^- [i]$ for $a \in \mathbf{N}$. By Lemma 1.11, if $x \in \mathcal{L}_\mathbf{Z}(\infty)$, then $x_{[i,a]} \in \mathcal{L}_\mathbf{Z}(\infty)$. Let $\mathcal{B}$ be a basis of $\mathbf{U}_q^-$. For $i \in I$ and $n \in \mathbf{N}$, set $\mathcal{B}_{i,n} = \{ b \in \mathcal{B} \mid \varepsilon_i(b) = n \}$. Thus we have a partition $\mathcal{B} = \bigsqcup_{n \geq 0} \mathcal{B}_{i,n}$.

We consider a basis $\mathcal{B}$ of $\mathbf{U}_q^-$ having the following properties:

(C1) $\mathcal{B}$ gives a $\mathbf{Z}[q]$-basis of $\mathcal{L}_\mathbf{Z}(\infty)$, and an $\mathbf{A}$-basis of $\mathbf{A} \mathbf{U}_q^-$.  
(C2) $\mathcal{B}$ is bar-invariant, namely, $\overline{b} = b$ for any $b \in \mathcal{B}$.  
(C3) $\mathcal{B}$ is almost orthonormal.  
(C4) For $\nu \in \mathbf{Q}_-$, set $\mathcal{B}_\nu = \mathcal{B} \cap (\mathbf{U}_q^-)_\nu$. Then we have a partition $\mathcal{B} = \bigcup_{\nu \in \mathbf{Q}_-} \mathcal{B}_\nu$, where $\mathcal{B}_\nu = \{ 1 \}$ if $\nu = 0$.  
(C5) If $b \in \mathcal{B}_{i,a}$ for $i \in I, a \geq 0$, then
\[
(1.12.1) \quad b \equiv b_{[i,a]} \mod q \mathcal{L}_\mathbf{Z}(\infty).
\]

(C6) $\bigcap_{i \in I} \mathcal{B}_{i,0} = \{ 1 \}$.  
(C7) Assume that $b \in \mathcal{B}_{i,0}$. Then for any $a > 0$, there exists a unique element $b' \in \mathcal{B}_{i,a}$ such that
\[
(1.12.2) \quad b' \equiv f_i^{(a)} b \mod f_i^{a+1} \mathbf{U}_q^-.
\]
The correspondence \( b \mapsto b' \) gives a bijection \( \pi_{i,a} : B_{i,0} \cong B_{i,a} \).

B is called the **canonical basis** of \( \mathbf{U}_q^- \). The word *canonical* is justified by the following lemma.

**Lemma 1.13.** The basis B of \( \mathbf{U}_q^- \) is unique if it exists.

**Proof.** Let B be a canonical basis of \( \mathbf{U}_q^- \). Assume that \( B' = \bigsqcup \nu B'_\nu \) is a basis satisfying similar properties. We show, by induction on \( |\nu| \), that \( B_\nu = B'_\nu \). If \( \nu = 0 \), this holds by (C4). Assume that \( \nu \neq 0 \) and that \( B_\nu = B'_\nu \) for any \( |\nu'| < |\nu| \).

Take \( b \in B'_\nu \). By (C6), there exists \( i \in I \) such that \( \varepsilon_i(b) = a > 0 \). Then there exists \( b' \in B'_{i,0} \) such that \( b \equiv f_i^{(a)} b' \mod f_i^{a+1} \mathbf{U}_q^- \) by (C7). \( b_{[i,a]} = (f_i^{(a)} b')_{[i,a]} \), and \( b \equiv (f_i^{(a)} b')_{[i,a]} \mod qL_\mathbf{Z}(\infty) \) by (C5). By induction, \( b' \in B_{i,0} \). Thus by applying (1.12.2) for B, there exists \( b_1 \in B_\nu \) such that \( b_1 \equiv f_i^{(a)} b' \mod f_i^{a+1} \mathbf{U}_q^- \). By a similar discussion as above, we see that \( b_1 \equiv (f_i^{(a)} b')_{[i,a]} \mod qL_\mathbf{Z}(\infty) \). Thus \( b = b_1 \) is written as \( b = b_1 = \sum_{b' \in B} a_{b'} b' \) with \( a_{b'} \in qL_\mathbf{Z} \). Since \( b = b_1 \) is bar-invariant, this implies that \( a_{b'} = 0 \) for any \( b' \), and so \( b = b_1 \in B_\nu \). Hence \( B'_\nu \subseteq B_\nu \). The opposite inclusion is obtained similarly, and we have \( B'_\nu = B_\nu \). The lemma is proved. \( \Box \)

1.14. Using the decomposition \( \mathbf{U}_q^- = \bigoplus_{n \geq 0} f_i^{(n)} \mathbf{U}_q^- [i] \) in Lemma 1.8, we write \( x = \sum_{n \geq 0} f_i^{(n)} x_n \) with \( x_n \in \mathbf{U}_q^- [i] \). We define \( E'_i, F'_i : \mathbf{U}_q^- \rightarrow \mathbf{U}_q^- \) by

\[
F'_i(x) = \sum_{n \geq 0} f_i^{(n+1)} x_n, \quad E'_i(x) = \sum_{n \geq 1} f_i^{(n-1)} x_n.
\]

\( E'_i, F'_i \) are called Kashiwara operators on \( \mathbf{U}_q^- \).

1.15. Assuming the existence of the canonical basis B for \( \mathbf{U}_q^- \), we define a map \( F_i : B_{i,a} \rightarrow B_{i,a+1} \) by

\[
F_i : B_{i,a} \xrightarrow{\pi_{i,a}^{-1}} B_{i,0} \xrightarrow{\pi_{i,a+1}} B_{i,a+1}.
\]

We define \( E_i : B_{i,a} \rightarrow B_{i,a-1} \) as the inverse of \( F_i \), if \( a > 0 \), and by \( E_i(b) = 0 \) if \( a = 0 \). The maps \( E_i, F_i : B \rightarrow B \cup \{0\} \) are called Kashiwara operators on B.

Note that B is an adapted basis of \( \mathbf{U}_q^- \) in the sense of [L2, 16.3.1]. Hence by [L2, Lemma 16.3.3, Prop. 16.3.5], we have the following.

**Proposition 1.16.** Let \( b \in B_{i,a} \) for \( i \in I \) and \( a \geq 0 \).

(i) \( F'_i(b) \in B_{i,a+1} + qL_\mathbf{Z}(\infty) \).

(ii) \( F'_i(b) \equiv F_i(b) \mod qL_\mathbf{Z}(\infty) \).

The following result was proved by Lusztig.

**Theorem 1.17.** Assume that X is symmetric. Then \( \mathbf{U}_q^- \) has the canonical basis B. The basis B has the property that \( *B = B \).
Proof. In [L1], [L2], an $A$-basis $B$ of $A \mathcal{U}_q^-$ was constructed by using the geometry of quivers associated to $X$, which satisfies the properties (C2), (C3), (C4), and (C6), (C7). By [L2, Lemma 16.2.5], $B$ is a $\mathbb{Z}_q$-basis of $\mathcal{L}_\mathbb{Z}(\infty)$, hence (C1) holds. We show that $B$ satisfies (C5). Take $b \in B$, and write it as $b = \sum_{n \geq 0} f_{(n)} x_n$ with $x_n \in \mathcal{U}_q^-$. Since $(b, b) + 1 \in q \mathcal{A}$, by Lemma 1.11, there exists a unique $a_0$ such that $f_{(n)} x_n \in q \mathcal{L}_\mathbb{Z}(\infty)$ for all $n \neq a_0$. By Proposition 1.16, $E^{a}_i b \equiv E^a b$ mod $q \mathcal{L}_\mathbb{Z}(\infty)$. Since $E^a_i b \in B$, $E^a_i b \in q \mathcal{L}_\mathbb{Z}(\infty)$. This implies that $a_0 = a$. Hence $b \equiv f_{(n)} x_n \equiv b_{(n)}$ mod $q \mathcal{L}_\mathbb{Z}(\infty)$, and (C5) holds. (This property is also proved in [L3, Prop. 1.8].) The property $*(B) = B$ was also proved in [L1], [L2], by using the geometric method.

1.18. We define a subset $\tilde{B}$ of $\mathcal{U}_q^-$ by

\[(1.18.1) \quad \tilde{B} = \{x \in \mathcal{U}_q^- \mid \bar{x} = x, (x, x) \in 1 + q \mathbb{Z}[q]\}.
\]

If there exists a basis $B$ of $\mathcal{U}_q^-$ such that $\tilde{B} = B \cup -B$, then $\tilde{B}$ is called the canonical signed basis.

In the case where $\mathcal{U}_q^-$ has the canonical basis $B$, then we have

\[(1.18.2) \quad \tilde{B} = B \cup -B,
\]

hence $\tilde{B}$ is the canonical signed basis.

In fact, since $B$ is almost orthonormal, $B \cup -B \subset \tilde{B}$. Conversely, take $x \in \tilde{B}$. By [L2, Lemma 16.2.5], there exists $b \in B$ such that $x \equiv b \mod q \mathcal{L}_\mathbb{Z}(\infty)$. Since $x$ and $b$ are bar-invariant, this implies that $x = \pm b \in B \cup -B$. Hence (1.18.2) holds.

The following result is immediate from (C6).

Proposition 1.19. Assume that the canonical basis $B$ exists for $\mathcal{U}_q^-$. Then for any $b \in B$, there exists a sequence $i_1, \ldots, i_N \in I$, and $c_1, c_2, \ldots, c_N \in \mathbb{Z}_{>0}$ such that $b = F^{i_1}_1 F^{i_2}_2 \cdots F^{i_N}_N 1$.

1.20. We review the theory of the modified quantized enveloping algebra $\hat{\mathcal{U}}_q$ (see [L2, Chap. 23]). Here we use a slightly different formulation. Consider the $\mathbb{Q}(q)$-vector space $\hat{\mathcal{U}}_q = \bigoplus_{\lambda \in P} \mathcal{U}_q^- \otimes \mathcal{U}_q^+ \otimes \mathbb{Q}(q)a_\lambda$, where $P$ is the weight lattice for $\mathcal{U}_q$ and $\mathbb{Q}(q)a_\lambda$ is a vector space generated by a vector $a_\lambda$ corresponding to $\lambda \in P$. We define a product on $\hat{\mathcal{U}}_q$ by the following rule:

\[(1.20.1) \quad a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda, \quad e_ia_\lambda = a_{\lambda+\alpha_i} e_i, \quad f_ia_\lambda = a_{\lambda-\alpha_i} f_i, \quad (e_if_j - f_je_i) a_\lambda = \delta_{ij}[h_i, \lambda] a_\lambda.
\]

Since $1 = \sum_{\lambda \in P} a_\lambda$ is not contained in $\hat{\mathcal{U}}_q$, the identity element does not exist in $\hat{\mathcal{U}}_q$.

A $\hat{\mathcal{U}}_q$-module $M$ is said to be unital ([L2, 23.1.4]) if

(i) for any $m \in M$, we have $a_\lambda m = 0$ for all but finitely many $\lambda \in P$. 


(ii) for any \( m \in M \), we have \( \sum_{\lambda \in P} a_\lambda m = m \).

Giving a unital \( \mathcal{U}_q \)-module is equivalent to giving a \( \mathcal{U}_q \)-module with weight space decomposition.

1.21. We define an \( A \)-submodule \( A\mathcal{U}_q \) of \( \mathcal{U}_q \) by

\[
A\mathcal{U}_q = \bigoplus_{\lambda \in P} A\mathcal{U}_q^- \otimes A\mathcal{U}_q^+ \otimes Aa_\lambda.
\]

It is known by ([L2, Lemma 23.2.2]) that \( A\mathcal{U}_q \) is an \( A \)-subalgebra of \( \mathcal{U}_q \) generated by \( e_i^{(n)} a_\lambda, f_i^{(n)} a_\lambda \) for various \( i \in I, n \geq 0, \lambda \in P \).

Let \( M(\lambda) \) be the Verma module for \( \mathcal{U}_q \) associated to \( \lambda \in P \), with highest weight vector \( v_\lambda \). Then \( M(\lambda) \cong \mathcal{U}_q^- v_\lambda \cong \mathcal{U}_q^- \) as \( \mathcal{U}_q^- \)-modules. Let \( _A M(\lambda) = _A \mathcal{U}_q^- v_\lambda \) be the \( A \)-submodule of \( M(\lambda) \). Note that \( M(\lambda) \) is a unital \( \mathcal{U}_q \)-module, and it can be verified ([L2, 23.3.2]) that \( _A M(\lambda) \) is stable under the actions of \( f_i^{(n)} a_\lambda, e_i^{(n)} a_\lambda \).

Hence \( _A M(\lambda) \) is an \( A\mathcal{U}_q \)-submodule of \( M(\lambda) \) generated by \( v_\lambda \).

1.22. Let \( R \) be a commutative ring with 1, and with an invertible element \( q \in R \). We fix a ring homomorphism \( \phi : A \to R \) such that \( \phi(q^n) = q^n \) for any \( n \in \mathbb{Z} \). We regard \( R \) as an \( A \)-algebra via \( \phi \), and consider

\[
_R \mathcal{U}_q^\pm = R \otimes_A \mathcal{U}_q^\pm, \quad _R \mathcal{U}_q^- = R \otimes_A \mathcal{U}_q^-.
\]

We have a direct sum decomposition \( _R \mathcal{U}_q^\pm = \bigoplus_{\nu \in Q} (_R \mathcal{U}_q^\pm)_\nu \), and \( _R \mathcal{U}_q^- \) is expressed as

\[
_R \mathcal{U}_q^- = \bigoplus_{\lambda \in P} _R \mathcal{U}_q^- \otimes _R \mathcal{U}_q^+ \otimes R a_\lambda
\]

Unital \( _R \mathcal{U}_q^- \)-modules are defined similarly to 1.20. Assume that \( M \) is a unital \( _R \mathcal{U}_q^- \)-module. Then \( M = \bigoplus_{\lambda \in P} M_\lambda \), where \( M_\lambda = a_\lambda M \), and \( M_\lambda \) becomes an \( R \)-module. Following [L2, 31.3], we introduce the notion of highest weight modules. Let \( M \) be a unital \( _R \mathcal{U}_q^- \)-module. \( M \) is called a highest weight module with highest weight \( \lambda \in P \) if there exists a vector \( m \in M_\lambda \) such that

(i) \( e_i^{(n)} m = 0 \) for any \( i \in I \) and \( n > 0 \).
(ii) \( M = _R \mathcal{U}_q^- m \).
(iii) \( M_\lambda \) is a free \( R \)-module of rank 1 with generator \( m \).

A unital \( _R \mathcal{U}_q^- \)-module \( M \) is said to be integrable if for any \( m \in M \) and any \( i \in I \), there exists \( n_0 \geq 1 \) such that \( e_i^{(n)} m = f_i^{(n)} m = 0 \) for all \( n \geq n_0 \).

1.23. For \( \lambda \in P \), let \( _A M(\lambda) \) be the Verma module defined in 1.21. Set \( _R M(\lambda) = _R \otimes_A _A M(\lambda) \). Then \( _R M(\lambda) \) is a highest weight \( _R \mathcal{U}_q^- \)-module with highest weight \( \lambda \). Since \( _A M(\lambda) \cong _A \mathcal{U}_q^- \) as \( A \)-modules, we have \( _R M(\lambda) \cong _R \mathcal{U}_q^- \) as \( R \)-modules.

Let \( P^+ \) be the set of dominant weights \( \lambda \) in \( P \), namely, \( \lambda \) such that \( \langle h_i, \lambda \rangle \geq 0 \) for any \( i \in I \). For \( \lambda \in P^+ \), let \( L(\lambda) \) be the integrable highest weight \( \mathcal{U}_q \)-module.
with highest weight $\lambda$ and highest weight vector $v_\lambda$. As an $U_q^-$-module, it is written as

$$L(\lambda) = U_q^- / \sum_{i,n>(h_i,\lambda)} U_q^- f_i^{(n)}.$$  

(1.23.1)

Let $A L(\lambda)$ be the $A U_q^-$-submodule of $L(\lambda)$ generated by $v_\lambda$. By using the commuting relation in [L2, 31.1.6], one can check that the actions of $f_i^{(n')} a_{x'} e_i^{(n')} a_{x'}$ ($i \in I, n' \in \mathbb{N}, \lambda' \in \mathbb{P}$) on $A U_q^-$ preserve $\sum_{i,n>(h_i,\lambda)} A U_q^- f_i^{(n)}$. Hence $A L(\lambda)$ is a unital $A U_q^-$-module. We define a unital $R U_q^-$-module $RL(\lambda)$ by $RL(\lambda) = R \otimes A L(\lambda)$. Then $RL(\lambda)$ is a highest weight module with highest weight $\lambda$, and $RL(\lambda)$ is integrable. $RL(\lambda)$ is a quotient of $RM(\lambda)$.

Assume that $R$ is any field such that $q_i = \phi(q_i)$ is not a root of unity. It is proved by Tanisaki [T, Thm. 5.5, Thm. 5.6] that the Weyl-Kac type character formula holds for $R U_q^-$ and in particular, any integrable highest weight $R U_q^-$-module is irreducible. By applying this to our situation, we have

(1.23.2) Let $R$ be as above. Then the integrable highest weight module $RL(\lambda)$ is irreducible.

1.24. Return to the case where $R$ is a commutative ring. We consider the map $r : A U_q^- \rightarrow A U_q^- \otimes A A U_q^-$, and $i r, r_i : A U_q^- \rightarrow A U_q^-$ as in 1.6. By tensoring with $R$, one can define maps $r : R U_q^- \rightarrow R U_q^- \otimes_R R U_q^-$ and $i r, r_i : R U_q^- \rightarrow R U_q^-$. Then $i r, r_i$ also satisfy similar properties as in (1.3.6). The following result is an $R U_q^-$-version of [L2, Prop. 3.1.6], and can be proved in a similar way by using (1.3.6).

**Lemma 1.25.** Assume that $q_i = \phi(q_i) \neq \pm 1$. Then for any $x \in R U_q^-, i \in I$, and $\lambda \in \mathbb{P}$, the following relation holds in $R U_q$.

$$\left( e_i x - x e_i \right) a_\lambda = \frac{q_i^{(h_i,\lambda)} r_i(x) - q_i^{-(h_i,\lambda+\text{wt} x+\alpha_i)} r_i(x)}{q_i - q_i^{-1}} a_\lambda.$$  

(1.25.1)

By making use of Lemma 1.25, we prove the following formula, which is a generalization of Lemma 1.5.

**Proposition 1.26.** Assume that $R$ is a field such that $q_i$ is not a root of unity. Consider maps $i r, r_i : R U_q^- \rightarrow R U_q^-$. Then we have

$$\bigcap_{i \in I} \text{Ker} r_i = R \cdot 1, \quad \bigcap_{i \in I} \text{Ker} r_i = R \cdot 1.$$  

(1.26.1)

Prove. We show the first formula. The second one follows from it by applying $. It is enough to show that $\bigcap_{i \in I} \text{Ker} r \cap \left( R U_q^- \right)_\nu = 0$ for any weight $\nu \neq 0$. We prove this by induction on $|\nu|$. Take $x \in \bigcap_{i \in I} \text{Ker} r \cap \left( R U_q^- \right)_\nu$. If $|\nu| = 1$, then $x = c f_i$ for some $i$, and $c \in R$. In this case, $r_i(x) = c = 0$, and $x = 0$. Assume that $|\nu| \geq 2$. By a similar discussion as in the proof of Lemma 1.4, it is shown that $i r$ and $r_j$ commute each other for any $i, j$. Thus $r_j(x) \in \bigcap_{i \in I} \text{Ker} r_i$. Hence by induction, $r_j(x) = 0$ for
any \( j \). We apply Lemma 1.25 for this \( x \). Then \((e_i x - x e_i) a_\lambda = 0\) in \( R \tilde{\mathcal{U}}_q\) for any \( i \in I, \lambda \in P \). We consider the highest weight \( R \tilde{\mathcal{U}}_q\)-module \( M = RL(\lambda)\) with highest weight vector \( v_\lambda \) as in (1.23.1). Then

\[ (e_i x - x e_i) a_\lambda v_\lambda = (e_i x - x e_i) v_\lambda = e_i x v_\lambda = 0. \]  

\( xv_\lambda \) is contained in the weight space \((RL(\lambda))_{\lambda'}\) with weight \( \lambda' \neq \lambda \). and \( e_i (xv_\lambda) = 0\) for any \( i \). Since \([n]_i \neq 0\) in \( R \), we also have \( e_i^{(n)}(xv_\lambda) = 0\) for any \( n \geq 1 \). Thus \( xv_\lambda \) generates a proper \( R \tilde{\mathcal{U}}_q\)-submodule if \( xv_\lambda \neq 0 \). Since \( RL(\lambda)\) is irreducible by (1.23.2), we have \( xv_\lambda = 0\). This is true for any \( \lambda \in P \) such that \( \langle h_i, \lambda \rangle \geq 0 \) for any \( i \in I \). But by the expression of \( RL(\lambda)\) in (1.23.1), if \( x \neq 0 \), one can find some \( \lambda \in P \) such that \( xv_\lambda \neq 0 \). This implies that \( x = 0 \). The proposition is proved. \( \Box \)

2. THE DIAGRAM AUTOMORPHISM

2.1. Let \( X = (I, ( , ) ) \) be a Cartan datum, and let \( \sigma : I \rightarrow I \) be a permutation such that \( (\alpha_{\sigma(i)}, \alpha_{\sigma(j)} ) = (\alpha_i, \alpha_j ) \) for any \( i, j \in I \). Such a \( \sigma \) is called a diagram automorphism on \( X \). Let \( \mathcal{I} \) be the set of \( \sigma \)-orbits in \( I \). \( \sigma \) is called admissible if for any orbit \( \eta \in \mathcal{I}, (\alpha_i, \alpha_j ) = 0 \) for any \( i \neq j \in \eta \). Hereafter, we assume that \( \sigma \) is an admissible diagram automorphism on \( X \).

Let \( \mathcal{I} \) be as above. We define a symmetric bilinear form \( ( , )_1 \) on \( \bigoplus_{\eta \in \mathcal{I}} Q a_\eta \) by

\[ (\alpha_\eta, \alpha_{\eta'})_1 = \begin{cases} (\alpha_i, \alpha_j ) |\eta|, & \text{if } \eta = \eta', \\ \sum_{i \in \eta, j \in \eta'} (\alpha_i, \alpha_j ) & \text{if } \eta \neq \eta'. \end{cases} \]  

Then \( (\alpha_\eta, \alpha_\eta )_1 \in 2\mathbb{Z}_{\geq 0} \) for each \( \eta \in \mathcal{I} \). Since

\[ \sum_{i \in \eta, j \in \eta'} (\alpha_i, \alpha_j ) = |\eta| \sum_{j \in \eta'} (\alpha_i, \alpha_j ) \quad \text{for } i \in \eta, \]

we have, for a fixed \( i \in \eta \),

\[ a_{\eta'} = 2 (\alpha_\eta, \alpha_{\eta'})_1 = \sum_{j \in \eta'} (\alpha_i, \alpha_j ) = \sum_{j \in \eta'} a_{ij} \in \mathbb{Z}_{\leq 0}. \]

Hence \( ( , )_1 \) satisfies (i), (ii) in 1.1, and \( X = (\mathcal{I}, ( , )_1 ) \) is a Cartan datum. The Cartan datum \( X \) is called the Cartan datum induced from \( (X, \sigma) \).

Note that a symmetric Cartan datum \( X \) is naturally identified with a finite graph \( \Gamma \) with multiple edges (for a given \( X \), \( I \) is the set of vertices in \( \Gamma \), \( i \) and \( j \) are joined by \( -(\alpha_i, \alpha_j ) \) edges for \( i \neq j \in I \)). The diagram automorphism on \( X \) corresponds to the graph automorphism. In the case where \( X \) is symmetric, the definition of \( ( , )_1 \) coincides with the one given in [L2, 14.1.1], and in [SZ2, 2.1].

2.2. It is known by [L2, Prop.14.1.2] that, for a given Cartan datum \( X \), there exists a symmetric Cartan datum \( \tilde{X} \), and an admissible diagram automorphism \( \sigma \) on \( \tilde{X} \) such that the Cartan datum induced from \( (\tilde{X}, \sigma) \) is isomorphic to \( X \). \( \tilde{X} \) is
constructed as follows; let \( X = (I, (\ , \ )) \) be a Cartan datum, where \((\ , \)\) is the bilinear form on the vector space \( \bigoplus_{i \in I} Q \alpha_i \). Recall that \( d_i = (\alpha_i, \alpha_i)/2 \) for each \( i \in I \). We consider a set \( D_i \) of cardinality \( d_i \), and let \( \sigma : D_i \to D_i \) be a cyclic permutation on \( D_i \). We consider a set \( D = \bigcup_{i \in I} D_i \), and a permutation \( \sigma \) on \( D \) induced from \( \sigma : D_i \to D_i \). Let \( \overline{D} \) be the set of \( \sigma \)-orbits on \( D \). Then \( \overline{D} \) is naturally in bijection with \( I \). We shall define a graph with vertex set \( D \). Fix \( i \neq j \in I \). Since \( (\alpha_i, \alpha_j) \) is divisible by \( d_i \) and \( d_j \), it is divisible by the smallest common multiple \( l(d_i, d_j) \) of \( d_i \) and \( d_j \). Choose \( x \in D_i \), \( y \in D_j \), and join \( x \) and \( y \) by \( c \)-fold edges, where \( c = -(\alpha_i, \alpha_j)/l(d_i, d_j) \). If \( (x', y') \in D_i \times D_j \) is \((\sigma \times \sigma)\)-conjugate to \( (x, y) \), we join \( x' \) and \( y' \) by the same number of edges. Hence the number of edges joining \( D_i \) and \( D_j \) is equal to \(- (\alpha_i, \alpha_j)\), which is independent from the choice of \( x, y \). We define a graph \( \Gamma = (D, \Omega) \), where \( \Omega \) is the set of edges defined above. Let \( \tilde{X} = (D, (\ , \)_{\tilde{\sigma}}) \) be the Cartan datum induced from \( \Gamma \), where \((\ , \)_{\tilde{\sigma}}\) is the bilinear form on \( \bigoplus_{x \in D} Q \alpha_x \) such that

\[
(\alpha_x, \alpha_y)_{\tilde{\sigma}} = \begin{cases} 
2 & \text{if } x = y, \\
-\frac{1}{l}\{\text{edges joining } x \text{ and } y \} & \text{if } x \neq y.
\end{cases}
\]

Then \( \tilde{X} \) is a Cartan datum of symmetric type, and \( \sigma : D \to D \) gives an admissible diagram automorphism on \( \tilde{X} \). The Cartan datum \( (\overline{D}, (\ , \)_{\overline{\sigma}}) \) induced from \( (\tilde{X}, \sigma) \) is isomorphic to \( X \).

### 2.3.

Let \( X = (I, (\ , \ )) \) be a Cartan datum, with an admissible automorphism \( \sigma \). Assume that the order of \( \sigma \) is \( n = am \), where \( a \) and \( m \) are prime each other. Then \( \sigma \) is decomposed as \( \sigma = \tau \tau' \), where \( \tau, \tau' \) are powers of \( \sigma \), and the order of \( \tau \) (resp. \( \tau' \)) is equal to \( a \) (resp. \( m \)). Let \( \overline{I} \) be the set of \( \sigma \)-orbits in \( I \) as before. Let \( I^{\tau} \) be the set of \( \tau \)-orbits in \( I \). Then \( \sigma \) permutes \( I^{\tau} \), whose action we denote by \( \sigma \). This action coincides with the induced action of \( \tau' \) on \( I^{\tau} \), and so the order of \( \sigma \) is equal to \( m \). The set of \( \overline{\sigma} \)-orbits in \( I^{\tau} \) is naturally identified with \( \overline{I} \).

The Cartan datum \( \overline{X} = (\overline{I}, (\ , \)_{\overline{\sigma}}) \) induced from \( (X, \sigma) \) is defined as in (2.1.1), namely

\[
(\alpha_{\eta}, \alpha_{\eta'})_{\overline{\sigma}} = \begin{cases} 
(\alpha_{i}, \alpha_{i})|\eta|, & \text{if } \eta = \eta', \\
\sum_{i \in \eta, j \in \eta'}(\alpha_i, \alpha_j) & \text{if } \eta \neq \eta'.
\end{cases}
\]

Let \( X^{\tau} = (I^{\tau}, (\ , \)_{\tau}) \) be the Cartan datum induced from \( (X, \tau) \). Then for \( \gamma, \gamma' \in I^{\tau} \) we have, again by (2.1.1),

\[
(\alpha_{\gamma}, \alpha_{\gamma'})_{\tau} = \begin{cases} 
(\alpha_{i}, \alpha_{i})|\gamma|, & \text{if } \gamma = \gamma', \\
\sum_{i \in \gamma, j \in \gamma'}(\alpha_i, \alpha_j) & \text{if } \gamma \neq \gamma'.
\end{cases}
\]

On the other hand, we consider the Cartan datum \( X^{\tau} \) and the admissible automorphism \( \overline{\sigma} \) on \( X^{\tau} \). We want to show that the Cartan datum \( X^{\tau} = (\overline{I}^{\tau}, (\ , \)_{\overline{\sigma}}) \) induced from \( (X^{\tau}, \overline{\sigma}) \) is canonically isomorphic to \( \overline{X} \). Note that \( \overline{I}^{\tau} \) is in bijection with \( \overline{I} \), which we denote by \( \eta \leftrightarrow \eta \). We consider the symmetric bilinear form \( (\ , \)_{\overline{\sigma}} \) on the vector space \( \bigoplus_{\bar{\eta} \in \overline{I}} Q \alpha_{\bar{\eta}} \). By (2.1.1), \( (\ , \)_{\overline{\sigma}} \) is defined by
(2.3.3) \[(\alpha_{\tilde{\eta}}, \alpha_{\tilde{\eta}'} )_2 = \begin{cases} (\alpha_{\gamma}, \alpha_{\gamma} |_{\tilde{\eta}} ) , & (\gamma \in \tilde{\eta} ) , \\ \sum_{\gamma \in \tilde{\eta}, \gamma' \in \tilde{\eta}'} (\alpha_{\gamma}, \alpha_{\gamma' } )_\tau , & (\gamma \notin \tilde{\eta} ) . \end{cases} \]

Let \( \tilde{\eta} \) be the \( \sigma \)-orbit of \( \gamma \in \mathcal{I}^\tau \). Then \(|\eta| = |\tilde{\eta}| \cdot |\gamma| \). Since \((\alpha_{\gamma}, \alpha_{\gamma} )_\tau = (\alpha_i, \alpha_i ) |_{\gamma} \) for \( i \in \gamma \), we obtain
\[
(\alpha_{\tilde{\eta}}, \alpha_{\tilde{\eta}} )_2 = (\alpha_{\eta}, \alpha_{\eta})_1 .
\]

Moreover, for \( \tilde{\eta} \neq \tilde{\eta}' \),
\[
(\alpha_{\tilde{\eta}}, \alpha_{\tilde{\eta}' } )_2 = \sum_{\gamma \in \tilde{\eta}, \gamma' \in \tilde{\eta}'} \sum_{i \in \gamma, j \in \gamma'} (\alpha_{i}, \alpha_{j} ) = \sum_{i \in \eta, j \in \eta'} (\alpha_{i}, \alpha_{j} ) = (\alpha_{\eta}, \alpha_{\eta'} )_1 .
\]

Thus under the identification \( \bigoplus_{\eta \in \mathcal{I}} Q\alpha_{\eta} \simeq \bigoplus_{\tilde{\eta} \in \mathcal{I}^\tau} Q\alpha_{\tilde{\eta}}, (\ , \ )_1 \) coincides with \((\ , \ )_2 \). Summing up the above discussion, we have

**Lemma 2.4.** The Cartan datum \((\mathcal{I}^\tau, (\ , \ )_2)\) induced from \((X, \sigma)\) is isomorphic to the Cartan datum \(\mathcal{X} = (\mathcal{I}, (\ , \ )_1)\) induced from \((X, \sigma)\).

**Proposition 2.5.** Let \( X \) be a Cartan datum. Let \( \tilde{X} \) be a Cartan datum of symmetric type with admissible \( \sigma : \tilde{X} \to X \) such that the Cartan datum induced from \((\tilde{X}, \sigma)\) is isomorphic to \( X \) as given in 2.2. Then there exists a sequence \( \tilde{X} = X_0, X_1, \ldots, X_k = X \) of Cartan data, and an admissible diagram automorphism \( \sigma_i : X_i \to X_i \) such that the Cartan datum induced from \((X_i, \sigma_i)\) is isomorphic to \(X_{i+1}\), where the order of \( \sigma_i \) is a prime power, and that \( \sigma = \sigma_{k-1} \cdots \sigma_1 \sigma_0 \).

**Proof.** Let \( n \) be the order of \( \sigma \). We prove the proposition by induction on \( n \). If \( n \) is a prime power, there is nothing to prove. Assume that \( n \) is not a prime power. We write \( n = am \), where \( m \) is a power of some prime number \( p \), and \( a \) is prime to \( p \). Write \( \sigma = \tau \tau' = \tau' \tau \) as in 2.3, where the order of \( \tau \) (resp. \( \tau' \)) is equal to \( a \) (resp. \( m \)). Then \( \tau \) is an admissible automorphism on \( \tilde{X} \), and we denote by \( X' \) the Cartan datum induced from \((\tilde{X}, \tau)\). \( \sigma \) induces an admissible automorphism on \( X' \), which we denote by \( \sigma' \). Here the order of \( \sigma' \) is equal to \( m \), and the order of \( \tau \) is equal to \( a \). By Lemma 2.4, the Cartan datum induced from \((X', \sigma')\) is isomorphic to \( X \). By applying the induction hypothesis on \( \tilde{X}, \tau \) and \( X' \), we obtain a sequence \( \tilde{X} = X_0, X_1, \ldots, X_{k-1} = X' \), and \( \sigma_i : X_i \to X_i \) such that \( \tau = \sigma_{k-2} \cdots \sigma_0 \) satisfying the condition. By setting \( \sigma_{k-1} = \sigma' \), we obtain the proposition. \( \square \)

3. The algebra \( \mathbf{V}_q \)

3.1. Let \( X = (I, (\ , \ )) \) be a Cartan datum, and \( \sigma \) an admissible diagram automorphism on \( X \). Let \( \mathbf{U}^- \) be the quantized enveloping algebra associated to \( X \). Then \( \sigma \) induces an algebra automorphism \( \sigma : \mathbf{U}^- \to \mathbf{U}^- \) by \( f_i \mapsto f_{\sigma(i)} \). Let \( \mathbf{U}_q^\sigma \) be the subalgebra of \( \mathbf{U}_q^- \) consisting of \( \sigma \)-fixed elements. If the canonical basis \( \mathbf{B} \) exists for \( \mathbf{U}_q^- \), then \( \sigma(\mathbf{B}) \) is also the canonical basis. Hence by the uniqueness property
(Lemma 1.13), $\sigma(B) = B$. $\sigma$ acts on $B$ as a permutation, and we denote by $B^\sigma$ the set of $\sigma$-fixed elements in $B$. Let $X = (I, (\cdot, \cdot)_1)$ be the Cartan datum induced from $(X, \sigma)$, and $U_q^-$ the associated quantized algebra. We will compare the algebra structure of $U_q^{-\sigma}$ and $U_q^-$, as in [SZ2].

3.2. From now on, throughout this section, we assume that the order of $\sigma$ is a power of a prime number $p$. We also assume that the canonical basis $B$ exists for $U_q^-$. (In the case where $p = 2$, we need to replace $B$ by the canonical signed basis $\tilde{B} = B \sqcup -B$. However since all the discussion in this section works well even in the case $p = 2$ under a suitable modification, we concentrate to the case where $B$ exists.)

$\sigma$ stabilizes $A U_q^-$, and we define $A U_q^{-\sigma} = U_q^\sigma \cap A U_q^-$, the subalgebra of $A U_q^-$ consisting of $\sigma$-fixed elements. Let $F = \mathbb{Z}/p\mathbb{Z}$ be a finite field of $p$ elements, and set $A' = F[q, q^{-1}]$. We consider the $A'$-algebra

$$A' U_q^{-\sigma} = A' \otimes A U_q^{-\sigma} \simeq A U_q^{-\sigma}/p(A U_q^{-\sigma}).$$

For each $x \in U_q^-$, we denote by $O(x)$ the orbit sum of $x$, namely $O(x) = \sum_{0 \leq i < k} \sigma^i(x)$, where $k$ is the smallest integer such that $\sigma^k(x) = x$. Hence $O(x)$ is $\sigma$-invariant. $O(x)$ is defined similarly for $A U_q^-, A' U_q^-$. Let $J$ be an $A'$-submodule of $A U_q^{-\sigma}$ generated by $O(x)$ for $x \in A U_q^-$ such that $\sigma(x) \neq x$. Then $J$ is a two-sided ideal of $A U_q^{-\sigma}$. We define an $A'$-algebra $V_q$ as the quotient algebra $A U_q^{-\sigma}/J$. Let $\pi : A U_q^{-\sigma} \to V_q$ be the natural projection. Note that $V_q$ is a generalization of the algebra $V_q$ introduced in [SZ1, SZ2].

3.3. For each $\eta \in I$ and $a \in \mathbb{N}$, set $f_{\eta}^{(a)} = \prod_{i \in \eta} f_i^{(a)}$. Since $f_i^{(a)}$ and $f_j^{(a)}$ commute each other for $i, j \in \eta$, we have $f_{\eta}^{(a)} \in A U_q^{-\sigma}$. We denote its image in $A U_q^{-\sigma}$ also by $\tilde{f}_\eta^{(a)}$. Thus we can define $g_\eta^{(a)} \in V_q$ by

$$g_\eta^{(a)} = \pi(\tilde{f}_\eta^{(a)}).$$

In the case where $a = 1$, we set $\tilde{f}_\eta^{(1)} = \tilde{f}_\eta = \prod_{i \in \eta} f_i$ and $g_\eta^{(1)} = g_\eta$.

Since $\ast$ commutes with $\sigma$, $\ast$ preserves $A U_q^{-\sigma}$, and acts on $A' U_q^{-\sigma}$, which induces an anti-algebra automorphism $\ast$ on $V_q$. Note that $\tilde{f}_{\eta}^{(a)}$ is $\ast$-invariant since $f_i$ and $f_j$ commute each other for $i, j \in \eta$. Thus $g_\eta^{(a)}$ is $\ast$-invariant.

Let $U_q^-$ be as above. The algebras $A U_q^-$, and $A' U_q^-$ are defined similarly to $U_q^-$. We denote by $f_{\eta} (\eta \in I)$ the generators of $U_q^-$, and $f_{\eta}^{(a)} (\eta \in I, a \in \mathbb{N})$ the generators in $A U_q^-$. The anti-algebra automorphism $\ast$ on $A U_q^-$ is inherited from $\ast$ on $U_q^-$. The following result is a generalization of Theorem 2.4 in [SZ2].

Theorem 3.4. Assume that $U_q^-$ has the canonical (signed) basis. The assignment $f_{\eta}^{(a)} \mapsto g_{\eta}^{(a)}$ gives an isomorphism $\Phi : A U_q^- \cong V_q$ of $A'$-algebras, which is compatible with $\ast$-operation.

First we note that
Proposition 3.5. The assignment $f^{(a)}_{ij} \mapsto g^{(a)}_{ij}$ gives a homomorphism $\Phi : A[U] \rightarrow V_q$.

The proof of Proposition 3.5 will be given in Section 5. Note that the corresponding result in the finite or affine case was proved in [SZ2, Prop. 2.6] by the computation using the PBW-bases. This method cannot be applied to our case, since the PBW-bases do not exist for Kac-Moody case. We prove Proposition 3.5, by a purely combinatorial argument, which is in some sense simpler than the computation by PBW-bases.

3.6. Assuming Proposition 3.5, we continue the discussion. Let $\widehat{A} = Z((q)) \cap Q(q)$ be the subring of $Q(q)$ containing $A = Z[q, q^{-1}]$. By (1.3.2), we have $(f_i, f_j) \in \widehat{A}$. Since $r$ preserves $\bigwedge_q^{-}$, we have $(\bigwedge_q^{-}, \bigwedge_q^{-}) \subset \widehat{A}$ by (1.3.5). Recall that $A' = F[q, q^{-1}]$, and $\bigwedge_q^{-} = \bigwedge_q^{-}/p(\bigwedge_q^{-})$. Let $F(q)$ be the field of rational functions of $q$ with coefficients in $F$. The bilinear form $(,)$ on $\bigwedge_q^{-}$ induces a bilinear form $(,)$ on $\bigwedge_q^{-}$, whose values are in $\widehat{A}/p\widehat{A} \subset F((q)) \cap F(q) = F(q)$.

Since $\sigma$ commutes with $r$, the bilinear form $(,)$ on $\bigwedge_q^{-}$ is $\sigma$-invariant, namely, $(\sigma(x), \sigma(y)) = (x, y)$ for any $x, y \in \bigwedge_q^{-}$. In particular, for $x, y \in \bigwedge_q^{-}$ such that $\sigma(x) \neq x$, we have

$$\begin{align*}
(3.6.1) \quad (O(x), O(y)) = \sum_{x' \in O(x)} (x', O(y)) = |O(x)|(x, O(y)) \in pZ((q)) \cap Q(q)
\end{align*}$$

since $|O(x)| > 1$ is a power of $p$. It follows that the bilinear form on $\bigwedge_q^{-}$ induces a bilinear form $(,)$ on $F(q) \bigwedge_q^{-} = F(q) \otimes A' \bigwedge_q^{-}$. Since $\sigma$ permutes $B$, $\pi(B^q)$ gives an $A'$-basis of $\bigwedge_q$, hence gives a basis of $F(q) \bigwedge_q^{-}$.

On the other hand, the bilinear form $(,)$ on $\bigwedge_q^{-}$ is defined similarly, and it induces a bilinear form $(,)$ on $F(q) \bigwedge_q^{-} = F(q) \otimes A \bigwedge_q^{-}$ with values in $F(q)$. Here we regard $F(q)$ as an $A$-algebra by a homomorphism $\phi : A \rightarrow F(q), q^n \mapsto q^n$. by applying Proposition 1.26 for $R = F(q)$, we obtain the following. Note that the existence of the canonical basis is not necessary for the proof of this fact.

Proposition 3.7. The bilinear form $(,)$ on $F(q) \bigwedge_q^{-}$ is non-degenerate.

Proof. Take $x \in (F(q) \bigwedge_q^{-})_\nu$. It is enough to show that if $(x, (F(q) \bigwedge_q^{-})_\nu) = 0$, then $x = 0$. We prove this by induction on $|\nu|$. This is clearly true for $\nu = 0$. We assume that $|\nu| \geq 1$, and that the claim holds for $\nu'$ such that $|\nu'| < |\nu|$. By (1.3.5), and by using $(f_i, f_i) \neq 0$, we see that $(y, r(x)) = 0$ for any $y$. Thus, by induction, $r(x) = 0$. Since this is true for any $i \in I$, by Proposition 1.26, we conclude that $x = 0$. The proposition is proved.

Proposition 3.8. Under the notation in 3.6, we have

(i) For any $x, y \in A' \bigwedge_q^{-}$, $(\Phi(x), \Phi(y)) = (x, y)$.

(ii) The map $\Phi : A' \bigwedge_q^{-} \rightarrow V_q$ is injective.

Proof. The map $\Phi$ can be extended to the map $\tilde{\Phi} : F(q) \bigwedge_q^{-} \rightarrow F(q) V_q$. To prove (ii), it is enough to show that $\tilde{\Phi}$ is injective. This follows from (i) since the bilinear form
on $F_q \mathbb{U}_q^{-}$ is non-degenerate by Proposition 3.7. Hence it is enough to prove (i).
The proof of (i) is done in an almost similar way as in the proof of [SZ2, Prop. 2.8].
We just give some additional remarks. We use the same notation as in the proof of Proposition 2.8 in [SZ2].
The definition of $J_1$ given there should be replaced by an $A$-submodule of $(\mathfrak{U}_q^{-} \otimes A \mathfrak{U}_q^{-})^\sigma$ generated by orbit sums $O(z)$ for $z \in A \mathfrak{U}_q^{-} \otimes A \mathfrak{U}_q^{-}$ such that $\sigma(z) \neq z$. Let $\eta_1 \in \mathfrak{U}$. The computation of $r(\tilde{f}_{\eta_1})$ in [SZ2, (2.8.2)] is done as follows; Take $\eta_1 \in \mathfrak{U}$. Then $f_i \otimes 1 + 1 \otimes f_i$ are commuting each other for $i \in \eta_1$, and we have

$$
r(\tilde{f}_{\eta_1}) = \prod_{i \in \eta_1} (f_i \otimes 1 + 1 \otimes f_i)
= \sum_{\zeta \subset \eta_1} \left( \prod_{i \in \zeta} f_i \otimes \prod_{j \in \eta_1 - \zeta} f_j \right)
= \tilde{f}_{\eta_1} \otimes 1 + 1 \otimes \tilde{f}_{\eta_1} \mod J_1.
$$

Since $\tilde{f}_{\eta_1} \otimes 1 + 1 \otimes \tilde{f}_{\eta_1} = \tilde{r}(\tilde{f}_{\eta_1})$, (2.8.1) in [SZ2] holds for $k = 1$. Then (2.8.2) is proved by a similar argument. For the proof of (2.8.7) in [SZ2], we use, for $z_1 \in U_q^{-} \otimes U_q^{-}$ such that $\sigma(z_1) \neq z_1$,

$$(O(z_1), \tilde{f}_{\eta_1} \otimes \tilde{y}' = \sum_{z_1 \in O(z_1)} (z_1, \tilde{f}_{\eta_1} \otimes \tilde{y}').$$

Since $|O(z_1)|$ is divisible by $p$, (2.8.7) follows. It remains to check, for $\eta, \eta' \in \mathfrak{U}$ with $i \in \eta$, that

$$(3.8.1) \quad (\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = \begin{cases} (1 - q_i^2)^{-|\eta|} & \text{if } \eta = \eta', \\ 0 & \text{if } \eta \neq \eta'. \end{cases}
$$

In fact, if $\eta \neq \eta'$, $\text{wt}(\tilde{f}_{\eta}) \neq \text{wt}(\tilde{f}_{\eta'})$. Hence $(\tilde{f}_{\eta}, \tilde{f}_{\eta'}) = 0$ by (1.3.3). Assume that $\eta = \eta'$. Write $\eta = \zeta \cup \{i\}$, and $\tilde{f}_\eta = f_i \tilde{f}_\zeta$, where $f_i = \prod_{j \in \zeta} f_i$. Since $f_i$ are commuting each other for $i \in \eta$, we have $r(\tilde{f}_\eta) = \tilde{f}_\zeta$. Then $(\tilde{f}_\eta, \tilde{f}_\eta) = (f_i, f_i)(\tilde{f}_\zeta, \tilde{f}_\zeta)$. By induction on $|\zeta|$ for any subset $\zeta_i \subset \eta$, we have

$$(\tilde{f}_\eta, \tilde{f}_\eta) = \prod_{i \in \eta} (f_i, f_i) = (1 - q_i^2)^{-|\eta|}.$$ 

Thus (3.8.1) holds. This proves (i). The proposition is proved. \[\square\]

**3.9.** In order to prove the surjectivity of $\Phi$, we need a preliminary for Kashiwara operators. Assume that $(\alpha_i, \alpha_j) = 0$. Since $i^r$ and $j^r$ commute each other, and the left action of $\tilde{f}_i$ commutes with $j^r$ by Lemma 1.4, $\Pi_{i,a}$ and $\Pi_{i,b}$ in (1.7.1) commute each other. Take $x \in U_q^{-}$, and write it as $x = \sum_{n \geq 0} f_i^{(n)} x_n$ with $x_n \in U_q^{-}[i]$. We
also write \( x_n = \sum_{m \geq 0} f_j^{(m)} x_{n,m} \) with \( x_{n,m} \in U^-_q[i] \), hence

\[
(3.9.1) \quad x = \sum_{n,m \geq 0} f_i^{(n)} f_j^{(m)} x_{n,m}.
\]

Then \( x_{n,m} = q_i^{n(n-1)/2} q_j^{m(m-1)/2} \Pi_{j,m} \Pi_{i,n}(x) \). Since \( \Pi_{j,m} \Pi_{i,n}(x) = \Pi_{i,n} \Pi_{j,m}(x) \), we have \( x_{n,m} = x_{m,n} \in U^-_q[i] \cap U^-_q[j] \). Since the expression of \( x \) in (3.9.1) is unique, we have a direct sum decomposition

\[
(3.9.2) \quad U^-_q = \bigoplus_{n,m \geq 0} f_i^{(n)} f_j^{(m)} (U^-_q[i] \cap U^-_q[j]).
\]

If \( x \in A U^-_q \), then \( x_{n,m} \in A U^-_q \). Hence (3.9.2) implies that

\[
(3.9.3) \quad A U^-_q = \bigoplus_{n,m \geq 0} f_i^{(n)} f_j^{(m)} A(U^-_q[i] \cap U^-_q[j]).
\]

By a similar argument as in the proof of (3.9.3), we have the following lemma.

**Lemma 3.10.** For \( \eta \in \mathbb{L} \), set \( U^-_q[\eta] = \bigcap_{i \in \eta} U^-_q[i] \). Then we have

\[
(3.10.1) \quad A U^-_q = \bigoplus_{(a_i) \in \mathbb{N}^\eta} \left( \prod_{i \in \eta} f_i^{(a_i)} \right) A(U^-_q[\eta]).
\]

**3.11.** Let \( B \) be the canonical basis of \( U^-_q \), and \( E_i, F_i : B \to B \cup \{0\} \) be Kashiwara operators on \( B \). We consider the partition \( B = \bigcup_{n \geq 0} B_{i:n} \). For \( b \in B \), we have \( \varepsilon_{\sigma(i)}(\sigma(b)) = \varepsilon_i(b) \) by definition. Hence we have \( \sigma(B_{i:n}) = B_{\sigma(i):n} \). This implies that

\[
(3.11.1) \quad \sigma \circ E_i \circ \sigma^{-1} = E_{\sigma(i)}, \quad \sigma \circ F_i \circ \sigma^{-1} = F_{\sigma(i)}.
\]

**Lemma 3.12.** Assume that \( (\alpha_i, \alpha_j) = 0 \). Then for \( b \in B_{i:0} \cap B_{j:0} \), we have \( \pi_{i:n} \pi_{j:m}(b) = \pi_{j:m} \pi_{i:n}(b) \). The map \( \pi_{i:n} \pi_{j:m} \) gives a bijection

\[
(3.12.1) \quad \pi_{i:n} \pi_{j:m} : B_{i:0} \cap B_{j:0} \to B_{i:n} \cap B_{j:m}.
\]

**Proof.** Take \( b \in B_{i:0} \cap B_{j:0} \). Set \( b_1 = \pi_{i:n}(b) \) and \( b_2 = \pi_{j:m}(b_1) \). Then \( b_1 \) is the unique element in \( B \) such that \( b_1 \equiv f_i^{(n)} b \mod f_i^{n+1} U^-_q \). Since \( \varepsilon_j(b_1) = 0 \) by (3.9.2), \( b_2 \) is the unique element in \( B \) such that \( b_2 \equiv f_j^{(m)} b_1 \mod f_j^{m+1} U^-_q \). By (C5), \( b_1 \equiv (b_1)_{[i:n]} = (f_i^{(n)} b)_{[i:n]} \mod qL_Z(\infty) \), and \( b_2 \equiv (b_2)_{[j:m]} = (f_j^{(m)} b_1)_{[j:m]} \mod qL_Z(\infty) \). By (3.9.3), this implies that

\[
(3.12.2) \quad b_2 \equiv (f_j^{(m)} f_i^{(n)} b)_{[n,m]} \mod qL_Z(\infty),
\]
where \(x_{[n,m]}\) denotes the projection of \(x \in A U_q^-\) onto \(f_i^{(n)} f_j^{(m)} A (U_q^- [i] \cap U_q^- [j])\). If we consider \(b'_2 = \pi_{i,n} \pi_{j;m} b\), we obtain a similar formula as (3.12.2) by replacing \(b_2\) by \(b'_2\). Since \(b_2\) and \(b'_2\) are bar-invariant, this implies that \(b_2 = b'_2\). Thus we have proved \(\pi_{i,n} \pi_{j;m} = \pi_{j;m} \pi_{i,n}\). \(\pi_{i,n} \pi_{j;m}\) gives a map \(B_{i,0} \cap B_{j,0} \rightarrow B_{i,0} \cap B_{j,0}\). This map is bijective since \(\pi_{j;m}\) gives a bijection \(B_{i,0} \cap B_{j,0} \rightarrow B_{i,0} \cap B_{j,0}\), and \(\pi_{i,n}\) gives a bijection \(B_{i,0} \cap B_{j,0} \rightarrow B_{i,0} \cap B_{j,0}\). The lemma is proved. □

**Proposition 3.13.** If \(b \in B^\sigma\), \(\varepsilon_i (b)\) is constant for \(i \in \eta\) by 3.11, which we denote by \(\varepsilon_\eta (b)\). We define

\[
B^{\sigma}_{\eta,a} = \{ b \in B^\sigma \mid \varepsilon_\eta (b) = a \}, \quad B^{\sigma}_{\eta,a'} = \bigcup_{a' > a} B^{\sigma}_{\eta,a'}.
\]

We have a partition \(B^\sigma = \bigsqcup_{a \geq 0} B^{\sigma}_{\eta,a}\). By Lemma 3.12, one can define a bijection \(\pi_{\eta,a} : B^{\sigma}_{\eta,0} \overset{\sim}{\rightarrow} B^{\sigma}_{\eta,a}\), where \(\pi_{\eta,a}\) is the restriction of \(\prod_{i \in \eta} \pi_{i,a}\) on \(B^{\sigma}_{\eta,0}\). We define Kashiwara operators \(\tilde{F}_\eta, \tilde{E}_\eta : B^\sigma \rightarrow B^\sigma \cup \{0\}\) in a similar way as 1.14 by using \(\pi_{\eta,a}\).

Note that \(\tilde{F}_\eta, \tilde{E}_\eta\) coincides with the restriction of \(\prod_{i \in \eta} F_i\) on \(B^\sigma\), and similarly for \(\tilde{E}_\eta\).

Take \(b \in B^{\sigma}_{\eta,0}\), and set \(b' = \pi_{\eta,a} b\). By using the decomposition (3.10.1) and the property (C7) of the canonical basis, we see that

\[
\tilde{f}_\eta^{(a)} b = \left(\prod_{i \in \eta} f_i^{(a_i)}\right) b \equiv b' \mod Z_{\eta,a}
\]

with \(Z_{\eta,a} = \sum (a_i) (\prod_{i \in \eta} f_i^{(a_i)}) A U_q^-\), where \((a_i)\) runs over all the elements in \(N^n\) such that \(a_i \geq a\) for all \(i \in \eta\) and that \(a_i > a\) for some \(i\). Note that \(Z_{\eta,a}\) is \(\sigma\)-invariant.

It follows from (1.9.1) and (C7), \(\bigcup_{n \geq 0} B_{;n'}\) gives an \(A\)-basis of \(\sum_{n \geq 0} f_i^{(n')} A U_q^-\). This implies that

\[
B^{\sigma}_{\eta,a} = \bigcup (a_i) \{ b \in B \mid \varepsilon_i (b) = a_i \}, \quad (a_i)\) runs over all the elements in \(N^n\) such that \(a_i \geq a\) for any \(i \in \eta\), and that \(a_i > a\) for some \(i\). Then \(B^{\sigma}_{\eta,a}\) gives an \(A\)-basis of \(Z_{\eta,a}\).

**Proposition 3.14.** The map \(\Phi : A U_q^- \rightarrow V_q\) is surjective.

*Proof.* The proposition is proved in a similar way as the proof of Proposition 2.10 in [SZ2]. We know that the image of \(B^\sigma\) gives a basis of \(V_q\). Thus it is enough to see, for each \(b \in B^\sigma\), that

\[
\pi(b) \in \operatorname{Im} \Phi.
\]

Take \(b \in B^\sigma\). Assume that \(b \in B_{\nu}\). If \(\nu = 0\), then \(b = 1\), and (3.14.1) holds. So assume that \(\nu \neq 0\), and by induction on \(|\nu|\), we may assume that (3.14.1) holds for \(b' \in B_{\nu'}\) such that \(|\nu'| < |\nu|\). There exists \(i \in I\) such that \(a = \varepsilon_i (b) > 0\) by (C6). We may also assume (by the backward induction on \(a\)) that, if \(b' \in B^\sigma \cap B_{\nu}\) is such that \(\varepsilon_i (b') > a\), then \(b'\) satisfies (3.14.1) (note that \(B^\sigma \cap B_{\nu}\) is a finite set, hence the set of such \(b'\) is empty if \(a >> 0\)). Let \(\eta\) be the \(\sigma\)-orbit containing \(i\). Then \(b \in B^{\sigma}_{\eta,a}\).
We consider \( \pi_{\eta}^{-1}(b) = b' \in B_{\eta,0}^\sigma \). By applying (3.13.2), there exists \( z \in Z_{\eta,>a}^\sigma \) such that
\[
b = f^{(a)}(b') + z.
\]
By induction, \( \pi(b') = \pi(\tilde{f}(a))b' \in \text{Im } \Phi \). Hence
\[
\pi(\tilde{f}(a))b' + z = \pi(b) \in \text{Im } \Phi.
\]
On the other hand, by (3.13.3), \( z \) is written as an \( A \)-linear combination of the orbit sum of \( b'' \in B_{\eta,>a}^\sigma \). If \( b'' \) is not \( \sigma \)-stable, its orbit sum is contained in \( J \), hence \( \pi(b'') = 0 \). If \( \sigma(b'') = b'' \), then \( b'' \in B_{\eta,>a}^\sigma \) satisfies the condition \( \varepsilon_i(b'') > a \) for \( i \in \eta \). Hence by induction hypothesis, \( \pi(b'') \in \text{Im } \Phi \). It follows that \( \pi(z) \in \text{Im } \Phi \), and we conclude that \( \pi(b) \in \text{Im } \Phi \). The proposition is proved. \( \square \)

3.15. Theorem 3.4 is now proved by Proposition 3.8 and Proposition 3.14 (modulo Proposition 3.5). Note that the discussion in the proof of Theorem 3.4 works also for the case where \( p = 2 \).

4. Canonical bases

4.1. We keep the setup in 3.2. Thus the order of \( \sigma : I \to I \) is a power of \( p \), and we consider \( U_q^- \) and \( U_q^-^{\sigma} \). We assume the existence of the canonical basis (resp. the canonical signed basis) if \( p \neq 2 \) (if \( p = 2 \)). Let \( B \) be the canonical basis of \( U_q^- \). Our aim is to construct the canonical basis \( B \) of \( U_q^- \) and a bijection \( B^\sigma \simeq B \), by making use of the isomorphism \( \Phi : A'U_q^- \simeq \sim V_q \). Note that in considering \( A'U_q^- \) or \( V_q \) with \( p = 2 \), there is no difference between the canonical basis and the canonical signed basis. In the discussion below, we basically consider the case where \( p \neq 2 \), but all the discussion works also for \( p = 2 \), except Lemma 4.23.

4.2. We consider the direct sum decomposition of \( A'U_q^- \) given in Lemma 3.10, and consider the action \( \sigma \) on \( A'U_q^- \). Then \( A(U_q^-[\eta]) \) is \( \sigma \)-stable, and \( \sigma \) permutes the factors \( \prod_{i \in \eta} f_i^{(a)} \). Those factors are \( \sigma \)-invariant if and only if \( a_i \) has a constant value \( a \) for any \( i \in \eta \), in that case \( \prod_i f_i^{(a)} = \tilde{f}_\eta^{(a)} \). If we replace \( A \) by \( A' \), a similar formula as in (3.10.1) holds for \( A' \). Thus we have
\[
(A'U_q^-)^{\sigma} = \bigoplus_{a \in \mathbb{N}} \tilde{f}_\eta^{(a)} A'(U_q[-\eta]^{\sigma}) \mod J.
\]
Recall that \( \pi : A'U_q^- \to V_q \) is the projection. We define a submodule \( V_q[\eta] \) of \( V_q \) by \( V_q[\eta] = \pi(A'U_q[-\eta]^{\sigma}) \). By using the \( A' \)-basis \( \pi(B^\sigma) \) of \( V_q \), the following lemma is easily obtained from (4.2.1).

Lemma 4.3. For \( \eta \in I \), we have
\[
V_q = \bigoplus_{a \in \mathbb{N}} g_\eta^{(a)} V_q[\eta].
\]
4.4. We consider the direct sum decomposition of $A\mathcal{U}_q^-$ as in (1.9.1). Since
$
\Phi^{-1}(\pi(B^x))$

induces a direct sum decomposition of $A'\mathcal{U}_q^-$, this induces a direct sum decomposition

\begin{equation}
A'\mathcal{U}_q^- = \bigoplus_{a \geq 0} f_q^{(a)} A\mathcal{U}_q^-[\eta].
\end{equation}

Comparing (4.3.1) and (4.4.1), together with Theorem 3.4, we have the following.

**Corollary 4.5.** Assume that $\eta \in L$. The map $\Phi : A'\mathcal{U}_q^- \to V_q$ induces an isomorphism of $A'$-modules,

\begin{equation}
\Phi : f_q^{(a)} A\mathcal{U}_q^-[\eta] \to g_q^{(\eta)} V_q[\eta].
\end{equation}

4.6. Recall that $\mathcal{L}_Z(\infty)$ is a $Z[q]$-submodule of $A\mathcal{U}_q^-$ spanned by $B$. Then $\sigma$ acts on $\mathcal{L}_Z(\infty)$, and $\pi(\mathcal{L}_Z(\infty)^{\sigma})$ gives an $F[q]$-submodule of $V_q$ spanned by $\pi(B^\sigma)$. We denote $\pi(\mathcal{L}_Z(\infty)^{\sigma})$ by $F[q]V_q$ and $\pi(B^\sigma)$ by $B^\sigma$. $B^\sigma$ gives an $F[q]$-basis of $F[q]V_q$.

Let $\eta \in L$. By (3.10.1), we have a direct sum decomposition

\begin{equation}
\tilde{f}_\eta^{(a)} A\mathcal{U}_q^- = \tilde{f}_\eta^{(a)} A(U_q^-[\eta]) \oplus \bigoplus_{(a_i)_{i \in \eta}} \left( \prod_{i \in \eta} f_i^{(a_i)} \right) A(U_q^-[\eta]),
\end{equation}

where $(a_i)_{i \in \eta}$ is such that $a_i \geq a$ for any $i$, and that $a_i > a$ for some $i$. Note that the second term coincides with $Z_{\eta;>a}$ given in (3.13.2). For $x \in \tilde{f}_\eta^{(a)} A\mathcal{U}_q^-$, let $x_{[\eta;a]}$ be the projection of $x$ onto $\tilde{f}_\eta^{(a)} A(U_q^-[\eta])$. By the discussion in 3.9, if $x \in \mathcal{L}_Z(\infty)$, then $x_{[\eta;a]}$ belongs to $\mathcal{L}_Z(\infty)$. $\tilde{f}_\eta^{(a)} U_q^-$ has a basis $B_{\eta;>a} = B_{\eta;>a} \sqcup B_{\eta;a}$, where $B_{\eta;a} = \bigcap_{i \in \eta} B_{i;a}$. Let $z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^-$ be the $Z[q]$-submodule of $\tilde{f}_\eta^{(a)} U_q^-$ spanned by $B_{\eta;>a}$. Then $z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^-$ is a $Z[q]$-submodule of $\mathcal{L}_Z(\infty)$. On the other hand, $\{b_{[\eta;a]} \mid b \in B_{\eta;a}\}$ gives an $A$-basis of $\tilde{f}_\eta^{(a)} U_q^-[\eta]$, and $B_{\eta;>a}$ gives an $A$-basis of $Z_{\eta;>a}$. We denote by $z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^-[\eta]$ the $Z[q]$-submodule of $\tilde{f}_\eta^{(a)} U_q^-[\eta]$ spanned by $\{b_{[\eta;a]} \mid b \in B_{\eta;a}\}$, and by $z_{[\eta]} Z_{\eta;>a}$ the $Z[q]$-submodule of $Z_{\eta;>a}$ spanned by $B_{\eta;>a}$. Thus we have a decomposition as $Z[q]$-submodules of $\mathcal{L}_Z(\infty)$,

\begin{equation}
z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^- = z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^-[\eta] \oplus z_{[\eta]} Z_{\eta;>a}.
\end{equation}

Note that $z_{[\eta]} \tilde{f}_\eta^{(a)} U_q^-[\eta]$ coincides with $\tilde{f}_\eta^{(a)} (z_{[\eta]} U_q^-[\eta])$. Also note that (4.6.2) holds if we replace $\eta$ by any subset $\gamma \subset \eta$, and define $\tilde{f}_\gamma^{(a)} U_q^-[\gamma], Z_{\gamma;>a}$ accordingly.

We show a lemma.

**Lemma 4.7.** Assume that $b \in B_{\eta;a}$. Then $b - b_{[\eta;a]} \in q\mathcal{L}_Z(\infty)$.

**Proof.** We consider the following statement.

(4.7.1) For any subset $\gamma \subset \eta$, $b - b_{[\gamma;\eta]} \in q\mathcal{L}_Z(\infty)$.

We prove (4.7.1) by induction on $|\gamma|$. If $|\gamma| = 1$, it is certainly true by (1.12.1). So assume that (4.7.1) holds for $\gamma \subset \eta$, and prove that it holds for $\gamma' = \gamma \cup \{j\} \subset \eta$. The proof is completed.
η. By using the induction hypothesis, and (4.6.2) for γ, one can write as $b = b_{[\gamma;a]} + \sum a \nu b'$, where $b' \in B_{\gamma;a}$ and $a \nu \in \mathbb{q}[q]$. We consider the decomposition $f_j^{(a)} U_q^{-} = f_j^{(a)} U_q^{-} [j] \oplus f_j^{(a+1)} U_q^{-}$. Write $b_{[\gamma;a]} = x_1 + x_2, b' = y_1 + y_2$ according to this decomposition, where $x_1, y_1 \in f_j^{(a)} U_q^{-} [j]$ and $x_2, y_2 \in f_j^{(a+1)} U_q^{-}$. Then $x_1$ coincides with $b_{[\gamma';a]}$, and $x_1 + \sum a \nu y_1$ coincides with $b_{[\gamma;a]}$. By (1.12.1), $b \equiv b_{[\gamma;a]} \mod q \mathcal{L}_q (\infty)$. Since $y_1 \in \mathcal{L}_q (\infty), \sum a \nu y_1 \in q \mathcal{L}_q (\infty)$. It follows that $b \equiv x_1 = b_{[\gamma';a]} \mod q \mathcal{L}_q (\infty)$, and (4.7.1) holds for $\gamma'$. The lemma is proved. □

4.8. Let $F[q] V_q$ be the $F[q]$-submodule of $V_q$ spanned by $B^\circ$ as in 4.6. Take $\eta \in \mathcal{J}$. We consider the decomposition of $V_q$ as in Lemma 4.3, and for $x \in V_q$, we denote by $x_{[\eta;\alpha]}$ the projection of $x$ onto $g_\eta^{(a)} V_q [\eta]$. Then $\{ b_{[\eta;\alpha]} \mid b \in B^\circ_{\eta;\alpha} \}$ gives an $\mathcal{A}'$-basis of $g_\eta^{(a)} V_q [\eta]$, and $B^\circ_{\eta;\alpha}$ gives an $\mathcal{A}'$-basis of $\sum a' > a g_\eta^{(a')} V_q$ (here set $B^\circ_{\eta;\alpha} = \pi (B^\sigma_{\eta;\alpha})$ and $B^\circ_{\eta;\alpha} = \pi (B^\sigma_{\eta;\alpha})$). For each $b \in B^\circ_{\eta;\alpha}$, $b_{[\eta;\alpha]}$ belongs to $F[q] V_q$. We denote by $F[q] g_\eta^{(a)} V_q [\eta]$ the $F[q]$-submodule of $F[q] V_q$ spanned by $b_{[\eta;\alpha]}$, which coincides with $g_\eta^{(a')} F[q] V_q [\eta]$. Thus we have a decomposition as $F[q]$-submodule of $F[q] V_q$,

$$
\sum_{a' \geq a} g_\eta^{(a')} F[q] V_q = g_\eta^{(a)} F[q] V_q [\eta] \oplus \sum_{a' > a} g_\eta^{(a')} F[q] V_q
$$

As a corollary to Lemma 4.7, we have the following.

Corollary 4.9. For $b \in B^\circ_{\eta;\alpha}$, we have $b - b_{[\eta;\alpha]} \in q (F[q] V_q)$.

4.10. We consider the $\mathcal{A}'$-algebra isomorphism $\Phi : \mathcal{A}' U_q^\circ \cong V_q$, and define $\mathcal{A}'$-basis $\mathcal{B}^\circ$ of $\mathcal{A}' U_q^\circ$ by $\mathcal{B}^\circ = \Phi^{-1} (B^\circ)$. Consider the decomposition $\mathcal{A}' U_q^\circ = \bigoplus_{a > a} \mathcal{A}' U_q^\circ [\eta]$ in (4.4.1). Then by Corollary 4.5, $\{ b_{[\eta;\alpha]} \mid b \in \mathcal{B}^\circ_{\eta;\alpha} \}$ gives an $\mathcal{A}'$-basis of $\mathcal{A}' U_q^\circ [\eta]$, and $\mathcal{B}^\circ_{\eta;\alpha}$ gives an $\mathcal{A}'$-basis of $\sum a' > a \mathcal{A}' U_q^\circ [\eta]$. Here we set $\mathcal{B}^\circ_{\eta;\alpha} = \Phi^{-1} (B^\circ_{\eta;\alpha})$, and $\mathcal{B}^\circ_{\eta;\alpha} = \Phi^{-1} (B^\circ_{\eta;\alpha})$. Let $\mathcal{L}_F (\infty)$ be the $F[q]$-submodule of $\mathcal{A}' U_q^\circ$ spanned by $\mathcal{B}^\circ$. Then for each $b \in \mathcal{B}^\circ_{\eta;\alpha}$, $b_{[\eta;\alpha]}$ belongs to $\mathcal{L}_F (\infty)$, and we denote by $F[q] f_j^{(a)} U_q [\eta]$ the $F[q]$-submodule of $\mathcal{A}' f_j^{(a)} U_q [\eta]$ spanned by $b_{[\eta;\alpha]}$ for $b \in \mathcal{B}^\circ_{\eta;\alpha}$, which coincides with $f_j^{(a')} F[q] U_q [\eta]$. Thus we have a decomposition as $F[q]$-submodules of $\mathcal{L}_F (\infty)$,

$$
\sum \mathcal{L}_F (\infty) = \bigoplus_{a' > a} \mathcal{L}_F (\infty) [\eta] \oplus \sum f_j^{(a')} F[q] U_q [\eta].
$$

Corollary 4.9 can be rewritten as follows;

Corollary 4.11. For $b \in \mathcal{B}^\circ_{\eta;\alpha}$, we have $b - b_{[\eta;\alpha]} \in q \mathcal{L}_F (\infty)$. 
4.12. Since $\mathbf{B}$ is almost orthonormal, $\mathbf{B}^\diamond$ gives an almost orthonormal basis of $V_q$ in the sense that, for $b, b' \in \mathbf{B}^\diamond$,

$$ (b, b') \in \delta_{b,b'} + (qF[[q]] \cap F(q)). $$

$\Phi$ is compatible with the bilinear forms on $A \cdot U_q^-$ and on $V_q$ by Proposition 3.8 (i). Hence $\mathbf{B}^\diamond$ gives an almost orthonormal basis of $A \cdot U_q^-$ in the sense of (4.12.1).

Moreover, since $\pi$ and $\Phi$ are compatible with the bar-involution, any $b \in \mathbf{B}^\bullet$ is bar-invariant.

Recall that $\mathbf{B}^\bullet_{\eta,a} = \Phi^{-1}(\mathbf{B}^\diamond_{\eta,a})$ for each $\eta \in \mathfrak{L}, a \geq 0$. Kashiwara operators $\bar{\mathcal{E}}_{\eta}, \bar{\mathcal{F}}_{\eta}$ on $\mathbf{B}^\diamond$ induces Kashiwara operators $\mathcal{E}_{\eta}, \mathcal{F}_{\eta}$ on $\mathbf{B}^\bullet$. $\mathcal{F}_{\eta}$ gives a bijection $\mathbf{B}^\bullet_{\eta,a} \simeq \mathbf{B}^\bullet_{\eta,a+1}$, and $\mathcal{E}_{\eta}$ gives the inverse map $\mathbf{B}^\bullet_{\eta,a+1} \simeq \mathbf{B}^\bullet_{\eta,a}$. Then (3.13.2) implies the following: for each $b \in \mathbf{B}^\bullet_{\eta,a}$, set $b' = \mathcal{E}_{\eta} b \in \mathbf{B}^\bullet_{\eta,a}$. Then we have

$$ f^{(a)}_\eta b \equiv b' \mod Z^\bullet_{\eta,a}, $$

where $Z^\bullet_{\eta,a} = \sum_{a' > a} f^{(a')}_{\eta} A \cdot U_q^-$. Moreover, the map $b \mapsto b'$ gives a bijection

$$ \pi_{\eta,a} : \mathbf{B}^\bullet_{\eta,0} \simeq \mathbf{B}^\bullet_{\eta,a}. $$

The $*$-operation on $U_q^-$ induces the $*$-operation on $V_q$, which leaves $f^{(a)}_\eta$ invariant. Hence $\Phi$ is compatible with $*$-operations on $A \cdot U_q^-$ and on $V_q$. Since $(\mathbf{B}^\diamond) = \mathbf{B}^\bullet$, we have $*(\mathbf{B}^\diamond) = \mathbf{B}^\diamond$, and so

$$ *(\mathbf{B}^\bullet) = \mathbf{B}^\bullet. $$

4.13. Let $Q = \bigoplus_{i \in I} \mathbf{Z} \alpha_i$ be the root lattice of $X$, and $Q = \bigoplus_{\eta \in \mathfrak{L}} \mathbf{Z} \alpha_\eta$ the root lattice of $\mathfrak{X}$. $\sigma$ acts on $Q$ by $\alpha_i \mapsto \alpha_{\sigma(i)}$, and we have $Q^\sigma \simeq Q$ under the map $\sum_{i \in \eta} \alpha_i \mapsto \alpha_\eta$. If $b \in \mathbf{B}^\sigma$, then the weight of $b$ is contained in $Q^\sigma$, and we have a partition $\mathbf{B}^\sigma = \bigcup_{\nu \in Q^\sigma} \mathbf{B}^\nu$, where $\mathbf{B}^\sigma = \mathbf{B}^\sigma \cap \mathbf{B}^\nu$. This gives a partition $\mathbf{B}^\diamond = \bigcup_{\nu \in Q^\sigma} \mathbf{B}^\nu$, where $\mathbf{B}^\nu = \pi(\mathbf{B}^\nu)$. It follows that $V_q$ has a weight space decomposition $V_q = \bigoplus_{\nu \in Q^\sigma} (V_q)_{\nu}$, where $(V_q)_{\nu}$ is an $A^\nu$-subspace of $V_q$ spanned by $\mathbf{B}^\nu$. On the other hand, the weight space decomposition $U_q^- = \bigoplus_{\nu \in Q^-} (U_q^-)_{\nu}$ induces a weight space decomposition $A \cdot U_q^- = \bigoplus_{\nu \in Q^-} (A \cdot U_q^-)_{\nu}$. The map $\Phi : A \cdot U_q^- \simeq V_q$ is compatible with those weight space decompositions under the identification $Q \simeq Q^\sigma$. We have a partition $\mathbf{B}^\bullet = \bigcup_{\nu \in Q^-} \mathbf{B}^\nu$, where $\mathbf{B}^\nu = \Phi^{-1}(\mathbf{B}^\nu) = \mathbf{B}^\nu \cap (A \cdot U_q^-)_{\nu}$.

4.14. Let $\mathbf{Z}_p$ be the ring of $p$-adic integers, and set $A_p = \mathbf{Z}_p[q, q^{-1}]$, which is a ring containing $A = \mathbf{Z}[q, q^{-1}]$. We have a natural surjective map $A_p \rightarrow A_p/pA_p \simeq A' = F[q, q^{-1}]$. Let us consider $A_p \cdot U_q^- = A_p \otimes A \cdot U_q^-$, which is an extension of $A \cdot U_q^-$. Let $\varphi : A \cdot U_q^- \rightarrow A_p \cdot U_q^- / (p(A \cdot U_q^-)) \simeq A \cdot U_q^-$ be the natural surjective map. $\varphi$ is extended to the surjective map $A_p \cdot U_q^- \rightarrow A \cdot U_q^-$, which we also denote by $\varphi$. $\varphi$ is compatible with the bar-involutions.
Let $M$ be an $A$-submodule of $A \bigoplus_{\eta}^{-}$ such that $\overline{M} = M$, and assume that $\varphi(M) = A'\overline{M}$ is spanned by a finite subset $B_{M}^{*}$ of $B^{*}$. Set $A_{p}M = A_{p} \otimes A \overline{M}$. We have a surjective map $\varphi : A_{p}M \rightarrow A'\overline{M}$. For each $b \in B_{M}^{*}$, choose $x \in M$ such that $\varphi(x) = b$. Since $b$ is bar-invariant, $x - \overline{x} \in pM$. Thus one can write $x - \overline{x} = py$ for some $y \in M$. $y$ satisfies the condition that $\overline{y} = -y$. Since $\varphi(y) \in A'\overline{M}$ is written as an $A'$-linear combination of the basis $B_{M}^{*}$, there exists $u \in A'\overline{M}$ such that $\varphi(y) = u - \overline{u}$. Hence $y$ is written as $y = y_{1} - \overline{y}_{1} + pz$ for some $y_{1}, z \in M$ such that $\overline{z} = -z$. Repeating this procedure, and taking the limit, one can find $c \in A'\overline{M}$ such that $\overline{c} = c$ and that $\varphi(c) = b$. For each $b \in B_{M}^{*}$, we choose $c$ as above, and let $B_{M}$ be the set of such $c \in A_{p}M$ obtained from $B_{M}^{*}$. We show that

**Lemma 4.15.** Let the notations be as above.

(i) $B_{M}$ gives an $A_{p}$-basis of $A_{p}M$.

(ii) Assume that $B_{M} \subset M$, and that they are almost orthonormal in the sense of 1.13. Then $B_{M}$ gives an $A$-basis of $M$.

**Proof.** Let $M_{0}$ be the $A_{p}$-submodule of $A_{p}M$ spanned by $B_{M}$. Then we have $A_{p}M = M_{0} + p(A_{p}M)$. Hence by Nakayama’s lemma, $M_{0} = A_{p}M$. We consider the relation that $\sum_{c \in B_{M}} a_{c}c = 0$ with $a_{c} \in A_{p}$. If this relation is non-trivial, we may assume that some of $a_{c}$ is not contained in $pA_{p}$. But then $\sum_{c} a_{c}\varphi(c) = 0$, where $a_{c} \in A'$ is the image of $a_{c}$, and so $a_{c} = 0$ for all $c$. This shows that all $a_{c} \in pA_{p}$, a contradiction. Thus $B_{M}$ is linearly independent. (i) holds.

Next assume that $B_{M} \subset M$, and they are almost orthonormal. Take $x \in M$ and write as $x = \sum_{c \in B_{M}} a_{c}c$, with $a_{c} \in A_{p}$. It is enough to show that $a_{c} \in A$ for any $c$. Let $t$ be the smallest integer $\geq 0$ such that $\sum_{c \in B_{M}} a_{c}c = 0$ with $a_{c} \in A_{p}$. Assume that $c_{1}$ is such that $\sum_{c \in B_{M}} a_{c}c_{1} = 0$. Then $a_{c_{1}} = \sum_{c \in B_{M}} a_{c}c_{1}$, and $\sum_{c \in B_{M}} a_{c}c = 0$. On the other hand, $\sum_{c \in B_{M}} a_{c}c = \sum_{c \in B_{M}} a_{c}c_{1}$, and $\sum_{c \in B_{M}} a_{c}c = 0$. Hence (ii) holds. The lemma is proved. □

**4.16.** Let $b \in B_{n,0}^{*}$, and choose $c \in A_{p}U_{q}^{-}$ such that $\overline{c} = c$ and that $\varphi(c) = b$. Set $b' = \pi_{\eta,0}b \in B_{n,0}^{*}$. We shall construct $c' \in A_{p}U_{q}^{-}$ satisfying a similar property as in (4.12.2). Take $x \in A\overline{U}_{q}^{-}$ such that $\varphi(x) = b'$. Then $\varphi(x - f_{\eta}(a)c) = b' - f_{\eta}(a)b \in Z_{\eta,>a}$ by (4.12.2). This element is bar-invariant. Hence by a similar argument as in 4.14 (applied for $M = Z_{\eta,>a} = \sum_{a' > a} f_{\eta}(a')A_{p}U_{q}^{-}$), there exists $z \in A_{p}Z_{\eta,>a} = \sum_{a' > a} f_{\eta}(a')A_{p}U_{q}^{-}$ with $\overline{z} = z$ such that $\varphi(z) = b' - f_{\eta}(a)b$. It follows that $x - f_{\eta}(a)c - z \in p(A_{p}U_{q}^{-})$. Then there exists $y \in A_{p}U_{q}^{-}$ such that $\overline{y} - y = py$. Here $y$ satisfies the condition that $\overline{y} = -y$. Thus again, by a similar argument as in 4.14, there exists $c' \in A_{p}U_{q}^{-}$ such that $\overline{c'} = c'$ and that $\varphi(c') = b'$. We consider the decomposition $A_{p}U_{q}^{-} = \bigoplus_{a \geq 0} f_{\eta}(a)A_{p}U_{q}^{-}[\eta]$. For $x \in A_{p}U_{q}^{-}$, we denote by $x[\eta,a]$ the projection of $x$ onto $f_{\eta}(a)A_{p}U_{q}^{-}[\eta]$. The following lemma holds.

**Lemma 4.17.** Assume that $c \in A_{p}U_{q}^{-}$ is such that $\overline{c} = c$ and that $\varphi(c) = b \in B_{n,0}^{*}$. 


(i) There exists \( c' \in A_pU_q^- \) with \( c' = c' \) such that \( \varphi(c') = \pi_{\eta}a \) and that

\[
(4.17.1) \quad c' \equiv f(q)^a c \mod A_pZ_{q;>a}.
\]

(ii) Under the notation in 4.16, we have

\[
(4.17.2) \quad (f(q)^a \eta c)[\eta; a] - (f(q)^a \eta c)[\eta; a] \in A_pZ_{q;>a}.
\]

Proof. (i) follows from the discussion in 4.16. We show (ii). By (4.17.1), \( (f(q)^a \eta c)[\eta; a] \equiv c' \mod A_pZ_{q;>a} \). Since \( c' = c' \) and \( A_pZ_{q;>a} \) is bar-invariant, we obtain (4.17.2). \( \square \)

We shall prove the following theorem.

**Theorem 4.18.** Assume that \( p \neq 2 \). Under the setup in 4.1, the following holds.

(i) \( U_q^- \) has the canonical basis \( B \).

(ii) The map \( \varphi : A U_q^- \rightarrow A U_q^- \) gives a bijection \( \varphi : B \cong B^\circ \). Hence the natural map \( A U_q^{-\sigma} \rightarrow V_q \rightarrow A U_q^- \) induces a unique bijection \( \xi : B^\sigma \cong B \) compatible with Kashiwara operators, where

\[
(4.18.1) \quad \xi : B^\sigma \xrightarrow{\pi} B^\circ \xrightarrow{\Phi^{-1}} B^\circ \xrightarrow{\Psi^{-1}} B.
\]

(iii) \( \ast(B) = B \).

**Remark 4.19.** In the case where \( p = 2 \), a weaker result than Theorem 4.18 holds. Assume that there exists a basis \( B \) of \( U_q^- \) such that \( \tilde{B} = B \sqcup -B \) is the canonical signed basis, satisfying the properties (C1) \( \sim \) (C6) in 1.12, and (C7') below instead of (C7).

(C7'): Assume that \( b \in B_{i;0} \). Then for \( a > 0 \), there exists a unique element \( b' \in B_{i;a} \) such that

\[
\pm b' \equiv f_i^{(a)} b \mod f_i^{a+1} U_q^-.
\]

The correspondence \( b \mapsto b' \) gives a bijection \( \pi : B_{i;0} \cong B_{i;a} \).

Then there exists a basis \( \tilde{B} \) of \( U_q^- \) such that \( \tilde{B} = \tilde{B} \sqcup -\tilde{B} \) is the canonical signed basis, satisfying the properties (C1) \( \sim \) (C6) and (C7'). Moreover, a similar result as in (ii) in the theorem holds, up to sign. In particular there exists a unique bijection \( \xi : \tilde{B}^\sigma \cong \tilde{B} \) compatible with Kashiwara operators.

4.20. In the discussion below, we basically follow the setup in 4.1. However, in the case where \( p = 2 \), we consider \( B \) and (C7') instead of \( B \) and (C7).

Note that the properties corresponding to (C1) \( \sim \) (C7) in 1.12 hold for \( B^\circ \) in \( A U_q^- \), by replacing \( Z[q] \)-module \( \mathcal{L}_Z(\infty) \) by \( F[q] \)-module \( \mathcal{L}_F(\infty) \). In fact, (C1) and (C4) follows from the discussion in 4.10 and 4.13, (C2), (C3) and (C7) follows from that of 4.12. (C5) follows from Corollary 4.11, and (C6) follows from Proposition 1.26, applied for \( R = F(q) \).
We shall construct $B_{v}$ by induction on $|\nu|$. If $\nu = 0$, $B_{v} = \{1\}$ satisfies all the properties. Thus we assume that $\nu \neq 0$, and that $B_{v'}$ are constructed for all $\nu' < |\nu|$. Note that we have a partition $B^{*} = \bigcup_{\nu \in \mathbb{Q}} B_{\nu}^{*}$. We shall construct $B_{\nu}$ such that $\varphi$ gives a bijection $B_{\nu} \to B_{\nu}^{*}$.

Take $b_{\bullet} \in B_{\nu}^{*}$. By (C6) for $B^{*}$, there exists $\eta \in I$ such that $\varepsilon_{\eta}(b_{\bullet}) = a \neq 0$. Then by (4.12.2) and (4.12.3), there exists $b'_{\bullet} \in B_{\nu,0}$ such that $b_{\bullet} \equiv f_{\eta}(a) b'_{\bullet} \mod \sum_{a' > a} f_{\eta}(a') A' \U_{\eta}^{-}$. Since $b'_{\bullet} \in B_{\nu}^{*}$, with $|\nu| < |\nu|$, by induction hypothesis, there exists $b' \in B_{\nu,0}$ such that $\varphi(b') = b'_{\bullet}$. Since $b' \in L_{Z}(\infty)$, $b'_{\nu,0} \in L_{Z}(\infty)$ by Lemma 1.11 (i). Then $(f_{\eta}(a) b')_{[\eta,a]} = f_{\eta}(b'_{\nu,0}) \in L_{Z}(\infty)$ by (1.9.2). Note that $(B_{\nu,0}^{'})_{\nu}$ is finite dimensional, $f_{\eta}(a) \U_{\eta}^{-} [\eta] \cap (U_{\eta}^{-})_{\nu} = \{0\}$ if $n > 0$. We shall construct $B_{\nu,0} \cap B_{\nu}$ by backward induction on $a$. So assume that $B_{\nu,0}^{' \nu}$ were already constructed for $a' > a$ and that $\bigcup_{a' > a} (B_{\nu,0}^{'} \cap B_{\nu})$ gives an $A$-basis of $\sum_{a' > a} f_{\eta}(a') A' \U_{\eta}^{-} \cap (U_{\eta}^{-})_{\nu}$. By the above discussion, $(f_{\eta}(a) b')_{[\eta,a]} \in L_{Z}(\infty)$. By applying Lemma 4.17 (ii), we have

$$f_{\eta}(a) b'_{\nu,0} \in L_{Z}(\infty).$$

Set $x = (f_{\eta}(a) b')_{[\eta,a]} - (f_{\eta}(a) b')_{[\eta,a]}$. Then $x \in \sum_{a' > a} f_{\eta}(a') A' \U_{\eta}^{-} \cap (U_{\eta}^{-})_{\nu}$. Hence we can write as $x = \sum_{a' > a} \sum_{b'' \eta'} A_{b''} \U_{\eta}^{-} \cap (U_{\eta}^{-})_{\nu}$. Since $\U = -x$, and all the $b''$ are bar-invariant, $A_{b''}$ is written as $a_{b''} = c_{b''} - \overline{c_{b''}}$ for some $c_{b''} \in qZ[q]$. Set $b = (f_{\eta}(a) b')_{[\eta,a]} + \sum_{b''} c_{b''} b''$. Then $b \in L_{Z}(\infty)$ and $\overline{b} = b$. Since $b_{[\eta,a]} = (f_{\eta}(a) b')_{[\eta,a]}$, we see that

$$b \equiv b_{[\eta,a]} \mod qL_{Z}(\infty).$$

We also note that

$$b \equiv f_{\eta}(a) b' \mod f_{\eta}^{a+1} \U_{\eta}^{-}.$$

In fact, $b \equiv b_{[\eta,a]} \mod f_{\eta}^{a+1} \U_{\eta}^{-}, f_{\eta}(a) b' \equiv (f_{\eta}(a) b')_{[\eta,a]} \mod f_{\eta}^{a+1} \U_{\eta}^{-}$. Since $b_{[\eta,a]} = (f_{\eta}(a) b')_{[\eta,a]}$, (4.20.3) follows.

Next we show that

$$\varphi(b) = b_{\bullet}.$$

In fact, note that

$$\varphi((f_{\eta}(a) b')_{[\eta,a]}) = (\varphi(f_{\eta}(a) b'))_{[\eta,a]} = (f_{\eta}(b')_{[\eta,a]} = (b_{\bullet})_{[\eta,a]}.$$
It follows that \( \varphi(b) = (b_*)_{[\eta:a]} + \sum_{b''} \varphi_{(b'')} \), where \( b'' = \varphi(b'') \) and \( \varphi_{(b'')} \in qF[q] \). Since \( b_* \equiv (b_*)_{[\eta:a]} \mod qL^*(\infty) \) by Corollary 4.11, and \( \varphi(b), b_* \) are bar-invariant, we obtain \( \varphi(b) = b_* \). Hence (4.20.4) holds.

Set \( Z_{[\eta:a]} = \sum_{a' > a} f_{[\eta':a]}(U)^{-} \). In the above discussion, for each \( b_* \in B_{[\eta:a]} \), we have constructed \( b \in L_z^\infty \) such that \( \varphi(b) = b_* \). We define \( B_{[\eta:a]} \cap B_{[\eta:b]} \) as the set of those \( b \) corresponding to \( b_* \). Set \( B_{[\eta:a]} \cap B_{[\eta:b]} = \bigcup_{a' > a} (B_{[\eta:a]} \cap B_{[\eta:b]}) \). We show

(4.20.5) The elements in \( B_{[\eta:a]} \cap B_{[\eta:b]} \) are almost orthonormal, and \( B_{[\eta:a]} \cap B_{[\eta:b]} \) gives a \( Z[q] \)-basis of \( Z_{[\eta:a]} \cap L_z^\infty \) \( \cap (U^-)^\nu \).

In fact, take \( b \in B_{[\eta:a]} \cap B_{[\eta:b]} \). Since \( b \equiv b_{[\eta:a]} \mod qL_z^\infty \) by (4.20.2), and since \( b_{[\eta:a]} \) is orthogonal to \( B_{[\eta:a]} \cap B_{[\eta:b]} \) by (1.12.2), \( b, b_0 \) \( \in qZ[q] \cap Q(q) \) for any \( b_0 \in B_{[\eta:a]} \cap B_{[\eta:b]} \). On the other hand, it follows from the construction, there exists \( b' \in B_{[\eta:a]} \cap B_{[\eta:b]} \) with \( |\nu'| < |\nu| \) such that \( b_{[\eta:a]} = (f_{[\eta:1} b'_{[\eta:a]} \) Here \( (f_{[\eta:1} b'_{[\eta:a]} = (f_{[\eta:1} b'_{[\eta:a]} \). Since \( b' \in B_{[\eta:a]} \cap B_{[\eta:b]} \), we have \( b' \equiv b'_{[\eta:a]} \mod qL_z^\infty \) by (4.20.2). Now take \( b_1 \in B_{[\eta:a]} \cap B_{[\eta:b]} \). There exists \( b_1' \in B_{[\eta:a]} \cap B_{[\eta:b]} \) satisfying similar properties as in the case of \( b \). Thus we have

\[
(b, b_1) \equiv (b_{[\eta:a]}, (b_1)_{[\eta:a]} \mod qZ[q] \cap Q(q)
= \left( (f_{[\eta:1} b'_{[\eta:a]} (f_{[\eta:1} b'_{[\eta:a]} \right)
= \left( f_{[\eta:1} b'_{[\eta:a]} (f_{[\eta:1} b'_{[\eta:a]} \right)
= c(b_{[\eta:a]} b'_{[\eta:a]} \right)
\text{ with } c \in 1 + qZ[q] \cap Q(q),
\]

where the last equality follows from (1.9.2). Since \( (b_{[\eta:a]}, (b_1)_{[\eta:a]} \equiv (b', b'_1) \), we have \( b, b_1 \equiv (b', b'_1) \mod qZ[q] \cap Q(q) \). Hence the almost orthonormality for \( b, b_1 \) follows from that for \( b', b'_1 \). Thus we have proved the almost orthonormality for \( B_{[\eta:a]} \cap B_{[\eta:b]} \). Now by applying Lemma 4.15 (ii) for \( M = \sum_{a' > a} (U^-)^\nu \cap (U^-)^\nu \), we give an \( A \)-basis of \( M \). Since \( \varphi(B_{[\eta:a]} \cap B_{[\eta:a]} \) gives an \( F[q] \)-basis of \( \varphi(M) \cap L_z^\infty \) \( \cap (U^-)^\nu \), we see that \( B_{[\eta:a]} \cap B_{[\eta:b]} \) gives a \( Z[q] \)-basis of \( M \cap L_z^\infty \). Thus (4.20.5) holds.

By the above procedure, one can construct \( B_{[\eta:a]} \cap B_{[\eta:b]} \) for any \( a > 0 \). But this method cannot be applied for constructing \( B_{[\eta:0]} \cap B_{[\eta:b]} \). In order to treat this case, we prepare a lemma.

**Lemma 4.21.** Let \( a \geq 0 \). Let \( B_{[\eta:a]} \cap B_{[\eta:b]} \) be the almost orthonormal basis of \( \sum_{a' > a} f_{[\eta:1} (U^-)^\nu \cap (U^-)^\nu \) constructed in 4.19. Assume that \( x \in L_z^\infty \) with \( (x, x) \in 1 + qA_q \). Further assume that \( x \sum_{a' > a} f_{[\eta:1} (U^-)^\nu \cap (U^-)^\nu \) and that \( \varphi(x) = b_* \) for some \( b_* \in B_{[\eta:a]} \). Then

(4.21.1) \( x \equiv x_{[\eta:a]} \mod qL_z^\infty \).

**Proof.** By Corollary 4.11, \( b_* \in B_{[\eta:a]} \) is written as \( b_* = (b_*)_{[\eta:a]} + \sum_{b'} a_{b'} b_* \), where \( b'_* \in B_{[\eta:a]} \) and \( a_{b'} \in qF[q] \). Take \( b' \in B_{[\eta:a]} \cap B_{[\eta:b]} \) such that \( \varphi(b') = b'_* \). Note that \( \varphi(x_{[\eta:a]} = (b_*)_{[\eta:a]} \). Then \( z = x - x_{[\eta:a]} - \sum_{b'} a_{b'} b'_* \in p(U^-) \), where \( b' \in
\( B_{q,a} \cap B_{\nu} \) and \( a_{\nu} \in qZ[q] \) is such that its image to \( qF[q] \) coincides with \( a_{\nu} \). Here \( z \in Z_{q,a} \cap L_z(\infty) \). Hence by (4.20.5), \( z \) can be written as \( z = \sum \nu c_{\nu} b' \), where \( b' \in B_{q,a} \cap B_{\nu} \) and \( c_{\nu} \in pZ[q] \). Suppose that \( z \notin qL_z(\infty) \). Then there exists \( b'_0 \) such that \( c_{\nu} b'_0 \in pZ[q] - qZ[q] \). This implies that \( c_{\nu} b'_0 \equiv d \mod qZ[q] \) for some \( 0 \neq d \in pZ \). We write \( x = \sum n_{\geq a} f^{(a)} x_n \), where \( x_n \in U_n[q] \). Then \( f^{(a)} x_a = x_{[q,a]} \). Since \( \varphi \) is compatible with bilinear forms, the image of \( (x_{[q,a]}, x_{[q,a]}) \) on \( F(q) \) coincides with \( \left((b_0), (b)_{[q,a]} \right) \) which is contained in \( 1 + (qF[q] \cap F(q)) \). It follows that \( (x_a, x_a) \notin qA_0 \). Then by Lemma 4.11 (i), \( (x_a, x_a) \in 1 + qA_0 \), and \( (x_{a'}, x_{a'}) \in qA_0 \) for all \( a' \neq a \). By the almost orthonormality of the basis \( B_{q,a} \cap B_{\nu} \), if there exists \( b'_0 \) such that \( c_{\nu} b'_0 \equiv d \mod qZ[q] \) with \( d \in pZ - \{0\} \), we must have \( (x_{a'}, x_{a'}) \notin qA_0 \) for \( a' \) such that \( b'_0 \in B_{q,a'} \). This is absurd, and we conclude that \( z \in qL_z(\infty) \). The lemma is proved.

\[4.22.\] We now construct \( B_{q,0} \cap B_{\nu} \). Take \( b_0 \in B_{q,0} \cap B_{\nu} \). There exists \( \eta' \in I \) such that \( \varepsilon_{\eta'}(b_0) = a > 0 \). By applying the discussion in 4.19 for \( \eta' \) and \( a > 0 \), we can find \( b \in B_{q,a} \cap B_{\nu} \) such that \( \varphi(b) = b_0 \). We define \( B_{q,0} \cap B_{\nu} \) as the set of those \( b \) such that \( \varphi(b) = b_0 \) for various \( b \in B_{q,a} \cap B_{\nu} \) (and for various \( \eta' \)). We set \( B_{\nu} = \bigcup_{a \geq 0} (B_{q,a} \cap B_{\nu}) \).

We know, by (4.20.5), that \( B_{q,0} \cap B_{\nu} \) gives an almost orthonormal basis of \( Z_{q,0} \cap L_z(\infty) \cap (U_0)^{\nu} \). Hence by applying Lemma 4.21 for \( a = 0 \), we see that, for \( b \in B_{q,0} \cap B_{\nu} \),

\[4.22.1 \] \( b \equiv b_{[q,0]} \mod qL_z(\infty) \).

Next we note that

\[4.22.2\] The elements in \( B_{\nu} \) are almost orthonormal, and \( B_{\nu} \) gives a \( Z[q]-basis of L_z(\infty) \cap (U_0)^{\nu} \).

We show (4.22.2). Take \( b \in B_{q,0} \). If \( b' \in B_{q',0} \cap B_{\nu} \), \( (b_{[q,0]}, b') = 0 \). Hence \( (b, b') \in qZ[q] \cap Q(q) \) by (4.21.1). So, assume that \( b, b' \in B_{q,0} \). Set \( b_0 = \varphi(b), b'_0 = \varphi(b'), \varepsilon_{\eta'}(b_0) = a, \varepsilon_{\eta'}(b'_0) = a' \). By our assumption, \( a > 0 \). If \( a' > 0 \), then by applying the discussion in (4.20.5) for \( B_{q',0} \), \( b \) and \( b' \) are almost orthonormal. Thus assume that \( a' = 0 \). By applying Lemma 4.21 for \( B_{q',0} \cap B_{\nu} \), we have \( b' \equiv b_{[q',0]} \mod qL_z(\infty) \). Since \( (b, b_{[q',0]}) = 0 \), we have \( (b, b') \in qZ[q] \cap Q(q) \). Hence \( B_{\nu} \) is almost orthonormal. The second assertion of (4.22.2) is shown as in (4.20.5). Thus (4.22.2) holds.

In the construction of \( B_{\nu} \), the choice of \( \eta, \eta' \), etc. is not unique. But the following lemma shows that \( B_{\nu} \) is determined up to the sign.

**Lemma 4.23.** Let \( B_{\nu} \) be a fixed basis of \( L_z(\infty) \cap (U_0)^{\nu} \) constructed as above.

(i) Let \( x \in L_z(\infty) \cap (U_0)^{\nu} \) be an element such that \( \overline{x} = x \) and that \( (x, x) \in 1 + qA_0 \). Assume further that \( \varphi(x) \in B_{\nu}^{*} \). Then \( x = b \) (resp. \( x = \pm b \)) if \( p \neq 2 \) (resp. \( p = 2 \)), where \( b \) is the unique element in \( B_{\nu} \) such that \( \varphi(x) = \varphi(b) \).

(ii) The set \( B_{\nu} \) is determined uniquely (resp. uniquely up to sign) if \( p \neq 2 \) (resp. \( p = 2 \)), independent from the construction process.
Proof. We show (i). Set \( \varphi(x) = b_\bullet \in \mathcal{B}_\nu^* \), and let \( b \in \mathcal{B}_\nu \) be such that \( \varphi(b) = b_\bullet \). Then \( x - b \in \mathcal{L}_Z(\infty) \cap (\mathcal{U}_q^-)_\nu \cap p(\mathcal{U}_q^-) \). Hence \( x - b \) is written as \( x - b = \sum_{b'} a_{b'b'} \), where \( b' \in \mathcal{B}_\nu \) and \( a_{b'} \in \mathbb{Z}[q] \cap pA \). But since \( x - b \) is bar-invariant, \( a_{b'} \in p\mathbb{Z} \) for any \( b' \in \mathcal{B}_\nu \). Then \( x = (a_b + 1)b + \sum_{b' \neq b} a_{b'b'} \). Since \( \mathcal{B}_\nu \) is almost orthonormal by (4.22.2), we have

\[
(4.23.1) \quad (x, x) \equiv (a_b + 1)^2 + \sum_{b' \neq b} a_{b'}^2 \mod qA_0.
\]

Since \( a_b \in p\mathbb{Z} \), \( a_b + 1 \neq 0 \). Since \( (x, x) \equiv 1 \mod qA_0 \), (4.23.1) implies that \( a_{b'} = 0 \) for any \( b' \in \mathcal{B}_\nu \) \( - \{b\} \), hence \( a_b + 1 = \pm 1 \). This implies that \( a_b + 1 = 1 \) and \( x = b \) if \( p \neq 2 \). While if \( p = 2 \), we have \( a_b + 1 = \pm 1 \), and so \( x = \pm b \). (i) is proved.

We show (ii). Let \( \mathcal{B}_\nu' \) be a set constructed in a similar way as \( \mathcal{B}_\nu \), but using different \( \eta \in \mathcal{I} \). Let \( x \in \mathcal{B}_\nu' \). Then \( x \) satisfies all the conditions in (i). Assume that \( p = 2 \). Then by (i), \( \pm x \in \mathcal{B}_\nu' \), and so \( \pm \mathcal{B}_\nu \subset \pm \mathcal{B}_\nu' \). Replacing the role of \( \mathcal{B}_\nu \) and \( \mathcal{B}_\nu' \), we have \( \pm \mathcal{B}_\nu \subset \pm \mathcal{B}_\nu' \). Hence \( \pm \mathcal{B}_\nu = \pm \mathcal{B}_\nu' \), and (ii) holds. The case \( p \neq 2 \) is similar. The lemma is proved.

\[\Box\]

4.24. In the case where \( p \neq 2 \), by Lemma 4.23, \( \mathcal{B}_\nu \) is defined canonically, independent of the construction process in 4.20. While if \( p = 2 \), Lemma 4.23 determines \( \mathcal{B}_\nu \) only up to sign. Now assume that \( p \neq 2 \). We define \( \mathcal{B} = \bigcup_{\nu \in Q} \mathcal{B}_\nu \).

We show that \( \mathcal{B} \) is the canonical basis of \( \mathcal{U}_-^- \). Clearly \( \mathcal{B} \) satisfies the properties (C1) \( \sim \) (C4) in 1.12. Since \( \varphi \) gives a bijection \( \varphi : \mathcal{B}_\nu \cong \mathcal{B}_\nu^* \) for any \( \nu \in Q_- \), it gives a bijection \( \varphi : \mathcal{B} \cong \mathcal{B}^* \). (C5) follows from Lemma 4.21. Next we show (C7). Take \( b \in \mathcal{B}_{\eta,0} \) for \( \eta \in \mathcal{I} \). If \( \eta \) is the one used for the construction of \( \mathcal{B} \), (4.20.3) holds. Since we can choose any \( \eta \in \mathcal{I} \) for the construction by the previous remark, (4.20.3) holds for this \( \eta \). By (4.12.3), \( b_\bullet \mapsto b'_\bullet \) gives a bijection \( \mathcal{B}_{\eta,0}^* \cong \mathcal{B}_{\eta,0}^* \). Hence \( b \mapsto b' \) gives a bijection \( \mathcal{B}_{\eta,0} \cong \mathcal{B}_{\eta,0}^* \). This proves (C7). (C6) follows from the corresponding property for \( \mathcal{B}^* \) (see 4.20) by using the bijection \( \varphi : \mathcal{B} \cong \mathcal{B}^* \). Thus \( \mathcal{B} \) is the canonical basis of \( \mathcal{U}^- \).

Now assume that \( p = 2 \). The above discussion shows that, there exists a basis \( \mathcal{B} \) such that \( \mathcal{B} = \mathcal{B} \uplus -\mathcal{B} \), satisfying the properties (C1) \( \sim \) (C6), and (C7').

4.25. We now prove Theorem 4.18. (i) is already shown. \( \ast \)-operation commutes with \( \varphi : \mathcal{A}\mathcal{U}^- \rightarrow \mathcal{A}\mathcal{U}^- \). By (4.12.4), we have \( \ast(\mathcal{B}^*) = \mathcal{B}^* \). Since \( \varphi \) induces a bijection \( \mathcal{B} \cong \mathcal{B}^* \), we obtain \( \ast(\mathcal{B}) = \mathcal{B} \). Hence (iii) holds. We show (ii). The bijection \( \varphi : \mathcal{B} \cong \mathcal{B}^* \) induces a bijection

\[
\xi : \mathcal{B}^\sigma \longrightarrow \mathcal{B}^\diamond \longrightarrow \mathcal{B}^* \longrightarrow \mathcal{B}.
\]

Kashiwara operators \( \tilde{F}_\eta : \mathcal{B}^\sigma \rightarrow \mathcal{B}^\sigma \) are obtained as the restriction of \( \prod_{i \in \eta} F_i \) on \( \mathcal{B}^\sigma \). Hence by Proposition 1.19, for any \( b \in \mathcal{B}^\sigma \), there exists a sequence \( \eta_1, \ldots, \eta_N \in \mathcal{I} \), and \( c_1, \ldots, c_N \in \mathbb{Z}_{>0} \) such that

\[
(4.25.1) \quad b = \tilde{F}_{\eta_1}^{c_1} \tilde{F}_{\eta_2}^{c_2} \cdots \tilde{F}_{\eta_N}^{c_N} 1.
\]
Kashiwara operators $F_\eta^*$ on $B^*$ are defined by using the bijection $\Phi^{-1} \circ \pi : B^\sigma \to B^\diamondsuit \to B^*$, and a similar property as (4.25.1) holds for $B^*$. Now Kashiwara operators $F_\eta$ on $B$ are defined for the canonical basis $B$ of $U_q^-$, and from the construction, $\varphi$ is compatible with those Kashiwara operators. Hence $\xi : B^\sigma \simeq B$ is compatible with Kashiwara operators $F_\eta$ and $F_\eta'$. Note that by Proposition 1.19, $B$ also satisfies a similar formula as (4.25.1) by replacing $F_\eta$ by $F_\eta'$. Thus such a bijection $\xi : B^\sigma \simeq B$ is unique. Hence (ii) holds. The proof of Theorem 4.18 is now complete.

4.26. We return to the general setup, and let $\sigma$ be an admissible diagram automorphism of $X$ of any order. Let $\underline{X}$ be the Cartan datum induced from $(X, \sigma)$. Let $U_q^-$ (resp. $U_q^+$) be the quantum enveloping algebra associated to $X$ (resp. $X$). We assume that $X$ is a symmetric Cartan datum. Then by Theorem 1.18, there exists the canonical basis $B$ for $U_q^-$, which satisfies the property that $\ast(B) = B$.

The following result is our main theorem.

**Theorem 4.27.** Under the notation in 4.26, let $B$ be the canonical basis of $U_q^-$. 
(i) Assume that the order of $\sigma$ is odd. Then there exists the canonical basis $\underline{B}$ of $U_q^-$, and a unique bijection $\xi : B^\sigma \simeq \underline{B}$ which is compatible with Kashiwara operators. Moreover, we have $\ast(B) = \underline{B}$.
(ii) Assume that the order of $\sigma$ is even. Then there exists a basis $\underline{B}$ of $U_q^-$ satisfying the properties as in Remark 4.19, and a bijection $\xi \circ \tilde{\xi} : \tilde{\underline{B}} \simeq \underline{B}$ compatible with Kashiwara operators (up to sign).

**Proof.** We prove the theorem in the case where the order of $\sigma$ is odd. The even case is similar. By Proposition 2.5, there exists a sequence $X = X_0, X_1, \ldots, X_k = \underline{X}$, and a diagram automorphism $\sigma_i : X_i \to X_i$ such that $X_{i+1}$ is isomorphic to the Cartan datum induced from $(X_i, \sigma_i)$. Moreover, the order of $\sigma_i$ is an odd prime power. Let $(i)U_q^-$ be the quantum algebra associated to $X_i$. By induction on $i$, we may assume that $(i)U_q^-$ has the canonical basis $(i)B$, which is stable by the $\ast$-operation. Then by Theorem 4.18, there exists the canonical basis $(i)B$ of $(i+1)U_q^-$, which is stable by the $\ast$-operation. and a bijection $\xi_i : (i)B^\sigma \simeq (i+1)B$, compatible with Kashiwara operators. Thus we obtain the canonical basis $\underline{B} = (k)B$ of $U_q^- = (k)U_q^-$, which is stable by the $\ast$-operation. From the construction, we have a bijection $\xi : B^\sigma = (\cdots (B^{\sigma_0})^{\sigma_1} \cdots)^{\sigma_k} \simeq \underline{B}$ as the composite of $\xi_0, \xi_1, \ldots, \xi_{k-1}$. Each $\xi_i$ is compatible with Kashiwara operators, hence $\xi$ is compatible with Kashiwara operators. In particular, $\xi$ is uniquely determined, independent of the expression $\sigma = \sigma_{k-1} \cdots \sigma_1 \sigma_0$. The theorem is proved. \qed

**Remark 4.28.** Recall, by 2.2, that for a Cartan datum $X$ of arbitrary type, there exists a symmetric Cartan datum $\tilde{X}$, and an admissible diagram automorphism $\sigma$ such that the Cartan datum induced from $(\tilde{X}, \sigma)$ is isomorphic to $X$. Thus Theorem 4.27 assures that the quantum enveloping algebra $U_q^-$ of any type has the canonical (signed) basis $B$ in the sense of 1.12 and Remark 4.19, which is stable by the $\ast$-operation.
5. The proof of Proposition 3.5

5.1. We follow the notation in Section 3. Note that $A_q U_q^-$ is the $A'$-algebra with generators $f^{(a)}_\eta (\eta \in I, a \in \mathbb{N})$ with fundamental relations

\[(5.1.1) \quad \sum_{k=0}^{1-a_{\eta \eta'}} (-1)^k \binom{1-a_{\eta \eta'}}{k} f^{(k)}_\eta f^{(1-a_{\eta \eta'})-k}_\eta = 0, \quad (\eta \neq \eta'),\]

\[(5.1.2) \quad [a]_{d_{\eta}} f^{(a)}_\eta = f^a_\eta, \quad (a \in \mathbb{N}),\]

where $d_\eta = (\alpha_\eta, \alpha_\eta)_{1/2}$. In order to prove Proposition 3.5, it is enough to show that $\tilde{f}_\eta$ satisfies the following relations in $A_q U_q^-\sigma$,

\[(5.1.3) \quad \sum_{k=0}^{1-a_{\eta \eta'}} (-1)^k \binom{1-a_{\eta \eta'}}{k} f^{(k)}_\eta f^{(1-a_{\eta \eta'})-k}_\eta \equiv 0 \mod J, \quad (\eta \neq \eta'),\]

\[(5.1.4) \quad [a]_{d_{\eta}} \tilde{f}^{(a)}_\eta = \tilde{f}^a_\eta, \quad (a \in \mathbb{Z}_{\geq 0}).\]

(5.1.4) is shown as follows. Since $|\eta|$ is a power of $p$, we have $([a]_{d_{\eta}})^{|\eta|} = [a]_{|\eta|d_{\eta}}$ in $A' = F[q, q^{-1}]$. Since $d_\eta = |\eta|d_{\eta}$ for $i \in \eta$, we have

\[\tilde{f}^{(a)}_\eta = \prod_{i \in \eta} f^{(a)}_i = (([a]_{d_{\eta}})^{-|\eta|} \prod_{i \in \eta} f^a_i = ([a]_{d_{\eta}})^{-1} \tilde{f}^a_\eta.\]

Hence (5.1.4) holds. The rest of this section is devoted to the proof of (5.1.3).

Recall that the following Serre relations hold in $U_q^-$. For $i \neq j \in I$,

\[(5.1.5) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} f^{(k)}_i f^{(1-a_{ij})-k}_j = 0.\]

We fix $\eta \neq \eta' \in I$, and write them as $\eta = \{1, \ldots, 1_m\}, \eta' = \{2, \ldots, 2_{n-1}\}$, here $|\eta| = m, |\eta'| = n - 1$.

5.2. Here we consider the special case where $|\eta| = 1$, and any $j \in \eta'$ is joined to $i = 1 \in \eta$. Thus $a_{ij}$ is independent of the choice of $j \in \eta'$, and we set $r = 1 - a_{ij}$. In this case, we have $-a_{\eta q'} = |\eta'| (-a_{ij}) = (n - 1)(r - 1)$. We set

\[(5.2.1) \quad L = 1 - a_{\eta q'} = (n - 1)(r - 1) + 1.\]

We have \(\tilde{f}_\eta = f_1, \tilde{f}_\eta' = f_2, \ldots f_{2n-1}.\) In order to verify the formula (5.1.3), we need to compute $f^{(k)}_\eta f^{(k)}_\eta f^{(1-a_{\eta \eta'})-k}_\eta$ for various $0 \leq k \leq L$. More generally, for each tuple $(a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $\sum_i a_i = L$, we consider the correspondence

\[(5.2.2) \quad (a_1, \ldots, a_n) \longleftrightarrow f^{a_1}_1 f^{a_2}_1 f^{a_2}_2 \cdots f^{a_{n-1}}_{2n-1} f^{a_n}_1 f^{a_{n-1}}_{2n-1} f^{a_n}_1 \in U_q^-.
The commuting relations for \( f_1 \) and \( f_2 \) are given by the Serre relations (5.1.5) for \( r = 1 - a_{ij} \) with \( i = 1, j = 2k \).

5.3. Based on the observation in 5.2, we introduce the following combinatorial object. We fix \( r \geq 2 \). Let \( V_n \) be a vector space over \( \mathbb{Q}(q) \) spanned by \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \), which satisfies the following relations for \( 1 \leq i \leq n - 1 \); if \( a_i \geq r \) for some \( i \), then \( a \) is written as

\[
\sum_{0 \leq j \leq r} (-1)^j \begin{bmatrix} r \\ j \end{bmatrix} a(j) = 0,
\]

where \( a(j) \in \mathbb{N}^n \) is given by

\[
a(j) = (a_1, \ldots, a_{i-1}, a_i - j, a_i + j, a_{i+2}, \ldots, a_n).
\]

In particular, \( a(0) = a \). For each \( m \geq 1 \), we denote by \( E_n(m) \) the subspace of \( V_n \) spanned by

\[
E_n(m) = \{ a = (a_1, \ldots, a_n) \mid \sum_{1 \leq i \leq n} a_i = m, a_i \in [0, r-1] \text{ for } 1 \leq i < n \}.
\]

It is clear if \( a = (a_1, \ldots, a_n) \in V_n \) is such that \( \sum a_i = m \), then \( a \in E_n(m) \).

We prove the following.

**Proposition 5.4.** Assume that \( n = 2 \). Then for any \( k \geq 0 \), the following formula holds in \( V_2 \).

\[
(r + k, \ell) = \sum_{s=1}^{r} (-1)^{s-1} \begin{bmatrix} r+k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} r+k \\ r-s \end{bmatrix} (r-s, \ell + s + k).
\]

5.5. (5.4.1) holds for \( k = 0 \) by (5.3.1). We shall prove (5.4.1) by induction on \( k \). Assume that (5.4.1) holds for \( k' < k \).

First assume that \( k \leq r \). In this case, we have

\[
(r + k, \ell) = \sum_{s=1}^{k} (-1)^{s-1} \begin{bmatrix} r \\ s \end{bmatrix} (r + k - s, \ell + s) + \sum_{s=k+1}^{r} (-1)^{s-1} \begin{bmatrix} r \\ s \end{bmatrix} (r + k - s, \ell + s)
\]

\[
= \sum_{s=1}^{k} (-1)^{s-1} \begin{bmatrix} r \\ s \end{bmatrix} \left( \sum_{t=1}^{r} (-1)^{t-1} \begin{bmatrix} t+(k-s)-1 \\ t-1 \end{bmatrix} \begin{bmatrix} r+(k-s) \\ r-t \end{bmatrix} (r-t, \ell + t + k) \right) + \sum_{t=1}^{r-k} (-1)^{t+k-1} \begin{bmatrix} r \\ t+k \end{bmatrix} (r-t, \ell + t + k)
\]
\[
= \sum_{t=1}^{r} (-1)^{t-1} A_t(r - t, \ell + t + k),
\]

where

\[
(5.5.1) \quad A_t = \left( \sum_{s=1}^{k} (-1)^{s-1} \binom{r}{s} \left[ \begin{array}{c} t + k - s - 1 \\ t - 1 \\ \end{array} \right] \left[ \begin{array}{c} r + k - s \\ r - t \\ \end{array} \right] \right) + (-1)^k \left[ \begin{array}{c} r \\ t + k \\ \end{array} \right].
\]

The last term appears only in the case where \( t + k \leq r \). A similar computation shows, in the case where \( k > r \), we have

\[
(5.5.2) \quad A_t = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \left[ \begin{array}{c} t + k - s - 1 \\ t - 1 \\ \end{array} \right] \left[ \begin{array}{c} r + k - s \\ r - t \\ \end{array} \right].
\]

We prove the following formula.

**Lemma 5.6.** Assume that \( 1 \leq t \leq r \) and \( k \in \mathbb{Z} \). Then we have

\[
(5.6.1) \quad \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} \left[ \begin{array}{c} t + k - s - 1 \\ t - 1 \\ \end{array} \right] \left[ \begin{array}{c} r + k - s \\ r - t \\ \end{array} \right] = 0.
\]

**Proof.** Note that, for any \( k \in \mathbb{Z} \), one can write as

\[
(5.6.2) \quad [k - s][k - s - 1] \cdots [k - s - r + 1] = \sum_{j=0}^{r} F_j(q)q^{s-2js},
\]

where \( F_j(q) \in \mathbb{Q}(q) \) is independent from \( s \). Applying (5.6.2) to our situation, we have

\[
\left[ \begin{array}{c} t + k - s - 1 \\ t - 1 \\ \end{array} \right] = \sum_{j=0}^{t-1} F_j(q)q^{s(t-1)-2js},
\]

\[
\left[ \begin{array}{c} r + k - s \\ r - t \\ \end{array} \right] = \sum_{j=0}^{r-t} G_j(q)q^{s(r-t)-2js},
\]

where \( F_j(q), G_j(q) \in \mathbb{Q}(q) \) are independent from \( s \). It follows that

\[
(5.6.3) \quad \left[ \begin{array}{c} t + k - s - 1 \\ t - 1 \\ \end{array} \right] \left[ \begin{array}{c} r + k - s \\ r - t \\ \end{array} \right] = \sum_{j=0}^{r-1} H_j(q)q^{s(r-1-2j)},
\]

where \( H_j(q) = \sum_{j' + j'' = j} F_{j'}(q)G_{j''}(q) \) is independent from \( s \).
Recall the formula (1.11.5),

\[(5.6.4)\quad \prod_{\ell=0}^{n-1} (1 + q^{2\ell}x) = \sum_{k=0}^{n} q^{k(n-1)} \binom{n}{k} x^k,\]

where \(x\) is another indeterminate. If we put \(x = -q^{-2j}\) for \(j = 0, \ldots, n-1\), the left hand side of (5.6.4) is equal to zero. Hence

\[(5.6.5)\quad \sum_{k=0}^{n} (-1)^k q^{k(n-1-2j)} \binom{n}{k} = 0 \quad \text{for} \quad j = 0, \ldots, n-1.\]

If we substitute (5.6.3) into the left hand side of (5.6.1), then the equality of (5.6.1) follows from (5.6.5). Thus the lemma is proved. \(\square\)

**5.7.** We consider the expansion of \((r + k, \ell)\) in 5.5. Now assume that \(k > r\). Then by (5.5.2) and Lemma 5.6, we see that

\[A_t = \binom{t + k - 1}{t - 1} \binom{r + k}{r - t}.\]

Hence (5.4.1) holds for \(k > r\).

Next assume that \(k \leq r\). In this case, \(A_t\) is given by (5.5.1). We consider the formula in Lemma 5.6. We fix \(s\) such that \(k + 1 \leq s \leq r\). In this case, \(t + k - s - 1 < t - 1\). Hence if \(t + k - s - 1 \geq 0\), then \(\binom{t + k - s - 1}{t - 1} = 0\). On the other hand, since \(r + k - s \geq 0\), if \(r + k - s < r - t\), then \(\binom{r + k - s}{r - t} = 0\). It follows that

\[(5.7.1)\quad \binom{t + k - s - 1}{t - 1} \binom{r + k - s}{r - t} \neq 0\]

only when \(t + k - s - 1 < 0\) and \(r + k - s \geq r - t\), namely only when \(s = t + k\). If \(s = t + k\), the left hand side of (5.7.1) is equal to \(\binom{-1}{t - 1} = (-1)^{t-1}\). Hence by Lemma 5.6, we have

\[\sum_{s=0}^{k} (-1)^s \binom{r}{s} \binom{t + k - s - 1}{t - 1} \binom{r + k - s}{r - t} + (-1)^{t+k} \binom{r}{t + k} (-1)^{t-1} = 0.\]

By (5.5.1), we obtain \(A_t = \binom{t + k - 1}{t - 1} \binom{r + k}{r - t}\). Thus (5.4.1) holds also for \(k \leq r\). The proof of Proposition 5.4 is now complete.

**5.8.** Proposition 5.4 holds for \(k \geq 0\). Now assume that \(-r \leq k < 0\). If \(s + k - 1 \geq 0\), then \(\binom{s + k - 1}{s - 1} = 0\), and if \(r + k < r - s\), then \(\binom{r + k}{r - s} = 0\). Hence
$A_s = \begin{bmatrix} s+k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} r+k \\ r-s \end{bmatrix} \neq 0$ only when $s+k-1 < 0$ and $r+k \geq r-s$, namely when $s = -k$. In that case, we have $A_s = \begin{bmatrix} -1 \\ s-1 \end{bmatrix} = (-1)^{s-1}$. Thus $(r+k, \ell)$ has an expansion as in (5.4.1), where all the coefficients of $(r-s, \ell + s+k)$ are zero unless $s = -k$, in which case, the coefficient is equal to 1. Hence (5.4.1) holds for $r+k \geq 0$. Set, for $1 \leq s \leq r$, (5.8.1)

$$A(k, s) = \begin{bmatrix} s+k-r-1 \\ s-1 \end{bmatrix} \begin{bmatrix} r \\ r-s \end{bmatrix}.$$

Then by replacing $r+k$ by $k$ in (5.4.1), we have

**Corollary 5.9.** Assume that $n = 2$, and $k \geq 0$. Then the following formula holds in $V_2$.

(5.9.1) $(k, \ell) = \sum_{s=1}^{r} (-1)^{s-1} A(k, s)(r-s, \ell + s+k - r)$.

By making use of Corollary 5.9, we obtain a formula for $V_n$ for any $n \geq 2$.

**Lemma 5.10.** Assume that $(k, 0, \ldots, 0, \ell) \in V_n$. Then we have

$(k, 0, \ldots, 0, \ell)$

$$= \sum_{a_1 + \cdots + a_n = k+\ell \atop a_1, \ldots, a_{n-1} \in [0, r-1]} (-1)^{a_1+\cdots+a_{n-1}+(n-1)(r-1)} \left( \prod_{1 \leq i \leq n-1} A(k-x_i, r-a_i) \right)(a_1, a_2, \ldots, a_n),$$

where $x_1 = a_1 + a_2 + \cdots + a_{i-1}$, and $(a_1, \ldots, a_n) \in \mathcal{E}_n(k+\ell)$.

**Proof.** We prove the lemma by induction on $n$. By (5.9.1), the lemma holds for $n = 2$. We assume that it holds for $n' < n$. By applying (5.9.1) for the first two terms $(k, 0)$ in $(k, 0, \ldots, 0, \ell)$, we have

$$(k, 0, \ldots, 0, \ell) = \sum_{s=1}^{r} (-1)^{s-1} A(k, s)(r-s, s+k-r, 0, \ldots, 0, \ell)$$

$$= \sum_{a_1 \in [0, r-1]} (-1)^{a_1+r-1} A(k, r-a_1)(a_1, k-a_1, 0, \ldots, 0, \ell).$$

Then by applying the induction hypothesis for $(k-a_1, 0, \ldots, 0, \ell) \in V_{n-1}$, we obtain the required formula. □

We prove the following formula.

**Proposition 5.11.** Set $L = (n-1)(r-1) + 1$. Then the following equality holds in $E_n(L)$. 

Proof. We apply Lemma 5.10 to the case where \( k + \ell = L \). In order to prove (5.11.1), it is enough to see, for a fixed \((a_1, \ldots, a_n) \in \mathcal{E}_n(L)\), that

\[
\sum_{k=0}^{L} (-1)^k \binom{L}{k} (k, 0, \ldots, 0, L - k) = 0.
\]

As in (5.6.3), but by replacing the role of \( s \) by \(-k\), we see that

\[
A(k - x_i, r - a_i) = \left[\frac{k - x_i - a_i - 1}{r - a_i - 1}\right] = \sum_{j=0}^{n-1} F_j(q) q^{-(2j)}
\]

for some \( F_j(q) \in \mathbb{Q}(q) \), which is independent from \( k \). It follows that

\[
\prod_{1 \leq i \leq n-1} A(k - x_i, r - a_i) = \sum_{j=0}^{n-1} G_j(q) q^{-(2j)}
\]

where \( G_j(q) \in \mathbb{Q}(q) \) is independent from \( k \). Note that by (5.6.5), we have

\[
\sum_{k=0}^{L} (-1)^k \binom{L}{k} q^{-k(L-1-2j)} = 0
\]

for \( j = 0, \ldots, L - 1 \). Since \( L = (n-1)(r-1) + 1 \), (5.11.3) implies (5.11.2). The proposition is proved.

Proof. We apply Lemma 5.10 to the case where \( k + \ell = L \). In order to prove (5.11.1), it is enough to see, for a fixed \((a_1, \ldots, a_n) \in \mathcal{E}_n(L)\), that

\[
\sum_{k=0}^{L} (-1)^k \binom{L}{k} (k, 0, \ldots, 0, L - k) = 0.
\]

5.12. Returning to the setup in 5.2, we consider \( \eta, \eta' \in \mathcal{I} \) with \( |\eta| = 1, |\eta'| = n - 1 \), where any element \( j \in \eta' \) is joined to \( i \in \eta \). Set \( r = 1 - a_{ij} \), and \( L = 1 - a_{\eta \eta'} = (n - 1)(r - 1) + 1 \). Set \( \tilde{f}_\eta = f_1 \) and \( \tilde{f}_\eta' = f_2, \ldots, f_{2n-1} \). Under the correspondence in (5.2.2), \((k, 0, \ldots, 0, L - k) \in \mathcal{E}_n(L)\) corresponds to \( \tilde{f}_\eta^k \tilde{f}_\eta' \tilde{f}_\eta^{L-k} \in U_q^- \). Thus in view of the discussion in 5.2, Proposition 5.11 can be translated to the following formula (by replacing \( q \) by \( q^{d_i} = q^{d_{\eta i}} \)), which proves (5.1.3) in this special case. Note that this formula holds without the assumption on the order of \( \sigma \), nor modulo \( J \).

**Proposition 5.13.** Under the notation above,

\[
\sum_{k=0}^{1-a_{\eta \eta'}} (-1)^k \left[ \frac{1-a_{\eta \eta'}}{k} \right] \tilde{f}_\eta^k \tilde{f}_\eta' \tilde{f}_\eta^{1-a_{\eta \eta'}-k} = 0.
\]
5.14. We shall extend Proposition 5.13 to the general setup as in 5.1. Hence we consider \( \eta \neq \eta' \in \mathcal{I} \) with \( |\eta| = n - 1, |\eta'| = m \), and write \( \eta = \{1_1, \ldots, 1_m\}, \eta' = \{2_1, \ldots, 2_{n-1}\} \) as in 5.1. For each \( i \in \eta \), let \( A_i \) be the set of elements in \( \eta' \) which is joined to \( i \). Set \( |A_i| = N - 1 \), which is independent of \( i \in \eta \). For \( i \in \eta, j \in A_i \), \( a_{ij} \) is independent of the choice of \((i, j)\). We set \( r = 1 - a_{ij} \). By (2.1.2), we have

\[
a_{\eta \eta'} = \sum_{j \in A_i} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \sum_{j \in A_i} a_{ij} = -(N - 1)(r - 1).
\]

We set

\[
L = 1 - a_{\eta \eta'} = (N - 1)(r - 1) + 1.
\]

We have \( \tilde{f}_\eta = f_{11} \cdots f_{1m}, \tilde{f}_\eta' = f_{21} \cdots f_{2_{n-1}} \). In this case, we need to compute \( \tilde{f}_\eta^k \tilde{f}_\eta' \tilde{f}_\eta'^{-k} = (f_{11} \cdots f_{1m})^k f_{21} \cdots f_{2_{n-1}}(f_{11} \cdots f_{1m})^{L-k} \). More generally, we consider

\[
f^{(1)}(\mathbf{a}) f^{(2)} f^{(n-1)} f^{(n)} = f_{21} f_{22} \cdots f_{2_{n-2}} f_{2_{n-1}} f_{2_{n-2}} \in \mathbb{U}_q
\]

for \( \mathbf{a}(k) \in \mathbb{N}^m (1 \leq k \leq n) \), where we set \( f^{a_1^1} = f_{a_1^1} \cdots f_{a_m^m} \) for \( \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m \). Corresponding to those elements, we consider the matrix \( \mathbf{a} = (\mathbf{a}(1), \ldots, \mathbf{a}(n)) = (a_i^{(k)}) \in M(m, n) \) (here we regard \( \mathbf{a}(k) = t(a_1^{(k)}, \ldots, a_m^{(k)}) \) as the \( k \)-th column vector, and \( \mathbf{a}_i = (a_i^{(1)}, \ldots, a_i^{(n)}) \) as the \( i \)-th row vector). The commuting relations for \( f_i \) and \( f_j \) \((i \in \eta, j \in \eta')\) are given by (5.1.5) if \( j \in A_i \), and \( f_i f_j = f_j f_i \) otherwise.

5.15. Taking the discussion in 5.14 into account, we generalize the combinatorial setting in 5.3 as follows. Fix \( m, n \) and \( N \leq n \), and consider \( M(m, n) \) as in 5.15. For each \( i \in [1, m] \), we fix a subset \( A_i \subset [1, n - 1] \) such that \( |A_i| = N - 1 \). Let \( V_{n,m} \) be a vector space over \( \mathbb{Q}(q) \) spanned by \( \mathbf{a} = (a_i^{(k)}) \in M(m, n) \) satisfying the following relations; for each \( a_i^{(k)} \), there exist \( \mathbf{a}(t) \) with \( \mathbf{a}(t) = (a(t)_i^{(k)}) \) for \( 1 \leq t \leq r \) such that

\[
a = \begin{cases} 
\sum_{1 \leq t \leq r} (-1)^{t-1} \begin{bmatrix} r \\ t \end{bmatrix} \mathbf{a}(t) & \text{if } k \in A_i \text{ and } a_i^{(k)} \geq r, \\
\mathbf{a}(1) & \text{if } k \notin A_i \text{ and } a_i^{(k)} \geq 1,
\end{cases}
\]

where

\[
\mathbf{a}(t)_i = (a_i^{(1)}, a_i^{(k-1)}, a_i^{(k)} - t, a_i^{(k+1)} + t, a_i^{(k+2)}, \ldots, a_i^{(n)}),
\]

and \( \mathbf{a}(t)_{i'} = \mathbf{a}(t') \) for \( i' \neq i \).

For each \( \mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{N}^m \), we denote by \( E_{n,m}(\mathbf{s}) \) the subspace of \( V_{n,m} \) spanned by

\[
\mathcal{E}_{n,m}(\mathbf{s}) = \{ \mathbf{a} = (a_i^{(k)}) \in M(m, n) \mid \sum_{1 \leq k \leq n} a_i^{(k)} = s_i, a_i^{(k)} \in [0, r - 1] \text{ for } 1 \leq k < n \}
\]
In the case where \( s = (L', \ldots, L') \in \mathbb{N}^m \) for some \( L' \geq 1 \), we set \( \mathcal{E}_{n,m}(s) = \mathcal{E}_{n,m}(L') \) and \( E_{n,m}(s) = E_{n,m}(L') \).

The following lemma is a generalization of Lemma 5.10. The proof is essentially reduced to the case where \( m = 1 \), which corresponds to Lemma 5.10.

**Lemma 5.16.** Assume that \( (k, 0, \ldots, 0, \ell) \in V_{n,m} \), where \( k = ^t(k_1, \ldots, k_m), \ell = ^t(\ell_1, \ldots, \ell_m) \in \mathbb{N}^m \) are column vectors. Then we have

\[
(5.16.1) \quad (k, 0, \ldots, 0, \ell) = \sum_{a \in \mathcal{E}_{n,m}(k+\ell)} \left( \prod_{1 \leq i \leq m} (-1)^{a_i^{(t_1)} + \cdots + a_i^{(N-1)} + (N-1)(r-1)} H(a_i, k_i) \right) a,
\]

where \( H(a_i, k_i) = \prod_{1 \leq j < N} A(k_i - x_{ij}, r - a_i^{(t_j)}) \) with \( x_{ij} = a_i^{(t_1)} + \cdots + a_i^{(t_{j-1})} \). (Here we write \( A_i = \{t_1, \ldots, t_{N-1}\} \) along the natural order.)

5.17 In the special case where \( s = (L, \ldots, L) \), we set \( k' = ^t(L - k_1, \ldots, L - k_m) \) for \( k = ^t(k_1, \ldots, k_m) \) with \( k_i \leq L \). Then (5.16.1) is written as

\[
(5.17.1) \quad (k, 0, \ldots, 0, k') = \sum_{a \in \mathcal{E}_{n,m}(L)} \left( \prod_{1 \leq i \leq m} (-1)^{a_i^{(t_1)} + \cdots + a_i^{(N-1)} + (N-1)(r-1)} H(a_i, k_i) \right) a.
\]

As in (5.11.3), \( H(a_i, k_i) \) can be written as

\[
(5.17.2) \quad H(a_i, k_i) = \prod_{1 \leq j < N} A(k_i - x_{ij}, r - a_i^{(t_j)}) = \sum_{j=0}^{(N-1)(r-1)} G_j^{(i)}(q) q^{-k_i \left((r-1)(N-1)-2j\right)},
\]

where \( G_j^{(i)}(q) \in \mathbb{Q}(q) \) is independent from \( k_i \). We shall prove the following formula.

**Proposition 5.18.** Set \( L = (N-1)(r-1) + 1 \). The following formula holds in \( E_{n,m}(L) \).

\[
(5.18.1) \quad \sum_{0 \leq k_1 \leq L} \cdots \sum_{0 \leq k_m \leq L} (-1)^{k_1 + \cdots + k_m} \begin{bmatrix} L \\ k_1 \end{bmatrix} \cdots \begin{bmatrix} L \\ k_m \end{bmatrix} (k, 0, \ldots, 0, k') = 0,
\]

where \( k = ^t(k_1, \ldots, k_m) \) and \( k' = ^t(L - k_1, \ldots, L - k_m) \).

**Proof.** As in the proof of Proposition 5.11, by (5.17.1) and (5.17.2), the proof of (5.18.1) is reduced to the following formula; for \( 0 \leq j_1, \ldots, j_m \leq L - 1 \),

\[
(5.18.2) \quad \sum_{0 \leq k_1 \leq L} (-1)^{k_1 + \cdots + k_m} q^{-\left((L-1-2j_1)k_1 + \cdots + (L-1-2j_m)k_m\right)} \begin{bmatrix} L \\ k_1 \end{bmatrix} \cdots \begin{bmatrix} L \\ k_m \end{bmatrix} = 0.
\]
But the left hand side of (5.18.2) is equal to
\[
\prod_{i=1}^{m} \left( \sum_{0 \leq k_i \leq L} (-1)^{k_i} q^{-(L-1-2j)k_i} \left[ \begin{array}{c} L \\ k_i \end{array} \right] \right),
\]
which is equal to zero by (5.6.5). Thus (5.18.2) holds, and the proposition follows. \( \square \)

5.19. Following the discussion in 5.14, we translate (5.18.1) to the original setup as in 5.1 and 5.14, namely, \( \eta \) and \( \tilde{\eta} \) which is equal to zero by (5.6.5). Thus (5.18.2) holds, and the proposition follows.

Proposition 5.20. Under the notation above, we have

\[
(5.20.1)
\sum_{0 \leq k_1 \leq L} \cdots \sum_{0 \leq k_m \leq L} (-1)^{k_1 + \cdots + k_m} \left[ \begin{array}{c} L \\ k_1 \end{array} \right] \cdots \left[ \begin{array}{c} L \\ k_m \end{array} \right] (f_1^{k_1} \cdots f_m^{k_m}) \tilde{f}_q (f_1^{k'_1} \cdots f_m^{k'_m}) = 0,
\]

where \( k'_a = L - k_a \) for \( a = 1, \ldots, m \), and \( i \in \eta \).

5.21. We consider the action of \( \sigma \) on \( I \), and on \( U_q^\sim \). Then \( \tilde{f}_q \) is \( \sigma \)-stable.

Since \( \sigma \) maps \( f_1^{k_1} \) to \( f_1^{k_1} \), \( f_1^{k_1} \cdots f_m^{k_m} \) is \( \sigma \)-stable if and only if \( k_1 = \cdots = k_m \), namely \( f_1^{k_1} \cdots f_m^{k_m} = \tilde{f}_q \) for some \( 0 \leq k \leq L \). Thus (5.20.1) can be rewritten as

\[
(5.21.1)
\sum_{0 \leq k \leq L} (-1)^{km} \left[ \begin{array}{c} L \\ k \end{array} \right] (f_q)^k \tilde{f}_q \tilde{f}_q \equiv 0 \mod J_1,
\]

where \( J_1 \) is a subspace of \( U_q^\sim \) spanned by the orbit sum \( O(x) \) for \( x \in U_q^\sim \) such that \( O(x) \neq x \). The following result proves (5.1.3), hence the proof of Proposition 3.5 is now complete.

Proposition 5.22. Under the setup in 5.1, (5.1.3) holds, namely, the equation

\[
(5.22.1)
\sum_{k=0}^{1-a_{q\eta'}} (-1)^k \left[ \begin{array}{c} 1 - a_{q\eta'} \\ k \end{array} \right] \tilde{f}_q \tilde{f}_q \tilde{f}_q \equiv 0 \mod J
\]

holds in \( A \cdot U_q^\sim \).

Proof. We follow the notation in 5.21. Since the order of \( \sigma \) is a power of \( p \), \( m = |\eta| \) is also a power of \( p \). Hence in the formula (5.21.1), we have an equality in \( A' \),

\[
\left( \begin{array}{c} L \\ k \end{array} \right)_{d_i} \equiv \left( \begin{array}{c} L \\ k \end{array} \right)_{md_i} \equiv \left( \begin{array}{c} L \\ k \end{array} \right)_{d_\eta}
\]
since $d_q = md_i$. Note that $(-1)^{km} = (-1)^k$ if $m$ is odd, and the term $(-1)^k$ is ignorable in $\mathfrak{A}U^+_q$ if $m$ is even. Since $L = 1 - a_{\eta\eta'}$, the proposition follows from (5.21.1).

□

Remark 5.23. In the case where $X$ is of finite or affine type, and the order of $\sigma$ is 2 or 3, the isomorphism $\Phi : \mathfrak{A}U^+_q \cong V_q$ was established in [SZ1, SZ2]. The proof of the fact that $\Phi$ is a homomorphism is reduced to the case where $X$ has rank 2, namely, $X$ is of type $A_1 \times A_1, A_2, B_2$ or $G_2$. In [SZ1, Prop.1.10], this was verified by case by case computation. The most complicated one is the case where $X$ is of type $D_4$ and $X$ is of type $G_2$. (3.5.1) in [SZ1] corresponds to the formula in Proposition 5.13, namely the case where $|\eta| = 1$, and $|\eta'| = 3$. (3.5.1) was proved after a hard computation by making use of PBW-bases of $U_q^+$ of type $D_4$. As stated in Remark 3.6 in [SZ1], the formula (3.5.1) holds without appealing modulo $J$ nor modulo 3, which corresponds to the statement in Proposition 5.13. For the proof of (3.4.1) in [SZ1], which is the case where $|\eta| = 3$ and $|\eta'| = 1$, we need to consider in $\mathfrak{A}U^+_q$ with modulo $J$. This corresponds to the situation in Proposition 5.22.

Note that Proposition 5.13 and Proposition 5.22 can be proved for the general setup, in a uniform way, and the discussion there is simpler, and more transparent than the one used in [SZ1].

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