Exact Analysis of δ-Function Attractive Fermions and Repulsive Bosons in One Dimension

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Abstract

The Gaudin integral equation for the ground state of a one-dimensional δ-function attractive spin-1/2 fermions is solved in the form of power series. The first few terms of the asymptotic expansions for both strong and weak coupling cases are calculated analytically. The physical quantities such as the ground state energy are expressed in terms of a single dimensionless parameter \( \gamma = c/D \), where \( c \) is the coupling constant and \( D \) is the number density. The results agree with those obtained from the perturbation calculations, which include the one in the classical electrostatics originally by Kirchhoff. In the strong coupling limit, the connection to the solutions of the Lieb-Liniger integral equation for the ground state of a one-dimensional δ-function repulsive bose gas is shown explicitly.

KEYWORDS: δ-function gas, one-dimensional integrable systems, BCS-BEC crossover, Lieb-Liniger integral equation, Gaudin integral equation, Tonks-Girardeau gas, hard-core bose gas

1 Introduction

We discuss a one-dimensional system of quantum \( N \) particles with δ-function interaction in a periodic box of length \( L \). The Hamiltonian of the system is

\[
H = -\sum_{j=1}^{N} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + c \sum_{i \neq j} \delta(x_i - x_j),
\]

where \( m \) is the mass and \( c \) is the coupling constant. Hereafter, we choose a unit, \( \hbar = 1 \) and \( 2m = 1 \) to simplify the expressions.

The system has been studied extensively by use of the Bethe ansatz method.\(^{1-9}\) For \( N \) repulsive (i.e. \( c > 0 \)) bosons, the ground state is described by the integral equation for the distribution function \( \rho(k) \) of the quasi-momenta,\(^1\)

\[
\rho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \int_{-K}^{K} \frac{\rho(q) dq}{c^2 + (k - q)^2},
\]

where \( K \) is a cut-off momentum, which is fixed by the normalization condition,

\[
\int_{-K}^{K} \rho(k) dk = D.
\]

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Here, $D \equiv N/L$ is the number density of the particles. The interval $[-K, K]$ is filled with quasi-momenta $k$. The ground state energy is given by

$$
\frac{E_B}{L} = \int_{-K}^{K} k^2 \rho(k) \, dk.
$$

(1.4)

We call the Lieb-Liniger integral equation.

In 1967, Yang proposed a method, which we now call the generalized Bethe ansatz, or “nested” Bethe ansatz, to study the system with arbitrary internal degree of freedom. For $N$ attractive (i.e. $c < 0$) spin-1/2 fermions, the ground state with total spin 0 is described by

$$
f(q) = 2 + \frac{c}{\pi} \int_{-Q}^{Q} \frac{f(q') \, dq'}{c^2 + (q - q')^2}.
$$

(1.5)

Here, $f(q)$ is the distribution of spins with spin-wave rapidity $q$, and a cut-off rapidity $Q$ is fixed by the normalization condition for $f(q)$:

$$
\frac{1}{\pi} \int_{-Q}^{Q} f(q) \, dq = D.
$$

(1.6)

while the ground state energy is given by

$$
\frac{E_F}{L} = \frac{1}{\pi} \int_{-Q}^{Q} \left( q^2 - \frac{c^2}{4} \right) f(q) \, dq.
$$

(1.7)

For our purpose, it is convenient to introduce the effective ground state energy,

$$
\frac{E_{\text{eff}}}{L} = \frac{E_F}{L} + \frac{c^2}{4} D = \frac{1}{\pi} \int_{-Q}^{Q} q^2 f(q) \, dq.
$$

(1.8)

We call the Gaudin integral equation.

By use of a method proposed in the previous paper to solve, we explicitly calculate the first few terms of the asymptotic expansions of $f(q)$ for both strong and weak coupling cases. It is essential to use the single dimensionless coupling constant $\gamma$, defined by

$$
\gamma = \frac{c}{D} = \frac{2mc}{\hbar^2 D}.
$$

(1.9)

Combining the results with those in the previous papers, we discuss in particular four cases, $\gamma \rightarrow -\infty$, $\gamma \rightarrow -0$, $\gamma \rightarrow +0$, and $\gamma \rightarrow +\infty$. We also study the physical quantities, such as the ground state energy and the chemical potential, as functions of $\gamma$.

This paper is organized as follows. In §2 we solve for large and small $|\gamma|$. The recent discussion on “BCS-BEC crossover” is made clear for these integrable models in one-dimension. For the small $|\gamma|$ case, it is known to be difficult to solve and the perturbation calculation in the classical electrostatics originally by Kirchhoff are discussed in §§3. The last section is devoted to the conclusions. To concentrate on physical discussions, most of the calculations are given in Appendices A, C.

2 Solutions of the Gaudin Integral Equation

2.1 Formulation

Following a method presented in the previous papers, we solve the Gaudin integral equation. In terms of new variables,

$$
q = Qx, \quad c = -Q\lambda, \quad q' = Qy,
$$

(2.1)

$$
f(Qx) = F(x),
$$

(2.2)
becomes
\[ F(y) + \frac{\lambda}{\pi} \int_{-1}^{1} \frac{F(x) \, dx}{\lambda^2 + (x - y)^2} = 2. \]  \tag{2.3}

We also refer to (2.3) as the Gaudin integral equation. As mentioned, we use the number density \( D \) and the dimensionless coupling constant \( \gamma \),

\[ D = \frac{N}{L}, \quad \gamma = \frac{c}{D}. \]  \tag{2.4}

The large (small) \( \lambda \) corresponds to the large (small) \( |\gamma| \).

By expanding
\[ F(x) = \sum_{n=0}^{\infty} a_n x^{2n}, \]  \tag{2.6}
we have
\[ \int_{-1}^{1} dx \frac{F(x)}{\lambda^2 + (x - y)^2} = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{m=0}^{p} \frac{a_n \cdot 2^{2m+1}}{(2m)! (p-m)!} \left[ \sum_{l=1}^{m+n} \frac{(-1)^{l+1} \lambda^{2(l-1)}}{2(m + n - l) + 1} + (-1)^{m+n} \lambda^{2(m+n)-1} \arctan \frac{1}{\lambda} \right] y^{2p}, \]  \tag{2.7}
where \( s = \lambda^2 \). Substituting (2.6) and (2.7) into (2.3), we get
\[ \sum_{p=0}^{\infty} a_p y^{2p} + \frac{\lambda}{\pi} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{m=0}^{p} \frac{a_n \cdot 2^{2m+1}}{(2m)! (p-m)!} \left[ \sum_{l=1}^{m+n} \frac{(-1)^{l+1} \lambda^{2(l-1)}}{2(m + n - l) + 1} + (-1)^{m+n} \lambda^{2(m+n)-1} \arctan \frac{1}{\lambda} \right] y^{2p} = 2. \]  \tag{2.8}

In order that (2.8) holds for any \( y \), \(-1 \leq y \leq 1\), we equate the terms of the same power of \( y^{2p} \). At \( y^0 \) and \( y^{2p} \ (p \geq 1) \), we respectively obtain
\[
\begin{align*}
a_0 &+ \frac{2}{\pi} \arctan \frac{1}{\lambda} \cdot a_0 + \frac{2\lambda}{\pi} \sum_{n=1}^{\infty} a_n \left( \sum_{l=1}^{n} \frac{(-1)^{l+1} \lambda^{2(l-1)}}{2(n-l) + 1} + (-1)^{n} \lambda^{2n-1} \arctan \frac{1}{\lambda} \right) = 2, \quad \tag{2.9a} \\
a_p &+ \frac{2\lambda}{\pi} \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} a_n \cdot 2^{2m} \frac{\partial^{m+p}}{\partial s^{m+p}} \left[ \sum_{l=1}^{m+n} \frac{(-1)^{l+1} \lambda^{2(l-1)}}{2(m + n - l) + 1} \right] = 0. \quad \tag{2.9b}
\end{align*}
\]

In this way, a solution of (2.9a) gives a solution \( F(x) \) of (2.3).

Until now, all the expressions are exact and no approximation is employed. We summarize here the method of solution:

1. Solve (2.9) to determine \( \{a_n\} \) of (2.0) in the case of large \( \lambda \) and small \( \lambda \).
2. Use (1.6) with (2.1), which can be rewritten as
\[ \gamma = -\frac{\pi \lambda}{\int_{-1}^{1} F(x) \, dx}, \]  
(2.10) to obtain \( \lambda \) as a function of \( \gamma \).

3. Calculate other physical quantities from the expressions of (1.6), (2.1), (1.8) and thermodynamic identities\(^2\) in the following form; 
\[ D = \frac{Q}{\pi} \int_{-1}^{1} F(x) \, dx, \]  
(2.11) 
\[ \frac{E_{\text{eff}}}{L} = \frac{Q^3}{\pi} \int_{-1}^{1} x^2 F(x) \, dx, \]  
(2.12) 
\[ \mu = D^2 \left( 3e - \frac{\gamma \, de}{d\gamma} \right), \]  
(2.13) 
\[ v = 2D \left( 3e - 2\gamma \frac{de}{d\gamma} + \frac{1}{2} \gamma^2 \frac{d^2e}{d\gamma^2} \right)^{1/2}, \]  
(2.14) 
where \( e \equiv E_F / (ND^2) \), \( \mu \) and \( v \) are the scaled energy of the ground state, the chemical potential and the sound velocity, respectively.

### 2.2 Strong coupling case

We consider the large \( \lambda \) case. In this case, only a few \( a_n \)'s are enough to have good approximations. Indeed, it can be shown that \( a_n = O(1/\lambda^{n+1}) \) for \( n \geq 1 \). Therefore, we only keep \( a_0, a_1 \), and set \( a_n \equiv 0 \) \((n \geq 2)\), 
\[ F(x) = a_0 + a_1 x^2. \]  
(2.15)
Equations (2.15) give 
\[ a_0 + \frac{2}{\pi} \arctan \frac{1}{\lambda} \cdot a_0 + \frac{2}{\pi \lambda} \left( \frac{1}{3} - \frac{1}{5\lambda^2} \right) a_1 = 2, \]  
(2.16a) 
\[ a_0 - \frac{2}{\pi \lambda^3} a_0 - \frac{2}{3\pi \lambda^3} a_1 = 0, \]  
(2.16b) 
and consequently, 
\[ a_0 = 2 \left( 1 - \frac{2}{\pi \lambda} + \frac{4}{\pi^2 \lambda^2} + \frac{2\pi^2 - 2}{3\pi^3 \lambda^3} \right), \]  
(2.17a) 
\[ a_1 = \frac{4}{\pi \lambda^3}. \]  
(2.17b) 
Using (2.16a) with (2.15) in (2.10), we obtain \( \lambda \) as a function of \( \gamma \). The result is 
\[ \lambda = -\frac{2(2\gamma + 1)}{\pi} \left( 1 - \frac{\pi^2}{6(2\gamma + 1)^3} \right), \]  
(2.18) 
The expressions of \( a_0 \) and \( a_1 \) in terms of \( \gamma \) are 
\[ a_0 = \frac{2\gamma + 1}{\gamma} \left( 1 - \frac{\pi^2 \gamma}{6(2\gamma + 1)^2} \right), \]  
(2.19a) 
\[ a_1 = -\frac{\pi^2}{2(2\gamma + 1)^3}. \]  
(2.19b)
The ground state energy is calculated from (2.12),

\[ \frac{E_{\text{eff}}^F}{L} = D^3 \frac{\pi^2}{12} \left( \frac{\gamma}{2\gamma + 1} \right)^2 \left( 1 + \frac{4\pi^2}{15(2\gamma + 1)^3} \right). \]  

(2.20)

The cut-off rapidity \( Q \), the chemical potential \( \mu \) and the sound velocity \( v \) are given by

\[ Q = \frac{\pi D}{2} \left( \frac{\gamma}{2\gamma + 1} \right) \left( 1 + \frac{\pi^2}{6(2\gamma + 1)^3} \right), \]

(2.21)

\[ \mu_{\text{eff}} = D^2 \frac{\pi^2}{12} \left( \frac{6\gamma + 1}{2\gamma + 1} \right)^2 \left( 1 + \frac{4\pi^2}{3(2\gamma + 1)^3} \right), \]

(2.22)

\[ v = 2\pi D \left( \frac{\gamma}{2\gamma + 1} \right)^2 \left( 1 + \frac{2\pi^2}{3(2\gamma + 1)^3} \right), \]

(2.23)

where we have introduced the effective chemical potential \( \mu_{\text{eff}} \), \( \mu_{\text{eff}} \equiv \mu + (1/4)D^2\gamma^2 \).

### 2.3 Weak coupling case

This is the small \( \lambda \) case. The lowest order approximation of (2.9) is

\[ 2a_0 = 2, \]  

(2.24a)

\[ 2a_p = 0, \quad p \geq 1. \]  

(2.24b)

A derivation of (2.24) from (2.9) is the same as shown before.\(^{10,11}\) Another derivation is given in Appendix A, where we treat the integral in a more direct manner in the small \( \lambda \) case. Equations (2.24) simply gives

\[ F(x) = 1. \]  

(2.25)

We list here the analysis based on (2.25):

\[ \lambda = \frac{2}{\pi} \gamma, \quad Q = \frac{\pi D}{2}, \quad \mu_{\text{eff}} = D^2 \frac{\pi^2}{4}, \quad v = \pi D. \]  

(2.26)

It should be noted that, in contrast to the solution of the Lieb-Liniger integral equation (1.2), all the coefficients \( a_n \) are finite at \( \lambda \to 0 \).\(^{10}\) This nice analytic property can be seen from (2.3), since the kernel \( \lambda/\{\pi[\lambda^2 + (x - y)^2]\} \) becomes an expression of \( \delta(x - y) \) in the limit \( \lambda \to 0 \).

The second order approximation of (2.9) is

\[ 2a_p + \frac{2}{\pi} \lambda \sum_{n=0}^{\infty} \frac{2p + 1}{2n - 2p - 1} a_n = 2\delta_{p0}, \quad p \geq 0. \]  

(2.28)

Here, \( \delta_{p0} \) is the Kronecker's delta. A derivation of (2.28) is also given in the previous papers\(^{10,11}\) and in Appendix A. We set

\[ a_n = \delta_{n0} + a_n^{(1)} + o(\lambda), \quad n \geq 0, \]  

(2.29)

where \( \delta_{n0} \) stands for the solution of (2.28) and \( a_n^{(1)} \) denote the corrections of the \( \lambda^1 \) order. Substitution of (2.28) into (2.28) yields

\[ 2a_p^{(1)} - \frac{2}{\pi} \lambda = 0, \quad p \geq 0, \]  

(2.30)

which leads to

\[ F(x) = 1 + \frac{\lambda}{\pi} \sum_{n=1}^{\infty} x^{2n} + o(\lambda) \]

\[ = 1 + \frac{\lambda}{\pi} \frac{1}{1 - x^2} + o(\lambda). \]  

(2.31)
This agrees with the approximation shown in the Gaudin's book. From (2.11), $F(x)$ is integrated to give

$$\gamma = -\pi \lambda \left(1 + \frac{2b}{\pi \lambda}\right)^{-1}, \quad (2.32)$$

where the divergent sum $b$ is defined by

$$b = \sum_{n=0}^{\infty} \frac{1}{2n+1}. \quad (2.33)$$

Equation (2.32) is solved to obtain $\gamma$ as a function of $\lambda$:

$$\lambda = -\frac{2}{\pi \gamma} \left(1 - \frac{2b}{\pi^2 \gamma}\right). \quad (2.34)$$

We proceed to calculate the physical quantities. From (2.11) and (2.34), we have

$$Q = \frac{\pi D}{2} \left(1 - \frac{2b}{\pi^2 \gamma}\right). \quad (2.35)$$

The ground state energy is obtained from (2.12), (2.31) and (2.35):

$$\frac{E_{\text{eff}}}{F} = \frac{Q^3}{2} \left(1 + \frac{4}{\pi^2 \gamma}\right), \quad (2.37)$$

This recovers the previous result where the integrals are directly treated without obtaining the explicit form of $F(x)$. It is remarkable that we observe the cancellation of the divergent sums as before. That is, the physical quantities are expressed in power series of $\gamma$ without any divergent sum. The chemical potential and the sound velocity can be calculated from the formulae (2.13) and (2.14),

$$\mu_{\text{eff}} = \frac{D^2 \pi^2}{4} \left(1 + \frac{4}{\pi^2 \gamma}\right), \quad (2.37)$$

The limit $\gamma \to -0$ gives the well-known results for free fermions.

In fig. 1, we plot the solutions with (2.15) and (2.31) with (2.34), for several values of $\gamma$. For comparison, we also plot the numerical solutions based on the classical quadrature method. In this figure, we choose a unit, $D = 1$ so that $Q/D$ takes a value between $\pi/4$ and $\pi/2$ (see (2.21) and (2.35)). For a fixed $Q$ and $-1 \leq x \leq 1$, $q$ and $f(q)$ are determined by (2.1) and (2.2), respectively. Numerical values of $\gamma$ are obtained through the relation $\gamma = -\lambda Q/D$. In the weak coupling case, it is fair to say that the error is large near the end-points, $|q|/D \lesssim Q/D$. This can be observed analytically when the solution (2.31) is substituted into the integral equation (2.2).

3 The BCS-BEC Crossover

Recently, there is a renewed interest in the crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluid with Cooper pairs to a Bose-Einstein Condensate (BEC) of molecules composed of
Figure 1: The normalized distribution function of spins $f(q)$ in the ground state for various values of $\gamma$. Numerical results are indicated by points (+, × etc.), while the analytical ones, (2.15) and (2.31), are by lines (—, — etc). Numerical solutions are obtained by a classical method.\textsuperscript{1,17} For small $\gamma$, analytical solutions are shown for the intervals $[-Q/D + |\gamma|/4, Q/D - |\gamma|/4]$ of $q/D$, which correspond to $[-1 + \lambda/4, 1 - \lambda/4]$ of $x$. For intermediate $\gamma$, $\gamma = -1.136$ and $\gamma = -2.239$, only the numerical results are shown. For large $\gamma$, analytical and numerical results are indistinguishable.
tightly bound fermion pairs.\textsuperscript{13,18} The experimental realizations of low-dimensional trapped Bose gases at ultracold temperatures also inspired the controversy even in one-dimension.\textsuperscript{12,14} The analytic solutions in §2 of the Gaudin integral equation and in the previous paper\textsuperscript{10} of the Lieb-Liniger (LL) integral equation enable us to make the quantitative discussion. To avoid confusion, physical quantities, such as the mass $m$, the number of particles $N$, the number density $D$, the coupling constants $\lambda, \gamma$, and the ground state energy $E$ are denoted with the subscripts F and B for a fermi gas with spin-1/2 and a bose gas, respectively.

The dimensionless form of the LL integral equation \textsuperscript{11,12} reads

$$g(y) - \frac{\lambda_B}{\pi} \int_{-1}^{1} \frac{g(x) \, dx}{\lambda_B^2 + (x-y)^2} = \frac{1}{2\pi},$$

(3.1)

where

$$g(x) = \rho(Kx), \quad k = Kx, \quad 2m_Bc = K\lambda_B.$$  

(3.2)

The relation between $\lambda_B$ and $\gamma_B$ is

$$\gamma_B = \frac{2m_Bc}{D_B} = \frac{\lambda_B}{\int_{-1}^{1} g(x) \, dx},$$

(3.3)

where $D_B = N_B/L$. The ground state energy is given by

$$\frac{E_B}{L} = \frac{K^3}{2m_B} \int_{-1}^{1} x^2 g(x) \, dx.$$  

(3.4)

As already mentioned in the Gaudin’s paper,\textsuperscript{3} strong coupling fermions described by the Gaudin integral equation \textsuperscript{23} may be regarded as the repulsive bosons of dimers. To examine whether it is indeed true, we establish a relation between the exact solutions of the Gaudin integral equation \textsuperscript{23} and the LL integral equation \textsuperscript{11,12}. In the following, we concentrate ourselves on the strong coupling case, i.e. the large $\lambda_F$ case for fermions and large $\lambda_B$ case for bosons are considered. For explicit comparison, we summarize here the results shown in §2.2 and the previous papers.\textsuperscript{10,11}

For $-\gamma_F \gg 1$ (fermions with strong attraction),

$$f(q) = \frac{2\gamma_F + 1}{\gamma_F} \left(1 - \frac{\pi^2 \gamma_F}{6(2\gamma_F + 1)^2}\right) - \frac{1}{\gamma_F^3 D_F^2 q^2},$$

(3.5a)

$$\lambda_F = -\frac{4\gamma_F + 2}{\pi} \left(1 - \frac{\pi^2}{6(2\gamma_F + 1)^3}\right),$$

(3.5b)

$$Q = \frac{\pi D_F}{2} \frac{\gamma_F}{2\gamma_F + 1} \left(1 + \frac{\pi^2}{6(2\gamma_F + 1)^3}\right),$$

(3.5c)

$$\frac{E_{\text{eff}}^F}{L} = \frac{D_F^2}{2m_F} \frac{\pi^2}{12} \left(\frac{\gamma_F}{2\gamma_F + 1}\right)^2 \left(1 + \frac{4\pi^2}{15(2\gamma_F + 1)^3}\right),$$

(3.5d)

$$\mu_{\text{eff}}^F = \frac{D_F^2}{2m_F} \frac{\pi^2}{12} \frac{6\gamma_F + 1}{2\gamma_F + 1} \left(\frac{\gamma_F}{2\gamma_F + 1}\right)^2 \left(1 + \frac{8\pi^2}{5(2\gamma_F + 1)^2(6\gamma_F + 1)}\right),$$

(3.5e)

$$v_F = \frac{2\pi D_F}{2m_F} \left(\frac{\gamma_F}{2\gamma_F + 1}\right)^2 \left(1 + \frac{2\pi^2}{3(2\gamma_F + 1)^3}\right).$$

(3.5f)
For $\gamma_B \gg 1$ (bosons with strong repulsion),

\[
\rho(k) = \frac{\gamma_B + 2}{2\pi\gamma_B} \left( 1 - \frac{2\pi^2}{3\gamma_B(\gamma_B + 2)^2} \right) - \frac{1}{\pi\gamma_B D_B^2} k^2, \quad (3.6a)
\]

\[
\lambda_B = \frac{\gamma_B + 2}{\pi} \left( 1 - \frac{4\pi^2}{3(\gamma_B + 2)^3} \right), \quad (3.6b)
\]

\[
K = \pi D_B \frac{\gamma_B}{\gamma_B + 2} \left( 1 + \frac{4\pi^2}{3(\gamma_B + 2)^3} \right), \quad (3.6c)
\]

\[
E_B \frac{L}{\pi} = \frac{D_B^3}{2m_B} \frac{\pi^2}{3} \left( \frac{\gamma_B}{\gamma_B + 2} \right)^2 \left( 1 + \frac{32\pi^2}{15(\gamma_B + 2)^3} \right), \quad (3.6d)
\]

\[
\mu_B = \frac{D_B^3}{2m_B} \frac{3\gamma_B + 2\pi^2}{\gamma_B + 2} \left( \frac{\gamma_B}{\gamma_B + 2} \right)^2 \left( 1 + \frac{64\pi^2}{5(\gamma_B + 2)^2(3\gamma_B + 2)} \right), \quad (3.6e)
\]

\[
v_B = \frac{2\pi D_B}{2m_B} \left( \frac{\gamma_B}{\gamma_B + 2} \right)^2 \left( 1 + \frac{16\pi^2}{3(\gamma_B + 2)^3} \right). \quad (3.6f)
\]

Identifying $N_F$ fermions with mass $m_F$ with $N_B$ bosons of dimers with mass $m_B$, we have

\[
m_B = 2m_F, \quad N_B = \frac{N_F}{2}, \quad D_B = \frac{D_F}{2}. \quad (3.7a)
\]

Hence, from (1.9) we have

\[
\gamma_B = 4\gamma_F. \quad (3.7b)
\]

Substitution of (3.7) into (3.5) gives

\[
\frac{E_{\text{eff}}}{L} = \frac{(2D_B)^3}{2m_B/2} \frac{\pi^2}{12} \left( \frac{\gamma_B/4}{\gamma_B/2 + 1} \right)^2 \left( 1 + \frac{4\pi^2}{15(\gamma_B/2 + 1)^3} \right)
\]

\[
= \frac{D_B^3}{2m_B} \frac{\pi^2}{3} \left( \frac{\gamma_B}{\gamma_B + 2} \right)^2 \left( 1 + \frac{32\pi^2}{15(\gamma_B + 2)^3} \right) = \frac{E_B}{L}. \quad (3.8)
\]

The effective ground state energies are exactly the same through the relation (3.7) in the strong coupling case. We can easily verify that other physical quantities are also smoothly connected through the relation (3.7) at the point $1/\gamma_F = 1/\gamma_B = 0$. In fact, in the case being considered, equations (3.5a) and (3.5c) can be treated in a unified way, if we regard $\lambda$ as a parameter and $D$ as a normalization condition for these integral equations. We perform such calculations in Appendix B.

We note that the quasi-momenta for bosons and fermions are related as

\[
K = 2Q, \quad k = 2q. \quad (3.9)
\]

These relations are consistent with the assumptions that lead us to the Gaudin integral equation. For negative $c$, quasi-momentum $\{k\}$ of the fermions (not to be confused with $k$ for the repulsive bosons in (3.9)) become imaginary. In the thermodynamic limit, two $k$'s with a real part $q$, which are complex conjugates of each other, form a bound state of two fermions with the binding energy $c^2/2$ (see (1.7)). This kind of argument usually starts with the $c = 0$ limit. Remarkably, it holds also in the strong coupling case.

Equations (3.6a) and (3.6b) need some explanation. Since we are seeking for the solution of $O(1/\lambda^3)$, the expansion must be in powers of either $1/\lambda \simeq 1/(\gamma_B + 2) \simeq 1/(\gamma_F + 1/2)$ or in $1/\gamma_B \simeq 1/\gamma_F$. However, it occurs that the solution of the first term should be $(\gamma_B/(\gamma_B + 2))^2 \simeq (|\gamma_F|/(\gamma_F - 1/2))^2$, not be simply 1. This factor can be counted as the reminiscent of the effective “diameter” $a$ of each particle, in the limit of the hard-core bose gas, or, in other words,
the quantized version of the classical Tonks gas. In fact, the energy at zero temperature for the hard-core bose gas is calculated as

$$E_{hc} = \frac{D_B^3 \pi^2}{2m_B} \frac{2}{3} \left( \frac{1}{1 - aD_B} \right)^2. \quad (3.10)$$

Besides, equating the scattering phase shifts in the low energy limit, we find

$$a = \frac{1}{2} \frac{2}{m_B c} = -\frac{1}{2} \frac{1}{m_F c}. \quad (3.11)$$

With the relation (3.11), we see that (3.10) and the first terms of (3.6d) and (3.5d) become exactly the same. The positiveness (negativeness) of the diameter $a$ for the attractive fermions (repulsive bosons) is due to its statistical property. The negativeness for the repulsive bosons can also be interpreted as the transparency effect for a finite $c > 0$. It is easy to verify that the chemical potential $\mu_{hc}$ and the sound velocity $v_{hc}$ of the hard-core bose gas are obtained by the substitution of (3.11) with (3.7) into either (3.5e) and (3.5f) or (3.6e) and (3.6f). Discarding the second term in the last brackets, we obtain

$$\mu_{hc} = \frac{D_B^2 \pi^2}{2m_B} \frac{3 - aD_B}{3} \left( \frac{1}{1 - aD_B} \right)^2, \quad (3.12)$$

$$v_{hc} = \frac{2\pi D_B}{2m_B} \left( \frac{1}{1 - aD_B} \right)^2. \quad (3.13)$$

Note that $\mu_{hc}$ corresponds to $2\mu_{ef}^F$.

Now we are ready to describe the BCS-BEC crossover in one-dimension. The discussion above for smooth connections in the strong coupling cases justifies a link between the two systems. In view of the physical quantities, we can map two fermions of opposite spins forming a bound state whose quasi-momenta are $q \pm ic/2$ onto one boson with quasi-momentum $k = 2q$. In such a way, the effective ground state energy per volume, the chemical potential and the velocity of sound can be identified. If we take $\gamma_F$ as the coupling constant common to the both systems and vary $1/\gamma_F$ from $-\infty$ to $+\infty$ through 0, we find the following asymptotic expressions of the effective energy:

$$E_{eff}^F = \frac{1}{N\epsilon_{FF}} \begin{cases} \frac{1}{3} \left( 1 + \frac{6}{\pi^2} \gamma_F \right), & -1/\gamma_F \gg 1, \\ \frac{1}{3} \left( \frac{\gamma_F}{2\gamma_F + 1} \right)^2 \left( 1 + \frac{4\pi^2}{15(2\gamma_F + 1)^3} \right), & 1/\gamma_F \sim 0, \\ \frac{\gamma_F}{\pi^2} \left( 1 - \frac{8}{3\pi} \gamma_F^{1/2} \right), & 1/\gamma_F \gg 1, \end{cases} \quad (3.14)$$

where $\epsilon_{FF}$ is the Fermi energy of the free fermions,

$$\epsilon_{FF} \equiv \frac{1}{2m_F} \left( \pi D_F \right)^2. \quad (3.15)$$

The meanings of the above three expressions are clear. They are the energy of the Hartree-Fock approximation for weakly interacting fermions, the energy of either the hard-core bose gas or the spinless non-interacting fermions, and the energy due to the Bogoliubov theory for weakly interacting bosons, for $-1/\gamma_F \gg 1$, $1/\gamma_F \sim 0$, and $1/\gamma_F \gg 1$, respectively. The asymptotic solutions of the LL and the Gaudin integral equation reproduce the results of these three “phases” rigorously. Two crossovers for the intermediate value of $\gamma_F$ are by no means a phase transition. The solutions are continuous and analytic functions of $1/\gamma_F$ except the two points $-\infty$ and $+\infty$. Those can be proved mathematically from the properties of the integral equations.
Figure 2: Asymptotics of the ground state energy $E_{\text{F}}^\text{eff}$ in units of the Fermi energy of the free fermions as a function of the inverse of the coupling constant, $1/\gamma_F$.

Figure 3: Three regimes of one-dimensional quantum gas: for $\gamma_F < 0$, spin-1/2 fermi gas and for $\gamma_F = \gamma_B/4 > 0$, $\delta$-function Bose gas.
Let us explain these three regimes in detail. When \(-1/\gamma_F \gg 1\), the ground state is a BCS-like state with Cooper pairs. The effective size of this bound state is much larger than the average interparticle distance. The existence of the thermodynamic limit is guaranteed by the Pauli exclusion principle; i.e. there is no bound state of more than two fermions.

The intermediate regime of the hard-core bose gas around \(1/\gamma_F = 0\) should be distinguished from the Girardeau gas, which corresponds only to the limit \(1/\gamma_F = 0\). Indeed, when we compare our result with the numerical one, as in fig. it traces well the sudden change around \(1/\gamma_F = 0\). In this region of the bosonic side, the strong mutual repulsion compels the bosonic particles to exhibit the fermionic properties due to one-dimensional kinematics. On the other hand, even the infinitely strong attraction cannot causes the fermions to condense. Namely, the particles to exhibit the fermionic properties due to one-dimensional kinematics. On the other hand, even the infinitely strong attraction cannot causes the fermions to condense. Namely, the distribution and the energy spectrum of the tightly bound fermion pairs can be identified with those of spinless hard-core bosons with a positive “diameter”.

In the regime of weakly interacting bose gas, the lowest order solution of the distribution \(\rho(k)\) is\(^{1,10}\)

\[
\rho(k) = \frac{1}{2\pi D_F \gamma_F} \left( D_F^2 \gamma_F - q^2 \right)^{1/2}. \tag{3.16}
\]

This indicates that the weaker the interaction, the narrower the quasi-momentum distribution region and the higher the peak of the distribution. In the recent experiment this behavior was observed for \(\gamma_B = 0.5^{21}\). In the limit \(1/\gamma_F \to +\infty\), all the particles condense into the zero quasi-momentum state.

In the same manner as we have done in the effective ground state energy, we list here the asymptotic expressions of the velocity of sound and the effective chemical potential:

\[
\frac{v_F}{v_{FF}} = \begin{cases} 
1 + \frac{2}{\pi^2}, & -1/\gamma_F \gg 1, \\
\frac{1}{2} \left( 1 + \frac{2\pi^2}{2(2\gamma_F + 1)} \right), & 1/\gamma_F \sim 0, \\
\frac{1}{2\pi} \gamma_F^{1/2} \left( 1 - \frac{1}{2\pi} \gamma_F^{1/2} \right), & 1/\gamma_F \gg 1.
\end{cases}
\tag{3.17}
\]

where \(v_{FF}\) is the Fermi velocity of the free fermions,

\[
v_{FF} = \frac{\pi D_F}{2m_F}. \tag{3.18}
\]

and

\[
\frac{\mu_F^e}{\epsilon_{FF}} = \begin{cases} 
1 + \frac{4}{\pi^2} \gamma_F, & -1/\gamma_F \gg 1, \\
\frac{1}{3} + \frac{1}{2\gamma_F + 1} \left( \frac{\gamma_F}{2\gamma_F + 1} \right)^2 \left( 1 + \frac{8\pi^2}{5(2\gamma_F + 1)^2(3\gamma_F + 1)} \right), & 1/\gamma_F \sim 0, \\
\frac{2}{\pi^2} \gamma_F \left( 1 - \frac{1}{2\gamma_F} \right), & 1/\gamma_F \gg 1.
\end{cases}
\tag{3.19}
\]

These expressions are plotted in fig. and fig. as functions of \(1/\gamma_F\).

### 4 Analogy with the Circular Plate Condenser

An intriguing fact is that the Gaudin integral equation and the LL integral equation also appear in a completely different subject, the circular plate condenser.\(^5,7,22\) We consider the condenser of the radius 1 whose two plates are separated by a distance \(\lambda\). Cylindrical coordinates are taken as follows: the radial distance from the common axis of the disks is taken to be \(\rho\), the distances from the disks to be \(\zeta, \zeta'\), respectively. E. R. Love\(^{22}\) (1949) proved that the potential \(\phi(\rho, \zeta, \zeta')\) due to the disks is expressed as

\[
\phi(\rho, \zeta, \zeta') = \frac{V_0}{\pi} \int_{-1}^{1} \left\{ \frac{1}{[\rho^2 + (\zeta + it)^2]^{1/2}} \pm \frac{1}{[\rho^2 + (\zeta' - it)^2]^{1/2}} \right\} h(t) \, dt, \tag{4.1}
\]
Figure 4: Asymptotics of the sound velocity $v_F$ in units of the Fermi velocity of the free fermions $v_{FF}$ as a function of the inverse of the coupling constant, $1/\gamma_F$.

Figure 5: Asymptotics of the chemical potential $\mu_{F_{\text{eff}}}$ in units of the Fermi energy of the free fermions $\epsilon_{FF}$ as a function of the inverse of the coupling constant, $1/\gamma_F$. 
where \( h(t) \) is determined by

\[
h(t) = \frac{\lambda}{\pi} \int_{-1}^{1} \frac{h(s) \, ds}{(t - s)^2 + \lambda^2} = 1. \tag{4.2}
\]

The capacitance \( C \) of the condenser is given by

\[
C = \frac{1}{2\pi} \int_{-1}^{1} h(t) \, dt. \tag{4.3}
\]

In the above expressions, the upper signs refer to the case of equally charged disks at potential \( V_0 \), and the lower signs to that of oppositely charged disks, namely, disks at potential \( 0 \) and \(-V_0\). The Love equation \( \text{[12]} \) is exactly in the same form as the Gaudin and the LL integral equation: to be precise, \( h(x) \) in \( \text{[12]} \) corresponds to \((1/2)F(x)\) and \(2\pi g(x)\) in \( \text{[23,24]} \) and \( \text{[23,24]} \), respectively.

In the case of oppositely charged disks (lower signs), the solution of \( \text{[12]} \) is calculated for small \( \lambda \) (small separation). By the heuristic argument, \( \text{[23,24]} \) away from the edges \( h(t) \) to the \( o(1) \) order is written as

\[
h(t) = \frac{1}{\lambda} (1 - t^2)^{1/2} + \frac{1}{2\pi} \left( \frac{16\pi e}{\lambda} - t \log \frac{1 + t}{1 - t} \right) + o(1), \tag{4.4}
\]

and with \( \text{[23,24]} \), the capacitance is

\[
C = \frac{1}{4\lambda} + \frac{1}{4\pi} \log \frac{16\pi e}{\lambda} + o(1). \tag{4.5}
\]

This result is originally by G. Kirchhoff (1877).\textsuperscript{25}

On the other hand, we have\textsuperscript{10,20}

\[
g(x) = \frac{1}{2\pi \lambda} \left(1 - x^2\right)^{1/2} - \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2m + 1)!! (2m - 1)!! (2n - 1)!!}{[(2m)!!]^2 (2n)!! (2n - 2m - 1)} x^{2n}, \tag{4.6}
\]

and

\[
C = \int_{-1}^{1} g(x) \, dx = \frac{1}{4\lambda} - \frac{1}{2\pi} + \frac{1}{4} b, \quad \text{where} \quad b = \sum_{m=0}^{\infty} \left[ \frac{(2m - 1)!!}{(2m)!!} \right]^2. \tag{4.7}
\]

We regularize the divergent sum \( b \) as follows. We set

\[
b(k) = \sum_{m=0}^{\infty} \left[ \frac{(2m - 1)!!}{(2m)!!} \right]^2 k^{2m} = \frac{2}{\pi} K(k). \tag{4.8}
\]

The function \( K(k) \) is the complete elliptic integral of the first kind with the modulus \( k \). Asymptotic behavior of \( K(k) \) is

\[
K(k) = \frac{1}{2} \log \frac{16}{1 - k^2} \quad \text{as} \quad k^2 \searrow 1. \tag{4.9}
\]

If we choose \( 1 - k^2 = \lambda/(\pi e) \) in \( \text{[12]} \) with \( \text{[12]} \), we get

\[
b = \lim_{k^2 \searrow 1} b(k) = \frac{1}{\pi} \log \frac{16\pi e}{\lambda}. \tag{4.10}
\]

Using the expression \( \text{[11,10]} \), we may rewrite \( \text{[10]} \) as

\[
g(x) = \frac{1}{2\pi \lambda} \left(1 - x^2\right)^{1/2} + \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{(2m - 1)!!}{(2m)!!} \right]^2 \frac{(2l - 1)!!}{(2l)!!} x^{2l} + \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2l - 1)!! 2l}{(2l)!! (2m)!!} \left[ \frac{(2m - 1)!!}{(2m)!!} \right]^2 \frac{1}{2m - (2l - 1)} x^{2l}
\]

\[
= \frac{1}{2\pi \lambda} \left(1 - x^2\right)^{1/2} + \frac{1}{4\pi} \left( \log \frac{16\pi e}{\lambda} - x \log \frac{1 + x}{1 - x} \right) \frac{1}{(1 - x^2)^{1/2}}. \tag{4.11}
\]
Here, we have used the power series formulae,
\[
\sum_{p=0}^{\infty} \frac{(2p-1)!!}{(2p)!!} x^{2p} = (1 - x^2)^{-1/2}, \quad |x| < 1,
\]
\[
\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = \frac{1}{2} \log \frac{1+x}{1-x}, \quad |x| < 1,
\]
and the following summation formula,
\[
\sum_{m=0}^{\infty} \left[ \frac{(2m-1)!!}{(2m)!!} \right]^2 \frac{1}{2m - (2l + 1)} = -\frac{2}{\pi} \frac{(2l)!!}{(2l+1)!!} \sum_{k=0}^{l} \frac{1}{2l - 2k + 1} \frac{(2k-1)!!}{(2k)!!},
\]
\[
l = 0, 1, 2, \ldots,
\]
which is proved in Appendix C.

Therefore, by the regularization of the divergent sum (4.10), we exactly reproduce Hutson’s solution (4.4) for $x \sim 0$ for $o(1)$. This coincidence is reasonable since we have started from the expansion of $g(x)$ around $x = 0$.

We may also calculate the physical quantities from (4.11). At our consideration, to the $\lambda^0$ order, we can integrate (4.11) for the whole interval $[-1,1]$. Indeed, the corrections come from near the edges, $x \in [-1, -1+\lambda]\cup[1-\lambda,1]$, and $g(x)$ is estimated to be of the order $1/\sqrt{\lambda}$ there. Then, the corrections are of the order $\sqrt{\lambda}$, and thus can be neglected.

The case of equally charged disks (upper signs in (4.1) and (4.2)) has not attracted much interests. So far, no prescription has been proposed to solve (4.2) asymptotically. Our solution (2.31) gives
\[
C = \frac{1}{4\pi} \int_{-1}^{1} F(x) \, dx = \frac{1}{2\pi} + \frac{\lambda}{4\pi^2} \log \frac{4\pi e}{\lambda} + O(\lambda).
\]
To obtain the last expression, the regularization of the divergent sum $b$ in the same manner as (4.10) is required:
\[
b = \sum_{n=0}^{\infty} \frac{1}{2m+1} = \lim_{\epsilon \to 0} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sum_{k=0}^{\infty} k^{2m} = \frac{1}{2} \log \frac{4\pi e}{\lambda},
\]
where (4.13) has been used to obtain the last expression. The first term in (4.15) was also referred to in the Kirchhoff’s paper. The second term agrees with the analysis by Gaudin except 4 in the logarithmic function.

## 5 Concluding Remarks

We have presented solutions of the Gaudin integral equation (4.1). We have solved the integral equation in the form of power series. Based on the solution, the strong coupling case is analyzed, in relation to the Lieb-Liniger integral equation for the bosons with repulsive interaction. Then, we have a unified view on $\delta$-function attractive fermions and repulsive bosons in one-dimension. Main results are summarized as follows.

i) As to the thermodynamic quantities, the expressions in the strong coupling cases $|\gamma| \gg 1$ are identical through the relations (3.7).

ii) The limit $1/\gamma = 0$ yields the Girardeau gas, in the sense that the energy spectra coincide with that of free fermions.

iii) Around the point $1/\gamma = 0$, the system behaves like a hard-core bose gas with a diameter $a$ given by (3.11). This indicates the existence of dimers of tightly bound fermions ($1/\gamma = -0$, $a > 0$), which can be regarded as bosons with strong repulsion ($1/\gamma = +0$, $a < 0$).
iv) The above regime of $\gamma$ can be smoothly connected to the weakly attractive fermi gas and the weakly repulsive bose gas at both ends with sufficiently small value of $|\gamma|$. This reflects the analyticity of the Love equation (4.2) in $\lambda$ on the whole real axis excluding the point $\lambda = 0$.

In the weak coupling case, we have again met the difficulty of divergences. Obviously, the solution (2.24) holds only for $x \in [-1 + \lambda, 1 - \lambda]$ (see Appendix A). Near the edges $|x| \approx 1$, a different method might be necessary to determine the precise behavior of the solution. Further, the relation between $\lambda$ and $\gamma$ again is important, since the regularization of $\lambda$ by $\gamma$ yields the finite expressions of the physical quantities. Up to now, our method is powerful enough to obtain the first two terms of the asymptotic expansions. We believe that other physical quantities, such as the dressed charge and the dressed energy, and other quantum integrable systems can be studied by use of this method. Those subjects remain as future problems.

**Appendices**

**A Another Derivation of (2.24) and (2.28)**

In this appendix we present the expansion of the integral,

\[ J(x, \lambda) \equiv \frac{\lambda}{\pi} \int_{-1}^{1} dy \frac{F(y)}{\lambda^2 + (x - y)^2} = \frac{\lambda}{\pi} \int_{-1}^{1} dy \sum_{m=0}^{\infty} \frac{a_m y^{2m}}{\lambda^2 + (x - y)^2}, \]  

(A.1)

in power series of $\lambda$ for the case $\lambda \ll 1$. Since

\[ \frac{1}{\lambda^2 + (x - y)^2} = \begin{cases} 
\frac{1}{(x - y)^2} & \text{if } |x - y| \gg \lambda, \\
\frac{1}{\lambda^2} \left( 1 + \frac{\lambda}{x - y} \right)^2 & \text{if } |x - y| \ll \lambda,
\end{cases} \]  

(A.2)

for a fixed $x \in [-1 + \lambda, 1 - \lambda]$, (A.1) becomes

\[
J(x, \lambda) = \frac{\lambda}{\pi} \left[ \int_{[-1,x-\lambda] \cup (x, x+\lambda)} + \int_{(x-\lambda, x+\lambda)} \right] \frac{dy}{\lambda^2 + (x - y)^2} \sum_{m=0}^{\infty} a_m y^{2m} \\
= -\frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{m} \lambda^{2l} \binom{2m}{2l} \frac{(-1)^n a_m x^{2(m-l)}}{2l - 2n - 1} \\
+ \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda^{2n+1} \binom{2m}{k} \frac{(-1)^n a_m x^{2m-k}}{k - 2n - 1} \left[ (-1)^k (1 + x)^{k-2n-1} + (1 - x)^{k-2n-1} \right] \\
- \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \lambda^{2n+1} \binom{2m}{2n+1} (-1)^n a_m x^{2(m-n)-1} \log \frac{1+x}{1-x} \\
+ \frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{m} \lambda^{2l} \binom{2m}{2l} \frac{(-1)^n a_m x^{2l}}{2n + 2m - 2l + 1}. 
\]  

(A.3)

In the above, $\sum_{k=0}^{2m}$ means that we exclude the $k = 2n + 1$ terms in the sum with respect to $k$. 
Let us rewrite (A.3) so that we can easily see the dependences of $\lambda$ and $x$. For the purpose, we change the variables and the orders of the multiple sums. To begin with, we consider the first term of the r.h.s. of (A.3). The order of the triple series can be changed as

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}. \tag{A.4}
\]

With a new variable $p = m - l$ instead of $m$, the first term of the r.h.s. of (A.3) can be written as

\[
-\frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{m} \lambda^{2l} \left(\frac{2m}{2l}\right) \frac{(-1)^n a_m x^{2(m-l)}}{2l - 2n + 1}.
\]

In exactly the same way, the fourth term becomes

\[
\frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{m} \lambda^{2(m-l)} \left(\frac{2m}{2l}\right) \frac{(-1)^n a_m x^{2l}}{2n + 2m - 2l + 1}.
\]

The third term is rewritten as follows:

\[
-\frac{1}{\pi} \sum_{n=1}^{m-1} \sum_{m=0}^{\infty} \lambda^{2n+1} \left(\frac{2m}{2n+1}\right) (-1)^n a_m x^{2(m-n+1)} \log \frac{1+x}{1-x}.
\]

The second term is rather complicated. Since

\[
(-1)^k (1+x)^{k-2n-1} + (1-x)^{k-2n-1} = (-1)^k \sum_{l=0}^{\infty} [1 + (-1)^k] \left(\frac{k-2n-1}{l}\right) x^l,
\]

we decompose the sum with respect to $k$ into four parts:

\[
\sum_{k=0}^{2m} = \sum_{k=0}^{2n} + \sum_{k=2n+2}^{2m} = \sum_{j=0}^{n-1} + \sum_{j=0}^{m} + \sum_{j=n+1}^{m} + \sum_{j=n+1}^{m-1}.
\]
For a fixed \( n \geq 0 \), the sum with respect to \( m \) can be decomposed into two parts:

\[
\sum_{m=0}^{\infty} = \sum_{m=0}^{n} + \sum_{m=n+1}^{\infty}.
\]

(A.10)

With (A.9) and (A.10), the double series with respect to \( m \) and \( k \) become

\[
\frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m} (2m) \left( \frac{-1)^n a_m x^{2m-k}}{k - 2n - 1} \right) \left( -1 \right)^k (1 + x)^{k-2n-1} + (1 - x)^{k-2n-1}
\]

Now we proceed to evaluate the second term of the r.h.s. of (A.3) using (A.9) and (A.11):

\[
\frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} \left\{ \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k=0}^{m} \left( \frac{2m}{2j} \right) \left( \frac{2n - 2j + 2p}{2p} \right) \frac{1}{2j - 2n - 1} x^{2m-2j+2p} \right. \\
+ \sum_{j=0}^{m-1} \left( \frac{2m}{2j+1} \right) \left( \frac{2n - 2j + 2p}{2p+1} \right) \frac{1}{2j - 2n - 1} x^{2m-2j+2p} \\
+ \sum_{m=n+1}^{\infty} \sum_{j=0}^{m-j-n-1} \sum_{p=0}^{m-j-n-1} \left( \frac{2m}{2j+1} \right) \left( \frac{2j - 2n - 1}{2p+1} \right) \frac{1}{2j - 2n - 1} x^{2m-2j+2p} \\
- \sum_{j=n+1}^{m-1} \sum_{p=0}^{m-j-n-1} \left( \frac{2m}{2j+1} \right) \left( \frac{2j - 2n - 1}{2p+1} \right) \frac{1}{2j - 2n - 1} x^{2m-2j+2p} \right\}
\]

(A.11)

\[
= \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} \left\{ \sum_{r=0}^{n} \sum_{q=0}^{2r} \sum_{l=0}^{q} a_q(-1)^l \left( \begin{array}{c} 2q - 2n - 1 - l \\ 2r - l \end{array} \right) \frac{1}{2q - 2n - 1 - l} x^{2r} \\
- \sum_{r=0}^{n-1} \sum_{q=0}^{2r} \sum_{l=2q+2n+3}^{2r} a_q(-1)^l \left( \begin{array}{c} 2q + 2n + 2 \\ 2r - l \end{array} \right) \frac{1}{2q + 2n + 2} x^{2r} \right\}
\]

(A.12)

To obtain the last expression from the middle, we have again changed the variables and the orders of the multiple sums. For instance, in the first square bracket \( [ \] \), we have replaced \( m, j \)
and $p$ with $q = m$, $l = m - j$, $r = m - j + p$, and used the relation
\[
\binom{n+k-1}{k} = (-1)^k \binom{-n}{k} \quad \text{for } n \geq 1, \; k \geq 0.
\] (A.13)

to collect the terms of $2l$ and $2l - 1$.

The combination of \(A.3\), \(A.12\), \(A.7\) and \(A.6\) yields the following form of \(A.3\):

\[
J(x, \lambda) = 2 \sum_{n=0}^{\infty} \lambda^{2n} [A_n(x) + B_n(x)] + \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \lambda^{2n+1} [C_n(x) + D_n(x) + E_n(x) + F_n(x)],
\] (A.14)

where $A_n(x)$, $B_n(x)$, $C_n(x)$, $D_n(x)$, $E_n(x)$ and $F_n(x)$ are defined as

\[
A_n(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \binom{2n+2p}{2n} \frac{(-1)^q a_{n+p}}{2q - 2n + 1} x^{2p},
\] (A.15)

\[
B_n(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \binom{2n+2p}{2n} \frac{(-1)^q a_{n+p}}{2q + 2n + 1} x^{2p},
\] (A.16)

\[
C_n(x) = \sum_{p=0}^{n} \left( \sum_{q=0}^{p} \sum_{r=0}^{2q} a_q A_{npqr} x^{2p} + \sum_{q=p+1}^{2n} \sum_{r=0}^{2q} a_q A_{npqr} x^{2p} \right) + \sum_{p=n+1}^{\infty} \sum_{q=0}^{n} \sum_{r=0}^{2q} a_q A_{npqr} x^{2p},
\] (A.17)

\[
D_n(x) = \sum_{p=2}^{\infty} \left( \sum_{q=n+1}^{p-1} \sum_{r=2q-2n}^{2q} a_q A_{npqr} x^{2p} + \sum_{q=p+1}^{n+p-1} \sum_{r=2q-2n}^{2q} a_q A_{npqr} x^{2p} \right) + \sum_{p=1}^{\infty} a_{n+p} A_{np,p+n,2p} x^{2p},
\] (A.18)

\[
E_n(x) = \sum_{p=0}^{\infty} \sum_{q=n+p+1}^{\infty} \sum_{r=0}^{2p} a_q A_{npqr} x^{2p},
\] (A.19)

\[
F_n(x) = \sum_{p=1}^{\infty} \sum_{q=1}^{p} \binom{2q+2n}{2n+1} \frac{a_{n+q}}{2q - 2p - 1} x^{2p}.
\] (A.20)

Here, the sequence

\[
A_{npqr} \equiv (-1)^r \binom{2q}{r} \binom{2q - 2n - 1 - r}{2p - r} \frac{1}{2q - 2n - 1 - r}, \quad n, q, p, r \in \mathbb{Z}_{\geq 0},
\] (A.21)

has been introduced to simplify the expressions. Note that sums in the above whose lower bounds are greater than its upper bounds, such as $\sum_{q=p+1}^{\infty}$, are treated as zero.

From \(A.4\), we obtain the expansion of $J(x, \lambda)$ with respect to $\lambda$ as follows:

\[
J(x, \lambda) = \sum_{n=0}^{\infty} \lambda^n J_n(x),
\] (A.22)

where the first few $J_n$ are

\[
J_0(x) = \frac{2}{\pi} [A_0(x) + B_0(x)],
\] (A.23)

\[
J_1(x) = \frac{2}{\pi} \left[ C_0(x) + D_0(x) + E_0(x) + F_0(x) \right],
\] (A.24)

\[
J_2(x) = \frac{2}{\pi} [A_1(x) + B_1(x)],
\] (A.25)

\[
J_3(x) = \frac{2}{\pi} [C_1(x) + D_1(x) + E_1(x) + F_1(x)].
\] (A.26)
From (A.23), (A.25) and
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = \frac{\pi}{4},
\]  
(A.27)
we easily verify that
\[
\mathcal{J}_0(x) = \sum_{p=0}^{\infty} a_p x^{2p} = F(x),
\]  
(A.28)
\[
\mathcal{J}_2(x) = - \sum_{p=0}^{\infty} (p + 1)(2p + 1) a_{p+1} x^{2p+1}.
\]  
(A.29)

Use of (A.28) in the Gaudin integral equation (2.3) leads to (2.24). To simplify the expressions of \( \mathcal{J}_1(x) \) and \( \mathcal{J}_3(x) \), we need a little calculation and some formulae. The results are,
\[
\mathcal{J}_1(x) = \frac{2}{\pi} \sum_{p=0}^{\infty} \left[ \sum_{n=0}^{\infty} \frac{2p + 1}{2n - (2p + 1)} a_n \right] x^{2p},
\]  
(A.30)
\[
\mathcal{J}_3(x) = -\frac{2}{\pi} \sum_{p=0}^{\infty} \left( \frac{2p + 3}{3} \right) \left[ \sum_{n=0}^{\infty} \frac{a_n}{2n - (2p + 3)} \right] x^{2p}.
\]  
(A.31)

Below, we shall explain these shortly. We observe that equation (A.30) is in accordance with (2.28).

Therefore, we conclude that our formal expansion (2.7) is also valid for small \( \lambda \) in the interval \([-1 + \lambda, 1 - \lambda]\).

**Calculation of \( \mathcal{J}_1(x) \)**

From (A.24) with (A.17)–(A.21), we have
\[
\int^0_1 \mathcal{J}_1(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q A_{0pq} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q A_{0pq} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q A_{0pq} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q A_{0pq} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q A_{0pq} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_q x^{2p}
\]  
(A.32)

Here, to obtain the third expression from the second, the following formula has been used:
\[
\sum_{l=0}^{p} (-1)^l \binom{m+1}{l} \binom{m-l}{p-l} \frac{1}{m-l} = (-1)^p \binom{p+1}{1} \frac{1}{m-p}, \quad m \geq p + 1.
\]  
(A.33)
Proof of (A.33)

Set

\[ \frac{P_p(m)}{m-p} = \sum_{l=0}^{p} (-1)^l \binom{m+1}{l} \binom{m-l}{p-l} \frac{1}{m-l}. \]  

It can be seen that

\[ P_p(m) \]  

is a polynomial of \( m \) whose degree is equal to or less than \( p \). (A.35)

It will be shown that \( P_p(m) \) is a constant, and actually,

\[ P_p(m) = (-1)^p (p+1). \]  

When \( p = 0 \) or \( p = 1 \), we can verify it directly, so we assume that \( p \geq 2 \).

First, let us denote the \( l \)-th term of the sum \( \sum_{l=0}^{p} \) in the r.h.s. of (A.34) as \( Q_l(m) \), namely,

\[ P_p(m) = \sum_{l=0}^{p} Q_l(m). \]  

Then, \( Q_l(m) \) is explicitly written as

\[ Q_l(m) = (-1)^l \binom{m+1}{l} \binom{m-l}{p-l} \frac{1}{m-l} \]

\[ = \begin{cases} 
\frac{(m-1)(m-2)\cdots(m-p)}{l!}, & \text{if } l = 0, \\
(-1)^l \frac{(m+1)m\cdots(m-l+1)(m-l)\cdots(m-p)}{(p-l)!}, & \text{if } 1 \leq l \leq p-1, \\
(-1)^p \frac{(m+1)m\cdots(m-p+2)}{p!}, & \text{if } l = p.
\end{cases} \]  

(A.38)

Here, \( \cdots \) denotes that such term is excluded from the sequence of the product.

From (A.37) and (A.38), \( P_p(0), P_p(1), \ldots, P_p(p) \) are calculated as follows:

\[ P_p(k) = \begin{cases} 
Q_k(k) + Q_{k+1}(k), & \text{if } 0 \leq k \leq p - 1, \\
Q_p(p), & \text{if } k = p, \\
\end{cases} \]

\[ = (-1)^p (p+1). \]  

(A.39)

Therefore, (A.39) with (A.35) assures that \( P_p(m) = (-1)^p (p+1) \) for all \( m \), which completes the proof.

Calculation of \( J_3(x) \)

This could be done in almost the same, but more complicated way than \( J_1(x) \). We sketch
the calculation. From (A.20) with (A.17)–(A.21), we have

\[ -\frac{\pi}{2} \mathcal{J}_3(x) = a_0 \sum_{p=0}^{\infty} A_{1p00} x^{2p} + a_1 A_{1010} x^0 + a_1 \sum_{p=1}^{2} \sum_{r=0}^{2} A_{1p1r} x^{2p} \]

\[ + \sum_{p=2}^{\infty} \sum_{q=2}^{2} \sum_{r=q-2}^{2q} a_q A_{1pqr} x^{2p} + \sum_{p=1}^{\infty} \left( \frac{2p+2}{2} \right) \frac{1}{-1} a_{p+1} x^{2p} \]

\[ + \sum_{p=1}^{\infty} \sum_{q=2}^{p+1} \left( \frac{2q}{3} \right) \frac{a_q}{2q-(2p+3)} x^{2p} \]

\[ + \sum_{p=0}^{\infty} \sum_{q=p+2}^{\infty} a_q \left[ \sum_{r=0}^{2p} (-1)^r \left( \frac{2q}{r} \right) \left( \frac{2q-3-r}{2p-r} \right) \frac{1}{2q-3-r} \right] x^{2p} \]

\[ = a_0 \sum_{p=0}^{\infty} \left( \frac{2p+2}{2} \right) \frac{1}{-3} x^{2p} + a_1 \sum_{p=0}^{\infty} \left( \frac{2p+3}{3} \right) \frac{1}{-(2p+1)} x^{2p} \]

\[ + \sum_{p=1}^{\infty} \sum_{q=2}^{p+1} \left( \frac{2p+3}{3} \right) \frac{a_q}{2q-(2p+3)} x^{2p} + \sum_{p=0}^{\infty} \sum_{q=p+2}^{\infty} \left( \frac{2p+3}{3} \right) \frac{a_q}{2q-(2p+3)} x^{2p} \]

\[ = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \left( \frac{2p+3}{3} \right) \frac{a_q}{2q-(2p+3)} x^{2p}. \tag{A.40} \]

In the above, the second expression is obtained from the first by applying the identity,

\[ \sum_{l=0}^{p} (-1)^l \left( \begin{array}{c} m + 3 \\ l \end{array} \right) \left( \begin{array}{c} m - l \\ p - l \end{array} \right) \frac{1}{m-l} = (-1)^p \left( \begin{array}{c} p + 3 \\ 3 \end{array} \right) \frac{1}{m-p}, \quad m \geq p + 1, \tag{A.41} \]

for the sum with respect to \( r \) in the last term. We note that (A.41) can be proved in a similar way as (A.33).

**B Solutions of the Lieb-Liniger and the Gaudin integral equations around 1/λ = 0**

In this appendix, we solve an integral equation,

\[ H(x, \eta) - \frac{\eta}{\pi} \int_{-1}^{1} \frac{H(y, \eta) \, dy}{1 + \eta^2 (x-y)^2} = \frac{1}{2\pi}, \tag{B.1} \]

around \( \eta = 0 \). We remark that, \( H(x, \eta) \) gives the solutions \( F(x) \) of (2.23) and \( g(x) \) of (3.4) in the strong coupling case (i.e. \( |1/\lambda| \ll 1 \)),

\[ F(x) = 4\pi H(x, -1/\lambda), \tag{B.2} \]

\[ g(x) = H(x, 1/\lambda). \tag{B.3} \]

The physical quantities are expressed as follows, for \( \eta \leq 0 \),

\[ \gamma_F = \left[ 4\eta \int_{-1}^{1} H(x, \eta) \, dx \right]^{-1}, \tag{B.4a} \]

\[ Q = D_F \left[ 4 \int_{-1}^{1} H(x, \eta) \, dx \right]^{-1}, \tag{B.4b} \]

\[ \frac{E_{\text{eff}}}{L} = \frac{D_F^3}{16} \int_{-1}^{1} x^2 H(x, \eta) \, dx \cdot \left[ \int_{-1}^{1} H(x, \eta) \, dx \right]^{-3}, \tag{B.4c} \]
for $\eta \geq 0$,

$$\gamma_B = \left[ \eta \int_{-1}^{1} H(x, \eta) \, dx \right]^{-1}, \quad (B.5a)$$

$$K = D_B \left[ \int_{-1}^{1} H(x, \eta) \, dx \right]^{-1}, \quad (B.5b)$$

$$\frac{E_B}{L} = D_B \int_{-1}^{1} x^2 H(x, \eta) \, dx \cdot \left[ \int_{-1}^{1} H(x, \eta) \, dx \right]^{-3}. \quad (B.5c)$$

Hereafter, we denote $H(x, \eta)$ as $H(x)$ for simplicity. With the expansions of $H(x)$ and the kernel,

$$H(x) = \sum_{p=0}^{\infty} b_p x^{2p}, \quad (B.6)$$

$$\frac{1}{1 + \eta^2 (x - y)^2} = \sum_{n=0}^{\infty} (-1)^n (x - y)^{2n} \eta^{2n}, \quad (B.7)$$

equation (B.1) becomes

$$\sum_{p=0}^{\infty} b_p x^{2p} - \frac{2}{\pi} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \eta^{2n+1} \frac{b_p}{2l + 2p + 1} \left( \frac{2n}{2l} \right) x^{2(n-l)} = \frac{1}{2\pi}, \quad (B.8)$$

which, as in §2, reduces to the following set of algebraic equations for $\{b_p\}$:

$$b_p - \frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=p}^{\infty} (-1)^n \eta^{2n+1} \frac{b_m}{2m + 2n - 2p + 1} \left( \frac{2n}{2m} \right) = \frac{1}{2\pi} \delta_{p0}. \quad (B.9)$$

From (B.9), we get

$$b_0 = \frac{1}{2\pi} \left[ 1 + \frac{2}{\pi} \eta + \frac{4}{\pi^2} \eta^2 + \frac{2}{\pi} \left( \frac{4}{\pi^2} - \frac{1}{3} \right) \eta^3 - \frac{4}{\pi^2} \left( \frac{1 - 4}{\pi^2} \right) \eta^4 + \frac{2}{\pi} \left( \frac{20}{\pi^2} \right) \frac{1}{\pi^4} - \frac{16}{\pi^4} \right] + O(\eta^6), \quad (B.10a)$$

$$b_1 = -\frac{1}{\pi^2} \eta^3 \left[ 1 + \frac{2}{\pi} \eta - 2 \left( \frac{1 - 2}{\pi^2} \right) \eta^2 \right] + O(\eta^6), \quad (B.11)$$

$$b_2 = \frac{1}{\pi^4} \eta^5 + O(\eta^6), \quad (B.12)$$

$$b_p = O(\eta^{2p+1}), \quad p \geq 3. \quad (B.13)$$

By use of the above solution, $\eta$ may be expressed as

$$\eta = \pi \xi + \frac{4}{3} \pi^3 \xi^4 - \frac{32}{15} \pi^5 \xi^6 + O(\xi^7), \quad (B.14)$$

where

$$\xi = \frac{1}{\gamma_B + 2} = \frac{1}{4\gamma_F + 2}. \quad (B.15)$$

In terms of $\xi$, we have

$$b_0 = \frac{1}{2\pi (1 - 2\xi)} \left( 1 - \frac{2}{3} \pi^2 \xi^3 \right) + O(\xi^6), \quad (B.16a)$$

$$b_1 = -\frac{\pi^3 \xi^3}{1 - 2\xi} + O(\xi^6), \quad (B.16b)$$

$$b_2 = \pi^3 \xi^5 + O(\xi^6). \quad (B.16c)$$
It follows from the above relations that for $\xi \geq 0$,

\[
K = \pi D_B (1 - 2\xi) \left( 1 + \frac{4}{3} \pi^3 \xi^3 \right) + O(\xi^5), 
\]

\[
E_B = \frac{D_B^3}{L} \left( 1 - 2\xi \right)^2 \left( 1 + \frac{32}{15} \pi^2 \xi^3 \right) + O(\xi^6).
\]

(B.17)  

(B.18)

Other physical quantities are readily obtained, which reproduce (3.5) and (3.6).

C Proof of (4.14)

We begin with the following expression of the complete elliptic integral of the first kind,

\[
K(k) = \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} \sum_{m=0}^{\infty} \left( \frac{(2m-1)!!}{(2m)!!} \right)^2 k^{2m}, \quad |k| < 1.
\]

(C.1)

If we consider the expansion such that

\[
\frac{2}{\pi} \int_x^1 \frac{dt}{t^{2l+2}} \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}} = \sum_{n=-\infty}^{\infty} c_n x^n, \quad \text{for } x > 0,
\]

(C.2)

we see from (C.1) that

\[
c_0 = \sum_{m=0}^{\infty} \left( \frac{(2m-1)!!}{(2m)!!} \right)^2 \frac{1}{2m - 2l - 1}.
\]

(C.3)

In the l.h.s. of (C.2), since the double definite integral is convergent for $x > 0$, we can invert the order of the integrations. To integrate with respect to $t$, we change the variable from $t$ to $\varphi$, where $\varphi$ is defined as $\varphi = \arcsin(t \sin \theta)$, $0 \leq \varphi \leq \pi/2$. Then the integration in the l.h.s. of (C.2) becomes

\[
\frac{2}{\pi} \int_x^1 \frac{dt}{t^{2l+2}} \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}} = \frac{2}{\pi} \int_0^{\pi/2} d\theta \int_{\arcsin(x \sin \theta)}^{\pi/2} d\varphi \frac{\sin^{2l+1} \theta}{\sin^{2l+2} \varphi}.
\]

(C.4)

By use of the formula,

\[
\int \frac{d\varphi}{\sin^{2l+2} \varphi} = -\frac{(2l)!!}{(2l+1)!!} \frac{\cos \varphi}{\sin^{2l+1} \varphi} \sum_{k=0}^{l} \frac{(2l - 2k - 1)!!}{(2l - 2k)!!} \sin^{2k} \varphi,
\]

(C.5)

the r.h.s. of (C.4) is rewritten as

\[
\frac{2}{\pi} \int_0^{\pi/2} d\theta \left\{ -\frac{(2l)!!}{(2l+1)!!} \cos \theta \sum_{k=0}^{l} \frac{(2l - 2k - 1)!!}{(2l - 2k)!!} \sin^{2k} \theta 
\]

\[
- \frac{1}{x^{2l+1}} \sqrt{1 - x^2 \sin^2 \theta} \sum_{k=0}^{l} \frac{(2l - 2k - 1)!!}{(2l - 2k)!!} x^{2k} \sin^{2k} \theta \right\}.
\]

(C.6)

Since the second term in the square bracket $[\ ]$ consists of odd powers of $x$, only the first term contributes to $c_0$, which gives

\[
c_0 = -\frac{2}{\pi} \frac{(2l)!!}{(2l+1)!!} \sum_{k=0}^{l} \frac{(2l - 2k - 1)!!}{(2l - 2k)!!} \frac{1}{2k + 1}.
\]

(C.7)
Therefore, we obtain

\[
\sum_{m=0}^{\infty} \left[ \frac{(2m-1)!!}{(2m)!!} \right]^2 \frac{1}{2m - (2l + 1)} = -\frac{2}{\pi (2l+1)!!} \sum_{k=0}^{l} \frac{(2l - 2k - 1)!!}{(2l - 2k)!} \frac{1}{2k+1}
\]

\[
= -\frac{2}{\pi (2l+1)!!} \sum_{k=0}^{l} \frac{1}{2l - 2k + 1} \frac{(2k-1)!!}{(2k)!!},
\]

\(l = 0, 1, 2, \ldots\), \hspace{1cm} (C.8)

which is nothing but (4.14).

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