Dynamical Twisting and the $b$ Ghost in the Pure Spinor Formalism

Nathan Berkovits

ICTP South American Institute for Fundamental Research
Instituto de Física Teórica, UNESP - Univ. Estadual Paulista
Rua Dr. Bento T. Ferraz 271, 01140-070, São Paulo, SP, Brasil

After adding an RNS-like fermionic vector $\psi^m$ to the pure spinor formalism, the non-minimal $b$ ghost takes a simple form similar to the pure spinor BRST operator. The N=2 superconformal field theory generated by the $b$ ghost and the BRST current can be interpreted as a “dynamical twisting” of the RNS formalism where the choice of which spin $\frac{1}{2}$ $\psi^m$ variables are twisted into spin 0 and spin 1 variables is determined by the pure spinor variables that parameterize the coset $SO(10)/U(5)$.

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1 e-mail: nberkovi@ift.unesp.br
1. Introduction

The pure spinor formalism for the superstring [1] has the advantage over the Ramond-Neveu-Schwarz (RNS) formalism of being manifestly spacetime supersymmetric and has the advantage over the Green-Schwarz (GS) formalism of allowing covariant quantization. However, the worldsheet origin of the pure spinor formalism is mysterious since its BRST operator and $b$ ghost do not arise in an obvious manner from gauge-fixing.

In the non-minimal pure spinor formalism, the BRST current and $b$ ghost can be interpreted as twisted $\hat{c} = 3$ N=2 superconformal generators [2]. But when expressed in terms of the d=10 superspace variables and the non-minimal pure spinor variables, the $b$ ghost and the resulting N=2 superconformal transformations are extremely complicated. In fact, the nilpotency of the $b$ ghost was only recently verified [3][4].

In this paper, it will be shown that the $b$ ghost dramatically simplifies when expressed in terms of a fermionic vector $\psi^m$ that is defined in terms of the other worldsheet variables. If one treats the ten $\psi^m$ variables as independent variables, 5 of the 16 $\theta^\alpha$ variables of d=10 superspace (and their conjugate momenta) can be eliminated [3]. The remaining 11 $\theta^\alpha$ variables and their conjugate momenta transform as the worldsheet superpartners of the pure spinor variables. The resulting N=2 superconformal field theory generated by the $b$ ghost and the BRST current can be interpreted as a “dynamically twisted” version of the RNS formalism.

In this dynamically twisted superconformal field theory, the N=2 generators are

\[ T = -\frac{1}{2} \partial x^m \partial x_m - \frac{(\lambda \gamma_m \gamma_n \lambda)}{2(\lambda \lambda)} \psi^m \partial \psi^n + ..., \]

\[ b = \frac{(\lambda \gamma_m \gamma_n \lambda)}{2(\lambda \lambda)} \psi^m \partial x^n + ..., \]

\[ j_{BRST} = -\frac{(\lambda \gamma_m \gamma_n \lambda)}{2(\lambda \lambda)} \psi^n \partial x^m + ..., \]

\[ J = -\frac{(\lambda \gamma_m \gamma_n \lambda)}{2(\lambda \lambda)} \psi^m \psi^n + ..., \]

where $\lambda^\alpha$ and $\bar{\lambda}_\alpha$ are the non-minimal pure spinor ghosts whose projective components parameterize the coset $SO(10)/U(5)$ that describes different twistings. The remaining terms ... in (1.1) are determined by requiring that $(\lambda^\alpha, \bar{\lambda}_\alpha)$ and their worldsheet superpartners transform in an N=2 supersymmetric manner.
So the resulting N=2 superconformal field theory is the sum of a dynamically twisted RNS superconformal field theory with an N=2 superconformal field theory for the pure spinor variables. This interpretation of the BRST operator and the \( b \) ghost as coming from dynamical twisting of an N=1 superconformal field theory will hopefully lead to a better geometrical understanding of the pure spinor formalism.

In section 2, the non-minimal pure spinor formalism is reviewed. In section 3, the \( b \) ghost in the pure spinor formalism is shown to simplify when expressed in terms of an RNS-like \( \psi^m \) variable. In section 4, dynamical twisting of the RNS formalism will be defined and the resulting twisted N=2 superconformal generators will be related to the \( b \) ghost and BRST current in the pure spinor formalism. And in section 5, the results will be summarized.

2. Review of Non-Minimal Pure Spinor Formalism

As discussed in [2], the left-moving contribution to the worldsheet action in the non-minimal pure spinor formalism is

\[
S = \int d^2z \left[ -\frac{1}{2} \partial x^m \overline{\partial} x_m - p_\alpha \overline{\partial} \theta^\alpha + w_\alpha \overline{\partial} \lambda^\alpha + \overline{\psi}^\alpha \overline{\partial} \overline{\lambda}_\alpha - s^\alpha \overline{\partial} r_\alpha \right] \tag{2.1}
\]

where \( x^m \) and \( \theta^\alpha \) are d=10 superspace variables for \( m = 0 \) to 9 and \( \alpha = 1 \) to 16, \( p_\alpha \) is the conjugate momentum to \( \theta^\alpha \), \( \lambda^\alpha \) and \( \overline{\lambda}_\alpha \) are bosonic Weyl and anti-Weyl pure spinors constrained to satisfy \( \lambda \gamma^m \lambda = 0 \) and \( \overline{\lambda} \gamma^m \overline{\lambda} = 0 \), and \( r_\alpha \) is a fermionic spinor constrained to satisfy \( \overline{\lambda} \gamma^m r = 0 \). Because of the constraints on the pure spinor variables, their conjugate momenta \( w_\alpha, \overline{\psi}^\alpha \) and \( s^\alpha \) can only appear in gauge-invariant combinations such as

\[
N^{mn} = \frac{1}{2}(w \gamma^{mn} \lambda), \quad J_\lambda = (w \lambda), \quad S^{mn} = \frac{1}{2}(s \gamma^{mn} \overline{\lambda}), \quad S = (s \overline{\lambda}), \tag{2.2}
\]

which commute with the pure spinor constraints.

The d=10 superspace variables satisfy the free-field OPE’s

\[
x^m(y)x^n(z) \to -\eta^{mn} \log |y - z|^2, \quad p_\alpha(y)\theta^\beta(z) \to (y - z)^{-1} \delta_\alpha^\beta, \tag{2.3}
\]

and, as long as the pure spinor conjugate momenta appear in gauge-invariant combinations and normal-ordering contributions are ignored, one can use the free-field OPE’s of pure spinor variables

\[
w_\alpha(y)\lambda^\beta(z) \to (y - z)^{-1} \delta_\alpha^\beta, \quad \overline{\psi}^\alpha(y)\overline{\lambda}_\beta(z) \to (y - z)^{-1} \delta_\alpha^\beta, \quad s^\alpha(y)r_\beta(z) \to (y - z)^{-1} \delta_\alpha^\beta. \tag{2.4}
\]
It is convenient to define the spacetime supersymmetric combinations

$$
\Pi^m = \partial x^m + \frac{1}{2}(\theta \gamma^m \partial \theta), \quad d_\alpha = p_\alpha - \frac{1}{2}(\partial x^m + \frac{1}{4}(\theta \gamma^m \partial \theta))(\gamma_m \theta)_\alpha
$$

which satisfy the OPE’s

$$
d_\alpha(y)d_\beta(z) \rightarrow -(y - z)^{-1}\Pi_m \gamma^{m}_{\alpha \beta}, \quad d_\alpha(y)\Pi^m(z) \rightarrow (y - z)^{-1}(\gamma^m \partial \theta)_\alpha.
$$

As shown in [2], the non-minimal BRST current forms a twisted \( \hat{c} = 3 \) \( \mathcal{N}=2 \) superconformal algebra with the stress tensor, a composite \( b \) ghost, and a \( \mathrm{U}(1) \) ghost-number current. These twisted \( \mathcal{N}=2 \) generators are

$$
T = -\frac{1}{2}(\partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha + \overline{w}^\alpha \partial \overline{\lambda}_\alpha - s^\alpha \partial r_\alpha),
$$

$$
b = s^\alpha \partial \overline{\lambda}_{\alpha} + \frac{\overline{\lambda}_{\alpha}(2\Pi^m(\gamma_m d)^{\alpha} - N_{mn}(\gamma^{mn} \partial \theta)^{\alpha} - J_{\lambda} \partial \theta^\alpha - \frac{1}{4}\partial^2 \theta^\alpha)}{4(\overline{\lambda} \lambda)}
$$

$$
-\frac{(\overline{\lambda} \gamma^{m np} r)(d \gamma^{m np} d + 24 N_{mn} \Pi_p)}{192(\overline{\lambda} \lambda)^2} + \frac{(r \gamma^{mnp} r)(\overline{\lambda} \gamma^m d) N^{np}}{16(\overline{\lambda} \lambda)^3} - \frac{(r \gamma^{mnp} r)(\overline{\lambda} \gamma^{pqr} r) N^{mn} N_{qr}}{128(\overline{\lambda} \lambda)^4},
$$

$$
J_{\text{BRST}} = \lambda^\alpha d_\alpha - \overline{w}^\alpha r_\alpha,
$$

$$
J_{\text{ghost}} = w_\alpha \lambda^\alpha - s^\alpha r_\alpha - 2(\overline{\lambda} \lambda)^{-1}[(\lambda \partial \overline{\lambda}) + (r \partial \theta)] + 2(\overline{\lambda} \lambda)^{-2}(\lambda r)(\overline{\lambda} \partial \theta).
$$

The terms \(-\frac{1}{16}(\overline{\lambda} \lambda)^{-1}\partial^2 \theta^\alpha\) in (2.8) and \(-2(\overline{\lambda} \lambda)^{-1}[(\lambda \partial \overline{\lambda}) + (r \partial \theta)] + 2(\overline{\lambda} \lambda)^{-2}(\lambda r)(\overline{\lambda} \partial \theta)\) in (2.10) are higher-order in \( \alpha' \) and come from normal-ordering contributions. To simplify the analysis, these normal-ordering contributions will be ignored throughout this paper. However, it should be possible to do a more careful analysis which takes into account these contributions.

3. Simplification of \( b \) Ghost

In this section, the complicated expression of (2.8) for the \( b \) ghost will be simplified by including an auxiliary fermionic vector variable which will be later related to the RNS \( \psi^m \) variable. The trick to simplifying the \( b \) ghost is to observe that the terms involving \( d_\alpha \) in (2.8) always appear in the combination

$$
\overline{\Gamma}^m = \frac{1}{2}(\overline{\lambda} \lambda)^{-1}(\overline{\lambda} \gamma^m d) - \frac{1}{8}(\overline{\lambda} \lambda)^{-2}(\overline{\lambda} \gamma^{m np} r) N_{np}.
$$
Note that only five components of $\Gamma^m$ are independent since $\Gamma^m(\gamma_m \lambda)^\alpha = 0$. In terms of $\Gamma^m$,

$$b = \Pi^m \Gamma_m - \frac{1}{4}(\lambda \bar{\lambda})^{-1}(\lambda \gamma^{mn} r) \Gamma_m \Gamma_n + s^\alpha \partial \bar{X}_\alpha + w_\alpha \partial \theta^\alpha - \frac{1}{2}(\lambda \bar{\lambda})^{-1}(w_m \lambda)(\lambda \gamma^m \partial \theta)$$

(3.2)

where terms coming from normal-ordering are being ignored and the identity

$$\delta^\gamma_\beta \delta^\delta_\alpha = \frac{1}{2} \gamma^m_{\alpha \beta} \gamma^\delta_m - \frac{1}{8}(\gamma^{mn})_{\alpha} (\gamma^mn)^\delta_\beta - \frac{1}{4} \delta^\gamma_\alpha \delta^\delta_\beta$$

(3.3)

has been used.

It is useful to treat (3.1) as a first-class constraint where $\Gamma^m$ is a new worldsheet variable which carries +1 conformal weight and satisfies the constraint $\Gamma^m(\gamma_m \lambda)^\alpha = 0$. Its conjugate momentum will be defined as $\Gamma^m$ of conformal weight zero and can only appear in combinations invariant under the gauge transformation generated by the constraint of (3.1). Note that $\Gamma^m$ and $\Gamma_m$ satisfy the OPE $\Gamma^m(y) \Gamma^m(z) \rightarrow (y - z)^{-1} \eta^{mn}$ and have no singular OPE’s with the other variables.

One can easily verify that the $b$ ghost of (3.2) is gauge-invariant since it has no singularity with (3.1). Furthermore, any operator $O$ which is independent of $\Gamma_m$ can be written in a gauge-invariant manner by defining $O_{inv} = e^R O e^{-R}$ where

$$R = \int \Gamma_m[\frac{1}{2}(\lambda \bar{\lambda})^{-1}(\bar{X} \gamma^m d) - \frac{1}{8}(\lambda \bar{\lambda})^{-2}(\bar{X} \gamma^{mn} r) N_{np}].$$

(3.4)

For example, the gauge-invariant version of the BRST current is

$$G^+ = e^R(\lambda^\alpha d_\alpha - \bar{w}^\alpha r_\alpha) e^{-R} = \lambda^\alpha d_\alpha - \bar{w}^\alpha r_\alpha$$

(3.5)

$$-\frac{1}{2} \Gamma^m(\lambda \bar{\lambda})^{-1}[\bar{X} \gamma_m \gamma_n \lambda) \Pi^n - (r \gamma_m \lambda) \bar{\Pi}^n]$$

$$+ \frac{1}{4} \Gamma^m \Gamma^n[(\lambda \bar{\lambda})^{-1}(\bar{X} \gamma_m \gamma_n \partial \theta) - (\lambda \bar{\lambda})^{-2}(\bar{X} \partial \theta)(\bar{X} \gamma_m \gamma_n \lambda)]$$

$$+ \frac{1}{8} \Gamma^m \Gamma^n(\lambda \bar{\lambda})^{-2}[(\bar{X} \gamma_m n r) r_p + (r \gamma_m n r) \bar{\Pi}^p]$$

$$- \frac{1}{24} \Gamma^m \Gamma^n \Gamma^p[2(\lambda \bar{\lambda})^{-3}(\bar{X} \partial \theta)(\bar{X} \gamma_m n r) - (\lambda \bar{\lambda})^{-2}(\bar{X} \gamma_m n r) \partial \lambda)]$$

where the constraint of (3.1) has been used to substitute $\Gamma^m$ for $\frac{1}{2}(\lambda \bar{\lambda})^{-1}(\bar{X} \gamma^m d) - \frac{1}{8}(\lambda \bar{\lambda})^{-2}(\bar{X} \gamma^{mn} r) N_{np}$. 

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One can also compute the gauge-invariant version of the stress tensor and U(1) current of (2.7) and (2.10) which are

\[ T = e^{R} \left( -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}^\alpha \partial \bar{\lambda}_\alpha \right) e^{-R} \]  

(3.6)

\[ = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}^\alpha \partial \bar{\lambda}_\alpha - \Gamma^m \partial \Gamma_m \]

and

\[ J = e^{R} \left( w_\alpha \lambda^\alpha + r_\alpha s^\alpha \right) e^{-R} = w_\alpha \lambda^\alpha + r_\alpha s^\alpha + \Gamma_m \Gamma^m. \]  

(3.7)

The operators of (3.6), (3.2), (3.5) and (3.7) form a set of twisted N=2 superconformal generators which preserve the first-class constraint of (3.1). The resulting N=2 superconformal field theory will be related to a dynamical twisting of the RNS formalism where the RNS fermionic vector variable \( \psi^m \) is defined as

\[ \psi^m = \Gamma^m + \frac{1}{2} (\lambda \bar{\lambda})^{-1} \Gamma_n (\lambda \gamma^m \gamma^n \bar{\lambda}). \]  

(3.8)

Note that \( \psi^m \) satisfies the usual OPE \( \psi^m(y) \psi^n(z) \rightarrow (y - z)^{-1} \eta^{mn} \) and commutes with the constraint \( \Gamma^m (\gamma_m \bar{\lambda})^\alpha = 0 \). Since this constraint eliminates half of the \( \Gamma^m \) variables and can be used to gauge-fix half of the \( \Gamma_m \) variables, the remaining 10 variables of \( \Gamma^m \) and \( \Gamma_m \) can be expressed in terms of \( \psi^m \).

Although \( w_\alpha \) and \( \bar{w}^\alpha \) have singular OPE’s with \( \psi^m \), one can define variables \( w'_\alpha \) and \( \bar{w}'^\alpha \) which have no singular OPE’s with \( \psi^m \) as

\[ w_\alpha = w'_\alpha - \frac{1}{4} \psi_m \psi_n [ (\lambda \bar{\lambda})^{-1} (\gamma^mn \bar{\lambda})^\alpha - \bar{\lambda}_n (\lambda \bar{\lambda})^{-2} (\lambda \gamma^mn \bar{\lambda})], \]  

(3.9)

\[ \bar{w}^\alpha - \frac{1}{2} \Gamma^m \Gamma^n (\lambda \bar{\lambda})^{-1} (\gamma_m \gamma_n \lambda)^\alpha = \bar{w}'^\alpha - \frac{1}{4} \psi_m \psi_n [ (\lambda \bar{\lambda})^{-1} (\gamma^mn \lambda)^\alpha - \lambda^\alpha (\lambda \bar{\lambda})^{-2} (\lambda \gamma^mn \lambda)]. \]

Note that \( \bar{w}'^\alpha \) always appears in the combination \( \bar{w}'^\alpha - \frac{1}{2} \Gamma^m \Gamma^n (\lambda \bar{\lambda})^{-1} (\gamma_m \gamma_n \lambda)^\alpha \) since it is this combination which commutes with the constraint \( \Gamma^m (\gamma_m \bar{\lambda})^\alpha = 0 \).

When expressed in terms of \( \psi^m \), \( w'_\alpha \) and \( \bar{w}'^\alpha \), the twisted N=2 generators of (3.6), (3.2), (3.5) and (3.7) take the form

\[ T = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + w'_\alpha \partial \lambda^\alpha - s^\alpha \partial r_\alpha + \bar{w}'^\alpha \partial \bar{\lambda}_\alpha \]  

(3.10)

\[ -\frac{1}{2} \psi^m \partial \psi_m - \frac{1}{4} \partial [ (\lambda \bar{\lambda})^{-1} (\lambda \gamma_m \gamma_n \bar{\lambda}) \psi^m \psi^n], \]
\[ G^- = \frac{1}{2}(\lambda \overline{\lambda})^{-1}(\lambda \gamma_m \gamma_n \overline{\lambda}) \psi^m \Pi^n + s^\alpha \partial \overline{\lambda}_\alpha + w'_\alpha \partial \theta^\alpha - \frac{1}{2}(\lambda \overline{\lambda})^{-1}(w' \gamma^n \overline{\lambda})(\lambda \gamma_m \partial \theta) \]

\[ + \frac{1}{4} \psi_m \psi_n (\lambda \overline{\lambda})^{-1}[(\overline{\lambda} \gamma^{mn} \partial \theta) + (\lambda \overline{\lambda})^{-1}(\overline{\lambda} \partial \theta)(\lambda \gamma^{mn} \overline{\lambda}) + (r \gamma^{mn} \lambda) + (\lambda \overline{\lambda})^{-1}(r \lambda))(\lambda \gamma^{mn} \overline{\lambda})], \]

\[ G^+ = -\frac{1}{2}(\lambda \overline{\lambda})^{-1}(\lambda \gamma_m \gamma_n \overline{\lambda}) \psi^m \Pi^n + \lambda^\alpha d_\alpha - \overline{w'}^\alpha r_\alpha \]

\[ + \frac{1}{4} \psi_m \psi_n (\lambda \overline{\lambda})^{-1}[(\overline{\lambda} \gamma^{mn} \partial \theta) + (\lambda \overline{\lambda})^{-1}(\overline{\lambda} \partial \theta)(\lambda \gamma^{mn} \overline{\lambda}) + (r \gamma^{mn} \lambda) + (\lambda \overline{\lambda})^{-1}(r \lambda))(\lambda \gamma^{mn} \overline{\lambda})], \]

\[ + G^- \left[ \frac{1}{24}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p \right], \]

\[ J = -\frac{1}{2}(\lambda \overline{\lambda})^{-1}(\lambda \gamma_m \gamma_n \overline{\lambda}) \psi^m \psi^n + w'_\alpha \lambda^\alpha + r_\alpha s^\alpha, \]

where \( G^- \left[ \frac{1}{24}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p \right] \) denotes the single pole in the OPE of \( G^- \) with \( \frac{1}{24}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p \) and is equal to the last two lines of (3.3).

Except for the extra term \( G^- \left[ \frac{1}{24}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p \right] \) in \( G^+ \), the generators of (3.10) have a very symmetric form. This asymmetry in \( G^+ \) and \( G^- \) can be removed by performing the similarity transformation \( \mathcal{O} \to e^R \mathcal{O} e^{-R} \) on all operators where

\[ R = -\frac{1}{24} \int (\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p. \] (3.11)

This similarity transformation leaves \( G^+ \) of (3.10) invariant but transforms \( T, G^- \) and \( J \) as

\[ T \to T + \frac{1}{24} \partial((\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p), \] (3.12)

\[ G^- \to G^- + G^- \left[ \frac{1}{24}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p \right], \]

\[ J \to J + \frac{1}{12}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp} r) \psi^m \psi^n \psi^p. \]

It also transforms the constraint of (3.11) into the constraint

\[ \frac{1}{2}(\lambda \overline{\lambda})^{-1}(\lambda \gamma^m \gamma^n \overline{\lambda}) \psi_n = \frac{1}{2}(\lambda \overline{\lambda})^{-1}(\overline{\lambda} \gamma^m d) - \frac{1}{8}(\lambda \overline{\lambda})^{-2}(\overline{\lambda} \gamma_{mnp}) N'_{np}, \] (3.13)

where \( N'_{np} = \frac{1}{2} w' \gamma_{np} \lambda. \)
After performing the similarity transformation of (3.11), the twisted N=2 generators preserve the constraint of (3.13) and take the symmetrical form

\[
T = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m - p_\alpha \partial \theta^\alpha + \frac{1}{2} (w_\alpha' \partial \lambda^\alpha - \lambda^\alpha \partial w_\alpha') \tag{3.14}
\]

\[-\frac{1}{2} (s^\alpha \partial r_\alpha + r_\alpha \partial s^\alpha) + \bar{\psi}^\alpha \partial \bar{\lambda}_\alpha + \frac{1}{2} \partial J,
\]

\[-G^+ + G^- = \psi_m \Pi^m - \lambda^\alpha d_\alpha + \bar{\psi}^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w_\alpha' \partial \theta^\alpha - \frac{1}{2} (\lambda \bar{\lambda})^{-1} (w' \gamma^m \bar{\lambda}) (\lambda \gamma_m \partial \theta),
\]

\[J = -\frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma_{mn} \bar{\lambda}) \psi^m \psi^n + \frac{1}{12} (\lambda \bar{\lambda})^{-2} (\lambda \gamma_{mn} \bar{\lambda}) \psi^m \psi^n \psi^p + w_\alpha' \lambda^\alpha + r_\alpha s^\alpha,
\]

\[G^+ + G^- = [-G^+ + G^-, J]
\]

\[= \psi_m \Pi^m - \lambda^\alpha d_\alpha + \bar{\psi}^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w_\alpha' \partial \theta^\alpha - \frac{1}{2} (\lambda \bar{\lambda})^{-1} (w' \gamma^m \bar{\lambda})
\]

\[+ \frac{1}{2} \psi_m \phi_n (\lambda \bar{\lambda})^{-1} [(\lambda \gamma_{mn} \partial \theta) + (\lambda \bar{\lambda})^{-1} (\lambda \partial \theta) (\lambda \gamma_{mn} \bar{\lambda}) + (\gamma_{mn} \lambda) + (\lambda \bar{\lambda})^{-1} (r \lambda)] (\lambda \gamma_{mn} \bar{\lambda})
\]

\[+ \frac{1}{4} \psi^m \psi^n \psi^p [\lambda \bar{\lambda}]^{-2} (\lambda \gamma_{mn} \bar{\lambda}) \Pi^p + \frac{1}{2} (\lambda \bar{\lambda})^{-3} (r \gamma_{mp} \rho \lambda) \psi_q
\]

\[+ \frac{1}{12} \psi^m \psi^n \psi^p [-2 (\lambda \bar{\lambda})^{-3} (\lambda \partial \theta) (\lambda \gamma_{mn} \bar{\lambda}) + (\lambda \bar{\lambda})^{-2} (\lambda \gamma_{mn} \partial \bar{\lambda})],
\]

where the last two lines in $G^+ + G^-$ is $G^- \left[ \frac{1}{12} (\lambda \bar{\lambda})^{-2} (\lambda \gamma_{mn} \bar{\lambda}) \psi^m \psi^n \psi^p \right]$. These N=2 generators of (3.14) will now be related to a dynamically twisted version of the RNS formalism.

### 4. Dynamical Twisting of the RNS Formalism

In this section, the RNS formalism will be “dynamically twisted” to an N=2 superconformal field theory by introducing bosonic pure spinor variables $\lambda^\alpha$ and $\bar{\lambda}_\alpha$ and their fermionic worldsheet superpartners. The corresponding twisted N=2 superconformal generators will then be related to the twisted N=2 generators of (3.14) in the pure spinor formalism.

Twisting the N=1 RNS superconformal generators

\[
T = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m, \quad G = \psi^m \partial x_m \tag{4.1}
\]
into \( \text{N}=2 \) superconformal generators usually involves choosing a \( \text{U}(5) \) subgroup of the Wick-rotated \( \text{SO}(10) \) Lorentz group and splitting the ten \( x^m \) and \( \psi^m \) variables into five complex pairs \((x^a, \overline{x}^a)\) and \((\psi^a, \overline{\psi}^a)\) for \( a = 1 \) to \( 5 \). One then defines the twisted \( \text{N}=2 \) superconformal generators as

\[
T_{RNS} = -\partial x^a \partial \overline{x}^a - \overline{\psi}^a \partial \psi^a, \tag{4.2}
\]

\[
G_{RNS}^- = \overline{\psi}^a \partial x^a, \quad G_{RNS}^+ = -\psi^a \partial \overline{x}^a,
\]

\[
J_{RNS} = -\overline{\psi}^a \psi^a,
\]

which satisfy the OPE \( G^+(y)G^-(z) \rightarrow (y - z)^{-2}J(z) + (y - z)^{-1}T(z) \).

To dynamically twist, one instead introduces pure spinor worldsheet variables \( \lambda^\alpha \) and \( \overline{\lambda}_\alpha \) satisfying

\[
\lambda \gamma^m \lambda = 0, \quad \overline{\lambda} \gamma^m \overline{\lambda} = 0, \tag{4.3}
\]

whose projective components parameterize the coset \( \text{SO}(10)/U(5) \). The \( \text{N}=2 \) superconformal generators of (4.2) can then be written in a Lorentz-covariant manner as

\[
T_{RNS} = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m - \frac{1}{4} \partial [ (\lambda \overline{\lambda})^{-1} (\lambda \gamma^m \gamma^n \overline{\lambda}) \psi_m \psi_n ], \tag{4.4}
\]

\[
G_{RNS}^- = \frac{1}{2} (\lambda \overline{\lambda})^{-1} (\lambda \gamma^m \gamma^n \overline{\lambda}) \psi_m \partial x_n, \quad G_{RNS}^+ = -\frac{1}{2} (\lambda \overline{\lambda})^{-1} (\lambda \gamma^m \gamma^n \overline{\lambda}) \psi_m \partial x_n,
\]

\[
J_{RNS} = -\frac{1}{2} (\lambda \overline{\lambda})^{-1} (\lambda \gamma^m \gamma^n \overline{\lambda}) \psi_m \psi_n.
\]

The next step is to introduce the fermionic worldsheet superpartners of the pure spinor variables \((\lambda^\alpha, \overline{\lambda}_\alpha)\) and their conjugate momenta \((w'_\alpha, \overline{w}'^\alpha)\). The fermionic superpartners of \( \lambda^\alpha \) and \( w'_\alpha \) will be denoted \( \tilde{\theta}^\alpha \) and \( \tilde{p}_\alpha \), and the fermionic superpartners of \( \overline{\lambda}_\alpha \) and \( \overline{w}'^\alpha \) will be denoted \( r_\alpha \) and \( s^\alpha \). They are constrained to satisfy

\[
\lambda \gamma^m \partial \tilde{\theta} = 0, \quad \overline{\lambda} \gamma^m r = 0, \tag{4.5}
\]

which will be the worldsheet supersymmetry transformation of the pure spinor constraints of (4.3). Because of the constraint \( \lambda \gamma^m \partial \tilde{\theta} = 0 \), \( \tilde{\theta}^\alpha \) is a constrained version of \( \theta^\alpha \) which only contains eleven independent non-zero modes. The corresponding twisted \( \text{N}=2 \) superconformal generators for these pure spinor multiplets are defined as

\[
T_{\text{pure}} = w'_\alpha \partial \lambda^\alpha - \tilde{p}_\alpha \partial \tilde{\theta}^\alpha + \overline{w}'^\alpha \partial \overline{\lambda}_\alpha - s^\alpha \partial r_\alpha, \tag{4.6}
\]
\[ G^\text{pure}_- = w'_\alpha \partial \tilde{\theta}^\alpha + s^\alpha \partial \tilde{\lambda}_\alpha, \quad G^\text{pure}_+ = \lambda^\alpha \tilde{p}_\alpha - \overline{w}' \alpha r^\alpha, \]

\[ J^\text{pure}_{} = w'_\alpha \lambda^\alpha + r^\alpha s^\alpha, \]

which preserve the pure spinor constraints of (4.3) and (4.5).

Finally, one adds the N=2 superconformal generators of (4.4) and (4.6) in a manner that preserves the N=2 algebra. This can be done by defining

\[ T = T_{RNS} + T_\text{pure}, \quad J = J_{RNS} + J_\text{pure}, \]

\[ -G^+ + G^- = (-G^+ + G^-)_{RNS} + (-G^+ + G^-)_{\text{pure}}, \]

and then defining \( G^+ + G^- \) using the commutator algebra

\[ G^+ + G^- = [-G^+ + G^-, J]. \]

Since \( G^\text{pure}_+ \) and \( G^\text{pure}_- \) do not commute with \( J_{RNS} \), \( G^+ + G^- \) is not the sum of \( (G^+ + G^-)_{RNS} \) and \( (G^+ + G^-)_{\text{pure}} \).

The resulting N=2 superconformal generators for the dynamically twisted RNS formalism are

\[ T = -\frac{1}{2} \partial x^m \partial x_m - \frac{1}{2} \psi^m \partial \psi_m - \tilde{p}_\alpha \partial \tilde{\theta}^\alpha + \frac{1}{2} (w'_\alpha \partial \lambda^\alpha - \lambda^\alpha \partial w'_\alpha) \]

\[ \quad - \frac{1}{2} (s^\alpha \partial r_\alpha + r_\alpha \partial s^\alpha) + \overline{w}' \alpha \partial \overline{\lambda}_\alpha + \frac{1}{2} \partial J, \]

\[ -G^+ + G^- = \psi^m \partial x_m - \lambda^\alpha \tilde{p}_\alpha + \overline{p}' \alpha r_\alpha + s^\alpha \partial \overline{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha, \]

\[ J = -\frac{1}{2} (\lambda \overline{\lambda})^{-1} (\lambda \gamma_{mn} \overline{\lambda}) \psi^m \psi^n + w'_\alpha \lambda^\alpha + r_\alpha s^\alpha, \]

\[ G^+ + G^- = [-G^+ + G^-, J] \]

\[ = \psi_m \partial x_n (\lambda \overline{\lambda})^{-1} (\lambda \gamma_{mn} \overline{\lambda}) + \lambda^\alpha \tilde{p}_\alpha - \overline{p}' \alpha r_\alpha + s^\alpha \partial \overline{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha \]

\[ + \frac{1}{2} \psi_m \psi_n (\lambda \overline{\lambda})^{-1} ([\overline{\lambda} \gamma_{mn} \partial \tilde{\theta}] + (\lambda \overline{\lambda})^{-1} (\overline{\lambda} \partial \tilde{\theta}) (\lambda \gamma_{mn} \lambda) + (r \lambda) (r \lambda) - (r \lambda)) (\lambda \gamma_{mn} \overline{\lambda})]. \]

The N=2 superconformal generators of (4.8) are obviously closely related to the N=2 generators of (3.14) in the pure spinor formalism, but there are three important differences. Firstly, the generators of (4.8) are not manifestly spacetime supersymmetric since they
The first difference is easily removed by performing the similarity transformation $\mathcal{O} \to e^R \mathcal{O} e^{-R}$ on all operators in (4.8) where

$$R = \frac{1}{2} \int (\lambda \gamma^m \tilde{\theta}) \psi_m.$$  

This similarity transformation does not affect $T$ or $J$ of (4.8) but transforms $-G^+ + G^-$ into the manifestly spacetime supersymmetric expression

$$-G^+ + G^- = \psi^m \tilde{\Pi}_m - \lambda^\alpha \tilde{d}_\alpha + \bar{\sigma}^\alpha r_\alpha + s^\alpha \partial \bar{\lambda}_\alpha + w'_\alpha \partial \tilde{\theta}^\alpha$$

where $\tilde{\Pi}_m = \partial x^m + \frac{1}{2} (\tilde{\theta} \gamma^m \partial \tilde{\theta})$ and $\tilde{d}_\alpha = \tilde{p}_\alpha - \frac{1}{2} (\partial x^m + \frac{1}{4} (\tilde{\theta} \gamma_m \partial \tilde{\theta}) (\gamma_m \tilde{\theta})_\alpha$, and transforms the $\psi_m \partial x_n (\lambda \bar{\lambda})^{-1} (\lambda \gamma_m \bar{\lambda})$ term in $G^+ + G^-$ into $\psi_m \tilde{\Pi}_n (\lambda \bar{\lambda})^{-1} (\lambda \gamma_m \bar{\lambda})$.

The second difference in the generators can be removed by modifying the definition of dynamical twisting in (4.3) so that the appropriate term is added to $J$. The generator $-G^+ + G^- = (G^+ + G^-)_{RNS} + (G^+ + G^-)_{pure}$ and the untwisted stress tensor $T - \frac{1}{2} \partial J = (T - \frac{1}{2} \partial J)_{RNS} + (T - \frac{1}{2} \partial J)_{pure}$ of (4.8) will be left unchanged. But $J$ will be modified so that after performing the similarity transformation of (4.3), the new $J$ includes the term $\frac{1}{12} (\lambda \bar{\lambda})^{-2} (\bar{X} \gamma_{mnpr}) \psi^m \psi^p \bar{\psi}^p$. And to preserve the N=2 algebra, $G^+ + G^-$ will be defined as the commutator $[-G^+ + G^-, J]$ using the new $J$.

Since $e^{-R} \psi^m e^R = \psi^m - \frac{1}{2} (\lambda \gamma^m \tilde{\theta})$, this means one should modify $J$ in (4.8) to

$$J = -\frac{1}{2} (\lambda \bar{\lambda})^{-1} (\lambda \gamma_m \bar{\lambda}) \psi_m \psi_n + w'_\alpha \lambda^\alpha + r_\alpha s^\alpha$$

$$+ \frac{1}{12} (\lambda \bar{\lambda})^{-2} (\bar{X} \gamma_{mnpr}) (\psi_m - \frac{1}{2} (\lambda \gamma_m \tilde{\theta})) (\psi_n - \frac{1}{2} (\lambda \gamma_n \tilde{\theta})) (\psi_p - \frac{1}{2} (\lambda \gamma_p \tilde{\theta})).$$

Although this modification of $J$ looks unnatural, it has the important consequence of breaking the abelian shift symmetry $\tilde{\theta}^\alpha \to \tilde{\theta}^\alpha + c^\alpha$ where $c^\alpha$ is any constant. This shift symmetry leaves invariant the generators of (4.8), but has no corresponding symmetry in the pure spinor formalism and should not be a physical symmetry.

After modifying $J$ in this manner and performing the similarity transformation of (4.3), the generators of (4.8) coincide with the generators of (3.14) except for the restriction that $\lambda \gamma^m \partial \tilde{\theta} = 0$. This final difference between the generators can be removed by interpreting $\lambda \gamma^m \partial \tilde{\theta} = 0$ as a partial gauge-fixing condition for the symmetry generated by
the first-class constraint of (3.13). After relaxing the restriction \( \lambda\gamma^m \partial \tilde{\theta} = 0 \) and adding the term \(-\frac{1}{2}(\lambda \bar{\lambda})^{-1}(w'\gamma^m \bar{\lambda})(\lambda\gamma^m \partial \theta) \) to \( G^- \), the generators of (3.14) coincide with those of (3.14) and therefore preserve the constraint of (3.13).

Since the generators preserve (3.13), it is consistent to interpret (4.8) as a partially gauge-fixed version of (3.14) where the symmetry generated by (3.13) is used to gauge-fix \( \lambda\gamma^m \partial \theta = 0 \). On the other hand, the original N=2 generators of (2.7) – (2.10) of the pure spinor formalism can be interpreted as a gauge-fixed version of (3.14) where the gauge-fixing condition is \( (\lambda\gamma^m \gamma^n \bar{\lambda})\psi_n = 0 \). This is easy to see since \( (\lambda\gamma^m \gamma^n \bar{\lambda})\psi_n = 0 \) implies that \( R = 0 \) in the similarity transformations of (3.5), (3.6) and (3.7).

5. Summary

In section 2, the \( b \) ghost of the pure spinor formalism was simplified by introducing the fermionic vector variable \( \Gamma^m \) of (3.1). After expressing \( \Gamma^m \) in terms of the RNS variable \( \psi^m \) using (3.8), the \( b \) ghost and BRST current form a symmetric set of twisted N=2 generators (3.14) which preserve the constraint of (3.13).

In section 3, the corresponding N=2 superconformal field theory was interpreted as a dynamically twisted version of the RNS formalism in which the pure spinors \( \lambda^\alpha \) and \( \bar{\lambda}_\alpha \) parameterize the \( SO(10)/U(5) \) choices of twisting. The dynamically twisted RNS generators are obtained from (3.14) using the constraint of (3.13) to gauge-fix \( \lambda\gamma^m \partial \theta = 0 \). And the twisted N=2 generators of the original pure spinor formalism are obtained from (3.14) using the constraint of (3.13) to gauge-fix \( (\lambda\gamma^m \gamma^n \bar{\lambda})\psi_n = 0 \).

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