ON THE LANGLANDS RETRACTION

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Abstract. Given a root system in a vector space $V$, Langlands defined in 1973 a canonical retraction $L : V \to V^+$, where $V^+ \subset V$ is the dominant chamber. In this note we give a short review of the material on this retraction (which is well known under the name of “Langlands’ geometric lemmas”).

The main purpose of this review is to provide a convenient reference for the work [DrGa], in which the Langlands retraction is used to define a coarsening of the Harder-Narasimhan-Shatz stratification of the stack of $G$-bundles on a smooth projective curve.

1. Introduction

Given a root system in a Euclidean space $V$, Langlands defined in [La2, Sect. 4] a certain retraction $L : V \to V^+$, where $V^+$ is the dominant chamber. Later this retraction was discussed in [BoVaI, Ch. IV, Subsect. 3.3] and [C, Sect. 1].

In this note we briefly recall the definition and properties of $L$. It has no new results compared with [La2] and [C]; my goal is only to provide a convenient reference for the work [DrGa] and possibly for some future works.

Following J. Carmona, we begin in Sect. 2 with the most naive definition of $L$ (which makes sense for a Euclidean space equipped with any basis $\{\alpha_i\}$): namely, $L(x)$ is the point of $V^+$ closest to $x$.

Starting with Section 3 we assume that $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$. The key point is that under this assumption $L$ can be characterized in terms of the usual ordering on $V$: namely, Corollary 3.2 says that $L(x)$ is the least element of the set

$$
\{ y \in V^+ \mid y \geq x \}.
$$

It is this characterization of $L$ that is important for most applications (in particular, it is used in [DrGa, Appendix B]). One can consider it as a definition of $L$ and Corollary 3.2 as a way to prove the existence of the least element of the set (1.1). In Section 4 we give another proof of this fact, which is independent of Sections 2-3 closely related to it are Remark 4.2 and Example 4.3.

In Section 5 we define the Langlands retraction as a map from the space of rational coweights of a reductive group to the dominant cone.

In Section 6 we make some historical remarks.

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2. The retraction defined by the metric

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite scalar product $\langle , \rangle$. Let $\{\alpha_i\}_{i \in \Gamma}$ be an arbitrary basis in $V$ and $\{\omega_i\}_{i \in \Gamma}$ the dual basis. Let $V^+ \subset V$ denote the closed convex cone generated by the $\omega_i$’s, $i \in \Gamma$. 

Following J. Carmona [C Sect. 1], we define the Langlands retraction \( \mathfrak{L} : V \to V^+ \) as follows: \( \mathfrak{L}(x) \) is the point of \( V^+ \) closest to \( x \) (such point exists and is unique because \( V^+ \) is closed and convex). It is easy to see that the map \( \mathfrak{L} \) is continuous.

Let us give another description of \( \mathfrak{L} \). For a subset \( J \subset \Gamma \) let \( K_J \) denote the closed convex cone generated by \( \omega_j \) for \( j \in \Gamma - J \) and by \( -\alpha_i \) for \( i \in J \). Clearly, each \( K_J \) is a simplicial cone of full dimension in \( V \). Let \( V_J \) denote the linear span of \( \alpha_j, j \in J \) (so \( V_J^+ \) is spanned by \( \omega_i, i \notin J \)). Let \( \text{pr}_J : V \to V \) denote the orthogonal projection onto \( V_J^+ \), so \( \ker (\text{pr}_J) = V_J \).

**Proposition 2.1.** (a) The map \( \mathfrak{L} \) is piecewise linear. The cones \( K_J \) are exactly the linearity domains of \( \mathfrak{L} \). For \( x \in K_J \) one has \( \mathfrak{L}(x) = \text{pr}_J(x) \).

(b) The cones \( K_J \) and their faces form a complete simplicial fan\(^1\) in \( V \), combinatorially equivalent to the coordinate fan\(^2\).

**Remark 2.2.** The wording in the above proposition was suggested to us by A. Zelevinsky.

The proposition immediately follows from the next lemma, whose proof is straightforward.

**Lemma 2.3.** Let \( x, y \in V \). Set \( J := \{j \in \Gamma \mid \langle \alpha_j, y \rangle = 0 \} \). Then the following are equivalent:

(a) \( y = \mathfrak{L}(x) \).

(b) \( x - y \) belongs to the the closed convex cone generated by \( -\alpha_j \) for \( j \in J \).

\[ (3.1) \quad \mathfrak{L}(x) \geq x, \quad x \in V. \]

**Theorem 3.1.** Assume that

\[ (3.2) \quad \langle \alpha_i, \alpha_j \rangle \leq 0 \text{ for } i \neq j. \]

Then the retraction \( \mathfrak{L} : V \to V^+ \) is order-preserving.

By (3.1), Theorem 3.1 implies the following statement, which characterizes \( \mathfrak{L} \) in terms of the order relation.

**Corollary 3.2.** If (3.2) holds then \( \mathfrak{L}(x) \) is the least element in \( \{y \in V^+ \mid y \geq x\} \).

Let us prove Theorem 3.1. To show that a piecewise linear map is order-preserving it suffices to check that this is true on each of its linearity domains. So Theorem 3.1 follows from Proposition 2.1(a) and the next proposition, which I learned from S. Schieder [Sch, Prop.3.1.2(a)].

**Proposition 3.3.** Assume (3.2). Then for each subset \( J \subset \Gamma \) the map \( \text{pr}_J : V \to V \) defined in Section 2 is order-preserving.

To prove the proposition, we need the following lemma.

**Lemma 3.4.** Let \( J \subset \Gamma \). Suppose that \( x \in V_J \) and \( \langle x, \alpha_j \rangle \geq 0 \) for all \( j \in J \). Then \( x \geq 0 \).

**Proof of the lemma.** We can assume that \( J = \Gamma \) (otherwise replace \( V \) by \( V_J \) and \( \Gamma \) by \( \Gamma_J \)). Then the lemma just says that \( V^+ \subset V^+_\text{pos} \). This is a well known consequence of (3.2).
Lemma 3.4, retraction \( L \subset V \):

Proposition 4.1. The existence of the least element in \( V \) follows from the next proposition.

Suppose that we have a family of vectors \( x \). Write \( x \in V \), for all \( i \).

Remark 4.2. The proof of Proposition 4.1 is identical to the proof of this classical statement.

Example 4.3. Consider the root system of \( SL(n) \). In this case \( V \) is the orthogonal complement of the vector \( v \) in the Euclidean space with orthonormal basis \( \varepsilon_1, \ldots, \varepsilon_n \), and \( \alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1 \). Let \( \omega_i \in V \) be the basis dual to \( \varepsilon_i \). For each \( v \in V \) define \( f_v : \{0, \ldots, n\} \to \mathbb{R} \) by

\[
    f_v(0) = f_v(n) = 0, \quad f_v(i) = \langle v, \omega_i \rangle \quad \text{for } 0 < i < n.
\]

Then the map \( v \mapsto f_v \) identifies \( V \) with the space of functions \( f : \{0, \ldots, n\} \to \mathbb{R} \) such that \( f(0) = f(n) = 0 \). Moreover, \( V^{pos} \) identifies with the subset of concave functions \( f \) and \( V^+ \) with the subset of concave functions \( f \). Thus the Langlands retraction assigns to a function \( f : \{0, \ldots, n\} \to \mathbb{R} \) the smallest concave function which is \( \geq f \).
5. Reductive groups

5.1. A remark on rationality. Suppose that in the situation of Sect. 2 one has \((\alpha_i, \alpha_j) \in \mathbb{Q}\) for all \(i, j \in \Gamma\). Then the \(\mathbb{Q}\)-linear span of the \(\alpha_i's\) equals the \(\mathbb{Q}\)-linear span of the \(\omega_i's\). Denote it by \(V^\mathbb{Q}\). Then \(V = V^\mathbb{Q} \otimes \mathbb{R}\). The cones \(K_j\), the subspaces \(V_j\), and the operators \(\text{pr}_j\) from Section 2 are clearly defined over \(\mathbb{Q}\). So by Proposition 2.1 one has

\[
\mathcal{L}(V^\mathbb{Q}) \subset V^\mathbb{Q}.
\]

5.2. The Langlands retraction for coweights. Now let \(G\) be a connected reductive group over an algebraically closed field. Let \(\Lambda_G\) be its coweight lattice, i.e., \(\Lambda_G = \text{Hom}(\mathbb{G}_m, T)\), where \(T\) is the maximal torus of \(G\). Set \(\Lambda_G^\mathbb{Q} := \Lambda_G \otimes \mathbb{Q}\). We have the simple coroots \(\check{\alpha}_i \in \Lambda_G\) and the simple roots \(\alpha_i \in \text{Hom}(\Lambda_G, \mathbb{Z})\). Let \(\Lambda_G^+, \Omega_G^\mathbb{Q}\) denote the dominant cone. Equip \(\Lambda_G^{+, \mathbb{Q}}\) with the following partial ordering: \(\lambda_1 \leq \lambda_2\) if \(\lambda_2 - \lambda_1\) is a linear combination of the simple coroots with non-negative coefficients.

Now define the Langlands retraction \(\mathcal{L}_G : \Lambda_G^\mathbb{Q} \rightarrow \Lambda_G^{+, \mathbb{Q}}\) as follows: \(\mathcal{L}_G(\lambda)\) is the least element of the set

\[
\{ \mu \in \Lambda_G^{+, \mathbb{Q}} \mid \mu \geq \lambda \}
\]

with respect to the \(\leq\) ordering.

**Corollary 5.3.** (i) \(\mathcal{L}_G(\lambda)\) exists.

(ii) \(\mathcal{L}_G(\lambda)\) is the element of \(\Lambda_G^{+, \mathbb{Q}}\) closest to \(\lambda\) with respect to any positive scalar product on \(\Lambda_G^{+, \mathbb{Q}} \otimes \mathbb{R}\) which is invariant with respect to the Weyl group.

(iii) \(\mathcal{L}_G(\lambda)\) is the unique element of the set \((5.2)\) with the following property: \((\mathcal{L}_G(\lambda), \alpha_i) = 0\) for any simple root \(\alpha_i\) such that the coefficient of \(\check{\alpha}_i\) in \(\mathcal{L}_G(\lambda) - \lambda\) is nonzero.

**Proof.** Combine Lemma 2.3, Corollary 3.2, and the inclusion (5.1). \(\square\)

5.4. Example: \(G = GL(n)\). In this case, just as in Example 4.3 one identifies \(\Lambda_G^\mathbb{Q}\) with the space of functions \(f : \{0, \ldots, n\} \rightarrow \mathbb{Q}\) such that \(f(0) = 0\) (while \(f(n)\) is arbitrary). Then the subset \(\Lambda_G^\mathbb{Q} \subset \Lambda_G^{+, \mathbb{Q}}\) identifies with the subset of concave functions \(f : \{0, \ldots, n\} \rightarrow \mathbb{Q}\) with \(f(0) = 0\). Just as in Example 4.3 the Langlands retraction assigns to a function \(f : \{0, \ldots, n\} \rightarrow \mathbb{Q}\) the smallest concave function which is \(\geq f\).

6. Some historical remarks

In [La2] R. Langlands defined the retraction \(\mathcal{L}\) and formulated his “geometric lemmas” (see [La2] Lemmas 4.4.4 and Corollary 4.6]) for the purpose of the classification of representations of real reductive groups in terms of tempered ones. However, much earlier he had formulated a closely related (and more complicated) combinatorial lemma\(^3\) in his theory of Eisenstein series, see [La1] Sect. 8. In this work Langlands considers Eisenstein series on quotients of the form \(G(\mathbb{A})/\Gamma\), where \(G\) is a reductive group over \(\mathbb{Q}\) and \(\Gamma\) is an arithmetic subgroup, but the same technique applies to quotients of the form \(G(\mathbb{A})/G(\mathbb{Q})\). Note that the stack \(\text{Bun}_G\) considered in [DrGa] is not far away from \(G(\mathbb{A})/G(\mathbb{Q})\), so the fact that the Langlands retraction is used in [DrGa] Appendix B) is not surprising.

\(^3\)An elementary introduction to this lemma can be found in [Cas1] [Cas2].
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