Supplementary Information for

Quantum Oscillations in the Magnetization and Density of States of Insulators

Animesh Panda, Sumilan Banerjee and Mohit Randeria

Corresponding author: Sumilan Banerjee, Mohit Randeria

E-mail: sumilan@iisc.ac.in, randeria.1@osu.edu

This PDF file includes:

- Supplementary text
- Figs. S1 to S4
- Table S1
- SI References
1. Table of symbols

| Symbol                                      | Expression                                                                 |
|---------------------------------------------|---------------------------------------------------------------------------|
| Effective masses $m_x$                      | $m_1, m_2 / (m_2 \pm m_1)$                                                |
| Mass ratio $m_x$                            | $m_x / m_+ = (m_2 - m_1) / (m_2 + m_1)$                                   |
| Unhybridized band crossing energy $\mu_0$  | $W / m_+$                                                                |
| Wave vector $k_F$ corresponding to $\mu_0$ | $(2m_+ W)^{1/2} / \hbar$                                                 |
| Direct gap $\Delta_D$                      | $2(2m_+ W)^{1/2} \nu / \hbar$                                           |
| Indirect gap $\Delta_I$                    | $\Delta_D \sqrt{m_1 m_2}$                                                |
| Impurity scattering rates $\Gamma$, $\gamma$, $\Gamma_c$, $\Gamma_r$ | $\Gamma_1 + \Gamma_2$, $\Gamma_1 - \Gamma_2$, $m_1 + m_2$, $m_1 - m_2$ |
| Cyclotron frequencies $\omega_1$, $\omega_2$, $\omega_\pm$, $\omega_c$ | $\pm \frac{eB}{m_1}, \pm \frac{eB}{m_2}, \pm \frac{eB}{m_1 + m_2}$ |
| Unhybridized oscillation frequency $F_0$    | $(\hbar / 2 \pi e) \pi k_F^2$                                           |
| Oscillation frequencies from band edges $F_\pm \pm \delta F$ | $(\hbar / 2 \pi e) \pi k_F^2 \pm (m_2 - m_1) \Delta_I / (4 \hbar e)$ |
2. Effective model in the semi-classical limit

In the energy eigenvalues $E_{0\pm}$ [Eq.2, Main text], the hybridization term $8l v^2 eB/\hbar$ becomes important for $l \approx l_F$ corresponding to the unhybridised band crossing, i.e. $\epsilon_{1l_F} \approx \epsilon_{2l_F} \approx \mu_0$. LL energies for $l$ farther from this energy tends to the original unhybridised energies $\epsilon_{1l} - i\Gamma_1$ and $\epsilon_{2l} - i\Gamma_2$. As a result, in the semiclassical limit $\mu_0 = Wm_+/m_1 \gg \hbar\omega_{c1}$, $l_F \simeq Wm_+/(\hbar eB) \gg 1$, and we get

$$8l v^2 eB/\hbar \approx 8l v^2 eB/\hbar \simeq \frac{8v^2 m_+}{\hbar^2} W = \Delta_D^2.$$  

The above leads to the semiclassical energy eigenvalues of Eq.3, Main text.

3. Critical field $B_c$ for field-induced insulator to metal transition

Here we give an estimate (1) of the critical field $B_c$ for $\Gamma = 0$. $B_c$ is obtained from the field at which the minimum, $E_-(B)$, and maximum, $E_+(B)$, of the energy eigenvalues $\mathcal{E}_{l\uparrow\downarrow}$ and $\mathcal{E}_{l\uparrow\downarrow}$, marked respectively in blue and yellow in Fig.1(b), Main text, coincide. We obtain $E_{\pm}$ from $\partial \mathcal{E}_{l\uparrow\downarrow}/\partial l = \partial \mathcal{E}_{l\uparrow\downarrow}/\partial l = 0$. To this end, for example, we rewrite

$$\mathcal{E}_{l\uparrow\downarrow} = \mu_0 + \frac{1}{2} \left[ \frac{m_+}{m_-} y - \sqrt{y^2 + \frac{a y + b}{\hbar \omega_{c\pm}}} + \frac{1}{2} \left( 1 - \frac{m_+^2}{m_-^2} \right) \right]$$

from Eq.2, Main text, using $y = (\hbar \omega_{c\pm}^{1/2} - \hbar \omega_{c\pm}^{1/2})/W$, $a = 8v^2 eB/\hbar^2 \omega_{c\pm}$, and $b = (8v^2 eB/\hbar^2 W, m_+ - 4v^2 eB/\hbar m_+ - m_-)$, where $\hbar \omega_{c\pm} = eBh/m_\pm$. Now, minimizing the above with respect to $y$ or $l$, we obtain

$$y = \frac{a/2}{\sqrt{m_+}} \frac{b - (a/2)^2}{1 - m_+^2/m_-^2}$$

for the maximum of $\mathcal{E}_{l\uparrow\downarrow}$ in the weak hybridization limit $W \gg m_+ v^2/\hbar$,

$$E_+(B) \simeq \mu_0 + \frac{1}{2} \left[ \frac{m_+}{m_-} \left( -\frac{4v^2 eB/\hbar}{\hbar \omega_{c\pm}} \right) - \frac{\hbar \omega_{c\pm}}{2} - \frac{1}{2} \left( 1 - \frac{m_+^2}{m_-^2} \right) \right]^{1/2} \frac{8W v^2 eB/\hbar}{\hbar \omega_{c\pm}}$$

Following similar steps, the minimum of $\mathcal{E}_{l\uparrow\downarrow}$ is obtained as

$$E_-(B) \simeq \mu_0 + \frac{1}{2} \left[ \frac{m_+}{m_-} \left( -\frac{4v^2 eB/\hbar}{\hbar \omega_{c\pm}} \right) - \frac{\hbar \omega_{c\pm}}{2} - \frac{1}{2} \left( 1 - \frac{m_+^2}{m_-^2} \right) \right]^{1/2} \frac{8W v^2 eB/\hbar}{\hbar \omega_{c\pm}}$$

Using the condition $E_+(B_c) = E_-(B_c)$, we obtain the critical field

$$B_c = \frac{2\Delta_D}{\hbar e} \frac{m_+ m_-}{\sqrt{m_-^2 - m_+^2}} = \sqrt{m_1 m_2} \Delta_D/\hbar e$$

4. Frequency of DOS oscillations at the gap edges

We show that the energy levels for non-zero magnetic field periodically crosses through the hybridization gap edges $\mathcal{E}_e$ and $\mathcal{E}_v$ [Fig.2(c),(d), Main text], i.e. the minimum of the conduction band $\mathcal{E}_+(k)$ and the maximum of valence band $\mathcal{E}_-(k)$, as a function of $1/B$. In the semiclassical limit $\mu_0 \gg \hbar \omega_{c1}$, we estimate $\mathcal{E}_{e/v}$ from energy dispersion

$$\mathcal{E}_{\pm}(k) = \frac{1}{2} \left[ W + \frac{\hbar^2 k^2}{2m_{\pm}} \pm \sqrt{\left( \frac{\hbar^2 k^2}{2m_+} - W \right)^2 + \Delta_D^2} \right],$$

which corresponds to the semiclassical eigenvalues $\mathcal{E}_{l\pm}$ in Eq.3, Main text. The wavevectors (magnitude) $k_e$ and $k_v$ at the energies $\mathcal{E}_{e/v}$ are obtained from $[\partial \mathcal{E}_+(k)/\partial k]_{k=k_e} = [\partial \mathcal{E}_-(k)/\partial k]_{k=k_v} = 0$ as $k_e^2 = k_F^2 \mp \Delta_I(m_2 - m_1)/(2\hbar^2)$. These lead to

$$\mathcal{E}_{l\pm}(k) = \mu_0 \pm \frac{\Delta_I}{2}$$

Equating the above with energy levels $\mathcal{E}_{l\pm}$ for two successive LL index, e.g. $l$ and $l + 1$, at two fields $B'$ and $B$ ($B < B'$), i.e. $\mathcal{E}_{l\pm}(B') = \mathcal{E}_{e/v}$ and $\mathcal{E}_{l+1\pm}(B) = \mathcal{E}_{e/v}$, we obtain

$$\frac{1}{F_{\pm}} = \frac{1}{B} - \frac{1}{B'} = \frac{\hbar e}{W + \Delta_D^2 / \sqrt{m_-^2 - m_+^2}} m_+ = \frac{1}{F_0} + \Delta_I(m_2 - m_1)/(4\hbar e)$$
The above proves the $1/B$-periodicity of the gap-edge crossing of the energy levels with frequencies $F_\omega$ even though the eigenvalues $E_{i\pm}$ do not have canonical equipressed LL form. The frequencies arise from the semiclassical orbits of areas $\pi k_\xi^2$ and $\pi k_\eta^2$ at energies $E_{i\pm}$ [Fig.2(c),(d), Main text]. The DOS oscillates with $1/B$ periodicity at the gap edges, which are the lowest energy excitations $\pm \Delta_I/2$ away from the chemical potential $\mu_0$. Thus the gap edges contribute to the oscillations of the LE-DOS of Eq.5, Main text with a thermally activated amplitude $\sim \exp(-\Delta_I/(2T))$ at low temperature in agreement with the low-$T$ saddle-point expression [Eq.9, Main text].

5. DOS in the semiclassical limit

Using the eigen energies in the semi-classical limit from Eq.3, Main text, the DOS can be written as

$$A(\xi) = -2 \left( \frac{N_B}{\pi} \right) \text{Im} \sum_{l, p = \pm} \frac{1}{{\xi + \mu_0 - E_{lp}}}$$

[9]

The factor of two is due to the degeneracy of the energy levels. Using the expressions for $E_{i\pm}$ we can write

$$\sum_{p = \pm} \left( \frac{1}{{\xi + \mu_0 - E_{lp}}} \right) = \sum_{p = \pm} \left( \frac{1}{{\xi + \mu_0 - \frac{\pi}{2} \left[ (\xi_l + e^{i2}) + p \sqrt{(\xi_l - e^{i2})^2 + \Delta_I^2} \right]} \right) = \frac{b_1 l + b_0}{a_2 l^2 + a_1 l + a_0}$$

where, $a_2 = -\hbar \omega_{c1} \hbar \omega_{c2}$, $a_1 = \hbar \omega_{c2}(W - i\Gamma_2) + \hbar \omega_{c2}(1 - (\xi + \mu_0) \hbar \omega_{c2}$, $a_0 = -\hbar \Gamma_1(W - i\Gamma_2) - (\xi + \mu_0)(W - \hbar \Gamma_1 - i\Gamma_2) + (\xi + \mu_0)^2 - \Delta_I^2/4$, $b_1 = -\hbar \omega_{c1}$ and $b_0 = 2(\xi + \mu_0) - W + i(\Gamma_1 + \Gamma_2)$. We can rewrite the above equation as

$$\sum_{p = \pm} \left( \frac{1}{{\xi + \mu_0 - E_{lp}}} \right) = -\frac{1}{2\pi B \hbar} \sum_{p = \pm} c_p$$

[10]

with $l_p(\xi) = (a_1 + \sqrt{a_1^2 - 4a_2a_0})/2a_2$ and $c_\pm = (-\pi B a_1) [b_1 \mp b_1(l_+ + l_-) + 2b_0]/(l_- - l_+).$ Since $\mu_0 \hbar \omega_{c1} = W \hbar \omega_{c1}$ and $\hbar^2 k_F^2/2m_1 = \mu_0$, we get

$$l_p(\xi) = \frac{F_0}{B} + x_p(\xi)$$

[11a]

$$x_p(\xi) = \frac{\iota(m_1 \Gamma_1 - m_2 \Gamma_2) + (m_1 - m_2) \xi + \iota \sqrt{\left( (m_1 + m_2) \xi + \iota(m_1 \Gamma_1 + m_2 \Gamma_2) \right)^2 - m_1 m_2 \Delta_I^2}}{2\pi B \hbar}$$

$$= \frac{1}{2\hbar \omega_{c1}} \left[ -m_r \xi + \iota \Gamma_r \pm \sqrt{((\xi + \iota \Gamma) / (\Delta_I / 2))^2} \right].$$

[11b]

Similarly, $c_p(\xi)$ is given by

$$c_p(\xi) = (m_1 - m_2) + \pi (m_1 + m_2) \left[ (m_1 + m_2) \xi - \iota(m_1 \Gamma_1 + m_2 \Gamma_2) / \sqrt{((m_1 + m_2) \xi + \iota(m_1 \Gamma_1 + m_2 \Gamma_2))^2 - m_1 m_2 \Delta_I^2} \right]$$

$$= (m_1 + m_2) \left( -m_r + \iota \frac{\xi + \iota \Gamma}{\sqrt{((\xi + \iota \Gamma) / (\Delta_I / 2))^2}} \right).$$

[11c]

Here $\omega_c = \varepsilon B/(m_1 + m_2)$, $m_r = (m_2 - m_1)/(m_2 + m_1)$, $\Gamma_r = (m_1 \Gamma_1 - m_2 \Gamma_2)/(m_1 + m_2)$ and $\Gamma_c = (m_1 \Gamma_1 + m_2 \Gamma_2)/(m_1 + m_2)$. Converting the LL sum over $l$ in Eq.9 into an integral using Poisson summation formula and evaluating the integrals using the poles $l_p(\xi)$ and the residues $c_p(\xi)$, we obtain the oscillatory part of the DOS as

$$A(\xi) = \frac{1}{\pi \hbar^2} \text{Im} \sum_{k \neq 0, p = \pm} \iota s_p(\xi)c_p(\xi)e^{2\pi i k s_p \xi l_p(\xi)}$$

[12]

with $s_p(\xi) = \text{sgn}[\text{Im}\{l_p(\xi)\}]$. 

4 of 15

Animesh Panda, Sumilan Banerjee and Mohit Randeria
6. LE-DOS oscillations at low temperatures

In the semiclassical limit, we rewrite the LE-DOS of Eq. 5, Main text using the DOS of Eq. (12) as

\[ D(T) = \frac{1}{2T} \int_{-\infty}^{\infty} d\xi \frac{1}{1 + \cosh(\frac{\xi}{T})} A(\xi) = \text{Re} \left[ \int_{-\infty}^{\infty} \frac{d\xi}{4T \pi \hbar^2} \sum_{k \neq 0, \pm} e^{2\pi i k_\sigma p(\frac{\xi}{T})} s_p(\xi) c_p(\xi) e^{-i k_p(\xi)} \right], \]  

where

\[ f_{k_p}(\xi) = -2\pi i k_\sigma p(\xi) x_p(\xi) + 2 \text{ln}[\cosh(\xi/2T)], \]

and we have used \( 2\cosh^2(\xi/2T) = 1 + \cosh(\xi/T) \). We split the integral in Eq. (13a) into three parts [Fig.S1] as

\[ \int_{-\infty}^{\infty} d\xi \left[ \ldots \right] = \left( \int_{-\infty}^{-\varepsilon_c} + \int_{-\varepsilon_c}^{\varepsilon_c} + \int_{\varepsilon_c}^{\infty} \right) d\xi \left[ \ldots \right] = I_+ + I_- + I_0, \]

where \( \xi_c \geq \Delta I/2 \) is an arbitrary cutoff. We show below that the integrals \( I_\pm (\xi = \pm) \) can be well approximated via a saddle-point method, whereas \( I_0 \) gets the main contribution from the region near \( \xi = 0 \) at low temperatures \( T \ll \Delta I \). The saddle-point contribution to the LE-DOS [Eq.9, Main text] is \( D_p(T) = I_+ + I_- \) and impurity induced in-gap DOS [Eq.10, Main text] \( D_0(T) = I_0 \).

A. LE-DOS oscillations from the gap edges. To carry out the saddle-point integration for \( I_\pm \), we expand \( 2\text{ln}[\cosh(\xi/2T)] \simeq \frac{\xi_\tau}{\pi} - 1 \) \( 2 \exp(-\xi/T) + \ldots \) at low temperature, where \( \zeta = 1 (+) \) for \( \xi > 0 \) and \( \zeta = -1 (-) \) for \( \xi < 0 \). Thus, approximating

\[ f_{k_p}(\xi) \approx -2\pi i k_\sigma p(\xi) x_p(\xi) + \xi_\tau \xi + 2 \text{ln} 2, \]

the saddle point is obtained from \( \partial f_p(\xi)/\partial \xi = 0 \) assuming that the sign \( s_p(\xi) \) does not vary around the saddle points. For, \( T \ll \hbar \omega_c \), we obtain two saddle points for each \( k, p \)

\[ \xi_{k_p} \approx -i\Gamma + \xi \Delta I/2 \left( 1 - \frac{k^2 \pi^2 T^2}{2\hbar^2 \omega_c^2} \right), \]

which are complex for \( \Gamma \neq 0 \). The saddle-point leads to

\[ x_p(\xi_{k_p}) \approx -\zeta \left( \frac{m_2 - m_1}{4\hbar e B} \right) \]  

\[ c_p(\xi_{k_p}) \approx -i p \left( \frac{\hbar B}{2kT} \right), \]

with

\[ \frac{1}{\tau_{k_p}} = \frac{1}{\hbar} \left[ \frac{2\gamma m_+}{m_2 + m_1} + p k_T \Delta I/\hbar \omega_c \right]. \]

where \( s_p(\xi_{k_p}) = \text{sgn}(\tau_{k_p}) \). The above implies that the saddle-point value of the pole

\[ \xi_0(\xi_{k_p}) = \frac{F_c}{B} + \frac{1}{2\omega_c \tau_{k_p}} \]

in the DOS [Eq. (12)] dominates the integrals \( I_\pm \) in the LE-DOS [Eq. (13a)] at low temperature. The real part of the saddle-point pole modifies the frequency of oscillations to \( F_c = F_0 = -\zeta(m_2 - m_1)\Delta I/(4\hbar e) \). As shown in Fig.S1, to evaluate the integrals \( I_\pm \) using the saddle-points in Eq. (16), we deform the integration contour from the real axis to the complex plane \( z = \xi + i\omega \) such that it goes through the saddle points. The deformed path is chosen such that, close to \( \xi_{k_p} \), the imaginary part of \( f_{k_p}(z) \) remains constant and the real part has a maximum at the saddle point along the path. This is achieved by the expansion \( f_{k_p}(z = \xi = \xi_{k_p} + \eta) \approx f_{k_p}(\xi_{k_p}) + 1/2(\partial^2 f_{k_p}/\partial z^2) \zeta \eta^2 \), where \( \partial^2 f_{k_p}/\partial z^2 \zeta = 2 s_{k_p} B(\delta \hbar \omega_c^2)/(\pi^2 k^2 T^3 \Delta I) \) with \( s_{k_p} = s_p(\xi_{k_p}) \), such that

\[ D_p(T) = \frac{1}{4T \pi \hbar^2} \text{Re} \left[ \sum_{k \neq 0} e^{2\pi i k_\sigma p(\xi_{k_p})} \frac{F_0}{s_{k_p} B(\delta \hbar \omega_c^2)/(\pi^2 k^2 T^3 \Delta I)} \int d\eta e^{\frac{1}{\hbar} \left( \frac{\partial^2 f_{k_p}}{\partial \eta^2} \right) \zeta \eta^2} \right]. \]

This leads to the Gaussian integral

\[ \int d\eta e^{\frac{1}{\hbar} \left( \frac{\partial f_{k_p}}{\partial \eta^2} \right) \zeta \eta^2} \simeq (-p s_{k_p})^{1/2} \int_{-\infty}^{\infty} d\eta e^{-\delta \hbar \omega_c^2/(\pi^2 k^2 T^3 \Delta I) \zeta \eta^2} = (-p s_{k_p})^{1/2} \frac{\pi^{3/2}}{2} \frac{T^3/2 \Delta I^{1/2}}{\hbar \omega_c}, \]

where the integration contour through the saddle-point is chosen via the variable transformation \( p s_{k_p} \eta^2 = -\tau^2 \). For \( \Gamma \to 0 \), \( s_{k_p} = p \) and the saddle-point paths are vertical [Fig.S1]. Finally, using the above and Eqs. 15,17 in Eq. (19) we obtain the expression for \( D_p(T) \) [Eq.9, Main text].
B. LE-DOS oscillations from impurity-induced in-gap DOS. The gap-edge oscillations coexist with the in-gap DOS oscillations in the presence of disorder ($\Gamma \neq 0$), and they can be separated from each other at low temperature since splitting of LE-DOS integral in Eq. (14) into three independent integrals is well controlled for $T \ll \xi_c \sim \Delta_I$. The in-gap DOS oscillations were derived in ref. (2). Here we briefly sketch the derivation for the sake of completeness.

The main effect of impurity-induced DOS arise near $\xi = 0$ at the chemical potential $\mu_0$ inside the gap. This is captured by the integral $I_0 = D_0(T) = \int_{-\xi_c}^{\xi_c} d\xi \ldots$ in the LE-DOS integral [Eq. (14)]. At low temperatures, for $\Gamma \neq 0$, due to the $[1 + \cosh(\xi/T)]^{-1}$ term in Eq. (13a), the main contribution to $I_0$ comes from the region near $\xi = 0$ along the real axis [Fig.S1]. Thus, by expanding $c_p(\xi) \simeq c_p(0) + c_p''(0)\xi$, $x_p(\xi) = x_p(0) + x_p'(0)\xi$, we can approximate $D_0(T)$ as

$$D_0(T) \simeq \frac{1}{2T\pi\hbar^2} \Re \left[ \sum_{kp} s_p(0) e^{2\pi iksp(0)} \left( \frac{\xi}{T} \right) \int_{-\infty}^{\infty} d\xi \frac{c_p(0)}{1 + \cosh \left( \frac{\xi}{T} \right)} e^{2\pi iksp(0)[x_p(0) + x_p'(0)\xi]} \right]$$

To evaluate the integral above we use the identity (2),

$$\int_{-\infty}^{\infty} d\xi \frac{1}{1 + \cosh \left( \frac{\xi}{T} \right)} e^{2\pi iksp(0)x_p'(0)\xi} = -\frac{4\pi^2 kT^2 x_p'(0)}{\sinh(2\pi^2 kTx_p'(0))}.$$

Moreover, from Eqs. 11b,11c

$$x_p(0) = \frac{i}{2\hbar\omega_c} \left[ \Gamma_r + p\sqrt{\Gamma_c^2 + (\Delta_I/2)^2} \right]$$
$$x_p'(0) = \frac{1}{2\hbar\omega_c} \left[ -m_r + \frac{p\Gamma_c}{\sqrt{\Gamma_c^2 + (\Delta_I/2)^2}} \right]$$
$$c_p(0) = \frac{p(m_1 + m_2)\Gamma_c}{\sqrt{\Gamma_c^2 + (\Delta_I/2)^2}} + (m_1 - m_2).$$

Using the above we obtain the expression for $D_0(T)$ given in Eq.10, Main text. Here $s_p(0) = \Im[x_p(0)] = p$ since $\sqrt{\Gamma_c^2 + (\Delta_I/2)^2} > \Gamma_r$. Moreover, it can be shown that next order in temperature correction appears at $O(T/\Delta_I, T/\Gamma)$ to $D_0(T)$. $D_0(T) \to 0$ as $\Gamma \to 0$, i.e. for the disorder-free case, as can be verified from Eq.10, Main text.
Fig. S1. Integration contour for the saddle-point approximation to LE-DOS: The original integration path (horizontal solid orange line) for LE-DOS energy integral is along the real axis $\xi$ on the complex plane $z = \xi + i\omega$. The contour is deformed to go through the saddle points $\tilde{\xi}_k \simeq \pm (\Delta_I/2) + i\Gamma_c$ and the origin $\xi = 0, \omega = 0$. Only the integrals $I_{\pm}$ along the saddle-point paths (vertical solid blue lines) close to the saddle points matter at low temperature $T \ll \Delta_I$. In addition, the integral $I_0$ from the region close to the origin contributes to LE-DOS for finite impurity scattering $\Gamma \neq 0$. The saddle-point paths are vertical for $\Gamma \to 0$, as shown here. The three integration regions $I_{\pm}, I_0$ are connected by arbitrary contours (dashed yellow lines) which have negligible contribution to LE-DOS at low $T$. 

$\frac{-\Delta_I}{2} + i\Gamma_c \quad I_-$

$I_0$

$\frac{\Delta_I}{2} + i\Gamma_c \quad I_+$

$\xi = 0$

$\omega$
7. Non-trivial temperature dependence of LE-DOS amplitude

Here we show that the LE-DOS oscillation amplitude at frequency \( F_0 \) can exhibit more complex temperature dependence at low temperature compared to that in Fig.Fig.4(a), Main text for different choices of disorder strengths. In Fig.S2, we show that for \( \Gamma_2 = 0.05\Delta_D \) and \( \Gamma_1 = 0.15\Delta_D, 0.25\Delta_D \), the amplitude \( \tilde{D} \) [normalized by \( \tilde{D}_{E=0}(T=0) \)] initially decreases with \( T \), following the LK-like form [Eq.10, Main text] due to in-gap DOS \( D_0(T) \), followed by activated increase expected from gap-edge contribution \( D_0(T) \) [Eq.9, Main text]. Also, due to this interplay of \( D_0(T) \) and \( D_0(T) \), the amplitude can sharply increase at low temperature, as shown for \( \Gamma_1 = 0.35\Delta_D, 0.45\Delta_D \). Here, with the increase in impurity scattering strength the zero temperature oscillation amplitude does not decrease monotonically, as one expects naively. This can be seen from the \( T = 0 \) oscillation amplitude for \( \Gamma_1 = 0.05\Delta_D, 0.15\Delta_D \) in Fig.S2 and \( \Gamma \to 0 \) case shown in Fig.4(a), Main text. In this range, \( \tilde{D} \) increases with \( \Gamma_1 \).

8. Magnetization

A. Magnetization oscillations at \( T = 0 \). At zero temperature the magnetization can be obtained from \( M = -\partial E(B)/\partial B \), where

\[
E(B) = N_B \sum_{l} (\tilde{E}_{l-} - \mu)
\]

is the total energy, and the chemical potential is inside the gap so that the sum above runs over all energy levels \( l \) in the valence band. Here we assume the semiclassical limit and use the energy eigenvalues of Eq.3, Main text. To see how the oscillations arise, \( \tilde{E}_{l-} \) can be split into two parts, i.e. \( \tilde{E}_{l-} = \tilde{E}_{l-v} + \tilde{E}_{l-e} \), with \( \tilde{E}_{l-e} = (W + heBl/m_+)/2 \) and

\[
\tilde{E}_{l-v} = -[(W - heBl/m_+)^2 + \Delta_D^2D_0]^1/2.
\]

Thus, we can write \( E = E_{\text{osc}} + E_{\text{osc}} \) with

\[
E_{\text{osc}} = \sum_{l} [\tilde{E}_{l-v} - \mu]
\]

\[
E_{\text{osc}} = \sum_{l} \tilde{E}_{l-e}
\]

\( E_{\text{osc}} \) above is exactly same as that of a completely filled valence band with usual equispaced LLs \( heBl/2m_- \). Hence, \( E_{\text{osc}} \) cannot give rise to any oscillations and it is a smooth monotonic function of \( B \). However, \( E_{\text{osc}} \) corresponds to the total energy due to completely occupied valence band of a particle-hole symmetric band structure [Fig.2(e),(f), Main text]. This fictitious band structure is similar to a particle-hole (PH) symmetric model of a hybridization-gap insulator that was considered in ref.(3). It was shown there (3) that such a PH symmetric insulator exhibits magnetization oscillations. Surprising, in our case, even though the original energy dispersion is not PH symmetric, only the PH symmetric part of the dispersion contributes to the magnetization oscillations. We refer the reader to ref.(3) for a detailed derivation of magnetization oscillations using Euler-MacLaurin expansion for a PH symmetric energy dispersion like \( \tilde{E}_\pm(k) \). Here we discuss a simple approximate derivation of the frequency of oscillations. This is further supported by our low-temperature saddle-point calculations discussed in the next sections.

Firstly, it is easy to see that energy levels \( \tilde{E}_{l-} \) periodically crosses the gap edge of the valence band \( \tilde{E}_-(k), \tilde{E}_+ = -\Delta_D/2 \) from the hole-like part of the band to the electron-like part [Fig.2(e),(f), Main text] with decreasing field. Here \( \tilde{E}_\pm \) is obtained from \( \partial \tilde{E}_\pm(k)/\partial k = 0 \). Considering two fields \( B \) and \( B' (B < B') \) such that \( \tilde{E}_{l-}(B') = -\Delta_D/2 \) and \( \tilde{E}_{l+1,-}(B) = -\Delta_D/2 \), we can find the periodicity

\[
\frac{1}{B} - \frac{1}{B'} = \frac{he}{m_+W} = \frac{1}{F_0}\]

A simple, albeit heuristic, understanding of how the above \( 1/B \)-periodic crossings affect the total energy can be obtained by neglecting the LLs for \( |W - heBl/m_+| \lesssim \Delta_D \) and approximating the energy levels as

\[
\tilde{E}_- \approx \frac{1}{2} \left( W - \frac{heBl}{m_+} \right) - \ldots \left( W - \frac{heBl}{m_+} \right) \gtrsim \Delta_D + \ldots
\]

\[
\approx \frac{1}{2} \left( W - \frac{heBl}{m_+} \right) - \ldots \left( \frac{heBl}{m_+} - W \right) \gtrsim \Delta_D + \ldots
\]

As a result

\[
E_{\text{osc}} \approx \frac{1}{2} \sum_{l \leq m_+ B/2heB} \left( W - \frac{heBl}{m_+} \right) + \frac{1}{2} \sum_{l > m_+ B/2heB} \left( W - \frac{heBl}{m_+} \right) = \sum_{l \leq m_+ B/2heB} \left( \frac{heBl}{m_+} - W \right) + \frac{1}{2} \sum_{l} \left( W - \frac{heBl}{m_+} \right)
\]

The second term in the last line above is monotonic function of \( B \), whereas the first term is an oscillatory function of \( 1/B \) with frequency \( F_0 \), exactly like the total energy of a metal with LLs \( heBl/m_+ \) and chemical potential \( W \). Thus, whenever an additional LL enters the electron-like part of the band \( \tilde{E}_-(k) \) from the hole-like part through the gap edge \(-\Delta_D/2\), the total energy sharply changes leading to \( 1/B \)-periodic oscillations of the magnetization.
Fig. S2. Complex temperature dependence of LE-DOS amplitude: LE-DOS oscillation amplitude $\tilde{D}$ at frequency $F_0$ [normalized by $\tilde{D}_{v=0}(T=0)$ for zero hybridization] as a function of temperature for scattering rates different from the ones in Fig. 4(a), Main text. Here $\Gamma_2 = 0.05 \Delta_D$, and the results are obtained using the energy eigenvalues $\xi_{l,\pm}$ in Eq. 2, Main text.
B. Oscillatory part of magnetization in the semiclassical limit. The grand potential (per unit area) of the model of Eq.1, Main text in the presence of impurity scattering can be written as

$$\Omega(T) = -TN_B \sum_{\omega_n} \text{Tr} \ln[-\beta G^{-1}(\omega_n)] e^{i\omega_n \theta}.$$  \[25\]

Here $G(\omega_n)$ is the single-particle Green’s function matrix in the combined LL index, band and spin space and the ‘Tr’ acts on the same space. For example, in the absence of magnetic field, $G(\omega_n)$ can be obtained from

$$G^{-1}(k, \omega_n) = \omega_n + \mu_0 - H(k) - i \[ \begin{array}{cc} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{array} \] \text{sgn}(\omega_n),$$  \[26\]

which implies $(\Gamma_1, \Gamma_2) \rightarrow (-\Gamma_1, -\Gamma_2)$ for $\omega_n \rightarrow -\omega_n$ $(\omega_n > 0)$. Since, $\mathcal{E}_\pm(k, -\Gamma_1, -\Gamma_2) = \mathcal{E}_\pm^\dagger(k, \Gamma_1, \Gamma_2)$, the Green function in the diagonal basis is $G_\pm^\dagger(k, \omega_n) = [\omega_n + \mu_0 - \mathcal{E}_\pm^\dagger(k)]^{-1}$ for $\omega_n > 0$ and $G_\pm(k, \omega_n) = [\omega_n + \mu_0 - \mathcal{E}_\pm(k)]^{-1}$ for $\omega_n < 0$. Similarly, for $B \neq 0$, $G_{l,bp}(\omega_n) = [\omega_n + \mu_0 - \mathcal{E}_{l,bp}]^{-1} \theta(\omega_n) + [\omega_n + \mu_0 - \mathcal{E}_{l,bp}^*]^{-1} \theta(-\omega_n)$. Thus, the grand potential can be written as

$$\Omega(T) = -TN_B \sum_{l,bp, \omega_n > 0} \ln(\mathcal{E}_{l,bp} - \mu_0 - \omega_n) e^{i\omega_n \theta} + \text{c.c.}$$  \[27\]

In the semi-classic limit($\mu_0 \gg \hbar \omega_c$) we replace the eigen energies $\mathcal{E}_{l,b}$ with $\mathcal{E}_{l,\pm}$ [Eq.3, Main text]. We convert the LL summation to an integral using Poisson summation formula and extract the oscillatory component of the $\Omega(T)$ through an integration by parts,

$$\Omega(T) \approx 2TN_B \sum_{\omega_n > 0} \sum_{k \neq 0} \int_{-\infty}^{\infty} \text{d}l \frac{e^{2\pi ikl}}{2\pi k} \frac{d}{dl} \left[ \mathcal{E}_+(l) - \mu_0 - \omega_n \right] \left[ \mathcal{E}_-(l) - \mu_0 - \omega_n \right] + \text{c.c.}$$

Using the quadratic nature of the function $[\mathcal{E}_+(l) - \mu_0 - \omega_n][\mathcal{E}_-(l) - \mu_0 - \omega_n]$, we obtain

$$\Omega(T) \approx 2TN_B \sum_{\omega_n > 0} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \text{d}l \frac{e^{2\pi ikl}}{2\pi k} \left( \frac{1}{l - l_+(\omega_n)} + \frac{1}{l - l_-(\omega_n)} \right) + \text{c.c.}$$

It is easy to verify that the poles $l_{\pm}(\omega_n) \equiv l_{\pm}(\omega_n)$ in the above equation is the same as the poles obtained in Eq.11a while calculating DOS, with $\xi \rightarrow \omega_n$ in the argument of $l_p(\xi)$. In the above, we have also extended the lower limit of the integral over $l$ to $-\infty$ since $\text{Re}[l_{\pm}(\omega_n)] \gg 1$ in the semiclassical limit. Performing the contour integration over $l$, we obtain

$$\Omega(T) \approx 4TN_B \sum_{p,k>0, \omega_n > 0} \frac{1}{k} e^{2\pi ikp(n)} l_p(n) + \text{c.c.}$$

where, $s_p(n) = \text{sgn}[\text{Im}(l_p(n))] = \text{sgn}[\text{Im}(x_p(n))]$. It can be seen from Eq.11a that $l_p(n) = (F_0/B) + x_p(n)$ and $l_p^*(n) = (F_0/B) - x_p(n)$ since $x_p(n)$ is purely imaginary. Thus, we get

$$\Omega(T) \approx 4TN_B \sum_{p,k>0, \omega_n > 0} \cos \left[ \frac{2\pi k}{k_0} \right] e^{2\pi ikp(n)} x_p(n)$$

We obtain the oscillatory component of the magnetization from $M = -\partial \Omega/\partial B$. The dominant, $O(\mu_0/\hbar \omega_c)$, contribution to magnetization in the semiclassical limit comes from the field derivative of the cosine term in the above equation and is given by

$$M \approx \frac{8\pi T \mu_0}{\hbar \omega_c} \sum_{p,k>0, \omega_n > 0} \sin \left[ \frac{2\pi k}{k_0} \left( \frac{F_0}{B} \right) \right] e^{2\pi ikp(n)} x_p(n).$$  \[28\]

Based on the low-temperature approximation discussed below it can be shown that the terms neglected above are smaller by factors of $O(\hbar \omega_c/\mu_0, T/\mu_0, \Gamma/\mu_0, \Delta_1/\mu_0)$.

C. Magnetization oscillations at low temperature. We rewrite the oscillatory part of magnetization given in Eq.28 as

$$M \approx \frac{8\pi \mu_0}{\hbar \omega_c} \sum_{p,k>0} T \sum_{n=0}^{\infty} F_{kp}(n)$$  \[29\]

where,

$$F_{kp}(n) = \sin \left[ 2\pi k \left( \frac{F_0}{B} \right) \right] e^{2\pi ikp(n)} x_p(n)$$
We use the Euler Maclaurin formula \( \sum_{n=a}^{b} f(n) = \int_{a}^{b} f(x)dx + (1/2)[f(a) + f(b)] + (1/12)[f'(b) - f'(a)] - \ldots \) to evaluate the sum at low temperature giving,

\[
T \sum_{n=0}^{\infty} F_{kp}(n) \approx T \int_{0}^{\infty} dn F_{kp}(n) + \frac{1}{2} TF_{kp}(0) + \ldots \]

where, we use the fact that \( F_{kp}(n \to \infty) \to 0 \). Doing a variable transformation \( \omega = (2n + 1)\pi T \) we get

\[
T \int_{0}^{\infty} dn F_{kp}(n) = \int_{0}^{\infty} \frac{d\omega}{2\pi} \sin \left[ 2\pi k \left( \frac{F_{0}}{B} \right) \right] e^{2\pi ik_{p}(\omega)x_{p}(\omega)}
\]

with

\[
x_{p}(\omega) = (\pi/2\hbar \omega) [m_{r}\omega + \Gamma_{r} + p\sqrt{(\omega + \Gamma_{c})^{2} + (\Delta_{I}/2)^{2}}] .
\]

The integral in Eq.31 does not depend on temperature and leads to a constant contribution to magnetization oscillations for \( T \to 0 \). We again evaluate the above integral by saddle point method. The condition \( \partial x_{p}(\omega)/\partial \omega = 0 \) gives

\[
\omega_{c} = \frac{m_{r}}{\sqrt{1 - m_{c}^{2}}} \frac{\Delta_{I}}{2}
\]

We only take the saddle point with \( \zeta = 1 \), denoted as \( \tilde{\omega} \), which falls on the path of the integration \( \int_{0}^{\infty} d\omega \). This leads to

\[
x_{p}(\tilde{\omega}) = \frac{i}{2\hbar \omega_{c}} \left( m_{r}\Gamma_{c} + \Gamma_{r} + (p + m_{c}^{2})\Delta_{I} \right) / 2\sqrt{1 - m_{c}^{2}} \]

\[
t_{p}(\tilde{\omega}) = \frac{F_{0}}{B} + \frac{i}{2\omega_{c} \tau_{p}} \]

with

\[
\frac{1}{\tau_{p}} = \frac{1}{\hbar} \left[ \frac{2m_{r}\gamma}{m_{1} + m_{2}} + \frac{(p + m_{c}^{2})\Delta_{I}}{2\sqrt{1 - m_{c}^{2}}} \right]
\]

and \( s_{p}(\omega) = \text{sgn}[\tau_{p}] \). Since, \( m_{r} < 1 \), for the limit \( \Delta_{I} \gg \gamma \), \( s_{p}(\omega) = p \). To perform the \( \omega \) integral in Eq.31 using the above saddle point, we expand around the saddle point \( \omega = \tilde{\omega} + \xi \), i.e.

\[
2\pi ik_{p}(\omega)x_{p}(\omega) \approx 2\pi kp \left[ x_{p}(\tilde{\omega}) + \frac{1}{2} \left( \frac{\partial^{2}x_{p}}{\partial \omega^{2}} \right)_{\omega = \tilde{\omega}} \xi^{2} \right]
\]

where,

\[
2\pi kp \left( \frac{\partial^{2}x_{p}}{\partial \omega^{2}} \right)_{\omega = \tilde{\omega}} = -\frac{2\pi k}{h \omega_{c} \Delta_{I}} (1 - m_{c}^{2})^{3/2}
\]

Now we perform the integral,

\[
\int d\xi \exp \left( -\frac{\pi k}{h \omega_{c} \Delta_{I}} (1 - m_{c}^{2})^{3/2} \xi^{2} \right) = \sqrt{\frac{h \omega_{c} \Delta_{I}}{k}} (1 - m_{c}^{2})^{3/4},
\]

finally to obtain

\[
T \int_{0}^{\infty} dn F_{kp}(n) \simeq \frac{1}{2\pi} \sqrt{\frac{h \omega_{c} \Delta_{I}}{k}} (1 - m_{c}^{2})^{3/4} \sin \left[ 2\pi k \left( \frac{F_{0}}{B} \right) \right] e^{-\pi k/\omega_{c} |\tau_{p}|},
\]

Here it is important to note that, unlike the saddle-point approximation for LE-DOS discussed in Sec.A, the saddle-point integral above is only controlled for \( m_{r} \to 1 \) i.e. \( m_{2} \gg m_{1} \), when the Gaussian integrand becomes sharply peaked around the saddle point. The temperature dependence of the magnetization oscillation amplitude comes from the second and higher order terms in Euler Maclaurin formula [Eq.12, Main text], namely

\[
\frac{1}{2} TF_{kp}(0) = \frac{T}{2} \sin \left[ 2\pi k \left( \frac{F_{0}}{B} \right) \right] e^{2\pi ik_{p}(0)x_{p}(0)}.
\]

Here, \( x_{p}(0) = (\tau_{2}^{2}/2\omega_{c} \tau_{1}(T)) \) with

\[
\frac{1}{\tau_{1}} = \frac{1}{\hbar} \left[ \sqrt{(\pi T + \Gamma_{c})^{2} + (\Delta_{I}/2)^{2}} + p(\Gamma_{r} - m_{r}\pi T) \right],
\]
and \( s_p(0) = p \) since \( \Gamma_r \leq \Gamma_c, \ m_r < 1 \). Thus we get
\[
\frac{1}{2} T F_{kp}(0) = \frac{T}{2} \sin \left( 2\pi k \left( \frac{F_0}{B} \right) \right) e^{-\pi k/\omega_c \tau_{1p}(T)}
\]  

[33]

From \( \partial \tau_{1p}^{-1}/\partial T = 0 \), we find out that \( \tau_{1p}^{-1}(T) \) has a minimum at some temperature \( T_{peak} \), and hence a peak for oscillation amplitude. This gives
\[
T_{peak} = \frac{1}{\pi} \left( -\Gamma_c \pm \frac{\Delta_l m_r}{2\sqrt{1 - m_r^2}} \right)
\]

Thus we see that there could be a peak oscillation amplitude for only one of the contributions `+' or `-`.

\[
T_{peak} = \frac{1}{\pi} \left( \frac{1}{2} \frac{m_2 - m_1}{m_1 + m_2} \Delta_D - \frac{m_1 \Gamma_1 + m_2 \Gamma_2}{m_1 + m_2} \right)
\]

For \( \Delta_l \gg \Gamma \), \( T_{peak} \sim \Delta_l / (2\pi) \) and it moves to lower temperature with increasing \( \Gamma_c \). Using Eqs.30,Eq. (32),33 in Eq.29, we obtain the expression for magnetization [Eq.13, Main text],
\[
M \approx \frac{4\mu_0}{\hbar \omega_c \sqrt{\hbar} \omega_c \Delta I} \sin \left( 2\pi k \left( \frac{F_0}{B} \right) \right) \sum_{p,k=1}^{\infty} \left[ \frac{(1 - m_c^2) \sqrt{m_c}}{\sqrt{k}} e^{-\pi k/\omega_c \tau_{1p}(T)} + \frac{\pi T}{\sqrt{\hbar} \omega_c \Delta I} e^{-\pi k/\omega_c \tau_{1p}(T)} \right]
\]

[34]

9. Numerical calculation of magnetization

In our numerical calculations for the disorder-free case \( \Gamma = 0 \), we compute \( M(T) \) using
\[
\Omega(T) = -\int_{-\infty}^{\infty} d\xi \frac{\partial \mathcal{V}_F(\xi, T)}{\partial \xi} \Omega(\xi, T = 0),
\]

[35]

for the grand potential at finite temperature and for the chemical potential \( \mu_0 \). Here
\[
\Omega(\xi, T = 0) = N_B \sum_{\xi > \mu_0 + \xi} (\mathcal{E}_{\xi, bp} - \mu_0 - \xi)
\]

is the grand potential or total energy at zero temperature for a chemical potential \( \mu_0 + \xi \). For numerically evaluating the above we put an upper cutoff \( \Lambda \) for the LL index \( l \). Furthermore, to extract the oscillatory part of the grand potential we subtract from \( \Omega(\xi, T = 0) \) a large non-oscillatory contribution \( 2 \sum_{\xi \neq 0} (\mathcal{E}_{=0} - \mu_0 - \xi) \) (factor 2 for the spin degeneracy), which is the grand potential for completely filled valence band in the absence of hybridization. The magnetization is obtained by numerical differentiation of \( \Omega(T) \) with respect to \( B \). We have verified that results obtained for \( M(T) \) are insensitive to the choice of \( \Lambda \) for sufficiently large \( \Lambda \). The results for magnetization oscillations are shown in Fig.3(b), Main text. The amplitude for the Fourier component at frequency \( F_0 \), \( \tilde{M}(T) \) shown in Fig.4(b), Main text as a function of \( T \), is obtained by fast Fourier transform (FFT) of \( M(T) \) with respect to \( 1/B \). We plot the amplitude \( \tilde{M}(T)/\tilde{M}_{\mu=0}(0) \) normalized by the \( T = 0 \) value \( \tilde{M}_{\mu=0}(0) \) for the zero hybridization case.

To evaluate \( M(T) \) for \( \Gamma \neq 0 \), we use the expression for grand potential given in Eq.27, and following steps similar to that discussed in Sec.B obtain the magnetization amplitude for the \( k = 1 \) harmonic of the fundamental frequency \( F_0 \), i.e.
\[
\tilde{M}(T) \approx \frac{8\pi T \mu_0}{\hbar \omega_c \sqrt{\hbar} \omega_c \Delta I} \sum_{bp, \omega_b > 0} e^{-2\pi |\text{int}[b, \omega_b]|}
\]

[37]

where \( b_{\pm}(n) (b = \uparrow, \downarrow) \) are the two poles of the function \((\mathcal{E}_{b=1} - \mu_0 - \omega_n)(\mathcal{E}_{b=1} - \mu_0 - \omega_n)^{-1} \). We perform the Matsubara summation above numerically with a cutoff for the largest Matsubara frequency. Note that we use the original energy eigenvalues of Eq.2, Main text, as opposed to the semiclassical eigenvalues [Eq.3, Main text] that are used in Sec.B. In Fig.4(b), Main text we have shown the result for \( \tilde{M}(T) \) for \( \Gamma_1 \geq \Gamma_2 \). We find qualitatively same result for \( \Gamma_1 < \Gamma_2 \) as shown in Fig.S3

10. Transport

Here we show that even in the presence of disorder, which induces finite DOS inside the gap, the system exhibits \( d\rho/dT < 0 \) at any finite temperature like an insulator. Nevertheless, the system, strictly speaking, remains a metal with finite resistivity \( \rho(T = 0) \) at zero temperature. To this end, we calculate the conductivity \( \sigma \) using the Kubo formula,
\[
\sigma = 2e^2 \pi h \int d\omega \left( -\frac{\partial \mathcal{V}_F}{\partial \omega} \right) \sum_{p = \pm} \int \frac{d^2 k}{(2\pi)^2} v_{2p}^2(k) A_p(k, \omega)^2
\]

[38]
Here $A_p(k, \omega) = -(1/\pi) \text{Im}[1/(\omega - \mathcal{E}_p(k))]$ is the spectral function, and we use the real part of the complex eigen energies to calculate the band velocity, i.e.,

$$v_{xp}(k) = \frac{\partial \text{Re} \left[ \mathcal{E}_p(k) \right]}{\partial (\hbar k_x)}.$$

We plot the resistivity $\rho = \sigma^{-1}$ as a function of temperature for several $\Gamma_1$ in Fig. S4. The insulating-like upturn ($d\rho/dT < 0$) with decreasing temperature is evident. Nevertheless, the resistivity eventually saturates to a finite value as $T \to 0$ implying that the system is actually metallic due to impurity-induced in-gap states.

References

1. L Zhang, XY Song, F Wang. Quantum oscillation in narrow-gap topological insulators. Phys. Rev. Lett. 116, 046404 (2016).
2. H Shen, L Fu. Quantum oscillation from in-gap states and a non-hermitian landau level problem. Phys. Rev. Lett. 121, 026403 (2018).
3. HK Pal, F Piéchon, JN Fuchs, M Goerbig, G Montambaux. Chemical potential asymmetry and quantum oscillations in insulators. Phys. Rev. B 94, 125140 (2016).
Fig. S3. $\Gamma_1 < \Gamma_2$: Magnetization amplitude plotted as a function of temperature for $\Gamma_1 < \Gamma_2$. We find the qualitative features remain unchanged in this case.
Fig. S4. Transport: Resistivity $\rho(T)$ as a function of temperature for several disorder strengths using Kubo formula for zero magnetic field case. The unit in the $y$-axis for resistivity is $h/e^2$. Here $\Gamma_2 = 0.1\Delta_D$. 