Concentration Behavior of Nonlinear Hartree-type Equation with almost Mass Critical Exponent

Yuan Li\textsuperscript{a} Dun Zhao\textsuperscript{a} Qingxuan Wang\textsuperscript{b}\textsuperscript{*}
\textsuperscript{a} School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, PR China
\textsuperscript{b} Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, Zhejiang, PR China

Abstract
We study the following nonlinear Hartree-type equation
\[-\Delta u + V(x)u - a\left(\frac{1}{|x|^{\gamma}} \ast |u|^2\right)u = \lambda u, \text{ in } \mathbb{R}^N,\]
where $a > 0$, $N \geq 3$, $\gamma \in (0, 2)$ and $V(x)$ is an external potential. We first study the asymptotic behavior of the ground state of equation for $V(x) \equiv 1$, $a = 1$ and $\lambda = 0$ as $\gamma \rightarrow 2$. Then we consider the case of some trapping potential $V(x)$, and show that all the mass of ground states concentrate at a global minimum point of $V(x)$ as $\gamma \rightarrow 2$, which leads to symmetry breaking. Moreover, the concentration rate for maximum points of ground states will be given.

Keywords: Hartree-type equation; Energy estimate; Concentration; Symmetry breaking; Almost mass critical

1 Introduction

In this paper, we consider the concentration behavior of the following nonlinear Hartree-type equation as $\gamma \rightarrow 2$
\[-\Delta u + V(x)u - a\left(\frac{1}{|x|^{\gamma}} \ast |u|^2\right)u = \lambda u, \text{ in } \mathbb{R}^N, \tag{1.1}\]
where $\gamma > 0$, $N \geq 3$, $a > 0$ is the parameter, $\lambda \in \mathbb{R}$, and $V(x)$ is an external potential. This equation can be used to describe the standing waves of the form $\psi(t, x) = e^{i\lambda t}u(x)$ of the focusing time-dependent equation
\[i\psi_t = -\Delta \psi + V(x)\psi - a\left(\frac{1}{|x|^{\gamma}} \ast |\psi|^2\right)\psi, \text{ in } \mathbb{R}^N \times \mathbb{R}_+. \tag{1.2}\]
Equation (1.2) can describe the geometry of stars and planets in celestial mechanics [15], and quantum mechanics for investigating Bose-Einstein condensates (BEC) and Thomas-Fermi type problems [2]. In particular, for \( a > 0 \) and \( \gamma = 1 \), the interaction is attractive Coulomb action, the model can describe the quantum mechanics of a polaron, see [17, 21].

There have been a great deal of papers devote to equation (1.1) in different aspects. For results about the existence of positive ground states, one can see [1, 13, 14, 18, 19, 20, 26]; Cingolani [29] and Clap [31] investigated the existence of multiple solutions; for the uniqueness of the ground state, see for example [13, 16, 22, 24, 25]; and the semi-classical analysis results, see [28, 30] and the references therein.

When \( \gamma = 2 \), the above Hartree-type nonlinearity in \( \mathbb{R}^N \) is corresponding to mass critical case for all \( N \geq 3 \). The authors in [3, 14, 27] considered a minimizing variational problem corresponding to (1.1), and proved that there exists a constant \( a^* \) such that (1.6) admits at least one minimizer if and only if \( a < a^* \), where \( a^* = \|Q_2\|_2^2 \), and \( Q_2 \) is the positive radially symmetric ground state of the following equation

\[
-\Delta u + u - a |u|^2 u = 0, \quad \text{in} \ \mathbb{R}^N.
\] (1.3)

Furthermore, the concentration and symmetry breaking of minimizers as \( a \nearrow a^* \) were also obtained for different potentials \( V(x) \). We mention that such kind of mass critical problems have been studied for cubic nonlinearity in \( \mathbb{R}^2 \) in the past few years, see [4, 5, 6, 7, 8] and the references therein.

Here we state the work of Guo, Zeng and Zhou [7], in which they studied concentration behavior of the following almost mass critical nonlinear Schrödinger equations as \( q \nearrow 2 \) (2 is mass critical exponent)

\[
-\Delta u + (V(x) + \lambda)u - a|u|^q u = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2,
\]

where \( a > a^* = \|Q\|_2^2 \), \( Q \) is the unique (up to translations) radially symmetric positive solution of the following limiting equation

\[
-\Delta u + u - |u|^2 u = 0, \quad \text{in} \ \mathbb{R}^2.
\] (1.4)

By using constrained variational method and energy estimates they present a detailed analysis of the concentration and symmetry breaking of the solutions for above equation as \( q \nearrow 2 \).

Inspired by [7], our focuses here will be on concentration behavior of nonlinear Hartree-type equation (1.1) as almost mass critical exponent \( \gamma \nearrow 2 \). Comparing with the results in [7], we should deal with the nonlocal term \( (\frac{1}{|x|^{\gamma}} \ast |u|^2)u \), and the almost mass critical exponent comes from Hartree action \( \frac{1}{|x|^{\gamma}} \), not the power exponent of \( u \), this will bring some difficulties; on the other hand, in this case we can consider in \( \mathbb{R}^N \) for all \( N \geq 3 \), while [7] only studied the cubic power exponent in \( \mathbb{R}^2 \).

Firstly, we study the asymptotic behavior of ground states of equation (1.1) for \( V(x) \equiv 1, a = 1 \) and \( \lambda = 0 \) as \( \gamma \nearrow 2 \), that is

\[
-\Delta u + u - (\frac{1}{|x|^{\gamma}} \ast |u|^2)u = 0.
\] (1.5)

We have the following Theorem.
Theorem 1.1. Let $Q_\gamma = Q_\gamma(|x|)$ be a positive radial ground state for equation (1.3) with $0 < \gamma < 2$. Then we have that

$$\|Q_\gamma\|_{L^2(\mathbb{R}^N)} \to \|Q_2\|_{L^2(\mathbb{R}^N)} \text{ as } \gamma \to 2.$$  

Here $Q_2 = Q_2(|x|) > 0$ is a positive radial ground state for equation (1.3).

This is not an easy result, we shall employ the mountain pass structure of corresponding energy functional to obtain the uniform bound for $\|Q_\gamma\|_{H^1}$, and then obtain the above convergence by Pohozaev identity.

Next we will show asymptotic behaviors of ground states of equation (1.1) for general $V(x)$. We consider a corresponding minimizing variational problem. Notice that, (1.1) is an Euler-Lagrange equation of following minimizing problem:

$$e_{a}(\gamma) = \inf_{\{u \in \mathcal{H}, \int_{\mathbb{R}^N} u^2 = 1\}} E_\gamma(u);$$  \hspace{1cm} (1.6)

$$E_\gamma(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) - \frac{a}{2} D_\gamma(u, u), \quad D_\gamma(u, u) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^2|u(y)|^2}{|x - y|^\gamma} \, dx \, dy.$$

Where

$$\mathcal{H} := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) |u|^2 \, dx < \infty\}.$$  

Here we assume that $V(x) : \mathbb{R}^N \to \mathbb{R}^+$ is locally bounded and satisfies $V(x) \to \infty$ as $|x| \to \infty$. Without loss of generality, by adding a suitable constant we may assume that

$$\inf_{x \in \mathbb{R}^N} V(x) = 0,$$

and $\inf_{x \in \mathbb{R}^N} V(x)$ can be attained. In this paper we are interested in addressing the limit behavior of minimizers for (1.6) when $\gamma \to 2$ and $a > a^*$. By [10] [11] we have the following Gagliardo-Nirenberg inequality:

$$D_\gamma(u, u) \leq C_\gamma \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^\frac{2}{2 - \gamma} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^\frac{4 - \gamma}{2}, \quad (1.7)$$

where the best constant $C_\gamma = \frac{4}{4 - \gamma} \left( \frac{4 - \gamma}{\gamma} \right)^\frac{1}{2} \frac{1}{\|Q_\gamma\|_2}$ and $N \geq 3$. $Q_\gamma$ is an optimal minimizer of above inequality, satisfying equation (1.5), and also a positive radial ground state for (1.3).

Theorem 1.2. Assume that $N \geq 3$, $a > a^* = \|Q_2\|_2^2$ and $Q_2$ is the positive radially symmetric ground state of (1.3). And also assume that

$$V \in C^1(\mathbb{R}^N), \quad \lim_{|x| \to \infty} V(x) = \infty \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} V(x) = 0.$$

Let $u_\gamma \in \mathcal{H}$ be a non-negative minimizer of (1.6) with $\gamma \in (0, 2)$. Then for each sequence $\{\gamma_k\}$ with $\gamma_k \to 2$ as $k \to \infty$, there exists a subsequence of $\{\gamma_k\}$, still denoted by $\{\gamma_k\}$, such that $u_{\gamma_k}$ concentrates at a global minimum point $y_0$ of $V(x)$ in the following sense: for each large $k$, $u_{\gamma_k}$ has a unique global maximum point $\bar{z}_k \in \mathbb{R}^N$, satisfies

$$\lim_{k \to \infty} \left( \frac{a}{\|Q_{\gamma_k}\|_2^2} \right)^{\frac{2}{2 - \gamma_k}} u_{\gamma_k} \left( \left( \frac{a}{\|Q_{\gamma_k}\|_2^2} \right)^{\frac{1}{2 - \gamma_k}} x + \bar{z}_k \right) = \frac{1}{\|Q_2\|_2^2} Q_2(|x|) \text{ in } H^1(\mathbb{R}^N), \quad (1.8)$$

where $\bar{z}_k \to y_0$ as $k \to \infty$.  

3
Remark 1. Under the assumption of $V(x)$ in Theorem 1.2, we have $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^N)$ is compact, where $p \in [2, \frac{2N}{N-2})$. The existence of the non-negative minimizer of (1.6) is similar to [17, 32].

Remark 2. It follows from [24] that the uniqueness of positive ground state of (1.3) holds for $N = 4$. If $N \geq 3$ and $N \neq 4$, we do not know that the positive ground state of equation (1.3) is unique. Therefore, if $N = 4$, we can obtain that $Q_\gamma \to Q_2$ strongly in $H^1(\mathbb{R}^4)$ and the right-hand side of the (1.8) is unique. However, if $N \neq 4$, we only know that there exists a positive radial ground state such that the above limit convergence to it.

In the following we shall assume that the external potential $V$ satisfies that there exist $n \geq 1$ distinct points $x_i \in \mathbb{R}^N$ with $V(x_i) = 0$, while $V(x) > 0$ otherwise. Moreover, there are numbers of $p_i > 0$ such that $V(x) = O(|x - x_i|^{p_i})$ near $x_i$, where $i = 1, 2, ..., n$, and $\lim_{x \to x_i} \frac{V(x)}{|x - x_i|^{p_i}}$ exists for all $1 \leq i \leq n$.

Let $p = \max\{p_1, p_2, ..., p_n\}$, and let $\lambda_i \in (0, \infty]$ be given by

$$\lambda_i = \lim_{x \to x_i} \frac{V(x)}{|x - x_i|^{p_i}}. \quad (1.10)$$

Define $\lambda = \min\{\lambda_1, ..., \lambda_n\}$ and let

$$\mathcal{Z} := \{x_i : \lambda_i = \lambda\}. \quad (1.11)$$

denote the locations of the flattest global minima of $V(x)$. By the above notations, we have the following result, which tells us some further about the concentration point $y_0$ given by Theorem 1.2.

Theorem 1.3. Under the assumptions of Theorem 1.2 and let $V(x)$ satisfy also the additional condition (1.9), then the unique concentration point $y_0$ obtained in Theorem 1.2 has the properties:

$$\lim_{k \to \infty} |z_k - y_0| \left(\frac{a}{\|Q_\gamma_k\|_2^2}\right)^{\frac{1}{2}} \to 0 \quad \text{and} \quad y_0 \in \mathcal{Z}. \quad (1.12)$$

This paper is organised as follows. Section 2 is devoted to the proof of Theorem 1.1 by using the mountain pass structure. In section 3, we prove Lemma 3.2 on energy estimates of minimizers for (1.6). Finally, we use Lemma 3.2 to prove Theorem 1.2 and then utilize the blowup analysis to complete the proof of Theorem 1.3 in section 4.

2 The proof of Theorem 1.1

First note from [19] that the equation (1.5) has a radially symmetric and monotone decreasing positive ground state solution $Q_\gamma$ which satisfies the following Pohozaev identity:

$$\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla Q_\gamma(x)|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} |Q_\gamma(x)|^2 dx = \frac{2N - \gamma}{4} D_\gamma(Q_\gamma, Q_\gamma), \quad (2.1)$$
for all $0 < \gamma < 4$, one can derive from (1.5) and (2.1) that $Q_{\gamma}(x)$ satisfies

$$
\frac{1}{\gamma} \int_{\mathbb{R}^N} |\nabla Q_\gamma(x)|^2 dx = \frac{1}{4 - \gamma} \int_{\mathbb{R}^N} |Q_\gamma(x)|^2 dx = \frac{1}{4} D_\gamma(Q_\gamma, Q_\gamma). \tag{2.2}
$$

Now we define the energy functional of equation (1.3) and (1.5) by

$$
J_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{4} D_\gamma(u, u), \quad \gamma \in (1, 2).
$$

Let

$$
G := \{Q_2(x) \in H^1(\mathbb{R}^N) : Q_2(x) \text{ is a positive radial ground state of (1.3)}\}.
$$

Combining (2.1) and (2.2), we get the ground state energy

$$
J_2(Q_2) = \frac{1}{2} \int_{\mathbb{R}^N} |Q_2(x)|^2 dx, \quad Q_2 \in G.
$$

Then we have $a^* = \int_{\mathbb{R}^N} |Q_2(x)|^2 dx$. That is, all positive ground states of equation (1.3) have the same $L^2$-norm. Similarly, we can obtain that all positive ground states of equation (1.5) have the same $L^2$-norm if $\gamma$ is fixed.

To prove Theorem 1.1, we divide the proof into several lemmas. The ground state energy level $J_\gamma$ satisfies the mountain pass characterization, i.e.,

$$
I_\gamma = \min_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_\gamma(tu).
$$

First, we have the following convergence holds.

**Lemma 2.1.** Let $\{\gamma_n\} < 2$ be a sequence converging to 2. Then for any $u \in H^1_{rad}(\mathbb{R}^N)$

$$
D_{\gamma_n}(u, u) \to D_2(u, u), \quad \text{as } n \to \infty.
$$

Let $\{u_n\} \subset H^1_{rad}(\mathbb{R}^N)$ be a sequence converging weakly in $H^1_{rad}(\mathbb{R}^N)$ to some $u_0 \in H^1_{rad}(\mathbb{R}^N)$. Then as $n \to \infty$

$$
D_{\gamma_n}(u_n, u_0) \to D_2(u_0, u_0).
$$

**Proof.** By combining Hardy-Littlewood-Sobolev inequality and the compact Sobolev embedding. It is easy to see the above convergence holds.

Next, we prove the following uniform estimate for the ground states $Q_{\gamma}$.

**Lemma 2.2.** Assume that $Q_{\gamma} \in H^1(\mathbb{R}^N)$ be the positive radial ground state of equation (1.5), where $\gamma \in (1, 2)$. Then there exists a positive constant $C > 0$ such that

$$
\|Q_{\gamma}\|_{H^1} \leq C.
$$
Proof. We claim that
\[ \limsup_{\gamma \to 2} I_\gamma \leq I_2. \]
Indeed, let \( Q_2 \) be a ground state of (1.3). Define \( t_\gamma \) by the positive number satisfying
\[ J_\gamma(t_\gamma Q_2) = \max_{t > 0} J_\gamma(tQ_2). \]
Elementary computation shows that
\[ t_\gamma = \left( \frac{\|Q_2\|^2_{H^1}}{D_\gamma(Q_2, Q_2)} \right)^{\frac{1}{2}} \]
and \( \lim_{\gamma \to 2} t_\gamma = 1. \) Now, by Lemma 2.1 we see from the mountain pass characterization of the ground states that as \( \gamma \to 2, \)
\[ J_\gamma(Q_\gamma) \leq J_\gamma(t_\gamma Q_2) \]
\[ = J_2(t_\gamma Q_2) + \frac{1}{4} \left( \int_{\mathbb{R}^N} \left( \frac{1}{|x|^2} * |t_\gamma Q_2|^2 \right)|t_\gamma Q_2|^2 - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\gamma} * |t_\gamma Q_2|^2 \right)|t_\gamma Q_2|^2 \right) \]
\[ \leq J_2(Q_2) + o(1). \]
Multiplying the equation (1.5) by \( Q_\gamma \) and integrating by part, we get
\[ \left( \frac{1}{2} - \frac{1}{4} \right)\|Q_\gamma\|_{H^1}^2 = J_\gamma(Q_\gamma). \]
By the above argument, we know that \( \|Q_\gamma\|_{H^1} \) is uniformly bounded for \( 0 < \gamma < 2. \)

The above Lemma 2.2 implies that \( \{Q_\gamma\} \) is a bounded sequence in \( H^1(\mathbb{R}^N). \) Therefore we can assume that, up to a subsequence, \( Q_{\gamma_n} \) converges weakly to a nonnegative radial function \( Q_\infty \in H^1_{rad}(\mathbb{R}^N), \) that is
\[ Q_{\gamma_n} \rightharpoonup Q_\infty \text{ in } H^1(\mathbb{R}^N). \]
Moreover, by the compact embedding \( H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \) for any \( 2 < q < \frac{2N}{N-2} \) (see Strauss [23]), we can assume that
\[ Q_{\gamma_n} \to Q_\infty \text{ in } L^q(\mathbb{R}^N) \]
for any \( 2 < q < \frac{2N}{N-2}, \) and
\[ Q_{\gamma_n} \to Q_\infty \text{ a.e. in } \mathbb{R}^N. \]
By Lemma 2.1 we easily deduce that
\[ \lim_{\gamma_n \to 2} \int_{\mathbb{R}^N} (|x|^{-\gamma_n} * |Q_{\gamma_n}|^2)|Q_{\gamma_n}|^2\,dy = \int_{\mathbb{R}^N} (|x|^{-2} * |Q_\infty|^2)|Q_\infty|^2\,dy. \]
Furthermore, we multiply the equation (1.5) by \( Q_\gamma \) and multiply the equation (1.3) by \( Q_\infty \) to get
\[ \|Q_{\gamma_n}\|_{H^1} = D_{\gamma_n}(Q_{\gamma_n}, Q_{\gamma_n}) \to D_2(Q_\infty, Q_\infty) = \|Q_\infty\|_{H^1}. \]
Combining this with the weak convergence of \( Q_{\gamma_n}, \) we obtain the strong convergence of \( Q_{\gamma_n} \) to \( Q_\infty \) in \( H^1(\mathbb{R}^N). \)
Lemma 2.3. It satisfies
\[
\liminf_{\gamma_n \to 2} I_{\gamma_n} \geq I_2.
\]

Proof. We see from the mountain pass characterization of ground states to (1.5), that for every \(t > 0\),
\[
I_{\gamma_n} \geq J_{\gamma_n}(tQ_{\gamma_n}) = J_2(tQ_{\gamma_n}) + \frac{t^4}{4} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^2} \cdot |Q_{\gamma_n}|^2 \right) \geq J_{\gamma_n}(tQ_{\gamma_n}) = J_2(tQ_{\gamma_n}) + o(1)t^4
\]
as \(\gamma_n \to 2\). Taking \(t = t_{\gamma_n}\) such that \(t_{\gamma_n} \) satisfies
\[
J_2(t_{\gamma_n}Q_{\gamma_n}) = \max_{t \geq 0} J_2(tQ_{\gamma_n})
\]
Elementary computation shows
\[
t_{\gamma_n} = \left( \frac{\|Q_{\gamma_n}\|_{H^1(\mathbb{R}^N)}}{D_{\gamma_n}(Q_{\gamma_n}, Q_{\gamma_n})} \right) = 1 \text{ as } \gamma_n \to 2
\]
as in the proof of Lemma 2.2. Then we get
\[
I_{\gamma_n} \geq J_2(t_{\gamma_n}Q_{\gamma_n}) + o(1) \geq I_2 + o(1),
\]
where the last inequality comes from the mountain pass characterization of \(I_2\). The proof is complete.

Proof of Theorem 1.1. Combining Lemma 2.2 and Lemma 2.3, we conclude
\[
\lim_{\gamma_n \to 2} I_{\gamma_n} = I_2.
\]
Hence we get
\[
J_2(Q_{\infty}) = I_2.
\]
In other words, \(Q_{\infty}\) is a positive radial ground state to (1.3), i.e. \(Q_{\infty} \in G\) (defined above).

On the other hand, from the identity (2.2), we know that
\[
I_1 = J_1(Q_{\gamma}) = \frac{1}{4 - \gamma} \int_{\mathbb{R}^N} |Q_{\gamma}(x)|^2 dx \text{ and } I_2 = J_2(Q_2) = \frac{1}{2} \int_{\mathbb{R}^N} |Q_2(x)|^2 dx.
\]
Therefore, from above we can obtain that
\[
\int_{\mathbb{R}^N} |Q_{\gamma}(x)|^2 dx \to \int_{\mathbb{R}^N} |Q_{\infty}(x)|^2 dx = a^*.
\]
Let \(\{Q_{\gamma}\} \subset H^1_{rad}(\mathbb{R}^N)\) be a family of positive radial ground state to (1.5) for \(\gamma\) near 2 and \(\gamma < 2\). Suppose \(\|Q_{\gamma}\|_{L^2(\mathbb{R}^N)}\) does not converge to the \(\|Q_2\|_{L^2(\mathbb{R}^N)}\). Then there exist a positive number \(\varepsilon_0\) and a sequence \(\{\gamma_k\} \to 2\) such that \(\|Q_{\gamma_k}\|_{L^2(\mathbb{R}^N)} - \|Q_2\|_{L^2(\mathbb{R}^N)} \geq \varepsilon_0\) which contradicts to Lemma 2.2 and Lemma 2.3. The proof of Theorem 1.1 is complete. \(\Box\)
3 Energy estimate

In this section, the main purpose is to establish Lemma 3.2. One can obtain from [19] that there exist positive constants \(\delta, C\) and \(R_0\) independent of \(\gamma \in (0, 2]\), such that

\[
|Q_\gamma| + |\nabla Q_\gamma| \leq Ce^{-\delta|x|} \text{ for } \gamma \in (0, 2].
\]

We next denote \(\tilde{E}_\gamma(u)\) the following energy functional without the potential:

\[
\tilde{E}_\gamma(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx - \frac{a}{2} D_\gamma(u, u) \quad u \in H^1(\mathbb{R}^N),
\]

and consider the associated minimization problem

\[
\tilde{e}_a(\gamma) = \inf \left\{ u \in H^1(\mathbb{R}^N): \|u\|_2^2 = 1 \right\} \tilde{E}_\gamma(u).
\]

Then the following Lemma gives refined information on the minimum energy \(\tilde{e}_a(\gamma)\) as well as its minimizers.

Lemma 3.1. Let \(\gamma \in (0, 2)\) and \(Q_\gamma\) be the radially symmetric positive solution of (1.3), then

\[
\tilde{e}_a(\gamma) = (1 - \frac{2}{\gamma})(\frac{4 - \gamma}{\gamma})\frac{a^{\frac{2}{2-\gamma}}}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}},
\]

and the positive minimizers of \(\tilde{e}_a(\gamma)\) must be of the form

\[
\tilde{Q}_\gamma(x) = \frac{\tau_{\gamma}^N/2}{\|Q_\gamma\|_2} Q_\gamma(\tau_\gamma x), \text{ where } \tau_\gamma = \sqrt{\frac{\gamma}{4 - \gamma} \left( \frac{a}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} \right)^{\frac{1}{2-\gamma}}}. \tag{3.4}
\]

Proof. By using the Gagliardo-Nirenberg inequality (1.7), it follows from (3.2) that

\[
\tilde{E}_\gamma(u) \geq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{2a}{4 - \gamma} \left( \frac{4 - \gamma}{\gamma} \right)^{\frac{1}{2-\gamma}} \frac{1}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{2}{2-\gamma}},
\]

for any \(u \in H^1(\mathbb{R}^N)\) and \(\int_{\mathbb{R}^N} u^2 = 1\).

Let

\[
g(s) = s - \frac{2a}{4 - \gamma} \left( \frac{4 - \gamma}{\gamma} \right)^{\frac{1}{2-\gamma}} \frac{1}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} s^{\frac{2}{2-\gamma}} \text{ for any } s \in [0, \infty). \tag{3.5}
\]

We known that \(g(s)\) attains its minimum at \(s = \left( \frac{\gamma}{4 - \gamma} \right)^{\frac{2}{2-\gamma}} \left( \frac{a}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} \right)^2\), i.e. \(s = \tau_\gamma^2\), which then implies that

\[
\tilde{E}_\gamma(u) \geq g(\tau_\gamma^2) = \left( 1 - \frac{2}{\gamma} \right) \left( \frac{\gamma}{4 - \gamma} \right) \left( \frac{a}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} \right)^{\frac{2}{2-\gamma}}.
\]

This yields that

\[
\tilde{e}_a(\gamma) \geq g(\tau_\gamma^2) = \left( 1 - \frac{2}{\gamma} \right) \left( \frac{\gamma}{4 - \gamma} \right) \left( \frac{a}{\|Q_\gamma\|_2^{\frac{2}{2-\gamma}}} \right)^{\frac{2}{2-\gamma}}. \tag{3.6}
\]
On the other hand, we introduce the following trial function

$$\psi_t^\gamma(x) = \frac{t^{\frac{N}{2}}}{\|Q_\gamma\|_2} Q_\gamma(tx)$$

and \(\int_{\mathbb{R}^N} |\psi_t^\gamma|^2 dx \equiv 1\) for all \(t \in (0, \infty)\). We then obtain from (2.2) that

$$\tilde{e}_a(\gamma) \leq \tilde{E}_\gamma(\psi_t^\gamma) = \frac{\gamma}{4 - \gamma} t^2 - \frac{a}{2} \left(\frac{4}{4 - \gamma}\right) \frac{t^\gamma}{\|Q_\gamma\|_2^2},$$

for any \(t \in (0, \infty)\).

Set

$$h(t) = \frac{\gamma}{4 - \gamma} t^2 - \frac{a}{2} \left(\frac{4}{4 - \gamma}\right) \frac{t^\gamma}{\|Q_\gamma\|_2^2}.$$ 

We then obtain its minimum

$$h_{\text{min}}(t) = \left(1 - \frac{2}{\gamma}\right) \left(\frac{\gamma}{4 - \gamma}\right) \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{2}{4 - \gamma}}.$$

This and (3.6) then imply the estimate (3.3). Moreover, \(\tilde{e}_a(\gamma)\) is attained at \(\tilde{Q}_\gamma(x) = \frac{x^{N/2}}{\|Q_\gamma\|_2} Q_\gamma(\tau_\gamma x)\), and the proof is complete.

\[\Box\]

**Remark 3.** For any fixed \(a > a^*\), from the Theorem 1.1 we know that there exists a constant \(\sigma > 1\), independent of \(\gamma\), such that \(\frac{a}{\|Q_\gamma\|_2^2} > \sigma > 1\) as \(\gamma\) is sufficiently close to \(2^-\). Therefore, we further have

$$\tau_\gamma = \sqrt{\frac{\gamma}{4 - \gamma}} \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{1}{4 - \gamma}} \to +\infty \text{ and } \tilde{e}_a(\gamma) \to -\infty \text{ as } \gamma \nearrow 2.$$  \hspace{1cm} (3.7)

By applying Lemma 3.1, we are able to establish the following estimates.

**Lemma 3.2.** Let \(a > a^*\) be fixed, and suppose that

\[V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N), \lim_{|x| \to \infty} V(x) = \infty \text{ and } \inf_{x \in \mathbb{R}^N} V(x) = 0.\]

Then

$$e_a(\gamma) - \tilde{e}_a(\gamma) \to 0, \text{ as } \gamma \nearrow 2.$$  \hspace{1cm} (3.8)

Furthermore, we have

$$\int_{\mathbb{R}^N} V(x)|u_\gamma(x)|^2 dx \to 0 \text{ as } \gamma \nearrow 2,$$  \hspace{1cm} (3.9)

and there exist two positive constants \(C_1\) and \(C_2\) independent of \(\gamma\), such that

$$C_1 \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{2}{4 - \gamma}} \leq \int_{\mathbb{R}^N} |\nabla u_\gamma(x)|^2 dx \leq C_2 \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{2}{4 - \gamma}} \text{ as } \gamma \nearrow 2,$$  \hspace{1cm} (3.10)

and

$$C_1 \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{2}{4 - \gamma}} \leq D_\gamma(u_\gamma, u_\gamma) \leq C_2 \left(\frac{a}{\|Q_\gamma\|_2^2}\right)^{\frac{2}{4 - \gamma}} \text{ as } \gamma \nearrow 2,$$

where \(u_\gamma(x)\) is a positive minimizer of (1.6).
Proof. We choose a suitable trial function to estimate the upper bound of $e_a(\gamma) - \tilde{e}_a(\gamma)$. For $R \geq 0$ fixed, let $\varphi_R(x) \in C^\infty_0(\mathbb{R}^N)$ be a cut-off function such that $\varphi_R(x) \equiv 1$ if $x \in B_R(0)$, $\varphi_R(x) \equiv 0$ if $x \in B^c_R(0)$, and $0 \leq \varphi_R(x) \leq 1$, $\nabla \varphi_R(x) \leq \frac{C}{R}$ for any $x \in B^c_R(0) \setminus B_R(0)$. Set

\[
\omega_{R,\gamma}(x) = A_{R,\gamma} \bar{\omega}_{R,\gamma}(x) = A_{R,\gamma} \varphi_R(x - x_0) \bar{Q}_\gamma(x - x_0) \quad \text{with} \quad x_0 \in \mathbb{R}^N,
\]

where $\bar{Q}_\gamma(x)$ defined in (3.4) is the positive minimizer of $\tilde{e}_a(\gamma)$, and $A_{R,\gamma} > 0$ is chosen so that $\|\omega_{R,\gamma}\|^2 = 1$. Now, we can calculate that

\[
0 \leq e_a(\gamma) - \tilde{e}_a(\gamma) \leq E_\gamma(\omega_{R,\gamma}(x)) - \tilde{e}_a(\gamma)
\]

\[
= E_\gamma(A_{R,\gamma} \bar{\omega}_{R,\gamma}(x)) - \bar{E}_\gamma(\bar{\omega}_{R,\gamma}(x)) + \bar{E}_\gamma(\bar{\omega}_{R,\gamma}(x)) - \bar{E}_\gamma(\bar{Q}_\gamma(x))
\]

\[
\leq (A^2_{R,\gamma} - 1) \int_{\mathbb{R}^N} |\nabla \bar{\omega}_{R,\gamma}(x)|^2 dx + \frac{a}{2} (A^4_{R,\gamma} - 1) D_\gamma(\bar{\omega}_{R,\gamma}, \bar{\omega}_{R,\gamma})
\]

\[
+ \int_{\mathbb{R}^N} V(x)|\omega_{R,\gamma}(x)|^2 dx + \int_{\mathbb{R}^N} |\nabla \bar{Q}_\gamma(x)|^2 dx - \int_{\mathbb{R}^N} |\nabla \bar{\omega}_{R,\gamma}(x)|^2 dx
\]

\[
+ \frac{a}{2} |D_\gamma(\bar{Q}_\gamma, \bar{Q}_\gamma) - D_\gamma(\bar{\omega}_{R,\gamma}, \bar{\omega}_{R,\gamma})|
\]

\[
= A_1 + A_2 + A_3 + A_4 + A_5.
\]

By using (3.1) and $\tau_\gamma \to \infty$ as $\gamma \nearrow 2$, we then have

\[
0 \leq A^2_{R,\gamma} - 1 \leq \frac{\int_{B^c_{R,\gamma}} |Q_\gamma(x)|^2 dx}{\int_{B_{R,\gamma}} |Q_\gamma(x)|^2 dx} \leq CR_\gamma e^{-2\delta R_\gamma} \leq Ce^{-\delta R_\gamma} \quad \text{as} \quad \gamma \nearrow 2,
\]

where $\delta > 0$ is as in (3.1). It hence follows from the above that

\[
1 \leq A^4_{R,\gamma} \leq (1 + Ce^{-\delta R_\gamma})^2 \leq 1 + 4Ce^{-\delta R_\gamma}.
\]

Direct calculations show that

\[
A_4 \leq \left| \int_{\mathbb{R}^N} |\nabla \bar{Q}_\gamma(x)|^2 dx - \int_{\mathbb{R}^N} (|\nabla \varphi_R|^2 |\bar{Q}_\gamma|^2 + |\varphi_R|^2 |\nabla \bar{Q}_\gamma|^2 + 2\nabla \varphi_R \varphi_R \nabla \bar{Q}_\gamma \bar{Q}_\gamma) dx \right|
\]

\[
\leq \int_{B^c_R} |\nabla \bar{Q}_\gamma(x)|^2 dx + \frac{C}{R^2} \int_{B^c_R} |\bar{Q}_\gamma|^2 dx + \frac{2C}{R} \int_{B^c_R} |\nabla \bar{Q}_\gamma| |\bar{Q}_\gamma| dx \leq Ce^{-\delta R_\gamma},
\]

where we use (3.1). One can also calculate that

\[
A_5 \leq Ce^{-\delta R_\gamma}.
\]

Moreover, we have

\[
\lim_{\gamma \nearrow 2} \int_{\mathbb{R}^N} V(x)|\omega_{R,\gamma}|^2 dx = \lim_{\gamma \nearrow 2} A^2_{R,\gamma} \int_{\mathbb{R}^N} V\left(\frac{x}{\tau_\gamma} + x_0\right) \bar{Q}_\gamma(x) dx = V(x_0)
\]

holds for almost every $x_0 \in \mathbb{R}^N$. Therefore, we choose $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, it follows from the above estimate that

\[
0 \leq e_a(\gamma) - \tilde{e}_a(\gamma) \leq Ce^{-\delta R_\gamma} + \int_{\mathbb{R}^N} V(x)|\omega_{R,\gamma}|^2 dx \to 0 \quad \text{as} \quad \gamma \nearrow 2,
\]

which then implies (3.8). Therefore, from (3.8), we can obtain (3.9). Nowadays the proof of (3.10) is standard, and therefore we omit it.


4 Concentration and symmetry breaking

This section is devoted to proving Theorem 1.2 and Theorem 1.3 on the concentration and symmetry breaking of minimizers for (1.6) as $\gamma \nearrow 2$, where $a > \|Q_2\|^2_2$ is fixed. Set

$$\varepsilon_\gamma := \varepsilon(\gamma) = \left(\frac{a}{\|Q_\gamma\|^2_2}\right)^{-\frac{1}{2}} > 0, \quad (4.1)$$

then $\varepsilon_\gamma \to 0$ by Remark 3. Define the $L^2(\mathbb{R}^N)$-normalized function

$$\tilde{v}_\gamma(x) := \varepsilon_\gamma^{\frac{N}{2}} u_\gamma(\varepsilon_\gamma x). \quad (4.2)$$

It then follows from Lemma 3.2 that there exist two positive constants $C_1$ and $C_2$, independent of $\gamma$, such that

$$C_1 \leq \int_{\mathbb{R}^N} |\nabla \tilde{v}_\gamma(x)|^2 \leq C_2 \quad \text{and} \quad C_1 \leq D_\gamma(\tilde{v}_\gamma, \tilde{v}_\gamma) \leq C_2. \quad (4.3)$$

By the above inequality (4.3), we easily obtain that there exist a sequence $\{y_{\varepsilon_\gamma}\}$, $R_0 > 0$ and $\eta > 0$ such that

$$\liminf_{\varepsilon_\gamma \to 0} \int_{B_{R_0}(y_{\varepsilon_\gamma})} |\tilde{v}_\gamma(x)|^2 dx \geq \eta > 0,$$

where $\tilde{v}_\gamma(x)$ is defined as (4.2). For the sequence $\{y_{\varepsilon_\gamma}\}$ given by above, set

$$v_\gamma(x) = \tilde{v}_\gamma(x + y_{\varepsilon_\gamma}) = \varepsilon_\gamma^{\frac{N}{2}} u_\gamma(\varepsilon_\gamma(x + y_{\varepsilon_\gamma})). \quad (4.4)$$

Then

$$\liminf_{\varepsilon_\gamma \to 0} \int_{B_{R_0}(0)} |v_\gamma(x)|^2 dx \geq \eta > 0, \quad (4.5)$$

which therefore implies that $v_\gamma(x)$ cannot vanish as $\gamma \nearrow 2$. We shall need the following technical result, proof of which can be found in [7].

**Lemma 4.1.** Assume $V(x) \in C^1(\mathbb{R}^N)$ satisfies $\lim_{|x| \to \infty} V(x) = \infty$ and $\inf_{x \in \mathbb{R}^N} V(x) = 0$. Then $\{\varepsilon_{\gamma_k} y_{\varepsilon_k}\}$ is bounded uniformly for $\gamma \nearrow 2$. Moreover, for any sequence $\{\gamma_k\}$ with $\gamma_k \to 2$ as $k \to \infty$, there exists a subsequence, still denoted by $\{\gamma_k\}$, such that $z_k := \varepsilon_{\gamma_k} y_{\varepsilon_k} \to k y_0$, where $\varepsilon_k := \varepsilon_{\gamma_k}$ is given by (7.1), and $y_0 \in \mathbb{R}^N$ is a global minimum point of $V(x)$, i.e., $V(y_0) = 0$.

Since $u_\gamma$ is a minimizer of (1.6), it satisfies the Euler-Lagrange equation

$$-\Delta u_\gamma(x) + V(x) u_\gamma(x) = \mu_\gamma u_\gamma(x) + a \left(\frac{1}{|x|^\gamma} * |u_\gamma|^2\right) u_\gamma(x) \quad \text{in} \quad \mathbb{R}^N, \quad (4.6)$$

where $\mu_\gamma \in \mathbb{R}$ is a Lagrange multiplier and satisfies

$$\mu_\gamma = e_a(\gamma) - \frac{a}{2} D_\gamma(u_\gamma, u_\gamma).$$
It then follows from Lemma 3.2 and (4.3) that there exist two positive constants $C_1$ and $C_2$, independent of $\gamma$, such that

$$-C_2 < \mu_\gamma \varepsilon_\gamma^2 < -C_1 \text{ as } \gamma \searrow 2.$$ 

By (4.1) and (4.6), $v_\gamma(x)$ satisfies

$$-\Delta v_\gamma(x) + \varepsilon_\gamma^2 V(\varepsilon_\gamma(x + y_\gamma))v_\gamma(x) = \varepsilon_\gamma^2 \mu_\gamma v_\gamma(x) + \|Q_\gamma\|_2^2 \left(\frac{1}{|x|^{2\gamma}} \ast |v_\gamma|^2\right)v_\gamma(x) \text{ in } \mathbb{R}^N.$$ 

Therefore, by passing to a subsequence if necessary, we can assume that, for some number $\beta > 0$,

$$\mu_\gamma \varepsilon_\gamma^2 \to -\beta^2 < 0 \text{ and } v_k := v_{\gamma_k} \to v_0 \geq 0 \text{ in } H^1(\mathbb{R}^N) \text{ as } \gamma_k \searrow 2,$$

for some $v_0 \in H^1(\mathbb{R}^N)$. By passing to the weak limit of (4.7), we deduce from Lemma 3.2 and Theorem 1.1 that the non-negative function $v_0$ satisfies

$$-\Delta v_0(x) = -\beta^2 v_0(x) + a^\ast \left(\frac{1}{|x|^2} \ast |v_0|^2\right)v_0(x) \text{ in } \mathbb{R}^N.$$ 

Furthermore, we infer from (4.5) that $v_0 \not\equiv 0$ in $\mathbb{R}^N$, and the strong maximum principle then yields that $v_0 > 0$ in $\mathbb{R}^N$. By the simple rescaling, we thus conclude from the positive ground state of (1.3) that

$$v_0 = \frac{\beta_N}{\|Q_0\|_2} Q_0(\beta |x - x_0|) \text{ for some } x_0 \in \mathbb{R}^N,$$

where $Q_0$ is the positive radially symmetric solution of equation (1.3). Since $J_2(Q_0) \geq J_2(Q_2) = I_2$, where $I_2$ is the ground state energy, and $J_2(Q_0) = \frac{1}{2} \|Q_0\|_2^2$, then we have $\|Q_0\|_2^2 \geq a^\ast$. Therefore $\int_{\mathbb{R}^N} |v_0(x)|^2 dx \geq 1$. On the other hand, it follows from the Fatou’s lemma that $\int_{\mathbb{R}^N} |v_0(x)|^2 dx \leq 1$. Then $\int_{\mathbb{R}^N} |v_0(x)|^2 dx = 1$, which implies that $\|Q_0\|_2 = \|Q_2\|_2 = a^\ast$. Thus, $Q_0$ is a positive radially symmetric ground state of equation (1.3). Note that $\|v_k\|_2^2 = 1$, then $v_k$ converges to $v_0$ strongly in $L^2(\mathbb{R}^N)$ and in fact, strongly in $L^p(\mathbb{R}^N)$ for any $2 \leq p < \frac{2N}{N-2}$ because of $H^1(\mathbb{R}^N)$ boundedness. Furthermore, since $v_k$ and $v_0$ satisfy (4.7) and (4.8) respectively, standard elliptic regularity theory gives that $v_k$ converges to $v_0$ strongly in $H^1(\mathbb{R}^N)$.

**Proof of Theorem 1.2.** Motivated by [7, 8], we are now ready to complete the proof of Theorem 1.1 by the following three steps.

**Step 1:** The decay property of $u_k := u_{\gamma_k}$. For any sequence $\{\gamma_k\}$. Let $v_k := v_{\gamma_k} \geq 0$ be defined by (4.4). The above analysis shows that there exists a subsequence, still denoted by $\{v_k\}$, satisfying (4.7) and $v_k \to v_0$ strongly in $H^1(\mathbb{R}^N)$ for some positive function $v_0$. Hence for any $2 < \alpha < \frac{2N}{N-2}$,

$$\int_{|x| \geq R} |v_k|^\alpha dx \to 0 \text{ as } R \to \infty \text{ uniformly for large } k.$$
Since $\mu_{\gamma_k} < 0$, it follows from (4.7) that

$$-\Delta v_k - c(x)v_k \leq 0,$$

where $c(x) = \|Q_\gamma\|_{L^2}^2 \frac{1}{|x|^\gamma} \ast |v_k|^2$.

Denote $\phi_{v_k}(x) = \int_{\mathbb{R}^N} \frac{|v_k(y)|^2}{|x-y|^{\gamma}} dy$. By the Riesz potential inequality, we then have

$$\|\phi_{v_k}(x)\|_{L^q(B_2(\xi))} \leq \|\phi_{v_k}(x)\|_{L^q(\mathbb{R}^N)} \leq C \|v_k\|^2_{L^p(\mathbb{R}^N)} = C \|v_k\|^2_{L^2p(\mathbb{R}^N)}$$

where $1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{N}$. In particular, if $q = \frac{N}{\gamma} > \frac{N}{2}$, then $p = 1$. Since $v_k \in H^1(\mathbb{R}^N)$, by the Sobolev embedding theorem, we get

$$\|\phi_{v_k}(x)\|_{L^q(B_2(\xi))} \leq \|v_k\|^2_{L^2p(\mathbb{R}^N)} < C < \infty.$$

Note that $q > \frac{N}{2}$, by applying De Giorgi-Nash-Moser theory (see [9], Theorem 4.1), we thus have

$$\max_{B_1(\xi)} v_k(x) \leq C(\int_{B_2(\xi)} |v_k(x)|^\alpha dx)^{\frac{1}{\alpha}}, \quad (4.10)$$

where $\xi$ is an arbitrary point in $\mathbb{R}^N$, and $C$ is a constant depending only on the bound of $\|\phi_{v_k}(x)\|_{L^p(B_2(\xi))}$. Hence we deduce from (4.10) that

$$v_k(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } k.$$

Since $v_k$ satisfies (4.7), one can use the comparison principle as in [12] to $v_k$ with $C e^{-\frac{\beta}{2}|x|}$, which then shows that there exists a large constant $R > 0$, independent of $k$, such that

$$v_k(x) \leq C e^{-\frac{\beta}{2}|x|} \text{ for } |x| > R \text{ as } k \to \infty. \quad (4.11)$$

By Lemma 4.1, we therefore obtain from (4.11) that the subsequence

$$u_k(x) := u_{\gamma_k}(x) = \frac{1}{\varepsilon_k^N} v_k \left( \frac{x - z_k}{\varepsilon_k} \right),$$

decays uniformly to zero for $x$ outside any fixed neighborhood of $y_0$ as $k \to \infty$, where $\varepsilon_k = \varepsilon_{\gamma_k}$, $z_k \in \mathbb{R}^N$ is defined as in Lemma 4.1 and $y_0 \in \mathbb{R}^N$ is a global minimum point of $V(x)$.

**Step 2**: The detailed concentration behavior. Let $\bar{z}_k$ be any local maximum point of $u_k$. By the definition of $\phi_{u_k}(\bar{z}_k)$, we have

$$\phi_{u_k}(\bar{z}_k) = \int_{\mathbb{R}^N} \frac{|u_k(y)|^2}{|\bar{z}_k - y|^{\gamma}} dy \leq u_k(\bar{z}_k) \Delta (\int_{|\bar{z}_k - y| < \delta} \frac{|u_k(y)|^2}{|\bar{z}_k - y|^{\gamma}} dy + \int_{|\bar{z}_k - y| \geq \delta} \frac{|u_k(y)|^2}{|\bar{z}_k - y|^{\gamma}} dy).$$

Since $0 < \gamma < 2$, then it follows from the H"older inequality that

$$\int_{|\bar{z}_k - y| < \delta} \frac{|u_k(y)|^2}{|\bar{z}_k - y|^{\gamma}} dy \leq \left( \int_{|\bar{z}_k - y| < \delta} |\bar{z}_k - y|^{-\frac{N\gamma}{2}} dy \right)^{\frac{\gamma}{N}} \left( \int_{|\bar{z}_k - y| < \delta} |u_k(y)|^{2+\frac{2}{N-2}} \right)^{\frac{N-2}{N}} \leq C < \infty,$$
where $C > 0$ is independent of $k$, since $u_k$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

On the other hand, combining Hölder inequality and Sobolev inequality yields that

$$\int_{|\tilde{z}_k - y| \geq \delta} \frac{|u_k(y)|^2}{|\tilde{z}_k - y|^{\frac{2}{N}}} \leq \left( \int_{|\tilde{z}_k - y| \geq \delta} |\tilde{z}_k - y|^{-(N+1)\gamma} \right)^{\frac{\gamma}{N+1}} \left( \int_{|\tilde{z}_k - y| \geq \delta} |u_k(y)|^{2+\frac{2N}{N+1}} dy \right)^{\frac{N+1-\gamma}{N+1}} \leq C < \infty,$$

where $C > 0$ is independent of $k$. According to the above two estimates, we deduce that

$$\phi_{u_k}(\tilde{z}_k) \leq C u_k(\tilde{z}_k)^{\frac{N}{\gamma}} \text{ as } \gamma \nearrow 2. \quad (4.12)$$

We then follows from (4.16) and (4.12) that

$$0 \leq \mu_\gamma u_\gamma(\tilde{z}_k) + \|Q_\gamma\|^2_2 \phi_{u_k}(\tilde{z}_k) u_\gamma(\tilde{z}_k) \leq \mu_\gamma u_\gamma(\tilde{z}_k) + \|Q_\gamma\|^2_2 Cu_\gamma^{\frac{N+1}{2}}(\tilde{z}_k) \text{ as } \gamma \nearrow 2,$$

which implies that

$$u_\gamma(\tilde{z}_k) \geq \left( \frac{-\mu_\gamma}{\|Q_\gamma\|^2_2 C} \right)^{\frac{N}{\gamma}} \geq C \varepsilon_\gamma^{-2N}.$$

This estimate and the above decay property thus imply that $\tilde{z}_k \rightarrow y_0$ as $k \rightarrow \infty$. Set

$$\tilde{v}_k = \varepsilon_k^N u_k(\epsilon_k x + \tilde{z}_k), \quad (4.13)$$

so that $\tilde{v}_k$ satisfies (4.3). It then follow from (4.17) that

$$-\Delta \tilde{v}_k(x) + \varepsilon_k^2 \nabla(\varepsilon_k x + \tilde{z}_k) \tilde{v}_k(x) = \varepsilon_k^2 \mu_\gamma \tilde{v}_k(x) + \|Q_\gamma\|^2_2 \left( \frac{1}{|x|^{\gamma}} \ast |\tilde{v}_k|^2 \right) \tilde{v}_k(x) \text{ in } \mathbb{R}^N. \quad (4.14)$$

The same argument as proving (4.8) yields that there exists a subsequence of $\{\tilde{v}_k\}$, still denoted by $\{\tilde{v}_k\}$, such that $\tilde{v}_k \rightarrow \tilde{v}_0$ as $k \rightarrow \infty$ in $H^1(\mathbb{R}^N)$ for some nonnegative function $\tilde{v}_0 \geq 0$, where $\tilde{v}_0$ satisfies (4.8) for some constant $\beta > 0$. We derive from (4.14) that

$$\tilde{v}_k(0) \geq \left( \frac{-\varepsilon_k^2 \mu_\gamma}{C \|Q_\gamma\|^2_2} \right)^{\frac{N}{\gamma}} \geq \left( \frac{\beta^2}{C \|Q_\gamma\|^2_2} \right)^{\frac{N}{\gamma}} \text{ as } k \rightarrow \infty,$$

which implies that $\tilde{v}_0(0) > 0$. Thus, the strong maximum principle yields that $\tilde{v}_0(x) > 0$ in $\mathbb{R}^N$. Since $x = 0$ is a critical point of $\tilde{v}_k$ for all $k > 0$, it is also a critical point of $\tilde{v}_0$. We therefore conclude from the positive radial solutions of equation (1.3) that $\tilde{v}_0$ is spherically symmetric about the origin, and

$$\tilde{v}_0 = \frac{\beta^N}{\|Q_2\|^2_2} Q_2(\beta |x|) \text{ for some } \beta > 0. \quad (4.15)$$

**Step 3** : The exact value of $\beta$ defined in (4.15). Let $\{\gamma_k\}$, where $\gamma_k \nearrow 2$ as $k \rightarrow \infty$, be the subsequence obtained in Step 2, and denoted $u_k := u_{\gamma_k}$. Recall from Lemma 3.2 that

$$e_a(\gamma_k) = \tilde{e}_a(\gamma_k) + o(1) = \left( 1 - \frac{2}{\gamma_k} \left( \frac{4 - \gamma_k}{\gamma_k} \right) \varepsilon_k^{-2} + o(1) \right) \text{ as } k \rightarrow \infty,$$
which yields that
\[ \lim_{k \to \infty} \frac{\gamma_k}{2 - \gamma_k} \varepsilon^2_k e_a(\gamma_k) = - \lim_{k \to \infty} \frac{4 - \gamma_k}{\gamma_k} = -1. \] (4.16)

On the other hand,
\[
e_a(\gamma_k) = \int_{\mathbb{R}^N} (|\nabla u_{\gamma_k}|^2 + V(x)|u_{\gamma_k}|^2)dx - \frac{a}{2} D_{\gamma_k}(u_{\gamma_k}, u_{\gamma_k})
= \varepsilon^2_k \left( \int_{\mathbb{R}^N} |\nabla \tilde{v}_k|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} D_{\gamma_k} (\tilde{v}_k, \tilde{v}_k) \right) + \int_{\mathbb{R}^N} V(x)|u_{\gamma_k}|^2 dx
\geq \varepsilon^2_k \left( \int_{\mathbb{R}^N} |\nabla \tilde{v}_k|^2 dx - \frac{2}{4 - \gamma_k} \left( \frac{4 - \gamma_k}{\gamma_k} \right)^{\frac{\gamma_k}{2}} \left( \int_{\mathbb{R}^N} |\nabla \tilde{v}_k|^2 dx \right)^{\frac{\gamma_k}{2}} \right) \] (4.17)

where \( \tilde{v}_k := \tilde{v}_{\gamma_k} \) is as in (4.13). Set \( \beta^2_{\gamma_k} := \int_{\mathbb{R}^N} |\nabla \tilde{v}_k|^2 dx \). Since \( \tilde{v}_k(x) \to \tilde{v}_0(x) \) strongly in \( H^1(\mathbb{R}^N) \), we have
\[ \lim_{k \to \infty} \beta^2_{\gamma_k} = \|\nabla \tilde{v}_0\|_2^2 = \beta^2, \] (4.18)

where (2.1) is used. Let \( f_k(t) = t - \frac{2}{4 - \gamma_k} \left( \frac{4 - \gamma_k}{\gamma_k} \right)^{\frac{\gamma_k}{2}} t^{\frac{\gamma_k}{2}} \), where \( t \in (0, \infty) \). A simple analysis shows that \( f_k(\cdot) \) attains its global minimum at the unique point \( t_k = \frac{\gamma_k}{4 - \gamma_k} \) and also \( f_k(t_k) = \frac{\gamma_k}{4 - \gamma_k} \). We hence deduce from (4.17) that
\[ \lim_{k \to \infty} \frac{\gamma_k - 2 - \gamma_k}{\varepsilon^2_k e_a(\gamma_k)} \geq \lim_{k \to \infty} \frac{\gamma_k - 2 - \gamma_k}{\beta^2_{\gamma_k}} \geq \lim_{k \to \infty} \frac{\gamma_k}{2 - \gamma_k} f_k(t_k) = -1, \]
Combine with (4.16), it follows that
\[ \lim_{k \to \infty} \frac{f_k(\beta^2_{\gamma_k})}{f_k(t_k)} = 1. \]

We then obtain that
\[ \lim_{k \to \infty} \beta^2_{\gamma_k} = \lim_{k \to \infty} t_k = 1, \]
and therefore we have \( \beta = 1 \) by applying (4.18), which, together with (4.13) and (4.15) give in (1.8). We thus complete the proof of Theorem 1.2.

Following the proof of Theorem 1.2, we next address Theorem 1.3 on the local properties of concentration points. Under the assumption (1.9), we first denoted
\[ V_i(x) = \frac{V(x)}{|x - x_i|^{p_i}}, \text{ where } i = 1, 2, \ldots, n, \]
so that the \( \lim_{x \to x_i} V_i(x) = V_i(x_i) \) is assumed to exist for all \( i = 1, 2, \ldots, n \).
Proof of Theorem 1.3. For convenience we still denoted \( \{ \gamma_k \} \) to be the subsequence obtained in Theorem 1.2. Choose a point \( x_{i_0} \in \mathcal{Z} \), where \( \mathcal{Z} \) is defined by (1.10), and let

\[
\omega_{R, \gamma_k}(x) = A_{R, \gamma_k} \varphi_R(x - x_{i_0}) \overline{Q}_{\gamma_k}(x - x_{i_0})
\]

be the trial function defined by (3.11). By (3.12), we know that

\[
e_a(\gamma_k) - \bar{e}_a(\gamma_k) \leq \int_{\mathbb{R}^N} V(x)|A_{R, \gamma_k}(x)|^2 dx + Ce^{-\delta R_{\gamma_k}}
\]

\[
\leq \frac{A^2_{R, \gamma_k}}{\tau_{\gamma_k}^p \|Q_{\gamma_k}\|^2_2} \int_{B_{2R_{\gamma_k}}} \overline{V}_{i_0}(\frac{x}{\tau_{\gamma_k} + x_{i_0}})|x|^2 Q^2_{\gamma_k}(x) dx + Ce^{-\delta R_{\gamma_k}}
\]

\[
\leq \frac{A^2_{R, \gamma_k}}{\tau_{\gamma_k}^p \|Q_{\gamma_k}\|^2_2} \int_{\mathbb{R}^N} \chi_{B_{2R_{\gamma_k}}} \overline{V}_{i_0}(\frac{x}{\tau_{\gamma_k} + x_{i_0}})|x|^2 Q^2_{\gamma_k}(x) dx + Ce^{-\delta R_{\gamma_k}}.
\]

(4.19)

where \( \tau_{\gamma_k} > 0 \) satisfies \( \tau_{\gamma_k} = \sqrt{\frac{\gamma}{1 - \gamma \varepsilon_k}} \) in view of Lemma 3.1 and (4.1), and \( \chi_{B_{2R_{\gamma_k}}} \) is the characteristic function of the set \( B_{2R_{\gamma_k}} \). Combining Theorem 1.1 and (3.1), we deduce that

\[
\chi_{B_{2R_{\gamma_k}}} \overline{V}_{i_0}(\frac{x}{\tau_{\gamma_k} + x_{i_0}})|x|^2 Q^2_{\gamma_k}(x) \leq \sup_{B_{2R}} \overline{V}_{i_0}(x + x_{i_0}) \cdot Ce^{-\delta |x|} \in L^1(\mathbb{R}^N),
\]

and

\[
\chi_{B_{2R_{\gamma_k}}} \overline{V}_{i_0}(\frac{x}{\tau_{\gamma_k} + x_{i_0}})|x|^2 Q^2_{\gamma_k}(x) \rightarrow \overline{V}_{i_0}(x_{i_0}) |x|^2 Q^2_{2}(x) \text{ a.e. } \mathbb{R}^N \text{ as } k \to \infty.
\]

Note that \( A_{R, \gamma_k} \to 1 \) as \( \gamma_k \not\to 2 \), we thus obtain from (4.19) and Lebesgue’s dominated convergence theorem that

\[
\lim_{k \to \infty} \frac{e_a(\gamma_k) - \bar{e}_a(\gamma_k)}{\varepsilon_k^p} \leq \frac{\overline{V}_{i_0}(x_{i_0})}{\|Q_2\|^2_2} \int_{\mathbb{R}^N} |x|^2 Q^2_2 dx.
\]

(4.20)

On the other hand, following the proof of Theorem 1.2 we denote \( \bar{z}_k \) to be the unique global maximum point of \( u_k \), and let \( \bar{v}_k \) be defined as in (4.13). Denote also \( y_0 \in \mathbb{R}^N \) to be the limit of \( \bar{z}_k \) as \( k \to \infty \). Since \( V(y_0) = 0 \), then there exists an \( x_j = y_0 \) for some \( 1 \leq j \leq n \). We claim that \( \{ \frac{\bar{z}_k - x_j}{\varepsilon_k} \} \) is bounded in \( \mathbb{R}^N \). Indeed, if there exists a subsequence, still denoted by \( \{ \gamma_k \} \), such that \( \frac{\bar{z}_k - x_j}{\varepsilon_k} \to \infty \) as \( k \to \infty \), it then follows from Fatou’s Lemma that, for any \( C > 0 \) sufficiently large,

\[
\lim_{k \to \infty} \frac{e_a(\gamma_k) - \bar{e}_a(\gamma_k)}{\varepsilon_k^p} \geq \lim_{k \to \infty} \int_{\mathbb{R}^N} \overline{V}_j(\varepsilon_k + \bar{z}_k) \left| x + \frac{\bar{z}_k - x_j}{\varepsilon_k} \right|^{p_j} \bar{u}_k^2 dx
\]

\[
\geq \int_{\mathbb{R}^N} \lim_{k \to \infty} \overline{V}_j(\varepsilon_k + \bar{z}_k) \left| x + \frac{\bar{z}_k - x_j}{\varepsilon_k} \right|^{p_j} \bar{u}_k^2 dx \geq C \overline{V}_j(x_j),
\]

which however contradicts (4.20) owing to \( p_j \leq p = \max\{p_1, p_2, ..., p_n\} \), and the claim is therefore true. Consequently, there exists a subsequence, still denoted by \( \{ \gamma_k \} \), such that

\[
\frac{\bar{z}_k - x_j}{\varepsilon_k} \to \bar{z}_0 \text{ for some } \bar{z}_0 \in \mathbb{R}^N.
\]
Since $Q_2$ is a radial decreasing function and decays exponentially as $|x| \to \infty$, we then deduce that
\[
\lim_{k \to \infty} \frac{\varepsilon_a(\gamma_k) - \tilde{\varepsilon}_a(\gamma_k)}{\varepsilon_k^p} \geq \lim_{k \to \infty} \int_{\mathbb{R}^N} \tilde{V}_j(\varepsilon_k + \tilde{z}_k) \left| x + \frac{\tilde{z}_k - x_j}{\varepsilon_k} \right|^{p_j} \tilde{v}_2 \, dx
\]
\[
\geq \tilde{V}_j(x_j) \int_{\mathbb{R}^N} |x + \tilde{z}_0|^{p_j} \tilde{v}_0^2 \, dx
\]
\[
\geq \frac{\tilde{V}_j(x_j)}{\|Q_2\|^2_2} \int_{\mathbb{R}^N} |x|^{p_j} Q_2^2 \, dx. \tag{4.21}
\]
where $\tilde{v}_0 > 0$ is as in (4.15), and “=” in the last inequality of (4.21) holds if and only if $\tilde{z}_0 = 0$.

Applying (4.20) and (4.21), it is not difficult to see that $p_j = p$ and then $V_j(x_j) = V_{i_0}(x_{i_0})$. Hence, $x_j = y_0 \in Z$ must be the flattest global minimum point of $V(x)$. Based on these facts, using (4.20) and (4.21) we see that (4.21) is essentially an equality. Therefore, $\tilde{z}_0 = 0$ and (1.12) holds. The proof of Theorem 1.3 is completed. \qed

References

[1] P. Choquard, J. Stubbe, M. Vuffray, Stationary solutions of the Schrödinger-Newton model–an ODE approach, Differential Integral Equations 21 (2008), 665-679.

[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999), 463-512.

[3] Y. B. Deng, L. Lu, W. Shuai, Constraint minimizers of mass critical Hartree energy functionals, existence and mass concentration, J. Math. Phys. 56 (2015), 061503.

[4] Y. J. Guo and R. Seiringer, On the mass concentration for Bose-Einstein condensates with attractive interactions, Lett. Math. Phys. 104 (2014), 141-156.

[5] Y. J. Guo, X. Y. Zeng, H. S. Zhou, Energy estimates and symmetry breaking in attractive Bose-Einstein condensates with ring-shaped potentials, Ann. Inst. H. Poincaré, 33 (2016), 809-828.

[6] Y. B. Deng, Y. J. Guo, L. Lu, On the collapse and concentration of Bose-Einstein condensates with inhomogeneous attractive interactions. Calc. Var. Partial Differential Equations 54 (2015), 99-118.

[7] Y. J. Guo, X. Y. Zeng, H. S. Zhou, Concentration behavior of standing waves for almost mass critical nonlinear Schrödinger equations, J. Differential Equations 256 (2014), 2079-2100.

[8] Q. X. Wang, D. Zhao, Existence and mass concentration of 2D attractive Bose-Einstein condensates with periodic potentials, J. Differential Equations 262 (2017), 2684-2704.
[9] Q. Han and F. H. Lin, Elliptic Partial Differential Equations, Courant Lecture Notes in Mathematics Vol. 1 (Courant Institute of Mathematical Science/AMS, New York, 2011).

[10] J. Huang, J. Zhang. Nonlinear Hartree equation in high energy-mass, Nonlinear Anal. Real world Appl. 34 (2017), 97-109.

[11] J. Huang, J. Zhang. Corrigendum to “nonlinear Hartree equation in high energy-mass” [Nonlinear Anal. RWA 34 (2017), 97-109][MR3567951], Nonlinear Anal. Real World Appl. 37 (2017), 512-513.

[12] O. Kavian and F. B. Weissler, Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation, Michigan Math. J. 41 (1994), 151-173.

[13] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010), 455-467.

[14] S. Li, J. L. Xiang, X. Y. Zeng, Ground states of nonlinear Choquard equations with muti-well potentials, J. Math. Phys. 57 (2016), 081515.

[15] L. Lichtenstein, Gleichgewichts Figuren der rotierenden Plässigkeiten (Springer, Berlin, 1933).

[16] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Stud. Appl. Math. 57 (1976/1977), 93-105.

[17] E. H. Lieb, B. Simon, The Hartree-Fock theory for Coulomb systems, Comm. Math. Phys. 53 (1977), 185-194.

[18] I. M. Moroz, R. Penrose, P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Classical Quantum Gravity. 15 (1998), 2733-2742.

[19] V. Moroz , J. Van Schaftingen, Ground states of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013), 153-184.

[20] V. Moroz, J. V. Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Amer. Math. Soc. 367 (2015), 6557-6579.

[21] S. Pekar, Untersuchungüüber die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.

[22] J. Seok, Limit profiles and uniqueness of ground state to the nonlinear Choquard equation, (2017) arXiv preprint [arXiv:1704.00126]

[23] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), 149-162.

[24] T. Wang, T. S. Yi, Uniqueness of positive solutions of the Choquard type equations, Appl. Anal. 96 (2017), 409-417.
[25] C. L. Xiang, Uniqueness and nondegeneracy of ground states for Choquard equations in three dimensions, Calc. Var. Partial Differential Equations 55 (2016), 134.

[26] G. B. Li, H. Y. Ye, The existence of positive solutions with prescribed $L^2$-norm for nonlinear Choquard equations, J. Math. Phys. 55 (2014), 121501.

[27] H. Y. Ye, Mass minimizers and concentration for nonlinear Choquard equations in $\mathbb{R}^N$, Topol. Methods Nonlinear Anal. 48 (2016), 393-417.

[28] V. Moroz, J. Van Schaftingen, Semi-classical states for the Choquard equation, Calc. Var. Partial Differential Equations 52 (2015), 199-235.

[29] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys. 63 (2012), 233-248.

[30] S. Cingolani, S. Secchi, M. Squassina, Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities. Proc. R. Soc. Edinb. Sect. A 140 (2010), 973-1009.

[31] M. Clapp, D. Salazar, Positive and sign-changing solutions to a nonlinear Choquard equation, J. Math. Anal. Appl. 407 (2013), 1-15.

[32] J. Van Schaftingen, J. Xia, Choquard equations under confining external potentials, NoDEA Nonlinear Differential Equations Appl. 24 (2017), 24.