Simulating chiral anomalies with spin dynamics

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Considering that the chiral kinetic equations of motion (CEOM) can be derived from the spin kinetic equations of motion (SEOM) for massless particles with approximations, we simulate the chiral anomalies by using the latter in a box system with the periodic boundary condition under a uniform external magnetic field. We found that the chiral magnetic effect is weaker while the damping of the chiral magnetic wave is stronger from the SEOM compared with that from the CEOM. In addition, effects induced by chiral anomalies from the SEOM are less sensitive to the decay of the magnetic field than from the CEOM due to the spin relaxation process.

The spin dynamics can induce many interesting phenomena in various systems with different types of spin-orbit couplings \textsuperscript{1}. For instance, in nucleonic systems where nucleons are massive compared with nuclear spin-related interactions, the nuclear spin-orbit coupling is responsible for the magic number in finite nuclei \textsuperscript{2,3} as well as various spin-related observables in heavy-ion reactions \textsuperscript{4}. In the latter case, the spin dynamics can be described by the spin-dependent equations of motion derived from the spin-dependent Boltzmann-Vlasov equation \textsuperscript{5}. In the massless limit, the Dirac equation is decoupled, and particles with different chiralities are affected by the Weyl spin-orbit coupling, which leads to chiral anomalies and various of interesting phenomena in not only material science (see, e.g., Ref. \textsuperscript{6}) but also heavy-ion physics \textsuperscript{7,8}.

Due to the chiral symmetry restoration of partons at high energy densities reached in relativistic heavy-ion collisions, the above massless limit is approximately satisfied. This can lead to the chiral magnetic effect (CME) \textsuperscript{9,11} induced by the non-zero axial charge density, leading to the net electric charge current along the direction of the magnetic field, and the resulting electric charge separation can be measured indirectly by comparing the correlation between particles of the same and opposite electric charge \textsuperscript{12–15}, although there are recently hot debates on the background contribution \textsuperscript{16–20}. A dual effect is called the chiral separation effect (CSE) \textsuperscript{21,22} induced by non-zero electric charge density, leading to the net axial charge current along the direction of the magnetic field. The interplay between the CME and the CSE can produce a collective excitation called the chiral magnetic wave (CMW) \textsuperscript{22,23}, which may generate an electric quadrupole moment in a quark matter, and can be responsible for the elliptic flow splitting between particles of opposite electric charges \textsuperscript{24,28}.

The various phenomena mentioned above originated from the Weyl spin-orbit coupling can be studied with transport simulations based on the chiral kinetic equations of motion (CEOM) derived from different approaches \textsuperscript{29,32}. It is interesting to see that the CEOM can also be derived from the spin kinetic equations of motion (SEOM) for massless particles by using an $\hbar$ expansion of the spin with respect to the momentum in the adiabatic limit \textsuperscript{33}. However, such expansion limits the validity of the CEOM only for particles with larger momenta \textsuperscript{29}. As shown in Ref. \textsuperscript{33}, an artificial truncation is needed in the momentum space in transport simulations, and this may underestimate the effects induced by chiral dynamics compared with the theoretical limits.

It is also a question how important are the higher-order $\hbar$ terms and how good is the adiabatic approximation in deriving the CEOM from the SEOM. In the present study, we do transport simulations directly according to the SEOM, and compare the results of the CME and CMW with those from the CEOM.

We start from the Hamiltonian for massless spin-1/2 particles with the charge number $q = \pm 1$ under an external magnetic field $\vec{B}$

$$H = c\vec{\sigma} \cdot \vec{k},$$

where $\vec{\sigma}$ are the Pauli matrices, $\vec{k} = \vec{p} - e\vec{A}$ is the kinematic momentum with $\vec{p}$ being the canonical momentum and $\vec{A}$ being the vector potential of the magnetic field $\vec{B}$. Considering $\vec{\sigma}$ as the expectation direction of the spin, the SEOM from Eq. \textsuperscript{11} can be written as

$$\dot{\vec{r}} = c\vec{\sigma},$$
$$\dot{\vec{k}} = c\vec{\sigma} \times qe\vec{B},$$
$$\dot{\vec{\sigma}} = \frac{2}{\hbar}\vec{\sigma} \times \vec{k}. \quad (4)$$

Equation (4) shows that the velocity is always the speed of light, Eq. (3) indicates the Lorentz force, and Eq. (4) describes the precessional motion of $\vec{\sigma}$ with respect to $\vec{k}$. It is easy to prove that the above SEOM conserve the single-particle energy $c\vec{\sigma} \cdot \vec{k}$. Using the following approximation $\textsuperscript{34,35}$

$$\vec{\sigma} \approx c\vec{k} - \frac{\hbar}{2k} \left( \vec{k} \times \dot{\vec{k}} \right),$$

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one gets the CEOM \[24, 32\]
\[
\sqrt{G} \dot{r} = \hat{k} + \hbar \left( \vec{c} \cdot \vec{k} \right) q \epsilon \vec{B},
\]
\[
\sqrt{G} \dot{k} = \hat{k} \times q \epsilon \vec{B},
\]
with \( \sqrt{G} = 1 + \hbar (q \epsilon \vec{B} \cdot \vec{b}) \), where \( \hat{k} = \vec{k}/k \) is a unit vector and \( \vec{b} = \vec{k}/(2k^3) \) is the Berry curvature for massless particles. Equation \( 6 \) quantifies the small deviation between \( \hat{\sigma} \) and \( \hat{c} \hat{k} \) in the \( \hbar \) order, and assumes that \( \hat{\sigma} \) evolves much faster with time than \( \hat{r} \) and \( \hat{k} \). With Eq. \( 4 \), the single-particle energy becomes \( c \hat{\sigma} \cdot \hat{k} = k \) for the CEOM.

Considering \( \hat{r}, \hat{k}, \) and \( \hat{\sigma} \) as free variables, the SEOM do not change the phase-space volume \[36\]. On the other hand, Eq. \( 7 \) in the CEOM modifies the phase-space integral \( d^3k/\langle 2\pi \rangle^3 \) to \( \sqrt{G} d^3k/\langle 2\pi \rangle^3 \) \[37\], so the average value of any statistical quantity is correspondingly calculated according to \( \langle A \rangle = \sum_i A_i \sqrt{G_i}/\sum_i \sqrt{G_i} \) by taking \( \sqrt{G_i} \) for the \( i \)th particle as a weight factor in transport simulations.

The charge and current densities of particles with the charge number \( q \) and the helicity \( c \) can be respectively expressed as
\[
\rho_{qc} = qN_c \int \frac{d^3k}{\langle 2\pi \rangle^3} f_{qc}(\vec{k}),
\]
\[
\vec{J}_{qc} = N_c \int \frac{d^3k}{\langle 2\pi \rangle^3} \vec{r} f_{qc}(\vec{k}),
\]
where \( N_c = 3 \) is the color degeneracy, and \( f_{qc}(\vec{k}) \) is the distribution function in momentum space. The electric charge density and current for particles with the right-handed (R) and the left-handed (L) chirality are defined as
\[
\rho_R = \rho_{q(+)c(+)} + \rho_{q(-)c(-)},
\]
\[
\rho_L = \rho_{q(+)c(-)} + \rho_{q(-)c(+)},
\]
\[
\vec{J}_R = \vec{J}_{q(+)c(+)} - \vec{J}_{q(-)c(-)},
\]
\[
\vec{J}_L = \vec{J}_{q(-)c(+)} - \vec{J}_{q(+)c(-)},
\]
and the total electric and axial charge density and current are respectively defined as
\[
\rho = \rho_R + \rho_L, \quad \rho_5 = \rho_R - \rho_L,
\]
\[
\vec{J} = \vec{J}_R + \vec{J}_L, \quad \vec{J}_5 = \vec{J}_R - \vec{J}_L.
\]
The initial and equilibrated distribution function in momentum space can be formally expressed as \( f_{qc}(\vec{k}) = \sqrt{G} \{ \exp[(k - q\mu_5)/T] + 1 \} \) for the CEOM, where \( \mu_5 = q\mu + c\mu_5 \) is the chemical potential with \( \mu(\mu_5) \) being the electric (axial) charge chemical potential, and \( T \) is the temperature of the system. The initial momentum distribution \( f_{qc}(\vec{k}) \) for the SEOM is isotropic and set as \( 1/\{ \exp[(k - q\mu_5)/T] + 1 \} \) with \( \hat{\sigma} = c\hat{k} \), while the equilibrated momentum distribution is anisotropic in the direction along and perpendicular to the magnetic field, but the analytical expression is not available. However, since the energy \( E = c\vec{\sigma} \cdot \vec{k} \) for each particle is conserved in the SEOM, \( \mu_5 \) and \( T \) are the same chemical potential and temperature at the equilibrated state for the SEOM, with the energy distribution \( 1/\{ \exp[(E - \mu_5)/T] + 1 \} \) unchanged during the evolution. The CME and the CSE are generally expressed by the following relations
\[
\dot{\vec{J}} \approx \frac{\alpha}{2\pi^2 \hbar^2} \mu_5 \epsilon \vec{B}, \quad (16)
\]
\[
\dot{\vec{J}}_5 \approx \frac{\alpha}{2\pi^2 \hbar^2} \mu_5 \epsilon \vec{B}, \quad (17)
\]
Taking \( f_{qc}(\vec{k}) \) as an isotropic Fermi-Dirac distribution times a \( \sqrt{G} \) factor in Eq. \( 9 \), the above relations are exact with \( \alpha = 1 \). In real simulations based on the CEOM, an artificial truncation to remove particles with too small \( k \) generally underestimates the CME and the CSE \[33\], leading to \( \alpha < 1 \). To understand the CME and the CSE from the SEOM, we take the second-order derivative of \( \dot{r} \) with respect to \( t \), i.e.,
\[
\frac{d^2 \dot{r}}{dt^2} = 2 \frac{\hbar q}{\epsilon^2} \dot{\vec{r}} \times \vec{B} + \frac{\epsilon}{\hbar^2} \vec{k} \times (\vec{k} \times \dot{\vec{r}}),
\]
where particles with different \( c \) and \( q \) become separated.

We will see that the linear relations between the currents and the chemical potentials are also approximately satisfied for the SEOM, but both the CME and the CSE are weaker, leading to \( \alpha < 1 \).

In the limit of \( \mu/T \ll 1 \) and \( \mu_5/T \ll 1 \), the electric and axial charge density are proportional to their corresponding chemical potential
\[
\rho \approx \frac{N_c T^2}{3\hbar^2} \mu, \quad (18)
\]
\[
\rho_5 \approx \frac{N_c T^2}{3\hbar^2} \mu_5. \quad (19)
\]
Combining Eqs. \( 16-19 \) and using the definitions of \( \vec{J}_{RL} \) and \( \rho_{RL} \), the following decoupled relation can be obtained:
\[
\vec{J}_{RL} \approx \pm \alpha \frac{3\hbar \epsilon \vec{B}}{2\pi^2 T^2} \rho_{RL}, \quad (20)
\]
with the upper (lower) sign for particles with the right-handed (left-handed) chirality. Under an external magnetic field in \(+y\) direction, and adding the diffusive term \(-D_L \vec{\nabla} \rho_{RL} \) \[24\], with \( D_L \) being the longitudinal diffusion constant, to the continuity equation \( \partial_t \rho_{RL} + \vec{\nabla} \cdot \vec{J}_{RL} = 0 \), the equation describing the CMW can be written as \[23, 24\]
\[
(\partial_t + v_p \partial_y - D_L \partial_y^2) \rho_{RL} \approx 0, \quad (21)
\]
where
\[
v_p \approx \alpha \frac{3\hbar \epsilon B}{2\pi^2 T^2} \quad (22)
\]
is the phase velocity.

The simulation was studied in a cubic box system with the periodic boundary condition under a uniform external magnetic field, with the side length of the cubic box $2l = 10 \text{ fm}$. Two scenarios of simulations have been done, i.e., by using the SEOM [Eqs. (2)-(4)] and using the CEOM [Eqs. (6)-(7)]. The momentum distribution for particles with the charge number $q$ and the helicity $c$ is sampled according to $1/\{\exp[(k - q\mu_c)/T] + 1\}$ as mentioned above, and all particles are uniformly distributed in the coordinate space. The density of the each particle species is determined by the temperature and the corresponding chemical potentials. The dynamics include not only the EOM but also two-body scatterings, with the isotropic scattering cross section $\sigma_{22}$ determined by the specific shear viscosity $\eta/s = h/4\pi$ (see the appendix of Ref. [33] for details). The collision probability for a pair of particles with the energy $E_1$ and $E_2$ in a volume $(\Delta x)^3$ and a time interval $\Delta t$ is

$$P_{22} = \frac{v_{rel}\sigma_{22}}{(\Delta x)^3} \Delta t,$$

where $v_{rel} = s/(2E_1E_2)$ is the Möller velocity with $s$ being the square of the invariant mass of the particle pair. In the simulation, only particle pairs in the same cell with the volume $(\Delta x)^3 = 1 \text{ fm}^3$ can collide with each other, and the time step is set as $\Delta t = 0.01 \text{ fm}/c$. The particle momentum after each scattering is sampled isotropically in the center-of-mass frame of the collision pair. In the scenario of the SEOM, the expectation direction of the particle spin $\vec{\sigma}'$ after each scattering is determined by the energy conservation condition $c\vec{\sigma} \cdot \vec{k} = c\vec{\sigma}' \cdot \vec{k}'$ and the maximum polarization condition, i.e., assuming that the spin always tends to be polarized in the magnetic field so that $qeB \cdot \vec{\sigma}'$ should have the maximum value. Whether an attempted scattering can become a successful one is decided by the Pauli blocking probability $1 - (1 - f_1')(1 - f_2')$. The occupation probability $f_{1/2}'$ is $1/\{\exp[(k - q\mu_c)/T] + 1\}$ for the CEOM scenario, while for the SEOM scenario it is calculated by counting the phase-space occupation probability for particles with different charge numbers and spin states with respect to the magnetic field. The temperature $T = 300 \text{ MeV}$ and the chemical potential remain unchanged due to the unchanged energy distribution during the time evolution. There is no truncation needed for the SEOM, and a truncation $0.3 < \sqrt{G} < 1.7$ is used for the CEOM in calculating the electric charge current $J$.

We first look at the cosine angle distribution between $\vec{k}$ and $\vec{\sigma}$, which is defined as $\arccos[(\vec{\sigma} \cdot \vec{k})/|k|]$ in the SEOM scenario and $\arccos[(\vec{c\sigma'} \cdot \vec{k})/|\vec{k}|]$ in the CEOM scenario, in Fig. 1. Due to the initialization for the SEOM scenario, $\cos(\vec{\sigma} \cdot \vec{k})$ is $\pm 1$ in the initial stage, while the distribution evolves gradually away from $\pm 1$ and becomes equilibrated in the final stage. It is seen that without the collision process, the time evolution of the cosine angle distribution has some oscillation behavior towards equilibrium, compared with the scenario with the collision process where the oscillation is largely damped. For the CEOM scenario, the cosine angle distribution doesn’t change with time, and is broader compared with that from the SEOM. The strength of effects induced by chiral anomalies is closely related to the angle distribution between $\vec{k}$ and $\vec{\sigma}$.

FIG. 1: (Color online) Time evolution of the cosine angle distribution between $\vec{k}$ and $\vec{\sigma}$ from the SEOM without the collision process (solid), the SEOM with the collision process (dashed), and the CEOM with the collision process (dotted) under the magnetic field in $+y$ direction $eB_y = 0.5 \text{ GeV}/\text{fm}$.

FIG. 2: (Color online) Left: Time evolution of the electric charge current from the initial axial chemical potential $\mu_5 = 20 \text{ MeV}$ under different strength of the magnetic field for the SEOM scenario without (a) and with (b) the collision process; Right: Dependence of the equilibrated electric charge current on the strength of the magnetic field from the SEOM, with those from the CEOM and the theoretical limit (solid line) also plotted for comparison.

The left panels of Fig. 2 displays the time evolution of the electric currents $J$ with different strength of the magnetic fields from an axial chemical potential $\mu_5 = 20 \text{ MeV}$ for the SEOM scenario. Since the momentum and the spin are sampled according to an isotropic distribution, $J$ is zero in the initial stage. As the time evolves, the
electric charge current is induced. A stronger oscillation behavior is observed for the SEOM scenario without the collision process as shown in Panel (a) compared with that with the collision process as shown in Panel (b), consistent with that observed in Fig. 1. Although the oscillation is stronger under a stronger magnetic field, the equilibrated values of \( J \) are reached at about \( t = 4 \text{ fm}/c \). For the CEOM scenario, the value of \( J \) doesn’t change with time, as shown in Fig. 1 of Ref. [32]. As shown in the right panel, the equilibrated value of \( J \) increases almost linearly with the strength of the magnetic field, and the values are similar for the SEOM scenario with and without the collision process. It is seen that the equilibrated values from the SEOM are still lower than that from the CEOM with the truncation \( 0.3 < \sqrt{G} < 1.7 \), with the latter lower than the theoretical limit from Eq. (10) with \( \alpha = 1 \).

In order to study the situation closer to that in relativistic heavy-ion collisions, we have also done simulations under a damping external magnetic field parametrized as

\[
eB_y(t) = \frac{eB_0}{1 + (t/\tau)^2},
\]

where \( eB_0 = 0.5 \text{ GeV}/\text{fm} \) is the magnetic field at \( t = 0 \) and \( \tau \) characterizes its life time. Figure 3 displays the time evolution of the electric currents \( J \) with different life times of the magnetic field for different scenarios, i.e., with the spin initialized as \( \hat{\sigma} = c\hat{k} \) for the SEOM (a), with the initial distribution taken as the equilibrated \( \hat{\sigma} \) and \( \hat{k} \) distribution at \( t = 4 \text{ fm}/c \) under \( eB_0 \) (b), and for the CEOM (c). The collision process is incorporated in all scenarios. It was found that the electric charge current \( J \) at later stage is not so sensitive to the initialization in the SEOM scenario, partially due to the collision process that damps oscillations. Although the initial \( J \) is larger for the CEOM, it damps more quickly compared with that for the SEOM. This is due to the immediate change of the spin polarization with the magnetic field for the CEOM, while some relaxation process is needed for the SEOM, especially under a rapidly damping magnetic field.

![FIG. 3: (Color online) Time evolution of the electric charge currents with different damping external magnetic fields for the SEOM initialized with \( \hat{\sigma} = c\hat{k} \) (a), the SEOM initialized with an equilibrated distribution (b), and the CEOM (c).](image)

To study the CMW, we set the density distributions in the initial stage as

\[
\rho_{RL}(y, 0) = \pm \frac{1}{2} A_c n \sin (\beta y),
\]

with the upper (lower) sign for particles with the right-handed (left-handed) chirality. In the above equation, \( n \) is the total number density of all particles, \( \beta = \pi/l \) is the wave number, and \( A_c = 0.1 \) represents the maximum chiral asymmetry. Applying the periodic boundary condition \( \rho_{RL}|_{y=-l} = \rho_{RL}|_{y=l} \), its expression at time \( t \) and position \( y \) can be obtained from Eq. (21) as

\[
\rho_{RL}(y, t) \approx \pm \frac{1}{2} A_c n e^{-D_L \beta^2 t} \sin [\beta (y \mp v_p t)].
\]

The electric and axial charge density can be respectively expressed as

\[
\rho = \rho_R + \rho_L \approx -A_c n e^{-D_L \beta^2 t} \sin (\beta v_p t) \cos (\beta y),
\]

\[
\rho_5 = \rho_R - \rho_L \approx +A_c n e^{-D_L \beta^2 t} \cos (\beta v_p t) \sin (\beta y).
\]

The time evolutions of distributions of various densities shown in Fig. 4 are consistent with the above formulas, from the SEOM scenario shown by dashed lines and the CEOM scenario shown by solid lines, with the collision process incorporated. The phase velocity \( v_p \) and the diffusion constant \( D_L \) can be obtained by fitting the time evolution of the distribution with the above formulas, and their dependence on the strength of the magnetic field are shown in Fig. 4. It is seen that \( v_p \) from the CEOM is slightly higher than that from the SEOM due to a weaker CME in the later scenario, while it is still lower than the theoretical limit obtained from Eq. (22).
with $\alpha = 1$. It is noteworthy that the net current is instantaneously induced by the net density for the CEOM, while a relaxation process is needed for the SEOM. The diffusion constant from the SEOM slightly decreases with the increasing $eB_y$, and it is larger than that from the CEOM, with the former (latter) decreasing (increasing) slightly with the increasing $eB_y$. The decreasing trend of $D_L$ with the increasing $eB_y$ from the SEOM is consistent with that observed in Ref. [2].

\[ D_{22} = \int \rho(r) \left( 3y^2 - r^2 \right) d^3r. \] (29)

Using Eq. (27), the above integral can be carried out as

\[ D_{22} = \frac{4A \kappa N}{\beta^2} e^{-D_L \beta t} \sin(\beta v_p t), \] (30)

where $N$ is the total particle number. The upper panels of Fig. 5 display the time evolution of the electric quadrupole moment under different constant magnetic fields. It is seen that $D_{22}$ increases more slowly with the SEOM due to the larger diffusion constant $D_L$ compared with the CEOM. The general feature that $D_{22}$ increases with the increasing strength of the magnetic field is observed as expected, but it is of interest to see how is it like in a damping magnetic field. As shown in the lower panels of Fig. 6, $D_{22}$ increases faster for the CEOM than for the SEOM under the magnetic field with the life time $\tau = 4 \text{ fm/c}$, but for a fast damping magnetic field with $\tau = 0.4 \text{ fm/c}$ its final value can be larger for the SEOM than for CEOM, as a result of the spin relaxation process in the SEOM scenario.

![Graph](image_url)

**FIG. 5:** (Color online) Dependence of the phase velocity $v_p$ (a) and the diffusion constant $D_L$ (b) on the strength of the magnetic field from the SEOM and the CEOM scenarios, with the solid line representing the theoretical limit.

The elliptic flow splitting between particles with opposite charges originated from the electric quadrupole moment is the key observational consequence of the CMW. To quantify the electric quadrupole moment in the box system, as can be seen from the third row of Fig. 4, we define the electric quadrupole moment as

\[ N \langle \Delta r^2 \rangle = \frac{1}{4} \mathbf{M}_r^2 \] (18)

where $\mathbf{M}_r$ is the magnetic moment of the proton.

\[ eB_y \Delta r^2 = \frac{e B_y}{m} \int \rho(r) \left( 3y^2 - r^2 \right) d^3r. \] (19)

Using Eq. (27), the above integral can be carried out as

\[ D = \frac{4A \kappa N}{\beta^2} e^{-D_L \beta t} \sin(\beta v_p t). \] (30)

where $N$ is the total particle number. The upper panels of Fig. 6 display the time evolution of the electric quadrupole moment under different constant (upper) and damping (lower) magnetic fields from the SEOM (left) and the CEOM (right) scenarios.

![Graph](image_url)

**FIG. 6:** (Color online) Time evolution of the reduced electric quadrupole moment under different constant (upper) and damping (lower) magnetic fields from the SEOM (left) and the CEOM (right) scenarios.

To summarize, the chiral magnetic effect and chiral magnetic wave have been studied in a box system with the periodic boundary condition under a uniform external magnetic field with the spin kinetic equations of motion for massless particles. Although results are qualitatively similar compared with those from the chiral kinetic equations of motion, the spin dynamics leads to weaker chiral effects, while it is less sensitive to the fast damping of the magnetic field due to the spin relaxation process. It is of great interest to do transport simulations according to the spin kinetic equations of motion for relativistic heavy-ion collisions under a more realistic space-time evolution of the magnetic field in the future study.

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