Boundary Layer Development Process under the Propagation of a Solitary Wave

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Abstract. When a generated solitary wave starts to propagate, it needs an establishment distance to create fully developed velocity profiles. We consider that the solitary wave moving in a viscous fluid under laminar flow. An important detail to state is the proper location to measure the solitary wave velocity profile due to its finite extent. This article discusses the establishment distance and the reattachment point on the bottom in the wave’s deceleration phase.

1. Introduction
Previous studies on water wave theory mostly focused on the potential flow regime, where the viscous effect is usually neglected. The bottom boundary layer under wave motion plays an important role in sediment transport and morphological evolutions in coastal zones. Solitary waves are commonly assumed to represent long sea waves as they approach a shallow water region. Keulegan \cite{1} pioneered the study of viscous effect on waves by deriving a set of analytical solutions for the laminar boundary layer under a solitary wave; this subject was later extended to cnoidal waves by Tanaka et al. \cite{2}. Recently, Liu and Orfila \cite{3} rederived an analytical solution of the laminar boundary layer under a solitary wave. The method presented in \cite{3} was extended by Liu et al. \cite{4} to examine the boundary layer characteristics for fully nonlinear laminar flows. They also performed particle image velocimetry to explore the flow fields and bottom shear stresses under a solitary wave. The turbulent boundary layer flow under a solitary wave was examined by Liu \cite{5} using Reynolds-averaged equations assuming the eddy viscosity to be a power function of the distance from the bottom. Vittori and Blondeaux \cite{6}, \cite{7} performed direct numerical simulation to investigate the quantitative information on the flows generated in a region close to the bottom. Later, Blondeaux and Vittori \cite{8} applied the two-equation ($k$-$\omega$) formulations in the RANS model to study this problem.

Similar to the concept of the entrance length for flows that reach a fully developed status when entering a pipe, a propagating solitary wave from its initial generation also requires a certain distance, which is defined as the establishment distance, to appear as a solitary wave with a fully developed velocity profile. In this study, a stream function–vorticity equations-based model combined with the application of free surface conditions is employed to effectively generate the flow field in streamline patterns. For the initial developing period, we consider that the wave-induced fluid motion is within a laminar flow regime. We focus on the evaluation of the establishment length and the development of a reattachment point for a solitary wave that is propagating along a constant depth channel.
2. Numerical model

Consider a two-dimensional solitary wave advancing along a long, uniform water channel. Figure 1 illustrates a typical streamline sketch of the flow field under a solitary wave. A dividing line is separated from the free surface and then reattached at the bottom with a distance $L_R$ measured from the wave peak location. The shear stress and surface tension on the liquid–air interface are assumed to be negligible in the model. The non-dimensional quantities refer to the length scale $H^*$ and the velocity scale $\sqrt{gH}$, in which $H$ is the undisturbed water depth and $g$ is the gravitational constant. One can describe the well-posed initial-boundary-valued problem in an evolving curvilinear ($\xi, \eta$)-coordinate system transformed from the ($x, y$)-Cartesian coordinates. The stream function ($\psi$) and vorticity ($\omega$), i.e., $\psi$-$\omega$ formulations expressed in the general curvilinear coordinate system, are

\[ \nabla^2 \psi = -\omega, \] 

\[ \frac{\nabla^2 \omega}{Re} = \omega_x + U\omega_{\xi} + V\omega_{\eta}, \]  

where $Re$ in Equation (2) is defined as $H^*\sqrt{gH}/\nu$ (here, $\nu$ is the kinematic viscosity coefficient), $(U, V) = ((\psi_x - x_y y_x + y_x x_y)/J, (-\psi_y + x_y y_x - y_x x_y)/J)$ is the convective velocity of contravariant components in the time-evolving ($\xi, \eta$) grid system, and

\[ \nabla^2 = g^{11} \frac{\partial^2}{\partial \xi^2} + 2g^{12} \frac{\partial^2}{\partial \xi \partial \eta} + g^{22} \frac{\partial^2}{\partial \eta^2} + f^1(\frac{\partial}{\partial \xi}) + f^2(\frac{\partial}{\partial \eta}), \]  

The subscripts in the above equations denote the partial derivatives. Here, $J = x_{\xi} y_{\eta} - x_{\eta} y_{\xi}$ is the Jacobian of the coordinate transformation, and

\[ g^{11} = (x_{\eta}^2 + y_{\eta}^2)/J^2, \]  

\[ g^{12} = -(x_{\xi} x_{\eta} + y_{\xi} y_{\eta})/J^2, \]  

\[ g^{22} = (x_{\xi}^2 + y_{\xi}^2)/J^2, \]
\[ f^1 = \left( J g^{11} \right)_x + \left( J g^{12} \right)_y \bigg/ J, \]
\[ f^2 = \left( J g^{12} \right)_x + \left( J g^{22} \right)_y \bigg/ J. \]

\( \psi \) and \( \omega \) are determined by solving Equations (1) and (2) with the specified initial and boundary conditions, respectively.

### 2.1 Initial conditions

The initial wave form adopted for the simulation is the analytic solution derived by Grimshaw [9], which is given as

\[ \zeta = A_0 \text{sech}^2 \left[ 1 - \frac{3}{4} A_0 \text{tanh}^2 k + A_0^2 \left( \frac{5}{8} \text{tanh}^2 k - \frac{101}{80} \text{sech}^2 k \text{ tanh}^2 k \right) \right] \tag{4} \]

where \( k = \sqrt{\frac{-1}{4^4}} \left( 1 - \frac{1}{4} A_0^2 \right) x \). Here, \( x \) denotes the \( x \) coordinate of the initial wave peak. The free surface boundary value of the stream function is given as \( \psi_f = C \zeta \), where

\[ C = 1 + \frac{1}{2} A_0 - \frac{3}{4} A_0^2 + \frac{3}{8} A_0^3, \tag{5} \]

The vorticity values in the entire fluid region are initially set to zero.

### 2.2 Boundary conditions

On the free surface as defined by \( y = \zeta(x, t) \), the dynamic and kinematic boundary conditions must be satisfied. The dynamic boundary condition means that the atmospheric pressure is uniform. Therefore, for 2D flow, the total derivative of pressure along the free surface line (\( \eta = \eta_{\text{max}} \)) is zero and can be expressed with the terms of the stream function and vorticity in the transformed coordinate system, which is shown below.

\[ \psi_x \left( J g^{12} \right) + \psi_y \left( J g^{22} \right) + \psi_x \tilde{A} + \psi_y \tilde{B} + (u-x_{\zeta}) \psi_x + (v-x_{\zeta}) \psi_y + \Omega \left[ (u-x_{\zeta}) x_{\zeta} - (v-x_{\zeta}) x_{\zeta} \right] + \frac{J}{Re} (g^{12} \Omega_{\zeta} + g^{22} \Omega_y) = 0 \tag{6} \]

where \( Re \) in Equation (5) is defined as \( H \sqrt{\rho \Omega / \nu} \) (here, \( \nu \) is the kinematic viscosity coefficient), and

\[ \tilde{A} = \left( -\frac{x_{\zeta}}{\tilde{f}}, x_{\zeta} - \frac{\zeta_{\zeta}}{\tilde{f}}, y_{\zeta} \right), \quad \tilde{B} = \left( \frac{x_{\zeta}}{\tilde{f}}, x_{\zeta} + \frac{\zeta_{\zeta}}{\tilde{f}}, y_{\zeta} \right). \]

The formulation of the kinematic boundary condition on the free surface is given as

\[ \psi_{\zeta} + \zeta_{\zeta} x_{\zeta} = x_{\zeta} y_{\zeta}. \tag{7} \]

In this study, a fixed and rigid bottom is assumed. A constant streamfunction is applied on the bottom, i.e., \( \psi_{\zeta} = 0 \). The vorticity on the horizontal bottom can be arranged as

\[ \omega_0 = -2g^{22} \psi_1 \tag{8} \]

A simple wave equation is chosen as the open boundary condition to radiate waves and all physical variables out of the left and right boundaries. The radiation boundary conditions are expressed as

\[ \partial_s \pm \sqrt{1 + \zeta \partial_s / x_{\zeta}} = 0, \tag{9} \]

where \( \partial_s \) is a dummy variable that can represent \( \psi, \omega \), and \( \zeta \); the “+” sign represents the downstream boundary; and the “−” sign denotes the upstream boundary.

### 2.3 Grid treatment and discretization
The unsteady free surface suggests the need to use a transient boundary-conformed grid system in the region of fluid motion. The range of the whole area in the horizontal direction is \(x = [-30, 50]\). A typical case of the grid structure for a solitary wave with \(A_0 = 0.4\) is shown in Figure 2, in which only the local range is shown. However, 484 \times 158 grid nodes are actually used in the whole computational domain. In this study, we simply distribute the vertical grid exponentially from the bottom upward, thereby increasing the vertical spacing. The minimum grid spacing in the \(y\)-direction is 0.003. Meanwhile, the horizontal grid lines are evenly distributed to lessen errors caused by the grid’s non-uniformity.

![Figure 2. Grid distribution for a solitary wave calculation](image)

The finite analysis method is employed to solve Equations (1) and (2) analytically in the local element to obtain the algebraic equations. This discretization consistently improves the solution accuracy and the scheme stability [10]. For a small cell, those geometric coefficients are updated iteratively with the free surface evolution for an instantaneous grid system.

3. Results

This article first shows the typical streamline for a solitary wave propagating from left to right along a uniform depth channel in Figure 3, which shows the streamline and velocity vector of the solitary wave moving toward the positive \(x\)-direction. The figure indicates that the velocity vector of the leading edge and that of the trailing edge moves to the upper right and to the lower right, respectively. The vertical velocity and horizontal velocity distributions are correspondingly plotted in Figures 4 and 5. The vertical velocity shown in Figure 4 shows a pattern of approximately symmetrical left and right sides at the peak, which is zero at the peak, positive at the right half, and negative at the left. The vertical velocity near the bottom is very weak. Figure 5 shows that the horizontal velocity is approximately vertically distributed, which is also a characteristic of shallow water waves. Therefore, horizontal velocity is treated as a depth-averaged quantity in the theory of shallow water waves. The peak and the area near the bottom of the bed are more inconsistent with theoretical assumptions.
Figure 3. Instant streamline distribution and velocity vectors for solitary wave propagation

Figure 4. Instant vertical velocity distribution for solitary wave propagation
Figure 5. Instant horizontal velocity distribution for solitary wave propagation

Figure 6. Comparisons of three horizontal velocity profiles in a flat-bottom boundary layer.

Liu et al. [4] studied the laminar bottom boundary layer flow induced by a solitary wave. Their typical Reynolds number in the experiments was about $10^5$. Given the relatively short action time for a solitary wave passing by a fixed point on the bottom, a much higher value of $Re$ is required for an unsteady flow to induce the fluid from its initial rest state to develop a motion of turbulence. Here, we consider only the laminar flow analysis. Figure 6 compares the present calculated horizontal velocity...
profiles with Liu’s measured data and their numerical results at phase \((P = t' = 0.83)\) for a solitary wave with wave height \(A_0 = 0.2\). The \(t'\) denotes the time to shift to the peak time occurred. Excellent agreement was observed between the presented results, thus indicating that the flows above the boundary layer always move in the same direction as the wave movement.

Figure 7. Horizontal velocity plotted versus the vertical coordinate at different phases and the development of establishment length for \(A_0 = 0.2\).

Figure 8. Reattachment lengths \((L_R)\) for different incident wave heights \((A_0)\): (a) variation of \(L_R\) versus time for various \(A_0\)'s; (b) fully developed \(L_R\) versus various \(A_0\)'s.

Figure 7 shows the horizontal velocity profiles at different phases \((t = 0.5 \text{ to } 7.5)\) and the development of the boundary layer for incident wave height \(A_0 = 0.2\). The boundary layer thickness is determined at the position where the flow velocity essentially reached 99% of the free-stream velocity. The
establishment length is about 6.5 (at the phase $t = 6.0$). Figure 8(a) shows the variation of reattachment length ($L_R$) versus the incident wave height ($A_0$). The $L_R$ for cases with various values of $A_0$ (0.2–0.6) decreases rapidly to an almost steady value as the time increases. A trend for the steady $L_R$ versus $A_0$ is plotted in Figure 8(b). A weak $A_0$ reflects a large $L_R$; this condition is related to the effective wave length for a solitary wave.

4. Conclusion
This study found that the establishment length of a solitary wave with $A_0 = 0.2$ is about $x = 6.5$, although it varies slightly among the cases with various incident wave heights. In other words, the measurement velocity profile in the laboratory is best measured in a region that is more than 6.5 times the still water depth after the wave is generated. In addition, we find that the reattachment length is shorter for a larger solitary wave.

5. References
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