PARABOLIC PROBLEM WITH FRACTIONAL TIME DERIVATIVE WITH NONLOCAL AND NONSINGULAR MITTAG-LEFFLER KERNEL

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Abstract. We prove Hölder regularity results for nonlinear parabolic problem with fractional-time derivative with nonlocal and Mittag-Leffler nonsingular kernel. Existence of weak solutions via approximating solutions is proved. Moreover, Hölder continuity of viscosity solutions is obtained.

1. Introduction. In this work we prove some Hölder regularity results for viscosity solutions of integro-differential equations in which the nonsingular kernel defining the fractional time operators is in terms of the Mittag-Leffler function. The spatial nonlocal operator kernel corresponds to the one for fractional Laplacian. One of our aims is to provide an analogous result using different type of fractional derivative of the previous results [1, 3] where the fractional time derivative in the sense of Caputo was used. In this sense, we would like to notice that we get similar results as those obtained in the aforementioned references.

The specific equation which is of interest is

\[ L u(t,x) - J u(t,x) = g(t,x), \quad \text{in } (-\infty, T) \times \mathbb{R}^n. \]  

(1)

The simplest example of the operators we study is

\[ J u(x) = \int_{\mathbb{R}^n} \delta_{h} u(x) K(x,h) dh, \]  

(2)

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where $\delta_h u(x) = u(x+h) + u(x-h) - 2u(x)$ denotes the central second order difference relation. For $\sigma \in (0, 2)$ the kernel
\[
K(x, h) = C(n, \sigma) |h|^{-n-\sigma},
\]
gives $\mathcal{J} = (-\Delta)^{\sigma/2}$ for an appropriate constant $C(n, \sigma)$.

We consider the kernel $K$ to be the one which allows to have large regions and comparable to $|h|^{-n-\sigma}$ from below. This kernel is treated without assuming any regularity in the $x$ variable through the assumption that it needs only to be above $|h|^{-n-\sigma}$ on a possibly small set as
\[
\lambda |h|^{n+\sigma} \leq K(x, h) \leq \Lambda |h|^{n+\sigma}, \quad h \in \mathbb{R}^n \setminus \{0\}. \quad (3)
\]

Furthermore, for $\alpha \in (0, 1)$ for all $t < a$ and for $n \in \mathbb{N}$, the fractional time derivative with nonsingular Mittag-Leffler kernel $\mathcal{L}$ involved in (1) and defined in [5] could be written as
\[
\mathcal{L}u(t) = C(n, \alpha) \int_{-\infty}^t \left[ u(t) - u(s) \right] \mathcal{T}(t, s)ds,
\]
where the kernel $\mathcal{T}(t, s) = (t-s)^{\alpha-1}E_{\alpha,\alpha} \left[ c(t-s)^{\alpha} \right]$ belongs to the class of kernels $\mathcal{T}_{sec}$ described by
\[
\mathcal{T}_{sec} = \left\{ \mathcal{T} : (-\infty, T) \to \mathbb{R} : \mathcal{T}(t, t-s) = \mathcal{T}(t+s, t), \quad -c\lambda \frac{(t-s)^{-\alpha-1}}{\Gamma(\alpha+1)} \leq \mathcal{T}(t, s) \leq -c\Lambda \frac{(t-s)^{-\alpha-1}}{\Gamma(\alpha+1)} \right\}. \quad (5)
\]

One could notice that this class of kernels contains a number of specific kernels such as the ones involved in Marchaud derivative [15], Caputo derivative [1, 4, 15, 19] and the fractional derivative with nonlocal and nonsingular Mittag-Leffler kernel [5]: if we consider the fractional time derivative in the sense of Caputo, defined as
\[
^cD^\alpha_a u(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \left[ u(t) - u(s) \right] (t-s)^{-\alpha-1}ds,
\]
the kernel $(t-s)^{-\alpha-1}$ belongs the class of kernels $\mathcal{T}_{sec}$. We recall that the case of the Caputo derivative has been used in [1, 2] to prove Hölder regularity of some parabolic problems in the non-divergence form. Further properties of the fractional time derivative will be given in Section 3.

Besides the mathematical satisfactions of the Caputo fractional derivative, the attentiveness for the derivative with nonlocal and nonsingular Mittag-Leffler kernel is based on its properties of portraying the behavior of orthodox viscoelastic materials, thermal medium, material heterogeneities and some structure or media with different scales [5]. The nonlocality of the new kernel allows a good description of the memory within structure and media with different scale, which cannot be described by classical fractional derivative. This derivative also takes into account power and exponential decay laws. The one with the Mittag-Leffler functions allows us to describe phenomena in processes that progress or decay too slowly to be represented by classical functions like the exponential function and its successors. The Mittag-Leffler function arises naturally in the solution of fractional integral equations, and especially in the study of the fractional generalization of the kinetic equation, random walks, Lévy flights, and so called super-diffusive transport.
Remark 1. There are various definitions—Riemann-Liouville, Caputo, Grunwald-Letnikov, Marchaud, Weyl, Riesz, Feller, and others—for fractional derivatives and integrals, (see e.g., [10, 11, 14, 15, 19] and references therein). This diversity of definitions is due to the fact that fractional operators take different kernel representations. So instead of studying the parabolic problem for different kernels in $T_{sec}$, we propose to study the problem for the time-derivative with nonsingular Mittag-Leffler kernel $T$ satisfying the properties given by (5).

In order to analyze the regularity of solutions to these nonlocal equations involving the fractional time derivative and the nonlocal spatial operator, some authors as [2, 13, 16, 19] studied the problem of Hölder continuity for solutions to master equations and Hölder continuity for parabolic equations with Riemann-Liouville derivative and the Caputo fractional time derivative and divergence form nonlocal operator. Furthermore very recently the author in [1] proves the Hölder continuity of viscosity solutions to (1) in the non divergence form, but using the generalized fractional time derivative of Marchaud or Caputo type under the appropriate assumptions. They obtained an estimate which remains uniform as the order of the fractional derivative $\alpha \to 1$. Moreover, maximum principles and extension theorems, and a quite unified approach to fractional time derivatives are presented in [6] and regularity theory for the fractional powers of the heat operator has been studied in [18].

With the clear intention to obtain a similar result as in [1, 2] where the generalized Marchaud or Caputo derivative was used to show the Hölder continuity of viscosity solutions to (1) in the non divergence form, but using the generalized fractional time derivative of Marchaud or Caputo type under the appropriate assumptions. They obtained an estimate which remains uniform as the order of the fractional derivative $\alpha \to 1$. Moreover, maximum principles and extension theorems, and a quite unified approach to fractional time derivatives are presented in [6] and regularity theory for the fractional powers of the heat operator has been studied in [18].

Remark 2. Despite the fact that we obtain similar results as those already presented in the literature, we consider that this approach with the kernel involving the Mittag-Leffler function is an additional contribution to the field.

To study regularity properties of solutions to equation (1), one could in some sense study the solution $u$ which simultaneously solve the two inequalities

$$\inf_{T} \left\{ L_T u(t,x) - J u(t,x) \right\} \leq C \quad \text{and}$$

$$\sup_{T} \left\{ L_T u(t,x) - J u(t,x) \right\} \geq -C \quad \text{in } (\infty, T) \times \Omega.$$

The problem of studying existence of solutions and regularity properties of parabolic problem with fractional nonlocal space-time operators such as (1) was presented in [1, 2, 19], using respectively Riemann fractional derivative and Caputo fractional derivative. We propose to study similar problem, but now with fractional time-derivative with nonsingular Mittag-Leffler kernel. Our main results are existence of weak solution and Hölder regularity estimate.

**Theorem 1.1 (Existence of weak solutions).** Let $\vartheta$ be a bounded and Lipschitz function on $(-\infty, T) \times \mathbb{R}^n$ and assume that the function $g$ is regular enough. So for a given smooth bounded initial data $u_0 > 0$, there exists a weak solution $u$ to the weak formulation

$$c_\alpha \int_{\mathbb{R}^n} \int_{-\infty}^{T} \left[ u(t,x) - u(s,x) \right] \left[ \vartheta(t,x) - \vartheta(s,x) \right] T(t,s,x) ds dt dx$$

bounded continuous viscosity solution in $B^\beta$ constants defined in (H"older Regularity) Theorem 1.2 then $u$ is H"older continuous in $B^1 \times [-1,0]$ and for $(x,t), (y,s) \in B^1 \times [-1,0]$ the following estimate holds
\[ |u(x,t) - u(y,s)| \leq C(||u||_{L^\infty} + \varepsilon_0^{-1}||g||_{L^\infty})|x-y|^{\alpha} + |t-s|^{\alpha/2} \] (7)
Furthermore $C$ remains bounded as $\alpha \to 1$.

**Remark 3.** Notice that in order to get solutions to Theorem 1.2, the solution of fractional differential equations involving the fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel is proposed so that it could be used as a test function for viscosity solutions. The case with the Caputo derivative was well presented in [1] showing that if $||x \in B^1 \times (-2,-1) : u(x,t) \leq 0|| \geq \mu_1$, then $u(t) \geq \mu_2$ if $t \in (-1,0)$.

We omit the proof for the fractional time derivative with nonlocal and non-singular Mittag-Leffler kernel since the idea to get the result is similar.

So with the H"older continuity estimates for ordinary differential equations involving the Caputo derivative [1], our class of weak solutions will be considered in the viscosity sense as described in Section 5. This will then allow us to get the similar result.

The organization of the article is as follows. In Section 2 we review some background related to Theorem 1.2. In Section 3 we collect notation, definitions, and preliminary results regarding (Theorem 1.1) and Theorem 1.2. Section 4 is dedicated to the sketch of proof of the existence of weak solutions via approximating solutions, mainly Theorem 1.1. Finally in Section 5 we discuss the pointwise estimates and put together the remaining pieces of the proof for the pointwise estimate and H"older Regularity.

2. **Background.** There are few collection of results related to Theorem 1.2 with both space and time fractional nonlocal operators. We will focus on the type of results which only depend on the ellipticity constants, $\lambda$ and $\Lambda$, for the spatial nonlocal operator as well as possibly the orders, $\alpha$, $\sigma$ and we shall try to see if we recover the results of Mark Allen et al. [1, 2, 3] by using fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel.

The problem of regularity of parabolic problem with fractional time derivative are new in the literature. As we can notice in the recent article [7], where the authors used the original method of De Giorgi to prove boundedness of solutions
and local Hölder regularity. Similar approach to prove apriori local Hölder estimates of solutions to the fractional parabolic type equation has also been used in [3], where they have followed the De Giorgi method as in [7] but now by taking into account the fractional time derivative in the sense of Caputo [15]. As an earlier result, the use of the fractional nature of the derivative made the estimates to not remain uniform as the fractional order $\alpha \to 1$.

In the same direction, we shall also bring the attention of the reader on the fact that similar result has been studied by Zacher in [19] but instead of using Caputo derivative the author used the Riemann-Liouville fractional time derivative and zero right hand side.

One should notice that these results based on the fractional space and time were obtained when both kernels of these operators are bounded. In the case of the fractional spatial nonlocal operator, there are a few interesting distinctions that are usually made: whether or not $K(x, h)$ is assumed to be even in $h$; whether or not the corresponding equations are linear; whether or not a Harnack inequality holds [12].

Regularity results (such as Theorem 1.2) as well as the Harnack inequality for linear equations with operators similar to (2) were obtained [12, 17]. Furthermore in [1] Hölder continuity of viscosity solutions to certain nonlocal parabolic equations that involve a generalized fractional time derivative of Marchaud or Caputo type as well is obtained under the assumption that kernel of the fractional time operator satisfied the symmetry condition $T(t, t-s) = T(t+s, t)$. The estimates are uniform as the order of the operator $\alpha$ approaches 1, so that the results recover many of the regularity results for local parabolic problem.

Finally, higher regularity in time type estimates were obtained in [3].

3. Preliminaries.

3.1. Notation. We first collect some notations which will be used throughout this article.

- $\mathcal{L}$ - the nonlocal fractional time-derivative with nonsingular Mittag-Leffler kernel.
- $\sigma \in (0, 2)$ - denote the order of the nonlocal spatial operator.
- $\alpha$ - will always denote the order of the space-time arbitrary order derivative.

\[ A_{sec} = \left\{ \mathcal{K} : \mathbb{R}^n \to \mathbb{R} : \mathcal{K}(-h) = \mathcal{K}(h), \quad \text{and} \quad \frac{\lambda}{|h|^{n+\sigma}} \leq \mathcal{K}(h) \leq \frac{\Lambda}{|h|^{n+\sigma}} \right\}, \quad (8) \]

\[ \delta_h u(x) = u(x + h) + u(x - h) - 2u(x), \quad (9) \]

\[ K_A u(x) = \int_{\mathbb{R}^n} \delta_h u(x) \mathcal{K}(h) dh, \]

\[ \mu(dh) = |h|^{-n-2\sigma} dh, \]

\[ Q_\varepsilon(x_0) = \left\{ x \in \mathbb{R}^n : |x - x_0|_\infty < \frac{\varepsilon}{2} \right\}, \]

\[ B_\varepsilon(x_0) = \left\{ x \in \mathbb{R}^n : |x - x_0| < \varepsilon \right\}. \]

We use $|\cdot|$ for the absolute value, the Euclidean norm, and the $n$-dimensional Lebesgue measure at the same time. Throughout this article $\Omega \subset \mathbb{R}^n$ denotes a bounded domain. For cubes and balls such that $x_0 = 0$ we write $Q_t$ instead of $Q_t(0)$ and similarly for $B_t$. It yields

$B_{1/2} \subset Q_1 \subset Q_3 \subset B_3 \subset B_2$. 

3.2. Definitions. We use the definitions and basic properties of the fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel and of viscosity solutions from [5] and [1], and for Hölder continuity [2, 3, 8, 17].

3.3. The fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel. In this section for the convenience of the reader, we recall some definitions of fractional time derivative with the nonlocal and nonsingular kernel as stated in [5] and state some of its new properties.

The fractional time derivative with nonlocal and Mittag-Leffler nonsingular kernel, recently introduced by Atangana and Baleanu and known as the Atangana-Baleanu derivative is useful in modelling equations arising in porous media. One formulation of the Atangana-Baleanu derivative is

\[ a^{BC}D_a^\alpha u(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_\alpha \left[ -c(t-s)^\alpha \right] u'(s)ds. \]  

(10)

The associate integral of the fractional time derivative with nonlocal and Mittag-Leffler nonsingular kernel, is defined as

\[ A^B I_t^\alpha u(t) = \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_t^a u(y)(t-y)^{\alpha-1}dy. \]  

(11)

In the above formulas \( B(\alpha) \) is a constant depending on \( \alpha \) such that \( B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \), \( c = -\frac{\alpha}{1-\alpha} \), and \( E_{\alpha,\beta}(z) \) is the two-parametric Mittag-Leffler function defined in terms of a series as the following entire function as

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \]  

and \( \beta > 0, \ z \in \mathbb{C}, \ E_\alpha(z) = E_{\alpha,1}(z) \).

The other representation of the Atangana-Baleanu fractional time derivative in the sense of Caputo holds point wise

\[ a^{BC}D_a^\alpha u(t) = \nu_\alpha E_\alpha \left[ -c(t-a)^\alpha \right] [u(t) - u(a)] + c_\alpha \int_a^t (t-s)^{\alpha-1}E_{\alpha,\alpha} \left[ -c(t-s)^\alpha \right] [u(t) - u(s)]ds, \]  

(12)

where \( \nu_\alpha = B(\alpha)(1-\alpha)^{-1} \) and \( c_\alpha = -c\nu_\alpha \). We set

\[ \mathcal{T}(t,s) = (t-s)^{\alpha-1}E_{\alpha,\alpha} \left[ -c(t-s)^\alpha \right], \]  

(13)

as our bounded kernel in time, satisfying the relation \( \mathcal{T}(t,t-s) = \mathcal{T}(t+s,t) \) and

\[ -c\lambda(t-s)^{\alpha-1} \leq \mathcal{T}(t,s) \leq -c\lambda(t-s)^{\alpha-1} \Gamma(\alpha + 1). \]  

(14)

In this setting, following the idea of [2, 3] we define \( u(t) = u(a) \) for \( t < a \),

\[ \mathcal{L}u(t) := c_\alpha \int_{-\infty}^t [u(t) - u(s)] \mathcal{K}(t,s) \ ds. \]  

(15)

One of the immediate consequence of the formulation (15) is that, it allows to drop out data and it is also useful for viscosity solutions [2].

The notion of viscosity solution and supersolutions for the initial values problem involving the time derivative with the non singular Mittag-Leffler kernel in (15) is
similar to the one of (1). For this purpose we state the following Proposition which shows that (15) is well defined.

**Proposition 1.** Let $u$ a continuous bounded function and $w \in C^{0,\beta}$ with $\alpha < \beta \leq 1$. If $w \geq (\leq)u$ on $[t_0 - \varepsilon, t_0]$ and $w(t_0) = u(t_0)$, then the integral

$$c_\alpha \int_{-\infty}^{t_0} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds$$

is well defined, so that $\mathcal{L}u(t_0)$ is well defined.

**Proof.** Let us assume that $w \geq u$ on $[t_0 - \varepsilon, t_0]$. From (15)

$$c_\alpha \int_{-\infty}^{t_0} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds =$$

$$c_\alpha \int_{-\infty}^{t_0 - \varepsilon} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds + c_\alpha \int_{t_0 - \varepsilon}^{t_0} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds$$

$$\geq c_\alpha \int_{-\infty}^{t_0 - \varepsilon} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds + c_\alpha \int_{t_0 - \varepsilon}^{t_0} [w(t_0) - w(s)] \mathcal{T}(t, s) \, ds$$

$$\geq c_\alpha \int_{-\infty}^{t_0 - \varepsilon} [u(t_0) - u(s)] \mathcal{T}(t, s) \, ds - cc_\alpha \Lambda \|w\|_{C^{0,\beta}} \int_{t_0 - \varepsilon}^{t_0} \left( t_0 - s \right)^{-\alpha + \beta - 1} \frac{\Gamma(1 + \alpha)}{\Gamma(1 + (\alpha + \beta))} \, ds$$

$$\geq c_\alpha \frac{e^{\Lambda \varepsilon^{-\alpha + \beta}}}{(\alpha + \beta) \Gamma(1 + \alpha)} \left( 2 \|u\|_{L^\infty} + \|w\|_{C^{0,\beta}} \right)$$

$$\geq \tilde{A}_{\alpha, \beta}.$$ 

Hence the integral is well defined. $\square$

Next we state the following estimate of bound of the fractional time derivative $\mathcal{L}g(t)$ that will be useful for the proof of H"older continuity in Section 5.

**Proposition 2.** Let us consider $g(t) = \max\{2|\nu t|^{\nu} - 1, 0\}$ under the condition that $\nu < \alpha$, and $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$.

If $t_1 \leq 0$ then

$$-d_{\alpha, \nu} \leq \mathcal{L}g(t_1) \leq 0$$

where the constant $d_{\alpha, \nu}$ depends on $\alpha$ and $\nu$.

**Proof.** From (10) and (15) the rescaled fractional time derivative takes the form

$$-\nu \int_{-\infty}^{t} E_\alpha [c(t-s)^{\alpha}] |rs|^{\nu-1} \, ds \geq -\nu \int_{-\infty}^{t} E_\alpha [c(t-s)^{\alpha}] |rs|^{\nu-1} \, ds.$$ 

One notice that $|rs|^{\nu-1}$ and the Mittag-Leffler function $E_\alpha [c(t-s)^{\alpha}]$ are increasing function of $s$, if $s < 0$, so

$$-\nu \int_{-\infty}^{t} E_\alpha [c(t-s)^{\alpha}] |rs|^{\nu-1} \, ds$$

is an increasing function of $t$.

Furthermore, if $t \leq -1$, then it follows that

$$\mathcal{L}g \geq -\nu \int_{-\infty}^{t} E_\alpha [c(t-s)^{\alpha}] |rs|^{\nu-1} \, ds \geq -\nu \int_{-\infty}^{-1} E_\alpha [c(-1-s)^{\alpha}] |rs|^{\nu-1} \, ds \geq -d_{\alpha, \nu}.$$
Now if \( t > -1 \), then
\[ Lq \geq -\nu \int_{-\infty}^{-1} E_{\alpha} [c(t-s)^{\alpha}] |rs|^{\nu-1} ds \geq -\nu \int_{-\infty}^{-1} E_{\alpha} [c(-1-s)^{\alpha}] |rs|^{\nu-1} ds \geq -d_{\alpha,\nu}. \]

Next, in order to get information over the various time, we are interested on the solution to the fractional differential equation involving the Atangana-Baleanu fractional derivative, in the form
\[ Lu(t) = -c_1 u(t) + c_0 h(t), \quad c_1, \ c_0 < \infty. \tag{16} \]

**Proposition 3.** Let \( u \in H^1(0,T), \ T > 0, \) and \( h \in \mathcal{C}^2, \) such that the Atangana-Baleanu fractional derivative exists. Then, the solution of differential equation (16), for \( c_1 = 0, \) is given by
\[ u(t) = u(0) + \frac{(1-\alpha)c_0}{B(\alpha)} h(t) + \frac{\alpha c_0}{B(\alpha) \Gamma(\alpha)} \int_0^t h(s)(t-s)^{\alpha-1} ds, \tag{17} \]
and for \( c_1 \neq 0 \) by
\[ u(t) = \zeta E_{\alpha} [-\gamma t^\alpha] u(0) + c_0 \zeta \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t E_{\alpha,\alpha} [-\gamma(t-s)^\alpha] (t-s)^{\alpha-1} h(s) ds \]
\[ + \frac{(1-\alpha)}{B(\alpha)} \left( h(t) - \gamma \int_0^t E_{\alpha,\alpha} [-\gamma(t-s)^\alpha] (t-s)^{\alpha-1} h(s) ds \right), \tag{18} \]
with \( \gamma = \frac{\alpha c_1}{(B(\alpha)+(1-\alpha)c_1)} \) and \( \zeta = \frac{B(\alpha)}{(B(\alpha)+\alpha c_1)}. \)

**Proof.** For \( c_1 = 0, \) we get that
\[ u(t) = u(0) + \frac{1-\alpha}{B(\alpha)} h(t) + \frac{\alpha c_0}{B(\alpha) \Gamma(\alpha)} \int_0^t h(s)(t-s)^{\alpha-1} ds. \]

We consider the proof of Proposition 3 for \( c_1 \neq 0 \) by using (11) and by applying the Laplace transform on (16). It comes that
\[ A^B I^\alpha_t \{ a B \hat{\alpha} C_{\alpha} u(t) \} = A^B I^\alpha_t \{ -c_1 u(t) + c_0 h(t) \} \]
\[ u(t) - u(0) = -c_1 \left\{ \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_0^t u(s)(t-s)^{\alpha-1} ds \right\} \quad \tag{19} \]

If we apply the Laplace transform to both sides of (19), it yields
\[ \left[ \frac{B(\alpha) + (1-\alpha)c_1}{B(\alpha)} + \frac{\alpha c_1}{B(\alpha) p^\alpha} \right] u(p) \]
\[ = \frac{1}{p} u(0) + c_0 \left( \frac{1-\alpha}{B(\alpha)} h(p) + \frac{\alpha c_0}{B(\alpha) \Gamma(\alpha)} \Gamma(\alpha) p^{-\alpha} h(p) \right) \]
\[ \frac{1}{B(\alpha)} \left[ p^\alpha \left( \frac{B(\alpha) + (1-\alpha)c_1}{B(\alpha)} + \frac{\alpha c_1}{B(\alpha) p^\alpha} \right) \right] u(p) \]
\[ = \frac{1}{p} u(0) + \frac{(1-\alpha)c_0}{B(\alpha)} h(p) + \frac{\alpha c_0}{B(\alpha) p^{-\alpha} h(p)}. \]
if we set

\[ \gamma = \frac{\alpha c_1}{(B(\alpha) + (1 - \alpha)c_1)} \quad \text{and} \quad \zeta = \frac{B(\alpha)}{(B(\alpha) + (1 - \alpha)c_1)}. \]

it comes that

\[
\begin{align*}
    u(p) &= \zeta \frac{p^{\alpha-1}}{p^\alpha + \gamma} u(0) + \frac{c_0 \zeta (1 - \alpha)}{B(\alpha)} \frac{p^\alpha}{p^\alpha + \gamma} h(p) + \frac{\alpha c_0 \zeta}{B(\alpha)} \frac{1}{p^\alpha + \gamma} h(p) \\
    &= \zeta \frac{p^{\alpha-1}}{p^\alpha + \gamma} u(0) + \frac{c_0 (1 - \alpha)}{B(\alpha)} \frac{1}{1 + (\gamma^{-\alpha} p)^{-\alpha}} h(p) + \frac{\alpha c_0 \zeta}{B(\alpha)} \frac{1}{p^\alpha + \gamma} h(p) \quad (20)
\end{align*}
\]

with \( d = \gamma^{-1/\alpha} \). We notice that

\[
\frac{1}{1 + (\gamma^{-\alpha} p)^{-\alpha}} = \sum_{k=1}^{\infty} (-1)^k d^{\alpha k} t^{\alpha k-1}/\Gamma(\alpha k).
\]

Hence by applying the inverse Laplace transform, we get the result as

\[
\begin{align*}
    u(t) &= \zeta E_\alpha \left[ -\gamma t^\alpha \right] u(0) + c_0 \zeta \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t E_{\alpha,\alpha} \left[ -\gamma (t-s)^\alpha \right] (t-s)^{\alpha-1} h(s) ds \\
    & \quad + \frac{(1 - \alpha)}{B(\alpha)} \left( h(t) - \gamma \int_0^t E_{\alpha,\alpha} \left[ -\gamma (t-s)^\alpha \right] (t-s)^{\alpha-1} h(s) ds \right).
\end{align*}
\]

\( \square \)

**Corollary 1.** Let \( g : [-2, 0] \to \mathbb{R} \) be a solution to \( ^{abc}D_{t}^{\alpha} g = -c_1 g + c_0 h(t) \) with \( g(-2) = 0, \ h \geq 0 \) and \( \int_{-2}^0 h(t) dt \geq \mu. \) Then

\[ g(t) \geq \frac{\alpha}{2} E_{\alpha,\alpha} \left[ -2 c_1 \right] c_0 \mu, \quad \text{for} \quad -1 \leq t \leq 0. \]

**Proof.** From Proposition 3 the solution of the differential equation for \( g \) can be computed explicitly

\[
\begin{align*}
    g(t) &= \zeta E_\alpha \left[ -\gamma t^\alpha \right] g(-2) + c_0 \zeta \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{-2}^t E_{\alpha,\alpha} \left[ -\gamma (t-s)^\alpha \right] (t-s)^{\alpha-1} h(s) ds \\
    & \quad + \frac{(1 - \alpha)}{B(\alpha)} \left( h(t) - \gamma \int_{-2}^t E_{\alpha,\alpha} \left[ -\gamma (t-s)^\alpha \right] (t-s)^{\alpha-1} h(s) ds \right).
\end{align*}
\]
If \( c_0 \) is small enough and \( c_1 \) is large, with the initial condition \( g(-2) = 0 \), and the fact that \( E_{\alpha,\alpha}(t) > 0 \), we have
\[
g(t) \geq \frac{\alpha c_0 \zeta}{2B(\alpha)} E_{\alpha,\alpha}([-2\gamma \int_{-2}^{t} h(s) ds \geq \frac{\alpha c_0 \zeta}{2B(\alpha)} E_{\alpha,\alpha}([-2c_1]c_0 \mu \\
\geq \frac{\alpha}{2} E_{\alpha,\alpha}([-2c_1]c_0 \mu.)
\]

4. **Existence of weak solutions via approximating solutions.** In this section we provide details of the proof of existence of a solution to the weak equation (1) via approximating solutions. We follow the idea of [3] to prove our result. To do this, we shall start with the weak formulation of the problem, then provide the discretization of the weak formulation that will enable us later to prove the existence of the unique solution and Hölder continuity.

We state the following integration by parts which will be used to prove weak formulation.

**Proposition 4.** Assume that \( \vartheta \) is bounded and Lipschitz function on \((a, T)\), then
\[
\int_{a}^{T} \vartheta(t) \mathcal{L}u(t) dt = c_{\alpha} \int_{a}^{T} u(t) \left( \int_{a}^{t} \mathcal{T}(t, s) [\vartheta(t) - \vartheta(s)] ds \right) dt \\
+ c_{\alpha} \int_{a}^{T} \int_{a}^{t} \mathcal{T}(t, s) [u(t) - u(s)] [\vartheta(t) - \vartheta(s)] ds dt \\
- \int_{a}^{T} E_{\alpha} [c(t - a)^{\alpha}] \left[ u(t) \vartheta(a) + u(a) \vartheta(t) \right] dt. \\
- \int_{a}^{T} u(t) \mathcal{L} \vartheta(t) dt
\]

**Proof.** The proof of the Proposition 4 follows from the direct computation of the first term before the equality.

Using the definition of the Atangana-Baleanu fractional derivative in the form of (15), and for \( t < a \), we have the following relation
\[
\int_{-\infty}^{T} \vartheta(t) \mathcal{L}u(t) dt = c_{\alpha} \int_{-\infty}^{T} \int_{-\infty}^{t} \left[ u(t) - u(s) \right] \left[ \vartheta(t) - \vartheta(s) \right] \mathcal{T}(t, s) ds dt \\
+ c_{\alpha} \int_{-\infty}^{T} \int_{-\infty}^{2t-T} u(t) \left[ \vartheta(t) - \vartheta(s) \right] \mathcal{T}(t, s) ds dt \\
- \int_{-\infty}^{T} u(t) \mathcal{L} \vartheta(t) dt.
\]

Now we state the weak formulation of the problem (1).

**Proposition 5.** Assume that \( \vartheta \) is a bounded and Lipschitz function on \((-\infty, T)\) for any \( t < a \) then the following weak formulation of the solutions (1) holds
\[
c_{\alpha} \int_{\mathbb{R}^{n}} \int_{-\infty}^{T} \int_{-\infty}^{t} \left[ u(t, x) - u(s, x) \right] \left[ \vartheta(t, x) - \vartheta(s, x) \right] \mathcal{T}(t, s, x) ds dx dt \\
+ \int_{a}^{T} \int_{\mathbb{R}^{n}} \mathcal{J}(t, x, \xi) \left[ u(t, x) - u(t, \xi) \right] \left[ \vartheta(t, x) - \vartheta(t, \xi) \right] dx d\xi dt
\]
\[ + c_a \int_{R^n} \int_{-\infty}^{T} \int_{-\infty}^{2t-T} u(t,x)\vartheta(t,x)T(t,s,x) ds \, dt \, dx \]

\[ = \int_{R^n} \int_{-\infty}^{T} f(t,x)\vartheta(t,x) dx \, dt. \]

**Proof.** Now to prove the statement of the Proposition 5 we use the equation (12) and we follow the idea introduced in [2].

Let \( \vartheta \) is a bounded and Lipschitz function on \((-\infty, T)\). Then

\[ \int_{a}^{T} \int_{a}^{t} T(t,s)[\vartheta(t,x) - \vartheta(s,x)] ds \, dt \]

\[ = \int_{a}^{T} \int_{a}^{t} T(t,s)\vartheta(t,x) - \int_{a}^{T} \int_{a}^{t} T(t,s)\vartheta(s) ds \, dt \]

\[ = \int_{a}^{T} \int_{a}^{T} T(t,s)\vartheta(t) - \int_{a}^{T} \int_{a}^{t} T(t,s)\vartheta(s) dt \, ds \]

\[ = \int_{a}^{T} \int_{a}^{t} T(t,s)\vartheta(t) - \int_{a}^{T} \int_{a}^{t} T(s,t)\vartheta(s) ds \, dt, \quad \text{(21)} \]

thanks to (5) the kernel \( K(t,s) \) satisfies that condition, therefore it can be written as

\[ T(t,t-s) = T(t+s,t) = s^{\alpha-1}E_{\alpha,\alpha}[cs^\alpha]. \quad \text{(22)} \]

Thus from (22), the equation (21) becomes

\[ \int_{a}^{T} \int_{a}^{t} T(t,s) [\vartheta(t) - \vartheta(s)] ds \, dt = \]

\[ \int_{a}^{T} \vartheta(t) \left( \int_{0}^{t-a} s^{\alpha-1}E_{\alpha,\alpha}[cs^\alpha] ds - \int_{0}^{T-t-a} s^{\alpha-1}E_{\alpha,\alpha}[cs^\alpha] ds \right) dt \]

\[ = \frac{1}{\alpha c_a} \int_{a}^{T} \vartheta(t) \left( E_{\alpha}[c(T-t)^\alpha] - E_{\alpha}[c(t-a)^\alpha] \right) dt \]

By making use of equation (23) and (21), it follows that

\[ c_a \int_{R^n} \int_{-\infty}^{T} \int_{-\infty}^{t} [u(t,x) - u(s,x)] [\vartheta(t,x) - \vartheta(s,x)] T(t,s,x) dsdtdx \]

\[ + \int_{a}^{T} \int_{R^n} \int_{R^n} J(t,x,\xi) [u(t,x) - u(t,\xi)] [\vartheta(t,x) - \vartheta(t,\xi)] dxd\xi dt \]

\[ + c_a \int_{R^n} \int_{-\infty}^{2t-T} u(t,x)\vartheta(t,x)T(t,s,x) dsdtdx - \int_{R^n} \int_{-\infty}^{T} u(t,x)\mathcal{L}\vartheta(t,x) dtdx \]

\[ = \int_{-\infty}^{T} \int_{R^n} f(t,x)\vartheta(t,x) dx \, dt. \quad \text{(23)} \]

This completes the proof of Proposition 5. \( \square \)

4.1. **Discretization of the problem.** In the following, we denote by \( \tau = T/\kappa \)

the time step which represents the subdivision of the interval \((a,T)\), where \( \kappa \in \mathbb{N} \)

denotes the number of time steps. Also, for \( 0 \leq k \leq \kappa, t = k\tau \). So the discrete form
of the Atangana-Baleanu fractional derivative in the sense of Caputo holds
\[ \mathcal{L}u(a + \tau k) = \tau^\alpha c_\alpha \sum_{-\infty < i < k} \frac{E_{\alpha,\alpha}[c\tau^\alpha (k-i)^\alpha]}{(k-i)^{1-\alpha}} \left[u(a + \tau k) - u(a + \tau i)\right]. \] (24)

Using (24) the discrete form of (10) takes the form
\[ \tau^\alpha c_\alpha \sum_{-\infty < i < k} \frac{E_{\alpha,\alpha}[c\tau^\alpha (k-i)^\alpha]}{(k-i)^{1-\alpha}} \left[u(a + \tau k) - u(a + \tau i)\right] \]
\[ = \int_{\mathbb{R}^n} \left[u(a + \tau k, \xi) - u(a + \tau k, x)\right] \mathcal{J}(a + \tau k, x, \xi) d\xi + g(a + \tau k, x). \]

Next we state the following Lemma

**Lemma 4.1.** Assume \( u(a) = u(a + \varepsilon j) = 0 \) for \( j < 0 \). Then the discrete integration by parts type estimate holds
\[ \sum_{k \leq j} u(a + \tau k) \mathcal{L}u(a + \tau k) \]
\[ \geq \frac{\tau^\alpha}{2} C_\alpha \sum_{0 \leq i < k \leq j} \frac{[u(a + \tau k) - u(a + \tau i)]^2}{(k-i)^{\alpha-1}} E_{\alpha,\alpha}[c\tau^\alpha (k-i)^\alpha] \]
\[ + \frac{\tau^\alpha}{2} C_\alpha \sum_{i \leq k \leq j} \frac{u^2(a + \tau k)}{(j-i)^{\alpha-1}} E_{\alpha,\alpha}[c\tau^\alpha (j-i)^\alpha]. \] (25)

**Proof:** The proof of this Lemma follows from the direct computation of the discrete integration by parts type estimate.

Now to get the discrete form of the weak formulation Proposition 5 we use the integration by parts type estimate given by Lemma 4.1
\[ c_\alpha \sum_{0 \leq i < k \leq j} \int_{\tau(k-1)}^{\tau k} \int_{\tau(i-1)}^{\tau i} \left[u_\tau(\tau k) - u_\tau(\tau i)\right] \left[\vartheta(\tau k) - \vartheta(\tau i)\right] \frac{E_{\alpha,\alpha}[c\tau^\alpha (k-i)^\alpha]}{(k-i)^{1-\alpha}}. \]
\[ + \int_{\tau(k-1)}^{\tau k} \int_{\mathbb{R}^n} \mathcal{J}(t, x, \xi) \left[u(\tau k, \xi) - u(\tau k, x)\right] \left[\vartheta(\tau k, x) - \vartheta(\tau k, \xi)\right] \]
\[ + c_\alpha \int_{\mathbb{R}^n} \tau \sum_{0 \leq i < k \leq j} \frac{u_\tau(\tau k)\vartheta(\tau k)}{(\tau(k-i))^{1-\alpha}} E_{\alpha,\alpha}[c\tau^\alpha (k-i)^\alpha] - \int_{\mathbb{R}^n} \tau \sum_{0 < k \leq j} u_\tau(\tau k) \mathcal{L}\vartheta(\tau k) \]
\[ = \int_{\tau(k-1)}^{\tau k} \int_{\mathbb{R}^n} g(\tau k, x) \vartheta(\tau k, x). \] (26)

Now we are ready to prove the Theorem 1.1 on existence of weak solutions following the idea of the authors in [2] by using the approximating method. For this purpose, we write the operators in (23) and (25) as \( \mathcal{H} \) and \( \mathcal{H}_\tau \) respectively.

**Proof:** Let \( \vartheta \) be a bounded and Lipschitz function on \((-\infty, T) \times \mathbb{R}^n\). There exists a sequence of solutions \( u_\tau \) to (23) with \( \tau \to 0 \), such that
\[ u_\tau \to u \in L^p((-\infty, T) \times \mathbb{R}^n), \]
with \( p \) obtained by embedding and defined in [3] as
\[ p = 2 \left( \frac{\alpha n + \beta}{\alpha n + (1-\alpha)\beta} \right). \]
For $\tau(k - 1) < t \leq \tau k$, we let $\mathcal{B}_\tau$ be the bilinear form associated with $K_\tau$. Our aim is to show that for $\vartheta$ bounded and Lipschitz function on $(-\infty, T) \times \mathbb{R}^n$,

$$\mathcal{H}(u, \vartheta) + \mathcal{H}_\tau(u_\tau, \vartheta) \rightarrow 0.$$ 

The first part of the proof where we shall consider the fractional Laplacian, we mean

$$\lim_{\tau \rightarrow 0} \int_0^T \mathcal{B}_\tau(u_\tau, \vartheta) \, dt = \lim_{\tau \rightarrow 0} \sum_{0 < k \leq j} \mathcal{B}(u_\tau(\tau k, x), \vartheta(\tau k, x))$$

has been proved by the authors in [3]. So next we shall focus on the pieces in time. To do this we start by showing that

$$\lim_{\tau \rightarrow 0} \int_{\mathbb{R}^n} \int_{-\infty}^T \int_{-\infty}^t \left[ u_\tau(t, x) - u_\tau(s, x) \right] \left[ \vartheta(t, x) - \vartheta(s, x) \right] \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(t - s)^{1-\alpha}} \, ds \, dt$$

$$= \lim_{\tau \rightarrow 0} \sum_{0 < i < k} \int_{\tau (k-1)}^{\tau i} \left[ u_\tau(\tau k) - u_\tau(\tau i) \right] \times \left[ \vartheta(\tau k) - \vartheta(\tau i) \right] \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(t - s)^{1-\alpha}}.$$

In order to achieve our goal, since $\vartheta$ is a bounded and Lipschitz function, then $\vartheta_\tau(t) \rightarrow \vartheta(t)$ and $u_\tau \rightarrow u \in L^p((-\infty, T) \times \mathbb{R}^n)$ we have that

$$\left| \int_{\mathbb{R}^n} \int_{-\infty}^T \int_{-\infty}^t \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(t - s)^{1-\alpha}} \times \left[ u_\tau(t) - u_\tau(s) \right] \left[ (\vartheta(t) - \vartheta(s)) - (\vartheta_\tau(t) - \vartheta_\tau(s)) \right] \, ds \, dt \right|$$

converges to 0 as $\tau \rightarrow 0$. We show also that

$$\lim_{\tau \rightarrow 0} \sum_{0 < i < k} (u_\tau(\tau k) - u_\tau(\tau i)) \left( \vartheta(\tau k) - \vartheta(\tau i) \right)$$

$$\times \int_{\tau (k-1)}^{\tau i} \left( \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(t - s)^{1-\alpha}} - \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(\tau - s)^{1-\alpha}} \right) = 0.$$ 

To do this we break up the integral over the sets $(t - s) \leq \tau^{1/2}$ and $(t - s) > \tau^{1/2}$. Then with the relation (5)

$$0 = \lim_{\tau \rightarrow 0} \int_{\mathbb{R}^n} \int_{t-s \leq \tau^{1/2}} \left| \frac{(u_\tau(t) - u_\tau(s)) (\vartheta(t) - \vartheta(s))}{(t - s)^{1-\alpha}} \right| E_{\alpha, \alpha} (\tau - s)\alpha$$

$$\geq \lim_{\tau \rightarrow 0} \int_{\mathbb{R}^n} \sum_{\tau (k-i) \leq \tau^{1/2}} \left| \frac{(u_\tau(\tau k) - u_\tau(\tau i)) (\vartheta(\tau k) - \vartheta(\tau i))}{\tau (k-i)^{1-\alpha}} \right|$$

$$\times \int_{\tau (k-1)}^{\tau i} \frac{E_{\alpha, \alpha} (\tau - s)\alpha}{(\tau - s)^{1-\alpha}}$$

For $t - s > \tau^{1/2}$, and $\tau (i-1) \leq s \leq \tau i$ and $\tau (k-1) \leq t \leq \tau k$, we can compute the estimate

$$\left| (t - s)^{\alpha - 1} E_{\alpha, \alpha} (\tau - s)\alpha - (\tau - i)^{\alpha - 1} \right| \leq (\tau^{1/2} - \tau)^{\alpha - 1}$$

$$\times E_{\alpha, \alpha} (\tau - s)\alpha - (\tau^{1/2} - \tau)^{\alpha - 1} E_{\alpha, \alpha} (\tau^{1/2} - \tau)^\alpha \leq T_{\alpha, \tau}$$
Hence the result follows for $\tau \to 0$. Next we consider the following
\[
\int_{\mathbb{R}^n} \int_{-\infty}^{T} u_\tau(t) \mathcal{L} \vartheta(t) \, dt \, dx - \int_{\mathbb{R}^n} \tau \sum_{0 < k \leq j} u_\tau(\tau k) \mathcal{L} \vartheta(\tau k).
\]
Similarly as in the previous case, since $\vartheta$ is a bounded Lipschitz function and $u_\tau \to u \in L^p((-\infty, T) \times \mathbb{R}^n)$, and from (25) one shows that this term also goes to zero.

The remaining pieces in time are handled in the similar manner. Thus the Theorem is proved.

5. **Pointwise estimates and Hölder regularity.** This section contains the some auxiliaries results, which are the key to prove Theorem 1.2. The proof of Proposition 8 uses the main contributions of this note. Once Proposition 8 is established, a priori Hölder regularity estimates follow by the classical method of diminishing oscillation given by Theorem 5.4.

Before going into this, we first collect the ingredients that will be useful. As one feature, we underline the viscosity solution. One of the useful property of viscosity solution is that viscosity sub-solutions themselves can be used to evaluate their corresponding equation classically at all of the points where the sub-solution can be touched from above by a smooth test function.

We need the following property
\[
\mathcal{L} u(t,x) - M_{x}^+ u(t,x) \leq g(t,x) \quad \text{in} \quad B_1 \times [-1,0],
\] (27)
Next we state the following proposition to clarify that $u$ is a solution on (27) and (1) in the viscosity sense by also making reference to Proposition 1.

**Proposition 6.** Let $u$ be a continuous bounded function on $(-\infty, T)$ and assume that for some $t \in (-\infty, T)$ there is a Lipschitz function touching $u$ by above at $t$. Then
\[
\int_{-\infty}^{t} [u(t,x) - u(s,x)] T(t,s) \, ds \geq g(t,x)
\]
if and only if $\mathcal{L} u(t,x) \geq g(t,x)$ in the viscosity sense.

**Proof.** The proof is standard and is based on the proof of [1, Proposition 2.3].

**Proposition 7.** Assume $u$ solves (27) in the viscosity sense. $\phi \geq u$ defined on the cylinder $\mathcal{C} := [t_0 - \varepsilon, t_0] \times B_{\varepsilon}(x_0)$ has a global maximum and touches $u$ from above at $(x_0, t_0) \in \mathcal{C}$ and we define $v$ as
\[
v(x,t) := \begin{cases} 
\phi(x,t) & \text{if} \ (x,t) \in \mathcal{C} \\
u(x,t) & \text{otherwise},
\end{cases}
\]
then $v$ is solution to (27) at $(x_0, t_0)$ or
\[
\mathcal{L} v(t_0,x_0) - J v(t_0,x_0) \leq g(t_0,x_0).
\]
So the solution is both a subsolution and supersolution.

The proof is straightforward.

Before we state the point evaluation proposition, we recall the comparison principle for which the proof is similar as in [1].
**Lemma 5.1.** (Comparison Principle). Let $u$ be bounded and upper semi-continuous and $w$ be bounded and lower semi-continuous $(-\infty, T_2)$. Let $g$ be a continuous function such that $L u \leq g \leq L w$ on $(T_1, T_2)$, with $u \leq w$ on $(-\infty, T_1)$. Then $u \leq w$ on $(-\infty, T_2)$, and if $u(t_0) = w(t_0)$ for some $t_0 \in (T_1, T_2)$, then $u(t) = w(t)$ for all $t \leq t_0$.

**Lemma 5.2.** Let $\phi(t)$ be continuous $(-\infty, T_1)$. Let $g$ be a continuous function on $(T_1, T_2)$. There exists a unique viscosity solution $u$ to, then

\[
\begin{cases}
Lu(t) = g(t) & \text{for } t \in [T_1, T_2] \\
u(t) = \phi(t) & \text{if } t \leq T_1
\end{cases}
\]
on $(T_1, T_2)$.

We start by recalling the definitions of Pucci’s extremal operators as defined in [12] for the spatial operator and next for the fractional-time derivative operator.

**Lemma 5.3** (Extremal Formula). Assume $u \in C^{1,1}(-\infty, T) \cap L^{\infty}(\mathbb{R}^n)$. Then we have the following elliptic spatial operator

\[
M^+_\alpha u(t, x) = \int_{\mathbb{R}^n} \left( \Lambda(\delta_h u(t, x))_+ - \Lambda(\delta_h u(t, x))_- \right) \mu(dh),
\]
and for the fractional-time derivative operator as

\[
M^+_T \theta u(t, x) := C(n, \alpha) \int_{-\infty}^t \left[ \Lambda(u(t, x) - u(s, x))_+ - \Lambda(u(t, x) - u(s, x))_- \right] \mathcal{T}_\sec.
\]

Next, in order to prove the H"older continuity, we use essentially the same ideas as the proof in [2, 17].

**Proposition 8.** (Point Estimate). Let $u \leq 1$ in $(\mathbb{R}^n \times [-2, 0]) \cup (B_1 \times [-\infty, 0])$ and assume it satisfies the following inequality in the viscosity sense in $B_2 \times [-2, 0]$

\[
Lu - M^+ u \leq \varepsilon_0.
\]
Assume also that $|\{u(x, t) \leq 0\} \cap (B_1 \times [-2, -1])| \geq \mu > 0$.

Then if $\varepsilon_0$ is small enough there exists $\theta > 0$ such that $u \leq (1 - \theta)$ in $B_1 \times [-2, 0]$.

The maximal value of $\theta$ as well as $\varepsilon_0$ depends on $\alpha, \Lambda, n$ and $\sigma$, but remain uniform as $\alpha \to 1$.

**Proof.** We consider the differential equation

\[
\begin{cases}
Lg(t) = c_0 \{ x \in B_1 : u(x, t) \leq 0 \} - c_1 g(t) \\
g(-2) = 0, \quad \text{for } t \leq -2
\end{cases}
\]

From Proposition 3 this ordinary differential equation can be computed explicitly and from Corollary 1 we have that

\[
g(t) \geq \frac{c_0 H}{2} E_{\alpha, \alpha}[-2c_1], \quad \text{for } -1 \leq t \leq 0.
\]
In the following we will show that if $c_0$ is small and $c_1$ is large, then $u \leq 1 - g(t) + \varepsilon_0 c_0 2^\alpha$ in $B_1 \times [-1, 0]$. The constant $c_\alpha$ is chosen such that $Lc_\alpha(2 + t)^\alpha = 1$ for $t \geq -2$. 

The maximal value of $\theta$ as well as $\varepsilon_0$ depends on $\alpha, \Lambda, n$ and $\sigma$, but remain uniform as $\alpha \to 1$.
Since for $t \in [-1, 0]$
\[ g(t) \geq \frac{\alpha}{2} E_{\alpha, \alpha} [-2c_1] c_0 \{ x : u(x, t) \leq 0 \} \cap B_1 \times [-2, -1] \]
\[ \geq \frac{\alpha}{2} E_{\alpha, \alpha} [-2c_1] c_0 \mu, \]
we set $\theta = \frac{\alpha \omega \mu}{2} E_{\alpha, \alpha} [-2c_1]$ for $\varepsilon_0$ small and finish the proof of the Lemma.

Next as in [2], let $\beta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth non-increasing function such that $\beta(x) = 0$ if $x \geq 2$. Let $\eta(x, t) = \beta(|x|)$. As a function of $x$ and $t$, $\eta(x, t)$ looks like a bump function for every fixed $t$. The main strategy of the proof is to show that the function $u(x, t)$ stays below $1 - g(t) \eta(x, t) + \varepsilon_0 c_\alpha (2 + t)^\alpha$.

In order to arrive to a contradiction, we assume that $\eta(x, t) > 1 - g(t) + \varepsilon_0 c_\alpha (2 + t)^\alpha$ for some point $(x, t) \in B_1 \times [-1, 0]$. Then at the point that $\tilde{\eta}$ realise it maximum
\[ \tilde{\eta}(x, t) = u(x, t) + g(t) \eta(x, t) - \varepsilon_0 c_\alpha (2 + t)^\alpha. \]
Assuming that there is one point $(x_0, t_0)$ in $B_1 \times [-1, 0]$ where $\tilde{\eta}(x, t) > 1$, $\tilde{\eta}$ must be larger than 1 at the point that realize the maximum of $\tilde{\eta}$. We mean by that
\[ \tilde{\eta}(x_0, t_0) = \max_{\mathbb{R}^n \times (-\infty, 0]} \tilde{\eta}(x, t). \]
Since $\tilde{\eta}(x_0, t_0) > 1$, the point $(x_0, t_0)$ must belong to the compact support $\eta$. Hence $|x| < 2$. Now on the remain of the proof is exactly as in [2].

Next we call $\varphi(x, t)$ the function such that
\[ \varphi(x, t) := \tilde{\eta}(x_0, t_0) - g(t) \eta(x, t) + c_\alpha (2 + t)^\alpha. \]
$\varphi$ touches $u$ from above at the point $(x_0, t_0)$. We define
\[ v(x, t) := \begin{cases} \varphi(x, t) & \text{if } x \in B_r, \\ u(x, t) & \text{if } x \notin B_r. \end{cases} \]
Then at the point that $\tilde{\eta}$ realise it maximum
\[ \mathcal{L} v(x_0, t_0) - \mathcal{M}^+ v(x_0, t_0) \leq \varepsilon_0. \tag{33} \]
So one can have
\[ \mathcal{L} v(x_0, t_0) - \mathcal{L} g(t_0) \eta(x_0) + \varepsilon_0. \tag{34} \]
Hence for $\mathcal{G} := \{ x \in B_1 \mid u(x, t_0) \leq 0 \}$, the following bound is obtained
\[ \mathcal{M}^+ v(x_0, t_0) \leq -g(t_0) \mathcal{M}^- \eta(x_0, t_0) - c_0 |\mathcal{G} \setminus B_r|. \tag{35} \]
Now if we insert the relation (35), (33), (34) into (32), we obtain
\[ -\mathcal{L} g(t_0) \eta(x_0, t_0) + \varepsilon_0 + c_0 |\mathcal{G} \setminus B_r| \leq \varepsilon_0, \]
or in its explicit form
\[ \left( -c_0 \{ x \in B_1 : u(x, t_0) \leq 0 \} \right) \eta(x_0) + \varepsilon_0 + c_0 |\mathcal{G} \setminus B_r| \leq \varepsilon_0. \]
But notice that for any $c_1 > 0$ this contradicts (32).

We now analyse the case where $\eta(x_0, t_0) > \beta_1$. As we said previously that $\eta$ is a smooth compactly supported function, then there exist some constant $\tilde{C}$ such that $|\mathcal{M}^{-}\eta| \leq \tilde{C}$. Then we have the bound
\[ \mathcal{M}^+ v(x_0, t_0) \leq -\tilde{C} g(t_0) + c_0 |\mathcal{G} \setminus B_r|. \tag{36} \]
Proof of Theorem 1.2. The proof is the adaptation of the proof in [1, 17]. We prove of the equation by little bit in the future. We then define the parabolic cylinders in terms of the scaling of the space is not modified if we make a parabolic dilation to make the profile a smaller cylinder. To do this, we renormalize the function \( u \) and by letting \( r \to 0 \), we obtain
\[
 c_0 (1 - \eta(x_0)) |G| + (c_1 \eta(x_0) - \tilde{C}) g(t_0) \leq 0,
\]
which can be written in the form
\[
 c_0 (1 - \eta(x_0)) |G| + (c_1 \beta_1 - \tilde{C}) g(t_0) \leq 0.
\]
Choosing \( c_1 \) large enough, we arrived to a contradiction. This ends the proof. \( \square \)

In the following we shall give an approach of the proof of the Hölder continuity. For this purpose, we state and prove the so called growth Lemma, which says that if a solution of the equation (1) in the unit cylinder \( Q_1 = B_1 \times [-1, 0] \) has oscillation one then its oscillation in a smaller cylinder \( Q_r \) is less that a fixed constant \((1 - \theta)\) [2, 17].

Lemma 5.4. (Diminish of Oscillation). Let \( u \) be a bounded continuous function which satisfies (1) or the following two inequalities in the viscosity sense in \( Q_1 \)
\[
 Lu - M^+ u \leq \varepsilon_0/2, \quad Lu - M^+ u \geq -\varepsilon_0/2.
\]
Then there are universal constants \( \theta > 0 \) and \( \nu > 0 \), depending on \( n, \sigma, \Lambda, \lambda \) and \( \alpha \), such that if
\[
 \| g \|_{L^\infty(Q)} \leq \varepsilon_0, \quad |u| \leq 1 \text{ in } Q_1
\]
\[
 |u(x,t)| \leq 2|tx|^{\nu} - 1 \text{ in } (\mathbb{R}^n \setminus B_1) \times [-1, 0]
\]
\[
 |u(x,t)| \leq 2|rt|^{\nu} - 1 \text{ in } B_1 \times [-\infty, -1]
\]
with \( r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\} \), then
\[
 \text{osc}_{\overline{Q}_r} u \leq (1 - \theta)
\]
Proof. With the Proposition 2 in hand the reader can finish the proof by following the idea of the authors in [2, 17]. \( \square \)

Having the diminish of oscillation Lemma in hand, where the proof is similar to the one proposed in [1], we are now going to prove one of our main results about Hölder continuity. The result requires the function to solve the equation only in a cylinder in order to have Hölder continuity in a smaller cylinder. To do this, we renormalize the function \( u \).

If \( u \) satisfies (1), then the rescaled function \( \varrho(x,t) = u(rx, r^{2\sigma/\alpha} t) \) satisfies also (1), with \( r \in (0, 1) \) such that \( r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\} \). This means that the structure of the space is not modified if we make a parabolic dilation to make the profile a little bit in the future. We then define the parabolic cylinders in terms of the scaling of the equation by
\[
 Q_r := B_r \times [-r^{2\sigma/\alpha}, 0].
\]

Now we state the result on Hölder continuity.

Proof of Theorem 1.2. The proof is the adaptation of the proof in [1, 17]. We prove \( C^\alpha \) estimate of (7) by proving a \( C^\alpha \) estimate for \( \varrho(x,t) \) at the point \((x_0, t_0) \in Q_1\).

For any point \((x_0, t_0)\) we consider the normalized function
\[
 \varrho(x,t) = \frac{u(x_0 + x, t + t_0)}{\|u\|_{L^\infty} + \varepsilon_0^{-1}\|g\|_{L^\infty}}.
\]
where $\varepsilon_0$ is a constant from Theorem 5.4, one can show that $\text{osc}_{\mathbb{R}^d \times [-1,0]} \varrho \leq 1$ and in $B_2 \times [-1,0]$, $\varrho$ is a solution to

$$\mathcal{L}\varrho - \mathcal{M}^+ \varrho \leq \varepsilon_0, \quad \mathcal{L}\varrho - \mathcal{M}^- \varrho \geq -\varepsilon_0.$$

Let $r \in (0, 1)$, such that $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$. To prove then the Hölder estimate, we prove by induction the following decay of the oscillation in cylinders for some $\kappa$ [16]

$$\text{osc}_{Q_{r^k}} \varrho \leq 2 r^{\kappa k} \quad \text{for } k = 0, 1, 2, \ldots$$

(37)

We construct two sequences $S_k \leq \varrho \leq T_k$ in $Q_{r^k}$, $T_k - S_k = 2 r^{\kappa k}$ with $S_k$ non-decreasing and $T_k$ nonincreasing. Indeed. for $k = 0$, by hypothesis it is true that $\text{osc}_{Q_{r^k}} \varrho = 1$. We assume that the sequences hold until certain value of $k$.

We scale once more by considering $z(x, t) = (\varrho(r^k x, r^{2k\sigma/\alpha} t) - (S_k + T_k)/2) r^{-\kappa k}$.

Then we have

$$|z| \leq 1 \quad \text{in } Q_1,$$

$$|z| \leq 2 r^{-\kappa k} - 1 \quad \text{in } Q_{r^{-k}},$$

and so

$$|z(x, t)| \leq 2|x|^{\nu} - 1 \quad \text{for } (x, t) \in B_1^c \times [-1, 0],$$

$$|z(x, t)| \leq 2|t|^{\nu} - 1 \quad \text{for } (x, t) \in B_1 \times (-\infty, -1).$$

For $\kappa < 2\sigma$, $z$ has right hand side bounded

$$||r^{(k-2\sigma)} \varrho||_{L^\infty} \leq \varepsilon_0 r^{\kappa(\kappa-2\sigma)} < \varepsilon_0.$$

If $\kappa$ small enough we can apply Theorem 5.4 to obtain

$$\text{osc}_{Q_{r^k}} z \leq 1 - \theta.$$

Furthermore if we choose $\kappa$ smaller than the one in Theorem 5.4 and also so that $r^{\kappa} \geq 1 - \theta$, we have that

$$\text{osc}_{Q_{r^{k+1}}} z \leq r^{\kappa}$$

which simply means $\text{osc}_{Q_{r^{k+1}}} z \leq r^{\kappa(k+1)}$ so we can find two sequences $S_{k+1}$ and $T_{k+1}$. This finishes the proof by induction.

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