Impossible solutions?

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October 18, 2000

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Abstract

We present $n$-dimensional vortex-ring-like and potential-like solutions with unusual properties related to some elliptical differential equations with compact sources. Solutions have almost 3- or 2-dimensional behaviour in the spaces with arbitrary (large) odd and even dimensions, respectively.

1 Introduction

It is well-known that in the $n$-dimensional $(nD)$ space the gravitational and Coulomb-like electric potentials possess the fall-off laws $\sim R^{-(n-2)}$ for $n > 2$, $R$ being a distance from a compact source, i.e., the source located in a confined space domain. In the case $n = 2$, the law has the form $\sim \ln(1/R)$. This basic statement is undoubtedly true. However, we can pose the question: Is it possible that in the $nD$ space some compact sources exist for which the fall-off law of a potential-like physical quantity is adequate to the law associated with the 3D or 2D space? - We give a positive answer to this question. The 3D or 2D behaviour of some quantities takes place in spaces with an arbitrary odd or even number of dimensions, respectively; moreover, we mean not only a far-zone asymptotics, but a near-zone one as well.

A situation can be explained as follows. Let us imagine that we split the $nD$ space onto 2D and $(n-2)D$ subspaces. Consider a function – a geometrical object – with a complicated transformation properties. It does not matter whether it is a scalar, or a vector orthogonal to the radial direction in the 2D space. Let it simultaneously be a component of some polyvector in the $(n-2)D$ space. The covariant operator $\nabla^2$ applied to such the function is not obliged to be an obvious scalar Laplace operator. It can has another appearance, and we call some operators, we dealt with, anti-Laplacians. And the solution of the corresponding Poisson-like equation with a compact source is not obliged to be $\propto R^{-(n-2)}$. In the 3D space such a situation is hardly distinguishable. Indeed, the remaining
single dimension (denote it by \( z \)) in the \((3 - 2)D\) space provides the term \( \partial^2 / \partial z^2 \) in \( \nabla^2 \), the same for a 1-vector and for a scalar, leading to the solution \( \propto R^{-1} \) in both the cases.

A physically meaningful aspect of this situation is that given such the fall-off laws, we cannot determine an exact dimensionality of our physical space. We can only extract a "best-fit" value \( n = 2 \) or \( n = 3 \) from observations and only state that the real dimensionality is odd or even. Thus, the effective dimensionality of our space can be dynamical in essence. Perhaps, such compact sources are just elementary particles, so that we have no contradictions with the physical picture of our world.

The author's intention here is not to deal with any special physical models. It seems to be enough to describe several mathematical facts which weaken the mentioned basic statement. In this paper, we undertake the systematic study of an outlined subject, which was originated from considering vortex rings: The notion of the anti-Laplacians appeared from the latter. The logic of our account repeats that of developing this study.

In Sec.2, we recall the reader how to obtain a current function describing an infinitely thin vortex ring. A traditional derivation of this subject in old and lovely manuals like [1, 2] has a drawback from the modern viewpoint: There is no appellation to \( \delta \) functions, although the latter being quite convenient for a researcher. Indeed, the vortex rings had been known well before P.A.M. Dirac invented his \( \delta \) function. We modify the traditional derivation by including the \( \delta \) functions. During this derivation we, for the first time, meet an operator belonging to the type of operators which we call anti-Laplacians.

In Sec.3, we consider an extension of equations suitable for vortex rings onto \( nD \) space. On the l.h.s. of these equations anti-Laplacian operators appear, however, we do not know a priori explicit forms of their r.h.s., i.e., those of ring sources. In this section we only give a constructive way of obtaining some solutions to homogeneous equations. The solutions are finite everywhere except of a ring set of points, and they could serve as those for the \( nD \) vortex rings. It turns out that the cases of odd and even \( n \) are principally distinguished, and so are their asymptotics. The odd-\( n \)D solutions have asymptotics of the 3D ones, whereas the even-\( n \)D solutions, omitting some details, have that of 2D ones. It only remains to learn explicit forms of the ring-like sources. However, in order to do it, we have to study some properties of anti-Laplacians (and Laplacians as well) in the next section.

Thus, we return in Sec.4 to the familiar Laplacians and Poisson equations with \( \delta \) sources possessing some symmetries in relevant coordinate frames. [In Sec.4, we speak about the standard solutions \( \propto R^{-(n-2)} \) only.] We have found some transformation of an \((n - 2)D\) solution into a \( nD \) one. Moreover, the same transformation, from \((n - 2)D\) to \( nD \), is suitable for both the l.h.s. and r.h.s. of the Poisson equations. Furthermore, the transformation is the same for both the odd-\( n \)D and even-\( n \)D cases. Thus, for symmetries considered, we can
construct any nD solution if we know the above transformation and the 3D and 2D solutions (or the 4D solution in a special symmetry). In Sec.4, we also show that the extraction of an nD δ source from the nD solution gives the same result, which can be obtained via applying our transformation to the 3D or 4D sources.

Sec.5 is an independent study without referring to vortex rings. We examine connections between Laplacians and anti-Laplacians which provide a possibility of finding (our nonstandard) solutions to Poisson-like equations similarly to Sec.2. We pay attention to one of the anti-Laplacians which surprisingly leads to potential-like nD solutions related to a one-point source. They certainly satisfy homogeneous equations everywhere except this point. It is remarkable that these solutions are simplified versions of those in Sec.3 and have the same properties. It is also remarkable that we also find (another) transformation from (n − 2)D quantities to nD ones suitable both for solutions and equations. Thus, we construct a transformation machinery which permits one to resolve the problem of finding points sources.

This is done in Sec.6. We obtain two different answers for the cases of odd-n and even-n dimensions. In the former case the source is an obvious δ-like one, however, in the latter case the situation is more intricate. In order to input the transformation machinery, we obliged to have in hand the case n = 4. It is simple enough to be calculated immediately, and we consider it in detail. The answer is that the point-like source is not a δ source, which contradicts, at first glance, to the known theorem. However, there are no true contradiction; some discussion on this subject is also given.

In Sec.7., we return to vortex-ring-like solutions of Sec.3 and exhibit their sources. We also describe some other ring-like solutions and give several concluding remarks. Two appendices with necessary mathematical information accomplish our work.

2 Recalling a 3D vortex ring

In the cylindrical coordinate frame (r, φ, z), consider an axially symmetric motion of incompressible fluid with the symmetry axis directed along z, r being the distance from z, and there is no dependence of φ (see [1, 2, 3] or any other suitable manuals). Let \( V_r(r, z) \) and \( V_z(r, z) \) be the r and z components of a fluid velocity field \( \vec{V} \), respectively. Let the continuity equation be satisfied everywhere, perhaps except some points:

\[
\vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial r}(rV_r) + \frac{\partial}{\partial z}(rV_z) = 0. \tag{2.1}
\]
In our symmetry assumptions, the only $\varphi$ component of the $\vec{V}$ curl survives, let it be a given value $\Omega_{\varphi}(r, z)$,

$$\left(\vec{\nabla} \times \vec{V}\right)_\varphi = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \Omega_{\varphi}. \quad (2.2)$$

It follows from (2.1) that the expression

$$r V_z dr - r V_r dz = d\psi$$

is a local differential, where the function $\psi(r, z)$ is called the Stokes current function\(^1\), and the previous definition of $\psi$ is equivalent to

$$V_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad V_z = \frac{1}{r} \frac{\partial \psi}{\partial r}. \quad (2.3)$$

Hence, equation (2.2) can be represented as that for $\psi$:

$$\Delta_r \psi \equiv r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = -r \Omega_{\varphi}(r, z). \quad (2.4)$$

The operator defined on the l.h.s. of (2.4) is not a Laplace operator. In keeping in mind further generalizations, we call it anti-$r$-Laplacian (we refer the reader to Sec.5).

In fact, the function $\psi$ makes the sense of the $\varphi$ component of a vector field $\vec{A}$. The identically vanishing divergence (2.1) means that $\vec{V}$ is a curl vector, i.e.,

$$\vec{V} = \vec{\nabla} \times \vec{A}.$$ 

In our assumptions about the vector $\vec{V}$, the only $A_\varphi$ component of $\vec{A}(r, z)$ is nonzero and

$$V_r = -\frac{1}{r} \frac{\partial A_\varphi}{\partial z}, \quad V_z = \frac{1}{r} \frac{\partial A_\varphi}{\partial r}. \quad (2.5)$$

The comparison of (2.3) and (2.5) allows us to impose $\psi = A_\varphi$;\(^2\) This fact helps one to solve equation (2.4) for $\psi$ by connecting the operator $\Delta_r$ with the Laplacian, see below. Our account is given following the familiar way of\[^3\, \[^3\].

In the Cartesian coordinate frame $x, y, z$, where $x = r \cos \varphi$ and $y = r \sin \varphi$, we take, e.g., the $y$ component of $\vec{A}$:

$$A_y = \frac{\cos \varphi}{r} A_\varphi$$

(we could choose the component $A_x$ that would give the same final result). In the above frame, the vector operator $\nabla^2$ applied to any component of $\vec{A}$ coincides

\(^1\)Our choice of signs in (2.3) coincides with that of\[^3\] and is opposite to that of\[^4, \[^4\].

\(^2\)This have a consequence $\psi = r |\vec{A}|$, cf. loc. cit. with our sign convention.
with the scalar operator, denoted by $\Delta_{\text{Cart}}$, applied to the same component, that is why
\[
\nabla^2 A_y = \Delta_{\text{Cart}} A_y \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_y =
\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) \cos \varphi \cos \varphi \frac{A_y}{r} \equiv \Delta_{(r,\varphi,z)} \cos \varphi \cos \varphi \frac{A_y}{r}.
\]

Then, in replacing back $A_\varphi$ by $\psi$, it is easy to establish the validity of the rearrangement
\[
\Delta_{(r,\varphi,z)} \cos \varphi \frac{\psi}{r} = \cos \varphi \cos \varphi \frac{\psi}{r} \Delta_r \psi.
\]

We stress once more that equation (2.6) is an important step which allows one to construct a solution for $\psi$ (or $A_\varphi$) generated by a $\delta$-like source using the fundamental solution of the Poisson equation.

Consider an infinitely thin vortex ring of the radius $a$ but with the finite vortex intensity $\kappa$: $\kappa = \Omega_\varphi dS$ where $dS$ is the square of the infinitesimal ring cross section. We should impose
\[
\Omega_\varphi = \kappa a \frac{\delta(r - a)}{r} \delta(z)
\]
in order for the integral of $\Omega_\varphi$ over the whole volume to be equal to $2\pi adS \Omega_\varphi = 2\pi \kappa a$, leading to equation (2.4) in the form
\[
\overline{\Delta}_r \psi = -\kappa a \delta(r - a) \delta(z).
\]

From (2.6) and (2.7), we obtain the Poisson-like equation:
\[
\Delta_{(r,\varphi,z)} \cos \varphi \frac{\psi}{r} = -\kappa a \cos \frac{\varphi}{r} \delta(r - a) \delta(z).
\]

Due to the compact character of the source on the r.h.s. of (2.8) the solution can be found by a standard way:
\[
\frac{\cos \varphi}{r} \psi = \frac{\kappa a}{4\pi} \int_0^\infty \frac{r}{r'} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi'}{2\pi} \int_{-\infty}^{\infty} \frac{dz'}{\infty} \frac{\cos \varphi'}{r'} \delta(r' - a) \delta(z') \left| \frac{1}{r} - \frac{1}{r'} \right|
\]

where
\[
|\overline{R} - \overline{R'}| = \sqrt{r^2 + r'^2 - 2r'r \cos(\varphi - \varphi') + (z - z')^2}
\]
is the distance between the points with the coordinates $(r, \varphi, z)$ and $(r', \varphi', z')$. Integrating (2.9) with respect to $r'$ and $z'$ gives
\[
\frac{\cos \varphi}{r} \psi = \frac{\kappa a}{4\pi} \int_0^{2\pi} \frac{d\varphi'}{r^2 + a^2 - 2ar \cos(\varphi - \varphi') + z^2} \left| \frac{1}{r} \right|
\]

\footnote{We denote by $\nabla^2$ an entirely covariant operator acting, e.g., on a vector. The notation $\Delta$ is reserved for an operator acting on a scalar (e.g., $\Phi$) only: $\nabla^2 \Phi \equiv \Delta \Phi$.}
In order to remove the factor $\cos \varphi$ on the l.h.s. of (2.10), we displace the origin of $\varphi' = \varphi + \alpha$, so that integration will be done with respect to $\alpha$. After using the equality

$$\cos \varphi' = \cos \varphi \cos \alpha - \sin \varphi \sin \alpha$$

we ensure that

$$\int_{0}^{2\pi} d\alpha \frac{\sin \alpha}{[r^2 + a^2 - 2ar \cos \alpha + z^2]^{1/2}} = 0,$$

and the above factor on the l.h.s. of (2.10) cancels with that on a r.h.s. of a final expression.

The required solution for the 3D vortex ring can now be written in the form

$$\psi = \frac{\kappa a}{2\pi} r \int_{0}^{\pi} d\alpha \frac{\cos \alpha}{[r^2 + a^2 - 2ar \cos \alpha + z^2]^{1/2}}.$$ (2.11)

And here we stop as yet. Our task was to only obtain (2.11) for comparison with further extensions on multidimensional spaces. It is well known how to express $\psi$ via elliptical functions, there are also known its various properties and asymptotics, see, e.g., already cited manuals and many others. Note that instead of the vortex ring, we could speak about a ring electric contour, $A_{\varphi}(= \psi)$ being the $\varphi$ component of the electromagnetic vector potential. – There are no difference from the mathematical viewpoint.

### 3 The $nD$ vortex-ring-like solutions

In order to describe an $n$-dimensional ring structure, which is a direct product of the ring of the radius $a$ on an infinitesimal $(n-1)$-dimensional ball, it is convenient to introduce an $(r, z)$ coordinate frame corresponding to the $2 + (n - 2)$ splitting of $R^n$: $R^n = R^2 \times R^{n-2}$. In each space, $R^2$ and $R^{n-2}$, we construct spherical coordinate frames with the radial coordinates $r$ and $z$, and the angle coordinates $\varphi$ and $\theta_1, \theta_2, \ldots, \theta_{n-3}$, respectively (see Appendix A).

From now we shall not confine ourselves by any physical interpretation. Consider a vector field $\vec{v}(r, z)$ which has the only $v_r(r, z)$ and $v_z(r, z)$ nonvanishing components. Let the divergence of this vector field be equal to zero almost everywhere,

$$\nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{zn-3} \frac{\partial}{\partial z} (zn^{-3}v_z) = 0 \quad (3.1)$$

[see (A.5), also note that in the frame selected $v^z = v_z, v^r = v_r$]. If we impose, as before in Sec.2,

$$v_r = -\frac{1}{rz^{n-3}} \frac{\partial \psi}{\partial z}, \quad v_z = \frac{1}{rz^{n-3}} \frac{\partial \psi}{\partial r}, \quad (3.2)$$
then the divergence (3.1) vanishes identically.

The extension of the curl operator to the $n$D space leads to a covariant 2-vector or a contravariant $(n-2)$-vector, see, e.g., [3]. However, this fact does not prevent us to propose that its only nonvanishing covariant component is unknown as yet value $\omega_{r,z}$:

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = \omega_{r,z}. \quad (3.3)$$

After substituting (3.2) into (3.3), an operator can be defined which we call anti-double-Laplacian and denote by $\overline{\Delta}^{(n)}$:

$$\overline{\Delta}^{(n)} \psi_n = \left( r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-3}} \frac{\partial}{\partial z} \right) \psi_n = -rz^{n-3}\omega_{r,z}. \quad (3.4)$$

Now we shall find solutions to the homogeneous (anti-double-Laplace) equations for all $n \geq 3$,

$$\overline{\Delta}^{(n)} \psi_n = 0, \quad (3.5)$$

keeping in mind that $\omega_{r,z}$ should be considered equal to zero almost everywhere, except of the set of points $r = a$, $z = 0$, because we suggest to search for the vortex-ring-like solutions. This is done below, separately for odd and even $n$.

### 3.1 The odd-$n$D solutions

Consider the probe function

$$\overline{\psi}_k = r \int_0^{\pi} d\alpha \cos \alpha \frac{z^{k-3}}{R^{k-2}} \quad (3.6)$$

where we denote

$$R = \sqrt{\rho^2 + z^2}, \quad (3.7a)$$

$$\rho^2 = r^2 + a^2 - 2ar \cos \alpha. \quad (3.7b)$$

In applying the operator $\overline{\Delta}^{(n)}$ to $\overline{\psi}_k$, we come to the equality

$$\overline{\Delta}^{(n)} \overline{\psi}_k = r \int_0^{\pi} d\alpha \cos \alpha \left[ (n-k)(k-2)\frac{z^{k-3}}{R^k} - (k-3)(n+1-k)\frac{z^{k-5}}{R^{k-2}} \right] \quad (3.8)$$

after using the integral identity

$$\int_0^{\pi} d\alpha \cos \alpha \left[ \frac{1}{R^{k-2}} - (k-2)\frac{ar \cos \alpha}{R^k} + k(k-2)\frac{a^2r^2 \sin^2 \alpha}{R^{k+2}} \right] =$$

$$\int_0^{\pi} d\alpha \frac{\partial}{\partial \alpha} \left[ \frac{\sin \alpha}{R^{k-2}} - (k-2)\frac{ar \sin \alpha \cos \alpha}{R^k} \right] = 0. \quad (3.9)$$


Let us explicitly write the formulae (3.8) for odd integer \( k \) between 3 and \( n \):

\[
\Delta^{(n)} \psi_3 = r \int_0^\pi d\alpha \cos \alpha \, (n - 3) \frac{1}{R^3}, \tag{3.10a}
\]

\[
\Delta^{(n)} \psi_5 = r \int_0^\pi d\alpha \cos \alpha \left[ -2(n - 4) \frac{1}{R^3} + 3(n - 5) \frac{z^2}{R^5} \right], \tag{3.10b}
\]

\[
\ldots \tag{3.10c}
\]

\[
\Delta^{(n)} \psi_{n-2} = r \int_0^\pi d\alpha \cos \alpha \left[ -3(n - 5) \frac{z^{n-7}}{R^{n-4}} + 2(n - 4) \frac{z^{n-5}}{R^{n-2}} \right], \tag{3.10c}
\]

\[
\Delta^{(n)} \psi_n = r \int_0^\pi d\alpha \cos \alpha \left[ -(n - 3) \frac{z^{n-5}}{R^{n-2}} \right]. \tag{3.10d}
\]

We see that the terms with the factors \((k - 3)\) and \((n - k)\) have disappeared on the r.h.s. of (3.10a) and (3.10d) for \( k = 3 \) and \( k = n \), respectively. The remaining terms can be compensated when taking suitable combinations of \( \psi_k \). The single term \((\propto 1/R^3)\) in (3.10a) can be compensated by the first term in (3.10b), and so on, the last \((\propto z^{n-5}/R^{n-2})\) term in (3.10c) can be finally compensated by the single term in (3.10d). Thus, the desired solution can be constructed as a finite series of \( \psi_k \),

\[
\psi_n = \sum_{k=3}^{n} a_{k,n} \psi_k = r \int_0^\pi d\alpha \cos \alpha \sum_{k=3}^{n} a_{k,n} \frac{z^{k-3}}{R^{k-2}}. \tag{3.11}
\]

where the prime over the sum indicates that the latter is taken with respect to the only odd \( k \). We omit henceforth any physically meaningful coefficients. As to the coefficients \( a_{k,n} \), the recurrence relation takes place:

\[
a_{k+2,n} = a_{k,n} \frac{(k - 2)(n - k)}{(k - 1)(n - k + 1)}. \tag{3.12}
\]

Solving (3.12) with the (arbitrary) choice \( a_{3,n} = 1 \) for every \( n \) yields

\[
a_{k,n} = \frac{1 \cdot 3 \cdot 5 \cdots (k - 4)(n - 3)(n - 5) \cdots (n - k + 2)}{2 \cdot 4 \cdot 6 \cdots (k - 3)(n - 4)(n - 6) \cdots (n - k + 1)}. \tag{3.13}
\]

More information about the properties of the coefficients \( a_{k,n} \) is given in Appendix B.
In addition to already given solution (2.11) for $n = 3$, we expose here the three subsequent solutions (3.11) for $n = 5, 7, 9$:

$$
\psi_3 = r \int_0^\pi d\alpha \cos \alpha \frac{1}{R},
$$

$$
\psi_5 = r \int_0^\pi d\alpha \cos \alpha \left( \frac{1}{R} + \frac{z^2}{R^3} \right),
$$

$$
\psi_7 = r \int_0^\pi d\alpha \cos \alpha \left( \frac{1}{R} + \frac{2}{3} \frac{z^2}{R^3} + \frac{z^4}{R^5} \right),
$$

$$
\psi_9 = r \int_0^\pi d\alpha \cos \alpha \left( \frac{1}{R} + \frac{3}{5} \frac{z^2}{R^3} + \frac{3}{5} \frac{z^4}{R^5} + \frac{z^6}{R^7} \right).
$$

The far-zone asymptotics of the obtained solutions is very interesting from the physical viewpoint. When $r \to \infty$ for any finite $z$, the term $\psi_k$ falls off as $\sim r^{2-k/2}$, so that for every odd-$n$D solution independently of $n$ the term with $k = 3, 1/R$ is asymptotically a leading one. When $z \to \infty$ for any finite $r$, all the terms $\psi_k$ have the same asymptotics $\sim z^{-1}$. Thus, far from a compact source, $\psi_n$ behaves almost like the 3D solution.

As for the near-zone asymptotics, the common feature of these solutions is that for $z = 0$ they exhibit the true 3D behaviour, and for every finite $z$ the integrand of any $\psi_n$ is $\sim z^{-1}$ when $r \to 0$.

### 3.2 The even-$n$D solutions

It is clear from the consideration in the previous subsection that any function of the (3.6)-kind is not suitable for the even-D case, hence we probe other functions. For even integers $l > 4$, we define

$$
\bar{\psi}_l = r \int_0^\pi d\alpha \cos \alpha \frac{z^{l-4}}{R^{l-4}},
$$

(3.14)

whereas for $l = 4$, we give a distinct definition

$$
\bar{\psi}_4 = r \int_0^\pi d\alpha \cos \alpha \ln \frac{1}{R}.
$$

(3.15)

We can find that, for $l > 4$

$$
\bar{\Delta}^{(n)} \bar{\psi}_l = r \int_0^\pi d\alpha \cos \alpha (l-2) \left[ (n-2-l) \frac{z^{l-2}}{R^{l-2}} - (n-l) \frac{z^{l-4}}{R^{l-4}} \right],
$$

(3.16)
after using the integral identity (3.9) with \( k \) replaced by \( l \), and for \( l = 4 \)

\[
\overline{\Delta}^{(n)}\overline{\psi}_4 = r \int_0^{\pi} d\alpha \cos \alpha \left[ -(n - 4) \frac{1}{R^2} \right]
\]

(3.17)

after using the following integral identity

\[
\int_0^{\pi} d\alpha \frac{\cos \alpha}{R^4} \left( \ln \frac{1}{R} - \frac{ar \cos \alpha}{R^2} + 2 \frac{a^2 r^2 \sin^2 \alpha}{R^4} \right) = \\
\int_0^{\pi} d\alpha \frac{\partial}{\partial \alpha} \left( \sin \alpha \ln \frac{1}{R} - \frac{ar \sin \alpha \cos \alpha}{R^2} \right) = 0.
\]

(3.18)

First of all, note that \( \psi_4 = \overline{\psi}_4 \) is a solution to (3.5) for \( n = 4 \), which is clear from (3.17): \( \overline{\Delta}^{(4)}\psi_4 = 0 \). As before, we also explicitly write (3.16):

\[
\overline{\Delta}^{(n)}\overline{\psi}_6 = r \int_0^{\pi} d\alpha \cos \alpha 2 \left[ (n - 4) \frac{1}{R^2} - (n - 6) \frac{z^2}{R^4} \right],
\]

(3.19a)

\[\ldots\]

\[
\overline{\Delta}^{(n)}\overline{\psi}_{n-2} = r \int_0^{\pi} d\alpha \cos \alpha (n - 6) \left[ 4 \frac{z^{n-8}}{R^{n-6}} - 2 \frac{z^{n-6}}{R^{n-4}} \right],
\]

(3.19b)

\[
\overline{\Delta}^{(n)}\overline{\psi}_n = r \int_0^{\pi} d\alpha \cos \alpha 2(n - 4) \frac{z^{n-6}}{R^{n-4}}.
\]

(3.19c)

For \( n \neq 4 \), the appearance of (3.17) and (3.19) also demonstrates that some linear combination of \( \overline{\psi}_l \) can give the solution to (3.5): The term \( \sim 1/R^2 \) in (3.17) can be compensated by the first term in (3.19a), and so on, the last term in (3.19b) being compensated by the single term in (3.19c). The solution can be written in the form

\[
\psi_n = a_{4,n} \overline{\psi}_4 + \sum_{l=6}^{n} a_{l,n} \overline{\psi}_l
\]

(3.20)

where the double prime means that the sum is taken with respect to the only even \( l \), and there is a simple recurrence relation for \( a_{l,n} \) when \( l \geq 6 \):

\[
a_{l+2,n} = a_{l,n} \frac{l - 4}{l - 2}.
\]

(3.21)

If we choose \( a_{4,n} = 1 \), then \( a_{6,n} = 1/2 \) from (3.17) and (3.19a), and (3.21) can be easily resolved for \( l \geq 6 \):

\[
a_{l,n} = \frac{1}{l - 4}.
\]

(3.22)
As a result of (3.20) and (3.22), the function
\[
\psi_n = r \int_0^\pi d\alpha \cos \alpha \left( \ln \frac{1}{R} + \sum_{l=6}^{n} \frac{1}{l - 4} z^{n-l} \right)
\] (3.23)
for \(n > 4\) is the solution we just searched for.

Here it is pertinent to consider the most degenerate case \(n = 2\) where the "z part" of the operator (3.4) is absent, that is
\[
\Delta^{(2)} \psi_2 = \Delta_r^{(2)} \psi_2 = r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \psi_2 = 0.
\] (3.24)
As expected, the function
\[
\psi_2 = r \int_0^\pi d\alpha \cos \alpha \ln \frac{1}{\rho}.
\] (3.25)
is a vortex-ring-like solution to (3.24) with \(\rho\) defined by (3.7b); during this derivation the integral identity similar to (3.18) is used:
\[
\int_0^\pi d\alpha \cos \alpha \left( \ln \frac{1}{\rho} - \frac{ar \cos \alpha}{\rho^2} + 2 \frac{a^2 r^2 \sin^2 \alpha}{\rho^4} \right) =
\int_0^\pi d\alpha \frac{\partial}{\partial \alpha} \left( \sin \alpha \ln \frac{1}{\rho} - \frac{ar \sin \alpha \cos \alpha}{\rho^2} \right) = 0.
\]
From the physical viewpoint, the solution (3.25) can be hardly interpreted as a proper vortex ring because there is no an orthogonal direction (z) to provide rotation around the ring axis. Nevertheless, it is worth including in our collection because it is required for a study described in Sec.7.

The distinct feature of the solution (3.23) is that the coefficients \(a_{l,n}\) do not involve \(n\), so that the \((n+2)D\) solution merely acquires an extra term as compared with the \(nD\) one. For example, the 4D solution and the three subsequent solutions with \(l \geq 6\) are as follows
\[
\psi_4 = r \int_0^\pi d\alpha \cos \alpha \ln \frac{1}{R},
\]
\[
\psi_6 = r \int_0^\pi d\alpha \cos \alpha \left( \ln \frac{1}{R} + \frac{1}{2} \frac{z^2}{R^2} \right),
\]
\[
\psi_8 = r \int_0^\pi d\alpha \cos \alpha \left( \ln \frac{1}{R} + \frac{1}{2} \frac{z^2}{R^2} + \frac{1}{4} \frac{z^4}{R^4} \right),
\]
\[
\psi_{10} = r \int_0^\pi d\alpha \cos \alpha \left( \ln \frac{1}{R} + \frac{1}{2} \frac{z^2}{R^2} + \frac{1}{4} \frac{z^4}{R^4} + \frac{1}{6} \frac{z^6}{R^6} \right).
\]
From the appearance of the solutions (3.15), (3.23) [and (3.25)], a far asymptotics is easily seen because the term $\ln(1/R)$ is always leading: When $r \to \infty$, the integrand of $\psi_n$ is always $\sim \ln(1/r)$ starting from $n = 2$ independently of $n$, and when $z \to \infty$, it is $\sim \ln(1/z)$ starting from $n = 4$ independently of $n$ as well. This is just the 2D asymptotics.

In the hypersurface $z = 0$, all these solutions behave like the 2D ones, $\sim \ln(1/r)$; whereas for every finite $z$, the integrand of $\psi_n$ is always $\sim \ln(1/z) + Const$ when $r \to 0$.

4 Known and unknown properties of Laplacians and relevant quantities

4.1 The case of spherical symmetry

It is widely known that the scalar Poisson equation in $n$ dimensions with the unit point source

$$\Delta^{(n)} \Phi_n = -\sigma_n \delta(\vec{R}) \equiv s(\vec{R}) \quad (4.1)$$

has solutions

$$\Phi_n = \frac{1}{n-2} \frac{1}{|\vec{R}|^{n-2}}, \quad \text{for} \quad n \geq 2,$$

$$\Phi_2 = \ln \frac{1}{|\vec{R}|}, \quad \text{for} \quad n = 2,$$

where $\vec{R}$ is the radius-vector originating from the point source, and

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (4.2)$$

with $\Gamma$ the gamma function is the surface of the $(n-1)$-dimensional sphere of unit radius. The expressions for equation (4.1) and its solutions are indeed covariant, without referring to any coordinate frame.

In the spherical coordinate frame (the $x$ frame), the entire expression for the Laplacian involving all the $n-1$ angles is given in Appendix A. Consider the spherically symmetric case so that $\Phi_n(\vec{x}) = \Phi_n(x)$ and the only radial dependence in (A.2) survives. We also replace $\delta(\vec{x})$ on $\delta(x)$ using the formula

$$\delta(\vec{x}) = \frac{\delta(x)}{\sigma_n x^{n-1}} \quad (4.3)$$

\(^4\)For accuracy, we add the index "+" to $\delta$, because one often defines $\int_0^\infty dx \delta(x)\phi(x) = \phi(0)/2$ for the singular point $x = 0$ coinciding with a limit of integration, see, e.g., \[.\]
provided by the definition

\[
\int_{x(n)} \tau^{(n)} \delta(\xi) \phi(x) = \int_0^\infty dx \, \delta_+(x) \phi(x) = \phi(0). \tag{4.4}
\]

Then, the truncated equation (4.1) acquires the form

\[
\Delta_x^{(n)} \Phi_n(x) \equiv \frac{1}{x^{n-1}} \frac{\partial}{\partial x} x^{n-1} \frac{\partial}{\partial x} \Phi_n(x) = - \frac{\delta_+(x)}{x^{n-1}} \equiv s_x^{(n)}(x), \tag{4.5}
\]

having solutions

\[
\Phi_n(x) = \frac{1}{n-2} \frac{1}{x^{n-2}}, \quad \text{for} \quad n \geq 2, \tag{4.6a}
\]

\[
\Phi_2(x) = \ln \frac{1}{x}, \quad \text{for} \quad n = 2. \tag{4.6b}
\]

Now consider a transformation which converts the \((n-2)\)D solution (4.6), \(\Phi_{n-2}(x)\), into the \(n\)D one, \(\Phi_n(x)\). For \(n \geq 4\):

\[
\Phi_n(x) = - \frac{1}{n-2} \frac{1}{x} \frac{\partial}{\partial x} \Phi_{n-2}(x). \tag{4.7}
\]

The equality (4.7) is evident. We see that odd-\(n\)D solutions are transformed into odd-\(n\)D ones starting from \(n = 5\), and even-\(n\)D solutions are transformed into even-\(n\)D ones starting from \(n = 4\). In other words, every \(n\)D solution can be obtained in applying the above transformation by required number of times to \(\Phi_3\) or to \(\Phi_2\).

Moreover, taking into account the following operator rearrangement

\[
\frac{1}{x} \frac{\partial}{\partial x} \Delta_x^{(n-2)} = \Delta_x^{(n)} \frac{1}{x} \frac{\partial}{\partial x} \tag{4.8}
\]

we obtain

\[
\Delta_x^{(n)} \Phi_n(x) = - \frac{1}{n-2} \frac{1}{x} \frac{\partial}{\partial x} \Delta_x^{(n-2)} \Phi_{n-2}(x). \tag{4.9}
\]

The equality (4.9) means that the l.h.s. of the Poisson equation (4.5) is transformed similar to the solution itself.

Independently, the direct calculation with the explicit form of \(s^{(n-2)}(x)\) [see the definition on the r.h.s. of (4.5)] shows that the similar situation takes place for sources:

\[
s_x^{(n)}(x) = - \frac{1}{n-2} \frac{1}{x} \frac{\partial}{\partial x} s_x^{(n-2)}(x), \tag{4.10}
\]

after using the formal equality \(x \delta_+(x) = -\delta_+(x)\) where the prime denotes differentiation.
4.2 The case of double-spherical symmetry

In the \((r, z)\) coordinate frame introduced before in Sec. 3, the central point source in (4.1) for the double-spherical symmetry can be expressed as follows \((n > 2)\)

\[
s^{(n)}(\vec{R}) = -\sigma_n \delta(r) \delta(z) = -\sigma_n \frac{\delta_+ (r)}{\sigma_2 r} \frac{\delta_+ (z)}{\sigma_{n-2} z^{n-3}} = -\frac{1}{n - 2} \frac{\sigma_+ (r)}{r} \frac{\delta_+ (z)}{z^{n-3}},
\]

(4.11)

where we have used (4.3), and the equality

\[
\sigma_n = \frac{\sigma_2 \sigma_{n-2}}{n - 2}
\]

(4.12)

which is easy to verify with referring to the definition (4.2), also note that \(\sigma_2 = 2\pi\) and \(\sigma_1 = 2\).

For the \((r, z)\) dependence of a solution, the truncated operator (A.6) combined with the source (4.11) gives the Poisson-like equation \((n > 2)\)

\[
\Delta_{r, z}^{(n)} \Xi_n(r, z) \equiv \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z} \right) \Xi_n(r, z) = -\frac{\delta_+ (r)}{r} \frac{\delta_+ (z)}{z^{n-3}} \equiv s^{(n)}_{r, z}
\]

(4.13a)

where the solution is

\[
\Xi_n(r, z) = \frac{1}{(r^2 + z^2)^{(n-2)/2}} \equiv \frac{1}{R^{n-2}}.
\]

(4.14a)

Note that due to the remarkable formula (4.12), the factor \((n - 2)\) is removed both on the r.h.s. of (4.13a) and in the solution (4.14a).

The case \(n = 2\) is degenerate because there are no the \(z\) space, and the Poisson equation

\[
\Delta_{r, z}^{(n)} \Xi_2(r) = \Delta^{(2)} \Xi_2(r) = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \Xi_2 = -\frac{\delta_+ (r)}{r}
\]

(4.13b)

has the trivial solution

\[
\Xi_2 = \ln \frac{1}{r}.
\]

(4.14b)

In the case of the \((r, z)\) frame, we can also introduce a transformation with properties similar to the above one despite of the presence of the two variables, \(r\) and \(z\). It is easy to verify that the required transformation is as follows \((n \geq 5)\)

\[
\Xi_n(r, z) = -\frac{1}{n - 4} \frac{1}{z} \frac{\partial}{\partial z} \Xi_{n-2}(r, z) \equiv f_{L}^{(n, n-2)} \Xi_{n-2}(r, z).
\]

(4.15)

The appearance of the factor \((n - 4)\) instead of \((n - 2)\) in (4.15) is due to that the solutions (4.6a) and (4.14a) have distinct coefficients. As before, the odd solutions are obtainable from the odd ones starting from \(n = 5\). However unlike the previous case, our transformation works starting from \(n = 6\) for even solutions. The point is that, as mentioned, the \(z\) dependence of the Laplace operator and
the corresponding solution disappears for \( n = 2 \), see (4.14b), that is why \( \Xi_{4} \) cannot be obtained from \( \Xi_{2} \).

Here, the rearrangement of the (4.8)-type also exists,

\[
\frac{1}{z} \frac{\partial}{\partial z} \Delta_{r,z}^{(n-2)} = \Delta_{r,z}^{(n)} \frac{1}{z} \frac{\partial}{\partial z},
\]

and leads to the equality analogous to (4.9):

\[
\Delta_{r,z}^{(n)} \Xi_{n} = f_{L}^{(n,n-2)} \Delta_{r,z}^{(n-2)} \Xi_{n-2}. \tag{4.16}
\]

For the \( \delta \)-source in (4.13a), we also have

\[
s_{r,z}^{(n)} = f_{L}^{(n,n-2)} s_{r,z}^{(n-2)}. \tag{4.17}
\]

We call the set of quantities \( \Xi_{n}, \Delta_{r,z}^{(n)} \Xi_{n}, \) and \( s_{r,z}^{(n)} \) the Laplace set \( L_{n} \):

\[
L_{n} = \{ \Xi_{n}, \Delta_{r,z}^{(n)} \Xi_{n}, s_{r,z}^{(n)} \}, \tag{4.18}
\]

so that the equalities (4.15), (4.16) and (4.17) acquire the compact notation

\[
L_{n} = f_{L}^{(n,n-2)} L_{n-2}. \tag{4.19}
\]

For odd and even \( n \), any quantity from the Laplace set can be expressed as follows

\[
L_{n} = f_{L}^{(n,n-2)} f_{L}^{(n-2,n-4)} \ldots f_{L}^{(5,3)} L_{3} = \frac{1}{(n-4)!!} \left( - \frac{1}{z} \frac{\partial}{\partial z} \right)^{(n-3)/2} L_{3} \tag{4.20}
\]

and

\[
L_{n} = f_{L}^{(n,n-2)} f_{L}^{(n-2,n-4)} \ldots f_{L}^{(6,4)} L_{4} = \frac{1}{(n-4)!!} \left( - \frac{1}{z} \frac{\partial}{\partial z} \right)^{(n-4)/2} L_{4}, \tag{4.21}
\]

respectively.

### 4.3 Methodological extraction of \( \delta \)-sources

In methodological goals and for future applications in Sec.6, we show how to prove that the nD solutions (4.14a) really provide corresponding \( \delta \) sources, i.e., to prove (4.13a). We shall demonstrate it by two ways. The first of them can be found in any suitable manual in functional analysis, see e.g., [7]. This is very trivial derivation in the case of the \( x \) frame, however it has some subtleties in the case of the \((r,z)\) frame. The second way, just invented, consists in using the introduced transformation \( f_{L}^{(n,n-2)} \) in order to construct a proof by a recurrent procedure.
Consider a function \( \phi(r, z) \in \mathcal{D}(\mathcal{G}) \), where \( \mathcal{D}(\mathcal{G}) \) denotes the class of trial functions, i.e. that of finite \( C^\infty(\mathcal{G}) \) functions in the domain \( \mathcal{G}(r, z) \): \( \text{supp} \ \phi \subset \mathcal{G}(r, z) \). Then,

\[
\int_{\tau^{(n)}} d\tau^{(n)} \Delta^{(n)}_{r,z} \Xi_n = \int_{\tau^{(n)}} d\tau^{(n)} (\Delta^{(n)}_{r,z} \phi) \Xi_n \tag{4.22}
\]

where \( d\tau^{(n)} = d\sigma_2 d\sigma_{n-2} r z^{n-3} dr dz \), see \((A.4)\), and integration is performed over a large enough volume \( \tau^{(n)} \), so that the ranges of values of the \( r \) and \( z \) coordinates contain \( \mathcal{G}(r, z) \).

**Traditional way.** Following \([7]\) and other manuals, we can write

\[
\int_{\tau^{(n)}} d\tau^{(n)} (\Delta^{(n)}_{r,z} \phi) \Xi_n = \lim_{\varepsilon, \eta \to 0} \int_{\tau^{(n)}_{\varepsilon, \eta}} d\tau^{(n)} (\Delta^{(n)}_{r,z} \phi) \Xi_n
\]

using the property \((4.22)\), where the volume \( \tau^{(n)}_{\varepsilon, \eta} \) means that we exclude from \( \tau^{(n)} \) the small domain near the point \( r = 0, z = 0; \tau^{(n)}_{\varepsilon, \eta}; r \geq \varepsilon, z \geq \eta \). The quantities \( J_r \) and \( J_z \) denote the surface integrals:

\[
J_r = J_{r1} + J_{r2},
\]

\[
J_{r1} = \int_{S_r} dS_r \left[ \phi \frac{\partial \Xi_n}{\partial z} \right]_{z=\eta} = -(n-2)\sigma_2 \sigma_{n-2} \eta^{n-2} \int_0^{\varepsilon} dr \phi(r, \eta) \frac{1}{(r^2 + \eta^2)^{n/2}}, \tag{4.24}
\]

\[
J_{r2} = -\int_{S_r} dS_r \left[ \frac{\partial \phi}{\partial z} \Xi_n \right]_{z=\eta} = -\sigma_2 \sigma_{n-2} \eta^{n-2} \int_0^{\varepsilon} dr \left. \frac{\partial \phi}{\partial z} \right|_{z=\eta} \frac{1}{(r^2 + \eta^2)^{(n-2)/2}},
\]

and

\[
J_z = J_{z1} + J_{z2},
\]

\[
J_{z1} = \int_{S_z} dS_z \left[ \phi \frac{\partial \Xi_n}{\partial r} \right]_{r=\varepsilon} = -(n-2)\sigma_2 \sigma_{n-2} \varepsilon^{n-2} \int_0^{\eta} dz \varepsilon z^{n-3} \phi(\varepsilon, z) \frac{1}{(\varepsilon^2 + z^2)^{n/2}}, \tag{4.26}
\]

\[
J_{z2} = -\int_{S_z} dS_z \left[ \frac{\partial \phi}{\partial r} \Xi_n \right]_{r=\varepsilon} = -\sigma_2 \sigma_{n-2} \varepsilon^{n-2} \int_0^{\eta} dz \varepsilon z^{n-3} \left. \frac{\partial \phi}{\partial r} \right|_{r=\varepsilon} \frac{1}{(\varepsilon^2 + z^2)^{(n-2)/2}}, \tag{4.27}
\]

where

\[
dS_r = d\sigma_2 d\sigma_{n-2} dr \eta^{n-3}
\]

and

\[
dS_z = d\sigma_2 d\sigma_{n-2} dz \varepsilon z^{n-3}
\]
are the \((n-1)\)-dimensional surface elements. The 3-dimensional analogue of \(S_r\) is the base surface of the cylinder \(0 \leq r \leq \varepsilon, -\eta \leq z \leq \eta\), and that of \(S_z\) is the lateral surface of the same cylinder. (Note that a 0-dimensional sphere here consists of the two isolated points \(z = -\eta\) and \(z = \eta\).)

In returning to (4.23), we note, first of all, that the prelimit term involving \(\Delta_{r,z}^{(n)} \Xi_n\) vanishes. Further, there are two ways of making a limit procedure depending on what limit (with respect to \(\varepsilon\) or \(\eta\)) is taken first. From the expressions (4.24), (4.25) and (4.26), (4.27) we see that always

\[
\lim_{\varepsilon \to 0} J_r = 0, \quad \lim_{\eta \to 0} J_z = 0
\]

meaning that when the order of taking limits in (4.23) is \(\varepsilon \to 0\) and then \(\eta \to 0\) (or \(\eta \to 0\), and then \(\varepsilon \to 0\)) the only integral \(J_z\) (or \(J_r\)) "works".

It can be easily shown that the following limit expressions of the integrals containing the derivatives of \(\phi\) vanish

\[
\lim_{\eta \to 0} J_{r2} = 0, \quad \lim_{\varepsilon \to 0} J_{z2} = 0.
\]

The calculation of the remaining integrals (4.24) and (4.26) will be exhibited in more details. The trivial integration in (4.24), mutual for both odd-\(n\)D and even-\(n\)D cases, leads to

\[
\lim_{\eta \to 0} J_{r1} = -\sigma_2 \sigma_{n-2} \bar{\phi}_\varepsilon
\]

with \(\bar{\phi}_\varepsilon\) some average value of \(\phi(r, 0)\) over the interval \(0 \leq r \leq \varepsilon\).

As to the integral in (4.26), we should use different handbook integrals for the odd-\(n\)D and even-\(n\)D cases. For \(n = 2q + 1\), the integral in (4.26) corresponds to the following one

\[
\int dz \frac{z^{2q-1}}{(\varepsilon^2 + z^2)^{q+1/2}} = \frac{\varepsilon^{2q-1}}{(2q-1)\varepsilon^2(\varepsilon^2 + z^2)^{q-1/2}},
\]

see [8], p.91, no.7. For even \(n\) it is more convenient to impose \(n = 2(m + 2)\), \(m = 0\) corresponding to \(n = 4\), then the integral in (4.26) can be represented as follows, see loc. cit., p.30, no.6,

\[
\int dz \frac{z^{n-3}}{(\varepsilon^2 + z^2)^{n/2}} = \frac{1}{2} \int du \frac{u^m}{(\varepsilon^2 + u)^{m+2}} = -\frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} \frac{(-\varepsilon^2)^{m-k}}{(m-k+1)(\varepsilon^2 + u)^{m-k+1}}
\]

where we have denoted \(u = z^2\) and

\[
\binom{m}{k} = \frac{m!}{k!(m-k)!}
\]

is the standard convention for a binomial coefficient. However, in both the odd-\(n\) and even-\(n\) cases, we have the same limit expression

\[
\lim_{\varepsilon \to 0} J_{z1} = -\sigma_2 \sigma_{n-2} \bar{\phi}_\eta
\]
with \( \bar{\phi}_\eta \) some average value of \( \phi(0, z) \) over the interval \( 0 \leq z \leq \eta \). (For the even-\( n \)D case we have required to additionally derive that
\[
\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{m + 1} = \frac{1}{m + 1} = \frac{2}{n - 2},
\]

note that the first sum in the square brackets is zero, see loc. cit, p.606, no.3.)

In returning to (4.23),
\[
\int d\tau^{(n)} \Delta^{(n)}_{r,z} \Xi_n = \lim_{\varepsilon \to 0} \left( \lim_{\eta \to 0} J_{r1} \right) = \lim_{\eta \to 0} \left( \lim_{\varepsilon \to 0} J_{z1} \right)
= -\sigma_2 \sigma_{n-2} \phi(0, 0) = -(n-2) \sigma_n \phi(0, 0),
\]

and our proof is finished. The appearance of the factor \( (n - 2) \) is due to that it was removed in the solution (4.14a). Thus, (4.28) is equivalent to equation (4.13a).

Below we shall use (4.28) in an equivalent form
\[
\int_0^\infty dr \int_0^\infty dz z^{n-3} \phi \Delta^{(n)}_{r,z} \Xi_n \equiv \left\langle r z^{n-3} \phi, \Delta^{(n)}_{r,z} \Xi_n \right\rangle = -\phi(0, 0).
\]

**The recurrent way.** Let us propose that we have proved, perhaps by the first method, that
\[
\Delta^{(3)}_{r,z} \Xi_3(r, z) = -\frac{\delta_+(r)}{r} \delta_+(z)
\]

and
\[
\Delta^{(4)}_{r,z} \Xi_4(r, z) = -\frac{\delta_+(r)}{r} \frac{\delta_+(z)}{z},
\]

after that we can use the mathematical induction method. We prove that if for \( k = 5, 6, 7, 8, \ldots, \)
\[
\Delta^{(k-2)}_{r,z} \Xi_{k-2}(r, z) = -\frac{\delta_+(r)}{r} \frac{\delta_+(z)}{z^{k-5}},
\]

then
\[
\Delta^{(k)}_{r,z} \Xi_k(r, z) = -\frac{\delta_+(r)}{r} \frac{\delta_+(z)}{z^{k-3}}.
\]

Equation (4.30) means that [see (4.29)]
\[
\left\langle r z^{k-5} \phi, \Delta^{(k-2)}_{r,z} \Xi_{k-2} \right\rangle = -\phi(0, 0)
\]

for any \( \phi(r, z) \supset D(G) \). The required proof is given below by a chain of relations with using the transformation \( f_{L}^{(k,k-2)} \):
\[
\left\langle r z^{k-3} \phi, \Delta^{(k)}_{r,z} \Xi_k \right\rangle = \left\langle r z^{k-3} \phi, f_{L}^{(k,k-2)}(k-2) \Delta^{(k-2)}_{r,z} \Xi_{k-2} \right\rangle.
\]
Here we recall that for every $\phi$ belonging to the mentioned class, first, any derivatives of $\phi$ belong to the same class, and, second, for any function $F(r, z)$ the equality

$$\left\langle \phi, \frac{\partial F}{\partial z} \right\rangle = -\left\langle F, \frac{\partial \phi}{\partial z} \right\rangle,$$

holds. Let us continue,

$$(4.33) = \frac{1}{k-4} \left\langle r \frac{\partial}{\partial z} (\phi z^{k-4}), \Delta^{(k-2)} r z^{k-2} \right\rangle =$$

$$\left\langle r z^{k-5} \left(\phi + \frac{z}{k-4} \frac{\partial \phi}{\partial z}\right), \Delta^{(k-2)} r z^{k-2} \right\rangle = -\phi(0, 0).$$

In the latter expression, the second term in parentheses vanishes when integrated with $\Delta^{(k-2)} r z^{k-2}$. This is due to the (formal) equality $z \delta_+(z) = 0$ after using (4.30), or this fact can be verified immediately. The first term there just gives (4.32). Thus, (4.31) is proved.

5 Anti-Laplacians: entirely unknown properties

5.1 Connections of anti-Laplacians with Laplacians

Until now we have already introduced the anti-$r$-Laplacian and anti-double-Laplacian operators. Here, we define the remaining anti-Laplacians and derive some relations connecting any anti-Laplacians with Laplacians. The way by which (2.6) was obtained suggests that we propose a relation given below.

Consider for simplicity the $n$D space in the Cartesian coordinate frame $x_1, x_2, \ldots, x_n$, let $x = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. We denote

$$\Delta^{(n)}(\text{Cart}) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

the $\Delta^{(n)}$ operator in the Cartesian frame. For any function $F$ depending only on $x$: $F = F(x)$, and for any coordinate $x_k$, the following proposition holds:

$$\Delta^{(n)}(\text{Cart}) \left(\frac{x_k}{x^n} F(x)\right) = \frac{x_k}{x^n} \Delta^{(n)} x F(x)$$

(5.1)

where the operator

$$\Delta_x^{(n)} \equiv x^{n-1} \frac{\partial}{\partial x} \frac{1}{x^{n-1}} \frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2} - \frac{n-1}{x} \frac{\partial}{\partial x}$$

should be called anti-Laplacian. (The reader has already understood that the prefix "anti" is given because of the inverse position of "$x$" and "$1/x$" in the first
expression and the opposite sign in the second expression as compared to the Laplacian).

In fact, Eq.(5.1) has a covariant meaning: it is independent of a coordinate frame. We reformulate it in the $x$ frame. Let $\lambda_1$ be a senior angle and $x_1$ be a Cartesian axis associated with $\lambda_1$ (see Appendix A. We could also take an arbitrary axis $x_k$ involving other angles, however this means dealing with more cumbersome expressions.) Then,

$$\Delta_{x,\lambda_1}^{(n)} \left( \frac{\cos \lambda_1}{x^{n-1}} F(x) \right) = \frac{\cos \lambda_1}{x^{n-1}} \Delta_x^{(n)} F(x) \quad (5.2)$$

where we have denoted

$$\Delta_{x,\lambda_1}^{(n)} = \frac{1}{x^{n-1}} \frac{\partial}{\partial x} x^{n-1} \frac{\partial}{\partial x} + \frac{1}{x^2 \sin^{n-2} \lambda_1} \frac{\partial}{\partial \lambda_1} \sin^{n-2} \lambda_1 \frac{\partial}{\partial \lambda_1},$$

the part of a full operator (A.2) acting on any function of $x$ and $\lambda_1$.

In the $(r,z)$ frame, the $n$D anti-$r$-Laplacian can be defined by analogy with (2.4):

$$\Delta_r^{(n)} = r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z}, \quad (5.3)$$

and we also define the operator

$$\Delta_z^{(n)} \equiv \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} \frac{1}{z^{n-3}} \frac{\partial}{\partial z}, \quad (5.4)$$

which following our "taxonomy" is natural to call anti-$z$-Laplacian and which plays a key role in our investigations. It accomplishes our collection of anti-Laplacians for a given splitting of the $n$D space.

Given the $(r, z)$ frame, the relation (5.2) can be written independently for the $r$ and/or $z$ spaces. For example, recall Eq.(2.6) in 3 dimensions. In $n$ dimensions, for any $F(r,z)$, the analogue of (2.6) is

$$\Delta_{r,\varphi,z}^{(n)} \left( \frac{\cos \varphi}{r} F(r,z) \right) = \frac{\cos \varphi}{r} \Delta_{r}^{(n)} F(r,z), \quad (5.5)$$

plus there are the relations in the $z$ space:

$$\Delta_{r,\varphi}^{(n)} \left( \frac{\cos \theta_1}{z^{n-3}} F(r,z) \right) = \frac{\cos \theta_1}{z^{n-3}} \Delta_{z}^{(n)} F(r,z) \quad (5.6)$$

and

$$\Delta_{r,\varphi,z}^{(n)} \left( \frac{\cos \varphi}{r} \frac{\cos \theta_1}{z^{n-3}} F(r,z) \right) = \frac{\cos \varphi}{r} \frac{\cos \theta_1}{z^{n-3}} \Delta_{r}^{(n)} F(r,z) \quad (5.7)$$

where we have denoted

$$\Delta_{r,\varphi,z}^{(n)} = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$
\[
\Delta^{(n)}_{r,z,\theta_1} = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z} + \frac{1}{z^2 \sin^{n-4} \theta_1} \frac{\partial}{\partial \theta_1} \sin^{n-4} \theta_1 \frac{\partial}{\partial \theta_1},
\]
and
\[
\Delta^{(n)}_{r,\varphi,z,\theta_1} = \Delta^{(n)}_{r,\varphi,z} + \frac{1}{z^2 \sin^{n-4} \theta_1} \frac{\partial}{\partial \theta_1} \sin^{n-4} \theta_1 \frac{\partial}{\partial \theta_1},
\]
the parts of the expression (A.6) involving required variables.

Besides of the relations (5.5), (5.6) and (5.7), there exist some evident differential relations. Let us turn again to the \(x\) frame. Then, for any function \(F(x)\),
\[
x^{n-1} \frac{\partial}{\partial x} \Delta_{x}^{(n)} F(x) = \overline{\Delta}^{(n)}_{x} \left( x^{n-1} \frac{\partial}{\partial x} F(x) \right) \tag{5.8a}
\]
and
\[
\frac{1}{x^{n-1}} \frac{\partial}{\partial x} \overline{\Delta}^{(n)}_{x} F(x) = \Delta_{x}^{(n)} \left( \frac{1}{x^{n-1}} \frac{\partial}{\partial x} F(x) \right). \tag{5.8b}
\]
The equalities (5.8) are in fact identities after using explicit forms of operators. Thus, we have demonstrated a principle of constructing such the relations.

In the \((r, z)\) frame, a variety of relations similar to (5.8a) arises, based on the fact of commutativity of the derivatives \(\partial/\partial r\) and \(\partial/\partial z\). For any \(F(r, z)\)
\[
r \frac{\partial}{\partial r} \Delta_{r,z}^{(n)} F(r, z) = \overline{\Delta}^{(n)}_{r} \left( r \frac{\partial}{\partial r} F(r, z) \right), \tag{5.9a}
\]
\[
z^{n-3} \frac{\partial}{\partial z} \Delta_{r,z}^{(n)} F(r, z) = \overline{\Delta}^{(n)}_{z} \left( z^{n-3} \frac{\partial}{\partial z} F(r, z) \right), \tag{5.9b}
\]
\[
z^{n-3} \frac{\partial}{\partial z} \overline{\Delta}^{(n)}_{r} F(r, z) = \overline{\Delta}^{(n)}_{z} \left( z^{n-3} \frac{\partial}{\partial z} F(r, z) \right). \tag{5.9c}
\]
There are no necessity to bring all the remaining relations which correspond to (5.8b). We shall use below one of them, namely
\[
\frac{1}{z^{n-3}} \frac{\partial}{\partial z} \overline{\Delta}^{(n)}_{r} F(r, z) = \Delta_{r,z}^{(n)} \left( \frac{1}{z^{n-3}} \frac{\partial}{\partial z} F(r, z) \right). \tag{5.10}
\]

Moreover, we can in principle combine the relations of the type (5.5)–(5.7) and those of the type (5.9) and (5.10).

5.2 Potential-like solutions of homogeneous equations

In keeping in mind that the point-like source is located at the point \(r = 0, z = 0\), it turns out that \(n\)D solutions to homogeneous equations involving the operator (5.4), i.e., the anti-\(z\)-Laplace equations
\[
\overline{\Delta}_{z}^{(n)} \Psi_n \equiv \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-3}} \frac{\partial}{\partial z} \right) \Psi_n = 0, \tag{5.11}
\]
are the simplified versions of the solutions (3.11) and (3.15), (3.23). As before, we should consider separately the cases of odd and even dimensions.

In the odd-$n$D case, the function

$$
\Psi_n = \sum_{k=3}^{n-1} a_{k,n} \frac{z^{k-3}}{R^{k-2}}
$$

(5.12)

is the solution to (5.11) with $a_{k,n}$ given by (3.12) and (3.13), recall that $R = \sqrt{r^2 + z^2}$. It is worth noting that the recurrence relation (3.12) for $a_{k,n}$ is a necessary and sufficient condition of the validity of (5.12), without any auxiliary construction like the integral identity (3.9).

In the even-$n$D case, the functions

$$
\Psi_4 = \ln \frac{1}{\sqrt{R}},
$$

(5.13a)

and

$$
\Psi_n = \ln \frac{1}{R} + \sum_{k=6}^{n-1} \frac{1}{k-4} \frac{z^{k-4}}{R^{k-4}}
$$

(5.13b)

for $n \geq 6$ are the solution to (5.11).

The solutions (5.13) have an interesting property:

$$
\frac{\partial}{\partial z} \Psi_n = -\frac{z^{n-3}}{R^{n-2}},
$$

(5.14)

which will be used for finding point-like sources in Sec.6.

### 5.3 Transformations from $(n - 2)$D quantities to $n$D ones

For anti-Laplacians there also exist a transformation converting an $(n - 2)$D solution to an $n$D one, mutual for odd-$n$D and even-$n$D cases. The transformation acts starting from $n \geq 5$:

$$
\Psi_n = f_A^{(n,n-2)} \Psi_{n-2} = -\frac{z^{n-3}}{n-4} \frac{\partial}{\partial z} \left( \frac{1}{z^{n-4}} \Psi_{n-2} \right).
$$

(5.15)

Using the differential rearrangement

$$
z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-4}} \Delta_z^{(n-2)} \Delta_z^{(n)} = \Delta_z^{(n)} z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-4}},
$$

we also find

$$
\Delta_z^{(n)} \Psi_n = f_A^{(n,n-2)} \Delta_z^{(n-2)} \Psi_{n-2}.
$$

(5.16)

As before, we introduce the anti-$z$-Laplace set $A_n$:

$$
A_n = \{ \Psi_n, \Delta_z^{(n)} \Psi_n, s_z^{(n)} \}.
$$

(5.17)
where $\bar{s}_z^{(n)}$ is a point-like source searched for, which gives rise to the solution $\Psi_n$:

$$\bar{\Delta}_z^{(n)} \Psi_n = \bar{s}_z^{(n)},$$

and whose explicit form we are just going to find using the transformation

$$\bar{s}_z^{(n)} = f_A^{(n,n-2)} \bar{s}_z^{(n-2)}. \quad (5.18)$$

This is in contrast to Sec.4, where the validity of the transformation $f_L^{(n,n-2)}$ acting on sources was merely checked. This will be done in the next sections separately for the odd-$n$D and even-$n$D cases. In compact notations, the equalities (5.15), (5.16) and (5.18) acquire the form

$$A_n = f_A^{(n,n-2)} A_{n-2}. \quad (5.19)$$

Any $n$D quantity from the anti-$z$-Laplace set (5.17) can be expressed via the 3D or 4D one. For odd $n$,

$$A_n = f_A^{(n,n-2)} f_A^{(n-2,n-4)} \ldots f_A^{(5,3)} A_3 = \frac{z^{n-3}}{(n-4)!!} \left( -\frac{\partial}{\partial z} \frac{1}{z} \right)^{(n-3)/2} A_3, \quad (5.19)$$

and for even $n$,

$$A_n = f_A^{(n,n-2)} f_A^{(n-2,n-4)} \ldots f_A^{(6,4)} A_4 = \frac{z^{n-3}}{(n-4)!!} \left( -\frac{\partial}{\partial z} \frac{1}{z} \right)^{(n-4)/2} \frac{A_4}{z}. \quad (5.20)$$

## 6 Obtaining the point sources for potential-like solutions to anti-$z$-Laplace equations

### 6.1 The simple odd-$n$D case

In the odd-$n$D case, the 3D quantities of the Laplace and anti-Laplace sets coincide: $A_3 = L_3$. Indeed, $\Delta_{r,z}^{(3)} = \bar{\Delta}_z^{(3)}$, $\Xi_3 = \Psi_3$ and $s_{r,z}^{(3)} = \bar{s}_z^{(3)}$. This fact outlines several ways to calculate $\bar{s}_z^{(n)}$. The first of them, the most simple and the most formal is to apply formula (5.19) to $\bar{s}_z^{(3)}$, the result being

$$\bar{s}_z^{(n)} = -\left( \sum_{k=3}^{n} a_{k,n} \right) \frac{\delta_+(r)}{r} \delta_+(z) = -\frac{(n-3)!!}{(n-4)!!} \frac{\delta_+(r)}{r} \delta_+(z), \quad (6.1)$$

see (B.3). Perhaps, anybody could say that there should be taken more care in dealing with formal expressions. That is why we give the second (improved) way for obtaining the same result.

If to compare the expressions (4.20) and (5.19) for the transformations of the Laplace and anti-Laplace sets, respectively, it is clear that the later differs
from the former by the presence of the factor \( z^{n-3} \) and by the permutation of
1/z and \( \partial/\partial z \). This fact enables us to express algebraically any quantity \( A_n \) via
\( L_3, L_5, \ldots, L_{n-2}, L_n \). The expected expression,

\[
A_n = \sum_{k=3}^{n} a_{k,n} z^{k-3} L_k,
\]

(6.2)
can be obtained by a recurrent way. Let us choose the second quantities from
the sets (4.18) and (5.17), then from (6.2)

\[
\Delta_z^{(n)} \Psi_n = \sum_{k=3}^{n} a_{k,n} z^{k-3} \Delta^{(k)} \Xi_k.
\]

(6.3)

Further, we multiply (6.3) on \( r\phi \), where \( \phi(r, z) \) belongs to the above-mentioned
class of trial functions, and perform integration with respect to \( r \) and \( z \), cf. (4.29):

\[
\langle r\phi, \Delta_z^{(n)} \Psi_n \rangle = \sum_{k=3}^{n} a_{k,n} \langle rz^{k-3} \phi, \Delta^{(k)} \Xi_k \rangle.
\]

(6.4)

The \( k \)-th integrals on the r.h.s. of (6.4) were already calculated, see (4.33) and
(4.34), thus

\[
\langle r\phi, \Delta_z^{(n)} \Psi_n \rangle = -\left( \sum_{k=3}^{n} a_{k,n} \right) \phi(0, 0),
\]

(6.5)

meaning the validity of (6.1).

At least, the third and the most accurate way is to examine the limit of the
(4.23)-type:

\[
\langle r\phi, \Delta_z^{(n)} \Psi_n \rangle = \lim_{\varepsilon, \eta \to 0} (\bar{J}_r + \bar{J}_z)
\]

with the following surface terms obtained with the use of (6.3):

\[
\bar{J}_r = \left[ \int_0^{\varepsilon} dr r \left( \phi \sum_{k=3}^{n} a_{k,n} z^{k-3} \frac{\partial \Xi_k}{\partial z} - \frac{\partial \phi}{\partial z} \Psi_n \right) \right]_{z=\eta},
\]

\[
\bar{J}_z = \left[ \int_0^{\eta} dz r \left( \phi \frac{\partial \Psi_n}{\partial r} - \frac{\partial \phi}{\partial r} \Psi_n \right) \right]_{r=\varepsilon}.
\]

The result, being independent of the order of taking limits with respect to \( \varepsilon \) and
\( \eta \), certainly coincides with (6.5).

In order to avoid a sum factor on the r.h.s. of (6.1) or (6.5), it is worth
replacing the coefficients \( a_{k,n} \) in (5.12) by the renormalized ones \( b_{k,n} \), see Appendix
B and especially (B.5). Thus, our final result is that for odd \( n \geq 3 \) the function

\[
\bar{\Psi}_n = \sum_{k=3}^{n} (k-4)!! (n-k-1)!! (k-3)!! (n-k)!! \frac{z^{k-3}}{R_{k-2}}
\]
satisfies the equation
\[
\Delta_z^{(n)} \Psi_n = \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-3}} \frac{\partial}{\partial z} \right) \Psi_n = -\frac{\delta_+(r)}{r} \delta_+(z).
\]

Due to this renormalization, the \(n\) dependence in the source on the r.h.s. of the latter equation has disappeared.

### 6.2 The difficult even-\(n\)D case

For even \(n\), the quantity \(A_n\) cannot be reduced to any combination of the quantities \(L_k\). Certainly, \(A_2 = L_2\) as a degenerate case, however our transformation starts to work with \(A_4 \neq L_4\), cf. (4.21) and (5.20). Fortunately, the solutions (5.13) possess the property (5.14). Let us recall the relation (5.10) and impose \(F(r,z) = \Psi_n\), then combining equations (5.10),(5.14) and (4.13) gives
\[
\Delta_z^{(n)} \Psi_n = \delta_+(z) + r \Theta_+(z) + z \Theta_+(z).
\]

After multiplying (6.6) on \(z^{n-3}\) and taking the antiderivative with the use of the formal equality \(\Theta_+(z) = \delta_+(z)\),
\[
\Delta_z^{(n)} \Psi_n = \frac{\delta_+(r)}{r} \Theta_+(z) + \tilde{f}(r)
\]
where \(\tilde{f}(r)\) is unknown as yet generalized function, and \(\Theta_+(z)\) is the Heaviside step function
\[
\Theta_+(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ 1, & \text{for } x > 0. \end{cases}
\]
The equality (6.7) suggests us to explicitly calculate the case \(n = 4\) with the \(\Theta\) source.

Let us find the function \(\tilde{\Psi}_4(r,z)\) which satisfies the equation
\[
\Delta_z^{(4)} \tilde{\Psi}_4 = \frac{\delta_+(r)}{r} \Theta_+(z).
\]
Combining (6.8) with (5.6) for \(n = 4\) and with \(F(r,z)\) replaced by \(\tilde{\Psi}_4(r,z)\), and redenoting \(\theta_1 = \theta\) lead to the equation
\[
\Delta_{r,z,\theta}^{(4)} \left( \frac{\cos \theta}{z} \tilde{\Psi}_4(r,z) \right) = \frac{\cos \theta}{z} \Delta_z^{(4)} \tilde{\Psi}_4(r,z) = \frac{\cos \theta}{z} \frac{\delta_+(r)}{r} \Theta_+(z) \equiv -\sigma_4 \mu(r,z,\theta).
\]

The fundamental solution to the Poisson equation (4.1) in the case \(n = 4\) is
\[
\Phi_4 = \frac{1}{2 |\vec{R} - \vec{R}'|^2}
\]
where

\[ |\vec{R} - \vec{R}'|^2 = r^2 + r'^2 - 2rr' \cos(\varphi - \varphi') + z^2 + z'^2 - 2z'z \cos(\theta - \theta'). \]

Thus, similarly to a derivation done in Sec. 2 and in accordance with a standard rule, the solution to (6.8) can be written as follows\footnote{Strictly speaking, the standard rule prescribes to deal with compact sources, although the source in (6.9) is not compact. Nevertheless, the integral (6.10) exists. For accuracy, we could make a limit procedure originating from integration over a confined space domain, and then coming to the whole space.}

\[
\frac{\cos \theta}{z} \tilde{\Psi}_4 = \frac{1}{2} \int_{\tau^{(4)}} \frac{d\tau^{(4)'}}{|\vec{R} - \vec{R}'|^2} = -\frac{1}{2\sigma_4} \int_0^\infty dr' \int_0^{2\pi} d\phi' \int_0^{2\pi} d\theta' \times
\frac{\cos \theta' \delta_+(r') \Theta_+(z')}{r^2 + r'^2 - 2rr' \cos(\varphi - \varphi') + z^2 + z'^2 - 2z'z \cos(\theta - \theta')}.
\]

(6.10)

For integration of (6.10) with respect to angles, we require the following handbook integrals (cf. [8], p.181, no.5 and p.414, no.22):

\[
\int_0^\pi \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}},
\]

(6.11)

\[
\int_0^\pi \frac{dx \cos x}{a + b \cos x} = \frac{\pi}{b} \left(1 - \frac{a}{\sqrt{a^2 - b^2}}\right).
\]

(6.12)

Now we shall perform the three subsequent operations: 1) integration with respect to \(\varphi'\) with the use of (6.11), 2) integration with respect to \(r'\), and 3) the procedure of removing the factor \(\cos \theta\) after imposing \(\theta' = \theta + \beta\) which was already described in Sec. 2. This leads to

\[
\tilde{\Psi}_4 = -\frac{1}{\pi} \int_0^\infty \int_0^{2\pi} \frac{d\beta \cos \beta}{r^2 + z^2 + z'^2 - 2z'z \cos \beta}.
\]

(6.13)

The next step is the integration of (6.13) with respect to \(\beta\) using (6.12):

\[
\tilde{\Psi}_4 = -\frac{1}{2} \int_0^\infty \frac{dz'}{z'} \left(\frac{r^2 + z^2 + z'^2}{[(r^2 + z^2)^2 + 2(r^2 - z^2)z'^2 + z'^4]^{1/2}} - 1\right).
\]

(6.14)

The integral (6.14) can be calculated with the use of the following handbook one (see loc. cit, p.102, no.8):

\[
\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln \left|\frac{2ax + b}{2\sqrt{a}} + \sqrt{ax^2 + bx + c}\right|.
\]
This final integration gives
\[
\Psi_4 = \ln \frac{1}{\sqrt{r^2 + z^2}} - \ln \frac{1}{r}.
\]

Now, we have obtained the solution corresponding to the \(\Theta\) source (6.8). It differs from the announced solution to the homogeneous equation by the only last term. However, the later is the degenerate 2D solution (4.14b) taken with an opposite sign, and that is why it is simultaneously the \(z\) independent solution to the equation [cf. (4.13b)]:
\[
\Delta_z^{(4)} \Xi_2 = \Delta_z^{(2)} \Xi_2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \Xi_2 \right) = -\frac{\delta_+(r)}{r}.
\]

Taking the sum of the solutions \(\tilde{\Psi}_4\) and \(\Xi_2\), we can state that the function (5.13a),
\[
\Psi_4 = \tilde{\Psi}_4 + \Xi_2 = \ln \frac{1}{R}, \quad (6.15)
\]
satisfies the equation with a point-like source
\[
\Delta_z^{(4)} \Psi_4 = -\frac{\delta_+(r)}{r} \left[ 1 - \Theta_+(z) \right] \equiv s_z^{(4)}. \quad (6.16)
\]

Indeed, for the \(z\) range of values: \(z \geq 0, \quad 1 - \Theta_+(z) = \begin{cases} 1, & \text{for } z = 0; \\ 0, & \text{for } z > 0. \end{cases} \quad (6.17)

We should remark that the situation is somewhat paradoxical. The function (6.17) is finite (nonsingular) in the point \(z = 0\); and the source in (6.16) although located at the point \(r = 0, z = 0\) is more "weak" than the true \(\delta\) source. However, the logarithmic divergency in (6.15) when reaching zero is more weak than the divergency \(\sim R^{-2}\) as it could be in the case \(n = 4\) for the true \(\delta\) source. If we tried to extract this source by the standard method involving trial functions as it was done before, we would obtain zero when integrating over the whole 4D space, although the derivative of (6.17) with respect to \(z\) is the \(-\delta_+\) function as usual.

There exists a theorem stating that every generalized function concentrated at a point is a combination of the \(\delta\) functions and their derivatives [3], the proof being done in terms of finite functionals on continuous functions. The theorem has a consequence that a solution to the Laplace equation with a power-law singularity is generated by this combination. It is also extended onto partial differential equations with constant coefficients (in a certain frame, if any). Our situation is more sophisticated. The author’s opinion is that, first, there are no finite nonzero functional determined by the generalized function (6.17) in any class of trial (continuous) functions. Second, the equations with anti-Laplacians for every \(n > 3\) and the even-\(n\)D solutions considered are not the equations and
solutions of the above type. Certainly, additional investigations to this situation from the viewpoint of functional analysis are required.

In order to extend the obtained 4D source on subsequent even \( n \), we can certainly use the transformation \( f^{(n,n-2)}_A \). One could easily ensure that it does not change the form of the source,

\[
f^{(n,n-2)}_A \left[ 1 - \Theta_+(z) \right] = - \frac{z^{n-3}}{n-4} \frac{\partial}{\partial z} \left( \frac{1}{z^{n-4}} \left[ 1 - \Theta_+(z) \right] \right) = 1 - \Theta_+(z),
\]

due to the formal equality \( z \delta_+(z) = 0 \). Our previous remark can be transferred mutatis mutandis to the \( n \)D case as well.

Thus, we establish that for even \( n \geq 6 \) the function (5.13b),

\[
\Psi_n = \ln \frac{1}{R} + \sum_{k=6}^{n} \frac{1}{k-4} \frac{z^{k-4}}{R^{k-4}}
\]

is the solution to the equation

\[
\overline{\Delta}^{(n)}\Psi_n = - \frac{\delta(r)_+}{r} \left[ 1 - \Theta_+(z) \right] \equiv \bar{s}_z^{(n)}.
\]

Fortunately, we have chose from the beginning such a mutual coefficient at \( \Psi_n \) that \( \bar{s}_z^{(n)} \) has no dependence on \( n \).

7 Returning to the vortex-ring-like solutions.

Conclusion

As before, introduce the anti-double-Laplacian set \( D_n \),

\[
D_n = \{ \psi_n, \overline{\Delta}^{(n)}\psi_n, \bar{s}_z^{(n)} \}.
\]

Given such a powerful tool as the transformation \( f^{(n,n-2)}_A \) in (5.15), we have no problems in constructing any quantity \( D_n \) from \( D_{n-2} \). Indeed, the distinction between corresponding \( A_n \) and \( D_n \) is entirely referred to their "\( r \) parts", the "\( z \) parts" being unchanged. Without such an explanation, it can be immediately verified that

\[
D_n = f^{(n,n-2)}_A D_{n-2}.
\]

Expected results can be formulated as before separately for the two cases.

In the odd-\( n \)D case, we should start with \( D_3 \): As to the operators, \( \overline{\Delta}^{(3)} \) coincides with \( \overline{\Delta}^{(3)} \), and the latter is the redenoted operator (2.4). Unlike Sec.2, where we have defined \( \Omega_\varphi \) by a traditional way, here we make another definition,

\[
\int_0^\infty dr \int_0^{2\pi} d\varphi \int_{-\infty}^\infty dz \Omega_\varphi = 4\pi, \quad (7.1)
\]
in order for the source $\mathcal{F}^{(3)}$ to have the form

$$\mathcal{F}^{(3)} = -\delta(r-a)\delta_+(z)$$  \hspace{1cm} (7.2)

instead of (2.7). [We have imposed $\delta(z) = \delta(\vec{z}) = \delta_+(z)/2$ in (7.2) according to the general formula (4.3).] Hence, the 3D vortex-ring solution can be rewritten in the form

$$\psi_3 = \frac{1}{\pi} r \int_0^\pi d\alpha \cos \alpha \frac{1}{R},$$

instead of (2.11), recall from (3.7) that

$$R = [r^2 + a^2 - 2ar \cos \alpha + z^2]^{1/2}.$$  

The final result is that, for odd $n \geq 3$, the function

$$\psi_n = \frac{1}{\pi} r \sum_{k=3}^{n} \left( \frac{k-4}{(k-3)!! (n-k-1)!!} \int_0^\pi d\alpha \cos \alpha \frac{z^{k-3}}{R^{k-2}} \right) \psi_n = -\delta(r-a) \delta_+(z) \equiv \mathcal{F}^{(n)}.$$  

In the even-$n$ case, the same problem as that in the subsection 6.2 arises, however, it can be resolved by one-to-one correspondence at each step of a derivation. We only recall that instead of the solution (4.13b) to equation (4.14b), we have to use the solution (3.25) to equation (3.24) with a source $\propto \delta(r-a)$. Thus, for $n = 4$ and even $n \geq 6$, the functions

$$\psi_4 = \frac{1}{\pi} r \int_0^\pi d\alpha \cos \alpha \ln \frac{1}{R}$$  \hspace{1cm} (7.3)

and

$$\psi_n = \frac{1}{\pi} r \int_0^\pi d\alpha \cos \alpha \ln \frac{1}{R} + \frac{1}{\pi} r \sum_{k=6}^{n} \frac{1}{k-4} \int_0^\pi d\alpha \cos \alpha \frac{z^{k-4}}{R^{k-4}},$$

respectively, are the solutions to the equation

$$\overline{\Delta}^{(n)} \psi_n = -\delta(r-a) [1 - \Theta_+(z)] \equiv \mathcal{F}^{(n)}.$$  

Now, several concluding remarks are in order.

1. Certainly, every $n$D solution to equations involving the anti-$z$-Laplacian and anti-double-Laplacian can in principle be obtained by the procedure described in Sec.2, namely, by using the relations (5.6) and (5.7) and by integrating
a fundamental solution to the Poisson equation multiplied by a corresponding source. However, such the way gives rise to great practical difficulties. In the \((r, z)\) frame, an immediate integration is still relatively easy for \(n = 4\), as it was demonstrated in the subsection 6.2, but even for \(n = 5\) an integrand arises which contains special functions with arguments involving trigonometric functions. These difficulties grows when coming to each subsequent \(n\). That is why the transformation \(f_{A}^{(n,n-2)}\) saves the situation: There are no problems in obtaining solutions with an arbitrary (large) \(n\).

2. All the obtained here solutions have one more advantage that they are some algebraic functions of \(z\) and \(\bar{R}\) (or \(R\)) plus the logarithmic function of \(\bar{R}\) (or \(R\)) for even \(n\). In the latter case both the anti-\(z\)-Laplace and anti-double-Laplace solutions corresponding to sources with \(\delta_+(z)\) instead of \(1 - \Theta_+(z)\) are also feasible. However, they involve inverse trigonometric functions of \(r\) and \(z\), and have no such remarkable announced properties. By virtue of the relation (5.9b), the odd-\(n\) solutions related to a \(\delta'_+(z)\) are also feasible, but this subject is outside the framework of our study.

3. It was technically more simple to work with the anti-\(z\)-Laplacians than with the anti-double-Laplacians. It was also more simple to extract the form of point sources than that of ring sources. Moreover, this study has clarified the fact that the solutions corresponding to both the operators are of a similar type – it does not matter whether their sources are ring-like or point-like ones.

However, there also exist ring-like solutions corresponding to the anti-\(z\)-Laplacians. We add them to the list of previous solutions. They are “scalar” in their “\(r\) parts” without changes in the “\(z\) parts”, so that the transformation \(f_{A}^{(n,n-2)}\) is working as before. As the result, for odd \(n\), the function

\[
\chi_n = \frac{1}{\pi} \sum_{k=3}^{n} \frac{(k - 4)!! (n - k - 1)!!}{(k - 3)!! (n - k)!!} \int_0^\pi d\alpha \frac{z^{k-3}}{R^{k-2}} \quad (7.4)
\]

\((n \geq 3)\) satisfies the equation

\[
\overline{\Delta}_z^{(n)} \chi_n = -\frac{\delta(r - a)}{r} \delta_+(z).
\]

While for \(n = 4\) and even \(n \geq 6\), the functions

\[
\chi_4 = \frac{1}{\pi} \int_0^\pi d\alpha \ln \frac{1}{R} \quad (7.5)
\]

and

\[
\chi_n = \frac{1}{\pi} \int_0^\pi d\alpha \ln \frac{1}{R} + \frac{1}{\pi} \sum_{k=6}^{n} \frac{1}{k - 4} \int_0^\pi d\alpha \frac{z^{k-4}}{R^{k-4}} \quad ,
\]

respectively, are the solutions to the equation

\[
\overline{\Delta}_z^{(n)} \chi_n = -\frac{\delta(r - a)}{r} [1 - \Theta_+(z)] .
\]
In both the cases, the integral identity
\[ \int_0^\pi d\alpha \left( \cos \alpha R_k - k \frac{ar^2}{R^{k+2}} \right) = \int_0^\pi d\alpha \frac{\partial}{\partial \alpha} \left( \cos \alpha \frac{R_k}{R^k} \right) = 0 \]
is necessary if one desires to immediately verify that the functions (7.4), (7.5) and (7.6) satisfy the homogeneous equations \( \overline{\Delta}_z^{(n)} \chi_n = 0 \).

4. Until now we have said nothing about solutions corresponding to the \((n > 3)\)-dimensional anti-\(r\)-Laplacians (5.3). The extension of equation (2.4) onto the \(n\)D space is
\[
\overline{\Delta}_r^{(n)} \xi_n = (r \frac{\partial}{\partial r} - 1 \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z}) \xi_n = -r \Omega^{(n)}_\varphi(r, z) \equiv \bar{s}_r^{(n)}. \tag{7.7}
\]
We define \(\Omega^{(n)}_\varphi\) by a way which generalizes (7.1) to the \(n\)D case:
\[
\int_{\tau^{(n)}} d\tau^{(n)} \Omega^{(n)}_\varphi = (n - 2)\sigma_n \tag{7.8}
\]
with \(d\tau^{(n)}\) determined by (A.4). (Here, both the odd-\(n\)D and even-\(n\)D cases can be considered in common.) After that the source in (7.7) acquires the form

\[
\bar{s}_r^{(n)} = -\sigma_{n-2} \delta(r - a) \delta(\vec{z}) = -\delta(r - a) \frac{\delta_+(z)}{z^{n-3}}.
\]
This case is very simple. It can be resolved using the relation (5.5) and dealing with the Cartesian frame in the \(z\) space where \(\delta(\vec{z}) = \delta(z_1)\delta(z_2) \ldots \delta(z_{n-2})\), see Appendix A. The answer is that the function
\[
\xi_n = \frac{1}{\pi} r \int_0^\pi d\alpha \cos \alpha \frac{1}{R^{n-2}}
\]
for all \(n \geq 3\) is the solution to the equation
\[
\overline{\Delta}_r^{(n)} \xi_n = -\delta(r - a) \frac{\delta_+(z)}{z^{n-3}}.
\]
We see that this solution has no attractive properties of those considered before. It exhibits a typical \(n\)D-space behaviour. Nevertheless, due to the choice of (7.8), the anti-\(r\)-Laplacian set \(B_n\):
\[
B_n = \{ \xi_n, \overline{\Delta}_r^{(n)} \xi_n, \bar{s}_r^{(n)} \}
\]
also has the property of the type (4.19):
\[
B_n = f_{L}^{(n,n-2)} B_{n-2}
\]
where the transformation is involved which is suitable for the Laplace set.

5. Emphasize one more that coming from a "Laplacian part" of an operator to an "anti-Laplacian part" for a given subspace means coming from scalars to other geometrical objects. Now it is already clear, by analogy with Sec.2, that in the case \( n = 4 \) the presented solutions, i.e. the functions (6.15) and (7.3), are the \( \theta \) components of vectors in the 2-dimensional \( z \) space orthogonal to the radial direction. For larger \( n \), the situation seems to be more intricate because components of some polyvector are dealt with. The author is going to devote to this subject further studies.

**Appendix A. The \( x \) and \( (r, z) \) frames**

In \( n \) dimensions, the spherical coordinate frame is given by the radial distance and the \( n - 1 \) angles, \( x \) and \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \) in our notations, respectively. We call this frame the \( x \) frame and \( \lambda_1 \) the senior angle. Note that the angle ranges of values are \( 0 \leq \lambda_k \leq \pi \) for \( k = 1, 2, \ldots, n-2 \) and \( 0 \leq \lambda_{n-1} \leq 2\pi \).

In transforming the \( x \) frame into a Cartesian one, we introduce the notations

\[
\begin{align*}
x_1 &= x \cos \lambda_1, \\
x_2 &= x \sin \lambda_1 \cos \lambda_2, \\
\vdots \quad &
\\x_{n-1} &= x \sin \lambda_1 \sin \lambda_2 \ldots \sin \lambda_{n-2} \cos \lambda_{n-1}, \\
x_n &= x \sin \lambda_1 \sin \lambda_2 \ldots \sin \lambda_{n-1} \sin \lambda_{n-1}.
\end{align*}
\]

The most compact expression for metric can be written as follows

\[
dl^2 = g_{\mu\nu}dx^\mu dx^\nu = dx^2 + x^2 d\varsigma_1^2
\]

(A.1)

\((\mu, \nu = 1, 2, \ldots, n)\) where

\[
\begin{align*}
d\varsigma_1^2 &= d\lambda^2_1 + \sin^2 \lambda_1 d\varsigma_2^2, \\
d\varsigma_2^2 &= d\lambda^2_2 + \sin^2 \lambda_2 d\varsigma_3^2, \\
\vdots \\
d\varsigma_{n-2}^2 &= d\lambda^2_{n-2} + \sin^2 \lambda_{n-2} d\varsigma_{n-1}^2 \\
d\varsigma_{n-1}^2 &= d\lambda^2_{n-1}.
\end{align*}
\]

In other words,

\[
\begin{align*}
g_{xx} &= 1, \\
g_{\lambda_1\lambda_1} &= x^2, \\
g_{\lambda_2\lambda_2} &= x^2 \sin^2 \lambda_1, \\
\vdots \\
g_{\lambda_{n-1}\lambda_{n-1}} &= x^2 \sin^2 \lambda_1 \sin^2 \lambda_2 \ldots \sin^2 \lambda_{n-2},
\end{align*}
\]

The covariant volume element for the metric (A.1) is

\[
d\tau^{(n)} = d\sigma_n x^{n-1} dx
\]
with
\[ d\sigma_n = \sin^{n-2}\lambda_1 \sin^{n-3}\lambda_2 \cdots \sin \lambda_{n-2} \, d\lambda_1 \, d\lambda_2 \cdots d\lambda_{n-2} \, d\lambda_{n-1}, \]

note that \( f \, d\sigma_n = \sigma_n \) defined by (4.2).

The scalar Laplacian for the metric (A.1) has the form
\[
\Delta^{(n)} = \frac{1}{x^{n-1}} \frac{\partial}{\partial x} x^{n-1} \frac{\partial}{\partial x} + \frac{1}{x^2} \left( \frac{1}{\sin^{n-2}\lambda_1} \frac{\partial}{\partial \lambda_1} \sin^{n-2}\lambda_1 \frac{\partial}{\partial \lambda_1} + \frac{1}{\sin^2 \lambda_1 \sin^{n-3}\lambda_2} \frac{\partial^2}{\partial \lambda_2 \partial \lambda_2} \right) \times
\]
\[
\frac{\partial}{\partial \lambda_2} \sin^{n-3}\lambda_2 \frac{\partial}{\partial \lambda_2} + \cdots + \frac{1}{\sin^2 \lambda_1 \sin^2 \lambda_2 \cdots \sin^2 \lambda_{n-2} \sin \lambda_{n-2}} \frac{\partial^2}{\partial \lambda_{n-1} \partial \lambda_{n-1}}. \quad (A.2)
\]

We introduce the \((r, z)\) frame which is the direct product of a 2-dimensional and \((n-2)\)-dimensional spherical frames with the coordinates \(r, \varphi\) and \(z, \theta_1, \theta_2, \ldots, \theta_{n-3}\), respectively. All the above quantities can be rewritten mutatis mutandis. We take a Cartesian frame as a direct product of the 2-dimensional and \((n-2)\)-dimensional Cartesian frames with the coordinates
\[
\begin{align*}
  r_1 &= r \cos \varphi, \\
  r_2 &= r \sin \varphi, \\
  z_1 &= z \cos \theta_1, \\
  z_2 &= z \sin \theta_1 \cos \theta_2, \\
  \vdots \\
  z_{n-3} &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-4} \cos \theta_{n-3}, \\
  z_{n-2} &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-4} \sin \theta_{n-3}.
\end{align*}
\]

In contrast to (A.1), we give below another (noncompact) expression for metric:
\[
dl^2 = dr^2 + r^2 d\varphi^2 + dz^2 + z^2 (d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 (\ldots + \sin^2 \theta_{n-2} d\theta_{n-3}^2 \ldots))). \quad (A.3)
\]

The volume element for the metric (A.3) is
\[
d\tau^{(n)} = d\sigma_2 d\sigma_{n-2} r z^{n-3} dr \, dz \quad \textit{(A.4)}
\]

with
\[
\begin{align*}
  d\sigma_2 &= d\varphi, \\
  d\sigma_{n-2} &= \sin^{n-4} \theta_1 \sin^{n-5} \theta_2 \cdots \sin \theta_{n-4} \, d\theta_1 \, d\theta_2 \cdots d\theta_{n-4} \, d\theta_{n-3}.
\end{align*}
\]

The divergence and the scalar Laplacian are respectively as follows,
\[
\langle \nabla \cdot \vec{v} \rangle = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g} v^\mu) = \frac{1}{r} \frac{\partial}{\partial r} (r v^r) + \frac{1}{r^2} \frac{\partial v^\varphi}{\partial \varphi} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} (z^{n-3} v^z) +
\]
\[
\frac{1}{\sin^{n-4} \theta_1} \frac{\partial}{\partial \theta_1} (\sin^{n-4} \theta_1 v^{\theta_1}) + \frac{1}{\sin^{n-5} \theta_2} \frac{\partial}{\partial \theta_2} (\sin^{n-5} \theta_2 v^{\theta_2})
\]

33
\[ + \ldots + \frac{1}{\sin \theta_{n-4}} \frac{\partial}{\partial \theta_{n-4}} (\sin \theta_{n-4} v^{\theta_{n-4}}) + \frac{\partial \theta_{n-3}}{\partial \theta_{n-3}} \] (A.5)

where \( g = \det g_{\mu\nu} \), and

\[
\Delta^{(n)}_{r,z} = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{z^{n-3}} \frac{\partial}{\partial z} z^{n-3} \frac{\partial}{\partial z} + \frac{1}{z^2} \left( \frac{1}{\sin^{n-4} \theta_1} \frac{\partial}{\partial \theta_1} \sin^{n-4} \theta_1 \frac{\partial}{\partial \theta_1} + \frac{1}{\sin^2 \theta_1 \sin^{n-5} \theta_2} \frac{\partial}{\partial \theta_2} \sin^{n-5} \theta_2 \frac{\partial}{\partial \theta_2} \right.
\]

\[
+ \ldots + \frac{1}{\sin^2 \theta_1 \sin^2 \theta_2 \ldots \sin^2 \theta_{n-5}} \sin \theta_{n-4} \frac{\partial^2}{\partial \theta_{n-3}^2} \right). (A.6)
\]

**Appendix B. More about \( a_{k,n} \)**

The coefficients \( a_{k,n} \) can be represented in the form equivalent to (3.13):

\[
a_{k,n} = \frac{(n-3)!! (k-4)!! (n-k-1)!!}{(n-4)!! (k-3)!! (n-k)!!}. \tag{B.1}
\]

Note that \((-1)!! = 1\) and \(0!! = 1\) by definition. It is quite clear from (B.1), that these coefficients possess the symmetry property

\[
a_{k,n} = a_{n-k+3,n}. \tag{B.2}
\]

In particular,

\[
a_{n,n} = a_{3,n} = 1.
\]

Besides the recurrence relation (3.12), there exists one more recurrence relation for \( k \leq n - 2 \):

\[
a_{k,n} = a_{k,n-2} \frac{(n-3)(n-k-1)}{(n-4)(n-k)}. \tag{B.3}
\]

The above formula permits one to rapidly calculate the triangle of the numerical values of \( a_{k,n} \). We give it below for \( n = 3, 5, 7, 9, 11, 13, 15 \).

\[
\begin{array}{cccccccc}
    n = 3 & 1 & & & & & & \\
    n = 5 & 1 & 1 & & & & & \\
    n = 7 & 1 & 2/3 & 1 & & & & \\
    n = 9 & 1 & 3/5 & 3/5 & 1 & & & \\
    n = 11 & 1 & 4/7 & 18/35 & 4/7 & 1 & & \\
    n = 13 & 1 & 5/9 & 10/21 & 10/21 & 5/9 & 1 & \\
    n = 15 & 1 & 6/11 & 5/11 & 100/231 & 5/11 & 6/11 & 1 \\
    \ldots & & & & & & & \\
\end{array}
\]

34
By immediate calculations for several odd $n$, we can check the three following heuristical relations. The first of them is the relation for the required sum,

$$\sum_{k=3}^{n} a_{k,n} = \frac{(n-3)!!}{(n-4)!!},$$

(B.3)

and the two other are the interesting although useless relations ($n > 3$)

$$\sum_{k=3}^{n} \frac{a_{k,n}}{k-4} = 0, \quad \sum_{k=3}^{n} \frac{a_{k,n}}{n-k-1} = 0,$$

the latter follows from the former when using (B.2).

In order to surely use (B.3), we must prove it. To do this, let us renormalize the coefficients $a_{k,n}$:

$$a_{k,n} = \frac{(n-3)!!}{(n-4)!!} b_{k,n},$$

(B.4)

$$b_{k,n} = \frac{(k-4)!! (n-k-1)!!}{(k-3)!! (n-k)!!}.$$  
(B.5)

According to (B.3) and (B.4), we must prove that

$$\sum_{k=3}^{n} b_{k,n} = 1.$$

Let us impose $k = 2l + 1$ and $n = 2q + 1$ and use the equality

$$\binom{-1/2}{m} = (-1)^m \frac{(2m - 1)!!}{(2m)!!},$$

see [8], p.772, where by definition

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}.$$

The coefficients (B.5) can be now expressed as follows

$$b_{k,n} = \tilde{b}_{l,q} = (-1)^{q-1} \binom{-1/2}{l-1} \binom{-1/2}{q-l}.$$

Let us write the expression for the following sum

$$\sum_{l=1}^{q} \binom{-1/2}{l-1} \binom{-1/2 + \epsilon}{q-l} = \binom{-1 + \epsilon}{q-1}$$

(B.6)

in accordance with the handbook equality for any complex $a$ and $b$ (see loc. cit., no.13 on p.616):

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}.$$
It is necessary to obtain the sum (B.6) for $\epsilon = 0$. However, in this case (B.6) contains an indeterminate form on its r.h.s., in order to evaluate it we take the limit $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \left( \frac{-1 + \epsilon}{q - 1} \right) = \lim_{\epsilon \to 0} \frac{\Gamma(\epsilon)}{\Gamma(q)\Gamma(1 - q + \epsilon)} = \frac{\Gamma(\epsilon)}{\Gamma(q - \epsilon)\Gamma(1 - q + \epsilon)} = \lim_{\epsilon \to 0} \frac{\sin \pi (q - \epsilon)}{\epsilon \pi} = (-1)^{q-1}$$

where we have replaced the finite value $\Gamma(q)$ on $\Gamma(q - \epsilon)$ under the sign of limit.

At least,

$$\sum_{k=3}^{n} b_{k,n} = \sum_{l=1}^{q} \tilde{b}_{l,q} = (-1)^{q-1} \lim_{\epsilon \to 0} \left( \frac{-1 + \epsilon}{q - 1} \right) = 1,$$

that finishes our proof. Perhaps, a more elegant proof is available, however, the author could not find it.

We think it is worth giving the triangle of the coefficients $b_{k,n}$ (similar to that of $a_{k,n}$) because our final expressions for odd-$n$D solutions contain just $b_{k,n}$.

\[
\begin{array}{cccc}
n = 3 & 1 \\
n = 5 & 1/2 & 1/2 \\
n = 7 & 3/8 & 1/4 & 3/8 \\
n = 9 & 5/16 & 3/16 & 3/16 & 5/16 \\
n = 11 & 35/128 & 5/32 & 9/64 & 5/32 & 35/128 \\
n = 13 & 63/256 & 35/256 & 15/128 & 15/128 & 35/256 & 63/256 \\
n = 15 & 231/1024 & 63/512 & 105/1024 & 25/256 & 105/1024 & 63/512 & 231/1024 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]

References

[1] H. Lamb, *Hydrodynamics* (Cambridge University press, Cambridge, 1895).

[2] P. Appell, *Traité de mecanique rationelle*, v.3, *Equilibre et mouvement des milieux continus* (Gauthier-Villars, Paris, 1928).

[3] M.A. Lavrent’yev and B.V. Shabat, *Problems of Hydrodynamics and Their Mathematical Models* (Nauka, Moscow, 1977, in Russian).

[4] L.M. Milne-Thomson, *Theoretical Hydrodynamics* (London–New-York, Macmillan and Co. LTD, 1960).

[5] J.A. Schouten, *Tensor Analysis for Physicists* (The Clarendon Press, Oxford, 1951).

[6] G.A. Korn and T.M. Korn, *Mathematical Handbook for Physicists and Engineers* (McCraw-Hill Book Company, New-York–London, 1961).
[7] I.M. Gel’fand and G.E. Shilov, *Generalized Functions and Operations with Them* (State Publishing House of Physical and Mathematical Literature, Moscow, 1958, in Russian).

[8] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series* (Nauka, Moscow, 1981, in Russian).

[9] I.M. Gel’fand and G.E. Shilov, *The Spaces of Trial and Generalized Functions* (State Publishing House of Physical and Mathematical Literature, Moscow, 1958, in Russian).