KÄHLERIAN FUNCTIONALS INVOLVING SCALAR CURVATURE AND
HOLOMORPHY POTENTIALS

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ABSTRACT. An Euler-Lagrange equation is obtained for a family of functionals whose integrands
involve scalar curvature and appropriately normalized holomorphy potentials, varying over Kähler
metrics associated with a fixed Kähler class. This is a simple generalization of a variational problem
considered by Calabi, which led to his definition of extremal Kähler metrics. Special cases that
are examined include holomorphically equivariant objects such as the Futaki invariant, and special
Kähler-Ricci potentials. A remaining open problem is indicated, along with associated iteration
problems.

1. Introduction

This note is an early off-shoot of the work on [3], and contains a simple generalization of a result
of Calabi [2]. The latter served to characterize via variational means extremal Kähler metrics,
in which the scalar curvature is a potential giving rise to a Killing vector field. Our motivation
came from metrics for which, instead, the scalar curvature was merely functionally dependent
on such a Killing potential. Metrics which are known to obey such a characterization include
Kähler metrics almost everywhere conformal to Einstein metrics, and their generalization, namely
metrics admitting a special Kähler-Ricci potential in complex dimension greater than two (see [3]).
However, the result has a broader scope. In fact, it yields an Euler-Lagrange condition which is
satisfied if, for example, the metric admits two nontrivial Killing potentials representing vector
fields which are generically linearly independent, and whose ratio is functionally dependent on a
possibly nonconstant scalar curvature. The main reason for this note is to raise the problem of
existence for such metrics on compact manifolds, the answer to which appears to be unknown, along
with associated iteration problems. A second, more minor reason is that this approach unifies such
generalizations of Calabi’s extremality condition, with Kähler class invariance proofs of quantities
akin to the Futaki invariant, obtained via considerations from holomorphic equivariant cohomology.
A subtlety involving normalizations of holomorphy potentials is a main theme of these invariance
proofs, as well as of the main theorem.

2. A variational characterization

In this section we review a lemma of Calabi and then generalize his result on the variation that
leads to, and in fact defines, extremal Kähler metrics. We assume throughout that $M^m$ is a compact
Kählerian manifold of complex dimension $m$, and also employ complex coordinates $z^a$, with commas
denoting complex covariant differentiation.
Lemma 2.1 (Calabi). If a (real or complex valued) function $\varphi$ satisfies the equation $\varphi^{ab}_{\bar{a}b} = 0$, then $\varphi_{\bar{a}b} = 0$. In other words, $\varphi$ is a holomorphy potential.

Note that a real holomorphy potential is automatically a Killing potential.

Proof. Write
\[ 0 \leq g^{\bar{a}c}g^{\bar{b}d} \varphi_{\bar{c}d} \varphi_{\bar{a}b} = (\varphi^{ab}_{\bar{a}a} \varphi - \varphi^{ab}_{\bar{a}a} \varphi)_{\bar{b}} + \varphi^{ab}_{\bar{a}b} \bar{\varphi} := v^b_{\bar{b}} + \varphi^{ab}_{\bar{a}b} \bar{\varphi}, \]
where the right hand side of the equality follows since $\varphi^{ab}_{\bar{a}b} = \varphi^{ba}_{\bar{b}a}$. Under the condition of the lemma, the first term on the right hand side is the divergence of the $(0, 1)$ part of the form corresponding to $\nu$:
\[ v^b_{\bar{b}} = -*d*(v_d z^b), \]
which vanishes upon integration, so that the integral of the left hand side also vanishes. That integral is the square of the $L_2$ norm of the tensor $\varphi_{\bar{a}b}$, hence the result follows. \(\square\)

Recall that a function $\varphi$ on a Kähler manifold $(M, g)$ is a $g$-holomorphy potential for a vector field $X$ if $X = (\partial \varphi)^{\#}$.

Theorem 2.2. Let $M^m$ be a compact complex manifold, with a triple $(g, X, \varphi)$ consisting of a (base) Kähler metric $g$ with Kähler form $\omega$ and a holomorphic vector field $X$ which admits zeros, for which $\varphi$ is a $g$-holomorphy potential. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{C} \to \mathbb{C}$ are $C^1$ functions. Consider the functional
\[ S := \int_{M^m} (f \circ s) \cdot (h \circ \varphi) \omega^m, \]
defined over the space of Kähler metrics (with Kähler forms) in the fixed Kähler class $[\omega]$, where, for a given such metric $g$ with Kähler form $\omega$ satisfying $\omega = d_{\partial} \omega$, the scalar curvature of $g$ is denoted $s$, while $\varphi$ is a $g$-holomorphy potential of $X$, normalized via $\int_{M^m} \varphi \omega^m = \int_{M^m} [\varphi + X] \omega^m$. A metric $g$ satisfies the Euler-Lagrange equation for $S$ if and only if the function
\[ (f' \circ s) \cdot (h \circ \varphi) \]
is also a $g$-holomorphy potential (and, of course, is Killing if it is real).

Here the prime denotes differentiation. The vector field $X$ is assumed to have zeros since only then it admits holomorphy potentials. Unless $h$ is chosen to be real valued, Theorem 2.2 describes a variation of a complex valued functional. Nonetheless, the main interest is in the case where the critical equation gives rise not just to a holomorphy potential, but to a holomorphic Killing one (for which the potential is real). Finally, the meaning of the normalization condition for $\varphi$ will be explained in (1) of §3.

Proof. We first recall Calabi’s basic variation scheme. As the variation takes place in a fixed Kähler class, the Kähler forms in a one parameter family $g_t$, may be considered to vary according to $\omega_t = \omega + \partial \partial u_t$, for purely imaginary functions $u_t$ (corresponding to $\varphi$ in the theorem) on the product of $M$ with an open interval. The $t$-derivative at zero of $u_t$ will be denoted $u$, and similar derivatives involving other geometric quantities will be denoted using the symbol $\delta$. One has: $\delta g_{\bar{a}b} = u_{\bar{a}b}$. Next, as variations of determinants involve traces, $\delta(\det g_{\bar{a}b}) = \Delta u \det g_{\bar{a}b}$, or $\delta(\omega^m) = \Delta u(\omega^m)$. Here
\[ \Delta is \text{ the } \bar{\partial}\text{-Laplacian, } \Delta u = -g^{ab} \nabla_a \nabla_b u. \] Consequently the Ricci forms \((\rho_t)_{ab} = - (\log \det (g_t))_{ab}\) admit the variation \(\delta \rho_{ab} = (\Delta u)_{ab}\).

Next, the variation of the scalar curvatures \(s_t = \langle \rho_t, \omega_t \rangle_t\) for elements of the one parameter family depends on \(\rho_t, \omega_t\) and the inner products \(\langle , \rangle_t\). For this notice that from \(g \cdot g^{-1} = Id\) it follows that \(\delta (g^{ac}) g_{bc} = -g^{ac} \delta (g_{bc})\), so, with \(R\) standing for the full curvature tensor,
\[
\delta s = g^{ab} \delta \rho_{ab} + \delta (g^{cd} g_{cd}) \rho_{ab} = -\Delta^2 u + (1 - 2) g^{cb} g^{ad} \delta (g_{cd}) \rho_{ab}
\]
\[
= -\Delta^2 u - u_{ab} \rho_{ab} - u_a b \rho_a - u_a b \rho_a
\]
\[
= (-u_{ab} + u_d R^d_{ab}) b - u_d b \rho_a = -u_{ab} + u_d b \rho_a + u_d b \rho_a - u_d b \rho_a
\]
\[
= -u_{ab} + u_d s d,
\]
where the symmetry of both the Hessian of \(u\) and \(\rho\) was used in the second and third lines, the Ricci-Weitzenböck formula in the third line, and the contracted Bianchi identity in the last line.

We proceed to the variation of the holomorphy potentials \(\varphi_t\) of \(g_t\), for the fixed vector field \(X\), with that of \(g\) denoted by \(\varphi\), as in the statement of the theorem. For Kähler forms \(\omega_t = \omega + \bar{\partial} \partial u_t\), we have \(i_X \omega_t = i_X \omega + i_X (\bar{\partial} \partial u_t) = \bar{\partial} \partial \varphi - i_X (\bar{\partial} \partial u_t) = \bar{\partial} \varphi + \bar{\partial} (i_X \partial u_t) = \bar{\partial} (\varphi + X u_t)\), where we have used the fact that \(i_X\) anti-commutes with \(\bar{\partial}\) when \(X\) is holomorphic. Hence \(\varphi_t = \varphi + X u_t + c_t\) for some \(t\)-dependent constant \(c_t\). But the normalization of the holomorphy potentials implies \(\int_{M^n} [\varphi + X u_t + c_t] \omega^m_t = \int_{M^n} [\varphi + X u_t] \omega^m_t\), so that \(\int_{M^n} c_t \omega^m_t = 0\). Hence, with this normalization, the constants \(c_t\) all vanish and \(\varphi_t = \varphi + X u_t\). Therefore the variation, i.e. the derivative at \(t = 0\), is
\[
\delta \varphi = Xu = \langle \partial \varphi, \partial u \rangle = \langle \partial \text{Re } \varphi, \partial u \rangle + i \langle \partial \text{Im } \varphi, \partial u \rangle,
\]
and the last two terms are, of course, \(\partial \text{Re } \varphi\) and \(\partial \text{Im } \varphi\), respectively.

Therefore, using \(f(s), h(\varphi)\) to denote \(f \circ s\) and \(h \circ \varphi\), respectively, and \(h_x, h_y\) denoting partial derivatives of \(h\) with respect to \(x = \text{Re } z\) and \(y = \text{Im } z\), we have
\[
\delta (f(s) h(\varphi) \omega^m) = \langle f'(s) \delta s h(\varphi) + f(s) [h_x(\varphi) \delta (\text{Re } \varphi) + i h_y(\varphi) \delta (\text{Im } \varphi)] + f(s) h(\varphi) \Delta u \rangle \omega^m
\]
\[
= \langle \frac{f'}{f}(s) (-u_{ab} + u_d s d) h(\varphi) + f(s) [h_x(\varphi) \langle \partial \text{Re } \varphi, \partial u \rangle + i h_y(\varphi) \langle \partial \text{Im } \varphi, \partial u \rangle] \rangle \omega^m
\]
\[
= \langle f(s) h(\varphi) \Delta u \rangle \omega^m
\]
\[
= \langle -f'(s) h(\varphi) u_{ab} \rangle \omega^m
\]
\[
= \langle f'(s) h(\varphi) \partial s + f(s) [h_x(\varphi) \partial \text{Re } \varphi + i h_y(\varphi) \partial \text{Im } \varphi], \partial u] \rangle + f(s) h(\varphi) \Delta u \rangle \omega^m
\]
\[
= \langle -f'(s) h(\varphi) u_{ab} + \text{div} (f(s) h(\varphi) \partial u) \rangle \omega^m,
\]
using only the Leibniz rule, with \(\text{div}\) denoting the divergence operator. Since the second summand is indeed a divergence, its integral vanishes. Therefore,
\[
\delta S = - \int_{M^n} \langle f'(s) \circ (h \circ \varphi) \cdot u_{ab} \rangle \omega^m = - \int_{M^n} \langle [f'(s) \cdot (h \circ \varphi)]_{ba} \cdot u \rangle \omega^m,
\]
as one sees integrating by parts four times. Since, equating this to zero is a requirement that must hold for every \( u \), one arrives at

\[
\left[(f' \circ s) \cdot (h \circ \varphi)\right]_{ba} = 0.
\]

The result now follows from Lemma 2.1 \( \square \)

3. Special Cases

The following is a description of special solutions to the above family of EL-equations.

(1) In some cases the resulting Euler-Lagrange equation gives the holomorphy potential of the zero vector field. Trivial ways to achieve this involve choosing \( f \) and \( h \) to be the zero function, or setting the initial vector field \( X \) to be the zero vector field and \( \varphi = 0 \). Another way is to have \( f \) be a nonzero constant function. In that case, the integrand of \( S \) is a function of \( \varphi \), while the Euler-Lagrange equation yields again the potential zero. As this is a holomorphy potential for any metric, any metric is critical, hence the functional \( S \) must be constant, i.e. an invariant associated with the Kähler class. This can be understood in the context of holomorphic equivariant cohomology (cf. [5]). Namely, the normalization condition can be rewritten in that language as follows:

\[
\int_{M^m} \omega + \varphi \wedge m^1 = \int_{M^m} \varphi \wedge m = \int_{M^m} \varphi + X u \wedge m = \int_{M^m} \varphi + X u \wedge m \varphi \wedge m + \int_{M^m} \varphi + X u \wedge m + 1
\]

In other words, as one varies the Kähler form in the Kähler class \([\omega]\), \( \varphi \) is constrained to vary so that the closed equivariant form \( \omega + \varphi \) changes by an equivariantly exact form. The statement is that with this variation, the integral of this closed equivariant form, as well as all the integrals with integrands of the form \( h \circ \varphi \) remain unchanged (note that \( h \) is not constrained to be polynomial (or exponential) as it does in equivariant cohomology). A discussion of Kähler class invariance of related integrals based on just such a variation of a closed equivariant form appears in [6, paragraph after (17)].

(2) If both \( f \) and \( h \) are the identity functions \((f(x) = x, h(z) = z)\), then the Euler-Lagrange equation of \( S \) is obeyed by metrics \( g \) for which \( \varphi = \varphi_g \) is a holomorphy potential. By the definition of \( \varphi \), this holds tautologically for every metric (with Kähler form in the Kähler class). Hence \( S \) is again an invariant associated with the Kähler class, provided that \( \varphi \) satisfied the normalization discussed in (1). Note that for this \( f \) (and nonconstant \( \varphi \)), there are no solution metrics if \( h \) is any non-affine function.

Combining this case with the previous one, one arrives at a simple proof of the invariance of the Futaki invariant, which, for a vector field with zeros, is given by \( \mathcal{F}_{[\omega]}(X) = \int_M (s - s_0) \varphi \omega^n \), with \( s_0 \) the average scalar curvature. However, this proof has the limitation of assuming a particular normalization on \( \varphi \), which is immediately seen to be unnecessary, as
the expression for $F[\omega](X)$ is independent of normalization. Associated closed equivariant related expressions for these invariants appear in \[7, \[6\].

(3) For $h$ a nonzero constant function, the functional varies, and the Euler-Lagrange equation requires $f' \circ s$ to be a holomorphy potential. This is perhaps the most obvious generalization of Calabi’s result.

(4) Suppose $h$ is nonconstant, and for the critical metric, $(f' \circ s) \cdot (h \circ \varphi)$ is also a holomorphy potential for the fixed vector field $X$, which we assume is nontrivial. Equivalently, the expression $(f' \circ s) \cdot (h \circ \varphi) - \varphi$ is constant, and one arrives at a functional relation involving $s$ and $\varphi$. As $h$ is nonconstant, one arrives at a case more involved than \( \[3\], and in particular more involved than the extremal metric case $s = \varphi$. Namely, it is in fact easily possible to define $S$ so that a metric for which $s = H \circ \varphi$, for some function $H : \mathbb{C} \to \mathbb{R}$, is critical. For example, choose $f$ to be any function whose derivative is nowhere zero, and then define $h(z) = z/f'(H(z))$. One natural choice which follows this prescription is to take $f$ to be the exponential function. This yields the functional $S = \int_M \exp[s - H \circ \varphi] \varphi \omega^m$. The case which is of most interest occurs when for the critical metric, $\varphi$ is in fact a Killing potential. In considering just such a metric and not the variation leading to it, one may restrict attention from the entire domain of $H$, and consider it instead to be just the real numbers.

As mentioned in the introduction, metrics for which $s = H \circ \varphi$, with $H$ a smooth function on the real numbers, which is arbitrary except for satisfying certain boundary conditions, do exist on compact Kähler manifolds. One example consists of metrics admitting a special Kähler-Ricci potential, at least when $m > 2$. See \[3, Lemma 11.1\] and \[4\]. In these examples, the function $\varphi$ is, in fact, a Killing potential. A special case of these consists of metrics almost everywhere conformal to Einstein metrics, where $H$ is, in general, a fairly complicated rational function.

(5) One remaining intriguing problem is whether there exists a choice of $f$, $h$ and nontrivial $X$, for which the resulting functional has a critical metric having the property that $(f' \circ s) \cdot (h \circ \varphi)$ is a holomorphy potential for a nontrivial vector field $Y$ that is \textit{(generically) linearly independent} from $X$. This amounts to the condition that a ratio of two distinct (functions of) holomorphy potentials, representing (generically) linearly independent vector fields, is a function of the scalar curvature. The author is not aware of any known examples on compact manifolds. A natural place to look for such examples seems to be the class of Kähler metrics admitting a Hamiltonian 2-form \[1\], and in particular, metrics on $CP^m$ admitting a special Kähler-Ricci potential. While the latter may be taken up in future work, note that it leaves open the question of whether such metrics exist that are not of the form described in item \( \[4\].

A fixed functional $S$ along with a fixed based metric $g$ allow one to consider the following iterative procedure: $X$ is regarded as an initial vector field denoted $X_0$ (with some preassigned $g$-holomorphy potential $\varphi$), with a critical metric $g_0$ distinguishing a second vector field $Y$, denoted $X_1$. Now consider $X_1$ as an initial vector field, with preassigned potential the one obtained in the previous variation stage. A critical metric $g_1$ now gives rise to a
new distinguished vector field $X_2$. Continuing we get a sequence of critical metrics $g_0, g_1, \ldots$ and vector fields $X_0, X_1, \ldots$. One can then investigate conditions on $\{g_i\}$ under which the sequence $\{X_i\}$ converges in the (finite dimensional) Lie algebra of vector fields with zeros.

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