Lower Bounds for the MMSE via Neural Network Estimation and Their Applications to Privacy

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Abstract

The minimum mean-square error (MMSE) achievable by optimal estimation of a random variable $Y \in \mathbb{R}$ given another random variable $X \in \mathbb{R}^d$ is of much interest in a variety of statistical settings. In the context of estimation-theoretic privacy, the MMSE has been proposed as an information leakage measure that captures the ability of an adversary in estimating $Y$ upon observing $X$. In this paper we establish provable lower bounds for the MMSE based on a two-layer neural network estimator of the MMSE and the Barron constant of an appropriate function of the conditional expectation of $Y$ given $X$. Furthermore, we derive a general upper bound for the Barron constant that, when $X \in \mathbb{R}$ is post-processed by the additive Gaussian mechanism and $Y$ is binary, produces order optimal estimates in the large noise regime. In order to obtain numerical lower bounds for the MMSE in some concrete applications, we introduce an efficient optimization process that approximates the value of the proposed neural network estimator. Overall, we provide an effective machinery to obtain provable lower bounds for the MMSE.

I. Introduction

The disclosure of individual data could pose severe privacy risks [2]. Even when the data being disclosed is not necessarily private, it could be correlated with sensitive information creating privacy vulnerabilities. To reduce risks, a widely adopted solution is to use a privacy mechanism to sanitize the non-private data prior to its disclosure, see, e.g., [3], [4]. In precise terms, we have a Markov chain $Y \rightarrow X \rightarrow \tilde{X}$ where $Y$ represents private information (e.g., gender), $X$ represents non-private information (e.g., height), and $\tilde{X}$ is a noisy version of $X$. Indeed, a common privacy mechanism for continuous non-private data is the so-called additive Gaussian mechanism that adds an independent Gaussian random variable $Z$ to the non-private data, i.e., $\tilde{X} = X + Z$.

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A prominent property of privacy is that it could be impossible to restore once it has been breached. For example, if an individual’s released data leads to the inference that he has a chronic disease, it is impossible to restore the privacy of his condition once this fact has been exposed. Hence, it is important to know beforehand the privacy risks associated with the release of data. To this end, many measures of information leakage have been proposed in the literature. Typically, information leakage measures are formulated to capture the ability of an adversary to make specific inferences about $Y$ upon observing $\tilde{X}$. For example, in the context of estimation-theoretic privacy, the minimum mean-square error (MMSE) in estimating $Y$ given $\tilde{X}$ captures the ability of the strongest adversary aiming to approximate $Y$ in the expected square-loss sense [4], [5]. In this context, privacy guarantees naturally come in the form of lower bounds for the MMSE, as they ensure that such an adversary cannot estimate $Y$ beyond a certain precision.

Given a privacy mechanism, it could be challenging to predict its performance in practical applications. Oftentimes, theoretical guarantees are obtained through worst-case analyses that result in loose bounds. There is an active research area that aims to overcome the latter challenge by analyzing, designing, and auditing privacy mechanisms in a data-driven manner. One way to implement this philosophy is by estimating the information leakage of $Y$ in $\tilde{X}$ based on samples of these random variables [6]–[9]. In the context of estimation-theoretic privacy, this amounts to establishing empirical lower bounds for the MMSE in estimating $Y$ given $\tilde{X}$.

In this work, we derive provable lower bounds for the MMSE in estimating $Y \in \mathbb{R}$ given $\tilde{X} \in \mathbb{R}^d$. These lower bounds are based on a neural network estimator of the MMSE and the Barron constant of (a function of) the conditional expectation of $Y$ given $X$. More specifically, we propose the minimum empirical square-loss attained by a two-layer neural network as an estimator of the MMSE. Furthermore, we derive a general upper bound for the Barron constant that, when $X \in \mathbb{R}$ is post-processed by the additive Gaussian mechanism and $Y$ is binary, produces order optimal estimates in the large noise regime. In order to obtain numerical lower bounds for the MMSE in some concrete applications, we also analyze some algorithmic aspects related to the computation of the proposed estimator of the MMSE and introduce an efficient optimization process to approximate it.

The rest of the paper is organized as follows. In the remainder of this section we discuss further related work and recall some common notation used through this paper. In Section II we present some elements of estimation-theoretic privacy, introduce our proposed estimator, recall Barron’s approximation theorem, and discuss some aspects of the additive Gaussian mechanism. We derive lower bounds for the MMSE upon the proposed estimator in Section III. In Section IV we derive a general bound for the Barron constant which, in Section V, yields order optimal estimates in the presence of the additive Gaussian mechanism. In Section VI we consider some numerical aspects related to the computation of the proposed estimator of the MMSE and instantiate our lower bounds in a particular example. We provide a summary and some final remarks in Section VII.

**Related Work.** The minimum mean-square error achievable by optimal estimation of a random variable given another one plays a key role in statistics and communications [10], [11], and it is closely related to fundamental information-theoretic concepts [12]. As such, the problem addressed in this paper is related to other fundamental

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1For ease of notation, we drop the tilde in $\tilde{X}$ and denote it just by $X$. 

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problems in information theory and statistics. In the special case when \( Y = X \), our setting is closely related to the problem of estimation in Gaussian channels as studied in [12]–[14]. Indeed, the problem considered in this work generalizes the aforementioned problem by considering finite samples and \( Y \neq X \).

There exists a vast literature on MMSE estimation techniques, including those relying on linear [10], kernel-based [15] and polynomial [16], [17] approximations of the conditional expectation. In line with the recent surge in neural network estimation methods for information measures, see, e.g., [18]–[20], we adopt neural network estimation in the context of the MMSE. It is important to remark that while neural network estimation is known to underperform in some settings, see, e.g., [21], our work is aligned with the theoretical nature of privacy where quantitative guarantees for the proposed methodologies are fundamental. This contrasts with existing MMSE estimation methodologies that focus mainly on empirical performance or provide only qualitative guarantees (e.g., convergence rates with unspecified constants).

There are several notions of privacy designed to capture the risks posed by a variety of adversaries, e.g., differential privacy quantifies the membership inference capabilities of an adversary in the context of database queries [22], maximal \( \alpha \)-leakage quantifies the capacity of an adversary to infer any (randomized) function of the private attribute [23], [24], probability of correctly guessing quantifies the probability of an adversary to guess the private attribute [25], to name a few notions. As mentioned before, our work belongs to research area dedicated to the data-driven estimation of information (leakage) measures, see, e.g., [6]–[9] and references therein. There is also a recent research effort dedicated to understanding the performance of an adversary with practical computational capabilities [26], [27]. From this perspective, our results compare the performance of a finite capacity adversary using a 2-layer neural network and finitely many samples to the performance of the strongest adversary capable of implementing any function and knowing the joint distribution of the private and disclosed data.

At a technical level, our starting point is Barron’s approximation theorem [28]. While there is a variety of works extending Barron’s result, see, e.g., [20], [29]–[34], most of them are asymptotic analyses in which the Barron constant is a fixed, yet unknown quantity. In contrast, in the present paper we show that this constant can be effectively controlled in the context of privacy under the additive Gaussian mechanism. To the best of the authors’ knowledge, this is the first time that a quantitative analysis of the Barron constant is performed at the proposed level of generality.

**Notation.** We let \((\Omega, \mathcal{F}, \mathbb{P})\) be the underlying probability space and \(\mathbb{E}\) be the corresponding expectation. We denote by \(1_E\) the indicator function of any set \(E \in \mathcal{F}\). We let \(\text{Unif}(\mathcal{U})\) be the uniform distribution over \(\mathcal{U}\). If \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) is a probability density function, we let \(\text{Supp}(f)\) be its support. If \(p \in [0, 1]\), we let \(\hat{p} = 1 - p\). For \(u, v \in \mathbb{R}^d\), we let \(u \cdot v = u_1v_1 + \cdots + u_dv_d\) and \(|u| = \sqrt{u \cdot u}\). Unless otherwise stated, we let \(\| \cdot \|_p\) be the \(p\)-norm in \(L^p(\mathbb{R})\) or \(L^p(\mathbb{R}^d)\), depending on the context. We say that \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) is \(\rho\)-Lipschitz if \(|f(u) - f(v)| \leq \rho|u - v|\) for all \(u, v \in \mathbb{R}^d\). Also, we say that \(f\) is of class \(C^m\) if it has continuous partial derivatives of order up to \(m\).

We write \(f(z) \sim g(z)\) to denote that \(f(z)/g(z) \rightarrow 1\) as \(z \rightarrow \infty\). We let \(\tanh : \mathbb{R} \rightarrow (-1, 1)\) be the hyperbolic tangent function. For \(B \subset \mathbb{R}^d\), we let \(\text{rad}(B) := \sup_{x \in B} |x|\). Recall that the gamma function is determined by \(\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx\) for \(z > 0\).
II. PROBLEM SETTING AND PRELIMINARIES

In this section we review some preliminary material on estimation-theoretic privacy, function approximation capabilities of two-layer neural networks, and the so-called additive Gaussian mechanism. Also, we introduce a neural network estimator of the MMSE that is used to derive the theoretical lower bounds for the MMSE in Section III.

A. Estimation-Theoretic Privacy

Given random variables $X \in \mathbb{R}^d$ and $Y \in \mathbb{R}$, the minimum mean square error in estimating $Y$ given $X$ is defined as
$$\text{mmse}(Y|X) := \inf_{h \text{ meas.}} \mathbb{E} \left[ (Y - h(X))^2 \right],$$
where the infimum is taken over all (Borel) measurable functions $h: \mathbb{R}^d \rightarrow \mathbb{R}$. The infimum in (1) is attained by the conditional expectation of $Y$ given $X$, i.e.,
$$\text{mmse}(Y|X) = \mathbb{E} \left[ (Y - \eta(X))^2 \right],$$
where $\eta(X) \overset{a.s.}{=} \mathbb{E}[Y|X]$. Note that if $Y \overset{a.s.}{=} h_0(X)$ for some function $h_0: \mathbb{R}^d \rightarrow \mathbb{R}$, then $\text{mmse}(Y|X) = 0$. Also, note that if $X$ and $Y$ are independent, then the MMSE is maximal and $\text{mmse}(Y|X) = \mathbb{E} \left[ (Y - \mathbb{E}[Y])^2 \right]$.

In the context of estimation-theoretic privacy, Asoodeh et al. [5] introduced the notion of $\epsilon$-weak estimation privacy to denote that
$$\text{mmse}(Y|X) \geq (1 - \epsilon) \mathbb{E} \left[ (Y - \mathbb{E}[Y])^2 \right].$$
Observe that, as defined in (1), $\text{mmse}(Y|X)$ quantifies the ability of an adversary to approximate $Y$, in the expected square-loss sense, upon observing $X$. Since a larger MMSE amounts to better privacy, estimation-theoretic privacy guarantees naturally come as lower bounds for $\text{mmse}(Y|X)$ as expressed in (3).

Also, when $Y$ is binary, the MMSE serves as a lower bound for the probability of error. Specifically, if $Y \in \{\pm 1\}$, then
$$P_{\text{error}}(Y|X) = \inf_{h: \mathbb{R}^d \rightarrow \{\pm 1\}} \mathbb{E} \left[ \mathbb{I}_{Y \neq h(X)} \right]$$
$$= \inf_{h: \mathbb{R}^d \rightarrow \{\pm 1\}} \mathbb{E} \left[ \frac{(Y - h(X))^2}{4} \right]$$
$$\geq \frac{1}{4} \inf_{h \text{ meas.}} \mathbb{E} \left[ (Y - h(X))^2 \right]$$
$$= \frac{1}{4} \text{mmse}(Y|X).$$
Thus, for binary $Y$, any lower bound for $\text{mmse}(Y|X)$ gives rise to a lower bound for $P_{\text{error}}(Y|X)$. This observation further illustrates the importance of studying lower bounds for the MMSE in the context of privacy, where probability of correctly guessing $(1 - P_{\text{error}})$ has also been used as an information leakage measure [9], [25], [35].

\footnote{Observe that this definition requires to have random variables with finite second moments. Since in this paper we always deal with bounded random variables, this requirement is immaterial.}
B. Neural Network-based MMSE Estimation

A sigmoidal function $\phi : \mathbb{R} \to [-1, 1]$ is a (measurable) function such that
\[
\lim_{z \to -\infty} \phi(z) = -1 \quad \text{and} \quad \lim_{z \to \infty} \phi(z) = 1.
\] (8)

Note that we are assuming that $|\phi(z)| \leq 1$ for all $z \in \mathbb{R}$. Let $\mathcal{H}_k^\phi$ be the hypothesis class associated with a two-layer neural network of size $k$ with activation function $\phi$. More specifically, $\mathcal{H}_k^\phi$ is the set of all functions $h : \mathbb{R}^d \to \mathbb{R}$ of the form
\[
h(x) = c_0 + \sum_{l=1}^{k} c_l \phi(a_l \cdot x + b_l),
\] (9)

where $a_l \in \mathbb{R}^d$ and $b_l, c_l \in \mathbb{R}$. In this work we propose the following neural network estimator of the MMSE of $Y$ given $X$. Given a random sample $\{(X_i, Y_i)\}_{i=1}^{n}$, we define
\[
\text{mmse}_{k,n}(Y \mid X) := \inf_{h \in \mathcal{H}_k^\phi} \frac{1}{n} \sum_{i=1}^{n} (Y_i - h(X_i))^2,
\] (10)
i.e., $\text{mmse}_{k,n}(Y \mid X)$ is the minimum empirical square-loss attained by a two-layer neural network. Observe that, optimization matters aside, $\text{mmse}_{k,n}(Y \mid X)$ can be obtained from the sample using a device capable of implementing a two-layer neural network of size $k$. In this paper we take an information-theoretic perspective and assume infinite computational power. Specifically, we assume that $\text{mmse}_{k,n}(Y \mid X)$ can be computed exactly.

Our goal is to establish a (probabilistic) bound of the form
\[
\text{mmse}_{k,n}(Y \mid X) - \epsilon_{k,n} \leq \text{mmse}(Y \mid X),
\] (11)
where $\epsilon_{k,n}$ is a positive number depending on the sample size $n$ and the neural network size $k$. In Section III we establish such a bound and, in addition, we derive an analogous result when the neural network has hyperbolic tangent as output activation function. Specifically, we replace $\mathcal{H}_k^\phi$ by $\tanh \circ \mathcal{H}_k^\phi$, i.e., the family of functions of the form $\tanh \circ h$ with $h \in \mathcal{H}_k^\phi$. In this case, the relevant MMSE estimator is the defined as
\[
\text{mmse}^*_{k,n}(Y \mid X) := \inf_{h \in \mathcal{H}_k^\phi} \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tanh(h(X_i)))^2.
\] (12)

Observe that, by definition, $\text{mmse}(Y \mid X)$ is the minimum expected square-loss attained by any measurable function. Hence, the bound in (11) differs from classical statistical learning results (e.g., Rademacher complexity bounds) for which the expected loss is minimized over the hypothesis class $\mathcal{H}_k^\phi$. In particular, we have to consider the so-called approximation error, which could be estimated via the function approximation theorem of Barron [28].

C. Barron’s Theorem

Let $B \subset \mathbb{R}^d$ be a bounded set such that $0 \in B$. We define $\Gamma_B$ as the set of all functions $h : B \to \mathbb{R}$ admitting an integral representation of the form
\[
h(x) = h(0) + \int_{\mathbb{R}^d} (e^{\imath \omega \cdot x} - 1) \hat{H}(d\omega),
\] (13)

\[3\text{From an applied perspective, this assumption is not trivial to guarantee. Indeed, it is known that training neural networks to optimality could be a computationally difficult problem, see, e.g., [36], [37]. We further discuss some computational aspects in Section VI.\]
for some complex-valued measure $\hat{H}$ such that $\int |\omega| |\hat{H}|(d\omega)$ is finite. Observe that, as pointed out by Barron \[28\] Sec. III, the right hand side of (13) defines an extension of $h$ to $\mathbb{R}^d$. However, it is important to remark that such an extension might not be unique as there might be multiple complex-valued measures $\hat{H}$ satisfying (13).

Given $h \in \Gamma_B$, its Barron constant $C_h$ is defined as

$$C_h := \inf_{\hat{H}} \int_{\mathbb{R}^d} |\omega| |\hat{H}|(d\omega),$$

where the infimum is over all complex-valued measures $\hat{H}$ satisfying (13) and

$$|\omega|_B := \sup_{x \in B} |\omega \cdot x|.$$

To the best of the authors’ knowledge, there is no known method to compute $C_h$ given an arbitrary $h \in \Gamma_B$. However, in practice, we can take any complex-valued measure $\hat{H}$ satisfying (13) and use it to evaluate the bound

$$C_h \leq \text{rad}(B) \int_{\mathbb{R}^d} |\omega| |\hat{H}|(d\omega),$$

where $\text{rad}(B) := \sup_{x \in B} |x|$.

Under mild assumptions, the Barron constant could be related to the Fourier transform. Recall that, for a function $h \in L^1(\mathbb{R}^d)$, its Fourier transform $\hat{h} : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\hat{h}(\omega) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} h(x) e^{-i \omega \cdot x} dx.$$ (17)

If, in addition, $\hat{h} \in L^1(\mathbb{R}^d)$ and $\int |\omega| |\hat{h}(\omega)|d\omega$ is finite, then the Fourier inversion theorem implies that $h|_B \in \Gamma_B$ and

$$C_{h|_B} \leq \frac{\text{rad}(B)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\omega| |\hat{h}(\omega)|d\omega.$$ (18)

The following proposition establishes, in a quantitative manner, the universal approximating capabilities of two-layer neural networks. Observe that the statement below is a translation of Barron’s original formulation \[28\] Theorem 1 to the case of sigmoidal functions as defined in Section II-B.4

Proposition 1 (Theorem 1, \[28\]). Let $B \subset \mathbb{R}^d$ be a bounded set containing 0. For every $h \in \Gamma_B$ and every probability distribution $P$ over $B$, there exists $h_k \in \mathcal{H}_k^\phi$ such that

$$\int_B |h_k(x) - h(x)|^2 P(dx) \leq \frac{(2C_h)^2}{k}.$$ (19)

Furthermore, the coefficients of $h_k$ may be restricted to satisfy $|c_0| \leq |h(0)| + C_h$ and $\sum_{t=1}^k |c_t| \leq C_h$.

D. Additive Gaussian Mechanism

To motivate the forthcoming applications, consider the following setting. Assume that $X \in \mathbb{R}^d$ are sensible features of an individual which are correlated with a private attribute $Y \in \{\pm 1\}$, e.g., $X$ could be height and $Y$ gender. Due to privacy concerns, a data analyst might not be able to observe $X$ but a sanitized version of $X$.
In this work we focus on the so-called additive Gaussian mechanism, a popular sanitization method in the information-theoretic and the differential privacy literature, see, e.g., [3]–[5]. Given \( \sigma > 0 \), we define

\[
X_i^\sigma := X_i + \sigma Z_i,
\]

where \( Z_1, \ldots, Z_n \) are i.i.d. standard Gaussian vectors. Since the Gaussian distribution has unbounded support, it is often convenient to further process extreme values of the random variables \( X_i^\sigma \). We consider two processing techniques.

1) Extreme Values Truncation: Let \( B \subset \mathbb{R}^d \) be a bounded set. Extreme values truncation is the data processing technique that discards all samples with \( X_i^\sigma \) outside the set \( B \). Let \( \tilde{f}_\pm^\sigma \) be the conditional density of \( X^\sigma \) after truncation given \( Y = \pm 1 \). It is straightforward to verify that

\[
\tilde{f}_\pm^\sigma (x) = \frac{(f_\pm * K_\sigma)(x)}{\mathbb{P}(X^\sigma \notin B | Y = \pm 1)} \mathbb{1}_{x \in B},
\]

where \( f_\pm \) is conditional density of \( X \) given \( Y = \pm 1 \), \( * \) is the convolution operator, and \( K_\sigma \) is the density of \( \sigma Z \).

2) Extreme Values Randomization: Let \( B \subset \mathbb{R}^d \) be a bounded set. Extreme values randomization is the data processing technique that takes each \( X_i^\sigma \) outside the set \( B \) and replaces it with a random value on \( B \). As before, let \( \tilde{f}_\pm^\sigma \) be the conditional density of \( X^\sigma \) after randomization given \( Y = \pm 1 \). It is straightforward to verify that

\[
\tilde{f}_\pm^\sigma (x) = \left[ (f_\pm * K_\sigma)(x) + \frac{\mathbb{P}(X^\sigma \notin B | Y = \pm 1)}{\text{vol}(B)} \right] \mathbb{1}_{x \in B},
\]

where \( \text{vol}(B) \) denotes the volume of \( B \) w.r.t. the Lebesgue measure on \( \mathbb{R}^d \).

Under mild assumptions on \( f_\pm \), e.g., bounded and compactly supported, both (21) and (22) define non-negative\(^6\) smooth functions on the interior of \( B \). If \( B \) is closed, a routine application of Whitney’s extension theorem \([38]\) shows that, for any smooth function \( h : \mathbb{R} \to \mathbb{R} \), the function \( h \circ (\tilde{f}_\pm^\sigma / \tilde{f}_\sigma^\pm) \) can be extended to a rapidly-decreasing smooth function over \( \mathbb{R}^d \) \([39\] Ch. 7] and, in particular, \( h \circ (\tilde{f}_\pm^\sigma / \tilde{f}_\sigma^\pm) \) belongs to \( \Gamma_B \).

III. MMSE LOWER BOUNDS

In this section we provide lower bounds for \( \text{mmse}(Y|X) \) based on \( \text{mmse}_{k,n}(Y|X) \) and \( \text{mmse}_{k,n}^*(Y|X) \), as envisioned in \([1]\).

A. Output Activation Function: Identity

The following theorem establishes a lower bound for the MMSE in estimating \( Y \) given \( X \) based on the estimator \( \text{mmse}_{k,n}(Y|X) \), as defined in \([10]\), and the Barron constant of the conditional expectation of \( Y \) given \( X \).

**Theorem 1.** Let \( k, n \in \mathbb{N} \) and \( B \subset \mathbb{R}^d \) be a bounded set containing 0. If \( Y \in [-1, 1] \), \( X \) is supported on \( B \), and the conditional expectation \( \eta(x) := \mathbb{E}[Y|X = x] \) belongs to \( \Gamma_B \), then, with probability at least \( 1 - \delta \),

\[
\text{mmse}_{k,n}(Y|X) - \epsilon_{k,n,\delta} \leq \text{mmse}(Y|X),
\]

\(^5\)Observe that this technique potentially reduces the sample size, although, the reduction is negligible when \( B \) is large. In any case, for ease of notation, we let \( n \) denote the effective sample size after truncation.

\(^6\)In fact, \( \tilde{f}_\pm^\sigma \) are bounded away from 0 on \( B \).
where
\[ \epsilon_{k,n,\delta} = 2(1 + C_\eta)2 \sqrt{\frac{2 \log(1/\delta)}{n}} + \frac{4C^2_\eta}{k} + \frac{8C_\eta}{\sqrt{k}}. \] (24)

**Proof.** For ease of notation, we define
\[ \Delta := \text{mmse}_{k,n}(Y|X) - \text{mmse}(Y|X). \] (25)

Also, we define
\[ L(h) := \mathbb{E}[(Y - h(X))^2] \]
and
\[ \hat{L}_n(h) := \frac{1}{n} \sum_{i=1}^{n} (Y_i - h(X_i))^2. \] (26)

Recall that the infimum defining \( \text{mmse}(Y|X) \) is attained by the conditional expectation \( \eta \), see (2). Thus, we have that
\[ \Delta = \inf_{h \in \mathcal{H}_k^\Phi} \hat{L}_n(h) - \inf_{h \text{ meas}} L(h) \] (27)
\[ = \inf_{h \in \mathcal{H}_k^\Phi} \hat{L}_n(h) - L(\eta). \] (28)

Since \( \eta \in \Gamma_B \) by assumption, Barron’s theorem (Proposition 1) implies that there exists \( \eta_k \in \mathcal{H}_k^\Phi \) such that
\[ \|\eta_k - \eta\|_2 \leq \frac{2C_\eta}{\sqrt{k}}, \] (29)
where \( \|\cdot\|_2 \) is the 2-norm w.r.t. the distribution of \( X \), i.e.,
\[ \|h\|_2^2 = \int_B |h(x)|^2 P_X(dx). \] (30)

Furthermore, if we let
\[ \eta_k(x) = c_0 + \sum_{i=1}^{k} c_i \phi(a_i \cdot x + b_i), \] (31)
the coefficients \( c_0, c_1, \ldots, c_k \) can be restricted to satisfy that \( c_0 \leq |\eta(0)| + C_\eta \) and \( \sum_{i=1}^{k} |c_i| \leq C_\eta \). Observe that, by (28),
\[ \Delta \leq \hat{L}_n(\eta_k) - L(\eta_k) + L(\eta_k) - L(\eta). \] (32)

Since \( Y \in [-1,1] \), we have that \( |\eta(x)| \leq 1 \) for all \( x \in B \). Recall that, by assumption, \( \phi(z) \in [-1,1] \) for all \( z \in \mathbb{R} \). Thus, by our choice of the coefficients \( c_0, c_1, \ldots, c_k \) in (31),
\[ |\eta_k(x)| \leq 1 + 2C_\eta, \quad x \in \mathbb{R}^d. \] (33)

Therefore, \((Y_i - \eta_k(X_i))^2 \leq 4(1 + C_\eta)^2\) for all \( i \in \{1, \ldots, n\} \). As a result, a routine application of Hoeffding’s inequality [40] Sec. 4.2] implies that, with probability at least \( 1 - \delta \),
\[ \hat{L}_n(\eta_k) - L(\eta_k) \leq 2(1 + C_\eta)^2 \sqrt{\frac{2 \log(1/\delta)}{n}}. \] (34)

It is straightforward to verify that
\[ L(\eta_k) - L(\eta) = \|\eta_k - \eta\|_2^2 + 2\mathbb{E}[(\eta(X) - \eta_k(X))(Y - \eta(X))]. \] (35)
Recall that $|\eta(x)| \leq 1$ for all $x \in B$. Therefore, an application of the Cauchy–Schwarz inequality implies that
\[
|L(\eta_k) - L(\eta)| \leq \|\eta_k - \eta\|_2 (4 + \|\eta_k - \eta\|_2).
\] (36)
By plugging (29) in (36), we conclude that
\[
|L(\eta_k) - L(\eta)| \leq 2C\eta \sqrt{\frac{k}{n}} (4 + 2C\eta \sqrt{\frac{k}{n}}).
\] (37)
The theorem follows by plugging (34) and (37) in (32).

Regarding the assumptions of the previous theorem, it is important to remark that, in general, it might be non-trivial to verify that the conditional expectation $\eta$ belongs to $\Gamma_B$. Nonetheless, as shown in Section V, this assumption is automatically satisfied when data is post-processed by the additive Gaussian mechanism.

Note that $\epsilon_{k,n,\delta}$, as defined in (24), is non-increasing in $k$. Since $\mathcal{H}_k^B \subseteq \mathcal{H}_{k+1}^B$ for all $k \in \mathbb{N}$, $\text{mmse}_{k,n}(Y|X)$ is also non-increasing in $k$. As a result, the lower bound in (23) does not necessarily improve by making $k$ larger. Indeed, it is known that if $k$ is large enough then $\text{mmse}_{k,n}(Y|X)$ is equal to 0, see, e.g., [41], [42], which makes the lower bound in (23) trivial. Together with the fact that the minimization defining $\text{mmse}_{k,n}(Y|X)$ becomes harder as $k$ increases, the previous observations reveal the non-trivial nature of finding the value of $k$ that produces the best numerical results. We expand on this discussion in Section VI.

B. Output Activation Function: Hyperbolic Tangent

The following theorem establishes a lower bound for the MMSE in estimating $Y$ given $X$ based on the estimator $\text{mmse}_{k,n}^*(Y|X)$, as defined in (12), and the Barron constant of the log-likelihood ratio defined in (38) below. Note that, unlike Theorem 1, the next theorem requires $Y$ to be binary.

**Theorem 2.** Let $k, n \in \mathbb{N}$ and $B \subset \mathbb{R}^d$ be a bounded set containing 0. Assume that $Y \in \{\pm 1\}$, $X$ is supported on $B$, and the conditional density of $X$ given $Y = \pm 1$, denoted by $f_{\pm}$, is positive on $B$. Let $p := \mathbb{P}(Y = 1)$ and
\[
\theta(x) := \frac{1}{2} \log \left( \frac{pf_+(x)}{pf_-(x)} \right), \quad x \in B.
\] (38)
If $\theta$ belongs to $\Gamma_B$, then, with probability at least $1 - \delta$,
\[
\text{mmse}_{k,n}^*(Y|X) - \epsilon_{k,n,\delta}^* \leq \text{mmse}(Y|X),
\] (39)
where
\[
\epsilon_{k,n,\delta}^* = 2\sqrt{\frac{2 \log(1/\delta)}{n}} + \frac{4C^2_\theta}{k} + \frac{8C_\theta}{\sqrt{k}}.
\] (40)

**Proof.** For ease of notation, we define
\[
\Delta^* := \text{mmse}_{k,n}^*(Y|X) - \text{mmse}(Y|X).
\] (41)
Also, we define $L^*(h) := \mathbb{E}[(Y - \tanh(h(X)))^2]$ and
\[
\hat{L}_{n}^*(h) := \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tanh(h(X_i)))^2.
\] (42)
Recall that the infimum defining $\text{mmse}(Y|X)$ is attained by the conditional expectation $\eta$, see (2). A straightforward computation shows that, for every $x \in B$,

$$\eta(x) = \frac{p_{f_+}(x) - p_{f_-}(x)}{p_{f_+}(x) + p_{f_-}(x)}$$

$$= \tanh \left( \frac{1}{2} \log \left( \frac{p_{f_+}(x)}{p_{f_-}(x)} \right) \right)$$

$$= \tanh(\theta(x)).$$

Therefore, we have that

$$\text{mmse}(Y|X) = \mathbb{E}[(Y - \eta(X))^2]$$

$$= \mathbb{E}[(Y - \tanh(\theta(X)))^2]$$

$$= L^*(\theta),$$

and, as a result,

$$\Delta^* = \inf_{h \in \mathcal{H}_k} \hat{L}_n^*(h) - L^*(\theta).$$

Since $\theta \in \Gamma_B$ by assumption, Barron’s theorem (Proposition 1) implies that there exists $\theta_k \in \mathcal{H}_k^\phi$ such that

$$\|\theta_k - \theta\|_2 \leq \frac{2C_\theta}{\sqrt{k}},$$

where $\|\cdot\|_2$ is the 2-norm w.r.t. the distribution of $X$. Furthermore, if we let

$$\theta_k(x) = c_0 + \sum_{l=1}^k c_l \phi(a_l \cdot x + b_l),$$

the coefficients $c_0, c_1, \ldots, c_k$ can be restricted to satisfy that $c_0 \leq |\theta(0)| + C_\theta$ and $\sum_{l=1}^k |c_l| \leq C_\theta$. Observe that, by (49),

$$\Delta \leq \hat{L}_n^*(\theta_k) - L^*(\theta_k) + L^*(\theta_k) - L^*(\theta).$$

Since $Y \in \{\pm 1\}$ and $\tanh : \mathbb{R} \to (-1,1)$, it is immediate to verify that $(Y_i - \tanh(\theta_k(X_i)))^2 \leq 4$ for all $i \in \{1, \ldots, n\}$. As a result, a routine application of Hoeffding’s inequality [40, Sec. 4.2] implies that, with probability at least $1 - \delta$,

$$\hat{L}_n^*(\theta_k) - L^*(\theta_k) \leq 2 \sqrt{\frac{2\log(1/\delta)}{n}}.$$  

It is straightforward to verify that

$$L^*(\theta_k) - L^*(\theta) = \|\tanh \circ \theta_k - \tanh \circ \theta\|_2^2 + 2\mathbb{E}[(\tanh(\theta(X)) - \tanh(\theta_k(X)))(Y - \tanh(\theta(X)))].$$

Thus, the Cauchy–Schwarz inequality leads to

$$|L^*(\theta_k) - L^*(\theta)| \leq \|\tanh \circ \theta_k - \tanh \circ \theta\|_2(4 + \|\tanh \circ \theta_k - \tanh \circ \theta\|_2).$$

Since $\tanh$ is a 1-Lipschitz function, it can be verified that $\|\tanh \circ \theta_k - \tanh \circ \theta\|_2 \leq \|\theta_k - \theta\|_2$. Therefore, we have that

$$|L^*(\theta_k) - L^*(\theta)| \leq \|\theta_k - \theta\|_2(4 + \|\theta_k - \theta\|_2).$$
By plugging (50) in (56), we conclude that
\[
|L^*(\theta_k) - L^*(\theta)| \leq \frac{2C_\theta}{\sqrt{k}} \left( 4 + \frac{2C_\theta}{\sqrt{k}} \right).
\] (57)
The theorem follows by plugging (53) and (57) in (52).

As with Theorem 1, the hypotheses of the previous theorem are automatically satisfied when data is post-processed by the additive Gaussian mechanism. We discuss this claim in detail in Section V.

Observe that, as established in (45), the conditional expectation \( \eta \) and the log-likelihood ratio \( \theta \) satisfy that
\[
\eta = \tanh \circ \theta.
\] (58)

In view of this relation, the choice of the neural network defining \( \text{mmse}_{k,n}^*(Y|X) \) becomes evident: the second layer approximates \( \theta \) while the output activation function is the hyperbolic tangent function.

**IV. A General Bound for the Barron Constant**

Theorem 1 establishes a lower bound for \( \text{mmse}(Y|X) \) based on the estimator \( \text{mmse}_{k,n}(Y|X) \) and the Barron constant \( C_\eta \) of the conditional expectation of \( Y \) given \( X \). While \( \text{mmse}_{k,n}(Y|X) \) can be computed from the sample, providing estimates for the Barron constant \( C_\eta \) might be challenging for two reasons: (i) the conditional expectation of \( Y \) given \( X \) depends on the distribution of \( X \) and \( Y \), which is typically unavailable in practice, and (ii) the Barron constant \( C_\eta \) is defined in terms of the Fourier transform of \( \eta \), which makes its computation unfeasible in most cases. (A similar remark applies, mutatis mutandis, to Theorem 2.) In this section we provide some results that alleviate the second issue; the discussion of the first issue is left for the following section.

**A. 1-Dimensional Bound**

In this section we focus on a special family of real-valued functions of a real variable whose Barron’s constant admits a relatively tractable representation.

Let \( h: \mathbb{R} \to \mathbb{R} \) be a differentiable function. If \( h' \in L^1(\mathbb{R}) \) and \( \hat{h}' \in L^1(\mathbb{R}) \), the Fourier inversion theorem implies that
\[
h'(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{h}'(\omega) e^{i\omega x} d\omega.
\] (59)

As pointed out by Barron [28 Appendix], (59) implies that \( h|_B \) belongs to \( \Gamma_B \) for every bounded set \( B \) containing 0 and
\[
C_{h|_B} \leq \frac{\text{rad}(B)}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{h}'(\omega)| d\omega.
\] (60)
Thus, by abuse of notation, we define
\[
C_h := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{h}'(\omega)| d\omega,
\] (61)
whenever \( h: \mathbb{R} \to \mathbb{R} \) satisfies that \( h', \hat{h}' \in L^1(\mathbb{R}) \).

**Theorem 3.** Let \( h: \mathbb{R} \to \mathbb{R} \) be a thrice differentiable function. If \( h', h'', h''' \in L^1(\mathbb{R}) \) and vanish at infinity, then
\[
C_h \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \log \left( \frac{\sqrt{\|h''\|_1 \|h'''\|_1}}{\|h'''\|_1} \right) \right) \|h''\|_1.
\] (62)
Proof. Let $0 < \lambda_1 < \lambda_2$. We split the integral in (61) as

$$I := \int_{-\lambda_1}^{\lambda_1} \left| \hat{h}'(\omega) \right| d\omega,$$

$$II := \left( \int_{\lambda_1}^{\lambda_2} + \int_{-\lambda_2}^{-\lambda_1} \right) \left| \hat{h}'(\omega) \right| d\omega,$$

$$III := \left( \int_{\lambda_2}^{\infty} + \int_{-\infty}^{-\lambda_2} \right) \left| \hat{h}'(\omega) \right| d\omega.$$  

First, observe that

$$I \leq 2\left\| \hat{h}' \right\|_{\infty} \lambda_1 \leq 2\|h'\|_1 \lambda_1,$$  

where we applied the inequality $\left\| \hat{h}' \right\|_{\infty} \leq \|h'\|_1$. Since $h'$ vanishes at infinity and $h'' \in L^1(\mathbb{R})$, $\hat{h}''(\omega) = i\omega \hat{h}'(\omega)$ for every $\omega \in \mathbb{R}$. Thus, we have that

$$II = \left( \int_{\lambda_1}^{\lambda_2} + \int_{-\lambda_2}^{-\lambda_1} \right) \left| \hat{h}'(\omega) \right| d\omega \leq 2\|h''\|_1 \log \left( \frac{\lambda_2}{\lambda_1} \right).$$

Similarly, $\hat{h}'''(\omega) = (i\omega)^2 \hat{h}'(\omega)$ for every $\omega \in \mathbb{R}$ and

$$III = \left( \int_{\lambda_2}^{\infty} + \int_{-\infty}^{-\lambda_2} \right) \left| \hat{h}'(\omega) \right| d\omega \leq 2\|h'''\|_1 \lambda_2.$$

By plugging (66), (69) and (70) in (61), we conclude that $\hat{h} \in L^1(\mathbb{R})$ and

$$C_h \leq \sqrt{\frac{2}{\pi}} \left( \|h'\|_1 \lambda_1 + \|h''\|_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) + \frac{\|h'''\|_1}{\lambda_2} \right).$$

By taking $\lambda_1 = \frac{\|h''\|_1}{\|h'\|_1}$ and $\lambda_2 = \frac{\|h'''\|_1}{\|h''\|_1}$, the result follows.

Observe that if we let $\lambda_1 = \lambda_2 = \sqrt{\|h'''\|_1/\|h''\|_1}$ in (71), we obtain

$$C_h \leq \sqrt{\frac{2\sqrt{\pi}}{\sqrt{\pi}}} \sqrt{\|h''\|_1 \|h'''\|_1}.$$

This bound is generalized for functions $h : \mathbb{R}^d \to \mathbb{R}$ in Theorem 4 below. Observe that while the bound in (72) is simpler than the one provided in Theorem 3, it is typically weaker in applications.

Since Theorem 3 will be applied to the conditional expectation $\eta$ and the log-likelihood ratio $\theta$, we need to compute the derivatives of these functions. The following lemma provides useful expressions for the first three derivatives of $\eta$ in the case when $Y$ is binary. Recall that if $f_{\pm} : B \to \mathbb{R}$ is the conditional density of $X$ given $Y = \pm1$ and $p = \mathbb{P}(Y = 1)$, then the conditional expectation of $Y$ given $X$ is given by

$$\eta(x) = \frac{pf_+(x) - pf_-(x)}{pf_+(x) + pf_-(x)}.$$
Lemma 1. If \( \eta \) is defined as in (73), then
\[
\eta' = \frac{2g_+g_- - g_+g_-'}{(g_+ + g_-)^2},
\]
\[
\eta'' = \frac{2g_+g_- - g_+g_-'}{(g_+ + g_-)^2} - 2\eta'g_+ + g_-',
\]
\[
\eta''' = \frac{2g_+g_- - g_+g_-'}{(g_+ + g_-)^2} - 2\eta'g_+ + g_-' - 3\eta''g_+ + g_-',
\]
where \( g_+ = pf_+ \) and \( g_- = pf_- \).

Proof. Equation (74) follows easily from the (73). By the quotient rule \( \left( \frac{h_1}{h_2} \right)' = \frac{h_1' h_2 - h_1 h_2'}{h_2^2} \), (75) follows from (74). Using similar arguments, (76) follows from (75).

Similarly, the following lemma provides useful expressions for the first three derivatives of the log-likelihood ratio
\[
\theta(x) = \frac{1}{2} \log \left( \frac{pf_+(x)}{pf_-(x)} \right).
\]

Lemma 2. If \( \theta \) is defined as in (77), then
\[
2\theta' = \frac{g_+'}{g_+} - \frac{g_-'}{g_-},
\]
\[
2\theta'' = \left[ \frac{g_+''}{g_+} - \left( \frac{g_+'}{g_+} \right)^2 \right] - \left[ \frac{g_-''}{g_-} - \left( \frac{g_-'}{g_-} \right)^2 \right],
\]
\[
2\theta''' = \left[ \frac{g_+'''}{g_+} - 3 \frac{g_+'' g_+'}{g_+^2} + 2 \left( \frac{g_+'}{g_+} \right)^3 \right] - \left[ \frac{g_-'''}{g_-} - 3 \frac{g_-'' g_-'}{g_-^2} + 2 \left( \frac{g_-'}{g_-} \right)^3 \right],
\]
where \( g_+ = pf_+ \) and \( g_- = pf_- \).

Proof. The identities in (78) – (80) follow from the quotient rule and the logarithmic derivative \( (\log h)' = h'/h \).

It is important to remark that, in general, \( \eta \) and \( \theta \) might not satisfy the assumptions of Theorem 3 i.e., having integrable derivatives that vanish at infinity. Nonetheless, as shown in Section V they satisfy the aforementioned assumptions when data is post-processed by the additive Gaussian mechanism. We verify this claim through a careful analysis of the derivatives of \( \eta \) and \( \theta \) given in Lemmas 1 and 2 respectively.

B. d-Dimensional Extension

In this section we focus on a special family of real-valued functions of \( d \)-real variables whose Barron’s constant admits a relatively tractable representation.

Let \( h : \mathbb{R}^d \to \mathbb{R} \) be a differentiable function. For ease of notation, we let \( h_{x_j} := \frac{\partial}{\partial x_j} h \). If \( h_{x_j}, \hat{h}_{x_j} \in L^1(\mathbb{R}^d) \) for every \( j \in \{1, \ldots, d\} \), the Fourier inversion theorem implies that
\[
\nabla h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{\nabla} h(\omega) e^{i\omega \cdot x} d\omega,
\]
where \( \nabla h = (h_{x_1}, \ldots, h_{x_d}) \) and \( \hat{\nabla} h = (\hat{h}_{x_1}, \ldots, \hat{h}_{x_d}) \). As pointed out by Barron [28, Appendix], (81) implies that \( h|_B \) belongs to \( \Gamma_B \) for every bounded set \( B \) containing 0 and
\[
C_{h|_B} \leq \frac{\text{rad}(B)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\hat{\nabla} h(\omega)| d\omega.
\]
Thus, by abuse of notation, we define
\[
C_h := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\hat{\nabla} h(\omega)| d\omega,
\]
whenever \( h : \mathbb{R}^d \to \mathbb{R} \) satisfies that \( h_{x_j} , \hat{h}_{x_j} \in L^1(\mathbb{R}^d) \) for every \( j \in \{1, \ldots, d\} \).

**Theorem 4.** Let \( h : \mathbb{R}^d \to \mathbb{R} \) be a function of class \( C^{d+2} \). If the partial derivatives of \( h \) of order up to \( d + 2 \) belong to \( L^1(\mathbb{R}^d) \) and vanish at infinity, then
\[
C_h \leq A_d N_1^{1/(d+1)} N_2^{d/(d+1)},
\]
where \( A_d = \frac{d+1}{d^{d/(d+1)} 2^{d/2} \Gamma(d/2+1)} \), \( N_1^2 = \sum_{j=1}^{d} ||h_{x_j}||^2_1 \) and
\[
N_2^2 = \sum_{j=1}^{d} \left( \sum_{j'=1}^{d} \left| \frac{\partial^{d+1}}{\partial x_{j'}^{d+1}} h_{x_j}(\omega) \right| \right)^2.
\]

**Proof.** For \( \lambda > 0 \), let \( B_\lambda \) be the \( d \)-dimensional ball of radius \( \lambda \). We split the integral in (83) as
\[
I := \int_{B_\lambda} |\hat{\nabla} h(\omega)| d\omega \quad \text{and} \quad II := \int_{B_\lambda^c} |\hat{\nabla} h(\omega)| d\omega.
\]

Observe that, for every \( \omega \in \mathbb{R}^d \),
\[
|\hat{\nabla} h(\omega)|^2 = \sum_{j=1}^{d} |\hat{h}_{x_j}(\omega)|^2 \leq \sum_{j=1}^{d} ||h_{x_j}||^2_1 =: N_1^2,
\]
where we applied the inequality \( ||\hat{h}_{x_j}||_\infty \leq ||h_{x_j}||_1 \). Therefore,
\[
I \leq N_1 \text{vol}(B_\lambda) = \frac{\pi^{d/2} N_1 \lambda^d}{\Gamma(d/2+1)},
\]
as \( \text{vol}(B_1) = \pi^{d/2} / \Gamma(d/2 + 1) \).

For \( \omega \in \mathbb{R}^d \), the generalized mean inequality asserts that
\[
\sqrt[d]{\frac{|\omega_1|^2 + \cdots + |\omega_d|^2}{d}} \leq \sqrt[d]{\frac{\sum_{j=1}^{d} |\omega_j|^{d+1}}{d}},
\]
which in turn leads to
\[
|\omega|^{d+1} \leq d^{(d-1)/2} \sum_{j'=1}^{d} |\omega_{j'}|^{d+1}.
\]
For ease of notation, we define \( \partial^{d+1}_{j'} := \frac{\partial^{d+1}}{\partial x_{j'}^{d+1}} \). Since the partial derivatives of \( h \) of order up to \( d + 2 \) belong to \( L^1(\mathbb{R}^d) \) and vanish at infinity, then, for every \( \omega \in \mathbb{R}^d \),
\[
\partial^{d+1}_{j'} h_{x_j}(\omega) = (i\omega_{j'})^{d+1} \hat{h}_{x_j}(\omega).
\]
Therefore, (90) and (91) imply that
\[
|\omega|^{d+1} |\widehat{h_{x_j}}(\omega)| \leq d^{(d-1)/2} \sum_{j'=1}^{d} |\omega|^{d+1} |\widehat{h_{x_j}}(\omega)|
\]
(92)
\[
= d^{(d-1)/2} \sum_{j'=1}^{d} |\partial^{d+1}_{x_j} h_{x_j}(\omega)|
\]
(93)
\[
\leq d^{(d-1)/2} \sum_{j'=1}^{d} \|\partial^{d+1}_{x_j} h_{x_j}\|_1,
\]
(94)
where we applied the inequality \(\|\partial^{d+1}_{x_j} h_{x_j}\|_\infty \leq \|\partial^{d+1}_{x_j} h_{x_j}\|_1\). Alternatively, we have that
\[
|\widehat{h_{x_j}}(\omega)| \leq d^{(d-1)/2} \sum_{j'=1}^{d} \|\partial^{d+1}_{x_j} h_{x_j}\|_1.
\]
(95)
As a result, we obtain that
\[
|\nabla h(\omega)| = \left( \sum_{j=1}^{d} |\widehat{h_{x_j}}(\omega)|^2 \right)^{1/2}
\]
\[
\leq d^{(d-1)/2} \left( \sum_{j=1}^{d} \left( \sum_{j'=1}^{d} \|\partial^{d+1}_{x_j} h_{x_j}\|_1 \right) \right)^{1/2}
\]
\[
= d^{(d-1)/2} \frac{\sum_{j'=1}^{d} \|\partial^{d+1}_{x_j} h_{x_j}\|_1}{\|\omega\|^{d+1}} N_2.
\]
(96)
(97)
(98)
Since \(\omega \mapsto 1/|\omega|^{d+1}\) is a radial function, we have that
\[
\Pi \leq d^{(d-1)/2} N_2 \int_{B_r^d} \frac{1}{|\omega|^{d+1}} d\omega
\]
(99)
\[
= d^{(d-1)/2} N_2 \int_{\lambda}^{\infty} \frac{1}{r^{d+1}} \frac{d\pi^{d/2} r^{d-1}}{\Gamma(d/2 + 1)} dr
\]
(100)
\[
= \frac{\pi^{d/2} d^{(d+1)/2} N_2}{\Gamma(d/2 + 1)} \int_{\lambda}^{\infty} \frac{1}{r^2} dr
\]
(101)
\[
= \frac{\pi^{d/2} d^{(d+1)/2} N_2}{\Gamma(d/2 + 1)} \frac{1}{\lambda}.
\]
(102)
By plugging (88) and (102) in (83), we conclude that
\[
C_h \leq \frac{1}{2 d^{2/2} \Gamma(d/2 + 1)} \left( N_1 \lambda^d + \frac{d^{d+1}}{\lambda} N_2 \right).
\]
(103)
By taking \(\lambda^{d+1} = \frac{d^{(d-1)/2} N_2}{N_1}\), the result follows.

Note that (84) generalizes (72), as it only involves derivatives of order 1 and \(d + 2\). While it is possible to establish bounds that more closely resemble Theorem 3 (e.g., by using derivatives of order 1, \(d + 1\) and \(d + 2\)), they are more convoluted than (84) and add little practical value.

Theorem 4 might not be straightforward to apply as it heavily depends on the higher order partial derivatives of \(h\). Furthermore, since \(d^{1/d} \sim 1\) and \(\Gamma(z + 1) \sim \sqrt{2\pi z} \left( \frac{z}{e} \right)^z\), we can show that
\[
A_d \sim \sqrt{\frac{e d}{\pi d}}.
\]
(104)
In particular, the bound in (84) has an exponential dependency on the dimension. Despite the negative nature of this observation, it is indeed natural in view of a similar comment made by Barron in [28, Sec. IX-9].

**Remark 1.** A line of research initiated by Breiman [29], and recently extended by Domingo-Enrich and Mroueh [34] building upon the results of Ongie et al. [33], focuses on approximation results for two-layer neural networks with ReLU activation functions. In the 1-dimensional case, the approximation results by Breiman [29] rely on a variation of the Barron constant given by

$$C'_h := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |w|^2 |\hat{h}(\omega)| d\omega. \quad (105)$$

It is important to remark that Theorems 3 and 4 can be generalized to $C'_h$ at the expense of increasing by one the order of the derivatives involved. This seemingly superficial change has a deep impact on the implementation of our techniques as the complexity of the derivatives of $\eta$ and $\theta$ increases drastically with the order (see Lemmas 7 and 2). As a result, it is unclear at the moment if our techniques could be effectively adapted to this case.

**V. ADDITIVE GAUSSIAN MECHANISM**

In this section we consider the situation in which $Y$ is binary and $X \in \mathbb{R}$ is post-processed by the additive Gaussian mechanism introduced in Section II-D. Specifically, we assume that $X$ is post-processed to produce a new random variable

$$X^\sigma := X + \sigma Z, \quad (106)$$

where $\sigma > 0$ and $Z$ is a standard Gaussian random variable independent of $X$ and $Y$. Also, we assume that the random variable $X^\sigma$ is further processed to remove extreme values, giving rise to a random variable $\tilde{X}^\sigma$. Specifically, we consider extreme values truncation and extreme values randomization as introduced in Sections II-D1 and II-D2 respectively. Given the different nature of these two processing techniques, below we provide estimates for the Barron constant of (i) the conditional expectation under truncation and (ii) the log-likelihood ratio under randomization.

**A. Extreme Values Truncation**

Consider extreme values truncation with $B = [-r, r]$ for some $r > 0$. As before, we let $p := \mathbb{P}(Y = 1)$ and $\tilde{f}_\pm^\sigma$ be the conditional density of $\tilde{X}^\sigma$ given $Y = \pm 1$. The conditional expectation of $Y$ given $\tilde{X}^\sigma$ is equal to

$$\tilde{\eta}^\sigma(x) = \frac{p \tilde{f}_+^\sigma(x) - \bar{p} \tilde{f}_-^\sigma(x)}{p \tilde{f}_+^\sigma(x) + \bar{p} \tilde{f}_-^\sigma(x)} \mathbb{I}_{|x| \leq r} \quad (107)$$

$$= \tanh \left( \frac{1}{2} \text{log} \left( \frac{p \tilde{f}_+^\sigma(x)}{\bar{p} \tilde{f}_-^\sigma(x)} \right) \right) \mathbb{I}_{|x| \leq r}. \quad (108)$$

As established in (21),

$$\tilde{f}_\pm^\sigma(x) = \frac{(f_\pm * K_\sigma)(x)}{\mathbb{P}(|X^\sigma| \leq r|Y = \pm 1)} \mathbb{I}_{|x| \leq r}, \quad (109)$$

where $f_\pm$ is the conditional density of $X$ given $Y = \pm 1$ and

$$K_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}, \quad x \in \mathbb{R}. \quad (110)$$
From [109], we conclude that \( x \mapsto \frac{p_{f_\sigma}^*(x)}{\hat{p}_{f_\sigma}^*(x)} \) is a non-negative smooth function over \([-r, r]\) and, as a result, \( \hat{\eta}_\sigma \) is a smooth function over the same domain as well. As pointed out by Barron [28, Sec. IX], this implies that \( \hat{\eta}_\sigma \) belongs to \( \Gamma_B \) and, in particular, \( C_{\hat{\eta}_\sigma} \) is finite. Our goal is to find a tractable, yet useful, upper bound for \( C_{\hat{\eta}_\sigma} \).

As discussed in (16), a first step in order to find an upper bound for the Barron constant of \( \hat{\eta}_\sigma \) is to find a function, say \( \eta_\sigma \), such that \( \eta_\sigma \) is defined over \( \mathbb{R} \) and \( \hat{\eta}_\sigma = \eta_\sigma |_B \). In this situation, we have that

\[
C_{\hat{\eta}_\sigma} \leq \frac{r}{\sqrt{2\pi}} \int_{\mathbb{R}} |\omega| |\hat{\eta}_\sigma(\omega)| d\omega =: rC_{\eta_\sigma}. \tag{111}
\]

Motivated by (107) and (109), we define \( \eta_\sigma : \mathbb{R} \to \mathbb{R} \) by

\[
\eta_\sigma(x) = \frac{\lambda_+ f_\sigma^+(x) - \lambda_- f_\sigma^-(x)}{\lambda_+ f_\sigma^+(x) + \lambda_- f_\sigma^-(x)}, \tag{112}
\]

where \( f_\pm^\sigma = f_\pm * K_\sigma \) and

\[
\lambda_\pm = \frac{1}{2} \pm \left(p - \frac{1}{2}\right) \frac{1}{\mathbb{P}(|X| \leq r) Y = \pm 1}. \tag{113}
\]

Note that, by large deviations arguments, \( \lambda_\pm \) can be estimated with relatively high precision as it only depends on the probabilities of the events \( \{Y = \pm 1\} \) and \( \{|X| \leq r, Y = \pm 1\} \). Furthermore, it can be shown that \( \lim_{\sigma \to \infty} \frac{\lambda_+}{\lambda_-} = \frac{p}{\bar{p}} \), making the estimation of \( \lambda_\pm \) unnecessary for large \( \sigma \).

To gain some intuition about the behavior of the Barron constant as a function of \( \sigma \), in the next proposition we compute \( C_{\eta_\sigma} \) in a simple case.

**Proposition 2.** If \( Y \sim \text{Unif}\{\pm 1\} \) and \( X = Y \), then,

\[
C_{\eta_\sigma} = \frac{1}{\sigma^2}, \quad \sigma > 0. \tag{114}
\]

**Proof.** By symmetry, we have that \( \lambda_+ = \lambda_- \). Thus, by (112),

\[
\eta_\sigma(x) = \frac{f_\sigma^+(x) - f_\sigma^-(x)}{f_\sigma^+(x) + f_\sigma^-(x)}. \tag{115}
\]

A direct computation shows that

\[
f_\pm^\sigma(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-(x \mp 1)^2/2\sigma^2}. \tag{116}
\]

Therefore, for all \( x \in \mathbb{R} \),

\[
\eta_\sigma(x) = \tanh \left( \frac{x}{\sigma^2} \right). \tag{117}
\]

Observe that \( (\eta_\sigma)'(x) = \frac{1}{\sigma^2} \text{sech}^2 \left( \frac{x}{\sigma^2} \right) \). Using contour integration, it can be verified that

\[
\text{sech}^2(\omega) = \sqrt{\frac{\pi}{2}} \omega \text{csch} \left( \frac{\pi}{2} \omega \right). \tag{118}
\]

In particular, \( (\eta_\sigma)' \), \( \hat{(\eta_\sigma)}' \) \( \in L^1(\mathbb{R}) \) and, by (61),

\[
C_{\eta_\sigma} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |(\hat{\eta}_\sigma)'(\omega)| d\omega. \tag{119}
\]

By (118), we have that \( (\hat{\eta}_\sigma)'(\omega) \) is non-negative for all \( \omega \in \mathbb{R} \). Therefore, by the Fourier inversion theorem, (119) implies that \( C_{\eta_\sigma} = \frac{1}{\sigma^2} \text{sech}^2(0). \)

\[
\Box
\]
The next theorem provides an upper bound for the Barron constant of \( \eta^\sigma \), as defined in (112), under minimal assumptions on the distribution of \( X \).

**Theorem 5.** If \( \text{Supp}(f_\pm) \subset [-1, 1] \), then, for every \( \sigma > 0 \),

\[
C_{\eta^\sigma} \leq \frac{2\sqrt{2}}{e\sqrt{\pi}} + \frac{16\sqrt{2}M_0^\sigma}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M_0^\sigma}{\sigma^8} \right) \right),
\]

(120)

where

\[
M_0^\sigma := \int_\mathbb{R} |x|^\alpha \frac{f_+^{\sigma}(x)\lambda_+ f_-^{\sigma}(x)}{(f_+^{\sigma}(x) + f_-^{\sigma}(x))^2} \, dx,
\]

(121)

\[
M_\sigma := M_0^\sigma (64M_2^\sigma + 176M_1^\sigma + (136 + 48\sigma^2)M_0^\sigma).
\]

(122)

Furthermore, if \( \sqrt{8eM_0^\sigma} \leq \sigma \), then

\[
C_{\eta^\sigma} \leq \frac{16\sqrt{2}M_0^\sigma}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M_0^\sigma}{\sigma^8} + 3\frac{M_1^\sigma}{M_0^\sigma} + 3 + \sigma^2 \right) \right).
\]

(123)

The proof of the previous theorem, which can be found in Appendix A, relies on Theorem 3 and careful estimates of the \( L^1 \)-norms of the derivatives of \( \eta^\sigma \). Specifically, we exploit the cancellations that occur between the terms in the numerators of (74) – (76).

Note that the bounds in the previous theorem only depend on the moment-like quantities \( M_0^\sigma \), as defined in (121). As we show below, in some canonical situations \( M_0^\sigma = O(\sigma^{2(1+\alpha)}) \) as \( \sigma \to \infty \). Therefore, in the large noise regime (\( \sigma \gg 1 \)),

\[
C_{\eta^\sigma} \leq O \left( \frac{\log(\sigma)}{\sigma^2} \right).
\]

(124)

In view of Proposition 3, we conclude that the previous bound is order optimal up to logarithmic factors. Below we also show that in some situations \( M_0^\sigma = O(1) \) as \( \sigma \to 0^+ \). Therefore, in the small noise regime (\( \sigma \ll 1 \)),

\[
C_{\eta^\sigma} \leq O \left( \frac{\log(1/\sigma)}{\sigma^4} \right).
\]

(125)

Although the order optimality of this bound is unclear, it is by no means trivial. Observe that, as \( \sigma \to 0^+ \), \( \eta^\sigma \) converges pointwise to \( \eta \) which in principle might have an unbounded Barron constant. Thus, (125) shows that even if \( C_{\eta^\sigma} \) diverges to infinity as \( \sigma \to 0^+ \), it does it polynomially in \( 1/\sigma \).

We end this section providing an upper bound for the moment-like quantities \( M_\alpha^\sigma \) under different structural properties of the support of \( f_\pm \). In the next proposition, we do so in the case where the supports of \( f_+ \) and \( f_- \) are well-separated by some margin.

**Proposition 3.** Let \( M_\alpha^\sigma \) be the quantities defined in (121). If there exist \( \gamma \in (0, 1) \) such that \( \text{Supp}(f_+) \subset [\gamma, 1] \) and \( \text{Supp}(f_-) \subset [-1, -\gamma] \), then, for every \( \sigma > 0 \),

\[
M_0^\sigma \leq 2 + \frac{\sigma^2 \lambda_+^2 + \lambda_-^2}{2\gamma \lambda_+ \lambda_-} e^{-2\gamma/\sigma^2},
\]

(126)

\[
M_1^\sigma \leq 2 + \left( \frac{\sigma^4}{4\gamma^2} + \frac{\sigma^2}{2\gamma} \right) \frac{\lambda_+^2 + \lambda_-^2}{\lambda_+ \lambda_-} e^{-2\gamma/\sigma^2},
\]

(127)

\[
M_2^\sigma \leq 2 + \left( \frac{\sigma^6}{4\gamma^3} + \frac{\sigma^4}{2\gamma^2} + \frac{\sigma^2}{2\gamma} \right) \frac{\lambda_+^2 + \lambda_-^2}{\lambda_+ \lambda_-} e^{-2\gamma/\sigma^2}.
\]

(128)
In particular, \( M_\alpha^\gamma = O(\sigma^{2(1 + \alpha)}) \) as \( \sigma \to \infty \) and \( M_\alpha^\sigma = O(1) \) as \( \sigma \to 0^+ \).

**Proof.** See Appendix [B](#). \( \square \)

In the next proposition we provide upper bounds for \( M_\alpha^\gamma \) in the case where the supports of \( f_\pm \) overlap but extreme values determine the value of \( Y \), i.e., there exists \( \gamma_0 \) such that if \( X > \gamma_0 \) then \( Y = 1 \) and if \( X < -\gamma_0 \) then \( Y = -1 \).

**Proposition 4.** Let \( M_\alpha^\gamma \) be the quantities defined in [121]. If there exist \( \gamma_0 \in (0, 1) \) such that

\[
[\gamma_0, 1] \subset \text{Supp}(f_+) \subset (-\gamma_0, 1),
\]

\[
[-1, -\gamma_0] \subset \text{Supp}(f-) \subset [-1, \gamma_0),
\]

then, for every \( \gamma \in (\gamma_0, 1) \) and \( \sigma > 0 \),

\[
M_0^\gamma \leq 2 + \frac{\sigma^2}{\gamma - \gamma_0} \Lambda,
\]

\[
M_1^\gamma \leq 2 + \left(\frac{\sigma^4}{(\gamma - \gamma_0)^4} + \frac{\sigma^2}{\gamma - \gamma_0}\right) \Lambda,
\]

\[
M_2^\gamma \leq 2 + \left(\frac{2\sigma^6}{(\gamma - \gamma_0)^6} + \frac{2\sigma^4}{(\gamma - \gamma_0)^3} + \frac{\sigma^2}{\gamma - \gamma_0}\right) \Lambda,
\]

where \( \Lambda = \frac{\delta_+\Lambda_+ + \delta_-\Lambda_-}{\delta_+\Lambda_0 + \delta_-\Lambda_0} \),

\[
\delta_+ = \int_\gamma^1 f_+(s) \, ds \quad \text{and} \quad \delta_- = \int_{-1}^{-\gamma} f_+(s) \, ds.
\]

In particular, \( M_\alpha^\gamma = O(\sigma^{2(1 + \alpha)}) \) as \( \sigma \to \infty \) and \( M_\alpha^\sigma = O(1) \) as \( \sigma \to 0^+ \).

**Proof.** See Appendix [C]. \( \square \)

### B. Extreme Values Randomization

Consider extreme values randomization with \( B = [-r, r] \) for some \( r > 0 \). As before, we let \( p := \mathbb{P}(Y = 1) \) and \( \tilde{f}_\pm^\sigma \) be the conditional density of \( \tilde{X}^\sigma \) given \( Y = \pm 1 \). Recall the definition of the log-likelihood function

\[
\tilde{\theta}^\sigma(x) = \frac{1}{2} \log \left( \frac{p \tilde{f}_+^\sigma(x)}{p \tilde{f}_-^\sigma(x)} \right) \mathbb{I}_{|x| \leq r}.
\]

As established in [22],

\[
\tilde{f}_\pm^\sigma(x) = \left[ (f_\pm + K_\sigma)(x) + \frac{\mathbb{P}(|X^\sigma| > r | Y = \pm 1)}{2r} \right] \mathbb{I}_{|x| \leq r},
\]

where \( f_\pm \) is the conditional density of \( X \) given \( Y = \pm 1 \) and

\[
K_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}, \quad x \in \mathbb{R}.
\]

From (136), we conclude that \( x \mapsto \frac{p \tilde{f}_+^\sigma(x)}{p \tilde{f}_-^\sigma(x)} \) is a positive smooth function over \( [-r, r] \) and, as a result, \( \tilde{\theta}^\sigma \) is a smooth function over the same domain as well. As pointed out by Barron [28, Sec. IX], this implies that \( \tilde{\theta}^\sigma \) belongs to \( \Gamma_B \) and, in particular, \( C_{\tilde{\theta}^\sigma} \) is finite. Our goal is to find a tractable, yet useful, upper bound for \( C_{\tilde{\theta}^\sigma} \).
As discussed in [16], a first step in order to find an upper bound for the Barron constant of \( \tilde{\theta}^\sigma \) is to find a function, say \( \theta^\sigma \), such that \( \theta^\sigma \) is defined over \( \mathbb{R} \) and \( \tilde{\theta}^\sigma = \theta^\sigma |_B \). In this situation, we have that

\[
C_{\tilde{\theta}^\sigma} \leq \frac{r}{\sqrt{2\pi}} \int_{\mathbb{R}} |\omega||\tilde{\theta}^\sigma(\omega)|d\omega =: rC_{\theta^\sigma}. \tag{138}
\]

Motivated by (135) and (136), we define \( \theta^\sigma : \mathbb{R} \to \mathbb{R} \) by

\[
\theta^\sigma(x) = \frac{1}{2} \log \left( \frac{pf^+_\sigma(x) + \lambda_+}{pf^-_\sigma(x) + \lambda_-} \right), \tag{139}
\]

where \( f^\pm_\sigma = f^*_\pm \ast K_\sigma \) and

\[
\lambda_\pm = \frac{\mathbb{P} \{ |X^\sigma| > r | Y = \pm 1 \}}{2r}. \tag{140}
\]

Note that, by large deviations arguments, \( \lambda_\pm \) can be estimated with relatively high precision as it only depends on the probabilities of the events \( \{ Y = \pm 1 \} \) and \( \{ |X^\sigma| > r, Y = \pm 1 \} \). Furthermore, it can be shown that \( \lim_{\sigma \to \infty} \lambda_\pm = \frac{1}{2r} \), making the estimation of \( \lambda_\pm \) unnecessary for large \( \sigma \).

The next theorem provides an upper bound for the Barron constant of \( \theta^\sigma \), as defined in (139), under minimal assumptions on the distribution of \( X \).

**Theorem 6.** If \( f^\pm_\sigma \) is a probability density function, then, for every \( \sigma > 0 \),

\[
C_{\theta^\sigma} \leq \frac{2\sqrt{2}}{e\sqrt{\pi}} + \frac{2\sqrt{2}N^\sigma_2}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{N^\sigma_1 N^\sigma_3}{N^\sigma_2} \right) \right), \tag{141}
\]

where

\[
N^\sigma_1 := \frac{\Lambda_1}{\sqrt{2\pi} \sigma}, \tag{142}
\]

\[
N^\sigma_2 := \frac{\Lambda_1}{\sigma^2} + \frac{\Lambda_2}{8\sqrt{\pi} \sigma^3}, \tag{143}
\]

\[
N^\sigma_3 := \frac{5\Lambda_1}{\sqrt{2\pi} \sigma^3} + \frac{3\sqrt{3}\Lambda_2}{8\sqrt{2\pi} \sigma^4} + \frac{\sqrt{2}\Lambda_3}{9\sqrt{\pi} \sigma^5}, \tag{144}
\]

with \( \Lambda_\alpha = \frac{\rho^\alpha}{\lambda^\alpha} + \frac{\bar{\rho}^\alpha}{\rho^\alpha} \). Moreover, if \( N^\sigma_2 \leq 1/e \), then

\[
C_{\theta^\sigma} \leq \frac{2\sqrt{2}N^\sigma_2}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{N^\sigma_1 N^\sigma_3}{(N^\sigma_2)^2} \right) \right). \tag{145}
\]

The proof of the previous theorem, which can be found in Appendix [D], relies on Theorem [3] and careful estimates of the \( L^1 \)-norms of the derivatives of \( \theta^\sigma \).

Note that the previous theorem does not assume anything about \( f^\pm_\sigma \) apart from its existence. Furthermore, (145) implies that, in the large noise regime \( (\sigma \gg 1) \),

\[
C_{\theta^\sigma} = O \left( \frac{1}{\sigma} \right). \tag{146}
\]

By [28, Sec. IX-14], it can be verified that in the context of Proposition [2] with \( r = \infty \), we have \( C_{\theta^\sigma} = 1/\sigma^2 \). Therefore, the previous bound is in fact order optimal. Similarly, (145) implies that, in the small noise regime \( (\sigma \ll 1) \),

\[
C_{\theta^\sigma} = O \left( \frac{\log(1/\sigma)}{\sigma^3} \right). \tag{147}
\]

As with (125), it is unclear if (147) is order optimal.
VI. NUMERICAL CONSIDERATIONS

In this section we explore some numerical aspects of our lower bounds for the MMSE. Specifically, we evaluate the upper bounds for the Barron constant produced by Theorems 5 and 6 in a particular setting. Also, we study the effect of the values of \(k\) and \(n\) on the proposed estimator, and propose an optimization method to approximate the value of \(\text{mmse}^*_{k,n}(Y|X)\) efficiently. We finish this section with a numerical illustration of our lower bounds for the MMSE.

A. Upper Bounds for the Barron Constant

In this section we evaluate the upper bounds for the Barron constant derived in Theorems 5 and 6. To this end, we consider the setting where \(Y \sim \text{Unif}([-1,1])\), \(X = Y\) and \(r = 2\).

Observe that under extreme value truncation, (111) and Proposition 2 imply that
\[
C \lesssim \frac{2}{\sigma^2}.
\]
(148)

Since this bound depends on the exact computation performed in Proposition 2, we use (148) as a benchmark for the upper bounds obtained using Theorems 5 and 6.

Note that in the current setting the assumption of Proposition 3 is satisfied with \(\gamma = 1\). Also, by symmetry,
\[
P(|X| \leq r|Y = \pm 1) = P(|X| \leq r|Y = -1),
\]
which implies that \(\lambda_+ = \lambda_-\). Thus, Proposition 3 leads to
\[
M_0^\sigma \leq 2 + \sigma^2 e^{-2/\sigma^2},
\]
(150)
\[
M_1^\sigma \leq 2 + \left(\frac{\sigma^4}{2} + \sigma^2\right) e^{-2/\sigma^2},
\]
(151)
\[
M_2^\sigma \leq 2 \left(\frac{\sigma^6}{2} + \sigma^4 + \sigma^2\right) e^{-2/\sigma^2}.
\]
(152)

Finally, (111) and Theorem 5 lead to the upper bound
\[
C \lesssim \frac{32 \sqrt{2} M_2^\sigma}{\sqrt{\pi} \sigma^4} \left(1 + \frac{1}{2} \log \left(\frac{M_2^\sigma}{M_0^\sigma} + 3 \frac{M_1^\sigma}{M_0^\sigma} + 3 + \sigma^2\right)\right),
\]
(153)
which holds true whenever \(8e(2 + \sigma^2 e^{-2/\sigma^2}) \leq \sigma^4\). It can be verified that the previous inequality holds for \(\sigma \geq 4.7\).

In the current setting under extreme value randomization, it can be shown that
\[
P(|X^\sigma| > 2|Y = \pm 1) = Q(1/\sigma) + Q(3/\sigma),
\]
(154)
where \(Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} \, dx\). In particular, in the notation of Theorem 6 we have that
\[
\lambda_\pm = \frac{Q(1/\sigma) + Q(3/\sigma)}{4},
\]
(155)
and
\[
\Lambda_\alpha = \frac{2^{\alpha+1}}{(Q(1/\sigma) + Q(3/\sigma))^{\alpha}}.
\]
(156)
Fig. 1. Bounds for the Barron constant produced by Proposition 2, Theorem 5 and Theorem 6. These bounds hold for $\sigma \geq 0$, $\sigma \geq 4.7$, and $\sigma \geq 4.25$, respectively. The bound produced by Proposition 2 depends on an exact computation and serves as a benchmark. Overall, Theorem 6 seems to produce better bounds in practice than Theorem 5.

Using the previous expressions, we can provide upper bounds for $N_1^\sigma$, $N_2^\sigma$ and $N_3^\sigma$ as defined in Theorem 6. The latter theorem and (138) lead to
\[ C_\theta \leq 4 \sqrt{2N_2^\sigma} \left( 1 + \frac{1}{2} \log \left( \frac{N_1^\sigma N_3^\sigma}{(N_2^\sigma)^2} \right) \right), \]
which holds true whenever $\Lambda_1^{\theta_2} + \Lambda_2^{\theta_3} \leq 1/\varepsilon$. It can be verified that the previous inequality holds for $\sigma \geq 4.25$.

The bounds (148), (153) and (157) are illustrated in Figure 1. We would like to remark that, in order to evaluate the bound produced by Theorem 6 we used the fact that $\gamma = 1$ to obtain the exact probability in (154). While this assumption is rather strong as it amounts to know that $X = Y$, a similar assumption was made to evaluate the bound produced by Theorem 5. Hence, the comparison of these bounds is fair and suggests that Theorem 6 produces better bounds in practice than Theorem 5. Hence, we focus on the numerical evaluation of $\text{mmse}^*_{k,n}(Y|X)$ for the remainder of this section.

B. Computation of $\text{mmse}^*_{k,n}(Y|X)$

A key difficulty to instantiate the proposed lower bounds for the MMSE is to determine the appropriate values of $k$ and $n$. In this section we study this problem and propose an optimization method to approximate the value of $\text{mmse}^*_{k,n}(Y|X)$ efficiently.

For each $k \in \mathbb{N}$, let $S_k$ be the set of functions $g : \mathbb{R} \to \mathbb{R}$ of the form
\[ g(x) = \sum_{l=0}^{k} s_l \mathbb{1}_{[t_l, t_{l+1})}(x), \]
where $s_l \in \{-1, 0, +1\}$ and
\[ -\infty = t_0 < t_1 < \cdots < t_k < t_{k+1} = \infty. \]
We refer to $t_1, \ldots, t_k$ as the threshold points of $g$. Observe that $S_k$ captures the set of class probability estimators \cite{43} that are either completely confident (i.e., $g(x) = \pm 1$) or completely uncertain (i.e., $g(x) = 0$) about their predictions.

The next lemma establishes a key structural property of the hypothesis class $\text{tanh} \circ \mathcal{H}_k^\phi$: any function $g \in S_k$ can be approximated, in the uniform norm outside a neighborhood of the threshold points of $g$, by functions on $\text{tanh} \circ \mathcal{H}_k^\phi$.
Lemma 3. Let \( k \in \mathbb{N} \) and \( \epsilon, \delta > 0 \). If \( g \in S_k \), then there exists \( h \in \mathcal{H}^\phi_k \) such that, for all \( x \in \mathbb{R} \setminus \bigcup_{l=1}^{k} (t_l - \delta, t_l + \delta) \),

\[
|g(x) - \tanh(h(x))| \leq \epsilon.
\]

(160)

The proof of the previous lemma, which can be found in Appendix E, relies on standard (uniform) approximation arguments. The next theorem establishes that the minimum empirical square-loss over \( S_k \) serves as an upper bound for \( \text{mmse}_{k,n}^*(Y|X) \).

Theorem 7. If \( k \in \mathbb{N} \), then

\[
\text{mmse}_{k,n}^*(Y|X) \leq \inf_{g \in S_k} \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))^2.
\]

(161)

Proof. Let \( g \in S_k \) be given as in (158). For each \( l \in [k] \), let

\[
\tilde{t}_l := t_l - \frac{1}{2} (1 \wedge \min \{t_l - X_i : X_i < t_l\}),
\]

(162)

where we take the minimum of the empty set as \( +\infty \). We define the function \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) as

\[
\tilde{g}(x) := \sum_{l=0}^{k} s_l \mathbb{1}_{[\tilde{t}_l, \tilde{t}_{l+1})}(x),
\]

(163)

where \( \tilde{t}_0 = -\infty \) and \( \tilde{t}_{k+1} = \infty \). It can be verified that \( \tilde{g} \in S_k, \) \( \{t_1, \ldots, t_k\} \cap \{X_1, \ldots, X_n\} = \emptyset \) and

\[
\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \tilde{g}(X_i))^2.
\]

(164)

Let \( \epsilon > 0 \). Take \( \delta > 0 \) such that, for every \( i \in [n] \),

\[
X_i \notin \bigcup_{l=1}^{k} (\tilde{t}_l - \delta, \tilde{t}_l + \delta).
\]

(165)

By Lemma 3 there exists \( h \in \mathcal{H}^\phi_k \) such that, for every \( i \in [n] \),

\[
|\tilde{g}(X_i) - \tanh(h(X_i))| \leq \epsilon.
\]

(166)

The previous inequality and the triangle inequality lead to

\[
\sum_{i=1}^{n} (Y_i - \tanh(h(X_i)))^2 \leq n \epsilon^2 + 2 \epsilon \sum_{i=1}^{n} |Y_i - \tilde{g}(X_i)| + \sum_{i=1}^{n} (Y_i - \tilde{g}(X_i))^2.
\]

(167)

Therefore, (164) and the fact that \( \epsilon > 0 \) is arbitrary imply that

\[
\text{mmse}_{k,n}^*(Y|X) \leq \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))^2.
\]

(168)

Since \( g \in S_k \) is also arbitrary, the conclusion follows.

The next corollary is a straightforward consequence of the previous theorem. Indeed, it follows by taking \( g \in S_k \) such that \( g(X_i) = Y_i \) for all \( i \leq k \) and \( g(X_i) = 0 \) for all \( i > k \).

Corollary 1. If \( k, n \in \mathbb{N} \) with \( k \leq n \), then

\[
\text{mmse}_{k,n}^*(Y|X) \leq 1 - \frac{k}{n}.
\]

(169)
Algorithm 1: Empirical Square-Loss Minimization over $S_k$

1: compute a permutation $\pi$ such that $X_{\pi(1)} < \cdots < X_{\pi(n)}$
2: set $L[l, s, i] = \infty$ for $0 \leq l \leq k$, $s \in \{-1, 0, +1\}$, $0 \leq i \leq n$ \(\triangleright\) minimal loss using $l$ thresholds up to $X_{\pi(i)}$ with $s_l = s$
3: set $L[0, s, 0] = 0$ for $s \in \{-1, 0, +1\}$ \(\triangleright\) setting $s_0 = s$
4: for $i = 1, \ldots, n$ do
5: \hspace{1em} $L[0, s, i] = L[0, s, i-1] + (Y_{\pi(i)} - s)^2$ for $s \in \{-1, 0, +1\}$
6: for $l = 1, \ldots, k$ do
7: \hspace{1em} $L[l, s, i] = (L[l, s, i-1] \wedge \min_{s' \neq s} L[l - 1, s', i-1]) + (Y_{\pi(i)} - s)^2$ for $s \in \{-1, 0, +1\}$
8: end for
9: end for
10: return $\frac{1}{n} \min_{l, s} L[l, s, n]$ \(\triangleright\) minimal loss over $S_k$

From (169) we conclude that it is necessary to have $k \ll n$ in order to obtain meaningful bounds for $\text{mmse}(Y | X)$. Note that the previous bound recovers the well-known fact that a two-layer neural network of size $k$ can memorize an entire sample of size $n$ whenever $k \geq n$, see, e.g., [42].

Motivated by Theorem 7, we propose the following optimization process to approximate the value of $\text{mmse}^*_{k,n}(Y | X)$: minimize the empirical square-loss around 0 using random initialization and gradient descent; minimize the empirical square-loss over $S_k$ using dynamic programming, as described in Algorithm 1 and take the minimum of those two empirical losses. While this combined minimization process is not guaranteed to find the exact value of $\text{mmse}^*_{k,n}(Y | X)$, it covers two important subsets of the hypothesis class $\text{tanh} \circ \mathcal{H}_k^\phi$.

C. Numerical Experiment

We end this section applying the tools developed so far in a concrete numerical example. We consider the setting introduced in Section VI-A where $Y \sim \text{Unif}(\{-1\})$, $X = Y$ and $r = 2$. As pointed out in that section, in this setting Theorem 6 produces better bounds for the Barron constant than Theorem 5. Hence, for the sake of illustration, we focus on the extreme values randomization setting introduced in Section II-D2.

Motivated by Corollary 1 in our numerical experiments we set $k = n/100$ for $n = 10,000$ and $n = 100,000$. Recall that our optimization strategy to approximate the value $\text{mmse}^*_{k,n}(Y | X)$ consists in (a) minimize the empirical square-loss around 0 using random initialization and gradient descent, (b) minimize the empirical square-loss over $S_k$ using Algorithm 1 and (c) take the minimum of those two empirical losses. In all of our experiments, the minimal empirical square-loss over $S_k$ was no larger than 0.9555, while the minimal empirical square-loss around 0 was no smaller than 0.9997. Thus, Algorithm 1 seems to perform significantly better than standard machine learning techniques for the task of minimizing the empirical square-loss.

We implicitly assume that $X_1, \ldots, X_n$ are pairwise different, which is the case in most practical cases, e.g., when the distribution of $X$ is absolutely continuous with respect to the Lebesgue measure.

We initialized the weights of the neural network at random with distribution $\mathcal{N}(0, 0.01)$. When a random initialization with empirical square-loss less than 1 was found, 100 iteration of gradient descent with step size equal to 0.1 were performed. For each value of $\sigma$, this experiment was conducted 5 times and the best set of parameters was stored.
In Figure 2 we plot our numerical results for \( n = 10,000 \) and \( n = 100,000 \), and a variety of values of \( \sigma \). Note that the quality of the lower bound for the MMSE improves as \( n \) and \( k \) increase. However, as suggested by Corollary 1, the ratio between \( k \) and \( n \) should remain bounded from below in order to get a meaningful bound.

We conjecture that the family \( S_k \) contains functions with relatively small empirical square-loss in the regime where \( k \ll n \). This seems to be the case since \( S_k \) models the functions in \( \tanh \circ \mathcal{H}_k^\phi \) that highly overfit to a portion of the data. Since Algorithm 1 has complexity \( O(kn) \), the minimal empirical square-loss over \( S_k \) provides a reasonable proxy for \( \text{mmse}_{k,n}^*(Y|X) \) that can be computed efficiently.

VII. SUMMARY AND FINAL REMARKS

Motivated by estimation-theoretic privacy, in this paper we have established provable lower bounds for the MMSE in estimating a random variable \( Y \in \mathbb{R} \) given another random variable \( X \in \mathbb{R}^d \) (Theorems 1 and 2). These bounds are based on a two-layer neural network estimator of the MMSE and the Barron constant of an appropriate function of the conditional expectation of \( Y \) given \( X \). More specifically, we have proposed the minimum empirical square-loss attained by a two-layer neural network of size \( k \) as an estimator of the MMSE. We considered two variations of this estimator: the first one, denoted by \( \text{mmse}_{k,n}(Y|X) \), uses the identity function as the output activation function; while the second one, denoted by \( \text{mmse}^*_{k,n}(Y|X) \), uses hyperbolic tangent as the output activation function.

Finding meaningful estimates for the Barron constant is challenging since (i) the underlying conditional expectation is rarely available in practice and (ii) the Barron constant is defined in terms of the Fourier transform of this conditional expectation. To alleviate the second issue, we provided an upper bound for the Barron constant of a function \( h : \mathbb{R} \rightarrow \mathbb{R} \) based on the \( L^1 \)-norms of its derivatives (Theorem 3). We have further generalized this result to multivariate functions \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) (Theorem 4), although the complexity of the result and the bound itself increase exponentially with \( d \). In addition, we have shown that one can circumvent the first issue in applications where the additive Gaussian mechanism is used (Theorems 5 and 6). In such applications, our estimates for the Barron constant are order optimal in the large noise regime.

In order to obtain numerical lower bounds for the MMSE in some concrete applications, we analyzed some algorithmic aspects related to the computation of the proposed estimator. First, we empirically found that the bounds for the Barron constant associated with the estimator \( \text{mmse}_{k,n}^*(Y|X) \) are tighter than those corresponding

![Fig. 2](image-url)
to m\text{mse}_{k,n}(Y|X). Building upon a structural property of the hypothesis class \(\tanh \circ \mathcal{H}_k^\theta\) (Lemma 3 and Theorem 7), we showed that the neural network size \(k\) should be significantly smaller than the sample size \(n\) in order to obtain meaningful lower bounds for the MMSE (Corollary 1). Moreover, motivated by the same structural property, we proposed an optimization process to approximate the value of \(m\text{mse}_{k,n}^*(Y|X)\) that performs better than standard machine learning techniques and that can be computed efficiently using dynamic programming. Overall, we developed an effective machinery to obtain theoretical lower bounds for the MMSE.

While we have only considered shallow neural networks, there are fundamental obstructions in trying to generalize the present work to deep neural networks.

- From a function approximation perspective, at the moment it seems that there is no analogue of Barron’s theorem for deep neural network.\(^9\) While there are many results explaining the approximation power of deep neural networks, see, e.g., [44–46], they are mainly qualitative and, hence, unfitted to produce concrete bounds.
- From a computational perspective, the optimization landscape of deep neural networks is significantly more complex than its shallow counterpart, see, e.g., [47–49]. As a result, it is harder to guarantee that a deep neural network has been trained to optimality, which is essential for the estimator proposed in this paper.
- As mentioned at the end of Section III-A, if \(k\) is sufficiently large then our lower bounds for the MMSE become trivial due to overfitting, i.e., \(m\text{mse}_{k,n}(Y|X)\) being equal to 0. Given the astonishing expressive power of deep neural networks, see, e.g., [44–46], [50], they seem likely to produce trivial lower bounds.

Overall, generalizing the present work to deep neural networks is highly non-trivial and, at the same, it is unclear if it will provide significantly better results.

In this work, we have shown that Barron’s approximation theorem could be used to derive non-trivial lower bounds for the MMSE. However, its implementation is challenging and, when data is post-processed by the additive Gaussian mechanism, it seems to work well only in the large noise regime. While Theorem 1 could be easily generalized to other families of approximating functions beyond neural networks, it is crucial to find a family with good approximation guarantees for the conditional expectations under consideration. We leave the search for such a family and approximation guarantees as future work.

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APPENDIX A

PROOF OF THEOREM 5

For each \(x \in \mathbb{R}\), we define
\[
g_{\pm}^*(x) := \lambda_{\pm}(f_{\pm} \ast K_\sigma)(x).
\] (170)

\(^9\)Lee et al. [31] have an important effort in this direction, although their results depend on a specific decomposition of the target function.
Observe that, with this notation,

$$\eta^\sigma(x) = \frac{g^\sigma_+(x) - g^\sigma_-(x)}{g^\sigma_+(x) + g^\sigma_-(x)}. \quad (171)$$

The following simple lemma provides useful expressions for the derivatives of $g^\sigma_{\pm}$.

**Lemma 4.** If $j \in \{0, 1, 2, 3\}$, then, for every $x \in \mathbb{R}$,

$$\frac{d^j g^\sigma_{\pm}}{dx^j}(x) = \int_{\mathbb{R}} \lambda_{\pm} f_{\pm}(s) P_j^\sigma(x - s) K_\sigma(x - s) ds, \quad (172)$$

where $P_0^\sigma(x) = 1$, $P_1^\sigma(x) = -x/\sigma^2$, $P_2^\sigma(x) = (x^2 - \sigma^2)/\sigma^4$ and $P_3^\sigma(x) = -(x^3 - 3\sigma^2 x)/\sigma^6$.

**Proof.** It can be verified that, for each $j \in \{0, 1, 2, 3\}$,

$$\frac{d^j K^{(j)}_\sigma}{dx^j}(x) = P_j^\sigma K_\sigma(x). \quad (173)$$

Observe that $g^\sigma_{\pm} = (\lambda_{\pm} f_{\pm})* K_\sigma$. Therefore, the lemma follows from the general formula $\frac{d^j}{dx^j}(h_1 * h_2) = h_1 * \frac{d^j}{dx^j}h_2$. \hfill \Box

In order to avoid cumbersome notation, we omit the superscript $\sigma$ when there is no risk of confusion, e.g., $g^\sigma_{\pm}$ is written as $g_{\pm}$ and $\eta^\sigma$ is written as $\eta$. In order to simplify our calculations, we introduce the following notation.

**Definition 1.** For each $n \in \mathbb{N}$, we define

$$K_{\delta_1, \ldots, \delta_n}(s_1, \ldots, s_n; x) := \prod_{i=1}^n \lambda_{\delta_i} f_{\delta_i}(s_i) K_\sigma(x - s_i), \quad (174)$$

where $\delta_1, \ldots, \delta_n \in \{\pm\}$, $s_1, \ldots, s_n \in \mathbb{R}$ and $x \in \mathbb{R}$. Also, for $j_1, \ldots, j_n \in \{0, 1, 2, 3\}$, we define

$$P_{j_1, \ldots, j_n}(s_1, \ldots, s_n; x) := \prod_{i=1}^n P_{j_i}(x - s_i). \quad (175)$$

With the above notation, Lemma 4 implies that for every $n \in \mathbb{N}$, $\delta_1, \ldots, \delta_n \in \{\pm\}$, $j_1, \ldots, j_n \in \{0, 1, 2, 3\}$ and $x \in \mathbb{R}$,

$$\prod_{i=1}^n g^{(j_i)}_{\delta_i}(x) = \int_{\mathbb{R}^n} P_{j_1, \ldots, j_n}(s; x) K_{\delta_1, \ldots, \delta_n}(s; x) ds, \quad (176)$$

where $s = (s_1, \ldots, s_n)$ and $ds = ds_1 \cdots ds_n$. In particular, by taking $j_i = 0$ for all $i \in [n]$,

$$\prod_{i=1}^n g_{\delta_i}(x) = \int_{\mathbb{R}^n} K_{\delta_1, \ldots, \delta_n}(s; x) ds. \quad (177)$$

Finally, observe that for every $x \in \mathbb{R}$,

$$\text{Supp}(K_{\delta_1, \ldots, \delta_n}(\cdot; x)) = \text{Supp}(f_{\delta_1}) \times \cdots \times \text{Supp}(f_{\delta_n}). \quad (178)$$

Now we derive a pointwise bound for $\eta'$.

**Lemma 5.** If $\text{Supp}(f_{\pm}) \subset [-1, 1]$, then, for all $x \in \mathbb{R}$,

$$|\eta'(x)| \leq \frac{4}{\sigma^2} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}, \quad (179)$$
Proof. In Lemma 1 we prove that, for all $x \in \mathbb{R}$,

$$
\eta'(x) = \frac{I(x)}{(g_+(x) + g_-(x))^2},
$$

(180)

where

$$
I(x) := 2 \left( g_+'(x)g_-(x) - g_+(x)g_-'(x) \right).
$$

(181)

The integral formula in (176) and (178) imply that

$$
g_+'(x)g_-(x) = \int_{-1}^{1} \int_{-1}^{1} P_{1,0}(s; x) K_{+,-}(s; x) \, ds.
$$

(182)

Mutatis mutandis, we have that

$$
g_+(x)g_-'(x) = \int_{-1}^{1} \int_{-1}^{1} P_{0,1}(s; x) K_{+,-}(s; x) \, ds.
$$

(183)

Thus, we have that

$$
I(x) = \int_{-1}^{1} \int_{-1}^{1} Q(s; x) K_{+,-}(s; x) \, ds,
$$

(184)

where $Q(s; x) = 2P_{1,0}(s; x) - 2P_{0,1}(s; x)$. By the definition of $P_{j_1,\ldots,j_n}$ in (175),

$$
2P_{1,0}(s; x) = \frac{-2}{\sigma^2} x + \frac{2s_1}{\sigma^2}, \quad -2P_{0,1}(s; x) = \frac{2}{\sigma^2} x + \frac{-2s_2}{\sigma^2}.
$$

(185)

(186)

As a result, $Q(s_1, s_2; x) = \frac{2(s_1 - s_2)}{\sigma^2}$ and (184) becomes

$$
I(x) = \int_{-1}^{1} \int_{-1}^{1} \frac{2(s_1 - s_2)}{\sigma^2} K_{+,-}(s_1, s_2; x) \, ds_1 ds_2.
$$

(187)

Since $|s_1 - s_2| \leq 2$ whenever $s_1, s_2 \in [-1, 1]$, we have that

$$
|I(x)| \leq \frac{4}{\sigma^2} \int_{-1}^{1} \int_{-1}^{1} K_{+,-}(s_1, s_2; x) \, ds_1 ds_2
$$

(188)

$$
= \frac{4}{\sigma^2} g_+(x)g_-(x),
$$

(189)

where the equality follows from (177). The lemma follows by plugging the previous inequality in (180). \(\square\)

Now we establish a similar upper bound for $\eta''$. 

\[P_{3,0}(s_1, s_2; x) = \frac{-x^3 + 3s_1x^2 + 3(\sigma^2 - s_1^2)x + s_1(s_1^2 - 3\sigma^2)}{\sigma^6}
\]

(190)

\[P_{2,1}(s_1, s_2; t) = \frac{-x^3 + (2s_1 + s_2)x^2 + (\sigma^2 - s_1^2 - 2s_1s_2)x + s_2(s_1^2 - \sigma^2)}{\sigma^6}
\]

(191)

\[-P_{1,2}(s_1, s_2; t) = \frac{x^3 - (s_1 + 2s_2)x^2 - (\sigma^2 - 2s_1s_2 - s_2^2)x - s_1(s_2^2 - \sigma^2)}{\sigma^6}
\]

(192)

\[-P_{0,3}(s_1, s_2; x) = \frac{x^3 - 3s_2x^2 - 3(\sigma^2 - s_2^2)x - s_2(s_2^2 - 3\sigma^2)}{\sigma^6}
\]

(193)
Lemma 6. If $\text{Supp}(f_{\pm}) \subset [-1, 1]$, then, for all $x \in \mathbb{R}$,
\[
|\eta''(x)| \leq \frac{8}{\sigma^4} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}.
\] (194)

Proof. In Lemma 1 we prove that
\[
\eta''(x) = 2g''_+g_+ - g + g''_- (g_+ + g_-)^2 - 4g'_+g_+g'_- + g'_+ + g'_-. \tag{195}
\]
In particular, we have that
\[
\eta''(x) = \frac{I_1(x) + I_2(x)}{(g_+(x) + g_-(x))^3}
\]
where
\[
I_1 = 2g_+g''_+g_- - 2g_+g'_+g'_- - 4g'_+g'_+g_- + 4g'_+g'_-g_+g'_-, \tag{196}
\]
\[
I_2 = 2g''_+g_-g_+ - 2g_+g'_+g'_- - 4g'_+g'_+g_- + 4g'_+g'_-g_+g'_-. \tag{197}
\]
The integral formula in (176) implies that, for all $x \in \mathbb{R}$,
\[
I_1(x) = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} Q_1(s; x)K_{+,+,-}(s; x)ds,
\] (198)
where $Q_1 = 2P_{0,0,2} - 2P_{0,0,2} - 4P_{1,1,0} + 4P_{1,0,1}$. By the definition of $P_{j_1,...,j_n}$ in (175),
\[
2P_{0,0,2}(s; x) = \frac{2}{\sigma^4} x^2 + \frac{-4s - \sqrt{x^2}}{\sigma^4} + \frac{2(s^2 - \sigma^2)}{\sigma^4}, \tag{199}
\]
\[-2P_{0,0,2}(s; x) = \frac{-2}{\sigma^4} x^2 + \frac{4s_3}{\sigma^4} x + \frac{2(\sigma^2 - s_3)^2}{\sigma^4}, \tag{200}\]
\[-4P_{1,1,0}(s; x) = \frac{-4}{\sigma^4} x^2 + \frac{4(s_1 + s_2)}{\sigma^4} x - \frac{4s_1 s_2}{\sigma^4}, \tag{201}\]
\[4P_{1,0,1}(s; x) = \frac{4}{\sigma^4} x^2 + \frac{-4(s_1 + s_3)}{\sigma^4} x + \frac{4s_1 s_3}{\sigma^4}. \tag{202}\]
As a result, we obtain that
\[
Q_1(s; x) = \frac{2(s^2 - s_3^2 - 2s_1 s_2 + 2s_1 s_3)}{\sigma^4}. \tag{203}\]
The inequality $2s_1 s_3 \leq s_1^2 + s_3^2$ implies that
\[
s_1^2 - s_3^2 - 2s_1 s_2 + 2s_1 s_3 \leq (s_1 - s_2)^2. \tag{204}\]
Similarly, the inequality $2s_1 s_2 \leq s_1^2 + s_2^2$ implies that
\[
s_1^2 - s_3^2 - 2s_1 s_2 + 2s_1 s_3 \geq -(s_1 - s_3)^2. \tag{205}\]
In particular, $|Q_1(s; x)| \leq \frac{8}{\sigma^4}$ whenever $s_1, s_2, s_3 \in [-1, 1]$. Therefore, (198) implies that
\[
|I_1(x)| \leq \frac{8}{\sigma^4} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} K_{+,+,-}(s; x)ds
\]
\[
= \frac{8}{\sigma^4} g_+(x)g_-(x), \tag{206}\]
where the equality follows from (177). Mutatis mutandis, it can be shown that
\[
I_2(x) = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} Q_2(s; x)K_{+,+,-}(s; x)ds, \tag{208}\]
\[\text{Recall that, by Lemma 1, } \eta' = 2g''_+g_- + \frac{g''_-}{(g_+ + g_-)^2}. \]
where
\[
Q_2(s; x) = \frac{2(s_1^2 - s_3^2 - 2s_1s_2 + 2s_2s_3)}{\sigma^4}.
\]
(209)

As before, (208) and (209) imply that, for all \( x \in \mathbb{R} \),
\[
|I_2(x)| \leq 8 \frac{g_+(x)g_-(x)}{\sigma^4(g_+(x) + g_-(x))^2}.
\]
(210)

Since \( \eta''(x) = \frac{I_1(x) + I_2(x)}{(g_+(x) + g_-(x))^3} \), (207) and (210) imply that
\[
|\eta''(x)| \leq 8 \frac{g_+(x)g_-(x)}{\sigma^4(g_+(x) + g_-(x))^2},
\]
(211)
as required.

Finally, we establish an upper bound for \( \eta''' \) akin to those in the previous lemmas.

**Lemma 7.** If \( \text{Supp}(f_{\pm}) \subset [-1, 1] \), then, for all \( x \in \mathbb{R} \),
\[
|\eta'''(x)| \leq \frac{16x^2 + 44|x| + 34 + 12\sigma^2}{\sigma^6} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}.
\]
(212)

**Proof.** In Lemma 1 we prove that
\[
\eta''' = \frac{1}{(g_+ + g_-)^2} - 2\eta' \frac{g''_+ + g''_-}{g_+ + g_-} - 3\eta'' \frac{g'_+ + g'_-}{g_+ + g_-},
\]
(213)

where
\[
I = 2(g''_+g_- + g''_+g'_- - g'_+g''_- - g_+g'''_-).
\]
(214)
The integral formula in (176) implies that, for all \( x \in \mathbb{R} \),
\[
I(x) = \int_{-1}^{1} \int_{-1}^{1} Q(s; x)K_{+,+,-}(s; x)ds,
\]
(215)
where \( Q = P_{3,0} + P_{2,1} - P_{1,2} - P_{0,3} \). By the definition of \( P_{j_1,...,j_n} \) in (175) and equations (190) – (193), we conclude that, for all \( x \in \mathbb{R} \),
\[
Q(s_1, s_2; x) = \frac{4(s_1 - s_2)x^2 + 4(s_2 - s_1)x}{\sigma^6} + \frac{(s_1 - s_2)(s_1 + s_2)^2 - 2(s_1 - s_2)\sigma^2}{\sigma^6}.
\]
(216)
In particular, for \( s_1, s_2 \in [-1, 1] \),
\[
|Q(s; x)| \leq \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6}.
\]
(217)

Therefore, (215) implies that
\[
|I(x)| \leq \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6} \int_{-1}^{1} \int_{-1}^{1} K_{+,+,-}(s; t)ds
\]
\[
= \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6} g_+(x)g_-(x),
\]
(218)
where the equality follows from (177).

Lemma 4 and the fact that \( \text{Supp}(f_{\pm}) \subset [-1, 1] \) imply that
\[
|g'_\pm(x)| \leq \frac{x^2 + 2|x| + 1 + \sigma^2}{\sigma^4} g_\pm(x).
\]
(220)
As a result, we obtain that
\[
\left| 2\eta'(x) \frac{g''_+(x) + g''_-(x)}{g_+(x) + g_-(x)} \right| \leq \frac{2x^2 + 4|x| + 2 + 2\sigma^2}{\sigma^4} |\eta'(x)|. \tag{221}
\]
Therefore, by Lemma 5
\[
\left| 2\eta'(x) \frac{g''_+(x) + g''_-(x)}{g_+(x) + g_-(x)} \right| \leq \frac{8x^2 + 16|x| + 8 + 8\sigma^2}{\sigma^6} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}. \tag{222}
\]
Lemma 4 and the fact that Supp(f_±) \subset [-1,1] imply that
\[
|g'_\pm(x)| \leq \frac{|x| + 1}{\sigma^2} g_\pm(x). \tag{223}
\]
Therefore, we conclude that
\[
\left| 3\eta''(x) \frac{g_+'(x) + g_-'(x)}{g_+(x) + g_-(x)} \right| \leq \frac{3(|x| + 1)}{\sigma^6} |\eta''(x)| \leq \frac{24(|x| + 1)}{\sigma^6} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}, \tag{224}
\]
where the last inequality follows from Lemma 6. By plugging (219), (222) and (223) in (213), the result follows. \(\square\)

It is possible to obtain slightly better constants than those in (212) by avoiding the use of Lemmas 5 and 6. However, the complexity of the proof increases considerably and the benefit is marginal given that the \(L^1\)-norm of \(\eta''\) only appears inside a logarithm.

**Proof of Theorem 5** By Theorem 5, we have that
\[
C_\eta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{4M_0}{\sigma^2} \right) \right) \frac{1}{\sigma^2} |\eta''(x)|. \tag{226}
\]
Recall the definition of \(M_\alpha\) in (121). By Lemmas 5 – 7, we have that
\[
||\eta'||_1 \leq \frac{4M_0}{\sigma^2}, \tag{227}
||\eta''||_1 \leq \frac{8M_0}{\sigma^2}, \tag{228}
||\eta''''||_1 \leq \frac{16M_2 + 44M_1 + (34 + 12\sigma^2)M_0}{\sigma^6}. \tag{229}
\]
As a result, for all \(\sigma > 0\),
\[
C_\eta \leq \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^8} \right) \right) - \frac{2\sqrt{2}}{\sqrt{\pi}} \log(||\eta''''||_1)||\eta''''||_1, \tag{230}
\]
where
\[
M := 64M_2M_0 + 176M_1M_0 + (136 + 48\sigma^2)M_0^2. \tag{231}
\]
Since \(-\log(z)z \leq 1/e\) for all \(z \in [0, \infty)\), (226) implies that, for all \(\sigma > 0\),
\[
C_\eta \leq \frac{2\sqrt{2}}{e\sqrt{\pi}} + \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^8} \right) \right). \tag{232}
\]
It is straightforward to verify that \(z \mapsto -\log(z)z\) is increasing over \([0,1/e]\). Thus, if \(\sqrt{8eM_0} \leq \sigma\), (230) implies that
\[
C_\eta \leq \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{64M_0^2} \right) \right). \tag{233}
\]
After some manipulations, (123) follows. \(\square\)
APPENDIX B

PROOF OF PROPOSITION 3

Recall that, for each $x \in \mathbb{R}$,
\[ g_\pm^\sigma(x) \coloneqq \lambda_\pm (f_\pm * K_\sigma)(x). \] (234)

The next lemma provides upper and lower bounds for $g_\pm^\sigma(x)$ under the assumptions of Proposition 3.

**Lemma 8.** In the context of Proposition 3 for all $x \in \mathbb{R}$,
\[ \lambda_+ \min_{s \in [x-1, x-\gamma]} K_\sigma(s) \leq g_+^\sigma(x) \leq \lambda_+ \max_{s \in [x-1, x-\gamma]} K_\sigma(s), \]
\[ -\lambda_- \min_{s \in [x+\gamma, x+1]} K_\sigma(s) \leq g_-^\sigma(x) \leq -\lambda_- \max_{s \in [x+\gamma, x+1]} K_\sigma(s). \]

**Proof.** By assumption $\text{Supp}(f_+) \subset [\gamma, 1]$, thus
\[ g_+^\sigma(x) = \lambda_+ \int_{\gamma}^{1} f_+(s) K_\sigma(x-s)ds. \] (235)

Therefore, for all $x \in \mathbb{R}$,
\[ \lambda_+ \left( \min_{s \in [\gamma, 1]} K_\sigma(x-s) \right) \int_{\gamma}^{1} f_+(s)ds \leq g_+^\sigma(x) \leq \lambda_+ \left( \max_{s \in [\gamma, 1]} K_\sigma(x-s) \right) \int_{\gamma}^{1} f_+(s)ds. \] (236)

Since $\int_{\gamma}^{1} f_+(s)ds = 1$, the inequalities for $g_+^\sigma$ follow. The inequalities for $g_-^\sigma$ are proved mutatis mutandis. \(\square\)

The next lemma provides an upper bound for $g_\pm^\sigma / (g_+^\sigma + g_-^\sigma)$.

**Lemma 9.** In the context of Proposition 3
- \[ \frac{g_\pm^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_+ e^{-2\gamma|x|/\sigma^2}}{\lambda_-}, \quad x < -1; \]
- \[ \frac{g_\pm^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_-}{\lambda_+} e^{-2\gamma|x|/\sigma^2}, \quad x > 1. \]

**Proof.** By Lemma 8, for all $x \in \mathbb{R}$,
\[ \frac{g_\pm^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_+ \max_{s \in [x-1, x-\gamma]} K_\sigma(s)}{-\lambda_- \min_{s \in [x+\gamma, x+1]} K_\sigma(s)}. \] (237)

When $x < -1$, we have that
\[ [x-1, x-\gamma], [x+\gamma, x+1] \subset (-\infty, 0). \] (238)

Since $K_\sigma$ is increasing on $(-\infty, 0)$,
\[ \frac{g_\pm^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_+ K_\sigma(x-\gamma)}{-\lambda_- K_\sigma(x+\gamma)} = \frac{\lambda_+ e^{-2\gamma|x|/\sigma^2}}{\lambda_-}. \] (239)

Mutatis mutandis, it can be shown that, for $x > 1$, we have the inequality
\[ \frac{g_\pm^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_-}{\lambda_+} e^{-2\gamma|x|/\sigma^2}. \] \(\square\)

Now we are in position to prove Proposition 3.

**Proof of Proposition 3** By definition, we have that
\[ M_\alpha^\sigma = \left( \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \right) |x|^p \frac{g_\pm^\sigma(x)g_\pm^\sigma(x)}{(g_+^\sigma(x) + g_-^\sigma(x))^2}dx. \] (240)
Lemma 9 and the trivial upper bound \( \frac{g_+^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq 1 \) imply that

\[
M_\sigma^* \leq 2 + \frac{\lambda_+^2 + \lambda_-^2}{\lambda_+ \lambda_-} \int_1^\infty x^p \exp \left\{ -2\gamma x / \sigma^2 \right\} \, dx.
\]

(241)

By the formulas,

\[
\int e^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta},
\]

(242)

\[
\int x e^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta^2} (\beta x + 1),
\]

(243)

\[
\int x^2 e^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta^3} (\beta^2 x^2 + 2\beta x + 2),
\]

(244)

the result follows.

\[
\square
\]

APPENDIX C

PROOF OF Proposition \[4\]

Recall that, for each \( x \in \mathbb{R} \), we define

\[
g_\pm^\sigma(x) := \lambda_\pm (f_\pm * K_\sigma)(x).
\]

(245)

The next lemma provides upper and lower bounds for \( g_\pm(x) \) under the assumptions of Proposition \[4\].

**Lemma 10.** In the context of Proposition \[4\] for all \( x \in \mathbb{R} \),

\[
\lambda_+ \delta_+ \min_{s \in [x-1,x-\gamma]} K_\sigma(s) \leq g_+^\sigma(x) \leq \lambda_+ \max_{s \in [x-1,x+\gamma_0]} K_\sigma(s),
\]

\[
\lambda_- \delta_- \min_{s \in [x+\gamma,x+1]} K_\sigma(s) \leq g_-^\sigma(x) \leq \lambda_- \max_{s \in [x-\gamma_0,x+1]} K_\sigma(s).
\]

**Proof.** By assumption \( \text{Supp}(f_+) \subset [-\gamma_0, 1] \), thus

\[
g_+^\sigma(x) = \lambda_+ \int_{-\gamma_0}^1 f_+(s) K_\sigma(x - s) \, ds.
\]

(246)

Since \( -\gamma_0 < \gamma < 1 \), for all \( x \in \mathbb{R} \),

\[
g_+^\sigma(x) \geq \lambda_+ \int_{-\gamma}^1 f_+(s) K_\sigma(x - s) \, ds
\]

(247)

\[
\geq \lambda_+ \int_{-\gamma}^1 f_+(s) \, ds \min_{s \in [\gamma, 1]} K_\sigma(x - s)
\]

(248)

\[
= \lambda_+ \delta_+ \min_{s \in [x-1,x-\gamma]} K_\sigma(s).
\]

(249)

Similarly, for all \( x \in \mathbb{R} \),

\[
g_-^\sigma(x) \leq \lambda_+ \int_{-\gamma_0}^1 f_+(s) \max_{s \in [-\gamma_0,1]} K_\sigma(x - s) \, ds
\]

(250)

\[
= \lambda_+ \max_{s \in [x-1,x+\gamma_0]} K_\sigma(s).
\]

(251)

The inequalities for \( g_-^\sigma \) are proved mutatis mutandis.

\[
\square
\]

The next lemma provides an upper bound for \( g_\pm^\sigma / (g_+^\sigma + g_-^\sigma) \).
Lemma 11. In the context of Proposition 4

- \( \frac{g^+_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq \frac{\lambda_+}{\delta_-\lambda_-} e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}, x < -1; \)
- \( \frac{g^-_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq \frac{\lambda_-}{\delta_+\lambda_+} e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}, x > 1. \)

Proof. By Lemma 10 for all \( x \in \mathbb{R}, \)

\[
\frac{g^+_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq \frac{g^+_\sigma(x)}{\delta_-\lambda_- \min_{x \in [x',x+1]} K_\sigma(s)} \leq \frac{\lambda_+ \max_{x \in [x',x+1]} K_\sigma(s)}{g^-_\sigma(x)} \leq \frac{\lambda_-}{\delta_+\lambda_+} e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}.
\]

When \( x < -1, \) we have that

\[
[x - 1, x + \gamma_0], [x + \gamma, x + 1] \subset (-\infty, 0).
\]

Since \( K_\sigma \) is increasing on \((-\infty, 0), \)

\[
\frac{g^+_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq \frac{\lambda_+}{\delta_-\lambda_-} K_\sigma(x + \gamma_0) \leq \frac{\lambda_-}{\delta_-\lambda_-} K_\sigma(x + \gamma) \leq \frac{\lambda_-}{\delta_-\lambda_-} e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}.
\]

\textit{Mutatis mutandis,} it can be shown that, for \( x > 1, \) we have the inequality \( \frac{g^-_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq \frac{\lambda_-}{\delta_+\lambda_+} e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}. \)

Now we are in position to prove Proposition 4.

\textbf{Proof of Proposition 4.} By definition, we have that

\[
M_p^\sigma = \left( \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \right) |x|^p \frac{g^+_\sigma(x)g^-_\sigma(x)}{(g^+_\sigma(x) + g^-_\sigma(x))^2} \, dx.
\]

Lemma 11 and the trivial upper bound \( \frac{g^+_\sigma(x)}{g^+_\sigma(x) + g^-_\sigma(x)} \leq 1 \) imply that

\[
M_p^\sigma \leq 2 + \frac{\delta_+\lambda^2_+ + \delta_-\lambda^2_-}{\delta_+\lambda_+\delta_-\lambda_-} \int_{1}^{\infty} x^p e^{-(\gamma-\gamma_0)(x-1)/\sigma^2} \, dx.
\]

By the formulas,

\[
\int e^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta},
\]

\[
\int xe^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta^2} (\beta x + 1),
\]

\[
\int x^2 e^{-\beta x} \, dx = -\frac{e^{-\beta x}}{\beta^3} (\beta^2 x^2 + 2\beta x + 2),
\]

the result follows.

\textbf{Appendix D}

\textbf{Proof of Theorem 5}

Let \( p_+ := p \) and \( p_- := 1 - p. \) For each \( x \in \mathbb{R}, \) we define

\[
g^\sigma_\pm(x) := p_\pm (f_\pm \ast K_\sigma)(x) + \lambda_\pm.
\]
Observe that, with this notation,

\[
\theta^\sigma(x) = \frac{1}{2} \log \left( \frac{g_x^\sigma(x)}{g_x^2(x)} \right).
\]  

(262)

The following lemma provides useful expressions for the $L^1$, $L^2$ and $L^3$-norms of the first derivative of $K_\sigma$.

**Lemma 12.** If $K_\sigma(x) = \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}}$, then $\|K_\sigma'\|_1 = \frac{\sqrt{2}}{\sqrt{\pi}\sigma}$, $\|K_\sigma'\|_2^2 = \frac{1}{4\sqrt{\pi}\sigma^3}$ and $\|K_\sigma'\|_3^3 = \frac{\sqrt{2}}{9\sqrt{\pi}\sigma^5}$.

**Proof.** It can be verified that, for all $x \in \mathbb{R}$,

\[
K_\sigma'(x) = -\frac{x}{\sigma^2}K_\sigma(x).
\]  

(263)

Thus, we have that

\[
\|K_\sigma'\|_1 = \frac{1}{\sigma^2} \int_{\mathbb{R}} |x| \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}} \, dx.
\]  

(264)

Note that the previous integral is the first absolute moment of a Gaussian random variable with mean 0 and variance $\sigma^2$. Therefore, we obtain that

\[
\|K_\sigma'\|_1 = \frac{\sqrt{2}}{\sqrt{\pi}\sigma}.
\]  

(265)

Similarly, we have that

\[
\|K_\sigma'\|_2^2 = \frac{1}{2\sqrt{\pi}\sigma^3} \int_{\mathbb{R}} x^2 \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}} \, dx.
\]  

(266)

Note that the previous integral is the second moment of a Gaussian random variable with mean 0 and variance $\sigma^2/2$. Therefore, we obtain that

\[
\|K_\sigma'\|_2^2 = \frac{1}{4\sqrt{\pi}\sigma^3}.
\]  

(267)

Finally, we have that

\[
\|K_\sigma'\|_3^3 = \frac{1}{2\sqrt{3}\pi\sigma^8} \int_{\mathbb{R}} |x|^3 \frac{e^{-3x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}/3} \, dx.
\]  

(268)

Note that the previous integral is the third absolute moment of a Gaussian random variable with mean 0 and variance $\sigma^2/3$. Therefore, we obtain that

\[
\|K_\sigma'\|_3^3 = \frac{\sqrt{2}}{9\sqrt{\pi}\sigma^5},
\]  

(269)

as required.

The following lemma provides useful expressions for the $L^1$ and $L^2$-norms of the second derivative of $K_\sigma$.

**Lemma 13.** If $K_\sigma(x) = \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}}$, then $\|K_\sigma''\|_1 \leq \frac{2}{\sigma^2}$ and $\|K_\sigma''\|_2 = \frac{\sqrt{3}}{2\sqrt{2}\sqrt[3]{\pi}\sigma^{5/2}}$.

**Proof.** It can be verified that, for all $x \in \mathbb{R}$,

\[
K_\sigma''(x) = \frac{x^2 - \sigma^2}{\sigma^4}K_\sigma(x).
\]  

(270)

Thus, we have that

\[
\|K_\sigma''\|_1 \leq \int_{\mathbb{R}} \left( \frac{x^2}{\sigma^4} + \frac{1}{\sigma^2} \right) \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}} \, dx
\]

\[
= \frac{1}{\sigma^4} \int_{\mathbb{R}} x^2 \frac{e^{-x^2}/2\sigma^2}{\sqrt{2\pi\sigma^2}} \, dx + \frac{1}{\sigma^2}.
\]  

(271)

(272)
Note that the last integral is the second moment of a Gaussian random variable with mean 0 and variance \( \sigma^2 \). Therefore,

\[
\| K'' \|_1 \leq \frac{2}{\sigma^2}.
\] (273)

Similarly, we have that

\[
\| K'' \|_2^2 = \int_{\mathbb{R}} \left( \frac{x^4}{2 \sqrt{2 \pi} \sigma^5} - \frac{x^2}{\sqrt{2 \pi} \sigma^7} + \frac{1}{2 \sqrt{2 \pi} \sigma^9} \right) e^{-x^2/\sigma^2} \, dx.
\] (274)

Note that the last integral is determined by the even moments of a Gaussian random variable with mean 0 and variance \( \sigma^2 / 2 \). Therefore, we obtain that

\[
\| K'' \|_2 = \left( \frac{3}{8 \sqrt{2 \pi} \sigma^5} - \frac{1}{2 \sqrt{2 \pi} \sigma^7} + \frac{1}{2 \sqrt{2 \pi} \sigma^9} \right)^{1/2}
\] (275)

\[
= \frac{\sqrt{3}}{2 \sqrt{2 \sqrt{2 \pi} \sigma^{5/2}}},
\] (276)

as required.

The following lemma provides useful expressions for the \( L^1 \)-norm of the third derivative of \( K_\sigma \).

**Lemma 14.** If \( K_\sigma(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \), then \( \| K''' \|_1 \leq \frac{5 \sqrt{2}}{\sqrt{2\pi} \sigma^3} \).

**Proof.** It can be verified that, for all \( x \in \mathbb{R} \),

\[
K'''_\sigma(x) = -\frac{x^3 - 3\sigma^2 x}{\sigma^6} K_\sigma(x).
\] (277)

Thus, we have that

\[
\| K''' \|_1 \leq \int_{\mathbb{R}} \left( \frac{|x|^3}{\sigma^8} + \frac{3|x|}{\sigma^4} \right) e^{-x^2/2\sigma^2} \, dx.
\] (278)

Note that the last integral is determined by the absolute moments of a Gaussian random variable with mean 0 and variance \( \sigma^2 \). Therefore,

\[
\| K''' \|_1 \leq \frac{2 \sqrt{2}}{\sqrt{2\pi} \sigma^3} + \frac{3 \sqrt{2}}{\sqrt{2\pi} \sigma^3} = \frac{5 \sqrt{2}}{\sqrt{2\pi} \sigma^3},
\] (279)

as required.

In order to avoid cumbersome notation, we omit the superscript \( \sigma \) when there is no risk of confusion, e.g., \( g_\sigma^\pm \) is written as \( g^\pm \) and \( \theta_\sigma^\pm \) is written as \( \theta^\pm \). The following corollary provides an upper bound for the \( L^1 \)-norm of \( \theta' \).

**Corollary 2.** If \( f_\pm \) is a probability density function, then

\[
\| \theta' \|_1 \leq \left( \frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-} \right) \frac{1}{\sqrt{2\pi} \sigma}.
\] (280)

**Proof.** In Lemma 2 we prove that, for all \( x \in \mathbb{R} \),

\[
2\theta'(x) = \frac{g'(x)}{g_+(x)} - \frac{g'(x)}{g_-(x)}.
\] (281)

By the triangle inequality, we have that

\[
2|\theta'(x)| \leq \frac{|g'(x)|}{g_+(x)} + \frac{|g'(x)|}{g_-(x)} \leq \frac{|g'_+(x)|}{\lambda_+} + \frac{|g'_-(x)|}{\lambda_-},
\] (282)

(283)
where the last inequality follows trivially from (261). Thus,\[ \|\theta''\|_1 \leq \frac{\|g'_+\|_1}{2\lambda_+} + \frac{\|g'_-\|_1}{2\lambda_-}. \] (284)

From (261), it is immediate to see that \( g'_+ = p_+(f_+ * K_\sigma)' \). Hence, the formula \((h_1 * h_2)' = h_1 * h_2'\) implies that\[ \|\theta''\|_1 \leq \frac{p_+\|f_+ * K_\sigma\|_1}{2\lambda_+} + \frac{p_-\|f_- * K_\sigma\|_1}{2\lambda_-}. \] (285)

Recall that Young’s convolution inequality establishes that\[ \|h_1 * h_2\|_r \leq \|h_1\|_{r_1} \|h_2\|_{r_2}, \] (286)
whenever \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} + 1 \). Hence, by taking \( r = r_1 = r_2 = 1 \),\[ \|\theta''\|_1 \leq \frac{p_+\|f_+\|_1\|K_\sigma\|_1}{2\lambda_+} + \frac{p_-\|f_-\|_1\|K_\sigma\|_1}{2\lambda_-}. \] (287)

Since \( f_{\pm} \) is a probability density function, we have that \( \|f_{\pm}\|_1 = 1 \). Therefore, Lemma [2] implies that\[ \|\theta''\|_1 \leq \left( \frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-} \right) \frac{1}{\sqrt{2\pi\sigma}}, \] (288)
as required.

The following corollary provides an upper bound for the \( L^1 \)-norm of \( \theta'' \).

**Corollary 3.** If \( f_{\pm} \) is a probability density function, then\[ \|\theta''\|_1 \leq \left( \frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-} \right) \frac{1}{\sigma^2} + \left( \frac{p^2_+}{\lambda_+^2} + \frac{p^2_-}{\lambda_-^2} \right) \frac{1}{8\sqrt{\pi\sigma^3}}. \] (289)

**Proof.** In Lemma [2] we prove that, for all \( x \in \mathbb{R}, \)
\[ 2\theta''(x) = \left[ \frac{g''_+(x)}{g_+(x)} - \left( \frac{g'_+(x)}{g_+(x)} \right)^2 \right] - \left[ \frac{g''_-(x)}{g_-(x)} - \left( \frac{g'_-(x)}{g_-(x)} \right)^2 \right]. \] (290)

By the triangle inequality, we have that\[ 2|\theta''(x)| \leq \left| \frac{g''_+(x)}{g_+(x)} \right| + \left| \frac{g'_+(x)}{g_+(x)} \right|^2 + \left| \frac{g''_-(x)}{g_-(x)} \right| + \left| \frac{g'_-(x)}{g_-(x)} \right|^2 \] \[ \leq \frac{|g''_+(x)|}{\lambda_+} + \frac{|g'_+(x)|^2}{\lambda_+^2} + \frac{|g''_-(x)|}{\lambda_-} + \frac{|g'_-(x)|^2}{\lambda_-^2}, \] (291)
where the last inequality follows trivially from (261). Thus,\[ \|\theta''\|_1 \leq \frac{\|g''_+\|_1}{2\lambda_+} + \frac{\|g'_+\|^2_1}{2\lambda_+^2} + \frac{\|g''_-\|_1}{2\lambda_-} + \frac{\|g'_-\|^2_1}{2\lambda_-^2}. \] (292)

From (261), it is immediate to see that \( g''_+ = p_+(f_+ * K_\sigma)' \). Hence, the formula \((h_1 * h_2)'' = h_1 * h_2''\) implies that\[ \|g''_{\pm}\|_1 = p_\pm\|f_+ * K_\sigma''\|_1 \leq p_\pm\|f_{\pm}\|_1\|K_\sigma''\|_1, \] (284)
where we applied Young’s convolution inequality (286) with \( r = r_1 = r_2 = 1 \). Since \( f_{\pm} \) is a probability density function, we have that \( \|f_{\pm}\|_1 = 1 \). Therefore, Lemma [13] implies that\[ \|g''_{\pm}\|_1 \leq \frac{2p_{\pm}}{\sigma^2}. \] (295)
Similarly, we have that
\[
\|g_+^{''}\|_2^2 = p_+^2 \|f_+ * K_*^{''}\|_2^2 \leq p_+^2 \|f_+\|_2^2 \|K_*^{''}\|_2^2, \tag{296}
\]
where we applied Young’s convolution inequality \(286\) with \(r = r_2 = 2\) and \(r_1 = 1\). Thus, Lemma \(12\) implies that
\[
\|g_+^{''}\|_2^2 \leq \frac{p_+^2}{4\sqrt{\pi}\sigma^3}. \tag{297}
\]
By plugging \(295\) and \(297\) in \(293\), we conclude that
\[
\|\theta^{''}\|_1 \leq \left(\frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-}\right) \frac{1}{\sigma^2} + \left(\frac{p_+^2}{\lambda_+^2} + \frac{p_-^2}{\lambda_-^2}\right) \frac{1}{8\sqrt{\pi}\sigma^3}, \tag{298}
\]
as required.

The following corollary provides an upper bound for the \(L^1\)-norm of \(\theta^{''}\).

**Corollary 4.** If \(f_\pm\) is a probability density function, then
\[
\|\theta^{''}\|_1 \leq \left(\frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-}\right) \frac{5}{2\sqrt{\pi}\sigma^3} + \left(\frac{p_+^2}{\lambda_+^2} + \frac{p_-^2}{\lambda_-^2}\right) \frac{3\sqrt{3}}{8\sqrt{\pi}\sigma^4} + \left(\frac{p_+^3}{\lambda_+^3} + \frac{p_-^3}{\lambda_-^3}\right) \frac{\sqrt{2}}{9\sqrt{\pi}\sigma^5}. \tag{299}
\]

**Proof.** In Lemma \(2\) we prove that
\[
2\theta^{''} = \left[\frac{g_+^{''}}{g_+} - 3\frac{g_+^{'''} g_+^3}{g_+^3} + 2\left(\frac{g_+^{''}}{g_+}\right)^3\right] - \left[\frac{g_-^{''}}{g_-} - 3\frac{g_-^{'''} g_-^3}{g_-^3} + 2\left(\frac{g_-^{''}}{g_-}\right)^3\right]. \tag{300}
\]
By the triangle inequality, we have that
\[
2|\theta^{''}| \leq \frac{|g_+^{''}|}{g_+} + 3\frac{|g_+^{'''} g_+^3|}{g_+^3} + 2\frac{|g_+^{''}|^3}{g_+^3} + 3\frac{|g_-^{''}|}{g_-} + 3\frac{|g_-^{'''} g_-^3|}{g_-^3} + 2\frac{|g_-^{''}|^3}{g_-^3} \leq \frac{|g_+^{''}|}{\lambda_+} + 3\frac{|g_+^{'''} g_+^3|}{\lambda_+^3} + 2\frac{|g_+^{''}|^3}{\lambda_+^3} + \frac{|g_-^{''}|}{\lambda_-} + 3\frac{|g_-^{'''} g_-^3|}{\lambda_-^3} + 2\frac{|g_-^{''}|^3}{\lambda_-^3}, \tag{301}
\]
where the last inequality follows trivially from \(261\). Thus,
\[
|\theta^{''}|_1 \leq \frac{2\|g_+^{''}|}{2\lambda_+} + \frac{3\|g_+^{'''} g_+^3\|_1}{2\lambda_+^3} + \frac{2\|g_+^{''}|^3\|_1}{2\lambda_+^3} + \frac{3\|g_-^{''}|}{2\lambda_-} + \frac{3\|g_-^{'''} g_-^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|_1}{\lambda_-^3}. \tag{302}
\]
From \(261\), it is immediate to see that \(g_+^{'''} = p_+(f_+ * K_*)^{'''}\). Hence, the formula \((h_1 * h_2)^{'''} = h_1 * h_2^{'''}\) implies that
\[
|\theta^{''}|_1 \leq \frac{2\|g_+^{''}|_1}{2\lambda_+} + \frac{3\|g_+^{'''} g_+^3\|_1}{2\lambda_+^3} + \frac{2\|g_+^{''}|^3\|_1}{2\lambda_+^3} + \frac{3\|g_-^{''}|_1}{2\lambda_-} + \frac{3\|g_-^{'''} g_-^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|_1}{\lambda_-^3}. \tag{303}
\]
From \(261\), it is immediate to see that \(g_+^{'''} = p_+(f_+ * K_*)^{'''}\). Hence, the formula \((h_1 * h_2)^{'''} = h_1 * h_2^{'''}\) implies that
\[
|\theta^{''}|_1 \leq \frac{2\|g_+^{''}|_1}{2\lambda_+} + \frac{3\|g_+^{'''} g_+^3\|_1}{2\lambda_+^3} + \frac{2\|g_+^{''}|^3\|_1}{2\lambda_+^3} + \frac{3\|g_-^{''}|_1}{2\lambda_-} + \frac{3\|g_-^{'''} g_-^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|^3\|_1}{2\lambda_-^3} + \frac{2\|g_-^{''}|_1}{\lambda_-^3}. \tag{304}
\]
where we applied Young’s convolution inequality \(286\) with \(r = r_1 = r_2 = 1\). Since \(f_\pm\) is a probability density function, we have that \(\|f_\pm\|_1 = 1\). Therefore, Lemma \(14\) implies that
\[
\|g^{'''}_\pm\|_1 \leq \frac{5\sqrt{2}p_{\pm}}{\sqrt{\pi}\sigma^3}. \tag{305}
\]
By Hölder’s inequality, we observe that
\[
\|g_+^{'''} g_+^{'''}\|_1 \leq \|g_+^{'''}\|_2 \|g_+^{'''}\|_2. \tag{306}
\]
As before, we have that
\[
\|g^{'''}_\pm\|_2 = p_{\pm} \|f_\pm * K_*^{'''}\|_2 \leq p_{\pm} \|f_\pm\|_1 \|K_*^{'''}\|_2, \tag{307}
\]
where we applied Young’s inequality \((286)\) with \(r = r_1 = 2\) and \(r_2 = 1\). Thus, Lemma 13 implies
\[
\|g_\pm''\|_2 \leq \frac{\sqrt{3}p_\pm}{2\sqrt{2}\sqrt{\pi\sigma^3}^{5/2}}.
\]
(308)

The previous inequality and \((297)\) lead to
\[
\|g_\pm'\|_4 \leq \frac{\sqrt{3}p_\pm}{4\sqrt{2}\pi\sigma^4}.
\]
(309)

Finally, we have that
\[
\|g_\pm'\|_3 = p_\pm\|f_\pm \ast K_\pm'\|_3 \leq p_\pm\|f_\pm\|_3\|K_\pm'\|_3,
\]
where we applied Young’s convolution inequality \((286)\) with \(r = r_1 = 3\) and \(r_2 = 1\). Thus, Lemma 12 implies that
\[
\|g_\pm'\|_3 \leq \frac{\sqrt{2}p_\pm^3}{9\sqrt{\pi^3}\sigma^5}.
\]
(311)

By plugging \((305)\), \((309)\) and \((311)\) in \((303)\), we conclude that
\[
\|\theta'''\|_1 \leq \left(\frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-}\right) \frac{5}{\sqrt{2}\pi\sigma^3} + \left(\frac{p_+^2}{\lambda_+^2} + \frac{p_-^2}{\lambda_-^2}\right) \frac{3\sqrt{3}}{8\sqrt{2}\pi\sigma^4} + \left(\frac{p_+^3}{\lambda_+^3} + \frac{p_-^3}{\lambda_-^3}\right) \frac{\sqrt{2}}{9\sqrt{\pi^3}\sigma^5}.
\]
(312)
as required.

Now we are in position to prove Theorem 6.

Proof of Theorem 6 By Theorem 3 we have that
\[
C_\theta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left(1 + \frac{1}{\sqrt{2}} \log (\|\theta''\|_1)\right) \|\theta''\|_1.
\]
(313)

Recall the definition of \(N_\alpha\) in (142) – (144). Corollaries 2 – 4 imply that \(\|\theta^{(\alpha)}\|_1 \leq N_\alpha\) for every \(\alpha \in \{1, 2, 3\}\). As a result,
\[
C_\theta \leq \frac{2\sqrt{2}N_2}{\sqrt{\pi}} \left(1 + \frac{1}{\sqrt{2}} \log (N_1N_3)\right) - \frac{2\sqrt{2}}{\sqrt{\pi}} \log (\|\theta''\|_1)\|\theta''\|_1.
\]
(314)

Since \(-\log(z)z \leq 1/e\) for all \(z \in [0, \infty)\), the previous inequality implies that
\[
C_\theta \leq \frac{2\sqrt{2}}{e\sqrt{\pi}} + \frac{2\sqrt{2}N_2}{\sqrt{\pi}} \left(1 + \frac{1}{\sqrt{2}} \log (N_1N_3)\right).
\]
(315)

It is straightforward to verify that \(z \mapsto -\log(z)z\) is increasing over \([0, 1/e]\). Thus, if \(N_2 \leq 1/e\), (314) implies that
\[
C_\theta \leq \frac{2\sqrt{2}N_2}{\sqrt{\pi}} \left(1 + \frac{1}{\sqrt{2}} \log \left(\frac{N_1N_3}{N_2^2}\right)\right),
\]
(316)
as required.

APPENDIX E

PROOF OF LEMMA 3

Proof of Lemma 3 Observe that, without loss of generality, we can assume that
\[
\delta < \frac{1}{2} \min \{t_2 - t_1, \ldots, t_k - t_{k-1}\}.
\]
(317)

Since \(\lim_{\zeta \to \infty} \tanh(\zeta) = 1\) and \(\tanh(-\zeta) = -\tanh(\zeta)\), there exists \(\zeta_\epsilon > 0\) such that, for all \(s \in \{-1, 0, +1\}\),
\[
|s - \tanh(\zeta_\epsilon s)| \leq \frac{\epsilon}{2}.
\]
(318)
Recall that $\phi$ satisfies that $\lim_{z \to \pm \infty} \phi(z) = \pm 1$. Hence, there exists $z_\epsilon > 0$ such that, for all $z \geq z_\epsilon$,

$$|1 - \phi(z)| \leq \frac{\epsilon}{2k\zeta_\epsilon} \quad \text{and} \quad |-1 - \phi(z)| \leq \frac{\epsilon}{2k\zeta_\epsilon}.$$  

(319)

Note that any $g \in S_k$ can be written as

$$g(x) = s_0 + \sum_{l=1}^{k} (s_l - s_{l-1}) \mathbb{1}_{[t_l, \infty)}(x),$$  

(320)

where $s_l \in \{-1, 0, +1\}$ and $t_1 < \cdots < t_k$. For such a $g \in S_k$, let $h \in \mathcal{H}_k^\phi$ be the function defined by

$$h(x) = \zeta_\epsilon \left( s_0 + \sum_{l=1}^{k} (s_l - s_{l-1}) \phi\left( \frac{z_\epsilon}{\delta} (x - t_l) \right) + 1 \right).$$  

(321)

In the sequel we show that $h$ satisfies (160).

Assume that $t_j + \delta \leq x \leq t_{j+1} - \delta$ for some $j \in \{0, \ldots, k\}$. In this case, (320) implies that $g(x) = s_j$ and, as a result,

$$|g(x) - \tanh(h(x))| \leq |s_j - \tanh(\zeta_\epsilon s_j)| + |\tanh(\zeta_\epsilon s_j) - \tanh(h(x))|$$  

\begin{align*}
\leq \frac{\epsilon}{2} + |\zeta_\epsilon s_j - h(x)|,  
(322)
\end{align*}

where we used (318) and the fact that $\tanh$ is 1-Lipschitz. A straightforward manipulation shows that

$$h(x) = \zeta_\epsilon s_j + \zeta_\epsilon \sum_{l=1}^{j} (s_l - s_{l-1}) \frac{\phi\left( \frac{z_\epsilon}{\delta} (x - t_l) \right) - 1}{2} + \zeta_\epsilon \sum_{l=j+1}^{k} (s_l - s_{l-1}) \frac{\phi\left( \frac{z_\epsilon}{\delta} (x - t_l) \right) + 1}{2}.$$  

(324)

In particular, we have that

$$|\zeta_\epsilon s_j - h(x)| \leq \zeta_\epsilon \sum_{l=1}^{j} \left| \phi\left( \frac{z_\epsilon}{\delta} (x - t_l) \right) - 1 \right| + \zeta_\epsilon \sum_{l=j+1}^{k} \left| \phi\left( \frac{z_\epsilon}{\delta} (x - t_l) \right) + 1 \right|.$$  

(325)

Note that, for all $l \leq j$,

$$\frac{z_\epsilon}{\delta} (x - t_l) \geq \frac{z_\epsilon}{\delta} (x - t_j) \geq z_\epsilon,$$  

(326)

and, for all $l \geq j + 1$,

$$\frac{z_\epsilon}{\delta} (x - t_l) \leq \frac{z_\epsilon}{\delta} (x - t_{j+1}) \leq -z_\epsilon.$$  

(327)

Therefore, (319) and (325) imply that

$$|\zeta_\epsilon s_j - h(x)| \leq \frac{\epsilon}{2},$$  

(328)

and, as a result, (323) becomes

$$|g(x) - \tanh(h(x))| \leq \epsilon,$$  

(329)

as required.
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