Optimal Quantum Subtracting machine

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The impossibility of undoing a mixing process is analysed in the context of quantum information theory. The optimal machine to undo the mixing process is studied in the case of pure states, focusing on qubit systems. Exploiting the symmetry of the problem we parametrise the optimal machine in such a way that the number of parameters grows polynomially in the size of the problem. This simplification makes the numerical methods feasible. For simple but non-trivial cases we computed the analytical solution, comparing the performance of the optimal machine with other protocols.

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A fundamental fact in quantum information theory is that not all maps between quantum states are possible: even before considering practical difficulties, quantum theory itself limits the operations that can be performed. A series of quantum no-go theorems [1–8] shows that transformations which would be very valuable from the point of view of information processing are in fact impossible. The most celebrated of them is the non-cloning theorem [2]: the impossibility of cloning makes many processing tasks (e.g. state estimation) non-trivial. Nonetheless, the importance of these impossible transformations drives the search for approximate implementations of them: optimal cloners [3] have been extensively studied, and similar efforts have been spent for other no-go theorems [7–8,10,12].

Here we introduce the no-subtracting theorem, which states the impossibility of undoing the mixing operation that involves a target state we wish to recover and an external noise source, and define the optimal subtractor operation which solves the problem with the best allowed approximation. This task is somehow related to those discussed in [13] and references therein, where one aims to perform quantum information processing of some sort (e.g. the recovery of the target state) when some classical knowledge (i.e. the reference frame for [13] and the amount of mixed noise for us) is replaced by bounded information encoded into the density matrix of an ancillary quantum system. Finding the optimal subtractor corresponds to a semidefinite program involving a number of variables that in principle grows exponentially with the input data (system copies). However, by exploiting the symmetry of the problem and a proper parametrisation of the N to 1 qubit covariant channels (analogous to those introduced in Refs. [13,15]), the number of effective parameters can be reduced to a subset which only scales polynomially. This reduction of the parameters makes the numerical optimisation feasible, and for small enough input data, allows also for analytical treatment.

Optimal Subtractor:– An Universal Quantum Subtracting machine UQS is a two-inputs/one-output transformation acting on two isomorphic quantum systems A and B. When provided by factorised input states of the form $(p\hat{\rho}_0 + (1-p)\hat{\rho}_1) \otimes \hat{\rho}_0$, with $p \in [0,1]$ assigned and $\hat{\rho}_0, \hat{\rho}_1 \in \mathcal{S}(\mathcal{H})$ arbitrary density matrices, it returns as output the system A into the state $\hat{\rho}_1$ realizing the mapping

$$\text{UQS}[\hat{\rho}_{\text{mix}}(p) \otimes \hat{\rho}_0] = \hat{\rho}_1,$$

which effectively allows one to recover $\hat{\rho}_1$ from the mixture $\hat{\rho}_{\text{mix}}(p) := p\hat{\rho}_0 + (1-p)\hat{\rho}_1$ by “removing” the perturbing state $\hat{\rho}_0$ and renormalizing the result. Equation (1) can be described as the formal inversion of the transformation $\text{IQA}[\hat{\rho}_0 \otimes \hat{\rho}_1] = \hat{\rho}_{\text{mix}}(p)$, which we may dub Incoherent Quantum Adder. At variance with the Coherent Quantum Adder analyzed in Refs. [7,10,11], an IQA can be easily implemented as it merely consists in creating a probability mixture out of two input configurations. In particular, IQA can be interpreted as an open quantum evolution [20,21,23] in which the state $\hat{\rho}_0$ of the input B, plays the role of the environment. In this scenario, the aim is to undo the action of IQA and recover $\hat{\rho}_1$ not having the full knowledge about the environment, the only information available being encoded through copies of $\hat{\rho}_0$.

Given the above premise it should be now clear that the possibility of constructing an UQS machine will have a profound impact in many practical applications, spanning from quantum computation [21,24], where it could be employed as an effective error correction procedure for certain kind of errors, to quantum communication [25], where instead it could be used as a decoding operation to distill the intended messages from the received deteriorated signals. Unfortunately the possibility of physically realizing an UQS machine for $p > 0$, turns out to be in contradiction with the basic requirements that any quantum evolution has to fulfil, see e.g. Ref. [20]. Indeed invoking linearity and using the fact that for $\hat{\rho}_1 = \hat{\rho}_0$ one has $\text{UQS}[\hat{\rho}_0 \otimes \hat{\rho}_0] = \hat{\rho}_0$, Eq. (1) can be cast in the following form $(1-p)\text{UQS}[\hat{\rho}_1 \otimes \hat{\rho}_0] = \hat{\rho}_1 - p\hat{\rho}_0$, which, as long as the parameter $p$ is strictly different from 0, will produce
unphysical non-positive results as soon as the support of \( \hat{\rho}_0 \) admits a non trivial overlap with the kernel of \( \hat{\rho}_1 \). Yet, as in the case of other better studied impossible quantum machines \([1]\), there could be still room for approximate implementations of the mapping \([1]\). In what follows we shall hence try to identify the implementation of an optimal UQS, i.e. a machine which, being physically realizable via a Completely Positive and Trace Preserving (CPTP) map \([20]\), would give us the best approximation alizable via a Completely Positive and Trace Preserving optimal UQS, i.e. a machine which, being physically re-implementable by means of the unitary set \( SU(d) \) to ensure a uniform distribution of the input copies. Before entering into the technical derivation, it is worth commenting that while the problem we are facing is the quantity we are going to study in the following.

As a figure of merit we shall consider the fidelity \([21]\) between the obtained output and the intended target states, properly averaged with respect to all possible inputs. To simplify the analysis in what follows we restrict ourself to the special case where both \( \hat{\rho}_0 \) and \( \hat{\rho}_1 \) are pure states of the \( d \)-dimensional space \( \mathcal{H} \), namely \( \hat{\rho}_1 = |\psi\rangle \langle \psi| \) and \( \hat{\rho}_0 = |\phi\rangle \langle \phi| \), where without loss of generality we adopt the parametrisation \( |\psi\rangle := U |\uparrow\rangle \) and \( |\phi\rangle := \hat{V} |\uparrow\rangle \), with \( |\uparrow\rangle \) being a fixed vector and \( \hat{U}, \hat{V} \) being arbitrary elements of the unitary set \( SU(d) \). Indicating hence with \( \Lambda \) the CPTP mapping that we want to test as a candidate for the implementation of UQS\((n_1,n_2)\), we evaluate its performance through the function

\[
F_{n_1,n_2}(\Lambda) := \iint d\mu |\psi\rangle \langle \psi| \Lambda (\hat{\rho}_{\text{mix}}^{\otimes n_2} \otimes \hat{\rho}_0^{\otimes n_1}) |\psi\rangle ,
\]

where the integral are performed via the Haar measure of \( SU(d) \) to ensure a uniform distribution of \( |\psi\rangle \) and \( |\phi\rangle \) on \( \mathcal{H} \). Before entering into the technical derivation, it is worth commenting that while the problem we are facing can be seen as a sort of purification procedure, it is definitely different from the task addressed by Cirac et al. in Ref. \([22]\), which is designed to remove the largest fraction of complete mixed state from \( \hat{\rho}_{\text{mix}} \) having access to some copies of it, but with no prior information on \( \hat{\rho}_0 \) or \( p \).

**Preliminary results:-** The maximum of Eq. \(3\) with respect to all possible CPTP transformations

\[
F_{n_1,n_2}^{(\text{max})} := \max_{\Lambda \in \text{CPTP}} F_{n_1,n_2}(\Lambda),
\]

is the quantity we are going to study in the following. Since one can always neglect part of the input copies, this functional is clearly non-decreasing in \( n_1 \) and \( n_2 \), i.e. \( F_{n_1,n_2}^{(\text{max})} \leq F_{n_1+1,n_2}^{(\text{max})} \leq F_{n_1,n_2+1}^{(\text{max)}, \text{with no ordering between the last two terms been foreseen from first principles. In particular, we are interested in comparing} F_{n_1,n_2}^{(\text{max})} \text{with the performances achievable via a trivial ‘doing nothing’ (DN) strategy in which one emulates the mapping \([2]\) by simply returning as output one of the qubits of the register A, i.e. the state} \hat{\rho}_{\text{mix}}(p) \). In this case, the associated average fidelity can be easily computed by exploiting the depolarizing identity \( \int d\mu |\psi\rangle \langle \psi| = \int d\mu \hat{U} |\uparrow\rangle \langle \uparrow| \hat{U}^\dagger = I/d \), obtaining \( F_{n_1,n_2}(\text{DN}) := 1 - p(d-1)/d \) by which, construction constitutes a lower bound for \( F_{n_1,n_2}^{(\text{max})} \), i.e. \( F_{n_1,n_2}^{(\text{max})} \geq 1 - p(d-1)/d \) (incidentally for the qubit case, \( F_{n_1,n_2}(\text{DN}) \) coincides with the average fidelity on would obtain by adapting the optimal protocol of the Cirac et al. scheme \([22]\) to our setting, see Ref. \([11]\)). Determining the exact value of \( F_{n_1,n_2}^{(\text{max})} \) is typically very demanding apart from the case where we have a single copy of A, i.e. for \( n_1 = 1 \). In this scenario in fact, irrespectively from the value of \( n_2 \), one can prove that the DN strategy is optimal, transforming the inequality \( F_{n_1,n_2}^{(\text{max})} \geq F_{n_1,n_2}(\text{DN}) \) into the identity \( F_{n_1,n_2}^{(\text{max})} = 1 - p(d-1)/d \).

One way to see this is to show that \([5]\) holds in the asymptotic limit of infinitely many copies of the B state, i.e. \( n_2 \to \infty \), and then invoke the monotonicity under \( n_2 \) to extent such result to all the other cases. As a matter of fact when \( n_2 \) diverges one can use quantum tomography to recover the classical description of B from the input data: accordingly the optimal implementation of UQS\((1,\infty)\) formally coincides with the optimal recovery map \([26]\) aiming to invert the CPTP transformation that takes a generic element \( \hat{\rho}_1 \in \mathcal{S}(\mathcal{H}) \) into \( \hat{\rho}_{\text{mix}}(p) \). In this case \([3]\) gets replaced by \( F_{1,\infty}(\Lambda) := \int d\mu |\psi\rangle \Lambda((1-p) |\psi\rangle \langle \psi| + p |\phi\rangle \langle \phi| |\psi\rangle , \) which thanks to the depolarizing identity can be easily shown to admit \( F_{n_1,n_2}(\text{DN}) \) not just as a lower bound but also as an upper bound, leading to \( F_{1,\infty}^{(\text{max})} = 1 - p(d-1)/d \) and hence to \([5]\).

As \( n_1 \) gets larger than \( 1 \), we aspect to see a non trivial improvement with respect to the DN strategy. This is clearly evident at least in the case where both \( n_1 \) and \( n_2 \) diverge (i.e. \( n_1, n_2 \to \infty \)). In this regime, similarly to the case of optimal quantum cloner \([2,27,31]\), Eq. \([2]\) becomes implementable by means of a simple measure-and-prepare (MP) strategy based on performing full quantum tomography on both inputs A and B, yielding the optimal value \( F_{\infty,\infty}^{(\text{max})} = 1 \) which clearly surpasses the DN threshold. In the next sections, we shall clarify a procedure that one can follow to solve the optimisation of Eq. \([3]\) for infinite values of the input copies. For the sake of simplicity we present it for the special cases where A and B are just qubit systems and we use such technique to analytically compute the exact value of \( F_{n_1,n_2}^{(\text{max})} \) for the simplest but non-trivial scenario where \( n_1 = 2 \) and \( n_2 = 1 \). Via numerical methods we also solve the optimisation problem for some selected values of \( p, n_1 \) and \( n_2 \), see Fig. \([2]\).

**Channel optimisation:-** The problem we are considering has special symmetries that allows for some sim-
plifications. Invoking the linearity of $\Lambda$ and the invariance of the Haar measure we can rewrite (3) as $F_{n_1,n_2}(\Lambda) = \langle \psi^n | \Lambda (\Omega_{n_1n_2}) | \psi^n \rangle$, where $\Omega_{n_1n_2}$ is the density operator

$$\Omega_{n_1n_2} := \int d\mu \left( p | \uparrow \rangle \langle \uparrow | + (1-p) \hat{V} | \uparrow \rangle \langle \uparrow | \hat{V} \right) \otimes (\hat{V} | \uparrow \rangle \langle \uparrow | \hat{V}) \otimes n_1 \otimes n_2 .$$

The channel $\Lambda_c$ appearing in the expression for $F_{n_1,n_2}(\Lambda)$ is obtained from $\Lambda$ through the following integral

$$\Lambda_c[\cdots] = \int d\mu U \Lambda [U \otimes \cdots \otimes SU(2) \right] ,$$

which ensures that $\Lambda_c$ is a $N$ qubits to 1 qubit covariant map, i.e. a CPTP transformation fulfilling the condition $U^\dagger \Lambda_c[\cdots] U = \Lambda_c[U \otimes \cdots \otimes SU(2)]$, $\forall U \in SU(2)$. Notice also that if $\Lambda$ is already covariant, then it coincides with its associated $\Lambda_c$, i.e. $\Lambda_c = \Lambda$. Exploiting these facts we can hence conclude that the maximisation of $F_{n_1,n_2}(\Lambda)$ can be performed by just focusing on this special set of transformations which, now we shall parametrise. The integral appearing in (7) motivates us to choose the total angular momentum eigenbasis as the basis for the Hilbert space $H_{\otimes N}$ where the channel operates. Specifically we shall write such vectors as $|j, m, g\rangle$ with $j$ the total angular momentum of $N$ spin $1/2$ particles, $m$ the total angular momentum in $z$ direction, and $g$ labelling different equivalent representations with total angular momentum $j$. Following the derivation presented in [41] we can then verify that, indicating with $\{j = \pm \frac{1}{2}, s\}_s = \pm \frac{1}{2}$ the angular momentum basis for a single qubit (no degeneracy being present), one has

$$\langle \frac{1}{2}, s_s | \Lambda_c | [j, m, g(j, m', g')] \rangle = (-1)^{m-m'} \times \delta_{s-m, s'-m'} \sum_{q \in Q_{j,j'}} C^q_{s_m, s'-m'} C^q_{s-s', m-m'} W^{j,j'}_{q,g,g'} ,$$

where the summation over the index $q$ runs over $Q_{j,j'} := \{ j \pm \frac{1}{2} \} \cap \{ j' \pm \frac{1}{2} \}$ (if the set is empty then the associated matrix element is automatically null), where $C^q_{j,m,M'} := \langle j, M, m | \Lambda_c | j, m' \rangle$ is a Clebsch-Gordan coefficient, and where finally $W^{j,j'}_{q,g,g'} := v^q_{g,g'} \cdot (v^q_{g,g'})^\dagger$ represents the scalar product between the complex row vectors $v^q_{g,g'}$ and $(v^q_{g,g'})^\dagger$ constructed from the Kraus operators of $\Lambda_c$ and explicitly defined in [41]. Equation (8) tells us which are the parameters characterizing $\Lambda_c$ that enter into the optimization problem. The number of $W^{j,j'}_{q,g,g'}$ grows exponentially in $n_1$ and $n_2$: the multiplicity of the representation with total angular momentum $j$ grows exponentially in general, therefore $g$ and $g'$ can take an exponential number of different values. It is worth observing that this quantity does not depend on $m, s, m', s'$ which only appear in the Clebsch-Gordan coefficients. Also, the structure of the covariant channels specified in Eq. (8) indicates that the action of $\Lambda_c$ on the off-diagonal elements in the total angular momentum basis is zero unless $|m - m'| = 1$ and $|j - j'| = 1$. In principle there is no selection rule on $g$ and $g'$, and at this level the number of variables of the problem still scales exponentially in $n_1$ and $n_2$. A dramatic simplification arises by using the symmetry properties of $\Omega_{n_1n_2}$. First of all $[\Omega_{n_1n_2}, J_j] = 0$, from which $(j, m, g) | \Omega_{n_1n_2} | j', m', g')$ is zero unless $m = m'$. Moreover, by Schur-Weyl duality [41] the Hilbert space of the problem can be decomposed as $\mathcal{H} = \bigoplus D_{j_1} \otimes D_{j_2}$, where $j_1$ and $j_2$ are the irreducible representations of $SU(2)$ and the symmetric group $\Sigma_n$, with Young diagram $D$. We notice that $\Omega_{n_1n_2}$ is symmetric under permutations acting independently on the first $n_1$ and the second $n_2$ quibits, hence by Schur’s lemma $\Omega_{n_1n_2}$ must have the form $\Omega_{n_1n_2} = \bigoplus D_{j_1} \otimes D_{j_2}$, where $D_{j_1}$ and $\alpha D_{j_2}$ are the irreducible representations of $SU(2)$ and the symmetric group $\Sigma_n$, with Young diagram $D$. From this observation it follows that $\Omega_{n_1n_2}$ is supported on a space spanned by orthonormal vectors labelled as $|j, m, g_j\rangle$, where we use the same conventions as before for the total angular momentum indices, and we simplify the notation using $g_j$, as a shortcut for the couple $(j, s_j, s_j')$ which indexes a basis of $\alpha D_{j_2}$. Putting all together we have proved that $(j, m, g_j, \Omega_{n_1n_2} | j', m', g_j') = \delta_{j, j'} \delta_{m, m'} \delta_{g_j, g_j'}$, with the function $\Omega_{n_1n_2}(j, j', m, j, p)$ depending only on $j, j', m, j, p$ and being explicitly computed in [41]. Exploiting these properties of $\Omega_{n_1n_2}$ the fidelity can then be expressed as

$$F_{n_1,n_2}(\Lambda) = \sum_{j,j',j_1,g} C^{j,j'}_{j_1j}(p) W^{j,j'}_{q,j_1} ,$$

where $C^{j,j'}_{q,j}(p)$ is a contraction of Clebsch-Gordan coefficients defined in [41], and $W^{j,j'}_{q,j}(p) := (\#g_j) W^{j,j'}_{q,j}(j,j_1) / \#g_j$, with $\#g_j = (\#g_j) / (\#g_j + 1)$ being the multiplicity of the representation $j_1$. Further constraints associated with the Completely Positive condition of $\Lambda_c$ are also automatically included in the parametrisation via $W^{j,j'}_{q,g,g'}$ through the connection between the vectors $v^q_{g,g'}$ and the Kraus operators of $\Lambda$. The trace preserving requirement reduces instead to $\sum_{s = \pm \frac{1}{2}} \langle \frac{1}{2}, s | \Lambda_c | [j, m, g(j, m', g') | \frac{1}{2}, s_s \rangle = \delta_{j,j'} \delta_{m,m'} \delta_{g,g'}$, which via some manipulations [41] can be cast in the equivalent form

$$\frac{2^{2+2|j|}}{(1+2j)} W^{j,j}_{q,j} | q = j \rangle = \frac{2j}{1+2j} W^{j,j}_{q,j} | q = j \rangle = \frac{1}{2} .$$

We notice that the linearity of $F_{n_1,n_2}(\Lambda)$ and the convexity of the set of channels allows us to restrict the

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We notice that the linearity of $F_{n_1,n_2}(\Lambda)$ and the convexity of the set of channels allows us to restrict the
becomes cumbersome and we resort to numerical analy-

sis using Mathematica 34 to compute the parameters of the problem and CVX, a package for specifying and solving convex programs 37–39 in Matlab, to calculate the maximum fidelity values. Results are reported in Fig. 2 for \( n_1 = 1, 2, \ldots, 10 \) and \( n_2 = 1, 2, \ldots, 10 \) and \( p = \frac{1}{5} \frac{p}{10} \).

**Conclusions:** The gap between \( F_{2,1}^{(\text{max})} \) and the DN strategy (which is optimal in the \( n_1 = 1 \) scenario and independent from the explicit value of \( n_2 \)) shows that even a small redundancies on the input \( A \), can be beneficial. On the contrary, the very small distance between \( F_{2,1}^{(\text{max})} \) and \( F_{1,1}^{(\text{max})} \) clarifies that gathering more information on the mixing term \( \rho_0 \) (the noise of the model) does not help too much. As can be seen from Fig. 2 for larger \( n_1 \) and \( n_2 \) one can instead see a noise-dependent separation line between two regions, one where it is indeed advantageous to increase \( n_1 \) instead of \( n_2 \) and the other where the opposite holds.

The symmetry of the problem allowed us to reduce exponentially the number of variables involved in the optimisation. The same analysis should be relevant also in a broader perspective for general noise models.

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[41] Details of the calculations are available in the Supplemental Material at [URL will be inserted by publisher], where [40] is cited.
SUPPLEMENTAL MATERIAL

Supplemental material is organised as following. First, we provide explicit derivation of the decomposition \([8]\) of the main text. Then using \([8]\) we derive the fidelity \([9]\) of the main text. Third, we present an explicit derivation of Eq. (10) of the main text. Then we analyze the application of the decomposition \([8]\) of the main text to the case where \(n_1 = n_2 = 1\), and in the following section we do the same for the case \(n_1 = 2, n_2 = 1\). Analytical optimisation is also done for the case \(n_1 = 2, n_\rightarrow \infty\). Then, we present the derivation of an upper bound for the average fidelity of the UQS realised via measurement and prepare strategies. Finally, we apply the method of Cirac et al. \([22]\) for case \(n_1 = 2\) and arbitrary \(n_2\).

Covariant Channel Characterisation

Here the calculations to derive the characterisation for covariant are presented. Introducing a Kraus decomposition for \(A\) in Eq. (7) of the main text we get

\[
\Lambda_c[\cdots] = \sum_k \int d\mu_U \hat{U} \hat{M}_k \hat{U}^\dagger [\cdots] \hat{U}^\dagger \hat{M}_k^\dagger \hat{U}^\dagger ,
\]

with \(\hat{M}_k\) the associated Kraus operators. Accordingly we can express the matrix element \([8]\)

\[
\langle \frac{1}{2}, s | \Lambda_c([j, m, g] \langle j', m', g'| \frac{1}{2}, s') \rangle = \sum_k \sum_{r, r'} \sum_{l, l'} \int d\mu_U D_{s,r}^{1/2}(\hat{U}) M^{k,j,g}_{r,l}(\hat{U}^\dagger) D_{s',r'}^{1/2}(\hat{U}^\dagger) ,
\]

where

\[
D_{s,r}^{1/2}(\hat{U}) = \langle j, l, g | \hat{U}^{\otimes N} | j, l, g \rangle , \quad M^{k,j,g}_{r,l} := \langle \frac{1}{2}, r | \hat{M}_k | j, l, g \rangle .
\]

We can write the multiplication of two Wigner matrices in the following form

\[
D_{s,r}^{1/2}(\hat{U}) D_{s',r'}^{1/2}(\hat{U}) = (-1)^{l-m} \langle \frac{1}{2}, s | \hat{U} | \frac{1}{2}, r \rangle \langle j, -m, g | \hat{U}^{\otimes N} | j, -l, g \rangle = (-1)^{l-m} \langle 1/2, s | \otimes (|j, -m, g| \hat{U}^{\otimes N+1} |1/2, r) \rangle \otimes |j, -l, g \rangle = (-1)^{l-m} \sum_{j-\frac{1}{2} \leq q \leq j+\frac{1}{2}} C_{j-s,j-m}^q C_{r-l}^q D_{s-m,r-l}^q (\hat{U}) ,
\]

where \(C_{j-s,j-m}^M = \langle J, M | j m \rangle \otimes | j' m' \rangle\) are the Clebsch-Gordan coefficients. Exploiting this we can hence rewrite Eq. (13) in the following form

\[
\langle \frac{1}{2}, s | \Lambda_c([j, m, g] \langle j', m', g'| \frac{1}{2}, s') \rangle = \sum_k \sum_{r, r'} \sum_{l, l'} \sum_q (-1)^{l-m'+m} \int d\mu_U \ C_{j-s,j-m}^q C_{r-l}^q \times D_{s-m,r-l}^q (\hat{U}) M^{k,j,g}_{r,l}(\hat{U}^\dagger) D_{s',r'}^{1/2}(\hat{U}^\dagger) .
\]

Remembering that following identity of Wigner matrices (Peter-Weyl theorem, see \([40]\))

\[
\int d\mu_U \ D_{m,j,l}^{i,j}(\hat{U}) D_{m',j',l'}^{i,j'}(\hat{U})^* = \frac{1}{2j+1} \delta_{j,j'} \delta_{m,m'} \delta_{l,l'}
\]

the integral in (16) can hence be simplified to

\[
\langle \frac{1}{2}, s | \Lambda_c([j, m, g] \langle j', m', g'| \frac{1}{2}, s') \rangle = \sum_k \sum_{r, r'} \sum_{l, l'} \sum_q (-1)^{l-m'+m'} \frac{1}{2j+1} \delta_{s-m,m'-m'} \delta_{r-l,r'-l'} \times C_{j-s,j-m}^q C_{r-l}^q \times C_{s'-j'-m'}^q C_{r'-l'}^q \times M^{k,j,g}_{r,l} M^{k,j',g'}_{r',l'} .
\]
Introducing then the variable \( p := r - l = r' - l' \), we can rewrite the above identity as
\[
\langle 1/2, s | \Lambda_c(|j, m, g \rangle \langle j', m', g' |) | 1/2, s' \rangle = \sum_{k, r, r'} \sum_{p} \sum_{q} \sum_{s} \frac{(-1)^{r-m'+r'-m'}}{2j+1} \delta_{s-m, s'-m'} C^q_{\frac{1}{2} s, j-m} C^q_{\frac{1}{2} s', j'-m'} M^{k,j,g}_{\frac{1}{2} r, p-r} C^q_{\frac{1}{2} r, p-r} M^{k,j',g'}_{\frac{1}{2} r', p-r'} \tag{19}
\]
which, defining the row vectors \( \mathbf{v}^{j}_{q,g} \) of components
\[
\mathbf{v}^{j}_{q,g}(k, p) := \frac{1}{2j+1} \sum_{r} (-1)^{r} C^q_{\frac{1}{2} r, p-r} M^{k,j,g}_{\frac{1}{2} r, p-r} \tag{20}
\]
and their associated scalar products
\[
W^{j,j'}_{q,g,q'} := \mathbf{v}^{j}_{q,g} \cdot (\mathbf{v}^{j'}_{q',g'})^\dagger = \sum_{k,p} \mathbf{v}^{j}_{q,g}(k, p) \left[ \mathbf{v}^{j'}_{q',g'}(k, p) \right]^* \tag{21}
\]
allows us to finally express Eq. (19) as in Eq. (8) of the main text
\[
\langle \frac{1}{2}, s | \Lambda_c [|j, m, g \rangle \langle j', m', g' |) | \frac{1}{2}, s' \rangle = (-1)^{m-m'} \delta_{s-m, s'-m'} \sum_{q \in Q_{j,j'}} C^q_{\frac{1}{2} s, j-m} C^q_{\frac{1}{2} s', j'-m'} W^{j,j'}_{q,g,q'} \tag{22}
\]

**Fidelity calculation for arbitrary \( n_1, n_2 \) and numerical optimisation**

Using Eq. (8) of the main text the average fidelity can be expressed as
\[
F_{n_1,n_2}(\Lambda_c) = \langle |\uparrow \rangle | \Lambda_c | \int d\mu_V \left( p |\uparrow \rangle \langle \uparrow | + (1-p) \hat{V} |\uparrow \rangle \langle \uparrow | \hat{V}^\dagger \right)^{\otimes n_1} \otimes \left( \hat{V} |\uparrow \rangle \langle \uparrow | \hat{V}^\dagger \right)^{\otimes n_2} |\uparrow \rangle \rangle , \tag{23}
\]
Knowing that \( \Omega_{n_1,n_2} \) is invariant under any permutation on the first \( n_1 \) qubits, we can write
\[
F_{n_1,n_2}(\Lambda_c) = \frac{1}{|S_{n_1}|} \langle |\uparrow \rangle | \Lambda_c | \sum_{\sigma} \hat{\Pi}_{\sigma} | \int d\mu_V \left( p |\uparrow \rangle \langle \uparrow | + (1-p) \hat{V} |\uparrow \rangle \langle \uparrow | \hat{V}^\dagger \right)^{\otimes n_1} \otimes \left( \hat{V} |\uparrow \rangle \langle \uparrow | \hat{V}^\dagger \right)^{\otimes n_2} \hat{\Pi}_{\sigma} |\uparrow \rangle \rangle , \tag{24}
\]
where \( \hat{\Pi}_{\sigma} \) is a permutation on the first \( n_1 \) qubits, and \( \sigma \) runs over all the elements of the symmetric group \( S_{n_1} \), and \( |S_{n_1}| \) is the number of elements of symmetric group. Then we can write
\[
F_{n_1,n_2}(\Lambda_c) = \frac{1}{|S_{n_1}|} \langle |\uparrow \rangle | \Lambda_c | \sum_{k=0}^{n_1} \sum_{\sigma} \left( \begin{array}{c} n_1 \\ k \end{array} \right) (1-p)^k p^{n_1-k} \hat{\Pi}_{\sigma} |\uparrow \rangle \langle \uparrow |^{\otimes k} \hat{A}_{N-k} |\uparrow \rangle \rangle , \tag{25}
\]
where \( \hat{A}_k := \int d\mu_V [\hat{V} |\uparrow \rangle \langle \uparrow |]^{\otimes k} \). Defining \( \hat{B}_k \), we carry on the calculation
\[
\hat{B}_k := |\uparrow \rangle \langle \uparrow |^{\otimes k} \otimes \hat{A}_{N-k} \tag{26}
\]
\[
= \sum_{m,m',s,s'} \frac{\delta_{m+s,m'+s'} C^{N-k}_{\frac{1}{2} m, \frac{1}{2} m', \frac{1}{2} m'} C^{N-k}_{\frac{1}{2} n_2, \frac{1}{2} n_2, \frac{1}{2} n_2, \frac{1}{2} s'}}{N-k+1} \langle \frac{N-k-n_2}{2}, m | \langle \frac{N-k-n_2}{2}, m' | \otimes \frac{n_2}{2}, s \rangle \langle \frac{n_2}{2}, s | \rangle . \tag{27}
\]
Note that here we do not need to sum over any multiplicity index for the states \( |\frac{N-k-n_2}{2}, m \rangle \) and \( |\frac{n_2}{2}, s \rangle \), because \( \hat{A}_{N-k} \) is supported on the completely symmetric subspace of \( N-k \) qubits, therefore it is also supported on the tensor product of the completely symmetric subspaces of \( N-k \) of \( n_2 \) qubits, which have multiplicity 1. Writing the first \( n_1 \) qubits in the total angular momentum basis we get
\[
\hat{B}_k = \sum_{m,m',s,s',j,j'} \delta_{m+s,m'+s'} \frac{C^{N-k}_{\frac{1}{2} m, \frac{1}{2} m', \frac{1}{2} m'} C^{N-k}_{\frac{1}{2} n_2, \frac{1}{2} n_2, \frac{1}{2} n_2, \frac{1}{2} s'}}{N-k+1} \langle j_1, k \rangle \langle j_1, k \rangle \langle j_1, k \rangle \otimes \langle \frac{n_2}{2}, s \rangle \langle \frac{n_2}{2}, s | .
\]
here the multiplicity $k$ indicates that we first wrote the $k$ qubits in the total angular momentum basis then we summed it up with $\ket{\frac{N-k-n_2}{2}, m} \bra{\frac{N-k-n_2}{2}, m}$. Schur's lemma implies
\[
\frac{1}{|S_{n_1}|} \sum_{\sigma} \hat{\Pi}_\sigma \ket{j_1, m, k} \bra{j_1, m', k} \hat{\Pi}_\sigma = \sum_{g_{j_1}} \frac{1}{\# g_{j_1}} \ket{j_1, m, g_{j_1}} \bra{j_1, m', g_{j_1}} \delta_{j_1, j_1'},
\]
where $g_{j_1}$ is the index for the multiplicity of $j_1$ and runs over all the possible values for a certain $j_1$, and $\# g_{j_1} = \frac{(n_1)!}{(j_1)!((n_1-j_1+1))!}$. Using Eq. (28) in Eq. (25) we get
\[
F_{n_1, n_2}(\Lambda_c) = \bra{\uparrow} \Lambda_c \sum_{j_1, j_1', j_1, k, g_{j_1}} \frac{1}{\# g_{j_1}} \left( \left( 1 - p \right) p_{n_1-j_1-k} \frac{\delta_{m+s, m'+s'}}{N-k+1} C_{N-k-n_2}^{\frac{N-k}{2}} m, s, s' \right)
\]
\[
= \bra{\uparrow} \Lambda_c \sum_{j_1, j_1', j_1, k, g_{j_1}} \frac{1}{\# g_{j_1}} \left( \left( 1 - p \right) p_{n_1-j_1-k} \frac{\delta_{m+s, m'+s'}}{N-k+1} C_{N-k-n_2}^{\frac{N-k}{2}} m, s, s' \right)
\]
Using the Eq. (8) of the main text we get
\[
F_{n_1, n_2}(\Lambda_c) = \sum_{j_1, j_1', j_1, k, g_{j_1}} \frac{1}{\# g_{j_1}} \left( \left( 1 - p \right) p_{n_1-j_1-k} \frac{\delta_{m+s, m'+s'}}{N-k+1} C_{N-k-n_2}^{\frac{N-k}{2}} m, s, s' \right)
\]
and the dependence of the coefficients of $W_{j_1' j_1', g_{j_1}}$ on the multiplicity index $g_{j_1}$ is only through $j_1$. So, we can define
\[
C_{q, j_1(p)} := \sum_{j_1, j_1', j_1, k} \left( \left( 1 - p \right) p_{n_1-j_1-k} \frac{\delta_{m+s, m'+s'}}{N-k+1} C_{N-k-n_2}^{\frac{N-k}{2}} m, s, s' \right)
\]
and write the fidelity as
\[
F_{n_1, n_2}(\Lambda_c) = \sum_{j_1, j_1', j_1, q} C_{q, j_1(p)} \frac{1}{\# g_{j_1}} W_{j_1' j_1', q, g_{j_1}}
\]
Because $\Omega_{n_2, n_2}$ is symmetric on the first $n_1$ qubits, we can always choose $\Lambda_c$ to be symmetric on the first $n_1$ qubits, therefore
\[
\Lambda_c[\hat{\rho}] = \frac{1}{|S_{n_1}|} \sum_{\sigma} \Lambda_c[\hat{\Pi}_\sigma \hat{\Pi}_\sigma^*] .
\]
Using (28) we derive
\[
W_{j_1' j_1', q, g_{j_1}, g_{j_1}} = \frac{1}{\# g_{j_1}} \sum_{g_{j_1}} W_{j_1' j_1', q, g_{j_1}, g_{j_1}}
\]
therefore
\[
W_{j_1' j_1', g_{j_1}, g_{j_1}} = W_{j_1' j_1', q, g_{j_1}, g_{j_1}} \forall g_{j_1}, g_{j_1'}
\]
so defining $W_{j_1' j_1', q, g_{j_1}, g_{j_1}} := \frac{1}{\# g_{j_1}} \sum_{g_{j_1}} W_{j_1' j_1', q, g_{j_1}, g_{j_1}}$, then we get
\[
F_{n_1, n_2}(\Lambda_c) = \sum_{j_1, j_1', j_1, q} C_{q, j_1(p)} W_{j_1' j_1', q, g_{j_1}}
\]
Now, the number of parameters i.e. $W_{q,j,j'}^{s,j,s,j'}$, scale polynomially with $n_1$ and $n_2$ because the multiplicity index is fixed to be $j_1$ and the number of different $j_1$ is $O(n_1)$. Without using the characterisation of covariant channels and writing $\Omega_{n_1,n_2}$ in the proper form, the number of parameters grows exponentially in $n_1$ and $n_2$. This exponential reduction of parameters makes the numerical optimisation feasible. In fact, this optimisation problem is exactly a semidefinite programming optimisation. To show this we first briefly review the semidefinite programming and then we define the parameters in the program.

A general semidefinite program can be defined as any mathematical program of the form

$$\max_{X \in \mathbb{S}^n} F_{n_1,n_2} (\hat{X}) = \text{Tr} \left[ \hat{C}^T \hat{X} \right]$$

subject to $\text{Tr} \left[ \hat{D}_k^T \hat{X} \right] \geq b_k$, $k = 1, \ldots, m$, and $\hat{X} \geq 0$

where $\mathbb{S}^n$ is the space of all real $n \times n$ matrices. $\hat{C}$ and $\hat{D}_k$ are $n \times n$ real matrices, and $b_k$ are real numbers and $\hat{X} \geq 0$ means that $\hat{X}$ is semidefinite.

In our problem, $C_{q,j_1}^{j,j'} (p)$ are the matrix elements of $\hat{C}$ which are all real since $C_{q,j_1}^{j,j'} (p)$ is the combination of the Clebsch-Gordan coefficients. Our constraints are equality constraints, each of which can be obtained from two inequalities. The matrix elements of $\hat{X}$ are $W_{q,j,j'}^{s,j,s,j'}$, and the elements of $\hat{D}_k$ and $b_k$ can be read from the coefficients in Eq. (10) of the main text.

To prove that our problem is a semidefinite program we should show that $\hat{X}$ is positive-semidefinite. $\hat{X}$ is positive-semidefinite if and only if there exists a set of vectors like $\{v_i\}$ such that $x_{m,n} = v_i^T v_n$. In the definition of $W_{q,g,g'}^{j,j'}$ in Eq. (21), we have

$$W_{q,g_1,g_2}^{j,j'} := v_{q,g_1}^j \cdot (v_{q,g_2}^{j'})^\dagger,$$

and using Eq. (35) we get

$$W_{q,j_1,j_1'}^{j,j'} := v_{q,j_1}^j \cdot (v_{q,j_1'}^{j'})^\dagger.$$

So, $\hat{X} \geq 0$, and our problem is a semidefinite program.

Note that in our maximisation problem the parameters in general can be complex numbers. However, the matrix elements of $\hat{C}$ are the contraction of Clebsch-Gordan coefficients which are all real, therefore without loss of generality we can assume that $W_{q,j,j'}$ are real.

**Derivation of Eq. (17)**

Here we give explicit derivation of the constraint (15) of the main text. The starting point to observe that by explicit substitution of Eq. (13) into Eq. (14) of the main text we get

$$\sum_{s=\pm \frac{1}{2}} \sum_{q} C_{s,j-m}^{q} C_{s,j'-m}^{q} W_{q,j,j'}^{s,j,s,j'} = \delta_{j,j'} \delta_{g,g'} .$$

Using the following symmetry property of Clebsch-Gordan coefficients

$$C_{j_1,j_2,j_3,j_4}^{M} = (-1)^{j_1-j_2} \sqrt{\frac{2 J + 1}{2 j_2 + 1}} C_{j_1,j_3,j_4,j}^{M} ,$$

we can observe that

$$\sum_{s=\pm \frac{1}{2}} C_{s,j-m}^{q} C_{s,j'-m}^{q} = \sum_{s=\pm \frac{1}{2}} 2 q + 1 \frac{1}{2 j + 1} C_{s,m-s}^{q} C_{s,m-s}^{q} = 2 q + 1 \left( j, m \right| \hat{\Pi}_m \left| j, m \right) = \frac{2 q + 1}{2 j + 1} \delta_{j,j'},$$

where $\hat{\Pi}_m$ is the projector on the the $j_z = m$ eigenspace. It follows hence that Eq. (40) is automatically fulfilled for $j \neq j'$, while for $j = j'$ instead it gives (17) of the main text

$$\frac{2 + 2 j}{1 + 2 j} W_{q,g,g'}^{j,j'} \bigg|_{q=j+j'} \left[ \frac{1}{2} \right] + \frac{2 j}{1 + 2 j} W_{q,g,g'}^{j,j'} \bigg|_{q=j-j'} \left[ \frac{1}{2} \right] = \delta_{g,g'} .$$

Using the definition of $W_{q,j,j'}^{j,j'} := \frac{1}{\# g_j} \sum_{g_j} W_{q,g,j,j'}$, and summing the equations (43) we get

$$\frac{2 + 2 j}{(1 + 2 j)} W_{q,j,j'}^{j,j'} \bigg|_{q=j+j'} \left[ \frac{1}{2} \right] + \frac{2 j}{1 + 2 j} W_{q,j,j'}^{j,j'} \bigg|_{q=j-j'} \left[ \frac{1}{2} \right] = 1. $$
Application of the formalism to the case $n_1 = n_2 = 1$

For $n_1 = n_2 = 1$, Eq. (18) of the main text explicitly yields

$$\hat{\Omega}_{1,1} = (1 - p) |\uparrow\rangle \langle \uparrow| \otimes \hat{I}/2 + p \hat{A}_2. \quad (45)$$

Notice that the term $\hat{A}_2$ is invariant under rotations hence it gets mapped by $\Lambda_c$ into a multiple of the identity operator: specifically noticing that $\text{Tr}[\hat{A}_2] = 1$ we have $\Lambda_c[\hat{A}_2] = \int d\mu_U \ U \Lambda[\hat{A}_2] U^\dagger = \hat{I}/2$ which implies $\langle \uparrow| \Lambda_c[\hat{A}_2] |\uparrow\rangle = 1/2$. On the contrary the first contribution to $\hat{\Omega}_{1,1}$ admits the following decomposition

$$|\uparrow\rangle \langle \uparrow| \otimes \hat{I} = |1,1\rangle \langle 1,1| + \frac{|1,0\rangle + |0,0\rangle}{\sqrt{2}}$$

where without loss of generality we identified $|\uparrow\rangle$ with the vector $|1,1\rangle$, and where in the r.h.s. appear states of the total angular momentum basis of two spin $1/2$ (no multiplicity being present). Using Eq. (8) of the main text and the table of Clebsch-Gordan coefficients we can then write

$$\langle \uparrow| \Lambda_c[|\uparrow\rangle \langle \uparrow| \otimes \hat{I}] |\uparrow\rangle = \frac{2}{3} v_{3/2}^0 + \frac{5}{6} v_{1/2}^1 + \frac{1}{2} v_{1/2}^0$$

where we dropped the index $g$ since here is no multiplicity in total angular momentum basis of two qubits. Similarly the constraints (10) of the main text becomes

$$\frac{4}{3} v_{3/2}^0 + \frac{2}{3} v_{1/2}^1 = 1, \quad 2 v_{1/2}^0 = 1. \quad (48)$$

Exploiting this we observe that fidelity of $F_{1,1}(\Lambda)$ for a generic map must fulfil the constraint

$$F_{1,1}(\Lambda) \leq p + \frac{1 - p}{2} \left[ \frac{2}{3} v_{3/2}^0 + \frac{5}{6} v_{1/2}^1 + \frac{1}{2} v_{1/2}^0 \right] \leq 1 - p/2,$$

the first inequality being obtained by forcing $v_{1/2}^0$ and $v_{1/2}^1$ to be collinear, while the second following directly from (48). By comparing this with the lower bound $F_{n_1,n_2}^{(\max)} \geq 1 - p(d - 1)/d$ discussed in the main text for the qubit case (i.e. $d = 2$) this allows us to recover the identity (5) of the main text, i.e.

$$F_{1,1}^{(\max)} = 1 - p/2,$$

the bound being achieved by employing the DN strategy.

Details of the Calculation for $n_1 = 2, n_2 = 1$

Here we present detailed calculation to derive Eq. (18) of the main text. Using the Eq. (30) we can write the fidelity as

$$F_{2,1}(\Lambda) = \frac{p^2}{p} + \frac{5(p(1-p)(3+5p)W_{2,1}^{3,2/3} + (1-p)(33+23p)W_{2,1}^{3,2/3})}{2}$$

$$+ \frac{(5(p-1)(15+11p)W_{2,1}^{3,2/3} + (1-p)(33+23p)W_{2,1}^{3,2/3})}{2} + \frac{(5(p-1)(15+11p)W_{2,1}^{3,2/3} + (1-p)(33+23p)W_{2,1}^{3,2/3})}{2}$$

$$+ \frac{(6-p)(1-p)W_{2,1}^{3,2/3} + (1-p)(6-5p)W_{2,1}^{3,2/3})}{36},$$

using the definition of $W_{g,j_1,1}^{j,j}$ := $\frac{1}{\#g_{j_1}} \sum_{g_{j_1}} W_{g,j_1,1}^{j,j}$ and the definition of $W_{g,j_1,1}^{j,j}$ in Eq. (21) we can write

$$F_{2,1}(\Lambda) = \frac{p^2}{p} + \frac{5(p(1-p)(3+5p)W_{2,1}^{3,2/3} + (1-p)(33+23p)W_{2,1}^{3,2/3})}{2} + \frac{(5(p-1)(15+11p)W_{2,1}^{3,2/3} + (1-p)(33+23p)W_{2,1}^{3,2/3})}{2} + \frac{(6-p)(1-p)W_{2,1}^{3,2/3} + (1-p)(6-5p)W_{2,1}^{3,2/3})}{36},$$

(52)
with constraints:

\[
\frac{5}{4}|v_{2,1}^2|^2 + 3|v_{3,1}^2|^2 = 1, \quad \frac{3}{2}|v_{1,2}^2|^2 + |v_{3,2}^2|^2 = 1,
\]

where \( g = 1, 2 \). Using the constraints we eliminate \( |v_{3,2}^2|^2 \), \( |v_{1,2}^2|^2 \), \( |v_{0,2}^2|^2 \), \( |v_{0,1}^2|^2 \), Eq. (52) becomes

\[
F_{2,1}(\Lambda) = \frac{3-2p(1-p)}{6} + \frac{p(1-p)|v_{1,2}^2|^2}{6} + \frac{(1-p)(3+p)|v_{3,1}^2|^2}{9} + \frac{(1-p)(3+p)|v_{1,1}^2|^2}{9\sqrt{2}} - \frac{(1-p)(6-7p)|v_{1,1}^2|^2}{18}.
\]

The coefficients of \( |v_{1,2}^2|^2 \), \( |v_{3,1}^2|^2 \) are positive everywhere, so to maximise the fidelity we put their maximum values

\[
|v_{1,2}^2|^2 = \frac{2}{3}, \quad |v_{3,1}^2|^2 = \frac{4}{9},
\]

obtaining

\[
F_{2,1}(\Lambda) = \frac{51-4p(7-p)}{54} + \frac{\sqrt{2}(1-p)(3+p)|v_{1,1}^2|^2}{9\sqrt{2}} - \frac{(1-p)(6-7p)|v_{1,1}^2|^2}{18}.
\]

This last expression has to be maximise with respect to \( |v_{1,1}^2|^2 \) considering that, according to the constraint (53) such variable has to belong to the interval \([0, \sqrt{2}/3] \). For the case \( p < \frac{2}{3} \) we take the derivative and put it equal to zero obtaining

\[
|v_{1,1}^2|^2 = \frac{\sqrt{2}(3+p)}{\sqrt{3}(6-7p)},
\]

which belongs to the allowed interval only when \( p < 3/8 \). Accordingly for these values of \( p \) we can use Eq. (56) obtaining

\[
F_{2,1}^{(\text{max})} = \frac{51-4p(7-p)}{54} + \frac{(1-p)(3+p)^2}{2(6-7p)}.
\]

For the case \( 1 > p > 3/8 \) (which incidentally also includes \( 1 > p > 6/7 \)), instead the maximum for (54) is always maximised for the maximum allowed value of \( |v_{1,1}^2|^2 \), i.e. \( |v_{1,1}^2|^2 = \sqrt{2}/3 \) yielding

\[
F_{2,1}^{(\text{max})} = \frac{(1-p)(51+23p)}{54} + \frac{p(1-p)}{3} + \frac{p^2}{2},
\]

which together with (57) gives us (18) of the main text.

**Case** \( n_1 = 2, n_2 = \infty \)

As we argued in the text,

\[
F_{n_1,\infty}^{(\text{max})} = \max_{\Lambda \in \text{CPTP}} \int d\mu(\Lambda) \left| \langle \phi | \Lambda | \psi \rangle \right|^2 = \max_{\Lambda \in \text{CPTP}} \int d\mu(\Lambda) \left| \langle \psi | \Lambda [(1-p) \psi \rangle + \psi \rangle \phi \rangle \rangle^2 | \psi \rangle \right|^2,
\]

This equality is consistent since \( F_{n_1,\infty}^{(\text{max})} \) does not depend on \( |\phi\rangle \) by virtue of the invariance property of the Haar measure, and therefore one can set \( |\phi\rangle = |0\rangle \) without loss of generality. In this case the optimal \( \Lambda \) is not covariant, since it depends on \( |0\rangle \) \( |0\rangle \), but we can still find the maximum fidelity through the standard Kraus representation of \( \Lambda \). For \( n_2 = 2 \), there is no need to distinguish between equivalent representations and the matrix elements \( M_{s,j,m}^{(k)} \) of a set of Kraus operators for \( \Lambda, \hat{M}_k \), satisfy

\[
\left\langle \frac{1}{2}, s | \Lambda | j, m \right| \left| j', m' \right\rangle \left| \frac{1}{2}, s' \right\rangle = \sum_k M_{s,j,m}^{(k)} M_{s',j',m'}^{(k)},
\]

\[
\sum_{s=-1/2}^{1/2} \left\langle \frac{1}{2}, s | \Lambda | j, m \right| \left| j', m' \right\rangle \left| \frac{1}{2}, s \right\rangle = \sum_{s=-1/2}^{1/2} \sum_k M_{s,j,m}^{(k)} M_{s',j',m'}^{(k)} = \delta_{j,j'} \delta_{m,m'}.
\]
The integral in (59) can be written as

$$\int d\mu_U \langle \psi | \Lambda((1-p) | \psi \rangle (\psi + p | 0 \rangle \langle 0 | ^{\otimes 2}) | \psi \rangle =$$

$$= \sum_{s,s',l,m,l',m'} \int d\mu_U \langle \frac{1}{2}, s | D_{\frac{1}{2}, s}^U (\hat{U}) \Lambda [\hat{\rho}_{\text{mix}}^{\otimes 2}(p)]_{l,m,l',m'} | \frac{1}{2}, l' \rangle \langle \frac{1}{2}, m | \otimes | \frac{1}{2}, m' \rangle | \frac{1}{2}, s' \rangle D_{\frac{1}{2}, s'}^U (\hat{U}) | \frac{1}{2}, s' \rangle,$$

$$= \sum_{s,s',l,m,l',m'} \int d\mu_U \hat{\rho}_{\text{mix}}^{\otimes 2}(p)_{l,m,l',m'} D_{\frac{1}{2}, s}^U (\hat{U}) D_{\frac{1}{2}, s'}^U (\hat{U}) \left( \sum_{k,j,l+m} M_{s,j,l+m}^{(k)} M_{s',j',l+m}^{(k)} C_{j,l+m}^{(k)} C_{j',l+m}^{(k)} \right),$$

where

$$\hat{\rho}_{\text{mix}}^{\otimes 2}(p)_{l,m,l',m'} = \left( D_{\frac{1}{2}, s}^U (\hat{U}) D_{\frac{1}{2}, s'}^U (\hat{U}) + \delta_{s,s'} \delta_{l,l'} \delta_{m,m'} \right).$$

After performing the integrations the result is

$$\int d\mu_U \langle \psi | \Lambda((1-p) | \psi \rangle (\psi + p | 0 \rangle \langle 0 | ^{\otimes 2}) | \psi \rangle =$$

$$= \frac{p(1-p)}{3} \Sigma_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2 + \frac{(1-p)^2}{6} \Sigma_k |M_{\frac{1}{2}, 0, 0}^{(k)}|^2 + \frac{8(1-p) - 5(1-p)^2}{12} \Sigma_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2 +$$

$$+ \frac{4(1-p) - 3(1-p)^2}{12} \Sigma_k |M_{\frac{1}{2}, 1, 0}^{(k)}|^2 + \frac{(1-p)^2}{4} \Sigma_k |M_{\frac{1}{2}, 1, -1}^{(k)}|^2 + \frac{(1-p)^2}{12} \Sigma_k |M_{\frac{1}{2}, 1, -1}^{(k)}|^2 +$$

$$+ \frac{(1-p)^2}{6 \sqrt{2}} \Sigma_k \text{Re}[M_{\frac{1}{2}, 0, 0}^{(k)} M_{\frac{1}{2}, 1, 1}^{(k)}] + \frac{1 - p}{6} \frac{\text{Re}[M_{\frac{1}{2}, 0, 0}^{(k)} M_{\frac{1}{2}, 1, 1}^{(k)}]}{\sqrt{2}}. \quad (64)$$

Using the constraints (10) and the positivity and magnitude of the coefficients most of the optimal parameter choices can be found:

$$\Sigma_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2 = 1, \quad \Sigma_k |M_{\frac{1}{2}, 0, 0}^{(k)}|^2 = 0, \quad \Sigma_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2 = 1,$$

$$\Sigma_k |M_{\frac{1}{2}, 1, 0}^{(k)}|^2 = 0, \quad \Sigma_k |M_{\frac{1}{2}, 1, -1}^{(k)}|^2 = 1, \quad \Sigma_k |M_{\frac{1}{2}, 1, -1}^{(k)}|^2 = 0.$$

Moreover, using the Cauchy-Schwartz inequality

$$|\text{Re}[M_{\frac{1}{2}, 0, 0}^{(k)} M_{\frac{1}{2}, 1, 1}^{(k)}]| \leq \sqrt{\sum_k |M_{\frac{1}{2}, 0, 0}^{(k)}|^2} \sqrt{\sum_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2} = \sqrt{\sum_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2},$$

$$|\text{Re}[M_{\frac{1}{2}, 1, 0}^{(k)} M_{\frac{1}{2}, 1, 1}^{(k)}]| \leq \sqrt{\sum_k |M_{\frac{1}{2}, 1, 0}^{(k)}|^2} \sqrt{\sum_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2} = \sqrt{1 - \sum_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2}, \quad (65)$$

one is left with the maximisation of a function of the variable $t := \sqrt{\sum_k |M_{\frac{1}{2}, 1, 1}^{(k)}|^2}$.

$$\frac{(1-p)(1+p)}{6 \sqrt{2}} (1 - t^2) + \frac{1-p}{6} t^2 + \frac{(1-p)^2}{6 \sqrt{2}} t + \frac{(1-p)(1+p)}{6 \sqrt{2}} \sqrt{1-t^2}. \quad (66)$$

The solution and the maximal value of the fidelity can be analytically determined, but they are quite cumbersome and we do not report them: instead we present the numerical plot in Fig. 2 of the main text.

**Upper Bound on Measurement and Prepare Protocols**

We have already observed that in the limit of large $n_1$ and $n_2$, MP protocols allows for optimal average fidelity. But what happens for finite number of copies? To answer this question we introduce an upper bound on the average fidelity.
attainable with MP protocols. Indeed, invoking once more the fact that for characterizing optimal performances one can restrict the analysis to transformations which are symmetric under the permutation of the first $n_1$ qubits. Using Eq. (25), the associated fidelity can be written as

\[
F_{n_1,n_2}(\Lambda_{MP}) := \sum_{k=0}^{n_1} \binom{n_1}{k} (1-p)^k p^{n_1-k} \int d\mu_U \langle \psi | \Lambda_{MP} (|\psi\rangle \langle \psi| \otimes \hat{A}_{N-k}) |\psi\rangle ,
\]

where now $\Lambda_{MP}$ is the optimal MP channel. Then, we can get the following upper bound by using an optimal MP for each independent part of the whole state

\[
F_{n_1,n_2}(\Lambda_{MP}) \leq \sum_{k=0}^{n_1} \binom{n_1}{k} (1-p)^k p^{n_1-k} \times \int d\mu_U \langle \psi | \Lambda_{k_{MP}} (|\psi\rangle \langle \psi| \otimes \hat{A}_{N-k}) |\psi\rangle ,
\]

where $\Lambda_{k_{MP}}$ is the optimal MP choice for $|\psi\rangle \langle \psi| \otimes k$. Using the known result for tomography of pure states we can then derive the following inequality

\[
F_{n_1,n_2}(\Lambda_{MP}) \leq \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{k+1}{k+2} (1-p)^k p^{n_1-k},
\]

where $\frac{k+1}{k+2}$ is the average fidelity in the optimal tomography of $k \geq 0$ copies of a pure state. Notice that the right-hand-side quantity does not depend explicitly on $n_2$, and that for $n_1 = 2$ reduces to the function (19) of the main text which we reported in Fig. 1.

**Performance of the Cirac, Ekert, Macchiavello (CEM) protocol as a subtracting machine**

Here we show that a direct application of the method of Ref. [22] to solve our problem for $n_1 = 2$ and arbitrary $n_2$ leads to the same average fidelity as the DN strategy, being hence sub-optimal for our purposes.

The method presented in Ref. [22] does not assume the possibility of operating on the noise signal, therefore the average fidelity one can achieve in this case does not depend on $n_2$. For case $n_1 = 2$, it consists of two steps first performing an orthogonal measurement on the system that discriminate the completely symmetric from the antisymmetric subspace of two qubits, and then tracing out on of the qubits. Adopting this procedure from Eq. (4) of the main text we get

\[
F_{n_1=2}^{CEM} = \int d\mu_U d\mu V \langle \psi | \Lambda_{CEM} (|\rho_{mix}(p)\rangle \langle \rho_{mix}(p)|) |\psi\rangle
\]

\[
= \int d\mu_U d\mu V \langle \psi | \Lambda_{CEM} [(1-p)^2 |\psi\rangle \langle \psi| + p^2 |\phi\rangle \langle \phi| + (1-p) |\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi| + |\phi\rangle \langle \phi| \otimes |\psi\rangle \langle \psi|)|\psi\rangle .
\]

Taking the integral on $\phi$ and using the fact that the method [22] is also covariant we can carry on the calculation

\[
F_{n_1=2}^{CEM} = (1-p)^2 + p^2 \int d\mu V \langle \psi | \Lambda_{CEM} [p(1-p) |\psi\rangle \langle \psi| \otimes \hat{f} + \hat{f} \otimes |\psi\rangle \langle \psi|)|\psi\rangle
\]

\[
= (1-p)^2 + p^2 + (0 |\Lambda_{CEM} [p(1-p) (0 |0 \rangle \langle 0| + \hat{f} \otimes |0\rangle |0\rangle)|0\rangle = 1 - \frac{p}{2} .
\]

where in the last inequality we use the fact that, as anticipated, $\Lambda_{CEM}$ consists in performing the measurements on the symmetric and antisymmetric subspace. We notice hence that $F_{n_1=2}^{CEM}$ exactly coincides with the fidelity one would get by simply adopting the DN strategy, i.e. $F_{n_1=2}^{CEM} = F_{n_1,n_2}^{DN}$ which is clearly not optimal in our case.