Online Learning in Fisher Markets with Unknown Agent Preferences

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Abstract

In a Fisher market, agents (users) spend a budget of (artificial) currency to buy goods that maximize their utilities, and producers set prices on capacity-constrained goods such that the market clears. The equilibrium prices in such a market are typically computed through the solution of a convex program, e.g., the Eisenberg-Gale convex program, that aggregates users’ preferences into a centralized social welfare objective. However, the computation of equilibrium prices using convex programs assumes that all transactions happen in a static market wherein all users are present simultaneously and relies on complete information on each user’s budget and utility function.

Since, in practice, information on users’ utilities and budgets is unknown and users tend to arrive over time in the market, we study an online variant of Fisher markets, wherein budget-constrained users enter the market sequentially. We focus on the setting where users have linear utilities with privately known utility and budget parameters drawn i.i.d. from a distribution $D$. In this setting, we develop a simple yet effective algorithm to set prices that preserves user privacy while achieving a regret and capacity violation of $O(\sqrt{n})$, where $n$ is the number of arriving users and the capacities of the goods scale as $O(n)$. Here, our regret measure represents the optimality gap in the objective of the Eisenberg-Gale convex program between the online allocation policy and that of an offline oracle with complete information on users’ budgets and utilities. To establish the efficacy of our approach, we show that even an algorithm that sets expected equilibrium prices with perfect information on the distribution $D$ cannot achieve both a regret and constraint violation of better than $\Omega(\sqrt{n})$. Finally, we present numerical experiments to demonstrate the performance of our approach relative to several benchmarks.

1 Introduction

In resource allocation applications, central planners face the problem of allocating resources to self-interested agents (users) who have heterogeneous preferences over the resources. While a utilitarian allocation, wherein the sum of the utilities of all users is maximized, achieves a system efficient outcome, such an allocation might not be implementable since it may discriminate against some users, who may accrue little or no utility, to maximize the utilitarian objective [1]. As a result, there has been a growing interest in defining an appropriate objective to allocate resources to users in a manner that is fair [2] across a wide range of applications, including vaccine allocation [3], traffic routing on road networks [4, 5], and the allocation of computational resources [6]. One such widely studied objective in the context of fair resource allocation is the (weighted) Nash social welfare objective, which entails maximizing the (weighted) geometric mean of the utilities of all users [7, 8].

While the (weighted) Nash social welfare objective has a range of desirable properties [9, 10], its appeal for fair resource allocation applications stems from its close connection to Fisher markets, which is one of the most fundamental models for resource allocation [11]. In Fisher markets, a central planner sets prices on capacity-constrained goods while each buyer spends a budget of (possibly artificial) currency to purchase a bundle of goods that maximizes their utility given the set prices. If the prices are set such that the market clears, i.e., all goods are sold when agents purchase their most favorable bundle of goods, then the
corresponding outcome is referred to as a *market equilibrium*. It is well known for a broad range of utility functions that the market equilibrium can be computed by maximizing the (weighted) Nash social welfare objective\(^1\) through the Eisenberg-Gale convex program \([12]\). While the solution to this convex program can be computed in polynomial time \([13, 14]\), computing equilibrium prices using the Eisenberg-Gale convex program relies on complete information on the users’ utilities and budgets and assumes that all transactions happen in a static market. In practice, however, information on users’ utilities and budgets is typically not known, and buyers tend to arrive sequentially into the market rather than being present all at once.

As a result, in this work, we study a generalization of Fisher markets to the setting of online user arrival wherein the utilities and budgets of users are unknown to the central planner. In particular, we consider the setting wherein users, with privately known utility and budget parameters, arrive sequentially in the market and purchase their most favorable bundle of goods given the set prices. While our setting models an inherently distributed market, wherein users make independent decisions on the goods they wish to purchase, this setting markedly differs from prior tatonnement \([15, 16]\) or proportional response \([17, 18]\) approaches to distributively setting prices in Fisher markets. In particular, tatonnement and proportional response approaches for Fisher markets typically involve a simulated setting wherein users repeatedly interact in the market, which enables the market converge to an equilibrium. On the other hand, our work models an online incomplete information setting, which closely resembles a real market wherein users arrive sequentially and do not have to repeatedly interact to enable the central planner to learn equilibrium prices. In this online incomplete information setting, we develop a novel algorithmic approach to adjust the prices of goods in the market when a user arrives solely based on the past observations of user consumption. Since user consumption is directly observable by a central planner, our algorithm helps maintain user privacy.

### 1.1 Our Contributions

In this work, we study the problem of setting equilibrium prices in an online variant of Fisher markets wherein budget-constrained users, with privately known utility and budget parameters, arrive into the market sequentially. In particular, we focus on the setting when users have linear utilities and their budget and utility parameters are independently and identically (i.i.d.) distributed according to some unknown probability distribution \(D\). Since the Eisenberg-Gale convex program \([12]\) cannot be directly solved to set market-clearing prices in this setting, we consider the problem of learning prices online to minimize (i) regret, i.e., the optimality gap in the objective of the Eisenberg-Gale convex program between the online allocation and that of an offline oracle with complete information on users’ budget and utility parameters, and (ii) constraint violation, i.e., the norm of the excess demand for goods beyond their capacity. For a detailed discussion on these performance measures, we refer to Section 3.

We develop a simple yet effective approach to set prices in the market that does not rely on any information on users’ utility and budget parameters, i.e., it preserves user privacy, and show that the regret and constraint violation of this algorithm is \(O(\sqrt{n})\), where \(n\) is the number of arriving users. We establish this result in the setting when the capacities of the goods scale with the number of users, i.e., the good capacities \(c_j = O(n)\) for all goods \(j\), which establishes a vanishing ratio between the constraint violation of our algorithm and the corresponding good capacities. Furthermore, we show that an algorithm that sets expected equilibrium prices with complete information on the distribution \(D\) from which users’ utility and budget parameters are drawn cannot achieve both a regret and constraint violation of better than \(\Omega(\sqrt{n})\). As a result, our online learning algorithm preserves user privacy while achieving an expected regret and constraint violation, up to constants, that is no more than that of an equilibrium pricing approach with complete information of the distribution \(D\).

We then evaluate the performance of our algorithm through numerical experiments, which validate our theoretical regret and constraint violation guarantees. Furthermore, we compare our algorithm to two benchmarks that have access to additional information on users’ utility and budget parameters, which elucidates a fundamental trade-off between regret and capacity violation. Finally, we also numerically compare our approach of adjusting prices through an additive price update step to an analogous approach wherein the prices are instead updated through a multiplicative update at each step.

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\(^1\)In the special that all buyers have the same budgets, the market equilibrium outcome maximizes the (unweighted) Nash social welfare objective and is known as a competitive equilibrium from equal incomes (CEEI) \([9]\).
Organization: This paper is organized as follows. Section 2 reviews related literature. Then, in Section 3, we present our model and performance measures to evaluate the efficacy of an algorithm in allocating goods to users online. We then introduce our privacy-preserving algorithm and bounds on its regret and constraint violation in Section 4 and evaluate its efficacy through numerical experiments in Section 5. Finally, we conclude the paper and provide directions for future work in Section 6.

2 Literature Review

Online resource allocation problems have been widely studied in the operations research and computer science communities, having found applications in domains including online advertising [19] and revenue management [20]. One of the most well-studied classes of online allocation problems is online linear programming (OLP), wherein columns of the constraint matrix and corresponding coefficients in the linear objective function are revealed sequentially to an algorithm designer. While the traditional approach to OLP problems has been to develop guarantees for adversarial inputs [21, 22], the often exceedingly pessimistic worst-case guarantees have prompted the study of beyond worst-case methodologies [23, 24] for such problems [25, 26].

The beyond worst-case approaches for OLP problems predominantly constitute the design and analysis of algorithms under (i) the random permutation and (ii) the stochastic input models. In the random permutation model, the constraints and objective coefficients arrive according to a random permutation of an adversarially chosen input sequence. In this context, [27] develop a two-phase algorithm, including model training on a small fraction of the input sequence and then using the learned parameters to make online decisions on the remaining input sequence. A dynamic variant of this approach was applied in the context of revenue management problems in [28]. On the other hand, in the stochastic input model, the input sequence is drawn i.i.d. from some potentially unknown probability distribution. In this setting, [29] study a general class of OLP models for which they investigate the convergence of the dual price vector and design algorithms using LP duality to obtain logarithmic regret bounds. Since the algorithms developed in [29] involve solving an LP at specified intervals, [30] developed a gradient descent algorithm wherein the dual prices are adjusted solely based on the allocation to users at each time step. More recently, [31] devised an adaptive allocation algorithm with a constant regret bound when the samples are drawn i.i.d. from a discrete probability distribution. As with some of the above works, we develop algorithms for online Fisher markets under the stochastic input model; however, in contrast to these works that assume a linear objective function, we develop regret guarantees under a non-linear concave objective.

Since non-linear objectives tend to arise in a range of online resource allocation problems, there has been a growing interest in the study of online convex optimization (OCO) [32]. In this context, [33] study a general class of OCO problems with concave objectives and convex constraints and develop dual-based algorithms with near-optimal regret guarantees under both the random permutation and stochastic input models. More recently, [34] extended the work of [33] by developing an online mirror descent algorithm that achieves sub-linear regret guarantees by ensuring that enough resources are left until the end of the time horizon. In line with these works, we also study an online resource allocation problem with a concave objective, which, in the Fisher market context, is the budget-weighted log utility objective [12], i.e., the sum of the logarithm of the utilities of all users weighted by their budgets. However, unlike [34] that assumes non-negativity and boundedness of the concave objective, we make no such assumption since the Fisher social welfare objective we consider involves a logarithm, which can be both unbounded and non-negative.

Online variants of Fisher markets have also been considered in the literature. For instance, [35, 36, 37] study the setting when goods arrive sequentially into the market and must be allocated irrevocably upon their arrival to a fixed set of users. While these works investigate this repeated allocation setting in Fisher markets from competitive [35, 36] and regret analysis [37] perspectives, Gorokh et al. [38] establish a connection between this setting and first-price auctions. In particular, [38] show that if the sequentially arriving goods are allocated using first-price auctions, then every user can obtain a constant fraction of their ideal utility irrespective of other user’s strategies. In contrast to these works that focus on the repeated allocation setting wherein goods arrive online, we consider the setting wherein agents enter the market sequentially and purchase a fixed set of limited resources.

In the context of online user arrival in Fisher markets, [39] study the problem of allocating a fixed set of limited resources to a random number of individuals that arrive over multiple rounds. While we also
consider the setting of online user arrival in Fisher markets, our work differs from that of [39] in several ways. First, [39] consider a setting where users belong to a finite set of types where all users have the same budgets. However, in the online Fisher market setting studied in this work, users’ preferences can be drawn from a continuous probability distribution, i.e., the number of user types may not be finite, and the budgets of the arriving users may not be equal. Next, we introduce and adopt different metrics to evaluate the performance of an algorithm as compared to that considered in [39]. In particular, our regret metric evaluates the optimality gap in the social welfare objective, i.e., the budget-weighted log utility objective, between the resulting allocation and that of an offline oracle that has complete information on the utilities and budgets of the arriving users. We refer to Section 3 for a detailed discussion of our chosen performance measures. Finally, as opposed to the two-phase algorithms that include a prediction and an optimization step proposed in [39], we develop a dual-based algorithm that adjusts prices in the market solely based on the consumption behavior of users that have previously arrived in the system.

Our algorithmic approach to adjusting the prices of the goods in the market is analogous to price update mechanisms that use information from interactions with earlier buyers to inform pricing decisions for future arriving buyers. Such price update mechanisms that use past user consumption information to guide future pricing decisions have been widely studied in revenue management [40], online load balancing [41], and traffic routing [42].

Finally, our work is also closely related to the line of literature on the design and analysis of artificial currency mechanisms [43, 44]. Such mechanisms have found applications in various resource allocation settings, including the allocation of food to food banks [45], the allocation of students to courses [46], and the allocation of public goods to people [47, 48]. More recently, there has been a growing interest in the design of artificial currency mechanisms for repeated allocation settings [49], which is also the main focus of this paper.

3 Model and Problem Formulation

In this section, we introduce our modeling assumptions and the individual optimization problem of users (Section 3.1), present the Eisenberg-Gale convex program used to compute equilibrium prices (Section 3.2), and performance measures used to evaluate the efficacy of an online pricing algorithm (Section 3.3).

3.1 Preliminaries and Individual Optimization Problem

We study the problem of allocating \( m \) divisible goods to a population of \( n \) users that arrive sequentially over time. Each good \( j \in [m] \) has a capacity \( c_j = n d_j \), where we denote \( c \in \mathbb{R}^m \) as the vector of good capacities and \( d \in \mathbb{R}^m \) as the vector of good capacities per user. Each user \( t \in [n] \) has a budget \( w_t \) of (artificial) currency and to model user’s preferences over the goods, we assume that each user’s utility is linear in their allocations. In particular, for a vector of allocations \( x_t \in \mathbb{R}^m \), where \( x_{ij} \) represents the consumption of good \( j \) by user \( t \), the utility function \( u_t(x_t) : \mathbb{R}^m \rightarrow \mathbb{R} \) is given by \( u_t(x_t) = \sum_{j=1}^{m} u_{ij} x_{ij} \), where \( u_{ij} \) is the utility received by user \( t \) for consuming one unit of good \( j \). That is, for a utility coefficient vector \( u_t \in \mathbb{R}^m \) for user \( t \), the utility of user \( t \) on consuming a bundle of goods \( x_t \) is \( u_t(x_t) = u_t^T x_t \). Then, for a given price vector \( p \in \mathbb{R}^m \) for the goods, the individual optimization problem for user \( t \) can be described as

\[
\begin{align}
\max_{x_t \in \mathbb{R}^m} & \quad u_t^T x_t = \sum_{j=1}^{m} u_{ij} x_{ij}, \\
\text{s.t.} & \quad p^T x_t \leq w_t, \\
& \quad x_t \geq 0, 
\end{align}
\]

where (1b) is a budget constraint, and (1c) are non-negativity constraints. Since each agent’s utility function is linear, a property of the optimal solution of Problem (1a)-(1c) is that given a price vector \( p \), users will only purchase goods that maximize their bang-per-buck. In other words, user \( t \) will purchase an affordable bundle of goods in the set \( S_t(p) = \{ j : j \in \arg \max_{j \in [m]} \frac{u_{ij}}{p_j} \} \).

The prices of the goods in the market that users best respond to through the solution of Problem (1a)-(1c) are set by a central planner whose goal is to set equilibrium prices in the market, defined as follows.
3.2 Offline Allocations Using the Eisenberg-Gale Convex Program

In this section, we present the Eisenberg-Gale convex program used to compute equilibrium prices and the corresponding desirable properties of this convex program.

Eisenberg-Gale Convex Program: In the offline setting, when complete information on the budgets and utilities of all users is known, the central planner can compute equilibrium prices through the dual
variables of the capacity constraints of the following Eisenberg-Gale convex program [12]

\[
\max_{x_t \in \mathbb{R}^m, \forall t \in [n]} U(x_1, \ldots, x_n) = \sum_{t=1}^{n} w_t \log \left( \sum_{j=1}^{m} u_{tj} x_{tj} \right),
\]

\[\text{s.t.} \quad \sum_{t=1}^{n} x_{tj} \leq c_j, \quad \forall j \in [m],\]

\[x_{tj} \geq 0, \quad \forall t \in [n], j \in [m],\]

where (2b) are capacity constraints, (2c) are non-negativity constraints, and the Objective (2a) represents a budget-weighted geometric mean of buyer’s utilities and is closely related to the Nash social welfare objective [8, 52]. If the prices in the market are set based on the dual variables of the capacity Constraints (2b) of Problem (2a)-(2c), then the optimal allocations of each user’s individual optimization Problem (1a)-(1c) can be shown to be equal to that of the social optimization Problem (2a)-(2c) [12]. That is, the dual variables of the capacity Constraints (2b) correspond to an equilibrium price in the market.

Furthermore, we note that the Eisenberg-Gale convex program has several computational advantages that make it practically feasible. In particular, the computational complexity of solving Problem (2a)-(2c) is identical to that of a linear program [53], i.e., Problem (2a)-(2c) can be solved in polynomial time [13, 14] with the same complexity as that of maximizing a linear objective function. Furthermore, if the utility coefficients, budgets, and good capacities are all rational, then the optimal solution of Problem (2a)-(2c) is also rational [54], i.e., an exact computation of the optimal solution of Problem (2a)-(2c) is possible. As a result, the optimal solution of Problem (2a)-(2c) can be efficiently computed.

Properties of the Eisenberg-Gale Convex Program and its Optimal Allocations: In addition to the computational advantages of the Eisenberg-Gale convex program in computing equilibrium prices, the weighted geometric mean objective has several desirable properties compared to optimizing other social welfare objectives. In particular, the weighted geometric mean objective results in an allocation that satisfies both Pareto efficiency, i.e., no user can be made better off without making another user worse off, and envy-freeness, i.e., each user prefers their allocation compared to that of other users. However, other social welfare objectives often only satisfy one of these desirable properties, e.g., the utilitarian welfare (weighted sum of user’s utilities) and egalitarian welfare (maximizing the minimum utility) objectives only achieve Pareto efficiency. Next, the weighted geometric mean objective achieves a natural compromise between the utilitarian and egalitarian objectives [55]. In particular, as compared to the utilitarian allocation, which may be unfair since some users may obtain zero utilities, under Objective (2a) all users receive a strictly positive utility. Furthermore, compared to the egalitarian objective that may result in highly inefficient outcomes through lower cumulative utilities for all users, the geometric mean objective is more robust since it provides a lower bound on the utilitarian objective.

3.3 Performance Measures in Online Setting

While the offline allocations corresponding to the solution of the Eisenberg-Gale convex program have several desirable properties, achieving such allocations will, in general, not be possible in the online setting when the central planner does not have access to information on the utility and budget parameters of users. As a result, in this work, we focus on devising algorithms that achieve good performance relative to an offline oracle that has complete information on users’ utilities and budgets. In particular, we evaluate the efficacy of an online allocation policy through two metrics: (i) expected regret, i.e., the optimality gap in the social welfare Objective (2a) of this allocation policy relative to the optimal offline allocation, and (ii) expected constraint violation, i.e., the degree to which the goods are over-consumed relative to their capacities. Here the expectation is taken with respect to the distribution \(D\) from which users’ budget and utility parameters are drawn.

Regret: We evaluate the regret of any online algorithm (pricing policy) \(\pi\) through the difference between the optimal objective of Problem (2a)-(2c) and that corresponding to the allocations resulting from the
online pricing policy \( \pi \). For a given set of utility and budget parameters for all users \( t \in [n] \), let \( U^*_{n} \) denote the optimal Objective (2a), i.e., \( U^*_{n} = U(x^*_1, \ldots, x^*_n) \), where \( x^*_1, \ldots, x^*_n \) are the optimal offline allocations corresponding to the solution of Problem (2a)-(2c). Further, let \( x_1, \ldots, x_n \) be the optimal consumption vectors of users given by the solution of the individual optimization Problem (1a)-(1c) for each user under the prices corresponding to the online pricing policy \( \pi \), i.e., for each user \( t \), \( x_t \) is the solution to Problem (1a)-(1c) given the price vector \( p^t \). Then, the social welfare Objective (2a) obtained by the policy \( \pi \) is \( U_n(\pi) = U(x_1, \ldots, x_n) \), and the corresponding expected regret of an algorithm \( \pi \) is given by

\[
R_n(\pi) = E_D [U^*_{n} - U_{n}(\pi)],
\]

where the expectation is taken with respect to the budget and utility parameters \((w, u)\) that are drawn i.i.d. from the distribution \( D \). In the rest of this work, with a slight abuse of notation, we will drop the subscript \( D \) in the expectation since we assume that all expectations are with respect to \( D \).

While the regret measure is defined with respect to the objective of the social optimization Problem (2a)-(2c), we note that regret guarantees derived for Objective (2a) directly translate into corresponding guarantees for the Nash social welfare objective, defined as \( NSW(x_1, \ldots, x_n) = \left( \prod_{t=1}^n u_t(x^*_t) \right)^{\frac{1}{n}} \). In particular, we now present a fundamental connection between the above defined regret measure, which applies to Objective (1a), and the ratio between the Nash social welfare objective of the optimal offline oracle and that corresponding to an online algorithm.

**Remark 1 (Regret and Nash Social Welfare).** We show that if the regret \( U^*_{n} - U_{n}(\pi) \leq o(n) \) for some algorithm \( \pi \), then \( \frac{NSW(x^*_1, \ldots, x^*_n)}{NSW(x_1, \ldots, x_n)} \to 1 \) as \( n \to \infty \). Here, \( x^*_1, \ldots, x^*_n \) are the optimal offline allocations, and \( x_1, \ldots, x_n \) are the optimal consumption vectors given by the solution of Problem (1a)-(1c) under the prices corresponding to the online pricing policy \( \pi \). Without loss of generality, consider the setting when the budgets of all users are equal. Note that if the budgets are not equal, then we can just re-scale the utilities of each user based on their budget. In this setting it holds that

\[
\frac{1}{n} U^*_{n} = \frac{1}{n} \sum_{t=1}^n \log(u_t(x^*_t)) = \frac{1}{n} \log \left( \prod_{t=1}^n u_t(x^*_t) \right) = \log \left( \left( \prod_{t=1}^n u_t(x^*_t) \right)^{\frac{1}{n}} \right),
\]

and \( \frac{1}{n} U_{n}(\pi) = \log \left( \left( \prod_{t=1}^n u_t(x_t) \right)^{\frac{1}{n}} \right) \). Then, it follows that

\[
\frac{NSW(x^*_1, \ldots, x^*_n)}{NSW(x_1, \ldots, x_n)} = \frac{e^{\frac{1}{n} U^*_{n}}}{e^{\frac{1}{n} U_{n}(\pi)}} = e^{\frac{1}{n} U^*_{n} - U_{n}(\pi)} \leq e^{o(n)}. \]

Observe that as \( n \to \infty \), the term \( e^{o(n)} \to 1 \). That is, if the regret of an algorithm \( \pi \) is \( o(n) \), then the ratio of the Nash social welfare objective of the algorithm \( \pi \) approaches that of the optimal offline oracle as \( n \) becomes large.

**Constraint Violation:** We evaluate the constraint violation of algorithm \( \pi \) through the norm of the expected over-consumption of the goods beyond their capacity. In particular, for the consumption bundles \( x_1, \ldots, x_n \) corresponding to the online pricing policy \( \pi \), the vector of excess demands is given by

\[
v(x_1, \ldots, x_n) = \left( \sum_{t=1}^n x_t - c \right)_+, \]

and the corresponding expected norm of the constraint violation is given by

\[
V_n(x_1, \ldots, x_n) = E [\|v(x_1, \ldots, x_n)\|_2].
\]

A few comments about the above regret and constraint violation metrics are in order. First, we reiterate that we define our regret metric with respect to the budget-weighted geometric mean Objective (2a). Our choice of this objective, as opposed to, for example, a linear utilitarian welfare objective, stems from the
superior properties of the Eisenberg-Gale convex program elucidated in Section 3.2 in the context of Fisher markets. Next, observe that the budget-weighted geometric mean Objective (2a) is nonlinear and unbounded. As a result, our regret metric is different from that considered in earlier work in the online linear programming and online convex optimization literature that either assumes a linear objective or a concave objective that is bounded and non-negative. Next, while our constraint violation metric is defined with respect to the $L_2$ norm, by norm-equivalence, any constraint violation guarantees obtained with respect to the $L_p$ norm can be extended to any $p$-norm, e.g., the $L_\infty$ norm. Finally, in our work, we jointly optimize for regret and capacity violation, as in [29], since achieving good performance on either one is typically an easy task. In particular, setting the prices of all goods to be very low will result in low regret but potentially lead to capacity violations because users will be able to purchase the goods at lower prices. On the other hand, setting exceedingly large prices will have the opposite effect, in that the capacity violations will be small but the regret is likely to be high. Due to this fundamental trade-off between regret and capacity violation [29], we focus on optimizing both metrics in this work.

4 Privacy Preserving Algorithm and Regret Guarantees

In this section, we present a privacy-preserving algorithm for online Fisher markets and its corresponding regret and constraint violation guarantees. To this end, to benchmark the performance of our algorithm, we first develop a lower bound on the expected regret and constraint violation of an algorithm that sets expected equilibrium prices with complete information on the distribution $\mathcal{D}$ from which the budget and utility parameters are drawn (Section 4.1). Next, to motivate our algorithm and perform our upper bound regret analysis, we present the dual of the Eisenberg-Gale convex Program (2a)-(2c) (Section 4.2). Then, we present a privacy-preserving algorithm, which follows from performing gradient descent on the dual of the Eisenberg-Gale convex program, that only relies on past observed consumption behavior of users to make future pricing decisions (Section 4.3). Finally, in Section 4.4, we establish an upper bound on both the regret and constraint violation of the privacy-preserving algorithm.

4.1 Lower Bound

In this section, we establish a $\Omega(\sqrt{n})$ lower bound on the expected regret and constraint violation of an algorithm that sets expected equilibrium prices with complete information on the probability distribution of the budget and utility parameters of the arriving users. To this end, we construct an example with $n$ users and two goods, with a capacity of $n$, to show that setting equilibrium prices based on the expected number of user arrivals will result in a constraint violation of $\Omega(\sqrt{n})$. On the other hand, if the central planner desired to satisfy the capacity constraints with high probability, we show that the corresponding expected equilibrium pricing policy will result in a regret of $\Omega(\sqrt{n})$.

**Theorem 1.** Suppose that the budget and utility parameters of users are drawn i.i.d. from a distribution $\mathcal{D}$. Then, there exists a market instance for which the expected constraint violation of an algorithm that sets equilibrium prices based on the expected number of user arrivals, i.e., the algorithm has complete information on the distribution $\mathcal{D}$, is $\Omega(\sqrt{n})$. Furthermore, the corresponding expected equilibrium prices that satisfy the capacity constraints of the goods with high probability will result in a regret of $\Omega(\sqrt{n})$.

**Proof (Sketch).** To prove this claim, we consider a setting with $n$ users with a fixed budget of one and two goods, each with a capacity of $n$. Furthermore, let the utility parameters of users be drawn i.i.d. from a probability distribution, where the users have utility $(1,0)$ with probability 0.5 and a utility of $(0,1)$ with probability 0.5. That is, users either only have utility for good one or for good two, each with equal probability. For this instance, we first derive a tight bound for the expected optimal social welfare objective, and show that $E[U^*] \geq n \log(2) - 1$. Next, we consider the equilibrium prices set by an algorithm that has complete information on the distribution $\mathcal{D}$ from which the utility parameters of users are drawn. In particular, since $\frac{n}{2}$ users of each utility type arrive in expectation, the expected equilibrium pricing policy corresponds to $p^* = (0.5, 0.5)$, which is the price vector shown to all users and meets the capacity constraints in expectation. We then use the central limit theorem to establish that the expected constraint violation under this equilibrium pricing policy is $\Omega(\sqrt{n})$. 

8
If, on the other hand, the central planner desired to satisfy the capacity constraints with high probability, then, using the derived lower bound on the expected optimal social welfare objective, it follows that setting the corresponding expected equilibrium prices using distributional information will result in an expected regret of $\Omega(\sqrt{n})$.

For a complete proof of Theorem 1, see Appendix A. Theorem 1 provides a benchmark for the performance of an online algorithm since it establishes a lower bound on the regret and constraint violation of an expected equilibrium pricing algorithm with perfect information on the distribution from which the utility and budget parameters of users are drawn. That is, even with complete information on the distribution of the budgets and utilities of the arriving users, setting expected equilibrium prices will result in either an expected regret or constraint violation of $\Omega(\sqrt{n})$. We reiterate that our lower bound result in Theorem 1 is not an algorithm independent lower bound but is a lower bound on the regret and constraint violation of an expected equilibrium pricing algorithm with complete knowledge of the distribution $\mathcal{D}$.

In the rest of this section, we develop an algorithm that does not use any distributional information in making pricing decisions and achieves a regret and constraint violation of $O(\sqrt{n})$. In other words, we develop an algorithm that preserves user privacy while matching, up to constants, the corresponding lower bound on the regret and constraint violation of an algorithm with complete distributional information on the utility and budget parameters of the arriving users.

### 4.2 Dual Formulation of Fisher Social Optimization Problem

Letting the price $p_j$ be the dual variable of the capacity Constraint (1b) corresponding to good $j$, the dual of the social optimization Problem (2a)-(2c) is

$$
\min_{p} \frac{1}{n} \sum_{t=1}^{n} w_t \log(w_t) - \sum_{t=1}^{n} w_t \log \left( \min_{j \in [m]} \frac{p_j}{w_{tj}} \right) + \frac{1}{n} \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t. \tag{3}
$$

For a derivation of the above dual using the Lagrangian of the social optimization Problem (2a)-(2c), we refer to Appendix C. We note that the above dual problem is the unconstrained version of the dual problem presented in [56] with the additional terms $\sum_{t=1}^{n} w_t \log(w_t)$ and $-\sum_{t=1}^{n} w_t$ in the objective. Observe that these terms in the objective are independent of the prices and thus do not influence the optimal solution of the dual problem but are necessary to analyze the regret of the algorithm we develop.

Since the utility and budget parameters of users are drawn i.i.d. from the same distribution, we note that the above dual problem can be formulated through the following equivalent sample average approximation (SAA) formulation

$$
\min_{p} D_n(p) = \sum_{j=1}^{m} p_j d_j + \frac{1}{n} \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j}{w_{tj}} \right) - w_t \right) \tag{4}
$$

by diving the dual objective in Problem (3) by the number of users $n$. Recall here that $d_j = \frac{c_j}{n}$. Note that each term in the second summation of the objective of the above problem is independent of each other under the i.i.d. assumption on the utility and budget parameters of users. We mention here that our privacy-preserving algorithm follows as a direct consequence of performing gradient descent on Problem (4) with respect to the prices, as is described in Section 4.3.

### 4.3 Privacy-Preserving Algorithm

In this section, we present an algorithm to dynamically update the prices of the goods in the market when the budget and utility parameters of the arriving users are drawn from an unknown probability distribution. Our algorithm adjusts the prices of the goods in the market when a user arrives based on whether the previous arriving user consumed more or less than their respective market share of each good. In particular, the price of a good $j$ is increased (decreased) if the previous arriving user consumed more (less) than $\frac{c_j}{n}$ units of good $j$. The prices are updated using a step size $\gamma_t$, which we specify in Section 4.4. Finally, in response to a given price vector $p^t$, each user best responds by purchasing an optimal bundle of goods corresponding
to the solution of their individual optimization Problem (1a)-(1c). This process of updating the prices based on the observed optimal consumption of users is presented formally in Algorithm 1.

A few comments about Algorithm 1 are in order. First, the price update step does not require any information on the utilities of users, and thus Algorithm 1 preserves user privacy. Next, Algorithm 1 is practically implementable with low computational overhead since the computational complexity of the price updates is only $O(m)$ at each time a user arrives. Note here that Phase I of Algorithm 1, wherein each arriving user solves their individual optimization problem, is a distributed step, and thus the central planner only incurs a cost when performing the price updates. Finally, we mention that for each user $t$, the price update step follows from performing gradient descent on the $t$th term of the dual Problem (4). In particular, if the optimal consumption set $S^*_t$ for user $t$, given the price vector $p^*_t$, consists of one good, then the sub-gradient of the $t$th term of the dual Problem (4) is given by

$$\partial_p \left( \sum_{j \in [m]} p_j d_j + w_t \log (w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j}{u_{tj}} \right) - w_t \right) \bigg|_{p=p^*_t} = d - x_t,$$

where $x_t$ is an optimal bundle corresponding to the solution of their individual optimization Problem (1a)-(1c) of agent $t$. Note here that $x_{tj^*} = \frac{w_t}{p_{tj^*}}$ for the good $j^*$ in its optimal consumption set $S^*_t$, which is of cardinality one, and $x_{tj} = 0$ for all goods $j \neq j^*$.

Given the above observation on the connection between gradient descent and the price update step, we note that other price update steps could also have been used in Algorithm 1 that are based on mirror descent. For instance, instead of adjusting the prices through an additive update, as in Algorithm 1, prices can be adjusted through a multiplicative update using the following widely studied [16, 34] update rule

$$p^{t+1} \leftarrow p^t e^{-\gamma_t (d - x_t)}.$$ (5)

In Section 5, we present a comparison between the regret and constraint violation of Algorithm 1 with the additive price update step and the corresponding algorithm with a multiplicative price update step through numerical experiments. For our theoretical analysis in Section 4.4, we focus on the additive price update step in Algorithm 1 and defer an exploration of the regret and constraint violation guarantees resulting from the multiplicative price update steps to future research. To this end, we do mention that this mirror descent-based price update rule achieves $O(\sqrt{n})$ regret guarantees in [34] for bounded and non-negative concave utilities. We believe that their analysis can also be extended to the budget-weighted log utility objective, i.e., Objective (2a) that can be negative and is unbounded, studied in this work.

### 4.4 Regret Upper Bound

In this section, we present an $O(\sqrt{n})$ upper bound on both the regret and constraint violation of Algorithm 1. To establish the regret and constraint violation bounds, we make the following assumption on the utility parameters of arriving users.

**Assumption 1** (Distributional Assumption). The support of the utility parameters of users $[u^{\text{min}}, u^{\text{max}}]$ is such that $u^{\text{max}} > 0$ and the distribution $\mathcal{D}$ is such that the probability of observing strictly positive utilities for each good is strictly greater than zero.
We note that Assumption 1 imposes a mild restriction on the set of allowable distributions from which the utility parameters of users are drawn. In particular, the assumption on the utility distribution implies that for each good, there are a certain fraction of the arriving users that have strictly positive utility for it. Note that if this assumption did not hold for certain goods, those goods could be removed from the market since no user wishes to purchase them as they have no utility for it. Assumption 1 ensures that the optimal equilibrium prices of all goods in the market must be strictly positive.

We now present the main result of this work, which establishes an $O(\sqrt{n})$ upper bound on both the regret and expected constraint violation of Algorithm 1.

**Theorem 2** (Regret and Constraint Violation Bounds for Privacy Preserving Algorithm). Suppose that the budget and utility parameters of users are drawn i.i.d. from a distribution $\mathcal{D}$. Furthermore, let $\pi$ denote the online pricing policy described by Algorithm 1 and let $x_1, \ldots, x_n$ be the corresponding allocations for the $n$ agents. In addition, suppose that the price vectors $p^t$ in Algorithm 1 for all users $t \in [n]$ satisfy $0 < p^t \leq \bar{p}$ with probability one, where $\bar{p}$ is a vector of constants, when the step size $\gamma_t = \frac{\bar{p}}{\sqrt{n}} \leq 1$ for all users $t \in [n]$ for some constant $\bar{D} > 0$. Then, under Assumption 1, the regret $R_n(\pi_3) \leq O(\sqrt{n})$, and constraint violation $V_n(\pi_3) \leq O(\sqrt{n})$ with probability one.

**Proof (Sketch).** To establish this result, we proceed in three steps. First, we first prove an $O(\sqrt{n})$ upper bound on the constraint violation for the price update rule in Algorithm 1 by showing that the constraint violation is upper bounded by $O(p \sqrt{n})$, where $p$ is an upper bound on the good prices, as in the statement of the theorem. Then, to establish an upper bound on the expected regret, we use duality (see Section 4.2) to derive a generic upper bound on the regret of any online algorithm as long as the prices $p^t$ are strictly positive and bounded for all users $t \in [n]$, which holds by assumption in the statement of the theorem. Finally, we apply the price update rule in Algorithm 1 and the generic regret upper bound to establish an $O(\sqrt{n})$ upper bound on the regret for a step size $\gamma = \frac{\bar{p}}{\sqrt{n}}$ for some constant $\bar{D} > 0$.

For a complete proof of Theorem 2, see Appendix B. Theorem 2 establishes that both the regret and constraint violation of Algorithm 1 are sub-linear in the number of arriving users $n$. Note that Theorems 1 and 2 jointly imply that Algorithm 1, while preserving user privacy, achieves expected regret and constraint violation guarantees, up to constants, that are no more than that of an expected equilibrium pricing approach with complete information on the distribution $\mathcal{D}$ from which the utility and budget parameters of users are drawn. We also note here that the assumption in Theorem 2 on strictly positive and bounded prices throughout the course of Algorithm 1 follows as a consequence of Assumption 1, which imposes a mild restriction on the utility distribution of the arriving users. In particular, the price of each good will be adjusted downward with probability one if its price is too high since users that have positive utility for goods with lower prices will eventually arrive. Analogously, since a certain fraction of users have strictly positive utility for each good, if the price of a good is too small, that good belongs to the optimal consumption set of such users. Note that the condition on the step size of the price updates ensures that the prices are updated in small increments at each step. This guarantees that users with strictly positive utilities for a particular good will arrive with probability one before its price drops below zero when the number of users $n$ is large. As a result, goods with low prices will eventually be purchased, thereby raising their prices and preventing them from falling below zero. We mention that the numerical experiments also verify the strict positivity and the boundedness of the prices through the operation of Algorithm 1 for appropriately chosen step sizes $\gamma$ (see Section 5).

### 5 Numerical Experiments

In this section, we compare the performance of Algorithm 1 to several benchmarks on both regret and constraint violation metrics. The results of our experiments not only validate the theoretical bounds obtained in Theorem 2 but also demonstrate the efficacy of Algorithm 1 as compared to two benchmarks that have access to additional information on users’ utility and budget parameters. In this section, we introduce two benchmarks to which we compare Algorithm 1 (Section 5.1), present the implementation details of Algorithm 1 and the benchmarks (Section 5.2), and present results to demonstrate the performance of Algorithm 1 (Section 5.3).
5.1 Benchmarks

In our experiments, we compare Algorithm 1 to two benchmarks that have access to additional information on users’ utility and budget parameters. In particular, the first benchmark assumes knowledge of the distribution $\mathcal{D}$ from which the budget and utility parameters are generated, as is the case for the expected equilibrium pricing algorithm in Theorem 1. The second benchmark assumes that users’ utility and budget parameters are revealed to the central planner when they enter the market and can be used to set prices for subsequent users. We mention that these algorithms are solely for benchmark purposes, and thus we do not discuss the practicality of the corresponding informational assumptions of these benchmarks. We also reiterate that, as opposed to these benchmarks, the price updates in Algorithm 1 only rely on the observed consumption of users rather than having access to additional information on their budget and utility parameters.

**Stochastic Program:** We begin with the benchmark wherein the distribution $\mathcal{D}$ from which the budget and utility parameters are generated i.i.d. is known. In this case, the SAA Problem (4) is related to the following stochastic program

$$\min_{p} D(p) = \sum_{j=1}^{m} p_j d_j + \mathbb{E}_{(w,u) \sim \mathcal{D}} \left[ \left( w \log (w) - w \log \left( \min_{j \in [m]} \frac{p_j}{u_j} \right) - w \right) \right],$$

which can be solved to give an optimal price vector $\mathbf{p}^*$. Given this price vector, all arriving users will purchase their most favorable bundle of goods by solving their individual optimization Problem (1a)-(1c).

Note here that the price vector $\mathbf{p}^*$ is computed before the online procedure, which is possible due to the prior knowledge of the distribution $\mathcal{D}$. For numerical implementation purposes, we consider a sample average approximation to compute the expectation in Problem (6).

**Dynamic Learning using SAA:** In this benchmark, we consider the setting wherein users’ budget and utility parameters are revealed to the central planner each time a user arrives. In this context, the prices are set based on the dual variables of the capacity constraints of the sampled social optimization problem with the observed budget and utility parameters of agents that have previously arrived. To improve on the computational complexity of this approach, the dual prices are updated at geometric intervals, as in earlier work [29, 28]. This price vector is then shown to all agents arriving in the corresponding interval, and these agents solve their individual optimization problems to obtain their most favorable goods under the set prices. The pseudo-code for this benchmark is presented formally in Appendix D.

5.2 Implementation Details

To numerically evaluate the performance of Algorithm 1 and the benchmarks, we consider a market setting of $m = 5$ goods, where each good has a capacity of 10 when there are $n$ users in the market. Each arriving user’s budget and utility parameters are generated i.i.d. from a probability distribution $\mathcal{D}$ specified as follows. In particular, each user’s budget can take on one of three values - 2, 5, or 10 - which represent users with low, medium, and high budgets, and a user can have either of these budgets with a probability of $\frac{1}{3}$. Furthermore, each user’s utility for the goods is independent of their budget, and their utility for each good is drawn uniformly at random between the range $[5, 10]$.

Under the above defined market instance, we implement Algorithm 1 using a step size of $\gamma = \gamma_t = \frac{1}{100 \sqrt{n}}$ for all users $t \in [n]$. Furthermore, to implement the stochastic programming benchmark, we compute the solution to the stochastic Program (6) using a sample average approximation with $K = 5000$ samples of budget and utility parameters generated from the aforementioned distribution $\mathcal{D}$ to evaluate the expectation.

Finally, to compare the performance of Algorithm 1 that has an additive price update rule to a corresponding approach with the multiplicative price update rule in Equation (5), we use the above defined market instance with two different step sizes - (i) $\gamma = \frac{1}{\sqrt{n}}$ and (ii) $\gamma = \frac{1}{50 \sqrt{n}}$ - which are used for both the price updates.
5.3 Results

Assessment of Theoretical Bounds: We first assess the theoretical bounds on the regret and constraint violation obtained in Theorem 2. To this end, Figure 1 depicts the infinity norm of the constraint violation (right) and a log-log plot of the regret of Algorithm 1. As expected, the black dots representing the empirically observed regret of Algorithm 1 on the market instance described in Section 5.2 are all very close to the theoretical $O(\sqrt{n})$ upper bound, which is represented by a line with a slope of 0.5 on the log-log plot. On the other hand, the empirical results for the constraint violation (right of Figure 1) of Algorithm 1 indicate that no good capacities are violated. As a result, the $O(\sqrt{n})$ upper bound on the constraint violation is also satisfied.

![Figure 1: Validation of theoretical regret and constraint violation upper bounds of Algorithm 1 on market instance described in Section 5.2. The regret of Algorithm 1 is plotted on a log-log plot (left), and the empirically observed performance, represented by the black dots, is very close to the theoretical $O(\sqrt{n})$ bound, represented by a line of slope 0.5. The infinity norm of the excess demand is zero for all instances and thus trivially satisfies the $O(\sqrt{n})$ upper bound on the constraint violation as well.](image)

Comparisons between Algorithm 1 and Benchmarks: We now compare Algorithm 1 and the two benchmarks on the regret and constraint violation metrics. Figure 2 (left) depicts the ratio of the regret and the optimal offline objective of the three algorithms while Figure 2 (right) depicts the ratio between their constraint violation and the capacities of the goods. From this figure, we observe that while Algorithm 1 incurs a higher regret as compared to the two benchmarks, it does not violate the capacity constraints. On the other hand, both the benchmarks violate the capacity constraints to achieve an overall lower level of regret. This observation highlights the fundamental trade-off between regret and constraint violation as well as the practical viability of Algorithm 1 since it achieves a social welfare efficiency loss of only about 5% for a market with 5000 users while not violating the capacity constraints. We reiterate here that Algorithm 1 achieves the performance depicted in Figure 2 without relying on the additional informational assumptions on the budget and utility parameters of users that the two benchmarks require.

Comparison between the Additive and Multiplicative Price Updates: We now compare Algorithm 1 that has an additive price update step to a corresponding algorithm with a multiplicative price update step, as in Equation (5). To this end, Figures 3 and 4 depict the regret and constraint violation for algorithms with the two price update steps under a step-size of the price update of $\gamma = \frac{1}{\sqrt{n}}$ (Figure 3) and $\gamma = \frac{1}{40\sqrt{n}}$ (Figure 4). We can observe from Figure 3 that for a larger step size of $\gamma = \frac{1}{\sqrt{n}}$, Algorithm 1 with an additive price update rule has a much higher regret and constraint violation as compared to the corresponding algorithm with a multiplicative price update rule. This observation implies the efficacy of the multiplicative price update rule in achieving good regret and constraint violation guarantees and motivates a deeper study of the regret and constraint violation bounds under the multiplicative price update rule.
As opposed to the results obtained for a step-size $\gamma = \frac{1}{\sqrt{n}}$ in Figure 3, we observe from Figure 4 that for a smaller step-size of $\gamma = \frac{1}{50\sqrt{n}}$ the regret of Algorithm 1 is smaller than that of the corresponding algorithm with a multiplicative price update rule. As a result, Figures 3 and 4 show that the choice of the step size $\gamma$ can be critical to the performance of both Algorithm 1 with an additive price update rule and the corresponding algorithm with a multiplicative price update rule.

6 Conclusion and Future Work

In this work, we studied an online variant of Fisher markets wherein users with linear utilities arrive into the market sequentially and have budget and utility parameters drawn i.i.d. from an unknown distribution. While a central planner can derive equilibrium prices by solving a convex optimization problem in classical Fisher markets in the offline setting, such an approach is practically infeasible in many real-world settings where users arrive in the market sequentially with privately known budgets and utilities. As a result, we proposed an online learning approach to set prices on the goods in the market without relying on any
information on each user’s budget and utility parameters. In particular, our algorithm adjusts the prices of the goods at each time a user arrives solely based on the past observations of user consumption, thereby preserving user privacy, and has a computationally efficient price update rule that makes it practically viable. Furthermore, we established that our approach achieves an $O(\sqrt{n})$ (where $n$ is the number of users) upper bound on the expected regret and constraint violation and that an algorithm that sets expected equilibrium prices with complete information on the distribution $D$ cannot achieve both a regret and constraint violation of better than $\Omega(\sqrt{n})$. We established these results in the setting when the capacities of the goods scale with the number of users, i.e., $c_j = O(n)$ for all goods $j$, which implies a vanishing ratio between the constraint violation of Algorithm 1 and the corresponding good capacities. Finally, we used numerical experiments to evaluate the efficacy of our proposed approach relative to several natural benchmarks. Our experiments validated our theoretical regret and constraint violation upper guarantees and highlighted a fundamental trade-off between the regret and constraint violation performance measures.

There are several directions for future research. First, it would be worthwhile to generalize the results obtained in this work to more general concave utility functions, e.g., homogeneous degree one utility functions, beyond linear utilities. Next, it would be valuable to investigate whether the regret and constraint violation guarantees obtained for Algorithm 1 also extend to settings beyond the stochastic input model, e.g., to the random permutation model of user arrival. Finally, it would also be interesting to investigate whether there are bandit problems where the weighted geometric mean social welfare objective is more applicable than the traditionally used linear objectives.

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A Proof of Theorem 1

Consider a setting with $n$ users with a fixed budget of one and two goods, each with a capacity of $n$. Further, let the utility parameters of users be drawn i.i.d. from a probability distribution, where the users have utility $(1, 0)$ with probability 0.5 and a utility of $(0, 1)$ with probability 0.5. That is, users either only have utility for good one or for good two, each with equal probability. For this instance, we first derive a
tight bound for the expected optimal social welfare objective. Next, we show that an expected equilibrium pricing policy results in an expected constraint violation is Ω(√n). If, on the other hand, the central planner desired to satisfy the capacity constraints with high probability, then using the lower bound on the expected optimal social welfare objective we then show that setting the corresponding expected equilibrium prices using distributional information will result in an expected regret of \( \Omega(\sqrt{n}) \).

## A.1 Tight Bound on Expected Optimal Social Welfare Objective

To obtain a bound on the expected optimal social welfare objective, we first find an expression for the objective given the number of arrivals \( s \) of users with the utility \((1, 0)\). To this end, for the defined problem instance, given \( s \) arrivals of users with the utility \((1, 0)\), we have the following offline social optimization problem

\[
\max_{x_{t1}, x_{t2}, \forall t \in [n]} \quad U(s) = s \log(x_{t1}) + (n - s) \log(x_{t2}),
\]  

(7a)

\[
\text{s.t.} \\
\sum_{i=1}^{n} x_{t1} \leq n,
\]  

(7b)

\[
\sum_{i=1}^{n} x_{t2} \leq n,
\]  

(7c)

\[
x_{tj} \geq 0, \quad \forall t \in [n], j \in [2].
\]  

(7d)

If \( 0 < s < n \), then the optimal solution of the above problem is to allocate \( x_t = (\frac{n}{s}, 0) \) to each user \( t \) with a utility of \((1, 0)\) and to allocate \( x_t = (0, \frac{n}{n-s}) \) to each user \( t \) with a utility of \((0, 1)\). In this case, the optimal objective value is given by

\[
U^*(s) = s \ln\left(\frac{n}{s}\right) + (n - s) \ln\left(\frac{n}{n-s}\right) = n \ln(n) - s \ln(s) - (n - s) \ln(n-s).
\]

We now develop a tight bound on the expected optimal objective \( U^*(s) \) using the fact that the number of arrivals \( s \) of users with utility \((0, 1)\) is binomially distributed with a probability of 0.5. That is, we seek to develop a tight bound for

\[
\mathbb{E}[U^*(s)] = \mathbb{E}[n \ln(n) - s \ln(s) - (n - s) \ln(n-s)] = n \ln(n) - \mathbb{E}[s \ln(s)] - \mathbb{E}[(n - s) \ln(n-s)].
\]

To this end, we present an upper bound for \( s \ln(s) \) and \( (n - s) \ln(n-s) \), which will yield a lower bound for \( U^*(s) \).

We begin by observing that the expectation of the binomial random variable is given by \( \mathbb{E}[s] = \frac{n}{2} \) and its variance is \( \mathbb{E}[(s - \frac{n}{2})^2] = \frac{n}{4} \). Next, letting \( \sigma = \frac{1}{\sqrt{n}}(s - \frac{n}{2}) \) we obtain the following upper bound on the term \( s \ln(s) \):

\[
s \ln(s) = s \ln\left(\frac{n}{2} + s - \frac{n}{2}\right),
\]

\[
= s \ln\left(\frac{n}{2} + \frac{2}{n}(s - \frac{n}{2})\right),
\]

\[
= s \ln\left(\frac{n}{2}\right) + s \ln\left(1 + \frac{2}{n}(s - \frac{n}{2})\right),
\]

\[
= s \ln\left(\frac{n}{2}\right) + s \ln(1 + \sigma),
\]

\[
\leq s \ln\left(\frac{n}{2}\right) + s \sigma.
\]

(8)

Observe that \( \sigma \) has zero mean and a standard deviation of \( \frac{1}{\sqrt{n}} \).

Similarly, we obtain the following upper bound for \( (n - s) \ln(n-s) \):

\[
(n - s) \ln(n-s) = (n - s) \ln\left(\frac{n}{2}\right) + (n - s) \ln(1 - \sigma) \leq (n - s) \ln\left(\frac{n}{2}\right) - (n - s)\sigma
\]

(9)
Adding Equations (8) and (9), we have that
\[ s \ln(s) + (n - s) \ln(n - s) \leq n \ln \left( \frac{n}{2} \right) + (2s - n)\sigma = n \ln \left( \frac{n}{2} \right) + n\sigma^2. \]
As a result, it holds that
\[ U^*(s) \geq n \ln(n) - n \ln \left( \frac{n}{2} \right) - n\sigma^2 = n \ln(2) - n\sigma^2 \]
for all \( 0 < s < n \). Next, since \( U^*(0) = n \ln(2) \) and \( U^*(n) = n \ln(2) \), we have that
\[ U^*(s) \geq n \ln(n) - n \ln \left( \frac{n}{2} \right) - n\sigma^2 = n \ln(2) - n\sigma^2 \]
holds for all \( 0 \leq s \leq n \) as \( n\sigma^2 \geq 0 \). As a result, we obtain the following lower bound on the expected optimal social welfare objective:
\[ \mathbb{E}[U^*(s)] \geq n \ln(2) - n\mathbb{E}[\sigma^2] = n \ln(2) - n\mathbb{E} \left( \frac{2}{n} (s - \frac{n}{2}) \right)^2 = n \ln(2) - 1 \]
Finally, using Jensen’s inequality for a concave function, we obtain the following upper bound on the expected optimal social welfare objective:
\[ \mathbb{E}[U^*(s)] \leq U^*(\mathbb{E}(s)) \leq n \ln(2). \]
As a result, we have shown the following tight bound on the expected optimal social welfare objective for the earlier defined instance:
\[ n \ln(2) - 1 \leq \mathbb{E}[U^*(s)] \leq n \ln(2). \]

A.2 \( \Omega(\sqrt{n}) \) bound on Expected Regret and Constraint Violation

**\( \Omega(\sqrt{n}) \) Constraint Violation:** Observe for this instance that in expectation \( \frac{n}{2} \) users of each type arrive. Thus, if the central planner knew the distribution \( D \) of utilities of the arriving users apriori then the expected equilibrium pricing policy used would be \( p^* = (0.5, 0.5) \). Note that setting \( p^* = (0.5, 0.5) \), it holds that all arriving users will purchase two units of their most preferred good yielding a utility of two for each user, and thus result in a social welfare objective of \( n \ln(2) \). Using the tight bound on the expected optimal social welfare objective, it follows that the regret of this pricing policy is \( O(1) \). While setting such equilibrium prices imply that the \( n \) units of both goods will be exactly consumed in expectation, the expected constraint violation of good one (and good two) is given by
\[ v_1 = \mathbb{E}[2(s - \frac{n}{2})_+] = 2\mathbb{E}[(s - \frac{n}{2})_+], \]
which is \( O(\sqrt{n}) \) by the central limit theorem. As a result, the norm of the constraint violation \( \Omega(\sqrt{n}) \). This establishes that setting perfect expected prices that use complete information on the distribution of the utilities of the arriving users has a constraint violation of \( \Omega(\sqrt{n}) \).

**\( \Omega(\sqrt{n}) \) Regret:** Next, we establish that if the central planner seeks to satisfy the capacity constraints with high probability, then the corresponding expected equilibrium pricing policy will result in a regret of \( \Omega(\sqrt{n}) \) on the above defined instance.

In this instance, users with utility \((1, 0)\) arrive with probability \( \frac{1}{\sqrt{n}} \), and thus the total number of users with utility \((1, 0)\) that arrive is binomially distributed as \( X_1 \sim \text{Binomial}(n, \frac{1}{\sqrt{n}}) \). By the symmetry of the problem, the total number of users with utility \((0, 1)\) that arrive is also binomially distributed as \( X_2 \sim \text{Binomial}(n, \frac{1}{\sqrt{n}}) \), where \( X_2 = n - X_1 \). In this setting by standard Chernoff bounds, it holds for any \( 0 \leq t \leq \sqrt{n} \) that
\[ \mathbb{P} \left[ X_1 \geq \frac{n}{2} + t \frac{\sqrt{n}}{2} \right] \leq e^{-t^2/2}. \]
Now, to satisfy the capacity constraints with high probability, i.e., with probability at least $1 - \epsilon$ for any $\epsilon > 0$, we consider a pricing policy that achieves an expected resource consumption of $n - D\sqrt{n}$ for both goods, where $D$ is some constant. Note that the corresponding expected equilibrium price vector that results in $n - D\sqrt{n}$ units of each good being consumed in expectation is $p = \left(\frac{1}{2 - \frac{D}{\sqrt{n}}}, \frac{1}{2 - \frac{D}{\sqrt{n}}}\right)$, since $\frac{n}{2}$ users of each utility type arrive in expectation. Furthermore, note that setting $t = 2\log(\frac{1}{\epsilon})$ and setting $D = 2\log(\frac{1}{\epsilon}) + \eta$, where $\eta > 0$, it holds that the capacity constraint for each resource will not be violated with probability at least $1 - \epsilon$.

Then, using the lower bound on the expected optimal social welfare objective of $\mathbb{E}[U^*_n] \geq n \log(2) - 1$, it follows that

$$
\text{Regret} \geq n \log(2) - 1 - \mathbb{E} \left[ \sum_{t=1}^{n} \log\left(\frac{1}{2 - \frac{2D}{\sqrt{n}}}\right) \right],
\geq n \log(2) - 1 - \mathbb{E} \left[ \sum_{t=1}^{n} \log\left(2 - \frac{2D}{\sqrt{n}}\right) \right],
\geq n \log(2) - 1 - n \log(2) - n \log\left(1 - \frac{D}{\sqrt{n}}\right),
\geq -1 + n D \sqrt{n},
= \Omega\left(\sqrt{n}\right).
$$

This establishes our claim that if the central planner seeks to satisfy the capacity constraints with high probability, then the corresponding expected equilibrium pricing policy will result in a regret of $\Omega(\sqrt{n})$ on the above defined instance.

### B Proof of Theorem 2

To establish this result, we proceed in three steps. First, we first prove an $O(\sqrt{n})$ upper bound on the constraint violation for the price update rule in Algorithm 1. Then, to establish an upper bound on the regret, we establish a generic bound on the regret of any online algorithm as long as the prices $p_t$ are strictly positive and bounded for all users $t \in [n]$. Finally, we apply the price update rule in Algorithm 1 to establish an $O(\sqrt{n})$ upper bound on the regret for $\gamma = \gamma_t = \frac{D}{\sqrt{n}}$ for all users $t \in [n]$ for some constant $\bar{D} > 0$.

**Expected Constraint Violation Bound:** To establish an $O(\sqrt{n})$ upper bound on the constraint violation, we utilize the price update rule in Algorithm 1 where $\gamma_t = \frac{\bar{D}}{\sqrt{n}}$ for some constant $\bar{D} > 0$. In particular, the price update step

$$
p^{t+1}_j = p^{t}_j - \frac{\bar{D}}{\sqrt{n}} (d_j - x_{tj})
$$

in Algorithm 1 can be rearranged to obtain

$$
x_{tj} - d_j = \frac{\sqrt{n}}{D} (p^{t+1}_j - p^{t}_j).
$$

Summing this equation over all arriving users $t \in [n]$, it follows that

$$
\sum_{t=1}^{n} x_{tj} - c_j \leq \frac{\sqrt{n}}{D} \sum_{t=1}^{n} (p^{t+1}_j - p^{t}_j) = \frac{\sqrt{n}}{D} (p^{n+1}_j - p^{1}_j) \leq \frac{\sqrt{n}}{D} p^{n+1}_j \leq \frac{\tilde{p}}{D} \sqrt{n},
$$
where the last inequality follows since $p_j^{n+1} \leq \bar{p}$ by the boundedness assumption on the price vector. Using this relation, the norm of the constraint violation can be bounded as

$$\left\| \sum_{t=1}^{n} x_t - c \right\|_2 \leq \left\| \sum_{t=1}^{n} x_t - c \right\|_2 = \sqrt{\sum_{t=1}^{m} \left( \sum_{j=1}^{n} x_{tj} - c_{j} \right)^2} \leq \sqrt{\sum_{j=1}^{m} \left( \frac{\bar{p}}{D} \right)^2 n} = \sqrt{m \left( \frac{\bar{p}}{D} \right)^2 n} \leq O(\sqrt{n}).$$

Taking an expectation of the above quantity, we obtain a $O(\sqrt{n})$ upper bound on the expected constraint violation, where $\mathbb{E}[V_n(x_1, \ldots, x_n)] \leq \frac{\bar{p}}{D} \sqrt{mn} = O(\sqrt{n})$.

**Generic Bound on the Regret:** We now turn to establishing a generic bound on the regret of any online algorithm for which the price vector $\mathbf{p}^t$ is strictly positive and bounded for each user $t \in [n]$. To perform our analysis, let $\mathbf{p}^*$ be the optimal price vector for the following stochastic program

$$\min_{\mathbf{p}} D(\mathbf{p}) = \sum_{j=1}^{m} p_j d_j + \mathbb{E} \left[ \left( w \log(w) - w \log \left( \min_{j \in [m]} \frac{p_j}{u_{tj}} \right) - w \right) \right]. \quad (10)$$

Then, by duality we have that the primal objective value $U_n^*$ is no more than the dual objective value with $\mathbf{p} = \mathbf{p}^*$, which gives the following upper bound on the optimal objective

$$U_n^* \leq \sum_{j=1}^{m} p_j^* c_j + \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right).$$

Then, taking an expectation on both sides of the above inequality, it follows that

$$\mathbb{E}[U_n^*] \leq \mathbb{E} \left[ \sum_{j=1}^{m} p_j^* c_j + \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right) \right],$$

$$= nD(\mathbf{p}^*),$$

by the definition of $D(\mathbf{p})$ in Problem (10). Finally, noting that $\mathbf{p}^*$ is the optimal solution to the stochastic Program (10), it follows that

$$\mathbb{E}[U_n^*] \leq nD(\mathbf{p}^*) \overset{(a)}{\leq} \sum_{t=1}^{n} \mathbb{E} \left[ D(\mathbf{p}^t) \right] \overset{(b)}{=} \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{j=1}^{m} p_j^* d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right],$$

$$\overset{(c)}{=} \mathbb{E} \left[ \sum_{t=1}^{n} \left( \sum_{j=1}^{m} p_j^* d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right) \right],$$

where (a) follows by the optimality of $\mathbf{p}^*$ for the stochastic Program (10), (b) follows by the definition of $D(\mathbf{p}^t)$, and (c) follows from the linearity of expectations.

Next, let $j_t$ be a good in the optimal consumption set $S_t^*$ for user $t$ given the price vector $\mathbf{p}^t$. Then, the true accumulated social welfare objective under an algorithm $\mathbf{p}$ can be expressed as

$$U_n(\mathbf{p}) = \sum_{t=1}^{n} w_t \log \left( \sum_{j=1}^{m} u_{tj} x_{tj} \right),$$

$$= \sum_{t=1}^{n} w_t \log \left( \sum_{j=1}^{m} u_{tj} 1_{j=j_t} \frac{w_t}{p_j^t} \right),$$

which follows since the utility when consuming any feasible bundle of goods in their optimal consumption set equals their utility when purchasing $\frac{w_t}{p_j^t}$ units of good $j_t \in S_t^*(\mathbf{p}^t)$. Finally combining the upper bound
on the expected optimal objective and the above obtained relation on the accumulated objective under an algorithm $\pi$, we obtain the following upper bound on the expected regret

$$
\mathbb{E}[U_n^* - U_n(\pi)] \leq \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^t}{u_{tj}} \right) \right] - \mathbb{E} \left[ \sum_{t=1}^{n} w_t \log \left( u_{tj} \frac{w_t}{p_j^t} \right) \right],
$$

(11)

$$
= \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t d_j - w_t \right].
$$

(12)

**Square Root Regret Bound:** We now use the generic regret bound derived in Equation (12) for any online algorithm with bounded prices that are always strictly positive for each $t \in [n]$ to obtain an $O(\sqrt{n})$ upper bound on the regret of Algorithm 1. In particular, we use the price update equation in Algorithm 1 to derive the $O(\sqrt{n})$ regret bound. We begin by observing from the price update equation that

$$
\|p^{t+1}\|^2 = \|p^t - \frac{D}{\sqrt{n}}(d - x_t)\|^2.
$$

Expanding the right hand side of the above equation, we obtain that

$$
\|p^{t+1}\|^2 \leq \|p^t\|^2 - \frac{2D}{\sqrt{n}} \left( \sum_{j=1}^{m} p_j^t d_j - \sum_{j=1}^{m} p_j^t x_{tj} \right) + \frac{\bar{D}^2}{2} \|d - x_t\|^2.
$$

We can then rearrange the above equation to obtain

$$
\sum_{j=1}^{m} p_j^t d_j - \sum_{j=1}^{m} p_j^t x_{tj} \leq \frac{\sqrt{n}}{2D} \left( \|p^t\|^2 - \|p^{t+1}\|^2 \right) + \frac{\bar{D}^2}{2\sqrt{n}} \|d - x_t\|^2.
$$

(13)

Finally, summing both sides of the above equation over $t \in [n]$, we get

$$
\sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t d_j - \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t x_{tj} \leq \frac{\sqrt{n}}{2D} \sum_{t=1}^{n} \left( \|p^t\|^2 - \|p^{t+1}\|^2 \right) + \frac{\bar{D}^2}{2\sqrt{n}} \|d - x_t\|^2,
$$

(14)

$$
\leq \frac{\sqrt{n}}{2D} \|p^t\|^2 + \frac{\bar{D}^2}{2\sqrt{n}} \sum_{t=1}^{n} m \left( \max_{j \in [m]} d_j + \bar{w} \right)^2,
$$

(15)

$$
\leq O(\sqrt{n}),
$$

(16)

where the (a) follows by the boundedness of the consumption vector for each agent, since the prices are strictly positive and bounded below by $p > 0$. Finally, noting that all agents completely spend their budget at the optimal solution of the individual optimization problem, i.e., $\sum_{j \in [m]} p_j^t x_{tj} = w_t$, we obtain from the generic regret bound in Equation (12) that

$$
\mathbb{E}[U_n^* - U_n(\pi)] \leq \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t d_j - w_t \right] = \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{j=1}^{m} p_j^t d_j - \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t x_{tj} \right],
$$

$$(a)

\leq \sqrt{n} \left( \frac{\|p^t\|^2}{2D} + \frac{\bar{D} m}{2} \left( \max_{j \in [m]} d_j + \bar{w} \right)^2 \right),
$$

$$
= O(\sqrt{n}),
$$

where (a) follows from Equation (15). Thus, we have proven the $O(\sqrt{n})$ upper bound on the expected regret of Algorithm 1 under the assumed conditions on the price vectors $p^t$ for all users $t \in [n]$. 

\[\square\]
C Derivation of Dual of Social Optimization Problem

In this section, we derive the dual of the social optimization Problem (2a)-(2c). To this end, we first consider the following equivalent primal problem

\[ \max_{x_t \in \mathbb{R}^m, u_t} \quad U(x_1, \ldots, x_n) = \sum_{t=1}^{n} w_t \log(u_t), \]  
\[ \text{s.t.} \quad \sum_{t=1}^{n} x_{tj} \leq c_j, \quad \forall j \in [m], \]  
\[ x_{tj} \geq 0, \quad \forall t \in [n], j \in [m], \]  
\[ u_t = \sum_{j=1}^{m} u_{tj} x_{tj}, \quad \forall t \in [n], \]  

where we replaced the linear utility \( \sum_{j=1}^{m} u_{tj} x_{tj} \) in the objective with the variable \( u_t \) and added the constraint \( u_t = \sum_{j=1}^{m} u_{tj} x_{tj} \). Observe that the optimal solution of this problem is equal to that of the social optimization Problem (2a)-(2c). We now formulate the Lagrangian of this problem and derive the first order conditions of this Lagrangian to obtain the dual Problem (3).

To formulate the Lagrangian of Problem (17a)-(17d), we introduce the dual variables \( p_j \) for each good \( j \in [m] \) for the capacity Constraints (17b), \( \lambda_{tj} \) for each user \( t \in [n] \) and good \( j \in [m] \) for the non-negativity Constraints (17c), and \( \beta_t \) for each user \( t \in [n] \) for the linear utility Constraints (17d). For conciseness, we denote \( p \in \mathbb{R}^m \) as the vector of dual variables of the capacity Constraints (17b), \( \Lambda \in \mathbb{R}^{n \times m} \) as the matrix of dual variables of the non-negativity Constraints (17c), and \( \beta \) as the vector of dual variables of the linear utility Constraints (17d). Then, we have the following Lagrangian:

\[ L((x_t, u_t)_{t=1}^{n}, p, \Lambda, \beta) = \sum_{t=1}^{n} w_t \log(u_t) - \sum_{j=1}^{m} p_j \left( \sum_{t=1}^{n} x_{tj} - c_j \right) - \sum_{t=1}^{n} \sum_{j=1}^{m} \lambda_{tj} x_{tj} - \sum_{t=1}^{n} \beta_t (u_t - \sum_{j=1}^{m} u_{tj} x_{tj}) \]

Next, we observe from the first order derivative condition of the Lagrangian that

\[ \frac{\partial L}{\partial u_t} = \frac{w_t}{u_t} - \beta_t = 0, \quad \forall t \in [n], \]  
\[ \frac{\partial L}{\partial x_{tj}} = -p_j - \lambda_{tj} + \beta_t u_{tj} = 0, \quad \forall t \in [n], j \in [m]. \]

Note that we can rearrange the first equation to obtain that \( u_t = \frac{w_t}{\beta_t} \) for all \( t \in [n] \). Furthermore, by the sign constraint that \( \lambda_{tj} \leq 0 \) for all \( t \in [n], j \in [m] \) it follows from the second equation that \( \beta_t u_{tj} \leq p_j \) for all \( t \in [n], j \in [m] \). Using the above equations, we can write the following dual problem:

\[ \min_{p, \beta} \quad \sum_{t=1}^{n} w_t \log(u_t) - \sum_{t=1}^{n} w_t \log(\beta_t) + \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t \]  
\[ \beta_t u_{tj} \leq p_j, \quad \forall t \in [n], j \in [m] \]

Note that at the optimal solution to the above problem \( \beta_t = \min_{j \in [m]} \left\{ \frac{p_j}{u_{tj}} \right\} \). Using this observation, we can rewrite the above problem as

\[ \min_p \quad \sum_{t=1}^{n} w_t \log(u_t) - \sum_{t=1}^{n} w_t \log \left( \min_{j \in [m]} \frac{p_j}{u_{tj}} \right) + \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t, \]

which is the dual Problem (3).
D Dynamic Learning Algorithm

In this section, we formally present the dynamic learning benchmark introduced in Section 5.1. In this benchmark, the prices $p_t$ at each time a user $t$ arrives is obtained based on the dual variables of the capacity constraints of the sampled social optimization problem with the observed budget and utility parameters of users that have previously arrived. The dual prices are updated at geometric intervals, as in earlier work [28, 29], and this price vector is then shown to all agents arriving in the corresponding interval. Furthermore, each agent solves their individual optimization problem to obtain their most favourable goods under the set prices. This process is presented formally in Algorithm 2.

**Algorithm 2: Dynamic Learning Algorithm**

**Input**: Vector of Capacities $c$

Set $\delta \in (1, 2]$ and $L > 0$ such that $\lfloor \delta^L \rfloor = n$

Let $t_k = \lfloor \delta^k \rfloor$, $k = 1, 2, \ldots, L - 1$ and $t_L = n + 1$

Initialize $p^{t_1} > 0$

Each user $t \in [t_1]$ purchases an optimal bundle of goods $x_t$ by solving Problem (1a)-(1c) given the price $p^{t_1}$

for $k = 1, \ldots, L - 1$ do

**Phase I: Set Price for Geometric Interval**

Set price $p^{t_k}$ based on dual variables of the capacity constraints of the sampled social optimization problem:

$$
\max_{x_t \in \mathbb{R}^m, \forall t \in [t_k]} U(x_1, \ldots, x_{t_k}) = \sum_{t=1}^{t_k} w_t \log \left( \sum_{j=1}^{m} u_{tj} x_{tj} \right), \tag{20a}
$$

s.t.

$$
\sum_{t=1}^{t_k} x_{tj} \leq \frac{t_k c_j}{n}, \quad \forall j \in [m], \tag{20b}
$$

$$
x_{tj} \geq 0, \quad \forall t \in [t_k], j \in [m], \tag{20c}
$$

**Phase II: Each User in Interval Consumes Optimal Bundle**

Each user $t \in \{t_k + 1, \ldots, t_{k+1}\}$ purchases an optimal bundle of goods $x_t$ by solving Problem (1a)-(1c) given the price $p^{t_k}$

end