Tree-Level Amplitudes in $\mathcal{N} = 8$ Supergravity

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Abstract

We present an algorithm for writing down explicit formulas for all tree amplitudes in $\mathcal{N} = 8$ supergravity, obtained from solving the supersymmetric on-shell recursion relations. The formula is patterned after one recently obtained for all tree amplitudes in $\mathcal{N} = 4$ super Yang-Mills which involves nested sums of dual superconformal invariants. We find that all graviton amplitudes can be written in terms of exactly the same structure of nested sums with two modifications: the dual superconformal invariants are promoted from $\mathcal{N} = 4$ to $\mathcal{N} = 8$ superspace in the simplest manner possible—by squaring them—and certain additional non-dual conformal gravity dressing factors (independent of the superspace coordinates) are inserted into the nested sums. To illustrate the procedure we give explicit closed-form formulas for all NMHV, NNHMV and NNNMV gravity super-amplitudes.

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I. INTRODUCTION

The past several years have witnessed dramatic progress in our understanding of gluon scattering amplitudes, especially in the maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory (SYM). These advances have provided a pleasing mix of theoretical insights, shedding light on the mathematical structure of amplitudes and their role in gauge/string duality, and more practical results, including impressive new technology for carrying out previously impossible calculations at tree level and beyond.

It has recently been pointed out [1] that there are reasons to suspect $\mathcal{N} = 8$ supergravity (SUGRA) to have even richer structure and to be ultimately even simpler than SYM. Despite great progress [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 32] however, our understanding of SUGRA amplitudes is still poor compared to SYM, suggesting that we are still missing some key insights into this problem.

Nowhere is the disparity between our understanding of SYM and SUGRA more transparent than in the expressions for what should be their simplest nontrivial scattering amplitudes, those describing the interaction of 2 particles of one helicity with $n-2$ particles of the opposite helicity. In SYM these maximally helicity violating (MHV) amplitudes are encapsulated in the stunningly simple formula conjectured by Parke and Taylor [26] and proven by Berends and Giele [27], which we express here (as throughout this paper) in on-shell $\mathcal{N} = 4$ superspace

$$A^{\text{MHV}}(1, \ldots, n) = \frac{\delta^{(8)}(q)}{(1\ 2)(2\ 3)\cdots(n\ 1)}.$$ (1.1)

In contrast, all known explicit formulas for $n$-graviton MHV amplitudes are noticeably more complicated. The first such formula was conjectured 20 years ago [28] and a handful of alternative expressions of more or less the same degree of complexity have appeared more recently [29, 30, 31, 32].

Beyond MHV amplitudes the situation is even less satisfactory, though the Kawai-Lewellen-Tye (KLT) relations [33] may be used in principle to express any desired amplitude as a complicated sum of various permuted squares of gauge theory amplitudes and other factors. These relations are a consequence of the relation between open and closed string amplitudes, but they remain completely obscure at the level of the Einstein-Hilbert Lagrangian [34, 35].
In this paper we present an algorithm for writing down an arbitrary tree-level SUGRA amplitude. Our result was largely made possible by combining and extending the results of two recent papers. In [36] an explicit formula for all tree amplitudes in SYM was found by solving the supersymmetric version [1, 40] of the on-shell recursion relation [41, 42], greatly extending an earlier solution [43] for split-helicity amplitudes only. We will review all appropriate details in a moment, but for now it suffices to write their formula for the color-ordered SYM amplitude

\[ A(1, \ldots, n) = A_{\text{MHV}}(1, \ldots, n) \sum_{\{\alpha\}} R_{\alpha}(\lambda_i, \tilde{\lambda}_i, \eta_i), \]  

(1.2)

where the sum runs over a collection of dual superconformal [37, 38, 39] invariants \( R_{\alpha} \). The set \( \{\alpha\} \) is dictated by whether \( A \) is MHV (in which case there is obviously only a single term, 1, in the sum), next-to-MHV (NMHV), next-to-next-to-MHV (NNMHV), etc.

Our second inspiration is an intriguing formula for the \( n \)-graviton MHV amplitude obtained by Elvang and Freedman [31] which has the feature of expressing the amplitude in terms of sums of squares of gluon amplitudes, in spirit similar to though in detail very different from the KLT relations. Their formula reads

\[ \mathcal{M}_{n}^{\text{MHV}} = \sum_{\mathcal{P}(2, \ldots, n-1)} [A_{\text{MHV}}(1, \ldots, n)]^2 G_{\text{MHV}}(1, \ldots, n), \]  

(1.3)

where the sum runs over all permutations of the labels 2 through \( n-1 \) and \( G_{\text{MHV}}(1, \ldots, n) \) is a particular ‘gravity factor’ reviewed below.

Our result involves a natural merger of (1.2) and (1.3), expressing an arbitrary \( n \)-graviton super-amplitude in the form

\[ \mathcal{M}_{n} = \sum_{\mathcal{P}(2, \ldots, n-1)} [A_{\text{MHV}}(1, \ldots, n)]^2 \sum_{\{\alpha\}} [R_{\alpha}(\lambda_i, \tilde{\lambda}_i, \eta_i)]^2 G_{\alpha}(\lambda_i, \tilde{\lambda}_i). \]  

(1.4)

Two important features worth pointing out are that the sum runs over precisely the same set \( \{\alpha\} \) that appears in the SYM case (1.2), rather than some kind of double sum as one might have guessed, and that the ‘gravity dressing factors’ \( G_{\alpha} \) do not depend on the fermionic coordinates \( \eta_i^A \) of the on-shell \( \mathcal{N} = 8 \) superspace. All of the ‘super’ structure of the amplitudes is completely encoded in the same \( R \)-factors that appear already in the SYM amplitudes.

We begin in the next section by reviewing some of the necessary tools for carrying out our calculation. In section III we provide detailed derivations of explicit formulas for MHV,
NMHV, and NNMHV amplitudes. Finally in section IV we discuss the structure of the gravity dressing factors $G_\alpha$ for more general graviton amplitudes.

II. SETTING UP THE CALCULATION

A. Supersymmetric Recursion

We will use the supersymmetric version \[1, 40\] of the on-shell recursion relation \[41, 42\]

$$\mathcal{M}_n = \sum_P \int \frac{d^8 \eta}{P^2} \mathcal{M}_L(z_P) \mathcal{M}_R(z_P)$$

(2.1)

where we follow the conventions of \[36\] in choosing the supersymmetry preserving shift

$$\lambda_1(z) = \lambda_1 - z \lambda_n,$$

$$\lambda_\mathcal{M}(z) = \lambda_n + z \lambda_1,$$

$$\eta_\mathcal{M}(z) = \eta_n + z \eta_1,$$

(2.2)

so that the sum in (2.1) runs over all factorization channels of $\mathcal{M}_n$ which separate particle 1 and particle $n$ (into $\mathcal{M}_L$ and $\mathcal{M}_R$, respectively). The value of the shift parameter

$$z_P = \frac{P^2}{[1|P|n]}$$

(2.3)

is chosen so that the shifted intermediate momentum

$$\hat{P}(z) = P + z \lambda_n \lambda_1, \quad P = -p_1 - \cdots = \cdots + p_n$$

(2.4)

go on-shell at $z = z_P$. The recursion relation (2.1) can be seeded with the fundamental 3-particle amplitudes \[1\]

$$\mathcal{M}^{\text{MHV}}_3 = \frac{\delta^{(8)}(\eta[2 3] + \eta_2[3 1] + \eta_3[1 2])}{([1 2][2 3][3 1])^2}, \quad \mathcal{M}^{\text{MHV}}_3 = \frac{\delta^{(16)}(q)}{(\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle)^2}.$$ 

(2.5)

B. Gravity Subamplitudes

Color-ordered amplitudes in SYM have a cyclic structure such that only those factorizations preserving the cyclic labeling of the external particles appear in the analogous recursion (2.1). In contrast, gravity amplitudes must be completely symmetric under the
\[ M_n = \sum_{P(2, \ldots, n-1)} M(1, \ldots, n), \quad (2.6) \]

FIG. 1: A diagrammatic representation of the relation (2.6) between a physical gravity amplitude \( \mathcal{M}_n \) and the sum over its ordered subamplitudes \( M(1, \ldots, n) \). We draw an arrow indicating the cyclic order of the indices between the special legs \( n \) and 1.

exchange of any particle labels, so vastly more factorizations contribute to (2.1). We can deal with this complication once and for all by introducing the notion of an ordered ‘gravity subamplitude’ \( M(1, \ldots, n) \). These non-physical but mathematically useful objects are related to the complete, physical amplitudes \( \mathcal{M}_n \) via the relation

\[ \mathcal{M}_n = \sum_{P(2, \ldots, n-1)} M(1, \ldots, n), \quad (2.6) \]
depicted graphically in Fig. 1. This decomposition only makes a subgroup of the full permutation symmetry manifest. However it is the largest subgroup that the recursion (2.1) allows us to preserve since two external lines are singled out for special treatment.

The relation (2.6) does not uniquely determine the subamplitudes for a given \( \mathcal{M}_n \), since one could add to \( M(1, \ldots, n) \) any quantity which vanishes after summing over permutations. We choose to define the subamplitudes \( M \) recursively via (2.1) restricted to factorizations which preserve the cyclic ordering of the indices, just like in SYM theory:

\[ M(1, \ldots, n) \equiv \sum_{i=3}^{n-1} \int \frac{d^8 \eta}{P^2} M(\hat{1}, 2, \ldots, i-1, \hat{P}) M(-\hat{P}, i, \ldots, n-1, \eta). \quad (2.7) \]

This recursion is also seeded with the three-point amplitudes (2.5) since there is no distinction between \( M(1, 2, 3) \) and \( \mathcal{M}_3 \). Note however that unlike the color-ordered SYM amplitudes \( A(1, \ldots, n) \), the gravity subamplitude \( M(1, \ldots, n) \) is not in general invariant under cyclic permutations of its arguments.

It remains to prove the consistency of this definition. That is, we need to check that the subamplitudes defined in (2.7), when substituted into (2.6), do in fact give correct expressions for the physical gravity amplitude \( \mathcal{M}_n \). This straightforward combinatorics exercise proceeds by induction, beginning with the \( n = 3 \) case which is trivial and then assuming that (2.6) is correct up to and including \( n - 1 \) gravitons. For \( n \) gravitons we then
The first line is the superrecursion for the physical amplitude, including a sum over all partitions of \( \{2, \ldots, n-1\} \) into two subsets \( A \) and \( B \), not just those which preserve a cyclic ordering. In the second line we have thrown in a spurious sum over all permutations of \( \{2, \ldots, n-1\} \) at the cost of dividing by \( (n-2)! \) to compensate for the overcounting. This is allowed since we know that \( \mathcal{M}_n \) is completely symmetric under the exchange of any of its arguments. Inside the sum over permutations we are then free to choose \( A = \{2, \ldots, i-1\} \) and \( B = \{i, \ldots, n-1\} \) as indicated on the third line, including the factor \( \left( \frac{n-2}{i-2} \right) \) to count the number of times this particular term appears. On the fourth line our prior assumption that (2.6) holds up to \( n-1 \) particles allows us to replace \( \mathcal{M}_a \to (a-2)!M_a \) inside the sum over permutations. The last line invokes the definition (2.7) and completes the proof that the physical \( n \)-graviton amplitude may be recovered from the ordered subamplitudes via (2.6) and the definition (2.7).

C. From \( \mathcal{N} = 4 \) to \( \mathcal{N} = 8 \) Superspace

The astute reader may have objected already to (1.3) in the introduction. The SYM MHV amplitude (1.1) involves the delta function \( \delta^{(8)}(q) \) expressing conservation of the total supermomentum

\[
q = \sum_{i=1}^{n} \lambda_{i}^{\alpha} \eta_{i}^{A}, \quad \alpha = 1, 2, \quad A = 1, \ldots, 4.
\]  

Since the square of a fermionic delta function is zero, it would seem that it makes no sense for the quantity \( [A_{\text{MHV}}(1, \ldots, n)]^2 \) to appear in (1.3).
Throughout this paper it will prove extremely convenient to adopt the convention that the square of an $\mathcal{N} = 4$ superspace expression refers to an $\mathcal{N} = 8$ superspace expression in the most natural way. For example, it should always be understood that

$$[\delta^{(8)}(q)]^2 = \delta^{(16)}(q),$$

(2.10)

where the $q$ on the right-hand side is given by the same expression (2.9) but with $A = 1, \ldots, 8$. This notation will prove especially useful for lifting results of Grassmann integration from $\mathcal{N} = 4$ to $\mathcal{N} = 8$ superspace. This trick works because we can break the SU(8) symmetry of a $d^8\eta$ integration into SU(4)$_a \times$ SU(4)$_b$ by taking $\eta_1, \ldots, \eta_4$ for SU(4)$_a$ and $\eta_5, \ldots, \eta_8$ for SU(4)$_b$. Then every $d^8\eta$ integral can be rewritten as a product of two SYM integrals and the SU(8) symmetry of the answer is restored simply by adopting the convention (2.10).

For a specific example consider the basic SYM integral

$$\int \frac{d^4\eta}{P^2} A_{\text{MHV}}^{\text{MHV}}(1, 2, \hat{P}) A_{\text{MHV}}^{\text{MHV}}(-\hat{P}, 3, \ldots, \pi) = \frac{\delta^{(8)}(q)}{(12)(23) \cdots (n1)}$$

(2.11)

which expresses the superrecursion for the case of MHV amplitudes. By ‘squaring’ this formula we immediately obtain the answer for a similar $\mathcal{N} = 8$ Grassmann integral,

$$\int \frac{d^8\eta}{P^2} [A_{\text{MHV}}^{\text{MHV}}(1, 2, \hat{P})]^2 [A_{\text{MHV}}^{\text{MHV}}(-\hat{P}, 3, \ldots, \pi)]^2 = P^2 \frac{\delta^{(16)}(p)}{(12)(23) \cdots (n1))^2}.$$  

(2.12)

Note the extra factor of $P^2$ which appears on the right-hand side because we have, for obvious reasons, chosen not to square the propagator $1/P^2$ on the left.

D. Review of SYM Amplitudes

Given the above considerations it should come as no surprise that we will be able to import much of the structure of SYM amplitudes directly into our SUGRA results. Therefore we now review the results of [36] for tree amplitudes in SYM. Here and in all that follows we use the standard dual superconformal [37, 38, 39] notation

$$x_{ij} = p_i + p_{i+1} + \cdots + p_{j-1},$$

$$\theta_{ij} = \lambda_i \eta_i + \cdots + \lambda_{j-1} \eta_{j-1},$$

(2.13)

where all subscripts are understood mod $n$. 

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We will base our expression for the SUGRA amplitudes on an expression for the SYM amplitudes which is equivalent to, but not exactly the same as the one presented in [36]. The reason is that the cyclic symmetry of the Yang-Mills amplitudes implies certain identities for the invariants $R_\alpha$ appearing in (1.2). This symmetry was used in [36] when solving the recursion relations. Instead it is helpful to have a different expression which is more suitable to the gravity case where the subamplitudes $M$ do not have cyclic symmetry.

To be precise we need to return to the construction of [36] and make sure that when considering the right-hand side of the BCF recursion relation we always insert the lower point amplitudes so that leg 1 of the left amplitude factor corresponds to the shifted leg $\hat{1}$. We also need to have the leg $n$ of the right amplitude factor corresponding to the shifted leg $\hat{n}$, but this was already the choice made in [36].

The expression for all $\mathcal{N} = 4$ SYM amplitudes is given in terms of paths in a particular rooted tree diagram. Here we will be using a different (but equivalent) diagram, shown in Fig. 2. Each vertex in the diagram, say with labels $a_1 b_1; a_2 b_2; \ldots; a_r b_r; ab$, corresponds to a particular dual conformal invariant. These invariants take the general form [36, 39]

$$R_{n; a_1 b_1; a_2 b_2; \ldots; a_r b_r; ab} = \frac{\langle a a - 1 \rangle \langle b b - 1 \rangle \delta^{(4)}(\langle \xi | x_{b_1 a} x_{ab} | \theta_{b_r} \rangle + \langle \xi | x_{b_r b} x_{ba} | \theta_{a_r} \rangle)}{x_{ab}^2 (\xi | x_{b_1 a} x_{ab} | b \langle b - 1 \rangle \langle \xi | x_{b_r b} x_{ba} | a \rangle \langle \xi | x_{b_r b} x_{ba} | a - 1 \rangle)},$$

where the chiral spinor $\xi$ is given by

$$\langle \xi | = \langle n | x_{n a_1} x_{a_1 b_1} x_{b_1 a_2} x_{a_2 b_2} \ldots x_{a_r b_r}.$$

As in [36] this expression needs to be slightly modified when any $a_i$ index attains the lower limit of its range. We indicate by means of a superscript on $R$ the nature of the appropriate modification. Specifically, $R^{1, \ldots, r}_{n; a_1 b_1; a_2 b_2; \ldots; a_r b_r; ab}$ indicates the same quantity (2.14) but with the understanding that when $a$ reaches its lower limit, we need to replace

$$\langle a - 1 \rangle \rightarrow \langle n | x_{n 1} x_{1 2} \ldots x_{r-1 r}.$$

We now have all of the ingredients necessary to begin assembling the complete amplitude, which is given by the formula

$$A_n = A_n^{\text{MHV}} P_n = \frac{\delta^{(8)}(q)}{(1 2) \ldots (n 1)} P_n,$$

In [36] it was also necessary to sometimes take into account modifications when indices reached the upper limits of their ranges, but this feature does not arise in our reorganized presentation of the amplitude.
where $P_n$ is given by the sum over vertical paths in Fig. 2 beginning at the root node. To each such path we associate a nested sum of the product of the associated $R$-invariants in the vertices visited by the path. The last pair of labels in a given $R$ are those which are summed first, these are denoted by $a_p b_p$ in row $p$ of the diagram. We always take the convention that $a_p$ and $b_p$ are separated by at least two ($a_p < b_p - 1$) which is necessary for the $R$-invariants to be well-defined. The lower and upper limits for the summation variables $a_p, b_p$ are indicated by the two numbers appearing adjacent to the line above each vertex.

The differences between the new diagram and the one of [36] are:

1. All pairs of labels in the vertices appear alphabetically in the form $a_i b_i$.

2. The edges on the extreme left of the diagram are labeled by $a_i$ rather than $a_i + 1$, and the summation variables must be greater than or equal to these lower limits $a_i$.

3. The edges on the extreme right of the diagram are labeled by $n$ rather than $n - 1$, and the summation variables must be strictly less than this upper limit $n$.

4. All superscripts on $R$-invariants which detail boundary replacements are left superscripts (i.e. for lower boundaries only). In a given cluster, e.g. the cluster shown in Fig. 3 the superscript associated to the left-most vertex is obtained from the sequence written in the vertex by deleting the final pair of labels and reversing the order of the last two labels which remain. Thus the sequence ends $b_i a_i$ for some $i$. Then proceeding to the right in the cluster, the next vertex has the same superscript, but with alphabetical order of the final pair, i.e. it ends $a_i b_i$. Going further to the right in the cluster one obtains the relevant superscripts by sequentially deleting pairs of labels from the right.

Given the complexity of this prescription it behooves us to illustrate a few cases explicitly. There is one path of length zero, whose value is simply 1 and this corresponds to the MHV amplitudes,

$$P_n^{MHV} = 1.$$  \hfill (2.18)

Then there is one path of length one which gives the NMHV amplitudes. We get $1 \times R_{n,a_1,b_1}$,

summed over the region $2 \leq a_1,b_1 < n$, as always with the convention that $a_i < b_i - 1$. 

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FIG. 2: An alternative rooted tree diagram for tree-level SYM amplitudes. The figure is the same as the tree diagram presented in [36] except that the labels in the vertices appear in a different order, meaning that the $R$-invariants appearing in the amplitude are slightly different. Also the limits, written to the left and right of each line, are treated differently.

FIG. 3: The rule for going from line $p - 1$ to line $p$ (for $p > 1$) in Fig. 2. For every vertex in line $p - 1$ of the form given at the top of the diagram, there are $r + 2$ vertices in the lower line (line $p$). The labels in these vertices start with $u_1 v_1; \ldots u_r v_r; a_{p-1} b_{p-1}; a_p b_p$ and they get sequentially shorter, with each step to the right removing the pair of labels adjacent to the last pair $a_p, b_p$ until only the last pair is left. The summation limits between each line are also derived from the labels of the vertex above. The left superscripts which appear on the associated $R$-invariants start with $u_1 v_1 \ldots u_r v_r b_{p-1} a_{p-1}$ for the left-most vertex. The next vertex to the right has the superscript $u_1 v_1 \ldots u_r v_r a_{p-1} b_{p-1}$, i.e. the same as the first but with the final pair in alphabetical order. The next vertex has the superscript $u_1 v_1 \ldots u_r v_r$ and thereafter the pairs are sequentially deleted from the right.
There are no boundary replacements so we have

$$\mathcal{P}^\text{NMHV}_n = \sum_{2 \leq a_1, b_1 < n} R_{n,a_1 b_1}. \quad (2.19)$$

The two paths of length two give the NNMHV amplitudes. This time we get superscripts on the $R$-invariants as dictated by the rules in point 4 above,

$$\mathcal{P}^\text{NNMHV}_n = \sum_{2 \leq a_1, b_1 < n} R_{n,a_1 b_1}
\left( \sum_{a_1 \leq a_2, b_2 < b_1} R_{n,a_1 b_2}^a b_1
+ \sum_{b_1 \leq a_2, b_2 < n} R_{n,a_2 b_2}^b a_1 \right). \quad (2.20)$$

Continuing to $N^3$MHV amplitudes we find five paths of length three, giving the following nested sums,

$$\mathcal{P}^\text{N}^3\text{MHV}_n = \sum_{2 \leq a_1, b_1 < n} R_{n,a_1 b_1}
\left[ \sum_{a_1 \leq a_2, b_2 < b_1} R_{n,a_1 b_2}^a b_1
\left( \sum_{a_2 \leq a_3, b_3 < b_2} R_{n,a_2 b_3}^a b_2
\left( \sum_{b_2 \leq a_3, b_3 < n} R_{n,a_3 b_3}^b a_2 \right) \right) + \sum_{b_1 \leq a_2, b_2 < n} R_{n,b_2 a_2}^b a_1 \right] \right]. \quad (2.21)$$

These examples hopefully serve to illustrate how to write a general SYM amplitude, though a more thorough discussion may be found in [36].

III. EXAMPLES OF GRAVITY AMPLITUDES

A. MHV Amplitudes

Elvang and Freedman have shown that the $n$-graviton MHV amplitude may be written in the form

$$\mathcal{M}^\text{MHV}_n = \sum_{\mathcal{P}(2, \ldots , n-1)} [A^\text{MHV}(1, \ldots , n)]^2 G^\text{MHV}(1, \ldots , n) \quad (3.1)$$

in terms of

$$G^\text{MHV}(1, \ldots , n) = x_{13}^2 \prod_{s=2}^{n-3} \langle s|x_{s,s+2} x_{s+2,n}|n\rangle \quad (3.2)$$

The formula (3.1) is valid for $n > 3$; $n = 3$ will always be treated as a special case with $G^\text{MHV}(1, 2, 3) = 1$.

\[^2\text{We have relabeled their indices according to } i \rightarrow 2 - i \text{ mod } n \text{ and have expressed the amplitude in } \mathcal{N} = 8 \text{ superspace.}\]
Comparison of (3.1) with (2.6) suggests that we should identify the MHV ordered sub-amplitude as
\[ M_{\text{MHV}}(1, \ldots, n) = [A_{\text{MHV}}(1, \ldots, n)]^2 G_{\text{MHV}}(1, \ldots, n). \] (3.3)
Let us now check that our definition (2.7) yields precisely the same expression for the subamplitude (they may have differed by terms which cancel out when one sums over all permutations in (2.6)).

We will again proceed by induction, assuming that (3.3) satisfies (2.7) for \( n-1 \) and fewer gravitons. To calculate \( M_{\text{MHV}} \) for \( n \) gravitons from the definition (2.7) we first note that only the single term \( i = 3 \) contributes, giving
\[ M_{\text{MHV}}(1, \ldots, n) = \int \frac{d^8 \eta}{P^2} M_{\text{MHV}}(1, 2, \hat{P}) M_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi) \] (3.4)
as shown in Fig. 4. The calculation is rendered essentially trivial by plugging in the relations
\[
M_{\text{MHV}}(1, 2, \hat{P}) = [A_{\text{MHV}}(1, 2, \hat{P})]^2,
\]
\[
M_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi) = [A_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi)]^2 G_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi)
\] (3.5)
between ordered graviton and Yang-Mills amplitudes. The \( G \) factor in (3.4) comes along for the ride as we perform the \( d^8 \eta \) integral using the square of the analogous Yang-Mills calculation as explained above (2.12). Therefore with no effort we find that (3.4) gives
\[ M_{\text{MHV}}(1, \ldots, n) = [A_{\text{MHV}}(1, \ldots, n)]^2 P^2 G_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi). \] (3.6)
A simple calculation using the shift (2.2) now reveals that
\[
P^2 G_{\text{MHV}}(-\hat{P}, 3, \ldots, \pi) = x_{13}^2 (-\hat{P} + p_3)^2 \prod_{s=3}^{n-3} \frac{\langle s|x_{s,s+2|x_{s+2}},|n\rangle}{\langle s\ n\rangle}
\]
\[
= x_{13}^2 \prod_{s=2}^{n-3} \frac{\langle s|x_{s,s+2|x_{s+2}},|n\rangle}{\langle s\ n\rangle}
\]
\[
= G_{\text{MHV}}(1, \ldots, n). \] (3.7)
This completes the inductive proof that the formula (3.3) obtained by Elvang and Freedman is precisely the MHV case of the ordered subamplitudes that we have defined in (2.7).

### B. NMHV Amplitudes

Next we turn our attention to the NMHV amplitude. The two kinds of diagrams which contribute to the recursion are shown in Fig. 5. Let us begin with \( n = 5 \), in which case the first diagram is absent and only the term \( i = 4 \) appears in the sum. According to the definition (2.7) we then have

\[
M_{\text{NMHV}}(1, \ldots, 5) = \int \frac{d^3 \eta}{P^2} M_{\text{MHV}}(\hat{1}, 2, 3, \hat{P}) M_{\text{MHV}}(-\hat{P}, 4, 5) = [A_{\text{NMHV}}(1, \ldots, 5)]^2 P^2 G_{\text{MHV}}(\hat{1}, 2, 3, \hat{P}) \equiv [A_{\text{NMHV}}(1, \ldots, 5)]^2 G_{\text{NMHV}}(1, \ldots, 5). \tag{3.8}
\]

Here, following the example set in the previous subsection, evaluating the Grassmann integral leads to the square of the analogous SYM result, times the gravity factor

\[
G_{\text{NMHV}}(1, \ldots, 5) = P^2 G_{\text{MHV}}(\hat{1}, 2, 3, \hat{P}) = (p_4 + p_5)^2 (p_1 + p_2)^2 = (p_4 + p_5)^2 \frac{|p_3 p_2|}{|41|}. \tag{3.9}
\]

One can check that this result is consistent with the known answer (for example, from the KLT relation).

Let us now turn to the general NMHV case. In the previous section we recalled the SYM result obtained in [36],

\[
A_{\text{NMHV}}(1, \ldots, n) = A_{\text{MHV}}(1, \ldots, n) \sum_{i=2}^{n-3} \sum_{j=i+2}^{n-1} R_{n;ij}. \tag{3.10}
\]

It was shown in [36] that the \( i = 2 \) term in (3.10) corresponds to the sum over MHV \( \times \) MHV diagrams in Fig. 5, while the \( i > 2 \) terms arise iteratively from the \( \overline{\text{MHV}} \times \text{NMHV} \) diagram.
1. Statement

Now we claim that the NMHV gravity subamplitude is given by

\[ M_{\text{NMHV}}^{1, \ldots, n} = \left| A_{\text{MHV}}^{1, \ldots, n} \right|^2 \sum_{i=2}^{n-3} \sum_{j=i+2}^{n-1} R_{n,ij}^2 G_{n,ij}^{\text{NMHV}} \]

(3.11)

where \( R \) is the same dual superconformal invariant \( (2.14) \) as in SYM and the NMHV gravity factor can be split for future convenience into three parts as follows,

\[ G_{n,ab}^{\text{NMHV}} = f_{n,ab} G_{n,ab}^L G_{n,ab}^R . \]

(3.12)

To express the gravity factor we introduce the notation

\[ P_{a_1, \ldots, a_r}^{l,u} = \prod_{k=l}^{u} \langle k| x_{k,k+2k+2a_1,k+2a_2,} \cdots x_{a_r-1,a_r}|a_r \rangle , \]

(3.13)

\[ Z_{b_1, \ldots, b_r}^{a_1, \ldots, a_u} = \frac{\langle a_1| x_{a_1,a_2} \cdots x_{a_{u-1},a_u}|a_u \rangle}{\langle b_1| x_{b_1,} x_{b_2,} \cdots x_{b_{r-1},c_r} \cdots x_{c_r-c_r}|c_r \rangle} , \]

(3.14)

which is overkill at the moment but will be fully utilized below when we move beyond the NMHV level. In the numerators only dual conformal chains of \( x \)-matrices appear, while in the denominators the chains are not dual conformal due to the break in the way the labels are arranged. The break is denoted by the semi-colon in the subscript of \( Z \) while in the denominator of \( P \) it is immediately after the left-most spinor \( \langle k \rangle \).

Then the first factor in (3.12) is given by

\[ f_{n,2b} = x_{1b}^2 , \]

(3.15)

\[ f_{n,ab} = x_{13}^2 (-Z_{n,a-1}^{2,a-2}) P_{n,a}^{2,a-2} \quad \text{for } a > 2 , \]

(3.16)

while the remaining two are

\[ G_{n,ab}^L = -Z_{n,b,a,n}^{n,a+1,b,a,n} P_{b,a,n}^{a,b-3} , \]

(3.17)

\[ G_{n,ab}^R = -Z_{n,b,a,n}^{n,b+1,b,a,n} P_{b,n-3}^{b,n-3} . \]

(3.18)

2. Proof

To check that the formula (3.11) is correct it is useful to first have a general formula for \( x_{1i}^2 \), where the shift is defined so that \( \tilde{P}_i^2 = x_{1i}^2 = 0 \). This tells us that the shift parameter is given by \( (2.3) \), i.e

\[ z_P = \frac{x_{1i}^2}{\langle n|x_{1i}|1 \rangle} . \]

(3.19)
Then we have
\[
\begin{align*}
x^2_{1v} &= x^2_{1v} - z_P \langle n|x_{1v}|1 \rangle \\
     &= x^2_{1v} \langle n|x_{1v}|1 \rangle - x^2_{1v} \langle n|x_{1v}|1 \rangle \\
     &= \frac{\langle n|x_{1v}(x_{1v} - x_{1i})x_{1i}|1 \rangle}{\langle n|x_{1i}|1 \rangle} \\
     &= \frac{\langle n|x_{1v}x_{1i}|1 \rangle}{\langle n|x_{1i}|1 \rangle} \\
     &= \frac{\langle n|x_{1v}x_{1i}x_{1i}|1 \rangle}{\langle n|x_{1i}|1 \rangle} \\
     &= -\frac{\langle n|x_{1v}x_{1i}x_{1i}x_{2n}|n \rangle}{\langle n|x_{1i}x_{2n}|n \rangle} \equiv -Z_{n;i,2,n}^{n,v,i,2,n} .
\end{align*}
\] (3.20)

Note that instead of writing (3.22) we could have alternatively written it as
\[
\begin{align*}
x^2_{1v} &= \frac{\langle n|x_{1i}(x_{1v} - x_{1i})x_{1v}|1 \rangle}{\langle n|x_{1i}|1 \rangle} \\
     &= \frac{\langle n|x_{1i}x_{1v}|1 \rangle}{\langle n|x_{1i}|1 \rangle} \\
     &= \frac{\langle n|x_{ni}x_{iv}x_{2n}|n \rangle}{\langle n|x_{2n}|n \rangle} \equiv Z_{n;i,2,n}^{n,i,v,2,n} .
\end{align*}
\] (3.25)

The freedom to write this factor in these two various forms is useful because in certain cases either one or the other form simplifies by cancelling factors from the numerator and denominator.

Finally we are set up to check our claim (3.12) for the NMHV $G$-factor. We first check the case $a = 2$ which comes entirely from MHV $\times$ MHV diagrams. From these diagrams we obtain
\[
\sum_{i=1}^{n-1} R^2_{n,2,i} G^\text{NMHV}_{n;2,i} = \sum_{i=1}^{n-1} R^2_{n,2,i} P^2 G^\text{MHV} (\hat{1}, \ldots, -\hat{P}) G^\text{MHV} (\hat{P}, \ldots, \pi) ,
\] (3.28)

from which we find
\[
G^\text{NMHV}_{n;2,i} = x^2_{1i} \left( x^2_{1i} \prod_{k=2}^{i-3} \frac{\langle k|x_{k,k+2x_{k+2,i}}|\hat{P} \rangle}{\langle k|\hat{P} \rangle} \right) \left( x^2_{1i+1} \prod_{l=i}^{n-3} \frac{\langle l|x_{l,l+2x_{l+2,n}}|n \rangle}{\langle l|n \rangle} \right) \\
= x^2_{1i} (-Z^{n,3,i,2,n}_{n;i,2,n} P^{2,i-3}_{i,2,n} (-Z^{n,i+1,i,2,n}_{n;i,2,n} P^{i,n-3}_n) ,
\] (3.29)

which is in agreement with equations (3.12) to (3.18) for the case $a = 2$.

For the case $a > 2$ we must consider diagrams of the form $\text{MHV}_3 \times \text{NMHV}_{n-1}$. From these diagrams we obtain
\[
\sum_{3 \leq a, b \leq n-1} R^2_{n,ab} G^\text{NMHV}_{n;ab} = \sum_{3 \leq a, b \leq n-1} R^2_{n,ab} P^2 G^\text{NMHV} (\hat{P}, 3, \ldots, \pi) .
\] (3.31)
The sum splits into two contributions, \( a = 3 \) and \( a > 3 \). The first gives

\[
G_{n;3b}^{\text{NMHV}} = x_{13}^2 x_{1b}^2 \left(-Z_{n,b,3,n}^{n,a,b,3,n} P_{b,a,n}^{a,b-3} \right) \left(-Z_{n,b,3,n}^{n,b+1,b,3,n} P_{n}^{b,n-3} \right) \tag{3.32}
\]

\[
= x_{13}^2 (-Z_{n,2}^{n,b,b}) \left(-Z_{n,b,3,n}^{n,a,b,3,n} P_{b,a,n}^{a,b-3} \right) \left(-Z_{n,b,3,n}^{n,b+1,b,3,n} P_{n}^{b,n-3} \right), \tag{3.33}
\]

in agreement with equations (3.12) to (3.18) for the case \( a = 3 \). To go from (3.32) to (3.33) we have used the fact that \( x_{1b}^2 = -Z_{n;3,2,n}^{n,b,3,2,n} = -Z_{n,2}^{n,b,2} \) where the simplification of the \( Z \)-factor is due to a cancellation between its numerator and denominator.

For the contributions to (3.31) where \( a > 3 \) we find

\[
G_{n;ab}^{\text{NMHV}} = x_{13} x_{14}^2 \left(-Z_{n;a-1}^{n,b,a-1} P_{n}^{3,a-2} \right) \left(-Z_{n,b,a,n}^{n,a+1,b,a,n} P_{b,a,n}^{a,b-3} \right) \left(-Z_{n,b,a,n}^{n,b+1,b,a,n} P_{n}^{b,n-3} \right) \tag{3.34}
\]

\[
= x_{13}^2 \left(-Z_{n,a-1}^{n,b,a-1} P_{n}^{3,a-2} \right) \left(-Z_{n,b,a,n}^{n,a+1,b,a,n} P_{b,a,n}^{a,b-3} \right) \left(-Z_{n,b,a,n}^{n,b+1,b,a,n} P_{n}^{b,n-3} \right), \tag{3.35}
\]

which is again in agreement with equations (3.12) to (3.18). The factor \( x_{14}^2 \) completes the factor \( P_{n}^{3,a-2} \) to \( P_{n}^{2,a-2} \) just as in the MHV case. This completes the verification of the formula (3.11) for NMHV graviton amplitudes. Appendix B contains some notes on extracting NMHV graviton amplitudes from the super-amplitude (3.11).

C. NNMHV Amplitudes

In this section we consider the NNMHV case as an exercise towards finding the general algorithm for all tree-level gravity amplitudes.

1. Statement

The structure of the result is just like in Yang-Mills and similar to the NMHV case (3.11) except that we now have two more subscripts on both the Yang-Mills \( R \)-factors and the gravity factors,

\[
M^{\text{NNMHV}}(1, \ldots, n) = \frac{\sum_{2\leq a,b\leq n-1} R_{n,ab}^2 \left[ \sum_{a\leq c,d<b} (R_{n,abcd}^{ba})^2 H_{n,abcd}^{(1)} + \sum_{b\leq c,d<n} (R_{n,abcd}^{ab})^2 H_{n,abcd}^{(2)} \right]}{A^{\text{MHV}}(1, \ldots, n)^2}. \tag{3.36}
\]

The factors \( H^{(1)} \) and \( H^{(2)} \) can be written in the form

\[
H_{n,abcd}^{(1)} = f_{n,ab}G_{n,ab}^R \tilde{f}_{n,abcd}G_{n,abcd}^L G_{n,abcd}^R, \tag{3.37}
\]

\[
H_{n,abcd}^{(2)} = f_{n,ab}G_{n,ab}^L \tilde{f}_{n,abcd}G_{n,abcd}^L G_{n,abcd}^R. \tag{3.38}
\]
In this formula \( f_{n;ab} \), \( G^L_{n;ab} \) and \( G^R_{n;ab} \) are defined as before in the case of the NMHV amplitude (see formulae \((3.16)\), \((3.17)\) and \((3.18)\)). The factor \( \tilde{f} \) in \( H^{(1)} \) is given by

\[
\tilde{f}_{n;ab,ad} = -Z_{n;ab,a,n}^{b,d,a,n},
\]

\[
\tilde{f}_{n;ab,cd} = (-Z_{n;ab,a,n}^{b,a+1,a,n})(-Z_{n;ab,a,n}^{c-1,d,b,a,n}) P_{b,a,n}^{a,c-2} \quad \text{for } c > a,
\]

and the factor \( \hat{f} \) in the second term in the parentheses is given by

\[
\hat{f}_{n;ab,bd} = -Z_{n;ab,a,n}^{n,d,b,a,n}
\]

\[
\hat{f}_{n;ab,cd} = (-Z_{n;ab,a,n}^{n,b+1,b,a,n})(-Z_{n;ab,a,n}^{n,d,c-1}) P_{b,c}^{b,c-2} \quad \text{for } c > b.
\]

Finally the new \( G \)-factors are given by

\[
G^L_{n;ab,cd} = -Z_{n;ab,b,c+1,d,c,b,a,n}^{n,a,b,c+1,d,c,b,a,n} P_{d,c,b,a,n}^{c,d-3},
\]

\[
G^R_{n;ab,cd} = -Z_{n;ab,b,d+1,d,c,b,a,n}^{n,a,b,d+1,d,c,b,a,n} P_{b,a,n}^{d,n-3}.
\]

2. Proof

Let us now check the claim \((3.36)\). As before we begin with the case \( a = 2 \) which comes purely from NMHV \( \times \) MHV diagrams and MHV \( \times \) NMHV diagrams. We start by calculating the former kind. From these diagrams we obtain

\[
\sum_{i=5}^{n-1} \sum_{2 \leq c,d < i} R_{n;2i}^{2} \sum_{2 \leq c,d < i} (R_{n;2i;cd}^{2})^2 H_{n;2i,cd}^{(1)}
\]

\[
= \sum_{i=5}^{n-1} \sum_{2 \leq c,d < i} (R_{n;2i;cd}^{2})^2 P^2 G^{\text{NMHV}}(\hat{1}, \ldots, -\hat{P}) G^{\text{MHV}}(\hat{P}, \ldots, \hat{n}).
\]

The sum over \( c \) splits into two pieces, \( c = 2 \) and \( c > 2 \). For the terms where \( c = 2 \) we have

\[
H_{n;2i,2d}^{(1)} = x_{2i}^{2} \left[ x_{1i}^{2} \left( -Z_{n;2i,d,2i,n}^{n,2i,d,2i,n} P_{d,2i,n}^{d,2i,d-3} \right) \left( -Z_{n;2i,i,d,2i,n}^{n,i,d,2i,n} P_{i,2i,n}^{d,2i,i-3} \right) \right] x_{i+1}^{2} P_{i,n-3}^{i,n-3}.
\]

Here as in the previous subsection we have used the fact that certain \( Z \)-factors simplify. For example, reading the \( Z \)-factor from the formula \((3.17)\) and taking into account the fact that the spinor \( \langle \hat{P} \rangle \) can be replaced in both the numerator and denominator of \( Z \) by \( \langle n | x_{n2} x_{2i} \rangle \), we would obtain \( Z_{n;2i,d,2i,n}^{n,2i,d,2i,n} \). The sequence of indices \( 2, i, 2 \) implies however that one can factor out \( x_{2i}^2 \). Since the sequence is present in both the numerator and the denominator, it
can simply be replaced by 2. Thus we arrive at the form of the $Z$-factor in the first set of parentheses in (3.46).

To verify that equation (3.46) is consistent with (3.37) it remains to substitute the $Z$-factors appropriate to the factors $x^2_{1d}$ and $x^2_{1,i+1}$. Doing so we obtain

$$H^{(1)}_{n;2i;2d} = x^2_{1i} \left[ -Z_{2i}^{n,d,i;2n} \left( -Z_{2i}^{n,2,i, d,2} P^{d,2-3}_{i,d,n} \right) \left( -Z_{2i}^{n,2,i, d,2} P^{d,2-3}_{i,d,n} \right) \left[ -Z_{2i}^{n,i+1,i,2n} P^{n,i,n-3}_{n,d,n} \right] \right].$$

(3.47)

The factor $x^2_{1i}$ gives the required contribution $f_{n;2i}$, while the factor in the second factor in square brackets is $G_{n;2i}^{R}$. The remaining factor in the first set of square brackets is the contribution from $\tilde{f}_{n;2i;2d}$ and the other $Z$ and $P$ factors in (3.37).

Now let us look at the terms where $c > 2$. We have

$$H^{(1)}_{n;2i;cd} = x^2_{1i} \left[ x^2_{13} \left( -Z_{2i}^{n,2,i,d,c-1} P^{d,c-2}_{i,d,n} \right) \left( -Z_{2i}^{n,2,i,d,c-1} P^{d,c-2}_{i,d,n} \right) \left[ -Z_{2i}^{n,i+1,i,2n} P^{n,i,n-3}_{n,d,n} \right] \right].$$

(3.48)

Again, substituting for $x^2_{13}$ and $x^2_{1,i+1}$ we find agreement with (3.37).

Now let us turn our attention to the latter kind of diagrams, namely the MHV $\times$ NMHV diagrams. From these diagrams we find

$$\sum_{i=4}^{n-3} R^2_{n;2i} \sum_{2\leq c,d<i} (R^2_{n;cd})^2 H^{(2)}_{n;2i;cd}$$

$$= \sum_{i=4}^{n-3} R^2_{n;2i} \sum_{2\leq c,d<i} (R^2_{n;cd})^2 P^2 G_{n;2i;cd}^{NMHV}(\hat{1}, \ldots, \hat{P}) G_{n;2i;cd}^{MHV}(\hat{P}, \ldots, \hat{P}).$$

(3.49)

As before the sum over $c$ splits into two pieces. For $c = i$ we find

$$H^{(2)}_{n;2i;2d} = x^2_{1i} \left[ x^2_{13} P^{2,i-3}_{i,2,n} \left[ x^2_{1d} \left( -Z_{n;2i}^{n,i+1,d,i,n} P^{i,d-3}_{d,i,n} \right) \left( -Z_{n;2i}^{n,i+1,d,i,n} P^{i,d-3}_{d,i,n} \right) \right] \right],$$

(3.50)

while for $c > i$ we find

$$H^{(2)}_{n;2i;cd} = x^2_{1i} \left[ x^2_{13} P^{2,i-3}_{i,2,n} \left[ x^2_{1,i+1} \left( -Z_{n;2i}^{n,d,c-1} P^{i,c-2}_{n,d,c,n} \right) \left( -Z_{n;2i}^{n,d,c-1} P^{i,c-2}_{n,d,c,n} \right) \left[ -Z_{n;2i}^{n,d,c-1} P^{n,i,n-3}_{n,d,c,n} \right] \right] \right].$$

(3.51)

Making the usual substitutions for the factors of the form $x^2_{1i}$ we find agreement with (3.37) in both cases.
To check the terms for \( a > 2 \) we need to consider MHV3 \( \times \) NNMHVn-1 diagrams. These diagrams give us

\[
\sum_{3 \leq a, b < n} R_{ab}^2 \left[ \sum_{a \leq c, d < b} (R_{abcd}^{ba})^2 H_{n;abcd}^{(1)} + \sum_{b \leq c, d < n} (R_{abcd}^{ab})^2 H_{n;abcd}^{(2)} \right]
\]

\[
= \sum_{3 \leq a, b < n} R_{ab}^2 \left[ \sum_{a \leq c, d < b} (R_{abcd}^{ba})^2 P^2 H^{(1)}(\hat{P}, \ldots, \pi) + \sum_{b \leq c, d < n} (R_{abcd}^{ab})^2 H^{(2)}(\hat{P}, \ldots, \pi) \right].
\]

As in the NMHV case, the sum over \( a \) splits into a part where \( a = 3 \) and a part where \( a > 3 \). The calculation is essentially the same as in the NMHV case, with the factor of \( P^2 = x_{13}^2 \) providing the necessary piece of \( f_{n;ab} \) in both cases. This completes the verification of the formula (3.36) for NNMHV amplitudes.

IV. DISCUSSION OF GENERAL TREE-LEVEL AMPLITUDES

Because of the association between vertices in the rooted tree diagram Fig. 2 with individual terms appearing in the iterative solution of the recursion relation (2.1), it is clear that the procedure applied in the previous section can be generalized to express an arbitrary \( N^p \)MHV n-graviton super-amplitude in the form

\[
\mathcal{M}_n = \sum_{p(2, \ldots, n-1)} [A^{\text{MHV}}(1, \ldots, n)]^2 \sum_{\{a\}} [R_\alpha(\lambda_i, \tilde{\lambda}_i, \eta_i)]^2 G_\alpha(\lambda_i, \tilde{\lambda}_i),
\]

where \( R_\alpha \) are precisely the same dual superconformal invariants (2.14) that appear in SYM and \( G_\alpha \) are some additional, non-dual conformally invariant, dressing factors. Explicit formulas for the MHV, NMHV, and NNMHV gravity factors are given respectively in (3.2), (3.12), and (3.37)–(3.38).

The gravity factors \( G_\alpha \) for a general amplitude can be worked out on a case-by-case basis. They always have the form

\[
G_{n;a_1b_1;\ldots} = f_{n;a_1b_1;\ldots},
\]

where \( \ldots \) is some combination of \( f \), \( G^R \) and \( G^L \) factors. The iterative construction of any desired amplitude is no more difficult than the examples we have already studied in detail. Actually one only needs to take care of the factor \( f_{n;a_1b_1} \), because the other parts just go from lower points to higher points automatically under the usual rules

\[
\langle n|x_{ny} \rightarrow \langle \hat{p}|x_{iy} \rightarrow \langle n|x_{nj}x_{ji}x_{iy},
\]

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and
\[ \langle n | x_{kl} \rightarrow \langle \hat{p} | x_{kl} \rightarrow \langle n | x_{nj} x_{ji} x_{kl} \rangle, \]
(4.4)
as, for example, in going from the NMHV formula (3.12) to the NNNHV formula (3.37) and (3.38). The $f$ factors arise at each level for the simple reason that an extra propagator $P^2$ appears in on-shell recursion for gravity as compared to the ‘square’ of the corresponding Yang-Mills result, a fact which we noted already back in (2.12). As we already explained carefully in previous section for the NMHV case, the factor $f_{n; a_1 b_1}$ is needed to satisfy the recursion relation.

Although it is simple to describe the algorithm for a general amplitude in words and by appealing to the examples detailed above, we have not identified a pattern which would allow us to write down a general explicit formula, as was done for SYM in (3.16). As noted above each $R_\alpha$ invariant comes with its own $f$-type factor, and each path in Fig. 2 which ends on a vertex with indices $a_1 b_1 \ldots ; a_p b_p$ leads to an associated factor of the form

\[ G_{a_1 b_1 ; \ldots ; a_p b_p}^R C_{a_1 b_1 ; \ldots ; a_p b_p}^L, \]
(4.5)
where the general $f$, $G^R$ and $G^L$ are suitably defined following the examples in the previous section. Specifically we have

\[ G_{n; a_1 b_1 ; \ldots ; a_r b_r ; a b}^L = -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, a b} P_{a_1 b_1, a_r b_r}^{a b}, \]
(4.6)

\[ G_{n; a_1 b_1 ; \ldots ; a_r b_r ; a b}^R = -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, a b} P_{a_1 b_1, a_r b_r}^{a b} \]
(4.7)
The $f$ factors can be of two types, $\tilde{f}$ and $\hat{f}$. The first type are defined as follows,

\[ \tilde{f}_{n; a_1 b_1 ; \ldots ; a_r b_r ; a b} = -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, a b} P_{a_1 b_1, a_r b_r}^{a b} \]
(4.8)

\[ \hat{f}_{n; a_1 b_1 ; \ldots ; a_r b_r ; a b} = \left( -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, a b} P_{a_1 b_1, a_r b_r}^{a b} \right) P_{a_1 a_r, b_1 b_r}^{a b} \]
(4.9)
The second type are given by

\[ \tilde{f}_{n; a_1 b_1 ; \ldots ; a_r b_r ; b_r b} = -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, b_r b} \]
(4.10)

\[ \hat{f}_{n; a_1 b_1 ; \ldots ; a_r b_r ; a_r a} = \left( -Z_{n, a_1 b_1 ; \ldots ; a_r b_r ; a_1, a_r}^{a_1 b_1 \ldots a_r b_r, b_r b} P_{a_1 a_r, b_1 b_r}^{a b} \right) P_{a_1 b_1, a_r b_r}^{a b} \]
(4.11)

In addition to the factors (4.5), other $G^R$ and $G^L$ factors also appear. If we try the simplest guess which is that we should be able to associate these factors to the vertices
in Fig. 2 such that every path ending on a given vertex picks up the factors associated to that vertex, then we find that:

1. for each cluster (see Fig. 3), the leftmost descendant vertex picks up the same factors as the parent vertex, and in addition a $G^R$ factor with the indices of the parent,

2. the next descendant vertex to the right is exactly the same, except that the additional factor is a $G^L$ instead of $G^R$, and

3. going further to the right along the descendant vertices, there is a more complicated structure $G^R$ and $G^L$ factors whose indices are modified from those of the parent.

We emphasize that we have attempted here only to illustrate some features of the general structure; in order to determine precisely the factors which appear for a given path it seems necessary to work out recursively which kinds of $N^aMHV \times N^bMHV$ factorizations that particular path corresponds to.

To stress that the algorithm can be simply exploited to generate higher and higher $N^pMHV$ amplitudes, we give here the formula for $N^3MHV$ amplitudes:

\[
M^{N^3MHV}(1, \ldots, n) = |A^{MHV}(1, \ldots, n)|^2 \sum_{2 \leq a_1, b_1 < n} R^2_{n,a_1b_1} \left[ \sum_{a_1 \leq a_2, b_2 < b_1} (R_{n,a_1b_1:a_2b_2}^{a_1b_1})^2 \left( \sum_{a_2 \leq a_3, b_3 < b_2} (R_{n,a_2b_2:a_3b_3}^{a_2b_2:a_3b_3})^2 G^{(1)}_{n,a_1b_1:a_2b_2:a_3b_3} + \sum_{b_2 \leq a_3, b_3 < b_1} (R_{n,a_2b_2:a_3b_3}^{a_2b_2:a_3b_3})^2 G^{(2)}_{n,a_1b_1:a_2b_2:a_3b_3} + \sum_{b_1 \leq a_3, b_3 < n} (R_{n,a_3b_3}^{a_3b_3})^2 G^{(3)}_{n,a_1b_1:a_2b_2:a_3b_3} \right) + \sum_{b_1 \leq a_2, b_2 < n} (R_{n,a_2b_2}^{a_2b_2})^2 \left( \sum_{a_2 \leq a_3, b_3 < b_2} (R_{n,a_2b_2:a_3b_3}^{a_2b_2:a_3b_3})^2 G^{(4)}_{n,a_1b_1:a_2b_2:a_3b_3} + \sum_{b_2 \leq a_3, b_3 < n} (R_{n,a_3b_3}^{a_3b_3})^2 G^{(5)}_{n,a_1b_1:a_2b_2:a_3b_3} \right) \right].
\]

(4.12)

The five different $G$-factors are in correspondence with the five different vertical paths from the root node to the vertices on the lowest row explicitly shown in Fig. 2. Explicitly they are given by

\[
G^{(1)}_{n,a_1b_1:a_2b_2:a_3b_3} = f_{n,a_1b_1} f_{n,a_1b_1:a_2b_2} f_{n,a_1b_1:a_2b_2:a_3b_3} G^{R}_{n,a_1b_1:a_2b_2} G^{L}_{n,a_1b_1:a_2b_2:a_3b_3},
\]

(4.13)

\[
G^{(2)}_{n,a_1b_1:a_2b_2:a_3b_3} = f_{n,a_1b_1} f_{n,a_1b_1:a_2b_2} f_{n,a_1b_1:a_2b_2:a_3b_3} G^{R}_{n,a_1b_1:a_2b_2} G^{L}_{n,a_1b_1:a_2b_2:a_3b_3},
\]

(4.14)

\[
G^{(3)}_{n,a_1b_1:a_2b_2:a_3b_3} = f_{n,a_1b_1} f_{n,a_1b_1:a_2b_2} f_{n,a_1b_1:a_2b_2:a_3b_3} G^{R}_{n,a_1b_1:a_2b_2:a_3b_3} G^{L}_{n,a_1b_1:a_2b_2:a_3b_3},
\]

(4.15)

\[
G^{(4)}_{n,a_1b_1:a_2b_2:a_3b_3} = f_{n,a_1b_1} f_{n,a_1b_1:a_2b_2} f_{n,a_1b_1:a_2b_2:a_3b_3} G^{R}_{n,a_1b_1:a_2b_2:a_3b_3} G^{L}_{n,a_1b_1:a_2b_2:a_3b_3},
\]

(4.16)

\[
G^{(5)}_{n,a_1b_1:a_2b_2:a_3b_3} = f_{n,a_1b_1} f_{n,a_1b_1:a_2b_2} f_{n,a_1b_1:a_2b_2:a_3b_3} G^{R}_{n,a_1b_1:a_2b_2:a_3b_3} G^{L}_{n,a_1b_1:a_2b_2:a_3b_3}.
\]

(4.17)
where $G$ is shorthand for $G^L \times G^R$ (with the same subscripts on both).

The expressions we have found can certainly be used in the calculation of loop amplitudes in supergravity. It is straightforward to apply the generalized unitarity technique in a manifestly supersymmetric way \cite{1,40,44}; the basic ingredients in this procedure are the tree-level super-amplitudes.

It would of course be extremely interesting to unlock the general pattern of $G$-factors to allow one to write down a general explicit formula. It would also be interesting to see if the SUGRA ‘bonus relations’ \cite{1,32} could be usefully exploited beyond MHV amplitudes. There is no doubt that much additional structure remains to be found. Hopefully, much simpler and more beautiful formulas await than the ones obtained here. Certainly this should be the case if the notion that SUGRA amplitudes are even simpler than those of SYM is to come to full fruition.

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APPENDIX A: CONVENTIONS

Here we give some formulae to establish the conventions we are using for the two-component spinors. We have

\begin{equation}
x_{\dot{a}\dot{a}} \equiv x_{\dot{a}\dot{a}} = (\sigma^\mu)_{\dot{a}\dot{a}} x_\mu, \quad \bar{x}^{\dot{\alpha}\dot{\alpha}} \equiv x^{\dot{\alpha}\dot{\alpha}} = (\bar{\sigma}^\mu)_{\dot{\alpha}\dot{\alpha}} x_\mu, \quad (A1)
\end{equation}

\begin{equation}
x^2 = x^\mu x_\mu = \frac{1}{2} x_{\dot{a}\dot{a}} x^{\dot{\alpha}\dot{\alpha}}, \quad x_{\dot{a}\dot{a}} x^{\dot{\alpha}\beta} = \delta_{\dot{a}}^{\dot{\alpha}} x^2, \quad x^{\dot{\alpha}\dot{\alpha}} x_{\dot{a}\dot{b}} = \delta^{\dot{\alpha}} x^2. \quad (A2)
\end{equation}

For the commuting spinors, the following notation has been used,

\begin{equation}
p^\alpha_{\dot{\alpha}} = \lambda^\alpha \bar{\lambda}_{\dot{\alpha}}, \quad \lambda^{\alpha}_{a} = \lambda^\alpha \epsilon_{\dot{\alpha}}, \quad \lambda^{\alpha}_{i} = \epsilon^{-a} \lambda^{\alpha}_{i}, \quad \bar{\lambda}_{a\dot{a}} = \bar{\lambda}_{a\dot{a}} = \bar{\lambda}_{\dot{a}\dot{a}}, \quad \bar{\lambda}^\alpha = \epsilon^{-\alpha} \bar{\lambda}_{\dot{a}\dot{a}}, \quad (A3)
\end{equation}
where $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ are antisymmetric tensors. For the contractions of these spinor variables we write for example

$$\langle i \vert j \rangle = \lambda_i^a \lambda_{ja}, \quad [i \vert j] = \bar{\lambda}_i^a \bar{\lambda}_{ja},$$

$$\langle i \vert x \vert j \rangle = \lambda_i^a x_{\alpha\dot{\alpha}} \bar{\lambda}_j^{\dot{a}}, \quad \langle i \vert x_1 \ldots x_{2m} \vert j \rangle = \lambda_i^a x_1 x_2 \ldots x_{2m} \bar{\lambda}_j^{\dot{a}} x_1 \ldots x_{2m}.$$  \hfill (A4)

For the dual coordinates we use

$$p_i^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}, \quad x_{n+1} \equiv x_1.$$ \hfill (A6)

Similarly for the Grassmann odd dual coordinates we have

$$q_i^{\alpha A} = \lambda_i^a \eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}. \hfill (A7)$$

### APPENDIX B: NMHV GRAVITON AMPLITUDES

Here we provide some additional details regarding the formula for general NMHV superamplitudes proven in section III.B,

$$\mathcal{M}_n^{\text{NMHV}} = \sum_{P(2, \ldots, n-1)} \left[ \frac{\delta^{(8)}(q)}{\langle 1 \vert 2 \rangle \ldots \langle n \vert 1 \rangle} \right]^{2n-3} \sum_{s=2}^{n-1} \sum_{t=s+2}^{n-1} R_{n, st}^2 G_{n, st}^{\text{NMHV}},$$ \hfill (B1)

where the $G$-factor is given in (3.12) and the dual superconformal invariant is

$$R_{n, st} = \frac{\langle s \vert s-1 \rangle \langle t \vert t-1 \rangle \delta^{(4)}(\Xi_{n, st})}{x_{nt}^2 \langle n \vert x_{ns} x_{st} \vert t \rangle \langle n \vert x_{nt} x_{ts} \vert s-1 \rangle \langle n \vert x_{nt} x_{ts} \vert s \rangle \langle n \vert x_{nt} x_{ts} \vert s-1 \rangle}.$$ \hfill (B2)

In terms of $[36, 39]$

$$\Xi_{n, st} = \langle n \vert \sum_{i=t}^{n-1} |i\rangle \eta_i + \sum_{i=s}^{n-1} |i\rangle \eta_i \rangle.$$ \hfill (B3)

In order to extract the $n$-particle NMHV graviton amplitude from this superspace expression we should perform the integral over $d^8 \eta_i$ for the three negative helicity gravitons $i$. It is convenient to choose particles 1 and $n$ to be two of these three since $\Xi_{n, st}$ does not depend on $\eta_1$ or $\eta_n$. These two variables appear only inside the supermomentum conserving delta function $\delta^{(16)}(q)$ which may be put into the form $[36, 44]$

$$\delta^{(16)}(q) = \langle 1 \vert n \rangle^8 \delta^{(8)} \left( \eta_n^A + \sum_{i=2}^{n-1} \frac{\langle n \vert i \rangle \eta_i^A}{\langle n \vert 1 \rangle \eta_i^A} \right) \delta^{(8)} \left( \eta_1^A + \sum_{i=2}^{n-1} \frac{\langle i \vert 1 \rangle \eta_i^A}{\langle n \vert 1 \rangle \eta_i^A} \right).$$ \hfill (B4)
The $d^8\eta_1d^8\eta_n$ integrals are then trivial, leading to
\[
\mathcal{M}(1^-,2^-,3^+,\ldots,n^-) = \int d^8\eta_2 \sum_{P(2,\ldots,n-1)} \sum_{s=2}^{n-3} \sum_{t=s+2}^{n-1} \left[ \frac{\langle 1 n \rangle^4 R_{n;st}}{\langle 1 2 \rangle \cdots \langle n 1 \rangle} \right] C_{n;st}^{\text{NMHV}}. \tag{B5}
\]
Here we have chosen, without loss of generality, particle 2 to be the third negative helicity graviton.

The analogous NMHV gluon amplitude simplifies further due to the fact that $R_{n;st}$ only depends on $\eta_2$ when $s = 2$; thus performing the integral eliminates the sum over $s$ \cite{13}. Here in the case of gravity it is unfortunately cumbersome to proceed analytically because the sum over permutations in \cite{B5} generates many terms, even for the simplest nontrivial case $n = 6$ where the sum over $s$ and $t$ produces just three terms and the corresponding $G$-factors simplify considerably,
\begin{align}
C_{6;24}^{\text{NMHV}} &= + (p_1 + p_2 + p_3)^2 \langle 2 \, 3 \rangle [3 \, 4] [4 \, 5] \langle 5 \, 6 \rangle \langle 6 \, 5 + 4 \, 3 \rangle \langle 4 \, 3 + 2 \, 1 \rangle \langle 6 \, 5 + 4 \, 1 \rangle \langle 6 \, 3 + 2 \, 1 \rangle, \\
C_{6;25}^{\text{NMHV}} &= + (p_5 + p_6)^2 \langle 2 \, 3 \rangle [3 \, 4] [4 \, 5] \langle 5 \, 6 \rangle [1 \, 5] \langle 6 \, 1 + 2 \, 3 \rangle \langle 4 \, 5 + 6 \, 1 \rangle \langle 2 \, 5 + 6 \, 1 \rangle, \\
C_{6;35}^{\text{NMHV}} &= - (p_1 + p_2)^2 \langle 3 \, 4 \rangle [4 \, 5 \, 6 \, \rangle [5 \, 6 \rangle [2 \, 3 + 4 \, 5 \rangle \langle 6 \, 1 + 2 \, 3 \rangle \langle 2 \, 6 \rangle \langle 6 \, 1 + 2 \, 3 \rangle \langle 4 \, 6 \rangle. \tag{B6}
\end{align}
Therefore we do not provide explicit analytic formulas for graviton amplitudes, which instead may be evaluated numerically as needed. We have checked numerically that our expression agrees with other representations for the $n = 6$ particle NMHV graviton amplitude in the literature (see for example \cite{7,16}).

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