Training Restricted Boltzmann Machines with Binary Synapses using the Bayesian Learning Rule

Xiangming Meng
Institute for Physics of Intelligence
The University of Tokyo
Tokyo, Japan
meng@g.ecc.u-tokyo.ac.jp

Abstract

Restricted Boltzmann machines (RBMs) with low-precision synapses are much appealing with high energy efficiency. However, training RBMs with binary synapses is challenging due to the discrete nature of synapses. Recently Huang (2019) proposed one efficient method to train RBMs with binary synapses by using a combination of gradient ascent and the message passing algorithm under the variational inference framework. However, additional heuristic clipping operation is needed. In this technical note, inspired from Huang (2019), we propose one alternative optimization method using the Bayesian learning rule, which is one natural gradient variational inference method. As opposed to Huang (2019), we update the natural parameters of the variational symmetric Bernoulli distribution rather than the expectation parameters. Since the natural parameters take values in the entire real domain, no additional clipping is needed. Interestingly, the algorithm in Huang (2019) could be viewed as one first-order approximation of the proposed algorithm, which justifies its efficacy with heuristic clipping.

1 Problem Formulation

Restricted Boltzmann machines (RBMs) with low-precision discrete synapses are much appealing due to high energy efficiency. However, compared to full-precision RBMs, they are more difficult to train, which is essentially a discrete optimization problem. In a recent paper Huang (2019), the author addressed the problem of training RBMs with binary synaptic connections. The problem is formulated as follows. Consider RBMs where the random visible variables $v = \{v_1, ..., v_N\}$ and hidden variables $h = \{h_1, ..., h_M\}$ only take binary values $\{-1, +1\}$. Then the joint distribution of this RBM model is given by the Gibbs distribution

$$p(v, h) = \frac{1}{Z} e^{-\beta E(v, h)},$$

(1)

where $Z$ is the normalization constant, $\beta$ is the temperature value, and $E(v, h)$ is the energy function defined as

$$E(v, h) = -\sum_{\mu=1}^{M} \sum_{i=1}^{N} w_{\mu i} h_{\mu} v_{i} - \sum_{i=1}^{N} b_{i} v_{i} - \sum_{\mu=1}^{M} c_{\mu} h_{\mu}. \quad (2)$$

For simplicity and without loss of generality, assume a simple case where the biases $b_i = 0, i = 1...N$ and $c_{\mu} = 0, \mu = 1...M$. The marginal distribution of $v$ could be obtained by marginalizing

---

*Most work performed when X. Meng was a postdoctoral researcher at RIKEN Center for Advanced Intelligence Project (AIP), Tokyo, Japan.

Technical Note. Work in progress.
out the hidden states $h$

$$p(v) = \frac{1}{Z(W)} \prod_{\mu=1}^{M} \cosh(\beta X_\mu)$$  \hfill (3)$$

$$X_\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{\mu i} v_i = \frac{1}{\sqrt{N}} w^T_\mu v$$  \hfill (4)$$

where $w^T_\mu$ is the $\mu$-th row of the synaptic connection matrix $W$, $X_\mu$ is the receptive field of the $\mu$-th hidden neuron, and $Z(W) = \sum_{v} \prod_{\mu=1}^{M} \cosh(\beta X_\mu)$ is the partition function depending on the synaptic connection matrix $W$.

When we have $D$ input data samples $\mathbb{D} = \{v_a\}_{a=1}^{D}$ which are weakly-correlated, then the likelihood distribution of data could be written as

$$p(\mathbb{D} | W) = \prod_{a=1}^{D} \frac{1}{Z(W)} \prod_{\mu=1}^{M} \cosh(\beta X^a_\mu),$$  \hfill (5)$$

where $X^a_\mu$ is the receptive field of the $\mu$-th hidden neuron for the $a$-th data sample $v_a$. From the Bayesian perspective, suppose that the prior distribution of $W$ is $p_0(W)$, according to Bayes’ rule, the posterior distribution could be obtained as

$$p(W | \mathbb{D}) = \frac{p(\mathbb{D} | W) p_0(W)}{p(\mathbb{D})},$$  \hfill (6)$$

where $p(\mathbb{D}) = \sum_W p(\mathbb{D} | W) p_0(W)$ is the partition function of the posterior and also known as the marginal data likelihood.

The goal of training RBMs with binary synapses is to learn the synaptic connection matrix $W$ from the observed data samples $\mathbb{D} = \{v_a\}_{a=1}^{D}$, subject to the discrete constraint that each element $w_{\mu i}$ in $W$ also takes binary value, i.e., $w_{\mu i} \in \{-1, +1\}$. If the posterior distribution $p(W | \mathbb{D})$ could be computed, then the learning problem is solved. However, exact computation of $p(W | \mathbb{D})$ is intractable.

For RBMs with full-precision synaptic connections, some classical training methods have been proposed such as the contrastive divergence (CD) algorithm \cite{Hinton2002}. However, in the case of RBMs with binary synaptic connections, it is essentially a challenging discrete optimization problem. As a result, the previous full-precision learning algorithms such as CD could not be used due to the discrete nature of the synapses.

\section{Review of Huang’s Method in \cite{Huang2019}}

Recently, \cite{Huang2019} addressed this challenging problem using a combination of gradient ascent\footnote{It could be also equivalently understood as minimizing the negative ELBO using gradient descent (GD).} and the message passing algorithm under the variational inference (VI) framework. Specifically, instead of computing the posterior directly, VI tries to find an approximate distribution $q_\lambda(W)$ that maximizes a lower bound of the log marginal likelihood $\log p(\mathbb{D})$, which is called the evidence lower bound (ELBO), i.e.,

$$\mathcal{L}(q_\lambda) = \mathbb{E}_{q_\lambda(W)}[\log p(\mathbb{D} | W)] - KL(q_\lambda(W) \parallel p_0(W)),$$  \hfill (7)$$

where $KL(q \parallel p) = \mathbb{E}_{q}[\log \frac{q}{p}]$ is the Kullback-Leibler (KL) divergence and $p_0(W)$ is the prior distribution which is assumed to be factorized as

$$p_0(W) = \prod_{\mu=1}^{M} \prod_{i=1}^{N} \left[ \frac{1 + m_{\mu i}}{2} \delta(w_{\mu i} = 1) + \frac{1 - m_{\mu i}}{2} \delta(w_{\mu i} = -1) \right],$$  \hfill (8)$$

where $m_{\mu i}$ is the prior mean of $w_{\mu i}$ and also controls the probability $p(w_{\mu i} = +1) = (1 + m_{\mu i})/2$. In practice, it is usually assumed that $m_{\mu i} = 0$ when no informative prior information is available about the synapses. Alternatively, $\mathcal{L}(q_\lambda)$ in (7) could be rewritten as

$$\mathcal{L}(q_\lambda) = \log p(\mathbb{D}) - KL(q_\lambda(W) \parallel p(W | \mathbb{D})),$$  \hfill (9)$$
so that $\mathcal{L}(q_\lambda) \leq \log p(D)$ and maximizing $\mathcal{L}(q_\lambda)$ is equivalent to minimizing the KL divergence $KL(q_\lambda(W) \parallel p(W | D))$. Hence, the problem of posterior inference problem in (6) is transformed to the optimization of $\mathcal{L}(q_\lambda)$ with respect to (w.r.t.) the variational parameters $\lambda$ of $q_\lambda(W)$, which is the core of VI.

To model the binary synaptic connections weights $W$, in Huang (2019) the variational distribution $q_\lambda(W)$ is chosen to be a mean-filed symmetric Bernoulli distribution

$$q_\lambda(W) = \prod_{\mu=1}^{M} \prod_{i=1}^{N} \left[ \frac{1 + \eta_{\mu i}}{2} \delta(w_{\mu i} = 1) + \frac{1 - \eta_{\mu i}}{2} \delta(w_{\mu i} = -1) \right],$$

where $\eta_{\mu i} \in [-1, 1]$ is the posterior mean of $w_{\mu i}$ and it controls the probability of the value of binary synaptic connection $w_{\mu i} \in \{-1, +1\}$, i.e., the probability of $w_{\mu i} = 1$ is $\frac{1 + \eta_{\mu i}}{2}$ while the probability of $w_{\mu i} = -1$ is $\frac{1 - \eta_{\mu i}}{2}$.

Then, Huang (2019) uses gradient ascent to update the variational parameters $\eta_{\mu i}$, i.e., in the $t$-th iteration, each parameter $\eta_{\mu i}$ is updated as

$$\eta_{\mu i}^{t+1} = \eta_{\mu i}^t + \alpha \nabla_{\eta_{\mu i}} \mathcal{L}(q_\lambda),$$

which seems easy to implement as long as the gradient term $\nabla_{\eta_{\mu i}} \mathcal{L}(q_\lambda)$ is obtained. However, in contrast to the case of supervised learning, it is far from trivial to obtain the gradient $\nabla_{\eta_{\mu i}} \mathcal{L}(q_\lambda)$.

To be clear, according to (7), the gradient consists of two terms

$$\nabla_{\eta_{\mu i}} \mathcal{L}(q_\lambda) = \nabla_{\eta_{\mu i}} E_{q_\lambda(W)} [\log p(D | W)] - \nabla_{\eta_{\mu i}} KL(q_\lambda(W) \parallel p_0(W)).$$

The gradient of the KL regularization term could be easily computed as

$$\nabla_{\eta_{\mu i}} KL(q_\lambda(W) \parallel p_0(W)) = - \sum_{x = \pm 1} \frac{x}{2} \left[ \log \frac{1 + x m_{\mu i}}{1 + x \eta_{\mu i}} - 1 \right].$$

However, the gradient of the expected log-likelihood term is intractable as it involves the computation of another log partition function $\log Z(W)$, i.e.,

$$\nabla_{\eta_{\mu i}} E_{q_\lambda(W)} [\log p(D | W)] = \nabla_{\eta_{\mu i}} E_{q_\lambda(W)} \left[ \sum_{a=1}^{D} \sum_{\mu=1}^{M} \log \cosh (\beta X^a_{\mu}) - D \log Z(W) \right].$$

To address this problem, Huang (2019) leverages the message passing algorithm to obtain an approximation of the log partition function. Specifically, as seen in (4), each $X^a_{\mu}$ is a sum of a large number of nearly independent random variables and hence, by the central limit theorem, follows a Gaussian distribution $\mathcal{N}(X^a_{\mu} ; G^a_{\mu} , \Xi^2_{\mu})$, where the mean and variance are defined as

$$G^a_{\mu} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \eta_{\mu i} v^a_i,$$

$$\Xi^2_{\mu} = \frac{1}{N} \sum_{i=1}^{N} (1 - \eta_{\mu i}^2),$$

As a result, similar to the local reparameterization trick Kingma et al. (2015), the expected log-likelihood could be approximated using the Monte-Carlo estimation

$$E_{q_\lambda(W)} \left[ \sum_{a=1}^{D} \sum_{\mu=1}^{M} \log \cosh (\beta X^a_{\mu}) - D \log Z(W) \right] \approx \frac{1}{S_1} \sum_{a,\mu,s} \log \cosh (\beta G^a_{\mu} + \beta \Xi_{\mu} z^s) - \frac{D}{S_2} \sum_{s} \log \prod_{\sigma} \cosh (\beta G^a_{\mu} + \beta \Xi_{\mu} z^s),$$

where $z^s$ are samples drawn from standard normal distribution, and $S_1$ and $S_2$ are the number of samples used to estimate different terms of the expected log-likelihood, respectively. However,
whose synaptic connections are \( \eta_{ii} \). Interestingly, as pointed out in Huang (2019), the term \( \log \sum_y \prod_j \cosh (\beta G_{ij} + \beta \Xi \cdot z^s_j) \) corresponds to the log partition function of an equivalent RBM whose synaptic connections are \( \eta_{ii} / \sqrt{N} \) and biases of hidden neurons are \( \Xi \cdot z^s_i \). As a result, the log \( \sum_y \prod_j \cosh (\beta G_{ij} + \beta \Xi \cdot z^s_j) \) could be efficiently computed by resorting to the message passing algorithm. To this end, denote by \( m_{i \rightarrow \mu} \) the message from visible neuron to hidden neuron and \( u_{\mu \rightarrow i} \) the message from hidden neuron to the visible neuron, respectively, then the message passing equation reads

\[
m_{i \rightarrow \mu} = \tanh \left( \sum_{v \in \partial_i \setminus \mu} u_{v \rightarrow i} \right),
\]

\[
u_{\mu \rightarrow i} = \tanh^{-1} \left( \tanh (\beta \chi_{\mu \rightarrow i} + \beta H_{\mu}) \tanh \left( \frac{\beta \eta_{ii}}{\sqrt{N}} \right) \right),
\]

where

\[
\chi_{\mu \rightarrow i} = \frac{1}{\sqrt{N}} \sum_j \eta_{ij} m_{j \rightarrow \mu},
\]

\[
H_{\mu} = \Xi \cdot z^s_i.
\]

After a few iterations, the log partition function \( \log Z(W) \) could be obtained approximately and thus the gradient of expected log-likelihood in (17) w.r.t. \( \eta \) could be approximated as (Huang, 2019)

\[
\nabla_{\eta_i} \mathbb{E}_{x \sim (W)} [\log p(D | W)] 
\approx \frac{\beta}{S_1 \sqrt{N}} \sum_{a,s} v^a_i \tanh (\beta G^a_{i} + \beta \Xi \cdot z^s_i) - \frac{\beta^2 \eta_{ii}}{S_1 N} \sum_{a,s} [1 - \tanh^2 (\beta G^a_{i} + \beta \Xi \cdot z^s_i)]
\]

\[- \frac{D\beta}{S_2 \sqrt{N}} \sum_s \left[ C_{\mu i} - \frac{\eta_{i} \cdot z^s_i}{\sqrt{N}} \hat{m}_\mu \right],
\]

where

\[
m_i = \tanh \left( \sum_{\mu \in \partial_i} u_{\mu \rightarrow i} \right)
\]

\[
\hat{m}_\mu = \int Dz \tanh \left( \beta \tilde{\chi}_\mu + \beta H_\mu + \beta \hat{\lambda}_\mu z \right)
\]

\[
C_{\mu i} = \hat{m}_\mu m_i + \beta \eta_{ii} \frac{1}{N} (1 - m_i^2) B_\mu
\]

\[
B_\mu = 1 - \int Dz \tanh^2 \left( \beta \tilde{\chi}_\mu + \beta H_\mu + \beta \hat{\lambda}_\mu z \right)
\]

and \( Dz \equiv e^{-z^2/2} / \sqrt{2\pi} dz \), \( \tilde{\chi}_\mu \equiv \frac{\sqrt{N}}{\sqrt{N}} \sum_{j i} \eta_{ij} m_i \), and \( \hat{\lambda}_\mu \equiv \frac{1}{N} \sum_{i \in \partial_\mu} \eta_{ii}^2 (1 - m_i^2) \).

Finally, the update equation in Huang (2019) for the variational parameters \( \eta_{ii} \) is

\[
\eta_{ii}^{t+1} = \eta_{ii}^{t} + \alpha \sum_{x = \pm 1} \left( \log \frac{1 + x m_{ii}}{1 + x \eta_{ii}} - 1 \right) + \alpha \frac{\beta}{S_1 \sqrt{N}} \sum_{a,s} v^a_i \tanh (\beta G^a_{i} + \beta \Xi \cdot z^s_i)
\]

\[- \frac{\beta^2 \eta_{ii}}{S_1 N} \sum_{a,s} [1 - \tanh^2 (\beta G^a_{i} + \beta \Xi \cdot z^s_i)] - \alpha \frac{D\beta}{S_2 \sqrt{N}} \sum_s \left[ C_{\mu i} - \frac{\eta_{i} \cdot z^s_i}{\sqrt{N}} \hat{m}_\mu \right].
\]

Since \( \eta_{ii} \in [-1, 1] \), the update in (27) could not guarantee such constraint. As a result, similar to Baldassi et al. (2018), a heuristic clipping operation is introduced in Huang (2019), which forces the \( \eta_{ii} = 1 \) when \( \eta_{ii} > 1 \) and \( \eta_{ii} = -1 \) when \( \eta_{ii} < -1 \). This trick is heuristic and but works well empirically. One natural question is that: are any principled explanations for the heuristic clipping operation? Or are there any other algorithms without such clipping operation?
3 Training RBMs with Binary Synapses using the Bayesian Learning Rule

In this section, we propose one alternative method to train RBMs with binary synaptic connections using the Bayesian Learning Rule [Khan & Lin (2017)], which is obtained by optimizing the variational objective by using natural gradient descent [Amari (1998); Hoffman et al. (2013); Khan & Lin (2017)]. As demonstrated in Khan & Rue (2019), the Bayesian learning rule can be used to derive and justify many existing learning-algorithms in fields such as optimization, Bayesian statistics, machine learning and deep learning. Note that recently the Bayesian learning rule has been applied in [Meng et al. (2020)] to train binary neural networks for supervised learning. Therefore, this note could be viewed as an extension of [Meng et al. (2020)] to the case of unsupervised learning.

Specifically, to optimize the variational objective in (7), the Bayesian learning rule [Khan & Rue (2019)] considers a class of minimal exponential family distribution

\[ q_{\lambda} (W) := h (\lambda) \exp \left[ \lambda^T \phi(W) - A (\lambda) \right] \] (28)

where \( \lambda \) is the natural parameter, \( \phi(W) \) is the vector of sufficient statistics, \( A(\lambda) \) is the log-partition function, and \( h(W) \) is the base measure. When the prior distribution \( p_0(W) \) follows the same distribution as \( q_{\lambda} (W) \) in (28), and the base measure \( h(W) = 1 \), the Bayesian learning uses the following update of the natural parameter [Khan & Rue (2019)]

\[ \lambda \leftarrow (1 - \alpha) \lambda + \alpha \left\{ \nabla_{\lambda} \mathbb{E}_{q_{\lambda} (W)} \left[ \log p (\mathcal{D} | W) \right] + \lambda_0 \right\} , \] (29)

where \( \alpha \) is the learning rate, \( \eta \) is the expectation parameter of \( q_{\lambda} (W) \), and \( \lambda_0 \) is the natural parameter of the prior distribution \( p_0(W) \). The main idea is to update the natural parameters using the natural gradient. Below we briefly show how to obtain the Bayesian learning rule; for more details, please refer to [Khan & Rue (2019); Khan & Lin (2017)].

To apply the Bayesian learning rule, the posterior approximation \( q_{\lambda} (W) \) is also chosen to be the fully factorized symmetric Bernoulli distribution in (10), which is in fact belonging to the minimal exponential family distribution. In particular, \( q_{\lambda} (W) \) in (10) could be reformulated as follows

\[ q_{\lambda} (W) = \prod_{\mu=1}^{M} \prod_{i=1}^{N} \left( 1 + \eta_{\mu i} \right)^{\frac{1 + \eta_{\mu i}}{2}} \left( 1 - \eta_{\mu i} \right)^{\frac{1 - \eta_{\mu i}}{2}} \] (30)

\[ = \prod_{\mu=1}^{M} \prod_{i=1}^{N} \exp \left\{ \frac{w_{\mu i}}{2} \log \left( 1 + \eta_{\mu i} \right) + \frac{1}{2} \log \left( 1 - \eta_{\mu i}^2 \right) \right\} \] (31)

\[ = \prod_{\mu=1}^{M} \prod_{i=1}^{N} q_{\lambda_{\mu i}} (w_{\mu i}) , \] (32)

where the natural parameter \( \lambda_{\mu i} \), sufficient statistics \( \phi(w_{\mu i}) \), log partition function \( A (\lambda_{\mu i}) \), and the associated expectation parameter \( \eta_{\mu i} = \mathbb{E}_{q_{\lambda_{\mu i}} (w_{\mu i})} [\phi(w_{\mu i})] \) are as follows

\[ \lambda_{\mu i} \equiv \frac{1}{2} \log \left( \frac{1 + \eta_{\mu i}}{1 - \eta_{\mu i}} \right) \] (33)

\[ \phi(w_{\mu i}) \equiv w_{\mu i} \] (34)

\[ A (\lambda_{\mu i}) \equiv -\frac{1}{2} \log \left( \frac{1 - \eta_{\mu i}^2}{4} \right) \] (35)

\[ \eta_{\mu i} \equiv \tanh (\lambda_{\mu i}) . \] (36)

As a result, instead of optimizing the expectation parameters \( \eta_{\mu i} \) using gradient ascent in (11) as Huang (2019), we could update the natural parameters \( \lambda_{\mu i} \) using the Bayesian learning rule in (29).

However, despite using the same Bayesian learning rule, the resultant algorithm for unsupervised learning in this note is quite different from that in Meng et al. (2020) for supervised learning.
Interestingly, as shown in (29), although the natural parameters $\lambda_{\mu i}$ are updated, the gradient is computed w.r.t. the expectation parameters $\eta_{\mu i} = \tanh (\lambda_{\mu i})$, which is already obtained in (22). When the prior $p_0 (W)$ is set to be the form in (8), each element of the natural parameters $\Lambda_0$ could be written as

$$\lambda_0^{\mu i} = \frac{1}{2} \log \left( \frac{1 + m_{\mu i}}{1 - m_{\mu i}} \right).$$

(37)

Therefore, substituting (22) into (29), the natural parameters $\lambda_{\mu i}$ could be updated as

$$\lambda_{\mu i}^{t+1} = \lambda_{\mu i}^t + \alpha \left( \lambda_0^{\mu i} - \lambda_{\mu i}^t \right) + \frac{\beta}{S_1 \sqrt{N}} \sum_{a,s} v_i^n \tanh \left( \beta G_{\mu i}^a + \beta \Xi_{\mu i} z_s^n \right) - \alpha \frac{\beta^2 \eta_{\mu i}}{S_1 N} \sum_{a,s} \left[ 1 - \tanh^2 \left( \beta G_{\mu i}^a + \beta \Xi_{\mu i} z_s^n \right) \right] - \alpha \frac{D \beta}{S_2 \sqrt{N}} \sum_s \left[ C_{\mu i} - \eta_{\mu i} z_s \hat{m}_{\mu i} \right].$$

(38)

It is easy to verify that

$$\lambda_{\mu i}^0 - \lambda_{\mu i}^t = \sum_{x = \pm 1} \frac{x}{2} \left( \log \frac{1 + x m_{\mu i}}{1 + x \eta_{\mu i}} - 1 \right).$$

(39)

Note that there is no need in (38) to explicitly compute the right hand side term of (39), which is different from (27). The resultant algorithm to train RBMs with binary synaptic connections with (38) is termed as Bayesian Binary RBMs (BayesBRBM). Note that in BayesBRBM, the update formula (38) is similar to (27) used in Huang (2019). However, there are two fundamental differences. First, BayesBRBM updates the natural parameters $\lambda_{\mu i}$ of the symmetric Bernoulli distribution while Huang (2019) updates the expectation parameters $\eta_{\mu i}$. One direct advantage is that since $\lambda_{\mu i} \in (-\infty, +\infty)$, no additional clipping operation is needed as Huang (2019). Second, although the update equations (38) and (27) appear the same, they actually correspond to two fundamentally different optimization methods: the former uses natural gradient ascent while the latter uses gradient ascent.

Interestingly, the algorithm in Huang (2019) could be viewed as one kind of first-order approximation of BayesBRBM. Specifically, using first-order Taylor expansion, the expectation parameters $\eta_{\mu i}$ could be approximated as

$$\eta_{\mu i} = \tanh (\lambda_{\mu i}) \approx \lambda_{\mu i}.$$  

(40)

Using the first-order approximation (40), the update equation in (38) is approximated as

$$\lambda_{\mu i}^{t+1} = \lambda_{\mu i}^t + \alpha \sum_{x = \pm 1} \frac{x}{2} \left( \log \frac{1 + x m_{\mu i}}{1 + x \eta_{\mu i}} - 1 \right) + \frac{\beta}{S_1 \sqrt{N}} \sum_{a,s} v_i^n \tanh \left( \beta G_{\mu i}^a + \beta \Xi_{\mu i} z_s^n \right) - \alpha \frac{\beta^2 \lambda_{\mu i}^t}{S_1 N} \sum_{a,s} \left[ 1 - \tanh^2 \left( \beta G_{\mu i}^a + \beta \Xi_{\mu i} z_s^n \right) \right] - \alpha \frac{D \beta}{S_2 \sqrt{N}} \sum_s \left[ C_{\mu i} - \lambda_{\mu i}^t z_s \hat{m}_{\mu i} \right].$$

(41)

where the relation in (39) is explicitly substituted for ease of comparison. It could be seen that the update formula in (41) has exactly the same form as (27) except the exchange of variables between $\lambda_{\mu i}$ and $\eta_{\mu i}$. Since $\eta_{\mu i} \in [-1, +1]$, using first-order approximation (40), the values $\lambda_{\mu i}$ should also be constrained into the range $[-1, +1]$ by using clipping, which is exactly the algorithm in Huang (2019). As a result, the proposed algorithm provides a different perspective on Huang (2019) which justifies its efficacy with heuristic clipping.

### 4 Summary

In this technical note, building on the work in Huang (2019), we propose one optimization method called BayesBRBM (Bayesian Binary RBM) to train RBM with binary Synapses using the Bayesian learning rule. Interestingly, the method in Huang (2019) could be viewed as a first-order approximation of BayesBRBM, which provides an alternative perspective and justifies its efficacy with heuristic clipping. One possible future work is to extend it to deep RBMs with binary synapses and make some detailed comparison of the two algorithms.
Acknowledgments

X. Meng would like to thank Haiping Huang (Sun Yat-sen University) for helpful discussions, and Mohammad Emtiyaz Khan (RIKEN AIP) for explanations on the Bayesian learning rule.

References

Amari, S.-I. Natural gradient works efficiently in learning. *Neural computation*, 10(2):251–276, 1998.

Baldassi, C., Gerace, F., Kappen, H. J., Lucibello, C., Saglietti, L., Tartaglione, E., and Zecchina, R. Role of synaptic stochasticity in training low-precision neural networks. *Physical review letters*, 120(26):268103, 2018.

Hinton, G. E. Training products of experts by minimizing contrastive divergence. *Neural computation*, 14(8):1771–1800, 2002.

Hoffman, M. D., Blei, D. M., Wang, C., and Paisley, J. Stochastic variational inference. *The Journal of Machine Learning Research*, 14(1):1303–1347, 2013.

Huang, H. How data, synapses and neurons interact with each other: a variational principle marrying machine learning and neuroscience. *Physical review letters*, 121(26):268103, 2018.

Kingma, D. P., Salimans, T., and Welling, M. Variational dropout and the local reparameterization trick. In *Advances in neural information processing systems*, pp. 2575–2583, 2015.

Khan, M. E. and Lin, W. Conjugate-computation variational inference: Converting variational inference in non-conjugate models to inferences in conjugate models. *AISTATS*, 2017.

Khan, M. E. and Rue, H. Learning-algorithms from Bayesian principles. 2019. https://emtiyaz.github.io/papers/learning_from_bayes.pdf

Meng, X., Bachmann, R., and Khan, M. E. Training binary neural networks using the Bayesian learning rule. In *International Conference on Machine Learning*, 2020.

Appendix

In this appendix, we briefly introduce the Bayesian learning rule. Please refer to [Khan & Rue (2019); Khan & Lin (2017)] for more details. According to the definition of natural gradient ascent, the update equation follows

$$
\lambda^{t+1} = \lambda^t + \alpha F(\lambda^t)^{-1} \nabla_\lambda \mathcal{L}(q_{\lambda^t}) = \lambda^t + \alpha \nabla_\lambda \mathcal{L}(q_{\lambda^t}),
$$  \hspace{1cm} (42)

where $F(\lambda^t)^{-1} \nabla_\lambda \mathcal{L}(q_{\lambda^t})$ denotes the natural gradient of $\mathcal{L}(q_{\lambda^t})$ with respect to (w.r.t) $\lambda$ at $\lambda = \lambda_t$, where $\nabla_\lambda \mathcal{L}(q_t)$ is the gradient of $\mathcal{L}(q_t)$ w.r.t $\lambda$ and $F(\lambda_t)$ is the Fisher information matrix (FIM)

$$
F(\lambda) \equiv \mathbb{E}_q(W) \left[ \nabla_\lambda \log q_{\lambda}(W) \nabla_\lambda \log q_{\lambda}(W)^T \right].
$$  \hspace{1cm} (43)

As a result, to update natural parameters using the natural gradient we need to compute the inverse FIM, which is intractable in general. Fortunately, for minimal exponential family distribution $q(W)$ in (28), there exists a concise result since $F(\lambda^t)^{-1} \nabla_\lambda \mathcal{L}(q_{\lambda^t}) = \nabla_\eta \mathcal{L}(q_{\lambda^t})$ where $\eta$ is the expectation parameter of exponential family distribution $q_{\lambda}(W)$. As a result, $\nabla_\lambda \mathcal{L}(q_{\lambda^t}) = \nabla_\eta \mathcal{L}(q_{\lambda^t})$ so that the natural gradient update in (42) could be equivalently written as

$$
\lambda^{t+1} = \lambda^t + \alpha \nabla_\mu \mathcal{L}(q_{\lambda^t}),
$$  \hspace{1cm} (44)

where, from the definition of $\mathcal{L}(q_{\lambda})$ in (7), there is

$$
\nabla_\eta \mathcal{L}(q_{\lambda^t}) = \nabla_\eta \mathbb{E}_{q_{\lambda^t}(W)}[\log p(D | W)] - (\lambda^t - \lambda_0).
$$  \hspace{1cm} (45)

Substituting (45) into (44) leads to the Bayesian learning rule in (29).