Norm of Gaussian integers in arithmetical progressions and narrow sectors

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Abstract. We proved the equidistribution of the Gaussian integer numbers in narrow sectors of the circle of radius $x^{\frac{1}{2}}$, $x \to \infty$, with the norms belonging to arithmetic progression $N(\alpha) \equiv \ell \pmod{q}$ with the common difference of an arithmetic progression $q$, $q \ll x^{\frac{2}{3} - \varepsilon}$.

Introduction

For the classical arithmetic functions $\tau(n)$ (the number of divisors for the positive integer $n$) and $r(n)$ (the number of representations for the positive integer $n$ as sum of two squares of integers) there were obtained the asymptotic formulas of the sums

$$\sum_{n \equiv \ell \pmod{q}} \tau(n) \quad \text{and} \quad \sum_{n \equiv \ell \pmod{q}} r(n),$$

where $q$ grows together with $x$ and they are nontrivial for $q \ll x^{\frac{2}{3} - \varepsilon}$.

For the function $\tau(n)$ K. Liu, I. Shparlinskii and T. Zhang ([2]) obtained the extended region of non-triviality.

In the present paper we investigate the distribution of points from complex plane $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$, $\varphi_1 < \arg(x + iy) \leq \varphi_2$, $\varphi_2 - \varphi_1 < 2010$ MSC: 11L07, 11T23.

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\[ \pi, x^2 + y^2 \equiv \ell \pmod{q}, x^2 + y^2 \leq N. \]
Using the property of Hecke Z-function of the quadratic field \( \mathbb{Q}(i) \) and the estimates of special exponential sums, we obtain a non-trivial asymptotic formula for the number of integer points under the circle’s sectorial region in arithmetic progression with the growing difference progression.

Throughout this paper we use the following notations.
- \( p \) denotes a prime number in \( \mathbb{Z} \);
- the Latin letters \( a, b, k, m, n, \ell \) be the positive integers;
- \( \Re z \) denotes the real part of \( z \) and \( \Im z \) be the imaginary part of \( z \);
- through \( \mathbb{Z} \) we denote the ring of integers;
- \( G = \mathbb{Z}[i] \) denotes the ring of Gaussian integers \( a + bi, a, b \in \mathbb{Z}, i^2 = -1 \);
- \( G_\gamma \) (respectively, \( G_\gamma^* \)) be the ring of residue classes modulo \( \gamma \) (respectively, the multiplicative group of inversive element in \( G_\gamma \));
- \( N(\omega) \) is the norm of \( \omega \in G, N(\omega) = |\omega|^2 \);
- \( Sp(\omega) \) is the trace of \( \omega \) from \( \mathbb{Q}(i) \) to \( \mathbb{Q} \), \( Sp(\omega) = 2\Re \omega \);
- symbols “\( \ll \)” and “\( O \)” are equivalent;
- \( s = \sigma + it \in \mathbb{C}, \Re s = \sigma, \Im s = t \);
- \( \chi \) denotes the Dirichlet character modulo \( q \) over \( \mathbb{Z} \);
- \( \gcd (a, q) \) in \( \mathbb{Z} \);
- \( \gcd (\alpha, \omega) \) in \( G \);

1. Auxiliary results

Let \( \delta_1, \delta_2 \in \mathbb{Q}(i) \) and \( s = \sigma + it \). For the rational integer number \( m \) let us define the function sized by absolutely convergent series into semiplane \( \Re s > 1 \):

\[
Z_m(s, \delta_1, \delta_2) := \sum_{\omega \in G} e^{4mi \arg \omega + \delta_1} e^{\pi i Sp(\delta_2 \omega)}.
\]

It is obvious that with \( m = 0 \) we get the Epstein zeta-function. With \( \delta_1, \delta_2 \in \mathbb{Q}_i \) we get the Hecke Z-function over the imaginary quadratic field \( \mathbb{Q}(i) \).

Let \( p > 2 \) be a prime rational number, \( n \in \mathbb{N} \). Denote

\[
E_{p^n} := \{ \alpha \in G_{p^n} \mid N(\alpha) \equiv \pm 1 \pmod{p^n} \}.
\]

It is also obvious that \( E_n \) is the subgroup of multiplicative group of residue classes modulo \( p^n \) over the ring \( G_{p^n} \).

We call \( E_{p^n} \) the norm group in \( G_{p^n}^* \).
Lemma 1. Let \( p \equiv 3 \pmod{4} \) and \( E_n \) be the norm group in \( G_{p^n} \). Then \( E_n \) is the cyclic group, \( |E_n| = 2(p+1)p^{n-1} \), and let \( u + iv \) be a generative element of \( E_n \). Then exist \( x_0, y_0 \in \mathbb{Z}_{p^n}^* \) such that

\[
(u + iv)^{2(p+1)} \equiv 1 + p^2x_0 +ipy_0, \quad 2x_0 + y_0^2 \equiv -2p^2x_0^2 \pmod{p^3}.
\]

Moreover, we have modulo \( p^n \) for any \( t = 4, 5, \ldots, p^{n-1} - 1 \),

\[
\Re \left( (u + iv)^{2(p+1)t} \right) = A_0 + A_1t + A_2t^2 + \cdots \quad \text{and} \quad \Im \left( (u + iv)^{2(p+1)t} \right) = B_0 + B_1t + B_2t^2 + \cdots,
\]

where

\[
A_0 \equiv 1 \pmod{p^4}, \quad B_0 \equiv 0 \pmod{p^4}, \\
A_1 \equiv p^2x_0 + \frac{1}{2}p^2y_0^2 \equiv -\frac{5}{2}x_0^2p^4 \pmod{p^5}, \\
B_1 \equiv p^2y_0(1 - p^2x_0) \pmod{p^4}, \\
A_2 \equiv -\frac{5}{2}x_0^2p^2 \pmod{p^5}, \quad B_2 \equiv \frac{5}{3}p^3x_0y_0 \pmod{p^4}, \\
A_j \equiv B_j \equiv 0 \pmod{p^3}, \quad j = 3, 4, \ldots
\]

(In greater details see [3])

Denote

\[
(u + iv)^k = u(k) + iv(k), \quad 0 \leq k \leq 2p + 1,
\]

\[
(u + iv)^{2(p+1)t+k} \equiv \sum_{j=0}^{n-1} (A_j(k) + iB_j(k))t^k \pmod{p^n}.
\]

It is obvious that

\[
A_j(k) = A_j u(k) - B_j v(k), \quad B_j(k) = A_j v(k) + B_j u(k).
\]

Thus from Lemma 1 we infer

Corollary. For \( k = 0, 1, \ldots, 2p + 1 \) we have

\[
u(0) = 1, \quad v(0) = 0, \quad (u(p+1), p) = 1, \quad p||v(p+1);
\]

\[
(u(k), p) = (v(k), p) = 1 \quad \text{for} \quad k \not\equiv 0 \pmod{\frac{p+1}{2}};
\]

\[
u(k) \equiv 0 \pmod{p}, \quad (v(k), p) = 1 \quad \text{if} \quad k = \frac{p+1}{2} \quad \text{or} \quad \frac{3p+1}{2};
\]

\[
u(k) \equiv u(-k), \quad v(k) \equiv -v(-k).
\]
Hence, for \( k \not\equiv 0 \pmod{\frac{p+1}{2}} \) we have

\[
A_0(k) \equiv u(k) \pmod{p}, \quad B_0(k) \equiv v(k) \pmod{p},
\]
\[
A_1(k) \equiv -py_0v(k), \quad B_1(k) \equiv py_0u(k) \pmod{p^2},
\]
\[
A_2(k) \equiv -\frac{5}{2}x_0^2p^2u(k), \quad B_2(k) \equiv -\frac{5}{2}x_0^2p^2v(k) \pmod{p^4}.
\]

For \( k = \frac{p+1}{2} \) or \( \frac{3p+1}{2} \) we obtain

\[
p||A_1(k), \quad p^2||B_1(k), \quad p^2||A_2(k), \quad B_2(k) \equiv 0 \pmod{p^3}.
\]

Moreover,

\[
A_1(0) \equiv -\frac{5}{2}x_0^2p^4 \pmod{p^5}, \quad B_1(0) \equiv 0 \pmod{p^4},
\]
\[
A_2(0) \equiv -\frac{5}{2}x_0^2p^2 \pmod{p^5}, \quad B_2(0) \equiv 0 \pmod{p^3}, \quad p^2||A_1(p+1),
\]
\[
p||B_1(p+1), \quad p^2||A_2(p+1), \quad B_2(p+1) \equiv 0 \pmod{p^3}.
\]

At last for all \( k = 0, 1, \ldots, 2p + 1 \)

\[
A_j(k) \equiv B_j(k) \equiv 0 \pmod{p^3}, \quad j = 3, 4, \ldots.
\]

Lemma 2. Let \( q = p^\ell \) with \( \ell \geq 1 \), \( g(y) \) is the polynomial in form

\[
g(y) = A_1y + pA_2y^2 + p^{\lambda_3}A_3y^3 + \cdots + p^{\lambda_k}A_ky^k, \quad k \geq 3,
\]

with \( A_j \in \mathbb{Z}, (A_j, p) = 1, \) \( j = 3, \ldots, k, \) \( 2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_k \). Then we have

\[
S_q := \sum_{y=1}^{q-1} e^{2\pi i \frac{g(y)}{p^\ell}} = p^{\left\lfloor \frac{\ell}{2} \right\rfloor} \sum_{y \in \mathbb{Z}_p^{\ell/2}} B_q(y),
\]

where

\[
B_q(y) = \begin{cases} 
0 & \text{if } (A_1, p) = 1, \\
1 & \text{if } \ell \equiv 0 \pmod{2}, \quad A_1 \equiv 0 \pmod{p}, \\
\sum_{z=0}^{p-1} e^{2\pi i \frac{(A_1 + 2A_2)z + z^2}{p}} & \text{if } \ell \equiv 1 \pmod{2}, \quad A_1 \equiv 0 \pmod{p}.
\end{cases}
\]

Proof. The proof of this assertion repeats the proofs of Lemmas 12.3 and 12.4 in [1].
For \( p \equiv 1 \pmod{4} \) or \( p = 2 \) the norm groups are not the cyclic groups. We shall use the description of the solutions \( x^2 + y^2 \equiv 1 \pmod{p^n} \) for these cases.

**Lemma 3.** Let \((x, y)\) is a solution of the congruence \( x^2 + y^2 \equiv 1 \pmod{p^\ell} \), \( p > 2 \) is a prime number. Then all solutions with \((x_0, p) = 1\) are described in the following manner

\[
x = x(0)f(y_0, t), \quad y = y_0 + pt, \quad t = 0, 1, \ldots, p^{\ell-1} - 1,
\]

where \( x(0) \) runs all solutions of the congruence

\[
x^2 \equiv 1 - y_0^2 \pmod{p^n},
\]

\( y_0 \) runs all solutions of the congruence

\[
x_0^2 + y_0^2 \equiv 1 \pmod{p}
\]

with \( x_0 \not\equiv 0 \pmod{p} \), and

\[
f(y_0, t) = 1 + p\frac{y_0}{y_0^2 - 1} t + p^2 \frac{1 - y_0^2}{y_0^2 - 1} t^2 + p^{\lambda_3}X_3(y_0)t^3 + \cdots + p^{\lambda_s}X_s(y_0)t^s,
\]

under conditions \((X_j(y_0), p) = 1, \lambda_j \geq 3, s \leq \left[ \frac{p-1}{p-2} \right] \).

For the solutions of the congruence \( x^2 + y^2 \equiv 1 \pmod{p^\ell} \) with \( x_0 \equiv 0 \pmod{p} \) we have

\[
x = pt, \quad y \equiv \pm \left( 1 - \frac{1}{2} p^2 t^2 \right) \pmod{p^4}.
\]

(Here, the multiplicative inverse for 2 and \( y_0^2 - 1 \) is considered modulo \( p^n \)).

**Lemma 3'.** Let \( s = \left[ \frac{\ell-1}{2} \right] \). There exists the polynomial

\[
f(t) = 1 + 2^{\lambda_1}A_1t + \cdots + 2^{\lambda_s}A_st^{2s},
\]

with \( A_j \equiv 1 \pmod{2}, \lambda_j \geq 2j + 1, j = 1, \ldots, s \), such that all solutions of the congruence \( x^2 + y^2 \equiv 1 \pmod{p^\ell} \) can be written as

\[
x = 4t, \quad y = \pm f(t) \quad \text{or} \quad x = 4t, \quad y = \pm (2^{\ell-1} - 1)f(t),
\]

\[
t = 0, 1, \ldots, 2^{\ell-2} - 1.
\]
Lemma 4. Let us $I(\ell, q)$ be the number of solutions of the congruence
\[ u^2 + v^2 \equiv a \pmod{q}, \quad (a, q) = \prod_{p|q} p^{t_0}. \]

Then we have
\[
I(a, q) = c(a, q)q \prod_{p^t || q} \left( 1 - \frac{\chi_4(p^{t_0+1})}{p} \right) \left( 1 - \frac{\chi_4(p^{t-t_0})}{p} \right) + \left( 1 - \frac{1}{p} \right) \sum_{b=t-t_0}^{t-1} \chi_4(p^{t-b}),
\]
where
\[
c(a, q) = \begin{cases} 
1 & \text{if } (q, 2) = 1, \\
1 & \text{if } 2 || q, \\
1 & \text{if } q \equiv 0 \pmod{4}, \; t_0 > t - 2, \\
2 & \text{if } q \equiv 0 \pmod{4}, \; t_0 < t - 2 \text{ and } \frac{a}{2q} \equiv 1 \pmod{4}, \\
0 & \text{if } q \equiv 0 \pmod{4}, \; t_0 \leq t - 2 \text{ and } \frac{a}{2q} \equiv 3 \pmod{4}.
\end{cases}
\]

This lemma follows from the equation
\[
I(a, p^t) = \sum_{u,v \in \mathbb{Z}_{p^t}} \frac{1}{p^t} \sum_{z \in \mathbb{Z}^*_p} e^{\frac{2\pi i (z^2 + v^2 - \ell)}{p^t}},
\]
and the values of the Gaussian sums $\sum_{x \in \mathbb{Z}_{p^t}} e^{\frac{2\pi i x^2}{p^t}}$.

Similarly, we obtain the description of the solutions of the congruence $x^2 + y^2 \equiv -1 \pmod{p^\ell}$, $p \equiv 1 \pmod{4}$. Indeed, let $c_0$ be the solution of the congruence $x^2 \equiv -1 \pmod{p^\ell}$. Then
\[
x = c_0 x(0) f_1(y_0, t), \quad y = y_0 + pt, \quad t = 0, 1, \ldots, p^\ell - 1 - 1,
\]
where $f_1(y_0, t)$ is as $f(y_0, t)$.

2. The main results

We consider the generalized Hecke $Z$-function of quadratic field $\mathbb{Q}(i)$
\[
Z_m(s; \delta_1, \delta_2) := \sum_{\substack{\omega \in G \\
\omega \neq \delta_1}} \frac{e^{4mi \arg(\omega + \delta_1)}}{N(\omega + \delta_1)} e^{\pi i Sp(\omega \delta_2)}, \quad (\Re s > 1),
\]
where $\delta_1, \delta_2 \in \mathbb{Q}(i)$, $m \in \mathbb{Z}$. This function satisfies the functional equation

$$
\pi^{-1} \Gamma(2|m| + s) Z_m(s; \delta_1, \delta_2) = \pi^{-(1-s)} \Gamma(2|m| + 1 - s) Z_{-m}(1 - s; -\delta_2, \delta_1) e^{\pi i S p(\delta_1 \delta_2)}.
$$

(9)
The function $Z_m(s; \delta_1, \delta_2)$ is an entire function except the case $m = 0$ and the Gaussian integer $\delta_2$ when $Z_m(s; \delta_1, \delta_2)$ is holomorphic for all complex $s$ exclusive $s = 1$ where it has a simple pole with residue $\pi$.

We define the multiplicative character modulo $q$ over $G_{p^f}$ as

$$
\chi(\omega) = \chi_{p^f}(N(\omega)),
$$

where $\chi_{p^f}$ is the character modulo $p^f$ in $\mathbb{Z}_{p^f}^\times$.

Let $\Xi_m(\omega) := e^{4 m i \arg \omega} \chi(\omega) = e^{4 m i \arg \omega} \chi_{p^f}(N(\omega))$. Then from (9) we have for $Z(s; \Xi_m) := \sum_\omega \frac{\Xi_m(\omega)}{N(\omega)}$ the following functional equation

$$
Z(s; \Xi_m) = \kappa(\Xi_m) \Psi(s, \Xi_m) Z(1 - s, \Xi_m),
$$

(10)

where

$$
\kappa(\Xi_m) = \left( N(p^f) \right)^{-\frac{1}{2}} \sum_{\tau \in G_{p^f}} \chi(N(\tau)) e^{\frac{S p^f}{p^f} \tau},
$$

$$
\Psi(s, \Xi_m) = \left( \frac{1}{\pi} N(p^f)^{\frac{1}{2}} \right)^{-2s} \frac{\Gamma(2|m| + 1 - s)}{\Gamma(2|m| + s)}.
$$

(11)

Denote

$$
r_m(n) = \sum_{u,v \in \mathbb{Z} \atop u^2 + v^2 = n} e^{4 m i \arg (u+iv)}.
$$

From this we have

$$
\sum_{n \leq x} r_m(n) \chi_{p^f}(n) = \sum_{u,v \in \mathbb{Z} \atop u^2 + v^2 = n \leq x} e^{4 m i \arg (u+iv)} \chi_{p^f}(n).
$$

Therefore,

$$
F_m(s) = \sum_{n=1}^{\infty} \frac{r_m(n)}{n^s} = \sum_{\chi} \chi_q(a) \cdot Z(s; \Xi_m).
$$
We get by the Perron’s formula on an arithmetic progression with $c > 1$, $T > 1$, $(a, p^\ell) = 1$, $0 < \varepsilon < \frac{1}{2}$ the following equality

$$
\sum_{n \equiv a \pmod{p^\ell}} r_m(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F_m(s) \frac{x^s}{s} ds + O \left( \frac{x^c}{T^{\ell}(c-1)} \right) + O(x^\varepsilon)
$$

$$
= \res_{s=0,1} \left( F_m(s) \frac{x^s}{s} \right) + \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} F_m(s) \frac{x^s}{s} ds + \max_{-\varepsilon \leq \Re s \leq c} \left| \frac{1}{s} F_m(s)x^s \right| + O \left( \frac{x^c}{T^{\ell}(c-1)} \right) + O(x^\varepsilon),
$$

(12)

where $\varepsilon$ is a positive arbitrary small number.

From the functional equation for $Z(s, \Xi)$, summing all over character $\chi_{p^\ell}$, we have for $\Re s < 0$

$$
F_m(s) = \pi^{-1+2s} \frac{\Gamma(2|m| + 1 - s)}{\Gamma(2|m| + s)} \times \sum_{\omega \in G \atop (\omega, p^\ell) = 1} \frac{e^{-4mi \arg \omega}}{N(\omega)^{1-s}} \sum_{\tau \in G_{p^\ell}^*} \frac{e^{Sp(\tau)}}{N(\tau) \equiv a N(\omega) \pmod{p^\ell}}.
$$

Consider the sum

$$
\sum_0 := \sum_{\tau \in G_{p^\ell}^* \atop N(\tau) \equiv a \pmod{p^\ell}} e^{\pi i Sp(\tau)},
$$

For $p \equiv 3 \pmod{4}$, we apply the representation of elements from the norm group $E_{p^\ell}$. Lemma 1 and its Corollary give

$$
\sum_0 = \sum_{k=0}^{2p+1} e^{2\pi i \frac{A_0'(k)}{p^\ell}} \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i \frac{A_1'(k)t + A_2'(k)t^2 + \ldots}{p^{\ell}}} + \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i \frac{A_1'(0)t + A_2'(0)t^2 + \ldots}{p^{\ell}}}
$$

$$
+ \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i \frac{A_1'(p+1)t + A_2'(p+1)t^2 + \ldots}{p^{\ell}}},
$$

where $A_0'(0)$ and $A_0'(p+1)$ differ from $A_j(j)$ and $A - j(p+1)$ only by the multiplier $N(\omega)a$. 

Now Lemma 3 gives
\[
E_0 = p^{\frac{\ell}{2}} \left( e^{2\pi i \frac{A_0'(0)}{p^\ell}} + e^{-2\pi i \frac{A_0'(p+1)}{p^\ell}} \right)
\]
\[\times \begin{cases} 
1 & \text{if } \ell \equiv 0 \pmod{2}, \\
e^{-2\pi i (2A_0')^{-1} \frac{A_0'(2)}{p}} & \text{if } \ell \equiv 1 \pmod{2}.
\end{cases}\] (13)

If \( p \equiv 1 \pmod{4} \) or \( p = 2 \) we use Lemma 1 and then obtain \( E_0 = O\left(p^\frac{3}{2}\right) \) with an absolute constant in the symbol "O".

Now we able to prove the main theorems.

Let us denote through \( A(x; \varphi_1, \varphi_2; a, p^\ell) \) the number of points \((u, v)\) in the circle \((u^2 + v^2) \leq x\) under conditions
\[
u, v \in \mathbb{Z}, \quad \varphi_1 < \text{arg}(u + iv) \leq \varphi_2, \\
u^2 + v^2 \equiv a \pmod{p^\ell}, \quad (a, p^\ell) = 1.
\] (14)

Theorem 1. For \( x \to \infty \) the following estimate holds,
\[
\sum_{\substack{n \equiv a \pmod{p^\ell} \\
n \leq x}} r_m(n) = \varepsilon \frac{\pi x}{p^\ell} k_0 \left( 1 - \frac{\chi_4(p)}{p} \right)
\]
\[+ O\left( \frac{x^{\frac{1}{2} + \varepsilon}}{p^\frac{\ell}{2}} M^{1+\varepsilon} \right) + O\left( p^\frac{\ell}{2} M^{1+\varepsilon} \right),\] (15)

holds, where \( \varepsilon_m = 0 \) if \( m \neq 0, \varepsilon_0 = 1, k_0 = 1 \) if \( p > 2, \) or \( k = 2 \) if \( p = 2, \) \( \ell \geq 3; \) \( M = |m| + 3, \varepsilon > 0 \) is an arbitrary small number; constants in the symbols can depend only on \( \varepsilon. \)

Proof. The function \( F_m(s) \) has a pole in \( s = 1 \) only if \( m = 0: \)
\[
\text{res}_{s=1} F_0(s) = \frac{\pi x}{p^\ell} k_0 \left( 1 - \frac{\chi_4(p)}{p} \right).
\]

The estimate for \( F_m(0) \) is easy proving by the Phragmen-Lindelöf principle and the estimates of \( Z_m(s) \) on the bounds of stripe \(-\varepsilon \leq \Re s \leq 1 + \varepsilon. \)
Therefore, we have
\[
\text{res}_{s=0} F_m(s) \ll p^\frac{\ell}{2} (|m| + 3) \log (|m| + 3).
\]
Hence,
\[
\sum_{n \equiv a \pmod{p^\ell}} r_m(n) = \varepsilon_m \frac{\pi x}{p^\ell} \sum_{u,v \in \mathbb{Z}_{p^\ell}} u^2 + v^2 \equiv a \pmod{p^\ell} 1 + O \left( p^\ell (|m| + 3) \log (|m| + 3) \right) + \\
+ \frac{1}{2\pi i} \int_{-\varepsilon - iT}^{-\varepsilon + iT} F_m(s) \frac{x^s}{s} ds + O \left( \frac{x^c}{Tp^\ell(c - 1)} + x^\varepsilon \right).
\]

Note that
\[
\varepsilon_m \frac{\pi x}{p^\ell} \sum_{u,v \in \mathbb{Z}_{p^\ell}} u^2 + v^2 \equiv a \pmod{p^\ell} 1 = \varepsilon_m \frac{\pi x}{p^\ell} k_0 \left( 1 - \frac{\chi_4(p)}{p} \right),
\]
where
\[
F_m(s) = \pi^{1 - 2s} \frac{\Gamma(2|m| + 1 - s)}{\Gamma(2|m| + s)} \times \sum_{\omega \in G_{p^\ell}^*} \frac{e^{-4mi \arg \omega}}{N(\omega)^{1-s}} \sum_{\tau \equiv a \pmod{p^\ell}} e^{\pi i \frac{Sp(\tau)}{p^\ell}}.
\]

Thus, using the estimate of the sum \(\sum_0\) and the Stirling formula for the gamma-function \(\Gamma(z)\), we at once obtain the estimate of the integral in (16)
\[
\frac{1}{2\pi i} \int_{-\varepsilon - iT}^{-\varepsilon + iT} F_m(s) \frac{x^s}{s} ds \ll T^{1+2\varepsilon} p^\ell \ll T^{1+2\varepsilon}.
\]

Choosing \(c = 1 + (\log x)^{-1}\), \(T = \frac{x^{1/2}}{p^{3/4}}\), we get assertion of Theorem 1. \(\square\)

The following theorems stem from this result and Vinogradov’s lemma (see, [4], Lemma 12, pp. 261-262).

**Theorem 2.** In the sectorial region \(u^2 + v^2 \leq x\), \(u^2 + v^2 \equiv a \pmod{p^\ell}\), \(\varphi_1 < \arg(u + iv) \leq \varphi_2\), \(\varphi_2 - \varphi_1 \gg x\) the following asymptotic formula holds:
\[
A(x; \varphi_1, \varphi_2; a, p^\ell) := \sum_{u,v \in \mathbb{Z}_{p^\ell}} 1 = \\
\sum_{u,v \equiv a \pmod{p^\ell}} \sum_{u^2 + v^2 \leq x} 1 = \\
\frac{\varphi_2 - \varphi_1}{2} \cdot k_0 x^{1/2} \left( 1 - \frac{\chi_4(p)}{p} \right) + O \left( \frac{x^{1/2+\varepsilon}}{p^{3/4}} \right).
\]
Theorem 3. Let \( p \) be a prime number, \( \ell \geq 3 \), and \( p^{\frac{3\ell}{4\kappa}} \leq x \leq p^{2\ell} \), \( 0 < \kappa \leq \frac{1}{8} - \frac{1}{4\ell} \), \( \varphi_2 - \varphi_1 \gg x^{-\kappa} \). Then we have

\[
A(x; \varphi_1, \varphi_2; a, p^\ell) = \frac{\varphi_2 - \varphi_1}{2} \cdot x \cdot \left( 1 - \frac{\chi_4(p)}{p} \right) + O \left( \frac{x^{1-\kappa}}{p^\ell} \log x^\kappa \right).
\]

Actually, in Vinogradov's lemma we take \( \Omega = \frac{\pi}{2} \), \( \delta = x^\kappa \), \( \Delta = x^{-\alpha} \) and let \( \Delta \leq \varphi_2 - \varphi_1 < \frac{\pi}{4} - 2\kappa \). Then \( f(\varphi_1, \varphi_2) \) be the function from that lemma.

Consider the function

\[
\Phi(\varphi_1, \varphi_2) = \frac{1}{4} \sum_{u^2+v^2 \leq x} f(\text{arg}(u + iv)).
\]

Then we have

\[
\Phi(\varphi_1, \varphi_2) = \sum_{\substack{u^2+v^2 \leq x \\ \text{mod} \ p^\ell}} \sum_{m=-\infty}^{\infty} a_m e^{4mi \text{arg}(u + iv)} = \\
\sum_{m=-\infty}^{\infty} a_m \sum_{n \equiv a \pmod{p^\ell}} r_m(n),
\]

(here \( a_m \) are the coefficients from the Vinogradov's lemma).

We take \( r = 3 \) (in the notation of the Vinogradov's lemma) and take into account that

\[
a_0 = \frac{1}{\Omega} (\varphi_2 - \varphi_1 + \Delta) \\
|a_m| \leq \begin{cases} \\
\frac{1}{\Omega} (\varphi_2 - \varphi_1 + \Delta) & \text{if} \ m \neq 0, \\
\frac{2}{\pi |m|} \left( \frac{r \Omega}{\pi |m| \Delta} \right)^r & \text{if} \ m = 0,
\end{cases}
\]

then after simple calculations we get Theorem 2 and Theorem 3.

Taking into account that Hecke characters and Gauss exponential sums have the multiplicative properties modulo \( q \), we have the following assertion.
Theorem 4. In the sectorial region \( u^2 + v^2 \leq x, u^2 + v^2 \equiv a \mod q \), \( \varphi_1 < \arg(u + iv) \leq \varphi_2, \varphi_2 - \varphi_1 \gg x \) the following asymptotic formula holds:

\[
A(x; \varphi_1, \varphi_2; a, q) := \sum_{\substack{u,v \mid u^2 + v^2 \equiv a \mod q \\ \varphi_1 < \arg(u + iv) \leq \varphi_2 \\ u^2 + v^2 \leq x}} 1 = \frac{\varphi_2 - \varphi_1}{2} \cdot \frac{k_0 x}{q} \prod_{p \mid q} \left( 1 - \frac{\chi_4(p)}{p} \right) + O \left( \frac{x^{1+\varepsilon}}{q^{1/4}} \right).
\]

Remark. The result of Theorem 1 can be improved in case \( p \equiv 3 \mod 4 \) and \( \ell \geq 3 \) in view of the fact that we have the precise meaning of the sum \( E_0 \) (see (13)).

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