HOMOGENIZATION OF A STOCHASTIC VISCOUS TRANSPORT EQUATION

IOANA CIOTIR*, NICOLAS FORCADEL AND WILFREDO SALAZAR

Normandie Univ, INSA de Rouen Normandie, LMI (EA 3226 - FR CNRS 3335), 76000 Rouen, France, 685 Avenue de l’Université, 76801 St Etienne du Rouvray cedex.

(Communicated by Viorel Barbu)

Abstract. In the present paper we prove an homogenisation result for a locally perturbed transport stochastic equation. The model is similar to the stochastic Burgers’ equation and it is inspired by the LWR model. Therefore, the interest in studying this equation comes from its application for traffic flow modelling. In the first part of paper we study the inhomogeneous equation. More precisely we give an existence and uniqueness result for the solution. The technical difficulties of this part come from the presence of the function ϕ under assumptions coherent for the model, which is giving the inhomogeneity with respect to the space variable, not present in the classical results. The second part of the paper is the homogenisation result in space.

1. Introduction.

1.1. General model. The goal of this paper is to present a homogenization result for a stochastic Burgers’ type equation perturbed by white noise. We consider the following perturbed transport equation with a viscosity term and with a random force which is a white noise in space and time,

\[ du^\delta(t,x) = \left( \mu \frac{\partial^2 u^\delta(t,x)}{\partial x^2} + \varphi \left( \frac{x}{\delta} \right) \frac{\partial}{\partial x} g(u^\delta(t,x)) \right) dt + dW, \quad x \in [0,1], \ t > 0. \ (1.1) \]

Here \( u \) represents a certain density in space and time and \( g : \mathbb{R} \to [0, +\infty) \) denotes the flux. Finally, \( \mu \) is a positive constant, the function \( \varphi : \mathbb{R} \to [0, +\infty) \) is a local perturbation (non-periodic) and \( W \) is a \( Q \)-Wiener process in \( L^2(0,1) \).

More precisely, we consider an operator \( Q \) which is a trace class non-negative operator on \( L^2(0,1) \) and we define

\[ W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t), \quad t \geq 0, \]

2020 Mathematics Subject Classification. Primary: 35B27, 60H15; Secondary: 76F25.

Key words and phrases. Homogenization, stochastic transport equation, mild solution.

This work was partially supported by the European Union with the European regional development fund (ERDF, HN0002137 and 18P03390/18E01750/18P02733) and by the Normandie Regional Council (via the M2NUM and M2SiNum projects). The first author was partially supported by the ANR Project QUTE-HPC Quantum Turbulence Exploration by High-Performance Computing (ANR-18-CE46-0013).

* Corresponding author: Ioana Ciotir.
where \( \{e_k\} \) is an orthonormal basis of \( L^2(0,1) \), \( \{\lambda_k\} \) is the family of eigenvalues of the operator \( Q \) (i.e. \( Qe_k = \lambda_k e_k, \ k \in \mathbb{N} \)), and \( \{\beta_k\}_k \) is a sequence of mutually independent real Brownian motions in a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), adapted to a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \).

The model is similar from the stochastic Burgers’ equation which is used to study turbulent flows in the presence of random forces. Note that, even if the deterministic model is not realistic because it does not display any chaos, the situation is different when the force is a random one, as in our case.

Several authors studied the stochastic Burgers equation in one dimension, driven by additive or multiplicative noise as model describing turbulences (see [4], [2], [7], [8]). A result concerning the homogenization was proved for the integro-differential deterministic Burgers equation in [1].

Equation (1.1) is also inspired to the LWR model (see for instance [9, 10]) for traffic flow. In fact, the LWR model is a transport equation on the density of vehicles \( \rho \):

\[
\rho_t(t,x) + \rho v(\rho) = 0 \quad \text{for all} \quad (t,x) \in [0, +\infty) \times \mathbb{R},
\]

where \( v \) is called the average speed. Notice that equation (1.1) contains an extra viscosity term \( \mu \rho_{xx} \). In the deterministic case, the LWR model with a viscosity term is not used since under certain conditions the vehicles can have negative velocities (see [6]). However, the stochastic case is different and we can consider \( \mu \) very small.

To the best of our knowledge, this is the first paper which treats the stochastic equation (1.1) and its homogenization.

The problem (1.1) shall be studied with the following boundary conditions,

\[
u^\delta(0, t) = u^\delta(1, t) = 0, \quad (1.2)
\]

and the initial condition,

\[
u^\delta(0, x) = u_0(x), \quad \text{for all} \quad x \in [0, 1]. \quad (1.3)
\]

We reformulate (1.1)-(1.2)-(1.3) as the following abstract stochastic equation,

\[
\begin{cases}
\frac{du^\delta}{dt} = (Au^\delta + B^\delta g(u^\delta)) dt + dW, \\
\quad u^\delta(0) = u_0,
\end{cases} \quad (1.4)
\]

where the self-adjoint operator \( A \) on \( L^p(0,1) \) is defined by

\[
Au = \mu \frac{\partial^2}{\partial x^2} u \quad \text{for} \quad u \in D(A) = \{ u \in H^2(0,1) \ s.t. \ u(0) = u(1) = 0 \}, \quad (1.5)
\]

and the operator \( B^\delta \) on \( L^p(0,1) \), for \( p \geq 3 \), is defined, for all \( \delta > 0 \), by

\[
B^\delta u = \varphi \left( \frac{x}{\delta} \right) \frac{\partial}{\partial x} u. \quad (1.6)
\]

In fact, since \( A \) is a self-adjoint negative operator in \( L^2(0,1) \), we have

\[
A e_k = -\alpha_k e_k, \quad n \in \mathbb{N},
\]

where \( \{\alpha_k\} \) is the sequence of positive eigenvalues.

We consider the following assumptions

(A1) (Uniform bound on \( \varphi \) and \( g \)). We have for all \( x \in \mathbb{R} \),

\[
|\varphi(x)| \leq ||\varphi||_{\infty} < +\infty, \quad \text{and} \quad |g(x)| \leq ||g||_\infty < +\infty.
\]

(A2) (Regularity). We have \( \varphi \in C^1(\mathbb{R}) \) and \( ||\varphi'||_\infty < +\infty \) and \( g \) is a Lipschitz continuous function and we denote \( L_g \) its Lipschitz constant.
(A3) (Asymptotic behaviour). There exists a constant \( \varphi \in \mathbb{R} \) such that
\[
\varphi \left( \frac{\cdot}{\delta} \right) \to \varphi, \quad \text{as} \quad \delta \to 0 \quad \text{in} \quad W^{1,p}(0,1).
\] (1.7)

(A4) (Regularity of the noise). The eigenvalues of the operators \( Q \) and \( A \) satisfy
\[
\sum_{k=1}^{\infty} \frac{\lambda_k}{\alpha_k^{1-\gamma}} < +\infty,
\]
for some \( \gamma \in (0,1) \).

**Remark 1.1** (Local perturbation). In Figure 1 we give a schematic representation of a function \( \varphi \) which satisfies assumptions (A1), (A2) and (A3). Function \( \varphi \) can oscillate near 0 but there must be a constant \( \overline{\varphi} \) such that \( \overline{\varphi} = \lim_{x \to +\infty} \varphi(x) \).

In this framework we shall prove existence and uniqueness of a solution to the equation describing the inhomogeneous model (1.4) for each \( \delta \) fixed, followed by the convergence of this solution, for \( \delta \to 0 \), to the one of the following homogeneous model.

\[
\begin{cases}
    d\overline{\varphi} = (A\overline{\varphi} + \overline{B}g(\overline{\varphi})) \, dt + dW, \\
    \overline{\varphi}(0) = u_0,
\end{cases}
\]

where \( \overline{B} = \varphi \frac{\partial}{\partial x} \overline{\varphi} \).

The organisation of the article is the following: Section 2 contains the definition of a mild solution and some preliminary results. Section 3 contains the proof of the result of existence and uniqueness of the solution while Section 4 contains the proof of the homogenization result.

2. Definitions and preliminary results.

2.1. **Definition of the mild solution for (1.4)**. Let us begin by introducing the definition of a mild solution. For the readers convenience, we recall that the unique solution of the linear problem

\[
\begin{cases}
    du = A ut + dW, \\
    u(0) = 0.
\end{cases}
\]

is given by the stochastic convolution,

\[
W_A(t) = \int_0^t e^{(t-s)A} dW(s).
\]
Keeping in mind that, for a constant \( C > 0 \), we have for the eigenvalues of \( A \) that
\[
|e_k(\xi)| \leq C, \quad k \in \mathbb{N}, \quad \xi \in [0, 1],
\]
and also considering the properties of the semi-group generated by the Laplace operator (see Lemma 2.2 below), we can apply [3, Theorem 5.24] and get that
\[
W_A \in C([0, T] \times [0, 1]) \cap C([0, T]; W^{1,p}(0, 1)),
\]
p \geq 3. Furthermore, by using the previous assumption (A4) we get by [3, Theorem 5.21] that \( W_A \) has a version which is an \( \alpha \)-Hölder continuous function for all \( \alpha \in (0, \gamma/2) \) with respect to the variables \( t \) and \( x \).

We can now give the definition of the solution.

**Definition 2.1.** A stochastic process \( u^\delta \in C([0, T]; L^p(0, 1)) \) is called mild solution of problem (1.4) on \([0, T]\) if for each starting point \( u_0 \in L^p(0, 1) \), \( p \geq 3 \), we have, for all \( t \in [0, T] \), that
\[
u^\delta(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}B^\delta g(v^\delta + W_A)ds + W_A \text{ a.s. in } \Omega.
\]

**2.2. Preliminary results.** The idea which shall be used for both existence and homogenization results consists in rewriting the stochastic equations as random differential equations.

In fact, for each fixed \( \omega \in \Omega \), we shall rewrite equation (1.4) in the form
\[
\begin{aligned}
\frac{dv^\delta}{dt} &= Av^\delta + B^\delta g(v^\delta + W_A), \\
v^\delta(0) &= u_0
\end{aligned}
\tag{2.3}
\]
by using the change of variable
\[
v^\delta(t) = u^\delta(t) - W_A(t), \quad \text{for all } t \geq 0.
\tag{2.4}
\]
The mild solution corresponding to (2.3) will be then given by
\[
v^\delta(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}B^\delta g(v^\delta + W_A)ds.
\tag{2.5}
\]

Using this form of the equation we shall prove that \( v^\delta \) (and implicitly \( u^\delta \)) exists and is unique. Like before, for the homogeneous problem (1.8) we can define \( \overline{\nu} = \overline{u} - W_A \), that satisfies
\[
\begin{aligned}
\frac{d\overline{\nu}}{dt} &= A\overline{\nu} + \overline{B}g(\overline{\nu} + W_A), \\
\overline{\nu}(0) &= u_0,
\end{aligned}
\tag{2.6}
\]
which mild solution is given by
\[
\overline{\nu} = e^{A\overline{t}}u_0 + \int_0^t e^{A(t-s)}u_0\overline{B}g(\overline{\nu} + W_A)ds.
\tag{2.7}
\]

The idea used to prove homogenization is then to obtain the convergence result of \( v^\delta \) to \( \overline{\nu} \), which will imply the same for \( u^\delta \) and \( \overline{u} \).

In order to see the properties of the stochastic convolution, we recall the following result concerning contraction semi-group on \( L^p(0, 1) \) (see for instance [11, Lemma 3, Part I]).
Lemma 2.2. For any \( s_1 \leq s_2 \in \mathbb{R} \), and \( r \geq 1 \), \( e^{At} : W^{s_1,r}(0,1) \to W^{s_2,r}(0,1) \), for all \( t > 0 \) and we have that there exists a constant \( C_1 \) depending only on \( s_1, s_2 \) and \( r \) such that
\[
|e^{At} z|_{W^{s_2,r}(0,1)} \leq C_1 \left( t^{\frac{s_2 - s_1}{2} + 1} |z|_{W^{s_1,r}(0,1)} \right), \quad \text{for all } z \in W^{s_1,r}(0,1).
\] (2.8)

Finally, we present a useful (deterministic) result that we will use several times in the rest of this paper.

Lemma 2.3. Assume \( \varphi \) continuous and derivable such that \( \varphi' \in L^p(0,1) \) and that \( g : \mathbb{R} \to \mathbb{R} \) is bounded and Lipschitz continuous. Then, there exists two constants \( C_1, C_2 > 0 \) such that for all \( t \geq 0 \) and \( v \in C([0,T]; L^p(0,1)) \) we have
\[
|e^{tA} \varphi(x) \frac{\partial}{\partial x} g(v)|_{L^p(0,1)} \leq C_1 C_2 \left( t^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) (|\varphi|_{L^p(0,1)} + |\varphi'|_{L^p(0,1)}) |g(v)|_{L^p(0,1)}.
\] (2.9)

Proof of Lemma 2.3. Using the Sobolev embedding theorem (the reader is referred to the Hitchhiker’s guide of Nezza, Palatucci and Valdinoci [5] for results concerning fractional Sobolev spaces), we have that there exists a constant \( C_2 > 0 \) such that
\[
|e^{tA} \varphi(x) \frac{\partial}{\partial x} g(v)|_{L^p(0,1)} \leq C_2 |e^{tA}Bg(v)|_{W^{\frac{p}{2},\frac{p}{2}}(0,1)} \leq C_1 C_2 \left( t^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) |Bg(v)|_{W^{-\frac{1}{2},\frac{p}{2}}(0,1)},
\] (2.10)
where we used Lemma 2.2 for the second inequality, with \( s_1 = -1 \), \( s_2 = 1/p \) and \( r = p/2 \). By definition, we have that
\[
|Bg(v)|_{W^{-\frac{1}{2},\frac{p}{2}}(0,1)} = \sup \left\{ \langle Bg(v), u \rangle / u \in W^{1,\frac{p}{2}}_0(0,1), |u|_{W^{1,\frac{p}{2}}_0(0,1)} \leq 1 \right\}.
\]

However, for all \( u \in W^{1,\frac{p}{2}}_0(0,1) \), \( |u|_{W^{1,\frac{p}{2}}_0(0,1)} \leq 1 \), we have
\[
|\langle Bg(v), u \rangle| = \left| \int_0^1 \varphi(x) \frac{\partial}{\partial x} g(v) u dx \right| = \left| \int_0^1 g(v) \left( \varphi'(x) u + \varphi \frac{\partial u}{\partial x} \right) dx \right| \leq \left| \int_0^1 g(v) \varphi'(x) u dx \right| + \left| \int_0^1 g(v) \varphi \frac{\partial u}{\partial x} dx \right| \leq \left( \int_0^1 |g(v)|^{p/2} |\varphi'|^{p/2} dx \right)^{2/p} \cdot \left( \int_0^1 |u|^{p/(p-2)} dx \right)^{(p-2)/p} \leq |g(v)|_{L^p(0,1)} \left( |\varphi|_{L^p(0,1)} + |\varphi'|_{L^p(0,1)} \right),
\]
where we have used for the fourth and fifth line the H"older inequality with coefficients \( p/2 \) and \( p/(p-2) \), for the sixth line the fact that \( |u|_{W^{1,\frac{p}{2}}_0(0,1)} \leq 1 \) and finally,
for the seventh line we used again H"{o}lder inequality with coefficients 2. Injecting this result in (2.10), we obtain (2.9). This concludes the proof of Lemma 2.3. □

3. Existence and uniqueness of the solution for the inhomogeneous and the limit equation. Our first main result is a result of existence and uniqueness of a solution for (1.4).

**Theorem 3.1 (Existence and uniqueness of a mild solution for (1.4)).** Assume (A1), (A2), and (A4). Let $u_0$ be given which is $\mathcal{F}_0$-measurable and such that $u_0 \in L^p(0,1)$ a.s. in $\Omega$, for some $p \geq 3$. Let $T > 0$ and $\delta > 0$. Then there exists a unique mild solution (see Definition 2.1) of equation (1.4), which belongs, a.s. in $\Omega$, to $C([0,T], L^p(0,1))$.

**Remark 3.2.** If $\psi$ is defined as in (A3), then Theorem 3.1 implies the existence of a unique mild solution for (1.8).

The proof of Theorem 3.1 is a direct consequence of combining (2.4) with the following lemma.

**Lemma 3.3 (Existence and uniqueness of a mild solution for (2.3)).** Assume (A1), (A2), and (A4). Let $u_0$ be given which is $\mathcal{F}_0$-measurable and such that $u_0 \in L^p(0,1)$ a.s. in $\Omega$, for some $p \geq 3$. Let $T > 0$ and $\delta > 0$. Then there exists a unique mild solution of (2.3), which belongs a.s. to $C([0,T], L^p(0,1))$.

The proof of Lemma 3.3 is done in two parts, first we prove that (2.3) admits a solution locally in time and then we prove the existence of a solution global in time. In order to simplify the notations, we drop the dependence in $\delta$ in the rest of the section.

We recall that (2.3) is a deterministic equation, for $\omega \in \Omega$ fixed.

3.1. Local existence in time.

**Lemma 3.4 (Local existence in time for (2.3)).** For any $p \geq 3$, and $m > |u_0|_{L^p(0,1)}$, there exists a stopping time $T^* > 0$ such that (2.3) has a unique mild solution of the form (2.5) in

$$
\Sigma_p(m,T^*) = \{ v \in C([0,T^*], L^p(0,1)) : |v(t)|_{L^p(0,1)} \leq m \forall t \in [0,T^*] \}.
$$

**Proof of Lemma 3.4.** This proof is done by a fixed point argument in $\Sigma_p(m,T^*)$. Therefore, we want to prove that for all $v \in \Sigma_p(m,T^*)$, the transformation $Gv = z$ defined by

$$
z(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} \varphi(x) \frac{\partial}{\partial x} g(v + W_A) ds,
$$

is a contraction from $\Sigma_p(m,T^*)$ into $\Sigma_p(m,T^*)$.

We define, for all $v \in \Sigma_p(m,T^*)$, the norm $\| \cdot \|_{\Sigma_p(m,T^*)}$ by

$$
\|v\|_{\Sigma_p(m,T^*)} = \sup_{t \in [0,T^*]} |v|_{L^p(0,1)}.
$$

Step 1: $G$ is stable in $\Sigma_p(m,T^*)$. Let us first prove that $z = Gv$ is in $\Sigma_p(m,T^*)$. First we have,

$$
|z(t)|_{L^p(0,1)} \leq |e^{tA}u_0|_{L^p(0,1)} + \int_0^t \left| e^{(t-s)A} Bg(v + W_A) \right|_{L^p(0,1)} ds. \tag{3.1}
$$
Let us now treat the term inside the integral, using Lemma 2.3, we obtain that
\[ \left| e^{(t-s)A} \varphi(x) \frac{\partial}{\partial x} g(v + W_A) \right|_{L_p(0,1)} \leq C_1 C_2 \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) (|\varphi|_{L_p(0,1)} + |\varphi'|_{L_p(0,1)}) |g|_{L_p(0,1)} \leq C_1 C_2 \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) (|\varphi|_{\infty} + |\varphi'|_{\infty}) |g|_{\infty}. \tag{3.2} \]
Since \( e^{At} \) is a contraction on \( L^p(0,1) \), we have that \( |e^{tA}u_0|_{L_p(0,1)} \leq |u_0|_{L_p(0,1)} \).

Injecting this result and (3.2) into (3.1) we obtain
\[ |z(t)|_{L_p(0,1)} \leq |u_0|_{L_p(0,1)} + C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) |g|_{\infty} \int_0^t \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) ds \leq |u_0|_{L_p(0,1)} + C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) |g|_{\infty} \left( \frac{2p}{p-1} T^\frac{1}{2} - \frac{1}{p} + T^* \right). \]
We can see that if \( m > |u_0|_{L_p(0,1)} \), given that \( p \geq 3 \), there exists a time \( T^* \) such that
\[ |u_0|_{L_p(0,1)} + C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) |g|_{\infty} \left( \frac{2p}{p-1} T^\frac{1}{2} - \frac{1}{p} + T^* \right) \leq m, \]
this implies that for all \( t \in [0,T^*], |z(t)|_{L_p(0,1)} \leq m \) and therefore \( z \in \Sigma_p(m,T^*) \).

Step 2: \( G \) is a contraction on \( \Sigma_p(m,T^*) \). Let us consider \( v_1, v_2 \in \Sigma_p(m,T^*) \). We set \( z = G(v_1 - v_2) \). Therefore,
\[ z(t) = \int_0^t e^{(t-s)A} B(g(v_1 + W_A) - g(v_2 + W_A)) ds. \]
Using Lemma 2.3, we get
\[ |z(t)|_{L_p(0,1)} \leq C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) \int_0^t \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) |g(v_1 + W_A) - g(v_2 + W_A)|_{L_p(0,1)} ds. \]
Using the regularity of \( g \), we obtain
\[ |z(t)|_{L_p(0,1)} \leq C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) \left| L_g \int_0^t \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) |v_1 - v_2|_{L_p(0,1)} ds \right| \leq C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) \left| L_g \sup_{t \in [0,T^*]} \left( |v_1 - v_2|_{L_p(0,1)} \right) \int_0^t \left( (t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1 \right) ds \right| \leq C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) \left( \frac{2p}{p-1} T^{*\frac{1}{2} - \frac{1}{p}} + T^* \right) L_g \sup_{t \in [0,T^*]} \left( |v_1 - v_2|_{L_p(0,1)} \right). \]
If we choose \( T^* \) small enough such that
\[ C_1 C_2 (|\varphi|_{\infty} + |\varphi'|_{\infty}) \left( \frac{2p}{p-1} T^{*\frac{1}{2} - \frac{1}{p}} + T^* \right) L_g < 1, \]
we obtain \( |z(t)|_{L_p(0,1)} < |v_1 - v_2|_{\Sigma_p(m,T^*)} \). By taking the supremum in time on the left hand side of the inequality we obtain the desired result. \( \square \)
3.2. Global existence in time.

**Lemma 3.5.** Let $T \in (0, +\infty)$. If $v \in C([0, T]; L^p(0, 1))$ satisfies (2.5), then there exists two constants $\alpha_{p, \varphi, g}$ and $\gamma_p$ depending only on $p$, $\|\varphi\|_\infty$, $\|\varphi'\|_\infty$ and $\|g\|_\infty$, such that, for all $t \in [0, T]$, we have

$$|v(t)|_{L^p(0, 1)} \leq (|u_0|_{L^p(0, 1)} + \alpha_{p, \varphi, g} \cdot T) \cdot \exp (\gamma_p \cdot t).$$

**Proof.** This proof is inspired by [4, Lemma 3.1], but for the readers convenience, we give the details.

We intend to work with regularized versions of the functions we considered before. For the initial condition, let $(u_0^n)_n$ be a sequence in $C^2(0, 1)$ such that

$$u_0^n \to u_0 \text{ as } n \to +\infty \text{ in } L^p(0, 1).$$

Let $W^n_A \subset C^2([0, T] \times [0, 1])$ be a sequence of regularised processes such that

$$W^n_A(t) = \int_0^t e^{(t-s)A}dW^n(s) \to W_A(t) \text{ in } C([0, T] \times [0, 1]) \text{ as } n \to +\infty,$$ \hspace{1cm} (3.4)

a.s. $\omega \in \Omega$.

Let $v_n$ be the solution of

$$v_n(t) = e^{At}u_0^n + \int_0^t e^{(t-s)A}Bg_n(v_n + W^n_A)ds,$$

provided by Lemma 3.4, with $g_n \in C^2(\mathbb{R})$ a sequence of regularised functions such that

$$g_n \to g \text{ in } C(\mathbb{R}).$$

Given (3.3)-(3.4), the function $v_n$ exists in a time interval $[0, T_n]$ with $T_n \to T^*$. Moreover, $v_n \in C^2([0, T] \times [0, 1])$ and converges to $v$ in $C([0, T^*], L^p(0, 1))$ a.s. $\omega \in \Omega$. We also have that the regular function $v_n$ satisfies

$$\frac{\partial v_n}{\partial t} + \frac{\partial^2 v_n}{\partial x^2} - \varphi(x) \frac{\partial}{\partial x}g_n(v_n + W^n_A) = 0 \text{ a.s. } \omega \in \Omega.$$ \hspace{1cm} (3.5)

Multiplying (3.5) by $|v_n|^{p-2}v_n$, and integrating over $[0, 1]$, we obtain

$$\frac{1}{p} \frac{\partial |v_n|_{L^p(0, 1)}^p}{\partial t} + (p - 1) \int_0^1 |v_n|^{p-2} \left( \frac{\partial v_n}{\partial x} \right)^2 dx$$

$$- \int_0^1 |v_n|^{p-2} v_n \varphi(x) \frac{\partial}{\partial x}g_n(v_n + W^n_A)dx = 0.$$ \hspace{1cm} (3.6)

Let us now consider the last term of the previous equation. We notice that it can be separated into two parts,

$$\int_0^1 |v_n|^{p-2} \varphi(x) \frac{\partial}{\partial x}g_n(v_n + W^n_A)dx$$

$$= -\int_0^1 \varphi'(x)g_n(v_n + W^n_A)|v_n|^{p-2} \cdot v_n dx$$ \hspace{1cm} (3.7)

$$- (p - 1) \int_0^1 \varphi(x)g_n(v_n + W^n_A)|v_n|^{p-2} \frac{\partial v_n}{\partial x} dx.$$ \hspace{1cm} (3.8)
Let us begin by considering (3.7),
\[
\left| \int_0^1 \varphi'(x)g_n(v_n + W_A^n)|v_n|^{p-2} \cdot v_n \, dx \right| \leq \|\varphi'\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^{p-1}
\]
\[
\leq \frac{1}{p} \|\varphi'\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p + \frac{p-1}{p} \|v_n\|_{L^p(0,1)}^p.
\]
where we have used for the first line the fact that \(\varphi'\) and \(g\) are bounded, for the second line we simply have used Hölder inequality with coefficients \(2\), for the fourth line Young inequality \((ab \leq (1/p)a^p + b^{p-1}(p-1)/p)\) and finally for the third line we have used the Sobolev embedding theorem. Let us now consider (3.8),
\[
\left| (p-1) \int_0^1 \varphi(x)g_n(v_n + W_A^n)|v_n|^{p-2} \frac{\partial v_n}{\partial x} \, dx \right|
\]
\[
\leq (p-1) \|\varphi\|_\infty \|g_n\|_\infty \int_0^1 \left| v_n \right|^{p-2} \left| \frac{\partial v_n}{\partial x} \right| \, dx
\]
\[
\leq (p-1) \|\varphi\|_\infty \|g_n\|_\infty \left( \int_0^1 \left| v_n \right|^{p-2} \left| \frac{\partial v_n}{\partial x} \right| \, dx \right)^{1/2} \left( \int_0^1 \left| v_n \right|^{p-2} \left( \frac{\partial v_n}{\partial x} \right)^2 \, dx \right)^{1/2}
\]
\[
\leq (p-1) \|\varphi\|_\infty^2 \|g_n\|_\infty^2 \left( \int_0^1 \left| v_n \right|^{p-2} \left( \frac{\partial v_n}{\partial x} \right)^2 \, dx \right)
\]
\[
\leq \frac{p-1}{4} \left( \frac{2}{p} \|\varphi\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p + \frac{p-2}{p} \|v_n\|_{L^{p-2}(0,1)}^p \right) + (p-1) \int_0^1 \left| v_n \right|^{p-2} \left( \frac{\partial v_n}{\partial x} \right)^2 \, dx
\]
\[
\leq \frac{p-1}{2p} \|\varphi\|_\infty^2 \|g_n\|_\infty^p + \frac{(p-2)(p-1)}{4p} \tilde{C}_1 \|v_n\|_{L^p(0,1)}^p + (p-1) \int_0^1 \left| v_n \right|^{p-2} \left( \frac{\partial v_n}{\partial x} \right)^2 \, dx.
\]
(3.10)

where we have used for the second line the fact that \(\varphi\) and \(g\) are bounded, for the third line we have used Hölder inequality with coefficients \(2\), for the fourth line Young inequality \((ab \leq a^2/4 + b^2)\), for the fifth line we have used Hölder inequality \((ab \leq (2/p)a^p + b^{p-1}(p-2)/p)\) and for the sixth line the Sobolev embedding theorem. Injecting (3.9) and (3.10) into (3.6), we obtain
\[
\frac{1}{p} \frac{\partial \|v_n\|_{L^p(0,1)}^p}{\partial t} \leq \left( \frac{\|\varphi\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p}{p} + \frac{(p-1)\|\varphi\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p}{2p} \right)
\]
\[
+ \left( \tilde{C}_1 \frac{(p-1)(p-2)}{4p} \tilde{C}_2 \frac{p-1}{p} \right) \|v_n\|_{L^p(0,1)}^p.
\]

Thanks to Gronwall’s lemma we obtain
\[
\|v_n\|_{L^p(0,1)} \leq \left( \|v_0\|_{L^p(0,1)} + \|\varphi\|_\infty \|g_n\|_\infty \right) \cdot \exp \left( \gamma_p \cdot t \right),
\]
(3.11)

with
\[
\alpha_{p,\varphi,g_n} = \|\varphi\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p + \frac{(p-1)\|\varphi\|_\infty \|g_n\|_\infty \|v_n\|_{L^p(0,1)}^p}{2p},
\]
and
\[
\gamma_p = \tilde{C}_1 \frac{(p-1)(p-2)}{4p} \tilde{C}_2 (p-1).
\]
Passing to the limit as \( n \) goes to infinity we obtain the desired result.

Combining Lemma 3.4 and 3.5 with a continuation argument, we obtain the global existence of the mild solution (Lemma 3.3) and it also implies the following result. If \( \varphi \) is defined as in (A3), then Theorem 3.1 implies also the existence of a unique mild solution of the equation (1.8), which describes the homogeneous model.

**Theorem 3.6 (Existence and uniqueness of a mild solution for (1.8)).** Assume (A1)-(A4). Let \( u_0 \) be given which is \( \mathcal{F}_0 \)-measurable and such that \( u_0 \in L^p(0,1) \) a.s. in \( \Omega \), for some \( p \geq 3 \). Let \( T > 0 \). Then there exists a unique mild solution of equation (1.8), which belongs a.s. in \( \Omega \) to \( C([0,T],L^p(0,1)) \).

4. **Homogenization.** Our second main result concerns the homogenization of (1.4).

**Theorem 4.1 (Homogenization result for (1.4)).** Assume (A1)-(A4). Let \( T > 0 \). Let \( (v^\delta)_\delta \) be the sequence of mild solutions of (1.4) in \([0,T]\), provided by Lemma 3.3. Let \( \varpi \) be the mild solution of (1.8) in \([0,T]\). Then

\[
v^\delta \to \varpi \text{ as } \delta \to 0 \text{ in } C([0,T],L^p(0,1)).
\]

The proof of Theorem 4.1 is a direct consequence of the following lemma, using (2.4) and the definition of \( \varpi \).

**Lemma 4.2.** Assume (A1)-(A2)-(A3). Let \( T > 0 \). Let \( (v^\delta)_\delta \) be the sequence of mild solutions of (2.5) in \([0,T]\), provided by Lemma 3.3. Let \( \varpi \) be the mild solution of (2.7) in \([0,T]\). Then

\[
v^\delta \to \varpi \text{ as } \delta \to 0 \text{ in } C([0,T],L^p(0,1)).
\]

**Proof of Lemma 4.2.** The idea of the proof is to compare the two mild solutions

\[
\varpi(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} \partial \frac{\partial}{\partial x}g(\varpi + W_A)ds,
\]

and

\[
v^\delta(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} \varphi \left( \frac{x}{\delta} \right) \partial \frac{\partial}{\partial x}g(v^\delta + W_A)ds.
\]

Combining (4.1) and (4.2), we obtain that

\[
|v^\delta - \varpi|_{L^p(0,1)} = \left| \int_0^t e^{(t-s)A} \left[ \varphi^\delta (x) \frac{\partial}{\partial x}g(v^\delta + W_A) - \varphi \frac{\partial}{\partial x}g(\varpi + W_A) \right] ds \right|_{L^p(0,1)}
\]

\[
\leq \int_0^t \left| e^{(t-s)A} \frac{\partial}{\partial x}g(v^\delta + W_A) (\varphi^\delta (x) - \varphi) \right|_{L^p(0,1)} ds
\]

\[
+ \int_0^t \left| e^{(t-s)A} \varphi \left[ \frac{\partial}{\partial x}g(v^\delta + W_A) - \frac{\partial}{\partial x}g(\varpi + W_A) \right] \right|_{L^p(0,1)} ds.
\]

Let us first treat the term inside the integral of (4.3). Using Lemma 2.3, we have that

\[
e^{(t-s)A} \frac{\partial}{\partial x}g(v^\delta + W_A) (\varphi^\delta (x) - \varphi) \left|_{L^p(0,1)} \right.
\]

\[
\leq C_1 C_2 \left( (t-s)^{-\frac{\alpha+1}{2\alpha}} + 1 \right) |g|_{\infty} \left( |\varphi^\delta - \varphi|_{L^p(0,1)} + |\varphi^\delta - \varphi|_{L^p(0,1)} \right).
\]

\[\text{(4.5)}\]
Let us now consider the term inside the integral of (4.4). Like before, using Lemma 2.3, we obtain
\[
\left| e^{(t-s)A}\frac{\partial}{\partial x} \left[ g(v^\delta + W_A) - g(\overline{v} + W_A) \right] \right|_{L^p(0,1)} \\
\leq C_1 C_2 (t-s)^{-\frac{\beta+1}{2p}} + 1) ||\overline{v}||_{\infty} L_g |v^\delta - \overline{v}|_{L^p(0,1)},
\]
Injecting (4.5) and (4.6) into (4.3)-(4.4), we obtain
\[
|v^\delta - \overline{v}|_{L^p(0,1)} \\
\leq C_1 C_2 \left\| g \right\|_\infty \left( \left| \varphi^{\prime \prime} - \varphi \right|_{L^p(0,1)} + \left| \varphi - \varphi \right|_{L^p(0,1)} \right) \int_0^t \left( (t-s)^{-\frac{\beta+1}{2p}} + 1 \right) ds \\
+ C_1 C_2 \left\| \varphi \right\|_{\infty} L_g \int_0^t \left( (t-s)^{-\frac{\beta+1}{2p}} + 1 \right) |v^\delta - \overline{v}|_{L^p(0,1)} ds \\
\leq C_1 C_2 \left\| g \right\|_\infty \left( \left| \varphi^{\prime \prime} - \varphi \right|_{L^p(0,1)} + \left| \varphi - \varphi \right|_{L^p(0,1)} \right) \left( \frac{2p}{p-1} T^{\frac{1}{2} - \frac{\beta}{p}} + T \right) \\
+ C_1 C_2 \left\| \varphi \right\|_{\infty} L_g \int_0^t \left( (t-s)^{-\frac{\beta+1}{2p}} + 1 \right) |v^\delta - \overline{v}|_{L^p(0,1)} ds \\
\leq \alpha_\delta(1) + C_1 C_2 \left\| \varphi \right\|_{\infty} L_g \int_0^t \left( (t-s)^{-\frac{\beta+1}{2p}} + 1 \right) |v^\delta - \overline{v}|_{L^p(0,1)} ds.
\]
Using Gronwall’s lemma, we obtain
\[
|v^\delta - \overline{v}|_{L^p(0,1)} \leq \alpha_\delta(1) \cdot \exp \left( 2C_1 C_2 \left\| \varphi \right\|_{\infty} L_g \int_0^t \left( (t-s)^{-\frac{\beta+1}{2p}} + 1 \right) ds \right) \\
\leq \alpha_\delta(1) \cdot \exp \left( 2C_1 C_2 \left\| \varphi \right\|_{\infty} L_g \left( \frac{2p}{p-1} T^{\frac{1}{2} - \frac{\beta}{p}} + T \right) \right) \\
\leq \alpha_\delta(1).
\]
Passing to the limit as \( \delta \) goes to 0, we obtain our result. \( \square \)

REFERENCES

[1] A. Amosov and G. Panasenko, Homogenization of the integro-differential burgers equation, in *Integral Methods in Science and Engineering*, Vol. 1, Birkhäuser Boston, Boston, MA, 2010, 1–8.
[2] G. Da Prato and D. Gatarek, Stochastic burgers equation with correlated noise, *Stochastics and Stochastic Reports*, 52 (1995), 29–41.
[3] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2008. [https://books.google.fr/books?id=JYiL8zz_nC8C](https://books.google.fr/books?id=JYiL8zz_nC8C).
[4] G. Da Prato, A. Debussche and R. Temam, Stochastic Burgers’ equation, *NoDEA Nonlinear Differential Equations Appl.*, 1 (1994), 389–402.
[5] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.
[6] M. Garavello and B. Piccoli, *Traffic Flow on Networks*, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.
[7] I. Gyoengy and D. Nualart, On the stochastic Burgers’ equation in the real line, *Ann. Probab.*, 27 (1999), 782–802.
[8] I. Hosokawa and K. Yamamoto, Turbulence in the randomly forced, one-dimensional Burgers flow, *J. Stat. Phys.*, 13 (1975), 245–272.
[9] M. J. Lighthill and G. B. Whitham, On kinematic waves II. A theory of traffic flow on long crowded roads, *Proc. Roy. Soc. London Ser. A*, 229 (1955), 317–345.
[10] P. I. Richards, Shock waves on the highway, *Operations Res.*, 4 (1956), 42–51.
[11] F. Rothe, *Global Solutions of Reaction-Diffusion Systems*, Vol. 1072, Springer-Verlag, Berlin, 1984.

Received for publication January 2020.

E-mail address: ioana.ciotir@insa-rouen.fr
E-mail address: nicolas.forcadel@insa-rouen.fr
E-mail address: wilfredo.salazar_elias@insa-rouen.fr