Equivalence of two BV classes of functions in metric spaces, and existence of a Semmes family of curves under a 1-Poincaré inequality

Estibalitz Durand-Cartagena, Sylvester Eriksson-Bique, Riikka Korte, Nageswari Shanmugalingam

September 12, 2018

Abstract

We consider two notions of functions of bounded variation in complete metric measure spaces, one due to Martio [M1, M2] and the other due to Miranda Jr. [Mi]. We show that these two notions coincide, if the measure is doubling and supports a 1-Poincaré inequality. In doing so, we also prove that if the measure is doubling and supports a 1-Poincaré inequality, then the metric space supports a Semmes family of curves structure.

Key words: AM-modulus, bounded variation, 1-Poincaré inequality, metric measure space, Semmes pencil of curves.

MSC classification: 26A45, 30L99, 31E05.

1 Introduction

Given $1 \leq p < \infty$, a function $u$ in $L^p(\mathbb{R}^n)$ is in $W^{1,p}(\mathbb{R}^n)$ if and only if $u$ has an $L^p$-representative that is absolutely continuous on almost every non-constant compact rectifiable curve in $\mathbb{R}^n$ with derivative in $L^p(\mathbb{R}^n)$, see [Va] for an in-depth discussion on this. Equivalently, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and there is a non-negative Borel function $g \in L^p(\mathbb{R}^n)$ such that for all
non-constant compact rectifiable curves $\gamma$ in $\mathbb{R}^n$,

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g \circ \gamma(t)|\gamma'(t)| \, dt.$$  \hspace{1cm} (1)

On the other hand, the class of BV functions on $\mathbb{R}^n$ has a more complicated analog; there should be a sequence $f_k \in W^{1,1}(\mathbb{R}^n)$, with $f_k \to u$ in $L^1(\mathbb{R}^n)$ and function $g_k$ associated with $f_k$ as in the inequality above, such that $\liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k \, dx$ is finite. Thus to verify that a function $u$ belongs to the class BV($\mathbb{R}^n$) we need a sequence of pairs of functions $(f_k, g_k)$ satisfying (1), where $f_k$ approximates $u$ in $L^1(\mathbb{R}^n)$, whereas to define a function in $W^{1,1}(\mathbb{R}^n)$ we only need a single energy function $g$ that satisfies (1).

The above complication carries through from $\mathbb{R}^n$ to more general metric measure spaces $X$, and so while we need only the energy function $g$ in order to know that $u$ is in the Sobolev class, to know that $u$ is in the BV class we need both, the approximating sequence $f_k$ as well as the corresponding energy functions $g_k$. To avoid this discrepancy, the recent work of Martio [M1, M2] proposed a new definition of BV functions in the Euclidean and general metric measure setting, denoted in the current paper by $BV_{AM}(X)$, see Definition 2.5. In this notion one needs a single sequence of “energy” functions $g_k$ associated with the function $u$ in a specific manner in order to determine whether $u \in BV_{AM}(X)$. The backbone of the construction of $BV_{AM}(X)$ is the notion of AM-modulus, and it appears that this modulus is better suited to the study of sets of finite perimeter than the standard 1-modulus. It is shown in [HMM2, Theorem 11] that Euclidean Borel sets $E$ are of finite perimeter if and only if the AM-modulus of the collection of all curves that cross the measure-theoretic boundary $\partial_s E$ of $E$ is finite; and in this case the perimeter measure of $E$ is precisely the AM-modulus of that collection of curves. This is a variant of the Federer characterization of sets of finite perimeter. Federer proved that a measurable set $E \subset \mathbb{R}^n$ is of finite perimeter if and only if $H^{n-1}(\partial_s E)$ is finite; a new, potential-theoretic proof of this characterization, valid even in the metric setting, can be found in [L].

The goal of this paper is to show that if the metric measure space $X$ is of controlled geometry, that is, if $X$ is complete, the measure $\mu$ is doubling and supports a 1-Poincaré inequality, then the notion of $BV_{AM}(X)$ from [M1, M2] gives the same function space as the BV class BV($X$) as defined by Miranda Jr. in [M]. To do so we also prove that if $\mu$ is doubling, then $X$ supports a 1-Poincaré inequality if and only if $X$ supports a Semmes family of curves corresponding to each pair $x, y \in X$ of points, that is, there is a family $\Gamma_{xy}$ of quasiconvex curves connecting $x$ to $y$ and a probability measure $\sigma_{xy}$ on $\Gamma_{xy}$ satisfying a Riesz-type inequality, see Definition 3.3 below. This auxiliary result is of independent interest. The notion of Semmes family of curves, first proposed in [Sc] (where clearly it was not termed a “Semmes family”), is known to imply the support of a 1-Poincaré inequality, see the discussion in [He, page 29]. In this paper we show that the converse also holds true, that is, if the measure is doubling and supports a 1-Poincaré inequality, then it supports a Semmes family of curves structure. Thus, our paper also characterizes the support of a 1-Poincaré inequality (in doubling complete metric measure spaces) via the existence of a Semmes family of curves. A recent preprint [FO] gives another characterization of the support of a 1-Poincaré inequality in terms of the existence of normal 1-currents for each pair of points $x, y \in X$, in the sense of Ambrosio and Kirchheim, such that the mass of the current is controlled by the Riesz measure $R_{xy}$, see [E] below. For the study
comparing BV-AM spaces with BV classes of functions, a Semmes family of curves seems to be more useful.

The equality of $\text{BV}(X)$ with $\text{BV}_{AM}(X)$ and the equivalence between the Semmes family of curves structure and the 1-Poincaré inequality form the two main results in this paper, see Theorem 3.10.

## 2 Two definitions of BV functions

In the rest of the paper, $(X, d, \mu)$ is a metric measure space, where $(X, d)$ is a complete metric space and $\mu$ is a Borel measure. We denote by $B$ an open ball in $X$ and by $\lambda B$ the ball with the same center as $B$ and radius $\lambda$ times the radius of $B$. Recall that the measure $\mu$ is said to be doubling if there is a constant $C \geq 1$ such that $\mu(2B) \leq C \mu(B)$ for every ball $B$ in $X$.

Given a compact interval $I \subset \mathbb{R}$, a curve $\gamma : I \to X$ is a continuous mapping. We only consider curves that are non-constant and rectifiable. A curve $\gamma$, connecting two points $x, y \in X$, is $C$-quasiconvex if its length is at most $C \, d(x, y)$.

### 2.1 $p$-Modulus and AM-modulus of a family of curves

**Definition 2.1** Given a family $\Gamma$ of curves in $X$, set $\mathcal{A}(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \, ds \geq 1 \text{ for all } \gamma \in \Gamma,$$

and set $\mathcal{A}_{\text{seq}}(\Gamma)$ to be the family of all sequences $(\rho_i)$ of non-negative Borel measurable functions $\rho_i$ on $X$ such that

$$\liminf_{i \to \infty} \int_\gamma \rho_i \, ds \geq 1 \text{ for all } \gamma \in \Gamma.$$

The integral $\int_\gamma \rho \, ds$ denotes the path-integral of $\gamma$ against the arc-length re-parametrization of $\gamma$, see for example the description in [He]. We define the $\infty$-modulus of $\Gamma$ by

$$\text{Mod}_{\infty}(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \| \rho \|_{L^\infty(X)},$$

and for $1 \leq p < \infty$ the $p$-modulus of $\Gamma$ is

$$\text{Mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p \, d\mu.$$

Following [M1], [M2], we define the approximate modulus (AM-modulus) of $\Gamma$ by

$$\text{AM}(\Gamma) = \inf_{(\rho_i) \in \mathcal{A}_{\text{seq}}(\Gamma)} \left\{ \liminf_{i \to \infty} \int_X \rho_i \, d\mu \right\}.$$
The notion of $AM_p(\Gamma)$ is defined analogously, with $\int_X \rho d\mu$ replaced by $\int_X \rho_p^\infty d\mu$. If a property holds for all except for a family $\Gamma$ of curves with $\text{Mod}_p(\Gamma) = 0$ (respectively with $AM(\Gamma) = 0$), then we say that the property holds for $p$-a.e. curve (respectively for $AM$-a.e. curve).

Note that $AM(\Gamma) \leq \text{Mod}_1(\Gamma)$. Thanks to Mazur’s lemma, it is a trivial consequence of the reflexivity of $L^p(X)$ that $AM_p(\Gamma) = \text{Mod}_p(\Gamma)$ when $1 < p < \infty$, see [HMM, Theorem 1]. It is also easy to see that for any family of curves $\Gamma$ we have $\text{AM}_\infty(\Gamma) = \text{Mod}_\infty(\Gamma)$. Indeed, let $\tau = \text{AM}_\infty(\Gamma)$. If $\tau = \infty$ there is nothing to prove, so let us assume that $\tau < \infty$. By definition, there is a sequence of non-negative Borel functions $(g_i^\infty) \in A_{seq}(\Gamma)$ such that

$$\liminf_{i \to \infty} \|g_i^\infty\|_{L^\infty(X)} < \tau + \varepsilon \quad \text{and} \quad \liminf_{i \to \infty} \int_{\gamma} g_i^\infty ds \geq 1 \text{ for each } \gamma \in \Gamma.$$ 

Let $\rho_\varepsilon := \sup_i g_i^\infty$. As $\rho_\varepsilon \geq g_i^\infty$ for each $i \in \mathbb{N}$, it follows that

$$1 \leq \liminf_{i \to \infty} \int_{\gamma} g_i^\infty ds \leq \int_{\gamma} \rho_\varepsilon ds,$$

and so $\text{Mod}_\infty(\Gamma) \leq \|\rho_\varepsilon\|_{L^\infty(X)} \leq \tau + \varepsilon$ and the result follows.

Note that if every curve in $\Gamma$ is contained in a fixed ball $B$, then

$$\text{AM}(\Gamma) \leq \text{Mod}_1(\Gamma) \leq \mu(B)^{1-1/p} \text{Mod}_p(\Gamma)^{1/p} \leq \mu(B) \text{Mod}_\infty(\Gamma),$$

and therefore

$$\limsup_{p \to \infty} [\text{Mod}_p(\Gamma)]^{1/p} \leq \text{Mod}_\infty(\Gamma).$$

The next example shows that it is possible to have $\text{Mod}_1(\Gamma) = \infty$ but $\text{AM}(\Gamma) = 1$. Further examples can be found in [HMM, Section 9]. The examples found there are families of curves that tangentially approach a smooth co-dimension one sub-manifold of $\mathbb{R}^n$.

**Example 2.2** Let $\Gamma$ be the collection of all rectifiable curves of length at most 1 in the plane, and start from the $x$-axis with $0 \leq x \leq 1$ and are parallel to the $y$-axis. Then there is no acceptable $\rho \in L^1(X)$ for computing $\text{Mod}_1(\Gamma)$, and hence $\text{Mod}_1(\Gamma) = \infty$. On the other hand, $\text{AM}(\Gamma)$ is finite but positive. To see this, for each positive integer let $\rho_n = n \chi_{[0,1] \times [0,1/n]}$. Then $\int_{\gamma} \rho_n ds \geq 1$ whenever $\gamma$ is in $\Gamma$ with length at least $1/n$, and as every curve in $\Gamma$ has positive length, we have that

$$\lim_{n \to \infty} \int_{\gamma} \rho_n ds \geq 1.$$ 

So the sequence $(\rho_n)$ is admissible for $\Gamma$, and thus

$$\text{AM}(\Gamma) \leq \limsup_{n \to \infty} \int_{\mathbb{R}^2} \rho_n d\mathcal{L}^2 = \limsup_{n \to \infty} n \left( \frac{1}{n} \times 1 \right) = 1.$$
To see that \( \text{AM}(\Gamma) > 0 \), we consider the sub-family \( \Gamma_{1/2} \) of all line segments in \( \Gamma \) with length \( 1/2 \), and let \((\rho_i) \in A_{\text{seq}}(\Gamma_{1/2}) \). Then by Fubini’s theorem, for each \( i \in \mathbb{N} \) we have
\[
\int_{\mathbb{R}^2} \rho_i \, d\mathcal{L}^2 \geq \int_0^1 \int_0^{1/2} \rho_i(x, y) \, dy \, dx = \int_0^1 \left( \int_0^{1/2} \rho_i(x, y) \, dy \right) \, dx.
\]
Now by Fatou’s lemma,
\[
\liminf_{i \to \infty} \int_{\mathbb{R}^2} \rho_i \, d\mathcal{L}^2 \geq \int_0^1 \left( \liminf_{i \to \infty} \int_0^{1/2} \rho_i(x, y) \, dy \right) \, dx \geq 1.
\]
It follows that \( \text{AM}(\Gamma) \geq \text{AM}(\Gamma_{1/2}) \geq 1 \).

### 2.2 BV functions based on the notion of AM-modulus.

**Definition 2.3** A nonnegative Borel function \( g \) on \( X \) is a 1-weak upper gradient of an extended real-valued function \( u \) on \( X \) if for 1-a.e. curve \( \gamma : [a, b] \to X \),
\[
|u(\gamma(b)) - u(\gamma(a))| \leq \int_{\gamma} g \, ds.
\]

Given a function \( u \) that has a 1-weak upper gradient in \( L^1(X) \), there is a minimal 1-weak upper gradient of \( u \), denoted \( g_u \), in the sense that whenever \( g \) is a 1-weak upper gradient of \( u \), we have \( g_u \leq g \) almost everywhere in \( X \).

The following notion of BV functions on \( X \) is due to Miranda Jr. [Mi].

**Definition 2.4 (BV functions)** For \( u \in L^1_{\text{loc}}(X) \), we define the total variation of \( u \) as
\[
\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \inf_{g_{u_i}} \int_X g_{u_i} \, d\mu : u_i \in \text{LIP}_{\text{loc}}(X), u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},
\]
where the second infimum is over all 1-weak upper gradients \( g_{u_i} \) of \( u_i \). We say that a function \( u \in L^1_{\text{loc}}(X) \) is of bounded variation, \( u \in BV(X) \) if \( \|Du\|(X) < \infty \). A measurable set \( E \subset X \) is said of finite perimeter if \( \|D\chi_E\|(X) < \infty \).

The following definition of BV\_AM class is from [M1].

**Definition 2.5 (BV-AM functions)** A function \( u \in L^1(X) \) is in the BV\_AM \( (X) \) class if there is a family \( \Gamma \) of rectifiable curves in \( X \) with AM(\( \Gamma \)) = 0, and a sequence \((g_i)\) of non-negative Borel measurable functions in \( L^1(X) \) such that whenever \( \gamma : [a, b] \to X \) is a non-constant compact rectifiable curve that does not belong to \( \Gamma \), we have that
\[
|u(\gamma(t)) - u(\gamma(s))| \leq \liminf_{i \to \infty} \int_{\gamma([s, t])} g_i \, ds.
\]
for $\mathcal{H}^1$-a.e. $s, t \in [a, b]$ with $s < t$, and

$$\liminf_{i \to \infty} \int_X g_i \, d\mu < \infty.$$ 

Such a sequence $(g_i)$ is said to be a BVAM-upper bound of $u$. We set

$$\|D_{AM} u\|(X) := \inf_{(g_i)} \liminf_{i \to \infty} \int_X g_i \, d\mu,$$

where the infimum is over all BVAM-upper bounds of $u$.

Notice that by [M2, Theorem 4.1], $BV(X) \subseteq BV_{AM}(X)$. This also follows from the next lemma. The following lemma holds even if $\mu$ is not doubling or does not support a 1-Poincaré inequality.

**Lemma 2.6** Assume that $u \in BV(X)$. Then there is a set $N \subset X$ with $\mu(N) = 0$ and a sequence $(g_i)$ of non-negative Borel measurable functions in $L^1(X)$ such that whenever $\gamma$ is a non-constant compact rectifiable curve with end-points $x, y \in X \setminus N$,

$$|u(y) - u(x)| \leq \liminf_{i \to \infty} \int_{\gamma} g_i \, ds$$

(that is, (2) holds) and

$$\liminf_{i \to \infty} \int_X g_i \, d\mu < \infty.$$

Note that the lemma gives a stronger control of $u$ than allowed by the BVAM-control. For functions in $BV_{AM}(X)$, we know that given a path $\gamma$ there is a set $N_{\gamma}$ with $\mathcal{H}^1(\gamma^{-1}(N_{\gamma})) = 0$ so that whenever $x, y$ lie in the trajectory of $\gamma$ with $x, y \not\in N_{\gamma}$, inequality (2) holds. Here we show that we can choose $N_{\gamma}$ to be independent of $\gamma$ and in addition with $\mu$-measure zero.

**Proof.** Given $u \in BV(X)$ there is a sequence $u_i \in \text{LIP}_{loc}(X)$ such that $u_i \to u$ in $L^1(X)$ and $\lim_{i \to \infty} \int_X g_i \, d\mu \leq M < \infty$ for a choice of upper gradients $g_i$ of $u_i$. By passing to a subsequence if necessary, we may also assume that $u_i \to u$ pointwise $\mu$-a.e. in $X$. Let $N$ be the set of all points $x \in X$ for which $\lim_{i \to \infty} u_i(x) \neq u(x)$. Then $\mu(N) = 0$. Let $\gamma$ be a non-constant compact rectifiable curve in $X$ with end points $x, y \in X \setminus N$. Then

$$|u(x) - u(y)| = \lim_{i \to \infty} |u_i(x) - u_i(y)| \leq \liminf_{i \to \infty} \int_{\gamma} g_i \, ds.$$

$\Box$

The main focus of this paper is to show that $BV_{AM}(X) = BV(X)$ when the measure on $X$ is doubling and supports a 1-Poincaré inequality.
2.3 The spaces $N^{1,1}(X)$ and $N^{1,1}_{AM}(X)$

Let $\tilde{N}^{1,1}(X,d,\mu)$, where $1 \leq p < \infty$, be the class of all $L^1$-integrable Borel functions on $X$ for which there exists a 1-weak upper gradient in $L^1(X)$. For $u \in \tilde{N}^{1,1}(X,d,\mu)$ we define

$$\|u\|_{\tilde{N}^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all 1-weak upper gradients $g$ of $u$. As usual, we can now define $N^{1,1}(X,d,\mu)$ equipped with the norm $\|u\|_{N^{1,1}(X)} = \|u\|_{\tilde{N}^{1,1}(X)}$.

Once we have the new concept of AM-a.e. curve, it is natural to define an upper gradient and a Sobolev class related to this notion.

**Definition 2.7 (Weak AM-upper gradient)** A nonnegative Borel function $g$ on $X$ is a weak AM-upper gradient of $u$ on $X$ if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds$$

for AM-a.e. curve $\gamma : [a, b] \to X$.

**Definition 2.8 ($N^{1,1}_{AM}$ functions)** Let $\tilde{N}^{1,1}_{AM}(X,d,\mu)$ be the class of all Borel functions in $L^1(X)$ for which there exists a weak AM-upper gradient in $L^1(X)$. For $u \in \tilde{N}^{1,1}_{AM}(X)$ we define

$$\|u\|_{\tilde{N}^{1,1}_{AM}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all weak AM-upper gradient $g$ of $u$. We can now define $N^{1,1}_{AM}(X)$ to be the class $\tilde{N}^{1,1}_{AM}(X,d,\mu)$, equipped with the norm $\|u\|_{N^{1,1}_{AM}(X)} = \|u\|_{\tilde{N}^{1,1}_{AM}(X)}$.

The following lemma proves that the first definition implies the second one. In some sense, the first definition is related to the Sobolev class $N^{1,1}$ while the second is related to the BV class.

**Lemma 2.9** If a function $u$ on $X$ has $g$ as a weak AM-upper gradient, then there exists a BV$_{AM}$-upper bound of $u$.

**Proof.** Assume that

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds$$

for AM-a.e. curve $\gamma : [a, b] \to X$. Let $\Gamma$ be the collection of curves for which (3) does not hold. By definition $AM(\Gamma) = 0$ and so by [HMM, Theorem 7] there is a sequence of non-negative Borel functions $\tilde{g}_i$ such that

$$\liminf_{i \to \infty} \|\tilde{g}_i\|_{L^1} < \infty \quad \text{and} \quad \liminf_{i \to \infty} \int_{\gamma} \tilde{g}_i \, ds = \infty \quad \text{for all} \ \gamma \in \Gamma.$$

Let $\Gamma_0$ be the collection of all non-constant compact rectifiable curves $\gamma$ in $X$ for which

$$\liminf_{i \to \infty} \int_{\gamma} \tilde{g}_i \, ds = \infty;$$
then $\text{AM}(\Gamma_0) = 0$. Observe that if $\gamma$ is a non-constant compact rectifiable curve in $X$ such that $\gamma \notin \Gamma_0$, then every sub-curve of $\gamma$ also does not belong to $\Gamma_0$. Now, for each $\varepsilon > 0$ the sequence of functions $g_i = g + \varepsilon \tilde{g}_i$ has the property that for $\gamma \notin \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds \leq \liminf_{i \to \infty} \int_{\gamma} (g + \varepsilon \tilde{g}_i) \, ds,$$

and for $\gamma \in \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \infty = \int_{\gamma} (g + \varepsilon \tilde{g}_i) \, ds,$$

so $|u(\gamma(a)) - u(\gamma(b))| \leq \liminf_{i \to \infty} \int_{\gamma} g_i \, ds$ holds for every curve $\gamma$. □

Note that we have more than just that the sequence $(g_i)$ forms a $\text{BV}_{\text{AM}}$-upper bound of $u$; the inequality holds for every subcurve of $\gamma$, not merely for $H^1$-almost every pair of points in the domain of $\gamma$.

From the above we know that for $1 < p < \infty$, $N^{1,1}(X) \subseteq N^{1,1}_{\text{AM}}(X) \subseteq \text{BV}_{\text{AM}}(X)$ and

$$\text{LIP}^\infty(X) \subseteq N^{1,\infty}(X) \subseteq N^{1,p}(X) \subseteq N^{1,1}(X) \subseteq \text{BV}(X) \subseteq \text{BV}_{\text{AM}}(X).$$

In Section 3 we will show that if $X$ supports a 1-Poincaré inequality then $\text{BV}_{\text{AM}}(X) = \text{BV}(X)$ and that $N^{1,1}_{\text{AM}}(X) = N^{1,1}(X)$.

**Remark 2.10** For $u \in \text{BV}_{\text{AM}}(X)$ and a sequence $(g_i)$ such that $\lim_{i \to \infty} \int_X g_i \, d\mu < \infty$, the sequence of measures $(g_i \, d\mu)$ is a bounded sequence. We can assume (by localizing the argument if need be) that $X$ is compact as well. Then there is a subsequence, also denoted $(g_i \, d\mu)$, and a Radon measure $\nu$ on $X$ such that the sequence of measures $(g_i \, d\mu)$ converges weakly* to $d\nu$ in $X$. As $X$ is compact, we see that $\|D_{\text{AM}}u\|(X) \leq \nu(X)$.

### 3 Equivalence of BV and AM-BV classes under Poincaré inequality

The aim of this section is to show the equivalence of the functional spaces $\text{BV}(X)$ and $\text{BV}_{\text{AM}}(X)$, under the additional hypothesis that the metric space supports a doubling measure and a 1-Poincaré inequality.

**Definition 3.1** The metric measure space $X$ supports a 1-Poincaré inequality if there are positive constants $C, \lambda$ such that whenever $B$ is a ball in $X$ and $g$ is an upper gradient of $u$,

$$\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \int_{\lambda B} g \, d\mu.$$

Here $u_B := \mu(B)^{-1} \int_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu$ is the average of $u$ on the ball $B$. 
With the notion of BV<sub>AM</sub> class, one could even define a stronger version of 1-Poincaré inequality.

**Definition 3.2** We say that X supports an **AM-Poincaré inequality** if there exist constants $C > 0$, $\lambda \geq 1$ such that for each measurable function $u$ on X, each BV-upper bound $(g_i)$ of $u$, and each ball $B \subset X$, we have

$$\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \liminf_{i \to \infty} \int_{\lambda B^i} g_i \, d\mu.$$ 

This should imply that

$$\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \|D_{AM}u\| (\lambda B^i) \mu(\lambda B^i).$$

On the other hand, notice that 1-Poincaré inequality implies

$$\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \|Du\| (\tau B^i) \mu(\tau B^i).$$

As a first step, in the following proposition we prove the equivalence of BV(X) and BV<sub>AM</sub>(X) under the hypotheses that the measure is doubling and the space supports an AM-Poincaré inequality. We will see in Theorem 3.10 that the support of an AM-Poincaré inequality is equivalent to the support of a 1-Poincaré inequality.

**Proposition 3.3** If X supports a AM-Poincaré inequality and $\mu$ is doubling, then the two classes BV<sub>AM</sub>(X) and BV(X) are equal, with comparable norms.

**Proof.** Note first that BV(X) $\subset$ BV<sub>AM</sub>(X), see Lemma 2.6.

Now let us prove that if $u \in$ BV<sub>AM</sub>(X), then $u \in$ BV(X). By the doubling property of $\mu$, for $\varepsilon > 0$ we can cover X by balls $B_i = B(x_i, \varepsilon)$ such that the balls $5\lambda B_i$ have bounded overlap. Let $\varphi_i^\varepsilon$ be a partition of unity subordinate to the cover $2B_i$. For $u \in$ BV<sub>AM</sub>(X) let

$$u_\varepsilon = \sum_i u_{B_i} \varphi_i^\varepsilon.$$ 

Recall that we have bounded overlap of the collection $5B_i$ with $X = \bigcup_j B_j$, $\mu$ is doubling, and that if $2B_i$ intersects $B_j$ then $5\lambda B_j \supset 2B_i$. Then we have for $x \in B_j \subset X$,

$$|u(x) - u_\varepsilon(x)| = \left| \sum_i |u_{B_i} - u(x)| \varphi_i^\varepsilon(x) \right| \leq \sum_i |u_{B_i} - u(x)| \varphi_i^\varepsilon(x)$$

$$\leq \sum_{i: x \in 2B_i} |u_{B_i} - u(x)|$$

$$\leq \sum_{i: 2B_i \cap B_j \neq \emptyset} |u_{B_i} - u(x)|$$

$$\leq C \sum_i |u_{5B_j} - u(x)|.$$
Therefore, by the AM-Poincaré inequality,
\[
\int_X |u - u_\varepsilon| \, d\mu \leq \sum_j \int_{B_j} |u - u_\varepsilon| \, d\mu \leq C \sum_j \int_{5B_j} |u_{5B_j} - u| \, d\mu
\]

\[
\leq C \sum_j \int_{5B_j} |u - u_{5B_j}| \, d\mu
\]

\[
\leq C \varepsilon \sum_j \|D_A u\|(5\lambda B_j).
\]

Since \(\|D_A u\|\) is a Radon measure ([M1 Theorem 3.4]) and \(5\lambda B_j\) have bounded overlap, we have
\[
\int_X |u - u_\varepsilon| \, d\mu \leq C \varepsilon \|D_A u\|(X) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+.
\]

Thus \(u_\varepsilon \rightarrow u\) in \(L^1(X)\), and we also know from the definition of \(u_\varepsilon\) that each \(u_\varepsilon\) is locally Lipschitz and hence in \(N_{loc}^{1,1}(X)\). Next, for \(x, y \in B_j\),
\[
|u_\varepsilon(x) - u_\varepsilon(y)| = \left| \sum_i u_{B_i} [\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)] \right| = \left| \sum_i [u_{B_i} - u_{B_j}] [\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)] \right|
\]

\[
\leq \frac{d(x, y)}{\varepsilon} \sum_{i: 2B_i \cap B_j \neq \emptyset} |u_{B_i} - u_{B_j}|
\]

\[
\leq C \frac{d(x, y)}{\varepsilon} \sum_{i: 2B_i \cap B_j \neq \emptyset} \int_{5B_j} |u - u_{5B_j}| \, d\mu.
\]

It follows from the bounded overlap of \(5B_i\) that
\[
\text{Lip}_u(x) \leq \frac{C}{\varepsilon} \int_{5B_j} |u - u_{5B_j}| \, d\mu
\]

whenever \(x \in B_j\). Integrating the above inequality over \(X = \bigcup_j B_j\), we obtain
\[
\int_X \text{Lip}_u \, d\mu \leq \frac{C}{\varepsilon} \sum_j \mu(B_j) \int_{5B_j} |u - u_{5B_j}| \, d\mu
\]

\[
\leq \frac{C}{\varepsilon} \sum_j \int_{5B_j} |u - u_{5B_j}| \, d\mu
\]

\[
\leq \frac{C}{\varepsilon} \sum_j \varepsilon \|D_A u\|(5\lambda B_j)
\]

\[
\leq C \|D_A u\|(X).
\]

Thus
\[
\liminf_{\varepsilon \rightarrow 0^+} \int_X g_{u_\varepsilon} \, d\mu \leq C \|D_A u\|(X) < \infty,
\]

and as \(u_\varepsilon \rightarrow u\) in \(L^1(X)\), it follows that \(u \in BV(X)\) with \(\|Du\|(X) \leq C \|D_A u\|(X)\).
We also have \( \|D_{AM}u\|(X) \leq \|Du\|(X) \), as we now show. Suppose now that \( \|Du\|(X) \) is finite, and let \( u_k \in BV(X) \) be such that \( u_k \to u \) in \( L^1(X) \) and \( \lim_{k\to\infty} \int_X g_{u_k} \, d\mu = \|Du\|(X) \). By passing to a subsequence if necessary, we may also assume that \( u_k \to u \) pointwise almost everywhere in \( X \) as well. For each \( k \in \mathbb{N} \) we choose an upper gradient \( g_k \) of \( u_k \) such that \( \int_X g_k \, d\mu \leq \int_X g_{u_k} \, d\mu + \varepsilon/2^k \). We set \( N \) to be the collection of all points \( x \in X \) at which \( u_k(x) \) does not converge to \( u(x) \). Then \( \mu(N) = 0 \), and so the 1-modulus of the collection \( \tilde{\Gamma}_N^+ \) of non-constant compact rectifiable curves \( \gamma \) in \( X \) for which \( \mathcal{H}(\gamma^{-1}(N)) > 0 \) is zero. Using \( [He, (7.8)] \), we know that the collection \( \Gamma^+_N \) of all non-constant compact rectifiable curves in \( X \) with a subcurve in \( \tilde{\Gamma}_N^+ \) is also of 1-modulus zero. Let \( \gamma \) be a non-constant compact rectifiable curve in \( X \) with \( \gamma \notin \Gamma^+_N \). By re-parametrizing if necessary, we now assume that \( \gamma : [a, b] \to X \) is arc-length parametrized; then \( \mathcal{H}^1([a, b] \setminus \gamma^{-1}(N)) = 0 \). For \( s, t \in [a, b] \setminus \gamma^{-1}(N) \) with \( s > t \) we have that

\[
|u(\gamma(t)) - u(\gamma(s))| = \lim_{k \to \infty} |u_k(\gamma(t)) - u_k(\gamma(s))| \leq \liminf_{k \to \infty} \int_{[\gamma][t,s]} g_k \, ds \leq \liminf_{k \to \infty} \int_{\gamma} g_k \, ds.
\]

This verifies that \( (g_k) \) is a \( BV_{AM} \)-upper bound for \( u \) in the sense of Definition \( 2.50 \). \( \Box \)

**Proposition 3.4** If \( X \) supports an AM-Poincaré inequality and \( \mu \) is doubling, then \( N_{AM}^{1,1}(X) = N^{1,1}(X) \) with comparable norms.

**Proof.** Note that \( N^{1,1}(X) \subset N_{AM}^{1,1}(X) \). Thus it suffices to prove the reverse inclusion. Let \( u \in N_{AM}^{1,1}(X) \), and let \( g \in L^1(X) \) be a weak AM-upper gradient of \( u \). Let \( \Gamma \) be the corresponding exceptional family; then AM(\( \Gamma \)) = 0. Then by the proof of Lemma \( 2.7 \) there is a sequence \( (\rho_i) \) of non-negative Borel functions in \( L^1(X) \) such that \( \int_X \rho_i \, d\mu \leq M < \infty \) for each \( i \in \mathbb{N} \) and for each \( \gamma \in \Gamma \) we have

\[
\lim_{i \to \infty} \int_{[\gamma]} \rho_i \, ds = \infty.
\]

Then for each \( \varepsilon > 0 \) we have that \( (g + \varepsilon \rho_i) \) forms a \( BV_{AM} \)-upper bound of \( u \), and so as \( X \) supports an AM-Poincaré inequality, whenever \( B \) is a ball in \( X \) we have

\[
\int_B |u - u_B| \, d\mu \leq \frac{C \text{rad}(B)}{\mu(B)} \liminf_{i \to \infty} \int_{\lambda B} [g + \varepsilon \rho_i] \, d\mu.
\]

As before, by passing to a subsequence if necessary, we may assume that \( \rho_i \, d\mu \) converges weakly to a Radon measure \( \nu \) on \( X \), and so the above turns into

\[
\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \left( \int_{\lambda B} g \, d\mu + \varepsilon \frac{\nu(2\lambda B)}{\mu(2\lambda B)} \right).
\]

Letting \( \varepsilon \to 0 \) we get

\[
\int_B |u - u_B| \, d\mu \leq C \text{rad}(B) \int_{\lambda B} g \, d\mu.
\]
We now know from Proposition 3.3 that \( u \in BV(X) \). Now an argument as in the proof of Proposition 3.3 up to and including (4), applied to open sets \( U \subset X \) with \( \mu(\partial U) = 0 \), we obtain that

\[
\|Du\|(U) \leq C \int_U g \, d\mu = \int_U g \, d\mu.
\]

Note that \( g \in L^1(X) \), and hence for each \( \eta > 0 \) there is some \( \varepsilon > 0 \) such that whenever \( K \subset X \) is measurable with \( \mu(K) < \varepsilon \), we have \( \int_K g \, d\mu < \eta \). Since whenever \( E \subset X \) with \( \mu(E) = 0 \), for each \( \varepsilon > 0 \) we can find an open set \( U_\varepsilon \supset E \) such that \( \mu(U_\varepsilon) < \varepsilon \) and \( \mu(\partial U_\varepsilon) = 0 \), it follows that \( \|Du\| \ll \mu \), and hence \( u \in N^{1,1}(X) \) by [HKLL] Theorem 4.6.

Note that if \( X \) does not support a 1-Poincaré inequality, we do not know the equivalence of \( N^{1,1}(X) \) with \( N^{1,1}_{AM}(X) \). Similar difficulties show up in comparing other alternative notions of \( N^{1,1}(X) \) as well, see for example [ADiM, Section 8]. We will prove in Theorem 3.10 that \( X \) supports a 1-Poincaré inequality if and only if it supports the a priori stronger AM-Poincaré inequality.

The key point in the above proof is that if \( u \in BV(X) \) and \( \|Du\| \ll \mu \), then \( u \in N^{1,1}(X) \); the validity of this point requires a doubling measure supporting a 1-Poincaré inequality. The following counterexample is from [ADiM, Example 7.4]. We do not have a counterexample for the statement \( \|Du\| \ll \mu \) implies \( u \in N^{1,1}(X) \) in the case \( \mu \) is doubling, but the measure \( \mu \) in the following example is asymptotically doubling.

**Example 3.5** Let \( X = \mathbb{R}^2 \) be equipped with the Euclidean metric and the measure \( \mu = \mathcal{L}^2 + \mathcal{H}^1|_C \) where \( C \) is the boundary of the unit disk \( D \) in \( \mathbb{R}^2 \) centered at the origin. Let \( u = \chi_D \). Then, by the approximations \( f_\varepsilon(x) = (1 - \varepsilon^{-1} \text{dist}(x, D))_+ \) of \( u \) we see that \( u \in BV(X) \) with \( \|Du\| \equiv \mathcal{H}^1|_C \). It follows that \( \|Du\| \ll \mu \). However, \( u \notin N^{1,1}(X) \): with \( \Gamma \) the collection of all line segments \( \gamma_x \), \(-1 < x < 1 \), given by \( \gamma_x : [-2, 2] \to X \) where \( \gamma_x(t) = (x, t) \), we have that \( u \circ \gamma_x \) is not absolutely continuous on \([-2, 2] \), and furthermore, \( \text{Mod}_1(\Gamma) > 0 \).

The existence of a Semmes family of curves provides a key tool for the proof that the AM-Poincaré inequality and the standard 1-Poincaré inequality are equivalent, which in turn allows us to prove equivalence of the two classes \( BV(X) \) and \( BV_{AM}(X) \) with just the assumption of a 1-Poincaré inequality in addition to the doubling property of \( \mu \). Thus we next prove that the existence of 1-Poincaré inequality in the doubling complete metric measure space \( X \) is equivalent to the existence of the following Semmes pencil of curves. See [FO] for a closely related characterization of the 1-Poincaré inequality in terms of 1-currents in the sense of Ambrosio and Kirchheim [AK].

If \( A \) is a Borel subset of \( X \) and \( \gamma \) is a rectifiable curve, we define \( \ell(\gamma \cap A) := \mathcal{H}^1(\gamma \cap A) \).

**Definition 3.6** ([Sa], [Ha]) A space \( X \) supports a *Semmes pencil of curves* if there exists a constant \( C > 0 \) such that for each pair of points \( x, y \in X \) with \( x \neq y \) there is a family \( \Gamma_{xy} \) of rectifiable curves in \( X \) equipped with a probability measure \( d\sigma = d\sigma_{x,y} \) so that each \( \gamma \in \Gamma_{xy} \) is...
a $C$-quasiconvex curve joining $x$ to $y$, and for each Borel set $A \subset X$, the map $\gamma \mapsto \ell(\gamma \cap A)$ is $\sigma$-measurable and satisfies

$$
\int_{\Gamma_{xy}} \ell(\gamma \cap A) \, d\sigma(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z) \, d\mu(z). \tag{5}
$$

In the previous inequality, for $C > 0$, $CB_{x,y} := B(x, Cd(x,y)) \cup B(y, Cd(x,y))$ and

$$
R_{xy}(z) := \frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))}. \tag{6}
$$

We next show that if the measure on $X$ is doubling and supports a 1-Poincaré inequality, then it supports a Semmes pencil of curves.

Denote

$$
\Gamma^C_{xy} := \{ (\gamma, I) : \text{ curve } \gamma : I \to X \text{ is 1-Lipschitz, } \gamma(0) = x, \gamma(\max(I)) = y \}, \tag{7}
$$

where $I$ are intervals contained in $[0, C d(x,y)]$ with left-hand end point 0. We equip $\Gamma^C_{xy}$ with the following metric. The elements of $\Gamma^C_{xy}$ can be identified with their graphs

$$
\Gamma_{\gamma} = \{ (t, \gamma(t)) : t \in I \} \subset \mathbb{R} \times X.
$$

We define a metric on $\Gamma^C_{xy}$ by setting

$$
d(\gamma, \gamma') := d_H(\Gamma_{\gamma}, \Gamma_{\gamma'}),
$$

where $d_H$ is the Hausdorff metric. Thanks to the Arzela-Ascoli theorem, this metric makes $\Gamma^C_{xy}$ into a complete and compact metric space because $X$ is complete and doubling and hence closed bounded subsets of $X$ are compact. For $f \in C(X)$, the functional $\Phi_f : \Gamma^C_{xy} \to \mathbb{R}$ given by

$$
\Phi_f((\gamma, I)) := \int_I f \circ \gamma \, dt,
$$

is continuous on $\Gamma^C_{xy}$.

We denote the Riesz measure by

$$
d\varpi^C_{xy}(z) = R_{xy} \, d\mu|_{CB_{x,y}}.
$$

**Theorem 3.7** If $(X, d, \mu)$ satisfies a 1-Poincaré inequality, then there exists $C \geq 1$ such that for every $x, y \in X$ with $x \neq y$, there exist a compact family of curves $\Gamma_{xy}$ and a Radon probability measure $\alpha_{xy}$ on $\Gamma_{xy}$ which constitutes a Semmes family of curves, i.e. for every Borel set $A$,

$$
\int_{\Gamma^C_{xy}} \int_{\gamma} \chi_A \, ds \, d\alpha_{xy}(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z) \, d\mu(z) = \varpi^C_{xy}(A).
$$
The proof of the above statement could be derived by a careful application of the techniques in [AMS] combined with the modulus estimates of [K]. However, the method in [AMS] directly works only for \( p > 1 \), and some additional care is necessary for \( p = 1 \). Further, the following proof is somewhat more direct than theirs. Our proof is more in line with the approaches in [B, S] in combination with the estimates from [K] to construct probability measures on the space of curve fragments. The papers [B, S] employ the Rainwater lemma from [R2, Theorem 9.4.3]. However, we are able to avoid this lemma by directly using the Min-Max theorem [R2, Theorem 9.4], restated below for the reader’s convenience.

**Proposition 3.8** (Min-Max Theorem [R2 Theorem 9.4.2]) Suppose that

(i) \( G \) is a convex subset of some vector space,

(ii) \( K \) is a compact convex subset of some topological vector space, and

(iii) \( F : G \times K \to \mathbb{R} \) satisfies

(a) \( F(\cdot, y) \) is convex on \( G \) for every \( y \in K \),

(b) \( F(x, \cdot) \) is concave and continuous on \( K \) for every \( x \in G \).

Then

\[
\sup_{y \in K} \inf_{x \in G} F(x, y) = \inf_{x \in G} \sup_{y \in K} F(x, y).
\]

**Proof.** [Proof of Theorem 3.7] Denote \( d(x, y) = r \). By the 1-Poincaré inequality and [K, Theorem 2], there exists a \( C \) such that we have

\[
\text{Mod}_{1,\mathcal{C}_{xy}}(\Gamma_{xy}^C) = \inf \int_X \rho \, d\mu_{xy}^C > \frac{1}{C},
\]

where the infimum is over non-negative Borel functions \( \rho \) with \( \int_\gamma \rho \geq 1 \) for every \( \gamma \in \Gamma_{xy}^C \). Note that the estimates in [K] give the modulus bound for the set of all rectifiable curves between \( x, y \), but the collection of curves that are longer than \( 4C^2d(x, y) \) has modulus less than \( 1/(2C) \), and can be excluded using the subadditivity of the modulus.

Another way of stating this estimate is that if \( f \) is a non-negative continuous function, and \( \int_X f \, d\mu_{xy}^C < \infty \), then for every \( \epsilon > 0 \) there exists a \( \gamma \in \Gamma_{xy}^C \) such that

\[
\int_\gamma f \, ds \leq (C + \epsilon) \int_X f \, d\mu_{xy}^C,
\]

for otherwise, \( \frac{1}{(C + \epsilon)} \int_X f \, d\mu_{xy}^C \) would be admissible with a too small a norm. In particular,  

\[
\inf_{(\gamma, I) \in \Gamma_{xy}^C} \int_\gamma f \, ds \leq C \int_X f \, d\mu_{xy}^C. \tag{8}
\]
Since \( f \) is continuous and \( \Gamma_{xy}^C \) is a compact family, the above infimum is a minimum. Parametrizing the curves \( \gamma \) by length we also get

\[
\int_{(\gamma,I) \in \Gamma_{xy}^C} \int_I f \circ \gamma(t) \, dt \, d\beta(\gamma,I) \leq C \int_X f \, d\mu_{xy},
\]

(9)

where \( \beta \) is the Dirac measure on \( \Gamma_{xy}^C \) based at any of the optimal choices \((\gamma,I)\) that achieves the infimum in (8).

Let \( K \) be the set of probability measures \( \alpha \) on \( \Gamma_{xy}^C \); thus \( K \) is a compact and convex set of measures with respect to weak* convergence. Set

\[
G = \{ f : X \to [0,1] \mid f \text{ is continuous} \} \subset C(X).
\]

Here \( C(X) \) is the set of all continuous functions equipped with the uniform topology and \( G \) is a closed convex subset thereof. Then define \( F : G \times K \to \mathbb{R} \) by

\[
F(f, \alpha) = C \int_X f \, d\mu_{xy} - \int_{\Gamma_{xy}^C} \int_I f(\gamma(t)) \, dt \, d\alpha(\gamma,I).
\]

Clearly \( F \) is continuous in \( \alpha \), since \( \Phi_f((\gamma,I)) = \int_I f(\gamma(t)) \, dt \) is continuous in \( \gamma \). Also, \( F(\cdot, \alpha) \) is convex for every \( \alpha \in K \), and \( F(f, \cdot) \) is affine and a fortiori concave for any \( f \in G \). Thus, we can apply Proposition 3.8 to obtain

\[
\sup_{\alpha \in K} \inf_{f \in G} F(f, \alpha) = \inf_{f \in G} \sup_{\alpha \in K} F(f, \alpha).
\]

Now, for \( f \in G \), by estimate (9) we have \( \sup_{\alpha \in K} F(f, \alpha) \geq 0 \). Thus, we get

\[
\sup_{\alpha \in K} \inf_{f \in G} F(f, \alpha) \geq 0.
\]

In particular, for every \( \epsilon > 0 \) and every \( f \in G \) there exists a \( \alpha_\epsilon \in K \), such that

\[
F(f, \alpha_\epsilon) \geq -\epsilon.
\]

Since for each \( f \in G \) the map \( K \ni \alpha \mapsto F(f, \alpha) \) is continuous, we can extract a weakly convergent sequence \( \alpha_{\epsilon_i} \to \alpha_{xy} \in K \) (with \( \epsilon_i \to 0 \) as \( i \to \infty \)), such that for every \( f \in G \)

\[
F(f, \alpha_{xy}) \geq 0.
\]

Now, recalling the definition of \( F \), for every \( f \in G \),

\[
\int_{\Gamma_{xy}^C} \int_I f(\gamma(t)) \, dt \, d\alpha_{xy}(\gamma,I) \leq C \int_X f \, d\mu_{xy}.
\]

Also, since the curves \( \gamma \) are 1-Lipschitz, it follows that \( \int_\gamma f \, ds \leq \int_I f(\gamma(t)) \, dt \), and \( \alpha_{xy} \) induces a measure (which we denote by the same symbol) on \( \Gamma_{xy} = \{ \gamma : (\gamma,I) \in \Gamma_{xy}^C \text{ for some } I \} \). With respect to this measure, we have for every \( f \in C(X) \) with \( 0 \leq f \leq 1 \) that

\[
\int_{\Gamma_{xy}} \int_\gamma f \, ds \, d\alpha_{xy}(\gamma) \leq C \int_X f \, d\mu_{xy}.
\]
By a limiting argument we obtain the same inequality for \( f = \chi_A \) corresponding to Borel sets \( A \subset X \), and thus the measure \( \sigma_{xy} = \alpha_{xy} \), which is supported on the compact set \( \Gamma_{xy} \), constitutes a Semmes family of curves in the sense of Definition 3.6 and the proof is complete. \( \square \)

Each Borel function in \( L^1_{loc}(X) \) can be approximated by simple Borel functions. Hence it follows from (5) that

\[
\int_{\Gamma_{xy}} \int g \, ds \, d\sigma(\gamma) \leq C \int_{CB_{x,y}} g(z) R_{xy}(z) \, d\mu(z),
\]

for Borel functions \( g : CB_{x,y} \to \mathbb{R} \). Doubling metric measure spaces supporting a Semmes pencil curves support a 1-Poincaré inequality (see e.g. the discussion following [Se, Definition 14.2.4]). In what follows we prove that they also support the AM-Poincaré inequality. Recall that

\[
I_A(u)(x) = \int_A \frac{u(z)d(x, z)}{\mu(B(x, d(x, z)))} \, d\mu(z)
\]

denotes the Riesz potential of a non-negative function \( u \) defined on \( X \) on a subset \( A \subset X \).

**Proposition 3.9** If \( X \) supports a Semmes pencil of curves, then \( X \) supports the AM-Poincaré inequality.

**Proof.** Let \( u \in L^1_{loc}(X) \) and let \((g_i)\) be a BV-upper bound of \( u \), and let \( N \) be the collection of all points \( x \in X \) for which

\[
\limsup_{r \to 0^+} \frac{\int_{B(x,r)} |u - u(x)| \, d\mu}{\mu(B(x,r))} > 0;
\]

Then \( \mu(N) = 0 \). We focus on points \( x, y \in X \setminus N \). Then for each \( \varepsilon > 0 \) we know that the sets

\[
E_\varepsilon(x) := \{ z \in X : |u(z) - u(x)| > \varepsilon \}, \quad E_\varepsilon(y) = \{ z \in X : |u(z) - u(y)| > \varepsilon \}
\]
satisfy

\[
\limsup_{r_i \to 0^+} \frac{\mu(B(x, r_i) \cap E_\varepsilon(x))}{\mu(B(x, r_i))} = 0 = \limsup_{r_i \to 0^+} \frac{\mu(B(y, r_i) \cap E_\varepsilon(y))}{\mu(B(y, r_i))}.
\]

We can inductively choose a strictly decreasing sequence \( r_i > 0 \) such that \( r_1 < d(x, y)/4 \), \( r_{i+1} < r_i/4 \), and

\[
\frac{\mu(B(x, r_i) \cap E_\varepsilon(x))}{\mu(B(x, r_i))} < \frac{2^{-i}}{2C_d}, \quad \frac{\mu(B(y, r_i) \cap E_\varepsilon(y))}{\mu(B(y, r_i))} < \frac{2^{-i}}{2C_d}.
\]

For each \( i \) let \( \Gamma_i(x) \) denote the collection of all \( \gamma \in \Gamma_{xy} \) such that

\[
\mathcal{H}^1(\gamma^{-1}(B(x, r_i) \setminus B(x, r_i/2)) \setminus E_\varepsilon(x)) = 0,
\]

16
and $\Gamma_i(y)$ the analogous family with $y$ playing the role of $x$. By the fact that $\Gamma_{xy}$ is a Semmes family and by the fact that $\mu$ is doubling, we have that

$$\frac{r_i}{2}\sigma_{xy}(\Gamma_i(x)) \leq \int_{\Gamma_{xy}} \ell(\gamma \cap E(x) \cap B(x, r_i/2)) \, d\sigma_{xy}(\gamma) \leq C_d \frac{r_i}{\mu(B(x, r_i))} \mu(E(x) \cap B(x, r_i)),$$

and so by the choice of $r_i$ we have

$$\sigma_{xy}(\Gamma_i(x)) \leq 2^{-i}.$$

Hence for each positive integer $n$,

$$\sigma_{xy}\left(\bigcup_{i=n}^\infty \Gamma_i(x)\right) \leq 2^{1-n},$$

and so with $\Gamma(x) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=n}^\infty \Gamma_i(x)$,

we have that $\sigma_{xy}(\Gamma(x)) = 0$. Note that if $\gamma \in \Gamma_{xy} \setminus \Gamma(x)$, then whenever $N_\gamma$ is a subset of the domain of $\gamma$ with $\mathcal{H}^1(N_\gamma) = 0$, we can find a sequence of points $x_i \in \gamma \setminus [E(x) \cup \gamma(N_\gamma)]$ such that $x_i \to x$ as $i \to \infty$. Let $\Gamma(y)$ be the analogous subfamily of curves with respect to the point $y$; then $\sigma_{xy}(\Gamma(x) \cup \Gamma(y)) = 0$. Let $(g_i)$ be a $\text{BV}_{\text{AM}}$-upper bound for $u$. For $\gamma \in \Gamma_{xy} \setminus (\Gamma(x) \cup \Gamma(y))$, we set $N_\gamma$ to be the set of points in the domain of $\gamma$ that forms the exceptional set in the condition (2), and we select the sequences $x_i, y_i$ as above. Then we have that

$$|u(x) - u(y)| - 2\varepsilon \leq \liminf_{i \to \infty} |u(x_i) - u(y_i)| \leq \liminf_{i \to \infty} \int_{\gamma} g_i \, ds.$$

Therefore, for $x, y \in X \setminus N$ and for each $\gamma \in \Gamma_{xy} \setminus (\Gamma(x) \cup \Gamma(y))$, we have

$$|u(x) - u(y)| - 2\varepsilon \leq \liminf_{i \to \infty} \int_{\gamma} g_i \, ds.$$

We then have by Fatou’s lemma and (10) that

$$|u(x) - u(y)| - 2\varepsilon \leq \int_{\Gamma_{xy}} \liminf_{i \to \infty} \int_{\gamma} g_i \, ds \, d\sigma_{xy}(\gamma) \leq \liminf_{i \to \infty} \int_{\Gamma_{xy}} \int_{\gamma} g_i \, ds \, d\sigma_{xy}(\gamma) \leq C \liminf_{i \to \infty} \int_{CB_{x,y}} g_i(z) R_{xy}(z) \, d\mu(z) \leq \int_{CB_{x,y}} R_{xy}(z) \, d\nu(z) \leq C(\int_{CB_{x,y}} \nu(x) + \int_{CB_{x,y}} \nu(y)),$$

where $\nu$ is the Radon measure as in Remark 2.10. The constant $C$ in the above does not depend on $\varepsilon$; hence, by letting $\varepsilon \to 0^+$ we obtain

$$|u(x) - u(y)| \leq C(\int_{CB_{x,y}} \nu(x) + \int_{CB_{x,y}} \nu(y)).$$
whenever $x, y \in X \setminus N$. For $x, y \in B$ with $R$ the radius of $B$, setting $B_i = B(x, 2^{-i}Cd(x, y))$ for $i \in \mathbb{N} \cup \{0\}$, we see that

$$I_{CB,y} \nu(x) \leq \int_{B(x,Cd(x,y))} \frac{d(x,z)}{\mu(B(x,d(z,x)))} d\mu(z) \leq C \sum_{i=0}^{\infty} 2^{-i}Cd(x,y) \frac{\nu(B_i)}{\mu(B_i)} \nu(B) \leq Cd(x,y) h_B(x) \sum_{i=0}^{\infty} 2^{-i},$$

where

$$h_B(x) = \sup_{0<r \leq CR} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$ 

Thus $h_B$ is a Hajłasz gradient of $u$ in $B$, that is,

$$|u(x) - u(y)| \leq Cd(x,y)[h_B(x) + h_B(y)]$$

for $\mu$-a.e. $x, y \in B$, and we have the weak inequality

$$\mu(\{x \in B : h_B(x) > t\}) \leq C \frac{\nu(B)}{t}$$

for $t > 0$.

Thus $h_B \in L^q(B)$ for $0 < q < 1$, and hence $u \in M^{1,q}(B)$ in the sense of [HajC], and so by [HajC] Corollary 8.9 of page 202 or by [KLS] Proposition 2.4, we know that

$$\int_B |u - u_B| d\mu \leq CR \frac{\nu(B)}{\mu(B)}.$$

The proof is then completed by taking a sequence of sequences $(g^j)_i$ that are BVAM-upper bound of $u$ with corresponding measures $\nu_j$ such that $\lim_j \nu_j(2B) = \|D_{AM}u\|(2B).$ \hfill $\square$

From Proposition 3.9, Theorem 3.7, and Proposition 3.3 we have the following.

**Theorem 3.10** Let $\mu$ be a doubling measure on $X$. Then the following are equivalent:

1. $X$ supports a 1-Poincaré inequality.
2. $X$ supports a Semmes pencil of curves.
3. $X$ supports an AM-Poincaré inequality.

In any (and therefore all) of the above, we have $BV(X) = BV_{AM}(X)$ and $N^{1,1}(X) = N_{AM}^{1,1}(X)$. 

18
References

[AK] L. Ambrosio, B. Kirchheim: Currents in metric spaces. Acta Math., 185 (1): 1–80, (2000).

[ADiM] L. Ambrosio and S. Di Marino: Equivalent definitions of BV space and of total variation on metric measure spaces. J. Funct. Anal. 266 (2014), no. 7, 4150–4188.

[AMS] L. Ambrosio, S. Di Marino, and G. Savaré: On the duality between $p$-modulus and probability measures. J. Eur. Math. Soc. (JEMS) (2015), vol. 17, issue 8, 1817–1853.

[B] D. Bate: Structure of measures in Lipschitz differentiability spaces. Journal of the American Mathematical Society (2014), 28(2), 421-482.

[FO] K. Fässler and T. Orponen: Metric currents and the Poincaré inequality. Preprint, https://arxiv.org/pdf/1807.02969.pdf

[HajC] P. Hajłasz: Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., 338, Amer. Math. Soc., Providence, RI (2003), 173–218.

[HKLL] H. Hakkarainen, H., Kinnunen, J., Lahti, P., and Lehtelä, P.: Relaxation and integral representation for functionals of linear growth on metric measure spaces. Anal. Geom. Metr. Spaces 4 (2016), 288–313.

[He] J. Heinonen: Lectures on Analysis on Metric Spaces. Springer (2001).

[HMM] V. Honzlová Exnerová, J. Malý, and O. Martio: Modulus in Banach function spaces. Ark. Mat. 55 (2017), no. 1, 105–130.

[HMM2] V. Honzlová Exnerová, J. Malý, J., O. Martio: Functions of bounded variation and the AM-modulus in $\mathbb{R}^n$, Nonlinear Analysis (2018), to appear.

[JJRRS] E. Järvenpää, M. Järvenpää, K. Rogovin, S. Rogovin, N. Shanmugalingam: Measurability of equivalence classes and $M E C_p$-property in metric spaces, Rev. Mat. Iberoamericana 23 no. 3 (2007) 811–830.

[K] S. Keith: Modulus and the Poincaré inequality on metric measure spaces. Mathematische Zeitschrift (2003), 245.2, 255-292.

[KLS] R. Korte, P. Lahti, N. Shanmugalingam: Semmes family of curves and a characterization of functions of bounded variation in terms of curves. Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1393–1424.

[L] P. Lahti: Federer’s characterization of sets of finite perimeter in metric spaces. Preprint, https://arxiv.org/abs/1804.11216 (2018).

[M1] O. Martio: The space of functions of bounded variation on curves in metric measure spaces. Conform. Geom. Dyn. 20 (2016), 81–96.
[M2] O. Martio: Functions of bounded variation and curves in metric measure spaces. *Adv. Calc. Var.* 9 (2016), no. 4, 305–322.

[Mi] M. Miranda Jr.: Functions of bounded variation on “good” metric spaces. *J. Math. Pures Appl.* (9) 82 (2003), no. 8, 975–1004.

[R2] W. Rudin: Function theory in the unit ball of $\mathbb{C}^n$, *Classics in Mathematics*, Springer-Verlag, Berlin, (2008). Reprint of the 1980 edition.

[S] A. Schioppa: Derivations and Alberti representations, *Adv. Math.* 293 (2016), 436–528.

[Se] S. Semmes: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math. (N.S.*) 2 (1996), 155–296.

[Vä] J. Väisälä: *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229, Springer-Verlag, Berlin-New York, 1971. xiv+144 pp.

Addresses:

E.D-C.: Departamento de Matemática Aplicada, ETSI Industriales, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain.
E-mail: edurand@ind.uned.es

S.E.-B.: UCLA Mathematics Department, P.O. 951555, Los Angeles, CA 90095–1555, U.S.A.
E-mail: syerikss@math.ucla.edu

R.K.: Aalto University, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland
E-mail: riikka.korte@aalto.fi

N.S.: Department of Mathematical Sciences, P.O.Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: shanmung@uc.edu