Cohomological properties of ruled symplectic structures

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1 Introduction

Donaldson’s work on Lefschetz pencils has shown that, after a slight perturbation of the symplectic form and a finite number of blow-ups, any closed symplectic manifold $(M, \omega)$ can be expressed as a singular fibration with generic fiber a smooth codimension 2 symplectic submanifold. Thus fibrations play a fundamental role in symplectic geometry. It is then natural to study smooth (non-singular) ruled symplectic manifolds $(P, \Omega)$, i.e. symplectic manifolds where $P$ is the total space of a smooth fiber bundle $M \hookrightarrow P \rightarrow B$ and the symplectic form $\Omega$ is such that its restriction $\omega$ to each $M$-fiber is nondegenerate.

In this survey, we present some of the recent results obtained by the authors and Leonid Polterovich in [9, 11, 10] that show that bundles endowed with such structures have interesting stability and cohomological properties. For instance, under certain topological conditions on the base, the rational cohomology of $P$ necessarily splits as the tensor product of the cohomology of $B$ with that of $M$.

As we describe below in §2, the bundle $M \hookrightarrow P \rightarrow B$ corresponding to a ruled symplectic manifold, has the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of $M$ for structure group if the base $B$ is simply connected. They can therefore be divided into two classes: those whose structural group belongs to a finite dimensional Lie subgroup of the group $\text{Ham}(M)$ of Hamiltonian diffeomorphisms of $M$, and those whose structural group is genuinely infinite dimensional. The first case belongs to the realm of classical symplectic geometry.

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and its study can be carried out by topological and Morse-theoretic methods, whereas the second case belongs to symplectic topology and requires analytic methods such as Gromov-Witten invariants and versions of quantum homology which are relevant in the context of bundles. Our results tend to confirm that many properties of the first case still hold in the second one. They therefore fit in with the principle mentioned by Reznikov that the group of symplectomorphisms behaves like a Lie group.

Our main conjecture is

The rational cohomology of a bundle \( M \to P \to B \), with \( \text{Ham}(M,\omega) \) for structure group, splits as the tensor product of the cohomology of the fiber with that of the base. In particular, the same splitting occurs for any ruled symplectic manifold \( (P,\Omega) \) over a simply connected base.

The section §3 of this survey presents the results related to the finite dimensional case, that were already known (Blanchard, Deligne, Kirwan, Atiyah-Bott). These results prove the above conjecture in the finite dimensional case and provide a good intuition of what should hold in general. The next section, §4, presents our results in the infinite dimensional case. These are established under some restrictions on the topology of the base, and lead to corollaries that apply to the general theory of ruled symplectic manifolds.

It is still unclear whether our conjecture holds in the full generality in which it is stated, that is to say when the structure group is infinite dimensional and when no restrictions are placed on the topology of the base and fiber. However, our methods are sufficiently general to yield all known results in the finite dimensional case. More details about many of the results presented here, together with some applications to the action of \( \text{Ham}(M,\omega) \) on \( M \), can be found in [10].

Before getting into these questions, it is useful to give a characterisation of ruled symplectic manifolds that holds when the base \( B \) is simply connected.

2 Characterizing ruled symplectic manifolds

A symplectic manifold \( (P,\Omega) \) is said to be ruled if \( P \) is a (locally trivial) fiber bundle over some base manifold \( B \) and the restriction of \( \Omega \) to each fiber is nondegenerate. It turns out that there is a close relation between such manifolds and Hamiltonian bundles.

A fiber bundle \( M \to P \to B \) is said to be symplectic (resp. Hamiltonian) if, for some symplectic form \( \omega \) on \( M \), its structural group reduces to the group of symplectomorphisms \( \text{Symp}(M,\omega) \) of \( (M,\omega) \) (resp. to the group \( \text{Ham}(M,\omega) \) of Hamiltonian diffeomorphisms of \( (M,\omega) \)). In both cases, each fiber \( M_b = \pi^{-1}(b) \) is equipped with a well defined symplectic form \( \omega_b \) such that \( (M_b,\omega_b) \) is symplectomorphic to \( (M,\omega) \). It is easy to see that every ruled symplectic manifold may be given the structure of a symplectic fiber bundle. However, it is
not so obvious that when the base is simply connected this bundle can be taken Hamiltonian.

Here is a geometric criteria for a symplectic bundle to be Hamiltonian, i.e. for the structural group to reduce to $\text{Ham}(M, \omega)$. Recall that the group $\text{Ham}(M, \omega)$ is a normal subgroup of the identity component $\text{Symp}_0(M, \omega)$ of the group of symplectomorphisms, and fits into the exact sequence

$$
\{id\} \to \text{Ham}(M, \omega) \to \text{Symp}_0(M, \omega) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})/\Gamma[\omega] \to 0,
$$

where $\Gamma[\omega]$ is a countable group called the flux group. In particular, $\text{Ham}(M, \omega)$ is connected, which means that a Hamiltonian bundle is trivial over the 1-skeleton of the base. The following proposition was proved in [13] Thm. 6.36 by a somewhat analytic argument and in [10] by a more geometric one.

**Proposition 2.1** A symplectic bundle $\pi : P \to B$ is Hamiltonian if and only if the following conditions hold:

(i) the restriction of $\pi$ to the 1-skeleton $B_1$ of $B$ is symplectically trivial, and

(ii) there is a cohomology class $a \in H^2(P, \mathbb{R})$ that restricts to $[\omega_b]$ on $M_b$.

When the spaces are smooth manifolds, a construction due to Thurston shows that the existence of the extension class $a$ in (ii) above is equivalent to the existence of a closed form $\tau$ on $P$ that extends the family of forms $\omega_b$ on the fibers. Such a form $\tau$ gives rise to a connection on the bundle $P \to B$ whose horizontal distribution consists of the $\tau$-orthogonals to the fibers. In this language, the previous result can be reformulated as follows.

**Proposition 2.2** A symplectic bundle $\pi : P \to B$ is Hamiltonian if and only if the forms $\omega_b$ on the fibers have a closed extension $\tau$ such that the holonomy of the corresponding connection around any loop in $B$ lies in the identity component $\text{Symp}_0(M)$ of $\text{Symp}(M)$.

When the base is itself a symplectic manifold, we can add that form $\tau$ to a sufficiently large multiple of the pull-back of the form on the base, which is then easily seen to be nondegenerate in all directions. This proves:

**Corollary 2.3** A Hamiltonian bundle over a symplectic base is a ruled symplectic manifold. Conversely, a ruled symplectic manifold over a simply connected base is a Hamiltonian bundle.

However the two categories are not the same: the trivial Hamiltonian bundle $M \times S^1$ is not ruled, while the Kodaira–Thurston manifold $X$ that fibers over $T^2$ with nontrivial monodromy is ruled but not Hamiltonian.
3 The finite dimensional case

We now consider the results in the finite dimensional case that were obtained in various contexts (Deligne in the algebraic case, Kirwan for circle actions using localisation techniques and Atiyah–Bott for torus actions).

Suppose that the structural group of the Hamiltonian bundle \((M, \omega) \hookrightarrow P \rightarrow B\) can be reduced to a compact Lie group \(G \subset \text{Ham}(M, \omega)\). This means that there is a representation of the group \(G\) in the group of all Hamiltonian diffeomorphisms of \(M\), that is to say a Hamiltonian action of \(G\) on \(M\). In this context, it is enough to consider the universal Hamiltonian \(G\)-bundle with fiber \(M\)

\[
M \rightarrow M_G = EG \times_G M \rightarrow B G.
\]

The cohomology of \(P = M_G\) is known as the equivariant cohomology \(H^*_G(M)\) of \(M\). Kirwan showed in [6] that if \(G\) is the circle, the rational cohomology of the total space \(M_G\) is isomorphic to the tensor product of the rational cohomology of \(M\) and the rational cohomology of \(BG\) (we will refer to this in the sequel by saying that the bundle \(M_G\) is \(c\)-split, i.e. it splits cohomologically). This was proved by localisation techniques. A different argument was given by Atiyah–Bott in [2], establishing the same result when \(G\) is a torus of arbitrary dimension. The result for a general compact Lie group \(G\) follows by a more or less standard argument, starting either from Atiyah–Bott’s result or from our infinite dimensional results (see next section). Here is the argument:

**Proposition 3.1** If \(G\) is a compact connected Lie group that acts in a Hamiltonian way on \(M\), then any bundle \(P \rightarrow B\) with fiber \(M\) and structural group \(G\) is \(c\)-split. In particular,

\[
H^*_G(M) \cong H^*(M) \otimes H^*(BG).
\]

**Proof:** It is enough to prove the second statement since

\[
M_G = EG \times_G M \rightarrow BG
\]

is the universal bundle. Every compact Lie group \(G\) is the image of a homomorphism \(T \times H \rightarrow G\), where the torus \(T\) maps onto the identity component of the center of \(G\) and \(H\) is the semi-simple Lie group corresponding to the commutator subalgebra \([\text{Lie}(G), \text{Lie}(G)]\) in the Lie algebra \(\text{Lie}(G)\). It is easy to see that this homomorphism induces a surjection on rational homology \(BT \times BH \rightarrow BG\). Therefore, we may suppose that \(G = T \times H\). Let \(T_{\text{max}} = (S^1)^k\) be the maximal torus of the semi-simple group \(H\). Then the induced map on cohomology \(H^*(BH) \rightarrow H^*(BT_{\text{max}}) = \mathbb{Q}[a_1, \ldots, a_k]\) takes \(H^*(BH)\) bijectively onto the set of polynomials in \(H^*(BT_{\text{max}})\) that are invariant under the action of the Weyl group, by the Borel-Hirzebruch theorem. Hence the maps \(BT_{\text{max}} \rightarrow BH\) and \(BT \times BT_{\text{max}} \rightarrow BG\) induce a surjection on homology. Therefore the desired result follows from part (ii) of the following lemma:
Lemma 3.2 Consider a commutative diagram

\[
P' \rightarrow P \\
\downarrow \downarrow \\
B' \rightarrow B
\]

where \( P' \) is the induced bundle. Then:

(i) If \( P \rightarrow B \) is c-split so is \( P' \rightarrow B' \).

(ii) (Surjection Lemma) If \( P' \rightarrow B' \) is c-split and \( H_*(B') \rightarrow H_*(B) \) is surjective, then \( P \rightarrow B \) is c-split.

Proof: (i): Use the fact that, by the Leray-Hirsch theorem, \( P \rightarrow B \) is c-split if and only if the map \( H_*(M) \rightarrow H_*(P) \) is injective.

(ii): The induced map on the \( E_2 \)-term of the cohomology spectral sequences is injective. Therefore the existence of a nonzero differential in the spectral sequence \( P \rightarrow B \) implies that the corresponding differential for the pullback bundle \( P' \rightarrow B' \) does not vanish either.

QED

Smooth projective bundles constitute an interesting special case of the above proposition. The result in this case can be derived from the Deligne spectral sequence, or more generally by the following argument due to Blanchard \[4\].

Let’s call a smooth fiber bundle \( M \hookrightarrow P \rightarrow B \) c-Hamiltonian if there is a class \( a \in H^2(P) \) whose restriction \( a_M \) to the fiber \( M \) is c-symplectic, i.e. \((a_M)^n \neq 0\) where \( 2n = \dim(M) \). Recall that a closed manifold \( M \) is said to satisfy the hard Lefschetz condition with respect to the class \( a_M \in H^2(M, \mathbb{R}) \) if the maps

\[
\cup(a_M)^k : H^{n-k}(M, \mathbb{R}) \rightarrow H^{n+k}(M, \mathbb{R}), \quad 1 \leq k \leq n,
\]

are isomorphisms. In this case, elements in \( H^{n-k}(M) \) that vanish when cupped with \((a_M)^{k+1}\) are called primitive, and the cohomology of \( M \) has an additive basis consisting of elements of the form \( b \cup (a_M)^\ell \) where \( b \) is primitive and \( \ell \geq 0 \).

Proposition 3.3 (Blanchard [4]) Let \( M \rightarrow P \rightarrow B \) be a c-Hamiltonian bundle such that \( \pi_1(B) \) acts trivially on \( H^*(M, \mathbb{R}) \). If in addition \( M \) satisfies the hard Lefschetz condition with respect to the c-symplectic class \( a_M \), then the bundle c-splits.

Proof: The proof is by contradiction. Consider the Leray spectral sequence in cohomology and suppose that \( d_p \) is the first non zero differential. Then, \( p \geq 2 \) and the \( E_p \) term in the spectral sequence is isomorphic to the \( E_2 \) term and so can be identified with the tensor product \( H^*(B) \otimes H^*(M) \). Because of the product structure on the spectral sequence, one of the differentials \( d_p^{0,i} \) must be nonzero. So there is \( b \in E_p^{0,i} \cong H^i(M) \) such that \( d_p^{0,i}(b) \neq 0 \). We may assume that \( b \) is primitive (since these elements together with \( a_M \) generate \( H^*(M) \)). Then \( b \cup a_M^{i-1} \neq 0 \) but \( b \cup a_M^{i+1} = 0 \).
We can write \( dp(b) = \sum_j e_j \otimes f_j \) where \( e_j \in H^*(B) \) and \( f_j \in H^\ell(M) \) where \( \ell < i \). Hence \( f_j \cup a_M^{n-i+1} \neq 0 \) for all \( j \) by the Lefschetz property. Moreover, because the \( E_p \) term is a tensor product

\[
(dp(b)) \cup a_M^{n-i+1} = \sum_j e_j \otimes (f_j \cup a_M^{n-i+1}) \neq 0.
\]

But this is impossible since this element is the image via \( d_p \) of the trivial element \( b \cup a_M^{n-i+1} \).

Here is another, perhaps easier, argument. Suppose \( d = d_p \) is the first non-vanishing differential. It vanishes on \( H^i(M) \) for \( i < p \) for reasons of dimension. Therefore, by the Lefschetz property it must vanish on \( H^{2n-j}(M) \) for these \( i \). But then it has to vanish on \( H^{2n-j}(M) \) for \( p \leq i < 2p \). For if not, take \( b \) in such \( H^i(M) \) such that \( d(b) \neq 0 \). By Poincaré duality there is \( c \in H^{2n-j}(M) \) for \( 0 \leq j < p \) such that \( d(b) \cup c \neq 0 \). But \( b \cup d(c) \neq 0 \), a contradiction. It follows that \( d \) vanishes on \( H^{2n-j}(M) \) for \( p \leq j < 2p \).

Now consider the next block of \( i \): \( 2p \leq i < 3p \) and so on.

QED

Another fundamental question about Hamiltonian bundles is that of their stability under small perturbations of the symplectic form on the fiber. If the bundle \( M \rightarrow P \rightarrow B \) has structure group \( \text{Ham}(M,\omega) \) and \( \omega' \) is some nearby form, an elementary argument (given in Lemma 4.6 below) shows that it can be given the structure group \( \text{Symp}_0(M,\omega') \). However, it is not at all obvious whether the latter group can be reduced to \( \text{Ham}(M,\omega') \). If this reduction is possible for all \( \omega' \) close to \( \omega \), the original Hamiltonian bundle is said to be stable. When the structural group is a compact Lie group, this clearly boils down to the following statement.

**Theorem 3.4 (Hamiltonian stability.)** Let \((M,\omega)\) be a closed symplectic manifold, and let \( \iota : G \rightarrow \text{Ham}(M,\omega) \) be a continuous homomorphism defined on a compact Lie group. Then, for each perturbation \( \omega' \) in some sufficiently small neighbourhood \( U \) of \( \omega \) in the space of all symplectic forms on \( M \), there is a continuous homomorphism

\[
\iota' : G \rightarrow \text{Ham}(M,\omega')
\]

that varies continuously as the form \( \omega' \) varies in \( U \).

**Proof:** We begin with a well-known averaging argument. Define \( \tau' \) to be the average of the forms \( \iota(g)^*(\omega') \), i.e. set

\[
\tau'(v,w) = \int_G \iota(g)^*(\omega')(v,w) \, d\mu_G, \quad v,w \in T^*(M),
\]

where \( d\mu_G \) is Haar measure. Since \( G \) is compact and \( \iota(g)^*(\omega) = \omega \) for all \( g \in G \), \( \tau' \) is a symplectic form whenever \( \omega' \) is sufficiently close to \( \omega \). Moreover, it is easy to see that \( \iota(g)^*(\tau') = \tau' \) for all \( g \in G \). Thus \( \iota \) maps \( G \) into \( \text{Symp}(M,\tau') \). But,
since \( \iota(G) \) is also contained in the connected group \( \text{Ham}(M, \omega) \), the elements of \( G \) must act trivially on \( H^2(M) \). Therefore \( \tau' \) is cohomologous to \( \omega' \) and hence equals \( f^*(\omega') \) where \( f \) is the time 1 map of some isotopy \( f_t \) (again assuming that \( \omega' \) is sufficiently close to \( \omega \).) Thus, the homomorphism

\[
\iota' : g \mapsto f \iota(g) f^{-1}
\]

takes \( G \) to \( \text{Symp}_0(M, \omega') \).

It remains to show that this homomorphism \( \iota' \) has image in \( \text{Ham}(M, \omega') \), in other words that the composite homomorphism

\[
G \xrightarrow{\iota'} \text{Symp}_0(M, \omega') \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})/\Gamma[\omega']
\]

is zero. Since the target is an abelian group and \( G \) is compact, it suffices to consider the case when \( G \) is the circle. Observe also that \( \iota' = \iota_{\tau'} \) and \( \iota = \iota_\omega \) are the same when considered as maps into \( \text{Diff}(M) \). Therefore the same vector field \( X \) generates both \( S^1 \)-actions and we just need to show that the closed 1-form \( \alpha_{\tau'} \) which is \( \tau' \)-dual to \( X \) is actually exact. But each of its Morse-Bott singular sets have the same topology and index as does the corresponding singular set for the \( \omega \)-dual \( \alpha_\omega \) of \( X \), since the underlying \( X \) is the same and \( \tau' \) is arbitrarily close to \( \omega \). Since \( \alpha_\omega \) is exact, all these indices are even dimensional. Now if \( x_0 \) is a global minimum of a primitive \( H_\omega \) of \( \alpha_\omega \), then it is a local minimum of any local primitive of \( \alpha_{\tau'} \). If \( \gamma \) is any loop based at \( x_0 \), it is easy to see that \( \int_\gamma \alpha_{\tau'} \) must vanish, because otherwise the map \( t \mapsto \int_{\gamma(t)} \alpha_{\tau'} \) would take different values at \( t = 1 \) and \( t = 0 \) and then a standard minimax argument over the loops homotopic to \( \gamma \) would yield a loop \( \gamma_0 \) such that the maximum value of the function

\[
[0, 1] \to \mathbb{R}, \quad t \mapsto \int_{\gamma_0(t)}^{\gamma_0(0)} \alpha_{\tau'}
\]

would occur at a critical point of \( \alpha_{\tau'} \) of odd index, a contradiction. QED

4 The general case: quantum homology and geometric induction on ruled symplectic manifolds

The methods developed in the finite dimensional case do not apply when the structural group \( \text{Ham}(M, \omega) \) of a Hamiltonian bundle \( M \hookrightarrow P \to B \) does not retract to a finite dimensional subgroup. Note that, even in examples as simple as \( S^2 \times S^2 \) endowed with a generic symplectic structure, the Hamiltonian group does not retract to any Lie subgroup (see (1)).

It turns out that quantum homology and the Gromov-Witten invariants can be used to derive properties concerning the ordinary homology of a Hamiltonian bundle in full generality, without any hypothesis on the structure group or on
the symplectic manifold \((M, \omega)\). It leads to a proof of the stability of ruled symplectic structures and other corollaries that we explain in the next sections.

Basically, pseudoholomorphic techniques (quantum homology, GW-invariants) apply when the fiber of the Hamiltonian bundle is any compact symplectic manifold and the base is a 2-sphere. This means that the bundle is given by a loop in the group \(\text{Ham}(M)\), which is generally not autonomous (i.e. it is a continuous map \(S^1 \to \text{Ham}(M)\) that need not be a homomorphism). In this case, as we explained in \(\S\) 2 there is a natural symplectic structure \(\Omega\) on the total space and we can first equip that space with an almost complex structure compatible both with \(\Omega\) and with its restriction to fibers (this can be done in general when the base is any symplectic manifold) and then study the pseudo-holomorphic sections of

\[
\pi : (P, \Omega, J) \to (S^2, j)
\]

where \(j\) is the complex structure of \(S^2\). The moduli space of these sections can be used to pair the quantum homology of the \(M\)-fiber at say the north pole with the one at the south pole. The main point is that this pairing is nondegenerate, i.e. it induces an isomorphism between the two quantum homologies. It is then easy to show that this implies that the GW-invariant attached to any ordinary cycle in the fiber at the north pole, considered as a cycle in the total space \(P\), cannot vanish if it does not vanish as a cycle in the fiber. In other words, the rational homology of the fiber injects inside the homology of \(P\), which implies by the Leray-Hirsch theorem that \(P\) is c-split.

We can then extend these results to Hamiltonian bundles defined over more general bases than \(S^2\) by using three types of arguments:

1) Analytic arguments that constitute a non-trivial generalisation of the analytic methods used over \(S^2\). Basically, these arguments prove the c-splitting of Hamiltonian fiber bundles over any base \(B\) that has enough \(J\)-holomorphic rational curves; more precisely, one asks that that \(B\) has at least one non-vanishing rational GW-invariant in some class \(A \in H_2(B)\) of the form \(n(pt, pt, c_1, \ldots, c_k; A)\) where \(k \geq 0\) and the \(c_i\)'s are any homology classes in \(B\) (see Proposition 4.11).

2) Geometric arguments needed to iterate bundles; they lead to a proof that a Hamiltonian bundle over some base \(P\) c-splits if \(P\) is itself a Hamiltonian bundle \(M \hookrightarrow P \to B\) over a simply connected base \(B\) and if all Hamiltonian bundles having \(M\) or \(B\) as base c-split.

3) Topological arguments based on properties of spectral sequences of symplectic bundles.

4.1 Bundles over \(S^2\)

We begin by explaining this in more detail in the case of a Hamiltonian bundle over \(S^2\) (this was proved in the semi-monotone case by the authors in collaboration with Polterovich in [9] and in the general case in [11].)
There is a correspondence between loops in the group of symplectic diffeomorphisms and symplectic bundles over $S^2$ with fiber $(M,\omega)$. The correspondence is given by assigning to each symplectic loop $\phi_{t \in [0,1]} \in \text{Symp}_0(M)$ the bundle $(M,\omega) \to P_\phi \to S^2$ obtained by gluing a copy of $D_2^+ \times M$ with another $D_2^- \times M$ along their boundary in the following way:

$$(2\pi t, x) \mapsto (-2\pi t, \phi_t(x)).$$

(Here $D_2$ is the closed disc of radius 1 of the plane.) In what follows we always assume that the base $S^2$ is oriented, and with orientation induced from $D_2^+$. Note that this correspondence can be reversed: given a symplectic bundle over the oriented 2-sphere together with an identification of one fiber with $M$, one can reconstruct the homotopy class of $\phi$.

An important topological tool for the study of such bundles is the Wang exact sequence:

$$\ldots \to H_{j-1}(M,\mathbb{Z}) \xrightarrow{\partial} H_j(M,\mathbb{Z}) \xrightarrow{i} H_j(P_\phi,\mathbb{Z}) \cap \lbrack M \rbrack \to H_j(M,\mathbb{Z}) \to \ldots$$

This sequence can be easily derived from the exact sequence of the pair $(P_\phi, M)$, where $M$ is identified with a fiber of $P_\phi$. The important point for us is that the boundary map $H_{j-1}(M) \to H_j(M)$ is precisely the trace homomorphism $\partial_\phi$ that assigns to each cycle $a$ the cycle $\{\phi_t(a) | t \in [0,1]\}$. Thus $\partial_\phi$ vanishes exactly when the inclusion $i$ is injective or, equivalently, when the restriction map $\cap \lbrack M \rbrack$ is surjective. By the Leray-Hirsch theorem, this is equivalent to the c-splitting of the bundle.

**Theorem 4.1** Let $\phi$ be a Hamiltonian loop on a closed symplectic manifold $(M,\omega)$. Then the homomorphism $i : H_*(M,\mathbb{Q}) \to H_*(P_\phi,\mathbb{Q})$ is injective; that is to say, the bundle c-splits.

We now briefly explain how the proof of this theorem proceeds. Since $p : P_\phi \to S^2$ is a Hamiltonian bundle it carries a natural deformation class of symplectic forms given by the weak coupling construction. Recall that the coupling class $u_\phi \in H^2(P_\phi,\mathbb{R})$ is the (unique) class whose top power vanishes, and whose restriction to a fiber coincides with the cohomology class of the fiberwise symplectic structure. Let $\tau$ be a positive generator of $H^2(S^2,\mathbb{Z})$. The deformation class above consists of symplectic forms $\Omega$ which represent the cohomology class of the form $u_\phi + \kappa p^*\tau$ ($\kappa >> 0$) and extend the fiberwise symplectic structure. It is always possible to choose $\Omega$ so that it is a product with respect to the given product structure near the fibers $M_0$ at $0 \in D_2^+$ and $M_\infty$ at $0 \in D_2^-$. Besides the coupling class $u_\phi$, the total space $P_\phi$ carries another canonical second cohomology class

$$c_\phi = c_1(TP_\phi^{\text{vert}}) \in H^2(P_\phi,\mathbb{R})$$

that is defined to be the first Chern class of the vertical tangent bundle.
Both classes $u_{\phi}, c_{\phi}$ behave well under compositions of loops. More precisely, consider two elements $\phi, \psi \in \pi_1(\Ham(M, \omega))$ and their composite $\psi * \phi$. This can be represented either by the product $\psi_t \circ \phi_t$ or by the concatenation of loops. It is not hard to check that the bundle $P_{\psi*\phi}$ can be realised as the fiber sum $P_\phi \# P_\psi$ obtained as follows. Let $M_{\phi, \infty}$ denote the fiber at $0 \in D_2^+$ in $P_\phi$ and $M_{\psi, 0}$ the fiber at $0 \in D_2^+$ in $P_\psi$. Cut out open product neighborhoods of each of these fibers and then glue the complements by an orientation reversing symplectomorphism of the boundary. The resulting space may be realised as

$$D_2^+ \times M \cup_{\alpha_{\phi,-1}} S^1 \times [-1, 1] \cup_{\alpha_{\psi,1}} D_2^- \times M,$$

where

$$\alpha_{\phi,-1}(2\pi t, x) = (2\pi t, -1, \phi_t(x)), \quad \alpha_{\psi,1}(2\pi t, 1, \psi_t(x)) = (2\pi t, x),$$

and this may clearly be identified with $P_{\psi*\phi}$. Set

$$V_\phi = D_2^+ \times M \cup S^1 \times [-1, 1/2], \quad V_\psi = S^1 \times (-1/2, 1] \cup D_2^- \times M.$$

The next lemma follows immediately from the construction of the coupling form via symplectic connections.

**Lemma 4.2** The classes $u_{\psi*\phi}$ and $c_{\psi*\phi}$ are compatible with the decomposition $P_{\psi*\phi} = V_\psi \cup V_\phi$ in the sense that their restrictions to $V_\psi \cap V_\phi = (-1/2, 1/2) \times S^1 \times M$ equal the pullbacks of $[\omega]$ and $c_1(TM)$.

We now explain the proof of Theorem 4.1 (see [9, 11] for more details).

Seidel’s maps $\Psi_{\phi, \sigma}$

We start with the definition of the quantum cohomology ring of $M$. To simplify our formulas we will denote the first Chern class $c_1(TM)$ of $M$ by $c$.

Let $\Lambda$ be the usual (rational) Novikov ring of the group $\mathcal{H} = H^2(M, \mathbb{Z})/\sim$ with valuation $\omega(.)$ where $B \sim \tilde{B}$ if $\omega(B - \tilde{B}) = c(B - \tilde{B}) = 0$, and let $\Lambda_R$ be the analogous (real) Novikov ring based on the group $\mathcal{H}_R = H^2(M, \mathbb{R})/\sim$. Thus the elements of $\Lambda$ have the form

$$\sum_{B \in \mathcal{H}} \lambda_B e^B,$$

where for each $\kappa$ there are only finitely many nonzero $\lambda_B \in \mathbb{Q}$ with $\omega(B) < \kappa$, and the elements of $\Lambda_R$ are

$$\sum_{B \in \mathcal{H}_R} \lambda_B e^B,$$

where $\lambda_B \in \mathbb{R}$ and there is a similar finiteness condition.\[1\] Set $QH_*(M) = H_*(M) \otimes \Lambda$ and $QH_*(M, \Lambda_R) = H_*(M) \otimes \Lambda_R$. Then $QH_*(M)$ is $\mathbb{Z}$-graded with

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1 In [7] Seidel works with a simplified version of the Novikov ring $\Lambda$ where the coefficients $\lambda_B$ affecting $e^B$, $B \in \mathcal{H}$, are elements of $\mathbb{Z}/2\mathbb{Z}$. However, his results extend in a straightforward way to the case of rational coefficients by taking into account orientations on the moduli spaces of pseudo-holomorphic curves. Let us emphasize that in our definition of $\Lambda_R$ not only the coefficients $\lambda_B$ are real, but also the exponents $B$ belong to a real vector space $\mathcal{H}_R$. 

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\[ \deg(a \otimes c^B) = \deg(a) - 2\alpha(B). \] It is best to think of \( QH_*(M, \Lambda_R) \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded with
\[ QH_{ev} = H_{ev}(M) \otimes \Lambda_R, \quad QH_{odd} = H_{odd}(M) \otimes \Lambda_R. \]

With respect to the quantum intersection product both versions of quantum homology are graded-commutative rings with unit \([M]\). Further, the units in \( QH_{ev}(M, \Lambda_R) \) form a group \( QH_{ev}(M, \Lambda_R)^\times \) that acts on \( QH_*(M, \Lambda_R) \) by quantum multiplication.

Now we describe how Seidel arrives at an action of the loop \( \phi \) on the quantum homology of \( M \). Denote by \( \mathcal{L} \) the space of contractible loops in the manifold \( M \). Fix a constant loop \( x_* \in \mathcal{L} \), and define a covering \( \tilde{\mathcal{L}} \) of \( \mathcal{L} \) with the base point \( x_* \) as follows. Note first that a path between \( x_* \) and a given loop \( x \) can be considered as a 2-disc \( u \) in \( M \) bounded by \( x \). Then the covering \( \tilde{\mathcal{L}} \) is defined by saying that two paths are equivalent if the 2-sphere \( S^2 \) obtained by gluing the corresponding discs has \( \omega(S) = c(S) = 0 \). Thus the covering group of \( \tilde{\mathcal{L}} \) coincides with the abelian group \( \mathcal{H} \).

Let \( \phi = \{ \phi_t \} \) be a loop of Hamiltonian diffeomorphisms. Because the orbits \( \phi_t(x), t \in [0,1], \) of \( \phi \) are contractible (see [3]), one can define a mapping \( T_\phi : \mathcal{L} \to \mathcal{L} \) which takes the loop \( \{ x(t) \} \) to a new loop \( \{ \phi_t(x(t)) \} \). Let \( \tilde{T}_\phi \) be a lift of \( T_\phi \) to \( \tilde{\mathcal{L}} \). To choose such a lift one should specify a homotopy class of paths in \( \mathcal{L} \) between the constant loop and an orbit of \( \{ \phi_t \} \). It is not hard to see that in the language of symplectic bundles this choice of lift corresponds to a choice of an equivalence class \( \sigma \) of sections of \( P_\phi \), where two sections are identified if their values under \( c_\phi \) and \( u_\phi \) are equal. Thus the lift can be labelled \( \tilde{T}_{\phi, \sigma} \).

Recall now that the Floer homology \( HF_*(M) \) can be considered as the Novikov homology of the action functional on \( \tilde{\mathcal{L}} \). Therefore \( \tilde{T}_{\phi, \sigma} \) defines a natural automorphism \( (\tilde{T}_{\phi, \sigma})_* \) of \( HF_*(M) \). Further, if \( \Phi : HF_*(M) \to HQ_*(M) \) is the canonical isomorphism constructed in Piunikhin–Salamon–Schwartz [14], there is a corresponding automorphism \( \Psi_{\phi, \sigma} \) of \( QH_*(M) \) given by
\[ \Psi_{\phi, \sigma} = \Phi \circ (\tilde{T}_{\phi, \sigma})_* \circ \Phi^{-1}. \]

This gives rise to an action of the group of all pairs \( (\phi, \sigma) \) on \( QH_*(M) \).

Seidel shows in [17] that when \( M \) satisfies a suitable semi-positivity condition the map
\[ \Psi_{\phi, \sigma} : QH_*(M) \to QH_*(M) \]

is in fact induced by quantum multiplication by an element of \( QH_{ev}(M)^\times \) that is formed from the moduli space of all \( J \)-holomorphic sections of \( P_\phi \). In our work we went backwards. We gave a new definition of the maps \( \Psi_{\phi, \sigma} \) that does not mention Floer homology, and proved that they are isomorphisms by a direct gluing argument. Besides being easier to work with for general \( M \), our version of the definition no longer restricts us to using the coefficients \( \Lambda \) via the covering \( \tilde{\mathcal{L}} \to \mathcal{L} \). Instead we will consider the extension \( \Lambda_R \), which allowed us to define an action of the group \( \pi_1(\text{Ham}) \) itself (see [4]).

Let \( \Omega \) be a symplectic form on \( P_\phi \) that extends \( \omega \) and is in the natural deformation class \( u_\phi + \kappa \rho^*(\tau) \). As above, define an equivalence relation on
the set of homology classes of sections of $P_\phi$ by identifying two such classes if their values under $c_\phi$ and $u_\phi$ are equal. Then, given a loop of Hamiltonian diffeomorphisms $\phi$ on $M$, and an equivalence class of sections $\sigma$ of $P_\phi$ with $d = 2c_\phi(\sigma)$, define a $\Lambda$-linear map

$$\Psi_{\phi, \sigma} : QH^*(M) \to QH^{*+d}(M)$$

as follows. For $a \in H_*(M, \mathbb{Z})$, $\Psi_{\phi, \sigma}(a)$ is the class in $QH^{*+d}(M)$ whose intersection with $b \in H_*(M, \mathbb{Z})$ is given by:

$$\Psi_{\phi, \sigma}(a) \cdot_M b = \sum_{B \in \mathcal{H}} n(i(a), i(b); \sigma + i(B)) e^B.$$ 

Here, we have written $i$ for the homomorphism $H_*(M) \to H_*(P)$ and $\cdot_M$ for the linear extension to $QH^*(M)$ of the usual intersection pairing on $H_*(M, \mathbb{Q})$. Thus $a \cdot_M b = 0$ unless $\dim(a) + \dim(b) = 2n$ in which case it is the algebraic number of intersection points of the cycles. Further, $n(v, w; D)$ denotes the Gromov–Witten invariant which counts isolated $J$-holomorphic stable curves in $P_\phi$ of genus 0 and two marked points that represent the equivalence class $D$ and whose marked points go through given generic representatives of the classes $v$ and $w$ in $H_*(P_\phi, \mathbb{Z})$. More precisely, one defines $n(v, w; D)$ to be the intersection of the virtual moduli cycle $ev : \overline{M}_{0,2}^\nu(P_\phi, J, D) \to P_\phi \times P_\phi$, that consists of all perturbed $J$-holomorphic genus 0 stable maps that lie in class $D$ and have 2 marked points, with a generic representative of the class $v \otimes w$ in $P_\phi \times P_\phi$. This definition is well understood provided $M$ is spherically monotone or has minimal spherical Chern number at least $n - 1$. In the general case, one uses a version of the virtual moduli cycle for Gromov-Witten invariants that is adapted to the fibered structure: see [11].

Note finally that, by Gromov compactness, there are for each given energy level $\kappa$ only finitely many homology classes $D$ with $\omega(D - \sigma) \leq \kappa$ that are represented by $J$-holomorphic curves in $P_\phi$. Thus $\Psi_{\phi, \sigma}(a)$ satisfies the finiteness condition for elements of $QH_*(M, \Lambda)$.

Since $n(i(a), i(b); D) = 0$ unless $2c_\phi(D) + \dim(a) + \dim(b) = 2n$, we have

$$\Psi_{\phi, \sigma}(a) = \sum a_B \otimes e^B,$$

where

$$\dim(a_B) = \dim(a) + 2c_\phi(D) = \dim(a) + 2c_\phi(\sigma) + 2c(B).$$

Observe also that

$$\Psi_{\phi, \sigma + B} = \Psi_{\phi, \sigma} \otimes e^{-B}.$$ 

When $M$ is spherically monotone or has minimal spherical Chern number at least $n - 1$ the following two results are proved by Seidel [17]. The general case is established in [11].

---

Footnote 2: The minimal spherical Chern number $N$ is the smallest nonnegative integer such that the image of $c = c_1(TM)$ on $H_2^S(M)$ is contained in $NZ$. The weakly exact case $N = 0$ is also tractable by these standard methods.
Lemma 4.3 If \( \phi \) is the constant loop \( * \) and \( \sigma_0 \) is the flat section \( pt \times S^2 \) of \( P_\phi = M \times S^2 \) then \( \Psi_{\phi, \sigma_0} \) is the identity map.

Proposition 4.4 Given sections \( \sigma \) of \( P_\phi \) and \( \sigma' \) of \( P_\psi \) let \( \sigma' \# \sigma \) be the union of these sections in the fiber sum \( P_\psi \# P_\phi = P_{\psi \# \phi} \). Then

\[
\Psi_{\psi, \sigma'} \circ \Psi_{\phi, \sigma} = \Psi_{\psi \# \phi, \sigma' \# \sigma}.
\]

The main step in the proof of these statements is to show that when calculating the Gromov-Witten invariant \( n(i(a), i(b); D) \) via the intersection between the virtual moduli cycle and the class \( i(a) \otimes i(b) \) we can assume the following:

— the representative of \( i(a) \otimes i(b) \) has the form \( \alpha \times \beta \) where \( \alpha, \beta \) are cycles lying in the fibers of \( P_\phi \);

— the intersection occurs with elements in the top stratum of \( \overline{M}_{0,2}(P_\phi, J, D) \) consisting of sections of \( P_\phi \).

In the semi-monotone case, Lemma 4.3 is then almost immediate \(^3\) and Proposition 4.4 can be proved by the well-known gluing techniques of \([16]\) or \([12]\).

Corollary 4.5 \( \Psi_{\phi, \sigma} \) is an isomorphism for all loops \( \phi \) and sections \( \sigma \).

With this in hand, we can establish the proof of Theorem 4.1 in the following way. The Gromov-Witten invariants are linear in each variable. Thus if \( i(a) = 0 \) for some \( a \neq 0 \), then \( \Psi_{\phi, \sigma}(a) = 0 \), a contradiction with the fact that \( \Psi_{\phi, \sigma} \) is an isomorphism.

QED

4.2 General stability

It turns out that the fact that all Hamiltonian bundles over \( S^2 \) are c-split is enough to establish the stability of general Hamiltonian bundles over any base. This is what we now explain.

Let \( \pi : P \to B \) be a symplectic bundle with closed fiber \( (M, \omega) \) and compact base \( B \). Moser’s homotopy argument implies that this bundle has the following stability property.

Lemma 4.6 There is an open neighborhood \( U \) of \( \omega \) in the space \( S(M) \) of all symplectic forms such that \( \pi : P \to B \) may be naturally considered as a symplectic bundle with fiber \( (M, \omega') \) for all \( \omega' \in U \).

Proof: First recall that for every symplectic structure \( \omega' \) on \( M \) there is a Serre fibration

\[
\text{Symp}(M, \omega') \to \text{Diff}(M) \to S_{\omega'},
\]

\(^3\)The proof of the first lemma is surprisingly hard in the general case. The difficulty lies in showing that invariants in classes \( A + B \) with \( B \neq 0 \in H_2(M) \) do not contribute. The reason is that such curves can never be isolated; they are graphs, and reparametrizations of the map to \( M \) give rise to families of graphs. However, to see this in the general case involves constructing a virtual moduli cycle that is invariant under an \( S^1 \)-action. See \([11]\).
whenever $\omega$ construct from it a smooth family of sections $\sigma$ on $V$ each the symplectic structure $\omega$ forms $\sigma$ on $M$ will lie in the fiber of $W$ the symplectic nondegeneracy condition will belong to the fiber of $S$. Section functions can be used to define the bundle $S$ isomorphism classes of symplectic structures on it with fiber $M,\omega$ to $\omega$ $S$ where $M,\omega$ choose them with values in $\text{Symp}(M,\omega)$. This is cohomologous to $\omega$ $S$ at $\omega$ the transition functions of $P$ are maps $\phi_{ij} : V_i \cap V_j \to \text{Diff}(M)$. We can of course choose them with values in $\text{Symp}(M,\omega)$ if the $T_i$’s are chosen compatible with the $\omega$-structure on the bundle, but this is not necessary. Then the same transition functions can be used to define the bundle $S \hookrightarrow W(\omega') \to B$, whatever the symplectic structure $\omega'$ may be.

Therefore, we are given a section $\sigma$ of $W(\omega)$ and our task is to show how to construct from it a smooth family of sections $\sigma_{\omega}$ of the bundles $W(\omega')$ for all $\omega'$ near $\omega$. Let $\sigma_i$ be the restriction of $\sigma$ over $V_i$. Then $(T_i)_* \sigma_i$ is a smooth map $V_i \to S$ (constant and equal to $\omega$ if the $\phi_{ij}$ are chosen in $\text{Symp}(M,\omega)$). For each $\omega'$ near $\omega$ and $b \in V_i$ consider the symplectic form

$$\sigma'_i(b) = (T_i)_*^{-1}(T_i)_*(\sigma_i(b)) + \omega' - \omega$$

on $M_b$ (this is simply $(T_i)_*^{-1}(\omega')$ if the transition functions have values in $\text{Symp}(M,\omega)$). This is cohomologous to $\omega'$, and because of the openness of the symplectic nondegeneracy condition will belong to the fiber of $W(\omega')$ at $b$, whenever $\omega'$ is sufficiently close to $\omega$. Therefore, $\sigma'_i$ is a section of $W(\omega')$ over $V_i$. Moreover, if $b \in \cap_{i \in I} V_i$ for some index set $I = I_b$, the convex hull of the forms $\sigma'_i(b)$, $i \in I_b$, will consist entirely of symplectic forms isotopic to $\omega'$ and so will lie in the fiber of $W(\omega')$ at $b$, again provided that $\omega'$ is sufficiently close to $\omega$. Hence if $\rho_i$ is a partition of unity subordinate to the cover $V_i$, the formula

$$\sigma' = \sum_i \rho_i \sigma'_i$$

defines a section of $W(\omega')$.

Thus the set $S_\pi(M)$ of symplectic forms on $M$, with respect to which $\pi$ is symplectic, is open. The aim of this paragraph is to show that a corresponding statement is true for Hamiltonian bundles, in other words that Hamiltonian bundles are stable. We begin with the following lemmas.

**Lemma 4.7** A Hamiltonian bundle $\pi : P \to B$ is stable if and only if the restriction map $H^2(P) \to H^2(M)$ is surjective.

**Proof:** If $\pi : P \to B$ is Hamiltonian with respect to $\omega'$ then by Proposition 2.3 $[\omega']$ is in the image of $H^2(P) \to H^2(M)$. If $\pi$ is stable, then $[\omega']$ fills out a neighborhood of $[\omega]$ which implies surjectivity. Conversely, suppose that we

Note that one uses Moser’s argument to prove that this is a Serre fibration.
have surjectivity. Then the second condition of Proposition 2.1 is satisfied. To check (i) let \( \gamma : S^1 \to B \) be a loop in \( B \) and suppose that \( \gamma^*(P) \) is identified symplectically with the product bundle \( S^1 \times (M, \omega) \). Let \( \omega_t, 0 \leq t \leq \varepsilon, \) be a (short) smooth path with \( \omega_0 = \omega \). Then, because \( P \to B \) has the structure of an \( \omega_t \)-symplectic bundle for each \( t \), each fiber \( M_b \) has a corresponding smooth family of symplectic forms \( \omega_{b,t} \) of the form \( g_{*}^* b, t(\omega_t) \), where \( g_b \) is a symplectomorphism \((M_b, \omega_b) \to (M, \omega)\). Hence, for each \( t \), \( \gamma^*(P) \) can be symplectically identified with \( \bigcup_{s \in [0,1]} \{ s \} \times (M, g_{s,t}^*(\omega_t)) \), where \( g_1^* t(\omega_t) = \omega_t \) and the \( g_{s,t} \) lie in an arbitrarily small neighborhood \( U \) of the identity in \( \text{Diff}(M) \). By Moser’s homotopy argument, we can choose \( U \) so small that each \( g_{1,t} \) is isotopic to the identity in the group \( \text{Symp}(M, \omega_t) \). This proves (i).

QED

Lemma 4.8

(i) Every Hamiltonian bundle over \( S^2 \) is stable.

(ii) Every symplectic bundle over a 2-connected base \( B \) is Hamiltonian stable.

Proof: (i) holds because every Hamiltonian bundle over \( S^2 \) is c-split, in particular the restriction map \( H^2(P) \to H^2(M) \) is surjective. (ii) follows from the fact that a symplectic bundle over a 2-connected base is automatically Hamiltonian since the relative homotopy groups \( \pi_i(\text{Symp}(M), \text{Ham}(M)), i \geq 2, \) all vanish. (See [10] for more details.) QED

Proposition 4.9

Every Hamiltonian bundle is stable.

Proof: First note that we can restrict to the case when \( B \) is simply connected. For the map \( B \to B \text{Ham}(M) \) classifying \( P \) factors through a map \( C \to B \text{Ham}(M) \), where \( C = B/B_1 \) as before, and the stability of the induced bundle over \( C \) implies that for the original bundle by Lemma 4.7.

Next note that by the same lemma a Hamiltonian bundle \( P \to B \) is stable if and only if the differentials \( d^0_{2} : E_{k}^{0,2} \to E_{k}^{k,3-k} \) in its Leray cohomology spectral sequence vanish on the whole of \( H^2(M) \) for \( k = 2, 3 \). But it is easy to see that we can reduce the statement for \( d^0_{2} \) to the case \( B = S^2 \). Thus \( d^0_{2} = 0 \) by Lemma 4.8(i). Similarly, we can reduce the statement for \( d^0_{3} \) to the case \( B = S^3 \) and then use Lemma 4.8(ii). For more details, see [10]. QED

It is then easy to extend the result of §3 to the case when the group \( \text{Ham}(M) \) does not retract to a finite dimensional Lie subgroup:

Theorem 4.10 Let \( (M, \omega) \) be a closed symplectic manifold, and let \( \iota : X \to \text{Ham}(M, \omega) \) be a continuous map defined on a finite CW-complex \( X \). Then, for each perturbation \( \omega' \) in some sufficiently small neighbourhood \( U \) of \( \omega \) in the space of all symplectic forms on \( M \), there is a map

\[ \iota' : X \to \text{Ham}(M, \omega') \]

that varies continuously as the form \( \omega' \) varies in \( U \).
Here is the proof. Given $\iota$, construct the Hamiltonian bundle $(M, \omega) \to P \to X \times S^1$ by considering the direct product $(X \times [0,1]) \times (M, \omega)$ and identifying the ends by the map

$$
\phi : (X \times \{0\}) \times (M, \omega) \to (X \times \{1\}) \times (M, \omega)
$$

$$(x, 0, y) \mapsto (x, 1, \omega(y))$$

Since $\iota$ is Hamiltonian, $\Phi$ is a Hamiltonian automorphism of $X \times (M, \omega)$ and therefore $P$ is a Hamiltonian bundle. By Proposition 4.5, the Hamiltonian structure on $P$ persists and varies continuously as $\omega$ varies in some open neighborhood $U$ of $S(M)$. For $\omega' \in U$, denote by $(P(\omega'), \pi, \{\sigma\})$ the corresponding Hamiltonian structure. Consider the restriction of $P(\omega')$ to each time $t \in S^1$.

This gives a Hamiltonian bundle $(M, \omega') \to P_t(\omega') \to X_t = X \times \{t\}$, for which $\omega' = \omega$ has the trivial (i.e. product) Hamiltonian structure. We showed in [9] that such structures are classified by maps $\pi_1(X_t) \to \Gamma_{\omega'}$ and in [10] that the rank of $\Gamma_{\omega'}$ is finite and locally constant. Thus because the map $\pi_1(X_t) \to \Gamma_{\omega'}$ is zero for $\omega' = \omega$ and since it depends continuously on $\omega'$, we conclude that it must be zero for all $\omega'$ and $t$, i.e. the induced Hamiltonian structure on $P_t(\omega')$ is trivial for each $t$ and in particular for $t = 0, 1$. This means that $P(\omega')$ is defined as the quotient of $(X \times [0,1]) \times (M, \omega')$ by an automorphism of the trivial Hamiltonian bundle

$$
\phi_{\omega'} : (X \times \{0\}) \times (M, \omega') \to (X \times \{1\}) \times (M, \omega'),
$$

which is homotopic to a map $\iota' : X \to \text{Ham}(M, \omega')$.

Here is a second more direct proof. First observe that if $K$ is any compact set of $\text{Symp}(M, \omega)$, then for all $\omega' \in S(M)$ sufficiently close to $\omega$, Moser’s argument gives a canonical map

$$
\Phi_{\omega, \omega'} : K \to \text{Symp}(M, \omega')
$$

defined by precomposing each map $f \in K$ by a diffeomorphism $g_f$ of $M$ such that $g^*_f(\omega') = \omega'$. Here $g$ is the diffeomorphism corresponding to the isotopy $\omega'_t = t \omega' + (1-t)\omega$. Because $K$ is compact, one can indeed choose a small enough neighborhood $U$ of $\omega$ in $S(M)$ so that such a segment is nondegenerate for all $\omega' \in U$.

Choosing $K$ to be the image of $\iota$, we get a map $\Phi_{\omega, \omega'} \circ \iota : B \to \text{Symp}(M, \omega')$. The map $\Phi_{\omega, \omega'} \circ \iota$ homotops into $\text{Ham}(M, \omega')$ if and only if the elements in $(\Phi_{\omega, \omega'} \circ \iota)_* (\pi_1(B))$ lie in the kernel of the homomorphism

$$
\text{Flux}_{\omega'} : \pi_1(\text{Symp}_B(M, \omega')) \to H^1(M, \mathbb{R}).
$$

In fact the flux homomorphism $\text{Flux}_{\omega'}$ is defined on $\pi_1(\text{Diff}(M))$, and so, since $\Phi_{\omega, \omega'} \circ \iota$ is homotopic to $\iota$ as maps to $\text{Diff}(M)$, it suffices to show that $\text{Flux}_{\omega'}$ vanishes on the elements of $\iota_* (\pi_1(B))$. But $\text{Flux}_{\omega}$ vanishes on $\iota_* (\pi_1(B))$ by construction, and so $\text{Flux}_{\omega'}$ also vanishes on these elements by the “stability of Hamiltonian loops” in [1]. This is just another way of expressing the stability of Hamiltonian structures over $S^2$. To see this, let $\phi = \iota_*(\gamma)$ be the image of a
loop $\gamma$ in $B$, and consider the associated bundle $P_\phi \to S^2$ constructed using $\phi$ as clusting map. Then, for any closed form $\tau$ on $M$, symplectic or not, $\text{Flux}_\tau(\phi)$ is nothing other than the value of the Wang differential $\partial_\phi$ of this bundle on the class $[\tau]$. The stability of $P_\phi \to S^2$ implies that $\partial_\phi([\omega']) = 0$, and therefore
\[
\text{Flux}_{\omega'}(\phi) = \partial_\phi([\omega']) = 0,
\]
as required.

### 4.3 From $S^2$ to more general bases, using analytic arguments

**Proposition 4.11** Let $(M,\omega)$ be a closed symplectic manifold, and $M \hookrightarrow P \to B$ a Hamiltonian bundle over a CW-complex $B$. Then the rational cohomology of $P$ splits if the base has the homotopy type of a symplectic manifold $W$ for which some spherical Gromov–Witten invariant $n_W(pt,pt,c_1,\ldots,c_k;A)$ does not vanish, where $k \geq 0$, $A \in H_2(W;\mathbb{Z})$ and the $c_i's$ are any cycles in $W$.

Note that spaces satisfying the above condition include all products of complex projective spaces and their blow-ups.

A special case of this proposition was proved in [10], and the general case will appear in [7]. The proof is a generalization of the arguments in [9, 11]. The idea is to show that moduli spaces of $J$-holomorphic curves in ruled symplectic manifolds $P$ behave like fibered moduli spaces, which implies that appropriate GW-invariants in $P$ are equal to the product of a GW-invariant of the base with a GW-invariant of the fiber. Indeed, suppose that $M \hookrightarrow P \to B$ is a Hamiltonian fiber bundle over a symplectic manifold $B$ and assume that $B$ contains a spherical class $A$ with a non-zero Gromov–Witten invariant of the form $n_B(pt,pt,c_1,\ldots,c_k;A)$. Recall that this invariant counts the number of $J$-curves in class $A$ that pass through two generic points and through generic representatives of the classes $c_1,\ldots,c_k$. Let $C$ be such a rational $J$-curve.

We have explained above that the restriction $P_C$ of $P$ to the curve $C$ is a Hamiltonian fiber bundle that $c$-splits. recall that this $c$-splitting was shown by taking two $M$-fibers $M_0, M_\infty$ in $P_C$ and by finding, for each cycle $a \in M_0$, a cycle $b = b(a) \in M_\infty$ such that the GW-invariant in $P_C$
\[
n_{P_C}(\iota(a),\iota(b);\sigma)
\]
does not vanish. (Here $\iota$ denotes the inclusion of the fiber in $P_C$, $\sigma$ is some homology class of sections of $P_C \to C = S^2$ and $n_{P_C}$ counts the number of $J$-holomorphic curves in class $\sigma$ passing through $\iota(a)$ and $\iota(b)$.) This implies that $\iota(a)$ cannot vanish, and therefore $P_C$ $c$-splits by the Leray-Hirsch theorem. Now take an almost complex structure $J'$ on $P$ such that the projection $\pi : P \to B$ is $(J', J)$-holomorphic and consider the invariant in $P$
\[
n_P(\iota'(a),\iota'(b),\pi^{-1}(c_1),\ldots,\pi^{-1}(c_k);\iota_{P_C},\rho(\sigma))
\]
where $\pi$ is the projection $P \to B$, $\iota$ denotes the inclusion of the fiber in $P$, and $\iota_{P,C,P}$ is the inclusion of $P_C$ in $P$. It is not hard to see, at least when the moduli spaces are well-behaved, that this last invariant must be equal to the sum, taken over the rational curves $C$ appearing in $n_B(pt,pt,c_1,\ldots,c_k;A)$, of the corresponding numbers $n_{P_C}(\iota(a),\iota(b);\sigma)$, with signs according to orientations in the moduli space. But because the Hamiltonian bundles $P_C$ and $P_C'$ are isomorphic when $C$ and $C'$ are homologous in $B$, this sum is actually the product

$$n_B(pt,pt,c_1,\ldots,c_k;A) \times n_{P_C}(\iota(a),\iota(b);\sigma)$$

which does not vanish.

Therefore $\iota'(a)$ cannot vanish either and the bundle $P$ c-splits.

### 4.4 Iterating bundles: geometric arguments

Let $M \hookrightarrow P \to B$ be a Hamiltonian bundle over a simply connected base $B$ and assume that all Hamiltonian bundles over $M$ as well as over $B$ c-split. We explain in this section that any Hamiltonian bundle over $P$ must also be c-split. This provides a powerful recursive argument that extends c-splitting results to much more general bases.

We begin with some trivial observations and then discuss composites of Hamiltonian bundles. The first lemma is true for any class of bundles with specified structural group.

**Lemma 4.12** Suppose that $\pi : P \to B$ is Hamiltonian and that $g : B' \to B$ is a continuous map. Then the induced bundle $\pi' : g^*(P) \to B'$ is Hamiltonian.

Recall that any extension $\tau$ of the forms on the fibers is called a connection form.

**Lemma 4.13** If $P \to B$ is a smooth Hamiltonian fiber bundle over a symplectic base $(B,\sigma)$ and if $P$ is compact then there is a connection form $\Omega^\kappa$ on $P$ that is symplectic.

**Proof:** The bundle $P$ carries a closed connection form $\tau$. Since $P$ is compact, the form $\Omega^\kappa = \tau + \kappa \pi^*(\sigma)$ is symplectic for large $\kappa$. QED

Observe that the deformation type of the form $\Omega^\kappa$ is unique for sufficiently large $\kappa$ since given any two closed connection forms $\tau, \tau'$ the linear isotopy

$$t\tau + (1-t)\tau' + \kappa \pi^*(\sigma), \quad 0 \leq t \leq 1,$$

consists of symplectic forms for sufficiently large $\kappa$. However, it can happen that there is a symplectic connection form $\tau$ such that $\tau + \kappa \pi^*(\sigma)$ is not symplectic for small $\kappa > 0$, even though it is symplectic for large $\kappa$. (For example, suppose $P = M \times B$ and that $\tau$ is the sum $\omega + \pi^*(\omega_B)$ where $\omega_B + \sigma$ is not symplectic.)

Let us now consider the behavior of Hamiltonian bundles under composition. If

$$(M,\omega) \to P \stackrel{\pi_I}{\longrightarrow} X,$$

and

$$(F,\sigma) \to X \stackrel{\pi_F}{\longrightarrow} B,$$

Then

$$(M,\omega) \to P \stackrel{\pi_I}{\longrightarrow} X \stackrel{\pi_F}{\longrightarrow} B,$$
are Hamiltonian fiber bundles, then the restriction

\[ \pi_P : W = \pi_P^{-1}(F) \to F \]

is a Hamiltonian fiber bundle. Since \( F \) is a manifold, we can assume without loss of generality that \( W \to F \) is smooth. Moreover, the manifold \( W \) carries a symplectic connection form \( \Omega_W^\kappa \), and it is natural to ask when the composite map \( \pi : P \to B \) with fiber \((W, \Omega_W^\kappa)\) is itself Hamiltonian.

**Lemma 4.14** Suppose that \( B \) is a simply connected CW-complex and that \( P \) is compact. Then \( \pi = \pi_X \circ \pi_P : P \to B \) is a Hamiltonian fiber bundle with fiber \((W, \Omega_W^\kappa)\), where \( \Omega_W^\kappa = \tau_W + \kappa \pi_P^\ast(\sigma) \), \( \tau_W \) is any symplectic connection form on \( W \), and \( \kappa \) is sufficiently large.

**Proof:** We may assume that the base \( B \) as well as the bundles are smooth. Let \( \tau_P \) (resp. \( \tau_X \)) be a closed connection form with respect to the bundle \( \pi_P \), (resp. \( \pi_X \)), and let \( \tau_W \) be its restriction to \( W \). Then \( \Omega_W^\kappa \) is the restriction to \( W \) of the closed form \( \Omega_P^\kappa = \tau_P + \kappa \pi_P^\ast(\tau_X) \).

By increasing \( \kappa \) if necessary we can ensure that \( \Omega_P^\kappa \) restricts to a symplectic form on every fiber of \( \pi \) not just on the the chosen fiber \( W \). This shows firstly that \( \pi : P \to B \) is symplectic, because there is a well defined symplectic form on each of its fibers, and secondly that it is Hamiltonian with respect to this form \( \Omega_W^\kappa \) on the fiber \( W \). Hence Lemma 4.7 implies that \( H^2(P) \) surjects onto \( H^2(W) \).

Now suppose that \( \tau_W \) is any closed connection form on \( \pi_P : W \to F \). Because the restriction map \( H^2(P) \to H^2(W) \) is surjective, the cohomology class \( [\tau_W] \) is the restriction of a class on \( P \) and so, by Thurston’s construction, the form \( \tau_W \) can be extended to a closed connection form \( \tau_P \) for the bundle \( \pi_P \). Therefore the previous argument applies in this case too. QED

Now let us consider the general situation, when \( \pi_1(B) \neq 0 \). The proof of the lemma above applies to show that the composite bundle \( \pi : P \to B \) is symplectic with respect to suitable \( \Omega_W^\kappa \) and that it has a symplectic connection form. However, even though \( \pi_X : X \to B \) is symplectically trivial over the 1-skeleton \( B_1 \) the same may not be true of the composite map \( \pi : P \to B \). Moreover, in general it is not clear whether triviality with respect to one form \( \Omega_W^\kappa \) implies that for another. Therefore, we may conclude the following:

**Proposition 4.15** If \( (M, \omega) \to P \xrightarrow{\pi} X \), and \( (F, \sigma) \to X \xrightarrow{\pi_X} B \) are Hamiltonian fiber bundles and \( P \) is compact, then the composite \( \pi = \pi_X \circ \pi_P : P \to B \) is a symplectic fiber bundle with respect to any form \( \Omega_W^\kappa \) on its fiber \( W = \pi^{-1}(pt) \), provided that \( \kappa \) is sufficiently large. Moreover if \( \pi \) is symplectically trivial over the 1-skeleton of \( B \) with respect to \( \Omega_W^\kappa \) then \( \pi \) is Hamiltonian.

In practice, we will apply these results in cases where \( \pi_1(B) = 0 \). We will not specify the precise form on \( W \), assuming that it is \( \Omega_W^\kappa \) for a suitable \( \kappa \).
Lemma 4.16 If \((M, \omega) \xrightarrow{\pi} P \to B\) is a compact Hamiltonian bundle over a simply connected CW-complex \(B\) and if every Hamiltonian fiber bundle over \(M\) and \(B\) is c-split, then every Hamiltonian bundle over \(P\) is c-split.

Proof: Let \(\pi_E : E \to P\) be a Hamiltonian bundle with fiber \(F\) and let

\[ F \to W \to M \]

be its restriction over \(M\). Then by assumption the latter bundle c-splits so that \(H_*(F)\) injects into \(H_*(W)\). Lemma 4.14 implies that the composite bundle \(E \to B\) is Hamiltonian with fiber \(W\) and therefore also c-splits. Hence \(H_*(W)\) injects into \(H_*(E)\). Thus \(H_*(F)\) injects into \(H_*(E)\), as required. QED

4.5 Topological arguments

We now put together the results and methods of the last subsections about c-splitting. For more details see [10].

Lemma 4.17 If \(\Sigma\) is a closed orientable surface then any Hamiltonian bundle over \(S^2 \times \ldots \times S^2 \times \Sigma\) is c-split.

Proof: Consider any degree one map \(f\) from \(\Sigma \to S^2\). Because \(\text{Ham}(M, \omega)\) is connected, \(B\text{Ham}(M, \omega)\) is simply connected, and therefore any homotopy class of maps from \(\Sigma \to B\text{Ham}(M, \omega)\) factors through \(f\). Thus any Hamiltonian bundle over \(\Sigma\) is the pullback by \(f\) of a Hamiltonian bundle over \(S^2\). Because such bundles c-split over \(S^2\), the same is true over \(\Sigma\) by Lemma 4.16(i).

The statement for \(S^2 \times \ldots \times S^2 \times \Sigma\) is now a direct consequence of iterative applications of Lemma 4.16 applied to the trivial bundles \(S^2 \times \ldots \times S^2 \times \Sigma \to S^2\). QED

Corollary 4.18 Any Hamiltonian bundle over \(S^2 \times \ldots \times S^2 \times S^1\) is c-split.

Proof: Consider the maps \(S^1 \to T^2 \to S^1\) given by inclusion on the first factor and projection onto the first factor. Their composition is the identity. Extend them to maps

\[ S^2 \times \ldots \times S^2 \times S^1 \to S^2 \times \ldots \times S^2 \times T^2 \to S^2 \times \ldots \times S^2 \times S^1. \]

by multiplying with the identity on the \(S^2\) factors. Then a Hamiltonian bundle \(P\) on \(S^2 \times \ldots \times S^2 \times S^1\) pulls-back to a c-split bundle \(P'\) on \(S^2 \times \ldots \times S^2 \times T^2\) by Lemma 4.17. By naturality, its pull-back \(P''\) to \(S^2 \times \ldots \times S^2 \times S^1\) is c-split. But \(P'' = P\). QED

Proposition 4.19 For each \(k \geq 1\), every Hamiltonian bundle over \(S^k\) c-splits.
Proof: By Lemma 4.17 and Corollary 4.18 there is for each \( k \) a \( k \)-dimensional closed manifold \( X \) such that every Hamiltonian bundle over \( X \) c-splits. Given any Hamiltonian bundle \( P \to S^k \) consider its pullback to \( X \) by a map \( f : X \to S^k \) of degree 1. Since the pullback c-splits, the original bundle does too by Lemma 3.2(ii). QED

By the Wang exact sequence, this implies that the action of the homology groups of \( \text{Ham}(M) \) on \( H_*(M) \) is always trivial.

Here are some other examples of situations in which Hamiltonian bundles are c-split.

Lemma 4.20 Every Hamiltonian bundle over \( \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_k} \) c-splits.

Proof: This is an obvious application of Lemma 4.16. QED

Lemma 4.21 Every Hamiltonian bundle over a compact CW-complex of dimension \( \leq 3 \) c-splits.

Proof: This is because one can first assume that \( B \) is simply-connected and then construct a homology surjection \( B' \to B \) where \( B' \) is a wedge of 2 and 3-spheres. QED

Proposition 4.22 Every Hamiltonian bundle over a product of spheres c-splits, provided that there are no more than 3 copies of \( S^1 \).

Proof: By hypothesis \( B = \prod_{i \in I} S^{2m_i} \times \prod_{j \in J} S^{2n_i+1} \times T^k \), where \( n_i > 0 \) and \( 0 \leq k \leq 3 \). Set

\[
B' = \prod_{i \in I} \mathbb{C}P^{m_i} \times \prod_{j \in J} \mathbb{C}P^{n_i} \times T^{|J|} \times T^\ell,
\]

where \( \ell = k \) if \( k + |J| \) is even and \( = k + 1 \) otherwise. Since \( \mathbb{C}P^{n_i} \times S^1 \) maps onto \( S^{2n_i+1} \) by a map of degree 1, there is a homology surjection \( B' \to B \) that maps the factor \( T^\ell \) to \( T^k \). By the surjection lemma, it suffices to show that the pullback bundle \( P' \to B' \) is c-split.

Consider the fibration

\[
T^{|J|} \times T^\ell \to B' \to \prod_{i \in I} \mathbb{C}P^{m_i} \times \prod_{j \in J} \mathbb{C}P^{n_i}.
\]

Since \( |J| + \ell \) is even, we can think of this as a Hamiltonian bundle. Moreover, by construction, the restriction of the bundle \( P' \to B' \) to \( T^{|J|} \times T^\ell \) is the pullback of a bundle over \( T^k \), since the map \( T^{|J|} \to B \) is nullhomotopic. (Note that each \( S^1 \) factor in \( T^{|J|} \) goes into a different sphere in \( B \).) Because \( k \leq 3 \), the bundle over \( T^k \) c-splits. Hence we can apply the argument in Lemma 4.16 to conclude that \( P' \to B' \) c-splits. QED
Corollary 4.23 Let $B$ be a simply connected Lie group, or more generally any $H$-space whose rational fundamental group has rank less than 4 and whose homotopy groups are finitely generated in each dimension. Then $c$-splitting holds for all Hamiltonian bundles over $B$.

Proof: Let $B$ be such a $H$-space. By the theory of minimal models (see [3] for instance) which applies in this case because the fundamental group of $B$ acts trivially on all higher homotopy groups, the rational cohomology of $B$ is generated as a $\mathbb{Q}$-vector space by cup-products of elements that pair non-trivially with spheres, i.e. each $a \in H^*(B; \mathbb{Q})$ can be written as a cup product $\cup_i a_i$’s where there is for each $i$ a spherical class $\alpha_i$ in rational homology with $a_i(\alpha_i) \neq 0$. If we denote by the same symbol $\alpha_i : S^{n_i} \to B$ a map that realises a non-zero multiple of the class $\alpha_i$, then the obvious map $\cup_i \alpha_i : \cup_i S^{n_i} \to B$ extends to a map $\phi_a$ defined on the product of these spheres that pulls back the element $a$ to a generator of the top rational cohomology group. If there were a Hamiltonian bundle $P$ over $B$ that did not $c$-split, there would be an element of lowest degree $a \in H^*(B; \mathbb{Q})$ with non-zero differential in the spectral sequence of $P$ and therefore the differential of the corresponding top element of $H^*(\Pi, S^{n_i})$ in the spectral sequence of the pull back bundle $\phi_a^*(P)$ would not vanish either. But this contradicts the $c$-splitting established in the previous proposition. QED

In particular, Hamiltonian fibrations $c$-split over the loop space $\Omega X$ of any simply connected CW-complex $X$ with $\pi_2 X$ of small enough rank. It is not at all clear how to go from this fact to $c$-splitting over $X$. The paper [10] contains an extensive discussion of what can be proved when $X$ has dimension 4.

Lemma 4.24 Every Hamiltonian bundle over a coadjoint orbit $c$-splits.

Proof: This follows immediately from the remarks in Grossberg–Karshon [5]§3 about Bott towers. A Bott tower is an iterated bundle $M_k \to M_{k-1} \to \cdots \to M_1 = S^2$ of Kähler manifolds where each map $M_{i+1} \to M_i$ is a bundle with fiber $S^2$. They show that any coadjoint orbit $X$ can be blown up to a manifold that is diffeomorphic to a Bott tower $M_k$. Moreover the blowdown map $M_k \to X$ induces a surjection on rational homology. Every Hamiltonian bundle over $M_k$ $c$-splits by repeated applications of Lemma 4.16. Hence the result follows from the surjection lemma. QED

5 Applications to ruled symplectic manifolds

Theorem 5.1 Obstructions to the existence of ruled symplectic structures. Let $M$ be a closed manifold and $P$ a smooth fiber bundle with fiber $M$ over a simply connected manifold $B$ and assume that $B$ is either a compact CW-complex of dimension less than 4 or is a product of complex projective spaces.

\hfill \footnote{We are grateful to Jaroslaw Kedra who pointed out a variant of this argument to us.}
of spheres and of coadjoint orbits of arbitrary dimensions. Denote by \( \iota \) the inclusion of the fiber in \( P \). Then the non-vanishing of the kernel of

\[ \iota_* : H_*(M) \to H_*(P) \]

is an obstruction to the existence of a ruled symplectic structure on \( P \).

By the Leray-Hirsch theorem, the vanishing of the kernel in the theorem above amounts to the cohomological splitting \( H_*(P) = H_*(B) \otimes H_*(M) \). Thus this last result may be stated as follows: under the given conditions on \( B \), a ruled structure exists on \( P \) only if \( P \) splits cohomologically. This imposes strong topological constraints on the construction of ruled symplectic manifolds by twisted products of two given ones.

Theorem 5.1 is an immediate corollary of our results about c-splitting and of the characterization of Hamiltonian fiber bundles over simply connected bases in terms of the existence of a closed extension to the total space of the symplectic forms on the fibers. This characterization also implies the following version of Hamiltonian stability.

**Theorem 5.2 Stability of ruled symplectic structures.** Let \( M \hookrightarrow P \xrightarrow{\pi} B \) be a smooth compact fiber bundle over a simply connected manifold \( B \). Suppose that \( P \) admits a ruled symplectic structure \( \Omega \), that restricts to \( \omega \) on the \( M \)-fiber. Then the ruled symplectic structure on \( P \) persists under small deformations of \( \omega \), i.e. there is a neighborhood \( \mathcal{U} \) of \( \omega \) in the space of all symplectic forms on \( M \) such that each \( \omega' \in \mathcal{U} \) extends to a ruled symplectic structure \( \Omega' \) on \( P \), which varies continuously as \( \omega' \) varies in \( \mathcal{U} \).

Observe that the above theorem remains true for arbitrary bases \( B \) provided that \( P \to B \) is symplectically trivial over the 1-skeleton of \( B \).

6 Concluding remarks

It is still unclear whether every Hamiltonian fiber bundle over any compact CW-complex c-splits. One of the simplest unknown cases is a Hamiltonian bundle \( (M, \Omega) \to P \to B \) with base the 4-torus and with fiber a symplectic 4-manifold that does not satisfy specific properties like the hard Lefschetz property. The problem here is that, if one tries to apply Lemma 4.16 to \( (M, \omega) \to P \xrightarrow{\rho} B \) with \( B = T^4 \) given itself as a bundle \( T^2 \to T^4 \xrightarrow{\pi} T^2 \), then nothing guarantees that the composite fibration \( W \to P \xrightarrow{\pi \circ \rho} T^2 \) is trivial over the 1-skeleton of the base. In fact, the structural group of the composite fibration may well be a disconnected subgroup of the symplectomorphism group of the fiber \( W = (\pi \circ \rho)^{-1}(pt) \). Note, however, that because all Hamiltonian fibrations over \( T^3 \) c-split, we do know that the elements of this subgroup act trivially on the cohomology of \( W \). This raises the interesting question of whether one can extend our results on c-splitting for Hamiltonian bundles to certain disconnected extensions of the Hamiltonian group. We have no techniques at present to deal with this question, since bundles over \( T^2 \) need not admit any \( J \)-holomorphic sections.
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