ISOGENIES OF ABELIAN VARIETIES OVER FINITE FIELDS

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1. Introduction

In this paper we give conditions under which two abelian varieties that are defined over a finite field $F$, and are isogenous over some larger field, are $F$-isogenous. Further, we give conditions under which a given isogeny is defined over $F$.

If $A$ and $B$ are abelian varieties defined over a field $F$, and $n$ is an integer not divisible by the characteristic of $F$ and greater than 2, then every homomorphism between $A$ and $B$ is defined over $F(A_n, B_n)$, the smallest extension of $F$ over which the $n$-torsion points on $A$ and $B$ are defined (see Theorem 2.4 of [11]). In this paper (and earlier in [12]) we consider the situation where we do not have full level $n$ structure, and give conditions under which isogenies between rigidified polarized abelian varieties, with level structure given by maximal isotropic subgroups of $n$-torsion points, are defined over fields of definition for the rigidified polarized abelian varieties. Our main results are Theorems 4.3, 5.4, and 5.5 and Corollary 5.6 below.

We emphasize that our assumptions do not require that all the points on the given maximal isotropic subgroups be defined over $F$, but only that the subgroups be defined over $F$ and that the restrictions of the isogeny to the subgroups be an isomorphism defined over $F$.

The results of this paper give improvements in the case of abelian varieties over finite fields, on results in [12] for abelian varieties over arbitrary fields. For example, Theorem 4.3 below does not hold in characteristic zero, even for elliptic curves. The results in this paper rely on the theory of abelian varieties over finite fields (see [15]), and our results in [12] and [13].

Abelian varieties over finite fields are useful in cryptography, especially in the case of elliptic curves and Jacobians of curves of genus two. Isogenies and torsion points arise, for example, when considering the so-called “distortion maps” in pairing-based cryptography, or when reducing the discrete log problem on one abelian variety to the discrete log problem on an isogenous one (see for example [16], [4], Chapter 24 of [2], Chapter 5 of [5]). These are settings where it can be important to know whether an isogeny is defined over the ground field, or is defined over an extension over which a given torsion point or subgroup is defined.

Next, we collect together hypotheses that are common to all our main results, and refer to them collectively as $\Phi(n)$.

**Definition 1.1.** The condition $\Phi(n)$ will mean that the following situation (a - h) holds:

(a) $A$ and $B$ are abelian varieties defined over a finite field $F$, 
(b) $n$ is a positive integer not divisible by the characteristic of $F$, 

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(c) \( f : A \to B \) is an isogeny of degree prime to \( n \),
(d) \( \mu \) is a polarization on \( B \), defined over \( F \),
(e) \( \tilde{B}_n \) is a subgroup of \( B_n \), defined over \( F \), and containing a maximal isotropic
subgroup of \( B_n \) with respect to the \( e_n \)-pairing induced by \( \mu \),
(f) \( \tilde{A}_n \) is a subgroup of \( A_n \), defined over \( F \), such that the restriction of \( f \) to
\( \tilde{A}_n \) is an isomorphism from \( \tilde{A}_n \) onto \( \tilde{B}_n \) defined over \( F \),
(g) \( L \) is a finite extension of \( F \) of degree \( m \) over which \( f \) is defined,
(h) \( \lambda \) is the polarization on \( A \) defined by \( \lambda = \tilde{f} \mu f \), and \( \lambda \) is defined over \( F \).

The collection of hypotheses \( \Phi(n) \) differs from the collection \( I(n) \) of our earlier work \cite{12} in that it contains the additional assumptions that the field \( F \) is finite and the polarizations \( \mu \) and \( \lambda \) are defined over \( F \). Also, the degree of \( L \) over \( F \) is now given the label \( m \).

A special case of our results is that if \( \Phi(n) \) holds for some \( n \geq 5 \) with \( m \leq 3 \), and if \( A \) (and thus \( B \)) is \( L \)-simple, then \( A \) and \( B \) are \( F \)-isogenous (see Corollary \cite{5,6}).

Additional notation is defined in \cite{3} below. In \S3 we give some lemmas about abelian varieties over finite fields that enable us to prove our main results in \S4. In \S5 we apply the results of \S4 and the Skolem-Noether Theorem to obtain additional information.

2. Notation and definitions

If \( F \) is a field, then \( \bar{F} \) denotes an algebraic closure of \( F \), and \( F^s \subset \bar{F} \) denotes a
separable closure of \( F \). A set \( C \) or a map \( g \) that is defined over \( F^s \) is defined over \( F \) if
and only if it is Gal(\( F^s/F \))-invariant (see p. 76 and p. 186 of \cite{18}). If \( A \) is an abelian
variety defined over \( F \), write \( A_n \) for the kernel of multiplication by \( n \) in \( A(\bar{F}) \), write
End(\( A \)) for the ring of \( \bar{F} \)-endomorphisms of \( A \), and let End\(^0\)(\( A \)) = End(\( A \)) \( \otimes \mathbb{Q} \).
If \( n \) is not divisible by the characteristic char(\( F \)), then \( A_n \) is a free \( \mathbb{Z}/n\mathbb{Z} \)-module of
rank 2dim(\( A \)), and \( A_n \subset A(F^s) \) (see Chapter II of \cite{7}). Let End\(_F^0\)(\( A \)) denote the
ring of endomorphisms of \( A \) defined over \( F \). Let End\(_F^0\)(\( A \)) = End\(_F\)(\( A \)) \( \otimes \mathbb{Q} \).

**Definition 2.1.** Let \( Z_F(A) \) denote the center of the semi-simple \( \mathbb{Q} \)-algebra End\(_F^0\)(\( A \)).

If \( \alpha \) is a homomorphism between abelian varieties that are defined over \( F \), then \( \alpha \) is
defined over a finite separable extension of \( F \) (see Corollary 1, p. 258 of \cite{1}). If \( \alpha \in \text{Hom}(A, B) \) is an isogeny, then \( \alpha' \) denotes the unique element \( \beta \) of Hom(\( B, A \)) \( \otimes \mathbb{Q} \)
such that \( \alpha \beta = 1 \) and \( \beta \alpha = 1 \). If \( A^* \) and \( B^* \) are respectively the Picard varieties of
abelian varieties \( A \) and \( B \), and \( \alpha \in \text{Hom}(A, B) \otimes \mathbb{Q} \), then \( \alpha^t \) denotes the transpose of \( \alpha \) (see p. 124 of \cite{6} or p. 3 of \cite{9}). Polarizations on \( A \) will
be viewed as isogenies from \( A \) onto \( A^* \).

If \( A \) is an abelian variety defined over a field \( F \), \( A \) is a polarization on \( A \), \( n \) is a positive integer not divisible by char(\( F \)), and \( \mu_n \) is the Gal(\( F^s/F \))-module of
\( n \)-th roots of unity in \( F^s \), then the \( e_n \)-pairing induced by the polarization \( \lambda \) is a
skew-symmetric bilinear map \( e_{\lambda,n} : A_n \times A_n \to \mu_n \) (see §75 of \cite{17} for a definition)
that satisfies:

(i) \( e(\sigma(x_1, x_2)) = e_{c_{\sigma}(\lambda),n}(\sigma(x_1), \sigma(x_2)) \) if \( \sigma \in \text{Gal}(F^s/F) \) and \( x_1, x_2 \in A_n \),
(ii) if \( f : A \to B \) is a homomorphism of abelian varieties, \( \lambda \) and \( \mu \) are polarizations on \( A \) and \( B \), respectively, \( \lambda = \tilde{f} \mu f \), and \( x_1, x_2 \in A_n \), then
\[ e_{\mu,n}(f(x_1), f(x_2)) = e_{\lambda,n}(x_1, x_2), \]
Lemma 3.4. Suppose we have

\[ f \] is a homomorphism, then \( f \) is defined over \( F \).

Proof. Let \( \tau \in \text{Gal}(F^s/F) \) denote the Frobenius element. By the definitions of \( \tau \), \( \pi_{F,A} \), and \( \pi_{F,B} \), we have \( \tau(f)\pi_{F,A} = \pi_{F,B}f \). Now \( f \) is defined over \( F \) if and only if \( \tau(f) = f \). Therefore, \( f \) is defined over \( F \) if and only if \( f\pi_{F,A} = \pi_{F,B}f \).

3. Lemmas

The results in this section will be used in later sections to prove our main results.

Proposition 3.1 (See Corollary 3.4 of [13]). Suppose \( n \) is an integer greater than 4, \( \mathcal{O} \) is an integral domain of characteristic zero such that no rational prime divisor of \( n \) is a unit in \( \mathcal{O} \), \( \alpha \in \mathcal{O} \), \( \alpha \) has finite multiplicative order, and \( (\alpha - 1)^2 \in n\mathcal{O} \). Then \( \alpha = 1 \).

The following proposition from [12] can be viewed as a variation on the Theorem on p. 17-19 of [8], which says that an automorphism of an abelian variety that has finite order, and is congruent to 1 modulo some integer \( n \geq 3 \), is the identity automorphism. Proposition 3.2 can be proved using Proposition 3.1.

Proposition 3.2 (Theorem 4.1b of [12]). Suppose \( B \) is an abelian variety, \( n \) and \( r \) are relatively prime positive integers, and \( n \) is at least 5 and is not divisible by the characteristic of a field of definition for \( B \). Suppose \( \alpha \) is an element of \( \text{End}(B) \otimes_{\mathbb{Z}} \mathbb{Z}[1/r] \), \( \alpha \) has finite multiplicative order, \( \tilde{B}_n \) is a subgroup of \( B_n \) on which \( \alpha \) induces the identity map, and \( (\alpha - 1)B_n \subseteq \tilde{B}_n \). Then \( \alpha = 1 \).

We now give some elementary lemmas concerning abelian varieties over finite fields, which we will make use of in later sections.

Lemma 3.3. If \( A \) and \( B \) are abelian varieties over a finite field \( F \), and \( f : A \to B \) is a homomorphism, then \( f \) is defined over \( F \) if and only if \( f\pi_{F,A} = \pi_{F,B}f \).

Proof. Let \( \tau \in \text{Gal}(F^s/F) \) denote the Frobenius element. By the definitions of \( \tau \), \( \pi_{F,A} \), and \( \pi_{F,B} \), we have \( \tau(f)\pi_{F,A} = \pi_{F,B}f \). Now \( f \) is defined over \( F \) if and only if \( \tau(f) = f \). Therefore, \( f \) is defined over \( F \) if and only if \( f\pi_{F,A} = \pi_{F,B}f \).

Lemma 3.4. Suppose we have \( \Phi(n) \) for some \( n > 0 \). Then:

(a) the elements \( \varphi(\pi_{F,B}) \) and \( \pi_{F,A} \) of \( \text{End}_F^0(A) \) are equal on \( \tilde{A}_n \), and

(b) the elements \( \varphi^{-1}(\pi_{F,A}) \) and \( \pi_{F,B} \) of \( \text{End}_F^0(B) \) are equal on \( \tilde{B}_n \).
Proof. By hypothesis (f) of assumption \( \Phi(n) \), we know that the restriction of the isogeny \( f \) to the subgroup \( \tilde{A}_n \) is defined over \( F \). As in Lemma 3.3, this means that \( f\pi_{F,A} = \pi_{F,B}f \) on \( \tilde{A}_n \). Therefore, we have (a). Since the restriction of \( f \) to \( \tilde{A}_n \) is an isomorphism from \( \tilde{A}_n \) onto \( \tilde{B}_n \), we have (b).

Suppose \( A \) is an abelian variety over a finite field \( F \). By Theorem 2a on p. 140 of [15], we have \( Z_F(A) = \mathbb{Q}[\pi_{F,A}] \) (with \( Z_F(A) \) defined as in Definition 2.1). Suppose now that \( L \) is a field extension of \( F \) of finite degree \( m \). Since \( \pi_{F,A}^{\omega} = \pi_{L,A} \), we have

\[
Z_F(A) = \mathbb{Q}[\pi_{F,A}] \supseteq \mathbb{Q}[\pi_{L,A}] = Z_L(A).
\]

Lemma 3.5. If \( A \) is an abelian variety over a finite field \( F \), and \( L \) is a finite field extension of \( F \), then the centralizer of \( Z_F(A) \) in \( \text{End}_L^0(A) \) is \( \text{End}_F^0(A) \).

Proof. The centralizer of \( Z_F(A) \) in \( \text{End}_L^0(A) \) contains \( \text{End}_F^0(A) \), since \( Z_F(A) \) is the center of \( \text{End}_F^0(A) \). Let \( \sigma \in \text{Gal}(L/F) \) denote the Frobenius element. If \( \beta \) is in the centralizer of \( Z_F(A) \) in \( \text{End}_L^0(A) \), then \( \beta \pi_{F,A} = \pi_{F,A} \beta \). But from the definitions of \( \sigma \) and \( \pi_{F,A} \), we have \( \sigma(\beta) \pi_{F,A} = \pi_{F,A} \beta \). Therefore, \( \sigma(\beta) = \beta \), and so \( \beta \in \text{End}_F^0(A) \).

Lemma 3.6. If \( A \) is an abelian variety over a finite field \( F \), \( L \) is a field extension of \( F \), and \( A' \) is an \( L \)-isotypic component of \( A \), then \( A' \) is defined over \( F \).

Proof. The \( L \)-isotypic components of \( A \) are the abelian subvarieties of \( A \) of the form \( (rp)(A) \) where \( p \) is a minimal idempotent of \( Z_L(A) \) (i.e., \( p \) is an idempotent of \( Z_L(A) \) such that \( pZ_L(A) \) is a field) and \( r \) is a positive integer such that \( rp \in \text{End}(A) \). Since \( Z_L(A) \subseteq Z_F(A) \), the abelian varieties \( (rp)(A) \) are defined over \( F \).

4. Fields of Definition for Isogenies

In §4.5 we determine fields of definition for isogenies of abelian varieties, under the hypothesis \( \Phi(n) \) (see Definition 2.1) and certain additional conditions. In Theorem 4.3 and Proposition 4.2 we give conditions under which the given isogeny \( f \) is defined over \( F \). In Proposition 4.1 we deal with an intermediate field of definition, between the fields \( F \) and \( L \), for the isogeny \( f \) (and the restrictions of \( f \) to \( L \)-isotypic components of \( A \)), and use this result to prove the others.

Proposition 4.1. Suppose we have \( \Phi(n) \) for some \( n \geq 5 \), \( j \) is a divisor of \( m \), \( F' \) is the degree \( j \) extension of \( F \) in \( L \), and \( \varphi \) is defined as in 1. Suppose either

(i) \( A' = A, B' = B, f' = f, \) and \( \varphi' = \varphi \), or

(ii) \( A' \) is an \( L \)-isotypic component of \( A, B' = f(A'), f' : A' \rightarrow B' \) is the isogeny induced by \( f \), and \( \varphi' : \text{End}_L^0(B') \rightarrow \text{End}_L^0(A') \) is defined by \( \varphi'(u) = (f')^{-1}uf' \).

If \( (\varphi')^{-1}(\pi_{F',A'}) \) commutes with \( \pi_{F,B'} \), then \( \varphi'(\pi_{F,B'}) = \pi_{F,A'} \) and \( f' \) is defined over \( F' \).

Proof. We have \( \pi_{F,Z} = \pi_{F',Z} \) for \( Z = A, A', B, \) and \( B' \). Let \( \sigma \in \text{Gal}(L/F') \) denote the Frobenius element, and let

\[
\alpha = \varphi^{-1}(\pi_{F',A})\pi_{F,B}^{-1} = f\sigma(f)^{-1} \in \text{End}_L^0(B).
\]

Let

\[
\alpha' = (\varphi')^{-1}(\pi_{F',A'})\pi_{F,B'}^{-1} = f'\sigma(f')^{-1} \in \text{End}_L^0(B').
\]
Since $(\varphi')^{-1}(\pi_{F',A'})$ commutes with $\pi_{F',B'}$, and $f'$ is defined over $L$, we have

$$(\alpha')^{m/j} = (\varphi')^{-1}(\pi_{L,A'})\pi_{L,B'}^{-1} = 1.$$ 

Since the isogeny $f$ has degree relatively prime to $n$, so does the isogeny $\sigma(f)$. Therefore, $\alpha \in \text{End}(B) \otimes \mathbb{Z}[1/r]$ for some positive integer $r$ relatively prime to $n$. By Lemma 2.2c of \cite{12}, we have $(\alpha - 1)B_n \subseteq \tilde{B}_n$. By Lemma 3.4 above, $\alpha = 1$ on $\tilde{B}_n$. Therefore, $(\alpha - 1)^2B_n = 0$, and so $(\alpha' - 1)^2(B'_n) = 0$. In case (i), Proposition 3.2 implies $\alpha = 1$, as desired. Suppose $\ell$ is a prime divisor of $n$ and let

$$T_\ell : \text{End}_0^\ell(B') \to \text{End}(V_\ell(B')) \cong M_{2\ell}(\mathbb{Q}_\ell)$$

be the natural map, where $q = \dim(B')$. Then $T_\ell(\alpha')$ is a matrix of finite order, and $(T_\ell(\alpha') - 1)^2 \in nM_{2\ell}(\mathbb{Z}_\ell)$, By Lemma 5.3 of \cite{14}, if $\lambda$ is an eigenvalue of $T_\ell(\alpha')$ then $(\lambda - 1)^2 \in n\tilde{\mathbb{Z}}$, where $\tilde{\mathbb{Z}}$ is the ring of algebraic integers in an algebraic closure of $\mathbb{Q}$. By Proposition 3.1, $\lambda = 1$. Therefore, $\alpha' = 1$. It follows that $\varphi'(\pi_{F',B'}) = \pi_{F',A'}$, and Lemma 3.3 implies $f'$ is defined over $F'$.

**Proposition 4.2.** Suppose we have $\Phi(n)$ for some $n \geq 5$. Suppose hypothesis (i) or (ii) of Proposition 4.1 holds. If $(\varphi')^{-1}(\pi_{F,A'})$ commutes with $\pi_{F,B'}$, then

$$\varphi'(\pi_{F,B'}) = \pi_{F,A'} \quad \text{and} \quad f' \text{ is defined over } F'.$$

**Proof.** This follows directly from Proposition \ref{prop:iso_cases} with $j = 1$. \hfill $\Box$

**Theorem 4.3.** Suppose we have $\Phi(n)$ for some $n \geq 5$. Suppose hypothesis (i) or (ii) of Proposition \ref{prop:iso_cases} holds. Then the following are equivalent:

(a) $f'$ is defined over $F$,

(b) $\varphi'(\pi_{F,B'}) = \pi_{F,A'}$,

(c) $\varphi'((\text{End}_{\mathbb{F}}(B')) = \text{End}^0_{\mathbb{F}}(A')$,

(d) $\varphi'(Z_F(B')) = Z_F(A')$,

(e) $\varphi'(Z_F(B')) \subseteq Z_F(A')$,

(f) $\varphi'(Z_F(B')) \supseteq Z_F(A')$.

**Proof.** Lemma 3.3 gives the equivalence of (a) and (b). The implications (b) $\Rightarrow$ (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (f) are straightforward.

To show that (d) $\Rightarrow$ (c), take $\gamma \in \text{End}_{\mathbb{F}}(B')$ and $\beta \in Z_F(A')$. Then (d) implies that $(\varphi')^{-1}(\beta) \in Z_F(B')$, and therefore $(\varphi')^{-1}(\beta)\gamma = (\gamma\varphi')^{-1}(\beta)\gamma = (\beta\varphi')^{-1}(\gamma)\beta = \varphi'((\gamma)\beta)$, and so $\varphi'((\gamma)\beta)$ is in the centralizer in $\text{End}^0_{\mathbb{F}}(A')$ of $Z_F(A')$. By Lemma 3.3, $\varphi'((\gamma)\beta) \in \text{End}^0_{\mathbb{F}}(A')$. Similarly, the reverse inclusion,

$$\text{End}_{\mathbb{F}}(A') \subseteq \varphi'((\text{End}_{\mathbb{F}}(B')),$$

follows from the fact that the centralizer in $\text{End}_{\mathbb{F}}(B')$ of $Z_F(B')$ is $\text{End}^0_{\mathbb{F}}(B')$.

To show that (f) $\Rightarrow$ (b), note that (f) implies that

$$\pi_{F,A'} \in Z_F(A') \subseteq \varphi'((Z_F(B'))$$

and therefore $(\varphi')^{-1}(\pi_{F,A'}) \subseteq Z_F(B')$, so $(\varphi')^{-1}(\pi_{F,A'})$ commutes with $\pi_{F,B'}$. We thus have (f) $\Rightarrow$ (b) from Proposition 4.1.

Similarly, to show that (e) $\Rightarrow$ (b), note that (e) implies that

$$\varphi'(\pi_{F,B'}) \in \varphi'((Z_F(B')) \subseteq Z_F(A').$$

Therefore, (e) implies that $\varphi'(\pi_{F,B'})$ commutes with $\pi_{F,A'}$, and so $\pi_{F,B'}$ commutes with $(\varphi')^{-1}(\pi_{F,A'})$. We thus have (e) $\Rightarrow$ (b) from Proposition 4.1. \hfill $\Box$
Remark 4.4. Under assumption (i) in Propositions 4.1 and 12 and in Theorem 4.3, we can allow $n$ to equal 4 if we assume in addition that $B_4 \cong (\mathbb{Z}/4\mathbb{Z})^b$ for some $b$. This is because Proposition 3.2 holds for $n = 4$ under the assumption $B_4 \cong (\mathbb{Z}/4\mathbb{Z})^b$ (see Theorem 4.1c of [14]).

Example 4.5. We give an example to show that one cannot drop the hypothesis that the ground field $F$ is finite. Let $A = B = E$ be an elliptic curve over $F = \mathbb{R}$ with complex multiplication, and take $g \in \text{End}_C(E) - \text{End}_K(E)$, where $C$ and $\mathbb{R}$ denote the fields of complex and real numbers, respectively. For each $n \in \mathbb{Z}_{>0}$ let $f = f' = 1 + ng \in \text{End}_C(E)$. Then $\text{End}_K^0(E) = \mathbb{Q}$, $E(\mathbb{R})$ contains a cyclic order $n$ subgroup $\tilde{E}_n = \tilde{A}_n = \tilde{B}_n$, $f$ acts on $\tilde{E}_n$ as the identity map, and $\varphi = \varphi'$ is the identity map. Here, conclusions (c–f) of Theorem 4.3 hold while conclusion (a) does not.

Example 4.6. Next we give an example where Theorem 4.3 applies, and statements (a–f) all fail to hold (i.e., $f'$ is not defined over $F$). Suppose $p$ is a prime, and either $p = 5$ or $p \geq 11$. Let $A = B = E$ be a supersingular elliptic curve over $F = \mathbb{F}_p$ (see Chapter V of [10] for basic facts about elliptic curves over finite fields, especially supersingular ones). Then $\#E(\mathbb{F}_p) = p + 1$. Writing $E(\mathbb{F}_p) \cong \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ with $a, b \in \mathbb{Z}_{>0}$, then $E[a] \subseteq E(\mathbb{F}_p) = \ker(\text{Frob} - 1)$, where $\text{Frob}$ denotes the Frobenius endomorphism. By Corollary III.4.11 of [10] there exists $\lambda \in \text{End}(E)$ such that $a\lambda = \text{Frob} - 1$. Since the characteristic polynomial of $\text{Frob}$ is $x^2 + p$, it follows that the characteristic polynomial of $\lambda$ is $x^2 + \frac{2}{q}x + \frac{1 + \alpha p}{a} \in \mathbb{Z}[x]$. Thus $a = 1$ or 2, and if $p \equiv 1 \pmod{4}$ then $a = 1$. Therefore either $E(\mathbb{F}_p)$ is cyclic (of order $p + 1$), or $p \equiv 3 \pmod{4}$ and $E(\mathbb{F}_p)$ has a cyclic subgroup of order $(p + 1)/2$. Let $\tilde{E}_n$ be a cyclic subgroup of $E(\mathbb{F}_p)$ of order $n \geq 5$ (such exist by the assumptions on $p$; further, $p \nmid n$). Pick $g \in \text{End}(E) - \text{End}_{\mathbb{F}_p}(E)$ and let $f = f' = 1 + ng \in \text{End}(E) - \text{End}_{\mathbb{F}_p}(E)$. Then $f$ acts on $\tilde{E}_n$, which is the identity map, $L = \mathbb{F}_{p^2}$, and $m = 2$. By Theorem 1.3, $\varphi(\mathbb{Z}_{\mathbb{F}_p}(E))$ and $\mathbb{Z}_{\mathbb{F}_p}(E)$ must be different quadratic subfields of the quaternion $\mathbb{Q}$-algebra $\text{End}_{\mathbb{F}_p}(E)$.

Remark 4.7. It follows directly from Proposition 4.2 that if we have $\Phi(n)$ for some $n \geq 5$, and $\text{End}_L(A)$ is commutative, then $f$ is defined over $F$. However, this result also follows directly from Theorem 1.3 and Remark 1.4 (see also Remark 1.5) of [12], which imply that if we have $\Phi(n)$ for some $n \geq 5$ and $\text{End}_{L}(A) = \text{End}_{F}(A)$ then $f$ is defined over $F$ (if $\text{End}_{L}(A)$ is commutative and $L$ is finite, then using [2] we have $\text{End}_{L}(A) = \mathbb{Z}_{L}(A) \subseteq \mathbb{Z}_{F}(A) \subseteq \text{End}_{F}(A)$).

5. Applications of the Skolem-Noether Theorem

In Theorem 1.6 of [12] we showed that if we have $\Phi(n)$ for some $n$ that does not divide $m'^2$, then $A$ and $B$ are $F$-isogenous. In Theorems 5.4 and 5.5 below we give other conditions on $m$ under which (with certain additional conditions on the abelian varieties) $A$ and $B$ are $F$-isogenous. First we state the Skolem-Noether Theorem, which we will use in Proposition 5.2. For a proof, see Theorem 3.62 on p. 69 of [3].

Theorem 5.1 (Skolem-Noether Theorem). Suppose $K$ is a field and suppose $\mathcal{N}$ and $\mathcal{N}'$ are simple $K$-subalgebras of a central simple $K$-algebra $\mathcal{M}$. Then every isomorphism of $K$-algebras $g: \mathcal{N} \rightarrow \mathcal{N}'$ can be extended to an inner automorphism...
of $\mathcal{M}$, that is, there exists an invertible element $a \in \mathcal{M}$ such that $g(b) = a^{-1}ba$ for every $b \in \mathcal{N}$.

The remaining theorems will all make use of the following result.

**Proposition 5.2.** Suppose $A$ and $B$ are abelian varieties defined over a finite field $F$, $f : A \to B$ is an isogeny defined over a field extension $L$ of $F$, and for each of $A$ and $B$ the number of $F$-isotypic components is equal to the number of $L$-isotypic components. If $A'$ is an $L$-isotypic component of $A$, let $B' = f(A')$, let $f' : A' \to B'$ be the isogeny induced by $f$, and define an isomorphism $\varphi'$ from $\text{End}_L^0(A')$ onto $\text{End}_L^0(A')$ by $\varphi'(u) = (f')^{-1}u f'$. Suppose that for every $L$-isotypic component $A'$ of $A$, there is an isomorphism $\delta : Z_F(B') \to Z_F(A')$ such that $\delta(\pi_{F,B'}) = \pi_{F,A'}$ and such that $\delta = \varphi'$ on $Z_L(B')$. Then $A$ and $B$ are $F$-isogenous.

**Proof.** The hypothesis about isotypic components implies that every $L$-isotypic component is also an $F$-isotypic component. Let $A'$ be an $L$-isotypic component of $A$. Then $Z_L(A')$, $Z_F(A')$, $Z_L(B')$, and $Z_F(B')$ are number fields. The map $\delta(\varphi')^{-1}$ defines an isomorphism from $\varphi'(Z_F(B'))$ onto $Z_F(A')$. In the Skolem-Noether Theorem, take

$$K = Z_L(A'), \quad \mathcal{M} = \text{End}_L^0(A'), \quad \mathcal{N} = \varphi'(Z_F(B')),$$

and $\mathcal{N}' = Z_F(A')$.

The algebra $\mathcal{M}$ is a central simple $K$-algebra, since by our assumption $A'$ is $F$-isogenous to a power of an $F$-simple abelian variety. Further, $\mathcal{N}$ is a simple $K$-subalgebra of $\text{End}_L^0(A')$, since $\varphi'(Z_F(B'))$ is isomorphic to the field $Z_F(B')$, which, because $L$ and $F$ are finite, by (2) contains the subfield $Z_L(B') \cong Z_L(A') = K$. The Skolem-Noether Theorem shows the existence (after multiplying, if necessary, by a sufficiently divisible integer) of an isogeny $u \in \text{End}_L^0(A')$ such that $\delta(\varphi')^{-1}(b) = u^{-1}bu$ for every $b \in \varphi'(Z_F(B'))$. Therefore,

$$\delta(a) = u^{-1}\varphi'(a)u = u^{-1}(f')^{-1}a f' u$$

for every $a \in Z_F(B')$. Thus, $f' u \delta(a) = a f' u$ for every $a \in Z_F(B')$, and in particular,

$$f' u \pi_{F,A'} = f' u \delta(\pi_{F,B'}) = \pi_{F,B'} f' u.$$

By Lemma [3.3](#3.3), the isogeny $f' u$ is defined over the field $F$. Summing the isogenies $f' u$ corresponding to the different $L$-isotypic components of $A$, we obtain an $F$-isogeny from the product of the $L$-isotypic components of $A$ to the product of the $L$-isotypic components of $B$. Since, under our hypotheses, $A$ and $B$ are each $F$-isogenous to the products of their $L$-isotypic components, we obtain the desired result.

**Remark 5.3.** If an abelian variety $A$ over a finite field $F$ is $F$-isogenous to a power of an $F$-simple abelian variety, and $L$ is a field extension of $F$, then the number of $F$-isotypic components of $A$ is equal to the number of $L$-isotypic components of $A$. This is because $A$ being $F$-isogenous to a power of an $F$-simple abelian variety is equivalent to $Z_F(A)$ being a number field. Since $Z_L(A) \subset Z_F(A)$, if $Z_F(A)$ is a number field so is $Z_L(A)$, and therefore $A$ is $L$-isogenous to a power of an $L$-simple abelian variety.

**Theorem 5.4.** Suppose we have $\Phi(n)$ for some $n \geq 5$, there is a $\mathbb{Q}$-algebra homomorphism from $\mathbb{Q}(\zeta_n)$ to $Z_L(B)$ that takes 1 to 1, and for each of $A$ and $B$ the number of $F$-isotypic components is equal to the number of $L$-isotypic components. Then $A$ and $B$ are $F$-isogenous.
Proof. Let $A'$ be an $L$-isotypic component of $A$, let $B' = f(A')$, let $f': A' \to B'$ be the isogeny induced by $f$, and let $\varphi': \operatorname{End}_L(B') \to \operatorname{End}_L(A')$ be defined by $\varphi'(u) = (f')^{-1}uf'$. Since $L$ and $F$ are finite fields, by [2] we have inclusions of number fields

$$Z_L(A') \subseteq Z_F(A'), \quad \mathbb{Q}((\zeta_m)) \subseteq Z_L(B') \subseteq Z_F(B').$$

Let $e$ be the largest divisor $j$ of $m$ such that $\pi_{L,B'}$ is a $j^\text{th}$ power in the field $Z_L(B')$. Let $\Pi_{B'} = \pi_{F,B'}^{m/e}$. Then $\Pi_{B'} = \pi_{L,B'}^{m} = \pi_{L,B'}^{m/e}$. Since some $e^\text{th}$ root of $\pi_{L,B'}$ is in $Z_L(B')$ by the definition of $e$, and $\mathbb{Q}((\zeta_m)) \subseteq Z_L(B')$, we know that $\Pi_{B'} \in Z_L(B')$.

We will now show that the minimal polynomial for $\pi_{F,B'}$ over $Z_L(B')$ is $h(t) = t^{m/e} - \Pi_{B'}$. It suffices to show that $h(t)$ is irreducible over $Z_L(B')$. Each root of $h(t)$ is the product of $\pi_{F,B'}$ by an $m^\text{th}$ root of unity. Suppose $g(t)$ is an irreducible monic factor of $h(t)$ over $Z_L(B')$. The constant term of $g(t)$ is an element of $Z_L(B')$ that is a product of roots of $h(t)$, so it is the product of $\pi_{F,B'}$ by an $m^\text{th}$ root of unity, where $c$ is the degree of the polynomial $g$. Since $\mathbb{Q}((\zeta_m)) \subseteq Z_L(B')$, we know that $\pi_{F,B'} \in Z_L(B')$. Let $d$ be the greatest common divisor of $c$ and $m$. Then $\pi_{F,B'}^d \in Z_L(B')$. But $(\pi_{F,B'}^{d^2})^{m/d} = \pi_{L,B'}$, so by the definition of $e$, we now have $m/d \leq e$. Therefore, $m/e \leq d \leq c = \operatorname{deg}(g)$. Since $m/e$ is the degree of the polynomial $h$, we have $h = g$ and $h$ is irreducible.

Let $\Pi_{A'} = \pi_{F,A'}^{m/e}$. The isomorphism $\varphi'$ induces an isomorphism of number fields $\varphi': Z_L(B') \to Z_L(A')$ such that $\varphi'(\pi_{L,B'}) = \pi_{L,A'}$. Therefore $e$ is also the largest divisor of $m$ such that $\pi_{L,A'}$ is a $j^\text{th}$ power in $Z_L(A') \equiv Z_L(B')$. We have $\Pi_{A'} \in Z_L(A')$, and the above reasoning shows that $t^{m/e} - \Pi_{A'}$ is the minimal polynomial of $\pi_{F,A'}$ over $Z_L(A')$.

Since $\Pi_{B'} \in Z_L(B')$, we have $\varphi'(\Pi_{B'}) = \Pi_{A'}$ by Proposition [4.1]. The composition of the natural isomorphisms

$$Z_F(B') = Z_L(B') \to Z_L(A') \cong Z_L(A') \equiv Z_L(B') \to Z_F(B'),$$

where the middle isomorphism is given by the map $\varphi'$, defines an isomorphism $\delta: Z_F(B') \to Z_F(A')$ such that $\delta(\pi_{F,B'}) = \pi_{F,A'}$ and such that $\delta = \varphi'$ on $Z_L(B')$. We can now apply Proposition [5.2].

**Theorem 5.5.** Suppose we have $\Phi(n)$ for some $n \geq 5$, $m$ is prime, and for each of $A$ and $B$ the number of $F$-isotypic components is equal to the number of $L$-isotypic components. Suppose also that for each $L$-isotypic component $A'$ of $A$, either $\mathbb{Q}((\zeta_m)) \subseteq Z_L(A')$, or $\mathbb{Q}((\zeta_m))$ and $Z_L(A')$ are linearly disjoint over $\mathbb{Q}$. Then $A$ and $B$ are $F$-isogenous.

**Proof.** We will apply Proposition [5.2]. Suppose $A'$ is an $L$-isotypic component of $A$, and let $B' = f(A')$. Let $Z = Z_L(A')$ and identify $Z_L(B')$ with $Z$ via the isomorphism $\varphi': Z_L(B') \to Z$ defined by $\varphi'(u) = (f')^{-1}uf'$ where $f': A' \to B'$ is the isogeny induced by $f$. It suffices to show there exists an isomorphism $\delta: Z_F(B') \to Z_F(A')$ over $Z$ such that $\delta(\pi_{F,B'}) = \pi_{F,A'}$. If $\mathbb{Q}((\zeta_m)) \subseteq Z$, then the proof of Theorem [5.4] produces the desired isomorphism $\delta$.

Suppose $\mathbb{Q}((\zeta_m))$ and $Z$ are linearly disjoint over $\mathbb{Q}$. We then have $\operatorname{Gal}(Z((\zeta_m))/Z) = \operatorname{Gal}(\mathbb{Q}((\zeta_m))/\mathbb{Q})$. Let $\pi = \pi_{L,A'} \in Z$. Then $\pi_{F,A'}^m = \pi = \pi_{L,B'} = \pi_{F,B'}^m$. Let $g(t)$
denote the minimal polynomial for \( \pi_{F,A'} \) over \( \mathbb{Z} \), and let \( d \) be the degree of \( g \). The coefficient of \( t^{d-1} \) in \( g(t) \) is an element of \( \mathbb{Z} \) of the form
\[
\pi_{F,A'} \sum_{j \in J} \zeta_{m}^j
\]
with \( J \) a subset of \( \{0, 1, \ldots, m-1\} \). If the sum is zero, then \( J = \{0, 1, \ldots, m-1\} \) and \( g(t) = t^m - \pi \), so \( t^m - \pi \) is an irreducible polynomial over \( \mathbb{Z} \) whose roots include \( \pi_{F,A'} \) and \( \pi_{F,B'} \), and therefore there is an isomorphism \( \delta \) as desired. So we can suppose the sum is non-zero. Therefore, \( \pi_{F,A'} \in \mathbb{Z}(\zeta_m) \).

Consider the inflation-restriction exact sequence (Kummer theory)
\[
H^1(\text{Gal}(\mathbb{Z}(\zeta_m)/\mathbb{Z}), \mu_m) \to H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Z}), \mu_m) \to H^1(\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Z}), \mu_m).
\]
Since \( m \) is prime, we have \( \#\text{Gal}(\mathbb{Z}(\zeta_m)/\mathbb{Z}) = m-1 \). Since \( \#\mu_m = m \) and \( \gcd(m, m-1) = 1 \), we have
\[
H^1(\text{Gal}(\mathbb{Z}(\zeta_m)/\mathbb{Z}), \mu_m) = 0.
\]
The cohomological long exact sequence now tells us that the natural group homomorphism
\[
\mathbb{Z}^x/(\mathbb{Z}^x)^m \to \mathbb{Z}(\zeta_m)^x/(\mathbb{Z}(\zeta_m)^x)^m
\]
is an inclusion. We have \( \pi \in \mathbb{Z}^x \). Since \( \pi_{F,A'} \in \mathbb{Z}(\zeta_m)^x \) and \( \pi_{F,A'} \in \mathbb{Z}^m \), we have \( \pi \in (\mathbb{Z}(\zeta_m)^x)^m \). Therefore, we can write \( \pi = v^m \) for some \( v \in \mathbb{Z}^x \). Then \( \pi_{F,A'} = v\zeta_m^a \) and \( \pi_{F,B'} = v\zeta_m^b \) with \( a, b \in \{0, 1, \ldots, m-1\} \). If \( a = 0 \) then \( \pi_{F,A'} \in \mathbb{Z} \subseteq \mathbb{Z}(\zeta_m) \), and by Theorem 4.3 we have \( \mathbb{Z}(\zeta_m)/\mathbb{Z} \cong \mathbb{Z}(\zeta_m)/\mathbb{Z}(\zeta_m)^x \), and we are done. We proceed similarly if \( b = 0 \). Now suppose \( a \) and \( b \) are both non-zero. Then \( \mathbb{Z}(\pi_{F,A'}) = \mathbb{Z}(\zeta_m) = \mathbb{Z}(\pi_{F,B'}) \), and (since \( \mathbb{Q}(\zeta_m) \) and \( \mathbb{Z} \) are linearly disjoint over \( \mathbb{Q} \)) there is an automorphism of \( \mathbb{Z}(\zeta_m) \) over \( \mathbb{Z} \) that takes \( \zeta_m^a \) to \( \zeta_m^b \), giving the desired isomorphism \( \delta \).

As an immediate corollary we have:

**Corollary 5.6.** Suppose we have \( \Phi(n) \) for some \( n \geq 5 \), for each of \( A \) and \( B \) the number of \( F \)-isotypic components is equal to the number of \( L \)-isotypic components, and \( m \leq 3 \). Then \( A \) and \( B \) are \( F \)-isogenous.

**Proof.** In the notation of Theorem 5.5 if \( m \leq 2 \) then \( \mathbb{Q}(\zeta_3) \subset \mathbb{Q} \). If \( m = 3 \), then \( \mathbb{Q}(\zeta_3)/\mathbb{Q} \) is quadratic, so every number field (including \( \mathbb{Z} \)) either contains a subfield isomorphic to \( \mathbb{Q}(\zeta_3) \) or is linearly disjoint from it over \( \mathbb{Q} \). In both cases, the desired result follows from Theorem 5.5.

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