Almost all positive linear functionals can be extended

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Abstract

Let $F$ be an ordered topological vector space (over $\mathbb{R}$) whose positive cone $F^+$ is weakly closed, and let $E \subseteq F$ be a subspace. We prove that the set of positive continuous linear functionals on $E$ that can be extended (positively and continuously) to $F$ is weak-$\ast$ dense in the topological dual wedge $E'_+$. Furthermore, we show that this result cannot be generalized to arbitrary positive operators, even in finite-dimensional spaces.

1 Introduction

Extension theorems for positive operators have been studied in great detail, and can be found in many textbooks on ordered vector spaces (e.g. [Sch99, §V.5], [AT07, §1.4–1.5]). A sufficient criterion is provided by the classical extension theorem of Kantorovich.

**Theorem (Kantorovich).** Let $F$ be an ordered vector space (over $\mathbb{R}$), $E \subseteq F$ a majorizing subspace, and $G$ a Dedekind complete Riesz space. Then every positive operator $E \to G$ has a positive extension $F \to G$.

For positive linear functionals, the following necessary and sufficient criterion was established independently by Bauer [Bau57, Théorème 1] and Namioka [Nam57, Theorem 4.4].

**Theorem (Bauer–Namioka).** Let $F$ be an ordered topological vector space (over $\mathbb{R}$) and let $E \subseteq F$ be a subspace. A positive continuous linear functional $\varphi \in E'_+$ has an extension in $F'_+$ if and only if there is a convex 0-neighbourhood $M \subseteq F$ such that $\varphi$ is bounded above on $E \cap (M - F^+)$. In this short note, we prove that almost all positive continuous linear functionals can be extended, provided that the positive cone is weakly closed.

**Theorem 1.** Let $F$ be a preordered topological vector space (over $\mathbb{R}$), and let $E \subseteq F$ be a subspace. If the topological dual $F'$ separates points on $F$, and if the positive wedge $F^+$ is weakly closed, then the set of all positive continuous linear functionals on $E$ that can be extended positively and continuously to $F$ is weak-$\ast$ dense in $E'_+$.

Note: if $F$ is locally convex, then $F^+$ is weakly closed if and only if $F^+$ is closed (because $F^+$ is a convex set). Additionally, note that, if $F^+$ is weakly closed and $F^+ \cap - F^+ = \{0\}$, then $\{0\}$ is also weakly closed, so the weak topology is Hausdorff. Thus, if $F^+$ is a cone, then the requirement that $F'$ separates points on $F$ is automatically met, and the statement from the abstract is recovered.

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The proof of Theorem 1 is deceptively simple, and it is likely that this has been proved as a lemma various times across different areas of mathematics (or physics, computer science, economics, etc.). However, the result appears to be unknown in the ordered vector spaces community, and the author has not been able to locate an earlier proof (or statement) of the main result.

We show in §4 that Theorem 1 cannot be generalized to arbitrary positive operators, even in the finite-dimensional case.

2 Proof of the main theorem

Proof of Theorem 1. Define $R := \{ \varphi | E : \varphi \in F'_+ \} \subseteq E'_+$, and note that $R$ is the wedge of positive continuous linear functionals $E \to \mathbb{R}$ that can be extended positively and continuously to $F$. Since $F_+$ is weakly closed, we have

$$F := \{ x \in F : \langle x, \varphi \rangle \geq 0 \text{ for all } \varphi \in F'_+ \}.$$

It follows that $E := F_+ \cap E = \{ x \in E : \langle x, \varphi \rangle \geq 0 \text{ for all } \varphi \in F'_+ \}$. This shows that $E$ is the predual wedge of $R$. Hence, by the bipolar theorem, $E'_+$ is the weak-$^*$ closure of $R$. □

3 Approximate duality between pushforwards and pullbacks

The main theorem can be seen as a special case of a more general duality, which we sketch here.

In this section, a subscript/superscript $w$ or $w^*$ refers to the weak or weak-$^*$ topology.

Let $F$ and $G$ be preordered topological vector spaces (over $\mathbb{R}$) whose topological duals separate points. We say that $T \in \mathcal{L}(F, G)$ is an approximate pushforward if $T[F_+]^w = G^w$, and an approximate pullback (or approximately bipositive) if $F^w = T^{-1}[G^w]$.

Recall that the adjoint of a continuous linear operator $T \in \mathcal{L}(F, G)$ restricts to a weak-$^*$ continuous operator $T' \in \mathcal{L}(G'_w, F'_w)$.

Proposition 2. Let $F$ and $G$ be as above, and let $T \in \mathcal{L}(F, G)$ be given. Then:

(a) $T$ is an approximate pushforward if and only if $T' : G' \to F'$ is bipositive;

(b) $T' : G'_w \to F'_w$ is a weak-$^*$ approximate pushforward if and only if $T$ is an approximate pullback.

This is a special case of a general result regarding pushforwards and pullbacks of polars (e.g. [Sch99, Proposition IV.2.3(a)]). For completeness, we give the (simple) proof here.

Proof of Proposition 2.

(a) By definition one has $T'\psi = \psi \circ T$, so the dual wedge of $T[F_+]$ is given by

$$(T[F_+])' = \{ \psi \in G' : \psi(Tx) \geq 0 \text{ for all } x \in F_+ \}$$

$$= \{ \psi \in G' : (T'\psi)(x) \geq 0 \text{ for all } x \in F_+ \}$$

$$= \{ \psi \in G' : T'\psi \in F'_+ \}$$

$$= (T')^{-1}[F'_+] = \{ T'\psi \in F'_+ \}.$$
(b) This follows immediately from (a), since $T''$ is equal to $T$, except with its domain and codomain equipped with the wedges $F_{w}^{w'}$ and $G_{w}^{w'}$ (after all, the bipolar wedge $F_{w}^{w} \subseteq (F_{w})' = F$ coincides with the weak closure $F_{w}^{w} \subseteq F$, by the bipolar theorem). \hfill $\square$

Theorem 1 can be recovered by applying Proposition 2(b) to the inclusion $T : E \hookrightarrow F$.

**Remark 3.** If $F$ is locally convex and if $E \subseteq F$ is closed, then the adjoint of the inclusion $E \hookrightarrow F$ is the quotient $F_{w}^{w} \to F_{w}^{w}/E_{w} = E_{w}^{+}$. By the duality from Proposition 2, for every positive continuous linear functional on $E$ to have a positive and continuous extension on $F$, it is necessary and sufficient that the pushforward of $F_{w}$ along the quotient $F'/F'/E_{w}^{+}$ is already weak-$*$ closed.

Note that, even in the finite-dimensional case, a quotient (i.e. projection) of a closed convex cone need not be closed, so even here the cone $R \subseteq E^{*}$ of extendable positive linear functionals is “only” dense in $E^{*}$. (In fact, Mirkil [Mir57, Corollary 1] proved that, for a finite-dimensional ordered vector space with a closed cone $E_{+}$, all positive linear functionals on all subspaces can be extended if and only if $E_{+}$ is polyhedral; see also Klee [Kle59, Theorem 4.13].) In other words, there is no exact duality between pushforwards and pullbacks, even if the spaces are finite-dimensional and the cones are closed.

### 4 No generalization to positive operators

One might ask if similar results can be obtained for arbitrary positive operators. Unfortunately, this is not the case. We will construct a counterexample below.

Recall that a wedge $K \subseteq E$ is a *simplex cone* (or *Yudin cone*) if there is an (algebraic) basis $B$ of $E$ such that $K$ is the wedge generated by $B$.

If $E$ and $G$ are vector spaces, and if $\varphi \in E^{*}$ and $z \in G$, then we write $z \otimes \varphi$ for the linear map $E \to G$, $x \mapsto \langle x, \varphi \rangle z$.

**Situation 4.** Let $E$ be a finite-dimensional vector space (over $\mathbb{R}$), and let $E_{+} \subseteq E$ be a generating polyhedral cone that is not a simplex cone. Let $\varphi_{1}, \ldots, \varphi_{m} \in E_{+}^{*}$ be representatives of the extreme rays of $E_{+}^{*}$, so that $E_{+} = \bigcap_{i=1}^{m} \{ x \in E : \langle x, \varphi_{i} \rangle \geq 0 \}$ and every positive linear functional is a positive combination of $\varphi_{1}, \ldots, \varphi_{m}$. Additionally, let $F := \mathbb{R}^{m}$ with the standard cone $F_{+} := \mathbb{R}^{m}_{\geq 0}$, so that the map $T : E \to F$, $x \mapsto (\varphi_{1}(x), \ldots, \varphi_{m}(x))$ is bipositive. We will identify $E$ with a subspace of $F$ via this map.

**Proposition 5.** In Situation 4, the positive linear maps $E \to E$ that can be extended to a positive linear map $F \to E$ are precisely the maps of the form $\sum_{i=1}^{k} x_{i} \otimes \psi_{i}$ with $x_{1}, \ldots, x_{k} \in E_{+}$ and $\psi_{1}, \ldots, \psi_{k} \in E_{+}^{*}$.

**Proof.** Let $T : E \to E$ be a positive linear map that can be extended to a positive linear map $S : F \to E$. Then we have

$$T(x) = S(\varphi_{1}(x), \ldots, \varphi_{m}(x)) = S(e_{1})\varphi_{1}(x) + \cdots + S(e_{m})\varphi_{m}(x),$$

so $T$ can be written as $T = \sum_{j=1}^{m} S(e_{j}) \otimes \varphi_{j}$.

Conversely, suppose that $T = \sum_{i=1}^{k} x_{i} \otimes \psi_{i}$ with $x_{1}, \ldots, x_{k} \in E_{+}$ and $\psi_{1}, \ldots, \psi_{k} \in E_{+}^{*}$. Every $\psi_{i}$ can be written as a positive combination of the $\varphi_{1}, \ldots, \varphi_{m}$, so after rearranging the terms we may write $T = \sum_{j=1}^{m} y_{j} \otimes \varphi_{j}$, where the $y_{j}$ are positive combinations of the $x_{i}$. In particular, $y_{1}, \ldots, y_{m} \in E_{+}$. Therefore the map $S : F \to E$, $e_{j} \mapsto y_{j}$ is a positive extension of $T$. \hfill $\square$

The following theorem of Barker and Loewy tells us that approximation by operators of the form described in Proposition 5 is not always possible.
Theorem 6 (Barker–Loewy, [BL75, Proposition 3.1]). Let $E$ be a finite-dimensional ordered vector space (over $\mathbb{R}$) whose positive cone $E_+$ is closed and generating. Then the identity $\text{id}: E \to E$ can be written as the limit of a sequence of operators of the form $\sum_{i=1}^{k} x_i \otimes \psi_i$ (with $x_1, \ldots, x_k \in E_+$ and $\psi_1, \ldots, \psi_k \in E_+^*$) if and only if $E_+$ is a simplex cone.

In particular, if $E$ and $F$ are as in Situation 4, then it follows from Proposition 5 and Theorem 6 that the identity $E \to E$ cannot be approximated by positive operators that can be extended to positive operators $F \to E$.

Remark 7. A more high-level interpretation of the preceding example is that the injective cone, much like the injective norm, does not preserve quotients/pushforwards. (For a detailed discussion of the similarities between the injective norm and the injective cone, see [Dob20].)

Let $E$ and $F$ be as in Situation 4. The cone of positive operators $F \to E$ coincides with the injective cone in the tensor product $F^* \otimes E$, and the adjoint of the inclusion $T: E \hookrightarrow F$ is the quotient $T^*: F^* \to E^*$. By Proposition 2(b), $T^*$ is an approximate pushforward. (In fact, all cones in this example are polyhedral, so $T^*$ is a pushforward.) However, $T^* \otimes \text{id}: F^* \otimes E \to E^* \otimes E$ is not an approximate pushforward (with respect to the injective cone), since not every positive operator $E \to E$ can be approximated by restrictions of positive operators $F \to E$.

In summary, Theorem 1 cannot be extended to positive operators precisely because the injective tensor product of cones does not preserve approximate pushforwards.

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