ON A COMPOSITION OF DIGRAPHS

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Abstract. Many “good” topologies for interconnection networks are based on line digraphs of regular digraphs. These digraphs support unitary matrices. We propose the property “being the digraph of a unitary matrix” as additional criterion for the design of new interconnection networks. We define a composition of digraphs, which we call diagonal union. Diagonal union can be used to construct digraphs of unitary matrices. We remark that diagonal union digraphs are state split graphs, as defined in symbolic dynamics. Finally, we list some potential directions for future research.

1. Introduction

Point-to-point interconnection networks for parallel and distributed systems are usually modeled by (di)graphs. Survey papers on interconnection networks are, e.g., (in chronological order) Feng [81], Hedemann [97] and Ferrero [99]. The vertices of the (di)graph correspond to the nodes of the network, that is processors, nodes with local memory, switches, etc.; the arcs correspond to communication wires. Performance of parallel and distributed systems is significantly determined by the choice of the network topology. The basic requirements are: (i) large numbers of nodes; (ii) small distance between any two nodes (small communication delay); (iii) limited number of wires. These requirements result in the need of finding (di)graphs, with, respectively: (i) large order; (ii) small diameter; (iii) bounded degree. Another important requirement is fault-tolerance, which is measured in the terms of several parameters, two of which are vertex-connectivity and arc(edge)-connectivity.

Many “good” topologies for interconnection networks are based on line digraphs of regular digraphs (e.g. de Bruijn digraphs, Reddy-Pradhan-Kuhl digraphs, butterflies digraphs, etc.), hypercubes (that are not line digraphs), and a number of their generalizations. Line digraphs have many remarkable properties. Line digraphs are used as tools in algorithmic applications (see, e.g., Gusfield [98]), as underlying digraphs of coined quantum random walks [03], as a survey paper on quantum random walks, in the study of spectral statistics and random matrix theory (see, e.g., Pakonski et al. [02]), in the study of growth functions of free groups (see, e.g., Rivin [99]).

Recall that, given an $n \times n$ matrix $M$ (over any field), a digraph $D$ is said to support $M$, or, equivalently, to be the digraph of $M$, if $D$ is on $n$ vertices, and its adjacency matrix, $M(D)$, has $ij$-th element

$$M(D)_{i,j} = \begin{cases} 1 & \text{if } M_{i,j} \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Recall that a complex $n \times n$ matrix $U$ is unitary if $U^\dagger U = I_n$, where $U^\dagger$ is the adjoint of $U$, and $I_n$ is the identity matrix of size $n$. The digraph of a unitary matrix is without cut-vertices and it is bridgeless, then it is 2-vertex-connected and 2-arc-connected. Note that this properties insure a certain degree of fault-tolerance when thinking about the digraph as a model of an interconnection network. Both, line digraphs of regular digraphs and hypercubes support unitary matrices [03]. This observation motivates the present paper, which is structured as follows.

A composition of digraphs is a method to construct digraphs “putting together” smaller digraphs and adding arcs according to specific rules. In §2 we define a composition of digraphs, which we call diagonal
union. The composition can be used to construct digraphs of unitary matrices. We observe: simple conditions under which a diagonal union digraph supports unitary matrices; that line digraphs of regular digraphs are diagonal union digraphs (§3); that diagonal union digraphs are state split graphs, as defined in symbolic dynamics (§3). Finally, in §4, we propose a few potential directions for future research.

Although the content of the paper is elementary, we wish to write it as self-contained as possible. For terms of graph theory not defined here, we refer the reader to the monograph by Bang-Jensen and Gutin [B-JG01].

2. A composition of digraphs

2.1. Set-up. A (finite) directed graph, for short digraph, consists of a non-empty finite set of elements called vertices and a (possibly empty) finite set of ordered pairs of vertices called arcs. Let us denote by $D = (V, A)$ a digraph with vertex-set $V(D)$ and arc-set $A(D)$. The adjacency matrix of a digraph $D$ on $n$ vertices, denoted by $M(D)$, is the $n 	imes n$ $(0, 1)$-matrix with $ij$-th entry

$$M_{i,j}(D) = \begin{cases} 1 & \text{if } (v_i, v_j) \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

A digraph $H$ is a subdigraph of a digraph $D$ if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. A subdigraph $H$ of a digraph $D$ is a spanning subdigraph of $D$ if $V(H) = V(D)$; in such a case, we say that $H$ spans $V(D)$. A decomposition of a digraph $D$ is a set $\{H_1, H_2, ..., H_k\}$ of subdigraphs whose arc-sets are exactly the classes of a partition of $A(D)$. A subdigraph of $D$ is called a factor of $D$ if it spans $V(D)$. A factorization of $D$ is a decomposition of $D$ into factors.

The Hadamard product (in the literature, sometimes called Schur product or entry-wise product) of $n \times m$ matrices $N$ and $M$, denoted by $N \circ M$, is defined as follows:

$$(N \circ M)_{i,j} = N_{i,j} \cdot M_{i,j}.$$ 

Definition 1 (Generalized Hadamard product). Let $M$ and $N$ be respectively an $m \times m$ and an $n \times n$ matrix (over any field). Let $m = n \cdot r$. The generalized Hadamard product of $N$ with $M$, denoted by $N \circ_G M$, is defined as follows:

(i) If $r = 1$ ($n = m$ and $N$ has the same size of $M$) then

$$N \circ_G M = N \circ M.$$ 

(ii) If $r > 1$ then we look at $M$ as a block-matrix of the form

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix},$$

where the $ij$-th block is $r \times r$. Then

$$N \circ_G \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} = \begin{bmatrix} N_{1,1}A_{1,1} & \cdots & N_{1,n}A_{1,n} \\ \vdots & \ddots & \vdots \\ N_{n,1}A_{n,1} & \cdots & N_{n,n}A_{n,n} \end{bmatrix}.$$ 

Example 1. Let

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. $$

Then

$$\sigma_x \circ_G \begin{bmatrix} A & B \\ D & C \end{bmatrix} = \begin{bmatrix} 0 & B \\ D & 0 \end{bmatrix} \text{ and } I_2 \circ_G \begin{bmatrix} A & B \\ D & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}.$$
2.2. Definition. Let $G$ be a finite group of order $n$. Let $W$ be a complex vector space of finite dimension. Let $GL(W)$ be the group of the bijective linear maps on $V$. A linear representation of $G$ is a homomorphism $\rho : G \rightarrow GL(W)$. To each $g_i \in G$, we associate the unit column vector $e_i \in W$ with $j$-th entry

$$e_{ij} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $GL(n, F)$ be the group of invertible $n \times n$ matrices over the field $F = GF(2)$. The (left) regular permutation representation of $G$, denoted by $\rho_{\text{reg}}$, is a homomorphism $\rho_{\text{reg}} : G \rightarrow GL(n, F)$. Then $\rho_{\text{reg}}(g_i)$ is a permutation matrix with $ij$-entry

$$\rho_{\text{reg}}(g_i)_{ij} = \begin{cases} 1 & \text{if } g_i g_j = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $\mathbb{Z}_n$ the additive abelian group of the integers modulo $n$. Let us denote by $J_n$ the $n \times n$ all-ones matrix, and by $M \oplus N$ the direct sum of matrices $M$ and $N$.

Definition 2 (Diagonal union). Let $D$ be a digraph on $n$ vertices and let $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ be a factorization of $D$. The digraph $D_{\mathcal{F}}$ is defined as follows:

$$M(D_{\mathcal{F}, 1}) = M(D_{\mathcal{F}}) = (J_k \otimes I_n) \bigoplus_{i=1}^{k} M(H_i).$$

Iteratively,

$$M(D_{\mathcal{F}, d}) = (J_k \otimes I_{n^{d-1}}) \bigoplus_{j=0}^{k-1} [\rho_{\text{reg}}(j) \circ G M(D_{\mathcal{F}, d-1})],$$

$\rho_{\text{reg}}$ is the (left) regular permutation representation of $\mathbb{Z}_k$. The digraph $D_{\mathcal{F}, d}$ is called $d$-diagonal union\(^1\) of $D$ in respect to $\mathcal{F}$.

Remark 1. If the factors $H_1, H_2, \ldots, H_k$ are digraphs of unitary matrices then $M(D_{\mathcal{F}, d})$ is also the digraph of a unitary matrix. This insures that $M(D_{\mathcal{F}, d})$ is 2-vertex-connected and 2-arc-connected [S03].

A concrete example might be useful to clarify Definition 2. Let us denote by $\overrightarrow{K_n}^+$ the complete symmetric digraph with a self-loop at each vertex.

Example 2. Let $\mathcal{F} = \{H_1, H_2\}$ be a factorization of a digraph $D$. In Figure 2 below, are represented the matrices $M(D_{\mathcal{F}})^T$, $M(D_{\mathcal{F}, 2})^T$, $M(D_{\mathcal{F}, 3})^T$ and $M(D_{\mathcal{F}, 4})^T$. The light and the dark shaded blocks represent $M(H_1)$ and $M(H_2)$, respectively; the white areas are of zeros only.

Let

$$D = \overrightarrow{K_2}^+, \quad M(H_1) = I_2 \quad \text{and} \quad M(H_2) = \sigma_x.$$

Then

$$M(D_{\mathcal{F}, k}) = B(2, k),$$

the binary $k$-dimensional de Bruijn digraph.

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\(^1\)On the choice of the terminology: The term “diagonal union” may be justified by the structure of $M(D_{\mathcal{F}, d})$. In fact, $M(D_{\mathcal{F}, d})$ is a block-matrix, with the blocks on the diagonal being the adjacency matrices of the factors of $D_{\mathcal{F}, d-1}$. 
3. Remarks

3.1. Extremal cases. Let $D$ be a digraph on $n$ vertices and $m$ arcs. Suppose that $\mathcal{F}$ is the trivial factorization of $D$, that is $\mathcal{F} = \{D\}$. In such a case, $D_\mathcal{F} = D$. A cycle factor of a digraph $D$ is a collection of pairwise vertex-disjoint dicycles spanning $D$. In other words, a cycle factor is a spanning 1-regular subdigraph of $D$. Note that the adjacency matrix of a cycle factor is a permutation matrix. A factorization in cycle factors is a factorization whose members are cycle factors. The line digraph of a digraph $D$, denoted by $\overrightarrow{L}D$, is defined as follows (see, e.g., [?]): the vertex-set of $\overrightarrow{L}D$ is $E(D)$; $(v_i, v_j), (v_k, v_l) \in A(\overrightarrow{L}D)$ if and only if $v_j = v_k$. Suppose that $\mathcal{F}$ is a factorization in cycle factors. In such a case, $D_{\mathcal{F},d} = \overrightarrow{L}^{d-1}D$.

So, note that the line digraph of a regular digraph is a diagonal union digraph.

3.2. State split graphs. The notion of state split graph, introduced by Williams [W74], is important in symbolic dynamics and coding. Lind and Marcus [LM95] is comprehensive for a monograph on symbolic dynamics. Given a digraph $D$, let $\mathcal{E}_i$ be the set of the arcs in which $v_i$ is tail. For each vertex $v_i \in V(D)$, let

$$\mathcal{E}_i = \bigcup_{k=1}^{m(i)} E_i^k, \quad m(i) \geq 1.$$ 

Let $\mathcal{P}$ be the resulting partition of $\mathcal{E}$, and let $\mathcal{P}_i$ the resulting partition restricted to $\mathcal{E}_i$. The state split graph $D[\mathcal{P}]$ formed from $D$ using $\mathcal{P}$ has vertices $v_i^1, v_i^2, ..., v_i^{m(i)}$, for all $v_i \in V(D)$. The arc $(v_h, v_l)^j \in A(D[\mathcal{P}])$, where $(v_h, v_l) \in A(D)$ and $1 \leq j \leq m(v_l)$. The arc $(v_h, v_l) \in \mathcal{E}_k$, for some $k$. The tail and the head of $(v_h, v_l)^j$ are respectively $v_k^l$ and $v_l^j$. If the construction of a state split graph uses a partition of the outgoing (ingoing) arcs (as above) then the obtained digraph is called out-split (in-split) graph.

**Remark 2.** A digraph $M(D_\mathcal{F})$ is a state split graph. In particular, $M(D_\mathcal{F})$ is a in-split graph formed from $D$ using the partition of $A(D)$ arising from the factorization $\mathcal{F}$.

**Example 3.** Consider the Cayley digraph $D = \text{Cay}(\mathbb{Z}_4, \{1, 2, 3\})$:

$$M(D) = \sum_{i=1}^{3} \rho_{reg}(i) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 \end{bmatrix},$$

where in the first column are the labels of the vertices.
Let \( F = \{D_1, D_2\} \), where
\[
M(D_1) = \rho_{reg}(1) + \rho_{reg}(3) \quad \text{and} \quad M(D_2) = \rho_{reg}(2).
\]
Then
\[
M(D_F) = [\rho_{reg}(1) \otimes \rho_{reg}(3)] \cdot (J_2 \otimes I_4)
\]
\[
= \begin{bmatrix}
0^4 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1^1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
2^4 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
3^1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0^2 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
2^2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
3^2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

Then \( D_F \) is an in-split graph formed from \( D \) using the partition arising from \( F \).

4. What’s next?

The following seem to be natural questions:

(i) To what extent the property “being the digraph of a unitary matrix” makes a digraph a “good” network topology?

(ii) Can we make use of diagonal union digraphs as bases of new network topologies?

(iii) We have seen that diagonal union digraphs are obtained via state-splitting. Are tools from symbolic dynamics and coding helpful in the design and analysis of interconnection networks, and in broadcasting problems?

There is also space for a legitimate speculation:

(i) The time-evolution of a quantum mechanical system, in a pure state, is induced by unitary matrices. Then interconnection networks based on digraphs of unitary matrices have something to do with quantum mechanical systems. What? In some way, these networks model certain classical systems, whose dynamics share some combinatorial structure with quantum systems. So, they are (partially?) combinatorially quantum. Can this idea be exploited in the context of quantum computation and communication?

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