We present a nonperturbative renormalization group solution of the Gell-Mann–Levy $\sigma$-model which was originally proposed as a phenomenological description of the dynamics of nucleons and mesons. In our version of the model the fermions are interpreted as quarks which interact via the $\sigma$ and $\pi$ mesons. We derive and numerically solve renormalization group (RG) flow equations to leading order in a derivative expansion to study the behavior of the model as it evolves from high to low momentum scales. We develop an expansion in chiral-symmetry-breaking which enables us to track this symmetry breaking with the evolution of the scale. We use infrared observables to constrain the phenomenology allowing predictions of other quantities such as $\pi - \pi$ scattering lengths. The results show improvement over the tree level calculation and are consistent with experiment and the results of alternate theoretical approaches such as chiral perturbation theory and lattice gauge theory.

I. INTRODUCTION

The complete elucidation of nuclear dynamics in terms of Quantum Chromodynamics (QCD) is still forthcoming. This elucidation involves a meeting in the middle of two complementary efforts. On the one hand, using mainly symmetry patterns and the wealth of low energy experimental data, nuclear physicists have been able to construct models valid for different regimes and deduce relationships between these models. On the other hand, since the early 1970s it has become increasingly clear that nuclear processes are the emergent phenomena of the dynamics of quarks and gluons (QCD). That these threads can be pursued independently has been demonstrated by the quantitative success of nuclear models without any knowledge of QCD and the subsequent widespread acceptance of QCD largely on the basis of high energy behavior that appears to have little or no bearing on nuclear phenomena. But in the middle there ought to be some confluence.

Indeed, though the details are still murky, QCD does dictate that quarks and gluons are confined within a region of about 1 fermi. Thus the QCD degrees of freedom at lower energies are transformed into the familiar nuclear degrees of freedom. One can then begin with a model of quarks and mesons at the confinement scale and ask what relationship each of the parameters has with QCD parameters. The form of the low energy dynamics is strongly constrained by chiral symmetry and its breaking in the hadronic sector. The small departure from chiral symmetry is a consequence of the smallness of the up and down quark masses with respect to the QCD scale $\Lambda_{QCD} \approx 200-300 MeV$. This can be exploited to construct an effective field theory for low energy QCD or what has been termed chiral perturbation theory ($\chi$PT). Here the picture is based on a sigma model whose parameters are determined phenomenologically. The question arises, however, how to relate the parameters of the effective low energy theory to those of short distance QCD. Intermediate to addressing this problem it is a desideratum to compute the scale dependence of parameters of simple models which incorporate the chiral symmetry breaking patterns of QCD.

In the present work we solve the Gell-Mann–Levy $\sigma$-model nonperturbatively using Renormalization Group (RG) flow equations where projections onto momentum independent couplings at each momentum step are made. This is equivalent to truncating the derivative expansion to leading order (LO), an approximation sometimes referred to as the “local potential approximation”. A small quark mass is introduced at the ultraviolet scale to break chiral symmetry. The free parameters of the model can be constrained by infrared observables allowing modest predictive power. The main result of the calculations, detailed below, are predictions of $\pi - \pi$ scattering lengths which are significant improvements over tree-level calculations and are consistent with experiment as well as with lattice and chiral perturbation theory results. The present paper summarizes many of the results reported in Ref. [4].

The organization of this paper is as follows. In section II we briefly outline the history of the RG approach to field
Theories. In section III, our derivation of the RG flow equations for the $\sigma$-model is sketched. The numerical technique for solving the flow equations is then outlined, our approach to constraining the parameters are discussed, and the results presented in section IV. A summary and some extensions to the model are considered in section VI. Three technical appendices include derivations of results used in the text.

II. REVIEW OF THE RG FLOW EQUATION

The renormalization group equation upon which this work is based was first derived by Wilson and Kogut [5] and Wegner and Houghton [6] in two different but equivalent ways. Wilson and Kogut derived their equation using a so-called “smooth cutoff” between integrated and unintegrated modes. Their rationale was to avoid nonlocalities that naturally arise in position space when one makes sharp divisions in momentum space. The price paid for this approach is that unconstrained smoothing functions must be introduced to facilitate the integrations. These smoothing functions complicate the equations and make their solution—even after drastic approximations (e.g. the leading order (LO) in the derivative expansion)—impossible analytically. There are also unphysical dependencies on the detailed form of the smoothing function. The so-called “sharp-cutoff” method of Wegner and Houghton [6] is simpler in that there are no smoothing functions to contend with, but nonlocal effects are now present. Another problem with this approach is that there appear to be ambiguities in the form of the RG equations that depend on a variable change at a particular point in the derivation [6]. However, at least to leading order (LO) in the derivative expansion, there is no ambiguity with the sharp cutoff approach which we employ in this paper.

One can imagine many different ways to approximate the exact RG equations. A natural approximation scheme involves expanding the action in powers of momentum, or—in real space—in powers of derivatives of the fields. After integrating over field components corresponding to momenta in the UV momentum shell, the leading order (LO) approximation is then obtained by setting all of the remaining components equal to zero except for the uniform component. (This is sometimes called the “local potential approximation” (LPA).) The next to leading order (NLO) result is obtained by keeping one small momentum and so on. The LO approximation to the full RG equations appears first in the work of Nicoll et al. [7].

Hasenfratz and Hasenfratz [8], in an early seminal work, showed that the RG equations approximated using a method equivalent to LO in the derivative expansion give interesting and nontrivial results. They solve for the RG flows of the effective potential in a pure scalar theory and extract critical exponents to compare with calculations performed by other means. For $d = 3$ they find impressive agreement for the critical exponents $\nu$ and $\omega$. Also for $d = 4$ they find no nontrivial fixed point solution which is consistent with the triviality of $\phi^4$ field theories. Their work showed that even the LO approximation gives a rich quantitative description of the critical properties of strongly-coupled scalar field theories.

There has been a recent resurgence of interest in the RG effective action approach to scalar field theories. Hasenfratz and Nager [9] study the cutoff dependence of the Higgs mass using RG methods. Wetterich and co-workers [10, 12], essentially extend the work of Hasenfratz and Hasenfratz to NLO use a smooth cutoff procedure. This allows them to include the effects of wavefunction renormalization and compute the critical exponent $\eta$ which they find to be in rough agreement with other calculations. Morris [13] studies approximations to the exact RG as derived using both smooth and sharp cutoffs and in Ref. 14 computes critical exponents to NLO noting that the scheme appears to converge. In Ref. 15 the so-called sharp cutoff ambiguities are treated in detail and the DE of the effective action is reviewed in 16. Alford [17], with an eye toward the electroweak phase transition, computes exponents with a sharp cutoff procedure and discusses some of the practical difficulties involved with extending the calculations to NLO. In an extensive RG and Monte Carlo analysis of general scalar field theories, Shepard et al. [18] derive “latticized” RG flow equations in LO and demonstrate impressive agreement with Monte Carlo results for a wide variety of cases, broken and unbroken phases, three and four dimensions, and O(1) and O(2) theories. Some related work appears here as Refs. [19], [20], [21].

In addition the inclusion of fermions has been addressed in the literature recently. Maggiore [22] includes fermions via a generalized Yukawa term and derives LO flows for the scalar and generalized Yukawa potentials. He finds no evidence that the fixed point structure of the theory is affected by the addition of fermions. In a more comprehensive study, Clark and co-workers [23] derive the exact RG equations for theories of arbitrary field content and derive from this equation the LO flows for the scalar and generalized Yukawa potentials which agree with the results of Ref. 22. In two related papers Clark et al. study the issue of computing mass bounds for scalars and fermions in the standard model [24] and the stability of fine tuned hierarchies [25]. In related work, Ellwanger and Vergara [26] use RG flow...
equations for generalized NJL models to study the Higgs top quark system to leading order in $1/N_c$. Other work pertaining to the Higgs top system appears as Ref. [27].

In an effort paralleling this work Jungnickel and Wetterich [12] have applied renormalization group methods to the generalized $\sigma$-model. In one sense their work is more ambitious than the present work. They have considered the general SU(N) sector with all allowable functions of the field invariants and included via a smooth cut-off the NLO quantities which allow treatment of the anomalous dimension. But the price paid for such generality is an abundance of free parameters, some non-physical, which are difficult to constrain. In contrast, the present work focuses exclusively on the SU(2) sector which requires only one field invariant, uses a sharp cut-off approach as in Ref. [8] truncated at leading order in the derivative expansion. While this precludes any treatment of the strange-quark sector or the anomalous dimension, there are fewer parameters which can be well constrained by experimental data allowing modest predictive power for the theory.

There is now ample evidence in the literature that the leading order approximation to the RG flow of a field theory incorporates correctly many of the intricacies of strong-coupled quantum field theories. The aim of the present work is to apply this powerful method to the the Gell-Mann–Levy $\sigma$-model in an attempt to understand better low energy nuclear phenomenology.

III. DERIVATION OF THE FLOW EQUATIONS

A. General

We first present a derivation of an exact RG equation for an arbitrarily complicated quantum field theory described by the action, $S^{(A)}[\Phi]$, regulated at some large momentum cutoff $\Lambda$. The field content of the theory can in principle be anything at all, $\Phi = \{\phi, \psi, A_\mu, F_{\mu\nu}, \ldots\}$ i.e. fields described by complex scalars, spinors, vectors and/or tensors, though we will only be concerned with theories containing scalars and spinors. We define the effective action at the momentum scale $\Lambda - \Delta \Lambda$, $S^{(A-\Delta A)}$ as,

$$ e^{-S^{(A-\Delta A)}} = \int_{\text{shell}} D\Phi D\Phi e^{-S^{(A)}}, \quad (1) $$

where the subscript “shell” indicates that only the Fourier components of the fields $\{\Phi_q\}$ with momenta in the shell, $\Lambda - \Delta \Lambda < |q| < \Lambda$ are integrated. Thus the notation for the measure means,

$$ D\Phi D\Phi = \prod_{q_1 \neq 0} d\Phi_{q_1} \prod_{q_2 \neq 0} d\Phi_{q_2}. \quad (2) $$

for $\Lambda - \Delta \Lambda < |q_1|, |q_2| < \Lambda$. $\overline{\Phi}$ is the “conjugate” of $\Phi$ (e.g. complex conjugate for the complex scalar field $\phi$, Dirac adjoint for the Dirac spinor $\psi$, Hermitian conjugate for matrix fields etc.). The actions $S^{(A-\Delta A)}$ and $S^{(A)}$ are considered equivalent in the sense that for $|q| \ll \Lambda$ they each give the same $n$-point functions. We now decompose $\Phi(x)$ into uniform and nonuniform pieces,

$$ \Phi(x) = \Omega_0 + \Omega(x) $$

$$ \Omega(x) = \sum_{q \neq 0} \Omega_q e^{iq\cdot x}. \quad (3) $$

Expanding $S^{(A)}$ in a functional Taylor series about the uniform field components gives,

$$ S^{(A)}[\overline{\Phi}, \Phi] = S^{(A)}[\overline{\Omega}_0, \Omega_0] + \sum_{q \neq 0} \left[ \frac{\delta S}{\delta \overline{\Omega}_q} \right]_{\overline{\Omega}_0} \Omega_q + \left[ \frac{\delta S}{\delta \Omega_q} \right]_{\Omega_0} \overline{\Omega}_q $$

$$ + \sum_{q_1, q_2 \neq 0} \left[ \frac{\delta^2 S}{\delta \Omega_{q_1} \delta \overline{\Omega}_{q_2}} \right]_{\Omega_0} \Omega_{q_1} \overline{\Omega}_{q_2} $$

$$ = S^{(A)}[\overline{\Omega}_0, \Omega_0] + J\overline{\Omega} + \overline{\Omega} J + \overline{\Omega} M \Omega. \quad (4) $$
The subscript zero means that all modes, Ω_q, with q in the shell are set to zero; and the primed summation symbol indicates that only momenta in the shell are included in the sum. For the purposes of deriving differential equations with respect to the independent variable Λ, we will eventually take the limit ∆Λ → 0, thus we can truncate the series after the quadratic term without approximation since all higher order contributions to the RG flows will be at least O(∆Λ^2). In the second equality several definitions have been made:

\[
\mathcal{J} = \begin{pmatrix}
\frac{\delta S}{\delta \Omega q_1} |_0 \\
\frac{\delta S}{\delta \Omega q_2} |_0 \\
\vdots
\end{pmatrix},
\]

and its conjugate are the “generalized source” column vectors and the supermatrix of second derivatives is defined as (see Appendix A for a review of the supermatrix formalism),

\[
M \equiv \left( \sum_A \frac{A F}{\mathcal{F}} \right).
\]

Also, where momentum subscripts are not present, matrix multiplication over the momentum indices is implied, e.g.,

\[
\mathcal{J} \Omega = \left( \frac{\delta S}{\delta \Omega q_1} |_0 \frac{\delta S}{\delta \Omega q_2} |_0 \ldots \right) \begin{pmatrix}
\Omega q_1 \\
\Omega q_2 \\
\vdots
\end{pmatrix} = \sum_{q \neq 0} \frac{\delta S}{\delta \Omega q} \bigg|_0 \Omega q.
\]

Now substituting Eq.(5) into Eq.(4) gives,

\[
e^{-S(\Lambda - \Delta \Lambda)} = e^{-S(\Lambda)} |_0 \int D\Omega D\Omega e^{-\left(\Omega M \Omega + \mathcal{J} \Omega + \overline{\mathcal{J}}\right)}
\]

\[
= e^{-S(\Lambda)} |_0 \left\{ e^{\mathcal{J} M^{-1} \mathcal{J}} \right\} s\text{det}^{-1} M.
\]

Irrelevant constant factors have been dropped wherever they appear. Using the identity,

\[
s\text{det}^{-1} M = e^{-\text{str} \ln M},
\]

the exact RG equation for this generalized system is easily obtained as,

\[
S^{(\Lambda - \Delta \Lambda)} = S^{(\Lambda)} |_0 + \text{str} \ln M - \mathcal{J} M^{-1} \mathcal{J} + O(\Delta \Lambda^2).
\]

This equation relates the action at momentum scale Λ - ∆Λ to the action at momentum scale Λ. We will not convert it directly into a functional differential equation (as Clark et al. [23] do) until we consider its form for a particular action. A terminology has grown up around equations such as (10). The “str ln” term is generally referred to as the “loop” term and the “\(\mathcal{J} M^{-1} \mathcal{J}\)” term is referred to as the “tree” term. This is because the contribution from the \(\mathcal{J} M^{-1} \mathcal{J}\) term is present in mean field theory whereas the contribution from the str ln term includes effects from loop integrations [29].

For the purposes of treating the σ-model we consider only real scalar and spinor fields (i.e. Φ = \{φ, ψ\}). If the flow equations are truncated to LO in the derivative expansion, many simplifications ensue. The source column vector becomes,

\[
\mathcal{J} = \begin{pmatrix}
J_{q_1} \\
J_{q_2} \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\eta_{q_1} \\
\eta_{q_2} \\
\vdots
\end{pmatrix},
\]

\[
\hline
4
\hline
where,

\[ J^*_q = \delta S \bigg|_{\delta \phi_q \rightarrow 0} \]
\[ \pi^*_q = \delta S \bigg|_{\delta \psi_q \rightarrow 0}. \]

(12)

The matrix of second derivatives is,

\[
M = \begin{pmatrix}
\Sigma & \mathcal{A} \\
\mathcal{A}^T & -\mathcal{F}
\end{pmatrix}
= \begin{pmatrix}
\delta^2 S \bigg|_{\delta \phi_q \rightarrow 0} & 0 \\
0 & \delta^2 S \bigg|_{\delta \psi_q \rightarrow 0}
\end{pmatrix}
\]

(13)

Now we can rewrite the superdeterminant appearing in Eq. (10) using some tricks from Appendix A (compare with Eq. (11)),

\[ \text{sdet}^{-1} M = (\text{det}^{-1/2} N)(\text{det} \mathcal{F}) \]  

(14)

(\text{where } N \equiv \Sigma - \mathcal{A} \mathcal{F}^{-1} \mathcal{A}) \text{ which gives,}

\[ \text{strln}M = \frac{1}{2} \text{trln}N - \text{trln} \mathcal{F}. \]

(15)

(The 1/2 coming from the counting in the momentum sums due to the real \( \phi \).) Now Eq. (10) becomes,

\[ S^{(\Lambda - \Delta \Lambda)} = S^{(\Lambda)} \bigg|_0 + \frac{1}{2} \text{trln}N - \text{trln} \mathcal{F} - \mathcal{J} M^{-1} \mathcal{J}. \]

(16)

Restricted to LO in derivatives the last ("tree") term in this equation may be neglected. One way to see this diagrammatically is to note that any particular contribution to the tree term in Eq. (13) or (14) will be comprised of external legs connected by a single internal propagator with momentum in the shell (see Fig. 1). Since to LO we set the momentum of all external legs to zero, these diagrams cannot conserve momentum at their vertices and therefore vanish. To make the argument analytically we write a particular term of \( S'_q \),

\[ \delta S \bigg|_{\delta \phi_{q'}} = \cdots + \frac{1}{2!} V''' \sum_{q_1, q_2 \neq 0} \phi_{q_1} \phi_{q_2} \delta_{q_1 + q_2 + q', 0} + \cdots, \]

(17)

where \( q' \) is in the shell. When, in going to LO we set all field components, \( \phi_q \), with nonzero momentum to zero, it’s clear that this term and all higher order terms will vanish. Thus we write Eq. (10) as,

\[ S^{(\Lambda - \Delta \Lambda)} = S^{(\Lambda)} \bigg|_0 - \frac{1}{2} \text{trln}N + \text{trln} \mathcal{F}. \]

(18)

This is the form of the RG flow equation that we use for our calculation in the present work.

**B. \( \sigma \)-model Flow Equations**

We now apply the results of the previous section to derive the flow equations for the \( \sigma \)-model. The incorporation of chiral symmetry breaking brings a substantial price in the complexity of the algebra. Many of the tedious details are relegated to the Appendices. Ref. [4] includes many details not related in this paper.

The Euclidean action for the \( \sigma \)-model is:
\[
S^{(\Lambda)} = \int d^4x \left[ V^{(\Lambda)}(\rho, \sigma) + \frac{1}{2} (\partial_{\mu} \phi^a)^2 + \overline{\psi} [G + U^{(\Lambda)}(\rho, \sigma, \mathcal{G})] \psi \right].
\]  

(19)

A few definitions to compactify the notation have been made:

\[
\phi^a = \left( \frac{\sigma}{\pi^i} \right), \quad a = 0, i; \quad i = 1, 2, \ldots, N
\]

\[
\Gamma^a = \left( i \gamma^5 \rho^i \right),
\]

\[
\mathcal{G} = \Gamma^a \phi^a = \sigma + i \gamma^5 \sigma \cdot \pi,
\]

\[
\rho^2 = \mathcal{G} \mathcal{G}^\dagger = (\phi^a)^2 = \sigma^2 + (\pi)^2,
\]

(20)

where the last term is the only field invariant when specifying \( \Gamma^a \) only to the \( N = 4 \) or \( O(4) \) case. We will often use the more general notation but will only be considering \( O(4) \) in this work. Recall that \( SU(2) \times SU(2) \) rotations of the fermions induce \( O(4) \) rotations in the boson sector. The scalar sector of the \( \sigma \)-model then is a four component field. The potentials \( V^{(\Lambda)}(\rho, \sigma) \) and \( U^{(\Lambda)}(\rho, \sigma, \mathcal{G}) \) include terms proportional to powers of the zeroth component of the scalar field, \( \sigma \), which will break chiral symmetry. To keep track of the symmetry breaking as the theory flows from high to low momentum we expand the potentials in the form,

\[
V(\rho, \sigma) = V_0(\rho) + \sigma V_1(\rho) + \frac{\sigma^2}{2} V_2(\rho) + \cdots
\]

\[
U(\rho, \sigma, \mathcal{G}) = m(\rho, \sigma) + \mathcal{G} g(\rho, \sigma)
\]

(21)

where \( m(\rho, \sigma) \) and \( g(\rho, \sigma) \) are expanded similarly to \( V(\rho, \sigma) \). Each of the functions \( V_k, m_k \) and \( g_k \) for \( k = 0, 1, 2 \) is \( SU(2) \times SU(2) \) symmetric since they only depend on \( \rho \). Chiral symmetry is broken by \( V_1, V_2, g_1, g_2 \) and \( m(\rho, \sigma) \).

The LO flow of the action Eq.(19) is given by Eq.(18) rewritten as,

\[
S^{(\Lambda, -\Delta \Lambda)} = S^{(\Lambda)} \bigg|_0 - \text{trln} \mathbb{N} + \text{trln} \mathbb{F}.
\]

(22)

where \( \mathbb{N} \) has components \( N^{ab} \),

\[
N^{ab} \equiv \Sigma^{ab} - \mathcal{A}^{a} \mathcal{F}^{-1} \mathcal{A}^{b}
\]

(23)

\[
\begin{pmatrix}
\Sigma \\
\mathcal{A} \\
\mathcal{F}
\end{pmatrix} =
\begin{pmatrix}
\frac{\delta^2 S}{\delta \phi_{b} \delta \phi_{a}} & 0 \\
0 & \frac{\delta^2 S}{\delta \psi_i \delta \phi_{a}} \\
0 & 0 & \frac{\delta^2 S}{\delta \psi_i \delta \psi_i}
\end{pmatrix}.
\]

The extra indices now refer to the components of the scalar field. We Taylor expand the potentials,

\[
V^{(\Lambda)}(\phi^a) = V^{(\Lambda)}(\phi^a_0) + V^{(\Lambda)}(\phi^a_0) \phi^a + \frac{1}{2} V^{(\Lambda)}(\phi^a_0) \phi^a + \cdots
\]

(24)

(and similarly for \( U \)). With normalization

\[
\int d^4x \ e^{iq \cdot x} = (\text{Vol}) \delta_{q,0}
\]

(25)

we get,

\[
\frac{S^{(\Lambda)}}{\text{Vol}} = V^{(\Lambda)}(\phi^a_0) + \overline{\psi}_0 U^{(\Lambda)}(\phi^a_0) \psi_0
\]

\[
+ \sum_{q \neq 0} \left[ \frac{1}{2} \phi^a_{-q} \left[ g^{ab} q^2 + V^{(\Lambda)}(\phi^a_0) \right] \phi^b_{q} + \overline{\psi}_{-q} \left[ i \gamma^5 + U^{(\Lambda)}(\phi^a_0) \right] \psi^a_{q} + \overline{\psi}^a_{-q} \phi^a_{q} \psi^a_0 \right] + \cdots
\]

(26)
From this expression we can compute all the matrices in Eq.(22),

$$\Sigma^{ab} = \delta^{ab} q^2 + V'^{ab}(\phi_0^a)$$

$$\Omega^{ab} = U'^{ab} - \frac{2U'^a U'^b}{iq + U}$$

$$\frac{F}{\text{Vol}} = \frac{1}{2}(i\dot{q} + U)$$

$$\frac{N^{ab}}{\text{Vol}} = \Sigma^{ab} + \psi_0 \Omega^{ab} \psi_0.$$  \hspace{1cm} (27)

With these definitions Eq.(22) becomes,

$$\frac{S^{(\Lambda-\Delta\Lambda)}}{\text{Vol}} = V^{(\Lambda-\Delta\Lambda)}(\phi_0^a) + \psi_0 U^{(\Lambda-\Delta\Lambda)}(\phi_0^a) \psi_0$$

$$= V^{(\Lambda)}(\phi_0^a) + \psi_0 U^{(\Lambda)}(\phi_0^a) \psi_0$$

$$- \frac{1}{2\text{Vol}} \text{tr} \ln \left[ \Sigma^{ab} + \psi_0 \Omega^{ab} \psi_0 \right]$$

$$+ \frac{1}{\text{Vol}} \text{tr} \ln (i\dot{q} + U^{(\Lambda)}(\phi_0^a)) + \cdots.$$  \hspace{1cm} (28)

We can write Eq.(28) as two coupled equations for the flow of $V$ and $U$. Before doing so, we make a number of modifications. First, we can write,

$$\frac{1}{i\dot{q} + U} \simeq \frac{U^\dagger}{D_F},$$

$$D_F \equiv q^2 + U U^\dagger = q^2 + m^2 + 2\sigma mg + \rho^2 g^2,$$  \hspace{1cm} (29)

where we’ve dropped the term proportional to $\dot{q}$ since it will vanish at LO when we take angle averages. Second, we can write the trace over the inverse fermion propagator term as,

$$\text{tr} \ln (i\dot{q} + U) = \frac{1}{2} n_f c_d \sum_q \ln D_F$$  \hspace{1cm} (30)

with $c_d = 2^{d/2}(2^{(d-1)/2})$ for $d$ even (odd), using the identity $|i\dot{q} + U|^2 = q^2 + U^2$; the $n_f c_d \sum_q$ factor comes from the trace over the flavor and Dirac indices. Also we can write,

$$\text{tr} \ln \left[ \Sigma^{ab} + \psi_0 \left( \Omega^{ab} \right) \psi_0 \right] = \ln \det \Sigma^{ab} + \text{tr} \left[ (\Sigma^{ac})^{-1} \psi_0 \Omega^{cb} \psi_0 \right],$$  \hspace{1cm} (31)

where the trace in the last term is only over the $ab$ indices. Putting all this together, we can write the $V$ and $U$ equations from Eq.(28) as,

$$V^{(\Lambda-\Delta\Lambda)} = V^{(\Lambda)} - \frac{1}{2\text{Vol}} \sum_q (\ln \det \Sigma - n_f c_d \ln D_F)$$

$$U^{(\Lambda-\Delta\Lambda)} = U^{(\Lambda)} - \frac{1}{2\text{Vol}} \sum_q \text{tr} \Sigma^{-1} \cdot \Omega.$$  \hspace{1cm} (32)

Now taking the limit $\Delta\Lambda \to 0$ we have,

$$\Lambda \frac{\partial V^{(\Lambda)}}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \left( \ln \det \Sigma - n_f c_d \ln D_F \right)$$

$$\Lambda \frac{\partial U^{(\Lambda)}}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \left( \Sigma^{-1} \cdot \Omega \right).$$  \hspace{1cm} (33)
where the determinant and trace are only over the $ab$ indices. In Eqs. (33) and (34),

$$\Sigma^{ab} = \delta^{ab} \Lambda^2 + V^{''ab}$$

$$\Omega^{ab} = U^{''ab} - 2U^{'a}U^{b'}} \frac{D_F}{\Lambda^2 + m^2 + \rho^2 + 2 \sigma mg.} \quad (35)$$

The matrices $V^{''ab}$, $U^{''ab}$, and $U^{a}$ are worked out in terms of derivatives with respect to $\rho$ in Appendix B. One glance at these expressions is adequate to impress the reader of the proliferation of algebraic complication in the extension of the simple Yukawa coupled fermions [4,22,23] to the case of broken chiral symmetry. The matrices can be simplified by a similarity transformation, however, since the determinant and the trace are invariant with respect to such transformations. As discussed in Appendix B, when similarity transformed, $\Sigma$ has only six nonzero elements,

$$\Sigma' = S \cdot \Sigma \cdot S^T = \begin{pmatrix} \Sigma_{00} & \Sigma_{01} & 0 & 0 \\ \Sigma_{10} & \Sigma_{11} & 0 & 0 \\ 0 & 0 & \Sigma_{22} & 0 \\ 0 & 0 & 0 & \Sigma_{33} \end{pmatrix}, \quad (36)$$

but $\Omega'$ still has every element nonzero although each of the elements is somewhat simpler. Notice, however, that since only the combination $tr \Sigma'^{-1} \cdot \Omega'$ appears in the flow equations, only six elements of $\Omega'$ will contribute so that,

$$tr \Sigma'^{-1} \cdot \Omega' = (\Sigma^{00})^{-1} \Omega^{00} + (\Sigma^{10})^{-1} \Omega^{01} + (\Sigma^{01})^{-1} \Omega^{01} + (\Sigma^{11})^{-1} \Omega^{11} + (\Sigma^{22})^{-1} \Omega^{22} + (\Sigma^{33})^{-1} \Omega^{33}, \quad (37)$$

where, e.g., $(\Sigma^{00})^{-1}$ means the 00 element of $\Sigma^{-1}$.

We can now consider the “chiral limit” where we set all chiral breaking terms $(m(\rho, \sigma), V_1, V_2, g_1$ and $g_2)$ to zero. Then,

$$\Sigma^{ab} \rightarrow (\Lambda^2 + V \rho V_0/\rho) \delta^{ab} + (V^{''0} - V^{0'}/\rho) \frac{\phi \phi^b}{\rho^2}, \quad (38)$$

which becomes, after a similarity transformation,

$$\Sigma'^{ab} = \begin{pmatrix} D_\phi & 0 & 0 & 0 \\ 0 & D_\pi & 0 & 0 \\ 0 & 0 & D_\pi & 0 \\ 0 & 0 & 0 & D_\pi \end{pmatrix}, \quad (39)$$

where $D_\phi = \Lambda^2 + V^{''0}$ and $D_\pi = \Lambda^2 + V^{0'}/\rho$. Thus,

$$(\Sigma'^{ab})^{-1} = \begin{pmatrix} D_\phi^{-1} & 0 & 0 & 0 \\ 0 & D_\pi^{-1} & 0 & 0 \\ 0 & 0 & D_\pi^{-1} & 0 \\ 0 & 0 & 0 & D_\pi^{-1} \end{pmatrix}, \quad (40)$$

and so in the chiral limit only the diagonal elements of $\Omega$ will contribute to the trace term. These can be shown to be,

$$\Omega^{00} = \mathcal{G}(g^{''0} + 2\rho g^{'}_0 \rho - \frac{2g^{'}_0}{D_F}(g^{2}_0 + 2g^{2}_0 \rho^2 + \rho^2 y^{2}_0)]$$

$$\Omega^{11} = \Omega^{22} = \Omega^{33} = \mathcal{G}(\frac{g^{'}_0}{\rho} + \frac{2g^{3}_0}{D_F}). \quad (41)$$

Thus from Eqs. (33) and (34) the $O(4)$ flow equations in the chiral limit are,
\[ \Lambda \frac{\partial V_0}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \left( \ln D_\phi + 3 \ln D_\pi - n_f c_d \ln D_F \right) \]  \hspace{1cm} (42)

\[ \Lambda \frac{\partial g_0}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \left\{ \frac{1}{D_\phi} \left[ g''_0 + \frac{2g'}{\rho} - \frac{2g_0}{D_F} (g_0^3 + 2g_0^2 \rho^2 + \rho^2 g_0^2) \right] + \frac{3}{D\pi} \left( \frac{g'}{\rho} + \frac{2g_0^3}{D_F} \right) \right\}. \]  \hspace{1cm} (43)

For \( O(N) \) symmetric theories there would be an \( N-1 \) in place of the 3s in front of the \( \ln D_\pi \) and the \( (1/D_\pi)(\cdots) \) terms. In Ref. [4,8,18], the flow equation for an \( O(N) \) scalar-only field theory is derived and is equivalent to Eq.(42) with \( n_f = 0 \) and \( N = 4 \). Also Eqs.(42,43) represent a generalization of the flow equations for the simple Yukawa coupled fermion models derived in Refs. [4,22,23] to a model with both scalar and pseudoscalar bosons coupled to fermions.

### IV. NUMERICAL RESULTS

We now discuss the numerical solution of the LO \( \sigma \)-model RG flow equations, (33) and (34). As mentioned in Appendix B, these equations is quite lengthy. Since we keep only \( O(\sigma) \) (first order in the chiral symmetry breaking parameter) in Eq. (21), there are six coupled flow equations, which for

\[ \det \Sigma' = F_0(\rho) + \sigma F_1(\rho) \]

\[ \text{tr } \Sigma'^{-1} \cdot \Omega = \Delta m_0(\rho) + G \Delta g_0(\rho) + \sigma \left[ \Delta m_1(\rho) + G \Delta g_1(\rho) \right] \]  \hspace{1cm} (44)

take the form:

\[ \Lambda \frac{\partial V_0}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \left( \ln F_0(\rho) - n_f c_d \ln D_F \right) \]

\[ \Lambda \frac{\partial V_1}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \frac{F_1(\rho)}{F_0(\rho)} \]

\[ \Lambda \frac{\partial m_0}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \Delta m_0(\rho) \]

\[ \Lambda \frac{\partial m_1}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \Delta m_1(\rho) \]

\[ \Lambda \frac{\partial g_0}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \Delta g_0(\rho) \]

\[ \Lambda \frac{\partial g_1}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^d \Delta g_1(\rho). \]  \hspace{1cm} (45)

Symbolic computing (we used Mathematica) greatly facilitates the determination of the functions \( F_0, F_1, \Delta m_0 \) etc. which are too lengthy to be reproduced here [‡]. (\( \Delta g_1(\rho) \) and \( \Delta m_1(\rho) \), for instance, require several pages of output!). Once these functions are determined, however, the numerical solution of Eqs.(45) proceeds exactly as in the scalar only case as reported, e.g., in Refs [8,18] except with six functions instead of two. There are a number of nonphysical subtleties associated with the numerical solution of Eq.(45) (e.g. the domain limits, the algorithm used to solve the equations, and the number of terms retained in the polynomial fits, etc.) These nonphysical sensitivities yield a small spread of output values for a given input. At present we have three independently constructed codes to solve the equations which give essentially the same results which gives us some confidence in the solutions presented here.

We set our “initial condition” by specifying the value of the parameters at the UV scale:

\[ V^{(\Lambda_0)}(\rho, \sigma) = \frac{1}{2} \mu_0^2 \rho^2 + \frac{1}{4} \lambda_0 \rho^4 \]

\[ U^{(\Lambda_0)}(\rho, \sigma, G) = m_q^0 + g_0 G. \]  \hspace{1cm} (46)

Then we expand in powers of \( \rho \),
\[ V_k^{(\Lambda)}(\rho) = \sum_{i=1}^{M} \frac{1}{2i} v_i^{(2i)}(\Lambda) \rho^{2i} \]
\[ m_k^{(\Lambda)}(\rho) = \xi_i^{(0)}(\Lambda) + \sum_{i=1}^{M} \frac{1}{2i} \xi_i^{(2i)}(\Lambda) \rho^{2i} \]
\[ g_k^{(\Lambda)}(\rho) = g_i^{(0)}(\Lambda) + \sum_{i=1}^{M} \frac{1}{2i} g_i^{(2i)}(\Lambda) \rho^{2i} \],

for \( k = 0, 1, 2 \) (see Eq. (2)). Thus at the UV scale we have

\[ v_0^{(2)}(\Lambda_0) = \mu_0^2 \]
\[ v_0^{(4)}(\Lambda_0) = \lambda_0 \]
\[ \xi_0^{(0)}(\Lambda_0) = m_0^q \]
\[ g_0^{(0)}(\Lambda_0) = g_0 \] (48)

with all higher order coefficients for \( k = 1, 2 \) set to zero. Just as in the previous section we perform a fit of the functions \( V_k, m_k, \) and \( g_k \) to a power series in \( \rho \) at each \( \Lambda \) step. Also, since we’re interested in spontaneous symmetry breaking, we will have the parameter \( f_\pi = <\phi>_{vac} = \sigma \), which sets the scale of the symmetry breaking. Thus we consider the set \( \{ m_\sigma, f_\pi, \mu_0^2, \lambda_0, m_0^q, g_0, \lambda_0, \Lambda_{IR} \} \) the input parameters to the model. This approach is philosophically different from that taken by Jungnickel and Wetterich (12) where \( f_\pi \) was the quantity they wished to predict.

The basic approach adopted in the present work for constraining the phenomenology is as follows. Since there are as yet no experimental or reliable theoretical constraints at the UV scale, we are forced to look to observables at the IR scale to determine the free parameters of the model. Perhaps the two parameters in the model most tightly constrained by low energy data are \( m_\sigma \) and \( f_\pi \). Thus we will tune other parameters of the model to get \( m_\sigma \) and \( f_\pi \) at their experimental values at the infrared scale. This still leaves four out of the six parameters unconstrained, and we must decide which out of \( \{ \mu_0^2, \lambda_0, m_0^q, g_0, \lambda_0, \Lambda_{IR} \} \) to fix and which to tune to \( m_\sigma \) and \( f_\pi \). Results for scalar-only calculations (3) as well as for Yukawa coupled fermions (12, 22, 23) indicate that the two parameters \( \lambda_0 \) and \( \mu_0^2 \) are not truly independent, i.e. we can tune to particular values of IR parameters with large number of values for \( \lambda_0 \) and \( \mu_0^2 \) so long as \( \lambda_0 > 0 \) and large. Thus we fix \( \lambda_0 = 10 \) and use \( \mu_0 \) to tune the IR parameters. From our experience choosing a different value of \( \lambda_0 \) will only require a re-tuning of \( \mu_0 \) to obtain equivalent results. Next there is the bare current quark mass, \( m_0^q \). Experimentally, there is about \( \pm 5 \) MeV spread of the values for \( m_u \) and \( m_d \) (30). We will tune \( m_0^q \) to get the average of the means of these values at the IR scale, i.e. \( m_0^{(\Lambda_{IR})} = m_q \equiv \frac{1}{2}(m_u + m_d) = 7.5 \) MeV. We also have the two cut-off scales \( \Lambda_0 \) and \( \Lambda_{IR} \), only one of which need be tuned with the other fixed since the RG equations only care about the ratio. We fix \( \Lambda_{IR} = m_{\pi}\bar{\pi} \approx 140 \) MeV and will allow \( \Lambda_0 \) to be adjusted. This leaves only \( g_0 \). We know that the quark-meson coupling is strong at these energy scales and the Goldberger-Treiman relation for the constituent quark model gives \( g \approx M_{\text{nucleon}}/3f_\pi \approx 3.366 \). But this is not an appropriate constraint in our zero-density model since we have no nucleons. We therefore perform three fits, one each for \( g >, \approx, < 3.366 \) and compare the results. To summarize, for each of these values of \( g_0 \) we tune \( \{ \mu_0, m_0^q, \Lambda_0 \} \) to \( \{ m_\sigma, f_\pi, m_q \} \). Once all the parameters are fixed, the model predicts other IR quantities such as \( m_\sigma \) (sigma mass), \( \lambda_{4-\text{pt}}, \lambda_{3-\text{pt}} \) (the sigma four- and three-point couplings), \( g, a_0^2, \) and \( a_2^2 \). Not all these quantities have experimental determinations however; the quantities best determined by experiment are the \( \pi\pi \) scattering lengths \( a_0^2 \) and \( a_2^2 \) discussed in more detail later (see Appendix C for background).

The results of the three fits for \( g >, \approx, < 3.366 \) are displayed in Table 3. The middle column is the result for fitting \( g_0 \) to give \( g \approx 3.36 \); the right and left columns are the results for arbitrarily choosing \( g_0 = 2.5 \) and 3.100 respectively. The last three rows (in the top section) contain the actual fits to \( m_\sigma, f_\pi, \) and \( m_q \). The point was to get \( \{ m_\sigma, f_\pi, m_q \} \approx (140 \text{ MeV}, 92.4 \text{ MeV}, 7.5 \text{ MeV}) \) (30). The \( \parallel \) and \( \perp \) indicate that \( m_0^q \) and \( \{ \mu_0, \Lambda_0 \} \) were used to fix \( m_q \) and \( \{ m_\sigma, f_\pi \} \). The values of \( m_0^q \) and \( m_q \) all fall within the uncertainty quoted in the particle data book (30). Rows 9 through 15 represent some of the predictions of the calculation. Since the sigma is less a “particle” and more a broad resonance, the values for \( m_\sigma, \lambda_{4-\text{pt}}, \) and \( \lambda_{3-\text{pt}} \) are hard to compare quantitatively with experiment. The values for the scalar density, \( <\bar{\psi}\psi> \) are close to other calculated values of \( <\bar{\psi}\psi> \sim -[(240 \pm 25) \text{MeV}]^2 \) (51). In addition the adjusted values of \( \Lambda_0 \sim 940 \) MeV are in the range expected.

Fig. 3 shows the boson potentials as a function of \( \rho \). As expected for weakly broken chiral symmetry, the first order term, \( V_1 \) is just a small correction. One can see clearly that the minimum is at \( \rho = <\sigma>_{vac} = f_\pi \approx 93 \) MeV (80).
Figs. 3 and 4 display the other functions computed in the model. At $\rho \approx 93$ MeV the values of $m$ and $g$ are just the IR values quoted in Table I. The relatively small contribution from $g_1$ and $m$ justify *ex post facto* our expansion in the (small) chiral symmetry breaking parameter $\sigma$ (Eq. (21)).

V. $\pi\pi$ SCATTERING LENGTHS

Another set of predictions from the RG solution of the $\sigma$-model come in the form of the parametrization of low energy $\pi\pi$ scattering. The expansion of the real part of the partial-wave amplitude can be written as

$$\text{Re } A^I_l(s) = 32\pi \left( \frac{q^2}{m^2_{\pi}} \right)^l \left( a^I_l + b^I_l \frac{q^2}{m^2_{\pi}} + \cdots \right),$$

(50)

where $I = 0, 1, 2$ denotes the isospin channel and $l$ is the partial wave index. (See, e.g., Refs. [32] section VI-4 and [34]. Appendix C contains a review of the perturbative calculation and the connection to our model.) So for low energy ($q^2 \ll m^2_{\pi}$) scattering $a^I_l$ and $b^I_l$ will be the most relevant quantities to study. Table I displays a comparison of our three fits and a number of different calculations and experimental values for $a^0_l$ and $a^2_l$ in dimensions of inverse pion mass. Also, the quantity $2a^0_l - 5a^2_l$ is included since for $s$-wave scattering it provides a constraint [2]. Each of the three fits give results that are consistent with experiment and with $\chi$PT and lattice QCD calculations. We discuss how the $\pi\pi$ scattering lengths are computed in our model in Appendix C.

VI. SUMMARY AND POSSIBLE FUTURE EXTENSIONS OF THE MODEL

We present a nonperturbative solution of the $\sigma$-model using the a sharp-cutoff RG equation truncated to LO in the derivative expansion. The model mimics the chiral symmetry of QCD. An important feature of our approach is that we can straight-forwardly track the chiral-symmetry-breaking. Including a small quark mass at the UV scale, we can follow the chiral symmetry breaking in the numerical solution as the scale is lowered. The parameters of the model can be constrained by low-energy data allowing predictions of, for example, $\pi - \pi$ scattering lengths. The values for $\pi\pi$ scattering lengths obtained in this way show an improvement over the perturbative calculation [2] and are essentially consistent with experiment [34] and other nonperturbative calculations [35,36]. It might be added that these calculations were performed at a substantially lower computational cost than those of Refs. [35,36]. This work contrasts with that of Ref. [12] in that we confine our attention to the $SU(2)\times SU(2)$ sector, use a sharp cut-off approach truncated at LO, and allow for explicit treatment of the chiral symmetry breaking via an expansion in the vacuum expectation value of the $\sigma$-field. We also adopt a different philosophy as regards the phenomenology constraining the free parameters.

There are a number of possible extensions to the present calculation. Perhaps the easiest is the extension to $O(\sigma^2)$. Indeed much of the analytical work has already been done with the expected result that, for small current quark masses, the second order potentials will be small corrections to the first order, testifying to the convergence of the method. We have performed $O(\sigma^2)$ calculations with only the bosonic potentials flowing (i.e. fixing $m(\rho) = m_{\eta}$ and $g(\rho) = g_0$ for all $\Lambda$) and confirmed that $V_2$ is small compared to $V_1$. The results for the other calculated parameters in the model are not expected to change substantially at $O(\sigma^2)$ since $m_{\eta}^0$ is small.

Another possible extension would be to allow for finite density which would permit an analysis of nuclear phenomena. Crudely, at finite density, there would be a momentum scale, $k_F$, below which the fermion parts of the flow equations would cease to contribute due to Pauli blocking, while the boson parts would still contribute to the flow. Such crude calculations using this show the qualitative restoration of chiral symmetry as $k_F$ increases. A proper handling of finite density with the RG, however, for relativistic field theories requires much more care so we defer offering any conclusions at the present time.

Yet another extension is to incorporate strangeness into the model by extending to $SU(3) \times SU(3)$. This requires the introduction of two more field invariants as treated in Ref. [12]. In this case, since $m_s \sim \Lambda_{QCD}$, chiral $SU(3) \times SU(3)$ is *strongly* broken and the expansion in the chiral symmetry breaking field may not converge rapidly enough and some other approach to following the chiral symmetry breaking may be required.
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APPENDIX A: SUPERMATRIX FORMALISM

In this appendix we review the supermatrix formalism. A “supermatrix” is a square matrix of the form,

$$M = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix},$$

(A1)

where the square submatrices $M_{BB}$ and $M_{FF}$ are both even elements of the Grassmann algebra while $M_{BF}$ and $M_{FB}$ are odd elements of the Grassmann algebra.

(Grassmann or anticommuting variables allow the incorporation of Fermi statistics into the path integral formalism of quantum field theory. An arbitrary function of a Grassmann variable can be Taylor expanded as

$$f(\theta) = \alpha + \beta \theta$$

(A2)

where $\alpha$ and $\beta$ are normal numbers $\theta$ obeys the Grassmann algebra,

$$\{\theta, \theta\} = \theta \theta + \theta \theta = 0 \Rightarrow \theta^2 = 0$$

(A3)

which is why the Taylor series terminates. For more details see, e.g., Ref. [37] p.214-219.)

The supertrace is defined as

$$\text{str} M = \text{tr} M_{BB} - \text{tr} M_{FF};$$

(A4)

so that the familiar commutative property of the normal trace still holds for the supertrace. The superdeterminant is defined as

$$\text{sdet} M = e^{\text{str} \ln M},$$

(A5)

which preserves the familiar property of determinants, $\text{det} MN = \text{det} M \text{det} N$.

Now consider the decomposition of $M$:

$$M = \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} \begin{pmatrix} 1 & M_{BB}^{-1} M_{BF} \\ 0 & N_{FF} \end{pmatrix}$$

(A6)

where $N_{FF} = M_{BF} - M_{FB} M_{BB}^{-1} M_{BF}$. Now it’s easy to show

$$\text{sdet} \begin{pmatrix} M_{BB} & 0 \\ M_{FB} & 1 \end{pmatrix} = \text{det} M_{BB}$$

(A7)

and,

$$\text{sdet} \begin{pmatrix} 1 & M_{BB}^{-1} M_{BF} \\ 0 & N_{FF} \end{pmatrix} = \text{det} N_{FF}$$

(A8)

so that,

$$\text{sdet} M = \text{det} M_{BB} \text{det}^{-1} N_{FF}.$$  

(A9)
Similarly we could have chosen the decomposition

\[
M = \left( \begin{array}{cc}
N_{BB} & M_{BF}M_{FF}^{-1} \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
M_{FB} & M_{FF}
\end{array} \right),
\]

(A10)

where \( N_{BB} = M_{BB} - M_{BF}M_{FF}^{-1}M_{FB} \). Then we would be led to

\[
\text{sdet}\, M = \det N_{BB}\text{det}^{-1}M_{FF}.
\]

(A11)

**APPENDIX B: DERIVATIVES OF THE CHIRAL FUNCTIONS**

In the derivation of the flow equations for the \( \sigma \)-model, derivatives with respect to \( \phi^a \) of functions of \( \rho, \sigma \), and \( G = \Gamma^a\phi^a \) are taken. In this appendix we work out what these derivatives are. Consider the potential

\[
U(\rho, \sigma, G) = m(\rho, \sigma) + Gg(\rho, \sigma)
\]

(B1)

where,

\[
m(\rho, \sigma) = m_0(\rho) + \sigma m_1(\rho) + \frac{\sigma^2}{2} m_2(\rho) + \cdots
\]

(B2)

and similarly for \( g \). Now

\[
\frac{\partial U}{\partial \phi^a} \equiv U'{}^a = m'{}^a + Gg'{}^a + \Gamma^a g
\]

\[
\frac{\partial^2 U}{\partial \phi^a \partial \phi^b} \equiv U''{}^{ab} = m''{}^{ab} + Gg''{}^{ab} + \Gamma^a \Gamma^b.
\]

(B3)

We need to work out the derivatives of functions of \( \rho \) and \( \sigma \). Consider \( f(\rho, \sigma), f = m \) or \( g \):

\[
\frac{\partial f}{\partial \phi^a} \equiv f'{}^a = f^{(0)}'{}^a + \sigma f^{(1)}'{}^a + \frac{\sigma^2}{2} f^{(2)}'{}^a + \delta^{ab} f^{(1)} + \sigma f^{(2)}
\]

\[
= \frac{\phi^a}{\rho} f' + \delta^{ab} f^{(1)} + \sigma f^{(2)}
\]

\[
\frac{\partial^2 f}{\partial \phi^a \partial \phi^b} \equiv f''{}^{ab} = f^{(0)}''{}^{ab} + \sigma f^{(1)}''{}^{ab} + \frac{\sigma^2}{2} f^{(2)}''{}^{ab} + \delta^{ab} f^{(1)} + \sigma f^{(2)}b
\]

\[
+ (f^{(1)}' + \sigma f^{(2)}') \delta^{ab} + \delta^{ab} \delta^{bc} f^{(2)}
\]

\[
= \delta^{ab} \frac{f'}{\rho} + \frac{\phi^a \phi^b}{\rho^2} f'' + \delta^{ab} \delta^{bc} f^{(2)}.
\]

(B4)

In the last of each of the above equalities we’ve used

\[
h'{}^a = \frac{\phi^a}{\rho} h'(
\]

\[
h''{}^{ab} = \delta^{ab} \frac{h'}{\rho} + \frac{\phi^a \phi^b}{\rho^2} (h'' - \frac{h'}{\rho})
\]

(B5)

where \( h = h(\rho), h = m^{(i)} \) or \( g^{(i)} \) for \( i = 0, 1, 2 \). Also
\[ f' (\rho) = f^{(0)'} + \sigma f^{(1)'} + \frac{\sigma^2}{2} f^{(2)'} \]
\[ f'' (\rho) = f^{(0)''} + \sigma f^{(1)''} + \frac{\sigma^2}{2} f^{(2)''}. \]

With these derivatives, we can compute the matrices in section III.B:

\[ \Sigma^{ab} \equiv \Lambda^2 \delta^{ab} + V^{''ab}(\rho, \sigma) \]
\[ = \left( \Lambda^2 + \frac{V'}{\rho} \right) \delta^{ab} + \left( V'' - \frac{V'}{\rho} \right) \frac{\delta^a \delta^b}{\rho^2} \]
\[ + \frac{V'}{\rho} \frac{\delta^{ab} \phi^a \phi^b + \phi^a \delta^{ab} \phi^b}{\rho} + V(2) \frac{\delta^{ab} \phi^b}{\rho} \]
\[ = \Sigma_{\delta} \delta^{ab} + \Sigma_{\phi \phi} \frac{\delta^a \phi^b}{\rho^2} + \Sigma_{\delta \phi} \frac{\delta^a \phi^b + \phi^a \delta^{ab} \phi^b}{\rho} + V(2) \frac{\delta^{ab} \phi^b}{\rho} \]  
\[ (B7) \]

and

\[ \Omega^{ab} \equiv U^{''ab}(\rho, \sigma, G) - \frac{2}{D_F} U^{'a}(\rho, \sigma, G) U^{'}(\rho, \sigma, G) U^{b}(\rho, \sigma, G) \]
\[ = \frac{U'}{\rho} \delta^{ab} + \left( \frac{U'' - \frac{U'}{\rho} - 2 \frac{U^{'}}{D_F} U^{''} \right) \frac{\phi^a \phi^b}{\rho^2} \]
\[ + \left( \frac{U'}{D_F} \frac{U}{\rho} \right) \frac{\delta^{ab} \phi^b + \phi^a \delta^{ab} \phi^b}{\rho} \]
\[ + \frac{2}{D_F} \frac{U^{'}}{D_F} \frac{G \cdot \delta^{ab} \phi^b}{\rho} + \frac{2}{D_F} \frac{U^{'}}{D_F} \frac{G \cdot m \cdot \phi^a \phi^b}{\rho} \]
\[ + \left( \frac{U(2)}{D_F} - \frac{2}{D_F} \frac{U^{'}}{D_F} \frac{U}{\rho} \right) \frac{\delta^{ab} \phi^b}{\rho} + \left( \frac{\delta^a \delta^b} {\rho} \right) \frac{\Gamma^a \phi^b + \phi^a \Gamma^b}{\rho} \]
\[ = \Omega_{\delta} \delta^{ab} + \Omega_{\phi \phi} \frac{\delta^a \phi^b}{\rho^2} + \Omega_{\delta \phi} \frac{\delta^a \phi^b + \phi^a \delta^{ab} \phi^b}{\rho} + \Omega_{\delta \phi} \frac{\phi^a \Gamma^b}{\rho} \]
\[ + \Omega_{\delta \phi} \frac{\delta^{ab} \phi^b}{\rho} + \Omega_{\phi \phi} \frac{\Gamma^a \phi^b + \phi^a \Gamma^b}{\rho} + \Omega_{\phi \phi} \frac{\delta^{ab} \phi^b}{\rho} - \Gamma^a \delta^{ab} \phi^b \]  
\[ (B8) \]

where

\[ \tilde{\mathcal{X}} = X^{(1)} + \sigma X^{(2)}, \quad X = V, U, m, g \]
\[ U' = m' + \mathcal{G} \tilde{g}. \]

(B9)

Note that the $\Sigma$s are all functions of $\rho$ and $\sigma$ and the $\Omega$s are all functions of $\rho$, $\sigma$ and $\mathcal{G}$. The ordering in all the terms containing $\mathcal{G}$s is nontrivial since the Pauli matrices don’t commute with each other.

Now using the similarity transformation matrix (defining $\pi^2_1 = \pi^2_1 + \pi^2_2$ and $\pi^2_2 = \pi^2_2 + \pi^2_3$),

\[ S = \frac{1}{\rho} \begin{pmatrix} \sigma & \pi_1 & \pi_2 & \pi_3 \\ -\pi_1 & \sigma \pi_1 & \pi_2 & \pi_3 \\ 0 & -\rho \pi_1 & \sigma \pi_1 & \pi_2 \\ 0 & 0 & -\rho \pi_2 & \pi_2 \end{pmatrix}, \]

\[ (B10) \]

(obtained by multiplying rotation matrices in 4-space—see Appendix D in Ref. 4) we can compute,

\[ \Sigma' = S \cdot \Sigma \cdot S^T \]
\[ \Omega' = S \cdot \Omega \cdot S^T, \]

(B11)
this amounts to performing similarity transformations on each of the tensors

\[
\begin{align*}
\delta_{ab}, & \frac{\phi^a \phi^b}{\rho^2}, & \frac{\phi^a \delta^b_0}{\rho}, & \frac{\delta^{a0} \phi^b}{\rho}, & \frac{\delta^{a0} \delta^b_0}{\rho} , \\
\Gamma^a \phi^b + \phi^a \Gamma^b, & (\Gamma^a \delta^b_0 + \delta^{a0} \Gamma^b), & \Gamma^a \Gamma^b ,
\end{align*}
\]  
(B12)

as can be read off from Eqs.(B7) and (B8). After performing the similarity transformation \( \Sigma' \) has the form

\[
\Sigma' = \begin{pmatrix}
\Sigma_{00} & \Sigma_{01} & 0 & 0 \\
\Sigma_{10} & \Sigma_{11} & 0 & 0 \\
0 & 0 & \Sigma_{22} & 0 \\
0 & 0 & 0 & \Sigma_{33}
\end{pmatrix},
\]  
(B13)

since \( \Sigma^{01} = \Sigma^{10} \) we can write

\[
\begin{align*}
\det \Sigma' &= (\Sigma^{00} \Sigma^{11} - (\Sigma^{01})^2) \Sigma^{22} \Sigma^{33} \\
\text{tr} \Sigma'^{-1} \cdot \Omega' &= (\Sigma^{00})^{-1} \Omega^{00} + (\Sigma^{10})^{-1} \Omega^{10} + (\Sigma^{01})^{-1} \Omega^{01} \\
&+ (\Sigma^{11})^{-1} \Omega^{11} + (\Sigma^{22})^{-1} \Omega^{22} + (\Sigma^{33})^{-1} \Omega^{33}.
\end{align*}
\]  
(B14)

Thus only 12 elements—6 from \( \Sigma' \) and 6 from \( \Omega' \)—need be computed to determine the flow equations for the \( \sigma \)-model. As is probably clear this substantially reduces the complexity of the expressions but they are still quite complicated. The derivation is facilitated by using Mathematica to compute the \( \Sigma \) and \( \Omega \) functions in Eqs.(B7) and (B8). Then Eqs.(B14) can be written in terms of these expressions.

**APPENDIX C: \( \pi \pi \) SCATTERING LENGTHS IN THE RG MODEL**

In this appendix we will sketch the calculation of \( \pi \pi \) scattering lengths. Before addressing the calculation in our model we first discuss the calculation using the perturbative \( \sigma \)-model. This was first done by Weinberg [2]; useful reviews appear here as Refs. [3,32].

Differential cross sections in field theory are computed by squaring the “invariant amplitude”, \( A \) which is usually computed to a given order in perturbation theory using diagrammatic techniques (for details on our conventions see e.g. Ref. [38] Appendix A-3):

\[
\frac{d\sigma}{d\Omega} \propto |A|^2.
\]  
(C1)

Consider the amplitude for processes involving two pions in and two pions out.

As indicated, it will depend on the isospin channel \( I = 0, 1, 2 \), the Mandelstam momentum variables \( s = (p_1 + p_2)^2 \), \( t = (p_1 + p_3)^2 \), \( u = (p_1 + p_4)^2 \) and the isospin indices of each of the pions, \( i, j, k, l \). (In the center-of-mass frame these become \( s = 4(m^2_\pi + \vec{k}^2) \), \( t = -2(1 - \cos \theta) \vec{k}^2 \), and \( u = -2(1 + \cos \theta) \vec{k}^2 \), where \( \vec{k} \) is the 3-momentum of the incident pion and \( \theta \) is the angle between the incident pion and the outgoing pion.) We can expand the amplitude in partial waves

\[
A^I(s, t, u) = \sum_{l=0}^\infty (2l + 1) A^I_l(s, t, u) P_l(\cos \theta),
\]  
(C2)

where \( P_l(\cos \theta) \) are the Legendre polynomials. Amplitudes for each of the isospin channels are not independent, however; since all the particles are bosons \( A^I_{ijkl}(s, t, u) \) is totally symmetric in all indices. This can be exploited to show that there is really only one independent amplitude \( A_l(s, t, u) \). Each of the amplitudes \( A^I_l \) can be written in terms of \( A_l(s, t, u) \):
We consider the process $\pi^+\pi^- \rightarrow \pi^0\pi^0$ with $\pi_0 = \pi^3$, $\pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm \pi^2)$, then we have
\[
V(\rho) = \cdots - \lambda \sigma_{\text{min}} \sigma(\pi_0^2 + 2\pi_+\pi_-) - \frac{\lambda}{4}(\pi_0^4 + 4\pi_+^2\pi_-^2 + 4\pi_0^2\pi_+\pi_-) + \cdots.
\] (C5)

The amplitude can be written as
\[
A_i(s,t,u) = -2i\lambda + 4(-i\lambda\sigma_{\text{min}})\frac{i}{s - m_\pi^2}
\]
\[
= -2i\lambda\left(1 + \frac{m_\pi^2 - m^2}{s - m_\pi^2}\right)
\]
\[
= \left(s - m_\pi^2\right)\left(m_\pi^2 - m^2 / m_\sigma^2 - s\right) \approx \frac{s - m_\pi^2}{f^2_\pi}
\]
where we used $m_\pi^2 \ll m_\sigma^2$ and $s = 4m_\pi^2$ (at threshold) in the last approximation. Also we used the relations
\[
\sigma_{\text{min}} = -f_\pi
\]
\[
\lambda = \frac{m_\sigma^2 - m_\pi^2}{2f^2_\pi}
\]
\[
\mu^2 = \frac{1}{2}(m_\sigma^2 - 3m_\pi^2)
\]
(C9)

to replace $(\sigma_{\text{min}}, \lambda, \mu^2)$ with the observable $(m_\sigma, m_\pi, f_\pi)$. From Eq. (C4), we can compute the amplitudes for the isospin channels,
\[
A_i^0(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)
\]
\[
A_i^1(s,t,u) = A(t,s,u) - A(u,t,s)
\]
\[
A_i^2(s,t,u) = A(t,s,u) + A(u,t,s).
\] (C10)

using Eq. (C7), $s + t + u = 4m_\pi^2$, $s = 4(m_\pi^2 + \vec{k}^2)$
\[
A_i^0 = \frac{2s - m_\pi^2}{f_\pi^2} = \frac{7}{3}m_\pi^2 - \frac{8m_\pi^2 \vec{k}^2}{f^2_\pi m_\pi^2}
\]
\[
A_i^1 = \frac{t - u}{f_\pi^2} = \frac{4m_\pi^2 \vec{k}^2}{3\pi f^2_\pi m_\pi^2}
\]
\[
A_i^2 = \frac{t + u - 2m_\pi^2}{f_\pi^2} = -\frac{2m_\pi^2 \vec{k}^2}{f^2_\pi} - \frac{4m_\pi^2 \vec{k}^2}{\pi f^2_\pi m_\pi^2}.
\] (C11)

The “scattering length” $a_i^f$ and “slope parameter” $b_i^f$ are defined by
\[
\text{Re } A_i^f(s) = 32\pi \left(\frac{q^2}{m_\pi^2}\right)^{\frac{1}{2}} \left[a_i^f + b_i^f \frac{q^2}{m_\pi^2} + O(\frac{\vec{k}^4}{m_\pi^2}) + \cdots\right],
\] (C12)
which gives the tree level values \( a_0^0 = \frac{7}{32\pi} \frac{m_\pi^2}{f_\pi^2} \simeq 0.16; \quad b_0^0 = \frac{8}{32\pi} \frac{m_\pi^2}{f_\pi^2} \simeq 0.18 \) \hspace{1cm} (C13)

\[
\begin{align*}
  a_1^1 &= \frac{1}{24\pi} \frac{m_\pi^2}{f_\pi^2} \simeq 0.0; \quad b_1^1 = 0 \\
  a_2^0 &= -\frac{1}{16\pi} \frac{m_\pi^2}{f_\pi^2} \simeq -0.045; \quad b_2^0 = -\frac{1}{16\pi} \frac{m_\pi^2}{f_\pi^2} \simeq -0.09
\end{align*}
\]

The calculation in our model proceeds similarly. Since we compute the potential \( V(\rho) \) we must relate the tree diagrams in the invariant amplitude to this potential. In the leading order (LO) approximation, all the momenta on the external legs of the diagrams are zero and therefore we can only compute the s-wave \((l = 0)\) scattering lengths.

We equate the vertex between two pions and one sigma with the third derivative of the potential and the 4-pion vertex with the fourth derivative.

Computation of these is straightforward using Eq.(B4) as a starting point,

\[
\begin{align*}
  \left. \frac{\partial^3 V}{\partial \sigma \partial \pi^i \partial \pi^j} \right|_{\rho, \sigma = f_\pi} &= \delta^{ij} \left( \frac{V''}{f_\pi} - \frac{V'}{f_\pi^2} \right) \tag{C14} \\
  \left. \frac{\partial^4 V}{\partial \pi^i \partial \pi^j \partial \pi^k \partial \pi^l} \right|_{\rho, \sigma = f_\pi} &= (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{jl} \delta^{ik}) \left( \frac{V'''}{f_\pi^2} - \frac{V'}{f_\pi^3} \right)
\end{align*}
\]

where

\[
V' = V_0'(\rho) + \sigma V_1'(\rho) + \frac{\sigma}{2} V_2'(\rho) \tag{C15}
\]

and similarly for \( V'' \). Defining \( F(\rho) = \frac{V''}{f_\pi} - \frac{V'}{f_\pi^2} \) and \( G(p) = F(\rho)/f_\pi + F^2(\rho)/(p - m_\pi^2) \) we can construct the invariant amplitude,

\[
A(s, t, u) = \delta^{ij} \delta^{kl} G(s) + \delta^{ik} \delta^{jl} G(t) + \delta^{il} \delta^{jk} G(u) \tag{C16}
\]

At threshold \( s = 4m_\pi^2, t = u = 0 \), so

\[
A(s) = \delta^{ij} \delta^{kl} \left( \frac{F(\rho)}{f_\pi} + \frac{F^2(\rho)}{4m_\pi^2 - m_\sigma^2} \right) \tag{C17}
\]

Thus \( A(t, s, u) = A(u, t, s) = A(0) \) and our crossing relations (Eq.(C11)) give,

\[
\begin{align*}
  A_0^0 &= 3A(s) + 2A(0) \\
  A_0^2 &= 2A(s)
\end{align*} \tag{C18}
\]

from which we have,

\[
\begin{align*}
  a_0^0 &= \frac{1}{32\pi} A_0^0 = \frac{1}{32\pi} (3A(s) + 2A(0)) \\
  a_2^0 &= \frac{1}{16\pi} A_0^2 = \frac{1}{16\pi} A(s) \tag{C19}
\end{align*}
\]

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‡ See Ref. [4] for full details. Soft copies of the expressions are available from the authors.

[1] M. Gell-Mann and M. Levy, Nuovo Cim. 16, 705, (1960).
[2] S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).
[3] D. Pocanić in Chiral Dynamics: Theory and Experiment eds. A. B. Bernstein and B. R. Holstein (Springer) 1994.
[4] A.S. Johnson, Ph.D. thesis, Colorado School of Mines, 1997.
[5] K. G. Wilson and J. Kogut, Phys. Rep. C. 12, 75 (1974).
[6] F.J. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973).
[7] J. F. Nicoll, T. S. Chang, H. E. Stanley, Phys. Rev. Lett., 33, 540 (1974).
[8] A. Hasenfratz and P. Hasenfratz, Nucl. Phys., B270, 685 (1986).
[9] P. Hasenfratz and J. Nager, Z. Phys. C, 37, 477, (1988).
[10] C. Wetterich, Phys. Lett., B301, 90 (1993).
[11] N. Tetradis and C. Wetterich, Nucl. Phys., B422, 541 (1994) and references cited therein.
[12] D. U. Jungnickel and C. Wetterich, Phys. Rev. 53, 5142 (1996).
[13] T. R. Morris, Int. J. Mod Phys., A9, 2411 (1994).
[14] T. R. Morris, Phys. Lett., B329, 241, (1994).
[15] T. R. Morris, Nucl. Phys., B458, 477 (1996).
[16] T. R. Morris, Nuc. Phys. Proc. Suppl. 42, 811 (1995).
[17] M. Alford, Phys. Lett., B336, 237, (1994).
[18] J.R. Shepard, V. Dmitrasinovic and J.A. McNeil, Phys. Rev. D51, 12, 7017 (1995) and hep-lat/9412111.
[19] S.-B. Liao and C. Gong, hep-ph/9404086 (unpublished).
[20] R. D. Ball, P. E. Haagensen, J. I. Lattorre, and E. Moreno, Phys. Lett., B347, 80, (1995).
[21] K. Halpern and K. Huang, Phys. Rev. Lett., 74, 3526 (1995).
[22] M. Maggiore, Z. Phys. C 41, 687, (1989).
[23] T.E. Clark, B. Haeri and S.T. Love, Nucl. Phys. B402, 628, (1993).
[24] T.E. Clark, B. Haeri, S.T. Love, M. A. Walker, and W. T. A. ter Veldhuis, Phys. Rev., D50, 606, (1994).
[25] T.E. Clark and S.T. Love, Phys. Lett., B344, 266, (1995).
[26] U. Ellwanger and L. Vergara, Nucl. Phys., B398, 52 (1993).
[27] A. Hasenfratz, P. Hasenfratz, K. Jensen, J. Kuti, and Y. Shen, Nucl. Phys. B365, 79 (1991).
[28] A. Hasenfratz, SCRI-88-63, in Proc. Lattice Higgs workshop (Tallahassee, FL 1988); Nucl. Phys. B (Proc. Suppl.) 9, 92, (1989) also see A. Hasenfratz and T. Neuhaus, Phys. Lett., 220, 439 (1989) and A. Hasenfratz, W. Liu, and T. Neuhaus, Phys. Lett., 236, 339 (1990).
[29] J. Polchinski, Nucl. Phys., B231, 269 (1984).
[30] Particle Data Group, Phys. Rev. D54, (1996).
[31] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. 27, 195 (1991).
[32] J. F. Donoghue, E. Golowich, and B. R. Holstein, Dynamics of the Standard Model (Cambridge University Press) 1992.
[33] R. G. Nelson, Scattering Theory of Waves and Particles (McGraw-Hill) 1966 section 11.2.
[34] B. R. Martin, D. M. Morgan, G. Shaw, Pion-Pion Interactions in Particle Physics (Academic Press, New York) 1976.
[35] J. Gasser and H. Leutwyler, Phys. Lett. 125B, 325 (1983).
[36] Y. Kuramashi, M. Fukugita, H. Mino, M. Okawa, A. Ukawa, Phys. Rev. Lett. 71, 2387 (1993).
[37] P. Ramond, Field Theory: A Modern Primer (Benjamin/Cummings, Reading, MA) 1981.
[38] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York) 1980.
TABLE I. Results for numerical solution of $\sigma$-model RG flow equations with 6 quark flavors for three different values of the UV quark-meson coupling $g_0$. The $\parallel$ and $\|$ indicate that $m_0^q$ and $(\mu_0, \Lambda_0)$ were used to fix $m_q$ and $(m_{\pi}, f_{\pi})$ respectively.

| Parameter | Fit 1 | Fit 2 | Fit 3 |
|-----------|-------|-------|-------|
| $g$       | 2.967 | 3.358 | 3.893 |
| $g_0$     | 2.500 | 2.765 | 3.100 |
| $\mu_0$ (MeV)$\parallel$ | 666.0 | 739.4 | 818.0 |
| $m_0^q$ (MeV)$\parallel$ | 6.54  | 6.42  | 6.36 |
| $\Lambda_0$ (MeV)$\parallel$ | 950.0 | 937.9 | 927.0 |
| $m_{\pi}$ (MeV)$\parallel$ | 140.0 | 140.0 | 140.0 |
| $f_{\pi}$ (MeV)$\parallel$ | 92.60 | 92.49 | 92.44 |
| $m_q$ (MeV)$\parallel$ | 7.48  | 7.51  | 7.54 |
| $m_{\sigma}$ (MeV) | 507.3 | 536.1 | 550.4 |
| $\lambda_{3-\text{pt}}$ | 43.0  | 50.8  | 64.2 |
| $< -\bar{\psi}\psi >^{1/3}$ (MeV) | 0.232 | 0.225 | 0.220 |
| $a_0^0(m_{\pi}^{-1})$ | 0.042 | -0.043 | -0.043 |
| $(2a_0^0 - 5a_2^0)(m_{\pi}^{-1})$ | 0.677 | 0.664 | 0.656 |

TABLE II. Comparison of $s$-wave $\pi\pi$ scattering lengths obtained by measurement and various calculations in dimensions of inverse pion mass. Experimental results in the first row are from Ref. [34]. The calculations in the second through the third row are quoted from Refs. [2,35,36] respectively and were performed using the perturbative $\sigma$-model, chiral perturbation theory (\chiPT), and lattice QCD respectively. In the last three rows are the results from our model for the three fits used in Table I. Comprehensive reviews of $\pi\pi$ scattering are given in Ref. [32] section VI-4 and Ref. [3].

|                | $a_0^0(m_{\pi}^{-1})$ | $a_0^2(m_{\pi}^{-1})$ | $2a_0^0 - 5a_2^0$ |
|----------------|-----------------------|-----------------------|-------------------|
| Experiment     | 0.26 ± 0.05           | -0.028 ± 0.012        | 0.66 ± 0.12       |
| Pert. $\sigma$-model | 0.16                     | -0.045                 | 0.56 |
| $\chi$PT     | 0.20                   | -0.042                 | 0.65 |
| Lattice QCD   | 0.22                   | -0.042                 | 0.65 |
| Fit 1         | 0.232                  | -0.42                  | 0.677 |
| Fit 2         | 0.225                  | -0.043                 | 0.664 |
| Fit 3         | 0.220                  | -0.043                 | 0.656 |
FIG. 1. Tree diagram

FIG. 2. RG boson potentials for the $\sigma$-model. The parameters of the calculation are displayed in Table 3.1
FIG. 3. Quark mass functions for the $\sigma$-model. The parameters of the calculation are displayed in Table 3.1
FIG. 4. Yukawa coupling functions for the $\sigma$-model. The parameters of the calculation are displayed in Table 3.1