On Maximum Contention-Free Interleavers and Permutation Polynomials over Integer Rings

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Abstract

An interleaver is a critical component for the channel coding performance of turbo codes. Algebraic constructions are of particular interest because they admit analytical designs and simple, practical hardware implementation. Contention-free interleavers have been recently shown to be suitable for parallel decoding of turbo codes. In this correspondence, it is shown that permutation polynomials generate maximum contention-free interleavers, i.e., every factor of the interleaver length becomes a possible degree of parallel processing of the decoder. Further, it is shown by computer simulations that turbo codes using these interleavers perform very well for the 3rd Generation Partnership Project (3GPP) standard.

Index Terms

Turbo code, interleaver, permutation polynomial, contention-free, algebraic, quadratic, parallel processing.
I. INTRODUCTION

Interleavers for turbo codes [1]–[12] have been extensively investigated. Recently, Sun and Takeshita [1] suggested the use of permutation polynomial-based interleavers over integer rings. In particular, quadratic polynomials were emphasized; this quadratic construction is markedly different from and superior to the one proposed earlier by Takeshita and Costello [7] in turbo coding applications. The algebraic approach in [1] was shown to admit analytical design of an interleaver matched to the constituent convolutional codes. The resulting performance was shown to be better than S-random interleavers [11] for relatively short block lengths and parallel concatenated turbo codes; we show in this correspondence that even for moderate block lengths (4096 information bits) an excellent performance can be obtained. An iterative turbo decoder needs both an interleaver and a deinterleaver. Ryu and Takeshita have also shown a necessary and sufficient condition for a quadratic permutation polynomial to admit a quadratic inverse [13]. Moreover, the simplicity of the algebraic construction in [1] implies efficient implementations as one witnesses in [14].

The decoding of turbo codes is performed by an iterative process in which the so-called extrinsic information is exchanged between sub-blocks of the iterative decoder. The parallel processing of iterative decoding of turbo codes is of interest for high-speed decoders. Aspects of implementations of parallel decoders in chips and expected performance are studied in [15]. Interleaving of extrinsic information is one important aspect to be addressed in parallel decoders because a memory access contention, as explained in this section, may appear during the exchange of extrinsic information between the sub-blocks of the iterative decoder [5]. The first approaches to solve the memory access contention problem simply avoided it by constraining the interleavers to be contention-free as in [2], [3], [5], [16]. For these type of constrained constructions of interleavers, Nimbalker et al. have shown that only a very small fraction of all interleavers are suitable for parallel processing of iterative decoding [2]. They have also proposed a new construction of a modified dithered relatively prime\(^1\) interleaver [8] (DRP) interleaver. If

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\(^1\)The construction in [7] generates interleavers typically with the performance and statistics of a randomly generated interleaver but with the advantage of a very simple generation.

\(^2\)There are typically two or more sub-blocks in an iterative turbo decoder, each implementing a soft-input soft-output decoding algorithm of a convolutional code.

\(^3\)The DRP interleaver construction is one of the best known for turbo codes with excellent error rate performance.
the interleaver is required to be left unconstrained (e.g., the interleaver cannot be modified because it is already part of a standard), then the memory contention problem can still be solved as shown in [17], [18] but at a cost of additional complexity.

In this correspondence, we approach the memory contention problem using constrained interleavers that are contention-free. The advantages of our approach are its low complexity induced from an algebraic solution but with no apparent error rate performance degradations against any good interleavers. The contention-free condition is illustrated in Fig. II through an arbitrary device (not necessarily a turbo decoder). The device has two sub-blocks. Each of the $N = 16$ cells in sub-block 0 needs to fetch data in a one-to-one fashion from the $N = 16$ cells in sub-block 1. If sub-block 0 processes data in a serial fashion, and its cells fetch data sequentially from left to right ($x = 0, x = 1, \ldots x = 15$) then the sequence $(f(0) = 0, f(1) = 7, \ldots, f(15) = 13)$ indicates the addresses of the cells in sub-block 1 from which data is extracted. The function $f(x)$ describes the interleaver. If sub-block 0 processes data in a parallel fashion using $M = 4$ processors, then sub-block 0 is split in windows of size $W = 4$. A cell in a window has an offset value $0 \leq j < W$ (different values of offsets are shown in different shades). Each of the four processors fetches data simultaneously always at a particular offset $j$. The contention-free property requires that for a fixed offset, exactly one cell is accessed from each of the four windows in sub-block 1. An example is shown in Fig. II for the offset $j = 2$. This implies that if cells in sub-block 1 are organized in four independent memory units for each of the windows A, B, C and D then we do need to worry about memory contention, i.e., two or more processors in sub-block 0 trying to simultaneously fetch data in the same memory unit in sub-block 1.

In turbo coding applications, this property is also desirable in the reverse order, i.e., when sub-block 0 and 1 switch roles. A mathematical description of the contention-free condition from [2] is now given. The exchange and processing of a sequence of $N = MW$ extrinsic information symbols between sub-blocks of the iterative decoder can be parallelized by $M$ processors working on window sizes of length $W$ in each sub-block without contending for memory access provided that the following condition holds for both the interleaver $f(x), 0 \leq x < N$ and deinterleaver $g(x) = f^{-1}(x)$:

$$[\pi(j + tW)/W] \neq [\pi(j + vW)/W]$$

(1)
where $0 \leq j < W$, $0 \leq t < v < N/W$, and $\pi(\cdot)$ is either $f(\cdot)$ or $g(\cdot)$.

If an interleaver is contention-free for all window sizes $W$ dividing the interleaver length $N$, it will be called a maximum contention-free interleaver. We show in this correspondence that permutation polynomials over integer rings always generate maximum contention-free interleavers.

This correspondence is organized as follows. In section II, we review a result for quadratic permutation polynomials [1], [13] over the integer ring $\mathbb{Z}_N$ and an elementary number theory proposition [19] needed for the main theorem. The main result is derived in section III, and examples and computer simulation results are given in section IV. Finally, conclusions are discussed in section V.

II. QUADRATIC PERMUTATION POLYNOMIALS OVER INTEGER RINGS

In this section, we establish notation, restate the criterion for existence of quadratic permutation polynomials over integer rings and, restate a result in number theory. The interested reader is referred to [1], [13] for further details. Given an integer $N \geq 2$, a polynomial

$$f(x) = f_1 x + f_2 x^2$$

\(^4\text{It can be shown that the exclusion of a constant coefficient } f_0 \text{ in } f(x) \text{ does not make this problem less general.}$$
\[ f_1 \text{ and } f_2 \text{ are non-negative integers, is said to be a quadratic permutation polynomial over the ring of integers } \mathbb{Z}_N \text{ when } f(x) \text{ permutes } \{0, 1, 2, \ldots, N-1\} \] [1], [20].

In this correspondence, let the set of primes be \( \mathcal{P} = \{2, 3, 5, 7, \ldots\} \). Then an integer \( N \) can be factored as \( N = \prod_{p \in \mathcal{P}} p^{n_{N,p}} \), where \( n_{N,p} \geq 1 \) for a finite number of \( p \)'s and \( n_{N,p} = 0 \) otherwise. For example, if \( N = 3888 = 2^4 \times 3^5 \) we have \( n_{3888,2} = 4 \) and \( n_{3888,3} = 5 \). For a quadratic polynomial \( f(x) = f_1 x + f_2 x^2 \pmod{N} \), we will abuse the previous notation by writing \( f_2 = \prod_{p \in \mathcal{P}} p^{n_{F,p}} \), i.e., the exponents of the prime factors of \( f_2 \) will be written as \( n_{F,p} \) instead of the more cumbersome \( n_{f_2,p} \) because we will be mainly interested in the factorization of the second degree coefficient.

Let us denote \( a \) divides \( b \) by \( a | b \) and by \( a \nmid b \) otherwise. The greatest common divisor of \( a \) and \( b \) is denoted by \( \gcd(a, b) \). The necessary and sufficient condition for a quadratic polynomial \( f(x) \) to be a permutation polynomial is given in the following proposition.

**Proposition 1:** [13] [1] Let \( N = \prod_{p \in \mathcal{P}} p^{n_{N,p}} \). The necessary and sufficient condition for a quadratic polynomial \( f(x) = f_1 x + f_2 x^2 \pmod{N} \) to be a permutation polynomial can be divided into two cases.

1) Either \( 2 \nmid N \) or \( 4 | N \) (i.e., \( n_{N,2} \neq 1 \))
   \[ \gcd(f_1, N) = 1 \text{ and } f_2 = \prod_{p \in \mathcal{P}} p^{n_{F,p}}, \forall p \text{ such that } n_{N,p} \geq 1. \]

2) \( 2 | N \) and \( 4 \nmid N \) (i.e., \( n_{N,2} = 1 \))
   \[ f_1 + f_2 \text{ is odd, } \gcd(f_1, \frac{N}{2}) = 1 \text{ and } f_2 = \prod_{p \in \mathcal{P}} p^{n_{F,p}}, \forall p \neq 2 \text{ such that } n_{N,p} \geq 1. \]

How many permutation polynomials are there? For example, if the interleaver length is \( N = 256 \) then we determine from case 1) of Proposition [1] that \( f_1 \in \{1, 3, 5, \ldots, 255\} \) (set of numbers relatively prime to \( N \)) and \( f_2 = \{2, 4, 6, \ldots, 254\} \) (set of numbers that contains 2 as a factor). This gives us \( 128 \times 127 = 16256 \) possible pairs of coefficients \( f_1 \) and \( f_2 \) that make \( f(x) \) a permutation polynomial; if \( N \) is a power of two then there are approximately \( N^2/4 \) possible pairs of coefficients. However, if \( N \) is a prime number then there are no polynomials of the form \( f(x) \) for a non-zero \( f_2 \). This may be perceived as a deficiency of the construction because certain interleaver lengths must be avoided. However, even restricting to powers of two gives plenty of possibilities and covers meaningful interleaver lengths. In general, the number of permutation polynomials is not a smooth function of \( N \).

Let us denote that \( x \) is congruent to \( y \) modulo \( N \) by \( x \equiv y \pmod{N} \); this means that there
exists an integer $k$ such that $x = y + kN$. The following elementary number theory proposition is used for deriving the main theorem (Theorem 1) of this correspondence.

**Proposition 2:** [19] Let $M$ be an integer. Suppose that $M|N$ and that $x \equiv y \pmod{N}$. Then $x \equiv y \pmod{M}$.

The proof follows by noting that $x = y + kN = y + kW_M$, where $W = N/M$.

### III. Maximum Contention-Free Permutation Polynomials Interleavers

The following defines a contention-free interleaver on its maximum extent.

**Definition 1:** An interleaver is maximum contention-free (MCF) when the interleaver is contention-free for every window size $W$ which is a factor of the interleaver length $N$.

It is natural that the previous definition implies that we potentially have a degree of parallel processing of any soft-input soft-output algorithm by $M = N/W$ processors, i.e., each factor of $N$ is a possible number of parallel processors. We show that quadratic permutation polynomials always generate interleavers that are MCF.

**Theorem 1:** Let $f(x) = f_1x + f_2x^2 \pmod{N}$ be a quadratic permutation polynomial. Then $f(x)$ generates a MCF interleaver.

**Proof:** We first verify condition (1) for $f(x)$ and then for $g(x) = f^{-1}(x)$.

Let

$$Q_t = \left\lfloor \frac{f(j + tW)}{W} \right\rfloor \quad \text{and} \quad Q_v = \left\lfloor \frac{f(j + vW)}{W} \right\rfloor$$

then

$$f(j + tW) = Q_tW + [f(j + tW) \pmod{W}] \quad \text{and} \quad f(j + vW) = Q_vW + [f(j + vW) \pmod{W}]$$

We must show that $Q_t \neq Q_v$ for $t - v \not\equiv 0 \pmod{M}$ and any $0 \leq j < W$.

Assume $Q_t = Q_v$. Then

$$Q_t - Q_v = \frac{f(j + tW) - [f(j + tW) \pmod{W}] - f(j + vW) + [f(j + vW) \pmod{W}]}{W} = 0$$

(2)
Using Proposition 2 and observing that
\[ f(j + tW) \equiv f_1 j + f_2 j^2 \pmod{W} \quad \text{and} \quad f(j + vW) \equiv f_1 j + f_2 j^2 \pmod{W}, \quad (3) \]
we conclude \([f(j + tW) \pmod{W}] = [f(j + vW) \pmod{W}]\) and therefore the absolute value of equation (2) can be simplified as
\[ |Q_t - Q_v| = \frac{|f(j + tW) - f(j + vW)|}{W} = 0 \quad (4) \]

By noting that \((j + tW) \neq (j + vW)\) and that \(f(x)\) is a permutation polynomial, we conclude \(f(j + tW) \neq f(j + vW)\) and we have a contradiction in (4).

To verify condition (1) for the inverse polynomial \(g(x)\), we start by observing that permutation polynomials form a finite group \(G\) under function composition, i.e., \(f(f(x))\) is a permutation polynomial and the inverse function can be found by a sufficient number of function compositions of \(f(x)\) to itself. In group theory parlance, \(f(x)\) generates the group \(G\). It now suffices to show that every element in \(G\), which includes the inverse function \(g(x)\), satisfies (1). This is easily shown by realizing that (3) implies \(f(x)\) permutes the set of indices \(A_j = \{j, j + W, j + 2W, \ldots, j + (M - 1)W\}\), i.e., indices belonging to every possible window at a particular offset \(j\), becomes mapped by \(f(x)\) to the set of indices \(B_k = \{k, k + W, k + 2W, \ldots, k + (M - 1)W\}\) where
\[ k \equiv f_1 j + f_2 j^2 \pmod{W}. \quad (5) \]
We conclude (5) must be a permutation polynomial, otherwise \(f(x)\) would not be a permutation polynomial.

Finally, one uses induction to find that every function obtained by successively composing \(f(x)\) (eventually generating the inverse function \(g(x)\)) generates a MCF interleaver.

We can observe from the previous proof that there exist MCF interleavers generated by permutation polynomials of degrees other than two. In fact, we have the following Corollary.

**Corollary 1:** Let \(f(x) = \sum_{i=0}^{K} f_i x^i \pmod{N}\) be a permutation polynomial of degree \(K\).
Then \(f(x)\) generates a MCF interleaver.
Proof: The proof is identical to as in Theorem 1 except that (3) is replaced by

\[ f(j + tW) \equiv \sum_{i=0}^{K} f_i j^i \pmod{W} \quad \text{and} \quad f(j + vW) \equiv \sum_{i=0}^{K} f_i j^i \pmod{W}, \quad (6) \]

and (5) is replaced by

\[ k \equiv \sum_{i=0}^{K} f_i j^i \pmod{W}. \quad (7) \]

We could had stated Corollary 1 as the main theorem and Theorem 1 as a special case but we present them in this order because the emphasis in this correspondence is on quadratic permutation polynomials. Naturally all linear interleavers [10] (also referred as circular interleavers [3], [11] and relatively prime (RP) interleavers [8]) are MCF. However, the error rate performance of turbo codes using linear interleavers are constrained by the linear interleaver asymptote [10].

The almost regular permutation (ARP) interleavers in [3] (closely related to linear interleavers and DRP interleavers) are mentioned to have a degree of parallel processing \( p_C \) dividing \( N \), where \( C \) is a design parameter also dividing \( N \) and \( p \) any integer. However, we believe many ARP interleavers (if not all) are MCF and therefore ARP interleavers are stronger with respect to the degree of parallel processing than what is stated in [3]. The advantage of our construction is a much simpler description of the interleaver by a single permutation polynomial, which we believe makes implementation simpler as well [14]. Moreover, the error performance is also not expected to degrade against any good interleavers as shown in the following section.

IV. Examples and Computer Simulation Results

We give four examples of interleavers generated by quadratic permutation polynomials in Table I. The respective inverse functions are also given for completeness and were computed using the theory in [13]. Because their MCF property is guaranteed by Theorem 1 regardless of the choice of the permutation polynomials, we only need to select permutation polynomials that yield interleavers with good error rate performance for turbo codes.

The interleavers in Examples 1 – 3 were found by a limited search for good polynomials using mainly the theory in [1], checking for the true minimum distance \( d_{\text{min}} \) of the associated turbo codes using the algorithm in [21] (the algorithm only finished within a reasonable amount...
TABLE I

EXAMPLES OF MCF INTERLEAVERS

| Example | $N$   | $f(x)$                              | $g(x)$                              | $D$  | $d_{\text{min}}$ |
|---------|-------|-------------------------------------|-------------------------------------|------|-------------------|
| 1       | 256   | $159x + 64x^2 \pmod{N}$            | $95x + 64x^2 \pmod{N}$              | 16   | 27                |
| 2       | 1024  | $31x + 64x^2 \pmod{N}$             | $991x + 64x^2 \pmod{N}$             | 32   | 27                |
| 3       | 4096  | $2113x + 128x^2 \pmod{N}$          | $4033x + 1920x^2 \pmod{N}$          | 64   | -                 |
| 4       | 15120 | $11x + 210x^2 \pmod{N}$            | $14891x + 210x^2 \pmod{N}$          | 20   | -                 |

of time for Examples 1 and 2), and finally running computer simulations. To the best of our knowledge, one of the most accepted indicators for a good interleaver with respect to error performance for parallel concatenated turbo codes is the spread factor [11], [22] defined as

$$D = \min_{i,j \in \{0,1,\ldots,N-1\}, i \neq j} \{|i-j| + |f(i) - f(j)|\}. \quad (8)$$

The upper bound on the spread factor was proved in [23] to be $\sqrt{2N}$ and was shown earlier [11] to be achievable or closely approximated with carefully chosen linear interleavers. The error rate performance of turbo codes using any linear interleaver is constrained by the linear interleaver asymptote [10]. Therefore, the maximization of the spread factor alone is not sufficient to guarantee a good error performance. Nevertheless, the spread factors $D$ are computed for our examples as a point of reference because many good constructions attempt a maximization of the spread factor. The spread factors obtained for Examples 1 – 3 (i.e., the codes simulated for this correspondence) are approximately 70% of the upper bound $\sqrt{2N}$ independent of the degree of parallel processing because we use a fixed interleaver. Interestingly, the spread factors obtained in [15] are also close to 70% of $\sqrt{2N}$ when the degree of parallel processing is $M = 1$ (serial processing) and with some small decrease as the degree of parallel processing increases; the interleavers therein found are all different for each degree of parallel processing and the search algorithm is designed to maximize the spread factor.

The interleaver in Example 4, chosen by the Jet Propulsion Laboratory, is being considered in [14] because of its excellent performance and ease of implementation. Example 4 is also interesting because $N = 2^4 \cdot 3^3 \cdot 5 \cdot 7$ is composed of several different prime factors whereas for Examples 1 – 3, the interleaver lengths are powers of two.
In all of the four examples, 16 is a factor of the interleaver length \( N \). This means that we can have a sub-block of an iterative decoder split into 16 parallel sections without causing memory access contention when exchanging extrinsic information with other sub-blocks.

We now demonstrate that the restriction of an interleaver generated by a quadratic polynomial to be \( MCF \) does not degrade the associated turbo code error performance. On the contrary, the quadratic interleavers generate turbo codes that have excellent error rate performance. The simulated turbo codes are of nominal rate 1/3 for the 3rd Generation Partnership Project (3GPP) standard [24] but using \( MCF \) interleavers generated by quadratic polynomials in Examples 1 – 3. We use BPSK modulation and assume an additive white Gaussian noise (AWGN) channel. The frame error rate (FER) performance curves are shown in Fig. 2. We used eight log-MAP decoding iterations and simulated until at least 100 frame errors had been counted. The typical benchmark S-random interleavers [11] were also simulated under the same conditions. In addition, the current 3GPP standard curves are plotted. Additional reference curves are available in [2].

It is observed from Fig. 2 that the FER performance curves of turbo codes using the quadratic permutation polynomial interleavers meet or exceed the performances of S-random interleavers down to an FER of at least \( 10^{-4} \). Moreover, from the slope of the curves, we again expect to meet or exceed the error performance against any other interleaver down to an FER of at least \( 10^{-4} \).

V. CONCLUSION

Nimbalker et al. proved that only a very small fraction of all interleavers are contention-free [2]. Therefore we have shown the remarkable fact that all permutation polynomials over integer rings generate \( MCF \) interleavers. This property is exceptionally important for a high-speed hardware implementation of iterative turbo decoders because it means a potential parallel processing of iterative decoding of turbo codes by \( M \) processors for any positive integer \( M \) dividing the interleaver length \( N \). Conversely, if one has a target of using \( M \) processors, then it suffices to choose an interleaver length \( N \) which is a multiple of \( M \). We have given examples of interleavers based on quadratic polynomials that are \( MCF \). These interleavers generate turbo

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5The curve was obtained from [2] but adjusted for any termination bits rate-loss as it was done in our curves. The curves therein had been simulated with eight decoding iterations and until at least 50 frame errors had been counted.

6The length 4096 quadratic polynomial curve is slightly worse for high FER’s compared with the S-random interleaver.
codes with error rate performances that are expected to meet or exceed any known interleavers for the 3GPP standard down to a frame error rate of at least $10^{-4}$. Moreover, $MCF$ interleavers based on quadratic permutation polynomials have virtually the simplest generation algorithm and the least number of input parameters among all known interleavers, which implies their very simple implementation in software or hardware and little memory requirements.

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