Responses of Multiple Nonlinear Tuned Vibration Absorbers under Harmonic Excitation

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Abstract. In this paper, a system consisting of multiple perfectly tuned identical translational vibration absorbers, having both hardening and softening springs, attached to a main mass under harmonic excitation is considered. The existence of absorbers' synchronous and non-synchronous responses is checked. The method of averaging is employed to reach to the averaged autonomous equations of motion that describe the dynamics of the absorbers. A graphical method is then employed to check the existence of different responses of the absorbers. It is found that for absorbers with hardening springs, only one synchronous response of the absorbers occurs and no other responses take place. However, for the case of absorbers with softening springs, other responses were found to exist. These include multi-valued synchronous responses and a jump instability. These findings are in agreement with those of another study by the author where a similar system was considered using different approach.

1. Introduction
Tuned vibration absorber (TVA) is widely used to reduce excessive steady state harmonic vibrations of a machine or system that arise when the frequency of the excitation force that causes vibrations nearly coincides with the natural frequency of the machine or the system, i.e., at resonance situation. The invention of TVA goes back to 1909 by Herman Frahm [1] (US Patent #989958, issued in 1911) and its use is considered to be one the most effective ways to achieve the goal of eliminating or reducing unwanted vibrations of engineering systems and their damaging effects. TVA has certain advantages over other vibration suppression methods. One of its big advantages is being external to the vibrating system, so no re-installation of equipment is necessary. Another big advantage of TVA is the fact that it does not require any external energy source to do its task. Vibration absorber, in its simplest form, consists of a mass attached to the vibrating system by an elastic element. The absorber mass and elastic element are chosen so that the natural frequency of the absorber is the same as the excitation frequency. In this situation, the absorber is said to be tuned to the excitation frequency. A damping element is usually added to the absorber in order to increase its effective bandwidth and to avoid system resonance and its damaging effects. Den Hartog [2] was the first to study TVA in depth and develop its theory which is very well established and can be found in any vibrations textbook, see for example [3].

Researchers considered nonlinear vibration absorbers in order to improve the absorbers' effectiveness. Samples of the research in this direction can be found in [4-10] and the references therein. Also, and due to some design constraints, designers may implement multiple linear or...
nonlinear absorbers instead of a single one to improve the absorbers' task of reducing vibrations and meet certain design requirements. The reader is referred to references [11-15] as examples of the research in that direction.

In a previous work, Alsuwaiyan [16] considered a system of multiple perfectly tuned nonlinear vibration absorbers attached to a primary system under harmonic excitation. The stability of the absorbers' synchronous response, which is the response where all absorbers move together during operation, and the existence of other responses were checked. The absorbers considered are translational type with both hardening and softening identical springs having a similar cubic nonlinearity in each of them. A mathematical approach was implemented. It was shown that when the absorbers' springs are of hardening type, the absorbers' synchronous response is stable and no other responses exist under practical excitation force levels. However, when the absorbers' springs are of softening type, multi-valued responses were found to exist and jump instability was recorded.

In the current work, the same system considered in [16] is reconsidered. The stability of the absorbers' synchronous response and the existence of other responses are checked for both hardening and softening springs type. Here, a mathematical/graphical approach is implemented instead of the pure mathematical approach used in [16]. The findings of this work are to support the corresponding findings presented in [16].

2. Mathematical Model

A system of $N$ translational vibration absorbers mounted on a primary mass, $M$, under harmonic excitation, as shown schematically in Fig. 1, is considered. The spring of each absorber has a linear spring stiffness, $k_i$ and a cubic spring stiffness coefficient, $\alpha_i$, i.e., the restoring force of each absorber spring is $F_{si} = k_i(x_i - y) + \alpha_i(x_i - y)^3$, where $i = 1, \ldots, N$.

\begin{equation} 
M \ddot{y} = F_0 \sin(\omega t) - c \dot{y} - ky + \sum_{i=1}^{N}(k_i z_i + \alpha_i z_i^3 + c_a \dot{z_i}) \tag{1} 
\end{equation}

\begin{equation} 
m_i \ddot{z_i} + c_a \dot{z_i} + k_a z_i + \alpha_i z_i^2 = -m_i \ddot{y} \tag{2} 
\end{equation}

Where,

\begin{equation} 
(\dot{\cdot}) = \frac{d}{dt}(\cdot), \quad (\ddot{\cdot}) = \frac{d^2}{dt^2}(\cdot), z_i = x_i - y, \quad i = 1, \ldots, N. \quad \text{The absorbers are assumed to be identical meaning that that their masses, damping and spring stiffness coefficients are the same, i.e.,} \quad m_i = m, c_i = c_a, k_i = k_a, \quad \alpha_i = \alpha \quad \forall i \in [1,N].
\end{equation}

Following the formulation procedure used in [16], one can reach to the following equation that describes the dynamics of each absorber uncoupled from the main mass dynamics:
\[ \ddot{z}_i + \omega^2 z_i = \epsilon \left[ -\bar{\mu}_a \dot{z}_i - \bar{\alpha}_a z_i^3 - f_o \sin(\omega t) - \omega_a^2 \sum_{j=1}^{N} z_j \right] + O(\epsilon^2) \]  

(3)

where, \( \epsilon = \frac{m}{M} \), \( \bar{\mu}_a = \frac{c_a}{m} \), \( \bar{\alpha}_a = \frac{\alpha_a}{\epsilon} \), \( \mu_a = \frac{\mu_a}{\epsilon} \), \( \alpha_a = \frac{\alpha}{m} \), \( \omega^2 = \frac{k}{M} \), \( \omega_a = \frac{\omega}{\epsilon} \), \( \omega_a = \frac{\omega}{\epsilon} \), and \( f_o = \frac{f_o}{m} \).

The right-hand side of Equation (3) is \((2\pi/\omega)\) periodic function of time. To put (3) into a form suitable for applying the method of averaging, the following invertible Vander Pol transformation can be used [17]:

\[
\begin{pmatrix}
  u_i \\
v_i
\end{pmatrix} = A \begin{pmatrix}
  z_i \\
  \dot{z}_i
\end{pmatrix}
\]

(4)

where

\[
A = \begin{pmatrix}
  \cos \omega t & -\frac{1}{\omega} \sin \omega t \\
  -\sin \omega t & -\frac{1}{\omega} \cos \omega t
\end{pmatrix}
\]

(5)

These equations can be expressed in polar coordinates by the usual transformation to polar coordinates, i.e.,

\[
\begin{aligned}
z_i &= r_i \sin(\tau + \theta_i), \quad \dot{z}_i = r_i \cos(\tau + \theta_i) \\
\end{aligned}
\]

(9)

where \( r_i = \sqrt{u_i^2 + v_i^2} \), \( \theta_i = \arctan\frac{v_i}{u_i} \).
Equations (7) and (8) are used to check the absorbers’ synchronous response and the existence of other absorber responses in the next section.

3. Synchronous and Non-Synchronous Absorbers’ Responses

To examine the existence of absorbers’ synchronous and non-synchronous responses, the $N$ absorbers are divided into two groups. The first consists of $Q$ absorbers and the other consists of the remaining $N-Q$ absorbers. The absorbers in each group vibrate in unison. The absorbers’ displacements in the first group are $(u_k, v_k)$, while the displacements of those in the other group are $(u_l, v_l)$, where $k=1, 2, ..., Q$ and $l=Q+1, Q+2, ..., N$.

For the first group, equations (7) and (8) will be:

\[
\begin{align*}
    u'_k &= \frac{\epsilon}{2} \left[ -\bar{\mu}_a u_k - \frac{3}{4} \bar{\alpha}_a v_k (u_k^2 + v_k^2) + f_o - Q v_k - (N-Q) v_l \right] \\
    v'_k &= \frac{\epsilon}{2} \left[ -\bar{\mu}_a v_k + \frac{3}{4} \bar{\alpha}_a u_k (u_k^2 + v_k^2) + Q u_k + (N-Q) u_l \right]
\end{align*}
\] (10) (11)

For the other group, equations (7) and (8) will be:

\[
\begin{align*}
    u'_l &= \frac{\epsilon}{2} \left[ -\bar{\mu}_a u_l - \frac{3}{4} \bar{\alpha}_a v_l (u_l^2 + v_l^2) + f_o - Q v_k - (N-Q) v_l \right] \\
    v'_l &= \frac{\epsilon}{2} \left[ -\bar{\mu}_a v_l + \frac{3}{4} \bar{\alpha}_a u_l (u_l^2 + v_l^2) + M u_k + (N-Q) u_l \right]
\end{align*}
\] (12) (13)

To find the absorbers’ steady state responses in both groups, the right hand sides of these equations, each equated to zero, should be solved for the absorbers displacements. This is done by employing the method used by Alsuwaiyan and Shaw [18] where they studied the existence and stability of the synchronous and non-synchronous responses in multiple centrifugal pendulum vibration absorbers. This method is as follows. One displacements pair is eliminated from two equations resulting in a two lengthy equations in the other pair. To be sure that all possible solutions are captured, the zero contours of the two equations are plotted in the displacement pair plane. The contour intersections are the solutions that represent the absorbers steady state responses. Absorbers’ displacements in each group can then be transformed to amplitude and phase angle coordinates for physical interpretation of the results if needed.

It should be noted here that solutions satisfying $(u_k, v_k) = (u_l, v_l)$ are the synchronous responses. In this case, the absorbers’ vibration amplitude is related to the excitation force by [16]:

\[
f_o = \sqrt{\left( \bar{\mu}_a \omega r \right)^2 + \left( \frac{3}{4} \bar{\alpha}_a r^2 + N \omega^2 r \right)^2}
\] (14)

To illustrate this procedure a numerical example is presented in the following section.

4. A Numerical Example

To compare the results of the current work with those in [16], the same example considered there is reconsidered here. It is a system of four absorbers ($N=4$) mounted on a main mass of 100 kg with a natural frequency of 25 rad/sec. The harmonic excitation force is applied to the main mass at resonance condition frequency and the absorbers are tuned to the excitation frequency with a mass and
a linear spring stiffness of each absorber of 6 kg and 3.75 kN/m respectively. The nonlinearity coefficient in the absorbers' spring are taken to be $I \text{kN/m}^3$ for hardening springs type and $-I \text{kN/m}^3$ for softening springs type.

4.1 Hardening Springs Type

The above described procedure is performed on absorbers with $\alpha_a = 1 \text{kN/m}^3$ and the contours are plotted for different applied force levels in Figs (2) to (5). It is clear that the only one synchronous response exists and no other responses are present. This is in agreement with what has been concluded in [16].

Fig. 2. $N=4$, $\alpha_a = 1 \text{kN/m}^3$, $F_o=0.5 \text{kN}$; (a) and (b): $Q=1$, (c): $Q=2$

Fig. 3. $N=4$, $\alpha_a = 1 \text{kN/m}^3$, $F_o=2 \text{kN}$; (a) and (b): $Q=1$, (c): $Q=2$
4.2 Softening Springs Type

Contour plots for different applied force levels for absorbers with $\alpha_a = 1 \text{ kN/m}^3$, $F_o = 6 \text{ kN}$; (a) and (b): $Q=1$, (c): $Q=2$
two responses disappear and the third continues to exist as can be seen from Fig.(9). This means that jump instability takes place at the force level where the two responses disappear. This instability should be taken into consideration when implementing multiple absorbers with softening springs. Again, this is in agreement with the conclusions made in [16].

Fig. 6. $N=4$, $\alpha_a = -1 \text{kN/m}^3$, $F_0 = 0.5 \text{kN}$; (a) and (b): $Q=1$, (c): $Q=2$

Fig. 7. $N=4$, $\alpha_a = -1 \text{kN/m}^3$, $F_0 = 2 \text{kN}$; (a) and (b): $Q=1$, (c): $Q=2$
5. Conclusion
In this paper, the responses of perfectly tuned multiple identical translational vibration absorbers with both hardening and softening springs type were checked. For absorbers with hardening springs type, this work showed that only one synchronous response of the absorbers takes place. For absorbers with
softening springs type, multi-valued synchronous responses and a jump instability were found to exist. These findings are in agreement with the recent work presented in [16] where the same dynamical system considered in this work was considered.

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