Spontaneous breaking of time translation symmetry in a spin-orbit-coupled fluid of light

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We study the interplay between intrinsic spin-orbit coupling and nonlinear photon-photon interactions in a nonparaxial, elliptically polarized fluid of light propagating in a bulk Kerr medium. We find that in situations where the nonlinear interactions induce birefringence, i.e., a polarization-dependent nonlinear refractive index, their interplay with spin-orbit coupling results in an interference between the two polarization components of the fluid traveling at different wave vectors, which entails the spontaneous breaking of translation symmetry along the propagation direction. This phenomenon leads to a Floquet band structure in the Bogoliubov spectrum of the fluid, and to characteristic oscillations of its intensity correlations. We characterize these oscillations in detail and point out their exponential growth at large propagation distances, revealing the presence of parametric resonances.

Spin-orbit coupling (SOC) in materials arises due to the interaction between the electron spin and its momentum, and lies at the heart of various phenomena and concepts such as spin Hall effects [1, 2], topological insulators [3], and Majorana fermions [4]. In the context of quantum fluids, significant progresses in the engineering of synthetic gauge fields have recently paved the way for intriguing quantum phenomena resulting from the interplay between SOC and particle interactions in ultracold atoms [5, 6]. In the ground states of Bose gases, e.g., this interplay gives rise to stripe-superfluid or lattice phases [7–12]. In degenerate Fermi gases, on the other hand, SOC can significantly impact the celebrated BEC-BCS crossover [13] or lead to topological superfluids [14, 15]. Beyond matter waves, SOC also exists in photonic systems. This has been pointed out, in particular, for exciton-polaritons in microcavities [16–20] or for photons tunneling in properly designed microstructures [21]. In those systems, effective photon-photon interactions also show up due to the interaction between the underlying excitons, even though their interplay with SOC has yet to be explored.

An alternative optical platform where photon interactions can be realized are fluids of light in the propagating geometry [22]. In this configuration, the propagation of light through a nonlinear medium becomes, in the paraxial limit, formally identical to the temporal evolution of a two-dimensional (2D) quantum fluid, the propagation axis playing the role of an effective time and the nonlinearity mediating the photon interactions. This remarkable analogy has been beautifully illustrated with measurements of the Bogoliubov dispersion [23, 24], the dynamical formation of optical condensates [25, 26], the spontaneous nucleation of vortices in a photonic lattice [27], or the temporal dynamics of correlation functions following a quench [28, 29]. Due to the absence of cavity or underlying microstructure, fluids of light in the propagating geometry do not seem to constitute, at first sight, a natural platform for achieving SOC. Nevertheless, recently a spin-orbit mechanism has been pointed out in this system [30], based on the fundamental coupling between the polarization and the trajectory of optical fields subjected to a refractive-index gradient [31–36]. In the presence of an optical nonlinearity, such a coupling is achieved by letting an inhomogeneous optical field propagate in the nonlinear medium and make it deviate from its regime of paraxial propagation. The inhomogeneous spatial profile gives rise to a nonlinear refractive-index gradient, which couples to the optical spin via the SOC terms emerging beyond paraxiality [30].

In this Letter, we show that elliptically polarized fluids of light propagating in media displaying a nonlinear birefringence exhibit a spontaneous breaking of translation symmetry along the optical axis direction (i.e., the effective time axis) due to SOC. This leads to the emergence of a Floquet band structure in the excitation spectrum, analogously to what is observed in driven systems [37], but here in a purely isolated optical fluid. The breaking of translation symmetry also gives rise to peculiar oscillations exhibited by several physical quantities, in particular the intensity-correlation function of the fluid of light. By characterizing these oscillations in detail, we further point out their exponential growth at large...
propagation distances. This showcases the existence of parametric resonances, an original manifestation of the interplay between nonlinearity and SOC in fluid systems.

Our setup consists of a bulk nonlinear medium infinitely extended along the x and y axes and the positive z direction (see Fig. 1). A monochromatic field \( \mathbf{E}(r, t) = \Re \{ \mathbf{E}(r)e^{-i\omega_0 t} \} \) propagates inside the material at frequency \( \omega_0 \). The components of the complex amplitude \( \mathbf{E} \) obey the nonlinear Helmholtz equation

\[
\nabla^2 \mathbf{E} - \nabla \left( \nabla \cdot \mathbf{E} \right) + \frac{\omega_0^2}{c^2} \left[ n_0^2 + 2n_0 \Delta n(\mathbf{E}) \right] \mathbf{E} = 0, \tag{1}
\]

with \( c \) the vacuum speed of light and \( n_0 \) the linear refractive index. The nonlinear refractive index [\( \Delta n(\mathbf{E})_{ij} = (n_2,d + n_2,s) |\mathbf{E}|^2 \delta_{ij} - n_2,s \varepsilon^* \varepsilon_j \)] is a tensor featuring two independent Kerr indices \( n_2,d \) and \( n_2,s \).

Equation (1) corresponds to the Euler-Lagrange equation for the action functional \( S = \int d^3 r \mathcal{L} \), where [30]

\[
\mathcal{L} = -\frac{1}{2}\beta_0 \left( \nabla \cdot \mathbf{E}^* \right) \nabla \cdot \mathbf{E} - \nabla \mathbf{E} \cdot \mathbf{E} \mathbf{E}^* - \beta_0 \mathbf{E} \mathbf{E}^* \mathbf{E} \mathbf{E}^* (2)
\]

Here summarization over repeated indices is implied, \( \beta_0 = n_0\omega_0/c \) is the propagation constant, and \( (S_k)_{ij} = -i \varepsilon_{ijk} \) denotes the \( k^{th} \) spin-1 matrix. The simultaneous presence of a spin-independent \( (g_d = -n_2,d \omega_0 / c) \) and a spin-dependent \( (g_s = -n_2,s \omega_0 / c) \) nonlinear coupling, both assumed positive, is typical of isotropic systems described by multicomponent fields, where the pairing in channels of different total spin can occur at different strengths. Besides nonlinear media [38] like optical fibers [39], this behavior is observed in microcavity exciton-polaritons [40] and atomic spinor Bose-Einstein condensates [41].

The mechanism of SOC of light, on the other hand, originates from the term \( \nabla \left( \nabla \cdot \mathbf{E} \right) \sim \nabla \left( \nabla \ln n_0 \mathbf{E} \right) \) in Eq. (1), which couples the fluid polarization to its trajectory (via the nonlinear refractive index). In the corresponding Lagrangian formalism, Eq. (2), the SOC effects are encoded in the term \( \nabla \cdot \mathbf{E} \mathbf{E}^* \mathbf{E} \mathbf{E}^* \). While naturally present in Maxwell equations, the latter is discarded within the paraxial approximation usually considered in the context of fluids of light [22]. In the following, we do not perform this approximation but work out the full Lagrangian (2).

Our aim is to determine the field amplitude inside the medium, given its value \( \mathbf{E}(r, z = 0) \) at the air-medium interface as a function of the transverse \( r_\perp = (x, y) \) coordinates. This can be regarded as the evolution problem of a 2D system with respect to the effective time \( z \) [22]. In the following, we assume that \( \mathbf{E} \) is the sum of a large homogeneous background and a small fluctuation, and treat the latter using Bogoliubov-Popov theory [42–45]. For that purpose, we write \( \mathbf{E} = (E_+, E_2, E_-) \) and employ the density-phase decomposition \( E_+ = \sqrt{I} \cos(\theta/2) e^{i(\theta + \chi/2)} \), \( E_- = \sqrt{I} \sin(\theta/2) e^{i(\theta - \chi/2)} \) of the field circular components \( E_\pm = \mp (E_x \mp iE_y)/\sqrt{2} \). Here \( I \) and \( \theta \) quantify the total optical intensity of the transverse components and their relative weight, respectively, while \( \Theta \) (\( \chi \)) is their total (relative) phase. We then split the field into a background and a fluctuating contribution, writing \( I = I_0 + \delta I, \theta = \theta_0 + \delta \theta, \) and \( \chi = \Delta k z + \delta \chi \), where \( \Delta k = k_+ - k_- \).

The wave numbers \( k_\pm \) of the two polarization components are imposed by the Helmholtz equation:

\[
k_{\pm} = \sqrt{\beta_0 - 2\beta_0 (g_d \pm g_s \cos \theta_0) I_0}. \tag{3}
\]

Notice that \( \Delta k = k_+ - k_- \neq 0 \) as soon as \( g_s \neq 0 \) and \( \cos \theta_0 \neq 0 \), i.e., when the background field is elliptically or circularly polarized. This defines the phenomenon of nonlinear birefringence, which will play a crucial role in the following.

Next, we insert the fluctuation variables into the Lagrangian (2) and determine the quadratic correction \( S^{(2)} \) to the background action. This is achieved by redefining \( E_\pm \rightarrow e^{i\varphi} E_\pm \) and \( \Theta \rightarrow \Theta + (k_+ + k_-) z/2 \), and expanding Eq. (2) with respect to \( \delta I, \delta \theta, \) and \( \delta \chi \). Note that the fluctuations of \( \Theta \) are, in contrast, possibly large in two dimensions [44], but its derivatives remain small, and so does \( \mathcal{E}_2 \). In this procedure, \( S^{(2)} \) turns out to be independent of the \( z \) derivatives of \( \mathcal{E}_2 \), so that one can use the Euler-Lagrange equation \( \delta S^{(2)}/\delta \mathcal{E}_2^* = 0 \) and its complex conjugate to express the longitudinal field as a function of the other variables [46].

The quadratic action \( S^{(2)} = \int dz \int d^3 q \delta \left( 2\pi \right)^3 \tilde{\mathcal{L}}^{(2)}(4) \) can be written in terms of a single column vector \( X = (\delta I/2 I_0 \delta \theta/2 \Theta \delta \chi/2)^T \) for the Fourier variables with respect to \( r_\perp \), e.g., \( \delta I(q, z) = \int d^2 r_\perp \delta I(r_\perp, z) e^{-i q \cdot r_\perp} \), with \( q_\perp = (q_\perp \cos \varphi_q, q_\perp \sin \varphi_q) \) the transverse momentum. We find:

\[
\tilde{\mathcal{L}}^{(2)}(4) = \dot{X}^\dagger \mathbf{A}_2 \dot{X} + \dot{X}^\dagger \mathbf{A}_1 X + \dot{X}^\dagger \mathbf{A}_T \dot{X} - X^\dagger \mathbf{A}_0 X. \tag{4}
\]

Here the dot refers to a derivative with respect to \( z \). The 4 \( \times \) 4 matrices \( \mathbf{A}_{0,1,2} \) are real \( \pi \)-periodic functions of the angular variable \( \varphi(z) = \varphi_0 + \Delta k z/2 \) (see [46]), which encodes a breaking of translation invariance along the effective time axis \( z \), the central result of the Letter. This dependence stems from the SOC term in Eq. (2), and is completely absent in the paraxial framework. In our description, the paraxial approximation corresponds to taking \( \dot{X}/\beta_0, (q_\perp/\beta_0)^2, \) and \( g_{d,s} I_0/\beta_0 \) small; the resulting expansion of \( \tilde{\mathcal{L}}^{(2)} \) up to first order becomes \( \varphi(z) \)-independent and formally identical to the Bogoliubov Lagrangian of symmetric binary mixtures of atomic Bose-Einstein condensates [48, 49].

In the same spirit as in Ref. [30], we define an Hamiltonian \( \mathcal{H}^{(2)} = \Pi^T \dot{X} + \dot{X} \Pi^T - \tilde{\mathcal{L}}^{(2)} \) depending on \( X \) and the conjugate momenta vector \( \Pi = \partial \tilde{\mathcal{L}}^{(2)}/\partial \dot{X}^T \). The effective-time evolution of \( X \) and \( \Pi \) is governed by the Hamilton equations \( \dot{X} = \partial \mathcal{H}^{(2)}/\partial \Pi^T \) and \( \dot{\Pi} = -\partial \mathcal{H}^{(2)}/\partial X^T \), yielding the eight coupled equations

\[
\begin{pmatrix}
\dot{X} \\
\dot{\Pi}^T
\end{pmatrix} = \begin{pmatrix}
-\Lambda_2^{-1} \Lambda_1 \\
(\Lambda_2^{-1} \Lambda_2^{-1} \Lambda_1 + \Lambda_0)^{-1} (\Lambda_2^{-1} \Lambda_1)^T
\end{pmatrix} \begin{pmatrix}
X \\
\Pi^T
\end{pmatrix}. \tag{5}
\]

Here the dot refers to a derivative with respect to \( z \).
In [30], Eq. (5) was solved in the $\Delta k = 0$ case (linearly polarized background field), where the matrix of coefficients is constant. Here on the contrary, we assume that the birefringence condition $\Delta k \neq 0$ is fulfilled. Hence, the coefficients of Eq. (5) oscillate in $z$ with period $2\pi/\Delta k$.

We stress that although this spontaneous breaking of time-translation symmetry is typical of Bogoliubov equations for periodically driven systems [50–54] (see also [55], here it occurs in a purely isolated system. The underlying mechanism is the interplay between nonlinear birefringence and SOC. The latter is responsible for the presence, in the Helmholtz Lagrangian (2), of interference terms between the two light polarization components propagating at relative wave vector $\Delta k$.

According to Floquet’s theorem [56, 57], the general solution of Eq. (5) has the form

$$X(q_{\perp}, z) = \sum_\ell C_\ell(q_{\perp}) \left[ X_{0,\ell}(q_{\perp}, \varphi) \right] \left[ \Pi_{0,\ell}^+(q_{\perp}, \varphi) \right] e^{-i\Omega_{\ell}(q_{\perp})z}. \tag{6}$$

Here the sum runs over eight independent solutions, labeled by the subscript $\ell$ and appearing with weight $C_\ell$. These Floquet solutions are characterized by their eigenfunctions $X_{0,\ell}$ and $\Pi_{0,\ell}$ and the corresponding quasi-frequencies, $\Omega_{\ell}$. Note that as is customary for Bogoliubov equations [58], for each solution with quasi-frequency $-\Omega_{\ell}$ associated with the same physical oscillation, hence there is a total of four Bogoliubov modes. In the paraxial regime, in contrast, one has only a density ($d$) and a spin ($s$) mode, characterized by in-phase and out-of-phase oscillations of the intensities of the two polarization components, respectively [46].

We first plot the real part of the quasi-frequency spectrum in Fig. 2(a). Because the $\Omega_{\ell}$'s are defined modulo $\Delta k$, it is sufficient to take their real part in the first Brillouin zone, $-\Delta k/2 < \text{Re} \Omega_{\ell} \leq \Delta k/2$ [46]. In the $q_{\perp} \to 0$ limit, the spectrum in panel (a) displays the usual four phononic bands $\pm \Omega_{d,s}(q_{\perp}) \approx \pm c_{d,s} q_{\perp}$, with two sound velocities $c_{d,s}$. Those correspond to the standard Bogoliubov modes in the paraxial regime, where $c_{d(s)}^2 = (g_d + g_s \pm \sqrt{g_d^2 + g_s^2 + 2g dg_s \cos 2\theta_0})/2\beta_0$. On the contrary, the other four bands tend to a finite value and describe light reflected at the air-medium interface.

At increasing $q_{\perp}$ the various bands cross one another at several points, giving rise to an involved structure. In particular, as a consequence of the Floquet structure some quasi-frequencies labeled by numbers in Fig. 2(a) -- develops a finite imaginary part at certain values of $q_{\perp}$, see Fig. 2(b) (notice that complex quasi-frequencies always occur in complex conjugate pairs). These points can be seen as avoided crossings in the complex frequency plane, and they reveal the presence of parametric resonances [59], analogous to those observed in periodically driven systems [50–54], inducing an exponential growth of the corresponding Bogoliubov modes. This important result will be discussed in more details below.

A central feature of Eq. (6) is that the modes $X_{0,\ell}$ and $\Pi_{0,\ell}$ are $\pi$-periodic functions of $\varphi(z)$. Since $\varphi(z)$ contains a term linear in $z$, in general this leads to periodic oscillations of specific observables in the effective time $z$.

To illustrate this phenomenon and as an application of the above formalism, we now consider a concrete scenario where a fluid of light is initially prepared in the form of a (two-component) plane-wave background plus a small fluctuating field (see Fig. 1):

$$\begin{bmatrix} E_+(r_{\perp}, z = 0) \\ E_-(r_{\perp}, z = 0) \end{bmatrix} = \sqrt{I_0} \begin{bmatrix} \cos \varphi_0 + \epsilon \varphi_+ (r_{\perp}) \\ \sin \varphi_0 + \epsilon \varphi_- (r_{\perp}) \end{bmatrix}, \tag{7}$$

where $0 < \epsilon \ll 1$, and $\varphi_0$ ($\alpha = \pm$) is a two-component random complex speckle field of two-point correlation $\langle \varphi_0(r_{\perp}) \varphi_0^* (r_{\perp} + \Delta r_{\perp}) \rangle = \gamma (\Delta r_{\perp}) \delta_{\sigma \sigma'}$, the brackets denoting statistical averaging. For definiteness we consider a Gaussian correlation $\gamma (\Delta r_{\perp}) = \exp(-|\Delta r_{\perp}|^2/4\sigma^2)$, with $\sigma$ the correlation length. In practice, an initial state of the form (7) was recently exploited experimentally [29] in order to realize an optical analogue of the quench dynamics of thermal fluctuations in a 2D Bose gas, following [36]. To access the state vector (6) at arbitrary effective time $z$, one needs to compute the mode weights $C_\ell(q_{\perp})$ from the initial condition. This is achieved by projecting Eq. (6) evaluated at $z = 0$ on each mode, making use of the mode orthogonality [46]. This procedure leads to a simple expression for the weights:

$$C_\ell(q_{\perp}) = \frac{i}{\mathcal{N}_\ell(q_{\perp})} \left[ X^\dagger_{0,\ell}(q_{\perp}, \varphi_0) \Pi^+_\ell(q_{\perp}, z = 0) \right] - \left[ \Pi^T_{0,\ell}(q_{\perp}, \varphi_0) X(q_{\perp}, z = 0) \right], \tag{8}$$

where $\mathcal{N}_\ell(q_{\perp})$ is the mode norm and $\ell$ is such that $\Omega_{\ell}(q_{\perp}) = \Omega^j_\ell(q_{\perp})$. The input conjugate momenta are
FIG. 3. Effective-time evolution of the intensity-intensity correlation function. In the main plot we compare the exact (solid blue curve) and paraxial (dashed curve) predictions for \( \beta_0 = 15 \) and the same parameters \( \theta_0 = \pi/4, g_z I_0/\beta_0 = 0.2, \) and \( g_z I_0/\beta_0 = 0.05 \) as in Fig. 2. \( z \) is measured in units of the nonlinear length \( z_{NL} = 1/2g_z I_0 \). In the inset we zoom on the large-\( z \) tail and include two additional curves showing the results for \( \beta_0 = 22 \) (red curve) and \( g_z I_0/\beta_0 = 0.035 \) (green curve), the other parameters being the same as above.

fixed requiring the vanishing of the weights of reflected modes in Eq. (6) [46]. Combining Eqs. (7) and (8) and inserting the weights into Eq. (6) gives access to any statistical observable. One of the simplest is the function \( g_2(z) \equiv \langle \delta I(r_\perp, 0) \delta I(r_\perp, z) \rangle \), which expresses how intensity fluctuations at a finite effective time \( z > 0 \) and at a given point in the transverse plane are correlated with their \( z = 0 \) value (see Fig. 1). We find [46]

\[
g_2(z) = \frac{1}{\epsilon^2 I_0^2} \sum q_\perp \int_0^\infty \frac{dq_\perp}{2\pi} \tilde{\gamma}(q_\perp) K(q_\perp, z)e^{-i\Omega(q_\perp)z},
\]

where \( \tilde{\gamma}(q_\perp) \) is the speckle power spectrum. In the paraxial regime \( q_\perp \to 0 \) and \( g_z I_0 \ll \beta_0 \), the coefficients for the density and the spin modes \( K_{d,s}(z) \approx 1/2 \pm (g_d + g_z \cos 2\theta_0)/2\sqrt{g_d^2 + g_z^2 + 2g_d g_z \cos 2\theta_0} \) reduce to constants independent of \( q_\perp \) and \( z \). The time correlation \( g_2 \) in the paraxial limit is shown in Fig. 3 as a function of \( z \) (dashed curve), at fixed background polarization, nonlinearities and correlation length. At \( z = 0, g_2(0)/\epsilon^2 I_0^2 = 2 \), which corresponds to the Rayleigh law of the speckle field. At large \( z \), the correlation is negative and approaches zero as \( z \to +\infty \) following \( g_2(z)/\epsilon^2 I_0^2 \approx - (\sqrt{g_d I_0}/\beta_0 z/2\sigma)^{-2} \). In between these two asymptotic limits \( g_2 \) changes sign, in agreement with the sum rule \( \int_0^\infty dz g_2(z) = 0 \).

Beyond the paraxial limit, the emerging Floquet structure of the spectrum gives rise to periodic \( z \)- oscillations of the coefficients \( K_\ell \). This manifests itself in the appearance of oscillations of \( g_2 \) about its background value at large \( z \), as seen in Fig. 3 (solid blue curve). The magnitude of these oscillations is directly controlled by two main physical parameters. First, the speckle correlation length \( \sigma \), which controls by how much the fluid of light deviates from paraxiality, a necessary condition for the system to exhibit SOC. Second, the coupling strength \( g_z \), which gives rise to nonlinear birefringence. These dependences are illustrated in the inset of Fig. 3: the oscillations amplitude decreases (increases) with \( \beta_0 \sigma (g_z I_0/\beta_0) \). In particular, at large correlation lengths \( \sigma \), the spectrum \( \tilde{\gamma}(q_\perp) \) becomes very narrow so that the integral \( \langle 9 \rangle \) becomes dominated by the small-\( q_\perp \) modes and \( K_\ell(q_\perp, z) \) can be replaced by its constant, \( q_\perp = 0 \) value. The oscillation frequency, in contrast, is practically independent of \( \sigma \). It is mainly governed by the momentum mismatch \( \Delta k \) of the two circularly-polarized components of the fluid, entailed by the nonlinear birefringence. From a Fourier analysis of \( g_2(z) \), we find that these oscillations are essentially harmonic at large enough \( \sigma \), with a frequency \( \sim 0.8|\Delta k| \) (see [46]).

A second remarkable consequence of the Floquet structure of the fluid’s spectrum is a phenomenon of parametric instability. At large enough \( z \), we observe an exponential growth of the fringe amplitude, as illustrated in Fig. 4(a). This parametric instability stems from the spontaneous emergence of imaginary frequencies in the Floquet spectrum, see Fig. 2(b). To explore more systematically the occurrence of these parametric instabilities in our system, we have analyzed their characteristic growth rate \( \Gamma \) for various interaction strength \( g_z I_0 \) and polarization \( \cos \theta_0 \) of the background field. To this aim, we have studied the evolution of \( g_2(z) \) within a fixed range of large values of \( z \) and fitted the data with an exponential law of the form \( \exp(\Gamma z) \) for many values of these parameters. The results are summarized by the diagram in Fig. 4(b). As expected, \( \Gamma \) vanishes when either \( \cos \theta_0 \to 0 \) (linearly polarized
background) or $g_s \rightarrow 0$ (no spin-dependent nonlinearity). At large enough $g_s$, Fig. 4(b) also reveals a sharp increase of $\Gamma$ as $\cos \theta_0$ is decreased from 1, pinpointed by the white dashed curve. This corresponds to the appearance of an additional imaginary frequency that starts to dominate the exponential growth in the chosen effective-time window, our fitting procedure yielding the largest $\Gamma$.

In conclusion, we have shown that two-component fluids of light in the presence of nonlinear birefringence spontaneously break translation symmetry along the effective-time direction, leading to the occurrence of a Floquet band structure in the Bogoliubov spectrum. This phenomenon fundamentally stems from the presence of intrinsic spin-orbit coupling, and cannot be captured by simple paraxial descriptions. We have shown that the emerging Floquet structure is visible in periodic oscillations of the intensity-intensity correlation function of the fluid, as well as by the presence of parametric resonances at large enough propagation distances.

We thank T. Bienaimé, D. Delande, N. Pavloff, K. Sacha, and J. Zakrzewski for fruitful discussions. We acknowledge financial support from the Agence Nationale de la Recherche (grant ANR-19-CE30-0028-01 CONFOCAL).

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Supplemental material:
Spontaneous breaking of time translation symmetry in a spin-orbit-coupled fluid of light

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In this supplemental material we provide further details about the calculations leading to the results of the main text, in particular the nonparaxial Bogoliubov action (Sec. I), the paraxial limit and the sound velocity (Sec. II), the solution of the Bogoliubov equations (Sec. III), the choice of the input conjugate momenta (Sec. IV), and the expression of the intensity-intensity correlation function (Sec. V).

I. NONPARAXIAL BOGOLIUBOV ACTION

The starting point for the formulation of the Bogoliubov theory for a nonparaxial fluid of light is represented by its Lagrangian density [Eq. (2) of the main text]. First, one has to rewrite this quantity in terms of the density-phase variables. The resulting expression, that can be found in Ref. [1], should then be expanded up to second order in the small fluctuations about the background field. After switching to the Fourier space, it is convenient to decompose the quadratic Lagrangian as \( \tilde{L}_z^{(2)} = \tilde{L}_z^{(2)} + \tilde{L}_z^{(2)} \), where \( \tilde{L}_z^{(2)} \) is the term that contains the longitudinal field \( \tilde{E}_z \), while \( \tilde{L}_z^{(2)} \) depends only on the transverse components. In the case of the elliptically polarized background considered in the present work one has

\[
\tilde{L}_z^{(2)} = -\frac{1}{2\beta_0} \left[ (q_z^2 - k_0^2) \tilde{E}_z^*(\mathbf{q}_\perp, z) \tilde{E}_z(\mathbf{q}_\perp, z) + \frac{\Delta k_0^2}{2} \tilde{E}_z(\mathbf{q}_\perp, z) \tilde{E}_z^*(-\mathbf{q}_\perp, z) \right] + \frac{i q_\perp}{2\beta_0} \sqrt{\frac{E_z}{2}} \left[ \mathcal{A}(\mathbf{q}_\perp, z) \tilde{E}_z^*(\mathbf{q}_\perp, z) - \mathcal{A}^*(\mathbf{q}_\perp, z) \tilde{E}_z(\mathbf{q}_\perp, z) \right],
\]

where \( k_0^2 = \beta_0^2 - 2\beta_0 g_\perp g_\| I_0, \Delta k_0^2 = 2\beta_0 g_\perp I_0 \sin \vartheta_0 \), and

\[
\mathcal{A}(\mathbf{q}_\perp, z) = \left( \cos \frac{\vartheta_0}{2} e^{i\varphi} - \sin \frac{\vartheta_0}{2} e^{-i\varphi} \right) \frac{\delta \hat{I}(\mathbf{q}_\perp, z)}{2I_0} - \left( \sin \frac{\vartheta_0}{2} e^{i\varphi} + \cos \frac{\vartheta_0}{2} e^{-i\varphi} \right) \frac{\delta \hat{\chi}(\mathbf{q}_\perp, z)}{2} + i \left( \cos \frac{\vartheta_0}{2} e^{i\varphi} - \sin \frac{\vartheta_0}{2} e^{-i\varphi} \right) \frac{\delta \hat{\chi}(\mathbf{q}_\perp, z)}{2} + \left( k_+ \cos \frac{\vartheta_0}{2} e^{i\varphi} - k_- \sin \frac{\vartheta_0}{2} e^{-i\varphi} \right) \frac{\delta \hat{\chi}(\mathbf{q}_\perp, z)}{2} - \left( k_+ \cos \frac{\vartheta_0}{2} e^{i\varphi} + k_- \sin \frac{\vartheta_0}{2} e^{-i\varphi} \right) \frac{\delta \hat{\chi}(\mathbf{q}_\perp, z)}{2}.
\]

We recall that \( \varphi = \varphi_0 + \Delta k z/2 \). Since the Bogoliubov Lagrangian density and the corresponding action \( S^{(2)} = \int dz \int d^2\mathbf{q}_\perp / (2\pi)^2 \tilde{L}^{(2)} \) do not depend on the effective-time derivatives of \( \tilde{E}_z \), the Euler-Lagrange equation \( \delta S^{(2)}/\delta \tilde{E}_z(\mathbf{q}_\perp) = 0 \) and its complex conjugate yield the relation

\[
\tilde{E}_z(\mathbf{q}_\perp, z) = \sqrt{\frac{I_0}{2}} q_\perp \tilde{E}_z^*(\mathbf{q}_\perp, z) \mathcal{A}(\mathbf{q}_\perp, z) + \Delta k_0^2 \mathcal{A}^*(\mathbf{q}_\perp, z),
\]

which allows us to eliminate the longitudinal field from the Lagrangian in favor of the variables related to the transverse components and their derivatives. The final result corresponds to Eq. (4) of the main text, i.e.,

\[
\tilde{L}^{(2)} = \hat{X}^\dagger \Lambda_2 \hat{X} + \hat{X}^\dagger \Lambda_1 X + X^\dagger \Lambda_1^T \hat{X} - X^\dagger \Lambda_0 X,
\]

\[\text{arXiv:2206.11714v1 [physics.optics] 23 Jun 2022}\]
where we have introduced three 4 × 4 matrices having the structure

$$
\Lambda_k = -\frac{I_0}{2\beta_0} \begin{bmatrix}
(\Lambda_k)_{1,1} & (\Lambda_k)_{1,2} & (\Lambda_k)_{1,3} & (\Lambda_k)_{1,4} \\
(\Lambda_k)_{2,1} & (\Lambda_k)_{2,2} & (\Lambda_k)_{2,3} & (\Lambda_k)_{2,4} \\
(\Lambda_k)_{3,1} & (\Lambda_k)_{3,2} & (\Lambda_k)_{3,3} & (\Lambda_k)_{3,4} \\
(\Lambda_k)_{4,1} & (\Lambda_k)_{4,2} & (\Lambda_k)_{4,3} & (\Lambda_k)_{4,4}
\end{bmatrix} \quad (k = 0, 1, 2).
$$

(5)

We now give the expressions of all the entries of these matrices. We first define

$$
A(q_\perp) = \frac{q_\perp^2 - k_0^2}{2(q_\perp^2 - k_0^2)^2 - (\Delta k_0^2)^2},
$$

(6a)

$$
B(q_\perp) = \frac{\Delta k_0^2}{2(q_\perp^2 - k_0^2)^2 - (\Delta k_0^2)^2}.
$$

(6b)

Then, the entries of $\Lambda_0$ are

$$
(\Lambda_0)_{1,1} = -\frac{q_\perp^2}{2} - 4\beta_0(g_d + g_* \cos^2 \vartheta_0)I_0 + A(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) - B(q_\perp) k_+ k_- \sin \vartheta_0 
- \frac{q_\perp^2}{2} \sin \vartheta_0 + A(q_\perp) k_+ k_- \sin \vartheta_0 - B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \sin 2\varphi,
$$

(7a)

$$
(\Lambda_0)_{1,2} = (\Lambda_0)_{2,1} = 2\beta_0 g_d I_0 \sin 2\vartheta_0 - A(q_\perp) \frac{k_+^2 - k_-^2}{2} \sin \vartheta_0 - B(q_\perp) k_+ k_- \cos \vartheta_0 
- \frac{q_\perp^2}{2} \cos \vartheta_0 + A(q_\perp) k_+ k_- \cos \vartheta_0 + B(q_\perp) \frac{k_+^2 - k_-^2}{2} \sin \vartheta_0 \cos 2\varphi,
$$

(7b)

$$
(\Lambda_0)_{1,3} = (\Lambda_0)_{3,1} = -B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} - k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \sin 2\varphi,
$$

(7c)

$$
(\Lambda_0)_{1,4} = (\Lambda_0)_{4,1} = \left[ \frac{q_\perp^2}{2} \sin \vartheta_0 + A(q_\perp) k_+ k_- \sin \vartheta_0 - B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \right] \sin 2\varphi,
$$

(7d)

$$
(\Lambda_0)_{2,2} = -\frac{q_\perp^2}{2} - 4\beta_0 g_d I_0 \sin^2 \vartheta_0 + A(q_\perp) \left( k_+^2 \sin^2 \frac{\vartheta_0}{2} + k_-^2 \cos^2 \frac{\vartheta_0}{2} \right) + B(q_\perp) k_+ k_- \sin \vartheta_0 
+ \frac{q_\perp^2}{2} \sin \vartheta_0 + A(q_\perp) k_+ k_- \sin \vartheta_0 + B(q_\perp) \left( k_+^2 \sin^2 \frac{\vartheta_0}{2} + k_-^2 \cos^2 \frac{\vartheta_0}{2} \right) \cos 2\varphi,
$$

(7e)

$$
(\Lambda_0)_{2,3} = (\Lambda_0)_{3,2} = \left[ \frac{q_\perp^2}{2} + A(q_\perp) k_+ k_- + B(q_\perp) \frac{k_+^2 + k_-^2}{2} \sin \vartheta_0 \right] \sin 2\varphi,
$$

(7f)

$$
(\Lambda_0)_{2,4} = (\Lambda_0)_{4,2} = \left[ \frac{q_\perp^2}{2} \cos \vartheta_0 + A(q_\perp) k_+ k_- \cos \vartheta_0 + B(q_\perp) \frac{k_+^2 - k_-^2}{2} \sin \vartheta_0 \right] \sin 2\varphi,
$$

(7g)

$$
(\Lambda_0)_{3,3} = -\frac{q_\perp^2}{2} + A(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) + B(q_\perp) k_+ k_- \sin \vartheta_0 
- \frac{q_\perp^2}{2} \sin \vartheta_0 + A(q_\perp) k_+ k_- \sin \vartheta_0 + B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \cos 2\varphi,
$$

(7h)

$$
(\Lambda_0)_{3,4} = (\Lambda_0)_{4,3} = -\frac{q_\perp^2}{2} \cos \vartheta_0 + A(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} - k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) 
- B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} - k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \cos 2\varphi,
$$

(7i)

$$
(\Lambda_0)_{4,4} = -\frac{q_\perp^2}{2} + A(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) - B(q_\perp) k_+ k_- \sin \vartheta_0 
+ \frac{q_\perp^2}{2} \sin \vartheta_0 + A(q_\perp) k_+ k_- \sin \vartheta_0 - B(q_\perp) \left( k_+^2 \cos^2 \frac{\vartheta_0}{2} + k_-^2 \sin^2 \frac{\vartheta_0}{2} \right) \cos 2\varphi.
$$

(7j)

The entries of $\Lambda_1$ are

$$
(\Lambda_1)_{1,1} = -A(q_\perp) \frac{k_+ - k_-}{2} \sin \vartheta_0 + B(q_\perp) \left( k_+ \cos^2 \frac{\vartheta_0}{2} - k_- \sin^2 \frac{\vartheta_0}{2} \right) \sin 2\varphi,
$$

(8a)
\[(A_{1})_{1,2} = \left[ A(q_{\perp}) \left( k_{+} \sin^{2} \frac{\vartheta_{0}}{2} + k_{-} \cos^{2} \frac{\vartheta_{0}}{2} \right) + B(q_{\perp}) \left( k_{+} \sin \vartheta_{0} \right) \right] \frac{k_{+} + k_{-}}{2} \sin 2 \varphi, \]  
\[(A_{1})_{1,3} = A(q_{\perp}) \left( k_{+} \cos^{2} \frac{\vartheta_{0}}{2} + k_{-} \sin^{2} \frac{\vartheta_{0}}{2} \right) + B(q_{\perp}) \frac{k_{+} + k_{-}}{2} \sin 2 \varphi, \]  
\[(A_{1})_{1,4} = A(q_{\perp}) \left( k_{+} \cos^{2} \frac{\vartheta_{0}}{2} - k_{-} \sin^{2} \frac{\vartheta_{0}}{2} \right) + B(q_{\perp}) \frac{k_{+} - k_{-}}{2} \sin \vartheta_{0} \]  
\[\begin{aligned}
(A_{2})_{1,1} = 1 - A(q_{\perp}) - B(q_{\perp}) \sin \vartheta_{0} + [A(q_{\perp}) \sin \vartheta_{0} + B(q_{\perp})] \cos 2 \varphi, 
\end{aligned}\]
\( (\Lambda_2)_{1,2} = (\Lambda_2)_{2,1} = -B(q_\perp) \cos \vartheta_0 + A(q_\perp) \cos \vartheta_0 \cos 2\varphi, \)
\( (\Lambda_2)_{1,3} = (\Lambda_2)_{3,1} = -B(q_\perp) \cos \vartheta_0 \sin 2\varphi, \)
\( (\Lambda_2)_{1,4} = (\Lambda_2)_{4,1} = -A(q_\perp) \sin \vartheta_0 \sin 2\varphi, \)
\( (\Lambda_2)_{2,2} = 1 - A(q_\perp) + B(q_\perp) \sin \vartheta_0 - [A(q_\perp) \sin \vartheta_0 - B(q_\perp)] \cos 2\varphi, \)
\( (\Lambda_2)_{2,3} = (\Lambda_2)_{3,2} = -[A(q_\perp) - B(q_\perp) \sin \vartheta_0] \sin 2\varphi, \)
\( (\Lambda_2)_{2,4} = (\Lambda_2)_{4,2} = -A(q_\perp) \cos \vartheta_0 \sin 2\varphi, \)
\( (\Lambda_2)_{3,3} = 1 - A(q_\perp) + B(q_\perp) \sin \vartheta_0 + [A(q_\perp) \sin \vartheta_0 - B(q_\perp)] \cos 2\varphi, \)
\( (\Lambda_2)_{3,4} = (\Lambda_2)_{4,3} = [1 - A(q_\perp)] \cos \vartheta_0 - B(q_\perp) \cos \vartheta_0 \cos 2\varphi, \)
\( (\Lambda_2)_{4,4} = 1 - A(q_\perp) - B(q_\perp) \sin \vartheta_0 - [A(q_\perp) \sin \vartheta_0 + B(q_\perp)] \cos 2\varphi. \)

**II. PARAXIAL LIMIT AND SOUND VELOCITIES**

As stated in the main text, the paraxial Bogoliubov Lagrangian can be deduced by expanding the nonparaxial one [Eq. (4)] up to first order in \( X/\beta_0, \ (q_\perp/\beta_0)^2, \) and \( g_{d,s} I_0/\beta_0. \) The final result is
\[
\mathcal{L}^{(2)}_{\text{par}} = \dot{X}^\dagger \Lambda_{\text{par},1} X + X^\dagger \Lambda_{\text{par},1}^T \dot{X} - X^\dagger \Lambda_{\text{par},0} X, \tag{10}
\]
where
\[
\Lambda_{\text{par},1} = I_0 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-\cos \vartheta_0 & \sin \vartheta_0 & 0 & 0
\end{bmatrix},
\]
\[
\Lambda_{\text{par},0} = I_0 \begin{bmatrix}
\frac{g_1^2}{2\beta_0^2} + 2(g_d + g_s \cos^2 \vartheta_0) I_0 & -g_s I_0 \sin 2\vartheta_0 & 0 & 0 \\
-g_s I_0 \sin 2\vartheta_0 & \frac{g_1^2}{2\beta_0^2} + 2g_s I_0 \sin^2 \vartheta_0 & 0 & 0 \\
0 & 0 & \frac{g_1^2}{2\beta_0^2} + g_s \cos 2\vartheta_0 & \frac{g_1^2}{2\beta_0^2} \cos \vartheta_0 \\
0 & 0 & \frac{g_1^2}{2\beta_0^2} \cos \vartheta_0 & \frac{g_1^2}{2\beta_0^2}
\end{bmatrix}. \tag{11}
\]
The Euler-Lagrange equation for \( X \) takes the simple form
\[
\dot{X} = -[(\Lambda_{\text{par},1} - \Lambda_{\text{par},1}^T)]^{-1} \Lambda_{\text{par},0} X. \tag{12}
\]
By making use of the Ansatz \( X(q_\perp, z) = X_0(q_\perp) e^{-i\Omega_1(q_\perp)z} \) one can reduce this equation to a four-dimensional eigenvalue problem. One finds four solutions characterized by the oscillation frequencies \( \pm \Omega_d \) and \( \pm \Omega_s \), which exhibit the standard Bogoliubov form
\[
\Omega_{d(s)}(q_\perp) = \sqrt{\frac{q_1^2}{2\beta_0^2} \left[ \frac{q_1^2}{2\beta_0^2} + 2\beta_0 \epsilon_{d(s)}^2 \right]}. \tag{13}
\]
These are the counterparts of the density and spin mode of a binary mixture of atomic Bose-Einstein condensates, featuring in-phase and out-of-phase oscillations of the densities of the two spin components, respectively. The corresponding sound velocities are given by
\[
c_{d(s)}^2 = \frac{(g_d + g_s \pm \sqrt{g_1^2 + g_s^2 + 2g_d g_s \cos 2\vartheta_0}) I_0}{2\beta_0}, \tag{14}
\]
where the upper (lower) sign refers to the density (spin) mode, respectively. In Fig. 1 we plot the velocity of (left) density and (right) spin sound waves as a function of the background polarization. The paraxial prediction (14) (dashed curves) is compared with the values obtained in the nonparaxial description of this work (solid curves) by numerically computing the slope of the linear bands appearing in the low-\( q_\perp \) part of the Bogoliubov spectrum (see Fig. 2 of the main text). We consider several values of the nonlinear coupling \( g_d I_0/\beta_0 \), keeping the ratio \( g_s/g_d \) fixed. We notice that the qualitative behavior does not change between the two frameworks. On the other hand, the quantitative discrepancy is negligible at small \( g_d I_0/\beta_0 \), but it grows significantly at increasing nonlinearity. We further mention that \( c_{d}^2 \) approaches the value \( g_d I_0/((\beta_0 - 3g_d I_0) \beta_0) \) predicted in Ref. [1] in the \( \cos \vartheta_0 \to 0 \) limit (linearly polarized
FIG. 1. Velocity of (left) density and (right) spin sound waves as a function of the background polarization quantified by \( \cos \vartheta_0 \). The solid curves represent the results of the full nonparaxial theory, while the dashed ones correspond to the paraxial predictions (14). We take \( g_s I_0/\beta_0 = 0.02 \) (blue lines), 0.1 (red lines), 0.2 (green lines), with the same ratio \( g_s/g_d = 0.25 \) for all the curves.

Concerning the spin sound velocity, we first recall that its value in the case of a linearly polarized background field was computed in Ref. [1] and found to be anisotropic, i.e., depending on \( \varphi \). Here we checked that \( c_s^2 \to g_s I_0[\beta_0 - 2(g_d + g_s)]I_0/[\beta_0 - 2g_d + 2g_s]I_0 \) as \( \cos \vartheta_0 \to 0 \), which is the prediction of Ref. [1] [see Eq. (47) therein] evaluated at \( \varphi = \pi/4 \). In this respect we point out that, unlike the frequency spectrum of a linearly polarized background field studied in Ref. [1], the quasi-frequency spectrum computed in the present work is independent of \( \varphi_q \), that is, it is isotropic. However, it should be noticed that frequency and quasi-frequency have different definitions that only make sense in the \( \Delta k = 0 \) and \( \Delta k \neq 0 \) case, respectively; hence, it is not inconsistent that the two spectra have qualitatively different features.

III. SOLUTION OF BOGOLIUBOV EQUATIONS. HILL’S METHOD

Let us rewrite Eq. (5) of the main text as

\[
\mathbf{B} \begin{bmatrix} X \\ \Pi^* \end{bmatrix} = \mathbf{B} \begin{bmatrix} X \\ \Pi^* \end{bmatrix} ,
\]

where the \( 8 \times 8 \) matrix

\[
\mathbf{B} = \mathbf{i} \begin{bmatrix} \mathbf{J} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{B} \end{bmatrix}
\]

is a \( \pi \)-periodic function of \( \varphi = \varphi_q + \Delta k z/2 \). As discussed in the main text, the Floquet theorem allows us to write the general integral of Eq. (15) as a linear combination of eight independent solutions of the form

\[
\begin{bmatrix} X_\ell(q_\perp, z) \\ \Pi^*_\ell(q_\perp, z) \end{bmatrix} = e^{-i\Omega_\ell(q_\perp)z} \begin{bmatrix} X_0,\ell(q_\perp, \varphi) \\ \Pi^*_0,\ell(q_\perp, \varphi) \end{bmatrix},
\]

where \( \Omega_\ell(q_\perp) \) is the quasi-frequency and the amplitudes \( X_0,\ell \) and \( \Pi^*_0,\ell \) are themselves \( \pi \)-periodic in \( \varphi \). In order to determine these solutions we adopt Hill’s method [2]. First, we formally represent \( \mathbf{B} \) as a Fourier expansion, \( \mathbf{B} = \sum_{n=-\infty}^{+\infty} \mathbf{B}_n e^{-2i\varphi} \), where the matrix expansion coefficients \( \mathbf{B}_n = 0 \) when \( n \neq 0, \pm1 \), and \( \mathbf{B}_{-1} = -\mathbf{B}_1^* \). Similarly, we Fourier expand the amplitudes entering the Ansatz (17), \( X_0,\ell(q_\perp, \varphi) = \sum_{n=-\infty}^{+\infty} X_0,\ell,n(q_\perp) e^{-2i\varphi} \) and \( \Pi^*_0,\ell(q_\perp, \varphi) = \sum_{n=-\infty}^{+\infty} \Pi^*_0,\ell,n(q_\perp) e^{-2i\varphi} \), where the \( X_0,\ell,n(q_\perp)'s \) and \( \Pi^*_0,\ell,n(q_\perp)'s \) are the 4-component expansion coefficients. Plugging these expansions into Eq. (15) and equating the terms on the two sides having the same oscillating behavior one finds

\[
\sum_{n'=\infty}^{+\infty} \mathcal{F}_{nn'} \begin{bmatrix} X_0,\ell,n' \\ \Pi^*_0,\ell,n' \end{bmatrix} = \Omega_\ell \begin{bmatrix} X_0,\ell,n \\ \Pi^*_0,\ell,n \end{bmatrix}.
\]
Here we have introduced the infinite-dimensional matrix $\mathcal{F}$ whose entries $F_{nn'} = B_{n-n'} - n\Delta k \delta_{n,n'}$ are themselves $8 \times 8$ matrices, and $\mathbb{I}_8$ the $8 \times 8$ identity matrix. Then, Eq. (18) is the eigenvalue problem for $\mathcal{F}$. One can immediately see that for each solution with quasi-frequency $\Omega_\ell$ there exists another one having quasifrequency $\Omega_\ell + n\Delta k$ for arbitrary $n \in \mathbb{Z}$. However, all these infinitely many solutions of Eq. (18) are physically equivalent as they correspond to a single Bogoliubov mode of the form (17). For this reason, one can restrict the real part of $\Omega_\ell$ in the first Brillouin zone, $-|\Delta k|/2 < \text{Re} \Omega_\ell \leq |\Delta k|/2$, as done in Fig. 2 of the main text. In addition, from the identity $\mathcal{F}_{-n,-n'} = -\mathcal{F}^*_{nn'}$ it follows that both $\Omega_\ell$ and $-\Omega_\ell^*$ belong to the spectrum of $\mathcal{F}$. Finally, the relation
\[
\begin{bmatrix}
0 & \mathbb{I}_4 \\
-i\mathbb{I}_4 & 0
\end{bmatrix} \mathcal{F}_{nn'} \begin{bmatrix}
0 & \mathbb{I}_4 \\
-i\mathbb{I}_4 & 0
\end{bmatrix}^{-1} = \mathcal{F}^\dagger_{-n,-n'},
\]
with $\mathbb{I}_4$ the $4 \times 4$ identity matrix, ensures that $\mathcal{F}$ and $\mathcal{F}^\dagger$ have the same spectrum, meaning that quasi-frequencies occur in complex conjugate pairs. Combining Eqs. (18) and (19), after a few appropriate manipulations, we obtain the amplitude orthonormalization relations
\[
\sum_{n',n=\pm\infty}^+ i \left[ X_{0,\ell',n'}^\dagger(q_\perp) \Pi_{0,\ell',n'}^\dagger(q_\perp) X_{0,\ell',n'}(q_\perp) - \Pi_{0,\ell',n'}^\dagger(q_\perp) X_{0,\ell',n'}(q_\perp) \right] = N_{\ell}(q_\perp) \delta_{\ell,\ell'} \delta_{n,0},
\]
where $\ell$ labels the solution with quasi-frequency $\Omega_\ell = \Omega_\ell^*$ and $N_{\ell}$ is the mode norm. From Eq. (20), we easily obtain the relation
\[
i[X_{0,\ell}^\dagger(q_\perp, \varphi) \Pi_{0,\ell}^\dagger(q_\perp, \varphi) - \Pi_{0,\ell}^\dagger(q_\perp, \varphi) X_{0,\ell}^\dagger(q_\perp, \varphi)] = N_{\ell}(q_\perp) \delta_{\ell,\ell'},
\]
holding at arbitrary $\varphi$. Then, taking Eq. (6) of the main text at $z = 0$ and projecting it onto each separate mode by means of Eq. (21) at $z = 0$, we end up with an expression for the weights as functions of the input field:
\[
C_{\ell}(q_\perp) = \frac{i}{N_{\ell}(q_\perp)} \left[ X_{0,\ell}^\dagger(q_\perp, \varphi_\ell) \Pi_\ell^\dagger(q_\perp, z = 0) - \Pi_{\ell}^\dagger(q_\perp, \varphi_\ell) X_{0,\ell}^\dagger(q_\perp, z = 0) \right],
\]
which is precisely Eq. (8) of the main text.

Up to now no approximation has been made. In order to perform the numerical computation of the Bogoliubov spectrum and amplitudes, the Hill method prescribes one to choose a (sufficiently large) positive integer $N$ and truncate $\mathcal{F}_{nn'}$ to the square block with $-(2N + 1) \leq n, n' \leq 2N + 1$. After checking the convergence of the numerical results, we have chosen $N = 20$ for the figures of the Letter.

IV. INPUT CONJUGATE MOMENTA

The study a nonparaxial light beam propagating inside a bulk nonlinear medium requires one to solve the Bogoliubov equations (15) for a given input field profile $X(q_\perp, z = 0)$, such as that of Eq. (7) of the main text. However, this is not enough to uniquely determine the solution to the problem, as the input value of the conjugate momenta $\Pi_\ell^\dagger(q_\perp, z = 0)$ is also required. In order to fix this value, we first recall that the Bogoliubov spectrum of the system includes eight branches, only half of which are associated with transmitted field modes propagating inside the medium; the remaining branches describe reflected modes at the air-medium interface which do not appear in the paraxial framework. To know whether a given branch $\ell$ is transmitted or reflected we define an average frequency $\bar{\Omega}_\ell = \Omega_\ell + N_{\ell}^{-1} \Delta k \sum_{n=-\infty}^{+\infty} \ln \left( X_{0,\ell,n}^\dagger \Pi_{0,\ell,n}^\dagger X_{0,\ell,n} - \Pi_{0,\ell,n}^\dagger X_{0,\ell,n} \right)$, which can be regarded as the expectation value of the effective energy operator $i\nabla_z$ [cfr. the orthonormalization condition (20)]. Transmitted (reflected) modes are those having the smallest (largest) value of $|\text{Re} \bar{\Omega}_\ell|$. We take $\Pi_\ell^\dagger(q_\perp, z = 0)$ such that only the transmitted modes, whose frequencies will be denoted by $\Omega_{-d}$ and $\Omega_{-s}$ (where $d$ and $s$ stand for “density” and “spin”, respectively), be excited; conversely, their reflected counterparts of quasi-frequency $\Omega_{+d}$ and $\Omega_{+s}$ do not appear in the solution of Eq. (15).

The practical implementation of the above criterion requires as a preliminary step to consider the operator evolving the solution of the Bogoliubov equations (15) with respect to effective time by one oscillation period $2\pi/\Delta k$,
\[
\begin{bmatrix}
X(q_\perp, z = 2\pi/\Delta k) \\
\Pi^*(q_\perp, z = 2\pi/\Delta k)
\end{bmatrix} = \mathcal{U}(2\pi/\Delta k, 0) \begin{bmatrix}
X(q_\perp, z = 0) \\
\Pi^*(q_\perp, z = 0)
\end{bmatrix}.
\]
One can decompose this operator as $\mathcal{U}(2\pi/\Delta k, 0) = \mathcal{P} \mathcal{U}_D \mathcal{P}^{-1}$, with
\[
\mathcal{U}_D = \begin{bmatrix}
e^{-2\pi i B_{+}/\Delta k} & 0 \\
0 & e^{-2\pi i B_{-}/\Delta k}
\end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix}\mathcal{P}_X & \mathcal{P}_X^- \\
\mathcal{P}_X^+ & \mathcal{P}_{X,-}
\end{bmatrix}, \quad \mathcal{P}^{-1} = \begin{bmatrix}(\mathcal{P}^{-1})_X & (\mathcal{P}^{-1})_X^- \\
(\mathcal{P}^{-1})_X^+ & (\mathcal{P}^{-1})_{X,-}
\end{bmatrix}.
\]
The $4 \times 4$ blocks entering $\mathcal{U}_D$ and $\mathcal{P}$ are given by

$$
\begin{align*}
B_{D\pm} &= \text{diag} \left( \Omega_{d,\pm}(q_{\perp}), -\Omega_{s,\pm}^*(q_{\perp}), -\Omega_{s,\pm}(q_{\perp}), \Omega_{s,\pm}^*(q_{\perp}) \right), \\
\mathcal{P}_X &= \left[ X_{d,\pm}(q_{\perp}, z = 0) \quad X_{s,\pm}(q_{\perp}, z = 0) \quad X_{d,\pm}^*(q_{\perp}, z = 0) \quad X_{s,\pm}^*(q_{\perp}, z = 0) \right], \\
\mathcal{P}_\Pi &= \left[ \Pi_{d,\pm}(q_{\perp}, z = 0) \quad \Pi_{s,\pm}(q_{\perp}, z = 0) \quad \Pi_{d,\pm}^*(q_{\perp}, z = 0) \quad \Pi_{s,\pm}^*(q_{\perp}, z = 0) \right].
\end{align*}
$$

(25a) (25b) (25c)

Notice that the columns of $\mathcal{P}$ correspond to the eight wave functions defined in Eq. (17) taken at $z = 0$. Consequently, the blocks of $\mathcal{P}^{-1}$ can be found from the orthonormalization conditions discussed in Sec. III. We now define the new variables

$$
\begin{bmatrix}
Y_{D+} \\
Y_{D-}
\end{bmatrix} = \mathcal{P}^{-1} \begin{bmatrix} X \\ \Pi \end{bmatrix},
$$

(26)

whose effective-time evolution over one oscillation period is simply given by a phase factor, $Y_{D\pm}(q_{\perp}, z = 2\pi/\Delta k) = e^{-2\pi i B_{D\pm}/\Delta k} Y_{D\pm}(q_{\perp}, z = 0)$. Requiring $Y_{D+}(q_{\perp}, z = 0) = 0$ is a sufficient condition to ensure that the reflected Bogoliubov modes are not excited. When expressing this constraint in terms of the original variables one obtains

$$
\Pi^*(q_{\perp}, z = 0) = - \left[ (\mathcal{P}^{-1})_{\Pi,+} \right]^{-1} (\mathcal{P}^{-1})_{X+} X(q_{\perp}, z = 0).
$$

(27)

We checked that in the paraxial limit this condition is consistent with the relation (12), connecting $X$ to its first-order derivative, taken at $z = 0$.

**V. INTENSITY-INTENSITY CORRELATION FUNCTION**

In order to compute the intensity-intensity correlation function one has to start from the input field [Eq. (7) of the main text] and write down the corresponding expression of the four-component vector $X$,

$$
X(q_{\perp}, z = 0) = \epsilon \sum_{\alpha=r,i} \sum_{\sigma = \pm} X_{\alpha,\sigma} \tilde{\phi}_{\alpha,\sigma}(q_{\perp}),
$$

(28)

where

$$
\begin{align*}
X_{r,+} &= \begin{bmatrix} \cos \frac{\Delta \alpha}{2} \\ -\sin \frac{\Delta \alpha}{2} \\ 0 \\ 0 \end{bmatrix}, & X_{r,-} &= \begin{bmatrix} \sin \frac{\Delta \alpha}{2} \\ \cos \frac{\Delta \alpha}{2} \\ 0 \\ 0 \end{bmatrix}, & X_{i,+} &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \cos \frac{\Delta \alpha}{2} \\ \frac{1}{2} \cos \frac{\Delta \alpha}{2} \end{bmatrix}, & X_{i,-} &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \sin \frac{\Delta \alpha}{2} \\ \frac{1}{2} \sin \frac{\Delta \alpha}{2} \end{bmatrix}.
\end{align*}
$$

(29)

The correlators of the Fourier transform of the speckle fields are given by

$$
\langle \hat{C}_{\ell}(q_{\perp}) \hat{C}_{\ell^*}(q_{\perp}') \rangle = \langle \tilde{\phi}_{\ell,i}(q_{\perp}) \tilde{\phi}_{\ell,i}^*(q_{\perp}') \rangle = (2\pi)^2 \delta(q_{\perp}' - q_{\perp}) \delta(q_{\perp}) \delta_{\ell \ell^*}/2 \quad \text{and} \quad \langle \tilde{\phi}_{r,i}(q_{\perp}) \tilde{\phi}_{r,i}^*(q_{\perp}') \rangle = 0, \quad \text{with} \quad \delta(q_{\perp}) = 4\pi \sigma^2 \exp(-\sigma^2 q_{\perp}^2).
$$

From the discussion of Sec. IV, we know that the input conjugate momenta take the form

$$
\Pi^*(q_{\perp}, z = 0) = \epsilon \sum_{\alpha=r,i} \sum_{\sigma = \pm} \Pi_{\alpha,\sigma}^*(q_{\perp}) \tilde{\phi}_{\alpha,\sigma}(q_{\perp}),
$$

(30)

where the $\Pi_{\alpha,\sigma}^*$’s can be directly deduced from Eq. (27). After solving the problem as discussed in the previous sections and in the main text, one can write the intensity fluctuation at arbitrary $r_{\perp}$ and $z$ as

$$
\delta I(r_{\perp}, z) = \int \frac{d^2 q_{\perp}}{(2\pi)^2} \sum_{\ell} C_{\ell}(q_{\perp}) \delta I_{0,\ell}(q_{\perp}, \varphi(z)) e^{i(q_{\perp} \cdot r_{\perp} - \Omega_{\ell}(q_{\perp}) z)},
$$

(31)

where $\delta I_{0,\ell}$ is (up to a factor $2I_0$) the first component of the amplitude vector $X_{0,\ell}$ [see Eq. (17)] and the weights are given by Eq. (22). To evaluate $g_\ell(z) \equiv \langle \delta I(r_{\perp}, 0) \delta I(r_{\perp}, z) \rangle$ one needs the expression of the weight correlator

$$
\langle C_{\ell}(q_{\perp}) C_{\ell}^*(q_{\perp}') \rangle = \epsilon^2 (2\pi)^2 \delta(q_{\perp}' - q_{\perp}) \gamma(q_{\perp}) \frac{\Xi_{\ell}(q_{\perp})}{2},
$$

(32)
As an example, in Fig. 2 we show the Fourier transform of $g_2$ (in modulus) as a function of the effective frequency $q_z$. The parameters are the same as the blue solid curves of Figs. 3 and 4(a) of the main text, namely the background polarization $\theta_0 = \pi/4$, the nonlinear couplings $g_d I_0/\beta_0 = 0.2$, $g_s I_0/\beta_0 = 0.05$, and the correlation length $\beta_0 \sigma = 15$. We can approximate $\Omega_{\ell}(q_\perp) \approx 0.05$, and the correlation length $\beta_0 \sigma = 15$. We can restrict to $q_z > 0$ because of symmetry. The parameters are the same as the blue solid curves of Figs. 3 and 4(a) of the main text, namely the background polarization $\theta_0 = \pi/4$, the nonlinear couplings $g_d I_0/\beta_0 = 0.2$, $g_s I_0/\beta_0 = 0.05$, and the correlation length $\beta_0 \sigma = 15$. We can approximate $\Omega_{\ell}(q_\perp) \approx 0.05$, and the correlation length $\beta_0 \sigma = 15$.

where

$$
\Xi_{\ell\ell'}(q_\perp) = \sum_{\alpha=\pm} \int_{-\pi}^{\pi} \frac{d\varphi_\perp}{2\pi} \sum_{\sigma} X_{0,\ell}(q_\perp, \varphi_\perp) \Pi_{\alpha,\sigma}^*(q_\perp) X_{\alpha,\sigma}
$$

$$
\times \left[ X_{0,\ell'}(q_\perp, \varphi_\perp) \Pi_{\alpha,\sigma}(q_\perp) - \Pi_{0,\ell'}(q_\perp, \varphi_\perp) X_{\alpha,\sigma} \right]^{*} [N_\ell(q_\perp) N_{\ell'}(q_\perp)]^{-1}.
$$

Then, a straightforward calculation yields

$$
g_2(z) = c^2 I_0^2 \int_0^\infty \frac{dq_\perp}{2\pi} q_\perp \hat{\gamma}(q_\perp) K_\ell(q_\perp, z) e^{-i\Omega_\ell(q_\perp)z},
$$

where we have introduced the modulated coefficients

$$
K_\ell(q_\perp, z) = \frac{1}{2I_0^2} \int_{-\pi}^{\pi} \frac{d\varphi_\perp}{2\pi} \sum_{\ell'} \Xi_{\ell\ell'}(q_\perp) \delta I_{0,\ell'}(q_\perp, \varphi_\perp) \delta I_{0,\ell}(q_\perp, \varphi(z)).
$$

Equation (34) also holds in the paraxial framework, with the coefficients taking the $q_\perp$- and $z$-independent values

$$
K_{d(s)} = \frac{1}{2} \pm \frac{g_d + g_s \cos 2\theta_0}{2\sqrt{g_d^2 + g_s^2 + 2gdg_s \cos 2\theta_0}}
$$

where the upper (lower) sign refers to the two density (spin) branches of the Bogoliubov spectrum. In addition, at large $z$ one can approximate $\Omega_{d(s)}(q_\perp) \approx c_{d(s)} q_\perp$ in the integral (34), with $c_{d(s)}$ the paraxial sound velocities (14), yielding the asymptotic behavior $g_2(z) \approx -c^2 I_0^2 (\sqrt{g_d I_0/\beta_0} z/2\sigma)^{-2}$.

In the nonparaxial regime the $K_\ell$’s are periodic in $z$, giving rise to the peculiar oscillating behavior shown in Figs. 3 and 4(a) of the main text. To identify the dominant oscillation frequencies we numerically evaluate $g_2(z)$ in a window of fairly large $z$ and then compute its (discrete) Fourier transform $\tilde{g}_2(q_z)$. At large $\sigma$ we find two symmetric peaks at some $q_z$ close to $\pm |\Delta k|$, meaning that the oscillations are practically harmonic. However, as $\sigma$ is decreased (at fixed values of the other parameters) additional peaks occur, and the oscillations become more and more anharmonic.

As an example, in Fig. 2 we show the Fourier transform of $g_2$ evaluated in the $1000 z_{NL} \leq z \leq 2000 z_{NL}$ range with the same parameters as the blue curves of Figs. 3 and 4(a) of the main text. One can clearly see the main peaks at $q_z \approx \pm 0.8 |\Delta k|$ and the secondary peaks at $q_z \approx \pm 0.5 |\Delta k|$ and $q_z \approx \pm 0.2 |\Delta k|$. However, we noticed that, each time a new peak appears, when further lowering $\sigma$ its position remains basically $\sigma$-independent.

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