Topological Semantics and Decidability

Dmitry Sustretov

March 30, 2022

Abstract

It is well-known that the basic modal logic of all topological spaces is $S4$. However, the structure of basic hybrid logics of classes of spaces satisfying various separation axioms was until present unclear. We give a direct proof of that modal logics of $T_0$, $T_1$ and $T_2$ topological spaces coincide and are $S4$. We also examine basic hybrid logics of these classes and prove their decidability; as part of this, we find out that the hybrid logics of $T_1$ and $T_2$ spaces coincide. Finally, we prove that logics of $T_0$ and $T_1$ spaces are PSPACE-complete.

1 Basic definitions

In this paper we are going to study modal logics that arise as sets of all formulas valid on certain classes of topological spaces. Thus the first definition in this paper is bound to be about how the modal formulas are interpreted on topological spaces (topological semantics was first introduced by Tarski [8]).

Definition 1 (Topological semantics). A topological space is a pair $(T, \tau)$ where $\tau \subseteq \mathcal{P}(T)$ such that $\emptyset, T \in \tau$ and $\tau$ is closed under finite intersections and arbitrary unions. Elements of $\tau$ are called opens, an open containing a point $x$ is called a neighborhood of the point $x$.

A topological model $\mathfrak{M}$ is a tuple $(T, \tau, V)$ where $(T, \tau)$ is a topological space and the valuation $V : \text{PROP} \to \mathcal{P}(T)$ sends propositional letters to subsets of $T$.

Truth of a formula $\phi$ (of the basic modal language) at a point $w$ in a topological model $\mathfrak{M}$ (denoted by $\mathfrak{M}, w \models \phi$) is defined inductively:

\[
\begin{align*}
\mathfrak{M}, w \models p & \quad \text{iff} \quad x \in V(p) \\
\mathfrak{M}, w \models \phi \land \psi & \quad \text{iff} \quad \mathfrak{M}, w \models \phi \quad \text{and} \quad \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \neg \phi & \quad \text{iff} \quad \mathfrak{M}, w \not\models \phi \\
\mathfrak{M}, w \models \lozenge \phi & \quad \text{iff} \quad \exists O \in \tau \quad \text{such that} \quad w \in O \quad \text{and} \quad \forall v \in O. (\mathfrak{M}, v \models \phi)
\end{align*}
\]

The basic modal language can be extended with nominals and $@$ operator (in this case we call it $H(\@)$) and universal modality $A$ (we denote the dual
modality $E$ and call the language $H(E)$). Nominals are a special kind of propositional letters: it is required that their valuation is a singleton set. The semantics of $@$ and $E$ is given below:

\[
\begin{align*}
\mathcal{M}, w \models @_i \varphi & \iff \exists v \mathcal{M}, v \models i \text{ and } \mathcal{M}, v \models \varphi \\
(\text{where } i \text{ is a nominal}) \\
\mathcal{M}, w \models E\varphi & \iff \exists v \mathcal{M}, v \models \varphi
\end{align*}
\]

Relational and topological semantics are not completely unrelated; it is possible to transform certain topological spaces into frames and vice versa in a satisfiability-preserving fashion.

**Proposition 1.** A topological space is called Alexandroff if every point of that space has a minimal neighborhood.

For any Alexandroff space $(T, \tau)$ there exists a binary relation $R$ such that for any valuation $V$ and for any formula $\varphi \in H(E)$, $(T, R, V), w \models \varphi$ iff $(T, \tau, V), w \models \varphi$.

For any transitive reflexive frame $(W, R)$ there exists a topology $\tau$ on $W$ such that for any valuation $V$ and for any formula $\varphi \in H(E)$, $(W, R, V), w \models \varphi$ iff $(W, \tau, V), w \models \varphi$.

**Proof.** See [9], section 2.4.

**Definition 2** (Topobisimulation). Let $(T, \tau, V)$ and $(S, \sigma, W)$ be two topological models and consider a relation $R \subseteq T \times S$. Denote

\[
\begin{align*}
R(X) & = \{ y \mid \exists x \in X, (x, y) \in R \} \\
R^{-1}(Y) & = \{ x \mid \exists y \in Y, (x, y) \in R \}
\end{align*}
\]

for any subset $X \subseteq T$, $Y \subseteq S$.

The relation $R$ is called a topobisimulation if

**Prop** if $Rxy$ then for all $p \in \text{Prop}$, $(T, \tau), V, x \models p$ iff $(S, \sigma), W, y \models p$

**Zig** for any $O \in \tau$, $R(O) \in \sigma$

**Zag** for any $U \in \sigma$, $R^{-1}(U) \in \tau$

A bisimulation is called total if for any $x \in T$ there is $y \in S$ such that $Rxy$ and for any $y \in S$ there is $x \in T$ such that $Rxy$.

A bisimulation is called an hybrid if additionally for any nominal $i$ if $x \in V(i)$ and $y \in W(i)$ then $Rxy$.

A map is called interior if it is open and continuous. Clearly, the graph of an interior map is satisfies Zig and Zag conditions.
In topological semantics just like in the relational semantics, two points connected by a topobisimulation satisfy the same formulas (if the topobisimulation is total, this is true for the formulas with universal modality). See [9] for the proofs.

It is well-known that the (basic modal) logic of all topological spaces is $S_4$. In what follows, we are going to deal with three classes of topological spaces, defined by the so-called separation axioms.

**Definition 3** (Separation axioms).

- $T_0$ for any two distinct points $x, y$ there is either an open neighborhood of $x$ that does not contain $y$, or an open neighborhood of $y$ that does not contain $x$.

- $T_1$ any singleton set is closed.

- $T_2$ any two distinct points $x, y$ can be separated by two open neighborhoods, i.e. there exist $O_x \ni x, O_y \ni y$ such that $O_x \cap O_y = \emptyset$.

There are necessary and sufficient conditions (given in [3]) of whether a class of spaces is definable in $H(\@)$ (and $H(E)$). Thus, axioms $T_0$ and $T_1$ are definable in $H(\@)$, the formulas are, respectively, $\@i \neg j \rightarrow (\@i \Box \neg j \lor \@j \Box \neg i)$ and $\Diamond i \rightarrow i$. On the other hand, [3] show that $T_2$ is not definable even in $H(E)$. The basic modal language is even less expressible: none of the separation axioms is definable in it. Nonetheless, although we know the boundaries of expressivity of modal and hybrid languages, we know very little about the structure of the logics. Are the logics of separation axioms distinct? Are they decidable? If yes, what is the complexity? In this paper we will address all those questions.

## 2 Modal logic with universal modality

The McKinsey-Tarski theorem [6], [8] states that the logic of every metrizable dense-in-itself (without isolated points) space is $S_4$. Thus, for example, the logic of rationals is $S_4$. It follows that the logics of all separation axioms that $\mathbb{Q}$ satisfies are all $S_4$. Shehtman [7] showed the same kind of result holds for the basic modal language enriched with universal modality. Denote by $\text{Log}(K)$ a set of formulas in basic modal language with universal modality which are valid on all topological spaces in class $K$. In this section we will give a direct proof of these facts, namely that $\text{Log}(T_0) = \text{Log}(T_1) = \text{Log}(T_2) = \text{Log}(\text{Top})$ where $\text{Top}$ is the class of all topological spaces. Our technique will be to build a topobisimulation between a finite topological space and a space from each respective class.

**Definition 4** (Finite model property). A logic $L$ has finite model property with respect to a class of topological models $K$ if $K \models L$ and for any $\varphi \notin L$ there exists a finite $\mathfrak{M} \in K$ such that $\varphi$ is satisfiable on $\mathfrak{M}$.
Proposition 2. The logic S4 has a finite model property.

Proof. The proof is Proposition 1 combined with the fact that S4 has a finite model property with respect to transitive reflexive frames.  

Theorem 3. \( \log(T_0) = \log(\text{Top}) \).

Proof. The inclusion \( S4 \subseteq \log(T_0) \) is obvious, so we only have to prove \( \log(T_0) \subseteq S4 \). Take an arbitrary topological space \( (T, \tau) \) and define an equivalence relation: \( x \sim y \) iff for all \( O \in \tau, x \in O \) iff \( y \in O \). The quotient set with the maximal topology that makes the natural projection continuous (this topological space is known as Kolmogorov quotient of \( T \)) is a \( T_0 \) space. The graph of the natural projection map is a topobisimulation. It follows that every formula, that is not an S4 validity is not a \( T_0 \) validity either.

Theorem 4. Any finite transitive and reflexive model is an interior image of a \( T_1 \) space.

Proof. Let \( (T, \tau, V) \) be a finite topological model, let us construct a topobisimilar model \( (S, \sigma, W) \) where \( (S, \sigma) \) is a \( T_1 \) topological space.

We will identify \( T \) with the initial segment of natural numbers, so \( T = \{1, \ldots, n\} \). First, let us introduce some notation:

\[
X_k = \{nx + k \mid 0 \leq x < \infty\}, 1 \leq k \leq n
\]

Let \( \sigma_0 \) be a cofinite topology on \( \mathbb{N} \), that is

\[
\sigma_0 = \{O \mid \mathbb{N} - O \text{ is finite}\}
\]

and for any subset \( O \subseteq T \) denote

\[
\tilde{O} = \{X_i \mid i \in O\}
\]

Then define the topology \( \sigma \) on \( S = \mathbb{N} \) to be generated by the set

\[
\beta = \sigma_0 \cup \{\tilde{O} \mid O \in \tau\}
\]

Define valuation to be

\[
W(p) = \overline{V(p)} \text{ for all } p \in \text{PROP}
\]

\((S, \sigma)\) is a \( T_1 \) space, because \( \sigma \) contains \( \sigma_0 \), hence all complements of singleton sets are open.

Let \( f : S \to T \) be a function that maps \( X_k \) to \( k \) for all \( k \in T \). It is an onto interior map.

First, let us prove that it is open. The topology generated by \( \beta \) consists of sets of the form \( \tilde{O} \cap U \), where \( U \) is an open set from \( \sigma_0 \) and \( O \in \tau \). We have \( f(\tilde{O} \cap U) = f(\cup_{k \in O}(X_k \cap U)) \) and since \( X_k \) are dense sets in the
topology $\sigma_0$, all intersections $X_k \cap U$ are non-empty, hence $f(X_k \cap U) = \{k\}$ and $f(O \cap U) = \bigcup_{k \in O}\{k\} = O$. That is, any open set is mapped by $f$ to an open set.

On the other hand, $f$ is continuous. The $f$-preimage of any open set $O \in \tau$ is $\tilde{O}$ by construction of $f$, and $\tilde{O}$ is open by definition of the topology $\sigma$.

A final note: consider the graph $R$ of $f$. The points connected by $R$ agree on propositional letters, hence $R$ is a topobisimulation. Since $f$ is defined on the whole $S$ and is onto, it is a total topobisimulation.

**Corollary 5.** $\text{Log}(T_1) = \text{Log}(\text{Top})$

*Proof.* By Proposition 2, $\text{Log}(\text{Top})$ has finite (topological) model property, i.e. every formula that is not valid on the class of all topological spaces can be refuted on a finite topological model. By the theorem we have just proved there exists a total bisimulation between any finite topological model and a model based on a topological space with $T_1$ topology. It follows that any $\text{Log}(\text{Top})$ non-theorem can be refuted on a $T_1$ model, hence $\text{Log}(T_1) \subseteq \text{Log}(\text{Top})$, hence $\text{Log}(T_1) = \text{Log}(\text{Top})$.

The main idea of the proof is that we could construct a space which is a disjoint union of dense subsets and then add necessary opens to the topology to get a space topobisimilar to $T$. We can exploit this idea in a more general setting, leading us to the following

**Theorem 6.** Any finite transitive and reflexive model is an interior image of a $T_2$ space.

*Proof.* We will use a construction by L. Feng and O. Masaveu. In the paper [2] they prove that for any cardinal $\alpha$ there exists a $T_2$ space which is a disjoint union of $\alpha$ dense subsets (such a space is called $\alpha$-resolvable). We will apply this statement for a finite $\alpha = n$, so let $(S, \sigma_0)$ be such a space and $S = \bigcup_{k=1}^{n}X_k$ where $X_k$ are disjoint dense subsets from the theorem of Feng and Masaveu.

Now define $\tilde{O}$, $\beta$, $\sigma$, the valuation $W$ and the function $f : S \to T$ the same way as in the proof of the previous theorem. $(S, \sigma)$ stays a $T_2$ space (because we have only added more opens to it). For any $O \in \tau$, $f^{-1}(O) = \tilde{O}$ and is open by the definition of topology $\sigma$.

In order to prove openness of $f$, we will use the same argument as in the Theorem 4. Indeed take any open $\tilde{O} \cap U$ from $\sigma$, where $\tilde{O} \in \tau$ and $U \in \sigma_0$. Since $X_k$ are dense sets in $\sigma_0$, all intersections $X_k \cap U$ are non-empty, hence $f(X_k \cap U) = O$, which is open.

Again, we have constructed the valuation in such a way that the points connected by the graph $R$ of $f$ agree on propositional letters, hence $R$ is a total topobisimulation.

5
Corollary 7. \( \text{Log}(T_2) = \text{Log}(\text{Top}) \)

In fact, nothing in the proof depends on the \( T_2 \) condition, except the existence of \( n \)-resolvable sets. That leads us to the following more general result.

Theorem 8. Let \( K \) be a class of topological spaces that contains an \( n \)-resolvable space for all finite \( n \) and for any space \( (T, \tau) \in K \), it is true that \( (T, \tau') \in K \) for all \( \tau' \supset \tau \) (i.e. \( K \) is closed under refinement of topology). Then \( \text{Log}(K) = \text{Log}(\text{Top}) \).

Proof. In this theorem we extract the properties of the class \( T_2 \) used in the proof of the Theorem 6. Indeed, we need an \( n \)-resolvable space to start our construction, then we add new opens to this space in order to obtain a space bisimilar to the given finite topological space. If the class of topological spaces in question is closed under refinement of topology, we can do it the same way as we have done in the Theorem 6.

\[ \square \]

3 Hybrid logic

In this section we will denote by \( \text{Log}(K) \) a set of formulas in the hybrid language \( H(E) \) (with nominals, \( \@ \) and universal modality) which are valid on all topological spaces in the class \( K \).

In the subsequent subsections we will prove decidability of logics of different separation axioms. Our main tool will be the notion of topological filtration, which allows to present the information relevant for the satisfiability of a formula in a finite structure.

Definition 5 (Topological filtration). Let \( \Sigma \) be a subformula-closed set of formulas and \( \mathcal{M} = (T, \tau, V) \) be a topological model. Define an equivalence relation \( \leftrightarrow \) on \( T \) as follows:

\[ w \leftrightarrow v \text{ iff } \forall \varphi \in \Sigma, w \models \varphi \text{ iff } M, v \models \varphi \]

A filtration of \( \mathcal{M} \) through \( \Sigma \) is a model \( \mathcal{N} = (S, \sigma, W) \), defined as follows. Let \( S = T/\leftrightarrow \Sigma \) and let us denote by \( [s] \) an equivalence class of \( \leftrightarrow \Sigma \) with a representative \( s \). For a formula \( \varphi \in \Sigma \) define

\[ [\varphi]^\mathcal{N} = \{ [x] \mid M, x \models \varphi \} \]

and \( W(p) = [p]^\mathcal{N} \). This is well-defined, because points from the same equivalence class satisfy the same formulas from \( \Sigma \).

Let \( \pi \) be a natural projection map \( t \mapsto [t] \). Define \( \sigma \) to be the finest topology that makes \( \pi \) continuous (that is, \( \sigma \) is the quotient topology).

Note that if \( \Sigma = \text{Cl}(\varphi) \) (all subformulas of a single formula \( \varphi \)), then any filtration by \( \Sigma \) is finite (there is only finite number of subsets of \( \text{Cl}(\varphi) \)).
3.1 $T_1$ spaces

The class $T_1$ does not have a finite model property with respect to the class of $T_1$ spaces: for example, the formula $i \rightarrow \Box i$ can only be falsified on an infinite model with $T_1$ topology. In order to prove decidability of $\text{Log}(T_1)$ we will introduce a class of finite topological models and prove that $T_1$ has the finite model property with respect to that class. Then we will show that for any formula the number of possible models from that class is bounded and that will imply decidability.

In fact, decidability of $\text{Log}(T_1)$ follows from the decidability of the logic of $T_1$ spaces for $H(E)$ extended with downarrow operator ([3], section 5.4). We can justify ourselves by the fact that the proof presented here provides us with concrete structures that represent what $T_1$ spaces are “from the point of view of hybrid logic” and that will help us later to prove complexity results.

**Definition 6** (Finite representation of a $T_1$ model). A $T_1$ model is called finitely representable if it is topobisimilar to a finite topological model. A finite topological model where the complement of any point named by a nominal is open is called finite representation of a $T_1$ model (we will say simply finite representation, when there is no confusion).

**Theorem 9.** A formula $\varphi$ has a $T_1$ model iff it has a finitely representable $T_1$ model.

**Proof.** The left-to-right direction is proved using filtrations. Indeed, a filtration of any $T_1$ space is a representation of a $T_1$ model, as follows from the fact known from general topology (see [1]) that the natural projection is open. Then it is left to apply the standard argument that filtration preserves satisfiability of all subformulas of $\varphi$.

To prove right-to-left direction we will construct a $T_1$ model $\mathcal{M} = (S, \sigma, W)$ such that the given finite representation $(T, \tau, V)$ is an interior image of $\mathcal{M}$.

We identify $T$ with the set of natural numbers $\{1, \ldots, n\}$. Suppose there are $m$ points $t_1, \ldots, t_m \in T$ named by a nominal. If $m = n$ then every point is named by a nominal and should be represented by a singleton. In this case the model construction process described below will produce a model with a finite discrete submodel.

Let the support of $\mathcal{M}$ be $\mathbb{N}$, the set of natural numbers. Denote $X_i = \{k\}$ for $i = t_k, 1 \leq k \leq m$ and let $X_i$ for $i \in T - t_1, \ldots, t_m$ form a partition of $\mathbb{N} - \{1, \ldots, m\}$ such that every $X_i$ is an infinite coinfinite set. Let $\sigma_0$ be a collection of cofinite subsets of $\mathbb{N}$ and for any subset $O \subseteq T$ denote

$$\hat{O} = \bigcup_{i \in O} X_i$$

Note that $X_k$ for $m + 1 \leq k \leq n$ are dense in $S$ in the topology $\sigma_0$. Then define the topology $\sigma$ on $\mathbb{N}$ to be generated by the following set:
\[ \sigma_0 \cup \{ \tilde{O} \mid O \in \tau \} \]

Note that by construction of \( \sigma \) if \( \tilde{O} \) is open then \( O \) is open.

The valuation is defined as follows:

\[ W(p) = \tilde{V}(p) \text{ for all } p \in \text{PROP} \cup \text{Nom} \]

The definition of topology and valuation looks similar to the definition in Section 2 (indeed, the only real difference is in the definition of \( X_k \)); however, the proof works differently because of nominals.

Define \( f : S \rightarrow T \) to be the map that maps \( X_k \) to \( k \) for all \( k \in T \). We will prove that \( f \) is an interior map and that its graph is a total hybrid topobisimulation.

Indeed, take an arbitrary open from \( \sigma \), it will have the form \( \tilde{O} \cup U \) where \( O \in \tau \) and \( U \in \sigma_0 \). It can be represented as a union \( \bigcup_{k \in O} (X_k \cap U) \). Here certain \( X_k \)-s are dense in \( S \) in the topology \( \sigma_0 \), in this case \( X_k \cap U \) is non-empty. All the other \( X_k \)-s are singletons that correspond to points named by nominals and in that case either \( X_k \cap U \) is just \( X_k \) or the intersection is empty. Denote by \( F \) the set of those \( k \)-s that have an empty intersection with \( U \). Then \( f(\tilde{O} \cap U) = O \setminus F = O \cap (T \setminus F) \). The set \( T \setminus F \) is open, because it is a complement of a set of points named by nominal, hence \( O \setminus F \) is open. We have proved that \( f \) is open.

The continuity of \( f \) follows easily from its definition and construction of \( \sigma \): indeed, \( f^{-1}(O) = \tilde{O} \), and if \( O \) is open, then \( \tilde{O} \) is open too.

It is easy to see that the graph \( R \) of \( f \) is a topobisimulation: it satisfies the \textit{Zig} and \textit{Zag} conditions, because it is a graph of a topobisimulation, it satisfies \textit{Prop} by construction of the valuation on \( S \). It is also total and connects all points named by the same nominal in two models. In other words \( R \) is a total hybrid topobisimulation.

Now, if \((T', \tau, V), k \models \varphi\) then \( \mathfrak{M}, v \models \varphi \) for all \( v \in X_k \), which finishes the proof of the right-to-left direction.

Note that the size of a filtration through \( \varphi \) is bounded by \( 2^{\text{Cl}(\varphi)} \), hence we have an upper bound on the size of finite representations of \( T_1 \) models necessary to refute non-theorems of \( \text{Log}(T_1) \). This allows us to deduce

**Theorem 10.** \( \text{Log}(T_1) \) is decidable.

### 3.2 \( \text{Log}(T_1) = \text{Log}(T_2) \)

In Theorem 6 we used the construction of Theorem 4 and replaced the naturals with cofinite topology with an \( n \)-resolvable \( T_2 \) space whose existence is guaranteed by the theorem of Feng and Masaveu. In a similar fashion we are going to reuse the notion of finite representation of a \( T_1 \) model and
modify the construction of Theorem 9 in order to construct $T_2$ models out of $T_1$ finite representations.

**Theorem 11.** A formula $\varphi$ is satisfiable on a $T_2$ space iff there exists a finite representation of a $T_1$ model where $\varphi$ is satisfiable.

**Proof.** Since all $T_2$ spaces are $T_1$, the same filtration argument as in Theorem 9 applies here.

Now suppose we are given a finite representation $\mathcal{M} = (T, \tau, V)$ such that $\varphi$ is satisfiable on it. Let $(S, \sigma_0)$ be an $(n-m)$-resolvable $T_2$ space, where $n = |T|$ and $m$ is the number of points in the finite representation named by a nominal. Let $X'_1, \ldots, X'_{n-m}$ be the dense subsets of $S$ which form the partition of $S$. Note that if $n > 1$ then these sets have empty interiors, because if one of them doesn’t then no other can be dense. Let $X_{n-m+1}, \ldots, X_n$ be arbitrary singleton subsets of $S$. Finally, denote

$$X_i = X'_i \setminus \bigcup_{j=n-m+1}^{n} X_j, \text{ for } 1 \leq i \leq 1, n-m$$

Since $S$ is a $T_1$ space, $X_1, \ldots, X_{n-m}$ are still dense in $S$.

As usual, denote

$$\tilde{O} = \bigcup_{i \in O} X_i$$

and consider a new topology $\sigma$ on $S$ generated by

$$\sigma_0 \cup \{ \tilde{O} \mid O \in \tau \}$$

and the valuation

$$W(p) = \tilde{V}(p) \text{ for all } p \in \text{PROP} \cup \text{NOM}$$

It is left to prove that this construction preserves satisfiability of subformulas of $\varphi$.

We use the same argument as in the proof of Theorem 9 here. Indeed, we are in the same setting: $X_k$ form a partition of $S$, some of them are singleton sets (named by nominals), others are dense in $T$ in $\sigma_0$. Consider the map $f : S \to T$ that maps $X_k$ to $k$ for all $k \in T$. It is continuous by its construction: the preimage of an open $O$ is $\tilde{O}$ which is open. It is open, because, like in Theorem 9 the image of any open $\tilde{O} \cap U$ from $\sigma$ is $O \setminus F$ where $F$ is a set of points named by nominals, and since that $T \setminus F$ is open follows definition of a finite representation of the $T_1$ model, we conclude that $f$ maps opens to opens. The graph of $f$ is a hybrid total bisimulation, which means that it preserves satisfiability of $H(E)$ formulas and the statement of the theorem follows.
Since every $T_2$ space is a $T_1$ space, we get the following corollary

**Theorem 12.** The logic of $T_2$ spaces coincides with the logic of $T_1$ spaces (and hence, is decidable).

### 3.3 $T_0$ spaces

In this section we will use a similar technique to prove one more representation/decidability result, this time for $T_0$ spaces.

**Proposition 13.** An Alexandroff space corresponding to a partial order by the Proposition 7 is $T_0$ and the frame that corresponds to a $T_0$ Alexandroff space is a partial order.

**Proof.** This is an easy consequence of Proposition 7. □

By the Proposition above, every $T_0$ validity is a partial order validity. The converse is not true.

Consider the countable topological space $(\mathbb{N}, \sigma)$ with cofinite topology. Construct a topological space $(T, \tau)$ as follows: let $T = \{ \ast \} \cup \mathbb{N}$ and $\tau = \{ U = \{ \ast \} \cup O \mid O \in \sigma \}$. This is a $T_0$ space. Now, introduce a valuation that names $\ast$ with a nominal $i$ and consider a formula $\varphi = \Diamond (\neg i \land \Diamond i)$. This formula is satisfied at $\ast$, but it is not satisfiable on any partial order. Hence $\text{Log}(T_0)$ is a strict subset of the logic of partial order.

Although the counterexample we have just mentioned tells us that $\text{Log}(T_0)$ is more complicated than the logic of partial orders, it will serve us as the source of ideas on how one might build a $T_0$ model out of a quasi-model. We will need a different notion of a finite representation of a model than one for $T_1$ and $T_2$ spaces (otherwise $\text{Log}(T_0)$ would coincide with $\text{Log}(T_1)$ which is impossible).

**Definition 7** (finite representation of a $T_0$ model). A **finitely representable $T_0$ model** is a $T_0$ topological model which is topobisimilar to a finite topological model. A finite topological model is called a **finite representation of a $T_0$ model** if for every pair of points $x, y$ named by nominals, there exists an open neighborhood $O_x$ of $x$ such that $y \notin O_x$ or there exists an open neighborhood $O_y$ of $y$ such that $x \notin O_y$ (we will say simply finite representation, when there is no confusion).

Once again we will describe a way to construct a topological space (this time a $T_1$ space) that satisfies a given formula given a finite representation that satisfies that formula. We will have as a consequence a

**Theorem 14.** $\text{Log}(T_0)$ is decidable.
Proof. What we really prove here is that a formula has a $T_0$ model iff it has a finitely representable $T_0$ model.

A filtration of a $T_0$ space through $\text{CL}(\varphi)$ gives a finite representation of a $T_0$ model, because the natural projection is an open map. This construction preserves satisfiability by the same argument, as the one that was mentioned in the previous sections.

The other direction of the proof goes as follows. Consider a finite representation $\mathfrak{M} = (T, \tau, V)$. We identify $T$ with natural numbers $1, \ldots, n$ and we will use such a numbering that $1, \ldots, m$ are named by a nominal. We construct a topological model $(S, \sigma, W)$ with a support $\{1, \ldots, m\} \cup \mathbb{N}$ and topology and valuation defined below. We will suppose further that $n \neq m$ since otherwise the finite representation is already a real $T_0$ that satisfies $\varphi$.

Partition $S$ into sets $X_1, \ldots, X_n$: let $X_k = \{k\}$ for $1 \leq k \leq m$ and let $X_{m+1}, \ldots, X_n$ be the sets of the form $\{k + j(n - m) \mid 0 \leq j < \infty\}$ for $m + 1 \leq k \leq n$.

As usual, denote

$$\hat{O} = \bigcup_{i \in O} X_i$$

for $O \subseteq T$. Define the topology $\sigma$ to be

$$\{\hat{O} \setminus F \mid O \in \tau, F \subseteq \mathbb{N} \text{ finite}\}$$

Valuation is also defined in a usual way

$$W(p) = \hat{V}(p) \text{ for all } p \in \text{PROP} \cup \text{NOM}$$

The model thus constructed is $T_0$. Any point $x$ from $\mathbb{N}$ can be separated from any other point by a set $S - \{x\}$. Since two points named by nominal can be separated by an open $O$ in the finite representation, $\hat{O}$ will separate them in $\mathfrak{M}$ (that is where we use the fact that $T$ is a finite representation of a $T_0$ model).

Consider the map $f : S \rightarrow T$ that maps $X_k$ to $k$ for all $k \in T$. This is an interior map and its graph is a total hybrid topobisimulation.

Indeed, note that $X_1, \ldots, X_n$ have non-empty intersection with all the sets of the form $T \setminus F$, where $F \subseteq \mathbb{N}$ is finite. Similarly, any open from $\sigma$ can be seen as a union $\cup_{k \in O}(X_k \cap (T \setminus F))$ for some $O \in \tau$ and some fixed finite $F \subseteq \mathbb{N}$. Thus the image of this set under $f$ will be $O$, which is open.

The continuity of $f$ follows from its construction and definition of the topology $\sigma$. The remaining conditions that make the graph of $f$ a hybrid bisimulation can be checked straightforwardly.

Since total hybrid bisimulations preserve satisfiability of $H(E)$ formulas, $(S, \sigma, W)$ satisfies the same formulas as $(T, \tau, V)$, which finishes the proof of the theorem.
3.4 Complexity

Now, when we know that $\text{Log}(T_0)$ and $\text{Log}(T_1)$ are decidable, the next natural question to ask is what the complexity is. The lower bound follows from the result of Ladner [4] that $S4$ has a PSPACE-complete satisfiability problem.

**Proposition 15.** $\text{Log}(T_0)$ and $\text{Log}(T_1)$ have a PSPACE-hard satisfiability problem.

To establish an upper bound we will present a two player game parametrized by a formula where one of the players has a winning strategy iff the formula is satisfied on a finite representation of a $T_0$ or $T_1$ model (and hence, is satisfiable on a $T_0$ or $T_1$ space). The amount of information on the board at the end of any play will be polynomial in the length of the formula. Thus, it is possible to build a polynomial space Turing machine that decides whether the game has a winning strategy by just repeatedly analyzing all possible plays.

We will present a different notion of a model, equivalent to the notions of finite representation of a $T_1$ (or $T_0$) model.

**Definition 8** (Hintikka set). Let $\Sigma$ be a set of formulas closed under subformulas and single negations. A set $A \subseteq \Sigma$ is called a Hintikka set if it is maximal subset satisfying the following conditions:

1. $\bot \notin A$
2. if $\neg \varphi \in \Sigma$ then $\varphi \in A$ iff $\neg \varphi \notin A$
3. if $\varphi \land \psi \in \Sigma$ then $\varphi \land \psi \in A$ iff $\varphi \in A$ and $\psi \in A$

**Definition 9** (Quasi-model). Let $\varphi$ be a formula and $\text{Cl}(\varphi)$ be its subformula closure. A tuple $(T, \tau, \lambda)$, where $(T, \tau)$ is a finite topological space and $\lambda$ is a function from $T$ to $\text{Cl}(\varphi)$ is called a quasi-model for $\varphi$ if the following holds:

1. $\lambda(t)$ is a Hintikka set for any $t \in T$
2. at least for one $t \in T$, $\varphi \in \lambda(t)$
3. for all $\Box \psi \in \text{Cl}(\varphi)$, $\Box \psi \in \lambda(t)$ iff there exists an open $O \ni t$ such that $\forall s \in O \psi \in \lambda(s)$

If we impose extra condition on the quasi model, we are then talking about $T_1$ or $T_0$ quasi-models:

$(T_1 \text{ condition for quasi-models})$ if $i \in \lambda(t)$ where $i$ is a nominal, then $T - \{t\}$ is open.
(T₀ condition for quasi-models) for every pair of points \( x, y \) named by nominals, there exists an open neighborhood \( O_x \) of \( x \) such that \( y \notin O_x \) or there exists an open neighborhood \( O_y \) of \( y \) such that \( x \notin O_y \).

**Lemma 16.** This definition is equivalent to the notion of a finite representation of a model in the following sense: a formula \( \varphi \) is satisfied on a \( T_1 \) (\( T_0 \)) model \( (S, \sigma, W) \) iff there exists a \( T_1 \) (\( T_0 \)) quasi-model for \( \varphi \).

**Proof.** To prove the left-to-right direction take a given finite topological space \( S, \sigma, W \) and define a mapping \( \lambda : S \to \text{Cl}(\varphi) \):

\[
\lambda(x) = \{ \psi \in \text{Cl}(\varphi) \mid (S, \sigma, W), x \models \psi \}
\]

Then \( (S, \sigma, \lambda) \) is a quasi-model for \( \varphi \).

Right-to-left direction: take \( (S, \sigma, \lambda) \) and define valuation \( W \):

\[
W(p) = \{ x \in S \mid p \in \lambda(x) \}
\]

Then \( (S, \sigma, W) \) is a finite representation. One can prove by induction on formula structure and using condition 3 in the definition 9 that for all formulas \( \psi \in \text{Cl}(\varphi), \psi \in \lambda(x) \) iff \( (S, \sigma, W), x \models \psi \).



The winning strategy in the game we are about to describe contains all the necessary information to build a quasi-model that satisfies the formula. During each play of the game a piece of model is constructed. Since the quasi-models are a special kind of finite topological spaces and by Proposition 1, finite topological spaces can be regarded as relational structures, we will think about the quasi-models as finite relational structures.

We will prove the upper bound for \( H(E) \) outright; it is not much harder than for \( H(\@) \) and the result is more general. One remark must be made, the quasi-model for \( H(E) \) should satisfy one extra condition:

**universal modality condition** if \( E\varphi \in \lambda(x) \) then there exists a point \( y \) such that \( \varphi \in \lambda(y) \).

**Theorem 17.** \( \text{Log}_{H(E)}(T_0) \) is \( \text{PSPACE}-\text{complete} \).

**Proof.** For the purposes of this proof we will consider \( \Diamond \) as a primitive operator and \( \Box \varphi \) as an abbreviation of \( \neg \Diamond \neg \varphi \). Every subformula of the form \( \@_i \varphi \) can be equivalently replaced by \( E(i \land \varphi) \) so we do not consider \( \@ \) either.

Here is the description of the game for a formula \( \varphi \). There are two players: \( \forall \)belard (male) and \( \exists \)loise (female). \( \exists \)loise plays by putting Hintikka sets on the board and defining a transitive reflexive relation \( R \) on them;
∀belard introduces challenges that she must meet. She starts the game by putting a set \{X_0, \ldots, X_k\} on the board and introducing a relation \(R\) among them (it will be updated after each move). The sets and the relation must satisfy the following conditions:

1. (ROOT) \(X_0\) contains \(\varphi\), \(k \leq |Cl(\varphi)|\),
2. (INIT-NOM) no nominal occurs in two different Hintikka sets,
3. (INIT-DIAMOND) for all \(\Diamond \chi \in Cl(\varphi)\), if \(RX_lX_j\) and \(\Diamond \chi \notin X_l\) and \(\chi \notin X_j\),
4. (INIT-UNIV) for all \(X_l\) and for all \(E\chi \in Cl(\varphi)\), \(E\chi \in X_l\) iff \(\chi \in X_j\) for some \(j\),
5. (INIT-CYCLES) \(R\) has no cycles.

If the conditions do not hold, \existsloise looses immediately. ∀belard’s turn consists of selecting a Hintikka set \(X_l\) and picking a formula \(\Diamond \psi\) out of it. \existsloise must meet the challenge by putting a Hintikka set \(Y\) on the board, such that the following conditions hold:

1. (DIAMOND) \(\psi \in Y\), \(RX_lY\) and for all \(\Diamond \chi \in Cl(\varphi)\), if \(\Diamond \chi \notin X_l\) then \(\Diamond \chi \notin Y\) and \(\chi \notin Y\),
2. (UNIV) for all \(X_l\) and for all \(E\chi \in Cl(\varphi)\), \(E\chi \in X_l\) iff \(\chi \in X_j\) for some \(j\),
3. (NOM) if \(i \in Y\) for some nominal \(i\) then \(Y\) is one of the Hintikka sets \existsloise played during the first move. If this is the case, the game stops and she wins (unless the next rule is violated, in which case she loses),
4. (CYCLES) \(R\) does not have cycles that involve Hintikka sets that contain nominals.

If \existsloise cannot find a \(Y\) that satisfies those conditions, then the game stops and \vbelard wins. Otherwise, \vbelard must choose a formula of the form \(\Diamond \psi\) from the last played set (that is, \(Y\)) and the game continues in a similar way. If \existsloise manages to meet all \vbelard’s challenges and if he has no more challenges to present, she wins. This does not guarantee that the game will stop at some point, so we introduce an extra rule. A list of formulas played by \vbelard is kept, if he plays a formula the second time, \existsloise must respond with the same Hintikka set as she did when she played the formula for the first time. If her set satisfies the conditions from the previous paragraph, \existsloise wins; otherwise, she loses. In any case, the game stops immediately.

We will now prove that \existsloise has a winning strategy in the game iff a formula \(\varphi\) has a quasi-model.

(LEFT-TO-RIGHT DIRECTION) Suppose that \existsloise has a winning strategy in the game. We build a quasi-model \((S, \sigma, \lambda)\) for \(\varphi\) as follows. Let \(S_0\) be the Hintikka sets played at the first move — \(\{X_0, \ldots, X_k\}\). Define sets \(\{S_i\}\) by induction; suppose \(S_i\) is defined, then \(S_{i+1}\) is a copy of the Hintikka sets played by \existsloise in reply to \vbelard moves when he picks sets form \(S_i\) (with an exception: we do not copy sets from the initial move when \existsloise plays
them further in the game). Let $S$ be the disjoint union of $S_i$. Set $Rxy$ iff for all formulas $\Diamond\psi \in Cl(\varphi)$, $\Diamond\psi \notin x$ implies $\Diamond\psi \notin y$ and $\psi \notin y$. Note that $R$ thus defined coincides with $R$ defined throughout the game. Note also that $R$ is reflexive, transitive and contains no cycles that involve Hintikka sets named by nominals. Let $\sigma$ consist of all upward closed sets (as in Proposition 1) and put $\lambda(x) = x$. The topology thus defined satisfies the $T_0$ condition for quasi-models (if it did not then $R$ would contain cycles with points named by nominals). The universal modality condition for quasi-models is taken care of by the rules of the game: namely, by conditions (INIT-UNIV) and (UNIV).

It is left to prove that condition 3 in the Definition 3 is satisfied. Suppose that $\Box\psi = \neg\Box\neg\psi \in Cl(\varphi)$ and $\Box\psi \in \lambda(t)$, then $\neg\psi \notin \lambda(t)$. Then the conditions (INIT-DIAMOND) and (DIAMOND) guarantee that for all $s$ in the minimal upward closed set $O \ni t$, $\neg\psi \notin \lambda(s)$ hence $\Box\psi = \neg\Box\neg\psi \in \lambda(s)$. By definition of $\sigma$, $O$ is open.

Suppose now that $t \in O$, $\forall s \in O \psi \in \lambda(s)$ where $O$ is open, or upward closed set. We need to prove that $\neg\psi \notin \lambda(t)$. We will prove it by contradiction: if $\neg\psi \in \lambda(t)$ then once $\forall$belard chooses this formula, $\exists$loise must respond with one of the Hintikka sets from $O$, but if she does that she breaks (DIAMOND) (because $\neg\psi \notin s$ for all $s \in O$) and loses. Hence, $\neg\psi \notin \lambda(t)$.

We have built a quasi-model from a winning strategy of $\exists$loise.

(right-to-left direction) Let us prove that $\exists$loise can read her winning strategy off a quasi-model $(S, \sigma, \lambda)$ for $\varphi$. Let $R$ be the relation of the corresponding relational structure obtained by the Proposition 1.

During her first move $\exists$loise picks a point $t$ such that $\varphi \in \lambda(t)$, for each nominal contained in $\varphi$ she picks a point named by that nominal, and for each subformula of $\varphi$ of the form $E\psi$ she picks a point $t$ such that $\psi \in \lambda(t)$. This move complies with the required conditions.

Next, when $\forall$belard chooses a point $X$ and a formula $\Box\psi$, $\exists$loise responds with a with a maximal (with respect to the relation $R$ understood as order relation) successor $Y$ of $X$ such that $Y$ contains $\psi$. Obviously this complies with (DIAMOND), (UNIV), (NOM) and (CYCLES) rules. It is always possible to find a maximal successor because quasi-models are finite. $\exists$loise needs to adopt this strategy to be able to successfully answer with the same Hintikka set when $\forall$belard will pick formula $\Box\psi$ again.

For suppose $\exists$loise played $Y$ in response for $\forall$belard 's challenge $\Box\psi$ from $X$ and suppose that later $\forall$belard picks the same formula $\Box\psi$ from a set $Z$, which is a successor of $X$. Since $RXZ$ any successor of $Z$ containing $\psi$ will be a maximal successor of $X$ containing $\psi$. So $Y$ is among successors of $Z$ and can be played again to fulfill the rules of the game.

\[\square\]

**Theorem 18.** $Log_{H(E)}(T_0)$ is PSPACE-complete.
Proof. The game for $T_1$ is the game for $T_0$ with the following modifications. (INIT-CYCLES) and (CYCLES) conditions are replaced with

(NO-INCOMING) points named by nominals have no incoming arcs

and (NOM) is dropped (it is has no effect because of (NO-INCOMING)).

We only have to prove that this new rule really correspond to $T_1$ quasi-models, the rest is taken care of in the proof for the $T_0$ case.

Indeed, in the model that we build out of the $\exists$Loise ‘s winning strategy no Hintikka set that contains a nominal has an incoming arc (because of the (NO-INCOMING) rule. Then a complement of any such point is a union of upward closed sets, hence open.

The converse is also true: in the relational counterpart of any $T_1$ quasi-model no nominal-named point has an incoming arc, because otherwise the complement of the point would not be open. Thus, when we build $\exists$Loise ‘s strategy based on a $T_1$ quasi-model, we will never break (NO-INCOMING) rule. 

\[\Box\]

4 Logics of concrete spaces

Up to now we have dealt with logics of separation axioms, that is logics of classes of spaces. In this section we will show that logics of certain concrete structures coincide with logics of classes of spaces, like the class $T_1$ spaces or the class of $T_1$ spaces without isolated points.

In basic modal logic the logics of all separation axioms coincide as a consequence of McKinsey-Tarski theorem. This theorem implies the inclusion $\text{Log}(T_n) \subset \text{Log}(\mathbb{Q}) = S4$ (where $\mathbb{Q}$ is the topological space of rational numbers) for $n = 0, 1, 2, 3, 4, 5$ which makes collapse the inclusion chain $S4 \subset \text{Log}(T_0) \subset \text{Log}(T_1) \subset \ldots \subset \text{Log}(T_5)$. The completeness results we are going to prove collapse a similar inclusion chain $\text{Log}_{H(E)}(T_1) \subset \text{Log}_{H(E)}(T_2) \subset \ldots \subset \text{Log}_{H(E)}(T_5)$.

But first let us present our representation result for $T_1$ under a slightly different angle.

Proposition 19. Any finite representation of a $T_1$ model is topobisimilar to a (topological space corresponding to) disjoint union of finite rooted models, such that each root is named by a nominal and no other points are named by nominals.

Proof. Indeed, from the relational perspective finite representations of $T_1$ models are models where points named by nominals have no incoming arcs from other points (the accessibility relation is reflexive so there is always an arc from a point to itself). We can restrain our attention to the submodel generated by the points named by nominals, so we suppose further that any point can be reached from a point named
by a nominal. Call this model \( M = (W, R, V) \). Take a point \( w_1 \) named by a nominal and take its minimal successor \( v_1 \) which has a predecessor \( w_2 \) named by another nominal. Let the domain of the generated submodel of \( v_1 \) be \( W_1 \). Construct a model \( M_1 = ((W \setminus W_1) \cup W'_1 \cup W''_1, R', V') \), where \( W'_1 \) and \( W''_1 \) are copies of \( W_1 \). Denote by \( w' \) and \( w'' \) members of \( W'_1 \) and \( W''_1 \) which correspond to \( w \in W_1 \), then define \( R' \) as follows:

\[
Rw'v \text{ and } Rw''v \text{ for all } w \in W_1, v \in W \text{ such that } Rwv
\]

and define \( V' \) as follows:

\[
w'' \in V'(p) \text{ and } w' \in V'(p) \text{ for all } w \in V(p)
\]

The points of the submodel of \( M_1 \) generated by \( w_1 \) have no predecessors from the submodel generated by \( w_2 \). At the same time \( M_1 \) is topobisimilar to \( M \). By repeating the described procedure repeatedly over \( w_1 \) and \( M_1 \) we will finally come to a model \( M_n \) such that successors of \( w_1 \) do not have predecessors named by nominals other than \( w_1 \). This way we can “peel off” rooted submodels which have other points named by nominals as their roots. At each step we have a model topobisimilar to initial model. The model that we get in the end is a disjoint union of rooted models, their roots named by nominals.

In what follows we will call a rooted model that contains more than one point (the root) a non-trivial model.

4.1 Completeness with respect to Cantor space

The proof we present here is a slight modification of the construction of [5].

There are many equivalent definitions of Cantor space. For example, one can regard Cantor space \( C \) as a set of paths in an infinite binary tree, or equivalently as a set of infinite strings over alphabet \( \{0, 1\} \): 0s correspond to the path turning left, 1s correspond to the path turning right. The topology is generated by the basic open sets of the form:

\[
B_X = \{y \mid X \text{ is a prefix of } y\}
\]

where \( X \) runs through all finite binary strings.

This definition is more formal than geometric, but in any case we will not need it so much, because our completeness result is built on top of the following result of [5].

**Definition 10.** A cluster in a relational structure \( (W, R) \) is a maximal set \( A \) such that \( R \) restricted to \( A \) is an equivalence relation. A cluster is called simple if it contains only one point, and proper otherwise.
**Theorem 20.** (Aiello, van Benthem, Bezhanishvili)

For any topological model corresponding to a finite, rooted, non-trivial, reflexive and transitive model where every point except the root is contained in a proper cluster there exists an interior map from the Cantor space $\mathcal{C}$ onto this model.

Denote $\log(T_{1}^{\text{no i.p.}})$ the logic of $T_{1}$ spaces without isolated points. This logic is different from $\log(T_{1})$ (consider a formula $\forall i \Diamond i$ which is satisfiable only in a model where $i$ names an isolated point). We can slightly modify the proof of Theorem 9 and prove that $\log(T_{1}^{\text{no i.p.}})$ is complete with respect to finite representations of $T_{1}$ models without isolated points, the Proposition 19 then is modified accordingly:

**Proposition 21.** Any finite representation of a $T_{1}$ model is topobisimilar to a (topological space corresponding to) disjoint union of non-trivial finite rooted models, such that each root is named by a nominal, no other points are named by nominals.

The interior map that is constructed in the proof of Theorem 20 has a nice property that it maps the root to exactly one point of the Cantor space, that means that its graph is a hybrid bisimulation (supposing that the root is named by a nominal). By Proposition 19 $\log(T_{1}^{\text{no i.p.}})$ is complete with respect to models which are disjoint unions of non-trivial finite rooted transitive and reflexive submodels. This is almost the same kind of models that we have in the Theorem 20. We only lack the condition that every point except the root belongs to a proper cluster.

**Theorem 22.** $\log(T_{1}^{\text{no i.p.}})$ is complete with respect to Cantor space.

**Proof.** First observe that the models form the Proposition 19 are topobisimilar to the models that have exactly the same properties but in addition every point (except points named by nominals) is a part of proper cluster. This can be done by taking all simple clusters and replacing them with, for example, two-point clusters, ensuring that all the incoming and outgoing edges from the initial cluster are copied to the new one. It is easy to see that this operation gives rise to a bisimulation.

Now we have that $\log(T_{1})$ is complete with respect to models that are disjoint unions of models that are suitable for Theorem 20. It is left to notice that Cantor space is homeomorphic to a disjoint union of any finite number of copies of itself to conclude the proof. 

**Corollary 23.** Denote by $\hat{\mathcal{C}}$ a disjoint union of $\mathcal{C}$ with countably many isolated points. Then $\log(T_{1}) = \log(\hat{\mathcal{C}})$.

**Proof.** One can map the non-trivial rooted submodels of a finite representation $\mathfrak{M}$ that refutes a non-theorem of $\log(T_{1})$ to $\mathcal{C}$ as in Theorem 22 and...
then map the isolated points of \( \mathcal{M} \) to isolated points of \( \mathcal{C} \). One can do that because there is only finitely many isolated points in \( \mathcal{M} \). The map obtained is interior.

\[ \text{Corollary 24. } \log(\mathcal{C}) = \log(T_n) \text{ for } n = 1, 2, 3, 4, 5. \]

4.2 Completeness with respect to rationals

In this section we will prove that the logic of \( T_1 \) spaces without isolated points is complete with respect to \( \mathbb{Q} \), the rational numbers. The proof is inspired by the proof of completeness of \( S4 \) with respect to rationals from \cite{9}. The proof uses the following statement.

**Theorem 25 (Cantor).** Every countable dense linear ordering without endpoints is isomorphic to rational numbers.

One of the consequences of Theorem 25 is

**Proposition 26.** The topological space of rational numbers is homeomorphic to disjoint union of any finite number of copies of itself.

\[ \text{Proof.} \] The finite disjoint union of copies of rationals can be seen as a linear order which is several copies of \( \mathbb{Q} \) (as on ordered set) juxtaposed. This linear order is dense, countable, without endpoints, hence order-isomorphic to rationals. But since the topology on both rationals and finite disjoint union of copies of \( \mathbb{Q} \) is completely determined by the order, the two spaces are homeomorphic.

We will turn each disjoint component of a finite representation of a \( T_1 \) model into an infinite \( n \)-ary tree, \( n \geq 2 \) and show that it is topobisimilar to rational numbers. We will then apply Proposition 26 to conclude that there exists a topobisimulation between any finite representation of a \( T_1 \) model without isolated endpoints and \( \mathbb{Q} \).

**Lemma 27.** Any finite, rooted, non-trivial, reflexive and transitive model with a root named by a nominal is bisimilar to the full infinite, reflexive and transitive \( n \)-ary tree, \( n \geq 2 \) and the root of the tree is mapped exactly to the root of the model by the bisimulation.

\[ \text{Proof.} \] Consider a model \( \mathcal{M} \) with the root \( w \) named by a nominal. First, remove the arc going from \( w \) to itself and consider the unraveling of the model obtained. This will be an infinite irreflexive, antisymmetric, non-transitive tree \( \mathcal{M}_1 \) of finite branching \( n \) (\( n \) is finite because the original model was finite). It is easy to see that \( \mathcal{M}_1 \) is bisimilar to \( \mathcal{M} \), moreover, \( w \) is mapped exactly to the root of \( \mathcal{M}_1 \) by the bisimulations.

However, it might be the case that \( \mathcal{M}_1 \) is not a full \( n \)-ary tree or it might be the case that \( n = 1 \) (\( n > 0 \) because \( \mathcal{M} \) is non-trivial). In the last case put
\( n = 2 \) as it is our desired number of successors for each node. Our strategy is to go inductively through the tree and repair it there where there is not enough successors. Suppose that we are standing at a point \( x \) that has \( m \) successors, \( m < n \). Take arbitrary successor of \( x, y \) and consider a subtree \( T_y \) with \( y \) as root. Make \( n - m \) copies of \( T_y \) and link \( x \) to their roots. By repeating this manipulation throughout the tree we will end up with a full infinite \( n \)-ary tree which is bisimilar to \( M_1 \). Consider its reflexive transitive closure. This tree satisfies the condition of the lemma.

\[ \square \]

**Lemma 28.** The full reflexive transitive infinite \( n \)-ary tree \( T_n \) is homeomorphic to \( Q \).

**Proof.** We will construct an interior map \( f \) from \( T_n \) onto a countable dense subborder \( X \) of \( Q \) which by Theorem 25 is order-isomorphic (hence, homeomorphic) to \( Q \).

First put \( f(r) = 0 \) for the root \( r \) of \( T_n \). Then define \( f \) inductively as follows. We will say that the root belongs to the level 0 of the tree, its immediate successors belong to the level 1 etc., in general, if a point belongs to level \( k \) then its immediate successors belong to level \( k+1 \). Now let \( f \) be defined on a point \( w \) of level \( k \) which has successors \( v_1, \ldots, v_n \). Define

\[
\begin{align*}
    f(v_1) &= f(w) - \frac{1}{(n+1)x}, \\
    f(v_m) &= f(w) + \frac{(m-1)}{(n+1)x} \quad \text{for } 2 \leq m \leq n
\end{align*}
\]

The map \( f \) thus constructed is an homeomorphism between \( T_n \) and \( X = f(T_n) \). It follows from construction that it is bijective. The proof that it is open and continuous is essentially the same as the proof in [9].

\[ \square \]

**Theorem 29.** \( \log(T_{n_0}^{\alpha_i, y_p}) \) is complete with respect to rational numbers.

**Proof.** Follows from Proposition 21, Lemma 27, Lemma 28 and Proposition 25.

\[ \square \]

**Corollary 30.** Denote by \( \hat{Q} \) a disjoint union of \( Q \) with countably many isolated points. Then \( \log(T_1) = \log(\hat{Q}) \).

**References**

[1] Nicolas Bourbaki, *Elements of mathematics; 3. General topology*, Hermann, Paris, 1966.

[2] L. Feng and O. Masaveu, *Exactly \( n \)-resolvable spaces and \( \omega \)-resolvability*, Math. Japonica 50 (1999), no. 3, 333–339.
[3] D. Gabelaia, B. ten Cate, and D. Sustretov, Modal languages for topology: Expressivity and definability, submitted to Annals of Pure and Applied Logic, preprint available at http://www.arxiv.org/abs/math.LO/0610357.

[4] R. Ladner, The computational complexity of provability in systems of modal logic, SIAM Journal of Computing 6 (1977), 467–480.

[5] J. van Benthem M. Aiello and G. Bezhanishvili, Reasoning about space: the modal way, Journal of Logic and Computation 13 (2003), no. 6, 889–920.

[6] J.C.C. McKinsey and A. Tarski, The algebra of topology, Annals of Mathematics 45 (1944), 141–191.

[7] V. Shehtman, “Everywhere” and “here”, Journal of Applied Non-Classical Logics 9 (1999), no. 2-3.

[8] A. Tarski, Der Aussagenkalkül und die Topologie, Fundamenta Mathematicae 31 (1938), 103–134.

[9] J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac, Multimodal logics of products of topologies, Studia Logica (2006), 369–392.