THE ENDPOINT FEFFERMAN-STEIN INEQUALITY FOR THE STRONG MAXIMAL FUNCTION

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Abstract. Let $M_n f$ denote the strong maximal function of $f$ on $\mathbb{R}^n$, that is the maximal average of $f$ with respect to $n$-dimensional rectangles with sides parallel to the coordinate axes. For any dimension $n \geq 2$ we prove the natural endpoint Fefferman-Stein inequality for $M_n$ and any strong Muckenhoupt weight $w$:

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda \}) \lesssim w_{\infty, n} \int_{\mathbb{R}^n} |f(x)| \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) M_n w(x) \, dx.$$

This extends the corresponding two-dimensional result of T. Mitsis.

1. Introduction

The strong maximal function. Let $\mathcal{R}_n$ denote the family of all rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For a locally integrable function $f$ on $\mathbb{R}^n$ we will denote by $M_n f$ the strong maximal function:

$$M_n f(x) := \sup_{\mathcal{R} \in \mathcal{R}_n, \mathcal{R} \ni x} \frac{1}{|\mathcal{R}|} \int_{\mathcal{R}} |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

Here $|S|$ denotes the $n$-dimensional Lebesgue measure of a set $S \subset \mathbb{R}^n$. We will sometimes use the same notation for the $(n-1)$-dimensional Lebesgue measure, but this will be clear from the context.

The endpoint behavior of $M_n$ close to $L^1$ is given by the classical theorem of Marcinkiewicz, Jessen and Zygmund, [8]:

$$(1.1) \quad |\{x \in \mathbb{R}^n : M_n f(x) > \lambda \}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) \, dx,$$

where $\log^+ t := \max(0, \log t)$. Inequality (1.1) implies the strong differentiation of the integral of all functions $f \in L(1 + (\log^+ L)^{n-1})(\mathbb{R}^n)$, that is, all functions $f$ on $\mathbb{R}^n$ such that:

$$\int_{\mathbb{R}^n} |f(x)| \left(1 + \left(\log^+ |f(x)|\right)^{n-1}\right) \, dx < +\infty.$$

The strong maximal theorem cannot be improved, that is, the function $t(1 + (\log^+ t)^{n-1})$ on the right hand side of (1.1) cannot be replaced by any slower increasing function. Remember

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that the usual Hardy-Littlewood maximal function is the maximal average of $f$ with respect to all $n$-dimensional Euclidean cubes, or balls and it maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. The important difference to be noted here is that the strong maximal function is an $n$-parameter maximal average, in contrast to the usual one-parameter Hardy-Littlewood maximal function, and this difference is reflected in the strong maximal theorem (1.1) which requires extra logarithmic scales of integrability. The original proof from [8] relies on the observation that $M_n$ can be viewed as a composition of $n$ one-dimensional Hardy-Littlewood maximal operators. One then appeals to the one-dimensional theory to get (1.1) together with its strong $L^p(\mathbb{R}^n)$ counterparts. A more geometric point of view was introduced by the work of Córdoba and R. Fefferman, [3], who gave a proof of (1.1) by means of a geometric covering argument. This is in a sense a dual point of view where the $n$-parameter composition of operators is replaced by induction on the dimension. The importance of the Córdoba-Fefferman geometric proof of the strong maximal theorem is highlighted by the fact that the usual Besicovitch covering argument fails when applied to families of rectangles having arbitrary eccentricities.

**Strong weights.** A weight $w$ will be a locally integrable, non-negative function on $\mathbb{R}^n$. We will say that $w$ belongs to the class $A^*_p$, $1 < p < \infty$, whenever

$$[w]_{A^*_p} := \sup_{R \in \mathcal{R}_n} \left( \frac{1}{|R|} \int_R w \right) \left( \frac{1}{|R|} \int_R w^{1-p'} \right)^{p-1} < +\infty.$$ 

In this case we will say that $w$ is a strong $A_p$-weight. For $p = 1$ the class $A_1^*$ is defined by the condition

$$\frac{1}{|R|} \int_R w \leq C \inf_{x \in R} w(x), \quad \text{for almost every } x \in R, \quad R \in \mathcal{R}_n,$$

which is equivalent to saying that $M_n w \leq Cw$ almost everywhere in $\mathbb{R}^n$. The smallest constant $C > 0$ in the previous inequality is the $A^*_1$-constant of the weight, denoted by $[w]_{A^*_1}$.

It follows by Hölder’s inequality that the $A^*_p$ classes are increasing, that is, for $1 \leq p \leq q < \infty$ we have $A^*_p \subset A^*_q$. We define the class $A_\infty^*$ by means of

$$A_\infty^* := \bigcup_{p>1} A^*_p.$$ 

It is equivalent to define the class $A_\infty^*$ by the following property: there exist constants $\delta, c > 0$ such that, given any rectangle $R \in \mathcal{R}_n$ and a measurable subset $S \subset R$, then

$$\frac{w(S)}{w(R)} \leq c \left( \frac{|S|}{|R|} \right)^{\delta}. \quad (1.2)$$

An important feature of strong $A_\infty^*$-weights is that if we fix any $t \in \mathbb{R}$ then the weight

$$w^t(x') := w(x', t), \quad x' \in \mathbb{R}^{n-1},$$

is an $A_\infty^*$-weight on $\mathbb{R}^{n-1}$, uniformly in $t \in \mathbb{R}$. In practice, uniformly means that all the constants connected with the properties of the $A_\infty^*$-weight $w^t$ can be taken to be independent of $t$. For these and other properties of strong Muckenhoupt weights see for example [2] or [6, Chapter IV].
Remark 1.3. Let $w \in A^*_\infty$. By the previous discussion we see that there exists some $\epsilon = \epsilon (w) > 0$ such that, for every rectangle $R \in \mathcal{R}_n$ and all measurable sets $F \subset \mathbb{R}^n$, we have

$$|R \cap F| \leq \epsilon |R| \Rightarrow w(R \cap F) \leq \frac{1}{2} w(R) \Rightarrow w(R \setminus F) \geq \frac{1}{2} w(R).$$

In fact, it suffices to choose $\epsilon > 0$ so that $c e^\delta \leq \frac{1}{2}$, where $c, \delta$ are the constants associated to $w \in A^*_\infty$ from (1.2). Since for any $t \in \mathbb{R}$ the weight $w^t := w(\cdot, t)$ is an $A^*_\infty$-weight on $\mathbb{R}^{n-1}$, uniformly in $t$, the $\epsilon > 0$ can be chosen sufficiently small so that we have the previous property also for $w^t$, rectangles $R' \in \mathcal{R}_{n-1}$ and sets $F' \subset \mathbb{R}^{n-1}$, uniformly in $t$. We will use this remark several times in what follows.

It is known that $M_n$ is bounded on $L^p(w)$, $1 < p < \infty$ if and only if $w \in A^*_p$. This result follows again by an appeal to the one dimensional theory. For the necessity of the $A^*_p$ condition, one argues as in the case of the usual $A_p$ weights. The corresponding endpoint bound is also true: the operator $M_n$ satisfies the distributional estimate

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}) \leq w, n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + (\log^+ |f(x)|)^{n-1}\right) w(x) dx$$

whenever $w \in A^*_1$. For these results see [2, Theorems 2.1 and 2.3].

**Weighted strong maximal function.** For $w \in A^*_\infty$ we will also consider the weighted strong maximal function $M^w_n$, defined with respect to $w$:

$$M_n^w f(x) := \sup_{R \in \mathcal{R}_n} \frac{1}{w(R)} \int_R |f(y)| w(y) dy.$$

For the weighted strong maximal function $M^w_n$, R. Fefferman showed in [5] that it maps $L^p(w)$ to $L^p(w)$ whenever $w \in A^*_\infty$:

$$\|M_n^w f\|_{L^p(w)} \leq w, n \ c_{p, n} \|f\|_{L^p(w)}, \quad 1 < p \leq \infty.$$  

 Furthermore we have the asymptotic estimate

$$c_{p, n} = O_n((p-1)^{-n}) \quad \text{as} \quad p \to 1^+.$$  

The behavior of the constants is not explicitly studied in [5] but follows by a close examination of the proof and the standard norm bounds from Marcinkiewicz interpolation. See also [10, Lemma 4]. The endpoint bound for $M^w_n$ is also true, namely $M^w_n$ satisfies

$$w(\{x \in \mathbb{R}^n : M^w_n f(x) > \lambda\}) \leq w, n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + (\log^+ |f(x)|)^{n-1}\right) w(x) dx$$

whenever $w \in A^*_\infty$. This was proved by Jawerth and Torchinsky, [7], and independently by Long and Shen [10].
**Fefferman-Stein inequality.** By this we mean in general an inequality of the form

$$\int_{\mathbb{R}^n} (Mf)^p w \lesssim_{w,n} \int_{\mathbb{R}^n} |f|^p Mwdx, \quad 1 < p < \infty,$$

where $M$ denotes some maximal operator. Inequalities of this type are important since, among other things, they can be used to derive the boundedness of vector-valued maximal operators. In fact this inequality was first proved for the Hardy-Littlewood maximal function by C. Fefferman and Stein, [4], for every non-negative, locally integrable weight $w$. The main application in [4] was exactly the vector-valued extension of the classical Hardy-Littlewood maximal theorem. For the strong maximal function the same inequality is true provided that $w \in A^*_\infty$; see [9] for a direct proof of this result and also [12], where the Fefferman-Stein inequality is obtained as a corollary of a more general two weight-norm inequality. Observe that, as in the case of (1.4), we need some extra assumption on the weight in order to prove the Fefferman-Stein inequality for the strong maximal function. This should be contrasted to the corresponding result for the Hardy-Littlewood weighted maximal function, as well as to the Fefferman-Stein inequality for the Hardy-Littlewood maximal function, where no assumption on the weight is needed.

The form of the endpoint Fefferman-Stein inequality depends on the corresponding unweighted endpoint properties of the maximal operator under study. For the usual Hardy-Littlewood maximal function $M_Q$ the right statement is

$$w(\{x \in \mathbb{R}^n : M_Qf(x) > \lambda\}) \lesssim_n \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_Q w(x) dx.$$ 

The natural endpoint Fefferman-Stein inequality for the strong maximal function was proved by Mitsis, [11], in dimension $n = 2$. In particular Mitsis showed that

$$w(\{x \in \mathbb{R}^2 : M_n f(x) > \lambda\}) \lesssim_w \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) M_n w(x) dx.$$ 

The main result of the current paper is the extension of the endpoint Fefferman-Stein inequality for the strong maximal function to all dimensions:

**Theorem.** Let $w \in A^*_\infty$. For all dimensions $n \geq 1$ we have

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}) \lesssim_n, w \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) M_n w(x) dx.$$ 

By interpolation, the Fefferman-Stein inequality of our main theorem above implies the strong $L^p$-version of the Fefferman-Stein inequality from [9], [12]. Furthermore, since every $A^*_1$-weight is an $A^*_\infty$-weight, we recover the endpoint inequality (1.6) for $A^*_1$-weights.

It should be noted that the proof of Mitsis in [11] uses the combinatorics of two-dimensional rectangles, which allow one to get favorable estimates for the measures

$$|\{x \in \mathbb{R}_k : \sum_{k=1}^N 1_{R_k}(x) = \ell\}|;$$
here \(\{R_k\}_{1 \leq k \leq N}\) is a sequence of rectangles which satisfy a certain sparseness property and \(\ell\) is any integer in \(\{1, 2, \ldots, N\}\). These combinatorics do not seem to be readily available in higher dimensions and so we adopt a different approach, which relies on the boundedness of the weighted strong maximal function \(M_w^n\) and the precise estimate for its norm, (1.5). In particular, our approach is inspired by the arguments in [10], a paper which seems to have been overlooked by most of the works on the weighted inequalities for the strong maximal function.

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2. Notation

We write \(A \lesssim B\) if \(A \leq CB\) for some numerical constant \(C > 0\). In order to indicate the dependence of the constant on some parameter \(n\) (say), we write \(A \lesssim_n B\). Similarly, \(A \simeq B\) means that \(A \lesssim B\) and \(B \lesssim A\).

3. Some geometry of \(n\)-dimensional rectangles

In this section we recall some sparseness properties of \(n\)-dimensional rectangles, introduced in [3]. Here we adopt the slightly different approach from [10]. In fact, both Lemmas in this section are mentioned in [10]. However, we present the proofs for the sake of completeness.

For \(t \in \mathbb{R}\) and \(E \subset \mathbb{R}^n\) we introduce the slice operator
\[
P_t(E) := \{(x', t) \in E \mid (x', t) \in E\}.
\]
Thus \(P_t\) is the ‘slice’ of \(E\) by a hyperplane perpendicular to the \(n\)-th coordinate axis at level \(t \in \mathbb{R}\). The \((n-1)\)-dimensional projection is
\[
P_{\parallel}(E) := \{x' \in \mathbb{R}^{n-1} : (x', t) \in E\} \quad \text{for some } t \in \mathbb{R}.
\]
We will also use the one-dimensional projection \(P_{\perp}\) defined for \(E \subset \mathbb{R}^n\) as
\[
P_{\perp}(E) := \{t \in \mathbb{R} : (x', t) \in E\} \quad \text{for some } x' \in \mathbb{R}^{n-1}.
\]
If \(R \in \mathbb{R}_n\) observe that we have
\[
R = P_{\parallel}(R) \times P_{\perp}(R) = P_t(R) \times P_{\perp}(R), \quad \text{for all } t \in P_{\perp}(R).
\]
For any interval \(I \subset \mathbb{R}\), let \(I^*\) be the interval with the same center and three times the length of \(I\), \(|I^*| = 3|I|\). For \(R \in \mathbb{R}_n\) we then use the notation
\[
R^* := P_{\parallel}(R) \times (P_{\perp}(R))^*.
\]
Thus \(R^*\) is the rectangle with the same center as \(R\) and whose sides parallel to the first \(n-1\) coordinate axes have the same lengths as the corresponding sides of \(R\); the side of \(R\) which is parallel to the \(n\)-th coordinate axis has length equal to three times the length of the corresponding side of \(R\).
Let \( \mathcal{R} = \{R_k\}_{1 \leq k \leq N} \) be a finite sequence of rectangles from \( \mathcal{R}_n \). We will say that \( \mathcal{R} \) satisfies the sparseness property \((P_2)\) if

\[
(P_2) \quad \begin{cases} 
P^\perp(R_1) \geq P^\perp(R_2) \geq \cdots \geq P^\perp(R_N), \\
|\mathcal{R}_k \cap \bigcup_{j < k} R^*_j| \leq \epsilon |\mathcal{R}_k|, \quad k = 1, 2, \ldots, N.
\end{cases}
\]

Here \( 0 < \epsilon < 1 \) will be assumed sufficiently small in various parts of the arguments below.

For \( t \in \mathbb{R} \) we now consider the collection \( \mathcal{J}(t) = \{\mathcal{P}_t(R_k)\}_{1 \leq k \leq N} \subset \mathcal{R}_{n+t} \) which is produced by slicing all the \( n \)-dimensional rectangles of \( \mathcal{R} \) by a hyperplane perpendicular to the \( n \)-th coordinate axis, at the level \( t \). The collection \( \mathcal{J} \) depends on \( t \) but we will many times suppress this fact in what follows. The main point about the collections \( \mathcal{R} \) and \( \mathcal{J} \) is contained in the following standard fact.

**Lemma 3.1.** Suppose that the sequence \( \mathcal{R} = \{R_k\}_{1 \leq k \leq N} \) has the sparseness property \((P_2)\). Then, for all \( t \in \mathbb{R} \), the \((n-1)\)-dimensional collection of rectangles \( \mathcal{J}(t) = \{\mathcal{P}_t(R_k)\}_{1 \leq k \leq N} \) has the sparseness property \((P_1)\), uniformly in \( t \):

\[
(P_1) \quad |\mathcal{P}_t(R_k) \cap \bigcup_{j < k} \mathcal{P}_t(R_j)| \leq \epsilon |\mathcal{P}_t(R_k)|, \quad k = 1, 2, \ldots, N.
\]

**Proof.** We fix some \( 1 \leq k \leq N \) and \( t \in \mathcal{P}^\perp(R_k) \). Denoting \( J := \{j < k : \mathcal{P}_t(R_k) \cap \mathcal{P}_t(R_j) \neq \emptyset\} \) we have by the second condition in \((P_2)\) that

\[
(3.2) \quad \epsilon |\mathcal{R}_k| \geq |\mathcal{R}_k \cap \bigcup_{j < k} R^*_j| = |\bigcup_{j < k} (\mathcal{R}_k \cap R^*_j)| \geq |\bigcup_{j \in J} (\mathcal{R}_k \cap R^*_j)|
\]

Observe that for \( j \in J \) we have that \( \emptyset \neq P^\perp(R_k) \cap P^\perp(R_j) \ni t \) and by the first condition in \((P_2)\) we have \( |P^\perp(R_j)| \geq |P^\perp(R_k)| \). A moment’s reflection shows that if \( I_1, I_2 \) are intervals in \( \mathbb{R} \), \( |I_2| \geq |I_1| \) and \( I_1 \cap I_2 \neq \emptyset \) then \( I_1 \subseteq I_2^* \). We conclude that \( P^\perp(R_k) \subseteq P^\perp(R_j^*) \). Thus the \( n \)-dimensional rectangle \( R_k \cap R_j^* \) is of the form \( P^\perp(R_k) \times P^\perp(R_k) \cap R_j^* \). However, \( j \in J \) implies that \( \mathcal{P}_t(R_k \cap R_j) = \mathcal{P}_t(R_k) \cap \mathcal{P}_t(R_j) \neq \emptyset \), so we conclude that \( P^\perp(R_k \cap R_j^*) = \mathcal{P}_t(R_k \cap R_j) \) and thus

\[
(3.3) \quad R_k \cap R_j^* = P^\perp(R_k) \times \mathcal{P}_t(R_k \cap R_j).
\]

Now estimate (3.2) and identity (3.3) give

\[
\epsilon |P^\perp(R_k)| \times |\mathcal{P}_t(R_k)| = \epsilon |\mathcal{R}_k| \geq |\bigcup_{j \in J} P^\perp(R_k) \times \mathcal{P}_t(R_k \cap R_j)|
\]

\[
= |P^\perp(R_k)| \times |\bigcup_{j \in J} \mathcal{P}_t(R_k \cap R_j)|
\]

\[
= |P^\perp(R_k)| \times |\mathcal{P}_t(R_k) \cap \bigcup_{j \in J} \mathcal{P}_t(R_j)|
\]

\[
= |P^\perp(R_k)| \times |\mathcal{P}_t(R_k) \cap \bigcup_{j < k} \mathcal{P}_t(R_j)|.
\]
This proves the lemma for \( t \in \mathbb{P}^+(R_k) \) while for \( t \notin \mathbb{P}^+(R_k) \) the conclusion follows trivially. \( \square \)

The next lemma gives a precise quantitative bound on the overlap of the rectangles in \( \mathcal{R} \) under the sparseness property \((P_2)\).

**Lemma 3.4.** Let \( w \in A^\infty_\infty \) and suppose that the finite sequence \( \mathcal{R} = \{R_k\}_{1 \leq k \leq N} \subset \mathcal{R}_n \) satisfies property \((P_2)\) with \( \epsilon \) sufficiently small, depending on the weight \( w \). We set \( \Omega := \bigcup_{k=1}^N R_k \). For \( 1 < p < \infty \) we have

\[
\left( \int_{\Omega} \left| \sum_{k=1}^N 1_{R_k} |w(x) dx \right|^p \right)^{\frac{1}{p}} \lesssim_{w,n} c_{p,n} w(\Omega)^{\frac{1}{p}}
\]

with \( c_{p,n} = O_n(p^{-1}) \) as \( p \to +\infty \).

**Proof.** For a sequence \( \{R_k\}_{1 \leq k \leq N} \) as before, consider the sequence \( \mathcal{J}(t) \) of \((n-1)\)-dimensional rectangles, by slicing the collection \( \mathcal{R} \) with a hyperplane perpendicular to the \( n \)-th coordinate axis, at level \( t \in \mathbb{R} \). Let \( \Omega_t := P_t(\Omega) \) denote the corresponding slice of \( \Omega \) at level \( t \) and set \( T_k := P_t(R_k) \) in order to simplify the notation. By Lemma 3.1 the collection \( \mathcal{J}(t) = \{T_k\}_{1 \leq k \leq N} \) has the property \((P_1)\). We set \( E_k := T_k \setminus \bigcup_{j<k} T_j \). For fixed \( t \in \mathbb{R} \), the function \( w^t(x') = w(x', t) \), \( x' \in \mathbb{R}^{n-1} \), is an \( A^\infty_\infty \)-weight in \( \mathbb{R}^{n-1} \), uniformly in \( t \in \mathbb{R} \); see [5]. By the property \((P_1)\) and the fact that \( w^t \in A^\infty_\infty \) uniformly in \( t \), we will have that \( w^t(T_k) = w^t(E_k) \geq \frac{\epsilon}{2} w^t(T_k) \) if \( \epsilon > 0 \) was selected sufficiently small in property \((P_2)\), and thus also in \((P_1)\), according to Remark 1.3.

Define the linear operator

\[
L_{w^t}(f)(x') := \sum_{k=1}^N \frac{1}{w^t(T_k)} \left( \int_{T_k} f(y') w^t(y') dy' \right) 1_{E_k}(x'), \quad x' \in \mathbb{R}^{n-1}.
\]

For any locally integrable function \( f \) on \( \mathbb{R}^{n-1} \) we have that \( L_{w^t}(f)(x') \leq M_{w^t}(f)(x') \), \( x' \in \mathbb{R}^{n-1} \). Also observe that for \( f, g \) locally integrable we have

\[
\int_{\Omega_t} L_{w^t}(f)(x') g(x') w^t(x') dx' = \int_{\Omega_t} \sum_{k=1}^N \frac{1}{w^t(T_k)} \left( \int_{E_k} g(y') w^t(y') dy' \right) 1_{R_k}(x') f(x') w^t(x') dx'
\]

\[
=: \int_{\Omega_t} L^*_{w^t}(g)(x') f(x') w^t(x') dx'.
\]

Furthermore \( L^*_{w^t}(1_{\Omega_t}) = \sum_{k=1}^N w^t(E_k) 1_{T_k} \geq \frac{1}{2} \sum_{k=1}^N 1_{T_k} \). For any locally integrable function \( g \) on \( \mathbb{R}^{n-1} \) we thus have

\[
\int_{\Omega_t} g(x') \sum_{k=1}^N 1_{T_k}(x') w^t(x') dx' \lesssim \int_{\Omega_t} g(x') L^*_{w^t}(1_{\Omega_t})(x') w^t(x') dx'
\]

\[
= \int_{\Omega_t} L_{w^t}(g)(x') w^t(x') dx' \leq \| M_{w^t} g \|_{L^{p'}(w^t; \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}} \lesssim_{w,n} (p' - 1)^{-(n-1)} \| g \|_{L^{p'}(w^t; \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}},
\]
by (1.5). Taking the supremum over $g \in L^p'(\mathbb{R}^{n-1})$ with $\|g\|_{L^p'(\mathbb{R}^{n-1})} \leq 1$ gives the estimate
\[
\int_{\Omega} \left| \sum_{k=1}^{N} 1_{T_k}(x') \right|^p w^t(x') \, dx \leq_{w,n} p^{(n-1)p} w^t(\Omega_t),
\]
as $p \to \infty$. It is essential to note here that this estimate is uniform in $t \in \mathbb{R}$. Thus integrating over $t \in P^\perp(\Omega)$, gives the claim. $\square$

4. Proof of the endpoint Fefferman-Stein inequality

We begin with a simple lemma.

**Lemma 4.1.** Let $\epsilon > 0$, $f$ be a locally integrable function and set $\mathcal{F} := \{x \in \mathbb{R}^n : M_n f(x) > 1\}$. There exists a finite collection of rectangles $\mathcal{R} = \{R_k^x\}_{1 \leq k \leq N}$ such that:

(i) The collection $\mathcal{R}$ has the property $(P_2)$ with parameter $\epsilon$.
(ii) For each $1 \leq k \leq N$ we have
\[
|R_k^x| < \int_{R_k^x} |f(y)| \, dy.
\]
(iii) We have the estimate
\[
w(\mathcal{F}) \leq_{\epsilon,w,n} w(\cup_k R_k^x).
\]

**Proof.** For every $x \in \mathcal{F}$ let $R_x \in \mathcal{R}_n$ be a rectangle such that
\[
\frac{1}{|R_x|} \int_{R_x} |f(y)| \, dy > 1.
\]
Without loss of generality we may assume that $\{R_x\}_{x \in \mathcal{F}}$ is a finite sequence $\{R_j\}_{1 \leq j \leq M}$, so that (ii) is satisfied and such that $w(\mathcal{F}) \leq w(\cup_{1 \leq j \leq M} R_j)$. From the collection $\{R_j\}_{1 \leq j \leq M}$ we will now choose a subcollection $\{R_k^x\}_{1 \leq k \leq N}$ so that (i) and (iii) are also satisfied. First we reorder the rectangles $R_j$ so that $P^\perp(R_1) \supseteq P^\perp(R_2) \supseteq \cdots \supseteq P^\perp(R_M)$. We choose $R_1^x := R_1$ and assume that the rectangles $R_1^x, R_2^x, \ldots, R_i^x$, have been selected. Also let $1 \leq j_o < M$ so that $R_j^x = R_j$. We then choose $R_{i+1}^x$ to be the rectangle with the smallest index among the rectangles $S \in \{R_{j_o+1}, \ldots, R_M\}$ that satisfy
\[
|S \cap \bigcup_{j \leq i} (R_j^x)^\ast| \leq \epsilon |S|.
\]
Since the collection $\{R_j\}_{1 \leq j \leq M}$ is finite, the selection process will end after a finite number of $N$ steps, and the collection $\{R_k^x\}_{1 \leq k \leq N}$ will automatically satisfy (i). Of course this subcollection still satisfies (ii). Now assume that some $S \in \{R_1, \ldots, R_M\}$ was not selected. We can then find some positive integer $K \in \{1, 2, \ldots, N\}$ such that
\[
|S \cap \bigcup_{j \leq K} (R_j^x)^\ast| > \epsilon |S|.
\]
Thus we get for all \( x \in S \)
\[
M_n(I_{1 \leq j \leq N(R_j^s)^c})(x) \geq M_n(I_{1 \leq k \leq K(R_k^s)^c})(x) > \epsilon,
\]
which means that
\[
\bigcup_{1 \leq j \leq N \atop R_j \text{ not selected}} R_j \subseteq \{ x : M_n(I_{1 \leq j \leq N(R_j^s)^c})(x) > \epsilon \}.
\]

However, since \( w \in A_\infty^* \) we know that \( M_n : L^{p_0}(w) \to L^{p_0,\infty}(w) \) for some \( p_0 > 1 \). We conclude that
\[
w\left( \bigcup_{1 \leq j \leq N \atop R_j \text{ not selected}} R_j \right) \leq \epsilon, w, n w(\bigcup_{1 \leq k \leq N} R_k^s) \approx w(\bigcup_{1 \leq k \leq N} R_k^s).
\]

Thus \( w(F) \leq w(\bigcup_{1 \leq j \leq M} R_j) \leq \epsilon, w, n w(\bigcup_{1 \leq k \leq N} R_k^s) \) as we wanted. \( \square \)

We are now ready to give the proof of our main result.

**Proof of the Main Theorem.** We assume that \( n \geq 2 \) since in one dimension \( M_1 \) is the usual Hardy-Littlewood maximal function and there is nothing (new) to prove. We henceforth write \( M \) for \( M_n \) since the dimension \( n \) is fixed throughout the proof. Furthermore, it suffices to prove the theorem for \( \lambda = 1 \). Let \( F := \{ x \in \mathbb{R}^n : M(f)(x) > 1 \} \) and consider the collection \( \mathcal{R} := \{ R_k^s \}_{k=1}^N \subset \mathcal{R}_n \) given by Lemma 4.1. By (i) of that Lemma the collection \( \mathcal{R} \) has the sparseness property \((P_2)\). We assume that \( \epsilon > 0 \) was chosen small enough in Lemma 4.1, and thus in \((P_2)\), so that Lemma 3.4 is valid. Observe that \((P_2)\) also implies that
\[
| R_k^s \cap \bigcup_{j < k} R_j^s | \leq \epsilon | R_k^s |.
\]

By choosing \( \epsilon > 0 \) small enough we can also assume that \( w(R_k^s \cap \bigcup_{j < k} R_j^s) \leq \frac{1}{2} w(R_k^s) \), according to Remark 1.3. Setting \( E_k := R_k^s \setminus \bigcup_{j < k} R_j^s \) we will thus have
\[
(4.2) \quad w(R_k^s) \geq w(E_k) \geq \frac{1}{2} w(R_k^s), \quad k = 1, 2, \ldots, N,
\]
and the choice of \( \epsilon > 0 \) depends only on the weight \( w \in A_\infty^* \). Denoting \( \Omega := \bigcup_{k=1}^N R_k^s \), we use (ii) and (iii) of Lemma 4.1 to estimate
\[
w(F) \leq \epsilon, w, n w(\Omega) \leq \sum_{k=1}^N w(R_k^s) \leq \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} \int_{R_k^s} |f(y)| dy
\]
\[
= \int_{\Omega} f(x) \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} 1_{R_k^s}(x) dx.
\]
Define the linear operators
\[ T_f(x) = \sum_{k=1}^{N} \frac{1}{|R_k^s|} \int_{R_k^s} f(y) dy \mathbf{1}_{E_k}(x), \quad T^*f(x) = \sum_{k=1}^{N} \frac{1}{|R_k^s|} \int_{E_k} f(y) dy \mathbf{1}_{R_k^s}(x), \quad x \in \mathbb{R}^n. \]

For locally integrable \( f, g \) we have
\[ \int_{\Omega} T_f(x) g(x) dx = \int_{\Omega} T^*g(x)f(x) dx, \quad T_f(x) \leq Mf(x), \quad x \in \mathbb{R}^n. \]

By (4.2) we have
\[ T^*w(x) = \sum_{k=1}^{N} \frac{w(E_k)}{|R_k^s|} \mathbf{1}_{R_k^s}(x) \equiv \sum_{k=1}^{N} \frac{w(R_k^s)}{|R_k^s|} \mathbf{1}_{R_k^s}(x) \]

thus we can estimate for any \( \delta > 0 \)
\[ w(\Omega) \leq \int_{\Omega} f T^*w \leq \int_{\{\Omega: T^*w \leq \delta Mw\}} f(x) T^*w(x) dx + \int_{\{\Omega: T^*w > \delta Mw\}} f(x) \frac{T^*w(x)}{Mw(x)} Mw(x) dx \]
\[ \leq \delta \int_{\mathbb{R}^n} |f(x)| Mw(x) dx + \int_{\{\Omega: T^*w > \delta Mw\}} f(x) \frac{T^*w(x)}{Mw(x)} Mw(x) dx. \]

We will use the following elementary estimate: For each \( \theta > 0 \) there exists a constant \( c_\theta > 0 \) such that for all \( s, t \geq 0 \) we have
\[ st \leq c_\theta s[1 + (\log^+ s)^{n-1}] + \exp(\theta t^{\frac{1}{n-t}}) - 1, \quad n \geq 2. \]

The interested reader can find a detailed proof of this classical inequality in [1]. Applying this pointwise estimate we get for every \( \theta > 0 \):
\[ w(\Omega) \leq (\delta + c_\theta) \int_{\mathbb{R}^n} |f(x)| \left( 1 + (\log^+ |f(x)|)^{n-1} \right) Mw(x) dx \]
\[ + \int_{\{\Omega: T^*w > \delta Mw\}} \left( \exp \left[ \theta \left( \frac{T^*w(x)}{Mw(x)} \right)^{\frac{1}{n-t}} \right] - 1 \right) Mw(x) dx. \]

We now estimate the last term,
\[ Q := \int_{\{\Omega: T^*w > \delta Mw\}} \left( \exp \left[ \theta \left( \frac{T^*w(x)}{Mw(x)} \right)^{\frac{1}{n-t}} \right] - 1 \right) Mw(x) dx \]
\[ = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \int_{\{\Omega: T^*w > \delta Mw\}} \left( \frac{T^*w(x)}{Mw(x)} \right)^{\frac{k}{n-t}} Mw(x) dx \leq \sum_{1 \leq k \leq n-1} + \sum_{k > n-1} =: I + II. \]

For I we just observe that since \( k/(n-1) \leq 1 \) and \( T^*w/(\delta Mw) > 1 \) we have the elementary estimate
\[ \left( T^*w/Mw \right)^{\frac{k}{n-t}} = \left( T^*w/(\delta Mw) \right)^{\frac{k}{n-t}} \delta^{\frac{k}{n-t}} \]

\[ \text{with } \delta := \frac{\delta}{\delta Mw} \text{ and } c := \sum_{1 \leq k \leq n-1} \int_{\{\Omega: T^*w > \delta Mw\}} \left( \frac{T^*w(x)}{Mw(x)} \right) \left( \frac{T^*w(x)}{Mw(x)} \right)^{\frac{k}{n-t}} Mw(x) dx. \]
\[
\begin{align*}
&= \delta^{\frac{k}{n}-1}(T^*w/(\delta Mw))^\frac{1}{n-1} T^*w \\
&\lesssim \delta^{\frac{k}{n}-1} \frac{T^*w}{Mw}.
\end{align*}
\]

So we have
\[
I \leq \sum_{1 \leq k \leq n-1} \frac{\theta^k}{k!} \int_{\Omega} T^*w(x) dx \leq \frac{\theta}{\delta} e^{\delta \frac{k}{n-1}} \int_{\Omega} T_1(x)w(x) \lesssim_{\delta,n} \theta w(\Omega).
\]

Here we abuse notation by denoting \( T_1, T^*1(x) \) the action of \( T, T^* \), respectively, on the constant function 1. For II we use the fact that \( T^*w \simeq \sum_{k=1}^N \frac{w(R_k)}{|R_k|} 1_{R_k} \leq Mw \sum_{k=1}^N 1_{R_k} \simeq Mw T^*1 \).

We have
\[
\begin{align*}
II &\leq \sum_{k>n-1} \frac{\theta^k}{k!} \int_{\Omega} \left( T^*1(x) \right)^{\frac{k}{n-1}-1} T^*w(x) dx \\
&\leq \sum_{k>n-1} \frac{\theta^k}{k!} \int_{\Omega} (T^*1(x))^{\frac{k}{n-1}} T^*w(x) dx \quad \text{(because \( T^*1 \simeq 1 \) on \( \Omega \))}
\end{align*}
\]

Since \( w \in A^*_p \) for some \( 1 < p_0 < \infty \) and \( T f \leq Mf \) we have \( \|Tf\|_{L^p(w)} \lesssim_{w,n} \|f\|_{L^p(w)}. \)

This together with Lemma 3.4 yields
\[
Q_k \lesssim_{w,n} w(\Omega) \frac{1}{n} \left( \int_{\Omega} |T^*1(x)|^{\frac{k}{n-1}}w(x) dx \right)^{\frac{1}{p_0}} \lesssim_{w,n} [k p_0/(n-1)]^k w(\Omega).
\]

Overall we get
\[
\begin{align*}
II &\lesssim_{w,n} \sum_{k>n-1} \frac{\theta^k}{k!} (kp_0)^k w(\Omega) \leq \sum_{k>n-1} \frac{(\theta e_{p_0}/(n-1))^k}{\sqrt{k}} w(\Omega) \\
&\leq \frac{(\theta e_{p_0}/(n-1))^n}{\sqrt{n}} w(\Omega),
\end{align*}
\]

if \( \theta \) is small enough. Thus \( Q \lesssim_{w,n} \theta w(\Omega) \). We have proved that for \( \theta > 0 \) small and fixing \( \delta = 1 \) (say) in the previous estimates we have
\[
w(\Omega) \lesssim_{w,n} \theta w(\Omega) + (1 + c_0) \int_{\mathbb{R}^n} |f(x)| \left( 1 + (\log^+ |f(x)|)^{n-1} \right) Mw(x) dx.
\]
Choosing \( \theta > 0 \) sufficiently small we thus have
\[
    w(F) \leq w(\Omega) \leq w_{n, n} \int_{\mathbb{R}^n} |f(x)| \left( 1 + \left( \log^+ |f(x)| \right)^{n-1} \right) Mw(x) \, dx,
\]
which is the desired estimate. \( \Box \)

We have actually proved the following weighted analogue of the Córdoba-Fefferman covering lemma from [3].

**Lemma 4.3.** Let \( w \in A^{\ast}_{\infty} \). Suppose that \( \{ R_j \}_{j \in J} \) is a finite sequence of rectangles from \( \mathcal{R}_n \). Then there exists a subcollection \( \{ R^+_k \}_{1 \leq k \leq N} \subset \bigcup_{j \in J} R_j \) such that

\( (i) \) \( w(\bigcup_{j \in J} R_j) \leq w_{n, n} w(\bigcup_{k=1}^N R^+_k) \).
\( (ii) \) For every \( \delta > 0 \) there exists \( \theta_\delta = \theta_\delta(\delta, w, n) > 0 \) such that, for every \( \theta < \theta_\delta \) we have
\[
    \int_{\Omega: T^*w(x) > \delta Mw(x)} \left( \exp \left( \theta \left( \frac{T^*w(x)}{Mw(x)} \right)^{\frac{1}{n-1}} \right) - 1 \right) Mw(x) \, dx \leq w_{n, n, \theta, \delta} w(\bigcup_{k=1}^N R^+_k).
\]

Here \( T^*w = \sum_{k=1}^N \frac{w(R^+_k)}{|R^+_k|} 1_{R^+_k} \) and \( M \) denotes the strong maximal function.

**References**

[1] Richard J. Bagby, *Maximal functions and rearrangements: some new proofs*, Indiana Univ. Math. J. 32 (1983), no. 6, 879–919, DOI 10.1512/iiumj.1983.32.32060. MR 721570 (86f:42010) †10

[2] Richard J. Bagby and Douglas S. Kurtz, *L(\log L) spaces and weights for the strong maximal function*, J. Analyse Math. 44 (1984/85), 21–31, DOI 10.1007/BF02790188. MR 801285 (87c:42018) †2, 3

[3] A. Córdoba and R. Fefferman, *A geometric proof of the strong maximal theorem*, Ann. of Math. (2) 102 (1975), no. 1, 95–100. MR 0379785 (52 #690) †2, 5, 12

[4] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115. MR 0284802 (44 #2026) †4

[5] R. Fefferman, *Strong differentiation with respect to measures*, Amer. J. Math. 103 (1981), no. 1, 33–40, DOI 10.2307/2374188. MR 601461 (83g:42009) †3, 7

[6] José García-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104. MR 807149 (87c:42023) †2

[7] Björn Jawerth and Alberto Torchinsky, *The strong maximal function with respect to measures*, Studia Math. 80 (1984), no. 3, 261–285. MR 783994 (87b:42024) †3

[8] B. Jessen, J. Marcinkiewicz, and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), 217–234. †1, 2

[9] Kai-Ching Lin, *HARMONIC ANALYSIS ON THE BIDISC*, ProQuest LLC, Ann Arbor, MI, 1984. Thesis (Ph.D.)–University of California, Los Angeles. MR 2633524 †4

[10] Rui Lin Long and Zhong Wei Shen, *A note on a covering lemma of A. Córdoba and R. Fefferman*, Chinese Ann. Math. Ser. B 9 (1988), no. 3, 283–291. A Chinese summary appears in Chinese Ann. Math. Ser. A 9 (1988), no. 4, 506. MR 968464 (91b:42037) †3, 5

[11] Themis Mitsis, *The weighted weak type inequality for the strong maximal function*, J. Fourier Anal. Appl. 12 (2006), no. 6, 645–652, DOI 10.1007/s00041-005-5060-3. MR 2275389 (2007i:42016) †4

[12] C. Pérez, *A remark on weighted inequalities for general maximal operators*, Proc. Amer. Math. Soc. 119 (1993), no. 4, 1121–1126, DOI 10.2307/2159974. MR 1107275 (94a:42016) †4
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