PATTERN RECOGNITION ON ORIENTED MATROIDS: 
κ*-VECTORS AND REORIENTATIONS

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Abstract. The components of κ*-vectors associated to a simple oriented matroid $M$ are the numbers of general or special tope committees for $M$. Using the principle of inclusion-exclusion, we determine how the reorientations of $M$ on one-element subsets of its ground set affect κ*-vectors.

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1. Introduction

Let $M := (E_t, T)$ be a simple oriented matroid on the ground set $E_t := \{1, \ldots, t\}$, with set of topes $T$; throughout we will suppose that it is simple, that is, it contains no loops, parallel or antiparallel elements.

See, e.g., [2, 3, 4, 5, 12, 13, 15] on oriented matroids.

Associated to each element $e \in E_t$ are the corresponding positive halfspace $T_e^+ := \{T \in T : T(e) = +\}$ and negative halfspace $T_e^- := \{T \in T : T(e) = -\}$ of $M$. If $T_e^* \subset T$ is a halfspace of $M$ then we denote by $\binom{T_e^*}{j}$ the family of $j$-subsets of the set $T_e^*$.

If $G \subseteq T$ is a subset of topes then $-G$ stands for the set of their opposites $\{-T : T \in G\}$.

If $A \subseteq E_t$ then $-_A M$ denotes the oriented matroid obtained from $M$ by reorientation on the set $A$; if $a \in E_t$ then we write $-_a M$ instead of $-(a) M$.

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A subset $K^* \subset T$ is called a tope committee for $M$ if for each element $e \in E_t$ it holds

$$|\{T \in K^* : \ T(e) = +\}| \geq \frac{1}{2}|K^*|,$$

see [7] [8] [9] [10]; in other words, if we replace the components $-$ and $+$ of the maximal covectors of the oriented matroid $M$ by the real numbers $-1$ and $1$, respectively, then a collection $K^* \subset T$ is a committee for $M$ iff the strict inequality

$$\sum_{T \in K^*} T > 0$$

holds componentwise.

Let $K^*_k(M)$ denote the family of tope committees, of cardinality $k$, for $M$, and let $K^*_1(M) := \bigcup_{1 \leq k \leq |T|-1} K^*_k(M)$ denote the family of all tope committees for $M$. By definition, the $k$th component $\kappa^*_k(M) := \#K^*_k(M)$ of the vector $\kappa^*(M) \in \mathbb{N}^{|T|/2}$, $1 \leq k \leq |T|/2$, is the number of committees in the family $K^*_k(M)$.

Similarly, we associate to each family $K^*_k(M)$, $1 \leq k \leq |T|/2$, of tope committees, of cardinality $k$, that contain no pairs of opposites, the $k$th component $\tilde{\kappa}^*_k(M) := \#\tilde{K}^*_k(M)$ of the vector $\tilde{\kappa}^*(M) \in \mathbb{N}^{|T|/2}$.

We always have $\kappa^*_2(M) = \kappa^*_2(M) = 0$. The oriented matroid $M$ is acyclic iff $\tilde{\kappa}^*_1(M) = \kappa^*_1(M) = 1$. If $M$ is not acyclic then $\tilde{\kappa}^*_1(M) = \kappa^*_1(M) = 0$ and $\tilde{\kappa}^*_3(M) = \kappa^*_3(M)$.

If $K^* \in K^*_1(M)$, for some $j$, $1 \leq j \leq |T|/2$, then there are $|T|/2 - j$ pairs of tope committees $\{T, -T\} \subset T$ such that $|K^* \cap \{T, -T\}| = 0$. If we add any such pairs of opposites to the set $K^*$ then the resulting set is a committee for $M$.

Thus, given an integer $k$ such that $j \leq k \leq |T|/2$ and the difference $k - j$ is even, in the family $K^*_k(M)$ there are exactly $2^{(|T|/2 - j)/2}$ tope committees which contain the committee $K^*$ as a subset.

We see that

$$\kappa^*_k(M) = \sum_{j \equiv k \pmod{2}} \binom{|T| - 2j}{k-j} \kappa^*_j(M), \quad 1 \leq k \leq |T|/2;$$

for example, $\kappa^*_3(M) = \frac{|T|-2}{2} \cdot \kappa^*_1(M) + \kappa^*_3(M)$, and $\kappa^*_5(M) = \frac{|T|-4(|T|-2)}{8} \cdot \kappa^*_1(M) + \frac{|T|-6}{2} \cdot \kappa^*_3(M) + \kappa^*_5(M)$.

The family $A^*(M)$ of anti-committees for the oriented matroid $M$ is defined as the family $\{-K^* : K^* \in K^*(M)\}$.

Let $A$ be any subset of the ground set $E_t$. The tope sets of the oriented matroids $-A \mathcal{M}$ and $-(E_t - A) \mathcal{M}$ coincide and, thanks to the composite bijection

$$K^*(A) \to A^*(A) \to A^*(E_t - A) \to K^*(E_t - A) \to K^*(E_t - A);$$

$$K^* \leftrightarrow -K^* \leftrightarrow -K^* \leftrightarrow K^*,$$
Lemma 2.1.

by restating expression (3.2)!

where

\[ \kappa^*(-A) = \kappa^*(-(E_i-A)) \]

and

\[ \kappa^*(-A) = \kappa^*(-(E_i-A)) \].

In this paper we compare \( \kappa^* \)-vectors of the oriented matroids \( M \) and \(-A\), where \( A := \{ a \} \) are one-element subsets of the ground set \( E_i \). In Section 2, we sum up the observations that concern general tope committees and committees containing no pairs of opposites, made in Sections 2 and 3, respectively.

2. The Number of Tope Committees

Consider general tope committees for the oriented matroid \( M \) and begin by restating expression \( \{7\} \) (3.2):

**Lemma 2.1.** The number \( \#K^*_k(M) \) of tope committees, of cardinality \( k \), \( 1 \leq k \leq |T| - 1 \), for the oriented matroid \( M := (E_i, T) \), is

\[
\#K^*_k(M) = \left( \frac{|T|}{|T|-\ell} \right) + \sum_{G \subseteq U \in E_t, (|T|)} (-1)^{\#G} \cdot \left( \frac{|T| - |U_{G \in G} G|}{|T| - \ell} \right),
\]

where \( \ell \in \{ k, |T| - k \} \).

Fix an integer \( k \), \( 1 \leq k \leq |T|/2 \), a ground element \( a \in E_t \), and an integer \( \ell \in \{ k, |T| - k \} \). If we set

\[
\alpha_k(a, M) := \left( \frac{|T|}{|T|-\ell} \right) + \sum_{G \subseteq U \in E_t(a), (|T|)} (-1)^{\#G} \cdot \left( \frac{|T| - |U_{G \in G} G|}{|T| - \ell} \right)
\]

then, according to (2.1), we have

\[
\kappa^*_k(M) = \alpha_k(a, M)
\]

\[
+ \sum_{G' \subseteq \{ T^+_{\ell+1/2} \} - \bigcup_{E_t - \{ a \}} (|T|)} (-1)^{\#G' + \#G''}
\]

\[
G' \subseteq \{ T^+_{\ell+1/2} \} - \bigcup_{E_t - \{ a \}} (|T|) \quad 1 \leq \#G' \leq (|T|), \quad |U_{G \in G} G| \leq \ell,
\]

\[
G'' \subseteq \bigcup_{E_t - \{ a \}} (|T|) \quad 0 \leq \#G'' \leq (|T| - \ell) - \#G', \quad |U_{G \in G''} G| \leq \ell
\]

\[
\left( \frac{|T| - |U_{G \in G''} G|}{|T| - \ell} \right). \quad (2.2)
\]

In an analogous expression for \( \kappa^*_k(-A) \) the families \( G' \) range over subfamilies of the family \( \{ T^+_{\ell+1/2} \} - \bigcup_{E_t - \{ a \}} (|T|) \).
3. The Number of Tope Committees Containing no Pairs of Opposites

Before proceeding to consider the tope committees that contain no pairs of opposites, we collect a few observations:

Let \( m \) be a positive integer, and \( \pm[1,m] \) the \( 2m \)-set \( \{-m, \ldots, -1, 1, \ldots, m\} \). If we fix a subset \( W \subseteq \pm[1,m] \) and denote by \( -W \) the set \( \{-w : w \in W\} \) then we have

\[
|\pm[1,m]| - |W| - 2\#\{i, -i \subseteq \pm[1,m] : \{(i, -i) \cap W = 0\}\} = |W \cup -W| - |W| \quad (3.1)
\]

and

\[
\#\{i, -i \subseteq \pm[1,m] : \{(i, -i) \cap W = 0\}\} = m - \frac{1}{2}|W \cup -W| \quad . \quad (3.2)
\]

Recall that the number of \( k \)-subsets \( V \subset \pm[1,m] \), such that

\[
v \in V \implies -v \notin V , \quad (3.3)
\]

is \( \binom{m}{k}2^k \) — this is the number of \( (k-1) \)-dimensional faces of an \( m \)-dimensional crosspolytope, see [4].

If \( W \neq \pm[1,m] \) then consider some nonempty \( k \)-set \( V \subset \pm[1,m] \) such that \( |V \cap W| = 0 \) and implication \( (3.3) \) holds. Let \( V = V' \cup V'' \) be the partition of \( V \) into two subsets with the following properties:

\[
v' \in V' \implies -v' \in W , \quad (3.4)
\]

\[
v'' \in V'' \implies -v'' \notin W . \quad (3.5)
\]

Let \( |V'| =: j \) and \( |V''| =: k - j \), for some \( j \). In fact, \( (3.1) \) and \( (3.2) \) imply that there are \( \binom{|W \cup -W| - |W|}{j} \) sets \( V' \subset \pm[1,m] \) such that \( |V'| = j \), \( |V' \cap W| = 0 \) and \( (3.4) \) holds; there are \( \binom{|W \cup -W|}{k-j}2^{k-j} \) sets \( V'' \subset \pm[1,m] \) such that \( |V''| = k - j \), \( |V'' \cap W| = 0 \) and \( (3.5) \) holds.

Let \( \mathbb{B}(2m) \) denote the Boolean lattice of subsets of the set \( \pm[1,m] \). The empty set of \( \pm[1,m] \) is denoted by 0. If \( b \in \mathbb{B}(2m) - \{0\} \) then we let \( -b \) denote the set of the negations of elements from \( b \).

Let \( r \) be a rational number, \( 0 \leq r < 1 \), and \( k \) an integer number, \( 1 \leq k \leq m \). If \( A \) is an antichain in \( \mathbb{B}(2m) \), such that \( |r \cdot k| + 1 \leq \min_{\lambda \in A} \rho(\lambda) \), then consider the subset

\[
\mathbf{I}_{r,k}(\mathbb{B}(2m), A) := \{ b \in \mathbb{B}(2m) : \rho(b) = k, \ b \land -b = 0, \ \rho(b \land \lambda) > r \cdot k \ \forall \lambda \in A \} \subset \mathbb{B}(2m)^{(k)} ,
\]

where \( \rho(\cdot) \) denotes the poset rank of an element in \( \mathbb{B}(2m) \), and \( \mathbb{B}(2m)^{(k)} := \{ b \in \mathbb{B}(2m) : \rho(b) = k \} \). The collection \( \mathbf{I}_{r,k}(\mathbb{B}(2m), A) \) is the set of relatively \( r \)-blocking elements \( b \in \mathbb{B}(2m)^{(k)} \) (with the additional property \( b \land -b = 0 \)) for the antichain \( A \) in the lattice \( \mathbb{B}(2m) \); relative blocking is discussed in [11].
Denote by $\mathcal{I}(\lambda)$ the principal order ideal of the lattice $\mathbb{B}(2m)$ generated by an element $\lambda \in \Lambda$. Using the principle of inclusion-exclusion [1, 14], we obtain

$$\left| \tilde{I}_{r,k}(\mathbb{B}(2m), \Lambda) \right| = \binom{m}{k} 2^k + \sum_{D \subseteq \min \bigcup_{\lambda \in \Lambda} (\mathbb{B}(2m)^{\rho(\lambda) - \lfloor r \cdot k \rfloor} \cap \mathcal{I}(\lambda)) \cap D \neq \emptyset} (-1)^{|D|} \cdot \sum_{0 \leq j \leq k} \left( \rho(\bigvee_{d \in D} d \vee - \bigvee_{d \in D} d) - \rho(\bigvee_{d \in D} d) \right) \cdot \left( m - \frac{1}{2} \rho(\bigvee_{d \in D} d \vee - \bigvee_{d \in D} d) \right) 2^{k-j},$$

(3.6)

where $\min \cdot$ denotes the set of minimal elements of a subposet.

Consider the lattice

$$\mathcal{E} := \left\{ \bigvee_{d \in D} d : D \subseteq \min \bigcup_{\lambda \in \Lambda} (\mathbb{B}(2m)^{\rho(\lambda) - \lfloor r \cdot k \rfloor} \cap \mathcal{I}(\lambda)), |D| > 0 \right\} \cup \{ \hat{0} \},$$

where $\hat{0}$ is a new least element adjoined. If we let $\mu_{\mathcal{E}}(\cdot, \cdot)$ denote the Möbius function of the lattice $\mathcal{E}$, then we have

$$\left| \tilde{I}_{r,k}(\mathbb{B}(2m), \Lambda) \right| = \binom{m}{k} 2^k + \sum_{z \in \mathcal{E} : z > \hat{0}} \mu_{\mathcal{E}}(\hat{0}, z) \cdot \sum_{0 \leq j \leq k} \left( \rho(z \vee -z) - \rho(z) \right) \left( m - \frac{1}{2} \rho(z \vee -z) \right) 2^{k-j},$$

(3.7)

where $\rho(z)$ denotes the poset rank of an element $z$ in the lattice $\mathbb{B}(2m)$.

It was shown in [7] that any tope committee $K^* \in K^k(\mathcal{M})$ for the oriented matroid $\mathcal{M}$ is a blocking $k$-set for the family $\bigcup_{e \in E} \left( \mathbb{B}(|T| - k + 1 / 2) \mathcal{K}^+ \right)$ of tope subsets, of cardinality $\lfloor (|T| - k + 1 / 2) / 2 \rfloor$, each of which is contained in some positive halfspace, see Lemma 2.1. As a consequence, the subfamily $\tilde{K}^k(\mathcal{M}) \subset K^k(\mathcal{M})$ is precisely the collection of blocking $k$-sets, that are free of opposites, for the family $\bigcup_{e \in E} \left( \mathbb{B}(|T| - k + 1 / 2) \mathcal{K}^+ \right)$. With the help of (3.6), we come to the following conclusion:
Lemma 3.1. The number $\#K_k^\circ(M)$ of tope committees, of cardinality $k$, $1 \leq k \leq |T|/2$, that contain no pairs of opposites, for the oriented matroid $M := (E_t, T)$, is

$$\#K_k^\circ(M) = \binom{|T|/2}{k} 2^k + \sum_{G \subseteq \bigcup_{e \in E_t} \left( \left( T + e \right) \cup \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right)} (-1)^{|G|} \cdot \sum_{0 \leq j \leq k} \left( \left| G \cup \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right) \cdot \left( \frac{1}{2} \left( \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right) \right) 2^{k-j}.$$  

(3.8)

If $G$ is a family of tope subsets then we denote by $E(G)$ the join-semilattice $\{ \bigcup_{F \subseteq G} F : F \subseteq G, \#F > 0 \}$ that consists of the unions of the sets from the family $G$ ordered by inclusion and augmented by a new least element $\hat{0}$ which is interpreted as the empty set. The Möbius function of the lattice $E(G)$ is denoted by $\mu_{E}(-, \cdot)$.

With the help of (3.7), Lemma 3.1 can be restated in the following way:

Proposition 3.2. The number $\#K_k^\circ(M)$ of tope committees which are free of opposites, of cardinality $k$, $1 \leq k \leq |T|/2$, for the oriented matroid $M := (E_t, T)$, is:

$$\#K_k^\circ(M) = \binom{|T|/2}{k} 2^k + \sum_{G \subseteq \bigcup_{e \in E_t} \left( \left( T + e \right) \cup \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right)} \mu_{E}(\hat{0}, G) \cdot \sum_{0 \leq j \leq k} \left( \left| G \cup \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right) \cdot \left( \frac{1}{2} \left( \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t} G} \right| \right) \right) 2^{k-j}.$$  

If an integer $k$, $1 \leq k \leq |T|/2$, and a ground element $a \in E_t$ are fixed, then we set

$$\beta_k(a, M) := \binom{|T|/2}{k} 2^k + \sum_{G \subseteq \bigcup_{e \in E_t - \{a\}} \left( \left( T + e \right) \cup \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t - \{a\}} G} \right| \right)} (-1)^{|G|} \cdot \sum_{0 \leq j \leq k} \left( \left| G \cup \left| \bigcup_{G \subseteq \bigcup_{e \in E_t - \{a\}} G} \right| \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t - \{a\}} G} \right| \right) \cdot \left( \frac{1}{2} \left( \left| T \cup \bigcup_{G \subseteq \bigcup_{e \in E_t - \{a\}} G} \right| - \left| \bigcup_{G \subseteq \bigcup_{e \in E_t - \{a\}} G} \right| \right) \right) 2^{k-j}.$$  

with the help of (3.7).
In view of \((3.8)\), we have
\[
\tilde{\kappa}_k^*(\mathcal{M}) = \beta_k(a, \mathcal{M}) + \sum_{0 \leq j \leq k} \left( \frac{1}{2} \left( |T| - |U \in G' \cup G'' G| - |U \in G' \cup G'' G| \right) \right) 2^{k-j}.
\]

In an analogous expression for \(\tilde{\kappa}_k^*(-a, \mathcal{M})\) the families \(G'\) range over subfamilies of the family \(\left( \frac{T^+_n(\mathcal{M})}{\sum T^+_n(\mathcal{M})} \right) - \bigcup_{e \in E_t - \{a\}} \left( \frac{T^+_n(\mathcal{M})}{\sum T^+_n(\mathcal{M})} \right)\).

4. \(\kappa^*\)-Vectors and Reorientations on One-Element Sets

To find the differences of the components of \(\kappa^*\)-vectors associated to the oriented matroid \(\mathcal{M}\) and to the oriented matroid \(-a, \mathcal{M}\) which is obtained from \(\mathcal{M}\) by reorientation on a one-element subset \(\{a\} \subset E_t\), we combine expressions \((2.2)\) and \((3.9)\) related to \(\mathcal{M}\) with analogous expressions related to \(-a, \mathcal{M}\):

**Proposition 4.1.** Let be an element of the ground set \(E_t\) of the oriented matroid \(\mathcal{M} := (E_t, T)\). For an integer \(k, 1 \leq k \leq |T|/2\), the sum

\[
\sum_{0 \leq \#G'' \leq \left\{ \left( \frac{|T|}{2} - k \right) \right\} - 1} \left( -1 \right)^{\#G''} \cdot Q(G', G'')
\]

\[
- \sum_{1 \leq \#G' \leq \left\{ \left( \frac{|T|}{2} - k \right) \right\} - \#G''} \left( -1 \right)^{\#G'} \cdot Q(G', G'')
\]

\[
- \sum_{|U \in G' G| \leq |T| - k} \left( -1 \right)^{\#G'} \cdot Q(G', G'')
\]

and the sum
\[ \sum_{G'' \in \mathcal{E}(\bigcup_{e \in Et - \{a\} \bigcup G'' \cup - (T + e(M)) \bigcup \mathcal{E}(T - a(M)))} \mu \mathcal{E}(0,G'') \cdot \Omega(G',G'') \]
\[ \sum_{G' \in \mathcal{E}(\bigcup_{e \in Et - \{a\} \bigcup G' \cup - (T + e(M)) \bigcup \mathcal{E}(T - a(M)))} \mu \mathcal{E}(0,G') \cdot \Omega(G',G'') \]
both calculate the difference
\[ \kappa^*_k(-aM) - \kappa^*_k(M) \]
under
\[ Q(G',G'') := \left| T - \bigcup_{G' \in G'} G \right| \quad \text{and} \quad \Omega(G',G'') := \left| T - \bigcup_{G' \in G'} G \right|. \]
These sums calculate the difference
\[ \kappa^*_k(-aM) - \kappa^*_k(M) \]
under
\[ Q(G',G'') := \sum_{0 \leq j \leq k} \left( \left| \bigcup_{G' \in G'} G \right| - \left| \bigcup_{G' \in G'} G \right| \right) \left( \frac{1}{2} \left( \left| T - \bigcup_{G' \in G'} G \right| \right) \right)^{2k-j} \]
and
\[ \Omega(G',G'') := \sum_{0 \leq j \leq k} \left( \left| (G' \cup G'') \cup - (G' \cup G'') \right| - \left| (G' \cup G'') \right| \right) \left( \frac{1}{2} \left( \left| T - (G' \cup G'') \cup - (G' \cup G'') \right| \right) \right)^{2k-j} \]

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