Quaternionic Superconformal Field Theory

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ABSTRACT

We develop a superfield formalism for N=4 superconformal two-dimensional field theory. A list is presented of minimal free superfields, i.e. of multiplets containing four bosons and four fermions. We show that the super-Poincaré algebra of the six-dimensional superstring in the light-cone gauge is essentially equivalent to a local N=4 superconformal symmetry, and outline the construction of octonionic superconformal field theory.
1. Introduction

There is a disturbing lack of understanding of string theory concerning the underlying gauge principles. This is especially true of superstring models, where a great obstacle to a covariant quantum theory is the presence of second-class constraints following from the so-called kappa-symmetry [1] for the covariant Green-Schwarz action [2]. The same is true for the superparticle [3]. Following the Dirac procedure [4], the second class constraints cannot be eliminated in a covariant way so to yield a covariant quantum theory [5]. In the superparticle case one now knows ways to circumvent this problem [6-11], most of which involve some kind of twistor formulation [13,7-10].

There have been several approaches to string theory within a twistor framework [14] (the term twistor is somewhat inappropriate due to the lack of space-time conformal invariance), but it was only recently that a twistorial superstring theory appeared [15]. Unfortunately it is again plagued by second-class constraints, which now appear in the bosonic sector. There is however another interesting feature – the presence of an extended superconformal symmetry, which tends to put rather strong constraints on string models. Clearly one should develop an appropriate superfield formalism and make that symmetry manifest. This is the main objective of this paper. We will restrict ourselves here to the case N=4, which pertains to the D=6 superstring. The structures we describe will carry over to N=8 and D=10.

The fact that the superstring can be treated in terms of the RNS model [16] implies a connection between spacetime and worldsheet supersymmetry. The relation is more intimate than that, since one can show that spacetime supersymmetry implies extended world-sheet supersymmetry [17,18]. There is however no explanation of why it should be so, presumably because the proofs were given in formulations where spacetime supersymmetry is not manifest. We will provide an explanation in the framework of the superstring on the light-cone: the super-Poincaré algebra is essentially equivalent to an extended superconformal algebra.

The manifestly supersymmetric string should hence be considered in a world-
sheet-supersymmetric setting. The idea of unifying the spacetime-supersymmetric and the worldsheet-supersymmetric incarnations of the superstring has occurred to many people in the field. The idea became a program with the realization that for certain examples of the superparticle and superstring with sufficiently extended worldsheet supersymmetry one can identify the kappa-symmetry as part of the conformal superdiffeomorphisms on the worldline or worldsheet [11,12,19]. This could mean that one can avoid second-class constraints by gauge-fixing that kind of action in the proper, yet unknown, fashion. We may then envisage a spacetime covariant superconformal gauge that features a finite number of free fields and first class constraints implemented à la BRST. If such a gauge does not exist then the new superstring formulations will probably not be terribly attractive. If it does we will obtain some kind of extended superconformal field theory.

Extended superconformal algebras were first constructed in refs. 20, 21 and have by now been studied in some detail [22]. Typically those algebras are much larger than the ones we are interested in, which contain the minimal number of generators. In addition, we are aiming at a formalism that allows at least the computation of the superstring S-matrix, and for that purpose it is surely convenient to have a linear algebra realized on a small number of free fields, with as much symmetry as possible manifest. Neither existing superfield formalisms [23,24] nor earlier studies of representations of N=4 supersymmetry in the context of hyperkähler sigma-models [25] are therefore tailormade for our purposes.

We believe that the division algebra formalism is a natural tool for the superconformal symmetries occurring in string theory. It it well known [26,27] that there exists a close relationship between the division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \( \mathbb{O} \) and N=1 supersymmetry in D=3,4,6,10 or, by dimensional reduction, N=1,2,4,8 supersymmetry in D=2. The reason is that those hypercomplex numbers very efficiently encode the important Gamma-matrix identity

\[
\gamma_{\mu(a} \gamma^\mu_{cd)} = 0 .
\]

Consequently they have been used in the analysis of the Green-Schwarz superstring
and twistor(-like) superstring models [15,31], and of course also in
the description of extended two-dimensional superconformal algebras [34]. We will
make extensive use of quaternions and intend this to be a stepping stone towards
octonions.

The paper is organized as follows: in section 2 we briefly discuss the quaternion
language we will use. A translation table to \(\gamma\)-matrix notation is given in the
Appendix. In section 3 we describe all free superconformal fields with minimal
field content, \(i.e.\) with 4 bosonic and 4 fermionic components, and describe their
energy-momentum superfields and correlation functions. The list is smaller than
expected, since one can show that minimal ghost systems with arbitrary conformal
weight do not exist. In section 4 we present actions for all the fields. Subsequently
we apply our technology to the Lorentz-algebra of the light-cone superstring in
\(D=6\). Our construction does not make reference to the coordinates of the internal
space, but only to the superconformal generators. We conclude with an outline of
the \(N=8\) construction.

2. Quaternionic formalism

We will often use a quaternionic language in this paper. This allows us to
dispense with complicated index structures and simplifies some calculations and
results. It is also important conceptually, since various transformations can be
clearly separated, and this makes the step from \(N=4\) to \(N=8\) almost straightforward.

In the Appendix we provide a translation table from quaternion notation to
\(\sigma\)-matrix notation that we will frequently use in the rest of the paper.

Our conventions are as follows. For an element \(h = \sum_{n=0}^{3} h_a e_a \in \mathbb{H}\) (the
quaternionic ring), we define \textit{conjugation} by

\[
h \rightarrow h^* = h_0 - \sum_{i=1}^{3} h_i e_i .
\]  

(2.1)
It is distinguished from complex conjugation, written as $z \rightarrow \overline{z}$. The complex imaginary unit is not affected by $\mathbb{H}$ conjugation. The \textit{real} or \textit{scalar} part of $h$ is

\[ [h] = \frac{1}{2} (h + h^*) , \]

and the \textit{imaginary} or \textit{vector} part

\[ \{h\} = \frac{1}{2} (h - h^*) . \]

The notation in the last two definitions is non-standard but useful. A quaternion is called a \textit{unit} quaternion if $hh^* = 1$, i.e. if it has unit norm.

2.1. Quaternionic representations

In this section, we will describe how the tensor algebra of some relevant compact Lie groups is expressed in terms of elements of the division algebra $\mathbb{H}$ of quaternions. The results presented can not be regarded as original (see e.g. refs. 27, 26, 8), but since our formalism depends so heavily on them, they deserve to be presented in some detail.

\underline{$SU(2)$ and $SU(2)^2$}

A \textit{spinor} $\lambda \in \mathbb{H}$ is defined to transform under $SU(2)$ as

\[ \lambda \rightarrow \lambda h_1 , \]

where $h_1$ is a unit quaternion, $h_1 h_1^* = 1$, i.e. a point on $S^3 \approx SU(2)$. It is obvious that the transformations

\[ \lambda \rightarrow h_2^* \lambda , \quad h_2 h_2^* = 1 , \]

parametrize another $SU(2)$ that commutes with the first one. We are naturally led to consider the group $SU(2)^2 \approx SO(4)$. We call the spinor representation “$4$”.
The reader will probably be used to complex spinorial representations of $SU(2)$. The quaternionic spinors (four real components) can be understood in terms of two-component complex spinors as follows. The imaginary unit quaternions fulfill the Clifford algebra of $SU(2)$:

$$e_i e_j + e_j e_i = -2\delta_{ij} .$$

(2.6)

They may therefore be realized as $i$ times the Pauli matrices. Let

$$\mathbf{H} \ni \lambda = \lambda_0 \mathbf{1} + i\lambda_j \sigma_j ,$$

(2.7)

where $\sigma_i$ are Pauli matrices. Then

$$\lambda = \begin{pmatrix} \lambda_0 + i\lambda_3 & i\lambda_1 + \lambda_2 \\ i\lambda_1 - \lambda_2 & \lambda_0 - i\lambda_3 \end{pmatrix} \equiv (\mu_1 \quad \mu_2) ,$$

(2.8)

and the two two-component spinors $\mu_\alpha$ are related by

$$\overline{\mu}_\alpha = i\sigma_2 \varepsilon_{\alpha\beta} \mu_\beta .$$

(2.9)

This is what in ref. 26 is called an $SU(2)$-Majorana condition. The representation $4$ is a real irreducible representation of $SU(2)^2$.

Using the Clifford algebra of eq. (2.6), one realizes that tensor products of representations are encoded in quaternionic multiplication. If we denote the $SU(2)$’s in eqs. (2.4) and (2.5) by $SU(2)_1$ and $SU(2)_2$ respectively, it is clear that multiplication of two spinors $\lambda, \mu \in 4$ yields

$$\{\lambda^* \mu\} \in 3_1 , \quad \{\lambda \mu^*\} \in 3_2 ,$$

(2.10)

with obvious meaning of the subscripts. With $A_1 \in 3_1, A_2 \in 3_2$ we also have

$$\lambda A_1 \in 4 , \quad A_2 \lambda \in 4 .$$

(2.11)

The vector representations in eq. (2.10) cover all six antisymmetric products in $4 \times 4$. In real formalism, we can form projection operators on the two orthogonal
(selfdual and anti-selfdual) subspaces in (2.10):

\[ \lambda_{[a\mu b]} = (\Pi^1_{abcd} + \Pi^2_{abcd}) \lambda_{c\mu d}, \]  

(2.12)

where

\[ \begin{align*}
\Pi^1_{abcd} &= \frac{1}{2}(\delta_{cd} - \frac{1}{2}\varepsilon_{abcd}) = -\frac{1}{4}[e^*_ae_be^*_ce_d] = \frac{1}{4}\sigma^M_{a|c}(1)\sigma^M_{d|b}(1), \\
&= \frac{1}{4}\sigma^J_{ab}(-1)\sigma^J_{cd}(-1) = \frac{1}{16}\sigma^{\mu\nu}(-1)\sigma^{\mu\nu}(-1) \\
\Pi^2_{abcd} &= \frac{1}{2}(\delta_{cd} + \frac{1}{2}\varepsilon_{abcd}) = -\frac{1}{4}[e_ae_be^*_ce^*_d] = \frac{1}{4}\sigma^H_{a|c}(-1)\sigma^H_{d|b}(-1) \\
&= \frac{1}{4}\sigma^J_{ab}(1)\sigma^J_{cd}(1) = \frac{1}{16}\sigma^{MN}_{ab}(1)\sigma^{MN}_{cd}(1).
\end{align*} \]

(2.13)

These projections will play a crucial rôle in defining the superfields. The various \( \sigma \)-matrices are defined in the Appendix, and are to be understood as chiral projections of \( \gamma \)-matrices.

**SU(2)\(^3\)**

The form of the projection operators in eq.(2.13) appears “non-covariant”, considering it contains an \( \varepsilon \)-symbol in “spinor” indices. However, if we reconsider the transformation rules (2.4) and (2.5), we find that \( \lambda \) is actually what is usually recognized as a vector of \( SO(4) \approx SU(2)_1 \otimes SU(2)_2 \). When we later specify the field content of our superconformal field theory, we will have bosons and fermions (four real degrees of freedom each) transforming under “vector” and “spinor” representations of \( SO(4) \) (the group of euclidean rotations on the six-dimensional light-cone superstring variables). The interpretation of the \( SU(2)^2 \) treated above is that \( SU(2)_1 \) is half the group of space rotations \( SO(4) \approx SU(2)_0 \otimes SU(2)_1 \), and that \( SU(2)_2 \) is an internal group (as in eq. (2.9)).

Considering the symmetry group \( SU(2)_0 \otimes SU(2)_1 \otimes SU(2)_2 \), we have three inequivalent quaternionic representations, which we denote \( 4_v, 4_s, \) and \( 4_c \). The transformation rules are (\( L \) and \( R \) denote left and right multiplication by a unit
quaternion and $L^*$ and $R^*$ by its conjugate):

\[
\begin{array}{cccc}
4_v & 4_s & 4_c \\
SU(2)_0 & R & L^* & 1 \\
SU(2)_1 & L^* & 1 & R \\
SU(2)_2 & 1 & R & L^*
\end{array}
\] (2.14)

The composition rules are symbolically, with $H$ multiplication,

\[
4_v4_s \rightarrow 4_c^* \text{ and cyclic permutations},
\] (2.15)

and express the triality of $SU(2)^3$, formally identical to the triality of $SO(8)$ as expressed in terms of octonionic multiplication [35]. This means that, as for $SO(8)$, the choice of “vector” representation is arbitrary, as seen from (2.14).

When $SU(2)^3 \rightarrow SU(2)^2$ by only considering group elements $h_0 \in SU(2)_0$ and $h_1 \in SU(2)_1$ with $h_0 = h_1$, (2.14) is reduced to

\[
\begin{array}{cccc}
4_v & 4_s & 4_c \\
\text{diag}(SU(2)_0, SU(2)_1) & L^*R & L^* & R \\
SU(2)_2 & 1 & R & L^*
\end{array}
\] (2.16)

so the the representations split as

\[
\begin{aligned}
4_v & \rightarrow 1 \oplus 3_1, \\
4_s & \rightarrow 4^*, \\
4_c & \rightarrow 4
\end{aligned}
\] (2.17)

as $SU(2)^3 \rightarrow SU(2)^2$ ($4^*$ is not inequivalent to $4$ — this is just to denote that the conjugate spinor obeys the transformation rules (2.4) and (2.5)). Observe again the exact formal analogue of the splitting of the three 8’s of $SO(8)$ under $SO(8) \rightarrow SO(7)$.
We may extend the transformation table (2.14) to four $SU(2)$’s:

\[ v \rightarrow h_1^* v h_0 \]
\[ s \rightarrow h_0^* s h_3 \]
\[ c^* \rightarrow h_1^* c^* h_2 \]
\[ w \rightarrow h_2^* w h_3 \]  \hspace{1cm} (2.18)

where $h_A$ are four unit quaternions parametrizing $SU(2)_A$. (2.14) is obtained by setting $h_2 = h_3$. Clearly $SO(4)_1 \approx SU(2)_0 \otimes SU(2)_1$ and $SO(4)_2 \approx SU(2)_2 \otimes SU(2)_3$ commute. We denote with $v^\mu$ and $w^M$ the corresponding vector representations. The $\sigma$-matrices satisfy

\[ \sigma^\mu_{ab}(-1)\sigma^N_{bc}(1) = \sigma^N_{ab}(1)\sigma^\mu_{bc}(-1) . \]  \hspace{1cm} (2.19)

Note that spinor indices may carry different chirality with respect to the two $SO(4)$’s.

Identity (2.19) makes some of the subsequent constructions possible. For $N=8$ it is valid only up to terms proportional to the associator of octonions.

It is difficult to write (2.19) using complex spinors. That is the reason the $SO(4)$-covariant form of the $N=4$ algebras we shall present was not written down already in ref. 21.
3. Superfields

3.1. Scalar Superfields

Consider a general scalar holomorphic field on the superspace spanned by the complex coordinate $z$ and the spinor $\theta = \theta_a e_a$. Its component expansion is

$$\Phi(z, \theta) = A(z) + \theta^* B(z) - \frac{1}{2}[\theta^* \theta C^*(z)] - \frac{1}{2}[\theta^* D^*(z)] + [\theta^3* E(z)] + \theta^4 F(z)$$

$$= A(z) + \theta_a B_a(z) - \frac{1}{8} \theta_a \sigma^{\mu \nu}_{ab} (-1) \theta_b C^{\mu \nu}(z) - \frac{1}{8} \theta_a \sigma^{MN}_{ab} (1) \theta_b D^{MN}(z)$$

$$+ \theta^3_a E_a(z) + \theta^4 F(z) ,$$

(3.1)

where $\theta^3 = -\frac{1}{6} \theta^\dagger \theta$ (i.e. $\theta^3_a = -\frac{1}{6} \varepsilon_{abcd} \theta^a \theta^b \theta^c \theta^d$) and $\theta^4 = \frac{1}{24} \theta^\dagger \theta^\dagger \theta^\dagger = \frac{1}{24} \varepsilon_{abcd} \theta^a \theta^b \theta^c \theta^d$. In the following, $z$-dependence will be suppressed when dealing with holomorphic fields. According to the discussion in section (3.1), the transformation properties of the component fields are: $A, F \in 1$; $B, E \in 4$; $C \in 3_1$ and $D \in 3_2$. In $SO(4)$-language, $C$ and $D$ are antisymmetric selfdual tensors. Note that the above interpretation implies a choice of spinor chiralities with respect to $SO(4)_1$ and $SO(4)_2$. We could just as well have made another choice by barring the appropriate $\sigma$-matrices in (3.1). We note that in ref. 21 some results similar to ours were derived using a complex formalism.

Constraints

We define the covariant spinor derivative

$$\mathcal{D} = \frac{\partial}{\partial \theta} + \theta \partial$$

(3.2)

with $\frac{\partial}{\partial \theta} = e_a \frac{\partial}{\partial \theta_a}$ and $\partial = \frac{\partial}{\partial z}$. In order to reduce the field content to four bosonic and four fermionic fields, we impose the following constraint on $\Phi$:

$$\{\mathcal{D} \mathcal{D}^* \} \Phi = 0 ,$$

(3.3)

i.e. we use the projection operators of eq.(2.13) and demand $\Pi^2_{abcd} \mathcal{D}_c \mathcal{D}_d \Phi = 0$. 

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This constraint is quite analogous to the selfduality condition in N=4 super-Yang-Mills theory [36]. The component expansion yields

\[ D = 0 , \]
\[ E = \partial B , \]
\[ F = \partial^2 A . \] (3.4)

According to eq. (3.2), \( w(\theta) = -\frac{1}{2} \) (\( w \) denotes conformal weight), and the constrained, antiselfdual, superfield

\[ \Phi = \varphi + [\theta^* \psi] - \frac{1}{2} [\theta^* \theta \chi^*] + [\theta^3 \partial \psi] + \theta^4 \partial^2 \varphi \] (3.5)

has by definition \( w(\Phi) = 0 \). Its \( \theta^2 \)-term is antiselfdual in the spinor indices. Hence the name. \( \Phi \) contains exactly the components corresponding to the light-cone superstring in D=6, provided we interpret the \( w = 1 \) vector as the derivative of a \( w = 0 \) field, \( \chi^j = \partial \varphi^j \). This assignment will be justified in section (4.2).

**Correlators**

For the moment, we assume that the field \( \Phi \) is “self-conjugate”, and we assign correlators to its components according to their conformal weights:

\[ \varphi(z) \varphi(\zeta) \sim \ln(z - \zeta) , \]
\[ \psi_a(z) \psi_b(\zeta) \sim \frac{\delta_{ab}}{z - \zeta} , \] (3.6)
\[ \chi^i(z) \chi^j(\zeta) \sim \frac{\delta_{ij}}{(z - \zeta)^2} . \]

These are collected into the supercorrelator

\[ \Phi(Z_1) \Phi(Z_2) \sim \Delta^{(e)}(Z_1, Z_2) \equiv \ln(z_{12} - [\theta^*_1 \theta^a_2]) - e \frac{\theta^4_{12}}{z_{12}} = \ln(Z_{12}) - e \frac{\theta^4_{12}}{Z_{12}^2} , \] (3.7)

where \( e = 1 \) for an antiselfdual field, \( Z = (z, \theta) \) and \( z_{12} = z_1 - z_2 \), while \( \theta_{12} = \theta^a_1 - \theta^a_2 \) and \( Z_{12} = z_1 - z_2 - \theta^a_1 \theta^a_2 \) are the supersymmetry-invariant distance functions. The last term ensures that the function \( \Delta^{(+)} \) satisfies the constraint (3.3). For a field of opposite duality it takes the opposite sign. We will say more about correlators when examining vector superfields.
3.2. The Energy-momentum Superfield

From the components of the selfdual superfield of eq.(3.5), we may construct the generators of an N=4 superconformal algebra:

\[ J = \frac{1}{2} \psi \psi^* \, , \]
\[ G = \psi (\partial \varphi - \chi) \, , \]
\[ L = \frac{1}{2} \partial \varphi \partial \varphi - \frac{1}{2} [\psi^* \partial \psi] + \frac{1}{2} \chi^* \chi \, . \]  

(3.8)

We have thus a set of \( SU(2)_2 \) Kac-Moody generators \( J^K = \frac{1}{2} \sigma^K_{ab} \psi_a \psi_b \in 3_2 \) \( (w = 1) \), four fermionic generators \( G_a = \sigma^H_{ab} (\psi_b \partial \varphi^\mu) \in 4 \) \( (w = \frac{3}{2}) \) and a Virasoro generator \( L = \frac{1}{2} \partial \varphi^\mu \partial \varphi^\mu - \frac{1}{2} \psi_a \partial \psi_a \in 1 \) \( (w = 2) \). As far as the energy-momentum algebra is concerned, it makes no difference whether we declare that \( (\partial \varphi, \partial \varphi^j) \) transforms as \( 1 + 3_1 \) or as \( 4_v^v \). The Kac-Moody currents transform as antisymmetric selfdual tensors under \( SO(4)_2 \) and that is the reason none of the \( \varphi^\mu \)'s have to appear in \( J \), making its construction feasible. The fact that the \( SO(4)'s \) commute assures the closure of the algebra.

The operator product expansions are

\[ J_u(z)J_v(\zeta) \sim \frac{-c/3}{(z - \zeta)^2} [u^*v] + \frac{1}{z - \zeta} J_{[u,v]} \, , \]
\[ J_u(z)G_\alpha(\zeta) \sim \frac{1}{z - \zeta} G_{u\alpha} \, , \]
\[ G_\alpha(z)G_\beta(\zeta) \sim \frac{2c/3}{(z - \zeta)^3} [\alpha\beta^*] + \frac{2}{(z - \zeta)^2} J_{(\alpha\beta^*)}(\zeta) \]
\[ + \frac{1}{z - \zeta} (\partial J_{(\alpha\beta^*)} + 2 [\alpha\beta^*] L) \, , \]
\[ L(z)J(\zeta) \sim \frac{1}{(z-\zeta)^2}J(\zeta) + \frac{1}{z-\zeta} \partial J, \]
\[ L(z)G(\zeta) \sim \frac{3}{2(z-\zeta)^2}G(\zeta) + \frac{1}{z-\zeta} \partial G, \] (3.9)
\[ L(z)L(\zeta) \sim \frac{c/2}{(z-\zeta)^4} + \frac{2}{(z-\zeta)^2}L(\zeta) + \frac{1}{z-\zeta} \partial L, \]

where a variable \( X \) in the quaternionic representation \( r \) is given an explicit (bosonic) quaternionic index \( \rho \in r \) by the contraction \( X_\rho = [\rho^* X] \) and the central terms are expressed in terms of the conformal anomaly \( c = 6 \). This is the \( N=4 \) superconformal algebra of ref. 21 in the notation of ref. 32.

We now want to gather the superconformal generators into a superfield of a well-defined type. If the superfield is not to contain inverse powers of \( \partial \), it should contain \( J \) as its lowest component, and thus as a whole belong to the representation \( 3 \cdot 2 \). It is not difficult to check that the field

\[ \mathcal{L} = \frac{1}{2} D\Phi D^*\Phi \] (3.10)

has the right properties. Its component expansion is

\[ \mathcal{L} = \{ J + \theta G^* + \theta \theta^* (\frac{1}{2} \partial J + L) - \theta^3 \partial G^* - \theta^4 \partial^2 J \}. \] (3.11)

Compared to a general superfield in \( 3 \cdot 2 \), with \( 3 \cdot 2^4 \) components, \( \mathcal{L} \) is highly constrained. The form of the constraint that projects down to the minimal superfield shown above is

\[ [D\Omega^*]\mathcal{L} + \{ D\Omega^* \mathcal{L} \} = 0, \] (3.12)

where \( \Omega \in 4 \) is a constant spinor. Its real-component version reads

\[ D_a \mathcal{L}^I - \frac{1}{2} \varepsilon^{IJK} \sigma_{ab}^I(1) D_b \mathcal{L}^K = 0, \] (3.13)

where we have used the Appendix.
The action of the stress tensor on a primary superfield \( \Xi_A(z, \theta) \) is given by:

\[
\mathcal{L}^J(Z_1) \Xi_A(Z_2) \sim \left( \delta^{JK} \frac{1}{Z_{12}} - \frac{1}{2} e^{JLK} \theta_{12} \sigma^L(e) \theta_{12} - 2 e \delta^{JK} \frac{\theta_{12}^4}{Z_{12}^4} \right) \delta^K \Xi_A(Z_2)
\]

\[
+ \left( \frac{\theta_{12a}}{Z_{12}} + e \frac{\theta_{12b}}{Z_{12}^2} \right) \sigma^J(\epsilon)_{ab} D_b \Xi_A(Z_2)
\]

\[
+ w \frac{\theta_{12} \sigma^J(\epsilon) \theta_{12}}{Z_{12}^2} \Xi_A(Z_2) + \frac{\theta_{12} \sigma^J(\epsilon) \theta_{12}}{Z_{12}} \partial \Xi_A(Z_2)
\]

(3.14)

Here \( e = 1 \), \( w \) denotes the conformal weight of \( \Xi_A \) and \( \delta^K \Xi_A \) describes the variation of \( \Xi_A \) under an infinitesimal \( SU(2) \)-rotation with index \( K \), for example: \( \delta^K D_a \Phi = -\sigma^K_{ab}(1) D_b \Phi \), \( \delta^K \Phi = 0 \) and \( \delta^K \mathcal{L}^L = 2 e^{KLJ} \mathcal{L}^J \). We note that the structure of (3.14) does not depend on \( \Xi_A \) being a minimal field with only four bosonic and four fermionic components. An example is provided by the \( SU(2) \)-scalar \( \Xi = \exp(ik \Phi) \), which is primary with \( w = -\frac{1}{2} k^2 \), but contains both selfdual and antiselfdual components at the \( \theta^2 \)-level.

Accordingly, the superspace rendition of (3.9) is

\[
\mathcal{L}^J(Z_1) \mathcal{L}^K(Z_2) \sim -\frac{c}{3} \left( \delta^{JK} \frac{1}{Z_{12}^2} + e^{JKL} \theta_{12} \sigma^L(e) \theta_{12} - 6 e \delta^{JK} \frac{\theta_{12}^4}{Z_{12}^4} \right)
\]

\[
+ \left( \delta^{IJ} \frac{1}{Z_{12}^2} - \frac{1}{2} e^{JLI} \theta_{12} \sigma^L(e) \theta_{12} - 2 e \delta^{IJ} \frac{\theta_{12}^4}{Z_{12}^4} \right) 2 e \delta^{KN} \mathcal{L}^N(Z_2)
\]

\[
+ \left( \frac{\theta_{12a}}{Z_{12}} + e \frac{\theta_{12b}}{Z_{12}^2} \right) \sigma^J(\epsilon)_{ab} D_b \mathcal{L}^K(Z_2)
\]

\[
+ 2 \frac{\theta_{12} \sigma^J(\epsilon) \theta_{12}}{Z_{12}^2} \mathcal{L}^K + \frac{\theta_{12} \sigma^J(\epsilon) \theta_{12}}{Z_{12}} \partial \mathcal{L}^K(Z_2)
\]

(3.15)

Again, \( e = 1 \). The other sign choice would be appropriate for an antiselfdual
energy-momentum multiplet. All the functions of \( Z_{12} \) and \( \theta_{12} \) that appear in (3.14) and (3.15) can be written as derivatives of the basic correlator (3.7).

3.3. Vector Superfields

Scalar superfields cannot describe the D=6 light-cone superstring. They have only one bosonic zero-mode. However, a look at the energy-momentum multiplet reveals that we may interpret the bosonic components as a vector of \( SO(4)_1 \). Then of course the \( SO(4) \) of space rotations for the six-dimensional superstring is “broken” in (3.5) to \( SU(2) \), and only \( SU(2)^2 \) of the \( SU(2)^3 \)-symmetry is manifest.

In order to restore the full symmetry we demand a superfield that transforms as \( 4_v \) under \( SU(2)^3 \). It must also contain no other components than this bosonic \( 4_v \) and a fermionic \( 4_s \) (or \( 4_c \)). We may construct it from \( \Phi \), once we observe that, although in this interpretation \( \Phi \) is non-covariant, the derivative \( D^a \Phi \equiv \Psi_a \) is:

\[
\Psi = \psi + \theta \partial X + \frac{1}{2} \theta \theta^* \partial \psi - \theta^3 \partial^2 X - \theta^4 \partial^2 \psi .
\]

(3.16)

Then the vector superfield \( \mathcal{X} \) can be obtained from \( \Phi \) by

\[
[\Omega^* D] \mathcal{X} = \Omega^* \Psi .
\]

(3.17)

This equation also encodes the constraints:

\[
D_a \mathcal{X} = \sigma^\mu_{ab} (-1) \Psi_b , \quad \iff \quad D_a \Psi_b = \sigma^\mu_{ab} (-1) \partial \mathcal{X} .
\]

(3.18)

The equivalence holds of course only up to zero modes of the differential operators involved. The equations above are quite useful for superspace calculations. Once we set \( X \in 4_v \), \( \psi \in 4_s^* \) and \( \theta \in 4_c \), the equivalent constraints

\[
\{ D^* \Omega \mathcal{X} \} = 0 ,
\]

(3.19)

are manifestly \( SU(2)^3 \)-covariant and imply the following component expansion of
the antiselfdual vectorfield:

$$\mathcal{X} = X + \theta^* \psi + \frac{1}{2} \theta^* \theta \partial X + \theta^3^* \partial \psi + \theta^4^* \partial^2 X. \quad (3.20)$$

Eq. (3.19) implies that the $\theta^2$-term is antiselfdual in the spinor indices. Note that in $\Psi$ the $\theta^2$-term is selfdual. (3.19) also relates the first and third components of the superfield. By construction we have $\Phi = \mathcal{X}^0$ and the energy-momentum superfield reads as for the antiselfdual scalar

$$\mathcal{L} = \frac{1}{2} \Psi \Psi^*. \quad (3.21)$$

It is straightforward to write down the correlator for the $\mathcal{X}$ field. One uses (3.18) to show $\mathcal{D}^* \mathcal{D} \Phi = 4 \partial \mathcal{X}$ and then derives from the $\Phi$-correlator (3.7)

$$\mathcal{X}_u(Z_1) \mathcal{X}_v(Z_2) \sim [v u^* \Delta^{(e)}(Z_1, Z_2)], \quad (3.22)$$

where $e = 1$ in the antiselfdual case and

$$\Delta^{(e)}(Z_1, Z_2) = \ln (Z_{12}) + \frac{1}{2} \frac{\theta^*_1 \theta^*_{12}}{Z_{12}} - \frac{e}{24} \frac{\theta^*_1 \theta^*_2 \theta^*_1 \theta^*_{12}}{Z_{12}^2}$$

$$= \ln \left( Z_{12} + \frac{1}{2} \theta^*_1 \theta^*_{12} + \frac{3 - e \theta^*_1 \theta^*_2 \theta^*_1 \theta^*_{12}}{24} \frac{1}{Z_{12}} \right). \quad (3.23)$$

The quaternionic logarithm is defined by its series expansion.

Since the scalar field is so closely related to the vector field, we can use the identities (3.18) also in that context. The scalar field correlator (3.7) is simply the real part of the vector correlator (3.23):

$$\Delta^{(e)}(Z_1, Z_2) = [\Delta^{(e)}(Z_1, Z_2)], \quad (3.24)$$
3.4. Ghost Systems

The superfields $\Phi$ and $\mathcal{X}$ may be thought of as matter multiplets of a gauge fixed version of some "covariant" theory (the case at hand, with $N=4$, corresponds to the six-dimensional superstring). The local superconformal symmetry may then be a remnant of the gauge symmetry of the covariant model. In order to treat this presumed gauge invariance, it is of interest to consider supermultiplets of ghost fields. It turns out that their $SO(4)$-index structure restricts the conformal weight assignments in a nontrivial manner.

Reparametrization Ghosts

We now consider conjugate pairs of ghosts $(B^J, C^K)$, $(\beta_a, \gamma_b)$ and $(b, c)$ of conformal weights $(\lambda - 1, 2 - \lambda)$, $(\lambda - \frac{1}{2}, \frac{3}{2} - \lambda)$ and $(\lambda, 1 - \lambda)$ respectively, with correlators

$$C^I(z)B^J(\zeta) \sim \frac{\delta^{IJ}}{z-\zeta},$$

$$\gamma_a(z)\beta_b(\zeta) \sim \frac{\delta_{ab}}{z-\zeta},$$

$$c(z)b(\zeta) \sim \frac{1}{z-\zeta}.$$

For $\lambda = 2$ this set of fields corresponds to the ghost system for super-reparametrizations.

We constructed the most general bilinear expressions for the currents $J$, $G$ and $L$, and demanded that they satisfy (3.9) for arbitrary central terms. Some algebra shows that this implies $\lambda = 2$ and $c = 12$, and up to trivial rescalings the currents are unique:

$$J^K = \beta_a \sigma^K_{ab}(1) \gamma_b - 2 \epsilon^{KLM} B^L C^M - \partial(B^K c)$$

$$G_a = - \partial \beta_a c - \frac{1}{2} \beta_a \partial c + 2 \gamma_a b + \sigma^J_{ab}(1) \beta_b C^J - 2 \sigma^J_{ab}(1) \partial \gamma_b B^J - \sigma^J_{ab}(1) \gamma_b \partial B^J$$

$$L = 2 \partial cb + c \partial b - \frac{3}{2} \partial \gamma_a \beta_a - \frac{1}{2} \gamma_a \partial \beta_a + \partial C^J B^J.$$

(3.26)
By $\oint \frac{dz}{2\pi i} G_a = \frac{\partial}{\partial \theta_a} - \theta_a \partial_z$ the above normalization implies the superfield expansions

$$C = c + 2\theta_a \gamma_a + \theta_a \sigma^J(1) \theta_b C^J - 2\theta_a^3 \partial \gamma_a - \theta^4 \partial^2 c$$

appropriate for a $w = -1$ selfdual scalar field obeying the constraint

$$\{D^*D\} C = 0,$$  \hspace{1cm} \text{(3.28)}

and

$$B^J = B^J + \theta_a \sigma^J(1) \beta_b + \theta_a \sigma^J(1) \theta_b b - \frac{1}{2} \theta_a \sigma^J(1)_{ab} \sigma^K(1)_{bc} \theta_c \partial B^K - \theta^3 \sigma^J(1) \partial \beta_b - \theta^4 \partial^2 B^J,$$  \hspace{1cm} \text{(3.29)}

which implies that $B$ is of the same type as $\mathcal{L}$ and hence carries $w = 0$ and satisfies

$$\{D \Omega^* B \} + [D \Omega^* ] B = 0.$$  \hspace{1cm} \text{(3.30)}

In order to obtain the supercorrelator, one forms the selfdual scalar field

$$B = -\frac{1}{24} \{D \Omega^* D^* \}$$

$$= \frac{1}{2} [\theta \partial \beta^*] - \frac{1}{4} [\theta \theta^* \partial^2 \beta^*] - \frac{1}{4} [\theta^3 \partial^2 B^*] - \theta^4 \partial^2 b.$$

Using the constraints (3.30) one derives $\{D \Omega^* \} B = 2\partial^2 B$, and from the scalar correlator $B(Z_1)C(Z_2) \sim \partial_1 \Delta(+) (Z_1, Z_2)$ one then gets

$$B(Z_1)C(Z_2) \sim \frac{\{D_1 D_1^* \}}{2\partial_1} \Delta(-)(Z_1, Z_2).$$  \hspace{1cm} \text{(3.31)}

The $\frac{1}{\partial}$ is superficial. The superfield identities that were used in this calculation are, by themselves, tricky. They are most easily understood by realizing that $(\partial c, -2C^J)$ and $2\gamma_a$ form, as far as the supersymmetry constraints are concerned, a (selfdual) vectormultiplet much like the (antiselfdual) multiplet $\partial \varphi^\mu$ and $\psi_a$ described previously. One then employs the analogs of (3.18). Similar considerations hold for the fields $(b, \partial B^J/2)$ and $\beta_a/2$. 

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The energy-momentum multiplet (3.26) reads in superfield language

\[ \mathcal{L}^{gh} = \left\{ \frac{1}{4} \{ \mathcal{D} \mathcal{D}^* \} \mathcal{C} \mathcal{B} + \frac{1}{4} \mathcal{D} \mathcal{C} \mathcal{D}^* \mathcal{B} + \partial (\mathcal{C} \mathcal{B}) \right\} . \] (3.32)

We also display the canonical N=4 super-reparametrization part of the BRST operator:

\[
Q = \oint \frac{dz}{2\pi i} \left( c \left( L^\Phi + \frac{1}{2} L^{gh} \right) - \gamma_a \left( G_a^\Phi + \frac{1}{2} G_a^{gh} \right) + C^K \left( J^K + \frac{1}{2} J^K \right) \right) \\
= \oint \frac{dz}{2\pi i} \left( c L^\Phi - \gamma_a G_a^\Phi + C^K J^K \right) \\
+ 2 \left( c \partial cb + c \partial C^K B^K - \epsilon^{IJK} C^I C^J B^K \right) \\
- c \partial \gamma_a \beta_a - \gamma_a \gamma_a b - \gamma_a \sigma_{ab}(1) \beta_b C^J + \gamma_a \sigma_{ab}(1) \partial \gamma_b B^J) \right)
\]

\[
= - \frac{1}{12} \oint \frac{dz}{2\pi i} \left[ \mathcal{D}_a \sigma_{ab}^J(1) \mathcal{D}_b \left( \mathcal{C} \left( L^\phi_J + \frac{1}{2} L^\phi_J \right) \right) \\
+ \frac{1}{2} \left( \mathcal{D}_a \sigma_{ab}^J(1) \mathcal{D}_b \mathcal{C} \right) \left( L^\phi_J + \frac{1}{2} L^\phi_J \right) \\
- \frac{1}{2} \mathcal{C} \mathcal{D}_a \sigma_{ab}^J(1) \mathcal{D}_b \left( L^\phi_J + \frac{1}{2} L^\phi_J \right) \right]_{\theta=0}
\] (3.33)

The BRST current is an unconstrained scalar superfield of conformal dimension 1, whose \( \theta_a \)-component is a total derivative. \( Q \) is therefore invariant under supersymmetry transformations. In order to achieve manifest supersymmetry we use

\[ \mathcal{L} = - \frac{1}{24} [ \mathcal{D} \mathcal{D}^* \mathcal{L}^* ] \]

and write

\[ Q = - \frac{1}{2} \oint \frac{dz}{2\pi i} d\theta^4 \left( \frac{1}{\partial^2} \mathcal{C} \right) \mathcal{L} \] (3.34)

where the inverse derivatives vanish in the component expansion after some partial integrations and the fermionic integral is normalized such that \( \int d\theta^4 \theta^4 = 1 \).

Since \( c^\Phi = 6 \) and \( c^{gh} = 12 \), we cannot hope for \( Q^2 = 0 \) beyond the classical level. If we choose the matter sector of our covariant theory to consist of, say, 6 selfdual scalar superfields, we must set constraints that restrict the physical field
content to just one of them. These constraints might be implemented in BRST-fasion with yet other ghosts, which then must make the total conformal anomaly vanish.

The above BRST current is not a component of a minimal field, and hence we cannot argue that the superstring, once properly gauge-fixed to $N=4$ superconformal gauge, should be built from a collection of minimal fields because those are the natural objects in conformal field theory. However, their simplicity and economy of size still makes them attractive as building blocks.

Other Minimal Ghosts

The somewhat surprising result that a minimal ghost system $\mathcal{C} \in \mathbf{1}$ and $\mathcal{B} \in \mathbf{3}$ has a unique conformal weight leads us to consider further representation assignments. The only remaining choices with minimal field content are $\mathcal{C} \in \mathbf{4}$ and $\mathcal{B} \in \mathbf{4}$, constrained like vector superfields:

\begin{align*}
\mathcal{D}_a & \mathcal{C}_v = [u^* v \Gamma] \\
\mathcal{D}_a & \mathcal{B}_w = [u^* \mathcal{B} w].
\end{align*}

(3.35)

We denote superfields according to their base components: the expansion of $\mathcal{C}$ starts with the fermion $c$, $\mathcal{B}$ has the bosonic base component $\beta$ etc. (3.35) can be interpreted as

\begin{align*}
\mathcal{D}_a & \mathcal{C}_b = \sigma^\mu_{ab}(-1) \Gamma^\mu \\
\mathcal{D}_a & \mathcal{B}_b = \sigma^\mu_{ab}(-1) \partial \mathcal{B}^\mu.
\end{align*}

(3.36)

or as

\begin{align*}
\mathcal{D}_a & \mathcal{C}^M = \sigma^M_{ab}(1) \Gamma_b \\
\mathcal{D}_a & \mathcal{B}^M = \sigma^M_{ab}(1) \partial \mathcal{B}^M.
\end{align*}

(3.37)

Any attempt to assign the fermionic variables $\theta$ to a vector representation results in breaking $SO(4)$-covariance. Therefore we do not consider that possibility any further. We assume free field correlators of the form (3.25), and superconformal
generators that are bilinear in the fields. The constraints (3.35) then determine the generators $G_a$ up to a total derivative term:

$$ G = -\partial c c\beta + b\gamma + \rho \partial(c\beta) $$  \hspace{1cm} (3.38)

We may actually read this equation as one in superspace, after simply replacing $c$ by $C$ and so on. Then (3.38) implies that the parameter $\rho$ parametrizes the conformal weights of the ghost multiplet. We now demand closure of the superconformal algebra and obtain the result that either $\rho = 0$ or $\rho = 1$, which in turn means that the component fields form a weight-$(1,0)$ and a weight-$(1/2,1/2)$ quadruplet.

We now give a list of the possible minimal vectorghost systems. In the following one may chose $e = \pm 1$ in each case, and of course also bar or unbar all the $\sigma$-matrices. The energy-momentum superfield can read off directly from the component expressions for the fields $J_{ab}$. Note that not all of those choices correspond to (3.35), since they require different quaternion products.

| $c$ | $a$, $b$ | $\gamma^\mu$ | $\beta^\mu$ |
|-----|---------|-------------|-------------|
| $-12$ | $0$, $1$ | $1/2$, $1/2$ |

$$ J_{ab} \equiv c_{ab}^k(e) J^k = -\overline{\sigma}_{ab}^{\mu
u}(e) \gamma^\mu \beta^\nu $$

$$ G_a = \overline{\sigma}_{ac}^\mu(e) b_c \gamma^\mu - \overline{\sigma}_{ab}^{\mu}(e) \partial c_b \beta^\mu $$

$$ L = \partial c_a b_a - \frac{1}{2} \partial \gamma^\mu \beta^\mu + \frac{1}{2} \gamma^\mu \partial \beta^\mu. $$

| $c$ | $a$, $b$ | $\gamma^\mu$ | $\beta^\mu$ |
|-----|---------|-------------|-------------|
| $12$ | $1/2$, $1/2$ | $1$, $0$ |

$$ J^K = b^a \sigma^K_{ab}(-e)c^b $$

$$ G_a = \overline{\sigma}_{ac}^\mu(e) b_c \gamma^\mu + \overline{\sigma}_{ab}^{\mu}(e) c_b \partial \beta^\mu $$

$$ L = \frac{1}{2} \partial c_a b_a - \frac{1}{2} c_a \partial b_a + \gamma^\mu \partial \beta^\mu. $$
The constraints (3.36) are appropriate for (3.39) and (3.40) with $e = -1$. Case (3.40) is somewhat trivial, since it corresponds to a pair of $\mathcal{X}^\mu$-type vector fields. The supercorrelator can be easily derived using (3.36):

$$C_a(Z_1)B^\nu(Z_2) \sim \frac{1}{4} \sigma^{\mu}_{ab}(e) D^b_1 \Theta^{-e\mu\nu}(Z_1, Z_2)$$
$$= \frac{1}{4} \sigma^{\mu}_{ab}(e) D^b_1 \left( \delta^{\mu\nu} \ln (Z_{12}) + \frac{1}{2} \theta_{12}^{\mu\nu} \theta_{12}^1 + e \delta^{\mu\nu} \theta_{12}^1 \right)$$
$$= \sigma^{\nu}_{ab}(e) \left( \frac{\theta_{12}^b}{Z_{12}} - e \frac{\theta_{12}^3}{Z_{12}^2} \right) .$$

This is the basic correlation function for the previous two ghost systems. Any other is computed by taking derivatives of it.

| $c$ | $\mu$ | $b^\nu$ | $\gamma_a$ | $\beta_a$ |
|-----|--------|--------|-------------|-------------|
| $w$ | 0      | 1      | 1/2         | 1/2         |

$$J^K = \beta^a \sigma^K_{ab}(-e) \gamma^b$$

$$G_a = \sigma^{\mu}_{ac}(e) \gamma_c b^\mu - \sigma^{\mu}_{ab}(e) \beta_b \partial c^\mu$$

$$L = \partial c^\mu b^\nu - \frac{1}{2} \partial \gamma_a \beta_a + \frac{1}{2} \gamma_a \partial \beta_a .$$

| $c$ | $\mu$ | $b^\nu$ | $\gamma_a$ | $\beta_a$ |
|-----|--------|--------|-------------|-------------|
| $w$ | 1/2    | 1/2    | 1           | 0           |

$$J_{ab} = \sigma^k_{ab}(e) J^k = \sigma^{\mu\nu}_{ab}(e) c^\mu b^\nu$$

$$G_a = \sigma^{\mu}_{ac}(e) \gamma_c b^\mu + \sigma^{\mu}_{ab}(e) \partial c^\mu$$

$$L = \frac{1}{2} \partial c^\mu b^\mu - \frac{1}{2} e^\mu \partial b^\mu + \gamma_a \partial \beta_a .$$

The constraints (3.37) with barred $\sigma$-matrices apply for (3.42) and (3.43) with $e = 1$. Case (3.43) is the triality-rotated version of (3.40). Here the bosons are
protected from the $J$-currents not by being a representation of the other $SO(4)$, but by their chirality. The supercorrelator reads:

$$C^\mu(Z_1)B_a(Z_2) \sim \sigma_{ab}^\mu(e)\left(\frac{\theta_{12b}}{Z_{12}} + e\frac{\theta_{12}^3}{Z_{12}^2}\right).$$ (3.44)

4. Actions

This section deals with the problem of finding proper actions for the types of fields discussed so far. There is one apparent problem present, associated with models with $N>2$. Namely, if one writes an action schematically,

$$S = \int d^2z \int d(\text{fermions})S,$$ (4.1)

and counts the scaling weights $s$, one obtains the relation ($s(S) = 0$)

$$0 = -2 + \frac{1}{2}n_f + s(S),$$ (4.2)

where $n_f$ is the number of fermionic variables appearing in the measure. Since $S$ should be constructed without inverse derivatives from a scalar field $\Phi$ with $s(\Phi) = 0$, there is a limit on $n_f$:

$$n_f \leq 4.$$ (4.3)

We also have to introduce a separate set of fermionic variables to accomodate the anti-holomorphic fields, so this means that we must restrict the fermionic integration from the “full” measure to a smaller one. One then has to check the supersymmetry invariance of the action separately.
4.1. Vector Fields

In order to describe a left- and a rightmoving vector field with common zero modes, we start with an unconstrained field $\mathcal{X}^\mu (z, \bar{z}, \theta, \bar{\theta})$. The first constraint we impose is

$$D_a \mathcal{X}^\mu = \sigma^\mu_{ab} (-1) \Psi_b .$$  \hspace{1cm} (4.4)

The next constraint then has to be

$$\bar{D}_a \mathcal{X}^\mu = \bar{\sigma}^\mu_{ab} (-1) \bar{\Psi}_b .$$  \hspace{1cm} (4.5)

This implies that $\theta \in 4c$ and $\bar{\theta} \in 4^*_s$ and we are thus constructing a type IIa superstring. If one unbars the $\sigma$-matrix in (4.5) to obtain left- as well as rightmoving fermions with the same chirality, the constraint system implies equations of motion and thus becomes illegal. If one replaces a $\sigma(-1)$ with $\sigma(1)$ one obtains equivalent results. (4.4) and (4.5) together imply the following component expansion:

$$\mathcal{X} = X + \theta^* \psi + \frac{1}{2} \theta \bar{\theta} \partial X + \ldots$$

$$- \bar{\psi}^* \bar{\theta} + \theta^* \eta \bar{\theta} - \frac{1}{2} \theta \partial \psi \bar{\theta} + \ldots$$

$$- \frac{1}{4} \partial \bar{X} \bar{\theta} \bar{\theta} - \frac{1}{2} \theta \partial \bar{\psi} \bar{\theta} - \frac{1}{4} \theta \partial \partial \bar{X} \bar{\theta} \bar{\theta} + \ldots$$

$$+ \ldots$$  \hspace{1cm} (4.6)

In this quaternion notation it is immediately obvious that $(\psi, \eta)$ is a vector multiplet in the $\bar{\theta}$-direction. In order to facilitate writing the Lagrangian, we note the action of derivatives on $\mathcal{X}$:

$$D_a \psi_b = \sigma^\mu_{ab} (-1) \partial \mathcal{X}^\mu$$

$$\bar{D}_a \bar{\psi}_b = \bar{\sigma}^\mu_{ab} (-1) \bar{\partial} \mathcal{X}^\mu$$

$$D_a \bar{\psi}_b = -\sigma^M_{ab} (1) N^M$$

$$\bar{D}_a \psi_b = \bar{\sigma}^M_{ab} (1) N^M$$  \hspace{1cm} (4.7)

$$D_a N^M = -\sigma^M_{ab} (1) \partial \bar{\psi}_b$$

$$\bar{D}_a N^M = \bar{\sigma}^M_{ab} (1) \bar{\partial} \psi_b$$

The $SO(4)_2$-vector superfield $N^M$ contains $\eta^M$ as its base component. The above equations may also be read as the supersymmetry transformations of the base
components of the relevant superfields. It is now easy to show that

\[ S = -\frac{1}{32} \int d^2z \, d\theta d\bar{\theta} \left( \mathcal{D}_a \mathcal{X}^\mu \tilde{\mathcal{D}}_b \mathcal{X}^\mu \right) \]

\[ = \frac{1}{2} \int d^2z \left( \partial X^\mu \overline{\partial} X^\mu + \eta^M \eta^M - \psi_a \overline{\partial} \psi_a - \tilde{\psi}_a \partial \tilde{\psi}_a \right) \]  

(4.8)

is off-shell supersymmetric and \( SU(2)^4 \)-invariant. We leave open the question of a type IIb vector field action that is manifestly N=4 supersymmetric and \( SO(4) \)-invariant.

4.2. Scalar Fields

The constraint system for the scalar field \( \Phi(z, \bar{z}, \theta, \bar{\theta}) \) reads:

\[ \mathcal{D}_a \sigma_{ab}^{MN}(1) \mathcal{D}_b \Phi = 0 \]  

(4.9)

\[ \tilde{\mathcal{D}}_a \sigma_{ab}^{MN}(1) \tilde{\mathcal{D}}_b \Phi = 0 \]  

(4.10)

\[ \mathcal{D}_a \tilde{\mathcal{D}}_b \Phi = \frac{1}{4} \delta_{ab} \mathcal{D}_c \tilde{\mathcal{D}}_c \Phi + \frac{1}{16} \sigma_{ab}^{MN}(1) \mathcal{D}_c \sigma_{cd}^{MN}(1) \tilde{\mathcal{D}}_d \Phi . \]  

(4.11)

Not surprisingly, the solution to these constraints is the real part of (4.6). The components \( \chi^j \) and \( \tilde{\chi}^j \) that appear at level \( \theta^2 \) and \( \bar{\theta}^2 \) due to (4.9) and (4.10) satisfy

\[ \partial \tilde{\chi}^j - \overline{\partial} \chi^j = 0 \]  

(4.12)

as a consequence of (4.11). Hence we may write

\[ \chi^j = \partial X^j \quad \text{and} \quad \tilde{\chi}^j = \overline{\partial} X^j , \]  

(4.13)

up to zero modes. As long as we only work with \( \Phi \) and its derivatives, that ambiguity does not matter. This is the promised justification of the assignment \( \chi^j = \partial \varphi^j \) in section (3.1).
The action is the same as that for the vector superfield, namely

\[ S = -\frac{1}{32} \int d^2z \ d\theta_a d\bar{\theta}_b \left( \sigma^\mu_{ac} (-1) \Psi^c \sigma^\mu_{bd} (-1) \bar{\Psi}^d \right) \]

and hence off-shell supersymmetric. In the first version of the action the left- and rightmoving fermions carry opposite $SO(4)_1$-chirality, in the second one they have the same. The equality in (4.14) should be understood component by component. The components are then interpreted as belonging to different $SO(4)$-covariant objects.

### 4.3. Heterotic Strings

For vector fields $X^\mu(z, \bar{z}, \theta)$ we set the constraints (4.4), which guarantee that the action

\[ S = \frac{1}{8} \int dz^2 d\theta_a \left[ D_a X^\mu \bar{\partial} X^\mu \right] \]

\[ = \frac{1}{2} \int dz^2 \left( \partial X^\mu \bar{\partial} X^\mu - \psi_a \bar{\partial} \psi_a \right) \] (4.15)

is off-shell supersymmetric.

For scalar fields we need to introduce $\Phi(z, \bar{z}, \theta)$ and the selfdual antisymmetric tensor field $X^{\mu\nu}(z, \bar{z}, \theta)$ and demand

\[ D_a \sigma_{ab}^{MN} (1) D_b \Phi = 0 \quad \text{and} \quad D_a \sigma_{ab}^{\mu\nu} (-1) D_b \Phi = 2 \partial X^{\mu\nu} . \] (4.16)

The latter condition implies that the $\theta^2$-component of $\Phi$ is the derivative of a scalar. Now it is not difficult to verify the supersymmetry of the action

\[ S = \frac{1}{8} \int dz^2 d\theta_a \left[ D_a \Phi \bar{\partial} \Phi + \frac{1}{4} D_a X^{\mu\nu} \bar{\partial} X^{\mu\nu} \right] . \] (4.17)

The introduction of $X^{\mu\nu}$ enables us to introduce four bosonic zeromodes into a theory that contains a $w = 0$ $SO(4)$ scalar. This trick works of course also for the nonheterotic string. The result looks component by component exactly like a vector field, but the interpretation in terms of $SO(4)$-representations is different.
4.4. Ghosts

The action for ghost-type fields is simpler since there are no common modes for the left- and rightmoving fields. We impose the constraints (3.28) and (3.30) on superfields \( C(z, \bar{z}, \theta) \) and \( B^J(z, \bar{z}, \theta) \). Then we may use our construction of the BRST-charge \( Q \) to immediately write down the off-shell supersymmetric action

\[
S = -\frac{1}{12} \int dz^2 \left[ \mathcal{D}_a \sigma^J_{ab}(1) \mathcal{D}_b \left( C \bar{B}^J \right) + \frac{1}{2} \left( \mathcal{D}_a \sigma^J_{ab}(1) \mathcal{D}_b C \right) \bar{\mathcal{B}}^J \right. \\
- \left. \frac{1}{2} C \mathcal{D}_a \sigma^J_{ab}(1) \mathcal{D}_b \bar{\mathcal{B}}^J \right]_{\theta=0}
\]

The same construction works for all the other minimal ghosts. We impose, for example, (3.36) on \( C_a(z, \bar{z}, \theta) \) and \( B^\mu(z, \bar{z}, \theta) \), and consequently

\[
S = -\frac{1}{4} \int dz^2 \ d\theta_a \ \bar{\sigma}^\mu_{ab}(\bar{1}) \ C_b \ \bar{\partial} B^\mu
\]

is off-shell supersymmetric. The same is of course true for

\[
S = -\frac{1}{4} \int dz^2 \ d\theta_a \ C^M \sigma^M_{ab}(1) \ \bar{\partial} B_b
\]

where the ghost fields obey (3.37).

The above actions are proper for the heterotic string. In order to obtain the field content necessary for closed strings with separate left- and rightmoving supersymmetries, we simply add in the appropriate ghost system. Strictly speaking our ghost actions are in that case not off-shell supersymmetric.
5. The Lorentz Algebra for the D=6 Superstring

We will now employ our formalism for analyzing the light-cone superstring. Specifically, we will construct the Lorentz algebra without making explicit reference to the internal coordinates. Only the internal N=4 superconformal generators $L$, $G_a$ and $J_{\mu\nu} = \frac{1}{4} \sigma^\mu_{\nu} (1) J_{ab}$ enter the construction. Their operator products are:

\[
G_a(z_1) G_b(z_2) = \frac{4 \delta_{ab}}{z_{12}^3} + \frac{2 \delta_{ab}}{z_{12}} [L(z_1) + L(z_2)] + \frac{1}{z_{12}^2} [J_{ab}(z_1) + J_{ab}(z_2)] + : G_a(z_1) G_b(z_2) :
\]

\[
J_{\mu\nu}(z_1) J^{\rho\tau}(z_2) = - \frac{4}{z_{12}^2} \left[ \delta_{\mu\rho} \delta_{\nu\tau} + \frac{1}{2} \epsilon_{\mu\nu\rho\tau} \right] - \frac{4}{z_{12}} \delta_{[\mu} \left( J_{\nu]}^\rho (z_1) + J_{\nu]}^\tau (z_2) \right) + : J_{\mu\nu}(z_1) J^{\rho\tau}(z_2) :
\]

\[
J_{\mu\nu}(z_1) G_a(z_2) = - \frac{1}{z_{12}} \sigma^\mu_{ab} G_b(z_2) + : J_{\mu\nu}(z_1) G_a(z_2) :
\]

The equations above should also be read as a definition of normal ordered currents. We emphasize that the $SU(2)$-current $J_{\mu\nu}$ carries transverse spacetime indices even though it contains no spacetime coordinates. The supersymmetry generators are

\[
Q_a^+ = 2^{1/4} \int \frac{dz}{2\pi i} \left( \frac{\alpha^+}{z} \right)^{1/2} \psi_a(z) \]

\[
Q_a^- = \frac{1}{2^{1/4}} \int \frac{dz}{2\pi i} \left( \frac{z}{\alpha^+} \right)^{1/2} \left( G_a^\tau(z) + G_a(z) \right) .
\]

They satisfy the anticommutation relations

\[
\{ Q_a^+, Q_b^+ \} = \sqrt{2} \delta_{ab} \alpha^+ ; \quad \{ Q_a^-, Q_b^+ \} = \sigma^\mu_{ab} \int \frac{dz}{2\pi i} \partial x^\mu(z)
\]

\[
\{ Q_a^-, Q_b^- \} = - \sqrt{2} \delta_{ab} \int \frac{dz}{2\pi i} \partial x^-(z)
\]

with $\partial x^- = -(\alpha^+)^{-1} z (L^x + L - \frac{1}{2} z^{-2})$. The generators of Lorentz rotations read,
at $x_0^+ = 0$,

$$J^{+-} = \frac{1}{4} \alpha^+ \frac{\partial}{\partial \alpha^+},$$

$$J^{+\mu} = -\frac{1}{4} x_0^+ \alpha^+.$$

$$J^{\mu\nu} = \oint \frac{dz}{2\pi i} \left( \frac{1}{2} x^\mu(z) \partial x^\nu(z) + \frac{1}{16} \psi(z) \sigma^{\mu\nu} (-1) \psi(z) + \frac{1}{8} J^{\mu\nu}(z) \right)$$

$$J^{-\mu} = \oint \frac{dz}{2\pi i} \frac{z}{\alpha^+} \left( \frac{1}{2} x^\mu(z) \left[ L^x(z) + L(z) - \frac{1}{2z^2} \right] + \frac{1}{8} \psi \sigma^\mu (-1) \left[ G^x(z) + 2G(z) \right] + \frac{1}{4} J^{\mu\nu}(z) \partial x^\nu(z) \right).$$

The $x^\mu$-sector of the algebra is of course identical to the usual bosonic algebra.

Note that the current $\frac{1}{2} \psi \sigma^{\mu\nu} (-1) \psi$ appearing in $J^{\mu\nu}$ is not the $SU(2)$-current appearing in the N=4 algebra of the $(x, \psi)$-system. We have written down only the open string algebra, or the leftmoving sector of the closed string algebra. It is fairly obvious how to complete the generators for the various closed string theories.

The commutator algebra of the operators defined in (5.4) yields the desired result if $c = 6$ for the internal algebra and provided the nullvector condition

$$G \sigma^{\mu\nu} G - 4LJ^{\mu\nu} - J^{[\mu} \partial J^{\nu]} + 2 \partial^2 J^{\mu\nu} = 0$$

(5.5)

is satisfied. More precisely, we need only require that the zero mode of that operator vanish. (5.5) is related by supersymmetry transformations to the nullvector

$$J^{\mu\nu} J^{\rho\tau} - \frac{1}{6} \left( \delta^{\mu\nu}_{\rho\tau} + \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \right) J^{\lambda\sigma} J^{\lambda\sigma} = 0,$$

(5.6)

and its zero mode restricts the allowed $SU(2)$-representations of the ground state to carry either spin 0 or spin 1/2. An explicit example is furnished by dimensional
reduction of the D=10 superstring:

\[
\begin{align*}
L &= \frac{1}{2} \partial \phi^N \phi^N - \frac{1}{2} \lambda_a \partial \lambda_a \\
G_a &= \sigma_{ab}^N (-1) \lambda_b \partial \phi^N \\
J^{\mu
u} &= \frac{1}{2} \lambda_a \sigma_{ab}^{\mu
u} (-1) \lambda_b
\end{align*}
\]

(5.7)

One has to pay attention to the fact that there is a difference between current normal ordered operators and operators normal ordered with respect to the modes of $\phi$ and $\lambda$.

It is striking how little room there is for changing (5.4). Let us assume that the internal sector appears only through only elements of the N=4 algebra. We may not actually break $SO(4)$, so we must have an internal contribution to $J^{\mu
u}$. Hence an internal N=4 algebra with $SU(2)$-currents $J^K(z) = \frac{1}{4} \sigma^K_{ab}(1) J_{ab}$ is impossible. While $J^x_{ab}$ is selfdual in $[ab]$, $J_{ab} = \frac{1}{4} \sigma_{ab}^{\mu
u} (-1) J^{\mu
u}$ must be antiselfdual. Higher dimension operators such as $G \sigma^{\mu\nu} G : \partial x^\nu$ in $J^{-\mu}$ are disallowed since by global scaling they appear with additional powers of $\alpha^+$, and the Lorentz algebra requires homogeneity in $\alpha^+$. One might also try to introduce a dimension 2 primary vector $A^\mu(z)$ that can be added to $J^{-\mu}$ and commutes with itself to the nullvector (5.5). This type of mechanism is in fact realized for D=4. In D=6 such an attempt fails since the superpartner $S^\mu_a(z)$ of $A^\mu(z)$ has to satisfy $\sigma_{ab}^{[\mu} S^\nu_b] = 0$, and this implies $S^\mu_a = 0$. We are thus tempted to conjecture that (5.4) is the most general form of the Lorentz algebra for the superstring in flat six-dimensional spacetime.

Lastly, we wish to point out that if we commute $J^{-\mu}$ with $\partial x^\mu$ and $\psi_a$, we obtain local operators that contain the internal N=4 current multiplet. In this sense the (global) algebra of (5.2) and (5.4) together with the algebra of spacetime currents $\partial x^\mu$ and $\psi_a$ allows a reconstruction of the local N=4 current algebra. Essentially, then, the supersymmetry- and Lorentz-algebra are an equivalent formulation of that extended superconformal algebra.
6. Conclusions. What happens for N=8?

We have developed a superfield formulation for minimal N=4 superconformal field theory in some detail, with emphasis on the division algebra structure. We have found that there is a N=4 superconformal symmetry in the six-dimensional light-cone superstring, and that it is very directly related to spacetime supersymmetry. Our superfields are tailor made for this application.

Of course, what we really are after is the ten-dimensional superstring, and more specifically its description in a spacetime covariant superconformal gauge. Spacetime covariant superstring actions that have a local extended worldsheet supersymmetry have been constructed [31,19], but it remains to be seen whether they can be gauge fixed to sufficiently simple covariant 2-D field theories. In particular, the question of second-class constraints needs to be settled. The solution to those problems presumably requires a working knowledge of the appropriate superconformal field theory, and we have provided the necessary machinery for D=6.

We have not yet completed the superfield formulation of N=8 superconformal field theory. The construction principles are clear however, and we will describe them in words. The N=8 model carries much of the structure of the octonion algebra. That algebra is nonassociative, and apart from that fact the generalization from N=4 to N=8 is straightforward. A general N=8 scalar superfield may be constrained analogously to eq.(3.3), and the component fields consist of eight bosons and eight fermions, transforming in the desired way. The projection operators analogous to (2.13) contain instead of the $\varepsilon$-symbol a tensor antisymmetric in four spinor indices [37], constructed from the octonionic structure constants and projecting onto 7- and 21-dimensional subspaces in the antisymmetric product.

The analogue of $SU(2) \approx S^3$ for N=4 is $S^7$, the round seven-sphere. Of course, $S^7$ is not a group manifold anymore, and that has to be taken properly into account. The quaternionic structure constants have to be regarded as components of the parallelizing torsion tensor, and consequently the gamma-matrices also become
tensors on $S^7$ [38,32]. The associator terms that spoil the naive generalization from $N=4$ to $N=8$ are then interpreted as covariant derivatives of the torsion tensor on the sphere. We emphasize that the superconformal algebra we construct is neither of the field-dependent type [32,33,15] nor of the exceptional type [39].

The relationship between the light-cone super-Poincaré algebra and the world-sheet superconformal algebra that we discovered for $N=4$ carries over to the $N=8$ construction outlined above.

A more comprehensive analysis of octonionic conformal field theory will be forthcoming, as well as an expanded discussion of the relation between light-cone super-Poincaré algebras and $N=2,4,8$ superconformal field theory.

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REFERENCES

1. W. Siegel, Phys.Lett. 128B (1983), 397; Class.Quant.Grav. 2 (1985) 195.
2. M.B. Green and J.H. Schwarz, Phys.Lett. 136B (1984), 367.
3. R. Casalbuoni, Nuovo Cim. 33A (1976), 389.
4. P.A.M. Dirac, Lectures on quantum mechanics, Belfer Graduate School of Science, Yeshiva Univ. New York (1964).
5. I. Bengtsson and M. Cederwall, ITP Göteborg preprint 84-21 (1984).
6. E. Nissimov, S. Pacheva and S. Solomon, Nucl.Phys. B297 (1988), 349; Y. Eisenberg and S. Solomon, Nucl.Phys. B309 (1988), 709; S.J. Gates Jr et. al., Phys.Lett. 225B (1989), 44; M.B. Green and C. Hull, Phys.Lett. 225B (1989), 57;
R. Kallosh, *Phys. Lett.* **224B** (1989), 273;
F. Bastianelli, G.W. Delius and E. Laenen, *Phys. Lett.* **229B** (1989), 223.

7. A.K.H. Bengtsson, I. Bengtsson, M. Cederwall and N. Linden,
   *Phys. Rev.* **D36** (1987) 1766.

8. I. Bengtsson and M. Cederwall, *Nucl. Phys.* **B302** (1988), 81.

9. N. Berkovits, *Phys. Lett.* **247B** (1990), 45.

10. M. Cederwall, *J. Math. Phys.* **33** (1992), 388.

11. D.P. Sorokin, V.I. Tkach, D.V. Volkov and A.A. Zheltukhin,
    *Phys. Lett.* **216B** (89) 302.

12. A. Galperin and E. Sokatchev, (1992), Phys. Inst. Univ. Bonn preprint
    BONN-HE-92-07, [hep-th/9203051](http://arxiv.org/abs/hep-th/9203051).

13. R. Penrose and M.A.H. McCallum, *Phys. Rep.* **6C** (1972), 241;
    A. Ferber, *Nucl. Phys.* **B132** (1978), 55;
    T. Shirafuji, *Progr. Theor. Phys.* **70** (1983), 18.

14. W.T. Shaw, *Class. Quantum Grav.* **3** (1986), 753;
    P. Budinich, *Comm. Math. Phys.* **107** (1986), 455;
    M. Cederwall, *Phys. Lett.* **226B** (1989), 45.

15. N. Berkovits, *Nucl. Phys.* **B358** (1991), 169.

16. P. Ramond, *Phys. Rev.* **D3** (1971) 2415;
    A. Neveu and J. H. Schwarz, *Nucl. Phys.* **B31** (1971) 86;
    F. Gliozzi, J. Scherk, and D. Olive, *Phys. Lett.* **65B** (1976) 282;
    *Nucl. Phys.* **B122** (1977) 253.

17. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten,
    *Nucl. Phys.* **B258** (1985) 46;
    C. Hull and E. Witten, *Phys. Lett.* **160B** (1985) 398.
18. T. Banks, L.J. Dixon, D. Friedan and E. Martinec,  
**Nucl.Phys. B299** (1988) 613;  
T. Banks and L.J. Dixon, *Nucl.Phys. B307* (1988) 93.

19. F. Delduc, A. Galperin, P. Howe and E. Sokatchev, Phys. Inst. Univ. Bonn preprint BONN-HE-92-19, hep-th/9207050.

20. M. Ademollo et. al., *Phys.Lett. B62* (1976), 105;  
*Nucl.Phys. B111* (1976) 77.

21. M. Ademollo et. al., *Nucl.Phys. B114* (1976), 297.

22. the literature on extended superconformal field theory is enormous. A classification of unitary representations of the algebras we are interested in can be found in:  
M. Yu, *Phys.Lett. 196B* (1987) 345;  
T. Eguchi and A. Taormina, *Phys.Lett. 200B* (1987) 315.

23. K. Schoutens, *Nucl.Phys. B295* (1988) 634.

24. S. Matsuda and T. Uematsu, *Phys.Lett. B220* (1989) 413;  
*Mod.Phys.Lett. A5* (1989) 841.

25. for a review see:  
N.J. Hitchin, A. Karlhede, U. Lindström and M Rocek,  
*Comm.Math.Phys. 108* (1987) 535.

26. T. Kugo and P. Townsend, *Nucl.Phys. B221* (1983), 357.

27. A.Sudbery, *J.Phys. A17* (1984), 939.

28. I. Bengtsson, *Class.Quantum Grav. 4* (1987), 1143.

29. D.B. Fairlie and C.A. Manogue, *Phys.Rev. D36* (1987), 475.

30. C.A. Manogue and A. Sudbery, *Phys.Rev. D40* (1989), 4073.

31. F. Delduc, E. Ivanov and E. Sokatchev, Phys. Inst. Univ. Bonn preprint BONN-HE-92-11 (1992), hep-th/9204071.
32. F. Englert, A. Sevrin, W. Troost, A. Van Proyen and Ph. Spindel,
   \textit{J. Math. Phys} \textbf{29} (1988) 281.

33. F. Defever, W. Troost and Z. Hasiewicz,
   \textit{Class.Quant.Grav.} \textbf{8} (1991) 253; 257.

34. see for example refs. 32, 33 and:
   P.Goddard, W. Nahm, D.I. Olive, H. Ruegg and A. Schwimmer,
   \textit{Commun.Math.Phys.} \textbf{112} (1987) 385.

35. R.D. Schafer, An introduction to nonassociative algebras, New York (1964).

36. L. Brink, O. Lindgren and B.E.W. Nilsson, \textit{Nucl.Phys.} \textbf{B212} (1983), 401.

37. B. de Wit and H. Nicolai, \textit{Nucl.Phys.} \textbf{B231} (1984), 506;
   F. Gürsey and C.H. Tse, \textit{Phys.Lett.} \textbf{127B} (1983), 191;
   L. Castellani and N.P. Warner, \textit{Phys.Lett.} \textbf{130B} (1983), 47;
   S. Fubini and H. Nicolai, \textit{Phys.Lett.} \textbf{155B} (1985), 369.

38. M. Rooman, \textit{Nucl.Phys.} \textbf{B238} (1984) 501.

39. E.S. Fradkin and V.Ya. Linetsky, ICTP Trieste preprint IC/91/348 (1991);
   P. Bowcock, Enrico Fermi Inst. Chicago preprint EFI 92-09 (1992),
   \url{hep-th/9202061}.
We define
\[ \sigma^0_{ab}(e) = \delta_{ab} \quad \text{and} \quad \sigma^j_{ab}(e) = -2e\delta^0_{[a} \delta^j_{b]} - \epsilon_{j\alpha\beta}, \] (1.1)
where \( a = (0, \alpha), e = \pm 1 \) and \([ab] = \frac{1}{2}(ab - ba)\). Note the selfduality property
\[ \sigma^j_{ab}(e) = \frac{1}{2}(\delta^a_{cd} + \frac{e}{2}
\epsilon_{abcd})\sigma^j_{cd}(e), \] (2.1)
where \( \delta^a_{cd} = \delta^c_{d[a} \delta^a_{b]} \). For \( \sigma^\mu = (\sigma^0, \sigma^j) \) and \( \overline{\sigma}^\mu = (\sigma^0, -\sigma^j) \) the chiral Lorentz generators are (anti)selfdual:
\[ \sigma^{\mu\nu}(e) \equiv \sigma^{\mu
u}(e) = \frac{1}{2} \left[ \delta^{\mu\nu}_{\rho\tau} + \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \right] \sigma_{\rho\tau}(e) \]
(3.1)
\[ \overline{\sigma}^{\mu\nu}(e) \equiv \overline{\sigma}^{\mu\nu}(e) = \frac{1}{2} \left[ \delta^{\mu\nu}_{\rho\tau} - \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \right] \overline{\sigma}_{\rho\tau}(e) \]
(3.2)
The generators of the four different \( SU(2)'s \) are obtained by (anti)selfduality projections on the spinor indices, i.e. by choices of \( e = \pm 1 \). We set \( e^\mu = (1, e^j) \) as well as \( e_a = (1, e_\alpha) \) with the quaternion units \( e_j \) satisfying the same algebra as the \( \sigma^j \):
\[ e^i e^j = -\delta^{ij} + e^{ijk} e^k. \] (4.1)
Then the following identities hold:
\[ e^*_a e_b = \sigma^\mu_{ab}(-1) e^\mu = \overline{\sigma}^{\mu}_{ab}(-1) e^{\mu*} \]
(5.1)
\[ e_a e^*_b = \sigma^\mu_{ab}(1) e^\mu = \overline{\sigma}^{\mu}_{ab}(1) e^{\mu*} \]
(5.2)
\[ e_a e^\mu = \sigma^\mu_{ab}(-1) e_b \quad e_a e^{\mu*} = \overline{\sigma}^{\mu}_{ab}(-1) e_b \]
(6.1)
\[ e^*_a e^\mu = \sigma^\mu_{ab}(1) e^*_b \quad e^*_a e^{\mu*} = \overline{\sigma}^{\mu}_{ab}(1) e^*_b \]
(6.2)
\[ e^\mu e^{\nu*} = \delta^{\mu\nu} + \frac{1}{2} \left[ \delta^{\mu\nu}_{\rho\tau} + \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \right] e^\rho e^{\tau*} = \delta^{\mu\nu} - 2 \left[ \delta^{\mu\nu}_{0j} + \frac{1}{2} \epsilon^{\mu\nu0j} \right] e^j \]
(7.1)
\[ e^{\mu*} e^\nu = \delta^{\mu\nu} - \frac{1}{2} \left[ \delta^{\mu\nu}_{\rho\tau} - \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \right] e^{\rho*} e^{\tau} = \delta^{\mu\nu} + 2 \left[ \delta^{\mu\nu}_{0j} - \frac{1}{2} \epsilon^{\mu\nu0j} \right] e^j \]
(7.2)
Note that \( e^\mu e_a = (e^*_a e^{\mu*})^* \). The above equations provide a handy translation table for conversion from quaternion notation to \( \sigma\)-matrix language.