Large deviation bounds for the Airy point process

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Abstract

In this paper, we establish large deviation bounds for the Airy point process, thus providing a partial answer to a question raised in [10]. Specifically, we establish exponential tightness, large deviation lower bound and large deviation upper bound for “nice tubes” (with definition in the main text). The proof is based on the approximation of the Airy point process using Gaussian unitary ensemble (GUE) up to exponentially small probability. To establish the corresponding large deviation bounds for GUE, we refine some edge rigidity results in [5] up to tail probabilities, and establish some estimates for the diffusion associated with the stochastic Airy operator introduced in [18].

1 Introduction

The Airy point process, introduced by Tracy and Widom in [21], is a simple determinantal point process on $\mathbb{R}$ which arises as a limit of rescaled eigenvalue configuration of the Gaussian unitary ensemble (GUE) near its spectral edge. The correlation kernel of Airy point process is given by

$$K^{Ai}(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y} = \int_0^\infty Ai(x + r)Ai(y + r)dr. \quad (1.1)$$

In the sequel, we denote by $a_1 > a_2 > \cdots$ the random points of the Airy point process. For an introduction to determinantal point processes and in particular Airy point process, we refer to [1], Section 4.2. Moreover, in [18], the Airy point process is related to the stochastic Airy operator $H_\beta$ with $\beta = 2$ (the point configuration of Airy point process has the same distribution as negated stochastic Airy spectrum), and is further related to a diffusion via Riccati transform. We provide a brief introduction to the stochastic Airy operator at the end of this introduction.

Various properties of the Airy point process have been studied. To name a few, in [21], the distribution of the extreme point $a_1$ is studied. In their
work, it is shown that \( F(s) = \mathbb{P}(a_1 < s) \) can be written in terms of Hastings-McLeod solution of Painlevé II. In [20], a central limit theorem for the number of particles in an interval is established. In [6], rigidity of the Airy point process is established. In [8], the generating function of the Airy point process is studied and is related to coupled Painlevé II equations.

Airy point process is also related to the KPZ equation:

\[
\partial_t h = \frac{1}{2} \xi_{xx} h + \frac{1}{2} (\partial_x h)^2 + \xi(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

with \( \xi(t, x) \) being the space-time white noise. The KPZ equation is originally introduced in [14] as a model for random surface growth. It is related to various physical phenomena such as directed polymer in random environment [13] and interacting particle systems (such as TASEP). We refer to [16], [9], [17], etc. for details. In [4], a moment-matching identity between KPZ equation and Airy point process is established. Using this result, it is proposed in [11] that the large deviation principle of Airy point processes can be used to derive the lower tail of the KPZ equation. However, the derivation of large deviation principle of Airy point processes in [11] is based on Coulomb gas ideas and is not mathematically rigorous. In [10], estimates for Airy point process are used to obtain upper and lower bounds for the lower tail probability. In a further development in [22], the author uses the stochastic Airy operator (introduced in [13]) to obtain exact lower tail large deviations. However, the problem of large deviation principle for Airy point process remains open.

The main task of this paper is to establish large deviation bounds for the Airy point process, thus providing a partial answer to the question raised in [10]. The main strategy is to use an approximation of Airy point process by Gaussian unitary ensemble (GUE) (which is established in Theorem 2.4 below). Then combining some rigidity estimates for the Airy point process from the diffusion characterization (Proposition 3.1, 3.2) and some improvements on edge rigidity for \( \beta \) ensembles from [5] (Theorem 5.1), the corresponding large deviation bounds for GUE can be established. We remark here that from our proof, the conclusion is actually valid for the spectrum of stochastic Airy operators \( H_{\beta} \) with general parameter \( \beta \in \mathbb{N}_+ \). However, in order to simplify the notations, we just present the \( \beta = 2 \) case in the main arguments.

Below, we briefly introduce the stochastic Airy operator as introduced in [15]. Let \( D = D(\mathbb{R}^+) \) be the space of generalized functions (continuous dual of the space \( C_0^\infty \) of smooth and compactly supported test functions with the topology of uniform convergence of all derivatives on compacts). Also denote by \( H_{1}^{\text{loc}} \) the space of functions from \( \mathbb{R}^+ \) to \( \mathbb{R} \) such that for any compact set \( I \subset \mathbb{R} \), \( f'_{1I} \in L^2 \). The stochastic Airy operator \( H_{\beta} \) is then defined as a map from \( D \) to \( H_{1}^{\text{loc}} \) such that

\[
H_{\beta} f = -f'' + xf + \frac{2}{\sqrt{\beta}} f B',
\]

where \( B \) denotes the Brownian motion, and \( f B' \) is understood as the derivative
of $-\int_0^y B f' dx + f(y) B_y - f(0) B_0$. Moreover, if we define

$$L^* = \{ f : f(0) = 0, \|f\| < \infty \},$$

(1.4)

where $\|f\|^2 = \int_0^\infty ((f')^2 + (1 + x)f^2)dx$, then the eigenvalue-eigenfunction pairs are $(\lambda, f) \in \mathbb{R} \times L^*$ such that $H_\beta f = \lambda f$ with both sides interpreted as distributions. We refer to [18] and Section 4.5 of [1] for more details.

2 Main results

In this section, we present the main results of this article. Recall that we denote by $a_1 > a_2 > a_3 > \cdots$ the random points of the Airy point process. In order to study large deviations of the Airy point process, we define the scaled and space-reversed Airy point process empirical measure as

$$\nu_k := \frac{1}{k} \sum_{i \geq 1} \delta_{\frac{\sqrt{x}}{k} - \frac{4}{3} a_i} - \frac{1}{\pi} \sqrt{x} 1_{x \geq 0} dx.$$  

(2.1)

Note that $\nu_k$ is a locally finite signed measure on $\mathbb{R}$.

Throughout this article, we will use the following notion of distance which is related to weak convergence on $[-R, R]$.

**Definition 2.1.** For any two compactly supported finite signed Borel measures $\mu, \nu$ on $\mathbb{R}$, we define the distance

$$d_R(\mu, \nu) := \sup_{\text{supp}(f) \subset [-R, R], \|f\|_{L^1} \leq 1, \|f\|_{L^{\infty}} \leq 1} \left| \int f \mu - \int f \nu \right|.$$  

(2.2)

Now we set up the topology. We fix $R_0 \geq 1$, and take the topological space $\mathcal{X}$ to be finite signed Borel measures on $[-R_0, R_0]$ equipped with metric $d_{R_0}$. We also take $\mathcal{Y}$ to be the space of compactly supported finite signed Borel measures on $\mathbb{R}$, and let

$$\mathcal{Z} := \{ \mu \in \mathcal{Y} : d\mu + \frac{1}{\pi} \sqrt{x} 1_{x \geq 0} dx \text{ is a positive measure on } \mathbb{R} \}.$$  

(2.3)

We also have the following definition which will be related to the rate function.

**Definition 2.2.** For any $\mu \in \mathcal{Z}$, we define

$$I_0(\mu) := -\int \log(|x - y|) d\mu(x) d\mu(y) + \frac{4}{3} \int_{-\infty}^0 |x|^\frac{2}{3} d\mu(x).$$  

(2.4)

We establish exponential tightness in the following theorem.
Theorem 2.1. [Exponential tightness] There exist absolute constants \(c, C, K > 0\), such that for any \(\eta \geq 20\), \(R \geq 1\) and \(k \geq K\),

\[
P(|\nu_k|((−\infty, R]) \geq \eta R^2) \leq C \exp(-c\eta R^3 k^2).
\] (2.5)

Specializing to our topological set-ups, for any \(\eta \geq 20\), we take \(K_\eta := \{\mu \in X : |\mu|([-R_0, R_0]) \leq \eta R_0^2\}\). We have that \(K_\eta\) is a compact set in \(X\), and

\[
\lim_{\eta \to \infty} \limsup_{k \to \infty} \frac{1}{k^2} \log P(\nu_k \notin K_\eta) = -\infty.
\] (2.6)

For large deviation lower bound, we have the following result.

Theorem 2.2. [LDP lower bound] For any \(\mu \in X\), we have the LDP lower bound result:

\[
\liminf_{\delta \to 0+} \liminf_{k \to \infty} \frac{1}{k^2} \log P(d_{R_0}(\nu_k, \mu) \leq \delta) \geq -I(\mu),
\] (2.7)

where

\[
I(\mu) := \inf_{\tilde\mu \in \mathcal{Z}, \mu(R) = 0, \tilde\mu([-R_0, R_0]) = \mu} \{I_0(\tilde\mu)\}.
\] (2.8)

We also have an LDP upper bound result for “nice tubes”, which are tubes around measures that have nice endings. Specifically, we have the following definition.

Definition 2.3. We define \(\mathcal{W}\) to be the set of finite signed measures \(\mu\) on \(\mathbb{R}\) with the property that \(|\mu|(\mathbb{R}) < \infty\) and \(\lim_{R \to \infty} \log(R)|\mu|((-R, R)^c) = 0\).

We have the following theorem on LDP upper bound for nice tubes.

Theorem 2.3. [LDP upper bound for nice tubes] We have the following LDP upper bound result: for any \(\mu \in \mathcal{W}\), if \(\mu(R) = 0\), we have

\[
\limsup_{R \to \infty} \limsup_{\delta \to 0+} \limsup_{k \to \infty} \frac{1}{k^2} \log P(d_{R}(\nu_k, \mu) \leq \delta)
\]

\[
\leq \liminf_{R \to \infty} \int_{[-R, R]_2} \log(\max(|x-y|, \frac{1}{R})) d\mu(x) d\mu(y) - \int_{-\infty}^{0} \frac{4}{3} |x|^\frac{3}{2} d\mu(x).
\]

For any \(\mu \in \mathcal{W}\), if \(\mu(R) \neq 0\), we have for sufficiently small \(\delta > 0\) (depending only on \(\mu\)),

\[
\limsup_{R \to \infty} \limsup_{k \to \infty} \frac{1}{k^2} \log P(d_{R}(\nu_k, \mu) \leq \delta) = -\infty.
\]

Remark. Here we control the probability for “nice cubes”. If the probability for other types of tubes is also controlled, then combined with Theorem 2.1, 2.2 and 2.3, a full LDP for the Airy point process can be established. We would like to discuss this issue in a future work.
During the process of proving the above large deviation bounds for the Airy point process, we also obtain an approximation result of the Airy point process (more generally, spectrum of $H_\beta$ for $\beta \in \mathbb{N}_+$) using Gaussian $\beta$ ensemble up to exponentially small probabilities, which might be of independent interest. We present the approximation result in the following theorem.

**Theorem 2.4.** We denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the sorted eigenvalues of Gaussian $\beta$ ensemble, and define the scaled eigenvalues $\tilde{\lambda}_i = (\lambda_i - 2\sqrt{n})n^{-\frac{1}{2}}$ for $1 \leq i \leq n$. Suppose that $a_1 > a_2 > \cdots$ are the particles of negated spectrum of $H_\beta$. Suppose that $k$ satisfies the bound $n^e \leq k \leq n^{10^{-3}}$ for some $e \in (0, \frac{1}{1000})$. Then for $\beta \in \mathbb{N}_+$, there is a coupling of Gaussian $\beta$ ensemble and the stochastic Airy operator $H_\beta$, such that for any $n$ and any $1 \leq i \leq k$,

$$|\tilde{\lambda}_i - a_i| \leq C_1 n^{-\frac{1}{2}}$$

with probability $\geq 1 - C_2 \exp(-ck^3)$ ($C_1, C_2, c$ are constants which only depend on $\beta$).

The rest of the sections are devoted to the proofs of Theorem 2.1-2.4.

### 3 Estimates for the Airy point process

In this section, we derive some estimates for the Airy point process (and more generally, spectrum of $H_\beta$). The main tool we will use is the diffusion characterization of [18]. Namely, the diffusion associated with the stochastic Airy operator $H_\beta$ is

$$dp(x) = (x - \lambda - p^2(x))dx + \frac{2}{\sqrt{\beta}}dB_x,$$

$$p(0) = \infty.$$

Here our sign for the term $\frac{2}{\sqrt{\beta}}dB_x$ is different from that of [18], but this will not make an essential difference.

The relation between stochastic Airy operator and the diffusion is that for almost all $\lambda$, the number $N(\lambda)$ of eigenvalues of $H_\beta$ that are less than or equal to $\lambda$ is the number of blow-ups of the diffusion $p(x)$ (see Proposition 3.4 in [18]).

We recall from Proposition 4.6 in [11] that the eigenvalues (denoted by $\gamma_i$ in this section only) of the Airy operator $A = -\partial_x^2 + x$ are given by

$$\gamma_i = \left(\frac{3\pi}{2}(i - \frac{1}{4} + R(i))\right)^{\frac{1}{2}},$$

where we have $|R(i)| \leq \frac{C}{i^3}$ for some absolute constant $C$. Below we denote by $N_0(\lambda)$ the number of eigenvalues of the Airy operator that are less than or equal to $\lambda$.

We establish the following two propositions which provide exponentially small upper bounds on the probability that the number of eigenvalues of $H_\beta$ in a certain region deviates too far from that of $A$. 

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Proposition 3.1. There exist constants $K, C, c > 0$ (which only depend on $\beta$), such that for any $\eta \geq 15$, $R \geq 1$ and $k \geq K$, we have
\[ P(N(Rk^2) \geq \eta R^3 k) \leq C \exp(-c\eta R^3 k^2). \] (3.2)

Proposition 3.2. There exist constants $K, C, c > 0$ (which only depend on $\beta$), such that for any $k \geq K$ and $k'$ satisfying $\log^3 k \leq k' \leq k$, if $\gamma_k \leq \lambda < \gamma_{k+1}$, then we have
\[ P(|N(\lambda) - N_0(\lambda)| \geq k') \leq \exp(Ck \log k) \exp(-ck'^2 \log(\frac{k'}{k})). \] (3.3)

Before the proof of Proposition 3.1 and 3.2, we state and prove several lemmas.

Lemma 3.1. Suppose we have a diffusion given by
\[ dp(x) = (x - a - p^2(x))dx + 2\sqrt{\beta}dB_x, \]
\[ p(0) = \infty, \]
where $a > (12\pi)^{\frac{4}{3}}$. We denote by $\Delta$ the blow-up time of this diffusion. Then for any $\epsilon, \delta \in (0, 1)$, on the event
\[ C_{\epsilon, \delta} := \{ \sup_{0 \leq x \leq \min\{\frac{2\pi}{\sqrt{a}}, \Delta\}} |B_x| \leq \sqrt{\frac{\beta a \epsilon \delta}{8}} \}, \]
we have
\[ \frac{\pi}{\sqrt{(1 + \epsilon)(1 + \delta)a}} \leq \Delta \leq \frac{\pi}{\sqrt{(1 - \epsilon)(1 - \delta)a - \frac{2\pi}{\sqrt{a}}}}. \]
Moreover, we have
\[ P(C_{\epsilon, \delta}) \leq 4 \exp\left(-\frac{\beta \delta a \epsilon \delta}{32\pi}\right). \]

Proof. We let $r(x) = p(x) - \frac{2}{\sqrt{\beta}}B_x$. Then $r(x)$ satisfies the ODE $dr(x) = (x - a - r(x) + \frac{2}{\sqrt{\beta}}B_x)^2 dx, r(0) = \infty$. For any $\epsilon \in (0, 1)$, we have $(r(x) + \frac{2}{\sqrt{\beta}}B_x)^2 \leq (1 + \epsilon)r^2(x) + (1 + \frac{1}{2})\frac{4B_x^2}{\beta}$. Thus on event $C_{\epsilon, \delta}$, when $x \in [0, \min\{\frac{2\pi}{\sqrt{a}}, \Delta\}]$, $r(x)$ is lower bounded by $w(x)$ defined by $dw(x) = (-1 + \delta)a - (1 + \epsilon)w^2(x)dx, w(0) = \infty$. Since $w(x)$ has blow-up time $\frac{\pi}{\sqrt{(1 + \epsilon)(1 + \delta)a}}$, we have $\Delta \geq \frac{\pi}{\sqrt{(1 + \epsilon)(1 + \delta)a}}$. Similarly, on event $C_{\epsilon, \delta}$, when $x \in [0, \min\{\frac{2\pi}{\sqrt{a}}, \Delta\}]$, $r(x)$ is upper bounded by $\tilde{w}(x)$ defined by $d\tilde{w}(x) = (\frac{2\pi}{\sqrt{a}} - (1 - \delta)a - (1 - \epsilon)\tilde{w}^2(x))dx, \tilde{w}(0) = \infty$. This leads to $\Delta \leq \frac{\pi}{\sqrt{(1 - \epsilon)(1 - \delta)a - \frac{2\pi}{\sqrt{a}}}}$. The probability estimate follows from standard estimate for Brownian motions. \qed
Lemma 3.2. Suppose we have a diffusion given by
\[ dp(x) = (x - a - p^2(x))dx + \frac{2}{\sqrt{\beta}} dB_x, \]
\[ p(0) = \infty, \]
where \( a < 0 \). We denote by \( \Delta \) the blow-up time of this diffusion. Then we have
\[ \mathbb{P}(\Delta < \frac{1}{2\sqrt{-a}}) \leq 4 \exp(-\frac{\beta(-a)^{\frac{3}{2}}}{32}). \]

Proof. We let \( r(x) = p(x) - \frac{2}{\sqrt{\beta}} B_x \). \( r(x) \) satisfies the ODE \( dr(x) = (x - a - (r(x) + \frac{1}{\sqrt{\beta}} B_x)^2)dx, r(0) = \infty \). We also let the event
\[ \mathcal{A} = \{ \sup_{0 \leq x \leq \frac{\sqrt{\beta\sqrt{-a}}}{4}} |B_x| \leq \frac{\sqrt{\beta\sqrt{-a}}}{4} \}. \]

On \( \mathcal{A} \), and for \( x \in [0, \min\{\Delta, \frac{1}{\sqrt{\beta\sqrt{-a}}}\}] \), \( r(x) \) is lower bounded by \( w(x) \) defined by \( dw(x) = (\frac{1}{2}a - 2w^2(x))dx, w(0) = \infty \). We let \( \Delta' = \inf\{x : w(x) = \sqrt{-a}\} \). By solving the ODE, we obtain \( \Delta' \geq \frac{1}{2\sqrt{-a}} \). Hence \( \Delta \geq \frac{1}{\sqrt{\beta\sqrt{-a}}} \).

Moreover, we have by standard estimate for Brownian motions,
\[ \mathbb{P}(\mathcal{A}^c) \leq 4 \exp(-\frac{\beta(-a)^{\frac{3}{2}}}{32}). \]  
(3.4)

This concludes the proof of the lemma. \( \square \)

Lemma 3.3. Consider the ODE
\[ dq(x) = (x - a - q^2(x))dx, \]
\[ q(0) = \infty. \]

where \( a > 0 \). Let \( \Delta > 0 \) be the minimal positive number such that \( p(\Delta) = -\infty \). Then we have \( \Delta \geq \frac{\pi}{\sqrt{a}} \). Moreover, if \( a \geq (4\pi)^{\frac{3}{2}} \), then we also have \( \frac{\pi}{\sqrt{a}} \leq \Delta \leq \frac{2\pi}{\sqrt{a} - \frac{a}{2\sqrt{a}}} \).

Proof. \( q(x) \) is lower bounded by \( r(x) \) which is defined as below:
\[ dr(x) = (-a - r^2(x))dx, \]
\[ r(0) = \infty. \]

Since \( r(x) \) has blow-up time \( \frac{\pi}{\sqrt{a}} \), we have \( \Delta \geq \frac{\pi}{\sqrt{a}} \).

Now suppose \( a \geq (4\pi)^{\frac{3}{2}} \), then \( q(x) \) is dominated by \( w(x) \) when \( 0 \leq x \leq \frac{2\pi}{\sqrt{a}} \).
\[ dw(x) = (\frac{2\pi}{\sqrt{a}} - a - w^2(x))dx, \]
\[ w(0) = \infty. \]

The blow-up time for \( w(x) \) is \( \frac{\pi}{\sqrt{a} - \frac{a}{2\sqrt{a}}} \leq \frac{2\pi}{\sqrt{a}} \), hence \( \Delta \leq \frac{2\pi}{\sqrt{a} - \frac{a}{2\sqrt{a}}} \) \( \square \).
Now we give the proof of Proposition 3.1 and 3.2.

**Proof of Proposition 3.1.** Let \( N_1 \) and \( N_2 \) be the number of blow-ups of the diffusion \( p(x) \) with \( \lambda = R^2 k \) in intervals \([0, 2R^2 k]\) and \((2R^2 k, \infty)\), respectively (we only count blow-ups of the diffusion that lies entirely in the interval). Then we have \( N(R^2 k) \leq N_1 + N_2 + 1 \). We define the stopping times \( \tau_1 < \tau_2 < \cdots \) to be the blow-up time of \( p(x) \), and we further define the stopping times \( \tau_1' < \tau_2' < \cdots \) to be the blow-up time of \( p(x) \) in \((2R^2 k, \infty)\). We also let \( \tau_0 = 0 \), and let

\[
\Delta_i = \tau_{i+1} - \tau_i.
\]

We define the events \( C_1 = \{ N_1 \geq \frac{1}{\epsilon} R^2 k \} \) and \( C_2 = \{ N_2 \geq \frac{1}{\epsilon} R^2 k \} \). Then

\[
\mathbb{P}(N(R^2 k) \geq \eta R^2 k) \leq \mathbb{P}(C_1) + \mathbb{P}(C_2).
\]

In Lemma 3.1 we take \( \epsilon = \delta = \frac{1}{6} \) and \( a = R^2 k \). On each \([\tau_i, \tau_{i+1}],\) conditioning on \( \mathcal{F}_{\tau_i} \), by the strong Markov property, we have that for \( x \in [0, \tau_{i+1} - \tau_i] \),

\[
p(x + \tau_i) \text{ is lower bounded by } q(x) \text{ defined as}
\]

\[
dq(x) = (-R^2 k - q^2(x))dx + \frac{2}{\sqrt{\beta}} dB_x,
\]

\[
q(0) = \infty,
\]

where \( B_x = B_{x + \tau_i} - B_{\tau_i} \).

Now we define the events

\[
\mathcal{A}_i = \{ \sup_{0 \leq x \leq \min(\frac{512}{\epsilon^2}, \tau_{i+1} - \tau_i)} |B_{x + \tau_i} - B_{\tau_i}| \leq \frac{1}{\delta} \sqrt{\frac{\beta a}{2}} \}.
\]

By Lemma 3.1 and strong Markov property, on the event \( \mathcal{A}_i, \Delta_i \geq \frac{4 \pi}{\sqrt{\beta a}} \). Moreover, \( \mathbb{P}(\mathcal{A}_i | \mathcal{F}_{\tau_i}) \leq 4 \exp\left(-\frac{\beta a}{3.12^2}\right) \). We denote \( p_0 := 4 \exp(-\frac{\beta a}{3.12^2}) \) below.

Now if we denote \( l_0 = \lfloor \frac{1}{\epsilon} R^2 k \rfloor \) and \( l = \lfloor \frac{1}{\epsilon} R^2 k - \frac{5}{2} R^2 k \rfloor (\geq \frac{1}{10} R^2 k) \), then

\[
C_1 \subset U_{0 \leq i_1 < \cdots < i_l \leq l_0}(\mathcal{A}^c_{i_1} \cap \cdots \cap \mathcal{A}^c_{i_l}),
\]

since otherwise we have

\[
2R^2 k \geq \sum_{1 \leq i \leq l_0} \Delta_i \geq \frac{5}{\pi} R^2 k \cdot \frac{4 \pi}{\sqrt{\beta a}} = 4R^2 k,
\]

a contradiction. Now by iterative conditioning we have for any \( 0 \leq i_1 < \cdots < i_l \leq l_0, \)

\[
\mathbb{P}(\mathcal{A}^c_{i_1} \cap \cdots \cap \mathcal{A}^c_{i_l}) = \mathbb{E}[\exp(|1_{\mathcal{A}^c_{i_1}} \cdots \exp(|1_{\mathcal{A}^c_{i_l}}|F_{\tau_i}) \cdots |F_{\tau_{i_l}}])]
\]

\[
\leq p_0 \mathbb{E}[\exp(|1_{\mathcal{A}^c_{i_1}} \cdots \exp(|1_{\mathcal{A}^c_{i_{l-1}}}|F_{\tau_{i_{l-1}}}) \cdots |F_{\tau_{i_l}}])]
\]

\[
\leq \cdots \leq p_0^l.
\]

Hence we have union bound

\[
\mathbb{P}(C_1) \leq 2^{l_0} p_0^l \leq 2^{\frac{1}{\epsilon}} \eta R^2 k \exp\left(-\frac{\beta a}{4096}\right),
\]

(3.5)
Similarly, on $[\tau'_1, \tau'_{i+1}]$, conditioning on $\mathcal{F}_{\tau'_i}$, by the strong Markov property, we have for $x \in [0, \tau'_{i+1} - \tau'_i]$, $p(x + \tau'_i)$ is lower bounded by $r(x)$ defined as

$$dr(x) = (Rk^2 - r^2(x))dx + \frac{2}{\sqrt{\beta}}d\tilde{B}_x, \quad r(0) = \infty,$$

where $\tilde{B}_x = B_{x+\tau'_i} - B_{\tau'_i}$. We also define the events $\mathcal{B}_i = \{\tau'_{i+1} < \infty\}$.

By iterative conditioning, we get

$$\lambda$$

eigenvalues of the Airy operator that are less than or equal to

$$\text{composition and comparison with Airy operator. Note that the numb}$$

er of

$$\text{Proof of Proposition 3.2. The proof of the proposition is based on dyadic de-}$$

composition and comparison with Airy operator. Note that the number of eigenvalues of the Airy operator that are less than or equal to $\lambda$ is determined by the number of blow-ups of the ODE

$$dq(x) = (x - \lambda - q^2(x))dx$$

$$q(0) = \infty.$$

In the following, we assume that $k$ is sufficiently large.

We fix $d = 120$ below. We denote by $N_1, N_2, N_3$ the number of blow-ups of $p(x)$ in intervals $[0, \lambda - \gamma(\frac{\lambda}{k})], [\lambda - \gamma(\frac{\lambda}{k}), \lambda + \gamma(\frac{\lambda}{k})]$ and $[\lambda + \gamma(\frac{\lambda}{k}), \infty)$, respectively. We also denote by $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$ the corresponding number of blow-ups for $q(x)$. We have $N(\lambda) = N_1 + N_2 + N_3$ and $N_0(\lambda) = \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3$. Note that $\tilde{N}_3 = 0$. Moreover, we have $\tilde{N}_2 \leq 4\lceil \frac{k}{T} \rceil \leq \frac{k}{20}$.

We first bound $\mathbb{P}(N_2 \geq \frac{k}{T})$. We denote by $\tau'_1 < \tau'_2 < \cdots < \tau'_{\lceil \frac{k}{T} \rceil}$, the first $\lceil \frac{k}{T} \rceil$ blow-up times of $p(x)$ in the interval $[\lambda - \gamma(\frac{\lambda}{k}), \lambda + \gamma(\frac{\lambda}{k})]$. Note that conditioning on $\mathcal{F}_{\tau'_i}$, for $x \in [0, \tau'_{i+1} - \tau'_i]$, $p(x + \tau'_i)$ is lower bounded by the diffusion

$$dr(x) = (x - \gamma(\frac{\lambda}{k}) - r^2(x))dx + \frac{2}{\sqrt{\beta}}d\tilde{B}_x, \quad r(0) = \infty,$$
where $\tilde{B}_q = B_{q+\tau_q} - B_{\tau_q}$. Note that $\frac{3\pi k'}{4} \leq (\gamma_{i,q})^\frac{2}{3} \leq \frac{3\pi k'}{2}$, by a similar argument as in the proof of Proposition 2.1, we have for some positive constants $c, C$,

$$\mathbb{P}(N_2 \geq \frac{k'}{4}) \leq C \exp(-ck'^2).$$

(3.6)

Then we bound $\mathbb{P}(N_3 \geq \frac{k'}{4})$. By a similar argument as in the proof of Proposition 2.1 using Tracy-Widom tail, we have

$$\mathbb{P}(N_3 \geq \frac{k'}{4}) \leq (C \exp(-\frac{2}{3} \beta(\gamma_{i,q}^\frac{2}{3}))^{\frac{k'}{4}} \leq C \exp(-ck'^2).$$

(3.7)

Finally, we bound $\mathbb{P}(|N_1 - \tilde{N}_1| \geq \frac{k'}{2})$. First we decompose the interval $[0, \lambda - \gamma_{i,q}]$ as follows. We denote by $I$ the integer such that $\gamma_{i,\frac{2I+1}{4}+1} \leq \lambda \leq \gamma_{i,\frac{2I+1}{4}}$. Note we have $I_1 + 1 - \log(\frac{\lambda}{\delta}) \leq 1$. For $0 \leq i \leq I$, we define the interval $J_i = [\lambda - \gamma_{i,\frac{2I+1}{4}} - 1, \lambda - \gamma_{i,\frac{2I+1}{4}} + 1]$, and we define $J_{I+1} = [0, \lambda - \gamma_{i,\frac{2I+1}{4}} + 1]$. We denote by $M_i$ the number of blow-ups of $p(x)$ in $J_i$, and $M_i$ the number of blow-ups of $q(x)$ in $J_i$. Moreover, we denote by $M_i'$ and $M_i''$ the number of blow-ups of the adapted versions $p_i(x)$, $q_i(x)$ of $p(x)$, $q(x)$. Here, by adapted versions we mean that for $x \in J_i = [a_i, b_i]$, $p_i(x)$ and $q_i(x)$ are given by $dp_i(x) = (x - \lambda - \tilde{p}_i(x))dx + \frac{\sqrt{h_i^2}}{\sqrt{h_i}}dB_x$, $p_i(a_i) = \infty$ and $dq_i(x) = (x - \lambda - q_i^2(x))dx$, $q_i(a_i) = \infty$. By monotonicity, we have $|M_i - M_i'| \leq 1, |M_i' - M_i''| \leq 1$. We also note that the $I + 1$ random variables $\{M_i'\}_{i=1}^{I+1}$ are mutually independent.

We let $\Delta_i = |M_i' - M_i''|$. We denote by $\alpha_i = \frac{2k'}{4}$. We conveniently let $t_i = M_i'$. By Lemma 5.3 there exists positive constants $c_1, c_2$ such that $c_1 \alpha_i \leq t_i \leq c_2 \alpha_i$ for $1 \leq i \leq I$, and $t_{I+1} \leq c_2 \alpha_{I+1}$. Now without loss of generality, we can combine $J_{I+1}$ into $J_I$. Thus on the event $\{|N_1 - \tilde{N}_1| \geq \frac{k'}{4}\}$, since $|M_i - M_i| \leq \Delta_i + 2$, we have $\frac{k'}{4} \leq |N_1 - \tilde{N}_1| \leq \sum_{i=1}^{I+1} \Delta_i + 2(I + 1)$. For $k$ sufficiently large, this leads to $\sum_{i=1}^{I+1} \Delta_i \geq \frac{k'}{8}$.

Now we do a dyadic decomposition. First note that for sufficiently large $M$, we have by Lemma 3.1

$$\mathbb{P}(\Delta_i \geq Mk) \leq \mathbb{P}(M_i' \geq Mk) \leq C \exp(-cMk^2).$$

(3.8)

By taking a union bound, we have

$$\mathbb{P}(\Delta_i \geq Mk \text{ for some } i) \leq Ck \exp(-cMk^2) \leq C \exp(-c\frac{k'^2}{\log(\frac{k'}{4})}).$$

(3.9)

Thus without loss of generality we can assume that either $\Delta_i = 0$, or

$$\delta_i \leq \Delta_i \leq 2\delta_i,$$

(3.10)

for some $\delta_i \in \{1, 2, \ldots, 2^{\log_2(Mk) - 1}\}$, as by the preceding argument the probability that there exists $i$ such that $\Delta_i$ does not fall in these ranges is
\(C \exp(-c \frac{k^2}{\log(k^2)})\). We also note that taking a union bound will only involve a factor of \((C \log k)^{C \log \frac{1}{\epsilon}} \leq \exp(C \log^2 k)\).

From our assumption, we have \(\sum_{i=1}^{I+1} \delta_i \geq \frac{k'}{40}\). Also we note that by independence,

\[
P(\delta_i \leq \Delta_i \leq 2\delta_i \text{ for all } 1 \leq i \leq I + 1) = \prod_{i=1}^{I+1} P(\delta_i \leq \Delta_i \leq 2\delta_i).
\]

Now we analyze the event \(C_i = \{\delta_i \leq \Delta_i \leq 2\delta_i\}\) on interval \(J_i\). We denote by \(A_i = C_i \cap \{M_i^t > M_i^\tau\}\), and \(B_i = C_i \cap \{M_i^t < M_i^\tau\}\). To bound \(P(A_i)\), we denote by \(\tau_1 < \tau_2 < \cdots < \tau_t\) the first \(t_i\) blow-up times of modified \(p(x)\) on \(J_i\). We also denote by \(\Delta_{j,1}, \Delta_{j,2}, \Delta_{j,3}\) the \(j\)th blow-up time of modified \(p(x)\), shifted \(q(x)\) (so that the starting point is \(\tau_j\)) and modified \(q(x)\). Again, we do a dyadic decomposition. For \(1 \leq j \leq t_i\), we set \(\delta_{i,j}\) such that

\[
\frac{1}{2} \delta_{i,j} \leq \sup_{0 \leq x \leq \min\{\frac{I}{\alpha_i}, \tau_j, |\tau_j| - \tau_j\}} |B_{x + \tau_j} - B_x| \leq \delta_{i,j}.
\]

Here, we take the range of \(\delta_{i,j}\) to be between \(\frac{1}{2\epsilon} \) and \(k\). The probability that \(\delta_{i,j} > k\) for some \(i, j\) has probability \(\leq C \exp(-ck^2)\) as before. For \(\delta_{i,j} < \frac{1}{2\epsilon}\), we simply modify \(\delta_{i,j}\) such that \(\delta_{i,j} = 0\). Similar to the proof of Proposition 3.1 if we take \(\epsilon = \delta = \epsilon_j := \min\{\frac{c_\delta}{\alpha_i}, \frac{1}{100}\}\) in Lemma 3.1 when \(\delta_{i,j} \neq 0\), conditioning on \(F_{\tau_j}\), with probability \(\geq 1 - 4 \exp(-c\epsilon_j^2\alpha_i)\), we have

\[
\frac{\pi}{(1 + \epsilon)\sqrt{\lambda - \tau_j}} \leq \Delta_{j,1} \leq \frac{\pi}{(1 - 2\epsilon)\sqrt{\lambda - \tau_j}}.
\]

Note that by Lemma 3.2
\[
\frac{\pi}{\sqrt{\lambda - \tau_j}} \leq \Delta_{j,2} \leq \frac{\pi}{\sqrt{\lambda - \tau_j} \sqrt{\lambda - \tau_j}}.
\]

To take into account the \(\delta_{i,j} = 0\) case, we further define \(\tilde{\epsilon}_j := \frac{\epsilon_j}{\alpha_i} + \frac{1}{k^2\alpha_i}\). Thus we have \(\Delta_{j,1} - \Delta_{j,2} \leq \frac{C\tilde{\epsilon}_j}{\alpha_i^2} + \frac{C}{\alpha_i^2}\).

We denote by \(K = \sum_{j=1}^{t_i} \tilde{\epsilon}_j\). We note that by Lemma 3.3

\[
|\Delta_{j,2} - \Delta_{j,3}| \leq \frac{C}{\alpha_i} \sum_{k=1}^{j-1} |\Delta_{k,2} - \Delta_{k,3}| + \frac{\sum_{k=j-1}^{j} \tilde{\epsilon}_k}{\alpha_i^2} + \frac{1}{\alpha_i^2}.
\]

If we denote by \(S_j = \sum_{k=1}^{j} |\Delta_{k,2} - \Delta_{k,3}|\), we have

\[
S_j \leq (1 + \frac{C}{\alpha_i})S_{j-1} + C\left(\frac{K}{\alpha_i} + \frac{1}{\alpha_i^2}\right).
\]
Thus by solving this recursion,
\[ S_{t_i} \leq (1 + \frac{C}{\alpha_i})e^t (K + \frac{1}{\alpha_i^2}) \leq C(K + \alpha_i^{-\frac{1}{3}}). \] (3.14)

Suppose that there are \( \geq \frac{\delta_i}{4} \) diffusions with blow-up time \( < \frac{1}{\alpha_i^3} \), similar to the proof of Proposition 3.1, this event will have probability \( \leq \exp(-c\alpha_i\delta_i) \leq \exp(-ck_i^2\delta_i) \). Otherwise, we have
\[ K + \alpha_i^{-\frac{1}{3}} \geq \frac{c\delta_i}{\alpha_i^3}, \] (3.15)
which leads to
\[ \sum_{j=1}^{t_i} \epsilon_j \geq c\delta_i - C. \] (3.16)

Similar to the proof of Proposition 3.1 and using Cauchy-Schwarz inequality,
\[ \mathbb{P}(A_i) \leq \exp(-ck_i^2\delta_i) + \exp(Ct_i \log \log k) \exp(-c\sum_{j=1}^{t_i} \epsilon_j^2\alpha_i) \]
\[ \leq \exp(-ck_i^2\delta_i) + \exp(Ct_i \log \log k) \exp(-c\delta_i^2). \]

For bounding \( \mathbb{P}(B_i) \), replacing \( t_i \) by \( t_i - \delta_i \) in the previous argument, and the same bound can be obtained. We thus have
\[ \mathbb{P}(C_i) \leq \exp(-ck_i^2\delta_i) + \exp(Ct_i \log \log k) \exp(-\delta_i^2). \] (3.17)

By taking a union bound and applying Cauchy-Schwarz inequality, we obtain the conclusion of the proposition. \( \square \)

4 Airy approximation

In this section, we present an approximation argument for the Airy point process via edge limits of Gaussian \( \beta \) ensembles up to exponentially small probability. We recall that (see, for example, [1]) the Gaussian \( \beta \) ensemble of size \( n \) can be realized by the eigenvalues of a tri-diagonal symmetric matrix \( H_{\beta,n} \) with entries \( H_{\beta,n}(i,j) = 0 \) if \( |i-j| > 1 \), \( H_{\beta,n}(i,i) = \sqrt{\frac{2}{\beta}} \xi_i \) and \( H_{\beta,n}(i,i+1) = \sqrt{\frac{2}{\beta}} Y_i \). Here, \( \xi_i \) are i.i.d. \( N(0,1) \), \( Y_i \sim \chi_{\sqrt{\beta}} \) are independent and independent of \( \{\xi_i\} \). The joint distribution of \( H_{\beta,n} \) is given by \( C_n(\beta) \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \exp(-\frac{\beta}{4} \sum_{i=1}^{n} \lambda_i^2) \). When \( \beta = 2 \), under proper scaling it has the same distribution as GUE (Gaussian unitary ensembles).

The main purpose of this section is to prove Theorem 2.4. The strategy of the proof is as follows. First we show exponential decay of top \( k \) eigenfunctions of the stochastic Airy operator \( H_{\beta} \) and \( \beta \) ensemble of size \( n \) \( H_{\beta,n} \) up to exponentially
small probability, which is achieved by analyzing the diffusion associated with stochastic Airy operator as well as a discrete analogue. Using this information, we can couple \(H_\beta\) and \(H_{\beta,n}\) so that their eigenvalues are close up to exponentially small probability. The rest of this section is devoted to the details of the proof of Theorem 2.4.

4.1 Exponential decay of \(H_\beta\) eigenfunctions

In this section, we show exponential decay of \(H_\beta\) eigenfunctions. Specifically, we have the following proposition.

**Proposition 4.1.** Suppose \(f_1, f_2, \ldots, f_k\) are the top \(k\) eigenfunctions of the stochastic Airy operator \(H_\beta\). Then there exist constants \(C, c, T_0\) which only depend on \(\beta\), such that when \(T \geq T_0\), there is a measurable event \(A\) with

\[
P(A^c) \leq C \exp(-c(kT)^{4})
\]

(4.1)

satisfying the properties below: on \(A\), for every \(kT \leq x_1 \leq x_2\) and \(1 \leq i \leq k\),

\[
|f_i(x_2)| \leq |f_i(x_1)| \exp(-\frac{1}{3}(x_2^2 - x_1^2)),
\]

(4.2)

and \(f_i(x_1)\) and \(f_i(x_2)\) have the same sign; moreover, for every \(x \geq kT\) and \(1 \leq i \leq k\),

\[
|f'_i(x)| \leq 4\sqrt{x}|f_i(x)|.
\]

(4.3)

To prove Proposition 4.1, we first set up some notations and lemmas. We recall that (see [18]) for the eigenfunction \(f(x)\) of \(H_\beta\) corresponding to eigenvalue \(\lambda\), the Ricatti transform \(p(x) = f'(x)f(x)\) can be written as a diffusion

\[
dp(x) = (x - \lambda - p^2(x))dx + \frac{2}{\sqrt{\beta}}dB_x.
\]

(4.4)

Below we fix \(K_1, K_2, L_1, L_2, L_3\) as constants, and set \(D_1 = \{ (x,y) : y \geq \frac{4}{3}\sqrt{x} \}, D_2 = \{ (x,y) : y \geq \frac{1}{2}\sqrt{x} \}, D_3 = \{ (x,y) : y \leq -\frac{1}{2}\sqrt{x} \}, D_4 = \{ (x,y) : y \leq -\frac{4}{3}\sqrt{x} \}.\) We also set up some events:

\[
A_1 = \{ -K_1 \leq \lambda \leq K_2 \},
\]

\[
A_2 = \{ \text{There exists } x_1 \geq L_1, \text{such that } (x, p(x)) \in D_2 \text{ for all } x \geq x_1 \},
\]

\[
A_3 = \{ \text{There exists } x_1 \geq L_1, \text{such that } (x_1, p(x_1)) \in D_1 \},
\]

\[
A_4 = \{ \text{There exists } x_1 \geq L_2, \text{such that } (x_1, p(x_1)) \in D_3 \},
\]

\[
A_5 = \{ \text{There exists } x_1 \geq L_1, \text{such that } p(x_1) \leq -4\sqrt{x_1} \}.
\]

The following lemma roughly says that once \(p(x)\) is “trapped” in the region \(D_1\), it will never leave \(D_2\) up to exponentially small probability.
Lemma 4.1. Suppose we take $L_1 > 32K_2$. Then there exist constants $C, c > 0$ which only depend on $\beta$, such that

$$
P(A_1 \cap A_2 \cap A_3) \leq C \exp(-cL_1^\frac{1}{4}).
$$

By definition of eigenfunctions, this has the consequence

$$
P(A_1 \cap A_4) \leq C_1 \exp(-c_2L_1^\frac{1}{4}).
$$

Proof. We take $\Delta = L_1^\frac{1}{4}$. The strategy of the proof is to break the interval $[L_1, \infty)$ into $I_i := [L_1 + i\Delta, L_1 + (i+1)\Delta]$ and control $p(x)$ inductively.

For $i \geq 0$, we let $f(i) = \frac{1}{4} - \frac{1}{16}(i+1)^{-\frac{1}{4}}$, $\epsilon(i) = \frac{1}{4} - \frac{1}{8}(i+1)^{-\frac{1}{4}}$. We also let $M_i = (f(i+1) - \epsilon(i))\sqrt{L_1 + i\Delta}$. Note that $M_i \geq \frac{1}{16}(i+2)^{-\frac{1}{4}}\sqrt{L_1 + i\Delta}$. We also set the events

$$
\mathcal{E}_i = \{ \sup_{0 \leq t \leq \Delta} \{|B_{L_1+i\Delta+t} - B_{L_1+i\Delta}| > \frac{\sqrt{\beta}}{8} M_i \},
$$

$$
\mathcal{F}_i = \left\{ \inf_{0 \leq t \leq \Delta} \left\{ \frac{1}{32}(L_1 + i\Delta)t + \frac{2}{\sqrt{\beta}}(B_{L_1+i\Delta+t} - B_{L_1+i\Delta}) \right\} \leq -\frac{1}{64}(i+2)^{-\frac{1}{4}}\sqrt{L_1 + i\Delta} \right\}.
$$

For

$$
\mathcal{G}_i = \{ \inf_{t \geq 0} \left\{ \frac{1}{32}(L_1 + i\Delta)t + \frac{2}{\sqrt{\beta}}(B_{L_1+i\Delta+t} - B_{L_1+i\Delta}) \right\} \leq -\frac{1}{64}(i+2)^{-\frac{1}{4}}\sqrt{L_1 + i\Delta} \},
$$

we have $\mathcal{F}_i \subset \mathcal{G}_i$. We further set $\mathcal{E} = \cup_{i=0}^\infty \mathcal{E}_i$ and $\mathcal{F} = \cup_{i=0}^\infty \mathcal{F}_i$.

We estimate the probability of these events. Using the inequality

$$
P(\sup_{0 \leq t \leq x} |B_t| \geq y) \leq 4 \exp\left(-\frac{y^2}{2x}\right),
$$

we obtain that $\mathbb{P}(\mathcal{E}_i) \leq 4 \exp\left(-\frac{M_i^2}{2L_1^\frac{1}{4}\Delta}\right) \leq 4 \exp\left(-\frac{c\sqrt{L_1 + i\Delta}}{\Delta}\right)$. Applying Girsanov Theorem to drifted Brownian motions, we have $\mathbb{P}(\mathcal{G}_i) \leq \exp(-c(i+2)^{-\frac{1}{4}}\sqrt{L_1 + i\Delta}) \leq \exp(-c(L_1 + i\Delta)^{\frac{1}{4}})$. Hence $\mathbb{P}(\mathcal{F}_i) \leq \exp(-c(L_1 + i\Delta)^{\frac{1}{4}})$. Thus we have by union bound

$$
P(\mathcal{E} \cup \mathcal{F}) \leq 4 \sum_{i=0}^\infty \exp\left(-\frac{c\sqrt{L_1 + i\Delta}}{\Delta}\right) + \sum_{i=0}^\infty \exp(-c(L_1 + i\Delta)^{\frac{1}{4}})
$$

$$
\leq 5 \exp(-cL_1^{\frac{1}{4}}) \sum_{i=0}^\infty \exp(-c((L_1 + i\Delta)^{\frac{1}{4}} - L_1^{\frac{1}{4}}))
$$

$$
\leq C \exp(-cL_1^{\frac{1}{4}}).
$$
Below, we assume that the event $\mathcal{E}^c \cap F^c \cap A_1 \cap A_3$ holds. Since event $A_3$ is true, there exists $i_0$, such that for some $x_1 \in [L_1 + i_0 \Delta, L_1 + (i_0 + 1)\Delta)$, $p(x_1) \geq \frac{1}{2}\sqrt[3]{x_1}$. We prove by induction that for any $i \geq i_0$ and any $x \in [L_1 + i\Delta, L_1 + (i + 1)\Delta) \cap [x_1, \infty)$, $p(x) \geq (1 - (i + 1)\sqrt{x})$.

For $x \in [x_1, L_1 + (i_0 + 1)\Delta)$, since event $A_1$ holds, we have that $p(x)$ is lower bounded by $r(x)$ defined by $dr(x) = (x - K_2 - r^2(x))dx + \frac{2}{\beta}dB_x$, $r(x_1) = p(x_1)$. If there exists $x_2 \in [x_1, L_1 + (i_0 + 1)\Delta]$, such that $p(x_2) \leq (1 - f(i_0 + 1))\sqrt{x}$ and $p(x) \geq (1 - f(i_0 + 1))\sqrt{x}$ for all $x \in [x_1, x_2]$. We take $x_3 = \inf\{x \in [x_1, x_2] : p(y) \leq (1 - \epsilon(i))\sqrt{y} \text{ for all } y \in [x, x_2]\}$. By continuity, $p(x_3) = (1 - \epsilon(i))\sqrt{x_3}$.

Let $r(x) = p(x) - (1 - (i + 1)\sqrt{x})$. For $x \in [x_3, x_2]$, by definition, $0 \leq r(x) \leq f(i + 1) - \epsilon(i)\sqrt{x}$. Note that $dr(x) = dp(x) - \frac{1}{2\sqrt{x}}(1 - f(i + 1))dx$, recalling that event $A_1$ is true, we have for $x \in [x_3, x_2]$, $r(x)$ is lower bounded by $w(x)$ defined by $dw(x) = \frac{1}{16}(L_1 + i\Delta)dx + \frac{2}{\beta}dB_x$, $w(x_3) = r(x_3)$. From this, we obtain by definition of $\mathcal{E}_i$,

$$r(x_2) - r(x_3) \geq w(x_2) - w(x_3) \geq \frac{2}{\sqrt[3]{x_2}}(B_{x_2} - B_{x_3}) \geq -\frac{4}{\sqrt[3]{\beta}}\sup_{0 \leq t \leq \Delta}|B_{L_1 + i_0\Delta + t} - B_{L_1 + i_0\Delta}| \geq -\frac{1}{2}(f(i_0 + 1) - \epsilon(i_0))\sqrt{L_1 + i_0\Delta}.$$

However, we also have $r(x_2) - r(x_3) = -(f(i_0 + 1) - \epsilon(i_0))\sqrt{x_3} \leq -(f(i_0 + 1) - \epsilon(i_0))\sqrt{L_1 + i_0\Delta}$, which combined with the above estimate gives $(f(i_0 + 1) - \epsilon(i_0))\sqrt{L_1 + i_0\Delta} \leq 0$, which is a contradiction. Thus for $x \in [x_1, L_1 + i_0\Delta]$, $p(x) \geq (1 - f(i_0 + 1))\sqrt{x}$.

Now we consider the induction step, i.e. for $i > i_0$. By induction hypothesis, we have $p(L_1 + i\Delta) \geq (1 - f(i))\sqrt{L_1 + i\Delta}$. If there exists $x_2 \in [L_1 + i\Delta, (L_1 + (i + 1)\Delta)$ such that for any $x \in [L_1 + i\Delta, x_2]$, $p(x) \geq (1 - f(i + 1))\sqrt{x}$, and $p(x_2) \geq (1 - (i + 1)\sqrt{x_2}$, then since event $\mathcal{E}_i^c$ happens, we have $p(x) \geq (1 - f(i + 1))\sqrt{x}$ for all $x \in I_i$ (the argument is similar for $i = i_0$ case).

Now suppose that there exists $x_2 \in I_i$, such that for any $x \in [L_1 + i\Delta, x_2]$, $(1 - f(i + 1))\sqrt{x} \leq p(x) \leq (1 - \epsilon(i))\sqrt{x}$, and $p(x_2) \leq (1 - f(i + 1))\sqrt{x_2}$. By estimates on the drift coefficient, we conclude that for $x \in [L_1 + i\Delta, x_2]$, $r(x)$ is lower bounded by $\tilde{w}(x)$ defined by $d\tilde{w}(x) = \frac{3}{16}(L_1 + i\Delta)dx + \frac{2}{\beta}dB_x$, $\tilde{w}(L_1 + i\Delta) = r(L_1 + i\Delta)$. If we denote by $\tilde{B}_t = \frac{3}{16}(B_{L_1 + i\Delta + t} - B_{L_1 + i\Delta}) + \frac{2}{\beta}(L_1 + i\Delta)t$ for $t \geq 0$, we have $r(x_2) - r(L_1 + i\Delta) \geq \tilde{B}_{x_2 - (L_1 + i\Delta)}$. We thus obtain that $\inf_{t \in [0, \Delta]}\tilde{B}_t \leq -(f(i + 1) - f(i))\sqrt{L_1 + i\Delta} \leq -\frac{1}{64}(i + 2)^{-2}\sqrt{L_1 + i\Delta}$. This is a contradiction to the fact that $\mathcal{E}_i$ happens.

Therefore, we conclude that $p(x) \geq (1 - f(i + 1))\sqrt{x}$ for $x \in I_i$. By induction, we conclude the proof that for any $i \geq i_0$ and any $x \in I_i \cap [x_1, \infty)$, $p(x) \geq (1 - f(i + 1))\sqrt{x}$. In particular, $p(x) \geq \frac{1}{2}\sqrt{x}$ for any $x \geq x_1$. This results in

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) \leq \mathbb{P}(E \cup F) \leq C \exp(-cL_1^4). \quad (4.7)$$
Now on $A_2$, suppose that for $x \geq x_1$, $p(x) \geq \frac{1}{2}\sqrt{x}$. Without loss of generality, we assume that $f(x_1) > 0$. Now for $x \geq x_1$, $\log(f(x)) \geq \log(f(x_1)) + \frac{1}{4}(x^{\frac{3}{2}} - x_1^{\frac{3}{2}})$. This is a contradiction to the fact that $\int_0^\infty f^2(x)dx = 1$. Hence
\[
P(A_1 \cap A_3) \leq C \exp(-cL_1^\frac{4}{7}). \quad (4.8)
\]

The next lemma roughly says that when $p(x)$ deviates a bit upward from the curve $(x, -\sqrt{x})$, it will hit $D_1$ up to exponentially small probability. This combined with Lemma 4.1 shows that in this case, $p(x)$ cannot be an eigenfunction except for exponentially small probability.

**Lemma 4.2.** Suppose we take $L_2 \geq \max\{100K_2, L_1, 10^8\}$. Then we have for $C,c > 0$ only depending on $\beta$,
\[
P(A_1 \cap A_3^c \cap A_4) \leq C \exp(-c\sqrt{L_2}). \quad (4.9)
\]
This combined with Lemma 4.1 shows that
\[
P(A_1 \cap A_4) \leq C \exp(-cL_1^\frac{4}{7}) + C \exp(-c\sqrt{L_2}). \quad (4.10)
\]

**Proof.** We take $\Delta = L_2^\frac{1}{3}$, and divide the interval $[L_2, \infty)$ into infinitely many sub-intervals $I_i = [L_2 + i\Delta, L_2 + (i + 1)\Delta]$ (where $i \geq 0$). We further divide each $I_i$ into $[\sqrt{L_2 + i\Delta}, \sqrt{L_2 + i\Delta}]$ many closed sub-intervals $I_{i,j}$ such that the length of each $I_{i,j}$ is within $[\frac{1}{\sqrt{2\sqrt{L_2 + i\Delta}}}, \frac{1}{\sqrt{2\sqrt{L_2 + i\Delta}}}]$.

Now for $I_{i,j} = [a_{i,j}, b_{i,j}]$, we define the event
\[
C_{i,j} = \{ \sup_{t \in I_{i,j}} |B_t - B_{a_{i,j}}| \geq 1 \}. \quad (4.11)
\]
We have $P(C_{i,j}) \leq 4 \exp(-\frac{1}{4}\sqrt{L_2 + i\Delta})$. If we denote by $C = \cup C_{i,j}$, we have by a union bound $P(C) \leq \sum_{i=0}^{\infty} \Delta\sqrt{L_2 + i\Delta}P(C_{i,j}) \leq C \exp(-c\sqrt{L_2})$.

Below we assume that event $C^c$ is true. Assuming that event $A_1 \cap A_4$ is true, there exists $x_2 \geq L_2$ such that $p(x_2) \geq -\frac{1}{2}\sqrt{x}$. Suppose that $x_2 \in [L_1 + i\Delta, L_1 + (i + 1)\Delta]$. Recall the diffusion SDE $dp(x) = (x - \lambda - p^2(x))dx + \sqrt{\rho}dB_x$.

When $p(x) \in [-\frac{1}{4}\sqrt{x}, \frac{1}{4}\sqrt{x}]$, we have $x - \lambda - p^2(x) \geq \frac{1}{4}(L_2 + i\Delta)$. Below we show that $p(x)$ will not hit $D_4$ for any $x \in [x_2, x_2 + \frac{\sqrt{L_2 + i\Delta}}{\sqrt{L_2 + i\Delta}}]$. Suppose that $p(x_3) = -\frac{1}{4}\sqrt{x_3}$, then there exists $x_4 \in [x_2, x_3]$ such that $p(x_4) \geq -\frac{1}{4}\sqrt{x_4}$ and $p(x) \in [-\frac{1}{4}\sqrt{x}, \frac{1}{4}\sqrt{x}]$ for all $x \in [x_1, x_3]$. Thus we have $p(x_3) - p(x_4) \geq \frac{1}{4}(L_2 + i\Delta)(x_3 - x_4) + \frac{2}{\sqrt{\rho}}(B_{x_3} - B_{x_4})$. Moreover, we have $p(x_3) - p(x_4) \leq \frac{1}{4}\sqrt{x_4} - \frac{1}{4}\sqrt{x_3} \leq \frac{1}{4}\sqrt{L_2 + i\Delta} + 1 - \frac{1}{4}\sqrt{L_2 + i\Delta} \leq -\frac{1}{4}\sqrt{L_2 + i\Delta}$. By the above two estimates we have $B_{x_3} - B_{x_4} \leq -\frac{1}{\sqrt{\rho}}\sqrt{L_2 + i\Delta} \leq -1000$. However, by definition of $C$, we obtain that $|B_{x_3} - B_{x_4}| \leq 66$, which is a contradiction.
Thus if \( p(x) \) does not hit \( D_1 \) for \( x \in [x_2, x_2 + 16\sqrt{L_2 + i\Delta}] \), we have

\[
p(x_2 + \frac{16}{\sqrt{L_2 + i\Delta}}) \geq p(x_2) + \frac{1}{4}(L_2 + i\Delta)\frac{16}{\sqrt{L_2 + i\Delta}} - 66
\]

\[
\geq -\frac{1}{2}\sqrt{L_2 + (i + 1)\Delta} + 4\sqrt{L_2 + i\Delta} - 66
\]

\[
\geq \frac{4}{5}\sqrt{L_2 + (i + 1)\Delta},
\]

which is a contradiction. Hence we have

\[
P(A_1 \cap A_3^c \cap A_4) \leq P(C) \leq C \exp(-c\sqrt{L_2}). \quad (4.12)
\]

\[\square\]

The next lemma bounds \( P(A_1^c) \).

**Lemma 4.3.** There exist constants \( T_0, K, C, c > 0 \) that only depend on \( \beta \), such that when \( k \geq K \) and \( T \geq T_0 \), we have

\[
P(\lambda_k > kT) \leq C \exp(-ck^3T^2). \quad (4.13)
\]

Moreover, there exist constants \( C, c \) that only depend on \( \beta \), such that for any \( K_1 \geq 1 \),

\[
P(\lambda_k \leq -K_1) \leq C \exp(-cK_1^3). \quad (4.14)
\]

**Proof.** By Proposition 3.2 we have

\[
P(\lambda_k > kT) \leq P(N(kT) \leq k - 1)
\]

\[
\leq P(|N(kT) - N_0(kT)| \geq c(kT)^{\frac{2}{3}})
\]

\[
\leq \exp(C(kT)^{\frac{2}{3}} \log(kT)) \exp(-c(kT)^{3})
\]

\[
\leq C \exp(-ck^3T^2).
\]

Now by the Tracy-Widom tail bound in [12],

\[
P(\lambda_k \leq -K_1) \leq P(\lambda_1 \leq -K_1) \leq C \exp(-cK_1^3). \quad (4.15)
\]

\[\square\]

We also have the following lemma lower bounding \( p(x) \).

**Lemma 4.4.** Suppose \( L_1 \geq \max\{K_1, \frac{10^6}{\beta}\} \). There exist constants \( C, c > 0 \) that only depend on \( \beta \), such that

\[
P(A_1 \cap A_3^c \cap A_5) \leq C \exp(-c\sqrt{L_1}). \quad (4.16)
\]
Proof. We do the partition as in Lemma 4.2 with $L_2$ replaced by $L_1$. $C_{i,j}$ are similarly defined. Let $C = \bigcup_{i,j} C_{i,j}$. Similarly we have

$$\mathbb{P}(C) \leq C \exp(-c\sqrt{L_1}). \quad (4.17)$$

Below we assume that $A_1 \cap A_3 \cap C^c$ is true. Suppose that there exists $x_1 \geq L_1$, such that $p(x_1) \leq -4\sqrt{x_1}$. We assume that $x_1 \in [L_1 + j\Delta, L_1 + (j + 1)\Delta)$, and let $q(x) = p(x) + 2\sqrt{x}$. Thus we have $q(x_1) \leq -2\sqrt{x_1}$. Moreover, $q(x)$ satisfies the SDE $dq(x) = (x - \lambda - (q(x) - 2\sqrt{x})^2 + \frac{1}{\sqrt{x}})dx + \frac{2}{\sqrt{x}}dB_x$. As $\lambda \geq -K_1$, for $x \geq x_1$ $q(x)$ is upper bounded by $\tilde{q}(x)$ which is defined by $d\tilde{q}(x) = (x + K_1 - (\tilde{q}(x) - 2\sqrt{x})^2 + \frac{1}{\sqrt{x}})dx + \frac{2}{\sqrt{x}}dB_x$. Let $\tau = \inf\{t \geq x_1 : \tilde{q}(x) = q(x_1)\}$. Let $x \in [x_1, \tau]$, $\tilde{q}(x)$ is upper bounded by $w(x)$ defined by $dw(x) = -w^2(x + dx) + \frac{2}{\sqrt{x}}dB_x$, $w(x_1) = q(x_1)$ (this uses the fact that $L_1 \geq K_1$).

Now let $r(x) = q(x) - \frac{2}{\sqrt{x}}(B_x - B_{x_1})$. We have $r(x) \leq -2\sqrt{x_1}$. For $x \in [x_1, x_1 + 16\sqrt{x_1 + j\Delta}]$, as $C^c$ is true, we have $|B_x - B_{x_1}| \leq 66$. Thus $r(x)$ is upper bounded by $\tilde{r}(x)$ defined by $d\tilde{r}(x) = -\frac{1}{2}(\tilde{r}(x) + L_1 + j\Delta)dx$. $\tilde{r}(x)$ blows up to $-\infty$ for some time in $[x_1, x_1 + 16\sqrt{x_1 + j\Delta}]$. Hence for some time in $[x_1, x_1 + 16\sqrt{x_1 + j\Delta}]$, $p(x)$ blows up to $-\infty$, and restarts at $\infty$. Thus $A_2$ is true. Hence we have

$$\mathbb{P}(A_1 \cap A_2^c \cap A_3) \leq \mathbb{P}(C) \leq C \exp(-c\sqrt{L_1}). \quad (4.18)$$

We proceed to the proof of Proposition 4.1 below.

Proof of Proposition 4.1. We take $K_1 = K_2 = \frac{K}{\log x}$ and $L_1 = L_2 = kT$. We take $A = A_1 \cap A_2^c \cap A_3$ (here $A_k$ ($1 \leq k \leq 5$) are defined as the intersection of the corresponding event defined above for $f_1, \ldots, f_k$). By Lemma 4.1-4.3 we have $\mathbb{P}(A^c) \leq C \exp(-c(KT)^{\delta})$.

Below we assume that $A$ is true. We denote by $p_i(x) = \frac{f_i(x)}{f_i(x)}$ for $1 \leq i \leq k$. By the definition of $A_5$, $p_i(x)$ never blows up to $-\infty$ for $x \geq kT$. Hence on $[kT, \infty)$, $f_i(x)$ ($1 \leq i \leq k$) doesn’t change sign. By the definition of $A_4$, we have $p_i(x) \leq -\frac{1}{2}\sqrt{x}$ for $x \geq kT$. Thus we have

$$|f_i(x_2)| \leq |f_i(x_1)| \exp(-\frac{1}{3}(x_2^3 - x_1^3)). \quad (4.19)$$

for any $x_2 \geq x_1 \geq kT$.

Moreover, by the definition of $A_4$, $A_5$, we have $|\frac{f_i(x)}{f_i(x)}| = |p_i(x)| \leq 4\sqrt{x}$ for $x \geq kT$. Thus $|f_i(x)| \leq 4\sqrt{x}|f_i(x)|$ for any $1 \leq i \leq k$ and $x \geq kT$. \qed
4.2 Exponential decay of $H_{\beta,n}$ eigenfunctions

In this section, we show exponential decay of eigenfunctions of Gaussian $\beta$ ensembles $H_{\beta,n}$. The main strategy is to give a discrete analogue of the previous argument for $H_{\beta}$. The result is recorded in the following proposition.

**Proposition 4.2.** Assuming $\beta \in \mathbb{N}_+$, and $n^e \leq k \leq n^{1600}$ for some fixed $e \in (0,1)$. Suppose $\phi_1, \phi_2, \cdots, \phi_k$ are the top $k$ eigenfunctions of Gaussian $\beta$ ensemble $H_{\beta,n}$. Then there exist constants $C, \epsilon$ which only depend on $\beta$, and a constant $K$ which only depends on $e$ and $\beta$, such that when $k \geq K$, there is a measurable event $A$ with

$$\mathbb{P}(A^c) \leq C \exp(-ck^3) + C \exp(-cn^{\epsilon}),$$

(4.20)

satisfying the following property: on $A$, for every $i_2 \geq i_1 \geq n^{\frac{e}{2}} k^{12}$ and $1 \leq j \leq k$,

$$|\phi_j(x_2)| \leq |\phi_j(x_1)| \exp(-\frac{1}{12}(i_2 n^{-\frac{1}{4}})^2 - (i_1 n^{-\frac{1}{4}})^2)),$$

(4.21)

and $\phi_j(i_1)$ and $\phi_j(i_2)$ have the same sign.

We use the notation introduced at the beginning of this section. We denote by $\hat{H}_{\beta,n} = n^2 (H_{\beta,n} - 2\sqrt{n}I)$, and let ($\lambda, \phi$) be an eigenvalue-eigenfunction pair, i.e. $\hat{H}_{\beta,n} \phi = \lambda \phi$. (Note that the eigenfunctions of $H_{\beta,n}$ and $\hat{H}_{\beta,n}$ are the same.) Expanding the equation, we obtain that for $2 \leq i \leq n - 1$,

$$n^{\frac{2}{3}} (\sqrt{\frac{\beta}{\beta}} \xi(i) + \frac{Y_{\beta-i}}{\sqrt{\beta}} \phi(i+1) + \frac{Y_{\beta-i+1}}{\sqrt{\beta}} \phi(i-1) - 2\sqrt{n} \phi(i)) = \lambda \phi(i).$$

(4.22)

We set $Z_i = \frac{Y_{\beta-i}}{\sqrt{\beta}} - \sqrt{n-i}$. As $\beta \in \mathbb{N}_+$, we write $Y_{\beta-i} = \sqrt{\sum_{l=1}^{\beta(n-i)} W_{i,l}^2}$ with $W_{i,l} \sim N(0,1)$. This gives $Z_i = \frac{\sum_{l=1}^{\beta(n-i)} (W_{i,l}^2 - 1)}{\sqrt{\beta(n-i)}}$. We also denote by $\mu_i = \mathbb{E}[Y_{\beta-i}^2], \gamma_i = \frac{Y_{\beta-i}}{\sqrt{\beta}} - \mu_i$. Note that by Cauchy-Schwarz inequality, $\mu_i \leq \frac{1}{\sqrt{\beta}} \mathbb{E}[Y_{\beta-i}^2] \leq \sqrt{n-i}$, and thus $n^{\frac{2}{3}} (\sqrt{n} - \mu_i) \geq \frac{1}{8} n^\frac{1}{2}.

4.2.1 Some preliminary bounds on partial sums of $Z_i$

In this part, we develop some preliminary bounds on partial sums of $Z_i$.

**Lemma 4.5.** For any $M > 0$ and $1 \leq i \leq n - 1$, we have the tail bond

$$\mathbb{P}(|Z_i| \geq M) \leq 2 \exp(-\frac{\beta \min\{M^2, \sqrt{n-i} M\}}{8}).$$

(4.23)

Moreover, if we denote the truncated version $\tilde{Z}_i = \min\{Z_i, -M\}, M\}$, we have the following bounds on moments

$$|E[\tilde{Z}_i] - E[Z_i]| \leq \frac{16}{\beta} \exp(-\frac{\beta M}{8}),$$

(4.24)

$$E[\tilde{Z}_i^2] \leq E[Z_i^2] \leq \frac{2}{\beta}.$$
Proof. Without loss of generality we only show the upper side of (4.23). By Markov’s inequality, we have
\[ P(Z_i \geq M) \leq \mathbb{P}\left( \sum_{l=1}^{(n-i)/\beta} (W_{i,l}^2 - 1) \geq \beta\sqrt{n-\beta n} \right) \]
\[ \leq \exp(-\theta \beta \sqrt{n-\beta n}) \prod_{l=1}^{(n-i)/\beta} \mathbb{E}[\exp(\theta(W_{i,l}^2 - 1))] \]
\[ = \exp(-\theta \beta \sqrt{n-\beta n} - \theta(n-i)\beta - \frac{1}{2} \log(1-2\theta)(n-i)\beta). \]

Taking \( \theta = \min\{\frac{M}{4n_{n-1}}, \frac{1}{4}\} \leq \frac{1}{4} \), we obtain (4.23).

We only show one side of (4.24). We have
\[ E[Z_i] \leq E[\tilde{Z}_i] + \int_{M}^{\infty} \frac{\beta x}{8} dx \leq E[\tilde{Z}_i] + \frac{16}{\beta} \exp(-\beta M/8). \quad (4.26) \]

For (4.25), we have
\[ E[Z_i^2] \leq \frac{1}{\beta^2(n-i)} \mathbb{E}\left[ \sum_{l=1}^{(n-i)/\beta} (W_{i,l}^2 - 1)^2 \right] \leq \frac{2}{\beta}. \quad (4.27) \]

The following lemma is a direct consequence of Bennett’s inequality.

Lemma 4.6. Let \( \tilde{S}_{m,n} = \sum_{i=m}^{n}(\tilde{Z}_i - E[\tilde{Z}_i]) \), we have for any \( t \geq 0 \),
\[ P(|\tilde{S}_{m,n}| \geq t) \leq 2 \exp\left(-\frac{\beta t^2}{4(n-m)} \psi_{\text{Benn}}(\frac{\beta M t}{2(n-m)}) \right). \quad (4.28) \]

Here for \( t \neq 0 \),
\[ \psi_{\text{Benn}} = \frac{(1 + t)\log(1 + t) - t}{t^2}, \quad (4.29) \]
and \( \psi_{\text{Benn}}(0) = 1 \).

In particular, when \( 0 \leq t \leq \frac{2(n-m)}{\beta M} \), we have
\[ P(|\tilde{S}_{m,n}| \geq t) \leq 2 \exp(-\frac{\beta t^2}{8(n-m)}). \quad (4.30) \]

As an application of Lemma 4.6, we have the following lemma.

Lemma 4.7. Suppose that \( m_2 - m_1 = \Delta n^\frac{1}{2} \) for some \( \Delta > 0 \). Let \( S_{m_1,m_2} = \sum_{i=m_1}^{m_2}(Z_i - E[Z_i]) \). Then for \( 0 \leq t \leq \frac{2\Delta n^{1/2}}{\beta} \),
\[ P(|S_{m_1,m_2}| \geq t + \frac{16\Delta n^{1/2}}{\beta} \exp(-\frac{8\beta n^{1/2}}{8}) \leq 2\Delta n^{1/2} \exp(-\frac{8\Delta n^{1/2}}{8}) + 2 \exp(-\frac{8\Delta n^{1/2}}{8}) \]. \quad (4.31)
Proof. We take $M = n^{\frac{1}{8}}$ in Lemma 4.5 and obtain
\[
| \sum_{i=m}^{m_2} (E[Z_i] - E[\hat{Z}_i]) | \leq \frac{16 \Delta n^{\frac{1}{2}}}{\beta} \exp\left(-\frac{\beta n^{\frac{1}{8}}}{8}\right),
\] (4.32)
\[
\mathbb{P}(\text{There exists } i \in [m_1, m_2] \cap \mathbb{N}_+, \text{such that } Z_i \neq \hat{Z}_i) \leq 2 \Delta n^{\frac{1}{8}} \exp\left(-\frac{\beta n^{\frac{1}{8}}}{8}\right).
\] (4.33)

Note that $|S_{m_1, m_2}| \leq |\hat{S}_{m_1, m_2}| + \sum_{i=m_1}^{m_2} (E[Z_i] - E[\hat{Z}_i])$ when $Z_i = \hat{Z}_i$ for all $m_1 \leq i \leq m_2$. By Lemma 4.6 and a union bound, we have for $0 \leq t \leq 2 \Delta n^{\frac{1}{8}}$,
\[
\mathbb{P}(|S_{m_1, m_2}| \geq t + \frac{16 \Delta n^{\frac{1}{2}}}{\beta} \exp\left(-\frac{\beta n^{\frac{1}{8}}}{8}\right))
\leq 2 \Delta n^{\frac{1}{8}} \exp\left(-\frac{\beta n^{\frac{1}{8}}}{8}\right) + \mathbb{P}(|\hat{S}_{m_1, m_2}| \geq t)
\leq 2 \Delta n^{\frac{1}{8}} \exp\left(-\frac{\beta n^{\frac{1}{8}}}{8}\right) + 2 \exp\left(-\frac{\beta t^2}{8 \Delta n^{\frac{1}{8}}}\right).
\]

\hfill \square

### 4.2.2 Proof of Proposition 4.2

In this part, we proceed to the proof of Proposition 4.2. Again, we set up some notations and events. Suppose $\phi$ is an eigenfunction of $H_{\beta, n}$, we take
\[
p(i) = \frac{u_1(i)}{\phi(i-1)} \text{ for } 2 \leq i \leq n.
\]
We fix $L_1 \in n^{-\frac{1}{2}} \mathbb{Z}$, $\Delta = [L_1^{\frac{1}{4}}]$ and $K_1$. We set up events $A_1 = \{ \lambda \geq -K_1 \}, A_2 = \{ |\gamma| \leq n^{\frac{1}{8}} \text{ for all } 1 \leq i \leq n-1 \}$,
\[
A_3 = \{ \text{There exists some } i \geq L_1 n^{\frac{1}{8}}, \text{ such that } p(i) \geq \frac{1}{8} \sqrt{\ln n^{\frac{1}{8}}} \},
\]
\[
A_4 = \{ \text{There exists some } i \geq L_1 n^{\frac{1}{8}}, \text{ such that } p(i) \geq -\frac{1}{8} \sqrt{\ln n^{\frac{1}{8}}} \}.
\]

The following lemma controls $\mathbb{P}(A_2)$.

**Lemma 4.8.** $\mathbb{P}(A_2) \leq C \exp\left(-cn^{\frac{1}{8}}\right)$ ($C, c$ are constants that only depend on $\beta$).

**Proof.** As $\gamma_i = Z_i - E[Z_i]$, we have $|\gamma_i| \leq |Z_i| + |E[Z_i]|$. By Lemma 4.3 $|E[Z_i]| \leq M + \frac{16}{\beta} \exp\left(-\frac{\beta M}{8}\right) \leq C$ for some constant $C$. Thus by Lemma 4.5 and union bound, we obtain that for $n$ sufficiently large, $\mathbb{P}(A_2) \leq \sum_{i=1}^{n-1} \mathbb{P}(|Z_i| \geq \frac{1}{8} n^{\frac{1}{8}}) \leq C \exp\left(-cn^{\frac{1}{8}}\right)$.

**Lemma 4.9.** Assume that $K_1 \leq \frac{1}{3} n^{\frac{1}{8}}$, and that $A_1$ holds. When $n$ is sufficiently large, for any $C_1 \geq 1$, with probability $\geq 1 - 4n \exp\left(-\frac{\beta_1 \sqrt{n}}{32} \right)$, the following property holds: for any $n - \sqrt{n} \leq i \leq n$, if $\phi(i-1), \phi(i) > 0$ and $\phi(i-1) \leq C_1 \phi(i)$, then $\phi(i+1) > 0$. 

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Proof. Suppose $\phi(i + 1) \leq 0$. From the eigen-equation
\[ \sqrt{\frac{2}{\beta}} \xi(i) + \frac{Y_{n-1}}{\sqrt{\beta}} \phi(i + 1) + \frac{Y_{n-1}+1}{\sqrt{\beta}} \phi(i - 1) - 2\sqrt{n} \phi(i) = \lambda n^{-\frac{1}{4}} \phi(i), \] (4.34)
we have
\[ \sqrt{\frac{2}{\beta}} \xi(i) + \frac{Y_{n-1}+1}{\sqrt{\beta}} \phi(i - 1) - 2\sqrt{n} \phi(i) \geq \frac{5}{3} \sqrt{n} \phi(i). \] (4.35)
Set
\[ C_1 = \{ \xi_i \leq \sqrt{\frac{3\beta}{2}} \text{ for any } n - \sqrt{n} \leq i \leq n \}, \]
\[ C_2 = \{ Y_{n-i+1} \leq \sqrt{\frac{3\beta}{2C_1}} \text{ for any } n - \sqrt{n} \leq i \leq n \}. \]
On $C_1 \cap C_2$, we have
\[ \frac{\sqrt{n} \phi(i)}{2} \geq \frac{\sqrt{n} \phi(i - 1)}{2C_1} \geq \frac{Y_{n-i+1} \phi(i - 1)}{\sqrt{\beta}} \geq \frac{2}{3} \sqrt{n} \phi(i), \] (4.36)
a contradiction. Hence we have $\phi(i + 1) > 0$ on $C_1 \cap C_2$.

Moreover, we have $P(C_1^c) \leq n \exp(-\frac{6n}{\sqrt{\beta}})$, and $P(C_2^c) \leq \sum_{i=n-\sqrt{n}}^n P(Z_i > \sqrt{\frac{6n}{2C_1}}) \leq 2n \exp(-\frac{9n}{2\sqrt{n}})$ for $n$ sufficiently large. The conclusion of the lemma follows. \( \square \)

Lemma 4.10. There exist constants $C, C', C'', c > 0$ which only depend on $\beta$, such that when $L_1 \geq \max\{C'', C'K_1, \log^8 n\}$ and $K_1 \leq \frac{L_1}{4n^{\frac{1}{4}}}$, we have
\[ P(A_1 \cap A_3) \leq C \exp(-cn^{\frac{1}{4}}) + C \exp(-cL_1^{\frac{1}{4}}). \] (4.37)
Proof. We assume that $A_1$ and $A_2$ hold. We set $I_j = [(L_1 + j\Delta)n^\frac{1}{4}, (L_1 + (j + 1)\Delta)n^\frac{1}{4}]$, $\epsilon(j) = \frac{1}{6b} + \frac{1}{6b}(j + 1)^{-\frac{1}{4}}$ and $f(j) = \frac{1}{6b} + \frac{1}{6b}(j + 1)^{-\frac{1}{4}}$. Note that $\frac{1}{6b} > \epsilon(j) > f(j) > f(j + 1)$. We also set up the bands $D_j = \{(i, y) : y \geq \epsilon(j) \sqrt{\text{in}^{-\frac{1}{4}}} \}$ and $D'_j = \{(i, y) : y \leq f(j + 1) \sqrt{\text{in}^{-\frac{1}{4}}} \}$. We denote by $Q_i = \sqrt{\frac{2}{\beta} \xi_i} + \gamma_i + \gamma_{i-1}$ and $\bar{Q}_i = \sqrt{\frac{2}{\beta} \xi_i} + Z_i + Z_{i-1}$ for $2 \leq i \leq n - 1$. We take $C_2 = 2$ in Lemma 4.10 and denote the event in that lemma by $B_1$; we have $P(B_1^c) \leq C \exp(-cn^{\frac{1}{4}})$. We denote by $B_2 = \{ \xi_i < \sqrt{\frac{2}{\beta} \text{in}}, Z_i < \frac{1}{\sqrt{m}} \text{ for all } i \}$; we have $P(B_2^c) \leq C \exp(-c\sqrt{n})$. Below we assume that $B_1$ and $B_2$ hold.

Moreover, we set for $1 \leq j \leq \lfloor \frac{n^\frac{1}{4}}{2} \rfloor$, $h_j := (f(j) - f(j + 1)) \sqrt{L_1 + j\Delta}$ and
\[
\mathcal{F}_j := \{ \text{For any } m_1, m_2 \text{ such that } 0 \leq m_2 - m_1 \leq \frac{200h_j n^\frac{1}{4}}{L_1 + j\Delta}, \}
\text{we have } |n^{-\frac{1}{4}} \sum_{i=m_1}^{m_2} Q_i| \leq \frac{h_j}{2}. \]
We let $F = \cap_j F_j$. By Lemma 4.7, \( P(F_j^c) \leq C n^2 (\exp(-cn^{\frac{1}{4}}) + \exp(-ch_j(L_1 + j\Delta))) \), and thus by union bound we have \( P(F^c) \leq C \exp(-cn^{\frac{1}{4}}) + C \exp(-cL_1^{\frac{1}{4}}) \).

We define two types of “bad” events on each $I_j$. Let $E_{j,1}$ be the event that for some $i_1 \in I_j$, $p(i_1) \geq f(j)\sqrt{i_1n^{-\frac{1}{4}}}$, and there exists $i_2 \in I_j$ such that $p(i) < \epsilon(j)\sqrt{i_1n^{-\frac{1}{4}}}$ for $i_1 \leq i \leq i_2$ and $p(i_2) < f(j+1)\sqrt{i_2n^{-\frac{1}{4}}}$. Let $E_{j,2}$ be the event that there exists $i_3, i_4 \in I_j$ such that $i_3 < i_4$, $(i_3, p(i_3)) \in D_j$, $(i, p(i)) \in D_j^c \cap (D_j')^c$ for all $i_3 < i < i_4$ and $(i_4, p(i_4)) \in D_j'$. We also let $E_j$ be the event that for any $i_1 \in I_j$ such that $p(i_1) \geq f(j)\sqrt{L_1 + j\Delta}$, we have $p(i) \geq f(j+1)\sqrt{i_1n^{-\frac{1}{4}}}$ for all $i \in [i_1, (L_1 + (j+1)\Delta)n^{\frac{1}{4}}]$. Finally we set $E = \cap_j E_j$.

By definition, we have $E_{j} \subset E_{j,1} \cup E_{j,2}$.

On $E \cap A_3$, suppose that for $i_0 \in [(L_1 + j_0\Delta)n^{\frac{1}{4}}, L_1 + (j_0 + 1)\Delta)n^{\frac{1}{4}}]$, we have $p(i_0) \geq \frac{1}{3}\sqrt{i_0n^{-\frac{1}{4}}}$. By the definition of $E_j$ and induction, we obtain that for all $i_0 \leq i \leq n$, $p(i) \geq \frac{1}{3}\sqrt{i_0n^{-\frac{1}{4}}}$. If we denote by $T = \{\text{For some } i_0 \geq L_1n^{\frac{1}{4}}, \text{we have } p(i) \geq \frac{1}{10}\sqrt{i_0n^{-\frac{1}{4}}}, \text{for all } i_0 \leq i \leq n\}$, we conclude that $E \cap A_3 \subset T$. (4.38)

Below we bound $P(E)$ by bounding $P(E_{j,1})$ and $P(E_{j,2})$. We conveniently let $\phi(n + 1) = 0$ below, so that (4.22) also holds for $i = n$.

We start with $P(E_{j,1})$. From (4.22), we obtain that for $2 \leq i \leq n$ if $\phi(i - 1)$ and $\phi(i)$ have the same sign,

$$
p(i) - p(i) \geq \frac{\phi(i) - \phi(i - 1)^2}{n^{-\frac{1}{4}}\phi(i)} + n^{-\frac{1}{4}}(\sqrt{n} - \mu_i)\phi(i) - \frac{1}{2}(i - 1)n^{-\frac{1}{4}}(n^{-\frac{1}{4}}\phi(i) - K_1 n^{-\frac{1}{4}} - n^{-\frac{1}{4}}Q_i)
\quad \quad \quad + \frac{1}{2}\phi(i - 1) - \gamma_i (\phi(i + 1) - \phi(i)) - n^{-\frac{1}{4}}\gamma_{i-1} (\phi(i - 1) - \phi(i)).
$$

We assume that $E_{j,1}$ holds. For $i_1 \leq i \leq i_2 - 1$, by assumption, $|\phi(i) - \phi(i - 1)| \leq n^{-\frac{1}{4}}\sqrt{i_0n^{-\frac{1}{4}}} \leq 1$. Hence $\phi(i - 1) > 0$. Without loss of generality, we assume that $\phi(i_1 - 1) > 0$, and thus have $\phi(i) > 0$ for all $i_1 - 1 < i \leq i_2 - 1$.

Now we show $\phi(i_2) > 0$. As $B_1$ holds, for $n - \sqrt{n} \leq i_2 \leq n$, we already have $\phi(i_2) > 0$. For $i_2 \leq n - \sqrt{n}$, as $A_1$ and $A_2$ hold, we have

$$
\frac{n^{-\frac{1}{4}}\phi(i_2)}{\phi(A_2 - 1)} - \frac{n^{-\frac{1}{4}}\phi(i_2 - 1)}{\phi(A_2 - 2)} \geq \frac{1}{50}(i_2n^{-\frac{1}{4}}n^{-\frac{1}{4}} - n^{-\frac{1}{4}}Q_{i_2 - 1}) - n^{-\frac{1}{4}}\frac{\phi(i_2)}{\phi(i_2 - 1)} \frac{\phi(i_2 - 1)}{\phi(i_2 - 1)}.\n$$

From (4.39), if $\phi(i_2) \leq 0$, then $Q_{i_2 - 1} \geq \frac{\sqrt{n}}{10}$. As $B_2$ holds, we have a contradiction. Thus we conclude that $\phi(i_2) > 0$. 23
Now from (4.39), for $i_1 \leq i \leq i_2 - 2$,
\[
p(i+1) - p(i) \geq \frac{1}{12} (in^{-\frac{3}{4}}) n^{-\frac{1}{4}} \frac{\phi(i-1)}{\phi(i)} - K_1 n^{-\frac{3}{4}} - n^{-\frac{1}{4}} Q_i \geq \frac{1}{2} n^{-\frac{3}{4}} (\gamma_i + \gamma_{i-1}) n^{-\frac{1}{4}} \sqrt{in^{-\frac{3}{4}}} \geq \frac{1}{50} (in^{-\frac{3}{4}}) n^{-\frac{1}{4}} - n^{-\frac{1}{4}} Q_i \geq \frac{1}{50} (L_1 + j\Delta)n^{-\frac{1}{4}} - n^{-\frac{1}{4}} Q_i.
\]
Similarly, for $i = i_2 - 1$,
\[
(1 + n^{-\frac{3}{4}}) p(i+1) - p(i) \geq \frac{1}{50} (L_1 + j\Delta)n^{-\frac{1}{4}} - n^{-\frac{1}{4}} Q_i.
\]
Thus we conclude that
\[
(1 + n^{-\frac{3}{4}}) p(i_2) - p(i_1) \geq \frac{1}{50} (L_1 + j\Delta)n^{-\frac{1}{4}} - n^{-\frac{1}{4}} (i_2 - i_1) - n^{-\frac{1}{4}} \left( \sum_{i=i_1}^{i_2} Q_i \right).
\]
We set $q(i) = p(i) - f(j + 1) \sqrt{in^{-\frac{3}{4}}}$, and get for $i_1 \leq i < j \leq i_2 - 2$,
\[
q(j) - q(i) \geq \frac{1}{100} (L_1 + j\Delta)(n^{-\frac{1}{4}} (j - i)) - n^{-\frac{1}{4}} \sum_{k=i}^{j} Q_k.
\]
Thus on event $\mathcal{F}_j \cap E_{j,1}$, within $\frac{200h}{L_1 + j\Delta}$ steps, $q(i)$ will move upward $\geq h_j$ without hitting lower than $\frac{h_j}{2}$; within next $\frac{200h}{L_1 + j\Delta}$ steps, $q(i)$ will move upward $\geq h_j$ without hitting lower than $\frac{3h_j}{2}$;... The process will either continue or stop when $(i, p(i))$ hits $\mathcal{D}$. This is a contradiction to the definition of $\mathcal{E}_{j,1}$. Therefore, we have $\mathcal{E}_{j,1} \cap \mathcal{A}_1 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \subset \mathcal{F}_j$. Thus we conclude that
\[
\mathbb{P}(\mathcal{A}_1 \cup \cup \mathcal{E}_{j,1}) \leq C \exp(-cn^{-\frac{1}{4}}) + C \exp(-cL_1^{-\frac{1}{4}}).
\]
Next we bound $\mathbb{P}(\mathcal{E}_{j,2})$. Assume that $p(i_3) \geq \frac{1}{2} \sqrt{i_3 n^{-\frac{3}{4}}}$. We will first show that $\phi(i_3 + 1) > 0$. For $i_3 \geq n - \sqrt{n}$, the previous argument using Lemma [19] works. For $i_3 < n - \sqrt{n}$, without loss of generality we assume that $\phi(i_3 - 1) > 0$ and $\phi(i_3) > 0$. Noting $0 \leq \frac{\phi(i_3) - \phi(i_3 - 1)}{\phi(i_3)} \leq 1$, we have
\[
\left( n^{-\frac{3}{4}} \sqrt{n} - i_3 - n^{-\frac{1}{4}} \right) \frac{\phi(i_3 + 1)}{\phi(i_3)} \geq \frac{1}{2} n^{-\frac{1}{4}} - n^{-\frac{1}{4}} Q_{i_3}. \]
From this we deduce that $Q_{i_3} \geq \frac{1}{2} \sqrt{n}$, which is a contradiction since event $\mathcal{B}_2$ holds.
Based on the preceding argument, we have
\[
(1 + n^{-\frac{3}{4}}) p(i_3 + 1) - (1 - n^{-\frac{3}{4}}) p(i_3) \geq -K_1 n^{-\frac{1}{4}} - n^{-\frac{1}{4}} Q_{i_3}.
\]
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Note that

\[
\frac{\phi(i_3) - \phi(i_3 - 1)}{\phi(i_3)} \geq 1 - \frac{1}{1 + n^{-\frac{1}{2}}(1 + \frac{1}{2} \sqrt{i_3 n^{-\frac{1}{2}}})}
\]

\[
\geq ((1 - n^{-\frac{1}{2}}) + (\frac{1}{4} - n^{-\frac{1}{2}})\sqrt{i_3 n^{-\frac{1}{2}}}) n^{-\frac{1}{2}}
\]

\[
\geq (1 + \frac{1}{4} - 2 n^{-\frac{1}{2}})\sqrt{i_3 n^{-\frac{1}{2}}}) n^{-\frac{1}{2}}
\]

\[
\geq (1 + \frac{1}{6} \sqrt{i_3 n^{-\frac{1}{2}}}) n^{-\frac{1}{2}}.
\]

Hence by Lemma 4.7 we have

\[
p(i_3 + 1) \geq 1 + \frac{1}{8} \sqrt{(i_3 + 1)n^{-\frac{1}{2}}},
\]

(which is a contradiction to the definition of \(E_{j,2}\)) up to probability \(\leq C \exp(-cn^{\frac{1}{2}})\).

Thus we assume below that \(p(i_3) \leq \frac{1}{8} \sqrt{i_3 n^{-\frac{1}{2}}}\). Similar to the above argument, we have \(\phi(i_4) > 0\). From the definition of \(E_{j,2}\), we have \(p(i_3) \geq \epsilon(j) \sqrt{i_3 n^{-\frac{1}{2}}} \) and \(p(i_4) \leq f(j + 1) \sqrt{i_4 n^{-\frac{1}{2}}}\). Hence we have \(q(i_4) - q(i_3) \leq -(\epsilon(j) - f(j + 1)) \sqrt{i_3 n^{-\frac{1}{2}}}\), from which we deduce that

\[
\sum_{k=i_3}^{i_4} Q_k \geq n^{\frac{1}{2}} (\epsilon(j) - f(j + 1)) \sqrt{i_3 n^{-\frac{1}{2}}},
\]

(4.45)

Since \(0 \leq i_4 - i_3 \leq \Delta n^{\frac{1}{2}}\), we conclude by Lemma 4.7 that

\[
P(A_1 \cap \cup_j E_{j,2})
\]

\[
\leq C \exp(-cn^{\frac{1}{2}}) + C \sum_{1 \leq j \leq \left\lfloor \frac{n}{L_1} \right\rfloor} n^2 \exp\left(-\frac{c(\epsilon(j) - f(j + 1)^2(L_1 + j\Delta)}{\Delta}\right)
\]

\[
\leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-cL_1^{\frac{1}{4}}).
\]

Therefore, we conclude that \(P(A_1 \cap E^c) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-cL_1^{\frac{1}{4}})\). Now on \(T\), using the preceding argument, we have to have \(\phi(n + 1) > 0\) (without loss of generality, we assume as before that \(\phi(n - 1), \phi(n) > 0\)). However, by definition \(\phi(n + 1) = 0\), which is a contradiction. Therefore we conclude that

\[
P(A_1 \cap A_3) \leq P(A_1 \cap E^c) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-cL_1^{\frac{1}{4}}).
\]

(4.46)

**Lemma 4.11.** The exist constants \(C, C', C'', c\) which only depend on \(\beta\), such that when \(L_1 \geq \max\{C'K_1, C'', \log^{\beta} n\}\), we have

\[
P(A_1 \cap A_3 \cap A_4) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-cL_1^{\frac{1}{4}}).
\]

(4.47)
Proof. We use similar arguments as in Lemma 4.12 and Lemma 4.10. Assume that $A_1 \cap A_3 \cap A_4$ hold. Let $G_{i,j}$ denote the event that there exists $(i_1, i_2)$ such that $i_1, i_2 \in I_j$, $0 < i_2 - i_1 \leq \frac{200n^{\frac{1}{2}}}{\sqrt{L_1 + j\Delta}}$, $p(i_1) \in \left[-\frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}} \right)$, $p(i) \in \left[\frac{1}{12} \sqrt{in^{-\frac{1}{2}}} \right)$ for $i_1 + 1 \leq i \leq i_2 - 1$ and $p(i_2) < -\frac{1}{2} \sqrt{i_2 n^{-\frac{1}{2}}}$. We set $G_1 = \cup_j G_{1,j}$. We also let $G_{2,j}$ denote the event that there exists $(i_1, i_2)$ with $i_1 \leq i_2$ and $i_1, i_2 \in I_j \cup I_{j+1}$ such that $n^{-\frac{3}{8}} \sum_{i=i_1}^{i_2} Q_i \geq \sqrt{L_1 + j\Delta}$, and set $G_2 = \cup_j G_{2,j}$.

On $G_{1,j}$, similar to the arguments in Lemma 4.10 we have

$$ (1 - n^{-\frac{3}{8}}) p(i_2) - p(i_1) \geq \frac{1}{50} (L_1 + j\Delta) n^{-\frac{3}{8}} (i_2 - i_1) - n^{-\frac{3}{8}} \sum_{k=i_1}^{i_2} Q_k, \quad (4.48) $$

and

$$ (1 - n^{-\frac{3}{8}}) p(i_2) - p(i_1) \leq -\frac{1}{4} \sqrt{i_2 n^{-\frac{1}{2}}} \leq -\frac{1}{4} \sqrt{L_1 + j\Delta} \quad (4.49) $$

for $n$ sufficiently large. From these we deduce that $n^{-\frac{3}{8}} \sum_{k=i_1}^{i_2} Q_k \geq \frac{1}{8} \sqrt{L_1 + j\Delta}$. Thus by Lemma 4.7 and a union bound over choices of $(i_1, i_2)$ and $j$, we obtain

$$ \Pr(G_1) \leq C \exp(-cn^\frac{1}{8}) + C \sum_{1 \leq j \leq \left(\frac{n^\frac{3}{8}}{200}\right)} \exp(-c(L_1 + j\Delta)^\frac{1}{2}) \leq C \exp(-cn^\frac{1}{8}) + C \exp(-cL^\frac{1}{2}). $$

Moreover, by Lemma 4.7 and a union bound we have

$$ \Pr(G_2) \leq C \exp(-cn^\frac{1}{8}) + C \sum_{1 \leq j \leq \left(\frac{n^\frac{3}{8}}{200}\right)} \exp(-c(L_1 + j\Delta)^\frac{1}{2}) \leq C \exp(-cn^\frac{1}{8}) + C \exp(-cL^\frac{1}{2}). $$

Below we assume that $G_1 \cap G_2$ holds. From the definition of $A_4$, there exists $i_1 \geq L_1 n^\frac{3}{8}$ such that $p(i_1) \geq -\frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}}$. As we are in $A_3$, we have

$$ p(i_1) \leq \frac{1}{8} \sqrt{i_1 n^{-\frac{1}{2}}}. $$

As we are in $G_1$, for $i_1 \leq i \leq i_1 + \frac{200n^{\frac{1}{2}}}{\sqrt{L_1 + j\Delta}}$, $p(i) \in \left[-\frac{1}{8} \sqrt{in^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{in^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{in^{-\frac{1}{2}}}, \frac{1}{8} \sqrt{in^{-\frac{1}{2}}} \right)$. Therefore with a similar argument as in Lemma 4.10

$$ p(i_2) - p(i_1) \geq \frac{1}{50} (L_1 + j\Delta) n^{-\frac{3}{8}} (i_2 - i_1) - n^{-\frac{3}{8}} \sum_{i=i_1}^{i_2} Q_i. \quad (4.50) $$

Take $i_2 \in [i_1 + \frac{100n^{\frac{1}{2}}}{\sqrt{L_1 + j\Delta}}, i_1 + \frac{200n^{\frac{1}{2}}}{\sqrt{L_1 + j\Delta}}]$. As we are in event $G_1$, $p(i_2) \leq \frac{1}{8} \sqrt{i_2 n^{-\frac{1}{2}}}$. Thus we obtain that $n^{-\frac{3}{8}} \sum_{i=i_1}^{i_2} Q_i \geq \sqrt{L_1 + j\Delta}$, which leads to a contradiction as we are actually in event $G_2$. 26
Hence we conclude that
\[ \mathbb{P}(A_1 \cap A_2^c \cap A_4) \leq \mathbb{P}(G_1) + \mathbb{P}(G_2) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-cL_1^{k_1}). \]  

**Lemma 4.12.** Suppose that there exists \( c \in (0,1) \), such that \( k \geq n^c \). Then there exist constants \( K, T_0, C, c > 0 \) which only depend on \( \beta \), such that for any \( T \geq T_0 \) and \( k \geq K \), and \( K_1 = k^{\frac{1}{2}} T_1 \),
\[ \mathbb{P}(\lambda_k \leq -K_1) \leq C \exp(-ckT_1^{\frac{1}{2}}). \]

**Proof.** We apply Theorem 5.1, which is proved independent of the rest part of this paper. Note that the scaling in Theorem 5.1 is different from that in this section. In this proof, we temporarily denote by \( \lambda_k^c \) the counterpart of \( \lambda_k \) in the notation of Theorem 5.1. We have for \( n \) sufficiently large,
\[ \mathbb{P}(\lambda_k \leq -K_1) \leq \mathbb{P}(\lambda_k^c \leq 2 - n^{-\frac{3}{5}} K_1) \leq \mathbb{P}(|\lambda_k^c - \gamma_n,k| \geq cn^{-\frac{3}{5}} K_1) \leq C \exp(-ckT_1^{\frac{1}{2}}). \]

Now we prove Proposition 4.2.

**Proof of Proposition 4.2.** By Lemma 4.8 and 4.12 if we take \( L_1 = k^{12}, K_1 = ck^{12} \) (where \( c \) is a small enough constant), and set \( A_0 = A_1 \cap A_4 \), then we have \( \mathbb{P}(A_0^c) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-ck^3) \).

Now we set
\[ \mathcal{H}_1 = \{ \text{For any } i_1, i_2 \geq L_1 n^{\frac{1}{2}}, \phi(i_1) \text{ and } \phi(i_2) \text{ have the same sign} \} \]

We also set \( \mathcal{H}_2 = \{ \text{For any } i \geq 1, |Z_i| \leq n^{\frac{1}{2}}, |\xi_i| \leq n^{\frac{1}{2}} \} \), and have \( \mathbb{P}(\mathcal{H}_2^c) \leq C \exp(-cn^{\frac{1}{2}}) \). Assume that \( A_1 \cap \mathcal{H}_2 \cap A_4^c \) is true. Suppose that for some \( i \geq L_1 n^{\frac{1}{2}} + 1, \phi(i-1) \text{ and } \phi(i) \text{ does not have the same sign} \). Without loss of generality, we assume that \( \phi(i-1) \geq 0 \) and \( \phi(i) \leq 0 \). If \( \phi(i) = \phi(i-1) = 0 \), then \( \phi(i) = 0 \) for any \( i \), which has 0 probability. If \( \phi(i) > 0 \) and \( \phi(i-1) \leq 0 \), then we have
\[ \sqrt{\frac{2}{\beta}} \xi_i + \frac{Y_{n-i}}{\sqrt{\beta}} \frac{\phi(i+1)}{\phi(i)} = 2\sqrt{n} - \frac{Y_{n-i-1}}{\sqrt{\beta}} \frac{\phi(i-1)}{\phi(i)} + \lambda n^{\frac{1}{2}} \geq 2\sqrt{n} - n^{\frac{1}{2}} K_1. \]

Now we obtain that \( p(i+1) = \frac{\phi(i+1) - \phi(i)}{\phi(i)} \geq \frac{1}{2} > 0 \), which is a contradiction as we are in event \( A_4 \). If \( \phi(i) = 0 \) and \( \phi(i-1) < 0 \), then we have \( \phi(i+1) > 0 \). The preceding argument works for \( (i, i+1) \). We can conclude that \( \mathcal{H}_2^c \subset \mathcal{A}_4 \cup \mathcal{A}_4 \cup \mathcal{H}_2^c \).

Let \( A = A_0 \cap \mathcal{H} \). We have \( \mathbb{P}(A^c) \leq C \exp(-cn^{\frac{1}{2}}) + C \exp(-ck^3) \).
Moreover, on $\mathcal{A}$ we have that for any $i \geq k^{12}n^{\frac{3}{4}}$, $p(i) = n^\frac{1}{8}\frac{\phi(i)-\phi(i-1)}{\phi(i)} \leq -\frac{1}{8}\sqrt{in^{-\frac{1}{2}}}$, and thus (without loss of generality we assume $\phi(i_1) > 0$ below) $\phi(i) \leq (1 - \frac{1}{8}n^{-\frac{1}{4}}\sqrt{in^{-\frac{1}{2}}})\phi(i-1)$. Therefore for any $i_1, i_2 \geq k^{12}n^{\frac{3}{4}}$ and $i_1 \leq i_2$, we have

$$\phi(i_2) \leq \prod_{k=i_1+1}^{i_2} (1 - \frac{1}{8}n^{-\frac{1}{4}}\sqrt{kn^{-\frac{1}{2}}})\phi(i_1) \leq \exp(-\frac{1}{8}n^{-\frac{1}{4}}\sum_{i_1+1 \leq k \leq i_2} k^{\frac{1}{2}})\phi(i_1) \leq \exp(-\frac{1}{8}n^{-\frac{1}{4}}\int_{i_1}^{i_2} x^{\frac{1}{2}}dx)\phi(i_1) \leq \exp(-\frac{(i_2n^{-\frac{1}{4}})^{\frac{1}{2}} - (i_1n^{-\frac{1}{4}})^{\frac{1}{2}}}{12})\phi(i_1).$$

Here $\phi$ can be any $\phi_j$ ($1 \leq j \leq k$). Note that in the preceding argument, we have already used the fact that $\phi(i)$ does not change sign for $i \geq k^{12}n^{\frac{3}{4}}$ on $\mathcal{A}$.

### 4.3 Proof of Airy approximation theorem

In this section, we prove Theorem 2.4. First we recall a lemma from [10].

**Lemma 4.13** (Lemma 4.7 in [10]). We define

$$Z = \sup_{x>0} \sup_{y \in [0,1)} \frac{|B_{x+y} - B_x|}{6\sqrt{\log(3+x)}}.$$  \hspace{1cm} (4.53)

Let $Q_x = B_{x+1} - B_x$ and $R_x = B_x - \int_x^{x+1} B_y dy$. Then

$$\max\{|R_x|, |Q_x|\} \leq 6Z \sqrt{\log(3+x)}.$$ \hspace{1cm} (4.54)

Moreover, there exist constants $C_1, C_2, s_0 > 0$ such that for all $s \geq s_0$,

$$\mathbb{P}(Z \geq s) \leq C_1 \exp(-C_2s^2).$$ \hspace{1cm} (4.55)

We proceed with the proof of Theorem 2.4.

**Proof of Theorem 2.4.** We proceed by approximating $H_\beta$ and $H_{\beta,n}$ along eigenfunctions. In this approximation, the exponential decay result plays a crucial role. We assume that all the eigenfunctions are normalized. We also take $L = k^{20}$ below.

First we show the approximation in one direction. We denote by $f_j$ the $j$th eigenfunction of $H_\beta$ for $1 \leq j \leq k$. We also define the discrete approximations $\hat{f}_j$ as follows: when $1 \leq i \leq Ln^{\frac{3}{4}}$, let $\hat{f}_j(i) = n^{-\frac{1}{4}}f_j(n^{-\frac{1}{4}}i)$; when $Ln^{\frac{3}{4}} < i < n$, let $\hat{f}_j(i) = 0$. 


Recall that we have defined the scaled version of \( H_{\beta,n} \) by \( \hat{H}_{\beta,n} = n^{\frac{1}{n}} (H_{\beta,n} - 2\sqrt{nI}) \). Thus \( \lambda_i \) are the eigenvalues of \( \hat{H}_{\beta,n} \). In order to facilitate presentation, below we denote by \( \psi = f_j \) and \( \phi = \hat{f}_j \) for a specific \( j \) that satisfies \( 1 \leq j \leq k \). We note that by definition,

\[
\phi^T \hat{H}_{\beta,n} \phi = n^{\frac{1}{n}} \sum_{i=1}^{n} \left( \frac{2}{\sqrt{\beta}} \xi_i \phi^2(i) + 2 \frac{Y_{n,i}}{\sqrt{\beta}} \phi(i) \phi(i + 1) \right) - 2n^{\frac{\beta}{n}} \sum_{i=1}^{n} \phi^2(i). \tag{4.56}
\]

Below we denote by \( L(\phi) := \phi^T \hat{H}_{\beta,n} \phi \). We start by simplifying the expression of \( L(\phi) \). Assuming that \( Y_{n,i} = \sqrt{\sum_{l=1}^{3(n-1)} W_{i,l}} \), with \( W_{i,l} \), i.i.d. \( \sim N(0,1) \), we denote

\[
\Delta_i = \frac{Y_{n,i}}{\sqrt{\beta}} - \sqrt{n-i} - \frac{1}{\sqrt{\beta}} \sum_{l=1}^{(n-i)\beta} (W_{i,l} - 1) \tag{4.57}
\]

Using the arguments of Lemma \ref{lem3}, we have with probability \( \geq 1 - C \exp(-cn^{\frac{1}{n}}) \) that for any \( 1 \leq i \leq Ln^{\frac{1}{n}} \),

\[
|\Delta_i| \leq \frac{1}{2\beta n^{\frac{1}{n}}} \frac{(\sum_{l=1}^{(n-i)\beta} (W_{i,l} - 1))^2}{\sqrt{(n-i)\beta}} \leq Cn^{\frac{-1}{n}}. \tag{4.58}
\]

Below we list some expressions that we will use later. Let

\[
A_1 = n^{\frac{1}{n}} \sum_{i=1}^{n} \frac{2}{\sqrt{\beta}} \xi_i \phi^2(i) + n^{\frac{1}{n}} \sum_{i=1}^{n} \frac{2}{\sqrt{\beta}} \sum_{k=1}^{(n-i)\beta} (X_{k,i} - 1) \phi^2(i), \tag{4.59}
\]

\[
A_2 = 2n^{\frac{1}{n}} \sum_{i=1}^{n} \sqrt{n-i} \phi(i) \phi(i+1) - 2n^{\frac{1}{n}} \sum_{i=1}^{n} \phi(i)^2, \tag{4.60}
\]

\[
B_1 = 2n^{\frac{1}{n}} \sum_{i=1}^{n} \frac{Y_{n,i}}{\sqrt{\beta}} \phi(i) (\phi(i+1) - \phi(i)), \tag{4.61}
\]

\[
B_2 = 2n^{\frac{1}{n}} \sum_{i=1}^{n} \Delta_i \phi^2(i). \tag{4.62}
\]

By Komlós-Major-Tusnády theorem \cite{13,77}, for \( 1 \leq i \leq Ln^{\frac{1}{n}} \), there exist i.i.d. \( N(0,1) \) variables \( \zeta_i \), such that

\[
\frac{\sum_{k=1}^{(n-i)\beta} (X_{k,i} - 1)}{\sqrt{2(n-i)\beta}} - \zeta_i \leq n^{\frac{-1}{n}}, \tag{4.63}
\]

with probability \( \geq 1 - C \exp(-cn^{\frac{1}{n}}) \).

Hence let

\[
C_1 = n^{\frac{1}{n}} \sum_{i=1}^{n} \frac{2}{\sqrt{\beta}} (\xi_i + \zeta_i) \phi^2(i), \tag{4.64}
\]

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we have

$$|A_1 - C_1| \leq n^{-\frac{1}{2}} \sqrt{\frac{2}{\beta}} \sum_{i=1}^{n} \phi^2(i).$$

(4.65)

Moreover, let

$$S_i = n^{-\frac{1}{2}} \sum_{k=1}^{i} (\xi_k + \zeta_k),$$

(4.66)

we can find a coupling so that $S_i = \sqrt{2B_{in}} - 1$.

By Abel transform, we obtain that (where $0 \leq \theta_i \leq 1$)

$$C_1 = \frac{2}{\sqrt{\beta}} n^{-\frac{1}{2}} \sum_{i=1}^{L_2} B_{in} \frac{\psi'(i + \theta_i) \psi(in^{-\frac{1}{2}}) \psi((i + 1)n^{-\frac{1}{2}})}{n^{-\frac{1}{2}} \sum_{k=1}^{i} \phi(i) \phi(i + 1)}.$$  (4.67)

Now let

$$B_3 = -\sum_{i=1}^{n} \frac{i n^{-\frac{1}{2}}}{\sqrt{n - i + \sqrt{n}}} \phi(i) \phi(i + 1),$$

(4.68)

$$C_2 = -\sum_{i=1}^{n} \left(\phi(i) - \phi(i + 1)\right)^2,$$

(4.69)

$$C_3 = -n^{-\frac{1}{2}} \sum_{i=1}^{n} i \phi(i) \phi(i + 1).$$

(4.70)

Then we have $A_2 = C_2 + C_3 + B_3$.

Now we show some estimates for $\psi(x)$. From Lemma 4.13 we have with probability $\geq 1 - C \exp(-ck^3)$,

$$|Q_x| \leq 6 \sqrt{\log(3 + x)} k^2,$$

(4.71)

$$|R_x| \leq 6 \sqrt{\log(3 + x)} k^2.$$  (4.72)

Thus

$$|\int_{0}^{\infty} Q_x \psi^2(x) dx| \leq \int_{L}^{\infty} 6 \sqrt{\log(3 + x)} k^2 \exp(-\frac{1}{6} x^2) dx + \int_{0}^{L} 6 \sqrt{\log(3 + x)} k^2 \psi^2(x) dx \leq Ck^2 \log k.$$

$$|\int_{0}^{\infty} R_x \psi(x) \psi'(x) dx| \leq \frac{8}{\sqrt{\beta}} \int_{0}^{\infty} |R_x|^2 \psi^2(x) dx + \frac{\sqrt{\beta}}{8} \int_{0}^{\infty} |\psi'(x)|^2 dx \leq Ck^4 \log k + \frac{\sqrt{\beta}}{8} \int_{0}^{\infty} |\psi'(x)|^2 dx.$$
Moreover, with probability $\geq 1 - C \exp(-ck^3)$, $|\lambda| \leq k^{12}$. Hence
\[
\int \psi'(x)^2 \, dx = \lambda - \int x \psi'^2(x) \, dx + \frac{2}{\sqrt{\beta}} \int Q_x \psi^2(x) \, dx \\
= -\frac{4}{\sqrt{\beta}} \int_0^\infty R_x \psi'(x) \, dx \\
\leq C k^{12} + \frac{1}{2} \int \psi'(x)^2 \, dx,
\]
which leads to
\[
\int \psi'(x)^2 \, dx \leq C k^{12}.
\] (4.73)

Thus for any $x, x_0$ such that $|x - x_0| \leq n^{-\frac{1}{3}}$, we have by Cauchy-Schwarz inequality,
\[
|\psi(x) - \psi(x_0)| = \left| \int_{x_0}^x \psi'(s) \, ds \right| \\
\leq \sqrt{\int_{x_0}^x \psi'(s)^2 \, ds} \sqrt{|x - x_0|} \\
\leq C k^6 n^{-\frac{1}{6}}.
\]
Hence we have $|\psi(x) - \psi(x_0)|^2 \leq C k^6 n^{-\frac{1}{6}} (2|\psi(x)| + k^6 n^{-\frac{1}{6}})$. Moreover, for any $x \in [0, L]$, we have $|\psi(x)| = |\int_0^L \psi'(x) \, dx| \leq (\int \psi'(x)^2 \, dx)^{\frac{1}{2}} L^{\frac{1}{2}} \leq C k^{16}$.

Now suppose $x_0 \in \{kn^{-\frac{1}{3}}\}_{k=1}^{Ln^{\frac{1}{3}}}$, and $|x - x_0| \leq n^{-\frac{1}{4}}$. By definition of eigenfunction of $H_\beta$, we have
\[
\psi'(x) - \psi'(x_0) = \int_{x_0}^x (\lambda + \theta) \psi(\theta) \, d\theta - \frac{2}{\sqrt{\beta}} B_x \psi(x) \\
+ \frac{2}{\sqrt{\beta}} B_{x_0} \psi(x_0) + \frac{2}{\sqrt{\beta}} \int_{x_0}^x B_\theta \psi'(\theta) \, d\theta.
\]
Now we have (without loss of generality we assume $x \geq x_0$) $|\int_{x_0}^x (\lambda + \theta) \psi(\theta) \, d\theta| \leq (\int_{x_0}^x \psi^2(\theta) \, d\theta)^{\frac{1}{2}} (\int_{x_0}^x (\lambda + \theta)^2 \, d\theta)^{\frac{1}{2}} \leq C k^{20} n^{-\frac{1}{4}}$. With probability greater than or equal to $1 - C \exp(-ck^3)$, sup $|x - x_0| \leq n^{-\frac{1}{4}} |B_x - B_{x_0}| \leq C k^2 n^{-\frac{1}{4}}$, and sup $x \in [0, L + 1] |B_x| \leq C k^{12}$. We thus have $|B_x \psi(x) - B_{x_0} \psi(x_0)| \leq |B_x| |\psi(x) - \psi(x_0)| + |B_x - B_{x_0}| |\psi(x_0)| \leq C k^{18} n^{-\frac{1}{4}}$. We also have
\[
|\int_{x_0}^x B_\theta \psi(\theta) \, d\theta| \leq (\int_{x_0}^x B^2_\theta \, d\theta)^{\frac{1}{2}} (\int_{x_0}^x \psi'(\theta)^2 \, d\theta)^{\frac{1}{2}} \leq C k^{18} n^{-\frac{1}{4}}.
\]
Thus we have $|\psi'(x) - \psi'(x_0)| \leq C k^{20} n^{-\frac{1}{4}}$.

With the above mentioned bounds, we start to bound the terms. We start with $B_1$. We have with probability $\geq 1 - C \exp(-ck^3)$, for $1 \leq i \leq Ln^{\frac{1}{4}}$, 31
\[ | \frac{Y_m}{\sqrt{n}} - \sqrt{n - 1} | \leq Ck^3. \] We also have
\[
| n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi^2(i n^{-\frac{1}{2}}) - \int_0^L \psi^2(x) dx | \leq C \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \int_{(i-1)n^{-\frac{1}{2}}}^{i n^{-\frac{1}{2}}} k^{22} n^{-\frac{1}{2}} dx \\
\leq C k^{42} n^{-\frac{1}{2}}.
\]
Similarly, we can obtain that
\[
| n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi'(i n^{-\frac{1}{2}})^2 - \int_0^L \psi'(x)^2 dx | \leq C k^{46} n^{-\frac{1}{2}}
\]
for any sequence of \( \{\theta_i\} \in [0, 1] \). Then we obtain by Cauchy-Schwarz inequality,
\[
|B_1| \leq C k^3 n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi(i n^{-\frac{1}{2}}) \psi'(i n^{-\frac{1}{2}}) \\
\leq C k^3 n^{-\frac{1}{2}} \left( n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi^2(i n^{-\frac{1}{2}}) \right)^{\frac{1}{2}} \left( n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi'(i n^{-\frac{1}{2}})^2 \right)^{\frac{1}{2}} \\
\leq C k^3 n^{-\frac{1}{2}}.
\]
We bound \( B_2 \). We have
\[
|B_2| \leq C n^{-\frac{1}{2}} \sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} n^{-\frac{1}{2}} \psi^2(i n^{-\frac{1}{2}}) \leq C n^{-\frac{1}{2}}.
\]
We bound \( A_1 \). As \( |A_1 - C_1| \leq n^{-\frac{1}{2}} \sqrt{\frac{2}{\beta} \sum_{i=1}^{\ln n} \phi^2(i)} \leq C n^{-\frac{1}{2}} \), it suffices to bound \( C_1 \). Note that if we take \( \mathcal{L}_j = \{ \sup_{x \in [j, j+1]} |B_x - B_j| \leq j \} \) for \( j \geq L \) and let \( \mathcal{L} := \cap \mathcal{L}_j \), by union bound we have \( \mathbb{P}(\mathcal{L}^c) \leq \sum_{j=L}^{\infty} \mathbb{P}(\mathcal{L}_j^c) \leq C \exp(-cL^2) \ll C \exp(-ck^3) \). On \( \mathcal{L} \) we have \( |B_x| \leq C x^2 \), hence
\[
| \int_L^\infty B_x \psi'(x) \psi(x) | \leq C \int_L^\infty x^2 \exp(-cx^2) dx \leq C \exp(-cL) \ll n^{-\frac{1}{2}}.
\]
We also have
\[
\sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} | B_{\frac{i n^{-\frac{1}{2}}}{(i-1)n^{-\frac{1}{2}}}} | \left| \psi(i n^{-\frac{1}{2}}) - \psi(x) \right| | \psi'(x) | dx \\
\leq C k^{18} n^{-\frac{1}{2}} L^{\frac{1}{2}} \sqrt{\int_0^L \psi'(x)^2 dx} \leq C k^{40} n^{-\frac{1}{2}},
\]
and
\[
\sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} | B_{\frac{i n^{-\frac{1}{2}}}{(i-1)n^{-\frac{1}{2}}}} | \left| \psi(i n^{-\frac{1}{2}}) \right| \left( \left| \psi'(x) - \psi'(i n^{-\frac{1}{2}}) \right| \right) dx \\
\leq C k^{32} n^{-\frac{1}{2}} L^{\frac{1}{2}} \sqrt{\sum_{i=1}^{\frac{\ln n}{\sqrt{2}}} \psi^2(i n^{-\frac{1}{2}})} \leq C k^{42} n^{-\frac{1}{2}},
\]

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Thus also, we have

\[ \int \sum_{i=1}^{\frac{Ln}{n}} \int_{i(n-\frac{1}{n})}^{\frac{Ln}{n}} |B_x - B_{m-\frac{1}{n}}| \psi(x) dx \leq Ck^2 n^{-\frac{1}{4}} \int_0^L (\psi'(x)^2 + \psi^2(x)) \, dx \leq Ck^2 n^{-\frac{1}{4}}. \]

We conclude that \( |C_1 - \frac{1}{\sqrt{n}} \int_0^\infty B_x \psi(x) \psi'(x) \, dx| \leq Ck^2 n^{-\frac{1}{4}} \), and also \( |A_1 - \frac{1}{\sqrt{n}} \int_0^\infty B_x \psi(x) \psi'(x) \, dx| \leq Cn^{-\frac{1}{4}}. \)

We bound \( A_2 \). We have \( |B_{i,4}| \leq L^2 n^{-\frac{3}{4}} \sum_{i=1}^{Ln} \phi(i)^2 \leq Ck^4 n^{-\frac{1}{4}}. \) We also have \( |C_2 + \int_0^\infty \psi'(x)^2 \, dx = |n^{-\frac{1}{4}} \sum_{i=1}^{Ln} \psi'(x)^2((i + \theta)n^{-\frac{1}{4}}) - \int_0^L \psi'(x)^2 \, dx + \int_0^\infty Cx \exp(-c x^\frac{3}{2}) \, dx \leq Ck^4 n^{-\frac{1}{4}}. \) Note that

\[ \int_0^\infty x^2 \, dx \leq \int_0^\infty Cx \exp(-c x^\frac{3}{2}) \, dx \leq C \exp(-cL) \ll n^{-\frac{1}{4}}. \]

Also,

\[ \left| \sum_{i=1}^{Ln} \frac{i}{n} \phi(i) \phi(i + 1) \right| \int_0^L x^2 \, dx \leq n^{-\frac{1}{4}} \int_0^L \psi^2(x) \, dx + L \sum_{i=1}^{Ln} \int_{(i-1)n^{-\frac{1}{4}}}^{in^{-\frac{1}{4}}} (\phi(i + 1)n^{-\frac{1}{4}} - \phi(i)n^{-\frac{1}{4}}) \, dx \]

\[ \leq Ck^6 n^{-\frac{1}{4}}. \]

Thus \( |A_2 + \int_0^\infty \psi'(x)^2 \, dx + \int_0^\infty x^2 \psi(x) \, dx| \leq Ck^6 n^{-\frac{1}{4}} \ll n^{-\frac{1}{4}}. \)

We thus obtain that for any \( 1 \leq i \leq k \), with probability \( \geq 1 - C \exp(-c k^3) \),

\[ |f^T_i \hat{H}_{\beta,n} \hat{f}_i + f^T_i H_{\beta} f_i| \leq C n^{-\frac{1}{4}}. \]  \hspace{1cm} (4.75)

Similarly we can show that for any \( 1 \leq i \neq j \leq k \),

\[ |\hat{f}^T_i \hat{H}_{\beta,n} \hat{f}_j| \leq C n^{-\frac{1}{4}}, \]  \hspace{1cm} (4.76)

\[ |\hat{f}^T_i \hat{f}_i - 1| \leq C n^{-\frac{1}{4}}, \]  \hspace{1cm} (4.77)

\[ |\hat{f}^T_i \hat{f}_j| \leq C n^{-\frac{1}{4}}. \]  \hspace{1cm} (4.78)

Now for any \( i \), let \( \phi = \sum_{1 \leq l \leq i} \theta_l \hat{f}_l \), such that

\[ \|\phi\|_2^2 = \sum_{1 \leq l \leq i} \theta_l^2 \|\hat{f}_l\|_2^2 + \sum_{1 \leq l \neq l' \leq i} \theta_l \theta_{l'} (f_l, f_{l'}) = 1. \]  \hspace{1cm} (4.79)
We have for the $i$th eigenvalue $\tilde{\lambda}_i$ of $\hat{H}_{\beta,n}$:

$$\tilde{\lambda}_i \geq \inf_{\theta} \phi^T \hat{H}_{\beta,n} \phi. \quad (4.80)$$

Let

$$\delta = 2kCn^{-\frac{1}{12}} \ll n^{-\frac{1}{12}}, \quad (4.81)$$

$$\epsilon = Cn^{-\frac{1}{12}}, \quad (4.82)$$

we have

$$1 \geq \sum_{1 \leq l \leq i} \theta_l^2 (1 - \epsilon) - \frac{1}{2} \sum_{1 \leq l \neq l' \leq i} (\theta_l^2 + \theta_l'^2) \epsilon$$

$$\geq (1 - (i + 1)\epsilon) \sum_{1 \leq l \leq i} \theta_l^2. \quad (4.83)$$

Hence

$$\sum_{1 \leq l \leq i} \theta_l^2 \leq 2.$$ 

Thereby we obtain with probability $\geq 1 - C \exp(-ck^3)$,

$$\phi^T \hat{H}_{\beta,n} \phi \geq \sum_{1 \leq l \leq i} \theta_l^2 (a_i - \epsilon) - \sum_{1 \leq l \neq l' \leq i} \epsilon |\theta_l \theta_l'|$$

$$\geq \sum_{1 \leq l \leq i} \theta_l^2 (a_i - \epsilon) - i\epsilon \sum_{1 \leq l \leq i} \theta_l^2$$

$$\geq \left((a_i - \epsilon) - \frac{1}{1 + \epsilon + \delta} - 2\delta\right)$$

$$\geq a_i - n^{-\frac{1}{12}}.$$

Now we deal with the other direction. Suppose that $g_j$ is the $j$th eigenfunction for $\hat{H}_{\beta,n}$ for $1 \leq j \leq k$. For $1 \leq j \leq k$, we introduce an approximation $\tilde{g}_j$ as follows: we let $\tilde{g}_j(0) = 0$, $\tilde{g}_j(kn^{-\frac{1}{4}}) = n^{\frac{1}{12}}g_j(k)$ for $k = 1, 2, \ldots, Ln^{-\frac{1}{4}}$, $\tilde{g}_j(L + n^{-\frac{1}{2}}) = 0$ and linearly interpolate in $[0, L + n^{-\frac{1}{2}}]$; we also let $\tilde{g}_j(x) = 0$ for $x \geq L + n^{-\frac{1}{4}}$. For convenience of notations, below we assume that $\phi$ is $g_j$ for a specific choice of $j$ with $1 \leq j \leq k$, and let $\psi = g_j$. Suppose the eigenvalue corresponding to $\phi$ is $\lambda$, we have

$$\lambda = -2n^{\frac{3}{2}} \sum_{i=1}^{n} \phi^2(i) + 2n^{\frac{3}{2}} \sum_{i=1}^{n} \phi(i) \phi(i + 1)$$

$$- 2n^{\frac{3}{2}} \sum_{i=1}^{n} (\sqrt{n} - \sqrt{n-i}) \phi(i) \phi(i + 1)$$

$$+ n^{\frac{3}{4}} \sum_{i=1}^{n} \sqrt{\frac{2}{\beta}} \xi \phi^2(i) + 2n^{\frac{3}{4}} \sum_{i=1}^{n} Z_{n-i} \phi(i) \phi(i + 1),$$

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where we adopt the convention that $\phi(0) = \phi(n+1) = 0$. By Theorem 5.4.1 (which is proved independently), we obtain that with probability $\geq 1 - C \exp(-c k^3)$, $|\lambda| \leq C k^20$. By Proposition 1.2 for $i \geq Ln^4$, $|\phi(i)| \leq C \exp(-cL^2i)$. Moreover, from tail bounds for $\xi_i$ and $Z_i$, for any $1 \leq i \leq n$, $|\xi_i| \leq k^2$, $|Z_i| \leq k^4$. Thus we have $\sum_{i=1}^{n} \sqrt{i} \phi^2(i) \leq C k^2 n^4$ and $\sum_{i=1}^{n} Z_{n-i} \phi(i) \phi(i+1) \leq k^4 n^4 \sum_{i=1}^{n} (\phi^2(i) + \phi^2(i+1)) \leq C k^4 n^4$. Moreover,

$$
|2n^{\frac{3}{8}} \sum_{i=1}^{n} (\sqrt{n} - \sqrt{n-i}) \phi(i) \phi(i+1)|
\leq 2n^{\frac{3}{8}} \sum_{i=1}^{n} i |\phi(i)||\phi(i+1)|
\leq 2n^{\frac{3}{8}} \sum_{i=1}^{Ln^4} \frac{i}{\sqrt{n}} |\phi(i)||\phi(i+1)| + C n^{\frac{3}{8}} \exp(-cL^2i)
\leq 2L \sum_{i=1}^{n} \phi^2(i) + C \exp(-ck^{10}) \leq C k^{20}.
$$

Hence we have $\sqrt{n} \sum_{i=0}^{n} (\phi(i+1) - \phi(i))^2 \leq C k^4$.

Moreover, from Theorem 4.2.2, we deduce that

$$
\sqrt{n}(\phi(i+1) + \phi(i-1) - 2\phi(i)) = \lambda n^{-\frac{3}{8}} \phi(i) + \phi(i+1)(\sqrt{n} - \sqrt{n-i})
+ \phi(i-1)(\sqrt{n} - \sqrt{n-i+1}) - \left(\frac{1}{\beta} \sum_{i} \xi_i \phi(i) + Z_i \phi(i+1) + Z_{i-1} \phi(i-1)\right).
$$

As $|\phi(i)| \leq 1$ for $1 \leq i \leq Ln^4$, we have that $|\phi(i+1)(\sqrt{n} - \sqrt{n-i}) + \phi(i-1)(\sqrt{n} - \sqrt{n-i+1})| \leq 2Ln^{-\frac{3}{8}}$. Moreover, by tail bounds for $\xi_i$ and $Z_i$, with probability $\geq 1 - C \exp(-c k^3)$, we have $|\sqrt{n} \xi_i \phi(i) + Z_i \phi(i+1) + Z_{i-1} \phi(i-1)| \leq C k^4(|\phi(i)| + |\phi(i+1)| + |\phi(i-1)|)$. Thus with probability $\geq 1 - C \exp(-c k^3)$, for any $1 \leq i \leq Ln^4$, we have

$$
\sqrt{n}(\phi(i+1) + \phi(i-1) - 2\phi(i)) \leq C(k^{20}n^{-\frac{3}{8}} + k^4(|\phi(i)| + |\phi(i+1)| + |\phi(i-1)|)).
$$

(4.84)

We still use notations such as $A_1, A_2, \cdots$ defined above. First we bound $B_1$ and $B_2$. We have with probability $\geq 1 - C \exp(-c k^3)$,

$$
|B_1| \leq 2n^{\frac{3}{8}} \sum_{i=1}^{n} |Z_i||\phi(i)||\phi(i+1) - \phi(i)|
\leq 2n^{\frac{3}{8}} k^4 \sqrt{\sum_{i=1}^{n} \phi^2(i)} \sqrt{\sum_{i=1}^{n} (\phi(i+1) - \phi(i))^2}
\leq Ck^{6} n^{-\frac{3}{8}},
$$

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\begin{align*}
|B_2| & \leq 2n^{\frac{1}{2}} \sum_{1 \leq i \leq \ln^\frac{1}{3} n} |\Delta_i| \phi^2(i) + Cn^3 \exp(-cL^\frac{2}{3}) \\
& \leq Cn^{-\frac{1}{6}}.
\end{align*}

We bound \( A_1 \). As \( |A_1 - C_1| \leq Cn^{-\frac{1}{6}} \), it suffices to bound \( C_1 \). We have
\[
n^\frac{1}{3} \sum_{i=1}^n \sqrt{\frac{2}{\beta} (\xi_i + \zeta_i)} \phi^2(i) \leq Cn^\frac{1}{3} k^2 \exp(-cL^\frac{2}{3}) \leq C \exp(-ck^3) \ll n^{-\frac{1}{6}}.
\]
Moreover, we have \( n^\frac{1}{3} \sum_{i=1}^n \sqrt{\frac{2}{\beta} (\xi_i + \zeta_i)} \phi^2(i) = \sqrt{\frac{2}{\beta} \sum_{i=1}^n B_{i-\frac{1}{6}} n^\frac{1}{3} (\phi(i) - \phi(i+1)) (\phi(i) + \phi(i-1))}. \) We also have
\[
\frac{4}{\sqrt{\beta}} \int_0^\infty \psi'(x) \psi(x) B_x dx = \frac{4}{\sqrt{\beta}} \sum_{i=0}^{\ln^\frac{1}{3} n} \int_{n^\frac{1}{3}}^{(i+1)n^\frac{1}{3}} B_x n^\frac{1}{3} (\phi(i+1) - \phi(i)) \psi(x) dx.
\]
Now let \( R_1 = \sum_{i=1}^{\ln^\frac{1}{3} n} \int_{n^\frac{1}{3}}^{(i+1)n^\frac{1}{3}} \sqrt{n} (\phi(i+1) - \phi(i))(B_x - B_{i-\frac{1}{6}}) \psi(x) dx, R_2 = \sum_{i=1}^{\ln^\frac{1}{3} n} \int_{n^\frac{1}{3}}^{(i+1)n^\frac{1}{3}} B_{i-\frac{1}{6}} \sqrt{n} (\phi(i+1) - \phi(i))(n^\frac{1}{3} (\phi(i) + \phi(i+1)) - 2\psi(x)) dx. \) We have with probability \( \geq 1 - C \exp(-ck^3) \),
\[
|R_1| \leq C k^2 n^\frac{1}{3} \sum_{i=1}^{\ln^\frac{1}{3} n} |\phi(i+1) - \phi(i)| |\phi(i)| = C k^2 n^\frac{1}{3} \sum_{i=1}^{\ln^\frac{1}{3} n} \phi^2(i) \sqrt{\sum_{i=1}^{\ln^\frac{1}{3} n} (\phi(i+1) - \phi(i))^2}
\leq C k^4 n^{-\frac{1}{6}}.
\]
\[
|R_2| \leq C k^2 n^\frac{1}{3} \sum_{i=1}^{\ln^\frac{1}{3} n} (\phi(i+1) - \phi(i))^2
\leq C k^4 n^{-\frac{1}{6}}.
\]
Therefore, we conclude that \( |A_1 - \frac{4}{\sqrt{\beta}} \int_0^\infty B_x \psi'(x) \psi(x)| \leq C n^{-\frac{1}{6}} \).

We bound \( A_2 \). The term \( B_3 \) can be handled similarly, and we have that \( |B_3| \leq C n^{-\frac{1}{6}} \). Now for \( C_2, \) \( |\sum_{i=1}^n B_{i-\frac{1}{6}} \int_{n^\frac{1}{3}}^{(i+1)n^\frac{1}{3}} \phi(i+1)^2 dx | \leq 2n^4 \exp(-cL^\frac{2}{3}) \leq n^{-\frac{1}{6}}. \)
Moreover, for \( 1 \leq i \leq \ln^\frac{1}{3} n, \int_{n^\frac{1}{3}}^{(i+1)n^\frac{1}{3}} \psi(x)^2 dx = (\frac{\phi(i) - \phi(i+1)}{n^\frac{1}{6}})^2. \) Hence
\[
|C_2 + \int_0^\infty \psi'(x)^2 dx| \leq C n^{-\frac{1}{6}}.
\]
For \( C_3 \), \( n^{-\frac{1}{3}} \sum_{i=1}^{L_n^{\frac{1}{3}}} i|\phi(i)\phi(i+1)| \leq C n^2 \exp(-cL^2) \leq n^{-\frac{1}{3}} \), and

\[
\left| n^{-\frac{1}{3}} \sum_{i=1}^{L_n^{\frac{1}{3}}} i\phi(i)^2 - \int_0^L x\phi(x)^2 \, dx \right|
\]

\[ \leq CL \sum_{i=1}^{L_n^{\frac{1}{3}}} (|\phi(i)| + |\phi(i+1)|)(n^{-\frac{1}{3}}(|\phi(i+1)| + |\phi(i)|) + in^{-\frac{1}{3}}|\phi(i+1) - \phi(i)|) \]

\[ \leq CLn^{-\frac{1}{3}} + CL^2 \sqrt{\sum_{i=1}^{L_n^{\frac{1}{3}}} \phi^2(i)} \sqrt{\sum_{i=1}^{L_n^{\frac{1}{3}}} (\phi(i+1) - \phi(i))^2} \]

\[ \leq Ck^{42}n^{-\frac{1}{3}}. \]

Hence for any \( 1 \leq j \leq k \), with probability \( \geq 1 - C \exp(-ck^3) \),

\[
|g_j^T \hat{H}_{\beta,n}g_j + \hat{g}_j^T H_{\beta}\hat{g}_j| \leq Cn^{-\frac{1}{36}}. \tag{4.85}
\]

Similarly we can show that for any \( 1 \leq i \neq j \leq k \),

\[
|\hat{g}_i^T H_{\beta}\hat{g}_j| \leq Cn^{-\frac{1}{36}}, \tag{4.86}
\]

\[
|\hat{g}_j^T \hat{g}_j - 1| \leq Cn^{-\frac{1}{36}}, \tag{4.87}
\]

\[
|\hat{g}_i^T \hat{g}_j| \leq Cn^{-\frac{1}{36}}. \tag{4.88}
\]

Thus using the same method, we can conclude that with probability \( \geq 1 - C \exp(-ck^3) \), \( |\hat{\lambda}_j - a_j| \leq C_k n^{-\frac{1}{36}}. \)

\[ \square \]

5 Results from edge rigidity

By adapting the proof of Proposition 6.2 of [5], we obtain the following edge rigidity result. Note that the scaling is different here (namely, the support of the equilibrium measure is shrunk by \( \frac{1}{\sqrt{n}} \)).

**Theorem 5.1.** Suppose the eigenvalues of \( \beta \) ensemble of size \( n \) is \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), and suppose the typical locations of eigenvalues are \( \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \). Then for any \( \epsilon \in (0,1) \), any \( a_0 \in (0,1) \), there exist positive constants \( N_0, \epsilon, C \) which only depend on \( \epsilon \) and \( a_0 \), such that for any \( n \geq N_0, a \in [a_0, 1] \) and any \( 1 \leq k \leq n \), we have

\[
\mathbb{P}(|\lambda_k - \gamma_k| \geq n^{-\frac{1}{3}+a(k)\frac{1}{2}}) \leq C \exp(-cn^{\frac{1}{2}a(1-\epsilon)}), \tag{5.1}
\]

where \( k = \min\{n - k, k\} \).
Proof. From large deviation results of empirical measure and extreme eigenvalues of $\beta$ ensembles (see for example, section 2.6 of [1] and equation (6.2) of [5]), we have that for any $\delta > 0$, for some positive constants $C, c$,

$$P(|\lambda_k - \gamma_k| \geq \delta) \leq C \exp(-cn). \quad (5.2)$$

Thus the conclusion holds for scale $a = 1$.

In order to prove the result for all $a \in [a_0, 1]$, we adapt a bootstrap argument in [5]. First we use the accuracy result in the proof of Theorem 2.4 in [5]. That is, for any $a \in [a_0, 1]$, there is a constant $N_0$ such that for any $n > N_0$ and any $1 \leq k \leq n$,

$$|\gamma_k - \gamma_k^{(n)}| \leq n^{-\frac{2}{3}+a} (\hat{k})^{-\frac{1}{3}}, \quad (5.3)$$

where $\gamma_k^{(n)}$ is defined by $E(\#\{\lambda_i \leq \gamma_k^{(n)}\}) = k$. As in the proof of Theorem 2.4 of [5], we just need to prove a concentration result for all $a \in [a_0, 1]$, that is, we need to show

$$P(|\lambda_k - E[\lambda_k]| \geq n^{-\frac{2}{3}+a} (\hat{k})^{-\frac{1}{3}}) \leq C \exp(-cn^{\frac{1}{2}a(1-\epsilon)}). \quad (5.4)$$

To show this concentration result, we adapt the bootstrap argument for Proposition 6.2 of [5]. Below we state the adjustment that we need to make to get the conclusion. From the proof of Lemma 6.10, we can conclude that it suffices to prove the result for convexified measure $\nu$ with right hand side of (5.4) replaced by $C \exp(-cn^{a(1-\epsilon)})$. We just need to apply the bootstrap argument to the convexified measure $\nu$. We fix $\epsilon \in (0, \frac{1}{10})$. Suppose concentration at some scale $a \in [a_0, 1]$ holds. Now we show for any $\epsilon' \in (0, \frac{1}{10})$, concentration at scale $(1-\epsilon')a$ holds. To avoid confusion, we denote by $\epsilon_0$ the $\epsilon$ that is used in the proof of Lemma 6.17 of [5], and here we take $\epsilon_0 = \frac{1}{2} \epsilon a$. For $M = n^{a-\epsilon_0}$, from the proof of Lemma 6.16 of [5], we see that

$$S_{\omega}[d\nu/d\omega(k,M)] \leq C \exp(-cn^{a(1-\epsilon)}).$$

In the proof of Lemma 6.18, replace $a$ by $a - \epsilon_0$, and take $r$ sufficiently small. This combined with the proof of Proposition 6.2 of [5] shows that concentration at scale $a(1-\epsilon')$ holds.

6 Proof of large deviation bounds for the Airy point process

In this section, we present the proof of the large deviation bounds for the Airy point process (Theorem 2.1, 2.2 and 2.3). The proof strategy is to first use Airy approximation result (Theorem 2.4) to transfer the problem to one about large deviation bounds for GUE. Then combining estimates for Airy point processes and edge rigidity results derived in previous sections, we arrive at the desired conclusions.

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6.1 Proof of Exponential tightness

In this part, we prove exponential tightness (Theorem 2.1).

Proof of Theorem 2.1. The first conclusion comes from Proposition 3.1 and the fact that $\nu_0((\infty, Rk^2]) = \frac{2}{\pi^2}kR^{\frac{3}{2}}$. Here, $dv_0 = \frac{1}{\pi} \sqrt{x} 1_{x \geq 0} dx$.

Now $[-R_0, R_0]$ is compact, and for any $\mu \in K_\eta, |\mu|([-R_0, R_0]) \leq \eta R_0^{\frac{3}{2}}$, hence by Theorem 8.6.2 of [3], $K_\eta$ is a compact set.

6.2 Proof of LDP upper bound for nice tubes

In this part, we first lay out a few lemmas and then finish the proof of Theorem 2.3.

In order to approximate the Airy point process by GUE, we introduce the corresponding rescaled version of empirical measure for GUE.

Definition 6.1. Suppose $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ are the ordered eigenvalues from GUE of size $n$. We denote by $b_i := \left(\frac{n}{3}\right)^{\frac{2}{3}} (2 - \lambda_i)$. We also denote by

$$d\mu_0(x) := \frac{1}{\pi} \sqrt{x} \sqrt{1 - \frac{1}{4} \left(\frac{k}{n}\right)^2 x} 1_{0 \leq x \leq 4 \left(\frac{n}{3}\right)^{\frac{2}{3}}} dx \quad (6.1)$$

the rescaled equilibrium measure. We thus define

$$\mu_{n,k} := \frac{1}{k} \sum_{i=1}^n \delta_{b_i} - \mu_0. \quad (6.2)$$

We also have the following definition concerning the classical locations of eigenvalues of GUE.

Definition 6.2. We define the classical locations $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ of GUE (of size $n$) by the following

$$\int_{\gamma_i}^2 \frac{\sqrt{4 - x^2}}{2\pi} dx = \frac{i}{n} \quad (6.3)$$

We have the following elementary lemma which estimates $\gamma_i$. Note that the eigenvalues of the Airy operator are denoted by $\gamma_i^{\text{Airy}}$ in this section.

Lemma 6.1. Assume that $1 \leq i \leq \frac{n}{2}$. We have

$$2 - \left(\frac{3\pi i}{\sqrt{2n}}\right)^{\frac{2}{3}} \leq \gamma_i \leq 2 - \left(\frac{3\pi i}{2n}\right)^{\frac{2}{3}}. \quad (6.4)$$

Moreover, we have

$$\gamma_i \geq 2 - \left(\frac{3\pi i}{2n}\right) \frac{1}{\sqrt{1 - \frac{1}{4} \left(\frac{3\pi i}{2\sqrt{2n}}\right)^2}}. \quad (6.5)$$
Recall that $\gamma_i^{Airy} = (\frac{2}{\pi}(i - \frac{1}{2} + R(i))^{\frac{3}{2}}$, where $|R(i)| \leq \frac{C}{i}$. Assume that $1 \ll i \ll n$, we have for $n, i$ sufficiently large,

$$\gamma_i^{Airy} \leq (2 - \gamma_i)n^{\frac{3}{2}} \leq \gamma_i^{Airy} + C1^{\frac{3}{2}}((\frac{i}{n})^{\frac{3}{2}} + \frac{1}{i}),$$

(6.6)

where $C$ is a constant.

**Proof.** By definition, we have $\gamma_i \geq 0$ for $1 \leq i \leq \frac{n}{2}$. Using the inequality $\sqrt{2} \leq \sqrt{2 + x} \leq 2$ when $0 \leq x \leq 2$, we have

$$\frac{1}{\sqrt{2\pi}} \int_{\gamma_i}^{2} \sqrt{2 - x} dx \leq \frac{1}{n} \int_{\gamma_i}^{2} \sqrt{2 - x} dx.$$

(6.7)

By computing the integrals, we arrive at the first conclusion. Now we have

$$\frac{i}{n} \geq \frac{1}{2\pi} \int_{\gamma_i}^{2} \sqrt{2 + \gamma_i} \int_{\gamma_i}^{2} \sqrt{2 - x} dx,$$

(6.8)

from which we obtain the second conclusion. The last conclusion follows from the bounds for $\gamma_i$ and $\gamma_i^{Airy}$.

We have the following lemma lower bounding the probability that $\mu_{n,k}$ and $0$ are close.

**Lemma 6.2.** Assume that $n^{\frac{1}{10000}} \leq k \leq n^{\frac{1}{10000}}$. For any fixed $R \geq 1, \delta > 0$, there exists $N$ depending only on $R$, such that for any $n \geq N$, we have

$$\mathbb{P}(d_R(\mu_{n,k}, 0) \geq C \frac{R}{\sqrt{k}}) \leq C \exp(-ck^{\delta}).$$

(6.9)

Here $C, c > 0$ are absolute constants.

**Proof.** By definition, we have $\mu_{n,k} = \mu_1 + \mu_2$, where

$$\mu_1 := \frac{1}{k} \sum_{i=1}^{n} \left( \delta(\frac{i}{n})^{\frac{3}{2}}(2 - \lambda_i) - \delta(\frac{i}{n})^{\frac{3}{2}}(2 - \gamma_i) \right),$$

(6.10)

$$\mu_2 := \frac{1}{k} \sum_{i=1}^{n} \delta(\frac{i}{n})^{\frac{3}{2}}(2 - \gamma_i) - \mu_0.$$

(6.11)

Thus $d_R(\mu_{n,k}, 0) \leq d_R(\mu_1, 0) + d_R(\mu_2, 0)$.

We take $\eta = 15$ in Proposition 3.1 and take $\mathcal{A}_1 := \{N(k^{\frac{3}{2}}(R + 1)) \leq \eta(R + 1)^{\frac{3}{2}}k\}$. Then by Proposition 3.1 we have $\mathbb{P}(\mathcal{A}_1^c) \leq C \exp(-c(R + 1)^{\frac{3}{2}}k^{\frac{3}{2}})$. We also take $\mathcal{A}_2 := \{\text{For any } 1 \leq i \leq \eta(R + 1)k, |(2 - \lambda_i)n^{\frac{3}{2}} + a_i| \leq Cn^{-\frac{4}{3}}\}$. By Theorem 2.4 we have $\mathbb{P}(\mathcal{A}_2^c) \leq C \exp(-ck^{\delta})$. On $\mathcal{A}_1 \cap \mathcal{A}_2$, taking $i_0 = [\eta(R + 1)^{\frac{3}{2}}k] + 1$, we have for $n$ sufficiently large (depending only on $R$), $(2 - \lambda_{i_0})n^{\frac{3}{2}} \geq -a_{i_0} - Cn^{-\frac{4}{3}} \geq k^{\frac{3}{2}}(R + 1) - Cn^{-\frac{4}{3}} \geq k^{\frac{3}{2}}R$. Thus on $\mathcal{A}_1 \cap \mathcal{A}_2$, we have $\#\{i : \lambda_i \geq 2 - (\frac{3}{2})^{\frac{3}{2}}R\} \leq \eta(R + 1)^{\frac{3}{2}}k$.
Now in Theorem 5.1 we take \( \epsilon = \frac{1}{4} \) and \( n^a = \sqrt{k} \). We thus have
\[
P(|\lambda_i - \gamma_i| \geq n^{-\frac{3}{8}K^\frac{1}{2}} 2^{-\frac{3}{4}}) \leq C \exp(-ck^\frac{3}{8}) \tag{6.12}
\]
for \( 1 \leq i \leq n \) and \( n \) sufficiently large. We denote by \( B := \{|\lambda_i - \gamma_i| \leq n^{-\frac{3}{8}K^\frac{1}{2}} 2^{-\frac{3}{4}} \} \) for \( 1 \leq i \leq n \). By union bound we have \( P(B^c) \leq C n \exp(-ck^\frac{3}{8}) \leq C \exp(-k^\frac{3}{8}) \). On \( B \cap A_1 \cap A_2 \), we have
\[
d_R(\mu_1, 0) \leq \frac{1}{k} \sum_{1 \leq i \leq 2n(R+1)^{\frac{1}{8}} k} (\frac{n}{k})^{\frac{1}{2}} |\lambda_i - \gamma_i|
\leq \frac{1}{k} \sum_{1 \leq i \leq 2n(R+1)^{\frac{1}{8}} k} (\frac{n}{k})^{\frac{1}{2}} (n^{-\frac{3}{8}K^\frac{1}{2}} 2^{-\frac{3}{4}})
\leq C \frac{R}{\sqrt{k}}.
\]
Now we bound \( d_R(\mu_2, 0) \). Take any \( \phi(x) \) such that \( ||\phi||_{sup} \leq 1 \) and \( sup(\phi) \subseteq (\infty, R] \). By change of variables, we have
\[
\int \phi d\mu_2 = \frac{1}{k} \sum_{i=1}^{\tilde{n}} \phi((\frac{n}{k})^{\frac{1}{2}} (2 - \gamma_i)) - \frac{n}{2\pi} \int_{-2}^{2} \sqrt{4 - t^2} \phi((2 - t)(\frac{n}{k})^{\frac{1}{2}}) dt. \tag{6.13}
\]
Now by Lemma 6.1 we have \( (2 - \gamma_i)(\frac{n}{k})^{\frac{1}{2}} \geq (\frac{3\pi i_0}{2n})^{\frac{1}{2}} (\frac{n}{k})^{\frac{1}{2}} = \frac{3\pi i_0}{2n} R > R \). Then again by Lemma 6.1 we have (we take \( i_0 := 2 \))
\[
\int \phi d\mu_2 \leq \frac{1}{k} \sum_{i=1}^{\tilde{n}} |\phi((2 - \gamma_i)(\frac{n}{k})^{\frac{1}{2}})| - \int_{\gamma_i}^{\gamma_i - 1} \frac{n}{2\pi} \sqrt{4 - t^2} \phi((2 - t)(\frac{n}{k})^{\frac{1}{2}}) dt
\leq \frac{1}{k} \sum_{i=1}^{\tilde{n}} (\gamma_i - 1 - \gamma_i)(\frac{n}{k})^{\frac{1}{2}} \leq \frac{1}{k} (\frac{n}{k})^{\frac{1}{2}} (2 - \gamma_i)
\leq \frac{1}{k} (\frac{n}{k})^{\frac{1}{2}} (\frac{3\pi i_0}{2n})^{\frac{1}{2}} \leq C \frac{R}{k}.
\]
Hence \( d_R(\mu_2, 0) \leq C \frac{R}{k} \).

We come to the desired conclusion by combining the two bounds. \( \square \)

We have the following lemma concerning the influence of far-away particles. First we set up some definitions. We fix \( R_1, R_2(0 < R_1 < \frac{1}{2} R_2) \) in the following. We also assume that \( R_1 \geq 10 \).

**Definition 6.3.** Given \( \mu_{n,k} \) and \( R_1, R_2 \), we define \( \mu_1 := \mu_{n,k}|_{[-R_1,R_1]} \), \( \mu_2 := \mu_{n,k}|_{[-R_2-R_1,R_2-R_1]} \), \( \mu_3 := \mu_{n,k}|_{[-R_2,R_2]} \) and \( \mu_3' := \mu_{n,k}|_{(-R_2,R_2)} \).

**Lemma 6.3.** We assume that \( n^\frac{1}{10000} \leq k \leq n^\frac{1}{10000} \), and fix \( R_1, R_2 \) as above. For \( |x| \leq R_1 \), we denote by
\[
E(x) := \int \frac{1}{x-y} d\mu_3(y), \tag{6.14}
\]
and
\[ E'(x) := \int \frac{1}{x-y} d\mu'_3(y). \quad (6.15) \]

Fixing any \( M, C_1 > 0 \), we also denote by
\[ C := \{ |E(x)| \leq \frac{C_1 \sqrt{M}}{\sqrt{R}} \text{ for all } |x| \leq R_1 \}, \]
\[ C' := \{ |E'(x)| \leq \frac{C_1 M}{R^2} \text{ for all } |x| \leq R_1 \}, \]
\[ C'' := \{ |\mu_{n,k}((-\infty, -R_2])| \leq \frac{C_1 M}{R^2} \}. \]

Then there exists absolute constant \( C_1 \) (which is used to define the events \( C, C', C'' \)) and \( C_2 \), such that for \( n \) sufficiently large (depending on \( R_1, R_2 \)),
\[ \mathbb{P}(C \cup C' \cup C'') \leq C_2 \exp(-Mk^2). \quad (6.19) \]

**Proof.** We have for \( |x| \leq R_1 \),
\[ \left| \int \frac{1}{x-y} d\mu'_3(y) \right| \leq \frac{\mu_{n,k}((-\infty, -R_2])}{R_2 - R_1} \leq \frac{2\mu_{n,k}((-\infty, -R_2])}{R_2}. \quad (6.20) \]

Using similar estimates as in Proposition 3.1, we have for any \( \eta > 0 \),
\[ \mathbb{P}(\nu_k((-\infty, -\frac{1}{2}R_2]) > \eta) \leq \left( C \exp(-cR_2^3 k) \right)^\eta \leq C^{\eta k} \exp(-cR_2^3 \eta k^2). \]

Take \( \eta = \frac{2M}{cR_2^2} \), we obtain that \( \mathbb{P}(\nu_k((-\infty, -\frac{1}{2}R_2]) > \frac{2M}{cR_2^2}) \leq C \exp(-Mk^2) \) for \( k \) sufficiently large (depending on \( R_1, R_2 \)). Now using the coupling of Theorem 2.3, we have that \( |k^\triangle b_i + a_i| \leq Cn^{-\frac{M}{5}} \) with probability \( \geq 1 - C \exp(-c k^3) \).

On the event \( \{|k^\triangle b_i + a_i| \leq Cn^{-\frac{M}{5}} \} \cap \{\nu_k((-\infty, -\frac{1}{2}R_2 k^\triangle]) \leq \frac{2M}{cR_2^2}\} \), taking \( i_0 := \frac{2M}{cR_2^2} + 1 \), we have \(-a_i \geq -\frac{1}{4}R_2 k^\triangle \), and \( b_i \geq -\frac{1}{2} R_2 - Cn^{-\frac{M}{5}} \geq -R_2 \) for \( n \) sufficiently large (depending on \( R_1, R_2 \)). Thus we obtain
\[ \mathbb{P}(\mu_{n,k}((-\infty, R_2]) \geq \frac{4M}{cR_2^2}) \leq C \exp(-Mk^2). \quad (6.21) \]

By (6.20), we have
\[ \mathbb{P}(|E'(x)| \geq \frac{8M}{cR_2^2} \text{ for some } |x| \leq R_1) \leq C \exp(-Mk^2). \quad (6.22) \]
we denote by \( k \leq k \), for any 1 \( k \) is sufficiently large, for any \( 1 \leq k \leq M \), with probability \( 1 \). Similarly, we have
\[
\frac{1}{k} \sum_{N_1+1 \leq i < k^N} \frac{1}{b_i - x} \leq \frac{k^{24}}{\gamma_{k^N} - R_1} \leq Ck^{-6},
\]
(6.23)
if \( k \leq N_1 \leq k^{48} + k^{25} \),
\[
\int_{\gamma_{k^N}^+}^{\gamma_{k^N}^-} \frac{d\mu_0(y)}{y - x} \leq \frac{k^{24}}{\gamma_{k^N} - R_1} \leq Ck^{-6}.
\]
(6.24)
Below we assume that \( |x| \leq R_1 \). First we bound \( M_3 \). By Theorem 5.1 when \( n \) is sufficiently large, for any \( 1 \leq i \leq n \), with probability \( 1 - C \) \( -ck^3 \), \( |\lambda_{k^i} - \gamma_i| \leq n^{-\frac{2}{3}}k^{4} \). Moreover, by Lemma 6.1 \( |\gamma_i - \gamma_i - \gamma_i - \gamma_i - \gamma_i| \leq Cn^{-\frac{2}{3}}k^{-4} \).
Therefore, for any \( k^{48} \leq i \leq n \), we have \( \frac{1}{b_i - x} - \frac{1}{\gamma_i - x} \leq C\frac{k^{14}}{i^{2\frac{4}{3}}} \). Thus we have
\[
\left| \frac{1}{k} \frac{1}{b_i - x} - \int_{\gamma_i}^{\gamma_i + 1} \frac{d\mu_0(y)}{y - x} \right| \leq \frac{1}{k} \max \left\{ \frac{1}{b_i - x} - \frac{1}{\gamma_i - x} , \frac{1}{b_i - x} - \frac{1}{\gamma_i + 1 - x} \right\} \leq \frac{Ck^{13}}{i^{2\frac{4}{3}}}
\]
Thus
\[
M_3 \leq C \sum_{n=1}^{k^{48}} \frac{k^{13}}{i^{2\frac{4}{3}}} + Ck^{-6} \leq Ck^{-6}.
\]
(6.25)
Next we bound \( M_1, M_2 \). The main tools are Theorem 2.4 and Proposition 3.2. For \( 1 \leq i \leq k^4 \), we denote by \( S_i := \mu_{n,k}((\gamma_i, \gamma_i),(-\infty, (1)] \). In order to bound \( S_i \), we introduce the related quantity \( \tilde{S}_i := \mu_k((\gamma_i, \gamma_i),(-\infty, (1)] \). We assume that \( \gamma_i \leq R < \gamma_i^{(i+1)} \), and by Lemma 6.1 we have \( |t - 2b_i^2k| \leq 2 \) when \( n \) is sufficiently large. By Lemma 6.1 and Proposition 3.2 with probability \( 1 - C \) \( -Mk^2 \), for \( \frac{1}{k} \leq i \leq k^4, |\tilde{S}_i| \leq C \sqrt{M} \log i \), and for \( k^\frac{4}{3} \leq i \leq k^4, |\tilde{S}_i| \leq C \log i \log k \max \{ \frac{1}{k}, 1 \} \).
By Theorem 2.4 with probability $\geq 1 - C \exp(-MK^2)$, for $\lfloor \frac{i}{k} \rfloor \leq i \leq k^{\frac{1}{2}}$, $|S_i| \leq C \sqrt{M} \log i$, and for $k^{\frac{1}{2}} \leq i \leq k^{47}$, $|S_i| \leq C \log i \sqrt{\log k \max\{\frac{1}{k}, 1\}}$.

Now we have (note that $|\gamma'_{(i+1)k} - \gamma'_{ik}| \leq C i^{-\frac{1}{2}}$)

$$
|\int_{[\gamma'_{ik}, \gamma'_{(i+1)k})} \frac{d\mu_{n,k}(y) - S_{i+1} - S_i}{\gamma'_{ik} - x}| \\
= |\int_{[\gamma'_{ik}, \gamma'_{(i+1)k})} \frac{d\mu_{n,k}(y)}{y - x} - \int_{[\gamma'_{ik}, \gamma'_{(i+1)k})} \frac{d\mu_{n,k}(y)}{\gamma'_{ik} - x}| \\
= |\int_{[\gamma'_{ik}, \gamma'_{(i+1)k})} \frac{(y - \gamma'_{ik})d\mu_{n,k}(y)}{(y - x)(\gamma'_{ik} - x)}| \\
\leq \frac{\int_{[\gamma'_{ik}, \gamma'_{(i+1)k})} (\gamma'_{i+1,k} - \gamma'_{ik})d(\mu_{n,k} + 2\mu_0)(y)}{(y - x)(\gamma'_{ik} - x)} \\
\leq \frac{(\mu_{n,k} + 2\mu_0)([\gamma'_{ik}, \gamma'_{(i+1)k}])}{i^\frac{1}{15}} \leq C(|S_{i+1} - S_i| + 1).
$$

We also note that with probability $\geq 1 - C \exp(-MK^2)$,

$$
\left| \sum_{i=\lfloor \frac{1}{k} \rfloor + 1}^{k^{47} - 1} \frac{S_{i+1} - S_i}{\gamma'_{ik} - x} \right| \\
\leq \left| \sum_{i=\lfloor \frac{1}{k} \rfloor + 1}^{k^{47} - 1} S_{i+1} \left( \frac{1}{\gamma'_{ik} - x} - \frac{1}{\gamma'_{(i+1)k} - x} \right) \right| + \frac{|S_{k^{47}}|}{\gamma'_{k^{48} - x}} + \frac{|S_{\lfloor \frac{k^{47}}{k} \rfloor + 1}|}{\gamma'_{\lfloor \frac{k^{47}}{k} \rfloor + 1} - x} \\
\leq C \sum_{i=\lfloor \frac{1}{k} \rfloor + 1}^{k^{47}} \frac{|S_i|}{i^\frac{1}{15}} + C \sqrt{M} \log R.
$$

For the interval $I_0 := (R, \gamma'_{\lfloor \frac{k}{k} \rfloor + 1})$, we have with probability greater than or equal to $1 - C \exp(-MK^2)$,

$$
\left| \int_{I_0} \frac{d\mu_{n,k}(y)}{y - x} \right| \leq 2 \int_{I_0} \frac{d(\mu_{n,k} + 2\mu_0)(y)}{R} \\
\leq \frac{C(\mu_{n,k}(I_0) + \mu_0(I_0))}{R} \leq \frac{C \sqrt{M} \log R}{R}.
$$

Combining the estimates, we have

$$
M_1 + M_2 \leq \frac{C \sqrt{M} \log R}{R} + \sum_{i=\lfloor \frac{1}{k} \rfloor + 1}^{k^{47}} \frac{|S_i| + 1}{i^\frac{1}{15}}. \quad (6.26)
$$
Now we have with probability \( \geq 1 - C \exp(-Mk^2) \),

\[
\sum_{i=\lfloor \frac{k}{2} \rfloor + 1}^{k^\frac{1}{2}} \frac{|S_i| + 1}{i^{\frac{3}{2}}} \leq \sum_{i=\lfloor \frac{k}{2} \rfloor + 1}^{k^\frac{1}{2}} \frac{C\sqrt{M}\log i}{i^{\frac{3}{2}}} \leq \frac{C\sqrt{M}}{\sqrt{R}},
\]
\[
\sum_{i=k^\frac{1}{2}}^{k} \frac{|S_i| + 1}{i^{\frac{3}{2}}} \leq \sum_{i=k^\frac{1}{2}}^{k} C\log i \sqrt{\log n} \leq C\frac{(\log n)\frac{3}{2}}{k^{\frac{3}{2}}} \leq Ck^{-\frac{3}{2}}.
\]
\[
\sum_{i=k^\frac{1}{2}}^{k^47} \frac{|S_i| + 1}{i^{\frac{3}{2}}} \leq \sum_{i=k^\frac{1}{2}}^{k^47} C\log i \sqrt{\log n} \sqrt{\frac{k}{i}} \leq C\frac{(\log n)\frac{3}{2}}{k^{\frac{3}{2}}} \leq Ck^{-\frac{3}{2}}.
\]

Hence with probability \( \geq 1 - C \exp(-Mk^2) \), \( |E(x)| \leq C\frac{\sqrt{M}}{\sqrt{R}} \).

To facilitate notations below, we introduce the following definition.

**Definition 6.4.** Denote by \( \Delta := \{(x,x) : x \in \mathbb{R}\} \). For any \( \mu \in \mathcal{Y} \), we define

\[
J(\mu) := -\int_{\mathbb{R}^2 \setminus \Delta} \log(|x-y|)d\mu(x)d\mu(y).
\]  

(6.27)

For any \( \mu_1, \mu_2 \in \mathcal{Y} \), we also define

\[
J(\mu_1, \mu_2) := -\int_{\mathbb{R}^2 \setminus \Delta} \log(|x-y|)d\mu_1(x)d\mu_2(y).
\]  

(6.28)

We also present a lemma lower bounding \( J(\mu_1), J(\mu_1, \mu_2) \) and \( J(\mu_1, \mu'_2) \).

**Lemma 6.4.** Suppose that \( d_R(\mu_n, k, \mu) \leq \delta \) for \( R \geq R_2 \). Assume also that \( |\mu|(\mathbb{R}) < \infty \) and

\[ \log R_1 |\mu|((-R_1, R_1) \epsilon) < \epsilon \]

for some \( \epsilon > 0 \). We assume that \( R_1 \geq 10, R_2 \geq R_1^5 \) and \( \delta < \frac{1}{2} \). Then there exists a constant \( C \) which only depends on \( \mu \), such that

\[
J(\mu_1) \geq -\int \log(\max\{|x-y|, \frac{1}{R_1}\})d\mu(x)d\mu(y) - C(\sqrt{\delta}\sqrt{R_1} + R_1^3\delta + R_1^6\delta^2 + \epsilon),
\]
\[
J(\mu_1, \mu_2 + \mu'_2) \geq -C(\sqrt{\delta}\sqrt{R_2} + R_2^3\delta + R_2^6\delta^2 + \epsilon).
\]  

(6.29)  

(6.30)
Proof. We start with the first bound. We denote by \( \gamma(\delta, R_1) := \frac{1}{R_1} \), and decompose as follows

\[
J(\mu_1) = \int -\log(\max\{|x-y|, \gamma(\delta, R_1)\})d\mu_1(x)d\mu_1(y) \\
+ \int -\log\left(\frac{|x-y|}{\gamma(\delta, R_1)}\right)1_{|x-y|\leq \gamma(\delta, R_1)}d\mu_1(x)d\mu_1(y) \\
:= K_1 + K_2.
\]

Now we have

\[
K_2 \geq 2 \int_{[-R_1, R_1]} d(\mu_1 + \mu_0)(x) \int_{|y-x| \leq \gamma(\delta, R_1)} d\mu_0(y) \log\left(\frac{|x-y|}{\gamma(\delta, R_1)}\right) \\
\geq 2 \sqrt{R_1} \int_{[-R_1, R_1]} d(\mu_1 + \mu_0)(x) \int_{|y-x| \leq \gamma(\delta, R_1)} dy \log\left(\frac{|x-y|}{\gamma(\delta, R_1)}\right) \\
= -4 \sqrt{R_1} \gamma(\delta, R_1)(\mu_1([-R_1, R_1]) + \mu_0([-R_1, R_1])) \\
\geq -4 \sqrt{R_1} \gamma(\delta, R_1)(|\mu|(\mathbb{R}) + \delta + R_1^2) \geq -\frac{C}{R_1}.
\]

Moreover, for any fixed \( x \in [-R_1, R_1] \),

\[\phi(x, y) := -\log(\max\{|x-y|, \gamma(\delta, R_1)\})\]

viewed as a function in \( y \) is \( \frac{1}{\gamma(\delta, R_1)} \)-Lipschitz and its absolute value is bounded by \( \log \frac{1}{\gamma(\delta, R_1)} \) for \( |y| \leq 2R_1 \). Now for \( \epsilon > 0 \) we define \( \psi(x) := 1_{[-R_1-\epsilon, R_1+\epsilon]} - \frac{x-R_1}{2\epsilon}1_{R_1 \leq x \leq R_1+\epsilon} + \frac{R_1+x}{2\epsilon}1_{-R_1-\epsilon \leq x \leq -R_1} \), and define \( \psi(x) := 1_{[R_1-2\epsilon, R_1+2\epsilon]} - \frac{x-(R_1+\epsilon)}{2\epsilon}1_{R_1+\epsilon \leq x \leq R_1+2\epsilon} + \frac{R_1+\epsilon+x}{2\epsilon}1_{-R_1-2\epsilon \leq x \leq -R_1-\epsilon} \).

We have

\[
K_1 = \int \phi(x, y)\psi(y)\psi(y)d\mu_{n,k}(x)d\mu_{n,k}(y) \\
+ \int \phi(x, y)(1_{[-R_1, R_1]}(x) - \psi(x))\psi(y)d\mu_{n,k}(x)d\mu_{n,k}(y) \\
+ \int \phi(x, y)1_{[-R_1, R_1]}(x)(1_{[-R_1, R_1]}(y) - \psi(y))d\mu_{n,k}(x)d\mu_{n,k}(y) \\
:= K_{1,1} + K_{1,2} + K_{1,3}.
\]

From the analysis above, we have that \( |\phi(x, y)\psi(y)| \leq \log \frac{1}{\gamma(\delta, R_1)} \). For any fixed \( y \in [-R_1, R_1] \), we have \( |\phi(x_1, y)\psi(x_1) - \phi(x_2, y)\psi(x_2)| \leq (\log \frac{1}{\gamma(\delta, R_1)} + \frac{1}{\gamma(\delta, R_1)})|x_1 - x_2| \). Thus we have \( |\int \phi(x, y)\psi(y)d(\mu_{n,k} - \mu)| \leq (\log \frac{1}{\gamma(\delta, R_1)} + \frac{1}{\gamma(\delta, R_1)})\delta|\mu|(\mathbb{R}). \)

\[
(6.31)
\]
Moreover, for any \((x_1, y_1), (x_2, y_2) \in [-2R_1, 2R_1]^2\), we have

\[
|\phi(x_1, y_1) - \phi(x_2, y_1) - \phi(x_1, y_2) + \phi(x_2, y_2)| \leq \frac{1}{\gamma(\delta, R_1)^2}|x_1 - x_2||y_1 - y_2|.
\]

Therefore for any \(y_1, y_2 \in [-2R_1, 2R_1]\),

\[
||| \phi(x, y) - \phi(x, y_2) \phi(x) |||_{Lip} \leq \left( \frac{1}{\gamma(\delta, R_1)^2} + \frac{1}{\gamma(\delta, R_1)^2} \right) ||y_1 - y_2||,
\]

and thus

\[
|\int (\phi(x, y) - \phi(x, y_2) \phi(x)) d(\mu_{n,k} - \mu)| \leq \left( \frac{1}{\gamma(\delta, R_1)^2} + \frac{1}{\gamma(\delta, R_1)^2} \right) ||y_1 - y_2||.
\]

(6.32)

We obtain

\[
|\int \phi(x, y) \psi(x) \psi(y) d(\mu_{n,k} - \mu) d(\mu_{n,k} - \mu)| \leq 2 \left( \frac{1}{\gamma(\delta, R_1)^2} + \frac{1}{\gamma(\delta, R_1)^2} \right) \delta^2.
\]

(6.33)

Therefore, we get

\[
|K_{1,1} - \int \phi(x, y) \psi(x) \psi(y) d\mu(x) d\mu(y)| \leq 2 \left( \frac{1}{\gamma(\delta, R_1)^2} + \frac{1}{\gamma(\delta, R_1)^2} \right) \delta \mu(\mathbb{R}) + 2 \left( \frac{1}{\gamma(\delta, R_1)^2} + \frac{1}{\gamma(\delta, R_1)^2} \right) \delta^2.
\]

Next we bound \(K_{1,2}\) and \(K_{1,3}\). We have

\[
|\mu_{n,k}([R_1 - \epsilon', R_1 + \epsilon'])| \leq (\mu_{n,k} + \mu_0)([R_1 - \epsilon', R_1 + \epsilon']) \leq 2 \sqrt{R_1} \epsilon' + \int \psi d(\mu_{n,k} + \mu_0) \leq 2 \sqrt{R_1} \epsilon' + \int \psi d(\mu + \mu_0) + \frac{\delta}{\epsilon'} \leq C(\sqrt{R_1} \epsilon' + \frac{\delta}{\epsilon'} + |\mu|([R_1/2, \infty))) \leq C(\sqrt{R_1} \epsilon' + \frac{\delta}{\epsilon'} + \frac{\epsilon}{\log R_1}).
\]

Similar bounds hold for \(|\mu_{n,k}([R_1 - \epsilon', -R_1 + \epsilon'])| \). Therefore

\[
K_{1,2} \leq C \log \frac{1}{\gamma(\delta, R_1)} |\mu(\mathbb{R})| |\mu_{n,k}|([R_1 - \epsilon', -R_1 + \epsilon'] \cup [R_1 - \epsilon', R_1 + \epsilon']) \leq C \log \frac{1}{\gamma(\delta, R_1)} |\mu(\mathbb{R})| (\sqrt{R_1} \epsilon' + \frac{\delta}{\epsilon'} + \frac{\epsilon}{\log R_1}).
\]

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The same bound holds for $K_{1.3}$. We also note that

$$\left| \int \phi(x,y)\psi(x)\psi(y)d\mu(x)d\mu(y) - \int_{[-R_1,R_1]} \phi(x,y)d\mu(x)d\mu(y) \right|$$

$$\leq \left| \int \phi(x,y)(\psi(x) - 1_{[-R_1,R_1]}(x))(\psi(y) - 1_{[-R_1,R_1]}(y))d\mu(x)d\mu(y) \right| + 2\left| \int \phi(x,y)(\psi(x) - 1_{[-R_1,R_1]}(x))1_{[-R_1,R_1]}(y)d\mu(x)d\mu(y) \right|$$

$$\leq 4\log \frac{1}{\gamma(\delta,R_1)}|\mu|(\mathbb{R})|\mu|([-R_1-\epsilon',R_1] \cup [R_1,R_1+\epsilon'])$$

$$\leq C|\mu|(\mathbb{R})\log \frac{1}{\gamma(\delta,R_1)}(\sqrt{R_1}\epsilon' + \frac{\delta}{c} + \frac{\epsilon}{\log R_1})$$

By taking $\epsilon' = \frac{\sqrt{\delta}}{R_1^2}$, we obtain that

$$J(\mu_1) \geq -\int \log(\max\{|x-y|, \frac{1}{R_1}\})d\mu(x)d\mu(y) - C(\sqrt{\delta} \sqrt{R_1} + R_1^3 \delta + R_1^6 \delta^2 + \epsilon).$$ 

(6.34)

Now we prove the remaining conclusions. We take $\gamma' = \frac{1}{R_1}$, $\epsilon' > 0$ and

$$\psi_1(x) := 1_{[R_1-\gamma',R_1+\epsilon']}(\sqrt{\frac{\epsilon}{\epsilon'}} 1_{[R_1,R_1+\epsilon']} + \frac{\epsilon - (R_1-\gamma')}{\epsilon'} 1_{[R_1-\gamma'-\epsilon',R_1-\gamma']}).$$

We have

$$|\mu_1([R_1-\gamma', R_1])| \leq (\mu_1 + \mu_0)([R_1 - \gamma', R_1]) + \mu_0([R_1 - \gamma', R_1])$$

$$\leq \int \psi_1d(\mu_1 + \mu_0) + \sqrt{R_1}\gamma'$$

$$\leq \int \psi_1d(\mu + \mu_0) + \frac{\delta}{c} + \sqrt{R_1}\gamma'$$

$$\leq |\mu|([\frac{1}{2}R_1, \infty]) + \sqrt{R_1}\gamma' + \sqrt{R_1}\epsilon' + \frac{\delta}{c}. $$

We take $\epsilon' = \frac{\sqrt{\delta}}{R_1^2}$, and obtain $|\mu_1([R_1 -\gamma', R_1])| \leq \sqrt{R_1}\gamma' + |\mu|([\frac{1}{2}R_1, \infty]) + R_1^4 \sqrt{\delta}$. Similarly, $|\mu_2([R_1, R_1 +\gamma']) \leq \sqrt{R_1}\gamma' + |\mu|([\frac{1}{2}R_1, \infty]) + R_1^4 \sqrt{\delta}$.

We have

$$J(\mu_1,\mu_2 + \mu_2') \geq -\int \log(\max\{|x-y|, \gamma'\})d\mu_1(x)d(\mu_2 + \mu_2')(y)$$

$$- \int \log(\frac{|x-y|}{\gamma'})1_{|x-y| \leq \gamma}d\mu_1(x)d\mu_2(y)$$

$$- \int \log(\frac{|x-y|}{\gamma'})1_{|x-y| \leq \gamma}d\mu_1(x)d\mu_2'(y) := L_1 + L_2 + L_3.$$

Now by a similar approach as above, we have

$$L_1 \geq -C(\sqrt{\delta} \sqrt{R_2} + R_2^3 \delta + R_2^6 \delta^2 + \epsilon).$$

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We also have that

\[ L_2 \geq -\int \log\left(\frac{|x-y|}{\gamma'}\right)1_{|x-y| \leq \gamma'}d\mu_0(x)d(\mu_2 + \mu_0)(y) \]

\[ -\int \log\left(\frac{|x-y|}{\gamma'}\right)1_{|x-y| \leq \gamma'}d(\mu_1 + \mu_0)(x)d\mu_0(y) \]

\[ \geq -\sqrt{R_1} \int_{[R_1 - \gamma', R_1] \times [R_1, R_1 + \gamma']} \log\left(\frac{|x-y|}{\gamma'}\right)dxd(\mu_2 + \mu_0)(y) \]

\[ -\sqrt{R_1} \int_{[R_1 - \gamma', R_1] \times [R_1, R_1 + \gamma']} \log\left(\frac{|x-y|}{\gamma'}\right)d(\mu_1 + \mu_0)(x)dy \]

\[ \geq -\sqrt{R_1} \gamma'(2\sqrt{R_1} + |\mu_1|([R_1 \gamma', R_1]) + |\mu_2|([R_1, R_1 + \gamma'])) \]

\[ \geq -C \sqrt{R_1} \gamma'(\sqrt{R_1} + \frac{\epsilon}{\log R_1} + R_1^\frac{1}{4} \sqrt{\delta}) \]

Moreover, using a similar argument as above,

\[ L_3 \geq -C \sqrt{R_1} \gamma'(\sqrt{R_1} + \frac{\epsilon}{\log R_1} + R_1^\frac{1}{4} \sqrt{\delta}). \tag{6.35} \]

Hence we have

\[ J(\mu_1, \mu_2 + \mu'_2) \geq -C(\sqrt{\delta} \sqrt{R_1} + R_2^\delta + R_2^\delta + \epsilon). \tag{6.36} \]

We also have the following definition.

**Definition 6.5.** We define

\[ \xi(x) := -\int \log|x-y|d\rho_0(y) + \frac{1}{4}x^2 - \frac{1}{2} \]

\[ \tag{6.37} \]

We note that by the proof of Lemma 2.7 in [2], \( \xi(x) \) satisfies \( \xi(x) = 0 \) for \( |x| \leq 2 \), \( \xi(x) = \xi(-x) \) and \( \xi'(x) = -\frac{\sqrt{x^2 - 4}}{2} \) for \( x > 2 \). We obtain that \( \xi(x) = \int_2^x \frac{\sqrt{s^2 - 4}}{2}ds \) \( (x \geq 2) \). Thus we have the estimates for \( x \geq 2 \)

\[ \xi(x) \geq \int_2^x \sqrt{s^2 - 4}ds = \frac{2}{3}(x - 2)\frac{1}{2} \]

\[ \tag{6.38} \]

and

\[ \xi(x) \leq \frac{\sqrt{x^2 + 2}}{3}(x - 2)^\frac{1}{2}. \tag{6.39} \]

Hence we have that

\[ \lim_{x \to 2^+0} \frac{\xi(x)}{\frac{2}{3}(x - 2)^\frac{1}{2}} = 1. \tag{6.40} \]

With the above lemmas, we finish the proof of Theorem 2.3. We first prove the \( \mu(\mathbb{R}) = 0 \) case.
Proof of Theorem 2.3. \( \mu(\mathbb{R}) = 0 \) case. Assume that \( \mu \in \mathcal{W} \). For any \( \epsilon > 0 \), we take \( R_1 \) sufficiently large that \( \log R_1 / \mu((−R_1, R_1)^c) < \epsilon \). We also take \( R_2 = R_1^6 \). We also assume that \( n \) sufficiently large.

For any given \( M > 0 \), we take \( \mathcal{D}_1(M) := \{ \text{For any } |x| \leq R_1, |\int \frac{dy}{y−x}| \leq C_1(\frac{M}{R_1^2}) \} \) (where \( C_1 \) is as in the statement of Lemma 6.3), \( \mathcal{D}_2 := \{ d_{−R_1,R}[−R_1,R_1](\mu_{n,k}, \mu) \leq \delta \} \), \( \mathcal{D}_3(M) := \{ \#\{ i : |b_i| \leq R_1 \} \leq CMR_1^2 k \} \). By Lemma 6.3 and Proposition 3.1, we have \( \mathbb{P}(\mathcal{D}_1(M)^c \cup \mathcal{D}_3(M)^c) \leq C \exp(−Mk^2) \). Below we let \( \mathcal{D}(M) := \mathcal{D}_1(M) \cap \mathcal{D}_2 \cap \mathcal{D}_3(M) \). Thus

\[
\mathbb{P}(\{ d_R(\mu_{n,k}, \mu) \leq \delta \} \cap \mathcal{D}) \leq C \exp(−Mk^2) + \mathbb{P}(\{ d_R(\mu_{n,k}, \mu) \leq \delta \} \cap \mathcal{D}^c).
\]

We proceed by conditioning on the particles outside of \([-R_1, R_1]\). For any given \( K \) and \( \alpha_1, \alpha_2, \cdots, \alpha_{n−K} \in [−R_1, R_1]^c \), we denote by

\[
\mathcal{L} := \{ \{ b_1, b_2, \cdots, b_n \} \cap [−R_1, R_1]^c = \{ \alpha, \cdots, \alpha_{n−K} \} \}.
\]

Below we will derive a uniform upper bound on the quantity

\[
R(\mathcal{L}) := \mathbb{P}(\{ d_R(\mu_{n,k}, \mu) \leq \delta \} \mid \mathcal{L})
\]

when \( \mathcal{L} \subseteq \mathcal{D} \).

We write \( \xi(x) := \frac{n}{2} \xi(2 − (\frac{x}{n})^2) \), and denote by

\[
J_0(\nu) := −\int_{\mathbb{R}^2} \log|x − y|d\nu(x)d\nu(y) + 2 \int_{−\infty}^{0} \tilde{\xi}(x)d\nu(x).
\]

The joint density of GUE eigenvalues is given by (see, for example, [1] and [19])

\[
\rho(x_1, \cdots, x_n) = K_n^{-1} \exp(−k^2 J_0(\mu_{n,k})),
\]

where \( K_n \) is normalizing constant, and \( \Delta = \{ (x, x) : x \in \mathbb{R} \} \).

We denote by \( \tilde{x} = (x_1, x_2, \cdots, x_K) \) the rescaled eigenvalues in the interval \([-R_1, R_1]\). For \( \mathcal{L} \subseteq \mathcal{D} \), by definition of \( \mathcal{D}_3 \), we have \( K \leq CMR_1^2 k \). By definition of \( \mu_{n,k} \), we have \( \mu_{n,k} = \frac{1}{K} \sum_{i=1}^{K} \delta_{x_i} + \frac{1}{K} \sum_{i=1}^{n−K} \delta_{\alpha_i} = \mu_0 \). For any given \( \mathcal{L} \), there is a constant \( C(\mathcal{L}) \) that only depends on \( \mathcal{L} \), such that for any measurable event \( \mathcal{A} \),

\[
\mathbb{P}(\mathcal{A} \mid \mathcal{L}) = C(\mathcal{L}) \int_{\tilde{x} \in \mathcal{A}} \exp(−k^2 J_0(\mu_{n,k}))d\tilde{x}.
\]

We thus have

\[
\mathbb{P}(\{ d_R(\mu_{n,k}, \mu) \leq \delta \} \mid \mathcal{L}) = C(\mathcal{L}) \int_{\tilde{x} \in \{ d_R(\mu_{n,k}, \mu) \leq \delta \}} \exp(−k^2 J_0(\mu_{n,k}))d\tilde{x} \leq C(\mathcal{L}) \exp(−k^2 \inf_{\tilde{x} \in \{ d_R(\mu_{n,k}, \mu) \leq \delta \}} \{ J_0(\mu_{n,k}) \})(2R_1)^K.
\]
Since \( \mu(\mathbb{R}) = 0 \) and \( \log R_1 |\mu([-R_1, R_1])| < \varepsilon \), we have that \( |\mu([-R_1, R_1])| < \frac{\varepsilon}{\log R_1} \). Using a similar argument as in the proof of Lemma 6.3, since \( d_R(\mu_{n,k}, \mu) \leq \delta \), we have \( |\mu_{n,k}([-R_1, R_1])| \leq C(\frac{\varepsilon}{\log R_1} + \delta \frac{\log R_1}{k}) := \Theta_1 \). We conclude that \( |\mu_{n,k}([0, R_1])| \leq \Theta_1 \). We now take a configuration \( \mathbf{c} := (c_1, c_2, \cdots, c_K) \) as follows: first we fix \( R_0 \in [1, 10] \) such that \( L := K - k \mu_0([R_0, R_1]) \in \mathbb{Z} \). For \( 1 \leq i \leq L \), we take \( c_i \) such that \( \mu_0([0, c_i]) = \frac{1}{L} \mu_0([0, R_0]) \), and for convenience we assume \( c_0 = 0 \); for \( 1 \leq i \leq K - L \), we take \( d_i \) such that \( \mu_0([R_0, d_i]) = \frac{i - 1}{k - L} \mu_0([R_0, R_1]) \), and take \( c_{i + L} = d_i \).

With the constructed \( \{c_i\}_{i=1}^K \), we define \( C := \{|\mathbf{x}| : |x_i - c_i| \leq \frac{1}{2n} \text{ for } 1 \leq i \leq K\} \). Therefore we have

\[
1 = C(L) \int_{\mathbf{x}: x_i \in [-R_i, R_i]} \exp(-k^2 J_0(\mu_{n,k})) \, d\mathbf{x} \\
\geq C(L) \int_{\mathbf{x} \in C} \exp(-k^2 J_0(\mu_{n,k})) \, d\mathbf{x} \\
\geq \frac{C(L)}{n^K} \exp(-k^2 \sup_{\mathbf{x} \in C} J_0(\mu_{n,k})).
\]

Moreover, from the definitions we have the following estimates: there exist constants \( C_1, C_2 > 0 \), such that for \( 1 \leq i < j \leq L \), \( c_j - c_i \geq \frac{C_1(j - i)}{k} \) and \( \frac{C_2}{k} \leq R_0 - c_K \leq \frac{C_2}{k} \). Moreover, we have \( c_{i+1} - c_i \leq \frac{C_2}{k} \). We also bound the discrepancy \( |\mu_0([c_{j-1}, c_j]) - \frac{1}{k}| \) for \( 1 \leq j \leq L \) as follows:

\[
|\mu_0([c_{j-1}, c_j]) - \frac{1}{k}| \leq \frac{1}{L + 1} |\mu_0([0, R_1]) - \frac{1}{k}| \\
\leq \frac{1}{L + 1} \mu_0([0, R_0]) - \frac{L}{k} + \frac{1}{(L + 1)k} \\
\leq \frac{\Theta_1}{L + 1} + \frac{1}{(L + 1)k} \\
\leq \frac{C(\frac{\varepsilon}{\log R_1} + \delta \frac{\log R_1}{k})}{k}
\]

for \( k \) sufficiently large (depending on \( R_1 \)).

For fixed \( i \), we also have the following estimate on \( \sum_{j: j \neq i, 1 \leq j \leq L} |\log(|c_j - c_i|)| \). First as \( |c_j - c_i| \leq R_0 \), we have \( \sum_{j: j \neq i, 1 \leq j \leq L} \log(|c_j - c_i|) \leq L \log R_0 \leq CK \). Second from the lower bound of \( c_j - c_i \), using the integral to approximate the sum we obtain

\[
\sum_{j: j \neq i, 1 \leq j \leq L} \log(|c_j - c_i|) \leq \sum_{j=1}^{L} \log(C_1j) \leq - \int_0^1 \log(C_1x) \, dx = \frac{k}{C_1}.
\]

Hence \( \sum_{j \neq i} |\log(|c_j - c_i|)| \leq CK \).

We bound \( \sup_{\mathbf{x} \in C} J_0(\mu_{n,k}) \) below. We only present the details for bounding \( J_0(\mu_{n,k}) \) when \( \bar{x} = \mathbf{c} \), and the same bound holds for general \( \bar{x} \in C \). Below we take \( \bar{x} = \mathbf{c} \).
First we bound $J(\mu_1)$. We denote by $\mu_{11} := \frac{1}{k} \sum_{i=1}^{L} \delta_{c_i} - \mu_0|_{[0,R_0]}$ and $\mu_{12} := \frac{1}{k} \sum_{i=1}^{K-L} \delta_{c_i} - \mu_0|_{(R_0,R_1]}$, which leads to $\mu_1 = \mu_{11} + \mu_{12}$.

For $1 \leq i \neq j \leq L$, we define

$$L_{ij} := \frac{1}{k} \log |c_i - c_j| - \int_{c_i}^{c_j} \log(|x - c_i|)d\mu_0(x).$$

We also define for $1 \leq i \leq L$ the quantity

$$L_i := \frac{1}{k} \sum_{1 \leq j \leq L, j \neq i} \log(|c_i - c_j|) - \int_{R_0}^{0} \log(|x - c_i|)d\mu_0(x).$$

We have $L_i = \sum_{1 \leq i \leq L, j \neq i} L_{ij} - \int_{c_i}^{R_0} \log(|x - c_i|)d\mu_0(x)$. We have $|\int_{c_i}^{R_0} \log(|x - c_i|)d\mu_0(x)| \leq \sqrt{R_0} \int_{c_i}^{R_0} \log(|x|)dx \leq 2 \sqrt{R_0} (R_0 - c_L) \log(R_0 - c_L) \leq \frac{C \log k}{k}$.

For $j \neq i, i+1$, $|L_{ij}| \leq \frac{1}{k} \log(|c_i - c_j|) + \max \{\log(|c_j - c_i|), \log(|c_j - c_{i+1}|), \log(|c_{i+1} - c_i|)\} \frac{1}{k}$ and $\sum_{j \neq i} |\log(|c_j - c_i|)|$.

$$\sum_{1 \leq j \leq L, j \neq i, i+1} |L_{ij}| \leq \frac{1}{k} (\log \frac{c_i}{c_i - c_{i-1}}) + \log \frac{c_L - c_i}{c_{i+1} - c_i}$$

$$+ \frac{C(\frac{\log R_1}{R_1} + \frac{\delta^2 R_1^2}{R_1})}{k} \sum_{j \neq i} |\log(|c_j - c_i|)|$$

$$\leq \frac{C \log k}{k} + C(\frac{\log R_1}{R_1} + \frac{\delta^2 R_1^2}{R_1})$$

$$\leq C(\frac{\log R_1}{R_1} + \frac{\delta^2 R_1^2}{R_1}),$$

for $k$ sufficiently large. Moreover, for $j = i+1$, $|L_{ij}| \leq \frac{1}{k} \log(|c_{i+1} - c_i|) + \sqrt{R_0} \int_{c_i}^{R_0} \log(|x - c_i|)|dx \leq \frac{C \log k}{k}$, Thus we have $|L_i| \leq C(\frac{\log R_1}{R_1} + \frac{\delta^2 R_1^2}{R_1})$ when $k$ is sufficiently large. Thus in a similar way, we obtain that $|J(\mu_{11})| \leq C(\frac{\log R_1}{R_1} + \frac{\delta^2 R_1^2}{R_1})$.

Now we bound $J(\mu_{12})$. Similar to the arguments above, we denote by

$$L'_i := \frac{1}{k} \sum_{1 \leq j \leq K-L, j \neq i} \log(|d_i - d_j|) - \int_{R_0}^{0} \log(|d_i - x|)d\mu_0(x).$$

Note that $\mu_0(|d_i, d_{i+1}|) = \frac{1}{k}$. We have

$$L'_i = \sum_{j \neq i-1, i} \left( \frac{1}{k} \log(|d_i - d_j|) - \int_{d_i}^{d_{j+1}} \log(|d_i - x|)d\mu_0(x) \right)$$

$$+ \frac{1}{k} \log(|d_{i+1} - d_i|) - \int_{d_{i-1}}^{d_{i+1}} \log(|d_i - x|)d\mu_0(x).$$
Similar to the arguments above, we have \( \frac{C(R_1) \log k}{k} \leq d_{i+1} - d_i \leq \frac{C(R_1)}{k} \). Thus

\[
\frac{1}{k} \log(|d_{i+1} - d_i|) \leq \frac{C(R_1) \log k}{k}
\]

and

\[
|\int_{d_{i-1}}^{d_{i+1}} \log(|d_i - x|) d\mu_0(x)| \leq \sqrt{R_1} \int_{d_{i-1}}^{d_{i+1}} \log(|d_i - x|) dx \leq \frac{C(R_1) \log k}{k}.
\]

Moreover, since

\[
\sum_{j \neq i, i} \left( \frac{1}{k} \log(|d_i - d_j|) - \int_{d_j}^{d_{j+1}} \log(|d_i - x|) d\mu_0(x) \right)
\]

\[
\leq \frac{1}{k} \sum_{j \neq i, i} \log\left( \frac{|d_i - d_j|}{|d_i - d_{j+1}|} \right)
\]

\[
\leq \frac{1}{k} \left( \log\left( \frac{|R_1 - d_i|}{|d_{i+1} - d_i|} \right) + \log\left( \frac{d_i - R_0}{d_i - d_{i-1}} \right) \right)
\]

\[
\leq \frac{C(R_1) \log k}{k}.
\]

we have \( |L_i| \leq \frac{C(R_1) \log k}{k} \). Similarly, we obtain that \( |J(\mu_{i,2})| \leq \frac{C(R_1) \log k}{k} \).

Below we bound \( J(\mu_{11}, \mu_{12}) \). For any \( 0 \leq x \leq R_0 \), we consider \( R(x) := \frac{1}{k} \sum_{i=1}^{K-L} \log(|d_i - x|) - \int_{R_0}^{R_1} \log(|x-y|) d\mu_0(y) \). For \( 0 \leq x \leq c_L \) we have \( |R(x)| \leq \frac{1}{k} \sum_{i=1}^{K-L} \log\left( \frac{|d_i - x|}{|d_i - d_{i-1}|} \right) \leq \frac{C(R_1) \log k}{k} \). Now since \( |\mu_{11}|([0, R_0]) \leq \frac{k}{k} + \mu_0([0, R_0]) \leq C \), we have \( |J(\mu_{11}, \mu_{12})| \leq \frac{C(R_1) \log k}{k} \). Similarly, when \( c_L \leq x \leq R_0 \), \( |\int_{c_L}^{R_0} R(x) d\mu_{11}(x) - \int_{c_L}^{R_0} R(x) d\mu_0(x)| \leq \frac{C(R_1) \log k}{k} \).

Now let \( Q(x) := \frac{1}{k} \log(|d_i - x|) - \int_{d_{i-1}}^{d_{i+1}} \log(|y-x|) d\mu_0(y) \). We have \( |Q(x)| \leq \frac{c}{k} \log(|d_i - x|) \) for \( c_L \leq x \leq R_0 \). We thus have \( |J(\mu_{11}, \mu_{12})| \leq \frac{C(R_1) \log k}{k} \).

Similarly, we bound \( J(\mu_{11}, \mu_{2}) \) and \( J(\mu_{12}, \mu_{2}) \). For \( J(\mu_{12}, \mu_{2}) \), noting that \( |\int_{c_L}^{R_1} \log(|x-y|) d\mu_0(y)| \leq \frac{C(R_1) \log k}{k} \) for \( x \geq R_1 \), and \( |\mu_2| \leq CR_2^2 \), using the same method for bounding \( J(\mu_{11}, \mu_{12}) \), we obtain that \( |J(\mu_{12}, \mu_{2})| \leq \frac{C(R_2) \log k}{k} \) and \( |J(\mu_{11}, \mu_{2})| \leq \frac{C(R_1) \log k}{k} \).

For \( J(\mu_{12}, \mu_{2}) \), similar to the proof of Lemma 6.4, we have \( |J(\mu_{11}, \mu_{12})| \leq \frac{C(\epsilon + \sqrt{R_1^2 \delta_1^2 + R_2^2 \delta_2^2})}{k} \). Similarly, \( |J(\mu_{11}, \mu_2')| \leq \frac{C(R_1) \log k}{k} \) and \( |J(\mu_{12}, \mu_2')| \leq \frac{C(\epsilon + \sqrt{R_1^2 \delta_1^2 + R_2^2 \delta_2^2})}{k} \).

Therefore, for \( k \) sufficiently large (depending on \( R_1 \)), we have (note that \( \mu_2, \mu_2', \mu_3, \mu_3' \) only depends on \( L \))

\[
\sup_{x \in \mathbb{C}} J_0(\mu_{0,k}) \leq C(\epsilon + \sqrt{R_1^2 \delta_1^2 + R_2^2 \delta_2^2}) + J(\mu_{12} + \mu_2 + \mu_3 + \mu_3') + \sup_{x \in \mathbb{C}} J(\mu_{11}, \mu_{12} + \mu_3') + 2 \sup_{x \in \mathbb{C}} \int_{-\infty}^{0} \tilde{\xi}(x) d\mu_{0,k}.
\]

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Now by Lemma 6.3 for $k$ sufficiently large (depending on $R_1$), if we denote by $\mathcal{E} := \{d_R(\mu_{n,k}, \mu) \leq \delta\}$, we have

$$
\inf_{\tilde{x} \in \mathcal{E}} J_0(\mu_{n,k}) \geq - \int_{[-R_1, R_1]^2} \log(\max |x-y|, \frac{1}{R_1}) \, \mu(x) \, \mu(y)
- \frac{C\varepsilon}{\sqrt{R_2}} \sqrt{\delta} + R_2^2 \delta + R_2^2 \delta^2
+ J(\mu_2 + \mu'_2 + \mu_3 + \mu'_3)
+ \inf_{\tilde{x} \in \mathcal{E}} J(\mu_1, \mu_3 + \mu'_3) + 2 \inf_{\tilde{x} \in \mathcal{E}} \int_{-\infty}^{0} \tilde{\xi}(x) \, d\mu_{n,k}.
$$

For $-R_1 \leq x \leq 0$, we have $|\tilde{\xi}(x)| \leq R_1^2$ and $|\tilde{\xi}'(x)| \leq 2\sqrt{R_1}$ for $k$ sufficiently large (depending on $R_1$). Now by definition, for any $\tilde{x} \in \mathcal{E}$, let $\psi(x) := 1_{[-R_1, \infty)}(x) + (x + R_1 - 1)1_{(-R_1, -R_1+1)}(x)$. We have $\int_{-R_1}^{0} \tilde{\xi}(x) \, d\mu_{n,k} \geq \int_{-\infty}^{0} \psi(x) \tilde{\xi}(x) \, d\mu_{n,k} \geq \int_{-\infty}^{0} \psi(x) \tilde{\xi}(x) \, d\mu(x) - 2R_1^2 \delta \geq \int_{-R_1+1}^{0} \tilde{\xi}(x) \, d\mu(x) - 2R_1^2 \delta. $ Thus

$$
2 \inf_{\tilde{x} \in \mathcal{E}} \int_{-\infty}^{0} \tilde{\xi}(x) \, d\mu_{n,k} - 2 \sup_{\tilde{x} \in \mathcal{E}} \int_{-\infty}^{0} \tilde{\xi}(x) \, d\mu_{n,k} \geq 2 \int_{-R_1+1}^{0} \tilde{\xi}(x) \, d\mu(x) - 4R_1^2 \delta.
$$

(6.45)

Note that by dominated convergence theorem (as $|\mu|(\mathbb{R}) < \infty$), for fixed $R_1$,

$$
\lim_{n \to \infty} 2 \int_{-R_1+1}^{0} \tilde{\xi}(x) \, d\mu(x) = \int_{-R_1+1}^{0} \frac{4}{3} |x|^2 \, d\mu(x).
$$

(6.46)

Now for any $\tilde{x} \in \mathcal{E}$ and $\tilde{y} \in \mathcal{E}$ (we denote by the corresponding quantities for $\tilde{y}$ by $\tilde{\mu}_i$, etc), we bound $|J(\mu_1, \mu_3 + \mu'_3) - J(\tilde{\mu}_1, \mu_3 + \mu'_3)|$ below. For $b_i, \tilde{b}_i \in [-R_1, R_1]$, when $\mathcal{E} \subset \mathcal{D}$, we have

$$
| \int (\log(|b_i - y|) - \log(|\tilde{b}_i - y|)) \, d(\mu_3 + \mu'_3)(y)|
\leq | \int_{b_i}^{\tilde{b}_i} dx | \int \frac{d(\mu_3)(y)}{|x-y|} + | \int \frac{d(\mu'_3)(y)}{|x-y|} |
\leq CR_1 \frac{\sqrt{M}}{\sqrt{R_2}} + \frac{M}{R_2^2} \leq CR_1 \frac{M}{\sqrt{R_2}}.
$$

Thus we have

$$
|J(\mu_1, \mu_3 + \mu'_3) - J(\tilde{\mu}_1, \mu_3 + \mu'_3)|
= \frac{1}{k} \sum_{i=1}^{K} | \int (\log(|b_i - y|) - \log(|\tilde{b}_i - y|)) \, d(\mu_3 + \mu'_3)(y)|
\leq \frac{CMR_1}{k\sqrt{R_2}} \leq \frac{CMR_1^2}{\sqrt{R_2}} \leq CM \frac{R_1}{\sqrt{R_2}}.
$$

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Hence
\[
|\sup_{\vec{x} \in C} J(\mu_1, \mu_3 + \mu_3') - \inf_{\vec{x} \in E} J(\mu_1, \mu_3 + \mu_3')| \leq \frac{CM}{\sqrt{R_1}}.
\] (6.47)

Combining all the bounds above, we obtain that
\[
\inf_{\vec{x} \in C} J_0(\mu_{n,k}) - \sup_{\vec{x} \in E} J_0(\mu_{n,k})
\geq - \int_{[-R_1,R_1]^2} \log(\max(|x-y|, \frac{1}{R_1}))d\mu(x)d\mu(y)
+ 2 \int_{-R_1+1}^{0} \xi d\mu(x) - C(\epsilon + \sqrt{R_1\sqrt{\delta} + R_1^3\delta + R_2^6\delta^2})
:= \Gamma.
\]

We also denote by
\[
\tilde{\Gamma} := - \int_{[-R_1,R_1]^2} \log(\max(|x-y|, \frac{1}{R_1}))d\mu(x)d\mu(y)
+ \int_{-R_1+1}^{0} \frac{4}{3}|x|^2 d\mu(x) - C(\epsilon + \sqrt{R_1\sqrt{\delta} + R_1^3\delta + R_2^6\delta^2}).
\]

Hence
\[
R(\mathcal{L}) \leq \exp(-k^2(\inf_{\vec{x} \in C} J_0(\mu_{n,k}) - \sup_{\vec{x} \in E} J_0(\mu_{n,k}))(2nR_1)^K)
\leq \exp(-k^2\Gamma)(2nR_1)^{CR_1^2k}.
\]

Thus
\[
P(d_R(\mu_{n,k}, \mu) \leq \delta) \leq C\exp(-Mk^2) + \exp(-k^2\Gamma)(2nR_1)^{CR_1^2k},
\] (6.48)

and
\[
\limsup_{k \to \infty} \frac{1}{k^2} \log P(d_R(\mu_{n,k}, \mu) \leq \delta) \leq \max\{-M, -\tilde{\Gamma}\}. 
\] (6.49)

By first sending \(\delta \to 0^+\) and then sending \(R \to \infty\) (hence we can send \(R_1 \to \infty\) and \(\epsilon \to 0\)), we obtain that (by monotone convergence theorem)
\[
\limsup_{R \to \infty} \limsup_{\delta \to 0^+} \limsup_{k \to \infty} \frac{1}{k^2} \log P(d_R(\mu_{n,k}, \mu) \leq \delta)
\leq \max\{-M, \liminf_{R \to \infty} \int_{[-R,R]^2} \log(\max(|x-y|, \frac{1}{R_1}))d\mu(x)d\mu(y)
- \int_{-\infty}^{0} \frac{4}{3}|x|^2 d\mu(x)\}.
\]

We can send \(M \to \infty\) to obtain
\[
\limsup_{R \to \infty} \limsup_{\delta \to 0^+} \limsup_{k \to \infty} \frac{1}{k^2} \log P(d_R(\mu_{n,k}, \mu) \leq \delta)
\leq \liminf_{R \to \infty} \int_{[-R,R]^2} \log(\max(|x-y|, \frac{1}{R_1}))d\mu(x)d\mu(y) - \int_{-\infty}^{0} \frac{4}{3}|x|^2 d\mu(x).
\]
Finally, we use Theorem 2.4 to transfer the result to the measure $\nu_k$. Note that $\mu_{n,k} = \frac{1}{k} \sum_{i=1}^{n} \delta_{b_i} - \mu_0$ and $\nu_k = \frac{1}{k} \sum_{i=1}^{\infty} \delta_{k^{-\frac{3}{8}} a_i} - \nu_0$. For any $\phi(x)$ supported in $(-\infty, R]$ with $\|\phi\|_{Lip} \leq 1, \|\phi\|_{\infty} \leq 1$, we have when $k \to \infty$,

$$| \int_{0}^{R} \phi(x) d(\mu_0 - \nu_0)(x) | \leq (1 - \sqrt{1 - \frac{1}{4} \frac{k}{n}}) \frac{2R^3}{3\pi} \to 0. \quad (6.50)$$

Moreover, we set $I_1 := \{i : b_i \in (-\infty, -R]\}$, $I_2 := \{i : k^{-\frac{3}{8}} a_i \in (-\infty, R]\}$ and $I := I_1 \cap I_2$. We have

$$\frac{1}{k} \sum_{i \in I_1} \phi(b_i) - \frac{1}{k} \sum_{i \in I_2} \phi(k^{-\frac{3}{8}} a_i) \leq \frac{1}{k} \sum_{i \in I} |\phi(b_i) - \phi(k^{-\frac{3}{8}} a_i)| + \frac{1}{k} \sum_{i \in I \setminus I_1} \phi(b_i)$$

$$+ \frac{1}{k} \sum_{i \in I \setminus I_2} \phi(k^{-\frac{3}{8}} a_i) =: P_1 + P_2 + P_3. \quad (6.51)$$

By Proposition 5.1, $\mathbb{P}(|I_2| \geq \eta R^{\frac{3}{8}} k) \leq C \exp(-cnR^3 k^2)$ for $\eta \geq 15$. Hence $|I| \leq |I_2| \leq CnR^3 k$ with probability $\geq 1 - C \exp(-\eta k^2)$. By a similar argument, we have $|I_1| \leq C \eta R^2 k$ with probability $\geq 1 - C \exp(-\eta k^2)$. Below we assume that these events hold. Now by Theorem 2.4 with probability $\geq 1 - C \exp(-ck^3)$ we have $|k^{-\frac{3}{8}} b_i - a_i| \leq Cn^{-\frac{3}{8}}$. Thus as $k \to \infty$,

$$P_1 \leq \frac{1}{k} \sum_{i \in I} |b_i - k^{-\frac{3}{8}} a_i| \leq \frac{C}{n^{\frac{3}{8}} k^\frac{3}{4}} \eta R^{\frac{3}{8}} \to 0. \quad (6.52)$$

Now for $i \in I_1 \setminus I$ or $i \in I_2 \setminus I$, we have $|b_i - R| \leq Cn^{-\frac{3}{8}} k^{-\frac{3}{8}}$ and $|k^{-\frac{3}{8}} a_i - R| \leq Cn^{-\frac{3}{8}} k^{-\frac{3}{8}}$. As $\text{supp}(\phi) \subset (-\infty, R]$, we have for any $\epsilon > 0$, $\phi(R + \epsilon) = 0$. Sending $\epsilon \to 0$ and using the continuity of $\phi$, we get $\phi(R) = 0$. Hence we have $|\phi(b_i)| \leq Cn^{-\frac{3}{8}} k^{-\frac{3}{8}}$ and $|\phi(k^{-\frac{3}{8}} a_i)| \leq Cn^{-\frac{3}{8}} k^{-\frac{3}{8}}$. Hence

$$P_2 + P_3 \leq \frac{C}{n^{\frac{3}{8}} k^\frac{3}{4}} \eta R^{\frac{3}{8}}. \quad (6.52)$$

Hence we have for any $\delta > 0$, $\mathbb{P}(d_R(\mu_{n,k}, \nu_k) \geq \delta) \leq C \exp(-\eta k^2)$ for any $\eta \geq 15$. Thus $\mathbb{P}(d_R(\nu_k, \mu) \leq \delta) \leq \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq 2\delta) + \mathbb{P}(d_R(\mu_{n,k}, \nu_k) \geq \delta)$. Using the result for measure $\mu_{n,k}$ and sending $\eta \to \infty$, we obtain that

$$\limsup_{R \to \infty} \limsup_{\delta \to 0} \limsup_{k \to \infty} \frac{1}{k^2} \log \mathbb{P}(d_R(\nu_k, \mu) \leq \delta) \leq \liminf_{R \to \infty} \int_{[-R, R]^2} \log(\max\{|x-y|, \frac{1}{R^3}\}) d\mu(x) d\mu(y) - \int_{-\infty}^{0} \frac{4}{3}|\phi|^2 d\mu(x).$$

Next we prove the $\mu(\mathbb{R}) \neq 0$ case for Theorem 2.3
Proof of Theorem 2.3. \(\mu(\mathbb{R}) \neq 0\) case. We recall that for every fixed \(\lambda\), the number of eigenvalues of \(H_\beta\) that is \(\leq \lambda\) is a.s. given by number of blow-ups of the diffusion \(dp(x) = (x - \lambda - p^2(x))dx + \frac{2}{\sqrt{\beta}}dB_x\), \(p(0) = \infty\). We denote by \(\kappa = \mu(\mathbb{R})\). Without loss of generality we assume that \(\kappa > 0\). By the conditions in the statement, we can choose sufficiently large \(R_1\) and sufficiently small \(\delta\) (depending only depends on \(\mu\)), such that for \(\nu_k \in U_{\mu, R, \delta}\) with \(R \geq 2R_1\) and \(\lambda \in [2R_1k^{\frac{3}{2}}, Rk^{\frac{3}{2}}]\), \(N(\lambda) \geq \frac{\kappa}{2}k\). We denote by \(A\) this event, and we will bound \(P(A)\) below.

The main difficulty is to leverage the dependence among \(N(\lambda)\) for different \(\lambda\). We take the following strategy. We take some \(K > 2R_1\) sufficiently large depending on \(R\) (the value will be chosen later). We let \(L_i = K^i k^{\frac{3}{2}}\) for \(1 \leq i \leq \log R \log K\). The diffusion corresponding to \(\lambda = L_i\) is denoted by \(p_i\). Note that the driving Brownian motion of these diffusions is the same. We will make use of the notations of Proposition 3.2.

We denote by \(A_i\) the event that for \(p_i\) there are \(\geq \frac{1}{K^i \log K} k^{\frac{3}{2}}k\) more or less blow-ups (than those of the Airy operator) for \(x \in [0, K^{i-1}k^{\frac{3}{2}}]\). From the proof of Proposition 3.2, we have \(P(A_i) \leq \exp(-cK^i k^{2})\).

Let \(C = \bigcap A_i\). By taking a union bound, we get \(P(C) \leq \exp(-cK^i k^{2})\). On the event \(C \cap A\), we have that on each interval \([K^{i-1}k^{\frac{3}{2}}, K^i k^{\frac{3}{2}}]\), \(p_i(x)\) will have \(\geq (1 - \frac{1}{K^i \log K}) \frac{1}{k} \geq \frac{\kappa}{4} k\) blow-ups, which will have restrictions on the Brownian motions. By Markov property of Brownian motion, we derive that (from Proposition 3.2)

\[
P(C \cap A) \leq C \prod_{i=1}^{\log R \log K} \exp(-c \frac{k^2}{i \log K}) \leq C \exp(-ck^2 \log \log R - \log \log K).
\]

By taking \(K = \log \log R\), we get

\[
P(C \cap A) \leq C \exp(-ck^2 \log \log \log R).
\]

By taking a union bound, we get

\[
P(A) \leq C \exp(-ck^2 \log \log \log R). \tag{6.53}
\]

We come to the conclusion by taking \(R \to \infty\).

6.3 Proof of LDP lower bound

In this part, we prove Theorem 2.2 (LDP lower bound).

Proof of Theorem 2.2. We use a similar strategy as in the proof of Theorem 2.3. We take \(R_0\) such that \(\text{supp}(\mu) \subset [-\frac{1}{2}R_0, \frac{1}{2}R_0]\), \(k\mu_0([0, R_0]) \in \mathbb{Z}\) and that \(R_0\) is uniformly bounded for fixed \(\mu\), and take \(R_0\) with \(5 \leq R_0 - R_0 \leq 10\). We also take \(R_1 \geq 2R_0\) and \(R_2 = R_1^6\) such that \(k\mu_0([R_0, R_1]) \in \mathbb{Z}\). We also take
\[ \mathcal{D} := \{ \text{For any } |x| \leq R_1, |\int \frac{\partial w(x)}{\partial x} \leq \frac{c \sqrt{R}}{\sqrt{k}}}, |\frac{d\nu(y)}{dy}| \leq \frac{c \sqrt{M}}{R_2^3} \}. \] 

We note that by Lemma 6.3, \[ \mathbb{P}(\mathcal{D}^c) \leq C \exp(-Mk^2). \]

Similar to the proof of Theorem 2.2, for any given \( K \) and \( \alpha_1, \ldots, \alpha_{n-K} \in [-R_1, R_1]^c \), we denote by

\[ \mathcal{L} := \{ \{ b_1, b_2, \ldots, b_n \} \cap [-R_1, R_1]^c = \{ \alpha_1, \ldots, \alpha_{n-K} \} \}. \] (6.54)

In the sequel of the proof, we will provide a uniform lower bound \( T_0 \) for all \( \mathcal{L} \subset \mathcal{D} \), such that

\[ \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta|\mathcal{L}) \geq T_0 \mathbb{P}(d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}}|\mathcal{L}), \] (6.55)

where \( C \) comes from the conclusion of Lemma 6.2. Assuming this result, by Lemma 6.2 for \( k \) sufficiently large depending on \( R \) and \( M \),

\[ \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta) \geq \mathbb{P}(\{ d_R(\mu_{n,k}, \mu) \leq \delta \} \cap \mathcal{D}) = \mathbb{E}[\mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta|\mathcal{L})|\mathcal{L}] \geq T_0 \mathbb{E}[\mathbb{P}(d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}}|\mathcal{L})|\mathcal{L}] = T_0 \mathbb{P}(\{ d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}} \}) \geq \frac{1}{2} T_0. \]

In the sequel we verify (6.55). Without loss of generality, we assume that \( \mathbb{P}(d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}}|\mathcal{L}) > 0 \). Similar to the proof of Theorem 2.2, we have

\[ \frac{\mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta|\mathcal{L})}{\mathbb{P}(d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}}|\mathcal{L})} = \frac{\int_{\mathcal{E} \subset \{ d_R(\mu_{n,k}, \mu) \leq \delta \}} \exp(-J_0(\mu_{n,k}))d\mathcal{E}}{\int_{\mathcal{E} \subset \{ d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}} \}} \exp(-J_0(\mu_{n,k}))d\mathcal{E}}. \]

Below we denote by \( \mathcal{E} := \{ d_R(\mu_{n,k}, 0) \leq \frac{CR}{\sqrt{k}} \} \). By Lemma 6.4, we have for \( \mathcal{E} \in \mathcal{E} \), \( J(\mu_1, J(\mu_1, \mu_2 + \mu_3) \geq -\frac{C(R)}{k^4} \), thus \( J_0(\mu_{n,k}) \geq -\frac{C(R)}{k^4} \). Without loss of generality we assume that \( \mu(\{ \mu_{n,k} \}) = 0 \). Below we construct a “nice” configuration that lies in \( \{ d_R(\mu_{n,k}, 0) \leq \delta \} \). In \([-R_0, R_0] \), we take points \( \{ \xi_i \}_{i=1}^L \) (where \( L = k \nu_0([0, R_0]) \)) such that \( (\mu + \nu_0)(R_0, c_i) = \frac{1}{k} \). Note that \( c_i = R_0 \), and we denote by \( c_0 := -R_0 \) for simplicity of notations.

Note that similar to the proof of Lemma 6.4, \( (R_0, c_i) \) lies in \( [-1, k] \). We take \( L' := K - L - k \mu_0([R_0, R_1]) \). In \((R_0, R_0) \), we take points \( \{ d_i \}_{i=1}^L \) such that \( \mu_0([R_0, d_i]) = \frac{1}{k+1} \mu_0([R_0, R_0]) \). We denote by \( K' := k \mu_0([R_0, R_1]) \). In \([R_0', R_1] \), we take points \( \{ e_i \}_{i=1}^k \) such that \( \mu_0([R_0, e_i]) = \frac{i-1}{k} \). In \([-1, k] \), the configuration is already determined by \( \mathcal{L} \).
We denote by \( C \) the set of configurations \( \mathbf{x} = (x_1, \ldots, x_K) \) such that when \( 1 \leq i \leq L, x_i = c_i + t_i \) with \( 0 < t_1 < t_2 < \cdots < t_L \leq \frac{1}{2n} \); when \( L+1 \leq i \leq L+L' \), \( |x_i - d_{i-L}| \leq \frac{1}{2n} \); and when \( L+L' + 1 \leq i \leq K \), \( |x_i - e_{i-(L+L')}| \leq \frac{1}{2n} \). It can be verified that for any \( \mathbf{x} \in C \), \( d_R(\mu_{n,k}, \mu) \leq \frac{C(R)}{k^{1/2}} \). Hence for \( k \) sufficiently large (depending on \( R, \delta \)), \( C \subset \{ d_R(\mu_{n,k}, \mu) \leq \delta \} \).

Now we bound \( J_0(\mu_{n,k}) \) for \( \mathbf{x} \in C \). We have
\[
- \int_{[-R_0, R_0]^2} \log(|x - y|) d(\mu + \nu_0)(x) d(\mu + \nu_0)(y)
\geq - \frac{1}{k^2} \sum_{i=1}^{L} \log(|c_i - c_{i-1}|) - \frac{2}{k^2} \sum_{1 \leq i < j \leq L} \log(|c_i - c_j|).
\]
Thus for any \( \mathbf{x} \in C \),
\[
- \frac{2}{k^2} \sum_{1 \leq i < j \leq L} \log(|x_i - x_j|)
\leq - \int_{[-R_0, R_0]^2} \log(|x - y|) d(\mu + \nu_0)(x) d(\mu + \nu_0)(y)
\leq - \frac{1}{k^2} \sum_{i=1}^{L} \log(|t_i - t_{i-1}|) + \frac{2}{k^2} \sum_{i=1}^{L} \log(|x_i|)
\leq - \int_{[-R_0, R_0]^2} \log(|x - y|) d(\mu + \nu_0)(x) d(\mu + \nu_0)(y)
- \frac{1}{k^2} \sum_{i=1}^{L} \log(|t_i - t_{i-1}|) + \frac{C}{k}.
\]

We also bound \( |J((\mu_{n,k} + \mu_0) - (\mu + \nu_0), \mu_0)| \) as follows. We denote by \( \phi(x) := \int_0^{R_0} \log(|x - y|) d\mu_0(y) \). By calculation, we conclude that \( \phi(x) \) is \( C(R) \log(1) \)-Lipschitz and \( C(R) \)-bounded on \([-R_1, R_1 - \delta] \). Hence \( |\int_{-R_1}^{R_1} \phi(x) d((\mu_{n,k} + \mu_0) - (\mu + \nu_0))| \leq \frac{C(R) \log \delta}{k^{1/2}} \). Take \( \delta = \frac{\varepsilon}{k} \) such that \( \{c_1, \ldots, c_L\} \cap [R_0 - \delta, R_0] = \emptyset \).

Now we have \( |\int_{-R_1}^{R_1} \phi(x) d\nu_0| \leq \frac{C \log^k k}{k} \). Thus we conclude that \( |J((\mu_{n,k} + \mu_0) - (\mu + \nu_0), \mu_0)| \leq \frac{C(R) \log k}{k} \).

With the above result, similar to the proof of Theorem 2.5, we obtain that for \( \mathbf{x} \in C \), \( J(\mu_1) \leq J(\mu) - \frac{1}{k^2} \sum_{i=1}^{L} \log(|t_i - t_{i-1}|) + \frac{C(R)}{k^{1/2}} \) and \( J(\mu_2, \mu_2) \leq \frac{C(R)}{k^{1/2}} \).

Thus for any \( \mathbf{x} \in C \), \( J_0(\mu_{n,k}) \leq J(\mu) + \frac{C(R)}{k^{1/2}} + J(\mu_2 + \mu_2 + \mu_3 + \mu_3) - \frac{1}{k^2} \sum_{i=1}^{L} \log(|c_i - c_{i-1}|) + \sup_{\mathbf{x} \in C} J(\mu_1, \mu_3 + \mu_3') + 2 \sup_{\mathbf{x} \in C} \int_{-\infty}^{0} \xi d\mu_{n,k}(x) \).

Now similar to the proof of Theorem 2.6
\[
|\sup_{\mathbf{x} \in C} J(\mu_1, \mu_3 + \mu_3') - \inf_{\mathbf{x} \in C} J(\mu_1, \mu_3 + \mu_3')| \leq \frac{CM}{\sqrt{R_1}} \quad \text{(6.56)}
\]
and
\[
2 \sup_{\vec{x} \in \mathbb{C}} \int_{-\infty}^{0} \tilde{\xi}(x) d\mu_{n,k} - 2 \inf_{\vec{x} \in \mathcal{E}} \int_{-\infty}^{0} \tilde{\xi}(x) d\mu_{n,k} \leq 2 \sup_{\vec{x} \in \mathbb{C}} \int_{-\infty}^{0} \tilde{\xi}(x) d\mu_{n,k} - 2 \int_{-\infty}^{0} \tilde{\xi}(x) d\mu_{n,k} + \frac{C(R)}{k^{10}}.
\]

Combining all the bounds, if we denote by
\[
\Gamma = J(\mu) + 2 \int_{-\infty}^{0} \tilde{\xi}(x) d\mu(x) + \frac{CM}{\sqrt{R_1}} + \frac{C(R)}{k^{10}},
\]
and
\[
\tilde{\Gamma} = J(\mu) + \int_{-\infty}^{0} \frac{4}{3} |x|^3 d\mu(x) + \frac{CM}{\sqrt{R_1}},
\]
then we have that
\[
\mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta | \mathcal{L}) \leq \mathbb{P}(d_R(\mu_{n,k}, 0) \leq \frac{1}{2k^{\frac{1}{2}}} | \mathcal{L}) \geq \frac{\exp(-k^2 \Gamma)}{(2nR_1)^{C(R)k}} \prod_{i=1}^{L} |t_i - t_{i-1}|.
\]

Thus we can take \( T_0 = \frac{\exp(-k^2 \Gamma)}{(2nR_1)^{C(R)k}} \left( \frac{1}{Cnk} \right)^{Ck} \) in (6.55).

Now we have
\[
\mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta) \geq \frac{\exp(-k^2 \Gamma)}{2(2nR_1)^{C(R)k}} \left( \frac{1}{Cnk} \right)^{Ck}.
\]

Thus
\[
\liminf_{n \to \infty} \frac{1}{k^2} \log \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta) \geq -\tilde{\Gamma}.
\]

Sending \( R \to \infty \), we get
\[
\liminf_{R \to \infty} \liminf_{n \to \infty} \frac{1}{k^2} \log \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \delta) \geq \int \log(|x - y|) d\mu(x) d\mu(y) - \int_{-\infty}^{0} \frac{4}{3} |x|^3 d\mu(x).
\]

Similar to the proof of Theorem 2.3, \( \mathbb{P}(d_R(\nu_k, \mu) \leq \delta) \geq \mathbb{P}(d_R(\mu_{n,k}, \mu) \leq \frac{1}{2} \delta) - \mathbb{P}(d_R(\mu_{n,k}, \nu_k) \geq \frac{1}{2} \delta) \), and \( \mathbb{P}(d_R(\mu_{n,k}, \nu_k) \geq \frac{1}{2} \delta) \leq C \exp(-\eta k^2) \) for any \( \eta \geq 15 \). Thus by sending \( \eta \to \infty \), we obtain that
\[
\liminf_{R \to \infty} \liminf_{n \to \infty} \frac{1}{k^2} \log \mathbb{P}(d_R(\nu_{n,k}, \mu) \leq \delta) \geq \int \log(|x - y|) d\mu(x) d\mu(y) - \int_{-\infty}^{0} \frac{4}{3} |x|^3 d\mu(x).
\]
Now for any fixed $R_0$, any $\mu \in \mathbb{Z}$ and any $\tilde{\mu} \in \mathbb{Z}$ such that $\tilde{\mu}(\mathbb{R}) = 0$, 
$\tilde{\mu}(-\infty, R_0] = \mu$, we have $P(d_{R_0}(\nu_k, \mu) \leq \delta) \geq P(d_R(\nu_k, \tilde{\mu}) \leq \delta)$. Hence we have

$$\liminf_{\delta \to 0+0} \liminf_{k \to \infty} \frac{1}{k^2} \log P(d_{R_0}(\nu_k, \mu) \leq \delta) \geq -I(\mu). \quad (6.61)$$

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