ON BARYCENTRIC TRANSFORMATIONS OF FANO POLYTOPES

DONGSEON HWANG AND YEON SU KIM

ABSTRACT. We introduce the notion of barycentric transformation of Fano polytopes, from which we can assign a certain type to each Fano polytope. The type can be viewed as a measure of the extent to which the given Fano polytope is close to be Kähler-Einstein. In particular, we expect that every Kähler-Einstein Fano polytope is of type $B_{\infty}$. We verify this expectation for some low dimensional cases. We emphasize that for a Fano polytope $X$ of dimension 1, 3 or 5, $X$ is Kähler-Einstein if and only if it is of type $B_{\infty}$.

1. Introduction

A Fano polytope of dimension $n$ is a full dimensional convex lattice polytope in $\mathbb{R}^n$ such that the vertices are primitive lattice points and the origin is an interior point. Note that the class of Fano polytopes of dimension $n$ up to unimodular transformation has a one-to-one correspondence with the class of toric Fano varieties of dimension $n$ up to isomorphism. A Fano polytope $P$ is said to be Kähler-Einstein if the dual polytope of $P$ has the origin as its barycenter. By [2, Theorem 1.2], a Fano polytope $P$ is Kähler-Einstein if and only if the associated toric Fano variety $X_P$ admits a Kähler-Einstein metric.

Batyrev and Selivanova introduced the notion of symmetric Fano polytopes to study Kähler-Einstein polytopes. A Fano polytope $P$ is said to be symmetric if the origin is the unique point of $\mathbb{R}^n$ fixed by every automorphism of $P$ onto itself.

Theorem 1.1 ([1, Theorem 1.1]). Every smooth symmetric Fano polytope is Kähler-Einstein.

There had been interests on whether the converse statement holds ([1], [15, Remark in p. 1257], [3, Remark 4.3], [5, p. 257]). For smooth Fano polytopes of dimension at most 8, an exhaustive investigation yields the following theorem.
Theorem 1.2 ([12, Proposition 2.1]). Let $P$ be a smooth Kähler-Einstein Fano polytope of dimension at most 8. Then $P$ is not symmetric if and only if $P$ is one of the three Fano polytopes $Q_1, Q_2, Q_3$ in [12] where $Q_2$ and $Q_3$ have dimension 8 and $Q_1$ has dimension 7.

We remark that, for each integer $n \geq 9$, there exists a smooth Kähler-Einstein Fano polytope of dimension $n$ that is not symmetric ([11, Corollary 5.3]). See also [7] for the discussion in the singular setting.

In this note, we shall describe a class of Fano polytopes that is expected to be very similar to the class of Kähler-Einstein Fano polytopes in the smooth case and have a close connection to symmetric or Kähler-Einstein Fano polytopes in general.

To be more precise, we shall introduce the notion of barycentric transformation, or B-transformation in short, of Fano polytopes. See Definition 2.2 for the precise definition. The B-transformation of a Fano polytope does not always produce a Fano polytope as we see in Example 2.2. By using this phenomenon of B-transformation, we define a type of Fano polytopes. Naively speaking, a Fano polytope is of type $B_k$ if it is possible to take the B-transformation $k$ times, and is of strict type $B_k$ if we can take the B-transformation at most $k$ times. See Definition 2.2 for the precise definition.

Theorem 1.3. Let $P$ be a smooth Fano polytope of dimension $n$.

1. Assume that $n$ is odd and $n \leq 5$. Then, $P$ is Kähler-Einstein if and only if $P$ is of type $B_\infty$.
2. Assume that $n$ is even and $n \leq 4$. If $P$ is Kähler-Einstein, then $P$ is of type $B_\infty$ except possibly for one case with GRDB ID 54 ([6] based on [13]).
3. If $P$ is a Kähler-Einstein Fano polytope of dimension 7 or 8 that is not symmetric, i.e., $P = Q_1, Q_2, Q_3$ in [12], then $P$ is of type $B_\infty$.

Moreover, the Kähler-Einstein property is preserved by the B-transformation in low dimension.

Theorem 1.4. Let $P$ be a smooth Fano polytope of dimension at most 6, or $P = Q_1, Q_2, Q_3$ in [12]. If $P$ is Kähler-Einstein, then so is $B(P)$.

Based on these results, we boldly pose the following conjecture.

Conjecture 1.5. Let $P$ be a smooth Fano polytope.

1. If $P$ has odd dimension, then $P$ is Kähler-Einstein if and only if $P$ is of type $B_\infty$.
2. If $P$ is Kähler-Einstein, then $B(P)$ is also a Kähler-Einstein Fano polytope. In particular, $P$ is of type $B_\infty$.
3. If $P$ is symmetric, then so is $B(P)$. In particular, $P$ is of type $B_\infty$.

See Question 3.1 for the discussion on the converse statement of Conjecture 1.5(2) in the even dimensional case.
One can ask whether Conjecture 1.5 holds true for the singular varieties. By [7, Theorem 1.3], every symmetric Fano polytope is Kähler-Einstein.

**Theorem 1.6.** Let $P$ be a Fano polygon.

1. If $P$ is Kähler-Einstein, then $P$ is of type $B_1$. If, moreover, $P$ is a triangle, then $B(P)$ is a Kähler-Einstein Fano triangle, hence $P$ is of type $B_{\infty}$.
2. If $P$ is symmetric, then so is $B(P)$, hence $P$ is of type $B_{\infty}$.

Every known example of a Kähler-Einstein Fano polygon that is not symmetric is a triangle. For this reason, the following was suggested.

**Conjecture 1.7 ([7, Conjecture 1.6]).** Let $P$ be a Kähler-Einstein Fano polygon. If $P$ is not symmetric, then it is a triangle.

Assuming Conjecture 1.7, Theorem 1.6 proves Conjecture 1.5 for surfaces. Theorem 3.13 shows that Conjecture 1.7, hence Conjecture 1.5, holds true for Fano polygons of index at most 17 without any assumption.

**Remark 1.8.** Not every Fano polytope of type $B_{\infty}$ becomes Kähler-Einstein after a suitable application of B-transformations. For example, consider a Fano polygon $P$ with GRDB ID 13118 ([6] based on [8]). Then, $P$ is 2-periodic, i.e., $B^2(P) = P$. But both $P$ and $B(P)$ are not Kähler-Einstein. Note that $P$ is of index 3, which is the smallest possible index with this property.

Section 2 provides rigorous definitions and examples of barycentric transformations. Main theorems are proven in Section 3. Section 4 presents some interesting observation on the orbits of a Fano polygon under B-transformations and discussion on Fano polygons with zero barycenter.

## 2. B-transformation: definition and examples

### 2.1. Notation

For a lattice point $v = (x, y)$, we define the primitive index $I(v)$ of $v$ by $I(v) = \gcd(x, y)$. A lattice point is called primitive if its primitive index is one. The order of the two dimensional cone $C$ spanned by two lattice points $v_i = (x_i, y_i)$ and $v_{i+1} = (x_{i+1}, y_{i+1})$, denoted by $\text{ord}(v_i, v_{i+1})$, is defined by

$$\text{ord}(v_i, v_{i+1}) = \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} = x_i y_{i+1} - y_i x_{i+1}.$$

### 2.2. B-transformations

To understand more about Kähler-Einstein Fano polytopes, we introduce the new notion called a barycentric transformation.

**Definition.** For a Fano polytope $P$, we can always obtain a lattice polytope by taking the convex hull of the barycenters of all maximal dimensional cones of $P$. See ([4, Exercise 11.1.10]) for the definition of the barycenter of a cone.
This association is called a \textit{barycentric transformation} of $P$ or, in short, a \textit{B-transformation} of $P$.

Note that the B-transformation is uniquely determined up to unimodular transformation.

We can easily compute the B-transformation of a Fano polygon using the following lemma, which immediately follows from definition.

\textbf{Lemma 2.1.} Let $P$ be a Fano polygon with vertices $v_1, \ldots, v_n$ written in counterclockwise order. Then,

$$B(P) = \text{conv}\left\{ \frac{v_1 + v_2}{T(v_1 + v_2)}, \ldots, \frac{v_n + v_1}{T(v_n + v_1)} \right\}.$$ 

\textbf{Example 2.2.} For a Fano polytope $P$, $B(P)$ is not a Fano polytope in general.

1. Take $P = \text{conv}\{(2, -1), (0, 1), (-1, 0)\}$. Then,

$$B(P) = \text{conv}\{(1, 0), (-1, 1), (1, -1)\}$$

is not a Fano polytope since it does not contain the origin as an interior point.

2. Take $P = \text{conv}\{(1, -2), (0, 1), (-1, -2)\}$. Then,

$$B(P) = \text{conv}\{(1, -1), (-1, -1)\}$$

is not a Fano polytope. Note that the dimension is decreased in this case.

\textbf{Definition.} Let $P$ be a Fano polytope of dimension $d$.

1. $P$ is said to be \textit{of type $B_m$} if $B_m(P)$ is a Fano polytope of dimension $d$.

2. $P$ is said to be \textit{of strict type $B_m$} if it is of type $B_m$ but not of type $B_{m+1}$.

3. $P$ is said to be \textit{of type $B_\infty$} if it is of type $B_m$ for every positive integer $m$.

By convention, every Fano polytope $P$ is of type $B_0$. Note that if a Fano polytope $P$ is type of $B_s$, then it is of type $B_t$ for every $t < s$.

We also introduce the following useful notion.

\textbf{Definition.} Let $P$ be a Fano polytope.

1. $P$ is said to be \textit{$B$-invariant} if $B(P)$ is unimodularly equivalent to $P$.

2. $P$ is called \textit{$k$-periodic} (or just \textit{periodic}) if there exists an integer $t$ such that $B^{k+t}(P) = B^t(P)$ for some integer $k \geq 1$.

3. $P$ is called \textit{pseudo-periodic} if the number of vertices of $B^k(P)$ is invariant under the B-transformation for some positive integer $k$.

Note that a periodic Fano polytope is of type $B_\infty$. 
Example 2.3. Let $P$ be a $d$-dimensional Fano cube. For example, if $d = 2$,
$$
P = \text{conv}\{e_1 + e_2, -e_1 + e_2, -e_1 - e_2, e_1 - e_2\}$$
where $e_1$ and $e_2$ are standard bases of $\mathbb{R}^2$. Then, $P$ is of type $B_{\infty}$. Indeed, one can easily see that $B(P) = \text{conv}\{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$ is a $d$-dimensional Fano bipyramid and $B^2(P) = P$, i.e., $P$ is 2-periodic. In this case, both $P$ and $B(P)$ are Kähler-Einstein. Note that $P$ is singular and $B(P)$ is smooth, from which we see that the B-transformation does not preserve the smoothness.

Example 2.4. Let $P = \text{conv}\{(0, 1), (3, -2), (-4, 1)\}$. One can compute that $B(P) = \text{conv}\{(3, -1), (-1, -1), (-2, 1)\}$, $B^2(P) = \text{conv}\{(-1, 0), (1, -1), (1, 0)\}$, from which we see that $P$ is of strict type $B_1$.

Example 2.5. Let $P = \text{conv}\{(-25, -12), (-5, -6), (25, 14)\}$. One can compute that $B(P) = \text{conv}\{(-5, -3), (5, 2), (0, 1)\}$ and
$$
B^2(P) = \text{conv}\{(0, -1), (5, 3), (-5, -2)\} = -B(P),
$$
thus $P$ is 2-periodic, and hence it is of type $B_{\infty}$. Now, it is easy to see that $P$ is not Kähler-Einstein but so is $B(P)$. Note also that $B^n(P)$ is not symmetric for every integer $n \geq 0$.

Lemma 2.6. For a Fano polygon, the number of vertices is not increasing under $B$-transformations.

Proof. It immediately follows from the fact that, in dimension two, the number of maximal dimensional cones is equal to the number of vertices. \qed

Example 2.7. The number of vertices is not invariant under $B$-transformation in general. Let
$$
P = \text{conv}\{(3, -1), (3, 1), (1, 2), (-3, 1), (-3, -1), (-1, -2)\}.
$$
Then, $P$ is both symmetric and Kähler-Einstein but
$$
B(P) = \text{conv}\{(4, 3), (-2, 3), (-4, -3), (2, -3)\}.
$$
It is easy to see that for a Kähler-Einstein Fano polygon with at most 4 vertices, the number of vertices is invariant under $B$-transformation. We do not know any example of such a Kähler-Einstein Fano polygon with 5 vertices.

Remark 2.8. Lemma 2.6 does not hold for higher dimensional Fano polytopes. See Example 2.3.

2.3. Condition to be of type $B_1$

We provide a sufficient condition for a given Fano polygon $P$ to be of type $B_1$.

Lemma 2.9. Let $P$ be a convex lattice polygon with primitive vertices $v_1, \ldots, v_n$ written in counterclockwise order. Then, $P$ contain the origin as an interior point if and only if $\text{ord}(v_i, v_{i+1}) > 0$ for all $i = 1, \ldots, n$. 

Proof. If \( P \) contain the origin as an interior point, then \( P \) is a Fano polygon. By the choice of the orientation of vertices, we have \( \text{ord}(v_i, v_{i+1}) > 0 \) for every \( i \). Conversely, suppose that \( \text{ord}(v_i, v_{i+1}) > 0 \) for all \( i = 1, \ldots, n \). Assume that \( P \) does not contain the origin as an interior point. Take a face \( F \) of \( P \) that is closest to the origin. Then, \( F \) has two vertices \( v_{i-1} \) and \( v_i \) of \( P \). Since \( v_i \) is a primitive point, we can map \( v_i \) to \((0,1)\) by an orientation-preserving unimodular transformation. Then, \( v_{i-1} = (a,-b) \) and \( v_{i+1} = (-c,d) \) for some positive integers \( a, b, c \) and \( d \). Since the origin is not an interior point of \( P \), \( v_{i+1} \) is above the line generated by \( v_{i-1} \) and \( v_i \). Indeed, if otherwise, it is easy to see that there exists another face of \( P \) that is closer to the origin. Now, the three vertices \( v_{i-1}, v_i, v_{i+1} \) are in clockwise order, which is a contradiction. \( \Box \)

Let \( P \) be a Fano polygon with vertices \( v_1, \ldots, v_n \) written in counterclockwise order. Then, for each \( i \) with \( 1 \leq i \leq n \), we consider the following function

\[
g(i) := \text{ord}(v_{i-1}, v_i) + \text{ord}(v_i, v_{i+1}) - \text{ord}(v_{i+1}, v_{i-1}).
\]

Proposition 2.10. Let \( P \) be a Fano polygon with vertices \( v_1, \ldots, v_n \) written in counterclockwise order. If \( g(i) > 0 \) for all \( i = 1, \ldots, n \), then \( P \) is of type \( B_1 \).

Proof. It is enough to show that \( B(P) \) contains the origin as an interior point. Since \( g(i) > 0 \) for all \( i = 1, \ldots, n \), it is easy to see that

\[
g(i) = \text{ord}(v_{i-1}, v_i) + \text{ord}(v_i, v_{i+1}) - \text{ord}(v_{i+1}, v_{i-1}) = \text{ord}(v_{i-1} + v_i, v_i + v_{i+1}) > 0.
\]

Note that \( \text{ord}(v_{i-1} + v_i, v_i + v_{i+1}) > 0 \) if and only if \( \text{ord}(\frac{v_{i-1} + v_i}{I_p(v_{i-1} + v_i)}, \frac{v_i + v_{i+1}}{I_p(v_i + v_{i+1})}) > 0 \). Then, the origin is an interior point of \( B(P) \) by Lemma 2.9. \( \Box \)

Corollary 2.11. Let \( P \) be a Fano triangle with \( \text{ord}(P) = \{a, b, c\} \) with \( a > b > c > 0 \). Then, \( P \) is of type \( B_1 \) if \( a < b + c \).

Remark 2.12. If the number of vertices of \( P \) is equal to that of \( B(P) \), then

\[
\text{ord}(B(P)) = \left\{ \frac{g(i)}{I_p(v_{i-1} + v_i)I_p(v_i + v_{i+1})} \right\}.
\]

3. Symmetric or Kähler-Einstein Fano polytopes under B-transformation

3.1. Smooth case

We shall prove Theorem 1.3 and Theorem 1.4.

We deal with Fano polytopes for each fixed dimension. Note that there is only one Fano polytope \( P = [-1,1] \) of dimension one, whose associated toric variety is the projective line \( \mathbb{P}^1 \), which is Kähler-Einstein. Note that \( P \) is \( B \)-invariant, hence it is of type \( B_\infty \).

There are 5 smooth Fano polygons. (See [9] and [10].) It turns out that they are all of type \( B_\infty \). More precisely, \( \mathbb{P}^2 \) is \( B \)-invariant and the other four smooth Fano polygons are 2-periodic. Among them, the three Fano polygons
corresponding to the projective plane \(\mathbb{P}^2\), the Hirzebruch surface \(\mathbb{F}_0\) of degree 0 and the del Pezzo surface of degree 6 are Kähler-Einstein.

Recall that smooth Fano polytopes of dimension at most 6 are completely classified in [6] based on the algorithm in [13]. From this, we have a list of smooth Fano polytopes of dimension at most 6. By the help of computer (e.g., [14]), one can compute the number of smooth Fano polytopes of fixed dimension which is of strict type \(B_k\) for each given \(k\). In particular, there are 18 smooth Fano 3-polytopes and we have the following table.

```
| Strict type | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(B_\infty\) | Total | KE |
|-------------|---------|---------|---------|---------|---------|-------------|-------|----|
| Number      | 2       | 3       | 7       | 0       | 1       | 5           | 18    | 5  |
```

We emphasize that the 5 Fano 3-polytopes of type \(B_\infty\) are precisely the 5 Kähler-Einstein smooth Fano 3-polytopes, and they are all 2-periodic, hence it is of type \(B_\infty\). Indeed, for a smooth Fano 3-polytope \(P\) of type \(B_5\), we have \(B^2(P) = P\) except for one case (GRDB ID = 18) in which case we have \(B^3(P) = B(P)\).

Similarly, we compute the number of smooth Fano 4-polytopes of strict type \(B_k\) for each given \(k\). Among 124 smooth Fano 4-polytopes, there are 14 Fano 4-polytope \(P\) of type \(B_4\), 11 of them being 2-periodic, hence of type \(B_\infty\). The remaining three cases (GRDB ID: 53, 54, 55) are at least of type \(B_20\). We expect that those three cases are of type \(B_\infty\) but we do not know the proof.

```
| Strict type | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(B_5\) | \(B_6\) | \(B_\infty\) | Total | KE |
|-------------|---------|---------|---------|---------|---------|---------|---------|-------------|-------|----|
| Number      | 28      | 33      | 47      | 2       | 0       | 0       | 0       | 0           | 124   | 12 |
```

There are precisely 12 smooth Kähler-Einstein Fano 4-polytopes where 11 of them are of \(B_\infty\) and the other one \(P\) has GRDB ID 54. We do not know whether \(P\) is periodic. But it seems that \(P\) is pseudo-periodic, which is not the case for the other two Fano polytopes (GRDB ID: 53, 55) of type \(B_20\).

**Question 3.1.** Let \(P\) be a smooth 4-Fano polytope of type \(B_\infty\). If \(P\) is pseudo-periodic, then is \(P\) Kähler-Einstein?

There are 866 smooth Fano 5-polytopes. We can similarly compute the number of smooth Fano 5-polytopes of strict type \(B_k\) for each given \(k\). It is worthwhile to note that the 23 Fano polytopes of type \(B_\infty\) are precisely the 23 Kähler-Einstein smooth Fano 5-polytopes.

```
| Strict type | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(B_\infty\) | Total | KE |
|-------------|---------|---------|---------|---------|---------|-------------|-------|----|
| Number      | 342     | 278     | 215     | 4       | 4       | 23          | 866   | 23 |
```

There are 7622 smooth Fano 6-polytopes. In this case, there are 88 smooth Fano 6-polytopes of type \(B_3\), but we did not completely determine how many of them are of strict type. See Remark 3.2. The 23 Fano 6-polytopes of type \(B_\infty\) are all 2-periodic. In fact, each \(P\) of them satisfies \(B^2(P) = B^4(P)\).

```
| Strict type | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(B_\infty\) | Total | KE |
|-------------|---------|---------|---------|---------|---------|-------------|-------|----|
| Number      | 3884    | 2510    | 1140    | \(\leq 65\) | \(\geq 23\) | 7622        | 51    |    |
```
Remark 3.2. For a smooth Fano 6-polytope $P$, $B^3(P)$ has a huge number of vertices and facets. For example, the Fano 6-polytope $P$ with GRDB ID 1787 has 12 vertices and 80 facets and is of type $B_3$, but $B^3(P)$ has 2772 vertices and 1614 facets. According to our estimation, it takes more than one month to precisely determine the number of smooth Fano 6-polytopes of strict type $B_3$, so we have stopped the computation.

Now, the following proposition will complete the proof of Theorem 1.3.

Proposition 3.3. $Q_1$, $Q_2$ and $Q_3$ in Theorem 1.2 are all of type $B_\infty$.

Proof. By computation, we see that $B_2(Q_1) = B_4(Q_1)$, $B_2(Q_2) = B_4(Q_2)$ and $B_2(Q_3) = B_4(Q_3)$, so $Q_1$, $Q_2$ and $Q_3$ are all periodic, hence they are of type $B_\infty$. □

Since we have computed $B(P)$ for each Kähler-Einstein Fano polytope $P$ of dimension at most 6 or $Q_1$, $Q_2$ and $Q_3$ in Theorem 1.2, it is easy to compute the barycenter of $B(P)$. This proves Theorem 1.4.

3.2. Singular case

We shall prove Theorem 1.6.

3.2.1. Symmetric Fano polygons.

Lemma 3.4. Let $P$ be a Fano polygon of type $B_1$. If $P$ admits a non-trivial automorphism $\sigma$, then $I(v_i + v_{i+1}) = I(\sigma(v_i) + \sigma(v_{i+1}))$ for all adjacent vertices $v_i$ and $v_{i+1}$ of $P$.

Proof. Since the lattice point $\frac{v_i + v_{i+1}}{I(v_i + v_{i+1})}$ is primitive, so is $\sigma\left(\frac{v_i + v_{i+1}}{I(v_i + v_{i+1})}\right) = \frac{\sigma(v_i) + \sigma(v_{i+1})}{I(\sigma(v_i) + \sigma(v_{i+1}))}$ by [7, Lemma 2.2], from which the result follows. □

Proposition 3.5. Let $P$ be a symmetric Fano polygon of type $B_3$. If $P$ admits a non-trivial rotation, then so does $B(P)$.

Proof. Let $v_1, \ldots, v_n$ be all vertices of $P$ written in counterclockwise order and $\sigma$ be a non-trivial rotation of $P$. By Lemma 3.4, $I_p(v_i + v_{i+1}) = I_p(\sigma(v_i) + \sigma(v_{i+1}))$ for all adjacent vertices $v_i$ and $v_{i+1}$ of $P$. Let

$$w_i = \frac{v_i + v_{i+1}}{I_p(v_i + v_{i+1})}$$

for every $i = 1, \ldots, n$ and $W = \{w_1, \ldots, w_n\}$ be an ordered set. Then, $\sigma(W) = W$ and $B(P) = \text{conv}(W)$. Note that a point of $W$ need not be a vertex of $B(P)$ anymore in general. It is enough to show that $\sigma$ preserves the vertex set of $B(P)$. Again, it is enough to show that if $w_i$ is not a vertex of $B(P)$, then so is not $\sigma(w_i)$. 

Suppose that \( w_i \) is not a vertex of \( B(P) \). Let \( w_j \) and \( w_k \) be the vertices of \( B(P) \) closest to \( w_i \). Define

\[
 f(w_i) = \det(w_j, w_i) + \det(w_i, w_k) + \det(w_k, w_j),
\]

where \( j < i < k \). Then, \( f(w_i) < 0 \). Since \( \sigma \) is a rotation,

\[
 f(\sigma(w_i)) = \det(\sigma(w_j), \sigma(w_i)) + \det(\sigma(w_i), \sigma(w_k)) + \det(\sigma(w_k), \sigma(w_j)) < 0.
\]

Thus, by [16, Proposition 5], \( \sigma(w_i) \) is not a vertex of \( W \). □

**Corollary 3.6.** If a matrix \( \sigma \) induces an automorphism of a symmetric Fano polygon \( P \), then it also induces an automorphism of \( B(P) \).

**Proof.** The result follows from Proposition 3.5. □

**3.2.2. Kähler-Einstein Fano polygons.** We first consider the triangle case.

**Theorem 3.7.** If \( P \) is a Kähler-Einstein Fano triangle, then \( P \) is of type \( B_\infty \).

Moreover, \( B^{2s+1}(P) = -P \) and \( B^{2s}(P) = P \) for all non-negative integer \( s \). Hence, \( B^s(P) \) is Kähler-Einstein for every non-negative integer \( s \).

**Proof.** By [7, Proposition 4.1], we may assume that

\[
 P = \text{conv}\{(a, -b), (0, 1), (-a, b - 1)\}.
\]

Then, it is easy to see that \( B(P) = \text{conv}\{(a, -b + 1), (-a, b), (0, -1)\} = -P \). The rest follows immediately. □

**Example 3.8.** The converse of Theorem 3.7 does not hold in general, i.e., there exists a Fano triangle of type \( B_\infty \) that is not Kähler-Einstein. Let

\[
 P = \text{conv}\{(-25, -12), (-5, -6), (25, 14)\}.
\]

Then, it is easy to see that \( P \) is not Kähler-Einstein. However, since

\[
 B(P) = \text{conv}\{(-5, -3), (5, 2), (0, 1)\}
\]

is Kähler-Einstein, by Theorem 3.7, \( B(P) \) is of type \( B_\infty \), thus so is \( P \).

We need the following technical lemma to prove Theorem 3.10.

**Lemma 3.9.** Let \( P \) be a Fano triangle with vertices \( (a, -b), (0, 1), (-c, d) \) written in counterclockwise order where \( a, b, c, d \) are positive integers. Let \( Q \) be the convex hull generated by \( P \) and \( v \) be a primitive lattice point that lies below the line spanned by \( (a, -b) \) and \( (c, -d) \). Then, we have the following.

1. The dual triangle \( P^* \) of \( P \) has vertices \( w_1, w_2, w_3 \) written in counterclockwise order such that \( w_1 \) and \( w_2 \) are strictly below the \( x \)-axis and \( w_3 \) is strictly above the \( x \)-axis.

2. The dual polygon \( Q^* \) of \( Q \) has vertices \( w_1, w_2, w'_3, w'_4 \) written in counterclockwise order such that \( w'_3 \) lies on the line segment connecting \( w_2 \) and \( w_3 \); and \( w'_4 \) lies on the line segment connecting \( w_1 \) and \( w_3 \).
Proof. Write \( v_1 = (a, -b), v_2 = (0, 1), v_3 = (-c, d), \) and \( v = (x, y) \). Then, we can compute that
\[
P^* = \text{conv}\left\{ \left(-\frac{b-1}{a}, -1\right), \left(1 - \frac{d}{c}, -1\right), \left(\frac{b + d}{bc - ad}, \frac{a + c}{bc - ad}\right) \right\}
\]
and
\[
Q^* = \text{conv}\left\{ \left(-\frac{b+1}{a}, -1\right), \left(1 - \frac{d}{c}, -1\right), \left(\frac{y - d}{cy + dx}, \frac{c + x}{cy + dx}\right), \left(-\frac{b+y}{bx + ay}, \frac{x-a}{bx + ay}\right) \right\}.
\]
Take
\[
w_1 = \left(-\frac{b-1}{a}, -1\right), \ w_2 = \left(1 - \frac{d}{c}, -1\right), \ w_3 = \left(\frac{b + d}{bc - ad}, \frac{a + c}{bc - ad}\right),
\]
\[
w'_3 = \left(\frac{y - d}{cy + dx} - \frac{c + x}{cy + dx}\right) \text{ and } w'_4 = \left(-\frac{b+y}{bx + ay}, \frac{x-a}{bx + ay}\right).
\]
Since \( v_1, v_2, v_3 \) are in counterclockwise order, we have \( bc - ad > 0 \). This proves (1).

Since \( Q \) is a Fano polygon,
\[
\text{ord}(v_3, v) + \text{ord}(v, v_1) + \text{ord}(v_1, v_3) = -cy - dx - bx - ay - (bc - ad) > 0.
\]
Thus,
\[
\text{ord}(w'_3, w_3) = \frac{-cy - dx - bx - ay - (bc - ad)}{(-cy - dx)(bc - ad)} > 0
\]
and
\[
\text{ord}(w_3, w'_4) = \frac{-cy - dx - bx - ay - (bc - ad)}{(bc - ad)(-bx - ay)} > 0.
\]
So, \( w'_3, w_3, w'_4 \) are written in counterclockwise order. Let \( S(w_i, w_j) \) be the slope between \( w_i \) and \( w_j \). Then,
\[
S(w_2, w_3) = S(w_2, w'_3) = \frac{c}{d} \quad \text{and} \quad S(w_1, w'_4) = S(w_1, w_3) = \frac{a}{b}.
\]
Thus, we see that \( w'_3 \) lies on the line segment connecting \( w_2 \) and \( w_3 \), and \( w'_4 \) lies on the line segment connecting \( w_1 \) and \( w_3 \).
It follows from Lemma 3.9 that the $y$-coordinate of the barycenter of $P'$ is less than that of $P$ since $w_1$ and $w_2$ does not depend on the choice of $v$.

**Theorem 3.10.** If $P$ is a Kähler-Einstein Fano polygon, then $P$ is of type $B_1$.

**Proof.** Let $P$ be a Kähler-Einstein Fano polygon. Assume that $P$ is not of type $B_1$. Then, by Proposition 2.10, there exists a vertex $v_i$ of $P$ such that $g(i) \leq 0$.

Up to a unimodular transformation, we may assume that $v_{i-1} = (a, -b), v_i = (0, 1)$ and $v_{i+1} = (-c, d)$ for some positive integers $a, b, c, d$ where $v_{i-1}, v_i, v_{i+1}$ are adjacent vertices of $P$ in counterclockwise order.

Let $P' = \text{conv}\{v_{i-1}, v_i, v_{i+1}\}$ and $(P')^*$ be its dual polygon. Note that the $y$-coordinate $y(P'^*)$ of the barycenter of $(P')^*$ is

$$y(P'^*) = \frac{1}{3} \left( \frac{a + c}{bc - ad} - 2 \right) = \frac{(a + c - (bc - ad)) - (bc - ad)}{3(bc - ad)}.$$

Since $g(i) = a + c - (bc - ad) \leq 0$, we have $y(P'^*) < 0$. By Lemma 3.9, if $P$ has another vertex $v$ other than $v_{i-1}, v_i$ and $v_{i+1}$, then $y_P^* \leq y(P'^*) < 0$. So, the barycenter of $P^*$ is not the origin. Thus, $P$ is not Kähler-Einstein, a contradiction. $\square$

**Theorem 3.11.** Let $P$ be a Fano polygon. If $P$ is symmetric, then so is $B(P)$ and $P$ is of type $B_\infty$.

**Proof.** Let $P$ be a symmetric Fano polygon. By [7, Theorem 1.3], $P$ is Kähler-Einstein, and hence it is of type $B_1$ by Theorem 3.10. By Proposition 3.5, $B(P)$ has an induced non-trivial rotation. Thus, $B(P)$ is also a symmetric Fano polygon of type $B_1$. This argument shows that $P$ is of type $B_\infty$. $\square$

Now, Theorem 3.10, Theorem 3.7 and Theorem 3.11 prove Theorem 1.6.

**Question 3.12.** Let $P$ be a Kähler-Einstein Fano polygon. Is $B(P)$ Kähler-Einstein?

If Question 3.12 is true, then Conjecture 1.5 is true for surfaces by Theorem 1.6.

**3.2.3. Fano polygons of index at most 17.** We determine the number of Fano polygons of given strict type $B_k$ using the database [6] based on [8] as we did in the smooth case. See Table 3.2.3.

Further direct computation based on Table 3.2.3 proves Conjecture 1.7 and Conjecture 1.5 for Fano polygons of index at most 17.

**Theorem 3.13.** Let $P$ be a Kähler-Einstein Fano polygon of index at most 17. Then,

1. $P$ is of type $B_\infty$ and $B(P)$ is again Kähler-Einstein.
2. If $P$ is not symmetric, then $P$ is a triangle.
Table 1. Fano polygons of index at most 17.

| Index | \( B_0 \) | \( B_1 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) | \( B_5 \) | \( B_6 \) | \( B_7 \) | \( B_\infty \) | Total | KE |
|-------|----------|----------|----------|----------|----------|----------|----------|----------|----------|-------|-----|
| 1     | 3        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 12    | 16  |
| 2     | 11       | 6        | 0        | 2        | 3        | 0        | 0        | 0        | 0        | 8     | 30  |
| 3     | 35       | 20       | 10       | 2        | 4        | 2        | 0        | 0        | 0        | 26    | 99  |
| 4     | 35       | 25       | 10       | 2        | 3        | 0        | 0        | 0        | 0        | 16    | 91  |
| 5     | 100      | 63       | 35       | 10       | 6        | 2        | 0        | 0        | 0        | 34    | 250 |
| 6     | 126      | 93       | 49       | 25       | 8        | 8        | 1        | 0        | 0        | 69    | 379 |
| 7     | 186      | 100      | 73       | 9        | 7        | 1        | 0        | 0        | 0        | 53    | 429 |
| 8     | 145      | 78       | 41       | 3        | 3        | 2        | 0        | 0        | 0        | 35    | 307 |
| 9     | 319      | 165      | 112      | 14       | 5        | 5        | 0        | 0        | 0        | 70    | 690 |
| 10    | 384      | 262      | 142      | 39       | 15       | 3        | 1        | 0        | 1        | 69    | 916 |
| 11    | 472      | 190      | 155      | 28       | 6        | 0        | 0        | 0        | 0        | 88    | 939 |
| 12    | 535      | 325      | 185      | 74       | 26       | 9        | 2        | 0        | 0        | 123   | 1279 |
| 13    | 563      | 227      | 205      | 27       | 11       | 2        | 0        | 0        | 0        | 107   | 1142 |
| 14    | 725      | 423      | 261      | 34       | 20       | 2        | 1        | 0        | 0        | 79    | 1545 |
| 15    | 1711     | 1119     | 784      | 305      | 129      | 24       | 5        | 1        | 0        | 234   | 4312 |
| 16    | 564      | 237      | 137      | 9        | 9        | 5        | 0        | 0        | 0        | 69    | 1030 |
| 17    | 1007     | 353      | 330      | 53       | 9        | 1        | 0        | 0        | 0        | 139   | 1892 |

4. Further discussions

In this section, we discuss the number of Fano polytopes that are obtained by a sequence of B-transformations of a given Fano polytope, and study the class of Fano polygons with barycenter zero.

4.1. Orbits of a Fano polytope under B-transformation

Definition. Let \( P \) be a Fano polytope. Then, a Fano polytope \( Q \) is called an orbit of \( P \) under B-transformation if it is obtained by applying a sequence of B-transformations of \( P \), i.e., \( Q = B^n(P) \) for some non-negative integer \( n \).

Theorem 4.1. Let \( P \) be a Kähler-Einstein Fano triangle. Then, the number of orbits of \( P \) under B-transformation is at most two.

Proof. It immediately follows from the proof of Theorem 3.7. \( \square \)

Question 4.2. Let \( P \) be a Fano polygon of type \( B_\infty \). Then, is the number of orbits of \( P \) under B-transformation at most two?

4.2. Fano polygons with barycenter zero

We discuss the generalization of the following proposition for Kähler-Einstein Fano triangles to arbitrary Kähler-Einstein Fano polygons.

Proposition 4.3 ([7, Proposition 4.1]). Let \( P \) be a Kähler-Einstein Fano triangle. Then, \( P \) has the origin as its barycenter.
Theorem 4.4. Let $P$ be a Fano polygon. If $P$ admits a non-trivial rotation, then its barycenter is the origin.

Proof. Let $v$ be the barycenter of $P$. Since the set of vertices of $P$ is invariant under the action of rotations, it follows that $\sigma(v) = v$. But, since the origin is the only fixed point of a rotation, we conclude that $v$ is the origin. \qed

Theorem 4.5. Assume that Conjecture 1.7 holds. Then, every Kähler-Einstein Fano polygon has the origin as its barycenter.

Proof. If $P$ is symmetric, then $P$ has a non-trivial rotation, so the barycenter of $P$ is the origin by Theorem 4.4. If otherwise, $P$ is a Kähler-Einstein Fano triangle by Conjecture 1.7. Now the result immediately follows from [7, Corollary 2.5]. \qed

Question 4.6. Is it possible to remove the assumption in Theorem 4.5?

Thanks to [6] based on [8], one can check that Kähler-Einstein Fano polygons are precisely the Fano polygons with barycenter zero if the index is at most 17. But there exist examples of Fano polygons with barycenter zero that are not Kähler-Einstein in higher index.

Example 4.7. The following examples have the origin as their barycenters but are not Kähler-Einstein.

1. Let $P_1 = \text{conv}\{-2, -1, -1.3, 1.2, 2.3\}$. Then, the barycenter of $P_1$ is the origin but $P_1$ is not Kähler-Einstein. Moreover, $P_1$ is of type $B_\infty$ and the index of $P_1$ is 280.
2. Let $P_2 = \text{conv}\{-5, -4, -1.8, 5.1, 8.2\}$. Then, the barycenter of $P_2$ is the origin but $P_2$ is not Kähler-Einstein. Moreover,

$$B^5(P_2) = \text{conv}\{-1, 0, 1, -1, 5, -3\}, \text{ and } B^6(P_2) = \text{conv}\{0, -1, 3, -2, 4, -3\}.$$

Thus, $P_2$ is of type $B_5$. Note that the index of $P_2$ is 270180.

Remark 4.8. The Fano polygon $P$ in Example 2.5 supports the example of a Fano polygon of type $B_\infty$ with nonzero barycenter that is neither symmetric nor Kähler-Einstein. The Fano polygon $P_1$ in Example 4.7(1) supports the example of a Fano polygon of type $B_\infty$ with barycenter zero that is neither symmetric nor Kähler-Einstein. The Fano polygon $P_2$ in Example 4.7(2) supports the example of a Fano polygon, not of type $B_\infty$, with barycenter zero that is neither symmetric nor Kähler-Einstein.

Acknowledgements. This research was supported by the Samsung Science and Technology Foundation under Project SSTF-BA1602-03.
References

[1] V. V. Batyrev and E. N. Selivanova, Einstein-Kähler metrics on symmetric toric Fano manifolds. J. Reine Angew. Math. 512 (1999), 225–236. https://doi.org/10.1515/crll.1999.054

[2] R. J. Berman and B. Berndtsson, Real Monge-Ampère equations and Kähler-Ricci solutions on toric log Fano varieties, Ann. Fac. Sci. Toulouse Math. (6) 22 (2013), no. 4, 649–711. https://doi.org/10.5802/afst.1386

[3] K. Chan and N. C. Leung, Miyaoka-Yau type inequalities for Kähler-Einstein manifolds, Comm. Anal. Geom. 15 (2007), no. 2, 359–379. http://dx.doi.org/10.4310/CAG.2007.v15.n2.a6

[4] D. A. Cox, J. B. Little, and H. K. Schenck, Toric varieties, Graduate Studies in Mathematics, 124, American Mathematical Society, Providence, RI, 2011. https://doi.org/10.1090/gsm/124

[5] A. Futaki, H. Ono, and Y. Sano, Hilbert series and obstructions to asymptotic semistability, Adv. Math. 226 (2011), no. 1, 254–284. https://doi.org/10.1016/j.aim.2010.06.018

[6] Graded Ring Database, http://www.grdb.co.uk/

[7] D. Hwang and Y. Kim, On Kähler-Einstein Fano polygons, arXiv:2012.13373v2.

[8] A. M. Kasprzyk, M. Kreuzer, and B. Nill, On the combinatorial classification of toric log del Pezzo surfaces, LMS J. Comput. Math. 13 (2010), 33–46. https://doi.org/10.1112/S1461157008000367

[9] A. M. Kasprzyk and B. Nill, Fano polytopes, in Strings, gauge fields, and the geometry behind, 349–364, World Sci. Publ., Hackensack, NJ, 2013.

[10] Y. Nakagawa, Combinatorial formulae for Futaki characters and generalized Killing forms of toric Fano orbifolds, in The Third Pacific Rim Geometry Conference (Seoul, 1996), 223–260, Monogr. Geom. Topology, 25, Int. Press, Cambridge, MA, 1998.

[11] M. Øbro, On the examples of Nill and Paffenholz, Internat. J. Math. 26 (2015), no. 4, 1540007, 15 pp. https://doi.org/10.1142/S0129167X15400078

[12] M. Øbro, An algorithm for the classification of smooth Fano polytopes, arXiv:0704.0049.

[13] SageMath, the Sage Mathematics Software System (Version 9.1), The Sage Developers, 2020, https://www.sagemath.org.

[14] J. Song, The $\alpha$-invariant on toric Fano manifolds, Amer. J. Math. 127 (2005), no. 6, 1247–1259.

[15] Y. Suyama, Classification of Toric log del Pezzo surfaces with few singular points, arXiv:1910.00206v1.