Small-$x$ physics in perturbative QCD

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Abstract
We review the parton model and the Regge approach to the QCD description of the deep-inelastic $ep$ scattering at the small Bjorken variable $x$ and demonstrate their relation with the DGLAP and BFKL evolution equations. It is shown, that in the leading logarithmic approximation the gluon is reggeized and the pomeron is a compound state of two reggeized gluons. The conformal invariance of the BFKL pomeron in the impact parameter space is used to investigate the scattering amplitudes at high energies and fixed momentum transfers. The remarkable properties of the Schrödinger equation for compound states of an arbitrary number of reggeized gluons in the multi-colour QCD are reviewed. The gauge-invariant effective action describing the gluon-Reggeon interactions is constructed. The known next-to-leading corrections to the QCD pomeron are discussed.

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1 Introduction

Recent measurements of the structure functions for the deep-inelastic ep scattering at HERA discovered their dramatic rise at the region of small \( x \approx 10^{-4} \) [1]. In the framework of the Bjorken-Feynman parton model [2] this experimental result implies the corresponding growth of the parton distributions \( n_i(x) \) inside the rapidly moving proton as functions of the decreasing parton momentum fraction \( x \) and the increasing photon virtuality \( Q^2 \).

In the framework of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [3, 4, 5, 6] the parton distributions grow at small \( x \) as a result of their \( Q^2 \)-evolution. In the framework of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [7] this growth is a consequence of their \( x \)-evolution. Within the double-logarithmic accuracy these equations coincide and the increase of the structure functions at small \( x \) is related with the singularities of the anomalous dimensions for the corresponding twist-2 operators at non-physical values \( j \to 1 \) of the Lorentz spin [2, 8]. The existing experimental data on structure functions agree with the DGLAP dynamics provided that the evolution equation in \( Q^2 \) is applied starting from rather small \( Q^2 = Q_0^2 \) [9]. The growth of the structure functions at small \( x \) can be also obtained with the use of the BFKL equation [10]. In this case a large uncertainty is related with the fact, that all next-to-leading corrections to this equation have not been calculated yet contrary to the case of the DGLAP equation where they are well known.

The additional information on the dynamics of the deep-inelastic scattering at small \( x \) is extracted from the study of the final state particles. An especially clean footprint of the BFKL pomeron can be found in the processes with the inclusive production of jets [11]. The quark-gluon structure of the pomeron can be investigated at the hard diffractive scattering when the hadrons are produced in the virtual photon fragmentation region [12]. The deep-inelastic process with a large rapidity gap for the final particle momenta was discovered at HERA [13]. The various theoretical models for its interpretation were suggested [14]. The other high energy processes with the large rapidity gaps were widely discussed to discriminate the dynamics related with the soft and hard pomerons [15].

In this review we consider the theory of the BFKL pomeron. Because this theory is related closely with the parton description of the deep-inelastic scattering in QCD and with the Regge model, we remind below the basic ideas of these two traditional approaches to the high energy physics (a more comprehensive information can be found in Refs [16], [17]). In the next section the basic properties of the solution of the BFKL equation are discussed in the framework of the impact parameter representation. The gluodynamics is known to be a low energy limit of the super-string model of elementary particles which includes the quantum gravity. On the contrary one can expect that QCD at high energies could be described in terms of an effective field theory for string-like objects. In the third section it will be demonstrated, that in the Regge limit of large energies \( \sqrt{s} \) and fixed momentum transfers \( \sqrt{-t} \) the gluon having the spin \( j = 1 \) at \( t = 0 \) lies on the Regge trajectory \( j = j(t) \). Such reggeization property was assumed to be typical for hadrons which are extended objects. We consider here a simple effective field model in which the Feynman vertices coincide with the QCD reggeon-particle couplings. It is shown, that in the leading logarithmic approximation the pomeron is a compound state of two reggeized gluons. However, for restoring the \( S \)-matrix unitarity one should
consider the contribution of the diagrams with an arbitrary number of the reggeized gluons in the $t$-channel. In the end of third section it will be demonstrated, that the equations for compound states of several reggeized gluons in the multi-colour QCD have remarkable properties: the conformal symmetry, the holomorphic factorization of their eigen functions and the existence of non-trivial integrals of motion in holomorphic and anti-holomorphic subspaces. The corresponding Hamiltonian turns out to be equivalent to the local Hamiltonian of the exactly solvable Heisenberg model with the spins being the generators of the conformal (Möbius) group. At high energies it is natural to reformulate QCD as an effective field theory for reggeized gluons. In the fourth section the gauge-invariant effective action for the interactions between the reggeized and usual gluons is constructed. The main results in the problem of finding next to leading corrections to the BFKL equation are reviewed in the fifth section. In Conclusion some unsolved problems are discussed.

### 1.1 Parton model in QCD

The deep-inelastic $ep$ scattering at large electron momentum transfers $q = p_e - p_{e'}$ and a fixed Bjorken variable $x = Q^2/(2p_{q}q)$ ($Q^2 = -q^2$) is a well investigated process for which the perturbative quantum chromodynamics (QCD) was traditionally and successfully applied. Before the QCD discovery the approximate scaling behaviour of the structure functions $W_{1,2}(x,Q^2)$ for the virtual photon-proton scattering at large $Q^2$ was derived in the framework of the Bjorken-Feynman parton model [2]. In this model the transverse momenta $k_{i\perp}$ of partons inside the moving proton are assumed to be independent of $Q^2$, the cross-section $\sigma_L$ for the longitudinally polarized virtual photon is zero and the cross-section $\sigma_T$ for the transversally polarized photon is expressed in the impulse approximation as a sum of photon-quark cross-sections averaged with the distributions of quarks $n_q(x)$ and anti-quarks $n_{\bar{q}}(x)$ in the proton [1]:

$$\sigma_T = \sum_q e_q^2 \frac{4\pi^2\alpha}{Q^2} x (n_q(x) + n_{\bar{q}}(x)), \quad (1)$$

where $e_q$ is the quark charge measured in the units of the electron charge $e$ and $\alpha = e^2/4\pi$. As a consequence of the charge and energy conservation the following sum rules for the parton distributions $n_i(x)$ inside the proton

$$1 = \sum_q e_q \int_0^1 dx \left( n_q(x) - n_{\bar{q}}(x) \right), \quad 1 = \sum_i \int_0^1 dx \, x \, n_i(x) \quad (2)$$

are valid.

In the renormalizable field theories the Bjorken scaling for structure functions is violated due to the logarithmic terms $(g^2 \ln Q^2)^n$ appearing in the perturbation theory. The leading logarithmic approximation (LLA) for the structure functions corresponds to the sum of all such contributions [3]. The physical reason of the scaling violation is that because of divergencies one should introduce an ultraviolet cut-off $\Lambda$ for the integrals over the transverse momenta $k_{i\perp}$ of partons being quanta of bare fields [4]:

$$(k_{i\perp})^2 < \Lambda^2. \quad (3)$$
Nevertheless, the parton model representation (1) for the $\gamma^* p$ cross-section in terms of the quark distributions $n_q(x)$, $n_{\bar{q}}(x)$ remains to be valid in LLA [4] if we identify the ultraviolet cut-off and the photon virtuality

$$\Lambda^2 = Q^2.$$  \hspace{1cm} (4)

Moreover, one can express the inclusive probabilities $n_i(x)$ for finding the parton $i$ with its momentum fraction $x = \frac{|k^\perp|}{|p^\perp|}$ inside the proton having the big momentum $p^\perp \rightarrow \infty$ through the partonic wave functions $\Psi_n(k_1^\perp, x_1; k_2^\perp, x_2; \ldots; k_n^\perp, x_n)$ as follows

$$n_i(x) = \sum_n \prod_k \frac{1}{n_k!} \int \prod_{r=1}^n \frac{d^2k_r^\perp}{(2\pi)^3} \frac{dx_r}{x_r} |\Psi_n|^2 \delta^2(\sum_r k_r^\perp) \delta(1 - \sum_r x_r) \sum_i \delta(x_{ri} - x_i).$$ \hspace{1cm} (5)

Here the index $r_i$ enumerates $n_i$ partons of the type $i$ in the state with $n = \sum_k n_k$ partons.

In expression (5) and in the normalization condition for $\Psi_n$:

$$1 = \sum_n \prod_i \frac{1}{n_i!} \int \prod_{r=1}^n \frac{d^2k_r^\perp}{(2\pi)^3} \frac{dx_r}{x_r} |\Psi_n|^2 (2\pi)^3 \delta^2(\sum_r k_r^\perp) \delta(1 - \sum_r x_r)$$ \hspace{1cm} (6)

the most essential integral contributions at large $\Lambda^2 = Q^2$ in LLA correspond to the branching processes in which each virtual particle can decay into two others having significantly bigger transverse momenta. The interference terms between the decay amplitudes describing the different branches are negligible in the physical light-cone gauge:

$$n_\mu v_\mu = 0, \quad n_\mu^2 = 0$$ \hspace{1cm} (7)

for the gluon field $v_\mu(x)$ and therefore we can use the probabilistic picture to calculate various characteristics of the partonic cascade [4]. From the renormalizability of the theory the $\Lambda$-dependence of $|\Psi_n|^2$ is known:

$$|\Psi_n|^2 \sim \prod_i (Z_i)^{n_i}, \quad \sum_i n_i = n,$$ \hspace{1cm} (8)

where $Z_i$ are the renormalization constants for the corresponding parton wave functions.

Using also the above probabilistic picture for essential contributions we obtain the following equation by differentiating the normalization condition (6) for the wave function $\Psi_n$ [4]:

$$0 = \sum_i N_i \left( Z_i^{-1} \frac{\partial}{\partial \ln Q^2} Z_i + P_i \right),$$ \hspace{1cm} (9)

where

$$N_i = \int_0^1 d x n_i(x)$$ \hspace{1cm} (10)

is the averaged number of partons $i$ in the proton. The quantity $P_i d \ln Q^2$ is the total probability of the parton decay during the “time” interval $d \ln Q^2$ and it depends on the effective charge $g(Q^2)$. The normalization condition (6) should be valid for the wave function $\Psi$ of each hadron, which due to relation (9) is compatible with the Callan-Symanzik equation [18] for the renormalization constants.
\[ \frac{\partial}{\partial \ln Q^2} Z_i = -P_i Z_i . \]  

Analogously by differentiating the partonic expression (5) for \( n_i(x) \) one can derive [4] the DGLAP equation [3-6]:

\[ \frac{\partial}{\partial \ln Q^2} n_i(x) = -P_i n_i(x) + \sum_k \int_x^{1} \frac{dx'}{x} P_{k \rightarrow i}(\frac{x}{x'}) n_k(x') , \quad P_i = \sum_r \int_0^{1} dy P_{i \rightarrow r}(y) \]  

governing the \( Q^2 \)-dependence of the parton distributions \( n_i(x) \). In the right-hand side of this equation the first term describes the decrease of the number of partons of the type \( i \) as a result of their decay to other partons. The second term corresponds to the increase of \( n_i(x) \) due to the decay of other partons into the states containing the partons of the type \( i \). The decay probabilities \( P_{k \rightarrow i}(x/x') \) and \( P_k = \sum_i \int_0^1 dy P_{k \rightarrow i}(y) \) are calculated in the form of the perturbative expansion over the running QCD coupling constant \( \alpha(Q^2) \) at large \( Q^2 \)

\[ \alpha(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{\beta_2 \ln \frac{Q^2}{\Lambda_{QCD}^2}} , \quad \beta_2 = \frac{11}{3} N_c - \frac{2}{3} n_f . \]  

Here \( \Lambda_{QCD} \approx 10^2 \text{ Mev} \) is the fundamental QCD constant, \( N_c = 3 \) is the rank of the gauge group \( SU(N_c) \) for QCD and \( n_f \) is the number of the quarks with masses smaller than \( \sqrt{Q^2} \). For example, in the lowest order corresponding to LLA one obtains [3-6]:

\[ P_{k \rightarrow i}(y) = \alpha(Q^2) \frac{w_k^i(y)}{4\pi} , \]

\[ w_{1/2}^1(y) = \frac{N_c^2 - 1 + y^2}{N_c} \frac{1}{1 - y} , \quad w_{1/2}^{-1} = 0 , \quad w_{1/2}^1(y) = \frac{1}{N_c} \frac{1}{y} , \quad w_{1/2}^{-1}(y) = \frac{1 - y^2}{N_c} \frac{1}{y} , \]

\[ w_{1/2}^1(y) = w_{1/2}^{-1}(1 - y) = y^2 , \quad w_1^i(y) = 2N_c \frac{1 + y^4}{y(1 - y)} , \quad w_1^{-1}(y) = 2N_c \frac{(1 - y)^3}{y} , \]

\[ w_q = \frac{N_c^2 - 1}{N_c} \int_0^1 dy \frac{1 + y^2}{1 - y} , \]

\[ w_g = \int_0^1 dy \left[ y^2 + (1 - y)^2 + 2N_c \frac{1 + y^4 + (1 - y)^4}{1 - y} \right] . \]  

where \( w_{\lambda_k}^\lambda(y) \) are proportional to the elementary inclusive probabilities for the transitions \( k \rightarrow i \) between the partons with definite helicities \( \lambda_k, \lambda_i \) (\( \lambda = \pm 1 \) for gluons and \( \lambda = \pm 1/2 \) for quarks; \( \lambda \) denotes the helicity \( \lambda \) for the anti-quark). Other inclusive probabilities can be obtained with the use of the parity and charge conservation. In evolution equation (12) the infrared divergency at \( y = \frac{\beta}{\beta'} \rightarrow 1 \) is cancelled. The dependence from interactions in the confinement region enters in \( n_i(x) \) only through the initial condition at some \( Q^2 = Q_0^2 \) assumed to be sufficiently large: \( Q_0^2 \gg \Lambda_{QCD}^2 \). One can apply the evolution equation also
for distributions of quarks and gluons inside a virtual parton of the type $k$ substituting $n_i(x) \rightarrow D^i_k(x)$ and assuming, that

$$D^i_k(x) = \delta_{ki} \delta(x - 1) \quad (15)$$

for $Q^2 = Q^2_0$. The parton distributions in the proton can be expressed through $D^i_k$ using the representation:

$$n_i(x) = \sum_k \int_0^1 d\beta f_k(\beta) D^i_k(\frac{x}{\beta}), \quad (16)$$

where the initial parton densities $f_k(\beta)$ are not calculated in the framework of the perturbation theory. The solution of the DGLAP equation for $D^i_k(x)$ can be written in the following form:

$$D^i_k(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\sigma}{2\pi i} \left(\frac{1}{x}\right)^j \sum_r M^r_k(j) M^r_i(j) \exp(\xi \gamma_r(j)), \quad (17)$$

where in LLA

$$\xi = \int_{Q^2_0}^{Q^2} \frac{d\sigma}{\sigma} \frac{\alpha(q^2)}{4\pi} = \frac{1}{\beta_2} \ln \left(\frac{\alpha(Q^2_0)}{\alpha(Q^2)}\right), \quad \alpha(Q^2) = \frac{4\pi}{\beta_2 \ln(Q^2/\Lambda^2_{QCD})}. \quad (18)$$

The integration contour $L = (\sigma - i\infty, \sigma + i\infty)$ in the complex plane of the t-channel angular momentum $j$ is situated to the right of the point $j = 1$. The anomalous dimensions $\gamma = \gamma_r(j)$ and couplings $M_i = M^r_i(j)$ are determined from the secular equation:

$$\gamma M_i = \gamma^i_k(j) M_k, \quad (19)$$

where the matrix $\hat{\gamma}(j)$ is defined as follows:

$$\hat{\gamma}^i_k(j) = \int_0^1 dy \left[ y^{j - 1} w^i_k(y) - \delta_{ki} \sum_r w^r_k(y) \right] \quad (20)$$

and its matrix elements are given below [19]:

$$\gamma^{1/2}_{1/2}(j) = \frac{N^2_c - 1}{N_c} \left[ \frac{3}{2} + \frac{1}{j(j+1)} - 2S_j \right], \quad \gamma^{-1}_{1/2}(j) = \frac{N^2_c - 1}{N_c} \left( \frac{2}{j(j^2 - 1)} \right),$$

$$\gamma^{1/2}_{1/2}(j) = \frac{N^2_c - 1}{N_c} \frac{1}{j - j}, \quad \gamma^{1/2}_{1/2}(j) = \frac{1}{j + 2}, \quad \gamma^{-1/2}_{1/2}(j) = \frac{2}{j(j+1)(j+2)},$$

$$\gamma^{1}_{1}(j) = 2N_c \left[ \frac{1}{6} + \frac{4j^2 + 4j - 2}{(j-1)j(j+1)(j+2)} - 2S_j \right] - \frac{2}{3} n_f,$$

$$\gamma^{-1}_{1}(j) = 2N_c \left( \frac{6}{(j-1)j(j+1)(j+2)} \right). \quad (21)$$

where

$$S_j = \psi(j+1) - \psi(1), \quad \psi(x) = \Gamma'(x)/\Gamma(x). \quad (22)$$
and $\Gamma(x)$ is the Euler $\Gamma$-function. Due to the charge and energy conservation the following sum rules are valid:

$$\gamma_{1/2}(1) = 0, \quad \gamma_{1/2}(2) + \gamma_{1/2}(2) + \gamma_{1/2}(2) = 0,$$

$$\gamma_{1}(2) + \gamma_{-1}(2) + 2 n_f \left( \gamma_{1/2}(2) + \gamma_{1/2}(2) \right) = 0. \quad (23)$$

Note, that for the $e^+e^-$ annihilation the inclusive cross-section of the hadron production can be written in the framework of the parton model in terms of the fragmentation function $\tilde{D}_h^i(x)$ which is the inclusive probability to find the hadron $h$ with the energy fraction $x$ inside the parton $i$. These fragmentation functions are expressed through the distributions $\tilde{D}_k^i(x)$ of dressed partons $k$ inside the bare partons $i$. In the leading logarithmic approximation the functions $D_k^i(x)$ and $\tilde{D}_k^i(x)$ coincide due to the Gribov-Lipatov relation [3]:

$$\tilde{D}_k^i(x) = D_k^i(x). \quad (24)$$

Using the arguments based on the crossing symmetry and analyticity one can expect that these two functions are related also by the Drell-Levy-Yan relation

$$\tilde{D}_k^i(x) = \pm x D_k^i \left( \frac{1}{x} \right), \quad (25)$$

where the sign in its right hand side depend on the parton types $i$ and $k$. In the gauge theory the point $x = 1$ is singular due to the Sudakov suppression of the quasi-elastic scattering. Nevertheless, this relation is fulfilled in a generalized form if in each order of the perturbation theory one would continue around $x = 1$ in $D(x)$ the polynomials in $\ln(1 - x)$ as the polynomials in $\ln(x - 1)$ [4, 20].

As it is seen from above formulas, for $j \to 1$ the anomalous dimensions $\gamma_q^g(j) = \gamma_{1/2}(j) + \gamma_{1/2}(j)$ and $\gamma_q^g(j) = \gamma_{1}(j) + \gamma_{-1}(j)$ have the pole singularities:

$$\gamma_q^g(j) \to 2 \frac{N_c - 1}{N_c} \frac{1}{j - 1}, \quad \gamma_q^g(j) \to 4 N_c \frac{1}{j - 1}. \quad (26)$$

Therefore for $x \to 0$ in the integrand of expression (17) for the distributions of gluons $n_g(x)$ and quarks $n_q(x)$ one obtains a saddle point at

$$j - 1 = 2 \sqrt{\frac{N_c \xi}{\ln \frac{1}{x}}}, \quad (27)$$

which leads after calculating the integral over $j$ to the rapid growth of the total cross-section for the $\gamma^* p$-scattering [3, 8]:

$$\sigma_T \sim \frac{1}{Q^2} \exp \sqrt{16 N_c \xi \ln \frac{1}{x}}. \quad (28)$$

In next sections we shall show, that for sufficiently small $x$ this asymptotics should be modified as follows

$$\sigma_T \sim \frac{1}{\sqrt{Q^2 \left( \frac{1}{x} \right)^{\omega_{BFKL}}}}, \quad \omega_{BFKL} = \frac{4 \alpha}{\pi} N_c \ln 2 \quad (29)$$
in accordance with the BFKL equation [7]:

\[ \frac{\partial}{\partial \ln \frac{1}{x}} n_g(x, k_\perp) = 2 \omega(k_\perp^2) n_g(x, k_\perp) + \int d^2k'_\perp K(k_\perp, k'_\perp) n_g(x, k'_\perp) \] (30)

written for more general distributions \( n_g(x, k_\perp) \) depending on the longitudinal \( x \) and transverse \( k_\perp \) gluon momenta.

The next to leading corrections to the anomalous dimension matrix and to the splitting kernels of the DGLAP evolution equation in QCD were calculated in Refs [21] and [22]. It turns out [22], that with taking into account these corrections the Gribov-Lipatov relation is violated.

Another approach to the deep-inelastic ep scattering is based on the Wilson operator product expansion of two electromagnetic currents [19]. In this approach the \( \gamma^* p \) scattering amplitude is presented at large \( Q^2 \) as a series over \( 1/x \) with the coefficients which are the moments of the structure functions. These moments can be expressed as a result of the operator product expansion through the product of the coefficient functions and the matrix elements of local operators \( O^i \) between the hadron states. The coefficient functions depend on \( Q^2 \) and on an intermediate parameter \( \mu \) being the normalization point for the running coupling constant \( \alpha(\mu^2) \). The matrix elements of \( O^i \) also depend on the normalization point \( \mu \) playing role of an ultraviolet cut-off in the corresponding Feynman integrals. In the product of the coefficient functions and the matrix elements the dependence from \( \mu^2 \) is cancelled. In LLA we can chose \( \mu^2 = Q^2 \). Then one should calculate the coefficient functions only in the lowest order of perturbation theory. In this case for the polarized particles only the operators \( O_q^{V \alpha_1 \alpha_2 ... \alpha_j} \) and \( O_q^{A \alpha_1 \alpha_2 ... \alpha_j} \) having the Lorentz spin \( j \) and constructed from the quark fields \( \psi(x) \) appear in the Wilson operator product expansion:

\[ O_q'^{\alpha_1 \alpha_2 ... \alpha_j} = \overline{S_{ym}} \psi(x) \gamma_1^{\alpha_1} D^2 D_3 ... D^{\alpha_j} \psi(x), \quad \gamma_1^{\alpha} = \gamma^{\alpha}, \quad \gamma_5^{\alpha} = \gamma^{\alpha} \gamma_5, \] (31)

where \( \overline{S_{ym}} \) means the symmetrization over the indices \( \alpha_1, \alpha_2 ... \alpha_j \) and the subtraction of traces. The covariant derivative \( D_\alpha \) equals

\[ D_\alpha = \frac{\partial}{\partial x^\alpha} + g v_\alpha(x) \] (32)

where the anti-hermitian matrix \( v_\alpha \) is expressed in terms of the gluon fields \( v_\alpha^a \) belonging to an adjoint representation of the gauge group \( SU(N_c) \)

\[ v_\alpha(x) = t^a v_\alpha^a, \quad [t^a, t^b] = f^{abc} t^c. \] (33)

Here \( t^a \) are anti-hermitian Gell-Mann matrices and \( f^{abc} \) are the structure functions of the gauge group. The commutator of the covariant derivatives is proportional to the strength \( G_{\alpha\beta}^a \) of the gluon field:

\[ [D_\alpha, D_\beta] = g G_{\alpha\beta}, \quad G_{\alpha\beta} = t^a G_{\alpha\beta}^a = \partial_\alpha v_\beta - \partial_\beta v_\alpha + g [v_\alpha, v_\beta]. \] (34)

Operators (31) constructed from the quark fields have the canonical dimension \( d_q = j + 2 \) and therefore their twist \( t = d - j \) equals \( t_q = 2 \). There are other twist-2 operators.
$O_G^V$ and $O_G^A$ constructed from the gluon fields:

$$O_G^{\alpha_1 \alpha_2 \ldots \alpha_j} = \sum_{\mu \nu} tr G^{\alpha_1 \sigma} D^{\alpha_2} D^{\alpha_3} \ldots D^{\alpha_{j-1}} G^{\alpha_j \sigma}, \quad G_V^{\alpha_1 \alpha_2} = G^{\alpha_1 \alpha_2}, \quad G_A^{\alpha_1 \alpha_2} = \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} G^{\alpha_3 \alpha_4},$$

(35)

where $\text{tr}$ implies the trace for the subsequent colour matrix. Under the renormalization the quark and gluon operators mix each with another and only certain linear combinations of them are multiplicatively renormalizable and diagonalize the evolution equations. It is convenient to use the light-cone gauge for the gluon field $v$:

$$v_\perp \equiv v_\alpha n_\alpha = 0, \quad n = \frac{q + xp}{2pq}, \quad n_\alpha^2 = 0.$$

(36)

In this gauge the light-cone components of the operators $O^{\alpha_1 \alpha_2 \ldots \alpha_j}$ appearing in the operator product expansion are the simplest examples of the quasi-partonic operators [23] which have two properties: a) they are universal for all possible interactions, which means in particular that the quasi-partonic operators do not depend on the QCD coupling constant $g$; b) their matrix element between hadronic states are expressed through the integrals from their partonic matrix elements on the mass shell averaged with the partonic correlators.

Let us consider the twist-2 operators for a general case, when their matrix element is calculated between the initial and final hadron states with different longitudinal momenta [24]. Then we can construct non-equivalent operators as bilinear combinations of fields with a different number of derivatives:

$$O(x) \equiv n_\alpha^1 n_\alpha^2 \ldots n_\alpha^j O_q^{\alpha_1 \alpha_2 \ldots \alpha_j} = \sum_{k=0}^{j-1} c_k \psi(i \not{\partial})^k \gamma_\alpha^I(i \not{\partial})^{j-k-1} \psi, \quad \gamma_\alpha^V = \gamma_\alpha, \quad \gamma_\alpha^A = \gamma_\alpha \gamma_5,$$

(37)

where $\not{\partial}$ means, that the derivative acts at the function situated in the left hand side from the differential operator. The coefficients $c_r, d_r$ can be fixed up to a common factor:

$$c_k = \frac{(-1)^k}{k!(k+1)!(j-k-1)!(j-k)!}, \quad d_k = \frac{(-1)^k}{(k-1)!(k+1)!(j-k+1)!(j-k-1)!}.$$

(38)

from the requirement, that the operators $O_q$ and $O_G$ are irreducible under the conformal transformations [25]. It is enough to verify their covariant properties under the inversion $x_\alpha \rightarrow x'_\alpha = x_\alpha/x^2$ with the simultaneous substitution of the fields:

$$\psi(x) \rightarrow \psi'(x') = x^2 \hat{x} \psi(x), \quad v_\mu(x) \rightarrow v'_\mu(x') = x^2 \Lambda_{\mu \nu}(x) v_\nu(x),$$

(39)

where

$$\Lambda_{\mu \nu}(x) = \delta_{\mu \nu} - 2 \frac{x_\mu x_\nu}{x^2}.$$

(40)

The free Green functions for the fields $\psi(x)$ and $F_{\alpha \beta}(x) = \partial_\alpha v_\beta(x) - \partial_\beta v_\alpha(x)$:

$$\langle 0 \left| T \psi(x) \bar{\psi}(y) \right| 0 \rangle \sim \frac{\hat{x} - \hat{y}}{(x-y)^4},$$

(41)
expressions. If we introduce the quantities gauge group the anomalous dimensions can be easily obtained from the above QCD ex-
also a gluino being the Majorana particle belonging to the adjoint representation of the 

\[ \sigma_{\mu\nu}(x-y) \frac{\Lambda_{\mu\nu}(x-y)}{(x-y)^4} \]

are transformed properly under the inversion. Analogously the matrix elements of the twist-2 operators (37) with coefficients (38) can be found in the free theory:

\[
\langle 0 \mid TF_{\mu}(x) F_{\alpha}(y) \mid 0 \rangle \sim \frac{\Lambda_{\mu\alpha}(x-y) \Lambda_{\nu\beta}(x-y) - \Lambda_{\mu\beta}(x-y) \Lambda_{\nu\alpha}(x-y)}{(x-y)^4}.
\]

and are transformed also correctly [25].

Conformally-covariant operators have simple multiplicative properties under the renormalization in LLA in accordance with the fact, that their matrix elements between the parton states in the momentum representation are proportional to the Gegenbauer polynomials which diagonalize partly the Brodsky-Lepage evolution equations generalizing the DGLAP equations for a non-zero momentum transfer [24]. These equations are invariant under a sub-group of the conformal group which includes the transformation

\[ x_\alpha \rightarrow x_\alpha + \epsilon n_\alpha x^2/1 + 2 \epsilon x_\sigma n_\sigma, \quad n_\sigma^2 = 0 \]

where \( n_\sigma \) is the above light-cone vector and \( \epsilon \) is the group parameter. One can verify, that the gauge condition \( \nu_\sigma n_\sigma = 0 \) is compatible with this conformal subgroup.

For the simplest supersymmetric Yang-Mills model containing apart from the gluon also a gluino being the Majorana particle belonging to the adjoint representation of the gauge group the anomalous dimensions can be easily obtained from the above QCD expressions. If we introduce the quantities

\[ \gamma^k_{\pm}(j) = \gamma^\lambda_{\pm}(j) \pm \gamma^\lambda_{\pm}(j) \]

related to the anomalous dimensions of the operators \( O^V \) and \( O^A \) the corresponding matrices can be parametrized by two functions \( \gamma_1(j) \) and \( \gamma_2(j) \):

\[
\begin{pmatrix}
\gamma^q_+(j) \\
\gamma^q_+(j)
\end{pmatrix} = \frac{\gamma_1(j)}{2j+1} \begin{pmatrix}
j - 1 & j - 1 \\
j + 2 & j + 2
\end{pmatrix} + \frac{\gamma_2(j)}{2j+1} \begin{pmatrix}
j + 2 & -j + 1 \\
j - 2 & j - 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
\gamma^q_-(j) \\
\gamma^q_-(j)
\end{pmatrix} = \frac{\gamma_2(j)}{2j+1} \begin{pmatrix}
j - 1 & j - 1 \\
j + 2 & j + 2
\end{pmatrix} + \frac{\gamma_1(j)}{2j+1} \begin{pmatrix}
j + 2 & -j + 1 \\
j - 2 & j - 1
\end{pmatrix}.
\]

This parametrization is a consequence of the fact [20], that there are four irreducible representations of the super-conformal group for the corresponding twist-2 operators with the degenerate anomalous dimensions proportional to \( \gamma_1 \) and \( \gamma_2 \):

\[ \gamma_1(j) = N_c \left[ 2 S_{j-2} - \frac{3}{2} + \frac{3}{j} \right], \quad \gamma_2(j) = N_c \left[ 2 S_{j+1} - \frac{3}{2} - \frac{3}{j} \right]. \]

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In particular from the above formulas we obtain the Dokshitzer relation [6] for the anomalous dimension matrix elements

$$\gamma_q^a(j) + \gamma_g^a(j) = - \gamma_g^a(j)$$

(47)

corresponding to a particular case of the Shmushkevich rule stating, that the probability of a process induced by a particle belonging to a group representation is equal for any member of this representation provided that all final states related by the symmetry transformation are taken into account. Because $\gamma_q^a(j), \gamma_g^a(j)$ and $\gamma_g^a(j)$ coincide in QED [4] and in QCD [5, 6] up to simple colour factors, from the Dokshitzer relation (47) we conclude, that the pure Yang-Mills anomalous dimension $\gamma_g^a(j)$ (and the corresponding splitting kernel $w_g^a(x)$) could be obtained from the QED results.

In the next-to-leading approximation the conformal invariance is violated due to the conformal anomaly related with the non-vanishing $\beta$-function [26]. Nevertheless the solution of the Brodsky-Lepage equations can be found if one would take into account the Ward identities of the conformal group [27].

The quasi-partonic operators [23] of an arbitrary twist seem to be a natural generalization of the above twist-2 operators related with the probabilistic picture. Their matrix elements can be presented only in terms of the parton correlators being the integrals from the product of initial and final state wave functions with a different number of partons. They do not mix with other operators under the renormalization. The evolution equations for the quasi-partonic operators have the form of the Schrödinger equations with a pair-wise interaction. The pair Hamiltonians are proportional to the splitting kernels for twist-2 operators with generally non-zero quantum numbers including colours, flavors, spins and momentum transfers. All these kernels are calculated in LLA [23]. They have a number of remarkable properties including their conformal invariance. The evolution equations are simplified at the small-$x$ region, which gives a possibility to find their solutions in the double-logarithmic approximation [28].

The higher twist contributions are important for finding screening corrections responsible for restoring the $S$-matrix unitarity. This problem will be discussed in this review later in more details. Here we want to remark only, that one can calculate correlators of parton densities within a certain accuracy using the probabilistic picture [29]. Let us introduce the exclusive multi-parton densities using the following definition:

$$f(x_1^{(1)}, x_2^{(1)}, \ldots, x_{n_1}^{(1)}; x_1^{(2)}, \ldots, x_{n_k}^{(k)}) = \frac{1}{\prod_r n_r!} \int \prod_i d^2 k_i^{(r)} (2\pi)^3 |\Psi_n|^2 (2\pi)^3 \delta^2(\sum k_i^{(r)}) \delta(1 - \sum x_i^{(r)}).$$

(48)

Then the generating functional for these quantities

$$I(\phi) = \sum \int \prod_r \prod_i d x_i^{(r)} \varphi_r(x_i) f(x_1^{(1)}, \ldots, x_{n_k}^{(k)})$$

(49)

satisfies an evolution equation in $Q^2$ which can be obtained from the above probabilistic picture. If we consider only the gluodynamics and take into account the most singular part of the corresponding splitting kernel in (14) by putting

$$w_g^a(x) = \frac{4 N_c}{x (1-x)} , \quad w = \int_0^1 \frac{4 N_c}{x (1-x)} d x ,$$

(50)
these equations can be easily solved:

$$I(\varphi) = \int_{i\infty}^{i\infty} \frac{dl}{2\pi i} \frac{\exp(-l) w \exp(-w\xi) (1 - \exp(-w\xi))^{-1}}{1 - \frac{4N_c}{w} (1 - \exp(-w\xi)) \int_0^\infty \frac{dx}{x} \varphi(x) \exp(-xl)}. \quad (51)$$

The inclusive multi-gluon densities are obtained from this expression by the change of its argument: $\varphi \rightarrow 1 + \chi$ and expanding the obtained generating functional in the series over $\chi(x)$. In particular for $\chi = 0$ we obtain: $I(1) = 1$, which corresponds to the normalization condition for the partonic $\Psi$-function. The generating functional $I(1 + \chi)$ for the inclusive partonic correlators can be used for finding the screening corrections at the quasi-Regge region $x \rightarrow 0$. A more general method taking into account the possibility of the parton merging was developed in ref.[30].

Below we survey the basic results of the Regge approach used in the BFKL theory.

### 1.2 Regge theory

The elastic scattering amplitude $A(s,t,u)$ for the process $a + b \rightarrow a' + b'$ is an analytic function of three invariants [17]:

$$s = (p_a + p_b)^2, \quad u = (p_a - p_b)^2, \quad t = (p_a - p_{a'})^2,$$

related as follows:

$$s + u + t = 4m^2 \quad (53)$$

for the case of spinless particles with an equal mass $m$.

The function $A(s,t,u)$ describe simultaneously three channels. In the $s$, $u$ and $t$-channels one has

$$s > 4m^2, \quad u < 0, \quad t < 0;$$

$$u > 4m^2, \quad s < 0, \quad t < 0 \quad (54)$$

and

$$t > 4m^2, \quad u < 0, \quad s < 0 \quad (55)$$

correspondingly. For example, the scattering amplitude in the $s$-channel (where $\sqrt{s}$ is the c.m. energy of colliding particles) is obtained as a boundary value of $A(s,t,u)$ on the upper side of the cut at $s > 4m^2$ in the complex $s$-plane.

The Regge kinematics of the scattered particles in the $s$-channel corresponds to the following region:

$$s \simeq -u \gg m^2 \approx -t = \overline{q}^2, \quad (56)$$

where $\overline{q}$ is the momentum transfer in the c.m. system ($q = p_a - p_{a'}$). From the $S$-matrix unitarity one can obtain the optical theorem for the total cross-section:

$$\sigma_{tot}(s) = \frac{1}{s} Im_s A(s,0), \quad (57)$$

where $Im_s A(s,t)$ is the imaginary part of the scattering amplitude in the $s$-channel. The whole amplitude is expressed through its imaginary parts in the $s$ and $u$ channels with the use of the dispersion relation:

$$A(s,t) = \frac{1}{\pi} \int_{4m^2}^\infty ds' \frac{1}{s' - s - i\epsilon} Im_{s'} A + \frac{1}{\pi} \int_{4m^2}^\infty du' \frac{1}{u' - u - i\epsilon} Im_{u'} A, \quad (58)$$
where we omitted the possible subtraction terms.

In the Regge kinematics (56) the essential $s$-channel angular momenta $l = \rho p$ ( $\rho$ is the impact parameter and $p$ is the c.m. momentum) are large. Therefore one can write the angular momentum expansion of $A$ in the form of its Fourier transformation:

$$A(s, t) = -2is \int d^2 \rho \left[ S(s, \rho) - 1 \right] e^{i \vec{q} \cdot \vec{\rho}}. \quad (59)$$

The $s$-channel partial wave $S(s, \rho)$ can be considered in some models as a two-dimensional $S$-matrix which is parametrized by the impact parameter $\vec{\rho}$ (see for example [31]). Usually it is written in terms of the eikonal phase $\delta(s, \rho)$:

$$S(s, \rho) = e^{i \delta(s, \rho)}. \quad (60)$$

In accordance with the $s$-channel unitarity one obtains $Im \delta > 0$. Because essential values of $\rho$ can not grow more rapidly than $\rho_{max} = c ln(s)$ the Froissart theorem is valid:

$$\sigma_{tot} < 4\pi c^2 ln^2(s). \quad (61)$$

If the scattering amplitude is pure imaginary at high energies one can derive from dispersion relations (58) the Pomeranchuck theorem for the particle-particle and particle-antiparticle total cross sections:

$$\sigma_{pp}^t = \sigma_{\bar{p}p}^t. \quad (62)$$

In a more general case when its real part is as big as possible this theorem is modified as follows

$$\sigma_{pp}^t / \sigma_{\bar{p}p}^t = 1, \quad Re(A) \propto Im(A) \propto s ln^2(s). \quad (63)$$

In the Regge model the asymptotics of the elastic scattering amplitude in region (56) has the following factorized form

$$A(s, t) = \sum_p \xi_p^j(t) g_1^p(t) g_2^p(t). \quad (64)$$

Here $g_{1,2}(t)$ are the Reggeon couplings with external particles, $\xi_{j(t)}^p$ is the signature factor (for the signature $p = \pm 1$):

$$\xi_j^p = i - \frac{\cos \pi j + p}{\sin \pi j} \quad (65)$$

and $j_p(t)$ is the Regge trajectory assumed to be linear:

$$j_p(t) = j_p^0 + \alpha_p^t t, \quad (66)$$

where $j_p^0$ and $\alpha_p^t$ are the reggeon intercept and slope correspondingly.

For the case, when the trajectory $j_p(t)$ passes through the physical value $j = n$ (different for two signatures $p = \pm 1$) corresponding to the integer (or half integer) spin $\sigma = n$ for an intermediate state, the amplitude takes the form:

$$A(s, t) \propto \frac{s^n}{t - t_n}. \quad (67)$$
where \( t_n \) is the squared mass of the compound state lying on the Regge trajectory [17]. Experimentally all hadrons constructed from light quarks belong to the Regge families with almost linear trajectories and an universal slope \( \alpha' \approx 1 \text{ GeV}^{-2} \). As it will be demonstrated below, in the perturbative QCD the gluons and quarks are reggeized, which means, that they lie on the corresponding Regge trajectories [7].

The Regge asymptotics seems to be natural [17] from the decomposition of the scattering amplitude in the sum of contributions of various angular momenta \( j \) in the \( t \)-channel where the scattering angle \( \theta_t \) is related with \( s \) as follows

\[
    z = \cos \theta_t = 1 + \frac{2s}{t - 4m^2}. \tag{68}
\]

For its symmetric and anti-symmetric parts

\[
    A(s, t) = A^+(s, t) + A^-(s, t), \quad A^\pm(-s, t) = \pm A^\pm(s, t), \tag{69}
\]

this decomposition continued analytically to big \( s \) and fixed negative \( t \) takes the simple form:

\[
    A^p(s, t) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dj}{2\pi i} s^j \phi^p_j(t) \tag{70}
\]

and satisfies asymptotically the dispersion relation (58). The functions \( \phi^p_j(t) \) are proportional to the \( t \)-channel partial waves \( f^p_j(t) \):

\[
    \phi^p_j(t) = c_j p (4m^2 - t)^{-j} f^p_j(t), \quad c_j = 16\pi^2 4^j \frac{\Gamma(j + \frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(j + 1)} \tag{71}
\]

and are real in the physical region of the \( s \)-channel.

Thus, for the imaginary parts of the signatured amplitudes \( A^p \) one obtains the simple formulae corresponding to the Mellin transformations:

\[
    \text{Im}_s A^p(s, t) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dj}{2\pi i} s^j \phi^p_j(t). \tag{72}
\]

The inverse Mellin transformations

\[
    \phi^p_j = \int_0^\infty d\xi e^{-j\xi} \text{Im}_s A^p(s, t), \quad \xi = \ln(s) \tag{73}
\]

are simplified versions of the Gribov-Froissart formulae for \( f^p_j(t) \) [17].

The \( t \)-channel elastic unitarity condition for partial waves analytically continued to complex \( j \) for \( t > 4m^2 \) takes the form

\[
    \phi^p_j(t + i\epsilon) - \phi^p_j(t - i\epsilon) = c_j^{-1}(t - 4m^2)^{j + \frac{1}{2}} t^{-\frac{1}{2}} \phi^p_j(t + i\epsilon) \phi^p_j(t - i\epsilon). \tag{74}
\]

Using the analytic continuation of this relation to complex \( t \) one can express the amplitude \( \phi^p_j \) on the physical sheet of the \( t \)-plane through its value in the same point on the second sheet and obtain for the \( t \)-channel partial wave \( f^p_j(t) \) the Regge poles [17]. But the \( t \)-channel partial wave could also have fixed square-root singularities of the type \( \phi_j(t) = a(t) + b(t) \sqrt{j - j_0} \) as it takes place in renormalizable field theories including QCD.
Total cross-sections for hadron-hadron interactions are approximately constant at high energies (up to possible logarithmic terms). To reproduce such behaviour in the Regge model a special reggeon is introduced. It is called the Pomeranchuck pole or the Pomeron. The Pomeron is compatible with the Pomeranchuck theorem (62) because it has the positive signature and the vacuum quantum numbers. Its trajectory is assumed to be close to 1:

$$j(t) = 1 + \omega(t) , \ \omega = \Delta + \alpha' t,$$

where $\Delta \approx 0.08$ and $\alpha' \approx 0.3 \text{ Gev}^{-2}$ [32]. It means, that the signature factor approximately equals $i$ and the real part of $A(s,t)$ is small in the agreement with experimental data. The above quantities $\Delta$ and $\alpha'$ are the bare parameters of the so called soft Pomeron. The BFKL Pomeron has a big value of $\Delta$.

In the $j$-plane there should be other moving singularities of $\phi^+_j(t)$ - the Mandelstam cuts arising as a result of multi-Pomeron contributions to the $t$-channel unitarity equations. V.Gribov constructed the reggeon diagram technique in which all possible Pomeron interactions are taken into account [33]. In the reggeon field theory the parameters of the initial Lagrangian are renormalized. The simple model for taking into account the contributions from the multi-Pomeron exchanges is based on the assumption that the phase in the eikonal representation can be calculated by the Fourier transformation of the amplitude written in the Regge form:

$$\delta(s,\rho) = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} e^{-i \vec{q} \cdot \vec{\rho}} i g^2(t) s^{\Delta - \alpha' \vec{q}^2}. \quad (76)$$

In this case the resulting amplitude satisfies the Froissart requirements. The analogous unitarization procedure can be used also in other cases when the scattering amplitude obtained in some approximation grows more rapidly than any power of $\ln(s)$ (see [34]).

The high energy theorems do not forbid the existence of another Regge pole with the vacuum flavour quantum numbers - the Odderon which has the negative signature and the negative charge parity [35]. It could be situated also near $j = 1$, which would lead to a large real part of scattering amplitudes at high energies and to a significant difference between proton-proton and proton-antiproton interactions. Such singularity appears in the perturbative QCD simultaneously with the Pomeranchuck singularity [36] and therefore the discovery of the Odderon effects would be very important.

Due to the optical theorem (57) the total cross-section is proportional to the imaginary part of the scattering amplitude, for which the Regge asymptotics is usually postulated. Therefore it is natural to ask what production processes are most probable at high energies. In the dual resonance model non-vacuum reggeons in the $t$-channel are obtained as a result of summing over the $s$ channel contributions from resonances with growing spins and masses and the Pomeron is dual to the $s$-channel background. In multi-peripheral models all reggeons are constructed as $t$-channel compound states of partons and from the $s$-channel point of view they describe the processes of the multi-particle production. The produced particles have the multi-peripheral kinematics: their transverse momenta $k_\perp^i$ are fixed and their longitudinal momenta are ordered in such way, that the invariant squared energies of neighbouring particles $s_i = (k_i + k_{i-1})^2$ are also fixed. Therefore the average number of particles in the multi-peripheral models grows as $\ln(s)$. 

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In QCD the situation looks to be simpler than in the multi-peripheral models. Namely, here the average transverse momenta of produced particles grow slowly with energy and therefore due to the asymptotic freedom the effective coupling constant decreases. As a result in QCD (at least in the perturbation theory) the most essential contribution to the total cross-section in the reaction

\[ a + b \rightarrow a' + d_1 + d_2 + \ldots + d_n + b', \quad a' = d_0, \quad b' = d_{n+1} \]  

arises from the multi-Regge kinematics for produced particle momenta:

\[ s \gg s_i \equiv (k_i + k_{i-1})^2 \gg m^2, \quad -t_i = \overrightarrow{q_i}^2 \sim m^2, \quad (k_i^\perp)^2 \sim m^2. \]  

There is the following constraint for energy invariants \( s_i \) resulting from the reality condition \( k_i^2 = m^2 \) for the final state particles

\[ \prod_{i=1}^{n+1} s_i = s \prod_{i=1}^{n} [m^2 + (k_i^\perp)^2]. \]  

The momentum transfers \( q_i \) in the crossing channels \( t_i \) are expressed through external particle momenta as follows:

\[ q_i = p_a - p_{a'} - \sum_{r=1}^{i-1} k_r. \]  

In terms of the Sudakov parameters for produced particles

\[ k_i = \beta_i p_a + \alpha_i p_b + k_i^\perp, \quad (k_i^\perp, p_{a,b}) = 0, \quad k_i^2 = s\alpha_i \beta_i - (k_i^\perp)^2 = m^2 \]  

the multi-Regge kinematics looks especially simple:

\[ 1 \gg \beta_1 > \beta_2 > \ldots > \beta_n \gg \frac{m^2}{s}, \quad \frac{m^2}{s} \ll \alpha_1 \ll \alpha_2 \ll \ldots \ll \alpha_n \ll 1, \]  

\[ s\alpha_i \beta_i \sim (k_i^\perp)^2 \sim m^2. \]  

The decomposition of the momentum transfers \( q_i \) in terms of the Sudakov parameters for produced particles is also simplified in the multi-Regge region:

\[ q_i = \beta_i p_a - \alpha_{i-1} p_b + q_i^\perp, \quad q_i^2 \simeq -(q_i^\perp)^2. \]  

Thus, momentum transfers with a good accuracy are transverse vectors. Finally the energy invariants also can be expressed through the Sudakov variables:

\[ s_i = s \beta_{i-1} \alpha_i \]  

and therefore the produced particle momenta in the multi-Regge kinematics are essentially longitudinal.

Now we review shortly the theoretical description of the multi-Regge processes in the framework of the Regge model. It is natural to generalize the formulae for the elastic processes in the form:

\[ A_{2\rightarrow n+2} = s_1^{j(t_1)} s_2^{j(t_2)} \ldots s_{n+1}^{j(t_{n+1})} \gamma(q_1, q_2, \ldots q_{n+1}). \]
Here the function \( \gamma(q_1, \ldots q_{n+1}) \) is not real and should contain something similar to signature factors for the elastic amplitude to satisfy analytic properties in the direct channels \( s_i \). These properties are significantly simplified in the multi-Regge regime, as it was shown by K. Ter-Martirosyan, H. Stapp, A. White, J. Bartels and others. Namely, the inelastic amplitude has only physical singularities in the channels where the real production of intermediate particles is possible. There can be simultaneous singularities only in non-overlapping channels.

Let us consider the amplitude \( A_{2\rightarrow 3} \) for the single particle production in the multi-Regge kinematics. The signatures in channels \( t_1 \) and \( t_2 \) are assumed to be equal to \( p_1 \) and \( p_2 \) correspondingly. The signatured amplitude has two contributions satisfying the double dispersion representation in the non-overlapping channels \( s_1 \) and \( s_2 \) correspondingly and can be written with the use of the double Watson-Sommerfeld transformation as follows:

\[
A_{2\rightarrow 3} = \int \frac{dj_1}{2\pi i} \frac{dj_2}{2\pi i} \left[ s_1^{j_1-j_2} \xi_{j_1-j_2} s_2^{j_2} \phi_1 + s_2^{j_2-j_1} \xi_{j_2-j_1} s_1^{j_1} \phi_2 \right],
\]

(87)

where two partial waves \( \phi_{j_1 j_2}^{1,2} \) are real functions. They are expressed through the spectral functions \( \rho(s_1, s) = Im_{s_1} Im_{s} A_{2\rightarrow 3} \) and \( \rho(s_2, s) = Im_{s_2} Im_{s} A_{2\rightarrow 3} \) with the inverse Mellin transformations:

\[
\phi_{j_1 j_2}^{1,2} = \int_0^\infty d\xi_{1,2} e^{-j_{1,2}\xi_{j_1 j_2}} \int_{\xi_{1,2}}^\infty d\xi_1 e^{-j_{1,2}\xi_1} \rho(s_{1,2}, s), \quad \xi_{1,2} = ln(s_{1,2}), \quad \xi = ln(s).
\]

(88)

Using the unitarity conditions in the direct channels the spectral functions can be presented in a non-linear way again through inelastic amplitudes, which leads to a set of non-linear equations. Such approach is convenient for finding the high energy asymptotics in non-abelian gauge theories [7].

In the framework of the Regge theory \( \phi_{j_1 j_2}^{1,2} \) are given in the factorized form:

\[
\phi_{j_1 j_2}^{1,2} = g(t_1) \frac{1}{j_1 - j_{p_1}(t_1)} \Gamma^{1,2}(t_1, t_2, k^\perp_2) \frac{1}{j_2 - j_{p_2}(t_2)} g(t_2).
\]

(89)

Here \( \Gamma \) is the reggeon-reggeon-particle vertex depending on the usual invariants \( t_{1,2} \) and the produced particle transverse momentum \( k^\perp_2 \) expressed through \( s_1, s_2 \) with the use of the reality condition

\[
\overrightarrow{k^2_\perp} = \frac{s_1 s_2}{s} - m^2.
\]

(90)

Thus, the production amplitude in the Regge model have the form

\[
A_{2\rightarrow 3} = s_1^{j_{p_1}(t_1)} s_2^{j_{p_2}(t_2)} \gamma(q_1, q_2),
\]

(91)

where the real and imaginary parts of \( \gamma \) are expressed in terms of the vertices \( g \) and \( \Gamma \).

In a general case of the amplitude (86) for \( n \)-particle production in the multi-Regge kinematics one should introduce many partial waves \( \phi_{j_1 \ldots j_{n+1}}^j \) describing different dispersion contributions from all non-overlapping channels.
2 BFKL pomeron in the impact parameter space

In this section we remind comparatively old results concerning the description of the BFKL Pomeron in the impact parameter representation [37].

The asymptotic behaviour of scattering amplitudes in the Born approximation is governed by the spin $\sigma$ of the particle exchanged in the crossing channel:

$$A_{\text{Born}} \sim s^\sigma$$

and the Regge asymptotics is a generalization of this rule to continuous values of the spin: $\sigma \to j = j(t)$. In higher orders of the perturbation theory the scattering amplitude behaves as $s^n$ (apart from possible logarithmic terms), where the power $n = 1 + \sum_i (\sigma_i - 1)$ grows linearly with the spins $\sigma_i$ of the particles in the $t$-channel intermediate states.

In QCD the gluon spin $\sigma$ is 1 and therefore here the most important high energy processes are caused by the gluon exchanges. For example, the Born amplitude for the parton-parton scattering is [7]

$$A(s, t) = 2s g\delta_{\lambda_a, \lambda_a'} T_{tA}^{c} \frac{1}{t} g\delta_{\lambda_b, \lambda_b'} T_{tB}^{c},$$

where $\lambda_i$ are helicities of the initial and final particles; $A, A', B, B'$ are their colour indices and $T_{ij}^c$ are colour group generators in the corresponding representation. The $s$-channel helicity for each colliding particle is conserved because the virtual gluon in the $t$-channel for small $q$ interacts with the total colour charge $Q^c$ commuting with space-time transformations.

2.1 Impact factors

Let us consider now the high energy amplitude for the colorless particle scattering described by the Feynman diagrams containing only two intermediate gluons with momenta $k$ and $q - k$ in the $t$-channel. With a good accuracy we can neglect the longitudinal momenta in their propagators (cf. (84)):

$$k^2 \simeq k_\perp^2, \quad (q - k)^2 \simeq (q - k)_\perp^2.$$

The polarization matrix for each gluon can be simplified at large energies $s = (p_a + p_b)^2 \gg m^2$ as follows

$$\delta^{\mu\nu} = \delta^{\mu\nu}_\parallel + \delta^{\mu\nu}_\perp \simeq \delta^{\mu\nu}_\parallel = \frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{p_a p_b}.$$

The projector to the longitudinal subspace $\delta^{\mu\nu}_\parallel$ can combine the large initial momenta $p_a, p_b$ in a big scalar product $s/2 = p_a p_b$. Moreover, if the indices $\mu$ and $\nu$ belong to the blobs with incoming particles $a$ and $b$ correspondingly, then with a good accuracy we have

$$\delta^{\mu\nu} \to \frac{p_a^\mu p_b^\nu}{p_a p_b}.$$

From the point of view of the $t$-channel unitarity for the partial waves $f_j(t)$ with complex $j$ this substitution has a rather simple interpretation. The most important
contribution in the $t$-channel appears from the nonsense intermediate state leading to a pole singularity of $f_j(t)$ at $j = 1$. For this nonphysical state the projection of the total spin $\vec{S}$ to the relative momentum of gluons in the c.m. system equals 2 which corresponds to the opposite sign of their helicities. It means, that here the complex polarization vectors $e_1$ and $e_2$ of the gluons coincide each with another: $e_1 = e_2 = e$. The projector to the nonsense state is $e_\mu e_\nu^* e_\mu^* e_\nu^*$ (note, that $e_\mu^2 = 0$). Due to the Lorentz and gauge conditions the vectors $e$ are orthogonal to both gluon momenta $k$ and $q - k$. Because these momenta are almost transverse at high energies (see (94)), after the analytic continuation from the $t$-channel to the $s$-channel the above projector should be proportional to the product $p_\mu b^\nu_p p_\mu^* p_\nu^* a^d$ in accordance with (96).

Using eq.(96) and introducing the Sudakov parameters

$$\alpha = -\frac{k p_a}{p_a p_b} = -\frac{s_a}{s}, \ \beta = -\frac{k p_b}{p_a p_b} = \frac{s b}{s}, \ \vec{k} = \frac{\vec{k}}{s}, \ d^4 k = d^2 k d s a d s b$$

(97)

for the virtual gluon momenta $k, q - k$ one obtains for the asymptotic contribution of the diagrams with two gluon exchanges the following factorized expression:

$$A(s, t) = 2 i |s| \frac{1}{2!} \int d^2 k \frac{1}{k^2} \frac{1}{(q - k)^2} \Phi^a(\vec{k}, \vec{q} - \vec{k}) \Phi^b(\vec{k}, \vec{q} - \vec{k}),$$

(98)

corresponding to the impact-factor representation [34]. Here the sum over the colour indices is implied and the factor $1/2!$ in front of the integral is related with the gluon identity and compensates the double number of Feynman diagrams appeared due to our definition of the impact factors $\Phi^{a, b}$ as integrals over the energy invariants $s_{a, b}$ from the photon-particle amplitudes $f^{a, b}$:

$$\Phi^{a, b}(\vec{k}, \vec{q} - \vec{k}) = \int_{-\infty}^{\infty} \frac{d s_{a, b}}{(2 \pi)^2 i s} \ f^{a, b}(s_{a, b}, \vec{k}, \vec{q} - \vec{k}).$$

(99)

The impact factors describe the inner structure of colliding particles. For large $\vec{k}$ they are proportional to the number of partons $N_i$ weighted with their colour group Casimir operators.

Note, that we neglected the longitudinal momenta in gluon propagators in representation (98) in accordance with (94) because in the essential integration region for the impact factors we have

$$s_{a, b} \sim m^2, \ \vec{k}^2 \sim (q - k)^2 \sim m^2$$

(100)

and therefore $k_\parallel^2 = \frac{s a s b}{s} \ll k_\perp^2$. The ultraviolet convergency of the integrals over $s_{a, b}$ follows from the fact, that due to the Ward identities $k^\mu f_{\mu \nu} = (q - k)^\nu f_{\mu \nu} = 0$ for the scattering amplitudes $f = f^{a, b}$ one can perform the substitution:

$$\frac{p_\mu b^\nu p_\mu a^d f^{a, b}}{s} \rightarrow \frac{k_\parallel^\mu}{s_{a, b}} \frac{(q - k)^\nu}{s_{a, b}} f^{a, b}.$$

(101)

Indeed, it has been assumed above that the amplitudes $f$ do not contain pure gluonic intermediate states in the $t$-channel and therefore $f^{a, b} \ll s_{a, b}$ at high energies, which
leads to the rapid convergency of the integrals over \( s_{a,b} \). Therefore one can enclose the integration contours around the right cuts of \( f^{a,b} \) in the complex \( s_{a,b} \) planes:

\[
\Phi^{a,b}(\vec{k}, \vec{q} - \vec{k}) = \int_{\text{th}} \frac{ds_{a,b}}{2\pi^2 s_{a,b}} \frac{k_{1}^\mu (q_{1}^\nu - k_{1}^\nu)}{s_{a,b}} \text{Im} s_{a,b} f_{\mu\nu}(s_{a,b}, \vec{k}, \vec{q} - \vec{k}).
\]  

(102)

after calculating its discontinuity. From this representation of \( \Phi \) we conclude that the impact factors are real functions of \( \frac{1}{k}, \vec{q} - \vec{k} \), vanishing for small \( |\vec{k}| \) and \( |\vec{q} - \vec{k}| \) in the case of the colorless particle scattering (e.g. photon-photon collisions), which is a consequence of the gauge invariance and of the absence of infrared divergencies in the integral over \( s_{a,b} \) for small \( |\vec{k}| \) and \( |\vec{q} - \vec{k}| \).

From the physical point of view the infrared stability follows from the fact, that for small \( |k| \) the virtual gluon interacts with the total colour charge of scattered particles. If this charge is zero, then the dipole and (generally) multipole interactions are proportional to powers of \( |k| \). In particular the total cross-section for the photon-photon interactions does not contain any infrared divergence in the integral over \( \vec{k} \). The various impact factors for the real and virtual photons are calculated [7, 34], which in particular gives us a possibility to estimate the hadron-hadron cross-sections using ideas of the QCD sum rules [38]. For hadrons the impact factors can be extracted phenomenologically for example from the inclusive lepton-hadron scattering because apart from the colour and charge factors these quantities are the same for the virtual gluons and photons.

### 2.2 Möbius invariance of the BFKL pomeron

It is convenient to present eq.(98) in the form of the Mellin transformation (70)

\[
A(s, t) = i |s| \int \frac{d\omega}{2\pi i} s^{\omega} f_{\omega}(q^{2}), t = -q^{2}
\]

(103)

and to pass to the impact parameter representation (cf. (59)) performing the Fourier transformation [38]:

\[
f_{\omega}(q^{2}) \delta^{2}(q - q') = \int \prod_{r=1,2} \frac{d^{2}\rho_{r} d^{2}\rho_{r'}}{(2\pi)^{4}} \Phi^{a}(\vec{p}_{1}, \vec{p}_{2}; \vec{q}) f_{\omega}(\vec{p}_{1}, \vec{p}_{2}; \vec{p}_{1}', \vec{p}_{2}) \Phi^{b}(\vec{p}_{1}', \vec{p}_{2}', \vec{q}').
\]

(104)

The quantity \( f_{\omega}(\vec{p}_{1}, \vec{p}_{2}; \vec{p}_{1}', \vec{p}_{2}') \) can be considered as a four-point Green function:

\[
f_{\omega}(\vec{p}_{1}, \vec{p}_{2}; \vec{p}_{1}', \vec{p}_{2}') = \langle 0 | \phi(\rho_{1}) \phi(\rho_{2}) \phi(\rho_{1}') \phi(\rho_{2}') | 0 \rangle
\]

(105)

where the field \( \phi(\rho) \) describes the (reggeized) gluons. In the Born approximation \( f_{\omega} \) is proportional to the product of free Green functions:

\[
f^{0}_{\omega}(\vec{p}_{1}, \vec{p}_{2}; \vec{p}_{1}', \vec{p}_{2}') = \frac{4\pi^{2}}{\omega} \ln |\rho_{11}'| \ln |\rho_{22}'| \delta_{k_{1}} \equiv \vec{p}_{1} - \vec{p}_{2},
\]

(106)

related with the gluon propagators

\[
\frac{1}{k^{2}} = - \int \frac{d^{2}\rho}{2\pi} \exp(i \vec{k} \cdot \vec{\rho}) \ln |\rho|.
\]

(107)
The functions $\Phi^{a,b}(\vec{p}, \vec{x}, \vec{q})$ are related with the impact factors by the Fourier transformations:

$$
\Phi^{a,b}(\vec{p}, \vec{x}, \vec{q}) = \int d^2k \, \Phi^{a,b}(\vec{k}, \vec{x}, \vec{q}) \exp(i\vec{k} \cdot \vec{p}) \exp(i\vec{x} - \vec{k}) \exp(i\vec{q} - \vec{k}) \rho_2^2). \tag{108}
$$

The vanishing of $\Phi^{a,b}(\vec{k}, \vec{x}, \vec{q})$ (102) at $\vec{k} \to 0$ or at $\vec{k} \to \vec{q}$ is equivalent to the following sum rules for $\Phi^{a,b}(\vec{p}, \vec{x}, \vec{q})$:

$$
\int \Phi^{a,b}(\vec{p}, \vec{x}, \vec{q}) \, d^2\rho_1 = \int \Phi^{a,b}(\vec{p}, \vec{x}, \vec{q}) \, d^2\rho_2 = 0. \tag{109}
$$

Therefore the expression for $f_\omega(q^2)$ is not changed if we add to $f_\omega(\vec{p}, \vec{x}, \vec{q})$ an arbitrary function which does not depend on $\vec{p}, \vec{x}, \vec{q}$ or $\vec{k}$. We can use this freedom related with the gauge invariance to modify $f_\omega(\vec{p}, \vec{x}, \vec{q})$ in the following way:

$$
f_\omega(\vec{p}, \vec{x}, \vec{q}) \to 2\pi^2 \omega \ln \left| \frac{\rho_{11'} \rho_{22'}}{\rho_{12'} \rho_{21'}} \right|, \quad \rho_{12'} \equiv \rho_{12} + \rho_{12}'. \tag{110}
$$

This expression is unique in comparison with all other physically equivalent expressions for $f_\omega$ because it depends only on two independent anharmonic ratios of the vectors $\vec{p}, \vec{x}, \vec{q}$ which can be chosen as follows:

$$
\alpha = \left| \frac{\rho_{11'} \rho_{22'}}{\rho_{12'} \rho_{21'}} \right|, \quad \beta = \left| \frac{\rho_{11'} \rho_{22'}}{\rho_{12'} \rho_{21'}} \right|, \quad \gamma = \frac{\alpha}{\beta} = \left| \frac{\rho_{12'} \rho_{12}}{\rho_{21'} \rho_{21}} \right|. \tag{111}
$$

Therefore $f_\omega$ is invariant under the conformal (Möbius) transformations:

$$
\rho_k \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rho_k, \quad a, b, c, d \in \mathbb{C}, \quad \rho_k = x_k + i y_k \notag
$$

for arbitrary complex $a$, $b$, $c$ and $d$ provided that we use the complex coordinates

$$
\rho_k = x_k + i y_k, \quad \rho_k^* = x_k - i y_k \tag{113}
$$

for all two-dimensional vectors $\vec{p}_k (x_k, y_k)$.

The solution of the BFKL equation obtained in the next section is also Möbius invariant in LLA and can be written in the form [37]:

$$
f_\omega(\vec{p}, \vec{x}, \vec{q}) = \sum_{n=-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{(\nu^2 + n^2/4)}{\nu^2 + (n-1)^2/4} \, d\nu \right] \frac{G_{\nu n}(\vec{p}, \vec{x}, \vec{q})}{\omega - \omega(\nu, n)}, \tag{114}
$$

where for $n = \pm 1$ the integral in $\nu$ is regularized as follows:

$$
\int_{-\infty}^{\infty} d\nu \, \frac{1}{\nu^2} \varphi(\nu) \equiv \lim_{\epsilon \to 0} \left( \int_{-\infty}^{\infty} d\nu \, \frac{\theta(\nu^2 - \epsilon^2)}{\nu^2} \varphi(\nu) - 2 \frac{\varphi(0)}{\epsilon} \right). \tag{115}
$$

In the "energy propagator" $(\omega - \omega(\nu, n))^{-1}$ the quantity $\omega(\nu, n)$ is the eigen value of the BFKL equation [7]:

-21
The coefficients $G$ and $\nu$ with real the Polyakov three-point function \[25\] for the case when the field $s$

For various physical applications of the four-point Green function (114) it is convenient to present $G_{\nu\nu}(\vec{p}_1^\nu, \vec{p}_2^\nu; \vec{p}_1^\nu, \vec{p}_2^\nu)$ (117) in terms of the hypergeometric functions taking into account its conformal invariance:

$$G_{\nu\nu}(\vec{p}_1^\nu, \vec{p}_2^\nu; \vec{p}_1^\nu, \vec{p}_2^\nu) = c_1 x^h x^{\tilde{h}} F(h, h, 2h; x) F(\tilde{h}, \tilde{h}, 2\tilde{h}; x^*) + c_2 x^{1-h} x^{1-\tilde{h}} F(1-h, 1-h, 2-2h; x) F(1-\tilde{h}, 1-\tilde{h}, 2-2\tilde{h}; x^*),$$

where $x$ is the complex anharmonic ratio:

$$x = \frac{\rho_{12} \rho_{1'2'}}{\rho_{11'} \rho_{22'}}$$

and $F(a, b, c; x)$ is defined by the series

$$F(a, b, c; x) = 1 + \frac{ab}{1!c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 + \ldots .$$

The coefficients

\[
\omega(\nu, n) = \frac{N_c g^2}{2\pi^2} \int_0^1 \frac{dx}{1-x} \left[ x^{(n-1)/2} \cos(\nu \ln x) - 1 \right] \\
= -\frac{N_c g^2}{2\pi^2} \Re \left( \psi \left( \frac{1+|n|}{2} + i\nu \right) - \psi(1) \right). \tag{116}
\]
\[ c_2 = \frac{b_{n,\nu}}{2\pi^2}, \quad \frac{c_1}{c_2} = \frac{b_{n,-\nu}}{b_{n,\nu}} = \frac{\Gamma(2-2h)\Gamma(2-2\bar{h})}{\Gamma(1-h)\Gamma(1-\bar{h})} \left[ \frac{\Gamma(h)\Gamma(\bar{h})}{\Gamma(2h)\Gamma(2\bar{h})} \right] \]  

(124)

and the factor \( b_{n,\nu} \)

\[ b_{n,\nu} = \pi^3 2^{4i\nu} \frac{\Gamma(-i\nu + (1 + |n|)/2) \Gamma(i\nu + |n|/2)}{\Gamma(i\nu + (1 + |n|)/2) \Gamma(1-i\nu + |n|/2)} \]  

(125)

are obtained from ref.[37]. The ratio \( c_1/c_2 \) also can be fixed from the condition, that \( G \) is a single-valued function of its arguments. With the use of the various relations among the hypergeometric functions in Appendix we derive other representations for the Green function \( G \), which gives us a possibility to continue it in the regions around points \( x = 1 \) and \( x = \infty \).

### 2.3 Operator product expansion of the \( t \)-channel partial waves

Let us consider now the properties of the solution (114) of the BFKL equation in the various interesting cases. It is convenient to use the mixed representation for the gluon-gluon scattering amplitude [37]:

\[ f_\omega^g(\vec{\rho}, \vec{\rho}') = \frac{1}{(2\pi)^2} \int d^2R \exp(i(\vec{R} - \vec{R}')\vec{q}) f_\omega(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1, \vec{\rho}_2), \]  

(126)

where \( R = (\rho_1 + \rho_2)/2, R' = (\rho'_1 + \rho'_2)/2. \)

Because the dependence of the impact factors \( \Phi^{a,b}(\vec{\rho}_1, \vec{\rho}_2, \vec{q}) \) from \( \vec{R} \) according to (108) is simple:

\[ \Phi^{a,b}(\vec{\rho}_1, \vec{\rho}_2, \vec{q}) = \Phi^a(\rho_{12}, \vec{q}) \exp(i\vec{R}\vec{q}), \]  

(127)

we have another representation for \( f_\omega(q^2) \) (104):

\[ f_\omega(q^2) = \int \frac{d^2\rho d^2\rho'}{(2\pi)^2} \Phi^a(\vec{\rho}, \vec{q}) f_\omega^g(\vec{\rho}, \vec{\rho}') \Phi^b(\vec{\rho}', \vec{q}). \]  

(128)

Here the function \( f_\omega^g(\vec{\rho}, \vec{\rho}') \) can be interpreted as the amplitude for the scattering of two composite objects with the sizes \( \vec{\rho} \) and \( \vec{\rho}' \). Further, the impact factors \( \Phi^a(\vec{\rho}, \vec{q}) \) and \( \Phi^b(\vec{\rho}', \vec{q}) \) describe the distributions over these sizes for fixed \( \vec{q} \). Note, that a simple space-time picture of the high energy scattering in QCD was developed in the framework of the dipole approach [40].

For small \( \vec{q} \) the essential values of \( \vec{\rho} \) and \( \vec{\rho}' \) do not depend on \( \vec{q} \). As for \( f_\omega^g(\vec{\rho}, \vec{\rho}') \), it has a weak singularity at small \( q^2 \) due to massless virtual gluons. Indeed, according to definition (122) for fixed \( \vec{\rho}, \vec{\rho}' \) and large \( R - R' \sim 1/q \) the anharmonic ratio \( x \) is small:

\[ x \simeq \frac{\rho \rho'}{|R - R'|^2} \ll 1 \]
and therefore expression (121) for $G_{\nu n}$ can be simplified:

$$G_{\nu n}(\rho^1_1, \rho^2_2; \rho^1_1, \rho^2_2) \rightarrow c_1 x^h x^{\tilde h} + c_2 x^{1-h} x^{1-\tilde h}.$$  \hspace{1cm} (129)

By putting it in eq.(114) we obtain, that at large distances $R - R'$ the function $f_\omega$ equals

$$f_\omega(\rho^1_1, \rho^2_2; \rho^1_1, \rho^2_2) \sim \left| \frac{\rho \rho'}{(R - R')^2} \right|^{1+2i\nu(\omega)}.$$  \hspace{1cm} (130)

Here $\nu(\omega)$ is a solution of the equation

$$\omega = \omega(\nu, 0)$$  \hspace{1cm} (131)

with $\text{Im}(\nu) \leq 0$. If analogously to (118) one will consider $f_\omega(\rho^1_1, \rho^2_2; \rho^1_1, \rho^2_2)$ as the four-point function of a two-dimensional theory:

$$f_\omega(\rho^1_1, \rho^2_2; \rho^1_1, \rho^2_2) = \langle 0 | \phi(\rho^1_1) \phi(\rho^2_2) \phi(\rho^1_1') \phi(\rho^2_2') | 0 \rangle$$  \hspace{1cm} (132)

then its asymptotics at small $\rho$ and $\rho'$ and fixed $R - R'$ is related with the anomalous dimension

$$\gamma(\omega) = \frac{1}{2} (h + \tilde h) = \frac{1}{2} + i \nu(\omega)$$  \hspace{1cm} (133)

of the operators $O_\omega$ appearing in the operator-product expansion:

$$\phi(\rho^1_1) \phi(\rho^2_2) \sim |\rho_{12}|^{2\gamma} O_\omega(\rho^1_1), \quad \phi(\rho^1_1') \phi(\rho^2_2') \sim |\rho_{1'2'}|^{2\gamma} O_\omega(\rho^1_1').$$  \hspace{1cm} (134)

The quantity $\gamma(\omega)$ is the anomalous dimension of the twist-2 operators constructed in a bilinear form from the gluon fields $G_{\mu\nu}(x)$. It can be calculated as an expansion in the parameter $g^2/\omega$ (cf. (26)):

$$\gamma(\omega) = \frac{N_c g^2}{4\pi^2 \omega} \zeta(3) \left( \frac{N_c g^2}{4\pi^2 \omega} \right)^4 + \ldots ,$$  \hspace{1cm} (135)

where $\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$ is the $\zeta$-function. In LLA where $\omega \sim g^2$ the anomalous dimension is of the order of unity and it has a square-root singularity

$$\gamma = \frac{1}{2} + \frac{\pi}{g} \sqrt{\frac{\omega - \omega_0}{14 N_c \zeta(3)}}$$  \hspace{1cm} (136)

at the point

$$\omega_0 = \frac{g^2}{\pi^2} N_c \ln 2.$$  \hspace{1cm} (137)

This value is the intercept of the Pomeranchuck singularity governing the asymptotics of the cross-sections $\sigma_{\text{tot}} \sim s^{\omega_0}$.

In the deep-inelastic regime of large $Q^2$ where $\rho_{1'2'} \ll \rho_{12} \sim \rho_{11'}$ one can again use the operator expansion for the product of the fields $\phi(\rho_{1'})$ and $\phi(\rho_{2'})$ and the asymptotics is governed by the anomalous dimensions $\gamma(\omega)$ (133) of the twist-2 operators with matrix
elements (118). As for the regime of large momentum transfers $|t| \gg |\rho_{12}|^2 \sim |\rho_{1'2'}|^2$, we can use here the operator expansion of the product of the fields $\varphi(\rho_i)$ and $\varphi(\rho_{i'})$, which corresponds to the asymptotic behaviour of $f_\omega$ (114) at $x \to \infty$ (see (122)). From the representations of $G_{\nu n}$ given in Appendix one can obtain the asymptotics of $f_\omega$ at large $x$:

$$f_\omega \sim c_\omega \ln |x|^2,$$

where $c_\omega$ does not depend on $\rho_{12}$ and $\rho_{1'2'}$. Therefore in the mixed representation we have at large $|t|$: 

$$f_q^\omega(\vec{\rho}, \vec{\rho}') \to \frac{1}{t} c_\omega,$$ 

which means, that in the momentum representation the function $f_\omega(\vec{k}, \vec{k}')$ contains the contributions proportional to $\delta^2(\vec{k}_1)$ and $\delta^2(\vec{k}_2)$. Using the normalization conditions for the wave functions of the initial and final particles, one can verify that at large momentum transfer the impulse approximation is valid and the hadron scattering amplitudes is expressed as a sum of the effective parton scattering amplitudes which do not contain infrared divergencies [39].

### 2.4 Asymptotic freedom and vacuum Regge poles

Because the singularity of $f_\omega(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')$ at large $R - R'$ is weak there is a finite limit for the scattering amplitude at small $q$. Indeed, this amplitude can be written at the mixed representation in the form:

$$f_q^\omega(\vec{\rho}, \vec{\rho}') = \frac{1}{16} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(\nu^2 + n^2/2) d\nu}{[\nu^2 + (n-1)^2/4][\nu^2 + (n+1)^2/4]} \frac{E_{\nu n}^*(\rho, q) E_{\nu n}(\rho', q)}{\omega - \omega(\nu, n)},$$

where the function $E_{\nu n}(\rho, q)$ is the Fourier transformation of $E_{\nu n}(\rho_1, \rho_2)$ in $\rho_0$. In particular, for $\rho q \to 0$ we obtain:

$$E_{\nu 0}(\rho, q) \sim \epsilon(\rho) = \left( |\rho|^{-2i\nu} + |q^2\rho|^{2i\nu} \exp i\delta(0, \nu) \right).$$

Here the phase $\delta(0, \nu)$ is the simple function of $\nu$ (given in ref. [37]) which for $\nu \to 0$ tends to the constant:

$$\delta(0, \nu) \to \pi.$$ 

The amplitude $\epsilon(\rho)$ is a solution of the homogeneous BFKL equation at small $\rho$:

$$\omega \epsilon(\rho) = \alpha \chi \left( \frac{i}{2} \rho \partial \right) \epsilon(\rho), \quad \chi(\nu) = \frac{2N_c}{\pi} \text{Re} \left( \psi(1) - \psi\left(\frac{1}{2} + i\nu\right) \right).$$

If we take into account the fact, that the QCD coupling constant is decreasing in the region of the large transverse momentum $k \sim 1/\rho \gg q$ this equation should be modified by the substitution (see (13))
\[ \alpha \to \frac{b}{\ln \frac{1}{\rho \Lambda^2}}, \quad b = \frac{4\pi}{\beta_2}. \]  

The solution of the modified equation can be found easily. At not too small \( \rho \) where \( \omega < \omega_0 \) it can be written in a semi-classical approximation as follow [37]

\[ \epsilon(\rho) \sim \cos \left( \frac{\pi}{4} + \varpi(\rho, \omega) \ln \frac{1}{|\rho|^2 \Lambda^2} - \frac{b}{\omega} \int_{0}^{\omega} d\nu' \chi(\nu') \right) \]  

where \( \varpi(\rho, \omega) \) satisfies the saddle point relation:

\[ \omega \ln \frac{1}{|\rho|^2 \Lambda^2} = b \chi(\varpi). \]  

By matching the logarithmic derivatives of two above expressions (141) and (145) for \( \epsilon(\rho) \) at \( \rho \sim 1/q \) and using the approximate expression for \( \chi(\nu) \) at small \( \nu \):

\[ \chi(\nu) = \frac{2 N_c}{\pi} \left( 2 \ln 2 \right. - 7 \zeta(3) \nu^2 \right) \]  

we obtain the following quantum spectrum of \( \varpi \) and \( \omega \) at small \( \nu \):

\[ \varpi_k(-q^2) = \left( \frac{3 \pi (k + 3/4) \ln 2}{7 \ln (q^2/\Lambda^2)} \right)^{1/3}, \]  

\[ \omega_k(-q^2) = \frac{2 N_c b}{\pi \ln \frac{q^2}{\Lambda^2}} \left( 2 \ln 2 - 7 \zeta(3) \varpi_k^2(-q^2) \right), \]  

where \( k = 0, 1, 2, ... \). Thus, as a consequence of the asymptotic freedom, the fixed square-root cut of the \( t \)-channel partial wave \( f_1^q \) (140) situated at \( \omega = \omega_0 \) is substituted by an infinite sequence of the Regge poles. It is possible, that the residues of some poles including leading one with \( k = 0 \) can be negligible small. For hadrons they are determined by the strong interactions at large distances. The dependence of the \( t \)-channel partial waves from the dynamics in the confinement region is even more significant for the fixed momentum transfer \( q^2 \sim \Lambda^2 \). Because the intercept of the soft pomeron is small \( \Delta \simeq 0.08 \ll 1 \) [32] one can attempt to calculate its trajectory at small \( q^2 \) in the perturbation theory [37]. In the region where \( \omega \ll \alpha(1/|\rho|^2) \ll 1 \) the BFKL equation does not depend on \( \omega \) and its solution has the simple form [37]:

\[ \epsilon(\rho) \sim \cos \left( \frac{\pi}{4} + \nu_1 \ln \frac{1}{|\rho|^2 \Lambda^2} - \frac{a}{\omega} \right), \quad a = b \int_{0}^{\nu_1} d\nu' \chi(\nu') \]  

where \( \nu_1 \simeq 0.637 \) is the root of the equation \( \chi(\nu) = 0 \) and \( a \simeq 1.28 \) for \( n_f = 3 \). It is resonable to expect, that one can neglect the \( \omega \)-dependence of the equation also in the confinement region. In this case the small-\( \rho \) asymptotics of the soft pomeron wave function should have the form

\[ \epsilon(\rho) \sim \cos \left( \frac{\pi}{4} + \nu_1 \ln \frac{1}{|\rho|^2 \Lambda^2} - \varphi(q^2) \right). \]  

26
Here $\varphi(-q^2)$ is a phase which is assumed to be a linear function $\varphi(-q^2) = \varphi_0 - ct$ with $c > 0$ at small $t = -q^2$. By matching above expressions for $\epsilon(\rho)$ we obtain the Regge trajectories [37]:

$$
\omega_k(-q^2) = \frac{a}{\pi k + \varphi(-q^2)}.
$$

(152)

Here without any restriction $\varphi_0$ is assumed to be in the interval $\pi > \varphi_0 > 0$. To obtain positive values for the pomeron intercepts we should put $k = 0, 1, \ldots$ and therefore the rightmost pole is situated at $\Delta > a/\pi = 0.4$ [37]. With increasing $t = -(q^2)$ the function $\varphi(-q^2)$ is decreasing and can reach the point $\varphi = 0$ where according to (152) the first Regge pole goes to the infinity. Near this point one should use more accurate expressions leading to a more moderate growth of $\omega_0(t)$. Nevertheless, to avoid the non-physical behaviour of the scattering amplitude at positive $t$ the residue of the first Regge pole should be small for all momentum transfers. In an analogous way the second pole with $k = 1$ also could have a small residue. For the position of the next poles the value of the unknown phase $\varphi$ is not essential. For example, the intercept of the third Regge pole approximately equals to the phenomenological intercept of the soft pomeron.

Note, that in the momentum representation at $t = 0$ the solution of the BFKL equation with a fixed QCD coupling constant can be written in the form:

$$
f(\omega, k, k') = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\nu \sum_{n=-\infty}^{\infty} \frac{|k|^{-1} |k'|^{-1} (k/k')^{i\nu+n/2} (k^*/k'^*)^{i\nu-n/2}}{\omega - \omega(\nu, n)},
$$

(153)

where $k$ and $k'$ are the transverse momenta of incoming and outgoing particles at the $t$-channel. This amplitude is normalized in such way, that at small $\omega$ we obtain:

$$
f(\omega, k, k') \to \frac{1}{\omega} \delta^2(k - k')
$$

Near the leading singularity $\omega = \omega_0$ the region of small $\nu$ is essential and we obtain the well known diffusion expression for the virtual gluon cross-section at high energies [7]

$$
\sigma(s, k^2, k'^2) \sim \int \frac{d\omega}{2\pi i} s^\omega f(\omega, k, k') \sim \frac{s^{\omega_0}}{\sqrt{\alpha \ln s}} \exp \left(-\frac{\ln^2(|k|^2/|k'|^2)}{c \ln s} \right)
$$

(154)

where

$$
c = 56 \frac{\alpha N_c}{\pi} \zeta(3)
$$

is the diffusion constant.

3 Multi-Regge processes in QCD

For the deep-inelastic scattering at small Bjorken variable $x$ the gluon distribution $g(x, k_\perp)$ depending on the longitudinal Sudakov component $x$ of the gluon momentum $k$ and on its transverse projection $k_\perp$ in the infinite momentum frame of the proton $|\vec{p}_A| \to \infty$ can be expressed in terms of the imaginary part of the gluon scattering amplitude at $t = 0$
in the Regge regime of high energies $\sqrt{s} = \sqrt{2p_A p_B}$ and fixed momentum transfers. The most probable process at large $s$ is the gluon production in the multi-Regge kinematics for final state particle momenta $k_0 = p_A', k_1 = q_1 - q_2, ... k_n = q_n - q_{n+1}, k_{n+1} = p_B'$ (see Introduction):

$$s \gg s_i = 2k_{i-1}k_i \gg t_i = q_i^2 = (p_A - \sum_{r=0}^{i-1} k_r)^2, \quad \prod_{i=1}^{n+1} s_i = s \prod_{i=1}^{n} k_i^2, \quad k_1^2 = -\nabla^2.$$  \hfill (155)

In LLA the production amplitude in this kinematics has the multi-Regge form [7]:

$$A_{2\to 2+n}^{LLA} = A_{2\to 2+n}^{tree} \prod_{i=1}^{n+1} s_i^{\omega(t_i)}. \hfill (156)$$

Here $s_i^{\omega(t_i)}$ are the Regge-factors appearing from the radiative corrections to the Born production amplitude $A_{2\to 2+n}^{tree}$. The gluon Regge trajectory $j = 1 + \omega(t)$ is expressed in terms of the quantity:

$$\omega(t) = -\frac{g^2 N_c}{16\pi^3} \int d^2 k \frac{-\nabla^2}{k^2 (\nabla^2 - k^2)^2}, \quad t = -\nabla^2. \hfill (157)$$

Infrared divergencies in the Regge factors cancel in $\sigma_{tot}$ with analogous divergencies in the contributions of real gluons. The production amplitude in the tree approximation has the following factorized form [7]

$$A_{2\to 2+n}^{tree} = 2g T_{r A}^{(c)} \frac{1}{t_1} g T_{e c_1}^{(d)} \frac{1}{t_2} g T_{c_{n+1}c_n}^{(d)} \frac{1}{t_{n+1}} g T_{t B B'}^{(r)} \Gamma_2. \hfill (158)$$

Here $A, B$ and $A', B', d_r \ (r = 1, 2, ... n)$ are colour indices for initial and final gluons correspondingly. $\nu_{r} = -if_{abc}$ are generators of the gauge group $SU(N_c)$ and $g$ is the Yang-Mills coupling constant. Further,

$$\Gamma_1 = \frac{1}{2} \epsilon_{\nu' \nu}^l \epsilon_{\lambda \lambda'}^l \Gamma_{\nu' \nu}, \quad \Gamma_{r+1, r} = -\frac{1}{2} \Gamma_\mu (q_{r+1}, q_r) \epsilon_{\mu}^{\lambda (k_r)} \hfill (159)$$

are the reggeon-particle-particle (RPP) and reggeon-reggeon-particle (RRP) vertices correspondingly. The quantities $\lambda_r = \pm 1$ are the s-channel gluon helicities in the c.m. system. They are conserved for each of two colliding particles: $\Gamma_1 = \delta_{\chi \lambda}$, which is not valid in the one loop approximation [41]. The tensor $\Gamma^{\nu' \nu}$ can be written as the sum of two terms:

$$\Gamma^{\nu' \nu} = \gamma^{\nu' +} - q^2 (n^+)_{\nu} \frac{1}{p_A^+} (n^+)_{\nu'}, \hfill (160)$$

where we introduced the light cone vectors

$$n^- = \frac{p_A}{E}, \quad n^+ = \frac{p_B}{E}, \quad E = \sqrt{s}/2, \quad n^+ n^- = 2 \hfill (161)$$

and the light cone projections $k^\pm = k^\sigma n^\pm_\sigma$ of the Lorentz vectors $k^\sigma$. The first term is the light cone component of the Yang-Mills vertex:

$$\gamma^{\nu' +} = (p_A^+ + p_{A'}^+) \delta^{\nu' -} - 2p_{A'}^+ (n^+)_{\nu'} - 2p_{A'}^+ (n^+)_{\nu}. \hfill (162)$$
The second (induced) term in (160) is a coherent contribution of the Feynman diagrams in which the pole in the \( t \)-channel is absent. Indeed, it is proportional to the factor \( q^2 \) cancelling the neighbouring propagator.

Similarly the effective RRP vertex \( \Gamma(q_2, q_1) \) can be presented as follows [7]
\[
\Gamma^\sigma(q_2, q_1) = \gamma^\sigma_{+} - 2q_1^\sigma (n^-)^\sigma + 2q_2^\sigma (n^+)^\sigma, \tag{163}
\]
where
\[
\gamma_{+-}^\sigma = 2q_2^\sigma + 2q_1^\sigma - 2(n^-)^\sigma k_1^+ + 2(n^+)^\sigma k_1^- \tag{164}
\]
is the light-cone component of the Yang-Mills vertex.

Due to the gluon reggeization the above expression for the production amplitude in LLA has the important property of the two-particle unitarity in each of the \( t_i \) channels (see (74)). Furthermore, it satisfies approximately the unitarity conditions in the direct channels \( s_i \) with the intermediate particles being in the multi-Regge kinematics [7, 37]. Note, that from the general analytic properties of the inelastic amplitudes in the multi-Regge kinematics discussed in Introduction (see (86, 87)) one can obtain dispersion relations in a differential form. Indeed, in accordance with the fact, that in LLA the real part of the production amplitudes for the negative signature is much bigger than their imaginary part we can simplify the signature factors for different dispersion contributions near the point \( j = 1 \). Using this simplification one can derive the following relation
\[
\sum_i Im(s_{ai})A_{2\to2+n} = \pi \frac{\partial}{\partial \ln(s_{a1})} Re A_{2\to2+n}. \tag{165}
\]
Here \( Im(s_{ai})A_{2\to2+n} \) means the imaginary part of the production amplitude in the channel \( s_{ai} = (p_{A^r} + \sum_{r=1}^i k_r)^2 \). Each of these imaginary parts can be calculated through the products of \( Re A_{2\to2+l} \) summed over all intermediate states and integrated over the produced particle momenta with the use of the unitarity conditions. Finally one can verify that the above multi-Regge production amplitudes in LLA satisfy the obtained "bootstrap" equations [7, 37].

Note, that \( \Gamma^\sigma \) has the important property:
\[
(k_1)^\mu \Gamma_\mu(q_2, q_1) = 0, \quad k_1 = q_1 - q_2, \tag{166}
\]
which gives us a possibility to chose an arbitrary gauge for each of the produced gluons. In the left (l) light cone gauge where \( p_Ae^l(k) = 0 \) the polarization vector \( e^l(k) \) is parametrized in terms of the two-dimensional vector \( e_\perp \)
\[
e^l = e_\perp + \frac{k_\perp e^l_\perp}{kp_A}p_A \tag{167}
\]
and satisfies the Lorentz condition \( k e^l = 0 \). The matrix element of the reggeon-reggeon-particle vertex \( \Gamma \) takes an especially simple form [42]
\[
\Gamma^l_{2,1} = Ce^* + C^*e, \quad C = \frac{q_1^* q_2}{k_1^*}, \tag{168}
\]
if we introduce the complex components
\[ e = e_x + i e_y, \quad e^* = e_x - i e_y; \quad k = k_x + i k_y, \quad k^* = k_x - i k_y \] (169)
for transverse vectors \( \vec{e}_\perp, \vec{k}_\perp \). The factors \( q_1^* \) and \( q_2 \) in expression (168) guarantee the vanishing of the inelastic amplitude at small momentum transfers, which is a consequence of the fact that the virtual gluons interact in this limit with the total colour charge, whose matrix elements are zero between the states with the different number of gluons. The singularity \( 1/k_1^* \) in \( C \) reproduces correctly the bremsstrahlung factor for the production amplitude in the soft gluon emission theorem.

The above complex representation was used in [42, 43] to construct the effective scalar field theory for multi-Regge processes which in particular can be applied for the derivation of the equations for compound states of several reggeized gluons [44, 45]. The corresponding effective action was obtained later from Yang-Mills one by integrating over the fields describing the highly virtual particles [43].

The effective action describing multi-Regge processes can be written in the form invariant under the abelian gauge transformations \( \delta V^a_\mu = i \partial_\mu \chi^a \) for the physical gluon fields \( V_\mu \) provided that the fields \( A_\pm \) corresponding to the reggeized gluons are gauge invariant (\( \delta A_\pm = 0 \)):

\[
S_{m,R} = \int d^4x \left\{ \frac{1}{4} (F^a_{\mu\nu})^2 + \frac{1}{2} (\partial_{\perp\sigma} A_+^a)(\partial_{\perp\sigma} A_-^a) + \right.
\]
\[
\frac{1}{2} g \left[ -A_+^a (F_{-\sigma} T^a i \partial_{-\sigma} F_{-\sigma}) - A_-^a (F_{+\sigma} T^a i \partial_{+\sigma} F_{+\sigma}) + (\partial_{-\sigma} F_{-\sigma})(A_- T^a i \partial_{+\sigma} A_+) + \right.
\]
\[
+ (\partial_{+\sigma} F_{+\sigma})(A_+ T^a i \partial_{-\sigma} A_-) + i \left( \frac{1}{\partial_{+\sigma}} \frac{1}{\partial_{-\sigma}} F^a_{-\sigma} T^a (\partial_{-\sigma} A_+) + i F^a_{-\sigma} (A_- T^a A_+) \right) \right\},
\] (170)

where \( F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \) and we introduced the light-cone components of the Lorentz tensors in accordance with (161). The fields \( A_\pm \) satisfy the kinematical constraints \( \partial_{\pm} A_\mp = 0 \) equivalent to the condition that in the multi-Regge kinematics the reggeized gluon takes the negligible part of energy from the colliding particles. The Feynman vertices which are generated by this action coincide on the mass shell with the effective vertices (160, 163) for the reggeon-gluon interactions. However, for virtual gluons these vertices are different, which leads to the inconsistency of this theory. Such drawback is absent in a more general nonabelian effective field theory which will be discussed in the next section.

3.1 BFKL Pomeron

Using the explicit expressions for production amplitudes in the multi-Regge kinematics one can calculate the imaginary part of the elastic scattering amplitude in LLA with the pomeron quantum numbers in the crossing channel related with the high energy asymptotics of the total cross-section. Its real part is small in accordance with the fact, that this amplitude has the positive signature. Due to the factorized form of the production amplitudes one can write down the Bethe-Salpeter equation for the vacuum \( t \)-channel partial wave \( f_\omega \) describing the pomeron as a compound state of two reggeized gluons [7]. The convenience of the \( \omega \)-representation is related with the angular momentum conservation.
in the $t$-channel. The contribution to the integral kernel of the BFKL equation from the real gluons is proportional to the product of the effective vertices calculated in the light cone gauge [42]:

$$C(p_1, p_1') C^*(p_2, p_2') + h.c. = \frac{p_1^* p_2 p_1' p_2'^*}{|k|^2} + h.c.$$ (171)

where $p_1, p_2$ and $p_1', p_2'$ are the corresponding complex transverse components of initial and final momenta in the $t$-channel ($q = p_1 + p_2 = p_1' + p_2'$). In turn, the contribution related with virtual corrections to the production amplitudes is proportional to the sum of the Regge trajectories of two gluons:

$$\omega(-\vec{p}_1^2) + \omega(-\vec{p}_2^2) \sim \ln |p_1|^2 + \ln |p_2|^2 + c,$$ (172)

where the constant $c$ contains the infrared divergent terms which are cancelled with the analogous terms from the real contribution after its integration in $k$. The final homogeneous equation for the wave functions of pomerons being the singularities of $f_\omega$ takes the form [46]

$$E \Psi = H_{12} \Psi,\ E = -\frac{8\omega \pi^2}{g^2 N_c}.$$ (173)

Here the "Hamiltonian" $H_{12}$ is [7, 47]

$$H_{12} = \ln |p_1|^2 + \ln |p_2|^2 + \frac{1}{|p_1|^2 |p_2|^2} (p_1^* p_2 \ln |\rho_{12}|^2 p_1 p_2^* + h.c.) - 4\psi(1)$$ (174)

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ and $\Gamma(x)$ is the Euler $\Gamma$-function. In the above expression $1/\vec{p}_i^2$ are the gluon propagators. We introduced the complex components $\rho_k = x_k + iy_k$ for the impact parameters canonically conjugated to the momenta $p_k = i \frac{\partial}{\partial \rho_k}$ and performed the Fourier transformation:

$$\frac{1}{|k|^2} \rightarrow \ln |\rho_{12}|^2,$$ (175)

where $\rho_{ik} = \rho_i - \rho_k$. The expressions

$$\ln |p_i|^2,\ |p_i|^{-2}$$ (176)

are the integral operators in the impact parameter representation. The Hamiltonian (174) has the property of the holomorphic separability [46]:

$$H_{12} = h_{12} + h^*_{12},\ E = \epsilon + \tilde{\epsilon},$$ (177)

where $\epsilon$ and $\tilde{\epsilon}$ are the energies correspondingly in the holomorphic and anti-holomorphic subspaces:

$$\epsilon \psi(\rho_1, \rho_2) = h_{12} \psi(\rho_1, \rho_2),\ \tilde{\epsilon} \tilde{\psi}(\rho_1^*, \rho_2^*) = h^*_{12} \tilde{\psi}(\rho_1^*, \rho_2^*),\ \Psi(\vec{\rho}_1, \vec{\rho}_2) = \psi \tilde{\psi}.$$ (178)
The holomorphic hamiltonian is [47]

\[ h_{12} = \frac{1}{p_1} \ln (\rho_{12}) p_1 + \frac{1}{p_2} \ln (\rho_{12}) p_2 + \ln(p_1 p_2) - 2\psi(1). \] (179)

One can verify the validity of another representation for \( H_{12} \):

\[ h_{12} = \rho_{12} \ln(p_1 p_2) \rho_{12}^{-1} + 2 \ln(\rho_{12}) - 2\psi(1). \] (180)

Further, using the following identities:

\[ 2 \ln \partial + 2 \ln \rho = \psi(-\rho \partial) + \psi(\partial \rho) = \psi(-\rho \partial) + \psi(1 + \rho \partial) \] (181)

and

\[ 2 \ln(\rho^2 \partial) - 2 \ln(\rho) = \psi(\rho \partial) + \psi(-\rho^2 \partial \rho^{-1}) = \psi(\rho \partial) + \psi(1 - \rho \partial), \] (182)

we can derive the following formulas [47]:

\[ h = \ln(\rho_{12}^2 p_1) + \ln(\rho_{12}^2 p_2) - 2 \ln(\rho_{12}) - 2\psi(1), \] (183)

\[ h = \frac{1}{2} \psi(\rho_{12} \partial_1) + \frac{1}{2} \psi(\rho_{21} \partial_2) + \frac{1}{2} \psi(1 + \rho_{21} \partial_1) + \frac{1}{2} \psi(1 + \rho_{12} \partial_2) - 2\psi(1). \] (184)

Here \( \psi(x) = \Gamma'(x)/\Gamma(x) \) and \( \Gamma(x) \) is the Euler \( \Gamma \)-function. From eq. (183) one can easily verify that \( h \) is invariant under the Möbius transformations:

\[ \rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d} \] (185)

for arbitrary complex values of \( a, b, c, d \). It means, that solutions of the homogeneous BFKL equation belong to irreducible unitary representations of the Möbius group. The generators of this group for a general case of \( n \)-particle interactions are

\[ M^z = \sum_{k=1}^n \rho_k \partial_k, \quad M^- = \sum_{k=1}^n \partial_k, \quad M^+ = -\sum_{k=1}^n \rho_k^2 \partial_k. \] (186)

Its Casimir operator is

\[ M^2 = (M^z)^2 - \frac{1}{2} (M^+ M^- + M^- M^+) = -\sum_{r<s} \rho_{rs} \partial_r \partial_s. \] (187)

For the wave function of two particles in the holomorphic subspace we can use the Polyakov ansatz (cf. (118) with the substitution \( h \rightarrow m \)):

\[ \psi_m(\rho_{10}, \rho_{20}) = \langle 0 | \varphi(\rho_1) \varphi(\rho_2) O_m(\rho_0) | 0 \rangle = (\frac{\rho_{12}}{\rho_{10} \rho_{20}})^m. \] (188)

The conformal weight \( m = \frac{1}{2} + i\nu + \frac{n}{2} \) of the composite operator \( O_m \) is related with its anomalous dimension \( d = 1/2 + iv \) and its conformal spin \( n \). This operator belongs to the basic series of the unitary representations provided that \( \nu \) is real and \( n \) is integer. In
turn, the fields $\varphi(\rho_i)$ describe the reggeized gluons and have the trivial quantum numbers $d = n = 0$. The holomorphic factor $\psi_m$ is an eigen function of the corresponding Casimir operator:

$$M^2 \psi_m = m(m-1)\psi_m.$$  \hspace{1cm} (189)

Simultaneously it is an eigen function of the BFKL equation in the holomorphic subspace:

$$h \psi_m = \epsilon \psi_m, \quad \epsilon = \psi(m) + \psi(1-m) - 2\psi(1).$$  \hspace{1cm} (190)

The eigen value $\epsilon$ can be obtained from representation (184) for $h$ if one would integrate the both sides of the corresponding Schrödinger equation in (178) over the coordinate $\rho_0$ with the use of the relation:

$$\int d\rho_0 \psi(\rho_{10},\rho_{20}) \sim \rho_{12}^{1-m}.$$  \hspace{1cm} (191)

The second Casimir operator $M^{2*}$ is expressed through the conformal weight $\tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$. The total energy is

$$E = \psi(m) + \psi(1-m) + \psi(\tilde{m}) + \psi(1-\tilde{m}) - 4\psi(1).$$  \hspace{1cm} (192)

After simple transformations one can rewrite $E$ as follows

$$E = 4 \Re \psi\left(\frac{1}{2} + i\nu + \left|\frac{n}{2}\right|\right) - 4\psi(1).$$  \hspace{1cm} (193)

The minimum of the energy is obtained for $\nu = n = 0$ and equals $E_0 = -8 \ln 2$. Therefore the total cross-section calculated in LLA grows as $s^\Delta$ (where $\Delta = (g^2N_c/\pi^2) \ln 2$), which violates the Froissart bound $\sigma_{\text{tot}} < c \ln^2 s$ [7]. One of the possible ways to improve LLA is to use the above effective field theory for multi-Regge processes [42, 43].

### 3.2 Compound states of reggeized gluons in multi-colour QCD

The simple method to unitarize the scattering amplitudes obtained in LLA is related with the solution of the BKP equation [44] for compound states of $n$ reggeized gluons:

$$E \Psi = \sum_{i<k} H_{ik} \Psi.$$  \hspace{1cm} (194)

Its eigen value $E$ is proportional to the position $\omega = j - 1$ of the singularity of the $t$-channel partial wave:

$$E = -\frac{8\pi^2}{g^2 N_c} \omega.$$  \hspace{1cm} (195)

Note, that a non-trivial example of the BKP dynamics is the equation for the Odderon which is a compound state of three reggeized gluons [36, 45].

The pair Hamiltonians $H_{ik}$ in (194) have the property of the holomorphic separability:

$$H_{ik} = -\frac{T^a}{N_c} (h_{ik} + h_{ik}^*).$$  \hspace{1cm} (196)
The group generators $T_i^a$ act on colour indices of the gluon $i$. The holomorphic pair hamiltonian is

$$h_{ik} = \frac{1}{p_i} \ln(\rho_{ik}) p_i + \frac{1}{p_k} \ln(\rho_{ik}) p_k + \ln(p_i p_k) - 2\psi(1).$$  \hfill (197)

Similar to the Pomeron case we introduce the complex coordinates $\rho_k = x_k + iy_k$ ($k = 1, 2, \ldots, n$) and their canonically conjugated momenta $p_k = i\frac{\partial}{\partial \rho_k}$ in the impact parameter space (note, that $\rho_{ik} = \rho_i - \rho_k$). The above Schrödinger equation for $\Psi$ is invariant \[46\] under the Möbius transformations:

$$\rho'_k = \frac{a\rho_k + b}{c\rho_k + d}$$

for any complex values of $a, b, c, d$.

According to t’Hooft for the multi-colour QCD ($N_c \rightarrow \infty$) only planar diagrams in the colour space are important. Because the colour structure of the eigen function at large $N_c$ is unique, due to eq. (196) the total hamiltonian $H$ can be written as a sum of the mutually commuting holomorphic and anti-holomorphic operators \[46\]:

$$H = \frac{1}{2}(h + h^*), \quad [h, h^*] = 0.$$  \hfill (198)

The colour factor $1/2$ appears because at large $N_c$ the neighbouring gluons are in the octet state. The holomorphic hamiltonian is

$$h = \sum_{i=1}^n h_{i, i+1}.$$  \hfill (199)

Thus, in the multi-colour QCD the solution of the Schrödinger equation (194) has the property of the holomorphic factorization:

$$\Psi = \sum c_k \psi_k(\rho_1, \ldots, \rho_n) \bar{\psi}_k(\rho^*_1, \ldots, \rho^*_n).$$  \hfill (200)

where $\psi$ and $\bar{\psi}$ are correspondingly the analytic and anti-analytic functions of their arguments and the sum is performed over all degenerate solutions of the Schrödinger equations in the holomorphic and anti-holomorphic subspaces:

$$\epsilon \psi = h \psi, \quad \epsilon^* \psi^* = h^* \psi^*, \quad E = \frac{1}{2}(\epsilon + \epsilon^*).$$  \hfill (201)

These equations have the nontrivial integrals of motion \[47\]:

$$t(\theta) = \text{tr} T(\theta), \quad [t(u), t(v)] = [t(\theta), h] = 0,$$  \hfill (202)

where $\theta$ is the spectral parameter of the transfer matrix $t(\theta)$. The monodromy matrix $T(\theta)$ is constructed from the product of the $L$-operators

$$T(\theta) = L_1(\theta)L_2(\theta)\ldots L_n(\theta)$$  \hfill (203)

expressed in terms of the Möbius group generators:
\begin{equation}
L_k(\theta) = \begin{pmatrix}
\theta + i\rho_k \partial_k & i\partial_k \\
-i\rho_k^2 \partial_k & \theta - i\rho_k \partial_k
\end{pmatrix}.
\end{equation}

The solution of the Schrödinger equation (201) in the holomorphic subspace is reduced to a pure algebraic problem of finding the representation of the Yang-Baxter commutation relation [47]:

\begin{equation}
T_{i_1i_2'}(u)T_{i_2i_3'}(v-u+iP_{12}) = (v-u+iP_{12})T_{i_3i_1'}(v)T_{i_1i_2'}(u),
\end{equation}

where the operator $P_{12}$ in its left and right hand sides transmutes correspondingly the right and the left indices of the matrices $T(u)$ and $T(v)$. Moreover [48], Hamiltonian (199) coincides with the local Hamiltonian for a completely integrable Heisenberg model with the spins belonging to an infinite dimensional representation of the non-compact Möbius group and all physical quantities can be expressed in terms of the Baxter function $Q(\lambda)$ satisfying the equation:

\begin{equation}
t(\lambda)Q(\lambda) = (\lambda + i)^nQ(\lambda + i) + (\lambda - i)^nQ(\lambda - i),
\end{equation}

where $t(\lambda)$ is an eigen value of the transfer matrix. The solution of the Baxter equation is known for $n = 2$. In a general case $n > 2$ one can present it as a linear combination of the solutions for $n = 2$ with a recurrence relation for the coefficients $d_k$. For $n = 3$ this relation takes the form:

\begin{equation}
Ad_k(A) = \frac{k(k+1)}{2(2k+1)}(k - m + 1)(k + m)(d_{k+1}(A) + d_{k-1}(A))
\end{equation}

with the initial conditions $d_0 = 0$, $d_1 = 1$. If one will consider for simplicity the integer values of the conformal weight $m$ the quantization condition for eigen values $A$ is $d_{m-1}(A) = 0$. Although the orthogonality and completeness conditions for the polynomials $d_k(A)$ are known:

\begin{equation}
\sum_{k=1}^{m-2} \frac{2(2k+1)}{k(k+1)(k - m + 1)(k + m)} d_k(A) d_k(\bar{A}) = \delta_{A\bar{A}}d'_{m-1}(A) d_{m-2}(A), \quad A \neq 0;
\end{equation}

\begin{equation}
\sum_{A \neq 0} d'_k(A) d'_k(A) = \delta_{kk'} \frac{k(k+1)(k - m + 1)(k + m)}{2(2k+1)} , \quad d'_k(A) = \frac{d}{dA} d_k(A),
\end{equation}

their full theory is not constructed yet. It does not allow us to calculate analytically the intercept and wave function of the Odderon in QCD [45, 47].

\section{Effective action for small $x$ physics in QCD}

All above results are based on calculations of effective Reggeon vertices and the gluon Regge trajectory in the first nontrivial order of perturbation theory. Up to now we do not know the region of applicability of LLA including the intervals of energies and momentum transfers fixing the scale for the QCD coupling constant. One should develop also a self-consistent approach for the unitarization of the BFKL pomeron. Therefore it is needed.
to generalize the effective field theory of ref. [42] to the quasi-multi-Regge processes in which the final state particles are separated in several groups consisting of an arbitrary number of gluons and quarks with a fixed invariant mass; each group is produced with respect to others in the multi-Regge kinematics. The production of two particles with a fixed invariant mass is the simplest example of such processes [49].

At high energies the rapidity \( y = \frac{1}{2} \ln \frac{k^+}{k^-} \) constructed from the light-cone components \( k^\pm = \kappa^\alpha n^\alpha_x \) of the particle momenta is similar to the time in quantum mechanics. The corresponding Hamiltonian is determined by the interaction of gluons and quarks with a nearly equal rapidity. Let us introduce the parameter \( \eta \) much smaller than \( \ln s \).

The gauge-invariant effective action \( S_{\text{eff}} \) local in a rapidity interval \((y_0 - \eta, y_0 + \eta)\) was constructed recently [51] and includes apart from the usual Yang-Mills action also the reggeon-particle interactions (cf. (170)):

\[
S_{\text{eff}}(v, A_\pm) = - \int d^4x \text{tr} \left[ \frac{1}{2} G_{\mu\nu}(v) + (A_-(v) - A_-) j_{\pm}^{\text{reg}} + (A_+(v) - A_+) j_{\pm}^{\text{reg}} \right] \quad (210)
\]

where the anti-hermitian \( SU(N_c) \) matrices \( v_\sigma \) and \( A_\pm \) describe correspondingly the usual and reggeized gluons. Because the action is local in the rapidity space, we omit temporally \( y \) as an additional argument of these fields. The reggeon current \( j_{\pm}^{\text{reg}} \) is expressed in terms of \( A_\pm \) as follows:

\[
j_{\pm}^{\text{reg}} = \partial_\sigma A_\pm, \quad (211)
\]

which guarantees, that the interaction disappears on the mass shell \( k^2 = 0 \).

The fields \( A_\pm \) are invariant

\[
\delta A_\pm = 0 \quad (212)
\]

under the infinitesimal gauge transformation

\[
\delta v_\sigma = [D_\sigma, \chi] \quad (213)
\]

with the gauge parameter \( \chi \) decreasing at \( x \to \infty \), but they belong to the adjoint representation of the global \( SU(N_c) \) group and are transformed at constant \( \chi \) as follows:

\[
\delta A_\pm = g [A_\pm, \chi] \quad (214)
\]

As usually,

\[
G_{\mu\nu}(v) = \frac{1}{g} [D_\mu, D_\nu] = \partial_\mu v_\nu - \partial_\nu v_\mu + g [v_\mu, v_\nu], \quad D_\sigma = \partial_\sigma + g v_\sigma. \quad (215)
\]

The fields \( A_\pm \) obey the additional kinematical constraints

\[
\partial_+ A_- = 0, \quad \partial_- A_+ = 0 \quad (216)
\]

meaning that the Sudakov components \( \alpha, \beta \) of the reggeon momentum \( q = \alpha p_B + \beta p_A + k_\perp \) are negligible small in comparison with the corresponding big components \( \alpha_k, \beta_{k-1} \) of the neighbouring particle momenta. Such simplification takes place at the quasi-multi-Regge kinematics where the gluons in the final and intermediate states are separated in several clusters. The invariant mass of each cluster is restricted from above by a
value proportional to $\exp(\eta)$ and the neighbouring clusters are significantly different in their rapidities: $y_{k-1} - y_k \gg \eta$. Further, the Sudakov components $\alpha_k, \beta_k$ of their total momenta are strongly ordered: $\alpha_k \gg \alpha_{k-1}, \beta_k \ll \beta_{k-1}$ and the transverse momenta $k_\perp$ are restricted. The effective action describes the self-interaction of real and virtual particles inside each cluster and their coupling with neighbouring reggeized gluons. Note, that because small fractions of the Sudakov parameters are transmitted from one to other clusters the constraints $\partial_\pm A_\pm = 0$ (216) are not absolute. The composite reggeon field $A_\pm(v)$ is given below

$$A_\pm(v) = v_\pm - gv_\pm \frac{1}{\partial_\pm} v_\pm + g^2 v_\pm \frac{1}{\partial_\pm} v_\pm \frac{1}{\partial_\pm} v_\pm - ...$$ (217)

and can be written in the explicit form

$$A_\pm(v) = v_\pm D_{\pm}^{-1} \partial_\pm = -\frac{1}{g} \partial_\pm U(v_\pm), U(v_\pm) = P \exp(-\frac{g}{2} \int_{-\infty}^{x_\pm} dx' v_\pm(x'),$$ (218)

where the integral operator $D_{\pm}^{-1} \partial_\pm$ is implied to act on an unit constant matrix from the left hand side and the symbol $P$ means the ordering of the fields $v$ in the matrix product in accordance with the increasing of their arguments $x_\pm$. We chose for the Wilson exponent $U(v_\pm)$ the following boundary condition: $U \to 1$ at $x \to -\infty$. This exponent does not have infrared singularities in the essential integration region in accordance with the fact, that the operators $\partial_\pm^{-1}$ are restricted from above for all particles inside the cluster. Because $j^{reg}$ (211) in the momentum representation contains the factor $t = q^2$ killing the pole in the neighbouring reggeon propagator, the corresponding scattering amplitudes satisfy the Steinman relations forbidding simultaneous singularities in the overlapping channels $t$ and $s$. The interaction terms describe contributions of the Feynman diagrams, in which the gluons in the given rapidity interval $(y_0 - \eta, y_0 + \eta)$ are coherently emitted by the neighbouring particles with essentially different rapidities.

The interaction terms of the action are gauge invariant due to the following relations

$$\left[D_\pm, j^{ind}_\mp\right] = 0, \quad j^{ind}_\mp = \frac{\partial}{\partial v_\mp} tr(A_\pm(v) \partial^2 A_\mp)$$ (219)

where $j^{ind}_\mp$ is the induced current. For the particles belonging to the same cluster the parameter $\eta$ plays a role of the ultraviolet cut-off in their relative rapidity.

Note, that action (210) is similar to the Legendre transformation for which the $A_\mp$-dependence of the current $j^{reg}_\pm$ would be fixed by the stationary condition $\delta F = 0$ for the free energy:

$$\exp(-iF(j^{reg}_\pm, A_\pm)) = \int \prod_x d v(x) \exp (-iS_{eff}(v, A_\pm))$$ (220)

with taking into account the Faddeev-Popov ghosts. Below some arguments supporting our choice (211) of $j^{reg}_\pm$ will be given. Because we do not use the Legendre transformation, there is a finite renormalization of the kinetic term for the fields $A_\pm$ in (210) even in the tree approximation:

$$- A_- j^{reg}_+ - A_+ j^{reg}_- \to (\partial_\sigma A_-)(\partial_\sigma A_+)$$ (221)
related with the transition \( A \rightarrow v \rightarrow A \) induced by the bilinear terms in the gluon and reggeon fields.

### 4.1 \( \eta \) - independence

The physical results should not depend on the auxiliary parameter \( \eta \) due to cancellations between the integrals over the invariant masses of the produced clusters and the integrals over their relative rapidities (for which \( \eta \) plays a part of the infrared cut-off). This criterion is very important for the self-consistency of the effective action (210). For big values of \( \eta \) the most essential polynomial contribution from integrals over particle rapidities inside each cluster corresponds to the production of an arbitrary number \( n \) of subgroups of real and virtual particles in the quasi-multi-Regge kinematics. We show below, that the interaction of the particles inside the whole cluster can be described by the same effective action for each subgroup with a smaller cut-off and the integrals over their relative rapidities reproduce correctly the contribution proportional to the polynomial \( P_{n-1}(\eta) \).

To begin with, let us consider only two subgroups of real and virtual particles with the fixed invariant masses being restricted from above by a value proportional to \( \exp(\eta_0) \) where \( \eta_0 \) is a new parameter which is assumed to be much smaller than initial one \( \eta \) for the whole cluster. The rapidities \( y^1 \) and \( y^2 \) of these subgroups satisfy the condition \( 0 < y^1 - y^2 < 2\eta \) leading to the integral contribution proportional to \( \eta \).

The field \( v_\alpha \) can be presented as the sum of three Fourier components:

\[
v = v^1 + v^2 + v^{12}.
\]

The fields \( v^1 \) and \( v^2 \) describe the particles inside two extracted subgroups and \( v^{12} \) is related with the strongly virtual gluons having the relatively big Sudakov parameters \( \beta \sim k^+ \) and \( \alpha \sim k^- \) of the order of value of \( \beta_1 \sim k^+_1 \) and \( \alpha_2 \sim k^-_2 \) for the first and second subgroups correspondingly. Because \( \alpha_i \) and \( \beta_i \) are strongly ordered in the quasi-multi-Regge kinematics: \( \alpha_2 \gg \alpha_1, \beta_1 \gg \beta_2 \), the virtuality of fields \( v^{12} \) is large: \( M^2 = S \alpha \beta \gg k^2_1 \).

Let us consider the functional integral over the heavy fields \( v^{12} \) for fixed \( v^1 \) and \( v^2 \) (cf. [43]). In expression (217) the integral operators \( \partial_{\pm}^{-1} \) correspond to the propagators of the strongly virtual neighbouring particles producing gluons within the chosen rapidity interval \( (y_0 - \eta, y_0 + \eta) \). Because these virtualities are significantly different for the emission of gluons inside above two subgroups we can simplify the composite fields \( A_\pm(v) \) by neglecting the more virtual fields:

\[
A_-(v) \simeq -\frac{\partial_-}{g} U(v^2) + ... , \quad A_+(v) \simeq -\frac{\partial_+}{g} U(v^1) + .... .
\]

The factors \( U(v^1_\pm) \) and \( U(v^2_\pm) \) in the above expansion lead to the corresponding interaction terms \( A_+(v^1) \partial^2_\alpha A_- \) and \( A_-(v^2) \partial^2_\alpha A_+ \) in the effective Lagrangians \( L^1 \) and \( L^2 \) for each of the above two subgroups. To get the other two terms in these Lagrangians one should calculate the functional integral over \( v^{12} \). It leads to the contributions depending only on \( v^1_+ \) and \( v^2_+ \) because they enter with the coefficients proportional to the big Sudakov parameters \( \alpha_2 \sim k^+_2 \) or \( \beta_1 \sim k^-_1 \). Due to the gauge invariance of the initial action
these contributions should be also gauge invariant. This property fixes their form almost completely: $\Delta L^1 = A_-(v_1) \partial_2^2 a_+$ and $\Delta L^2 = A_+(v_2^2) \partial_2^2 a_-$ up to two arbitrary functions $a_\pm (x)$ satisfying the kinematical constraints $\partial_- a_+ = \partial_+ a_- = 0$. Further, one can integrate the obtained expression $\exp(iS_1^{int} + iS_2^{int})$ over $a_\pm$ with an arbitrary weight depending on $a_\pm$ but to achieve an agreement with the perturbation theory the total Lagrangian should contain at least the quadratic term $-2 (\partial_\alpha a_+)(\partial_\alpha a_-)$ leading to a correctly normalized propagator of fields $a_\pm$ responsible for the interaction of the particles with the relative rapidities smaller than $\eta$ (and bigger than the new intermediate parameter $\eta_0$).

The integration over $a_\pm$ with the subsequent integration over the relative rapidity gives the result proportional to $\eta$ in accordance with our expectations. Non-linear interactions of the fields $a_\pm$ would lead to the contributions having higher powers of $\eta$ and therefore they should be absent for the two-group kinematics. Note, that these higher power contributions appear if we shall consider several subgroups of real and virtual particles in the quasi-multi-Regge kinematics inside the given cluster. In this more general case one can also verify, that the integration over momenta of the strongly virtual particles gives the interaction terms linear in $a_\pm$ for each subgroup. It means, that the functions $a_\pm$ coincide in fact with the fields $A_\pm$ and describe the interaction between the subgroups of particles within the additional interval $(\eta_0, \eta)$ of their relative rapidity. Thus, in the framework of the effective action approach the $\eta$-dependence of integrals over particle momenta inside each group is compensated by the $\eta$-dependence of integrals over the relative rapidities of these groups. In particular for $\eta = y$ the fields $A_\pm$ are absent and we return to the initial Yang-Mills theory.

### 4.2 Classical equations

One can verify, that the composite fields $A_\pm (v)$ are changed under the infinitesimal transformations of the gauge group $\delta v_\sigma = [D_\sigma, \chi]$ according to the abelian law: $\delta A_\pm (v) = \partial_\pm \varphi$. Therefore after integrating by parts and using the kinematical constraint $\partial_{\pm} A_{\mp} = 0$ we conclude, that the effective action $S_{eff}$ is gauge invariant for $\chi$ decreasing at $\infty$ provided that the reggeon fields are fixed: $\delta A_{\pm} = 0$. For constant $\chi$ the action is also invariant because the fields $A_{\pm}$ belong to the adjoint representation of the gauge group $SU(N_c)$.

The effective action $S_{eff}$ has a nontrivial stationary point $v = \bar{v}$ satisfying the following Euler-Lagrange equations [51]:

$$[D_\sigma, G_{\sigma \pm}] = j^{ind}_{\pm}, \quad [D_\sigma, G_{\sigma \rho}] = 0, \quad \text{(224)}$$

where the induced current $j^{ind}_{\sigma}$ equals

$$j^{ind}_{\pm} = \frac{1}{D_\mp} \partial_\mp (\partial_\mp^2 A_\pm) \partial_\pm \frac{1}{D_\mp} \partial_\pm j^{ind}_{\pm} = 0 \quad \text{(225)}$$

and due to (219) satisfies the covariant conservation law:

$$[D_\sigma, j^{ind}_{\sigma}] = 0. \quad \text{(226)}$$

We remind, that the integral operators $D_\mp^{-1} \partial_\mp$ and $\partial_\pm D_\mp^{-1}$ are implied to act on unit constant matrices correspondingly from the left and right hand sides (in the second case
after integrating by parts the operators \( \partial_{\mp} \) should be substituted by \(-\partial_{\mp}\) acting on \( A_\pm \).

Using the following expression for the Yang-Mills current

\[
j^{YM}_\mu = -\left[ D_\sigma, G_{\sigma\mu} \right] = \partial_\mu G_{\sigma\mu} = \partial_\mu G_{\sigma\mu} = \partial_\sigma G_{\sigma\mu} - \partial_\sigma^2 v_\mu + g \left( [v_\sigma, \partial_\mu v_\sigma] - [\partial_\sigma v_\sigma, v_\mu] - 2 [v_\sigma, \partial_\sigma v_\mu] \right) - g^2 [v_\sigma, [v_\sigma, v_\mu]]
\]

(227)

we can rewrite the above equations in the form:

\[
\partial_\sigma G_{\sigma\mu} = j_\mu,
\]

(228)

where

\[
j_\mu^\perp = -g [v_\sigma, G^\perp_{\sigma\mu}],
\]

(229)

and

\[
j_\pm = -g [v_\sigma, G_{\sigma\pm}] + j^{\text{ind}}_\pm.
\]

(230)

The above equations are self-consistent because the current \( j \) is conserved:

\[
\partial_\mu j_\mu = \frac{1}{2} \left[ D_-, j^{\text{ind}}_+ \right] + \frac{1}{2} \left[ D_+, j^{\text{ind}}_- \right] = 0.
\]

(231)

We can construct a perturbative solution \( v = \overline{v} \) of the classical equations. For example in the Landau gauge where

\[
\partial_\sigma \overline{v}_\sigma = 0
\]

(232)

one obtains

\[
\overline{v}_\sigma = \sum_{n=0}^{\infty} g^n f^n_\sigma, \quad f^0_\pm = A_\pm, \quad f^0_{\perp_\sigma} = 0.
\]

(233)

and

\[
\overline{j}^{\text{ind}}_\sigma = \sum_{n=0}^{\infty} g^n \Delta^n_\sigma, \quad \Delta^0_\pm = \partial^2_\alpha A_\pm, \quad \Delta^n_{\perp_\sigma} = 0.
\]

(234)

The higher order coefficients \( f^n_\sigma, \Delta^n_\sigma \) satisfy the recurrence relations:

\[
\partial^2_\alpha f^n_\sigma = \Delta^n_\sigma - \sum_{i=0}^{n-1} \left[ f^{n-1-i}_\alpha, 2 \partial_\alpha f^i_\sigma - \partial_\sigma f^i_\alpha + \sum_{j=0}^{i-1} \left[ f^{i-j}_\alpha, f^j_\sigma \right] \right]
\]

(235)

and

\[
\partial_{\mp} \Delta^n_\pm = - \sum_{i=0}^{n-1} \left[ f^{n-1-i}_{\mp}, \Delta^i_\pm \right],
\]

(236)

where we used the gauge properties (219) of \( j^{\text{ind}}_{\pm} \).

The coefficients \( \Delta^n_\sigma \) can be presented in the explicit form:

\[
\Delta^n_\pm = \sum_{k=0}^{n} \left( \frac{1}{D_{\mp}} \partial_{\mp} \right)_k \partial^2_\sigma A_\pm \left( \partial_\mp \frac{1}{D_{\mp}} \right)_{n-k}, \quad \Delta^n_{\sigma\perp} = 0
\]

(237)

where

\[
\left( \frac{1}{D_{\mp}} \partial_{\mp} \right)_k = \sum_{\text{perm}} \prod_{r=0}^{\infty} \left( -\frac{1}{\partial_{\mp}} f^r_{\mp} \right)^{i_r}, \quad \sum_{r=0}^{\infty} (r+1)i_r = k,
\]

(238)
\[(\partial_\mp \frac{1}{D_\mp})_k = \sum_{\text{perm}} \prod_{r=0}^{\infty} (-f_\mp^r \frac{1}{\partial_\mp})^{i_r}; \sum_{r=0}^{\infty} (r+1)i_r = k. \quad (239)\]

In these expressions the summation is performed over all permutations of the non-commuting integral operators \((-\frac{1}{\partial_\mp} f_\mp^\sigma)\) and \((-f_\mp^\sigma \frac{1}{\partial_\mp})\) correspondingly and the following recurrence relations are valid:\n
\[
\partial_\mp (\frac{1}{D_\mp} \partial_\mp) n = -\sum_{i=0}^{n-1} f_\mp^{n-1-i} (\frac{1}{D_\mp} \partial_\mp)_i, \quad (240)\]
\[
\partial_\mp (\frac{1}{D_\mp} \partial_\mp) n = -\sum_{i=0}^{n-1} (\partial_\mp \frac{1}{D_\mp} \partial_\mp)_i; f_\mp^{n-1-i}. \quad (241)\]

Using equations (235, 236), we can construct \(f^n\) and \(\Delta^n\) in several orders of perturbation theory taking into account that \(\partial_\mp A_\pm = 0\). Namely,

\[
f_\pm^0 = A_\pm, \quad f_\sigma^{0\perp} = 0, \quad \Delta_\pm^0 = \partial_\sigma^2 A_\pm, \quad \Delta_\perp^\sigma = 0; \quad (242)\]
\[
\partial_\alpha^2 f_\pm^1 = \Delta_\pm^1 - \frac{1}{2} [A_\mp, \partial_\pm A_\pm], \quad \Lambda_\pm^1 = [\partial_\alpha^2 A_\pm, \partial_\mp^{-1} A_\mp]; \quad (243)\]
\[
\partial_\alpha^2 f_\sigma^{1\perp} = \frac{1}{2} [A_+, \partial_\sigma^1 A_-], \quad (244)\]

and

\[
\partial_\alpha^2 f_\sigma = \frac{1}{2} ([A_+, \partial_\sigma f_\mp^1] + [A_-, \partial_\sigma f_\mp^1] + [f_\mp^1, \partial_\sigma A_-] + [f_\mp^1, \partial_\sigma A_+])
- [A_+, \partial_- f_\sigma^1] - [A_-, \partial_+ f_\sigma^1] + \Omega_\sigma, \quad (245)\]

where \(\Omega_\sigma^{\perp} = 0\) and

\[
\Omega_\pm = \Delta_\pm^2 - 2 [f_\alpha^1, \partial_\sigma A_\pm] - \frac{1}{2} [A_\pm, [A_\mp, A_\pm]], \quad (246)\]
\[
\Delta_\perp^2 = -\partial_\mp^{-1} \left( [f_\mp^1, \partial_\sigma A_\pm] + [A_\mp, \Delta_\perp^1] \right). \quad (247)\]

These perturbative terms \(f^n_\sigma\) satisfy the Landau gauge condition \(\partial_n f^n_\alpha = 0\). For the quasi-elastic kinematics when \(A_+ = 0\) (\(A_- = 0\)) the solution of the recurrence equations is trivial: \(\bar{v}_- = A_- (\bar{v}_+ = A_+)\).

Note, that the Euler-Lagrange equations for the effective action \(S_{eff}\) can be obtained also from the usual Yang-Mills action in the form

\[
[D_\pm, [D_\sigma, G_{\sigma\pm}]] = 0, \quad [D_\sigma, G_{\sigma\rho}^{\perp}] = 0 \quad (248)\]

if one will restrict oneself to the quasi-gauge variations \(\delta v_\pm = [D_\pm, \chi_\pm]\) under which the interaction terms are stationary. Therefore the reggeon fields \(A_\pm\) can be considered as some parameters of the solutions of eqs (248) and the effective theory can be derived from the gluodynamics by imposing certain constraints for the initial Yang-Mills fields: \(A_\pm(v) = \varphi_\pm\) where \(\varphi_\pm\) is an abelian field.
4.3 Reggeon action in the semi-classical approximation

To obtain the reggeon action in the tree approximation we should substitute \( v_\sigma \rightarrow v_\sigma \) in \( S_{\text{eff}} \) [51]. The interaction terms in \( S_{\text{eff}} \) can be expressed in terms of the Wilson integrals with the contours displaced along two light-cone lines:

\[
S_{\text{int}}^{\text{eff}} = \frac{1}{g} \int d^2 x_\perp \ tr \left( \int_{-\infty}^{\infty} dx^+(T(v_-) - 1) \partial_\sigma^2 A_+ + \int_{-\infty}^{\infty} dx^-(T(v_+) - 1) \partial_\sigma^2 A_- \right),
\]

\[
T(v_\pm) = P \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dx^\pm v_\pm \right) = \lim_{x^\pm \to \infty} \frac{1}{D^\pm} \partial_\pm,
\]

where we used the constraints \( \partial_\pm A_\mp = 0 \). Note, however, that because this constraint is not absolute and \( T(v_\pm) \) contains singular operators \( \partial_\pm^{-1} \) there could be uncertainties. In this case the initial representation for the action is preferable.

The expression \( \partial_\pm D^\pm_\pm \) appearing in expression (225) has the following integral representation:

\[
\partial_\pm D^\pm_\pm = P \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dx^\pm v_\pm \right).
\]

Therefore it tends to 1 at \( x^\pm \to \infty \) and the interaction term can be expressed in terms of the induced current \( j^{\text{ind}}_\pm \) (225):

\[
S_{\text{eff}} = \frac{1}{g} \int d^2 x_\perp d x^+ \ tr \left( j^{\text{ind}}_+(\infty) - \partial_\sigma^2 A_+ \right) + \frac{1}{g} \int d^2 x_\perp d x^- \ tr \left( j^{\text{ind}}_-(\infty) - \partial_\sigma^2 A_- \right).
\]

For this choice of the boundary conditions for the perturbative coefficients \( \Delta_n^\pm \) of \( j^{\text{ind}}_\pm \) the factors \( \left( \frac{1}{D_\pm} \partial_\pm \right)^k \) and \( \left( \partial_\pm D_\pm \right)^k \) are defined as the coefficients of the perturbative expansion of the following expressions:

\[
\frac{1}{D_\pm} \partial_\pm = P \prod_{n=0}^\infty \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dx^\pm f_\pm^n \right),
\]

\[
\partial_\pm D_\pm = P \prod_{n=0}^\infty \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dx^\pm f_\pm^n \right).
\]

In the particular case of the quasi-elastic process, when there are only two clusters, after integration over the fields \( A_\pm \) we obtain for the effective action

\[
S_{\text{eff}}^{\text{qe}} = -\frac{1}{2} \int d^4 x \ tr \ G^2_{\mu\nu}(v) + \Delta S,
\]

where \( \Delta S \) is the action for a two-dimensional \( \sigma \)-model:

\[
\Delta S = -\frac{2}{g^2} \int d^2 x_\perp tr \left( (\partial_\perp T(v_-)) (\partial_\perp T(v_+)) \right),
\]

introduced firstly by E. Verlinde and H. Verlinde using other arguments [52]. In our approach the Yang-Mills interaction of the fields \( v \) inside of two separated clusters is also essential.
By inserting the perturbative solution $v_{\sigma} = \sigma_{\nu}, j_{\pm} = j_{\pm \nu}^{\text{ind}}$ of classical equations in $S_{\text{eff}}$ one can write the reggeon action in the tree approximation as follows:

$$S_{\text{eff}}^{\text{tree}} = - \int d^4 x \text{ tr} \left( s_2 + g s_3 + g^2 s_4 + \ldots \right), \quad (256)$$

where

$$s_2 = \sigma^\perp_\sigma A_+ \sigma^\perp_\sigma A_-, \quad (257)$$

$$s_3 = - (\sigma^2_\sigma A_-) A_+ \sigma^\perp_\sigma A_+ - (\sigma^2_\sigma A_+) A_- \sigma^\perp_\sigma A_-, \quad (258)$$

$$s_4 = - (\sigma^2_\mu f_\nu^1)^2 - \frac{1}{4} [A_+, A_-]^2 +$$

$$+ (\sigma^2_\sigma A_-) A_+ \sigma^\perp_\sigma A_+ \sigma^\perp_\sigma A_+ + (\sigma^2_\sigma A_+) A_- \sigma^\perp_\sigma A_- \sigma^\perp_\sigma A_- \quad (259)$$

and $f_\nu^1$ is determined by eq. (243).

The term $s_2$ leads to the correctly normalized propagator

$$\langle 0 | T (A_+^b(x))_b^\dagger (A_+^{y'}(x'))_y^\dagger | 0 \rangle = \Lambda_{bb'}^{\gamma\gamma'} \theta(y - y' - \eta) \delta^2(x|| - x'||) \frac{1}{2\pi i} \ln (x - x')^2 \quad (260)$$

of the reggeon fields $(A_+^b)_b$ ($b, c$ are colour indices and $\Lambda_{bb'}^{\gamma\gamma'} = \delta^c_b \delta^{\gamma'}_{c'} - N_c^{-1} \delta^c_b \delta^{\gamma'}_{c'}$ is the projector to the adjoint representation of the colour group). Note, that we introduced $y$ as an additional argument enumerating the Fourier components of the fields and included the factor $\theta(y - y' - \eta)$ in the Green function to remind that $A_{\pm}$ describe the interaction of the clusters with the relative rapidity bigger than $\eta$. One can modify the term $s_2$ to guarantee this property:

$$s_2 \rightarrow \int_{-\infty}^{+\infty} dy \, \sigma^\perp_\sigma A_+^{y'} \frac{\partial}{\partial y} \sigma^\perp_\sigma A_-^{y' - \eta} \quad (261)$$

Here we took into account the interactions for all rapidities contrary to the case of the initial action defined in the fixed rapidity interval $(y_0 - \eta, y_0 + \eta)$.

The term $s_3$ corresponds to the triple reggeon coupling. This reggeon vertex was introduced earlier by A. White [53]. Because the signature of the gluon is negative, $s_3$ does not give any contribution to the elastic scattering amplitude due to the Gribov signature conservation rule but this vertex can be essential for some inclusive processes.

The term $s_4$ describes the four-linear interaction of the reggeons. The first two contributions correspond to their elastic scattering in the $t$-channel and last ones give the gluon transitions $1 \Leftrightarrow 3$ which are not suppressed by the Gribov rule. The quantity $f_\sigma^1$ is conserved: $\sigma^\perp_\sigma f_\sigma^1 = 0$ and is directly related with the effective RRP vertex $C_{\sigma}(q_1, q_2)$. Because for the big squared mass $\kappa \sim \sigma^2_\sigma$ of the intermediate state and fixed transverse momenta we have

$$f_\nu^1 \sigma^\perp_\sigma f_\nu^1 \simeq \frac{1}{4} [A_+, A_-]^2,$$

there is a significant cancellation between two first terms in $s_4$ leading to a better convergence of the corresponding integral: $\int \frac{d\kappa}{\kappa}$ where $\ln \kappa < \eta$. For the colorless state in the $t$-channel this integral is convergent in the ultraviolet region and by taking the residue
at $\kappa = 0$ one can obtain the result, corresponding to the real contribution to the BFKL kernel.

The term $s_5$ also can be expressed in terms of the above calculated quantities $f^1$ and $f^2$. The higher order term $s_6$ describes in particular the reggeon transition $2 \rightarrow 4$ related with the triple pomeron vertex [54].

In a general case to cancel the infrared divergencies one should take into account apart from the classical expressions for the corresponding transition vertices also the contributions from quantum fluctuations near classical solutions. For example in LLA they are responsible for the gluon reggeization.

Let us write down the field $v$ as a sum of its classical component $\overline{v}$ and the small variation $\epsilon$ describing its fluctuations near the classical solution:

$$v = \overline{v} + \epsilon$$  \hspace{1cm} (262)

and expand the action in $\epsilon$:

$$\Delta S = S_{\text{eff}} - S_{\text{eff}}^{\text{tree}} = - \int d^4x \, tr \left\{ [D_\mu, \epsilon_\nu]^2 - [D_\mu, \epsilon_\nu][D_\nu, \epsilon_\mu] + g G_{\mu\nu} [\epsilon_\mu, \epsilon_\nu] + \frac{1}{2} (\epsilon_- \frac{\partial}{\partial v_-}) (\epsilon_- \frac{\partial}{\partial v_-}) j^{\text{ind}}(v_-) A_+ + \frac{1}{2} (\epsilon_+ \frac{\partial}{\partial v_+}) (\epsilon_+ \frac{\partial}{\partial v_+}) j^{\text{ind}}(v_+) A_- + O(\epsilon^3) \right\}.$$  \hspace{1cm} (263)

The reggeon action in the semiclassical approximation can be obtained if one would calculate the functional integral over the quantum fluctuations $\epsilon$ (with taking into account the Faddeev-Popov ghosts). Here we use the perturbative solution of the classical equations to write down $\Delta S$ only up to quadratic terms in $\epsilon$ and bilinear in $A_{\pm}$:

$$\Delta S = - \int d^4x \, tr \left\{ (\partial_\mu \epsilon_\nu)^2 - (\partial_\mu \epsilon_\nu)(\partial_\nu \epsilon_\mu) + g \{ 2(\partial_\mu \epsilon_\nu)[\overline{v}_\mu, \epsilon_\nu] - 2(\partial_\nu \epsilon_\mu)[\overline{v}_\mu, \epsilon_\nu] \right\}$$

$$+ 2(\partial_\nu \overline{v}_\mu)[\epsilon_\nu, \epsilon_\mu] - (\partial_{\perp \sigma} A_+ + \epsilon_+ \frac{1}{\partial_+} \epsilon_- - (\partial_{\perp \sigma} A_-) \epsilon_+ \frac{1}{\partial_+} \epsilon_- \}$$

$$+ g^2 \{ [A_+, \epsilon_\nu][A_-, \epsilon_\nu] - \frac{1}{2} [A_+, \epsilon_+] [A_-, \epsilon_-] - \frac{1}{4} [A_+, \epsilon_+]^2 - \frac{1}{4} [A_-, \epsilon_-]^2$$

$$\left[ - \frac{1}{2} [A_+, A_-][\epsilon_+, \epsilon_-] + (\partial_{\perp \sigma} A_+ + \epsilon_+ \frac{1}{\partial_+} \epsilon_- - \epsilon_- - A_+ \frac{1}{\partial_+} \epsilon_+ + A_- \frac{1}{\partial_-} \epsilon_- \right]$$

$$\left[ (\partial_{\perp \sigma} A_-)(\epsilon_- \frac{1}{\partial_-} \epsilon_+ + \frac{1}{\partial_+} A_- + A_+ \frac{1}{\partial_+} \epsilon_- + A_- \frac{1}{\partial_-} \epsilon_+ \right) \right\},$$  \hspace{1cm} (264)

where one should substitute $\overline{v}$ by the classical solution. The terms bilinear simultaneously in $A_\pm$ and in $\epsilon$ can be used to find the next-to-leading corrections to the BFKL pomeron. In comparison with the dispersion approach [49] the method based on the effective action could provide a better infrared convergency of intermediate expressions.

We consider below only the contributions which are linear in $A_{\pm}$ and bilinear in $\epsilon_\sigma$ with the singularities $\partial_{\perp \pm}^{-1}$. Expanding the integrand $\exp(-iS_{\text{eff}})$ up to the order containing linearly both $A_+$ and $A_-$ and substituting the products of $\epsilon_+$ and $\epsilon_-$ by the free propagators in the Feynman gauge

$$< \epsilon_-(x) \epsilon_+(0) > = -2i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i0} \exp(ikx),$$  \hspace{1cm} (265)
we obtain effectively the box diagram because apart from the gluon propagators \((k^2 + i0)^{-1}\) and \(((q - k)^2 + i0)^{-1}\) there are also the propagators \(k_\pm^{-1}\) and \(k_-^{-1}\) appearing from the corresponding singular factors \(\partial_\pm^{-1}\) and \(\partial_-^{-1}\) in the vertices. The contribution of the Faddeev-Popov ghosts is small. As usual, one can obtain the logarithmic term \(\sim \ln s\) as a result of integration over \(k_+\) and \(k_-\) restricted by the constraint \(k_+ k_- \sim k_\perp^2\). We interpret it as the integral over the rapidity \(y\) of the \(t\)-channel gluons with momenta \(k\) and \(q - k\) and write down the corresponding one-loop contribution to the effective action in the form

\[
S_{eff}^1 = -\int d^4x_\perp \int d^2x'_\perp \int d\gamma tr (\partial_\perp A_+^\gamma(x_-,x_\perp)) (\partial_\perp A_-^\gamma \eta(x_+,x'_\perp)) \beta(x_\perp - x'_\perp) \tag{266}
\]

where \(\beta(x_\perp)\) is obtained by the Fourier transformation

\[
\beta(x_\perp) = \int \frac{d^2q}{(2\pi)^2} \omega(-q^2) \exp(i x_\perp \cdot q) \tag{267}
\]

from the gluon Regge trajectory:

\[
\omega(-q^2) = \frac{g^2}{16 \pi^2} N_c \int \frac{(-q^2)}{k^2} \frac{d^2k}{(q - k)^2}. \tag{268}
\]

Note, that in LLA \(\omega(t)\) does not depend on the intermediate parameter \(\eta\) because it does not contain the ultraviolet divergencies in the rapidity.

With taking into account apart from \(s_2\) also the one loop correction \(S_{eff}^1\) the renormalized correlation function contains the Regge factor \(\exp(\omega(q^2)(y - y'))\) in the momentum representation:

\[
\int d^2 x_\perp \exp(i x_\perp q) < A_+^\gamma(x) A_-^\gamma(0) >_{ren} \sim \theta(y - y' - \eta) \exp(\omega(q^2)(y - y')). \tag{269}
\]

Analogously starting from the term \(-((\partial_\mu f_\nu)^2)\) in the contribution \(s_4\) (259) one can reduce the integrals over the momenta \(k_\perp\) of the real intermediate gluon to the integral over its rapidity \(y\) and derive the corresponding term of the BFKL kernel.

### 4.4 Effective vertices

Because the classical extremum of the effective action is situated at non-zero fields \(v_\pm = A_\pm + ...\), it is natural to parametrize \(v_\pm\) as follows [51]:

\[
v_\pm = V_\pm + U(V_\pm) A_\pm U^{-1}(V_\pm) \, , \, U(V_\pm) = \frac{1}{1 + g\partial_\pm^{-1} V_\pm}, \tag{270}
\]

where \(V_\pm\) is transformed under the gauge group similar to \(v_\pm\) and \(\nabla_\pm = 0\) for small \(g\). For this parametrization the expansion of \(S_{eff}\) in the series over \(A_\pm\) is gauge invariant but the coefficients are rather complicated and some contributions to scattering amplitudes contain the simultaneous singularities in overlapping channels (cancelling after the use of equations of motion). It is the reason, why we shall use the simpler parametrization of \(v_\pm\):

\[
v_\pm = V_\pm + A_\pm. \tag{271}
\]
In this case one has also $\nabla_{\pm} = 0$ at $g = 0$ in the Landau gauge $\partial_\sigma v_\sigma = 0$ and the homogeneous polynomials $L^i$ of fields $A_\pm$ appearing in the expansion of $S_{eff}$:

\[
S_{eff}(V, A_\pm) = -\int d^4x \ tr \sum_{i=0}^{\infty} L^i,
\]

are compatible with the Steinman relations. The terms $L^i$ do not have simple gauge properties but the corresponding scattering amplitudes are invariant under the gauge transformation after using equations of motion.

In the above expansion $L^0$ describes the pure gluonic interaction:

\[
L^0 = \frac{1}{2} G_{\mu\nu}^2(V) = \frac{1}{2} \left( \partial_\mu V_\nu - \partial_\nu V_\mu \right)^2 + 2g [V_\mu, V_\nu] \partial_\mu V_\nu + \frac{g^2}{2} [V_\mu, V_\nu]^2.
\]

The next term $L^1$ contains the reggeon fields $A_\pm$ linearly and in particular it governs the quasi-elastic processes:

\[
L^1 = j_+ A_- + j_- A_+,
\]

where the currents $j_\pm$ are given below:

\[
j_\pm = -[D_\sigma, G_{\sigma\pm}] - \frac{1}{g} \partial_{1\sigma} A_\pm \frac{1}{D_\pm} \partial_\pm
\]

\[
= g \left( [V_\sigma, \partial_\pm V_\nu] - [\partial_\sigma V_\nu, V_\pm] - 2 [V_\sigma, \partial_\sigma V_\pm] - \partial_{2\sigma} V_\pm \partial^{-1}_\pm V_\pm \right) + g^2 \left( [V_\sigma, [V_\nu, V_\pm]] + \partial_{1\sigma} V_\pm \partial_{-1\pm} V_\pm \partial_{+1\pm} V_\pm \right) - g^3 \partial^2_{1\sigma} V_\pm \partial_{-1\pm} V_\pm \partial_{+1\pm} V_\pm + ...
\]

The quadratic term of the expansion

\[
L^2 = \left[ D^\perp_\sigma, A_+ \right] \left[ D^\perp_\sigma, A_- \right] - \frac{1}{4} \left( [D_+, A_-] - [D_-, A_+] \right)^2 + \frac{g}{2} G_{\pm \pm} [A_-, A_+]
\]

\[
+ \left( \partial_+ \frac{1}{D_+} A_+ \frac{1}{D_+} \partial_+ - A_+ \right) \partial_{1\sigma} A_- + \left( \partial_- \frac{1}{D_-} A_- \frac{1}{D_-} \partial_- - A_- \right) \partial_{1\sigma} A_+
\]

\[
= \partial^2_{\perp\sigma} A_+ \partial_{1\sigma} A_- + g \left\{ \left[ V^\perp_\sigma, A_+ \right] \partial_{\perp\sigma} A_- + \frac{1}{2} [A_+, A_-] \partial_+ V_- - \left[ V_+, \partial_{-1\sigma} A_+ \right] \partial_{1\sigma} A_- + \left[ V^\perp_\sigma, A_- \right] \partial_{\perp\sigma} A_+ + \frac{1}{2} [A_+, A_-] \partial_- V_+ - \left[ V_-, \partial_{-1\sigma} A_- \right] \partial_{1\sigma} A_+ \right\}
\]

\[
+ g^2 \left\{ \left[ V^\perp_\sigma, A_+ \right] \left[ V^\perp_\sigma, A_- \right] - \frac{1}{4} \left( [V_+, A_+] - [V_-, A_-] \right)^2 + \frac{1}{2} [V_+, V_-] [A_-, A_+] + \left( (V_+ \partial_{-1} A_+ + A_+ (\partial_{-1} V_+)^2 + V_+ \partial_{-1} A_+ \partial_{+1} V_+ \right) \partial^2_{1\sigma} A_-
\]

\[
+ \left( (V_- \partial_{-1})^2 A_- + A_- (\partial_{-1} V_-)^2 + V_- \partial_{-1} A_- \partial_{-1} V_+ \right) \partial^2_{1\sigma} A_+ \right\} + ...
\]

describes the gluon production due to the fusion of two reggeized gluons at the central rapidity region.
The following terms non-linear in fields $A_{\pm}$ are responsible for more complicated processes

$$L^3 = \frac{g}{2} ([D_+, A_-] - [D_-, A_+]) [A_-, A_+] - \frac{g}{2} \left( \partial_+ \frac{1}{D_+} A_+ \frac{1}{D_+} A_+ \partial_\perp A_- + \partial_- \frac{1}{D_-} A_- \frac{1}{D_-} A_- \partial_\perp A_+ \right)$$

$$= -\frac{g}{2} (A_+ \partial_+ A_+ \partial_\perp A_- + A_- \partial_- A_- \partial_\perp A_+) + O(g^2). \quad (277)$$

$$L^4 = -\frac{g^2}{4} [A_+, A_-]^2 + \frac{g^2}{6} \left( \partial_+ \frac{1}{D_+} A_+ \frac{1}{D_+} A_+ \frac{1}{D_+} A_+ \partial_+ \partial_\perp A_- + \partial_- \frac{1}{D_-} A_- \frac{1}{D_-} A_- \frac{1}{D_-} \partial_\perp A_+ \right). \quad (278)$$

For $i > 4$ we have

$$L^i = (-1)^i \frac{g^{i-2}}{(i-1)!} \left( \partial_+ \frac{1}{D_+} \left( A_+ \frac{1}{D_+} \right)^{i-1} \partial_\perp A_- + \partial_- \frac{1}{D_-} \left( A_- \frac{1}{D_-} \right)^{i-1} \partial_\perp A_+ \right). \quad (279)$$

The effective vertices obtained according to the Feynman rules from the above action are sums of the usual Yang-Mills couplings and some nonlocal induced terms. Below we construct the gluon production amplitudes in the quasi-multi-Regge kinematics using these effective vertices. One can find in the framework of this approach also the perturbative expansion of the reggeon action $S_{\text{regg}}$ defined as follows

$$\exp(-iS_{\text{regg}}(A_{\pm})) = \int DV \exp(-iS_{\text{eff}})$$

and depending on the reggeon fields $A_{\pm}$.

The subsequent functional integration over $A_{\pm}$ corresponds to the solution of the reggeon field theory defined in the two-dimensional impact parameter subspace with the rapidity playing the role of time. This theory is obtained after the integration of the multi-reggeon couplings over the reggeon longitudinal momenta within the fixed rapidity interval $y_0 - \eta, y_0 + \eta$, which is equivalent to calculating certain limits of these couplings in the longitudinal subspace. It is important, that in the above approach the $t$-channel dynamics of the reggeon interactions turns out to be in the agreement with the $s$-channel unitarity of the $S$-matrix in the initial Yang-Mills model. In the Hamiltonian formulation of this reggeon calculus the wave function will contain the components with an arbitrary number of reggeized gluons. Nevertheless, one can hope that at least some of the remarkable properties of the BFKL equation [46-48] will remain in the general case of the non-conserving number of reggeized gluons.

Note, that to build the effective action for the multi-Regge kinematics one should take into account only two first terms of the perturbative expansion of $L$:

$$L_{mR} = \frac{1}{2} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \partial_\perp A_+ \partial_\perp A_- + g b_3(A_+, A_-, V),$$

47
\[ b_3 = -\frac{1}{2} A_+ \partial_+^{-1} A_+ \partial_\perp^2 A_- - \frac{1}{2} A_- \partial_-^{-1} A_- \partial_\perp^2 A_+ + F_+ \left[ A_-, A_+ \right] \\
- \left( \partial_+^{-1} \partial_-^{-1} F_+ \right) \left[ \partial_+ A_- - \partial_+ A_+ \right] + \left( \partial_-^{-1} F_- \right) \left[ A_-, \partial_\sigma A_+ \right] + \left( \partial_+^{-1} F_\sigma \right) \left[ A_+, \partial_\sigma A_- \right] \\
- A_+ \left[ F_- \sigma, \partial_\perp^{-1} F_- \sigma \right] - A_- \left[ F_+ \sigma, \partial_\perp^{-1} F_+ \sigma \right], \quad (281) \]

where we introduced the abelian strength tensor:

\[ F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \quad (282) \]

and omitted in the last line some terms containing the factors \( \partial_\sigma V_\sigma \) and \( \partial_\perp^2 V_\mu \) vanishing for the real gluons. The Feynman vertices of this theory coincide with the effective reggeon-particle vertices of the leading logarithmic approximation.

The obtained multi-Regge action is invariant under the abelian gauge transformations:

\[ V_\sigma \rightarrow V_\sigma + \partial_\sigma \phi. \quad (283) \]

Its disadvantage is that the contributions of loop diagrams can contain the terms incompatible with the Steinman relations forbidding the simultaneous singularities in the overlapping channels. Such unpleasant properties are absent for the full effective action (210) invariant under nonabelian gauge transformations.

### 4.5 Gluon and quark pair production in the quasi-multi-Regge kinematics

As an example of the general approach we consider the quasi-elastic process in which several gluons are produced with a fixed invariant mass in the fragmentation region of the initial gluon \( A \) provided that the momentum of the other particle \( B \) is almost conserved: \( p_{B'} \simeq p_B \) [49, 51]. It is convenient to denote the colour indices of the produced gluons by \( a_1, a_2, \ldots, a_n \) \( (a_i = 1, 2, \ldots, N_c^2 - 1) \) leaving the index \( a_0 \) for the particle \( A \). Further, the momenta of the produced gluons and of the particle \( A \) are denoted by \( k_1, k_2, \ldots, k_n \) and \( -k_0 \) correspondingly. The momentum transfer \( q = -\sum_{i=0}^n k_i \) is fixed and \( t = q^2 \). Omitting the polarization vectors \( e_{\nu i}(k_i) \) for the gluons we can write the production amplitude related with the one-gluon exchange in the \( t \)-channel in the factorized form:

\[ A_{a_0 a_1 \ldots a_n B'B} = -\phi^{\mu_1 \nu_1 \ldots \mu_n}_{a_0 a_1 \ldots a_n c} + \frac{1}{t} g \bar{p}_{B'} T_{B'B}^c \delta_{\lambda B'} \lambda_B. \quad (284) \]

Here the form-factor \( \phi \) depends on the invariants constructed from the momenta \( k_0, \ldots, k_n \).

In the simplest case of the elastic scattering the tensor \( \phi \) equals

\[ \phi^{\mu_1 \nu_1}_{a_0 a_1 c} = g \Gamma^{\mu_1 \nu_1}_{a_0 a_1 c} \quad (285) \]

where after extracting the colour group generators

\[ \Gamma^{\mu_1 \nu_1}_{a_0 a_1 c} = T_{a_1 a_0}^{c \nu_1} \Gamma^{\mu_1 \nu_0}(k_1, -k_0) \quad (286) \]
the quantity $\Gamma^{\nu_1 \nu_0^+}$ can be presented as the sum of two terms:

$$\Gamma^{\nu_1 \nu_0^+}(k_1, -k_0) = \gamma^{\nu_1 \nu_0^+}(k_1, -k_0) + \Delta^{\nu_1 \nu_0^+}(k_1, -k_0).$$

(287)

Here $\gamma^{\nu_1 \nu_0^+}$ is the light-cone component of the Yang-Mills vertex:

$$\gamma^{\nu_1 \nu_0^+}(k_1, -k_0) = \delta^{\nu_1 \nu_0} (k_1^+ + k_0^+) + (n^+)\nu_0 (2k_0 + k_1)^\nu_1 + (n^+)\nu_1 (-2k_1 - k_0)^\nu_0$$

(288)

and $\Delta^{\nu_1 \nu_0^+}$ is the induced vertex:

$$\Delta^{\nu_1 \nu_0^+}(k_1, -k_0) = -t (n^+)\nu_1 \frac{1}{k_1^+} (n^+)\nu_0, \ k_0^+ + k_1^+ = 0.$$  

(289)

For the case of the production of one extra gluon the amplitude $\phi$ was calculated several years ago [49] and in accordance with the above effective action (272) it can be written in the gauge invariant form

$$\phi^{\nu_0 \nu_1 \nu_2^+} = g^2 \{(\Gamma^{\nu_0 \nu_1 \nu_2^+}_a)_{a_0 a_1 a_2} - T^a_{a_0 a} T^c_{a_2 a} \frac{\gamma^{\nu_1 \nu_0^+}(k_1, -k_0) \Gamma^{\nu_2 \nu_1^+}(k_2, k_2 + q)}{(k_0 + k_1)^2}$$

$$- T^a_{a_2 a_0} T^c_{a_1 a} \frac{\gamma^{\nu_1 \nu_0^+}(k_2, -k_0) \Gamma^{\nu_1 \nu_0^+}(k_1, k_1 + q)}{(k_0 + k_2)^2}$$

$$- T^a_{a_2 a_1} T^c_{a_0 a} \frac{\gamma^{\nu_1 \nu_0^+}(k_2, -k_1) \Gamma^{\nu_0 \nu_1^+}(k_0, k_0 + q)}{(k_0 + k_2)^2}\}}.$$  

(290)

The last three terms in the brackets correspond to the Feynman diagram contributions constructed from the gluon propagator combining the usual Yang-Mills vertex $\gamma$ and the effective reggeon-gluon-gluon vertex $\Gamma$. The first term is the sum of two terms

$$\Gamma^{\nu_0 \nu_1 \nu_2^+}_{a_0 a_1 a_2} = \gamma^{\nu_0 \nu_1 \nu_2^+}_{a_0 a_1 a_2} + \Delta^{\nu_0 \nu_1 \nu_2^+}_{a_0 a_1 a_2},$$  

(291)

where $\gamma$ is the light-cone projection of the usual quadri-linear Yang-Mills vertex

$$\gamma^{\nu_0 \nu_1 \nu_2^+}_{a_0 a_1 a_2} = T^a_{a_0 a} T^c_{a_1 a} (\delta^{\nu_1 \nu_2} \delta^{\nu_0^+} - \delta^{\nu_1^+} \delta^{\nu_0})$$

$$+ T^a_{a_0 a} T^c_{a_1 a} \delta^{\nu_1 \nu_0^+} - \delta^{\nu_2^+} \delta^{\nu_0^+}) + T^a_{a_0 a} T^c_{a_1 a} (\delta^{\nu_2 \nu_0} \delta^{\nu_1^+} - \delta^{\nu_2^+} \delta^{\nu_1^+})$$

(292)

and $\Delta$ is the induced vertex appearing in the effective action (see (275)):

$$\Delta^{\nu_0 \nu_1 \nu_2^+}_{a_0 a_1 a_2}(k_0^+, k_1^+, k_2^+) = -t (n^+)\nu_0 (n^+)\nu_1 (n^+)\nu_2 \left\{ \frac{T^a_{a_0 a} T^c_{a_1 a}}{k_1^+} + \frac{T^a_{a_2 a} T^c_{a_0 a}}{k_0^+} \right\}. $$  

(293)

Note that due to the Jacobi identity

$$T^a_{a_2 a_0} T^c_{a_1 a} - T^a_{a_2 a_1} T^c_{a_0 a} = T^a_{a_1 a_0} T^c_{a_2 a},$$  

(294)

and the momentum conservation law

$$k_0^+ + k_1^+ + k_2^+ = 0$$  

(295)
valid in the quasi-elastic kinematics the tensor $\Delta$ has the Bose symmetry with respect to the simultaneous transmutation of momenta, colour and Lorentz indices of the gluons 0, 1, 2. The above expression for $\phi^{\nu_0\nu_1\nu_2+}$ is significantly simplified if we use the light-cone gauge $e^+(k_i) = 0$ for the gluon polarization vectors because in this gauge all induced terms disappear.

In a general case $n > 2$ for the gauge invariance of $\phi$ one should take into account apart from the usual Yang-Mills vertices $\gamma$ also an arbitrary number of the induced vertices $\Delta$ appearing in $j^+$ (275) and satisfying the recurrence relation:

$$
\Delta^\nu_0\nu_1...\nu_r+(k_0^+, k_1^+, ... k_r^+) = \frac{(n^+)^{\nu_r}}{k_r^+} \sum_{i=0}^{r-1} T_{\alpha_i, \alpha_{i+1}} \Delta^\nu_0...\nu_{r-1}+(k_i^+, ... k_{i-1}^+, k_i^+ + k_r^+, k_{r-1}^+,... k_1^+, k_0^+). 
$$

(296)

These induced vertices are invariant under arbitrary transmutations of indices $i$ :

$$
\Delta^\nu_0\nu_1...\nu_r+(k_{i_0}^+, k_{i_1}^+, ... k_{i_r}^+) = \Delta^\nu_0\nu_1...\nu_r+(k_0^+, k_1^+, ... k_r^+) 
$$

(297)

due to the Jacobi identity for the colour group generators $T$ and the energy-momentum conservation

$$
\sum_{i=0}^{r} k_i^+ = 0.
$$

(298)

One can calculate easily also amplitudes of the quark production in the quasi-elastic kinematics because the reggeized and usual gluons interact with quarks in a similar way.

Let us consider now the multi-gluon production in the central rapidity region [49, 51]. The following kinematics of the final state particles is essential: the gluons almost along the momenta of the initial gluons and the energy-momentum decomposition

$$
q_1 = p_A - p_{A'} \quad \text{and} \quad q_2 = p_{B'} - p_B \quad \text{in this regime have the decomposition}
$$

$$
q_1 = q_1^{\perp} + \beta p_A, \quad q_2 = q_2^{\perp} - \alpha p_B 
$$

(299)

where $\beta$ and $\alpha$ are the Sudakov parameters of the total momentum $k = \sum_{i=1}^{n} k_i$ of the produced gluons:

$$
k = k^{\perp} + \beta p_A + \alpha p_B, \quad \kappa = k^2 = \alpha \beta + (q_1 - q_2)^2_{\perp}
$$

(300)

and $\sqrt{\kappa}$ is their invariant mass which is assumed to be fixed at high energies: $\kappa \ll s$. In this kinematical region the production amplitude has the factorized form:

$$
A^{\nu_1\nu_2...\nu_n+}_{d_1d_2...d_nA'AB'B} = -g p_A^+ T_{\lambda A'\lambda A}^{c_1} \frac{1}{t_1} \psi^{\nu_1\nu_2...\nu_n+}_{d_1d_2...d_n c_2 c_1} \frac{1}{t_2} g p_B^+ T_{\lambda B'\lambda B'}^{c_2} \delta_{\lambda B'\lambda B'}. 
$$

(301)

For the simplest case of one gluon emission we have

$$
\psi^{\nu+}_{d_1c_2 c_1} = g \Gamma^{\nu+}_{d_1c_2 c_1},
$$

(302)
where $\Gamma$ is the sum of two terms

$$
\Gamma_{d_1c_2c_1}^{\nu_1^+-} = \gamma_{d_1c_2c_1}^{\nu_1^+-} + \Delta_{d_1c_2c_1}^{\nu_1^+-}.
$$

The first term is the contribution from the tri-linear Yang-Mills vertex

$$
\gamma_{d_1c_2c_1}^{\nu_1^+-} = T_{d_1c_2c_1}^{\nu_1^+-}, \quad \gamma = 2(q_2 + q_1) - 2k_1^+n^- + 2k_1^-n^+.
$$

The second term is the induced one

$$
\Delta_{d_1c_2c_1}^{\nu_1^+-} = T_{d_1c_2c_1}^{\nu_1^+-} \Delta_{d_1c_2c_1}^{\nu_1^+-}, \quad \Delta = -2t_1 \frac{n^-}{k_1^1} + 2t_2 \frac{n^+}{k_2^1}.
$$

Due to the relation

$$
\gamma + \Delta = -2C,
$$

we obtain the known result for the multi-Regge kinematics.

For the more complicated case of the two gluon production in the central rapidity region the amplitude was calculated also and the quantity $\psi$ is given below [49, 51]:

$$
\psi_{d_1d_2c_2c_1}^{\nu_1\nu_2^+-} = g^2 \left\{ \Gamma_{d_1d_2c_2c_1}^{\nu_1\nu_2^+-} - \frac{T_{d_1d_2c_2c_1}^{\nu_1\nu_2^+}\sigma(k_2, -k_1)}{(k_1 + k_2)^2} \frac{\Gamma_{d_1d_2c_2c_1}^{\nu_1^+-\sigma}(q_2, q_1)}{\sigma(q_1 - k_1)^2} \right. \left. \frac{\Gamma_{d_1d_2c_2c_1}^{\nu_1^+-\sigma}(k_1, k_1 - q_1)\Gamma_{d_1d_2c_2c_1}^{\nu_1^+-\sigma}(k_2, k_2 + q_2)}{(q_1 - k_1)^2} \right. \left. \frac{\Gamma_{d_1d_2c_2c_1}^{\nu_1^+-\sigma}(k_2, k_2 - q_1)\Gamma_{d_1d_2c_2c_1}^{\nu_1^+-\sigma}(k_1, k_1 + q_2)}{(q_1 - k_2)^2} \right. \right\}.
$$

The second term in the brackets describes the production of a pair of gluons through the decay of the virtual gluon in the direct channel. This contribution is a product of the effective vertex $\Gamma$, the usual YM vertex $\gamma$ and the gluon propagator. Analogously, the third and fourth contributions are products of two effective vertices $\Gamma$ having the light cone components $\pm$ and of the gluon propagator in the crossing channels.

The first term in the brackets can be presented as the sum of two terms

$$
\Gamma_{d_1d_2c_2c_1}^{\nu_1\nu_2^+-} = \gamma_{d_1d_2c_2c_1}^{\nu_1\nu_2^+-} + \Delta_{d_1d_2c_2c_1}^{\nu_1^+-\nu_2^+-},
$$

where the contribution $\gamma$ is the light cone component of the quadri-linear Yang-Mills vertex

$$
\gamma_{d_1d_2c_2c_1}^{\nu_1\nu_2^+-} = T_{d_1c_1}^{\nu_1\nu_2^+} T_{d_2c_2}^{\nu_2^+}(\delta^{\nu_1\nu_2^+} - \delta^{\nu_1^+\nu_2^-}) +
$$

$$
+ T_{d_2c_2}^{\nu_2^+} T_{d_1c_1}^{\nu_1\nu_2^+}(\delta^{\nu_1\nu_2^+} - \delta^{\nu_2^+\nu_1^-}) + T_{d_2c_1}^{\nu_2^+} T_{d_1c_2}^{\nu_1\nu_2^+}(\delta^{\nu_1\nu_2^+} - \delta^{\nu_2^+\nu_1^-})
$$

and $\Delta$ is the new induced vertex appearing in the effective action (see (276)):

$$
\Delta_{d_1d_2c_2c_1}^{\nu_1^+-\nu_2^+-} = -2t_2 (n^+)^{\nu_1\nu_2} \frac{T_{d_1d_2c_2c_1}^{\nu_1\nu_2^+\nu_2^+}(k_1^+, k_2^+)}{k_1^1 + k_2^1} + \frac{T_{d_1d_2c_2c_1}^{\nu_1^+-\nu_2^+\nu_2^-}(k_1^-, k_2^-)}{(-k_1^1 - k_2^1) k_2^1}
$$

$$
- 2t_1 (n^-)^{\nu_1\nu_2} \frac{T_{d_1d_2c_2c_1}^{\nu_1^+-\nu_2^+\nu_2^-}(k_1^-, k_2^+)}{(-k_1^1 - k_2^1) k_2^1} + \frac{T_{d_1d_2c_2c_1}^{\nu_1^+-\nu_2^+\nu_2^-}(k_1^-, k_2^-)}{(-k_1^1 - k_2^1) k_2^1}.
$$

One can verify that $\Delta$ is symmetric under simultaneous transmutations of $n^-, t_1, d_1$ and $n^+, t_2, d_2$.
Note, that in the general case of multi-gluon production one should take into account
the induced vertices with an arbitrary number of external legs which are expressed in
terms of the light cone projections of the vertices introduced above for the quasi-elastic
case
\[ \Gamma_{d_1 \ldots d_n c_1}^{α_1 \ldots α_n + +} = \Delta_{α_1 d_1 \ldots d_n c_2}^{+ +} (k_0^+ k_1^+ \ldots k_n^+) + \Delta_{c_2 d_1 \ldots d_n c_1}^{− +} (k_0^- k_1^- \ldots k_n^-), \]  
(311)
where \( k_0^+, k_0^- \) are determined by the momentum conservation
\[ \sum_{i=0}^{n} k_i^+ = \sum_{i=0}^{n} k_i^- = 0. \]  
(312)

The tensor \( ψ_{d_1 d_2 c_1 c_2}^{α_1 α_2 + −} \) appearing in the two gluon production amplitude \( A_{2→4} \) can be written in the other form:
\[ ψ_{d_1 d_2 c_1 c_2}^{α_1 α_2 + −} = 2 g^2 \left[ T_{d_1 d}^{v_1} T_{d_2 d}^{v_2} A^{α_1 α_2}(k_1, k_2) + T_{d_1 d}^{v_1} T_{d_2 d}^{v_2} A^{α_2 α_1}(k_2, k_1) \right], \]  
(313)
where
\[ A^{α_1 α_2} = -\frac{\Gamma^{α_1 β}(k_1, k_1 - q_1) \Gamma^{α_2 β}(k_2, k_2 + q_2)}{2(k_1 - k_2)^2} - \frac{\gamma^{α_2 α_1}(k_2, -k_1) \Gamma^{α_1 β}(q_2, q_1)}{2(k_1 + k_2)^2} \]
\[ + n^{α_1} n^{− α_2} - g^{α_1 α_2} - \frac{1}{2} n^{α_2} n^{− α_1} + t_1 \frac{n^{− α_1} n^{− α_2}}{k_1^− (k_1^- + k_2^-)} + t_2 \frac{n^{α_1} n^{α_2}}{k_2^+ (k_1^+ + k_2^+)}. \]  
(314)

One can verify the following gauge property of \( A^{α_1 α_2} \):
\[ k_1^{α_1} A^{α_1 α_2} = \frac{1}{2} k_2^{α_2} \left( \frac{k_1^- k_1^+}{k_2^+ (k_2^+ + q_2)^2} - \frac{k_1^β \Gamma^{α_1 β}(q_2, q_1)}{(k_1^- + k_2^-)^2} \right), \]  
(315)
which implies, that the production amplitude \( A_{2→4} \) multiplied by the gluon polarization
vectors \( e(k_i) \) is gauge invariant. To calculate the matrix element of \( A^{α_1 α_2} \) it is convenient
to use the different light-cone gauges for two produced gluons:
\[ e^−(k_1) = 0, \ e^+(k_2) = 0. \]  
(316)
The corresponding polarization vectors can be parametrized by the two-dimensional vectors \( e_{1⊥}, e_{2⊥} \) as follows
\[ e(k_1) = e_{1⊥} - \frac{(k_1 e_{1⊥})}{k_1^−} n^−, \ e(k_2) = e_{2⊥} - \frac{(k_2 e_{2⊥})}{k_2^+} n^+. \]  
(317)
and the matrix element of the tensor \( A^{α_1 α_2} \) can be expressed in terms of a new tensor \( c^{α_1 α_2} \) with pure transverse components according to the definition:
\[ e_{α_1}^∗(k_1) e_{α_2}^∗(k_2) A^{α_1 α_2} \equiv 4 e_{1⊥}^{α_1}(k_1) e_{1⊥}^{α_2}(k_2) c^{α_1 α_2}. \]  
(318)
Note, that the corresponding tensor \( c_{α_2 α_1} \) for \( A^{α_2 α_1}(k_2, k_1) \) is obtained from \( c \) by the transmutation of gluons \( 1 ⇔ 2 \) with the simultaneous multiplication by the matrices
\[ \Omega^{α_1 α_2} = g^{α_1 α_2} - 2 \frac{k_{1⊥} k_{2⊥}^{α_1}}{k_{2⊥}^2}, \]
interchanging two light-cone gauges (316).

Using also the reality condition \( k^+ k^- = |k_i|^2 \) for the produced gluons \( i = 1, 2 \) to exclude the light-cone momenta \( k_i^- \) one can present the tensor \( c^{α1α2} \) only in terms of transverse momenta \( \vec{k}_1^\perp, \vec{q}_i^\perp \) and the relative Sudakov parameter \( x = \frac{k_1^+}{k^+ + k_2^+} \) [55]:

\[
c^{α1α2} = \frac{\delta^{α1α2}}{2Z} \vec{q}_1^2 x(1-x) - xk_1^α \vec{q}_2^2 \frac{k_1^β}{k_1^2} + \frac{\Delta^{α2} Q_1^{α2} - (1-x)^{-1}(\vec{Q}_1^2 - \vec{k}_1^2)k_2^{α2}}{\kappa k_1^2}
\]

\[
- \frac{1}{k_1^2} k_1^α Q_1^{α2} + \frac{\Delta^{α1} q_1^{α2} + \delta^{α1α2} q_1^α (\vec{k}_1^\perp + xq_2^\perp) - (1-x)^{-1} q_1^{α1} (k_1^α - x\Delta^{α2})}{\kappa}
\]

\[
+ \frac{Q_1^α Q_1^{α2} - \frac{1}{2}(1-x)(\vec{Q}_1^2 - \vec{k}_1^2)\delta^{α1α2}}{t} + xq_2^2 \frac{k_2^{α1} k_2^{α2} + \delta^{α1α2} (1-x)\vec{k}_2^\perp \vec{q}_2}{\kappa Z},
\]

where \( \delta^{α1α2} = -\delta_α^1 \delta_β^2 \) is the Kroniker tensor \((\delta^{11} = \delta^{22} = 1)\) and

\[
\bar{\Delta} \equiv \vec{q}_1^\perp - \vec{q}_2^\perp = \vec{k}_1^\perp + \vec{k}_2^\perp, \quad \vec{Q}_1 = \vec{q}_1^\perp - \vec{k}_1^\perp, \quad \kappa = \frac{(\vec{k}_1^\perp - x\bar{\Delta})^2}{x(1-x)}
\]

\[
Z = -x(1-x)(\bar{\Delta}^2 + \kappa), \quad t = -\frac{(\vec{k}_1^\perp - x\vec{q}_1^\perp)^2 + x(1-x)\vec{q}_1^2}{x}.
\]

Analogously to the gluon case, the amplitude of the production of the quark and anti-quark with their momenta \( k_1 \) and \( k_2 \) in the central rapidity region also can be written as a sum of two terms being the matrices in the spin and colour spaces [55]:

\[
\psi_{c_2c_1} = -g^2 \left( t^{c_1} t^{c_2} b(k_1, k_2) - t^{c_2} t^{c_1} b^T(k_2, k_1) \right).
\]

Here \( t^c \) are the colour group generators in the fundamental representation and the expressions for \( b(k_1, k_2) \) and \( b^T(k_1, k_2) \) are constructed according to the Feynman rules including the effective vertices:

\[
b(k_1, k_2) = \gamma^- \frac{\hat{q}_1 - \hat{k}_1}{(q_1 - k_1)^2} \gamma^+ - \frac{\gamma^β \Gamma^{+β}(q_2, q_1)}{(k_1 + k_2)^2},
\]

\[
b^T(k_1, k_2) = \gamma^+ \frac{\hat{q}_1 - \hat{k}_2}{(q_1 - k_2)^2} \gamma^- - \frac{\gamma^β \Gamma^{+β}(q_2, q_1)}{(k_1 + k_2)^2}.
\]

One can calculate the matrix elements of these expressions between the spinors \( u^{(±)}(k_1) \) and \( v^{(±)}(k_2) \) describing the quark and anti-quark with helicities \( \lambda = ±\frac{1}{2} \):

\[
\bar{u}^{(+)}(k_1) b(k_1, k_2) v^{(-)}(k_2) = \frac{1}{2} c_{±−}(k_1) d_{−+}(k_2) b^{(−)}(k_1, k_2),
\]

\[
\bar{u}^{(−)}(k_1) b^T(k_2, k_1) v^{(−)}(k_2) = \frac{1}{2} c_{±+}(k_1) d_{++}(k_2) b^{(−)}(k_2, k_1),
\]

\[
\bar{u}^{(−)}(k_1) b(k_1, k_2) v^{(+)}(k_2) = \frac{1}{2} c_{−−}(k_1) d_{+−}(k_2) b^{(−)}(k_1, k_2),
\]

\[
\bar{u}^{(−)}(k_1) b^T(k_2, k_1) v^{(+)}(k_2) = \frac{1}{2} c_{−+}(k_1) d_{++}(k_2) b^{(−)}(k_2, k_1),
\]

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\[
\mp^{(-)}(k_1)b^T(k_2,k_1)v^{(+)}(k_2) = \frac{1}{2} c^{*}_{-}(k_1)d_{++}(k_2) b^{(+)}(k_2,k_1) .
\] (324)

The introduced complex wave functions \( c \) and \( b \) satisfy the relations

\[
c_{++}(k_1) = -\frac{k_1^-}{k_1^+} c_{--} = -\frac{k_1}{k_1^*} c_{--} , \quad d_{++}(k_2) = -\frac{k_2^-}{k_2^+} d_{--} = -\frac{k_2}{k_2^*} d_{--} ,
\]

\[
c_{--}(k_1) = \frac{k_1}{k_1^*} c_{++} = \frac{k_1^+}{k_1} c_{++} , \quad d_{--}(k_2) = \frac{k_2}{k_2^*} d_{++} = \frac{k_2^+}{k_2} d_{++} ,
\] (325)

and are normalized as follows

\[
| c_{\pm}(k_1) |^2 = 2k_1^\mp , \quad | d_{\pm}(k_2) |^2 = 2k_2^\mp .
\] (326)

in accordance with the usual normalization condition for the spinors:

\[
\bar{u}(k_1)\gamma^\alpha u(k_1) = 2k_1^\alpha , \quad \bar{v}(k_2)\gamma^\alpha v(k_2) = 2k_2^\alpha .
\] (327)

The amplitudes \( b \) can be written in the form

\[
b^{(+\pm)}(k_1,k_2) = (b^{(-\pm)}(k_1,k_2))^* = -4 \frac{q_1 \mp k_1}{(q_1 \mp k_1)^2} - j^\beta \Gamma^{+-\beta}(q_2,q_1) ,
\] (328)

where the quark current \( j \) is given below

\[
j = n + n \frac{k_2^-}{k_2^+} \frac{k_1}{k_1^*} - n \frac{k_1^-}{k_1^*} - n + \frac{k_2^-}{k_2^+} .
\] (329)

Due to the relations

\[
n^\alpha k^\alpha = k , \quad n^{*\alpha} k^{\alpha} = k^* , \quad n^{\pm\alpha} k^\alpha = k^\pm
\]

this current is conserved:

\[
(k_1^\beta + k_2^\beta) j^\beta = 0 .
\] (330)

5 Next-to-leading corrections to the BFKL pomeron

5.1 Corrections from the gluon and quark pair production

The imaginary part of the elastic scattering amplitude calculated with the use of the
\( s \)-channel unitarity condition through the squared production amplitude \( A_{2\rightarrow 4} \) (301, 313)
contains the infrared divergences at small \( k_{\perp}^2 \) and \( \kappa \). To avoid such divergencies the
dimensional regularization

\[
\frac{d^4k}{(2\pi)^4} \rightarrow \mu^{4-D} \frac{d^Dk}{(2\pi)^D}
\] (331)
is used in the gauge theories (where \( \mu \) is the normalization point). It is important, that
in the \( D \)-dimensional space the gluon has \( D - 2 \) degrees of freedom.
The total cross-section of the gluon production is proportional to the integral from the squared amplitude $\psi^{\alpha_1\alpha_2+\lambda+\mu}_{d_1d_2c_1}$ summed over all indices. It is expressed in terms of the bilinear combinations of the tensor $c^{\alpha_1\alpha_2}$ with transverse components [55]:

$$R \equiv (c^{\alpha_1\alpha_2}(k_1, k_2))^2 + (c^{\alpha_1\alpha_2}(k_2, k_1))^2 + c^{\alpha_1\alpha_2}(k_1, k_2) c^{\alpha_2\alpha_1}(k_2, k_1) \Omega^{\alpha_1\alpha'_1} \Omega^{\alpha_2\alpha'_2},$$

(332)

where the matrix $\Omega^{\alpha_1\alpha'_1}$ interchanges the left and right gauges:

$$\Omega^{\alpha_1\alpha'_1} = g^{\alpha_1\alpha'_1}_\perp - 2 \frac{k_{1\perp} k_{2\perp}^{\alpha_1\alpha'_1}}{k_{2\perp}^2}.$$  

In expression (332) we took into account that the colour factor for the interference term is two times smaller than one for the direct contributions. The generalized BFKL equation describing the pomeron as a compound state of two reggeized gluons is valid also in the next-to-leading approximation. The equation for the total cross-section of the reggeon scattering can be written in the integral form as follows

$$\sigma(q_1^+, q_2^+) = \sigma_0(q_1^+, q_2^+) + \int \frac{d q_2^+}{q_2^+} \mu^{4-D} \int d^2q_2 K_\delta(q_1^+, q_2) \sigma(q_2^+, q_2^+).$$

(333)

Here the integration region for the longitudinal momentum $q_2^+$ is restricted from above by the value proportional to $q_1^+$:

$$q_2^+ < \delta q_1^+.$$  

(334)

The intermediate infinitesimal parameter $\delta > 0$ is introduced instead of the above cut-off $\eta$ to arrange the particles in the groups with strongly different rapidities. The integral kernel $K_\delta(q_1^+, q_2)$ takes into account the interactions among the particles with approximately equal rapidities. The kernel $K_\delta$ can be calculated in the perturbation theory:

$$K_\delta(q_1^+, q_2) = \sum_{r=1}^{\infty} \left( \frac{g^2}{2(2\pi)^{D-1}} \right)^r K_\delta^{(r)}(q_1^+, q_2^+).$$

(335)

We remind, that the real contribution to the kernel in the leading logarithmic approximation is proportional to the square of the effective vertex:

$$K_{\text{real}}^{(1)}(q_1^+, q_2^+) = - \frac{N_c}{4 q_1^2 q_2^2} \left( \Gamma^{+\beta}(q_1, q_2) \right)^2 = N_c \frac{4}{(q_1^2 - q_2^2)^2}.$$  

(336)

The corresponding virtual contribution is determined by the gluon Regge trajectory $\omega(-q_1^2)$.

The next-to-leading term in $K_\delta$ related with the two gluon production is given below [55]

$$K_{\text{gluons}}^{(2)} = \int d\kappa \frac{d^Dk_1}{\mu^{D-4}} \delta(k_1^2) \delta(k_2^2) \frac{16 N_c^2 R}{q_1^2 q_2^2} = \frac{16 N_c^2}{2 q_1^2 q_2^2} \int_{\delta}^{1-\delta} \frac{dx}{x(1-x)} \int \frac{d^D-2k_1^\perp}{\mu^{D-4}} R.$$  

(337)

The limits in the integral over $x$ correspond to a restriction from above for the invariant mass $\sqrt{\kappa}$ of the produced gluons. In the solution of the generalized BFKL equation the dependence from $\delta$ should disappear.
For the physical value $D = 4$ of the space-time dimensions one can express $R$ as the following sum:

$$R = R(+-) + R(++)$$

(338)

where $R(+-)$ and $R(++)$ are the contributions from the production of two gluons with the same and opposite helicities correspondingly.

$$R(+-) = \frac{1}{2} \left( |c^{+-}(k_1, k_2)|^2 + |c^{+-}(k_2, k_1)|^2 + Re c^{+-}(k_1, k_2) c^{+-}(k_2, k_1) \frac{k_1^* k_2}{k_1 k_2} \right),$$

$$R(++) = \frac{1}{2} \left( |c^{++}(k_1, k_2)|^2 + |c^{++}(k_2, k_1)|^2 + Re c^{++}(k_1, k_2) c^{++}(k_2, k_1) \frac{k_1^* k_2^*}{k_1 k_2} \right).$$

(339)

Here we have used the following relation between the polarization vectors (317) in the right and left gauges [42]:

$$e^\perp_1(k) = - (e^\perp_1(k))^* \frac{k}{k^*}$$

(340)

to express amplitudes (319) in terms of two complex functions $c^{+-}(k_1, k_2)$ and $c^{++}(k_1, k_2)$ given below:

$$c^{+-}(k_1, k_2) \equiv c^{11} + ic^{21} - ic^{12} + c^{22} = c^{--}(k_1, k_2) = -x \frac{q_2 q_1^*}{(k_1 - x \Delta) k_1^*},$$

$$c^{++}(k_1, k_2) = c^{11} + ic^{21} + ic^{12} - c^{22} = c^{--}(k_1, k_2)$$

$$= -\frac{x (Q_1)^2}{(k_1^2 - x q_1^2) (1 - x)^2} + \frac{x q_1^2 (k_2)^2}{\Delta^2 (k_1^2 - x \Delta)^2} - \frac{x q_1^* k_2 q_2^*}{\Delta^* (k_1 - x \Delta) k_1^*} - \frac{x q_1^* k_2 q_2^*}{\Delta^* (k_1 - x \Delta) k_1^*} + \frac{x q_2^2 Q_1}{\Delta^* k_1^*}.$$ (341)

These expressions were obtained independently also in Ref. [56].

In the Regge regime of small $1 - x$ and fixed $k_i, q_i$ amplitudes $c^{\alpha_1 \alpha_2}$ are simplified as follows

$$c^{+-}(k_1, k_2) \rightarrow \frac{q_1^* q_2}{k_1^* k_2}, \quad c^{++}(k_1, k_2) \rightarrow \frac{q_1^* q_2}{k_1^* k_2} - \frac{k_1 - k_1^*}{k_1^*}$$ (342)

and are proportional to the product of the effective vertices $\Gamma^{+-\beta}$ in the light-cone gauge [42]. For $x \rightarrow 0$ the amplitudes $c^{+-}(k_1, k_2)$ and $c^{++}(k_1, k_2)$ vanish, but for simultaneously small $k_1$ or $k_1 - x \Delta$ one obtains a nonzero integral contribution because in this region there are poles:

$$c^{++}(k_1, k_2) \rightarrow \frac{q_1 q_2^* \Delta}{q_1^* q_2 \Delta^*} c^{+-}(k_1, k_2) \rightarrow -\frac{x q_1 q_2^* \Delta}{\Delta^* (k_1 - x \Delta) k_1^*}.$$ (343)

For large $k_1$ and fixed $q_i, x$ we obtain

$$c^{+-}(k_1, k_2) \rightarrow -x \frac{q_1 q_2}{k_1^2}, \quad c^{++}(k_1, k_2) \rightarrow -x (1 - x) \frac{q_1 q_2}{k_1^2} - x^3 \frac{q_1 q_2 k_1}{(k_1^3)}$$ (344)
and therefore the integrals for the cross section of the gluon production do not contain any ultraviolet divergency.

As it is seen from above formulas, $c^{a_1a_2}$ has infrared poles at small $k_1^\rightarrow$:

$$c^{+-}(k_1, k_2) \to \frac{q_1^* q_2}{\Delta k_1^\rightarrow}, \quad c^{++}(k_1, k_2) \to \frac{q_1 q_2^*}{\Delta^* k_1^\rightarrow}$$

and at small $k_1 - x \Delta$:

$$c^{+-} \to -\frac{q_1^* q_2}{\Delta^*(k_1 - x\Delta)}, \quad c^{++} \to -(1 - x)^2 \frac{q_1 \Delta q_2^*}{(\Delta^*)^2(k_1 - x\Delta)} - x^2 \frac{q_1^* q_2}{\Delta^*(k_1^\rightarrow - x\Delta^*)}. \quad (346)$$

Let us return now to the quark-antiquark production. The total cross-section of this process in accordance with the normalization condition for spinors is proportional to the integral from the expression:

$$K_{\text{quarks}}^{(2)} = \int_0^1 \frac{dx \mu^{4-D}}{x(1-x)} \int d^Dk_1^\rightarrow \frac{8 L}{q_1^2 q_2^2} \left( \frac{\Delta}{\Delta^*} \right)$$

where we put $\delta = 0$ because the integral in $x$ is convergent at $x = 0$ and $x = 1$.

The expression for $L$ with corresponding colour factors for $D = 4$ is given below:

$$L = \frac{N_c^2 - 1}{4N_c} \left( \frac{1 - x}{x} \right) |c(k_1, k_2)|^2 + \frac{x}{1 - x} |c(k_2, k_1)|^2 + \frac{1}{2N_c} \text{Re} \ c(k_1, k_2) c(k_2, k_1) \quad (348)$$

where two equal contributions from two different helicity states of the quark-anti-quark pair were taken into account. The amplitude $c$ is determined by the equation (see (328)):

$$b^{+-}(k_1, k_2) = \frac{4}{k_1^\rightarrow} c(k_1, k_2). \quad (349)$$

and is given below:

$$\frac{1}{x} c(k_1, k_2) = \frac{(1 - x)q_1 \Delta k_1^\rightarrow - x\Delta^* k_1 + xq_1^2}{(k_1^\rightarrow - xq_1^\rightarrow)^2 + x(1 - x)q_1^2}$$

$$- \frac{\Delta^2}{\Delta} \frac{(1 - x)\Delta k_1^\rightarrow - x\Delta^* k_1 + x\Delta^2}{(k_1^\rightarrow - x\Delta)^2 + x(1 - x)\Delta^2}$$

$$+ \frac{(1 - x)q_1 q_2^*}{\Delta^*(k_1 - x\Delta)} - \frac{xq_1^* q_2}{\Delta^*(k_1^\rightarrow - x\Delta^*)}. \quad (350)$$

Note, that in the limit of large $N_c$ the quark contribution is smaller than gluon one and the interference term is suppressed by the factor $1/N_c^2$ in comparison with the direct contribution.

### 5.2 Infrared and collinear divergencies

The gluon and quark production cross-sections contain infrared divergencies which should be cancelled with the virtual corrections to the multi-Regge processes. For example from the products of amplitudes $c^{+-}$ we can calculate $R^{+-}$ for $D = 4$ and obtain the following expression:

$$R^{+-} = \frac{q_1^2 q_2^2}{4} \left( \frac{1}{k_1^2 (k_1 - \Delta)^2} + \frac{x^2}{k_1^2 (k_1 - x\Delta)^2} + \frac{(1 - x)^2}{(k_1^\rightarrow - \Delta)^2 (k_1^\rightarrow - x\Delta)^2} \right) \quad (351)$$

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from which the presence of infrared and collinear singularities is obvious.

As for \( R(++) \), we can write it as the sum of singular and regular terms [55]

\[
R(++) = R_{\text{sing}}(++) + R_{\text{reg}}(++)
\]

(352)

where for \( R_{\text{sing}}(++) \) the following expression is chosen:

\[
R_{\text{sing}}(++) = R(+-) + \frac{1}{4} (r(k_1, x) + r(\Delta - k_1, 1 - x)).
\]

(353)

and

\[
r(k_1, x) = \frac{q_1^2 q_2^2}{\Delta^2} \left( x(1 - x) - 2 \right) \frac{2x^2(1 - x)\Delta}{k_1^2 (k_1 - x \Delta)^2} + 2 \text{Re} \frac{x^3(1 - x)^2 q_1^2 q_2^2 \Delta}{\Delta^2 (k_1 - x \Delta)^2 k_1}.
\]

(354)

Therefore for the total sum \( R_{\text{sing}} = R(+-) + R_{\text{sing}}(++) \) we have

\[
R_{\text{sing}} = 2 R(+-) + \frac{1}{4} (r(k_1, x) + r(\Delta - k_1, 1 - x)).
\]

(355)

The term \( R_{\text{sing}} \) contains all infrared and Regge singularities of \( R \) and is generalized to the case of arbitrary \( D \). It decreases rapidly at large \( k_1 \).

One can obtain the following contribution from \( R_{\text{sing}} \) after its dimensional regularization (and taking into account \( D - 4 \) non-physical polarizations for each gluon):

\[
\frac{\mu^{4-D}}{\pi} \int_1^\Delta \frac{dx}{x(1 - x)} \int d^{D-2}k_1 R_{\text{sing}} = \frac{1}{6} \text{Re} \frac{q_1^2 q_2^2}{\Delta^2} + \frac{\pi^2}{\mu^2} \left( \frac{67}{18} - \frac{\pi^2}{6} \right)
\]

\[
+ \frac{q_1^2 q_2^2}{\Delta^2} \left( \frac{4D-1}{1} \right) \frac{\pi^{D-3}}{\Gamma(\frac{D-3}{2})} \sin \left( \frac{\pi D}{2} \right) \left( \frac{2}{D-4} + 4 \ln \frac{1}{\delta} - \frac{11}{6} \right),
\]

(356)

where it is implied, that the terms of the order of value of \( D - 4 \) should be omitted. The infrared divergencies at \( D \to 4 \) in the above formulas should be cancelled with the contribution from one-loop corrections to the Reggeon-Reggeon-particle vertex which will be considered below.

It is important to investigate the region of small \( \Delta \):

\[
|\Delta| \ll |q_1| \simeq |q_2|.
\]

(357)

Here the essential integration region corresponds to the soft gluon transverse momenta:

\[
k_1 \sim k_2 \sim \Delta \ll q
\]

(358)

where \( q \) means \( q_1 \) or \( q_2 \).

The total contribution of the singular and regular terms in the soft region \( \Delta \to 0 \) can be calculated [55]:

\[
\frac{\mu^{4-D}}{\pi} \int_{1-\delta}^1 \frac{dx}{x(1 - x)} \int d^{D-2}k_1 R = \frac{q_1^4 q_2^2}{\Delta^2} \left( \frac{\pi \Delta^2}{\mu^2} \right)^{\frac{D-4}{2}} \Gamma(3 - \frac{D}{2}) \Gamma^2(\frac{D}{2} - 1) \times
\]

58
\[
\left[ \frac{4}{(D-4)^2} + \frac{D-2}{2(D-4)(D-1)} + \frac{2(D-3)}{D-4} \left( 4 \ln \frac{1}{\delta} + \psi(1) + \psi\left(\frac{D}{2} - 1\right) - 2\psi(D-3) \right) \right].
\]

In this expression three first terms of the expansion in \( \epsilon = D - 4 \) including one of the order of \( \epsilon \) are correct. Such accuracy of calculations is needed because the integration over \( \Delta \) in the generalized BFKL equation leads to the infrared divergency at \( \Delta = 0 \) for \( D \to 4 \). It means, that to obtain the final result after cancellations of the infrared divergencies one should know the real and virtual corrections to the BFKL kernel up to terms \( \sim \epsilon \).

Note, that it is possible to modify the definition of the singular and regular parts of \( c^{++}(k_1, k_2) \):

\[
c^{++}(k_1, k_2) = \tilde{c}^{++}_{\text{sing}}(k_1, k_2) + \tilde{c}^{++}_{\text{reg}}(k_1, k_2),
\]

\[
\frac{1}{x} \tilde{c}^{++}_{\text{sing}}(k_1, k_2) = \frac{q_1^* q_2 k_1}{\Delta^2} \left( \frac{(k_1^* - x\Delta)^2}{(k_1^* - x\Delta)^2 + x(1-x)\Delta^2} \right) - \frac{q_1^* q_2 k_1}{\Delta^2} - (2-x) \frac{q_1 q_2^*}{\Delta^* k_1^*}
\]

\[
\frac{1}{x} \tilde{c}^{++}_{\text{reg}}(k_1, k_2) = - \frac{Q_1^2}{(k_1^* - x\Delta)^2 + x(1-x)\Delta^2} + \frac{q_1^* k_1 (1-x) - (2-x)q_1 \Delta^* k_1 + q_1 \Delta^2}{\Delta^* \left( (k_1^* - x\Delta)^2 + x(1-x)\Delta^2 \right)} - \frac{q_1 q_2^*}{\Delta^* k_1^*}.
\]

in such way to include in the comparatively simple expression for \( \tilde{c}^{++}_{\text{sing}}(k_1, k_2) \) all singular terms without the loss of its good behaviour at large \( k_1 \). In the Regge limit \( x \to 1 \) and fixed \( k_i \) we obtain

\[
\tilde{c}^{++}_{\text{sing}} \to \frac{q_1 q_2^*}{k_1 k_2}
\]

and therefore in this region the contributions of the exact amplitude \( c^{++} \) and approximate one \( \tilde{c}^{++}_{\text{sing}} \) to the differential cross-section also coincide. However, as a result of the singularity at \( k_1 = x\Delta \) after integration over \( k_1 \) these contributions to the total cross-section in the Regge limit \( x \to 1 \) turn out to be different.

We return now to the quark production amplitude \( c(k_1, k_2) \) and present it in the form analogous to the gluon case:

\[
c(k_1, k_2) = c_{\text{sing}}(k_1, k_2) + c_{\text{reg}}(k_1, k_2),
\]

where the singular and regular terms are chosen as follows

\[
\frac{1}{x} c_{\text{sing}}(k_1, k_2) = \frac{(1-x)q_1 q_2^*}{\Delta^* (k_1^* - x\Delta)} - \frac{x q_1^* q_2}{\Delta (k_1^* - x\Delta^*)} - \frac{(1-x)\Delta q_1 q_2^* k_1^* - x\Delta^* q_1^* q_2 k_1 + x\Delta^2 q_1^* q_2}{\Delta^2 \left( (k_1^* - x\Delta)^2 + x(1-x)\Delta^2 \right)}.
\]

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The term \( c_{\text{reg}} \) does not lead to any divergence and gives a regular contribution to the BFKL kernel at the soft region \( \Delta \to 0 \).

We obtain the following contributions for the bilinear combinations of \( c_{\text{sing}} \) taking into account the dimensional regularization of the singularity at \( \vec{k}_1 \to x \vec{\Delta} \) [55]:

\[
\int_0^1 \frac{dx}{x^2} \int \frac{d^{D-2}k_1}{\mu^{D-4}\pi} |c_{\text{sing}}(k_1, k_2)|^2 = \int_0^1 \frac{dx}{x(1-x)} \int \frac{d^{D-2}k_1}{\mu^{D-4}\pi} Re c_{\text{sing}}(k_1, k_2)c_{\text{sing}}(k_2, k_1) \\
= \frac{1}{\Delta^2} \left( \frac{\pi\Delta^2}{\mu^2} \right)^{\frac{D}{2}-2} \Gamma(3 - \frac{D}{2}) \Gamma(\frac{D}{2}) \Gamma(\frac{D}{2}) 4 \left( \frac{6 - D}{D - 4} \right) \frac{q_1^2 q_2^2}{(q_1, q_2)^2} .
\]

The integration over \( \Delta \) in the generalized BFKL equation leads to the infrared divergence at \( \Delta = 0 \) for \( D \to 4 \):

\[
-16N_c n_f \pi^\frac{D-1}{2} \left( \frac{g^2 \mu^{4-D}}{2(2\pi)^{D-1}} \right)^2 \Gamma(2 - \frac{D}{2}) \frac{\Gamma(\frac{D}{2}) \Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2})} \int d^{D-2} \Delta \left( \Delta^2 \right)^{\frac{D}{2}-3} .
\]

The quark correction to the RRP vertex expressed in terms of the bare coupling constant does not give such singular contribution at small \( \Delta \). This divergence is cancelled with the doubled quark contribution to the gluon Regge trajectory (see below).

### 5.3 One-loop corrections to reggeon-particle-particle vertices

To find next-to-leading corrections to the BFKL equation one should calculate also the analogous corrections to the gluon production amplitudes in the multi-Regge kinematics [41, 49]. These amplitudes in LLA are constructed from the RPP and RRP vertices combined by the reggeized gluon propagators expressed in terms of the Regge trajectory. In the next-to-leading approximation this structure of the production amplitude remains and therefore one should calculate one-loop contributions to the reggeon-particle vertices and two-loop corrections to the gluon Regge trajectory.

To begin with, we consider the elastic amplitude for the gluon-gluon scattering. In the Regge model it can be written in the form:

\[
M = T_{A^c A}^c \Gamma_{A^c A^c}(t) \frac{s}{t} \left[ \left( \frac{s}{t} \omega(t) \right) \left( \frac{-s}{t} \omega(t) \right) \right] T_{B^c B}^c \Gamma_{B^c B^c}(t) \quad t = q^2 ;
\]

where \( \Gamma_{A^c A^c}(t) \) due to the parity conservation can be presented as the sum of two terms:

\[
\Gamma_{A^c A^c}(t) = \Gamma^{(1)}(t) \delta_{A^c A^c} + \Gamma^{(2)}(t) \delta_{A^c A^c} \delta_{A^c - A^c}.
\]
\[
\Gamma_{\alpha'\alpha}(t) = \Gamma^{(1)}(t) (-\delta_{\alpha'\alpha}^{+\perp}) + \Gamma^{(2)}(t) (\delta_{\alpha'\alpha}^{\perp\perp} - 2\frac{q_{\alpha}^\perp q_{\alpha}^\perp}{q^2})
\]
(368)
in the tensor representation. Here
\[
-\delta_{\alpha'\alpha}^{\perp\perp} = \sum_{\lambda=\pm} e_{\alpha'}^\lambda (p_{A'}) e_{\alpha}^{\lambda*} (p_A), \ e_{A} p_{B} = 0.
\]
(369)
is the projector to the physical gluon states in the light-cone gauge. The vectors \(q_{\alpha}^\perp\) and \(q_{\alpha'}^\perp\) are orthogonal to the momenta \(k, p_B\) and \(k', p_B\) correspondingly. For finding the next-to-leading correction to \(\Gamma^{(i)}\) it is enough to calculate in the one-loop approximation the scattering amplitude at high energies. The \(t\)-channel unitarity is convenient for this purpose. In the framework of this approach the \(t\)-channel imaginary part of the gluon-gluon scattering amplitude in the pure gluodynamics is obtained as a product of two Born amplitudes with summing over the colour and Lorentz indices and integrating over momenta \(-k, q + k\) of the intermediate gluons in the octet state. Each of these amplitudes is presented as a sum of two terms after projecting to the octet state. For example,
\[
A_{AA'CC'}^{\alpha\alpha'\gamma\gamma'}(k, q + k, p_A) = g^2 (-T_A^{d} T_{B'}^{d}) \left(A^{(a)} + A^{(n)}\right)^{\gamma\gamma'}_{\alpha\alpha'},
\]
(370)
where \(A^{(a)}\) is the factorized contribution containing the correct asymptotic behaviour of the total amplitude:
\[
A_{\alpha\alpha'}^{(a)\gamma\gamma'} = \delta_{\alpha\alpha'}^{\perp\perp} A^{\gamma\gamma'}(k, k + q, p_A)
\]
(371)
and \(A^{(n)}\) is the non-asymptotic term which is small in the Regge kinematics. The factor \(A^{\gamma\gamma'}(k, k', p)\) is proportional to the effective RPP vertex:
\[
A^{\gamma\gamma'}(k, k', p) = -2 \delta_{\gamma\gamma'}^{\perp\perp} \frac{k p + k' p}{t} + 2 \left(1 + \frac{2}{t}\right) p^\gamma k'^{\gamma'}
+ 2 \left(1 + \frac{2}{t}\right) k'^\gamma p^{\gamma'} + 2 \left(1 + \frac{2}{t}\right) p^\gamma p^{\gamma'}, \ k' = k + q.
\]
(372)
This tensor has the simple gauge properties on the mass-shell \(k^2 = k'^2 = 0\):
\[
A^{\gamma\gamma'}(k, k', p) k^{\gamma'} = 0, \ A^{\gamma\gamma'}(k, k', p) k'^{\gamma'} = 0
\]
(373)
and therefore the Faddeev-Popov ghosts are absent in the \(t\)-channel imaginary part of the one-loop amplitude. Further, the product of two non-asymptotic terms \(A^{(n)}\) does not give any contribution of the order of \(s\) to the elastic scattering amplitude. The integral contribution from the product of the terms \(A^{(a)}\) is given below in the \(D\)-dimensional space-time [41]:
\[
A_{\alpha\alpha'\beta\beta'}^{(aa)\alpha'\beta\beta'}(k, q + k, p_A) = \frac{g^4}{t} g^4 \delta_{\alpha\alpha'}^{\perp\perp} \delta_{\beta\beta'}^{\perp\perp} N_c T_A^{d} T_{B'}^{d} T_{B'}^{c} \frac{2 s}{(-t)^{2-D/2}} \frac{\Gamma(2-D/2) [\Gamma(D/2 - 1)]^2}{(4\pi)^{D/2} \Gamma(D - 2)} \times
\]

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\[ \left[ \frac{1}{2} - \frac{4}{D - 4} - \frac{9}{2} + (D - 3) \left( \ln \left( \frac{-s}{t^2} \right) + 2\psi(3 - D/2) - 4\psi(D/2 - 2) + 2\psi(1) \right) \right]. \] (374)

In an analogous way the contribution from the interference term between \( A^{(n)} \) and \( A^{(n)} \) equals [41]

\[
A^{(n)\alpha'\beta'\gamma'}_{A'B'B'} = g_4^4 \delta_{\perp\perp}^\rho \delta_{\perp\perp}^\sigma \frac{2\Gamma(2 - D/2) \Gamma(D/2 - 1)}{(4\pi)^{D/2} \Gamma(D - 2)} \left( \frac{1}{D - 1} + \frac{2}{D - 4} + \frac{q_4^a q_4^{a'}}{\frac{D - 4}{D - 1}} \right) \] (375)

Using the following regularized expression for the gluon Regge trajectory

\[
\omega(t) = g^2 N_c \frac{2}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2) \Gamma(D/2 - 1)}{(D - 3)}, \] (376)

we obtain for the unrenormalized form-factors \( \Gamma^{(1)}(t) \) and \( \Gamma^{(2)}(t) \) of the reggeon coupling with gluons [41]:

\[
\Gamma^{(1)}(t) = g + g^3 \frac{N_c}{(t)^{2-D/2}} \frac{\Gamma(2 - D/2) \Gamma(D/2 - 1)}{(4\pi)^{D/2} \Gamma(D - 2)} \times \left[ (D - 3)(\psi(3 - D/2) - 2\psi(D/2 - 2) + \psi(1)) - \frac{7}{4} - \frac{1}{4} \right] \] (377)

and

\[
\Gamma^{(2)}(t) = g^3 \frac{N_c}{(t)^{2-D/2}} \frac{\Gamma(3 - D/2) \Gamma(D/2 - 1)}{(4\pi)^{D/2} \Gamma(D - 2)} \frac{1}{D - 1}. \] (378)

Therefore in the pure gluodynamics in the one-loop approximation the s-channel helicity of colliding particles is not conserved even in the limit \( D \rightarrow 4 \). The contribution to the reggeon-gluon-gluon vertex from the virtual quarks was calculated in Ref [57]. For \( n_f \) massless quarks the result is

\[
\Delta \Gamma^{(1)}(t) = 2n_f g^3 (t)^{D/2-2} \frac{\Gamma(2 - D/2) \Gamma(D/2)}{(4\pi)^{D/2} \Gamma(D)},
\]

\[
\Delta \Gamma^{(2)}(t) = -2n_f g^3 (t)^{D/2-2} \frac{\Gamma(3 - D/2) \Gamma(D/2 - 1)}{(4\pi)^{D/2} \Gamma(D)}. \] (379)

For \( D \rightarrow 4 \) we obtain for \( \Delta \Gamma^{(2)} \):

\[
\Delta \Gamma^{(2)}(t) \rightarrow \Gamma^{(2)}(t) \left( -\frac{n_f}{N_c} \right). \]

The total contribution of quarks and gluons to the vertex \( \Gamma^{(2)} \) responsible for the helicity non-conservation is zero for \( n_f = N_c = 3 \). This fact is a consequence of the supersymmetry. Indeed, in the case of the super-symmetric Yang-Mills theory the helicity of
the gluon is conserved because it takes place for the gluino belonging to the same supermultiplet. On the other hand, this conservation in the one-loop approximation appears as a result of the cancellation between contributions of the virtual gluon and gluino. The gluino contribution can be obtained from quark one by its multiplication by the factor \(N_c/n_f\) because the gluino belongs to the adjoint representation and it is invariant under the charge conjugation. It leads to the vanishing of the helicity non-conserving transition at \(n_f = N_c = 3\).

The one-loop correction to the reggeon-quark-quark vertex was calculated in ref. [57]. In the massless case we have:

\[
\Gamma_\ell(t) = g + g^3 (-t)^{D/2-2} \frac{\Gamma(2 - D/2) \left[\Gamma(D/2 - 1)\right]^2}{(4\pi)^{D/2} \Gamma(D - 2)} \left\{ \frac{n_f}{D - 1} \left[\frac{3}{2} \left(-\frac{D}{2}\right) - \frac{2}{D - 4}\right]\right. \\
+ \left. N_c \left[ (D - 3) \left( \psi(3 - \frac{D}{2}) - 2\psi\left(\frac{D}{2} - 2\right) - \psi(1) \right) + \frac{1}{4} \frac{3}{D - 1} - \frac{2}{D - 4} - \frac{7}{4} \right] \right\}. \tag{380}
\]

The infrared and ultraviolet divergencies can be extracted easily from expressions (317-380) and are in the agreement with the general considerations.

### 5.4 Loop corrections to the reggeon-reggeon-particle vertex

The amplitude \(M_{2\to 3}\) for the production of a gluon in the central rapidity region can be written in the Regge model as follows [41]

\[
M_{2\to 3} = s T_{A'A}^{e_1} \Gamma_{\lambda'\nu\lambda A}(t_1) \frac{1}{t_1} T_{2c_1}^d \frac{1}{t_2} T_{2B'B}^{e_2} \Gamma_{\lambda'\nu\lambda B}(t_2) f(s, s_1, s_2, q_1^\perp, q_2^\perp),
\]

where \(j_{1,2} = 1 + \omega_{1,2}\) (for \(\omega_i = \omega(t_i)\)) are the gluon Regge trajectories with the negative signature at the crossing channels \(t_1, t_2\). It is important, that this amplitude does not have the simultaneous singularities in the overlapping channels \(s_1 = 2kp_A\) and \(s_2 = 2kp_B\).

In LLA one obtains:

\[
R(t_1, t_2, \vec{k}_\perp^2) + L(t_1, t_2, \vec{k}_\perp^2) \to 2g C_\mu(q_2, q_1) e_\mu^*(k) \tag{382}
\]

where \(C_\mu\) is the RRP vertex. One can calculate the discontinuities of \(M_{2\to 3}\) in the \(s, s_1\) and \(s_2\) channels for small \(g\) with the use of the unitarity condition in the one-loop approximation in the \(D\)-dimensional space:

\[
R - L = \frac{C_\mu(q_2, q_1) e_\mu^*(k)}{\omega_1 - \omega_2} \frac{4 N_c g^3}{(4\pi)^{D/2}} \Gamma(3 - \frac{D}{2}) \left[ \ln \frac{1}{\vec{k}_\perp^2} - \frac{2}{D - 4} \right]. \tag{383}
\]

Note, that this quantity is real in all physical channels. By adding the analogous contribution of the order of \(g^3\) to the sum \(R + L\) in such way to provide correct analytic properties
of the production amplitude in \( s, s_1 \) and \( s_2 \) channels we obtain for the one-loop correction to \( M_{2 \to 3} \) \[41\]:

\[
M_{2 \to 3}^{\text{loop}} = M_{2 \to 3}^{\text{Born}} N_c \frac{g^2}{(4\pi)^{D/2}} \Gamma(3 - \frac{D}{2}) \frac{1}{2} \left[ \frac{1}{4 - D} \ln \frac{(-s_1)^2 s_1^2 (-s_2)^2 s_2^2}{(-s)^3 s^3 \mu^4} \right. \\
- \frac{1}{4} \left( \ln^2 \frac{(-s_1)(-s_2)}{-s} + \ln^2 \frac{s_1 s_2}{-s} + \ln^2 \frac{s_1(-s_2)}{s} + \ln^2 \frac{(-s_1)s_2}{s} \right) \left] \right\} + \Delta M_{2 \to 3}, \tag{384}
\]

where \( \Delta M_{2 \to 3} \) is an polynomial in \( k_{\perp}^2 = s_1 s_2 / s \). One can reproduce this analytic structure of \( M_{2 \to 3} \) using the approach based on the unitarity conditions in \( t_1 \) and \( t_2 \) channels and obtain the following result \[41\]:

\[
R + L = 2g C_\mu(q_2, q_1) \epsilon^*_\mu(k) f_1 + \left( \frac{p_B}{s_2} - \frac{p_A}{s_1} \right) \epsilon^*_\mu(k) 4g f_2,
\]

\[
f_1 = \left( 1 + c_1 \frac{N_c g^2}{(4\pi)^{D/2}} \Gamma(3 - \frac{D}{2}) \right), \quad f_2 = c_2 \frac{N_c g^2}{(4\pi)^{D/2}} \Gamma(3 - \frac{D}{2});
\]

\[
c_1 = \frac{1}{2} \ln^2 \frac{k_{\perp}}{3} \left( -\frac{t_1 + t_2}{(t_1 - t_2)^2} + \frac{1}{6} \ln \frac{t_1}{t_2} \left( 11 \ln \frac{t_1 + t_2}{t_1 - t_2} + 4 \frac{k_{\perp}^2}{(t_1 - t_2)^2} \right) + \frac{\pi^2}{2} \right);
\]

\[
c_2 = \frac{1}{3} \ln \frac{t_1}{t_2} \left( 11 - \frac{k_{\perp}^2}{(t_1 - t_2)^2} \right) \frac{t_1 t_2}{t_1 - t_2} - \frac{k_{\perp}^2}{6} \left( 1 + \frac{t_1 + t_2}{(t_1 - t_2)^2} \right) \left( 2 \frac{k_{\perp}^2}{(t_1 - t_2)^2} + t_1 + t_2 \right). \tag{385}
\]

The contribution to the RRP vertex from the massless quark loop was also calculated \[58\]:

\[
\frac{N_c}{n_f} \Delta c_1 = \frac{2/3}{D - 4} - \frac{k_{\perp}^2}{3} \ln \frac{t_1 + t_2}{(t_1 - t_2)^2} + \frac{1}{3} \ln \frac{t_1}{t_2} \left( \frac{t_1 + t_2}{t_1 - t_2} + 2 \frac{k_{\perp}^2}{(t_1 - t_2)^2} \right), \quad \frac{N_c}{n_f} \Delta c_2 =
\]

\[
= \frac{1}{3} \ln \frac{t_1}{t_2} \left( -2 - \frac{k_{\perp}^2}{3} \ln \frac{t_1 + t_2}{(t_1 - t_2)^2} \right) \frac{t_1 t_2}{t_1 - t_2} + \frac{k_{\perp}^2}{6} \left( 1 + \frac{t_1 + t_2}{(t_1 - t_2)^2} \right) \left( 2 \frac{k_{\perp}^2}{(t_1 - t_2)^2} + t_1 + t_2 \right) \tag{386}
\]

and therefore for the super-symmetric Yang-Mills (where effectively \( n_f = N_c \)) the total gluon and quark contribution to \( c_2 \) is significantly simplified.

### 5.5 Two-loop corrections to the gluon Regge trajectory

For finding the next-to-leading corrections to the gluon Regge trajectory one can use the unitarity relations in the \( t \) or \( s \) channels. Initially the \( t \)-channel unitarity approach was advocated \[49\]. In the framework of this approach the universality of the Regge trajectory obtained from the different high energy processes is obvious. To find the next-to-leading
correction to the gluon trajectory one should calculate the imaginary part of the elastic scattering amplitude in the two-loop approximation taking into account the terms of the order of \( \ln^2 s \) and \( \ln s \). The contribution from the two-particle intermediate state in the \( t \)-channel can be expressed in terms of the product of two elastic amplitudes. One of these amplitudes is taken in the Born approximation and another one corresponds to the helicity conserving part of the above calculated one-loop elastic amplitude in the Regge kinematics. As for the contribution from the three-particle intermediate state in the \( t \)-channel, one should take the product of two amplitudes \( A_{2\to3} \) for the quasi-multi-Regge kinematics for each of two inelastic amplitudes \( A_{2\to3} \). Because in the leading logarithmic approximation the Regge expression for the scattering amplitude has only two-particle intermediate states, it is enough to calculate only the asymptotic behaviour of this three-particle contribution.

One can use also the \( s \)-channel unitarity to find the asymptotic behaviour of the scattering amplitude in the two-loop approximation [59]. In this case the imaginary parts of the amplitude in the \( s \) and \( u \) channels should be calculated with taking into account the terms of the order of \( s \ln s \) and \( s \). Subtracting from the result the known one-loop corrections to the reggeon-particle-particle vertices, one can calculate the two-loop corrections to the Regge trajectory. For the case of the gluodynamics it is given below [59]:

\[
\omega^{(2)}(t) = \frac{N_c^2 n_f g^4 t}{4(2\pi)^{2D-2}} \int \frac{d^{D-2}q_1}{\overline{q}_1^2} \int \frac{d^{D-2}q_2}{\overline{q}_2^2} \ f
\]

\[
f = \frac{\overline{q}^2}{(\overline{q} - \overline{q}_2)^2(\overline{q} - \overline{q}_1)^2} \ln \frac{\overline{q}^2}{(\overline{q} - \overline{q}_1)^2} + \frac{2}{(\overline{q} - \overline{q}_1 - \overline{q}_2)^2} \ln \frac{\overline{q}^2}{(\overline{q} - \overline{q}_1)^2} + \left( 2\psi(D-3) + \psi(3 - \frac{D}{2}) - 2\psi(\frac{D}{2} - 2) - \psi(1) + \frac{1}{D-3} \left( \frac{1}{4(D-1)} - \frac{2}{D-4} - \frac{1}{4} \right) \right).
\]

Note, that the logarithms in \( f \) appear as a result of integration over longitudinal momenta.

For the two-loop including the quark loop we obtain the following result [59]:

\[
\omega_q^{(2)}(-\overline{q}^2) = \frac{N_c n_f \pi^{D-2} g^4}{(2\pi)^{2D-2}} \Gamma(2 - \frac{D}{2}) \frac{\Gamma^2(D)}{\Gamma(D)} \int \frac{d^{D-2}q_1}{\overline{q}_1^2} \frac{\overline{q}^2}{(\overline{q} - \overline{q}_1)^2} \left( 2(\frac{\overline{q}_1}{\mu})^{D-4} - (\frac{\overline{q}}{\mu})^{D-4} \right).
\]

### 6 Conclusion

As it was shown above, the theory of the high-energy scattering in QCD is based on the BFKL equation summing the leading logarithmic terms in the perturbation theory. Next-to-leading terms were calculated using the \( k_L \)-factorization only for the anomalous dimensions describing the transition of quarks to gluons [60]. Above we expressed the next-to-leading real contribution to the BFKL kernel in terms of squares of the amplitudes for the production of gluons and quarks with definite helicities. All infrared divergencies
were extracted from these expressions in an explicit form and are regularized in the $D$-dimensional space. These divergencies cancel with the analogous divergencies from the virtual corrections to the BFKL equation which were also written above. It is known [55], that the total next-to-leading corrections to the integral kernel can be expressed in terms of the dilogarithm integrals and will be available soon. The effective field theory for the quasi-multi-Regge processes reviewed above can be used in particular for the unitarization program. Because the effective action is expressed in terms of the Wilson contour integrals, one can attempt to include into consideration also some non-perturbative effects. The effective field theory can be generalized to the case of the high energy behaviour of the amplitudes with non-singlet $t$-channel exchanges including one with the baryon quantum numbers and to the small-$x$ asymptotics of the structure functions for the polarized $ep$ deep-inelastic scattering [61].

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Appendix

There are the following helpful relations for hypergeometric functions:

$$F(h, h, 2h; x) = \frac{\Gamma(2h)}{[\Gamma(h)]^2} (-\ln(1-x) + 2\psi(1) - 2\psi(d) - \partial_a - \partial_b - 2\partial_c) F(h, h, 1; 1-x),$$

$$x^h (\partial_a + \partial_b + 2\partial_c) F(h, h, 1; 1-x) = x^{1-h} (\partial_a + \partial_b + 2\partial_c) F(1-h, 1-h, 1; 1-x),$$

$$x^h F(h, h, 1; 1-x) = x^{1-h} F(1-h, 1-h, 1; 1-x),$$

where the derivatives $\partial_a$, $\partial_b$ and $\partial_c$ act on the corresponding arguments of the function $F$ (122) with the subsequent substitution of $a, b$ and $c$ by their given values. Using these relations one can continue analytically $G_{\nu n}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1^\dagger, \vec{\rho}_2^\dagger)$ to the region near $x = 1$:

$$c_1^{-1} \frac{[\Gamma(h)\Gamma(h)]^2}{\Gamma(2h)\Gamma(2h)} x^{-h} x^{\ast -\vec{h}} G_{\nu n}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1^\dagger, \vec{\rho}_2^\dagger)$$

$$= 2\pi \cot(\pi h) (\ln |1-x|^2 - 4\psi(1) + \partial_a + \partial_b + 2\partial_c) F(h, h, 1; 1-x) F(\vec{h}, \vec{h}, 1; 1-x)$$

$$+ 4 \left( \psi(h) \psi(\vec{h}) - \psi(1-h) \psi(1-\vec{h}) \right) F(h, h, 1; x) F(\vec{h}, \vec{h}, 1; x^\ast).$$

Analogously according to the relations

$$F(h, h, 2h; x) = \frac{\Gamma(2h)}{[\Gamma(h)]^2} (-x)^h (\ln(-x) + 2\psi(1) - 2\psi(d) - \partial_a - \partial_b - 2\partial_c) F(h, 1-h, 1; 1),$$

$$x^h (\partial_a + \partial_b) F(h, h, 1; 1-x) = x^{1-h} F(1-h, 1-h, 1; 1-x),$$

$$x^h F(h, h, 1; 1-x) = x^{1-h} F(1-h, 1-h, 1; 1-x),$$

$$\left( \psi(1) - \psi(1-\vec{h}) \right) F(h, h, 1; x) F(\vec{h}, \vec{h}, 1; x^\ast).$$
\[(\partial_a + \partial_b + 2\partial_c) F(h, 1 - h, 1; 1/x) = (\partial_a + \partial_b + 2\partial_c) F(1 - h, h, 1; 1/x),
F(h, 1 - h, 1; 1/x) = F(1 - h, h, 1; 1/x)\]

one can continue $G_{\nu n}(p_1^\perp, p_2^\perp, \tilde{p}_1^\perp, \tilde{p}_2^\perp)$ at large $x$:

\[
c_1^{-1} \frac{[\Gamma(h)\Gamma(\tilde{h})]^2}{\Gamma(2h)\Gamma(2\tilde{h})} G_{\nu n}(p_1^\perp, p_2^\perp, \tilde{p}_1^\perp, \tilde{p}_2^\perp)
= -2\pi \cot(\pi h) \left( \ln |x|^2 + 4\psi(1) - \partial_a - \partial_b - 2\partial_c \right) F(h, 1 - h, 1; 1/x) F(\tilde{h}, 1 - \tilde{h}, 1; 1/x).
+ 4 \left( \psi(h)\psi(\tilde{h}) - \psi(1 - h)\psi(1 - \tilde{h}) \right) F(h, 1 - h, 1; 1/x) F(\tilde{h}, 1 - \tilde{h}, 1; 1/x).
\]

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