Data-driven Output Regulation via Gaussian Processes and Luenberger Internal Models

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Abstract: This paper deals with the problem of adaptive output regulation for multivariable nonlinear systems by presenting a learning-based adaptive internal model-based design strategy. The approach builds on the recently proposed adaptive internal model design techniques based on the theory of nonlinear Luenberger observers, and the adaptation side is approached as a probabilistic regression problem. In particular, Gaussian processes priors are employed to cope with the learning problem. Unlike the previous approaches in the field, here only coarse assumptions about the friend structure are required, making the proposed approach suitable for applications where the exosystem is highly uncertain. The paper presents performance bounds on the attained regulation error and numerical simulations showing how the proposed method outperforms previous approaches.

Keywords: Nonlinear Output Regulation, Adaptive Control Systems, Gaussian Processes, Nonparametric Methods, Identification for Control

1. INTRODUCTION

In this paper we consider a class of nonlinear systems of the form
\[ \dot{x} = f(x, w, u), \quad e = h(x, w), \]  
with state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), regulation error \( e \in \mathbb{R}^n \), and with \( w \in \mathbb{R}^n \) an exogenous signal. As customary in the literature of output regulation, we assume that the exogenous signal \( w \) belongs to the set of solutions of an exosystem of the form
\[ \dot{w} = s(w), \] 
originating in a compact invariant subset \( W \) of \( \mathbb{R}^n \). For the class of systems (1) (2), in this paper we consider the problem of design an output-feedback regulator of the form
\[ \dot{x}_c = f_x(x_c, e), \quad u_c = k_c(x_c, e), \] 
that ensures boundedness of the closed-loop trajectories and asymptotically removes the effect of \( w \) from the regulated output \( e \), thus ideally obtaining \( e(t) \to 0 \) as \( t \to \infty \). More precisely, the sought regulator must ensure
\[ \limsup_{t \to \infty} \|e(t)\| \leq \epsilon \] 
with \( \epsilon \geq 0 \) possibly a small number measuring the regulator’s asymptotic performance. In this work we focus on the specific case of adaptive approximate regulation, where the aforementioned control objective is relaxed to the case with \( \epsilon > 0 \), and a learning technique is employed to cope with uncertainties in the exosystem, and in the plant dynamics. In particular, the adaptation side is approached in a system identification fashion where the Gaussian process regression is used to infer the internal model dynamics directly out of the collected data.

Related Works Most of the work in the field of output regulation can be traced-back to Francis and Wonham (1976) and Davison (1976) who firstly formalize and solve the asymptotic regulation problem in the context of linear systems. Asymptotic results was given also in the field of Single-Input-Single-Output (SISO) nonlinear systems, first in a local context (Byrnes et al. (1997), Isidori and Byrnes (1990)), and later in a purely nonlinear framework (Byrnes and Isidori (2003), Marconi et al. (2007)), based on the “non-equilibrium” theory (Byrnes et al. (2003)). Recently, asymptotic regulators have been also extended to some classes of multivariable nonlinear systems (Wang et al. (2016), Wang et al. (2017)). The major drawbacks of asymptotic regulators reside in their complexity and fragility (Bin et al. (2018)). Indeed, sufficient conditions under which asymptotic regulation is ensured are typically expressed by equations whose analytic solution becomes a hard task even for relative simple problems. Moreover, even if a regulator can be build, asymptotic regulation may be lost at front of exosystem perturbation and plant uncertainties. The aforementioned problems motivates the researchers to move toward more robust solutions, introducing the concept of adaptive and approximate regulation. Among the approaches to approximate regulation it is worth mentioning Marconi and Praly (2008) and Astolfi et al. (2015), whereas practical regulators can be found in Isidori et al. (2012) and Freidovich and Khalil (2008). Adaptive designs of regulators can be found in Priscoli et al. (2006) and Pyrkin and Isidori (2017), where linearly parametrized internal models are constructed in the con-
text of adaptive control, in Bin et al. (2019) where discrete-time adaptation algorithms are used in the context of multivariable linear systems, and in Forte et al. (2016), Bin and Marconi (2019), Bin et al. (2020), where adaptation of a nonlinear internal model is approached as a system identification problem.

Learning dynamics models is also an active research topic. In particular, Gaussian Processes (GPs) are increasingly used to estimate unknown dynamics (Kocijan et al. (2016), Buisson-Fenet et al. (2020)). Unlike other nonparametric models, GPs represent an attractive tool in learning dynamics due to their flexibility in modeling nonlinearities and the possibility to incorporate prior knowledge (Rasmussen (2003)). Moreover, since GPs allow for analytical formulations, theoretical guarantees on the a posteriori can be drawn directly from the collected data (Umlauf and Hirche (2020), Lederer et al. (2019)). Recently, GP models spread inside the field of nonlinear optimal control (Sforini et al. (2021)), with several applications to the particular case of Model Predictive Control (MPC) (Torrette et al. (2021), Kabzán et al. (2019)), and inside the field of nonlinear observers (Buisson-Fenet et al. (2021)).

Contributions In this paper, we propose a data-driven adaptive output regulation scheme, built on top of the recently published works Bin et al. (2020) and Gentilini et al. (2022), in which the problem of approximate regulation is solved by means of a regulator embedding an adaptive internal model. Unlike previous approaches, here the high flexibility of Gaussian process priors (Rasmussen (2003)) is used to adapt an internal model unit in a discrete-time system identification fashion, enabling the possibility to handle a possibly infinite class of input signals needed to ensure zero regulation error (the so-called friend, Isidori and Byrnes (1990)). Compared to Bin et al. (2020), where the identifier is related to a particular choice of class of functions, to which the friend may (or may not) belongs, the proposed approach aims to perform probabilistic inference in a possibly infinite-dimensional space. Unlike Gentilini et al. (2022), the proposed regulator relies on non-high-gain stabilising actions and Luenberger-like internal models that lead to a fixed choice of the model order. The latter property, jointly with the black-box nature of Gaussian process methods, makes the proposed approach suitable for those applications where the exosystem dynamics is highly uncertain and the friend structure is not a priori known. Theoretical performance bounds on the attained regulation error are analytically established.

The paper unfolds as follows. In Section 2 we briefly describe the problem at hand along with the standing assumptions over the presented results build. Section 2.1 reviews the most recent advancements in the output regulation field, and introduces the barebone regulator adapted for this work, while Section 2.2 introduces the basics of Gaussian process inference. In Section 3 we present the proposed regulator and state the main result of the paper. Finally, in Section 4 a numerical example is presented.

2. PROBLEM SET-UP & PRELIMINARIES
In this section, we first detail the subclass of problems that this work focuses on, along with the constructive assumptions. Then, a Luenberger-like internal model design technique is reviewed, together with the adaptive regulator of Bin et al. (2020). Finally, basic concepts behind the notion of Gaussian process regression are introduced.

2.1 Approximate Nonlinear Regulation
In this paper, we focus on a subclass of the general regulation problem presented in Section 1, by considering systems of the form

\[
\dot{z} = f_0(w, z, e), \\
\dot{e} = Ae + B(q(w, z, e) + b(w, z, e)u), \tag{3}
y = Ce,
\]

in which \(z \in \mathbb{R}^n\) together with the error dynamics \(e \in \mathbb{R}^n\) represent the overall state of the plant. The quantities \(u \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\) are the control input and the measured output respectively, while \(w \in \mathbb{R}^{\nu_w}\) is an exogenous input, \(f_0 : \mathbb{R}^{\nu_w} \times \mathbb{R}^{n_x} \times \mathbb{R}^n \to \mathbb{R}^n\), \(q : \mathbb{R}^{\nu_w} \times \mathbb{R}^{n_x} \times \mathbb{R}^n \to \mathbb{R}^m\), \(b : \mathbb{R}^{\nu_w} \times \mathbb{R}^{n_x} \times \mathbb{R}^n \to \mathbb{R}^{\nu_y \times n_y}\) are continuous functions, and \(A, B,\) and \(C\) are defined as

\[
A = \begin{pmatrix}
0_{(r-1)n_x \times n_y} & I_{(r-1)n_y} \\
0_{n_x \times n_y} & 0_{n_y \times (r-1)n_y}
\end{pmatrix}, \quad B = \begin{pmatrix}
0_{(r-1)n_x \times n_y} \\
I_{n_y}
\end{pmatrix}, \\
C = \begin{pmatrix}
I_n & 0_{n \times (r-1)n_y}
\end{pmatrix},
\]

for some \(r \in \mathbb{N}\), consisting in a chain of \(r\) integrators of dimension \(n_y\). The aforementioned framework embraces a large number of use-cases addressed in literature. In particular, all systems presenting a well-defined vector relative degree and admitting a canonical normal form, or that are strongly invertible and feedback linearisable fit inside the proposed framework. Nevertheless, this approach limits to systems having an equal number of inputs and controlled outputs (\(n_y\)). The results presented in the next sections are grounded over the following set of standing assumptions.

Assumption 1. The function \(f_0\) is locally Lipschitz and the functions \(q\) and \(b\) are \(C^1\) functions, with local Lipschitz derivative.

Assumption 2. There exists a \(C^1\) map \(\pi : \mathcal{P} \subset \mathbb{R}^n \mapsto \mathbb{R}^{\nu_w}\), with \(\mathcal{P}\) an open neighborhood of \(\mathcal{W}\), satisfying

\[
L^{w}_{\pi}(w)\pi(w) = f_0(w, \pi(w), 0),
\]

with \(L^{w}_{\pi}(w)\pi(w) = \partial_{w} \pi(w) s(w), \) such that the system

\[
\dot{w} = s(w), \quad \dot{z} = f_0(w, z, e),
\]

is Input-to-State Stable (ISS) with respect to the input \(e\), relative to the compact set \(\mathcal{A} = \{w, z \in \mathbb{W} \times \mathbb{R}^{n_x} : z = \pi(w)\}\).

Assumption 3. There exists a known constant nonsingular matrix \(b \in \mathbb{R}^{\nu_y \times n_y}\) such that the inequality

\[
\|b(w, z, e) - b(w, z, 0)\| \leq 1 - \mu_0,
\]

holds for some known scalar \(\mu_0 \in (0, 1)\), and for all \((w, z, e) \in \mathbb{W} \times \mathbb{R}^{n_x} \times \mathbb{R}^n\).

Remark 2. Although not necessary (see Byrnes and Isidori (2003)), Assumption 2 is a minimum-phase assumption customary made in the literature of output regulation (see Isidori (2017), Pavlov et al. (2006)). In particular, Assumption 2 is asking that the zero dynamics

\[
\dot{w} = s(w), \quad \dot{z} = f_0(w, z, 0),
\]

has a steady-state of the kind \(z = \pi(w)\), compatible with the control objective \(y = 0\). As a consequence, the ideal input \(u^*\) making the set \(\mathcal{B} = \mathcal{A} \times \{0\}\) invariant for (3) reads as

\[
u^* (w, \pi(w)) = -b(w, \pi(w), 0)^{-1} q(w, \pi(w), 0).
\]
The ability of the regulator to generate such an input is generally referred to as the internal model property. With a little abuse of notation, from now on we refer to $u^*(w, \pi(w))$ with $u^*(w)$.

**Remark 3.** Assumption 1 asks for some Lipschitz conditions on maps that play a fundamental role in the stability analysis. In particular, Lipschitz continuity is required as long as high-gain-based observers are employed inside the regulator structure, later detailed in (3). Furthermore, even if in this work we deal with data-driven adaptive control techniques, that ideally require smoothness assumptions on the function to be identified $u^*$, in practice the adaptation of the internal model structure proposed by Marconi et al. (2007) makes the problem solvable without any further assumption. The details about this issue are more deeply discussed in Section 3.

**Remark 4.** Assumption 3 is a stabilizability assumption asking that $b(w, z, e)$ is always invertible whatever $(x, z, e)$ is (see Wang et al. (2017)). Moreover, the designer is required to have access to an estimate $b$ of $b(w, z, e)$ which captures enough information about its behavior.

In this framework, we now recall two results based on Marconi et al. (2007) and (Bin et al., 2020, Theorem 1).

**Lemma 1.** Let Assumption 2 hold and let $n_{\eta} = 2(n_{w} + n_{z} + 1)$. Then, for any choice of controllable pair $(F, G)$, with $F$ a Hurwitz matrix, there exist two maps $\tau : \mathbb{R}^{n_{w}} \rightarrow \mathbb{R}^{n_{w}}$, and $\gamma : \mathbb{R}^{n_{w}} \rightarrow \mathbb{R}^{n_{w}}$ such that for all $w$ in $\mathcal{W}$

$$\gamma \circ \tau(w) = u^*(w),$$

and the system

$$\dot{w} = s(w),$$

$$\dot{z} = f_0(w, z, e),$$

$$\dot{\eta} = F\eta + G\gamma^*(w) + \delta,$$

is ISS relative to the set $\mathcal{E} = \{ (w, z, \eta) \in \mathcal{A} \times \mathbb{R}^{n_{w}} : \eta = \tau(w) \}$ and with respect to the input $(e, \delta)$.

Let Assumption 1, 2, and 3 hold, and let $\mathcal{M} = \{ (\psi, \theta, \cdot) : \mathbb{R}^{n_{w}} \rightarrow \mathbb{R}^{n_{w}} | \theta \in \Theta \}$, with $\Theta$ a finite-dimensional normed vector space, be a finite-dimensional model set where $\gamma$ is supposed to range. Consider the following regulator structure \(^1\)

$$\begin{align*}
\dot{\zeta} &= 1 \\
\dot{\eta} &= F\eta + G\gamma^*(w) \\
\dot{\xi} &= A\xi + B(\dot{\xi} + Bu) + \Lambda(l)H(y - \hat{e}_1) \\
\dot{\zeta} &= -B\psi(\theta, \eta, u) + \Gamma^{++}H_{r+1}(y - \hat{e}_1) \\
\dot{\zeta} &= 0 \\
(\zeta, \eta, \xi, \zeta, \theta) &\in C_{\zeta} \times \mathbb{R}^{n_{w} + n_{w} + n_{w}} \times \mathbb{R} \times \mathbb{R}_{\zeta, \eta, \xi}, \dot{\zeta}, \xi, \eta, \xi, \zeta, \theta, y) &\in D_{\zeta} \times \mathbb{R}^{n_{w} + n_{w} + n_{w}} \times \mathbb{R} \times \mathbb{R}_{\zeta, \eta, \xi}, \dot{\zeta}, \xi, \eta, \xi, \zeta, \theta, y,
\end{align*}$$

with $\theta = \vartheta(\zeta)$ and output $u = b^{-1}(sat(\xi + \kappa_{\delta}(\zeta)))$. Where $A, B, b$ are the same in (3) and Assumption 3, while $F, G$ and $n_{\eta}$ are the same of Lemma 1, and $\Sigma$ finite-dimensional normed vector space. The sets $C_{\zeta}, D_{\zeta}$ are defined as

$$C_{\zeta} = [0, T], D_{\zeta} = [T, T],$$

with $T$, $T \in \mathbb{R}^+$, satisfying $0 < T < T$. Furthermore, $\Lambda(l) = \text{diag}(l_{1}, l_{2}, \ldots, l_{n_{w}})$, $H = \text{diag}(H_{1}, \ldots, H_{e})$, and $H_{i} = \text{diag}(h_{i 1}, \ldots, h_{i n_{w}})$ with $\{h_{i 1}, h_{i 2}, \ldots, h_{i n_{w}}\}$ for all $j = 1, \ldots, n_{\eta}$ coefficients of a Hurwitz polynomial, and $l \in \mathbb{R}_{>0}$ is a control parameter. Let the tuple $(M, \Sigma, \psi, \Theta, \vartheta)$ be such that the identifier requirements, relative to a given cost function $J$, are satisfied. Namely there exist $\beta_{\zeta} \in \mathcal{K}_{\zeta}$, locally Lipschitz $\rho_{\zeta}, \rho_{\eta} \in \mathcal{K}$, a compact set $\Sigma_{\ast} \subset \Sigma$ and, for each solution pair $((\zeta, w, \varsigma, \theta), (d_{\eta}, d_{\eta}))$ to

$$\begin{align*}
\dot{\zeta} &= 1 \\
\dot{\eta} &= s(w) \\
\dot{\xi} &= 0 \\
(\zeta, w, \zeta, \theta, d_{\eta}, d_{\eta}) &\in C_{\zeta} \times W \times \Theta \times \mathbb{R}_{\zeta, \eta, \xi, \tau} \times \mathbb{R}_{\zeta, \eta, \xi, \tau}, \\
\dot{\zeta} &= 0 \\
\dot{\xi} &= \vartheta(\zeta, \tau(w) + d_{\eta}, \gamma(\tau(w) + d_{\eta})) \\
(\zeta, w, \zeta, \theta, d_{\eta}, d_{\eta}) &\in D_{\zeta} \times W \times \Theta \times \mathbb{R}_{\zeta, \eta, \xi, \tau} \times \mathbb{R}_{\zeta, \eta, \xi, \tau}, \end{align*}$$

with $\theta = \vartheta(\zeta)$, there exists a pair $(\varsigma_{\ast}, \theta_{\ast})$ and a $j_{\ast} \in \mathbb{N}$, such that $((\zeta, w, \varsigma, \theta_{\ast}), (0, 0))$ is a solution pair to (4) satisfying $\varsigma(\ast) \in \Sigma_{\ast}$ for all $j \geq j_{\ast}$ and the following properties hold:

1. **Optimality:** For each $j \geq j_{\ast}$

$$\theta_{\ast} = \arg\min_{\theta \in \Theta} J(j, \theta).$$

2. **Stability:** For each $j$

$$|\varsigma(j) - \varsigma(\ast)| \leq \max \left\{ \beta_{\zeta}(\varsigma(0) - \varsigma(\ast), 0, j) \right\},$$

$$\rho_{\zeta} \left( (d_{\eta}, d_{\eta}) \right).$$

3. **Regularity:** The function $\vartheta$ satisfies

$$|\vartheta(\zeta) - \vartheta(\zeta_{\ast})| \leq \rho_{\eta}(|\varsigma - \varsigma_{\ast}|),$$

for all $(\zeta, \varsigma_{\ast}) \in \Sigma \times \Sigma_{\ast}$, the map $\psi(\theta, \eta, u)$ is $C^{1}$ with locally Lipschitz derivative in the argument $\eta$.

Then, for each compact sets $Z_{0} \subset \mathbb{R}^{n_{w}}$, $E_{0} \subset \mathbb{R}^{n_{w}}$, and $S_{0} \subset \mathbb{R}^{n_{w} + n_{w}}$ of initial conditions for $z, e$, and $(\dot{e}, \dot{b})$ respectively, there exists $l_{T} > 0$ such that if $l > l_{T}$, then the aggregate state $x = (\zeta, w, z, e, \eta, \xi, \zeta, \theta)$ of the closed-loop system is bounded. Moreover, there exists a $\alpha_{\pi} > 0$ and for each $T > 0$, an $l_{T} > 0$, such that if $l > l_{T}$ then

$$\limsup_{t \to \infty} \sup_{t \in [t, t+T]} |u^*(w) - \psi(\theta_{\ast}, \tau(w))|.$$
is the space of functions where $\gamma$ is supposed to range. Due to implementation constraints, it is customary to focus on finite-dimensional sets, which allows the parametrization of $\gamma$ by a parameter $\theta$ ranging in a finite-dimensional vector space $\Theta$. This, in turn, limits the flexibility of the proposed approach, especially when the structure of the friend is not a priori known. For this reason, we drop the assumption about $\mathcal{M}$ by performing regression in the space of universal approximators made by Gaussian processes.

2.2 Gaussian Process Inference

The key idea behind the proposed approach consists in modeling the unknown function $\gamma$ as the realization of a Gaussian process. A GP is a stochastic process such that any finite number of outputs is assigned a joint Gaussian distribution with prior mean function $m: \mathbb{R}^n \mapsto \mathbb{R}^n$ and covariance defined through the kernel $\kappa: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ (Rasmussen (2003)). While there are many possible choices of mean and covariance functions, in this work we keep the formulation of $\kappa$ general, with the only constraint expressed by Assumption 5 below. Yet we force, without loss of generality, $m(\eta) = 0$, for any $\eta \in \mathbb{R}^n$. Thus, we assume that

$$\gamma \sim \mathcal{GP}(0, \kappa(\cdot, \cdot)).$$

Supposing to have access to a data-set of samples collected at different time instants $t_i \in \mathbb{R}_{>0}$, $\mathcal{D} \subseteq \{(\eta, u) \in \mathbb{R}^n \times \mathbb{R}^n : \eta = \eta(t_i), u = u(t_i)\}$ for each pair $(\eta, u) \in \mathcal{D}$ obtained as $u(t_i) = \eta(t_i) + \varepsilon(t_i)$ with $\varepsilon(t_i) \sim N(0, \sigma^2_n I_{n_h})$ while Gaussian noise with known variance $\sigma^2_n$, the regression is performed by conditioning the prior GP distribution on the training data $\mathcal{D}$ and a test point $\eta$. Denoting $\eta = (\eta(t_1), \ldots, \eta(t_N))^T$ and $u = (u(t_1), \ldots, u(t_N))^T$, the conditional posterior distribution given the data-set is still a Gaussian process with mean $\mu$ and variance $\sigma^2$ given by (Rasmussen (2003))

$$\mu(\eta) = \kappa(\eta, \eta)^T (\mathbf{K} + \sigma^2_n I_{n_h})^{-1} u,$$

$$\sigma^2(\eta) = \kappa(\eta, \eta) - \kappa(\eta)^T (\mathbf{K} + \sigma^2_n I_{n_h})^{-1} \kappa(\eta),$$

where $\mathbf{K} \in \mathbb{R}^{N \times N}$ is the Gram matrix whose $(k,h)$-th entry is $\kappa(\eta_k, \eta_h)$, with $\eta_k$ the $k$-th entry of $\eta$, and $\kappa(\eta) \in \mathbb{R}^N$ is the kernel vector whose $k$-th component is $\kappa_k(\eta) = \kappa(\eta, \eta_k)$. The problem of inferring an unknown function from a finite set of noisy data can be seen as a special case of ridge regression where the prior assumptions (mean and covariance) are encoded in terms of smoothness of $\mu$. In particular, let $\mathcal{H}$ be a RKHS associated with the kernel function $\kappa$, then an estimation of $\gamma$ can be inferred by minimizing the functional

$$\mathcal{J} = \frac{\lambda}{2} \|\mu\|^2_{\mathcal{H}} + Q(u, \mu(\eta)),$$

where $\|\mu\|^2_{\mathcal{H}}$ is the RKHS norm and represents the smoothness assumptions on $\mu$ (this term plays the role of regularizer), while $Q$ assesses the quality of the prediction $\mu(\eta)$ with respect to the observed data $u$ (Rasmussen (2003)). According to the Representer Theorem (O’sullivan et al. (1986)), each minimizer $\mu \in \mathcal{H}$ of $\mathcal{J}$ takes the form $\mu(\eta) = \kappa(\eta)\alpha$, with $\alpha$ which depends on the particular choice of the prediction error. In the particular case in which $Q(u, \mu(\eta))$ corresponds to a negative log-likelihood of a Gaussian model with variance $\sigma^2_n$, namely

$$Q(u, \mu(\eta)) = \frac{1}{2\sigma^2_n} \|u - \mu(\eta)\|^2_2,$$

the value of $\alpha$ recovers the expression in Equation (5) as

$$\alpha = (\mathbf{K} + \sigma^2_n I_N)^{-1} u.$$

From now on we suppose that the following standing assumptions hold (see Buisson-Fenet et al. (2021), Lederer et al. (2021))

**Assumption 4.** The unknown function $\gamma$ has a bounded norm in the RKHS $\mathcal{H}$ generated to the kernel $\kappa$.

**Assumption 5.** The kernel function $\kappa$ is isotropic and Lipschitz continuous with constant $L_\kappa$, with a locally Lipschitz derivative of constant $L_\kappa$.

Although any kernel fulfilling Assumption 5 can be a valid candidate, in the following, we exploit the commonly adopted squared exponential kernel as prior covariance function, which can be expressed as

$$\kappa(\eta, \eta') = \sigma^2_n \exp \left( - \frac{(\eta - \eta')^T \Lambda^{-1} (\eta - \eta')}{} \right)$$

for all $\eta, \eta' \in \mathbb{R}^n$, where $\Lambda = \text{diag}(2\lambda^2_1, \ldots, 2\lambda^2_n)$, $\lambda_n \in \mathbb{R}_{>0}$ is known as characteristic length scale relative to the $i$-th signal, and $\sigma^2_n$ is usually called amplitude (Rasmussen (2003)).

**Remark 6.** Assumption 5 is asking some Lipschitz continuity property of the unknown function that makes it well-representable by means of a Gaussian process prior. Nevertheless, it represents a very strong assumption, difficult to be checked even if the unknown function is known. Assumption 5 can be relaxed to the condition that $\gamma$ is a sample from the Gaussian process $\mathcal{GP}(0, \kappa(\cdot, \cdot))$, which, in turn, leads to a larger pool of possible unknown functions and it is easier to be check. As an example, the pool generated by the squared exponential kernel Equation (7) is equal to the space of continuous functions.

**Remark 7.** The isotropic kernel structure is a customary (although not necessary) assumption in the literature of Gaussian process regression. In this respect, the following results can be generalized for any Lipschitz continuous kernel by means of well-known arguments (see Lederer et al. (2021)).

We conclude this section by recalling two results based on Lederer et al. (2021).

**Lemma 3.** Consider a zero-mean Gaussian process defined through a kernel $\kappa: \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$, satisfying Assumption 5 on a compact subset $\mathcal{X}$ of $\mathbb{R}^n$, and $N \in \mathbb{N}$ observations $\mathcal{D} \subseteq \{(x^i, y^i) : i = 1, \ldots, N\}$, with $y^i = f(x^i) + \epsilon^i$, where $\epsilon^i \sim N(0, \sigma^2_n I_{n_h})$. Then, the posterior variance is bounded as

$$\sigma^2(x) \leq \sigma^2(0) - \frac{\kappa(0)}{\kappa(0) + \frac{\sigma^2_n}{\|\epsilon\|^2_{\mathbb{R}^n}}} \forall x \in \mathcal{X},$$

where $B_\rho(x) = \{x' \in \mathcal{D} : \|x - x'\| \leq \rho\}$ denotes the training data-set restricted to a ball around $x$ with radius $\rho \in \mathbb{R}_{>0}$, and $|\cdot|$ denotes the cardinality.

**Lemma 4.** Consider a zero-mean Gaussian process defined through a kernel $\kappa: \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$, satisfying Assumption 5 on the compact set $\mathcal{X}$. Furthermore, consider a continuous unknown function $f: \mathcal{X} \mapsto \mathbb{R}$ with Lipschitz constant $L_f$, and $N \in \mathbb{N}$ observations $y^i = f(x^i) + \epsilon^i$, with

$^2$Isotropic kernels are functions depending only on the Euclidean distance of their arguments. In this respect, the compact notation $\kappa(x, x') = \kappa(||x - x'||)$ is commonly used.
\[ \varepsilon_i \sim N(0, \sigma_i^2 I_{n_i}). \] Then, there exists \( \rho \in \mathbb{R}_{>0} \) such that the posterior mean \( \mu \) and posterior variance \( \sigma^2 \) conditioned on the training data \( \mathcal{D} = \{(x^1, y^1), \ldots, (x^N, y^N)\} \) are continuous with Lipschitz constants \( L_\mu \) and \( L_{\sigma^2} \) on \( \mathcal{X} \), respectively, satisfying
\[ L_{\mu} \leq L_{\mu} \sqrt{N} \left\| (\mathbf{K} + \sigma_i^2 I_N)^{-1} y \right\|, \]
\[ L_{\sigma^2} \leq 2\rho L_\mu \left( 1 + N \left\| (\mathbf{K} + \sigma_i^2 I_N)^{-1} \right\| \max_{x, x' \in \mathcal{X}} \kappa(x, x') \right), \]
with \( x = (x^1, \ldots, x^N)^T \) and \( y = (y^1, \ldots, y^N)^T \). Moreover, pick \( \delta \in (0, 1) \) and set
\[ \beta(\rho) = 2 \log \left( \frac{M(\rho, \mathcal{X})}{\delta} \right), \]
\[ \alpha(\rho) = (L_f + L_\mu) \rho + \sqrt{\beta(\rho)} \rho \]
with \( M(\rho, \mathcal{X}) \) the \( \rho \)-covering number \( 3 \) related to the set \( \mathcal{X} \). Then, the bound
\[ |f(x) - \mu(x)| \leq \sqrt{\beta(\rho)} \sigma^2(x) + \alpha(\rho) \qquad \forall x \in \mathcal{X} \]
holds with probability at least \( 1 - \delta \).

3. THE PROPOSED REGULATOR

The proposed regulator reads as follows
\[
\begin{align*}
\dot{\xi} &= 1 \\
\dot{\eta} &= Fu + Gu \\
\dot{\xi} &= A\bar{\xi} + B(\bar{\xi} + bu) + \Lambda(l)H(y - \bar{e}_1) \\
\dot{\xi} &= 0 \\
\zeta, \eta, \xi, \theta, y &\in \mathcal{C}, \\
\zeta^+ &= 0, \\
\eta^+ &= \eta, \\
\xi^+ &= \xi, \\
\zeta^+ &= (S \otimes I_N) \zeta + (B \otimes I_N) \left[ \eta, u \right]^T, \\
(\zeta, \eta, \xi, \theta, y) &\in \mathcal{D},
\end{align*}
\]
with \( \theta = \partial(\zeta) \) and output \( u = b^{-1} \text{sat}(-\hat{\xi} + \kappa_{\alpha}(\hat{\xi})) \). Where \( A, B, \) and \( b \) are the same as in (3) and Assumption 3, \( F, G, \) and \( n_0 \) are the same as Lemma 1, and \( \Lambda(l), H \) are defined as in Section 2.1 with \( l \in \mathbb{R}_{>0} \) a free control parameter fixed later to a sufficiently large number, while the matrices \( S \in \mathbb{R}^{N(n_n+n_n) \times N(n_n+n_n)} \) and \( B \in \mathbb{R}^{N(n_n+n_n) \times N(n_n+n_n)} \) have the shift form, denoting \( n_c = N(n_n+n_n) \) and \( n_y = N(n_n+n_n) \),
\[ S = \begin{pmatrix} 0_{a(n_n-1) \times 1} & I_{n_n-1} \\ 0_{1 \times n_x-1} \end{pmatrix}, B = \begin{pmatrix} 0_{a(n_n-1) \times 1} \\ 1 \end{pmatrix}. \]
The flow and jump set are defined as \( \mathcal{C} = \{ (\zeta, \eta, \xi, \zeta, \theta, y) \in \mathbb{R}_{>0} \times \mathbb{R}^{n_x+n_x} \times \mathbb{R}^{n_x+n_x} : 0 \leq \zeta \leq \sqrt{T}, \sigma^2, \sigma^2(\eta, \xi, \theta) \leq \sigma^2_{\text{thr}} \} \) and \( \mathcal{D} = \{ (\zeta, \eta, \xi, \zeta, \theta, y) \in \mathbb{R}_{>0} \times \mathbb{R}^{n_x+n_x} \times \mathbb{R}^{n_x+n_x} : 0 \leq \zeta \leq \sqrt{T}, \sigma^2, \sigma^2(\eta, \xi, \theta) \leq \sigma^2_{\text{thr}} \} \) respectively, where \( \Sigma \subset \mathbb{R}^{N(n_n+n_n)} \), \( \Theta \subset \mathbb{R}^{N(n_n+n_n)} \), with \( N \in \mathbb{N}_{>0} \), and \( \mathcal{K}, \mathcal{K}, \sigma^2_{\text{thr}} \in \mathbb{R}_{>0} \) satisfying \( \mathcal{B} \subseteq \mathcal{K} \) and \( \sigma^2_{\text{thr}}(\sigma^2 + \sigma^2) \leq \sigma^2_{\text{thr}} + \sigma^2_{\text{thr}} \leq \sigma^2_{\text{thr}} \). The functions \( \mu(\eta, \zeta, \theta) \) and \( \sigma^2(\eta, \zeta, \theta) \) are the a posteriori GP estimate mean and variance, respectively.

\footnote{The minimum number satisfying \( \min_{x \in \mathcal{X}} \max_{x' \in \mathcal{D}S} \| x - x' \| \leq \rho. \)

after the collection of \( N \) samples. According to Section 2.2, denoting \( \varsigma = (\varsigma_0, \varsigma_0)^T \), the latter functions read to
\[ \mu(\eta, \varsigma, \theta) = \kappa(\eta)^T \theta, \]
\[ \sigma^2(\eta, \varsigma, \theta) = \kappa(\eta, \varsigma) - \kappa(\eta)^T (\mathbf{K} + \sigma^2_{\text{thr}})^{-1} \kappa(\eta) \]
with \( \theta = (\mathbf{K} + \sigma^2_{\text{thr}})^{-1} \varsigma_0 \). In this settings, \( \mathbf{K} \) and \( \kappa \) are evaluated with respect to the data-set \( \mathcal{D} \) as defined in Section 2.2.

Claim 1. Let Assumptions 4 and 5 hold, then the tuple \( (\Sigma, \mu, \theta) \) satisfies the identifier requirements relative to the functional (6).

Claim 2. Let Assumptions 1-5 hold and consider the regulator (3), then for each compact sets \( \Omega_0, \Omega_0 \), there exists \( \alpha_\in > 0 \) and \( \ell_0 > 0 \), and for any choice of \( \sigma^2_{\text{thr}} \in \mathbb{R}_{>0} \), and \( N \in \mathbb{N}_{>0} \), and for each initial condition \( w_0 \in \mathcal{W} \), a \( \rho^*(w_0) > 0 \) such that if \( l > \ell_0 \), then the bound
\[ \limsup_{t \to \infty} |y(t)| \leq \alpha_\in \left( \sqrt{\beta} \left[ \sigma^2_{\text{thr}} \kappa(\theta) + \sigma^2_{\text{thr}} \right] + \alpha(\rho^*) \right) \]
with \( \beta \) and \( \alpha \) defined as
\[ \beta = 2 \log \left( \frac{N}{\delta} \right), \alpha(\rho^*) = (L_f + L_\mu) \rho^* + \sqrt{\beta L_{\sigma^2} \rho^*}, \]
holds with probability at least \( 1 - \delta \).

Remark 8. The quantity \( \rho^* \) in Claim 2 represents a notion of coverage of the set \( \mathcal{E} \) by the collected data-set. In particular, the lower \( \rho^* \) is, the better the set \( \mathcal{E} \) is covered.

As long as it approaches to zero, the regulation error approaches the lower bound
\[ \limsup_{t \to \infty} |y(t)| \leq \alpha_\in \left( \sqrt{\beta} \left[ \sigma^2_{\text{thr}} \kappa(\theta) + \sigma^2_{\text{thr}} \right] \right), \]
driven by the measurement noise \( \sigma^2_{\text{thr}} \).

4. NUMERICAL SIMULATION

To test the proposed regulator performances against state-of-the-art output regulation solutions, we consider the same problem proposed by Bin et al. (2020) where the output of a Van der Pol oscillator, with unknown parameter, must be synchronized with a triangular wave with unknown frequency. The forced Van der Pol oscillator is described by the following equations
\[ \ddot{x}_1 = x_2, \]
\[ \ddot{x}_2 = -x_1 + a (1 - \chi_1^2) x_2 + u, \]
with a scalar unknown parameter regulating the system damping. Furthermore, a triangular wave can be generated by an exosystem of the form
\[ \dot{w}_1 = w_2, \quad \dot{w}_2 = -gw_1, \]
with output
\[ x^*(w) = \frac{2 \sqrt{w_1^2 + w_2^2}}{\pi} \arcsin \left( \frac{w_1}{\sqrt{w_1^2 + w_2^2}} \right), \]
with scalar parameter \( g \) the unknown oscillating frequency. The goal is to steer the output \( x_1 \) of (8) to the reference \( x^*(w) \). The error coordinates \( e \) are thus defined as
\[ (e_1, e_2) = \left( \chi_1 - x_1^*(w), \chi_2 - L_{\alpha(\in)} \chi_2^*(w) \right), \]
and the error system reads as
...
\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= -e_1 - \chi^* - L_{s(w)}^2 \chi^* \\
&\quad + a \left( 1 - (e_1 + \chi^*(w))^2 \right) \left( e_2 + L_{s(w)}^2 \chi^*(w) \right).
\end{align*}
\]

The system (9) is in the same form of (3) with Assumption 2 trivially fulfilled since the \( z \) dynamics is absent. Furthermore, Assumption 1 and Assumption 3 hold with \( b = 1 \) and any \( \mu \in (0,1) \). To be compliant with the results presented by Bin et al. (2020) we exploit the same controller parameters

1. \( \kappa_1(\ell) = K\dot{e} \) with \( K \) such that \( \sigma(A - BK) = \{-1,-2\} \), and the input \( u \) has been saturated inside the interval \([-100,100]\).
2. The internal model dimension is \( n = 2(n_w + 1) = 6 \), and the matrices \( F \) and \( G \) has been fixed as

\[
F = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}, \quad G = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}.
\]

3. The control parameters has been chosen as \( l = 20 \), \( h_1 = 6 \), \( h_2 = 11 \), \( h_3 = 6 \), and \( \mathbf{T} = \mathbf{T} = 0.1 \).

The simulations reported in Figures 1, 2, and 3 show the proposed regulator applied with \( a = \rho = 2 \) in three cases with \( N = 50 \), \( N = 100 \), and \( N = 200 \). The obtained results are then compared with the regulator proposed by Bin et al. (2020) where the identifier is chosen as a least-squares identifier working on the model set \( \mathcal{M} = \{\psi(\theta, \eta) : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{\eta} | \psi(\theta, \eta) = \theta^\top \eta, \theta \in \Theta \subset \mathbb{R}^{n_u} \} \). In all simulations the GP parameters has been kept fixed at \( \sigma_n^2 = 0.01 \) and \( \sigma_{\text{bias}}^2 = \sigma_p^2 = 1 \), while the kernel hyper-parameters \( \lambda = (\lambda_{\eta_1}, \ldots, \lambda_{\eta_2}) \) has been estimated via log-likelihood minimization (Rasmussen (2003)) yielding to the values of \( \lambda = (7.7, 34.3, 19.9, 0.4, 133.6, 1.2) \). As emerges from Figure 1, the proposed approach reduces the maximum error of more than 100 times compared to the case with least-square identifier.

5. CONCLUSION

We presented a learning-based technique to design internal model-based regulators for a large class of nonlinear systems. The flexibility of the proposed approach makes the regulator able to deal with highly uncertain shapes of the optimal steady-state control input useful to make zero the output error. Thanks to the fact that only coarse and qualitative knowledge about the friend is required, the proposed approach may be employed as solution to many of the output regulation problems addressed in literature. The paper also derives probabilistic bounds on the attained performances and presents numerical simulations showing how the proposed method outperforms previous approaches when the regulated plant or the exogenous disturbances are subject to unmodeled perturbations. Future research directions will be aimed at exploring deeper the Gaussian process flexibility by focusing on the injection of possibly a priori knowledge of the friend structure, and at investigating how the proposed performance bound changes. We also aim to investigate if Gaussian process-based internal models may deal with non minimum-phase systems.

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