NILPOTENCE IN NORMED MGL-MODULES

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Abstract. We establish a motivic version of the May Nilpotence Conjecture: if \( E \in \text{NAlg}(\mathcal{SH}(S)) \) satisfies \( E \vee \mathbb{H}Z \simeq 0 \), then also \( E \vee \text{MGL} \simeq 0 \). In words, motivic homology detects vanishing of normed modules over the algebraic cobordism spectrum.

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1. Introduction

A guiding principle of stable homotopy theory is the celebrated Nilpotence Theorem of Devinatz, Hopkins, and Smith [DHS88]:

**Theorem** (DHS Nilpotence). Suppose \( R \) is a homotopy ring spectrum. Then an element \( x \in \pi_\ast(R) \) is nilpotent if and only if its Hurewicz image in \( MU_\ast(R) \) is nilpotent.

When \( R \) is highly structured, a stronger nilpotence criterion, due to Mathew, Noel, and Naumann, is available [MNN17]:

**Theorem** (May Nilpotence Conjecture). Suppose \( R \) is an \( \mathcal{E}_\infty \)-ring spectrum. Then an element \( x \in \pi_\ast(R) \) is nilpotent if and only if its Hurewicz image in \( \mathbb{H}Z_\ast(R) \) is nilpotent.
The proof of the May Nilpotence Conjecture in [MNN17] runs by reduction to the DHS Nilpotence Theorem. As such, the authors prefer to phrase the May Nilpotence Conjecture in the following equivalent form:

**Theorem** (May Nilpotence Conjecture). Suppose that $R$ is an $\mathcal{E}_\infty$-ring spectrum. Then $R \wedge MU \simeq 0$ if and only if $R \wedge \mathbb{H} \simeq 0$.

**Remark 1.1.** To see that the first version of the May Nilpotence Conjecture implies the second, note that $R \simeq 0$ if and only if 1 is nilpotent in $\pi_*(R)$. To see that the second version implies the first, note that $x$ is nilpotent in $\pi_*(R)$ if and only if $R[x^{-1}] \simeq 0$. Here $R[x^{-1}]$ denotes the mapping telescope (sequential colimit), which one may show is also an $\mathcal{E}_\infty$-ring.

The present paper is part of a larger project to understand which nilpotence theorems continue to hold in the stable motivic category $\mathcal{S}H(S)$, where $S$ is a scheme. While $\mathcal{S}H(S)$ is a symmetric monoidal stable $\infty$-category, and hence admits a notion of $\mathcal{E}_\infty$-ring object, there is additionally the stronger notion of a normed spectrum, which in the notation of [BH21] is an object of $NAlg(\mathcal{S}H(S))$. Many of the most important $\mathcal{E}_\infty$-rings in the classical stable homotopy category, such as $MU$, $KU$, and $\mathbb{H}$, have normed analogs, such as $MGL$, $KGL$, and $\mathbb{H} \in NAlg(\mathcal{S}H(S))$. Here, we will be particularly interested in $\mathbb{H}$, which denotes Spitzweck’s motivic cohomology spectrum [Spi12], and $MGL$, which denotes the algebraic cobordism spectrum (see, e.g., [BH21, §16]).

Our main result is the following normed analog of the May Nilpotence Conjecture:

**Theorem 1.2.** Let $S$ be a noetherian scheme of finite dimension, and write $S$ for the set of primes not invertible on $S$. Let $E \in NAlg(\mathcal{S}H(S))$ and suppose that

$$E \wedge \mathbb{H} [S^{-1}] \simeq 0.$$

Then also

$$E \wedge MGL [S^{-1}] \simeq 0.$$

We deduce Theorem 1.2 from the following slightly stronger result:

**Theorem 1.3.** Let $S$ be a noetherian scheme of finite dimension, $\ell$ a prime invertible on $S$, and $E \in NAlg(\mathcal{S}H(S))$. Suppose that $\ell^n = 0 \in \pi_0(E)$ for some $n$, and also that

$$E \wedge \mathbb{H} / \ell \simeq 0.$$

Then also

$$E \wedge MGL / \ell \simeq 0.$$

**Remark 1.4.** In [MNN17], Mathew, Noel, and Naumann prove the May Nilpotence Conjecture by studying power operations in Morava $E$-theories. It is beyond the range of present technology to produce a normed spectrum structure on any direct analog of Morava $E$-theory in $\mathcal{S}H(S)$. Our proof uses, as replacements for Morava $E$-theories, normed spectra $R_n$ that are motivic analogs of

$$MU[v_n^{-1}] \wedge_{k_0, k_1, \ldots, k_{n-1}}.$$

Since the $R_n$ spectra are built out of MGL by inverting and completing, but not by quotienting, they are easily seen to be normed spectra. Other than the change of context from a height $n$ Morava $E$-theory to $R_n$, our proof is similar in spirit to that of Mathew, Noel, and Naumann, though with a few additional complications. In particular, we must deal with the fact that Morava $K$-theories are not homotopy ring spectra over arbitrary bases $S$.

1.1. An $\mathcal{E}_\infty$ $MGL$-algebra with vanishing motive. To understand the significance of Theorem 1.2, it is important to note that the assumption that $E$ be normed cannot be weakened to $E$ being merely $\mathcal{E}_\infty$:

**Example 1.5** (a non-zero, torsion $\mathcal{E}_\infty$ $MGL$-algebra with vanishing motive). Let $k = C$. Denote by $K(n)^{mot}$ the motivic Morava $K$-theory spectrum, which we can define as

$$K(n)^{mot} = MGL_{(\ell)}[t_n^{-1}] \otimes \prod_{i \neq t_n^{-1}} MGL_{(\ell)}/t_i;$$

see e.g. §3.4 for a definition of the $t_i$.

First, note that $K(n)^{mot} \wedge \mathbb{H} \simeq 0$. Indeed, this is an orientable homotopy ring spectrum whose coefficients $\pi_{2*+}$ form a ring of characteristic $\ell$ over which the formal group law of $K(n)^{mot}$ becomes isomorphic to the additive one, and hence must have infinite height [Rav86, Lemma A2.2.9]. This is only possible if $v_n$ maps to zero; since $v_n$ also maps to a unit we conclude that this must be the zero ring.
Denote by $\mathbb{I}_{\tau}/\tau$ the $E_\infty$ ring spectrum that is the motivic cofiber of $\tau$ [Ghe17]. Note that $K(n)_{\text{mot}}/\tau \neq 0$ (e.g., since we know its homotopy groups, which are computed below). We shall construct an $E_\infty$-algebra structure on $K(n)_{\text{mot}}/\tau$. There exists an equivalence of categories between certain cellular MGL$^\wedge_{/\tau}$/modules and the bounded derived category of $\text{HU}_*$-modules [GWX, Theorem 3.7], which is in fact symmetric monoidal. Let $X$ be a cellular MGL$^\wedge_{/\tau}$/module such that $\pi_{\ast}X$ is concentrated in degrees $(2g, \ast)$. Then under this equivalence, $X$ is sent to the MU$_* = \text{MGL}_{2g, \ast}$-module $\pi_{\ast}X$. We have $\pi_{\ast}K(n)_{\text{mot}}/\tau = \mathbb{Z}/[v_n, v_n^{-1}]$ with $|v_n| = (2^{2n-2}, 2^n-1)$, and consequently $K(n)_{\text{mot}}/\tau$ corresponds to the MU$_*$-module $\pi_{\ast}K(n) = \mathbb{Z}/[v_n, v_n^{-1}]$. This admits an obvious commutative MU$_*$-algebra structure, and hence defines an $E_\infty$-ring in the derived category of MU$_*$-modules. The above equivalence transports this to an $E_\infty$-ring structure on $K(n)_{\text{mot}}/\tau \in \text{MGL}_{/\tau}$.

1.2. Normed nilpotence, related work, and open questions. It has long been known that a naive analog of Devinatz–Hopkins–Smith nilpotence fails in motivic stable homotopy theory, even in the category $SH(C)$. The prototypical example of this is that the element $\eta$ is not nilpotent in the homotopy of the unit object $\mathbb{I} \in SH(C)$, despite the fact that $\eta$ has trivial Hurewicz image in $\text{MGL}_{/\tau}$. In fact, $\mathbb{I}^{-1}$ is an $E_\infty$-ring object in $SH(C)$ that is non-zero, though it becomes trivial after tensoring with $\text{MGL}$. There remains, however, the intriguing fact that $\mathbb{I}^{-1}$ is not a normed spectrum in $SH(C)$:

Example 1.6 (Example 12.11 in [BH21]). Suppose $S$ is pro-smooth over a field of characteristic $\neq 2$, and that $R \in \text{NA}lg(SH(S))$. If the multiplication by $\eta$ is a map on $R$, then $R \simeq 0$.

In light of the above example, it seems reasonable to make the following definition:

**Definition 1.7.** Suppose $R \in \text{NA}lg(SH(S))$ and $x \in \pi_{\ast}R$. We say that $x$ is normed nilpotent if, whenever $R \to R'$ is a map of normed spectra such that the multiplication by $x$ map is an equivalence on $R'$, $R' \simeq 0$.

**Question 1.8.** For what class of normed spectra $R$ is it true that normed nilpotence in $R$ is detected under the Hurewicz map $R \to R \wedge \mathbb{H}Z$?

Our work here can be thought of as the beginnings of an answer to the above question. For example, we have the following:

**Theorem 1.9.** Let $S$ be a noetherian scheme of finite dimension, and write $S$ for the set of primes not invertible on $S$. Suppose that $R \in NA_{\text{lg}}(SH(S))$ is also a homotopy left unital MGL$[S^{-1}]$-algebra, with no assumed relationship between the normed spectrum structure and the homotopy left unital MGL$[S^{-1}]$-algebra structure. Then normed nilpotence in $R$ is detected by the Hurewicz map

$$R \to R \wedge \mathbb{H}Z[S^{-1}].$$

**Proof.** Let $x$ be an element of $\pi_{\ast}R$ such that $x$ is normed nilpotent in $R \wedge \mathbb{H}Z[S^{-1}]$. Our task is to prove that $x$ is normed nilpotent in $R$. To do so, suppose that $R \to R'$ is a map of normed spectra such that the multiplication by $x$ map is an equivalence on $R'$. By assumption $\mathbb{H}Z[S^{-1}] \wedge R' \simeq 0$, and so by Theorem 1.2 we conclude that $MGL[S^{-1}] \wedge R' \simeq 0$. However, the composite map $MGL[S^{-1}] \to R \to R'$ gives $R'$ a homotopy left unital MGL$[S^{-1}]$-algebra structure, whence $R'$ is a retract of $R' \wedge MGL[S^{-1}]$, and hence zero.

Forthcoming work [BEH22] of the first author, Elden Elmanto and Jeremiah Heller answers Question 1.8 in another case.

**Theorem 1.10.** Let $S$ be a scheme with $1/2 \in S$, $R$ a normed spectrum over $S$, and $x \in \pi_{2g+1, \ast}R$ an element in odd simplicial degree. Then normed nilpotence of $x$ is detected under the Hurewicz map

$$R \to R \wedge HZ,$$

and in fact even by the Hurewicz map

$$R \to R \wedge HF_2.$$

**Remark 1.11.** The above theorem provides a proof, slightly different from that in [BH21], that $\eta$ is normed nilpotent in $\mathbb{I} \in \text{NA}lg(SH(S))$.

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1One way of seeing this is as follows. The work of Pstragowski identifies cellular motivic spectra over $\mathbb{C}$ with (even) synthetic spectra, and this equivalence is symmetric monoidal [Pst18, Theorem 7.34, Remark 7.35]. The modules over cofiber $\tau$ correspond in the two categories, and are symmetric monoidally equivalent to the stable category of MU$_*$-MU-comodules [Pst18, Proposition 4.53, Remark 4.55]. This identifies the symmetric monoidal category of modules of $MGL/\tau$ with modules over MU$_*MU$ in the stable category of MU$_*MU$-comodules. This is symmetric monoidally equivalent to the derived category of MU$_*$-modules, by taking “extended” comodules.
Remark 1.12. Possibly after a finite separable extension, the Koszul sign rule will imply that \( x^2 \) is 2-torsion in \( \pi_* R \). Theorem 1.10 arises from a motivic analog of Mahowald’s stable homotopy theory theorem that an \( \mathbb{F}_2 \)-ring with \( 2 = 0 \) is an \( \mathbb{H}\mathbb{F}_2 \)-algebra [MNN17, Theorem 4.18].

When it comes to answering Question 1.8 in general, the authors feel a great deal of progress would be made if one could understand the answer to the following:

Question 1.13. Suppose that \( R \) is a normed spectrum such that some power \( \tau^k \) of \( \tau \) is 0 in \( \pi_* R \). Must \( R \) be null?

Remark 1.14. In stable homotopy theory, the May Nilpotence Conjecture can be viewed as a strong restriction on the mixed characteristic of an \( E_\infty \)-ring. Specifically, one learns that if \( H\mathbb{Q} \wedge R \simeq 0 \), then \( K(n) \wedge R \simeq 0 \) for every Morava \( K \)-theory. In [Hah16], the second author observed that, even if \( H\mathbb{Q} \wedge R \not\simeq 0 \), one may deduce that \( K(n) \wedge R \simeq 0 \) whenever \( K(n-1) \wedge R \simeq 0 \).

Question 1.15. For \( R \) a normed spectrum in \( \text{NAlg}(\mathbb{S}H(S)) \), does \( K(n-1) \wedge R \simeq 0 \) imply that \( K(n) \wedge R \simeq 0 \)? Are there similar relationships involving the \( K(\beta_i) \) of [Kra18]?

1.3. Organization of the paper. In §2 we define the universal transfer and universal norm of elements in the homotopy groups of (normed) motivic spectra; the former exists because spectra are built out of infinite loop spaces, and the latter is a multiplicative version of the former that is special to normed spectra. We establish some of their standard properties, which are largely analogous to the classical case.

In the next two sections, we restrict to the base scheme being a field. After establishing some preliminary results in §3, we study in §4 the norm and transfer in the very special normed MGL-module \( R = R_n \). Our key result is that any torsion normed \( R \)-module is zero, which is morally related to Tate vanishing in the \( K(n) \)-local category. Along the way we compute the homotopy groups \( \pi_* R \), the cohomology groups \( R^{2*} \), and we show that the completed homology \( \pi_* \hat{\mathbb{B}}_\mu \wedge R \) is finitely generated free.

In the final §5 we first establish some general techniques for showing that a normed spectrum is zero. For example, we show that this may be tested after an arbitrary separable field extension. Then we combine all our work to deduce the main theorems.

1.4. Conventions. In §3 and §4 we fix a base field \( k \) of exponential characteristic \( e \neq \ell \), and all our spectra will be implicitly \( e \)-periodized. We make some additional technical assumptions on \( k \) in §4, elaborated on at the beginning of that section.

We will use the notion of normed spectrum as set out in [BH21]. We deviate from the terminology in this reference in one aspect: we will only deal with “\( \text{Sm} \)-normed spectra”, and just call them normed spectra; we denote the category of normed spectra over a scheme \( S \) by \( \text{NAlg}(\mathbb{S}H(S)) \).

Throughout we freely use the language of \( \infty \)-categories, as set out in [Lur09, Lur16].

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2. Norms and Transfers in Cohomology

If \( E \in \mathbb{S}H(S) \), then \( E \) represents a cohomology theory on smooth \( S \)-schemes, with all the usual structures. For example, if \( f: X \to Y \in \text{Sm}_S \) is a finite étale morphism, then there is a transfer map \( \text{tr}_{X/Y}: E^0(X) \to E^0(Y) \) (to be contrasted with the pullback map \( E(Y) \to E(X) \) which of course also exists), which is an additive homomorphism. If \( E \in \text{NAlg}(\mathbb{S}H(S)) \), then the cohomology theory is multiplicative, and in particular there is a norm map \( N_{X/Y}: E^0(X) \to E^0(Y) \), which is a morphism of multiplicative monoids. The definition of these maps only uses that \( E \) has “incoherent” addition and multiplication maps, i.e. for example \( N_{X/Y} \) is obtained using the multiplication map \( f_\# E|_X \to E|_Y \), and similarly for the transfer. But \( E \) actually carries coherently commutative addition and multiplication maps, and we can use this structure to build a more complicated transfer map, essentially by taking the homotopy colimit over all transfers along finite étale extensions of a certain type. For example, the map \( E\Sigma_n \to \mathbb{B}\Sigma_n \) is the “universal degree \( n \) finite étale morphism”, and so one might expect to build a transfer (and norm) \( E^0(E\Sigma_n) \to E^0(\mathbb{B}\Sigma_n) \). Since \( E\Sigma_n \) is contractible, thus just takes the form

\[
\text{Tr}_*: E^0(S) \to E^0(\mathbb{B}\Sigma_n).
\]

In the current section we implement this idea.
2.1. Extended powers. We make use of the motivic extended powers $D_{C_n}$, defined in [BEH21, §5]. In particular these are functors $D_{C_n} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ satisfying $D_{C_n}(1) \simeq \Sigma_{\infty}^{\infty} BC_n$, where $BC_n$ denotes the geometric or étale classifying space [MV99, §4]. If $E \in \text{NAlg}(\mathcal{SH}(S))$, then $E$ comes equipped with multiplication maps $m : D_{C_n}(E) \to D_{C_n}(E) \to E$.

We need to be a little bit more precise. Given $E \in \mathcal{SH}(S)$, the evident $\Sigma_n$-action on $E^{\wedge n}$ refines to a “genuine motivic” action, yielding $E^{\wedge n} \in \mathcal{SH}_{\Sigma_n}(E)$ [BEH21, §6.1]. Let us denote by $E^\Sigma_{C_n} \in \mathcal{SH}_{\Sigma_n}(S)$ the result of forgetting from the $\Sigma_n$-action to the $C_n$-action. Then we have the formula

$$D_{C_n}(E) = E^\Sigma_{C_n} \wedge_{C_n} \mathbb{E}C_n,$$

where $\mathbb{E}C_n$ is the universal motivic space with a free $C_n$-action [GH19, §3.1], and $(-) \wedge_{C_n} \mathbb{E}C_n$ denotes the composite [GH19, §5.2]

$$\mathcal{SH}^{\Sigma_n}(S) \xrightarrow{\wedge \mathbb{E}C_n} \mathcal{SH}^{\Sigma_n}(S) \xrightarrow{(\sim)/C_n} \mathcal{SH}(S).$$

Let us also recall that in order to have a good six functors formalism for $\mathcal{SH}^{C_n}(-)$, one needs to assume that $1/|C_n| \in S$ [Hoy17, §1.2]. This is needed for example if we want finite étale $S$-schemes (with non-trivial action) to be self-dual in $\mathcal{SH}^{C_n}(S)$.

2.2. Transfer. We assume throughout that $1/n \in S$.

**Definition 2.1.** Let $E \in \mathcal{SH}(S)$ be arbitrary.

1. The universal transfer map $\Sigma_{\infty}^{\infty} BC_n \to 1$ is obtained by applying $(\sim) \wedge_{C_n} \mathbb{E}C_n$ to the transfer map $* \to \Sigma_{\infty}^{\infty} BC_n \in \mathcal{SH}_{\Sigma_n}(S)$ (obtained using the Wirthmüller/ambidexterity isomorphism).

2. The stable transfer for $E$ is the map

$$\text{Tr} : E \to E^{BC_n}$$

obtained by applying $\text{map}(-, E)$ to the universal transfer map.

3. The restriction for $E$ is the map

$$\text{Res} : E^{BC_n} \to E$$

obtained by applying $\text{map}(-, E)$ to the restriction $* \simeq \text{Be} \to BC_n$.

**Example 2.2.** We are mostly interested in the effect of the transfer on homotopy groups. There it takes the form

$$\text{Tr} : E^{**} \to E^{**}(BC_n),$$

and the restriction map just becomes the pullback map

$$\text{Res} : E^{**}(BC_n) \to E^{**}(\text{Be}) \simeq E^{**}.$$
which is just given by the sum of the maps \( m_a : C_n \to C_n \), \( x \mapsto xa \) for \( a \in C_n \). Since \( m_a \times C_n \| \mathbb{E} C_n \simeq \text{id}_* \) for every \( a \), the result follows.

(4) We need to show that the two longest paths in the following diagram represent homotopic maps

\[
\begin{array}{ccc}
\Sigma_+^\infty BC_n & \xrightarrow{\Delta} & \Sigma_+^\infty BC_n \wedge \Sigma_+^\infty BC_n \\
\text{Tr} \downarrow & & \text{Tr} \wedge \text{id} \downarrow \\
\mathbb{1} & \xrightarrow{\text{Res}} & \mathbb{1} \wedge \Sigma_+^\infty BC_n \xrightarrow{\pi \wedge \text{id}} E \wedge E \xrightarrow{m} E.
\end{array}
\]

For this it suffices to show that the left hand square commutes. It is obtained by applying \( \Sigma_+^\infty [(-) \wedge C_n \times C_n \mathbb{E} S_2] \) to the following diagram in \( \text{Span}(\text{Sm}_{\mathbb{S}}^0, \{\text{fet}, \text{all}\}) \)

\[
\begin{array}{ccc}
C_n \times C_n / \Delta & \xrightarrow{} & \ast \times \ast \\
\downarrow & & \downarrow \\
C_n \times C_n & \xrightarrow{} & C_n \times \ast.
\end{array}
\]

Here \( \Delta \subset C_n \times C_n \) denotes the diagonal subgroup, the horizontal maps are the obvious ones (just maps of sets), and the vertical ones are the obvious transfers. The two composites are represented by the spans \( C_n \times C_n / \Delta \leftarrow (C_n \times C_n / \Delta) \times (C_n \times \ast) \to C_n \times \ast \) and \( C_n \times C_n / \Delta \leftarrow C_n \times C_n \to C_n \times \ast \).

These are easily verified to be isomorphic. \( \square \)

### 2.3. Norm.

**Definition 2.6.** Let \( E \in \text{NAlg}(SH(S)) \). The norm map for \( E \)-cohomology is the map \( N : E^0 \to E^0(BC_n) \) which sends \( x : \mathbb{1} \to E \) to the composite

\[
\Sigma_+^\infty BC_n \simeq D_{C_n}(\mathbb{1}) \xrightarrow{D_{C_n}(x)} D_{C_n}(E) \xrightarrow{m} E.
\]

**Remark 2.7.** The norm map is obtained by applying \( \pi_0 \) to the total power operation \( P_{C_n} : E_0 \to \text{Map}(BC_n, E_0) \) of [BH21, Example 7.25].

**Lemma 2.8.** Let \( n = p \) be a prime, and \( 1/p \in S \). There exists an equivalence of functors \( SH(S) \times SH(S) \to SH(S) \) as follows

\[
D_{C_p}(E \vee F) \simeq D_{C_p}(E) \vee D_{C_p}(F) \vee \bigvee_s (E, F)^{\wedge s}.
\]

Here the sum at the end is over strings of length \( p \) in \( \{E, F\} \) containing at least one copy of \( E \) and one copy of \( F \), up to cyclic permutation, and \( (E, F)^{\wedge s} \) denotes the corresponding smash product of \( E \)s and \( F \)s.\(^2\) This decomposition has the following properties.

1. If \( x : E \to E' \) and \( y : F \to F' \) are morphisms, then the induced map \( D_{C_p}(x \vee y) : D_{C_p}(E \vee F) \to D_{C_p}(E' \vee F') \) is given by

\[
D_{C_p}(x) \vee D_{C_p}(y) \vee \bigvee_s (x, y)^{\wedge s}.
\]

2. If \( E = F \) and \( \Delta : E \to E \vee E \) is the diagonal map, then the map

\[
D_{C_p}(E) \xrightarrow{D_{C_p}(\Delta)} D_{C_p}(E \vee E) \simeq D_{C_p}(E) \vee D_{C_p}(E) \vee \bigvee_s E^{\wedge p}
\]

is given in components by \( (\text{id}, \text{id}, \text{tr}, \text{tr}, \text{tr}, \ldots, \text{tr}) \), where \( \text{tr} : D_{C_p}(E) \to E^{\wedge p} \) is the generalized universal transfer of Remark 2.4.

3. If \( E = F \) and \( \nabla : E \vee E \to E \) is the fold map, then the map

\[
D_{C_p}(E) \vee D_{C_p}(E) \vee \bigvee_s E^{\wedge p} \simeq D_{C_p}(E \vee E) \xrightarrow{D_{C_p}(\nabla)} D_{C_p}(E)
\]

is given in components by \( (\text{id}, \text{id}, c, c, \ldots, c) \), where \( c : E^{\wedge p} \to D_{C_p}(E) \) is the canonical map.

\(^2\)Strictly speaking, in order to make this into a well-defined functor, we need to pick a representative for each equivalence class. We tacitly assume that this has been done.
Proposition 3.1. Let $k$ be a field, $E \in \mathrm{NAlg}(SH(k))$ and $\alpha : E \rightarrow L$ be an $E$-module map for some $L \in \pi_0 \mathrm{Pic}(E\text{-Mod}) = \Gamma$. 

1. Suppose that for every finite separable field extension $L/K$ over $k$, the element $N_{L/K}(\alpha_L)$ is a unit in the $\Gamma$-graded commutative ring $\pi_*(E_K)[\alpha_K^{-1}]$. Then $E[\alpha^{-1}]$ is a normed spectrum.

2. Suppose that for every finite separable field extension $L/K$ over $k$, the element $\alpha_K$ is a unit in the $\Gamma$-graded commutative ring $\pi_*(E_K)[N_{L/K}(\alpha_L)^{-1}]$. Then $E^\wedge_\alpha$ is a normed spectrum.

Proof. Examination of the proofs of [BH21, Proposition 12.6, Proposition 12.14] shows that it suffices to establish the following: for every finite (surjective) étale morphism $f : X \rightarrow Y \in \mathrm{Sm}_k$, the functor $f_\otimes : E_X\text{-Mod} \rightarrow E_Y\text{-Mod}$ preserves $\alpha$-periodic equivalences (respectively $\alpha$-complete equivalences). Using [Bac18, Corollary 14] and compatibility of norms with base change [BH21, Proposition 5.3], for this we may assume that $Y$ is the spectrum of a field. Then the desired result follows from our assumption, exactly as in the proofs of [BH21, Proposition 12.6, Proposition 12.14].

Remark 3.2. If $E$ is oriented and $L = T^n \wedge E$ for some $n$, then $\pi_{f_\otimes L}(E) \simeq \pi_{T^n[L,K]}(E)$ canonically, and the notion of a $\Gamma$-graded commutative ring can be dispensed with in favor of the ordinary commutative graded ring $\pi_{2*}(E)$.

Example 3.3. If $\pi_{2*}(E)[\alpha_K^{-1}] = \pi_{2*}(N_{L/K}(\alpha_L)^{-1})$, then the Proposition applies and $E[\alpha^{-1}]$, $E^\wedge_\alpha$ are normed spectra. This happens in particular if $N_{L/K}(\alpha_L) = \alpha_K^{-1}$.

3. Completions, localizations, and quotients of MGL

We collect some preliminary results. Starting from §3.2, $k$ is a field of exponential characteristic $e \neq \ell$, and all objects are implicitly $e$-periodized.

3.1. Localization and completion of normed spectra.

Proposition 3.4. Let $E \in \mathrm{NAlg}(SH(k))$ and $\alpha : E \rightarrow L$ be an $E$-module map for some $L \in \pi_0 \mathrm{Pic}(E\text{-Mod}) = \Gamma$. 

1. Suppose that for every finite separable field extension $L/K$ over $k$, the element $N_{L/K}(\alpha_L)$ is a unit in the $\Gamma$-graded commutative ring $\pi_*(E_K)[\alpha_K^{-1}]$. Then $E[\alpha^{-1}]$ is a normed spectrum.

2. Suppose that for every finite separable field extension $L/K$ over $k$, the element $\alpha_K$ is a unit in the $\Gamma$-graded commutative ring $\pi_*(E_K)[N_{L/K}(\alpha_L)^{-1}]$. Then $E^\wedge_\alpha$ is a normed spectrum.

Proof. Examination of the proofs of [BH21, Proposition 12.6, Proposition 12.14] shows that it suffices to establish the following: for every finite (surjective) étale morphism $f : X \rightarrow Y \in \mathrm{Sm}_k$, the functor $f_\otimes : E_X\text{-Mod} \rightarrow E_Y\text{-Mod}$ preserves $\alpha$-periodic equivalences (respectively $\alpha$-complete equivalences). Using [Bac18, Corollary 14] and compatibility of norms with base change [BH21, Proposition 5.3], for this we may assume that $Y$ is the spectrum of a field. Then the desired result follows from our assumption, exactly as in the proofs of [BH21, Proposition 12.6, Proposition 12.14].

Remark 3.2. If $E$ is oriented and $L = T^n \wedge E$ for some $n$, then $\pi_{f_\otimes L}(E) \simeq \pi_{T^n[L,K]}(E)$ canonically, and the notion of a $\Gamma$-graded commutative ring can be dispensed with in favor of the ordinary commutative graded ring $\pi_{2*}(E)$.

Example 3.3. If $\pi_{2*}(E)[\alpha_K^{-1}] = \pi_{2*}(N_{L/K}(\alpha_L)^{-1})$, then the Proposition applies and $E[\alpha^{-1}]$, $E^\wedge_\alpha$ are normed spectra. This happens in particular if $N_{L/K}(\alpha_L) = \alpha_K^{-1}$. 
3.2. The homotopy of MGL. Recall our convention that the exponential characteristic of $k$ has been inverted throughout; without this assumption the following result is (at the time of writing) not known.

Lemma 3.4 (Spitzweck). We have $\text{MGL}_{2q}(K) = L_{2q}$ (where $L_{2q} = \pi_{2q}\text{MU}$), $\text{MGL}_{2q+1, q}(K) = K^\times \otimes L_{2q+2}$ and $\pi_{p,q}(\text{MGL})(K) = 0$ for $p < 2q$ (and also for $p < q$).

Proof. See [Spi14, Proposition 7.1, Corollary 7.4 and Corollary 7.5].

3.3. Localizations and completions of MGL.

Corollary 3.5. Let $a_1, \ldots, a_r, b_1, \ldots, b_s \in \text{MGL}_{2*,*}(k)$ and $E \in \text{NAlg}(\text{MGL-Mod})$. Then

$$E[b_1^{-1}, \ldots, b_s^{-1}]_{a_1, \ldots, a_r}$$

is a normed spectrum.

Similarly with $\text{MGL}_{(t)}$ in place of $\text{MGL}$.

Proof. Since tensoring with an invertible object preserves limits (and colimits), $b_i$-periodic spectra are stable under limits and colimits. Since also $a_j$-complete spectra are stable under limits, it suffices to treat the localization or completion at one element $u$. By Example 3.3, it suffices to show that $N_{L/K}(u) = u^{[L:K]}$. Consider the presheaf $F = \text{MGL}_{2*,*}(-)$, say on $\text{ET}_K$, viewed as a presheaf of multiplicative monoids. It follows from Lemma 3.4 that this is a constant sheaf, and hence in particular an étale sheaf. We deduce from this and [BH21, Corollary C.13] that there is a unique normed structure on $F$ compatible with the multiplication. Since both the norms coming from $\text{MGL} \in \text{NAlg}(SH(k))$ and the norms coming from the assignment $N_{L/K}(a) = a^{[L:K]}$ are compatible with the multiplication in $F$, they must agree. The case of $\text{MGL}_{(t)}$ is completely analogous. This concludes the proof.

3.4. $\text{MGL}_{(t)}$, $\text{BPGL}$, and generators. Recall that there is a retraction of homotopy commutative rings [Vez01, Definition 5.3]

$$\text{BPGL} \rightarrow \text{MGL}_{(t)} \rightarrow \text{BPGL}.$$  

We shall fix for all time an orientation of $\text{BPGL}$, which therefore also gives an orientation of $\text{MGL}_{(t)}$. This orientation induces an isomorphism of rings [Vez01, Proposition 3.5]

$$\text{BPGL}^{2*,*}(\mathbb{P}^\infty) \cong \text{BPGL}_{2*,*}[x],$$

and the latter ring comes equipped with a formal group law.

Proposition 3.6. It is possible to choose elements $v_i \in \pi_{2^e-2, e-1} \text{BPGL}$ such that:

1. The map

$$\mathbb{Z}(t)[v_1, v_2, \cdots] \rightarrow \pi_{2*,*} \text{BPGL}$$

is an isomorphism of rings.

2. We have the formula, in $\text{BPGL}^{2*,*}(\mathbb{P}^\infty)$,

$$[t]^i(x) \equiv v_i x^{e^i} \text{ modulo } t, v_1, \cdots, v_{i-1}, x^{e^i} + 1.$$  

Here and throughout, $[t]^i(x)$ denotes the $t$-series of the formal group law [Rav86, Definition A2.1.19].

Proof. The formal group law on $\pi_{2*,*} \text{MGL}$ induces $\text{MU}_{2*} \rightarrow \pi_{2*,*} \text{MGL}$, which is an isomorphism by Lemma 3.4. By construction, the same holds for $\pi_{2*} \text{BP} \rightarrow \pi_{2*,*} \text{BPGL}$, so these claims reduce to their classical analogs, which are well-known. ∎

We denote by $t^e - 1$ the image of $v_i$ in $\pi_{2^e-2, e-1} \text{MGL}_{(t)}$.

Proposition 3.7. It is possible to choose, for all $i \neq e^n - 1$, generators $t_i \in \pi_{2i, i} \text{MGL}_{(t)}$ such that:

1. $\pi_{2*,*} \text{MGL}_{(t)} \cong \mathbb{Z}(t)[t_1, t_2, \cdots]$, and

2. $h(t_i) \equiv b_i$ modulo decomposables ($i \neq e^n - 1$), where $h : \pi_{*,*} \text{MGL}_{(t)} \rightarrow \text{HZ}_{*,*} \text{MGL}_{(t)} \cong Z(t)[b_1, b_2, \cdots]$ is the Hurewicz map.

Proof. Note that for (2), it suffices to satisfy the stronger analogous condition for the $\text{MGL}$-Hurewicz map $\pi_{*,*} \text{MGL}_{(t)} \rightarrow \text{MU}_{2*,*} \text{MGL}_{(t)}$. The formal group law on $\pi_{2*,*} \text{MGL}_{(t)}$ induces a morphism of Hopf algebroids

$$(\pi_{2*} \text{MU}_{(t)}, \text{MU}_2 \text{MU}_{(t)}) \rightarrow (\pi_{2*,*} \text{MGL}_{(t)}, \text{MGL}_2 \text{MGL}_{(t)}$$

preserving the $b_i$ on both sides (see e.g. [Hoy15, p. 23]). This reduces the claims (in the strengthened form) to their classical analogs, which are well-known. ∎
Lemma 3.9. We have claim reduces to the classical analog, which is well-known. □

Let \( C \) be a presentably symmetric monoidal \( \infty \)-category and \( X \in C \). We can form the free \( A_\infty \)-ring on \( X \) [Lur16, Proposition 4.1.1.14], denoted by

\[
\mathllbracket C, X \rrbracket \simeq \bigvee_{n \geq 0} X \wedge^n.
\]

This ring is homotopy commutative if and only if the switch map on \( X \wedge X \) is homotopic to the identity. This applies in particular if \( C = \text{MGL-Mod} \) and \( X = \text{MGL}(i)[j] \), provided that \( j \) is even. We may thus form the homotopy commutative ring and map of homotopy commutative rings

\[
(1) \quad \text{BPGL}[T_i | i \neq \ell^n - 1] := \text{BPGL} \otimes_{\text{MGL}} \bigotimes_i \text{MGL}[T_i] \to \text{MGL}(\ell),
\]

where \( T_i \) is in degree \( (2i, i) \) and is sent to \( t_i \in \pi_* \text{MGL}(\ell) \).

Lemma 3.8. The map (1) is an equivalence.

Proof. Let us denote the map by \( \alpha \). Since \( \alpha \) is a map of very effective homotopy \( \text{MGL}(\ell) \)-modules and \( \bigoplus_n (\text{MGL}(\ell)) \simeq \text{HZ}(\ell) \), it suffices to show that we \( \alpha \wedge \text{HZ}(\ell) \) is an equivalence. Since \( \text{MGL} \wedge \text{HZ} = \text{HZ}[\mathbb{Z}_2, \mathbb{Z}_2, \ldots] \) is a sum of objects of the form \( \text{HZ}(i)[2]^\infty \) for \( i \in \mathbb{Z} \), it suffices to show that \( \pi_{n,1}(\alpha \wedge \text{HZ}(\ell)) \) is an isomorphism for each \( i \in \mathbb{Z} \). Since \( \pi_{2,1}(\text{MGL} \wedge \text{HZ}) \simeq \pi_{2,1}(\mu_1 \wedge \text{HZ}) \) and similarly for \( \text{BPGL} \), our claim reduces to the classical analog, which is well-known. □

Lemma 3.9. We have

(1) \( \text{MGL}(\ell)/(t_1, t_2, \ldots) \simeq \text{HZ}(\ell) \),
(2) \( \text{MGL}(\ell)/(t_i | i \neq \ell^n - 1) \simeq \text{BPGL} \),
(3) \( \text{BPGL}/(v_1, v_2, \ldots) \simeq \text{HZ}(\ell) \),
(4) \( \text{MGL}(\ell)/(v_1, v_2, \ldots) \simeq \text{HZ}(\ell)/(i | i \neq \ell^n - 1) \).

Remark 3.10. Note that a priori we only obtain \( \text{BPGL} \) as an object of \( \mathcal{SH}(k) \), not \( \text{MGL-Mod} \). In particular (3) does not really make sense. However (2) exhibits a lift of \( \text{BPGL} \) to \( \text{MGL-Mod} \), allowing us to make sense of the iterated (infinite) quotient.

Proof. (1) This is a minor adaptation of the main theorem of [Hoy15]. It suffices to show that we have an equivalence after \( \wedge \text{HZ} \) and \( \wedge \text{HZ}/\ell \). The arguments of [Hoy15, Lemma 7.8 and Lemma 7.9] go through essentially unchanged (the key fact we use here is that our generators \( t_{\ell^n-1} \) are adequate in the sense of [Hoy15, Definition 7.1]).

(2) We claim that the composite \( \beta : \text{BPGL} \to \text{MGL}(\ell) \to \text{MGL}(\ell)/(t_i | i \neq \ell^n - 1) \) is an equivalence. As in the proof of Lemma 3.8, it suffices to prove that \( \beta \wedge \text{HZ}(\ell) \) is an equivalence. By construction, \( \text{HZ}(\ell)/\text{MGL}(\ell)/(t_i | i \neq \ell^n - 1) \) is a polynomial ring on generators in the same bidegrees as the \( v_i \). This implies that \( \text{HZ} \wedge \text{MGL}(\ell)/(t_i | i \neq \ell^n - 1) \) is a sum of objects of the form \( \text{HZ}(\ell)(i)[2]^\infty \) for \( i \in \mathbb{Z} \). Hence suffices to check that \( \text{HZ}_{2,1} \beta \) is an equivalence, whence the claim reduces to the classical analog, which is well-known.

(3) Follows from (1) and (2) since the formation of the quotients is independent of the order of elements being killed.

(4) Follows from (3) and Lemma 3.8. □

3.5. Ring structures on quotients of \( \text{MGL} \).

Definition 3.11. We call a field \( k \) \( \ell \)-good if \( cd(k) = 0 \) and \( k \) contains all \( \ell^n \)-th roots of unity for all \( n \).

Lemma 3.12. Suppose that \( k \) is \( \ell \)-good. Then

\[
\pi_* \text{MGL}_k^\ell \simeq L_{\ell}^\ell[\tau] \text{ and } \pi_* \text{MGL}/\ell^n \simeq L/\ell^n[\tau],
\]

where \( |\tau| = (0, -1) \).

Proof. The first statement follows from the second (for all \( n \)). Our assumptions imply that \( H^{**}(k, \mathbb{Z}/\ell^n) = \mathbb{Z}/\ell^n[\tau] \); this follows from the Bloch-Kato conjecture [Voe11, Theorem 6.17]. Noting that \( s_q(\text{MGL}) = \Sigma^{2q} \wedge_{\mathcal{H}} L_{2q} \) [Spi10, Corollary 4.7] (see also [Spi14, Theorem 3.1]), the result follows from the strongly convergent slice spectral sequence for \( \text{MGL}/\ell^n \) [Hoy15, Theorem 8.12], which degenerates at the \( E_1 \)-page without extension problems, since everything is concentrated in even degrees and every degree only has elements in a unique filtration. □
Definition 3.13. Let $C$ be a symmetric monoidal category. We say that $E \in C_1$ admits a homotopy left unital multiplication if there exists $m : E \otimes E \to E$ such that the following diagram commutes

$$
\begin{array}{ccc}
E \otimes E & \longrightarrow & E \otimes E \\
\downarrow & & \downarrow m \\
E & \longrightarrow & E.
\end{array}
$$

In the lemma below, we work with $C$ the category of $MGL$-modules:

Lemma 3.14. Let $x_1, \ldots, x_n \in \pi_{2\ast} MGL_\ast$.

(1) Suppose that $k$ is $\ell$-good. Then $MGL_\ast/(x_1, \ldots, x_n)$ admits a homotopy left unital $MGL$-algebra structure.

(2) Suppose that $x_1 = \ell^N$. Then there exists a finite separable extension $k' \mid k$ such that $MGL_\ast/(x_1, \ldots, x_n)|_{k'}$ admits a homotopy left unital $MGL$-algebra structure.

Proof. (1) It suffices to deal with $n = 1$. Note that by Lemma 3.12, $x = x_1$ is not a zero-divisor in $\pi_{2\ast} MGL_\ast$; in particular both $MGL_\ast$ and $MGL_\ast/x$ have homotopy groups concentrated in even (simplicial) degrees. We can now argue exactly as in [Str99, Lemma 3.2, Corollary 3.3, Lemma 3.4]: the evenness of $MGL_\ast$ implies that $[MGL_\ast/x, MGL_\ast/x]_{2\ast} \xrightarrow{\text{can}} [MGL_\ast, MGL_\ast/x]_{2\ast}$ is an injection, and in particular $x$ acts by zero on the left hand side (since it does so on the right hand side). It follows that $MGL_\ast/x \wedge MGL_\ast/x \simeq MGL_\ast/x \vee \Sigma MGL_\ast/x$ (non-canonically), and we can take a projection to the first summand as the multiplication map. Injectivity of $\alpha$ implies that any such multiplication is unital (even on both sides), as desired.

(2) Let $R = MGL_\ast/(x_1, \ldots, x_n)$. We need to find $m : R \otimes R \to R$. Let $k'/k$ be an algebraic separable $\ell$-good extension (e.g. a separable closure of $k$). Then $m' : R \otimes R|_{k'} \to R|_{k'}$ exists, by our assumptions and (1). Since $R \otimes R$ is a compact $MGL_{(i)}$-module, by continuity [Hoy15, Lemma A.7(1)] there exists a finite subextension $k'/l/k$ and $m : R \otimes R|_l \to R|_l$ such that $m|_{k'} \simeq m'$. Replacing $k$ by $l$ if necessary, we may thus assume that $m$ exists. We need $m$ to satisfy certain properties. They are expressed as commutativity of certain further diagrams of compact $MGL_{(i)}$-modules, and hold for $m|_{k'}$ by construction. Hence using continuity again, they hold after passing to a possible further finite subextension. This concludes the proof.

Example 3.15. The finite extension cannot be avoided in general. For example, if $k = \mathbb{R}$, then not every quotient $MGL_\ast/t_i$ can admit a homotopy left unital $MGL$-algebra structure. If they all did, then their $C_2$-equivariant realizations would all be homotopy MU_{2\ast}$-algebras, and it would follow that the Real Morava $K$-theories were $C_2$-equivariant homotopy MU_{2\ast}$-algebras. That this cannot be was pointed out by Hu-Kriz [HK01, p. 332].

In general, as the second author first learned from Xiaolin Danny Shi, it is impossible for $BP_{2\ast}/v_i$ to be a homotopy ring when $i > 0$. If this were possible, then the Postnikov truncation map $BP_{2\ast}/v_i \to \mathbb{F}_2(2) \to \mathbb{F}_2$ would induce a map of ordinary rings $\pi_* (\Phi(BP_{2\ast}/v_i)) \to \pi_* (\Phi(\mathbb{F}_2))$. Here, $\Phi$ is the $C_2$ geometric fixed points functor. The codomain of this map is a polynomial ring $\mathbb{F}_2[t]$, but the image consists of just the unit and $a^2$. In particular, the image is not a subring of the codomain.

4. THE TRANSFER OVER AN ADEQUATE FIELD

Consider the spectrum $MGL_{(i)} \in SH(k)$. By [BH21, Proposition 12.8], this is a normed spectrum, and by Lemma 3.4 we have $\pi_{2\ast} MGL_{(i)} \cong L_i \otimes \mathbb{Z}_{(i)}$. In Proposition 3.7 we chose generators of this latter ring such that

$$
\pi_{2\ast} MGL_{(i)} \cong \mathbb{Z}_{(i)}[t_1, t_2, \ldots].
$$

The map $\pi_{2\ast} BPGL \to \pi_{2\ast} MGL_{(i)}$ sends $v_i$ to $t_{n_i-1}$. When referring to elements of $\pi_{2\ast} MGL_{(i)}$, we will use $v_i$ and $t_{n_i-1}$ interchangeably.

Definition 4.1. Fix for the remainder of this section an integer $n > 0$. We call a field $k$ adequate if in $SH(k)$, the $MGL$-module $MGL/(\ell, v_1, \ldots, v_n)$ admits the structure of a homotopy left unital $MGL$-algebra.

Note that, by Lemma 3.14, every field $k$ admits a finite separable extension $k'$ that is adequate. Moreover $\ell$-good fields (such as separably closed fields) are adequate, and so are extensions of adequate fields.
Convention 4.2. We will assume for the remainder of this section that the field $k$ is adequate.

Definition 4.3. We define the normed spectrum $R$ (depending on the implicit integer $n$) to be

$$R = \text{MGL}(\ell)[v_n^{-1}]_{(\ell, v_1, v_2, \ldots, v_{n-1})}.$$ 

Note that $R$ is a normed spectrum by Corollary 3.5.

Our goal will be to understand the completed $R$-module

$$(R \wedge B\mu)^{\wedge}_{(\ell, v_1, v_2, \ldots, v_{n-1})}.$$ 

We will see that it is a free, finitely-generated $R$-module, and we will name a basis for it in terms of elements in $R_{2*}^{2*}(B\mu)$. If the field $k$ has a primitive $\ell$th root of unity, there is an equivalence of classifying spaces $B\mu \simeq BC_{\ell}$. We will end by giving (under this assumption) a formula for the stable transfer

$$R_{2*}^{2*} \to R_{2*}^{2*}(BC_{\ell}) \cong R_{2*}^{2*}(B\mu).$$

4.1. The homotopy of $R$. 

Definition 4.4. For a tuple of non-negative integers $K = (k_0, k_1, \ldots, k_{n-1})$, we define

$$M_K = R/(\ell, v_1^{k_1}, \ldots, v_{n-1}^{k_{n-1}}).$$

We will use $M$ to denote

$$M = M_{(1,1,\ldots,1)} \simeq R/(\ell, v_1, \ldots, v_{n-1}) \simeq \text{MGL}[v_n^{-1}]/(\ell, v_1, \ldots, v_{n-1}).$$

Since

$$M \simeq \text{MGL}/(\ell, v_1, \ldots, v_{n-1}) \otimes_{\text{MGL}} R,$$

the assumption that $k$ is an adequate field ensures that $M$ is a homotopy left unital $R$-algebra.

Lemma 4.5. Suppose $K$ is a tuple of non-negative integers. Then

$$\pi_{2*+i} M_K \cong (\pi_{2*+i} \text{MGL})[v_n^{-1}]/(\ell^{k_0}, v_1^{k_1}, \ldots, v_{n-1}^{k_{n-1}}),$$

and $\pi_{p,q} M_K = 0$ for $p < 2q$. Furthermore, if $K'$ is another tuple of integers such that $k'_i \geq k_i$ for all $0 \leq i \leq n - 1$, then the map $\pi_{2*+1, i} M_{K'} \to \pi_{2*+1, i} M_K$ is surjective.

Proof. For $0 \leq i \leq n - 1$, let $A_i$ denote $\text{MGL}[v_n^{-1}]/(\ell^{k_0}, \ldots, v_i^{k_i})$. Then $A_0 = \text{MGL}[v_n^{-1}]$ and $A_{n-1} = M_K$. We understand all the relevant homotopy groups of $A_0$ by Lemma 3.4, and we may inductively understand the homotopy groups of $A_{i+1}$ via the long exact sequence associated to the cofiber sequence

$$\Sigma^{k_i+1} v_{n+i} A_i \to A_i \to A_{i+1}.$$ 

To be explicit, we have the long exact sequence

$$\pi_{(p,q)+(i+1)(2,1)} A_i \xrightarrow{v_{n+i}^{\wedge i}} \pi_{p,q} A_i \to \pi_{p,q} A_{i+1} \to \pi_{(p-1,q)+(i+1)(2,1)} A_i \xrightarrow{v_{n+i}^{\wedge i}} \pi_{p-1,q} A_i.$$ 

The lemma follows from the inductive calculations

$$\pi_{p,q} A_i \cong 0 \text{ if } p < 2q,$$

$$\pi_{2*+i} A_i \cong L[v_n^{-1}]_{2*}/(\ell^{k_0}, v_1^{k_1}, \ldots, v_i^{k_i}),$$

and

$$\pi_{2*+1, i} A_i \cong (k^{k_i}/\ell^{k_0}) \otimes L[v_n^{-1}]_{2*+2}/(v_1^{k_1}, \ldots, v_i^{k_i}).$$

Proposition 4.6. We have the formula

$$\pi_{2*} R \cong (\pi_{2*} \text{MGL}[v_n^{-1}])_{(\ell, v_1, v_2, \ldots, v_{n-1})}.$$ 

In particular, none of $\ell, v_1, \ldots, v_{n-1}$ are 0-divisors inside of $\pi_{2*} R$.

Proof. By definition, $R$ is the homotopy limit of the $M_K$. The Milnor exact sequence [GJ09, Proposition VI.2.15] thus says that

$$0 \to \lim^1 \pi_{2*+1} M_K \to \pi_{2*} R \to \lim K \pi_{2*} M_K \to 0,$$

and the final clause of Lemma 4.5 guarantees that the $\lim^1$ term vanishes.

The following useful lemma is one of the main reasons we assumed the field $k$ to be adequate.
Lemma 4.7. Let \( x \in \pi_{2*,*} R \) be an element of the ideal \((\ell, v_1, v_2, \ldots, v_{n-1})\). Then the \( R \)-module homomorphism
\[
x : \Sigma^{|x|} R \rightarrow R
\]
becomes trivial after tensoring down to \( M \). In other words, the \( R \)-module and homotopy \( M \)-module map
\[
\Sigma^{|x|} M \simeq \Sigma^{|x|} R \otimes_R M \xrightarrow{\delta} R \otimes_R M \simeq M
\]
is nullhomotopic.

Proof. This map is a map of homotopy \( M \)-modules, and hence determined by a class in \( \pi_{|x|} M \). Thus, it suffices to check that \( x \) maps to 0 under the unit map
\[
\pi_{2*,*} R \rightarrow \pi_{2*,*} M.
\]
This is clear by construction. \( \square \)

4.2. The completed \( R \)-homology of \( BP_{\mu} \). Our chosen orientation of \( BPGL \) induces an orientation of \( R \), giving an isomorphism
\[
R^{**}(\mathbb{P}^\infty) \simeq R^{**}[\ell]
\]
as well as a formal group law. We denote by \([\ell](x) \in R^{**}[\ell]\) the \( \ell \)-series of this formal group law. Note that
\[
[\ell](x) = \ell x + \cdots
\]
is a power series in \( x \), congruent to \( \ell x \) modulo \( x^2 \), and with coefficients in \( BPGL_{2*,*} \).

Lemma 4.8. There is a cofiber sequence of \( MGL \)-modules
\[
BP_{\mu_+} \wedge MGL \rightarrow \mathbb{P}^\infty \wedge MGL \xrightarrow{[\ell]} \mathbb{P}^\infty \wedge MGL(1)[2].
\]
The map \( \mathbb{P}^\infty \wedge MGL \xrightarrow{[\ell]} \mathbb{P}^\infty \wedge MGL(1)[2] \) is determined by its dual, which in \( MGL^{**}(\mathbb{P}^\infty) \) is multiplication by \([\ell](x)\).

Proof. The geometric construction of \( BP_{\mu} \) shows that this space can be obtained as \( O(-\ell)_{\mathbb{P}^\infty} \setminus 0 \) \cite[Lemma 6.3]{V}, and consequently there is a cofiber sequence
\[
BP_{\mu} \simeq O(-\ell)_{\mathbb{P}^\infty} \setminus 0 \rightarrow O(-\ell)_{\mathbb{P}^\infty} \rightarrow Th(O(-\ell)_{\mathbb{P}^\infty}).
\]
Smashing with \( MGL \) and using the Thom isomorphism we obtain a cofiber sequence of \( MGL \)-modules
\[
BP_{\mu_+} \wedge MGL \rightarrow \mathbb{P}^\infty \wedge MGL \rightarrow \mathbb{P}^\infty \wedge MGL(1)[2].
\]
The final statement comes from the relation of the Euler class of \( O(-\ell) \) with the \( \ell \)-series in the formal group law, which is essentially the definition of the formal group law. \( \square \)

Lemma 4.9. There is an equivalence of \( R \)-modules
\[
(R \wedge \Sigma^\infty BP_{\mu})_{\ell, v_1, \ldots, v_{n-1}} \simeq \bigvee_{k=0}^{\ell-1} \Sigma^{2k+1} R.
\]

Proof. Lemma 4.8 implies that there is a cofiber sequence
\[
BP_{\mu_+} \wedge R \rightarrow \mathbb{P}^\infty \wedge R \xrightarrow{[\ell]} \mathbb{P}^\infty \wedge R(1)[2].
\]
Here \( \mathbb{P}^\infty \wedge R \simeq \bigvee_{i \geq 0} R_\beta \), where \( \beta \) in bidegree \((2i, i)\) is dual to \( x^i \), and the map \([\ell] \) is dual to multiplication by the power series \([\ell](x) \). Let \( M \) denote \( M(1,1,\ldots,1) \). The key property of our preferred generators \( t_i \) of \( L_+ \otimes \mathbb{Z}(\ell) \) is that
\[
[\ell](x) \equiv v_n x^\ell \mod (x^{\ell+1}, \ell, v_1, v_2, \ldots, v_{n-1}).
\]
This implies, using Lemma 4.7, that \([\ell] \otimes_R M \) is the homotopy \( M \)-module map specified by
\[
\begin{align*}
\beta_0 &\mapsto 0 \\
\beta_1 &\mapsto 0 \\
\beta_{2\ell-1} &\mapsto 0 \\
\beta_{2\ell+1} &\mapsto v_{n}\beta_0 \\
\beta_{2\ell+2} &\mapsto v_{n}\beta_1+?\beta_0 \\
&\vdots
\end{align*}
\]
Since $v_0$ is a unit in $R$, it follows that if we split $M \wedge \mathbb{P}^\infty = P_1 \vee P_2$, where $P_1 = \bigvee_{k=0}^{n-1} M\{\beta_k\}$ and $P_2 = \bigvee_{k \geq n} M\{\beta_k\}$, then the restriction of $[\ell] \otimes_R M$ to $P_1$ is zero whereas the restriction to $P_2$ is an isomorphism $P_2 \to M \wedge \mathbb{P}^\infty$. In particular, if we similarly set $R \wedge \mathbb{P}^\infty = Q_1 \vee Q_2$, where $Q_1 = \bigvee_{k=0}^{n-1} R$ and $Q_2 = \bigvee_{k \geq n} R$, then the restriction of $[\ell]$ to $Q_2$ is an equivalence onto $R \wedge \mathbb{P}^\infty$ after completion at the ideal $(\ell, v_1, \ldots, v_{n-1})$. It follows that

\[ (R \wedge \mathbb{P}^\infty \mathcal{B} \mu_\ell)^{[\ell], v_1, \ldots, v_{n-1}} \cong Q_1. \]

\[ \square \]

4.3. The $R$-cohomology of $\mathcal{B} \mu_\ell$. We calculate $R^{2*}([\ell]; \mathcal{B} \mu_\ell)$ by copying the classical argument for $E^*([\ell] \mathcal{B} C_\ell)$, where $E$ is a complex-oriented ring spectrum [HKR00, §5.4].

Lemma 4.10. In the ring $R^{2*}[x]$, we have $[\ell](x) = ux$, where $u$ is a unit and
g(x) = a_{n-1} x^{n-1} + \cdots + \ell x,
for some coefficients $a_i \in R^{2*}$ such that $a_{n-1}$ is a unit.

Proof. The key property of our preferred generators $t_i$ of $L_2 \otimes \mathbb{Z}(\ell)$ is that

\[ [\ell](x) \equiv t_{\ell-1} x^{\ell} \mod (x^{\ell+1}, \ell, t_1, \ldots, t_{\ell-2}). \]

Since $t_{\ell-1}$ is a unit in $R_{2*}$, we may apply the Weierstrass Preparation Theorem [O’M72, Theorem 2.10] to the power series \( \frac{d(x)}{x} \), which yields an expression

\[ \frac{[\ell](x)}{x} = h(x)v, \]

where $v$ is a unit in $R^{2*}[x]$ and $h(x)$ is a monic degree $\ell^n$ polynomial with coefficients in $R^{2*}$ and constant term $a_{n-1} \ell$ for some unit $a_{n-1} \in R^{2*}$. We define $g(x) = a_{n-1} x h(x)$ and $u = a_{n-1} \ell$. This concludes the proof.

\[ \square \]

Corollary 4.11. There are canonical isomorphisms

\[ R^{2*}([\ell]; \mathcal{B} \mu_\ell) \cong R^{2*}([\ell]; \mathbb{P}^\infty) \cong R^{2*}[x]/g(x). \]

In particular, $R^{2*}([\ell]; \mathcal{B} \mu_\ell)$ is a finite free $R^{2*}$-module with basis $1, x, \ldots, x^{n-1}$.

Proof. The first statement follows from Lemma 4.8 via the long-exact sequence

\[ R^{2*}([\ell]; \mathcal{B} \mu_\ell) \cong R^{2*}([\ell]; \mathbb{P}^\infty) \cong R^{2*}[x]/g(x). \]

The key point is that $R^{2*}([\ell]; \mathcal{B} \mu_\ell)$ is a finite free $R^{2*}$-module with basis $1, x, \ldots, x^{n-1}$.

The second statement then follows from Lemma 4.10.

\[ \square \]

4.4. The transfer and its section. For this section, we assume furthermore that $k$ contains a primitive $\ell$-th root of unity. Thus $\mu_\ell \cong C_\ell$.

Our first goal will be to determine a formula for the stable transfer

\[ R \to R^{B C_\ell}. \]

Specifically, applying $\pi_{2*}$ gives an $R^{2*}$-linear map

\[ \text{Tr}(-) : R^{2*} \to R^{2*}[x]/g(x) \cong R^{2*}[x]/[\ell](x) \cong R^{2*}[x]/g(x), \]

and we seek a formula for this homomorphism.

Lemma 4.12. The stable transfer map

\[ \text{Tr}(-) : R^{2*} \to R^{2*}[x]/g(x) \]

is the unique $R^{2*}$-linear homomorphism such that

\[ \text{Tr}(1) = \frac{g(x)}{x}. \]

Proof. The argument is exactly analogous to that in [HKR00, 6.15]; we recall it for the convenience of the reader. Note that the restriction map $R^{2*}([\ell] \mathcal{B} C_\ell) \to R^{2*}([\ell] \mathcal{B} e)$ is the $R^{2*}$-algebra homomorphism sending $x$ to $0$. The projection formula of Lemma 2.5 thus implies that $x \text{Tr}(1) = 0$. Corollary 4.11 implies via a straightforward calculation that the only classes in $R^{2*}([\ell] \mathcal{B} C_\ell)$ that are killed by $x$ are the $R^{2*}$-multiples of $\frac{g(x)}{x}$. Thus

\[ \text{Tr}(1) = r \frac{g(x)}{x}. \]
for some element \( r \in R^{2n} \). Since the constant term of \( g(x)/x \) is \( \ell \), Lemma 2.5(3) implies that \( r\ell = \ell \).

By Proposition 4.6, \( \ell \) is not a 0-divisor in \( \pi_{2n} \ast R \), and it follows that \( r = 1 \). This concludes the proof. \( \square \)

Recall that the stable transfer arises from a map
\[
\Sigma^\infty_+ BC_\ell \to \mathbb{1}.
\]
After tensoring this map with \( R \), we obtain a map
\[
R \wedge \Sigma^\infty_+ BC_\ell \to R.
\]
Since the codomain \( R \) is complete, we may complete the domain of the above to obtain
\[
(R \wedge \Sigma^\infty_+ BC_\ell)_{\ell, v_1, \ldots, v_{n-1}} \to R.
\]

**Corollary 4.13.** There is an \( R \)-module homomorphism
\[
\gamma : R \to (R \wedge \Sigma^\infty_+ BC_\ell)_{\ell, v_1, \ldots, v_{n-1}}
\]
such that the composite of \( \gamma \) with the transfer
\[
(R \wedge \Sigma^\infty_+ BC_\ell)_{\ell, v_1, \ldots, v_{n-1}} \to R
\]
is the identity on \( R \).

**Proof.** Since \((R \wedge BC_\ell)_{\ell, v_1, \ldots, v_{n-1}}\) is a finitely generated free \( R \)-module, we may specify the map \( \gamma \) by specifying its \( R \)-linear dual. This dual element is specified by a class in \( R^{2n+1} \) generated by \( g(x)/g(x) \), which we know to be a free \( R^{2n+1} \)-module generated by \( 1, x, \ldots, x^{\ell-1} \). Recall that \( g(x) = a_{\ell} x^\ell + a_{\ell-1} x^{\ell-1} + \cdots + a_1 x + a_0 \) is a unit in \( R^{2n+1} \). We choose \( \gamma \) to be dual to the class \( a_{-\ell} x^{\ell-1} \). The result then follows from the formula
\[
\text{Tr}(1) = \frac{g(x)}{x},
\]
the fact that the coefficient of \( x^{\ell-1} \) in \( \frac{g(x)}{x} \) is \( a_{-\ell} \), and the meaning of specifying a map via its dual. \( \square \)

The following is the main result of this section.

**Corollary 4.14.** Let \( k \) be an adequate field (see Definition 4.4) of exponential characteristic \( e \neq \ell \) containing a primitive \( e \)-th root of unity.

Let \( A \in \mathbb{N}Alg(SH(S))_{R/} \), and assume that \( \ell^k = 0 \in \pi_0(A) \) for some \( k \geq 0 \). Then \( A \cong 0 \).

**Proof.** If \( k = 0 \), then this is just the statement that a ring \( A \) is zero if and only if \( 1 = 0 \in \pi_0(A) \).

In general, it will suffice by induction to prove that \( \ell^{k-1} = 0 \in \pi_0(A) \). Using Corollary 2.10 (and naturality of norms) we find that
\[
0 = N(\ell^k) = -\ell^{k-1}\text{Tr}(1) \in A^0(BC_\ell).
\]
This implies that the composite
\[
\Sigma^\infty_+ BC_\ell \xrightarrow{\text{Tr}(1)} \mathbb{1} \xrightarrow{R} \ell^{k-1} A
\]
is nullhomotopic, where the last map is \( \ell^{k-1} \) times the unit map. Using the fact that \( R \) is complete, we obtain that the composite of \( R \)-module maps
\[
(R \wedge \Sigma^\infty_+ BC_\ell)_{\ell, v_1, \ldots, v_{n-1}} \xrightarrow{\text{Tr}(1)} R \xrightarrow{\ell^{k-1}} A
\]
is null. Consequently
\[
0 = \ell^{k-1}\text{Tr}(1)\gamma = \ell^{k-1} \in A^0,
\]
making use of Corollary 4.13. \( \square \)

5. Detecting zero rings

5.1. Detection in abstract \( \infty \)-categories. Recall the notion of a homotopy left unital ring from Definition 3.13.

**Lemma 5.1.** Let \( E \in \mathcal{C} \) carry a homotopy left unital multiplication. Then \( E \cong 0 \) if and only if the unit map \( 1 : \mathbb{1} \to E \) is null homotopic.

**Proof.** By definition, the identity map on \( E \) factors as \( E \cong \mathbb{1} \wedge E \xrightarrow{1 \wedge \text{Id}} E \wedge E \xrightarrow{m} E \) and hence is null homotopic as soon as \( 1 \) is. The converse is clear. \( \square \)

**Lemma 5.2.** Let \( \mathcal{C} \) be a stable symmetric monoidal \( \infty \)-category which has filtered colimits compatible with \( \otimes \). Let \( \{L_i \in \mathcal{C}\}_{i} \) be invertible objects and \( \{x_i : L_i^{-1} \to \mathbb{1}\} \) maps. Put

\[
A_n := 1/(x_1, \ldots, x_n) = \mathbb{1}/x_1 \otimes \cdots \otimes \mathbb{1}/x_n;
\]
Proof. (1) Let us show that 

\[ A_\infty := \mathbb{I}/(x_1, x_2, \ldots) = \text{colim}_n \mathbb{I}/(x_1, \ldots, x_n); \]

\[ B_{n+1} := \mathbb{I}/(x_1, \ldots, x_n)[x_{n+1}^{-1}] = \mathbb{I}/(x_1, \ldots, x_n) \otimes [x_{n+1}^{-1}]. \]

Let \( E \in \mathcal{C} \).

(1) If \( E \land A_M \simeq 0 \) and \( E \land B_{n+1} \simeq 0 \) for \( n = N, N+1, \ldots, M-1 \). Then \( E \land A_N \simeq 0 \).

(2) Suppose that \( \mathbb{I} \in \mathcal{C} \) is compact and both \( E \) and each \( A_n \) (for \( n \geq N \)) admit a homotopy left unital multiplication. Suppose further that \( (i) E \land A_\infty \simeq 0 \) and \( (ii) E \land B_{n+1} \simeq 0 \) for \( n = N, N+1, N+2, \ldots \). Then \( E \land A_N \simeq 0 \).

Proof. (1) Let us show that \( E \land A_{n+1} \simeq 0 \) and \( E \land B_{n+1} \simeq 0 \) together imply \( E \land A_n \simeq 0 \). This will imply the result by induction. We have the cofiber sequence

\[ L^{-1}_{n+1} \land E \land A_n \xrightarrow{x_{n+1} \land id_E \land id_{A_n}} E \land A_n \rightarrow \mathbb{I}/x_{n+1} \land E \land A_n \simeq E \land A_{n+1} \simeq 0, \]

from which we conclude that \( x_n \land id_E \land id_{A_n} \) is an equivalence. By definition, \( E \land B_{n+1} = E \land A_n \land \mathbb{I}[x_{n+1}^{-1}] \) is obtained as the filtered colimit of

\[ E \land A_n \xrightarrow{x'_{n+1} \land id_E \land id_{A_n}} L^{-1}_{n+1} \land E \land A_n \xrightarrow{id_{L^{-1}_{n+1}} \land x'_{n+1} \land id_E \land id_{A_n}} L^{-2}_{n+1} \land E \land A_n \rightarrow \ldots, \]

where

\[ x'_{n+1} = id_{L^{-1}_{n+1}} \land x_{n+1} : \mathbb{I} \simeq L^{-1}_{n+1} \land L^{-1}_{n+1} \rightarrow L^{-1}_{n+1} \land L^{-1}_{n+1} \simeq L^{-1}_{n+1}. \]

Consequently, this is a filtered colimit along equivalences, and the colimit (which is 0 by assumption) is equivalent to any one object. Hence \( E \land A_n \simeq 0 \), as desired.

(2) Let \( C, C' \in \mathcal{C} \) have homotopy left unital multiplications. Then so does \( C \otimes C' \). If \( C_1, C_2, \ldots \in \mathcal{C} \) have homotopy left unital multiplications then so does \( C_\infty := \text{colim}_i C_i \). In particular, \( C_\infty \simeq 0 \) if and only if \( C_i \simeq 0 \) for some \( n \) (using that \( \mathbb{I} \) is compact and Lemma 5.2).

We deduce that \( A_n, A_\infty, E \land A_n, E \land A_\infty \) have homotopy left unital multiplications, and \( E \land A_n \simeq 0 \) for \( n \) sufficiently large. If \( n \leq N \) then the unit \( \mathbb{I} \rightarrow E \land A_n \) factors as \( \mathbb{I} \rightarrow E \land A_n \rightarrow E \land A_N \). Since the middle term is zero we conclude that \( E \land A_N \simeq 0 \) as desired. We may thus assume that \( n > N \). We have reduced to (1). This concludes the proof.

5.2. Detection in MGL-modules. In this subsection we assume again that all spectra are \( \ell \)-periodized, where \( \ell \neq e \) is the exponential characteristic of \( k \).

Lemma 5.3. Let \( E \in SH(k) \) be a homotopy left unital ring spectrum with \( E \land HZ/\ell = 0 \) and also \( E \land MGL[\mathfrak{v}_i^{-1}]/(\ell, v_1, \ldots, v_i) = 0 \) for all \( i \geq 0 \).

(1) If \( k \) is \( \ell \)-good, then \( E \land MGL/\ell \simeq 0 \).

(2) In general, there exists a finite separable extension \( k'/k \) such that \( E \land MGL/\ell|_{k'} \simeq 0 \).

Proof. By Lemma 3.9(4) we know that \( E \land MGL/(\ell, v_1, \ldots) \simeq 0 \). It follows that the unit map \( \mathbb{I} \rightarrow E \land MGL/(\ell, v_1, \ldots, v_N) \) is null, for \( N \) sufficiently large. By Lemma 3.14, possibly after replacing \( k \) by a finite separable extension, we may assume that \( MGL/(\ell, v_1, \ldots, v_N) \) has a homotopy left unital multiplication. Hence by Lemma 5.1 we know that \( E \land MGL/(\ell, v_1, \ldots, v_N) \simeq 0 \). Thus we conclude by Lemma 5.2(1).

5.3. Detecting zero normed spectra.

Lemma 5.4. Let \( K/k \) be a (not necessarily algebraic) separable field extension and \( E \in NAlg(SH(k)) \) with \( E|_{K} \simeq 0 \). Then \( E \simeq 0 \).

Proof. It is enough to show that \( 1 = 0 \in \pi_0(E) \). Since the functor \( K \mapsto \pi_0(E|_K) \) preserves filtered colimits [Hoy15, Lemma A.7(1)], we may assume that \( K/k \) is finitely generated. We may further reduce (by induction) to either a simple purely transcendental extension or a finite separable extension. For the simple purely transcendental case, we use that

\[ \pi_0(E) = \pi_{0,0}(E)(k) \simeq \pi_{0,0}(E)(k[x]) \rightarrow \pi_{0,0}(E)(k(x)) = \pi_0(E|_{K(x)}), \]

i.e. homotopy invariance and unramifiedness of homotopy sheaves of motivic spectra [Mor05, Theorem 6.2.7 and Lemma 6.4.4]. Finally if \( \ell/k \) is finite separable and \( E|_{\ell} \simeq 0 \) then the norm \( N_{\ell/k}(E|_{\ell}) \rightarrow E \) [BH21, §7.2] is a ring map with vanishing source and hence vanishing target. This concludes the proof.
5.4. Proof of the main theorem. We first prove a version of our main theorem over fields.

**Theorem 5.5.** Let $k$ be a field, $\ell$ a prime invertible in $k$, and $E \in \text{NAlg}(\mathcal{SH}(S))$. Suppose that $\ell^n = 0 \in \pi_0(E)$ for some $n$, and also that

$$E \wedge \mathbb{H}/\ell \simeq 0.$$  

Then also

$$E \wedge \text{MGL}/\ell \simeq 0.$$  

**Proof.** It is necessary and sufficient to show that $(E \wedge \text{MGL})^\ell \simeq 0$. Let

$$R_i = \text{MGL}(\ell^{i-1}(v_1, \ldots, v_i)),$$

for each $i$. There exists a finite separable extension $k_i/k$ containing a primitive $\ell$-th root of unity which is adequate (in the sense of Definition 4.4) for $R_i$. Since base change along a finite separable extension commutes with completion (being a right adjoint), $R_i$ is stable under finite separable base change, and hence $R_i \wedge E|_{k_i} \simeq 0$ by Corollary 4.14. Thus by Lemma 5.4 we find that $R_i \wedge E \simeq 0$, and hence $\text{MGL}^{\ell^{i-1}}((\ell, v_1, \ldots, v_i)) \wedge E \simeq 0$ for all $i$. By Lemma 5.3, there exists a further finite separable extension $k'/k$ such that $E \wedge \text{MGL}/\ell|_{k'} \simeq 0$, whence $(E \wedge \text{MGL})^\ell|_{k'} \simeq 0$. We conclude by a final application of Lemma 5.4. \hfill \square

One may prove that normed spectra are stable under arbitrary base change; this is explained in the forthcoming work [BEH22]. Using this, the above easily implies our result over more general bases.

**Theorem 5.6.** Let $S$ be a noetherian scheme of finite dimension, $\ell$ a prime invertible on $S$, and $E \in \text{NAlg}(\mathcal{SH}(S))$. Suppose that $\ell^n = 0 \in \pi_0(E)$ for some $n$, and also that

$$E \wedge \mathbb{H}/\ell \simeq 0.$$  

Then also

$$E \wedge \text{MGL}/\ell \simeq 0.$$  

**Proof.** Pullback to fields is conservative [Bac18, Corollary 14], formation of normed spectra is compatible with arbitrary base change [BEH22], and MGL is stable under base change (since the Grassmannians it is built out of are). \hfill \square

**Theorem 5.7.** Let $S$ be a noetherian scheme of finite dimension, and write $\mathcal{S}$ for the set of primes not invertible on $S$. Let $E \in \text{NAlg}(\mathcal{SH}(S))$ and suppose that

$$E \wedge \mathbb{H}[S^{-1}] \simeq 0.$$  

Then also

$$E \wedge \text{MGL}[S^{-1}] \simeq 0.$$  

**Proof.** It suffices to show that for every $\ell \not\in \mathcal{S}$ we have $E \wedge \text{MGL}(\ell) \simeq 0$, whence replacing $E$ by $E \wedge \text{MGL}(\ell)$ we may assume that $E$ is an $\ell$-local MGL-module. Under this assumption $E[1/\ell] \simeq E_{\mathbb{Q}} \simeq E \wedge \mathbb{Q} \simeq 0$, and hence $\ell^n = 0 \in \pi_0(E)$ for some $n$. Since also $E \wedge \mathbb{H}/\ell \simeq 0$, the theorem implies that $E \wedge \text{MGL}/\ell \simeq 0$. From this we deduce that also $E \wedge \text{MGL}/\ell^n \simeq 0$ for all $N$. Finally since $\ell^n = 0$ in $E$, $E \wedge \text{MGL}$ is a wedge summand of $E \wedge \text{MGL}/\ell^n$, which is zero. This concludes the proof. \hfill \square

**References**

[Bac18] Tom Bachmann. Motivic and real étale stable homotopy theory. *Compositio Mathematica*, 154(5):883–917, 2018. arXiv:1608.08855.

[Bac22] Tom Bachmann. Motivic spectral mackey functors. arXiv:2205.13926, 2022.

[BEH21] Tom Bachmann, Elden Elmanto, and Jeremiah Heller. Motivic colimits and extended powers. arXiv:2104.01057, 2021.

[BEH22] Tom Bachmann, Elden Elmanto, and Jeremiah Heller. Splitting results for normed spectra. arXiv:2104.01057, 2022.

[BH21] Tom Bachmann and Marc Hoyois. Splitting results for normed spectra. in preparation, 2022.

[DHS88] Ethan S Devinatz, Michael J Hopkins, and Jeffrey H Smith. Nilpotence and stable homotopy theory i. *Annals of Mathematics*, 128(2):207–241, 1988.

[GH19] David Gepner and Jeremiah Heller. The tom dieck splitting theorem in equivariant motivic homotopy theory. *Journal of the Institute of Mathematics of Jussieu*, pages 1–70, 2019.

[Ghe17] Bogdan Gheorghe. The motivic collier of $\tau$. *arXiv preprint arXiv:1701.04877*, 2017.

[GJ09] Paul Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[GWX] Bogdan Gheorghe, Guozhen Wang, and Zhong Xu. The special fiber of the motivic deformation of the stable homotopy category is algebraic (2018). arXiv:1809.09290.

[Hah16] Jeremy Hahn. On the Bousfield classes of $H_n$-ring spectra. *arXiv preprint arXiv:1612.04386*, 2016.

[HK01] Po Hu and Igor Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology*, 40(2):317–399, 2001.
[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[Hoy15] Marc Hoyois. From algebraic cobordism to motivic cohomology. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015(702):173–226, 2015.

[Hoy17] Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Advances in Mathematics*, 305:197–279, 2017.

[Kra18] Achim Krause. Periodicity in motivic homotopy theory and over $\mathbb{BP}^*\mathbb{BP}$. *PhD thesis, Max Planck Institute for Mathematics*, 2018.

[Lur09] Jacob Lurie. *Higher topos theory*. Number 170. Princeton University Press, 2009.

[Lur16] Jacob Lurie. Higher algebra, May 2016.

[MNN17] A. Mathew, N. Naumann, and J. Noel. Nilpotence and descent in equivariant stable homotopy theory. *Advances in Mathematics*, 305:994–1084, 2017.

[Mor05] Fabien Morel. The stable $\mathbb{A}^1$-connectivity theorems. *K-theory*, 35(1):1–68, 2005.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 90(1):45–143, 1999.

[O’M72] Matthew O’Malley. On the weierstrass preparation theorem. *The Rocky Mountain Journal of Mathematics*, 2(2):265–273, 1972.

[Pst18] Piotr Pstragowski. Synthetic spectra and the cellular motivic category. *arXiv preprint arXiv:1803.01804*, 2018.

[Rav86] Douglas C Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121. Academic press New York, 1986.

[ Spi10] Markus Spitzweck. Relations between slices and quotients of the algebraic cobordism spectrum. *Homology, Homotopy and Applications*, 12(2):335–351, 2010.

[ Spi12] Markus Spitzweck. A commutative $S^1$-spectrum representing motivic cohomology over dedekind domains. *arXiv preprint arXiv:1207.4078*, 2012.

[ Spi14] Markus Spitzweck. Algebraic cobordism in mixed characteristic. 2014.

[Str99] Neil Strickland. Products on MU-modules. *Transactions of the American Mathematical Society*, 351(7):2569–2696, 1999.

[Vez01] Gabriele Vezzosi. Brown-peterson spectra in stable $\mathbb{A}^1$-homotopy theory. *Rendiconti del Seminario Matematico della Università di Padova*, 106:47–64, 2001.

[Voe03] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 98:1–57, 2003.

[Voe11] Vladimir Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Annals of mathematics*, 174(1):401–438, 2011.

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