Stochastic effects of waves on currents in the ocean mixed layer

Darryl D. Holm and Ruiao Hu
Department of Mathematics
Imperial College London SW7 2AZ, UK
d.holm@ic.ac.uk and ruiao.hu15@ic.ac.uk

Abstract
This paper introduces an energy-preserving stochastic model for studying wave effects on currents in the ocean mixing layer. The model is called stochastic forcing by Lie transport (SFLT). The SFLT model is derived here from a stochastic constrained variational principle, so it has a Kelvin circulation theorem. The examples of SFLT given here treat 3D Euler fluid flow, rotating shallow water dynamics and the Euler-Boussinesq equations. In each example, one sees the effect of stochastic Stokes drift and material entrainment in the generation of fluid circulation. We also present an Eulerian-averaged SFLT model (EA SFLT), based on decomposing the Eulerian solutions of the energy-conserving SFLT model into sums of their expectations and fluctuations.

1 Introduction

Wave effects on currents (WEC). In studies of the ocean mixing layer (OML) the problem of wave effects on currents (WEC) arises. For example, surface gravity waves can drive Langmuir circulations which have important influences on near-surface currents [50]. Langmuir circulations are horizontally oriented pairs of oppositely circulating vortex tubes aligned generally along the direction of the wind, as reviewed, e.g., in [46, 26, 70, 24]. Because they represent arrays of organised fluid transport, Langmuir circulation patterns can produce vertical transport which strongly entrains sediment and detritus into the OML from both above and below. This means they can have a strong effects, for example, on the dispersion of oil spills in a shallow sea, [69].

Most previous studies of turbulence in the OML have been performed in the context of wave-averaged dynamics. For a recent review, see [1]. The underlying assumption is that the surface gravity waves represent the fastest component in the system and are only weakly modulated by the other components (turbulence and currents). Averaging the stratified Euler equations in three spatial dimensions over a time scale longer than the wave period produces a modified set of equations, known as the CL equations, after Craik and Leibovich, [15].

The temporal-averaging basis of the CL equations introduces Stokes-drift effects which represent additional wave-averaged forces and material advection terms that emerge from multi-scale asymptotic theories [15, 61]. Remarkably, the ideal CL equations preserve many of the properties of the original stratified Euler equations. For example, the ideal CL equations conserve energy and have a Hamiltonian formulation [34].

The aim of the present paper is to develop a stochastic theory of WEC which encompasses the Craik-Leibovich theory and still preserves the original energy. Indeed, in deriving the stochastic theory, we will take an approach which can be adapted to select whichever primary conservation laws of the deterministic theory are desired. Specifically, our approach will use a stochastic version of a method from classical mechanics known as the Reduced Lagrange-d’Alembert-Pontryagin (RLDP) formulation of constrained dynamics which is reduced by a symmetry of the Lagrangian in Hamilton’s principle. Stochastic applications of the RLDP formulation are discussed, e.g., in [10, 27].

RLDP result for the stochastic Craik-Leibovich theory for the Euler-Boussinesq equations. Specific results of the theoretical developments in this paper can be assessed by simply examining the example of applying the RLDP approach to the ideal CL equations themselves in section 4.3.

One finds in this example that the energy-preserving stochastic EB equations in (4.31) produce a stochas-
tic contribution to the vortex force in the Craik-Leibovich equations [15] whose deterministic formulation with Hamilton’s principle is given in [34]. Namely, they reduce as follows, where \( du \) denotes the Stratonovich stochastic time differential of Eulerian fluid velocity \( u \), the Stokes drift velocity is \( u^S(x) \), the Coriolis parameter is \( 2\Omega = \text{curl} \mathbf{R}(x) \), the pressure is \( p \), the volume element is \( D \), the buoyancy is \( b \) with gravitational constant \( g \), and the stochastic spatial modes obtained via data calibration are denoted \( f^k(x) \).

\[
du - u \times \text{curl}(u - u^S(x) + R(x))dt = - \nabla \left( p + \frac{1}{2}u^2 + u \cdot u^S \right)dt - gb \hat{z} \, dt + \sum_{k>0} \left( u \times \text{curl} f^k - \nabla (u \cdot f^k) \right) \circ dW^k_t, \tag{1.1}
\]

where one interprets the semimartingale \( du^S = u^S(x) \, dt + \sum_k f^k(x) \circ dW^k_t \) as a stochastic augmentation of the usual steady prescribed Stokes drift velocity and one recovers the deterministic Craik-Leibovich equations when the stochastic terms proportional to \( dW^k_t \) are absent.

The energy-preserving stochastic EB equations in (4.31) imply the following equation for potential vorticity density, defined by \( q := (\text{curl} \mathbf{m}) \cdot \nabla = (\omega + 2\Omega) \cdot \nabla b \), where \( \mathbf{m} = u + R \) for \( D = 1 \). Namely,

\[
dq + u \cdot \nabla q \, dt = - \sum_k \text{div} J^k \circ dW^k_t, \tag{1.2}
\]
in which the “J-fluxes of PV” on the right-hand side are discussed, e.g., in [32, 56]. See also [8] for LES turbulence interpretations of these fluxes.

In summary, while the energy-preserving stochastically-augmented CL vortex force and entrainment effects in this example can locally create stochastic Langmuir circulations, the total volume-integrated potential vorticity \( Q = \int q \, dV \) will be preserved for appropriate boundary conditions.

### 1.1 Motivating question and main results of the paper

The present paper addresses the geometric interplay between energy and circulation, when stochasticity is introduced into fluid dynamics, in both the incompressible flow of an ideal Euler fluid and in the flows of ideal fluids with advected quantities. This is a burgeoning area of research in fluid dynamics. For recent introductory surveys of stochastic fluid dynamics with applications, see, e.g., [17, 23, 28].

The question underlying the present work is, “What types of noise perturbations can be added to fluid dynamics which will preserve its fundamental properties of energy conservation, Kelvin-Noether circulation dynamics and conserved properties resulting from invariance under Lagrangian particle relabelling?” Since these properties all arise from the geometric structure of fluid dynamics, the noise perturbations we consider will be introduced in a geometrical framework.

The RLDP formulation of fluid dynamics leads to a constrained variational principle for stochastic ideal fluid dynamics through which noise may be introduced as a prescribed stochastic force. By choosing an appropriate form of the stochastic force, the RLDP formulation can be designed to preserve whatever conservation law one may desire among those of the deterministic ideal fluid equations in any number of dimensions.

For applications in fluid dynamics, the RLDP formulation results in the procedure mentioned above, called Stochastic Forcing by Lie Transport (SFLT). This is constructed by choosing the stochastic forces to Lie derivative of the transport velocity vector field acting on some prescribed one-form density which conserves the ideal fluid energy. Potential applications include a stochastic version of the Craik-Leibovich (CL) vortex force which presumably could generate stochastic Langmuir circulation. This result is introduced for the Euler fluid equations for 3D incompressible flow in Example 2.1 and Proposition 2.2. The corresponding theory for ideal flows in general is also treated in section 2.2 where stochastic advected quantities are introduced. Using RLDP, the existence of Kelvin-Noether theorem is automatic due to the variational nature of the formulation. When the advection of the fluid density is assumed, the Kelvin circulation theorem becomes a simple corollary of the Kelvin-Noether theorem as proven in Theorem 2.7.

**SFLT has dual design capabilities.** Besides adding stochastic forces which drive the fluid motion equation, the SFLT framework can also be designed to distribute stochastic sources in the advective transport equations,
as discussed in section 2.2. The stochastic sources distributed in the advective transport equations by SFLT can be designed to model, for example, stochastic changes in the material properties of inertial fluid parcels which may be embedded in the flow. In particular, one can use SFLT to model the stochastic dynamics of a mixture of heavier and lighter parcels whose fluid paths deviate from passive tracers, which are carried by the drift flow velocity. In this case, the mass density would be changing stochastically in the material frame of the flow drift velocity, because of entrainment or detrainment of sediments. One can observe the entrainment or detrainment of sediments in Langmuir circulations, see for example, [50]. One can also imagine using SFLT to model the transport of a stochastically evolving, spatially distributed, algae bloom which has a distribution of shapes, so it is only partially embedded in a flow around an obstacle, such as an island in the ocean [30, 63].

Thus, SFLT has dual design capabilities. It can transport stochastic sources in the material frame, and it can also impose stochastic non-inertial forces arising from stochastic changes of the frame of motion, such as the Craik-Leibovich vortex force [15, 34]. The dual design capabilities of SFLT could potentially lead to a variety of important applications. For example, the use of SFLT for modelling entrainment and detrainment of various materials into flows in the ocean mixed layer (OML) can in principle be instrumental in modelling some important components of natural processes such as gas and nutrient exchanges. Besides its potential importance in modelling natural processes in the transport of materials such as sediment or algae blooms in the OML, the SFLT stochastic modelling framework may also find a role in data assimilation for predicting the transport of pollution such as oil droplets, microplastics, etc.

The energy preserving SFLT framework is extended to the Eulerian Averaged SFLT (EA SFLT) framework by applying an Eulerian Average on the Eulerian quantities in the equations. These systems are non-local in probability space in the sense that the expected momentum density and expected advected quantities are assumed to replace the drift momentum density and advected quantities respectively. These equations retain the fundamental properties of fluid dynamics and in addition, they have potential uses in climate change science since the quantities of interests have a clear sense of expectation and fluctuation, and the expected fluid motion is deterministic. For quadratic Hamiltonians, we find closed-form equations for the dynamics of the expectations and fluctuations of the Eulerian fluid variables. Total energy conservation enables us to show that, over time, the energy of the expected quantities is converted into the energy of the fluctuations, while the sum remains the same.

To illustrate the dual design capability of the proposed SFLT framework and the EA SFLT framework, explicit applications for the motion of fluids under gravity with SFLT noise will be given in the last part of the paper. Section 4.1 deals with the heavy top, which is a finite degree-of-freedom subsystem of fluid motion [33]; in section 4.2 for rotating shallow water dynamics; and in section 4.3 for the stochastic Euler-Boussinesq equations.

List of abbreviations
- Geophysical Fluid Dynamics (GFD)
- Wave Effects on Currents (WEC)
- Reduced Lagrange d’Alembert Principle (RLDP)
- Craik-Leibovich (CL)
- Generalised Lagrangian Mean (GLM)
- Stochastic Forcing by Lie Transport (SFLT)
- Eulerian Averaged Stochastic Forcing by Lie Transport (EA SFLT)
- Stochastic Advection by Lie Transport (SALT)
- Lagrangian Averaged Stochastic Advection by Lie Transport (LA SALT)
- Stochastic Forced Euler-Poincaré (SFEP)
- Stochastic Forced Lie-Poisson (SFLP)
- Rotating Shallow Water (RSW)
- Euler-Boussinesq (EB)
In section 2 we introduce SFLT noise in the RLDP form and show that this formulation is flexible enough to accommodate a variety of different types of noise perturbations with potential applications to stochastic fluid dynamics. In section 3 we introduce the EA SFLT framework as an extension of the SFLT framework. We show that this modification to SFLT have potential applications to climate change science. In section 4 we present several examples of conservative noise types for semidirect-product coadjoint motion. These include the finite-dimensional case of the heavy top (which is a gyroscopic analog for collective motion of a stratified fluid [33]), and the infinite-dimensional fluid cases of rotating shallow water (RSW) dynamics in 2D and Euler-Boussinesq (EB) dynamics in 3D. The paper emphasises the utility of the RLDP formulation in introducing a variety of stochastic perturbations which may be chosen to preserve the properties of the deterministic solutions of the fluid dynamics equations that are of most concern and value to the modeller. In particular, RLDP admits a unified variational formulation which combines the variational principle used in [38] and SFLT, as explained in remark 2.16.

Three appendices have been provided. These appendices are meant to supply supporting details without interfering with the main flow of the paper. They contain further discussions of the following topics related to the main part of the paper: A Coadjoint operator of semidirect-product Lie-Poisson brackets; B Stochastic advection by Lie transport (SALT); C Itô form of the SFLP equation;

Slow + Fast decompositions of fluid flows. In modelling geophysical fluid dynamics (GFD) in ocean, atmosphere or climate science, one tends to focus on balanced solution states which are near certain observed equilibrium [59, 24]. These equilibria include hydrostatic and geostrophic balanced states, for example, in both the ocean and the atmosphere. Upsetting these balances can introduce both fast and slow temporal behaviour. The response depends on the range of the frequencies in the spectrum of excitations of the system away from equilibrium under the perturbations. When a separation in time scales exists in the response of the system to perturbations of one of its equilibria, then one may propose to average over the high frequency response and retain the remaining slow dynamics which remains near the equilibrium. This happens, for example, in the quasigeostrophic response to disturbances of geostrophic equilibria in the 2D rotating shallow water equations. Averaging over the high frequencies also often produces a slow ponderomotive force, due for example to a slowly varying envelope which modulates the high frequency response.1 In most situations in GFD, though, the solution only stays near the low-frequency slow manifold for a rather finite time before developing a high-frequency response, See, e.g., Lorenz [52, 53, 54]. In practice, for example in numerical weather prediction, the emergence of the high-frequency disturbances of the devoutly wished slow manifold introduces undeniable uncertainty which historically has often been handled by some sort of intervention, such as nonlinear re-initialisation [47]. Sometimes the effects of the emergence of high frequencies can be treated to advantage in computational simulations. For example, the stochastic back-scatter approach of Leith [38] is commonly used in computational simulations to feed energy from the burgeoning unstable development of high-wavenumber excitations into large-scale coherent structures at low frequencies, so as to enhance the formation of eddies in ocean flows, [6].

Hamilton’s principle and Kelvin’s circulation theorem. An opportunity for further theoretical understanding of the interactions of disparate scales in GFD (and, for example, in astrophysics and planetary physics [65]) arises when the non-dissipative part of the fluidic system dynamics under consideration in a domain $D$ can be derived from Hamilton’s principle, $\delta S = 0$, for an action time-integral given by $S = \int_0^t \ell(u,a)dt$, where the fluid Lagrangian $\ell(u,a)$ depends on Eulerian fluid variables comprising the fluid velocity vector field $u \in X(D)$ and some set of advected quantities, $a \in V^*(D)$, dual in $L^2$ pairing to a vector space, or tensor space, $V$, defined over the domain $D$ with appropriate boundary conditions. This approach leads directly to a Kelvin-Noether circulation theorem arising from the symmetry of the Lagrangian $\ell(u,a)$, written in terms of Eulerian fluid variables, under transformations of the initial material labels which preserve the initial conditions of the advected quantities along the particle trajectories in the flow [41]. In this case, the averaging over high frequencies in the solution may be applied by substituting a WKB (slowly varying complex amplitude times a fast but slowly varying phase) decomposition of the fluid parcel trajectory into the Lagrangian, then averaging over the fast phase before taking variations. A prominent example of this approach for applications in GFD is the General

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1In taking these averages over the fast behaviour of the dynamics one must also deal properly with any resonances which would occur. However, the effects of resonances will be neglected in our discussion here.
Lagrangian Mean (GLM) phase-averaged description of the interaction of fluctuations with a mean flow introduced in [2] and developed further in [29, 35, 36, 71, 1]. Many of the ideas underlying GLM are also standard in the stability analysis of fluid equilibria in the Lagrangian picture. See, e.g., the classic stability analysis papers of [7, 25, 64, 31].

Stochastic variational principles. The present paper discusses yet another opportunity for introducing a slow-fast decomposition for the sake of further understanding of GF. This opportunity arises when the separation in time scales in the symmetry-reduced Lagrangian $\ell(u,a)$ for Eulerian fluid variables can be posed as the decomposition of the fluid velocity $u$ into the sum of a deterministic drift velocity modelling the computationally resolvable scales and a stochastic velocity vector field modelling the correlates at the resolvable scales of the computationally unresolvable sub-grid scales of motion. A stochastic variational principle using the slow-fast decomposition of the Lagrangian flow map led to the derivation of stochastic Euler–Poincaré (SEP) equations in [38]. In [38], the fast motion of the Lagrangian trajectory is represented by a stochastic process, whose correlate statistics are to be calibrated from data as in [13, 12]. The stochastic decomposition proposed in [38] was later derived using multi-time homogenization by Cotter et al. [14]. Transport of fluid properties along an ensemble of these stochastic Lagrangian trajectories is called Stochastic Advection by Lie Transport (SALT). The well-posedness of the Euler fluid version of the Euler–Poincaré SALT equations in three dimensions was established in [16] for initial conditions in appropriate Sobolev spaces. The mathematical framework of semimartingale-driven stochastic variational principles was established in [66].

The geometric interplay between energy and circulation in ideal fluid dynamics. Fluid dynamics transforms energy into circulation. As it turns out, this transformation is quite geometric. In particular, the solutions of Euler’s fluid equations for ideal incompressible flow describe geodesic curves parametrised by time on the manifold of volume-preserving diffeomorphisms (smooth invertible maps). These geodesic curves are defined with respect to the metric provided by the fluid’s kinetic energy, defined on the smooth, divergence-free, velocity vector fields which comprise the tangent space of the volume-preserving diffeomorphisms $\text{SDiff}(D)$ acting on the flow domain $D$. This 1966 result of V. I. Arnold [4] introduced a fundamentally new geometric way of understanding energy and circulation in fluid dynamics. The incompressible Euler fluid case in [4] possesses the well-known conservation laws of energy and circulation, both of which arise via Noether’s theorem from symmetries of the Lagrangian in Hamilton’s variational principle under the right action of $\text{SDiff}(D)$. Later, Arnold [5] noticed the topological nature of another conservation law for the Euler fluid equations which is known as helicity. In particular, the conserved helicity measures the topological linkage number of the vorticity field lines in Euler fluid dynamics.

Some of this geometric interplay between energy and circulation in ideal fluid dynamics already shows up in Kelvin’s circulation theorem [43] for ideal Euler fluids, which emerges as the Kelvin-Noether theorem from right-invariance (relabelling symmetry) of the Lagrangian in Hamilton’s principle when written in terms of the Eulerian representation, [41]. In physical fluids, the particle-relabelling symmetry of the fluid Lagrangian in the Eulerian representation is broken from the full diffeomorphism group $G = \text{Diff}(D)$ to its subgroup $G_a = \text{Diff}(D)|_{a_0}$ which leaves invariant the initial conditions $a_0$ for advected fluid variables, denoted $a$, such as the mass density and thermodynamic properties. The Lagrangian histories $x_t = g_t x_0$ evolve as $\dot{x}_t = \dot{g}_t x_0 = u(g_t x_0, t) = u(x_t, t)$, where the Eulerian velocity vector field given by $u := \dot{g} g^{-1}$ is right-invariant under the particle-relabelling $x_0 \rightarrow y_0 = h_0 x_0$ for any fixed $h_0 \in G$.

In this geometrical setting for fluids in the Eulerian representation, the Legendre transform maps the Lagrangian variational formulation to the Hamiltonian formulation in which a Lie-Poisson bracket governs the motion generated by the Hamiltonian. This Lie-Poisson bracket is defined on the dual space of the Lie algebra of divergence-free vector fields $\mathcal{X}_{\text{div}}(D)$ for Euler fluids. For ideal fluids with advected quantities, the Lie-Poisson bracket is defined on the dual space of the SDP Lie algebra $\mathcal{X}(D)\otimes V(D)$. As discussed in [41], the fluid motion in each case represents the coadjoint action of the corresponding Lie algebra on its dual space. For more details in the present context, refer to appendices A and B, particularly B.3.

Geometric formulation of advective transport. The action of the full diffeomorphism group $G = \text{Diff}(D)$ on its order parameter variables $a \in V(D)$ represents fluid advection. Advection occurs by push-forward of functions on $V(D)$ (by right action by the inverse of the Lagrange-to-Euler map). This means the time-
dependence of an advected quantity is given by the push-forward relation for composition of functions; namely,

\[ a(t) = g_t \cdot a_0 := a_0 g_t^{-1} . \]

Thus, an advected quantity \( a(t) \) evolves by the Lie chain rule

\[ \partial_t a = - \mathcal{L}_{\dot{g} g_t^{-1}} a_t = - \mathcal{L}_{u_t} a , \]

in which \( \mathcal{L}_{u_t} a \) denotes the Lie derivative of the advected quantity \( a \in V(D) \) by the time-dependent Eulerian velocity vector field, \( u_t := \dot{g} g_t^{-1} \in \mathcal{X}(D) \). For more details about performing this type of calculation for Lie transport, see appendix A. The applications of these ideas in developing the SALT approach are discussed in appendix B.

2 Stochastic forcing by Lie transport (SFLT)

After briefly surveying in section 2.1 a few of the available capabilities arising from the RLDP principle for the sake of other potential research directions, we will continue in section 2.2 toward our primary objective to develop the energy preserving SFLT theory for applications to semidirect-product fluid motion and material entrainment in the remainder of the paper.

2.1 Reduced Lagrange d’Alembert Pontryagin (RLDP) Principle

Stochastic non-inertial frames. Newton’s law of motion in a non-inertial frame redefines momentum so as to introduce an additional force. The canonical example is the Coriolis force, which arises from redefining the momentum in a rotating frame of motion in terms of the velocity as viewed from an inertial frame. The CL vortex force arises in the fluid momentum equation by the addition of Stokes drift velocity to the frame of motion of the fluid momentum variable \( m \in \mathbb{X}^* \). In the present setting, it is natural to consider a stochastic addition to the momentum which corresponds to a stochastic change of reference frame.

One variational principle which permits such additional forces is the reduced Lagrange d’Alembert Pontryagin (RLDP) principle. The general construction of fluid dynamics using the RLDP formulation is as follows [27].\(^2\) Consider the Lie group \( G = \text{Diff}(D) \) with an associated Lie algebra \( \mathbb{X} \), the RLDP principle for reduced Lagrangian \( \ell : \mathbb{X} \to \mathbb{R} \) and external force \( F \in \mathbb{X}^* \) is given by

\[ \delta \int_a^b \ell(u) + \langle m, \dot{g} g^{-1} - u \rangle \, dt - \int_a^b \langle F, \delta g g^{-1} \rangle \, dt = 0 , \]

(2.1)

where \( g \in G \), the variations \( \delta g, \delta u, \delta m \) are arbitrary with \( \delta g \) vanishes at the end points \( t = a, b \). The stationary condition (2.1) yields the forced Euler-Poincaré equation with force \( F \), as a 1-form density equation,

\[ \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \text{ad}^*_u \delta \ell + F = 0 . \]

(2.2)

Adding stochasticity to the RLDP principle with external forces can be done, as follows. For reduced Lagrangian \( \ell : \mathbb{X} \to \mathbb{R} \) and the set of external forces \( F_i \in \mathbb{X}^* \), stochastic RLDP is given by

\[ \delta \int_a^b \ell(u) + \langle m, \delta g g^{-1} - u \rangle \, dt - \sum_i \int_a^b \langle F_i, \eta \rangle \circ dW^i_t = 0 , \]

(2.3)

where \( g \in G \), the variations \( \delta g, \delta u, \delta m \) are arbitrary, \( \delta g \) vanishes at the end points, \( t = a, b \), and \( \eta := \delta g g^{-1} \). The stationary condition of the variational principle (2.3) yields the following stochastically forced Euler-Poincaré (SFEP) equation,

\[ d \frac{\delta \ell}{\delta u} + \text{ad}^*_u \frac{\delta \ell}{\delta u} \, dt + \sum_i F_i \circ dW^i_t = 0 . \]

(2.4)

\(^2\)See [10] for the corresponding result for the finite-dimensional Euler-Lagrange equation in the absence of symmetry.
Lie-Poisson Hamiltonian formulation. Upon passing to the Hamiltonian side via the Legendre transform
\[ \ell(u) = \langle m, u \rangle - h(m), \]
where \( m = \frac{\delta h}{\delta u} \) and \( h \) is the reduced Hamiltonian, one finds the reduced Hamilton-
d’Alembert Pontryagin phase space principle given by
\[ \delta \int_b^a \left[ \langle m, dg^{-1} \rangle - h(m) \right] dt - \sum_i \int_b^a \left( F_i, \eta \right) dW_i^t = 0. \tag{2.5} \]
As before, \( g \in G \), the variations \( \delta g, \delta m \) are arbitrary and \( \delta g \) vanishes at the endpoints in time, \( t = a, b \). The resulting stochastically forced Lie-Poisson (SFLP) equation is
\[ dm + ad^*_u m dt + \sum_i F_i \circ dW_i^t = 0, \quad \text{where } u = \frac{\delta h}{\delta m}. \tag{2.6} \]
Within the construction of class of SFLP equations, choices of the external forces \( F_i \) exists. With specific
choices of the external forces \( F_i \), the resulting equations can be either energy-preserving, or Casimir preserving
as shown by subsequent examples.

Example 2.1 (Energy-preserving SFLP equations). Let \( F_i = ad^*_u f_i \) where \( f_i \in \mathcal{X}^* \), then the SFLP equation (2.6) becomes
\[ dm + ad^*_u m dt + \sum_i ad^*_u f_i \circ dW_i^t = 0, \quad \text{where } u = \frac{\delta h}{\delta m}, \tag{2.7} \]
in agreement with [21]. Energy preservation may now be immediately verified, since
\[ dh = \left< dm, \frac{\delta h}{\delta m} \right> = \left< -ad^*_u m dt - \sum_i ad^*_u f_i \circ dW_i^t, u \right> = 0, \tag{2.8} \]
in which the last equality follows because of the anti-symmetry of the commutator, \( ad_u u = -[u, u] = 0 \). However,
the Casimirs are no longer conserved, because the Lie-Poisson operator in equation (2.7) has been changed by
the addition of noise. For completeness, the Itô form of the equation (2.7) is presented in Appendix C.

Proposition 2.2 (Vortex force). The energy preserving SFLP equation (2.7) contains stochastic vortex forces.

Proof. The stochastic term in (2.7) can be expressed as
\[ \sum_i ad^*_u f_i \circ dW_i^t = \sum_i \left( u \times \text{curl} f_i - \nabla (u \cdot f_i) - f_i \cdot \text{div} u \right) \cdot dx \circ dW_i^t, \tag{2.9} \]
where \( u \) and \( f_i \) are the coefficients of the vector field \( u \) and 1-form densities \( f_i \), i.e. \( u = u \frac{\partial}{\partial x} \) and \( f_i = f_i \cdot dx \cdot d^3x \).
Thus, a stochastic version of the CL vortex force \( \sum_i u \times \text{curl} f_i \cdot dW_i^t \) with a stochastic contribution to the pressure \( \sum_i \nabla (u \cdot f_i) \circ dW_i^t \) emerges in the energy-preserving form of the RLDP equations in (2.7), cf. [15].
Note that the term \( f_i \cdot \text{div} u \) vanishes for incompressible flow. \( \square \)

For a specific choice of Hamiltonian, the forces presented in example 2.1 may not be the only forces that
conserve energy. However, the \( ad_u f_i^* \) form of the noise is geometric, so the energy preserving property does
apply to all fluid systems that have a variational formulation. Thus, it is possible to construct a class of energy
preserving stochastic systems using the geometric formulation. As discussed further in section 2.2, the noise
terms in this class of energy preserving stochastic systems appear the form of a ‘frozen’ (constant coefficient)
Lie-Poisson bracket, whose properties are discussed, e.g., in Appendix B of [42].

Example 2.3 (Casimir-preserving SFLP equations). Given a Casimir function \( C(m) \), the Casimir preserving
forces satisfy \( F_i = ad^*_{zm} f_i \) where \( f_i \) are arbitrary. Then
\[ dC = \left< dm, \frac{\delta C}{\delta m} \right> = \left< -ad^*_u m dt - \sum_i ad^*_{zm} f_i \circ dW_i^t, \frac{\delta C}{\delta m} \right> = 0, \]
provided both \( \left< ad^*_u m, \frac{\delta C}{\delta m} \right> = 0 \) and \( \left< ad^*_{zm} f_i, \frac{\delta C}{\delta m} \right> = 0 \), because of the degeneracy of the LP bracket and the
anti-symmetry of the commutator, respectively. One concludes that the choice of external forces \( F_i = ad^*_{zm} f_i \)
cannot in general preserve all of the Casimirs seen in the unperturbed LP equation, unlike the case with SALT,
where all of the Casimirs are preserved. Note that the energy are no longer conserved, since
\[ \left< dm, \frac{\delta h}{\delta m} \right> = \left< -ad^*_u m dt - \sum_i ad^*_{zm} f_i \circ dW_i^t, \frac{\delta h}{\delta m} \right> = \left< -\sum_i ad^*_{zm} f_i \circ dW_i^t, \frac{\delta h}{\delta m} \right>, \]
does not vanish trivially.
Proposition 2.4 (Helicity preservation for a stochastic 3D Euler fluid).

The deterministic 3D Euler fluid equations preserve the helicity,

\[ C(m) = \langle m, \mathbf{d}m \rangle = \frac{1}{2} \int m \cdot \text{curl} \mathbf{m} \, d^3x, \]

where \( m = m \cdot dx \) is the circulation 1-form. The SFLT model will preserve the helicity \( C(m) \) when the constraint force is taken to be

\[ F_i = \text{ad}^*_u f_i \in \mathfrak{X}^* \]

where \( \frac{\delta C}{\delta m} = \mathbf{d}m \) is the vorticity 2-form, \( \mathbf{d}m = (\text{curl} \mathbf{m}) \cdot dS \) and .

Proof. In vector calculus terms, one computes

\[ \langle \text{ad}^*_u m, \frac{\delta C}{\delta m} \rangle = \langle -u \times \text{curl} \mathbf{m} + \nabla (u \cdot m), \text{curl} \mathbf{m} \rangle = \oint_{\partial D} (u \cdot m) \text{curl} \mathbf{m} \cdot \hat{n} dS = 0, \]

which is obtained after an integration by parts in the second summand and taking the boundary term to vanish as usual for the helicity.

Example 2.5 (Proto-SALT SFLP equation). Let us choose \( F_i = \text{ad}^*_{f_i(x)} m \) in equation (2.6), where \( f_i \in \mathfrak{X} \) are arbitrary functions taking values in the Lie algebra of smooth vector fields \( \mathfrak{X} \). Then, the SFLP equation (2.6) recovers an expression similar to the momentum terms in the SALT fluid motion equation (B.10); namely,

\[ \mathbf{d}m + \text{ad}^*_{\mathbf{dx}_t} m = 0, \quad \text{where} \quad u = \frac{\delta h}{\delta m} \quad \text{and} \quad \mathbf{dx}_t = u \, dt + \sum_i f_i \circ dW^i_t. \quad (2.10) \]

This is the SALT fluid motion equation (B.10) in the absence of advected quantities.

2.2 Energy preserving SFLT for semidirect-product motion and material entrainment

We next extend the SFLT approach to case of coadjoint motion of fluids carrying advected quantities such as mass and heat which are associated with potential energy. The motion of advected quantities arises from the semidirect-product action of the diffeomorphisms on the vector spaces containing the advected quantities. The vector spaces containing the advected quantities comprise coset spaces obtained from symmetry-breaking of the full diffeomorphism group to the remaining isotropy subgroup of the initial conditions of the advected quantities.

Keeping in sight the objective of the paper to derive a class of energy-preserving stochastic models of fluid dynamics, the same energy-preserving noise as the Euler fluids in Example 2.1 and its vortex force 2.2 will be chosen in the construction that follows. This extension enables the derivation of stochastic vortex forces which model the uncertainty of unresolved slow-fast interaction effects as energy-preserving stochastic perturbations of fluid models which possess a potential energy. In addition to vortex forces, the introduction of stochasticity in the passive advection relations allows the modelling of quantities that do not passively follow the drift velocity field. Physical applications of this type of stochastic modification are discussed in Remark 2.12. For symmetry-reduced semidirect-product motion, considering the energy preserving noise of the form \( F_i = \text{ad}^*_u f_i \), the stochastic RLDP principle in (2.3) becomes

\[ 0 = \delta S = \delta \int_a^b \ell(u, a) \, dt + \langle \mathbf{m}, \mathbf{d}g^{-1} - u \, dt \rangle + \langle \mathbf{db}, a_0 g^{-1} - a \rangle - \int_a^b \sum_i \langle \text{ad}^*_u f_i, \delta g g^{-1} \rangle \circ dW^i_t, \quad (2.11) \]

where the variations \( \delta \mathbf{g}, \delta u, \delta \mathbf{m}, \delta \mathbf{db}, \delta a \) are arbitrary with \( \delta g \) vanishing at the endpoints, \( t = a, b \). Taking the

3Advelled quantities are also known as order parameters in condensed matter physics.
indicated variations in (2.11) yields the following result,

\[ 0 = \int_a^b \left( \frac{\delta \ell}{\delta u} , \delta u \right) dt + \left( \frac{\delta \ell}{\delta a} , \delta a \right) dt + \left( \delta m , \delta g^{-1} - u dt \right) + \left( db , a_0 \delta g^{-1} - \delta a \right) + \left( \delta dB , a_0 g^{-1} - a \right) \]

\[ + \left( m , \delta (dg^{-1}) - ud\right) - \sum_i \left( \text{ad}^*_u f_i , \delta g^{-1} \right) \circ dW^i_t \]

\[ = \int_a^b \left( \frac{\delta \ell}{\delta u} - m , \delta u \right) dt + \left( \frac{\delta \ell}{\delta a} , \delta a \right) dt + \left( \delta m , \delta g^{-1} - u dt \right) + \left( db , -a\eta - \delta a \right) + \left( \delta dB , a_0 g^{-1} - a \right) \]

\[ + \left( m , \delta \eta - \text{ad}_{dg^{-1}} \eta \right) - \sum_i \left( \text{ad}^*_u f_i , \eta \right) \circ dW^i_t , \]

where we denote \( \eta = \delta g^{-1} \), as before. Vanishing of the coefficients of the variations implies the following relations,

\[ \frac{\delta \ell}{\delta u} = m , \quad \delta g^{-1} = ud, \quad \delta b = \frac{\delta \ell}{\delta a} dt , \]

\[ \delta m = -\text{ad}^*_{dg^{-1}} m + db + a - \sum_i \text{ad}^*_u f_i \circ dW^i_t , \quad \delta a = -\mathcal{L}_{dg^{-1}} a . \quad (2.12) \]

Assembling these relations yields the SFEP equations with advected quantities

\[ \frac{d\delta \ell}{du} = -\text{ad}^*_u \frac{\delta \ell}{\delta u} dt - \sum_i \text{ad}^*_u f_i \circ dW^i_t + \frac{\delta \ell}{\delta a} \circ a dt , \quad \delta a = -\mathcal{L}_u a dt . \quad (2.13) \]

Now, an application of the Legendre transform \( h(m,a) = \langle m, u \rangle - \ell(u,a) \), followed by calculations similar to those made in the deterministic case arrives at the SFLP equations with advected quantities

\[ \delta m + \text{ad}^*_{h_{m}} m dt + \sum_i \text{ad}^*_{h_{m}} f_i \circ dW^i_t + \frac{\delta h}{\delta a} \circ a dt = 0 , \quad \delta a = -\mathcal{L}_u a dt . \quad (2.14) \]

These equations can also be written in the form of a Poisson operator, as

\[ d \left( \begin{array}{c} m \\ a \end{array} \right) = - \left[ \begin{array}{c} \text{ad}^* \left( m dt + \sum_i f_i \circ dW^i_t \right) \\ \delta \ell \circ a dt \end{array} \right] \left[ \begin{array}{c} \delta h/\delta m \\ \delta h/\delta a \end{array} \right] . \quad (2.15) \]

**Remark 2.6** (Alternative formulation for a reduced Hamilton-d’Alembert principle with advected quantities). The SFLP equation with advected quantities (2.14) can also be derived from the following reduced Hamilton-d’Alembert phase space principle with advected quantities,

\[ 0 = \delta S = \delta \int_a^b \left( \delta m , \delta g^{-1} \right) + \left( db , a_0 g^{-1} - a \right) - h(m,a) dt - \int_a^b \sum_i \left( \text{ad}^*_{h_{m}} f_i , \delta g^{-1} \right) \circ dW^i_t . \]

The proof of this statement is a direct calculation following the same pattern as for the EP derivation.

**Theorem 2.7** (SFLT Kelvin-Noether theorem). The Kelvin-Noether quantity \( \langle K(g(t)c_0,a(t)) , m(t) \rangle \) associated with equation (2.14) satisfies the following stochastic Kelvin-Noether relation.

\[ d \langle K(g(t)c_0,a(t)), m(t) \rangle = \left( K(g(t)c_0,a(t)) \right) - \frac{\delta h}{\delta a} \circ a dt - \sum_i \text{ad}^*_u f_i \circ dW^i_t \right) . \quad (2.16) \]

where the identification \( d\delta g^{-1} = \frac{dh}{\delta m} dt \) is obtained from the stochastic RLDP constrained variational principle in (2.11).

**Remark 2.8.** Upon assuming that the fluid density \( D \) is also advected by the flow, so that \( \partial_t D + \mathcal{L}_u D = 0 \), the Kelvin circulation theorem may be expressed as

\[ d \oint_{c(u)} \frac{m}{D} = \oint_{c(u)} \frac{1}{D} \left[ \frac{\delta h}{\delta a} \circ a dt - \sum_i \text{ad}^*_u f_i \circ dW^i_t \right] , \quad (2.17) \]

where the notation of Lagrangian loop \( c(u) \) denotes the Lagrangian loop moving with the deterministic fluid velocity \( u \), as in the deterministic case.
Natural generalisation of SFLT  Note that the SFLP equations with advected quantities in (2.14) modifies the momentum equation alone, while keeping the advection equation of $a \in V^*$ unchanged. In terms of co-adjoint motion, the natural generalisation of (2.7) to semidirect-product Lie group action would be

$$d(m, a) = -ad^a(m, a) \, dt - \sum_i ad^a_i (f^m_i, f^a_i) \circ dW_i^i,$$  \hfill (2.18)

where $(m, a), (f^m_i, f^a_i) \in g^*$, $(\frac{\delta h}{\delta m}, \frac{\delta h}{\delta a}) \in s$ and $ad^*$ is the coadjoint operator on $g^*$. The maps $f^m_i : \mathcal{X} \times V^* \to \mathcal{X}^*$ and $f^a_i : \mathcal{X} \times V^* \to V^*$ are arbitrary for all $i$. Using definition of semidirect product coadjoint action discussed in appendix A, the individual equations are then

$$dm + ad^m \frac{\delta h}{\delta a} \circ a \, dt + \sum_i ad^m_i (f^m_i, dW_i^i) + \frac{\delta h}{\delta a} \circ f^a_i \circ dW_i^i = 0,$$

$$da + \mathcal{L}_{\frac{\delta h}{\delta a}} a \, dt + \sum_i \mathcal{L}_{\frac{\delta h}{\delta a}} f^a_i \circ dW_i^i = 0.$$  \hfill (2.19)

These individual equations can be written in the form of a Poisson operator, as follows,

$$d \begin{bmatrix} m \\ a \end{bmatrix} = - \begin{bmatrix} ad^m & \mathcal{L}_{\frac{\delta h}{\delta a}} \\ \mathcal{L}_{\frac{\delta h}{\delta a}} & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta h}{\delta m} \\ \frac{\delta h}{\delta a} \end{bmatrix}.$$

Energy is conserved since the Poisson operator is skew-symmetric. Thus,

$$dh(m, a) = \langle d(m, a), \begin{bmatrix} \frac{\delta h}{\delta m} \\ \frac{\delta h}{\delta a} \end{bmatrix} \rangle = ad^m \frac{\delta h}{\delta a} \, dt - \sum_i ad^m_i (f^m_i, dW_i^i) \circ dW_i^i = 0,$$

where the last equality uses the anti-symmetry of the $ad$ operator. The class of SFLP equations in (2.18) can be obtained via a phase-space variational principle, as we discuss next.

Variational principle for semidirect product SFLP equation  The SFLP equation (2.18) can be derived from a reduced Hamilton-d’Alembert phase space variational principle in terms of the full semidirect product group $S = G \mathcal{S} V$ with the associated semidirect-product Lie algebra $s = \mathcal{X} \mathcal{S} V$. This phase-space variational principle reads

$$\delta \int_{t_1}^{t_2} \langle (m, a), (du, db) \rangle_s - h(m, a) \, dt - \sum_i \int_{t_1}^{t_2} \langle ad^m_i (f^m_i, f^a_i), (\eta, w) \rangle_s \circ dW_i^i = 0,$$  \hfill (2.21)

for arbitrary variations of $\delta(m, a) \in g^*$ and constrained variations $\delta(du, db)$ of $(du, db) \in s$. The constrained variation of $(du, db)$ takes the form

$$\delta du = d\eta + [du, \eta] \quad \text{and} \quad \delta db = dw - db \eta + wdu,$$

in which $\eta \in \mathcal{X}$ and $w \in V$ are arbitrary. Here the notation $\langle \cdot, \cdot \rangle_s$ is the semidirect product pairing where

$$\langle (m, a), (u, b) \rangle_s := \langle m, u \rangle_{\mathcal{X}} + \langle a, b \rangle_V.$$

In the following we will continue to suppress the pairing subscript when the context is clear. Note that the constrained variations of $du$ and $db$ are related to the adjoint action of $s$ by

$$\delta(du, db) = \langle d\eta + [du, \eta], dw - db \eta + wdu \rangle = \delta(\eta, w) - ad^*(du, db)(\eta, w).$$

Computing the variations and applying the constrained variations yields

$$0 = \int_{t_2}^{t_1} \langle du - \frac{\delta h}{\delta m} dt, \frac{\delta h}{\delta a} \circ a \rangle + \langle db - \frac{\delta h}{\delta a} dt, \delta a \rangle + \langle m, \eta \rangle + \langle [du, \eta], dw - db \eta + wdu \rangle - \langle ad^m_i (f^m_i, f^a_i), \eta \rangle \circ dW_i^i - \langle \mathcal{L}_{\frac{\delta h}{\delta a}} f^a_i, w \rangle \circ dW_i^i$$

$$= \int_{t_2}^{t_1} \langle du - \frac{\delta h}{\delta m} dt, \delta m \rangle + \langle db - \frac{\delta h}{\delta a} dt, \delta a \rangle + \langle -d\eta - ad^* m, \eta \rangle - \langle -da - a du, w \rangle + \langle a \circ db, \eta \rangle - \langle ad^m_i (f^m_i, f^a_i), \eta \rangle \circ dW_i^i - \langle \mathcal{L}_{\frac{\delta h}{\delta a}} f^a_i, w \rangle \circ dW_i^i.$$
Consequently, one may collect terms to find the following system of motion and advection equations

\[
\begin{align*}
\frac{dm}{dt} &+ ad^* m + db \odot a + ad^* f^m_i \odot dW^i_t + \frac{\delta h}{\delta a} \circ f^n_i \odot dW^i_t = 0, \\
du &= \frac{\delta h}{\delta m} dt, \quad db = \frac{\delta h}{\delta a} dt, \quad da + \mathcal{L}_{ad} a + \mathcal{L}_{\frac{\delta}{\delta f^m_i}} f^n_i \odot dW^i_t = 0.
\end{align*}
\]

(2.22)

**Remark 2.9** (Reduced Hamilton-d’Alembert Pontryagin phase space variation principle). The choice of variation principle (2.21) is not the Hamiltonian version of the RLDP variation principle used previous sections. It is, however, equivalent to the reduced Hamilton-d’Alembert Pontryagin phase space variation principle

\[
\delta \int_{t_1}^{t_2} \langle (m, a), d(g, v) (g, v)^{-1} \rangle - h(m, a) dt - \sum_i \int_{t_1}^{t_2} \left\langle ad^* \left( \frac{\delta h}{\delta f^m_i} \circ f^n_i + \delta h \circ f^n_i \right) \right\rangle \odot dW^i_t = 0,
\]

(2.23)

where the variations \(\delta (g, v) \in S\) and \(\delta (m, a) \in s\) are arbitrary.

**Proposition 2.10** (Itô form of semidirect product SFLP equation). The Itô form of (1.19) are

\[
\begin{align*}
\frac{dm}{dt} + ad^* m + db \odot a + ad^* f^m_i \odot dW^i_t + \frac{\delta h}{\delta a} \circ f^n_i \odot dW^i_t &= 0, \\
da + \mathcal{L}_{ad} a + \mathcal{L}_{\frac{\delta}{\delta f^m_i}} f^n_i \odot dW^i_t &= 0,
\end{align*}
\]

(2.24)

where one defines

\[
\sigma_i := \left( \frac{\delta^2 h}{\delta m^2}, -ad^* f^m_i - \frac{\delta h}{\delta a} \circ f^n_i \right) \quad \text{and} \quad \theta_i := \left( \frac{\delta^2 h}{\delta a^2}, -\mathcal{L}_{ad} f^n_i \right).
\]

Again the notation \((\cdot, \cdot)\) denotes contraction, not \(L^2\) pairing.

The proof is similar to the case without advected quantities are it is given in appendix C.

**Remark 2.11.** The external forces \(ad^* f^m_i \odot dW^i_t\) introduced in (2.21) are energy preserving only. For a general sets of forces \(F^m_i\) and \(F^n_i\), the semidirect product SFLP equation will be derived from the variational principle

\[
\delta \int_{t_1}^{t_2} \langle (m, a), (du, db) \rangle - h(m, a) dt - \sum_i \int_{t_1}^{t_2} \langle (F^m_i, F^n_i), (\eta, \omega) \odot dW^i_t \rangle = 0,
\]

where the variations have the same condition as principle (2.21). The resulting equations are the following

\[
\begin{align*}
\frac{dm}{dt} + ad^* m dt + \frac{\delta h}{\delta a} \odot a dt + \sum_i F^m_i \odot dW^i_t &= 0, \\
da + \mathcal{L}_{ad} a dt + \sum_i F^n_i \odot dW^i_t &= 0.
\end{align*}
\]

(2.25)

**Remark 2.12** (Stochastic material entrainment modelled in equation (2.19)). When stochastic processes \(f^n_i\) are included in the Lie-derivative action on the fluid variables, \(a(t)\), then one can no longer say that \(a(t)\) is passively advected by the flow \(g \in G\), i.e., \(a(t) \neq a \circ g^{-1}(t)\). Consider the case that the variable \(a(t)\) represents the mass density of the fluid. Then, the class of stochastic equations in (2.19) or (2.20) could model a fluid containing parcels whose density does not quite passively follow the drift velocity flow. Examples of such deviations from passive transport might include inertial fluid parcels whose density has a certain probability of being heavier or lighter than the ambient (or average, or expected) density. The motion of these inertial parcels would then be uncertain, as modelled by a stochastic variation in their density, relative to parcels undergoing passive advection by the flow. We include this feature of equation (2.19) because it may provide a useful single-fluid approach to dealing with stochastic material entrainment of material particles into fluid flows such as Langmuir circulations. This stochastic model of material entrainment into fluid flows introduces probabilistic aspects into the theory, rather than dealing with the intricacies of multiphase flow models. The applications of this feature may include, for example, ice slurry in the Arctic Ocean, or dust clouds, or debris in tornadoes, or fluid flows with gas bubbles, or well-mixed oil spills, or transport of algae, or plastic detritus in the ocean. In the case where \(f^n_i = 0\) for all \(i\), then \(a(t)\) would be passively advected and would satisfy the standard relation \(a(t) = a \circ g^{-1}(t)\). For a recent review of deterministic LES turbulent models of this type of mixed-buoyancy fluid transport, see [11].
Remark 2.13 (SFLT Kelvin-Noether theorem). Note that the inclusion of forcing terms $f^a_i$ on advected quantities $a$ implies $a(t) \neq a_0 - 1$. This means one cannot take the pull back the Kelvin-Noether quantity $(\langle K(g(t)c_0, a(t)), m(t) \rangle$ by $g(t)$ to the Kelvin-Noether quantity defined by the initial conditions $c_0$ and $a_0$, i.e. $(\langle K(g(t)c_0, a(t)), m(t) \rangle \neq (\langle K(c_0, a_0), Ad^*_g m(t) \rangle)$. Hence there is no simple modification of the Kelvin-Noether theorem associated with (2.19). For fluids, the map $K$ is the circulation integral around a material loop $c$ which is independent of the advected quantities $a$. Thus a Kelvin circulation theorem exists as formulated below.

**Theorem 2.14** (SFLT Kelvin circulation theorem with entrainment). Upon assuming that the fluid density $D$ is also advected by the flow, so that $\partial_t D + \mathcal{L}_u D = 0$, the Kelvin circulation theorem associated with (2.19) may be expressed as

$$d \int_{c(u)} \frac{m}{D} = \int_{c(u)} \frac{1}{D} \left[ - \sum_i \delta a^*_u f^m_i \circ dW_i^t - \frac{\delta h}{\delta a} \circ \left( a dt + \sum_i f^a_i \circ dW_i^t \right) \right] ,$$

(2.25)

where the notation of Lagrangian loop $c(u)$ denotes the Lagrangian loop moving with the deterministic fluid velocity $u$, as in the deterministic case.

**Remark 2.15.** Note that the presence of the stochastic material entrainment terms $f^a_i$ in equation (2.25) may have a significant effect as a source of circulation of the fluid flow.

**Remark 2.16** (An unified variational approach). By introducing stochastic Hamiltonians into the variational principle (2.21), one can formulate a stochastic RLDP principle which encompasses both SALT and SFLT. The augmented stochastic RLDP principle becomes

$$\delta \int_{t_1}^{t_2} \langle (m, a), (du, db) \rangle - \sum_i h_i(m, a) \circ dW_i^t$$

$$- \sum_i \int_{t_1}^{t_2} \langle (F^m_i, F^a_i), (\eta, w) \rangle \circ dW_i^t = 0 .$$

(2.26)

Here variations of $\delta (m, a)$ of $(m, a) \in \mathfrak{s}^*$ are arbitrary and variations $\delta (g, v)$ of $(g, v) \in S$ are given by

$$\delta du = d\eta + [du, \eta] \text{ and } \delta db = dw - db \eta + wdu ,$$

where $(\eta, w) \in \mathfrak{s}$ are arbitrary are vanishes at the boundaries. The set of Hamiltonians $h_i$ generates SALT type noise and the set of forces $(F^m_i, F^a_i)$ generates SFLT type of noise.

3 Eulerian Averaged SFLT

Ed Lorenz captured the essence of the climate science problem in his celebrated unpublished paper [55] in which he motivated his discussion by invoking the old adage that “Climate is what you expect. Weather is what you get.”

Lorenz’s lesson was that climate science is fundamentally probabilistic. Much later, this lesson inspired the derivation of the LA SALT fluid model, which also exploited an idea introduced in [21] to apply Lagrangian-averaging (LA) in probability space to the fluid equations governed by stochastic advection by Lie transport (SALT) which were introduced in [38]. The general theory of LA SALT and applications to 2D Euler-Boussinesq equations can be found in [22] and [3] respectively.

Here, we apply the probabilistic approach to derive the corresponding Eulerian-averaged SFLT model (EA SFLT) by decomposing the Eulerian solutions of the energy-preserving SFLT model into the sums of their expectations and their fluctuations. As for the energy-preserving SFLT models, the EA SFLT equations admit a Kelvin circulation theorem and preserve the deterministic energy. For systems resulting from quadratic Hamiltonians, this modification of the SFLT model allows the dynamics of the statistical properties of the solutions of EA-SFLT such as the evolution of the expectation of energy to be considered explicitly. As for the LA-SALT models, the EA-SFLT models can be viewed as the interaction of the expected quantities and the fluctuations in a conservative system. In this system, the energy of the expected quantities is dynamically converted into the energy of fluctuations whilst keeping the total energy invariant as shown in Theorem 3.2. The Kelvin circulation theorem for EA-SFLT differs from that of SFLT by the presence of an additional forcing. This feature makes the EA-SFLT approach particularly apt for the examples of quadratic fluid Hamiltonians which are discussed in section 4.
3.1 EA SFLT

In Eulerian Averaged SFLT, the expectation of the Eulerian quantity \( m \) from equation (2.7) is the taken over the underlying probability space, and the Eulerian Averaged Stochastic Forced Lie Poisson (EA SFLP) equation is proposed, as

\[
dm + \text{ad}^*_u E[m] \ dt + \sum_i \text{ad}^*_u f^i \circ dW_i = 0, \quad u = \frac{\delta h}{\delta m}
\]  

(3.1)

This modification still preserves the deterministic energy of the original SFLP equation since

\[
dh(m) = \langle dm, u \rangle = -\left( \text{ad}^*_u E[m] \ dt + \sum_i \text{ad}^*_u f^i \circ dW_i, u \right) = 0,
\]  

(3.2)

by anti-symmetry of of the ad operation. The evolution of the expectation \( E[m] \) can be determined by considering the Itô form of the equation (3.1)

\[
dm + \text{ad}^*_u E[m] \ dt + \sum_i \text{ad}^*_u f^i dW_i^i + \frac{1}{2} \sum_i \text{ad}^*_u f^i dt = 0,
\]  

(3.3)

where \( \sigma_i = \left( \frac{\delta^2 h}{\delta m^2}, -\text{ad}^*_u f^i \right) \). The proof of the Itô form is similar to the case of the SFLP equation in appendix C. Taking the expectation of (3.3) yields the following partial differential equation (PDE) for the expectation of the momentum density, \( E[m] \),

\[
\partial_t E[m] + \text{ad}^*_u E[m] + \frac{1}{2} \sum_i \text{ad}^*_u[s_i] f^i = 0.
\]  

(3.4)

The expectation of (3.3) has produced a deterministic PDE because \( \text{ad}^* \) is a linear operation and the expectation of \( dW_i^i \) vanishes by Itô’s Lemma. The PDE (3.4) closes, whenever the Hamiltonian is quadratic in \( m \), i.e. \( h = \frac{1}{2} \langle m, \ I^{-1} m \rangle \) where \( I : \mathfrak{g} \to \mathfrak{g}^* \) is a constant invertible symmetric operator which commutes with taking the expectation. Furthermore, \( \mathfrak{l} \) is assumed to be positive definite. In this case, one finds the vector field

\[
\mathbb{E}[\sigma_i] = \left( -\text{ad}^*_u f^i, \frac{\delta^2 h}{\delta m^2} \right) = -\mathbb{I}^{-1} \text{ad}^*_u f^i.
\]

Assuming quadratic Hamiltonian \( h \), one can computes also the Hamiltonian function of the expectation of \( m \), \( E[m] \) to have

\[
\partial_t h(E[m]) = \langle E[u], \partial_t E[m] \rangle = -\frac{1}{2} \left( \mathbb{I}^{-1} \text{ad}^*_u f^i, \text{ad}^*_u f^i \right) < 0,
\]  

(3.5)

Define fluctuations \( u' = u - E[u], \ m' = m - E[m] \) and \( \sigma'_i = \sigma_i - E[\sigma_i] \) and obtain the fluctuation dynamics of \( m \) by subtracting (3.4) from (3.3) to obtain

\[
dm' + \text{ad}^*_u E[m] dt + \sum_i \text{ad}^*_u f^i dW_i^i + \frac{1}{2} \sum_i \text{ad}^*_u f^i dt = 0.
\]  

(3.6)

The time derivative of the Hamiltonian of the fluctuation of \( m \), \( h(m') \) is found as

\[
dh(m') = \left( \frac{\delta h}{\delta m'}, \ dm' \right) + \frac{1}{2} d \left( \frac{\delta h}{\delta m'} \right) dt
\]

\[
= \left( \frac{\delta h}{\delta m'}, -\text{ad}^*_u E[m] dt - \sum_k \text{ad}^*_u f^k dW_k^i + \frac{1}{2} \text{ad}^*_u f^i dt \right) + \frac{1}{2} d \left( m', \frac{\delta h}{\delta m'} \right) dt
\]  

(3.7)

This expression simplifies dramatically when the Hamiltonian is quadratic, where

\[
\frac{\delta h}{\delta m'} = \mathbb{I}^{-1} m' = u', \quad \sigma'_i = -\mathbb{I}^{-1} \text{ad}^*_u f^i.
\]
A dynamical expression can be calculated as

$$ dh(m') = \left< u', -\text{ad}_a^* E[m] \right> dt - \sum_k \left< \text{ad}_a^* f^k \cdot dW_i^k - \frac{1}{2} \text{ad}_a^* f^k \cdot dW^k_i \right> dt + \frac{1}{2} \sum_k \left< \text{ad}_a^* f^k, \Gamma^{-1} \text{ad}_a^* f^k \right> dt $$

$$ = \sum_k \left< -\text{ad}_a^* f^k \cdot dW_i^k - \frac{1}{2} \text{ad}_a^* f^k \cdot dW^k_i, u' \right> + \frac{1}{2} \left< \text{ad}_a^* f^k, \Gamma^{-1} \text{ad}_a^* f^k \right> dt $$

$$ = \sum_k \left< u', -\text{ad}_a^* f^k \cdot dW_i^k + \left< \text{ad}_a^* f^k, \frac{1}{2} \text{ad}_a^* f^k \right> dt + \frac{1}{2} \left< \text{ad}_a^* f^k, \Gamma^{-1} \text{ad}_a^* f^k \right> dt $$

$$ = \sum_k \left< u', -\text{ad}_a^* f^k \cdot dW_i^k + \left< -\text{ad}_a^* f^k + \Gamma^{-1} \text{ad}_a^* f^k, \frac{1}{2} \text{ad}_a^* f^k \right> dt + \frac{1}{2} \left< \text{ad}_a^* f^k, \Gamma^{-1} \text{ad}_a^* f^k \right> dt $$

$$ = \sum_k \left< u', -\text{ad}_a^* f^k \cdot dW_i^k + \frac{1}{2} \left< \text{ad}_a^* f^k, \Gamma^{-1} \text{ad}_a^* f^k \right> dt - \frac{1}{2} \left< -\text{ad}_a^* f^k + \Gamma^{-1} \text{ad}_a^* f^k, \text{ad}_a^* f^k \right> dt $$

$$ + \frac{1}{2} \left< \Gamma^{-1} \text{ad}_a^* f^k, \text{ad}_a^* f^k \right> dt . $$

Taking the expectation yields

$$ \partial_t E[h(m')] = \frac{1}{2} \left< \Gamma^{-1} \text{ad}_a^* f^k, \text{ad}_a^* f^k \right> . $$ (3.9)

The above calculation with (3.5) have proved the following.

**Theorem 3.1 (Energy balance for EA SFLT).** The sum of energies $h(E[m]) + E[h(m')]$ is preserved by the dynamics of EA SFLT in (3.1),

$$ \partial_t h(E[m]) + \partial_t E[h(m')] = 0 . $$ (3.10)

**Proof.** The previous calculation demonstrates this theorem by direct calculation. Having done so, an alternative proof suggests itself, for quadratic Hamiltonians, one has $h(m) = h(E[m] + m') = const \Rightarrow E[h(m)] = h(E[m]) + E[h(m')]$. \qed

### 3.2 EA SFLT with advected quantities

It is straightforward to extend the EA SFLT framework to SFLT systems with advected quantities. Starting with the energy conserving SFLT equation with advected quantities (2.19), one can take the average of $m$ and $a$ in the underlying probability space to have the EA SFLT equation with advected quantities

$$ \text{dm} + \text{ad}_{a}^* \mathbb{E}[m] \cdot dt + b \circ \mathbb{E}[a] \cdot dt + \sum_i \left( \text{ad}_{a}^* f^i \circ dW_i^i + b \circ f^i \circ dW^i_i \right) = 0 $$

$$ \text{da} + \mathcal{L}_a \mathbb{E}[a] \cdot dt + \sum_i \mathcal{L}_a f^a_i \circ dW_i^i = 0 \quad \text{(u, b, a)} \left( \frac{\partial h}{\partial m} \frac{\partial h}{\partial a} \right) , $$ (3.11)

which can be written more succinctly by using the ad* action on the semidirect product Lie algebra, more specifically,

$$ \text{d}(m, a) + \text{ad}_{(u, b)}^* \mathbb{E}[(m, a)] \cdot dt + \sum_i \text{ad}_{(u, b)}^* (f^i, f^a_i) \circ dW_i^i = 0 . $$ (3.12)

Energy preservation is inherited from the SFLT equation with advected quantities,

$$ dh(m, a) = \left< \text{d}(m, a), \left( \frac{\partial h}{\partial m} \frac{\partial h}{\partial a} \right) \right> = - \left< \text{ad}_{(u, b)}^* \mathbb{E}[(m, a)] \cdot dt + \sum_i \text{ad}_{(u, b)}^* (f^i, f^a_i) \circ dW_i^i, (u, b) \right> = 0 , $$ (3.13)

where the last equality uses the anti-symmetry of ad* of the semidirect product Lie algebra. The Itô form of (3.11) is

$$ \text{dm} + \text{ad}_{a}^* \mathbb{E}[m] \cdot dt + b \circ \mathbb{E}[a] \cdot dt + \sum_i \left( \text{ad}_{a}^* f^i \circ dW_i^i + b \circ f^i \circ dW^i_i + \frac{1}{2} \left( \text{ad}_{a}^* f^i + \theta_i \circ f^i \right) \cdot dt \right) = 0, $$

$$ \text{da} + \mathcal{L}_a \mathbb{E}[a] \cdot dt + \sum_i \left( \mathcal{L}_a f^a_i \circ dW_i^i + \frac{1}{2} \mathcal{L}_a f^a_i \circ dt \right) = 0 , $$ (3.14)
where one defines
\[ \sigma_i := \left( \frac{\delta^2 h}{\delta m^2}, -\partial_{x} f_i + \frac{\delta h}{\delta a} \circ f_i^a \right) \quad \text{and} \quad \theta_i := \left( \frac{\delta^2 h}{\delta a}, -\mathcal{L}_{\frac{\delta h}{\delta a}} f_i^a \right). \]

The proof is similar to the case of SFLP equation with advected quantities which is included in appendix C. Taking expectation of (3.14) to have temporal evolution of the expectation of \( m \) and \( a \). These equations are deterministic because of the linearity of \( \partial_{\sigma} \) and \( \partial \) operators, as well as the \( dW_t^a \) terms vanishing by Itô’s Lemma.

\[
\begin{align*}
\partial_t \mathbb{E}[m] + \partial_{x} f_i^m \mathbb{E}[m] + \mathbb{E}[b] \circ \mathbb{E}[a] + \frac{1}{2} \sum_i \left( \partial_{x} f_i^m \mathbb{E}[\sigma_i] f_i^a + \mathbb{E}[\theta_i] \circ f_i^a \right) &= 0 \\
\partial_t \mathbb{E}[a] + \mathcal{L}_{\mathbb{E}[a]} \mathbb{E}[a] + \frac{1}{2} \sum_i \mathcal{L}_{\mathbb{E}[\sigma_i]} f_i^a &= 0.
\end{align*}
\]

Equations (3.15) closes when the Hamiltonian is assumed to be quadratic in \( m \) and \( a \), i.e., \( \frac{\delta^2 h}{\delta m^2} = \mathbb{I}^{-1} \) and \( \frac{\delta^2 h}{\delta a^2} = \mathbb{J}^{-1} \), where the \( \mathbb{I} : \mathbb{g} \rightarrow \mathbb{g}^* \) and \( \mathbb{J} : V \rightarrow V^* \) are the inertia tensors for the vector fields and advected quantities respectively, which are also assumed to be positive definite.

\[
\frac{\delta h}{\delta m'} = \mathbb{I}^{-1} m' = u', \quad \frac{\delta h}{\delta a'} = \mathbb{J}^{-1} a' = b'
\]

In this case one have the vector fields
\[
\begin{align*}
\mathbb{E}[\sigma_i] &= -\mathbb{I}^{-1} \left( \partial_{x} f_i^m + \mathbb{E}[b] \circ f_i^a \right) \\
\mathbb{E}[\theta_i] &= -\mathbb{J}^{-1} \left( \mathcal{L}_{\mathbb{E}[a]} f_i^a \right)
\end{align*}
\]

and the equations (3.15) closes. Using the evolution of the expectations \( \mathbb{E}[m] \) and \( \mathbb{E}[a] \), we have the following time derivative of a quadratic Hamiltonian of \( \mathbb{E}[m] \) and \( \mathbb{E}[a] \)

\[
\begin{align*}
\partial_t h(\mathbb{E}[m], \mathbb{E}[a]) &= -\sum_i \frac{1}{2} \left( \langle \mathbb{E}[u], \partial_{x} f_i^m + \mathbb{E}[\sigma_i] \circ f_i^a \rangle + \langle \mathbb{E}[b], \mathcal{L}_{\mathbb{E}[\theta_i]} f_i^a \rangle \right) \\
&= -\sum_i \frac{1}{2} \left( \langle \mathbb{I}^{-1} \left( \partial_{x} f_i^m + \mathbb{E}[b] \circ f_i^a \right), \left( \partial_{x} f_i^m + \mathbb{E}[b] \circ f_i^a \right) \rangle + \langle \mathbb{J}^{-1} \mathcal{L}_{\mathbb{E}[a]} f_i^a, \mathcal{L}_{\mathbb{E}[a]} f_i^a \rangle \right) < 0
\end{align*}
\]

(3.17)

The fluctuation \( m' = m - \mathbb{E}[m] \) and \( a' = a - \mathbb{E}[a] \) are computed in Itô form using (3.14) and (3.15) as

\[
\begin{align*}
\text{d}m' + \partial_{x} f_i^m \text{d}m' + b' \circ \mathbb{E}[a] \text{d}t + \sum_i \left( \partial_{x} f_i^m dW_i^m + b \circ f_i^a dW_i^a + \frac{1}{2} \left( \partial_{x} f_i^m + b' \circ f_i^a \right) dt \right) &= 0 \\
\text{d}a' + \mathcal{L}_{\mathbb{E}[a]} \mathbb{E}[a] \text{d}t + \sum_i \left( \mathcal{L}_{u} f_i^a dW_i^m + \frac{1}{2} \mathcal{L}_{\sigma_i} f_i^a dt \right) &= 0
\end{align*}
\]

(3.18)

(3.19)

Taking expectation gives

\[
\begin{align*}
\partial_t \mathbb{E}[h(m', a')] &= \sum_i \frac{1}{2} \left( \langle \mathbb{I}^{-1} \left( \partial_{x} f_i^m + \mathbb{E}[b] \circ f_i^a \right), \left( \partial_{x} f_i^m + \mathbb{E}[b] \circ f_i^a \right) \rangle + \langle \mathbb{J}^{-1} \mathcal{L}_{\mathbb{E}[a]} f_i^a, \mathcal{L}_{\mathbb{E}[a]} f_i^a \rangle \right).
\end{align*}
\]

(3.20)

The above calculations with (3.17) proves the following theorem
\textbf{Theorem 3.2} (Energy balance for EA SFLT with advected quantities). Equations (3.11) lead to the energy balance,

\[ \partial_t h(E[m], E[a]) + \partial_m h(m', a^\prime) = 0. \] (3.21)

\textit{Proof.} The previous calculation demonstrates this theorem by direct calculation. Having done so, an alternative proof suggests itself for quadratic Hamiltonians, since in this case one has \( h(m, a) = h(E[m] + m', E[a] + a^\prime) = \text{const} \implies E[h(m, a)] = h(E[m], E[a]) + E[h(m', a')]. \)

\textbf{Theorem 3.3} (EA SFLT Kelvin circulation theorem). Upon assuming that the fluid density \( D \) is also advected by the flow, so that \( \partial_t D + \mathcal{L}_u D = 0 \), the Kelvin circulation theorem may be expressed as

\[ \frac{d}{dt} \int_{c(u)} \frac{m}{D} = \int_{c(u)} - \mathcal{L}_u D dt + \frac{1}{D} \left[ \int m \mathcal{L}_u D dt - \text{ad}^* \mathcal{E}[m] dt - b \mathcal{E}[a] dt - \sum_i (\text{ad}^* f^i \circ dW^i + b \circ f^i \circ dW^i) \right], \] (3.22)

where the notation of Lagrangian loop \( c(u) \) denotes the Lagrangian loop moving with the deterministic fluid velocity \( u \), as in the deterministic case.

\textit{Proof.} When \( m \in \mathbb{R}^3 \), one can identify the \( \text{ad}^* \) action as the Lie derivative, i.e. \( \text{ad}^* m = \mathcal{L}_u m \) for all \( u \in u \). Let \( m = D\alpha \) where \( \alpha \in \Lambda^1 \), the stochastic time derivative of \( m \) can be written as

\[ dm = \alpha \, dD + D \, d\alpha = -\alpha \mathcal{L}_u D dt + D \, d\alpha, \] (3.23)

thus

\[ d\alpha = \frac{\alpha}{D} \mathcal{L}_u D dt - \frac{1}{D} \left( \text{ad}^* \mathcal{E}[m] dt + b \mathcal{E}[a] dt + \sum_i (\text{ad}^* f^i \circ dW^i + b \circ f^i \circ dW^i) \right). \] (3.24)

Inserting the expression of \( d\alpha \) into the circulation integral to have the result.

\[ \frac{d}{dt} \int_{c(u)} \alpha = \int_{c(u)} (d + \mathcal{L}_u dt) \alpha \]

\[ = \int_{c(u)} \mathcal{L}_u \alpha dt + \frac{\alpha}{D} \mathcal{L}_u D dt - \frac{1}{D} \left( \text{ad}^* \mathcal{E}[m] dt + b \mathcal{E}[a] dt + \sum_i (\text{ad}^* f^i \circ dW^i + b \circ f^i \circ dW^i) \right). \] (3.25)

\textbf{Remark 3.4.} This section has responded to Lorentz’s lesson in [55] that climate is essentially probabilistic. Accordingly, one might imagine that climate change science would be predicated on the dynamics of variances and higher moments of the fluctuations, which might even apply to the considerations introduced here. Investigations of the potential for applications of the EA SFLT formulation to climate science have been left for future work.

\section{Examples of SFLT applications}

\subsection{Heavy Top}

The motion of a heavy top under gravity is a good first example application for SFLT, because it is a finite degree-of-freedom subsystem of Euler-Boussinesq fluid motion [33]. Following, e.g., [37], the configuration space of the heavy top can be taken as the semidirect-product Lie group of Euclidean motions by rotation and translation, \( S = \text{SO}(3) \ltimes \mathbb{R}^3 \). The associated Lie algebra and its dual are, respectively,

\[ s = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \simeq \mathbb{R}^3 \ltimes \mathbb{R}^3 \quad \text{and} \quad s^* = \mathfrak{so}(3)^* \ltimes \mathbb{R}^3 \simeq \mathbb{R}^3 \ltimes \mathbb{R}^3. \]

The natural pairing between \( \mathbb{R}^3 \) and its dual is the dot-product, for the pairing of Euclidean vectors in \( \mathbb{R}^3 \). The heavy top Hamiltonian \( H(\Pi, \Gamma) \) comprises the sum of its rotational kinetic energy and its gravitational potential energy,

\[ H(\Pi) = \frac{1}{2} \Pi \cdot \Pi^{-1} \Pi + mg \Gamma \cdot \chi. \] (4.1)
The Hamiltonian for the heavy top is written in the body frame, in terms of the body angular momentum, \( \mathbf{\Pi} \), and the vertical unit vector as seen from the body, \( \mathbf{\Gamma} = \mathbf{z} \mathbf{O}(t)^{-1} \), where \( \mathbf{O}(t) \in SO(3) \) is the time-dependent rotation from the reference configuration of the top to its current configuration. Here, \( \mathbb{I} \) is the moment of inertia in the body frame and \( \mathbf{\chi} \) is the vector from the point of support to the centre of mass in the body frame. Finally, \( m \) is the mass of the body and \( g \) is the constant acceleration of gravity.

The variational derivatives of the heavy top Hamiltonian are

\[
\frac{d\mathcal{H}}{d\mathbf{\Pi}} = \mathbb{I}^{-1} \mathbf{\Pi} = \mathbf{\Omega} \quad \text{and} \quad \frac{d\mathcal{H}}{d\mathbf{\Gamma}} = mg\mathbf{\chi}. 
\]

Upon using the ‘hat’ map isomorphism \([37]\) to identify the Lie derivative and \( \text{ad}^* \) operation as the cross product in the \( \mathbb{R}^3 \) representation, the deterministic LP equation is written as

\[
\frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi} \times \mathbf{\Omega} + \Gamma \times mg\mathbf{\chi}, \quad \frac{d\mathbf{\Gamma}}{dt} = \Gamma \times \mathbf{\Omega},
\]

which can also be written in Poisson operator form

\[
\frac{d}{dt} \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi} \times & \mathbf{\Gamma} \times \\
\mathbf{\Gamma} \times & 0 \end{bmatrix} \begin{bmatrix} \delta\mathcal{H}/\delta\mathbf{\Pi} \\ \delta\mathcal{H}/\delta\mathbf{\Gamma} \end{bmatrix}. 
\]

Defining the force vectors \( \mathbf{f}_i^m, \mathbf{f}_i^a \in \mathbb{R}^3 \), the SFLP equations corresponding to (2.19) may be written as

\[
\frac{d\mathbf{\Pi}}{dt} = \left( \mathbf{\Pi} dt + \sum_i \mathbf{f}_i^m \circ dW_i^t \right) \times \mathbf{\Omega} + \left( \mathbf{\Gamma} dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times mg\mathbf{\chi}, \\
\frac{d\mathbf{\Gamma}}{dt} = \left( \mathbf{\Gamma} dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times \mathbf{\Omega}, 
\]

When written in Poisson bracket form, these equations become

\[
\frac{d}{dt} \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi} dt + \sum_i \mathbf{f}_i^m \circ dW_i^t \times & (\mathbf{\Gamma} dt + \sum_i \mathbf{f}_i^a \circ dW_i^t) \times \\
n(\mathbf{\Gamma} dt + \sum_i \mathbf{f}_i^a \circ dW_i^t) \times & 0 \end{bmatrix} \begin{bmatrix} \delta\mathcal{H}/\delta\mathbf{\Pi} \\ \delta\mathcal{H}/\delta\mathbf{\Gamma} \end{bmatrix}. 
\]

One observes that these equations preserve the Hamiltonian in (4.1), which follows because of skew symmetry of the matrix Poisson operator.

**Eulerian averaged heavy top** Consider the EA SFLP equations associated to (4.2), which read

\[
\frac{d\mathbf{\Pi}}{dt} = \left( \mathbf{E} \left[ \mathbf{\Pi} \right] dt + \sum_i \mathbf{f}_i^m \circ dW_i^t \right) \times \mathbf{\Omega} + \left( \mathbf{E} \left[ \mathbf{\Gamma} \right] dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times mg\mathbf{\chi}, \\
\frac{d\mathbf{\Gamma}}{dt} = \left( \mathbf{E} \left[ \mathbf{\Gamma} \right] dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times \mathbf{\Omega}. 
\]

Energy conservation is immediate since the Poisson structure is preserved. In Itô form, these equation reads

\[
\frac{d\mathbf{\Pi}}{dt} = \left( \mathbf{E} \left[ \mathbf{\Pi} \right] dt + \sum_i \mathbf{f}_i^m \circ dW_i^t \right) \times \mathbf{\Omega} + \left( \mathbf{E} \left[ \mathbf{\Gamma} \right] dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times mg\mathbf{\chi} + \frac{1}{2} \sum_i \mathbf{f}_i^m \times \mathbb{I}^{-1} (\mathbf{f}_i^m \times \mathbf{\Omega} + \mathbf{f}_i^a \times mg\mathbf{\chi}), \\
\frac{d\mathbf{\Gamma}}{dt} = \left( \mathbf{E} \left[ \mathbf{\Gamma} \right] dt + \sum_i \mathbf{f}_i^a \circ dW_i^t \right) \times \mathbf{\Omega} + \frac{1}{2} \sum_i \mathbf{f}_i^m \times \mathbb{I}^{-1} (\mathbf{f}_i^m \times \mathbf{\Omega} + \mathbf{f}_i^a \times mg\mathbf{\chi} 
\]

Taking expectations to have the time evolution of expectations \( \mathbf{E} \left[ \mathbf{\Pi} \right] \) and \( \mathbf{E} \left[ \mathbf{\Gamma} \right] \) as

\[
\delta \mathbf{E} \left[ \mathbf{\Pi} \right] = \mathbf{E} \left[ \mathbf{\Pi} \right] \times \mathbf{E} \left[ \mathbf{\Omega} \right] + \mathbf{E} \left[ \mathbf{\Gamma} \right] \times mg\mathbf{\chi} + \frac{1}{2} \sum_i \mathbf{f}_i^m \times \mathbb{I}^{-1} (\mathbf{f}_i^m \times \mathbf{E} \left[ \mathbf{\Omega} \right] + \mathbf{f}_i^a \times mg\mathbf{E} \left[ \mathbf{\chi} \right]), \\
\delta \mathbf{E} \left[ \mathbf{\Gamma} \right] = \mathbf{E} \left[ \mathbf{\Gamma} \right] \times \mathbf{E} \left[ \mathbf{\Omega} \right] + \frac{1}{2} \sum_i \mathbf{f}_i^m \times \mathbb{I}^{-1} (\mathbf{f}_i^m \times \mathbf{E} \left[ \mathbf{\Omega} \right] + \mathbf{f}_i^a \times mg\mathbf{E} \left[ \mathbf{\chi} \right]).
\]
As the Hamiltonian is quadratic, the above equations close and we have

\[ \partial_t H(\mathbb{E}[\Pi], \mathbb{E}[\Gamma]) = -\frac{1}{2} \left( f_i^m \times \mathbb{E}[\Omega] + f_i^a \times mg\mathbb{E}[\chi] \right) \mathbb{I}^{-1} \left( f_i^m \times \mathbb{E}[\Omega] + f_i^a \times mg\mathbb{E}[\chi] \right). \]  

(4.7)

If the inertial tensor \( I \) is positive definite, the energy of the expectations \( \mathbb{E}[\Pi] \) and \( \mathbb{E}[\Gamma] \) will decay to zero. Defining the fluctuation of \( \Pi \) and \( \Gamma \) as \( \Pi' := \Pi - \mathbb{E}[\Pi] \) and \( \Gamma' := \Gamma - \mathbb{E}[\Gamma] \) respectively, the evolution of the fluctuation can be written in Itô form as

\[
d\Pi' = \left( \Pi' dt + \sum_i f_i^m dW_i \right) \times \Omega' + \left( \Gamma' dt + \sum_i f_i^a dW_i \right) \times mg \chi' \\
+ \frac{1}{2} \sum_i f_i^m \times \mathbb{I}^{-1} \left( f_i^m \times \Omega' + f_i^a \times mg \chi' \right),
\]

(4.8)

\[
d\Gamma' = \left( \mathbb{E}[\Gamma] dt + \sum_i f_i^a dW_i \right) \times \Omega' + \frac{1}{2} \sum_i f_i^m \times \mathbb{I}^{-1} \left( f_i^m \times \Omega' + f_i^a \times mg \chi' \right).
\]

The energy of the fluctuations \( h(\Pi', \Gamma') \) has evolution equation after using Itô’s Lemma

\[
dh(\Pi', \Gamma') = \sum_i \Omega' \cdot \left( f_i^m \times \Omega' + f_i^a \times mg \chi' \right) dW_i + mg \chi' \cdot f_i^a \times \Omega' dW_i \\
- \frac{1}{2} \left( f_i^m \times \Omega' + f_i^a \times mg \chi' \right) \mathbb{I}^{-1} \left( f_i^m \times \Omega' + f_i^a \times mg \chi' \right) \\
+ \frac{1}{2} \left( f_i^m \times \Omega + f_i^a \times mg \chi \right) \mathbb{I}^{-1} \left( f_i^m \times \Omega + f_i^a \times mg \chi \right)
\]

(4.9)

Taking the expectation yields

\[
\partial_t \mathbb{E}[h(\Pi', \Gamma')] = \frac{1}{2} \left( f_i^m \times \mathbb{E}[\Omega] + f_i^a \times mg \mathbb{E}[\chi] \right) \mathbb{I}^{-1} \left( f_i^m \times \mathbb{E}[\Omega] + f_i^a \times mg \mathbb{E}[\chi] \right).
\]

(4.10)

Together with the expression of \( \partial_t H(\mathbb{E}[\Pi], \mathbb{E}[\Gamma]) \), we have exemplified Theorem 3.2 for the EA HT dynamics.

### 4.2 Energy-preserving stochastic rotating shallow water equations (RSW)

Let \( S = \text{Diff}(D) \otimes \text{Den}(D) \), where \( \text{Diff}(D) \) denotes the group of diffeomorphisms acting on the planar domain \( D \). Let \( \eta \in \mathcal{F}(D) \), and let \( \eta dV \in \text{Den}(D) \) denote the density on \( D \). The associated Lie algebra and its dual are \( \mathfrak{s} = \mathfrak{X}(D) \otimes \text{Den}(D) \) and \( \mathfrak{s}^* = (\Lambda^1(D) \otimes \text{Den}(D)) \otimes \text{Den}(D) \). We denote \( u \cdot \frac{\partial}{\partial x} = u \in \mathfrak{X}(D) \) and \( m = m \cdot dx \otimes dV \in \Lambda^1(D) \otimes \text{Den}(D) \). In this notation, the LP equations can be written in Cartesian coordinates as follows, the partial differential equations,

\[
\frac{\partial}{\partial t} \left( \frac{\mathbf{m}}{\eta} \right) + (\mathbf{u} \cdot \nabla) \frac{\mathbf{m}}{\eta} + \frac{m_{ij}}{\eta} \mathbf{u} \cdot \mathbf{v}^j + \nabla \frac{\delta H}{\delta \eta} = 0, \quad \frac{\partial}{\partial t} \eta + \nabla \cdot (\eta \mathbf{u}) = 0,
\]

(4.11)

and the LP operator can be written as

\[
\frac{\partial}{\partial t} \left[ m_i \right] = - \left[ \frac{\partial_j m_i + m_i \partial_j}{\eta} \frac{\eta \partial_i}{\delta \eta} \right] \left[ \frac{\delta H}{\delta m_j} \right].
\]

(4.12)

For rotating shallow water equations (RSW) in \( \mathbb{R}^2 \), the Hamiltonian \( H \) is given by

\[
H = \int_D \frac{1}{2\epsilon \eta} \left( \mathbf{m} - \eta \mathbf{R} \right)^2 + \frac{(\eta - B)^2}{\epsilon \mathcal{F}} d^2 x,
\]

(4.13)

in which \( \epsilon \ll 1 \) denotes Rossby number and \( \mathcal{F} = O(1) \) denotes the Froude number. The mean depth is \( B \) and the surface elevation is \( (\eta - B) \). The reduced Legendre transform yields \( \mathbf{u} = \delta H / \delta \mathbf{m} = (\mathbf{m} - \eta \mathbf{R}) / \epsilon \eta \). The variational derivatives of the RSW Hamiltonian are obtained as

\[
\delta H = \int_D \mathbf{u} \cdot \delta \mathbf{m} + \left( \frac{\eta - B}{\epsilon \mathcal{F}} - \frac{\epsilon}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{R} \right) \cdot \delta \eta d^2 x.
\]

Substituting into (4.11) and using the relation \((\text{curl} \mathbf{u}) \times \mathbf{v} + \nabla (\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{u} \times \nabla \times \mathbf{v} \) yields the standard set of RSW equations governing motion and continuity,

\[
\frac{\partial \mathbf{u}}{\partial t} + (\text{curl} \mathbf{R} + \epsilon \text{curl} \mathbf{u}) \times \mathbf{u} + \nabla \left( \frac{\eta - B}{\epsilon \mathcal{F}} + \frac{\epsilon}{2} |\mathbf{u}|^2 \right) = 0, \quad \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) = 0.
\]

(4.14)
Consider the Poisson operator of the form (2.20) applied to \( s^* \). It reads

\[
\frac{d m_i}{\eta} = - \left[ \frac{\partial_j m_i + m_j \partial_i \eta}{\partial_j \eta} \right] dt - \sum_k \left[ \frac{\partial_j f^k_i + f^k_j \partial_i \eta}{\partial_j \eta} \right] \left( \frac{\delta H/\delta m_j}{\delta H/\delta \eta} \right) dW_t^k ,
\]

where \( f^k \) are the components of \( F^k \) such that \( f^k = f^k \cdot dx \otimes dV \in \Lambda^1(D) \otimes \text{Den}(D) \) and \( g^k \in \mathcal{F}(D) \) for all \( k \).

Then the SFLP equations become

\[
dm_i + (\partial_j (m_i w^j) + m_j \partial_i w^j) dt + (\partial_j (f^k_i w^j) + \sum_k f^k_j \partial_i w^j) \circ dW_t^k + \eta \partial_i \eta\frac{\delta H}{\delta \eta} dt + g^k \partial_i \frac{\delta H}{\delta \eta} \circ dW_t^k = 0 ,
\]

\[
d\eta + \nabla \cdot (\eta u) dt + \sum_k \nabla \cdot (g^k u) \circ dW_t^k = 0 .
\]

**Remark 4.1.** When noise is introduced into the continuity equation for density \( \eta \), one can still write the momentum equation in terms of \( m/\eta \). Namely, it reads

\[
d\eta \frac{\delta H}{\delta \eta} dt + \sum_k \eta dV ((-\frac{m}{\eta} \partial_j (g^k w^j) + \partial_j (f^k_i w^j) + f^k_j \partial_i w^j + g^k \partial_i \frac{\delta H}{\delta \eta}) \circ dW_t^k = 0 ,
\]

where the inhomogenous term \(-\eta^2 m_j \partial_i (g^k w^j)\) in the second line of (4.17) is due to the modified advection relation for \( \eta dV \). The SFLP equations for RSW dynamics in (4.16) written more succinctly using Lie derivatives and denoting the momentum 1-form as \( \alpha = m \cdot dx \), then

\[
d\alpha/\eta + \left( L_u \alpha/\eta + d \frac{\delta H}{\delta \eta} \right) dt + \sum_k \eta dV \left( -\frac{\alpha}{\eta} L_u g^k + L_u f^k + \frac{\delta H}{\delta \eta} g^k \circ dW_t^k = 0 ,
\]

\[
d(\eta dV) + \nabla \cdot (\alpha/\eta dV) dt + \sum_k L_u (g^k dV) \circ dW_t^k = 0 ,
\]

In vector calculus notation, the stochastic RSW equation obtained by substituting the variational derivatives of \( H \) into equation (4.15) yields

\[
ces u + (\text{curl} \mathbf{R} + \epsilon \text{curl} u) \times u dt + \nabla \left( \frac{\eta - B}{\epsilon F} + \frac{\epsilon}{2} |u|^2 \right) dt
\]

\[+ \sum_k \frac{1}{\eta} \left( -\left( \epsilon \mathbf{u} + \mathbf{R} \right) \nabla \cdot (g^k u) + (\text{curl} f^k) \times u + \nabla (f^k \cdot u) + f^k \nabla u + g^k \nabla \pi_{RSW} \right) \circ dW_t^k = 0 ,
\]

\[
d\eta + \nabla \cdot (\eta u) dt + \sum_k \nabla \cdot (g^k u) \circ dW_t^k = 0 ,
\]

where \( \pi_{RSW} = \left( \frac{\eta - B}{\epsilon F} - \frac{\epsilon}{2} |u|^2 - u \cdot \mathbf{R} \right) \).

The corresponding Kelvin circulation theorem follows easily from (4.18) in geometric form, as

\[
d \oint_{c(u)} \frac{\alpha}{\eta} dt = \frac{\delta H}{\delta \eta} \circ dW_t^k .
\]

The stochastic RSW equations in (4.19) enable the Kelvin circulation equation (4.20) to be written in vector calculus form as

\[
d \oint_{c(u)} (\epsilon u + \mathbf{R}) \cdot dx
\]

\[= - \oint_{c(u)} \frac{1}{\eta} \left( -\left( \epsilon \mathbf{u} + \mathbf{R} \right) \nabla \cdot (g^k u) + (\text{curl} f^k) \times u + \nabla (f^k \cdot u) + f^k \nabla u + g^k \nabla \pi_{RSW} \right) \cdot dx \circ dW_t^k .
\]

Here, one sees the effects of the material entrainment terms proportional to \( g^k \) appearing in the generation of Kelvin circulation. Setting \( g^k = 0 \) simplifies equation (4.20) to

\[
d \oint_{c(u)} \frac{\alpha}{\eta} dt = - \oint_{c(u)} \frac{1}{\eta} \circ dW_t^k .
\]
The vector form of the motion equation also simplifies for \( g^k = 0 \) to
\[
\epsilon \text{du} + (\text{curlR} + \epsilon \text{curlu}) \times \text{u} \, dt + \nabla \left( \frac{\eta - B}{cF} + \frac{\epsilon}{2} |\text{u}|^2 \right) dt = -\sum_k \frac{1}{\eta} ((\text{curlf}^k) \times \text{u} + \nabla(f^k - \text{u}) + f^k \nabla \cdot \text{u}) \circ dW_t^k,
\]
whose right hand side may be regarded as a compressible version of the CL vortex force. The Ito form is
\[
\epsilon \text{du} + (\text{curlR} + \epsilon \text{curlu}) \times \text{u} \, dt + \nabla \left( \frac{\eta - B}{cF} + \frac{\epsilon}{2} |\text{u}|^2 \right) dt + \sum_k \frac{1}{\eta} ((\text{curlf}^k) \times \text{u} + \nabla(f^k - \text{u}) + f^k \nabla \cdot \text{u}) \, dW_t^k = \frac{1}{2} \sum_k \frac{1}{\eta} ((\text{curlf}^k) \times \sigma^k + \nabla(f^k - \sigma^k) + f^k \nabla \cdot \sigma^k) \, dt,
\]
where \( \sigma^k = \frac{1}{\eta} ((\text{curlf}^k) \times \text{u} + \nabla(f^k - \text{u}) + f^k \nabla \cdot \text{u}) \). By taking the exterior derivative of equation (4.18) and noting \( d^2_q = \eta q \, dV \) where \( q \) is the potential vorticity and \( dV \) is the area element, one finds that the vorticity density \( \eta q \, dV \) satisfies
\[
d(\eta q \, dV) + L_u(\eta q \, dV) \, dt + \sum_k \frac{d}{\eta} \frac{1}{dV} \left[ -\frac{\alpha}{\eta} L_u g^k + L_u f^k + g^k \frac{d\delta H}{d\eta} \right] = 0.
\]

**Remark 4.2.** In coordinates, the last two summands in equation (4.25) can be written as divergences, so one finds the pathwise conservation law \( d \int_D(\eta q \, dV) = 0 \) for appropriate (homogeneous, or periodic) boundary conditions.

### 4.3 Euler-Boussinesq (EB) equations

Consider the case where \( S = \text{Diff}(D) \circ \mathcal{F}(D) \circ \text{Den}(D) \), where \( D \, dV \in \text{Den}(D) \) and \( b \in \mathcal{F}(D) \). The Poisson operator form of the LP equation in this case is
\[
\frac{\partial}{\partial t} \begin{bmatrix} m_i \\ D \\ b \end{bmatrix} = -\begin{bmatrix} \partial_j m_i + m_j \partial_i D \partial_k - b_j \\ \partial_j D \\ b_j \end{bmatrix} \begin{bmatrix} \delta H/\delta m_j \\ \delta H/\delta D \end{bmatrix}.
\]

The stochastic extension of equation (2.20) to include buoyancy reads
\[
d \begin{bmatrix} m_i \\ D \\ b \end{bmatrix} = -\begin{bmatrix} \partial_j m_i + m_j \partial_i D \partial_k - b_j \\ \partial_j D \\ b_j \end{bmatrix} \begin{bmatrix} \delta H/\delta m_j \\ \delta H/\delta D \end{bmatrix} \, dt - \sum_k \begin{bmatrix} \partial_j f^k + f^k_j \partial_i g^k - a^k_j \\ a^k_j \end{bmatrix} \begin{bmatrix} \delta H/\delta m_j \\ \delta H/\delta b \end{bmatrix} \circ dW_t^k.
\]

Here, \( f^k \) are the components of \( f^k \) such that \( f^k = f^k \cdot dx \otimes dV \in \Lambda^1(D) \circ \text{Den}(D) \), \( g^k \in \mathcal{F}(D) \) and \( a^k \in \mathcal{F}(D) \) for all \( k \). For simplicity, let us consider incompressible flows with \( D = 1 \) and neglect the stochastic part of the advection of \( D \). That is, we set \( g^k = 0 \). This case will yield the standard incompressibility condition, \( \nabla \cdot \text{u} = 0 \).

The EB Hamiltonian is
\[
H = \int_D \left[ \frac{1}{2} |\text{u}|^2 - GBz + p(D - 1) \right] \, d^3 x,
\]
where the momentum density \( \text{m} = D(\text{u} + \text{R}(x)) \) and the pressure \( p \) is a Lagrange multiplier which enforces incompressibility. The variational derivatives of \( H \) are given by
\[
\delta H = \int_D \text{u} \cdot \delta \text{m} + \delta D (gbz + p - \frac{1}{2} |\text{u}|^2 - \text{u} \cdot \text{R}(x)) + (gDz) \delta b \, d^3 x.
\]

The deterministic EB equations then follow as
\[
\partial_t \text{u} - \text{u} \times \text{curl}(\text{u} + \text{R}(x)) = -\nabla \left( p + \frac{1}{2} |\text{u}|^2 \right) - gb \hat{z},
\]
\[
\partial_t D + \text{div}(D \text{u}) = 0, \quad \text{with} \quad D = 1,
\]
\[
\partial_t b + \text{u} \cdot \nabla b = 0.
\]
One can obtain the energy preserving stochastic EB equations by substituting the variational derivatives into stochastic Poisson operator in (4.27) to find
\[ d\mathbf{u} - \mathbf{u} \times \text{curl}(\mathbf{u} + \mathbf{R}) \, dt + \nabla \left( dp + \frac{1}{2}|\mathbf{u}|^2 \, dt \right) + gb \mathbf{z} \, dt + \sum_k \left[ -\mathbf{u} \times \text{curl}(\mathbf{f}^k) + \nabla(\mathbf{f}^k \cdot \mathbf{u}) - gz \cdot \nabla a^k \right] \circ dW^k_t = 0, \]
\[ \partial_t D + \text{div}(D \mathbf{u}) = 0, \quad \text{with} \quad D = 1, \]
\[ db + \mathbf{u} \cdot \nabla b \, dt + \sum_k \mathbf{u} \cdot \nabla a^k \circ dW^k_t = 0. \]
\( (4.31) \)

**Remark 4.3.** Following [66], the change of pressure \( p \rightarrow dp = p \, dt + \sum_i p_i \circ dW^i_t \) has been made, so that the incompressibility condition, \( \nabla \cdot \mathbf{u} = 0 \), will be enforced in both the drift and stochastic parts of \( \mathbf{u} \).

Similar to the case of \( S = \text{Diff}(\mathcal{D}) \otimes \text{Den}(\mathcal{D}) \) for energy-preserving stochastic RSW equation, a convenient way of considering the Kelvin circulation theorem for the Euler-Boussinesq equations is to write the associated SFLP equations in terms of the momentum 1-form \( \alpha = \mathbf{m} \cdot d\mathbf{x} \). These equations then read
\[ d\alpha + \left( L_u \alpha - \frac{\delta H}{\delta b} db \right) \, dt + d \left( dp - \frac{1}{2} |\mathbf{u}|^2 \, dt - \mathbf{u} \cdot D \mathbf{u} \, dt \right) + \sum_k \frac{1}{D} \left( L_u \beta^k - \frac{\delta H}{\delta b} da^k \right) \circ dW^k_t = 0, \]
\[ \partial_t (D \mathbf{v}) + L_u(D \mathbf{v}) = 0, \quad \text{with} \quad D = 1, \]
\[ db + \mathbf{u} \cdot \nabla b \, dt + \sum_k \mathbf{u} \cdot \nabla a^k \circ dW^k_t = 0, \]
\( (4.32) \)

where the 1-form \( \beta^k := \mathbf{f}^k \cdot d\mathbf{x} \). The Kelvin circulation theorem for these equations is immediate, as
\[ d \int_{c(u)} \alpha = \int_{c(u)} \frac{\delta H}{\delta b} db \, dt - \sum_k \int_{c(u)} \left( L_u \beta^k - \frac{\delta H}{\delta b} da^k \right) \circ dW^k_t. \]
\( (4.33) \)

In vector calculus notation, this Kelvin circulation theorem is written as
\[ d \int_{c(u)} (\mathbf{u} + \mathbf{R}) \cdot d\mathbf{x} = \int_{c(u)} g \mathbf{z} \cdot \nabla b \cdot d\mathbf{x} \, dt - \sum_k \int_{c(u)} \left( (\text{curl} \mathbf{f}^k) \times \mathbf{u} - g \mathbf{z} \cdot \nabla a^k \right) \cdot d\mathbf{x} \circ dW^k_t \]
\( (4.34) \)

**Remark 4.4.** In the case where \( \mathbf{f}^0 = -\mathbf{u}^S(\mathbf{x}), \) \( dW^p_t = dt, \) and \( a^k = 0 \) for \( k = 0, \ldots, \) one finds that the energy-preserving stochastic EB equations in (4.31) produce a stochastic contribution to the vortex force in the Craik-Leibovich equations [15] whose formulation with Hamilton’s principle is discussed in [34]. Namely, they reduce as follows,
\[ d\mathbf{u} - \mathbf{u} \times \text{curl}(\mathbf{u} - \mathbf{u}^S(\mathbf{x}) + \mathbf{R}(\mathbf{x})) \, dt = - \nabla \left( p + \frac{1}{2}|\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{u}^S \right) \, dt - gb \mathbf{z} \, dt + \sum_{k>0} \left( \mathbf{u} \times \text{curl} \mathbf{f}^k - \nabla(\mathbf{f}^k \cdot \mathbf{u}) \right) \circ dW^k_t, \]
\[ \partial_t D + \text{div}(D \mathbf{u}) = 0, \quad \text{with} \quad D = 1, \]
\[ \partial_t b + \mathbf{u} \cdot \nabla b = 0, \]
(4.35)

where one interprets the semimartingale \( d\mathbf{u}^S = \mathbf{u}^S(\mathbf{x}) \, dt + \sum_k \mathbf{f}^k(\mathbf{x}) \circ dW^k_t \) as a stochastic augmentation of the usual steady prescribed Stokes drift velocity.

**Physical interpretation of the functions \( \mathbf{f}^k \) and \( a^k \) in terms of stochastic PV fluxes** The energy-preserving stochastic EB equations in (4.31) imply the following equation for potential vorticity density, \( q \, dV \), defined by \( q \, dV := d\alpha \wedge db = (\text{curl} \mathbf{m}) \cdot \nabla b \, dV = (\omega + 2\mathbf{f}) \cdot \nabla b \, dV, \) where \( \mathbf{m} = \mathbf{u} + \mathbf{R} \) for \( D = 1 \). Namely,
\[ d(q \, dV) + L_u(q \, dV) \, dt = d(\alpha + L_u \alpha \, dt) \wedge db + d(\alpha \wedge db + L_u b \, dt) \]
\[ = - \sum_k d \left[ (L_u \beta^k - g \mathbf{z} \, da^k) \wedge db - (L_u a^k) \, d\alpha \right] \circ dW^k_t, \]
\( (4.36) \)

where we recall that \( \beta^k := \mathbf{f}^k \cdot d\mathbf{x} \) is a 1-form. In vector calculus notation, after using the incompressibility condition \( \nabla \cdot \mathbf{u} = 0, \) the potential vorticity equation (4.36) can be written in terms of \( q = (\text{curl} \mathbf{m}) \cdot \nabla b \) as, cf.
\[ dq + u \cdot \nabla q \, dt = \sum_k \text{div} \left[ (u \times \text{curl} f^k + gz \nabla a^k) \times \nabla b + (\omega + 2\Omega) \cdot \nabla (u \cdot \nabla a^k) \right] \, dW^k_t \]

\[ = \sum_k \text{div} \left[ F^k \times \nabla b + (\omega + 2\Omega) \cdot (\nabla D^k) \right] \, dW^k_t =: -\sum_k \text{div} J^k \, dW^k_t, \tag{4.37} \]

where \( \text{curl} \, m = \omega + 2\Omega \) is the total vorticity, and the quantities \( F^k \) and \( D^k \) are defined as

\[ F^k := u \times \text{curl} f^k + gz \nabla a^k \quad \text{and} \quad D^k := u \cdot \nabla a^k. \tag{4.38} \]

The summands in \( J^k = -F^k \times \nabla b - (\omega + 2\Omega) \cdot (\nabla D^k) \) are called the “\( J \)-fluxes of PV” and are identified with “frictional” and “diabatic” effects, respectively, in [32, 56]. See also [8] for LES turbulence interpretations of these fluxes.

In summary, while the energy-preserving stochastically-augmented CL vortex force and entrainment effects in equations (4.31) or (4.35) can locally create stochastic Langmuir circulations, the total volume-integrated potential vorticity \( Q = \int_D q \, dV \) will be preserved for appropriate boundary conditions. See [67] for more information about Langmuir circulations and their importance in the mixing processes in the upper ocean boundary layer. The sub-mesoscale excitations created by the \( J \)-fluxes of PV are a subject of intense present research aimed at understanding the effects of turbulence on oceanic frontogenesis, as well as wave forcing which transports materials such as sediment, gases, algae (carbon), oil spills and plastic detritus, [59, 57, 58, 60, 51, 9, 20, 68].

**Remark 4.5** (Eulerian averaged Euler Boussinesq equation). For completeness, we mention that the corresponding Eulerian averaged equation to the stochastic EB equation (4.31). Namely, the EA SFLT EB equation is given by

\[
\begin{align*}
\frac{d}{dt} u - u \times \text{curl} \mathbb{E}[u + R] &+ \nabla \left( dp - \frac{1}{2} |u|^2 \, dt - u \cdot R \, dt + \mathbb{E}[u + R] \cdot u \, dt \right) - gz \nabla \mathbb{E}[b] \, dt \\
+ \sum_k \left[ -u \times \text{curl} f^k + \nabla (f^k \cdot u) - gz \cdot \nabla a^k \right] \, dW^k_t &= 0, \\
\partial_t D + \text{div}(Du) &= 0, \quad \text{with} \quad D = 1, \\
db + u \cdot \nabla b &+ \sum_k u \cdot \nabla a^k \, dW^k_t = 0.
\end{align*}
\tag{4.39}
\]

The time evolution of expectation of \( u \) and \( b \) can be found by passing to the Hamiltonian side and we have

\[ \partial_t \mathbb{E}[m], \mathbb{E}[b] = -\frac{1}{2} \sum_i \int_D \left[ -\mathbb{E}[u] \times \text{curl} f^i + \nabla (\mathbb{E}[u] \cdot f^i) - \mathbb{E}[gz] \cdot \nabla a^i \right]^2 \, d^3 x \tag{4.40} \]

and

\[ \partial_t \mathbb{E}[H(m', b')] = \frac{1}{2} \sum_i \int_D \left[ -\mathbb{E}[u] \times \text{curl} f^i + \nabla (\mathbb{E}[u] \cdot f^i) - \mathbb{E}[gz] \cdot \nabla a^i \right]^2 \, d^3 x \tag{4.41} \]

which agrees with Theorem 3.2.

## 5 Conclusion and outlook

The motivation of this paper has been to determine what type of stochastic perturbations can be added to fluid dynamics that will preserve the fundamental properties of energy conservation, Kelvin circulation theorem and conserved quantities arising from the Lagrangian particle relabelling symmetry. The geometric framework employed in this paper introduces stochastic forcing by Lie transport (SFLT) noise as a series of perturbations which automatically produce a Kelvin circulation theorem and can be chosen to satisfy either energy, or Casimir preservation. In this paper, we have mainly focused on preserving energy conservation. These stochastic external forces can be seen as the slow + fast decomposition of external forces corresponding to the slow + fast decomposition of fluid flow. For Euler fluid equations, the stochastic CL vortex force will always be energy conserving and its physical interpretation as a wave-averaged forces fits well into the external forces considered in the reduced Lagrange-d’Alembert-Pontryagin (RLDP) principle. In comparison with the location uncertainty (LU) approach by Mémin [62], the present paper gives an alternative set of fluid equations which are
energy preserving. The relation to LU has been left for future work. Numerical simulations of the stochastically forced Lie-Poisson (SFLP) equations (2.19) will be needed to classify solution behaviours of these new stochastic extension of classical fluid equations. As in the applications of the stochastic advection by Lie transport (SALT) and LU approaches, computational simulations of Langmuir fluid circulations and their material entrainment using equations (4.31) and (4.35) will require the calibration of the functions \( f^i_0, f^i_0 \), perhaps via data analysis methods similar to those used in the approach detailed in [12, 13]. Computational simulations of the equations resulting from the SFLT and SFLP modelling approaches introduced here, as well as simulations of the EA SFLT equations in section 3 have all been left for future work.

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Data availability

No data was created or used in writing this paper.

A Coadjoint operator of semidirect-product Lie-Poisson brackets

Following [41] and [18], consider a Lie Group \( G \) which acts from the left by linear maps on a vector space \( V \) which induces a left action of \( G \) on \( V^* \). In the right representation of \( G \) on the vector space \( V \), the semidirect product group \( S = G \ltimes V \) has group multiplication

\[
(g_1, v_1)(g_2, v_2) = (g_1g_2, v_2 + v_1g_2),
\]

where the action of \( G \) on \( V \) is denoted by concatenation \( vg \). The identity element in \( S \) is \((e, 0)\) where \( e \) is the identity in \( G \). The inverse of an element in \( S \) is given by

\[
(g, v)^{-1} = (g^{-1}, -vg^{-1})
\]

The Lie algebra bracket on the semidirect product Lie algebra \( \mathfrak{s} = \mathfrak{g} \ltimes V \) is given by

\[
[[\xi_1, v_1], (\xi_2, v_2)] = ([\xi_1, \xi_2], v_2\xi_1 - v_1\xi_2)
\]

where the induced action of \( \mathfrak{g} \) on \( V \) is denoted by concatenation \( v\xi \). The operation \( \text{AD} : S \times S \to S \) is defined by

\[
\text{AD}_{(g_1, v_1)}(g_2, v_2) = (g_1, v_1)(g_2, v_2)(g_1, v_1)^{-1} = (g_1g_2g_1^{-1}, -v_1g_1^{-1} + v_2g_1^{-1} + v_1g_2g_1^{-1}).
\]

Taking the time derivatives of \( g_2 \) and \( v_2 \), then evaluating them at the identity \( t = 0 \) yields the Adjoint operation \( \text{Ad} : S \times \mathfrak{s} \to \mathfrak{s} \) which is defined by

\[
\text{Ad}_{(g, v)}(\xi, a) = \left. \frac{d}{dt} \right|_{t=0} \text{AD}_{(g, v)}(\tilde{g}(t), \tilde{v}(t)) = (g\xi g^{-1}, (a + v\xi)g^{-1}),
\]

where \( \frac{d}{dt} \big|_{t=0} \tilde{g}(t) = \xi \) and \( \frac{d}{dt} \big|_{t=0} \tilde{v}(t) = a \). The coAdjoint operation \( \text{Ad}^* \) is the formal adjoint of \( \text{Ad} \) with respect to the pairings \( \langle \cdot, \cdot \rangle_\mathfrak{g} \) and \( \langle \cdot, \cdot \rangle_V \), which can be computed as

\[
\text{Ad}^*_{(g, v)}(\mu, a) = (g\mu + (vg^{-1}) \circ (ag^{-1}), ag^{-1}),
\]

where the diamond operator \( \circ \) is defined as

\[
\langle \beta \circ a, \xi \rangle_\mathfrak{g} := \langle b, -a\xi \rangle_V.
\]
The notation $a g^{-1}$ denotes the inverse of the dual isomorphism defined by $g \in G$ (so that $g \rightarrow a g^{-1}$ is a right action). Note that the adjoint and coadjoint actions are left actions. In this case, the $g$-actions on $g^*$ and $V^*$ are defined as before to be minus the dual map given by the $g$-actions on $g$ and $V$ and are denoted, respectively, by $\xi \mu$ (left action) and $a \xi$ (right action). Taking time derivative of $(g, v)$ in the definition of $\text{Ad}_g(v)$ and evaluating at the identity gives the adjoint operator $ad$ which coincides with the Lie algebra bracket. One computes the formal adjoint of $ad$ with respect to the pairing

$$\langle \text{ad}_{(\xi, b)}(\xi, b), (\mu, a) \rangle = \left( \langle \text{ad}_\xi \xi, b \xi - b \xi \rangle, (\mu, a) \right) = \left( \langle \text{ad}^*_\xi \mu + b \circ a, -a \xi \rangle, (\xi, b) \rangle \right),$$

where in the first equality we have used the left Lie algebra action in (A.2).

Proposition B.1 (Euler-Poincaré theorem [41]). The Euler-Poincaré (EP) equations of a reduced Lagrangian $\ell(u, a), \ell : \mathcal{X} \times V^* \rightarrow \mathbb{R}$ defined over the space of smooth vector fields with elements $u \in \mathcal{X}$ acting by Lie derivative on elements $a \in V^*$ of a vector space $V^*$ are written as [41]

$$\frac{d}{dt} \ell + ad^*_a \frac{\delta \ell}{\delta a} - \frac{\delta \ell}{\delta u} \circ a = 0, \quad \frac{d}{dt} a - L_u a = 0. \quad \text{(B.1)}$$

In the EP equation in (B.1), the variational derivative is defined as usual by,

$$\delta \ell(u, a) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \ell(u + \epsilon a, a) = \left( \frac{\delta \ell}{\delta u}, \frac{\delta \ell}{\delta a} \right)_u + \left( \frac{\delta \ell}{\delta a}, \frac{\delta \ell}{\delta a} \right)_a. \quad \text{(B.2)}$$

The quantities $a \in V^*$ and $\delta \ell/\delta a \in V$ in (B.2) are dual to each other under the $L^2$ pairing $\langle \cdot, \cdot \rangle_V : V \times V^* \rightarrow \mathbb{R}$. Likewise, the velocity vector field $u \in \mathcal{X}$ and momentum 1-form density $m := \delta \ell/\delta u \in \mathcal{X}^*$ are dual to each other under the $L^2$ pairing $\langle \cdot, \cdot \rangle_\mathcal{X} : \mathcal{X} \times \mathcal{X}^* \rightarrow \mathbb{R}$. In terms of these pairings and the Lie derivative operator $L_u$ with respect to the vector field $u \in \mathcal{X}$, the coadjoint operator $ad^*_u$ and the diamond operator $(\diamond)$ in (B.1) are defined by

$$\langle \text{ad}^*_u \frac{\delta \ell}{\delta u}, v \rangle_\mathcal{X} := \left( \frac{\delta \ell}{\delta a}, -L_u v \right)_\mathcal{X} := \left( \frac{\delta \ell}{\delta u}, ad_u v \right)_\mathcal{X}, \quad \text{(B.3)}$$

where $v \in \mathcal{X}$, and $ad : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is defined as $ad_u v := -[u, v] := -(v^j u^i - v^i u^j) \partial_i$, and

$$\langle b \circ a, v \rangle_{\mathcal{X}} := \left( b, -L_v a \right)_V. \quad \text{(B.4)}$$

As we shall see later, the coadjoint operator $ad^*_u$ and the diamond operator $(\diamond)$ enter together in (B.1) as a form of coadjoint motion for semidirect product action of the Lie algebra.
Proposition B.2 (The EP equation (B.1) also follows from the HPVP). The EP equation in (B.1) can be derived by direct calculation from the following Hamilton-Pontryagin variational principle (HPVP)

\[ 0 = \delta S = \delta \int_a^b \left( \ell(u, a) + \langle m \cdot \delta g^{-1} - u \rangle + \langle db \cdot a_0 g^{-1} - a \rangle \right) dt, \quad (B.5) \]

where \( m \in \mathbb{X}^* \), \( b \in V \), \( g \in G \) and the variations are taken to be arbitrary.

### B.2 Stochastic semidirect-product coadjoint fluid motion with SALT noise

To add noise in the SALT form to the deterministic HPVP in (B.5), we introduce the following stochastic variational principle, cf. [38],[4]

\[ 0 = \delta S = \delta \int_a^b \left( \ell(u, a) dt + \langle m \cdot \delta g^{-1} - u dt \rangle + \langle db \cdot a_0 g^{-1} - a \rangle - \sum_i h_i(m, a) \circ dW_i^i \right), \quad (B.6) \]

For brevity of notation, we will suppress the subscript labels in the pairings whenever the meaning is clear. The symbol \( \delta \) in (B.6) abbreviates stochastic time integrations. The action integral \( S \) in (B.6) is defined in the framework of variational principles with semimartingale constraints which was established in [56]. As we shall see below, the semimartingale nature of a Lagrange multiplier which imposes one of these semimartingale constraints emerges in the context of the full system of equations, which is obtained after the variations have been taken. The Hamiltonian functions \( h_i(m, a) \) on \( \mathbb{X}^* \times V^* \) in (B.6) will be prescribed here without discussing how they would be obtained in practice, e.g., via data assimilation. The data assimilation procedure for SALT is discussed, e.g., in [12, 13].

Euler-Poincaré (EP) Lagrangian formulation. Taking arbitrary variations in the stochastic HPVP in equation (B.6) yields the following determining relations among the variables,

\[
\begin{align*}
0 &= \int_a^b \frac{\delta \ell}{\delta u} \delta u \, dt + \frac{\delta \ell}{\delta a} \delta a \, dt + \delta m \cdot \delta g^{-1} - u \, dt + \langle db \cdot a_0 g^{-1} - \delta a \rangle + \langle \delta db \cdot a_0 g^{-1} - a \rangle \\
&\quad - \left( \frac{\delta h_i}{\delta m} \circ dW_i^i \right) + \left( \frac{\delta h_i}{\delta a} \circ dW_i^i \right) + \langle m \cdot (\delta (d g^{-1}) - \delta u) \rangle \\
&= \int_a^b \left( \frac{\delta \ell}{\delta u} \delta u \, dt + \frac{\delta \ell}{\delta a} \delta a \, dt + \delta m \cdot \delta g^{-1} - u \, dt + \langle db \cdot -a \eta - \delta a \rangle + \langle \delta db \cdot a_0 g^{-1} - a \rangle \\
&\quad - \left( \frac{\delta h_i}{\delta m} \circ dW_i^i \right) + \left( \frac{\delta h_i}{\delta a} \circ dW_i^i \right) + \langle m \cdot (d \eta - ad \cdot a_0 g^{-1} \cdot \eta - \delta u) \rangle.
\end{align*}
\]

Here, \( \eta = \delta g^{-1} \) and natural boundary terms have been assumed. Collecting terms among the variational relations gives the following set of equations, which turn out to involve four semimartingales,

\[
\begin{align*}
\frac{\delta \ell}{\delta u} &= m, \quad \frac{\delta \ell}{\delta a} = b, \quad \delta g^{-1} = u \, dt + \sum_i \frac{\delta h_i}{\delta m} \circ dW_i^i, \quad \delta b = \frac{\delta \ell}{\delta a} \, dt - \sum_i \frac{\delta h_i}{\delta a} \circ dW_i^i, \\
\frac{\delta m}{\delta u} &= -ad \cdot a_0 g^{-1} \cdot m + db \circ a, \quad \frac{\delta a}{\delta a} = -L \cdot d g^{-1} \cdot a.
\end{align*}
\]

Thus the SALT EP equations are found to be

\[
\begin{align*}
d\frac{\delta \ell}{\delta u} &= -ad \cdot a_0 g^{-1} \cdot \frac{\delta \ell}{\delta u} + \left( \frac{\delta \ell}{\delta a} \, dt - \sum_i \frac{\delta h_i}{\delta a} \circ dW_i^i \right) \circ a, \quad da = -L \cdot d g^{-1} \cdot a,
\end{align*}
\]

where the definition of \( dg^{-1} \) are taken from (B.7). For a similar, but more rigorous approach to the derivation of these equations, see [56].

Lie-Poisson (LP) Hamiltonian formulation. Using the Legendre transform, \( h(m, a) = \langle m \cdot u \rangle - \ell(u, a) \) and taking variations yields

\[
\begin{align*}
\frac{\delta \ell}{\delta u} &= m, \quad \frac{\delta h}{\delta m} = u, \quad \frac{\delta h}{\delta a} = -\frac{\delta \ell}{\delta a}.
\end{align*}
\]

\footnote{Note: we can choose separate (uncorrelated) Brownian motions in the \( m \) and \( a \) equations in (B.6) by choosing \( h_i(m, a) = h_i^m(m) + h_i^a(a) \) for the Stratonovich noise, \( dW_i^i \). The choice of Stratonovich noise enables the standard calculus chain rule and product rule to be used for the operations of differentiation and integration by parts, in which variational principles are defined.}
and the corresponding SALT Lie-Poisson (LP) Hamiltonian equations obtained after a Legendre transform are
\[
\frac{\partial m}{\partial t} = -\text{ad}^*_{\frac{\partial}{\partial m}} m \left( \frac{\delta h}{\delta a} \, dt + \sum_i \frac{\delta h_i}{\delta a} \, dW^i_t \right) \circ a, \quad \frac{\partial a}{\partial t} = -L_{\frac{\partial}{\partial a}}, \tag{B.10}
\]
where the Lagrangian path \( dx_t = dg \, g^{-1} \) Legendre-transforms to the Hamiltonian side as
\[
dx_t = \frac{\delta h}{\delta m} \, dt + \sum_i \frac{\delta h_i}{\delta m} \circ dW^i_t. \tag{B.11}\]

By using the \( \text{ad}^* \) operator for semidirect product Lie algebras defined in appendix A, the LP equations (B.10) can be written equivalently in the following compact form,
\[
d(m, a) = -\text{ad}^*_{(dx_t, db)} (m, a), \quad \text{where } db = -\frac{\delta h}{\delta a} \, dt - \sum_i \frac{\delta h_i}{\delta a} \circ dW^i_t.
\]

**Remark B.3** (Stochastic reduced Hamiltonian phase space variational principle for SALT). The SALT equations (B.10) can be derived from a stochastic reduced Hamilton phase-space variational principle, namely,
\[
0 = \delta S = \delta \int_a^b \langle m, dg \, g^{-1} \rangle + \langle db, a_0 g^{-1} - a \rangle - h(m, a) \, dt - \sum_i h_i(m, a) \circ dW^i_t,
\]
where the variations \( \delta m, \delta g, \delta a \) and \( \delta (b) \) are taken to be arbitrary.

The SALT Hamiltonian equations in (B.10) can be arranged into the Lie-Poisson (LP) operator form
\[
\frac{d}{dt} \begin{bmatrix} m \\ a \end{bmatrix} = -\left[ \frac{\partial}{\partial m} \begin{bmatrix} m \\ a \end{bmatrix} \bigg|_{dx_t, db} \right] \begin{bmatrix} \frac{\delta h}{\delta m} \, dt + \sum_i \frac{\delta h_i}{\delta m} \circ dW^i_t \\ \frac{\delta h}{\delta a} \, dt + \sum_i \frac{\delta h_i}{\delta a} \circ dW^i_t \end{bmatrix}. \tag{B.12}
\]

The Lie-Poisson operator in (B.12) preserves the Casimirs of its deterministic counterpart, since the Poisson structures remains the same. However, the Hamiltonian is now a semimartingale, so energy depends explicitly on time and, hence, is no longer preserved.

**Kelvin-Noether theorem.** The SALT Lie-Poisson (LP) Hamiltonian equations in (B.10) also possess a Kelvin-Noether theorem. To understand this statement, consider the following \( G \)-equivariant map \( \mathcal{K} : C \times V^* \to \mathfrak{X}^* \) as explained in [41],
\[
\langle \mathcal{K}(gc, ag^{-1}) \circ \text{Ad}_{g^{-1}}^* v, v \rangle = \langle \mathcal{K}(c, a) \circ v, v \rangle, \quad \text{for all } g \in G,
\]
for a manifold \( C \) on which \( G \) acts from the left. For fluid dynamics, \( C \) is the space of loops the fluid domain \( \mathcal{D} \) and the map \( \mathcal{K} \) is the circulation around the loop. More specifically, for all \( \alpha \in \Lambda^1 \),
\[
\langle \mathcal{K}(c, a) \circ \alpha, \alpha \rangle := \oint_c \alpha. \tag{B.13}
\]

**Theorem B.4** (SALT Kelvin-Noether theorem). Given solutions \( m(t), a(t) \) satisfying the SALT LP equations (B.10) and fixed \( c_0 \in C \), the associated Kelvin-Noether quantity \( \langle \mathcal{K}(g(t)c_0, a(t)), m(t) \rangle \) satisfies the following stochastic Kelvin-Noether relation.
\[
\frac{d}{dt} \langle \mathcal{K}(g(t)c_0, a(t)), m(t) \rangle = \left\langle \mathcal{K}(g(t)c_0, a(t)), \left( \frac{\delta h}{\delta a} \, dt - \sum_i \frac{\delta h_i}{\delta a} \circ dW^i_t \right) \circ a \right\rangle \tag{B.14}
\]
where we identify \( dg \, g^{-1} = : dx_t \).

**Remark B.5.** The proof of theorem B.4 is in appendix B.3. The fluid mechanics counterpart of the Kelvin-Noether theorem is expressed as
\[
\oint_{c_t} \frac{m}{D} = \oint_{c_t} \frac{1}{D} \left( \frac{\delta h}{\delta a} \, dt - \sum_i \frac{\delta h_i}{\delta a} \circ dW^i_t \right) \circ a, \quad \text{where } D = m / \text{mass density of the fluid which is also advected as } D(t) = g_t * D_0. \quad \text{That is, the mass density } D \text{ satisfies the stochastic continuity equation } dD + L_{dx_t} D = 0.
\]
Remark B.6 (SALT Hamiltonians). In [38], the SEP equations are derived from a stochastic Clebsch variational principle where the advection of phase space Lagrangian variables are through a stochastic vector field $dx_t = u dt + \sum_i \xi_i(x) \circ dW^i_t$. Compared to the stochastic vector field defined via $dg^{-1}$ in (B.7), they coincide with those in [38] when the noise Hamiltonians $h_i$ are linear in $m$, i.e. $h_i(m, a) = \langle \xi_i(x), m \rangle$. The SEP equations then becomes

$$\frac{d \delta f}{du} = -ad^*_{dx_t} \frac{d \delta f}{da} + \frac{d \delta f}{a} \circ a dt,$$  
$$da = -L_{a, x} dx_t,$$  
$$dx_t = u dt + \sum_i \xi_i(x) \circ dW^i_t.$$  

For data assimilation purposes, the choice $h_i(m, a) = \sum_i \langle m, \xi_i(x) \rangle$ was made in [13, 12].

B.3 SALT Kelvin theorem via the Kunita-Itô-Wentzell theorem

A pair of results in Kunita [44, 45] provided the key to working with stochastic advection by Lie transport (SALT) in ideal fluid dynamics. In particular, if we choose the diffeomorphism $\phi_t$ as the stochastic process obtained by homogenisation in [14]

$$d\phi_t(x) := u(\phi_t(x), t) dt + \sum_i \xi_i(\phi_t(x)) \circ dW^i_t,$$

then the Kunita Itô-Wentzell change of variables formula discussed below leads to the following differential form leads to the stochastic advection law,

$$d(\phi^*_t K(t, x)) = \phi^*_t \left( dK(t, x) + L_{\phi_t(x)} K(t, x) \right) = 0, \quad a.s.$$  

where $L_{\phi_t(x)}$ is the Lie derivative by the vector field $d\phi_t(x)$ whose time integral $\int_0^t d\phi_s(x) = \phi_t(x) - \phi_0(x)$ generates the semimartingale flow $\phi_t$ acting on semimartingale k-form, $dK(t, x) = G(t, x) dt + H(t, x) \circ dW_t$. One may recall that the Lie derivative $L_{\phi_t(x)} K$ has both a dynamic and a geometric definition,

$$L_{\phi_t(x)} K = \lim_{\Delta x \to 0} \frac{1}{\Delta x} (\phi^*_t K - K) = d\phi_t \star dK + d(d\phi_t \star K)$$

in which the latter formula is attributed to Cartan.

Here is a simplified statement of the theorem for applying the Kunita Itô-Wentzell change of variables formula to stochastic advection of differential k-forms proved in [19], based on [44, 45]. See also [49].

Theorem B.7 (Kunita-Itô-Wentzell (KIW) formula for k-forms). Consider a sufficiently smooth k-form $K(t, x)$ in space which is a semimartingale in time

$$dK(t, x) = G(t, x) dt + \sum_{i=1}^M H_i(t, x) \circ dW^i_t,$$

(B.15)

where $W^i_t$ are i.i.d. Brownian motions. Let $\phi_t$ be a sufficiently smooth flow satisfying the SDE

$$d\phi_t(x) = b(t, \phi_t(x)) dt + \sum_{i=1}^N \xi_i(t, \phi_t(x)) \circ dB^i_t,$$

in which $B^i_t$ are i.i.d. Brownian motions. Then the pull-back $\phi^*_t K$ satisfies the formula

$$d(\phi^*_t K)(t, x) = \phi^*_t G(t, x) dt + \sum_{i=1}^M \phi^*_i H_i(t, x) \circ dW^i_t$$  
$$+ \phi^*_t L_{b, x} K(t, x) dt + \sum_{i=1}^N \phi^*_i L_{\xi_t} K(t, x) \circ dB^i_t.$$  

(B.16)

Formulas (B.15) and (B.16) are compact forms of equations in [19] which are written in integral notation to make the stochastic processes more explicit.

To understand the distinction between integral and differential notation for SPEs, one may begin by writing the stochastic ‘fundamental theorem of calculus’ as

$$\phi^*_t K(t, x) - \phi^*_0 K(0, x) := K(t, \phi_t(x)) - K(0, x) = \int_0^t d(\phi^*_s K_s).$$
In the integral notation, the Kunita-Itô-Wenzell (KIW) formula is written as
\[
\int_t^s \mathbf{d} (\phi^*_s K_s) = \int_t^s \phi^*_s (\mathbf{d} K(s, x) + \mathbf{L}_{ad}(x) K(s, x)).
\]

So, in the differential notation the KIW formula ‘transfers’ to the equivalent differential form
\[
\mathbf{d} (\phi^*_s K(t, x)) = \phi^*_t \left( \mathbf{d} K(t, x) + \mathbf{L}_{ad}(x) K(t, x) \right), \quad \text{a.s.}
\]

**Remark B.8.** In applications, one sometimes expresses equation (B.16) using the differential notation
\[
\mathbf{d} (\phi^*_s K)(t, x) = \phi^*_s (\mathbf{d} K + \mathbf{L}_{ad}(x) K)(t, x),
\]
where \(dx_t\) is the stochastic vector field \(dx_t(x) = b(t, x)dt + \sum_{i=1}^N \xi_i(t, x) \circ dB^i_t\). Importantly for fluid dynamics, this formula is also valid when \(K\) is a vector field rather than a k-form.

**Stochastic Kelvin circulation theorem for the SALT theory.** Having understood the differential notation for stochastic integrals, now we may assemble the stochastic fluid equations via the stochastic Kelvin circulation theorem for the SALT theory [38]. For this purpose, we shall make the argument that the stochastic Kelvin circulation theorem is fundamentally a stochastic form of Newton’s law of motion,
\[
\mathbf{d} \int_{\mathcal{c}(\phi_t)} \mathbf{v} \cdot dx = \int_{\mathcal{c}(\phi_t)} (\mathbf{d} + \mathbf{L}_{ad}) (\mathbf{v} \cdot dx) = \int_{\mathcal{c}(\phi_t)} \mathbf{f} \cdot dx.
\]

This formula corresponds to the motion equation derived from Hamilton’s principle
\[
(\mathbf{d} + \mathbf{L}_{ad}) \left( \frac{1}{2} \frac{\delta^2}{\delta \mathbf{u}} \right) dx = \mathbf{f} \cdot dx,
\]
along with the law of advection of mass expressed in KIW form
\[
(\mathbf{d} + \mathbf{L}_{ad}) (Dd^3 x) = 0,
\]
where the flow velocity is given by the stochastic vector field
\[
d\phi_t(x) := u(\phi_t(x), t)dt + \sum_i \xi_i(\phi_t(x)) \circ dB^i_t.
\]
Thus, the stochastic Kelvin’s circulation theorem for SALT simply describes the rate of change of momentum of a stochastically moving material loop.

**C Itô form of the SFLP equation**

**Proposition C.1.** The Itô form of the SFLP equation (2.7)
\[
\mathbf{d} m + \mathbf{ad}_{m}^* m dt + \sum_i \mathbf{ad}_{m}^* f_i \circ dW^i_t = 0,
\]
is given by
\[
\mathbf{d} m + \mathbf{ad}_{m}^* m dt + \frac{1}{2} \sum_i \mathbf{ad}_{w}^* f_i dt + \sum_i \mathbf{ad}_{m}^* f_i dW^i_t = 0, \quad \sigma_i = \left( -\mathbf{ad}_{m}^* f_i, \frac{\delta^2 h}{\delta m^2} \right), \quad \text{(C.1)}
\]
where the brackets \((\cdot, \cdot)\) in the definition of \(\sigma_i\) denotes contraction, not \(L^2\) pairing.

**Proof.** This can be shown via direct computation. We ignore the drift term and suppress the indices on constants \(f_i\) for ease of notation, choosing an arbitrary \(\phi = \phi(x) \in \mathcal{X}\), we have the Stratonovich stochastic equation
\[
\mathbf{d} \langle m, \phi \rangle = \left( -\mathbf{ad}_{m}^* f, \phi \right) \circ dW_t = \left( \mathbf{ad}_{m}^* f, \frac{\delta h}{\delta m} \right) \circ dW_t \quad \text{(C.2)}
\]
The corresponding Itô form is then
\[
\langle d\phi, m \rangle = \left\langle ad_\phi^* f, \frac{\delta h}{\delta m} \right\rangle dW_t + \frac{1}{2} \left\langle d\phi, f \right\rangle, dW_t \right]\]
\[
= \left\langle ad_\phi^* f, \frac{\delta h}{\delta m} \right\rangle dW_t + \frac{1}{2} \left\langle d\phi, f \right\rangle, dW_t \right]\]
\[
= \left\langle ad_\phi^* f, \frac{\delta h}{\delta m} \right\rangle dW_t + \frac{1}{2} \left\langle -\frac{\delta^2 h}{\delta m^2}, \frac{\delta h}{\delta m} f \right\rangle dW_t, dW_t \right]\]
\[
= -\left\langle ad_\phi^* f, \phi \right\rangle dW_t - \frac{1}{2} \left\langle ad_\phi^* f, \phi \right\rangle [dW_t, dW_t] \right)
\]
\[
= -\left\langle ad_\phi^* f, \phi \right\rangle dW_t - \frac{1}{2} \left\langle ad_\phi^* f, \phi \right\rangle [dW_t, dW_t] \right)
\]
\[
= \left\langle ad_\phi^* f, \phi \right\rangle dW_t + \frac{1}{2} \left\langle ad_\phi^* f, \phi \right\rangle [dW_t, dW_t] \right)
\]
\[
= \left\langle ad_\phi^* f, \phi \right\rangle dW_t - \frac{1}{2} \left\langle ad_\phi^* f, \phi \right\rangle [dW_t, dW_t] \right)
\]

Since \(\phi\) is arbitrary, the Itô form of \((2.7)\) is
\[
dm + ad_m^* m dt + \sum_i ad_m^* f_i dW_i + \frac{1}{2} \sum_{i,j} ad_m^* f_i [dW_i, dW_j] = 0. \tag{C.4}
\]

For Brownian motion, the quadratic variation term simplifies to \([dW_i, dW_j] = \delta_{ij} dt\) which recovers Itô form as presented.

**Proposition C.2.** The Itô form of the SFLP equation with advected quantity \((2.19)\)
\[
dm + ad_m^* m dt + \frac{\delta h}{\delta a} \circ a dt + \sum_i ad_m^* f^i \circ dW^i + \frac{\delta h}{\delta a} \circ g^i \circ dW^i = 0 \tag{C.5}
\]
\[
da + L \circ a dt + \sum_i L \circ g^i \circ dW^i = 0
\]
is given by
\[
dm + ad_m^* m dt + \frac{\delta h}{\delta a} \circ a dt + \sum_i ad_m^* f^i \circ dW^i + \frac{\delta h}{\delta a} \circ g^i \circ dW^i + \frac{1}{2} \sum_i (ad_{\sigma_i} f^i + \theta_i \circ g^i) dt = 0 \tag{C.6}
\]
\[
da + L \circ a dt + \sum_i L \circ g^i \circ dW^i + \frac{1}{2} \sum_i L_{\sigma_i} g^i dt = 0
\]
where \(\sigma_i\) and \(\theta_i\) are given as
\[
\sigma_i := \left( \frac{\delta^2 h}{\delta m^2}, \frac{\delta^2 h}{\delta a^2}, -L_{\delta h} f^i \right) \quad \text{and} \quad \theta_i := \left( \frac{\delta^2 h}{\delta a^2}, -L_{\delta h} f^i \right)
\]

**Proof.** Consider similarly by ignoring drift terms and suppress the indices on \(f^i \in X^*, g^i \in V^*, \sigma_i \in X\) and \(\theta_i \in V\). Consider constants \(\phi = \phi(x) \in X\) and \(\psi = \psi(x) \in V\), the following Stratonovich stochastic equations hold
\[
d\langle m, \phi \rangle = -\left\langle ad_m^* f + \frac{\delta h}{\delta a} \circ g, \phi \right\rangle \circ dW_t \quad \text{and} \quad d\langle a, \psi \rangle = -\left\langle L \circ g, \psi \right\rangle \circ dW_t = 0 \tag{C.7}
\]
The Itô form thus satisfies

\[
\begin{align*}
    d\langle m, \phi \rangle &= -\left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> dW_t - \frac{1}{2} \left[ d \left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> , dW_t \right] \\
    &= -\left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> dW_t + \frac{1}{2} \left[ d \left< \frac{\partial}{\partial m} f \circ g, \phi \right> + d \left< L_\phi g, \frac{\delta h}{\delta a} \right> , dW_t \right] \\
    &= -\left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> dW_t + \frac{1}{2} \left[ d \left< \frac{\partial}{\partial m} f \circ g, \frac{\delta h}{\delta a} \right> + d \left< L_\phi g, \frac{\delta h}{\delta a} \right> , dW_t \right] \\
    &= -\left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> dW_t + \frac{1}{2} \left[ \left< \frac{\partial}{\partial m} f \circ g, \sigma \right> + \left< L_\phi g, \theta \right> , dW_t \right] \\
    &= -\left< \frac{\partial}{\partial a} m f + \frac{\delta h}{\delta a} \circ g, \phi \right> dW_t - \frac{1}{2} \left< \frac{\partial}{\partial m} f + L_\phi g, \phi \right> [dW_t, dW_t]
\end{align*}
\]

Since \( \phi \) and \( \psi \) are arbitrary, the Itô form of \( dm \) and \( da \) with drift are

\[
\begin{align*}
    dm + \frac{\partial}{\partial a} m dt + \frac{\delta h}{\delta a} \circ a dt + \sum_i \left( \frac{\partial}{\partial m} f_i + \frac{\delta h}{\delta a} \circ g_i \right) dW_t^i + \frac{1}{2} \sum_{i,j} \left( \frac{\partial}{\partial m} f_i \circ g_j + \theta_j \circ g^j \right) \left[ dW_t^i , dW_t^j \right] &= 0 \\
    da + \frac{\partial}{\partial m} a dt + \sum_{i,j} L_{\sigma,j} g^j \left[ dW_t^i , dW_t^j \right] &= 0
\end{align*}
\]

(C.8)

For Brownian motion, the quadratic variation term simplifies to \( \left[ dW_t^i , dW_t^j \right] = \delta^{ij} dt \) which completes the proof.

\[\square\]

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