Weak Convex Domination in Hypercubes

S. V. Padmavathi
Assistant Professor of Mathematics, Ramanujan Research Center in Mathematics Saraswathi Narayanan College, Madurai, Tamil Nadu, India

K. Krishnan
Vice-Principal and Associate Professor of Mathematics Saraswathi Narayanan College, Madurai, Tamil Nadu, India

Abstract
The n-cube $Q_n$ is the graph whose vertex set is the set of all n-dimensional Boolean vectors, two vertices being joined if and only if they differ in exactly one coordinate. The n-star graph $S_n$ is a simple graph whose vertex set is the set of all $n!$ permutations of {1, 2, ..., n} and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i, i \neq 1$.

In this paper we determine weak convex domination number for hypercubes. Also convex, weak convex, $m$-convex and $l_1$-convex numbers of star and hypercube graphs are determined.

Keywords: Convexity number, Weak convexity number, Weak convex domination, $m$-convexity number, $l_1$-convexity number.

Mathematics Subject Classification: 05C12

Introduction
Graphs considered here are connected, simple. Akers and Krishnamurthy introduced the n-star graph $S_n$ [1]. The vertex set of $S_n$ is the set of all permutations of {1, 2, ..., n} and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i, i \neq 1$.

The n-star graph is an alternative to n-cube with superior characteristics. Day and Tripathi have compared the topological properties of the n-star and the n-cube in [5]. Arumugam and Kala have determined some domination parameters of star graph and obtained bounds for $\gamma$, $\gamma^i$, $\gamma^i_c$, and $\gamma_p$ in n-cube for $n \geq 7$ in [2].

Let $G$ be a simple connected graph. A subset $S$ of $V$ is called a convex set if for any $u, v$ in $S$, $S$ contains all the vertices of every $u - v$ geodesic in $G$. A subset $S$ of $V$ is called a weak convex set if for any $u, v$ in $S$, $S$ contains all the vertices of a $u - v$ geodesic in $G$.

A subset $S$ of $V$ is called a $m$-convex set if for any $u, v$ in $S$, $S$ contains all the vertices of every $u - v$ induced path in $G$.

A subset $S$ of $V$ is called a $l_1$-convex set if it is convex and has a vertex which is adjacent to the rest of the vertices of $S$. Maximum
cardinality of a proper convex set is the convexity number of $G$. In a similar way we define weak convex number, $m$-convex number and $l_1$-convex is the maximum of $\{\text{Con}< N[x]\}/x \in V(G)\}$.

A subset $S$ of $V$ is called a domination set if every vertex in $V-S$ is adjacent to at least one vertex in $S$. A dominating set is a weak convex dominating set if it is weak convex. So far exact value of domination number for large $n$ in $Q_n$ has not been determined. Here we determine weak convex domination number of $Q_n$ for any $n$.

**Results on Convexity Number Parameters**

For $Q_n$

$\text{Con}(Q_n) = 2^{n-1}$ for all $n$.

$\text{wcon}(Q_n) = 2^{n-1}$ for all $n$.

$l_1 - \text{con}(Q_n) = 2$ for all $n$.

$m\text{con}(Q_n) = 2^{n-1}$ for all $n$.

For $S_n$

$\text{Con}(S_n) = 3$.

$\text{wcon}(S_n) = 4$.

$m\text{con}(S_n) = 2$.

$l_1 - \text{con}(S_n) = 2$.

$\text{Con}(S_3) = 6$.

$\text{wcon}(S_3) = 18$.

$m\text{con}(S_3) = 6$.

$l_1 - \text{con}(S_3) = 3$.

In general

$\text{Con}(S_n) = 6$ for all $n$.

$\text{wcon}(S_n) = 2n$ for all $n$.

$m\text{con}(S_n) = 6$ for all $n$.

$l_1 - \text{con}(S_n) = n$ for all $n$, since girth of $S_n$ is $C_6$ and circumference of $S_n$ is $C_{2n}$. $S_n$ is $n$-regular.

**Main Results**

Weak convex domination number in $Q_n$

Theorem 2.1 $\gamma_{wc}(Q_n) = 2^{n-1}$ for all $n$.

Proof: $Q_n$ is a connected graph. For $n = 1$, $Q_1 = K_2$ and $\gamma_{wc}(Q_1) = 1 = 2^{1-1}$.

For $n = 2$, $Q_2 = C_4$ and $\gamma_{wc}(Q_2) = 2 = 2^{2-1}$.

For $n = 3$, $\gamma_{wc}(Q_3) = 4 = 2^{3-1}$.

For $n = 4$ the structure of $Q_4$ is given below.

![Graph of Q_4](image)

We know that $\gamma(Q_3) = 2$. Either $\{3, 5\}$ or $\{1, 8\}$ can be chosen that is diametrically opposite vertices are chosen. Therefore their distance is three and hence $\gamma_{wc}(Q_3) = 4$. Let $\gamma_{wc}(Q_3)$ set be $\{1,$
3, 4, 5}= A (say). A dominates \{9, 11, 12, 13\}. Therefore vertices to be dominated in Q4 are \{10, 14, 15, 16\}. Minimum two vertices are required to dominate \{10, 14, 15, 16\}. Therefore \{1, 3, 4, 5, 14, 16\} dominate Q4. These eight vertices can be chosen in any manner from the two layers of Q3. Hence we observe that for Qn, 2n−1 vertices are required for a weak convex dominating set which can be got in any manner from the two layers of Qn−1. Now we claim that 2n−1 is the minimum number of vertices for a weak convex dominating set in Qn.

Let k + l = 2n−1 where k,l are the number of vertices chosen in two layers of Qn−1 for a weak convex dominating set in Qn.

Without loss of generality assume l < k. Let Q1n−1 and Q2n−1 denote the first and second layers of Qn−1. Choose k vertices in Q1n−1 in such a way that they form a weak convex dominating set in Qn−1. Suppose we take l − 1 vertices in Q2n−1. Now we claim that k + l − 1 vertices do not form a weak convex dominating set in Qn.

Case (i)

l − 1 vertices are private neighbors of k vertices. Consider Q2n−1. k vertices are dominated by k vertices of Q1n−1. Rest of 2n−1 − k vertices in Q2n−1 must be dominated. Choose l − 1 vertices among vertices of Q2n−1 so that weak convexity is maintained among l − 1 vertices. Suppose there are m vertices from k vertices in Q1n−1 that are for domination in Q1n−1 then these m vertices have private neighbors in Qn−1 itself. So private neighbors of these m vertices in Q2n−1 must be chosen so that domination is not violated in Q2n−1.

Thus m + x = l − 1. Now k − (l − 1) private neighbors in Q2n−1 of k vertices are not chosen. Since l < k and we choose l − 1 vertices, there are atleast two vertices in k − (l − 1).

Let k −(l −1) = 2. Let these vertices be u, v. Let private neighbors of u and v be u1 and v1 respectively in Q1n−1 u, v are not among l − 1 vertices of Q2n−1, u1, v1 do not contribute for domination in Q1n−1. Suppose u1, v1 are adjacent to both u and v. Then single vertex that dominates u and v is either u1 or v1. Therefore weak convexity is violated between u(v) and a vertex among k vertices which is adjacent to private neighbors of u(v) in Q1n−1. Thus a contradiction.

If private neighbors of u and v do not form an edge in Q1n−1 and N(u)\cap N(v) = φ then weak convexity is violated in Q2n−1 which is a contradiction.

Case (ii)

None of l − 1 vertices are private neighbors of k vertices. Clearly weak convexity is violated between any vertex of Q1n−1 and Q2n−1.

Case (iii)

Some of l − 1 vertices are private neighbors of k vertices. By Case (i) we get the result. Interchanging k and l we get the result for k < l.
Conclusion

In this paper we determined weak convex domination number for hypercube graphs. We also determined convex, weak convex, m - convex and $l_1$-convex numbers of star and hypercube graphs. Other domination parameters for hypercubes are under study in our group.

References

1. S. B. Akers, D. Harei, B. Krishnamurthy,: The Star Graph: an Attractive Alternative to the n- Cube. In: Proceedings of the International Conference on parallel Processing.pp. 393-400 (1987)
2. S. Arumugam, R. Kala.: Domination parameters of hypercubes. Journal of the Indian Math. Soc. Vol.65, No.1-4, pp.31-38 (1998).
3. F. Buckley and F. Harary, Distance in Graphs, Redwood City: Addison-Wesley, (1990).
4. G. Chartrand, E. Wall Curtiss, Ping Zhang, The Convexity Number of a Graph, Graphs and combinatorics, 18, 209-217, (2002).
5. K. Day, A. Tripathi.: A comparative Study of Topological Properties of Hypercubes and Star Graphs, IEEE Trans. On Parallel Distributed Systems 5(1), 31-38 (1994).
6. Douglas B. West, Introduction to Graph Theory, Second Edition Prentice - Hall, (2001).
7. Eric Degreel, The Convex sum and direct sum space, A quarterly J. of Pure and applied Maths., 56, 1-2, (1982).
8. M. Farber and R.E. Jamison, On Local Convexity in Graphs, Discrete Mathematics, 66, 231-247, (1987).
9. Gerald Sierksma, Convexity on union of sets, composite mathematica, 42, 391-400, (1981).
10. J. Gimbel, Some Remarks on the Convexity Number of a Graph, Graphs and Combinatorics, 19, 351-361, (2003).
11. F. Harary, Graph Theory, Addison Wesley, reading Mass, (1969).