IFGF-accelerated integral equation solvers for acoustic scattering

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Abstract

We present an accelerated and hardware parallelized integral-equation solver for the problem of acoustic scattering by a two-dimensional surface in three-dimensional space. The approach is based, in part, on the novel Interpolated Factored Green Function acceleration method (IFGF) that, without recourse to the Fast Fourier Transform (FFT), evaluates the action of Green function-based integral operators for an $N$-point surface discretization at a complexity of $O(N \log N)$ operations instead of the $O(N^2)$ cost associated with nonaccelerated methods. The IFGF algorithm exploits the slow variations of factored Green functions to enable the fast evaluation of fields generated by groups of sources on the basis of a recursive interpolation scheme. In the proposed approach, the IFGF method is used to account for the vast majority of the computations, while, for the relatively few singular, nearly-singular and neighboring non-singular integral operator evaluations, a high-order rectangular-polar quadrature approach is employed instead. Since the overall approach does not rely on the FFT, it is amenable to efficient shared- and distributed-memory parallelization; this paper demonstrates such a capability by means of an OpenMP parallel implementation of the method. A variety of numerical examples presented in this paper demonstrate that the proposed methods enable the efficient solution of large problems over complex geometries on small parallel hardware infrastructures. Numerical examples include acoustic scattering by a sphere of up to 128 wavelengths, an 80-wavelength submarine, and a turbofan nacelle that is more than 80 wavelengths in size, requiring, on a 28-core computer, computing times of the order of a few minutes per iteration and a few tens of iterations of the GMRES iterative solver.

1 Introduction

We present a boundary-integral method for the simulation of time-harmonic acoustic-scattering by general surfaces in three-dimensional space. As is well known, boundary integral methods provide a number of advantages (notably, they only require discretization of the scattering surfaces, and they inherently enforce the condition of radiation at infinity), but they do give rise to certain computational challenges, concerning the accurate evaluation of the associated singular integrals, the $O(N^2)$ computational cost that results from straightforward numerical implementations (for an $O(N)$-sized grid, with, in particular, large $N$ for high frequencies), the generation of high-fidelity surface grids, and well known difficulties associated with their shared- and distributed-memory parallelization.

This paper addresses all of these challenges. In particular, the proposed approach provides high-order convergence (and thus, accurate results with minimal computational grids), and it employs a certain “IFGF” acceleration technique discussed below which, not relying on the Fast Fourier Transform (FFT), reduces the computing cost to $O(N \log N)$ operations in a manner that lends itself to effective parallelization under both large and small shared memory and distributed memory hardware infrastructures [1, 2]. In this work we focus on an OpenMP parallel implementation of this method, suitable for computer systems containing relatively small number of computing cores; the development of a massively parallel scattering solver is left for future work. A variety of numerical examples presented in this paper demonstrate that the proposed methods enable the efficient solution of large problems over complex geometries on small parallel hardware infrastructures. Numerical examples include acoustic scattering by a sphere
of up to 128 wavelengths, an 80-wavelength submarine, and a turbofan nacelle that is more than 80 wavelengths in size, each one requiring, on a 28-core computer, computing times of the order of a few minutes per iteration and a few tens of iterations of the GMRES iterative solver.

In this work the scattering surface is represented by means of a set of non-overlapping surface patches. The associated integral operators, which require careful treatment to ensure high-order accuracy, are tackled by means of the Nyström approach presented in [8], at least in the case of singular and near-singular Green function interactions, which, on the basis of a rectangular-polar change of variables, operator kernel precomputations, and Chebyshev expansions of the underlying densities, achieves high-order accuracy in a manner that leads to seamless integration with the aforementioned IFGF acceleration approach. The IFGF acceleration method mentioned above (Interpolated Factored Green Function), in turn, evaluates the action of Green function based integral operators at a cost of $O(N \log N)$ operations for an $N$-point surface mesh. The efficiency of the IFGF approach is not based on previously-employed acceleration elements such as the Fast Fourier Transform (FFT), special-function expansions, high-dimensional linear-algebra factorizations, translation operators, equivalent sources, or parabolic scaling [3, 4, 5, 9, 11, 15, 20, 24, 25, 27, 33]. Instead, the IFGF method relies on an interpolation scheme of factored forms of the operator kernels which, when applied recursively to larger and larger groups of Green function sources, gives rise to the desired $O(N \log N)$ accelerated evaluation.

This paper is organized as follows: integral equation formulations for exterior Dirichlet boundary-value problems are presented in Section 2. Section 3 describes the proposed non-overlapping patch surface representation approach, and Section 4 provides a brief description of the rectangular-polar Chebyshev-based solver [8] we use for the small subset of interactions that are not treated by the IFGF method. Section 5 then presents the IFGF algorithm, including the details necessary for its coupling with the rectangular-polar approach. The numerical results presented in Section 6 demonstrate the efficiency of the overall IFGF-based solver by means of a variety of numerical experiments. Section 7, finally, presents a few concluding remarks.

2 Integral equations for acoustic scattering

2.1 Scattering boundary-value problem

We consider wave propagation in a homogeneous isotropic medium with density $\rho$, speed of sound $c$, and no damping [13]. Scattering obstacles are represented by a bounded set $\Omega \subset \mathbb{R}^3$ which is the open complement of an unbounded domain. For time-harmonic acoustic waves, the wave motion can be obtained from the velocity potential $U(x, t) = \text{Re} \{u(x)e^{-i\omega t}\}$, where $\omega > 0$ is the angular frequency, and the spatially-dependent complex-valued part $u(x)$ satisfies the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^3 \setminus \bar{\Omega},$$

with $k = \omega/c$; the corresponding acoustic wavelength is given by $\lambda = 2\pi/k$. Denoting the boundary of $\Omega$ by $\Gamma$, the sound-soft obstacle case that we consider requires that $u = 0$ on $\Gamma$. Writing the total field $u(x) = u^i(x) + u^s(x)$, where $u^i(x)$ is a given incident field which also satisfies Helmholtz equation, leads to an exterior Dirichlet boundary-value problem for the scattered field $u^s(x)$

$$\begin{align*}
\Delta u^s(x) + k^2 u^s(x) &= 0, & x &\in \mathbb{R}^3 \setminus \bar{\Omega}, \\
u^s(x) &= -u^i(x), & x &\in \Gamma, \\
|\mathbf{x}| \left( \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla u^s(x) - iku^s(x) \right) &= 0, & |\mathbf{x}| &\to \infty.
\end{align*}$$

2.2 Integral representations and integral equations

The solutions to the acoustic scattering problem can be obtained in terms of an integral equation posed on the obstacle boundary. To present the integral formulation succinctly, we first define a few auxiliary
boundary integral operators.

Recall that the fundamental solution to the Helmholtz equation with positive wave number $k$ is given by

$$
\Phi_k(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y.
$$

Given a complex scalar function $\varphi \in C(\Gamma, \mathbb{C})$ and a point $x$ on $\Gamma$, we define the single- and double-layer integral operators, respectively, as

$$
S_k[\varphi](x) := \int_{\Gamma} \Phi_k(x, y) \varphi(y) dS(y),
$$

$$
D_k[\varphi](x) := \int_{\Gamma} \frac{\partial \Phi_k(x, y)}{\partial \nu(y)} \varphi(y) dS(y).
$$

The solution to (2) can be expressed as a combined-layer potential

$$
u^s(x) = \int_{\Gamma} \left\{ \frac{\partial \Phi_k(x, y)}{\partial \nu(y)} - i\gamma \Phi_k(x, y) \right\} \varphi(y) dS(y), \quad x \in \mathbb{R}^3 \setminus \bar{\Omega},
$$

for a real coupling parameter $\gamma \neq 0$, where the density $\varphi$ is a solution to the integral equation

$$
\frac{1}{2} \varphi(x) + D_k[\varphi](x) - i\gamma S_k[\varphi](x) = f(x), \quad x \in \Gamma,
$$

with $f(x) = -u^i(x)$.

## 3 Surface representation

Following [8], we partition a scattering surface as the disjoint union of a set of non-overlapping parametrized component “patches”. We also refer to a surface patch as a “logical quadrilateral” (LQ) since it is assumed to be the image of a rectangular reference domain. Given a scattering surface $\Gamma$, we thus utilize a number $Q$ of smooth parametrizations

$$
y^q : R \rightarrow \mathbb{R}^3, \quad (q = 1, \ldots, Q),
$$

from a $uv$-plane reference domain $R := (-1, 1)^2$ onto an LQ patch $\Gamma^q \subset \mathbb{R}^3$ such that

$$
\Gamma^q = y^q(R) \quad \text{and} \quad \Gamma = \bigcup_{q=1}^{Q} \Gamma^q.
$$

A general integral operator defined over $\Gamma$ can then be evaluated component-wise over each patch $\Gamma^q$.

We discretize the patch $\Gamma^q$ by means of a surface grid containing $N_u^q \times N_v^q$ points given by the image of the tensor-product discretization

$$
\{ u_i = s_i \mid i = 0, \ldots, N_u^q - 1 \} \times \{ v_j = s_j \mid j = 0, \ldots, N_v^q - 1 \},
$$

under the parametrization $y^q$, where the nodes $s_j$ and associated integration weights $w_j$ are given by Fejér’s first quadrature rule:

$$
s_j = \cos \left( \frac{\pi 2j + 1}{2J} \right),
$$

$$
w_j = \frac{2}{J} \left[ 1 - 2 \sum_{\ell=1}^{\lfloor J/2 \rfloor} \frac{1}{4\ell^2 - 1} \cos \left( \ell \pi \frac{2j + 1}{J} \right) \right],
$$
for \( j = 0, \ldots, J - 1 \) and \( J \) is either \( N_u^q \) or \( N_v^q \). The set of all surface discretization points will be denoted by

\[
\Gamma_N := \bigcup_{q=1}^Q \Gamma_{N_u^q, N_v^q},
\]

where \( N \) denotes the total number of grid points over all patches.

\section{Chebyshev-based rectangular-polar integral equation solver}

This section presents a brief description of the high-order integral equation solver presented in \cite{8}. In that approach, a general integral operator \( I^q \) with singular kernel \( K^q \) and density \( \varphi^q \) defined over a component patch \( \Gamma^q \) is expressed in the parametric form

\[
(I^q \varphi)(x) = \int_R K^q(x, u, v)\varphi^q(u, v)J^q(u, v) du dv,
\]

for \( x \in \Gamma \), where \( K^q(x, u, v) := K(x, y^q(u, v)) \) and \( \varphi^q(u, v) := \varphi(y^q(u, v)) \), and where \( J^q(u, v) du dv \) denotes the element of area. We often refer to \( x \) as the “evaluation” or “target” point.

To compute (12) accurately we use two different high-order methods depending on whether the target point \( x \) is less than or greater than some “proximity distance” \( \delta \) to the integration patch. In detail, letting

\[
\text{dist} (x, \Gamma^q) := \inf \{ |x - y| \mid y \in \Gamma^q \},
\]

denote the distance from a point \( x \) to a patch \( \Gamma^q \) (where \( | \cdot | \) denotes the Euclidean distance), the set of target points gives rise to “singular” and “nearly-singular” over \( \Gamma^q \) is defined by

\[
\Omega^s_\delta := \{ x \in \Gamma \mid \text{dist} (x, \Gamma^q) \leq \delta \}.
\]

In contrast, the set of regular (non-singular) target points is defined by

\[
\Omega^r_\delta := \{ x \in \Gamma \mid \text{dist} (x, \Gamma^q) > \delta \}.
\]

We say that the interaction of an integration patch \( \Gamma^q \) with a target point is singular or regular/non-singular, according to whether the target point lies in \( \Omega^s_\delta \) or \( \Omega^r_\delta \), respectively.

\subsection{Integration algorithm for singular interactions}

To evaluate (12) at a singular or near-singular target point \( x \in \Omega^s_\delta \), we proceed as follows. First, we form the Chebyshev expansion of the density \( \varphi^q \) over \( \Gamma^q \):

\[
\varphi^q(u, v) \approx \sum_{m=0}^{N_u^q-1} \sum_{n=0}^{N_v^q-1} a_{n,m}^q T_n(u)T_m(v),
\]

where, in view of the discrete orthogonality property satisfied by Chebyshev polynomials at the Fejér nodes, we have

\[
a_{n,m}^q = \frac{\alpha_n \alpha_m}{N_u^q N_v^q} \sum_{j=0}^{N_u^q-1} \sum_{i=0}^{N_v^q-1} \varphi^q(u_i, v_j)T_n(u_i)T_m(v_j), \quad \alpha_n := \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}.
\]
Replacing the density \(\varphi^q\) by its Chebyshev expansion \((16)\), in the proposed scheme the integral \((12)\) is numerically approximated by

\[
(T^q \varphi)(x) \approx \int_R K^q(x, u, v) \left( \sum_{m=0}^{N_q^q-1} \sum_{n=0}^{N_q^q-1} a_{n,m}^q T_n(u)T_m(v) \right) J^q(u, v) \, du \, dv
\] \tag{18a}

where

\[
\beta_{n,m}^{q,\ell} = \sum_{m=0}^{N_q^q-1} \sum_{n=0}^{N_q^q-1} a_{n,m}^q \beta_{n,m}^{q,\ell},
\] \tag{19}

Note that the double integral in \((18b)\) does not depend on the density; it depends only on the kernel, a product of Chebyshev polynomials, and the geometry. Once this integral has been computed to the desired accuracy, the proposed method stores its value and uses it as needed.

We write the value of \(T^q\) at all target points \(x_\ell \in \Omega_{p,\delta}^q\) succinctly as

\[
(T^q \varphi)(x_\ell) = \sum_{m=0}^{N_q^q-1} \sum_{n=0}^{N_q^q-1} a_{n,m}^q \beta_{n,m}^{q,\ell},
\] \tag{20}

To compute \((20)\) at an evaluation point \(x_\ell\), we first identify its corresponding integration patch node \((\bar{u}_\ell^q, \bar{v}_\ell^q)\). If the target point \(x_\ell\) is itself a grid point of \(\Gamma^q\), then finding its node is straightforward: \(x_\ell = y^q(\bar{u}_\ell^q, \bar{v}_\ell^q)\) for some point \((\bar{u}_\ell^q, \bar{v}_\ell^q)\) in the \(uv\)-plane reference domain for \(\Gamma^q\). On the other hand, if \(x_\ell \in \Omega_{p,\delta}^q \setminus \Gamma^q\), then we search for a \(\Gamma^q\) node such that

\[
(\bar{u}_\ell^q, \bar{v}_\ell^q) = \arg \min_{(u,v) \in [-1,1]^2} \|x_\ell - y^q(u,v)\|.
\] \tag{21}

As in \((8)\), for robustness and simplicity we solve the minimization problem \((21)\) by means of the golden section search algorithm.

Next, we apply a one-dimensional change of variables to each coordinate in the \(uv\)-parameter space to construct a clustered grid around each given target node. To this end we consider the following one-to-one, strictly monotonically increasing, and infinitely differentiable function \(w : [0, 2\pi] \to [0, 2\pi]\), with parameter \(d \geq 2\) (proposed in \((14)\) Section 3.5]),

\[
w(\tau; d) := 2\pi \frac{[\nu(\tau)]^d}{[\nu(\tau)]^d + [\nu(2\pi - \tau)]^d}, \quad 0 \leq \tau \leq 2\pi,
\] \tag{22}

where

\[
\nu(\tau; d) := \left( \frac{1}{d} - 1\right) \left( \frac{\pi - \tau}{\pi} \right)^3 + \frac{1}{d} \left( \frac{\tau - \pi}{\pi} \right) + \frac{1}{2}.
\] \tag{23}

It can be shown that \(w\) has vanishing derivatives up to order \(d - 1\) at the interval endpoints. Then, the following change of variables

\[
\xi_\alpha(\tau; d) := \begin{cases} 
\alpha + \left( \frac{\text{sgn}(\tau) - \alpha}{\pi} \right) w(\pi|\tau|; d), & \text{for } \alpha \neq \pm 1, \\
\alpha + \left( \frac{1 - \pi}{\pi} \right) w(\pi|\tau + 1|; d), & \text{for } \alpha = -1, \\
\alpha + \left( \frac{\pi - 1}{\pi} \right) w(\pi|\tau + 1|; d), & \text{for } \alpha = 1,
\end{cases}
\] \tag{24}

has the effect of clustering points around \(\alpha\). Fejér’s rule applied to the integral \((20)\), transformed using the change of variables \((24)\), yields the approximation

\[
\beta_{n,m}^{q,\ell} \approx \sum_{j=0}^{N_\delta^q - 1} \sum_{i=0}^{N_\beta^q - 1} K^q(x_\ell, u_i^{q,\ell}, v_i^{q,\ell})T_n(u_i^{q,\ell})T_m(v_i^{q,\ell})J^q(u_i^{q,\ell}, v_i^{q,\ell})w_i^{u,q,\ell}w_i^{v,q,\ell},
\] \tag{25}
where

\begin{align}
  u_i^{q,\ell} &= \xi_{q}(s_i; d), & w_i^{u,q,\ell} &= \frac{d\xi_{q}(s_i; d)}{d\tau} w_i, \\
  v_j^{q,\ell} &= \xi_{q}(s_j; d), & w_j^{v,q,\ell} &= \frac{d\xi_{q}(s_j; d)}{d\tau} w_j,
\end{align}

for \( i, j = 0, \ldots, N_d - 1 \). To avoid division by zero, we set the kernel \( K^q \) to zero at integration points where the distance to the target point is less than some prescribed tolerance, usually on the order of \( 10^{-14} \).

### 4.2 Integration algorithm for non-singular interactions

Together with the singular integration method discussed in the previous subsection, the (non-accelerated) high-order solver \( [8] \) evaluates the integral operator \( \{[12] \) at all regular target points \( x_\ell \in \Omega^{\ell,q} \) simply by means of Fejér's first quadrature rule:

\[
(T^q \varphi)(x_\ell) \approx \sum_{j=0}^{N_d^q-1} \sum_{i=0}^{N_d^q-1} K^q(x_\ell, u_i, v_j) \varphi^q(u_i, v_j) J^q(u_i, v_j) w_i w_j.
\]

It is not difficult to show that, asymptotically, the regular interactions dominate the integral operator computation (see \( [3] \) Section 4.4]). Evaluating all non-singular interactions using (28) leads to an algorithm with complexity \( O(N^2) \) operations. For acoustically-large problems such a computational cost becomes prohibitively expensive. To deal with this difficulty we use, instead, the recent IFGF acceleration method \( [1] \) to accelerate the evaluation which is described in the following section. As indicated in \( [1] \) for simple discrete operators, and is confirmed for full scattering problems by the numerical results in Section \( [3] \), the IFGF method leads to an overall algorithm that runs at computing cost of \( O(N \log N) \) operations.

### 5 IFGF method for fast evaluation of non-singular interactions

#### 5.1 General overview of IFGF method

The IFGF method provides a fast (\( O(N \log N) \)) method for the accelerated evaluation of discrete integral operators of the form

\[
I(x_\ell) := \sum_{m=1}^{N} a_m G(x_\ell, x_m), \quad \ell = 1, \ldots, N,
\]

where, letting \( \Gamma_N \subset \mathbb{R}^3 \) denotes an \( N \)-point discretization of the surface \( \Gamma \), \( x_\ell \in \Gamma_N (\ell, m = 1, \ldots, N) \) are pairwise different surface discretization points, \( a_m \in \mathbb{C} \) are arbitrary complex coefficients and \( G \) denotes a Green function such as the one displayed in \( [3] \).

Given \( D \in \mathbb{N} \), the IFGF method is based on a \( D \)-level octree partition of the discrete surface \( \Gamma_N \), where the first octree level consists of a single box that contains all of the surface discretization points. Starting from the first level, the algorithm partitions \( \Gamma_N \) recursively into axis-aligned boxes \( B_k^d \subset \mathbb{R}^3 \) (\( d = 1, \ldots, D \)) where \( k \in \mathbb{N}^3 \) is a multi-index that describes the three-dimensional position of the box. Using this notation, we see that the first octree level (\( d = 1 \)) consists of the single box \( B^{1}_{(1,1,1)} \).

The boxes are defined iteratively starting from the single box \( B^{1}_{(1,1,1)} \supset \Gamma_N \) of side-length \( H_1 \). The boxes on subsequent levels \( d = 2, \ldots, D \), are defined through a partition of each of the level (\( d - 1 \)) boxes into eight equi-sized and disjoint boxes of side \( H_d = H_{d-1}/2 \) resulting in the level \( d \) boxes \( B_k^d \) (\( k \in \{1, \ldots, 2^{d-1}\}^3 =: I_d^d \)). Each box \( B_k^d \) on level \( d (2 \leq d \leq D) \) therefore is contained in a parent box on
level $d - 1$, which we denote by $\mathcal{P}B_k^d$. Figure 1(a) illustrates the two-dimensional equivalent hierarchical octree structure in the three-level ($D = 3$) case.

To achieve the desired acceleration, the IFGF method only considers interactions in the relevant boxes $\mathcal{R}_B$ set, which are defined as the boxes that contain at least one surface discretization point. More precisely, the relevant boxes are defined by

$$\mathcal{R}_B := \{ B^d_k : \Gamma_N \cap B^d_k \neq \emptyset, 1 \leq d \leq D, k \in I^d \}. \quad (30)$$

Furthermore, the following notation is introduced for the neighbors $\mathcal{N}B^d_k$ and the cousins (non-neighboring boxes who are children of the parents neighbors) $\mathcal{M}B^d_k$ of a box $B^d_k$ on level $d$ together with the sets of neighbor points $\mathcal{U}B_k^d$ and cousin points $\mathcal{V}B_k^d$.

$$\mathcal{N}B^d_k := \{ B^d_j \in \mathcal{R}_B : ||j - k||_{\infty} \leq 1 \}, \quad (31a)$$

$$\mathcal{M}B^d_k := \{ B^d_j \in \mathcal{R}_B : B^d_j \notin \mathcal{N}B^d_k \land \mathcal{P}B^d_j \in \mathcal{N} \mathcal{P}B^d_k \} \quad (31b)$$

$$\mathcal{U}B_k^d := \left( \bigcup_{B \in \mathcal{N}B^d_k} B \right) \cap \Gamma_N \quad (31c)$$

$$\mathcal{V}B_k^d := \left( \bigcup_{B \in \mathcal{M}B^d_k} B \right) \cap \Gamma_N. \quad (31d)$$

Figure 1(a) illustrates the concept of neighbors (white boxes) and cousins (gray boxes) in two dimensions for the single box $B^3_{(2,1)}$.

The IFGF algorithm accelerates the evaluation of (29) by pairwise interactions of level-$d$ cousin boxes, for $d = D, \ldots, 3$. (Indeed, the algorithm stops at level $d = 3$ since, per construction of the boxes and definition of cousins, at that stage all boxes are cousins and therefore all interactions have
already been performed.) The IFGF method evaluates these interactions by means of simple piece-wise interpolation of the Green function analytic factor $g_k^d(x,y)$, defined below, in box-centered spherical coordinates, resulting in a set of so-called cone segments $C_{k_{uv}}^d$ for each box $B_k^d$—each one of which represents the piece-wise interpolation domain in spherical coordinates—and a set of interpolation points $x^d\in \mathbb{R}^3$ within the cone segment $C_{k_{uv}}^d$. (The problem-dependent multi-index $\nu \in \mathbb{N}^3$ labels the cone segments associated with a single box.) More precisely, to achieve fast computation times, a certain quadrature presented in Section 4.2 is used instead. We emphasize that local interactions are computed case the source points and the target point in neighboring boxes are a distance greater than $\delta$. While the procedure is straight-forward, it requires a lengthy discussion which is not presented here but which was covered in detail in [1]. A two-dimensional sketch of two box-centered cone segments for a box $B_k^d$ is illustrated in Figure 1(b). To achieve the stated acceleration and perform the computation of (29) in $O(N \log N)$ operations, the IFGF algorithm uses iterated interpolation to evaluate the analytic factor $G(x,y)$, the wavenumber and, possibly, the Green function $G$. While the procedure is straight-forward, it requires a lengthy discussion which is not presented here but which was covered in detail in [1]. A two-dimensional sketch of two box-centered cone segments for a box $B_k^d$ and its parent $PB_k^d$ is illustrated in Figure 1(b). To achieve the stated acceleration and perform the computation of (29) in $O(N \log N)$ operations, the IFGF algorithm uses iterated interpolation to evaluate the analytic factor at the interpolation points in consecutive levels. Moreover, analogously to the set of relevant boxes, the method only considers the set of relevant cone segments $R_{CB_k^d}$ centered at the box $B_k^d$. Using the aforementioned factorization centered at $y_k^d$ yields

$$I_k^d(x) = \sum_{y \in B_k^d \cap \Gamma_N} a(y)G(x,y) = G(x,y_k^d)F_k^d(x), \quad F_k^d(x) := \sum_{y \in B_k^d \cap \Gamma_N} a(y)g_k^d(x,y), \quad (32)$$

where $a(y)$ denotes the coefficient $a_{ym}$ in (29) that corresponds to the point $y \in \Gamma$. The IFGF interpolation procedure is then only performed to evaluate $F_k^d$. In [1] it was shown—for the special case of Helmholtz Green function—that the analytic factor is analytic up to and including infinity in $\mathbb{R}^3 \setminus NB_k^d$ and can therefore be interpolated accurately everywhere in $\mathbb{R}^3 \setminus NB_k^d$ using only a finite and small number of interpolation points. It follows that the analyticity properties and the interpolability transfer to $F_k^d$ since it is a finite sum of analytic factors.

The cone segments $C_{k_{uv}}^d$ are defined iteratively for each box—similarly to the boxes—but in reversed order starting from $d = D$ moving upwards the tree to $d = 3$. The choice of the cone segments depends on the level $d$, the surface $\Gamma_N$, the wavenumber and, possibly, the Green function $G$. While the procedure is straight-forward, it requires a lengthy discussion which is not presented here but which was covered in detail in [1]. A two-dimensional sketch of two box-centered cone segments for a box $B_k^d$ and its parent $PB_k^d$ is illustrated in Figure 1(b). To achieve the stated acceleration and perform the computation of (29) in $O(N \log N)$ operations, the IFGF algorithm uses iterated interpolation to evaluate the analytic factor at the interpolation points in consecutive levels. Moreover, analogously to the set of relevant boxes, the method only considers the set of relevant cone segments $R_{CB_k^d}$ centered at the box $B_k^d$.

$$R_{CB_k^d} := \emptyset \quad \text{for} \quad d \in \{1,2\}, \quad (33a)$$

$$R_{CB_k^d} := \left\{ \begin{array}{l}
C_{k_{uv}}^d : \nu \in I_C^d, \quad C_{k_{uv}}^d \cap \cup_{C \in \mathbb{R}_{CB_k^d}} C \neq \emptyset \\
\end{array} \right\} \quad \text{for} \quad d \geq 3. \quad (33b)$$

The IFGF method is summarized in Algorithm 1. Note that the algorithm does not compute the “local interactions”, that is, the interactions between neighboring boxes on the finest level ($d = D$). For the scattering solver proposed in this paper, such interactions are evaluated by means of two separate methods, as illustrated in Figure 2 and described in what follows. If the distance from a box’s source points to a neighboring box’s target point $x_i$ is less than or equal to the proximity distance $\delta$ defined in Section 4.1 the local interactions are evaluated by means of the algorithm described in that section. In case the source points and the target point in neighboring boxes are a distance greater than $\delta$, in turn, the quadrature presented in Section 4.2 is used instead. We emphasize that local interactions are computed only in the finest level ($d = D$). Figure 2(a) shows a four-level ($D = 4$) IFGF domain decomposition for a sphere, where a source patch and its neighbors patches are highlighted in red and yellow, respectively. A close-up view of these patches is shown in Figure 2(b); source-to-neighbor near-singular evaluation points are drawn in blue, while regular evaluation points are depicted in green. All evaluations from the source patch to itself are singular and are also computed using the algorithms from Section 4.1.
Figure 2: (a) IFGF four level \((D = 4)\) domain decomposition for a sphere (only the finest level boxes are shown); a source surface patch and its neighboring patches are highlighted in red and yellow, respectively. (b) Near-singular and regular points neighboring the red patch are depicted in blue and green, respectively. All evaluations from the red patch to itself are singular and, together with the near-singular and neighboring regular interactions are computed by means of the rectangular-polar algorithm described in Section 4.

Algorithm 1: IFGF Algorithm

1: Direct evaluations of the highest-level local interactions.
2: for \(B_k^D \in R_B\) do
3: for \(C_{k,\nu}^\nu \in R_C B_k^D\) do
4: for \(x \in \chi C_{k,\nu}^\nu\) do
5: Evaluate and store \(F_k^D(x)\).
6: end for
7: end for
8: end for
9: \end for

10: Interpolation onto surface discretization points and parent interpolation points.

11: for \(d = D, \ldots, 3\) do
12: for \(B_k^d \in R_B\) do
13: for \(x \in \chi B_k^d\) do
14: Evaluate \(I_k^d(x)\) by interpolation
15: end for
16: if \(d > 3\) then
17: Determine parent \(B_{k}^{d-1} = \mathcal{P} B_k^d\)
18: for \(C_{j,\nu}^{d-1} \in R_C B_{k}^{d-1}\) do
19: for \(x \in \chi C_{j,\nu}^{d-1}\) do
20: Evaluate and add \(F_k^d(x) G(x, y_k^d) / G(x, y_{j,\nu}^{d-1})\)
21: end for
22: end for
23: end if
24: end for
25: end for
6 Numerical results

This section presents numerical results that demonstrate the accuracy and efficiency of the proposed IFGF-accelerated acoustic scattering solvers. For comparison, results obtained using the $O(N^2)$ nonaccelerated Chebyshev-based scattering solvers introduced in [8] are also included. Both the accelerated and nonaccelerated solver are implemented using OpenMP for shared-memory parallelism.

After solving (7) for the density $\varphi$, the far field pattern $u^\infty$ can be obtained from

$$u^\infty(\hat{x}) = \frac{1}{4\pi} \int_\Gamma \left\{ \frac{\partial}{\partial y} e^{-ik\hat{x} \cdot y} - i\gamma e^{-ik\hat{x} \cdot y} \right\} \varphi(y) \, dS(y), \quad \hat{x} \in \mathbb{S}^2,$$

where $\mathbb{S}^2$ denotes the unit sphere and $\Gamma$ is the scatterer’s boundary. The far field is computed over a uniformly-spaced unit spherical grid

$$\mathbb{S}_N^2 := \{(\phi_m, \theta_n) \in [0, \pi] \times [0, 2\pi] \mid 1 \leq m \leq N_\phi, 1 \leq n \leq N_\theta\},$$

with $\phi_m = (m - 1)\Delta\phi$, $\theta_n = (n - 1)\Delta\theta$ and where the spacings are defined as $\Delta\phi = \pi/(N_\phi - 1)$ and $\Delta\theta = 2\pi/(N_\theta - 1)$, respectively; specific values of $N_\phi$ and $N_\theta$ are given in each example’s subsection. Given the exact (or reference) far field modulus $|u^\infty|$ and an approximate far field modulus $|\tilde{u}^\infty|$, the maximum far field relative error $\varepsilon_{far}$ over $\mathbb{S}_N^2$ given by

$$\varepsilon_{far} = \max_{(m,n) \in \mathbb{S}_N^2} \left\{ \frac{|u^\infty_{m,n} - |\tilde{u}^\infty_{m,n}|}{|u^\infty_{m,n}|} \right\},$$

is reported in each case.

Similarly, using the solution $\varphi$ in the combined-layer representation [6] we evaluate and display the scattered field $u^s$ over near field planes that are parallel to the $xy$-, $xz$-, or $yz$-planes. For example, we evaluate fields (incident, scattered, and total) at every point of a uniformly-spaced two-dimensional $xy$-planar grid $\mathbb{P}_N^{xy}(z_0)$ at $z = z_0$ defined by

$$\mathbb{P}_N^{xy}(z_0) := \{(x_m, y_m, z) \in [x_{min}, x_{max}] \times [y_{min}, y_{max}] \times \{z_0\} \mid 1 \leq m \leq N_x, 1 \leq n \leq N_y\},$$

where the grid points are given by $x_m = (m - 1)\Delta x$, $y_n = (n - 1)\Delta y$ and the grid spacings are $\Delta x = (x_{max} - x_{min})/(N_x - 1)$ and $\Delta y = (y_{max} - y_{min})/(N_y - 1)$. Near field planar grids parallel to the $xz$- and $yz$-plane are defined analogously. Denoting the exact (or reference) and approximate modulus of the total field at each point of $\mathbb{P}_N^{xy}(z_0)$ by $v_{m,n} (= |u^s_{m,n} + u^t_{m,n}|)$ and $\tilde{v}_{m,n} (= |\tilde{u}^s_{m,n} + \tilde{u}^t_{m,n}|)$, respectively, we compute the near field (total magnitude) relative error $\varepsilon_{near}$ over $\mathbb{P}_N^{xy}(z_0)$ as

$$\varepsilon_{near} = \max_{(m,n) \in \mathbb{P}_N^{xy}(z_0)} \left\{ \frac{|v_{m,n} - \tilde{v}_{m,n}|}{|v_{m,n}|} \right\}.$$

The numerical results presented in what follows were obtained using a single Intel Xeon Platinum 8276 2.20 GHz computer using 28 cores. Solutions to the complex-coefficient linear systems that arise from discretizations of the boundary integral equation (7) were obtained with a complex-arithmetic GMRES iterative solver [28]. Following [9], we set the combined-layer equation (7) coupling parameter $\gamma = \max\{3, A/\lambda\}$, where $A$ is the diameter of the scatterer; computational results indicate that, to reach a given residual tolerance, this value reduces the number of GMRES iterations by a factor of $5 - 10$ compared with $\gamma = k$. Plots were generated using the visualization software VisIt [12].

6.1 Scattering by a sphere

We consider plane wave scattering by a sphere of various acoustical sizes. For a sound-soft acoustic sphere, the well-known closed-form far field expression is used to compute relative errors [7]. Table 1
summarizes the accuracy and efficiency of the IFGF-accelerated solver and nonaccelerated solver for a sphere of diameter ranging from 4 to 128 wavelengths. For each problem, the number of IFGF levels is selected so that the finest-level IFGF box side length is approximately $0.5\lambda$. All computations are performed using a GMRES residual tolerance set to $10^{-4}$. We report the total number of unknowns, the size of the sphere in wavelengths, the time required to compute one GMRES iteration as well as the total number of iterations required to achieve the prescribed residual, and the far field relative error. Far field relative errors are computed over the spherical grid $\{55\}$ with $(N_\phi, N_\theta) = (200, 200)$.

Table 1: Comparison of IFGF-accelerated solver and nonaccelerated solver for acoustic scattering by a sphere of acoustical sizes ranging from 4 to 128 wavelengths. The table summarizes the total number of surface unknowns, sphere size in wavelengths, maximum number of IFGF levels, time required to compute one GMRES iteration, total number of iterations, and far field relative error $\varepsilon_{far}$. In all cases the GMRES residual tolerance was set to $10^{-4}$.

| Unknowns | Size  | Time (1 iter.) | Tot. iter. | $\varepsilon_{far}$ | IFGF levels | Time (1 iter.) | Tot. iter. | $\varepsilon_{far}$ |
|----------|-------|----------------|------------|--------------------|-------------|----------------|------------|--------------------|
| 13,824   | 4\lambda | 0.5 s          | 12         | 1.1e-4             | 4           | 0.2 s         | 12         | 1.3e-4             |
| 55,296   | 8\lambda | 7.4 s          | 14         | 8.9e-5             | 5           | 1.0 s         | 14         | 1.1e-4             |
| 221,184  | 16\lambda| 116.4 s        | 14         | 2.6e-5             | 6           | 4.6 s         | 14         | 6.3e-5             |
| 884,736  | 32\lambda| 1862.4 s (est.)| –          | –                  | 7           | 19.4 s        | 16         | 2.9e-5             |
| 3,538,994| 64\lambda| 8.3 h (est.)   | –          | –                  | 8           | 83.1 s        | 18         | 6.0e-5             |
| 14,155,776| 128\lambda| 132.8 h (est.)| –          | –                  | 9           | 443.2 s       | 21         | 3.8e-4             |

Table 1 shows that the time per iteration required by the nonaccelerated algorithm grows by a factor of around $14.8 - 15.7$ as the number of points per dimension in each surface patch is doubled (so that the overall number of unknowns is quadrupled), which is consistent with the expected quadratic complexity of the algorithm. For the IFGF-based solver, on the other hand, the computing costs scale like $O(N \log N)$, where $N$ is the total number of discretization points. The reduced complexity of the IFGF-based algorithm has a significant impact on computing times. At 128 wavelengths, the nonaccelerated solver takes more than 1000 times longer than the accelerated method for each GMRES iteration; for larger problems, the difference in compute times grows as expected from the complexity estimates for the two methods. Note that the total number of GMRES iterations necessary to satisfy the residual tolerance is the same for both the nonaccelerated and accelerated solvers. Additionally, the errors for both algorithms are comparable: the nonaccelerated solver yields solutions for the 4, 8 and 16 wavelength problems with an average relative error of $\mathbf{1.1 \cdot 10^{-4}}$, while errors obtained with the accelerated method average $\mathbf{1.3 \cdot 10^{-4}}$ across the entire 4 to 128 wavelength range.

6.2 Scattering by a submarine geometry

In this section we present acoustic scattering simulations for a realistic submarine configuration of up to 80 wavelengths in acoustical size. Due to its importance in detection and tracking applications, methods for efficient and accurate scattering simulations are the subject of ongoing research [23, 29, 19, 17, 32, 30, 31].

The submarine model used in subsequent simulations, which is comprised of the main hull, sail, diving planes, rudders, and a five-blade propeller, is depicted in Figure 3. The complete submarine geometry is contained in the bounding region $[-3.2, 3.2] \times [-1.9, 2.8] \times [-19.2, 10.9]$. Figures 3(b) and 3(c) show a surface mesh of 4,560 patches, each of which is represented by $6 \times 6$ points.

We consider plane wave scattering for two cases: a) head-on incidence and b) oblique incidence. The
Figure 3: Submarine model and surface mesh with a total of 164,160 points. The submarine hull is aligned with the z-axis and the sail is parallel to the +y-axis; the front of the vessel points in the +z-direction.

Incident field is a plane wave $u^i$ that travels along the wave direction $\hat{k}$ and is given by

$$u^i(x) = e^{i \hat{k} \cdot x}, \quad \hat{k} = \begin{pmatrix} \cos \theta \sin \phi \\ \sin \theta \sin \phi \\ \cos \phi \end{pmatrix},$$

(39)

where the position vector $x = (x_1, x_2, x_3)$, $k > 0$ is a given wavenumber, and $(\theta, \phi) \in [0, 2\pi) \times [0, \pi]$. Since the bow of the submarine points in the $+z$-direction, “head-on” incidence corresponds to $(\theta, \phi) = (0, \pi)$ in (39). For the oblique incidence case we set $(\theta, \phi) = (0, 5\pi/4)$.

To verify the accuracy of the IFGF-accelerated solver in the present case, we conducted convergence studies for the submarine structure at $10\lambda$, $20\lambda$, and $40\lambda$ in acoustical size (measured from the bow to the propeller cap). In all cases the number of IFGF levels was chosen so that the side length of the smallest, finest-level, boxes is around $0.8\lambda$. The GMRES residual tolerance was set to $10^{-3}$ in all cases. We start with a $10\lambda$ vessel whose geometry is represented by 1,140 surface patches, each of which has $6 \times 6$ points. As the size of the problem is doubled, the geometry is partitioned from the previous size so that every patch is split into four subpatches while keeping the same number of points per patch. Thus, for example, the $20\lambda$ problem uses four times as many surface points as the $10\lambda$ case. This is admittedly a suboptimal strategy, in this case, (as the smaller patches on the propeller, rudders and diving planes which already fully discretize the wavelength do not require additional partitioning), which, however, simplifies the code implementation. Additionally, this distribution of surface points makes suboptimal use of the present version of the IFGF algorithm. As indicated in [1], the IFGF method can be extended to incorporate a box octree algorithm that adaptively partitions a geometry until each box contains a (small) prescribed number of points, thus eliminating this difficulty. While such an addition is left for future work, as demonstrated in Table 2 even the simple uniform-partition IFGF algorithm we use in this contribution is sufficient to simulate scattering by a realistic submarine geometry for up to 80 wavelengths in size with several digits of accuracy and using only modest computational resources. For example, the 656,640...
unknowns, 40\(\lambda\) run for head-on incidence, required a computing time of 313 seconds per iteration and a total of 78 iterations. The fully adaptive version of the IFGF algorithm, which, as mentioned above, is not pursued in this paper, should yield for the submarine geometry computing times consistent with those shown in Tables 1 and 3 for the sphere and nacelle geometries.

Table 2: Convergence study of IFGF-accelerated acoustic solver for the submarine geometry, with acoustical sizes ranging from 10 to 40 wavelengths. In all cases the residual tolerance was set to \(10^{-3}\).

| Unknowns   | Size | IFGF levels | \(\varepsilon_{\text{near}}\) Front Incidence | \(\varepsilon_{\text{near}}\) Oblique Incidence |
|------------|------|-------------|-----------------------------------------------|-----------------------------------------------|
| 41,040     | 10\(\lambda\) | 5           | 2.4e-4                                        | 6.8e-4                                        |
| 164,160    | 20\(\lambda\) | 6           | 1.8e-4                                        | 6.0e-4                                        |
| 656,640    | 40\(\lambda\) | 7           | 2.1e-4                                        | 1.7e-4                                        |
| 2,626,560  | 80\(\lambda\) | 8           | 2.1e-4 (est.)                                | 4.8e-4 (est.)                                |

Near field relative errors for front (head-on) and oblique plane wave incidence are shown in Table 2. For each problem, we estimate \(\varepsilon_{\text{near}}\) over \(P_{xy}^{xz}(z_0)\), where \([x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}] = [-12, 12]^2\), \(z_0 = -25\) and \(N_x = N_y = 260\), using (38) with a reference solution obtained with the same number of surface patches as the target discretization but using \(8 \times 8\) points per patch and a residual tolerance of \(10^{-5}\). (Thus, the reference solution uses nearly twice as many discretization points and it satisfies a more stringent convergence condition.) The numerical results indicate that the solution accuracy is consistent for both front and oblique incidence and for all acoustical sizes considered. For front and oblique incidence, the relative errors for the 10\(\lambda\), 20\(\lambda\), and 40\(\lambda\) problems achieve an average accuracy of \(2.1 \cdot 10^{-4}\) and \(4.8 \cdot 10^{-4}\), respectively, and we use these values to estimate the expected relative errors in the 80-wavelength case.

In Figure 4 we present pseudocolor near field plots of the total field magnitude \(|u(x)| = |u^i(x) + u^s(x)|\) for front plane wave incidence for an 80-wavelength submarine. The field is plotted over a uniform 1040 \(\times\) 1760 point planar grid \(P_{xy}^{xz}(y_0)\) for \((x, z) \in [-12, 12] \times [-25, 15]\) and \(y_0 = 0\). The incident plane wave impinges on the vessel head-on and we see in Figures 4(a) and 4(b) that the strongest interaction occurs around the bow and diving planes (also known as hydroplanes) of the ship. Shadow regions are visible immediately behind the hydroplanes as well as along the hull, particularly in the aft of the ship where the body tapers. Figure 4(c) shows that the wider sections of the ship obstruct the propeller from most incoming waves and, as a consequence, there is minimal interaction in this region.

Figure 5 shows near field pseudocolor plots for the same 80-wavelength submarine but this time for oblique plane wave incidence. The total field magnitude is plotted over the uniform grid \(P_{xy}^{xz}(y_0)\) described in the previous paragraph. In this case the wave interaction is markedly different. We see the expected shadow region in the opposite side of the incoming wave but there is now clear evidence of wave interaction between the hull and diving planes as well as around the rudders and propeller. In addition to multiple scattering, the close-up views of Figures 5(b) and 5(c) show the formation of bright spots near the junction of the left hydroplane and hull and in the vicinity of the propeller.

6.3 Scattering by an aircraft nacelle

The simulation of aircraft engine noise has been the subject of intense research for the past several decades due to its importance in civil aviation applications [22, 16, 10, 18, 26]. In this section we present simulations of sound propagation in and around the turbofan engine nacelle model shown in Figure 6. According to the nacelle wall liner case study [21], under typical operating conditions, engine nacelle noise occurs in the 125 – 5650 Hz frequency range. For a typical airliner engine that is around 5 m long, these frequencies correspond to acoustical sizes between 2 and 82 wavelengths.

The engine nacelle geometry used in the simulations that follow is depicted in Figures 6(a) and 6(b); it is comprised of an outer housing and a center shaft. The entire two-piece nacelle structure is contained
Figure 4: Total field magnitude $|u(x)| = |u^i(x) + u^s(x)|$ pseudocolor plots for an 80-wavelength submarine. The field is plotted over a uniform grid of 1040 × 1760 points for $(x, z) \in [-12, 12] \times [-25, 15]$. In this case the incident plane wave impinges on the vessel head-on, which corresponds to the wave direction $\hat{k}$ inside the bounding region $[-1.5, 1.5] \times [-1.5, 1.5] \times [-3.27, 3.27]$. The center shaft is aligned with the z-axis, with the tip of the shaft pointing towards the positive direction. A discretization with 8,576 surface patches with 6 × 6 points per patch is shown in Figure 6(c); for future reference, note that the inset image shows that the mesh is not rotationally symmetric near the tip of the shaft.

Two types of incident fields are used in simulations: a) a plane wave that travels towards the $-z$-axis, so that it impinges on the nacelle head-on and b) a set of eight point sources placed inside the housing around the center shaft. As in the submarine example, the plane wave incident field is given by (39) with $(\theta, \phi) = (0, \pi)$. The incident field b), on the other hand, serves as a simple model for fan noise generation inside the nacelle and is given by

$$u^i(x) = \sum_{j=1}^{8} e^{ik|x - x^j|}, \quad \text{with point source locations} \quad x^j = (x_1^j, x_2^j, x_3^j) = (\cos \alpha_j, \sin \alpha_j, 2), \quad (40)$$
Figure 5: Total field magnitude $|u(x)| = |u^i(x) + u^s(x)|$ pseudocolor plots for an 80-wavelength submarine. The field is plotted over a uniform grid of $1040 \times 1760$ points for $(x,z) \in [-12,12] \times [-25,15]$. In this case the incident plane wave impinges on the vessel at an oblique angle, which corresponds to the wave direction $\hat{k}$ in (39) with $(\theta, \phi) = (0, 5\pi/4)$.

where $\alpha_j = (j - 1)\Delta\alpha + \pi/8$, for $j = 1, \ldots, 8$, and $\Delta\alpha = \pi/4$.

Table 3 tabulates the results of a convergence study for both plane wave and point source incidence for a nacelle of 10.2, 20.5 and 40.9 wavelengths in size. We also include results for an 81.8-wavelength nacelle. The number of IFGF levels is selected so that the finest-level IFGF box side length is approximately $0.6\lambda$ in all cases. All computations were performed with a GMRES residual tolerance equal to $10^{-3}$. The total near field magnitude relative error $\varepsilon_{\text{near}}$ was estimated over a near field planar grid $P_{N_x N_y}^N(z_0)$, where $[x_{\text{min}}, x_{\text{max}}] \times [y_{\text{min}}, y_{\text{max}}] = [-4,4]^2$, $z_0 = -5$ and $N_x = N_y = 400$, by computing (38) with a reference solution obtained with the same number of surface patches as the target discretization but using $8 \times 8$ points per patch and a residual tolerance of $10^{-5}$. In addition to the near field relative error, Table 3 also includes the total number of unknowns, the size of the nacelle in wavelengths, the time required to compute one GMRES iteration and the total number of GMRES iterations required to satisfy the $10^{-3}$ residual tolerance. Thus, as the problem size increases from $10.2\lambda$ to $20.5\lambda$, $20.5\lambda$ to $40.9\lambda$, and $40.9\lambda$ to $81.8\lambda$, and the number of unknowns is quadrupled in each case, the computing cost per iteration increases.
Figure 6: (a) Aircraft engine nacelle model, (b) translucent view of the geometry where the center shaft is visible, and (c) 8,576-surface-patch discretization with $6 \times 6$ points per patch (for clarity, only every other mesh point is plotted).

Table 3: Convergence study of IFGF-accelerated acoustic solver for a nacelle geometry of $10.2, 20.5$ and $40.9$ wavelengths for plane wave and point source scattering. (The table also includes data for an $81.8\lambda$ nacelle but in this case the near field relative error is estimated using the average relative errors of the three previous problems.) The table summarizes the total number of surface unknowns, nacelle size in wavelengths, maximum number of IFGF levels, time required to compute one GMRES iteration, total number of iterations, and near field relative error $\varepsilon_{\text{near}}$. In all cases the GMRES residual tolerance was set to $10^{-3}$.

| Unknowns     | Size    | IFGF levels | Time (1 iter.) | Plane Wave Incidence | Point Source Incidence |
|--------------|---------|-------------|----------------|----------------------|------------------------|
|              |         |             |                | Tot. iter.           | $\varepsilon_{\text{near}}$ | Tot. iter. | $\varepsilon_{\text{near}}$ |
| 77,184       | $10.2\lambda$ | 6           | 2.0 s          | 33                   | 1.5e-3                 | 39         | 4.1e-3                 |
| 308,736      | $20.5\lambda$ | 7           | 8.7 s          | 47                   | 2.2e-3                 | 59         | 4.0e-3                 |
| 1,234,944    | $40.9\lambda$ | 8           | 40.4 s         | 55                   | 6.7e-4                 | 115        | 2.4e-3                 |
| 4,939,776    | $81.8\lambda$ | 9           | 176.4 s        | 65                   | 1.5e-3 (est.)          | 219        | 3.5e-3 (est.)          |

by a factor of only $4.4, 4.6$ and $4.4$, respectively (which is consistent with an $O(N \log N)$ complexity), and not the $16$-fold cost increase per wavelength doubling that would result from a nonaccelerated algorithm with quadratic complexity. This scaling of the IFGF-accelerated combined-layer solver is consistent with the IFGF method computations presented in [1], which did not include singular local interactions, and suggests that the partitioning and discretization of the geometry makes optimal use of the IFGF algorithm. The results also indicate that the discretization and $10^{-3}$ residual tolerance is sufficient to produce solutions for the $10.2, 20.5$ and $40.9$ wavelength cases with an average error of $1.5 \cdot 10^{-3}$ for plane wave scattering and $3.5 \cdot 10^{-3}$ for point source scattering. The average relative error values are used to estimate the accuracy of the $81.8\lambda$ simulation, which also converged to the same GMRES tolerance as the smaller problems. Note that, as reported in Table 3, the number of iterations required for convergence increases by only $8 - 14$ iterations.

The total near field magnitude $|u(x)| = |u^t(x) + u^s(x)|$ for the $81.8$-wavelength plane wave scattering case is displayed in Figure 7. The field magnitude is plotted over the $xz$-planar grid $P_{Nz}^N(y_0)$ (recall the planar grid definition (37)), where $[x_{\text{min}}, x_{\text{max}}] \times [z_{\text{min}}, z_{\text{max}}] = [-4, 4] \times [-5, 6], y_0 = 0$, with $N_x = 2800$ and $N_z = 4000$. Along most of the exterior circumference of the nacelle housing, the total field forms a relatively uniform stratified pattern. In other regions, intricate multiple-scattering patterns develop, particularly in the region around the intake and throughout the inside of the nacelle. For a closer examination, Figures 7(b) and 7(c) display top views of the field but with the scattering surfaces removed. It is evident that the strongest reflection occurs directly in front of the tip of the nacelle shaft. Note the symmetry in the detail of the near field shown in Figure 7(b), which results in spite of the lack...
of symmetry in the geometry discretization illustrated in the inset in Figure 6(c).

Near fields for the eight-point source, 81.8-wavelength, incident field are displayed in Figure 8. The total field magnitude is plotted over the same \(xz\)-planar grid \(P_{xz}^N(y_0)\) used for the plane wave scattering case. In Figure 8(a), the point-source generated fields can be seen to scatter and exit the front inlet and rear exhaust. The close-up view in Figure 8(b) highlights the location of four of the eight point sources, drawn as red spheres for emphasis; the remaining four sources are obstructed from view by the near field plane. In Figures 8(c) and 8(d) the geometry is removed so we can examine the field interaction within the scatterer in greater detail. Both images exhibit complex multiple scattering and a high degree of symmetry throughout the interior of the structure and in the regions outside that surround the nacelle assembly. In contrast to the plane wave scattering case, where the incident wave travels mostly parallel to the housing and shaft, placing sources between the shaft and nacelle walls guarantees that most waves scatter multiple times before exiting the geometry.

The far field magnitudes are shown in Figure 9 for both plane wave and point source incident fields. Figures 9(a) and 9(b) present the far field for a 40.9\(\lambda\) plane wave, while Figures 9(c) and 9(d) present the far field for an 81.8-wavelength plane wave. Using (34), the far field \(\tilde{u}_\infty\) is computed over \(S_N^2\) (recall the spherical grid definition (35)) with \((N_\phi, N_\theta) = (2000, 600)\) in the 40.9\(\lambda\) case and \((N_\phi, N_\theta) = (3200, 800)\) for the 81.8 wavelength plane wave. More points are used at higher frequencies to resolve the far field lobes that are visible in Figures 9(b) and 9(d). The far field plots for both 40.9\(\lambda\) and 81.8\(\lambda\) plane wave scattering once again show that most of the wave reflection occurs in the region directly in front of the
Figure 8: Total field magnitude $|u(x)| = |u^i(x) + u^s(x)|$ pseudocolor plots for an 82-wavelength aircraft nacelle. The field is plotted over a uniform grid of 2800 × 4000 points for $(x,z) \in [-4,4] \times [-5,6]$. The nacelle is aligned with the $z$-axis and the front points towards the $+z$-direction. The incident field is given by the sum (40) of eight point sources within the nacelle around the center shaft, four of which are shown as small red spheres in panel (b).

The maximum magnitude of the far field increases by a factor of approximately 1.5 for the 81.8λ wave compared with the 40.9λ case. Figures 9(e-g) show the far field magnitude for the eight-point source incident field defined in (40) at 40.9λ and 81.8λ wavelengths. Figure 9(e) displays a side view of the 40.9λ far field magnitude $|u(x)|$ including the nacelle geometry, for reference, with the intake pointing left. Figures 9(f) and 9(g), where the geometry is not included, present the far field, with the positive $z$ direction pointing out of the page, for the 40.9λ and 81.8λ cases, respectively.

7 Conclusions

We presented an accelerated acoustic scattering boundary integral solver based on a novel Interpolated Factored Green Function method (IFGF) for the efficient evaluation of regular interactions and a high-order rectangular-polar quadrature algorithm for local singular and near-singular operator evaluations. Unlike standard nonaccelerated methods which, for an $N$-point surface discretization, evaluate the action of Green function-based integral operators at a complexity of $O(N^2)$, the IFGF method performs the same computation at a cost of $O(N \log N)$ operations. The IFGF accelerator relies on a recursive interpolation
Figure 9: Far-field magnitudes for the nacelle geometry under plane wave and eight point-source incident fields. Panels (a) and (b) present the far field for a 40.9\(\lambda\) plane wave and panels (c) and (d) present the far field for an 81.8-wavelength plane wave. Panels (e)-(g) display far fields for the eight-point source incident field defined in (40), for 40.9\(\lambda\) in panels (e) and (f) and for 81.8\(\lambda\) in panel (g).

scheme that enables the fast evaluation of slowly-varying factored Green functions for groups of sources. The IFGF-based acceleration approach does not rely on Fast Fourier Transforms and this, in turn, gives rise to efficient shared- and distributed-memory parallelization of the underlying algorithms. The rectangular-polar local interaction algorithm, on the other hand, evaluates integral operators at singular and near-singular target points accurately and independently of the IFGF accelerator.

Although in this work we considered only a shared-memory implementation of acoustic IFGF-based integral equation solvers, the numerical examples presented show that even in this case our methods enable the efficient solution of problems with millions of unknowns over complex geometries using only a single computer. In addition to simulations of acoustic scattering by a sphere of up to 128 wavelengths in diameter, we demonstrated the versatility of our numerical algorithms with computational results presented for realistic geometries of relevance in acoustic applications, including acoustic scattering by a submarine geometry and a turbofan nacelle.

CRediT authorship contribution statement

Oscar P. Bruno: Conceptualization, Methodology, Validation, Investigation, Resources, Writing, Supervision, Funding acquisition. Edwin Jimenez: Conceptualization, Methodology, Software, Validation, Investigation, Writing, Visualization. Christoph Bauinger: Conceptualization, Methodology, Software, Validation, Investigation, Writing, Visualization.
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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