Dynamics of dissipative coupled spins: decoherence, relaxation and effects of a spin-boson bath

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Abstract. We study the reduced dynamics of interacting spins, each coupled to its own bath of bosons. We derive the solution in analytic form in the white-noise limit and analyze the rich behaviors in diverse limits ranging from weak coupling and/or low temperature to strong coupling and/or high temperature. We also view the single spin as being coupled to a spin-boson environment and consider the regimes in which it is effectively nonlinear and in which it can be regarded as a resonant bosonic environment.
1. Introduction

Comprehension of the phenomenon of decoherence in open quantum systems has always attracted much attention, in particular as a prerequisite for understanding the transition from quantum to classical behavior. The dissipative two-state or spin-boson model has been thoroughly studied in wide regions of the parameter space with diverse methods and techniques since the 1980s [1, 2].

In the last decade, the subject of decoherence has experienced a renaissance following the growing interest in the field of quantum state manipulation and quantum computation [3]. Any noise source sensitively leads to a narrowing of the quantum coherence domain. This entails severe limitations for coupled qubits to perform logic quantum operations. For this reason, an extensive understanding of the decoherence mechanisms is indispensable.

In this work, we focus on a model that is a generalization of the single spin-boson model to the case of two spins that mutually interact via an Ising-type coupling and are coupled to independent environments made up of bosons. The first analysis of this model relying on the influence functional method was given by Dubé and Stamp [4]. They obtained results for the dynamics in analytic form in restricted regions of the parameter space by omitting certain classes of path contributions and bath correlations. Several other previous studies on the same or related models relied on the master equation and/or perturbative Redfield approach [5]–[7]. Besides the weak-coupling assumption, often the secular approximation [8] is made, which breaks down, however, when the spectrum becomes degenerate.

The model allows, for instance, to study decoherence and relaxation of two coupled qubits [5, 7], or the influence of a bistable impurity on the qubit dynamics ([9] and references...
therein). The latter may significantly degrade coherence in Josephson phase qubits \cite{10}. Another possible application is study of coherence effects in coupled molecular magnets \cite{11}.

In earlier works, the model has been analyzed in the pure dephasing regime both by the Feynman–Vernon method \cite{12} and by the Lindblad approach \cite{13}. Here, we extend the work in \cite{12} beyond the pure dephasing regime and include the full dynamics of the qubit. In particular, we are interested in the competition between decoherence and relaxation to the equilibrium state. Here we focus on the white-noise regime. We shall derive the exact solution for the reduced density matrix (RDM) without restriction on the parameters of the model and analyze it in the coherent and incoherent domains and in the crossover regions in between.

The model and relevant dynamical quantities of the two-spin system are introduced in section 2. The exact formal solution for the reduced dynamics within the Feynman–Vernon influence functional method is presented in section 3. In section 4, the path sum is carried out in the white-noise domain without any further approximation. This is achieved in Laplace space by first tracking down the kernels resulting from the sum of all irreducible path segments and then taking the sum of the infinite geometrical series of these objects. The resulting expressions for the relevant expectation values are in the form of ratios of two polynomials in the Laplace variable. After an overview of the qualitative features of the dynamics in section 5, we present in section 6 explicit expressions for decoherence and relaxation in the various parameter regimes ranging from low temperature and/or weak coupling to high temperature and/or strong coupling. Comparison of the results with those of the perturbative Redfield approach is made. Finally, we study in section 7 the influence of a nonlinear spin-boson environment on the second spin in the various limits. We demonstrate that it behaves in the weak-coupling limit as a bosonic (linear) bath with a resonant spectral structure.

2. Model

We consider two two-state systems that are coupled to each other via an Ising-type coupling and to independent bosonic environments. In pseudospin representation, we choose the generalized spin-boson Hamiltonian (we use units where $\hbar = k_B = 1$)

$$H = -\frac{\Delta_1}{2} \sigma_x - \frac{\Delta_2}{2} \tau_x - \frac{v}{2} \sigma_z \tau_z - \frac{1}{2} \sigma_z X_1 - \frac{1}{2} \tau_z X_2 + \sum_{\xi=1,2} \sum_\alpha \omega_{\xi,\alpha} b_{\xi,\alpha}^\dagger b_{\xi,\alpha}. \tag{1}$$

In the basis formed by the localized eigen states $|R\rangle$ and $|L\rangle$ of $\sigma_z$ and $\tau_z$, respectively, $\Delta_1$ and $\Delta_2$ represent the tunneling couplings between the localized states, and the coupling term $-\frac{1}{2} v \sigma_z \tau_z$ acts as a mutual bias energy of strength $v$. The collective bath modes $X_\xi(t) = \sum_\alpha c_{\xi,\alpha} [b_{\xi,\alpha}(t) + b_{\xi,\alpha}^\dagger(t)]$ ($\xi = 1, 2$) represent fluctuating bias forces. The Hamiltonian is very rich in content and may model diverse physical situations. It may describe two coupled qubits or a qubit $\sigma$ in contact with a complex environment formed by a bistable dissipative impurity $\tau$. Other possible realizations are coupled molecular magnets of which the low-energy states can be viewed as a spin \cite{11}.

For the model \cite{1}, all effects of the environments are captured by the power spectrum of the collective bath modes

$$S_{\xi,\xi'}(\omega) = \text{Re} \int_{-\infty}^{\infty} dt \, e^{it\omega} \langle X_\xi(t) X_{\xi'}(0) \rangle_\beta = \delta_{\xi,\xi'} S_\xi(\omega) = \delta_{\xi,\xi'} \pi G_\xi(\omega) \coth \left( \frac{\beta \omega}{2} \right). \tag{2}$$

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with the spectral density of the coupling [1, 2]
\[ G_\zeta(\omega) = \sum_a c_{\zeta,a}^2 \delta(\omega - \omega_{\zeta,a}) = 2K_\zeta \omega e^{-|\omega|/\omega_c}, \quad \zeta = 1, 2. \] (3)

Here, the second form represents the Ohmic case with a high-frequency cutoff \( \omega_c \). Alternatively, one may choose that the two spins are coupled to a common bath [7]. Here, we study the effects of independent environments. This case is realistic in most physical systems of actual interest.

The density matrix of a single spin has four matrix elements, the two populations that we shall label as RR \( \equiv 1 \) and LL \( \equiv 3 \) and the two coherences with labels LR \( \equiv 2 \) and RL \( \equiv 4 \). The two-spin density matrix has 16 matrix elements \( \rho_{n,m}(t) \). We choose for convenience that the first (second) index refers to the states \( n = 1, \ldots, 4 \) \( (m = 1, \ldots, 4) \) of the \( \sigma \)-spin (\( \tau \)-spin). The matrix elements can be expressed in terms of expectation values of 15 operators, \( \langle \sigma_i \otimes 1 \rangle_t, \langle 1 \otimes \tau_i \rangle_t = \langle \tau_i \rangle, \) and \( \langle \sigma_i \otimes \tau_j \rangle_t = \langle \sigma_i \tau_j \rangle_t \) \( (i = 1, 2, 3 \) and \( j = 1, 2, 3 \) \). The four pure populations may then be written as

\[
\rho_{1,1}(t) = \left[ 1 + \langle \sigma_z \rangle_t + \langle \tau_z \rangle_t + \langle \sigma_z \tau_z \rangle_t \right]/4,
\]
\[
\rho_{1,3}(t) = \left[ 1 + \langle \sigma_z \rangle_t - \langle \tau_z \rangle_t - \langle \sigma_z \tau_z \rangle_t \right]/4,
\]
\[
\rho_{3,1}(t) = \left[ 1 - \langle \sigma_z \rangle_t + \langle \tau_z \rangle_t - \langle \sigma_z \tau_z \rangle_t \right]/4,
\]
\[
\rho_{3,3}(t) = \left[ 1 - \langle \sigma_z \rangle_t - \langle \tau_z \rangle_t + \langle \sigma_z \tau_z \rangle_t \right]/4.
\] (4)

Corresponding expressions hold for the four pure coherences and the eight hybrid states. For instance, we have

\[
\rho_{2,4}(t) = \left[ \langle \sigma_x \tau_x \rangle_t + \langle \sigma_y \tau_y \rangle_t + i\langle \sigma_x \tau_y \rangle_t - i\langle \sigma_y \tau_x \rangle_t \right]/4.
\] (5)

Here, we are predominantly interested in the populations. Throughout we will choose that the reduced system starts out from the initial state \( \rho_{1,1}(t = 0) = 1 \), while the heat reservoirs are in thermal equilibrium at temperature \( T \).

In the absence of the environment, the Hamiltonian \( H = H_0 \) can be easily transformed into diagonal form

\[ \tilde{H}_0 = -\frac{\Omega}{2} (\sigma_z \otimes 1) - \frac{\delta}{2} (1 \otimes \tau_z). \] (6)

The eigenfrequencies are

\[ \omega = \frac{1}{2}(\Omega_+ + \Omega_-), \quad \delta = \frac{1}{2}(\Omega_+ - \Omega_-), \] (7)

with

\[ \Omega_\pm = \sqrt{\Delta_1 \pm \Delta_2} + v^2, \] (8)

and they obey the Vieta relations

\[ \Omega^2 + \delta^2 = \Delta_1^2 + \Delta_2^2 + v^2, \quad \Omega^2 \delta^2 = \Delta_1^2 \Delta_2^2, \]
\[ \Omega^2 \Omega_+^2 = 2(\Delta_1^2 + \Delta_2^2 + v^2), \quad \Omega_+^2 \Omega_-^2 = (\Delta_1^2 + \Delta_2^2 + v^2)^2 - 4\Delta_1^2 \Delta_2^2. \] (9)

The Liouville equations \( \dot{W}_j(t) = -i[H, W_j(t)] \) \( (j = 1, \ldots, 15) \), where the set \{\( W_j(t) \)\} represents the above 15 operators, yield 15 coupled equations. These are conveniently solved in
Within the Feynman–Vernon influence functional method, the exact formal expression for the reduced dynamics is given by

\[ \langle \sigma_z(\lambda) \rangle = \frac{\lambda (v^2 + \Delta_1^2 + \lambda^2)}{(\lambda^2 + \Omega^2)(\lambda^2 + \delta^2)}, \]

\[ \langle \tau_z(\lambda) \rangle = \frac{\lambda (v^2 + \Delta_1^2 + \lambda^2)}{(\lambda^2 + \Omega^2)(\lambda^2 + \delta^2)}, \]

\[ \langle \sigma_z \tau_z(\lambda) \rangle = \frac{(v^2 + \lambda^2)(v^2 + \Delta_1^2 + \Delta_2^2 + \lambda^2)}{\lambda (\lambda^2 + \Omega^2)(\lambda^2 + \Omega^2)}. \]

Here, we are interested in the evolution of the two-spin system without restricting ourselves to weak damping. Therefore, we refrain from employing the perturbative Redfield approach. Rather we calculate the reduced dynamics by using the Feynman–Vernon influence functional method. We show that the solution is available in analytic form in the white-noise limit for general parameters \( \Delta_1, \Delta_2, v \) and \( T \).

### 3. Formal solution for the RDM

Within the Feynman–Vernon influence functional method, the exact formal expression for the RDM of the two-spin system is the quadruple path integral

\[ \rho_{\sigma \sigma', \tau \tau'}(t) = \int \mathcal{D}\sigma \mathcal{D}\sigma' \mathcal{D}\tau \mathcal{D}\tau' A[\sigma] A^*[\sigma'] A^*[\tau] A[\tau'] B[\sigma, \sigma'; \tau, \tau'] \mathcal{F}[\sigma, \sigma'; \tau, \tau'] \]

with appropriately chosen boundary values for the spin paths. Here, each of the paths \( \sigma(t'), \sigma'(t'), \tau(t') \) and \( \tau'(t') \) starts out from the localized state \( |R \rangle \) at time zero. They end up at time \( t \) in the states \( |\sigma_t \rangle, |\sigma'_t \rangle, |\tau_t \rangle \) and \( |\tau'_t \rangle \), respectively, where \( \sigma_t, \sigma'_t, \tau_t, \tau'_t \in R, L \).

The functional \( A[\sigma] \) is the amplitude for the free spin \( \sigma \) to follow the path \( \sigma(t') \), the functional \( B[\sigma, \sigma'; \tau, \tau'] \) represents the coupling of the two spins (see below), and the functional \( \mathcal{F}[\sigma, \sigma'; \tau, \tau'] \) introduces the environmental influences.

For uncorrelated baths, we have

\[ \mathcal{F}[\sigma, \sigma'; \tau, \tau'] = \mathcal{F}_1[\xi_1, \eta_1] \mathcal{F}_2[\xi_2, \eta_2], \]

where

\[ \ln \mathcal{F}_\zeta[\xi_\zeta, \eta_\zeta] = \int_0^t dt' \int_0^{\prime t} dt'' \left[ \xi_\zeta(t') Q_\zeta(t' - t'') \eta_\zeta(t'') + i \xi_\zeta(t') Q''_\zeta(t' - t'') \eta_\zeta(t'') \right]. \]

Here, we have introduced symmetric and antisymmetric spin paths,

\[ \xi_1(t') = \frac{1}{2}[\sigma(t') - \sigma'(t')], \quad \eta_1(t') = \frac{1}{2}[\sigma(t') + \sigma'(t')], \]

\[ \xi_2(t') = \frac{1}{2}[\tau(t') - \tau'(t')], \quad \eta_2(t') = \frac{1}{2}[\tau(t') + \tau'(t')]. \]

The correlator \( Q_\zeta(t) = Q'_\zeta(t) + i Q''_\zeta(t) \) is the second integral of the force autocorrelation function \( \langle X_\zeta(t) X_\zeta(0) \rangle_\beta \) (see equation (2)). In the Ohmic scaling limit, we have

\[ Q_\zeta(t) = 2 K_\zeta \left[ \ln \left( \frac{\beta \omega_\zeta}{\pi} \right) + \ln \sinh \left( \frac{\pi |t|}{\beta} \right) \right] + i \pi K_\zeta \sgn(t), \quad \zeta = 1, 2. \]

Here, \( K_\zeta \) is the usual dimensionless Ohmic coupling strength for the spin \( \zeta \), and \( \beta = 1/T \) is the inverse temperature.

To handle the quadruple path integral (11), we follow the procedure for the single spin-boson problem [1, 2] and write it as an integral over two paths, one for each spin. Each such
path visits the diagonal ‘sojourn’ states and the off-diagonal ‘blip’ states of the respective spin. A path that starts and ends in a sojourn state must contain an even number of transitions with amplitude $\mp i \Delta \zeta /2$ for each flip of spin $\zeta$. The flips occur at times $t_j$ for spin 1 and at times $s_j$ for spin 2. Upon labeling the sojourn and blip states with charges $\eta_{1,j}, \xi_{1,j}$ (for spin 1) and $\eta_{2,j}, \xi_{2,j}$ (for spin 2), each with values $\pm 1$, the paths with $2n_1$ and $2n_2$ transitions, respectively, may be written as

$$\eta_{1}^{(n_1)}(t') = \sum_{j=0}^{n_1} \eta_{1,j} \left[ \theta(t' - t_{2j}) - \theta(t' - t_{2j+1}) \right],$$

$$\xi_{1}^{(n_1)}(t') = \sum_{j=1}^{n_1} \xi_{1,j} \left[ \theta(t' - t_{2j-1}) - \theta(t' - t_{2j}) \right],$$

$$\eta_{2}^{(n_2)}(t') = \sum_{j=0}^{n_2} \eta_{2,j} \left[ \theta(t' - s_{2j}) - \theta(t' - s_{2j+1}) \right],$$

$$\xi_{2}^{(n_2)}(t') = \sum_{j=1}^{n_2} \xi_{2,j} \left[ \theta(t' - s_{2j-1}) - \theta(t' - s_{2j}) \right].$$

Upon introducing the notation $Q_{\zeta;j,k} = Q_{\zeta}(t_j - t_k)$, we may write the bath correlations between the blip pair $\{j, k\}$ of spin $\zeta$ in the compact form

$$\Lambda_{\zeta;j,k} = Q_{\zeta;2j,2k-1} + Q_{\zeta;2j-1,2k} - Q_{\zeta;2j,2k} - Q_{\zeta;2j-1,2k-1}.$$  

With this, the influence function for the paths (13) reads

$$\mathcal{I}_{\zeta}^{(n_{\zeta})} = \exp \left[ - \sum_{j=1}^{n_{\zeta}} Q_{\zeta;2j,2j-1} \right] \exp \left[ - \sum_{j=2}^{n_{\zeta}} \sum_{k=1}^{j-1} \xi_{j}^{(j,k)} \Lambda_{\zeta;j,k} \right] \exp \left[ i \pi K_{\zeta} \sum_{k=0}^{n_{\zeta}-1} \xi_{\zeta,k+1} \eta_{\zeta,k} \right].$$

Here the first and second terms represent the intrablip and interblip correlations, respectively. The phase term is specific to the Ohmic scaling limit and represents correlations of the sojourns with their subsequent blips.

The sum over all paths now means (i) to sum over all possible intermediate sojourn and blip states of the two spins the paths with a given number of transitions can visit, (ii) to integrate over the (for each spin) time-ordered jumps of these paths, and (iii) to sum over the possible number of transitions the two spins can take,

$$\sum_{\text{all paths}} \cdots \sum_{n_1=0}^{\infty} \sum_{\{|\xi_{1,i}|=\pm 1\}} \sum_{\{|\xi_{2,i}|=\pm 1\}} \int_{t_i}^{t_f} \int_{s_{2n_1}}^{s_{2n_1+1}} \cdots \int_{s_1}^{t_f} \cdots$$

$$\times \sum_{n_2=0}^{\infty} \sum_{\{|\xi_{2,i}|=\pm 1\}} \sum_{\{|\xi_{2,i}|=\pm 1\}} \int_{t_i}^{t_f} \int_{s_{2n_2}}^{s_{2n_2+1}} \cdots \int_{s_1}^{t_f} \cdots$$

For coherences, the number of transitions in the respective spin path is odd.

Next, we rewrite the double path sum (21) with time-ordering for each spin in terms of a single path over the 16 possible states with time-ordering of all the flip times $\{t_j\}, \{s_i\}$. In this
representation, the system lingers for some period in a particular state of the double-spin RDM and then it flips with amplitude \( \pm i \Delta_1/2 \) or \( \pm i \Delta_2/2 \) to another state. For the longitudinal spin–spin coupling \(-\frac{1}{2} \sigma_z \tau_z\) in (1), the coupling factor \( B \) in equation (11) is unity when the system dwells on one of the four pure sojourn or one of the four pure blip states, and it is sensitive to the coupling when it stays, say for a period \( u_j \), in one of the eight hybrid states. In detail, we have

\[
B[j] = e^{iu_j}, \quad \text{for } j \in (1, 2), (2, 1), (4, 3), (3, 4),
\]
\[
B[j] = e^{-iu_j}, \quad \text{for } j \in (1, 4), (4, 1), (2, 3), (3, 2).
\]

Combination of the above expressions yields the exact formal solution for the dynamics of the RDM of the two-spin system in the Ohmic scaling limit. Evidently, because of the nonconvolutive form of the bath correlations in the influence functional (20), the path sum cannot be performed in analytic form. Alternatively, one may recast the exact formal series expression for the populations in the form of generalized master equations in which the kernels, by definition, are the irreducible components of path segments with diagonal initial and final states [2]. In the general case, the kernels are given by an infinite series in \( \Delta_1 \) and \( \Delta_2 \), with the time integrals in each summand being again in nonconvolutive form. Additional difficulties in performing the path sum (21) arise from the spin–spin coupling factor \( B \) given in equation (22).

4. Exact solution in analytic form in the white-noise limit

In the white-noise limit of equation (2),

\[
S_\zeta(\omega \ll 1/\beta) = 2 \theta_\zeta, \tag{23}
\]

where

\[
\theta_\zeta = 2\pi K_\zeta T \tag{24}
\]

is a scaled thermal energy, the bath correlation function takes the form.

\[
Q_\zeta(t) = 2K_\zeta \left[ \ln \left( \frac{\omega_c}{2\pi T} \right) + \theta_\zeta \right] + \theta_\zeta |t| + i\pi K_\zeta \text{sgn}(t). \tag{25}
\]

This expression emerges directly from equation (14) in the high temperature or long-time limit \( t/\beta \gg 1 \). The first term in equation (25) leads to an adiabatic (Franck–Condon-type) renormalization factor made up of modes in the frequency range \( 2\pi T < \omega < \omega_c \). It is natural to assimilate this term, together with the phase term, into an effective temperature-dependent tunneling matrix element,

\[
\Delta_\zeta^2 = \Delta_\zeta^2 \cos(\pi K_\zeta), \quad \text{with} \quad \Delta_\zeta^2 = (2\pi T/\Delta_\zeta t)^{2K_\zeta} \Delta_\zeta^2 e^{-2K_\zeta \gamma_\zeta}, \tag{26}
\]

where \( \Delta_\zeta^{1-K_\zeta} = \Delta_\zeta/\omega_c^K_\zeta \) is the standard renormalized tunneling matrix element.

All dynamical effects of the environmental coupling are captured by the second term \( \theta_\zeta |t| \) in equation (25). From this we see that the weight of the thermal energy relative to the systems energies \( \Delta_\zeta \) and \( \nu \) is assessed by the scaled thermal energy \( \theta_\zeta \).

The term \( \gamma_\zeta \) accounts for deviation of the actual high-frequency behavior of \( G_\zeta(\omega) \) from the exponential cutoff form in equation (3) [2].
Based on our experience with the single spin-boson system, we should expect that the form (25) is a remarkably good approximation in the parameter range $K_\zeta \lesssim 0.3$ and $\Omega_\pm \lesssim T \ll \omega_c$.

and this is corroborated indeed by our study. For a single unbiased spin with Ohmic damping $K \ll 1$, the coherent–incoherent ‘phase’-transition is at temperature $T = T^*$ with $T^* = \Delta_z/(\pi K)$ [2]. Therefore we should expect that, for $K_\zeta \ll 1$, the white-noise form (25) is valid not only in the incoherent regime but also in a sizeable domain of the coherent regime. This shall be confirmed subsequently.

For the form (25) of $Q_\zeta(t)$, the blip-pair correlation (19) vanishes, $\Lambda_{\zeta; j, k} = 0$, so that the influence function (20) reduces to

$$F_\zeta^{(n)} = \exp \left[ -\sum_{j=1}^{n_\zeta} Q'_{\zeta; 2j, 2j-1} \right] \exp \left[ i\pi K_\zeta \sum_{k=0}^{n_\zeta-1} \xi_{\zeta, k+1} \eta_{\zeta, k} \right].$$

(28)

With this expression for the influence function, each term of the infinite series for the elements of the RDM becomes a convolution. This feature readily makes the path sum accomplishable. We shall now exemplify, by taking $\langle \sigma_z \rangle_t$ and $\langle \tau_z \rangle_t$ as examples, that the path sum for the Laplace transform of the RDM can be carried out exactly in analytic form. This is achieved in two steps. First, we calculate the kernels and then we sum up the respective infinite geometrical series of these objects.

4.1. The expectations $\langle \sigma_z \rangle_t$ and $\langle \tau_z \rangle_t$

In the master equation approach, the key quantities are the kernels [2]. Iterative solution of the master equation in Laplace space yields an infinite geometrical series of the kernels, which can easily be summed in analytic form. Within the path sum approach presented here, the kernels correspond to the sum of all irreducible path segments that interpolate between pure sojourn states. Irreducibility means that these segments cannot be separated into uncorrelated pieces without removing bath correlations. Consider now first the kernel $K(\lambda)$ of the expectation $\langle \sigma_z(\lambda) \rangle$. The analysis yields that every contribution to the kernel $K(\lambda)$ of $\langle \sigma_z(\lambda) \rangle$ displays initially and finally a transition of spin $\sigma$ with any number of even hops of spin $\tau$ at intermediate times. Thus, we have $K(\lambda) = \sum_{n=0}^{\infty} K_{1,n}(\lambda)$, where $K_{1,n}(\lambda)$ is of the order of $\Delta_1^2 \Delta_2^{2n}$.

The first three terms of the infinite series in $\Delta_2^2$ are graphically depicted in figure 1. For all other types of irreducible diagrams, one may think of, e.g., where either the first or the last
polynomials of degree four. We get
\[ K_{1,0}(\lambda) = -\Delta_1^2 \int_0^\infty d\tau_1 e^{-(\lambda+\vartheta_1)\tau_1} \cos(\pi K_1 + v\tau_1) \]
\[ = -(\lambda + \vartheta_1 + \vartheta_2) \left[ 1 - \tan(\pi K_1) \frac{v}{\lambda + \vartheta_1} \right] \alpha_{11}(\lambda), \] 
(30)

\[ K_{1,1}(\lambda) = \Delta_1^2 \Delta_2^2 \int_0^\infty d\tau_1 d\tau_2 d\tau_3 e^{-(\lambda+\vartheta_1)(\tau_1+\tau_2+\tau_3)} e^{-\vartheta_2 \tau_2} \]
\[ \times \frac{1}{2} [\cos(\pi K_1 + v\tau_1 + v\tau_3) - \cos(\pi K_1 + v\tau_1 - v\tau_3)] \cos(\pi K_2) \]
\[ = -(\lambda + \vartheta_1 + \vartheta_2) \left[ \frac{v^2}{(\lambda + \vartheta_1)^2} + \frac{v}{\lambda + \vartheta_1} \tan(\pi K_1) \right] \alpha_{11} \alpha_{12}, \] 
(31)

\[ K_{1,p}(\lambda) = K_{1,1}(\lambda) \left( -\Delta_2^2 \int_0^\infty d\tau_1 d\tau_3 e^{-(\lambda+\vartheta_1)\tau_1} e^{-(\lambda+\vartheta_1+\vartheta_2)\tau_3} \cos(v\tau_1) \right)^{n-1} \]
\[ = K_{1,1}(\lambda) \left( -\alpha_{12}(\lambda) \right)^{n-1}. \] 
(32)

Here, we have taken into account that, for the white-noise form (25), a correlation of bath \( \zeta \) stretching over an interval between neighboring hops effectively leads to a shift of the Laplace variable in the respective time integral, \( \lambda \to \lambda + \vartheta_{\xi} \). It is convenient to split the kernels into the contributions which are even and odd in the coupling \( v \), \( K_\xi(\lambda) = K^{(+)}(\lambda) + K^{(-)}(\lambda) \). The resulting expressions may be written as

\[ K^{(+)}(\lambda) = -(\lambda + \vartheta_1 + \vartheta_2) \left[ 1 + \frac{v^2}{(\lambda + \vartheta_1)^2} \frac{\alpha_{12}(\lambda)}{1 + \alpha_{12}(\lambda)} \right] \alpha_{11}(\lambda), \] 
(33)

\[ K^{(-)}(\lambda) = (\lambda + \vartheta_1 + \vartheta_2) \tan(\pi K_1) - \frac{v}{\lambda + \vartheta_1} \frac{1}{1 + \alpha_{12}(\lambda)} \alpha_{11}(\lambda). \] 
(34)

Paths that visit a pure sojourn state at intermediate times yield reducible contributions. Taking into account all possibilities of such visits yields a geometrical series in the kernel \( K^{(+)}(\lambda) \), while \( K^{(-)}(\lambda) \) occurs only once as initial irreducible contribution. Performing the sum we readily obtain for \( \langle \sigma_z(\lambda) \rangle \) the concise expression

\[ \langle \sigma_z(\lambda) \rangle = \frac{1}{\lambda} \left[ 1 + \frac{K^{(-)}(\lambda)}{\lambda K^{(+)}(\lambda)} \right] = \frac{1}{\lambda} \frac{N_1(\lambda)}{D_1(\lambda)}. \] 
(35)

Simple algebraic manipulation yields the second form, in which \( N_1(\lambda) \) and \( D_1(\lambda) \) are polynomials of degree four. We get

\[ N_1(\lambda) = \lambda^4 + (3\vartheta_1 + \vartheta_2)\lambda^3 + (v^2 + \Delta_1^2 + 3\vartheta_1^2 + 2\vartheta_1 \vartheta_2)\lambda^2 + (v^2 \vartheta_1 + \Delta_2^2 \vartheta_1 + \vartheta_1^3 + v^2 \vartheta_2 + \vartheta_1^2 \vartheta_2)\lambda \]
\[ + \Delta_1 \tan(\pi K_1)(\vartheta_1 + \vartheta_2 + \lambda), \] 
(36)
Figure 2. Sketch of the irreducible kernels $A_1^{(a)}(\lambda)$ (left) and $A_1^{(b)}(\lambda)$ (right). The intervals are dressed by self-energy contributions of spin $\sigma$ (black circle) and spin $\tau$ (black square), as sketched in figure 3.

\[
D_1(\lambda) = \lambda^4 + (3\vartheta_1 + 2\vartheta_2)\lambda^3 + (2\vartheta_1^2 + 3\vartheta_2^2 + 2\vartheta_1\vartheta_2)\lambda^2 + (v^2\vartheta_1 + 2\Delta_1^2\vartheta_1 + \Delta_2^2\vartheta_1 + \vartheta_1^3) + v^2\vartheta_2 + \Delta_1^2\vartheta_1 + \Delta_2^2\vartheta_2)\lambda + \Delta_1^2\vartheta_1^2 + \Delta_2^2\vartheta_1\vartheta_2 + \Omega^2\bar{\delta}^2.
\]  

(37)

Here, we have introduced the eigenfrequencies $\Omega$, $\delta$ of the undamped coupled two-spin system. The bar denotes adiabatic renormalization in equation (7) according to equation (26).

The pole at $\lambda = 0$ in equation (35) determines the thermal equilibrium value of $\langle \sigma_z \rangle$ taken over at time infinity,

\[
\lim_{t \to \infty} \langle \sigma_z \rangle_t = \langle \sigma_z \rangle_{eq} = -\frac{K(-)(0)}{K(+)(0)} = \frac{v}{2T} \frac{\vartheta_1(\vartheta_1 + \vartheta_2)}{\Delta_1^2 + \vartheta_1(\vartheta_1 + \vartheta_2)}.
\]  

(38)

This shows that $\langle \sigma_z \rangle_{eq}$ is negligibly small for $\Delta_2 \neq 0$, while, as $\Delta_2 \to 0$, it takes the proper (white noise) equilibrium value $v/(2T)$ of the single biased spin-boson system.

The positions $\lambda_j$ ($j = 1, \ldots, 4$) of the dynamical poles are the solutions of a quartic equation with real coefficients, $D_1(\lambda) = 0$. They appear in complex conjugate or real pairs situated in the left half-plane, $\text{Re} (\lambda) < 0$.

Upon collecting the various pole contributions, we obtain in the time domain

\[
\langle \sigma_z \rangle_t = \sum_{i=1}^{4} A_i e^{\lambda_i t} + \langle \sigma_z \rangle_{eq}.
\]  

(39)

The temperature dependence of the $\{\lambda_j\}$ is discussed below in section 5.

For completeness, we remark that the expectation $\langle \tau_z(\lambda) \rangle$ may be expressed in a similar form,

\[
\langle \tau_z(\lambda) \rangle = \frac{1}{\lambda} \frac{N_2(\lambda)}{D_2(\lambda)}.
\]  

(40)

Evidently, the polynomials $N_2(\lambda)$ and $D_2(\lambda)$ follow from the expressions (36) and (37) by interchange of the indices 1 and 2.

4.2. The expectation $\langle \sigma_z \tau_z \rangle$,

Consider first contributions to the kernel of $\langle \sigma_z \tau_z(\lambda) \rangle$ in which the first and the last flip are flips of spin $\sigma$. The diagrams with flip sequences $\Delta_1\Delta_1$ and $\Delta_1\Delta_2\Delta_1\Delta_2\Delta_1$ are sketched in figure 2. Higher order diagrams of this class have flip sequence $\Delta_1(\Delta_2\Delta_1\Delta_2)^{n+1}\Delta_1$, where $n \geq 1$. The characteristic feature of these diagrams is that in all intermediate time intervals either
spin $\sigma$ or spin $\tau$ or both spins dwell in off-diagonal states. Consider now first the bare diagrams, in which the black box and black circles are absent. Figure 2(a) yields

$$\mathcal{A}_{1, \text{bare}}^{(a)}(\lambda) = \mathcal{K}_{1,0}(\lambda) \sum_{n=0}^{\infty} \left( -\frac{\Delta^2}{2} \int_{0}^{\infty} d\tau_1 d\tau_2 e^{-(\lambda + \vartheta_1)\tau_1} e^{-(\lambda + \vartheta_1 + \vartheta_2)\tau_2} \cos(v\tau_1) \right)^n$$

$$= - (\lambda + \vartheta_1 + \vartheta_2) \left[ 1 - \frac{v}{\lambda + \vartheta_1} \tan(\pi K_1) \right] \alpha_{11}(\lambda),$$

$$\mathcal{A}_{1, \text{bare}}^{(b)}(\lambda) = \mathcal{K}_{1,0}(\lambda) = - (\lambda + \vartheta_1 + \vartheta_2) \left[ 1 - \frac{v}{\lambda + \vartheta_1} \tan(\pi K_1) \right] \alpha_{11}(\lambda),$$

where the function $\alpha_{11,\epsilon}(\lambda)$ is given in equation (29).

The nested bare diagram in figure 2(b) gives the additional factor $\alpha_{11}(\lambda)$, $\alpha_{11}(\lambda)$

$$\mathcal{A}_{1, \text{bare}}^{(b)}(\lambda) = \mathcal{A}_{1, \text{bare}}^{(a)}(\lambda) \alpha_{11}(\lambda).$$

Higher-order nested bare diagrams of the type sketched in figure 2 class into a geometrical series. All these terms are readily summed up to the contribution

$$\mathcal{A}_{1, \text{bare}}(\lambda) = \mathcal{A}_{1, \text{bare}}^{(a)}(\lambda) \frac{1}{1 - \alpha_{22}(\lambda) \alpha_{11}(\lambda)},$$

Next, we take into account that every time interval in which spin $\zeta$ dwells in a sojourn state is dressed by self-energy corrections. These contributions are schematically sketched for spin $\sigma$ (circle) and spin $\tau$ (square) in figure 3. They lead to modifications of the functions $\alpha_{11}(\lambda)$ and $\alpha_{22}(\lambda)$ according to

$$\alpha_{11}(\lambda) \rightarrow \alpha_1(\lambda) = \frac{\alpha_{11}(\lambda)}{1 + \alpha_{12}(\lambda)},$$

$$\alpha_{22}(\lambda) \rightarrow \alpha_2(\lambda) = \frac{\alpha_{22}(\lambda)}{1 + \alpha_{21}(\lambda)}.$$

Thus, the dressed diagrams of the type sketched in figure 2 yield the expression

$$\mathcal{A}_{1}^{(a)}(\lambda) = (\lambda + \vartheta_1 + \vartheta_2) \left[ 1 - \frac{v}{\lambda + \vartheta_1} \tan(\pi K_1) \right] \frac{\alpha_1(\lambda)}{1 - \alpha_1(\lambda) \alpha_2(\lambda)}.$$

Similarly, the dressed diagrams (a) and (b) in figure 4 give

$$\mathcal{B}_{1}^{(a)}(\lambda) = (\lambda + \vartheta_1 + \vartheta_2) \left[ 1 - \frac{v}{\lambda + \vartheta_1} \tan(\pi K_1) \right] \alpha_1(\lambda) \alpha_2(\lambda),$$

$$\mathcal{B}_{1}^{(b)}(\lambda) = \mathcal{B}_{1}^{(a)}(\lambda) \alpha_1(\lambda) \alpha_2(\lambda).$$
We readily get for $C_1$ again described by the kernel $N$ and

$$\Delta_1 = \frac{1}{1 - \alpha_1(\lambda)\alpha_2(\lambda)}.$$  

Clearly, we must add those contributions resulting from the terms $A_1(\lambda)$ and $B_1(\lambda)$ by interchange of the spins $\sigma$ and $\tau$. The analysis shows that there are no other contributions. Again, we split the kernel $C(\lambda)$ into the parts which are even and odd in the coupling $v$. We readily get for $C^{(\pm)}(\lambda) = \sum_{\varepsilon=1,2}[A_1^{(\varepsilon)}(\lambda) + B_1^{(\varepsilon)}(\lambda)]$ the forms

$$C^{(\varepsilon)}(\lambda) = -(\lambda + \vartheta_1 + \vartheta_2) \frac{\alpha_1(\lambda) + \alpha_2(\lambda) - 2\alpha_1(\lambda)\alpha_2(\lambda)}{1 - \alpha_1(\lambda)\alpha_2(\lambda)}.$$  

$$C^{(-)}(\lambda) = (\lambda + \vartheta_1 + \vartheta_2) \sum_{\varepsilon=1,2} \frac{v}{\lambda + \vartheta_1^\varepsilon} \tan(\pi K_1^\varepsilon) \frac{\alpha_1(\lambda) - \alpha_1(\lambda)\alpha_2(\lambda)}{1 - \alpha_1(\lambda)\alpha_2(\lambda)}.$$  

These expressions represent the entity of irreducible path segments. Next, we observe that the sum of two-spin paths with any number of interim visits of pure sojourn states yields a geometrical series of these objects. In the part that is odd in $v$, the first irreducible path section is again described by the kernel $C^{(-)}(\lambda)$. Thus we get

$$\langle \sigma_z \tau_z(\lambda) \rangle = \frac{1 + C^{(-)}(\lambda)/\lambda}{\lambda - C^{(-)}(\lambda)/\lambda} = \frac{1}{\lambda} N(\lambda).$$  

The second form is a simple fraction with polynomials $N(\lambda)$ and $D(\lambda)$ of sixth order,

$$N(\lambda) = v^3(\vartheta_1 + \vartheta_2 + \lambda)[\tan(\pi K_1)\Delta_1^2 + \tan(\pi K_2)\Delta_2^2] + v \tan(\pi K_1)\Delta_1^2(\vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_2^2 - \Delta_1^2 + (\vartheta_1 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)] + \lambda v^2(\vartheta_1 + \vartheta_2 + \lambda)$$

$$\times[\Delta_1^2 - \Delta_2^2 + (\vartheta_2 + \lambda)(\vartheta_1 + \vartheta_2 + \lambda)].$$  

Figure 4. Sketch of the irreducible kernels $B_1^{(a)}(\lambda)$ (left) and $B_1^{(b)}(\lambda)$ (right). Again, the intervals are dressed as sketched in figure 3.
$D(\lambda) = \lambda^6 + 3(\vartheta_1 + \vartheta_2)\lambda^5 + (\bar{\Omega}_0^2 + \bar{\Omega}_1^2 + 3\vartheta_1^2 + 8\vartheta_1\vartheta_2 + 3\vartheta_2^2)\lambda^4 + (\vartheta_1 + \vartheta_2)$

$\times (2\bar{\Omega}_0^2 + 2\bar{\Omega}_1^2 + \vartheta_1^2 + 6\vartheta_1\vartheta_2 + \vartheta_2^2)\lambda^3 + [\bar{\Omega}_0^2 \bar{\Omega}_1 + \vartheta_1 \vartheta_2 \vartheta_2(2\vartheta_1 + \vartheta_2)(\vartheta_1 + 2\vartheta_2)]$

$+ v^2(3\vartheta_1^2 + 4\vartheta_1\vartheta_2 + 3\vartheta_2^2) + \Delta_1^2(2\vartheta_1 + \vartheta_2)(\vartheta_1 + 3\vartheta_2) + \Delta_2^2(2\vartheta_2 + \vartheta_1)(\vartheta_2 + 3\vartheta_1)]\lambda^2$

$+ (\vartheta_1 + \vartheta_2)[\bar{\Omega}_0^2 + \vartheta_1^2 + \vartheta_2^2 + (v^2 + \Delta_1^2)\vartheta_1^2 + (v^2 + \Delta_2^2)\vartheta_2^2 + 3\vartheta_1\vartheta_2(\Delta_1^2 + \Delta_2^2)]\lambda$

$+ v^2(\vartheta_1 + \vartheta_2)(\Delta_1^2\vartheta_1 + \Delta_2^2\vartheta_2) + \vartheta_1\vartheta_2[(\Delta_1^2 - \Delta_2^2)^2 + (\Delta_2^2 \vartheta_1 + \Delta_1^2 \vartheta_2)(\vartheta_1 + \vartheta_2)]$.  (53)

The odd powers of the pole function can be removed with a shift. Putting $D[\lambda = x - (\vartheta_1 + \vartheta_2)/2] \equiv \bar{D}(x)$, we obtain a polynomial with even powers,

$\bar{D}(x) = x^6 + [\bar{\Omega}_0^2 + \bar{\Omega}_1^2 - \frac{1}{2}(\vartheta_1^2 + \vartheta_2^2) - \frac{1}{2}(\vartheta_1 - \vartheta_2)^2]x^4 + [\bar{\Omega}_0^2 \bar{\Omega}_1 + (\bar{\Delta}_1^2 + \bar{\Delta}_2^2 - 2v^2)\vartheta_1\vartheta_2$

$- \Delta_1^2\vartheta_1^2 - \Delta_2^2\vartheta_2^2 + \frac{1}{16}(\vartheta_1 - \vartheta_2)^2(3\vartheta_1^2 + 2\vartheta_1\vartheta_2 + 3\vartheta_2^2)]x^2$

$- \frac{1}{4}v^2(\vartheta_1 + \vartheta_2) + (\Delta_1^2 - \Delta_2^2)(\vartheta_2 - \vartheta_1) + \frac{1}{4}(\vartheta_1^2 - \vartheta_2^2)(\vartheta_1 - \vartheta_2)]^2$.  (54)

Taking into account the various pole contributions, we obtain in the time domain

$$\langle \sigma_z \tau_z \rangle_t = \sum_{j=1}^{6} B_e e^{\lambda_j t} + \langle \sigma_z \tau_z \rangle_{eq},$$

where the $\{\lambda_j\}$ are the zeros of $D(\lambda) = 0$, and the equilibrium value is

$$\langle \sigma_z \tau_z \rangle_{eq} = -\frac{C^{(-)}(0)}{C^{(+)}(0)} = \frac{v}{2T}.$$  (56)

The expressions (35)–(40) and (51)–(54) are the main results of this work. They represent the exact analytical solutions for $\langle \sigma_z(\lambda) \rangle$, $\langle \tau_z(\lambda) \rangle$ and $\langle \sigma_z \tau_z(\lambda) \rangle$, in the white-noise limit for general coupling $v$ and general effective reservoir couplings $\vartheta_1$ and $\vartheta_2$. Except for use of the form (25), no other approximation has been made. We remark that for all other initial and final states of the RDM, we would find the same pole functions (37) and (53). Only the numerator function would be different.

**5. Qualitative features**

The behaviors of the four dynamical poles of $\langle \sigma_z \rangle_t$ and the six dynamical poles of $\langle \sigma_z \tau_z \rangle_t$, and the respective amplitudes are quite multifarious. In this section, we sketch the characteristics for the symmetric system, $\Delta_1 = \Delta_2 \equiv \Delta$ and $\vartheta_1 = \vartheta_2 \equiv \vartheta$.

**5.1. $\langle \sigma_z \rangle_t$**

In the coupling range $v < v_{uc} = \Delta/\sqrt{2}$, there are three crossover temperatures, denoted by $\vartheta_0^c$, $\vartheta_1^c$ and $\vartheta_2^c$ (see figures 5 and 6).

In the regime $\vartheta < \vartheta_0^c$ the dynamics is coherent and described by a superposition of two damped oscillations. For $\vartheta < \vartheta_0^c$, the oscillations have different frequencies and the same damping rate, and the amplitudes are comparable in magnitude. On the other hand, in the
range $\vartheta_0^* < \vartheta < \vartheta_1^*$, they have the same frequency, but different decrements and the amplitude belonging to the larger decrement is negligibly small.

In the temperature regime $\vartheta > \vartheta_1^*$, the dynamics is incoherent. In the regime $\vartheta_1^* < \vartheta < \vartheta_2^*$, the four poles are real, and the two smallest rates have largest amplitudes and dominate the relaxation process.

In the so-called Kondo regime $\vartheta > \vartheta_2^*$, the dominant pole is real and approaches $-\bar{\Delta}/\vartheta$, and its residuum goes to $1 - \langle \sigma_z \rangle_{\text{eq}}$, as temperature is increased. The other real pole takes the value $-2\vartheta$, while its residuum drops to zero. There is also a damped oscillation of which the frequency and rate approach asymptotically $\nu$ and $\vartheta$, but the amplitude becomes negligibly small. The phenomenon that, in the Kondo regime for $K < \frac{1}{2}$, incoherent relaxation slows down with increasing temperature, is already well known in the single spin-boson problem [2].

Figure 7 shows the transition from coherent to incoherent dynamics as $\vartheta$ is raised. One can also see that at high $\vartheta$ the effective damping decreases with increasing $\vartheta$.

For $\nu > \nu_{\text{cr}}$, there is only one crossover. It separates the regime with two complex conjugate poles from the regime with one pair of complex conjugate poles and two real poles. Above $\vartheta_2^*$, the relaxation is governed by the Kondo pole.

5.2. $\langle \sigma_z \tau_z \rangle_t$

For a symmetric system, two poles are cancelled. The reduced pole equation reads

$$\lambda^4 + 4\vartheta \lambda^3 + (\bar{\Omega}_1^2 + 5\vartheta^2)\lambda^2 + 2\vartheta (\bar{\Omega}_1^2 + \vartheta^2)\lambda + 4\vartheta^2 \bar{\Delta}^2 = 0. \quad (57)$$
The characteristic behavior of the poles as a function of the scaled temperature $\vartheta$ is shown in figure 8. At low $\vartheta$, all poles contribute to the dynamics. In the Kondo regime $\vartheta \gtrsim 3 \bar{\Delta}$, the leading pole behaves as $-2 \bar{\Delta}^2 / \vartheta$, and the amplitudes of the other contributions are negligibly small.

6. Dynamics in the various parameter regimes for differing spins

6.1. Low temperature behavior

Below the first crossover temperature, $\Omega_{\pm} \lesssim T \lesssim T_0^*$, the real parts of the $\{\lambda_j\}$ are determined by the one-boson exchange contribution. As a result, they vary linearly with $T$ in the white-noise limit. This is the regime in which the perturbative Redfield approach for white noise would yield the same results. The respective parameter regime for $\langle \sigma_z \rangle$, corresponds to the region below the $\vartheta_0^*$-curve in figure 5. For $\vartheta > \vartheta_0^*$, i.e. $T > T_0^*$, multiboson contributions, resulting in nonlinear $T$-dependence of the rates, become relevant, as is directly visible in figure 8. In this regime, the perturbative Redfield approach fails.

In the remainder of this subsection, we assort the expressions for the $\{\lambda_j\}$ and the respective amplitudes in the low-temperature regime $\Omega_{\pm} \lesssim T \lesssim T_0^*$. 

Figure 7. $\langle \sigma_z \rangle$: Plots of $\langle \sigma_z \rangle$ in the Fourier regime (a) and time regime (b), $v = 0.5$, $\bar{\Delta} = 1$ and $K = 0.05$. Red (small-dashed): $\vartheta = 0.2$; green (dashed): $\vartheta = 1.2$; blue (medium-dashed): $\vartheta = 2.1$; and grey (long-dashed): $\vartheta = 2.8$

Figure 8. $\langle \sigma_z \tau_z \rangle$: Real (a) and imaginary (b) part of $\lambda_j$, $v = 0.5$, $\bar{\Delta} = 1$. 

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6.1.1. $\langle \sigma_z \rangle_z$. There is a superposition of two damped oscillations with frequencies

$$
\lambda_{1,2} = \pm i \tilde{\Omega} - \frac{\hat{\Omega}^2 + \hat{\Delta}_1^2 - 2\tilde{\delta}^2}{2(\Omega^2 - \tilde{\delta}^2)} \vartheta_1 - \frac{\hat{\Omega}^2 - \hat{\Delta}_2^2}{2(\Omega^2 - \delta^2)} \vartheta_2 + O(\vartheta^2),
$$

$$
\lambda_{3,4} = \pm i \tilde{\Omega} - \frac{2 \tilde{\delta}^2 - \hat{\Delta}_1^2}{2(\Omega^2 - \delta^2)} \vartheta_1 - \frac{\hat{\Delta}_2^2 - \delta^2}{2(\Omega^2 - \delta^2)} \vartheta_2 + O(\vartheta^2),
$$

and amplitudes

$$
A_{1,2} = \frac{1}{2} \frac{\hat{\Delta}_2 + \nu^2 - \tilde{\delta}^2}{\Omega^2 - \tilde{\delta}^2} + O(\vartheta),
$$

$$
A_{3,4} = \frac{1}{2} \frac{\hat{\Omega}^2 - \nu^2 - \hat{\Delta}_2^2}{\Omega^2 - \delta^2} + O(\vartheta). \tag{58}
$$

6.1.2. $\langle \sigma_z \tau_z \rangle_z$. There is a superposition of two damped oscillations and two contributions describing incoherent relaxation toward the equilibrium value $\langle \sigma_z \tau_z \rangle_{eq}$,

$$
\lambda_{1,2} = \pm i \tilde{\Omega} - \frac{\hat{\Omega}^2 - \hat{\Delta}_1^2}{2(\Omega^2 - \delta^2)} \vartheta_1 \frac{\hat{\Omega}^2 - \hat{\Delta}_2^2}{2(\Omega^2 - \delta^2)} \vartheta_2, \quad \lambda_{3,4} = \pm i \tilde{\Omega} - \frac{\hat{\Omega}^2 - \hat{\Delta}_1^2}{2(\Omega^2 - \delta^2)} \vartheta_1 \frac{\hat{\Omega}^2 - \hat{\Delta}_2^2}{2(\Omega^2 - \delta^2)} \vartheta_2, \tag{59}
$$

where terms of order $O(\vartheta^2)$ are disregarded. In leading order, the amplitudes are

$$
B_{1,2} = \frac{1}{4} \left(1 - \frac{\nu^2}{\Omega^2} \right), \tag{60}
$$

$$
B_{3,4} = \frac{1}{4} \left(1 - \frac{\nu^2}{\Omega^2} \right),
$$

$$
B_5 = \frac{\nu^2 \tilde{\delta}^2}{(\Omega^2 - \tilde{\delta}^2)^2} + \frac{\tilde{\delta}^2 \langle \sigma_z \tau_z \rangle_{eq}}{\Omega^2 - \delta^2}, \tag{61}
$$

$$
B_6 = \frac{\nu^2 \hat{\Omega}^2}{\Omega^2 - \delta^2} - \frac{\Omega^2 \langle \sigma_z \tau_z \rangle_{eq}}{\Omega^2 - \delta^2}. \tag{62}
$$

6.2. The regimes of large coupling and/or high temperature

When the coupling $\nu$ and/or the scaled temperatures $\vartheta_{1,2}$ are large compared with the other frequencies, the amplitudes of three (five) pole contributions to $\langle \sigma_z \rangle_z$, $(\langle \sigma_z \tau_z \rangle_z)$ are negligibly small, and among the dynamical poles only the real pole with the smallest modulus is relevant. Hence the two spins essentially behave as a single spin which relaxes incoherently to the equilibrium state according to

$$
\langle \sigma_z \rangle_z = (1 - \langle \sigma_z \rangle_{eq}) e^{-\gamma_{\sigma} t} + \langle \sigma_z \rangle_{eq},
$$

$$
\langle \sigma_z \tau_z \rangle_z = (1 - \langle \sigma_z \tau_z \rangle_{eq}) e^{-\gamma_{\sigma} t} + \langle \sigma_z \tau_z \rangle_{eq}. \tag{63}
$$
6.2.1. $\langle \sigma_z \rangle_t$. The relaxation rate is found from $D_1(\lambda) = 0$ with the form (37) as

$$
\gamma_\sigma = \frac{\bar{\Delta}_1^2 \bar{\Delta}_2^2 + \vartheta_1 (\vartheta_1 + \vartheta_2)}{\vartheta_1^2 + \vartheta_2^2}.
$$

(64)

This reduces in the parameter regime $\vartheta_1 \gg v, \bar{\Delta}_2$ to

$$
\gamma_\sigma = \frac{\bar{\Delta}_2^2}{\vartheta_1} \propto T^{2K_1-1}.
$$

(65)

In this regime, $\langle \sigma_z \rangle_t$ is independent of the coupling $v$ and hence independent of the dynamics of the $\tau$-spin. The temperature dependence $\gamma_\sigma \propto T^{2K_1-1}$ distinguishes the so-called Kondo regime, in which, for $K_1 < \frac{1}{2}$, the relaxation dynamics slows down as temperature is increased.

On the other hand, when $v \gg \bar{\Delta}_{1,2}$, the two spins are locked together [4], and the effective tunneling matrix element is $\bar{\delta} = \bar{\Delta}_1 \bar{\Delta}_2 / v$, as follows from (7) with (8). We then get from equation (64)

$$
\gamma_\sigma = \frac{\bar{\delta}^2}{\vartheta_1 + \vartheta_2} \left[ 1 + \frac{\vartheta_1 (\vartheta_1 + \vartheta_2)}{\bar{\Delta}_2^2} \right].
$$

(66)

This yields the limiting expressions

$$
\gamma_\sigma = \frac{\bar{\Delta}_1^2}{v^2} \vartheta_1 \quad (\bar{\Delta}_2 \ll \vartheta_{1,2}) \quad \text{and} \quad \gamma_\sigma = \frac{\bar{\delta}^2}{\vartheta_1 + \vartheta_2} \quad (\bar{\Delta}_2 \gg \vartheta_{1,2}).
$$

(67)

The former is the relaxation rate of the biased single spin-boson system at low $\vartheta_1$. The latter describes Kondo-like joint relaxation of the locked spins.

6.2.2. $\langle \sigma_z \tau_z \rangle_t$. In the incoherent regime, the relaxation rate $\gamma_{\sigma \tau}$ of the effective single spin receives rate contributions from both the $\sigma$- and the $\tau$-spin as if these were independent biased spins in contact with their own heat reservoir.

In the large-coupling limit, $v \gg \bar{\Delta}_{1,2}$, $\vartheta_{1,2}$, the relaxation rate is found as

$$
\gamma_{\sigma \tau} = \frac{\bar{\Delta}_1^2}{v^2} \vartheta_1 + \frac{\bar{\Delta}_2^2}{v^2} \vartheta_2.
$$

(68)

The individual contributions are single-spin rates in the large-bias regime. In the high temperature limit $\vartheta_{1,2} \gg v, \bar{\Delta}_{1,2}$, on the other hand, both rate contributions are Kondo-like,

$$
\gamma_{\sigma \tau} = \frac{\bar{\Delta}_1^2}{\vartheta_1} + \frac{\bar{\Delta}_2^2}{\vartheta_2}.
$$

(69)

Consider next the regime $\vartheta_1 \gg \bar{\Delta}_1, \bar{\Delta}_2, v$, in which spin $\sigma$ behaves Kondo-like, as in equation (65). Hence the dynamics of the $\sigma$-spin is slow compared with that of the $\tau$-spin. Thus we should expect that $\langle \sigma_z \tau_z \rangle_t$ approaches the dynamics of the biased single spin-boson case as $\vartheta_1$ is increased. Taking into account terms of linear order in $\gamma_\sigma$ in the pole equation, the expression (51) with (52) and (53) assumes the form

$$
\langle \sigma_z \tau_z (\lambda) \rangle = \frac{\lambda^2 + 2 \vartheta_2 \lambda + v^2 + \vartheta_2^2 + \pi K_z v \bar{\Delta}_2^2 / \lambda}{\lambda^3 + 2 (\vartheta_2 + \gamma_\sigma) \lambda^2 + (\bar{\Delta}_2^2 + v^2 + \vartheta_2^2 + 3 \gamma_\sigma \vartheta_2) \lambda + \bar{\Delta}_2^2 \vartheta_2 + \gamma_\sigma (v^2 + \vartheta_2^2)}.
$$

Indeed, in the limit $\gamma_\sigma \to 0$, this form reduces just to the analytic expression for $\langle \tau_z (\lambda) \rangle$ of the biased single spin-boson system in the white-noise limit [2].

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Figure 9. $\langle \sigma_z \rangle$ with the spin-boson environment: real (a) and imaginary (b) part of $\lambda_j$. At low $\vartheta_2$, $\langle \sigma_z \rangle$ is a superposition of two damped oscillations. At high $\vartheta_2$, there is one damped oscillation and one relevant relaxation contribution. The parameters are $v = 0.8$, $\Delta_1 = 1$ and $\bar{\Delta}_2 = 1.5$.

7. Dynamics of a spin coupled to a spin-boson environment

Let us now view spin $\tau$ with reservoir 2 as an environment for spin $\sigma$. This complex environment is in general non-Gaussian and non-Markovian [14]. Recently, the same model has been studied numerically using a Markovian master equation approach [15]. To proceed, we first note that in the absence of bath 1, $\vartheta_1 = 0$, the pole equation $N_1(\lambda) = 0$ is still of fourth order. There is no reduction in the general case. In figure 9, we show plots of the four poles as functions of $\vartheta_2$ for a particular set of parameters.

7.1. High temperature limit

Simplification occurs, however, when $\vartheta_2$ is very large compared with the other frequencies. In this regime, the kernel (33) reduces to the form

$$K_{\text{ht}}^{(r)}(\lambda) = -\lambda \frac{\Delta_1^2}{\lambda^2 + v^2} \left(1 + \frac{v^2}{\lambda} \frac{\gamma_\tau}{\lambda + v^2 + \gamma_\tau \lambda}\right),$$

where $\gamma_\tau = \bar{\Delta}_2^2/\vartheta_2$ is the relaxation rate of spin $\tau$ in the Kondo regime. With this high-temperature expression for the kernel, the quantity $\langle \sigma_z(\lambda) \rangle$ is found to read

$$\langle \sigma_z(\lambda) \rangle = \frac{1}{\lambda + K_{\text{ht}}^{(r)}(\lambda)} = \frac{\lambda^2 + \gamma_\tau \lambda + v^2}{\lambda^3 + \gamma_\tau \lambda^2 + (\Delta_1^2 + v^2)\lambda + \bar{\Delta}_2^2 \gamma_\tau}.$$  (71)

This expression describes the dynamics of spin $\sigma$ coupled to a spin-boson environment, where the latter is in the Kondo regime. To leading order in $\gamma_\tau$, the poles of the expression (71) are (see figure 9 at large $\vartheta_2$)

$$\lambda_{1,2} = \pm i \sqrt{v^2 + \Delta_1^2} - \frac{v^2}{2(v^2 + \Delta_1^2)} \bar{\Delta}_2^2 \vartheta_2,$$

$$\lambda_3 = -\frac{\Delta_1^2}{v^2 + \Delta_1^2} \frac{\bar{\Delta}_2^2}{\vartheta_2},$$  (72)

and the amplitudes read

$$A_{1,2} = \frac{\Delta_1^2}{2(v^2 + \Delta_1^2)} + O\left(\frac{1}{\vartheta_2}\right), \quad A_3 = \frac{v^2}{v^2 + \Delta_1^2} + O\left(\frac{1}{\vartheta_2}\right).$$  (73)
The expressions (71)–(73) may now be compared with the corresponding ones of a fictive single biased spin-boson system with parameters $\Delta_1$ and $v$ in the white noise limit at scaled temperature $\tilde{\vartheta}$. The part that is symmetric in the bias reads

$$\langle \sigma_z^{(ab)}(\lambda) \rangle_s = \frac{\lambda^2 + 2\tilde{\vartheta} \lambda + v^2 + \tilde{\vartheta}^2}{\lambda^3 + 2\tilde{\vartheta} \lambda^2 + (\Delta_1^2 + v^2 + \tilde{\vartheta}^2)\lambda + \Delta_1^2 \tilde{\vartheta}}.$$  (74)

We see that with the identification $\tilde{\vartheta} \equiv \nu = \Delta_2^2/\vartheta$, the expressions (71) and (74) are quite similar. Observe, however, that the damping rate of the oscillation is somewhat different because of the term $2\tilde{\vartheta} \lambda^2$ in equation (74) instead of $\tilde{\vartheta} \lambda^2$ in equation (71) [2]. Most importantly and interestingly in the correspondence, temperature maps on the inverse of it.

7.2. Linear response limit: spin-boson environment as a structured bosonic bath

We should expect that, in the weak-coupling limit, the spin-boson environment is Gaussian and can be represented by a resonant power spectrum of a bath of bosons. The Gaussian approximation of the spin-boson environment is found by matching the power spectrum of the coupling of the $\sigma$-spin to the spin-boson environment [14] (with normalization as in equation (2))

$$S_{\tau}(\omega) = \frac{v^2}{2\pi} \text{Re} \int_{-\infty}^{\infty} dt \langle [\tau_z(t) \tau_z(0) + \tau_z(0) \tau_z(t)] e^{i\omega t} \rangle,$$  (75)

with that of a harmonic oscillator bath. In the white-noise limit of an unbiased spin, the symmetrized equilibrium correlation function $\text{Re} \langle \tau_z(t) \tau_z(0) \rangle$ coincides with the expectation $\langle \tau_z \rangle_t$. Thus we obtain

$$S_{\tau}(\omega) = (2/\pi) v^2 \text{Re} \langle \tau_z(\lambda = -i\omega) \rangle \text{ with } \langle \tau_z(\lambda) \rangle = \frac{\lambda + \tilde{\vartheta}}{\lambda(\lambda + \tilde{\vartheta}) + \Delta_2^2}. \quad (76)$$

The resulting power spectrum is that of a structured bath of bosons with a resonance of width $\tilde{\vartheta}_2$ at frequency $\omega = \Delta_2$,

$$S_{\tau}(\omega) = \frac{2\tilde{\vartheta}_2}{\omega^2 (\Delta_2^2 - \vartheta^2)^2 + \tilde{\vartheta}_2^2 \omega^2}. \quad (77)$$

Due to the coupling to the spin-boson environment, the spin $\sigma$ performs damped oscillation, $\langle \sigma_z \rangle_t = \cos(\Delta_1 t) e^{-\gamma_{\text{dec}} t}$. Upon calculating the decoherence rate in order $v^2$, i.e. the so-called one-boson-exchange contribution of the effective boson bath, we obtain

$$\gamma_{\text{dec}} = \frac{\pi}{4} S_{\tau}(\Delta_1) = \frac{v^2 \tilde{\vartheta}_2}{2[(\Delta_2^2 - \Delta_1^2)^2 + \tilde{\vartheta}_2^2 \Delta_1^2]}.$$  (78)

The analysis is completed by observing that this form emerges also upon calculating directly $\gamma_{\text{dec}}$ from the pole equation $D_1(\lambda) = 0$ with the form (37).

8. Conclusions

We have studied the dynamics of a spin or qubit $\sigma$ coupled to another spin, which could be, for instance, another qubit, or a bistable impurity, or a measuring device. We have solved the dynamics exactly for white-noise reservoir couplings, and we have studied the rich behaviors
of the dynamics in diverse limits ranging from weak coupling and/or low temperatures to strong coupling and/or high temperature. We have also analyzed the effects of a spin-boson environment on the spin dynamics in the Gaussian and non-Gaussian domains.

This paper has not attempted to perform applications to already available experiments; instead we have tried to make some general points on complementary regimes and on the crossovers in between.

One possible simple generalization beyond the white noise limit would be to replace the white-noise bath correlations in time intervals in which the Laplace variable is irrelevant by the full quantum noise correlation, for instance

$$\tilde{\Delta}_\zeta \frac{\partial \zeta}{v^2 + \partial \zeta} \rightarrow \Delta^2 \cos(\pi K_{\zeta}) \int_0^\infty ds \cos(\nu s) e^{-Q_{\zeta}(s)}.$$  \quad (79)

The advantage would be twofold: (i) the noise integral is known in analytic form for the Ohmic correlation function \cite{2}, and (ii) with this substitution, the algebraic form of the pole equation and residua would be left unchanged.

One further generalization of the Hamiltonian (1) is an applied bias acting on one or both of the spins, e.g. of the form \epsilon_1 \sigma_z and \epsilon_2 \tau_z. This important extra ingredient could be taken into account exactly in the white-noise regime, and would lead to additional shifts of the Laplace variable in all blip states of the \sigma- and \tau-spin. Then one would end up with expressions of the form (35) and (51) with polynomials ramped up by bias terms. This extension will be discussed elsewhere.

Finally, extension of the analysis of the dynamics to the colored quantum noise regime requires us to revert to the original quantum mechanical power spectrum (2) and compute its effect perturbatively in the one-boson-exchange approximation. This can be done either with the self-energy method presented in \cite{2} or with the Redfield approach. In the end, one finds that in the expressions (58) and (60), the scaled temperature \zeta (\zeta = 1, 2) is replaced by a particular linear combination of \sigma (\Omega_\tau) and \tau (\Omega_\sigma), where \sigma (\omega) is the power spectrum (2). Furthermore, one obtains the actual equilibrium state of \langle \sigma_\tau \rangle_{eq} for \zeta \ll 1 in the form

$$\langle \sigma_\tau \rangle_{eq} = \frac{v}{\Omega^2} \left[ \tilde{\Omega} \tanh(\beta \tilde{\Omega}/2) - \tilde{\delta} \tanh(\beta \tilde{\delta}/2) \right].$$  \quad (80)

These expressions reduce for \text{T} > \tilde{\Omega}_\sigma to the previous form (56) found in the white-noise regime (23). The respective analysis of the two-spin problem in the quantum-noise regime for weak coupling will be reported elsewhere.

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