An Algorithm for the Closed-Form Solution of Certain Classes of Volterra–Fredholm Integral Equations of Convolution Type

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Abstract: In this paper, a direct operator method is presented for the exact closed-form solution of certain classes of linear and nonlinear integral Volterra–Fredholm equations of the second kind. The method is based on the existence of the inverse of the relevant linear Volterra operator. In the case of convolution kernels, the inverse is constructed using the Laplace transform method. For linear integral equations, results for the existence and uniqueness are given. The solution of nonlinear integral equations depends on the existence and type of solutions of the corresponding nonlinear algebraic system. A complete algorithm for symbolic computations in a computer algebra system is also provided. The method finds many applications in science and engineering.

Keywords: integral equations; Volterra–Fredholm equations; nonlinear equations; closed-form solution; convolution kernels; Laplace transform

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1. Introduction

Mathematical modeling in physics, life sciences and engineering very often leads to integral equations. Examples of specific fields include electromagnetism, biology, population dynamics, genetics, epidemiology, heat transfer, elasticity, viscoelasticity, hydrodynamics and telecommunications [1–4]. Integral equations (IE) may have an integral term with variable limits (Volterra IE), fixed limits (Fredholm IE) or both (Volterra–Fredholm IE), with the unknown function appearing only in the integrand (IE of the first kind) or in the integrand as well as outside the integral (IE of the second kind). The Volterra–Fredholm integral equations (VFIE) appear in two forms [5], namely in a form where Volterra and Fredholm type integrals are disjoint [6–90] and in a mixed form where these two integrals are together in a multiple integral, (see [91–103] and the references therein). As an example of the two types of one-dimensional linear VFIE of the second kind, we have

\[ y(x) = f(x) + \int_a^x k_1(x,t)y(t)\,dt + \int_a^b k_2(x,t)y(t)\,dt, \quad a \leq x \leq b, \]

and

\[ y(x) = f(x) + \int_a^x \int_a^b k(s,t)y(t)\,dt\,ds, \quad a \leq x \leq b, \]

respectively, where \( f(x) \), \( k_i(x,t) \), \( i = 1, 2 \), and \( k(x,t) \) are given continuous functions and \( y(x) \) is the unknown function.

Due to their complexity, VFIE are mainly solved by numerical methods. In recent decades, many different computational methods have been developed to solve approximately both linear and nonlinear VFIE. Taylor series expansion methods have been developed in [5,10,12,15,30]. The Adomian decomposition method [5,95] and the modified decomposition method [5,17,49,96] are very powerful methods and, in some cases, can give the exact solution. Collocation methods are very popular. They are based on splines [21,29,92,93], general approximate functions [22,82], Chebyshev polynomials [16],...
We elaborate on the Volterra operators of convolution type (CVFIE) and their inversion by with the closed-form solution of the Fredholm-type integro-differential equations [105–108], will relate to specific categories of VFIE. Methods for solving Volterra IE and Fredholm IE wavelet [68], and fixed point methods in extended b-metric space [71] have been used successfully are given in [104], which describes the method for integral transforms, the method exactly are given in [104], which describes the method for integral transforms, the method to the nature of the problem, especially in the nonlinear case, and whether they exist they solution of VFIE. It is understandable that such methods are very difficult to develop due to the existence and uniqueness results are considered in [6–9,11,20,83,91].

The above analysis shows that there are no procedures for the exact closed-form solution of VFIE. It is understandable that such methods are very difficult to develop due to the nature of the problem, especially in the nonlinear case, and whether they exist they will relate to specific categories of VFIE. Methods for solving Volterra IE and Fredholm IE exactly are given in [104], which describes the method for integral transforms, the method for differentiation and the direct method for integral equations in both the Urysohn form and the Hammerstein form. Motivated by this work and based on our previous experience with the closed-form solution of the Fredholm-type integro-differential equations [105–108], we consider the VFIE of the second kind in the form

\[ y(x) = f(x) + \sum_{i=1}^{n} \int_0^x k_i(x,t)y(t)dt + \sum_{j=1}^{m} \int_0^b \tilde{k}_j(x,t,y(t))dt, \quad 0 \leq x \leq b, \]

(1)

where

\[ k_i(x,t,y(t)) = \tilde{q}_i(x,t)\bar{\varphi}_j(t,y(t)), \quad j = 1,2,\ldots,m. \]

(2)

The kernels \( k_i(x,t) \), \( i = 1,2,\ldots,n \), and \( \tilde{q}_i(x,t) \), \( j = 1,2,\ldots,m \), and the input function \( f(x) \) are given continuous functions, and \( y(x) \) is the unknown function. Equation (1) with a kernel of the form (2) is called the Hammerstein equation of the second kind. When all functions \( \bar{\varphi}_j(t,y(t)) \), \( j = 1,2,\ldots,m \), are linear in \( y \), then Equation (1) is a linear integral equation. We formulate (1) in a convenient operator form and derive its closed-form solution when the inverse of the associated integral Volterra operator of the second kind is available in explicit closed form, and the kernels \( \tilde{q}_j(x,t) \), \( j = 1,2,\ldots,m \), are degenerate. In the linear case, i.e., when all functions \( \bar{\varphi}_j(t,y(t)) \), \( j = 1,2,\ldots,m \), are linear in \( y \), we establish results of the existence and uniqueness. We elaborate on the Volterra operators of convolution type (CVFIE) and their inversion by

shifted Chebyshev polynomials [81], Bernstein polynomials [34,63], Chelyshkov Polynomials [36], Lagrange polynomials [42], Taylor polynomials [47], Fibonacci polynomials [57], Bell polynomials [65], first Boubaker polynomials [60], Müntz–Legendre polynomials [70], generalized Lucas polynomials [87], Jacobi polynomials [89], block-pulse functions [27], hybrid of block-pulse functions and Lagrange polynomials [26], hybrid block-pulse function and Taylor polynomials [41] (see also [73]), block-pulse functions and Bernoulli polynomials [45], hybrid block-pulse functions and Bernstein polynomials [62], Haar wavelets [4,66], rationalized Haar functions [19], Legendre wavelets [13,97], triangular functions [23,100], fuzzy transforms [32], Sinc function [37,40], radial basis functions [35], pseudospectral integration matrices [54] and shifted piecewise cosine basis functions [75].
the Laplace transform method. A complete algorithm for the proposed solution method is provided.

The rest of the paper is organized as follows: In Section 2, an operator formulation of the problem is given, and the solution technique is explained. In Section 3, the case of convolution-type kernels is studied, and an algorithm is presented. In Section 4, we solve various illustrative examples to show the efficiency of the method. Lastly, some conclusions are given in Section 5.

2. Direct Operator Method for Solving Linear and Nonlinear VFIE

In the space $X = C[0,b]$, $b \in \mathbb{R}^+$, let $K : X \rightarrow X$ be the linear Volterra integral operator defined by

$$K_y(x) = \sum_{i=1}^{n} \int_{0}^{x} k_i(x,t)y(t)dt,$$

where each of the kernels $k_i(x,t) \in X \times X$, $i = 1, \ldots, n$, and $y(x) \in X$.

Let the Fredholm type integrals

$$\bar{k}_j(x,t,y(t)) = \int_{0}^{b} \bar{q}_j(x,t)\bar{\varphi}_j(t,y(t))dt,$$  

where the kernels $\bar{q}_j(x,t)$ are supposed to be degenerate in the form $\bar{q}_j(x,t) = g_j(x)h_j(t)$ and $g_j(x)$, $h_j(t) \in X$. Define the vectors

$$g = (g_1, g_2, \ldots, g_m), \quad g_j = g_j(x), \quad j = 1, 2, \ldots, m,$$

and note that $g \in X^m$, and $\Phi$ is vector of $m$ nonlinear functionals $\Phi_j$ defined on $X$.

By using (3)–(6) define the Volterra–Fredholm integral operator $I : X \rightarrow X$ as

$$Iy = y - Ky - \sum_{j=1}^{m} g_j\Phi_j(y) = y - Ky - g\Phi(y), \quad y = y(x) \in X,$$

and write the VFIE in (1) in the symbolic form

$$Iy = f, \quad f = f(x) \in X.$$

Let $\Phi(g)$ be the $m \times m$ matrix

$$\Phi(g) = \begin{pmatrix} \Phi_1(g_1) & \cdots & \Phi_1(g_m) \\ \vdots & \ddots & \vdots \\ \Phi_m(g_1) & \cdots & \Phi_m(g_m) \end{pmatrix},$$

where the element $\Phi_j(g_j)$ is the value of the functional $\Phi_j$ on the element $g_j$, $I_m$ the $m \times m$ identity matrix, $c$ a $m \times 1$ column constant vector and $0$ the $m \times 1$ column zero vector.
2.1. Linear VFIE

First, we consider the case of linear VFIE where without loss of generality we assume that \( \Phi_j(t, y(t)) \equiv y(t), \ j = 1, 2, \ldots, m \). In this case, the Fredholm type functionals \( \Phi_j(y) \) in (6) become

\[
\Phi_j(y) = \int_0^b h_j(t)y(t)dt, \quad j = 1, 2, \ldots, m,
\]

which are bounded linear functionals, and the vector \( \Phi \in [X^*]^m \) where \( X^* \) is the space of all bounded linear functionals on \( X \). Moreover, for a vector of functions \( g \), such as in (5), and a constant vector \( c \), the following relation holds

\[
\Phi(gc) = \Phi(g)c. \tag{8}
\]

**Theorem 1** (Linear VFIE). Let the linear Volterra–Fredholm integral operator of the second kind \( I : X \to X \) be defined by

\[
Iy = Jy - g\Phi(y), \quad y = y(x) \in X, \tag{9}
\]

where the Volterra-type integral operator of the second kind \( J : X \to X \) is defined as

\[
Jy = y - Ky, \quad y = y(x) \in X, \tag{10}
\]

and the operator \( K \) as in (3), the vector of functions \( g \) as in (5) and \( \Phi \) is a vector of linear bounded functionals of the kind (7). If the integral operator \( J \) is bijective on \( X \), and its inverse is \( J^{-1} \), then the operator \( I \) is injective if and only if

\[
\det W = \det \left[ I_m - \Phi \left( J^{-1}g \right) \right] \neq 0. \tag{11}
\]

Furthermore, the unique solution of the linear VFIE

\[
Iy = f, \quad \text{for all } \ f = f(x) \in X, \tag{12}
\]

is given by

\[
y = I^{-1}f = J^{-1}f + J^{-1}gW^{-1}\Phi \left( J^{-1}f \right). \tag{13}
\]

**Proof.** Let us assume that \( \det W \neq 0 \) and take an element \( y \in \ker I \). Then

\[
Iy = Jy - g\Phi(y) = 0,
\]

and since the operator \( J \) is bijective, we have

\[
y = J^{-1}g\Phi(y). \tag{14}
\]

Acting by the vector \( \Phi \) on both sides of (14) and by using (8), we get

\[
\Phi(y) = \Phi \left( J^{-1}g\Phi(y) \right) = \Phi \left( J^{-1}g \right)\Phi(y),
\]

and hence

\[
\left[ I_m - \Phi \left( J^{-1}g \right) \right] \Phi(y) = W\Phi(y) = 0.
\]

Since \( \det W \neq 0 \), we have \( \Phi(y) = 0 \). Then, from (14), it follows that \( y = 0 \), i.e., \( \ker I = \{0\} \), which implies that the operator \( I \) is injective. Conversely, we assume \( I \) is injective, and we will show that \( \det W \neq 0 \), or, equivalently, we assume \( \det W = 0 \), and we
will prove that \( I \) is not injective. In this case, there exists a nonzero constant vector \( c \) such that \( Wc = 0 \). Let the element \( y_0 = \mathcal{J}^{-1}gc \in X \). Note that \( y_0 \neq 0 \) since \( y_0 = 0 \), we have

\[
Wc = \left[ I_m - \Phi \left( \mathcal{J}^{-1}g \right) \right] c = c - \Phi \left( \mathcal{J}^{-1}g \right) c = c - \Phi \left( \mathcal{J}^{-1}gc \right) = c - \Phi (y_0) = c = 0,
\]

which contradicts the assumption that \( c \) is nonzero. Then

\[
Iy_0 = \mathcal{J}y_0 - g\Phi (y_0) = gc - g\Phi (\mathcal{J}^{-1}gc) = gc - g\Phi (\mathcal{J}^{-1}g)c = g \left[ I_m - \Phi \left( \mathcal{J}^{-1}g \right) \right] c = gWc = 0.
\]

This means that \( \ker I \neq \{0\} \), and, therefore, \( I \) is not injective.

Let now the VFIE in (12), viz.

\[
Iy = \mathcal{J}y - g\Phi (y) = f, \quad f \in X.
\]  

(15)

By applying the operator \( \mathcal{J}^{-1} \) on (15), we get

\[
y = \mathcal{J}^{-1}f + \mathcal{J}^{-1}g\Phi (y).
\]  

(16)

Acting above by the vector \( \Phi \), we have

\[
\Phi (y) = \Phi \left( \mathcal{J}^{-1}f \right) + \Phi \left( \mathcal{J}^{-1}g \right) \Phi (y),
\]

and hence

\[
\left[ I_m - \Phi \left( \mathcal{J}^{-1}g \right) \right] \Phi (y) = \Phi (\mathcal{J}^{-1}f),
\]

\[
\Phi (y) = \left[ I_m - \Phi \left( \mathcal{J}^{-1}g \right) \right]^{-1} \Phi (\mathcal{J}^{-1}f) = W^{-1} \Phi \left( \mathcal{J}^{-1}f \right).
\]  

(17)

Substitution of (17) into (16) yields

\[
y = \mathcal{J}^{-1}f + \mathcal{J}^{-1}gW^{-1} \Phi \left( \mathcal{J}^{-1}f \right),
\]

which is the solution of the linear VFIE in (12) for every \( f \in X \). \( \Box \)

2.2. Nonlinear VFIE

Consider now the nonlinear VFIE in the Hammerstein form (1) where the Fredholm type functionals \( \Phi_j (y) \) are nonlinear and are given in (6).

**Theorem 2** (Nonlinear VFIE). Let the nonlinear Volterra–Fredholm integral operator of the second kind \( I : X \to X \) be defined by

\[
Iy = \mathcal{J}y - g\Phi (y), \quad y = y(x) \in X,
\]  

(18)

where the Volterra-type integral operator of the second kind \( \mathcal{J} : X \to X \) is defined as

\[
\mathcal{J}y = y - \mathcal{K}y, \quad y = y(x) \in X,
\]  

(19)

and the operator \( \mathcal{K} \) as in (3), and the vector of functions \( g \) and the vector of nonlinear functionals \( \Phi \) as in (5) and (6), respectively. If the integral operator \( \mathcal{J} \) is bijective on \( X \), and its inverse is denoted by \( \mathcal{J}^{-1} \), then the exact solution of the nonlinear VFIE
is given by
\[ y = J^{-1}f + J^{-1}g\mathbf{a}^*, \tag{21} \]
for every vector \( \mathbf{a}^* = \Phi(y) \), which is a solution of the nonlinear algebraic (transcendental) system of the \( m \) equations
\[ \mathbf{a} = \Phi \left( J^{-1}f + J^{-1}g\mathbf{a} \right). \]

Proof. Applying the inverse operator \( J^{-1} \) to the nonlinear VFIE in (20), we get
\[ y = J^{-1}f + J^{-1}g\Phi(y). \tag{22} \]

Acting then by the vector \( \Phi \) on both sides, we have
\[ \Phi(y) = \Phi \left( J^{-1}f + J^{-1}g\Phi(y) \right), \]
where it is reminded that the elements of the vector \( \Phi \) are nonlinear functionals, and no linear operations are allowed. By defining \( \mathbf{a} = \Phi(y) \), we obtain the nonlinear algebraic (transcendental) system
\[ \mathbf{a} = \Phi \left( J^{-1}f + J^{-1}g\mathbf{a} \right). \]

Let \( \mathbf{a}^* \) be a solution of this system. Substitution then of \( \mathbf{a}^* \) into (22) yields (21). The solution \( y(x) \) satisfying \( \mathbf{a}^* = \Phi(y) \) is a solution of the nonlinear VFIE (21).

3. Solving VFIE of Convolution Type

The method described in the previous section requires the existence and construction in closed form of the inverse operator \( J^{-1} \) of the linear Volterra operator of the second kind \( J \). Finding the exact inverse is not an easy task, and in many cases it is not possible. Polyanin and Manzhirov [104] provide the closed-form solution for several classes of linear Volterra integral equations (VIE) of the second kind. Here, we distinguish and deal with the special case of the VIE of convolution type that can be explicitly solved by applying the Laplace transform method.

It is well known that the Laplace transform of a function \( f(x) \) defined for all \( x \geq 0 \) is a function \( F(s) \) of the variable \( s \) defined by
\[ F(s) = \mathcal{L}(f) = \int_0^\infty e^{-sx}f(x)dx, \]
where \( \mathcal{L} \) denotes the Laplace transform operator, which is linear and injective. Suppose \( f(x) \) is continuous and of exponential order; that is, there exist real constants \( \gamma \) and \( M \) such that \( |f(x)| \leq Me^{\gamma x} \) for all \( x \geq 0 \). Then the Laplace transform of \( f \) exists for all \( s > \gamma \). The inverse of the Laplace transform is \( f(x) = \mathcal{L}^{-1}(F) \). The convolution of two functions of exponential order \( f, g \) is defined by
\[ (f * g)(x) = \int_0^x f(x-t)g(t)dt, \]
and \( \mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \) (see for example [109,110]).

Let the linear Volterra integral operator \( \mathcal{K} \) in (3) be of convolution type, i.e.,
\[ \mathcal{K}y(x) = \sum_{i=1}^n \int_0^x k_i(x-t)y(t)dt = \sum_{i=1}^n (k_i * y)(x), \tag{23} \]
and let the convolution-type Volterra integral equation (CVIE) of the second kind

$$\mathcal{J} y(x) = y(x) - K y(x) = y(x) - \sum_{i=1}^{n} (k_i \ast y)(x) = r(x), \quad r \in X. \tag{24}$$

By applying the Laplace transform operator on both sides of (24) and utilizing the convolution theorem, we get

$$\mathcal{L}\{\mathcal{J} y\} = \mathcal{L}\{y - \sum_{i=1}^{n} (k_i \ast y)\} = Y(s) - Y(s) \sum_{i=1}^{n} K_i(s) = R(s),$$

where $Y(s) = \mathcal{L}\{y\}, K_i(s) = \mathcal{L}\{k_i\}, i = 1, 2, \ldots, m$, and $R(s) = \mathcal{L}\{r\}$. Multiplication by the transfer function

$$Q(s) = \frac{1}{1 - (K_1(s) + K_2(s) + \cdots + K_n(s))},$$

yields

$$Y(s) = R(s) Q(s).$$

Taking the inverse Laplace transform, we obtain the solution of (24), namely

$$y = \mathcal{J}^{-1} r = \mathcal{L}^{-1}\{R(s) Q(s)\}. \tag{25}$$

Since the Equation (25) holds for every $r(x) \in X$, it also implies that the operator $\mathcal{J}$ is bijective.

To summarize and facilitate the implementation of the methods presented in the previous section and here, we provide the following algorithm in Algorithm 1.
Algorithm 1: Algorithm for the implementation of the Laplace transform to construct the inverse Volterra operator $J^{-1}$ and the Theorems 1 and 2 to obtain the exact solution of the linear (12) and nonlinear VFI (20), respectively.

begin
input
$k_1(x), k_2(x), \ldots, k_n(x),$
$g_1(x), g_2(x), \ldots, g_m(x), h_1(x), h_2(x), \ldots, h_m(x), b,$
$f(x)$
compute
$K_i(s) := L\{k_i\}, \quad i = 1, 2, \ldots, n$
$Q(s) := 1 - (K_1(s) + K_2(s) + \cdots + K_n(s))$
$G_j(s) := L\{g_j\}, \quad j = 1, 2, \ldots, m$
$J^{-1}g_j(x) := L^{-1}\{G_j(s)Q(s)\}, \quad j = 1, 2, \ldots, m$
$J^{-1}g := (J^{-1}g_1(x) \cdots J^{-1}g_m(x))$
$F(s) := L\{f\}$
$J^{-1}f := L^{-1}\{F(s)Q(s)\}$
linear case ($\Phi_i(t, y(t)) = y(t)$):
$\Phi_1(J^{-1}g_1) := \int_a^b h_1(t)J^{-1}g_1(t)dt, \quad i, j = 1, 2, \ldots, m$
$\Phi(J^{-1}g) := \begin{pmatrix}
\Phi_1(J^{-1}g_1) & \cdots & \Phi_1(J^{-1}g_m)
\vdots & \ddots & \vdots \\
\Phi_m(J^{-1}g_1) & \cdots & \Phi_m(J^{-1}g_m)
\end{pmatrix}$
$W := I_m - \Phi(J^{-1}g)$
if $\det W \neq 0$ compute
$\Phi_i(J^{-1}f) := \int_a^b h_i(t)J^{-1}f(t)dt, \quad i = 1, 2, \ldots, m$
$\Phi(J^{-1}f) := \begin{pmatrix}
\Phi_1(J^{-1}f) \\
\vdots \\
\Phi_m(J^{-1}f)
\end{pmatrix}$
y := $J^{-1}f + J^{-1}gW^{-1}\Phi(J^{-1}f)$
print $y(x)$
else
print 'There is no unique solution'
end
nonlinear case (define $\Phi_i(t, y(t))$):
a := col($a_1 \cdots a_m$)
v(t) := $J^{-1}f + J^{-1}ga$
$\Phi_1(v) := \int_a^b h_1(t)\Phi_1(t, y(t))dt, \quad i = 1, 2, \ldots, m$
$\Phi(v) := \begin{pmatrix}
\Phi_1(v) \\
\vdots \\
\Phi_m(v)
\end{pmatrix}$
solve the system
$a = \Phi(v)$
for every solution $a^*$ compute
$y := J^{-1}f + J^{-1}ga^*$
if $y$ satisfies $a^* - \Phi(y) = 0$ print the solution $y(x)$
end
end
4. Examples

In this section, we solve three linear and three nonlinear problems with VFIE of the convolution type to show the capabilities, ease of use and cost-effectiveness of the proposed method.

4.1. Example 1

The following illustrative example of linear CVFIE is solved in [5] by means of the Taylor series solution method and in [65] numerically by means of a collocation method based on Bell polynomials,

\[ u(x) = -x^4 - x^3 + 12x^2 - x - 5 + \int_0^x (x - t)u(t)dt + \int_0^1 (x + t)u(t)dt, \quad 0 \leq x \leq 1. \]  

(26)

Here, we solve it using the proposed direct operator method. Since the kernel of the Fredholm integral operator is degenerate, we write (26) in the form

\[ u(x) - \int_0^x (x - t)u(t)dt - x \int_0^1 u(t)dt - \int_0^1 tu(t)dt = -x^4 - x^3 + 12x^2 - x - 5. \]  

(27)

Juxtaposing expression (27) with (12) in Theorem 1, we take

- \( k_1(x) = x, \)
- \( g_1(x) = x, \quad g_2(x) = 1, \quad h_1(t) = 1, \quad h_2(t) = t, \quad \phi_1(t, u(t)) = \phi_2(t, u(t)) = u(t), \)  

(28)

\[ f(x) = -x^4 - x^3 + 12x^2 - x - 5. \]

The convolution kernel \( k_1(x) \) and the functions \( g_1(x), g_2(x) \) and \( f(x) \) are continuous and of exponential order, and so, their Laplace transforms \( \mathcal{L}\{k_1(x)\}, \mathcal{L}\{g_1(x)\}, \mathcal{L}\{g_2(x)\} \) and \( \mathcal{L}\{f(x)\} \), respectively, exist. As a result, the Volterra integral operator of the second kind \( J \) is invertible.

When entering the functions (28) in the Algorithm 1 and executing, we get the exact solution

\[ u(x) = 12x^2 + 6x. \]

4.2. Example 2

Let the linear CVFIE of the second kind

\[ u(x) = e^{x+1} - 4e^x + x^2 + \int_0^x e^{x-t}u(t)dt - 3 \int_0^1 e^{x-t}u(t)dt, \quad 0 \leq x \leq 1. \]  

(29)

Note that the kernel of the Fredholm integral operator in (29) is degenerate, and therefore Equation (29) can be written as follows

\[ u(x) - \int_0^x e^{x-t}u(t)dt + 3e^x \int_0^1 e^{t}u(t)dt = e^{x+1} - 4e^x + x^2. \]  

(30)

Comparing (30) with (12) in Theorem 1, we set \( n = m = 1, \quad b = 1 \) and

- \( k_1(x) = e^x, \)
- \( g_1(x) = -3e^x, \quad h_1(t) = e^t, \quad \phi_1(t, u(t)) = u(t), \)
- \( f(x) = e^{x+1} - 4e^x + x^2, \)  

(31)

The convolution kernel \( k_1(x) \) and the functions \( g_1(x) \) and \( f(x) \) are continuous and of exponential order, and therefore their corresponding Laplace transforms exist, which in turn implies that the Volterra integral operator of the second kind \( J \) is invertible. Then, by giving the functions (31) as input in Algorithm 1, and after performing the calculations, we get the exact solution

\[ u(x) = \frac{1}{4}\left( e^{2x-2} + 2x^2 - 2x - 1 \right). \]
4.3. Example 3

The Fredholm integral equation of the second kind

\[ u(x) = f(x) + \frac{1}{2} \int_0^1 e^{-|x-t|} u(t) dt, \quad 0 \leq x \leq 1, \]  

(32)

appears in the one-dimensional transport process [111] and the bending analysis of nonlocal integral models of Euler–Bernoulli nanobeams [112–114].

This equation, by removing the modulus in the integrand, can be converted to a Volterra–Fredholm integral equation of convolution type, namely

\[ u(x) = f(x) + \frac{1}{2} \left[ \int_0^x e^{-(x-t)} u(t) dt + \int_x^1 e^{x-t} u(t) dt \right] \]

(33)

This linear CVFIE can now be solved by the direct operator method presented in previous sections. In particular, comparing (33) with (12) in Theorem 1 it is natural to take \( n = 2, m = 1, b = 1 \) and

\[ K u(x) = \frac{1}{2} \left[ \int_0^x e^{-(x-t)} u(t) dt - \int_0^x e^{x-t} u(t) dt \right], \]

which is a convolution Volterra integral operator of the form (23) with

\[ k_1(x) = \frac{1}{2} e^{-x}, \quad k_2(x) = -\frac{1}{2} e^x. \]  

(34)

Additionally, we take

\[ g_1(x) = \frac{1}{2} e^x, \quad h_1(t) = e^{-t}, \quad \phi_1(t, u(t)) = u(t), \]  

(35)

and let

\[ f(x) = \frac{1}{2} \left( e^{-x} + e^{-(1-x)} \right), \]  

(36)

as in [111].

The functions \( k_1(x), k_2(x), g_1(x) \) and \( f(x) \) are continuous and of exponential order, and therefore the Volterra integral operator of the second kind \( \mathcal{J} \) is invertible. By entering the functions in (34)–(36) in the Algorithm 1 and executing, we get the exact solution of the CVFIE (33), viz.

\[ u(x) = 1, \]

which is the solution of the given Fredholm integral Equation (32) with \( f(x) \) as in (36).

In Table 1, we give the exact solution corresponding to three other types of the input function \( f(x) \).
where the Fredholm integral operator is nonlinear and degenerate. We write Equation (37)
which has the two real solutions

\[
\begin{align*}
\alpha x^2 + \beta x + \gamma, & \quad \alpha, \beta, \gamma \in \mathbb{R} \\
\sin(\pi kx), & \quad k \in \mathbb{Z}^+ \\
x^3 \cosh(x) &
\end{align*}
\]

in the form

\[
\frac{3\alpha x^4 + 6\beta x^3 + (18\gamma - 36\alpha)x^2 - (18\gamma + 44\beta + 5\alpha)x - 54\gamma - 8\beta - 5\alpha}{16}
\]

\[
\frac{3\pi^2 k^2 + 3}{3\pi^2 k^2} \sin(\pi k) - [\sin(\pi k) + \pi k \cos(\pi k) + \pi k^2] x - \pi k \cos(\pi k) + 2\pi k
\]

Comparing (38) with (20) in Theorem 2, we take

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

Comparing (38) with (20) in Theorem 2, we take \( n = m = 1, b = 2 \) and

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

Comparing (38) with (20) in Theorem 2, we take \( n = m = 1, b = 2 \) and

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

Comparing (38) with (20) in Theorem 2, we take \( n = m = 1, b = 2 \) and

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

Comparing (38) with (20) in Theorem 2, we take \( n = m = 1, b = 2 \) and

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

According to Theorem 2, we have \( n = m = 1, b = 1 \) and

\[
4.5. \text{Example 5}
\]

4.4. Example 4

Consider the nonlinear CVFIE of the second kind

\[
u(x) = -\frac{x}{4} + \int_0^x \sin(x - t) u(t) dt - \frac{1}{10} \int_0^2 x tu^2(t) dt, \quad 0 \leq x \leq 2,
\]

(37)

where the Fredholm integral operator is nonlinear and degenerate. We write Equation (37)
in the form

\[
u(x) - \int_0^x \sin(x - t) u(t) dt + \frac{x}{10} \int_0^2 tu^2(t) dt = -\frac{x}{4}.
\]

(38)

Comparing (38) with (20) in Theorem 2, we take \( n = m = 1, b = 2 \) and

\[
\begin{align*}
k_1(x) & = \sin x, \\
g_1(t) & = -\frac{x}{10}, \\
h(t) & = t, \\
\phi_1(t, u(t)) & = u^2(t),
\end{align*}
\]

Since the convolution kernel \( k_1(x) \) and the functions \( g_1(x) \) and \( f(x) \) are continuous and
of exponential order, the corresponding Laplace transforms exist, and the Volterra integral
operator of the second kind \( \mathcal{J} \) is invertible. Entering the functions (39) in Algorithm 1, we
get the second order algebraic equation

\[
76a^2 - 520a + 475 = 0,
\]

which has the two real solutions

\[
a_1^* = \frac{130 - 15\sqrt{35}}{38}, \quad a_2^* = \frac{130 + 15\sqrt{35}}{38}.
\]

For each of them the Algorithm 1 delivers the following corresponding solution of the
given nonlinear CVFIE

\[
u_1(x) = \frac{\sqrt{35} - 15}{152} \left( x^3 + 6x \right) \quad \text{and} \quad \nu_2(x) = -\frac{\sqrt{35} + 15}{152} \left( x^3 + 6x \right).
\]

4.5. Example 5

Solve the nonlinear CVFIE of the second kind

\[
u(x) - \int_0^x e^{x-t} u(t) dt - \int_0^1 \frac{1}{u(t)} dt = 1, \quad 0 \leq x \leq 1.
\]

(40)

According to Theorem 2, we have \( n = m = 1, b = 1 \) and

Table 1. Exact solution of the linear CVFIE (33) for three different cases of the input function \( f(x) \).

| \( f(x) \)                           | \( u(x) \)                                      |
|-------------------------------------|------------------------------------------------|
| \( ax^2 + \beta x + \gamma \) \quad \alpha, \beta, \gamma \in \mathbb{R} | \( -3ax^4 + 6\beta x^3 + (18\gamma - 36\alpha)x^2 - (18\gamma + 44\beta + 5\alpha)x - 54\gamma - 8\beta - 5\alpha \) |
| \( \sin(\pi kx) \), \quad k \in \mathbb{Z}^+ | \( \frac{3\pi^2 k^2 + 3}{3\pi^2 k^2} \sin(\pi k) - [\sin(\pi k) + \pi k \cos(\pi k) + \pi k^2] x - \pi k \cos(\pi k) + 2\pi k \) |
| \( x^3 \cosh(x) \)                    | \( \frac{18x^7 - 54x + 72}{x^6} - (18x^2 + 54x + 72) e^{-x} (13e - 12 - 33e^{-1}) x - 13e + 24 + 33e^{-1} \) |
We set \( n \) which has the two real solutions

\[
\begin{align*}
    k_1(x) &= e^x, \\
    g_1(x) &= 1, \\
    h_1(t) &= 1, \\
    \phi_1(t, u(t)) &= \frac{1}{u(t)}, \\
    f(x) &= 1.
\end{align*}
\] (41)

The functions \( k_1(x), g_1(x) \) and \( f(x) \) are continuous and of exponential order, and therefore the Volterra integral operator \( J \) can be inverted. Entering the functions (41) in Algorithm 1, we get the algebraic equation

\[
a^2 + a + \ln(e^2 + 1) - \ln 2 - 2 = 0,
\]

which admits the two real solutions

\[
a_1^a = -\frac{-\sqrt{4\ln(e^2 + 1) + 4\ln 2 + 9} + 1}{2}, \quad a_2^a = \frac{-\sqrt{4\ln(e^2 + 1) + 4\ln 2 + 9} - 1}{2}.
\]

The corresponding solutions of the given nonlinear CVFIE (40) are

\[
\begin{align*}
    u_1(x) &= -\frac{-\sqrt{4\ln(e^2 + 1) + 4\ln 2 + 9} + 1}{4}(e^x + 1), \\
    u_2(x) &= \frac{-\sqrt{4\ln(e^2 + 1) + 4\ln 2 + 9} - 1}{4}(e^x + 1).
\end{align*}
\)

4.6. Example 6

As a last example, we consider the nonlinear CVFIE of the second kind

\[
u(x) - \int_0^x \sin(x-t)u(t)dt - \int_0^1 te^{x-t}(t-u(t))^2dt = \sin x - e^x, \quad 0 \leq x \leq 1,
\] (42)

The kernel \( \bar{q}(x,t) = e^{x-t} = e^x e^{-t} \) is degenerate, and therefore Theorem 2 is applicable. We set \( n = m = 1, b = 1 \) and

\[
\begin{align*}
    k_1(x) &= \sin x, \\
    g_1(x) &= e^x, \\
    h_1(t) &= e^{-t}, \\
    \phi_1(t, u(t)) &= t(t-u(t))^2, \\
    f(x) &= \sin x - e^x.
\end{align*}
\] (43)

The functions \( k_1(x), g_1(x) \) and \( f(x) \) are continuous and of exponential order, and hence the inverse Volterra operator \( J^{-1} \) exists. Entering (43) in Algorithm 1, we get the algebraic equation

\[
(35e - 84)a^2 + (168 - 73e)a + 35e - 84 = 0,
\]

which has the two real solutions

\[
a_1^a = -\frac{35e\sqrt{143e - 336} - 73e + 168}{70e - 168}, \quad a_2^a = \frac{35e\sqrt{143e - 336} + 73e - 168}{70e - 168}.
\]

Therefore, the given nonlinear CVFIE (42) has two real solutions that are

\[
\begin{align*}
    u_1(x) &= -\frac{-6e^{x+1} + \sqrt{35e\sqrt{143e - 336}(2e^x - x - 1) + (168 - 73e)x + 3e}}{70e - 168}, \\
    u_2(x) &= \frac{6e^{x+1} + \sqrt{35e\sqrt{143e - 336}(2e^x - x - 1) + (76e - 168)x - 3e}}{70e - 168}.
\end{align*}
\]
5. Conclusions

Many different approximate and numerical methods are available in the literature for solving linear and nonlinear VFIE. However, there is no effective direct procedure for solving them in closed form.

In this paper, a direct operator method for the closed-form solution of some classes of VFIE was presented. An algorithm was developed for the case of VFIE with a linear Volterra convolution-type operator. The algorithm was implemented into the Maxima Computer Algebra System, and several problems were solved from the literature.

The main advantages of the technique are that it calculates the exact solution of VFIE and avoids problems related to approximate and numerical methods, such as computer accuracy errors and method convergence and stability. In the case of nonlinear VFIE, it finds all the solutions, rather than only one, as is the case with most numerical methods.

The only drawback of the method is that it requires the inverse of the Volterra operator and that all integrations must be performed analytically in general. In the special case of Volterra convolution-type operators, it depends upon the existence of the inverse Laplace transform of the involved functions. These limit the scope of the method to certain categories of functions, which makes perfect sense given the complexity of VFIE.

Solving some problems has proven that the method will be useful in several fields of science and engineering. The method can be extended to other categories of Volterra operators, the mixed type VFIE and the VFIE in two or three dimensions.

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Abbreviations

The following abbreviations are used in this manuscript:

IE Integral Equation(s)
VIE Volterra Integral Equation(s)
VFIE Volterra–Fredholm Integral Equation(s)
CVIE Convolution-type Volterra Integral Equation(s)
CVFIE Convolution-type Volterra-Fredholm Integral Equation(s)

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