Regular matchstick graphs

Sascha Kurz∗
Fakultät für Mathematik, Physik und Informatik, Universität Bayreuth, Germany

Rom Pinchasi†
Mathematics Dept., Technion—Israel Institute of Technology, Haifa 32000, Israel

November 9, 2021

Abstract. A graph $G = (V,E)$ is called a unit-distance graph in the plane if there is an injective embedding of $V$ in the plane such that every pair of adjacent vertices are at unit distance apart. If additionally the corresponding edges are non-crossing and all vertices have the same degree $r$ we talk of a regular matchstick graph. Due to Euler’s polyhedron formula we have $r \leq 5$. The smallest known 4-regular matchstick graph is the so called Harborth graph consisting of 52 vertices. In this article we prove that no finite 5-regular matchstick graph exists and provide a lower bound for the number of vertices of 4-regular matchstick graphs.

Keywords: unit-distance graphs
MSC: 52C99, 05C62

1 Introduction

One of the possibly best known problems in combinatorial geometry asks how often the same distance can occur among $n$ points in the plane. Via scaling we can assume that the most frequent distance has length 1. Given any set $P$ of points in the plane, we can define the so called unit-distance graph in the plane, connecting two elements of $P$ by an edge if their distance is one. The known bounds for the maximum number $u(n)$ of edges of a unit-distance graph in the plane, see e. g. [3], are given by

$$\Omega\left(ne^{\frac{\log n}{\log \log n}}\right) \leq u(n) \leq O\left(n^3\right).$$

For $n \leq 14$ the exact numbers of $u(n)$ were determined in [8], see also [3].

If we additionally require that the edges are non-crossing, then we obtain another class of geometrical and combinatorial objects: A matchstick graph is graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect, see Figure 2 for an example.

For matchstick graphs the known bounds for the maximum number $\bar{u}(n)$ of edges, see e. g. [3], are given by

$$\left|3n - \sqrt{12n - 3}\right| \leq \bar{u}(n) \leq 3n - O\left(\sqrt{n}\right),$$

where the lower bound is conjectured to be exact.

We call a matchstick graph $r$-regular if every vertex has degree $r$. In [5] the authors consider $r$-regular matchstick graphs with the minimum number $m(r)$ of vertices. Obviously we have $m(0) = 1,$
$m(1) = 2$, and $m(2) = 3$, corresponding to a single vertex, a single edge, and a triangle, respectively. The determination of $m(3) = 8$ is an amusement left to the reader. For degree $r = 4$ the exact determination of $m(4)$ is unsettled so far. The smallest known example is the so called Harborth graph, see e. g. [4] for a drawing, yielding $m(4) \leq 52$. Here we prove $m(4) \geq 20$.

Due to the Euler polyhedron formula every finite planar graph contains a vertex of degree at most five so that we have $m(r) = \infty$ for $r \geq 6$. Examples are given by the regular triangular lattice or an infinite 6-regular tree.

For degree 5 it is announced at several places that no finite 5-regular matchstick graph does exist. In [1] the authors list five publications where they have mentioned an unpublished proof for the non-existence of a 5-regular matchstick graph and state that up to their knowledge the problem is open. Currently there is no published proof available.\(^2\)

The aim of this article is indeed to bridge this gap.

## 2 5-regular matchstick graphs

**Theorem 2.1** No finite 5-regular matchstick graph does exist.

**Proof.** Suppose to the contrary that there is such a graph and let $M$ be the planar map which is drawing of this graph in the plane such that every edge is a unit length straight line segment and no two edges cross.

Without loss of generality we assume that this graph is connected and denote by $V$ the number of its vertices, by $E$ the number of its edges, and by $F$ the number of faces in the planar map $M$. By Euler’s formula we have $V - E + F = 2$. For every $k \geq 3$ we denote by $f_k$ the number of faces in $M$ with precisely $k$ edges.

We observe that $2E = \sum k f_k = 5V$ and $F = \sum f_k$. Therefore,

$$-6 = -3V + E + 2E - 3F = -3V + \frac{5}{2}V + \sum k f_k - 3 \sum f_k = -\frac{1}{2}V + \sum(k - 3)f_k. \quad (1)$$

We begin by giving a charge of $-\frac{1}{2}$ to each vertex and by giving a charge of $k - 3$ to each face in $M$ with precisely $k$ edges. By (1) the total charge of all the vertices and faces is negative. We will reach a contradiction by redistributing the charge in such a way that eventually each vertex and each face will have a non-negative charge.

We redistribute the charge in the following very simple way. Consider a face $T$ of $M$ and a vertex $x$ of $T$. Let $\alpha$ denote measure of the internal angle of $T$ at $x$. Only if $\alpha > \frac{\pi}{3}$ we take a charge of $\min\left(\frac{1}{2}, \frac{2}{3}(\alpha - \frac{\pi}{2})\right)$ from $T$ and move it to $x$.

We now show that after the redistribution of charges every vertex and every face have a non-negative charge. Consider a vertex $x$. Let $\ell$ denote the number of internal angles at $x$ that are greater than $\frac{\pi}{3}$. As the degree of $x$ equals to 5 we must have $\ell > 0$. If due to one of these $\ell$ angles we transferred a charge of $\frac{1}{2}$ to $x$, then the charge at $x$ is non-negative. Otherwise, note that the sum of these $\ell$ angles is at least $2\pi - (5 - \ell)\frac{\pi}{3} = \frac{\pi}{3}(\ell + 1)$. Hence the total charge transferred to $x$ due to these angles is at least $\frac{1}{2} \cdot \frac{\pi}{3}(\ell + 1) - \frac{\ell}{2} = \frac{\pi}{6}$. Here again we conclude that the charge at $x$ is non-negative.

Consider now a face $T$ in $M$ with $k \geq 3$ edges. Assume first that $T$ is a bounded face. The initial charge of $T$ is $k - 3 \geq 0$. Therefore, if the charge at $T$ becomes negative this implies that one of the internal angles of $T$ is greater than $\frac{\pi}{3}$. In particular $T$ cannot be a triangle and thus $k \geq 4$. If $k = 4$, then $T$ is a rhombus. If only two internal angles of $T$ are greater than $\frac{\pi}{3}$, then at most a total charge of 1

---

1In the meantime the temporarily lost manuscript was recovered and after some minor corrections the arguments turned out to be valid. A retyped and slightly edited electronic version can be found at http://www.wm.uni-bayreuth.de/fileadmin/Sascha/aartb.pdf.

2There is a further unpublished manuscript containing a rather technical and long proof [6].
was deduced from the initial charge of $T$, leaving its charge non-negative. If all internal angles of $T$ are greater than $\frac{\pi}{3}$, then the total charge deduced from $T$ is at most $\frac{3}{2\pi} \cdot 2\pi - \frac{2}{9} = 1$, leaving the charge at $T$ non-negative.

If $k = 5$ and the charge of $T$ is negative after we redistribute the charges, then each of the internal angles of $T$ must be greater than \( \frac{\pi}{3} \) and as the sum of the internal angles of $T$ is equal to $3\pi$, the charge deduced from $T$ amounts to at most $3\pi - \frac{2}{3} = 2$, leaving the charge at $T$ non-negative.

Finally if $k \geq 6$, then the charge deduced from $T$ is at most $k \cdot 2\pi$, leaving a charge of at least $k - 3 - \frac{k}{2} \geq 0$.

It is left to consider the unbounded face $S$ of $M$. If the number of edges of $S$ is at least 6, we are done as in the case of a bounded face. The cases where the unbounded face consist of at most 5 edges can be easily excluded. Another way to settle this issue is to observe that if $S$ consists of at most 5 edges, then the total charge deduced from $S$ is at most $\frac{5}{2}$ leaving the charge of $S$ at least $-\frac{5}{2}$ (and in fact at least $-\frac{3}{2}$).

We still obtain a contradiction as the sum of all charges should be equal to $-6$ while only the unbounded face may remain with a negative charge that is not smaller than $-\frac{5}{2}$.

\[\text{Figure 1: Infinite 5-regular match stick graph.}\]

In Figure 1 we have drawn a fraction of an infinite 5-regular matchstick graph. Since we can shift single rows of such a construction there exists an uncountable number of these graphs (even in the combinatorial sense). As mentioned by Bojan Mohar there are several further examples.

Starting with the infinite 6-regular triangulation we can delete several vertices at different rows and obtain an example by suitably removing horizontal edges to enforce degree 5 for all vertices. Taking an arbitrary planar unit-distance graph with maximum degree 5, whose vertices not being part of the unbounded face have degree exactly 5, and continuing the boundary vertices with trees gives another set of examples.

\section{4-regular matchstick graphs}

In this section we show that every 4-regular matchstick graph must consist of at least 20 vertices. Although this is a little far from the best known construction with 52 vertices (\cite{1,4,5}), we bring some arguments that may lead the way to obtain improved bounds and remark, that the whole proof is free from computer calculations and massive case differentiations. For a determination of the lower bound $m(4) \geq 34$, based on massive computer calculations, we refer the interested reader to \cite{7}.

\textbf{Lemma 3.1} For every connected 4-regular matchstick graph in the plane we have $n \geq \frac{3k-1+12}{5}$, where $n$ denotes the number of vertices, $k$ the number of edges of the unbounded face, and $l \leq k - 3$ the number of internal angles of the unbounded face that are equal to $\pi$.

\textbf{Proof.} Assume $G$ is a 4-regular matchstick graph. Let us denote by $V$ the number of its vertices, by $E$ the number of its edges, and by $F$ the number of its faces in the planar map $M$ realizing $G$ as a matchstick graph.\[\]
We begin by giving a charge of $-1$ to each vertex and by giving a charge of $r - 3$ to each face in $M$ with precisely $r$ edges. By (2) the total charge of all the vertices and faces is $-6$.

We now redistribute the charge in the following simple way. Consider a face $T$ of $M$ and a vertex $x$ of $T$. Let $\alpha$ denote the measure of the internal angle of $T$ at $x$. If $T$ is an equilateral triangle, we do nothing. If $\alpha = \pi$ we move a charge of $\frac{\pi}{3}$ from $T$ to $x$. Otherwise, we take a charge of $\frac{2}{\pi}\alpha - \frac{1}{15}$ from $T$ and move it to $x$.

We now show that after the redistribution of charges every vertex is at least $-\frac{1}{5}$ and every bounded face has a nonnegative charge.

For the first part consider a vertex $x$. Let $p$ denote the number of internal angles at $x$ which are equal to $\frac{\pi}{3}$. Clearly, $0 \leq p \leq 3$. If $l = 3$, then one of the internal angles at $x$ is equal to $\pi$. It follows that the charge at $x$ is at least $-1 + \frac{2}{5}$. This is because $x$ never gets a negative contribution from internal angles that are equal to $\frac{\pi}{3}$.

If $l \leq 2$, then the charge at $x$ is at least

$$-1 + \frac{9}{8\pi} \left(2\pi - \frac{l\pi}{3}\right) - \frac{5(4 - l)}{16} = -\frac{l}{16} \geq -\frac{1}{8}.$$

Consider now a bounded face $T$ in $M$ with $r \geq 3$ edges. We will show that the charge of $T$ after we have redistributed the charges is nonnegative. The initial charge of $T$ is $r - 3 \geq 0$. If $T$ is a (necessarily equilateral) triangle, then the charge of $T$ does not change and it remains $0$. If $T$ is a quadrilateral and hence a rhombus, then clearly no internal angle of $T$ can be equal to $\pi$ and the total charge removed from $T$ is $\frac{2}{\pi} \cdot 2\pi - \frac{2}{15} \cdot 4 = 1$. This leaves $T$ with a total charge of $0$.

Suppose now that $T$ has $r \geq 5$ edges. The sum of all internal edges of $T$ is equal to $\pi(r - 2)$. Let $p$ denote the number of internal angles of $T$ that are equal to $\pi$. Clearly, $p \leq r - 3$. The total charge removed from $T$ is equal to

$$\frac{7}{8}l + \frac{9}{8\pi} \cdot \pi(r - 2 - p) - \frac{5(r - p)}{16} = \frac{p}{16} + \frac{13}{16}r - \frac{9}{4}.$$

Because $p \leq r - 3$ this charge is at most

$$\frac{r - 3}{16} + \frac{13}{16}r - \frac{9}{4} = \frac{7}{8}r - \frac{39}{16}.$$

This is smaller than the initial charge of $T$ which is $r - 3$ for all $r \geq 5$.

We have thus shown that every bounded face of $M$ remains with a nonnegative charge. But what about the unbounded face? Suppose $T$ is the unbounded face and it has $k$ edges. The sum of all internal edges of $T$ is equal to $\pi(k + 2)$. Let $l$ denote the number of internal angles of $T$ that are equal to $\pi$. Here too we have $l \leq k - 3$.

The total charge removed from $T$ is equal to

$$\frac{7}{8}l + \frac{9}{8\pi} \cdot \pi(k + 2 - l) - \frac{5(k - l)}{16} = \frac{l}{16} + \frac{13}{16}k + \frac{9}{4}.$$

This leaves $T$ with a total charge of $\frac{3}{16}k - \frac{l}{16} - \frac{21}{4}$. 
The total charge of all vertices and faces should be equal to $-6$. The total charge of all vertices is at least $-\frac{n}{8}$ and every bounded face has a nonnegative charge. Therefore, we have

$$\frac{3}{16} k - \frac{l}{16} - \frac{21}{4} - \frac{n}{8} \leq -6,$$

which is equivalent to

$$n \geq \frac{3k - l + 12}{2}.$$  

From $2E = \sum rf_r = 4V$, $F = \sum f_r$, and Euler’s formula $V - E + F = 2$ we conclude

$$\sum_{r=3}^{\infty} (4 - r)f_r = f_3 - f_5 - 2f_6 - 3f_7 - 4f_8 - \cdots = 8, \quad (3)$$

using the notation from the proof of Lemma 3.1. Thus we have $f_3 \geq k + 4$, where $k$ denote the number of edges of the unbounded face. The maximum area of an equilateral $k$-gon, whose edges have length 1, is given by

$$A_{\text{max}}(k) = \frac{k}{4} \cdot \cot \left( \frac{\pi}{k} \right). \quad (4)$$

Thus $k$ must be big enough in order for the complement of the unbounded face to contain at least $k + 4$ triangles. In Table 1 we have listed the maximum area of an equilateral $k$-gon measured in units of equilateral triangles.

| $k$  | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $A_{\text{max}}(k)$ | 1.000 | 2.310 | 3.974 | 6.000 | 8.393 | 11.151 | 14.277 | 17.770 | 21.630 | 25.857 |
| $A_{\text{max}}(3)$ | 1.000 | 2.310 | 3.974 | 6.000 | 8.393 | 11.151 | 14.277 | 17.770 | 21.630 | 25.857 |

Table 1: Maximum number of equilateral triangles inside an equilateral $k$-gon.

Solving $\frac{k}{4} \cdot \cot \left( \frac{\pi}{k} \right) \geq (k + 4) \cdot \frac{\sqrt{3}}{2}$ or using Table 1 yields $k \geq 9$ which can be improved by some further arguments.

**Lemma 3.2** If a 4-regular matchstick graph contains an inner vertex but no bounded faces with $s \geq 5$ edges, then it contains one of the configurations of Figure 2.

**PROOF.** Due to the premises all inner faces are triangles and quadrangles. An inner vertex is part of four inner faces. Due to angle sums at most two of them can be triangles.

Figure 2: Configurations of triangles and quadrangles.

**Lemma 3.3** The total area of the quadrangles in each of the three configurations of Figure 2, where the sides have length 1, is greater than $\frac{\sqrt{3}}{2}$.
PROOF. If we denote the angles of the quadrangles at the central vertex by $\alpha_i$, the area of the quadrangles is given by $\sum_i |\sin \alpha_i|$. A little calculus using $0 \leq \alpha_i \leq \pi$ yields the stated result.

**Lemma 3.4** For integers $s \geq 1$ the minimum area of an equilateral $(2s + 1)$-gon with side lengths 1 is at least $\frac{\sqrt{3}}{4}$.

**PROOF.** See [2].

Thus we can conclude $\frac{k}{4} \cdot \cot \left( \frac{\pi}{k} \right) \geq (k + 6) \cdot \frac{\sqrt{3}}{4}$ yielding $k \geq 10$.

**Lemma 3.5** For the number of edges $k$ of a connected 4-regular matchstick graph in the plane we have $k \geq 11$.

**PROOF.** Let us assume $k = 10$. Due to Equation (3), Lemma 3.3, Lemma 3.4, and Table 1 we have the following possible nonzero values for the $f_i$:

(a) $f_3 = 14$, $f_4 = t \in \mathbb{N}$, $f_{10} = 1$, or

(b) $f_3 = 15$, $f_4 = t \in \mathbb{N}$, $f_5 = 1$, $f_{10} = 1$, or

(c) $f_3 = 16$, $f_4 = t \in \mathbb{N}$, $f_6 = 1$, $f_{10} = 1$.

Due to an angle sum of $(10 - 2)\pi$ at most $3 \cdot 10 - 7 = 23$ of the inner angles of the outer face can be part of triangles. In a similar manner we conclude that at most $3s - 3$ outer angles of an inner $s$-gon can be part of triangles. Since $15 \cdot 3 - 23 - (3 \cdot 5 - 3) = 10 > 0$ and $16 \cdot 3 - 23 - (3 \cdot 6 - 3) = 10 > 0$ there must exist one of the configurations of Figure 2 in the cases (b) and (c), which is a contradiction to the area argument.

The number of outer and inner angles of one of the configurations in Figure 2, which can be part of triangles, is at most 15. Thus we can conclude from $14 \cdot 3 - 23 - 1 \cdot 15 = 4 > 0$ that in case (a) there must exist at least two subgraphs as in Figure 2, which contradicts the area argument.

**Theorem 3.6** Every 4-regular matchstick graph in the plane contains at least 20 vertices.

**PROOF.** Without loss of generality we can assume that the graph is connected. If the number of edges $k$ of the unbounded face is at least 12, we can use Lemma 3.1 and $l \leq k - 3$ to conclude $n \geq 20$.

If $k = 11$ and $l \leq k - 5$ then Lemma 3.1 gives $n \geq 20$. In the remaining cases we have $k = 11$ and $l \in \{k - 4, k - 3\}$ so that the area of the unbounded face is at most $\max \left( \left( \frac{11}{4} \right)^2, \left( \frac{11}{3} \right)^2, \frac{\sqrt{3}}{4} \right) = \frac{121}{16} < 18 \cdot \frac{\sqrt{3}}{4}$. Due to Equation (3), Lemma 3.3, Lemma 3.4, and Table 1 we have the following possible nonzero values for the $f_i$ in this case:

(a) $f_3 = 15$, $f_4 = t \in \mathbb{N}$, $f_{11} = 1$, or

(b) $f_3 = 16$, $f_4 = t \in \mathbb{N}$, $f_5 = 1$, $f_{11} = 1$, or

(c) $f_3 = 17$, $f_4 = t \in \mathbb{N}$, $f_6 = 1$, $f_{11} = 1$.

Using similar arguments as in the proof of Lemma 3.5 we obtain a contradiction in all three cases.
References

[1] J.-P. Bode, H. Harborth, and C. Thürmann, Minimum regular rectilinear plane graph drawings with fixed numbers of edge lengths, Congr. Numer. 169 (2004), 193–198.

[2] K. Böröczky, G. Kertész, and E. Makai Jr., The minimum area of a simple polygon with given side lengths, Period. Math. Hung. 39 (1999), no. 1-3, 33–49.

[3] P. Brass, W. Moser, and J. Pach, Research problems in discrete geometry, Springer, 2005.

[4] E. H.-A. Gerbracht, Minimal polynomials for the coordinates of the Harborth graph, preprint available at http://arxiv.org/abs/math.CO/0609360, 2006.

[5] H. Harborth, Match sticks in the plane, The Lighter Side of Mathematics (Washington, DC) (R. K. Guy and R. E. Woodrow, eds.), Math. Assoc. Amer., 1994, pp. 281–288.

[6] S. Kurz, No finite 5-regular matchstick graph exists, available at http://www.wm.uni-bayreuth.de/index.php?id=226, 2009.

[7] S. Kurz, Fast recognition of planar non unit distance graphs - Searching the minimum 4-regular planar unit distance graph, Geombinatorics Quarterly 21 (2011), no. 1, 25–33.

[8] C. Schade, Exakte Maximalzahlen gleicher Abstände, Master’s thesis, TU Braunschweig, 1993.