The binary-outcome detection loophole

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Abstract

The detection loophole problem arises when quantum devices fail to provide an output for some runs. If treating these devices in a device-independent manner, failure to include the unsuccessful runs in the output statistics can lead to an adversary falsifying security i.e. Bell inequality violation. If the devices fail with too high frequency, known as the detection threshold, then no security is possible, as the full statistics cannot violate a Bell inequality. In this work we provide an intuitive local hidden-variable strategy that the devices may use to falsify any two-party, binary-outcome no-signalling distribution up to a threshold of $2(m_A + m_B - 8)/(m_A m_B - 16)$, where $m_A, m_B$ refer to the number of available inputs choices to the two parties. This value is the largest analytically predicted lower bound for no-signalling distributions. We strongly conjecture it gives the true detection threshold for $m_A = m_B$, and for computationally tractable scenarios we provide the Bell inequality which verifies this. We also prove that a non-trivial detection threshold remains, even when allowing one party an arbitrary number of input choices.

1. Introduction

Due to the scales on which it operates, quantum technology faces the challenge of single photons or electrons being lost to the environment. This can result in devices failing to give any output. Ignoring these failures leads to the ‘detection loophole’ [1–3] security flaw. This is where a preprogrammed ‘hidden-variable’ device can falsely appear to exhibit non-local behaviour. Non-locality is necessary for the security proofs of device-independent quantum cryptography [4–11], therefore understanding and preventing the detection loophole is an extremely relevant problem.

One important question to consider is how low the rate of successful detection events (the efficiency) can be before all observed correlations are describable by a local realistic model. Knowing this threshold allows one to set minimum requirements for commercial devices and benchmark current technology. However, obtaining this bound for quantum states is generally difficult due to the infinite set of extremal quantum correlations, and only a few optimal constructions are known [12, 13].

In this article we present an intuitive local hidden-variable (LHV) construction for two parties, arbitrary inputs, and binary outputs, which will be able to reproduce any no-signalling distribution obtained by the successful runs, up to a detection efficiency dependent on the number of inputs. This provides a lower bound on the threshold for quantum measurements in the same scenario. When both parties have the same number of inputs into their device, this construction achieves numerically known thresholds (for general no-signalling distributions) leading us to conjecture it is optimal for this symmetric case. We furthermore show that in cases with an asymmetric number of measurements, increasing the number of Bob’s measurements $m_B$ above $2 \log_2 m_A$ provides no additional power in verifying non-local correlations.

Bell’s seminal theorem [14] and its subsequent generalisations [15–17] give fundamental constraints on the correlations exhibited by any local realistic model; constraints that quantum theory can violate. These violations have been confirmed experimentally [18, 19]. Due to limitations on technology however, to show Bell violations they relied on a ‘fair-sampling’ assumption; that the device failures were non-malicious and the successful detections were representative of the underlying system. In cryptographic protocols however,
we cannot make that assumption, allowing an adversary (Eve) to pre-program the device to fail. It was not until much later that loophole-free violations, with no fair-sampling assumptions, were experimentally demonstrated [20–22]. The difficulty involved in closing this loophole highlights the importance of obtaining the best theoretical thresholds possible, so that minimal technological developments are required to perform secure protocols.

2. Preliminaries

In this paper, we are working in the device-independence framework. We assume that two parties (named Alice and Bob) have been distributed a joint system, on which they can make measurement choices, also referred to as inputs (labelled by $x$ for Alice, and $y$ for Bob) and receive outcomes (labelled $a$ for Alice and $b$ for Bob). We characterise the joint system only by the conditional probability distribution $p(ab|xy)$, making no assumptions about the underlying state or measurements made. This is known as a black box description. However, we do assume that Alice and Bob can isolate their systems, also referred to here as devices, from communicating with each other. This imposes the no-signalling conditions

\[
\sum_b p(ab|xy) = \sum_b p(ab|x'y') \quad \forall \, a, x, y, y', \quad (1)
\]

\[
\sum_a p(ab|xy) = \sum_a p(ab|x'y) \quad \forall \, b, x, x', y. \quad (2)
\]

When the number of inputs and outputs are finite, so that $x \in \{0 \ldots m_A - 1\}$, $y \in \{0 \ldots m_B - 1\}$, $a \in \{0 \ldots n_A - 1\}$, $b \in \{0 \ldots n_B - 1\}$, then we may express any no-signalling probability distribution via the vector $p := [p(00) \ldots p(n_A - 1 m_B - 1 m_A - 1 m_B - 1)]$. The set of such vectors forms a convex set with finitely many extremal points, known as the no-signalling polytope, $N\mathcal{S}$. This restriction is known as the $(m_A, n_A, n_B)$-scenario.

Within this set is a strict subset [23] of quantumly realisable distributions, $Q$. Unlike the full no-signalling space, $\mathcal{Q}$ has an infinite number of extremal points, making it more difficult to deal with computationally. Strictly contained within $Q$ is the set of local distributions, $L$. Any distribution $p(ab|xy)$ within $L$ has a LHV model of the form

\[
p(ab|xy) = \int_{\lambda} d\lambda \rho(\lambda)p(a|x, \lambda)p(b|y, \lambda).
\]

These distributions may always be expressed as convex combinations of deterministic distributions $p(ab|xy) = \delta_{ab, \theta_{ab}}$, which are finite in number. Geometrically, this means the structure of $L$ is also a polytope.

$L$ may be equivalently described by a set of Bell inequalities, linear inequalities of the form

\[
\sum_{a,b,c,d} s_{abcd} p(ab|xy) \leq k, \quad \text{where} \quad p(ab|xy) \quad \text{is our input-conditional joint distribution [24]. There is a finite set of facet Bell inequalities; if all facets are satisfied by} \quad p(ab|xy) \quad \text{it must have a LHV model i.e. it belongs to} \quad L. \quad \text{Thus violation of a Bell inequality is used to prove the impossibility of a LHV model. We will also often denote a Bell inequality by a vector} \quad s = (s_{00} \ldots s_{m_A-1 m_B-1}), \quad \text{though one must also state the sign and magnitude of the inequality.}
\]

The typical detection loophole model; and the one considered in this article, is one in which the devices fail to detect with equal probability independently of each other [25]. Whilst not completely general, it is how we would expect the device to behave if the failures were ‘honest’; if we see autocorrelations, or correlations between the joint failures; this is a clear signal of adversarial manipulation. The model considered here adds an extra output to both parties to alter the original distribution $p(ab|xy)$ in the following way:

\[
\begin{align*}
p_\eta(ab|xy) &= \eta^7 p(ab|xy), \\
p_\eta(Fb|xy) &= \eta(1-\eta)p(b|y), \\
p_\eta(aF|xy) &= \eta(1-\eta)p(a|x), \\
p_\eta(FF|xy) &= (1-\eta)^2.
\end{align*}
\]

One can see this as a linear map $D_\eta : p \rightarrow p_\eta$, from the set of no-signalling distributions in the $(m_A, n_B, n_A, n_B)$-scenario to those in the $(m_A, m_B, n_A + 1, n_B + 1)$-scenario. The quantity we are interested in is the (quantum) critical detection efficiency, $\eta_c := \inf \{\eta | \exists p \in Q, p_\eta \notin L\}$, where $Q, L$ are considered in the $(m_A, m_B, n_A, n_B)$-scenario and $(m_A, m_B, n_A + 1, n_B + 1)$-scenario respectively.
Alexa’s input; say their input choices. Between themselves, Alexa and Boris first randomly choose a leader, with bias towards threshold $\eta$ at an arbitrary distribution the detection threshold of that particular distribution, (see the appendix for details) gives and Boris as the names of their devices, whose goal is to falsify an arbitrary non-local distribution.

To emphasise the idea that Alice and Bob’s devices are working against them, we introduce Alexa with some probability $\beta$. To emphasise the idea that Alice and Bob’s devices are working against them, we introduce Alexa with some probability $\beta$. T o emphasise the idea that Alice and Bob’s devices are working against them, we introduce Alexa with some probability $\beta$. T o emphasise the idea that Alice and Bob’s devices are working against them, we introduce Alexa with some probability $\beta$.

Meanwhile, Boris receives his input and returns $b \in \{0, \ldots, n_B - 1\}$ according to $p(ab|ky)/p(a|k)$ regardless. Notice that they never jointly output a failure, so in order to fully reproduce inefficient statistics they must with some probability $\beta$ agree to both output $F$, regardless of input. This strategy gives rise to the statistics:

\begin{equation}
\begin{aligned}
p^{LHV}(ab|xy) &= (1 - \beta) \left( \frac{\alpha}{m_A} + \frac{1 - \alpha}{m_B} \right) p(ab|xy), \\
p^{LHV}(Fb|xy) &= (1 - \beta) \alpha \frac{m_A - 1}{m_A} p(b|y), \\
p^{LHV}(aF|xy) &= (1 - \beta)(1 - \alpha) \frac{m_B - 1}{m_B} p(a|x), \\
p^{LHV}(FF|xy) &= \beta.
\end{aligned}
\end{equation}

3. Pre-existing local hidden variable constructions

In order to understand our explicit construction, it is first useful to compare it to a LHV construction for the detection loophole introduced in [25]. Valid for any number of outputs, the construction is simple yet elegant. To emphasise the idea that Alice and Bob’s devices are working against them, we introduce Alexa and Boris as the names of their devices, whose goal is to falsify an arbitrary non-local distribution. Beforehand they may agree a strategy (using the LHV $\lambda$) but cannot communicate once they have received their input choices. Between themselves, Alexa and Boris first randomly choose a leader, with bias towards $\alpha \in [0, 1]$; let us suppose for this run Alexa is chosen. They then generate uniformly a prediction for Alexa’s input; say $k \in \{0, \ldots, m_A - 1\}$. Finally they agree on an output $a \in \{0, \ldots, n_A - 1\}$ for Alexa according to her desired marginal probability $p(a|k)$. When separated, once Alexa receives her input, if they have guessed correctly she will return outcome $a$. If the input received from Alice does not match their prediction, then Alexa outputs a failed detection $F$. It is clear this occurs with probability $(m_A - 1)/m_A$. Meanwhile, Boris receives his input and returns $b \in \{0, \ldots, n_B - 1\}$ according to $p(ab|ky)/p(a|k)$ regardless. Notice that they never jointly output a failure, so in order to fully reproduce inefficient statistics they must with some probability $\beta$ agree to both output $F$, regardless of input. This strategy gives rise to the statistics:
We will look to improve this strategy on extremal binary-output points, thereby bounding the threshold for the entire space. To do this, we need to understand better the extremal points themselves. Fortunately, for binary outputs a complete characterisation has been provided in [27]. One can see their threshold for the entire space. To do this, we need to understand better the extremal points themselves.

\[ \eta \leq \frac{m_A + m_B - 2}{m_A m_B} \]

By comparison to results in table 1, one can easily check for e.g. \( m_A = m_B = 3 \) this is not optimal.

4. A new local hidden variable construction

4.1. The model

We will look to improve this strategy on extremal binary-output points, thereby bounding the threshold for the entire space. To do this, we need to understand better the extremal points themselves. Fortunately, for binary outputs a complete characterisation has been provided in [27]. One can see their general form in figure 1. They may also be expressed in the simple form

\[
S = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}, \quad K = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}, \quad L = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}, \quad M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Figure 1. The structure of an extremal non-local distribution \( NS \), where \( g \in \{0, 1, \ldots, m_A - 2\}, h \in \{0, 1, \ldots, m_B - 2\} \). If \( g \) or \( h \) is non-zero, then the distribution is a lower input-number extremal point with local deterministic inputs appended to it. As such, its detection threshold cannot be lower than that of the \( m_A - h, m_B - g \) case; and is not generally optimal for \( m_A, m_B \) inputs.

One can equate equations (3) and (5) to find this LHV strategy can reproduce statistics up to \( \eta \leq \frac{m_A + m_B - 2}{m_A m_B} \). By comparison to results in table 1, one can easily check for e.g. \( m_A = m_B = 3 \) this is not optimal.

\[
p(ab|xy) = \begin{cases} 
1/2 & \text{if } a \oplus b = G(x, y) = \sum_{i=1}^{n_x} Q_i(x) R_i(y) \equiv \sum_{i=1}^{n_y} S_i(y) T_i(x) \\
0 & \text{otherwise.}
\end{cases}
\]

where \( Q_i(x) \) are polynomials in the binary digits\(^1\) of \( x \), which we label \( x_1 \), and \( R_i(y) \) are monomials in the binary digits of \( y \) (labelled \( y_1 \)). Similarly, \( S_j(y) \) are polynomials of \( y_2 \) and \( T_j(x) \) monomials. \( n_x = \lceil \log_2 m_A \rceil \) is the length of \( x_2 \) and similarly for \( n_y \). The most famous example of this is the (generalised) PR box [23], which has the form

\[
p(ab|xy) = \begin{cases} 
1/2 & \text{if } a \oplus b = x_2 \cdot y_2 \mod 2 \\
0 & \text{otherwise.}
\end{cases}
\]

For all the numerically evaluated cases presented in table 1, the generalised PR box achieves the no-signalling threshold \( \eta^* \).

In particular, given any extremal \( NS \) point, the conditional output distribution for two input pairs either match exactly or are exactly anti-matching. This allows the following strategy: Alexa and Boris with probability \( \alpha \) randomly choose a leader; suppose it is Alexa. They generate uniformly a prediction for \( x \), say \( k_1 \in \{0 \ldots m_A - 1\} \); then another from the remaining \( m_A - 1 \) choices a second prediction, \( k_2 \in \{0 \ldots m_A - 1\} \setminus \{k_1\} \). They also with probability 1/2 decide whether they will use a matching or unmatching strategy. Finally, they decide uniformly on a value for \( a, a_l \in \{0, 1\} \). Once Alexa receives her input, if it matches \( k_1 \) she returns outcome \( a_l \). If she receives \( k_2 \), for the matching strategy she returns \( a_l \), and if they are following the unmatching strategy \( a_l \oplus 1 \). If her input does not match \( k_1 \) or \( k_2 \), then she outputs a failed detection \( F \). It is clear this occurs with probability \( (m_A - 2)/m_A \). Meanwhile, Boris receives his input \( z \in \{0 \ldots m_B - 1\} \) and checks if \( G(k_1, z) = G(k_2, z) \). If these values match \( \text{and} \) they chose the matching strategy he outputs \( a_l \oplus G(k_1, z) \), otherwise he outputs \( F \). If the two values are unequal and they chose the unmatching strategy he outputs \( a_l \oplus G(k_1, z) \), and \( F \) otherwise. They still with some\(^1\) \( \oplus \) denotes addition modulo 2.
We now prove that the no-signalling detection threshold cannot be improved by increasing asymmetrically.

### 4.2. Asymptotic power of the model

We now prove that the no-signalling detection threshold cannot be improved by increasing asymmetrically one party’s possible measurements beyond the limit \( m_B = 2^{\log_2 m_A} \). One may express any extremal point as having \( p(ab|x) = 1/2 \) when \( a \oplus b = G(x, y) = \sum_{j=1}^{2^{m_A}} S_j(y) T_j(x) \), with \( n_x = \lfloor \log_2 m_A \rfloor \). In particular this implies there are at most \( 2^{2 m_A} \) functions of \( x \) defined by the inputs of Bob. Equivalently, it implies that for any extremal point of a scenario with \( m_B > 2^{m_A} \), then for any input choice \( y > 2^{m_A} \) the joint distribution \( p(ab|x) \) is identical to the joint distribution \( p(ab|y) \) of some \( y' \leq 2^{m_A} \), \( \forall a, b, x \). Therefore, if one has a valid LHV strategy for \( m_A, m_B = 2^{2 \log_2 m_A} \) inputs up to efficiency \( \eta \), one also has a valid strategy for all \( m_A, m_B > 2^{2 \log_2 m_A} \) which will also achieve efficiency \( \eta \). This strategy simply treats \( y > 2^{2 \log_2 m_A} \) identically to the corresponding \( y' \leq 2^{2 \log_2 m_A} \).

### 4.3. Comparison to numerically known no-signalling thresholds

Although the bound derived in the previous section holds for all pairs \((m_A, m_B)\), we see from the numerical evidence in table 1 it is not generally tight. In the case where \( m_A = 3, m_B = 4 \), we know the no-signalling detection threshold to be \( \eta^* = 5/9 \); however, the hidden variable strategy we have proposed only simulates arbitrary distributions up to \( \eta = 1/2 \). To reproduce correlations up to \( \eta^* \), one can mix our strategy with the pre-existing one [25] presented earlier in this paper. By choosing the pre-existing strategy, which guesses a single input, 20% of the time and our strategy, predicting two inputs, 80% of the time, and by choosing Alexa solely as the leader for both strategies one can achieve \( \eta \leq 5/9 \). This mixing of strategies does not extend to higher dimensional asymmetric scenarios though; for \( m_A = 5, m_B = 6 \) no combination of the two strategies beats the bound given by equation (8).

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2 The simplified bound also holds for \( m_A = m_B = 4 \), since cancellation prevents the denominator vanishing.
As the number of input choices increases, one could propose a more general variation; in which the leader (say Alexa) chooses many input predictions \( k_1 \ldots k_n \in \{0 \ldots m_A - 1\} \), \( n \leq m_A \). With this strategy, they must beforehand predict whether \( G(zk) \) will coincide with \( G(z,k_i) \), for each \( i = 2 \ldots n \). This is analogous to the ‘matching/unmatching’ choice seen earlier. The probability of guessing this correctly scales as \( 2^{n-1} \). However, the benefit of predicting additional inputs only scales as \( n/m_A \). This implies the probability of a correct output will scale as \( n/m_A \), which takes its maximal value at \( n = 1, 2 \) only. Trying to incorporate this strategy to simulate \( m_A = 5, m_B = 6 \) distributions, our optimisation never chose strategies with \( n > 2 \). This suggests for the asymmetric case a more nuanced joint strategy is required. However, we stress that when \( m_A = m_B \), the bound predicted by this model matches all numerically obtained bounds.

In order to prove that our conjecture of \( \eta_{\ast} = 4/(m + 4) \) for \( m_A = m_B = m \) is correct, one would need to provide an extremal \( \mathcal{N}/\mathcal{S} \) distribution \( p^{\text{NS}} \), and corresponding Bell inequality \( s_{xy}^{{\eta}} \), such that \( \sum_{a',b',x,y} s_{a'b'xy}^{{\eta}} p_{xy}^{\text{NS}}(a'b'|xy) \leq k \), \( \forall \eta > \eta_{\ast} \). Here we have used \( a',b' \) to explicitly remind the reader that \( a' \) ranges both in the original values of \( a \) and \( F \); that is, it is a three-outcome inequality. From numerical results, the generalised PR box is the best candidate for the extremal point, but we found no obvious generalisation of the witnessing Bell inequalities, which are provided for evaluated cases in the appendix.

4.4. Comparison to quantumly realisable thresholds

As stated above, to prove the no-signalling threshold \( \eta_{\ast} = 2/3 \) is achievable via qubits using the \( m_A = m_B = m \) CHSH inequality [12], whilst testing ququarts with a \( m_A = m_B = 4 \) inequality allows a critical efficiency of \( (\sqrt{5} - 1)/2 \approx 0.618 \) [13]. The respective Bell inequalities verifying non-locality for efficiencies higher than the critical efficiency, when applied to the generalised PR box achieve the no-signalling detection threshold, \( \eta_{\ast} \), for their respective scenarios. These inequalities are somewhat special in that they are ‘lifted inequalities’; they are of the form

\[
\sum_{a',b',x,y} s_{a'b'xy}^{{\eta}} p_{xy}^{{\eta}}(a'b'|xy) \leq k, \quad \forall \eta > \eta_{\ast}.
\]

For quantum correlations, a critical efficiency of \( \eta_{\ast} = 2/3 \) is achievable via qubits using the \( m_A = m_B = 3 \) CHSH/qubit construction [12], whilst testing ququarts with a \( m_A = m_B = 4 \) inequality allows a critical efficiency of \( (\sqrt{5} - 1)/2 \approx 0.618 \) [13]. The respective Bell inequalities verifying non-locality for efficiencies higher than the critical efficiency, when applied to the generalised PR box achieve the no-signalling detection threshold, \( \eta_{\ast} \), for their respective scenarios. These inequalities are somewhat special in that they are ‘lifted inequalities’; they are of the form

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\]

\[
s_{a'b'xy}^{{\eta}} = s_{a'b'xy}, \quad \forall \eta > \eta_{\ast}.
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\]
construction which predicts $n$ inputs in a $k$-output scenario to scale as $\frac{n}{m_k^{1-\varepsilon}}$, which for $k > 2$ achieves optimal integer value only at $n = 1$. This suggests for higher output-number scenarios the construction of [25], defining equation (5), may be optimal.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Local weight linear program

In order to calculate the linear weight of an arbitrary distribution $q$, we solve the following problem:

$$\text{Maximise } \sum_i \alpha_i, \text{ subject to: } \sum_i \alpha_i q^L_i \leq q, \quad \alpha_i \geq 0.$$ 

where $q^L_i$ are the extremal points of the polytope $L$. By rearranging the inequality, we see that the leftover distribution $q' := q - \sum_i \alpha_i q^L_i$ has all positive entries, and satisfies the no-signalling constraints since so too do $q, q^L_i$. Therefore it is a valid (sub-normalised) distribution. This linear program therefore looks to optimise the total weight of the local extremal points over all decompositions of $q$.

It is also worth mentioning that every linear program has a dual with the same optimal value [30]. The dual of the above function gives us a vector $b$ such that:

$$b^T q = \sum_i \alpha_i, \quad b^T q^L_i \geq 1 \quad \forall i$$

we see immediately that if $\sum_i \alpha_i < 1$, this gives us a Bell inequality violated by $q$.

Appendix B. Bell inequalities which verify the threshold

In this supplemental file, the optimal Bell inequalities are provided to achieve the detection loophole threshold for the generalised PR box. They are presented in matrix format:

$$S = \begin{pmatrix} s_{0,0} & \ldots & s_{0,n_B} & \ldots & s_{0,m_B - 1} & \ldots & s_{0,n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{n_A,0} & \ldots & s_{n_A,n_B} & \ldots & s_{n_A,m_B - 1} & \ldots & s_{n_A,n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{n_A,0} & \ldots & s_{n_A,n_B} & \ldots & s_{n_A,m_B - 1} & \ldots & s_{n_A,n_B} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{n_A,0} & \ldots & s_{n_A,n_B} & \ldots & s_{n_A,m_B - 1} & \ldots & s_{n_A,n_B} \\ \end{pmatrix}$$

(10)

where the solid lines delineate different inputs. All presented inequalities have local bound $\geq 1$. Note that there are $n_A + 1 (n_B + 1)$ outputs to account for the additional output $F$. 


B.1. Optimal inequality for two-inputs
As mentioned in the main body of the paper, this inequality is a ‘lifting’ of the CHSH inequality. For all measurements failure to output is treated identically to 0. Since other liftings of the same CHSH inequality achieve the optimal value, we can see generally there is not a single unique inequality that witnesses the threshold.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

(11)

B.2. Optimal inequality for three-inputs
Unlike the previous case; this inequality is a ‘true’ three-input, three-output inequality; it cannot be created from lifting a previous, lower dimensional inequality. What is interesting to note is that, for the first two inputs for each party, failure is again treated identically to 0—it is only the final input which treats failure differently.

\[
\begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\end{pmatrix}
\]

(12)

B.3. Optimal inequality for four-inputs
This inequality is also a lifting of a four-input, two-output inequality; however in this instance the choice of treating failure as 0 or 1 depends on the input.

\[
\begin{pmatrix}
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\
\end{pmatrix}
\]

(13)

B.4. Optimal inequality for an asymmetric case: Alice 3 inputs, Bob 4 inputs
For this asymmetric case we can again provide a Bell inequality which achieves the optimal threshold for the generalised PR Box. Like the previous cases, the inequality we provide here is a facet inequality; that is a maximally dimensional face of the local polytope. This is the first inequality provided where the failure outcome is treated differently from the valid outcomes for all input choices; we leave open the question
whether this is necessary, or an artefact of this particular inequality.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(15)

B.5. Optimal inequality for five-inputs

The previous inequalities provided were all calculated using exact arithmetic. Unfortunately this takes much longer than floating point methods, particularly as the dimension increases. Therefore, we are only able to provide a Bell inequality here which is accurate up to 6 s.f. and moreover, not a facet inequality. However, it still verifies the detection loophole threshold, and is included for completeness.

\[
\begin{pmatrix}
0 & 144.708 & 0.0476284 & 0 & 147.002 & 0.05601450 & 0 & 147.002 & 0.05601450 & 0 \\
147.002 & 0.05601450 & 0 & 148.982 & 0.0504226 & 144.708 & 0 & 0 & 0.0476284 & 147.002 & 0 \\
0.05601450 & 147.002 & 0 & 0 & 0.0560145 & 148.982 & 0 & 0 & 0.0504226 & 0 \\
0.0476284 & 0.0476284 & 0 & 0 & 0.0504226 & 0.0326552 & 0.0494062 & 0.0494062 & 0.0326552 & 0.0494062 & 0.0494062 \\
0.0494062 & 0.0326552 & 0.0712469 & 0.0712471 & 0.0001059 & 0 & 0 & 0.0476284 & 147.002 & 0 & 0.0494062 & 147.860 & 0 \\
0.0582717 & 0 & 0.0510420 & 147.860 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 145.578 & 0.0573425 & 0 & 0 \n0.0582717 & 0 & 0.0510420 & 147.860 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 145.578 & 0.0573425 & 0 & 0 \n0.0582717 & 0 & 0.0510420 & 147.860 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 145.578 & 0.0573425 & 0 & 0 \n0.0582717 & 0 & 0.0510420 & 147.860 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 0.0582717 & 0 & 0 & 145.578 & 0.0573425 & 0 & 0 \n\end{pmatrix}
\]

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