On the Quintessence Scalar Field Potential

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(Dated: June 20, 2008)

Abstract

In this work we propose a new analytical method for determining the scalar field potential $V(\phi)$ in FRW type cosmologies containing a mixture of perfect fluid plus a quintessence scalar field. By assuming that the equation of state parameters of the perfect fluid, $\gamma - 1 \equiv p_\gamma/\rho_\gamma$ and the quintessence, $\omega \equiv p/\rho$ are constants, it is shown that the potential for the flat case is $V(\phi) = A\rho_\phi \sinh^B(\lambda \phi)$, where $A$, $B$ and $\lambda$ are functions of $\gamma$ and $\omega$. This general result is a pure consequence of the Einstein field equations and the constancy of the parameters. Applying the same method for closed and open universes, the corresponding scalar field potentials are also explicitly obtained for a large set of values of the free parameters $\gamma$ and $\omega$. A formula yielding the transition redshift from a decelerating to an accelerating regime is also determined and compared to the $\Lambda$CDM case.

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I. INTRODUCTION

A large number of recent observational data strongly suggest that we live in a flat, accelerating Universe composed of $\sim 1/3$ of matter (baryonic + dark) and $\sim 2/3$ of an exotic component with large negative pressure, usually named Dark Energy or Quintessence. The basic set of experiments includes: the luminosity distance from SNe Ia [1], temperature anisotropies of the cosmic background radiation [2], large scale structure, X-ray data from galaxy clusters [3], age estimates of globular clusters [4] and the ages from old high redshift galaxies [5].

A traditional candidate for the missing energy component is the vacuum energy density or cosmological constant ($\Lambda$) which is equivalent to a perfect fluid obeying the equation of state $p_v = -\rho_v$. Due to the cosmological constant problem [6], some authors have also considered that the vacuum energy density, due to its coupling with the other matter fields, can be a time-dependent function ($\Lambda(t)$ - models) [7].

A more generic possibility is a dynamical, time-dependent scalar field $\phi$ evolving slowly in its potential $V(\phi)$, which is usually referred to as Quintessence field [8]. Actually, due to its simplicity, a scalar field works like a kind of paradigm in particle physics (including string theory), and these can act as dark energy candidates. Although still lacking experimental evidence of its existence scalar fields are needed in all unification theories.

The Quintessence cosmological model considered here is defined by the action $S = m_{pl}^2/16\pi \int d^4x \sqrt{-g}[R - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) + L_m]$, where $R$ is the Ricci scalar and $m_{pl} \equiv G^{-1/2}$ is the Planck mass. The scalar field minimally coupled to gravity is assumed to be homogeneous, such that $\phi = \phi(t)$ and the Lagrangian density $L_m$ includes the additional perfect fluid component.

In a cosmological setting it proves convenient to characterize the scalar field by an effective equation of state (EoS) parameter, $w(t) \equiv p/\rho$, measuring the ratio between its pressure and energy density which is different from baryons, dark matter, neutrinos or radiation. Depending on the form of the potential $V(\phi)$, $w$ can be constant, monotonically increasing (decreasing) or even oscillatory [9]. Actually, a wide variety of scalar field models with many disparate applications in cosmology have been proposed in the literature [8, 9, 10, 11, 12].

Whether the EoS parameter $w$ of the Quintessence is constant and satisfies $\omega \geq -1$, the quintessence has been termed “X-matter” [13, 14], which also includes the cosmological con-
stant (ΛCDM) models as a limiting case. For these “X-matter” models the first constraints on the free parameters were obtained by Perlmutter, Turner and M. White \[15\]. The latest constraints from cosmic microwave background anisotropies \[2\] alone yields \(\Omega_x = 0.73^{+0.10}_{-0.11}\) and \(\omega = -1.06^{+0.41}_{-0.42}\) (95% C.L.). The large scale structure \[3, 16\] provides \(\Omega_m = 0.28 \pm 0.06\) and \(\omega = -1.14 \pm 0.31\) for a flat cosmology while more tight limits are obtained from SN Ia data \[1\] \(\omega = -1.023 \pm 0.090(stat) \pm 0.054(sys)\). When \(\omega \leq -1\) the quintessence field is called a phantom fluid \[17\]. The basic difficulties and the main advantages underlying the physics of the different dark energy candidates has been reviewed by several authors \[18\].

In this article we focus our attention to a Quintessence dark energy in its X-matter version. The main aim here is to determine the general analytic form of the scalar field potential which is simultaneously compatible with the “X-matter” constraint and the symmetries of the FRW line element. Actually, this problem has previously been considered in the literature, however, only special solutions has been derived \[19, 20, 21\].

As we shall see, if the “X-matter” interacts only gravitationally, that is, in the absence of decaying process or energy transference among the components, only a very restricted class of potentials is mathematically allowed by the Einstein Field Equations (EFE). In this case, the complete spectrum of solutions (for the flat case) can be fully determined with basis on the new method proposed here. In particular, for specific values of the free parameters, the solutions are slightly different from some expressions recently obtained in the literature. For closed and hyperbolic Universes, analytical solutions are also presented for specific values of the free parameters.

### II. BASIC EQUATIONS

We shall restrict our analysis to homogeneous and isotropic cosmologies described by the FRW line element

\[
ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right]
\]

where \(R(t)\) is the scale factor and \(k = 0, \pm 1\) is the curvature parameter.

Let us now consider a Universe filled with a perfect fluid plus a decoupled scalar field \(\phi\). In the background \(\Box\), the Einstein’s field equations (EFE) can be written as

\[
\frac{8\pi}{m_{pl}^2} (\rho_\gamma + \rho_\phi) = 3 \frac{\dot{R}^2}{R^2} + 3 \frac{k}{R^2},
\]

(1)
\[ \frac{8\pi}{m_{pl}^2} (p_\gamma + p_\phi) = -2 \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2}, \]  

(3)

where an overdot means time derivative, and \( m_{pl}^2 = 1/G \) is the Planck mass. The quantities \( \rho_\gamma, \rho_\phi, p_\gamma \) and \( p_\phi \) are the energy densities and pressures of the perfect fluid and scalar field \( \phi \), respectively.

It will be assumed that the perfect fluid and scalar field \( \phi(t) \) obeys the following equation of state

\[ p_\gamma = (\gamma - 1)\rho_\gamma \quad \text{and} \quad p_\phi = w\rho_\phi \]  

(4)

where \( p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \) and \( \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \). Moreover, the constant parameter \( \gamma \in [0, 2] \) and \( w \) take values within the interval \((-1, 0)\) because the phantom case is not being considered.

On the other hand, since the energy of each component is separately conserved, the energy densities of both components satisfy

\[ \dot{\rho}_\gamma + 3\gamma H \rho_\gamma = 0, \]  

(5)

\[ \dot{\rho}_\phi + 3(1+w)H \rho_\phi = 0, \]  

(6)

where \( H = \dot{R}/R \) is the Hubble parameter. These equations can explicitly be integrated giving

\[ \rho_\gamma = \rho_{\gamma 0} \left( \frac{R}{R_0} \right)^{-3\gamma} \quad \text{and} \quad \rho_\phi = \rho_{\phi 0} \left( \frac{R}{R_0} \right)^{-3(1+w)}, \]  

(7)

where \( \rho_{\gamma 0}, \rho_{\phi 0} \) and \( R_0 \) are the values of these parameters at the present time \((t = t_0)\). Naturally, the second solution is valid only for constant values of \( w \). As usual, inserting the expressions of \( \rho_\phi \) and \( p_\phi \) into the energy conservation law for the scalar field (or more directly from the field Lagrangian), one obtains the equation of motion

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \]  

(8)

If \( V(\phi) \) is given a priori, one may follow the standard approach by integrating directly the above equation. In what follows we explore the latter approach for the case of “X-matter”.

III. EVOLUTION OF THE SCALE FACTOR AND THE QUINTESSENCE POTENTIAL

In order to find the scalar field potential corresponding to a generic constant EoS parameter in the presence of a perfect fluid, we start by combining the expressions of \( p_\phi \) and \( \rho_\phi \) defined in (7). One finds
\[ V(\phi) = \frac{(1-w)}{2(1+w)} \dot{\phi}^2 \quad \text{and} \quad \rho_\phi = \frac{1}{(1+w)} \dot{\phi}^2, \quad (9) \]

showing that \( V(\phi) \) and \( \rho_\phi \) may be readily determined if \( \dot{\phi}^2 \) is known as a function of \( \phi \).

Now, substituting the derivative of \( V(\phi) \) with respect to \( \phi \) into the equation \( (8) \), one obtains the following differential equation

\[ \frac{\ddot{\phi}}{\dot{\phi}} + \frac{3(1+w)}{2} \frac{\ddot{R}}{\dot{R}} = 0, \quad (10) \]

and a straightforward integration yields

\[ \dot{\phi} = \sqrt{(1+w)\rho_\phi} \left( \frac{R}{R_0} \right)^{-\frac{3(1+w)}{2}} = \sqrt{(1+w)\rho_\phi} x^{-\frac{3(1+w)}{2}} \quad (11) \]

where the variable \( x = \frac{R}{R_0} \) has been introduced in the second equality. From \( (11) \), we see that \( w = -1 \) implies \( \dot{\phi} = 0 \). This special case corresponds to the cosmological constant. The above expression tell us that the solution of our problem will be obtained only if the scale factor is determined as a function of \( \phi \).

On the other hand, by combining the set of equations \( (2) - (4) \) and \( (7) \) one finds the differential equation governing the behavior of the scale factor \( R(t) \) in the presence of a perfect fluid \( \gamma \) plus the dark “X-matter” energy

\[ R\ddot{R} + \Delta \dot{R}^2 + \Delta k + \frac{3}{2} H_0^2 (1 - \gamma + w) \Omega_\gamma_0 R_0^{3(1+w)} R^{-(1+3w)} = 0, \quad (12) \]

whose first integral can be written as

\[ \dot{R}^2 = \frac{A}{R^{3\gamma-2}} - k + H_0^2 \Omega_\phi R_0^{3(1+w)} R^{-(1+3w)}, \quad (13) \]

where \( A = H_0^2 \Omega_\phi R_0^{3\gamma} \) is an integration positive constant and \( \Delta \equiv \frac{3\gamma-2}{2} \). In the absence of a quintessence component \( (\Omega_\phi \equiv 0) \) the above Eq. \( (12) \) reduces to the general FRW differential equation as discussed by Assad and Lima \[24\].

On the other hand, from the first EFE \( (2) \) one find that \( k \) parameter satisfies \( \Omega_\gamma_0 + \Omega_\phi - 1 = \frac{k}{H_0^2 R_0^2} \). As one may check, inserting the values of \( A \) and \( k \) into the above equation, and by introducing the variable \( x = R/R_0 \) it follows that

\[ \frac{dt}{dx} = \frac{H_0^{-1}}{\sqrt{1 - \Omega_\gamma_0 - \Omega_\phi + \Omega_\gamma_0 x^{-(3\gamma-2)} + \Omega_\phi x^{-(1+3w)}}, \quad (14) \]
where $H_0$ is the Hubble parameter at the current time ($t = t_0$). Thus, substituting (14) into (11) we obtain

$$
d\phi = H_0^{-1} \sqrt{(1 + w)\rho_\phi} \frac{x^{-\frac{2}{3}(1+w)} \, dx}{\sqrt{1 - \Omega_{\gamma 0} - \Omega_{\phi 0} + \Omega_{\gamma 0} x^{-(3\gamma - 2)} + \Omega_{\phi 0} x^{-(1+3w)}}}. \tag{15}
$$

The integration and inversion of the above equation yields $R(\phi)$ and from Eq. (9) one obtains the scalar field potential. However, it cannot be analytically solved for arbitrary values of the curvature parameter. As discussed below, a general solution exists only for the flat case. Solutions for $k = \pm 1$ are also possible for specific values of the pair $(\gamma, \omega)$.

A. Solution of the flat case

For $k = 0$ we see that $\Omega_{\gamma 0} + \Omega_{\phi 0} = 1$. Inserting this into (15) and introducing the auxiliary coordinate $\theta = \frac{\Omega_{\phi 0}}{\Omega_{\gamma 0}} x^{3(\gamma - w - 1)} = \sinh^2 \theta$, one finds

$$
R(\phi) = R_0 \left( \frac{\Omega_{\gamma 0}}{\Omega_{\phi 0}} \right)^{\frac{1}{2(\gamma - w - 1)}} \sinh^{\frac{2}{2(\gamma - w - 1)}} \left[ \frac{3 (\gamma - w - 1)\sqrt{8 \pi}}{2 \sqrt{3(1 + w)}} \frac{\phi}{m_{pl}} \right], \tag{16}
$$
or equivalently,

$$
\frac{\phi(R)}{m_{pl}} = \frac{2 \sqrt{3(1 + w)}}{3 (\gamma - w - 1) \sqrt{8 \pi}} \arcsinh \left[ \sqrt{\frac{\Omega_{\phi 0}}{\Omega_{\gamma 0}}} \left( \frac{R}{R_0} \right)^{\frac{3(\gamma - w - 1)}{2}} \right], \tag{17}
$$

where the integration constant has been fixed equal to zero. Now, inserting (16) into (11) and using (8), we obtain the following expression for the scalar field potential

$$
V(\phi) = \frac{1 - w}{2} \rho_\phi \left( \frac{\Omega_{\phi 0}}{\Omega_{\gamma 0}} \right)^{\frac{1}{\gamma - w - 1}} \sinh^{-\frac{2}{2(\gamma - w - 1)}} \left[ \frac{3 (\gamma - w - 1)\sqrt{8 \pi}}{2 \sqrt{3(1 + w)}} \frac{\phi}{m_{pl}} \right]. \tag{18}
$$

The corresponding energy densities for the perfect fluid $\gamma$ and the scalar field $\phi$ are given by

$$
\rho_\gamma(\phi) = \rho_\gamma \left( \frac{\Omega_{\phi 0}}{\Omega_{\gamma 0}} \right)^{\frac{\gamma - w - 1}{\gamma - w - 1}} \sinh^{-\frac{2}{2\gamma}} \left[ \frac{3 (\gamma - w - 1)\sqrt{8 \pi}}{2 \sqrt{3(1 + w)}} \frac{\phi}{m_{pl}} \right], \tag{19}
$$

$$
\rho_\phi(\phi) = \rho_\phi \left( \frac{\Omega_{\phi 0}}{\Omega_{\gamma 0}} \right)^{\frac{1}{\gamma - w - 1}} \sinh^{-\frac{2}{2(\gamma - w - 1)}} \left[ \frac{3 (\gamma - w - 1)\sqrt{8 \pi}}{2 \sqrt{3(1 + w)}} \frac{\phi}{m_{pl}} \right]. \tag{20}
$$

Relations (16) - (20) are the general and unified solutions describing the main physical quantities for a flat universe filled with perfect fluid plus a “X-matter” component characterized by the pair $(\gamma, \omega)$. Thus, all known solutions are peculiar cases of it through an
adequate choice of the corresponding parameters. In particular, they allow us to calculate the expressions at different epochs. For example, substituting $\gamma = 1$ (dust) and $\gamma = \frac{4}{3}$ (radiation) into (18), one gets, respectively:

$$V(\phi) = \frac{1}{2} \rho_{\phi_0} \left( \frac{\Omega_{M0}}{\Omega_{\phi_0}} \right)^{\frac{1}{1+w}} \sinh \frac{2(1+w)}{w} \left[ \frac{-3w \sqrt{8 \pi}}{2 \sqrt{3(1+w)}} \frac{\phi}{m_{pl}} \right],$$  \hspace{1cm} (21)

$$V(\phi) = \frac{1}{2} \rho_{\phi_0} \left( \frac{\Omega_{r0}}{\Omega_{\phi_0}} \right)^{\frac{3(1+w)}{(1-3w)}} \sinh \frac{6(1+w)}{(1-3w)} \left[ \frac{(1-3w) \sqrt{8 \pi}}{2 \sqrt{3(1+w)}} \frac{\phi}{m_{pl}} \right].$$  \hspace{1cm} (22)

Solution (21) was independently discovered by several authors using different methods \cite{21,22,23}. However, our general solution (18) display analytically the influence of the different regimes on the behavior of the potential $V(\phi)$, as can be seen from the above expression for the radiation phase. More information may also be obtained taking the limit of (18) at early times. For $R << R_0$ the scalar fields satisfies the condition $\frac{3(\gamma-w-1) \sqrt{8 \pi}}{2 \sqrt{3(1+w)}} \frac{\phi}{m_{pl}} \ll 1$ and from (18) we obtain

$$V(\phi) \sim \frac{1}{2} \rho_{\phi_0} \left( \frac{\Omega_{\phi_0}}{\Omega_{\gamma 0}} \right)^{\frac{3(1+w)}{(1-3w)^2}} \sinh \left[ \frac{3(\gamma-w-1) \sqrt{8 \pi}}{2 \sqrt{3(1+w)}} \frac{\phi}{m_{pl}} \right]^{-\frac{2(1+w)}{(1-3w)^2}},$$  \hspace{1cm} (23)

where $\dot{\phi}_0$ was substituted by $\sqrt{(1+w)\rho_{\phi_0}}$. In particular, for $\gamma = 1$ and $\gamma = \frac{4}{3}$ the potentials scale as

$$V(\phi) \sim \left[ \frac{-3w \sqrt{8 \pi} \Omega_{M0}}{2 \sqrt{3(1+w)} \Omega_{\phi_0}} \frac{\phi}{m_{pl}} \right]^{\frac{2(1+w)}{w}},$$  \hspace{1cm} (24)

$$V(\phi) \sim \left[ \frac{(1-3w) \sqrt{8 \pi} \Omega_{r0}}{2 \sqrt{3(1+w)} \Omega_{\phi_0}} \frac{\phi}{m_{pl}} \right]^{-\frac{6(1+w)}{(1-3w)}},$$  \hspace{1cm} (25)

which could be obtained directly from equations (21) and (22). As far as we know, the limiting case above for radiation (25) was not presented in the literature, whereas (24) was first obtained by Ureña-López and Matos \cite{23}.

\section*{B. Solution $k \neq 0$ (open and closed Universes)}

As remarked before, in this case equation (15) has no general analytical solution. Thus we consider arbitrary values of $\gamma$ with specific values of $\omega$ and vice-versa. As an example,
let us consider arbitrary $\gamma$ and $w = -\frac{1}{3}$. In this case, for $w = -\frac{1}{3}$ and introducing the coordinate $[\frac{1-\Omega_{\gamma 0}}{\Omega_{\gamma 0}}] x^{3\gamma-2} = \sinh^2 \theta$ in (15), it reduces to

$$d\phi = H_0^{-1} \sqrt{\frac{2}{3} \rho_{\phi 0}} \frac{x^{-1}\, dx}{\sqrt{1 - \Omega_{\gamma 0} + \Omega_{\gamma 0} x^{-(3\gamma-2)}}}$$

(26)

with solution

$$\frac{\phi(R)}{m_{pl}} = \frac{1}{(3\gamma - 2)\sqrt{\pi}} \sqrt{\frac{\Omega_{\phi 0}}{1 - \Omega_{\gamma 0}}} \arcsinh \left[ \sqrt{\frac{1 - \Omega_{\gamma 0}}{\Omega_{\gamma 0}}} \left( \frac{R}{R_0} \right)^{\frac{3\gamma - 2}{2}} \right],$$

(27)

or equivalently,

$$R(\phi) = R_0 \left( \frac{\Omega_{\gamma 0}}{1 - \Omega_{\gamma 0}} \right)^{\frac{1}{3\gamma - 2}} \sinh^{\frac{2}{3\gamma - 2}} \left[ (3\gamma - 2)\sqrt{\pi} \frac{1 - \Omega_{\gamma 0}}{\Omega_{\phi 0}} \frac{\phi}{m_{pl}} \right].$$

(28)

On the other hand, inserting (28) into (7) we get the expressions for the energy densities of the two components in terms of $\phi$, that is

$$\rho(\phi) = \rho_{\gamma 0} \left( \frac{1 - \Omega_{\gamma 0}}{\Omega_{\gamma 0}} \right)^{\frac{3\gamma}{3\gamma - 2}} \sinh^{-\frac{6}{3\gamma - 2}} \left[ (3\gamma - 2)\sqrt{\pi} \frac{1 - \Omega_{\gamma 0}}{\Omega_{\phi 0}} \frac{\phi}{m_{pl}} \right],$$

(29)

$$\rho_{\phi}(\phi) = \rho_{\phi 0} \left( \frac{1 - \Omega_{\gamma 0}}{\Omega_{\gamma 0}} \right)^{\frac{2}{3\gamma - 2}} \sinh^{-\frac{4}{3\gamma - 2}} \left[ (3\gamma - 2)\sqrt{\pi} \frac{1 - \Omega_{\gamma 0}}{\Omega_{\phi 0}} \frac{\phi}{m_{pl}} \right].$$

(30)

Now, substituting (30) into (9), one obtains the potential $V(\phi)$, i.e.,

$$V(\phi) = \frac{2}{3} \rho_{\phi 0} \left( \frac{1 - \Omega_{\gamma 0}}{\Omega_{\gamma 0}} \right)^{\frac{2}{3\gamma - 2}} \sinh^{-\frac{4}{3\gamma - 2}} \left[ (3\gamma - 2)\sqrt{\pi} \frac{1 - \Omega_{\gamma 0}}{\Omega_{\phi 0}} \frac{\phi}{m_{pl}} \right].$$

(31)

Naturally, the behavior for different epochs may be obtained by an appropriate choice of $\gamma$. In particular, taking $\gamma = 1$ in (31), one may see that the potential $V(\phi)$ reduces to the one found by Di Pietro and Demaret [21].

### IV. TRANSITION EPOCH

Let us now consider the transition redshift, $z_*$, at which the Universe switches from deceleration to acceleration or, equivalently, the redshift at which the deceleration parameter vanishes. In order to derive the general expression of the transition redshift let us consider the deceleration parameter written as $q(R) = -\frac{\ddot{R}}{R}$. Now, by considering the general differential equation governing the behavior of the scale factor $R(t)$ in the presence of a
perfect fluid $\gamma$ plus the “X-matter” energy, i.e., equation (12), it is straightforward to show that, the transition redshift is given by

$$z_* = \left( \frac{3}{2\Delta} (\gamma - \omega - 1) - 1 \right) \left( \frac{\Omega_{\phi_0}}{\Omega_{\gamma_0}} \right)^{-\frac{1}{3(1-\gamma+\omega)}} - 1.$$  

As one may check, by assuming the values $\gamma = 1$, $\omega = -1$, $\Omega_{\phi_0} = 0.7$ and $\Omega_{\gamma_0} = 0.3$ we obtain $z_* = 0.66$ which is in fully agreement with the result for the cosmic concordance $\Lambda$CDM model. In addition, by slightly modifying the pressure term $\gamma = 1 + \epsilon$, where $\epsilon << 1$, we have

$$z_* = \left[(1 - 3\epsilon) \left( \frac{\Omega_{\phi_0}}{\Omega_{\gamma_0}} \right) \right]^{\frac{1}{3(1-\epsilon)}} - 1.$$  

In particular, if $\epsilon \sim 0.1$ we find $z_* \sim 0.59$, in accordance with a recent kinematic determination based on two different Supernovae type Ia samples [1, 25].

V. CONCLUSIONS

We have discussed FRW cosmologies with a perfect simple fluid plus a dark energy component. If the Quintessence fluid is represented by a scalar field with constant equation of state parameter, $\omega$, the EFE determine univocally the form of the scalar field potential. In other words, we cannot postulate simultaneously an arbitrary form for the potential and the “X-matter” condition.

The general solution of $V(\phi)$ has been explicitly derived by assuming a FRW flat Universes for generic values of the pair of parameters $(\gamma, \omega)$. In this case, our general solutions (Eqs. 16 - 20) tell us how the potential behaves at different epochs. As remarked there, the asymptotic behavior of the potential at early universe (radiation era) was not previously derived by another authors. The explicit analytic solution for the potential is accompanied by the others relevant quantities like $R(\phi)$, or equivalently $\phi(R)$, $\dot{\phi}(R)$ and $\rho_\gamma$ and $\rho_\phi$. Naturally, the cosmological model discussed here may be useful for universes filled with only two dominant components. The efficiency of the method has also been exemplified by considering closed and hyperbolic FRW spacetimes for particular values of the EOS parameter $\omega$ and arbitrary values of $\gamma$.

In the search for a more realistic description of the Quintessence field presumably filling the Universe, models with more than one type of fluid component, such as a combination
of radiation and non-relativistic matter must also be considered. The possibility of a time-dependent $\omega$ and interacting scalar fields as discussed by many authors in the literature \[26, 27, 28\] need also to be investigated in light of the method proposed in the present work.

**Acknowledgments**

This work was partially supported by the Fondo de Desarrollo Universitario, UNAC (Perú) and by the Conselho Nacional de Desenvolvimento Científico e Tecnológico-CNPq (Brazilian Research Agency). JASL is also partially supported by FAPESP (No. 04/13668-0).

[1] A. G. Riess *et al.*, *Astron. J.* 116, 1009 (1998); S. Perlmutter *et al.*, *Astrophys. J.* 517, 565 (1999); P. Astier *et al.*, *Astron. Astrophys.* 447, 31 (2006); A. G. Riess *et al.*, *Astrophys. J.* 659, 98 (2007); T. M. Davis *et al.*, *Astrophys. J.* 666, 716 (2007); M. Kowalski *et al.*, arXiv:0804.4142 [astro-ph].

[2] D. N. Spergel *et al.*, *Astron. J. Suppl.* 148, 175 (2003); D. N. Spergel et al. *Astrophys. J. Suppl. Ser.* 170, 377 (2007); J. Dunklei *et al.*, arXiv:0803.0586 [astro-ph]; E. Komatsu *et al.*, arXiv:0803.0547 [astro-ph].

[3] S. W. Allen, R. W. Schmidt and A. C. Fabian, *MNRAS* 334, L11 (2002); S. Ettori, P. Tozzi and P. Rosati, A&A 398, 879 (2003); J. A. S. Lima, J. V. Cunha and J. S. Alcaniz, *Phys. Rev. D* 68, 023510 (2003), astro-ph/0303388; S. W. Allen et. al., arXiv:0706.0033v1 (2007).

[4] F. Ponto *et al.* *Astron. Astrophys.* 329, 87 (1998); R. Cayrel *et al.*, *Nature* 409, 691 (2001); L. M. Krauss and B. Chaboyer, *Science* 299, 65 (2003).

[5] J. S. Dunlop *et al.*, *Nature* 381, 581 (1996); H. Spinrad *et al.*, *ApJ* 484, 581 (1997); L. Krauss, *ApJ* 480, 466 (1997); J. S. Alcaniz and J. A. S. Lima, *ApJ* 521, L87 (1999); *ibidem*, *ApJ* 550, L133 (2001); J. A. S. Lima and J. S. Alcaniz, *MNRAS* 317, 893 (2000), astro-ph/0005441; G. Hasinger, N. Schartel and S. Komossa, *Astrophys. J.* 573, L77 (2002); J. S. Alcaniz, J. A. S. Lima and J. V. Cunha, *MNRAS* 340, L39 (2003), astro-ph/0301226; J. V. Cunha and R. C. Santos, *Int. J. Mod. Phys.* D13, 1321 (2004), astro-ph/0402169; A. C. S. Friaça, J. S. Alcaniz and J. A. S. Lima, *MNRAS* 362, 1295 (2005), astro-ph/0504031; D. Jain and A. Dev, *Phys. Lett. B* 633, 436 (2006); J. F. Jesus, astro-ph/0603142.
[6] Ya. B. Zeldovich, Sov. Phys. Usp. 11, 381 (1968); S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).
[7] M. Özer and M. O. Taha, Phys. Lett. B 171, 363 (1986); J. A. S. Lima and J. M. F. Maia, Phys. Rev. D49, 5597 (1994); J. A. S. Lima and J. C. Carvalho, Gen. Rel. Grav. 26, 909 (1994); J. A. S. Lima, Phys. Rev. D54, 2571 (1996), \texttt{gr-qc/9605055}; J. M. Overduin and F. I. Cooperstock, Phys. Rev. D58, 043506 (1998); J. S. Alcaniz and J. A. S. Lima, Phys. Rev. D72, 063516 (2005), \texttt{astro-ph/0507372}; J. F. Jesus et al., \texttt{arXiv:0806.1366} [astro-ph].
[8] C. Wetterich, Astron. & Astrophys. 301, 321 (1995); B. Ratra and P. J. E. Peebles, Phys. Rev D37, 3406 (1988); P. J. E. Peebles and B. Ratra, Astrophys. J. Lett. 325, L17 (1988); R. R. Caldwell, R. Dave, P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998); S. M. Carroll, Phys. Rev. Lett., 81, 3067 (1998); I. Zlatev, L-M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999); P. G. Ferreira and M. Joyce, Phys. Rev. D58, 023503(1998); R. R. Caldwell and E. V. Linder, \textit{Phys. Rev. Lett.} 95, 141301 (2005); K. R. S. Balaji and R. H. Brandenberger, Phys. Rev. Lett. 94, 031301 (2005); J. M. F. Maia and J. A. S. Lima, Phys. Rev. D 65, 083513 (2002), \texttt{astro-ph/0112091}; V. Faraoni and M. N. Jensen, Class. Quant. Grav. 23, 3005 (2006); F. C. Carvalho et al., Phys. Rev. Lett. 97, 081301 (2006), \texttt{astro-ph/0608439}; ibdem, \texttt{arXiv:0704.3043} (2007).
[9] S. Dodelson, M. Kaplinghat and E. Stewart, Phys. Rev. Lett. 85, 5276 (2000); B. Feng B, M. Z. Li and X. M. Zhang, Phys. Lett. B 634 101 (2006); Z. Wen, Chin. Phys. 16, 2830 (2007).
[10] V. Sahni and Y. Shtanov, JCAP 0311, 014 (2003); R. R. Caldwell and E. V. Linder, Phys. Rev. Lett. 95, 141301 (2005); J. D. Barrow, A. R. Liddle and C. Pahud, Phys. Rev. D 74, 127305 (2006).
[11] M. P. Dabrowski, T. Stachowiak, and M. Szydlowski, Phys. Rev. D68, 103519 (2003); L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. 91, 211301 (2003); P. K. Townsend and M. N. R. Wohlfarth, Phys. Rev. Lett. 91, 061302 (2003); I. P. Neupane and D. L. Wiltshire, Phys. Rev. D72, 083509 (2005); R. Rosenfeld and J. A. Frieman, J. Cosmol. Astropart. Phys. 09, 003 (2005); T. Matos, J. A. Vazquez and J. Magana, \texttt{arXiv:0806.0683} [astro-ph].
[12] S. Tsujikawa, Phys. Rev. D. 72, 083512 (2005); M. Sahlén, A. R. Linde and D. Parkinson, Phys. Rev. D 72, 083511 (2005); Z. K. Guo, N. Ohta and Y. Z. Zhang, Phys. Rev. D 72, 023504 (2005); O. Sergijenko and B. Novosyadlyi, \texttt{arXiv:0805.3782} [astro-ph].
[13] M. S. Turner and M. White, Phys. Rev. D 56, R4439 (1997); T. Chiba, N. Sugiyama and T. Nakamura, MNRAS, 289, L5 (1997).
[14] J. V. Cunha, L. Marassi and R. C. Santos, IJMP D 16, 403 (2007), astro-ph/0608686.

[15] S. Perlmutter, M. S. Turner and M. White, Phys. Rev. Lett. 83, 670 (1999). See also G. Efstathiou, MNRAS, 310, 842 (1999).

[16] M. Tegmark et al., Astrophys. J. 606, 702 (2004); W. J. Percival et al., MNRAS 327, 1297 (2001).

[17] T. Padmanabhan and T. R. Choudhury, Mon. Not. Roy. Astron. Soc. 344, 823 (2003); P. T. Silva and O. Bertolami, Astrophys. J. 599, 829 (2003); J. A. S. Lima, J. V. Cunha and J. S. Alcaniz, Phys. Rev. D 68, 023510 (2003); Z.-H. Zhu and M.-K. Fujimoto, Astrophys. J. 585, 52 (2003); S. Nesseris and L. Perivolaropoulos, Phys. Rev. D 70, 043531 (2004); Y. Wang and M. Tegmark, Phys. Rev. Lett. 92, 241302 (2004); T. R. Choudhury and T. Padmanabhan, Astron. Astrophys. 429, 807 (2005); S. Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007); R. C. Santos and J. A. S. Lima, Phys. Rev. D77, 023519 (2008), arXiv:0803.1865 [astro-ph].

[18] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000); P. J. E. Peebles and B. Ratra Rev. Mod. Phys. 75, 559 (2003); T. Padmanabhan, Phys. Rept. 380, 235 (2003); J. A. S. Lima, Braz. J. Phys. 34, 194 (2004), astro-ph/0402109; E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D15, 1753 (2006); J. S. Alcaniz, Braz. J. Phys. 36, 1109 (2006), astro-ph/0608631.

[19] T. Matos, F. S. Guzmán and L. A. Ureña-López CQG 17, 1707 (2000).

[20] P. F. González Díaz, Phys. Rev. D62, 023513 (2000).

[21] E. Di Pietro and J. Demaret, IJMPD 10, 231 (2001).

[22] L. P. Chimento and A. S. Jakubi, IJMPD 10, 231 (1996).

[23] L. A. Ureña-López and T. Matos, Phys. Rev. D 62, 081302 (2000).

[24] M. J. D. Assad and J. A. S. Lima, Gen. Rel. Grav. 20, 527 (1988); J. A. S. Lima, Am. J. Phys. 69, 1245 (2001), astro-ph/0109215.

[25] J. V. Cunha and J.A.S. Lima, arXiv:0805.1261 [astro-ph].

[26] G. Efstathiou, Mon. Not. R. Astron. Soc. 310 842 (1999); A. R. Cooray and D. Huterer, Astrophys. J. 513 L95 (1999); J. Weller and A. Albrecht, Phys. Rev. D 65 103512 (2002); I. Maor et al., Phys. Rev. D 65 123003(2002); E. V. Linder, Phys. Rev. Lett. 90 091301 (2003).

[27] L. Amendola, Phys. Rev. D62 043511 (2000); J. A. S. Lima, A. I. Silva and S. M. Viegas, Mon. Not. Roy. Astron. Soc. 312, 747 (2000); J. A. S. Lima and J. A. Espichan Carrillo,
A. Nunes and J. P. Mimoso, Phys. Lett. B 488, 423 (2000); L. P. Chimento et al., Phys. Rev. D67, 083513 (2003); S. Carneiro and J. A. S. Lima, Int. J. Mod. Phys. A20, 2465 (2005), gr-qc/0405141; J. D. Barrow and T. Clifton, Phys. Rev. D73, 103520 (2006); gr-qc/0604063.

[28] R. Horvat and D. Pavon, Phys. Lett. B653, 373 (2007), arXiv:0707.2299 [gr-qc]; M. Cataldo and J. Saavedra, Phys. Lett. B 662, 314 (2008), arXiv:0803.1086 [hep-th]. See also Miguel Quartin et al., JCAP 0805, 007 (2008); arXiv:0802.0546 [astro-ph].