SURFACE, CURVES AND THE LAKSHMANAN EQUVALENT COUNTERPARTS OF THE SOME MYRZAKULOV EQUATIONS

G.N.Nugmanova

Centre for Nonlinear Problems, PO Box 30, 480035, Almaty-35, Kazakstan

Abstract

The Lakshmanan equivalent counterparts of the some Myrzakulov equations are found.
I. INTRODUCTION

In a series of papers [1] were presented the new class integrable and nonintegrable spin equations and the unit spin description of soliton equations. In this paper using the geometrical method we will find the Lakshmanan equivalent [2] (according to the terminology of ref. [1]) counterparts of the some Myrzakulov equations.

Consider an n-dimensional space with the basic unit vectors: \( \vec{e}_1 = \vec{S}, \vec{e}_2, ..., \vec{e}_n \). Then the M-0 equation has the form

\[
\vec{S}_t = \sum_{i=2}^{n} a_i \vec{e}_i
\]  

(1)

where \( a_i \) are real functions, \( \vec{S} = (S_1, S_2, ..., S_n) \), \( \vec{S}^2 = E = \pm 1 \). This equation admit the many interesting class integrable and nonintegrable reductions [1]. Below we will find the L-equivalent counterparts of the some integrable reductions and only for the cases \( n = 2, 3 \). To this end, starting from the results of the ref. [8] we consider the motion of surface in the 3-dimensional space which generated by a position vector \( \vec{r}(x, y, t) = r(x^1, x^2, t) \). According to the C-approach from the ref. [1], the main elements of which we present in this section, let \( x \) and \( y \) are local coordinates on the surface. The first and second fundamental forms in the usual notation are given by

\[
I = d\vec{r}d\vec{r} = Edx^2 + 2F dx dy + Gdy^2
\]  

(2)

\[
II = -d\vec{r}d\vec{n} = Ldx^2 + 2Md x dy + Ndy^2
\]  

(3)

where

\[
E = \vec{r}_x \vec{r}_x = g_{11}, F = \vec{r}_x \vec{r}_y = g_{12}, G = \vec{r}_y \vec{r}_y = g_{22},
\]

\[
L = \vec{n} \vec{r}_{xx} = b_{11}, M = \vec{n} \vec{r}_{xy} = b_{12}, N = \vec{n} \vec{r}_{yy} = b_{22}, \vec{n} = \frac{(\vec{r}_x \wedge \vec{r}_y)}{|\vec{r}_x \wedge \vec{r}_y|}
\]

In the C-approach [1], the starting set of equations reads as

\[
\vec{r}_t = a\vec{r}_x + b\vec{r}_y + c\vec{n}
\]  

(4a)

\[
\vec{r}_{xx} = \Gamma^1_{11} \vec{r}_x + \Gamma^2_{11} \vec{r}_y + L\vec{n}
\]  

(4b)

\[
\vec{r}_{xy} = \Gamma^1_{12} \vec{r}_x + \Gamma^2_{12} \vec{r}_y + M\vec{n}
\]  

(4c)

\[
\vec{r}_{yy} = \Gamma^1_{22} \vec{r}_x + \Gamma^2_{22} \vec{r}_y + N\vec{n}
\]  

(4d)

\[
\vec{n}_x = p_1 \vec{r}_x + p_2 \vec{r}_y
\]  

(4e)

\[
\vec{n}_y = q_1 \vec{r}_x + q_2 \vec{r}_y
\]  

(4f)

where \( \Gamma^k_{ij} \) are the Christoffel symbols of the second kind defined by the metric \( g_{ij} \) and \( g^{ij} = (g_{ij})^{-1} \) as

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)
\]  

(5)
The coefficients $p_i, q_i$ are given by

$$p_i = -b_{1j}g^{ji}, \quad q_i = -b_{2j}g^{ji}$$  \hspace{1cm} (6)

The compatibility conditions $\vec{r}_{xxy} = \vec{r}_{xyx}$ and $\vec{r}_{yyx} = \vec{r}_{xyy}$ yield the following Mainardi-Peterson-Codazzi equations (MPCE)

$$R^l_{ijk} = b^l_{ij}b^l_{kj} - b^l_{ik}b^l_{lj}$$  \hspace{1cm} (7a)

$$\frac{\partial b_{ij}}{\partial x^k} - \frac{\partial b_{ik}}{\partial x^j} = \Gamma^s_{ik}b_{js} - \Gamma^s_{ij}b_{ks}$$  \hspace{1cm} (7b)

where $b_i^j = g^{jl}b_{il}$ and the curvature tensor has the form

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ij}\Gamma^l_{ks} - \Gamma^s_{ik}\Gamma^l_{js}$$  \hspace{1cm} (8)

Let $\vec{Z} = (\vec{r}_x, \vec{r}_y, \vec{n})^t$. Then

$$\vec{Z}_x = A\vec{Z}$$  \hspace{1cm} (9)

$$\vec{Z}_y = B\vec{Z}$$  \hspace{1cm} (10)

where

$$A = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ p_1 & p_2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \\ q_1 & q_2 & 0 \end{pmatrix}$$

Hence we get the new form of the MPCE(7)

$$A_y - B_x + [A, B] = 0$$  \hspace{1cm} (11)

Let us introduce the orthogonal trihedral[1]

$$\vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \frac{\vec{r}_y}{\sqrt{F}}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 = \vec{n}$$  \hspace{1cm} (12a)

or

$$\vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \vec{n}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2$$  \hspace{1cm} (12b)

Let $\vec{r}_x^2 = E = \pm 1, \quad F = \vec{r}_x \vec{r}_y = 0$. Then from the previous equations after some algebra we get[1]

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$  \hspace{1cm} (13a)
\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_y =
D
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}
\tag{13b}
\]

Here
\[
C = \begin{pmatrix}
0 & k & 0 \\
-Ek & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix},
\]
\[
D = \begin{pmatrix}
0 & m_3 & -m_2 \\
-Em_3 & 0 & m_1 \\
Em_2 & -m_1 & 0
\end{pmatrix}
\]

Now the MPCE (7) and/or (11) becomes
\[
C_y - D_x + [C, D] = 0
\tag{14}
\]

Hence we obtain
\[
(m_1, m_2, m_3) = (\partial_x^{-1}(\tau y + km_2), m_2, \partial_x^{-1}(ky - \tau m_2))
\tag{15a}
\]
\[
m_2 = \partial_x^{-1}(\tau m_3 - km_1)
\tag{15b}
\]

The time evolution of \(\vec{e}_i\) we can write in the form[1]
\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_t =
G
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}
\tag{16}
\]
with
\[
G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-E\omega_3 & 0 & \omega_1 \\
E\omega_2 & -\omega_1 & 0
\end{pmatrix},
\]

So, we have
\[
C_t - G_x + [C, G] = 0
\tag{17a}
\]
\[
D_t - G_y + [D, G] = 0
\tag{17b}
\]

II. THE L - EQUIVALENT COUNTERPARTS OF THE SOME MYRZAKULOV EQUATIONS: the 2-dimensional case

In this case the M-0 equation(1) becomes
\[
\vec{S}_t = a_2 \vec{e}_2
\tag{18}
\]
and $\vec{S} = (S_1, S_2), \vec{S}^2 = E = \pm 1, \tau = c = 0$. So, we have[1]

\[
\begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix}_x = C \begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix},
\tag{19a}
\]

\[
\begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix}_y = D \begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix},
\tag{19b}
\]

Here

\[
C = \begin{pmatrix}
0 & k \\
-Ek & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & m_3 \\
-Em_3 & 0
\end{pmatrix}.
\]

Now from the MPCE (14) we obtain

\[
m_3 = \partial_x^{-1} k_y
\tag{20}
\]

For the time evolution we get[1]

\[
\begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix}_t = G \begin{pmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{pmatrix},
\tag{21}
\]

where

\[
G = \begin{pmatrix}
0 & \omega_3 \\
-E\omega_3 & 0
\end{pmatrix}.
\]

Now already we ready using this formalism to construct the L-equivalent counterparts of the some Myrzakulov equations.

**Examples**

1) The M-IV equation has the following L-equivalent counterpart

\[
\phi_t + \phi_{xxy} - 3E(\phi\partial_x^{-1}(\phi^2)_y)_x = 0
\tag{22a}
\]

which is the 2+1 dimensional mKdV[9].

2) The M-XXI equation has the following L-equivalent counterpart

\[
\phi_t + \phi_{xxy} - 3E(\phi\partial_x^{-1}\phi_y)_x = 0
\tag{22b}
\]

which is the 2+1 dimensional KdV[9].

**III. THE L - EQUIVALENT COUNTERPARTS OF THE SOME MYRZAKULOV EQUATIONS:** the 3-dimensional case

In this case work equations(12)-(16) and the M-0 equation becomes

\[
\vec{S}_t = a_2\bar{e}_2 + a_3\bar{e}_3
\]
and $\vec{S} = (S_1, S_2, S_3), \vec{S'^2} = E = \pm 1$. Using these equations we construct the L-equivalent counterparts of the some Myrzakulov and Ishimori equations[1]. Below we present only the final results.

**Examples**

1) The Myrzakulov-III equation

$$\vec{S_t} = (\vec{S} \wedge \vec{S_y} + u\vec{S})_x + 2b(cb + d)\vec{S}_y - 4cv\vec{S}_x$$  \hspace{1cm} (23a)

$$u_x = -\vec{S}_x(\vec{S} \wedge \vec{S_y}), v_x = \frac{1}{4(2bc + d)^2}(\vec{S}^2_x)_y$$  \hspace{1cm} (23b)

in this case

$$(m_1, m_2, m_3) = (\partial^{-1}_x(\tau_y + km_2), -\frac{ux}{k}, \partial^{-1}_x(k_y - \tau m_2))$$  \hspace{1cm} (24)

and the L-equivalent is the following set of equations [1]

$$i\phi_t = \phi_{xy} - 4ic(V\phi)_x + 2d^2V\phi, V_x = 2E(|\phi|^2)_y.$$  \hspace{1cm} (25)

Note that equations (23) and (25) admit the following integrable reductions: a) the M-I[1] and the Zakharov[4] equations, as c=0; b) the M-II[1] and Strachan[9] equations, as d=0, respectively [1].

2) The Myrzakulov-VIII equation looks like[1]

$$iS_t = \frac{1}{2}[S_{xx}, S] + q_{xx}$$  \hspace{1cm} (26a)

$$u_{xy} = \frac{1}{4i}tr(S[S_y, S_x])$$  \hspace{1cm} (26b)

where the subscripts denote partial derivatives and S denotes the spin matrix ($r^2 = \pm 1$)

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix},$$  \hspace{1cm} (27)

$$S^2 = I$$

Equations (26) are integrable, i.e. admits Lax representation and different type soliton solutions [1]. The Lakshmanan equivalent counterpart of the M-VIII equation (26) has the form[1]

$$iq_t + q_{xx} + vq = 0,$$  \hspace{1cm} (28a)

$$ip_t - q_{xx} - vq = 0,$$  \hspace{1cm} (28b)

$$v_y = 2(pq)_x$$  \hspace{1cm} (28c)

where $p = Eq$. On the other hand, in [5] was shown that eqs.(26) and (28) are gauge equivalent each other.

3) The Ishimori equation

$$iS_t + \frac{1}{2}[S, M_{10}S] + A_{20}S_x + A_{10}S_y = 0$$  \hspace{1cm} (29a)
\[ M_{20} u = \frac{\alpha}{4i} tr(S_y, S_x) \]  \hspace{1cm} (29b)

where \( \alpha, b, a = \text{consts and} \)

\[ M_{j0} = M_j, \; A_{j0} = A_j \; \text{as} \; a = b = -\frac{1}{2}. \]

The L-equivalent counterpart has the form [1]

\[ iq_t + M_{10} q + vq = 0 \]  \hspace{1cm} (30a)

\[ ip_t - M_{10} p - vp = 0 \]  \hspace{1cm} (30b)

\[ M_{20} v = M_{10}(pq) \]  \hspace{1cm} (30c)

which is the Davey-Stewartson equation, where \( p = Eq. \) As well known these equations are too gauge equivalent each other[10].

4) The Myrzakulov-IX equation has form[1]

\[ iS_t + \frac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0 \]  \hspace{1cm} (31a)

\[ M_{20} u = \frac{\alpha}{4i} tr(S_y, S_x) \]  \hspace{1cm} (31b)

where \( \alpha, b, a = \text{consts and} \)

\[ M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(b - a) \frac{\partial}{\partial x} + (a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}; \]

\[ M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a + 1) \frac{\partial}{\partial x} + a(a + 1) \frac{\partial^2}{\partial x^2}; \]

\[ A_1 = i\alpha\{2ab + a + b\}u_x - (2b + 1)\alpha u_y \]

\[ A_2 = i\{\alpha(2ab + a + b)u_y - (2a^2b + a^2 + 2ab + b)u_x\}. \]

Eqs.(31) admit the two integrable reductions. As \( b=0, \) eqs. (31) after the some manipulations reduces to the M-VIII equation (26) and as \( a = b = -\frac{1}{2} \) to the Ishimori equation(29). In general we have the two integrable cases: the M-IXA equation as \( \alpha^2 = 1, \) the M-IXB equation as \( \alpha^2 = -1. \) We note that the M-IX equation is integrable and admits the following Lax representation [1]

\[ \alpha \Phi_y = \frac{1}{2}[S + (2a + 1)I]\Phi_x \]  \hspace{1cm} (32a)

\[ \Phi_t = \frac{i}{2}[S + (2b + 1)I]\Phi_{xx} + \frac{i}{2}W\Phi_x \]  \hspace{1cm} (32b)

where

\[ W_1 = W - W_2 = (2b + 1)E + (2b - a + \frac{1}{2})SS_x + (2b + 1)FS \]

\[ W_2 = W - W_1 = FI + \frac{1}{2}S_x + ES + \alpha SS_y \]


\[ E = -\frac{i}{2\alpha}u_x, \quad F = \frac{i}{2}\left(\frac{2a + 1}{\alpha}u_x - 2u_y\right) \]

Hence as \( b = 0 \) we get the Lax representations of the M-VIII(26) as \( b = 0 \) and for the Ishimori equation (29) \( a = b = -\frac{1}{2} \). The M-IX equation (31) admit the different type exact solutions (solitons, lumps, vortex-like, dromion-like and so on)[7]. As shown in [1] eqs. (31) have the following L-equivalent counterpart

\[ iq_t + M_1 q + vq = 0 \quad (33a) \]
\[ ip_t - M_1 p - vp = 0 \quad (33b) \]
\[ M_2 v = M_1 (pq) \quad (33c) \]

where \( p = E\bar{q} \). As well known these equations are too integrable [4] and as in the previous case, equations (33) have the two integrable reductions: equations(28) as \( b = 0 \) and the Davey-Stewartson equation(30) as \( a = b = -\frac{1}{2} \).

5) The Myrzakulov-XXII equation has form[1]

\[ -iS_t = \frac{1}{2}((S, S_y) + 2iuS)_x + \frac{i}{2}V_1 S_x - 2ia^2 S_y \quad (34a) \]
\[ u_x = -\bar{S}(\bar{S}_x \wedge \bar{S}_y) \quad (34b) \]
\[ V_{1x} = \frac{1}{4a^2}(\bar{S}^2_x)_y \quad (34c) \]

The L-equivalent of these equations are given by[1]

\[ q_t = iq_{yx} - \frac{1}{2}((V_1 q)_x - qV_2 - qrq_y] \]
\[ r_t = -ir_{yx} - \frac{1}{2}((V_1 r)_x - qrr_y + rV_2] \]
\[ V_{1x} = (qr)_y \]
\[ V_{2x} = r_{yx}q - rq_{yx} \]

where \( r = E\bar{q} \). Both these set of equations are integrable and the corresponding Lax representations were presented in[1].

IV. CONCLUSION

In this paper using the C-approach[1] we have presented the L-equivalent soliton equations of the Ishimori and some Myrzakulov equations. Finally we note that using the C-approach we can see to the older problems from the new point of view. For example, the isotropic Landau-Lifshitz equation

\[ \vec{S}_t = \vec{S} \wedge \vec{S}_{xx} \]

and the NLSE

\[ iq_t + q_{xx} + 2E\bar{q}^2\bar{q} = 0 \]
are L-equivalence each other[2]. In our case $E = \pm 1$. At the same time the 1+1 dimensional M-IV and M-XXI equations have the following L-equivalents: the 1+1 dimensional mKdV

$$q_t + q_{xxx} + 6Eq^2q_x = 0$$

the 1+1 dimensional KdV

$$q_t + q_{xxx} + 6Eqq_x = 0$$

respectively.

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