**Abstract.** We prove that any reduced amalgamated free product C*-algebra is KK-equivalent to the corresponding full amalgamated free product C*-algebra. The main ingredient of its proof is Julg–Valette’s geometric construction of Fredholm modules with Connes’s view for representation theory of operator algebras.

1. Introduction

In [2][3] Cuntz gave a strategy of computing the K-theory of the reduced C*-algebra C_{red}(Γ) of a given discrete group Γ. The strategy consists of two parts:

1. proving that the canonical surjection λ: C*(Γ) → C_{red}(Γ) (where C*(Γ) denotes the full C*-algebra of Γ) gives a KK-equivalence, that is, has an inverse in KK-theory, and
2. computing the K-theory of C*(Γ).

In fact, usual computations in K-theory are made by establishing suitable exact sequences, and the full group C*-algebra C*(Γ) is easier to handle than the reduced one C_{red}(Γ). By the strategy, Cuntz indeed gave a much simpler proof of celebrated Pimsner–Voiculescu’s result of the K-theory of C_{red}(F_n) ([15]). Then Julg and Valette [11] achieved part (1) of the strategy when Γ acts on a tree with amenable stabilizers. In the direction, Pimsner gave in [13] an optimal result, but his strategy looks different from Cuntz’s one.

It is very natural (at least for us) to try to adapt Cuntz’s strategy to amalgamated free product C*-algebras. Part (2) of the strategy was achieved by Thomsen [17] under a very weak assumption. Thus, our main problem is part (1) of the strategy. It was Germain [5][6] who first tried to examine the strategy for plain free product C*-algebras, and he obtained the desired KK-equivalence result for plain free product C*-algebras of nuclear C*-algebras. Following Germain’s idea in [5][7] we recently proved in [9] (also see [8]) that the canonical surjection onto a given reduced amalgamated free product C*-algebra from the corresponding full one gives a KK-equivalence under the assumption that every free component is “strongly relative nuclear” against the amalgamated subalgebra. This was is a byproduct of our attempt to seek for a suitable formulation of “relative nuclearity” for inclusions of C*-algebras.

In this paper we adapt, unlike [5][7][8][9], Julg–Valette’s geometric idea to the problem, and establish the optimal KK-equivalence result for amalgamated free product C*-algebras. We emphasize that the core part of the proof is very simple and just 3 pages long (though this paper is rather self-contained). To state our main result precisely, let us give a few terminologies. Let \( \{ (B \subset A_i, E_{A_i}^B) \}_{i \in I} \) be a countable family of quasi-unital inclusions of separable C*-algebras with nondegenerate conditional expectations from \( A_i \) onto \( B \). Here \( B \subset A_i \) is quasi-unital if \( BA_iB \) is norm-dense in \( A_i \), and also \( E_{A_i}^B : A_i \to B \) is nondegenerate if the associated GNS representation is faithful. Let \( (A,E) = \bigstar_{i \in I} (A_i, E_{A_i}^A) \) be the reduced amalgamated free product and we call \( A \) the reduced amalgamated free product C*-algebra. Also, let \( \mathfrak{A} = \bigstar_{i \in I} A_i \) be the full amalgamated free product C*-algebra. With the notation we will prove the following:

**Key words and phrases.** amalgamated free product, KK-theory.
Theorem A. The canonical surjection $\lambda : \mathcal{A} \to A$ gives a $KK$-equivalence without any extra assumption.

The proof is done by translating the “geometric” construction of Fredholm modules due to Julg–Valette (and its quantum group analog due to Vergnioux [20]) into a C*-algebraic language following Connes’s view of correspondences. This is similar to our previous work [9] on relative nuclearity. More precisely, we will easily prove that the canonical surjection $\lambda$ gives a $KK$-subequivalence like Julg–Valette [11] and Vergnioux [20]. Then we will directly prove that $\lambda$ indeed gives a $KK$-equivalence. The latter is unnecessary in the amenable (quantum) group case [11] [20] thanks to the existence of counits, and is the most original part of the present paper. As a bonus of the present approach we obtain the following corollary:

Corollary B. Both $\mathcal{A}$ and $A$ are $K$-nuclear if all the $A_i$ are nuclear.

Throughout this paper, we employ the following standard notation: For a Hilbert space $H$, we denote by $\mathcal{B}(H)$ the bounded linear operators on $H$ and by $\mathcal{K}(H)$ the compact ones on $H$. For C*-algebras $A$ and $B$, $A \otimes B$ stands for the minimal tensor product. We use the symbol $\odot$ for algebraic tensor products. For a subset $S$ of a normed space $X$, we denote by $[S]$ the closed linear subspace of $X$ generated by $S$.

2. PRELIMINARIES

2.1. C*-correspondences. For the theory of Hilbert C*-modules, we refer to Lance’s book [12]. Let $A$ and $B$ be C*-algebras. An $A$-$B$ C*-correspondence is a pair $(X, \pi_X)$ such that $X$ is a Hilbert $B$-module and $\pi_X$ is a *-homomorphism from $A$ into the C*-algebra $L_B(X)$ of right $B$-linear adjointable operators on $X$. We denote by $\mathcal{L}_B(X)$ the C*-ideal of $L_B(X)$ generated by “rank one operators” $\theta_{\xi,\eta}$, $\xi, \eta \in X$ defined by $\theta_{\xi,\eta}(\zeta) := \xi \langle \eta, \zeta \rangle$. The identity C*-correspondence over $A$ is the pair $(A, \lambda_A)$, where $A$ is equipped with the $A$-valued inner product $\langle x, y \rangle = x^*y$ for $x, y \in A$ and $\lambda_A : A \to L_A(A)$ is defined by the left multiplication. It is known that $L_A(A)$ is naturally isomorphic to the multiplier algebra $M(A)$ of $A$.

We use the following two notions of tensor products for Hilbert C*-modules. Let $X$ be a Hilbert $B$-module and $(Y, \phi)$ be a $B$-$C$ C*-correspondence. We denote by $X \otimes \phi Y$ the interior tensor product of $X$ and $(Y, \phi)$, which is given by separation and completion of $X \otimes Y$ with respect to the $C$-valued inner product $\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \phi(\langle \xi, \xi' \rangle)\eta \eta'$. There is a canonical map $L_B(X) \to L_C(X \otimes \phi Y)$ sending $T$ to the operator $T \otimes \phi 1_Y : \xi \otimes \eta \mapsto (T\xi) \otimes \eta$. For a given *-homomorphism $\pi_X : A \to L_B(X)$ we define $\pi_X \otimes \phi 1_Y : A \to L_C(X \otimes \phi Y)$ by $(\pi_X \otimes \phi 1_Y)(a) = \pi_X(a) \otimes \phi 1_Y$. When no confusion may arise, we use the notations $X \otimes B Y$ and $\pi_X \otimes B 1_Y$ for short.

For a Hilbert $D$-module $Z$, we denote by $X \otimes Z$ the exterior tensor product of $X$ and $Y$, which is the completion of $X \otimes Y$ with respect to the $B \otimes D$-valued inner product $\langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle = \langle \xi, \xi' \rangle \otimes \langle \zeta, \zeta' \rangle$. When $(Z, \pi_Z)$ is a C-$D$ C*-correspondence, there is a natural *-homomorphism $\pi_X \otimes \pi_Z : A \otimes C \to L_{B \otimes D}(X \otimes Z)$ so that $(X \otimes Z, \pi_X \otimes \pi_Z)$ is an $A \otimes C$-$B \otimes D$ C*-correspondence.

Let $B \subset A$ be a quasi-unital inclusion of C*-algebras (i.e., $B A B$ is norm-dense in $A$) with conditional expectation $E : A \to B$. We denote by $L^2(A, E)$ the Hilbert $B$-module given by separation and completion of $A$ with respect to the $B$-valued inner product $\langle x, y \rangle = E(x^*y)$ for $x, y \in A$, and by $\pi_E : A \to L_B(L^2(A, E))$ the *-homomorphism induced from the left multiplication. The conditional expectation $E$ is said to be nondegenerate if $\pi_E$ is faithful. We denote by $1_A$ the unit of the multiplier algebra of $A$. Since the inclusion $B \subset A$ is quasi-unital, $B$ contains an approximate unit for $A$. In particular, $A$ is unital if and only if so is $B$, and they should have a common unit. Thus, we can uniquely extend $E$ to a conditional expectation $\tilde{E} : A + C1_A \to B + C1_A$. Let $\xi_E$ be the vector in $L^2(A + C1_A, \tilde{E})$ corresponding to $1_A$. We always identify $L^2(A, E)$ with $[\pi_E(A)\xi_E] \subset L^2(A + C1_A, \tilde{E})$ and call the triple $(L^2(A, E), \pi_E, \xi_E)$ the GNS-representation associated with the conditional expectation $E$. Notice that $\xi_E$ need not to be in $L^2(A, E)$ when $A$ is non-unital.
2.2. **KK-theory.** Throughout this subsection, all $C^*$-algebras are assumed to be separable for simplicity. We refer the reader to [1] for $KK$-theory.

**Definition 2.1.** For (trivially graded) $C^*$-algebras $A$ and $B$, a Kasparov $A$-$B$ bimodule is a triple $(X, \phi, F)$ such that $X$ is a countably generated graded Hilbert $B$-module, $\phi: A \to L_B(X)$ is a *-homomorphism of degree 0, and $F \in L_B(X)$ is of degree 1 and satisfies the following condition:

- $[F, \phi(a)] \in K_B(X)$ for $a \in A$,
- $(F - F^*)\phi(a) \in K_B(X)$ for $a \in A$,
- $(1 - F^2)\phi(a) \in K_B(X)$ for $a \in A$.

When $[F, \phi(a)] = (F - F^*)\phi(a) = (1 - F^2)\phi(a) = 0$ holds for every $a \in A$, we say that $(X, \phi, F)$ is degenerate. We denote by $E(A, B)$ and $D(A, B)$ the corrections of Kasparov $A$-$B$ bimodules and degenerate ones, respectively.

We say that two Kasparov $A$-$B$ bimodules $(X, \phi, F)$ and $(Y, \psi, G)$ are unitarily equivalent, denoted by $(X, \phi, F) \cong (Y, \psi, G)$, if there exists a unitary $U \in L(X, Y)$ of degree 0 such that $\psi = \text{Ad} U \circ \phi$ and $G = UFU^*$.

For any Hilbert $B$-module $X$, we set $IX := C([0, 1]) \hat{\otimes} X$. In particular, we set $IB = C([0, 1]) \hat{\otimes} B$. For each $t \in [0, 1]$ we still denote by $t$ the surjective *-homomorphism $IB \cong C([0, 1], B) \ni f \mapsto f(t) \in B$. Note that we have a natural isomorphism $IX \otimes_t B \cong X$ for every $t \in [0, 1]$.

**Definition 2.2.** Two Kasparov $A$-$B$ bimodules $(X_0, \phi_0, F_0)$ and $(X_1, \phi_1, F_1)$ are said to be homotopic if there exists a Kasparov $A$-$IB$ bimodule $(Y, \psi, G)$ such that $(Y \otimes_1 B, \psi \otimes_1 1_B, G \otimes_1 1_B) \cong (X_t, \phi_t, F_t)$ for $t = 0, 1$. The $KK$-group $KK(A, B)$ is the set of homotopy equivalence classes of all Kasparov $A$-$B$ bimodules.

The next technical lemma will be used later.

**Lemma 2.3.** Let $P, Q$ and $R$ be separable $C^*$-algebras and let $(X, \psi_1, F) \in E(Q, R)$ be given for $i = 0, 1$. Suppose that there exist a surjective *-homomorphism $\pi: P \to Q$ and a family of Kasparov $P$-$R$ bimodules $(X, \phi_t, F)$ for $t \in [0, 1]$ satisfying

(i) the mapping $[0, 1] \ni t \mapsto \phi_t(a)$ is strictly continuous for each $a \in P$;
(ii) $\phi_t$ factors through $\pi: P \to Q$ for every $t \in [0, 1]$;
(iii) $\phi_0 = \psi_1 \circ \pi$ holds for $i = 0, 1$.

Then, $(X, \psi_0, F)$ and $(X, \psi_1, F)$ are homotopic.

**Proof.** By assumption, there exists a *-homomorphism $\phi: P \to L_{IR}(IX)$ such that $(IX, \phi, F \otimes 1_{C([0,1])}) \in E(P, IR)$ and $\phi \otimes 1_R = \phi_t$ for $t \in [0, 1]$. Since one has $\|\phi(a)\| = \sup_{0 \leq t \leq 1}\|\phi_t(a)\| \leq \|\pi(a)\|$ for $a \in P$, there exists $\psi: Q \to L_{IR}(IX)$ such that $\phi = \psi \circ \pi$. We then have $(IX, \psi, F \otimes 1_{C([0,1])}) \in E(Q, IR)$ and the evaluations of this Kasparov bimodule at endpoints are exactly $(X, \psi_1, F)$, $i = 0, 1$. \qed

The $KK$-group becomes an additive group in the following way: For $\alpha, \beta \in KK(A, B)$ implemented by $(X, \phi, F), (Y, \psi, G)$, respectively, $\alpha + \beta$ is the element implemented by $(X + Y, \phi + \psi, F \oplus G)$. All degenerate Kasparov bimodules are homotopic to the trivial bimodule $0 = (0, 0, 0)$ and define the zero element in $KK(A, B)$. Let $X_0$ and $X_1$ be the even and odd parts of $X$ so that $X = X_0 \oplus X_1$ and let $-X$ be the graded Hilbert $B$-module with the even part $X_1$ and the odd part $X_0$. The inverse of $\alpha$ is implemented by $(-X, \text{Ad} U \circ \phi, UFU^*)$, where $U: X \to -X$ is the natural unitary.

For any *-homomorphism $\phi: A \to B$, we have $(B \oplus 0, \phi \oplus 0, 0) \in E(A, B)$ and still denote by $\phi$ the corresponding element in $KK(A, B)$.

For $\alpha \in KK(A, B)$ and $\gamma \in KK(B, C)$, the Kasparov product of $\alpha$ and $\gamma$ is denoted by $\gamma \circ \alpha$ (or $\alpha \otimes_B \gamma$). When one of $\alpha$ and $\beta$ comes from a *-homomorphism, the construction of the Kasparov product is very simple (and we will use Kasparov products only in these special cases). Indeed, if $\gamma$ comes from a *-homomorphism $\gamma: B \to C$ with $[\gamma(B)C] = C$ and $\alpha$ is implemented

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by \((X, \phi, F)\), then the Kasparov product \(\gamma \circ \alpha\) is implemented by \((X \otimes_x C, \phi \otimes \gamma 1_C, F \otimes \gamma 1_C)\). Similarly, when \(\alpha\) is a \(\ast\)-homomorphism from \(A\) into \(B\) and \(\gamma\) is implemented by \((Y, \psi, G)\) with \([\psi(B)Y] = Y\), the Kasparov product \(\gamma \circ \alpha\) is implemented by \((Y, \psi \circ \alpha, G)\).

**Definition 2.4.** An element \(\alpha \in KK(A, B)\) is said to be a **KK-equivalence** if there exists \(\beta \in KK(B, A)\) such that \(id_A = \beta \circ \alpha\) and \(id_B = \alpha \circ \beta\). In this case, \(A\) and \(B\) are said to be **KK-equivalent**.

Note that KK-equivalence between \(A\) and \(B\) implies \(KK(A, C) \cong KK(B, C)\) and \(KK(C, A) \cong KK(C, B)\) for any separable C*-algebra \(C\).

Finally, we recall the notion of \(K\)-nuclearity in the sense of Skandalis [16].

**Theorem 2.5 ([16 Theorem 1.5]).** Let \(A\) and \(B\) be separable C*-algebras and let \(\pi : A \rightarrow \mathbb{B}(H)\) be a faithful and essential representation on a separable Hilbert space \(H\). For a given \(A\)-\(B\) C*-correspondence \((X, \sigma)\) with \(X\) countably generated, the following are equivalent:

(i) For any unit vector \(\xi \in X\) the c.c.p. (completely contractive positive) map \(A \ni a \mapsto (\xi, \sigma(a)\xi) \in B\) is nuclear.

(ii) For any \(x \in \mathbb{K}_B(X)\) of norm 1, the c.c.p. map \(A \ni a \mapsto x^*\sigma(a)x \in \mathbb{K}_B(X)\) is nuclear.

(iii) There exists a sequence of isometries \(V_n \in L_B(X, H \otimes B)\) such that \(\sigma(a) - V_n^* (\pi(a) \otimes 1_A) V_n \rightarrow 0\) for all \(a \in A\).

When any of these three conditions holds, we say that \((X, \sigma)\) is nuclear.

Note that any C*-correspondence of the form \((X \otimes_B Y, \pi_X \otimes_B 1_Y)\) is nuclear whenever \(B\) is nuclear (see e.g. [9] Remark 2.11).

**Definition 2.6.** A separable C*-algebra \(A\) is said to be \(K\)-nuclear if \(id_A\) in \(KK(A, A)\) is implemented by a Kasparov bimodule \((X, \phi, F)\) such that \((X, \phi)\) is nuclear.

2.3. **Amalgamated free products.** Let \(\{B \subset A_i\}_{i \in \mathcal{I}}\) be a family of inclusions of C*-algebras. Recall that the **full amalgamated free product** of \(\{A_i\}_{i \in \mathcal{I}}\) over \(B\) is a C*-algebra \(\mathfrak{A}\) generated by the images of injective \(\ast\)-homomorphisms \(f_i : A_i \rightarrow \mathfrak{A}\) such that \(f_i|_B = f_j|_B\) for \(i, j \in \mathcal{I}\) and satisfying the following universal property: for any C*-algebra \(C\) and \(\ast\)-homomorphisms \(\pi_i : A_i \rightarrow C\) satisfying \(\pi_i|_B = \pi_j|_B\) for \(i, j \in \mathcal{I}\), there exists a unique \(\ast\)-homomorphism \(\bigstar_{i \in \mathcal{I}} \pi_i : \mathfrak{A} \rightarrow C\) such that \((\bigstar_{i \in \mathcal{I}} \pi_i) \circ f_i = \pi_i\) for \(i \in \mathcal{I}\). Since the full amalgamated free product is unique up to isomorphism, we denote it by \(\bigstar_{B, i \in \mathcal{I}} A_i\). We identify \(A_i\) with \(f_i(A_i)\) so that \(A_i \subset \bigstar_{B, i \in \mathcal{I}} A_i\) for every \(i \in \mathcal{I}\).

Further assume that, the inclusion \(B \subset A_i\) is quasi-unital and there exists a nondegenerate conditional expectation \(E_{B|A_i} : A_i \rightarrow B\) for each \(i \in \mathcal{I}\). In [21], Voiculescu introduced reduced amalgamated free products of unital inclusions of C*-algebras with conditional expectations. To reduce Theorem [A] to the case when \(\mathcal{I} = \{1, 2\}\), we need to extend Voiculescu’s definition to quasi-unital inclusions. For any \(m \in \mathbb{N}\), set \(\mathcal{I}_m := \{i : \{1, \ldots, m\} \rightarrow \mathcal{I} | i(k) \neq i(k + 1)\) for \(1 \leq k \leq m - 1\). Recall that the **reduced amalgamated free product** of \(\{(A_i, E_{B|A_i}^A)\}_{i \in \mathcal{I}}\) is a pair \((A, E)\) such that

- \(A\) is a C*-algebra generated by the images of injective \(\ast\)-homomorphisms \(g_i : A_i \rightarrow A\) such that \(g_i|_B = g_j|_B\) for \(i, j \in \mathcal{I}\);
- \(E\) is a nondegenerate conditional expectation from \(A\) onto \(g_i(B)\) (independent of \(i\));
- one has \(E(g_{i(1)}(x_1)g_{i(2)}(x_2) \cdots g_{i(m)}(x_m)) = 0\) for any \(m \geq 1, i \in \mathcal{I}_m, \) and \(x_k \in ker E_{B|A_i}^{A_i}\) for \(1 \leq k \leq m\).

We will also identify \(A_i\) with \(g_i(A_i)\) for every \(i \in \mathcal{I}\). Since the pair \((A, E)\) is determined by the above three conditions, we will write \(\bigstar_{B, i \in \mathcal{I}} (A_i, E_{B|A_i}^A) := (A, E)\). Clearly, we have a canonical surjection \(\lambda : \mathfrak{A} \rightarrow A\) satisfying that \(\lambda \circ f_i = g_i\) for every \(i \in \mathcal{I}\).

We recall the construction of \((A, E)\) ([21]). Let \((X_i, \pi_{X_i}, \xi_i)\) be the GNS-representation associated with \(E_{B|A_i}^A\) (see §2) and set \(A_i^\perp := ker E_{B|A_i}^A\) and \(X_i^\perp = X_i \ominus \xi_i B = [\pi_{X_i}(A_i^\perp) \xi_i]\) for
Proof. From the fact that $E_1^i$ is associated with $E_1$, let $(X, \xi_1)$ be the GNS-representation associated with $\Gamma$. We may assume that $E_1$ is the GNS-representation associated with $E_1$, and hence $E_1$ must be nondegenerate.

The following lemma is probably well-known, but we give its proof for the reader’s convenience.

Lemma 2.7. Let $\mathfrak{A} = \bigstar_{i \in I} A_i$ and $(A, E) = \bigstar_{i \in I} (A_i, E_i^A)$. Suppose that for each $i \in I$ there exists a subset $S_i \subseteq A_i^\circ$ generating $A_i^\circ$ as a normed space, and there exists a cyclic subspace $\Gamma \subset Z$ (i.e., $[\pi_Z(\mathfrak{A}) \Gamma C] = Z$ holds) which satisfies the freeness condition: for any $m \in \mathbb{N}$, $\xi, \eta \in \Gamma$, one has $\langle \xi, \pi_Z(x_1 x_2 \ldots x_m) \eta \rangle = 0$. Then, $\pi_Z$ factors through $\lambda : \mathfrak{A} \to A$.

Proof. We will show that ker $\lambda \subset \ker \pi_Z$. Choose and fix $z \in \ker \lambda$ arbitrarily. By assumption, it suffices to show that $\langle \xi, \pi_Z(xzy) \eta \rangle = 0$ for all $x, y, \xi, \eta \in \Gamma$. We may assume that $x$ and $y$ are in $\ast$-alg$(\bigcup_{i \in I} A_i)$ such that $\lim_{n \to \infty} \| z - z_n \| = 0$. For each $n \geq 1$ there exists $b_n \in B$ such that $x_n y - b_n$ is a sum of some elements of the form $x_1 \cdots x_m$ for some $m \geq 1$, $x, y \in \Gamma$, and $x_1, \ldots, x_m \in \bigcup_{i \in I} A_i^\circ$ (for $1 \leq k \leq m$ so that $b_n = E(x_n y)$. The assumption on $\Gamma$ implies that $\langle \xi, \pi_Z(x_n y - b_n) \eta \rangle = 0$, and hence we have $\| \langle \xi, \pi_Z(xzy) \eta \rangle \| = \lim_{n \to \infty} \| \langle \xi, \pi_Z(b_n) \eta \rangle \| \leq \limsup_{n \to \infty} \| \langle \xi, \pi_Z(b_n) \eta \rangle \| = 0$.

3. Proof

3.1. Case of two free components. We first deal with the case when $I = \{1, 2\}$. Let $(A, E) = (A_1, E_1^A) * (A_2, E_2^A)$, $\mathfrak{A} = A_1 * A_2$ and $\lambda : \mathfrak{A} \to A$ be as in Theorem A. As in the previous section, let $(X, \pi_X, \xi_0)$ be the GNS-representation associated with $E$ and identify $X_i := L^2(A_i, E_i^A)$ with $\xi_0 B \otimes X_i^\circ$ for $i = 1, 2$. Let $E_{A_i} : A \to A_i$ be the canonical conditional expectation given by the compression map to $X_i$ and let $(Y_i, \pi_Y, \eta_i)$ be the GNS-representation associated with $E_{A_i}$ for $i = 1, 2$. Note that any vector of the form $\xi_0 \otimes a$ with $a \in A_i$ sits in $X(r, i) \otimes B A_i$ for each $i \in I$ thanks to the assumption that $B \subset A_i$ is quasi-unital. The following lemma can be shown in the same manner as [21] Lemma 3.1], but we give its proof for the reader’s convenience.

Lemma 3.1. The exists a unitary $S_i : X(r, i) \otimes B A_i \to Y_i$ satisfying that $S_i(\xi_0 \otimes y) = \eta_i y$ and $S_i(x_1 \cdots x_m \xi_0 \otimes y) = x_1 \cdots x_m \eta_i y$ for all $y \in A_i$ and $m \in \mathbb{N}$, $i \in I_m$ with $\lambda (m) \neq i$, and $x_k \in A_i^\circ$ for $1 \leq k \leq m$.

Proof. Note that if $S_i$ is bounded, then it must be surjective. Thus, it suffices to show that $S_i$ is an isometry. By the polarization trick, we only have to verify that $E_{A_i} (x \ast x) = E(\pi_Y x)$ for all $x = x_1 \cdots x_m$ with $m \in \mathbb{N}$, $i \in I_m$, $\lambda(m) \neq i$ and $x_k \in A_i^\circ$ for $1 \leq k \leq m$. When $m = 1$, this follows from the fact that $E_{A_i}$ is given by the compression to $X_i$. Assume that we have shown for $k = 1, \ldots, m$. For $i \in I_{m+1}$ with $\lambda(m+1) \neq i$, take $x_k \in A_i^\circ$. If we put $y = x_2 \cdots x_{m+1}$
and \( b = E(x_1^* x_1) \), then the induction hypothesis implies that \( E_A(y^* b y) = E(y^* b y) \). Thus, we have \( E_A(x^* x) = E_A(y^* E(x_1^* x_1) y) + E_A(y^* (x_1^* x_1 - E(x_1^* x_1)) y) = E_A(y^* b y) = E(y^* b y) = E(y^* E(x_1^* x_1) y) + E(y^* (x_1^* x_1 - E(x_1^* x_1)) y) = E(x^* x) \). Hence, we are done. \( \square \)

Consider two \( A \mathfrak{A} \) \( C^* \)-correspondences \((Z^+, \pi^+) := (X \otimes_B \mathfrak{A}, \pi_X \otimes_B 1)\) and \((Z^-, \pi^-) := \bigoplus_{i=1}^2 (Y_i \otimes_A \mathfrak{A}, \pi_Y \otimes_A 1)\). Notice that the vector \( \zeta_i := \eta_i \otimes 1_{\mathfrak{A}} \) is not necessarily in \( Z^- \), but one has \( \zeta_i \mathfrak{A} \subset Z^- \). Define the isometry \( S : Z^+ \to Z^- \) by

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\begin{align*}
S_1 \otimes A_1 & : X(r, 1) \otimes_B \mathfrak{A} \to Y_1 \otimes_A \mathfrak{A}; \\
S_2 \otimes A_2 & : X(r, 2) \otimes_B \mathfrak{A} \to Y_2 \otimes_A \mathfrak{A}.
\end{align*}
\]

**Lemma 3.2** (c.f. [20] Theorem 3.3 (2))). \( \text{The operator } S \text{ satisfies that } \ker S^* = \zeta_i \mathfrak{A} \text{ and } \pi^-(a) S - S \pi^+(a) \text{ is compact for all } a \in A. \text{ Consequently, the triple } (Z^+ \oplus Z^-, \pi^+ \oplus \pi^-, \left[ \begin{smallmatrix} 0 & S^* \\ S & 0 \end{smallmatrix} \right] ) \text{ is an } A \mathfrak{A} \text{ Kasparov bimodule.} \)

**Proof.** The first assertion is obvious. Thus, it suffices to show \( \pi^-(x) S - S \pi^+(x) \) is compact for all \( x \in A_1 \cup A_2 \). In fact, since each \( x \in A_1 \) enjoys \( x X(r, 1) \subset X(r, 1) \) and \( x X(r, 2) \subset X(r, 2) \), one has \( \pi^-(x) S = S \pi^+(x) \) for \( x \in A_1 \). If we define \( S^* : Z^+ \to Z^- \) by \( S^* \xi_0 \otimes a = \zeta_1 Z \) for \( a \in \mathfrak{A} \) and by \( S \) on \( X^+ \otimes B \mathfrak{A} \), then \( S^* \) intertwines the actions of \( A_2 \) by the above argument. Since \( S \) is a compact perturbation of \( S^* \), we are done. \( \square \)

**Remark 3.3.** Recall that the Bass–Serre tree associated with an amalgamated free product group \( G = G_1 \ast_H G_2 \) is a graph whose vertex and edge sets are given by \( \Delta^0 = G_1 \cup G \cap G_2 \) and \( \Delta^1 = G/H \), respectively. Consider the unitary representations of \( G \) on \( \ell^2(\Delta^0) \) and \( \ell^2(\Delta^1) \) induced from the action of \( G \) on \( (\Delta^0, \Delta^1) \). In [9], we saw that \( C^* \)-correspondences play a role of unitary representations for groups. In our theory, the canonical representation \( G \) on \( \ell^2(G/H) \) corresponds to \( (L^2(A, E) \otimes_B \mathfrak{A}, \pi_E \circ \lambda \otimes_B 1) \) (c.f. [11]). Thus, the \( C^* \)-correspondences \((Z^+, \pi^+ \circ \lambda)\) and \((Z^-, \pi^- \circ \lambda)\) should play a role of the Bass–Serre tree in \( C^* \)-algebra theory. Also, the adjoint of \( S \) corresponds to the co-isometry of Julg–Valette in [11].

Here is the main technical result of this paper.

**Theorem 3.4.** With the notation above, let \( \alpha \) be the element in \( KK(A, \mathfrak{A}) \) implemented by \((Z^+ \oplus Z^-, \pi^+ \oplus \pi^-, \left[ \begin{smallmatrix} 0 & S^* \\ S & 0 \end{smallmatrix} \right] ) \). Then, we have \( \alpha \circ \lambda + id_{\mathfrak{A}} = 0 \) and \( \lambda \circ \alpha + id_A = 0 \).

**Proof.** We first prove that \( \alpha \circ \lambda + id_{\mathfrak{A}} = 0 \) following the proof of [20] Theorem 3.3 (3)]. Set \( \rho^+ := \pi^+ \circ \lambda \) and \( \rho^- := \pi^- \circ \lambda \). Define the unitary \( U : Z^+ \oplus \mathfrak{A} \to Z^- \) by \( S \) on \( Z^+ \) and by \( U(0 \oplus a) := \zeta_1 a \) for \( a \in \mathfrak{A} \). Since \( S \) is a compact perturbation of \( U \), \( \alpha \circ \lambda + id_{\mathfrak{A}} \) is implemented by \( ((Z^+ \oplus \mathfrak{A}) \oplus Z^-) = (\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A}) \) (see [22]). Take a norm continuous path \( (v_t)_{0 \leq t \leq 1} \) of unitaries in \( M_2(\mathbb{C}) \) such that \( v_0 = 1 \) and \( v_1 = \left[ \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right] \). With the natural identification \( M_2(\mathbb{C}) \otimes M(\mathfrak{A}) = M_2(\mathfrak{A} \otimes \mathfrak{A}) \), we define the unitary \( u_t \in L_\mathfrak{A}(Z^-) \) by \( v_t \) on \( \zeta_1 \mathfrak{A} \otimes \zeta_2 \mathfrak{A} \) and by the identity operator on \( Z^- \ominus (\zeta_1 \mathfrak{A} \oplus \zeta_2 \mathfrak{A}) \). Since the restriction of \( B \) to \( \zeta_1 \mathfrak{A} \otimes \zeta_2 \mathfrak{A} \) is just \( \mathfrak{A} \otimes \mathfrak{A} \) with the above identification, the family \( (u_t)_{0 \leq t \leq 1} \) forms a norm continuous path of unitaries in \( \pi^-(B)^\prime \cap (\mathfrak{A} \otimes \mathfrak{A}) \) satisfying that \( u_0 = 1 \) and \( u_1 \) switches \( \zeta_1 \mathfrak{A} \) and \( \zeta_2 \mathfrak{A} \) for each \( a \in \mathfrak{A} \). Let \( i : A_1 \to A \) be the inclusion map for \( i = 1, 2 \). Since \( \pi^- \circ i_1 \) agrees with \( Ad u_t \circ \pi^- \circ i_2 \) on \( B \), we have the natural *-homomorphism \( \phi_t := (\pi^- \circ i_1) \ast (Ad u_t \circ \pi^- \circ i_2) : \mathfrak{A} \to L_\mathfrak{A}(Z^-) \) thanks to the universality of \( \mathfrak{A} \). Then, the Kasparov bimodules

\[
((Z^+ \oplus \mathfrak{A}) \oplus Z^-) = (\rho^+ \oplus \lambda \mathfrak{A}) \oplus \phi_t, \left[ \begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix} \right], \quad t \in [0, 1]
\]

satisfy conditions (i) and (ii) in Lemma 23 (with \( P = Q = \mathfrak{A} \)), and its evaluation at \( t = 0 \) implements \( \alpha \circ \lambda + id_{\mathfrak{A}} \). Thus, we need to show that \( ((Z^+ \oplus \mathfrak{A}) \oplus Z^-) = (\rho^+ \oplus \lambda \mathfrak{A}) \oplus \phi_t, \left[ \begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix} \right] \) is degenerate, that is,

\[
U(\rho^+(x) \oplus \lambda \mathfrak{A}(x)) = \phi_t(x) U \quad \text{for } x \in \mathfrak{A}.
\]

Since \( U \) is unitary, we may assume that \( x \) is in \( A_1 \cup A_2^2 \). When \( x \) is in \( A_1 \), the above equation is trivial because \( S \) intertwines \( \pi^+(x) \) and \( \pi^-(x) \). Let \( S' \) be as in the proof of the previous lemma.
Then, we have $u_1^* U = S^0$ on $Z^+$ and $u_1^* U (0 \oplus a) = \zeta_a$ for $a \in \mathfrak{A}$. Since $S^0$ intertwines the actions of $A_2$, we have $U (\pi^+ (x) \oplus \lambda_A (x)) = u_1 \pi^+ (x) u_1^* U$ for every $x \in A_2$. Thus we obtain equation (11), and hence Lemma 2.3 shows $\alpha \circ \lambda + i d_\mathfrak{A} = 0$.

We next prove that $\lambda \circ \alpha + i d_A = 0$ in $KK(A,A)$. Note that $\lambda \circ \alpha + i d_A$ is implemented by the Kasparov $A \to B$ bimodule

$$
\left( \left((Z^+ \otimes \lambda \ A) \oplus A \right) \oplus (Z^- \otimes \lambda \ A), ((\pi^+ \otimes \lambda \ A) \oplus \lambda_A) \oplus (\pi^- \otimes \lambda \ A), \left[ \begin{array}{cc} 0 & U \otimes_{\lambda A} U \otimes 1_A \\ U^* \otimes_{\lambda A} U \otimes 1_A & 0 \end{array} \right] \right)
$$

(see §2.2). We observe that the family of Kasparov $\mathfrak{A} \to B$ bimodules

$$
\left( \left((Z^+ \otimes \lambda \ A) \oplus A \right) \oplus (Z^- \otimes \lambda \ A), ((\rho^+ \otimes \lambda \ A) \oplus \lambda) \oplus (\phi_t \otimes \lambda A), \left[ \begin{array}{cc} 0 & U \otimes_{\lambda A} U \otimes 1_A \\ U^* \otimes_{\lambda A} U \otimes 1_A & 0 \end{array} \right] \right), \ t \in [0,1]
$$

satisfy conditions (i) and (ii) in Lemma 2.3 (with $P = \mathfrak{A}$ and $Q = A$) and its evaluations at endpoints implement $(\lambda \circ \alpha + i d_A) \circ \lambda$ and $0$. Thus, by Lemma 2.3 and the fact that $\pi_X : A \to L_B (X)$ is faithful, it suffices to show that $\phi_t \otimes_{\pi_X \otimes \lambda} 1_X : \mathfrak{A} \to L_B (Z^- \otimes_{\pi_X \otimes \lambda} X)$ factors through $\lambda : \mathfrak{A} \to A$ for every $t \in [0,1]$. If we set $\sigma := \bigoplus_{t=1}^2 \pi_Y \otimes \lambda, 1_X : A \to L_B ((Y \otimes \lambda A, X) \oplus (Y_2 \otimes A_2 X))$ and $w_t := u_t \otimes_{\pi_X \otimes \lambda} 1_X \in L_B ((Y \otimes \lambda A, X) \oplus (Y_2 \otimes A_2 X))$, then $\phi_t \otimes_{\pi_X \otimes \lambda} 1_X$ coincides with $\psi_t := (\sigma \circ \lambda) \ast (\Ad u_t) \circ \sigma \circ i_2$. Note that $\psi_0 = \sigma \circ \lambda$ and $\psi_1 \equiv (\rho^+ \otimes A_1 X) \oplus \pi_X \circ \lambda$ apparently factor through $\lambda$. Thus, we only have to deal with $0 < t < 1$ and we write $w = w_t$ for short.

For the convenience, we identify $X(r, i) \otimes_B X$ with $Y_i \otimes A_i X$ via $S_i \otimes A_i 1$ as right $B$-modules. To distinguish between vectors in $X(r, 1) \otimes_B X$ and $X(r, 2) \otimes_B X$, we use the symbols $\otimes$ and $\otimes$ as markers in such a way that, for $\zeta \in X$ we denote by $\xi_0 \otimes \zeta \in X(r, 1) \otimes_B X$ and $\xi_0 \otimes \zeta \in X(r, 2) \otimes_B X$ the vectors corresponding to $\eta_1 \otimes \zeta$ and $\eta_2 \otimes \zeta$, respectively. Thanks to Lemma 2.7 the proof will be completed by proving the following claim:

**Claim 3.5.** The subspace $\Gamma := w (\xi_0 \otimes_B X (r, 1)) + \xi_0 \otimes_B X (r, 2)$ satisfies the assumption of Lemma 2.7.

We first show that $\Gamma$ is cyclic for $\psi_t (\mathfrak{A})$. Let $\Lambda := [\psi_t (\mathfrak{A})]\Gamma$. We set $\xi_0 := \xi_0 \otimes_B X (r, 1) \otimes_B X (r, 2), \xi_0 := \bigoplus_{k=0}^m \xi_0 (r, 2) \otimes_B X (m, k) \oplus \bigoplus_{k=0}^m \xi_0 (r, 2) \otimes_B X (m, k)$. We will show this by induction. When $m = 0$, this is trivial because $\xi_0 (r, 2) = \xi_0 \otimes_B X (r, 2) = w (\xi_0 \otimes_B X (r, 2)) + \xi_0 \otimes_B X (r, 2)$. Suppose that $\Lambda$ contains $\zeta_k (r, 2)$ for $0 \leq k \leq m$. Since $w$ is equal to $1$ on the complement of $\xi_0 \otimes_B X (r, 2)$, it is easily seen that

$$
\left( \bigoplus_{k=0}^m \xi_0 (r, 2) \otimes_B X (m+1-k) \right) \oplus \left( \bigoplus_{k=0}^m \xi_0 (r, 2) \otimes_B X (m+1-k) \right) \subset \Lambda.
$$

Thus, we only have to check that $\Lambda$ contains the following six subspaces:

$$
X_2 \otimes_B \xi_0 \otimes_B X (r, 1), \quad \xi_0 \otimes_B X (r, 1), \quad \xi_1 \otimes_B X (r, 1), \quad \xi_0 \otimes_B X (r, 2), \quad \xi_1 \otimes_B X (r, 2).
$$

By assumption of induction, one has $w (\xi_0 \otimes_B X (r, 1)) \subset \xi_0 \otimes_B X (r, 1)$, and hence $X_2 \otimes_B \xi_0 \otimes_B X (r, 1) \subset \xi_0 \otimes_B X (r, 1)$. We also have $X_1 \otimes_B X (r, 1) = [\sigma (A_1) \xi_0 \otimes_B X (r, 1)] \subset \Lambda$. We observe that $w (\xi_0 \otimes_B X (r, 1)) \subset \xi_0 \otimes_B X (r, 1)$, and $w (\xi_0 \otimes_B X (r, 2)) \subset \Gamma$ by the definition of $\Gamma$. Thus, one has

$$
\xi_0 \otimes_B X (r, 1) \subset \xi_0 \otimes_B X (r, 1) \subset \Lambda.
$$

Finally we obtain that $\xi_0 \otimes_B X (r, 2) \subset \xi_0 \otimes_B X (r, 2) \subset \Lambda$ by the definition of $\Gamma$ again. Therefore, by induction, it follows that $\Gamma$ is cyclic for $\psi_t (\mathfrak{A})$. 

We next show that $\Gamma = w(\xi_0 B \hat{\otimes}_B X(\ell, 1)) + \xi_0 B \hat{\otimes}_B X(\ell, 2)$ satisfies the freeness condition. Let $\Gamma_1 := \xi_0 B \hat{\otimes}_B X(\ell, 2)^\circ$ and $\Gamma_2 := w(\xi_0 B \hat{\otimes}_B X(\ell, 1)^\circ)$. We then claim that the following inclusions hold:

$$\sigma(A_1^2)^\star \Gamma \subset \Gamma_1 + X_1^\perp \hat{\otimes}_B X \quad \text{and} \quad \sigma(A_2^2)^w \Gamma \subset \Gamma_2 + X_2^\perp \hat{\otimes}_B X.$$  \hfill (2)

Indeed, for any $x \in A_1^2$ one has

$$\sigma(x)w(\xi_0 B \hat{\otimes}_B X(\ell, 1)) \subset \sigma(x)(\xi_0 B \hat{\otimes}_B X(\ell, 1) + \xi_0 B \hat{\otimes}_B X(\ell, 1))$$

$$\subset \xi_0 B \hat{\otimes}_B X(\ell, 2)^\circ + X_1^\perp \hat{\otimes}_B X(\ell, 1)$$

$$\subset \Gamma_1 + X_1^\perp \hat{\otimes}_B X$$

and

$$\sigma(x)(\xi_0 B \hat{\otimes}_B X(\ell, 2)) \subset X_1^\perp \hat{\otimes}_B X(\ell, 2).$$

Similarly, for any $y \in A_2^2$ one has

$$\sigma(y)^w w(\xi_0 B \hat{\otimes}_B X(\ell, 1)) \subset \sigma(y)(\xi_0 B \hat{\otimes}_B X(\ell, 2) + \xi_0 B \hat{\otimes}_B X(\ell, 2))$$

$$\subset w(X_2^\perp \hat{\otimes}_B X(\ell, 2)) + w(\xi_0 B \hat{\otimes}_B X(\ell, 1)^\circ)$$

$$= X_2^\perp \hat{\otimes}_B X(\ell, 2) + \Gamma_2.$$  \hfill ($\blacksquare$)

The subspaces on the right hand side in both equations (2) are apparently orthogonal to $\Gamma$, and one can easily verify that $\sigma(A_1^2)^\star \Gamma_2 \subset \Gamma_1 + X_1^\perp \hat{\otimes}_B X$ and $\sigma(A_2^2)^w \Gamma_1 \subset \Gamma_2 + X_2^\perp \hat{\otimes}_B X$. Since $w = 1$ on the complement of $(\xi_0 B \hat{\otimes}_B X) \oplus (\xi_0 B \hat{\otimes}_B X)$, the above observations show that for any $x_k \in A_{i(k)}$ for $k = 1, \ldots, m$ with $i \in I_m$, the subspace $\psi(t(x_1 \cdots x_m))\Gamma$ is contained in $\Gamma_1 + X(\ell, 2)^\circ \hat{\otimes}_B X + X(\ell, 2)^\circ \hat{\otimes}_B X$ when $i(1) = 1$, and in $\Gamma_2 + X(\ell, 1)^\circ \hat{\otimes}_B X + X(\ell, 1)^\circ \hat{\otimes}_B X$ when $i(1) = 2$. This implies that $\Gamma$ satisfies the freeness condition.

3.2. Case of countably many free components. Let $\mathcal{I}$ be a general countable set and let $\mathfrak{A} := \mathbf{A}_{B_{i \in \mathcal{I}}} A_i$ and $(A, E) := \mathbf{A}_{B_{i \in \mathcal{I}}} (A_i, E^A_{B_i})$ be as in Theorem 3.1. We set $c_0 := c_0(\mathcal{I})$ and $\mathcal{K} := \mathcal{K}(\ell(\mathcal{I}))$.

**Proposition 3.6.** With the notation above, there exist nondegenerate conditional expectations $\sum_i E_{A_i}^A : \sum_i A_i \rightarrow c_0 \hat{\otimes} B$ and $E_{c_0} \hat{\otimes} id_B : \mathcal{K} \hat{\otimes} B \rightarrow c_0 \hat{\otimes} B$. If we set $\mathfrak{A} := (\sum_i A_i) \ast c_0 \hat{\otimes} B (K \hat{\otimes} B)$ and $(A, E) := (\sum_i A_i, \sum_i E_{A_i}^A) \ast c_0 \hat{\otimes} B (K \hat{\otimes} B, E_{c_0} \hat{\otimes} id_B)$, then there exist isomorphisms $\pi : \mathfrak{A} \rightarrow \mathcal{K} \hat{\otimes} \mathfrak{A}$ and $\pi_{\text{red}} : A \rightarrow \mathcal{K} \hat{\otimes} A$ such that the following diagram

$$\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\pi} & \mathcal{K} \hat{\otimes} \mathfrak{A} \\
\lambda \downarrow & & \downarrow \text{id}_{\mathcal{K} \hat{\otimes} \mathfrak{A}} \\
\bar{A} & \xrightarrow{\pi_{\text{red}}} & \mathcal{K} \hat{\otimes} A
\end{array}$$

commutes, where $\lambda$ is the canonical surjection.

Proof. Since the proof in the case when $\mathcal{I}$ is finite is essentially same as (and easier than) the case when $\mathcal{I} = \mathbb{N}$, we may (and do) assume that $\mathcal{I} = \mathbb{N}$. Let $\{e_{ij}\}_{i,j \geq 1}$ be the system of matrix units for the canonical basis $\{\delta_i\}_{i \geq 1}$ of $\ell(\mathbb{N})$, and set $f_n := e_{nm}$. We realize $\sum_{n \geq 1} A_n$ and $c_0 \hat{\otimes} B$ inside $\mathcal{K} \hat{\otimes} A$ as

$$\sum_{n \geq 1} A_n = C^* \{f_n \circ a \mid n \geq 1, a \in A_n\} \quad \text{and} \quad c_0 \hat{\otimes} B = C^* \{f_n \circ b \mid n \geq 1, b \in B\}.$$  

Consider two conditional expectations $\sum_{n \geq 1} E_{A_n}^A : \sum_{n \geq 1} A_n \rightarrow c_0 \hat{\otimes} B$ and $E_{c_0} \hat{\otimes} id_B : \mathcal{K} \hat{\otimes} B \rightarrow c_0 \hat{\otimes} B$ defined by $(\sum_{n \geq 1} E_{A_n}^A)(f_i \circ a) = f_i \circ E_{A_i}^A(a)$ and $(E_{c_0} \hat{\otimes} id_B)(e_{ij} \circ b) = \delta_{ij} f_i \circ b$ for $i, j \in \mathbb{N}$, $a \in A_i$ and $b \in B$. Set $\mathfrak{A} := (\sum_{n \geq 1} A_n) \ast c_0 \hat{\otimes} B (K \hat{\otimes} B)$ and $(A, E) := (\sum_{n \geq 1} A_n, \sum_{n \geq 1} E_{A_n}^A) \ast c_0 \hat{\otimes} B (K \hat{\otimes} B, E_{c_0} \hat{\otimes} id_B)$ and let $\lambda : \mathfrak{A} \rightarrow A$ be the canonical surjection.
The inclusion maps $\sum_n A_n \hookrightarrow K \otimes A$ and $K \otimes B \hookrightarrow K \otimes A$ induce a $*$-homomorphism $\pi : \tilde{A} \to K \otimes A$. For any $n, i, j \in \mathbb{N}$, $a \in A_n$ and $b, c \in B$, one has $e_i(a) \otimes b c = \pi (e_i(a) \otimes b) \pi (f_n \otimes a) \pi (e_{n,i} \otimes c) \in \pi (\tilde{A})$. Since $[B A_n B] = A_n$ holds, $\pi$ is surjective. Define $\sigma_n : A_n \to \tilde{A}$ by $\sigma_n (a b c) = (e_i(a) \otimes b) \pi (e_{n,i} \otimes c)$ for $a \in A_n$ and $b, c \in B$. We then obtain $\sigma = \bigoplus_n \sigma_n : A \to \tilde{A}$.

We use the notation in the proof of Proposition 3.6. By the minimal projection $\lambda$ for every $i, j \geq 1$, $i \neq 1$, $b \in B$ as normed spaces, respectively. We represent $K \otimes A$ on the Hilbert $B$-module $L^2(N) \otimes L^2(A, E)$ faithfully and show that $C_b \otimes \xi E B$ satisfies the assumption of Lemma 2.7 for $S_1$ and $S_2$. Let $m \geq 1$, $x_1, \ldots, x_m \in S_1$ and $y_1, \ldots, y_n \in S_2$. Therefore, Lemma 2.7 guarantees that $(a \otimes E \lambda) \otimes \pi$ factors through $\lambda$, and hence $\pi$ is surjective.

Similarly, representing $\tilde{A}$ on $L^2(D, E)$ faithfully we observe that $\xi E (\tilde{A}, E)$ satisfies the freeness condition for $S' = B A_n B$, $n \geq 1$. Indeed, for $m \geq 1$, $i \in \mathbb{N}$, $y \in A_{k_i}(k)$, and $b, c \in B$, we have $\sigma (b_1 b_2 \cdots b_n c) = (e_i(c) \otimes b_1)(f_1 \otimes y_1)(c_1 \otimes b_2) \cdots (f_m \otimes y_m)(c_m \otimes b_m)$, which belongs to $\ker E$. Thus, $\lambda \circ \pi : A \to \tilde{A}$ factors through $\lambda : A \to A$, which implies that $\lambda \circ \pi$ factors through $\lambda : A \to A$.

The following general fact is well-known (see, e.g., [1, Proposition 17.8.7]).

**Proposition 3.7.** Let $K$ be as above and let $\iota : K \hookrightarrow B (l^2(\mathbb{N}))$ be the inclusion map. Fix a minimal projection $e \in K$. For any separable $C^*$-algebras $A$ and $B$, the mapping $\mathcal{E} (A, B) \ni (X, \phi, F) \mapsto (K \otimes X, \lambda \otimes \phi, 1_{K} \otimes F) \in \mathcal{E} (K \otimes A, K \otimes B)$ induces an isomorphism $\iota : KK (A, B) \to KK (K \otimes A, K \otimes B)$. The inverse of $\iota$ is given by the mapping $\mathcal{E} (K \otimes A, K \otimes B) \ni (Y, \psi, G) \mapsto (Y \otimes \lambda \otimes 1_{K} (l^2(\mathbb{N})) \otimes B), (\psi \otimes \lambda \otimes 1_{K}) \circ (\sigma, G \otimes \lambda \otimes 1_{K}) \in \mathcal{E} (A, B)$, where $\sigma (a) = e \otimes a$ for $a \in A$. We are now ready to prove Theorem 4 and Corollary 5.2.

**Proof of Theorem 4 and Corollary 5.2.** We use the notation in the proof of Proposition 3.6. By Theorem 4 and Proposition 3.6, there exists $\beta \in KK(K \otimes A, K \otimes A)$ such that $\beta \circ (id \otimes \lambda) = id \otimes \lambda$ and $(id \otimes \lambda) \circ \beta = id \otimes A$. Let $\tau$ be as in Proposition 3.7. We then have $id_A = \tau^{-1} (id \otimes \lambda) = \tau^{-1} (\beta \circ (id \otimes \lambda)) = \tau^{-1} (\beta)$ and $\iota_A = \tau^{-1} (id \otimes \lambda) = \tau^{-1} (id \otimes \lambda) \circ \beta = \lambda \circ \tau^{-1} (\beta)$. Thus, $\lambda$ gives a $KK$-equivalence.

Moreover, by Theorem 3.3 and Proposition 3.7, again, $\tau^{-1} (\beta)$ is implemented by a Kasparov $\mathbb{A} \otimes \mathbb{A}$ bimodule whose “$C^*$-correspondence part” is the direct sum of three $C^*$-correspondences of the form $(Y \otimes D Z, \pi_Y \otimes \iota_2)$, where $D$ is either $\mathbb{C} \otimes B$, $\mathbb{Z}$ or $\mathbb{C} \otimes B$. Thus, if $A_i$ is nuclear for every $i \in I$, then $id = \tau^{-1} (\beta)$ is also implemented a Kasparov bimodule consisting of a nuclear $C^*$-correspondence (see the remark just after Theorem 2.5), and hence $\mathbb{A}$ is $K$-nuclear.

**Remark 3.8.** Theorem 4 generalizes the previous $K$-amenability results for amalgamated free products of amenable discrete (quantum) groups [11] and [20]. However, we should remark that our result does not imply Pimsner’s result that $K$-amenability is closed under amalgamated free products. Similarly, Corollary 5.2 does not imply that $K$-nuclearity is closed under amalgamated free products (even for plain free products). The latter seems a interesting next question in the direction.
4. SIX-TERM EXACT SEQUENCES

Let \((A, E) = (A_k, E_k^A)\) as in Theorem 3.4. We denote by \(i_k : B \rightarrow A_k\) and \(j_k : A_k \rightarrow A, k = 1, 2\) the inclusion maps. As we mentioned in the introduction, our KK-equivalence and K-nuclearity results with Thomsen’s result [17] imply the following:

**Corollary 4.1.** With the notation above, there is a cyclic six-term exact sequence

\[
\begin{array}{c}
K_0(B) \xrightarrow{(i_1, i_2)} K_0(A_1) \oplus K_0(A_2) \xrightarrow{j_{12}} K_0(A) \\
\uparrow \\
K_1(A) \xrightarrow{j_{12}} K_1(A_1) \oplus K_1(A_2) \xrightarrow{(i_1, i_2)} K_1(B)
\end{array}
\]

If \(A_1\) and \(A_2\) are further assumed to be nuclear, then for any separable C\(^*\)-algebras \(D\) there is a cyclic exact sequence

\[
\begin{array}{c}
KK(B, D) \xrightarrow{i_{12}} KK(A_1, D) \oplus KK(A_2, D) \xrightarrow{j_{12}} KK(A, D) \\
\downarrow \\
KK(A, SD) \xrightarrow{j_{12}} KK(A_1, SD) \oplus KK(A_2, SD) \xrightarrow{i_{12}} KK(B, SD)
\end{array}
\]

Note that the second exact sequence of KK-groups is new even in the full case. We also obtain the next corollary from Theorem 3.4 and [4].

**Corollary 4.2.** With the notation above, suppose that \(B\) is a direct sum of finite dimensional C\(^*\)-algebras. Then, for any separable C\(^*\)-algebra \(D\) there are two cyclic exact sequences:

\[
\begin{array}{c}
KK(D, B) \xrightarrow{(i_1, i_2)} KK(D, A_1) \oplus KK(D, A_2) \xrightarrow{j_{12}} KK(D, A) \\
\uparrow \\
KK(SD, A) \xrightarrow{j_{12}} KK(SD, A_1) \oplus KK(SD, A_2) \xrightarrow{(i_1, i_2)} KK(SD, B) \\
KK(B, D) \xrightarrow{i_{12}} KK(A_1, D) \oplus KK(A_2, D) \xrightarrow{j_{12}} KK(A, D) \\
\downarrow \\
KK(A, SD) \xrightarrow{j_{12}} KK(A_1, SD) \oplus KK(A_2, SD) \xrightarrow{i_{12}} KK(B, SD)
\end{array}
\]

Finally, we would like to point out that a similar result holds for HNN extensions. We refer the reader to [18, 19] for HNN extensions of C\(^*\)-algebras. The next corollary follows from the C\(^*\)-version of Proposition 3.1”, Proposition 3.3 and Proposition 4.2 in [19] and Theorem 3.4.

**Corollary 4.3.** Let \(B \subseteq A\) be unital inclusion of separable C\(^*\)-algebras with nondegenerate conditional expectation \(E : A \rightarrow B\), and \(\theta : B \rightarrow A\) be an injective \(*\)-homomorphism whose image is the range of a conditional expectation \(E_\theta : A \rightarrow \theta(B)\). Then, the full HNN-extension \(A \ast_B^\text{min} \theta\) and the reduced one \((A, E) \ast_B (\theta, E_\theta)\) are KK-equivalent via the canonical surjection, and there is a six-term exact sequence:

\[
\begin{array}{c}
K_0(B) \xrightarrow{(\theta_* - i_B)} K_0(A) \xrightarrow{i_A} K_0((A, E) \ast_B (\theta, E_\theta)) \\
\uparrow \\
K_1((A, E) \ast_B (\theta, E_\theta)) \xleftarrow{i_A} K_1(A) \xrightarrow{(\theta_* - i_B)} K_1(B)
\end{array}
\]

Here \(i_B : B \rightarrow A\) and \(i_A : A \rightarrow (A, E) \ast_B (\theta, E_\theta)\) are inclusion maps. Further assume that \(A\) is nuclear. Then, these HNN-extensions are K-nuclear.
Remark 4.4. Using Proposition 3.7 we can generalize the results in this section to amalgamated free products and HNN extensions of countably many $C^*$-algebras and countably many injective $*$-homomorphisms, respectively. Such generalizations for HNN extensions include Pimsner–Voiculescu’s six-term exact sequence for crossed products by free groups ([14],[15]) as special cases (see also [19]).

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