PARTIAL REGULARITY FOR \( \mathcal{A} \)-QUASICONVEX FUNCTIONALS

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Abstract. We establish partial Hölder regularity for (local) generalised minimisers of variational problems involving strongly quasi-convex integrands of linear growth, where the full gradient is replaced by a first order homogeneous differential operator \( \mathcal{A} \) with constant coefficients. Working under the assumption of \( \mathcal{A} \) being \( C \)-elliptic, this is achieved by adapting a method recently introduced in [33, 32].

1. Introduction

1.1. Variational Problems. The analysis of functionals taking the form \( \mathcal{F} = \int_{\Omega} f(\nabla u) \, dx \) is a major task in the calculus of variations with a long standing tradition. Let us suppose that \( \Omega \subset \mathbb{R}^n \) is open and bounded and that \( u \) is a weakly differentiable \( \mathbb{R}^N \)-valued map. The growth condition on the integrand \( f \in C(\mathbb{R}^{N \times n}) \) determines the functional analytic environment in which we analyse \( \mathcal{F} \). A standard growth assumption, which has been studied intensively in the field, constitutes the following: There exist \( p \in [1, \infty) \) and a constant \( L > 0 \) such that for all \( z \in \mathbb{R}^{N \times n} \) we have

\[
|f(z)| \leq L \left( 1 + |z|^p \right).
\]

The existence of minimisers within a given class of functions (or maps) is a fundamental question. Concretely, we want to minimise the functional \( \mathcal{F} \) in \( g + W^{1,p}_0(\Omega; \mathbb{R}^N) \) for a prescribed Dirichlet boundary datum \( g \in W^{1,p}(\Omega; \mathbb{R}^N) \). In the super-linear growth case \( p > 1 \), this task can immediately be tackled by means of the direct method, a lower semi-continuity method dating back to Tonelli. Consequently, the question arises under which assumptions on \( f \) is the functional \( \mathcal{F} \) sequential weakly lower semi-continuous in \( W^{1,p}(\Omega; \mathbb{R}^N) \). Towards this question, convexity certainly suffices, but in the vectorial case \( (N > 1) \) it is seen to not be a necessary condition. Ball and Murat [8] have shown that Morrey’s notion of quasi-convexity [43], i.e., for all \( z \in \mathbb{R}^{N \times n} \) and all \( \zeta \in C^1_\text{c}(B; \mathbb{R}^N) \) we have

\[
f(z) \leq \int_B f(z + \nabla \zeta) \, dx,
\]

turns out to be necessary. In fact, for unsigned integrands \( f \), quasi-convexity is also a sufficient condition, as Acerbi and Fusco have shown in [1].

Once having addressed the issue of existence of minimisers, we would like to know what further information about a minimiser we can extract. This question dates back to David Hilbert [35] and is today known by the name of regularity theory in the calculus of variations. There are many different notions of regularity and in our setting, we are interested whether a minimiser is (locally) of the class \( C^{1,\alpha} \). In the scalar case \( (N = 1) \) the notions of convexity and quasi-convexity coincide and the regularity theory, at least in the quadratic growth case, reduces to the regularity of solutions of elliptic equations established by De Giorgi [18], Nash [45] and Moser [44]. However, in the vectorial case, the regularity of minimisers can no longer be extracted from the Euler-Lagrange equation only, because, as various
counterexamples show [20, 46, 29], there is no such theory for elliptic systems in general. Furthermore, full Hölder regularity can no longer be expected since minimisers may be unbounded within a small set. Adapting ideas from geometric measure theory developed by Almgren [4] and De Giorgi [19], Evans established in the non-parametric setting a fundamental partial regularity result assuming a stronger notion of quasi-convexity [27]. This means that a minimiser enjoys Hölder regularity outside a small set. We stress that partial regularity is a feature of the vectorial case. After the quadratic case, the super-quadratic case \( p \geq 2 \) was established by Acerbi and Fusco in [2]. Later on, the sub-quadratic case \( 1 < p \leq 2 \) was resolved partially by Carozza and Passarelli di Napoli in [14] and then fully by Carozza, Fusco, and Mingione in [13]. Some time later, even the Orlicz growth case has been resolved by Diening et al. in [22]. An overview of related results can be found in [41, 42, 9, 30].

The case of linear growth for quasi-convex integrands, however, had remained an open problem, since the classical methods were bound to fail due to the lack of weak compactness. Only in the recent years, partial local Hölder regularity of the (distributional) gradient of (local) BV-minimisers in the quasi-convex setting has been established by Gmeineder and Kristensen in [33].

Let us try to roughly describe the underlying ideas on how to obtain partial Hölder regularity of the weak gradient of a minimiser \( u \) of \( F \). The main objective is to prove a decay estimate for the excess of \( u \). The excess is a quantity that, similar to Campanato’s semi-norm, measures by means of integrals the rate of oscillation of \( \nabla u \). The goal is to show that if the excess is small enough, it decays with any rate \( \alpha \in (0, 1) \). By means of a Caccioppoli Inequality, one passes from the excess, which depends on the gradient, to a quantity depending merely on \( u \). The Caccioppoli Inequality, in turn, builds on the combination of minimality and a stronger notion of quasi-convexity by means of Widman’s hole-filling trick. On the level of order 0, the strategy then is to approximate the minimiser by a \( B \)-harmonic map \( h \), where \( B \) denotes a strongly Legendre-Hadamard elliptic bilinear form on \( \mathbb{R}^{N \times n} \). The map \( h \) has a good decay since it solves a homogeneous elliptic system. The difficulty is to construct \( h \) in such a way that \( u - h \) has good decay as well. Classically, the construction of \( h \) follows an indirect approach utilising compactness, for example, of the embedding \( W^{1,2}(\mathcal{B}) \rightarrow L^{2}(\mathcal{B}) \). This kind of approximation goes by the name of \( \mathcal{B} \)-Harmonic Approximation Lemma [19, 23, 25].

Many generalisations of the functional \( F \) have been studied. Here, we are going to replace the full gradient with a first order homogeneous differential operator \( \mathcal{A} \). The two most prominent examples are the symmetrised gradient \( \xi \) and the trace-free symmetric gradient \( \bar{\xi} \). Let \( V \) and \( W \) be two real and finite dimensional Hilbert spaces. We are going to consider differential operators of the form

\[
\mathcal{A} = \sum_{\alpha=1}^{n} \mathcal{A}_{\alpha} \partial_{\alpha}, \quad \mathcal{A}_{\alpha} \in \mathcal{L}(V; W).
\]

For \( \xi \in K^{n}, K = \mathbb{R}, C \), the linear map \( \mathcal{A}[\xi] v = \sum_{\alpha=1}^{n} \xi_{\alpha} \mathcal{A}_{\alpha} v \), modulo a factor of \(-i\), is called the symbol map associated to the differential operator \( \mathcal{A} \). We say that \( \mathcal{A} \) is \( K \)-elliptic if the symbol map \( \mathcal{A}[\xi] \) is one-to-one for all \( \xi \in K^{n} \setminus \{0\} \) ([50, 49, 36]).

The notion of \( R \)-ellipticity has been characterised by means of Fourier multipliers [40] and singular integrals [11] that the Korn-type Inequality

\[
\forall p \in (1, \infty) \exists C > 0 \forall \xi \in C_{c}^{\infty}(\mathbb{R}^{n}; V) : \| \nabla \xi \|_{L^{p}} \leq C \| \mathcal{A} \xi \|_{L^{p}}
\]

is satisfied by the differential operator \( \mathcal{A} \). In the super-linear growth case, Conti and Gmeineder showed that this allows to reduce the question of partial Hölder
regularity of a local minimiser of a functional of the form
\[ \mathcal{F}[u; \Omega] = \int_{\Omega} f(\mathcal{A}u) \, dx, \]
where \( f \in C(W) \) is of \( p \)-growth (\( p > 1 \)), to the full gradient case [17]. Ornstein’s Non-Inequality [47, Theorem 1], [16, 37], stating that there is no non-trivial Korn Inequality in the \( L^1 \)-setting, implies that such a reduction is impossible in the linear growth regime. Consequently, it had constituted a highly non-trivial task to adapt the full gradient case [33] to the symmetric gradient case [32] in the linear growth regime.

1.2. Partial Hölder Regularity in the Linear Growth Regime. In the convex case, partial regularity for linear growth functionals has been known in the convex context following the work of Anzellotti and Giaquinta [6] (also see [48, 31] for variations of this theme). However, the methods employed therein are confined to the convex case. In the linear growth context, the key difficulty to overcome is the lack of weak compactness. This concerns the existence of minimisers as much as their regularity theory. In particular, this excludes indirect methods like the by now classical \( B \)-Harmonic Approximation Lemma [23, 25, 24]. A direct approach was needed to construct a \( B \)-harmonic approximation. Gmeineder and Kristensen solved this problem by showing that the traces of BV-maps on spheres of radius \( R \) enjoy for \( L^1 \)-almost all sufficiently small radii \( R \) more regularity than the default \( L^1 \)-regularity. This is called a Fubini-type property of BV-maps and it was in effect the key point to construct a \( B \)-harmonic approximation by solving the elliptic system
\[
\begin{aligned}
- \text{div} (B \nabla h) &= 0 &\text{in } B_R(x_0) \\
h &= u &\text{on } \partial B_R(x_0)
\end{aligned}
\]
We note that the solution operator
\[
(C^0 \cap L^1)(\partial B_R(x_0); \mathbb{R}^N) \ni u \mapsto h \in W^{1,1}(B_R(x_0); \mathbb{R}^N)
\]
associated to this system cannot be bounded as an operator from \( L^1 \) to \( W^{1,1} \). In other words, without more regularity of \( \text{Tr}_{B_R(x_0)}(u) \) we lack tools to precisely measure how close the \( B \)-harmonic map \( h \) is to \( u \). This is why the Fubini-type property of BV-maps is essential for a direct approach in the linear growth regime.

Gmeineder was able to adapt the ideas used in [33] to the scenario where the full gradient is replaced by the symmetric gradient [32]. The main difficulties were to prove a Fubini-type property and, building on the latter, to prove precise estimates for \( u - h \), where \( h \) denotes a suitable \( B \)-harmonic approximation of \( u \).

1.3. The Main Theorem. Our scope is to show that the method for the symmetric gradient case extends to an entire class of first order homogeneous differential operators with constant coefficients, namely the class of \( C \)-elliptic operators.

In line with the full [33] and symmetric gradient case [32], we will work from now on under the following assumptions:

- \( \text{(H0)} \) The differential operator \( \mathcal{A} \) is \( C \)-elliptic.
- \( \text{(H1)} \) The integrand \( f \in C^{2,1}_{\text{loc}}(W) \) is of linear growth.
- \( \text{(H2)} \) The integrand \( f \) is strongly \( V_1-\mathcal{A} \)-quasi-convex, where \( V_1 \) denotes the reference integrand to be defined in the the upcoming section on preliminaries:

There exists \( \nu > 0 \) such that \( F = f - \nu V_1 \circ | \cdot | \) is \( \mathcal{A} \)-quasi-convex, i.e., for all \( w \in W \) and all \( \zeta \in C^\infty_c(B) \) we have
\[
F(w) \leq \int_B F(w + \mathcal{A}\zeta) \, dx.
\]
For $\omega \subset \mathbb{R}^n$ open and bounded we associate to the integrand $f$ the functional $\mathcal{F}[u; \omega] = \int_{\omega} F(\mathcal{A}u(x)) \, dx$, where $u \in W^{1,1}(\omega)$.

We recall that to any $\mathbb{R}$-elliptic potential $\mathcal{A}$ there exists an annihilator, see [51], $\mathcal{A} = \sum_{\alpha=1}^n \mathcal{A}_\alpha \partial_\alpha$ with $\mathcal{A}_\alpha \in \mathcal{L}(W^1; V)$ such that the symbol complex

$$V \xrightarrow{\mathcal{A}} W \xrightarrow{\mathcal{A}^*} V$$

is exact for all $\xi \in \mathbb{R}^n \setminus \{0\}$. This shows that the notion of $\mathcal{A}$-quasi-convexity is equivalent to the more widely known notion of $\mathcal{A}^*$-quasi-convexity which is strongly linked to weak sequential lower semi-continuity of the functional $\mathcal{F}$, which Fonseca and Müller showed in [28].

We fix $\Omega \subset \mathbb{R}^n$ an open and bounded Lipschitz domain and $g \in W^{1,1}(\Omega)$ a prescribed boundary datum. Due to the lack of weak compactness, analogously to the full gradient case, the task to minimise $\mathcal{F}$ within a given Dirichlet class $g + W_0^{1,1}(\Omega)$ cannot be tackled by plainly applying the direct method. Hence, we pass from the Sobolev-type space $W^{1,1}$ to the BV-type space $BV^1$, hoping for better compactness with respect to the weak$^*$-topology on the latter space. Already in the full gradient case, this forces us to somehow relax our functional $\mathcal{F}$ to this larger space $BV^1$. Without the assumption of $C$-ellipticity, this relaxation procedure cannot be implemented analogously: We recall that in the full gradient case, Alberti’s rank-one result [3] for BV-maps has proven to be essential in order to obtain an integral representation of the Lebesgue-Serrin extension [5]. Using that quasi-convexity implies rank-one convexity, Alberti’s result ensures that the strong recession function of a quasi-convex integrand with linear growth is well-defined on the rank-one cone. The requisite result paralleling Alberti’s result in the $\mathbb{R}$-elliptic case has been established by De Philippis and Rindler in [21]. Weak$^*$-compactness of closed, norm bounded sets in $BV^1$, as well as the existence of a strictly continuous linear trace operator $Tr_\Omega: BV^1(\Omega) \rightarrow L^1(\partial \Omega)$ for open and bounded Lipschitz domains are two features exclusive to the $C$-elliptic case [10, Theorem 1.1], [34, Theorem 1.1]. Since the trace-operator on $BV^1$ is discontinuous with respect to weak$^*$-convergence, the boundary condition is no longer reflected by the space $BV^1$, but rather by the relaxed functional itself. The boundary condition then is encoded by a so called penalty term, which already pops up in the full gradient case [38]:

$$P_{f,\Omega,g}[u] = \int_{\partial \Omega} f^{\infty}(\nu_{\partial \Omega} \otimes _{\mathcal{A}} Tr_\Omega(u - g)) \, dH^{n-1},$$

where $f^{\infty}(w) = \lim_{t \rightarrow -\infty} \frac{f(t w)}{t}$ denotes the strong recession function and $\xi \otimes _{\mathcal{A}} v = \sum_{\alpha=1}^n \xi_\alpha \mathcal{A}_\alpha v$ for $\xi \in \mathbb{R}^n$ and $v \in V$. We stress that it is necessary to assume that $\mathcal{A}$ is $C$-elliptic in order for this integral expression to be well-defined. The relaxed functional then takes the form ([10, Section 5], [7])

$$\bar{\mathcal{F}}_g[u; \Omega] = \int_{\Omega} f(\mathcal{A}u) + P_{f,\Omega,g}[u].$$

We note that (H2) implies that there exist $b \in \mathbb{R}$ and $c > 0$ such that for all $\zeta \in g + W_0^{1,1}(\Omega)$ we have $\mathcal{F}[\zeta; \Omega] \geq cV_1(\mathcal{F}_0(\zeta) \, dx) + b$. This can be inferred from an extension and gluing argument similar to the argument in [15, pp. 218–219]. Identifying $\bar{\mathcal{F}}_g[u; \Omega]$ as the Lebesgue-Serrin extension [10, Section 5]

$$\bar{\mathcal{F}}_g[u; \Omega] = \inf \left\{ \lim_{j \rightarrow -\infty} \inf \mathcal{F}[u_j; \Omega]: (u_j) \subset g + W_0^{1,1}(\Omega), \, u_j \xrightarrow{BV^1} u \right\}$$

yields coercivity of the relaxation. Hence, under our assumption, generalised minimisers subject to a given Dirichlet boundary condition exist by means of the direct
method. Since we are only striving for a local regularity result, it is natural to consider the class of local generalised minimisers:

**Definition.** We call a BV$_{\text{loc}}^1(\Omega)$-map $u$ a local generalised minimiser of $\mathcal{F}$ if for any $\omega \Subset \Omega$ open and bounded Lipschitz domain and all $\zeta \in \text{BV}^1(\omega)$ we have

$$\mathcal{F}_u[u; \omega] \leq \mathcal{F}_u[\zeta; \omega].$$

At this stage, we are ready to formulate the main theorem:

**Theorem 1.1.** Let us assume that (H0), (H1), and (H2) hold. Furthermore, let $u \in \text{BV}_{\text{loc}}^1(\Omega)$ be a local generalised minimiser of the to $f$ associated functional $\mathcal{F}$. Let $\alpha \in (0, 1)$, let $M > 0$ and let $B = B_r(x_0) \Subset \Omega$ be a ball. Then there exists $\varepsilon > 0$ depending on $M, \alpha, F, n, d_V, d_W$ and $\frac{\mu}{\rho}$ such that whenever we have

$$|(\mathcal{A}u)_B| \leq M, \quad \text{and} \quad \int_B V_1(\mathcal{A}u - (\mathcal{A}u)_B) < \varepsilon,$$

then $u|_B$ belongs to the class $C^{1, \alpha}(B; V)$. In particular, the singular set $\Sigma_u$ defined by

$$\left\{ x \in \Omega : \limsup_{r \to 0} \| (\mathcal{A}u)_{B_r(x)} \| = +\infty \right\} \bigcup \left\{ x \in \Omega : \liminf_{r \to 0} \int_B V_1(\mathcal{A}u - (\mathcal{A}u)_B) > 0 \right\}$$

is a relatively closed Lebesgue-null-set and we have $u \in C^{1, \alpha}_{\text{loc}}(\Omega \setminus \Sigma_u; V)$ for all $\alpha \in (0, 1)$.

We wish to point out that for the present paper, the assumption of $C$-ellipticity is crucial and visible on several stages (so e.g. in the very definition of the functionals where boundary traces come into play); the elliptic case, however, seems to require refined methods.

2. Preliminaries

2.1. **Notation.** For a finite dimensional real vector space $Z$ we use the shorthand notation $d_Z = \text{dim}_R Z$. Furthermore, we will suppress the target vector space when dealing with different function spaces. For example, if $u : \Omega \to Z$ is a $L^1$-map, we simply write $u \in L^1(\Omega)$ instead of $u \in L^1(\Omega; Z)$. Within the context, it will be clear which target vector space we are referring to. Furthermore, by $| \cdot |$ we will denote any norm of a finite dimensional, normed vector space such as $L(V; W)$, $V$ or $L(V \times V; W)$. Since all norms of a finite dimensional normed vector space are equivalent, this is an non-problematic convention. Throughout, we fix an orthonormal basis $(e_j)_{j=1}^N$ of $V$, i.e., $N = d_V$. Furthermore,

$$(e_{jk})_{j=1,k=1}^{N,k=n} = \delta_{jk}$$

denotes the standard basis of $\mathbb{R}^{N \times n}$.

Integration with respect to the $n$-dimensional Hausdorff-measure $\mathcal{H}^{n-1}$ will be denoted by $d\sigma_x$, where $x$ is integration variable.

As usual, $B_r(x_0) \subset \mathbb{R}^n$ denotes the open ball with centre $x_0$ and radius $r > 0$. Often we will suppress the centre of the ball if it is clear within the context and we will simply write $B_r$. Furthermore, we put $\mathbb{B} = B_1(0)$ and $\partial B$.

By $\mathcal{M}(\Omega; Z)$ we denote the space of $Z$-valued finite Radon measures on $\Omega$. For $\mu \in \mathcal{M}(\Omega)$ and open, bounded subsets $\omega \subset \Omega$, the total variation-measure of $\mu$ will be denoted by $|\mu|$ and the average of $\mu$ on $\omega$ with respect to the Lebesgue measure will be written as $(\mu)_\omega := \frac{\mu(\omega)}{\mathcal{H}^n(\omega)}$.

Let $V_1(t) = \sqrt{1 + t^2 - 1}$ denote the reference integrand. We will use the shorthand notation $V_1(z) = V_1(|z|)$.
For a $C^1$-function $G: Z \to \mathbb{R}$ and $z_0 \in Z$ we denote by

$$(G)_{z_0} : Z \to \mathbb{R}, \ z \mapsto G(z_0 + z) - \left( G(z_0) + DG(z_0)[z] \right)$$

the linearisation of $G$ at $z_0$.

For $k \in \mathbb{N}$ we denote by $\mathcal{P}_k(Z)$ the vector space of all polynomials

$$p \in \mathbb{R}[X_1, \ldots, X_n] \otimes \mathbb{R} Z$$

of degree at most $k$.

By $C$ we will denote a generic constant which may vary from line to line. Since it is very important throughout on which parameters a constant depends on, we will write for example $C(M, p)$ if the constant depends on $M$ and $p$.

For $\xi \in \mathbb{R}^n$ and $v \in V$ we put $\xi \otimes_A v = \sum_{n=1}^N \xi_a A_n v$. Furthermore, we denote by

$$\mathcal{A}(A) := \text{span}\{\xi \otimes_A v : \xi \in \mathbb{R}^n, v \in V\}$$

the effective range of $A$ and we call

$$\mathcal{N}(A) := \{u \in \mathcal{A}^*: A u = 0\}$$

the null-space of $A$. The formally adjoint operator $A^*$ is here defined by the formula

$$A^* = -\sum_{\alpha=1}^n A_{\alpha}^* \partial_{\alpha}.$$

Let $\Omega \subset \mathbb{R}^n$ be open and let $M \subset \mathbb{R}^n$ be an embedded $(n-1)$-dimensional $C^1$-submanifold of $\mathbb{R}^n$. For $\alpha \in (0,1)$ and $p \in [1, \infty)$, we recall the definition of the fractional Sobolev space (semi)-norms on $\Omega$ and $M$, respectively:

- $[u]_{W^{\alpha,p} (\Omega)} = \left( \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha p}} \, dx \, dy \right)^{\frac{1}{p}}$, $[u]_{W^{\alpha,p} (M)} = \left( \int_{M} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha p}} \, d\sigma_x \, d\sigma_y \right)^{\frac{1}{p}}$,

2.2. Space of maps of bounded $A$-variation. We are going to collect prerequisites on $A$-weakly differentiable maps. In the spirit of [10], we define Sobolev- and BV-type spaces as follows:

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open and let $p \in [1, \infty]$. We define:

- $W^{A,p} (\Omega) := \{ u \in L^p(\Omega; V) : Au \in L^p(\Omega; W) \}$, and
- $BV^A (\Omega) := \{ u \in L^p(\Omega; V) : Au \in \mathcal{M}(\Omega; W) \}$.

These spaces can be equipped with the obvious norms making them Banach spaces. Also the spaces $W^{A,p} (\Omega)$ are as usual defined as the closure of $C_c^\infty (\Omega; V)$ with respect to the according norm.

Let $u \in BV^A_{\text{loc}} (\Omega)$. Then we consider the Radon-Nikodým decomposition of $Au$ with respect to the Lebesgue measure $Au = A^0 u + A^u u$, where $A^0 u$ denotes the absolutely continuous part and $A^u u$ the singular part. Next, we recall different notions of convergence in BV:

Definition 2.2. Let $u \in BV^A (\Omega)$ and $(u_j) \subset BV^A (\Omega)$. Then $u_j$ converges to $u$ in the

(i) $A$-weak*-sense ($u_j \overset{\ast}{\rightharpoonup} u$) if $u_j \to u$ strongly in $L^1(\Omega)$ and $Au_j \overset{\ast}{\rightharpoonup} Au$ in the weak*-sense of $W$-valued Radon measures on $\Omega$.

(ii) $A$-strict sense ($u_j \to u$) if $u_j \to u$ strongly in $L^1(\Omega)$ and $|Au_j|(\Omega) \to |Au|(\Omega)$.
(iii) $\mathcal{A}$-area-strict sense $(u_j \rightharpoonup u)$ if $u_j \to u$ strongly in $L^1(\Omega)$ and
\[
\int_{\Omega} \sqrt{1 + \frac{|dA^\alpha_0u_j|^2}{dZ^n}} \, dx + |A^\alpha_0u_j|(\Omega) \to \int_{\Omega} \sqrt{1 + \frac{|dA^\alpha_0u|^2}{dZ^n}} \, dx + |A^\alpha_0u|(\Omega)
\]

Lemma 2.3. [10, Theorem 2.8, Lemma 4.15] Let $\Omega \subset \mathbb{R}^n$ open. Then $(C^\infty \cap BV^\alpha)(\Omega)$ is dense in $BV^\alpha(\Omega)$ with respect to the strict and area-strict topologies. If $\Omega$ is additionally a bounded Lipschitz domain, then $C^\infty(\Omega)$ is dense in $BV^\alpha(\Omega)$ with respect to the strict and area-strict topologies. Let $u_0 \in W^{\alpha,1}(\Omega)$. For each $u \in BV^\alpha(\Omega)$ there exists a sequence $(u_j) \subset u_0 + C^\infty_c(\Omega)$ such that $\|u_j - u\|_{L^1(\Omega)} \to 0$ and
\[
\int_{\Omega} \sqrt{1 + \frac{|dA^\alpha_0u_j|^2}{dZ^n}} \, dx + |A^\alpha_0u_j|(\Omega) 
\]
and
\[
\int_{\Omega} \sqrt{1 + \frac{|dA^\alpha_0u|^2}{dZ^n}} \, dx + |A^\alpha_0u|(\Omega)
\]
\[
\int_{\Omega} |(\text{Tr}(u) - \text{Tr}(u_0)) \otimes A \nu| \, d\mathcal{H}^{n-1} \quad \text{for} \ j \to \infty.
\]

The proof of the last assertion is analogous to the BD-case [32].

Lemma 2.4. [10, Theorem 3.2] Let $B \subset \mathbb{R}^n$ be an open ball of radius $r > 0$ and let $\Pi_B$ denote the $L^2(B)$-projection onto $\mathcal{M}(\mathcal{A})$. Then there exists a constant $C > 0$ such that for all $u \in BV^\alpha(B)$ we have
\[
\|u - \Pi_B u\|_{L^1(B)} \leq Cr|\mathcal{A}u|(B).
\]

Lemma 2.5. [10, Theorem 1.2] Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain. Then there exists a linear and strictly continuous operator $\text{Tr}_\Omega : BV^\alpha(\Omega) \to L^1(\partial\Omega)$ such that for all $u \in C^1(\Omega)$ we have $\text{Tr}_\Omega u = u|_{\partial\Omega}$. For an open and bounded Lipschitz subset $\Omega' \subset \Omega$, we consider so called interior and exterior traces of $u$ denoted by
\[
\text{Tr}_{\Omega'}(u) := \text{Tr}_{\Omega'}(u|_{\Omega'}) \quad \text{and} \quad \text{Tr}_{\Omega'}^+(u) := \text{Tr}_{\Omega'}(u|_{\Omega'}).
\]

One can explicitly compute
\[
\lim_{r \to 0} \int_{B^+(x,r)} |u(y) - \text{Tr}_{\partial B}^\pm(u)(x)| \, dy = 0
\]
for $\mathcal{H}^{n-1}$-a.e. $x \in \partial B$, where $B^+(x,r) := \{y \in B_r(x) \mid |y - x, \nu(x)| \leq 0\}$. Here, $\nu(x)$ designates the outer unit normal vector to the sphere $\partial B$ at point $x$.

Proposition 2.6. [10, Proposition 5.1] Let $\Omega \subset \mathbb{R}^n$ be open, bounded and let $g : \mathbb{W} \to \mathbb{R}$ be an $\mathcal{A}$-quasi-convex integrand of linear growth. Then the functional
\[
\mathcal{F} : BV^\alpha(\Omega) \to \mathbb{R}, u \mapsto \int_{\Omega} g(Au) := \int_{\Omega} g\left(\frac{A^\alpha u}{dZ^n}\right) \, dx + \int_{\Omega} g^\infty\left(\frac{dA^\alpha u}{|dA^\alpha u|}\right) \, d|A^\alpha u|
\]
is $\mathcal{A}$-area strictly continuous and sequentially lower semi-continuous with respect to weak* convergence.

We will use the shorthand notation $\mathcal{F}_{\Omega} g(Au) = \int_{\Omega} g(Au) = \frac{\int_{\Omega} g(Au)}{\mathcal{H}^n(\Omega)}$.

Lemma 2.7. Let $n \geq 2$, $\alpha \in (0,1)$, let $B_{2r} \subset \mathbb{R}^n$ be a ball of radius $2r > 0$ and let $p := \frac{n}{n-1+\alpha}$. Then there exists a constant $C > 0$ independent of the radius $r$ such that for every ball $B_{r} \subset \mathbb{R}^n$ and every $u \in BV^\alpha(\mathbb{R}^n)$, there exists some $b \in \mathcal{M}(\mathcal{A})$ with
\[
\left(\int_{B_r^+} \frac{|u_b(x) - u_b(y)|^p}{|x-y|^{n+\alpha p}} \, dx \, dy\right)^{\frac{1}{p}} \leq Cr^{1-\alpha} \int_{B_{2r}} |Au|,
\]
where $u_b := u - b$. 

Proof. Let \( \tilde{u} := u \varphi \), where \( \varphi \in C^\infty_c(B_{2r}; [0, 1]) \) is a bump function with \( 1_{B_{r}} \leq \varphi \leq 1_{B_{2r}} \) and \( |\nabla \varphi| \leq C/r \). Then \( \tilde{u} \in BV^k(\mathbb{R}^n) \) and

\[
(2.3) \quad \| \Lambda \tilde{u} \|_{L^1(B_{2r})} \leq C(r) \| u \|_{BV^k(B_{2r})}.
\]

Noting that \( \text{Tr}(\tilde{u}) = 0 \) on \( \partial B_{2r} \), there exists a sequence \( (\tilde{u}_j) \subset C^\infty_c(B_{2r}; \mathbb{R}^N) \) such that \( \tilde{u}_j \to \tilde{u} \) strictly. Since \( \Lambda \) is \( C \)-elliptic it is in particular \( \mathbb{R} \)-elliptic and canceling, see [34]. Applying [51, Proposition 8.11] we obtain

\[
\| \tilde{u}_j \|_{W^{\alpha,p}(B_r)} \leq \| \tilde{u}_j \|_{W^{\alpha,p}(\mathbb{R}^n)} \leq C \| \Lambda \tilde{u}_j \|_{L^1(\mathbb{R}^n)}
\]

By passing to a subsequence we may assume that \( (\tilde{u}_j) \) converges \( \mathcal{L}^n \)-almost everywhere. By Fatou’s Lemma and the strict convergence we obtain

\[
(2.4) \quad \| u \|_{W^{\alpha,p}(B_r)} \leq \| u \|_{BV^k(B_{2r})}.
\]

We put \( b := \Pi_{B_r} u \), \( (u - b)_r := (u - b)(rx) \) for \( x \in B \) and eventually we obtain by Poincaré’s Inequality 2.4, scaling and applying 2.4 for \( r = 1 \):

\[
\left( \int_{B_r} \int_{B_r} \frac{|u_0(x) - u_0(y)|^p}{|x - y|^{n + p}} \, dx \, dy \right)^{1/p} \leq C r^{-\alpha} \left( \int_{B_r} |\Lambda u| \right)^{1/p}.
\]

\[\square\]

**Lemma 2.8.** Let \( B \Subset \Omega \) be a ball and \( u \in BV^k_{loc}(\Omega) \). Then there exists a linear map \( a : \mathbb{R}^n \to V \) such that \( \Lambda a = (\Lambda u)_B \).

**Proof.** Let \( u_j \in C^\infty(\overline{B}; V) \) such that \( u_j \to u \) as \( j \to \infty \). Clearly, we have \( \sup_j \| (\Lambda u_j)_B \| < \infty \) and \( (\Lambda u_j)_B \in \mathcal{A}(\mathcal{A}) \), since \( \Lambda u_j (B) = \sum_{\alpha=1}^n \alpha_{\alpha} \int_B \partial_{\alpha} u_j \, dx \).

After extraction of a non-relabeled sub-sequence we find \( w \in \mathcal{A}(\mathcal{A}) \) such that \( (\Lambda u_j)_B \to w \) in \( W \) as \( j \to \infty \). We find \( v_\alpha \in V \) such that \( w = \sum_{\alpha=1}^n \alpha_{\alpha} v_\alpha \) and put \( a[x] = \sum_{\alpha=1}^n x_\alpha v_\alpha \). This yields the claim. \(\square\)

2.3. Linearisation and the Reference Integrand.

**Lemma 2.9.** Let \( f : W \to \mathbb{R} \) satisfy (H1) and (H2), let \( m > 0 \). Then there exists a constant \( C(m) > 0 \) such that for all \( w_0 \in W \) with \( |w_0| \leq m \), all \( \xi \in \mathbb{R}^n \) and all \( v \in V \) we have

\[
(2.5) \quad D^2 f(w_0)[\xi \otimes_A v, \xi \otimes_A v] \geq \frac{C(m)}{\nu} |\xi \otimes_A v|^2,
\]

\[
|D^2 f(w_0)[w, \cdot] - Df(w_0)(w)| \leq C(m) V_1(w).
\]

Also, it is worth noting that we have

\[
(2.6) \quad 0 < \inf_{t \in (0, 1)} \frac{V_1(t)}{t^2} \leq \sup_{t \in (0, 1)} \frac{V_1(t)}{t^2} < \infty,
\]

\[
(2.7) \quad V_1(rt) \leq r V_1(t) \text{ for } r \in (0, 1), \quad V_1(rt) \leq r^2 V_1(t) \text{ for } r \in (1, \infty),
\]

\[
(2.8) \quad V_1(|s| + |t|) \leq 4 (V_1(|s|) + V_1(|t|)).
\]

The proof is analogous to the proof of [33, Lemma 4.2] and [32, Lemma 5.1].
2.4. Caccioppoli Inequality of the second kind. The Caccioppoli Inequality is indispensable for our proof of partial regularity. However, it is line for line analogous to the full and symmetric gradient case [33, Proposition 4.3], [32, Proposition 5.2], replacing $\nabla$ or $\varepsilon$ with $\hat{A}$, respectively: by exploiting the strong $V_1\hat{A}$-quasi-convexity and minimality of $u$, we apply Widman’s hole-filling trick and iterate the resulting inequality.

**Proposition 2.10.** We assume that (H0), (H1), and (H2) hold. Let $u \in \operatorname{BV}^n_{\text{loc}}(\Omega)$ be a local generalised minimiser of the functional $F$ and let $a: \mathbb{R}^n \rightarrow \mathbb{R}^N$ be an affine map with $|Aa| \leq m$ for some $m > 0$. Then there exists a constant $C = C(d_V; d_W, \frac{2}{p}) \in (1, \infty)$ such that
\[
\int_{B_{r/2}(x_0)} V(\hat{A}(u - a)) \leq C \int_{B_r(x_0)} V_1 \left( \frac{u - a}{r} \right) \, dx
\]
for any ball $B_r(x_0) \subset \Omega$.

2.5. The Ekeland Variational Principle.

**Lemma 2.11.** [26, Theorem 1.1] Let $(X, d)$ be a complete metric space and let $G: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function for the metric topology, bounded from below and taking a finite value at some point. Assume that for some $x \in X$ and some $\epsilon > 0$ we have
\[
G(u) \leq \inf_X G + \epsilon.
\]
Then, there exists $\tilde{x} \in X$ such that
\begin{enumerate}[(i)]
\item $d(x, \tilde{x}) \leq \sqrt{\epsilon}$,
\item $G(\tilde{x}) \leq G(x)$,
\item $G(y) + \sqrt{\epsilon}d(\tilde{x}, y)$ for all $y \in X$.
\end{enumerate}

2.6. Estimates for Elliptic systems.

**Lemma 2.12.** We are going to consider a strongly $\hat{A}$-Legendre-Hadamard elliptic bilinear form $\mathcal{B}: (\mathcal{H}^0(\hat{A}))^2 \rightarrow \mathbb{R}$, i.e., there exist $\alpha, \beta > 0$ such that for all $\xi \in \mathbb{R}^n$ and $v \in V$ we have
\[
\mathcal{B}[\xi \otimes_{\hat{A}} v, \xi \otimes_{\hat{A}} v] \geq \alpha|\xi \otimes_{\hat{A}} v|^2, \quad \text{and} \quad |\mathcal{B}| \leq \beta.
\]
(i) For every $g \in W^{1, \frac{n+1}{n}}(\mathbb{S}; V)$ there exists a unique weak solution $h \in W^{1, \frac{n+1}{n}}(\mathbb{B}; V)$ of the elliptic system:
\[
\begin{cases}
\hat{A}^*(\mathcal{B}h) = 0 & \text{in } \mathbb{B} \\
h = g & \text{on } \mathbb{S}.
\end{cases}
\]
Furthermore, there exists a positive constant $C = C(d_V, d_W, n, \frac{2}{p})$ such that we have the estimates
\begin{align*}
(2.9) \quad & \|h\|_{W^{1, \frac{n+1}{n}}(\mathbb{B})} \leq C \|g\|_{W^{1, \frac{n+1}{n}}(\mathbb{S})} \quad \text{and} \quad \|\nabla h\|_{L^{\frac{n+1}{n}}(\mathbb{B})} \leq C \|g\|_{W^{1, \frac{n+1}{n}}(\mathbb{S})}. \\
(ii) \quad & \text{For every } g \in L^\infty(\mathbb{B}; W) \text{ and every } p > n \text{ there exists a unique solution } u \in (W^{1, \infty} \cap W_0^{1,p})(\mathbb{B}; V) \text{ of the elliptic system:}
\begin{cases}
\hat{A}^*(\mathcal{B}u) = g & \text{in } \mathbb{B} \\
u = 0 & \text{on } \mathbb{S}.
\end{cases}
\end{align*}
Furthermore, there exists a positive constant $C = C(p, d_V, d_W, n, \frac{2}{p})$ such that $\|u\|_{W^{1, \infty}(\mathbb{B})} \leq C \|g\|_{L^p(\mathbb{B})}$. 

(iii) Moreover, if \( h \in W^{1,1}(\Omega; V) \) satisfies
\[
\mathcal{K}^*(\mathcal{B}h) = 0 \quad \text{in } \mathcal{D}'(\Omega; V),
\]
then \( u \in C^\infty(\Omega; V) \) and
\[
(2.11) \sup_{B_{r/2}} |\nabla u| + r \sup_{B_{r/2}} |\nabla^2 u| \leq C \int_{B_r} |\nabla u| \, dx
\]
for all balls \( B_r := B_r(x_0) \subseteq \Omega \), where \( C = C(n, d_V, d_W, \| \frac{\partial}{\partial R} \|) > 0 \) is constant.

**Proof.** We define the bilinear form \( \mathcal{B} : (\mathbb{R}^{N \times n})^2 \to \mathbb{R} \) by the relations
\[
\mathcal{B}[j_{1}, j_{2}] = \mathcal{B}[A_{k_{1}} j_{1}, A_{k_{2}} j_{2}] \quad \text{for } j_{1}, j_{2} = 1, ..., N, \ k_{1}, k_{2} = 1, ..., n.
\]

Note that by construction we have that \( \mathcal{B} \) is strongly Legendre-Hadamard elliptic, i.e., we have for all \( z \in \mathcal{C}(N, n): \)
\[
\mathcal{B}[z, z] \geq \alpha c_h |z|^2, \quad \text{and } |\mathcal{B}| \leq \beta n \sup_{\alpha = 1} |A_{\alpha}|^2.
\]

Applying [33, Proposition 2.11] in combination with Morrey’s Inequality yields (i) and (ii). For the gradient estimate in (i) we note that we have \( \nabla h = \nabla (h - f_{g} g d\mathcal{H}^{n-1}) \) and that we have \( \| g - f_{g} g d\mathcal{H}^{n-1} \|_{L^1} \leq C[g] \) for a constant independent of \( g \). The third item may be derived by means of the difference quotient method as it has been carried out on the level of first order derivatives in the proof of [13, Proposition 2.10].

**Corollary 2.13.** Let \( B = B_R(x_0), \ r \in (\frac{38R}{|a|}, \frac{39R}{4|a|}), \ \tilde{B} = B_r(x_0) \) and let \( g \in W^{\frac{n+1}{n-1}}(\tilde{B}; V) \). We suppose that \( h \in W^{1, \frac{n+1}{n-1}}(\tilde{B}; V) \) solves the elliptic system:
\[
(2.12) \quad \left\{ \begin{array}{l}
\mathcal{K}^*(\mathcal{B}h) = 0 \quad \text{in } \tilde{B} \\
u = g \quad \text{on } \partial \tilde{B}.
\end{array} \right.
\]

Then there exists \( C(n, d_V, d_W, \| \frac{\partial}{\partial R} \|) > 0 \) such that for all \( \sigma \in (0, \frac{1}{|a|}) \) we have for \( A_k[z] = h(x_0) + (\nabla h(x_0), x - x_0) \):
\[
(2.13) \quad \int_{B_{2\pi R}} V_1 \left( \frac{\lambda - A_k}{\sigma R} \right) \, dx \leq C \sigma^\alpha R^\alpha V_1 \left( \sigma r^{-\frac{\alpha}{\alpha+1}} [g]_{W^{\frac{n+1}{n-1}}(\partial \tilde{B})} \right)
\]
Furthermore, we have:
\[
(2.14) \quad \sup_{\tilde{B}} |\nabla h| \leq C \sigma^{-\frac{\alpha}{\alpha+1}} [g]_{W^{\frac{n+1}{n-1}}(\partial \tilde{B})}.
\]

**Proof.** By Taylor’s formula, we obtain in conjunction with 2.11 and Jensen’s Inequality
\[
\sup_{B_{2\pi R}(x_0)} \left| \frac{h - A_k}{\sigma R} \right| \leq \sigma R \sup_{B_{\frac{\pi R}{N}}(x_0)} |\nabla^2 h| \leq C \sigma \left( \int_{B_{\frac{\pi R}{N}}(x_0)} |\nabla h|^{\frac{n+1}{n-1}} \, dx \right)^{\frac{n-1}{n+1}}.
\]
Combining this with the estimate 2.9 yields the corollary. \( \square \)

### 2.7. Auxiliary Measure Theory.

**Lemma 2.14.** Let \(-\infty < a < b < \infty \) and let \( J \subseteq (a, b) \) be a measurable subset with \( \mathcal{L}^1((a, b) \setminus J) = 0 \). Then for every \( g \in L^1((a, b); \mathbb{R} \geq 0) \), there exists a Lebesgue point \( \xi_0 \) in \( J \) for \( g \) such that
\[
g^*(\xi_0) = \lim_{r \searrow 0} \int_{\xi_0-r}^{\xi_0+r} g \, dx \leq \frac{2}{b - a} \int_{a}^{b} g \, dx,
\]
where \( g^* \) is the precise representative of \( g \).
The Lemma can be verified by means of a contraposition argument. One may assume without loss of generality that \( \int_{a}^{b} g(x) \, dx > 0 \). Then one integrates the reversed inequality to derive a contradiction to the latter integral being positive.

3. Fubini-type theorem

In this section, we are going to establish a Fubini-type property for BV\(A\)-maps. Later on, this will prove essential in order to construct a \(\mathcal{B}\)-harmonic approximation of a given local generalised minimiser. We have seen that certain semi-norms of a fractional Sobolev space on some ball may be estimated from above by the total variation on a larger ball. We will prove that for \( \mathcal{L}\)-almost every sufficiently small radii \( R \), an \((n-1)\)-dimensional analogous estimate holds on a sphere of radius \( R \).

**Theorem 3.1.** Let \( n \geq 2 \) and \( \alpha \in (0, 1) \). Let further be \( x_0 \in \mathbb{R}^n \), \( R > 0 \) and \( u \in \text{BV}_{\text{loc}}^\alpha(\mathbb{R}^n) \). Then for \( \mathcal{L}^1\)-a.e. radius \( r \in (0, R) \), the restrictions \( u|_{\partial B_r(x_0)} \) are well-defined and belong to the space \( W^{\alpha,p}(\partial B_r(x_0); V) \), where \( p := \frac{n}{n - 1 + \alpha} \).

Moreover, there exists a constant \( C = C(\mathcal{A}, n, \alpha) > 0 \), independent of \( x_0 \), \( R \) and \( u \), such that for all \( 0 < s < r < R \) there exists \( t \in (s, r) \) with

\[
\left( \int_{\partial B_t(x_0)} \int_{\partial B_t(x_0)} \frac{|u(x) - u(y)|^p}{|x - y|^{n-1+\alpha p}} \, d\sigma_x \, d\sigma_y \right)^{\frac{1}{p}} \leq C \frac{r^n}{t^{n-1+\alpha p}} \int_{B_{2r}(x_0)} |\mathcal{A}u|,
\]

for some suitable \( b \in \mathcal{N}(\mathcal{A}) \).

**Remark.** In the following constellation \( p = \frac{n+1}{n+1} \), \( \alpha = \frac{1}{n+1} \) and \( s > Cr \), the inequality 3.1 then takes the form

\[
[u_b]_{W^{\alpha+1, \frac{n+1}{n}}(\partial B_r)} \leq Cr^{n+1} \int_{B_{2r}} |\mathcal{A}u|.
\]

**Proof.** The proof is analogous to the one for [32, Theorem 4.1] in the BD-case and only requires minor modifications in the second and the third step.

Let \( \theta \in (0, 1), \varphi \in [1, \infty) \) and \( u \in (W^{\theta,q}(\mathcal{C})(\mathbb{R}^n; V) \). Then it has been established [32, Theorem 4.1] that there is a constant \( C = C(n, \theta, q) > 0 \) such that for all \( R > 0 \) we have

\[
\int_0^R \int_{\partial B_r \times \partial B_r} \frac{|u(x) - u(y)|^q}{|x - y|^{n-1+\alpha q}} \, d\sigma_x \, d\sigma_y \, dr \leq C \int_{B_R \times B_R} \frac{|u(x) - u(y)|^q}{|x - y|^{n+\alpha q}} \, dx \, dy.
\]

The aim now is to establish that \( u \) may be explicitly evaluated \( \mathcal{H}^{n-1}\)-a.e. point wisely on \( \mathcal{L}^1\)-a.e. sphere centred at the origin. For that matter, let \( u \in \text{BV}^\alpha(\mathbb{R}^n) \) and let \( 0 < R_1 < R_2 < \infty \) be arbitrary. The set

\[ I := \{ t \in (R_1, R_2) \mid |\mathcal{A}u|(\partial B_t) > 0 \} \]

is at most countable and hence a \( \mathcal{L}^1\)-nullset. Now let \( t \in (R_1, R_2) \setminus I \). Then by [10, Corollary 4.21],

\[ \mathcal{A}u \mathcal{L}_{\partial B_t} B_t = (\text{Tr}_{B_t}^蒲(u) - \text{Tr}_{B_t}^蒲(u)) \otimes_A \nu_{\partial B_t} \mathcal{H}^{n-1} \mathcal{L}_{\partial B_t} B_t, \]

where \( \nu_{B_t} \) denotes the outer unit normal to \( \partial B_t \). Hence, using that \( t \in (R_1, R_2) \setminus I \) in the last step,

\[ \int_{\partial B_t} |(\text{Tr}_{B_t}^蒲(u) - \text{Tr}_{B_t}^蒲(u)) \otimes_A \nu_{\partial B_t}| \, d\sigma = |\mathcal{A}u|(\partial B_t) = 0. \]
As a result, we have $|\text{Tr}_{B_r}^+(u) - \text{Tr}_{B_r}^-(u)| = 0 \mathcal{H}^{n-1}$-a.e. on $\partial B_1$. Furthermore, writing $\tilde{u}(x) := \text{Tr}_{B_r}^+(u)(x) = \text{Tr}_{B_r}^-(u)(x)$ for such $x \in \partial B_1$, (2.1) implies that

$$
\lim_{r \searrow 0} \int_{B_r(x) \cap \partial B_1} |u - \tilde{u}(x)| \, dy = \lim_{r \searrow 0} \int_{B_r(x) \cap \partial B_1} |u - \tilde{u}(x)| \, dy = 0.
$$

In consequence, we have

$$
\lim_{r \searrow 0} \int_{B_r(x)} |u - \tilde{u}(x)| \, dy = 0.
$$

But this means that $\mathcal{H}^{n-1}$-a.e. $x \in \partial B_1$ is a Lebesgue point of $u$ for $L^1$-a.e. radius $t \in (R_1, R_2)$.

Let $\alpha \in (0, 1)$ be arbitrary and set $p := n/(n - 1 + \alpha)$. Let further $u \in BV^L_\text{loc}(\mathbb{R}^n)$ and consider for $\varepsilon > 0$ a family of standard mollifiers $u_\varepsilon(x) := (\rho_\varepsilon * u)(x)$.

Note that for each Lebesgue point $x \in \mathbb{R}^n$ of $u$, one has $u_\varepsilon(x) \to u^*(x)$ as $\varepsilon \to 0^+$, where $u^*$ is the precise representative of $u$.

Now invoking Lemma 2.7 for $u_\varepsilon$ provides an element $b_\varepsilon \in \mathcal{N}(\hat{A})$ such that

\begin{equation}
(3.5) \quad \left( \int_{B_r} \int_{B_r} \frac{|u_{b_\varepsilon}(x) - u_{b_\varepsilon}(y)|^p}{|x - y|^{n+\alpha p}} \, dx \, dy \right)^{\frac{1}{p}} \leq C r^{1-\alpha} \int_{B_{2r}} |\Delta u_{b_\varepsilon}|,
\end{equation}

where $u_{b_\varepsilon} := u_\varepsilon - b_\varepsilon$. We note also that $b_\varepsilon \in C^\infty_c(\mathbb{R}^n; V)$ are in fact the $L^2$-orthogonal projections of $u_\varepsilon$ onto $\mathcal{N}(\hat{A})$ and satisfy the $L^1$-stability estimate [10, Section 3.1]:

$$
||b_\varepsilon||_{L^1(B_r)} \leq C ||u_\varepsilon||_{L^1(B_r)} \to ||u||_{L^1(B_r)}.
$$

Since $\hat{A}$ is $\mathbb{C}$-elliptic, the nullspace $\mathcal{N}(\hat{A})$ is of finite dimension, so one can find a subsequence $(b_{\varepsilon_j}) \subset (b_\varepsilon)$ and some $b \in \mathcal{N}(\hat{A})$ such that $b_{\varepsilon_j} \to b$ in $\mathcal{N}(\hat{A})$. Consequently, denoting $u_{b_\varepsilon}^*$ to be the precise representative of $u_\varepsilon$, one can estimate

$$
\int_s^r \int_{\partial B_t \times \partial B_t} \frac{|u_{b_\varepsilon}^*(x) - u_{b_\varepsilon}^*(y)|^p}{|x - y|^{n+\alpha p}} \, d\sigma_x \, d\sigma_y \, dt \leq \liminf_{\varepsilon_j \searrow 0} \int_s^r \int_{\partial B_{t_j} \times \partial B_{t_j}} \frac{|u_{b_{\varepsilon_j}}(x) - u_{b_{\varepsilon_j}}(y)|^p}{|x - y|^{n+\alpha p}} \, d\sigma_x \, d\sigma_y \, dt
$$

(by (3.3))

$$
\leq C \liminf_{\varepsilon_j \searrow 0} \int_{B_r} \int_{B_r} \frac{|u_{b_{\varepsilon_j}}(x) - u_{b_{\varepsilon_j}}(y)|^p}{|x - y|^{n+\alpha p}} \, d\sigma_x \, d\sigma_y
$$

(by (3.5))

$$
\leq C \liminf_{\varepsilon_j \searrow 0} r^n \left( r^{1-\alpha} \int_{B_{2r}} |\Delta u_{b_{\varepsilon_j}}| \right)^p \leq C r^n \left( r^{1-\alpha} \int_{B_{2r}} |\Delta u| \right)^p.
$$

Next, towards employing the auxiliary Lemma 2.14, consider the set

$$
J := \{ t \in (s, r) \mid |\Delta u| (\partial B_t) = 0 \}
$$

and let $g : (s, r) \to \mathbb{R}_{\geq 0}$ be defined by

$$
g(t) := \begin{cases} 
\int_{\partial B_t \times \partial B_t} \frac{|u_{b_\varepsilon}^*(x) - u_{b_\varepsilon}^*(y)|^p}{|x - y|^{n+1+\alpha}} \, d\sigma_x \, d\sigma_y, & t \in J, \\
0, & \text{else.}
\end{cases}
$$

Then, by Step 2 and Lemma 2.14, there is a $t \in J$ such that

$$
g^*(t) \leq \frac{2}{r - s} \int_s^r g(t) \, dt \leq C r^n \frac{\left( r^{1-\alpha} \int_{B_{2r}} |\Delta u| \right)^p}{r - s}.
$$
Now plugging the definition of $g(t)$ in the above inequality finally produces
\[
\left( \int_{\partial B} \int_{\partial B} \frac{|u^*_i(x) - u^*_j(y)|^p}{|x-y|^{n+1-\alpha p}} \, dx \, dy \right)^{\frac{1}{p}} \leq C \frac{r^{1-\alpha}}{(r-s)^{\frac{n}{p}} \epsilon} \int_{B_\epsilon} |Au|,
\]
which is the desired estimate (3.1). Note that throughout the computations, the generic constant $C$ did not depend on $u$ or $r$, completing the proof. \qed

4. $\mathcal{B}$-harmonic Approximation

In this section, we are going to construct a $\mathcal{B}$-harmonic Approximation $h$ for a given local generalised minimiser $u$. We will solve an elliptic system, which later on will be a linearisation of the Euler-Lagrange equation at a given average $(\mathcal{A}u)_B$, where $B \subset \Omega$ is a ball. We will assume that $u$ behaves nicely on the boundary $\partial B$ and we will give a precise estimation of $u - h$ in terms of the excess associated to $u$. The Ekeland variational principle allows us to prove a corresponding estimate for all minimisers close to $u$ and in the limit the estimate is inherited to $u$ itself. The precise control of $u - h$ will play a crucial role in the excess decay estimates.

**Theorem 4.1.** Assuming (H0), (H1) and (H2), let $u \in BV^{A}_{\text{loc}}(\Omega)$ be a generalised local minimiser of $\mathcal{F}$, let $M > 0$, $q \in (1, \frac{1}{n-1})$, and let $B = B_r(x_0) \subset \Omega$ such that
\[
u|_{\partial B} = \text{Tr}^{-1}_B(u) = \text{Tr}^{+}_B(u) \in W^{\frac{n+1}{n}, q+1}((\partial B)).
\]
then there exists a constant $C = C(M, d_V, d_W, n, q, \frac{L}{\alpha}) > 0$ such that
\[
\int_B V \left( \frac{u - a - h}{r} \right) \, dx \leq C \left( \int_B V(\mathcal{A}(u-a)) \right)^q.
\]

**Proof.** We confine ourselves to merely sketching the proof, since it follows the lines of the proof [32, Proposition 5.4]. To start with we fix some notation $\tilde{u} = u - a$, $\tilde{f} = (f)_{\mathcal{A}h}$ and $X = \{ v \in W^{1,1}(\Omega; \mathbb{R}^N) \mid \text{Tr}(v) = \text{Tr}(\tilde{u}) \}$. Let $\epsilon > 0$. There is a $u_\epsilon \in X$ such that
\[
\int_B \left| u_\epsilon - \tilde{u} \right| + \int_B V_1(\mathcal{A}u_\epsilon) - V_1(\mathcal{A}\tilde{u}) \leq \epsilon^2, \quad \int_B \tilde{f}(\mathcal{A}u_\epsilon) \leq \int_B \tilde{f}(\mathcal{A}\tilde{u}) + \epsilon^2.
\]
Noting that $\tilde{u}$ is a local generalised minimiser of the functional $\mathcal{F}_0[\zeta; \omega] = \int_\omega \tilde{f}(\zeta) \, dx$ and taking proposition 2.6 into account allows to apply the Ekeland variational principle 2.11 to $(X, d)$, $x = u_\epsilon$ and $\mathcal{F} = \mathcal{F}_0$, where $d(\zeta_1, \zeta_2) = \| \mathcal{A}(\zeta_1 - \zeta_2) \|_{L^1(B)}$. This in conjunction with Poincaré’s Inequality, yields $u_\epsilon \in X$ such that for all $\zeta \in W^{1,1}_0(B)$ we have
\[
\left| \int_B D\tilde{f}(\mathcal{A}u_\epsilon)[\mathcal{A}\zeta] \, dx \right| \leq C \int_B |\mathcal{A}\zeta| \, dx \text{ and } \| u_\epsilon - \tilde{u}_\epsilon \|_{W^{1,1}(B)} \leq C \epsilon^{(r^n + r^{n+1})}
\]
for a constant $C > 0$ independent of $u_\epsilon, \tilde{u}_\epsilon$ and $r$. Writing $\mathcal{B}[\tilde{u}_\epsilon, \cdot] = \Lambda[\mathcal{A}\tilde{u}_\epsilon, \cdot] + D\tilde{f}(\mathcal{A}\tilde{u}_\epsilon)$ and applying the pointwise estimate 2.5 to the first term then yields for some constant $C(M, L) > 0$ and all $\zeta \in W^{1,1}_0(B)$
\[
\left| \int_B \mathcal{B}[\mathcal{A}\tilde{u}_\epsilon, \mathcal{A}\zeta] \, dx \right| \leq C \int_B V_1(\mathcal{A}\tilde{u}_\epsilon)|\mathcal{A}\zeta| \, dx + \epsilon \int_B |\mathcal{A}\zeta| \, dx.
\]
At this stage, we scale things to the unit ball $\mathbb{B}$: For a measurable map $p$ defined on $B$ we put $S[p](x) = r^{-1}p(rx)$. Furthermore, we put $\Psi_\varepsilon = S[\hat{u}_\varepsilon - h]$ and $U_\varepsilon = S[\hat{u}_\varepsilon]$. We will now truncate the map $\Psi_\varepsilon$. To this aim, we put

$$T(w) = \begin{cases} w, & \|w\| \leq 1 \\ \frac{w}{\|w\|}, & \|w\| > 1. \end{cases}$$

We fix $p > n$ and let $\Phi_\varepsilon \in W^{1,\infty}(\mathbb{B}; V)$ be the solution of the elliptic system

$$\Phi_\varepsilon = 0 \quad \text{on} \quad S.$$

Now testing the system with $\hat{\Psi}_\varepsilon$ we put

$$\tilde{\Psi}_\varepsilon := \left(\varepsilon \Phi_\varepsilon\right)\circ \hat{\Psi}_\varepsilon - \varepsilon \Phi_\varepsilon.$$

In this section, we will display the most important step towards proving an excess decay for a given local generalised minimiser $u$. First, we will invoke Caccioppoli’s Inequality and then, using our Fubini-type theorem for BV $\hat{A}$-maps, we will construct a $\hat{A}$-harmonic approximation. Then we will show separately that $h$ and $u - h$ have a good decay. Here, the excess of $u$ is defined as follows:

$$E(u, x, r) := \int_{B_r(x)} V_1(a u - (a u)_{B_r(x)}) \quad \text{and} \quad \tilde{E}(u, x, r) := \frac{E(u, x, r)}{\mathcal{L}^n(B_r(x))}.$$

**Lemma 5.1.** Assuming (H0), (H1) and (H2), let $u \in BV^A_{\text{loc}}(\Omega)$ be a generalised local minimiser of $\mathcal{F}$. Furthermore, let $M > 0$ and $q \in (1, \frac{np}{n-1})$. There exists a constant $C(M, A, q, d_V, d_W, \frac{2}{q}) > 0$ with the property: If we have $\tilde{B} = B(x_0, R) \subset \Omega$, 

$$|(a u)| < M \quad \text{and} \quad \int_{\tilde{B}} |a u - (a u)_{\tilde{B}}| \leq 1,$$

then for all $\sigma \in (0, 1]$ we have

$$\tilde{E}(u; x_0, \sigma R) \leq c\left(\sigma^2 + \sigma^{-n-2}\left(\tilde{E}(u; x_0, R)\right)^{q-1}\right)\tilde{E}(u; x_0, R).$$

**Proof.** We are going to set up the proof as follows:

- Due to Lemma 2.8 we find $\tilde{a} \in \mathcal{L}(\mathbb{R}^n; V)$ such that $\hat{a} = (\hat{a} u)_{B_r}$. We put $\tilde{u} = u - a$, $\tilde{f} = (f)(\hat{a} u)^\#$, and $\tilde{u} = D^2 \tilde{f}(0)$.

- Due to Theorem 3.1 there exists a radius $r \in \left(\frac{38R}{40}, \frac{39R}{40}\right)$ such that $\tilde{u}|_{\partial B_r(x_0)}$ is well-defined and belongs to the space $W^{1,\infty}(\mathbb{B})$ with $\tilde{B} = B_r$. Furthermore, for some $b \in \mathcal{N}(\hat{a})$, we have

$$\left|\tilde{u}_b\right|_{W^{1,\infty}(\mathbb{B})} \leq C \int_{B_{2r}(x_0)} |\hat{a} u| \leq C \tilde{E}(u; x_0, R)$$

for a constant $C(A, n, \alpha)$.
• Let $h$ be the solution of the elliptic system
\[
\begin{cases}
\mathcal{A}^*(\mathcal{A}h) = 0 & \text{in } \hat{B} \\
h = u|_{\partial \hat{B}} - a & \text{on } \partial \hat{B}.
\end{cases}
\]

• In view of Corollary 2.13, we put $a_0 = a + A\mathcal{A}^{-b}$ and let $\sigma \in (0, \frac{1}{\text{w}^*})$ be arbitrary but fixed. Since we have $\int_B V_1(\mathcal{A}u) \leq 1$, we may estimate by means of Lemma 2.9 and Jensen’s Inequality
\[
V_1(\sigma \int_B \mathcal{A}u) \leq C\sigma^2 \left( \int_B \mathcal{A}u \right)^2 \leq C\sigma^2 V_1(\int_B \mathcal{A}u) \leq C\sigma^2 \mathcal{E}(u; x_0, R).
\]

Combining the estimates 5.3, 2.13 and 5.2 yields
\[
\int_{B_{2R}} V_1(\frac{h - b - A\mathcal{A}^{-b}}{2\sigma R}) \leq CR^n\sigma^{n+2}\mathcal{E}(u; x_0, R).
\]

We note that we have due to Corollary 2.13 $|\mathcal{A}u| \leq M_0 + C\int_{B_{2R}} |\mathcal{A}u| \leq M_0 + C := m$ for a constant $C(m, \mathcal{A}, dW, dW, \frac{1}{\nu})$. Especially, $u - b$ is a generalised local minimiser as well. By Caccioppoli’s Inequality, Proposition 5.2, we estimate
\[
\mathcal{E}(u; x_0, \sigma R) \leq C \int_{B_{2R}} V_1(\frac{u - b - a_0}{\sigma R}) \leq C \sigma^{-2} \int_{B_{2R}} V_1(\frac{\hat{u} - h}{r}) + \int_{B_{2R}} V_1(\frac{h - b - A\mathcal{A}^{-b}}{2\sigma R}) \leq C \sigma^{-2} R^n (\mathcal{E}(u; x_0, R))^n + R^n\sigma^{n+2}\mathcal{E}(u; x_0, 2R)
\]
for a constant $C(m, \mathcal{A}, dV, dW, \frac{1}{\nu})$. In the second step, we have exploited Theorem 4.1 and the estimate 5.4.

**Proposition 5.2.** Assuming $(H0)$, $(H1)$ and $(H2)$, let $u \in \text{BV}_{\text{loc}}(\Omega)$ be a generalised local minimiser of $\mathcal{F}$, where $f$ satisfies $(H1)$ and $(H2)$. Let $\alpha \in (0, 1)$ and $M > 0$. Then there exist constants $\gamma(M, \mathcal{A}, \alpha, dV, dW, \frac{1}{\nu})$ and $\epsilon(M, \mathcal{A}, \alpha, dV, dW, \frac{1}{\nu}) > 0$ with the property: If we have $B = B(x_0, R) \Subset \Omega$, $|\mathcal{A}u| < M$ and $\mathcal{E}(u; x_0, R) < \epsilon$, then we have for all $\partial B(x_0)$:
\[
\mathcal{E}(u; x_0, \partial R) \leq \gamma \theta^n \mathcal{E}(u; x_0, R).
\]

The proof is analogous to the proof of [33, Proposition 4.8] or [32, Proposition 5.7] since we already have established Lemma 5.1. At this stage, we sketch the proof of the main theorem. We will pay special attention to the final step, which needs to be modified in regards to [32]:

**Proof.** Using [39, 1.6.1, Theorem 1, 1.6.2, Theorem 3], we observe that $\mathcal{L}^n(S_\sigma) = 0$.

Let $x_0 \in \Omega \setminus \mathcal{S}_\sigma$ and $\alpha \in (0, 1)$. For $M > 0$, we denote by $\gamma_M$ and $\epsilon_M$ the constants determined by Proposition 5.2. Then there exists $M > 0$ and a radius $R > 0$ with $B_R(x_0) \subset \Omega$ such that for all $x \in B = B_{\hat{R}}(x_0)$ we have
\[
|\mathcal{A}u_B| \leq M \quad \text{and} \quad \mathcal{E}(u; x; \frac{R}{2}) \leq \epsilon_M.
\]

This can be inferred from a similar argument to the one in [32, p. 32]. In particular, we then have for all $x \in B = B_{\hat{R}}(x_0)$ and all $0 < r < \frac{R}{2}$,
\[
\mathcal{E}(u; x; r) \leq \gamma_M\epsilon_M \left( \frac{2}{R} \right)^\alpha r^\alpha.
\]

From this decay, it can be deduced by a simple covering argument that the measure $|\mathcal{A}u|_{L_B}$ is absolutely continuous with respect to the Lebesgue measure.
Towards the Hölder continuity of the distributional gradient $Du$, we invoke Campanato’s characterisation [12, Theorem 5.1]: Let $k = \dim_{\mathbb{R}} \mathcal{N}(A) \geq 1$ and $q$ be a linear $V$-valued polynomial such that $Aq = (Au)_{B(x,r)}$. Furthermore, we put $p = q + \Pi(u - q) \in \mathcal{P}_k(\mathbb{R}^N)$. Now estimate for all $x \in B$ and all $0 < r < \frac{R}{2}$

$$\int_{B(y,r)} |u - p| \, dx \leq C_{\text{Poin}} r \int_{B(y,r)} |A(u - q)| \, dx \leq Cr^{1+\alpha}.$$ 

$\square$
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