A gapped generalization of
Kingman’s subadditive ergodic theorem

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Abstract

We state and prove a generalization of Kingman’s ergodic theorem on a measure-preserving dynamical system \((X, \mathcal{F}, \mu, T)\) where the \(\mu\)-almost sure subadditivity condition
\[
f_{n+m} \leq f_n + f_m \circ T^n
\]
is relaxed to a \(\mu\)-almost sure, “gapped”, almost subadditivity condition of the form
\[
f_{n+\sigma_m+m} \leq f_n + \rho_n + f_m \circ T^{n+\sigma_n}
\]
for some nonnegative \(\rho_n \in L^1(d\mu)\) and \(\sigma_n \in \mathbb{N} \cup \{0\}\) that are suitably sublinear in \(n\). This generalization has a first application to the existence of specific relative entropies for suitably decoupled measures on one-sided shifts.

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1 Introduction

In this short note, we consider a general measurable space \((X, \mathcal{F})\), on which a probability measure \(\mu\) is invariant for some measurable transformation \(T : X \to X\), and state and prove an ergodic theorem for sequences \((f_n)_{n \in \mathbb{N}}\) of measurable functions on \((X, \mathcal{F})\) satisfying what we will call a “gapped almost subadditivity condition”. It is a generalization of Kingman’s subadditive ergodic theorem \([Ki68]\) that — to the author’s knowledge — has not been pointed out in the literature. The basic theorem, Theorem 3.1, yields \(\mu\)-almost sure convergence of \(\frac{1}{n} f_n\) as \(n \to \infty\), but may be strengthened to convergence in \(L^1(d\mu)\) under stronger assumptions using Theorem 3.3.

For the author of the present note, the interest in such a generalization stemmed from the study of estimators of specific relative entropies for pairs of shift-invariant Borel probability measures on the space \(A^\mathbb{N}\) of sequences \(x = (x_k)_{k \in \mathbb{N}}\) with values in some finite alphabet \(A\) \([CDEJRa, CDEJRb]\). The focus there is on measures satisfying decoupling-type conditions, already exploited in works such as \([Pf02]\) by Ch.-É. Pfister as well as \([CJPS19]\) by N. Cuneo, V. Jakšić, C.-A. Pillet and A. Shirikyan on the theory of large deviations. Such conditions become gapped subadditivity-type conditions upon taking logarithms. As a concrete example, consider the following upper-decoupling property for a probability measure \(Q\) on \(A^\mathbb{N}\):

There exist \(o(n)\)-sequences \((c_n)_{n \in \mathbb{N}}\) and \((\tau_n)_{n \in \mathbb{N}}\) such that
\[
Q\{x : x_1^n = a, x_{n+\tau_n+1}^{n+\tau_n+m} = b\} \leq e^{c_n} Q\{x : x_1^n = a\} Q\{x : x_{n+\tau_n+1}^{n+\tau_n+m} = b\}
\]
for all \(a \in A^n\), \(n \in \mathbb{N}\), \(b \in A^m\) and \(m \in \mathbb{N}\).
Indeed, the functions $f_n : x \mapsto \log Q_n(x^n)$ then define a gapped, almost subadditive sequence in the sense of the present note. Hence, by corollary of Theorems 3.1 and 3.3 (and the Shannon–McMillan–Breiman theorem), we have the following result. We refer to [CJPS19, CDEJRa, CDEJRb] for thorough discussions of the range of applicability of the upper decoupling condition and its generalizations, including adaptations to countably infinite alphabets.

**Corollary** (Special case of [CDEJRa, §5]). Let $P$ and $Q$ be shift-invariant probability measures on $\mathcal{A}^\mathbb{N}$. If $Q$ satisfies the above upper-decoupling condition, then the (possibly infinite) $P$-almost sure limit

$$h_{P,Q}(x) := \lim_{n \to \infty} \frac{1}{n} \log Q_n \{ y : y_1^n = x_1^n \}$$

exists, and so does the (possibly infinite) specific relative entropy of $P$ with respect to $Q$.

## 2 Gapped subadditivity and Fekete’s lemma

The main difference with previous works that the author is aware of is that the subadditivity-type condition under consideration for the ergodic theorem below allows for what we will call “gaps”. Both as an illustration and as a technical building block, we note the following generalization of Fekete’s lemma.

**Lemma 2.1.** Let $(F_n)_{n \in \mathbb{N}}$ be a sequence\(^1\) in $[\infty, \infty)$ and suppose that there exist $o(n)$-sequences $(\sigma_n)_{n \in \mathbb{N}}$ and $(R_n)_{n \in \mathbb{N}}$ of nonnegative integers such that

$$F_{n+\sigma_n+m} \leq F_n + R_n + F_m, \quad (2.1)$$

for all $n, m \in \mathbb{N}$. Then, with

$$F := \inf_{n \in \mathbb{N}} \frac{F_n + R_n}{n + \sigma_n} \quad (2.2)$$

understood in $[\infty, \infty)$, we have

$$\lim_{n \to \infty} \frac{F_n}{n} = F.$$

**Proof.** Since the assumptions on $(R_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ and the definition of $F$ imply that

$$\liminf_{n \to \infty} \frac{F_n}{n} = \liminf_{n \to \infty} \frac{F_n + R_n}{n + \sigma_n} \geq \inf_{n \in \mathbb{N}} \frac{F_n + R_n}{n + \sigma_n} = F,$$

it suffices to show that

$$\limsup_{n \to \infty} \frac{F_n}{n} \leq F. \quad (2.3)$$

To this end, let $F' > F$ be arbitrary. By definition of the infimum under consideration, there exists $r \in \mathbb{N}$ such that

$$F_r + R_r \leq F'(r + \sigma_r).$$

Now, note that an arbitrary number $n \in \mathbb{N}$, may be written as $n = k_n(r + \sigma_r) + q_n$ for some $k_n \in \mathbb{N}$ and $q_n \in \{1, 2, \ldots, r + \sigma_r\}$. Hence, using (2.1) repeatedly,

$$F_n \leq k_n(F_r + R_r) + F_{q_n} \leq k_n F'(r + \sigma_r) + \max \{ F_{q_n} : q_n \in \{1, 2, \ldots, r + \sigma_r\} \}.$$

Therefore,

$$\frac{F_n}{n} \leq \frac{k_n(r + \sigma_r)}{k_n(r + \sigma_r) + q_n} F' + \frac{\max \{ F_{q_n} : q_n \in \{1, 2, \ldots, r + \sigma_r\} \}}{n}.$$

\(^1\)We choose the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$. 

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Because \( k_n \to \infty \) as \( n \to \infty \) and because \( q_n \leq r + \sigma_r \) independently of \( n \), we may deduce that
\[
\limsup_{n \to \infty} \frac{F_n}{n} \leq F'.
\]
Since \( F' > F \) was arbitrary, we conclude that the inequality (2.3) indeed holds.

In what follows, the terms “gap” and “gapped” will refer to the possibly nonzero integers \( \sigma_m \) that appear in conditions such as (2.1) — this has nothing to do with the spectral-theoretic notion of “gap”. In this terminology, the usual statement of Fekete’s lemma is the gapless case.

### 3 Main result

Several decades after M. Fekete’s lemma, J.F.C. Kingman’s ergodic theorem highlighted the role of (gapless) subadditivity in ergodic theory and dynamical systems [Ki68]. Starting in the 1980s, new proofs of the original result paved the way for several key (gapless) generalizations, most notably by Y. Derriennic [De83] and by K. Schürger [Sc91]. Our main result is a gapped version of such a generalization of J.F.C. Kingman’s theorem.

**Theorem 3.1.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions with positive parts \( f_{n,+} \in L^1(d\mu) \) for all \( n \in \mathbb{N} \). Suppose in addition that there exists an \( o(n) \)-sequence \((\sigma_n)_{n \in \mathbb{N}}\) of nonnegative integers with \( \sigma_1 = 0 \) and a sequence \((\rho_n)_{n \in \mathbb{N}}\) of nonnegative functions in \( L^1(d\mu) \) such that
\[
f_{n+\sigma_n+m} \leq f_n + \rho_n + f_m \circ T^{n+\sigma_n}
\]
\( \mu \)-almost surely for all \( n,m \in \mathbb{N} \), and such that
\[
\lim_{n \to \infty} \frac{\rho_n}{n} = 0
\]
\( \mu \)-almost surely. Then, the limit
\[
f = \lim_{n \to \infty} \frac{f_n}{n}
\]
exists \( \mu \)-almost surely.

Before we proceed with the proof of this almost sure convergence — in steps mostly inspired by K. Schürger’s sequence of arguments in [Sc91, §2], but also to some extent by J.M. Steele [St89, §2] and by A. Avila and J. Bochi [AB] —, we briefly comment on its hypotheses: it is condition (3.1) that we call a gapped almost subadditivity condition, and the condition \( \sigma_n = o(n) \) controls the size of the gaps in this gapped condition. Let us now proceed.

**Proof.** We set
\[
f := \liminf_{n \to \infty} \frac{f_n}{n}
\]
and want to show that, \( \mu \)-almost surely, this limit inferior coincides with the corresponding limit superior. To do so, we introduce an arbitrarily small parameter \( \epsilon > 0 \).

**Step 1** Let us first show that, as a consequence of (3.1) with \( n = 1 \) and \( \sigma_1 = 0 \), we have the identity
\[
f \circ T = f
\]
in the almost sure sense. First, note that taking \( m \to \infty \) there, we find
\[
f \leq \liminf_{m \to \infty} \left( \frac{f_{1,+}}{m} + \frac{\rho_1}{m} + \frac{f_m \circ T}{m} \right)
\]
\[
= f \circ T
\]
in the almost sure sense. But then, we have the inclusion \( \{ x : f(Tx) \geq y \} \supseteq \{ x : f(x) \geq y \} \) for all \( y \in [-\infty, \infty) \), and since
\[
\mu \{ x : f(Tx) \geq y \} = (\mu \circ T^{-1})\{ x : f(x) \geq y \} = \mu \{ x : f(x) \geq y \}
\]
by \( T \)-invariance of the measure \( \mu \), we conclude that
\[
\mu(\{ x : f(Tx) \geq y \} \triangle \{ x : f(x) \geq y \}) = 0
\]
for all \( y \in [-\infty, \infty) \), and therefore that (3.4) indeed holds.

**Step 2** Let us show that, for \( r \in \mathbb{N} \) large enough, almost surely, there exists \( k \) such that
\[
\frac{f_{kr} + \rho_{kr}}{kr + \sigma_{kr}} \leq \max \{ f, -\epsilon^{-1} \} + \epsilon.
\]
(3.5)

Given any \( n \in \mathbb{N} \), there exists a natural number \( k_n \) such that
\[
(k_n - 1)r \leq n + \sigma_n < k_n r,
\]
and then, by (3.1), we have the almost sure inequality
\[
f_{k_n r} \leq f_n + \rho_n + f_{k_n r - n - \sigma_n} \circ T^{n + \sigma_n}
\]
(3.7)

Note that
\[
f_{k_n r - n - \sigma_n} \circ T^{n + \sigma_n} \leq \sum_{q=1}^{r} f_{q,+} \circ T^{n + \sigma_n}.
\]
by (3.6). Hence, dividing (3.7) by \( n \) and taking \( n \to \infty \),
\[
\liminf_{n \to \infty} \frac{f_{k_n r}}{n} \leq \liminf_{n \to \infty} \left( \frac{f_n}{n} + \frac{\rho_n}{n} + \frac{\sum_{q=1}^{r} f_{q,+} \circ T^{n + \sigma_n}}{n} \right),
\]
almost surely. In view of (3.2), (3.6), and a standard consequence of Birkhoff’s theorem\(^2\), we have the almost sure inequality
\[
\liminf_{n \to \infty} \frac{f_{k_n r}}{k_n r} \leq \liminf_{n \to \infty} \frac{f_n}{n}.
\]
Recall that the right-hand side is the definition of \( f \). Hence, for \( r \) large enough, there almost surely exist infinitely many \( k \) such that
\[
\frac{f_{kr}}{kr + \sigma_{kr}} \leq \max \{ f, -\epsilon^{-1} \} + \epsilon.
\]
The criterion on \( r \) can be determined as a function of \( \epsilon \) and \( (\sigma_r)_{r \in \mathbb{N}} \) only. But, again almost surely,
\[
\frac{\rho_{kr}}{kr + \sigma_{kr}} \leq \epsilon
\]
for all \( k \) large enough in view of (3.2) and nonnegativity of \( \rho_{kr} \). This gives the desired conclusion.

**Step 3** Let us show that, for \( r \in \mathbb{N} \) large enough, the sets
\[
D^{r,K,\epsilon} := \bigcap_{k=1}^{K} \left\{ x : \frac{f_{kr}(x) + \rho_{kr}(x)}{kr + \sigma_{kr}} > \max \{ f(x), -\epsilon^{-1} \} + \epsilon \right\}
\]

\(^2\)See e.g. Lemma 2 in [AB] for a direct proof. This lemma is to be applied to each of the finitely many integrable functions of the form \( f_{q,+} \) using the fact that \( \sigma_n = o(n) \).
are such that
\[
\lim_{K \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} (1_{D_r,K}(1 + f_{r,+} + \rho_r)) \circ T^j = 0
\]
almost surely. To do so, we omit some indices and let
\[
\psi_{n,K} := \frac{1}{n} \sum_{j=0}^{n-1} (1_{D_K}(1 + f_{r,+} + \rho_r)) \circ T^j.
\]
By Birkhoff’s theorem, the limit \( \psi_K = \lim_{n \to \infty} \psi_{n,K} \)
almost surely exists and
\[
\int \psi_K \, d\mu = \int_{D_K} (1 + f_{r,+} + \rho_r) \, d\mu.
\]
On the other hand, Lebesgue dominated convergence guarantees that
\[
\int \lim_{K \to \infty} \psi_K \, d\mu = \lim_{K \to \infty} \int \psi_K \, d\mu.
\]
Therefore, because \( \psi_K \) is nonnegative, we need only show that the right-hand side vanishes. But, using once again Lebesgue dominated convergence,
\[
\lim_{K \to \infty} \int \psi_K \, d\mu = \lim_{K \to \infty} \int_{D_K} (1 + f_{r,+} + \rho_r) \, d\mu = \int_{\bigcap_{k \in \mathbb{N}} D_k} (1 + f_{r,+} + \rho_r) \, d\mu,
\]
so this follows from Step 2 provided that \( r \) is large enough.

**Step 4** Let us show that
\[
\limsup_{n \to \infty} \frac{f_n(x)}{n} \leq \max \{ f_r, -\epsilon^{-1} \} + \epsilon.
\]
almost surely. To do so, fix \( r \) large enough and \( x \) in a set of measure 1 where conclusions of Steps 1 and 3 hold, and let us construct a gapped Steele-type collection \((I_\ell)_{\ell \in \mathbb{N}}\) of ordered disjoints intervals in \( \mathbb{N} \) based on the behaviour of the trajectory starting at \( x \).

**Base case** Set \( I_0 = \{0\} \).

**Induction** Suppose that we have constructed \( I_0 \) up to \( I_\ell \) and let \( m_\ell \) be the maximum of \( I_\ell \).

**Case 1.** If \( T^{m_\ell} x \notin D_r,K,\epsilon \), let \( I_{\ell+1} = [m_\ell + 1, m_\ell + k_{\ell+1}r + \sigma_{k_{\ell+1}}] \cap \mathbb{N} \), where \( k_{\ell+1} \) is the smallest \( k \in \{1,2,\ldots,K\} \) such that
\[
\frac{f_{kr}(T^{m_\ell}x) + \rho_{kr}(T^{m_\ell}x)}{kr + \sigma_{kr}} \leq \max \{ f(T^{m_\ell}x), -\epsilon^{-1} \} + \epsilon. \tag{3.8}
\]

**Case 2.** If \( T^{m_\ell}x \in D_r,K,\epsilon \), let \( I_{\ell+1} = [m_\ell + 1, m_\ell + r + \sigma_r] \cap \mathbb{N} \).

Given \( n \in \mathbb{N} \) large enough, we may split relevant indices into two categories\(^3\)
\[
G_n = \{ \ell : I_\ell \subseteq [1, n-1] \text{ and } I_\ell \text{ follows Case 1} \}.
\]
and
\[
B_n = \{ \ell : I_\ell \subseteq [1, n-1] \text{ and } I_\ell \text{ follows Case 2} \}.
\]
Now, we set \( M_n := m_{\max(B_n \cup G_n)} \) and use (3.1) repeatedly along these intervals to write
\[
f_n(x) \leq \sum_{\ell \in G_n} (f_{kr} + \rho_{kr})(T^{m_{\ell-1}}x) + \sum_{\ell \in B_n} (f_r + \rho_r)(T^{m_{\ell-1}}x) + f_{n-M_n}(T^{M_n}x); \tag{3.9}
\]
Figure 1: Illustration of the gapped Steele-type collection of intervals in Step 4 of the proof of Theorem 3.1: green corresponds to Case 1 (deemed “good”) and red corresponds to Case 2 (deemed “bad”). The construction of these blocks along which we exploit the gapped almost subadditivity condition depends on $x, r, K$ and $\epsilon$.

see Figure 1. Note that

$$\sum_{\ell \in G_n} (f_{k\ell r} + \rho_{k\ell r})(T^{m_{\ell-1}}x) \leq \left( \max \{ f(x), -\epsilon^{-1} \} + \epsilon \right) \sum_{\ell \in G_n} (k\ell r + \sigma_{k\ell r})$$

by (3.8) and (3.4),

$$\sum_{\ell \in B_n} (f_r + \rho_r)(T^{m_{\ell-1}}x) \leq \sum_{j=0}^n (1_{D_r, K, \epsilon}(f_{r, +} + \rho_r))(T^j x)$$

by nonnegativity, and

$$f_{n-M_n, +}(T^{M_n}x) \leq \sum_{q=1}^{K_r + \sigma_{K_r}} f_{q, +}(T^{M_n}x),$$

where $\sigma_{K_r} := \max\{\sigma_{k\ell r} : k = 1, 2, \ldots, K\}$ is independent of $n$. Therefore, appealing to Step 3 and Birkhoff’s theorem as used in Step 2, we have the bound

$$\limsup_{n \to \infty} \frac{f_n(x)}{n} \leq \limsup_{K \to \infty} \limsup_{n \to \infty} \left( \max \{ f(x), -\epsilon^{-1} \} + \epsilon \right) \sum_{\ell \in G_n} (k\ell r + \sigma_{k\ell r}).$$

Note that, by construction,

$$n \leq \sum_{\ell \in G_n} (k\ell r + \sigma_{k\ell r}) + \sum_{\ell \in B_n} (r + \sigma_r) + (n - M_n)$$

$$\leq \sum_{\ell \in G_n} (k\ell r + \sigma_{k\ell r}) + \sum_{j=0}^n (r + \sigma_r) 1_{D_r, K, \epsilon} \circ T^j + K_r + \sigma_{K_r},$$

so that

$$n - (r + \sigma_r) \sum_{j=0}^n 1_{D_r, K, \epsilon} \circ T^j - K_r - \sigma_{K_r} \leq \sum_{\ell \in G_n} (k\ell r + \sigma_{k\ell r}) \leq n.$$

Therefore,

$$\limsup_{n \to \infty} \frac{f_n(x)}{n} \leq \max \{ f(x), -\epsilon^{-1} \} + \epsilon$$

$$+ \epsilon^{-1} \limsup_{K \to \infty} \limsup_{n \to \infty} \left( \frac{K_r + \sigma_{K_r}}{n} + \frac{r + \sigma_r}{n} \sum_{j=0}^n 1_{D_r, K, \epsilon} \circ T^j(x) \right).$$

Taking first $n \to \infty$, and then $K \to \infty$ using again the conclusion of Step 3, we obtain the desired inequality.

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3Informally, $\ell \in G_n$ is “good” because the term $f_{k\ell r}$ appearing in the estimate below can be bounded in terms of $f$ and $\epsilon$ directly; $\ell \in B_n$ is “bad” because the corresponding term $f_{r, +}$ cannot — but this is rare enough for $n \gg K \gg 1$ in view of Step 3.
The fact that the conclusion of Step 4 holds \( \mu \)-almost surely for an arbitrarily small \( \epsilon > 0 \), together with the fact that \( f \) was initially defined as the corresponding limit inferior, gives the existence of the desired limit, \( \mu \)-almost surely.

\[ \square \]

**Remark 3.2.** Note that the requirement that \( \sigma_1 = 0 \) was only used in deriving the \( \mu \)-almost sure inequality \( f \leq f \circ T \) and can therefore be discarded if one knows \textit{a priori} that this inequality holds \( \mu \)-almost surely, \textit{e.g.} because \( f_{n+1} \leq f_n \circ T \) for all \( n \in \mathbb{N} \) large enough, as is the case in [CDEJRa, CDEJRb] and in the corollary presented in the Introduction, where \( f_n \) is the logarithm of the \( n \)-th marginal of a probability measure. In fact, there \( f_{n+1} \leq f_n \) as well and this simplifies Steps 2 and 4 of the proof.

An additional assumption on the sequence \( (\rho_n)_{n \in \mathbb{N}} \) of error terms then allows to prove convergence at the level of the integrals. As announced in the introduction, the theorem we are about to state and prove gives convergence in \( L^1(\mu) \) using the Riesz–Scheffé lemma if the functions \( f_n \) all have a definite sign and if \( f \in L^1(\mu) \)—the latter being equivalent in this context to the requirement that \( \inf_{n \in \mathbb{N}} \frac{1}{n} \int f_n \, d\mu > -\infty \).

**Theorem 3.3.** If, in addition to the hypotheses of Theorem 3.1, the sequence \( (\rho_m)_{m \in \mathbb{N}} \) satisfies

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \int \rho_n \, d\mu = 0, \tag{3.10}
\end{equation}

then

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu = \int f \, d\mu, \tag{3.11}
\end{equation}

understood in \([-\infty, \infty]\).

**Proof.** We prove the two inequalities behind the desired equality (3.11) separately using a temporary cutoff depending on an arbitrarily small parameter \( \epsilon > 0 \).

**Step 1** Integrating (3.1) and exploiting invariance of the measure, we define a sequence of integrals\(^4\) by \( F_n := \int f_n \, d\mu \) and apply Lemma 2.1 with the error terms defined by \( R_n := \int \rho_n \, d\mu \) being \( o(n) \) thanks to (3.10). This observation yields

\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu &= \inf_{n \in \mathbb{N}} \frac{1}{n + \sigma_n} \left( \int f_n \, d\mu + \int \rho_n \, d\mu \right) \\
&= \inf_{n \in \mathbb{N}, \epsilon > 0} \frac{1}{n + \sigma_n} \left( \int \max \{ f_n, -n\epsilon^{-1} \} \, d\mu + \int \rho_n \, d\mu \right) \\
&= \inf_{\epsilon > 0} \lim_{n \to \infty} \frac{1}{n + \sigma_n} \left( \int \max \{ f_n, -n\epsilon^{-1} \} \, d\mu + \int \rho_n \, d\mu \right)
\end{align*}

in view of Lebesgue monotone convergence in \( \epsilon \) at fixed \( n \) and integrability of the positive part of \( f_n \). Now, using again Lemma 2.1 to deal with the inner infimum on the right-hand side,

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \int \max \{ f_n, -n\epsilon^{-1} \} \, d\mu. \tag{3.12}
\end{equation}

**Step 2** At fixed \( \epsilon > 0 \), Fatou’s lemma applies with \(-\epsilon^{-1}\) as an integrable lower bound, yielding

\begin{equation}
\liminf_{n \to \infty} \frac{1}{n} \int \max \{ f_n, -n\epsilon^{-1} \} \, d\mu \geq \int \max \{ f, -\epsilon^{-1} \} \, d\mu. \tag{3.13}
\end{equation}

Taking the positive part of both sides of (3.1) and using the obvious bound for the positive part of a sum, we see that the sequence \( (f_{n+})_{n \in \mathbb{N}} \) of positive parts also satisfies (3.1). Therefore, the same argument and the fact that the limit inferior on the left-hand side can be written as an infimum guarantee that \( f_+ \in L^1(\mu) \).

\( ^4 \)Since we have assumed in Theorem 3.1 that \( f_{n+} \) is integrable, the integral of \( f_n \) is well defined in \([-\infty, \infty]\), which suffices for Lemma 2.1.
Step 3 To obtain the opposite bound, we argue separately for the negative and positive parts. For the negative parts, the Lebesgue dominated convergence theorem applies at fixed $\epsilon > 0$ and yields
\[
\lim_{n \to \infty} \frac{1}{n} \int \max \{ -f_{n,-}, -n\epsilon^{-1} \} \, d\mu = \int \max \{ -f_{-}, -\epsilon^{-1} \} \, d\mu.
\] (3.14)

By our previous comment on the positive parts, we may take the analogue of (3.9) for positive parts, integrate and divide by $n$ in order to derive the bound
\[
\frac{1}{n} \int f_{n,+} \, d\mu \leq \int f_{+} \, d\mu + \epsilon + \int_{D_{r,K}^{+}} (f_{r,+} + \rho_{r}) \, d\mu + \frac{1}{n} \sum_{q=1}^{K_{r}^{+}+\sigma_{K}} \int f_{q,+} \, d\mu.
\]

Taking $n \to \infty$,
\[
\limsup_{n \to \infty} \frac{1}{n} \int f_{n,+} \, d\mu \leq \int f_{+} \, d\mu + \epsilon + \int_{D_{r,K}^{+}} (f_{r,+} + \rho_{r}) \, d\mu.
\]

Taking $K \to \infty$ using Step 3 of the proof of the previous theorem applied to the positive parts, we find
\[
\limsup_{n \to \infty} \frac{1}{n} \int f_{n,+} \, d\mu \leq \int f_{+} \, d\mu + \epsilon.
\] (3.15)

Combining (3.14) and (3.15), we find
\[
\limsup_{n \to \infty} \frac{1}{n} \int \max \{ f_{n}, -n\epsilon^{-1} \} \, d\mu \leq \int \max \{ f, -\epsilon^{-1} \} \, d\mu + \epsilon.
\] (3.16)

Combining (3.12) with (3.13) and (3.16), we find
\[
\lim_{n \to \infty} \frac{1}{n} \int f_{n} = \lim_{\epsilon \to 0} \int \max \{ f_{-}, -\epsilon^{-1} \} \, d\mu.
\]

Because $f_{+}$ has already been shown to be integrable, we now only need the Lebesgue monotone convergence theorem to conclude that
\[
\lim_{\epsilon \to 0} \int \max \{ f_{-}, -\epsilon^{-1} \} \, d\mu = \int f \, d\mu,
\]
and thus that the theorem holds.

Remark 3.4. While it is certainly natural to seek generalizations of the main results of this note along the lines of the passage from (DS) to (AS) in [Sc91], one must be aware that the gaps that are allowed here complicate the construction of the Steel-type intervals for a fixed $x$ in Step 4 of the proof of Theorem 3.1. For example, it is already unclear to the author how to adapt the construction if one replaces (3.1) with
\[
f_{n+\sigma_{m}+m} \leq f_{n} + (f_{m} + \rho_{m}) \circ T^{n+\sigma_{m}}.
\]
In the theory of large deviations for measures on one-sided shifts mentioned in the Introduction, the role of such “exchanges of $n$ and $m$” is still not completely understood.

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