Fermionic Contributions to the Free Energy of Noncommutative Quantum Electrodynamics at High Temperature

F. T. Brandt, J. Frenkel and C. Muramoto

Instituto de Física, Universidade de São Paulo, 05508-090, São Paulo, SP, BRAZIL

Abstract

We consider the fermionic contributions to the free energy of noncommutative QED at finite temperature $T$. This analysis extends the main results of our previous investigation where we have considered the pure bosonic sector of the theory. For large values of $\theta T^2$ ($\theta$ is the magnitude of the noncommutative parameters) the fermionic contributions decrease the value of the critical temperature, above which there occurs a thermodynamic instability.

Key words: Noncommutative QED, Thermal field theory

PACS: 11.10.Wx

1. Introduction

The formulation of Quantum Electrodynamics in noncommutative space (NCQED) [1,2,3] opens the interesting possibility of new dynamics for the gauge fields, including self-interactions, without the introduction of additional internal degrees of freedom. The gauge field self-interactions of this theory contain trigonometric factors like $\sin(p_\mu \theta^{\mu\nu} q_\nu)$ where $p_\mu$ and $q_\nu$ are the momenta and $\theta^{\mu\nu}$ are the noncommutative parameters introduced via the commutation relation of the coordinates

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad (1)$$

where, in general, the Grönewold-Moyal $\star$-product [1,2] between two functions $f(x)$ and $g(x)$ is given by

$$f(x) \star g(x) = f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu \right) g(x), \quad (2)$$

In the limit when the coordinates can be considered commutative, $\theta^{\mu\nu} \rightarrow 0$, all the self-interactions vanish and the usual theory of non-interacting photons is recovered.

When we consider a system at finite temperature $T$[4,5,6], the thermal Green functions may become dependent on the quantity $\tau \equiv \theta T^2$, where $\theta$ is the typical magnitude of the components $\theta^{\mu\nu}$. In general the dependence on $\tau$ may be very involved. However, for asymptotic values of $\tau$ one can have a simpler understanding of the behavior of thermal Green functions. It is also possible that other mass scales may be present, so that one could have quantities like $\theta p T$, where $p$ represents the magnitude of some external momenta in a given Green function. There is however an important quantity, namely the free-energy, which can only depend on $\tau$. This happens because the diagrams which contribute to the free-energy do not have external legs.

Previous investigations on this subject [7] have revealed interesting properties already at the lowest non-trivial order (two-loop order). Corrections to the free energy at higher orders than two-loops, in thermal field theories, require for a self-consistent treatment to take into consideration an infinite series of diagrams. This happens because the fields acquire, through their interactions, an effective thermal mass. This so-called plasmon effect is known...
to occur for instance in non-abelian gauge theories like the Yang-Mills theory at high temperature [4,5,6]. Inasmuch NCQED is a theory of self-interacting gauge fields, it is natural to consider the possibility that similar non-perturbative effects may arise when we take into account all the higher order corrections to the free energy. In this case, only the leading hard thermal contributions are relevant. Consequently, other interesting phenomena such as the UV/IR mixing [8], which are important at $T = 0$, do not arise in the present context since the temperature does provide an UV regularization of the thermal graphs.

In a recent paper the free-energy of thermal NCQED (TNCQED) has been analyzed by taking into account all the contributions which arise from the pure bosonic sector of TNCQED [9]. In the regime when $\tau \gg 1$, we have summed the ring-diagrams, which consist of an arbitrary number of self-energy insertions and yield the leading contributions. A remarkable property revealed by this analysis was the emergence of a thermodynamic instability above the critical temperature

$$T_c = \sqrt{\frac{1}{e\beta} \frac{3}{2\pi}} \approx 1.229 \sqrt{\frac{1}{e\beta}} \quad (3)$$

where $e$ is the coupling constant.

In the present paper we consider the possibility that the critical temperature $T_c$ may be modified by the inclusion of fermions. If we consider only Dirac fermions in the fundamental representation, there would be no modification of $T_c$, because, as we have argued in [9], the contribution from fermion loops are the same as in the commutative theory and the existence of $T_c$ is a consequence of the noncommutative magnetic mode, which is absent in the commutative theory. However, we may additionally include fermions in the “adjoint” representation so that the fermionic action, for massless fermions, is given by

$$S_{\text{term}} = \int \! \! d^4x \left[ \bar{\psi} \gamma^\mu \partial_\mu \psi - i e \gamma_\mu A^\mu \psi \right] + \bar{\chi} \gamma^\mu \partial_\mu \chi,$$

where $A^\mu$ is the gauge field. The fields $\psi$ and $\chi$ are the fundamental and “adjoint” fermions, which transform respectively as [3]

$$\psi \to e^{i\alpha(x)} \psi \quad \text{and} \quad \chi \to e^{i\alpha(x)} \chi \gamma^\mu e^{-i\alpha(x)}, \quad (5)$$

where $\alpha(x)$ is the parameter of the U(1) group. In the limit when $\theta_{\mu\nu} \to 0$ the adjoint fermions decouple and we recover the usual electron interaction of commutative QED.

In section 2 we describe the method employed to identify the leading contributions to the free-energy in terms of ring diagrams. Then, in section 3, we present the result for the free-energy, in a form which allows us to derive the critical temperature, which turns out to be smaller than the one in the pure bosonic sector. The instability which occurs above the critical temperature may be understood, as discussed in section 4, in terms of the behavior of the noncommutative thermal magnetic mass. Many technical details, which are relevant in the context of these calculations, can be found in our previous paper [9].

2. Basic Approach

Let us review the main steps which lead us to find the critical temperature $T_c$ in [9]. First we have to consider a non-perturbative (a very comprehensive analysis of the nonpertubative regime has been carried out in [10] using the lattice formulation) contribution to the free-energy which is generated by the ring diagrams, as follows

$$\tilde{\Omega}'(T, \theta) = -\frac{1}{2} \left[ \frac{1}{2} \left( \bar{\psi} \gamma^\mu \psi \right)^n \frac{1}{3} \left( \bar{\phi} \gamma^\mu \phi \right)^n + \cdots \right]. \quad (6)$$

The quantity $\tilde{\Omega}'$ stands for all the contributions to the one-loop photon self-energy. In [9] we have shown that at any given order in the loop expansion, Eq. (6) gives the leading contribution in the limit $\tau \gg 1$. Therefore, it is implicit in (6) that we are considering only the zero mode contribution of the photon propagator (denoted by wavy lines), so that $k^2 = -|k|^2$. The reason for this is because each photon propagator in (6) introduces a factor

$$\frac{1}{k^2} = \frac{1}{(2\pi n)^2 + |k|^2/\tau^2}, \quad (7)$$

so that the zero mode, $n = 0$, yields the following leading contribution for large values of $\tau$

$$\frac{1}{k^2} \approx - \frac{\tau^2}{|k|^2} \quad (8)$$

(here and in all the expressions which follows we have performed the rescaling $(k_4, \mathbf{k}) \to (T(k_4, \mathbf{k})/\tau)$ as well as $(p_4, \mathbf{p}) \to (T(p_4, \mathbf{p})$ where $p^\mu$ stands for the internal momentum of $\Pi_{\mu\nu}$). On the other hand, possible contributions to (6) with fermionic rings (photon lines replaced by fermion lines) would be subleading, because the inverse of the fermionic propagator is nonzero for $n = 0$ and $|k| = 0$ (the Matsubara frequencies are all multiples of odd integers).
where we have employed the following decomposition

\[ u = \text{(decomposition)} \]

part done, yielding the following results for the thermal self-energy in NCQED.

\[ \Pi_{\mu\nu}(k) = \frac{1}{2} \left( \frac{4e^2 T^2}{(2\pi)^2} \right) \left( \frac{\pi^2}{6} + \frac{\pi}{2k} \coth(\pi k) + \frac{\pi^2}{2} \coth^2(\pi k) \right), \]

(12a)

\[ \Pi_{\mu\nu}^\text{boson}(k) = -\frac{1}{2} \left( \frac{4e^2 T^2}{(2\pi)^2} \right) \left( \frac{\pi^2}{2} - \frac{\pi}{2k} \coth(\pi k) - \frac{\pi^2}{2} \coth^2(\pi k) + \frac{1}{k^2} \right) \]

(12b)

and \( \Pi_{\mu\nu}^\text{boson} = \Pi_{\mu\nu}^\text{boson} = 0 \).

Let us now apply the same technique to the computation of the last diagram in Fig. 1. In fact, this diagram represents the sum of the adjoint and the fundamental fermion loops. The contribution of the fundamental fermion loop is well known and gives the following result in the high temperature limit

\[ \Pi_{\mu\nu}^\text{fund}(k) = -\frac{\epsilon^2 T^2}{3} u_\mu u_\nu. \]

As we have already anticipated this contribution of fermions in the fundamental representation does not modify the structure function \( \Pi_{\mu\nu}^\text{boson} \); it just gives the usual Debye screening mass of QED [4]. The contribution of the adjoint fermion can be readily obtained from the Feynman rules generated by the first term in the action (4). Proceeding in the same way as in the case of the bosonic contribution, we consider the limit of large \( \tau \), which amounts to set \( k_4 = |k| = 0 \) except inside the trigonometric factor. This leads to (reminding that the momenta have been scaled as \( p \to pT \))

\[ \Pi_{\mu\nu}^\text{adj}(k) = 4e^2 T^2 \int \frac{d^3p}{(2\pi)^3} \sin^2 \left( \frac{\mathbf{k} \cdot \mathbf{p}}{2} \right) \times \sum_{n=0}^{\infty} \text{tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \right]_{p4=(2n+1)\pi}, \]

(14)

Using \( \text{tr} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta = 4 (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) \), we obtain

\[ \Pi_{\mu\nu}^\text{adj}(k) = -4e^2 T^2 \int \frac{d^3p}{(2\pi)^3} \sin^2 \left( \frac{\mathbf{k} \cdot \mathbf{p}}{2} \right) \times \sum_{n=0}^{\infty} \tilde{\Pi}_{\mu\nu}(p, k) |_{p4=(2n+1)\pi}, \]

(15)

where \( \tilde{\Pi}_{\mu\nu} \) is given by (10). It is remarkable that the result for the fermionic integrand in Eq. (15) coincides, up to a factor, with the bosonic result in Eq. (10).
Our explicit calculation in [9] shows that there are two kinds of sums in terms of which the structure functions in (11) can be expressed, which are $\sum 1/p^2$ and $\sum 1/p^4$. The thermal part of these sums can be expressed in terms of the Bose-Einstein distributions $N_B$ for $p_4 = 2n\pi$. On the other hand, in the present fermionic case, with $p_4 = (2n + 1)\pi$, the Matsubara sums yield a result which is simply related to the bosonic result in [9] by the replacement $N_B \rightarrow -N_F$, where $N_F$ is the Fermi-Dirac distribution.

Proceeding similarly to the bosonic case, we can now express $\Pi_{\mu\nu}^{adj}(k)$ in terms of four structure functions, using the same tensor basis as in Eq. (11). Then, the structure functions can be explicitly computed performing the elementary angular integrations as well as the slightly more involved integrations over $|p|$ [11]. As a result of these straightforward steps we obtain

\[
\Pi_{00}^{adj} = \frac{(4\pi^2) T^2}{(2\pi)^2} \left[ F(\hat{k}) - G(\hat{k}) - \frac{\pi^2}{6} \right] \quad (16a)
\]

\[
\Pi_{\mu\nu}^{adj} = \frac{(4\pi^2) T^2}{(2\pi)^2} \left[ F(\hat{k}) + G(\hat{k}) + \frac{1}{k^2} \right] \quad (16b)
\]

and $\Pi_{11}^{adj} = \Pi_{22}^{adj} = 0$, where, for notational convenience we have introduced

\[
F(\hat{k}) \equiv \frac{\pi}{4k} \left[ \tanh \left( \frac{\pi \hat{k}}{2} \right) - \coth \left( \frac{\pi \hat{k}}{2} \right) \right] \quad (17)
\]

and

\[
G(\hat{k}) \equiv \frac{\pi^2}{8} \left[ \tanh^2 \left( \frac{\pi \hat{k}}{2} \right) - \coth^2 \left( \frac{\pi \hat{k}}{2} \right) \right] \quad (18)
\]

The full result for the relevant part of the photon self-energy can now be obtained adding the expressions in Eqs. (16) with the corresponding ones in (12). Using the notation $\Pi_{00} = \Pi_{00}^{boson} + \Pi_{00}^{fund} + \Pi_{00}^{adj}$ and $\Pi_{\mu\nu} = \Pi_{\mu\nu}^{boson} + \Pi_{\mu\nu}^{adj}$ we can write the full expression for the photon self-energy as

\[
\Pi_{\mu\nu} = \Pi_{00} u_\mu u_\nu + \Pi_{\mu\nu}^{adj} \frac{\hat{k}_\mu \hat{k}_\nu}{k^2}. \quad (19)
\]

3. The Free Energy

We have now all the ingredients to compute the free-energy in the limit of large $\tau$. Taking into account the orthonormality and idempotency of the quantities $u_\mu u_\nu$ and $\frac{\hat{k}_\mu \hat{k}_\nu}{k^2}$ and inserting Eq. (19) into Eq. (6) (using $\sum_{n=2}^{\infty} x^n/n = -x - \log(1 - x)$ in order to perform the sum of all the rings), we find

\[
\Omega^f(T, \theta) = \frac{1}{2} \frac{T^4}{(2\pi)^3} \int d^4k \left[ \frac{(e\tau)^2 \Pi_{00}(\hat{k})}{|k|^2} \right.
\]

\[
\left. + \log \left( 1 - \frac{(e\tau)^2 \Pi_{00}(\hat{k})}{|k|^2} \right) \right]
\]

\[
+ \frac{(e\tau)^2 \Pi_{\mu\nu}(\hat{k})}{|k|^2} \left( \frac{e\tau}{|k|^2} \right)^2 \right] \quad (20)
\]

where we have introduced the quantities $\Pi_{00}(\hat{k}) \equiv \Pi_{00}(\hat{k})/(eT)^2$ and $\Pi_{\mu\nu}(\hat{k}) \equiv \Pi_{\mu\nu}(\hat{k})/(eT)^2$. This expression constitutes a contribution to the free-energy which is non-analytic in the coupling constant, since it is not proportional to a simple power of $e\tau$. There are many interesting physical consequences which can be drawn from Eq. (20). Usually, in the case of commutative theories, the components of the static self-energy are constants independent of the momentum $|k|$. For instance, even in the case of thermal gravity [12,13], the components of the graviton self-energy, in the high temperature limit, are constants (thermal masses) which can be positive or negative. When these constants have a positive sign, the free-energy picks up an imaginary part from the logarithm, which can be interpreted as an instability [14]. In the present case of the TNCQED, the self-energy is a function of the momentum $|k|$ and also of the direction of the vector $\theta_i \equiv \frac{1}{2} \epsilon_{ijk} \theta_{jk}$, except in the case of the contribution of the fundamental fermion. Therefore, it is important to investigate the behavior of $\Pi_{00}(k)$ and $\Pi_{\mu\nu}(k)$ ($k = |k|$ sin $\alpha$ are $\alpha$ is the angle between $k$ and $\theta$). Let us first obtain the asymptotic behavior for small and large $\hat{k}$. The Taylor expansion of expressions (12) and (16) for small $k$ gives

\[
\lim_{k \to 0} \Pi_{00}^{boson}(\hat{k}) = -\frac{4\pi^2}{45} \hat{k}^2 + \frac{4\pi^4}{315} \hat{k}^4 + \cdots, \quad (21a)
\]

\[
\lim_{k \to 0} \Pi_{00}^{fund}(\hat{k}) = -\frac{2\pi^2}{45} \hat{k}^2 - \frac{8\pi^4}{945} \hat{k}^4 + \cdots, \quad (21b)
\]

\[
\lim_{k \to 0} \Pi_{00}^{adj}(\hat{k}) = -\frac{7\pi^2}{45} \hat{k}^2 + \frac{31\pi^4}{1260} \hat{k}^4 + \cdots, \quad (22a)
\]

\[
\lim_{k \to 0} \Pi_{\mu\nu}^{boson}(\hat{k}) = -\frac{7\pi^2}{90} \hat{k}^2 - \frac{31\pi^4}{1890} \hat{k}^4 + \cdots. \quad (22b)
\]
For large $\tilde{k}$ these functions behave as

$$\lim_{\tilde{k} \to \infty} \Pi_{00}^{\text{boson}}(\tilde{k}) = -\frac{2}{3},$$  \hspace{2cm} (23a)

$$\lim_{\tilde{k} \to \infty} \Pi_{00}^{\text{nc}}(\tilde{k}) = 0,$$  \hspace{2cm} (23b)

$$\lim_{\tilde{k} \to \infty} \Pi_{00}^{\text{adj}}(\tilde{k}) = -\frac{2}{3},$$  \hspace{2cm} (24a)

$$\lim_{\tilde{k} \to \infty} \Pi_{00}^{\text{adj}}(\tilde{k}) = 0.$$  \hspace{2cm} (24b)

We may consider $\tilde{k} \to \infty$ as the limit when the two opposite charges in the dipole are very far apart from each other. This can be seen expressing $\tilde{k}$ in terms of the original dimensionful momenta so that $\tilde{k} \sim |k| \tau/T \sin \alpha = \theta T |k| \sin \alpha$ and considering the limit $\theta T \to \infty$. Both the adjoint fermion as well as the noncommutative photon can be pictured as an extended dipole of effective length $L \sim \theta T$ [15].

In this case, loop contributions of photons and adjoint fermions reduce to the sum of two fundamental fermion loops, corresponding to each charge in the dipole. Since the self-energy is proportional to $e^2$, the two contributions add to twice the result obtained for the fundamental fermion loop. This interesting physical picture is indeed verified when we compare Eq. (13) with Eqs. (23a) and (24a).

Adding all the contributions (including also the contribution from Eq. (13) such that $\Pi_{00}^{\text{fund}} = \Pi_{00}^{\text{fund}}/(e^2 T^2) = -\frac{4}{3}$), we obtain

$$\lim_{k \to 0} \Pi_{00}(k) = -\frac{1}{3} \left( \frac{\alpha}{45} \right) k^2 + \frac{47 \pi^4}{1260} k^4 + \cdots,$$  \hspace{2cm} (25a)

$$\lim_{k \to \infty} \Pi_{00}(k) = -\frac{5}{3},$$  \hspace{2cm} (25b)

$$\lim_{k \to 0} \Pi_{nc}(k) = \frac{11 \pi^2}{90} k^2 - \frac{47 \pi^4}{1890} k^4 + \cdots,$$  \hspace{2cm} (26a)

$$\lim_{k \to \infty} \Pi_{nc}(k) = 0.$$  \hspace{2cm} (26b)

We also show in the figure 2 the plots of the functions $\Pi_{00}$ and $\Pi_{nc}$ (full lines) as well as their bosonic and fermionic components (dotted and dashed lines).

From the behavior of $\Pi_{nc}$ and $\Pi_{00}$ we can now investigate the thermodynamic stability as follows.

Figure 2a shows that the contribution of the bosonic sector to $\Pi_{00}$ does not induce an imaginary part to the free-energy in (20), because $\Pi_{00}^{\text{boson}}$ (doted line) is always negative. Similarly, the contribution to $\Pi_{00}$ from the adjoint fermions (dashed line) is also negative. Hence, the longitudinal 00 mode is always stable.

In the case of the transverse mode, figure 2b shows that the bosonic sector (dotted line) is positive, yielding the instability already discussed in ref. [9]. The adjoint fermion contribution is also positive. Therefore, the sums of the fermionic and the bosonic contributions produce a larger positive result for the noncommutative transverse mode, which enhances the instability.

Finally, we can investigate how the critical value of the temperature given in (3) is modified by the inclusion of fermions. This can be done by realizing
that the second logarithm in (20) has an imaginary part when
\[
(e\tau)^2 > \frac{k^2}{\Pi_{nc}(k)} > \frac{k^2}{\Pi_{nc}(k)}.
\] (27)

This condition can be solved analytically and leads to the following critical temperature
\[
T_c = \sqrt{\frac{1}{e\theta}} \sqrt{\frac{10}{11}} \approx 0.95\sqrt{\frac{1}{e\theta}}
\] (28)

which is associated with the noncommutative transverse mode. Comparing with (3) we conclude that the inclusion of the adjoint fermion has decreased the critical temperature.

4. Discussion

In conclusion, our results reveal that the possible excitation of degrees of freedom of the adjoint fermion renders the system more unstable, due to a decrease in the value of the critical temperature, when compared to the one obtained in [9]. This happens because the thermal contribution of the adjoint fermion is qualitatively similar to the one of the (noncommutative) photon. This behavior might be expected since one may picture all such particles as having a likewise dipole structure with zero net electric charge [3]. Consequently, the total contribution from the photon and the adjoint fermion to the transverse noncommutative magnetic mode makes \( \Pi_{nc} \) even larger, which enhances the instability of the system.

One may understand this instability, if we consider the noncommutative thermal magnetic mass defined in a self-consistent way as
\[
m_{nc}^2 = -\Pi_{nc}(k_0 = 0, k)|_{k^2 = m_{nc}^2}.
\] (29)

Proceeding similarly as in [9], we find that this equation admits negative solutions for \( m_{nc}^2 \). This occurs in the region \( e\tau > e\tau_c \), which is where the argument of the second logarithm in Eq. (20) becomes negative. When \( e\tau \) is close to its critical value, we can obtain an analytic solution of Eq. (29), using for \( \Pi_{nc} \) the expression given by Eq. (26a). After a simple calculation, we find that the noncommutative thermal magnetic mass can be written in this region in the form
\[
m_{nc}^2 = \frac{847}{705}e^2(T_c^2 - T^2),
\] (30)

where \( T_c \) is given by Eq. (28). This solution shows explicitly that \( m_{nc}^2 \) becomes negative when \( T > T_c \).

It is interesting to note that a negative value of the squared magnetic mass is reminiscent of the negative squared Jeans mass \( M^2 \sim -GT^4 \) which arises in quantum gravity [12,13]. Such a mass leads to the appearance of an imaginary part in the free energy, which indicates that the system becomes unstable and may undergo a phase transition [14].

Acknowledgment

The authors would like to thank CNPq and FAPESP, Brazil, for financial support.

References

[1] M. R. Douglas, N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977–1029.
[2] R. J. Szabo, Phys. Rept. 378 (2003) 207–299.
[3] M. Hayakawa, Phys. Lett. B478 (2000) 394–400.
[4] J. I. Kapusta, Finite Temperature Field Theory, Cambridge University Press, Cambridge, England, 1989.
[5] M. L. Bellac, Thermal Field Theory, Cambridge University Press, Cambridge, England, 1996.
[6] A. Das, Finite Temperature Field Theory, World Scientific, NY, 1997.
[7] G. Arcioni and M. A. Vazquez-Mozo, JHEP 0001 (2000) 028.
[8] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002 (2000) 020.
[9] F. T. Brandt, J. Frenkel, C. Muramoto, Nucl. Phys. B754 (2006) 146–177.
[10] W. Bietenholz, J. Nishimura, Y. Suzuki, J. Volkholz, JHEP 10 (2006) 042.
[11] I. S. Gradshteyn, M. Ryzhik, Tables of Integral Series and Products, Academic, New York, 1980.
[12] D. J. Gross, M. J. Perry, L. G. Yaffe, Phys. Rev. D25 (1982) 330–355.
[13] A. Rebhan, Nucl. Phys. B351 (1991) 706–734.
[14] I. Affleck, Phys. Rev. Lett. 46 (1981) 388.
[15] K. Landsteiner, E. Lopez, M. H. G. Tytgat, JHEP 06 (2001) 055.