FLAT RELATIVE MITTAG-LEFFLER MODULES AND ZARISKI LOCALITY

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Abstract. The ascent and descent of the Mittag-Leffler property were instrumental in proving Zariski locality of the notion of an (infinite dimensional) vector bundle by Raynaud and Gruson in [25]. More recently, relative Mittag-Leffler modules were employed in the theory of (infinitely generated) tilting modules and the associated quasi-coherent sheaves, [1, 21]. Here, we study the ascent and descent along flat and faithfully flat homomorphisms for relative versions of the Mittag-Leffler property. In particular, we prove the Zariski locality of the notion of a locally f-projective quasi-coherent sheaf for all schemes, and for each $n \geq 1$, of the notion of an $n$-Drinfeld vector bundle for all locally noetherian schemes.

I. INTRODUCTION

Relative Mittag-Leffler modules were introduced by Rothmaler in [26]. His approach was model theoretic: Mittag-Leffler modules were shown to be the counterparts of pure-injective modules in the sense that the former are atomic (i.e., they realize only the finitely generated pp-types) while the latter are saturated (i.e., they realize all pp-types). The adjective ‘relative’ referred to restricting to theories of modules induced by definable subclasses of Mod–$R$. Much later, the important role of relative Mittag-Leffler modules for (infinite dimensional) tilting theory was recognized by Angeleri and Herbera [1]; this in turn led to a proof of finite type of all 1-tilting modules in [4].

Flat Mittag-Leffler modules played a key role in proving Zariski locality of the notion of an (infinite dimensional) vector bundle in the classic work of Raynaud and Gruson, [25, Seconde partie]. The locality follows by the Affine Communication Lemma (see e.g. [30, 5.3.2]), whose assumptions are guaranteed by the ascent and descent of projectivity along flat ring homomorphisms, and faithfully flat ring homomorphisms, respectively.

Once a structure theory of tilting modules over commutative rings was developed in [2] and [20], it was possible to generalize the classic results to proving Zariski locality for various notions of quasi-coherent sheaves associated with tilting, [21]. Another generalization, employing the notion of a restricted flat Mittag-Leffler module, proved the Zariski locality of restricted Drinfeld vector bundles in [14].

Our goal here is to refine the classic result on the ascent and descent of flat Mittag-Leffler modules to the relative setting. The main technical tools needed for this purpose are presented in Section 3. In Section 4, we apply these tools and prove Zariski locality of the corresponding notions of flat quasi-coherent sheaves. In particular, we prove the Zariski locality of the notion of a locally f-projective quasi-coherent sheaf for all schemes, and for each $n \geq 1$, of the notion of an $n$-Drinfeld vector bundle for all locally noetherian schemes.
2. Preliminaries

Let $R$ be an (associative, unital) ring and Mod–$R$ the category of all (unitary right $R$-) modules. The elements of Mod–$R$ will often be referred to simply as modules. Further, $R$–Mod will denote the category of all (unitary) left $R$-modules.

Let $n \geq 0$. A module $M$ is an FP$_n$ module provided that $M$ possesses a projective resolution $\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ such that all the modules $P_i$ ($i \leq n$) are finitely generated. So FP$_0$ modules are just the finitely generated modules, FP$_1$ modules are the finitely presented ones, etc. Notice that the ring $R$ is right noetherian, iff the classes of FP$_n$ modules coincide for all $n \geq 0$ , while $R$ is right coherent, iff the classes of FP$_n$ modules coincide for all $n \geq 1$.

We will denote by $\cal P_n$, $\cal F_n$, and $\cal I_n$ the classes of all modules of projective, weak, and injective dimension $\leq n$, respectively.

Let $\cal B$ be a class of modules. Then $^\perp \cal B$ denotes the class of all modules $A$ such that $\text{Ext}_R^1(A, B) = 0$ for each $B \in \cal B$. Similarly, $\cal B^+$ is the class of all modules $C$ such that $\text{Ext}_R^1(B, C) = 0$ for all $B \in \cal B$. Further, $\cal B^\perp$ denotes the class of all left $R$-modules $D$ such that $\text{Tor}_1^R(B, D) = 0$ for all $B \in \cal B$. Similarly, for a class of left $R$-modules $\cal D$, $^\perp \cal D$ denotes the class of all modules $C$ such that $\text{Tor}_1^R(C, D) = 0$ for all $D \in \cal D$.

For a class of modules $\cal C$ we denote by $\varprojlim \cal C$ the class of all modules that are direct limits of direct systems consisting of modules from $\cal C$. For example, $\cal F_0 = \varprojlim \cal P_0$ for any ring $R$. Also, $\cal P I$ will denote the class of all pure-injective modules.

We will need the following consequence of [15, Theorem 8.40 and Corollary 8.42]:

Lemma 2.1. Let $R$ be a ring and $\cal C$ be a class of FP$_2$-modules closed under extensions, direct summands and containing $R$. Let $\cal B = C^\perp$. Then $\varprojlim \cal C = ^\perp (\cal B \cap \cal P I)$.

We also recall the following identities satisfied by the Tor bifunctor.

Lemma 2.2. Let $\varphi : R \rightarrow S$ be a flat ring homomorphism of commutative rings.

1. For all modules $A$ and $B$, there is an $S$-isomorphism $\text{Tor}_1^R(A, B) \otimes_R S \cong \text{Tor}_1^S(A \otimes_R S, B \otimes_R S)$.

2. If $A$ is a module and $B$ is an $S$-module, then there is an $S$-isomorphism $\text{Tor}_1^S(A \otimes_R S, B) \cong \text{Tor}_1^R(A, B)$.

Proof. (1) is a particular instance of [12, Theorem 2.1.11], and (2) a particular instance of [9, Proposition VI.4.1.2].

The central notion of our paper is that of a relative Mittag-Leffler module:

Definition 2.3. Let $R$ be an arbitrary ring, $M \in \text{Mod–}R$ and $Q \subseteq R$–$\text{Mod}$. Then $M$ is Q-Mittag-Leffler (or Mittag-Leffler relative to $Q$), provided that the canonical morphism $\psi_M : M \otimes_R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$ defined by $\psi_M(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I}$ is injective for any family $(Q_i | i \in I)$ consisting of elements of $Q$.

As mentioned above, relative Mittag-Leffler modules were introduced in [20]. Further results on these modules were proved in [11], and in the more recent papers [17] and [24]. Following [19], we will denote by $\cal D_Q$ the class of all flat $Q$-Mittag-Leffler modules.

The two borderline cases of Definition 2.3 occur for $Q = \emptyset$, when $\cal D_Q = \cal F_0$, and for $Q_0 = R$–$\text{Mod}$, when $\cal D_Q = \cal F M$ is the class of all flat Mittag-Leffler modules. As $Q_0 = (\cal F_0)^\perp$, the latter setting can be extended as follows: for each $n \geq 0$, we let $Q_n = (\cal F_n)^\perp$. We will call the modules $M \in \cal D_{Q_n}$ flat n-Mittag-Leffler.

Another case of interest is when $Q = \{R\}$, or equivalently, $Q$ is the class of all flat left $R$-modules. Then the flat $Q$-Mittag-Leffler modules coincide with the f-projective modules, that is, the modules $M$ such that each homomorphism from a finitely generated module to $M$ factorizes through a free module, see [16] or [5, §3].
Denoting the class of all \( f \)-projective modules by \( \mathcal{FP} \), we have the following chain of classes of modules

\[
(*) \quad \mathcal{P}_0 \subseteq \mathcal{FM} = \mathcal{D}_{\mathcal{Q}_0} \subseteq \cdots \subseteq \mathcal{D}_{\mathcal{Q}_n} \subseteq \mathcal{D}_{\mathcal{Q}_{n+1}} \subseteq \cdots \subseteq \mathcal{FP} \subseteq \mathcal{T}_0.
\]

The inclusions in the chain \((*)\) need not be strict in general. For example, if \( R \) has weak global dimension \( \leq n \), then \( \mathcal{D}_{\mathcal{Q}_n} = \mathcal{D}_{\mathcal{Q}_{n+1}} = \cdots = \mathcal{FP} \). If \( R \) is a right perfect ring, then all the classes in the chain \((*)\) coincide.

**Remark 1.** 1. Other variants of the notion of a flat Mittag-Leffler module, called **restricted flat Mittag-Leffler modules**, were introduced in [14]. Their classes form a chain located between the classes \( \mathcal{P}_0 \) and \( \mathcal{FM} \).

2. The following generalization of the notion of an \( f \)-projective module goes back to Simson [28]: given a cardinal \( \kappa \geq \aleph_0 \), a module \( M \) is \( \kappa \)-projective if each homomorphism from a \( \kappa \)-generated module to \( M \) factorizes through a free module. Denote by \( C_\kappa \) the class of all \( \kappa \)-projective modules. Since \( C_\kappa = \mathcal{FP} = \mathcal{D}(R) \) one may wonder whether the classes \( C_\kappa \) fit in the setting of flat relative Mittag-Leffler modules also for \( \kappa > \aleph_0 \).

It is easy to see that \( \mathcal{FM} \subseteq C_\kappa \subseteq \mathcal{FP} \) for any ring \( R \), and that \( C_\kappa = \mathcal{FM} \) when \( R \) is right hereditary or von Neumann regular (cf. [15] 3.19 and [5] 3.7(iii)). Also, for each \( \kappa \geq \aleph_0 \), all \( \kappa \)-generated modules in the class \( C_\kappa \) are projective. In particular, for \( \kappa = \aleph_1 \), the classes \( C_{\aleph_1} \) and \( \mathcal{FM} \) contain the same countably presented modules (namely the projective ones), so if \( C_{\aleph_1} \neq \mathcal{FM} \), then \( C_{\aleph_1} \neq \mathcal{D}(Q) \) for any class of left \( R \)-modules \( Q \) by [5] 2.5(i). If \( R \) is not right perfect, then the class \( \mathcal{FM} \) contains \( \aleph_1 \)-generated non-projective modules (cf. [15] 3.19 and [11] VII.1.3)), so for each \( \kappa > \aleph_1 \), \( \mathcal{FM} \not\subseteq C_\kappa \), whence again \( C_\kappa \neq \mathcal{D}(Q) \) for any class of left \( R \)-modules \( Q \).

**Definition 2.4.** Let \( R \) be a commutative ring.

1. Let \( \mathcal{B} \) be a property of modules. Then \( \mathcal{B}(\text{Mod-}R) \) denotes the class of all modules satisfying the property \( \mathcal{B} \).

2. Let \( \mathcal{R} \) be a class of commutative rings. Let \( X \) be a scheme and \( (\mathcal{R}(u) \mid u \subseteq X, u \text{ open affine}) \) be its structure sheaf. Then \( X \) is a **locally \( \mathcal{R} \)-scheme** provided that \( \mathcal{R}(u) \in \mathcal{R} \) for each open affine set \( u \) of \( X \).

The main properties \( \mathcal{B} \) of modules that we will be interested in here are the flatness, projectivity, and various properties related to Mittag-Leffler conditions that are in general weaker than projectivity, but stronger than flatness. We will work with general schemes, but in our final application, we will restrict ourselves to **locally noetherian** schemes, that is, the locally \( \mathcal{R} \)-schemes where \( \mathcal{R} \) is the class of commutative noetherian rings.

Recall that given two commutative rings \( R \) and \( S \), a ring homomorphism \( \varphi : R \to S \) is **flat**, provided that \( S \) is a flat \( R \)-module (where the \( R \)-module structure on \( S \) is induced by \( \varphi \)), that is, the functor \( F = - \otimes_R S \) is exact.

Moreover, \( \varphi \) is **faithfully flat** provided that \( \varphi \) is flat, and \( N \otimes_R S \neq 0 \), whenever \( 0 \neq N \in \text{Mod-}R \). Faithful flatness of \( \varphi \) is equivalent to the following property of the functor \( F \): for each complex \( C \) of \( R \)-modules, \( C \) is exact in \( \text{Mod-}R \), if and only if \( F(C) \) is exact in \( \text{Mod-}S \), [23] Theorem 7.2.

A useful characterization of faithfully flat ring homomorphisms of commutative rings goes back to [6] Chap. I, §3, Proposition 9] (see also [3] Lemma 2):

**Lemma 2.5.** A flat ring homomorphism \( \varphi : R \to S \) of commutative rings is faithfully flat, if and only if \( \varphi \) - viewed as an \( R \)-homomorphism – is a pure monomorphism.

Next, we recall the classic notions of ascent and descent, cf. [22] 10.82 or [24].

**Definition 2.6.** Let \( \mathcal{B} \) be a property of modules, and \( \mathcal{R} \) a class of commutative rings.
(1) $\Psi$ is said to ascend along flat morphisms in $\mathfrak{R}$, provided that for each flat ring homomorphism $\varphi : R \to S$, such that $R, S \in \mathfrak{R}$, and each $M \in \Psi(\text{Mod}-R)$, also $M \otimes_R S \in \Psi(\text{Mod}-S)$.

(2) $\Psi$ is said to descend along faithfully flat morphisms in $\mathfrak{R}$, provided that for each faithfully flat ring homomorphism of commutative rings, $\varphi : R \to S$, such that $R \in \mathfrak{R}$ and $S$ is a finite direct product of rings from $\mathfrak{R}$, and for each $M \in \text{Mod}-R$, such that $M \otimes_R S \in \Psi(\text{Mod}-S)$, also $M \in \Psi(\text{Mod}-R)$.

(3) $\Psi$ is an ad-property in $\mathfrak{R}$, provided that $\Psi$ ascends along flat morphisms in $\mathfrak{R}$, descends along faithfully flat morphisms in $\mathfrak{R}$, and, moreover, $\Psi$ is compatible with finite ring direct products in the following sense: if $R = \prod_{i \in n} R_i$ is a finite ring direct product of rings with $R_i \in \mathfrak{R}$ for each $i < n$, and $(M_i \mid i < n)$ satisfy $M_i \in \Psi(\text{Mod}-R_i)$ for each $i < n$, then $M = \prod_{i \in n} M_i \in \Psi(\text{Mod}-R)$.

In the case when $\mathfrak{R}$ is the class of all commutative rings, we will omit the attribute ‘in $\mathfrak{R}$’ and say simply that $\Psi$ ascends, descends, and $\Psi$ is an ad-property.

Let $\Psi$ be a property of modules. If $X$ is an affine scheme, i.e., $X = \text{Spec}(R)$ for a commutative ring $R$, then $\text{Qcoh}(X) = \text{Mod}-R$, so $\Psi$ is at the same time a property of quasi-coherent sheaves on $X$. For general schemes $X$, one can extend $\Psi$ to a property of quasi-coherent sheaves $M$ on $X$ algebraically, by requiring property $\Psi$ to hold for each module of sections of $M$:

**Definition 2.7.** Let $\Psi$ be a property of $R$-modules, $X$ a scheme, and $(\mathcal{R}(u) \mid u \subseteq X, u \text{ open affine})$ be its structure sheaf. A quasi-coherent sheaf $M$ on $X$ is a locally $\Psi$-quasi-coherent sheaf on $X$ in the case when for each open affine set $u$ of $X$, the $\mathcal{R}(u)$-module of sections $M(u)$ satisfies $\Psi$. That is, $M(u) \in \Psi(\mathcal{R}(u))$.

If $\Psi$ is the property of being a projective module, then the locally $\Psi$-quasi-coherent sheaves are the (infinite dimensional) vector bundles, see [10]. When $\Psi$ denotes the property of being a flat Mittag-Leffler module (a restricted flat Mittag-Leffler module) then by [13], the locally $\Psi$-quasi-coherent sheaves are called Drinfeld vector bundles (restricted Drinfeld vector bundles). Extending this notation to $n \geq 0$, we will call a quasi-coherent sheaf $M$ an $n$-Drinfeld vector bundle in case it is a locally $\Psi_n$-quasi-coherent sheaf where $\Psi_n$ is the property of being a flat $n$-Mittag-Leffler module. Thus, $0$-Drinfeld vector bundles are just the Drinfeld vector bundles from [13].

A basic question concerning the various algebraic notions of locally $\Psi$-quasi-coherent sheaves defined above is whether these notions are also geometric, independent on a particular choice of affine coordinates on $X$, that is, whether the notions are Zariski local:

**Definition 2.8.** Let $\mathfrak{R}$ be a class of commutative rings, and $\mathfrak{C}$ be the class of all locally $\mathfrak{R}$-schemes.

The notion of a locally $\Psi$-quasi-coherent sheaf is Zariski local on $\mathfrak{C}$ provided that for each $X \in \mathfrak{C}$, each open affine covering $X = \bigcup_{v \in V} v$ of $X$, and each quasi-coherent sheaf $M$ on $X$, the following implication holds true: if $M(v) \in \Psi(\mathcal{R}(v))$ for all $v \in V$, then $M$ is locally $\Psi$-quasi-coherent.

ad-properties of modules are important, because they guarantee Zariski locality:

**Lemma 2.9.** Let $\mathfrak{R}$ be a class of commutative rings. Let $\Psi$ be an ad-property in $\mathfrak{R}$. Then the notion of a locally $\Psi$-quasi-coherent sheaf is Zariski local on the class of all locally $\mathfrak{R}$-schemes.

**Proof.** This is proved via the Affine Communication Lemma [30, 5.3.2], see [22, 27.21.2] or [14] Lemma 3.5].

It is well-known that the properties of being a projective, flat, flat Mittag–Leffler, and restricted flat Mittag–Leffler module, are ad-properties in the class of all commutative
rings. Thus the corresponding notions of an (infinite dimensional) vector bundle, flat quasi-coherent sheaf, Drinfeld vector bundle, and restricted Drinfeld vector bundle, are Zariski local on the class of all schemes (see [25 Seconde partie], [24 §§8-9], and [14]). Further instances of ad-properties, related to tilting and silting, have recently been introduced in [7] and [21].

Our goal here is to investigate the ascent and descent for flat relative Mittag-Leffler modules, i.e., the flat \( Q \)-Mittag-Leffler modules where \( Q \) is a subclass of \( R \)-\( \text{Mod} \). Then we will apply the results obtained to proving Zariski locality for the corresponding notions of quasi-coherent sheaves.

For further unexplained terminology, we refer to [12] and [15].

3. The Algebraic Background of Ascent and Descent for Flat Relative Mittag-Leffler Modules

First, we recall some connections between the Mittag-Leffler property and stationarity.

**Definition 3.1.** Let \( R \) be an arbitrary ring and \( B \) be a module.

1. Let \( (I, \leq) \) be an upper directed poset. A direct system \( (M_i, f_{ij} \mid i \leq j \in I) \) of modules is said to be \( B \)-stationary provided that the induced inverse system
   \[
   (\text{Hom}_R(M_i, B), \text{Hom}_R(f_{ij}, B) \mid i \leq j \in I)
   \]
   satisfies the Mittag-Leffler condition, that is, for each \( i \in I \) there exists \( i \leq j \in I \) such that \( \text{Im} \text{Hom}_R(f_{ki}, B) = \text{Im} \text{Hom}_R(f_{ji}, B) \) for all \( j \leq k \in I \).
2. A module \( M \) is said to be \( B \)-stationary if there exists a \( B \)-stationary direct system of finitely presented modules \( (M_i, f_{ji} \mid i \leq j \in I) \) such that \( M = \text{lim} M_i \).
3. Let \( B \) be a class of right \( R \)-modules. We say that a direct system \( (M_i, f_{ji} \mid i \leq j \in I) \), or a right \( R \)-module \( M \), is \( B \)-stationary, if it is \( B \)-stationary for all \( B \in B \).

Recall that a class of modules is said to be definable provided that it is closed under direct limits, direct products and pure submodules. For each class of modules \( Q \) there is the least definable class of modules containing \( Q \), called the definable closure of \( Q \) and denoted by \( \text{Def} Q \). It is obtained by closing \( Q \) first by direct products, then direct limits, and finally by pure submodules, cf. [17 Lemma 2.9 and Corollary 2.10]. Note that each definable class is also closed under direct sums, pure extensions, and pure-epimorphic images (see e.g. [15 Lemma 6.9]).

There is a duality between definable classes of left and right \( R \)-modules: given a definable class \( Q \) of left (right) \( R \)-modules, the dual definable class \( Q^\perp \) of \( Q \) is the least definable class of right (left) \( R \)-modules containing the character modules \( Q^\perp = \text{Hom}_R(M, Q/\mathbb{Z}) \) of all modules \( M \in Q \). Then \( Q = (Q^\perp)^\perp \) for any definable class of left (right) modules \( Q \), see e.g. [27] §2.5.

FP\(_2\) modules are important sources of mutually dual definable classes of left and right modules:

**Example 3.2.** Let \( S \) be a class of FP\(_2\) modules. Then \( S^\perp \) is a definable class in \( \text{Mod}–R \) (see [15 Example 6.10]), and \( S^\perp \) is a definable class of left \( R \)-modules. Indeed, \( S^\perp \) is always closed under direct limits and pure submodules, and since \( S \) consists of FP\(_2\) modules, \( S^\perp \) is also closed under products (cf. [12 Theorem 3.2.26] and [8 §VIII.5]). Since \( M^\perp \in S^\perp \) for each \( M \in S^\perp \), and \( N^\perp \in S^\perp \) for each \( N \in S^\perp \) by [15 Lemma 2.16(b) and (d)], the definable classes \( S^\perp \) and \( S^\perp \) are mutually dual.

The classes of left \( R \)-modules \( Q \) of the form \( Q = S^\perp \) for a class \( S \) consisting of FP\(_2\) modules will be called of finite type.

For example, when \( R \) is a right coherent ring and \( S \) the class of all finitely presented modules, then the class \( S^\perp \) of all absolutely pure modules is definable in \( \text{Mod}–R \), and its dual definable class of all flat left \( R \)-modules, \( S^\perp \), is of finite type.
Proposition 3.3. [17] Proposition 1.7 and Theorem 2.11] Let $R$ be a ring. Let $Q$ be a definable class of left $R$-modules and $B = Q^\perp$ be its dual definable class. Let $M$ be a right $R$-module. Then the following conditions are equivalent:

1. $M$ is $Q$-Mittag-Leffler.
2. $M$ is $Q$-Mittag-Leffler for all $Q \in Q$.
3. $M$ is $Q^\perp$-stationary for all $Q \in Q$.
4. $M$ is $B$-stationary.

While studying flat $Q$-Mittag-Leffler modules, one can actually restrict to definable classes of modules $Q$.

Proposition 3.4. [17] Corollary 2.10] Let $Q$ be a class of left $R$-modules. Let $M$ be a $Q$-Mittag-Leffler module. Then $M$ is also Def $Q$-Mittag-Leffler.

Now we will turn to the ascent for flat relative Mittag-Leffler modules, so we will again restrict ourselves to commutative rings.

Lemma 3.5. Let $\varphi : R \to S$ be a flat homomorphism of commutative rings and $Q$ be any class of modules. If $M$ is a flat $Q$-Mittag-Leffler module, then $M \otimes_R S$ is a flat $(Q \otimes_R S)$-Mittag-Leffler $S$-module.

Proof. Since $M$ is a flat module, the functor $(M \otimes_R S) \otimes_S - : \text{Mod-}S \to \text{Mod-}\mathbb{Z}$ is a composition of two exact functors

$$(M \otimes_R S) \otimes_S - = (M \otimes_R -)(S \otimes_S -).$$

Thus $M \otimes_R S$ is a flat $S$-module. Assume that $M$ is a $Q$-Mittag-Leffler module and let $(Q_i \mid i \in I)$ be a family of elements of $Q$. First, note that $Q \otimes_R S \subseteq \text{Def}(Q)$ as classes of modules. Indeed, since $S$ is a flat module, we can write it as a direct limit of finitely generated free modules, say $S = \lim_{\to} R^a_i$. Therefore, $Q \otimes_R \lim_{\to} R^a_i \cong \lim_{\to} Q \otimes_R R^a_i \in \text{Def}(Q)$. By our assumption on $M$ and by Proposition 3.3 we infer that the canonical map $\psi_M : M \otimes_R \prod_{i \in I}(Q_i \otimes_R S) \to \prod_{i \in I}(M \otimes_R Q_i \otimes_R S)$ is monic.

We have the following commutative diagram whose horizontal maps are isomorphisms:

\[
\begin{array}{ccc}
(M \otimes_R S) \otimes_S \prod_{i \in I}(Q_i \otimes_R S) & \xrightarrow{\sim} & M \otimes_R \prod_{i \in I}(Q_i \otimes_R S) \\
\phi_{M \otimes_R S} & & \phi_M \\
\prod_{i \in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S) & \xrightarrow{\sim} & \prod_{i \in I} M \otimes_R (Q_i \otimes_R S)
\end{array}
\]

Here, the left vertical map $\psi_{M \otimes_R S}$ is the canonical morphism $\psi_{M \otimes_R S} : (M \otimes_R S) \otimes_S \prod_{i \in I}(Q_i \otimes_R S) \to \prod_{i \in I}(M \otimes_R S) \otimes_S (Q_i \otimes_R S)$. Thus $\psi_{M \otimes_R S}$ is monic. This proves that $M \otimes_R S$ is a $(Q \otimes_R S)$-Mittag-Leffler $S$-module.

The descent of flatness is well-known, we include a proof here for the sake of completeness.

Lemma 3.6. Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings, and let $M$ be a module such that the $S$-module $M \otimes_R S$ is flat. Then $M$ is a flat module.

Proof. First, since $S$ is a flat module, also $M \otimes_R S$, viewed as an $R$-module, is flat. Indeed, the functor $(M \otimes_R S) \otimes_R -$ is a composition of two exact functors as follows: $M \otimes_R (S \otimes_R S) \otimes_R - = ((M \otimes_R S) \otimes_R -)(S \otimes_R -)$. So for each short exact sequence $C$ of modules, $C \otimes_R (M \otimes_R S)$ is a short exact sequence of $S$-modules. Hence, by faithful flatness of $\varphi$, $C \otimes_R M$ is exact in $\text{Mod-R}$, whence $M$ is a flat module.

Recently, a short proof of the descent of the (absolute) flat Mittag-Leffler property along all pure (and hence all faithfully flat) ring homomorphisms was presented in [3, Lemma 5]. We include this short proof here as it works also in our relative setting. (We refer to [18] for a broader context and further applications.)
Lemma 3.7. Let $\varphi : R \to S$ be a pure monomorphism of commutative rings. Let $Q$ be a class of modules. Let $M$ be a flat module such that $M \otimes_R S$ is a $(Q \otimes_R S)$-Mittag-Leffler $S$-module. Then $M$ is a $Q$-Mittag-Leffler module.

Proof. Let $(Q_i \mid i \in I)$ be a family consisting of modules from $Q$. Since $\varphi$ is pure, the canonical morphism $g_i : Q_i \cong Q_i \otimes_R R \to Q_i \otimes_R S$ is monic for each $i \in I$, and so is $g = \prod g_i : \prod Q_i \to \prod Q_i \otimes_R S$.

Let $M$ be a flat module such that $M \otimes_R S$ is a $(Q \otimes_R S)$-Mittag-Leffler $S$-module. Since $M$ is flat, the morphism $M \otimes_R \prod Q_i \to M \otimes_R \prod-Q_i \otimes_R S$ is monic. Moreover, we have the canonical isomorphism $\psi : M \otimes_R \prod Q_i \otimes_R S \cong M \otimes_R (S \otimes S \prod(Q_i \otimes_R S)) = (M \otimes_R S) \otimes_S \prod Q_i \otimes_R S$. Since $M \otimes_R S$ is a $(Q \otimes_R S)$-Mittag-Leffler $S$-module, the canonical isomorphism $h : (M \otimes_R S) \otimes_S \prod Q_i \otimes_R S \to \prod M \otimes_R S \otimes_S (Q_i \otimes_R S)$ is monic. Thus the composite morphism $k = h \psi (M \otimes_R S)$ is monic.

Notice that $k(m \otimes_R (q_i)_{i \in I}) = ((m \otimes_R 1) \otimes_S (q_i \otimes_R 1))_{i \in I}$, so $k$ can also be expressed as the composition of another triple of canonical morphisms: $k = \psi' g' h'$, where $h' : M \otimes_R \prod Q_i \to \prod M \otimes_R (Q_i \otimes_R S)$, $g'$ is the monomorphism $\prod g_i : \prod Q_i \to \prod Q_i \otimes_R S$, and $\psi'$ the isomorphism $\prod Q_i \otimes_R (Q_i \otimes_R S) \to \prod M \otimes_R S \otimes_S Q_i \otimes_R S$. Since $k$ is monic, so is $h'$. The latter says that $M$ is a $Q$-Mittag-Leffler module.

Now, we can easily prove the descent for flat relative Mittag-Leffler modules:

Theorem 3.8. Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings. Let $Q$ be a class of modules. Let $M$ be a module such that $M \otimes_R S$ is a flat $(Q \otimes_R S)$-Mittag-Leffler $S$-module. Then $M$ is a flat $Q$-Mittag-Leffler module.

Proof. By Lemma 3.6, we can assume that $M$ is a flat module. By Lemma 3.7, $\varphi$ is a pure monomorphism, so Lemma 3.7 applies and shows that $M$ is a $Q$-Mittag-Leffler module.

It is worth noting that for countably presented flat modules, Mittag-Leffler conditions relative to definable classes of modules can be expressed in terms of vanishing of the Ext functor, following [17, §1].

Lemma 3.9. Let $R$ be any ring. Let $M$ be a countably presented flat module, $Q$ be a definable class of left $R$-modules, and $B = Q^\vee$. Then $M$ is $Q$-Mittag-Leffler, if and only if $M \in ^{\perp}B$.

Proof. If $M$ is $Q$-Mittag-Leffler, then $M$ is $B$-stationary by Proposition 3.3. Since $M$ is a countable direct limit of finitely presented free modules and $B$ is closed under countable direct sums, we infer from [15, Corollary 2.23] and [17, Lemma 1.11(3)] that $\text{Ext}^n_M (M, B) = 0$ for each $B \in B$. The converse implication follows by [17, Lemma 1.11(1)] and Proposition 3.3.

Remark 2. Lemma 3.9 does not extend to uncountably presented modules in general. Just consider any non-right perfect ring $R$ and let $Q = R$-Mod. Then $B = Q^\vee = \text{Mod}$-$R$, and so $^{\perp}B = \mathcal{P}_0 \subseteq \mathcal{FM}$ (though, as correctly claimed by Lemma 3.9, the countably presented modules in $\mathcal{P}_0$ and $\mathcal{FM}$ are the same).

Theorem 3.8, Proposition 3.4, and Lemmas 3.5 and 3.9 yield the following

Corollary 3.10. Let $\varphi : R \to S$ be a faithfully flat homomorphism of commutative rings. Let $Q$ be a definable class of modules and $B = Q^\vee$. Let $Q'$ denote the least definable class of $S$-modules containing $Q \otimes_R S$, and $B'$ its dual definable class.

Let $M$ be a countably presented flat module. Then $M \in ^{\perp}B$, if and only if $M \otimes_R S \in ^{\perp}B'$.

In the particular setting of definable classes arising from kernels of Tor functors (such as the definable classes of finite type from Example 3.2), we have the following relation between definable closures:
Lemma 3.11. Let \( \varphi : R \to S \) be a flat homomorphism of commutative rings and \( C \) be a class of \( R \)-modules. Then Def \((C \otimes_R S)^\top) = \text{Def}(C^\top \otimes_R S)\).

In particular, if \( C \) consists of FP\(_2\) modules, then Def \((C^\top \otimes_R S) = (C \otimes_R S)^\top\).

Proof. First, \((C^\top) \otimes_R S \subseteq (C \otimes_R S)^\top\) by Lemma 2.2(1), whence Def \((C^\top \otimes_R S) \subseteq \text{Def}(C \otimes_R S)^\top\).

For the opposite inclusion, note that by Lemma 2.2(2), \((C \otimes_R S)^\top\) is the class of all \( S \)-modules \( N \) satisfying the following condition: \( N \), viewed as an \( R \)-module, is an element of \((C^\top)^\top\). Then again \( N \otimes_R S \in (C \otimes_R S)^\top\) by Lemma 2.2(1). Since the canonical homomorphism \( f : n \mapsto n \otimes 1 \) from \( N \) to \( N \otimes_R S \) is an \( S \)-homomorphism, and the \( S \)-homomorphism \( g : N \otimes_R S \to N \) defined by \( g : n \otimes s \mapsto n.s \) satisfies \( g f = 1_N \), we infer that \( N \) is isomorphic to a direct summand in \( N \otimes_R S \) as an \( S \)-module. Thus \((C \otimes_R S)^\top\) consists of \( S \)-modules isomorphic to direct summands of the modules from \((C^\top) \otimes_R S\), whence \((C \otimes_R S)^\top \subseteq \text{Def}(C^\top \otimes_R S)\), proving the opposite inclusion.

If \( C \) consists of FP\(_2\) modules, then also \( C \otimes_R S \) consists of FP\(_2\) \( S \)-modules, whence \((C \otimes_R S)^\top\) is a definable class by Example 3.2. \( \Box \)

4. Zariski Locality of Quasi-coherent Sheaves Associated with Flat Relative Mittag-Leffler Modules

In this section, we will apply the results of Section 3 to prove Zariski locality of flat relative \( Q \)-Mittag-Leffler modules in various particular settings.

We start with a direct general application to quasi-coherent sheaves associated with f-projective modules. Recall that a module \( M \) is f-projective if \( M \) is flat and \((R)\)-Mittag-Leffler, or equivalently, \( M \) is a flat \( Q \)-Mittag-Leffler module where \( Q \) is the class of all flat left \( R \)-modules, [16] (see also Proposition 3.4 and [5], §3). In accordance with our Definition 2.7, we call a quasi-coherent sheaf \( M \) on a scheme \( X \) locally f-projective in case for each open affine set \( u \) in \( X \), the \( \mathcal{R}(u) \)-module of sections \( M(u) \) is an f-projective \( \mathcal{R}(u) \)-module.

Theorem 4.1. The notion of a locally f-projective quasi-coherent sheaf is Zariski local on the class of all schemes.

Proof. By Lemma 2.9 it suffices to prove that the property of being an f-projective module is an ad-property in the class of all commutative rings. However, its ascent and descent follows for \( Q = \{ R \} \) immediately by Lemma 3.5 and Theorem 3.8 respectively. The compatibility with finite ring direct products is obvious (cf. Definition 2.6(3)). \( \Box \)

For the rest of this section, \( R \) will denote a commutative ring, \( C_R \) a class of modules, and \( Q_R \) the definable class \( Q_R = \text{Def}(C_R^\top)\). In particular, \( Q_R = C_R^\top\) in case \( C_R \) consists of FP\(_2\) modules.

The relevant property \( \mathfrak{P} \) of modules is defined as follows: if \( M \) is a module, then \( M \in \mathfrak{P}(\text{Mod–}R) \), iff \( M \) is a flat \( Q_R \)-Mittag-Leffler module.

In order to prove locality of the induced notions of quasi-coherent sheaves in this setting, we will need compatibility of the properties \( \mathfrak{P} \) for commutative rings \( R \) connected by flat, and faithfully flat, morphisms. More precisely, we will require the following compatibility conditions (C1), (C2) and (C3):

Definition 4.2. Let \( \mathcal{R} \) be a class of commutative rings.

(C1) For each flat ring homomorphism \( \varphi : R \to S \) with \( R, S \in \mathcal{R} \), \( C_R \otimes_R S \subseteq C_S\).

(C2) For each faithfully flat ring homomorphism \( \varphi : R \to S \) where \( R \in \mathcal{R} \) and \( S \) is a finite direct product of rings in \( \mathcal{R} \), Def \((C_S)^\top = \text{Def}(C_R \otimes_R S)^\top\).

(C3) If \( S = \prod_{i=1}^n R_i \) where \( R_i \in \mathcal{R} \) for each \( i < n \), then \( C_S = \prod_{i=1}^n C_{R_i} \).

Notice that (C1) implies the inclusion \( C_S^\top \subseteq (C_R \otimes_R S)^\top \), and hence Def \((C_S)^\top = \text{Def}(C_R \otimes_R S)^\top\).
Lemma 4.3. Let $\mathcal{R}$ be a class of commutative rings such that condition (C1) holds. Then the property $\mathcal{P}$ ascends along flat morphisms in $\mathcal{R}$.

Proof. Let $\varphi : R \to S$ be a flat ring homomorphism with $R, S \in \mathcal{R}$ and $M$ be a flat $Q_R$-Mittag-Leffler module. By Lemma 3.5, $M \otimes_R S$ is a flat $(Q_R \otimes_R S)$-Mittag-Leffler $S$-module, and hence a flat $\text{Def}(Q_R \otimes_R S)$-Mittag-Leffler $S$-module by Proposition 3.4. Condition (C1) and Lemma 3.11 give

$$Q_S = \text{Def} C_S^\top \subseteq \text{Def}(Q_R \otimes_R S)^\top = \text{Def}(Q_R \otimes_R S).$$

Thus, $M \otimes_R S$ is a flat $Q_S$-Mittag-Leffler $S$-module. □

Lemma 4.4. Let $\mathcal{R}$ be a class of commutative rings such that condition (C2) holds. Then the property $\mathcal{P}$ descends along faithfully flat morphisms in $\mathcal{R}$.

Proof. Let $\varphi : R \to S$ be a faithfully flat ring homomorphism, where $R \in \mathcal{R}$ and $S$ is a finite direct product of rings in $\mathcal{R}$. Let $M$ be a module such that $M \otimes_R S$ is a flat $Q_S$-Mittag-Leffler $S$-module. Condition (C2) and Lemma 3.11 yield

$$Q_S = \text{Def} C_S^\top = \text{Def}(C_R \otimes_R S)^\top = \text{Def}(Q_R \otimes_R S),$$

so $M \otimes_R S$ is a flat $(Q_R \otimes_R S)$-Mittag-Leffler $S$-module. By Theorem 3.8, $M$ is a flat $Q_R$-Mittag-Leffler module. □

Thus, we obtain

Theorem 4.5. Let $\mathcal{R}$ be a class of commutative rings such that conditions (C1), (C2) and (C3) hold. Then $\mathcal{P}$ is an ad-property in $\mathcal{R}$, whence the notion of a locally $\mathcal{P}$-quasi-coherent sheaf is Zariski local on the class of all locally $\mathcal{R}$-schemes.

Proof. By condition (C3), $\mathcal{P}$ is compatible with finite ring direct products, so the ad-property of $\mathcal{P}$ follows by Lemmas 4.3 and 4.4. The final claim follows by Lemma 2.20. □

We finish this section by noting several applications of Theorem 4.5.

4.1. Applications. 1. Let $\mathcal{R}$ be the class of all commutative rings and $C_R = \{0\}$, so $Q_R = R$-Mod. In this case, Theorem 4.5 yields the Zariski locality of the notion of a Drinfeld vector bundle (= locally flat Mittag-Leffler quasi-coherent sheaf) proved in [13].

2. Let $\mathcal{R}$ be the class of all commutative rings and $C_R$ the class of all finitely presented modules. Then $Q_R = \text{Def} C_R^\top = \text{Def} F_0$. By Proposition 3.5, a module $M$ has property $\mathcal{P}$, iff $M$ is $f$-projective. Conditions (C1) and (C3) clearly hold true.

Condition (C2) holds even in the stronger form of $C_R^\top = (C_R \otimes_R S)^\top$ whenever $\varphi : R \to S$ is a faithfully flat homomorphism of commutative rings. Indeed, $C_R^\top$ is the class of all flat $S$-modules. Let $M \in (C_R \otimes_R S)^\top$. By Lemma 2.2, $\text{Tor}_1^R(C_R, M) = 0$, whence $M$ is a flat $R$-module. Then $M \otimes_R S$ is a flat $S$-module, by (the proof of) Lemma 3.5. However, the $S$-module $M$ is isomorphic to a direct summand in $M \otimes_R S$ (cf. the proof of Lemma 3.11), whence $M$ is a flat $S$-module. This proves the inclusion $(C_R \otimes_R S)^\top \subseteq C_S^\top$; the other inclusion is a consequence of condition (C1).

Thus, Theorem 4.1 is just a particular instance of Theorem 4.5 for $C_R = \text{the class of all finitely presented modules.}$

3. A more involved application of Theorem 4.5 concerns the case when $C_R = F_\alpha$ for some $n \geq 1$. In this case, we will verify conditions (C1) - (C3) for $\mathcal{R}$ = the class of all noetherian rings.

Condition (C1) holds since $C_R \otimes_R S \subseteq C_S$ when $S$ is a flat module, and (C3) is obvious. As in Application 2, it only suffices to prove the inclusion $(C_R \otimes_R S)^\top \subseteq C_S^\top$ for each faithfully flat homomorphism of commutative noetherian rings $\varphi : R \to S$.

Recall that for an $S$-module $M$, $M^\alpha = \text{Hom}_S(M, \mathbb{Q}/\mathbb{Z})$ denotes the $S$-module of characters of $M$, and for a class of $S$-modules $\mathcal{E}$, $\mathcal{E}^\alpha = \{M^\alpha \mid M \in \mathcal{E}\}$. We claim that
\[ \mathcal{F}_n^+ = I_n \cap (\text{Mod--}\mathcal{S})^+ \]. Since character modules of flat modules are injective, the \( \subseteq \) inclusion holds. Conversely, let \( N = M^+ \in I_n \). Since \( \mathcal{S} \) is noetherian, character modules of injective modules are flat (e.g., by \cite{Herbera-2021} Lemma 2.16(d)), so \( N^+ = M^{**} \in \mathcal{F}_n \). As the class \( \mathcal{F}_n \) is closed under pure submodules and the embedding \( M \hookrightarrow M^{**} \) is pure, \( M \in \mathcal{F}_n \) and the claim is proved. Using \cite{Herbera-2021} Lemma 2.16(b) and the fact that the pure embedding \( M \hookrightarrow M^{**} \) splits for any pure-injective module \( M \), we get \( (\mathcal{F}_n)^+ = \perp (I_n \cap \mathcal{P} I) \).

Let \( S_{n,R} \) denote the class of all finitely generated modules that appear as \( n \)-th syzygies in some projective resolution, \( \mathcal{P} \), of a finitely generated module such that \( \mathcal{P} \) consists of finitely generated modules. Then \( R \in S_{n,R}, \) and since \( R \) is noetherian, \( S_{n,R} = I_n \) by the Baer Test of Injectivity and by dimension shifting. Let \( D_{n,R} \) denote the class of all modules \( M \) that are isomorphic to direct summands of finite extensions of the modules from \( S_{n,R} \). Then the class \( D_{n,R} \) is closed under extensions, direct summands, and contains \( R \). Moreover, \( D_{n,R} = I_n \). By Lemma \ref{1}, \( \lim D_{n,R} = \perp (I_n \cap \mathcal{P} I) \).

Finally, let \( M \in (C_R \otimes R S)^+ \). Then Lemma \ref{2} gives \( \text{Tor}^R_1(C_R, M) = 0 \). By the above, \( M \), viewed as an \( R \)-module, is an element of \( \lim D_{n,R} \). Since \( S \) is a flat module, \( S_{n,R} \otimes_R S \subseteq S_{n,S} \), whence also \( D_{n,R} \otimes_R S \subseteq D_{n,S} \). Moreover, the tensor product commutes with direct limits, so \( M \otimes_R S \in \lim D_{n,S} = C^+_S \). As \( M \) is isomorphic to a direct summand in \( M \otimes_R S \) as an \( S \)-module, also \( M \in C^+_S \), and the inclusion \( (C_R \otimes R S) \subseteq C^+_S \) is proved.

Recall that if \( n \geq 0 \) and \( Q_n = (\mathcal{F}_n)^+ \), then the flat \( Q_n \)-Mittag-Leffler modules are called flat \( n \)-Mittag-Leffler, and the corresponding quasi-coherent sheaves are the \( n \)-Drinfeld vector bundles. Thus, we have the following consequence of Theorem \ref{4} for \( C_R = \mathcal{F}_n \):}

**Theorem 4.6.** For each \( n \geq 1 \), the notion of an \( n \)-Drinfeld vector bundle is Zariski local on the class of all locally noetherian schemes.

**Remark 3.** If \( R \) is a non-right perfect ring (e.g., a commutative noetherian ring of Krull dimension \( \geq 1 \)), then there is a gap between the classes \( \mathcal{F} M \) of all flat Mittag-Leffler modules and \( \mathcal{F} \) of all flat modules. In fact, for each class \( Q \) of left \( R \)-modules we have \( \mathcal{F} M \subseteq D_Q \subseteq \mathcal{F} \). Since \( D_Q = D_{\mathcal{Def}}(Q) \) by Proposition \ref{5} and there is only a set of definable classes of modules, there is also only a set of such intermediate classes \( D_Q \) between \( \mathcal{F} M \) and \( \mathcal{F} \) (see also \cite{Huet-Posp} Theorem 3.5(iii)).

Of course, the variety of classes of modules between \( \mathcal{F} M \) and \( \mathcal{F} \) translates directly into the same variety of classes of locally \( \mathcal{P} \)-quasi-coherent sheaves in the class of all flat quasi-coherent sheaves on the affine scheme \( X = \text{Spec}(R) \), where \( R \) is any commutative non-perfect ring (since in this case, \( \text{Qcoh}(X) \) is equivalent to \( \text{Mod--R} \)). Moreover, all these classes contain a flat generator, as they contain all vector bundles on \( X \).

However, the picture for non-affine schemes may be different, depending on further properties of the schemes. For example, by \cite{Herbera-2021}, if \( X \) is a quasi-compact and quasi-separated scheme, then \( \text{Qcoh}(X) \) contains a flat generator, if and only if \( X \) is semiseparated (i.e., the intersection of any two open affine sets is affine).

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