Abstract

Analysis of (partial) groundness is an important application of abstract interpretation. There are several proposals for improving the precision of such an analysis by exploiting type information, including our own work with Hill and King [SHK00], where we had shown how the information present in the type declarations of a program can be used to characterise the degree of instantiation of a term in a precise and yet inherently finite way. This approach worked for polymorphically typed programs as in Gödel or HAL. Here, we recast this approach following Codish, Lagoon and Stuckey [CL00, LS01]. To formalise which properties of terms we want to characterise, we use labelling functions, which are functions that extract subterms from a term along certain paths. An abstract term collects the results of all labelling functions of a term. For the analysis, programs are executed on abstract terms instead of the concrete ones, and usual unification is replaced by unification modulo an equality theory which includes the well-known ACI-theory. Thus we generalise [CL00, LS01] w.r.t. the type systems considered and relate those two works.

1 Introduction

Analysing logic programs for properties such as sharing and (partial) groundness is important in compiler optimisations and program development tools. Programs are usually analysed using abstract interpretation [CC77]. In this paper, we consider in particular the framework of abstract compilation [CD95]: a program is abstracted by replacing each unification with an abstract counterpart, and then the abstract program is evaluated just like a concrete program.

It has been recognised for some time that abstract interpretation can be used for type analysis, and conversely, that type information available a priori can improve the precision of other analyses [BM95, CD94, CL00, GW94, JB92, VCL93]. For example, being able to say that \([L, X]\) is a list skeleton with possibly uninstantiated elements is more precise than only being able to distinguish a ground from a possibly non-ground term. Underlying all those works is a descriptive view of types: types are not part of the programming language (in particular, no program is rejected for not being “well-typed”), but rather introduced to analyse an arbitrary, say Prolog, program. In such approaches, it is natural that there is no sharp line between type analysis and mode (groundness, instantiation) analysis. For example, saying that a term is a list has two aspects: it is a list as opposed to, say, an integer; it is a list, as opposed to an uninstantiated variable.

Underlying this paper is a prescriptive view of types, i.e., types are a part of the programming language. We analyse programs written in typed logic programming

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languages such as Gödel [HL94], HAL [DMGMS99], or Mercury [SHC96]. This implies that the types do not have to be analysed since they are given beforehand by declarations or inference. In particular, unlike e.g. [CL00], we do not have to deal with “ill-typed” terms such as [1|2], since these can never occur.

We are aware of only two other works following this view: our own [SHK00] and the recent work by Lagoon and Stuckey [LS01]. This paper is a synthesis of those two works and [CL00], which, although designed for a descriptive view of typing, can be adapted to prescriptive typing. The generalisation w.r.t. [CL00, LS01] concerns polymorphism, which is disregarded in [LS01] and considered in [CL00] only in a restricted form. We recast our own previous work using some aspects of their formalisms. In particular, the use of the notions of grammar and variables labelling non-terminals [LS01] should improve the understanding of what properties of terms our analysis captures, whereas the use of ACI-unification [CL00] may provide the basis for an implementation using well-studied algorithms. Also, we hope that our work will prove to be applicable to analyses previously not envisaged by us, such as sharing analysis [LS01].

In the intuitive explanations that follow, we refer to a set of possible characterisations of the instantiation of a term as abstract domain.

The standard example to illustrate the benefits of an instantiation analysis using types is the ubiquitous APPEND program. For example, for the query append([A], [B], C), a typed analysis is able to infer that any answer substitution will bind C to a list skeleton. However, this example is unfit to explain the advance of this paper over previous works.

We therefore give another example. A table is a data structure containing an ordered collection of nodes, each of which has two components, a key of type string, and a value, of arbitrary type. That is to say, the type constructor table is parameterised by the type of the values. For any type τ, table(τ) is the type of tables whose values have type τ. Tables are implemented in Gödel as an AVL-tree [Emd81]: a non-leaf node has a key argument, a value argument, arguments for the left and right subtrees, and an argument which represents balancing information. For a term of type table(τ), our abstract domain characterises the instantiation of all key arguments, all value arguments, and all the arguments representing the balancing information.

The characterisation of the instantiation of the value arguments depends on τ. Hence, our analysis supports parametric polymorphism. In devising an analysis for polymorphically typed programs, there are two main problems: the construction of an abstract domain for table(τ) should be truly parametric in τ, and the abstract domains should be finite for a given program and query. We only briefly illustrate what these points mean here. Explaining why these requirements are non-trivial is technically too involved for this introduction.

The statement that the construction of an abstract domain for table(τ) is truly parametric in τ, means, for example, that the abstract domain for table(string) relates to string in exactly the same way as the abstract domain for table(integer) relates to integer. This implies that the abstraction of a table can be defined in a generic way.

Lagoon and Stuckey formalise types as regular tree grammars. Each type is identified with a non-terminal in the grammar, and it is assumed that there are only finitely many types. Finiteness is crucial for the termination of an analysis. When we extend this approach to polymorphism, finiteness becomes a problem, since there are infinitely many types, e.g. list(integer), list(list(integer)), .... Nevertheless, under certain conditions, it can be ensured that for a given query and program, there are only finitely many types. Note that this is in contrast to [CL00] where it is proposed that termination of analyses of polymorphic programs should be enforced by imposing

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1 The journal article [CL00] is based on an earlier article [CL96]. However there are some interesting differences, and therefore we will also sometimes refer to the earlier article.

2 We abbreviate string by str and integer by int.
an ad-hoc bound on the depth of types.

The rest of this paper is organised as follows. The next section provides some preliminaries. In Sec. 2, following [LS03], we show how the type of a term gives rise to characterising its degree of instantiation in a structured way. In Sec. 3, following [CL00], we define abstract terms based on the ACI1 equality theory. In Sec. 4, we formalise how abstract terms capture the degree of instantiation of concrete terms, thereby linking the two preceding sections, and also linking [LS03] with [CL00]. Section 5 lifts the abstraction of terms to an abstraction of programs, and relates the semantics of a concrete program and its abstraction in the sense of abstract interpretation. Section 6 makes some comments on a possible future implementation, and Sec. 7 discusses our results.

2 Preliminaries

The reader is assumed to be familiar with the basics of logic programming [Llo87]. We use a type system for logic programs with parametric polymorphism [DMGMS99, HL94, SHC90].

Let \( K \) be a finite set of (type) constructors, each \( c \in K \) with an arity \( n \geq 0 \) associated (by writing \( c/n \)), and \( U \) be a denumerable set of parameters. The set of types is the term structure \( T(K, U) \). A type substitution is an idempotent mapping from parameters to types which is the identity almost everywhere. We define the order \( \prec \) on types as the order induced by some (for example lexicographical) order on constructor and parameter symbols, where parameter symbols come before constructor symbols.

The set of parameters in a syntactic object \( o \) is denoted by \( \text{pars}(o) \). Parameters are denoted by \( u, v \), in concrete examples by \( U, V \). A tuple of distinct parameters ordered with respect to \( \prec \) is denoted by \( \bar{u}, \bar{v} \).

Let \( V \) be a denumerable set of variables. The set of variables in a syntactic object \( o \) is denoted by \( \text{vars}(o) \). Variables are denoted by \( x, y \), in concrete examples by \( X, Y \). A tuple of distinct variables is denoted by \( \bar{x}, \bar{y} \).

A variable typing is a mapping from a finite subset of \( V \) to \( T(K, U) \), written as \( \{x_1 : \tau_1, \ldots, x_n : \tau_n \} \).

Let \( F \) (resp. \( P \)) be a finite set of function (resp. predicate) symbols, each with an arity and a declared type associated with it, such that: for each \( f \in F \), the declared type has the form \((\tau_1, \ldots, \tau_n, \tau)\), where \( n \) is the arity of \( f \), \((\tau_1, \ldots, \tau_n, \tau) \in T(K, U)^{n+1} \), and \( \tau \) satisfies the transparency condition [HT92]: \( \text{pars}(\tau_1, \ldots, \tau_n) \subseteq \text{pars}(\tau) \); for each \( p \in P \), the declared type has the form \((\tau_1, \ldots, \tau_n)\), where \( n \) is the arity of \( p \) and \((\tau_1, \ldots, \tau_n) \in T(K, U)^n \). We often indicate the declared types by writing \( f_{\tau_1, \ldots, \tau_n : \tau} \) and \( p_{\tau_1, \ldots, \tau_n} \).

Throughout this paper, we assume \( K, F \), and \( P \) arbitrary but fixed. The typed language, i.e. a language of terms, atoms etc. based on \( K, F \), and \( P \), is defined by the rules in Table 1. All objects are defined relative to a variable typing \( \Gamma \), and \( \vdash \ldots \) stands for “there exists \( \Gamma \) such that \( \Gamma \vdash \ldots \)”. Actually, we will rarely refer to the type system explicitly, but it should be noted that any objects we will come across in the context of analysing a typed program will be correctly typed according to those rules. This is guaranteed because typed programs have the subject reduction property [HT92]. Concerning semantics, we entirely follow [CL00]. The set of atoms is denoted by \( B \), and elements of \( 2^B \) are called interpretations. For a syntactic object \( o \) and a set of objects \( I \), we denote by \( \langle C_1, \ldots, C_n \rangle \ll o, I \) that \( C_1, \ldots, C_n \) are elements of \( I \) renamed apart from \( o \) and from each other. So the analysis we shall propose is generic and independent of any particular (say top-down or bottom-up) concrete semantics, but examples will be given using the s-semantics, i.e. the semantics based on the
We denote by $\{ x : \tau, \ldots \} \vdash x : \tau$.

Table 1: Rules defining a typed language ($\Theta$ is a type substitution)

| (Var)   | $\{ x : \tau, \ldots \} \vdash x : \tau$ |
|---------|-----------------------------------------|
| (Func)  | $\Gamma \vdash t_1 : \tau_1 \Theta \ldots \Gamma \vdash t_n : \tau_n \Theta$ |
|         | $\Gamma \vdash f_{t_1 \ldots t_n} : \tau_1 \ldots \tau_n : \tau$ |
| (Atom)  | $\Gamma \vdash t_1 : \tau_1 \Theta \ldots \Gamma \vdash t_n : \tau_n \Theta$ |
|         | $\Gamma \vdash p_{t_1 \ldots t_n} : (t_1, \ldots, t_n)$ |
| (Head)  | $\Gamma \vdash t_1 : \tau_1 \ldots \Gamma \vdash t_n : \tau_n$ |
|         | $\Gamma \vdash p_{t_1 \ldots t_n} (t_1, \ldots, t_n)$ |
| (Query) | $\Gamma \vdash A_1 \text{Atom} \ldots \Gamma \vdash A_n \text{Atom}$ |
|         | $\Gamma \vdash A_1, \ldots, A_n \text{Query}$ |
| (Clause) | $\Gamma \vdash A \text{Head} \ldots \Gamma \vdash Q \text{Query}$ |
|         | $\Gamma \vdash A \text{--} Q \text{Clause}$ |
| (Program) | $\vdash C_1 \text{Clause} \ldots \vdash C_n \text{Clause}$ |
|         | $\vdash \{ C_1, \ldots, C_n \} \text{Program}$ |

non-ground $T_P$-operator, defined as follows:

$$T_P(I) = \{ H \theta | C = H \leftarrow B_1, \ldots, B_n \in P, \{ A_1, \ldots, A_n \} \not\leq_C I, \theta = \text{MGU}((B_1, \ldots, B_n), \{ A_1, \ldots, A_n \}) \}.$$ 

We denote by $[P]_\sigma$ the least fixpoint of $T_P$.

We denote by $t_1 \leq t_2$ that $t_1$ is an instance if $t_2$. The domain of a substitution $\theta$ is denoted as $\text{dom}(\theta)$.

3 The Structure of Terms and Types

In this section, we show how the type of a term gives rise to a certain way of characterising its structure, and in particular, how its degree of instantiation can be characterised in a structured way. We alternate between recalling the formalism of [LS01], and adapting it to polymorphic types, thereby linking to [SHK00].

3.1 Regular Types [LS01]

Definition 3.1 A top-down deterministic finite tree automaton (top-down DFTA) is a tuple $\mathcal{A} = \langle q_0, Q, \Sigma, \Delta \rangle$, where $Q$ is a set of states, $q_0 \in Q$ is an initial state and $\Delta$ is a set of transition rules of the form $q(f(x_1, \ldots, x_n)) \rightarrow f(q_1(x_1), \ldots, q_n(x_n))$, such that no two rules have the same left-hand side.

Top-down DFTA’s accept the class of languages called regular types.

Definition 3.2 A regular tree grammar is a tuple $\mathcal{G} = \langle S, W, \Sigma, \Delta \rangle$, where $W$ is a finite set of non-terminal symbols, $S \in W$ is a starting non-terminal, $\Delta$ is a set of productions in the form $X \rightarrow f(Y_1, \ldots, Y_n)$ s.t. $X, Y_1, \ldots, Y_n \in W$ and $f/n \in \Sigma$. A regular tree grammar is deterministic if for any non-terminal $X$ and any two productions $X \rightarrow f(Y_1, \ldots, Y_n)$ and $X \rightarrow g(Z_1, \ldots, Z_m)$, we have $f/n \neq g/m$.

It has been pointed out that the two formalisms above define the same class of languages. Transitions of the automaton can be converted to grammar productions and vice versa by identifying each non-terminal symbol with a corresponding state of the automaton.

Example 3.3 The DFTA $\langle q_L, \{ q_L, q_E \}, \{ \text{nil}/0, \text{cons}/2, a/0, b/0 \}, \Delta \rangle$, where

$$\Delta = \{ q_L(\text{nil}) \rightarrow \text{nil}, q_L(\text{cons}(x, y)) \rightarrow \text{cons}(q_E(x), q_L(y)), q_E(a) \rightarrow a, q_E(b) \rightarrow b \},$$

accepts ground lists of a’s and b’s.

The grammar $L \rightarrow \text{nil} | \text{cons}(E, L), \ E \rightarrow a | b$ defines the same language.
a particular segment of a single path in a derivation tree starting from root $G = f(\ldots, t_n) \rightarrow f(N_1(t_1), \ldots, N_n(t_n))$, where $N \rightarrow f(N_1, \ldots, N_n)$ is a production of the grammar ($n \geq 0$). Using this notation, we say that the grammar $G = (S, W, \Sigma, \Delta)$ accepts a term $t$ if $S(t) \rightarrow^* t$. Sometimes we are interested in a particular segment of a single path in a derivation tree starting from root $S$ and reaching a non-terminal $N$ with a subterm $t'$ of $t$, i.e., in derivations $S(t) \rightarrow^* s[N(t')]$, where the notation $s[N(t')]$ means that $s$ has $N(t')$ as a subterm. Abusing notation, we write $S(t) \rightarrow^* N(t')$ in this case.

**Example 3.4** Given the grammar in Ex. 3.3, we have

$$L(\text{cons}(a, \text{nil})) \rightarrow \text{cons}(E(a), L(\text{nil})) \rightarrow \text{cons}(a, L(\text{nil})) \rightarrow \text{cons}(a, \text{nil}).$$

We also write $L(\text{cons}(a, \text{nil})) \rightarrow^* E(a)$ and $L(\text{cons}(a, \text{nil})) \rightarrow^* L(\text{nil})$, using the above notation.

This notation can also be applied to non-ground terms. For example, we have $L(\text{cons}(a, Y)) \rightarrow^* E(a)$ and $L(\text{cons}(X, Y)) \rightarrow^* L(Y)$.

It is also convenient to depict a grammar graphically as a type graph, defined previously as a graph whose nodes are labelled with types or functions $\mathcal{VCL93}$. We simplify that definition by leaving out the function nodes. Thus a type graph for $G = (S, W, \Sigma, \Delta)$ is a directed graph whose nodes are labelled by non-terminals, and there is an edge from $N$ to $N'$ if and only if there is a production $N \rightarrow f(\ldots, N', \ldots)$ is $G$. We call the node labelled $S$ the starting node. Figure 1 shows the type graph for Ex. 3.3.

### 3.2 Regular Types and Polymorphism

Converting the type declarations of a typed language such as Mercury into grammar rules has been considered straightforward $\mathcal{LS01}$, footnote 1]. This seems justified, albeit only in the absence of polymorphism. Since $\mathcal{K}$ is a finite set of type constants, we can identify each type constant with a non-terminal, and each function $f_{\tau_1, \ldots, \tau_n \rightarrow \tau} \in \mathcal{F}$ is translated into a production $\tau \rightarrow f(\tau_1, \ldots, \tau_n)$. In that way, each type (constant) corresponds to a grammar with that type as starting non-terminal. Table 2 summarises the correspondences between the four formalisms we effectively identify in this paper.

Note that in Sec. 2, we have specified that each $f$ has exactly one declaration; in other words, there is no overloading. This is a sufficient condition for the grammar to be deterministic. One could allow some overloading, by specifying: if $f_{\tau_1, \ldots, \tau_n \rightarrow \tau} \in \mathcal{F}$ and $f_{\sigma_1, \ldots, \sigma_m \rightarrow \sigma} \in \mathcal{F}$ and $f_{\tau_1, \ldots, \tau_n \rightarrow \tau} \neq f_{\sigma_1, \ldots, \sigma_m \rightarrow \sigma}$, then either $\tau \neq \sigma$, or $n \neq m$. This would strictly include the overloading allowed in Gödel. We prefer however to disallow overloading to avoid unnecessary complication.

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Table 2: Correspondences between formalisms

| Automata | Grammars | Types | Graphs |
|----------|----------|-------|--------|
| state    | non-terminal | type | node  |
| transition rule | production | $\approx$ type declarations | edge |

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Although one might be confused by the fact that $\mathcal{LS01}$ also says that there is a grammar for each program variable, but this is simply a matter of renaming.
We now give a pseudo-definition of a grammar corresponding to a polymorphic type — “pseudo” because the set of non-terminals may be infinite. It is trivial to generalise the definition of a grammar to infinite sets of non-terminals, but in the end, it is not desirable, since it would undermine our goal of characterising (approximating) properties of a term of arbitrary type in a finite way. We will later impose a condition to ensure finiteness.

**Definition 3.5** Consider a typed language given by $\mathcal{K}, \mathcal{F}$ and a type $\phi$. The grammar corresponding to $\phi$, denoted $G(\phi)$, is the grammar $\langle \phi', W, F, \Delta \rangle$, where $W$ is inductively defined as follows

- $\phi' \in W$,
- $f_{\tau_1, \ldots, \tau_n} \rightarrow \tau \in F$ and $\tau' \in W$ for some type substitution $\Theta$ implies $\tau_1 \Theta', \ldots, \tau_n \Theta' \in W$,

and $\Delta = \{ \tau \Theta' \rightarrow f_{\tau_1, \ldots, \tau_n} (\tau_1 \Theta', \ldots, \tau_n \Theta') \mid \tau \Theta \in W \}$.

We put types in quotation marks to indicate that when looking at the grammar, types are just non-terminal symbols. Type graphs are defined as before. In Fig. 2, we give some type graphs to which we will refer frequently.

It is also useful to have names and a notation for the relations holding between the types in a type graph.

**Definition 3.6** A type $\sigma$ is a direct subterm type of $\phi$ (denoted as $\sigma \triangleleft \phi$) if there is $f_{\tau_1, \ldots, \tau_n} \rightarrow \tau \in F$ and a type substitution $\Theta$ such that $\tau \Theta = \phi$ and $\tau_i \Theta = \sigma$ for some $i \in \{1, \ldots, n\}$. The transitive, reflexive closure of $\triangleleft$ is denoted as $\triangleleft^*$. If $\sigma \triangleleft^* \phi$, then $\sigma$ is a subterm type of $\phi$.

We now discuss two problems related to the generalisation to polymorphism, including that of finiteness mentioned above.

**Example 3.7** Whenever we give a particular typed language, $\mathcal{K}$ is given implicitly as the set of all type constructors occurring in the type subscripts in $\mathcal{F}$.

One would hope that even if a typed language contains an infinite set of types, the type graph taking a fixed type as starting node should be finite. However, consider $\mathcal{F} = \{ f_{c(\bar{c}(0))} \rightarrow c(0) \}$. The type graph of $c(0)$ is shown in Fig. 2 (a). As can be seen, it is infinite.

We impose the following restriction on any typed language to ensure finiteness.

**Reflexive Condition:** For all $c \in \mathcal{K}$ and types $\sigma = c(\bar{\sigma})$, $\tau = c(\bar{\tau})$, if $\sigma \triangleleft^* \tau$, then $\sigma$ is a sub“term” (in the syntactic sense) of $\tau$.
Clearly, this condition is violated by the example above, where \( c(c(U)) \subset c(U) \). With this condition in place, it is easy to see that any type graph for a given starting node is finite.

In the introduction we mentioned another problem, namely that the construction of abstract domains should be “truly parametric”.

**Example 3.8** Consider \( \mathcal{F} = \{ f_{\mathit{u} \rightarrow \mathit{k(U)}}, g_{\mathit{int} \rightarrow \mathit{k(str)}} \} \). Figure 3 (b) shows the type graph for \( k(U) \). Essentially, the type graph for any instance \( k(\tau) \) is obtained by replacing the node \( k(U) \) with \( k(\tau) \) and the node \( U \) with the type graph for \( \tau \). However, there is one exception to this: if \( \tau = \mathit{str} \), then the type graph is the one shown in Fig. 3 (c). For this example, it would clearly be wrong to say that “\( k(U) \) relates to \( U \) in the same way as \( k(\mathit{str}) \) relates to \( \mathit{str} \)”.

Again, we rule out this anomaly. First we define:

**Definition 3.9** A flat type is a type of the form \( c(\bar{u}) \), where \( c \in \mathcal{K} \).

We now impose the following condition on any typed language.

**Flat Range Condition:** For all \( f_{r_1, \ldots, r_m \rightarrow \tau} \in \mathcal{F}, \tau \) is a flat type.

In Mercury (and also in functional languages such as ML or Haskell), this condition is enforced by the syntax. In Gödel, it is possible to violate the condition, but this can be regarded as an artefact of that syntax.

Thus we assume from now on that any typed language we consider meets the two conditions above.

### 3.3 Labelling [LS01]

Labellings can be used to characterise the degree of instantiation of a term taking its type into account, i.e., analyse a term on a per-role basis [LS01].

**Definition 3.10** A variable \( x \) in a term \( t \) labels a non-terminal \( N \) of a grammar \( G \) if \( S(t) \rightarrow^* N(x) \), where \( S \) is the starting non-terminal of \( G \).

We denote by \( \zeta(S, N, t) \) the function which returns the set of variables \( x \) such that \( S(t) \rightarrow^* N(x) \) (one could also write \( \zeta(G, N, t) \) [LS01]).

**Example 3.11** The grammar \( LL \rightarrow \mathit{nil}|\mathit{cons}(L, LL), \ L \rightarrow \mathit{nil}|\mathit{cons}(E, L), \ E \rightarrow a|b \) accepts ground lists of lists of \( a \)’s and \( b \)’s. Note that the use of \( \mathit{cons} \) and \( \mathit{nil} \) could be regarded as overloading, but this is not forbidden by [LS01] as it is not in contradiction to the grammar being deterministic.

We use the usual list notation for ease of reading. The type graph of \( LL \) is shown in Fig. 3. We are interested in the labelling of all non-terminals reachable from \( LL \). Let \( t = [[a], [b]] \). Then \( \zeta(LL, E, t) = \zeta(LL, L, t) = \zeta(LL, LL, t) = \emptyset \). Now let \( t = [[a], [X]] \). Then \( \zeta(LL, E, t) = \{X\} \) and \( \zeta(LL, L, t) = \zeta(LL, LL, t) = \emptyset \). Now let \( t = [[a], X] \). Then \( \zeta(LL, E, t) = \emptyset, \zeta(LL, L, t) = \{X\} \) and \( \zeta(LL, LL, t) = \emptyset \).

### 3.4 Labelling and Polymorphism

We now want to adapt the idea of labelling to the case when we have polymorphism. With polymorphism, one can have infinitely many types, and even though the type graph for a fixed type as starting node is finite, it can become arbitrarily large, i.e., it can have an arbitrary number of non-terminals reachable from the
starting node. Also, it would clearly be desirable to describe the labellings for say
\texttt{list(int)}, \texttt{list(list(int))}, \ldots \text{ in a uniform way. This motivates defining a hierarchy in the type graph.}

**Definition 3.12** A type \( \sigma \) is a **recursive type** of \( \phi \) (denoted as \( \sigma \triangleright \phi \)) if \( \sigma \triangleright^* \phi \) and \( \phi \triangleright^* \sigma \). We write \( \mathcal{M}(\phi) \) for the tuple of recursive types of \( \phi \) other than \( \phi \) itself, ordered by \( \prec \) (see Sec. 3).

A type \( \sigma \) is a **non-recursive subterm type** (NRS) of \( \phi \) (denoted as \( \sigma \bowtie \phi \)) if \( \phi \triangleright \sigma \) and there is a type \( \tau \) such that \( \sigma \bowtie \tau \) and \( \tau \triangleright \phi \). We write \( \mathcal{S}( \phi ) \) for the tuple of non-recursive subterm types of \( \phi \), ordered by \( \prec \).

It follows immediately from the definition that, for any types \( \phi, \sigma \), we have \( \phi \triangleright \sigma \) and, if \( \sigma \bowtie \phi \), then \( \sigma \not\bowtie \phi \). Consider the type graph for \( \phi \). The recursive types of \( \phi \) are all the types in the strongly connected component (SCC) containing \( \phi \). The non-recursive subterm types of \( \phi \) are all the types \( \sigma \) not in the SCC but such that there is an edge from the SCC containing \( \phi \) to \( \sigma \).

**Example 3.13** Consider Fig. 2. Let \( \mathcal{F}_{\text{LISTS}} = \{ \text{nil} \rightarrow \text{list(U)}, \text{cons}_{U, \text{list(U)} \rightarrow \text{list(U)}} \} \). We have \( \text{list(U)} \bowtie \text{list(U)} \) and \( U \bowtie \text{list(U)} \).

Let \( \mathcal{F}_{\text{NESTS}} = \mathcal{F}_{\text{LISTS}} \cup \{ \text{ev} \rightarrow \text{nest}(V), \text{List}_{\text{nest}(V)} \rightarrow \text{nest}(V) \} \). The NESTS language implements rose trees \( \text{tree} \), i.e., trees where the number of children of each node is not fixed. We have \( \text{list}(\text{nest}(V)) \bowtie \text{nest}(V) \) and \( \text{nest}(V) \bowtie \text{nest}(V) \) and \( V \bowtie \text{nest}(V) \).

Suppose \( \mathcal{F}_{\text{STRINGS}} \) contains all strings with \( \rightarrow \text{str} \) as type subscript. Let \( \mathcal{F}_{\text{TABLES}} = \mathcal{F}_{\text{STRINGS}} \cup \{ \text{lh} \rightarrow \text{bal}, \text{rh} \rightarrow \text{bal}, \text{eq} \rightarrow \text{bal}, \text{null} \rightarrow \text{table(U)}, \text{node}_{\text{table}(U), \text{str}, U, \text{bal}, \text{table}(U) \rightarrow \text{table}(U)} \} \).

The type \( \text{bal} \) contains three constants representing balancing information. We have \( \text{table}(U) \bowtie \text{table}(U) \) and \( \bowtie(\text{table}(U)) = (U, \text{bal}, \text{str}) \).

An NRS of a flat type is often just a parameter of that type, as in \( U \bowtie \text{list(U)} \). However, this is not always the case, as witnessed by \( \text{str} \bowtie \text{table}(U) \).

Instead of looking at the labellings of all non-terminals reachable from some starting node without distinction \( [LS01] \), we classify them according to the recursive types and the NRSs of that node. This will be reflected in the construction of abstract domains.

**Definition 3.12** is obviously applicable in particular in the monomorphic case and thus to the grammars as in \( [LS01] \). Figure 3 shows that the recursive types and NRSs may not be all types reachable from a starting node. In that example, we have \( LL \bowtie LL \) and \( L \bowtie LL \). In the approach of \( [LS01] \), we may also be interested in \( \zeta(\text{LL}, E, t) \) for some term \( t \), so in the labellings of \( E \). In the approach proposed here, the domain construction for \( LL \) depends on \( E \) only indirectly, via the abstract domain for \( L \). Without such an inductive approach to domain construction, we would not know how to deal with polymorphism.

The key to devising a “parametric” abstract domain construction is to focus on type constructors, or equivalently, on flat types \( c(\bar{u}) \). So for example, we should focus on \( \text{list}(U) \) and not a particular instance such as \( \text{list(int)} \). This may not be surprising, but it has two consequences which may not be obvious.

First, note that the relation \( \bowtie \) is not stable under instantiation of types. This can be seen by comparing \( \text{LISTS} \) with \( \text{NESTS} \). We have \( U \bowtie \text{list}(U) \), but \( \text{nest}(V) \bowtie \text{list}(\text{nest}(V)) \). The abstract domain for \( \text{list}(\text{nest}(V)) \) however, being derived from the abstract domain for \( \text{list}(U) \), must relate to \( \text{nest}(V) \) as if \( \text{nest}(V) \) was an NRS of \( \text{list}(\text{nest}(V)) \). In contrast, the abstract domain for \( \text{nest}(V) \) must reflect that \( \text{list}(\text{nest}(V)) \bowtie \text{nest}(V) \). One could paraphrase this by saying: \( \text{LISTS} \) does not know about \( \text{NESTS} \), but \( \text{NESTS} \) does know about \( \text{LISTS} \).
The second consequence can be illustrated using TABLES. The type table(\mathcal{U}) has three NRSSs, and each of them will be dealt with in the construction of the abstract domain. However, the instance table(string) has only two NRSSs, as \mathcal{U} becomes instantiated to string. The domain for table(\tau) will be based on assuming three NRSSs, i.e., it will deal with the value and the key arguments separately, even if by coincidence \tau is equal to string.

We now define a function \mathcal{Z} in analogy to \zeta. In the approach of [LS01], we could safely identify a grammar with its starting non-terminal. In what follows, we will always assume a grammar \mathcal{G}(\phi) where \phi is flat (Def. 3.3). However, it is also useful to consider productions of that grammar starting from some other non-terminal than the "official" starting non-terminal. Therefore \mathcal{Z} has four arguments, the additional first one specifying the grammar and the second the starting symbol.

Unlike \zeta, the function \mathcal{Z} also collects non-variable terms.

**Definition 3.14** Let \phi be a flat type, \tau be a type such that \tau \approx \phi, and \sigma a type such that either \sigma \ni \phi or \sigma \leq \phi.

We denote by \mathcal{Z}(\phi, \tau, \sigma, t) the function which returns the set of all terms \epsilon such that \text{`\tau'}(\epsilon) \rightarrow^* \text{`\sigma'}(s)\text{ in the grammar } \mathcal{G}(\phi).

The function \mathcal{Z} is lifted to sets (in the fourth argument) in the obvious way.

**Example 3.15** Let \mathcal{F} = \mathcal{F}_{\text{LISTS}} \cup \{\text{a}\_\text{char}, \text{b}\_\text{char}\}. We have

\[
\mathcal{Z}\left(\text{list}(\mathcal{U}), \text{list}(\mathcal{U}), \text{list}(\mathcal{U}), [a, X]\right) = \{[a, X], [X], []\}
\]

\[
\mathcal{Z}\left(\text{list}(\mathcal{U}), \text{list}(\mathcal{U}), \text{list}(\mathcal{U}), [a, X]\right) = \{[a, X]\}
\]

\[
\mathcal{Z}\left(\text{list}(\mathcal{U}), \text{list}(\mathcal{U}), \text{list}(\mathcal{U}), [a|X]\right) = \{[a|X,X]\}
\]

Note that unlike \zeta (Ex. 3.11), \mathcal{Z} cannot be used to extract from the term \text{[a|X]} the subterm X directly.

Now consider the NESTS example, assuming that \mathcal{F}_{\text{NESTS}} is augmented with the integers \text{l}\_\text{ist}, \ldots. We have

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), n([e(7)])) = \{n([e(7)]), e(7)\}
\]

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), \text{list}(\text{nest}(\mathcal{V})), n([e(7)])) = \{[e(7)], []\}
\]

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), \mathcal{V}, n([e(7)])) = \{e(7)\}
\]

(1)

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{list}(\text{nest}(\mathcal{V})), \text{nest}(\mathcal{V}), n([e(7)])) = \{n([e(7)]), e(7)\}
\]

(2)

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{list}(\text{nest}(\mathcal{V})), \mathcal{V}, n([e(7)])) = \{[e(7)], [e(7)], [], []\}
\]

(3)

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{list}(\text{nest}(\mathcal{V})), \mathcal{V}, n([e(7)])) = \{n([e(7)]), e(7)\}
\]

(4)

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), \mathcal{V}, e(7)) = \{7\}
\]

(5)

\[
\mathcal{Z}(\text{nest}(\mathcal{V}), \text{nest}(\mathcal{V}), \mathcal{V}, e(7)) = \{7\}
\]

(6)

Note the difference between the labellings obtained for \text{[e(7)]} depending on whether we use the grammar for nest(V) (\textbf{[] [] []}), or the grammar for list(U) (\textbf{[ [] ]}).
 Lemma 3.16 Let $\phi$ and $\tau$ be flat types such that $\tau \Theta \vorw \phi$ for some $\Theta$. Let $t = f_{r_1...r_n} \rightarrow \langle t_1, \ldots, t_n \rangle$ be a term such that $\vdash t : \tau \Theta \Theta'$ for some $\Theta'$. Then

$$Z(\phi, \tau \Theta, \sigma, t) = \begin{cases} \{t_i \mid \tau_i \Theta = \sigma\} \cup \bigcup_{\tau_i, \Theta \vdash \phi} Z(\phi, \tau_i \Theta, \sigma, t_i) & \text{if } \sigma \nvdash \phi \\ \bigcup_{\tau_i, \Theta \vdash \phi} Z(\phi, \tau_i \Theta, \sigma, t_i) & \text{if } \sigma \vdash \phi, \sigma \neq \phi \\ \{t\} \cup \bigcup_{\tau_i, \Theta \vdash \phi} Z(\phi, \tau_i \Theta, \sigma, t_i) & \text{if } \sigma = \phi. \end{cases}$$

Proof. Follows from the fact that for each $i \in \{1, \ldots, n\}$, we have $\mu \tau' (\langle t_i \rangle) \rightarrow \mu \tau (\langle t_i \rangle)$ in $G(\phi)$.

4 Abstract Terms

In this section, we define an abstraction of terms using the notions of recursive type and non-recursive subterm type. This amounts to a generalisation of [CL00].

4.1 Abstract Domains and Terms

We first introduce the formalism of set logic programs shown to be powerful for program analyses [LS01]. Consider a language based on a set of variables $\mathcal{V}$ and a set of functions $\mathcal{F}^\oplus = \{\emptyset, \oplus\}$, where $\emptyset/0$ represents the empty set and $\oplus/2$ is a set constructor. Set expressions are elements of the term algebra $T(\mathcal{F}^\oplus, \mathcal{V})$ modulo the ACI1 equality theory, consisting of:

\[
\begin{align*}
(x \oplus y) \oplus z &= x \oplus (y \oplus z) & \text{(associativity)} \\
x \oplus y &= y \oplus x & \text{(commutativity)} \\
x \oplus \emptyset &= x & \text{(unity)}
\end{align*}
\]

(Lagrange and Stuckey [LS01] now proceed by regarding each non-terminal in the grammar corresponding to a variable $x$ as an abstract variable. The instantiation of that abstract variable obtained from the execution of the abstract program gives us information about the labels of the non-terminal after the concrete execution.

We do not see how this approach could work in the presence of polymorphism. Instead, we follow [CL00]. We now introduce new function symbols, one $c^A$ for each type constructor $c \in \mathcal{K}$, in addition to $\emptyset$ and $\oplus$. These are used to collect the information corresponding to the different non-terminals in a structure, which we call abstract term. The arity of $c^A$ is given by the arity of $\forall(c(\bar{u}))$ plus the arity of $\exists(c(\bar{u}))$.

Definition 4.1 We define

$$\mathcal{F}^A := \mathcal{F}^\oplus \cup \{c^A/m \mid c \in \mathcal{K}, m = \#(\forall(c(\bar{u}))) + \#(\exists(c(\bar{u})))\}.$$ 

Now let $\tau = c(\bar{u})$, $\forall(c(\bar{u})) = \langle \rho_1, \ldots, \rho_{m'} \rangle$, and $\exists(c(\bar{u})) = \langle \rho_{m'+1}, \ldots, \rho_m \rangle$. For a term $t = f_{r_1...r_n} \rightarrow \langle t_1, \ldots, t_n \rangle$, we define

$$\alpha(t) = c^A \left( \bigoplus_{\tau_i = \rho_1} \alpha(t_i), \ldots, \bigoplus_{\tau_i = \rho_m} \alpha(t_i) \right) \oplus \bigoplus_{\tau_i = \tau} \alpha(t_i).$$

For a variable $x$ we define $\alpha(x) = x$. We call the image of $\alpha$ the domain of abstract terms, or simply the abstract domain.
In [CL96], the abstraction function is denoted type, and it is essentially a special case of the above definition for \( \#(\varphi) = 1 \) and \( \#(\mathcal{M}(\varphi)) = 0 \). Using our terminology, they assume that there is a function, which they denote by \( \alpha \), returning for each \( f \cdot \cdots \cdot (\varphi) \) the type constructor \( c \); moreover, there is a function \( \pi \) returning the set of argument positions of \( f \) that have declared type \( u \), and all other argument positions are assumed to have declared type \( u \). However, since their typing is descriptive, \( \alpha \) and \( \pi \) have to be provided by the user.

**Example 4.2** Consider again Ex. 3.15,

\[
\begin{align*}
\alpha(7) &= \text{int}^A \\
\alpha([7]) &= \text{list}^A(\alpha(7)) \oplus \alpha(\text{nil}) = \text{list}^A(\text{int}^A) \oplus \text{list}^A(\emptyset) \\
\alpha([e(7)]) &= \text{nest}^A(\alpha(7), \emptyset) = \text{nest}^A(\text{int}^A, \emptyset) \\
\alpha(\text{nil}) &= \text{nest}^A(\emptyset, \alpha([e(7)])) = \text{nest}^A(\emptyset, \text{list}^A(\text{nest}^A(\text{int}^A, \emptyset))).
\end{align*}
\]

Note how it comes into play that the empty \( \oplus \)-sequence is naturally defined as the neutral element \( \emptyset \). There is a notable difference between [CL96] and [CL00] at this point. In the latter, there is no neutral element. This means in particular that \( \text{nil} \) cannot be abstracted as \( \text{list}(\emptyset) \). Instead, it is abstracted as \( \text{nil} \), and as a consequence, the list \([7]\) is abstracted as \( \text{list}(\text{int}) \oplus \text{nil} \). While it is argued that such an abstraction simplifies the implementation of abstract unification, we believe that an object \( \text{list}(\text{int}) \oplus \text{nil} \) mixes types/abstract terms on one hand and concrete terms on the other hand in a way which is undesirable.

In fact, from the design of our abstract domains and the fact that we are analysing prescriptively typed programs, it follows that whenever an expression \( c^A(\ldots) \oplus c'^A(\ldots) \) occurs, then \( c = c' \). This also explains why in the definition of \( \alpha \), the abstraction of those \( t_i \) such that \( \tau_i \propto \tau \) but \( \tau_i \neq \tau \) is included in reserved argument positions of \( c^A(\ldots) \), whereas the abstraction of those \( t_i \) such that \( \tau_i = \tau \) is directly conjoined (using \( \oplus \) with the whole expression \( c^A(\ldots) \)).

Looking at Ex. 4.2, one might expect that \( \text{list}^A(\text{int}) \oplus \text{list}^A(\emptyset) \) can be simplified to \( \text{list}^A(\text{int}) \). Maybe less obvious, one might also expect that the abstract term \( \text{nest}^A(\emptyset, \text{list}^A(\text{nest}^A(\text{int}^A, \emptyset))) \) can be simplified to \( \text{nest}^A(\text{int}^A, \emptyset) \). We now extend ACII by further axioms for this purpose.

**Definition 4.3** For each \( c^A/m \in \mathcal{F}^A \), the **distributivity axiom** is defined as follows:

\[
c^A(x_1, \ldots , x_m) \oplus c^A(y_1, \ldots , y_m) = c^A(x_1 \oplus y_1, \ldots , x_m \oplus y_m)
\]

(8)

Moreover, consider a flat type \( \varphi = d(\varphi) \) such that \( \varphi(\varphi) = \langle \sigma_1, \ldots , \sigma_\ell \rangle \), \( \mathcal{M}(\varphi) = \langle \sigma_{\ell+1}, \ldots , \sigma_l \rangle \). For each \( j \in \{ l' + 1, \ldots , l \} \), we have \( \sigma_j = \tau \Theta \) for some flat type \( \tau = c(\bar{u}) \) and some \( \Theta \). Suppose \( \varphi(\tau) = \langle \rho_1, \ldots , \rho_m' \rangle \), \( \mathcal{M}(\tau) = \langle \rho_{m'+1}, \ldots , \rho_m \rangle \). We define the **extraction axiom** for \( d^A \) and \( \sigma_j \) as follows:

\[
d^A(x_1, \ldots , x_{j-1}, c^A(y_1, \ldots , y_m) \oplus x_j, x_{j+1}, \ldots , x_l) =
\]

\[
d^A\left(x_1 \bigoplus_{\rho_k \Theta = \sigma_1} y_k, \ldots , x_{j-1} \bigoplus_{\rho_k \Theta = \sigma_{j-1}} y_k, x_j, x_{j+1} \bigoplus_{\rho_k \Theta = \sigma_{j+1}} y_k, \ldots , x_m \bigoplus_{\rho_k \Theta = \sigma_l} y_k \right)
\]

Let \( ACI1DE \) be the theory given by the axioms in (3) and the distributivity and extraction axioms. We abbreviate ACI1DE by \( AC+ \) and denote equality modulo \( AC+ \) as \( =_{AC+} \).

Note that applying a distributivity or extraction axiom from left to right decreases the number of occurrences of function symbols by 1.
Example 4.4 Consider LISTS, NESTS, respectively. The extraction axiom for \texttt{nest}(v) and \texttt{list(nest}(v)) is
\[
\text{\texttt{nest}}^A(x_1, \text{\texttt{list}}^A(y) \oplus x_2) = \text{\texttt{nest}}^A(x_1, x_2) \oplus y.
\]
In AC+, we have
\[
\begin{align*}
\text{\texttt{list}}^A(\texttt{int}^A) & \oplus \text{\texttt{list}}^A(\emptyset) = \text{AC}_+ \text{\texttt{list}}^A(\texttt{int}^A \oplus \emptyset) = \text{AC}_+ \text{\texttt{list}}^A(\texttt{int}^A) \\
\text{\texttt{nest}}^A(\emptyset, \text{\texttt{nest}}^A(\texttt{int}^A, \emptyset)) & = \text{AC}_+ \text{\texttt{nest}}^A(\emptyset, \emptyset) \oplus \text{\texttt{nest}}^A(\texttt{int}^A, \emptyset) = \text{AC}_+ \\
\end{align*}
\]
In [CL00], it is mentioned that one might have chosen to add a distributivity axiom but the authors argue the case for not doing so. The extraction axiom has no equivalent in [CL00].

4.2 Normal Abstract Terms
Following [CL00], we have defined the abstraction by structural induction on a term. This definition is a good basis for our analysis, but it is still unsatisfactory: as it is stated (i.e. without applying any axioms), even for ground terms, the abstraction of a term is proportional in size to the term itself; consequently, the abstraction is not in a form that makes it convenient to read any properties of the concrete term from it.

We first show that using AC+, any abstract term can be converted in a normal form. To this end, it is useful to view abstract terms as typed terms according to the rules of Table 1. For each \(\tau\), to this end, it is useful to view abstract terms as typed terms according to the rules of Table 1. For each \(\tau\), we declare \(\text{\texttt{nest}}^A(\emptyset, \emptyset) \oplus \text{\texttt{nest}}^A(\text{\texttt{int}^A, \emptyset}) = \text{AC}_+ \text{\texttt{nest}}^A(\text{\texttt{int}^A, \emptyset})\).

In [CL00], it is mentioned that one might have chosen to add a distributivity axiom but the authors argue the case for not doing so. The extraction axiom has no equivalent in [CL00].

Proposition 4.5 The \(\lhd\) and \(\gg\)-relations based on the declared types of the “abstract functions” in \(\mathcal{F}^A\) are the same as the \(\lhd\) and \(\gg\)-relations based on \(\mathcal{F}\).

To distinguish type judgements in the concrete and abstract language, we use \(\vdash^A\) for the latter. The following proposition says that the abstraction of a term has the same type as the concrete term itself. Its proof is straightforward by structural induction.

Proposition 4.6 If \(\Gamma \vdash t : \tau\) then \(\Gamma \vdash^A \alpha(t) : \tau\).

The next lemma says that application of the equality axioms preserves well-typing. The interesting axiom is the extraction axiom.

Lemma 4.7 If \(\Gamma \vdash^A a : \bar{\tau} \) and \(a =_{\text{AC}_+} b\) then \(\Gamma \vdash^A b : \bar{\tau}\).

Proof. We only consider the extraction axiom. Assume the notations of Def. 4.3, and consider an abstract term
\[
a = d^A(b_1, \ldots, b_{j-1}, c^A(a_1, \ldots, a_m) \oplus b_j, b_{j+1}, \ldots, b_l)
\]
where \(\vdash^A a : \phi\Theta'\) for some \(\Theta'\) (note that the typing rules are such that \(a\) must have a type that is an instance of \(\phi\), but not necessarily \(\phi\) itself). By the rules of Table 1, we must have \(\vdash^A b_{j'} : \sigma_{j'}\Theta'\) for all \(j' \in \{1, \ldots, l\}\) and \(\vdash^A a_k : \rho_k\Theta'\) for all \(k \in \{1, \ldots, m\}\). Therefore it follows that for \(j' \in \{1, \ldots, l\} \setminus \{j\}\), we have
\[
\vdash^A \left( \bigoplus_{\rho_k \Theta' = \phi} a_k \right) : \phi\Theta'.
\]
\[\text{These declarations violate the Simple Range Condition, and in any case would not be permissible in any existing typed programming language since a range type must not be a parameter. However, this causes no problems for our theoretical considerations.}\]
This implies that if \( b \) is obtained from \( a \) by applying the extraction axiom in the \( j \)th position, then \( \vdash \overline{A} b : \phi \Theta' \).

We now define normal forms for abstract terms. To simplify the notation, we denote a variable sequence \( x_1 \oplus \cdots \oplus x_n \) as \( x \oplus \), and of course, this is \( \emptyset \) if \( n = 0 \). Note that the following definition is by structural induction: a normal abstract term for \( \tau \Theta \) is defined based on a normal abstract term for some \( \tau_i \Theta \). The well-foundedness of this induction is not obvious but has been stated in \( \text{[SHK00, Lemma 4.3]} \).

**Definition 4.8** For a parameter, the only normal abstract term is \( \emptyset \).

Now let \( \tau = c(\overline{u}) \) be a flat type such that \( \triangledown(\tau) = \langle \rho_1, \ldots, \rho_m \rangle \) and \( \chi(\tau) = \langle \rho_{m'+1}, \ldots, \rho_m \rangle \), and \( \Theta \) be any type substitution. A normal abstract term for \( \tau \Theta \) is defined based on a normal abstract term for some \( \tau_i \Theta \). The following definition is by structural induction: a normal abstract term for \( \tau \Theta \) is defined based on a normal abstract term for some \( \tau_i \Theta \). The well-foundedness of this induction is not obvious but has been stated in \( \text{[SHK00, Lemma 4.3]} \).

**Theorem 4.9** For any \( t \) with \( \vdash t : \phi \), \( \alpha(t) \) has a representative which is a normal abstract term for \( \phi \).

**Proof.** By Prop. 4.6 and Lemma 4.7 we have \( \vdash a : \phi \) if \( a = \overline{AC+} \alpha(t) \).

Assume the notations of Def. 4.3. The fact that an abstract term is typed according to the rules of Table 1 means in particular that if it has the form \( d^A(b_1, \ldots, b_{j-1}, \cdots \oplus c^A(\cdots) \oplus \cdots b_j, \ldots, b_l) \) where \( j \in \{l' + 1, \ldots, l\} \), i.e., it is not in normal form, then \( c' = c \) and an extraction axiom is applicable, possibly after several applications of associativity and commutativity.

Likewise, if an abstract term has the form \( \ldots d^A(\ldots) \oplus \cdots \oplus d'^A(\ldots) \ldots , \) then \( d = d' \) and the distributivity axiom is applicable.

Since the distributivity and extraction axioms can only be applied a finite number of times, it follows that successive application of them yields an abstract term in normal form.

Example 4.4 shows the conversion of two abstract terms to their normal forms.

We can make some further observations. The first follows from the definition of \( \alpha \).

**Proposition 4.10** For a term \( t \), \( \alpha(t) \) contains variables if and only if \( t \) contains variables.

The next proposition follows from the fact that by the transparency condition, no subterm type of a ground type \( \phi \) can contain a parameter.

**Proposition 4.11** If \( \phi \) is a ground type, then there is exactly one normal abstract term for \( \phi \) not containing variables and not containing \( \emptyset \).

Let \( a \) be this abstract term. Then any other normal abstract term for \( \phi \) not containing variables is obtained by replacing some subterms in \( a \) with \( \emptyset \).

The previous three statements tell us that the size of the abstraction of a ground term depends only on its type and not on the size of the term itself. However, it would be wrong to conclude that all ground terms of the same type have the same abstraction.
The problem is due to polymorphism: one term can have several types. For example, it is correct to say that both \([\texttt{[]}}\) and \([\texttt{[7]}\) are of type \texttt{list(int)}, but \(\alpha([\texttt{[]}}) = \texttt{list}(\texttt{int}^4)\) and \(\alpha([\texttt{[7]}\)) = \texttt{list}(\texttt{int}^4)\). However, we can state:

**Lemma 4.12** If \(\vdash s : \phi\) and \(\vdash t : \phi\) and \(s, t\) are ground, then both \(\alpha(s)\) and \(\alpha(t)\) are obtained from the unique normal abstract term as mentioned in Prop. 4.11 by replacing zero or more subterms with \(\emptyset\).

5 Relating the Abstraction and the Labels

Now that we know that each abstract term has a representative in a compact normal form, we state a theorem which relates the abstraction of a term to the labels as defined in Sec. 3. Actually, it would have been possible to have the following theorem as definition of \(\alpha\), and have our present definition as a lemma. This is effectively what we did in [SHK00]. The theorem also links [CL00] with [LS01]. Note that \(\alpha\) is lifted to sets as follows: \(\alpha(S) := \bigoplus_{t \in S} \alpha(t)\).

Before we can show the theorem, we extend the definition of \(Z\) to abstract terms, typed as shown in the previous section. We state the following lemma.

**Lemma 5.1** Let \(\phi\) and \(\tau\) be flat types such that \(\tau \Theta \gg \phi\) for some \(\Theta\). Let \(t\) be a term such that \(\vdash t : \tau \Theta \Theta'\) for some \(\Theta'\). Then for any \(\sigma \ll \phi\), or \(\sigma \gg \phi\), we have \(\alpha(Z(\phi, \tau \Theta, \sigma, t)) = Z(\phi, \tau \Theta, \sigma, \alpha(t))\).

**Proof.** The proof is by induction on the depth of \(t\). First suppose that \(t \in V\). Then we have to distinguish whether \(\sigma = \tau \Theta\) or \(\sigma \neq \tau \Theta\). In the first case, \(\alpha(Z(\phi, \tau \Theta, \sigma, t)) = t = Z(\phi, \tau \Theta, \sigma, \alpha(t))\). In the second case, \(\alpha(Z(\phi, \tau \Theta, \sigma, t)) = \emptyset = Z(\phi, \tau \Theta, \sigma, \alpha(t))\).

Now suppose that \(t\) is a constant. Again, we have to distinguish whether \(\sigma = \tau \Theta\) or \(\sigma \neq \tau \Theta\). In the first case, \(\alpha(Z(\phi, \tau \Theta, \sigma, t)) = c^A(\emptyset, \ldots, \emptyset) = Z(\phi, \tau \Theta, \sigma, \alpha(t))\). In the second case, \(\alpha(Z(\phi, \tau \Theta, \sigma, t)) = \emptyset = Z(\phi, \tau \Theta, \sigma, \alpha(t))\).

Now consider a term \(t = f_{\tau_1 \cdots \tau_n}(t_1, \ldots, t_n)\) and suppose the result has been proven for \(t_1, \ldots, t_n\). Suppose \(\phi(\tau) = (\rho_1, \ldots, \rho_m')\) and \(\boxplus(\tau) = (\rho_{m' + 1}, \ldots, \rho_{m'})\). Consider first \(\sigma \ll \phi\), and in the following equation sequence, \(\ast\) marks steps that use simple rearrangements such as lifting a function to sets.

\[
\begin{align*}
\alpha(Z(\phi, \tau \Theta, \sigma, t)) &= (\text{Lem. 3.10}) \\
\alpha \left( \{ t_i \mid \tau_i \Theta = \sigma \} \cup \bigcup_{\tau_i \Theta \gg \phi} Z(\phi, \tau_i \Theta, \sigma, t_i) \right) &= \\
\bigoplus_{\tau_i \Theta = \sigma} \alpha(t_i) \oplus \bigoplus_{\tau_i \Theta \gg \phi} \alpha(Z(\phi, \tau_i \Theta, \sigma, t_i)) &= (\ast) \\
\bigoplus_{\tau_i \Theta = \sigma} \alpha(t_i) \oplus \bigoplus_{\tau_i \Theta \gg \phi} Z(\phi, \tau_i \Theta, \sigma, \alpha(t_i)) &= (\text{ind. hyp.}) \\
\bigoplus_{\rho_j \Theta = \sigma} \alpha(t_i) \oplus \bigoplus_{\rho_j \Theta \gg \phi} Z(\phi, \rho_j \Theta, \sigma, \alpha(t_i)) &= (\ast) \\
\bigoplus_{\tau_i \Theta = \sigma} \bigoplus_{\tau_i \Theta = \tau} Z(\phi, \tau_i \Theta, \sigma, \alpha(t_i)) &= (\ast) \\
\left\{ \bigoplus_{\tau_i = \rho_j} \alpha(t_i) \mid \rho_j \Theta = \sigma \right\} \cup \bigcup_{\rho_j \Theta \gg \phi} Z(\phi, \rho_j \Theta, \sigma, \bigoplus_{\tau_i = \rho_j} \alpha(t_i)) \oplus \bigoplus_{\tau_i = \tau} Z(\phi, \tau \Theta, \sigma, \bigoplus_{\tau_i = \tau} \alpha(t_i)) &= (\text{Lem. 3.10})
\end{align*}
\]
\[
Z \left( \phi, \tau \Theta, \sigma, c^A \left( \bigoplus_{\tau_i = \rho_1} \alpha(t_i), \ldots, \bigoplus_{\tau_i = \rho_m} \alpha(t_i) \right) \right) \oplus \nabla\nabla \left( \phi, \tau \Theta, \sigma, c^A \left( \bigoplus_{\tau_i = \tau} \bigoplus_{\tau_i = \rho_1} \alpha(t_i), \ldots, \bigoplus_{\tau_i = \rho_m} \alpha(t_i) \right) \right) = (*)
\]

\[
Z (\phi, \tau \Theta, \sigma, \alpha(t)) = (Def. 4.1)
\]

The remaining cases, that either \( \sigma \ni \phi \) and \( \sigma \neq \phi \), or \( \sigma = \phi \), are very similar and hence omitted.

We can now state the theorem.

**Theorem 5.2** Let \( \tau = c(\bar{u}) \) be a flat type such that \( \preceq(\tau) = \langle \rho_1, \ldots, \rho_{m'} \rangle \) and \( \bowtie(\tau) = \langle \rho_{m'+1}, \ldots, \rho_m \rangle \). For any term \( t = f_{\tau_1 \ldots \tau_n}(\ldots) \), we have

\[
\alpha(t) =_{AC} c^A \left( \alpha(Z(\tau, \tau, \rho_1, t)), \ldots, \alpha(Z(\tau, \tau, \rho_{m'}, t)), \alpha(Z(\tau, \tau, \rho_{m'+1}, t)) \cap V, \ldots, \alpha(Z(\tau, \tau, \rho_m, t)) \cap V \right) \oplus (Z(\tau, \tau, \tau) \cap V).
\]

**Proof.** Suppose the normal form of \( \alpha(t) \) is \( c^A(a_1, \ldots, a_m) \oplus a \).

Consider some \( j \in \{1, \ldots, m'\} \). Since \( a \) as well as \( a_k \) for all \( m' < k \leq m \), only consist of variables, it follows by Prop. 4.5 and Lemma 3.16 that

\[
Z (\tau, \tau, \rho_j, c^A(a_1, \ldots, a_m) \oplus a) = a_j.
\]

At the same time, by Lemma 5.1.

\[
Z (\tau, \tau, \rho_j, \alpha(t)) = \alpha(Z(\tau, \tau, \rho_j, t)).
\]

Now consider some \( j \in \{m'+1, \ldots, m\} \). Since \( a \) as well as \( a_k \) for all \( m' < k \leq m \), only consist of variables, it follows by Prop. 4.3 and Lemma 3.14 that

\[
Z (\tau, \tau, \rho_j, c^A(a_1, \ldots, a_m) \oplus a) \cap V = a_j.
\]

At the same time, by Lemma 5.1.

\[
Z (\tau, \tau, \rho_j, \alpha(t)) = \alpha(Z(\tau, \tau, \rho_j, t)).
\]

Finally consider \( \tau \). Since \( a \) as well as \( a_k \) for all \( m' < k \leq m \), only consist of variables, it follows by Prop. 4.3 and Lemma 3.14 that

\[
Z (\tau, \tau, \tau, c^A(a_1, \ldots, a_m) \oplus a) \cap V = a.
\]

At the same time, by Lemma 5.1.

\[
Z (\tau, \tau, \tau, \alpha(t)) = \alpha(Z(\tau, \tau, \tau, t)).
\]

Thus we have shown that the normal form of \( \alpha(t) \) is as stated. \qed
Example 5.3 Consider LISTS. We have

\[\alpha([X],[7]) = \text{list}^4(\alpha(\mathcal{Z}(\text{list}(U),\text{list}(U),\mathcal{U},[[X],[7]]))) + \mathcal{Z}(\text{list}(U),\text{list}(U),\mathcal{U},[[X],[7]])) \cap V) = \text{list}^4(\alpha([X],[7])) + (\{[[X],[7]], [7], []\} \cap V) = \text{list}^4(\text{list}^4(X \oplus \text{int}^4)) \oplus \emptyset = \text{list}^4(\text{list}^4(X \oplus \text{int}^4)).\]

The theorem tells us how to read the abstract term. First, the absence of variable on the highest level (i.e. \(\alpha([X],[7])\)) is not of the form \(x \oplus \ldots\) means that \(\mathcal{Z}(\text{list}(U),\text{list}(U),\mathcal{U},[[X],[7]])\) contains no variables, or, to use the notation of \(\text{LS01}\) applied to the grammar in Ex. 3.11, \(\zeta(\mathcal{L}L, \mathcal{L}L, [[X],[7]])\) is empty. Likewise, the theorem tells us that the argument of the outermost \(\text{list}^4\) contains the abstraction of all subterms of \([X],[7]\] returned by \(\mathcal{Z}(\text{list}(U),\text{list}(U),\mathcal{U},[[X],[7]])\), and again in terms of \(\text{LS01}\), the absence of variables at this level tells us that \(\zeta(\mathcal{L}L, L, [[X],[7]])\) is empty.

6 The Analysis

In this section, we show how an entire program is abstracted based on an abstraction of the fundamental operation, namely unification. The abstract program is then given a semantics. This semantics describes, in a well-defined sense, the semantics of the concrete program. Moreover, under reasonable assumptions, it is finitely computable.

6.1 Abstract Interpretation

In this subsection, we link our formalism to the standard definitions of abstract interpretation.

Our abstract terms may contain variables, and hence it is only natural to define substitutions as for concrete terms, only that the range of those substitutions will contain abstract terms. The instantiation order \(\leq_{\text{AC}^+}\) is defined as follows: \(a \leq_{\text{AC}^+} b\) if \(b \theta^A =_{\text{AC}^+} a\) for some \(\theta^A\). It is lifted to substitutions: \(\theta^A \leq_{\text{AC}^+} \theta^A\) if \(a b \theta^A \leq_{\text{AC}^+} a b \theta^A\) for all \(a\). We write \(a \approx b\) for \(a \leq_{\text{AC}^+} b \land b \leq_{\text{AC}^+} a\). One should not confuse \(=_{\text{AC}^+}\) ! Our notation follows \(\text{CL00}\), not \(\text{LS01}\). An abstract atom is an atom using abstract terms. We denote the set of abstract atoms by \(\text{B}^A\).

In order to define and relate semantics for concrete and abstract programs in the framework of abstract interpretation, we consider sets of (abstract) atoms with a suitable notion of ordering and equivalence. We consider the lower power domain or Hoare domain \(\text{S90}\). For sets of abstract atoms \(I_1^A\) and \(I_2^A\), we define

\[I_1^A \leq_{AC^+} I_2^A \iff \forall A_1^A \in I_1^A \exists A_2^A \in I_2^A : A_1^A \leq_{AC^+} A_2^A,\]

and \(I_1^A \approx I_2^A\) if \(I_1^A \leq_{AC^+} I_2^A\) and \(I_2^A \leq_{AC^+} I_1^A\). The elements of \(\text{B}^A\) are called abstract interpretations. Abusing notation, we denote \(\text{B}^A\) by \(\text{B}^A\).

We call a set of abstract atoms \(I^A\) downwards-closed if \(A^A \in I^A\) implies \(\check{A}^A \in I^A\) for all \(A^A \leq_{AC^+} A^A\). The order relation \(\approx\) is defined in such a way that each \(I^A \in \text{B}^A\) is equivalent to a downwards-closed set. This observation implies the following lemma.

Lemma 6.1 \(\text{CL00}\), Lemma 3.1 \((\text{B}^A, \leq_{AC^+})\) is a complete lattice.

Definition 6.2 A Galois insertion is a quadruple \((\mathcal{A}, \subseteq_A), \alpha, (\mathcal{B}, \subseteq_B), \gamma)\) where

1. \((\mathcal{A}, \subseteq_A)\) and \((\mathcal{B}, \subseteq_B)\) are complete lattices of concrete and abstract domains, respectively;
2. $\alpha : A \to B$ and $\gamma : B \to A$ are monotonic functions called abstraction and concretisation, respectively; and

3. $a \sqsubseteq \gamma(\alpha(a))$ and $\alpha(\gamma(b)) = b$ for every $a \in A$ and $b \in B$.

In the above definition, the $\alpha$ is a priori not the $\alpha$ of Def. [4.1] but of course, we have used the same letter because we link the two in the natural way.

**Theorem 6.3** [CL00, Theorem 3.3] Define $\alpha$ and $\gamma$ as follows:

\[
\begin{align*}
\alpha : 2^B &\to 2^{B^A}, \quad \alpha(I) = \{\alpha(A) \mid A \in I\}, \\
\gamma : 2^{B^A} &\to 2^B, \quad \gamma(I^A) = \cup\{I \mid \alpha(I) \leq_{AC^+} I^A\}.
\end{align*}
\]

Then $\langle 2^B, \alpha, 2^{B^A}, \gamma \rangle$ is a Galois insertion.

**Definition 6.4** An abstract term $a$ describes a concrete term $t$, denoted $a \prec t$, if $\alpha(t) \leq_{AC^+} a$ (and likewise for atoms).

Note that for an atom $A$ and an abstract atom $A^A$, we have $A^A \prec A$ if and only if $I \in \gamma(\{A^A\})$. For an interpretation $I$ and an abstract interpretation $I^A$, we define $I^A \prec I$ if $I \subseteq \gamma(I^A)$, or equivalently, $\alpha(I) \leq_{AC^+} I^A$.

### 6.2 Abstract Unification

In this subsection, we show how abstract unification describes concrete unification. First, we can relate abstraction and application of a substitution as follows.

**Lemma 6.5** [CL00, Lemma 4.1] Let $t$ be a term an $\theta$ a substitution. Then $\alpha(t\theta) =_{AC^+} \alpha(t)(x/\alpha(x) \mid x \in dom(\theta))$.

**Proof.** By structural induction on the term. Let $\theta^A = \{x/\alpha(x) \mid x \in dom(\theta)\}$.

If $t \in \mathcal{V}$ and $t \notin dom(\theta)$, the result trivially holds. If $t \in \mathcal{V}$ and $t \in dom(\theta)$, then $\alpha(t\theta) = t\{t/\alpha(t\theta)\} = \alpha(t)\theta^A$.

Now consider $t = f_{\tau_1, \ldots, \tau_n}(t_1, \ldots, t_n)$, where $\tau = c(\bar{u})$, $\Phi(\tau) = \langle \rho_1, \ldots, \rho_m \rangle$, and $\Psi(\tau) = \langle \rho_{m+1}, \ldots, \rho_m \rangle$, and assume that the result holds for $t_1, \ldots, t_n$. We have

\[
\begin{align*}
\alpha(t\theta) &= c^A \bigoplus_{\tau_i = \rho_{p_1}} \alpha(t_i\theta) \ldots \bigoplus_{\tau_i = \rho_{p_m}} \alpha(t_i\theta) \\
&= c^A \bigoplus_{\tau_i = \rho_{p_1}} \alpha(t_i)\theta^A \ldots \bigoplus_{\tau_i = \rho_{p_m}} \alpha(t_i)\theta^A \oplus \bigoplus_{\tau_i = \tau} \alpha(t_i)\theta^A = \alpha(t)\theta^A.
\end{align*}
\]

**Example 6.6** By Lemma 6.5

\[
\begin{align*}
\alpha([X|Y]\{X/7, Y/\text{nil}\}) &= \\
\alpha([7]) &= \\
\text{list}^A(\text{int}^A) &=_{AC^+} \\
\text{list}^A(\text{int}^A) \oplus \text{list}^A(\emptyset) &= \\
\{\text{list}^A(X) \oplus Y\}\{X/\text{int}^A, Y/\text{list}^A(\emptyset)\} &= \\
\alpha([X|Y])\{X/\alpha(7), Y/\alpha(\text{nil})\} &= .
\end{align*}
\]

The following theorem is a straightforward consequence.
Theorem 6.7 [CL00, Thm. 4.2] Let $t_1, t_2$ be terms. If $t_1 \leq t_2$ then $\alpha(t_1) \leq_{AC^+} \alpha(t_2)$ (and likewise for atoms).

Definition 6.8 We denote by $cU_{AC^+}(o_1, o_2)$ a complete set of $AC^+$-unifiers of syntactic objects $o_1, o_2$, i.e., a set of abstract substitutions such that for each $\theta^A \in cU_{AC^+}(o_1, o_2)$, we have $o_1 \theta^A =_{AC^+} o_2 \theta^A$, and for any $\theta^A$ such that $o_1 \theta^A =_{AC^+} o_2 \theta^A$, we have $\theta^A \leq_{AC^+} \theta^A$ for some $\theta^A \in cU_{AC^+}(o_1, o_2)$.

The next theorem says that unification of abstract terms is a correct abstract unification. The second part of the statement says that an abstract substitution corresponding to a concrete substitution as stated correctly mimics that concrete substitution, for any atom.

Theorem 6.9 [CL00, Thm. 4.4] Let $A_1, A_2$ be atoms that are unifiable with MGU $\theta$, and $A^i_1, A^i_2$ be abstract atoms such that $A^i_1 \propto A_1$ and $A^i_2 \propto A_2$. Then there exists a unifier $\theta^A \in cU_{AC^+}(A^i_1, A^i_2)$ such that $A^i_1 \theta^A \propto A_1 \theta$, and moreover for any atom $B$, we have $\alpha(B) \theta^A \propto B \theta$.

Proof. Consider the pairs $\langle B, A_1 \rangle$, $\langle B, A_2 \rangle$ where $B$ is an arbitrary atom. Since $\langle B, A_i \rangle \theta \leq \langle B, A_i \rangle$, we have by Thm. 6.7 and the definition of $\propto$

$$\alpha(\langle B, A_i \rangle \theta) \leq_{AC^+} \alpha(\langle B, A_i \rangle) \leq_{AC^+} \langle \alpha(B), A^i_1 \rangle \quad (i = 1, 2).$$

Since $A_1 \theta = A_2 \theta$, it follows that $\alpha(\langle B, A_1 \rangle \theta) = \alpha(\langle B, A_2 \rangle \theta)$ and so $\alpha(\langle B, A_1 \rangle \theta)$ is a common $AC^+$-instance of $\langle \alpha(B), A^i_1 \rangle$ and $\langle \alpha(B), A^i_2 \rangle$. Hence

$$cU_{AC^+}(\langle \alpha(B), A^i_1 \rangle, \langle \alpha(B), A^i_2 \rangle)$$

contains a $\theta^A$ such that $\alpha(\langle B, A_1 \rangle \theta) \leq_{AC^+} \alpha(\langle B, A^i_1 \rangle \theta^A)$ and so $\alpha(B) \theta^A \leq_{AC^+} \alpha(B) \theta^A$ and $\alpha(A_{1\theta}) \leq_{AC^+} A^i_1 \theta^A$. Now $cU_{AC^+}(\langle \alpha(B), A^i_1 \rangle, \langle \alpha(B), A^i_2 \rangle)$ is also a complete set of unifiers of $A^i_1$ and $A^i_2$, and so the claim follows. \qed

In addition, we also have that $AC^+$-unification is optimal. To make this notion precise, consider two abstract atoms $A^i_1, A^i_2$, and let $I_i = \gamma(\{A^i_1\}) (i = 1, 2)$. For two atoms $A_1 \in I_1, A_2 \in I_2$, any common instance is in $I_1 \cap I_2$, since $I_1, I_2$ are downwards-closed. Now let $cI_{AC^+}(A^i_1, A^i_2) = \{A^i_\theta \propto A^i_\gamma \mid \theta^A \in cU_{AC^+}(A^i_1, A^i_2) \}$. We might call $cI_{AC^+}(A^i_1, A^i_2)$ a complete set of common $AC^+$-instances of $A^i_1, A^i_2$. Optimality of abstract unification means that $cI_{AC^+}(A^i_1, A^i_2)$ describes only the atoms in $I_1 \cap I_2$.

Theorem 6.10 [CL00, Thm. 4.6] For $i = 1, 2$, let $A^i_1, A^i_2$ be abstract atoms and $I_i = \gamma(\{A^i_1\})$. Then $cI_{AC^+}(A^i_1, A^i_2)$ describes only the atoms in $I_1 \cap I_2$.

Proof. To derive a contradiction, assume that $B$ is an atom such that for some $B^A \in cI_{AC^+}(A^i_1, A^i_2)$, we have $B^A \propto B$ but $B \not\in I_1 \cap I_2$. This implies that $A^i_1 \propto B$ or $A^i_2 \propto B$. On the other hand, $B^A$ is a common instance of $A^i_1, A^i_2$, which implies $A^i_1 \propto B$ and $A^i_2 \propto B$. Contradiction. \qed

6.3 Abstraction of Programs

In [LS01, SHK00], programs were assumed to be in normal, also called canonical, form. In [LS01], the abstraction of a unification constraint $x = y$ involves computing the intersection of the two grammars corresponding to $x$ and $y$. In addition, the abstraction of a unification constraint $y = f(x_1, \ldots, x_n)$ involves computing a grammar for $f(x_1, \ldots, x_n)$ from the $n$ grammars for $x_1, \ldots, x_n$. In [SHK00], no such operations are performed, but still the abstraction of unification constraints is not obvious. Each
unification constraint \( y = f(x_1, \ldots, x_n) \) is abstracted as a call \( f_{\text{rel}}(y, x_1, \ldots, x_n) \), where \( f_{\text{rel}} \) is a predicate that expresses the relationship between the abstraction of a term and its subterms.

In the framework we have set up here following [CL00], abstracting a program is much simpler. We have designed the domains so that Thm. 6.9 holds, and so we can abstract a program simply by replacing each term with its abstraction. Thus \( \alpha \) is lifted in the obvious way to atoms, clauses, programs and queries.

The semantics of the abstract program will be defined using an AC+-enhanced version of the \( T_P \)-operator. Formally

\[
T^A_P(I^A) = \{ \alpha(H) \theta^A | C = H \leftarrow B_1, \ldots, B_n \in P, \langle A_1^A, \ldots, A_n^A \rangle \ll C I^A, \theta^A \in cU_{AC+}(\langle \alpha(B_1), \ldots, \alpha(B_n) \rangle, \langle A_1^A, \ldots, A_n^A \rangle) \}\.
\]

We denote by \([P^A]_{AC+}\) the least fixpoint of \( T^A_P \), which exists by [CL00, Cor. 5.2]. The following theorem says that this abstract semantics correctly describes the concrete semantics.

**Theorem 6.11** [CL00, Thm. 5.4] Let \( P \) be a program. Then \([\alpha(P)]_{AC+}\cong [P]_s\).

We could make further statements about the semantics, e.g. call and answer patterns; for that purpose, we would use the magic set transformation. However, those results would be completely along the lines of [CL96, CL00], as are the proofs of the results we gave in this subsection.

### 6.4 Finiteness

In [CL00] we find a result that the abstract semantics of a program is finite provided that the type abstraction is monomorphic. The result does not hold anymore for polymorphic type abstractions, and the authors give the program

\[
P_1 := \{ p([X]) \leftarrow p(X)., \ p(1). \}
\]

as an example. As a solution, the authors propose a depth-k abstraction, i.e., some ad-hoc bound on the depth of types.

It is understandable that a descriptive view of types leads to the conviction that infinity of the abstract semantics is inherent in a polymorphic type abstraction and cannot reasonably be avoided.

Instead of \( P_1 \), consider the program

\[
P_2 := \{ p([X]) \leftarrow p(X)., \ p([]). \}
\]

For the argument in [CL00], \( P_2 \) is completely equivalent to \( P_1 \), i.e., the authors could just as well have chosen \( P_2 \). However, using \( P_2 \) allows us to make a stronger point than using \( P_1 \). The reason is that both programs are not typable by the rules of Table 1 (that is to say, in a prescriptive approach to typing), but for \( P_2 \), this is much less obvious. The program \( P_2 \) is forbidden due to the head condition \([HT92]\), i.e., the special typing rule Head which is different from rule Atom (see Table 1).

**Proposition 6.12** Assuming \( F_{\text{LISTS}} \) (see Ex. 3.13), there exists no variable typing \( \Gamma \) such that \( \Gamma \vdash p([X]) \leftarrow p(X) \) Clause, regardless of what the declared type of \( p \) is.

**Proof.** To derive \( \Gamma \vdash p([X]) \leftarrow p(X) \) Clause, we need to derive \( \Gamma \vdash p([X]) \) Head, and in turn,

\[
\Gamma \vdash [X] : \sigma
\]
for some type \( \tau \). By rule \( \text{Func} \) and the fact that the declared range type of \( \text{Cons} \) is \( \text{List}(U) \), it follows that \( \sigma = \text{List}(\tau) \) for some \( \tau \), and so for \( \Gamma \vdash p([X]) \) Head to be a valid type judgement, the declared type of \( p \) must be \( \text{List}(\tau) \), so we write \( p_{\text{List}(\tau)} \).

To derive \( \Gamma \vdash p([X]) \leftarrow p(X) \) Clause, we also need to derive \( \Gamma \vdash p_{\text{List}(\tau)}([X]) : \text{Atom} \), and in turn, \( \Gamma \vdash X : \text{list}(\tau)\Theta \) for some type substitution \( \Theta \). This implies that \( X : \text{list}(\tau)\Theta \in \Gamma \) and hence \( \Gamma \not\vdash [X] : \text{list}(\text{list}(\tau))\Theta \), and in particular, \( \Gamma \not\vdash [X] : \text{list}(\tau) \). This is a contradiction to [4], showing that there exists no \( \Gamma \) such that \( \Gamma \vdash p([X]) \leftarrow p(X) \) Clause.

We want to show that disregarding such programs, the abstract semantics is always finite. We first need the following lemma about concrete programs, stating that the arguments of any atom in \([P]_s\) are of the declared type.

**Lemma 6.13** Let \( P \) be a typed program. For any atom \( p_{\tau_1...\tau_n}(t_1, \ldots, t_n) \in [P]_s \), we have \( \vdash (t_1, \ldots, t_n) : (\tau_1, \ldots, \tau_n) \).

**Proof.** Suppose \( I \) is a set if atoms having the property stated for \([P]_s\). We show that an application of the \( T_P \)-operator to \( I \) preserves the property. This immediately implies the result.

Consider some clause \( C = p_{\tau_1...\tau_n}(t_1, \ldots, t_n) \leftarrow B_1, \ldots, B_m \) in \( P \), and suppose that \( \langle A_1, \ldots, A_m \rangle \ll C I \), such that \( \theta = MGU(\langle B_1, \ldots, B_m \rangle, \langle A_1, \ldots, A_m \rangle) \). By the rules in Table 1, in particular Head, we have \( \vdash (t_1, \ldots, t_n) : (\tau_1, \ldots, \tau_n) \). For each \( B_i = q_{\sigma_1, \ldots, \sigma_n}(s_1, \ldots, s_{n'}) \), by the rules in Table 1, we have \( \vdash (s_1, \ldots, s_{n'}) : (\sigma_1, \ldots, \sigma_{n'})\Theta \) for some \( \Theta \). Let \( A_i = q(r_1, \ldots, r_{n'}) \). By assumption about \( I \), we have \( \vdash (r_1, \ldots, r_{n'}) : (\sigma_1, \ldots, \sigma_{n'})\Theta \). By standard results [HT92, Thm. 1.4.1, Lemma 1.4.2], it follows that \( \vdash (t_1, \ldots, t_n)\theta : (\tau_1, \ldots, \tau_n) \). Since the choice of \( A_i \) was arbitrary, it follows that each atom in \( C\theta \) can be typed using the same types as for the corresponding atom in \( C \). This applies in particular for the clause head, and so \( \vdash (t_1, \ldots, t_n)\theta : (\tau_1, \ldots, \tau_n) \).

By Prop. 44, the above lemma applies also to the abstraction of a program.

**Corollary 6.14** Let \( P \) be a typed program. For any atom \( p_{\tau_1...\tau_n}(a_1, \ldots, a_n) \in [\alpha(P)]_{AC+} \), we have \( \vdash^A (a_1, \ldots, a_n) : (\tau_1, \ldots, \tau_n) \).

The following lemma states that for a given type \( \phi \), there are only finitely many different abstract terms for \( \phi \).

**Lemma 6.15** For any type \( \phi \), the set of abstract terms \( \{a \mid \vdash^A a : \phi\} \) is finite modulo \( \approx \).

**Proof.** Since the claim is that the set is finite modulo \( \approx \), it is clearly sufficient to restrict our attention to normal abstract terms. It is useful to recall the notations and definitions of Subsec. 2.2.

The proof is along the lines of the proof of [CL00, Thm. 3.2], but matters are slightly more complicated since our abstract terms are nested.

We define by structural induction:

- for each abstract term \( a \oplus x^\oplus \), \( \epsilon \) is a path, and any variable in \( x^\oplus \) is a variable occurring in \( a \oplus x^\oplus \) at \( \epsilon \);
- for an abstract term \( c^A(a_1, \ldots, a_m) \oplus \ldots \), if \( \zeta \) is a path for \( a_j \) and \( x \) is a variable occurring in \( a_j \) at \( \zeta \), then \( j, \zeta \) is a path for \( c^A(a_1, \ldots, a_m) \oplus \ldots \), and \( x \) is a variable occurring in \( c^A(a_1, \ldots, a_m) \oplus \ldots \) at \( j, \zeta \).
By the well-definedness of normal abstract terms, it follows that there is a maximum number of paths that a normal abstract term for \( \phi \) can have. Let \( n \) be this number for our given \( \phi \).

Suppose that a normal abstract term \( a \) for \( \phi \) contains more than \( 2^n - 1 \) distinct variables. By a simple combinatorial argument, one sees that there must be at least two variables, say \( x \) and \( y \), occurring at exactly the same paths in \( a \). Consider \( a' = a\{x/z, y/z\} \). Trivially \( a' \leq a\). On the other hand, we have \( a'\{z/x \oplus y\} = a \), and thus \( a \leq a' \). So we have \( a \approx a' \), and \( \#(\text{vars}(a')) = \#(\text{vars}(a)) - 1 \).

By iterating this argument, it follows that any normal abstract term for \( \phi \) is \( \approx \)-equivalent to a term containing no more than \( 2^n - 1 \) variables, and thus the claim follows.

The following is a simple corollary.

**Corollary 6.16** Let \( p_{\tau_1,...,\tau_n} \) be a predicate and \( \Theta \) a type substitution. Modulo \( \approx \), there are only finitely many abstract atoms \( p(a_1,...,a_n) \) such that \( (a_1,...,a_n) \) is a vector of normal abstract terms for the type vector \( (\tau_1,...,\tau_n)\Theta \).

**Theorem 6.17** Let \( P \) be a typed program. Then \( \|\alpha(P)\|_{AC^+} \) is finite.

**Proof.** By Corollaries 6.14 and 6.16.

As it stands, the theorem depends critically on the fact that we assume a bottom-up semantics. To explain this, consider the program

\[
P_3 = \{ p(\langle X \rangle) \leftarrow p([X]), p([]) \}.
\]

which at first look is very similar to the program \( P_2 \) given at the beginning of this subsection. However, assuming that \( p \) has declared type \( \text{list}(U) \), the program \( P_3 \) is typed according to the rules of Table 1. Therefore, of course, Thm. 6.17 applies to this program.

Note that \( P_3 \), when called with the query \( p(Y) \), gives rise to infinitely many calls \( p(Y), p([Y]), p([[[Y]])], \ldots \), with abstractions \( p(Y), p(\text{list}^4(Y)), p(\text{list}^3(\text{list}^4(Y))), \ldots \). So the set of calls cannot be described (in the technical sense, using \( \approx \)) finitely.

We make two observations about \( P_3 \):

- The magic set version \( [CD95] \) of the program contains the clause
  \[
p^*(\langle X \rangle) \leftarrow p^*(X),
\]
  which is to be read as “in order for \( p(\langle X \rangle) \) to be called, \( p(X) \) must be called. This clause is not typable according to the rules of Table 1 as the head condition is violated. Thus Thm. 6.17 is not applicable. This is not surprising given the fact that the very purpose of the magic set transformation is to characterise calls.

- In the literature on prescriptive typing, the behaviour exhibited by \( P_3 \) has been called **polymorphic recursion** \( [Hen93] \). It is by no means common. In fact, it is very much on the borderline of what is allowed in prescriptively typed programming languages. In ML for example, it is forbidden, as it breaks the capabilities of the type inference procedure.

It has previously been suspected that there is an interesting relationship between the head condition and polymorphic recursion, which deserves some profound investigation \( [DS01] \). The two observations above add weight to this.

In this paper, we do not want to study the difference between top-down and bottom-up semantics in detail. Nevertheless, we now formulate what it means for a program...
not to use polymorphic recursion, in which case we say that it only uses monomorphic recursion.

For simplicity, we assume that a program only contains direct recursion, i.e., no recursion of the form \( p \leftarrow \ldots q, q \leftarrow \ldots p \). It is straightforward to generalise our considerations otherwise. In the following, we use \( \vec{t} (\vec{\tau}) \) to denote vectors of terms (types).

**Definition 6.18** A typed program uses only monomorphic recursion if for any clause \( p(\vec{t}) \leftarrow \ldots p(\vec{s}) \ldots \), we have \( \vdash \vec{t} : \vec{\tau}, \vec{s} : \vec{\tau} \) for some vector of types \( \vec{\tau} \).

Thus monomorphic recursion means that in each clause, the type of any recursive call must be identical to the type of the head. Since the head condition is still in force, this implies that any recursive call must have a type which is identical to the declared type of its predicate.

For programs using only monomorphic recursion, it should be possible to devise a variant of the magic set transformation such that the transformed program is typed according to the rules in Table 1, and therefore has a finite abstract semantics.

# 7 Towards an Implementation

So far, we have not implemented the analysis proposed in this paper. As far as computing the semantics of the abstract program is concerned, the only difference with the implementations mentioned in [CL96, CL00] is that instead of ACI or ACII we have the equality theory AC+. The former theories are finitary [BS98], and the corresponding unification problems are NP-complete. Obviously, AC+ cannot behave any better. Studying AC+ is a topic for future work, but we would certainly expect it to be finitary as well.

There is an implementation of the analysis we proposed in [SHK00], which essentially aims at the same degree of precision we have here, but the framework is different. In fact, this paper relates to [SHK00] in the same way as [CL96, CL00] relates to [CD94]. This is interesting because the authors mention that an implementation using ACI-unification turned out to be much faster than the implementation in [CD94].

Note that to compute the abstraction of a program, in [CD94, CL00], it is the user who has to provide information about the particular type language used in a program (see paragraph after Def. 4.1), whereas in our analysis, this information is extracted from the declared types. We had previously shown [SHK00] that analysing the type declarations (computing the NRSs and recursive types) is viable even for some contrived, complex type declarations, which one would never expect in practice, since good programs have small types [Hen93].

In [CL96], there is some speculation as to why abstract unification is not as bad as it seems by the theoretical result that it is NP-complete. It is said that usually unifications involve “few” variables. Here we want to substantiate that claim somewhat, but it remains speculation all the same.

- The first argument is that abstract terms (in normal form) are likely to be linear. Recall that the abstract terms are designed in such a way that different positions correspond to different subterm types. Since we use prescriptive types, one is tempted to conclude that abstract terms must be linear, since the same variable cannot have different types. That however would be a fallacy, since via

\[ \text{However, this should not be understood as a contradiction to our } \text{NESTS or TABLES examples. Henglein comes from a functional programming background, and in that community, those type declarations would by all means still qualify as “small”}\]
instantiation of types, different subterm types can become equal. For example, the term node(null, X, X, eq, null) of type table(string) has the abstraction table^A(X, bal^A, X) which is not linear. Nevertheless, this should be an exception.

On the whole, logic programs are based on a very simple notion of modes. Of course, it is the very exceptions to this rule that justify developing a complex instantiation analysis like the one of this paper or [CL00, LS01], but still, often one deals with simple assignments rather than full unification in concrete programs, and this carries through to abstract programs.

To give at least one example of the advance of our analysis over [CL00], we use table(int). Suppose there is a predicate insert/4 whose arguments represent: a table t, a key k, a value v, and a table obtained from t by inserting the node whose key is k and whose value is v. From the abstract semantics of the program, it is possible to read that a query whose abstraction is

\[
\text{insert(table}^A(\text{int}^A, \text{bal}^A, \text{str}^A), \text{str}^A, V2, T),
\]

i.e., a query to insert an uninstantiated value into a ground table, yields an answer whose abstraction is

\[
\text{insert(table}^A(\text{int}^A, \text{bal}^A, \text{str}^A), \text{str}^A, V2, \text{table}^A(\text{int}^A \oplus V2, \text{bal}^A, \text{str}^A)),
\]

i.e., the result is a table whose values may be uninstantiated.

8 Discussion

In this paper, we have proposed a formalism for deriving abstract domains from the type declarations of a typed program. Effectively, we have recast our previous work [SHK00] using important parts of the formalisms of [CL00, LS01]. We now compare this paper with those two works under several aspects.

The type system. Using the terminology introduced in this paper, one can say that [CL00] uses a polymorphic type system with the following assumptions: types are either monomorphic or unary, and the only subterm types of a unary type \(c(u)\) are \(c(u)\) itself (and \(c(u) \supset c(u)\)) and \(u\) (and \(u \sq supp c(u)\)). This is the simplest thinkable scenario of proper polymorphism; in fact, only lists and trees are covered. Our TABLES or let alone the NESTS example are not covered. In contrast, [LS01] assumes regular types without polymorphism. Thus there are only finitely many types the analysis has to deal with. However, those may be very complex; e.g., one can easily construct a grammar that corresponds to the type table(int). So the type systems of [CL00, LS01] are not formally comparable, but the type system we assume in this paper is a strict generalisation of both.

Descriptive vs. prescriptive types. According to the authors' claims, [CL00] takes a descriptive view of typing, whereas [LS01] takes a prescriptive view. However, we find that the formalism of [CL00] can very well be adapted to prescriptive typing. On the other hand, we find that some aspects of [LS01] belong rather to a descriptive view of typing.

First, the fact that the typing approach of [CL00] is descriptive rightly accounts for the fact that they must consider “ill-typed” terms such as [1|2]. In this paper, all terms are “well-typed”, and so are the abstract terms.

In [LS01] it is assumed that a unique type (or equivalently, grammar) is associated with each program variable. A unification constraint in a program gives rise to
operations such as computing the intersection of two types and computing the type of a term from the types of its subterms. Such operations can improve the precision of an analysis, e.g. if \( X \) has the declared type “list with an even number of elements” and \( Y \) has the declared type “list with a number of elements divisible by 3”, then a unification \( X = Y \) implies that both variables have the type “list with a number of elements divisible by 6”. In our opinion, the presence of such operations introduces an aspect of type inference into their formalism which is somewhat in contradiction to a prescriptive approach to typing. While such type inference may be useful, the authors do not give a convincing example for it being so.

Labellings. Labellings are useful to formalise which aspects of the structure of a concrete term we want to capture by our analysis, and so it is natural that we used them mainly in Sec. 3. In \([CL00]\), they are absent, although they may have been useful (see Sec. 3). In \([SHK00]\), there were similar functions called extractors and termination functions.

First note that \( \zeta \) only collects variables, whereas \( \mathcal{Z} \) also collects non-variable terms. This generalisation allows us to describe the relation between a term and its abstraction as we did in Sec. 3.

The labelling function \( \zeta \) in \([LS01]\) has three arguments: a grammar (which however can be identified with its starting non-terminal), a non-terminal to be labelled, and a labelling term. Our labelling function \( \mathcal{Z} \) has four arguments. We found it useful to have as first argument a flat type (e.g. \( \text{nest}(V) \)) which gives us a certain grammar, but also allow for productions of that grammar starting from some other non-terminal (e.g. \( \text{list(nest}(V)) \)). Actually, it may well be the case that the first argument is redundant, i.e. that the grammar can be derived from the starting non-terminal (e.g. \( \text{nest}(V) \) from \( \text{list(nest}(V)) \)). We prefer however to keep this useful intuitive explanation: one argument to indicate the grammar, one to indicate the starting non-terminal. The difference between our labelling function and that of \([LS01]\) is due to polymorphism.

Abstract terms. In \([LS01]\), the abstraction of terms is not actually made explicit, but effectively, given a program variable \( x \), its abstraction is the (somehow ordered) tuple of non-terminals of the grammar of \( x \). Non-terminals are thought of as abstract variables (see the paragraph after Equation (7)). Our abstraction of terms, denoted \( \alpha \), is designed in such a way that the abstraction type in \([CL00]\) is essentially a special case of it. We do not introduce abstract variables but rather collect the labellings for a term in a structure called abstract term. The reason for this decision is that it allows us to deal with arbitrarily large grammars/type graphs.

Type hierarchies. Given a function \( f_{\ldots c(u)} \), the abstraction type in \([CL00]\) distinguishes between the argument positions of declared type \( u \) and the “recursive” argument positions. Via type instantiation, this already gives rise to a certain hierarchy of arbitrary depth, as reflected for example in the abstract term \( \text{list}^2(\text{list}^2(\text{int}^2)) \). Our concepts of non-recursive subterm type and recursive type generalise this idea. An NRS of a flat type is not necessarily a parameter, and \( \tau \) can have other recursive types than \( \tau \) itself. This hierarchy allows us to deal with the fact that through instantiation of types, a polymorphic language gives rise to arbitrarily big type graphs (grammars).

In contrast, in \([LS01]\), all non-terminals (types) reachable from the starting node of a grammar are treated in the same way. This approach is viable since the size of the grammars is fixed beforehand.

Equality theory. The equality theory for evaluating abstract terms in \([LS01]\) is ACII. Distributivity is not applicable. In \([CL00]\), the equality theory is ACI, so there is no neutral element. The authors mention distributivity but decide against it. This
is in contrast to \[\text{CL96}\] where the equality theory is \(\text{ACI1} + \text{distributivity}\). Our extraction axioms are only relevant for a language where some type has a recursive type other than itself, and so it is not applicable to \[\text{CL96, CL00}\]. We believe that at least conceptually, both a neutral element and the distributivity axioms are very natural, even if at the level of an implementation, it might be preferable not to have them.

**Types = abstract terms?** In \[\text{CL00}\], there is no distinction between a type constructor \(c\) and the function \(c^A\) to build abstract terms. Also, the equivalent of our function \(\alpha\) is called *type abstraction* and denoted by \(\text{type}\), which highlights the fact that in descriptive approaches to typing, *type* analysis and *mode* analysis are blurred almost to the extent of being considered to be the same thing. However, such an identification only works because the assumptions about the type system are so restrictive.

Thus we have generalised \[\text{CL00, LS01}\] by considering a type system which almost (see below) corresponds to the type system of existing typed programming languages. We have given several examples in Sec. 3 hoping to convince the reader that such a generalisation is non-trivial. In particular, there are two natural requirements: the construction of an abstract domain for a polymorphic type should be truly parametric, and the abstract domains should be finite for a given program and query. We had to impose two conditions on the type declarations to ensure these requirements. On a technical level, the fact that the SCCs of a type graph are not stable under instantiation makes it difficult to meet those requirements.

We now very briefly recall some other related work. We refer to the discussions in \[\text{CL00, LS01, SHK00}\] for more details.

Both this paper and \[\text{CL00, LS01}\] build on ideas presented originally in \[\text{CD94}\].

Recursive modes \[\text{FL97}\] characterise that the left spine, right spine, or both, of a term are instantiated. This seems ad-hoc but often coincides with characterising that all recursive subterms of a term are instantiated.

A system for type analysis of Prolog is presented in \[\text{VCL93}\]. It takes a descriptive approach to typing, and the abstract domains are, in general, infinite. Therefore, widenings must be used. Similarly, in \[\text{JB92}\], the finiteness of abstract domains and terms is ensured by imposing an ad-hoc bound on the number of symbols.

It would probably be possible to express the abstraction of terms proposed here as application of a particular *pre-interpretation* \[\text{GBS95}\].

A classical instantiation analysis is not interesting for Mercury \[\text{SHC96}\] as the language is strongly moded. However, our work might also have applications for Mercury.

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