Mass radius relation of compact stars in the braneworld

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Abstract. The braneworld scenario, based on the fact that the four dimension space-time is a hyper-surface of a five dimensional manifold, was shown to deal in a satisfactory way with the hierarchy problem. In this work we study macroscopic stellar properties of compact stars from the braneworld point of view. Using neutron star equations of state, we test the possibility of extra dimensions by solving the brane Tolman-Oppenheimer-Volkoff equations obtained for three kinds of possible compact objects: hadronic, hybrid and quark stars. By comparing the macroscopic solutions with observational constraints, we establish a brane tension lower limit and the value for which the Tolman-Oppenheimer-Volkoff equations in the braneworld converge to the usual Tolman-Oppenheimer-Volkoff equations.

Keywords: extra dimensions, neutron stars, massive stars

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Dedicated to my family, Isabel and Alexander (L.B. Castro)
1 Introduction

The four fundamental nature interactions (gravity, electromagnetism, strong and weak nuclear forces) were investigated separately during their initial studies and developments. At some point, it became obvious that an unified theory could help the understanding of the primordial universe and its evolution. Two of these fundamental forces (electromagnetism and weak nuclear forces) were then joined in an unique formalism known as the electroweak interaction by Abdus Salam, Sheldon Glashow and Steven Weinberg, but the unification of the other two is still pursued nowadays. Around 1990’s, string theory came up as a possible candidate for the theory of everything or M-theory because it presented the advantage of conciliating quantum mechanics with gravity by putting aside the idea of particles as the elementary bricks of the universe and explaining them as particular quantum states of strings. String theory started with bosons only and later incorporated fermions and the requirement of extra dimensions was a key point in its development. M-theory assumes the existence of 11 dimensions, some of them treated as hidden dimensions. These extra dimensions of the space-time have since then been used as an attempt to explain several open problems like the existence of dark matter, dark energy, the accelerated expansion of the universe, etc.

The braneworld scenario is based on the assumption that the four dimensional space-time is a hyper-surface of a five dimensional manifold. The gravity in the braneworld is a five dimensional phenomenon and the other fundamental interactions are confined to the brane. The braneworld scenario has attracted a lot of interest, because it tackles the hierarchy problem \[1\] in an effective way. Another attractive property is that the Newtonian law of gravity with a correction is also given in this braneworld scenario \[2\]. In the Randall-Sundrum (RS) model \[1\], one can further add scalar fields \[3\] with usual dynamics and allow them to interact with gravity in the standard way. In this scenario, the smooth character of the solutions generate thick branes with a diversity of structures \[4-7\]. In the braneworld scenario an import issue is how gravity and different observable matter fields of the Standard Model of particle physics are localized on the brane. Recently, the localization of fermions and bosons on the brane have received considerable attention in the literature \[8-28\].

On the other hand, in \[29-31\], the idea that neutron stars could be a laboratory for testing extra dimensions was first explored. It was found that the star in the braneworld is less compact that in general relativity and an astrophysical lower limit for the brane tension was established. In \[32\], brane theory was utilized to estimate the correction in the waves
radiated by a binary system and the observational masses of the PSR B1913+16 were then used to constrain the brane tension.

Original neutron star models assume that the dense matter in its interior is composed of hadrons (protons and neutrons only or the complete lowest lying baryonic octet) and leptons, responsible for ensuring charge neutrality and $\beta$-equilibrium [33]. On the other hand, the Bodmer-Witten conjecture [34–36] states that quarks can be deconfined from the hadrons, forming a stable quark matter under certain conditions. Hence, compact stars can be constituted of pure quark matter or perhaps of hybrid matter, containing in their core a pure quark phase or a mixed phase of quarks and hadrons [33, 37–41]. In order to rule out improbable stellar configurations, observational constraints have been used. While most neutron stars have masses of the order of $1.4\,M_\odot$, at least two pulsars, PSR J1614-2230 [42] and PSR J0348+0432 [43] were confirmed to be very massive objects, with masses of the order of $2\,M_\odot$. The theoretical calculation of macroscopic stellar properties, as the masses and radii, is done by solving the Tolman-Oppenheimer-Volkoff (TOV) [44] equations, which use equations of state (EOS) as input.

In this work, we study the effects of the braneworld scenario on the macroscopic stellar properties of compact stars. In that spirit, we revisit the idea of solving a TOV-like system of equations in the braneworld (brane-TOV) and see if the resulting mass and radius results survive the known observational constraints. We also check weather, with more realistic equations of state than the perfect fluid one used in [29, 32], limits for the brane tension can be established. Furthermore, we show that the star becomes more compact and that the radii can be adjusted to smaller values depending of the branе-TOV parameters.

The paper is organized as follows: in section 2, we present the standard TOV formalism. In section 3, we give a brief review of the brane-TOV formalism. In section 4, we use different EOS as input to the brane-TOV equations, present and discuss our results. Finally, in section 5, we draw our final conclusions.

2 Tolman-Oppenheimer-Volkoff equations

The Tolman-Oppenheimer-Volkoff (TOV) equations for static and spherical stars are derived from the standard 4D general relativity [44]

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa^2 T_{\alpha\beta}, \quad (2.1)$$

where $\kappa^2 = 8\pi G$, $g_{\alpha\beta}$ is the metric, $R_{\alpha\beta}$ is the Ricci tensor, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature and $T_{\alpha\beta}$ is the energy-momentum tensor. The differential element for a spherical relativistic star is given by

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.2)$$

Solving Einstein’s field equations for a perfect fluid matter, we obtain

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad (2.3)$$

$$\frac{dp}{dr} = -(\epsilon + p)\frac{d\phi}{dr}, \quad (2.4)$$

$$\frac{d\phi}{dr} = \frac{Gm + 4\pi Gr^3p}{r(r - 2m)}, \quad (2.5)$$
where \( m(r) \) and \( p(r) \) are respectively the gravitation mass and pressure both defined as star radius functions and \( \epsilon \) is the energy density. In this case, we have 3 equations and 4 unknown functions, which are \( m(r) \), \( \epsilon(r) \), \( p(r) \) and \( \phi(r) \). For this reason, to solve these equations, we need an equation of state \( p(\epsilon) \) and appropriate boundary conditions, which we choose as:

\[
\begin{align*}
m(0) &= 0, \\
p(0) &= p_c, \\
m(R) &= M , \\
p(R) &= 0,
\end{align*}
\]

where \( p_c \) is central pressure, \( R \) is the star radius and \( M \) is the total gravitation mass of the star.

## 3 Tolman-Oppenheimer-Volkoff equations in the braneworld

According to the current idea, our observable universe may be confined on some hypersurface (4D) called brane which, in turn, is embedded in some multidimensional space (5D) called bulk. In this context, only gravitation could propagate in the bulk (extra-dimension). Within the braneworld scenario, the field equations induced on the brane are derived via an elegant geometric approach developed in ref. [45]. The basic idea of this approach is to project the 5D curvature along the brane and the result is a modification of the standard Einstein’s equations, with the new terms carrying bulk corrections onto the brane. The bulk corrections to Einstein’s equations on the brane can be consolidated into an effective total energy-momentum tensor [45]

\[
G_{\alpha\beta} = \kappa^2 T_{\text{eff}}^{\alpha\beta},
\]

where \( G_{\alpha\beta} \) is the usual Einstein’s tensor and

\[
T_{\text{eff}}^{\alpha\beta} = T^{\alpha\beta} + \frac{6}{\lambda} S^{\alpha\beta} - \frac{1}{\kappa^2} E^{\alpha\beta},
\]

where \( \lambda \) is the brane tension, which correspond to the vacuum energy density on the brane. The effective total energy-momentum tensor (3.2) show two key modifications to the standard 4D Einstein’s field equations. The bulk corrections can be classified as local and non-local corrections. The local correction is carried via the tensor \( S^{\alpha\beta} \) (matter corrections), while the non-local correction is carried via the projection \( E^{\alpha\beta} \) of the bulk Weyl tensor. The geometric tensor \( E^{\alpha\beta} \) transmit non-local gravitational degrees of freedom from the bulk to the brane.

For a perfect fluid or minimally coupled scalar field the expressions for \( T^{\alpha\beta} \) and \( S^{\alpha\beta} \) are given by [46, 47]

\[
\begin{align*}
T^{\alpha\beta} &= \rho u^{\alpha} u^{\beta} + p h^{\alpha\beta}, \\
S^{\alpha\beta} &= \frac{1}{12} \rho u^{\alpha} u^{\beta} + \frac{1}{12} \rho (\rho + 2p) h^{\alpha\beta},
\end{align*}
\]

where \( u^{\alpha} \) is the four-velocity and \( h^{\alpha\beta} = g^{\alpha\beta} + u^{\alpha} u^{\beta} \) is the projection orthogonal to \( u^{\alpha} \). Further, assuming static spherical symmetry the expression for \( E^{\alpha\beta} \) becomes [29]:

\[
E^{\alpha\beta} = -\frac{6}{\kappa^2 \lambda} \left[ \mathcal{U} u^{\alpha} u^{\beta} + \mathcal{P} r^{\alpha} r^{\beta} + \frac{(\mathcal{U} - \mathcal{P})}{3} h^{\alpha\beta} \right],
\]

where \( r^{\alpha} \) is a unit radial vector, \( \mathcal{U} \) and \( \mathcal{P} \) are respectively the non-local energy density and non-local pressure on the brane. The terms \( \mathcal{U} \) and \( \mathcal{P} \) may be really interpreted as an energy density and pressure respectively, the label of “non-local” in the Weyl terms are associated...
with the fact that they have an extra dimensional origin and are commonly referred in the literature as “dark radiation” $\mathcal{U}$ and “dark pressure” $\mathcal{P}$ \cite{47, 48}. From (3.5), we see that $\varepsilon_{\alpha\beta} \rightarrow 0$ as $\lambda^{-1} \rightarrow 0$. Applying this limit in eq. (3.2), we obtain that $T_{\alpha\beta} = T_{\alpha\beta}$, therefore the standard 4D general relativity is regained.

Solving Einstein’s equations on the brane, we obtain the modified TOV equations

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon_{\text{eff}}, \tag{3.6}
\]
\[
\frac{dp}{dr} = - (\epsilon + p) \frac{d\phi}{dr}, \tag{3.7}
\]
\[
\frac{d\phi}{dr} = \frac{Gm + 4\pi Gr^3 (p_{\text{eff}} + \frac{4P}{\kappa^4 \lambda})}{r(r - 2Gm)}, \tag{3.8}
\]
\[
\frac{dU}{dr} + (4U + 2P) \frac{d\phi}{dr} = - 2(4\pi G)^2 (\epsilon + p) \frac{d\epsilon}{dr} - 2 \frac{dp}{dr} - \frac{6}{r} \frac{dP}{dr}, \tag{3.9}
\]

where

\[
\epsilon_{\text{eff}} = \epsilon + \frac{e^2}{2\lambda} + \frac{6}{\kappa^4 \lambda} \mathcal{U}, \tag{3.10}
\]
\[
p_{\text{eff}} = p + \frac{pe}{\lambda} + \frac{e^2}{2\lambda} + \frac{2}{\kappa^4 \lambda} \mathcal{U}. \tag{3.11}
\]

In contrast to standard 4D general relativity, in this case we have 4 equations and 6 unknown functions, which are $m(r)$, $\epsilon(r)$, $p(r)$, $\phi(r)$, $U(r)$ and $\mathcal{P}(r)$. In order to solve satisfactorily these system of equations, we need an equation of state $p = p(\epsilon)$ and additionally an equation of state-like relation $\mathcal{P} = \mathcal{P}(U)$. We assume that the Weyl terms (non-local energy density and non-local pressure on the brane) obey the simplest relation $\mathcal{P} = w U$, in such a way that the system of equations (3.6), (3.7), (3.8) and (3.9) become:

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon_{\text{eff}}, \tag{3.12}
\]
\[
\frac{dp}{dr} = - (\epsilon + p) \frac{d\phi}{dr}, \tag{3.13}
\]
\[
\frac{d\phi}{dr} = \frac{Gm + 4\pi Gr^3 (p_{\text{eff}} + \frac{4w}{\kappa^4 \lambda} \mathcal{U})}{r(r - 2Gm)}, \tag{3.14}
\]
\[
\frac{dU}{dr} = - \frac{2}{1 + 2w} \left[ (4\pi G)^2 (\epsilon + p) \frac{d\epsilon}{dr} + A \right], \tag{3.15}
\]

where $A = \frac{3w}{r} U + (2 + w) U \frac{d\phi}{dr}$. Finally, we need appropriate boundary conditions, which we choose as

\[
m(0) = 0, \quad p(0) = p_c, \quad p(R) = 0, \quad m(R) = M, \quad U(0) = 0. \tag{3.16}
\]

Other choices for the initial condition on $U$ are possible, but the one used and written above is numerically convenient and it may be interpreted as a condition that does not influence the non-local energy density at the center of the compact star. It is worth mentioning that different values of the initial condition for $U$ lead to the same final results after just few iterations, as explained in \cite{30}.\footnote{1In this case, iterations refers to radial integration steps of the modified TOV equations.} We have checked that this is numerically correct.
To solve the Tolman-Oppenheimer-Volkoff (TOV) equations numerically in the brane-world we use mixed units \[^{33}\] which are useful in stellar calculations, where pressure and energy density are given in km\(^{-2}\) units and \(w\) is a dimensionless quantity. The unit of the brane tension \(\lambda\) is also km\(^{-2}\), but for convenience we present our results in dyn/cm\(^2\), the unit generally used in the literature (see \[^{32}\], for instance).

The solutions from equations (3.12), (3.13), (3.14) and (3.15) can be separated into internal and external solutions. The internal solution is the result obtained by integrating the system from the core of the star until the null pressure point, at its surface. The external solution is obtained from the null pressure until the Weyl term \(U\) becomes approximately null. In the present work we restrict ourselves to the study of the internal solutions.

4 Testing the equations of state

We next choose some EOS as input to the brane-TOV equations. As mentioned in the Introduction, depending on their possible interior composition, neutron stars can be classified as hadronic stars with or without hyperons \[^{33, 49}\], hybrid stars containing hadronic and quark phases \[^{37–39, 50}\] or hadronic and pion or kaon condensates \[^{51–54}\] and quark (also known as strange) stars \[^{55, 56}\]. In the following all three possibilities are analyzed. We start from the case in which just the solution related to the interior of the star is considered. For hadronic and hybrid stars, a crust is always expected and hence, we add the Baym-Pethick-Sutherland (BPS) \[^{57}\] equation of state for very low densities. Due to the recent discoveries of massive stars \[^{42, 43}\] we start our analysis from EOS that we know are capable of generating high maximum masses when used as input to the usual TOV equations.

In figure 1 we display the brane-TOV solutions for hadronic stars whose equations of state include only nucleons and leptons, necessary to enforce \(\beta\)-equilibrium and charge neutrality conditions. The equation of state was obtained with the relativistic non-linear Walecka model as in \[^{39, 58}\]. We have fixed both the brane tension \(\lambda = 10^{37}\) dyn/cm\(^2\) and \(\lambda = 10^{38}\) dyn/cm\(^2\) and varied \(w\). It is import to mention that values of \(w\) have been chosen in such way that the values of radii are in accordance with some recent estimates \[^{59–61}\]. For a broader range of \(w\) values, all the mass-radius curves fall within the same interval as the ones obtained within our chosen range \((-3 < w < 2)\), so that with the chosen range, all possible mass-radius results are contemplated. Moreover, when examining equation (3.15) for a large value of \(|w|\), one sees that the derivative of \(U\) with respect to \(r\) becomes very small and hence, its contribution in equation (3.14) becomes negligible. If the central value of \(U\) is different from zero, then the contribution becomes a constant, but the qualitative results are equivalent.

A first analysis of the mass-radius curves for \(\lambda = 10^{37}\) dyn/cm\(^2\), displayed in figure 1a shows that the stellar masses vary in an oscillatory way with the change of \(w\) and for some values around \(w = -0.6\), the solutions may become unstable. A necessary condition for stability is that the derivative of the stellar mass with respect to the central energy density must be positive \[^{33}\].

As already stated in the Introduction, the star becomes more compact, in the sense that the masses and radii decrease for all \(w\) values as compared with the usual TOV solution. Our results of stellar properties for hadronic stars obtained with \(\lambda = 10^{37}\) dyn/cm\(^2\) are summarized in table 1. From figure 1c, we see that the central energy density is practically the same for all \(w\) values until the maximum mass star is reached. Only after this point, the solutions deviate from each other. For the sake of completeness the radii in function
Figure 1. Stellar properties for hadronic stars obtained with $\lambda = 10^{37}$ dyn/cm$^2$ (left figures) and $\lambda = 10^{38}$ dyn/cm$^2$ (right figures) and different values of $w$: (a), (b) mass-radius curves, (c), (d) mass in function of the central energy density and (e), (f) radius versus central energy density.

of the central energy density is depicted in figure 1e. We then analyze the situation when $\lambda = 10^{38}$ dyn/cm$^2$ shown in figures 1b, 1d and 1f. All brane-TOV solutions still represent slightly more compact stars than the TOV one, but for this value of the brane tension, the stellar properties for a family of stars are almost independent of $w$. Moreover, for this value of the brane tension, the results are very similar to the ones obtained with the standard TOV solution. One has to bear in mind that these solutions are not 100% precise due to small
Table 1. Stellar properties for hadronic stars obtained with $\lambda = 10^{37}$ dyn/cm$^2$ and different values of $w$.

| Hadronic Stars | $\lambda = 10^{37}$ (dyn/cm$^2$) | $\lambda = 10^{38}$ (dyn/cm$^2$) | $\lambda = 10^{39}$ (dyn/cm$^2$) |
|----------------|---------------------------------|---------------------------------|---------------------------------|
| $M_{\text{max}}$ ($M_0$) | $w = -3$ | $w = -1$ | $w = -0.6$ | $w = -0.1$ | $w = -0.2$ | $w = 2$ | standard TOV |
| 1.73 | 1.72 | 1.82 | 1.69 | 1.74 | 1.74 | 2.04 |
| 10.64 | 11.14 | 12.48 | 9.07 | 9.81 | 10.32 | 10.70 |
| 6.41 | 6.56 | 11.92 | 5.10 | 5.89 | 6.25 | 7.33 |

Table 2. Stellar properties for hadronic stars obtained with $\lambda = 10^{38}$ dyn/cm$^2$ and different values of $w$.

| Hadronic Stars | $\lambda = 10^{38}$ (dyn/cm$^2$) | $\lambda = 10^{39}$ (dyn/cm$^2$) |
|----------------|---------------------------------|---------------------------------|
| $M_{\text{max}}$ ($M_0$) | $w = -3$ | $w = -1$ | $w = -0.6$ | $w = -0.1$ | $w = -0.2$ | $w = 2$ | standard TOV |
| 2.00 | 2.00 | 2.00 | 1.98 | 2.00 | 2.00 | 2.04 |
| 10.70 | 10.75 | 10.88 | 10.46 | 10.60 | 10.67 | 10.70 |
| 7.21 | 7.24 | 7.24 | 6.70 | 7.04 | 7.16 | 7.33 |

Figure 2. Mass radius curves obtained from the same relativistic equation of state as in figure 1 for different values of $\lambda$ and two values of $w$.

numerical uncertainties. Our results of stellar properties for hadronic stars obtained with $\lambda = 10^{38}$ dyn/cm$^2$ are summarized in table 2.

We then vary $\lambda$ from $10^{35}$ to $10^{39}$ dyn/cm$^2$ for two fixed values of $w$, one negative ($w = -3$) and one positive ($w = 2$). The mass-radius curves are shown in figure 2, from where we see that the stellar masses increase with the increase of $\lambda$ and tend to converge to their maximum values around $\lambda = 10^{38}$ dyn/cm$^2$. On the other hand, when this value is achieved, the results depend only slightly on the $w$ values, as already seen in figures 1b, 1d and 1f. An interesting aspect related to these solutions is the fact that $\lambda$ clearly controls the values of the maximum star masses, while $w$ influences the corresponding radii. For small values of the brane tension, the brane-TOV solutions become either unstable or produce very low maximum masses, what is not expected from astronomical observations. Thus, we can
obtain both a lower limit for the brane tension and a value for which the usual TOV results are obtained, as already discussed in [29, 32], in a simplified context, where a perfect fluid was used instead of a more realistic equation of state. In the present work, we establish a range for $\lambda$ in between $3.89 \times 10^{36} < \lambda < 10^{38}$ dyn/cm$^2$, the lower limit is obtained in such a way that at least a 1.44 $M_\odot$ star can be achieved. Of course, as discussed above and seen in figure 3, these tensions depend on the $w$ values.

Next we move to the study of hybrid stars. The equation of state we use as input to the brane-TOV equations is obtained from the non-linear Walecka model for the hadron phase, the Nambu-Jona-Lasinio model for the quark phase and a Gibbs construction for the mixed phase as in [37, 38]. In face of the results obtained for hadronic stars, we have chosen $\lambda = 10^{38}$ dyn/cm$^2$. Our results of stellar properties for hybrid stars are summarized in table 3. From the figure 4, we can see that the general qualitative behavior is the same as for hadronic stars, except for the kinks in all curves related to the transitions from one phase to the other inside the star structure. However, for certain values of $w$ ($w = -0.1$, for instance), there are still sensitive deviations from the standard TOV results, which means that, even for the limit brane tension, the results still depend on the value of $w$ if hybrid stars are considered.

To finish our analysis, we look at an equation of state that is used to describe quark stars. A model for quark matter that yields maximum masses of the order of $2 M_\odot$ is the quark mass density dependent model [62, 63] and we use an equation of state taken from [56]. Once again we see, from figure 5, that for the chosen value of $\lambda = 10^{38}$ dyn/cm$^2$, the TOV

| Hybrid Stars | $\lambda = 10^{38}$ (dyn/cm$^2$) |
|--------------|---------------------------------|
| $w = -3$     | 1.88                            |
| $w = -1$     | 1.89                            |
| $w = -0.6$   | 1.86                            |
| $w = -0.1$   | 1.87                            |
| $w = -0.2$   | 1.88                            |
| $w = 2$      | 1.91                            |
| $M_{\text{max}}$ ($M_\odot$) |                    |
| 12.65         | 12.68                           |
| 12.72         | 12.31                           |
| 12.56         | 12.62                           |
| 12.67         | 12.67                           |
| $R$ (km)      |                                 |
| 12.65         | 12.68                           |
| 12.72         | 12.31                           |
| 12.56         | 12.62                           |
| 12.67         | 12.67                           |
| $\varepsilon_c$ (fm$^{-4}$) |               |
| 4.86          | 4.95                            |
| 5.00          | 4.11                            |
| 4.58          | 4.84                            |
| 4.97          |                                 |

Table 3. Stellar properties for hybrid stars obtained with $\lambda = 10^{38}$ dyn/cm$^2$ and different values of $w$. 

Figure 3. Mass radius curves with maximum mass 1.44 $M_\odot$ obtained from the same relativistic equations of state as in figure 1.

Figure 4. Mass radius curves with maximum mass 1.44 $M_\odot$ obtained from the same relativistic equations of state as in figure 1.
Figure 4. Hybrid star properties obtained with $\lambda = 10^{38}$ dyn/cm$^2$ and different values of $w$: (a) mass-radius curves, (b) mass in function of the central energy density and (c) radius versus central energy density.

| Quark Stars | $\lambda = 10^{38}$ (dyn/cm$^2$) $w = -3$ | $w = -1$ | $w = -0.6$ | $w = -0.1$ | $w = -0.2$ | $w = 2$ | standard TOV |
|-------------|---------------------------------|----------|-------------|-------------|-------------|--------|-------------|
| $M_{\text{max}}$ ($M_0$) | 2.29 | 2.29 | 2.30 | 2.24 | 2.28 | 2.29 | 2.32 |
| $R$ (km) | 11.96 | 11.97 | 12.01 | 11.77 | 11.90 | 11.94 | 11.98 |
| $\epsilon_c$ (fm$^{-4}$) | 4.50 | 4.56 | 4.56 | 4.12 | 4.41 | 4.49 | 4.60 |

Table 4. Stellar properties for quark stars obtained with $\lambda = 10^{38}$ dyn/cm$^2$ and different values of $w$.

solutions are practically reproduced for a certain range of $w$ values, as in the case of hybrid stars. Our results of stellar properties for quark stars are summarized in table 4. A point worth mentioning is the behavior of the quark star radius as a function of the energy density shown in figure 5c, very similar to the mass-energy density, due to the fact that quark stars present a finite density at their surface contrary to hadronic and hybrid stars, which have zero densities at zero pressure surface points. These different physical structures are seen when one compares figures 1a and 4a with figure 5a.
Figure 5. Quark star properties obtained with $\lambda = 10^{38}$ dyn/cm$^2$ and different values of $w$: (a) mass-radius curves, (b) mass in function of the central energy density and (c) radius versus central energy density.

5 Final conclusions

In the present work we have revisited the brane-TOV solutions discussed in [29, 32] for realistic equations of state normally used in the literature and compared the results with the ones obtained from the standard TOV solutions. In order to solve satisfactorily the brane-TOV equations we need an equation of state $p = p(\varepsilon)$ and additionally an equation of state-like relation $P = P(U)$. We have assumed that the Weyl terms obey the simplest relation $P = wU$. We have then chosen appropriate EOS for the description of hadronic, hybrid and quark stars as input to the brane-TOV equations. An interesting aspect related to our results is the fact that the brane tension $\lambda$ clearly controls the values of the maximum star masses, while $w$ influences the corresponding radii. We have established a range for $\lambda$ in between $3.89 \times 10^{36} < \lambda < 10^{38}$ dyn/cm$^2$. The lower limit is obtained in such a way that at least a 1.44 $M_\odot$ star can be achieved. On the other hand, there is a value for which the solutions encountered reproduce the standard TOV solutions. This fact means that, as far as the equations of state survive the observational constraints when the macroscopic properties are computed from the usual TOV equations, they are also suitable as input to the brane-TOV equations. Once the maximum brane tension value is attained, the results are practically independent of the value of $w$ for hadronic stars and very little dependent for hybrid and quark stars.
It is very important to make some comments on the possible values of neutron stars radii. Based on chiral effective theory, the authors of ref. [59] estimate the radii of the canonical $1.4 M_\odot$ neutron star to lie in the range $9.7–13.9$ km. More recently, two different analysis of five quiescent low-mass X-ray binaries in globular clusters resulted in different ranges for neutron star radii. The first one, in which it was assumed that all neutron stars have the same radii, predicted that they should lie in the range $R = 9.1^{+1.3}_{-1.5}$ [60]. The second calculation, based on a Bayesian analysis, foresees radii of all neutron stars to lie in between 10 and 13.1 km [61]. If one believes those are definite constraints, all hadronic, hybrid and quark stars with the choice of EOS studied in the present work survive the observational constraints for values of $w$ in between $-3 < w < 2$ (excluding an interval of $w$ values around $-0.6$) for $\lambda = 10^{37}$ dyn/cm$^2$). For other values of $w$, the brane-TOV solutions produce mass-radius results which fall within the same interval as the ones obtained within our chosen range. We can conclude that one advantage of using the brane-TOV equations is that the radii can be adjusted to smaller values, as seen, for instance in figure 1a.

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