BESSEL K SERIES FOR THE RIEMANN ZETA FUNCTION

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Abstract. This paper provides some expansions of the Riemann xi function, \( \xi \), as a series of Bessel K functions.

1. Introduction

Some expansions of the Riemann \( \xi \) function (and expansions of related functions) are provided in this article as sums of the form,

\[
\sum_{j=1}^{\infty} c_j(s) K_s(x_j),
\]

where \( K_s(x) \) denotes the Bessel K function. Theorem 2.1 and Theorem 3.1 use transformation formulas of theta functions. Conjecture 6.1 is based on a partition of unity different from the one Riemann used in his memoir. Partitions of unity, such as that employed by Riemann, are more or less equivalent in the derivation of the meromorphy and functional equation of \( \zeta(s) \), but we found only one of them that gave Bessel expansions. We suspect that among algebraic functions of \( \lambda \) there are others. One of these is given in the appendix.

Much of the appendix is a discussion of a curious zeros phenomenon which occurred in relation to partitions of unity that arose in the use of Hecke operators. Some of the numerical data is given. It hinted that an eigenform for Hecke operators would be interesting. We were able to construct an eigenform for weight one half Hecke operators indexed over odd indices, but we could not show that it converged.

Conjecture 6.1 is a provisional result, based on a conjectured upper bound on certain partition coefficients. The result holds up under numerical tests, and it looks like an elementary proof, somewhat lengthy, would establish the upper bounds in the same way as the elementary proofs establish the upper bounds of the standard partition function.

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As usual,

\[ \theta_2 = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} \]

\[ \theta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} \]

\[ \theta_4 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \]

\[ \lambda = \left( \frac{\theta_2}{\theta_3} \right)^4. \]

2. A first \( \zeta(s) \) Bessel Series

**Theorem 2.1.** Define a multiplicative function, \( a_n \), on prime powers by,

\[ a_n = \sigma_1(n) \]

\[ = 2^f \]

\[ n = p^f, \ p > 2 \]

\[ n = 2^f. \]

Then,

\[ \frac{s}{4\pi} \left( 2^{s+1} - 2^{\frac{s+1}{2}} \right) \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \sum_{n=1}^{\infty} a_n n^{-\frac{1+s}{2}} \sum_{m=0}^{\infty} (2m+1)^{\frac{s+1}{2}} K_{\frac{s+1}{2}} \left( \pi \sqrt{n(2m+1)} \right). \]
Proof. For $\sigma > 0$,

\[
(1 - 2^{1-s})\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{2} \int_0^\infty y^{\frac{s}{2}-1}(\theta_4 - 1)\,dy
\]

\[
= \frac{1}{2} \int_0^\infty y^{\frac{s}{2}-1}(-\theta_4 + 1)\,dy
\]

\[
= \frac{1}{s} \int_0^\infty y^\frac{s}{2}\frac{\theta_4}{\theta_4}\,dy
\]

\[
= \frac{1}{s} \int_0^\infty y^\frac{1-s}{2}\frac{\theta_4}{\theta_4}(y)\frac{1}{\sqrt{y}}\frac{\theta_2}{\theta_2}\left(\frac{1}{y}\right)\,dy
\]

\[
= \frac{1}{s} \int_0^\infty y^{\frac{1-s}{2}}\frac{\theta_4}{\theta_4}(y)\frac{\theta_2}{\theta_2}\left(\frac{1}{y}\right)\,dy
\]

\[
= \frac{1}{s} \sum_{n=1}^\infty (2\pi) a_n \sum_{m=0}^\infty 2 \int_0^\infty y^{\frac{1+s}{2}-1} e^{-\pi (ny + (2m+1)^2)}\,dy
\]

\[
= \frac{4\pi}{s} \sum_{n=1}^\infty a_n \sum_{m=0}^\infty \left(\frac{2m+1}{2\sqrt{n}}\right)^{\frac{s+1}{2}} \int_0^\infty x^{\frac{1+s}{2}-1} e^{\pi\sqrt{(2m+1)}(x+\frac{1}{2})}\,dx
\]

\[
= \frac{4\pi}{s} \sum_{n=1}^\infty a_n \sum_{m=0}^\infty \left(\frac{2m+1}{2\sqrt{n}}\right)^{\frac{s+1}{2}} 2K_{\frac{s+1}{2}}(\pi\sqrt{n}(2m+1))
\]

\[
= \frac{8\pi}{s} \sum_{n=1}^\infty a_n \sum_{m=0}^\infty \left(\frac{2m+1}{2\sqrt{n}}\right)^{\frac{s+1}{2}} K_{\frac{s+1}{2}}(\pi\sqrt{n}(2m+1)),
\]

and the result follows on multiplying both sides by

\[
\frac{s}{8\pi} 2^{\frac{s+1}{2}}.
\]

The result provides a rapidly convergent series for $\zeta(s)$. How it would compare with the methods currently in use calculating zeros we cannot say. We include this result, and some others like it, as a comparison to the kinds of expansions using a partition of unity mentioned in the introduction.

3. A second $\zeta(s)$ Bessel Series

Define multiplicative functions, $A = a_n$ and $B = b_m$, on prime powers by,

\[
a_n = \sigma_1(n) \quad n = p^f, \quad p > 2 \\
= 2^f \quad n = 2^f,
\]
Define
\[ c_j(s, r) = \sum_{d|j} a_r(d)b_r \left( \frac{j}{d} \right) \left( \frac{j}{d^2} \right)^{s/2} \]

In this case,
\[ c_j(s) = \prod_{p^f||j} c_{p^f}(s) \]
is multiplicative in \( j \) for all \( s \) and we have,

**Theorem 3.1.**
\[
\frac{s(s + 1)}{32\pi^2 \sqrt{2}} \left( 2^{\frac{s+1}{2}} - 2^{\frac{s}{2}} \right) \left( 2^{\frac{s+1}{2}} - 2^{\frac{s-1}{2}} \right) \zeta^*(s) \zeta^*(s+1) = \sum_{j=1}^{\infty} c_j(s) K_s(2\pi\sqrt{j}) \]

**Proof.** The proof of theorem 3.1 is very much like the proof of theorem 2.1. Replace \( \theta_4 \) with \( \theta_4^4 \) on the right hand side. Use the formula for the number of representations of an integer as the sum of four squares on the left hand side. \( \Box \)

The asterisks indicate the usual,
\[
\zeta^*(s) = \pi^{-\frac{s}{2}} \Gamma\left( \frac{s}{2} \right) \zeta(s)
\]
\[
\zeta^*(s + 1) = \pi^{-\frac{s+1}{2}} \Gamma\left( \frac{s+1}{2} \right) \zeta(s + 1),
\]
and the compound zeta function is essentially the zeta function of \( r = 4 \), where \( r \) is the integer variable introduced in section 4. We mention also another BesselK expansion of zeta expansion which originates from \( \theta_4^8 \). We skip the details.

Regarding theorem 3.1, there is an additional result that we can prove about the zeros of \( c_k(s) \).

**Theorem 3.2.** For any \( n \), all the zeros of \( c_n(s) \) are pure imaginary.

**Proof.** Since \( c_n(s) \) is multiplicative, we only need to look at the zeros of \( c_{p^k}(s) \) where \( p \) is prime. If \( p = 2 \) there are no zeros of
\[
c_{2^m}(s) = 2^m 2^{-ms/2}.
\]
For \( p \) an odd prime, the equation for \( c_{p^k}(s) \) is as follows
\[
c_{p^k}(s) = \sum_{j=0}^{k} \frac{p^{j+1} - 1}{p - 1} \frac{p^{k-j+1} - 1}{p - 1} p^{(k-2j)s/2}.
\]
Now we will perform a couple of changes of variable in turn:

\[ z = p^{s/2} \]
\[ e^{i\phi} = z \]

Note that \( s \) is pure imaginary exactly when \( z \) is on the unit circle which also corresponds to real \( \phi \) between 0 and 2\( \pi \). We will use the following approximation of \( c_{pk}(s) \),

\[
c_{pk}(s) \approx \frac{p^{k+2}}{(p - 1)^2} \sum_{j=0}^{k} z^{(k-2j)} = \frac{p^{k+2}}{(p - 1)^2} \frac{z^{k+1} - z^{-k-1}}{z - z^{-1}}.
\]

Note that the formula on the right has all its zeros when \( z \) is on the unit circle. We will attempt to use this approximation to show that the zeros of \( c_{pk}(s) \) only occur when \( z \) is on the unit circle. To make this argument we use the following estimate for all \( z \) on the unit circle:

\[
\left| \frac{(p - 1)^2}{p^{k+2}} \frac{z - z^{-1}}{2i} c_{pk}(s) - \frac{z^{k+1} - z^{-k-1}}{2i} \right| = \left| \frac{z - z^{-1}}{2i} \sum_{m=0}^{k} \left( \frac{1}{p^{k+2}} - \frac{1}{p^{m+1}} - \frac{1}{p^{k-m+1}} \right) z^{k-2m} \right| < \sum_{m=0}^{k} \left( \frac{1}{p^{k+2}} - \frac{1}{p^{m+1}} - \frac{1}{p^{k-m+1}} \right) < 1
\]

If we now make the change of variable

\[ z = e^{i\phi} \]

then this inequality indicates that

\[
\frac{(p - 1)^2}{p^{k+2}} \sin \phi c_{pk}(s)
\]

differs from

\[
\sin (k + 1)\phi
\]

by less than one. This means that for each integer \( m \) in the interval

\[ 0 < m < 2(k + 1) \]

where \( m \neq k + 1 \) we have a zero for \( c_{pk}(s) \) for some \( \phi \) in the range

\[
\frac{1}{k + 1} \left( m - \frac{1}{2} \right) \pi < \phi < \frac{1}{k + 1} \left( m + \frac{1}{2} \right) \pi.
\]

Since \( c_{pk}(s) \) is \( z^{-k} \) times a degree \( 2k \) polynomial in \( z \) this accounts for all the zeros of \( c_{pk}(s) \). □
4. A BRIEF NOTE

For the purposes of this paper, it will be convenient to define the functions, $\theta_2$, $\theta_3$, $\theta_4$ and $\lambda$ as a function of $y$ where $\tau = iy$ is always pure imaginary. For a positive integer, $r$, put

$$\theta^r = 1 + 2r \sum_{n=1}^{\infty} u_r(n) q^n \quad u_r(1) = 1 \quad q = e^{i\pi \tau},$$

and its associated zeta function (valid for $\sigma > 1$),

$$\zeta^*(s, r) = \frac{1}{2r} \int_{0}^{\infty} y^{\frac{rs}{2} - 1} (\theta^r - 1) dy \quad (\tau = iy, y > 0)$$

$$= \pi^{-\frac{sr}{2}} \Gamma\left(\frac{rs}{2}\right) \sum_{n=1}^{\infty} \frac{u_r(n)}{n^{\frac{rs}{2}}} \quad (u_r(1) = 1)$$

$$= \pi^{-\frac{sr}{2}} \Gamma\left(\frac{rs}{2}\right) \zeta(s, r).$$

Central to much that follows are properties of the modular function, $\lambda$. It provides a particular partition of unity that allows an analytic continuation of the $\zeta^*(s, r)$ to a form that readily transitions to Bessel K functions. We examined many $\lambda$ related partitions of unity and found only one that returned a Bessel expansion like Conjecture 6.1. (There may be others. This will be discussed in an appendix.)

5. A Partition of Unity

The function, $\zeta^*(s, r)$, may be expressed,

$$\zeta^*(s, r) = \frac{1}{2r} \int_{0}^{\infty} y^{\frac{rs}{2} - 1} \left(\theta^r - 1 - \frac{1}{y^{\frac{rs}{2}}}\right) dy \quad (0 < \sigma < 1)$$

on the critical strip, and we combine this representation with the partition of unity,

$$1 = (1 - \lambda(y)) + \lambda(y)$$

$$= \lambda \left(\frac{1}{y}\right) + \lambda(y),$$

so that if

$$F_r = \theta^r(1 - \lambda),$$
then two integrations by parts gives two meromorphic expressions in lines 3 and 4 below:

\[
2r\zeta^*(s, r) = \int_0^\infty y^\frac{r}{2} \left( \theta^r (1 - \lambda) - 1 \right) dy + \int_0^\infty y^{-\frac{r}{2}} (\theta^r \lambda - \frac{1}{y^2}) dy
\]

\[
= \int_0^\infty y^\frac{r}{2} - 1 (F_r(y) - 1) dy + \int_0^\infty y^{r(1-s)-1} (F_r(y) - 1) dy
\]

\[
= -\frac{2}{rs} \int_0^\infty y^\frac{r}{2} \hat{F}_r dy - \frac{2}{r(1-s)} \int_0^\infty y^{r(1-s)} \hat{F}_r dy
\]

\[
= \frac{4}{r^2 s (s-1)} \int_0^\infty \left( y^{-\frac{(1-s)}{2}} + y^{-\frac{r}{2}} \right) (\hat{F}_r y^{\frac{r}{2}+1})' dy.
\]

We have chosen not to work with the entire function that comes from line 4,

\[
Z(s, r) = \frac{r^3}{2} s(s-1) \zeta^*(s, r)
\]

\[
= \int_0^\infty \left( y^{-\frac{(1-s)}{2}} + y^{-\frac{r}{2}} \right) (\hat{F}_r y^{\frac{r}{2}+1})' dy
\]

\[
\quad = \int_0^\infty \left( y^{\frac{r}{2}+1} + y^{r(1-s)+1} \right) \hat{F}_r dy
\]

\[
\quad \quad + \left( \frac{r}{2} + 1 \right) \int_0^\infty \left( y^{\frac{r}{2}} + y^{r(1-s)} \right) \hat{F}_r dy,
\]

but rather with,

\[
r^2\zeta^*(s, r) = -\left( \frac{\Psi_r(s)}{s} + \frac{\Psi_r(1-s)}{1-s} \right),
\]

where

\[
\Psi_r(s) = \int_0^\infty y^\frac{r}{2} \hat{F}_r dy,
\]

is the entire function that comes from line 3 (and one integration by parts) in the sequence above for \( \zeta^*(s, r) \). One integration by parts will introduce one set of coefficients, \( a_r(n) \), to consider in upcoming formulas, while two integrations involves two sets of coefficients, \( a_r(n), a'_r(n) \), and a bit more bookkeeping. In the body of the paper we stick with the simpler single integration, and present three expansions for \( r = 1, 2, 3 \). While two integrations give pure Bessel expansions in all cases, in the absence of any insight, we go for the method with fewer coefficients.
Define sequences, \( a_r(n) \) and \( b_r(m) \), as the coefficients in the expansions,

\[
\begin{align*}
4 \frac{\theta_4'}{\theta_4} - (4 - r) \frac{\theta_3'}{\theta_3} &= \pi \sum_{n=1}^{\infty} a_r(n) e^{-\pi n y} \quad (a_r(1) = -16 + 2r) \\
\theta_3' \lambda &= \sum_{m=1}^{\infty} b_r(m) e^{-\pi m y}, \quad (b_r(1) = 16)
\end{align*}
\]

where the derivatives are taken with respect to \( y \). We note that the coefficients of the log derivatives of \( \theta_3 \) and \( \theta_4 \) have a very simple form:

\[
a_r(n) = (-4 - (4 - r)(-1)^{n+1}) \sigma_1^*(n)
\]

where

\[
\sigma_1^*(n) = \begin{cases} 
  n & \text{if } n \text{ is a power of 2} \\
  \sigma_1(n) & \text{otherwise}
\end{cases}
\]

6. A Provisional Result

**Conjecture 6.1.**

\[
\zeta^*(s, r) = -\left( \frac{\Psi_r(s)}{s} + \frac{\Psi_r(1 - s)}{1 - s} \right),
\]

where,

\[
\Psi_r(s) = 2\pi \sum_{j=1}^{\infty} c_j(s, r) K_{\frac{r + 2 - r}{2}}(2\pi \sqrt{j})
\]

\[
c_j(s, r) = \sum_{d|j} a_r(d) b_r \left( \frac{j}{d} \right) \left( \frac{j}{d^2} \right)^{\frac{r + 2 - r}{4}} \quad j \geq 1
\]
Proof. We have,
\[
\Psi_r(s) = \int_0^\infty y^{\frac{r}{2}} \tilde{F}_r(y) \, dy
\]
\[
= \int_0^\infty y^{\frac{r}{2}} \left( \frac{\theta_4}{\theta_4} - (4 - r) \frac{\theta_3}{\theta_3} \right) \tilde{F}_r(y) \, dy
\]
\[
= \int_0^\infty y^{\frac{r}{2}} \left( \frac{\theta_4}{\theta_4} - (4 - r) \frac{\theta_3}{\theta_3} \right) \frac{1}{y^2} \theta^r \left( \frac{1}{y} \right) \lambda \left( \frac{1}{y} \right) \, dy
\]
\[
= \sum_{m,n=1}^{\infty} a_r(n) b_r(m) \int_0^\infty y^{\frac{r+2-r}{2}-1} e^{-\pi (ny + \frac{m}{y})} \, dy
\]
\[
= \sum_{m,n=1}^{\infty} a_r(n) b_r(m) \left( \frac{m}{n} \right)^{\frac{r+2-r}{4}} \int_0^\infty y^{\frac{r+2-r}{2}-1} e^{-\pi \sqrt{mn}(y + \frac{1}{y})} \, dy
\]
\[
= 2 \sum_{m,n=1}^{\infty} a_r(n) b_r(m) \left( \frac{m}{n} \right)^{\frac{r+2-r}{4}} K_{\frac{r+2-r}{2}} \left( 2\pi \sqrt{mn} \right),
\]
where we recall,
\[
2K_w(x) = \int_0^\infty u^{w-1} e^{-\frac{x}{2} (u + \frac{1}{u})} \, du, \quad x > 0.
\]

The result follows from Dirichlet convolution, collecting \(m\) and \(n\) according to, \(j = mn\). \(\Box\)

The result is provisional upon the justification of the interchange of the sum and integral in the fourth step.

7. Upper Bounds

We conjecture that for any \(\epsilon > 0\), then
\[
b_1(m) \leq e^{\pi \sqrt{3m}(1+\epsilon)} \quad m \geq m(\epsilon)
\]
\[
b_2(m) \leq e^{\pi \sqrt{2m}(1+\epsilon)} \quad m \geq m(\epsilon)
\]
\[
b_3(m) \leq e^{\pi \sqrt{m}(1+\epsilon)} \quad m \geq m(\epsilon).
\]

We have numerical evidence for this conjecture and it conforms with the elementary estimates as obtained in [2].

We recall the estimate,
\[
K_w(2\pi \sqrt{m}) \sim \left( \frac{1}{4\sqrt{m}} \right)^{\frac{1}{4}} e^{-2\pi \sqrt{m}},
\]
so that the $-2\pi\sqrt{m}$ swamps the $\pi\sqrt{3m}$, the $\pi\sqrt{2m}$, and the $\pi\sqrt{m}$, and the absolute convergence of,

$$\sum_{m=1}^{\infty} b_r(m) K_{s+\frac{1}{2}}(2\pi\sqrt{m}),$$

holds for $r = 1, 2, 3$, which justifies the interchange of sum and integral.

Had we applied the inversion,

$$y \rightarrow \frac{1}{y}$$

to $\lambda$ alone, the interchange of limits could not be justified. The coefficients, $b_0(m)$,

$$\lambda(\tau) = \sum_{m=1}^{\infty} b_0(m) q^m,$$

of $\lambda$, as given in Simons paper [4], (eq 4), satisfy

$$b_0(m) \sim \frac{\pi}{8\sqrt{m}} I_1(2\pi\sqrt{m}).$$

As

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \quad x \rightarrow \infty,$$

then,

$$\sum_{m=1}^{\infty} b_0(m) K_w(2\pi\sqrt{m})$$

does not converge absolutely.

8. $r = 1$

From the formulas above for general $r \geq 1$, there follows for $r = 1$,

$$\Psi_1(s) = \int_{0}^{\infty} y^{\frac{s}{2}} F_1 dy$$

$$= \pi \int_{0}^{\infty} y^{\frac{s+1}{2}} \sum_{n=1}^{\infty} a_1(n)e^{-\pi ny} \sum_{m=1}^{\infty} b_1(m)e^{-\pi m} dy$$

$$= \pi \int_{0}^{\infty} y^{\frac{s-1}{2}} \sum_{n=1}^{\infty} a_1(n) \sum_{n=1}^{\infty} b_1(m)e^{-\pi (ny + \frac{m}{n})} dy$$

$$= \pi \sum_{n=1}^{\infty} a_1(n) \sum_{m=1}^{\infty} b_1(m) \left(\frac{m}{n}\right)^{s+\frac{1}{2}} \int_{0}^{\infty} y^{\frac{s+1}{2}} e^{-\pi \sqrt{mn}(y+\frac{1}{n})} dy$$

$$= 2\pi \sum_{n=1}^{\infty} a_1(n) \sum_{m=1}^{\infty} b_1(m) \left(\frac{m}{n}\right)^{s+\frac{1}{4} + \frac{1}{2}} K_{s+\frac{1}{2}}(2\pi\sqrt{mn}).$$
The inclusion of the weight one half form, $\theta$, as a factor of $\lambda$, reduces the growth of the $b_1(m)$ significantly, and, as we have conjectured, the sum,

$$\sum_{m=1}^{\infty} b_1(m) K_{\frac{s+1}{2}}(2\pi \sqrt{m})$$

is absolutely convergent. Having obtained a Bessel K expansion of the zeta function, the coefficients, $b_1(m)$, are then of prime interest and we followed the method of Simons paper to obtain the partition expression of these coefficients. Here we encountered major and minor differences in the $r = 1$ calculations from Simons $r = 0$ calculation (and from the classical partition expansion).

In contrast with the $r = 0$ case the sum for $r = 1$ in the next section does not converge absolutely. In fact we did not prove that it converges. Numerical tests indicate that it does. We evaluated the arithmetic components of the terms in elementary terms; i. e., sines and hyperbolic sines. The case of $r = 2$ is like $r = 1$ though the arithmetic components were not all elementary.

**Conjecture 8.1.** The coefficients of $\theta \lambda$ are given by,

$$b_1(m) = \frac{2\pi}{8\sqrt{2}} \left( \frac{3}{m} \right)^{\frac{1}{4}} \sum_{k \equiv 2 \pmod{4}} A_k(m) I_{\frac{1}{2}} \left( \frac{2\pi \sqrt{3m}}{k} \right) + \delta_1(m)$$

where,

$$A_k(m) = -i \sum_{(h,k)=1} \kappa(\gamma^{-1}) e^{-2\pi i \left( \frac{4mh + 3h'}{4k} \right)}$$

$$\delta_1(m) = \begin{cases} 
1 & \text{if } m \text{ is a square} \\
0 & \text{otherwise}
\end{cases}$$

and where, $\kappa(\gamma^{-1})$, is a theta multiplier.

This conjecture was obtained using a method similar to that used in [4].

**Theorem 8.2.** Let $k = 2l$, $l$ odd. The $A_k(m)$ may be factored as,

$$A_k(m) = i^{\frac{(l+3)(l-1)}{4}} (-1)^{m-1} S(m, -3B_l^4, l) \quad (B_l = \frac{l+1}{2})$$

where $S$ is a Salie sum defined as,

$$S(a, b, l) = \sum_{(h,l)=1} \left( \frac{h}{l} \right) e^{2\pi i \left( ah + bh \right)}.$$
Proof. Using Rademacher’s explicit transformation formula \[3\] for the \(\theta_3\)-multiplier, it is a straight-forward calculation to show that the formula holds.

The Bessel function, \(I_{\frac{1}{2}}\), is a hyperbolic sine. This fact combined with conjecture 8.1 and theorem 8.2, gives a final form for \(b_1(m)\):

\[
b_1(m) = \frac{\pi}{4\sqrt{2}} \left( \frac{3}{m} \right)^{\frac{1}{4}} (-1)^{m-1} \sum_{l \geq 1 \text{ odd}} i^{(l+3)(l-1)} \frac{1}{2l} S(m, -3B_4^4, l) I_{\frac{1}{2}} \left( \frac{\pi \sqrt{3m}}{l} \right) + \delta_1(m) \]

\[
= \frac{1}{8\sqrt{m}} (-1)^{m-1} \sum_{j=0}^{\infty} i^{j(j+2)} \frac{1}{\sqrt{2j+1}} S(m, -3(j+1)^4, 2j+1) \sinh \left( \frac{\pi \sqrt{3m}}{2j+1} \right) + \delta_1(m). \]

The bulk of the remaining calculations are the determination of the Salie sums as trigonometric expressions.

9. \(S(a, b, l)\) Evaluation

When the Salie sums are not zero they are simple trigonometric functions after a \(\sqrt{l}\) is extracted. We saw that the exponential sum,

\[
S(a, b, l) = \sum_{(h, l) = 1} \left( \frac{h}{l} \right) e^{\frac{2\pi i (ah + bh)}{l}},
\]

evaluates the \(A_k(m), k = 2l,\) as,

\[
A_k(m) = i^{(l+3)(l-1)} (-1)^{m-1} S(m, -3B_4^4, l).
\]

We must consider all choices of \(m\) and \(l\), in which \(m\) and \(l\) might or might not share factors of 3 and might or might not share an odd factor other than 3. All possibilities can occur in the \(A_k(m)\). Thus,

\[
m = 3^e \times l_1 \times a \quad (a, 3l) = 1
\]
\[
(l_1 l_1', 3) = 1,
\]

where \(l_1\) and \(l_1'\) are the factors of \(m\) and \(l\) that share primes other than 3. The combinations of \(e\) and \(f\) that modify the values of \(S\) do so in more complicated arrangements for \(p = 3\) than for the other primes, due to the, 3, in \(S(m, -3B_4^4, l)\). As we saw, this 3 comes from the first power of \(\theta\) in the formula for the coefficients of \(\theta \lambda\). 3 is no longer special for any higher power of \(\theta\). At an exceptionally primitive level of understanding, it is strange that any prime would mediate differently between the coefficients of \(\theta^r\) and the zeros of its Mellin transform.
The effect of $3$ disappears for $\theta^2$, which also looks to satisfy a Riemann hypothesis.

10. SOME GENERAL RESULTS

We observe the following general results:

1. $S(a, b, l) = S(b, a, l)$
2. $S(ca, b, l) = \left(\frac{c}{l}\right)S(a, cb, l)$ \quad ($\left(\frac{c}{l}\right) = 1$)
3. $S(a, b, l) = S(a + ml, b + nl, l)$
4. $S(a, b, l) = S(a\bar{l}_1, b\bar{l}_1, l_2)S(a\bar{l}_2, b\bar{l}_2, l_1)$, \quad ($l = l_1l_2, (l_1, l_2) = 1$)

It follows from 1 and 2 that if $a = b = g$, then

$$S(cg, g, l) = 0 \quad \text{if} \left(\frac{c}{l}\right) = -1.$$ 

If $c = g_1g$, $(g_1g, l) = 1$, then

$$S(g_1, g, l) = 0 \quad \text{if} \left(\frac{g_1g}{l}\right) = -1.$$ 

11. A USEFUL THEOREM

A result of [1] evaluates many cases:

Theorem 11.1. Let $(l, 2b) = 1$. Then,

$$S(a, b, l) = \left(\frac{b}{l}\right)\epsilon_1\sqrt{l} \sum_{y^2 \equiv ab \pmod{l}} e^{\frac{2\pi iy}{l}}$$

Corollary 11.2. Let $p \geq 3$ be a prime. If $(a, p) = 1$, $e \geq 1$, and $f \geq 2$, then,

$$S(a, pf^b, pf^f) = 0$$

12. $p > 3$

For a prime, $p > 3$ dividing $l_1$ and $l'_1$, a sum of the form, $S(p^e a, 3b, pf^f), (b, p) = 1$, will be a factor of $S(m, -3B^4, l) = S(3^e l_1a, -3B^4, 3f \times l'_1 \times c)$. The few cases are contained in the following result.

Theorem 12.1. Let $p > 3$ be a prime and let $S = S(p^e a, 3b, pf^f), (p, ab) = 1$. Then,
\[e \geq 1, \ f \geq 2, \ S = 0\]
\[e \geq 1, \ f = 1, \ S = \left(\frac{3b}{p}\right)\epsilon_p\sqrt{p}\]
\[e = 0, \ f \geq 1, \ S = \left(\frac{3b}{p}\right)\epsilon_p\sqrt{p} \sum_{y^2 \equiv -3ab(p^f)} e^{\frac{2\pi iy}{p^f}}\]

**Proof.** All of these cases are contained in the Iwaniec formula. We separated the ones that vanished, and the simple Gauss sum, from the last case. \(\square\)

13. \(p = 3\)

The combination of powers of 3 dividing \(m\) and \(l\) can be grouped according to \(e \geq 0, 1, 2\), where we have let,

\[m = 3^e \times l_1 \times a\quad (a, 3l) = 1\]
\[l = 3^f \times l'_1 \times c\quad (c, 3m) = 1\quad (l_1l'_1, 3) = 1.\]

The (twisted) multiplicativity result, 4, of the note, shows that a sum of the type, \(S = S(3^e a, 3b, 3^f)\), is a factor of \(S(3^e l_1 a, -3B^4, 3^f \times l'_1 \times c)\). The situation regarding this factor is given in the following result.

**Theorem 13.1.** Let \(S = S(3^e a, 3b, 3^f)\). Then,

\[e \geq 2, \ f = 2, \ S = -3\]
\[e \geq 2, \ f \geq 3, \ S = 0\]
\[e \geq 1, \ f = 1, \ S = 0\]
\[e = 1, \ f = 2, \ S = 6, \ \left(\frac{ab}{3}\right) = -1\]
\[e = 1, \ f = 2, \ S = -3, \ \left(\frac{ab}{3}\right) = 1\]
\[e = 1, \ f \geq 3, \ S = 0 \quad \text{ab is not a square mod } 3^f\]
\[e = 1, \ f = 2n, n \geq 2, \ S = -\left(\frac{g}{3}\right) 2 \times 3^n \sqrt{3} \sin\left(\frac{4\pi g}{32^{n-1}}\right) \quad g^2 \equiv ab(3^{2n})\]
\[e = 1, \ f = 2n - 1, n \geq 2, \ S = 2i \left(\frac{g}{3}\right) 3^n \sqrt{3} \sin\left(\frac{4\pi g}{32^{n-2}}\right) \quad g^2 \equiv ab(3^{2n-1})\]
\[e = 0, \ f \geq 2, \ S = 0\]
\[e = 0, \ f = 1, \ S = \left(\frac{a}{3}\right) \sqrt{3}i\]
Proof. The two $e = 0$ cases are direct results of Theorem 11.1. The results for $e \geq 1$ follow from a formula that generalizes Theorem 11.1 in an obvious way. We worked out each of the cases that can occur and presented them in their simplest form. □

14. $r = 2$

With $r = 2$, then $\alpha_2(s) = s$, and

$$
\zeta^*(s, 2) = \frac{1}{4} \int_0^\infty y^{s-1}(\theta^2 - 1)dy
= \pi^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{u_2(n)}{n^s}
= \pi^{-s} \Gamma(s) \zeta(s, 2)
= \pi^{-s} \Gamma(s) \zeta(s)L(s, \chi),
$$

Theorem 14.1.

$$
4\zeta^*(s, 2) = -\left(\frac{\Psi_2(s)}{s} + \frac{\Psi_2(1-s)}{1-s}\right),
$$

$$
\Psi_2(s) = 2\pi \sum_{j=1}^{\infty} c_j(2, s) K_s(2\pi \sqrt{j})
$$

$$
c_j(2, s) = \sum_{d|j} a_2(d)b_2 \left(\frac{j}{d}\right) \left(\frac{j}{d^2}\right)^{\frac{s}{2}}
$$

Conjecture 14.2.

$$
b_2(m) = \frac{\pi}{2} \sum_{k \equiv 2 (mod 4)} \frac{A_k(m)}{k} I_0 \left(\frac{2\pi \sqrt{2m}}{k}\right) + 2\delta_2(m)
$$

where

$$
\frac{\theta^2 - 1}{4} = \sum_{n=1}^{\infty} \delta_2(m) q^m,
$$

and

$$
A_k(m) = -\sum_{(h,k)=1} \kappa^2(\gamma^{-1}) e^{-2\pi i \frac{h'}{\pi} + \frac{m h}{k}}
$$

is the arithmetic factor with the same multiplier, $\kappa(\gamma^{-1})$, as before, but squared.
Theorem 14.3.  

\[ A_k(m) = (-1)^{(l-1)/2}(-1)^{m-1}T(-2mA^2, A^2, l), \quad (k = 2l, A = \frac{l+1}{2}) \]

where \( T \) is the Kloosterman sum:  

\[ T(a, b, l) = \sum_{(h,k)=1} e^{\frac{2\pi i}{l}(ah+bh)}. \]

Combining conjecture 14.2 and theorem 14.3 above provides a final evaluation for the \( b_2 \):  

\[ b_2(m) = \frac{\pi}{4}(-1)^{m-1} \sum_{j=0}^{\infty} \frac{T(-2m(j+1)^2, (j+1)^2, 2j+1)}{2j+1} + 2\delta_2(m). \]

15. \( r = 3 \)

The function,  

\[ \zeta^*(s, 3) = \frac{1}{6} \int_0^\infty y^{s-1} (\theta^3 - 1) dy \]

has the Bessel K series expansion,  

Theorem 15.1.  

\[ 6\zeta^*(s, 3) = -\left( \frac{\Psi_3(s)}{s} + \frac{\Psi_3(1-s)}{1-s} \right), \]

\[ \Psi_3(s) = 2\pi \sum_{j=1}^{\infty} c_j(3, s) K_{\frac{3s-1}{2}}(2\pi j) \]

\[ c_j(3, s) = \sum_{d|j} a_3(d) b_3 \left( \frac{j}{d} \right) \left( \frac{j}{d^2} \right)^{\frac{3s-1}{2}} \]

Following the same steps in the calculations for \( b_1 \) and \( b_2 \) we obtained the expression,  

\[ b_3(m) = \sqrt{2\pi} \left( m^{\frac{1}{2}} \sum_{k=2 \text{(mod 4)}} \frac{A_k(m)}{k} I_{-\frac{1}{2}} \left( \frac{2\pi \sqrt{m}}{k} \right) \right) + 3\delta_3(m). \quad (15.1) \]

where  

\[ \frac{\theta^3 - 1}{6} = \sum_{n=1}^{\infty} \delta_3(m) q^m, \]

and  

\[ A_k(m) = -i \sum_{(h,k)=1} \kappa^3(\gamma^{-1}) e^{-2\pi i (k/m + m/h)} \]

We do not believe that the sum in equation 15.1 converges.
Theorem 15.2.

\[ A_k(m) = (-1)^{m-1} i^{l+1} S(-4mA^3, A^3, l) \quad (A = \frac{l+1}{2}, k = 2l) \]

where

\[ S(a, b, l) = \sum_{(h,l)=1} \left( \frac{h}{l} \right) e^{2\pi i (ah+bh)}. \]

Appendix A.

Assume that for arithmetic sequences \( [a_n] \) and \( [b_n] \), the function

\[ \Psi(s) = \sum_{j=1}^{\infty} c_j(s) K_s(2\pi \sqrt{j}), \]

is entire, where, as defined earlier,

\[ c_j(s) = \sum_{d|j} a_j b_{d/j} \left( \frac{j}{d^2} \right)^{\frac{s}{2}} \quad j \geq 1 \]

For the special sequences that appear in Theorem 3.1, numerical tests for the zeros of \( \Psi \) gave what was expected: zeros on four vertical lines, \( \sigma = \frac{1}{2}, \frac{1}{2}, 0, 1 \). For \( a_n = b_n = 1 \), the zeros of \( \Psi \) that we found were pure imaginary to within machine precision. Other pairs of multiplicative sequences gave zeros that were sporadically distributed. A rationale for which sequences produced highly organized zeros for the associated \( \Psi \) is an attractive puzzle. We considered briefly an operator that would encompass the second order equation satisfied by \( K_s \), but as that approach would seem to produce a complete understanding of primes, we dropped it except for a curious property of Hecke operators which we take up in Appendix B.

Appendix B. Other partitions of unity

There are other partitions of unity that may be used to provide Bessel expansions for the Riemann Zeta function. For a modular function of weight zero to be a partition of unity it must satisfy the functional equation

\[ f(\tau) + f\left( -\frac{1}{\tau} \right) = 1. \]

If \( f \) is such a partition of unity then this leads to a partition of \( \xi \) of the form

\[ 2\xi(s) = (1-s)\Psi_f(s) + s\Psi_f(1-s) \quad (B.1) \]
where
\[ \Psi_f(s) = \int_0^\infty x^{s/2} \left( \frac{d}{dx} \theta_3(ix)f(ix) \right) dx. \] (B.2)

One thing that we considered when we examined partitions of unity was patterns of the zeros of the associated \( \Psi(s) \). As a first example, we will consider the zeros of \( \Psi_{1-\lambda} \):

\[ -6.85993 + 18.5302i \quad 8.13082 + 18.4858i \\
-7.17924 + 28.5267i \quad 8.33654 + 28.5203i \\
-8.228 + 36.2544i \quad 9.40533 + 36.2529i \\
-8.20765 + 42.7431i \quad 9.33192 + 42.7147i \\
-8.69688 + 49.6138i \quad 9.82173 + 49.6313i \\
-9.19901 + 55.0839i \quad 10.3377 + 55.0643i \]

A large class of partitions of unity may be obtained by using weight one-half Hecke operators acting on weight one-half modular forms. For an odd prime, \( p \), the Hecke operator acting on a weight one-half acting on a modular function, \( g \), may be expressed as

\[ H_p(g)(\tau) = g(p^2\tau) + \frac{1}{\sqrt{p} \epsilon_p} \sum_{m=1}^{p-1} \left( \frac{m}{p} \right) g \left( \tau + \frac{2m}{p} \right) + \frac{1}{p} \sum_{m=0}^{p^2-1} g \left( \frac{\tau + 2m}{p^2} \right) \]

where
\[ \epsilon_p = \begin{cases} 1 & p \equiv 1(4) \\ i & p \equiv 3(4) \end{cases} \]

and
\[ \left( \frac{m}{p} \right) \]

is a Jacobi symbol. If \( \phi \) is a partition of unity then
\[ \frac{1}{p+1} \frac{1}{\theta_3} H_p(\theta_3\phi) \]

is also a partition of unity.

For example, if \( f_1(\tau) = \theta_3(\tau)(1 - \lambda(\tau)) \) then
\[ \frac{1}{4} \theta_3(f_1)(\tau) = \]
\[ = \frac{1}{4} \left( f_1(9\tau) - \frac{i}{\sqrt{3}} f_1 \left( \tau + \frac{2}{3} \right) + \frac{i}{\sqrt{3}} f_1 \left( \tau + \frac{4}{3} \right) + \frac{1}{3} \sum f_1 \left( \frac{\tau + 2k}{9} \right) \right) \]
\[ = \theta_3(\tau)(1 - \lambda(\tau))(1 - 12000\lambda(\tau) + 441792\lambda(\tau)^2 \]
\[ - 3350528\lambda(\tau)^3 + 9224192\lambda(\tau)^4 \]
\[ - 10485760\lambda(\tau)^5 + 4194304\lambda(\tau)^6). \]
equations in $\lambda(\tau)$ whose roots are partitions of unity.

At this point, we began some experiments with numeric evaluations of the $\Psi$ functions for these partitions of unity. We used Mathematica’s numeric integration technology to perform the integrations for equation B.2. We were a bit worried about the accuracy of such calculations because as the prime, $p$, gets larger, the behavior of the integrand becomes extremely oscillatory. For example, for $p = 3$ (see figure 1), the integrand goes between $-20,000$ to $+20,000$ but the curve is pretty smooth. But as the prime $p$ gets larger the behavior of the integrand
becomes more extreme. For example, figure 2 shows a plot of the integrand when $p = 11$.

We were a bit concerned that Mathematica could not accurately numerically calculate the value of $\Psi$ when the integrand was behaving in such an extreme manner. However when we tested our numerical values for $\Psi$ in the equation B.1 on page 17 and the results were very encouraging. For example, when $s = .5 + i$ numeric calculations showed

$$\xi(s) = -0.485757$$
$$\Psi_{11}(s) = 4.738132 - 2.85482i$$
$$2\xi(s) - ((1 - s)\Psi_{11}(s) + s\Psi_{11}(1 - s)) = 0$$

These results continued to be encouraging for other larger values of $s$. For example at $s = 50 + 50i$ we obtained

$$\xi(s) = -8.08211 \times 10^9 + 1.715833 \times 10^9i$$
$$\Psi_{11}(s) = -5.135 \times 10^{38} - 3.14968 \times 10^{38}i$$
$$2\xi(s) - ((1 - s)\Psi_{11}(s) + s\Psi_{11}(1 - s)) = -5.63493 \times 10^{-19} + 3.19468 \times 10^{-19}i$$

This seemed like pretty convincing evidence to us because it did not seem likely that Mathematica was using integration by parts in such a manner as to obtain erroneous results that were consistent with B.1.

It may be worth noting that in order to avoid errors in Mathematica we had to use some rather extreme values on the numeric integration precision. For example, we set Mathematica up to perform integration with a working precision of 200 places of accuracy and a precision goal of 30 places of accuracy. We noticed that with these settings Mathematica seemed to return results that were significantly more accurate than requested.

One thing that we looked at for partitions of unity generated by a Hecke operator was that in some cases, the zeros clustered around the one-half line. This behavior was not particularly apparent when we considered the $\Psi$-function associated with $\mathcal{H}_3$ of $1 - \lambda$. In this example we searched for zeros in the square between -50 and 50 + 40i:

$$0.235237 + 0.77205i$$
$$0.516168 + 3.6727i$$
$$0.498618 + 6.13523i$$
$$-21.1074 + 19.7716i$$
$$22.1074 + 19.7716i$$
$$-23.4598 + 29.768i$$
$$24.4599 + 29.768i$$
$$-25.2761 + 37.4269i$$
$$26.2762 + 37.4269i$$
But when we consider the Ψ-function associated with \( H_5 \) of \( 1 - \lambda \) and searched for zeros in the square from -50 to 50 + 50i:

\[
\begin{align*}
0.233242 + 0.437961i \\
0.519924 + 2.43977i \\
0.49621 + 4.48092i \\
0.500645 + 6.21101i \\
0.49978 + 9.29704i \\
0.500002 + 11.2699i \\
0.5 + 13.5882i \\
0.5 + 17.5795i \\
-34.6505 + 21.523i & 35.6505 + 21.523i \\
0.5 + 22.4764i \\
-38.8576 + 31.7487i & 39.8576 + 31.7487i \\
-41.6848 + 39.6315i & 42.6848 + 39.6315i \\
-43.8901 + 46.6857i & 44.8901 + 46.6857i \\
\end{align*}
\]

In this example, the occurrences of a real part of .5 does not mean that the zero is exactly on the one-half line; it is just means that such a zero is within machine precision of the one-half line. The zeros of Ψ-function associated with \( H_7 \) of \( 1 - \lambda \) also show this behavior. Here are the results of search for zeros in the box between -50 and 50 + 50i:

\[
\begin{align*}
0.234502 + 0.298452i \\
0.519389 + 2.01942i \\
0.495116 + 3.64979i \\
0.501157 + 5.24145i \\
0.49982 + 6.98016i \\
0.500017 + 9.04737i \\
0.499999 + 11.023i \\
0.5 + 12.9791i \\
0.5 + 15.0838i \\
-1.01888 + 18.9618i & 2.01888 + 18.9618i \\
0.5 + 22.496i \\
-47.6872 + 23.3824i & 48.6872 + 23.3824i \\
0.5 + 25.9916i \\
0.5 + 29.3926i \\
0.5 + 33.1953i \\
0.5 + 37.4991i \\
0.5 + 42.5267i \\
0.5 + 48.8834i \\
\end{align*}
\]

Finally, for the examples that we will present here the results of a search for the zeros of Ψ-function associated with \( H_{11} \) of \( 1 - \lambda \) in the box from
We wondered what would happen if there was a function that was eigenvector of some set of Hecke operators. We found a sum, which, formally, was an eigenvector for the Hecke operators for all odd primes but we could not prove that the series converged.

There are also partitions of unity that come from pieces of the Hecke operator. For example,

\[ f(\tau) = 1 - \lambda \left( \frac{\tau + a}{p} \right) \]

is a partition of unity if \( a \) is even, \( p \) is an odd prime and\[ a^2 = -1 \text{ mod } p.\]

This example can be generalized.

Such functions are algebraic functions of \( \lambda \) although their coefficients grow too quickly produce a Bessel K expansion of the \( \zeta \) function. Another class of algebraic functions of \( \lambda \) are also partitions of unity may be obtained using the lemma below. In some cases, these have the advantage that their coefficients go to infinity slowly enough that they may be used provide Bessel K expansions for the zeta function.
Lemma B.1. Suppose that \( R(x) \) and \( S(x) \) are polynomials of odd and even degrees respectively with real coefficients such that

- \( R(1-x) + R(x) = S(x) \)
- The polynomial \( R(x)S'(x) - R'(x)S(x) \) is zero only when \( x = 0 \) or \( x = 1 \)
- \( S(x) \) has no real zeros when \( 0 < x < 1 \)
- \( S(1) = R(1) \neq 0 \)

then there is a modular function over a finite index subgroup of the \( \Lambda \) group, \( \phi \), which satisfies the following conditions:

\[
R(\phi(\tau)) - \lambda(\tau)S(\phi(\tau)) = 0 \\
\phi(\tau) + \phi(-1/\tau) = 1 \\
\phi(0) = 1 \\
\phi(i\infty) = 0 \\
\phi(i) = 1/2.
\]

Proof. For any fixed \( y \), the equation,

\[ R(x) - yS(x) = 0, \]

will only have a double root when

\[ R'(x) - yS'(x) = 0. \]

This means that,

\[ R'(x)S(x) - R(x)S'(x) = 0, \]

so \( x \) must be either zero or one. It then follows that \( y \) is either zero or one.

Now if we consider \( x \) and \( y \) to be functions of \( \tau \) where \( x = \phi(\tau) \) and \( y = \lambda(\tau) \) then there are no double roots in the \( x \) variable of

\[ R(x) - yS(x) = 0, \]

when \( \tau \) is in the upper half plane. In addition, since the degree of \( R(x) \) is at least one more than the degree of \( S(x) \), \( x \) cannot go to infinity for \( \tau \) in the upper half plane.

Thus any solution for \( \phi(\tau) \), for any given \( \tau \) in the upper half plane, of the equations

\[ R(\phi(\tau)) - \lambda(\tau)S(\phi(\tau)) = 0 \quad (B.3) \]

can be extended to a single-valued root for all \( \tau \) in the upper half plane.

Now we will consider the behavior of

\[ \frac{R(x)}{S(x)} \]
as $x$ goes from zero to one in order to fix attention on a specific root of
\[ R(\phi(\tau)) - \lambda(\tau)S(\phi(\tau)) = 0. \]
We have
\[ \frac{R(1)}{S(1)} = 1 \]
and
\[ \frac{R(0)}{S(0)} = 0. \]
In addition for $x$ in the open interval from 0 to 1,
\[ \frac{d}{dx} \left( \frac{R(x)}{S(x)} \right) = \frac{R'(x)S(x) - R(x)S'(x)}{S(x)^2} \neq 0. \]
This means that
\[ \frac{R(x)}{S(x)} \]
is monotonically increasing for $x$ in the open interval from zero to one.
In addition,
\[ \frac{R(x)}{S(x)} + \frac{R(1-x)}{S(1-x)} = 1 \]
So
\[ \frac{R(1/2)}{S(1/2)} = 1/2. \]
We thus can choose the root of
\[ R(\phi(\tau)) - \lambda(\tau)S(\phi(\tau)) = 0. \]
where
\[ \phi(i) = 1/2. \quad (B.4) \]
Now we will use [B.4] to show that the solution, $\phi$, of equation [B.3] must be a partition of unity. The conditions of lemma [B.1] have been crafted so that if
\[ R(x_0) - y_0S(x_0) = 0 \]
then
\[ R(1 - x_0) - (1 - y_0)S(1 - x_0) = 0 \]
This means that
\[ R(1 - \phi(-1/\tau)) - (1 - \lambda(-1/\tau))S(1 - \phi(-1/\tau)) = 0 \]
or
\[ R(1 - \phi(-1/\tau)) - \lambda(\tau)S(1 - \phi(-1/\tau)) = 0 \]
So, by looking at what happens in a neighborhood of $\tau = i$ we get
\[ \phi(\tau) = 1 - \phi(-1/\tau). \]
This completes the proof. □

The algebraic equations

\[2\phi(\tau)^3 - 3\phi(\tau)^2 + \lambda(\tau) = 0\]
\[\phi(\tau)^5 - \lambda(\tau)(5\phi(\tau)^4 - 10\phi(\tau)^3 + 10\phi(\tau)^2 - 5\phi(\tau) + 1) = 0\]
\[2\phi(\tau)^7 - 7\phi(\tau)^5 + \lambda(\tau)(35\phi(\tau)^4 - 70\phi(\tau)^3 + 63\phi(\tau)^2 - 28\phi(\tau) + 5) = 0\]
\[2\phi(\tau)^{11} - 11\phi(\tau)^{10} + \lambda(\tau)(165\phi(\tau)^8 - 600\phi(\tau)^7 + 1386\phi(\tau)^6 - 1848\phi(\tau)^5 + 1650\phi(\tau)^4 - 990\phi(\tau)^3 + 385\phi(\tau)^2 - 88\phi(\tau) + 9) = 0\]

satisfy the conditions of lemma [B.1]. At this time, we don’t know if there are other similar equations that can be derived using lemma [B.1] or if a more sophisticated lemma would produce more roots.

We will now provide some expanded analysis of the first of these equations:

\[2\phi(\tau)^3 - 3\phi(\tau)^2 + \lambda(\tau) = 0 \quad (B.5)\]

For \(\tau\) in the upper half plane, this equation has no double roots and thus it defines three single valued functions, \(\phi_1, \phi_2\) and \(\phi_3\), on the upper half plane.

The partition of unity root of equation [B.1] is the root, \(\phi_1\), where

\[\phi_1(i) = \frac{1}{2}.\]

It is easy to see (see figure 3) that for the \(\phi_1\) root we have

\[\phi_1(0) = 1\]
\[\phi_1(\infty) = 0\]
\[\phi_1(\tau) + \phi_1(-1/\tau) = 1.\]

The other two roots can be characterized by their evaluation at \(\tau = i\):

\[\phi_2(i) = \frac{1 - \sqrt{3}}{2}\]
\[\phi_3(i) = \frac{1 + \sqrt{3}}{2}\]
It is easy to see (see figure 3) that

\[ \phi_2(0) = -\frac{1}{2}, \]
\[ \phi_3(0) = 1, \]
\[ \phi_2(i\infty) = 0, \]
\[ \phi_3(i\infty) = \frac{3}{2}, \]
\[ \phi_2(\tau) + \phi_3(-1/\tau) = 1. \]

From this last identity, we could also use \( \phi_2 \) and \( \phi_3 \) to provide partitions of unity.

The \( \lambda \)-group \((\Gamma[2])\) permutes \( \phi_1 \), \( \phi_2 \) and \( \phi_3 \). If

\[ \lambda_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \]
\[ \lambda_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \]
then

\[
\begin{align*}
\phi_1 \circ \lambda_1 &= \phi_2 \\
\phi_2 \circ \lambda_1 &= \phi_1 \\
\phi_3 \circ \lambda_1 &= \phi_3 \\
\phi_1 \circ \lambda_2 &= \phi_3 \\
\phi_2 \circ \lambda_2 &= \phi_2 \\
\phi_3 \circ \lambda_2 &= \phi_1
\end{align*}
\]

It then follows from combinatorial group theory that the subgroup of \( \Gamma[2] \) that leaves \( \phi_1 \) invariant is freely generated by the elements

\[
\begin{align*}
\lambda_1^2 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\
\lambda_2^2 &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \\
\lambda_1 \lambda_2 \lambda_1^{-1} &= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
\lambda_2 \lambda_1 \lambda_2^{-1} &= \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}.
\end{align*}
\]

Simon’s method may be applied to \( \phi_1 \) and used to give another BesselK expansion for \( \zeta(s) \). The main term Simon’s formula for the coefficients of \( \phi_1 \) gives an order of magnitude of the mth-coefficient of \( \phi_1(q) \) as

\[
\frac{1}{m} \exp \left( 2\pi \sqrt{m/3} \right).
\]

This order of magnitude estimate allows the interchange of summation and integration.

This order of magnitude estimate is consistent with numeric calculations of the coefficients wherein, for example, the 1000th coefficient of \( \phi_1(q) \) is

\[-2.21563 \times 10^{46}\]

and

\[
\frac{1}{1000} \exp \left( 2\pi \sqrt{1000/3} \right) = 6.60664 \times 10^{46}.
\]
We searched for zeros of $\Psi_{\phi_1}$ inside the box extending to -20 on the left and +20 on the right and going up from 0 to 40:

$$\begin{align*}
-0.196822 + 9.837i & \quad 0.773957 + 12.8053i \\
-0.305477 + 19.939i & \quad 1.67883 + 19.3617i \\
-1.51671 + 26.1875i & \quad 2.14412 + 25.9197i \\
-0.729122 + 31.5098i & \quad 1.73372 + 31.7993i \\
-1.18117 + 36.8074i & \quad 2.50435 + 36.6545i
\end{align*}$$

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