POSITIVE SCALAR CURVATURE AND A NEW INDEX THEORY FOR NONCOMPACT MANIFOLDS

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Abstract. In this article, we develop a new index theory for noncompact manifolds endowed with an admissible exhaustion by compact sets. This index theory allows us to provide examples of noncompact manifolds with exotic positive scalar curvature phenomena.

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1. Introduction

If $M$ is an $n$-dimensional manifold endowed with a Riemannian metric $g$, then its scalar curvature $\kappa: M \to \mathbb{R}$ satisfies the property that, at each point $p \in M$, there is an expansion

$$\text{Vol}_M(B_\varepsilon(p)) = \text{Vol}_{\mathbb{R}^n}(B_\varepsilon(0)) \left( 1 - \frac{\kappa(p)}{6(n + 2)} \varepsilon^2 + \ldots \right)$$

for all sufficiently small $\varepsilon > 0$. A complete Riemannian metric $g$ on a manifold $M$ is said to have uniformly positive scalar curvature if there is a fixed constant $\kappa_0 > 0$ such that $\kappa(p) \geq \kappa_0 > 0$ for all $p \in M$. For compact manifolds, obstructions to such metrics are largely achieved in one of two ways: (1) the minimal surface techniques in dimensions at most 7 by Schoen-Yau [38] and in dimension 8 by Joachim and Schick [21]; (2) the Dirac index method for spin manifolds by Atiyah-Singer and its generalizations by Connes-Moscovici, Hitchin, Gromov, Lawson, Roe and Rosenberg, among others.

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In the realm of noncompact manifolds it is now well recognized that the original approach by Gromov-Lawson \cite{17} and Schoen-Yau \cite{38}, which proves that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature, is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds. Connes and Moscovici \cite{9} develop a higher index theory that proves that any aspherical manifold whose fundamental group is hyperbolic does not have a metric of positive scalar curvature. Roe \cite{31} subsequently introduces a coarse index theory to study positive scalar curvature problems for noncompact manifolds. Block and Weinberger \cite{3} investigate the problem of complete metrics for noncompact symmetric spaces when no quasi-isometry conditions are imposed. They prove that, if $G$ is a semisimple Lie group with maximal compact subgroup $K$ and irreducible lattice $\Gamma$, then the double quotient $M = \Gamma \backslash G/K$ can be endowed with a complete metric of uniformly positive scalar curvature if and only if $\Gamma$ is an arithmetic group with $\text{rank}_Q \Gamma \geq 3$. This theorem includes, in light of the work of Borel and Harish-Chandra \cite{4}, previous results of Gromov-Lawson \cite{17} in rational rank 0 and 1. In the case when the rational rank exceeds 2, Chang proves that any metric on $M$ with uniformly positive scalar curvature fails to be coarsely equivalent to the natural one \cite{5}.

The Gromov-Lawson-Rosenberg conjecture states that a spin closed manifold $M^n$ with $n \geq 5$ has a metric of positive scalar curvature if, and only if, its Dirac index vanishes in $KO_\ast(C_r^\ast \pi)$, where $\pi = \pi_1(M)$. While this conjecture is known to be false in general, it has been verified in number of cases. To study compact manifolds $(M, \partial M)$ with boundary with respect to a positive scalar curvature metric that is collared at the boundary, one would ideally like to produce a $C^\ast$-algebra that encodes information about both $\pi_1(M)$ and $\pi_1(\partial M)$. In this paper we show that such an algebra can be constructed with the appropriate properties, and apply it to obtain information about noncompact manifolds.

In the first section, we use the notion of localization algebras \cite{43} and generalized asymptotic morphisms to define a relative group $C^\ast$-algebra $C^\ast_{max}(\pi_1(M), \pi_1(\partial M))$ along with a homomorphism

$$\mu_{max}: KO_\ast(M, \partial M) \to KO_\ast(C^\ast_{max}(\pi_1(M), \pi_1(\partial M)))$$

which we call the maximal relative Baum-Connes map. The usual Baum-Connes conjecture has many different guises, the simplest of which is that the homomorphism $KO_\ast(C^\ast_r \Gamma) \to KO_\ast(C^\ast_r \Gamma)$ is an isomorphism. One may similarly hope that the map $\mu_{max}$ above is an injection if $M$ and $\partial M$ are both aspherical. In line with the compact case, we show that, if $M$ has a metric of positive scalar curvature that is collared near the boundary, then the relative index of the Dirac operator in $KO_\ast(M, \partial M)$ belongs to the kernel of $\mu_{max}$. In this section, we also formulate a relative Gromov-Lawson-Rosenberg conjecture for manifolds with boundary and show that the converse to the above statement holds when the relative Gromov-Lawson-Rosenberg conjecture holds for torsion-free amenable groups satisfying certain conditions on their cohomological dimensions.
In the next sections, we offer a new index theory for noncompact manifolds with so-called admissible exhaustions. We combine this theory with the machinery built in the first part of the paper to give various geometric applications: we first construct a noncompact manifold \( M \) with an exhaustion \( \bigcup_{i=1}^{\infty} (M_i, \partial M_i) \) by compact sets with boundary such that each \((M_i, \partial M_i)\) has a metric of positive scalar curvature collared at the boundary, but \( M \) itself has no metric of uniformly positive scalar curvature. Next, we construct a noncompact manifold \( N \) whose space \( PS(N) \) of uniformly positive scalar curvature metrics has uncountably many connected components.

A companion paper [6] will use the techniques of this paper and more complicated topology to obtain a contractible manifold that has a positively curved exhaustion, but no metric of positive scalar curvature.

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2. The relative group \( C^* \)-algebra and the relative Gromov-Lawson-Rosenberg conjecture

In this section, we introduce the concept of relative group \( C^* \)-algebras and formulate a relative version of the Gromov-Lawson-Rosenberg conjecture. The \( K \)-theory of the relative group \( C^* \)-algebras serves as the receptacle of the relative higher index of the Dirac operators.

In this paper all \( C^* \)-algebras are real. We deal only with metric spaces \( X \) that are locally compact and metrically locally simply connected; i.e. for all \( \varepsilon > 0 \) there is \( \varepsilon' \leq \varepsilon \) such that every ball in \( X \) of radius \( \varepsilon' \) is simply connected.

If \( G \) is a discrete group, denote by \( C^*_r(G) \) and \( C^*_{max}(G) \) the usual reduced and maximal real \( C^* \)-algebras of \( G \), respectively. Let \( Y \subseteq X \) both be compact (metric) spaces. We wish to define a Baum-Connes map from the relative \( KO \)-homology group \( KO^f(Y) \) to the \( KO \)-theory of some relative \( C^* \)-algebra encoding the fundamental groups of both \( Y \) and \( X \) and the homomorphism between them. Let \( \phi: \pi_1(Y) \to \pi_1(X) \) be the map induced by the homomorphism \( j_*: \pi_1(Y) \to \pi_1(X) \). Consider the mapping cone \( C^* \)-algebra of \( \phi \) given by

\[
C_{\phi, max} = \{(a, f): f \in C_0([0,1), C^*_{max}(\pi_1(X))), a \in C^*_{max}(\pi_1(Y)), f(0) = \phi(a)\}.
\]

Define \( C^*_{max}(\pi_1(X), \pi_1(Y)) \) to be the seventh suspension \( S^7C_{\phi, max} \) of \( C_{\phi, max} \), i.e. \( C_{\phi, max} \otimes C_0(\mathbb{R}^7) \), where \( C_0(\mathbb{R}^7) \) is the \( C^* \)-algebra of continuous real-valued functions on \( \mathbb{R}^7 \) which vanish at infinity. The seventh suspension is chosen because
KO-theory is eight-periodic. We call this algebra the maximal relative group $C^*$-algebra of $(\pi_1(X), \pi_1(Y))$. If in fact the homomorphism $j_*$ is an injection, we can likewise define a reduced relative $C^*$-algebra $C^\ast_{\text{red}}(\pi_1(X), \pi_1(Y))$.

If $M$ is a metric space, we say that a Hilbert space $H$ is an $M$-module if there is a representation of the continuous functions $C_0(M)$ in $H$, that is, a $C^*$-homomorphism $C_0(M) \to B(H)$. We will say that an operator $T : H \to H$ is locally compact if, for all $\varphi \in C_0(M)$, the operators $T \varphi$ and $\varphi T$ are compact on $H$. We define the support $\text{Supp}(\varphi)$ of $\varphi \in H$ as the smallest closed set $K \subseteq M$ such that, if $f \in C_0(M)$ and $f \varphi \neq 0$, then $f|_K$ is not identically zero. An operator $T : H \to H$ on an $M$-module $H$ has finite propagation if there is $R > 0$ such that $\varphi T \psi = 0$ whenever $\varphi$, $\psi \in C_0(M)$ satisfy $d(\text{Supp}(\varphi), \text{Supp}(\psi)) > R$.

Recall that a locally compact metric space $Z$ is said to have bounded geometry if there is a discrete subset $Y \subseteq Z$ such that (1) $Y$ is $c$-dense for some $c \geq 0$, i.e. $d(z, Y) \leq c$ for all $z \in Z$; (2) for all $r > 0$ there is $N$ such that, for all $p \in Y$, we have $\# \{y \in Y : d(y, p) \leq r\} \leq N$. In the remainder of the article, we assume that all spaces have bounded geometry.

**Definition 2.1.** Let $Z$ be a locally compact metric space. If $H$ is a Hilbert space, we denote by $B(H)$ the algebra of bounded operators with on $H$.

1. Denote by $\mathbb{R}(Z)$ the Roe algebra, i.e. the algebra of locally compact, finite propagation operators on some ample Z-module $H$ by way of a representation $\rho : C_0(Z) \to B(H)$ (see Roe [31, Definition 4.5]).

2. Denote by $C^\ast_{\text{red}}(Z)$ and $C^\ast_{\text{max}}(Z)$ the completions of $\mathbb{R}(Z)$ with respect to the reduced and maximal norm completions, respectively. Here we define the maximal norm in the following way. If $a \in \mathbb{R}(Z)$, then let $||a||_{\text{max}} = \sup_\psi ||\psi(a)||$, where the supremum is taking over all *-homomorphisms $\psi : \mathbb{R}(Z) \to B(W)$, where $W$ is real Hilbert space. By the bounded geometry assumption, the quantity $||a||_{\text{max}}$ is finite by Gong-Wang-Yu [15, Lemma 3.4]. Note that, if $Z$ is compact, then the two completions are the same and coincide with $\mathcal{K}$, the $C^\ast$-algebra of compact operators, as $\mathbb{R}(Z)$ is already all of $\mathcal{K}$.

3. Let $\pi_1(Z)$ act on $\hat{Z}$ by deck transformations and let $\mathbb{R}(\hat{Z})^{\pi_1(Z)}$ be the algebra of operators in $\mathbb{R}(\hat{Z})$ that are invariant under this action. We endow $\mathbb{R}(\hat{Z})^{\pi_1(Z)}$ with a maximal norm by defining $||a||_{\text{max}} = \sup_\psi ||\psi(a)||$, where the supremum is taken over all *-homomorphisms $\psi : \mathbb{R}(\hat{Z})^{\pi_1(Z)} \to B(H)$, where $H$ is a Hilbert space. Note that, although $\mathbb{R}(\hat{Z})^{\pi_1(Z)}$ is a subalgebra of $\mathbb{R}(\hat{Z})$, this maximal norm is different than the one defined in (2) because the domain of $\psi$ is different.

**Definition 2.2.** For continuous maps $g : [0, \infty) \to \mathbb{R}(Z)$, we define norms $||g||_{\text{red}} = \sup_{t \in [0, \infty)} ||g(t)||_{\text{red}}$ and $||g||_{\text{max}} = \sup_{t \in [0, \infty)} ||g(t)||_{\text{max}}$. Suppose that (a) $g$ is uniformly bounded and uniformly continuous, and (b) the propagation of $g(t)$ tends to 0 as $t \to \infty$. We define the following sets:
(1) Denote by $\mathbb{R}_L(Z)$ the collection of maps $g$ satisfying (a) and (b).

(2) Denote by $C^*_L,\text{red}(Z)$ the closure of $\mathbb{R}_L(Z)$ with respect to $\| \cdot \|_{\text{red}}$, called the reduced localization algebra of $X$.

(3) Denote by $C^*_L,\text{max}(Z)$ the closure of $\mathbb{R}_L(Z)$ with respect to $\| \cdot \|_{\text{max}}$, called the maximal localization algebra of $X$. Here the maximal norm is taken as in (2) in the previous definition.

(4) Denote by $C^*_\text{red}(\mathbb{Z})^{\pi_i(Z)}$ and $C^*_\text{max}(\mathbb{Z})^{\pi_i(Z)}$ the closure of the algebra $\mathbb{R}(\mathbb{Z})^{\pi_i(Z)}$ with respect to the reduced and maximal norms, respectively. Here the maximal norm is taken as in (3) in the previous definition.

(5) Denote by $C^*_L,\text{red}(\mathbb{Z})^{\pi_i(Z)}$ and $C^*_L,\text{max}(\mathbb{Z})^{\pi_i(Z)}$ the closure of the algebra $\mathbb{R}_L(\mathbb{Z})^{\pi_i(Z)}$ with respect to the reduced and maximal norms, respectively. Here the maximal norm is taken as in (3) in the previous definition.

Remark 2.3. When $Z$ is compact, then the two localization algebras in (2) and (3) coincide.

For the rest of this paper, we will simplify notation and simply write $C^*_L(Z)$ for either the reduced and maximal localization algebra.

Let $X$ be a locally compact metric space. We shall briefly recall the local index map $\text{ind}_L : KO^L_0(X) \to KO_*(C^*_L(X))$, first introduced by Yu in [33]. We assume that $* \equiv 0 \mod 8$. The other cases can be handled in a similar way with the help of suspensions. Here $KO^L_0(X) \equiv KO^*(C_0(X))$.

Let $(H,F)$ represent a cycle for $KO^L_0(X)$, where $H$ is a standard nondegenerate $X$-module and $F$ is a bounded operator acting on $H$ such that $F^*F - I$ and $FF^* - I$ are locally compact, and $\phi F - F \phi$ is compact for all $\phi \in C_0(X)$. For each positive integer $n$, let $\{U_{n,i}\}_i$ be a locally finite and uniformly bounded open cover of $X$ such that $\text{diam}(U_{n,i}) < \frac{1}{n}$. Let $\{\phi_{n,i}\}_i$ be a continuous partition of unity subordinate to the open cover. Define

$$F(t) = \sum_i ((n-t)^{\frac{1}{n}} \phi_{n,i}^\frac{1}{n} F \phi_{n,i}^\frac{1}{n} + (n-t+1)^{\frac{1}{n}} \phi_{n+1,i}^\frac{1}{n} F \phi_{n+1,i}^\frac{1}{n})$$

for all positive integers $n$ and $t \in [n-1, n]$, where the infinite sum converges in the strong topology. If $\text{prop}$ denotes the propagation of an operator (again see Roe [31 Definition 4.5]), then notice that $\text{prop}(F(t)) \to 0$ as $t \to \infty$.

Observe that $F(t)$ is a multiplier of the localization algebra $C^*_L(X)$ and is invertible modulo the localization algebra. Hence the standard index construction in $K$-theory gives

$$\text{ind}_L([H,F]) = [P_F] - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in KO_0(C^*_L(X)),$$

where $P_F$ is an idempotent in the matrix algebra of $C^*_L(X)^+$. We call this class $\text{ind}_L([H,F])$ the local index of $F$. In fact, we choose $P_F(t)$ to be the matrix

$$
\begin{pmatrix}
F(t)F^*(t) + (1 - F(t)F^*(t))F(t)F^*(t) + (1 - F(t)F^*(t))F(t)(1 - F^*(t)F(t)) & F(t)(1 - F^*(t)F(t)) + (1 - F(t)F^*(t))F(t)(1 - F^*(t)F(t)) \\
(1 - F^*(t)F(t))F^*(t) & (1 - F^*(t)F(t))^2
\end{pmatrix}
$$
See also Willett-Yu [41]. We write $P_p$ for $P_p(t)$ for simplicity. For the rest of this paper, we also abbreviate $[(H,F)]$ as $[F]$ and $\text{ind}_L[(H,F)]$ as $\text{ind}_L[F]$.

The following isomorphism is demonstrated in Yu [43, Theorem 3.2] in the case when $X$ is a CW complex and for general metric space $X$ in Qiao-Roe [28, Theorem 3.4].

**Proposition 2.4.** The local index map $\text{ind}_L: KO_*(X) \to KO_*(C^*_L(X))$ is an isomorphism.

**Definition 2.5.** Let $Y \subseteq X$ be compact metric spaces. In the definitions of $C^*_L(Y)$ and $C^*_L(X)$, we choose the $Y$-module and $X$-module to be $\ell^2(Z_Y) \otimes H$ and $\ell^2(Z_X) \otimes H$ such that $Z_Y \subseteq Z_X$ are countable dense subsets of $Y$ and $X$, respectively, and $H$ is a separable and infinite-dimensional Hilbert space. The inclusion isometry from $\ell^2(Z_Y) \otimes H$ to $\ell^2(Z_X) \otimes H$ induces a homomorphism $i: C^*_L(Y) \to C^*_L(X)$.

**Remark 2.6.** The homomorphism $i$ is not canonical. However it induces a homomorphism at the $KO$-theory that is canonical.

Let $C_i$ be the mapping cone of $i$ given by

$$C_i = \{(a,f) : f \in C_0([0,1), C^*_L(X)), a \in C^*_L(Y), f(0) = i(a)\}.$$  

Define the relative $KO$-homology group of $(X,Y)$ to be $KO_*(X,Y) \equiv KO_*\left(S^7 C_i\right)$. This definition of relative $KO$-homology allows us to have a relative long exact sequence

$$\cdots \to KO_*(Y) \to KO_*(X) \to KO_*(X,Y) \to \cdots.$$  

**Lemma 2.7.** Let $X$ be a compact space and let $\mathcal{K}$ be the $C^*$-algebra of compact operators on a separable, infinite-dimensional Hilbert space. Then there is an isomorphism $C^*_{red}(\tilde{X})^{\pi_1(X)} \cong C^*_L(\pi_1(X)) \otimes \mathcal{K}$ and $C^*_max(\tilde{X})^{\pi_1(X)} \cong C^*_{max}(\pi_1(X)) \otimes \mathcal{K}$.

**Proof.** In Roe [32, Lemma 2.3] the $*$-isomorphism $(\mathbb{R}\tilde{X})^{\pi_1(X)} \cong (\mathbb{R}\pi_1(X)) \otimes \mathcal{K}$ is proved. This algebraic $*$-isomorphism extends to the required $*$-isomorphism in both the reduced and maximal case, since $\mathcal{K}$ is a nuclear $C^*$-algebra. 

**Proposition 2.8.** Let $X$ be a compact metric space with universal cover $\tilde{X}$. There is $\varepsilon > 0$ depending only on $X$ such that, if $b$ is an operator in $\mathbb{R}(X)$ with propagation at most $\varepsilon$, then $b$ lifts to a $\pi_1(X)$-invariant operator $\tilde{b}$ in $\mathbb{R}(\tilde{X})$.

**Proof.** In the definition of $\mathbb{R}(X)$, we choose the $X$-module to be $\ell^2(Z_X) \otimes H$ such that $Z_X$ is a countable dense subset of $X$ and $H$ is a separable and infinite-dimensional Hilbert space. Let $p: \tilde{X} \to X$ be the projection map. We define $Z_{\tilde{X}} = p^{-1}(Z_X)$. We choose the $\tilde{X}$-module to be $\ell^2(Z_{\tilde{X}}) \otimes H$ in the definition of $\mathbb{R}(\tilde{X})$. Every operator $b \in \mathbb{R}(X)$ can be represented by a kernel $k(\cdot, \cdot)$ such that $k(x,y)$ belongs to $\mathcal{K}$ for all $(x,y) \in Z_X \times Z_X$ and $\text{Supp}(k)$ is contained in $\{(x,y) \in X \times X : d(x,y) < r\}$ for some $r > 0$. The smallest such $r$ is the propagation of $b$. Now let $k'(x', y') = k(p(x'), p(y'))$ for all $(x',y') \in Z_{\tilde{X}} \times Z_{\tilde{X}}$ satisfying $d(x',y') < r$ and $k'(x', y') = 0$ for all $(x', y') \in Z_{\tilde{X}} \times Z_{\tilde{X}}$ satisfying $d(x', y') \geq r$. 


By the compactness of $X$, there is $\varepsilon > 0$ such that, if $b$ has propagation at most $\varepsilon$, then $k'$ represents an element $b'$ of $\mathbb{R}(X)$ and $b$ has the same propagation as $b$.

This discussion shows that there exists $\varepsilon > 0$ such that, if $b \in \mathbb{R}(X)$ and $\text{prop}(b) < \varepsilon$, then there is a unique lifting of $b$ in $\mathbb{R}(X)$ to $\phi(b)$ in $\mathbb{R}(\tilde{X})$. \hfill $\square$

Note that, if the propagations $\text{prop}(b_1), \text{prop}(b_2) < \varepsilon/2$, then this lifting respects multiplication and addition, i.e. $\phi(b_1 b_2) = \phi(b_1)\phi(b_2)$ and $\phi(b_1 + b_2) = \phi(b_1) + \phi(b_2)$.

**Definition 2.9.** Let $s \in [0, \infty)$ and let $X$ be a compact metric space. For all $b \in \mathbb{R}_L(X)$, denote by $b_s \in \mathbb{R}_L(X)$ the operator given by $b_s(t) = b(s + t)$ for all $t \in [0, \infty)$. Let $\varepsilon$ be as in the above proposition. For each $b \in \mathbb{R}_L(X)$, there is $s_b > 0$ such that $\text{prop}(b_s) < \varepsilon$ when $s > s_b$. We define $\phi_s(b) = b_s \in \mathbb{R}_L(\tilde{X})^{\pi_1(X)}$ when $s > s_b$.

The next result indicates that $\phi_s$ is an asymptotic morphism in the following generalized sense.

**Lemma 2.10.** Let $X$ be a compact metric space. For all $b \in \mathbb{R}_L(X)$, let $s_b$ be given as in the previous definition.

1. There is $C > 0$ such that, for all $b \in \mathbb{R}_L(X)$, if $s > s_b$, then $||\phi_s(b)||_{\text{red}} \leq C||b||_{\text{red}}$ and $||\phi_s(b)||_{\text{max}} \leq C||b||_{\text{max}}$.

2. For all $b \in \mathbb{R}_L(X)$, if $s > s_b$, then $\phi_s(b)^* = \phi_s(b^*)$.

3. For all $b_1, b_2 \in \mathbb{R}_L(X)$, the operator $\phi_s(b_1 b_2) - \phi_s(b_1)\phi_s(b_2)$ is zero when $s > \max\{s_{b_1}, s_{b_2}, s_{b_1b_2}\}$.

**Proof.** Let $\{U_i\}_{i=1}^N$ be a finite open cover of $X$ such that, for each $i$, the diameter of the union of all $U_j$ satisfying $U_j \cap U_i \neq \emptyset$ is less than $\varepsilon$, where $\varepsilon$ is as in Proposition 2.8. Let $\{\varphi_i\}_i$ be the continuous partition of unity subordinate to $\{U_i\}$. We have $\phi_s(b) = \sum_{i=1}^N \phi_s(\varphi_i b)$. By the definition of $\phi_s$ and the choice of $\varphi_i$, we have $||\phi_s(\varphi_i b)|| = ||\varphi_i b|| \leq ||b||$. It follows that $||\phi_s(b)|| \leq N||b||$ if $s > s_b$. This proves (1). The proofs of (2) and (3) are straightforward. \hfill $\square$

There is a pushdown $\mathbb{R}_L(\tilde{X})^{\pi_1(X)} \to \mathbb{R}_L(X)$ for operators with small propagation. Such a pushdown induces homomorphisms

$$KO_*(C_{L,\text{max}}^*(\tilde{X})^{\pi_1(X)}) \to KO_*(C_L^*(X))$$

and

$$KO_*(C_{L,\text{red}}^*(\tilde{X})^{\pi_1(X)}) \to KO_*(C_L^*(X)),$$

which are inverses to the homomorphisms induced by the liftings. This lemma implies that the liftings $\phi_s$ induce isomorphisms $KO_*(C_L^*(X)) \to KO_*(C_{L,\text{max}}^*(\tilde{X})^{\pi_1(X)})$ and $KO_*(C_L^*(X)) \to KO_*(C_{L,\text{red}}^*(\tilde{X})^{\pi_1(X)})$. 
Definition 2.11. Let \( j_* : \pi_1(Y) \to \pi_1(X) \) be the homomorphism induced by the inclusion \( Y \to X \). Then \( j_* \) induces a unique map \( \eta: \tilde{Y} \to \tilde{X} \) such that \( \eta(gy) = i_*(g)\eta(y) \) for all \( g \in \pi_1(Y) \) and \( y \in \tilde{Y} \).

Note that such \( \eta \) exists because \( X \) and \( Y \) are metrically locally simply connected.

Let \( p \) be the covering map \( \tilde{X} \to X \) and let \( Y' = p^{-1}(Y) \). Let \( p' : \tilde{Y} \to Y \) be the covering map from the universal cover \( \tilde{Y} \). Let \( Y'' \) be the Galois covering of \( Y \) whose deck transformation group is \( j_*\pi_1(Y) \), and let \( p'' : Y'' \to Y \) be the covering map.

We have \( Y'' = \pi_1(X) \times_{j_*\pi_1(Y)} Y'' \). This decomposition gives rise to a natural \(*\)-homomorphism

\[
\psi': C^*_\text{max}(Y'')^{j_*\pi_1(Y)} \to C^*_\text{max}(Y')^{\pi_1(X)}.
\]

Choose countable dense subsets \( Z_Y \) of \( Y \) and \( Z_X \) of \( X \) such that \( Z_Y \subseteq Z_X \). Let \( H \) be a separable and infinite-dimensional Hilbert space. We use the modules \( \ell^2(p^{-1}(Z_Y)) \otimes H \), \( \ell^2(p^{-1}(Z_X)) \otimes H \), and \( \ell^2((p')^{-1}(Z_Y)) \otimes H \) and \( \ell^2((p'')^{-1}(Z_Y)) \otimes H \), respectively, to define \( C^*_\text{max}(Y'')^{\pi_1(X)} \), \( C^*_\text{max}(X^{\pi_1(X)} \), \( C^*_\text{max}(Y')^{\pi_1(X)} \) and \( C^*_\text{max}(Y'')^{j_*\pi_1(Y)} \).

Lemma 2.12. There exists a \(*\)-homomorphism

\[
\psi'' : C^*_\text{max}(\tilde{Y})^{\pi_1(Y)} \to C^*_\text{max}(Y'')^{j_*\pi_1(Y)}
\]

such that there is \( \varepsilon > 0 \) for which, if \( k \in C^*(\tilde{Y})^{\pi_1(Y)} \) is an operator with propagation at most \( \varepsilon \) and is represented as a kernel \( k \) on \((p'')^{-1}(Z_Y)\) with values in \( K \), then there is a unique kernel \( k_Y \) on \( Z_Y \) with values in \( K \) such that \( k(x, y) = k_Y(p(x), p(y)) \) for all \( x, y \in p^{-1}(Z_Y) \) satisfying \( d(x, y) \leq \varepsilon \) and \( \psi''(k) \) is represented by a kernel \( k'' \) on \((p'')^{-1}(Z_Y)\) with values in \( K \) such that \( k''(x, y) = k_Y(p''(x), p''(y)) \) for all \( x, y \in (p'')^{-1}(Z_Y) \) satisfying \( d(x, y) \leq \varepsilon \).

Proof. Let \( H \) be the kernel of the homomorphism \( j_* : \pi_1(Y) \to \pi_1(X) \). Let \( k \) be an operator in \( R(\tilde{Y})^{\pi_1(Y)} \) represented by a kernel \( k(x, y) \) on \((p'')^{-1}(Z_Y)\). We define a kernel \( k_a(x, y) \) on \((p'')^{-1}(Z_Y)\) by the formula \( k_a(x, y) = \sum_{h \in H} k(hx, y) \) for all \( x, y \in (p'')^{-1}(Z_Y) \). Note that the above sum is finite since \( k \) has finite propagation.

We have \( k_a(h_1x, h_2y) = k_a(x, y) \) for all \( h, h_1, h_2 \in H \) and \( x, y \in (p'')^{-1}(Z_Y) \). For each \( x, y \in (p'')^{-1}(Z_Y) \), let \([x], [y] \) be the corresponding pair of equivalence classes in \((p'')^{-1}(Z_Y) = (p^{-1}(Z_Y))/H \). We let \( k''([x], [y]) = k_a(x, y) \). Note that \( k'' \) is well-defined. We now define a \(*\)-homomorphism \( \psi'' : R(\tilde{Y})^{\pi_1(Y)} \to R(Y'')^{j_*\pi_1(Y)} \) given by \( \psi''(k) = k'' \). By maximality, this map \( \psi'' \) extends to a \(*\)-homomorphism

\[
C^*_\text{max}(\tilde{Y})^{\pi_1(Y)} \to C^*_\text{max}(Y'')^{j_*\pi_1(Y)}.
\]

We choose \( \varepsilon > 0 \) small enough such that \( d(hx, x) > 10\varepsilon \) for all \( h \neq e \) in \( H \) and all \( x \in \tilde{Y} \). If \( d([x], [y]) > \varepsilon \), then \( d(hx, y) > \varepsilon \) for all \( h \in H \). Therefore if \( k \) has propagation at most \( \varepsilon \), then \( k'' \) has propagation at most \( \varepsilon \). If \( \varepsilon \) is small enough, there is a unique kernel \( k_Y \) on \( Z_Y \) such that \( k(x, y) = k_Y(p(x), p(y)) \) for all \( x, y \in p^{-1}(Z_Y) \) satisfying \( d(x, y) \leq \varepsilon \) and \( k_Y \) has propagation at most \( \varepsilon \). If \( d(x, y) \leq \varepsilon \), then \( d(hx, y) < \varepsilon \) for all \( h \neq e \) in \( H \) and \( x, y \in \tilde{Y} \). Therefore \( k_a(x, y) = k(x, y) \) if \( d(x, y) \leq \varepsilon \). It follows that \( k''(x, y) = k_Y(p''(x), p''(y)) \) for all \( x, y \in (p'')^{-1}(Z_Y) \) satisfying \( d(x, y) \leq \varepsilon \).
Let $\psi''$ be as in Lemma 2.12 above and let $\psi'$ be as previously defined. We now define a $*$-homomorphism

$$\psi_{\text{max}} = \psi' \circ \psi'' : C^*_\text{max}(\tilde{Y})^{\pi_1(Y)} \to C^*_\text{max}(\tilde{X})^{\pi_1(X)}.$$  \hspace{1cm} (2.13)

This homomorphism in turn induces a $*$-homomorphism

$$\psi_{L,\text{max}} : C^*_{L,\text{max}}(\tilde{Y})^{\pi_1(Y)} \to C^*_{L,\text{max}}(\tilde{X})^{\pi_1(X)}.$$  

Let $C_{\psi,\text{max}}$ be the mapping cone of $\psi_{L,\text{max}}$ given by

$$C_{\psi,\text{max}} = \{(a, f) : f \in C_0([0, 1), C^*_{L,\text{max}}(\tilde{X})^{\pi_1(X)}), a \in C^*_{L,\text{max}}(\tilde{Y})^{\pi_1(Y)}, f(0) = \psi_{L,\text{max}}(a)\}.$$  

Recall that $i : C^*_L(Y) \to C^*_L(X)$ is the homomorphism induced by the inclusion $Y \to X$, and $C_i$ is its mapping cone. For each $(b, f) \in C_i$ satisfying $\text{prop}(b) < \infty$ and $\text{prop}(f(t)) < \infty$ for all $t \in [0, 1]$, there is $s_{(b, f)} > 0$ such that $\text{prop}(b_s) < \varepsilon$ and $\text{prop}(f(t)) < \varepsilon$ for all $s > s_{(b, f)}$. We define

$$\chi_{s,\text{max}}(b, f) = (\phi_s(b_s), \phi_s(f(\cdot)_{s})) \in C_{\psi,\text{max}}$$

for all $s > s_{(b, f)}$, where $\phi_s$ is as in Lemma 2.10. By the same lemma, we then know that $\chi_{s,\text{max}}$ induces a homomorphism

$$(\chi_{s,\text{max}})_* : KO_*(S^7C_i) \to KO_*(S^7C_{\psi,\text{max}}).$$

Let $e$ be the evaluation homomorphism induced by the evaluation maps at 0 from $C^*_{L,\text{max}}(\tilde{X})^{\pi_1(X)}$ to $C^*_{L,\text{max}}(\tilde{Y})^{\pi_1(Y)}$ and from $C^*_L(\tilde{Y})^{\pi_1(Y)}$ to $C^*_L(\tilde{Y})^{\pi_1(Y)}$. This homomorphism induces a map $e_* : KO_*(S^7C_{\psi,\text{max}}) \to KO_*(S^7C_{\psi,\text{max}})$ at the level of $KO$-theory.

Define $\mu_{\text{max}}$ to be the composition given by

$$KO_*(S^7C_i) \xrightarrow{(\chi_{s,\text{max}})_*} KO_*(S^7C_{\psi,\text{max}}) \xrightarrow{e_*} KO_*(S^7C_{\psi,\text{max}}).$$

Equivalently $\mu_{\text{max}}$ is a map

$$\mu_{\text{max}} : KO_*(X, Y) \to KO_*(C^*_\text{max}(\pi_1(X), \pi_1(Y)))$$

which we call the maximal relative Baum-Connes map. A reduced relative Baum-Connes map

$$\mu_{\text{red}} : KO_*(X, Y) \to KO_*(C^*_\text{red}(\pi_1(X), \pi_1(Y)))$$

can be similarly constructed if the homomorphism $j$ from $\pi_1(Y)$ to $\pi_1(X)$ is injective.

Remark 2.14. By homotopy invariance, both $\mu_{\text{max}}$ and $\mu_{\text{red}}$ are independent of the choices of the liftings.

Conjecture 2.15. Let $Y \subseteq X$ and suppose that $X$ and $Y$ are both aspherical compact spaces.

1. (Relative Novikov conjecture) The maximal relative Baum-Connes map

$$\mu_{\text{max}} : KO_*(X, Y) \to KO_*(C^*_\text{max}(\pi_1(X), \pi_1(Y)))$$

is an injection.
(2) **(Relative Baum-Connes conjecture)** If \( j : \pi_1(Y) \to \pi_1(X) \) is an injection, then the reduced relative Baum-Connes map

\[
\mu_{\text{red}} : KO_*(X,Y) \to KO_*(C_{\text{red}}^*(\pi_1(X),\pi_1(Y)))
\]

is an isomorphism.

**Remark 2.16.** If the classic Baum-Connes conjecture holds for \( \pi_1(X) \) and \( \pi_1(Y) \), then statement (2) is true for the pair \( (\pi_1(X),\pi_1(Y)) \). In general the maximal relative Baum-Connes conjecture may not be an isomorphism. The real version \((KO)\) of the Baum-Connes conjecture follows from the classic (complex version) of the Baum-Connes conjecture (see Baum-Karoubi [2]). After inverting 2, even the injectivity of the complex Baum-Connes map implies the injectivity of the real Baum-Connes map (see Schick [37, Corollary 2.13]).

Recall that the notion of K-amenable was formulated by Cuntz [10, Definition 2.2]. This notion can be extended to the KO-setting.

**Theorem 2.17.** Suppose that \( Y \subseteq X \) are aspherical compact spaces such that \( \pi_1(Y) \) and \( \pi_1(X) \) are K-amenable and satisfy the Baum-Connes conjecture.

1. Then \( \mu_{\text{max}} \) is an isomorphism.
2. Assume also that \( \pi_1(Y) \to \pi_1(X) \) is an injection. Then \( \mu_{\text{red}} \) is an isomorphism.

**Proof.** By the definition of K-amenable, the natural homomorphisms \( C_{\text{max}}^*(\pi_1(X)) \to C_{\text{max}}^*(\pi_1(Y)) \) and \( C_{\text{max}}^*(\pi_1(Y)) \to C_{\text{max}}^*(\pi_1(Y)) \) induce KK-equivalences. If \( \pi_1(X) \) and \( \pi_1(Y) \) are K-amenable and satisfy the Baum-Connes conjecture, and if \( \pi_1(Y) \) injects into \( \pi_1(X) \), then the KO-theory of the reduced relative group C*-algebra coincides with the KO-theory of the maximal relative group C*-algebra.

The theorem is proved from the following commutative diagram and the five-lemma.

We now prove that the existence of positive scalar curvature implies that a particular index vanishes in the KO-theory of the relative group C*-algebra. For the rest
of this section, the $C^*$-algebras involved are maximal. If the reduced relative group $C^*$-algebra is well defined, then the rest of this section extends to the reduced case as well. We will use $C^*(\pi_1(X), \pi_1(Y))$ to denote both the reduced and maximal relative group $C^*$-algebra when the use of such a notation does not cause confusion.

Let $M$ be a spin manifold with boundary $N = \partial M$. We assume that the dimension of $M$ is $0 \mod 8$. The other cases can be handled in a similar way with the help of suspensions. More specifically, in dimension $k \mod 8$ for some $0 \leq k < 8$, we consider the manifold $M \times \mathbb{R}^{8-k}$. We can define a relative higher index of the Dirac operator associated to $KO_0(C^*(\pi_1(M), \pi_1(\partial M)) \otimes C^*_L(\mathbb{R}^k))$.

We can apply the same argument below to show that this relative index vanishes in $KO_k(C^*(\pi_1(M), \pi_1(\partial M)))$ if $(M, \partial M)$ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary. This relative higher index corresponds to the relative index of the Dirac operator associated to $M$ under the isomorphism

$$KO_0(C^*(\pi_1(M), \pi_1(\partial M)) \otimes C^*_L(\mathbb{R}^k)) \cong KO_k(C^*(\pi_1(M), \pi_1(\partial M))).$$

As a consequence, the relative index of the Dirac operator associated to $M$ vanishes in $KO_k(C^*(\pi_1(M), \pi_1(\partial M)))$ if $(M, \partial M)$ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary.

Extend the manifold by attaching a cylinder $W = N \times [0, \infty)$ to the boundary, forming a noncompact manifold $Z$. Let $D$ be the Dirac operator on $Z$. Let $f$ be an odd smooth chopping function in the sense of Roe on the real line satisfying the following conditions: (1) $|f(x)| \leq 1$ for all $x$ and $f(x) \to \pm 1$ as $x \to \pm \infty$; (2) $g = f^2 - 1 \in S(\mathbb{R})$, the space of Schwartz functions, (3) if $\hat{f}$ and $\hat{g}$ are the Fourier transforms of $f$ and $g$, respectively, then $\text{Supp}(\hat{f}) \subseteq [-1, 1]$ and $\text{Supp}(\hat{g}) \subseteq [-2, 2]$.

Such a chopping function exists (cf. Roe [30, Lemma 7.5]). We define

$$F_D = f(D) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) \exp(itD) dt.$$

By condition (3) above, it follows that the propagation of $F_D$ is at most 1. Let $[F_D]$ be its homology class in $KO_0^L(Z) = KO^0(C_0(Z))$. We simplify the notation by replacing $F_D$ with $F$ and $P_{F_D}$ with $P_D$. We write

$$\text{ind}_L([F]) = [P_D] - \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] \in KO_0(C^*_L(Z)),$$

where $P_D$ is an idempotent in the matrix algebra of $C^*_L(Z)$ and $\text{ind}_L$ is the local index map. The element $P_D - \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right]$ belongs to the matrix algebra of the localization algebra $C^*_L(Z)$.

Let $v$ be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing the generator in $KO_{-1}(C_0(\mathbb{R}^7)) \cong KO_0(C_0(\mathbb{R}^8))$ (see Atiyah [1] or Schröder [35 Proposition 1.4.11]). Let $\tau_D = v \otimes P_D + I \otimes (I - P_D)$. Then we have $\tau_D^{-1} = v^{-1} \otimes P_D + I \otimes (I - P_D)$. If $\chi_M$ is the characteristic function on $M$, let $\tau_{D,M} = (1 \otimes \chi_M)\tau_D(1 \otimes \chi_M)$.
and \((\tau^{-1}_D)_M = (1 \otimes \chi_M) \tau^{-1}_D (1 \otimes \chi_M)\). In the future pages, we will simply write \(\chi_M\) for \(1 \otimes \chi_M\). For all \(s \in [0, 1]\), define \(w_{D,M}(s)\) to be the product
\[
\begin{pmatrix}
1 & (1-s)\tau_{D,M} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-(1-s)(\tau^{-1}_D)_M & I
\end{pmatrix}
\begin{pmatrix}
I & (1-s)\tau_{D,M} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}.
\]
Define \(q_{D,M}(s) = w_{D,M}(s) \begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} w^{-1}_{D,M}(s)\).

Now define \(C^*_L(N \subseteq M)\) be the closed two-sided ideal of \(C^*_L(M)\) generated by \(C^*_L(N)\) considered as a subalgebra of \(C^*_L(M)\). Then \(\tau_{D,M}\) and \((\tau^{-1}_D)_M\) both lie in \(C^*_L(M) \otimes C_0(\mathbb{R}^7)\). Both \((\tau_{D,M}\tau^{-1}_D)_M - I\) and \((\tau^{-1}_D)_M\tau_{D,M} - I\) lie in \(C^*_L(N \subseteq M)\). As a consequence \(q_{D,M}(0)\) is an element in the matrix algebra of \((C^*_L(N \subseteq M) \otimes C_0(\mathbb{R}^7))^+\). We define \([q_D]\) to be the \(KO\)-theory element
\[
\left[\left[ q_{D,M}(0), q_{D,M}(\cdot) \right] - \left[ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right] \right]
\]
in \( KO_0(S^7C_j)\), where \(C_j\) is the mapping cone associated with \(j: C^*_L(M, N) \rightarrow C^*_L(M)\) and \(S^7C_j = C_j \otimes C_0(\mathbb{R}^7)\). The inclusion map \(C^*_L(N) \rightarrow C^*_L(N \subseteq M)\) induces an isomorphism
\[
KO_*(C^*_L(N)) \cong KO_*(C^*_L(N \subseteq M)),
\]
which can be proved by a standard Mayer-Vietoris sequence and a five-lemma argument. As a consequence, we have the isomorphism \( KO_0(S^7C_j) \cong KO_0(M, N)\).

We call the class \([q_D]\) the \textit{relative \(KO\)-homology class} of \(D\). We define the \textit{relative higher index} of \(D\) to be \(\mu(q_D) \in KO_0(C^*(\pi_1(M), \pi_1(\partial M)))\).

\[\text{Theorem 2.18.} \text{ If } (M, \partial M) \text{ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary, then the relative higher index of the Dirac operator is zero in } KO_0(C^*(\pi_1(M), \pi_1(\partial M))).\]
Proof. As before, let \( N = \partial M \) and \( Z = M \cup N \ (N \times [0, \infty)) \). Denote by \( Z_n \) and \( Z_n' \) the truncations \( Z_n = M \cup N \ (N \times [0, n]) \), \( Z_n' = M \cup N \ (N \times [0, \frac{n}{7}]) \), and let \( T_n \) be the subset of \( Z_n \) given by \( T_n = N \times \left[ \frac{n}{7}, n \right] \). We assume that the dimension of \( Z \) is 0 mod 8. The other cases can be handled in a similar way with the help of suspensions (refer back to the section after Theorem 2.17).

Let \( u \in [1, \infty) \) and write

\[
\text{ind}_L(uD) = [P_{uD}] - \left[ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right] \in KO_0(C^*_L(Z)).
\]

We define \( w_{D,Z_n}(s) \) and \( q_{D,Z_n}(s) \) by replacing \( M \) with \( Z_n \) in the definitions of \( w_{D,M}(s) \) and \( q_{D,M}(s) \), respectively, before Theorem 2.18. By the finite propagation of \( D \), we know that the propagation of \( \exp(itD) \) is less than or equal to \(|t|\). It follows that the propagation of \( P_{uD} \) is less than or equal to 100\( u \). This estimate is based on the matrix formula before Proposition 2.4.

Claim 2.19. For all \( u > 0 \), there exists \( N_u > 0 \) such that, for all \( n > N_u \), we have

\[
\chi_{Z_n'} \left( q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right) \chi_{Z_n'} = 0,
\]

\[
\chi_{T_n} \left( q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right) \chi_{T_n} = 0,
\]

\[
\chi_{Z_n'} \left( q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right) \chi_{T_n} = 0.
\]

Proof. Let \( \alpha = \tau_{uD,Z_n} \) and \( \beta = (\tau_{uD})Z_n \). We can compute

\[
w_{uD,Z_n}(0) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} w_{uD,Z_n}(0)^{-1} = \begin{pmatrix} (2\alpha - \alpha \beta) \beta & (2\alpha - \alpha \beta)(I - \beta \alpha) \\ (I - \beta \alpha) \beta & (I - \beta \alpha)^2 \end{pmatrix}.
\]

We note that \((2\alpha - \alpha \beta) \beta = \alpha(I - \beta \alpha) \beta + (\alpha \beta - I) + I\). Let \( N_u = 100u \). Using the formulas for \( \alpha \) and \( \beta \), and the fact that \( P_{uD} \) has propagation at most 100\( u \), we know that the elements \( \chi_{Z_n'}(\alpha \beta - I), \chi_{Z_n'}(\beta \alpha - I), (\alpha \beta - I)\chi_{Z_n'}, (I - \beta \alpha)\beta \chi_{Z_n'} \) and \( \beta \alpha - I - \beta \alpha \) are all zero when \( n > N_u \). Now our claim follows.

Let \( P^{(n)}_{uD} = \chi_{Z_n} P_{uD} \chi_{Z_n} \), where \( \chi_{Z_n} \) is the characteristic function on \( Z_n \). By the construction of \( P_{uD} \), we have \( ||P_{uD}|| \leq 10 \). As a result, we have \( ||P^{(n)}_{uD}|| \leq 10 \), giving an upper bound for \( ||q_{uD,Z_n}|| \). Together with the above claim, this implies that \( \prod_n q_{uD,Z_n}(0) \in \prod_n (S^1 C^*_L(Z_n))^+ / \bigoplus_n (S^1 C^*_L(Z_n))^+ \) belongs to the image of the inclusion map

\[
\prod_n (S^1 C^*_L(T_n))^+ / \bigoplus_n (S^1 C^*_L(T_n))^+ \rightarrow \prod_n (S^1 C^*_L(Z_n))^+ / \bigoplus_n (S^1 C^*_L(Z_n))^+,
\]

where \((S^1 C^*_L(Z_n))^+\) and \((S^1 C^*_L(T_n))^+\) are respectively obtained from \( S^1 C^*_L(Z_n) \) and \( S^1 C^*_L(T_n) \) by adjoining a unit. We identify \( \prod_n q_{uD,Z_n}(0) \) with an element in
$\prod_n (S^7 C_7^*(T_n))^+ / \bigoplus_n (S^7 C_7^*(T_n))^+$ gives an element in the matrix algebra of $\prod_n (S^7 C_7^*(T_n))^+ / \bigoplus_n (S^7 C_7^*(T_n))^+$, where $s \in [0,1]$ is the variable and $C_j$ is the mapping cone of the homomorphism $j_n : S^7 C_7^*(T_n) \to S^7 C_7^*(Z_n)$.

Recall that $W = N \times [0, \infty)$ and let $W' = N \times \mathbb{R}$ be the double of $W$. Let $D'$ be the Dirac operator on $W'$. Let $\tilde{W}'$ be the universal cover of $W'$ and $\tilde{D}'$ be the lifting of $D'$ to $\tilde{W}'$. Let $D$ be the lifting of $D$ to $\tilde{Z}$, the universal cover of $Z$. We know that $P_{u\tilde{D}'}(0)$ and $P_{u\tilde{D}}(0)$ are respectively the liftings of $P_{uD'}(0)$ and $P_{uD}(0)$. Define

$$x_{n,u}(s) = q_{u\tilde{D},\tilde{Z}}(s),$$

where $P_{u\tilde{D}} = \chi_{\tilde{Z}} P_{\tilde{D}} \chi_{\tilde{Z}}$. Define

$$y_{n,u} = q_{u\tilde{D}',\tilde{W}'}(0),$$

where $W'_n = N \times (-\infty, n]$. By an argument similar to the proof of Claim 2.19 we know that $[\prod_n y_{n,u}]$ is an operator in the image of the inclusion map:

$$\prod_n (S^7 C_7^*(\tilde{T}_n))^{(N)} + / \bigoplus_n (S^7 C_7^*(\tilde{T}_n))^{(N)} \to \prod_n (S^7 C_7^*(\tilde{W}'))^{(N)} + / \bigoplus_n (S^7 C_7^*(\tilde{W}'))^{(N)}.$$

We identify $[\prod_n y_{n,u}]$ with an element in $\prod_n (S^7 C_7^*(\tilde{T}_n))^{(N)} + / \bigoplus_n (S^7 C_7^*(\tilde{T}_n))^{(N)}$.

By Lemma 2.12 and Formula 2.13 there is a natural $*$-homomorphism

$$\phi_n : C_7^*(\tilde{T}_n)^{(N)} \to C_7^*(T'_n)^{(M)}.$$

For each $n$, the map $\phi_n$ induces a natural $*$-homomorphism

$$S^7 C_7^*(\tilde{T}_n)^{(N)} \to S^7 C_7^*(\tilde{Z}_n)^{(M)},$$

which we still denote by $\phi_n$. We have

$$\left[ \prod_n \phi_n(y_{n,u}) \right] = \left[ \prod_n x_{n,u}(0) \right]$$

in $\prod_n (S^7 C_7^*(\tilde{Z}_n))^{(M)} + / \bigoplus_n (S^7 C_7^*(\tilde{Z}_n))^{(M)}$. Denote by $C_{\phi_n}$ the mapping cone of the map $\phi_n$. The element $[\prod_n (y_{n,u}, x_{n,u}(s))]$ gives a $KO$-theory element

$$\left[ \prod_n (y_{n,u}, x_{n,u}(s)) \right]$$

in $KO_0 \left( \prod_n S^7 C_{\phi_n} / \bigoplus_n S^7 C_{\phi_n} \right)$.

Let $V_{1,n} : L^2[0,n] \to L^2[0,1]$ be the isometry given by $f(\cdot) \mapsto \frac{1}{\sqrt{n}} f(\cdot)$ for all $f \in L^2[0,n]$. Let $V_{2,n} : L^2[\frac{n}{2},n] \to L^2[\frac{1}{2},1]$ be the isometry given by $f(\cdot) \mapsto \frac{1}{\sqrt{n}} f(\cdot)$ for all $f \in L^2[\frac{n}{2}, \frac{3}{2}]$. We can naturally construct isometries $V_{1,n}^\prime : L^2(\tilde{Z}_n) \to L^2(\tilde{Z}_1)$ and $V_{2,n}^\prime : L^2(\tilde{T}_n) \to L^2(\tilde{T}_1)$. Conjugations by $V_{1,n}^\prime$ and $V_{2,n}^\prime$ give us a $*$-isomorphism $C_{\phi_n} \to C_{\phi_1}$, where $C_{\phi_1}$ is naturally $*$-isomorphic to $C_{\phi}$. Identifying $(S^7 C_{\phi_n})^+$ with
(\text{S}^7C_\phi)^+ \text{ for all } n, \text{ we see that } [(y_{n,u}, x_{n,u}(s))] = [\mu(q_D)] \text{ in } KO_0(\text{S}^7C_\phi). \text{ It follows that there is a natural isomorphism } \\
\psi: KO_0\left(\prod_n S^7C_\phi_n / \bigoplus_n S^7C_\phi_n\right) \to KO_0\left(\prod_n S^7C_\phi / \bigoplus_n S^7C_\phi\right) \\
such that \\
\psi\left(\prod_n (y_{n,u}, x_{n,u}(s))\right) = \prod_n [\mu(q_D)], \\
where \mu(q_D) \in KO_0(\text{S}^7C_\phi) \cong KO_0(C^*(\pi_1(M), \pi_1(N))) \text{ was defined as the relative higher index of } D \text{ before Theorem 2.18 and } \\
KO_0(\prod_n S^7C_\phi / \bigoplus_n S^7C_\phi) \text{ is identified with} \\
\prod_n KO_0(\text{S}^7C_\phi) / \bigoplus_n KO_0(\text{S}^7C_\phi). \\

When \( M \) has a metric of uniform positive scalar curvature, then by the Lichnerowicz formula we know that \\
P^{(n)}_{uD}(0) \text{ converges to } \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ in the operator norm as } \\
u \to \infty, n \to \infty \text{ and } n \geq N_u. \text{ As a consequence, we know that} \\
\tau_{uD,\tilde{W}_n} \to v \otimes \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + I \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\
\text{and} \\
y_{n,u} \to \exp\left(2\pi i \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) = I \\
in the operator norm as } u \to \infty, n \to \infty \text{ and } n \geq N_u. \text{ Together with the formula for } \chi_{n,u}(s), \text{ we then have } \\
[\prod_n (y_{n,u}, x_{n,u}(s))] = 0 \text{ in } KO_0(\prod_n S^7C_\phi / \bigoplus_n S^7C_\phi). \text{ Therefore it follows that } \mu(q_D) = 0. \square

As mentioned in the introduction, the Gromov-Lawson-Rosenberg conjecture states that a closed spin manifold \( M^n \) with \( n \geq 5 \) has a metric of positive scalar curvature if, and only if, its Dirac index vanishes in \( KO_*(C^*_r\pi) \), where \( \pi = \pi_1(M) \). We formulate now a relative version of this conjecture.

**Conjecture 2.20.** (Relative Gromov-Lawson-Rosenberg) Let \((N, \partial N)\) be a compact spin manifold with boundary. Let \( n \) be the dimension of \( N \). If the relative higher index of the Dirac operator \( D \) is zero in \( KO_*(C^*_r(\pi_1(N), \pi_1(\partial N))) \), then there is a metric of positive scalar curvature on \( N \) that is collared near \( \partial N \).

**Remark 2.21.**

(1) This conjecture is made by analogy to surgery theory, where obstructions to surgery for degree one normal maps have this formal structure.

(2) By the Gromov-Lawson surgery theorem [17, Theorem A] and Schoen-Yau [38, Corollary 6] as improved by Gajer [14], the \( \pi-\pi \) case of the conjecture is correct.
Because of the failure of stability for the ordinary Gromov-Lawson conjecture Schick [36, Example 2.2] and Dwyer-Schick-Stolz [11], we recognize that, in general, this statement cannot be true as stated. One should either interpret it as a stable conjecture (i.e. after crossing with some number of Bott manifolds, see Rosenberg-Stolz [33]) or as a guide to prove the correct statement in the unstable situation. We hope to address this matter in a future paper.

In Rosenberg-Stolz [33], the index map \( \alpha : \Omega^{\text{spin}}_n(B\pi) \to KO_n(C^*_r\pi) \) is factored in the following way:

\[
\Omega^{\text{spin}}_n(B\pi, B\pi) \xrightarrow{D} ko_n(B\pi) \xrightarrow{p} KO_n(B\pi) \xrightarrow{A} KO_n(C^*_r\pi),
\]

where \( p : ko_n(B\pi) \to KO_n(B\pi) \) is the canonical map from connective to periodic \( KO \)-homology and \( A \) is the standard assembly map. This sequence can be generalized to pairs. Let \( (N, \partial N) \) be a manifold with boundary and let \( \pi = \pi_1(N) \) and \( \pi^\infty = \pi_1(\partial N) \). Then we have a composition

\[
\Omega^{\text{spin}}_n(B\pi, B\pi^\infty) \xrightarrow{D} ko_n(B\pi, B\pi^\infty) \xrightarrow{p} KO_n(B\pi, B\pi^\infty) \xrightarrow{A} KO_n(C^*(\pi, \pi^\infty)).
\]

Let \( \text{Pos}^{\text{spin}}_n(B\pi, B\pi^\infty) \) be the subgroup of \( \Omega^{\text{spin}}_n(B\pi, B\pi^\infty) \) consisting of bordism classes represented by pairs \( (M^n, \partial M^n, f) \) for which \( M \) admits a metric of positive scalar curvature that is collared near the boundary.

There is a map from \( \partial M \) to \( B\pi^\infty \) classifying its universal cover \( \tilde{\partial M} \). By elementary homotopy theory, the composite map to \( B\pi \) commutes up to homotopy with the map \( M \to B\pi \) classifying its universal cover \( \tilde{M} \). The homotopy extension principle then implies that we have a map of pairs \( (M, \partial M) \to (B\pi, B\pi^\infty) \).

**Theorem 2.22.** Let \( (M, \partial M) \) be a spin manifold with boundary of dimension \( \geq 6 \). Let \( \pi = \pi_1(M) \) and \( \pi^\infty = \pi_1(\partial M) \). Let \( u : (M, \partial M) \to (B\pi, B\pi^\infty) \) be the map described above. Then \( (M, \partial M) \) has a positive scalar curvature which is collared near the boundary \( \partial M \) if, and only if, the index \( D_*[\Omega_0(M, \partial M), u] \) lies in \( \text{Pos}^{ko}_n(B\pi, B\pi^\infty) \), where \( \text{Pos}^{ko}_n(B\pi, B\pi^\infty) \) is the image of \( D_* \) restricted to \( \text{Pos}^{\text{spin}}_n(B\pi, B\pi^\infty) \).

**Proof.** First we will explain that the capacity of a spin manifold \( M^n \) for \( n \geq 6 \) to admit a positive scalar curvature metric depends only on its spin cobordism class. As in the closed case, this result follows from the Gromov-Lawson surgery theorem, or equivalently the reduction to spin cobordism. For manifolds \( (M, \partial M) \) with boundary whose boundary is collared, there is a relative surgery theorem that follows from an improvement by Gajer [14] of the usual Gromov-Lawson surgery theorem that provides a collared neighborhood for the trace of the surgery. We remind the reader that the Gromov-Lawson theorem holds if the spin cobordism respects fundamental group and the dimension is at least 5. The proof of our theorem requires two applications of the Gajer/Gromov-Lawson Theorem, as we will now demonstrate. Let \( (M, \partial M) \) and \( (M', \partial M') \) be cobordant and suppose that \( (M, \partial M) \) has a metric of positive scalar curvature that is collared near the boundary.
We first apply the surgery theorem to the restricted cobordism between $\partial M$ and $\partial M'$ to endow $\partial M'$ with a metric of positive scalar curvature. Conner [7] allows us to execute a rounding maneuver that replaces metric pieces isometric to a product $\partial M \times Q$ of the boundary with a quadrant $Q$ with metric pieces isometric to the product $\partial M \times H$ of the boundary with a half-space $H$ (and vice versa). Finally we use the Gromov-Lawson theorem to endow $M'$ with a metric of positive scalar curvature that is collared near $\partial M'$. We note that the original proof of the Gromov-Lawson theorem is sufficiently local as to be unchanged by this slight additional generality.

As a next step, we need to show that all the elements of the kernel of the map from relative spin bordism to relative $ko$ have positive scalar curvature, i.e. that $\ker D_n \subseteq \text{Pos}_n^{\text{spin}}(B\pi, B\pi^{\infty})$. Both away from the prime 2 and at the prime 2, the inclusion can be obtained from the relative versions of existing theorems. Away from 2, the result holds by readapting the result of Führing [13] on Baas-Sullivan theory. This result was stated in Rosenberg-Stolz [34] as unpublished work of Jung. Führing proves that a smooth spin closed manifold $M$ of dimension $n \geq 5$ admits a metric of positive scalar curvature if its orientation class in $ko_n(B\pi)$ lies in the subgroup consisting of elements which contain positive representatives. At the prime 2, we can extend Theorem B (2) of Stolz [39]. Here he proves the following. Let $X$ be a topological space. Suppose that $T_n(X)$ is the subgroup of $\Omega_n^{\text{spin}}(X)$ consisting of bordism classes $[E, f \circ p]$, where $p: E \to B$ is an $\mathbb{H}P^2$-bundle over a spin closed manifold $B$ of dimension $n - 8$ and $f$ is a map $B \to X$. Then the map $\Omega_n^{\text{spin}}(X)/T_n(X) \to ko_n(X)$ is a 2-local isomorphism. In the papers of both Führing and Stolz it is effectively shown that the kernel of $D_n$ is a homology theory. As such we can extend these results to pairs.

\begin{corollary}
Let $p: ko_n(B\pi, B\pi^{\infty}) \to KO_n(B\pi, B\pi^{\infty})$ and $A: KO_n(B\pi, B\pi^{\infty}) \to KO_n(C^*(\pi_1, \pi_1^{\infty}))$ be as above, with $n \geq 6$. The Relative Gromov-Lawson-Rosenberg conjecture holds if $p$ and $A$ are both injective.
\end{corollary}

\begin{theorem}
Let $n \geq 6$. Let $N^n$ be a manifold with boundary such that $\pi_1(N)$ and $\pi_1(\partial N)$ are both amenable. Suppose that $\pi_1(\partial N) \to \pi_1(N)$ is an injection and that the cohomological dimensions of $\pi_1(N)$ and $\pi_1(\partial N)$ are less than $n$. If the classifying spaces $B\pi_1(N)$ and $B\pi_1(\partial N)$ are finite complexes, then the Relative Gromov-Lawson-Rosenberg conjecture holds for the pair $(N, \partial N)$.
\end{theorem}

\begin{proof}
Let $A = \pi_1(N)$ and $B = \pi_1(\partial N)$. The $E^2$ term for the Atiyah-Hirzebruch spectral sequence for $KO_n(K(A, 1), K(B, 1))$ is $H_p(A, B; KO_q)$. Similarly the $E^2$ term for $ko_n(K(A, 1), K(B, 1))$ is $H_p(A, B; ko_q)$. The groups coincide when $q \geq 0$. There is a comparison map between the spectral sequences from the $ko$-sequence to the $KO$-sequence which is an isomorphism on $E^2$ for $q \geq 0$. The reason that this map may fail to be an isomorphism on $E^\infty$ is that there are differentials for the $KO$-sequence that can start in the fourth quadrant and end in the first. For this reason, a nonzero element in $ko_n$ can vanish in $KO_n$. But if $n > \max\{cd(A), cd(B)\}$,
differentials can only come from the line \( p + q = n + 1 \) with \( p \leq \max\{cd(A), cd(B)\} \). But then \( q \) is positive and the map is therefore an isomorphism.

Using Higson-Kasparov \cite{19} Theorem 1.1 extended into the \( KO \) setting, we see that the \( KO \)-theory groups of \( C^*_\text{max}(\pi) \) and \( C^*_\text{max}(\pi_\infty) \) are given by the \( KO \)-theories of their classifying spaces. Thus the relative assembly map \( A: KO_n(B\pi, B\pi_\infty) \to KO_n(C^*(\pi_1, \pi_1^\infty)) \) is an isomorphism. The rest of the proof is as the last paragraph of Theorem 2.22.

\[ \square \]

**Remark 2.25.** This unstable version of Conjecture 2.20 for large \( n \) obviously implies the stable version of the conjecture for all \( n \).

## 3. A New Index Theory for Noncompact Manifolds

In this section we will develop a new index theory for a noncompact manifold. Our index theory will depend on a choice of an exhaustion.

**Definition 3.1.** Let \((Y, d)\) be a noncompact, complete metric space. Suppose that \( Y \) is also metrically locally simply connected; i.e. for all \( \varepsilon > 0 \) there is \( \varepsilon' \leq \varepsilon \) such that every ball in \( X \) of radius \( \varepsilon' \) is simply connected. Let \( Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots \) be a sequence of connected compact subsets of \( Y \). We say that \( \{Y_i\} \) is an **admissible exhaustion** if

1. \( Y = \bigcup_{i=1}^{\infty} Y_i \);
2. for each \( j > i \), there is a connected compact subset \( Y_{i,j} \subseteq Y \) such that \( Y_j = Y_{i,j} \cup Y_i \) and \( Y_{i,j} \cap Y_i = \partial Y_i \), where \( \partial Y_i = Y_i - \mathring{Y}_i \) for all \( i \) and \( \mathring{Y}_i \) denotes the interior of \( Y_i \);
3. \( d(\partial Y_i, \partial Y_j) \to \infty \) as \( |j - i| \to \infty \).

Often we will write \( \{Y_i; Y_{i,j}\} \) for the exhaustion.

Let \( \{Y_i; Y_{i,j}\} \) be an admissible exhaustion of \( Y \). Define \( D^*_i \) to be the \( C^* \)-algebra inductive limit given by

\[ D^*_i = \lim_{j \to \infty, j > i} C^*_\text{max}(\pi_1(Y_j), \pi_1(Y_{i,j})) \otimes K, \]

where \( K \) is the \( C^* \)-algebra of compact operators. Let

\[ \prod_{i=1}^{\infty} D^*_i = \left\{ (a_1, a_2, \ldots) : a_i \in D^*_i, \sup_i ||a_i|| < \infty \right\}. \]

There is a natural homomorphism \( \rho_{i+1}: D^*_{i+1} \to D^*_i \) induced by the group homomorphisms given by inclusions of the corresponding spaces. Let \( \rho \) be the homomorphism from \( \prod_{i=1}^{\infty} D^*_i \) to \( \prod_{i=1}^{\infty} D^*_i \) mapping \( (a_1, a_2, \ldots) \) to \( (\rho_2(a_2), \rho_3(a_3), \ldots) \).

We now define the \( C^* \)-algebra \( A(Y) \) by:

\[ A(Y) = \left\{ a \in C \left([0, 1], \prod_{i=1}^{\infty} D^*_i \right) : \rho(a(0)) = a(1) \right\}. \]
Notice that $A(Y)$ is the $C^*$-algebra inverse limit of the $D^*_i$ in a certain homotopical sense. In particular, this $C^*$-algebra encodes dynamical information about how the fundamental groups of the pieces of the exhaustion interact with each other. We emphasize that the definition of $A(Y)$ depends on the exhaustion $\{Y_i\}$ of $Y$. We will now define an index map $\sigma : KO_i^j(Y) = KO^0(C_0(Y)) \to KO_*(A(Y))$.

There exists $\varepsilon_0 > 0$ such that, for any closed subspace $Z$ of $Y$, any operator on a $Z$-module with propagation less than or equal to $\varepsilon_0 > 0$ can be lifted to the universal cover of $Z$. One can prove that the above constant $\varepsilon_0$ exists because $Y$ is metrically locally simply connected (as defined in the beginning of Section 2). The proof is similar to that of Proposition 2.8.

If an operator $F$ represents a class in $KO_0^j(Y)$, for each $\varepsilon < \frac{\varepsilon_0}{100}$, we can choose another operator $F_\varepsilon$ representing the same $KO$-class such that the propagation of $F_\varepsilon$ is smaller than $\varepsilon$. Let

$$\text{ind}_L([F_\varepsilon]) = [P_{F_\varepsilon}] - \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] \in KO_0(C^*_L(Y)),$$

where $P_{F_\varepsilon}$ is the idempotent in the matrix algebra of $C^*_L(Y)^+$ as given in the definition of the local index whose propagation is less than $100\varepsilon < \varepsilon_0$.

Let $P_{F_\varepsilon}^{(j)} = \chi_{Y_j}F_{\varepsilon}\chi_{Y_j}$ and let $\tilde{P}_{F_\varepsilon}^{(j)}$ be the lifting of $P_{F_\varepsilon}^{(j)}$ to $\tilde{Y}_j$, the universal cover of $Y_j$. Let $v$ be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing the generator in $KO_{-1}(C_0(\mathbb{R}^7)) \cong KO_0(C_0(\mathbb{R}^8))$ (see Atiyah [1] or Schröder [35, Proposition 1.4.11]). Let $\tau_{F_\varepsilon}^{(j)} = v \otimes \tilde{P}_{F_\varepsilon}^{(j)} + I \otimes (I - \tilde{P}_{F_\varepsilon}^{(j)})$ and let $(\tau_{F_\varepsilon}^{-1})^{(j)} = v^{-1} \otimes \tilde{P}_{F_\varepsilon}^{(j)} + I \otimes (I - \tilde{P}_{F_\varepsilon}^{(j)})$. For all $s \in [0, 1]$, define $w_{F_\varepsilon}^{(j)}(s)$ to be the product

$$\left( \begin{array}{cc} I & (1-s)\tau_{F_\varepsilon}^{(j)} \\ 0 & I \end{array} \right) \left( \begin{array}{cc} I & 0 \\ -(1-s)(\tau_{F_\varepsilon}^{-1})^{(j)} & I \end{array} \right) \left( \begin{array}{cc} I & (1-s)\tau_{F_\varepsilon}^{(j)} \\ 0 & I \end{array} \right).$$

For each $k$, there exist $j_k > k$ and a sequence of positive numbers $\{\varepsilon_k\}$ converging to 0 such that $100\varepsilon_k < \varepsilon_0$ and $y_k = w_{F_{\varepsilon_k}}^{(j_k)}(0) = (w_{F_{\varepsilon_k}}^{(j_k)}(0))^{-1}$ has propagation less than $\varepsilon_0$ for all $k$, and there is $z_k$ in the matrix algebra of $(S^7C^*_{\text{max}}(\tilde{Y}_{k,j_k})\pi_1(Y_{k,j_k}))^+$ such that $y_k = \phi_{k,j_k}(z_k)$, where $\phi_{k,j_k}$ is the $*$-homomorphism from the matrix algebra of $(S^7C^*_{\text{max}}(\tilde{Y}_{k,j_k})\pi_1(Y_{k,j_k}))^+$ to the matrix algebra of $(S^7C^*_{\text{max}}(\tilde{Y}_{j_k})\pi_1(Y_{j_k}))^+$. Note that the existence of such a $*$-homomorphism follows from Lemma 2.12 and Formula 2.13. The existence of such $z_k$ is a result of the following claim.

**Claim 3.2.** Let $\tilde{Y}_{j_k}$ be the universal cover of $Y_{j_k}$ and let $\pi_k : \tilde{Y}_{j_k} \to Y_{j_k}$ be the covering map. Then we have

$$y_k = \chi_{\pi_k^{-1}(Y_{k,j_k})} y_k \chi_{\pi_k^{-1}(Y_{k,j_k})} \oplus \chi_{\tilde{Y}_{j_k} - \pi_k^{-1}(Y_{k,j_k})}$$

when $k$ and $j_k$ are sufficiently large.

**Proof.** This proof is identical to that of Claim 2.19. \[\square\]
Let $\lambda \in [0, 1]$. We define $z_k' (\lambda)$ by replacing $P^{(j_k)}_{F_{e_k}} (0)$ with $(1 - \lambda) P^{(j_k)}_{F_{e_k}} (0) + \lambda P^{(j_{k+1})}_{F_{e_{k+1}}} (0)$ in the definition of $z_k$. Define $y'_k (\lambda) = \phi_k; j_k (z_k' (\lambda))$. Let $\psi_k$ be the natural homomorphism $\psi_k : S^7 C^*_{\max} (Y_{j_k}) \pi_i (Y_{j_k}) \rightarrow S^7 C^*_{\max} (Y_{j_{k+1}}) \pi_i (Y_{j_{k+1}})$. Let

$$\tau_k (\lambda) = v \otimes \left( (1 - \lambda) \psi_k (P^{(j_k)}_{F_{e_k}} (0)) + \lambda P^{(j_{k+1})}_{F_{e_{k+1}}} (0) \right) + I \otimes I - \left( (1 - \lambda) \psi_k (P^{(j_k)}_{F_{e_k}} (0)) + \lambda P^{(j_{k+1})}_{F_{e_{k+1}}} (0) \right)$$

and

$$\tau'_k (\lambda) = v^{-1} \otimes \left( (1 - \lambda) \psi_k (P^{(j_k)}_{F_{e_k}} (0)) + \lambda P^{(j_{k+1})}_{F_{e_{k+1}}} (0) \right) + I \otimes I - \left( (1 - \lambda) \psi_k (P^{(j_k)}_{F_{e_k}} (0)) + \lambda P^{(j_{k+1})}_{F_{e_{k+1}}} (0) \right)$$

for all $\lambda \in [0, 1]$. Again, the existence of $\psi_k$ follows from Lemma 2.12 and Formula 2.13. For all $s, \lambda \in [0, 1]$, define $(w_k (s))(\lambda)$ to be the product

$$\begin{pmatrix}
I & (1 - s) \tau_k (\lambda) \\
0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-(1 - s) \tau'_k (\lambda) & I
\end{pmatrix}
\begin{pmatrix}
I & (1 - s) \tau_k (\lambda) \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}.
$$

Define

$$(c_k (s))(\lambda) = (w_k (s))(\lambda) \begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} \frac{1}{(w_k (s))(\lambda)}.$$

By the definition of $z_k'$ and $c_k$, for each $\lambda \in [0, 1]$, the pair $(z_k' (\lambda), (c_k (\cdot))(\lambda))$ lies in $(S^7 D^*_k)^+$, where $D^*_k$ is as in the definition of $A(Y)$. Let $a_k = (z_k', c_k)$. We finally define the index of $F$ in $KO_0 (A(Y))$ to be

$$\sigma ([F]) = [(a_1, a_2, \ldots)] - \left[ \begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} \right] \in KO_0 (A(Y)).$$

One can similarly define the index map $\sigma : KOB^{n} (Y) \rightarrow KO_n (A(Y))$ when $n \neq 0 \mod 8$ with the help of suspensions (refer back to the section after Theorem 2.17).

The proof of the following vanishing theorem contains some of the same elements as are found in Section 2, but now in the context of a noncompact manifold $M$.

**Theorem 3.3.** Let $Y$ be a noncompact space with an admissible exhaustion $\{Y_i\}$. Let $M$ be a noncompact manifold. Assume that there is a uniformly continuous proper coarse map $f : M \rightarrow Y$ with an admissible exhaustion $\{M_i; M_{i,j}\}$ of $M$ such that each $M_i$ is a compact manifold with boundary $\partial M_i$, $f^{-1}(Y_i) = M_i$, $f^{-1}(Y_{i,j}) = M_{i,j}$ and $f^{-1}(\partial Y_i) = \partial M_i$. Suppose that $M$ is spin and let $D_M$ be the Dirac operator on $M$. If $M$ admits a metric of uniform positive scalar curvature, then the index $\sigma (f_* [D_M])$ of $D_M$ is zero in $KO_*(A(Y))$, where $f_* : KOB^{n} (M) \rightarrow KO^{n} (Y)$ is the homomorphism induced by $f$.

**Proof.** We assume that the dimension of $M$ is $0 \mod 8$. The other cases can be handled in a similar way with the help of suspensions (refer back to the section after Theorem 2.17).

Let $Y_i$, $M_i$, $Y_{i,j}$ and $M_{i,j}$ be given as in the statement of the theorem. In this proof, all $C^*$-algebras are the maximal ones.

Let $f$ be an odd smooth chopping function on the real line satisfying the following conditions: (1) $f(x) \rightarrow \pm 1$ as $x \rightarrow \pm \infty$; (2) $g = f^2 - 1 \in S(\mathbb{R})$, the space of Schwartz
functions, (3) if \( \hat{f} \) and \( \hat{g} \) are the Fourier transforms of \( f \) and \( g \), respectively, then \( \text{Supp}(\hat{f}) \subseteq [-1, 1] \) and \( \text{Supp}(\hat{g}) \subseteq [-2, 2] \). As stated earlier, such an odd chopping function exists (cf. Roe \[30, Lemma 7.5\]).

Let \( D_M \) be the Dirac operator on \( M \). We define
\[
F = f(D_M) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) \exp(itD_M) dt.
\]

Let
\[
\text{ind}_L([F]) = [P_F] - \left[ \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] \in KO_0(C^*_L(M)),
\]

where \( P_F \) is the idempotent in the matrix algebra of \( (C^*_L(M))^+ \) as given in the definition of the local index.

Recall that there exists \( \varepsilon_0 > 0 \) such that, for any closed subspace \( Z \) of \( Y \), any operator on a \( Z \)-module with propagation less than or equal to \( \varepsilon_0 \) can be lifted to the universal cover of \( Z \). Define \( P_F^{(j)} = \chi_{M_j} P_F \chi_{M_j} \), where \( \chi_{M_j} \) is the characteristic function of \( M_j \). Let \( n_0 \) be the smallest natural number such that \( n_0 > \frac{2}{\varepsilon_0} \). We write
\[
\exp(itD_M) = \exp\left( \frac{it}{n_0} D_M \right) \cdots \exp\left( \frac{it}{n_0} D_M \right).
\]

Let \( j' > j \) be the smallest integer such that
\[
d(M - M_{j'}, M_j) > 10n_0\varepsilon_0.
\]

Here \( n_0\varepsilon_0 \) is roughly 2. Let \( \tilde{M}_{j'} \) be the universal cover of \( M_{j'} \). Using the formula for \( P_F \) in terms of \( \exp(itD_M) \), the identity (*) and the fact that \( \exp\left( \frac{it}{n_0} D_M \right) \) has propagation less than \( \varepsilon_0 \) for all \( t \in [-2, 2] \), we can lift \( P_F^{(j)} \) to an element \( \tilde{P}_F^{(j)} \) in \( (C^*_L(\tilde{M}_{j'}))^+ \).

For any \( i < j \), let \( m_{i,j} = i + [\frac{i-j}{2}] \) and \( m_{i,j}' = i + [\frac{j-i}{2}] \), where \( [\frac{i-j}{2}] \) and \( [\frac{j-i}{2}] \) are respectively the integer parts of \( \frac{i-j}{2} \) and \( \frac{j-i}{2} \).

We define
\[
P_F^{(i,j)} = \chi_{M_{m_{i,j}'} \cdot j} P_F \chi_{M_{m_{i,j}'} \cdot j},
\]

where \( \chi_{M_{m_{i,j}'} \cdot j} \) is the characteristic function of \( M_{m_{i,j}'} \cdot j \). Let \( v \) be an invertible element in the matrix algebra of \( C_0(\mathbb{R}^7)^+ \) representing the generator in \( KO_{-1}(C_0(\mathbb{R}^7)) \cong KO_0(C_0(\mathbb{R}^8)) \).

Define
\[
\tau_{i,j} = v \otimes P_F^{(i,j)}(0) + I \otimes (I - P_F^{(i,j)}(0))
\]
and
\[
\tau_{i,j}' = v^{-1} \otimes P_F^{(i,j)}(0) + I \otimes (I - P_F^{(i,j)}(0)).
\]

Define
\[
x_{i,j} = \begin{pmatrix} I & \tau_{i,j} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\tau_{i,j}' & I \end{pmatrix} \begin{pmatrix} I & \tau_{i,j} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}
\]
and

\[ u_{i,j} = x_{i,j} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} x_{i,j}^{-1}. \]

Let \(|j - i|\) be large enough such that

\[ d(M_i, M - M_{m_{i,j}}) > 10n_0 \varepsilon_0. \]

We define \(v_{i,j} \in (S^7C^*(M_{i,m_{i,j}}))^+\) and \(w_{i,j} \in (S^7C^*(M_{i,m_{i,j}}))^+\) by:

\[ v_{i,j} = \chi_{M_{i,m_{i,j}}} u_{i,j} \chi_{M_{i,m_{i,j}}}, \]

\[ w_{i,j} = \chi_{M_{m_{i,j}}} u_{i,j} \chi_{M_{m_{i,j}}}, \]

where the identities in \((S^7C^*(M_{i,m_{i,j}}))^+\) and \((S^7C^*(M_{i,m_{i,j}}))^+\) are respectively identified with the multiplication operators by \(\chi_{M_{i,m_{i,j}}}\) and \(\chi_{M_{m_{i,j}}}\). By the propagation of \(P_F\) and the formula for \(u_{i,j}\), we have

\[ u_{i,j} = v_{i,j} \oplus w_{i,j}. \]

This equality is proved exactly in the same manner as Claims 2.19 and 3.2. From the definitions of \(v_{i,j}\) and \(w_{i,j}\), we then have \(\text{prop}(v_{i,j}) < 100n_0 \varepsilon_0\) and \(\text{prop}(w_{i,j}) < 100n_0 \varepsilon_0\).

Let \(\tilde{M}_{i,j'}\) be the universal cover of \(M_{i,j'}\) and let \(\pi_{i,j'}\) be the covering map from \(\tilde{M}_{i,j'}\) of \(M_{i,j'}\). Again using the identity (*) and the formula for \(P_F\) in terms of \(\exp(itD_M)\) and the small propagation of \(\exp(tD_M)\), we can lift \(P_F^{(i,j)}\) to an element \(\tilde{P}_F^{(i,j)}\) in \((C^*(\tilde{M}_{i,j'}))^+\), where \(j'\) is defined as in the construction of the lifting of \(P_F^{(j)}\). Let \(\tilde{u}_{i,j}\) be the lifting of \(u_{i,j}\) to \(\tilde{M}_{i,j'}\). Let \(|j - i|\) be large enough such that

\[ d(M_i, M - M_{m_{i,j}}) > 100n_0 \varepsilon_0. \]

We define \(\tilde{v}_{i,j} \in (S^7C^*(\tilde{M}_{i,m_{i,j}}))^+\) and \(\tilde{w}_{i,j} \in (S^7C^*(\tilde{M}_{m_{i,j},j'}))^+\) to be the liftings of \(v_{i,j}\) and \(w_{i,j}\) respectively, where \(\tilde{M}_{i,m_{i,j}} = \pi_{i,j}^{-1}(M_{i,m_{i,j}})\) and \(\tilde{M}_{m_{i,j},j'} = \pi_{i,j'}^{-1}(M_{m_{i,j},j'})\). We have

\[ \tilde{u}_{i,j} = \tilde{v}_{i,j} \oplus \tilde{w}_{i,j}. \]

Next we shall represent the index class \(\sigma([D_M])\) as a KO-theory element explicitly constructed using the above liftings.

Let \(\{j_k\}\) be a sequence of integers such that \(j_k > k\) for each \(k\) and \(j_k - k \to \infty\) as \(k \to \infty\). Let \(z_k\) be the image of \(\tilde{w}_{k,j_k}\) under the inclusion map from \((S^7C^*(\tilde{M}_{m_{k,j_k},j'}))^+\) to \((S^7C^*(\tilde{M}_{k,j_k}))^+\).

Let \(\pi_{k,j'_k}\) be the covering map from the universal cover of \(\tilde{M}_{j_k}\) to \(\tilde{M}_{j_k}\). Let \(y_k\) be the element in the image of the inclusion map from \((S^7C^*(\tilde{M}_{k,j_k}))^+\) to \((S^7C^*(\tilde{M}_{j_k}))^+\) defined by

\[ y_k = \phi_{k,j'_k}(z_k), \]

where \(\phi_{k,j'_k}\) is the homomorphism from \((S^7C^*(\tilde{M}_{k,j'_k})^+\) to \((S^7C^*(\tilde{M}_{j'_k})^+\).

Note that the existence of this homomorphism follows from Lemma 2.12 and Formula 2.13.
As before, let $\psi_k$ be the natural map $: S^7 C^*(\tilde{M}_{j_k}) \rightarrow S^7 C^*(\tilde{M}_{j_{k+1}})$. We similarly define $z_k'(\lambda)$ by replacing $P_F^{(j_k)}(0)$ with $(1 - \lambda)P_F^{(j_k)}(0) + \lambda P_F^{(j_{k+1})}(0)$ in the definition of $z_k$. Define $y_k'(\lambda) = \phi_{k,j_k}(z_k'(\lambda))$.

Let 

$$\tau_k(\lambda) = v \otimes \left( (1 - \lambda)\psi_k(\tilde{P}_F^{(j_k)}(0)) + \lambda \tilde{P}_F^{(j_{k+1})}(0) \right) + I \otimes \left( I - \left( (1 - \lambda)\psi_k(\tilde{P}_F^{(j_k)}(0)) + \lambda \tilde{P}_F^{(j_{k+1})}(0) \right) \right),$$

and 

$$\tau_k'(\lambda) = v^{-1} \otimes \left( (1 - \lambda)\psi_k(\tilde{P}_F^{(j_k)}(0)) + \lambda \tilde{P}_F^{(j_{k+1})}(0) \right) + I \otimes \left( I - \left( (1 - \lambda)\psi_k(\tilde{P}_F^{(j_k)}(0)) + \lambda \tilde{P}_F^{(j_{k+1})}(0) \right) \right)$$

for all $\lambda \in [0, 1]$. For all $s, \lambda \in [0, 1]$, define $(w_k(s))(\lambda)$ to be the product

$$(c_k(s))(\lambda) = (w_k(s))(\lambda) \left( \begin{array}{cc} I & 0 \\ 0 & (w_k(s))(\lambda)^{-1} \end{array} \right).$$

Note that, for each $\lambda \in [0, 1]$, the pair $(z_k'(\lambda), (c_k(\cdot))(\lambda))$ lies in $(S^7 D_k^*)$, where $D_k^*$ is as in the definition of $A(Y)$.

Let $a_k = (z_k', c_k)$. By a homotopy invariance argument, we have

$$\sigma([D_M]) = [(a_1, a_2, \ldots)] - \left[ \begin{array}{cc} I \\ 0 \end{array} \right] \in KO_0(A(Y)).$$

In the above construction, for each $\alpha \geq 1$, we can replace respectively the Dirac operator $D_M$ by $\alpha D_M$, $n_0$ by $[\alpha n_0] + 1$, and $j_k'$ by another natural number $j_k'$ satisfying $d(M - M_{j_k'}) > 10\alpha n_0 \in \mathbb{Z}$ to obtain the index of $\alpha D_M$:

$$\sigma([\alpha D_M]) = [(a_1, a_2, \alpha, \ldots)] - \left[ \begin{array}{cc} I \\ 0 \end{array} \right] \in KO_0(A(Y)).$$

Notice that the $KO$-theory class $\sigma([\alpha D_M]) \in KO_0(A(Y))$ is independent of the choice of $\alpha$.

For all $k$, we write $a_{k,\alpha} = (z_k', c_{k,\alpha})$. Let $\tau_{k,\alpha}$ by replacing with $D$ with $\alpha D$ in the definition of $\tau_k$. By the assumption that $M$ has uniform positive scalar curvature and the local nature of the Lichnerowicz formula, we have

$$\tau_{k,\alpha} \rightarrow v \otimes \left( \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) + I \otimes \left( \begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right)$$

in the operator norm when $\alpha \rightarrow \infty$. This result implies the vanishing of $\sigma([D_M])$. \qed
Using the above notation $D^*_i$, we note that, by Guentner-Yu [18], there is a Milnor exact sequence given by

$$0 \rightarrow \limleft KO_*(D^*_i) \rightarrow KO_*(A(Y)) \rightarrow \limleft KO_*(D^*_i) \rightarrow 0. \quad (*)$$

This sequence gives rise to a commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & \limleft KO_*(Y, \partial Y) & \rightarrow & KO_*(Y) & \rightarrow \limleft KO_*(D^*_i) & \rightarrow 0 \\
0 & \rightarrow & \limleft KO_*(D^*_i) & \rightarrow & KO_*(A(Y)) & \rightarrow & \limleft KO_*(D^*_i) & \rightarrow 0 \\
\end{array}$$

where the map $\phi: KO_*(A(Y)) \rightarrow \limleft KO_*(D^*_i)$ is induced by the $\ast$-homomorphism $\pi_i: A(Y) \rightarrow D^*_i$ from

$$A(Y) \equiv \left\{a \in C \left([0, 1], \prod_{i=1}^{\infty} D^*_i\right) : \rho(a(0)) = a(1)\right\}$$

to $D^*_i$ obtained from the $i$-th component of the evaluation at 0. We will use this diagram in the next section.

4. A MANIFOLD WITH EXOTIC POSITIVE SCALAR CURVATURE BEHAVIOR

We will now construct a noncompact manifold $M$ endowed with a nested exhaustion of compact subsets $M_i$, such that the $M_i$ can be endowed with positive scalar metrics which are in totality incompatible in the sense that $M$ itself has no metric of uniformly positive scalar curvature.

In the last section we introduced a Milnor exact sequence with a $\limleft$ term. We quickly review some properties of this functor. If $\{G_i\}$ is an inverse system of abelian groups indexed by the positive integers together with a coherent family of maps $f_{j,i}: G_j \rightarrow G_i$ for all $j \geq i$, then $\limleft G_i$ is categorically defined to be the first derived functor of $\limleft$. Eilenberg-Moore [12] also provides a description in the following. If $\Psi: \prod G_i \rightarrow \prod G_i$ is defined by $\Psi(g_i) = (g_i - f_{i+1,i}(g_i))$, then $\limleft G_i$ is defined by $\limleft G_i \equiv \text{coker}(\Psi)$. Gray [16] proves that, if each $G_i$ is countable, then $\limleft G_i$ is either zero or uncountable.

An example of an inverse system with a nontrivial $\limleft$ term is

$$S = \left\{\mathbb{Z}^{\times 3} \xleftarrow{\times 3} \mathbb{Z}^{\times 3} \xleftarrow{\times 3} \cdots \right\}$$

in which case we have the uncountable group $\limleft S = \hat{\mathbb{Z}}_3 / \mathbb{Z}$. Let $S^1$ denote the standard circle. Consider the composite mapping cylinder $B_S$ of the infinite composite

$$S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \cdots$$

which is capped off at the left end (see picture below), where each map takes $z \in S^1$ to $z^3$. 

Let $Y_j$ be the given exhaustion of $B_S$. For each $i$, let $\phi_i: (Y_{i+1}, \partial Y_{i+1}) \to (Y_i, \partial Y_i)$ be the obvious collapse map. Notice that $Y_i$ is contractible and that $\partial Y_j$ is a circle for all $j$. Consider the sequence

$$0 \to \lim_{\leftarrow}^1 K\!O_n(Y_j, \partial Y_j) \to K\!O_n^f(B_S) \to \lim_{\leftarrow}^1 K\!O_n(Y_j, \partial Y_j) \to 0.$$ 

Proposition 4.1. The group $\lim_{\leftarrow}^1 K\!O_0(Y_j, \partial Y_j)$ is nontrivial.

Proof. We have an exact sequence

$$K\!O_2(Y_i) \to K\!O_2(Y_i, \partial Y_i) \cong K\!O_2(Y_i, \partial Y_i) \to K\!O_1(\partial Y_i) \to K\!O_1(Y_i).$$

By the contractibility of $Y_i$, we have $\widetilde{K\!O}_2(Y_i) = 0$ and $K\!O_1(Y_i) = 0$. Therefore $K\!O_2(Y_i, \partial Y_i) \to K\!O_1(\partial Y_i)$ is an isomorphism. Consider the commutative square:

$$\begin{array}{c}
K\!O_2(Y_i, \partial Y_i) \xrightarrow{\partial} K\!O_1(\partial Y_i) \\
\downarrow \phi \quad \quad \quad \downarrow \times 3 \\
K\!O_2(Y_{i-1}, \partial Y_{i-1}) \xrightarrow{\partial} K\!O_1(\partial Y_{i-1})
\end{array}$$

Clearly it follows that $\phi_*$ is multiplication by 3, so that at the level of quotients we have a map $\phi_*: K\!O_2(Y_{i+1}, \partial Y_{i+1}) \to K\!O_2(Y_i, \partial Y_i)$. □

Theorem 4.2. Let $[c] \in K\!O_n^f(B_S)$, where $B_S$ is endowed with an exhaustion by compact sets $\{Y_i\}$ as above. There is $(M, f) \in \Omega^{spin}(B_S)$ such that

1. $f_*[D_M] = [c]$;
2. the inverse images $(M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)$ are compact manifolds with boundary such that the induced maps $\pi_1(M_i) \to \pi_1(Y_i)$ and $\pi_1(\partial M_i) \to \pi_1(\partial Y_i)$ are all isomorphisms.
Proof. We consider a proper map \( g: W^n \to B_S \) with \( n \geq 5 \) from a noncompact manifold \( W^n \) to \( B_S \). Notice that every spin cobordism class in \( \Omega_n^{\text{spin}}(pt) \) with \( n \geq 2 \) has a simply connected manifold representative, and consequently for any compact polyhedron \( Z \) and any \( n > \dim(Z) + 1 \), every class in \( \Omega_n^{\text{spin}}(Z) \) has a manifold representative \( N \) with \( \pi_1(N) \cong \pi_1(Z) \). Now we apply this observation to all of the pieces of the exhaustion of \( X \) (i.e. first apply it for the inverse images \( g^{-1}(S^1) \) of the separating circles, and then relatively to the inverse images \( g^{-1}(Y_i - Y_{i-1}) \) of the annular regions \( Y_i - Y_{i-1} \)). These pieces can be assembled to procure the required \( (M, f) \in \Omega_n^{\text{spin}}(B_S) \).

\[ \text{Proof.} \]

\[ \text{We have a commutative diagram} \]

\[ \begin{array}{ccccccccc}
0 & \to & \lim KO_{n+1}(M_i, \partial M_i) & \to & KO^f(M) & \to & \lim KO_n(M_i, \partial M_i) & \to & 0 \\
0 & \to & \lim KO_{n+1}(Y_i, \partial Y_i) & \to & KO^f(B_S) & \to & \lim KO_n(Y_i, \partial Y_i) & \to & 0 \\
0 & \to & \lim KO_{n+1}(D^*_i) & \to & KO_n(A(B_S)) & \to & \lim KO_n(D^*_i) & \to & 0.
\end{array} \]

By the definition of \( D^*_i \) and homotopy invariance, we have

\[ D^*_i \cong C_{\text{max}}^*(\pi_1(Y_i), \pi_1(\partial Y_i)) \otimes K. \]

Since the \( Y_i \) are contractible and the \( \partial Y_i \) are circles, the outer vertical arrows from the second to third row are isomorphisms by Theorem 2.17 so the map \( KO^f_n(B_S) \to KO_n(A(B_S)) \) is also an isomorphism. Note that by construction the element \( \xi \) in \( KO^f_n(B_S) \) will lift to the Dirac class \([D_M]\) in \( KO^f(M) \). By the commutativity of the diagram, the image of \([D_M]\) is zero in \( \lim KO_n(M_i, \partial M_i) \) so it is zero in \( \lim KO_n(D^*_i) \). Therefore it is zero in each \( KO_n(D^*_i) \).

Finally we establish the veracity of the relative Gromov-Lawson-Rosenberg conjecture in our case. Indeed, with respect to the multiplication map \( f: Z \to \mathbb{Z}_2 \), we have \( \Omega_n^{\text{spin}}(f) = \Omega_{n-1}(\ast) \otimes \mathbb{Z}_3 \). Consider the index map of the relative Dirac operator \( D: \Omega_n^{\text{spin}}(f) \to \mathbb{Z}_3 \). When \( n \not\equiv 1 \mod 4 \), we have \( \ker(D) = 0 \). Otherwise, when \( n \equiv 1 \mod 4 \), the kernel \( \ker(D) \) is \( S^1 \times \ker(\Omega_{n-1}(\ast) \to \mathbb{Z}) \). This latter kernel \( \ker(\Omega_{n-1}(\ast) \to \mathbb{Z}) \) consists of closed manifolds of positive scalar curvature according to Stolz. Therefore there is a positive scalar curvature metric on each piece \( M_i \) of the exhaustion that is collared around the boundary. Moreover the image of
$[D_M]$ is nonzero in $KO_n(A(B_S))$ so $M$ itself has no metric of uniformly positive scalar curvature by Theorem 3.3.

5. A manifold with uncountably many connected components of positive scalar curvature metrics

In this section we use the previously developed theory to identify a connected noncompact manifold $M$ such that $PS(M)$, the space of complete positive scalar curvature metrics on $M$ equipped with the $C^\infty$-topology, has uncountably many connected components.

In various spin cases, it can be shown using index theory that $PS(M)$ has infinitely many concordance classes. In fact, one can prove that the 7-sphere $S^7$ is such a manifold (see Gromov-Lawson [17] or Lawson-Michelsohn [24]). With separability and the openness of positivity, it is an argument in point-set topology to see that $PS(M)$ has at most countably many components when $M$ is compact. These properties may fail in the noncompact case. In the compact open topology, positivity is not necessarily an open condition. In the uniform topology, we typically do not have separability.

In the proof of the following theorem, we refer the reader to the paper of Xie-Yu [42, Theorem A], which develops the notion of a relative higher index $\text{ind}_{D_{g_1,g_2}}$ on a spin closed manifold $N$ with two Riemannian metrics $g_1$ and $g_2$. This relative index is defined to be the higher index of the Dirac operator $D_{g_1,g_2}$ on the infinite cylinder $N \times \mathbb{R}$, where the cross section $N \times \{x\}$ is endowed with $g_1$ if $x < -1$ and with $g_2$ if $x > 1$ and the metric in $N \times [-1, 1]$ can be chosen to be arbitrary. The nonvanishing of this relative index in $KO_*(C^*_\pi_1(N))$ gives information about the concordance classes of positive scalar curvature metrics on $N$. In the case when the manifold $M$ is not compact but has an admissible exhaustion by compact sets, a similar theory shows that a relative higher index can be constructed in $KO_*(A(M))$, where $A(M)$ is the algebra constructed in section 3.

Prior to the theorem we also make the following observation. Let $\pi$ be a fixed finitely presented group with generators $g_1, \ldots, g_s$ and relations $r_1, \ldots, r_t$. Let $n \geq 5$ and execute $s$ successive 0-surgeries on $\mathbb{S}^n$ to produce a manifold $K'$ with fundamental group $F_s$, the free group on $s$ generators. The process of surgery on maps (see Ranicki [29] for explicit details) shows that one can then perform a 1-surgery on $K'$ to produce a manifold with fundamental group $F_s/\langle r_1 \rangle$, where $\langle r_1 \rangle$ is the subgroup of $F_s$ normally generated by $r_1$. After performing these 1-surgeries successively with respect to $r_2, \ldots, r_t$, we obtain a manifold $K$ with fundamental group $\pi$. Since $K$ is constructed from the sphere $\mathbb{S}^n$ from surgeries of codimension at least 3, it follows from the Gromov-Lawson surgery theorem [17, Theorem A] that $K$ also has a metric of positive scalar curvature.
An easy example of a manifold with uncountably many components of positive scalar curvature metrics is the disjoint union of countably many copies of $S^7$. Here we present a connected example.

**Theorem 5.1.** There is a connected noncompact manifold $M$ for which the set $PS(M)$ of components of positive scalar curvature metrics on $M$ is uncountable.

**Proof.** We provide a general construction that provides a host of examples. Let $W$ be an $(n+1)$-dimensional spin manifold with nontrivial higher $\hat{A}$-genus. For example, we may take $W$ to be the torus $T^{n+1}$. By the discussion above, we can produce a manifold $N^n$ with a positive scalar curvature metric $\alpha$ and $\pi_1(N) = \pi$. We can perform a $0$-surgery on the disjoint union of $N \times I$ and $W$ to create a connected manifold $X'$. Let $\pi' = \pi_1(X')$ with a classifying map $\alpha: \pi' \to B\pi'$. Let $[\beta]$ represent the class of $\beta$ in $\Omega_{n+1}^{spin}(B\pi')$. Execute additional surgeries (via surgery on maps) on $X'$ to arrive at a manifold $X$ with fundamental group $\pi$ and two boundaries components both homeomorphic to $N$. (In other words, since $\pi' = \pi \ast \pi$, we kill each element of $\pi'$ of the form $g_1g_2^{-1}$, where $g_1$ and $g_2$ represent the same element of $\pi$. The fundamental group of $X$ may not be $\pi'$, but, since $X$ is cobordant to $X'$, there is a map $X \to B\pi'$ which is also represented by $[\beta]$ in $\Omega_{n+1}^{spin}(B\pi')$.

Let $\alpha'$ be the positive scalar curvature metric on $N$ as the other boundary component of $X$, as constructed by the Gromov-Lawson surgery theorem. Let $D_{\alpha,\alpha'}$ be the Dirac operator on $N \times \mathbb{R}$. Here the Riemannian metric on $N \times (-\infty, -1)$ is defined using the product metric of $\alpha$ on $N$ and the standard metric on $(-\infty, -1)$, and the metric on $N \times (1, \infty)$ is defined using the product metric of $\alpha$ on $N$ and the standard metric on $(1, \infty)$. The metric on $N \times [-1, 1]$ can be arbitrary. From the nontriviality of the higher $\hat{A}$-genus for $W$, we can infer that the relative higher index $\text{ind} D_{\alpha,\alpha'}$ of $N$ is nonzero in $KO_*(C^*_\pi \mathbb{R})$.

This nonzero condition implies that $\alpha$ and $\alpha'$ lie in different connected components of $PS(N)$. Form the infinite connected sum $M = N \# N \# \cdots$ with the obvious exhaustion $M_i = (N \# \cdots \# N) - \mathbb{D}^n$. Apply the Gromov-Lawson surgery theorem to modify the metric near each glueing so that $M$ is positively curved at every point. On each summand $N$ we make a choice to endow $N$ with either $\alpha$ or $\alpha'$. Clearly the number of metrics on $M$ constructed in this way is uncountable, and these metrics are all in different connected components of $PS(M)$ by an application of our relative higher index, which lies in $KO_*(A(M))$ as explained in the following.

If $\beta, \beta'$ are two distinct metrics on $M$ defined in this way, let $D_{\beta,\beta'}$ be the Dirac operator on the product $M \times \mathbb{R}$. Here the metric on $M \times (-\infty, -1)$ is defined using the product metric of $\alpha$ on $N$ and the standard metric on $(-\infty, -1)$, and the metric on $M \times (1, \infty)$ is defined using the product metric of $\alpha$ on $N$ and the standard metric on $(1, \infty)$. The metric on $M \times [-1, 1]$ can be arbitrary. We can define a relative higher index $\text{ind}(D_{\beta,\beta'})$ of $D_{\beta,\beta'}$ in $KO_*(A(M))$. By the relative higher index theorem in Xie-Yu [12, Theorem A], the relative higher index $\text{ind}(D_{\alpha,\alpha'})$ does not lie in the image of the map $i_*: KO_*({\mathbb{R}}) \to KO_*(C^*_\pi \mathbb{R})$, where $\mathbb{R}$ is
the one-dimensional real $C^*$-algebra and \( i : \mathbb{R} \to C^*_\pi(N) \) is the inclusion map. The Pimsner Theorem (see [27, Theorem 18]) allows us to compute \( KO_*(D^*_i) \). The above facts and the relative index theorem of [42] imply that if \( \beta, \beta' \) are two distinct metrics on \( M \) defined as above, then \( (\pi_i)_*(\text{ind}(D_{\beta, \beta'})) \) is nonzero in \( KO_*(D^*_i) \) for \( i \) sufficiently large. Here \( \pi_i : A(M) \to D^*_i \) is the *-homomorphism defined in Section 3. The Mihajlov sequence \((*)\) given after Theorem 3.3 tells us that the index is nonzero in \( KO_*(A(M)) \). Therefore \( \beta \) and \( \beta' \) are two metrics of \( M \) that lie in different connected components of \( PS(M) \). Since \( \beta \) and \( \beta' \) are arbitrary, it follows that \( PS(M) \) has uncountably many components.

\[ \square \]

In a sequel to this paper, we will provide examples of noncompact contractible spaces with exotic positive scalar curvature behavior.

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