Abstract. In this paper, we consider the problem of learning a (first-order) theorem prover where we use a representation of beliefs in mathematical claims instead of a proof system to search for proofs. The inspiration for doing so comes from the practices of human mathematicians where a proof system is typically used after the fact to justify a sequence of intuitive steps obtained by “plausible reasoning” rather than to discover them.

Towards this end, we introduce a probabilistic representation of beliefs in first-order statements based on first-order distributive normal forms (dnfs) devised by the philosopher Jaakko Hintikka. Notably, the representation supports Bayesian update and does not enforce that logically equivalent statements are assigned the same probability—otherwise, we would end up in a circular situation where we require a prover in order to assign beliefs. We then examine (1) conjecturing as (statistical) model selection and (2) an alternating-turn proving game amenable (in principle) to self-play training to learn a prover that is both complete in the limit and sound provided that players maintain “reasonable” beliefs. Dnfs have super-exponential space requirements so the ideas in this paper should be taken as conducting a thought experiment on “learning to prove”. As a step towards making the ideas practical, we will comment on how abstractions can be used to control the space requirements at the cost of completeness.

§1. Introduction. The process of discovering a mathematical proof can be seen as a perfect information game where the goal is to show that a path exists (i.e., the proof) between a given starting state (i.e., the axioms) and ending state (i.e., the claim) using a predefined collection of rules (i.e., deduction). Like other perfect information games such as Go and Chess, the complexity of the theorem proving game involves managing the combinatorial nature of the search space. We can do this, for instance, by identifying useful heuristics and patterns. This is one sense in which players can learn and improve from their experiences playing the game.

The idea of “learning from experience” suggests that we can apply machine learning to learn these heuristics and patterns as opposed to distilling them manually from human experience. Towards this end, researchers have demonstrated that machine learned algorithms can navigate the search spaces of Go [70] and Chess [71] at a level exceeding human experts (i.e.,
consistently defeat the best human players). Researchers have also experimented with applying machine learning to theorem provers (e.g., see [47, 44, 25, 15, 69, 43, 41, 50, 45, 38]), although the problem is much more difficult compared to Go and Chess when quantifiers are involved.

In this paper, we consider the problem of learning a prover for first-order theories, a well-understood setting with quantification, where we use a representation of beliefs in mathematical claims instead of a proof system to search for proofs. The inspiration for doing so comes from the practices of human mathematicians where a proof system is typically used after the fact to justify a sequence of intuitive steps obtained by “plausible reasoning” rather than to discover them.

We start by showing how distributive normal forms (dnfs) of first-order logic (Section 2) devised by the philosopher Jaakko Hintikka can be used to formulate a representation of beliefs in the validity of first-order mathematical statements (Section 3). Notably, the representation supports Bayesian update and does not enforce that logically equivalent statements are assigned the same probability—otherwise, we would end up in a circular situation where we require a prover in order to assign beliefs. The idea of assigning weights to dnfs has been proposed by Hintikka [33] in the context of inductive philosophy so the idea is not new. Our contribution here is to extract and formalize some of these ideas for the purposes of “learning to prove”.

Next, we consider two applications that our shift in viewpoint from deductive inference in a proof system to probabilistic inference on beliefs has for “learning to prove”. First, we identify conjecturing as a form of (statistical) model selection (Section 4). Second, we introduce an alternating-turn game that agents can play to learn a prover that is both complete in the limit and sound provided that players maintain “reasonable” beliefs (Section 5). The game involves determining the consistency of statements, and

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1 The state spaces of Chess and Go, albeit large, are finite. In contrast, quantifiers can range over infinite domains.
2 First-order logic along with the axioms of set theory are expressive—they are in principle sufficient to encode most of modern mathematics, although humans generally work at a higher level of abstraction and within a natural language extended with mathematical concepts as opposed to a formal language.
3 Automated theorem provers typically implement a proof-theoretic approach. The literature on automated theorem proving is expansive (e.g., see [18] for a survey of first-order methods which are often based on proof theory).
4 Pólya has written extensively on plausible reasoning, i.e., the heuristic and non-deductive aspects of mathematical reasoning, including (1) weighing evidence for and against a conjecture, (2) making physical analogies, and (3) reasoning from randomness (e.g., see [58, 59, 60]).
5 The non-deductive aspects of mathematical reasoning has been recognized by mathematicians and philosophers (e.g., see [28, 11, 56, 68, 53]).
hence, agents that play the game well should assign high beliefs to theorems. However, as we might expect, the strategy corresponding to optimal play is not computable. The game is amenable (in principle) to self-play training, a technique that has demonstrated success in learning expert-level play for the games of Go and Chess. Implementing and empirically testing self-play for these games is technically challenging and beyond the scope of this paper.

The ideas in this paper should be taken with one major caveat: the space complexity of dnfs is (highly) super-exponential so that they are not practically implementable without modification. Thus, our analysis in its current form should only be seen as conducting a thought experiment. As a step towards making the ideas here more practical, we will comment on how to control the sizes of the representations at the cost of completeness by treating certain combinations of properties as observationally indistinguishable, i.e., by making abstractions and lazily considering more properties as needed (Section 6). Thus abstractions provide a path towards implementation (e.g., for the proving game).

As one final qualification concerning the ideas in this paper, we acknowledge that we have taken a somewhat narrow view of “learning to prove”. First, we restrict ourselves to a first-order axiomatic view of mathematics. Second, we consider only a probabilistic aspect of plausible reasoning. Finally, we emphasize that our work is not human-style theorem proving (e.g., see [22]) even though we take inspiration from human mathematicians. In spite of these limitations and shortcomings, we believe that the ideas presented here offer a compelling viewpoint of “learning to prove” that cohesively accounts for aspects of the proving process and presents it in a form amenable (in principle) to learning via techniques that have demonstrated empirical success in other (difficult) domains.

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6Although the axiomatic approach to mathematics is widely adopted, mathematicians typically do not carry out the paradigm to its full extent and write completely formal proofs. When they do, there are a variety of formal languages they can choose from in addition to first-order logic (e.g., higher-order logic and type theories). The practicality of formalizing mathematics has been aided by the development of tools called interactive theorem provers (e.g., see the formalization of Feit-Thompson’s theorem [26]). There are interactive theorem provers based on first-order logic (e.g., see [80]), higher-order logic (e.g., see [77]), and type theories (e.g., see [79]). An interesting direction of future work would be to see how the ideas in this paper apply to higher-order and type-theoretic settings.

7The use of probabilistic reasoning to model plausible reasoning is not a new idea (e.g., see probabilistic graphical models [57] and work on inductive inference [73, 74, 42]). The field of automated reasoning (e.g., see [64, 65] for a survey) contains work on other forms of non-deductive reasoning including reasoning by induction (e.g., see [62, 4, 9]), abduction (e.g., see [10, 52, 19, 14]), and analogy (e.g., see [12, 1, 66]).
§2. Preliminaries. We begin by setting up the notation and terminology we will use throughout this paper (Section §2.1). Next, we provide intuition for Hintikka's dnfs (Section §2.2) and then introduce them formally for first-order logic without equality (Section §2.3). For more background on dnfs, we refer the reader to [31, 34, 55].

2.1. Notation and Background. Let $\mathbb{2} \triangleq \{0, 1\}$. We will interchangeably use $\bot$ for 0 (false) and $\top$ for 1 (true). $\mathbb{N}$ denotes the set of naturals and $\mathbb{N}^+$ denotes the set of positive naturals. $\mathbb{R}$ denotes the set of reals and $\mathbb{R}^+$ denotes the set of positive reals.

We write $\text{Set}(X) \equiv X \to \mathbb{2}$ to indicate the power set of $X$. We will often write a binary relation such as $\sim : X \times X \to \mathbb{2}$ in infix notation as $x \sim y \triangleq \sim(x, y)$ for $x \in X$ and $y \in X$. The notation $\{x \mid P(x)\}$ where $P : X \to \mathbb{2}$ is a predicate on a set $X$ indicates a set comprehension. We also write set comprehensions for indexed sets as $\{x_i \mid P(i)\}$ where $P : I \to \mathbb{2}$ is a predicate on an index set $I$ that indexes $X$.

When order matters, we use $\langle \cdot \rangle$ for sequences instead of $\{\cdot\}$. We write $\text{Seq}^n(X) \triangleq \{\langle x \mid x \in X \rangle \mid |\langle x \mid x \in X \rangle| = n\}$ for the set of length $n$ sequences comprised of elements from $X$. We write $\text{Str}^n(X)$ for the set of length $n$ strings comprised of elements from $X$.

We will use ellipsis notation “…” frequently in this paper. As usual, it means “fill in the dots with all the missing elements in between”. For example, $x_1, \ldots, x_n$ gives the elements $x_1, x_2$, and so on until $x_n$. When the commas are omitted as in $x_1 \ldots x_n$, the notation indicates a string of those elements instead.

2.1.1. First-order logic. The syntax of first-order logic (without equality) is summarized below.

$M \triangleq x \mid c \mid f^n(M_1, \ldots, M_n) \mid P^n(M_1, \ldots, M_n)$

$\phi \triangleq M \mid \neg \phi \mid \phi \vee \phi \mid (\exists x) \phi$

We use the metavariable $M$ to refer to terms. A term is either a variable $x$, a constant $c$, a $n$-ary function $f^n(M_1, \ldots, M_n)$ applied to $n$ terms, or a $n$-ary predicate $P^n(M_1, \ldots, M_n)$ on $n$ terms. We use the metavariable $\phi$ to refer to formulas. A formula is either a term ($M$), the logical negation of a formula ($\neg \phi$), the logical or of two formulas ($\phi \vee \phi$), or an existential quantification ($\exists x \phi$). As usual, we encode logical and as $\phi_1 \land \phi_2 \triangleq$

The restriction to first-order logic without equality is for simplicity: dnfs are defined for first-order logic with equality as well. All results given here apply to dnfs in both cases with the appropriate modifications. The difference between the two is between an inclusive treatment of quantifiers (without equality) and an exclusive treatment of quantifiers (with equality). As usual, note that we can include a binary predicate that encodes equality in first-order logic without equality, the difference with the case of first-order logic with equality being that structures may not necessarily be normal.
\(\neg(\neg \phi_1 \lor \neg \phi_2)\) and universal quantification as \((\forall x)\phi \triangleq \neg(\exists ! \neg \phi)\) where we assume the usual precedence and use additional (meta-level) parentheses to aid the parsing of formulas. The meta-level notation \((\pm)^b \phi\) where \(b \in \mathbb{2}\) either negates the formula \((\pm)^0 \phi \triangleq \neg \phi\) or leaves it alone \((\pm)^1 \phi \triangleq \phi\).

We write a formula with free variables as \(\phi[x_1, \ldots, x_n]\) where \(x_1, x_2, \ldots\) is a supply of free variables. A formula without free variables is called a sentence.

We use the standard semantics of first-order logic based on structures. A structure is a tuple \(\mathcal{M} \triangleq (D, \Sigma, [\cdot])\) where \(D\) is a (potentially empty) set called the domain, \(\Sigma\) is a signature (the functions and relations of the language), and \([\cdot]\) is an interpretation of the signature. Note that an empty domain cannot be used to interpret a language with constants. We say that a formula is satisfiable in a structure \(\mathcal{M}\) if \(\mathcal{M} \models \phi[a_1, \ldots, a_n]\) for every \(a_1, \ldots, a_n \in D\) where \(\models\) is the usual satisfaction relation defined by induction on the structure of formulas and we overload \(\phi[a_1, \ldots, a_n]\) to mean that the interpretation of the variable \(x_m\) in \(\phi[x_1, \ldots, x_m, \ldots, x_n]\) is \(a_m\). A formula is logically valid if it is satisfiable in every structure.

2.1.2. Graphs and trees. A directed graph is a tuple \((V, E)\) where \(V\) is a set of vertices and \(E \subseteq \{(v_1, v_2) \mid v_1, v_2 \in V\}\) is a set of edges. Because we only consider directed graphs in this paper, we will abbreviate directed graph as graph. A path in a graph \((V, E)\) is a graph \((V', E')\) of the form \(V' \triangleq \{v_1, \ldots, v_k\} \subseteq V\) and \(E' \triangleq \{(v_1, v_2), \ldots, (v_{k-1}, v_k)\} \subseteq E\) where all \(v_i\) are distinct. We refer to \(v_1\) and \(v_k\) as the endpoints of the path.

A (rooted) tree is a tuple \((V, E, v_R)\) where \((V, E)\) is a graph such that any two vertices are connected by a unique path and \(v_R \in V\) is a vertex designated as a root. Because there is only one path between any two vertices, a path between \(v_1 \in V\) and \(v_k \in V\) can be identified by the traversed vertices \(\{v_1, \ldots, v_k\}\), or simply the two endpoints \(v_1\) and \(v_k\). We say that \(v_p \in V\) is a parent of \(v_c \in V\), and \(v_c\) is a child of \(v_p\), if there is a path \(\{v_R, \ldots, v_p, v_c\}\). We write child : \(V \rightarrow \text{Set}(V)\) so that \text{child}(v)\) obtains the set of children of \(v\).\ We say that \(v_a \in V\) is an ancestor of \(v_d \in V\), and \(v_d\) is a descendant of \(v_a\), if there is a path \(\{v_R, \ldots, v_a, \ldots, v_d\}\). We write anc : \(V \rightarrow \text{Set}(V)\) so that \text{anc}(v)\) obtains the set of ancestors of \(v\) (\text{desc} : \(V \rightarrow \text{Set}(V)\) for descendants).

2.2. Distributive Normal Forms: Intuition. The role of a dnf of a first-order formula is analogous to that of a disjunctive normal form of a propositional formula in that the dnf of a formula is a disjunction of mutually exclusive possibilities. That we can describe mutually exclusive possibilities in the first-order setting is not obvious as the domain of quantification can be infinite and individuals in the domain can become related to one another as more individuals are considered. We start with an example to illustrate the basic problem and solution due to Hintikka.
Consider a first-order theory with one binary predicate $M$, where $x M y \triangleq M(x, y)$ is infix for “$x$ is the biological mother of $y$”, for describing individuals and their ancestral relations with one another. We can look at what the normal form of the statement “every individual has a mother”, encoded in this language as

$$(\forall x)(\exists m)m M x$$

could be. Assuming that we have a constant that names each element in the domain of quantification, a first attempt would be to translate each $\forall$ into a conjunction (over the domain of individuals) and each $\exists$ into a disjunction (over the domain of individuals), and use a propositional normal form. That is, we convert the result of translating the quantifiers away

$$\bigwedge_x \left( \bigvee_m m M x \right)$$

into disjunctive normal form. Unfortunately, the domain of quantification can be infinite, so the resulting formula may be of infinite size. The “trick” for circumventing this is to enumerate how the predicates at hand can describe the relationships between $k$ individuals (uniformly in $k$) instead of enumerating tuples of individuals. We can then identify possible kinds of worlds by listing which kinds of individuals exist or not.

To see how this works, we rewrite the original statement as

$$\neg(\exists x)(\neg(\exists m)(m M x)) .$$

(In words, it is impossible to find an individual that does not have a mother.) In this form, we can think of the normal form of a statement with quantification as describing whether kinds of individuals with certain relations to one another exist or not. What is missing from the above formula is considering all the cases in which $x$ and $m$ can related to one another that are consistent with the original formula. For example, $x M x \land m M x \land \neg(x M m) \land m M m$ and $\neg(x M x) \land m M x \land \neg(m M x) \land \neg(m M m)$ where the former allows individuals to be their own biological mother.

We can see this better in our specific example by introducing notation that enumerates all descriptions of two free individuals $x_1$ and $x_2$ describable by the predicate $M$.

$$P_{a_1a_2a_3a_4}(x_1, x_2) \triangleq (\pm)^{a_1}(x_1 M x_1) \land (\pm)^{a_2}(x_1 M x_2) \land (\pm)^{a_3}(x_2 M x_1) \land (\pm)^{a_4}(x_2 M x_2)$$

The subscript $a_1a_2a_3a_4$ indexes each $P$. For example,

$$P_{0100}(x_1, x_2) = \neg(x_1 M x_1) \land x_1 M x_2 \land \neg(x_2 M x_1) \land \neg(x_2 M x_2) .$$
Next we enumerate all combinations of whether such individuals exist or not.

\[ \delta_{b_1 \ldots b_{16}} \triangleq (\exists x_1) (\pm b_1 [P_{0000}(x_1, x_2)] \land \cdots \land (\pm b_{16} [P_{1111}(x_1, x_2)]) \]

The possible kinds of worlds described by our original formula is then any \( \delta_{0b_2 \ldots b_{16}} \)

where \( b_2 \in 2, \ldots, b_{16} \in 2 \)—there are individuals of every kind except when \( P_{0000}(x_1, x_2) \) is false, i.e., when \( x_1 \) and \( x_2 \) are not related in any way via the mother relationship.

The example deserves some remarks. First, note that we really have enumerated all the possibilities. The possibility \( \delta_{0 \ldots 0} \) describes one extreme where there are no individuals (and hence the original statement is vacuously true), the possibility \( \delta_{01 \ldots 1} \) requires every other permutation (of \( x \) and \( m \)) to hold, and the possibility \( \delta_{0010 \ldots 0} \) requires every other permutation (of \( x \) and \( m \)) to fail. Second, note that the number of possibilities even in this small example (two individuals and one predicate) is quite large number \( (2^{2^4}) \) of them. Third, the formula given here is not fully reduced with respect to scoping as it will be in the actual definition. In particular, observe that the atomic formulae of the form \( x_1 M x_1 \) in \( \delta_{1 \ldots 1} \) can be pulled out from under \( (\exists x_2) \).

2.3. Distributive Normal Forms: Background. Define the set

\[ S(\{\phi_1, \ldots, \phi_k\}) \triangleq \{ \bigwedge_{i \in \{1, \ldots, k\}} (\pm b_i \phi_i \mid b_1 \in 2, \ldots, b_k \in 2) \} . \]

An element of \( S(\{\phi_1, \ldots, \phi_k\}) \) is a conjunction of every \( \phi_i \) or its negation.

Let \( A[y_1, \ldots, y_k] \) denote the set of all atomic formula involving the free individual terms (i.e., constants or variables) \( y_1, \ldots, y_k \). Let \( B[y_1, \ldots, y_k] \) denote the subset of \( A[y_1, \ldots, y_k] \) that mentions \( y_k \) at least once.

2.3.1. Attributive constituents. An attributive constituent with \( k \) free individual terms \( y_1, \ldots, y_k \) of depth 0 is an element of \( S(B[y_1, \ldots, y_k]) \). We write \( \Gamma^{(0)}[y_1, \ldots, y_k] \triangleq S(B[y_1, \ldots, y_k]) \) for the set of all attributive constituents with \( k \) free individual terms \( y_1, \ldots, y_k \) of depth 0. By convention, we set \( \Gamma^{(0)[]} = \{ \top \} \). An attributive constituent of depth 0 is a formula of the form

\[ \gamma^{(0)}_{r}[y_1, \ldots, y_k] = \bigwedge_{i \in \{1, \ldots, \ell_i\}} (\pm b_i B_i[y_1, \ldots, y_k] \]

Note that traditional presentations of first-order model theory disallow empty domains although this restriction is not necessary. On the syntactic side, we will need to modify proof rules (e.g., the rule \( (\forall x) \rightarrow (\exists x) \) used in converting formula to prenex normal form no longer holds) to maintain soundness and completeness.
where $\ell^R_k \triangleq |B[y_1, \ldots, y_k]|$, each $b_k \in \mathcal{B}$, and each $B_i[y_1, \ldots, y_k] \in B[y_1, \ldots, y_k]$. The subscript $r$ indexes the attributive constituent and can be identified with the string $b_1 \ldots b_{\ell^R_k}$. Let $G_k^0 \triangleq \mathbf{Str}^{\ell^R_k}(2)$ be an index set for attributive constituents with $k$ free individual terms of depth 0. We have $G_k^0 \cong \Gamma(0)[y_1, \ldots, y_k]$. The superscript (0) indicates the depth of the formula, i.e., the maximal number of nested quantifiers in the formula. Hence a depth of 0 indicates that there are no quantifiers.

The set of attributive constituents $\Gamma^{[d]}[y_1, \ldots, y_k]$ of depth $d > 0$ is defined by induction on $d$. More concretely, we have an attributive constituent with $k$ free individual terms $y_1, \ldots, y_k$ of depth $d > 0$ has the form

$$\gamma_{r,s}^{[d]}[y_1, \ldots, y_k] = \gamma_{r}^{(0)}[y_1, \ldots, y_k]$$

$$\wedge \begin{cases} 
\bigwedge_{r' \in G_{k+1}^d} (\pm)^{s(r')} (\exists x)^{y_1, \ldots, y_k} \gamma_{r'}^{(0)}[y_1, \ldots, y_k, x] & d = 1 \\
\bigwedge_{(r',s') \in G_{d+1}^{d-1}} (\pm)^{s(r',s')} (\exists x)^{\gamma_{r',s'}^{(d-1)}}[y_1, \ldots, y_k, x] & d > 1 
\end{cases}$$

where we will explain the undefined notation below. Let $G_k^d \triangleq G_k^0 \times (G_{k+1}^{d-1} \rightarrow 2) \cong \Gamma^{[d]}[y_1, \ldots, y_k]$ be an index set for attributive constituents of depth $d > 0$ with $k$ free individual terms $y_1, \ldots, y_k$. The subscript $(r, s) \in G_k^d$ is a pair of $r \in G_k^0$ and a function $s : G_{d+1}^{d-1} \rightarrow 2$ indicating whether the appropriately indexed attributive constituent (of depth $d - 1$ with $k + 1$ free individual terms) exists or not. When the indices do not matter, we will abbreviate $\delta_{r,s}^{[d]}[y_1, \ldots, y_k]$, $\delta^{[d]}[y_1, \ldots, y_k]$. When we refer to two distinct attributive constituents whose indices do not matter, we will overload the subscripts as in $\delta_{i}^{[d]}$ and $\delta_{j}^{[d]}$ to distinguish them.

An attributive constituent with $k$ free individual terms $y_1, \ldots, y_k$ of depth $d \geq 0$ can equivalently be defined as

$$\gamma_{r,s}^{[d]}[y_1, \ldots, y_k] = \gamma_{r}^{(0)}[y_1, \ldots, y_k]$$

$$\wedge \bigwedge_{(r',s') \in G_{d+1}^{d-1}^+} (\exists x)^{\gamma_{r',s'}^{(d-1)}}[y_1, \ldots, y_k, x]$$

$$\wedge \left( \forall x \right) \bigvee_{(r',s') \in G_{d+1}^{d-1}^+} \gamma_{r',s'}^{(d-1)}[y_1, \ldots, y_k, x]$$

where $G_{d+1}^{d-1}^+ \triangleq \{(r',s') \mid s(r',s') = 1\}$ is the index set restricted to the positive ones as given by the function $s$.

### 2.3.2. Constituents

A constituent with $k$ free individual terms $y_1, \ldots, y_k$ of depth $d \geq 0$ is a formula of the form

$$\delta_{q,r,s}^{[d]}[y_1, \ldots, y_k] = A_q[y_1, \ldots, y_{k-1}] \wedge \gamma_{r,s}^{[d]}[y_1, \ldots, y_k]$$

where $A_q \in \mathbf{S}(A[y_1, \ldots, y_k])$. Let $\Delta^{[d]}[y_1, \ldots, y_k]$ be the set of constituents of depth $d$ with $k$ free individual terms. By convention, we set $\Delta^{(0)}[\mathcal{V}] = \{ \top \}$. 
We write \( D_k^d \equiv \Delta^d[y_1, \ldots, y_k] \) for the set indexing \( \Delta^d[y_1, \ldots, y_k] \). We use the same abbreviation scheme for the indices of constituents as we did for attributive constituents. Note that a constituent is an attributive constituent with an additional \( A_q[y_1, \ldots, y_{k-1}] \). Thus attributive constituents and constituents can be identified when there are 0 free individual terms.

2.3.3. Distributive normal forms. A distributive normal form (dnf) with \( k \) free individual terms \( y_1, \ldots, y_k \) is a disjunction of constituents

\[
\bigvee_{\delta^d[y_1, \ldots, y_k] \in D} \delta^d[y_1, \ldots, y_k]
\]

for some subset \( D \subseteq \Delta^d[y_1, \ldots, y_k] \) of constituents.

2.3.4. Properties. Attributive constituents, constituents, and dnfs have the following useful properties.

**Proposition 2.1 ([31]).**

**Exclusivity:** Any two constituents and attributive constituents of the same depth are mutually exclusive, i.e., \( \delta_i^d \implies \neg \delta_j^d \) for any \( \delta_i^d \neq \delta_j^d \).

**Expansion:** Every constituent \( \delta^d[y_1, \ldots, y_k] \) can be written as a disjunction of its expansion constituents, i.e., there is a function \( \text{expand} : \mathbb{N} \times \Delta^d \rightarrow \text{Set}(\Delta^{d+e}) \) such that

\[
\delta^d[y_1, \ldots, y_k] = \bigvee_{\delta^{d+e}[y_1, \ldots, y_k] \in \text{expand}(e, \delta^d[y_1, \ldots, y_k])} \delta^{d+e}[y_1, \ldots, y_k].
\]

**Existence:** Every formula \( \phi[y_1, \ldots, y_k] \) (of depth \( d \)) has a distributive normal form (of depth \( d \)), i.e., there is a function \( \text{dnf} : \mathcal{L}[y_1, \ldots, y_k] \rightarrow \text{Set}(\Delta^d[y_1, \ldots, y_k]) \) such that

\[
\phi^d[y_1, \ldots, y_k] = \bigvee_{\delta^d[y_1, \ldots, y_k] \in \text{dnf}(\phi[y_1, \ldots, y_k])} \delta^d[y_1, \ldots, y_k]
\]

where \( \mathcal{L}[y_1, \ldots, y_k] \) is the set of well-formed first-order sentences with free individual terms \( y_1, \ldots, y_k \).

Any \( \delta^{d+e}[y_1, \ldots, y_k] \in \text{expand}(e, \delta^d[y_1, \ldots, y_k]) \) refines (or is a refinement of) \( \delta^d[y_1, \ldots, y_k] \)\( ^{10} \). We write \( \delta^d[y_1, \ldots, y_k] \leq \delta^{d+e}[y_1, \ldots, y_k] \) when \( \delta^{d+e}[y_1, \ldots, y_k] \) refines \( \delta^d[y_1, \ldots, y_k] \). Let

\[
\Delta[y_1, \ldots, y_k] \equiv \bigcup_{d \in \mathbb{N}} \Delta^d[y_1, \ldots, y_k].
\]

\(^{10}\)The original terminology that Hintikka uses is subordinate. We prefer the term refinement because it evokes the intuition that \( \delta^{d+e}[y_1, \ldots, y_k] \in \text{expand}(e, \delta^d[y_1, \ldots, y_k]) \) describes the possibility described by \( \delta^d[y_1, \ldots, y_k] \) in finer detail.
Then the refinement relation $\leq : \Delta[y_1, \ldots, y_k] \times \Delta[y_1, \ldots, y_k] \rightarrow 2$ is a partial order and $(\Delta[y_1, \ldots, y_k], \geq)$ is a poset.

It is well-known that validity of first-order formulas is undecidable. Consequently, the consistency of constituents in a dnf is undecidable. There is a weaker notion called trivial inconsistency that is decidable. There are several notions of trivial inconsistency (e.g., see [34, 55]), although the exact form is not important for our purposes.

**Proposition 2.2 (Completeness [31]).** An attributive constituent is inconsistent iff all of its expansions at some depth are trivially inconsistent. Thus, an inconsistency at depth $d$ will eventually manifest itself as trivially inconsistent at some depth $e \geq d$, although the depth $e$ is not recursively computable. The main idea is show that a consistent attributive constituent always has an expansion that is not trivially inconsistent; the result follows from an application of König’s tree lemma.

§3. Representing Beliefs in Mathematical Knowledge. In this section, we introduce a probabilistic representation of beliefs in the validity of first-order statements. More concretely, we formalize a method for assigning beliefs to constituents (i.e., possible kinds of mathematical worlds) following the idea of assigning weights to constituents described by Hintikka [33, pg. 274–282] (Section 3.1). The representation supports Bayesian update and does not enforce that logically equivalent statements are assigned the same probability so that the beliefs of agents that are not logically omniscient can be encoded. Next, we show that the representation induces a distribution on first-order sentences (Section 3.2). In particular, we will see that there is a computable representation that induces a deductively consistent distribution on first-order sentences in the limit, although the limit itself is not computable.

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11There are notions of trivial inconsistency that are not strong enough to ensure completeness as noted by Nelte [55].

12An agent is logically omniscient if it knows all the logical consequences that follow from a set of axioms. Consequently, we would like logical omniscience to fail in the context of learning a prover—there are no theorems to prove if an agent already knows all of them.

The problem of logical omniscience is an issue encountered in epistemic logic (e.g., see [72, 16, 29]) where we reason about the knowledge of agents. One solution for weakening logical omniscience involves modeling impossible possible worlds, i.e., modeling worlds that an agent considers possible but are eventually revealed to be impossible. Hintikka argues that dnfs provide a model of impossible possible worlds—an impossible possible world is an inconsistent constituent that is not trivially inconsistent at some depth and revealed to be trivially inconsistent at a later depth (by completeness) [35]. Thus the application of dnfs to address the problem of logical omniscience has also been hinted at by Hintikka.
Remark (Simple first-order languages). For simplicity, we restrict attention to first-order languages with a finite number of predicates, no function symbols, and no constants unless stated otherwise. The constant-free restriction simplifies the form of constituents and dnfs we will need to consider. As a reminder, every first-order formula \( \phi[y_1, \ldots, y_k] \) with \( k \) free individual terms \( y_1, \ldots, y_k \) has a dnf of depth \( d \) constituents. In a constant-free setting, the free individual terms \( y_1, \ldots, y_k \) are all variables. Thus the original formula is equivalent to its universal closure (i.e., a sentence). Consequently, we only need to consider the set of constituents \( \Delta_0^{(0)} \), abbreviated \( \Delta^{(0)} \), of depth \( 0 \) with 0 free individual terms. We have that \( \Delta_0^{(0)} \doteq \{ \top \} \) by convention.

3.1. Hintikka Trees. We formalize a representation of beliefs in mathematical claims called a Hintikka tree (HT) in this section. A HT is a representation of beliefs in constituents. As constituents can be thought of as describing possible kinds of mathematical worlds, the idea behind the representation is the standard one: list all mutually exclusive possibilities and assign beliefs to them that sum to one. Because constituents of different depths do not denote mutually exclusive possibilities when they are related according to the refinement partial order, we require bookkeeping to keep track of this relation, which has a tree structure.

3.1.1. Refinement tree. Let the set of vertices be the set of constituents of any depth \( \Delta \). Let the set of edges \( \xi \doteq \{ (\delta^{(d)}, \delta^{(d+1)}) \mid \delta^{(d)} \leq \delta^{(d+1)} \} \) consist of the refinement relation omitting reflexive relations. Then \((\Delta, \xi)\) is a graph that encodes the refinement relation (minus the reflexive edges).

The graph \((\Delta, \xi)\) is not a tree is because the expansions of two distinct constituents can share refining constituents, although the shared constituents are necessarily inconsistent.

**Proposition 3.1.** Suppose \( \delta_i^{(d)} \neq \delta_j^{(d)} \). If \( \delta^{(d+e)} \in \text{expand}(e, \delta_i^{(d)}) \cap \text{expand}(e, \delta_j^{(d)}) \), then \( \delta^{(d+e)} \) is inconsistent.

**Proof.** Assume additionally for the sake of contradiction that \( \delta^{(d+e)} \) is consistent. Then there exists a structure \( M \) such that \( M \models \delta^{(d+e)} \). Thus

---

13 As a reminder, the effect of equality is to give an exclusive interpretation of quantifiers. All the results that hold on constituents in first-order logic without equality also hold on constituents in first-order logic with equality with the appropriate modifications.

Note that functions can be encoded as predicates in first-order logic with equality. Observe also that the current setting actually admits a finite number of constants. More concretely, we can associate each constant \( c \) with a monadic predicate \( P_c \) where the interpretation of \( P_c(x) \) is “\( x \) is the constant \( c \)”. Any formula \( \phi[c] \) that refers to the constant \( c \) can thus be translated to \((\exists x)P_c(x) \land \phi[x]\) where \( x \) is not free in \( \phi \) and we add the additional axiom \((\exists x)P_c(x)\) to the theory. Hence, we are roughly working with first-order languages with a finite number of predicates, functions, and constants.
Figure 1. A depiction of a refinement tree $(\Delta, E)$. Each vertex represents a constituent and each edge indicates a refinement relation. For example, the constituent $\delta_{11}^{(2)}$ occurs in the expansion of $\delta_1^{(1)}$. We assume that each constituent’s set of refinements is an indexed set so that $\delta_{11}^{(2)}$ indicates that we take the first refinement of $\delta_0^{(0)}$ and then take the first refinement of $\delta_1^{(1)}$.

we have that $M \models \delta_i^{(d)}$ and $M \models \delta_j^{(d)}$ (because $\delta^{(d+e)} \in \text{expand}(e, \delta_i^{(d)}) \cap \text{expand}(e, \delta_j^{(d)})$ by another assumption), which contradicts that $\delta_i^{(d)}$ and $\delta_j^{(d)}$ are mutually incompatible (by exclusivity in Proposition 2.1). $\Box$

By the proposition above, we can associate any shared constituent that is a refinement of two parent constituents to either parent constituent and disassociate it with the other without changing the consistency of either parent constituent. In other words, we can remove one edge. We can use this observation to convert $(\Delta, \xi)$ into a tree. We call $(\Delta, \xi_R)$ a refinement tree where $\xi_R$ is the set of edges obtained after the pruning procedure described above is applied. Throughout the rest of this paper, we will assume that we have chosen one such pruning and will write $(\Delta, \xi_R)$ as $(\Delta, \xi)$. We will also overload $\leq$ to refer to the pruned refinement partial order. We have the following obvious relationships between the (pruned) refinement partial order and (pruned) refinement tree.

**Proposition 3.2.**
1. $\delta^{(d)} \leq \delta^{(d+1)}$ iff $\delta^{(d+1)} \in \text{child}(\delta^{(d)})$.
2. $\delta^{(d)} \leq \delta^{(e)}$ iff $\delta^{(d)} = \delta^{(e)}$ or $\delta^{(d)} \in \text{anc}(\delta^{(e)})$ (equivalently $\delta^{(e)} \in \text{desc}(\delta^{(d)})$).
ON LEARNING TO PROVE

Proof. Straightforward.

A path between constituents \( \delta^d \) and \( \delta^{d+e} \) is the sequence of constituents and their refinements \( \delta^d \leq \delta^{d+1} \leq \ldots \leq \delta^{d+e} \). Because there is only one path between any two vertices in a tree, we can identify a constituent (i.e., a node in a refinement tree) with the path taken through a refinement tree starting at the root node \( \delta^{(0)} \) to reach it. Figure [II] gives an illustration of a refinement tree where constituents are indexed by their paths. The root constituent \( \delta^e(0) \) of the tree is indexed by the empty path \( e \).

3.1.2. Hintikka trees. We assign beliefs to constituents by attaching a weight to each node of the refinement tree. Because the assignment of weights needs to respect the refinement partial order, we introduce a notion of coherence between levels of the refinement tree. Let \( \Delta^{(\leq d)} \triangleq \bigcup_{e=0}^{d} \Delta^{(e)} \).

Definition 3.1. \( F^d : \Delta^{(\leq d)} \rightarrow [0, 1] \) is coherent with \( F^{d+1} : \Delta^{(\leq d+1)} \rightarrow [0, 1] \) if

1. \( F^d(\delta(e)) = F^{d+1}(\delta(e)) \) for every \( \delta(e) \in \Delta^{(\leq d-1)} \) and
2. \( F^d(\delta(d)) = \sum_{\delta^{(d+1)} \geq \delta(d)} F^{d+1}(\delta^{(d+1)}) \)

for every \( \delta(d) \in \Delta^{(d)} \).

We say that \( F^d : \Delta^{(\leq d)} \rightarrow [0, 1] \) is a coherently constructed function if there is a tuple of functions \( (F^0, \ldots, F^d) \) such that (1) adjacent elements are coherent with each other and (2) \( F^d(\delta(e)) = F^e(\delta(e)) \) for \( e < d \). We also say that \( F : \Delta \rightarrow [0, 1] \) is coherently constructed if there is a sequence of functions \( (F^d)_{d \in \mathbb{N}} \) such that (1) adjacent elements are coherent with each other and (2) \( F(\delta(d)) = F^d(\delta(d)) \) for any \( d \in \mathbb{N} \).

An assignment of weights to the refinement tree that is coherently constructed constrains our beliefs to be consistent across depth. Thus it provides a static representation of an agent’s beliefs. Naturally, an agent may encounter a situation where it realizes that its beliefs need to be revised. For example, upon further inspection of all the expansions of a parent constituent, the agent may realize that they are all inconsistent so the belief in the parent constituent needs to be redistributed. Intuitively, this may occur because the increase in depth corresponds to the construction of an object (i.e., an introduction of an existential) and the consideration of this extra object changes the valuation of the consistency of the parent possibility. Indeed, such a situation arises from the constituent completeness theorem: inconsistent constituents are eventually revealed to be trivially inconsistent at some depth even if they are not trivially inconsistent at shallower depths.
We define an appropriate renormalization to account for the dynamics of belief revision next. We will need to specify (1) which constituents to redistribute beliefs to and (2) the amount of belief to redistribute to those constituents. The idea for redistributing beliefs is given in one paragraph by Hintikka [33, pg. 281] which we interpret as an application of Bayes rule.

Throughout the scope of describing renormalization, we restrict attention to the first $d$ levels of the refinement tree $(\Delta(\leq d), \xi(\leq d))$.

We start with the first task and begin by identifying which constituents to redistribute beliefs to. Define the support function $\text{supp} : (\Delta(\leq d) \to 2) \to \Delta(\leq d) \to 2$ as (1) $\text{supp}(S)(\delta(e)) = \top$ if there is some $\delta(d) \geq \delta(e)$ such that $S(\delta(f)) \neq \bot$ for any $\delta(f)$ such that $\delta(e) \leq \delta(f) \leq \delta(d)$ and (2) $\text{supp}(S)(\delta(e)) = \bot$ otherwise. Let $S : \Delta(\leq d) \to 2$. We say that $\delta(e)$ is supported (through depth $d + 1$) with respect to $\text{supp}(S)$ if $\text{supp}(S)(\delta(e)) = \top$ and unsupported (by depth $d + 1$) with respect to $\text{supp}(S)$ otherwise.

Suppose we would like to discontinue beliefs in $\delta(d)$ in a function $G : \Delta(\leq d) \to [0, 1]$. Define the function $\text{supp}_{G, \delta(d)}^0 : \Delta(\leq d) \to 2$ such that (1) $\text{supp}_{G, \delta(d)}^0(\delta(d)) = \top$ if $\delta(d) = \delta(d)$ or $G(\delta(d)) = 0$ and (2) $\text{supp}_{G, \delta(d)}^0(\delta(d)) = \bot$ otherwise. Thus $\text{supp}_{G, \delta(d)}^0(\delta(e)) = \top$ if there is a path from $\delta(e)$ to $\delta(d)$ such that $G$ assigns positive weight to each vertex of that path (excluding the vertex $\delta(d)$).

The idea is that we will transfer beliefs assigned to unsupported constituents over to the appropriate supported constituents. Define the abbreviations $S^+_{G, \delta(d)} \triangleq \{ \delta(e) \mid e \leq d, \text{supp}_{G, \delta(d)}^0(\delta(e)) = \top \}$ and $S^-_{G, \delta(d)} \triangleq \Delta(d) \setminus S^+_{G, \delta(d)}$. Thus $S^-_{G, \delta(d)}$ and $S^+_{G, \delta(d)}$ partition $\Delta(\leq d)$. Define a $d$-redistribution point as

$$
\rho_{G, \delta(d)} \triangleq \max_{0 \leq r \leq d} \{ \delta(e) \mid \delta(e) \leq \delta(d), \text{ some } \delta(d) \in \text{desc}(\delta(e)) \cap S^+_{G, \delta(d)} \},
$$

which is the closest (i.e., deepest by depth) ancestor constituent that is supported through depth $d + 1$. A $d$-redistribution point identifies a vertex of the refinement tree that has descendants supported through depth $d + 1$ to redistribute beliefs in unsupported constituents to.

We turn our attention towards the second task concerning the amount of belief to redistribute to each constituent now. Let $D^+_{G, \delta(d)} \triangleq \text{child}(\rho_{G, \delta(d)}) \cap S^+_{G, \delta(d)}$ be the children of $\rho_{G, \delta(d)}$ that are supported through depth $d + 1$. Then

$$
Z^+_{G, \delta(d)} \triangleq \sum_{\delta(e) \in D^+_{G, \delta(d)}} G(\delta(e))
$$
is the positive renormalization constant and
\[ Z_{G,\delta(d)}(d) \triangleq \sum_{\delta(e) \in \text{child}(\rho_{G,\delta(d)})} G(\delta(e)) \]
is the total renormalization constant.

**Definition 3.2.** The renormalization of \( G \) with respect to \( \delta(d) \) is a function \( \text{renorm}_{\delta(d)} : (\Delta(\leq d) \rightarrow 2) \rightarrow \Delta(\leq d) \rightarrow [0,1] \) defined as

\[
\text{renorm}_{\delta(d)}(G)(\delta(e)) = \begin{cases} 
Z_{G,\delta(d)}(d) & \delta(e) \in \text{desc}(\rho_{G,\delta(d)}) \cap S^+_{G,\delta(d)} \\
Z_{G,\delta(d)}^{-}(d) & \delta(e) \in \text{desc}(\rho_{G,\delta(d)}) \cap S^-_{G,\delta(d)} \\
G(\delta(e)) & \text{otherwise}.
\end{cases}
\]

Note that \( \text{renorm}_{\delta(d)}(G) \) has a subtree property: \( \text{renorm}_{\delta(d)}(G)(\delta(e)) = F_d(\delta(e)) \) for \( \delta(e) \notin \text{desc}(\rho_{G,\delta(d)}) \).

Here are some properties of renormalization for coherently constructed functions.

**Proposition 3.3.** Suppose \( F_d : \Delta(\leq d) \rightarrow [0,1] \) is coherently constructed. Then the following holds:

- **Coherence:** \( \text{renorm}_{\delta(d)}(F_d) \) is coherently constructed provided there is some \( \delta(d) \neq \delta(d) \) such that \( F_d(\delta(d)) > 0 \);
- **Preservation:**
  \[ F_d(\rho_{F_d,\delta(d)}) = \sum_{\delta(r+e) \in \text{expand}(e,\rho_{F_d,\delta(d)})} \text{renorm}_{\delta(d)}(F_d)(\delta(r+e)) \]
  ; and
- **Commutative:** \( \text{renorm}_{\delta(d)} \circ \text{renorm}_{\delta(d)}(F_d) = \text{renorm}_{\delta(d)} \circ \text{renorm}_{\delta(d)}(F_d) \)
  provided there is some \( \delta(d) \neq \delta(d) \) and \( \delta(d) \neq \delta(d) \) such that \( F_d(\delta(d)) > 0 \).

**Proof.** See Section 3.3 as the proof is straightforward but tedious. \( \dashv \)

We put the pieces together to assign beliefs to constituents at every depth in a manner consistent with a refinement tree. Let \( (F_d, T_d) \) be a pair where \( F_d : \Delta(\leq d) \rightarrow [0,1] \) and \( T_d = \{\delta_1(d), \ldots, \delta_n(d)\} \). We write \( \text{renorm}_{T_d} \triangleq \)
renorm$_{\delta(d)} \circ \cdots \circ$ renorm$_{\delta(d)}$ which is well-defined because renorm is invariant under permutation (renormalization is commutative by Proposition 3.3). We say that $(F^d, T^d)_{d \in \mathbb{N}}$ is a sequence of coherent renormalizations if

$$\text{renorm}_T F^d(\delta(d)) = \sum_{\delta(d+1) \geq \delta(d)} F^{d+1}(\delta(d+1))$$

for every $d$ and $\delta(d) \in \Delta(d)$. Suppose $F : \Delta \to [0,1]$ where $F(\delta(d)) \triangleq F^d(\delta(d))$ for every $d$. We say that $F$ is the coherent renormalization of $(F^d, T^d)_{d \in \mathbb{N}}$ when $(F^d, T^d)_{d \in \mathbb{N}}$ is a sequence of coherent renormalizations.

**Definition 3.3.** A Hintikka tree $(HT)^{14}$ is a tuple $(\Delta, E, H)$ where $(\Delta, E)$ is a refinement tree and $H : \Delta \to [0,1]$ is a function on constituents satisfying

**Initial initial beliefs:** $H(\delta(0)) = 1$;

**Infinite supported path:** there is an infinite refinement path

$$\delta^{(d)}_{j_1} \leq \delta^{(d)}_{j_2} \leq \ldots$$

where $H(\delta^{(d)}_{j_d}) > 0$ for every $d \in \mathbb{N}^+$; and

**Coherent renormalization:** $H$ is the coherent renormalization of some sequence $(F^d, T^d)_{d \in \mathbb{N}}$.

We write $HT(\mathcal{L})$ for the set of HTs defined with respect to the first-order simple language $\mathcal{L}$.

The first condition states that we start off with initial beliefs. The second condition guarantees that there is an infinite path through the refinement tree that has positive belief. This disallows a degenerate tree where belief in every possibility disappears. The last condition relaxes the notion of coherence so that agents are not required to have strictly coherent beliefs across depth.

**Proposition 3.4.** The beliefs assigned to constituents at each depth $d \in \mathbb{N}$ by a HT $(\Delta, E, H)$ are normalized:

$$\sum_{\delta(d) \in \Delta(d)} H^d(\delta(d)) = 1.$$

---

14 Naturally, the definition of a HT given here is much more explicit than the one Hintikka gives. For example, Hintikka devotes one paragraph to describing renormalization whereas it takes us a substantial portion of the section. As another example, we also explicitly make the connection between the redistribution of weights for unsupported constituents and Bayes rule.

We note that our definition of a HT is a bit less restrictive than Hintikka’s in that constituents that are trivially inconsistent can be assigned positive weight. The less restricted definition is desirable in the context of learning because it would enable an agent to “learn” that a constituent is trivially inconsistent.
Proof. Suppose $\mathcal{H}$ is the coherent renormalization of some sequence $(F^d, T^d)_{d \in \mathbb{N}}$. We proceed by induction on $d$. The base case follows from initial initial beliefs. In the inductive case, we have to show that

$$\sum_{\delta^{(d+1)} \in \Delta^{(d+1)}} \mathcal{H}(\delta^{(d+1)}) = 1.$$ 

We have that

$$\sum_{\delta^{(d+1)} \in \Delta^{(d+1)}} F^{d+1}(\delta^{(d+1)}) = \sum_{\delta^{(d)} \in \Delta^{(d)}} \sum_{\delta^{(d+1)} \geq \delta^{(d)}} F^{d+1}(\delta^{(d+1)})$$

$$= \sum_{\delta^{(d)} \in \Delta^{(d)}} \text{renorm}_{T_d}(F^d)(\delta^{(d)}).$$

Now $\sum_{\delta^{(d+1)} \in \Delta^{(d+1)}} F^d(\delta^{(d)}) = 1$ by the inductive hypothesis. As we have a coherent renormalization, the result follows as $\text{renorm}_{T_d}(F^d)(\delta^{(d)})$ is also coherently constructed (by Proposition 3.3).

Figure 2 gives an example of a HT. We end with two special examples of HTs that assign beliefs in concordance with inconsistency and trivial inconsistency.

Example 1. A HT is a depth Hintikka tree if $\mathcal{H}$ is constrained so that inconsistent constituents are assigned 0. Inconsistency is undecidable so that a depth HT is not computable. If a theorem proving agent represents mathematical knowledge with a depth HT, then the agent is logically omniscient. Note that a depth HT has no $d$-renormalization points. To make the connection with standard semantics, we have that $\models \phi^{(d)}$ iff $\sum_{\delta^{(d)} \in \text{dnf}(\phi^{(d)})} \mathcal{H}(\delta^{(d)}) = 1$ for some depth HT $(\Delta, E, \mathcal{H})$.

Example 2. A HT is a surface Hintikka tree if $\mathcal{H}$ is constrained so that trivially inconsistent constituents are assigned 0. Trivial inconsistency is decidable so that a surface HT is computable. Note that a surface HT has $d$-renormalization points due to the constituent completeness theorem. We can convert a surface HT $(\Delta, E, \mathcal{H})$ into a depth HT $(\Delta, E, \tilde{\mathcal{H}})$ as

$$\tilde{\mathcal{H}} = \delta^{(d)} \mapsto \min_{e \in \mathbb{N}} \max_{\delta^{(d+e)} \in \text{expand}(e, \delta^{(d)})} \mathcal{H}(\delta^{(d+e)}).$$

To make the connection with standard semantics, we have that $\mathcal{M} \models \phi^{(d)}$ for a structure $\mathcal{M}$ iff $\sum_{\delta^{(d)} \in \text{dnf}(\phi^{(d)})} \mathcal{M}(\delta^{(d)}) = 1$ for some $\delta^{(d)} \in \text{dnf}(\phi^{(d)})$ and surface HT. In words, a formula $\phi^{(d)}$ is satisfiable in a structure $\mathcal{M}$ iff there is a constituent in its dnf whose refinement at every depth is consistent.

---

15 The terminology is inspired by depth information [33]
16 The terminology is inspired by surface information [33]
Figure 2. A drawing of an example Hintikka tree (HT). Vertices with 0 belief are not shown. When transitioning from depth 1 to depth 2, $\delta^{(1)}_b$ becomes an unsupported constituent so $\delta^{(0)}_e$ is a 1-renormalization point as it is the closest constituent with supported descendants at depth 2. The 1/3 belief assigned to $\delta^{(1)}_b$ is redistributed according to Bayes rule across the 1-renormalization point’s descendants at depth 2 (i.e., $\delta^{(1)}_a$ and $\delta^{(1)}_c$). When transitioning from depth 2 to depth 3, $\delta^{(1)}_{ad}$ and $\delta^{(1)}_{ce}$ become unsupported constituents, so $\delta^{(0)}_e$ is a 2-renormalization point. Note that the total belief in the branch starting with $\delta^{(1)}_c$ is 1/2 at depth 1, 3/4 at depth 2, and 1 at depth 3.

3.2. Probabilities on First-Order Sentences. As every depth $d$ first-order sentence can be written as a depth $d$ dnf, a HT induces a distribution on first-order sentences. The method of assigning probabilities to first-order sentences will not be used in the rest of the paper and is included for the sake of completeness.

The basic idea is that the probability of a first-order sentence is the sum of the beliefs of the constituents comprising its dnf at “infinite” depth. At a high-level, the limit exists because of Bayes rule’s “refute and rescale dynamics”: either belief in a constituent is refuted and belief in all of its refinements converges to 0 or the belief in a constituent is rescaled by belief lost in refuted constituents so that belief in all of its refinements
is a monotonically increasing and bounded sequence. We say that \( \delta^{(d)} \) is \textit{eventually unsupported} with respect to \( H \) if there exists an \( e \in \mathbb{N} \) such that all of its depth \( d + e \) expansions \( \delta^{(d+e)} \) have \( H(\delta^{(d+e)}) = 0 \). We say that \( \delta^{(d)} \) is always supported otherwise.

**Proposition 3.5.** For any \( \delta^{(d)} \) and HT \( H \), let

\[
\theta_e^{\delta^{(d)}} = \sum_{\delta^{(d+e)} \in \text{expand}(e, \delta^{(d)})} H(\delta^{(d+e)})
\]

for every \( e \in \mathbb{N} \). Then

1. \( \lim_{e \to \infty} \theta_e^{\delta^{(d)}} = 0 \) when \( \delta^{(d)} \) is eventually unsupported; and
2. \( (\theta_e^{\delta^{(d)}})_{e \in \mathbb{N}} \) is a monotonically increasing sequence bounded by 1 when \( \delta^{(d)} \) is always supported so that \( \lim_{e \to \infty} \theta_e \) exists; and
3. \( \lim_{e \to \infty} \theta_e^{\delta^{(d)}} = \sum_{\delta^{(d+1)} \geq \delta^{(d)}} \lim_{e \to \infty} \theta_e^{\delta^{(d+1)}} \).

**Proof.**

1. If \( \delta^{(d)} \) is eventually unsupported, then there is an \( e \) such that \( H(d+e)(\delta^{(d+e)}) = 0 \) for every refinement \( \delta^{(d+e)} \). Hence the series converges and is 0.

2. It is easy to see that

\[
\sum_{\delta^{(d+e)} \in \text{expand}(e, \delta^{(d)})} H(\delta^{(d+e)}) \leq \sum_{\delta^{(d+e+1)} \in \text{expand}(e+1, \delta^{(d)})} H(\delta^{(d+e+1)})
\]

when \( \delta^{(d)} \) is always supported by the preservation property of renormalization. That we have a monotonically increasing sequence follows by induction on \( e \). The sequence is bounded by 1 because a HT is normalized at every depth. Thus the limit exists.

3. We have

\[
\lim_{e \to \infty} \theta_e^{\delta^{(d)}} = \lim_{e \to \infty} \sum_{\delta^{(d+1)} \geq \delta^{(d)}} \theta_e^{\delta^{(d+1)}} = \sum_{\delta^{(d+1)} \geq \delta^{(d)}} \lim_{e \to \infty} \theta_e^{\delta^{(d+1)}}
\]

where the first equality follows by definition and the second equality follows because the sequence is dominated by 1 (by items 1 and 2).
We now define a probability measure on the refinement tree. Let $\Psi^d \triangleq \{\delta^{(0)} \ldots \delta^{(d)} \mid \delta^{(0)} \leq \ldots \leq \delta^{(d)}\}$ be the set of length $d$ paths of the refinement tree. Let $(\Psi^\omega, \mathcal{O})$ be a topological space\(^{17}\) where $\mathcal{O}$ is a topology generated by the basis of open sets $\mathcal{B} \triangleq \{\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega \mid \delta^{(0)} \ldots \delta^{(d)} \in \Psi^d\} \cup \{\emptyset\}$, where each basic open $\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega$ contains every infinite refinement path that begins with $\delta^{(0)} \ldots \delta^{(d)} \in \Psi^d$. We can associate each basic open $\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega \in \mathcal{O}$ contains $\delta^{(d)}_{j_1 \ldots j_d}$ with exactly one $(\theta^d_{e})_{e \in \mathbb{N}}$ sequence.

**Definition 3.4.** The belief $\mathbb{B}_B : \mathcal{B} \to [0, 1]$ in a basic open $j_1 \ldots j_d \Psi^\omega \in \mathcal{B}$ with respect to a HT $\mathbb{H}$ is defined as

$$\mathbb{B}_B(\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega) \triangleq \lim_{e \to \infty} \theta_{e}^{(d)}$$

$$\mathbb{B}_B(\emptyset) \triangleq 0.$$  

**Proposition 3.6.** The basic opens have consistent assignments:

$$\mathbb{B}_B(\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega) = \sum_{\delta^{(d)}_{j_1 \ldots j_d} \geq \delta^{(d)}} \mathbb{B}_B(\delta^{(0)} \ldots \delta^{(d)}_{j_1 \ldots j_d} \Psi^\omega).$$

**Proof.** This follows directly from Proposition 3.5 item 3. \(\square\)

Thus we have a finitely additive set function.

We extend the belief in a basic open to the measurable space\(^{18}\) $(\Psi^\omega, \sigma(\mathcal{O}))$ where $\sigma(\mathcal{O})$ is the Borel $\sigma$-algebra in the standard way.

**Proposition 3.7.** The belief $\mathbb{B}_B$ in a basic open defines a unique probability measure on the measurable space $(\Psi^\omega, \sigma(\mathcal{O}))$.

**Proof.** Observe that $(\Psi^\omega, \mathcal{O})$ has a countable basis $\mathcal{B}$ so that the Borel $\sigma$-algebra is generated by the basis $\mathcal{B}$. Moreover, the basis $\mathcal{B}$ is a $\pi$-system (i.e., closed under finite intersections). The result follows as a finitely additive set function on a $\pi$-system (Proposition 3.2) can be uniquely extended to a set function on a $\sigma$-algebra when it is $\sigma$-finite. \(\square\)

Finally, we define a distribution on first-order sentences.

**Definition 3.5.** The belief in first-order sentences is given by

$$\mathbb{B}(\phi^{(d)}) \triangleq \sum_{\delta^{(d)} \in \text{dnf}(\phi^{(d)})} \mathbb{B}_B(\delta^{(0)} \ldots \delta^{(d)} \Psi^\omega)$$

where $\mathbb{B}_B$ is the belief in basic opens with respect to a HT $\mathbb{H}$.

The belief in a first-order formula of depth $d$ with $k$ free variables is the $(d+k)$-belief in the closed first-order formula obtained via universal closure (which increases the depth to $d+k$).

\(^{17}\)For background on topology, we refer the reader to Munkres [54].

\(^{18}\)For background on measure-theoretic probability, we refer the reader to Kallenberg [46].
**Proposition 3.8.** The probability on first-order sentences has the following properties:

1. \( B(\neg \phi) = 1 - B(\phi) \);
2. \( B(\phi_1 \land \phi_2) \leq \min(B(\phi_1), B(\phi_2)) \);
3. \( \max(B(\phi_1), B(\phi_2)) \leq B(\phi_1 \lor \phi_2) \);
4. \( B((\forall x)\phi) \leq \min_{\delta \in \text{dnf}((\exists x)\neg \phi)} \{1 - B(\delta)\} \); and
5. \( \max_{\delta \in \text{dnf}((\exists x)\phi)} B(\delta) \leq B((\exists x)\phi) \).

**Proof.** These all follow from set-theoretic manipulations. \( \lceil \)

For the case of universal and existential quantification, the minimum and maximum are taken over constituents (i.e., possible kinds of individuals) as opposed to individuals in the domain of quantification. Note that this differs with the Gaifman condition [20] which defines the probability of a universal or existential in as the infimum or supremum over individuals in the domain.

The beliefs possessed by a logically omniscient agent are not computable.

**Proposition 3.9.**

1. The beliefs with respect to a depth HT satisfy \( B(\phi) = 1 \) when \( \models \phi \) and \( B(\phi) = 0 \) when \( \not\models \phi \).
2. A depth HT and a surface HT both induce the same distribution on first-order sentences.
3. Neither depth nor surface beliefs are computable.

**Proof.**

1. When \( \not\models \phi^{(d)} \), then the dnf of \( \phi^{(d)} \) contains only inconsistent constituents so that \( B(\phi^{(d)}) = 0 \).

   To see that \( B(\phi^{(d)}) = 1 \) when \( \models \phi^{(d)} \), recall a formula \( \phi^{(d)} \) is logically valid iff its dnf contains all consistent constituents at depth \( d \). By Proposition 3.5 item 3, we have that the sum \( \sum_{\delta^{(d)}} \) of consistent \( \mathbb{H}(\delta^{(d)}) = 1 \) so that \( B(\phi^{(d)}) = 1 \) when \( \models \phi^{(d)} \) as required.

2. We claim that \( \delta^{(d)} \) is eventually unsupported in a depth HT iff it is eventually unsupported in a surface HT. To see this, observe that (1) \( \delta^{(d)} \) is eventually unsupported in a depth HT iff it is inconsistent and (2) \( \delta^{(d)} \) is eventually unsupported in a surface HT iff there is a depth \( d + e \) such that all of its refinements are trivially inconsistent at depth \( d + e \), which occurs iff \( \delta^{(d)} \) is inconsistent by the constituent completeness theorem.
It is easy to see that $\delta^{(d)}$ is always supported in a depth HT iff it is always supported in a surface HT.

Thus, a depth HT and a surface HT induce the same distribution on first-order sentences.

3. Suppose for the sake of contradiction that depth beliefs are computable. As a constituent is eventually unsupported if it is inconsistent and always supported if it is consistent, we thus have a decision procedure for validity of first-order logic, a contradiction.

Unsurprisingly, probabilities in first-order sentences represented by a depth HT are not computable (item 2). Perhaps somewhat more surprising, probabilities in first-order sentences represented by a surface HT are also not computable and are equivalent to those given by a depth HT (item 3). Thus we can obtain a deductively consistent distribution as the limit of a computable representations given by a surface HT, although the limit is not computable.

3.3. Supplementary on Renormalization. We give the supplementary proof of the properties of renormalization (Proposition 3.3) below.

Proof.

Coherence: We show this by case analysis on whether $\delta^{(e)} \in \text{desc}(\rho_{F^d, \delta^{(d)}})$ or $\delta^{(e)} \notin \text{desc}(\rho_{F^d, \delta^{(d)}})$. We have to show that

$$\text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e)}) = \sum_{\delta^{(e+1)} \geq \delta^{(e)}} \text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e+1)}).$$

We proceed by case analysis on whether $\delta^{(e)} \in S^-_{F^d, \delta^{(d)}}$ or $\delta^{(e)} \in S^+_{F^d, \delta^{(d)}}$.

In case of the former, we have that $\text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e)}) = 0$ and $\sum_{\delta^{(e+1)} \geq \delta^{(e)}} \text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e+1)}) = 0$ as required.

In case of the latter, we have that $\text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e)}) = 0$ and

$$\sum_{\delta^{(e+1)} \geq \delta^{(e)}} \text{renorm}_{\delta^{(e)}}(F^d)(\delta^{(e+1)}) = \frac{Z_{F^d, \delta^{(e)}}}{Z_{F^d, \delta^{(e)}}} \sum_{\delta^{(e+1)} \geq \delta^{(e)}} F^d(\delta^{(e)})$$

by rearranging (We need the hypothesis that there is at least one supported constituent, otherwise we divide by zero). The result follows by the coherence of $F^d$. 
Suppose $\delta^{(e)} \notin \text{desc}(\rho_{F^d, \delta^{(d)}})$. The only non-trivial case occurs when $\delta^{(e)} = \rho_{F^d, \delta^{(d)}}$. We have

$$\sum_{\delta^{(e+1)} \geq \delta^{(e)}} \text{renorm}_{\delta^{(d)}}(F^d(\delta^{(e+1)})) = \frac{Z_{F^d, \delta^{(d)}}}{Z_{F^d, \delta^{(d)}}} \sum_{\delta^{(e+1)} \in \text{child}(\delta^{(e)}) \cap S^+_{F^d, \delta^{(d)}}} F^d(\delta^{(e)})$$

by substituting definitions. The result follows by observing that

$$\sum_{\delta^{(e+1)} \in \text{child}(\delta^{(e)}) \cap S^+_{F^d, \delta^{(d)}}} F^d(\delta^{(e)})$$

is exactly $Z_{\delta^{(d)}}^+$ so the result follows.

**Preservation:** By induction on $e$. The base case is trivial. In the inductive case, we have to show that

$$F^d(\rho_{F^d, \delta^{(d)}}) = \sum_{\delta^{(r++)} \in \text{expand}(e+1, \rho_{F^d, \delta^{(d)}})} \text{renorm}_{\delta^{(d)}}(F^d(\delta^{(r++1)})).$$

Rewriting the right hand side, we obtain

$$\sum_{\delta^{(r++)} \in \text{expand}(e, \rho_{F^d, \delta^{(d)}})} \sum_{\delta^{(r++++)} \geq \delta^{(r++)}} \text{renorm}_{\delta^{(d)}}(F^d(\delta^{(r++++)}))$$

$$= \sum_{\delta^{(r++)} \in \text{expand}(e, \rho_{F^d, \delta^{(d)}})} \text{renorm}_{\delta^{(d)}}(F^d(\delta^{(r++)}))$$

where the equality follows by coherence (Proposition coherence). The result follows by the induction hypothesis.

**Commutative:** The proof is quite tedious so we give the intuition first: renorm is commutative because renorm applies Bayes rule to rescale a subtree of $(\Delta^{(\leq d)}, \xi^{(\leq d)})$ and that rescaling by Bayes rule is commutative. The proof follows in two parts. First, we show that the two subtrees (i.e., descendants of the two $d$-redistribution point) we apply rescaling to via Bayes rule to are identical no matter which order we apply renormalization in. Second, it suffices to show that the rescaling on the two subtrees is commutative (due to the subtree property of renormalization).

We start with part one. We claim that the two $d$-redistribution points encountered are identical no matter which order we carry the renormalization. We show this by a direct (and tedious) case analysis. Suppose we apply $\text{renorm}_{\delta^{(d)}}$ first. We perform case analysis on
whether (1) \( \rho_{_Fd,\delta_1^{(d)}} \in \text{anc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \), (2) or \( \rho_{_Fd,\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \), or (3) \( \rho_{_Fd,\delta_1^{(d)}} \in \text{desc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \) where \( \delta_1^{(d)} \sqcup \delta_2^{(d)} \) is the deepest common ancestor of \( \delta_1^{(d)} \) and \( \delta_2^{(d)} \).

Consider the first case \( \rho_{_Fd,\delta_1^{(d)}} \in \text{anc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \). Observe that 
\[ \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} = \rho_{_Fd,\delta_1^{(d)}} \]  
Otherwise, it would contradict that \( \rho_{_Fd,\delta_1^{(d)}} \) has supported children. We need to show that we encounter the same renormalization point applying \( \text{renorm}_{\delta_2^{(d)}} \) first. To see this, \( \rho_{_Fd,\delta_1^{(d)}} \geq \delta_1^{(d)} \sqcup \delta_2^{(d)} \) contradicts that \( \rho_{_Fd,\delta_1^{(d)}} \) has supported children. Thus \( \rho_{_Fd,\delta_1^{(d)}} \in \text{anc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \). Thus we conclude that \( \rho_{_Fd,\delta_1^{(d)}} = \rho_{_Fd,\delta_1^{(d)}} \) because both give the deepest common ancestor with supported children in a tree. Finally, we conclude that 
\[ \rho_{_Fd,\delta_2^{(d)}} = \rho_{\text{renorm}_{\delta_2^{(d)}}(F_d),\delta_1^{(d)}} \]  
as required.

Consider the second case \( \rho_{_Fd,\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \). There are two subcases to consider: either \( \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} \in \text{anc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \) or 
\[ \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \]  

Consider the first subcase \( \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} \in \text{anc}(\delta_1^{(d)} \sqcup \delta_2^{(d)}) \). We conclude that the path from \( \rho_{_Fd,\delta_1^{(d)}} \) to \( \delta_2^{(d)} \) is the only path that is positively supported after applying \( \text{renorm}_{\delta_1^{(d)}}(F_d) \). We see that we encounter the same renormalization points applying \( \text{renorm}_{\delta_1^{(d)}} \) first by performing an even deeper case analysis: either (1) \( \rho_{_Fd,\delta_1^{(d)}} = \rho_{_Fd,\delta_1^{(d)}} \) which occurs when the path from \( \rho_{_Fd,\delta_1^{(d)}} \) to \( \delta_2^{(d)} \) is the only path that is positively supported after applying \( \text{renorm}_{\delta_1^{(d)}}(F_d) \) or (2) 
\[ \rho_{_Fd,\delta_2^{(d)}} = \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} \]  
Thus we conclude that the result holds in this subcase.

Consider the second subcase 
\[ \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \]  
We show that we encounter the same renormalization points applying \( \text{renorm}_{\delta_1^{(d)}} \) first. Observe that 
\[ \rho_{_Fd,\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \]  
Otherwise, it would contradict that \( \rho_{\text{renorm}_{\delta_1^{(d)}}(F_d),\delta_1^{(d)}} \) has supported children. Similarly, observe that 
\[ \rho_{\text{renorm}_{\delta_2^{(d)}}(F_d),\delta_1^{(d)}} = \delta_1^{(d)} \sqcup \delta_2^{(d)} \]  
Thus the result follows.
Consider the third case $\rho_{F,d,\delta_1} \in \text{desc}(\delta_1 \cup \delta_2)$. Observe that $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} \geq \delta_1 \cup \delta_2$ because $\rho_{F,d,\delta_1}$ has supported children.

There are two subcases to consider: either (1) $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} = \delta_1 \cup \delta_2$ or (2) $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} \in \text{desc}(\delta_1 \cup \delta_2)$.

Consider the first subcase $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} = \delta_1 \cup \delta_2$. We show that we encounter the same renormalization points applying $\text{renorm}_{\delta_1}$ first. Observe that $\rho_{F,d,\delta_2} = \delta_1 \cup \delta_2$. Otherwise, it would contradict that $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2}$ has supported children.

Next, we observe that $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} = \rho_{F,d,\delta_1}$ as required.

Consider the second subcase $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2} \in \text{desc}(\delta_1 \cup \delta_2)$. Observe that the subtrees of $\rho_{F,d,\delta_1}$ and $\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2}$ are non-overlapping so that the result follows.

Consequently, there are two $d$-distribution points $\rho_{\delta_1}^{\text{renorm}}$ and $\rho_{\delta_2}^{\text{renorm}}$ and three cases to consider to see that renormalization is commutative for part two of the proof: either (1) $\rho_{\delta_1}^{\text{renorm}}$ and $\rho_{\delta_2}^{\text{renorm}}$ are not ancestors of each other, (2) $\rho_{\delta_1}^{\text{renorm}} = \rho_{\delta_2}^{\text{renorm}}$, or (3) $\rho_{\delta_1}^{\text{renorm}}$ is an ancestor of $\rho_{\delta_2}^{\text{renorm}}$ without loss of generality. The first case is straightforward and second case can be seen as a special case of the third.

Consider the third case where $\rho_{\delta_1}^{\text{renorm}}$ is an ancestor of $\rho_{\delta_2}^{\text{renorm}}$. We show that the result holds by another (tedious) case analysis. Let $X^+ \triangleq X \cup \bigcup_{x \in X} \text{desc}(x)$. We perform a further case analysis on the position of $\delta^{(c)}$ with respect to the support function. Note that the renormalization points may be encountered in the same order or different order. If they are encountered in the same order, then the values are obviously identical. Thus we consider the case when they are encountered in a different order. It suffices to consider the case where $\rho_{\delta_1}^{\text{renorm}}$ is encountered first followed by $\rho_{\delta_2}^{\text{renorm}}$ by symmetry.

Let $D_{\rho_{\delta_1}^{\text{renorm}}} \triangleq D_{\rho_{F,d,\delta_1}}^{\text{renorm}}$ and $D_{\rho_{\delta_2}^{\text{renorm}}} \triangleq \text{child}(\rho_{F,d,\delta_2}) \setminus D_{\rho_{\delta_1}^{\text{renorm}}}$. Let $D_{\rho_{\delta_2}^{\text{renorm}}} \triangleq D_{\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2}}^{\text{renorm}}$ and $D_{\rho_{\delta_1}^{\text{renorm}}} \triangleq \text{child}(\rho_{\text{renorm}_{\delta_1} (F,d),\delta_2}) \setminus D_{\rho_{\delta_2}^{\text{renorm}}}$. 


Case $\delta^{(c)} \in$ 

| $(D^{+}_{\rho^{(d)}_{a}})\dagger$ | $0$ | $0$ |
| $(D^{+}_{\rho^{(d)}_{b}} \setminus \text{child}(\delta^{(d)}_{b}))\dagger$ | $Z^{(d)}_{\delta^{(d)}_{a}} F^{d}(\delta^{(c)})$ | $Z^{(d)}_{\delta^{(d)}_{a}} F^{d}(\delta^{(c)})$ |
| $(D^{-}_{\rho^{(d)}_{a}})^\dagger$ | $0$ | $0$ |
| $(D^{+}_{\rho^{(d)}_{b}})^\dagger$ | $Z^{(d)}_{\delta^{(d)}_{a}} Z^{(d)}_{\delta^{(d)}_{b}} F^{d}(\delta^{(c)})$ | $Z^{(d)}_{\delta^{(d)}_{a}} Z^{(d)}_{\delta^{(d)}_{b}} F^{d}(\delta^{(c)})$ |
| $(D^{-}_{\rho^{(d)}_{a}})^\dagger$ | $Z^{(d)}_{\delta^{(d)}_{a}} F^{d}(\delta^{(c)})$ | $Z^{(d)}_{\delta^{(d)}_{a}} F^{d}(\delta^{(c)})$ |

After substituting definitions, we see that

$$Z^{(d)}_{\delta^{(d)}_{a}} = \frac{\sum_{\delta^{(f)} \in \text{child}(\rho^{(d)}_{\delta^{(d)}_{a}})} F^{d}(\delta^{(f)})}{\sum_{\delta^{(f)} \in D^{+}_{\delta^{(d)}_{a}}} F^{d}(\delta^{(f)})}$$

and

$$Z^{(d)}_{\delta^{(d)}_{b}} = \frac{\sum_{\delta^{(f)} \in \text{child}(\rho^{(d)}_{\delta^{(d)}_{b}})} \text{renorm}^{(d)}_{\delta^{(d)}_{b}}(F^{d})(\delta^{(f)})}{\sum_{\delta^{(f)} \in D^{+}_{\delta^{(d)}_{b}}} \text{renorm}^{(d)}_{\delta^{(d)}_{b}}(F^{d})(\delta^{(f)})}$$

are identical. Similarly, we obtain that $Z^{(d)}_{\delta^{(d)}_{a}}$ and $Z^{(d)}_{\delta^{(d)}_{b}}$ are also identical. Thus the result follows.

---

§4. On Conjecturing. We examine conjecturing as (statistical) model selection in this section. Although conjecturing does not directly lead to a proof, it is an integral part of proving in practice: we require interesting conjectures to prove or disprove and attempting a proof may lead to interesting conjectures. When we introduce a theorem proving game, we will see how conjecturing can be applied as a strategy for playing the game (Section 5).

A conjecture, in its barest form, is a well-formed mathematical statement that we (1) do not have a proof for and (2) consider “interesting”. The first criterion is obviously necessary. The second criterion is also necessary but is inherently subjective. Thus we will not attempt to identify sufficient conditions. With these two criterion in mind, we define a conjecturer now.

**Definition 4.1.** A conjecturer is a function

$$\text{conj} : \text{HT}(\mathcal{L}) \rightarrow \prod_{d \in \mathbb{N}} \text{Perm}(\text{Set}(\Delta^{(d)}))$$

where \text{HT}(\mathcal{L}) is the set of HTs over the language \mathcal{L} and \text{Perm}(X) is the set of permutations on the finite set \(X\).
A conjecturer maps a HT $\mathcal{H}$ and a depth $d$ to a permutation on the powerset of depth $d$ constituents. A depth $d$ conjecture is a depth $d$ dnf, i.e., it is a subset $X \subseteq \Delta^{(d)}$ of depth $d$ constituents. By convention, $\emptyset$ corresponds to conjecturing $\bot$. A permutation on the powerset of $\Delta^{(d)}$ thus provides a ranking of depth $d$ conjectures that we use as a proxy for ranking how interesting depth $d$ conjectures are. We explore how to construct rankings next.

4.1. “Interesting” as Model Selection. We convert the problem of quantifying how interesting a conjecture is to a model selection problem. Thus we take a statistical viewpoint of conjecturing. We accomplish this in two stages.

Model class: First, we identify each depth $d$ conjecture with a model $m$ from some class of models $\mathcal{H}$. That is, we define a surjection $m : \text{Set}(\Delta^{(d)}) \to \mathcal{H}$ from conjectures of any depth to the model class $\mathcal{H}$.

Model scoring: Second, we define a scoring function $S : \mathcal{H} \to \mathbb{R}$ for the model class $\mathcal{H}$, potentially subject to regularization. Given a scoring function $S : \mathcal{H} \to \mathbb{R}$, we can create a ranking on $\mathcal{H}$ as $h_1 \leq h_2$ when $S(h_1) \leq S(h_2)$ with ties broken arbitrarily.

We identify a subclass of finite distributions as an example model class for conjectures.

**Example 3.** Let $\mathbb{D}_D \triangleq \{ \delta^{(d)} \mapsto \mathcal{H}(\delta^{(d)}) \mid \delta^{(d)} \in D \}$ for $D \in \text{Set}(\Delta^{(d)})$ and

$$
\mathbb{D}_D^\dagger = \mathbb{D}_D \cup \{ \star \mapsto 1 - \sum_{(\delta^{(d)} \mapsto b) \in \mathbb{D}_D} b \}
$$

be the distribution that adds a unique element $\star$ representing the remaining unassigned belief. We call the class $\mathcal{G} \triangleq \{ \mathbb{D}_D^\dagger \mid D \in \text{Set}(\Delta^{(D)}) \}$ of finite distributions a distribution conjecture class.

Now that we have a model class for conjectures, we can define scoring functions for models to rank conjectures. We give an example scoring function below.

**Example 4.** A likelihood-entropy scoring function for the distribution conjecture class $\mathcal{G}$ scores conjectures as a function of their likelihood and entropy: we have

$$
\mathcal{L}(\mathbb{D}_D^\dagger) \triangleq \frac{c(|D|) L(\mathbb{D}_D^\dagger)}{H[\mathbb{D}_D^\dagger]} \cdot \begin{cases} H(\mathbb{D}_D^\dagger) & \text{when } D^+ \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
$$

\[19\] We use model in the statistical sense and not the model-theoretic sense for which have used the word structure instead.
where $D^+ \triangleq \{ \delta^{(d)} \mid \mathbb{D}_D(\delta^{(d)}) > 0 \}$, $H$ is entropy, $H^{|X|}$ is the entropy of the uniform distribution over $|X|$ elements, $\ell(\mathbb{D}_D) = \sum_{(x \rightarrow b) \in \mathbb{D}_D, x \neq *} b$ is the total probability except for $*$, and $c : \{1, \ldots, |X|\} \rightarrow \mathbb{R}^+$ is a positive and concave function over $\{1, \ldots, |X|\}$ ordered by $\leq$. We calculate the entropy of the modified distribution $\mathbb{D}_{D^+}$ that only considers the constituents with positive probability. The factor

$$\frac{c(|D|) \ell(\mathbb{D}(D)^\dagger)}{H^{|D|}}$$

is a form of regularization. First, recall that the entropy of a uniform distribution increases as the set of support increases so that an unnormalized measure would be biased towards selecting larger dnfs. Thus we normalize by the entropy of the uniform distribution of the appropriate size. Second, we want to encourage dnfs that include enough possibilities. This is what the concave function $c$ achieves. Lastly, we want to ensure that the conjecture captures enough of what we believe to be true. Otherwise, we would be encouraged towards selecting the least believable constituents because these provide the most information. A likelihood-entropy score can be thought of as measuring how “informative” a conjecture is weighted by how likely it is.

We check the conjectures generated by a conjecturer using a likelihood-entropy scoring function with beliefs $\mathbb{H}$ under two extremes.

**Maximally ignorant:** Given a HT $\mathbb{H}$ that assigns beliefs uniformly to constituents at every depth, the ranking produced by a likelihood-entropy scoring ranks conjectures is solely a function of the number of constituents in their dnf. This follows directly from the definition of a likelihood-entropy score.

**Maximally knowledgeable:** Given a depth HT $\mathbb{H}$, a likelihood-entropy score ranks conjectures at every depth containing consistent constituents higher than conjectures mentioning no consistent constituents. To see this, observe that if a conjecture has no consistent constituents, then $D^+ = \emptyset$ and thus gets assigned score 0. Moreover, a conjecture that contains consistent constituents has $D^+ \neq \emptyset$ so that it gets assigned positive score. A conjecturer that knows all logical truths will rank true statements according to the regularization factor and higher than any false statement.

**4.2. Top-down versus Bottom-up Regularization.** The regularization given by the concave function $c$ in likelihood-entropy scoring can be seen as a form of bottom-up regularization in that we control the size of the conjecture so that it describes just enough possibilities. We can also impose top-down regularization where we control the sizes and form of the first-order sentences in addition to the sizes of their dnfs. The intuition
for additionally considering top-down regularization is that we would like
the conjectures to be “compact and structured enough” to write down in
addition to describing just enough possibilities. Let \( K \) be a finite set of
first-order sentences, \( d_{\text{min}} \) be the minimum depth of formulas in \( K \), and
\( D^d_K \triangleq \{ \delta^{(d)} \mid \delta^{(d)} \in \text{dnf}(\phi^{(d)}), \phi^{(d)} \in K \} \).

**Definition 4.2.** A \( K \)-regularized conjecturer is a function
\[
\text{conj}_K : \text{HT}(\mathcal{L}) \to \prod_{d \geq d_{\text{min}}} \text{Perm} \left( \text{Set} (D^d_K) \right).
\]
Thus a conjecturer is a \( \mathcal{L} \)-regularized conjecturer. The definition of a \( K \-
regularized conjecturer allows any kind of subset, although it may be useful
to use the regularization to restrict the form of the sentences.

*Example 5.* Any finite subset of \( \Sigma^0_n \) or \( \Pi^0_n \) can be used to form a \( K \-
regularized conjecturer. Recall that these sentences constrain \( \forall \) and \( \exists \)
to occur in an alternating sequence.

*Example 6.* The singleton set \( K \triangleq \{ \delta^{(d)} \} \) can be used to form a \( K \-
regularized conjecturer. In particular, such a conjecturer only generates
conjectures that are refinements of \( \delta^{(d)} \).

\section*{§5. On Games and Proving.}
We return to the connection between
game playing and proving in this section. In particular, we introduce an
alternating-turn game where agents with “reasonable” beliefs (i.e., consider
“enough” possible kinds of worlds to be possible) can play to learn a prover
that is both complete in the limit (that their beliefs converge to true beliefs)
and sound. The game involves learning how to recognize the consistency of
statements in a HT. Thus agents that play this game well should assign high
beliefs to true statements and low beliefs to false statements. The game is
amenable to self-play training setups similar to those used to learn Chess
and Go, although the challenging task of implementation and empirical
testing self-play is beyond the scope of this paper. One reason for the
technical difficulty is that dnfs have intractable space requirements. We
will comment on how to reduce the space complexity by using *abstractions*
(Section 6).

\subsection*{5.1. From Beliefs to Proofs.}
Before we introduce the game, we explain how to extract a proof attempt from beliefs represented by a HT \( \mathbb{H} \).
Define a function \( \text{prove}_{\mathbb{H}} : \mathcal{L} \to \{\top, \bot\} \) as
\[
\text{prove}_{\mathbb{H}}(\phi^{(d)}) = \begin{cases} 
\top & \sum_{\delta^{(d)} \in \text{dnf}(\phi^{(d)})} \mathbb{H}(\delta^{(d)}) = 1 \\
\bot & \text{otherwise.}
\end{cases}
\]

We say that a belief \( \mathbb{H} \) is *reasonable* if \( \mathbb{H}(\delta^{(d)}) > 0 \) whenever \( \delta^{(d)} \) is not
trivially inconsistent.
Example 7. One easy way to obtain reasonable beliefs is to be maximally ignorant, i.e., assign every depth $d$ constituent the same probability. The only theorem that an agent with maximally ignorant beliefs can prove is $\top$. Thus the agent is sound and maximally incomplete.

Example 8. Both a surface HT and depth HT are reasonable.

We show that the function $\text{prove}_H$ converts reasonable beliefs into proofs.

Proposition 5.1.

Sound: The procedure $\text{prove}_H$ is sound if $H$ is reasonable.

Complete: The procedure $\text{prove}_H$ is complete whenever $H$ is a depth HT.

Proof.

1. Suppose for the sake of contradiction that $\text{prove}_H(\phi^{(d)}) = \top$ but $\phi^{(d)}$ is inconsistent. Thus there is at least one consistent constituent $\delta^{(d)} \not\in \text{dnf}(\phi^{(d)})$. We conclude that $H(\delta^{(d)}) = 0$ when $\text{prove}_H(\phi^{(d)}) = \top$ because $\sum_{\delta^{(d)} \in \text{dnf}(\phi^{(d)})} H(\delta^{(d)}) = 1$ and by the normalization property of HTs. This contracts the assumption that $H(\delta^{(d)}) > 0$ when it is not trivially inconsistent.

2. Recall that a formula is logically valid iff its dnf contains all consistent constituents.

The first part shows that agents are not required to possess true beliefs in order to obtain a sound prover. The second part of the proposition above indicates that we are only at risk of losing completeness. The situation intuitively makes sense: so long as our beliefs deviate from true beliefs, we will not be able to prove every true theorem. We turn our attention now towards learning beliefs $H$.

5.2. Pathfinder: A Proving Game. Pathfinder is an alternating-turn game where the goal of the game is to recognize inconsistent constituents. Because we can extract a prover given a HT $H$ as above, agents that learn to recognize the consistency of constituents well will learn to be a better theorem prover (i.e., be able to prove more theorems).

A player is given a depth $d$ constituent $\delta^{(d)}$ and allowed to make one of two moves:

1. A player can select a refinement constituent $\delta^{(d+1)} \geq \delta^{(d)}$ and pass play to the other player. The select move introduces an existential which intuitively corresponds to the construction of an auxiliary object that may be useful for the proof.

As a concrete instance, consider proofs in Euclidean geometry. These proofs involve constructing the appropriate points, lines, and circles so that the conclusion is “obvious”. This method of proof contrasts with the design of many first-order automated theorem provers where quantifiers are lifted to the head of the formula and eliminated.
2. A player can issue a *challenge* meaning that the player believes the constituent to be inconsistent. If $\delta^{(d)}$ is revealed to be inconsistent, the player issuing the challenge wins. Otherwise, if $\delta^{(d)}$ is revealed to be consistent, then the player issuing the challenge loses.

In order to play the game well, the players need to develop an intuition about which constituents “look” inconsistent. Figure 3 illustrates the flow of an example game. We describe the game more formally now.

Let $* \equiv \Delta \cup \{\ast\}$ denote the *states* of Pathfinder. We write $x \in X$ to denote a state generically or $\delta^{(d)} \in X$ when it is a constituent. Define the *positions* of Pathfinder to be the set

$$P \equiv \bigcup_{d \in \mathbb{N}} \text{Seq}^d(X)$$

of all finite sequences of states.

Let the two players be $O$ for odd and $E$ for even. Define the *turn order* function $T : P \to \{O, E\}$ as

$$T(x_1 \ldots x_n) = \begin{cases} O & \text{n even} \\ E & \text{n odd} \end{cases}$$

Thus player $O$ plays the positions that have even length (resulting in a position that has odd length) and player $E$ plays the positions that have odd length.
Algorithm 1 Strategy for rational agent $A_H$.

```
1: function step($\delta^{(d)}$)
2:     if $\sum \delta^{(d+1)} \geq \delta^{(d)}$ then
3:         challenge
4:     else
5:         if flip($1 - H(\delta^{(d)})$) = true then
6:             challenge
7:         else
8:             select $\delta^{(d+1)} \geq \delta^{(d)}$ with probability $\frac{H(\delta^{(d+1)})}{\sum \delta^{(d+1)} \geq \delta^{(d)}}$
```

Next, we define transition relation $\rightsquigarrow: P \rightarrow P \rightarrow 2$ to indicate the legal moves. We give the inference rules generating $\rightsquigarrow$ below.

**Select:** $\delta^{(0)} \ldots \delta^{(d)} \rightsquigarrow \delta^{(0)} \ldots \delta^{(d)} \delta^{(d+1)}$ whenever $\delta^{(d)} \leq \delta^{(d+1)}$

**Challenge:** $\delta^{(0)} \ldots \delta^{(d)} \rightsquigarrow \delta^{(0)} \ldots \delta^{(d)} *$

The player whose turn it is to move chooses either select or challenge.

Finally, we define the function $W: P \rightarrow \{O, E\}$ which determines which player wins:

$$W(x_1 \ldots x_{n-1}*) = \begin{cases} 
O & n \text{ odd and } x_{n-1} \text{ inconsistent} \\
E & n \text{ odd and } x_{n-1} \text{ consistent} \\
O & n \text{ even and } x_{n-1} \text{ inconsistent} \\
E & n \text{ even and } x_{n-1} \text{ consistent}
\end{cases}$$

We can define the game now that we have all the requisite components.

**Definition 5.1.** The Pathfinder game is given by the tuple $\mathcal{P} \equiv (P, T, \rightsquigarrow, W)$.

**5.3. Playing Pathfinder.** As we have just seen, the rules for Pathfinder game are quite simple. Nevertheless, like many other games whose rules are easy to state, playing Pathfinder “well” is difficult because it reduces to first-order theorem proving. We can analyze the plays made by agents (i.e., what it means to play “well”) using the formalization above. Towards this end, we model an agent as using a HT to guide their game play for Pathfinder.

Suppose an agent $A_H$ playing Pathfinder uses a HT $H$ to represent its uncertainty in mathematical statements. Intuitively, we should be able to derive a strategy for playing Pathfinder that is compatible with the agent’s beliefs $H$. In essence, it should issue challenges and select constituents in proportion to the probability $H$ assigns to the consistency of each constituent. More formally, a strategy for a player says for each position what the next position to play is when it is that player’s turn. We say that the
agent $A_H$ is rational if it plays the strategy given by Algorithm 1 (hence it plays a mixed strategy). In words, the agent first checks that it does not believe all continuations are inconsistent as $\sum_{\delta'(d+1) \geq \delta(d)} H(\delta'(d+1)) \neq 0$ and challenges if it is (lines 2–3). If it is not, then the agent challenges with probability $1 - H(\delta(d))$ (lines 5–6). With the remainder of the probability, it selects a constituent $\delta'(d+1) \geq \delta(d)$ in proportion to its belief in its consistency

$$H(\delta(d+1)) \sum_{\delta'(d+1) \geq \delta(d)} H(\delta'(d+1))$$

(lines 7–8).

As we might expect, a rational agent with perfect knowledge is able to achieve optimal play: (1) only challenge inconsistent constituents and (2) only select consistent constituents.

**Proposition 5.2.** A rational agent $A_H$ where $H$ is a depth HT achieves optimal play.

**Proof.** By assumption, $H$ is a depth HT so it assigns inconsistent constituents probability 0. We proceed by case analysis. $\delta(d)$ is inconsistent when $\sum_{\delta'(d+1) \geq \delta(d)} H(\delta'(d+1)) = 0$ so a challenge is issued. If $H(\delta(d)) = 0$, then the agent challenges with probability 1. If $H(\delta(d)) > 0$, then the agent selects only consistent constituents and passes play to the second player. $\square$

The game of Pathfinder continues ad infinitum with optimal play, i.e., is drawn. Recall that a strategy is winning if the strategy always produces a win no matter what the other player does.

**Proposition 5.3.** There are no winning strategies.

**Proof.** The contrapositive of the completeness theorem for constituents gives that every consistent constituent has a refinement. Hence both players always have a non-losing continuation. $\square$

Of course, optimal play is not computable as a depth HT is not computable. A surface HT is computable although it does not lead to optimal play. In particular, all the refinements of a constituent that are not trivially inconsistent can be revealed to be trivially inconsistent at greater depth. In this case, the agent will issue a challenge, resulting in a loss.

5.4. **Incorporating Conjecturing.** We can incorporate conjecturing into the playing of Pathfinder. We say that an agent is a conjecturing agent if it plays the (mixed) strategy given in Algorithm 2. The rules for challenging are identical to the rules played by a rational agent. The difference occurs in the selection of the next constituent to play (lines 7–9). In this case, the agent uses a $K$-regularized conjecturer where $K = \text{expand}(\delta(d))$ to generate a ranking of conjectures $\pi$. Next, the agent selects
Algorithm 2 Strategy for conjecturing agent $A_H$.

1: function $\text{step}(\delta(d))$
2: if $\sum_{\delta'(d+1) \geq \delta(d)} H(\delta'(d+1)) = 0$ then
3: challenge
4: else
5: if flip$(1 - H(\delta(d))) = \text{true}$ then
6: challenge
7: else
8: $\pi \leftarrow \text{conj}_{\delta(d)}(H)(d+1)$
9: select $\delta(d+1)$ with probability $\frac{H(\delta'(d+1))}{\sum_{\delta'(d+1) \in \pi_1, \delta'(d+1) \neq *} H(\delta'(d+1))}$

the highest ranked conjecture $\pi_1$ and selects the constituent from that excluding $*$ following the probabilities given by $H$.

**Proposition 5.4.** A conjecturing agent $A_H$ using a likelihood-entropy scoring function where $H$ is a depth HT achieves optimal play.

**Proof.** The only difference is the select case. We claim that $\pi_1$ contains consistent constituents. Assume for the sake of contradiction that it does not. As a likelihood-entropy scoring function ranks conjectures containing consistent constituents higher than those that contain none, then $\pi_1$ contains no consistent constituents. But this means that $\delta(d)$ is inconsistent because it contains no refinement constituents that are consistent, a contradiction. As a depth HT assigns inconsistent constituents belief 0 and $\pi_1$ contains consistent constituents, an agent selecting constituents in proportion to their beliefs will select a consistent constituent as required. ⊥

5.5. A Note on Self-Play for Pathfinder. We note that self-play training similar to those described in the literature (e.g., see [78, 70, 71]) is applicable to Pathfinder as it is an alternating-turn game with symmetric play. We recall the standard setup here to make the idea concrete. Of course, the implementation and empirical testing of self-play setups are the most challenging and non-trivial portions of the task, which we do not address in this paper.

Let $x_1 \ldots x_t$ be a sequence of Pathfinder positions where $x_t = \delta(0) \ldots \delta(d) \ast$ is a terminal board state. We truncate games so that they take at most $N$ steps. If no challenge is issued within $N$ steps, we say that the game is drawn. Define a reward signal $z_O$ for player $O$ as $z_O \triangleq 1$ when $O$ wins, $z_O \triangleq -1$ when $O$ loses, and $z_O \triangleq 0$ when there is a draw. As usual, the reward signal $z_E$ for the other player $E$ is the negation $z_E = -z_O$.
Define a parameterized function
\[ f_\theta : \prod_{d \in \mathbb{N}} \Psi(d) \rightarrow \text{Dist}(\Delta^{(d+1)} \cup \{\text{challenge}\}) \times [0, 1] \]
where \( \text{Dist}(X) \) gives the collection of finite distributions on \( X \) which takes a current depth \( d \) and a path through the refinement tree, and produces a distribution on constituents to select or to challenge paired with an estimate of the expected value (with respect to the move probabilities) of winning for player \( O \) starting at the current path. Suppose we have taken the refinement path \( \delta^{(0)} \leq \ldots \leq \delta^{(d)} \) and that \( f(\delta^{(0)} \ldots \delta^{(d)}) = (p_1, \ldots, p_{K+1}, p_{\text{challenge}}, \hat{z}_O) \). A self-play game can be generated by selecting the move in proportion to \( (p_1, \ldots, p_{K+1}, p_{\text{challenge}}) \). We can learn the parameters \( \theta \), for instance, by reinforcement learning (e.g., see [76]) using appropriately defined estimates of the true move probabilities and expected value of winning. The hope is that self-play learns \( \pi_1(f_\theta) \approx \mathbb{H} \) where \( \mathbb{H} \) is a depth HT.

**Remark** (Initialization with reasonable beliefs). The initialization of initial beliefs \( \mathbb{H} \) is important. Notably, if \( \mathbb{H} \) is not reasonable, we will obtain neither a sound or complete theorem prover. We will not obtain a sound prover because an agent that is not reasonable assigns all of its beliefs to inconsistent possibilities. Moreover, we will also not obtain a complete prover in the limit because Bayesian dynamics can never rescale zero probability possibilities to have positive probability. Provided that we have initialized a self-play agent with reasonable beliefs and ensure that beliefs in constituents are never zeroed unless they are known to be inconsistent, then the agent will maintain reasonable beliefs.

§6. On Abstraction. Both conjecturing and Pathfinder are not practically implementable as currently presented because there are a super-exponential number of constituents at each depth resulting in a HT having a super-exponential branching factor. The reason that there are so many depth \( d \) constituents is because they provide the finest grained view of possible kinds of worlds describable with respect to \( d \) individuals. However, for most intents and purposes, we can take a coarser grained view that captures the details that we care about. In other words, we can treat certain possibilities as observationally indistinguishable to reduce the space complexity of a dnf, i.e., make abstractions and lazily consider more details as needed. At the end of the section, we will introduce Trailblazer, a modification of the Pathfinder game, that utilizes abstractions and laziness to trade-off completeness for on-demand space requirements.

6.1. Filtrations. The basic idea we have in mind is to control the “resolution” at which constituents distinguish possibilities by partitioning each
Figure 4. An illustration of the set of depth $d$ ($d = 1, 2, \text{ or } 3$) constituents (i.e., the universe of possibilities by depth) where each cell corresponds to a constituent, the dimension of the cell corresponds to the depth of the constituents, and the refinement relation is encoded as projection. Cells colored light blue indicate that the attributive constituent asserts that individuals satisfying that description exist. At each depth $d$, the collection of all depth $d$ attributive constituent corresponds to all possible ways of coloring the cells. Defining a filtration corresponds to treating certain combinations of cells as one super cell.

set of depth $d$ constituents in a compatible manner across depth. Each cell of the partition describes all of the possibilities identified by that cell’s member constituents.

Let $\{C_i^{(d)}\}$ be a partition of $\Delta^{(d)}$. For each $C_i^{(d)}$, define the super constituent $\sigma_i^{(d)}$ with respect to a partition $\{C_i^{(d)}\}$ as

$$\sigma_i^{(d)} \triangleq \bigvee_{\delta \in C_i^{(d)}} \delta^{(d)}.$$ 

A super constituent collapses multiple distinct possibilities into one possibility, and thus, can be viewed as a form of abstraction. Let $S^{(d)}$ be the set of super constituents with respect to the partition $\{C_i^{(d)}\}$. Naturally, a super constituent is said to be trivially inconsistent if all of its members are trivially inconsistent.

**Definition 6.1.** We say $\mathcal{F} = (\{C_i^{(d)}\})_{d \in \mathbb{N}}$ where each $\{C_i^{(d)}\}$ is a partition of $\Delta^{(d)}$ is a filtration of $(\Delta, \xi)$ if adjacent elements satisfy the following
condition: for every cell \( C_j^{(d)} \in \{C_i^{(d)}\} \), there exists a subset \( D \subseteq \{C_i^{(d+1)}\} \)

such that \( C_j^{(d)} = \bigcup_{C_k^{(d+1)} \in D} C_k^{(d+1)} \).

In words, we have a filtration if the partition at depth \( d + 1 \) of \( \Delta^{(d+1)} \) can be used to form a partition of each cell at depth \( d \). A filtration induces a corresponding set of super constituents.

We can lift the refinement partial order on partitions to filtrations. Let \( \mathcal{F} \) be the set of all filtrations. We have \((\mathcal{F}, \sqsubseteq)\) is a partial order where \( \mathcal{F}_1 \sqsubseteq \mathcal{F}_2 \) if each depth \( d \) partition in \( \mathcal{F}_2 \) is finer than the corresponding depth \( d \) partition in \( \mathcal{F}_1 \). At one extreme, we have a filtration consisting of one cell that contains every constituent so that it has the lowest resolution. At the other extreme, each filtration assigns each constituent to its own set so that we have the highest resolution possible so that no space savings is gained. We can intuitively think of the “resolution” of a filtration \( \mathcal{F} \) as the height in the Hasse diagram of \( \mathcal{F} \). Naturally, some resolutions are incomparable.

Super constituents possess some of the same properties as constituents.

**Proposition 6.1.**

**Exclusivity:** Any two super constituents of the same depth are mutually exclusive.

**Expansion:** Every depth \( d \) super constituent can be written as a disjunction of super constituents of greater depth.

**Completeness:** A super constituent is inconsistent if and only if all of its refinements at some depth are trivially inconsistent.

**Proof.** These properties all follow directly from the properties of partitions. \( \square \)

In general, we lose existence of super constituents: there are depth \( d \) sentence \( \phi^{(d)} \) that cannot be written as a disjunction of depth \( d \) super constituents. For example, the super constituents obtained from the trivial filtration cannot express logically invalid statements. We say that a filtration is complete if it assigns every consistent constituent to its own cell.

**Proposition 6.2 (complete existence).** Every depth \( d \) sentence can be written as a disjunction of depth \( d \) super constituents defined with respect to a complete filtration \((\{C^{(d)}\})_{d \in \mathbb{N}}\).

**Proof.** Recall that we can adjoin inconsistent constituents to a dnf without affecting its satisfiability. \( \square \)

As we might expect by now, a complete filtration is not computable.
6.2. Choosing Filtrations. For pragmatic purposes, we will need to cleverly choose a filtration. One method for constructing filtrations uses the fact that depth \(d\) constituents indicate which depth \(d-1\) attributive constituents (with 1 free individual term) exist or not. As a reminder, constituents are defined in terms of attributive constituents as

\[
\delta^{(d)}_s = \bigwedge_{(r_1, s_1) \in \mathcal{G}^{d-1}_1} (\pm)^{s(r_1, s_1)} (\exists x_1) \gamma^{(d)}_{r_1, s_1}[x_1].
\]

Let \(o : \mathcal{G}^{d-1}_1 \rightarrow \{O, I\}\) be an observation of depth \(d\) constituents where \(o(r_1, s_1) = O\) means that we observe position \((r_1, s_1)\) and \(o(r_1, s_1) = I\) means that we ignore position \((r_1, s_1)\). Then we can define an equivalence class on constituents \(\delta^{(d)}_s \sim_o \delta^{(d)}_t\) if \(o(r_1, s_1) = t(r_1, s_1)\) whenever \(o(r_1, s_1) = O\). The collection of equivalence classes forms a filtration. Whenever \(o(r_1, s_1) = I\), we have that every super constituent contains both \(\neg(\exists x_1) \gamma^{(d)}_{r', s'}[x_1]\) and \((\exists x_1) \gamma^{(d)}_{r', s'}[x_1]\) so that we can no longer tell the two possibilities apart. When \(o(r_1, s_1) = I\) for every \((r_1, s_1) \in \mathcal{G}^{d-1}_1\), the induced filtration produces exactly one super constituent. When \(o(r_1, s_1) = O\) for every \((r_1, s_1) \in \mathcal{G}^{d-1}_1\), the induced filtration assigns each constituent to its own set.
We can further break down the construction of filtrations by constructing an observation of depth $d$ constituents using their substructure. Unfolding the recursive definition of a constituent $\delta^d$ by depth, we see that it is a formula of the form

$$\delta^d = \bigwedge_{(r_1,s_1) \in G^d_{d-1}} (\pm)^{s(r_1,s_1)}(\exists x_1)^{\gamma_{(d-1)}}(x_1) \land \ldots$$

$$\land \bigwedge_{r_d \in G_d^d} (\pm)^{s_d-1}(r_d) (\exists x_d)^{\gamma_{(d-1)}}(x_1, \ldots, x_d).$$

Figure 5 gives an illustration of a depth $d$ attributive constituent tree. In this unfolded form, we see that a depth $d$ attributive constituent is a tree where nodes are existential formulas (except for the root node which is $\top$) of the form $(\pm)^b(\exists x_e)^{\gamma_{(0)}}[x_1, \ldots, x_e]$ and edges indicate the scope of the quantifier. Each partial description $(\pm)^{s(r_1,s_1)}(\exists x_1)^{\gamma_{(d-1)}}(x_1)$ corresponds to a subtree in the attributive constituent of depth $d$ indicating which nested sequences of individuals described by the appropriate depth $0$ attributive constituents exist or not. We can thus construct an observation by indicating which subtrees to observe or ignore.

6.3. Trailblazer: Game Play with Super Constituents. We can play Pathfinder using super constituents instead of constituents in the obvious way. When Pathfinder is played with super constituents obtained from a filtration that is not a complete filtration, agents will only be able to learn beliefs that enable them to prove a subset of the first-order theorems. This situation makes intuitive sense: we cannot prove certain theorems if we use inappropriate abstractions, even if we have infinite compute. This brings us to a variation of Pathfinder called Trailblazer where agents can additionally choose abstractions during game play.

A player is given a depth $d$ super constituent $\sigma^{(d)}$ and allowed to make one of three moves: select, challenge, or refine. The first two are similar to the corresponding ones in Pathfinder. For the last move, a refine move takes a super constituent and breaks it into smaller super constituents and chooses one of the smaller super constituents to continue the game. This corresponds to increasing the resolution at which that player would like to continue the game at. We describe the game more formally now.

Let $\ast$ represent the terminal state reached after a challenge is issued. Let the dependent sum $X \triangleq \sigma_{\mathcal{F},\mathcal{G}}(S_{\mathcal{F}} \cup \{\ast\})$ denote the states of Trailblazer which pairs a filtration $\mathcal{F}$ with the super constituents $S_{\mathcal{F}}$ obtained from filtration $\mathcal{F}$. Define the positions of Trailblazer to be the set

$$P \triangleq \bigcup_{d \in \mathbb{N}} \text{Seq}^d(X)$$

of all finite sequences of states.
The turn order for Trailblazer is identical to that of Pathfinder. However, whereas player $O$ selects constituents of odd depth and player $E$ selects constituents of even depth in Pathfinder, this is not the case in Trailblazer due to the refine move.

The transition relation $\rightsquigarrow: P \rightarrow P \rightarrow 2$ for Trailblazer has an additional clause for refine. For the sake of completeness, we give all the inference rules generating $\rightsquigarrow$ below.

**Select:** \( (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) \rightsquigarrow (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) (\mathcal{F}_{d+1}, \sigma(d+1)) \) whenever \( \sigma(d) \leq \sigma(d+1) \)

**Challenge:** \( (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) \rightsquigarrow (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) (\mathcal{F}_d, *) \)

**Refine:** \( (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) \rightsquigarrow (\mathcal{F}_0, \sigma(0)) \ldots (\mathcal{F}_d, \sigma(d)) (\mathcal{F}_d', \sigma'(d)) \) whenever \( \mathcal{F}_d \subseteq \mathcal{F}_d' \) and \( C'(d) \subseteq C(d) \) where \( C(d) \) and \( C'(d) \) are the cells corresponding to \( \sigma(d) \) and \( \sigma'(d) \) respectively

The player whose turn it is to move chooses either select, challenge, or refine.

The winning condition \( W: P \rightarrow \{O, E\} \) is the similar to that of Pathfinder where we use inconsistency of super constituents as opposed to inconsistency of constituents.

**Definition 6.2.** The Trailblazer game is given by the tuple \( \mathfrak{P} \equiv (P, T, \rightsquigarrow, W) \).

As before, there are no winning strategies in Trailblazer and the winning strategy is not computable. Note that we can start game play in Trailblazer with any filtration \( \mathcal{F} \) including the minimal one (i.e., the trivial filtration). In this case, the game is always drawn unless players “take risks” and play refine moves. In particular, the super constituent that is selected on a refine move may turn out to be inconsistent, in which case the player that recognizes this first will win the game. Like Pathfinder, Trailblazer is also amenable to self-play training.

**§7. Related Work.** We review related work relevant to each section encountered in the body of the paper. We apologize in advance for missing connections to the literature.

**7.1. Representing Beliefs in Mathematical Statements.** As a reminder, the inspiration for the definition of a HT comes from our reading of [33, pg. 274–282]. To the best of our knowledge, the application of HTs to assigning probabilities to first-order sentences is new.

There have been several approaches proposed for assigning probabilities to statements with first-order quantifiers and probabilistic assertions. One approach defines measures on a suitable space of structures where the probability of a statement is the measure of the set of structures that satisfy the
statement (e.g., see [20, 67] and [39] for the case of higher-order logic). Logically valid statements are satisfied in every structure so they are assigned measure 1. We are not concerned with the ability to express probabilistic assertions in the logic because we simply use the logic to encode mathematics as opposed to empirical propositions. However, we are concerned with weakening the requirement that logically equivalent statements are assigned the same probability. Demski [13] proposes another approach (that enforces logical omniscience) that assigns probabilities to sentences based on algorithmic probability.

There have been several approaches developed with learning in mind that assign probabilities to statements based on a measure on structures. A Markov logic network [63] is a representation designed with probabilistic inference in mind that assigns probabilities to statements expressed in a first-order language interpreted in models with finite domains. The restriction to finite domains means that the setting for Markov logic networks is effectively propositional because an existential quantifier can be encoded as a finite disjunction (similarly, a universal quantifier can be encoded as a finite conjunction). Thus the quantifiers in Markov logic can be translated away at the cost of increasing the sizes of the formulas considered. Blog [3] is a representation that combines first-order logic and probabilities designed with Bayesian inference in mind that assigns probabilities to statements based on a possible worlds semantics. Thus the representation also enforces logical omniscience.

Logical induction [24] is a method for assigning sequences of probabilities to first-order sentences that only enforces logical omniscience in the limit. Thus it is identical in its objective of weakening logical omniscience for the purpose of assigning probabilities to mathematical statements. The method uses a market mechanism to assign “prices” to sentences and uses a no Dutch book argument to make the connection between “prices” and probabilities. Like our method, the sequence of probabilities assigned to first-order sentences is computable while the limit is not computable. Our method also enforces logical omniscience in the limit when given a depth or surface HT. Unlike logical induction which treats each formula as an atomic unit, we use the structure of formulas to assign probabilities.

Another approach to weakening logical omniscience in the context of assigning probabilities to logical sentences is to syntactically model an agent’s knowledge (e.g., restrict logical omniscience to a subset of sentences [21] or introducing syntax for a new modality to distinguish implication from provability [23]).

\[\text{Note that this differs from assigning probabilities to possible kinds of worlds as we have done which does not directly consider the individuals in the domain of quantification.}\]
It is also possible to adapt a syntactic approach where an agent’s reasoning capability is modeled as bounded (e.g., see [40] for more on bounded reasoning and [2] for length-bounded reasoning) to assigning probabilities to statements as the probability of its provability. This idea is in essence that of using a stochastic proof system where inference rules are applied non-deterministically. One issue with this approach is that there can be multiple proofs (or refutations) of the same fact so some notion of minimal length proof is required. In contrast, the approach taken in this paper directly models an agent’s beliefs independent of the difficulty of its provability.

7.2. Conjecturing. Larson [48] provides a nice survey of the field of automatic conjecture generation. Many of these programs generate conjectures by enumerating syntax (generated from a production system) and pruning them by a combination of hand-crafted heuristics and model checking [49, 17, 27, 8]. Some methods such as the one implemented in Graffiti [17] based on the idea of generating the strongest conjecture that has no known counter-example have produced “research-level” conjectures (e.g., see [7]). In contrast to these operational descriptions of conjecturing, our description of conjecturing is denotational. One advantage of a denotational approach is that it is not sensitive to the order in which syntax is enumerated.

One form of conjecturing concerns making generalizations from special cases. In short, given that we have seen that \( P(a_1), \ldots, P(a_N) \) where each \( a_i \) is a constant that identifies a unique element in the domain of quantification and \( P \) is a unary predicate, to what degree do we believe that \( (\forall x)P(x) \) is true? This form of conjecturing has been studied in inductive philosophy (e.g., Carnap [5] studies inductive generalizations in monadic first-order logic and Hintikka [32] studies inductive generalizations on constituents). We do not address this form of conjecturing. In particular, each \( P(a_i) \) results in a depth 0 dnf whereas \( (\forall x)P(x) \) results in a depth 1 dnf so that we would need to compare conjectures across depth. It would be an interesting direction of future work to analyze the notion of conjecturing while taking depth into account. We note that we can apply any method of inductive generalization defined on constituents (e.g., Hintikka’s [32]) to our setting.

7.3. On Games and Proving. The connection between games and proving first-order theorems has been recognized since the development of modern first-order logic. The philosopher Peirce casts first-order theorem proving as a non-alternating-turn game on existential graphs [6, 75]. That the semantics of first-order logic can be given in terms of games has also been recognized in the literature (e.g., see [30, 34, 51, 36]). The connection between games and other logics (especially modal logic) has also been recognized (e.g., see [81]).
We are not aware of any alternating-turn games for proving the validity of first-order theorems. There are alternating-turn games that can be played on first-order structures. For instance, the well-known Ehrenfeucht-Fraïssé game, also known as a back-and-forth game, can be used to determine the elementary equivalence of first-order structures. The game-theoretic semantics of first-order logic gives rise to a game for checking the satisfiability of first-order formulas in a given first-order structure, and is alternating-turn game when played on dnf of the second kind.\[22\]

7.4. On Abstraction. Hintikka and Tuomela [37] study the concept of definition, a form of abstraction, using constituents. They show a la analysis on constituents that a theory employing definitions that are explicitly definable in a first-order logic can reveal the trivial inconsistency of sentences at shallower depths compared to a theory not employing those definitions. The idea is that definitions are useful, even if they can be translated away, because they make certain theorems easier to prove. In contrast, we consider abstraction as a method for controlling the sizes of constituents, and as a cost, give up the ability to prove certain theorems.

A form of abstraction, namely proofs with cut (i.e., proofs where we can use lemmas) can be used to reduce the sizes of proofs in first-order proof calculi [61]. Notably, first-order proofs with cut-elimination increases the sizes of proofs by a super-exponential amount.

§8. Conclusion. In summary, we consider the problem of learning a first-order theorem prover where we use a representation of beliefs in mathematical claims instead of a proof system to search for proofs. Towards this end, we introduce a representation of beliefs in the validity of first-order statements based on first-order dnf. The probabilistic view enables us to cast conjecturing as (statistical) model selection and create an alternating-turn proving game that is (in principle) amenable to self-play training for learning a prover that is both complete in the limit and sound provided

\[22\]The game semantics of first-order logic is defined by induction on the structure of formulas. It is a game between two players: Eloise who controls the positive fragment of the logic and Abelard who controls the negative fragment of the logic. Negations correspond to switching who controls the positive and negative fragments of the logic. Eloise has a winning strategy if the formula is satisfiable in a structure $M$ whereas Abelard has a winning strategy if the formula is not satisfiable in $M$.

To see that game play on a dnf results in alternating-turn move order, recall that a constituent $\delta^{(d)}[y_1, \ldots, y_k]$ of the second kind is a formula of the form $\wedge(\exists x)\delta^{(d-1)}[y_1, \ldots, y_k, x] \land (\forall x)\delta^{(d-1)}[y_1, \ldots, y_k, x]$. Thus, either (1) Abelard picks a conjunct from $\wedge$ and passes play to Eloise to instantiate an existential $\exists$ or (2) Abelard instantiates a universal and passes play to Eloise to play a disjunct from $\lor$. By an induction on $d$, we see that this results in an alternating-turn play. Play begins with Eloise selecting a disjunct.
that players maintain “reasonable” beliefs. Along the way, we also give another method for assigning probabilities to first-order statements.

We have left numerous questions answered. Perhaps the most interesting ones are related to efficient implementation and empirical testing of self-play for Trailblazer (i.e., the variation of the Pathfinder proving game using abstractions). In particular, (1) can we efficiently implement HTs by selecting clever abstractions and using lazy representations, (2) what machine learning representations are effective for representing HTs, and (3) do self-play learning systems for the game learn stable and meaningful evaluation functions that can be used to build actual theorem provers? It is unclear to us how and if these technical issues can be resolved. In spite of the numerous technical difficulties, we are intrigued by this direction of future work and hope to pursue it.

It is often said that mathematics is not a spectator sport, that one learns mathematics by doing mathematics. Pólya expresses this sentiment in conjunction with the necessity of “plausible reasoning”:

The result of the mathematician’s creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning [emphasis added] of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference. [58, pg. vi]

If we agree with Pólya, then the implication for applying machine learning to proving is that we require both plausible and demonstrative reasoning in training and in inference. Put another way, we will be missing many modes of mathematical reasoning that are useful (at least for humans) for the discovery of proofs if we constrain ourselves to an exclusively proof-theoretic view of proving during training.

In shifting our view on proving from sound inference in a proof system to probabilistic inference on beliefs about possible kinds of mathematical worlds, we have seen that (1) proving requires the construction of individuals with certain properties (as opposed to the strategy of eliminating existentials), (2) conjecturing can be defined independently of enumerating syntax and can be employed for proving, and (3) abstractions (and laziness) are necessary for managing complexity although we (potentially) lose completeness. Thus what we have accomplished is largely philosophical in that we provide an account of the mathematical process where one “learns” mathematics by “doing” mathematics. We hope that the thought experiment conducted in this paper has shed some additional light on the activity of “learning to prove” that can inspire further attempts to learn theorem proving systems.
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