THE QUANTUM LEFSCHETZ PRINCIPLE FOR VECTOR BUNDLES
AS A MAP BETWEEN GIVENTAL CONES

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Abstract. Givental has defined a Lagrangian cone in a symplectic vector space which encodes all
genus-zero Gromov–Witten invariants of a smooth projective variety $X$. Let $Y$ be the subvariety
in $X$ given by the zero locus of a regular section of a convex vector bundle. We review arguments
of Iritani, Kim–Kresch–Pantev, and Graber, which give a very simple relationship between the
Givental cone for $Y$ and the Givental cone for Euler-twisted Gromov–Witten invariants of $X$. When
the convex vector bundle is the direct sum of nef line bundles, this gives a sharper version of the
Quantum Lefschetz Hyperplane Principle.

1. Gromov–Witten Invariants and Twisted Gromov–Witten Invariants

Given a smooth projective variety $X$, one can define Gromov–Witten invariants of $X$ [17,18]:

(1) \[ \langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n} \rangle_{g,n,d}^X := \int_{[X_{g,n,d}]_{vir}} \prod_{i=1}^n \text{ev}_i^* \gamma_i \cup \psi_i^{k_i} \]

Notation here is by now standard; a list of notation and definitions can be found in Appendix A.

Given a class $A \in H^*(X_{g,n,d}; \mathbb{Q})$, we can include it in the integral (1), writing:

(2) \[ \langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n}; A \rangle_{g,n,d}^X := \int_{[X_{g,n,d}]_{vir}} A \cup \prod_{i=1}^n \text{ev}_i^* \gamma_i \cup \psi_i^{k_i} \]

In particular, we can consider twisted Gromov–Witten invariants [8]. Let $E \to X$ be a vector
bundle, and let $c(\cdot)$ be an invertible multiplicative characteristic class. We can evaluate $c$ on classes
in K-theory by setting $c(A \ominus B) = \frac{c(A)}{c(B)}$. The twisting class $E_{g,n,d} \in K^0(X_{g,n,d})$ is defined by
$E_{g,n,d} = \pi_1 \text{ev}^* E$, where

\[ C \xrightarrow{\text{ev}} X \]
\[ \pi \]
\[ X_{g,n,d} \]

is the universal family over the moduli space of stable maps. $(c, E)$-twisted Gromov–Witten invariants of $X$ are intersection numbers of the form:

(3) \[ \langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n}; c(E_{g,n,d}) \rangle_{g,n,d}^X \]

Consider the $S^1$-action on vector bundles $V \to B$ which rotates the fibers of $V$ and leaves the
base $B$ invariant. The $S^1$-equivariant Euler class $e(\cdot)$ is invertible over the field of fractions $\mathbb{Q}(\lambda)
$ of $H^{*}_{S^1}(\{\text{point}\}) = \mathbb{Q}[\lambda]$. Taking $c = e$, we refer to twisted Gromov–Witten invariants [3] as
Euler-twisted Gromov–Witten invariants.
Givental has defined a Lagrangian cone \( L_X \) in a symplectic vector space \( H_X \) which encodes all genus-zero Gromov–Witten invariants of \( X \). Fix a basis \( \{ \phi_i \} \) for \( H^*(X; \mathbb{Q}) \), and let \( \{ \phi^\epsilon \} \) denote the dual basis with respect to the Poincaré pairing \((\cdot, \cdot)\) on \( H^*(X) \), so that \((\phi^\mu, \phi^\nu) = \delta^\mu_\nu \). Let \( \Lambda_X \) denote the Novikov ring of \( X \); this is defined in Appendix A. Consider the vector space (or rather, free \( \Lambda_X \)-module):

\[
\mathcal{H}_X := H^*(X; \Lambda_X) \otimes \mathbb{C}(z^{-1})
\]

equipped with the symplectic form (or rather, \( \Lambda_X \)-valued symplectic form):

\[
\Omega_X(f, g) := \text{Res}_{z=0} (f(-z), g(z)) \, dz
\]

Let \( t(z) = t_0 + t_1 z + t_2 z^2 + \cdots \), where \( t_i \in H^*(X; \Lambda_X) \). A general point on Givental’s Lagrangian cone \( L_X \subset \mathcal{H}_X \) has the form:

\[
J_X(t) := -z + t(z) + \sum Q^d \frac{1}{n!} \langle t_{k_1} \psi_1^{k_1}, \ldots, t_{k_n} \psi_n^{k_n}, \phi^\epsilon \psi_{n+1}^m \rangle_X \phi_\epsilon (-z)^{-m-1}
\]

where the sum runs over non-negative integers \( n \) and \( m \), multi-indices \( k = (k_1, \ldots, k_n) \) in \( \mathbb{N}^n \), degrees \( d \in H_2(X; \mathbb{Z}) \), and basis indices \( \epsilon \). Knowing the Lagrangian submanifold \( L_X \) is equivalent to knowing all genus-zero Gromov–Witten invariants \( I(X) \) of \( X \).

A similar Lagrangian cone encodes all genus-zero Euler-twisted Gromov–Witten invariants of \( X \). Consider the twisted Poincaré pairing \((\alpha, \beta)_\epsilon = \int_X \alpha \cup \beta \cup \epsilon(E)\), and the twisted symplectic form:

\[
\Omega_\epsilon(f, g) := \text{Res}_{z=0} (f(-z), g(z)) \, e \, dz
\]

on \( \mathcal{H}_X \). Let \( \{ \phi^\epsilon \} \) denote the basis dual to \( \{ \phi_\epsilon \} \) with respect to the twisted Poincaré pairing, so that \((\phi^\mu, \phi^\epsilon) = \delta^\mu_\epsilon \). A general point on the Lagrangian cone \( L_\epsilon \subset (\mathcal{H}_X, \Omega_\epsilon) \) has the form:

\[
J_\epsilon(t) := -z + t(z) + \sum Q^d \frac{1}{n!} \langle t_{k_1} \psi_1^{k_1}, \ldots, t_{k_n} \psi_n^{k_n}, \phi^\epsilon \psi_{n+1}^m \rangle X \phi_\epsilon (-z)^{-m-1}
\]

where the sum runs over the same set as above. Knowing \( L_\epsilon \) is equivalent to knowing all genus-zero Euler-twisted Gromov–Witten invariants of \( X \). In this expository note, we describe a close relationship, in the case where the vector bundle \( E \) is convex, between Euler-twisted invariants of \( X \) and Gromov–Witten invariants of the subvariety \( Y \subset X \) defined by a regular section of \( E \). We prove:

**Theorem 1.1.** Let \( X \) be a smooth projective variety. Let \( E \to X \) be a convex vector bundle, let \( Y \) be the subvariety in \( X \) defined by a regular section of \( E \), and let \( i: Y \to X \) be the inclusion map. Let \( J_\epsilon \) denote the general point \( 5 \) on the Lagrangian cone \( L_\epsilon \) for Euler-twisted Gromov–Witten invariants of \( X \). Let \( J_Y \) denote the general point on the Lagrangian cone \( L_Y \) for genus-zero Gromov–Witten invariants of \( Y \), as in \( 1 \). Then the non-equivariant limit \( J_\epsilon \big|_{\lambda=0} \) is well-defined and satisfies:

\[
i^* J_\epsilon(t) \big|_{\lambda=0} = J_Y(i^* t)
\]

In particular, \( i^* L_\epsilon \big|_{\lambda=0} \subset L_Y \).

Throughout here we have applied the homomorphism \( Q^\delta \mapsto Q^{i* \delta} \) to the Novikov ring of \( Y \).

**Remark 1.2.** A vector bundle \( E \to X \) is called convex if and only if \( H^1(C, f^* E) = 0 \) for all stable maps \( f: C \to X \) that the curve \( C \) has genus zero. Globally generated vector bundles are automatically convex, as are direct sums of nef line bundles.

**Remark 1.3.** If the dimension of \( Y \) is at least 3 then, by the Lefschetz theorem, the homomorphism of Novikov rings \( \Lambda_Y \to \Lambda_X \) given by \( Q^\delta \mapsto Q^{i* \delta} \) is an isomorphism.

**Remark 1.4.** In the non-equivariant limit, the map \( i^*: H_X \to H_Y \) becomes symplectic; it satisfies \( i^* \Omega_\epsilon \big|_{\lambda=0} = \Omega_Y \). Thus Theorem 1.1 fits neatly into a general story that encompasses the Crepant Resolution Conjecture [9,10], Brown’s toric bundle theorem [2], and so on: geometrically-natural
operations in Gromov–Witten theory give rise to symplectic transformations of Givental’s symplectic space that preserve the Lagrangian cones.

**Key Remark 1.5.** Only the statement of Theorem 1.1 is new. As we will see, the proof is a very minor variation of an argument by Iritani [15, Proposition 2.4]. Iritani’s result in turn builds on arguments by Kim–Kresch–Pantev [16] and Graber [21, §2].

**Remark 1.6.** Theorem 1.1 improves upon [8, formula 19], which roughly speaking, in the special case where $E$ is the direct sum of nef line bundles, relates $J_e(t)|_{\lambda=0}$ to $i_*J_Y(i^*t)$. The improved version determines invariants of $Y$ with one insertion (that at the last marked point) involving an arbitrary cohomology class on $Y$, whereas the original version determined only invariants of $Y$ such that all insertions are pullbacks of cohomology classes on $X$. When combined with the Lee–Pandharipande reconstruction theorem [19] this determines, under moderate hypotheses on $Y$, the big quantum cohomology of $Y$. This should be compared with §0.3.2 of ibid., which gives a reconstruction result for Gromov–Witten invariants of $Y$ such that all insertions are pullbacks of cohomology classes on $X$. One can use the same approach together with the Abelian/Non-Abelian Correspondence with bundles [4, §6.1] to determine the genus-zero Gromov–Witten invariants of many subvarieties of flag manifolds and partial flag bundles.

**Remark 1.7.** The formulation in Theorem 1.1 is well-suited to proving mirror theorems for toric complete intersections or subvarieties of flag manifolds. One first obtains a family $t \mapsto I_e(t, z)$ of elements of $L_e$, by combining the Mirror Theorem for toric varieties or toric Deligne–Mumford stacks [3,5,12] with the Quantum Lefschetz theorem [8] or the Abelian/Non-Abelian Correspondence with bundles [4, §6.1]. After taking the non-equivariant limit $\lambda \to 0$ and applying Theorem 1.1, one can then argue as in [8, §9] or [6, Example 9].

2. **The Proof of Theorem 1.1**

2.1. **The Non-Equivariant Limit Exists.** For the remainder of this note, we consider only stable maps of genus zero. Since $E$ is convex, we have that $R^1\pi_* ev^* E = 0$ and hence that $E_{0,n+1,d}$ is a vector bundle. The fiber of $E_{0,n+1,d}$ over a stable map $f: C \to X$ is $H^0(C, f^* E)$, and thus there is an exact sequence of vector bundles:

$$0 \longrightarrow E_{0,n+1,d} \longrightarrow E_{0,n+1,d} \stackrel{ev_{n+1}}{\longrightarrow} ev_{n+1}^* E \longrightarrow 0$$

This implies that $e(E_{0,n+1,d}) = e(E'_{0,n+1,d})e(ev_{n+1}^* E)$. The Projection Formula, together with the fact that $\phi^e = \phi^e_e(E)$, gives that:

$$J_e(t) = -z + t(z) + \sum \frac{Q^d}{n!} (ev_{n+1})_* \left[ X_{0,n+1,d} \cap e(E'_{0,n+1,d}) \cup \psi_{n+1}^m \cup \prod_{i=1}^n ev_{k_i}^* t_k \cup \psi_{k_i}^l \right] (-z)^{m-1}$$

This makes it clear that the non-equivariant limit $J_e(t)|_{\lambda=0}$ exists. Let us write $e(\cdot)$ for the non-equivariant Euler class, noting that $e(\cdot)$ is the non-equivariant limit of $e(\cdot)$.
2.2. A Comparison of Virtual Fundamental Classes. Consider the diagram:

\[
\begin{array}{ccc}
\prod_{\delta: i_{*}, \delta = d} Y_{0,n+1,\delta} & \xrightarrow{G} & Z \\
\downarrow ev & & \downarrow F \\
Y^{n+1} & \xrightarrow{g} & X^{n} \times Y \\
\downarrow q & & \downarrow f \\
Y & \xrightarrow{i} & X
\end{array}
\]

where \(p, q,\) and \(r\) are projections onto the last factor of their domains (which are products); \(f\) and \(g\) are induced by the inclusion \(i: Y \to X;\) the maps \(ev\) in the first and third columns are the evaluation maps \(ev_{1} \times \cdots \times ev_{n+1};\) the upper right-hand square is Cartesian; the composition \(G \circ F\) is the union of canonical inclusions \(Y_{0,n+1,\delta} \to X_{0,n+1,d};\) and the map \(G\) is defined by the universal property of the fiber product \(Z.\) The stack \(Z\) consists of those stable maps in \(X_{0,n+1,d}\) such that the last marked point lies in \(Y;\) it is the zero locus of the section \(ev_{n+1}^{*} s \in \Gamma(X_{0,n+1,d}, ev_{n+1}^{*} E).\) The map \(ev\) in the second column is also given by \(ev_{1} \times \cdots \times ev_{n+1}^{*}.

**Proposition 2.1.** With notation as above, we have:

(A) \[
f^{!} \left( e(E_{0,n+1,d}^{t}) \cap [X_{0,n+1,d}]^{vir} \right) = \sum_{\delta: i_{*}, \delta = d} G_{*}[Y_{0,n+1,\delta}]^{vir}
\]

(B) For any \((k_{1}, \ldots, k_{n+1}) \in \mathbb{N}^{n+1}:
\[
f^{*} ev_{*} \left( \psi_{1}^{k_{1}} \cup \cdots \cup \psi_{n+1}^{k_{n+1}} \cap e(E_{0,n+1,d}^{t}) \cap [X_{0,n+1,d}]^{vir} \right) = \sum_{\delta: i_{*}, \delta = d} g_{*} ev_{*} \left( \psi_{1}^{k_{1}} \cup \cdots \cup \psi_{n+1}^{k_{n+1}} \cap [Y_{0,n+1,\delta}]^{vir} \right)
\]

**Proof.** Let \(0_{X}: X_{0,n+1,d} \to E_{0,n+1,d}, 0_{X}' : X_{0,n+1,d} \to E_{0,n+1,d}^{t}, 0_{Z} : Z \to E_{0,n+1,d}^{t} \mid _{Z}\) denote the zero sections. Consider the Cartesian diagram:

\[
\begin{array}{ccc}
\prod_{\delta: i_{*}, \delta = d} Y_{0,n+1,\delta} & \xrightarrow{G} & Z \\
\downarrow & & \downarrow = |Z \\
Z & \xrightarrow{0_{Z}'} & E_{0,n+1,d}^{t} |_{Z} \\
\downarrow & & \downarrow \tilde{s} \\
X_{0,n+1,d} & \xrightarrow{0_{X}'} & E_{0,n+1,d}^{t} \\
\end{array}
\]

\[
\begin{array}{ccc}
\prod_{\delta: i_{*}, \delta = d} Y_{0,n+1,\delta} & \xrightarrow{G} & Z \\
\downarrow & & \downarrow = |Z \\
Z & \xrightarrow{0_{Z}'} & E_{0,n+1,d}^{t} |_{Z} \\
\downarrow & & \downarrow \tilde{s} \\
X_{0,n+1,d} & \xrightarrow{0_{X}'} & E_{0,n+1,d}^{t} \\
\end{array}
\]


where \( j \) is the inclusion from (6) and \( \bar{s} \) is the section of \( E_{0,n+1,d} \) induced by the section \( s: X \to E \) that defines \( Y \). Note that, on the bottom row, \( \partial X \circ j = 0_X \). We have:

\[
\sum_{\delta:i,\delta=d} G_*[Y_{0,n+1,\delta}]_\text{vir} = \sum_{\delta:i,\delta=d} G_*[X_{0,n+1,\delta}]_\text{vir} \quad \text{(functoriality [16])}
\]

\[
= \sum_{\delta:i,\delta=d} G_*[0'_X]^j[X_{0,n+1,\delta}]_\text{vir} \quad \text{(functoriality [11] Theorem 6.5)}
\]

\[
= \sum_{\delta:i,\delta=d} (0'_X)^*(\bar{s}|_Z)^*j[X_{0,n+1,\delta}]_\text{vir} \quad \text{(by [11] Theorem 6.2)}
\]

\[
e \big( E'_{0,n+1,d}\big) \cap j[X_{0,n+1,\delta}]_\text{vir}
\]

\[
= j^! \big( E'_{0,n+1,d} \big) \cap [X_{0,n+1,\delta}]_\text{vir}
\]

\[
f^! \big( E'_{0,n+1,d} \big) \cap [X_{0,n+1,\delta}]_\text{vir}
\]

This proves (A). Since \( f^* \text{ev}_* = \text{ev}_* f^! \text{[11] Theorem 6.2} \) and \( g_* \text{ev}_* = \text{ev}_* G_* \), and since the classes \( \psi_i \) on \( Z \) and on \( Y_{0,n+1,\delta} \) are pulled back from the class \( \psi_i \) on \( X_{0,n+1,d} \), (A) implies (B). \( \square \)

### 2.3. Applying the Projection Formula

We now deduce Theorem 2.1 from Proposition 2.1. This amounts to repeated application of the Projection Formula. Recall the diagram (7). The non-equivariant limit \( J_{e}(t)\big|_{\lambda=0} \) is equal to:

\[
-z + t(z) + \sum \frac{Q_n}{n!} \big( \text{ev}_n + 1 \big) \big[ X_{0,n+1,\delta} \big]_\text{vir} \cap e(E'_{0,n+1,d}) \cup \psi^{m}_{n+1} \cup \prod_{i=1}^{n} t_{k_i} \cup \psi_{k_i}
\]

\[
= -z + t(z) + \sum \frac{Q_n}{n!} (-z)^{m+1} Q_n \big[ X_{0,n+1,\delta} \big]_\text{vir} \cap e(E'_{0,n+1,d}) \cup \psi^{m}_{n+1} \cup \prod_{i=1}^{n} t_{k_i} \cup \psi_{k_i}
\]

Using \( i^*p_* = q_* f^* \), we see that the pullback \( i^*J_{e}(t)\big|_{\lambda=0} \) is:

\[
-z + i^*t(z) + \sum \frac{Q_n}{n!} (-z)^{m+1} Q_n \big[ X_{0,n+1,\delta} \big]_\text{vir} \cap e(E'_{0,n+1,d}) \cup \psi^{m}_{n+1} \cup \prod_{i=1}^{n} t_{k_i} \cup \psi_{k_i}
\]

Proposition 2.1(B) now gives:

\[
i^*J_{e}(t)\big|_{\lambda=0} = -z + i^*t(z) + \sum' \frac{Q_n}{n!} (-z)^{m+1} Q_n \big[ X_{0,n+1,\delta} \big]_\text{vir} \cap e(E'_{0,n+1,d}) \cup \psi^{m}_{n+1} \cup \prod_{i=1}^{n} t_{k_i} \cup \psi_{k_i}
\]

where the sum \( \sum' \) runs over non-negative integers \( n \) and \( m \), multi-indices \( k = (k_1, \ldots, k_n) \) in \( \mathbb{N}^n \), degrees \( \delta \in H_2(Y; \mathbb{Z}) \), and basis indices \( \epsilon \). Applying the Projection Formula again, we see that:

\[
i^*J_{e}(t)\big|_{\lambda=0} = -z + i^*t(z) + \sum' \frac{Q_n}{n!} (-z)^{m+1} Q_n \big[ X_{0,n+1,\delta} \big]_\text{vir} \cap e(E'_{0,n+1,d}) \cup \psi^{m}_{n+1} \cup \prod_{i=1}^{n} t_{k_i} \cup \psi_{k_i}
\]

The Theorem is proved. \( \square \)
Remark 2.2. Let $X$ be a smooth Deligne–Mumford stack with projective coarse moduli space, let $E \to X$ be a convex vector bundle, let $Y$ be the substack in $X$ defined by a regular section of $E$, and let $i: Y \to X$ be the map of inertia stacks induced by the inclusion $Y \to X$. The analog of Theorem 1.1 holds in this context, with the same proof: cf. [15, Proposition 2.4]. Note that a convex line bundle on a Deligne–Mumford stack is necessarily the pullback of a line bundle on the coarse moduli space [7].

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### APPENDIX A. Notation

What follows is a list of notation and definitions: first for symbols in Roman font, then for Greek symbols, then for miscellaneous symbols.

| Symbol | Description |
|--------|-------------|
| $c$    | an invertible multiplicative characteristic class |
| $e$    | the $S^1$-equivariant Euler class; see page 1 for the definition of the $S^1$-action |
| $e'$   | the non-equivariant Euler class |
| $E$    | a convex vector bundle over $X$ |
| $E_{g,n,d}$ | the twisting class $E_{g,n,d} \in K^0(X_{g,n,d})$; see page 1 |
| $E'_{0,n+1,d}$ | a sub-bundle of $E_{0,n+1,d}$; see page 3 |
| $\ev_i$ | the evaluation map $X_{g,n,d} \to X$ at the $i$th marked point |
| $\mathcal{H}_X$, $\mathcal{H}_Y$ | Givental’s symplectic vector space; see page 2 |
| $\mathcal{L}_e$ | Givental’s Lagrangian cone for Euler-twisted invariants of $X$; see page 2 |
| $\mathcal{L}_X$, $\mathcal{L}_Y$ | Givental’s Lagrangian cone for $X$, $Y$; see page 2 |
| $i$    | the inclusion map $Y \to X$ |
| $j$    | the inclusion map $E'_{0,n+1,d} \to E_{0,n+1,d}$ |
| $j_e(t)$ | a general point on $\mathcal{L}_e$; see (5) |
| $j_X(t)$ | a general point on $\mathcal{L}_X$; see (4) |
| $k_0$  | a non-negative integer |
| $Q^d$ | the representative of $d \in H_2(X;\mathbb{Z})$ in the Novikov ring $\Lambda_X$ |
| $t$    | $t(z) = t_0 + t_1z + t_2z^2 + \cdots$ where $t_i \in H^*(X)$ |
| $t_i$  | a cohomology class on $X$ |
| $X$    | a smooth projective variety |
| $X_{g,n,d}$ | the moduli space of stable maps to $X$, from genus-$g$ curves with $n$ marked points, of degree $d \in H_2(X;\mathbb{Z})$ [17][18] |
| $[X_{g,n,d}]^\text{vir}$ | the virtual fundamental class of the moduli space of stable maps to $X$ [1][20] |
| $Y$    | a subvariety of $X$ cut out by a regular section of $E$ |
| $Y_{g,n,d}$ | the moduli space of stable maps to $Y$, from genus-$g$ curves with $n$ marked points, of degree $d \in H_2(Y;\mathbb{Z})$ [17][18] |
| $[Y_{g,n,d}]^\text{vir}$ | the virtual fundamental class of the moduli space of stable maps to $Y$ [1][20] |
| $\gamma_i$ | a cohomology class on $X$ |
| $\lambda$ | the generator of $H^*_S(\{\text{point}\})$ given by the first Chern class of $\mathcal{O}(1) \to \mathbb{C}P^\infty \cong BS^1$ |
| $\Lambda_X$ | the Novikov ring of $X$; this is a completion of the group ring $\Lambda_X$ with respect to the valuation $v(Q^d) = \int Q^d \omega$, where $Q^d$ is the representative of $d \in H_2(X;\mathbb{Z})$ in the group ring and $\omega$ is the Kähler form on $X$ |
| $\phi$ | an element of the basis $\{\phi_x\}$ for $H^*(X;\mathbb{Q})$ |
| $\phi'$ | an element of the dual basis $\{\phi_x\}$ for $H^*(X;\mathbb{Q})$, so that $(\phi_x, \phi_y') = \delta_y^x$ |
| $\psi_i$ | the first Chern class of the universal cotangent line bundle $L_i \to X_{g,n,d}$ at the $i$th marked point |
| $\Omega_X$, $\Omega_e$, $\Omega_Y$ | the symplectic forms on $\mathcal{H}_X$, $\mathcal{H}_X$, and $\mathcal{H}_Y$ respectively; see page 2 |
| $0_X$, $0'_X$, $0'_Z$ | zero section maps; see page 4 |
| $(\cdot, \cdot)$ | the Poincaré pairing on $H^*(X)$, $(\alpha, \beta) = \int X \alpha \cup \beta$ |
| $(\cdot, \cdot)_e$ | the twisted Poincaré pairing on $H^*(X)$, $(\alpha, \beta) = \int X \alpha \cup \beta \cup e(E)$ |
| $(\cdot, \cdot)_g,n,d$ | Gromov–Witten invariants or twisted Gromov–Witten invariants of $X$; see (1)[3] |

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