Lamperti Type Laws: Positive Linnik, Bessel Bridge Occupation and Mittag-Leffler Functions

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Abstract: This paper explores various distributional aspects of random variables defined as the ratio of two independent positive random variables where one variable has an $\alpha$ stable law, for $0 < \alpha < 1$, and the other variable has the law defined by polynomially tilting the density of an $\alpha$ stable random variable by a factor $\theta > -\alpha$. When $\theta = 0$, these variables equate with the ratio investigated by Lamperti [22] which remarkably was shown to have a simple density. This variable arises in a variety of areas and gains importance from a close connection to the stable laws. This rationale, and connection to the PD($\alpha, \theta$) distribution, motivates the investigations of its generalizations which we refer to as Lamperti type laws. We identify and exploit links to random variables that commonly appear in a variety of applications. Namely, Linnik, generalized Pareto and $z$–distributions. In each case we obtain new results that are of potential interest. As some highlights, we then use these results to (i) obtain integral representations and other identities for a class of generalized Mittag-Leffler functions, (ii) identify explicitly the Lévy density of the semigroup of stable continuous state branching processes (CSBP) and hence corresponding limiting distributions derived in Slack, Zolotarev[23, 24], which is related to the recent work by Berestycki, Berestycki and Schweinsberg, and Bertoin and Le Gall[25, 26] on beta coalescents, (iii) We obtain explicit results for the occupation time of generalized Bessel bridges and some interesting stochastic equations for PD($\alpha, \theta$)-bridges. In particular we obtain the best known results for the density of the time spent positive of a Bessel bridge of dimension $2 - 2\alpha$.

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1. Introduction

Let $S_{\alpha}$, for $0 < \alpha < 1$ denote a positive stable random variable having Laplace transform

$$ \mathbb{E}[e^{-\lambda S_\alpha}] = e^{-\lambda^\alpha}. $$

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Additionally, for $\theta > -\alpha$ define variables $S_{\alpha,\theta}$ independent of $S_\alpha$ whose laws follow a polynomially tilted stable distribution having density proportional to $t^{-\theta} f_\alpha(t)$. When $\theta = 0$, $S_{\alpha,0} := S'_\alpha \overset{d}{=} S_\alpha$. In this case Lamperti (66), (see also Zolotarev (96), and Chaumont and Yor (21)), showed that, despite the general intractability of $f_\alpha$, the ratio $X_\alpha \overset{d}{=} \frac{S_\alpha}{S'_\alpha}$ has a remarkably simple density given as

$$f_{X_\alpha}(y) = \frac{\sin(\pi \alpha)}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^{\alpha}\cos(\pi \alpha) + 1} \text{ for } y > 0.$$ (1.1)

This variable arises in many important and often seemingly unrelated contexts. For instance, (17; 28; 27; 71; 70; 76). Inspired by these facts and connections to the $(\alpha, \theta)$ Poisson Dirichlet family of distributions discussed in Pitman and Yor (82) leads us to investigate properties of variables defined as

$$X_{\alpha,\theta} \overset{d}{=} \frac{S_\alpha}{S_{\alpha,\theta}}.$$

We refer to these variables as being Lamperti variables or variables having Lamperti type laws. Our purpose, from a broad perspective, is to demonstrate that these variables have strong connections to more familiar random variables that appear in a variety of applications in probability, statistics and related fields. In other words, the Lamperti variables, albeit often hidden, appear in many important contexts. Furthermore, we show how to utilize these links to both deduce properties of $X_{\alpha,\theta}$, and develop new non-trivial results related to the linked variables. These results can also be potentially used to expand modeling capabilities. Our results are suggestive of an active beta-gamma-stable calculus that extends the notion often associated with beta and gamma variables via Lukacs’ (69) characterization.

1.1. Outline

We now present an outline of this paper highlighting specifics. More detailed references can be found in each section. We note that each section contains new results of a non-trivial nature that in some cases are generalizations of existing results. In addition, combined, they represent a nice partial survey of linked variables. Section 2 consists of essentially two parts. The first develops a series of pertinent distributional results for $X_{\alpha,\theta}$ and for a more broader class defined by multiplying the Lamperti variables by beta variables. One shall notice the class of random variables we denote as $X_{\alpha,1}^{(\sigma)}$ plays a major role throughout the sections. This multiplication is based on ideas we developed in (45). The second constitutes a natural progression of ideas, each section building on the previous one. Specifically, section 2.3 establishes links with positive Linnik variables. Where, as highlights, we obtain expressions for the density of Linnik
variables and also establish an interesting gamma identity. Section 2.4, exploits
this identity in connection with generalized Pareto distributions. Albeit brief,
the main result is used to identify an unknown limiting distribution obtained by
Zolotarev (97) and Slack (87), which we discuss in section 4. Section 2.4, influ-
enced by (14; 42; 54; 95), uses the characterization in the previous sections to
demonstrate how one can develop a calculus of sorts involving \( \alpha \)-distributions.
In particular, we identify new classes of random variables, arising as solutions to
stochastic equations involving \( \alpha \)-distributions, having both hyperbolic character-
istic functions and variables whose density can be computed explicitly. Section
3 obtains results for a generalization of Mittag-Leffler functions that can be ex-
pressed as Laplace transforms of \( S_{\alpha, \theta} \) or \( X_{\alpha, \theta} \) and can be represented in terms
of densities of Linnik variables. Section 4 solves a fairly hard problem, identifying
the explicit Lévy density of the semigroup of stable continuous state branching
processes. This section continues with the idea of using results from previous
sections. Results in sections 5 and 6, with the exception of \( \alpha = \frac{1}{2} \), present the
best known results for occupation times of various quantities including times
spent positive on certain random subsets. We also develop a series of interest-
ing stochastic equations. As one highlight we obtain results for the otherwise
evens case of Bessel bridges of dimension \( 2 - 2\alpha \). For some related works see
(12; 51; 52; 59; 94). Section 7 discusses aspects of Brownian time changed models
where we close by exploiting an interesting, yet not well known, representation
of symmetric stable variables of index \( 0 < 2\alpha \leq 1 \), found in (26).

1.2. Some notation and background

Here we briefly recount some notation and background related to Bessel pro-
cesses and the Poisson Dirichlet family of laws. See Pitman (79; 80) for a more
precise exposition. Let \( \mathcal{B} := (B_t, t > 0) \) denote a strong Markov process on \( \mathbb{R} \)
whose normalized ranked lengths of excursions, \((P_i) \in \mathcal{P} = \{ s = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} s_i = 1 \} \), follow a Poisson Dirichlet law with pa-
rameters \((\alpha, 0)\) for \(0 < \alpha < 1\), as discussed in Pitman and Yor (82). Denote
this law as PD\((\alpha, 0)\). We hereafter consider \( \mathcal{B} := (B_t, 0 \leq t \leq 1) \). Let \((L_t, t > 0) \)
denote its local time starting at 0, and let \( \tau_\ell = \inf\{ t : L_t > \ell \}, \ell \geq 0 \) denote its
inverse local time. In this case \( \tau \) is an \( \alpha \)-stable subordinator where we choose
\( \tau_1 \overset{d}{=} S_\alpha \). Writing \( L_1 = L(t) \), there is the scaling identity (see (81)),

\[
L_1 \overset{d}{=} \frac{L(t)}{\ell^\alpha} \overset{d}{=} \left( \frac{s}{\tau(s)} \right)^\alpha \overset{d}{=} S_{\alpha, \theta}^{\alpha},
\]

where the inverse local time up to time 1, \( L_1 \), satisfies

\[
L_1 := \Gamma(1 - \alpha)^{-1} \lim_{\ell \to 0} \epsilon^\alpha \{ i : P_i \geq \epsilon \} \text{ a.s.}
\]

and is said to follow a Mittag-Leffler distribution. This shows that \((L_t, \tau_t)\) have
distributions determined by PD\((\alpha, 0)\). Furthermore, independent of \((P_i)\), we
suppose that for a fixed \( 0 < p < 1 \), \( \mathcal{B} \) is symmetrized so that \( \mathbb{P}(B_t > 0) = p \).
Under these specifications $\mathcal{B}$ could be a $p$-skewed Bessel process of dimension $2 - 2\alpha$. In particular when $p = 1/2, \alpha = 1/2$ then $\mathcal{B}$ behaves like a brownian motion. An interesting aspect of $\mathcal{B}$ is the time its spends on certain subsets of $\mathbb{R}$. Let $A_t^+ = \int_0^t \mathbb{1}_{(B, > 0)}(s)ds$ and $A_t^- = \int_0^t \mathbb{1}_{(B, < 0)}(s)ds$, such that $t = A_t^+ + A_t^-$. Denote the time $\mathcal{B}$ spends positive and negative respectively up till time $t$. Remarkably, by excursion theory, the time changed-processes $(A_t^+; \ell > 0)$ and $(A_t^+; \ell > 0)$ are independent $\alpha$-stable subordinators such that $A_t^+ \overset{d}{=} p^{1/\alpha} S_\alpha$ and $A_t^- \overset{d}{=} (1-p)^{1/\alpha} S_\alpha^\prime$, for $S_\alpha^\prime \overset{d}{=} S_\alpha$. This leads to,
\[
X_\alpha \overset{d}{=} \frac{A_{\tau_1}^+}{A_{\tau_1}^-} \overset{d}{=} \frac{S_\alpha}{S_\alpha^\prime} \text{ and } \frac{cX_\alpha}{cX_\alpha + 1} \overset{d}{=} \frac{A_{\tau_1}^+}{\tau_1} \overset{d}{=} A_{\tau_1}^+. \tag{1.3}
\]

Hereafter, denote the law that governs $\mathcal{B}$ and related functionals under the above specifications as $\mathbb{P}_{\alpha,0}^{(p)}$. Denote the corresponding expectation operator as $\mathbb{E}_{\alpha,0}^{(p)}$. Thus writing $\mathbb{P}_{\alpha,0}^{(p)}(A_1^+ \in dx)/dx$ equates with the density of the time spent positive on $[0, 1]$ of a $p$-skewed Bessel process. As noticed by, \cite{[3],[81]}, this law was originally obtained by Lamperti \cite{[66]} and from \cite{[13]} it equates with,
\[
\mathbb{P}_{\alpha,0}^{(p)}(A_1^+ \in dx)/dx = \mathbb{P}(cX_\alpha/(cX_\alpha + 1) \in dx)/dx.
\]

Now for $\theta > 0$ let $\mathbb{P}_{\alpha,\theta}^{(p)}$ and $\mathbb{E}_{\alpha,\theta}^{(p)}$, denote the law and expectation operator of functionals connected to a $p$-skewed process whose excursion lengths follow PD$(\alpha, \theta)$. In particular if $(P_i)$ is distributed according to PD$(\alpha, \theta)$, then it satisfies, for measurable $H$,
\[
\mathbb{E}_{\alpha,\theta}^{(p)}[H((P_i))] = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \mathbb{E}_{\alpha,0}^{(p)}[H((P_i))\tau_1^{-\theta}].
\]

When $\theta = \alpha$, this corresponds to the case of a Bessel bridge of dimension $2 - 2\alpha$. We use the notation $A_1^{(br)}$ for the variable that satisfies
\[
\mathbb{P}_{\alpha,\theta}^{(p)}(A_1^{(br)} \in dx) = \mathbb{P}_{\alpha,\theta}^{(p)}(A_1^+ \in dx).
\]

Note also that under $\mathbb{P}_{\alpha,\theta}^{(p)}$, $L_1 \overset{d}{=} S_{\alpha,\theta}^{-\alpha}$, which is also equivalent in distribution to the $\alpha$-diversity of a PD$(\alpha, \theta)$ law. Let now $U_1, U_2, \ldots$, denote a sequence of iid Uniform$[0,1]$ random variables, for $(P_i)$ distributed according to PD$(\alpha, \theta)$ and $0 \leq u \leq 1$, the class of PD$(\alpha, \theta)$ random cumulative distribution functions are defined as
\[
P_{\alpha,\theta}(u) \overset{d}{=} \sum_{k=1}^{\infty} P_k \mathbb{1}_{(u_k \leq u)}.
\]

Furthermore under $\mathbb{P}_{\alpha,\theta}^{(u)}$,
\[
A_1^+ \overset{d}{=} P_{\alpha,\theta}(u)
\]

for a fixed $u$. See Bertoin \cite{[8]} for applications to coagulation/fragmentation phenomena where it is called a PD$(\alpha, \theta)$-bridge and Ishwaran and James \cite{[44]}
[see also Pitman (77)] for applications to Bayesian statistics, where in particular $P_{\alpha,\theta}$ is referred to as a Pitman-Yor process. Under this name the process has also been applied to problems arising in natural language processing, see Teh(89).

When $0 > \theta$ and $\alpha = 0$ $P_{\theta}$ is a Dirichlet process made popular by Ferguson (34).

For basic notation, we write $\gamma_a$ and $\beta_{a,b}$ to denote a gamma random variable with shape $a$ and scale 1, and a beta random variable with parameters $(a,b)$.

Additionally $\xi_\sigma$ denotes a Bernoulli variable with success parameter $0 < \sigma \leq 1$.

If $X$ and $Y$ are random variables we will assume that $XY$ is a product of independent random variables unless otherwise specified, if we write $X,X'$ this will mean $X \overset{d}{=} X'$ but they are not equal. Lastly we always consider $e^{\alpha} = p/q = p/(1-p)$ where $q = 1-p$ unless otherwise specified.

2. Distributional Results for $X_{\alpha,\theta}$

In this section we shall derive various distributional properties of $X_{\alpha,\theta}$. For $\tau > 0$ and $0 < \sigma \leq 1$, we will sometimes work with the parametrization $\tau \sigma$, to accommodate values such as $\tau \sigma = \theta > 0$ and $\tau \sigma = \theta + \alpha$. First, we briefly discuss some pertinent properties of random variables referred to as Dirichlet means and the related class of infinitely divisible random variables whose distributions are generalized gamma convolutions (GGC), as they will play a significant role in our exposition. For more details and related notions, one may consult (16; 22; 23; 24; 32; 47; 48) and, in particular for this exposition, (45).

For a generic positive random variable $M$, let $C_{\tau \sigma}(\lambda; M) = \mathbb{E}[(1 + \lambda M)^{-\tau \sigma}]$ denote its Cauchy-Stieltjes transform of order $\tau \sigma$. Hereafter we shall refer to this as Cauchy transform. Similar to Laplace transforms, $C_{\tau \sigma}(\lambda; M)$ uniquely characterizes the law of $M$. Let $R$ denote a non-negative random variable with distribution function $F_R$. A random variable $M$, depending on parameters $(\tau \sigma, R)$, is said to be a Dirichlet mean of order $\tau \sigma$ if

$$-\log C_{\tau \sigma}(\lambda; M) = \tau \sigma \mathbb{E}[\log(1 + \lambda R)] := \tau \sigma \psi_R(\lambda) < \infty.$$  (2.1)

Equivalently $M$ satisfies the stochastic equation,

$$M \overset{d}{=} \beta_{\tau \sigma,1} M + (1 - \beta_{\tau \sigma,1}) R.$$  

We denote such a variables as $M \overset{d}{=} M_{\tau \sigma}(F_R)$. Importantly, Cifarelli and Regazzini (23) (see also (24)), apply inversion formula to obtain the distributional formula for $M_{\tau \sigma}(F_R)$. In general these are expressed in terms of Abel-type transforms. An exception is the case of $\tau \sigma = 1$, where the density of $M_1(F_R)$ can be expressed as,

$$\frac{1}{\pi} \sin(\pi F_R(x))e^{-\Phi_R(x)},$$  (2.2)

where $\Phi_R(x) = \mathbb{E}[\log |x - R| I_{(R \neq x)}]$. Additionally,

$$\tau \sigma \psi_R(\lambda) = \tau \sigma \int_0^\infty (1 - e^{s \lambda}) s^{-1} \mathbb{E}[e^{-s/R}] ds$$
is also the Lévy exponent of an infinitely divisible random variable with Lévy density $\tau \sigma s^{-1} E[e^{-s/R}]$. We say that such a random variable is GGC($\tau \sigma, R$) and may be represented in distribution as a gamma scale mixture

$$
\gamma_{\tau \sigma} M_{\tau \sigma}(F_R) = \gamma_{\tau, \tau} \beta_{\tau, (1-\sigma)} M_{\tau \sigma}(F_R).
$$

(2.3)

Highly relevant to (2.3), and our exposition, is a result by James (45), that for each $0 < \sigma \leq 1$,

$$
\beta_{\tau, \tau} M_{\tau \sigma}(F_R) = M_{\tau}(F_{R \xi_{\sigma}}),
$$

(2.4)

where $\xi_{\sigma}$ is a Bernoulli variable with success probability $\sigma$. Note also that $\beta_{\tau, \tau} M_{\tau \sigma}(F_R)$.

One consequence is that a GGC($\tau \sigma, R$) variable is also a GGC($\tau, \mu R \xi_{\sigma}$) variable. In other words, for a fixed $\theta > 0$, a GGC($\theta, R$) variable is a GGC($\theta', R \mu R \xi_{\sigma}$) variable, for all $\theta' > \theta$. As pointed out in (45), one significant point about these multiple representations is that if $0 < \theta = \sigma \leq 1$, then one can set $\theta' = 1$ and use the explicit density formula for Dirichlet means of order 1, established by Cifarelli and Regazzini (23) to obtain an explicit representation of the density of such a GGC($\sigma, R$) variable. See (45) for its precise form and further details.

2.1. Identities

For the case of $X_{\alpha, \theta}$, one can show that for $\theta > 0$,

$$
C_0(\lambda; X_{\alpha, \theta}) = (1 + \lambda^\alpha)^{-\frac{\alpha}{\theta}} = E[e^{-\gamma_{\theta} X_{\alpha, \theta}}],
$$

(2.5)

and for $\theta > -\alpha$,

$$
C_{1+\theta}(\lambda; X_{\alpha, \theta}) = E[e^{-\gamma_{\theta+1} X_{\alpha, \theta}}] = (1 + \lambda^\alpha)^{-\frac{\alpha + \theta}{\theta}} = C_{\theta+\alpha}(\lambda; X_{\alpha, \theta+\alpha}).
$$

(2.6)

We will use (2.5) and (2.6) to more easily establish the next series of results. However, we note that the expressions in (2.5), (2.6) are not obvious. We will provide justification for (2.5), when we discuss Linnik variables in the next section. Assuming that (2.5) is true, (2.6) then follows from an identity due to Perman, Pitman and Yor (73),

$$
\frac{1}{S_{\alpha, \theta}} \xrightarrow{\text{d}} \frac{\beta_{\theta+\alpha, 1-\alpha}}{S_{\alpha, \theta+\alpha}},
$$

(2.7)

for $\theta > -\alpha$. (2.7) is another highly relevant component to our exposition and shows that $X_{\alpha, \theta} \xrightarrow{\text{d}} \beta_{\theta+\alpha, 1-\alpha} X_{\alpha, \theta+\alpha}$.

Proposition 2.1. The random variables $X_{\alpha, \theta}$ are Dirichlet means having the following properties: For $\theta > 0$,

$$
X_{\alpha, \theta} \xrightarrow{\text{d}} \beta_{\theta, 1} X_{\alpha, \theta} + (1 - \beta_{\theta, 1}) X_{\alpha} \xrightarrow{\text{d}} M_{\theta}(F_{X_{\alpha}}),
$$

(2.8)
and for $\theta > -\alpha$ and $\sigma = (\theta + \alpha)/(1 + \theta)$,
\[
X_{\alpha,\theta} \overset{d}{=} \beta_{\theta+\alpha,1-\alpha} X_{\alpha,\theta+\alpha} \overset{d}{=} \beta^{1/\alpha}(\frac{1}{\theta + \alpha}) X_{\alpha,1+\theta} = M_{1+\theta}(F_{X_{\alpha}X_{\sigma}}),
\] (2.9)

with $X_{\alpha,\theta} \overset{d}{=} \beta_{1+\theta,1} X_{\alpha,\theta} + (1 - \beta_{1+\theta,1}) X_{\alpha} X_{\sigma}$. As special cases of (2.9),

(i) $X_{\alpha,1} \overset{d}{=} \beta_{1+\alpha,1-\alpha} X_{\alpha,1+\alpha}$

(ii) $X_{\alpha,1-\alpha} \overset{d}{=} \beta_{1-\alpha,1-\alpha} X_{\alpha,1} \overset{d}{=} \beta^{1/\alpha}(1 - \alpha) X_{\alpha,2-\alpha}$.

(iii) $X_{\alpha} \overset{d}{=} \beta_{1-\alpha,1} X_{\alpha,\alpha} \overset{d}{=} \beta^{1/\alpha}(1 - \alpha) X_{\alpha,1} = M_{1}(F_{X_{\alpha}X_{\sigma}})$, which yields the identity

\[
X_{\alpha} \overset{d}{=} U X_{\alpha} + (1 - U) X_{\alpha} X_{\sigma}
\]

for $X_{\alpha} \overset{d}{=} X_{\alpha}$.

Proof. (2.8) follows from $C_{1+\theta}(\lambda; X_{\alpha},\theta) = C_{\theta}(\lambda; X_{\alpha},\theta) C_{1}(\lambda; X_{\alpha})$. For (2.9), we again calculate $C_{1+\theta}(\lambda; X_{\alpha},\theta)$. The first equality is easily checked. For the second we use,
\[
C_{1+\theta}(\lambda; \beta^{1/\alpha}(\frac{1}{\theta + \alpha}) X_{\alpha,1+\theta}) = C_{1+\theta}(\lambda, \beta^{1/\alpha}(\frac{1}{\theta + \alpha}) X_{\alpha,1+\theta}).
\] (2.10)

In order to establish the equivalence to $M_{1+\theta}(F_{X_{\alpha}X_{\sigma}})$, first note that (2.8) establishes $X_{\alpha,\theta+\alpha} \overset{d}{=} M_{\theta+\alpha}(F_{X_{\alpha}})$. The result is then concluded by an application of (2.4) for $\tau = 1 + \theta$, $\sigma = (\theta + \alpha)/(1 + \theta)$, and $R = X_{\alpha}$.

The next result establishes results for the larger class of variables defined with (2.3) and (2.4) in mind. Equation (2.9) of Proposition 2.1 is an important special case.

**Proposition 2.2.** Define, for $\tau > 0$ and $0 < \sigma \leq 1$, random variables $X_{\alpha,\tau}^{(\sigma)} \overset{d}{=} \beta_{\tau,\tau,\tau(1-\sigma)} X_{\alpha,\tau}$. Then
\[
X_{\alpha,\tau}^{(\sigma)} \overset{d}{=} \beta_{\tau,\tau,\tau(1-\sigma)} X_{\alpha,\tau} = \beta^{1/\alpha}(\frac{\tau(1-\sigma)}{\alpha}) X_{\alpha,\tau} \overset{d}{=} M_{\tau}(F_{X_{\alpha}X_{\sigma}}),
\]

which leads to the identity
\[
\frac{\beta_{\tau,\tau,\tau(1-\sigma)}}{S_{\alpha,\tau}} \overset{d}{=} \beta^{1/\alpha}(\frac{\tau(1-\sigma)}{\alpha}) \frac{S_{\alpha,\tau}}{S_{\alpha,\tau}}.
\] (2.11)

Proof. The result is easily checked by following arguments similar to those used to establish (2.9). Hence we just note that one uses the calculation, $C_{\tau}(\lambda; X_{\alpha,\tau}^{(\sigma)}) = C_{\tau,\sigma}(\lambda; X_{\alpha,\tau})$, in place of (2.10). The equality (2.11) follows immediately since stable random variables $S_{\alpha}$ are simplifiable, see (21).

Note that Propositions 2.1 and 2.2 show that
\[
-\log C_{\theta}(\lambda; X_{\alpha,\theta}) = \frac{\theta}{\alpha} \log(1 + \lambda^\alpha) = \theta \mathbb{E}[\log(1 + \lambda X_{\alpha})].
\]
2.2. Densities and explicit mixture representations

We first describe some more pertinent features of \( X_\alpha \), see also (48).

**Proposition 2.3.** Let \( X_\alpha \overset{d}{=} S_\alpha/S'_\alpha \), having density (4.1). Then,

(i) The cdf of \( X_\alpha \) can be represented explicitly as

\[
F_{X_\alpha}(x) = 1 - \frac{1}{\pi \alpha} \cot^{-1} \left( \cot(\pi \alpha) + \frac{x^\alpha}{\sin(\pi \alpha)} \right)
\]  

(ii) Its inverse is given by

\[
F^{-1}_{X_\alpha}(y) = \left[ \frac{\sin(\pi \alpha(y))}{\sin(\pi \alpha(1 - y))} \right]^{1/\alpha}
\]  

(iii) The equations (2.12) and (2.13) yield the identity,

\[
\sin(\pi \alpha F_{X_\alpha}(y)) = \frac{y^\alpha \sin(\pi \alpha(1 - F_{X_\alpha}(y)))}{y^\alpha \sin(\pi \alpha)} = \frac{y^\alpha \sin(\pi \alpha)}{[y^{2\alpha} + 2y^\alpha \cos(\pi \alpha) + 1]^{1/2}}.
\]  

(iv) Additionally,

\[
\cos(\pi \alpha F_{X_\alpha}(y)) = \frac{y^\alpha \cos(\pi \alpha) + 1}{[y^{2\alpha} + 2y^\alpha \cos(\pi \alpha) + 1]^{1/2}}.
\]

**Proof.** This derivation of the cdf is influenced by arguments in Fujita and Yor (36) where it becomes clear that it is easier to work with the density of \((X_\alpha)^\alpha\). Specifically the density of \((X_\alpha)^\alpha\) is given by

\[
\frac{\sin(\pi \alpha)}{\pi \alpha} \frac{1}{y^2 + 2y \cos(\pi \alpha) + 1} \quad \text{for } y > 0.
\]

It then easy to obtain the form of the cdf of \((X_\alpha)^\alpha\) by direct integration. Now using the fact that this equates with \( F_{X_\alpha}(y^{1/\alpha}) \) yields statement [(i)]. Statement [(ii)] then follows by using properties of the inverse cotangent. In order to establish [(iii)], use (2.13) which yields the identity,

\[
y = F^{-1}_{X_\alpha}(F_{X_\alpha}(y)) = \left[ \frac{\sin(\pi \alpha(F_{X_\alpha}(y)))}{\sin(\pi \alpha(1 - F_{X_\alpha}(y)))} \right]^{1/\alpha}.
\]  

Hence statement [(iii)] follows.

We now focus on obtaining explicit distributional formulae for the pertinent random variables based on their representations as Dirichlet means. In relation to this Proposition 2.3 gives precise details on the pertinent cdf \( F_{X_\alpha} \), it then remains to obtain a nice expression for the quantity

\[
\Phi_\alpha(x) := \Phi_{X_\alpha}(x) = \mathbb{E}[\log |x - X_\alpha|].
\]
for $x > 0$. The key to calculating $\Phi_\alpha(x)$ is the fact that we showed that $X_\alpha$ is a mean functional of the type $M_1(F_\xi, X_\alpha)$, as described in Proposition 2.1. This sets up an equivalence between the form of the density of $X_\alpha$ obtained by Lamperti (66) and that of $M_1(F_\xi, X_\alpha)$, obtained from (2.2). Hence we have the following calculation.

**Proposition 2.4.** For $0 < \alpha < 1$, and $x > 0$,
\[
\Phi_\alpha(x) = \frac{1}{2\alpha} \log(x^{2\alpha} + 2x^\alpha \cos(\alpha \pi) + 1).
\]

**Proof.** Since $X_\alpha \overset{d}{=} M_1(F_\xi, X_\alpha)$, it follows by using (2.2) that the density of $X_\alpha$ satisfies the equivalence,
\[
f_{X_\alpha}(x) = \frac{1}{\pi} \sin(\pi \alpha) e^{-\alpha \Phi_\alpha(x)} x^{\alpha - 1}.
\]

Where on the left hand side we use the expression in (1.1). Now applying the identity in (2.14) shows that,
\[
f_{X_\alpha}(x) = \frac{x^{\alpha - 1} \sin(\pi \alpha)}{\pi} e^{-\alpha \Phi_\alpha(x)}.
\]

Solving this expression for $\Phi_\alpha(x)$ concludes the result. \qed

Set $\rho_{\alpha, \tau}(x^\alpha) = \frac{\tau}{\alpha} \arctan \left( \frac{\sin(\pi \alpha)}{\cos(\pi \alpha) + x^\alpha} \right) = \pi \tau [1 - F_{X_\alpha}(x)]$ and define the function,
\[
\Delta_{\alpha, \tau}(x) = \frac{x^{\tau - 1}}{\pi} \sin(\rho_{\alpha, \tau}(x^\alpha)) \sin(\pi \alpha u) \sin(\pi \alpha (1-u)), 0 < u < 1.
\]

We next obtain density formula for a key class of random variables that includes the case of $X_\alpha$ and $X_{\alpha, 1}$.

**Proposition 2.5.** For $0 < \sigma \leq 1$, and $x > 0$, the densities of the random variables
\[
X_{\alpha, 1}^{(\sigma)} \overset{d}{=} \beta_{\sigma, 1-\sigma} X_{\alpha, 1} \overset{d}{=} [\beta_{\sigma, 1-\sigma}]^{\frac{1}{\sigma}} X_{\alpha, 1},
\]
with $X_{\alpha} \overset{d}{=} X_{\alpha, 1}^{(\alpha)}$ and $X_{\alpha, 1} \overset{d}{=} X_{\alpha, 1}^{(1)}$, can be expressed as $\Delta_{\alpha, \sigma}(x)$, given in (2.16). Furthermore,

(i) $X_{\alpha, 1}^{(\sigma)} \overset{d}{=} F_{X_\alpha}^{-1}(U_{\alpha, \sigma})$ where $U_{\alpha, \sigma} \overset{d}{=} F_{X_\alpha}([X_{\alpha, 1}^{(\sigma)}])$ has density
\[
\frac{\sin(\pi \alpha u)}{\sin(\pi \alpha)} \frac{\sin(\pi \sigma (1-u))}{\sin(\pi \sigma (1-u))}, 0 < u < 1.
\]

(ii) If $0 < \sigma \leq \alpha$, then $X_{\alpha, 1}^{(\sigma)} \overset{d}{=} [\beta_{\sigma, 1-\sigma}]^{\frac{1}{\sigma}} X_{\alpha}$. 


Proof. The representations of $X^{(e)}_{\alpha,1}$ is just a special case of Proposition 2.2. The density is calculated based on Proposition 2.4 and the results discussed in (12) and (23), as mentioned previously. Statement [(i)] takes advantage of the properties of $F_{X_{\alpha}}$ and is otherwise straightforward to obtain. Statement [(ii)] is just a manipulation of the beta random variables.

One important aspect of the previous result is that we can use it to obtain density/mixture representations for the following Lamperti random variables. This is facilitated by the identity (2.9).

**Proposition 2.6.** Suppose that $0 \leq \theta \leq 1 - \alpha$, then $\alpha \leq \sigma^* = \theta + \alpha \leq 1$ and there is the distributional identity,

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+1-\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \beta_{1,\theta} X_{\alpha,1}^{(\sigma^*)},$$

In particular, $X_{\alpha,1-\alpha} \stackrel{d}{=} \beta_{1-\alpha} X_{\alpha,1}$. 

(i) Hence for $0 \leq \theta \leq 1 - \alpha$, the density of $X_{\alpha,\theta}$, can be written as,

$$f_{X_{\alpha,\theta}}(x) = \theta \int_{0}^{1} \frac{\Delta_{\alpha,\sigma^*}(x/u)}{u(1-u)^{1-\theta}} du, \quad x > 0,$$

(2.17)

where $\Delta_{\alpha,\sigma^*}(x) \geq 0$, is the density of $X_{\alpha,1}^{(\sigma^*)}$. When $\theta = 0$, the density is $\Delta_{\alpha,\alpha}(x)$ equating with (1.1).

(ii) As a special case, when $\alpha \leq 1/2$, $X_{\alpha,\alpha} \stackrel{d}{=} B_{1,\alpha} X_{\alpha,1}^{(2\alpha)}$, where $X_{\alpha,1}^{(2\alpha)}$ has density,

$$\frac{\sin(\pi \alpha) 2\alpha 2^{2\alpha-1} \cos(\pi \alpha) + x^\alpha}{\pi \left[ x^{2\alpha} + 2x^\alpha \cos(\alpha \pi) + 1 \right]^2}. \quad (2.18)$$

Proof. The result follows from Propositions 2.2 and 2.5 by writing

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+1-\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \beta_{1,\theta} \beta_{\theta+1-\alpha} X_{\alpha,\theta+\alpha}.$$

The simplification in (2.18) follows from,

$$\sin(2\pi \alpha [1 - F_{X_{\alpha}}(x)]) = \frac{\sin(2\pi \alpha) + 2x^\alpha \sin(\pi \alpha)}{1 + 2x^\alpha \cos(\pi \alpha) + x^{2\alpha}}.$$

The previous results allow one to obtain simple mixture representations or densities for $X_{\alpha,\theta}$ in the range $0 \leq \theta \leq 1 - \alpha$, and $\theta = 1$. The fact that we obtain such results for a continuous range of $\theta$ is significant, as shown in the next result.

**Proposition 2.7.** Set $\theta = \sum_{j=1}^{k} \theta_j$ where $\theta_j > 0$. Furthermore, let $(D_1, \ldots, D_k)$ denote a Dirichlet random vector having density proportional to $\prod_{i=1}^{k} \theta_i^{\theta_i}$. That is each $D_i \stackrel{d}{=} \beta_{\theta_i,\theta-\theta_i}$. Then,

$$X_{\alpha,\theta} \stackrel{d}{=} \sum_{j=1}^{k} D_j X_{\alpha,\theta_j}$$
where $X_{\alpha,\theta_j}$ are mutually independent and independent of $(D_1, \ldots, D_k)$. When $\theta_j$ are chosen such that $0 < \theta_j \leq 1 - \alpha$, each $X_{\alpha,\theta_j}$ has an explicit density $f_{X_{\alpha,\theta_j}}$ described in (2.17). When $\theta = k$, one can use $\theta_j = 1$.

Proof. Since we have shown that $X_{\alpha,\theta} \overset{d}{=} M_\theta(F_{X_{\alpha}})$, this result follows directly as a special case of Hjort and Ongaro (41), Proposition 9.

2.3. Positive Linnik variables

For $\theta > 0$,

$$\chi_{\alpha,\theta} \overset{d}{=} \gamma^{1/\alpha} S_\alpha$$

(2.19)

denotes the class of generalized Linnik variables as considered in [66, 16, 51, 28, 68, 43]. The results in the previous section depend on the validity of the transforms in (2.5) and (2.6). It is evident, and known, that the expression is the Laplace transform of $\chi_{\alpha,\theta}$ for $\theta > 0$. Hence (2.5) is verified if one shows that that $\chi_{\alpha,\theta} \overset{d}{=} \gamma_\theta X_{\alpha,\theta}$, for $\theta > 0$. It is already known, using a result of Devroye (27) combined with (2.7), that

$$\chi_{\alpha,\theta} \overset{d}{=} \gamma_\theta X_{\alpha,\theta}.$$  

(2.20)

Furthermore, from Bondesson (16, p.38), it follows that $\chi_{\alpha,\theta}$ are GGC$(\theta, F_{X_{\alpha}})$. In the next result we will verify the usage of (2.5), and use the $X_{\alpha,\theta}^{(\sigma)}$ to obtain explicit density representations. In this regard, it is is important to note that we do not need explicit results for $X_{\alpha,\theta}$ to get corresponding results for $\chi_{\alpha,\theta}$. In addition we obtain some interesting identities.

**Proposition 2.8.** For all $\theta > 0$, $\chi_{\alpha,\theta}$ is a GGC$(\theta, F_{X_{\alpha}})$ variable that satisfies,

$$\chi_{\alpha,\theta} \overset{d}{=} \gamma_\theta X_{\alpha,\theta}.$$  

For $0 < \theta = \sigma \leq 1$, $\chi_{\alpha,\sigma} \overset{d}{=} \gamma_1 X_{\alpha,1}^{(\sigma)}$, and hence has the density,

$$f_{\chi_{\alpha,\sigma}}(x) = \int_0^\infty e^{x/y} y^{-1} \Delta_{\alpha,\sigma}(y) dy.$$  

(2.21)

See also (2.24) for $\theta > 0$. Additionally,

(i) For $\theta > -\alpha$,

$$\chi_{\alpha,\theta + \alpha} \overset{d}{=} \gamma_{\theta + \alpha} X_{\alpha,\theta + \alpha} \overset{d}{=} \gamma_{1 + \theta} X_{\alpha,\theta}.$$  

(2.22)

(ii) Hence, for $\theta > -\alpha$,

$$\gamma^{1/\alpha}_{\theta + \alpha} = \gamma_{\theta + \alpha} \frac{S_{\alpha,\theta + \alpha}}{S_{\alpha,\theta}}.$$  

(2.23)

(iii) For $-\alpha < \theta \leq k$, $k = 0, 1, 2, \ldots$,

$$\chi_{\alpha,\theta + \alpha} = \gamma_{k + 1} X_{\alpha,k} \beta^{1/\alpha}_{\theta + \alpha}.$$  

(2.24)
(iv) For \( \theta = \sum_{i=1}^{k} \theta_i > 0 \), \( \chi_{\alpha,\theta} \) are independent.

Proof. From (2.19) and using the identity

\[
e^{-\frac{z^\alpha}{\alpha}s} = E[e^{-\frac{z^\alpha}{\alpha}S_\alpha}],
\]

it is easy to see that the density can be expressed as

\[
f_{\chi_{\alpha,\theta}}(x) \propto x^{\theta-1} \int_0^\infty e^{-\frac{z^\alpha}{\alpha}s} s^{-\theta} f_\alpha(s) ds
\]

yielding the equivalence with \( \gamma_{\theta}X_{\alpha,\theta} \). The expression in (2.21) is due to Proposition 2.5. Statement [(i)] follows from (2.7). Statement [(ii)] follows by removing \( S_\alpha \), which is justified since it is a simplifiable variable. For (iii), apply Proposition 2.2. [(iv)] follows from infinite divisibility.

Remark 2.1. It is not difficult to show that a general expression for the density of \( \chi_{\alpha,\theta} \), for all \( \theta > 0 \), is obtained by replacing \( \Delta_{\alpha,\sigma} \) by \( \Delta_{\alpha,\theta} \) as follows,

\[
f_{\chi_{\alpha,\theta}}(z) = \frac{1}{\pi} \int_0^\infty \frac{e^{-z x} \sin(\pi \theta F_{X_{\alpha,\theta}}(x)) dx}{[x^{2\alpha} + 2x^\alpha \cos(\alpha \pi) + 1]^\frac{\theta}{\alpha}}.
\]

However, \( \Delta_{\alpha,\theta} \) can take negative values when \( \theta > 1 \), so this does not in general yield a mixture representation for \( \chi_{\alpha,\theta} \). Nonetheless, it may not be difficult to evaluate numerically, which is relevant for the Mittag-Leffler functions discussed in section 3.

Remark 2.2. The gamma identity in statement (2.23) of the previous proposition is quite remarkable, and as we shall see below has some interesting implications. We note that although not obvious, our result coincides with a variation of Bertoin and Yor ((11), Lemma 6). Checking moments one can see that, in their notation, \( J_{s,s/\alpha} \) follows from (2.7). Statement [(ii)] follows by removing \( S_\alpha \), which is justified since it is a simplifiable variable. For (iii), apply Proposition 2.2. [(iv)] follows from infinite divisibility.

2.4 Generalized Pareto Laws

Influenced in part by (2.23), we next look at relationships between the Lamperti laws and a class of generalized Pareto distributions. We note that the next result also plays an important role in section 4, when discussing continuous state branching processes. Define random variables,

\[
W_{1/\alpha}^{1/\alpha,\theta} := \left( \frac{U}{1-U^{\frac{\theta}{\alpha}}} \right)^{\frac{1}{\alpha}}.
\]
These represent a sub-class of generalized Pareto distributions with cdf and density given as

\[ F_{W^{1/\alpha}}(y) = \frac{y^\theta}{1 + y^\theta} \quad \text{and} \quad f_{W^{1/\alpha}}(y) = \frac{\theta y^{\theta-1}}{(1 + y^\theta)^{(\theta+\alpha)/\alpha}}. \]

**Proposition 2.9.** Let \( U \) denote a Uniform \([0,1]\) random variable, then for \( \theta > 0 \),

(i) there is the identity,

\[ W^{1/\alpha}_{\alpha,\theta} \overset{d}{=} \left( \frac{U^\theta}{1 - U^\theta} \right)^{1/\alpha} \overset{d}{=} \frac{\gamma_1^\theta}{\gamma_1} \overset{d}{=} \frac{\gamma_1^\theta}{\gamma_1} X_{\alpha,\theta}. \]

(ii) For \( 0 < \sigma \leq 1 \), the random variable \( \Sigma_{\alpha,\sigma} \overset{d}{=} \frac{\gamma_1}{X^{(\sigma)}_{\alpha,1}} \) has Laplace transform,

\[ E[e^{-\lambda \Sigma_{\alpha,\sigma}}] = 1 - \lambda^\sigma (1 + \lambda^\alpha)^{-\sigma/\alpha}. \] (2.25)

**Proof.** Statement [(i)] is an application of (2.23) For statement[(ii)] notice that

\[ P\left( \frac{\gamma_1}{\Sigma_{\alpha,\sigma}} > \lambda \right) = E[e^{-\lambda \Sigma_{\alpha,\sigma}}], \]

but this is the survival function of the random variable

\[ \frac{\gamma_1}{\gamma_1} X^{(\sigma)}_{\alpha,1} \overset{d}{=} \frac{\gamma_1}{\gamma_1} X^{(\sigma)}_{\alpha,1} \overset{d}{=} W^{1/\alpha}_{\alpha,\sigma}. \]

\( \square \)

**Remark 2.3.** As we shall discuss in section 4, Statement [(ii)], (2.25) serves to identify explicitly the (unknown) limiting distribution obtained by (97) and (87) corresponding to \( \sigma = 1 \). It is relevant also to note that the density of \( W^{1/\alpha}_{\alpha,1} \) is the only case that corresponds to a Laplace transform. So here we see a distinguishing feature of \( X_{\alpha,1} \).

### 2.5. z-variables and hyperbolic laws

Proposition 2.9, along with the works of (14; 84; 95; 42), motivates us to consider several questions related to \( z \)-distributions, which are distributed as the logarithm of independent gamma variables. We also believe that some of the variables we discuss will be of interest in terms of applications along the lines discussed in (4) and Hu(42). In fact, (42) suggests the use of a class of variables that turn out to be equivalent in distribution to \( \log(X_{\alpha}) \). Naturally we do this in the spirit of highlighting what one can do with Lamperti laws. We also obtain additional information about these variables.

In particular, for illustration, we consider the following generic type of problem. Suppose for generic variables \( X, Y, Z \) with \( Z \) and \( Y \) independent there is the following relation

\[ X \overset{d}{=} Y + Z. \]
One natural question is to ask given explicit information about $X$ and $Y$, what $Z$ satisfies the above equation? In addition does $Z$ have an explicit density or mixture representation? Notice also that if the characteristic function of $Z$ is not known then we can use $X$ and $Y$ to obtain this. We will consider $Z$ that are variants of Lamperti laws.

We first give a brief discussion on $z$-distributions and cite some relations to $\log(X^\alpha)$ that exist in the literature but are probably not well known. Following (95), the class of $z$-distributions are defined as $\pi^{-1} \log(\gamma_{\theta_1}/\gamma_{\theta_2})$, having characteristic function

$$E[e^{i\lambda \log(\gamma_{\theta_1}/\gamma_{\theta_2})}] = \frac{\Gamma(\theta_1 + i\lambda)\Gamma(\theta_2 - i\lambda)}{\Gamma(\theta_1)\Gamma(\theta_2)}.$$  

As special cases, the variables, for $0 < \sigma < 1$, $M_\sigma \overset{d}{=} \pi^{-1} \log(\gamma_\sigma/\gamma_{1-\sigma})$, have a Meixner distributions with characteristic function

$$E[e^{-i\lambda M_\sigma}] = \frac{\cos(\varepsilon_\sigma)}{\cosh(\lambda - i\varepsilon_\sigma)},$$  

where $\varepsilon_\sigma = \pi(\sigma - 1/2)$. Note that a Meixner distributed is usually defined as $(1/2)M_\sigma$. $S_1 \overset{d}{=} \pi^{-1} \log(U/(1-U))$ has a logistic distribution with characteristic function,

$$E[e^{i\lambda S_1}] = \frac{\lambda}{\sinh(\lambda)},$$  

and $C_1 \overset{d}{=} \pi^{-1} \log(\gamma_{1/2}/\gamma_{1/2})$ has the hyperbolic distribution with characteristic function,

$$E[e^{i\lambda C_1}] = \frac{1}{\cosh(\lambda)}.$$  

It is known that the variables $S_1$ and $C_1$ satisfy

$$C_1 \overset{d}{=} S_1 + T_1,$$  

where $T_1$ is an independent variable having characteristic function

$$E[e^{i\lambda T_1}] = \frac{\tanh(\lambda)}{\lambda}.$$  

Biane and Yor (15) showed that the density of $T_1$ is

$$f_{T_1}(x) = \frac{1}{\pi} \log(\coth(\frac{4}{\pi} |x|)), -\infty < x < \infty.$$  

In regards to this and other considerations note that the characteristic function of $\alpha \pi^{-1} \log(X_\alpha)$ is equivalent to

$$E[e^{\frac{i\alpha\lambda}{\pi} \log(X_\alpha)}] = \frac{\sinh(\alpha\lambda)}{\alpha \sinh(\lambda)}.$$
This expression can be found in Chaumont and Yor ([21], p. 147). In addition we see that this characteristic function agrees with the class of generalized secant hyperbolic distributions discussed for instance in [12]). It follows from the first Laplace transform given in Biane, Pitman and Yor ([14], p. 459, eq. (6.1)), that
\[ \alpha \pi^{-1} \log(X_\alpha) \begin{array}{c} \overset{d}{=} \\
\end{array} B(T_1^\alpha(R_3)), \]
where \( B(\cdot) \) is a Brownian motion and, independent of \( B(\cdot), T_1^\alpha(R_3) \) is the first hitting time of 1 by a Bessel process \( R_3 \) of dimension 3 starting at \( \alpha \). From Chaumont and Yor [21, p.155 and p.169], \( \lim_{\alpha \to 0} X_\alpha \overset{d}{=} U/(1 - U) \). Hence,

\[ \lim_{\alpha \to 0} \alpha \pi^{-1} \log(X_\alpha) \overset{d}{=} \lim_{\alpha \to 0} B(T_1^\alpha(R_3)) \overset{d}{=} B(T_1^0(R_3)) \overset{d}{=} S_1. \]

Note however that \( \pi^{-1} \log(X_{1/2}) \overset{d}{=} C_1 \overset{d}{=} B(T_1^0(R_1)), \) for \( R_1 \) a Bessel process of dimension 1.

Now using (2.7) and (2.11), one can write

\begin{equation}
\log(X_\alpha) = \log(S_{\alpha,1}/S'_{\alpha,1}) + \log(\beta_{\alpha,1}/\beta'_{\alpha,1}) \tag{2.28}
\end{equation}

Notice that when \( \alpha = 1/2, S_1 \overset{d}{=} \pi^{-1} \log(S_{1/2,1/2}/S'_{1/2,1/2}), \) hence the first equality in (2.28) coincides with (2.27) suggesting that this line is the natural generalization. Furthermore, it follows that

\[ T_1 \overset{d}{=} \pi^{-1} \log(\beta_{1/2,1/2}/\beta'_{1/2,1/2}). \]

This illustration can obviously be applied to other variables using (2.7) and (2.11). However we will not attempt to develop a thorough analysis here. Rather we will present two results that give a sense of what else can be done. In the first result below we are able to obtain a description of the characteristic function of \( X_{\alpha,\theta} \) and \( X_{\alpha,1} \).

**Proposition 2.10.** From Proposition 2.9, it follows that,

(i) for \( \theta > 0, \)

\[ \frac{1}{\alpha} \log \left( \frac{\gamma_\theta}{\gamma_1} \right) \begin{array}{c} \overset{d}{=} \\
\end{array} \log \left( \frac{\gamma_\theta}{\gamma_1} \right) + \log(X_{\alpha,\theta}). \]

(ii) For \( \theta > -\alpha \)

\[ \mathbb{E}[e^{i\lambda \pi^{-1} \log(X_{\alpha,\theta})}] = \frac{\Gamma(\frac{\theta + \alpha}{\alpha} + i\frac{\lambda}{\alpha})\Gamma(1 - \frac{\lambda}{\alpha})\Gamma(1 + \theta)}{\Gamma(1 + \theta + i\frac{\lambda}{\alpha})\Gamma(1 - \frac{\lambda}{\alpha})\Gamma(\frac{\theta + \alpha}{\alpha})}. \]
(iii) For $0 < \sigma \leq 1$,
\[
\frac{1}{\alpha} \log \left( \frac{\gamma_\sigma}{\gamma_1} \right) \overset{d}{=} \log \left( \frac{\gamma_1}{\gamma_1} \right) + \log(X^{(\sigma)}_{\alpha,1})
\]
where $\log(X^{(\sigma)}_{\alpha,1})$, has density
\[
\frac{1}{\pi} \sin \left( \rho_{\alpha,\sigma}(e^{z}) \right) \left[ e^{-2z\alpha} + 2e^{-z\alpha} \cos(\alpha\pi) + 1 \right]^{-\frac{1}{2}}, \quad -\infty < z < \infty.
\]
and characteristic function
\[
E[e^{i\lambda \log(X^{(\sigma)}_{\alpha,1})}] = \frac{\sinh(\lambda\pi)}{\lambda\pi} \frac{\Gamma(\frac{\alpha}{\theta} + i\frac{\alpha}{\theta})\Gamma(1 - i\frac{\alpha}{\theta})}{\Gamma(\frac{\alpha}{\theta})}. 
\]

Proof. This follows as a simple consequence of our previous results and the characteristic functions of $z$-distributions.

The next result illustrates some cases where the characteristic functions are based on hyperbolic functions and all random variables have explicit densities. Define,
\[
H_{\alpha,\sigma} \overset{d}{=} X^{(\sigma)}_{\alpha,1} / X_{\alpha,1} = \frac{\beta^{1/\alpha}_{\sigma,1-\sigma,\alpha}}{\beta^{1/\alpha}_{1-\sigma,\alpha}} X_{\alpha},
\]
where the equality follows from (2.11). In addition for $\alpha \delta \leq \theta \leq \alpha(1 - \delta)$, for $\delta \leq 1/2$, define
\[
L_{\alpha,\theta}^{(\delta)} \overset{d}{=} \left( \frac{\beta^{1/\alpha}_{\delta,\theta-\alpha\delta}}{\beta^{1/\alpha}_{1-\sigma,\alpha(1-\delta)-\theta}} \right) \frac{S_{\alpha,1-\theta}}{S_{\alpha,\theta}},
\]
where one can easily check that the density of $S_{\alpha,1-\theta}/S_{\alpha,\theta}$, denoted as $f_{1-\theta,\theta}$, satisfies
\[
f_{1-\theta,\theta}(x) = c_{\alpha,\theta} x^{-(1-\theta)} f_{X_{\alpha,1}}(x) = c_{\alpha,\theta} x^{-(1-\theta)} \Delta_{\alpha,1}(x),
\]
for
\[
c_{\alpha,\theta} = \frac{\Gamma(1/\alpha + 1)\Gamma(\theta + 1)\Gamma(2 - \theta)}{\Gamma(\theta/\alpha + 1)\Gamma((1-\theta)/\alpha + 1)}.
\]

Proposition 2.11. For $0 < \sigma < 1$, $\delta \leq 1/2$ and $\alpha \delta \leq \theta \leq \alpha(1 - \delta)$, there are the following relationships:

(i) $\frac{1}{\alpha} \log \left( \frac{\gamma_\sigma}{\gamma_1} \right) \overset{d}{=} \log \left( \frac{\gamma_1}{\gamma_1} \right) + \log(H_{\alpha,\sigma}).$ Hence,
\[
E[e^{i\lambda \log(H_{\alpha,\sigma})}] = \frac{\cos(\varepsilon_{\sigma}) \sinh(\alpha\lambda)}{\lambda \alpha \cosh(\lambda - i\varepsilon_{\sigma})} = \frac{\sinh(\alpha\lambda)}{\alpha \sinh(\lambda)} \frac{\cos(\varepsilon_{\sigma}) \sinh(\lambda)}{\lambda \cosh(\lambda - i\varepsilon_{\sigma})}
\]
where $\varepsilon_{\sigma} = \pi(\sigma - 1/2)$.
(ii) \( \frac{1}{\alpha} \log \left( \frac{\gamma_1}{\gamma_1 - \theta} \right) \overset{d}{=} \log \left( \frac{\gamma_2}{\gamma_1 - \theta} \right) + \log(L_{\alpha,\theta}^{(i)}) \). With,

\[
E[e^{\frac{\lambda}{\alpha} \log(L_{\alpha,\theta}^{(i)})}] = \frac{\cos(\epsilon_\theta) \cosh(\lambda - i\epsilon_\theta)}{\cos(\epsilon_\theta) \cosh(\lambda - i\epsilon_\theta)}.
\]

(iii) When \( \delta = 1/2 \), then \( \theta = \alpha/2 \), and \( L_{\alpha,\alpha/2}^{(1/2)} \overset{d}{=} S_{\alpha,1-\alpha/2}/S_{\alpha,\alpha/2} \). where \( \log(L_{\alpha,\alpha/2}^{(1/2)}) \) has density

\[
\frac{e^{\frac{\epsilon}{\alpha}} \sin(\rho_{\alpha,1}(e^{\epsilon_\alpha})))}{\pi \left[ e^{2z\alpha} + 2e^{\epsilon_\alpha} \cos(\alpha\pi) + 1 \right]^{1/2}}, \quad -\infty < z < \infty,
\]

and characteristic function

\[
E[e^{\frac{\lambda}{\alpha} \log(L_{\alpha,\alpha/2}^{(1/2)})}] = \frac{\cosh(\lambda - i\epsilon_{\alpha/2})}{\cos(\epsilon_{\alpha/2}) \cosh(\lambda/\alpha)}.
\]

Proof. All the characteristic functions follow from that of \( S_1 \) and (2.20). In order to establish [(i)] we use the identity \( \gamma_1/S_\alpha = \gamma_1^{1/\alpha} \) and apply this to the second definition of \( H_{\alpha,\sigma} \). Statement [(iii)] follows from a manipulation of (2.23) to force the form of the first two variables. The choice of \( 1-\theta \) and \( \theta \) in the definition of \( L_{\alpha,\theta}^{(i)} \) was deliberately made so that we could get an explicit expression of the density of the relevant ratios of stable variables.

3. Mittag-Leffler functions

In this section we obtain integral representations, and other identities for a generalization of the Mittag-Leffler function given by

\[
E^{(\theta/\alpha + 1)}_{\alpha,1+\theta}(-\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{(\theta/\alpha + 1)_k}{\Gamma(\alpha k + \theta + 1)} \quad \text{for } \theta > -\alpha \tag{3.1}
\]

where

\[
(\theta/\alpha + 1)_k = \frac{\Gamma(\theta/\alpha + 1 + k)}{\Gamma(\theta/\alpha + 1)}.
\]

So when \( \theta = 0 \), one recovers the Mittag-Leffler function as,

\[
E_{\alpha,1}(-\lambda) = E^{1}_{\alpha,1}(-\lambda) = E^{(0)}_{\alpha,0}(-\lambda).
\]

The function (3.1) is a special case of the function introduced by Prabhakar (86),

\[
E^\gamma_{\rho,\mu}(-\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)}, \quad \rho, \mu, \gamma \in \mathbb{C}, \Re(\rho) > 0 \tag{3.2}
\]

where \( (\rho, \mu, \gamma \in \mathbb{C}, \Re(\rho) > 0) \). That is the case where \( \gamma = (\theta + \alpha)/\alpha \) and \( \mu = \theta + 1 \). The quantity (3.1) represents a special sub-class of yet more general
Mittag-Leffler type functions which are discussed, for instance in Kilbas, Saigo and Megumi (55). See also (2; 7; 5; 20; 30; 40; 57; 58; 59).

Note that using simple cancelations involving gamma functions it is easy to show that for \( \theta > 0 \),
\[
E_{\alpha,1+\theta}^{(\theta/\alpha+1)}(-z) = \frac{\Gamma(\theta)}{\Gamma(1+\theta)} E_{\alpha,\theta}^{(\theta/\alpha)}(-z).
\]

(3.3)

Recall that the Mittag-Leffler function can be expressed as the Laplace transform of \( S_{\alpha}^{-\theta} \). One can show that by taking a Taylor expansion and calculating moments that
\[
E[e^{-z/S_{\alpha,\theta}^{\alpha}}] = \Gamma(\theta + 1) E_{\alpha,\theta+1}^{(\theta/\alpha+1)}(-z),
\]

(3.4)

One consequence of this observation is that one can use a Monte-Carlo method based on \( S_{\alpha,\theta} \) to evaluate this quantity. The next result develops more connections with \( X_{\alpha,\theta} \) and \( \chi_{\alpha,\theta} \).

**Proposition 3.1.** For \( \theta > -\alpha \),

(i) \( f_{1/X_{\alpha,\theta}}(x) = x^{-\theta} f_{X_{\alpha,\theta}}(x) \)

(ii) \( f_{\gamma_{1}/X_{\alpha,\theta}}(x) = \Gamma(\theta + 1) x^{-\theta} f_{X_{\alpha,\theta+\alpha}}(x) \)

(iii) For \( \theta > 0 \),
\[
\mathbb{P} \left( \frac{\gamma_1}{X_{\alpha,\theta}} > x \right) = \mathbb{E}[e^{-x X_{\alpha,\theta}}] = E_{\alpha,\theta}^{(\theta/\alpha)}(-x^\alpha) = \Gamma(\theta) x^{1-\theta} f_{X_{\alpha,\theta}}(x).
\]

Hence if \( \theta = \sigma \) then these expressions are explicitly determined by (2.21), otherwise one might use (2.24).

(v) Applying Proposition 2.7, it follows that for \( \sum_{i=1}^{k} \theta_i = \theta \), where \( \theta_i > 0 \),
\[
E_{\alpha,\theta}^{(\theta/\alpha)}(-z) = \int_{S_k} \prod_{i=1}^{k} E_{\alpha,\theta_i}^{(\theta/\alpha_i)}(-zz_{i}^{\alpha_i}) x_i^{\theta_i - 1} dx_i
\]
where \( S_k = \{(x_1, \ldots, x_k) : 0 < \sum_{i=1}^{k} x_i \leq 1 \} \).

The result is fairly straightforward using the basic definitions as ratios of stable variables and the identities in Proposition 2.5. We omit the details.

**Remark 3.1.** Recall that for \( \alpha = 1/2 \), \( X_{1/2,\theta} \overset{d}{=} \gamma_{\theta+1/2}/\gamma_{1/2} \). Using the notation in Pitman (79), eq. (88), eq(98) and Lemma 15, and noting (79), eq. (104)), the conditional moments of the meander length \( (1 - G_1) \), of Brownian motion on \( [0,1] \) conditioned on its local time \( L_1 \) is related to the generalized Mittag-Leffler function as follows,
\[
\mathbb{E}_{1/2,0}((1 - G_1)^{\theta+1/2}|L_1 = \sqrt{2} \lambda) = \mathbb{E}[e^{-\lambda X_{1/2,\theta}}] = \mathbb{E}|B_1|^{\theta+1/2} |h_{-(2\theta+1)}(\lambda)
\]
\[
= \mathbb{E}_{1/2,0}^{\{2\theta\}}(-\frac{\lambda}{\sqrt{2}}) = \mu(\theta + 1/2||\lambda),
\]
which corresponds to the moments of the structural distribution of the Brownian excursion partition. $h_{-2q}(\lambda)$ is a Hermite function, here $q = \theta + 1/2$.

4. The explicit Lévy density of Stable CSBPs and the Zolotarev-Slack distribution

We now show how our results lead to an explicit identification of the Lévy density of the semigroup of stable continuous state branching processes of index $1 < \delta < 2$, that is $\delta = 1 + \alpha$ and the limiting distributions first obtained by Zolotarev [87] and Slack [87]. We also mention briefly its connection to the work of [6, 9] on beta coalescents. From Lamperti, [65] continuous state branching processes (CSBP) are Markov processes that can be characterized as limits of Galton-Watson branching processes when the population size grows to infinity. A $(1+\alpha)$ stable (CSBP) process $(Y_t, t > 0)$ is a Markov process whose semigroup is specified by,

$$E^a[e^{-\lambda Y_t}] = E[e^{-\lambda Y_t}|Y_0 = a] = e^{-a\nu_t(\lambda)} (4.1)$$

where

$$\nu_t = \int_0^\infty (1 - e^{-s\lambda})\nu_t(ds) = \lambda(\alpha t + \lambda^\alpha)^{-1/\alpha}.$$ 

Furthermore, see for instance [6, 9], there exists a process $(Y(t, a); t > 0, a > 0)$ such that for each $t$, $Y(t, \cdot)$ is a compound Poisson process with intensity $\nu_t$. Related to this result, Kawazu and Watanabe [53] show that all continuous state branching processes with immigration arise as limits of Galton-Watson processes with immigration. Analogous to (4.1), Theorem 2.3 of their work yields the limiting $(1+\alpha)$ stable (CSBP) with immigration $(\hat{Y}_t, t > 0)$ satisfying,

$$E^a[e^{-\hat{Y}_t}] = (1 + \alpha\lambda^\alpha t)^{-\frac{1}{\alpha}} E^a[e^{-Y_t}]. (4.2)$$

It is evident from (4.2), that the entrance laws of $\hat{Y}$ are positive Linnik distributions. However, the intensity $\nu_t$, which plays a fundamental role in (4.1) is only known up to its Laplace transform. It is known that this Laplace transform coincides, up to some scaling factors, with the Laplace transforms of the limiting distributions obtained by Zolotarev [97] and Slack [87]. Before identifying these expressions we recount the limiting distributions obtained by Slack [87] and Berestycki, Berestycki and Schweinsberg [6]. As described, for instance, in [6, 9], Slack’s result describes the limiting distribution, say $\mu_\alpha$, of the number of offspring in generation $n$ of a critical Galton Watson process, rescaled to have mean 1 and conditioned to be positive, when the offspring distribution is in the domain of attraction of a stable law of index $1 < \delta < 2$. This result complements Yaglom’s [92] well known result for the case where the offspring distribution has finite variance. In that case the limiting distribution is exponential with mean 1. Precisely, following the exposition in [72], we state a variation of Slack’s result.

**Proposition 4.1.** (Slack(1968) [87]). Let $Z = (Z_n, n > 0)$ denote a supercritical Galton Watson process initiated by a single process. Furthermore, sup-
pose the non-extinction probability \( Q_n = P(Z_n > 0) \), satisfies

\[
Q_n = n^{-1/\alpha} L(n)
\]

where \( L(x) \) is a slowly varying function. Then,

\[
\lim_{n \to \infty} P(Q_n Z_n \leq x | Z_n > 0) = \mu_\alpha([0, x])
\]

where for each \( 0 < \alpha < 1 \), \( \mu_\alpha \) is the distribution of a random variable \( \Sigma_\alpha \) satisfying

\[
\int_0^\infty e^{-\lambda w} \mu_\alpha(dw) = \mathbb{E}[e^{-\lambda \Sigma_\alpha}] = 1 - \lambda(1 + \lambda^\alpha)^{-1/\alpha}.
\]

Zolotarev ([97], Theorem 7) also obtained this limit in the case of a class of continuous parameter regular branching processes. However, prior to our work, an explicit description of its density or corresponding random variable was not known. It is now evident from (4.4) that \( \Sigma_\alpha \overset{d}{=} \Sigma_{\alpha,1} \) in Proposition 2.9, as we mentioned previously. These limits and discussions related to (4.1), (4.2) appear more recently in for instance, [6, 9, 33, 63, 64, 74]. Before we summarize our results we shall say a bit more about the context of (6). Random variables with law \( \mu_\alpha \) arise in the work of Berestycki, Berestycki and Schweinsberg [6], see also [9], in connection with Beta \((2 - \delta, \delta)\) coalescents for \( 1 < \delta < 2 \). See in particular ([6], Theorem 1.2.) Equivalently these are Beta \((1 - \alpha, 1 + \alpha)\) coalescents.

In addition, there is a related result of Berestycki, Berestycki and Schweinsberg [6] that, with some work, would have otherwise allowed us describe the law \( \mu_\alpha \). We quote their result below,

**Proposition 4.2** (Berestycki, Berestycki, and Schweinsberg ([6], Proposition 1.5)). Let \((\Pi(t), t > 0)\) denote a Beta \((1 - \alpha, 1 + \alpha)\) coalescent where \( 0 < \alpha < 1 \), and let \( K(t) \) denote the asymptotic frequency of the block of \( \Pi(t) \) containing 1. Then

\[
(\Gamma(\alpha + 2)t^{-1})^{\frac{1}{2}} K(t) \quad \overset{d}{\to} \quad \zeta_\alpha \quad \text{as} \quad t \downarrow 0,
\]

where \( \zeta_\alpha \) is a random variable satisfying,

\[
\mathbb{E}[e^{-\lambda \zeta_\alpha}] = (1 + \lambda^\alpha)^{-(1+\alpha)/\alpha}.
\]

Furthermore, as noted in [6], \( \zeta_\alpha \) has the size biased distribution

\[
\mathbb{P}(\zeta_\alpha \in dx) = x \mu_\alpha(dx).
\]

We now summarize our result which again demonstrates the relevance of \( X_{\alpha,1} \)

**Proposition 4.3.** For \( 0 < \alpha < 1 \), recall that \( X_{\alpha,1} \overset{d}{=} S_\alpha/S_{\alpha,1} \), has explicit density \( \Delta_{\alpha,1} \). Then,
(i) \( \zeta_\alpha \), the random variable described in (4.5) and (4.6), satisfies
\[
\zeta_\alpha d = \gamma_2 X_{\alpha,1} \chi_{\alpha,1+\alpha}.
\]

(ii) Let \( \Sigma_\alpha \) and \( \mu_\alpha \) be as in (4.3) and (4.4), then
\[
\Sigma_\alpha d = \gamma_1 X_{\alpha,1} \mu_\alpha(\chi_{\alpha,1}) = \chi_{\alpha,1+\alpha}.
\]

(iii) Furthermore, for each \( x > 0 \),
\[
\mathbb{P}(\Sigma_\alpha > x) = \mu_\alpha(\chi_{\alpha,1+\alpha}) = \mathbb{E}(1/\alpha) \chi_{\alpha,1+\alpha}(-x\alpha) = f_{\gamma_1 X_{\alpha,1}}(x).
\]

(iv) Set \( \theta_t = (\alpha t)^{-1} \), then the Lévy density \( \nu_t \) corresponding to the \( (1 + \alpha) \) (CSBP) specified by (4.1) is,
\[
\nu_t(x) = (\alpha t)^{-1} \chi_{\alpha,1+\alpha} e^{(-x\alpha)/(\alpha t)}
= (\alpha t)^{-1} \int_{0}^{-\infty} e^{-\alpha y} y \Delta_{\gamma_1 X_{\alpha,1}}(y) dy.
\]

Proof. Statements [(i)] and [(ii)] are now quite obvious from Propositions 2.8 and 2.9. Statements [(iii)] and [(iv)] are deduced from Propositions 2.5 and 3.1, along with the specifications for \( \nu_t \), in terms of \( \theta_t \) described in ([6], Lemma 2.2).

5. Occupation times of generalized Bessel bridges

We now show how our results for \( X_{\alpha,\theta} \) can be used to obtain new results related to \( A_+^{1} \), which is equivalent in distribution to \( P_{\alpha,\theta}(p) \) under \( P_{\alpha,\theta}(p) \). This can be seen as a continuation of a subset of the work of [47], who looked at more general \( PD(\alpha,\theta) \) mean functionals, where, with the exception of \( \alpha = 1/2 \), the best results for describing the density of \( P_{\alpha,\theta}(p) \) were obtained for \( \theta = 1 \), and \( \alpha = 1 - \alpha \). The results for \( \alpha = 1/2 \) are classic. For \( \alpha = 1/2 \), and \( p = 1/2 \), Lévy ([67]) showed that \( A_+^{1} \) under \( (1/2,0) \) and \( (1/2,1/2) \), follow the Arcsine and Uniform\([0,1]\) distributions respectively. A general formula for \( (1/2,\theta) \), for all \( \theta > -1/2 \) can be found in Carlton [19], see also [54]. We also obtain results for time spent positive on certain random subsets of \([0,1]\), and also develop some interesting stochastic equations. As a highlight, we obtain explicit results for the case of \( \theta = \alpha \), corresponding to the the time spent positive of a Bessel bridge on \([0,1]\). In this case the best previous expressions were obtained independently in [14, 93].

Consider now the following stochastic equations and Cauchy transforms that can be found in ([47]) with further references, for \( \theta > 0 \)
\[
P_{\alpha,\theta}(p) = \beta_{\theta,1} P_{\alpha,\theta}(p) + (1 - \beta_{\theta,1}) P_{\alpha,0}(p)
\]

(5.1)
and for $\theta > -\alpha$,
\[ P_{\alpha, \theta}(p) = \beta_{\theta + \alpha, 1 - \alpha} P_{\alpha, \theta + \alpha}(p) + (1 - \beta_{\theta + \alpha, 1 - \alpha}) \xi_p. \tag{5.2} \]

Additionally there are the Cauchy transforms for $\theta > 0$,
\[ C_\theta(\lambda; P_{\alpha, \theta}(p)) = (q + (1 + \lambda)^{\alpha} p)^{-\frac{\alpha}{\lambda}} e^{-\theta \psi_{\alpha, \theta}^{(p)}(\lambda)} \tag{5.3} \]
where
\[ \psi_{\alpha, \theta}^{(p)}(\lambda) = \mathbb{E}[\log(1 + \lambda P_{\alpha, \theta}(p))] = \mathbb{E}_{\alpha, \theta}^{(p)}[\log(1 + \lambda A_{\theta, \alpha}^+)]. \]

and for $\theta < -\alpha$,
\[ C_{1 + \theta}(\lambda; P_{\alpha, \theta}(p)) = \frac{(1 + \lambda)^{\alpha - 1} p + (1 - p)}{(q + (1 + \lambda)^{\alpha} p)^{-\frac{\alpha}{\lambda}}} \tag{5.4} \]

The first equation (5.1) shows that $P_{\alpha, \theta}(p) \overset{d}{=} M_\theta(F_{\alpha, \theta}(p))$ for $\theta > 0$. The second equation (5.2), see for instance (10), (29), (44), (80), for some other interpretations and applications, can be traced to Pitman and Yor ((81), Theorem 1.3.1), and Perman, Pitman and Yor ((81), Theorem 3.8, Lemma 3.11) as follows: Let $A_{G_1}^+ = \int_G^1 \mathbb{I}_{(B_t > 0)} dt$ denote the time spent positive of $B$ up till time $G_1$, which is the time of the last zero of $B$ before time 1. Then under $\mathbb{P}_{\alpha, \theta}$ there is the equivalence
\[ (A_{G_1}^+, G_1) \overset{d}{=} (G_1 A_{1}^{(br)}, G_1) \overset{d}{=} (\beta_{\theta + \alpha, 1 - \alpha} P_{\alpha, \theta + \alpha}(p), \beta_{\theta + \alpha, 1 - \alpha}) \]

This shows that (5.2) can be rewritten in terms of the following decomposition,
\[ A_1^+ \overset{d}{=} A_{G_1}^+ + (1 - G_1) \xi_p \overset{d}{=} G_1 A_{1}^{(br)} + (1 - G_1) \xi_p. \]

See for example Enriquez, Lucas, and Simenhaus (31) for an interesting recent application of this expression.

We now show that the density of $P_{\alpha, \theta}(p)$ can be expressed in terms of the density of $X_{\alpha, \theta}$. Hereafter, define $r_p(y) = y/(c(1 - y))$, for $c^\alpha = p/(1 - p)$.

**Proposition 5.1.** For $\theta > -\alpha$, let $R_{\alpha, \theta} = cX_{\alpha, \theta}/(cX_{\alpha, \theta} + 1)$. Then
\[ \mathbb{P}_{\alpha, \theta}^{(p)}(A_1^+ \in dy) = \frac{(1 - y)^{\theta}}{(1 - p)^{\theta}} \mathbb{P}(R_{\alpha, \theta} \in dy) \]
where $A_1^+ \overset{d}{=} P_{\alpha, \theta}(p)$. Hence as special cases, using Propositions 2.5 and 2.6,

(i) $\mathbb{P}_{\alpha, 1}^{(p)}(A_1^+ \in dy)/dy = (1 - y)^{-1} p^{-1/\alpha} \Delta_{\alpha, 1}(r_p(y)) = \Omega_{\alpha, 1}(y)$.
(ii) For $0 < \theta \leq 1 - \alpha$, and $\sigma^* = \theta + \alpha$,
\[ \mathbb{P}_{\alpha, \theta}^{(p)}(A_1^+ \in dy)/dy = \frac{\theta(1 - y)^{\theta - 2}}{1 - p} \cdot \int_0^1 \frac{\Delta_{\alpha, \sigma^*}(r_p(y)/u)}{u(1 - u)^{1-\sigma}} du. \]
where $\Delta_{\alpha, \sigma^*}(x) \geq 0$ is the density of $X_{\alpha, 1}^{(\sigma^*)}$.
For Proposition 5.2, interesting properties of generalizations of this variable. From (1.3) it follows that for measureable functions $g$,

$$E[g(cX_{\alpha,\theta})] = \frac{(1-p)^{\theta/\alpha}}{E[S_{\alpha}^{-\theta}]} E^{(p)}[g \left( \frac{A^+}{A^{-}_{\tau_1}} \right)](A^{-}_{\tau_1})^{-\theta}].$$

The result is concluded by showing that

$$E^{(p)}_{\alpha,\theta}[g(A^+_1)] := \left. \frac{1}{E[S_{\alpha}^{-\theta}]} E^{(p)}[g(A^+_1/\tau_1)] \right|_{\tau_1}^{\theta}$$

is equal to $(1-p)^{-\theta/\alpha}E[g(R_{\alpha,\theta})(1 + X_{\alpha,\theta})^{-\theta}].$ But this follows from $\tau_1 = (A^+_1/A^+_{\tau_1} + 1)A^{-}_{\tau_1}$.

Pitman and Yor (83), Proposition 15) establishes an interesting relationship between the densities of $A^+_1$ and $A^+_{\tau_1}$ under the law $E^{(p)}_{\alpha,\theta}$. Making no changes to the essence of their clever argument, one can easily extend this result to all $(\alpha, \theta)$. Combining this with Proposition 5.1 yields the relationships,

$$E^{(p)}_{\alpha,\theta}(A^+_1 \in dy) = \frac{1 - y}{(1-p)^{1+\theta}} P^{(p)}_{\alpha,\theta}(A^+_1 \in dy) \quad (5.5)$$

$$= \left. \frac{1 - y}{(1-p)^{1+\theta}} P_{\alpha,\theta}(R_{\alpha,\theta} \in dy) \right|_{\tau_1}^{\theta}.$$

Recall that under $P^{(p)}_{\alpha,\theta}$, $A^+_{\tau_1} \overset{d}{=} \beta_{\theta+\alpha,1-\alpha} P_{\alpha,\theta}(p)$. The next result describes interesting properties of generalizations of this variable.

**Proposition 5.2.** For $\tau > 0$ and $0 < \sigma \leq 1$, let $V \overset{d}{=} \beta_{\tau/(1-\sigma),\tau/\sigma} V'$ and hence is a Dirichlet mean satisfying the stochastic equation,

$$V \overset{d}{=} \beta_{\tau/(1-\sigma),\tau/\sigma} V' + (1 - \beta_{\tau/(1-\sigma)})\xi_\sigma,$$

for $V \overset{d}{=} V'$. Then for $0 < p \leq 1$,

$$\beta_{\tau,\tau(1-\sigma)} P_{\alpha,\tau}(p) \overset{d}{=} P_{\alpha,\tau}(pV) = M_{\tau}(F_{P_{\alpha,\theta}(p\xi_\sigma)}). \quad (5.6)$$

This leads to the stochastic equations, for $0 < p \leq 1$

$$P_{\alpha,\tau}(pV) \overset{d}{=} \beta_{\tau,1} P_{\alpha,\tau}(pV') + (1 - \beta_{\tau,1})P_{\alpha,0}(p\xi_\sigma) \quad (5.7)$$

In the first expression $P'_{\alpha,\tau}(pV')$ denotes a random variable equivalent only in distribution to $P_{\alpha,\tau}(pV)$. However in the second equation $V$ is the same variable.

**Proof.** First note that $\beta_{\tau,\tau(1-\sigma)} P_{\alpha,\tau}(p) \overset{d}{=} M_{\tau}(F_{P_{\alpha,\theta}(p\xi_\sigma)})$, follows from (2.4).

Note also by the definition of $P_{\alpha,\theta}(p)$ it is easy to see that $P_{\alpha,\theta}(p)\xi_\sigma \overset{d}{=} P_{\alpha,\theta}(p\xi_\sigma).$
In order to establish the rest of (5.6) we can check Cauchy transforms of order \( \tau \) using (5.3). For the variable appearing on the left of (5.6) this is easy to calculate. Applying this to \( P_{\alpha,\tau}(pV) \) conditioned on \( V \), its final evaluation rests on the simple equality
\[
(1 - pV + (1 + \lambda)^{\alpha}pV) = 1 + [(1 + \lambda)^{\alpha} - 1]pV.
\]
Taking the Cauchy transform of order \( \tau/\alpha \) for \( V \) yields the result. The second equality in (5.8) is then due to (5.1).

Next, is one of our main distributional results, which is an analogue of Proposition 2.6, but also highlights the role of various randomly skewed processes.

**Proposition 5.3.** For \( 0 < \sigma \leq 1 \), set \( R^{(\sigma)}_{\alpha,1} = cX_{\alpha,1}^{(\sigma)}/(cX_{\alpha,1}^{(\sigma)} + 1) \), then the density of \( \beta^{\sigma}_{\alpha,1}(\sigma)p_{\alpha}^{\sigma} \) is for \( 0 < y < 1 \), equivalent to \( (1 - p)^{-\sigma} (1 - y)^{\alpha} \) \( \Omega_{\alpha,\sigma}(y) = \frac{1}{\pi} \frac{y^{\alpha-1} \sin (\rho_{\alpha,\sigma}(y))}{y^{2\alpha q^2 + 2qy\alpha(1-y)^{\alpha} \cos(\alpha\pi) + (1-y)^{2\alpha}q^2} \frac{1}{\pi}} \), (5.8)

In particular \( \Omega_{\alpha,1}(y) \) is the density of \( P_{\alpha,1}(p) \) and \( \Omega_{\alpha,\alpha}(y) \) is the density of \( \beta_{\alpha,1-P_{\alpha,\alpha}}(p) \). In addition,

(i) if \( 0 \leq \theta \leq 1 - \alpha \), then \( \beta^{\sigma}_{\alpha,1}(\sigma)p_{\alpha}^{\sigma} \)

\[
\beta_{\alpha+\sigma,1-\alpha}P_{\alpha+\sigma,\alpha}(p) \overset{d}{=} P_{\alpha+\sigma,1}(\rho_{\alpha+\sigma}(\frac{1}{\alpha+\sigma})) \overset{d}{=} \beta_{\alpha,1}\beta_{\alpha+\sigma,1}(\rho_{\alpha+\sigma}(\frac{1}{\alpha+\sigma})),
\]

and, using (5.6), there is the explicit formula determining the densities of \( A_{1}^{\tau} \) and \( A_{1}^{\tau} \),

\[
\mathbb{P}_{\alpha,\sigma}(A_{1}^{\tau} \in dy)/dy = \frac{1 - y}{(1 - p)} \mathbb{P}_{\alpha,\sigma}(A_{1}^{\tau} \in dy)/dy = \int_{0}^{1} \frac{\theta \Omega_{\alpha,\sigma}(y/u) du}{u(1-u)^{1-\sigma}}.
\]

When \( \theta = 0 \), (5.5) is \( \Omega_{\alpha,\sigma}(y) \) which agrees with Pitman and Yor([83], Proposition 15).

(ii) As in Proposition 2.7, for any \( \theta > 0 \), set \( \theta = \sum_{j=1}^{k} \theta_{j} \) for some integer \( k \) and \( \theta_{j} > 0 \). This leads to the representation,

\[
P_{\alpha,\theta}(p) \overset{d}{=} \sum_{j=1}^{k} \theta_{j} P_{\alpha,\theta_{j}}(p)
\]

for independent variables \( P_{\alpha,\theta_{j}}(p) \) and \((D_{1}, \ldots, D_{k}) \) a Dirichlet vector as in Proposition 2.7.
variable has $\mathbb{P}^{(p)}_{\alpha,\theta}(A_t^+ \in dy)$ given by (5.9) with $\sigma^* = \theta_j + \alpha$. It suffices to choose $\theta_i = \theta/k$, for $0 < \theta \leq k(1 - \alpha)$. If $\theta = k$, one may set $\theta_j = 1$ and use $\Omega_{\alpha,1}$.

**Proof.** The various representations of the random variables are due to Proposition 5.2 and otherwise an application of the beta/gamma calculus. The density $\Omega_{\alpha,\sigma}$ is obtained from $\Delta_{\alpha,\sigma}$, which is justified by the exponential tilting relationships discussed in James ((45), section 3), see also (46).

**Remark 5.1.** The random variable $P_{\alpha,\tau}(pV)$ described in Proposition 5.2 has law

$$
\mathbb{P}(P_{\alpha,\tau}(pV) \in dx) = \mathbb{P}^{(pV)}_{\alpha,\tau}(A_t^+ \in dx) := \int_0^1 \mathbb{E}^{(pV)}_{\alpha,\tau}(A_t^+ \in dx) f_V(u) du.
$$

That is, it may be read as the time spent positive up till one of a process whose excursion lengths, conditional on $V$, follow a $PD(\alpha, \tau)$ distribution and is otherwise randomly skewed by $pV$. See also Aldous and Pitman (1), section 5.1) for connections with $T$-partitions. This is made clear, as follows; For $(L_t; 0 \leq t \leq 1)$ governed by $PD(\alpha, \theta)$, and letting $\bar{L}_t = L_t/L_1$, there is the equivalence,

$$
P_{\alpha,\theta}(u) \overset{d}{=} \inf\{t : \bar{L}_t \geq u\}, 0 \leq u \leq 1.
$$

In other words, letting $P_{\alpha,\theta}^{(-1)}(\cdot)$ denote the random quantile function of $P_{\alpha,\theta}$, it follows that $\bar{L}_t \overset{d}{=} P_{\alpha,\theta}^{(-1)}(t), 0 \leq t \leq 1$. See the next section, section 6, for more general $V$.

**Remark 5.2.** We note that in reference to Propositions 5.3 and 5.4, setting $Q_{\alpha,\tau}(\sigma, p) = \beta_{\tau,\tau(1-\sigma)} P_{\alpha,\tau\sigma}(p)$ leads to a well defined bivariate process $(Q_{\alpha,\tau}(\sigma, p) : 0 \leq p \leq 1, 0 < \sigma \leq 1)$, that has some natural connections to the coagulation operations discussed in Pitman (78). This observation may be deduced from the subordinator representation given in Pitman and Yor (82), Proposition 21). When $p = 1$, $Q_{\alpha,\tau}(\sigma, 1)$ is a Dirichlet process, which corresponds to the operation of coagulating $PD(\alpha, \tau)$ by $PD(0, \tau/\alpha)$. In general one may write,

$$
Q_{\alpha,\tau}(\sigma, p) \overset{d}{=} P_{0,\tau}(\sigma) P_{\alpha,\tau\sigma}(p) = P_{\alpha,\tau}(pP_0, \sigma(\sigma)).
$$

We will not elaborate on this here except to note that in connection with results for the standard $U$-coalescent, setting $\tau = \alpha, p = 1$ one recovers ((78), Corollary 33, and Proposition 32). In a distributional sense, that is without the nice interpretation, one also recovers ((78), Corollary 16) by setting $\sigma = \alpha = e^{-t}$ and $\tau = 1, p = 1$. When $p \neq 1$, we expect that one can obtain new, but related, interpretations of $Q_{\alpha,\tau}(\sigma, p)$.
5.1. Some special cases

Note that, as in (47) (combined with Proposition 5.2), one can rewrite (5.2) as,

\[ P_{\alpha,\theta}(p) = \beta_{\theta+\alpha,1-\alpha}(1-\xi_p) + (1 - \beta_{\theta+\alpha,1-\alpha}(p))\xi_p \]

Besides giving an alternate mixture representation in terms of easily interpreted random variables, this also suggests that one can obtain the density of \( P_{\alpha,\theta}(p) \) if one knows the density of \( P_{\alpha,\theta+\alpha}(p) \). In (47), it was noted that this could be applied for \( \theta + \alpha = 1 \), which yields an expression for the density of \( P_{\alpha,1-\alpha}(p) \).

In view of Proposition 5.3 we see that such a density representation can be extended to any \( 0 \leq \theta \leq 1 - \alpha \). Of course in terms of a density representation this is not as good as the expression one can obtain from (5.9), since it would have to be used twice. In this section we look at some specific cases of random variables that have either appeared in the literature or we anticipate might be of some interest.

Example 5.1.1 \(((\alpha, 1-\alpha)) A distribution relevant to phylogenetic models).

As noted in (47) the case of \( P_{\alpha,1-\alpha}(p) \) equates in distribution to the limit of a phylogenetic tree model appearing in ((38), Proposition 20). Here using Proposition 5.3 we obtain a slight improvement over the density given in [(47), Corollary (6.1)]. Since under \( P_{\alpha,1-\alpha}(p) \), \( A_G^+ \frac{d}{dy} = \beta_{1,1-\alpha}P_{\alpha,1}(p) \), we have,

\[ P_{\alpha,1-\alpha}(A_G^+ \in dy)/dy = \frac{1-y}{1-p} P_{\alpha,1-\alpha}(A_G^+ \in dy)/dy = \int_0^1 (1-\alpha)\Omega_{\alpha,1}(y/u) \frac{u}{u(1-u)^\alpha} du. \]

Example 5.1.2 (The case of \( \beta_{1+\alpha,1-\alpha}P_{\alpha,1+\alpha}(p) \)).

Under \( P_{\alpha,1}(p) \), \( A_G^+ \frac{d}{dy} = \beta_{1+\alpha,1-\alpha}P_{\alpha,1+\alpha}(p) = \beta_{1,1-\alpha}(p\beta_{\alpha,1-\alpha}(1-\xi_p)) \). Hence its density is given by

\[ P_{\alpha,1}(A_G^+ \in dy)/dy = \frac{1-y}{1-p} \Omega_{\alpha,1}(y). \]

In view of the literature related to section 4 we believe this variable will be of interest.

We now address some harder cases.

Example 5.1.3 \(((\alpha, \alpha)) Occupation time of a Bessel Bridge).

Obtaining density expressions for the general case of \( A_G^+ \) when \( B \) is a Bessel bridge has been difficult, except for the case of \( \alpha = 1/2 \). Due to the importance of the PD(\( \alpha, \alpha \)) family this quantity arises in many contexts, see for instance Aldous and Pitman [1]. The best results were obtained independently by Yano [92] and (47), who give expressions in terms of Abel type transforms. That is to say integrals of possibly non-negative functions. Hence this does not yield a mixture
representation for $P_{\alpha,\alpha}(p) \overset{d}{=} A_1^+$. Here we show how our results in the previous section can be used to achieve this. Under $\mathbb{P}_{\alpha,\alpha}^{(p)}$,

$$A_1^+ \overset{d}{=} \beta_{2\alpha,1-\alpha} P_{\alpha,2\alpha}(p) \overset{d}{=} P_{\alpha,1+\alpha}(p\beta(2,1,\alpha)).$$

Hence for $\alpha \leq 1/2$, we can apply statement[(i)] of Proposition 5.3 writing,

$$A_1^+ \overset{d}{=} \beta_{1,\alpha} P_{\alpha,1}(p\beta(2,1,\alpha))$$

to get

$$\mathbb{P}_{\alpha,\alpha}^{(p)}(A_1^+ \in dy)/dy = \frac{1 - y}{(1 - p)\alpha,\alpha} \mathbb{P}_{\alpha,\alpha}^{(p)}(A_1^+ \in dy)/dy = \int_0^1 \frac{\alpha,\alpha(\alpha,\alpha)}{u(1 - u)^{1-\alpha}} du. \tag{5.11}$$

Where

$$\alpha,\alpha(\alpha,\alpha) = \frac{2 \sin(\pi \alpha)}{\pi} py^{\alpha-1}(1 - y)^\alpha [qy^\alpha + \cos(\pi \alpha)p(1 - y)^\alpha]$$

$$= \frac{py^{\alpha-1} + 2qy^\alpha(1 - y)^\alpha \cos(\pi \alpha) + (1 - y)^{2\alpha} p^2}{y^{\alpha+1}}$$

is the density of $P_{\alpha,1}(p\beta(2,1,\alpha))$.

When $\alpha > 1/2$, we, at present, need to resort to statement [(ii)] of Proposition 5.3. So, for instance, for $\alpha \leq 3/2$, it follows that

$$P_{\alpha,\alpha}(p) \overset{d}{=} \beta_{\alpha,2\alpha/2,\alpha/2}(p) + (1 - \beta_{\alpha,2\alpha/2,\alpha/2})(p)$$

where $P_{\alpha,\alpha}(p)$ and $P_{\alpha,\alpha/2}(p)$ are iid variables having distribution $\mathbb{P}_{\alpha,\alpha/2}^{(p)}(A_1^+ \in dy)$ obtainable from (5.9).

Example 5.1.4 ($(\alpha, \alpha - 1)$, and fragmentation equations.)

Suppose that we are interested in the case of $\theta = \alpha - 1$, that under $\mathbb{P}_{\alpha,\alpha-1}^{(p)}$, $P_{\alpha,\alpha-1}(p) \overset{d}{=} A_1^+$. Of course this only makes sense for $\alpha > 1/2$. Notice that

$$A_1^+ \overset{d}{=} \beta_{2\alpha-1,1-\alpha} P_{\alpha,2\alpha-1}(p)$$

so we can apply (5.9) directly if $\alpha \leq 2/3$. It is then interesting to note what other quantities we can obtain. We can use another stochastic equation that takes the form,

$$P_{\alpha,\theta}(p) \overset{d}{=} \beta_{\theta + \alpha \delta,1-\alpha \delta} P_{\alpha,\theta + \alpha \delta}(p) + (1 - \beta_{\theta + \alpha \delta,1-\alpha \delta})P_{\alpha,-\alpha \delta}(p).$$

for $\theta > -\alpha \delta$, $0 < \delta \leq 1$. Note that this equation is not well known but it is simple to check. Furthermore, a close inspection shows that it is a nice way to code Pitman’s (78) fragmentation. As as special case, set $\delta = (1 - \alpha)/\alpha$ and $\theta = \alpha$, so we obtain

$$P_{\alpha,\alpha}(p) \overset{d}{=} \beta_{1,\alpha} P_{\alpha,1}(p) + (1 - \beta_{1,\alpha})P_{\alpha,-1}(p).$$
Another consideration is the choice of $\theta = 1 - \alpha$, that might be relevant to (5.2), yields

$$P_{\alpha,1-\alpha}(p) \overset{d}{=} \beta_{2(1-\alpha),\alpha}P_{\alpha,2(1-\alpha)}(p) + (1 - \beta_{2(1-\alpha),\alpha})P_{\alpha,\alpha-1}(p).$$

6. Power scaling property and randomly skewed processes

We saw that in the previous section the random processes $P_{\alpha,\tau}(pV)$, where $V$ is a beta variable occurs naturally and plays an interesting role. An interpretation in terms of occupation times of randomly skewed processes is mentioned in Remark 5.1., and an interpretation via coagulation processes is hinted at in Remark 5.2. Also there is the surprising stochastic equation in (5.8). There is also a related result given in Proposition 2.2.. One may wonder if properties of this sort only hold for beta random variables. We show in the next result, which was first obtained in (46), that there is a considerable generalization which leads to some interesting stochastic equations.

**Proposition 6.1.** Let $\mathcal{R} \overset{d}{=} M_{\tau/\alpha}(F_R)$ and $\mathcal{Q} \overset{d}{=} M_{\tau/\alpha}(F_Q)$ denote Dirichlet means with parameters $(\tau/\alpha, R)$ and $(\tau/\alpha, Q)$ where $R$ is a non-negative random variable, and $Q$ is a random variable taking values in $[0,1]$. Equivalently,

$$\mathcal{R} = \beta(1)R + (1 - \beta(1))R; \text{ and } \mathcal{Q} = \beta(1)Q + (1 - \beta(1))Q. \quad (6.1)$$

Which implies that $\mathcal{Q} \overset{d}{=} M_{\tau/\alpha}(F_Q)$ takes it values in $[0,1]$. If $Q$ is a constant then $M_{\tau/\alpha}(F_Q) = Q$. Then the following results hold.

(i) $\mathcal{R}^{1/\alpha}X_{\alpha,\tau} \overset{d}{=} M_{\tau}(F_{X_{\alpha,R^{1/\alpha}}})$, that is,

$$\mathcal{R}^{1/\alpha}X_{\alpha,\tau} \overset{d}{=} \beta_{\tau,1}\mathcal{R}^{1/\alpha}X_{\alpha,\tau} + (1 - \beta_{\tau,1})R^{1/\alpha}X_{\alpha}.$$

(ii) $P_{\alpha,\tau}(Q)$ is a Dirichlet mean with parameters $(\tau, P_{\alpha,0}(Q))$, and satisfies,

$$P_{\alpha,\tau}(Q) \overset{d}{=} \beta_{\tau,1}P'_{\alpha,\tau}(Q') + (1 - \beta_{\tau,1})P_{\alpha,0}(Q) \quad (6.2)$$

where $P'_{\alpha,\tau}(Q')$ is equivalent only in distribution to $P_{\alpha,\tau}(Q)$, but $Q$ is the same variable.

**Proof.** The first result follows by noting

$$\tau\mathbb{E}[(1 + \lambda X_{\alpha}R^{1/\alpha})] = (\tau/\alpha)\mathbb{E}[(1 + \lambda^\alpha R)],$$

which gives the negative logarithm of $C_{\tau}(\lambda; R^{1/\alpha}X_{\alpha,\tau}) = C_{\tau/\alpha}(\lambda^\alpha; R)$. For the second, evaluate $C_{\tau}(\lambda, P_{\alpha,\tau}(Q))$ conditional on $Q$ and then notice similar to Proposition 5.2, that the transform of order $\tau$ coincides with the exponential of

$$\log C_{\tau/\alpha}((1 + \lambda)^\alpha - 1; Q) = -\frac{\tau}{\alpha}\mathbb{E}[(Q^Q)].$$

It remains then to apply (5.1), to get the second equality in (6.3).
Notice that setting $R = \xi_{\sigma}$ and $Q = p\xi_{\sigma}$ we recover Propositions 2.2 and Proposition 5.2. Setting $R = X_{\delta}$ for $0 < \delta < 1$, leads to the identity $X_{\delta,\tau}^{1/\alpha} X_{\alpha,\tau} \overset{d}{=} X_{\alpha,\tau}^{1/\alpha}$, since it follows from known properties of stable random variables that $X_{\delta,\tau}^{1/\alpha} X_{\alpha} \overset{d}{=} X_{\alpha,\delta,\tau}$. Furthermore, if one chooses $Q := Q(u)$ such for each fixed $u$ it satisfies (6.1), and for $0 < u < 1$ it is an exchangeable bridge, that is a random cumulative distribution function, then $P_{\alpha,\tau}(Q(u))$ identifies a coagulation operation as described in Pitman (85), Lemma 5.18, see also Bertoin (8). In particular, one recovers Pitman’s (78) coagulation as follows. Setting $Q := P_{\beta,\tau/\alpha}(u)$, means that $Q = P_{\delta,0}(u)$, leading easily to, $P_{\alpha,\tau}(P_{\delta,0}(u)) \overset{d}{=} P_{\alpha\delta,0}(u)$, which implies

$$P_{\alpha,\tau}(P_{\beta,\tau/\alpha}(u)) \overset{d}{=} P_{\alpha\delta,\tau}(u).$$

We shall discuss other applications of Proposition 6.1 and related identities elsewhere.

7. Subordinators and symmetric generalized Linnik laws and processes

Related to the previous section we first briefly discuss some properties of their corresponding GGC subordinators which shows that they can be used in applications where stable and exponentially tilted stable subordinators are used but are generally much more easy to handle from a practical point of view. This is especially true for the tilted stable subordinators, whose practical usage often requires non-trivial simulation techniques.

Similar to the case of $\chi_{\alpha,\tau}$, it follows that

$$T_{\alpha}(\tau) \overset{d}{=} \gamma_{\tau} R^{1/\alpha} X_{\alpha,\tau} \overset{d}{=} \chi_{\alpha,\tau} R^{1/\alpha} \overset{d}{=} \gamma_{\tau} M_{\tau}(F_{X_{\alpha,\tau}^{1/\alpha}}),$$

and

$$\hat{T}_{\alpha}(\tau) \overset{d}{=} \gamma_{\tau} P_{\alpha,\tau}(Q)$$

are GGC $(\tau, X_{\alpha} R^{1/\alpha})$ and GGC$(\tau, P_{\alpha,0}(Q))$ variables, respectively. Where we are suppressing the fact that both $R$ and $Q$ depend on $(\alpha, \tau)$. In fact $T_{\alpha}(\tau)$ and $\hat{T}_{\alpha}(\tau)$ are GGC subordinators varying in $\tau > 0$. Let $S_{\alpha}(t)$ denote a positive stable subordinator such that $S_{\alpha}(1) \overset{d}{=} S_{\alpha}$, and let $\hat{S}_{\alpha}(t)$ denote the subordinator with

$$-\log \mathbb{E}[e^{-\lambda S_{\alpha}(t)}] = t[(1 + \lambda)^\alpha - 1],$$

so that $\hat{S}_{\alpha}(1) \overset{d}{=} \hat{S}_{\alpha}$ is a random variable with density $e^{-t f_{\alpha}(t)}$. We note that $\hat{S}_{\alpha}$ is much more challenging to simulate than $S_{\alpha}$.

It follows that,

$$S_{\alpha}(\gamma_{\tau} R) \overset{d}{=} T_{\alpha}(\tau) \text{ and } \hat{S}_{\alpha}(\gamma_{\tau} Q) \overset{d}{=} \hat{T}_{\alpha}(\tau).$$

It is evident that all such $T_{\alpha}(\tau)$ involve $X_{\alpha,\tau}$, and one can work with $\chi_{\alpha,\tau}$ instead of $S_{\alpha}$. For $\hat{T}_{\alpha}(\tau)$ the extra randomization allows one to avoid sampling variables.
such as \( \hat{S}_\alpha \). For instance, assuming one can sample \( Q \), then first conditioning on \( Q = p \) one can sample the variables \((\gamma_\tau, P_{\alpha,\tau}(p), Q)\) However, in many cases it is not necessary to sample \( P_{\alpha,\tau}(p) \). For example, using Proposition 5.3, for any \( 0 < \sigma \leq 1 \)

\[
\gamma_\sigma P_{\alpha,\sigma}(p) \overset{d}{=} \gamma_1 P_{\alpha,1}(p\beta(1-\sigma))
\]

In this case one can work with \( \gamma_1 \) and a variable with density \( \Omega_{\alpha,\sigma} \). This also implies that, due to independent increments, one has an explicit description of the finite dimensional distributions. Of course if one can sample \( R \) and \( Q \) directly these variables need not correspond to Dirichlet means. In this sense \( X_{\alpha,\tau} \) and \( P_{\alpha,\tau}(p) \) are basic to \( T_\alpha \) and \( \hat{T}_\alpha \). Naturally if \( R \) and \( Q \) are Dirichlet means, this often leads to additional simplicities, and in many cases it is not necessary to sample \( Q \) or \( R \). This is particularly true if \( Q \) and \( R \) are not too complicated and \( 0 < \tau \leq 1 \). One can also use James (45) to obtain explicit densities. Lastly if \( \lim_{\alpha \to 0} Q \overset{d}{=} Q_0 \) exists then by continuity properties of \( P_{\alpha,\tau} \), (82), section 5.2), for fixed \( \tau \),

\[
\lim_{\alpha \to 0} Q \overset{d}{=} Q_0, \quad \text{and} \quad \lim_{\alpha \to 0} \hat{T}_\alpha(\tau) \overset{d}{=} \gamma_\tau P_{0,\tau}(Q_0).
\]

Hence these \( \text{Lévy} \) processes are attractive in terms of potential applications arising for instance in finance, financial econometrics or Bayesian statistics. With applications to finance in mind, it is quite natural to use these processes as Brownian time changes creating process \( B(T_\alpha(\cdot)) \) and \( B(\hat{T}_\alpha(\cdot)) \), for \( B(\cdot) \) an independent Brownian motion, we take to have log characteristic function \(-\lambda^2\). The characteristic functions of \( B(T_\alpha(\tau)) \) and \( B(\hat{T}_\alpha(\tau)) \), can be expressed as

\[
C_{\hat{\tau}}(\lambda^2, \hat{\sigma}; R) = e^{-\hat{\tau} \psi \hat{\beta}(\lambda^2\hat{\sigma})} \quad \text{and} \quad C_{\hat{\tau}}((1+\lambda^2)^\alpha - 1; Q) = e^{-\hat{\tau} \psi Q((1+\lambda^2)^\alpha - 1)}.
\]

Recall that, \( B(S_\alpha(\cdot)) \) is a symmetric stable process of index \((0, 2]\) and \( B(\hat{S}_\alpha(\cdot)) \), is a process that includes the NIG process when \( \alpha = 1/2 \). When \( \alpha = 0 \) and \( Q = p \), \( B(T_0(\cdot)) \), is a variance-gamma (VG) process. The case of \( B(\chi_{\alpha,\theta}) \), corresponds to generalized \( \text{Linnik} \) processes considered by Pakes (73), see also (27), (61). We will focus on this case.

It suffices to examine the random variables,

\[
B(\chi_{\alpha,\theta}) \overset{d}{=} N \sqrt{2^{1/\alpha} \gamma_{\alpha,\theta}} = N \sqrt{\chi_{\alpha,\theta}},
\]

where \( N \) is a standard Normal random variable. For general \( \alpha \) and \( \theta > 0 \) the extra randomization by \( N \) does not add much beyond our results for \( \chi_{\alpha,\theta} \).

However, when \( \alpha \leq 1/2 \) we are able to obtain some interesting results which we describe below. In this case, we will use a result of Devroye (28) which yields a tractable mixture representation for symmetric stable random variables of index between 0 and 1. Devroye’s (28) result is not well known but as we shall show can be used to obtain a nice description of the density of \( B_\alpha(T_\alpha(\theta)) \) for all fixed \( \theta > 0 \). We do however stress that there are many applications requiring \( \alpha > 1/2 \).
7.1. $\alpha \leq 12$, Results based on Fejer-de la Vallee Poussin mixtures

For symmetric stable random variables $N\sqrt{2S_\alpha}$ for $0 < \alpha \leq 1/2$, Devroye [26] shows that

$$N\sqrt{2S_\alpha} \overset{d}{=} Y/Z,$$

where $Y$ has a Fejer-de la Vallee Poussin density,

$$\omega(x) = \frac{1}{2\pi} \left(\frac{\sin(x/2)}{x/2}\right)^2, -\infty < x < \infty$$

and $Z \overset{d}{=} \gamma_1(1-\xi_{2\alpha}) + \gamma_2\xi_{2\alpha}$. It follows that the density of a symmetric stable of index between $[0, 1/2]$ is,

$$2\alpha \int_0^\infty \omega(xy)y^{2\alpha}e^{-\theta^2(1-2\alpha) + 2\alpha\theta^2}dy.$$

Hence as a mild extension of Devroye ([26], Example B), that is the simple symmetric Linnik variable corresponding to $\theta = \alpha$, we have for all $\theta > 0$,

$$N\sqrt{2\chi_{\alpha,\theta}} \overset{d}{=} Y/\tilde{W}$$

where $\tilde{W} \overset{d}{=} \frac{\gamma_1}{\gamma_{\alpha,\theta}}(1-\xi_{2\alpha}) + \frac{\gamma_2}{\gamma_{\alpha,\theta}}\xi_{2\alpha}$

is a mixture of Pareto variables, having density for $0 < \alpha \leq 1/2$

$$\frac{\theta[(1+2\theta)w + (1-2\alpha)]}{\alpha(1+w)^{2+\theta/\alpha}}, w > 0.$$

Naturally this representation extends to all $\mathcal{B}(T_\alpha(\cdot))$, provided that $\alpha \leq 1/2$. In particular,

$$\mathcal{B}(T_\alpha(\cdot)) \overset{d}{=} Y/\tilde{W}$$

where now $\tilde{W} \overset{d}{=} \frac{\gamma_1}{\gamma_{\alpha,\theta}}(1-\xi_{2\alpha}) + \frac{\gamma_2}{\gamma_{\alpha,\theta}}\xi_{2\alpha}$.

Quite interestingly the density of $\tilde{W}$ only requires information about the Laplace transform of $\gamma_{\theta/\alpha,\theta}$. Let $\psi_R^{(1)}(x)$ and $\psi_R^{(2)}(x)$ denote the first and second derivatives of $\psi_R(x)$. Then the density of $\tilde{W}$ is given by

$$\eta_{\alpha,\theta}(x) = \frac{\theta}{\alpha} e^{-\frac{\theta}{\alpha}\psi_R(x)}[\psi_R^{(1)}(x)(1-2\alpha) + x2\alpha(\psi_R^{(1)}(x))^2\frac{\theta}{\alpha} - \psi_R^{(2)}(x)].$$

From this, we close with an interesting identity,

**Proposition 7.1.** For $0 < \alpha \leq 1/2$, and $\theta > 0$, let $V \overset{d}{=} \beta_{1/2,1/2}$, then for $-\infty < x < \infty$

$$\Phi_{\alpha,\theta}(x) = \mathbb{E} \left[ \frac{|x|}{2V\chi_{\alpha,\theta}} e^{-\frac{x^2}{2V\chi_{\alpha,\theta}}} \right] = \sqrt{\frac{2}{\pi}} \mathbb{E} \left[ |\sqrt{2\chi_{\alpha,\theta}}| e^{-\frac{x^2}{2\chi_{\alpha,\theta}}} \right] = \int_0^\infty \omega(xy)\theta y^{2\alpha}[(1+2\theta)y^{2\alpha} + (1-2\alpha)]dy.$$
Which is just the density of $N\sqrt{2\chi_2^2}$. Additionally, for all fixed $\theta > 0$, the density of $B(T_\alpha(\theta)) \overset{d}{=} N\sqrt{2\chi_2^2}R^{\frac{1}{2}\alpha}$, satisfies

$$\mathbb{E} \left[ \Phi_{\alpha,\theta}(\frac{x}{R^{\frac{1}{2}\alpha}}) \right] = 2\alpha \int_0^\infty \omega(xy) y^{2\alpha} \eta_{\alpha,\theta}(y^{2\alpha}) dy.$$ 

Proof. The result follows from the derivation of the density described above using $\omega$, in combination with a derivation of the density based on $N^2 \overset{d}{=} 2\gamma_1 V$, and additionally Pitman and Yor ([85], eq. (29)).

8. General remark about rational case

When $\alpha$ is rational it is known, see ([21], [91], [96]), that $S_\alpha$ is equivalent in distribution to a product of independent beta and gamma variables. Extending an argument in Chaumont and Yor ([21], p.143-144) using the gamma duplication formula, it follows that for $\alpha = m/n$ for integers, $m, n$, such that $m < n$, and all $\theta > -m/n$,

$$(X_m,\theta)^m \overset{d}{=} \left( \frac{S_m}{S_m,\theta} \right)^m \overset{d}{=} \left( \prod_{k=1}^{m-1} \beta_{\theta} \frac{\frac{m}{n}+\frac{k}{n}}{\frac{m}{n}+\frac{k}{n},\frac{1}{n}} \right) \left( \prod_{k=m}^{n-1} \gamma_{\theta} \frac{\frac{m}{n}+\frac{k}{n}}{\frac{m}{n}+\frac{k}{n}} \right),$$

where all random variables are independent. Additionally

$$(\frac{m}{S_m,\theta})^m \overset{d}{=} n^m \left( \prod_{k=1}^{m-1} \beta_{\theta} \frac{\frac{m}{n}+\frac{k}{n}}{\frac{m}{n}+\frac{k}{n},\frac{1}{n}} \right) \left( \prod_{k=m}^{n-1} \gamma_{\theta} \frac{\frac{m}{n}+\frac{k}{n}}{\frac{m}{n}+\frac{k}{n}} \right).$$

An implication of these relationships is that one may use the result of Springer and Thompson ([88]) to express their densities in terms of Meijer-G functions. In many cases these are equivalent to expressions in terms of generalized Gauss hypergeometric functions. Furthermore, it is known that Laplace transforms of Meijer-G functions are also Meijer-G functions, with known arguments. Hence, Proposition 3.1 and Proposition 5.1 show that in the rational case of $\alpha = m/n$, one may express the generalized Mittag-Leffler functions and densities for $P_{\alpha,\theta}(p)$ in terms of Meijer-G functions. Such representations are not entirely appealing in many respects, for instance the density $\Delta_{m/n,\sigma}$ is a much more desirable expression than its Meijer-G counterpart. However, from a computational viewpoint they are significant. This is due to the fact that Meijer-G functions, which constitute many special functions, are available as built-in functions in mathematical computational packages such as Mathematica or Maple. Naturally many of the quantities we discussed for general $\alpha$ can be expressed as the more general Fox-H functions ([51], [57], [52], [30], [35], [53]). However, in general, computations for these expressions are not yet available. Hence another contribution of our work is to give new explicit identities for a class of Meijer-G and Fox-H functions. That is to say quantities such as $\Delta_{\alpha,\sigma}$ give an explicit form to their corresponding Fox-H representation. We omit details of this representation, but it is not difficult to obtain.
9. An example from time series

This last section is based on a question posed by Richard Davis which illustrates another case where additional randomization is helpful, albeit not obviously so. Davis and Resnick (25) [see also (18), Theorem 13.3.1, p. 538-539.], shows that in the infinite variance case, the sample correlation function of a moving average process converges to a variable of the form,

$$\frac{B(S_0)}{S_0} = N \sqrt{S_0},$$

for all $0 < \alpha < 1$, and where we omit various constants. The appearance of the square root seems to complicate matters. Looking at the square we get

$$N^2 \frac{S'_\alpha}{S_\alpha} = 2^{\gamma_1/2} \frac{S'_\alpha}{S_\alpha} \frac{1}{2} \frac{S_\alpha}{S_{\alpha/2}},$$

where the last equality follows by noting that $4^{\gamma_1/2} = 1/S_{1/2}$, and $S'^2_{\alpha} S_{1/2} \overset{d}{=} S_{\alpha/2}$. There are essentially two options left. Writing $S_{\alpha/2} \overset{d}{=} S'_{\alpha} S_{1/2} \overset{d}{=} S_{1/2}^{1/\alpha}$, one gets

$$\frac{S'_\alpha}{S_{\alpha/2}} \overset{d}{=} \frac{S'_\alpha}{S_{1/2}^{1/2} S_{\alpha}} \overset{d}{=} S_{1/2}^{-1/\alpha} X_\alpha \overset{d}{=} 4^{1/\alpha} \gamma_1^{1/2} X_\alpha.$$

Notice that the last two expressions consists of variables with known densities. So this illustrates another situation where how one randomizes matters.

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