Uncertainty Principle for Quantum Instruments and Computing

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The notion of quantum instruments is formalized as statistical equivalence classes of all the possible quantum measurements and mathematically characterized as normalized completely positive map valued measures under naturally acceptable axioms. Recently, universally valid uncertainty relations have been established to set a precision limit for any instruments given a disturbance constraint in a form more general than the one originally proposed by Heisenberg. One of them leads to a quantitative generalization of the Wigner-Araki-Yanase theorem on the precision limit of measurements under conservation laws. Applying this, a rigorous lower bound is obtained for the gate error probability of physical implementations of Hadamard gates on a standard qubit of a spin 1/2 system by interactions with control fields or ancilla systems obeying the angular momentum conservation law.

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I. INTRODUCTION

Heisenberg’s uncertainty principle in its original formulation has been understood to set a limitation on measurements by asserting a lower bound of the product of the imprecision of measuring one observable and the disturbance caused in another noncommuting observable. However, the mathematical formulation established by Kennard, Robertson, and Heisenberg merely represents the trade-off between standard deviations of noncommuting observables in a given state and does neither allow such an interpretation, nor has served to provide a universally valid precision limit of measurements. Although such a state of the art has been undoubtedly resulted from the lack of reliable general measurement theory, the recent development of the theory has made possible to establish desirable operational uncertainty relations universally valid for the most general class of quantum measurements, which will be useful for precision measurements, quantum information, and quantum computing. This paper reports the development on the operational uncertainty relations, and their applications to operational decoherence of quantum logic gates based on the authors recent work.

II. QUANTUM INSTRUMENTS

Since von Neumann’s axiomatization of quantum mechanics, we have definite answers to questions as to what are general states and what are general observables. However, the question was left unanswered for long time as to what are general measurements. Towards this problem, Davies and Lewis (DL) introduced the mathematical formulation of the notion of “instrument” as normalized positive map valued measures (DL instruments), and showed that this notion generally describes the statistical properties of measurement, so that joint probability distributions of any sequence of measurements are determined by their corresponding DL instruments. However, the question was left open for some time as to whether every DL instrument corresponds to a possible measuring apparatus. In order to solve this question, Refs. introduced a general class of mathematical models of measuring processes (indirect measurement models) and showed that the statistical properties given by any such model is described by a normalized completely positive map valued measure (CP instrument), and conversely that any CP instrument arises in this way. Thus, we can naturally conclude that measurements are represented by CP instruments, just as states are represented by density operators and observables are represented by self-adjoint operators.

Ref. introduced the notion of statistical equivalence of measurements so that two measuring apparatuses are statistically equivalent if and only if they are interchangeable without affecting joint probability distributions of any sequences of measurements, and reformulated the above characterization of measurements under the following naturally acceptable axioms.

(i) Mixing law: If two apparatuses are applied to a single system in succession, the joint probability distribution of outputs from those two apparatuses depends affinely on the input state.

(ii) Extendability axiom: Every apparatus measuring one system can be trivially extended to an apparatus measuring a larger system including the original system without changing the statistics.

(iii) Realizability postulate: Every indirect measurement model corresponds to an apparatus whose measuring process is described by that model.
Under the above axioms (i)–(iii), it was proven in Ref. 13 that the statistical equivalence classes of apparatuses are in one-to-one correspondence with the CP instruments. Thus, we established the notion of “instrument” as the function of a measuring apparatus by the mathematical notion “CP instrument” that represents the statistical equivalence class of a measuring apparatus. In this paper, we shall thus define “instruments” as CP instruments.

Let \( \mathcal{H} \) be a Hilbert space. A map \( \Pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \) is called a probability operator valued measure (POVM) for \((\mathcal{H}, \mathcal{R}^d)\), where \( \mathcal{B}(\mathcal{H}) \) stands for the Borel \( \sigma \)-field of the Euclidean space \( \mathcal{H} \) and \( \mathcal{L}(\mathcal{H}) \) stands for the space of bounded linear operators on \( \mathcal{H} \), if it satisfies the following conditions: (i) For any disjoint sequence \( \Delta_1, \Delta_2, \ldots \in \mathcal{B}(\mathcal{H}) \), we have \( \sum_{i=1}^{\infty} \Pi(\Delta_i) = \sum_{i=1}^{\infty} 1_{\Pi(\Delta_i)} \), where the sum is convergent in the weak operator topology. (ii) \( \Pi(\mathcal{R}^d) = I \).

A linear transformation \( T : \mathcal{H} \otimes \mathcal{R}^d \to \mathcal{H} \otimes \mathcal{R}^d \) is called completely positive (CP), where \( \tau_c(\mathcal{H}) \) stands for the space of trace class operators on \( \mathcal{H} \), if it satisfies the following conditions: (i) For any disjoint sequence \( \Delta_1, \Delta_2, \ldots \in \mathcal{B}(\mathcal{H}) \), the relation

\[
\Pi(\Delta_i) = \Pi(\Delta_i) \otimes I
\]

defines a probability measure on \( \mathcal{B}(\mathcal{H}) \), which called the output probability distribution of \( \Pi \) in \( \rho \). Let \( \Delta \in \mathcal{B}(\mathcal{H}) \). The CP map \( \Pi(\Delta) \) is called the operation of \( \Pi \) given \( \Delta \), and \( \Pi(\mathcal{R}^d) \) is called the nonselective operation of \( \Pi \). For any Borel set \( \Delta \in \mathcal{B}(\mathcal{H}) \) and state \( \rho \) with \( \text{Tr}[\Pi(\Delta)\rho] > 0 \), the state

\[
\rho \Delta = \frac{\Pi(\Delta)\rho}{\text{Tr}[\Pi(\Delta)\rho]}
\]

is called the output state of \( \Pi \) for the input state \( \rho \) given \( \Delta \).

A finite set \( \{A_1, \ldots, A_n\} \) of observables are called compatible, if

\[
\left[ E^{A_i}(\Delta_1), E^{A_j}(\Delta_2) \right] = 0
\]

for all \( i, j = 1, \ldots, n \) and \( \Delta_1, \Delta_2 \in \mathcal{B}(\mathcal{H}) \), where \( E^{A_i} \) stands for the spectral measure corresponding to \( A_j \).

In this case, we shall write \( [A_i, A_j] = 0 \) for any \( i, j \).

In this paper, by a measuring process we shall generally mean an experiment described as follows. Let \( \mathbf{P} \) be a quantum system, called a probe system, described by a Hilbert space \( \mathcal{K} \). The system \( \mathbf{P} \) is coupled to the system \( \mathbf{S} \) during a finite time interval \((t, t + \Delta t)\). Denote by \( U \) the unitary operator on \( \mathcal{H} \otimes \mathcal{K} \) corresponding to the time evolution of the system \( \mathbf{S} \otimes \mathbf{P} \) for the time interval \((t, t + \Delta t)\). At time \( t \), the time of measurement, the probe system \( \mathbf{P} \) is prepared in a fixed state \( \sigma \). At time \( t + \Delta t \), the time just after the measuring interaction, the systems \( \mathbf{S} \) and \( \mathbf{P} \) are separated and a compatible observables \( M_1, \ldots, M_d \) of the system \( \mathbf{P} \), called the meter observables, are measured precisely. Thus, any measuring process is characterized by a \((3+d)\)-tuple \( \mathcal{M} = (\mathcal{K}, \sigma, U, M_1, \ldots, M_d) \) consisting of a Hilbert space \( \mathcal{K} \), a density operator \( \sigma \) on \( \mathcal{K} \), a unitary operator \( U \) on \( \mathcal{H} \otimes \mathcal{K} \) and a compatible sequence \( (M_1, \ldots, M_d) \) of self-adjoint operators on \( \mathcal{K} \). Each measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, U, M_1, \ldots, M_d) \) determines a unique instrument \( \mathcal{I} : \mathcal{B}(\mathcal{R}^d) \to \mathcal{CP}_K \), called the instrument of \( \mathcal{M} \), by the following relation

\[
\mathcal{I}(\Delta_1 \times \cdots \times \Delta_d)\rho = \text{Tr}_{\mathcal{K}} \left\{ [1 \otimes E^{M_1}(\Delta_1) \cdots E^{M_d}(\Delta_d)] U(\rho \otimes \sigma) U^\dagger \right\},
\]

for all \( \rho \in \tau_c(\mathcal{H}) \) and \( \Delta_1, \ldots, \Delta_d \in \mathcal{B}(\mathcal{R}) \), where \( \text{Tr}_{\mathcal{K}} \) stands for the partial trace operation of \( \mathcal{K} \).
Theorem II.1 Every instrument $I : B(\mathbb{R}^d) \rightarrow CP_{\mathcal{H}}$ has at least one realization $\mathcal{M} = (K, \sigma, U, M_1, \ldots, M_d)$ such that $\sigma$ is a pure state.

The following theorems$^{11,12}$ are immediate consequences from the above theorem; Theorem II.3 was also obtained by Kraus$^8$ independently.

Theorem II.2 For any POVM $\Pi : B(\mathbb{R}^d) \rightarrow \mathcal{L}(\mathcal{H})$, there exists a measuring process $\mathcal{M} = (K, |\xi\rangle \langle \xi|, U, M_1, \ldots, M_d)$ satisfying the relation

$$\Pi(\Delta_1 \times \cdots \times \Delta_d) = \langle \xi | U^\dagger [I \otimes M_1(\Delta_1) \cdots M_d(\Delta_d)] U | \xi \rangle$$

for all $\Delta_1, \ldots, \Delta_d \in B(\mathbb{R}^d)$, where $\langle \xi | \cdots | \xi \rangle$ stands for the partial inner product such that $\langle \psi | \xi \cdots | \xi \rangle \psi = (\psi \otimes \xi) \cdots (\psi \otimes \xi)$.

Theorem II.3 For any trace preserving CP map $T : \tau c(\mathcal{H}) \rightarrow \tau c(\mathcal{H})$, there exist a Hilbert space $K$, a unit vector $\xi \in K$, and a unitary operator $U$ on $\mathcal{H} \otimes K$ satisfying the relation

$$T \rho = \text{Tr}_K[U(\rho \otimes |\xi\rangle \langle \xi|)U^\dagger]$$

for all $\rho \in \tau c(\mathcal{H})$.

Theorem II.4 For any normal unit preserving CP map $T : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$, there are a Hilbert space $K$, a unit vector $\xi \in K$, and a unitary operator $U$ on $\mathcal{H} \otimes K$ satisfying the relation

$$T a = \langle \xi | U^\dagger (a \otimes I) U | \xi \rangle$$

for all $a \in \mathcal{L}(\mathcal{H})$.

Let $\Pi$ be a POVM for $(\mathcal{H}, \mathcal{B})$. Let $f(x)$ be a real Borel function on $\mathbb{R}$. Denote by $\int f(x) d\Pi(x)$, or $\int f d\Pi$ for short, the symmetric operator defined by

$$\langle \xi | \int f(x) d\Pi(x) | \eta \rangle = \int f(x) d\langle \xi | \Pi(x) | \eta \rangle$$

for any $\xi, \eta \in \text{dom}(\int f(x) d\Pi(x))$, where the domain is defined by

$$\text{dom} \left( \int f(x) d\Pi(x) \right) = \left\{ \xi \in \mathcal{H} \mid \int f(x)^2 d\langle \xi | \Pi(x) | \xi \rangle < \infty \right\}.$$  

The $n$-th moment operator of $\Pi$, denoted by $O^{(n)}(\Pi)$, is defined by

$$O^{(n)}(\Pi) = \int x^n d\Pi(x).$$

We shall write $O(\Pi) = O^{(1)}(\Pi)$. The mean $\langle O(\Pi) \rangle$ and the standard deviation $\Delta(\Pi)$ of POVM $\Pi$ is given, if the integral converges, by

$$\langle O(\Pi) \rangle = \text{Tr}[O(\Pi) \rho],$$

$$\Delta(\Pi) = \left( \langle O^{(2)}(\Pi) \rangle - \langle O(\Pi) \rangle^2 \right)^{1/2}. $$

For any observable $A$, we have $A = O(E^A)$ and the mean of $A$ in state defined by $\langle A \rangle = \text{Tr}[A \rho]$ satisfies $\langle A \rangle = \langle O(E^A) \rangle$. The standard deviation of $A$ in state defined by $\Delta A = (\langle A^2 \rangle - \langle A \rangle^2)^{1/2}$ satisfies $\Delta A = \Delta(E^A)$. By the Robertson uncertainty relation$^{9}$ for any state $\rho$ and any pair of observables $A, B$ with $\Delta A, \Delta B < \infty$, we have

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|,$$

where $\langle [A, B] \rangle = \text{Tr}\{[A, B] \rho\}$.

### III. OPERATIONAL UNCERTAINTY RELATIONS

In this section, we generalize Heisenberg’s noise-disturbance uncertainty relation to a relation that holds for any instruments, from which conditions are obtained for measuring instruments to satisfy Heisenberg’s relation$^{11,12,13}$. In particular, every instrument with the noise and the disturbance uncorrelated with the measured object is proven to satisfy Heisenberg’s relation$^{13}$.
A. Uncertainty relations for joint direct measurements

Let \( \mathcal{W} \) be a Hilbert space and \( \sigma \) be a density operator on \( \mathcal{W} \). Let \( A, B \) be two observables on \( \mathcal{W} \). The noise operator \( N(C, A) \) for \( C \) in measuring \( A \) and the root-mean-square noise \( \epsilon(C, A, \sigma) \) for \( C \) in measuring \( A \) in \( \sigma \) are defined by

\[
N(A, C) = C - A, \\
\epsilon(A, C, \sigma) = \text{Tr}[N(A, C)^2]^{1/2}.
\]

Under the above definitions, we have the following.\(^{11}\)

**Theorem III.1** For any four observables \( A, B, C, D \) on \( \mathcal{W} \), if \( C \) and \( D \) are commuting, we have

\[
\epsilon(A) \epsilon(B) + \epsilon(A) \Delta B + \Delta A \epsilon(B) \geq \Delta N_A \Delta N_B + \Delta N_A \Delta B + \Delta A \Delta N_B \\
\geq \frac{1}{2} |\langle [N_A, B] \rangle| + \frac{1}{2} |\langle [A, N_B] \rangle| \\
\geq \frac{1}{2} |\langle [A, B] \rangle| \tag{21}
\]

for any state \( \sigma \) for which all the relevant terms are finite, where \( N_A = N(A, C) \), \( N_B = N(B, D) \), \( \epsilon(A) = \epsilon(A, C, \sigma) \), \( \epsilon(B) = \epsilon(B, D, \sigma) \), \( \Delta \) stands for the standard deviation in \( \sigma \), and \( \langle \cdots \rangle \) stands for the mean value in \( \sigma \).

**Proof.** By definition, we have

\[
C = A + N_A, \\
D = B + N_B. \tag{22, 23}
\]

From \( [C, D] = 0 \), we have the following commutation relation for noise operators,

\[
[N_A, N_B] + [N_A, B] + [A, N_B] = -[A, B]. \tag{24}
\]

Taking the modulus of means of the both sides and applying the triangular inequality, we have

\[
|\langle [N_A, N_B] \rangle| + |\langle [N_A, B] \rangle| + |\langle [A, N_B] \rangle| \geq |\langle [A, B] \rangle|. \tag{25}
\]

By Robertson’s inequality, Eq. \(10\), we have

\[
\Delta N_A \Delta N_B \geq \frac{1}{2} |\langle [N_A, N_B] \rangle|, \\
\Delta A \Delta N_B \geq \frac{1}{2} |\langle [A, N_B] \rangle|, \\
\Delta N_A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|, \tag{26-28}
\]

so that inequalities \(20\) and \(21\) follow. Since the variance is not greater than the mean square, we have

\[
\epsilon(A) \geq \Delta N_A, \tag{29}
\]

\[
\epsilon(B) \geq \Delta N_B, \tag{30}
\]

and hence inequality \(19\) follows. \( \Box \)

B. Uncertainty relations for joint indirect measurements

Let \( \mathcal{H} \) be a Hilbert space and \( \rho \) be a density operator on \( \mathcal{H} \). Let \( A \) be an observable on \( \mathcal{H} \) and let \( \Pi^A \) be any POVM on \( \mathcal{H} \). The mean noise operator \( n(A, \Pi^A) \) for \( \Pi^A \) in measuring \( A \), the mean noise \( \bar{n}(A, \Pi^A) \), the root-mean-square noise \( \epsilon(\Pi^A, A, \rho) \), and the standard deviation \( \Delta N(\Pi^A, A, \rho) \) of the noise for \( \Pi^A \) in measuring \( A \) in \( \rho \) are defined by

\[
n(A, \Pi^A) = O(\Pi^A) - A, \tag{31}
\]

\[
\bar{n}(A, \Pi^A, \rho) = \langle n(A, \Pi^A) \rangle, \tag{32}
\]

\[
\epsilon(A, \Pi^A, \rho) = \langle O^2(\Pi^A) - O(\Pi^A)A - AO(\Pi^A) + A^2 \rangle^{1/2}, \tag{33}
\]

\[
\Delta N(A, \Pi^A, \rho) = [\epsilon(A, \Pi^A, \rho)^2 - \bar{n}(A, \Pi^A)^2]^{1/2}. \tag{34}
\]
where $\langle \cdots \rangle$ stands for the mean value in the state $\rho$, i.e., $\langle \cdots \rangle = \text{Tr}[\cdots \rho]$.

By the Naimark theorem, there exist a Hilbert space $W$, an isometry $V : H \to W$, and a self-adjoint operator $C$ such that

$$\Pi^A(\Delta) = V^\dagger E^C(\Delta)V$$

(35)

for every Borel set $\Delta$. We shall call any triple $(W, V, C)$ satisfying Eq. (35) a Naimark extension of $\Pi^A$. Then, we have the following.

**Theorem III.2** For any Naimark extension $(W, V, C)$ of a POVM $\Pi^A$ on $H$, we have

$$n(A, \Pi^A) = V^\dagger CV - A,$$  
(36)

$$\bar{n}(A, \Pi^A) = \text{Tr}[(V^\dagger CV - A)\rho],$$  
(37)

$$\epsilon(A, \Pi^A, \rho) = ||CV\sqrt{\rho} - VA\sqrt{\rho}||_{HS},$$  
(38)

$$\Delta N(A, \Pi^A, \rho) = ||CV\sqrt{\rho} - VA\sqrt{\rho} - \bar{n}(A, \Pi^A)\sqrt{\rho}||_{HS},$$  
(39)

where $||\cdots||_{HS}$ stands for the Hilbert-Schmidt norm.

The following theorem characterizes POVMs with zero-noise.

**Theorem III.3** For any POVM $\Pi^A$ on $H$ and any observable $A$ on $H$, the following conditions are equivalent.

(i) $\Pi^A = E^A$.

(ii) $\epsilon(A, \Pi^A, \rho) = 0$ for any state $\rho$.

(iii) $\epsilon(A, \Pi^A, \rho) = 0$ for a faithful state $\rho$.

(iv) $\epsilon(A, \Pi^A, |n\rangle) = 0$ for any $|n\rangle$ in an orthonormal basis $\{|n\rangle\}$.

(v) $\epsilon(A, \Pi^A, \psi) = 0$ for any state vector $\psi \in H$.

We call any POVM for $(H, R^2)$ the joint POVM for $H$. The marginal POVMs $(\Pi^A, \Pi^B)$ of joint POVM $\Pi$ are defined by $\Pi^A(\Delta) = \Pi(\Delta \times R)$ and $\Pi^B(\Gamma) = \Pi(R \times \Gamma)$ for any $\Delta, \Gamma \in B(R)$.

Under the above definitions, we have the following.

**Theorem III.4** For any two observables $A, B$ on $H$, and joint POVM $\Pi$ for $H$ with marginal POVMs $(\Pi^A, \Pi^B)$, we have

$$\epsilon(A) \epsilon(B) + \epsilon(A) \Delta B + \Delta A \epsilon(B)$$

$$\geq \Delta N_A \Delta N_B + \Delta N_A \Delta B + \Delta A \Delta N_B$$

(40)

$$\geq \frac{1}{2}||[n_A, B]|| + \frac{1}{2}||[A, n_B]||$$

(41)

$$\geq \frac{1}{2}||[A, B]||$$

(42)

for any state $\rho$ for which all the relevant terms are finite, where $n_A = n(A, \Pi^A)$, $n_B = n(B, \Pi^B)$, $\epsilon(A) = \epsilon(A, \Pi^A, \rho)$, $\epsilon(B) = \epsilon(B, \Pi^B, \rho)$, $\Delta N_A = \Delta N(A, \Pi^A, \rho)$, $\Delta N_B = \Delta N(B, \Pi^B, \rho)$, while $\Delta A, \Delta B$ stand for the standard deviations in $\rho$, and $\langle \cdots \rangle$ stands for the mean value in $\rho$.

**Proof.** Let $(K, \xi, U, M_1, M_2)$ be a realization of $\Pi$ given in Theorem II.2. By defining $C = U^\dagger(I \otimes M_1)U$ and $D = U^\dagger(I \otimes M_2)U$ in Eq. (38), we have commuting observables $C, D$ on $H \otimes K$ such that $\Pi(\Delta \times \Gamma) = \langle \xi | E^C(\Delta)E^D(\Gamma) | \xi \rangle$ for any $\Delta, \Gamma \in B(R)$. Then, from Theorem II.2 we have

$$n(A, \Pi^A) = \langle \xi | N(\hat{A}, C) | \xi \rangle,$$  
(43)

$$\epsilon(A, \Pi^A, \rho) = \epsilon(\hat{A}, C, \rho \otimes |\xi\rangle\langle\xi|),$$  
(44)

$$\Delta A = \Delta \hat{A},$$  
(45)

$$\Delta N(A, \Pi) = \Delta N(\hat{A}, C)$$  
(46)

and analogous relations for $B$ and $D$. By the relations

$$\langle N(\hat{A}, C) \hat{B} \rangle = \text{Tr}\{N(\hat{A}, C)(B \rho) \otimes |\xi\rangle\langle\xi|)\}$$

$$= \text{Tr}\{\text{Tr}_K[N(\hat{A}, C)(I \otimes |\xi\rangle\langle\xi|)](B \rho)\}$$

$$= \text{Tr}[|\xi\rangle N(\hat{A}, C) |\xi\rangle B \rho]$$

$$= \langle n(A, \Pi^A) B \rangle,$$
we have
\[
\langle [N(\hat{A}, C), \hat{B}] \rangle = \langle [n(A, \Pi^A), B] \rangle.
\] (47)
Similarly, we also have
\[
\langle [\hat{A}, N(\hat{B}, D)] \rangle = \langle [A, n(B, \Pi^B)] \rangle.
\] (48)
Therefore, by substituting the above relations, the assertion follows from Theorem III.1. QED

From the above, if \( \Pi \) precisely measures \( A \), i.e., \( \epsilon(A) = 0 \), we have
\[
\Delta A \epsilon(B) \geq \frac{1}{2} |\langle [A, B] \rangle|.
\] (49)

We say that POVM \( \Pi^A \) has uncorrelated noise for \( A \), if the mean noise \( \bar{n}(A, \Pi^A, \rho) \) does not depend on the input state \( \rho \), or equivalently, if the mean noise operator \( n(A, \Pi) \) is a constant operator, i.e., \( n(A, \Pi) = rI \) for some \( r \in \mathbb{R} \). We say that POVM \( \Pi^A \) makes an unbiased measurement of \( A \), if \( n(A, \Pi) = 0 \), so that if \( \Pi \) makes an unbiased measurement of \( A \), then \( \Pi \) has uncorrelated noise for \( A \). For the \( B \) measurement, the corresponding definitions on uncorrelated noise and unbiased measurements are introduced analogously.

The relations \( n_A = rI \) and \( n_B = r'I \) obviously imply \( [n_A, B] = [A, n_B] = 0 \), and hence from Theorem III.4 we conclude the following.

**Theorem III.5** If the marginal observables \( (\Pi^A, \Pi^B) \) of a joint POVM \( \Pi \) have uncorrelated noises for \( A \) and \( B \), respectively, then we have
\[
\epsilon(A)\epsilon(B) \geq \Delta N_A \Delta N_B \geq \frac{1}{2} |\langle [A, B] \rangle|.
\] (50)
for any state \( \rho \).

The above relations were previously proven for the unbiased case in Refs. 23, 24.

If \( \Pi \) has uncorrelated noise for both \( A \) and \( B \), we have
\[
\Delta(\Pi^A)^2 = (\Delta A)^2 + (\Delta N_A)^2 \geq 2\Delta A \Delta N_A,
\] (51)
\[
\Delta(\Pi^B)^2 = (\Delta B)^2 + (\Delta N_B)^2 \geq 2\Delta B \Delta N_B,
\] (52)
and hence apply Eq. (10) and Eq. (50) to the product of the above two inequalities, we have
\[
\Delta(\Pi^A)\Delta(\Pi^B) \geq |\langle [A, B] \rangle|.
\] (53)
The above relation has been previously proven for the unbiased case in Ref. 25.

**C. Uncertainty relations for instruments**

Let \( \mathcal{H} \) be a Hilbert space and \( \rho \) be a density operator on \( \mathcal{H} \). Let \( B \) be an observables on \( \mathcal{H} \). Let \( T \) be a trace-preserving operation for \( \mathcal{H} \). The POVM \( T^*E^B \) is defined by
\[
(T^*E^B)(\Delta) = T^*[E^B(\Delta)].
\] (54)
We have \( T^*(B^n) = O(n)(T^*E^B) \), if \( B \) is bounded. The mean disturbance operator \( d(B, T) \) of \( B \) for \( T \), the root-mean-square disturbance \( \eta(B, T, \rho) \) of \( B \) for \( T \) in \( \rho \), and the standard deviation \( \Delta D(B, T, \rho) \) of the disturbance of \( B \) for \( T \) in \( \rho \) are defined by
\[
d(B, T) = n(B, T^*E^B),
\] (55)
\[
\eta(B, T, \rho) = \epsilon(B, T^*E^B),
\] (56)
\[
\Delta D(B, T, \rho) = \Delta N(B, T^*E^B, \rho).
\] (57)
Under the above definitions, we have the following universal noise-disturbance uncertainty relations.11
Theorem III.6 Let $A, B$ be two observables on $\mathcal{H}$. For any instrument $I$ with POVM $\Pi$ and nonselective operation $T$, we have
\begin{equation}
\epsilon(A) \eta(B) + \epsilon(A) \Delta B + \Delta A \eta(B) \\
\geq \Delta N_A \Delta D_B + \Delta N_A \Delta B + \Delta A \Delta D_B \\
\geq \Delta N_A \Delta D_B + \frac{1}{2}|\langle [n_A, B]\rangle| + \frac{1}{2}|\langle [A, d_B]\rangle| \\
\geq \frac{1}{2}|\langle [A, B]\rangle| 
\end{equation}
for any state $\rho$ for which all the relevant terms are finite, where $n_A = n(A, \Pi)$, $d_B = n(B, T)$, $\epsilon(A) = \epsilon(A, \Pi, \rho)$, $\eta(B) = \epsilon(B, T, \rho)$, and $\Delta D_B = \Delta D(B, T, \rho)$.

Let $I$ be an instrument with POVM $\Pi$ and nonselective operation $T$. We say that instrument $I$ has uncorrelated noise for $A$, if the POVM $\Pi$ has uncorrelated noise, i.e., $n(A, \Pi) = rI$ for some $r \in \mathbb{R}$. We say that instrument $I$ has uncorrelated disturbance for $B$, if the mean disturbance operator $d(B, T)$ is a constant operator, i.e., $d(B, T) = rI$ for some $r \in \mathbb{R}$.

We say that an instrument $I$ makes an unbiased measurement of $A$, if $n(A, \Pi) = 0$ and it makes an unbiased disturbance of $B$, if $d(B, T) = 0$.

The universal noise-disturbance uncertainty relations lead to rigorous conditions on what instrument satisfies Heisenberg’s noise-disturbance uncertainty relation, as follows.

Theorem III.7 Let $A$ and $B$ be a pair of observables. An instrument $I$ satisfies Heisenberg’s noise-disturbance uncertainty relation, i.e.,
\begin{equation}
\epsilon(A) \eta(B) \geq \frac{1}{2}|\langle [A, B]\rangle| 
\end{equation}
for any state $\rho$ for which all the relevant terms are finite, if one of the following conditions holds:

(i) The mean noise operator commutes with $B$ and the mean disturbance operator commutes with $A$, i.e.,
\begin{align}
[n_A, B] &= 0, \\
[d_B, A] &= 0.
\end{align}

(ii) The instrument $I$ has both uncorrelated noise for $A$ and uncorrelated disturbance for $B$.

(iii) The instrument $I$ makes both unbiased measurement of $A$ and unbiased disturbance of $B$.

For the general case, we have the following trade-off relations for precise $A$ measurements or $B$-non-disturbing measurements.

Theorem III.8 For any instrument $I$ and observables $A$ and $B$, if $\eta(B) = 0$, we have
\begin{equation}
\epsilon(A) \Delta B \geq \frac{1}{2}|\mathrm{Tr}([A, B]\rho)| 
\end{equation}
for any state $\rho$ for which all the relevant terms are finite.

Theorem III.9 For any apparatus $A(x)$ and observables $A$ and $B$, if $\epsilon(A) = 0$, we have
\begin{equation}
\Delta A \eta(B) \geq \frac{1}{2}|\langle [A, B]\rangle| 
\end{equation}
for any state $\rho$ for which all the relevant terms are finite.

IV. WIGNER-ARAKI-YANASE THEOREM

Every interaction brings an entanglement in the basis of a conserved quantity, so that measurements, and any other quantum state controls such as quantum information processing, are subject to the decoherence induced by conservation laws. One of the earliest formulations of this fact was given by the Wigner-Araki-Yanase (WAY) theorem.
stating that any observable which does not commute with an additive conserved quantity cannot be measured with absolute precision.

It is natural to expect that the WAY theorem can be derived by Heisenberg’s uncertainty principle. However, Heisenberg’s relation concludes that if the measurement does not disturb the total momentum, the position cannot be measured even with finite precision, despite that we can do with finite or even arbitrarily small noise. Actually, the WAY theorem does not conclude unmeasurability of any observables, but merely sets the accuracy limit of the measurement with size limited apparatus in the presence of bounded conserved quantities.

We show that the above new formulation of the universal noise-disturbance uncertainty relation can be used to derive the quantitative expression of the WAY theorem as follows.

**Theorem IV.1** Let $\mathcal{M} = (K, \sigma, U, M)$ be an indirect measurement model for $\mathcal{H}$ and let $\epsilon(A)$ be the root-mean-square noise for this measurement in a state $\rho$, i.e., $\epsilon(A) = \epsilon(A, \Pi_0, \rho)$, where $\Pi(\Delta) = \text{Tr}_K [U^\dagger (I \otimes E^M(\Delta)) U (I \otimes \sigma)]$. Let $L_1$ and $L_2$ be a pair of additive conserved quantities on $\mathcal{H}$ and $\mathcal{K}$, respectively, i.e., $[U, \hat{L}_1 + \hat{L}_2] = 0$, where $\hat{L}_1 = L_1 \otimes I_K$ and $\hat{L}_2 = I_H \otimes L_2$. Suppose that the meter observable $M$ commutes with the conserved quantity, i.e., $[M, L_2] = 0$. Then, we have

$$\epsilon(A)^2 \geq \frac{\langle [A, L_1] \rangle^2}{4(\Delta L_1)^2 + 4(\Delta L_2)^2},$$

where the mean and standard deviations are taken in the state $\rho \otimes \sigma$.

**Proof.** Let $W = L^2(\mathbb{R})$ be the Hilbert space of one-dimensional mass with position $\hat{q}$ and momentum $\hat{p}$. Let $\alpha > 0$ be an arbitrary positive number and let $\xi$ be a state vector in $W$ such that $\langle \xi | \hat{q}^2 | \xi \rangle < \alpha^2$. Consider the indirect measurement model

$$\mathcal{M}_0 = [W, |\xi\rangle \langle \xi|, (I_K \otimes e^{-iM \otimes \hat{p}/\hbar})(U \otimes I_W), \hat{q}]$$

for $\mathcal{H} \otimes \mathcal{K}$ and let $I_0$ be the corresponding instrument with POVM $I_0$ and nonselective operation $T_0$. Then, we have

$$T^*_0[E^{\hat{L}_1 + \hat{L}_2}(\Delta)] = \text{Tr}_W [U^\dagger (E^{\hat{L}_1 + \hat{L}_2}(\Delta) \otimes I_W)U_2U_1]$$

where $U_1 = U \otimes I_W$ and $U_2 = I_H \otimes e^{-iM \otimes \hat{p}/\hbar}$. By assumption, we have $[U_1U_2, (\hat{L}_1 + \hat{L}_2) \otimes I_W] = 0$, so that we have

$$\eta(\hat{L}_1 + \hat{L}_2, T_0, \rho \otimes \sigma) = 0.$$  

Thus, from Theorem [11,13] we have

$$\epsilon(\hat{A}, \Pi_0, \rho \otimes \sigma) \Delta(\hat{L}_1 + \hat{L}_2) \geq \frac{1}{2} \langle [\hat{A}, \hat{L}_1 + \hat{L}_2] \rangle,$$

where $\hat{A} = A \otimes I_K$. We have $\langle [\hat{A}, \hat{L}_1 + \hat{L}_2] \rangle = \langle [A, L_1] \rangle$ and $\langle \Delta(\hat{L}_1 + \hat{L}_2) \rangle^2 = (\Delta L_1)^2 + (\Delta L_2)^2$. Thus, we have

$$\epsilon(\hat{A}, \Pi_0, \rho \otimes \sigma)^2 \geq \frac{\langle [A, L_1] \rangle^2}{4(\Delta L_1)^2 + 4(\Delta L_2)^2}.$$  

Let $\rho_0 = \rho \otimes \sigma \otimes |\xi\rangle \langle \xi|$. Then, we have

$$\|U^\dagger_1U^\dagger_2 (I_H \otimes I_K \otimes \hat{q}) U_2U_1 \sqrt{\rho_0} - U^\dagger_1 (I_H \otimes I_W) U_1 \sqrt{\rho_0} \|_{HS}$$

$$= \|U^\dagger_1 (I_H \otimes I_W + I_H \otimes I_K \otimes \hat{q}) U_1 \sqrt{\rho_0} - U^\dagger_1 (I_H \otimes I_W) U_1 \sqrt{\rho_0} \|_{HS}$$

$$= \|U^\dagger_1 (I_H \otimes I_K \otimes \hat{q}) U_1 \sqrt{\rho_0} \|_{HS}$$

$$= \langle \xi | \hat{q}^2 | \xi \rangle^{1/2}$$

$$< \alpha.$$  

We also have

$$\|U^\dagger_1 (I_H \otimes I_W) U_1 \sqrt{\rho_0} - (A \otimes I_K \otimes I_W) \sqrt{\rho_0} \|_{HS}$$

$$= \|U^\dagger_1 (I_H \otimes I_W) U_1 \sqrt{\rho_0} \|_{HS} - \|U^\dagger_1 (A \otimes I_K \otimes I_W) \sqrt{\rho_0} \|_{HS}$$

$$= \epsilon(A).$$
It follows that we have
\[
\epsilon(A \otimes I_K, \Pi_0, \rho \otimes \sigma) \\
= \| U_1^\dagger U_1^\dagger (I_H \otimes | 0 \rangle \langle 0 |) U_2 U_1 \sqrt{\rho_0} - \rho \|_{HS} \\
\leq \| U_1^\dagger (I_H \otimes M) U \sqrt{\rho} \otimes \sigma - (A \otimes I_K) \sqrt{\rho} \otimes \sigma \|_{HS} \\
+ \| U_1^\dagger U_1^\dagger (I_H \otimes | 0 \rangle \langle 0 |) U_2 U_1 \sqrt{\rho_0} + U_1^\dagger (I_H \otimes M \otimes I_W) U_1 \sqrt{\rho_0} \|_{HS} \\
< \epsilon(A) + \alpha.
\]
Since \( \alpha \) is arbitrary, we have
\[
\epsilon(A) \geq \epsilon(A \otimes I_K, \Pi_0, \rho \otimes \sigma).
\] (70)
Therefore, the assertion follows from Eqs. (59) and (60). QED

\[\text{V. OPERATIONAL DECOHERENCE IN QUANTUM LOGIC GATES}\]

The current theory of fault-tolerant quantum computing suggests that the most formidable obstacle for realizing a scalable quantum computer is the demand for the high operation precision for each quantum logic gate, rather than the environment induced decoherence on quantum memories. One of the main achievements of this field is the threshold theorem stating that provided the noise in individual quantum gates is below a certain threshold, it is possible to implement quantum Fourier transforms without error correction, so that the required error probability for each Hadamard gates is at most \( O(\log L) \) elementary gates in quantum Fourier transform without error correction, that the required error probability for each Hadamard gates is below \( 1/\sqrt{O(\log L)} \) in average. This suggests that the accuracy of Hadamard gates is indeed a demanding factor in implementing Shor’s algorithm. In what follows, we shall consider the accuracy of implementing Shor’s algorithm. In what follows, we shall show that Hadamard gates are no easier to implement under the angular momentum conservation law than CNOT gates.

Let \( Q \) be a spin 1/2 system as a qubit with computational basis \{0, 1\} encoded by \( S_z = (\hbar/2)(|0\rangle \langle 0| - |1\rangle \langle 1|) \), where \( S_i \) is the \( i \) component of spin for \( i = x, y, z \). Let \( H = 2^{-1/2}(|0\rangle \langle 0| + |1\rangle \langle 1|) \) be the Hadamard gate \( Q \).

Let \( \alpha = (U, \xi) \) be a physical implementation of \( H \) defined by a unitary operator \( U \) on the system \( Q + A \), where \( A \) is a quantum system called the ancilla, and a state vector \( |\xi\rangle \) of the ancilla, in which the ancilla is prepared at the time at which \( U \) is turned on. The implementation \( \alpha = (U, \xi) \) defines a trace-preserving quantum operation \( E_{\alpha} \) by
\[
E_{\alpha}(\rho) = \text{Tr}_{A}[U(\rho \otimes |\xi\rangle \langle \xi|)U^\dagger]
\] (71)
for any density operator \( \rho \) of the system \( Q \), where \( \text{Tr}_{A} \) stands for the partial trace over the system \( A \). On the other hand, the gate \( H \) defines the trace-preserving quantum operation \( \text{ad}H \) by
\[
\text{ad}H(\rho) = H \rho H^\dagger
\] (72)
for any density operator \( \rho \) of the system \( Q \).

How successful the implementation \( (U, \xi) \) has been is most appropriately measured by the completely bounded (CB) distance between two operations \( E_{\alpha} \) and \( \text{ad}H \) defined by
\[
D_{CB}(E_{\alpha}, H) = \sup_{n, \rho} D(E_{\alpha} \otimes \text{id}_n(\rho), \text{ad}H \otimes \text{id}_n(\rho)),
\] (73)
where \( n \) runs over positive integers, \( \text{id}_n \) is the identity operation on an \( n \)-level system \( S_n \), \( \rho \) runs over density operators of the system \( Q + S_n \), and \( D(\sigma_1, \sigma_2) \) stands for the trace distance of two states \( \sigma_1 \) and \( \sigma_2 \). Since the trace distance of
the above two states can be interpreted as an achievable upper bound on the so-called total variation distance of two probability distributions arising from measurements performed on the two output states of the corresponding gates. We interpret $D_{CB}(E_\alpha, H)$ as the worst error probability of operation $E_\alpha$ in simulating the gate $H$ on any input state of any circuit including those two gates. We shall call $D_{CB}(E_\alpha, H)$ the gate error probability of the implementation $\alpha$ of the gate $H$.

Another measure, which is more tractable in computations, is the gate fidelity defined by

$$F(E_\alpha, H) = \inf_{|\psi\rangle} F(|\psi\rangle)$$

(74)

where $|\psi\rangle$ varies over all state vectors of $Q$, and $F(|\psi\rangle)$ is the fidelity of two states $H|\psi\rangle$ and $E_\alpha(|\psi\rangle)H|\psi\rangle$ given by

$$F(|\psi\rangle) = \langle \psi | H^\dagger E_\alpha(|\psi\rangle) | H | \psi \rangle^{1/2}. $$

(75)

By the relation

$$1 - F(E_\alpha, H)^2 \leq D_{CB}(E_\alpha, H),$$

(76)

any lower bound of $1 - F(E_\alpha, H)^2$ gives a lower bound of the gate error probability. The operator $U$ and the operation $E_\alpha$ is generally described by the following actions on computational basis states

$$U|a\rangle|\xi\rangle = \sum_{b=0} |b\rangle E_b^\alpha$$

(77)

$$E_\alpha(|a\rangle|a'\rangle) = \sum_{b,b'=0} |b\rangle E_b^\alpha |b'\rangle $$

(78)

for $a, a' = 0, 1$, where $|E_b^\alpha\rangle$ is not necessarily normalized. It follows that the fidelity is given by

$$F(|0\rangle)^2 = \frac{1}{2} ||E_0^0||^2 + ||E_1^0||^2 = 1 - \frac{1}{2} ||E_0^1||^2 - ||E_1^1||^2,$$

$$F(|1\rangle)^2 = \frac{1}{2} ||E_1^0||^2 - ||E_1^1||^2 = 1 - \frac{1}{2} ||E_0^1||^2 + ||E_1^1||^2.$$  

(79)

(80)

We consider implementations $(U, |\xi\rangle)$ such that $U$ satisfies the angular momentum conservation law. For simplicity, we only assume that the $x$ component of the total angular momentum is conserved, i.e,

$$[U, \hat{S}_x + \hat{L}_x] = 0,$$

(81)

where $L_x$ is the $x$ component of the total angular momentum of the ancilla.

Now, we consider the following process of measuring the operator $S_z$ of $Q$: (i) to operate $U$ on $Q + A$, and (ii) to measure $S_z$ of $Q$ by a projective measurement. Since $S_z = H^\dagger S_z H$, if $U = H$ the above process would measure $S_z$ precisely. Since each step does not disturb $\hat{S}_x + \hat{L}_x$, we can apply Eq. (80) to this measurement. Precisely, we consider the instrument $I$ for the system $Q + A$ defined by

$$I(a)\rho = E^{S_z} a \rho U^\dagger E^{S_z} a$$

(82)

for any state $\rho$ of $Q + A$, where $a = \pm \hbar/2$. Then, the nonselective operation $T$ of $I$ satisfies

$$T[(\hat{S}_z + \hat{L}_x)^n] = \sum_{a=\pm \hbar/2} U^\dagger E^{S_z} a (\hat{S}_z + \hat{L}_x)^n E^{S_z} a U$$

(83)

$$= (\hat{S}_z + \hat{L}_x)^n.$$  

(84)

Thus, we have

$$\eta(\hat{S}_z + \hat{L}_x, T, \rho) = 0.$$  

(85)

Thus, the POVM $\Pi$ of $I$ satisfies

$$\epsilon(\hat{S}_z, \Pi, \rho) \geq \frac{|[\hat{S}_z, \hat{S}_z]|}{2\Delta(\hat{S}_x + \hat{L}_x)},$$

(86)
Let $\rho = |\psi\rangle\langle\psi| \otimes |\xi\rangle\langle\xi|$. Then, we have $\langle[\tilde{S}_z, \tilde{S}_z]\rangle = \langle[S_x, S_z]\rangle, [\Delta(\tilde{S}_z + \tilde{L}_z)]^2 = (\Delta S_z)^2 + (\Delta L_z)^2$, and $\epsilon(\tilde{S}_z, \Pi, \psi \otimes \xi)^2 = \epsilon(S_z, \Pi_0, \psi)$, where $\Pi_0\{a\} = \text{Tr}_A [\Pi\{a\} (I \otimes |\xi\rangle\langle\xi|)]$ for $a = \pm \hbar/2$, so that we have

$$\epsilon(S_z)^2 \geq \frac{|\langle[S_x, S_z]\rangle|^2}{4(\Delta S_z)^2 + 4(\Delta L_z)^2}, \tag{87}$$

where $\epsilon(S_z) = \epsilon(S_z, \Pi_0, \psi)$. Now, we have

$$\epsilon(S_z)^2 = \langle\psi|O^{(2)}(\Pi_0) - O(\Pi_0)S_z - S_zO(\Pi_0) + S_z^2|\psi\rangle$$

$$= \langle\psi \otimes \xi|(U^T \tilde{S}_z U - \tilde{S}_z^2)|\psi \otimes \xi\rangle$$

$$= \|\tilde{S}_z U|\psi \otimes \xi\rangle - U \tilde{S}_z|\psi \otimes \xi\rangle\|^2$$

$$= \frac{\hbar^2}{2} |\langle 0|\psi\rangle|^2 \|E_0^0\| - |E_1^0\|^2 + \frac{\hbar^2}{2} |\langle 1|\psi\rangle|^2 \|E_0^1\| + |E_1^1\|^2.$$

Thus, from Eq. (89) and Eq. (90), we have

$$\frac{\epsilon(S_z)^2}{\hbar^2} = 1 - |\langle 0|\psi\rangle|^2 F(|0\rangle)^2 + |\langle 1|\psi\rangle|^2 F(|1\rangle)^2 \tag{88}$$

It follows that we have

$$1 - F(E_\alpha, H)^2 = 1 - \inf_{|\psi\rangle} F(|\psi\rangle)^2 \tag{89}$$

$$\geq 1 - |\langle 0|\psi\rangle|^2 F(|0\rangle)^2 + |\langle 1|\psi\rangle|^2 F(|1\rangle)^2 \tag{90}$$

$$= \frac{\epsilon(S_z)^2}{\hbar^2} \tag{91}$$

For the input state $\psi = (|0\rangle + i|1\rangle)/\sqrt{2} = |\psi_y = \hbar/2\rangle$, the numerator $|\langle S_z, S_z\rangle|^2$ of the lower bound (87) is maximized as

$$1 - F(E_\alpha, H)^2 \geq \frac{\epsilon(S_z)^2}{\hbar^2} \geq \frac{1}{4 + 4(2\Delta L_x/\hbar)^2}. \tag{92}$$

Similar result on CNOT gates were previously obtained in Ref. 6 (see also, Ref. 9,10). Here, we have shown that the Hadamard gate, a single qubit gate, has the unavoidable error probability equivalent to that for the CNOT.

In the following, we shall interpret the above relation for bosonic control systems and fermionic control systems separately. In current proposals, the external electromagnetic field prepared by laser beam is considered to be a feasible candidate for the controller $A$ to be coupled with the computational qubits $Q$ via the dipole interaction. In this case, the ancilla state $|\xi\rangle$ is considered to be a coherent state, for which we have $\langle \Delta N \rangle^2 = \langle \xi | N | \xi \rangle = \langle N \rangle$, where $N$ is the number operator. We assume that the beam propagates to the $x$-direction with right-hand-circular polarization. Then, we have $L_x = \hbar N$, and hence

$$(2\Delta L_x/\hbar)^2 = (2\Delta N)^2 = 4\langle N \rangle \tag{93}$$

Thus, from Eq. (92) we have

$$1 - F(E_\alpha, H)^2 \geq \frac{1}{4 + 16\langle N \rangle}. \tag{94}$$

Thus, we cannot implement Hadamard gates within the error probability $(4 + 16\langle N \rangle)^{-1}$ on a qubit represented by a spin component of a spin $1/2$ system controlled by the dipole interaction with external electromagnetic field with average photon number $\langle N \rangle$. Enk and Kimble22 and Gea-Banacloche23 also showed that there is unavoidable error probability in this case inversely proportional to the average strength of the external field by calculations with the model obtained by rotating wave approximation. Here, we have shown the same result only from the angular momentum conservation law.

We now assume that the ancilla $A$ comprises $n$ spin $1/2$ systems. Then, we have

$$\Delta L_x \leq \|L_x\| = \frac{n\hbar}{2}. \tag{95}$$
Thus, we have the following lower bound of the gate error probability

$$1 - F(\mathcal{E}_o, H)^2 \geq \frac{1}{4 + 4n^2}. \quad (96)$$

Thus, it has been proven that if the computational basis is represented by the $z$-component of spin, we cannot implement Hadamard gates within the error probability $(4 + 4n^2)^{-1}$ with $n$ qubit ancillas by rotationally invariant interactions such as the Heisenberg exchange interaction. Thus, for the error probability $\sim 10^{-5}$, we need the ancilla consisting of at least $\sim 100$ physical qubits. This result shows a drastic contrast with the new universal encoding of the computational qubit recently proposed by DiVincenzo et al. In their encoding, each computational qubit is encoded into three physical qubits, instead of one spin 1/2 system, and they showed that any quantum gates for $n$ logical qubits are implemented with arbitrary accuracy by rotationally invariant interactions on $3n$ physical qubits, so that Hadamard gates are implemented only on three physical qubits with required accuracy.

In the above discussion, we have assumed that the control system can be prepared in an entangled state. However, it is also interesting to estimate the error in the case where we can prepare the control system only in a separable state. In this case, we have

$$\left(\Delta L_x \right)^2 \leq \sum_{j=1}^{n} \left(\Delta S^{(j)}_x \right)^2 \leq n\|S_x\|^2 = \frac{nh^2}{4}, \quad (97)$$

where $S^{(j)}_x$ is the spin component of the $j$th ancilla qubit so that $L_x = \sum_{j=1}^{n} S^{(j)}_x$. Thus, we have the following lower bound of the gate error probability

$$1 - F(\mathcal{E}_o, H)^2 \geq \frac{\epsilon(S_x)^2}{\hbar^2} \geq \frac{1}{4 + 4n}. \quad (98)$$

Thus, the error probability is lower bounded by $(4 + 4n)^{-1}$, and hence the achievable error can be considered to be inversely proportional to $4n^2$ for entangled control system but $4n$ for separable control system. Note that even if the ancilla is in a separable mixed state, the relation $(4 + 4n)^{-1} \leq \epsilon(S_x)^2/\hbar^2$ still holds, since $\epsilon(S_x)^2$ is an affine function of the ancilla state.

If the field is in a number state $|n\rangle$, then

$$(2\Delta L_x/\hbar)^2 = (2\Delta N)^2 = 0, \quad (99)$$

so that we have

$$\frac{\epsilon(S_x)^2}{\hbar^2} \geq \frac{1}{4}. \quad (100)$$

Thus, if the field state is a mixture of number states such as the thermal state, i.e., $\sigma = \sum_n p_n |n\rangle\langle n|$, we have also the lower bound $\epsilon(S_x)^2/\hbar^2 \geq 1/4$. Thus, it seriously matters whether the control field is really in a coherent state or a mixture of number states.

VI. CONCLUSIONS

The notion of quantum instruments is formalized by normalized completely positive map valued measures to represent statistical equivalence classes of all the possible quantum measurements. Universally valid operational uncertainty relations are established to set a precision limit for any instrument given a disturbance constraint. The Heisenberg relation on the lower bound for the product of the root-mean-square noise and disturbance is derived for those instruments with uncorrelated noise and disturbance from a universal uncertainty relation. A new precision bound for nondisturbing instruments follows immediately from the universal uncertainty relation and leads to a quantitative generalization of the Wigner-Araki-Yanase theorem on the precision limit of measurements under conservation laws. Applying this, a rigorous lower bound is obtained for the gate error probability of any physical realizations of the Hadamard gate under the constraint that the computational basis is represented by a component of spin of a spin 1/2 system, and that physical implementation obeys the angular momentum conservation law. The lower bound is shown to be $1/(4 + 16\langle N \rangle)$ for the external control field with average photon number $\langle N \rangle$ in a coherent state, whereas it amounts to $1/4$ for the field in the thermal state. For fermionic control, the lower bound is $1/(4 + 4n^2)$ for $n$ qubit ancilla in an entangled state, and $1/(4 + 4n)$ in a separable states. All of these lower bounds have been obtained from rigorous calculations without any approximations under the sole assumption of the angular momentum conservation law. Physical significance of those fundamental lower bounds deserve further investigations and will be discussed elsewhere.
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