ON RESIDUALLY FINITE SEMIGROUPS OF CELLULAR AUTOMATA

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Abstract. We prove that if $M$ is a monoid and $A$ a finite set with more than one element, then the residual finiteness of $M$ is equivalent to that of the monoid consisting of all cellular automata over $M$ with alphabet $A$.

1. Introduction

In a concrete category, a finite object is an object whose underlying set is finite. A finiteness condition is a property relative to the objects of the category that is satisfied by all finite objects. Finiteness is a trivial example of a finiteness condition. Hopficity and co-Hopficity provide examples of finiteness conditions that are non-trivial and worth studying in many concrete categories, e.g., the category of groups, the category of rings, the category of compact Hausdorff spaces, etc. (see the survey paper [12] and the references therein). We recall that an object $X$ in a concrete category $C$ is called Hopfian if every surjective endomorphism of $X$ is injective and co-Hopfian if every injective endomorphism of $X$ is surjective. Another interesting finiteness condition is residual finiteness. An object $X$ in a concrete category $C$ is said to be residually finite if, given any two distinct elements $x_1, x_2 \in X$, there exists a finite object $Y$ of $C$ and a $C$-morphism $\rho: X \to Y$ such that $\rho(x_1) \neq \rho(x_2)$.

Suppose now that we are given a monoid $M$ and a finite set $A$. We say that a map $\tau: A^M \to A^M$ is a cellular automaton over the monoid $M$ and the alphabet $A$ if $\tau$ is continuous for the prodiscrete topology on $A^M$ and $M$-equivariant with respect to the shift action of $M$ on $A^M$ (see Section 2 for more details). It is clear from this definition that the set $\text{CA}(M, A)$, consisting of all cellular automata $\tau: A^M \to A^M$, is a monoid for the composition of maps.

The main result of the present note is the following statement which yields a characterization of residual finiteness for monoids in terms of cellular automata.

**Theorem 1.1.** Let $M$ be a residually finite monoid and let $A$ be a finite set with more than one element. Then the following conditions are equivalent:

(a) the monoid $M$ is residually finite;
(b) the monoid $\text{CA}(M, A)$ is residually finite.
Residual finiteness is obviously hereditary, in the sense that every subobject of a residually finite object is itself residually finite. Thus, an immediate consequence of implication (a) \(\Rightarrow\) (b) in Theorem 1.1 is the following:

**Corollary 1.2.** Let \(M\) be a residually finite monoid and let \(A\) be a finite set. Then every subsemigroup of \(\text{CA}(M, A)\) is residually finite. \(\square\)

In [9], it was shown by Mal’cev that every finitely generated residually finite semigroup is Hopfian and has a residually finite monoid of endomorphisms. Combining Corollary 1.2 with these results of Mal’cev, we get the following.

**Corollary 1.3.** Let \(M\) be a residually finite monoid and let \(A\) be a finite set. Then every finitely generated subsemigroup of \(\text{CA}(M, A)\) is Hopfian. \(\square\)

**Corollary 1.4.** Let \(M\) be a residually finite monoid and let \(A\) be a finite set. Suppose that \(T\) is a finitely generated subsemigroup of \(\text{CA}(M, A)\). Then the monoid \(\text{End}(T)\) of endomorphisms of \(T\) is residually finite. \(\square\)

The next section precises the terminology used and collects some background material. For the convenience of the reader, we have also included a proof of the results of Mal’cev mentioned above. The proof of Theorem 1.1 is given in the final section.

## 2. Preliminaries

### 2.1. Semigroups and monoids.

A **semigroup** is a set equipped with an associative binary operation. We shall use a multiplicative notation for the operation on semigroups. If \(S\) and \(T\) are semigroups, a **semigroup morphism** from \(S\) to \(T\) is a map \(\varphi : S \to T\) such that \(\varphi(s_1s_2) = \varphi(s_1)\varphi(s_2)\) for all \(s_1, s_2 \in S\). We denote by \(\text{Mor}(S, T)\) the set consisting of all semigroup morphisms from \(S\) to \(T\). A relation \(\gamma\) on a semigroup \(S\) is called a **congruence relation** if there exist a semigroup \(T\) and a semigroup morphism \(\varphi : S \to T\) such that \(\gamma\) is the kernel relation associated with \(\varphi\), i.e., the equivalence relation defined by \(\gamma := \{(s_1, s_2) \in S \times S : \varphi(s_1) = \varphi(s_2)\}\).

Equivalently, an equivalence relation \(\gamma \subset S \times S\) on \(S\) is a congruence relation if and only if \((s_1, s_2) \in \gamma\) implies \((ss_1, ss_2) \in \gamma\) and \((s_1s, s_2s) \in \gamma\) for all \(s, s_1, s_2 \in S\).

Suppose that \(\gamma\) is a congruence relation on a semigroup \(S\). Then there is a natural semigroup structure on the quotient set \(S/\gamma\). This semigroup structure is the only one for which the canonical map from \(S\) onto \(S/\gamma\) (i.e., the map sending each \(s \in S\) to its \(\gamma\)-class \([s] \in S/\gamma\)) is a semigroup morphism. Moreover, \(\gamma\) is the kernel relation associated with this semigroup morphism. One says that the congruence relation \(\gamma\) is of **finite index** if the quotient semigroup \(S/\gamma\) is finite.

A **monoid** is a semigroup admitting an identity element. The identity element of a monoid \(M\) is denoted \(1_M\). If \(M\) and \(N\) are monoids, a **monoid morphism** from \(M\) to \(N\) is a semigroup morphism from \(M\) to \(N\) that sends \(1_M\) to \(1_N\). Suppose that \(\gamma\) is a congruence relation on a monoid \(M\). Then the quotient semigroup \(M/\gamma\) is a monoid. Moreover, the canonical semigroup morphism from \(M\) onto \(M/\gamma\) is a monoid morphism.
2.2. **Residually finite semigroups.** It is clear from the general definition of residual finiteness given in the Introduction that a group is residually finite as a group if and only if it is residually finite as a monoid and that a monoid is residually finite as a monoid if and only if it is residually finite as a semigroup.

The class of residually finite semigroups includes all free groups and hence (since residual finiteness is a hereditary property) all free monoids and all free semigroups, all polycyclic groups [6] and hence all finitely generated nilpotent groups, all finitely generated commutative semigroups [10] (see also [7] and [2]), all finitely generated semigroups that are both regular in the sense of von Neumann and nilpotent in the sense of Mal’cev [8], and all finitely generated semigroups of matrices over commutative rings [9], [11].

The following two fundamental results about finitely generated residually finite semigroups are due to Mal’cev [9] (see also [4]).

**Theorem 2.1** (Mal’cev). Every finitely generated residually finite semigroup is Hopfian.

**Proof.** Let \( S \) be a finitely generated residually finite semigroup. Suppose that \( \psi: S \to S \) is a surjective endomorphism of \( S \). Let \( s_1 \) and \( s_2 \) be distinct elements in \( S \). Since \( S \) is residually finite, there exists a finite semigroup \( T \) and a semigroup morphism \( \rho: S \to T \) such that \( \rho(s_1) \neq \rho(s_2) \). Consider the map

\[
\Phi: \text{Mor}(S, T) \to \text{Mor}(S, T)
\]

defined by \( \Phi(u) = u \circ \psi \) for all \( u \in \text{Mor}(S, T) \). Observe that \( \Phi \) is injective since \( \psi \) is surjective. On the other hand, as \( S \) is finitely generated and \( T \) is finite, the set \( \text{Mor}(S, T) \) is finite. Therefore \( \Phi \) is also surjective. In particular, there exists a morphism \( u_0 \in \text{Mor}(S, T) \) such that \( \rho = \Phi(u_0) = u_0 \circ \psi \). Since \( \rho(s_1) \neq \rho(s_2) \), this implies that \( \psi(s_1) \neq \psi(s_2) \). We deduce that \( \psi \) is injective. This shows that \( S \) is Hopfian. \( \square \)

**Theorem 2.2** (Mal’cev). Let \( S \) be a finitely generated residually finite semigroup. Then the monoid \( \text{End}(S) \) is residually finite.

**Proof of Theorem 2.2.** Let \( S \) be a residually finite semigroup. Then the monoid \( \text{End}(S) \) is residually finite.

Let us first establish the following auxiliary result.

**Lemma 2.3.** Let \( S \) be a semigroup. Suppose that \( \gamma_1 \) and \( \gamma_2 \) are congruence relations of finite index on \( S \). Then the congruence relation \( \gamma := \gamma_1 \cap \gamma_2 \) is also of finite index on \( S \).

**Proof.** Two elements in \( S \) are congruent modulo \( \gamma \) if and only if they are both congruent modulo \( \gamma_1 \) and modulo \( \gamma_2 \). Therefore, there is an injective map from \( S/\gamma \) into \( S/\gamma_1 \times S/\gamma_2 \) given by \( [s] \mapsto ([s]_1, [s]_2) \), where \( [s] \) (resp. \( [s]_1 \), resp. \( [s]_2 \)) denotes the class of \( s \in S \) modulo \( \gamma \) (resp. \( \gamma_1 \), resp. \( \gamma_2 \)). As the sets \( S/\gamma_1 \) and \( S/\gamma_2 \) are finite by our hypothesis, we deduce that \( S/\gamma \) is also finite, that is, \( \gamma \) is of finite index on \( S \). \( \square \)

**Proof of Theorem 2.2.** Let \( \alpha_1, \alpha_2 \in \text{End}(S) \) such that \( \alpha_1 \neq \alpha_2 \). Then we can find an element \( s_0 \in S \) such that \( \alpha_1(s_0) \neq \alpha_2(s_0) \). As \( S \) is residually finite, there exist a finite semigroup \( T \) and a semigroup morphism \( \rho: S \to T \) satisfying \( \rho(\alpha_1(s_0)) \neq \rho(\alpha_2(s_0)) \). Consider the set \( \gamma \subset S \times S \) defined by

\[
\gamma := \bigcap_{\psi \in \text{Mor}(S, T)} \gamma_\psi,
\]
where $\gamma_\psi$ denotes the kernel congruence relation associated with the semigroup morphism $\psi : S \to T$. Observe first that $\gamma$ is a congruence relation on $S$ since it is the intersection of a family of congruence relations on $S$. On the other hand, for every $\alpha \in \text{End}(S)$ and $(s_1, s_2) \in \gamma$, we have that $(\alpha(s_1), \alpha(s_2)) \in \gamma$ since $\psi \circ \alpha \in \text{Mor}(S, T)$ for every $\psi \in \text{Mor}(S, T)$. We deduce that $\alpha$ induces an endomorphism $\overline{\alpha}$ of $S/\gamma$, given by $\overline{\alpha}([s]) = [\alpha(s)]$, for all $s \in S$ (here $[s]$ denotes the $\gamma$-class of $s$). The map $\alpha \mapsto \overline{\alpha}$ is clearly a morphism from $\text{End}(S)$ into $\text{End}(S/\gamma)$. Now the set $\text{Mor}(S, T)$ is finite since $S$ is finitely generated and $T$ is finite. Moreover, as the semigroup $T$ is finite, the congruence relation $\gamma_\psi$ is of finite index on $S$ for every $\psi \in \text{Mor}(S, T)$. By applying Lemma 2.3, we deduce that the congruence relation $\gamma$ is of finite index on $S$. Thus, the semigroup $S/\gamma$ is finite and hence the monoid $\text{End}(S/\gamma)$ is also finite. On the other hand, we have that
\[
\overline{\alpha_1}([s_0]) = [\alpha_1(s_0)] \neq [\alpha_2(s_0)] = \overline{\alpha_2}([s_0])
\]
since $\gamma \subset \gamma_\rho$ and $\rho(\alpha_1(s_0)) \neq \rho(\alpha_2(s_0))$. Therefore $\overline{\alpha_1} \neq \overline{\alpha_2}$. This shows that the monoid $\text{End}(S)$ is residually finite.

2.3. Shift spaces. Let $A$ be a finite set, called the alphabet, and let $M$ be a monoid. The set $A^M$, consisting of all maps $x : M \to A$, is called the set of configurations over the monoid $M$ and the alphabet $A$. We equip $A^M$ with its prodiscrete topology, i.e., the product topology obtained by taking the discrete topology on each factor $A$ of $A^M = \prod_{m \in M} A$. Observe that $A^M$ is a compact Hausdorff totally disconnected space since it is a product of compact Hausdorff totally disconnected spaces. We also equip $A^M$ with the $M$-shift, that is, the action of the monoid $M$ on $A^M$ given by $(m, x) \mapsto mx$, where
\[
mx(m') = x(m'm)
\]
for all $x \in A^M$ and $m, m' \in M$.

Let $\gamma$ be a congruence relation on $M$. We define the subset $\text{Inv}(\gamma) \subset A^M$ by
\[
\text{Inv}(\gamma) := \{ x \in A^M : m_1x = m_2x \text{ for all } (m_1, m_2) \in \gamma \}.
\]
Observe that $\text{Inv}(\gamma)$ is $M$-invariant, i.e., $mx \in \text{Inv}(\gamma)$ for all $m \in M$ and $x \in \text{Inv}(\gamma)$. One immediately checks that $\text{Inv}(\gamma)$ consists of all configurations $x \in A^M$ that are constant on each $\gamma$-class. This implies in particular that the set $\text{Inv}(\gamma)$ is finite whenever $\gamma$ is of finite index.

A configuration $x \in A^M$ is called periodic if its orbit
\[
Mx := \{ mx : m \in M \}
\]
is finite.

Residually finite monoids are characterized by the density of periodic configurations in their shift spaces. More precisely, we have the following result (see [3, Proposition 2.14]).

**Theorem 2.4.** Let $M$ be a monoid and let $A$ be a finite set with more than one element. Then the following conditions are equivalent:

(a) the monoid $M$ is residually finite;

(b) the set of periodic configurations of $A^M$ is dense in $A^M$ for the prodiscrete topology.
2.4. Cellular automata. Let $M$ be a monoid and let $A$ be a finite set. A cellular automaton over the monoid $M$ and the alphabet $A$ is a map $\tau : A^M \to A^M$ that is continuous for the prodiscrete topology on $A^M$ and commutes with the shift action, i.e., satisfies $\tau(mx) = m\tau(x)$ for all $m \in M$ and $x \in A^M$. We denote by $\text{CA}(M, A)$ the set consisting of all cellular automata $\tau : A^M \to A^M$. It is clear from the above definition that $\text{CA}(M, A)$ is a monoid for the composition of maps.

Example 2.5. If $m \in M$, one immediately checks that the map $\tau_m : A^M \to A^M$, defined by $\tau(x) = x \circ L_m$ for all $x \in A^M$, where $L_m : M \to M$ denotes the left-multiplication by $m$, is a cellular automaton. Moreover, the map $m \to \tau_m$ yields an monoid anti-morphism from $M$ into $\text{CA}(M, A)$. This means that $\tau_1$ is the identity map on $A^M$ and that $\tau_{m_1 m_2} = \tau_{m_2} \circ \tau_{m_1}$ for all $m_1, m_2 \in M$. This monoid anti-morphism is injective as soon as the alphabet $A$ has more than one element. Indeed, let $m_1, m_2 \in M$ with $m_1 \neq m_2$. Suppose that $a$ and $b$ are distinct elements in $A$ and consider the configuration $x \in A^M$ defined by $x(m_1) = a$ and $x(m) = b$ for all $m \in M \setminus \{m_1\}$. We then have $\tau_{m_1}(x) \neq \tau_{m_2}(x)$ since $\tau_{m_1}(1_M) = x(m_1) = a \neq b = x(m_2) = \tau_{m_2}(x)(1_M)$, and hence $\tau_{m_1} \neq \tau_{m_2}$.

3. Proof of the main result

In this section, we give the proof of Theorem 1.1.

Proof of (a) $\implies$ (b). Suppose that $M$ is residually finite. Let $\tau_1, \tau_2 \in \text{CA}(M, A)$ be two distinct cellular automata.

Since $M$ is residually finite, the periodic configurations in $A^M$ are dense in $A^M$ (see Theorem 2.4). As $\tau_1$ and $\tau_2$ are continuous and $A^M$ is Hausdorff, this implies that there exists a periodic configuration $x_0 \in A^M$ such that $\tau_1(x_0) \neq \tau_2(x_0)$. Consider the orbit $Y := Mx_0$ of $x_0$ under the $M$-shift. As the set $Y$ is $M$-invariant, the equivalence relation $\gamma$ defined by

$$\gamma := \{(m_1, m_2) \in M \times M : m_1 y = m_2 y \text{ for all } y \in Y\} \subset M \times M$$

is a congruence relation on $M$. Moreover, $\gamma$ is of finite index since $Y$ is finite. Consider now the associated $M$-invariant subset

$$X := \text{Inv}(\gamma) = \{x \in A^M : m_1 x = m_2 x \text{ for all } (m_1, m_2) \in \gamma\} \subset A^M.$$ 

Note that $X$ is finite since the congruence relation $\gamma$ is of finite index. As every cellular automaton $\tau \in \text{CA}(M, A)$ is $M$-equivariant, restriction to $X$ yields a monoid morphism $\rho : \text{CA}(M, A) \to \text{Map}(X)$, where $\text{Map}(X)$ denotes the symmetric monoid of $X$, i.e., the set consisting of all maps $f : X \to X$ with the composition of maps as the monoid operation. Observe that the monoid $\text{Map}(X)$ is finite since $X$ is finite. On the other hand, as $x_0 \in Y \subset X$ and $\tau_1(x_0) \neq \tau_2(x_0)$, we have that $\rho(\tau_1) \neq \rho(\tau_2)$. This shows that $\text{CA}(M, A)$ is residually finite. □
Proof of (b) ⇒ (a). First observe that a semigroup is residually finite if and only if its opposite semigroup is (this trivially follows from the fact that a semigroup is finite if and only if its opposite semigroup is). Suppose now that the monoid \( \text{CA}(M, A) \) is residually finite. Since there is an injective monoid anti-morphism \( M \to \text{CA}(M, A) \) (see Example 2.5) and residual finiteness is hereditary, we deduce that the opposite monoid of \( M \) is residually finite. By the above observation, the monoid \( M \) is itself residually finite. \( \square \)

Remark 3.1. Let us observe that Corollary 1.3 and Corollary 1.4 become false if we drop the hypothesis that the subsemigroup of \( \text{CA}(M, A) \) is finitely generated, even if we restrict to the case where \( M \) is the group \( \mathbb{Z} \) of integers (the classical case studied in symbolic dynamics). Indeed, let \( A \) be a finite set with more than one element. It can be shown, using the technique of markers introduced in [5], that the free group on two generators can be embedded in \( \text{CA}(\mathbb{Z}, A) \) (see [11] Theorem 2.4 for a more general statement). It follows that the free group \( F_\infty \) on infinitely many generators \( g_i, i \in \mathbb{N} \), can be also embedded in \( \text{CA}(\mathbb{Z}, A) \). Now, the group \( F_\infty \) is not Hopfian since the unique endomorphism \( \psi \in \text{End}(F_\infty) \) satisfying \( \psi(g_i) = g_{i-1} \) if \( i \geq 1 \) and \( \psi(g_0) = g_0 \) is clearly surjective but not injective. On the other hand, by using automorphisms of \( F_\infty \) induced by permutations of its generators, one sees that the automorphism group of \( F_\infty \) contains a copy of the symmetric group \( \text{Sym}(\mathbb{N}) \) (the group of permutations of \( \mathbb{N} \)). The group \( \text{Sym}(\mathbb{N}) \) is not residually finite since, by Cayley’s theorem, every countable group can be embedded in \( \text{Sym}(\mathbb{N}) \) and there exist countable groups that are not residually finite (e.g., the additive group \( \mathbb{Q} \) of rational numbers or the Baumslag-Solitar group \( BS(2, 3) := \langle a, b : ba^2b^{-1} = a^3 \rangle \)). Therefore, the monoid \( \text{End}(F_\infty) \) is not residually finite either.

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