Learning Approximate and Exact Numeral Systems via Reinforcement Learning

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Abstract

Recent work (Xu et al., 2020) has suggested that numeral systems in different languages are shaped by a functional need for efficient communication in an information-theoretic sense. Here we take a learning-theoretic approach and show how efficient communication emerges via reinforcement learning. In our framework, two artificial agents play a Lewis signaling game where the goal is to convey a numeral concept. The agents gradually learn to communicate using reinforcement learning and the resulting numeral systems are shown to be efficient in the information-theoretic framework of Regier et al. (2015), Gibson et al. (2017). They are also shown to be similar to human numeral systems of same type. Our results thus provide a mechanistic explanation via reinforcement learning of the recent results in Xu et al. (2020) and can potentially be generalized to other semantic domains.

Keywords: efficient communication; reinforcement learning; numeral systems

Introduction

Why do languages partition mental concepts into words the ways they do? A recent influential body of work suggests language is shaped by a pressure for efficient communication which involves an information-theoretic tradeoff between cognitive load and informativeness (Kemp and Regier, 2012; Gibson et al., 2017). This means that language is under pressure to be simultaneously informative, to support effective communication, while also being simple, in order to minimize the cognitive load.

While the information-theoretic framework is insightful and has broad explanatory power across a variety of domains, see the reviews by Kemp et al. (2018); Gibson et al. (2019), a fundamental question that is left unaddressed is if there is mechanistic explanation for how such efficient communication schemes could arise. We address this question here from a learning-theoretic viewpoint: is there a computational learning mechanism that leads to efficient communication?

We can relate our approach to previous work using the influential “three levels of analysis” framework posited by David Marr (Marr, 1982) which has been described as one of the most enduring constructs of twentieth century cognitive science and computational neuroscience. While the previous work such as Kemp and Regier (2012); Kemp et al. (2018); Gibson et al. (2019) is situated at the first or “theory” level of Marr, our analysis is at the representation and algorithmic level. In particular, we propose very natural reinforcement learning mechanisms that are able to learn such efficient communication schemes. The learning aspect is emphasised by Tomaso Poggio (Poggio, 2012) in an update of Marr:

it is ... important to understand how an individual organism, and in fact a whole species, learns and evolves [the computations and the representations used by the brain] from experience of the natural world ... a description of the learning algorithms and their a priori assumptions is deeper, more constructive, and more useful than a description of the details of what is actually learned ... the problem of learning is at the core of the problem of intelligence and of understanding the brain ... learning should be added to the list of levels of understanding ...

Recent research gives evidence that the style of learning algorithms we consider here seem to be centrally implicated in exploration strategies used by humans (Schulz and Gershman, 2019).

Reinforcement learning has been proposed recently as a mechanistic explanation for how efficient communication arises in the colour domain (Kägebäck et al., 2020; Chaabouni et al., 2021) and it was observed that this approach could potentially be applied to other domains. Here we investigate the reinforcement learning approach in the domain of numeral systems. It has been shown recently that numeral systems across languages reflect a need for efficient communication (Xu et al., 2020). Numeral systems come in many shapes, some are recursive like English and can express any numerosity while other non-recursive systems only consists of a small set of words (Comrie, 2013). These non-recursive systems could be either exact restricted - in the sense that exact numerosities can only be expressed on a restricted range, or approximate like in the language Mundurukú where most numeral words have an imprecise meaning (Pica et al., 2004). Here we only consider non-recursive systems.

We show that reinforcement learning mechanisms can indeed be used to learn exact and approximate numeral systems which are near-optimal in an information-theoretic sense and similar in structure to human numeral systems of the same complexity. Unlike Kägebäck et al. (2020), who use a policy-gradient method, we use a Q-learning algorithm with an implicit Thompson Sampling exploration scheme (Sutton and Barto, 1998).
Reinforcement Learning is an area of machine learning which for how precise the listener’s reconstruction has to be. The reward functions which success depends on how well the listener reconstructed the number the sender had in mind. The reward function is given states as to maximize a reward signal $r$ and samples a model $f$ from $F$ using dropout and conveys the word $w$ giving highest reward according to $f$. The listener proceeds in similar fashion, given $w$ it samples a model $f_L$ from $F_L$ and guesses the number $\hat{n}$ that yields most reward according to $f_L$. A shared reward is given to both agent based on how close $\hat{n}$ is to $n$.

Learning to Communicate: Signalling Games

We consider the communication framework developed in Regier et al. (2015); Xu et al. (2020) which consists of a sender and a listener. The sender has a concept in mind and wishes to convey this to a listener over a discrete communication channel. The listener then tries to reconstruct the concept. This is illustrated schematically in Figure 1.

We extend this setup to a Lewis signaling game (Lewis 1969), by considering two artificial agents starting tabula rasa and gradually learning to communicate efficiently via a reinforcement learning algorithm (introduced in detail in later sections) by playing several rounds of the game. In each round of the game, a number $n \in \mathcal{N}$ from the interval $\mathcal{N}$ is sampled according to a need probability of the environment, $p(n)$, which represent how often a numeral concept has to be referred to in the environment. The sampled number $n$ is then given to the sender which has to pick a word $w$ from the vocabulary $\mathcal{W}$ and utter to the listener. Having received a word $w$, the listener guesses a number $\hat{n} \in \mathcal{N}$ and a shared reward, $r(n,\hat{n})$, is given to both agents based on the distance between the guess $\hat{n}$ and the true number $n$. Here we explore three different reward functions, one linear, one inverse and one exponential

$$r_{\text{linear}}(n, \hat{n}) = 1 - \frac{|n - \hat{n}|}{\mathcal{N}},$$
$$r_{\text{inverse}}(n, \hat{n}) = (1 + |n - \hat{n}|)^{-1},$$
$$r_{\text{exp}}(n, \hat{n}) = e^{-|n - \hat{n}|}.$$

One round of the signaling game is visualized in Figure 2 and one could interpret it as follows: the agents are playing a cooperative game which involves solving a common task in which success depends on how well the listener reconstructed the number the sender had in mind. The reward functions considered were chosen in order to model different pressure for how precise the listener’s reconstruction has to be.

Reinforcement Learning for Efficient Communication

Reinforcement learning is an area of machine learning which studies how agents in an environment can learn to pick actions...
Given a new round of our signaling game each agent samples a smaller network $f_{S} \sim F_{S}$ and $f_{L} \sim F_{L}$ from its neural network using the regularization technique dropout (Srivastava et al., 2014) which means that the activation at each neuron in the network is randomly set to 0 with probability $p$. In this way the agents sample, via dropout, one out of all possible models of the expected rewards spanned by $F_{S}$ and $F_{L}$. Hence, the networks $f_{S}$ and $f_{L}$ become the current internal models of the expected reward of the speaker and listener. Given an input, each agent acts greedily w.r.t. the smaller networks $f_{S}$ and $f_{L}$: given the number $n$, the sender conveys the word $\hat{w}$ yielding highest expected reward according the sampled model

$$\hat{w} = \arg\max_{w \in W} f_{S}(n, w)$$

Similarly, given the word $\hat{w}$, the listener guesses the number $\hat{n}$ satisfying

$$\hat{n} = \arg\max_{n' \in \mathbb{N}} f_{L}(\hat{w}, n').$$

After playing the game for $m$ rounds, each agent update the weights in $F_{S}$ (or respectively $F_{L}$) by finding the values which minimize the mean-squared error (MSE)

$$\text{MSE}_{S} = \frac{1}{m} \sum_{i} (f_{S}(\hat{w}_{i}, n_{i}) - r_{i})^{2},$$

$$\text{MSE}_{L} = \frac{1}{m} \sum_{i} (f_{L}(\hat{n}_{i}, \hat{w}_{i}) - r_{i})^{2}.$$ 

It should be noted that this game is only partially observable—in each round of the game the sender observes the tuple $(n, \hat{w}, r)$ while the listener observes $(\hat{w}, \hat{n}, r)$.

**Numerical Systems**

We study two of the three types of numeral systems presented in Xu et al. (2020). First, we consider the exact restricted systems, or simply exact systems, where exact numerosities can only be expressed on a restricted range. An example of this is the numeral system one, two, three and more than three. With this system precise communication can only be achieved for the first three numerals and it is clear which part of the number line each numeral word corresponds to.

The second type is approximate numeral systems where the meaning of numerals are approximate. Example of such inexact numerals are a few and many which do not cover a precise restricted range.

We do not address recursive numeral systems in this work since it require a different way of modelling the agents and we leave it for future work.

**Artificial Numerical Systems**

Given that a sender-listener pair has played the signaling game in Figure 2 for a certain number of rounds we would like to compute the resulting numeral system. We do this by first estimating the conditional probability $p(w|n)$, i.e the probability that the sender refers to the number $n$ with the word $w$, by running $m = 1000$ rounds of the game, without updating the agents, with the number $n$ given to the sender and count how many times each word is used. Hence, we do the following Monte-Carlo estimation

$$p(w|n) \approx \frac{1}{m} \sum_{i=1}^{m} 1(w = \arg\max_{\hat{w}} f_{S,i}(\hat{w}, n))$$

where $1(\cdot)$ is the indicator function. We check if the resulting conditional distribution is peaked, i.e if it for each $n$ assigns more than 0.90 probability mass to one token $w$, if not we interpret it as an approximate numeral system. Moreover, we consider the mode of $p(w|n)$ to be an exact numeral system.

**Complexity and Communication Cost**

We measure complexity of a numeral system simply as the number of words used in the system. In Xu et al. (2020) a grammar based complexity measure was used. This is not needed here since we do not consider recursive numeral systems and for exact and approximate systems there is no pressure for systematicity.

Given a sender distribution $S$ and a listener distribution $L_{w}$ we measure the communicative cost of conveying a number $n$ as the information lost in the listener’s reconstruction of the sender distribution given the numeral $w$. As has been done in previous studies (Xu et al., 2020), we model this as the Kullback-Leibler divergence (KL) between $S$ and $L_{w}$. Under sender certainty, $S(n) = 1$, this reduces to the surprisal

$$\text{KL}(S||L_{w}) = \sum_{i} S(i) \log \frac{S(i)}{L_{w}(i)} = -\log L_{w}(n),$$

which can be viewed as how surprised the listener would be by the fact that the sender uttered $w$ if they knew the true number $n$.

In order to measure the full communication cost of a numeral system we compute the expected surprisal as

$$C = - \sum_{n,w} p(w|n)p(n) \log L_{w}(n),$$

where $L_{w}(n)$ is computed using Bayes rule

$$L_{w}(n) = \frac{p(w|n)p(n)}{\sum_{n'} p(w|n')p(n')}.$$

Here $p(w|n)$ denotes the sender partition of the number line and $p(n)$ the need probability of the environment. The measure of the total communication cost of a numeral system used here is exactly the measure of communication cost used in Gibson et al. (2017) and by taking a deterministic sender, i.e a sender which for each $n$ assigns all probability mass to a single word $w$, we get the measure of communication cost used in Xu et al. (2020).

Note that we use the speaker model to compute the listener distribution, instead of the listener model, because given a
number the sender is forced to assign positive probability to at least one word while the listener can choose to never guess on a number no matter which word is conveyed from the sender. For example the word “many” might refer to a large, or possibly infinite, of numbers while the listener may choose to only guess on small subset of these numbers given that “many” has been uttered. Another argument for computing the listener distribution using Bayes rule is because, given a sender distribution, it minimizes the communication cost in the information bottleneck framework presented in Zaslavsky et al. (2018). The proof of this is presented in the supplementary files of Zaslavsky et al. (2018).

Experiments

We consider the interval $\mathcal{N} = [1, 20]$ and each agent is modelled as a feed-forward neural network with one hidden layer consisting of 50 hidden neurons with a dropout rate of $p = 0.3$ and with ReLu activation\(^2\). The agents starts with a vocabulary $\mathcal{W}$ of size 10 and is trained for 10000 updates where each update is over a batch of 100 rounds of the signaling game. The weights in the neural networks are updated using a version of stochastic gradient descent called Adam (Kingma and Ba, 2014) with an initial learning-rate of 0.001. The dropout rate, learning rate and batch size are in the range of what is commonly used in machine learning. However, we also performed experiments varying these parameters and found the downstream results to be robust.

We estimate the need probability in four different ways and the priors are shown in Figure 5a. The power-law prior is computed by first taking the normalized frequencies of English numerals in the Google ngram corpus English 2000 (Michel et al., 2011) and smoothing the frequencies using a power-law distribution as done in Xu et al. (2020). We also derive need probabilities using the capacity-achieving prior (CAP) method (Zaslavsky et al., 2018), which infer a prior directly from naming data, and by using the maximum-entropy

\(^2\)This interval was chosen since the need distributions are exponentially decaying and very little probability mass lies beyond 20, see Figure 5a.

\(^3\)The size of the vocabulary $\mathcal{W}$ was taken to be equal to the largest number of terms among the human systems analyzed in Xu et al. (2020), which are presented in Table I.

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Figure 3: Term usage vs communication cost. Note that our agents are not restricted to model the words as Gaussian distributions and can create other probability distributions. This explains why the line goes below the convex hull, for 2 terms, which was computed assuming Gaussian distributions. We plot the numeral systems from the human languages presented in Table I and since many of them are very similar we only get a few distinct points for human languages in the figure.
(MaxEnt) method \cite{Zaslavsky2019}, which given a naming distribution $p(w|n)$ and word frequencies $p(w)$ computes the maximum-entropy achieving prior $p(n)$ given these constraints. We obtain a universal CAP by first computing a CAP for each exact numeral system presented in Table 1 and then averaging them together. Further, to compute a MaxEnt prior we consider the language Gooniyandi, which has four number terms translated to one, two, three, many, and the corpus data available for the language Gooniyandi \cite{McGregor2004} p. 204). When computing the MaxEnt prior the fourth term, many, is modelled as a Gaussian distribution with mean $\mu = 5$ and standard deviation $\sigma = 0.31 \times \mu$. Lastly, we consider an uniform prior which was also done in \cite{Xu2020} and the authors showed that human systems are less optimal under this prior compare to the more skewed power-law prior, illustrating that the near-optimality patterns found in human numeral systems depend critically on the need probability.

We start by training 6000 independent sender-listener pairs under the power-law prior, for each reward function. We then fix the reward function to be linear and train 6000 independent sender-listener pairs for each of the priors CAP, MaxEnt and Uniform. Note that the agents are free to decide how many terms from the vocabulary that are actually used during communication and it is possible for the agents to converge to a numeral system with less than 10 terms. Thus, the actual number of terms in the final numeral system will vary over sender-listener pairs due to randomness in the initialization of the neural networks and the sampling from the need probability.

Following \cite{Xu2020}, we compute the convex hull of hypothetical approximate and exact numeral systems to use as baselines. For exact systems this is done using an approach where we start from a random numeral system and greedily updates the system until a local optima is encountered w.r.t. communication cost. For approximate systems we proceed in similar fashion but model a numeral word as Gaussian with a mean $\mu_w$ and a standard deviation $\sigma = 0.31 \times \mu_w$ following \cite{Xu2020}. We start from randomly chosen means and perform greedy updates until a local optima is reached. For both types of systems we solve for both the best and worst performing numeral system and the optimization procedures are repeated 1000 times for each number of terms.

Further, we compare the numeral systems developed by our agents to the human approximate and exact restricted numeral systems considered in \cite{Xu2020} which are presented in Table 1. Most of this data was collected from \cite{Comrie2013} except for Chiquitano, Fuyuge, Krenák which comes from \cite{Hammarstrom2010} and Mundurukú which comes from \cite{Pica2004}.

**Approximate systems:**
Chiquitano, Fuyuge, Gooniyandi, Mundurukú, Pirahã, Waru

**Exact restricted systems:**
Achagua, Araona, Awa Pit, Barasano, Baré, Hixkaryana, Inmona, Kayardild, Krenák, Mangarrayi, Martuthunira, Pitjantjatjara, Rama, Yidiny, !Xóó

Table 1: Human numeral systems considered in Figure 3

In Figure 3 we present the performance of our agents, w.r.t communication cost, relative to numeral systems found in human languages and the convex hull of hypothetically possible numeral systems, for the different need probabilities and various reward functions. We observe that our agents produce numeral systems that are near-optimal for all need probabilities and reward functions. For the left-skewed priors we observe that the communication cost of our agents are close to the communication cost of human systems.

Furthermore, in Figure 5b we plot the relative frequency of term usages between the sender-listener pairs when using the
linear reward function and varying the need probability. As
expected, we observe that a more skewed distribution gen-
ernally results in fewer terms used by the agents which indi-
cates that numeral systems with few terms can be suf-
cient to achieve a near-optimal reward while we observe a pressure
for using more terms under the uniformed need probability.

We use Correlation Clustering (Bansal et al., 2004) to find
the consensus numeral system for each number of terms over
all experiments. Correlation Clustering is a method for find-
ing the optimal clustering, w.r.t. a similarity measure. We
create a $20 \times 20$ matrix and each time two numbers $i$ and
$j$ belongs to the same partition, or word, over two different
sender-listener pairs we increase the element $(i,j)$ of the ma-
trix by 1 otherwise we decrease it with 1. We apply Corre-
lation Clustering to the final matrix to get a consensus sys-
tem and this will be an exact numeral system. The resulting
systems for the experiments using the power-law prior are
presented in Figure 4 and we observe some similarities be-
tween the consensus systems and human systems with the
same number of terms. The main difference seems to be
that our agents produce systems that tends to be slightly less
precise for smaller numbers, especially for the linear reward
function, and this could be a result of having reward functions
that gives a fair amount of reward for imprecise reconstruc-
tion of the number the sender had in mind.

In addition, we compare the representation of numbers de-
veloped by our agents to the Gaussian model used in Xu et al.
(2020), which is inspired by the the formalization of the ap-
proximate number line presented in Pica et al. (2004). The
model assumes that a numeral word, $w$, is represented as a
Gaussian distribution with some mean $\mu_w$ and standard de-
ivation $\nu = \nu \times \mu_w$ where $\nu$ is the Weber fraction. We fit
this model to the distributions produced by our agents by
first computing, for each sender-listener pair $i$, the expected
number $\mu_w$ given a word $w$ under the listener distribution
$\mu_w = E_i\left\{ n | w \right\}$. We then compute a distribution according to

$$p_i^j(n|w) \propto e^{-\frac{(|n-\mu_w|)^2}{2\times\nu_i^2}}$$

and search for $\nu$ in $[0.05, 2]$, with a granularity of 0.01, that
minimizes the the MSE w.r.t the listener distribution of pair $i$.
The best fitting Weber fractions along with the corresponding
MSEs are presented in Table 2 and the Gaussian model fits the
listener distribution well with an average MSE in the interval
$[0.0032, 0.0076]$ over all the sender-listener pairs. These errors
are of the same magnitude as the error reported between the
Gaussian model and the numeral system of Mundurukú in
Xu et al (2020) and with similar Weber fraction as reported
for Mundurukú adults in Piazza et al. (2013). Hence, our
agents produce approximate numeral systems via reinforce-
ment learning which exhibit similar behavior as the Gaussian
models used in Xu et al. (2020) and Pica et al. (2004) without
being explicitly programmed to do so.

| Reward   | Best $\nu$ | MSE            |
|----------|------------|----------------|
| Linear   | 0.31       | 0.0042 ± 0.0036 |
| Inverse  | 0.31       | 0.0032 ± 0.0042 |
| Exponential | 0.44     | 0.0076 ± 0.0063 |

Table 2: The Weber fractions corresponding the Gaussian
model that on average fits the listener distribution best along
with the average MSE ±1 standard deviation for that Weber
fraction, averaged over all sender-listener pairs trained using
the particular reward function.

Conclusions and future work

We have shown that artificial agents can develop exact and
approximate numeral systems, via interaction and reinforce-
ment learning, which are near-optimal in an information-
theoretic sense and similar to human systems. Our work
offers a mechanistic explanation via reinforcement learning
of the results in Xu et al. (2020). More generally, it offers
a powerful framework to address fundamental questions of
cognition across a wide range of semantic domains using a
learning theoretic approach that complements the normative
approaches summarized in Kemp et al. (2018); Gibson et al.
(2019).

In the numerals domain, there are still several questions
that remain to be explored: Would the results be the same if
we increase the range of numbers? Can approximate arith-
metic be learned in the same way? Could the recursive sys-
tems described in Xu et al. (2020) be learned via interaction?
An interesting topic for future work is to establish a rigorous
connection between reward function and communication cost
in our setup.

In this work our artificial agents have been completely
driven by the reward signal. In the future we would like to
add a pragmatic reasoning scheme to our model, similar to
RSA (Frank and Goodman, 2012), and explore what effect
this has on the emergent behavior.

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