Laser-induced Rotation of a Trapped Bose-Einstein Condensate

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In this letter, atom optic techniques are proposed to control the excitation of a Bose-Einstein condensate in an atomic trap. We show that by employing the dipole potential induced by four highly detuned travelling-wave laser beams with appropriate phases and frequencies, one can coherently excite a trapped Bose-Einstein condensate composed of ultracold alkali atoms into a state rotating around the trap center. The connection to vortex states is discussed.

The experimental realizations of Bose-Einstein condensation in an atomic trap have led to a broad interest in the properties of coherent ultracold atomic gases. The comparison of trapped condensates to superfluid Helium suggests the possibility to study and analyse the rotational properties of the condensate. This is usually made by exciting states in which the atoms move collectively around one or several vortex lines. Recently the properties of vortex states of a trapped Bose-Einstein condensate have been studied theoretically.

In this letter we propose to employ the atom optic techniques to study the excitation of rotating states, being a coherent superposition of vortex states, from the ground state of the trap. In atom optics, the center-of-mass motion of atoms can be manipulated by light-induced gradient forces. We apply the same idea to a trapped Bose-Einstein condensate which can be described by a single macroscopic wave function. The interaction of light waves with ultracold atomic samples composed of Bose-Einstein condensates has well been studied in recent years. The vector nonlinear stochastic Schrödinger equations describe the dynamics of the ground- and excited-state atomic wave packets in the laser fields. For large laser detuning, adiabatic elimination of the excited-state component leads to a nonlinear scalar Schrödinger equation for the ground-state macroscopic wave function $\psi_g(\vec{x},t)$ of the condensate.

$$i\hbar \frac{d\psi_g(\vec{x},t)}{dt} = \left( H_{c.m.} + V_L(\vec{x}) \right) \psi_g(\vec{x},t) + \lambda |\psi_g(\vec{x},t)|^2 \psi_g(\vec{x},t).$$

The parameter $\lambda$ is connected to the s-wave scattering length of the atoms in the presence of the laser field. For simplicity we ignore the effects of the nonlinearity by setting $\lambda = 0$ in this paper. The incorporation of the nonlinear interaction in the proposed scheme will be the subject of a further publication. The center-of-mass Hamiltonian $H_{c.m.}$ includes the kinetic energy and the harmonic potential of the trap. According to the usual treatment of the quantum mechanical harmonic oscillator the Hamiltonian can be expressed as

$$H_{c.m.} = \hbar \omega \{ a_x^+ a_x + a_y^+ a_y + 1 \} + \hbar \omega_z \{ a_z^+ a_z + 1/2 \},$$

where $a_i := [x_i/R_i + i R_i p_i/\hbar]/\sqrt{2}$, $i = x, y, z$, is the annihilation operators for the harmonic oscillator in the i-direction and $R_i := \sqrt{\hbar/(M \omega_i)}$ is the trap size parameter. $M$ is the mass of the atoms, $\omega_\perp$ is the trap frequency for the x- and y-direction, and $\omega_z$ is the trap frequency in the z-direction. Because the trap is cylindrically symmetric around the z-axis, the Hamiltonian $H_{c.m.}$ commutes with the orbital angular momentum operator $J_3 = xp_y - yp_x$. This is a natural result of the conservation of orbital angular momentum along the z-direction in the cylindrically symmetric trap in the absence of the laser fields. To excite the rotating states from a Bose-Einstein condensate occupying the ground state of the trap which is the ground state $|0\rangle$ of the harmonic oscillators, one must introduce an additional external force in the x-y plane to break down the rotational symmetry of the ground state around the z-axis and create an angular momentum for a rotation of the ground state. The laser-induced dipole potential $V_L(\vec{x})$ in equation (1) can be used for this purpose. Before working out the exact form of the laser-induced dipole potential required for the excitation of vortex states, we need to understand the properties of the rotating states for the ground-state condensate. A vortex state rotating around the axis of symmetry is a state in which the macroscopic wave function $\psi_g(\vec{x})$ has the form $\exp(\im \phi) \chi(r,z)$, where $\varphi$ is the angle and $r$ is the radius in the x-y-plane and $n$ is an integer. It therefore can be characterized as being an eigenstate of $J_3$ with eigenvalue $n\hbar$, and of $H_{c.m.}$ as well. To deal with rotating states it is convenient to use the complete set of states $|n\rangle = (c_n^\dagger |0\rangle)^m / \sqrt{m!}$ to decompose the wavefunction of the condensate. Here $|0\rangle$ denotes the common ground state of the three-dimensional harmonic trap, and the vortex annihilation operators $c_{\pm}$ are given by

$$c_{\pm} = \frac{1}{\sqrt{2}} \{ a_x \pm \im a_y \}. $$

Using their commutators with $H_{c.m.}$ and $J_3$ one can show that $c_{\pm}$ both annihilate one quantum $\hbar \omega_\perp$ of phonon energy and change the angular momentum by $\pm \hbar$. 


In terms of the properties of the rotating states, we conclude that if the laser-induced dipole potential contains the appropriate combination of the vortex operators \((c_\perp, c_\perp^\dagger)\), one can create a superposition of vortex states from the ground-state condensate. To obtain such a dipole potential, we propose to employ four travelling-wave laser beams with the spatial configuration as shown in Fig.1.

The dipole potential induced by these laser beams has the following form:

\[
V_L(\vec{x}) = \frac{\hbar}{4(\Delta + \gamma/2)} \sum_{a,b=1}^{4} \Omega_a(\vec{x})\Omega_b^\dagger(\vec{x}) e^{-i(\omega_a - \omega_b)t} \tag{4}
\]

where we assume that all lasers are highly detuned from the transition frequency between the internal ground and excited state. In this case the excited state has only a small population and can be adiabatically eliminated. In equation (4) the symbols \(\Omega_a(\vec{x}) := \vec{d} \cdot \vec{E}_a^{(+)}(\vec{x})/\hbar\), \(a = 1, \ldots, 4\) denote the Rabi frequencies for the laser beams with frequencies \(\omega_a\). \(\vec{d}\) is the matrix element of dipole moment of the two-level atoms, and \(\vec{E}_a^{(+)}(\vec{x})\) is the positive frequency part of the \(a\)th laser’s electric field, and the average detuning of the lasers is given by \(\bar{\Delta}\). We assume \(\omega_a - \omega_b\) to be much smaller than \(\bar{\Delta}\) so that the difference between the true detuning of a laser and \(\bar{\Delta}\) is of higher order in \(1/\bar{\Delta}\). The spontaneous emission rate \(\gamma\) describes the incoherent loss of atoms from the condensate. The loss can be neglected in the time scale considered here for the case \(\Delta \gg \gamma\).

For the configuration shown in Fig.1, the two laser beams propagating along the \(x\)- and \(y\)-direction have the same frequency \(\omega_1 = \omega_2 \equiv \omega\). Their phases are chosen to be equal at the origin so that the Rabi frequencies of the two lasers have the form \(\Omega_1(\vec{x}) = \Omega_0\exp[ikx]\) and \(\Omega_2(\vec{x}) = \Omega_0\exp[iky]\). The other two laser beams are chosen to have the frequency \(\omega_3 = \omega_4 \equiv \omega'\) and propagate in directions slightly different from the \(x\)- and \(y\)-direction with a small angle \(\theta\). The frequency difference \(\Delta \omega := \omega - \omega'\) is assumed to be so small that the difference in the wavelength between the two pairs of laser beams can be neglected. In terms of the geometry shown in Fig.1, the wavevector of the third laser beam is given by \(k \cos(\theta)e_x - k \sin(\theta)e_y \approx \delta k e_x - \delta k e_y\), where \(\delta k := \kappa \theta\) denotes the deviation of the wavevector. Similarly the 4th laser’s wavevector is given by \(\delta k e_y - \delta k e_x\). The phases are chosen so that laser 4 is in phase with lasers 1 and 2 at the origin whereas the phase of laser 3 is shifted by \(-\pi/2\). This phase shift is essential for the scheme to work since in the final Hamiltonian it will lead to a \(\pi/2\) phase difference between \(a_x\) and \(a_y\), which exactly realizes the vortex operators \(\vec{c}\). The third Rabi frequency is then given by \(\Omega_3(\vec{x}) = \exp[-i\pi/2]\Omega_0\exp[i(kx - \delta ky)]\) and the fourth by \(\Omega_4(\vec{x}) = \Omega_0\exp[i(ky - \delta kx)]\). This arrangement of laser beams produces the dipole potential

\[
V_L(\vec{x}) = \frac{\hbar|\Omega_0|^2}{4\Delta} \left\{ 2 + e^{-i(kx-ky)}[1 + ie^{-i\delta k(x-y)}] \right. \\
+ e^{-i\Delta \omega t}[e^{i\delta kx} + ie^{i\delta ky} + ie^{-i(kx-ky) + i\delta kx}] + H.c. \right\}. \tag{5}
\]

To deduce explicitly from Eq. (5) an interaction that can create rotating states we consider the case that the laser intensity is so weak that the effective Rabi frequency \(|\Omega_0|^2/(4\Delta)|\) is much smaller than the trap frequency \(\omega_\perp\). This results in the following consequences: (i) If the atoms absorb and emit photons within a pair of laser beams with the same frequency (first square bracket in Eq. (5)) transitions between states of different energies \(H_{e.m.}\) are off-resonant and can therefore be neglected; (ii) If the atoms absorb a photon from one pair of laser beams and emit it into the other pair with different frequency, we have to take into account the frequency difference \(\Delta \omega\) between the two photons with different frequencies. It produces an energy shift so that transitions between states having the energy difference \(\pm \Delta \omega\) are resonantly enhanced whereas other off-resonant transitions are suppressed. Here we choose

\[
\Delta \omega = \omega_\perp \tag{6}
\]

so that only transitions between neighboring states of the harmonic oscillators are dominant.

Mathematically this can be incorporated by switching to the interaction picture with respect to \(H_{e.m.}\), so that the annihilation operators \(a_i\) are replaced by \(a_i \exp[-i\omega_\perp t]\) for \(i = x, y\). In the interaction picture, by performing the rotating wave approximation with respect to \(\omega_\perp\), the exponential \(\exp[ik(x-y)]\) has the form

\[
e^{ik(x-y)} \approx e^{-\eta^2/2} : J_0(2\eta\sqrt{q}) : \tag{7}
\]

\[
e^{-i\Delta \omega t} e^{ik(x-y)} \approx ie^{-\eta^2/2} : J_1(2\eta\sqrt{q}) q^m : , \tag{8}
\]

where for notational convenience we have introduced the operator \(q := (a_x - a_y)/\sqrt{2}\). The normally ordered Bessel functions are defined as

\[
J_n(\zeta \sqrt{q}) := \left( \frac{1}{2} \zeta q^{1/2} \right)^n \sum_{m=0}^{\infty} \frac{(-\zeta^2/4)^m}{m!(m+n)!}(q^1)^m q^m . \tag{9}
\]

Since the initial state for the problem discussed here is the condensate occupying the ground state of the trap, the Lamb-Dicke parameter \(\eta := k R_\perp\) is generally larger than one for the current realizable condensates \([9]\) which usually have transverse sizes \(R_\perp > 1\mu\text{m}\). As a result, Eqs. (7) and (8) imply that the terms containing \(\exp[\pm ik(x-y)]\) in \(V_L\) of Eq. (5) are exponentially suppressed for large \(\eta\) and therefore can be omitted for realistic cases. A physical interpretation of this effect can be
given by realizing that \( \exp(ikx) = \exp(i\eta(a_x + a_x^\dagger)) \) acts as a displacement operator on the trap states. Thus, for a large value of \( \eta \), it would cause a large displacement of the initial ground state of the trap. But since the interaction is weak the energy required for this displacement is not fully provided so that only the low-energy part of the displaced ground state can be realized.

On the other hand, the Lamb-Dicke parameter \( \delta \eta := \delta k R_\perp \) corresponding to the exponentials \( \exp[i\delta k x] \) and \( \exp[i\delta ky] \) in Eq. (3) depends on the angle \( \theta \) between the laser beams through \( \delta k = \theta R_\perp \) and can therefore be adjusted over a wide range. Here we require \( \theta \) to be very small so that \( \delta \eta \ll 1 \) is valid. This allows us to approximate the two exponentials by \( 1 + i\delta k x \) and \( 1 + i\delta k y \), respectively. The small angle between the laser beams lead to a simple dipole potential \( V_L \) which linearly depends on the combination of the creation and annihilation vortex operators of the trap.

In addition, in the interaction picture, the term \( \exp[-i\omega t\{1 + i\delta k x\}] \) can be approximated by \( i\delta k R_\perp a_x^\dagger/\sqrt{2} \) under the rotating-wave-approximation. A similar expression is obtained for \( \exp[-i\delta k y\{1 + i\delta k y\}] \) so that the final expression for the complete Hamiltonian in the interaction picture reads

\[
H_{\text{int}} = \frac{\hbar \Omega_0^2}{4\Delta} \left\{ 4 + i\delta k R_\perp \left[ c_x^\dagger - c_- \right] \right\} . \tag{10}
\]

The physical content of Eq. (10) is very clear. The second term in Eq. (10) exactly excites atoms in the ground state \( |0\rangle \) into the rotating \( c_- \) mode. Therefore the initial cylindrical symmetry is broken by the interaction Hamiltonian (10) and a rotating state of the center-of-mass motion of the ground state atoms in the trap is created with each atom gaining an angular momentum \( \hbar \). Thus, if we start from the ground state \( |0\rangle \) of the trap, this term produces a vortex state with angular dependence \( \exp(i\omega \cdot \vec{r}) \) in the coordinate representation. The term proportional to \( c_- \) is the corresponding Hermitian conjugate term and decreases the angular momentum by \( \hbar \). By ignoring the unimportant constant phase shift in Eq. (10), and by returning to the Schrödinger picture, the time evolution operator \( U(t) \) of the condensate has the simple form

\[
U(t) = \exp \left\{ \frac{t}{T_v} \left[ c_x^\dagger \exp(i\omega_\perp t) - c_- \exp(-i\omega_\perp t) \right] \right\} , \tag{11}
\]

where the time factor

\[
T_v := \frac{4\Delta}{|\Omega_0|^2 \delta k R_\perp} . \tag{12}
\]

characterises the time scale required for the creation of rotating states. When the time \( t \) is much smaller than \( T_v \), the ground state evolves into the state \( U(t)|0\rangle \) which approximately is a linear superposition of the first excited vortex state and the initial condensate occupying the ground state \( |0\rangle \) of the trap. With increasing interaction time \( t \), higher vortex states are created. Finally we have a coherent state for the vortex operator \( c_- \) for a long interaction time. This rotating coherent state is a coherent superposition of vortex states circulating in the same direction at different angular velocities. From the coordinate representation of the vortex states, \( (\vec{x}(t)|n\rangle = 2^{n/2}(x + iy)^n R_\perp^{-(n+1)} \exp[-((x^2 + y^2)/(2R_\perp^2))]) \), we can deduce the probability density for the coherent rotating state,

\[
\rho(t) = |\sqrt{N}\langle \vec{x}|U(t)|0\rangle|^2 = \frac{N}{\pi R_\perp^2} \exp\left\{ -\frac{(\vec{x} - \vec{x}_0(t))^2}{R_\perp^2} \right\} , \tag{13}
\]

where \( \vec{x}_0(t) \equiv \langle \vec{x} \rangle = (R_\perp t/T_v) \cos(\omega_\perp t) \vec{e}_x - \sin(\omega_\perp t) \vec{e}_y \) determines the center-of-mass trajectory of the condensate. Eq. (13) describes a condensate starting at the center of the trap and rotating with increasing amplitude thereby preserving its shape (see Fig. 2).

So far we have theoretically presented a simple scheme to create rotating states from a trapped ground-state Bose-Einstein condensate by a light-induced dipole potential. To analyze the experimental feasibility, we use the experimental data of \(^{23}\)Na Bose-Einstein condensate created by the MIT group [10] as an example. The mass of the \(^{23}\)Na atoms is \( M = 3.8 \times 10^{-23} \) g and the wavelength of the \(^{23}\)Na D-line transition is 589 nm so that the wave number of the laser beams is given by \( k = 1.06 \cdot 10^5 \text{ cm}^{-1} \). For the MIT trap the trap frequency in the x-y-plane is \( \omega_\perp \approx 2\pi \cdot 320\text{ Hz} \) so that the transversal trap size parameter is found to be \( R_\perp \approx 1.17 \mu \text{m} \). With these parameters the Lamb-Dicke parameter \( \eta = k R_\perp \) is approximately 12.4 so that the right hand side of Eqs. (13) can safely be ignored for low lying trap states. Hence the \(^{23}\)Na Bose-Einstein condensate is an appropriate candidate to satisfy the conditions required above.

The small Lamb-Dicke parameter \( \delta \eta = k\theta R_\perp \) is determined by the angle \( \theta \) and we can realize a value \( \delta \eta = 0.1 \) by adjusting \( \theta \) around 0.46 degrees. In addition, to make sure that only resonant interaction is included, the effective Rabi frequency \( \Omega_{\text{eff}} := |\Omega_0|^2/(4\Delta) \) must be much smaller than the trap frequency \( \omega_\perp \). This can be experimentally realized if the laser intensity is chosen so weak that \( \Omega_{\text{eff}} \) is about 200 Hz. In this case, the characteristic time scale for vortex excitation \( T_v \) is about 50 ms which is experimentally reasonable since it is much shorter than the life-time of the currently realized trapped condensates.

In conclusion we have shown that it is possible to create rotating states from a trapped ground-state Bose-Einstein condensate in a trap by using light-induced forces. Technically we show that dipole potential induced by four travelling-wave laser beams with an appropriate configuration in space, phase and frequency can be used to realize such a purpose. The important features of the
The proposed scheme are the following: (i) The low intensity of the laser beams to guarantee that only resonant transitions can occur; (ii) The small frequency difference between the two pairs of lasers to provides the energy difference required to bring the atoms from the ground state of the trap into a superposition of the first excited states. (iii) The small angle $\theta$ between the laser beams avoids the problem that the Lamb-Dicke parameter $\eta$ is so large that the transitions are strongly suppressed if the difference between the absorbed and re-emitted momentum is not very small. (iv) The phase difference between the laser beams leads to a special form of the light-induced potential $V_L$ which exactly results in a creation of vortex states.

Although the nonlinear interatomic interaction between the atoms is ignored in our discussion, it can be shown that the total Hamiltonian including the nonlinear interatomic interaction commutes with the second quantized form of the angular momentum operator $J_z$. Hence in principle the interatomic interaction does not affect the creation of rotating states, and only affects the spatial profile of the macroscopic wave function in the radical direction. The detailed discussions on the effect of the nonlinear interatomic interaction will be covered in a long publication [9].

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\[1\] M. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollet and R. Hulet, Phys. Rev. Lett. 75, 1687 (1995); M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend and W. Ketterle, Phys. Rev. Lett. 77, 416 (1996); D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 77, 420 (1996); M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend and W. Ketterle, Phys. Rev. Lett. 77, 988 (1996).

\[2\] S. Stringari, Phys. Rev. Lett. 76, 1405 (1996); F. Dalfovo and S. Stringari, Phys. Rev. A 53, 2477 (1996).

\[3\] Weiping Zhang, Phys. Lett. A 176, 225 (1993).

\[4\] G. Lenz, P. Meystre and E. M. Wright, Phys. Rev. Lett. 71, 3271 (1993).

\[5\] Weiping Zhang, D. F. Walls, and Barry Sanders, Phys. Rev. Lett. 72, 60 (1994); Weiping Zhang and D. F. Walls, Phys. Rev. A 49, 3799 (1994).

\[7\] L. You, M. Lewenstein and J. Cooper, Phys. Rev. A 50, R3565 (1994).

\[8\] J. Javanainen, Phys. Rev. Lett. 75, 1927 (1995).

\[9\] Karl-Peter Marzlin and Weiping Zhang, to be published.

\[10\] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
Figure Captions:

**Fig. 1:** Proposed scheme to create a rotating state being a coherent superposition of vortex states of the condensate. The laser beams 1 and 2 are oriented along the x- and y-axis. A frequency modulator produces a slight frequency difference between the beams (1,2) (solid line) and (3,4) (dashed line). A small angle between beam 1 and 3, and beam 2 and 4 is arranged by the adjustable mirror.

**Fig. 2:** The trajectory of the center-of-mass of the condensate in the x-y-plane under the influence of the four laser beams described above. x and y are given in units of the trap size $R_\perp$. During the rotation the shape of the condensate remains a Gaussian profile with a width of $2R_\perp$. 
