Bending, buckling and vibration analyses of MSGT microcomposite circular-annular sandwich plate under hydro-thermo-magneto-mechanical loadings using DQM

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ABSTRACT
The purpose of this paper is to investigate the bending, buckling, vibration analyses of microcomposite circular-annular sandwich plate with CNT reinforced composite facesheets under hydro-thermo-magneto-mechanical loadings are presented using first order shear deformation theory (FSDT) and modified strain gradient theory (MSGT) that includes three material length scale parameters. Also, an isotropic homogeneous core is considered for microcomposite circular-annular sandwich plate. The generalized rule of mixture is employed to predict mechanical, moisture and thermal properties of microcomposite sandwich plate. By using Hamilton’s principle, governing equations are solved by differential quadrature method (DQM) for a circular annular sandwich plate. The predicted results are validated by carrying out the comparison studies for the FGM plates by modified couple stress theory (MCST). The obtained results are given to indicate the influence of the material length scale parameter, core- to-facesheet thickness ratios, magnetic effect, thermal and moisture effects on the dimensionless deflection, critical buckling load, and natural frequency of microcomposite circular sandwich plate. The results can be employed in solid-state physics, materials science, nano-electronics, and nano electro-mechanical devices such as microactuators, and microsensor.

1. Introduction
In recent years, the polymeric composite material is used in various industrial aspects, but they have some weaknesses that limited them in particular applications. For increasing their material properties, various reinforcements can be added to them or emphasize the structures by changing of the shapes. This reinforced microcomposite material can be employed in junction micropipes connections, solid-state physics, material science, microtransducers, microactuators, microsensor and also due to their low specific weight, bending rigidity, excellent vibration characteristics and good fatigue properties can be widely considered in the field of transportation (helicopter blades, ship’s hull, etc.), urban services, and any other fields [1].

Experimental investigations showed that CNTs have extraordinary mechanical properties over carbon fibers [2,3]. Thereby, to improve the characteristics, the face sheets can be
laminated composites \[4\], functionally graded materials \[5\] or polymer matrix with reinforcements \[6\]. Thostenson and Chou \[7\] illustrated that the addition of nanotubes enhances the tensile modulus, yield strength and ultimate strengths of the polymer films. Their study has also showed that the polymer films with aligned nanotubes as reinforcements yield superior strength to randomly oriented nanotubes.

Moreover, utilization of single-walled carbon nanotubes (SWCNT) or double-walled carbon nanotubes (DWCNT) improve the characteristics of structures \[8–16\]. Belkorissat et al. \[17\] worked on vibration behavior of functionally graded nano-plate using a new nonlocal refined four variable model and they discussed on effects of nonlocal parameter, the thickness, the aspect ratio, and various materials on the dynamic response of the FG nano-plate. Zhu et al. \[18\] studied the static and free vibration of CNT reinforced plate based on the first-order shear deformation theory (FSDT). They considered polymer matrix with CNT reinforcement. It is predicted that the CNT volume fraction has greater influence on the fundamental frequency and the maximum center deflection. Ghorbanpour-Arani et al. \[19\] studied the buckling and post-buckling characteristics of CNT reinforced plates using the finite element method (FEM). It is shown that the reinforcement with CNT increasing, the load carrying capacity of the plate enhances. In the other work, they \[20\] used differential quadrature method (DQM) for non-linear vibration analysis of laminated composite Mindlin micro/nano-plates resting on orthotropic Pasternak medium, they showed that considering elastic medium increases the nonlinear frequency of system and the effect of boundary conditions becomes lower at higher nonlocal parameters. Mohammadimehr et al. \[21\] considered biaxial buckling and bending of smart nanocomposite plate reinforced by CNT using extended mixture rule approach. It is depicted that nonlocal deflection of smart nanocomposite plate decreases with an increase in the magnetic field intensity and also, the stability of smart nanocomposite plate increases in the presence of elastic foundation.

Some researchers are investigated different plate shapes based on variety of plate theories \[22–35\]. Bourada et al. \[36\] analyzed a new simple shear and normal deformations theory for functionally graded beams. They detected the effect of the inclusion of transverse normal strain on the deflections and stresses. Tounsi et al. \[37\] performed a new 3-unknowns non-polynomial plate theory for buckling and vibration of functionally graded sandwich plate. Bouazza et al. \[38\] inquired a refined hyperbolic shear deformation theory for thermal buckling analysis of cross-ply laminated plates. Ramezani \[39\] developed buckling and vibration of micro plate based on SGT and FSDT. He also showed that the size dependent effect at the micro scale is significant. Ke et al. \[40\] examined the vibration analysis of micro plate based on MCST and FSDT. They concluded that as the thickness is close to the material length scale parameter, the size effect on the natural frequency of micro plate is important. Sahmani and Ansari \[41\] investigated the vibration analysis of FG micro plate based on SGT and showed that as the thickness reaches to the material length scale parameter, the natural frequency increases significantly.

In general, the structures are subjected to different kind of loading such as mechanical, thermal, and electrical loading. Then, it is important to analyze the behavior of structural elements subjected to diversity loads or a combination of them \[42–51\]. Chikh et al. \[52\] presented thermal buckling analysis of cross-ply laminated plates using a simplified HSST and they illustrated the proposed HSST is accurate and simple in solving the thermal buckling behavior of laminated composite plates. Mohammadimehr and Salemi \[53\] developed SGT for bending and buckling analysis of FG Mindlin nanoplate. They concluded that considering SGT
leads to increase stiffness of nanoplate. Mohammadimehr and Shahedi [54] presented high-order buckling and free vibration analysis of two types sandwich beam including AL or PVC-foam flexible core and CNTs reinforced nanocomposite face sheets using generalized differential quadrature method (GDQM). Nonlinear bending of spherical shell under hydro-thermo-mechanical loads is examined by Lal et al. [55]. They used higher-order shear deformation theory (HSDT) and micro-mechanical model in their study and examined the hydrothermal effects are more detrimental as the working temperature increases and reaches closer to the glass transition temperature. Reddy and Berry [56] used Kirchhoff and FSDT to analyze the bending of circular plate based on MCST. They presented the analytical solutions of bending, buckling, and free vibration for the linear case.

At micro and nano scales, many researchers are demonstrated that the effect of size dependent is significant in mechanical properties of structures. It is well-known that classical continuum mechanics cannot indicate the size influences at micro and nano-scale structures. In order to overcome this problem, many nonlocal theories that consider additional material constants, such as the nonlocal elasticity theory [57,58], the strain gradient theory (SGT), modified couple stress theory (MCST) [59,60], and modified strain gradient theory (MSGT) [61–64] have been developed to characterize the size effect at micro, nano-scale structures by introducing an intrinsic length scale in the constitutive relations. After this, Al-Basyouni et al. [65] inquired size dependent bending and vibration analysis of functionally graded micro beams based on MCST and neutral surface position. Also Trinh et al. [66] established size dependent behaviour of FG microbeams using various shear deformation theories based on MCST. Mohammadimehr et al. [67] considered MSGT Reddy rectangular plate model for biaxial buckling and bending analysis of double-coupled piezoelectric polymeric nanocomposite reinforced by FG-SWCNT. A microstructure-dependent sinusoidal plate model based on the strain gradient elasticity theory developed for bending, buckling and free vibration of microplate by Akgöz and Civalek [68]. Ashoori Movassagh and Mahmoodi [69] determined a microscale modeling of Kirchhoff plate based on MSGT and they revealed that the differences between the deflections predicted by various theories are large when the plate thickness is small and comparable to the material length scale parameters. Wang et al. [70] investigated the size-dependent Kirchhoff micro-plate model using SGT. They showed that these size effects on the buckling load are not noticeable if the plate thickness is about 15 times larger than the material length scale parameter.

Sandwich construction has more attractive to the introduction of advanced composite materials for the face sheets, e.g. Fiber reinforced composites especially CNTs reinforced composite facesheets [71–74]. Sankar et al. [75] presented panel flutter characteristics of sandwich plates with CNT reinforced facesheets using an accurate higher-order theory. They considered bringing out the efficacy of the higher-order model. Zenkour and Sobhy [76] used FSDT, sinusoidal shear deformation theory (SSDT) and third-order shear deformation theory (TSDT) for thermal buckling of various types of FG sandwich plates. Their numerical examples covered the effects of the gradient index, plate aspect ratio, side-to-thickness ratio, loading type and sandwich plate type on the critical buckling load for sandwich plates. Sobhy [77] investigated buckling and free vibration of exponentially graded sandwich plates resting on elastic foundations under various boundary conditions. He considered the effect of the inhomogeneity parameter, and the foundation parameters on the natural frequencies and critical buckling loads. Based on the new first-order shear deformation theory, Thai et al. [78] considered the analysis of FG sandwich plates. Ansari et al. [79] performed bending, buckling
and free vibration analysis of size-dependent functionally graded circular annular microplates based on the modified strain gradient elasticity theory.

In the present work, based on MSGT that includes three material length scale parameters, the bending, buckling, vibration analyses of microcomposite circular-annular sandwich plate with CNT reinforced composite facesheets under hydro-thermo-magneto-mechanical loadings are presented using first order shear deformation theory (FSDT) are investigated. The generalized rule of mixture is employed to predict mechanical, moisture and thermal properties of microcomposite sandwich plate. By using Hamilton’s principle, governing equations are solved by differential quadrature method (DQM) for microcomposite circular annular sandwich plate. Also, influences of various parameters such as: material length scale parameter, magnetic field, percentage of moisture, thermo and mechanical loadings, core-to-facesheet thickness effect, and different kind of plate models are considered.

2. Theoretical formulation

Consider a circular annular sandwich plate composed of three layers as shown in Figure 1. It is assumed that the CNT reinforced layers as top and bottom facesheets are made from a mixture of FG SWCNT distribution in the thickness direction and the matrix is considered as an isotropic material, while a core is made of an isotropic homogeneous material as a middle layer. The core-to-facesheet thickness ratio is $h_c/h_f$, where, $h_c$ is the core thickness and $h_f$ is the facesheet thickness. The inner radius and outer radius of the sandwich plate are $a$ and $b$, respectively.

The effective properties of reinforced structures can be used by Mori-Tanaka scheme or by the rule of mixtures. In this article, it is used the rule of mixtures with correction factors to estimate the effective material properties of CNT reinforced matrix. The effective material properties of the CNT reinforced matrix are given by [80]:

$$
\begin{align*}
E_{11} &= \eta_1 V_{CNT} E_{11}^{CNT} + V_m E_m \\
\eta_2 &= \frac{V_{CNT}}{E_{22}^{CNT}} + \frac{V_m}{E_m} \\
\eta_3 &= \frac{V_{CNT}}{G_{12}^{CNT}} + \frac{V_m}{G_m}
\end{align*}
$$

where $\eta_i (i = 1, 2, 3)$ denotes force transformation between SWCNTs and polymeric matrix and $E_{11}, E_{22}$ and $G_{12}$ are the Young’s moduli and the shear modulus of the CNT, and $V_{CNT}$ and $V_m$ are the volume fraction of the CNT and the matrix, respectively. Others properties such as density, poison ratios, thermal, and hydro are written as [73]:

$$
\begin{align*}
\rho &= \rho_{CNT} V_{CNT} + \rho_m V_m \\
\nu_{12} &= \nu_{12}^{CNT} V_{CNT} + \nu_m V_m \\
\alpha_{11} &= \alpha_{11}^{CNT} V_{CNT} + \alpha_m V_m \\
\alpha_{22} &= (1 + \nu_{12}^{CNT}) \alpha_{22}^{CNT} V_{CNT} + (1 + \nu_m) \alpha_m V_m - \nu_{12} \alpha_{11} \\
\beta_{11} &= \beta_{11}^{CNT} V_{CNT} + \beta_m V_m \\
\beta_{22} &= (1 + \nu_{12}^{CNT}) \beta_{22}^{CNT} V_{CNT} + (1 + \nu_m) \beta_m V_m - \nu_{12} \beta_{11}
\end{align*}
$$

(2)
The volumes fractions are related by $V_{CNT} + V_m = 1$. The sandwich plate is made up of a homogeneous core with core thickness $h_c$ and two facesheets with thickness $h_f$. The facesheets are supposed to be reinforced with CNTs. It is supposed the volume fraction $V_{CNT}$ for the top facesheet as [81]:

$$V_{CNT} = 2 \left( \frac{t_1 - z}{t_1 - t_0} \right) V_{CNT}$$

and for the bottom facesheet as follows:

$$V_{CNT} = 2 \left( \frac{z - t_2}{t_3 - t_2} \right) V_{CNT}$$

where

$$V_{CNT}^* = \frac{w_{CNT}}{w_{CNT} + \left( \frac{\rho_{CNT}}{\rho_m} \right) (1 - w_{CNT})}$$

where $w_{CNT}$ is the mass fraction of the nanotubes. $\rho_{CNT}$ and $\rho_m$ are the mass densities of carbon nanotube and the matrix, respectively.

3. Kinematics equation of circular sandwich plate

The cylindrical coordinate system ($r$, $\theta$, $z$) and axial symmetry in loading and geometry are taken into account. The displacement field based on FSDT is expressed as follows [82]:

**Figure 1.** Schematic view of circular-annular sandwich plate with SWCNT reinforced composite facesheets (a) three dimensional (b) two dimensional.
\[ u_r(r, t, z) = u_0(r, t) + z\psi_r(r, t) \]
\[ u_\theta(r, t, z) = 0 \]
\[ u_z(r, t, z) = w_0(r, t) \]

where \( u_0, v_0 \) and \( w_0 \) denote middle surface displacements of the plate in \( r, \) and \( z \) directions. \( \psi_r \) is the rotation of middle surface at \( z = 0. \)

Strain–displacement relations according to FSDT can be expressed as follows:
\[ \varepsilon_{rr} = u_{rr} = u_{0,rr} + z\psi_{r,r} \]
\[ \varepsilon_{r\theta} = \frac{1}{r}u_r = \frac{1}{r}(u_0 + z\psi_r) \]
\[ \varepsilon_{rz} = \frac{1}{2}(u_{rr} + u_{z,r}) = \frac{1}{2}(\psi_r + w_{0,r}) \]

A term 'microcomposite sandwich plate' means the composite sandwich plate at micro scale. Using Hook's law, the constitutive equations for microcomposite circular-annular sandwich plate based on hydro-thermo-mechanical loadings can be stated as follows [50]:

\[
\begin{align*}
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{r\theta} \\
\sigma_{rz} \\
\sigma_{\theta\theta} \\
\sigma_{\theta z} \\
\sigma_{zr}
\end{bmatrix}
&= \begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{21} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & Q_{55} & 0 \\
0 & 0 & 0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rr} - \alpha_{11}\Delta T - \beta_{11}\Delta H \\
\varepsilon_{r\theta} - \alpha_{22}\Delta T - \beta_{22}\Delta H \\
\varepsilon_{rz} \\
\varepsilon_{\theta\theta} - \alpha_{44}\Delta T - \beta_{44}\Delta H \\
\varepsilon_{r\theta}
\end{bmatrix}
\end{align*}
\]

where \( \Delta T \) and \( \Delta H \) denote temperature and moisture changes, respectively, and \( Q_{ij} \) is the stiffness coefficient matrix for each layer that \( k = 1, 2, \) and \( 3, \) are defined as follows:
\[
\begin{align*}
Q_{11}^k &= \frac{E_{11}}{1-\nu_{12}\nu_{21}}, & Q_{22}^k &= \frac{E_{22}}{1-\nu_{12}\nu_{21}}, & Q_{12}^k &= \frac{\nu_{12}E_{11}}{1-\nu_{12}\nu_{21}}, \\
Q_{44}^k &= G_{23}, & Q_{55}^k &= G_{13}, & Q_{66}^k &= G_{12}, & Q_{16}^k &= Q_{26}^k = 0;
\end{align*}
\]

Hamilton’s principle is used herein to derive equations of motion. The principle can be obtained as follows:
\[
\int_0^T (\delta U + \delta V - \delta K)dt = 0
\]

where, \( \delta U, \delta V, \) and \( \delta K \) are the variations of strain energy, work done, and kinetic energy, respectively. The variation of kinetic energy for FSDT is given by:
\[
\delta K = \frac{1}{2} \int_A \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dzdA
\]
\[
= \int_A \left( -l_0(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w) - l_1(\ddot{u}\delta \psi_r + \ddot{v}\delta u + \ddot{w}\delta v) - l_2(\ddot{\psi}_r, \delta \psi_r) \right) dA
\]

The work done by external and shear forces can be stated as follows:
\[
\delta V = -\int_A [q\delta w + N_{rr}(w_r, \delta w_r)] r dr d\theta = \int_A \left( -q + \frac{1}{r}N_{rr} \frac{\partial}{\partial r}(rw_r) \right) r w r dr d\theta
\]

where \( N_{rr}, \) and \( q \) are shear forces and transverse load, respectively.

The magnetic field can be used into equations in two different ways, in a constitutive law or as a body force that in this work magnetic field is used as a body force [80]. The Maxwell relations and variation of external work done by magnetic fields for the microcomposite sandwich plate can be written as follows [80]:
\[
\vec{h} = \vec{V} \times (\vec{U} \times \vec{H}) \\
\vec{J} = \vec{V} \times \vec{h} \\
f_L = \eta (\vec{J} \times \vec{H}) \\
\delta V^H = \int_A (f_L \delta u_d)dz d\theta d\phi 
\]

where \( \vec{H}, \vec{h}, \vec{U}, \vec{J}, f_L, \eta \) and \( \delta V^H \) are magnetic intensity vector, perturbation of magnetic field vector, displacement vector, electric current density vector, Lorentz force, magnetic permeability and the external work variation, respectively.

By substituting Equation (6) into Equation (13) yields for magnetic intensity vector in z direction (0, 0, 0, Hz) variation of external work done by magnetic fields can be obtained as:

\[
\delta V^H = \int_A (f_L \delta u_0 + f_L z \delta \psi_t)dz d\theta d\phi \\
\delta V^H = \int_A \left( F_{rL}^0 \delta u_0 + F_{rL}^1 \delta \psi_t \right)dz d\theta d\phi 
\]

Thus, variation of total external work done can be stated as:

\[
\delta V = \int_A \left( \left( -q + \frac{1}{r} \sum_{n} \frac{\partial}{\partial r} (rw_{r' t}) \right) \delta w + (F_{rL}^0 \delta u_0 + F_{rL}^1 \delta \psi_t) \right)dz d\theta d\phi 
\]

The classical strain gradient theory is summarized. Mindlin developed a higher-order elastic theory in 1960s. Fleck and Hutchinson modified it to a second-order version in 1997. The major difference between the strain gradient theory and the conventional elastic theory is that the modified strain energy density, \( f \), depends on both the conventional strain (the symmetric part of the first-order deformation gradient) and on the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor as follows:

\[
f = f(\varepsilon_{ij}, \gamma_i, \eta_{ijk}, \chi_{ij}) \tag{16a}
\]

where the components of the strain tensor \( \varepsilon_{ij} \), dilatation gradient tensor \( \gamma_i \), deviatoric stretch gradient tensor \( \eta_{ijk} \) and symmetric rotation gradient tensor \( \chi_{ij} \).

The variation of strain energy based on MSGT can be obtained as follows:

\[
\delta U = \int_v \left( \sigma_{ij} \varepsilon_{ij} + p_i \gamma_i + \tau_{ijk} \eta_{ijk} + T_{ij} \chi_{ij} \right)dv \tag{16b}
\]

where the components of the strain tensor \( \varepsilon_{ij} \), dilatation gradient tensor \( \gamma_i \), deviatoric stretch gradient tensor \( \eta_{ijk} \) and symmetric rotation gradient tensor \( \chi_{ij} \) are defined as follows [84]:

\[
\varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \\
\gamma_i = \varepsilon_{mm,i} \\
\eta_{ijk} = \frac{1}{3} (\varepsilon_{ij,k} + \varepsilon_{jk,i} + \varepsilon_{ki,j}) - \frac{1}{15} \left( \delta_{ij} (\varepsilon_{mm,k} + 2 \varepsilon_{mk,m}) + \delta_{jk} (\varepsilon_{mm,i} + 2 \varepsilon_{mi,m}) + \delta_{ki} (\varepsilon_{mm,j} + 2 \varepsilon_{mj,m}) \right) \\
\chi_{ij} = \frac{1}{2} (e_{ipq} \varepsilon_{qij} + e_{ipq} \varepsilon_{qij})
\]
where \( u_i, \delta_{ij}, \) and \( e_{ijk} \) are the components of the displacement vector, the kronecker delta, and permutation tensor, respectively.

For a linear elastic material, the constitutive equation can be expressed by the components of kinematic parameters effective on the strain energy density as follows [79]:

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G e_{ij}
\]
\[
p_i = 2Gl_0^2 y_i
\]
\[
\tau_{ijk} = 2Gl_0^2 \eta_{ijk}
\]
\[
T_{ij} = 2Gl_0^2 x_{ij}
\]

where \( l_0, l_1 \) and \( l_2 \) are three material length scale parameters [84].

According to FSDT and substituting Equations (6) and (7) into Equation (17), dilatation gradient tensor, symmetric rotation gradient tensor, and deviatoric stretch gradient tensor can be derived as following form, respectively:

\[
\gamma_r = \frac{1}{4} \left( \psi_{r,r} - w_{0,rr} + \frac{1}{r} (w_{0,r} - \psi_r) \right)
\]

and

\[
\chi_{r\theta} = \frac{1}{4} \left( \psi_{r,r} - w_{0,rr} + \frac{1}{r} (w_{0,r} - \psi_r) \right)
\]

and

\[
\eta_{rrr} = \frac{1}{5} \left( 2u_{0,rr} - \frac{1}{r} u_{0,r} + \frac{u_0}{r^2} \right) + \frac{z}{5} \left( 2\psi_{r,rr} - \frac{1}{r} \psi_{r,r} + \frac{1}{r^2} \psi_r \right)
\]
\[
\eta_{zzz} = -\frac{1}{5} \left( 2\psi_{r,r} + \frac{1}{r} \psi_r - \frac{1}{r} w_{0,rr} \right)
\]
\[
\eta_{rrz} = \eta_{rr} = \frac{1}{15} \left( 8\psi_{r,r} + 4w_{0,rr} - \frac{1}{r} \psi_r \right)
\]
\[
\eta_{r\theta\theta} = \eta_{\theta r\theta} = \eta_{\theta\theta} = \frac{1}{15} \left( -3u_{0,rr} + \frac{4}{r} u_{0,r} - \frac{4}{r^2} u_0 \right) + \frac{z}{15} \left( -3\psi_{r,rr} + \frac{4}{r} \psi_{r,r} - \frac{4}{r^2} \psi_r \right)
\]
\[
\eta_{\theta r\theta} = \eta_{r\theta\theta} = \frac{1}{15} \left( -2\psi_{r,r} + \frac{4}{r} \psi_r - w_{0,rr} \right)
\]
\[
\eta_{zzz} = \eta_{zrr} = \frac{1}{15} \left( -3u_{0,rr} - \frac{1}{r} u_{0,r} + \frac{1}{r^2} u_0 \right) + \frac{z}{15} \left( -3\psi_{r,rr} + \frac{1}{r} \psi_r - \frac{1}{r} \psi_{r,r} \right)
\]

Also the following relations for stresses can be defined as [85]:

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G e_{ij}
\]
\[
p_i = 2Gl_0^2 y_i
\]
\[
\tau_{ijk} = 2Gl_0^2 \eta_{ijk}
\]
\[
T_{ij} = 2Gl_0^2 x_{ij}
\]
where $\kappa_s = \pi^2/12$ is the shear correction factor for circular plate [82].

Using Equation (A1) in Appendix 1 and separation of variables, the following governing equations of motion for microcomposite sandwich plate reinforced by CNTs based on MSGT and FSDT are derived as follows:

For $\delta u_0$:

$$
-M^0_{rr,r} - \frac{1}{r} M^0_{rr} + \frac{1}{r} M^0_{\theta\theta} + P^0_{r,rr} + \frac{1}{r} P^0_{r,r} - \frac{1}{r^2} P^0_{r} - M^1_{rr,r} - \frac{1}{r} M^1_{rr} + \frac{1}{r} M^1_{\theta\theta} + P^1_{r,rr} + \frac{1}{r} P^1_{r,r} - \frac{1}{r^2} P^1_{r}
$$

$$
+ \frac{1}{5} \left( 2 T^0_{rr,rr} + \frac{4}{r} T^0_{rr,\theta} + \frac{1}{r^2} T^0_{rr} + \frac{1}{r^2} T^0_{\theta\theta} + 2 T^1_{rr,rr} + \frac{4}{r} T^1_{rr,\theta} + \frac{1}{r^2} T^1_{rr} + \frac{1}{r^2} T^1_{\theta\theta} \right)
$$

$$
+ \frac{3}{15} \left( \frac{-4}{r^2} T^0_{r\theta \theta} - \frac{4}{r} T^0_{\theta \theta,rr} - 3 T^0_{r\theta \theta,\theta} - \frac{6}{r^2} T^0_{r\theta \theta,\theta} - 3 T^0_{r \theta \theta,rr} - \frac{6}{r^2} T^0_{r \theta \theta,rr} - \frac{6}{r} T^0_{r \theta \theta,rr} - \frac{1}{r} T^0_{r \theta \theta,rr} + \frac{1}{r} T^0_{r \theta \theta,rr} \right)
$$

$$
F^0_{rl} = -I_0 u - I_1 \psi_r
$$

(23)

For $\delta w_0$:

$$
-M^0_{rz,\theta} - \frac{1}{r} M^0_{rz} + \frac{1}{r} \left( T^0_{zzz,rr} - \frac{2}{r} T^0_{zzz,\theta} + 4 T^0_{rz,rr} + \frac{8}{r} T^0_{rz,\theta} - T^0_{z\theta z,rr} - \frac{2}{r} T^0_{z\theta z,\theta} \right)
$$

$$
+ \frac{1}{2} \left( -y^0_{r\theta \theta,rr} - \frac{3}{r} y^0_{r\theta \theta,\theta} \right) - q + \sum_{n} \left( w_n \frac{1}{r} w_r \right) = -I_0 \ddot{w}
$$

(24)

For $\delta \psi_r$:

$$
-M^0_{rr} - \frac{1}{r} M^0_{rr} + \frac{1}{r} M^0_{\theta\theta} + P^0_{r,rr} + \frac{1}{r} P^0_{r,r} - \frac{1}{r^2} P^0_{r} - M^1_{rr,r} - \frac{1}{r} M^1_{rr} + \frac{1}{r} M^1_{\theta\theta} + P^1_{r,rr} + \frac{1}{r} P^1_{r,r} - \frac{1}{r^2} P^1_{r}
$$

$$
+ \frac{1}{5} \left( 2 T^0_{rr,rr} + \frac{4}{r} T^0_{rr,\theta} + \frac{1}{r^2} T^0_{rr} + \frac{1}{r^2} T^0_{\theta\theta} + 2 T^1_{rr,rr} + \frac{4}{r} T^1_{rr,\theta} + \frac{1}{r^2} T^1_{rr} + \frac{1}{r^2} T^1_{\theta\theta} \right)
$$

$$
+ \frac{3}{15} \left( \frac{-4}{r^2} T^0_{r\theta \theta} - \frac{4}{r} T^0_{\theta \theta,rr} - 3 T^0_{r\theta \theta,\theta} - \frac{6}{r^2} T^0_{r\theta \theta,\theta} - 3 T^0_{r \theta \theta,rr} - \frac{6}{r^2} T^0_{r \theta \theta,rr} - \frac{6}{r} T^0_{r \theta \theta,rr} - \frac{1}{r} T^0_{r \theta \theta,rr} + \frac{1}{r} T^0_{r \theta \theta,rr} \right)
$$

$$
+ \frac{1}{2} \left( -y^0_{r\theta \theta,rr} - \frac{3}{r} y^0_{r\theta \theta,\theta} \right) + F^1_{rl} = -I_0 \ddot{u} - I_1 \ddot{\psi}_r
$$

(25)

By using of Equations (9) and (11), the stiffness component and inertia related terms for microcomposite sandwich plate are introduced as follows:
\begin{equation}
(A_{ij}, B_{ij}, D_{ij}) = \sum_{n=1}^{3} \frac{h_n}{h_{n-1}} \int (1, z, z^2) Q_{ij}^{(n)}(z) dz
\end{equation}

\begin{equation}
(l_0, l_1, l_2) = \sum_{n=1}^{3} \frac{h_n}{h_{n-1}} \int (1, z, z^2) \rho_{ij}^{(n)}(z) dz
\end{equation}

4. The governing equations of motion

Substituting Equation (26) into Equations (23)–(25) and by introducing the following non-dimensional quantities, we have:

\begin{align*}
\zeta &= \frac{r}{b}, \quad \eta = \frac{b}{h}, \quad U = \frac{u_0}{h}, \quad W = \frac{w_0}{h}, \quad \ell_i = \frac{l_i}{h} \\
\alpha_{ij} &= \frac{A_{ij}}{A_{110}}, \quad \alpha_{ij}^a = \frac{A_{ij}^a}{A_{110}}, \quad \alpha_{ij}^b = \frac{A_{ij}^b}{A_{110}}, \quad \alpha_{ij}^c = \frac{A_{ij}^c}{A_{110}}, \quad \alpha_1 = \frac{a_1}{A_{110}} \\
\beta_{ij} &= \frac{B_{ij}}{A_{110}h}, \quad \beta_{ij}^a = \frac{B_{ij}^a}{A_{110}h}, \quad \beta_{ij}^b = \frac{B_{ij}^b}{A_{110}h}, \quad \beta_{ij}^c = \frac{B_{ij}^c}{A_{110}h}, \quad \alpha_1 = \frac{b_1}{A_{110}h} \\
\gamma_{ij} &= \frac{D_{ij}}{A_{110}h^2}, \quad \gamma_{ij}^a = \frac{D_{ij}^a}{A_{110}h^2}, \quad \gamma_{ij}^b = \frac{D_{ij}^b}{A_{110}h^2}, \quad \gamma_{ij}^c = \frac{D_{ij}^c}{A_{110}h^2}, \quad \gamma_1 = \frac{d_1}{A_{110}h^2} \\
\psi &= \frac{q_0 b^2}{A_{110}h}, \quad N_{ij} = \frac{N_{ij}}{A_{110}}, \quad \tau = \frac{t}{b} \sqrt{\frac{A_{110}}{I_{10}}}
\end{align*}

The non-dimensional forms of equations of motion for microcomposites based on the first-order shear deformation theory and MSGT in terms of displacements have been defined in Appendix 2. It is observed from that for a symmetric sandwich plate \( b_{ij} = \overline{b_{ij}} = 0 \).

The associated boundary conditions at the inner and outer edges of the annular microcomposite plate can be handled as a same way and it can be written for clamped (C) edges as:

\begin{equation}
U(\zeta) = U(\zeta)_{,\zeta} = W(\zeta) = W(\zeta)_{,\zeta} = \psi(\zeta) = 0 \quad \text{at} \quad \zeta = a, 1
\end{equation}

5. Solution procedure

5.1. Solution method

DQM is employed to solve the governing equations and associated boundary conditions. In this method, derivative of any arbitrary function in arbitrary point can be rewritten in all intervals. In this article, this method is employed to discretize the governing equations and associated boundary conditions and the grid points are located based on Chebyshev points as:

\begin{equation}
\varphi_i = a + \frac{1 - a}{2} \left(1 - \cos \left(\frac{m - 1}{n - 1}\right)\right); \quad m = 1, 2, 3, \ldots, n
\end{equation}
where \( a = R_o/R_b \) is the inner-to-outer radius ratio and \( n \) is the total number of nodes along the radial direction. Also according to DQM, the displacement components \( u, w, \) and \( \psi \) can be described as follows:

\[
\begin{align*}
U &= \{u_1, u_2, \ldots, u_m\}, \quad W = \{w_1, w_2, \ldots, w_m\}, \quad \Psi = \{\psi_1, \psi_2, \ldots, \psi_m\} \\
\frac{\partial^k}{\partial \zeta^k} \{u, w, \psi\} = \sum_{m=1}^{n} C_{km} \{u_m, w_m, \psi_m\}
\end{align*}
\]

where \( u_m = u(\zeta_m, t), w_m = w(\zeta_m, t), \psi_m = \psi(\zeta_m, t) \), and \( C_{km} \) is the weighting coefficient.

The governing in Appendix 2 conjugating with the associated boundary condition yield the following matrix from equation

\[
M\ddot{d} + (K_e + N_{r0}K_g)d + \bar{q}_0 + \bar{p}_0 = 0
\]

where \( M, K_e, \) and \( K_g \) denote the mass, stiffness and geometric stiffness matrices, respectively. Also \( \bar{q}_0, \bar{p}_0, \) and \( d \) are the transverse load, the external distributed loads by magnetic field, and displacement vector, respectively. The Equation (31) can be used for the static bending, static buckling and free vibration analysis of microcomposite annular sandwich plate.

### 5.2. Static bending

By neglecting the inertia term and radial load \( \bar{N}_{r0} \), Equation (31) can be reduced to a static bending problem of microcomposite annular sandwich plate as:

\[
K_e d + \bar{q}_0 + \bar{p}_0 = 0
\]

The solutions of Equation (32) give the deflection of the microcomposite annular sandwich plate.

### 5.3. Static buckling

In the absent of the inertia term and all external loads, Equation (31) can be reduced to a static buckling eigenproblem of microcomposite annular sandwich plate as follows:

\[
(K_e + \bar{N}_{r0}K_g)d = 0
\]

By setting \( \bar{N}_{r0} = -P_{cr} \), the critical buckling load can be obtained. Where \( P_{cr} \) represent the critical buckling load.

### 5.4. Free vibration

By dropping all external loads and expressing the dynamic displacement vector \( d \) in form of \( d = d'e^{i\omega t} \), Equation (31) can be reduced to a free vibration of eigenvalue system of microcomposite annular sandwich plate as follows:

\[
(M\omega^2 + K_e)d' = 0
\]

where \( \omega \) and \( d' \) are the dimensionless natural frequency and the corresponding mode shape, respectively.
6. Numerical result and discussion

The numerical results of bending, buckling and free vibration of microcomposite annular sandwich plate with CNTs reinforced facesheet are presented. In this case, we take Polymethyl methacrylate (PMMA) as the matrix in which the CNTs are used as reinforcements [73]. The materials properties of which are assumed to be: \( \rho_m = 1150 \text{ Kg/m}^3 \), \( \nu_m = 0.34 \), \( \alpha_m = 45(1 + 0.0005\Delta T) \times 10^{-6}/K \), \( \beta_m = 2.67(1 + 0.0005\Delta T) \times 10^{-6}/K \), and Young’s modulus, \( E_m = (3.51 – 0.0047T)\text{GPa} \), where \( T = T_0 + \Delta T \), \( T_0 = 300K \) (room temperature). It must be noted that temperature throughout the thickness is constant. Single walled CNTs are used as reinforcements and the material properties are given in Table 1. The following CNT efficiency parameters are defined [73]: \( \eta_1 = 0.137 \), \( \eta_2 = 1.002 \) and \( \eta_3 = 0.175 \) for \( V_{\text{CNT}} = 0.12 \); \( \eta_1 = 0.149 \), \( \eta_2 = 1.381 \), and \( \eta_3 = \eta_2 \) for \( V_{\text{CNT}} = 0.17 \); \( \eta_1 = 0.141 \), \( \eta_2 = 1.585 \) and \( \eta_3 = 1.109 \) for \( V_{\text{CNT}} = 0.28 \) that it is assumed for this study, and also \( G_{12} = G_{13} = G_{23} \) is supposed. For the homogeneous core, we use Ti-6Al-4V Titanium alloy. This properties are: \( E_c = 122.56(1 – 4.586 \times 10^{-4}T)\text{GPa} \), \( \nu_c = 0.29 \), \( \alpha_c = 7.5788(1 + 6.638 \times 10^{-4}T – 3.147 \times 10^{-6}T^2) \times 10^{-6}/K \), and \( \rho_c = 4429 \text{ Kg/m}^3 \).

It should be mentioned that the experimental data is needed to evaluate the length scale parameters for homogeneous or composite microstructures. The length scale parameters of an isotropic homogeneous microbeam are experimentally obtained as \( L_1 = 17.6\mu m \) and \( i = 0,1,2 \) by Lam et al. [84]. However, there is no available experimental data relevant to the microcomposite structures in open literature. In order to quantitatively analyze the size effect of the microcomposite sandwich plate, the values of length scale parameters are approximately assumed to be equal to \( L_1 = 15\mu m \).

The accuracy of the present formulation is investigated through example of functionally graded annular micro plate based on MCST. Then, a parametric study is carried out to calculate the static bending, buckling and free vibration analysis of microcomposite annular sandwich plate. For the present study, five different core-to-facesheet thickness ratios, five length scale parameters, and different plate theories are considered. The temperature-dependent material properties of armchair SWCNTs (10, 10) as reinforcement are listed in Table 1 [53].

6.1. Convergence and comparison studies

Since there are no published results available for annular sandwich plate with CNT reinforced composite facesheets in open literature, the results of a FGM circular plate based on MCST are used for comparison.

For ensuring the accuracy and validity of this research, the results of first dimensionless natural frequency of FGM circular plate for different ratio of thickness to scale parameter are compared based on modified couple stress model in Table 2.

### Table 1. Material properties depended on temperature for (10, 10) SWCNT (R = 0.68 nm, L = 9.2 6nm, \( t = 0.067 \text{ nm} \), \( \rho = 1400 \text{ Kg/m}^3 \), \( \nu_{12} = 0.175 \)) [50].

| Temperature (K) | \( E_{11}^{\text{CNT}} \) (TPa) | \( E_{22}^{\text{CNT}} \) (TPa) | \( G_{12}^{\text{CNT}} \) (TPa) | \( a_{11}^{\text{CNT}} \) (1/GPa.K) | \( a_{22}^{\text{CNT}} \) (1/GPa.K) |
|-----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 300             | 5.6466                      | 7.08                        | 1.9445                      | 3.4584                      | 5.1682                      |
| 500             | 5.5308                      | 6.9348                      | 1.9643                      | 4.5361                      | 5.0189                      |
| 700             | 5.4744                      | 6.8641                      | 1.9644                      | 4.6677                      | 4.8943                      |
The results of the dimensionless critical buckling loads to number of nodes distributed along the radial direction of microcomposite circular sandwich plate in different ratio of thicknesses of core to facesheet are listed in Table 3 for three kinds of plate theories (the modified strain gradient, the modified couple stress, and classical theories). This results are obtained with differential quadratic method be choosing of $N$ and $N_c = 19$ is used in all of the following calculation.

6.2. Bending, buckling, and free vibration analyses of sandwich plate

In Table 4, the results of the first two dimensionless natural frequencies of microcomposite circular-annular sandwich plate with CNT reinforced composite facesheets in different ratio of thicknesses of core thickness to total thickness ratios and in three kind of theories are shown for the modified strain gradient theory, the modified couple

| N  | CT       | MCST     | MSGT     |
|----|----------|----------|----------|
| 7  | 1.178439 | 1.608476 | 2.480370 |
| 9  | 1.178244 | 1.580606 | 2.386449 |
| 11 | 1.178097 | 1.581314 | 2.390096 |
| 13 | 1.178000 | 1.581326 | 2.389746 |
| 15 | 1.177935 | 1.581341 | 2.389809 |
| 17 | 1.177890 | 1.581347 | 2.389797 |
| 19 | 1.177854 | 1.581347 | 2.389799 |
| 21 | 1.177834 | 1.581347 | 2.389799 |

Table 4. The first two dimensionless natural frequencies of microcomposite circular-annular sandwich plate with CNT reinforced composite facesheets for $V_{CNT} = 0.28$, $r_a/r_b = 0.1$, $\eta = 12$.

| h/l | $h_c/h$ | $\omega_1$ | $\omega_2$ | $\omega_1$ | $\omega_2$ | $\omega_1$ | $\omega_2$ |
|-----|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| 4   | 0.9     | 0.7006      | 1.7827      | 0.7653      | 1.9531      | 0.8680      | 2.1683      |
| 0.8 | 0.7781  | 1.9461      | 0.8333      | 2.0922      | 0.9171      | 2.2609      |
| 0.7 | 0.8470  | 2.0805      | 0.8947      | 2.2077      | 0.9626      | 2.3411      |
| 0.6 | 0.9080  | 2.1883      | 0.9495      | 2.3002      | 1.0040      | 2.4068      |
| 6   | 0.9     | 0.7006      | 1.7827      | 0.7302      | 1.8612      | 0.7829      | 1.9756      |
| 0.8 | 0.7781  | 1.9461      | 0.8033      | 2.0133      | 0.8454      | 2.1008      |
| 0.7 | 0.8470  | 2.0805      | 0.8688      | 2.1391      | 0.9024      | 2.2074      |
| 0.6 | 0.9080  | 2.1883      | 0.9270      | 2.2400      | 0.9538      | 2.2945      |
| 8   | 0.9     | 0.7006      | 1.7827      | 0.7174      | 1.8275      | 0.7489      | 1.8968      |
| 0.8 | 0.7781  | 1.9461      | 0.7924      | 1.9844      | 0.8173      | 2.0370      |
| 0.7 | 0.8470  | 2.0805      | 0.8594      | 2.1139      | 0.8792      | 2.0946      |
| 0.6 | 0.9080  | 2.1883      | 0.9188      | 2.2179      | 0.9346      | 2.0446      |
stress theory ($l_0 = l_1 = 0$), and classical theory ($l_0 = l_1 = l_2 = 0$). In this Table and all following result, the total thickness assumed to constant value and according to this assumption, reducing of core thickness increases the dimensionless natural frequencies because the sandwich plate strength will increase. Also it can be seen that by increasing of dimensionless length scale parameter in MCST and MSGT, the results are near to CT.

Figure 2 illustrated the dimensionless deflection curve of microcomposite circular sandwich plate with CNT reinforced composite facesheets subjected to a uniformly distributed transverse load $q_0 = 1 MN$ and a magnetic field effect. The obtained result from the modified strain gradient theory is evident that the effect of the magnetic field leads to lower dimensionless deflection or in the other word, the subjected plate will be more stiffness by increasing of magnetic field.

Figures 3 and 4 depict the effects of moisture and temperature changes on the dimensionless deflection of the microcomposite sandwich plate versus to inner-to-outer ratios. Dimensionless deflection leads to smaller value by rising of percentage of moisture and temperature. It is observed that the effect of temperature and moisture changes have approximately same trend but the influence of temperature is remarkably greater, and also they are shown that microcomposite sandwich plate become softer by increasing of percentage of moisture and temperature.

In Figure 5, dimensionless deflection of microcomposite sandwich plate to radius ratios is demonstrated for various dimensionless length scale parameter ratios ($h/l$). $V_{CNT} = 0.28$, $h/c/h = 0.8$, and $\eta = 6$ is assumed. It is observed that dimensionless deflection decreased by decreasing of $h/l$ and it means that by mounting number of scale parameter ratios specially more than 10 (as previously mentioned in Table 4), behavior of plate leads to CT.

Figure 6 shows the effect of inner-to-outer radius ratio on the dimensionless deflection of microcomposite circular sandwich plate with CNT reinforced facesheet with

![Figure 2. Dimensionless deflection of circular sandwich plate with CNT reinforced composite facesheets based on MSGT with magnetic field ($l_0 = 15\mu m$, $\eta = 6$, $h/l = 2$).](image-url)
predicted by different plate models. It is clear that among different kind of plate models, CT, MCST, and MSGT predict the maximum and minimum values of deflection, respectively, and this prediction is the same for all ratios of inner-to-outer radius.

**Figure 3.** The moistures effect on a dimensionless deflection for different thickness ratio.

**Figure 4.** The temperatures effect on a dimensionless deflection for different thickness ratio.
Figure 5. Effect of $h/l$ on the dimensionless deflection curve of microcomposite circular sandwich plate with CNT reinforced facesheets with predicted by MSGT.

Figure 6. Effect of inner-to-outer radius on the dimensionless deflection curve of circular sandwich plate with CNT reinforced composite facesheets corresponding to different plate models ($l_0 = 15\mu m$, $\eta = 6$, $h_c/h = 2$, $h/l = 2$).
The results of the dimensionless critical buckling loads of microcomposite circular-annular sandwich plate with CNT reinforced composite facesheets in different core-to-total thickness ratios and in three kinds of plate theories (the modified strain gradient, the modified couple stress, and classical theories) are tabulated in Table 5. By increasing of core thickness, the dimensionless critical buckling loads of microcomposite sandwich plate will be decreased and also the results for different dimensionless length scale parameter are same as natural frequencies and it means that by growing up of $h/l$ theories of MCST and MSGT lead to CT.

Figure 7 presents dimensionless critical buckling loads on the inner-to-outer radius ratios of microcomposite circular sandwich plate with CNT reinforced facesheet with

| $h/l$ | $h_z/h$ | $N_{cr}$ |
|-------|---------|---------|
|       |         | CT      | MCST    | MSGT    |
| 4     | 0.9     | 0.0325  | 0.0386  | 0.0503  |
|       | 0.8     | 0.0373  | 0.0426  | 0.0520  |
|       | 0.7     | 0.0410  | 0.0454  | 0.0528  |
|       | 0.6     | 0.0433  | 0.0469  | 0.0525  |
| 6     | 0.9     | 0.0325  | 0.0352  | 0.0407  |
|       | 0.8     | 0.0373  | 0.0397  | 0.0441  |
|       | 0.7     | 0.0410  | 0.0429  | 0.0464  |
|       | 0.6     | 0.0433  | 0.0449  | 0.0475  |
| 8     | 0.9     | 0.0325  | 0.0340  | 0.0372  |
|       | 0.8     | 0.0373  | 0.0386  | 0.0412  |
|       | 0.7     | 0.0410  | 0.0421  | 0.0441  |
|       | 0.6     | 0.0433  | 0.0442  | 0.0457  |

Figure 7. Effect of the dimensionless length scale parameter on the dimensionless critical buckling loads versus the inner-to-outer radius ratios.
various values of dimensionless length scale parameter. It is shown that despite the enhancement in dimensionless length scale parameter, increasing of the value of $\frac{r_a}{r_b}$ leads to higher critical buckling load.

Figure 8 illustrates a comparison between the dimensionless critical buckling loads predicted by different size dependent effects for various inner-to-outer radius ratios. It shows that for all values of inner-to-outer radius ratios, MSGT and CT predict the maximum and minimum values of critical buckling loads, respectively, and critical buckling loads is directly related to the inner-to-outer radius ratios.

Figure 9 shows the effect of inner-to-outer radius ratio on the dimensionless critical buckling load of microcomposite circular sandwich plate for different outer radius-to-thickness ratios. It is found that higher value of inner-to-outer radius ratio leads to the higher critical buckling load and also lower value of $\eta$ belongs to the higher critical buckling load.

Figure 10 shows the comparison between the dimensionless natural frequencies predicted by different size dependent effects for various dimensionless length scale parameter. It can be seen that for all values of dimensionless length scale parameter, MSGT and CT predict the maximum and minimum values of natural frequencies, respectively. Also increasing of the value of $h/l$ leads to lower natural frequencies and tends to the value predicted by CT.

The results of dimensionless natural frequencies on the inner-to-outer radius ratios of microcomposite circular sandwich plate are plotted in Figure 11 with various values of outer radius-to-thickness ratios. It is clear that natural frequencies have a same behavior with critical buckling load and it means that by increasing of inner-to-outer radius ratio, dimensionless natural frequencies leads to higher value and effect of values of $\eta$ have the same as behavior.

Figure 8. Comparison of the dimensionless critical buckling load predicted by different size dependent effects versus to the inner-to-outer radius ratios.
Figure 9. Effect of outer radius-to-thickness ratio on the dimensionless critical buckling load of microcomposite circular sandwich plates versus the inner-to-outer ratio ($l_0 = 15\mu m$, $h/l = 2$).

Figure 10. Comparison of the dimensionless natural frequency predicted by different size dependent effects versus to various values of dimensionless length scale parameter ($h_c/h_l = 8$).
The effect of inner-to-outer radius ratio on the dimensionless natural frequencies of microcomposite circular sandwich plate for different dimensionless length scale parameter is demonstrated in Figure 12. It is observed that higher value of inner-to-outer
radius ratio leads to the higher natural frequencies and the amount of inner-to-outer radius ratio will further impact the mount of dimensionless length scale parameter.

7. Conclusion

Based on modified strain gradient theory, bending, buckling, and free vibration of circular annular sandwich plate with carbon nanotubes reinforced composite facesheets was developed. The governing equations were derived by using Hamilton’s principle and according to first order shear deformation theory. To obtain the deflection, critical buckling loads and natural frequencies of microcomposite circular sandwich plate, the DQ method was utilized to discretize the governing differential equations along with clamped-clamped boundary conditions. It is observed that the effect of the magnetic field leads to lower dimensionless deflection, and also microcomposite sandwich plate become softer by increasing of percentage of moisture and temperature. It was depicted that by increasing the value of dimensionless length scale parameter, the maximum deflection of microcomposite circular sandwich plate increases. Furthermore, it was revealed that increasing of the value of $h/l$ leads to lower critical buckling load and natural frequency and also deflection of microcomposite circular sandwich plate increases and it tends to the value of predicted by CT. Moreover, it was found that higher value of inner-to-outer radius ratio leads to the higher natural frequency and critical buckling load. For all types of analysis, decreasing of core-to-total thickness ratios (by considering of constant value for total thickness) leads to strengthen the structures.

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The variation of strain energy for MSGT and FSDT microcomposite circular annular sandwich plate reinforced by CNTs subjected to hydro-thermo-mechanical loadings is written as:

\[ \text{Strain Energy} = \text{Function of CNT reinforcement and loadings} \]
\[
\delta U = \int_A \left\{ \left( -M_{rr,r} - \frac{1}{r} M_{rr} + \frac{1}{r} M_{r\theta} + \frac{1}{r^2} p_{r,r} - \frac{1}{r^2} p_{r} - M_{r,rr} + \frac{1}{r} M_{r,\theta} + M_{\theta,rr} + \frac{1}{r} p_{r,rr} - \frac{1}{r^2} p_{r,\theta} \right) + \frac{1}{5} \left( 2T_{rr,rr} + \frac{4}{r} T_{rr,\theta} + \frac{1}{r} T_{r\theta,rr} + 2T_{r\theta,\theta} + \frac{4}{r^2} T_{r\theta,\theta} + \frac{1}{r^2} T_{\theta,rr} \right) \right\} \, dr d\theta
\]

**Appendix 2**

The non-dimensional forms of equations of motion for microcomposite sandwich plate based on the first-order shear deformation theory and MSGT in terms of displacements can be expressed as follows:

\[
\begin{align*}
& a_{11} U_{,\xi\xi} + \frac{a_{12}}{\zeta} (U_{,\xi}) - \frac{a_{12}}{\zeta^2} (U) + b_{11} \psi_{,\xi,\xi} + \frac{b_{12}}{\zeta} \psi_{,\xi} - \frac{b_{12}}{\zeta^2} \psi = 0, \\
& + \frac{1}{\zeta} \left( a_{11} U_{,\xi} + \frac{a_{12}}{\zeta} (U) + b_{11} \psi_{,\xi} + \frac{b_{12}}{\zeta} \psi \right) - \frac{1}{\zeta} \left( a_{12} U_{,\xi} + \frac{1}{\zeta} a_{22} (U) + b_{12} \psi_{,\xi} + \frac{b_{22}}{\zeta} \psi \right) \\
& - \frac{2}{\zeta} \frac{a_{35}}{\eta^2} \left( U_{,\xi,\xi,\xi,\xi} + \frac{2}{\zeta} U_{,\xi,\xi,\xi} - \frac{3}{\zeta^2} U_{,\xi,\xi} + \frac{3}{\zeta^3} U_{,\xi} - \frac{3}{\zeta^4} (U) \right) \\
& - \frac{2}{\zeta} \frac{b_{35}}{\eta^2} \left( \psi_{,\xi,\xi,\xi,\xi} + \frac{2}{\zeta} \psi_{,\xi,\xi,\xi} - \frac{3}{\zeta^2} \psi_{,\xi,\xi} + \frac{3}{\zeta^3} \psi_{,\xi} - \frac{3}{\zeta^4} (\psi_{,\xi}) \right) \\
& + \frac{4}{5} \frac{a_{35}}{\eta^2} \left( -U_{,\xi,\xi,\xi,\xi} - \frac{2}{\zeta} U_{,\xi,\xi,\xi} + \frac{1}{3} \left( -\frac{1}{\zeta^2} U_{,\xi,\xi} + \frac{1}{\zeta^3} U_{,\xi} - \frac{1}{\zeta^4} (U) \right) \right) \\
& + \frac{4}{5} \frac{b_{35}}{\eta^2} \left( -\psi_{,\xi,\xi,\xi,\xi} - \frac{2}{\zeta} \psi_{,\xi,\xi,\xi} + \frac{1}{3} \left( -\frac{1}{\zeta^2} \psi_{,\xi,\xi} + \frac{1}{\zeta^3} \psi_{,\xi} - \frac{1}{\zeta^4} (\psi_{,\xi}) \right) \right) \\
& - \frac{a_1}{2} (U_{,\xi}) - b_1 (\psi_{,\xi}) - \frac{\eta}{\zeta} \left( a_{11} a_{11} + a_{12} a_{12} + a_{11} a_{11} + b_{11} b_{11} + b_{12} b_{12} \right) \\
& - \frac{\eta}{\zeta} \left( a_{12} a_{12} + a_{12} a_{12} + a_{12} a_{12} + b_{12} b_{12} + b_{12} b_{12} + b_{12} b_{12} \right) = \bar{I}_0 U_{,rr} + \bar{I}_1 \psi_{,rr}
\end{align*}
\]
$$k_s a_{55} (\eta \psi_r, \zeta + W, \zeta) + \frac{k_s a_{55}}{\zeta} (\eta \psi_r + W, \zeta)$$
$$- \frac{8 a_{55} \ell_2^2}{15 \eta^2} (W_{\psi r, \zeta} + 2 \zeta W, \zeta) - \frac{8 a_{55} \ell_1^2}{15 \eta} \left( 2 \psi_r, \zeta + \frac{15}{4 \zeta} \psi_r, \zeta \right)$$
$$+ \frac{a_{55} \ell_2^2}{4 \eta^2} \left( \eta \psi_r, \zeta + \frac{2}{\zeta} (\eta \psi_r, \zeta) - \frac{1}{\zeta^2} (\eta \psi_r) + \frac{1}{\zeta^3} (\eta \psi_r) - W_{\psi r, \zeta} - 2 \zeta W, \zeta + \frac{1}{\zeta^2} W_r, \zeta - \frac{1}{\zeta^3} W, \zeta \right)$$
$$+ \left( \sum_{\psi} \left( \frac{1}{\zeta} W_r, \zeta + W, \zeta \right) \right) + q = l_0 W_{\psi r}$$

$$b_{11} U, \zeta + b_{12} \frac{1}{\zeta^2} (U, \zeta) - b_{12} \frac{1}{\zeta^2} (U) + \left( d_{11} \psi_r, \zeta + d_{12} \frac{1}{\zeta} (\psi_r, \zeta) - d_{12} \frac{1}{\zeta^2} (\psi_r) \right)$$
$$+ \frac{1}{\zeta^3} \left( b_{11} U_r, \zeta + b_{12} \frac{1}{\zeta} (U) + \left( d_{11} \psi_r, \zeta + d_{12} \frac{1}{\zeta} (\psi_r) \right) \right)$$
$$- \frac{1}{\zeta^2} \left( b_{12} U_r, \zeta + b_{22} \frac{1}{\zeta} (U) + \left( d_{12} \psi_r, \zeta + d_{22} \frac{1}{\zeta} (\psi_r) \right) \right)$$
$$- k_s a_{55} (\eta^2 \psi_r + \eta W, \zeta)$$
$$- 2 \frac{b_{55} \ell_0^2}{\eta^2} \left( U, \zeta + \frac{2}{\zeta} U, \zeta + \frac{3}{\zeta^2} U_r, \zeta + \frac{3}{\zeta^3} U, \zeta \right)$$
$$- \frac{2}{\zeta^2} \psi_r, \zeta + \frac{2}{\zeta^2} \psi_r, \zeta + \frac{3}{\zeta^3} \psi_r, \zeta + \frac{3}{\zeta^4} \psi_r$$
$$- \frac{4 a_{55} \ell_0^2}{\eta^2} \left( \psi_r, \zeta + \frac{1}{\zeta} \psi_r, \zeta - \frac{1}{\zeta^2} \psi_r \right)$$
$$- \frac{4 b_{55} \ell_1^2}{5 \eta^2} \left( U, \zeta + \frac{2}{\zeta} U, \zeta + \frac{1}{3} \left( \frac{1}{\zeta^2} U_r, \zeta + \frac{1}{\zeta^3} U, \zeta \right) \right)$$
$$- \frac{4 d_{55} \ell_1^2}{5 \eta^2} \left( \psi_r, \zeta + \frac{2}{\zeta} \psi_r, \zeta + \frac{1}{3} \left( \frac{1}{\zeta^2} \psi_r, \zeta + \frac{1}{\zeta^3} \psi_r \right) \right)$$
$$+ \frac{4 a_{55} \ell_2^2}{15 \eta^2} \left( 8 W, \zeta + \frac{9}{\zeta} W_r, \zeta + 4 \eta \left( 4 \psi_r, \zeta + \frac{4}{\zeta} \psi_r, \zeta - \frac{1}{\zeta^2} \psi_r \right) \right)$$
$$+ \ell_2^2 \frac{a_{55}}{4 \eta} \left( \eta \psi_r, \zeta + \frac{\eta}{\zeta} \psi_r, \zeta - \frac{\eta}{\zeta^2} \psi_r - W_{\psi r, \zeta} - \frac{1}{\zeta} W, \zeta + \frac{1}{\zeta^2} W, \zeta \right)$$
$$- \tilde{b}_1 (U, \zeta) - d_1 (\psi_r, \zeta) - \frac{\eta}{\zeta} \left( b_{11} + b_{12} + b_{11}^\alpha + b_{12}^\alpha + d_1^\alpha + d_2^\alpha + d_1^\beta + d_2^\beta \right)$$
$$- \frac{\eta}{\zeta} \left( b_{12} + b_{22} + b_{22}^\beta + d_2^\beta + d_2^\beta + d_2^\beta + d_2^\beta \right) = \tilde{I}_{1} U_{\psi r} + \tilde{I}_{2} \psi_r_{\psi r}$$