Strata Separation for the Weil-Petersson Completion and Gradient Estimates for Length Functions

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Abstract

In general, it is difficult to measure distances in the Weil-Petersson metric on Teichmüller space. Here we consider the distance between strata in the Weil-Petersson completion of Teichmüller space of a surface of finite type. Wolpert showed that for strata whose closures do not intersect, there is a definite separation independent of the topology of the surface. We prove that the optimal value for this minimal separation is a constant \( \delta_{1,1} \) and show that the it is realized exactly by strata whose nodes intersect once. We also give a nearly sharp estimate for \( \delta_{1,1} \) and give a lower bound on the size of the gap between \( \delta_{1,1} \) and the other distances. A major component of the paper is an effective version of Wolpert’s upper bound on \( \langle \nabla \ell_a, \nabla \ell_b \rangle \), the inner product of the Weil-Petersson gradient of length functions. We also obtain new lower bounds on the systole for the Weil-Petersson metric on the moduli space of a punctured torus.

1 Strata separation

There are several natural quantities associated to the Weil-Petersson metric on Teichmüller and moduli space. One is the length of closed geodesics on moduli space or, equivalently, the translation length of pseudo-Anosovs on Teichmüller space. Another is the distance between strata on the boundary of Teichmüller space. Boundary strata are determined by a multi-curve on the underlying surface and two strata will have intersecting closures if and only if the associated multi-curves have positive intersection. Wolpert has shown that there is a definite separation (independent of the surface) between two strata whose closures don’t intersect. The key tool in the proof of this theorem are upper bounds on the gradients of length functions. In this note we will improve on Wolpert’s gradient estimates and use this to show that, as aspected, the minimal distance is realized when the multi-curves intersect exactly once. We will also see that nearly sharp bounds on the this distance follow easily for our gradient estimates.

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We begin with some setup before stating our results more precisely. Let $S$ be a hyperbolic surface of finite type and $\text{Teich}(S)$ the associated Teichmüller space. We let $\text{Teich}(S)$ be the completion with respect to the Weil-Petersson metric.

There is a natural stratification of $\text{Teich}(S)$ which can be described via length functions.

Given a closed curve (or multi-curve) $\alpha$ in $S$ we have the length function $\ell_\alpha: \text{Teich}(S) \rightarrow (0, \infty)$ given by letting $\ell_\alpha(X)$ be the length of the geodesic representative of $\alpha$ in $X$. Then $\ell_\alpha$ extends to a continuous function $\ell_\alpha: \overline{\text{Teich}(S)} \rightarrow [0, \infty]$. Given a multi-curve $\tau$ on $S$, we define the associated stratum $S_\tau(S) = \{ X \in \overline{\text{Teich}(S)} such that \ell_\alpha(X) = 0 if and only if \alpha \subseteq \tau \}.$

Points in $S_\tau(S)$ are noded hyperbolic structures on $S$ where the multi-curve $\alpha$ is the set of nodes.

We note that if $\sigma \subseteq \tau$ then $S_\tau(S) \subseteq S_\sigma(S)$ and it follows easily that $i(\sigma, \tau) = 0 if and only if d_{WP}(S_\sigma(S), S_\tau(S)) = 0.$

Wolpert proved the following:

**Theorem 1.1 (Wolpert Strata Separation, [Wo2])** There is a universal constant $\delta_0 > 0$ such that if $S_\sigma(S), S_\tau(S)$ are two strata with geometric intersection number $i(\sigma, \tau) \neq 0$ then $d_{WP}(S_\sigma(S), S_\tau(S)) \geq \delta_0.$

Wolpert does not give an explicit value for the constant $\delta_0$. We will give the optimal value for $\delta_0$.

We let $T$ be a punctured torus and $\alpha, \beta$ two curves on $T$ with $i(\alpha, \beta) = 1.$ Observe that there is an element of the mapping class group (i.e. an isometry of $\overline{\text{Teich}(T)}$) that takes any other pair of curves on $T$ that intersect once to $\alpha$ and $\beta$ so the constant $\delta_{1,1} = d_{WP}(\mathcal{S}_\alpha(T), \mathcal{S}_\beta(T))$ is well defined.

Using estimates on the Weil-Petersson gradient of the length functions $\ell_\alpha$ along with Wolpert’s description of the Alexandrov tangent cone for the Weil-Petersson completion, we prove that Wolpert’s constant $\delta_0$ is exactly $\delta_{1,1}.$ More precisely:

**Theorem 1.2** Let $S_\sigma(S), S_\tau(S)$ be two strata in $\text{Teich}(S).$ Then one of the following holds:

1. $i(\sigma, \tau) = 0$ and $d_{WP}(S_\sigma(S), S_\tau(S)) = 0.$
2. $i(\sigma, \tau) = 1$ and $d_{WP}(S_\sigma(S), S_\tau(S)) = \delta_{1,1}.$
3. $i(\sigma, \tau) > 1$ and $d_{WP}(S_\sigma(S), S_\tau(S)) \geq 7.61138.$

We note that it is not hard to see that the set of distances between strata (even for the punctured torus) is not a discrete set and Wolpert’s original theorem does not give that the constant $\delta_0$ is attained.

If $S$ is a punctured sphere then intersecting curves intersect at least twice and this setting needs a slightly separate analysis. See section 6.
Gradient estimates

Riera gave a beautiful formula for the inner product of the Weil-Petersson gradient of length functions $\ell_{\alpha}$ and $\ell_{\beta}$ (see Theorem 3.1). Using this formula Wolpert obtained the following estimate:

**Theorem 1.3 (Wolpert, \cite{Wol3})** Let $\ell_{\alpha}, \ell_{\beta}$ be geodesic length functions on $\text{Teich}(S)$, then

\[
\frac{2}{\pi} \ell_{\alpha}(X) \delta_{\beta}^{\alpha} \leq \langle \nabla \ell_{\alpha}, \nabla \ell_{\beta} \rangle \leq \frac{2}{\pi} \ell_{\alpha}(X) \delta_{\beta}^{\alpha} + O(\ell_{\alpha}(X)^2 \ell_{\beta}(X)^2)
\]

where $\delta_{\beta}^{\alpha}$ is the Kronecker delta function and where for $\ell > 0$ the term $O(\ell_{\alpha}(X)^2 \ell_{\beta}(X)^2)$ is uniform for $\ell_{\alpha}(X), \ell_{\beta}(X) < \ell$.

Following the same basic strategy of Wolpert’s proof we obtain the following effective form of Wolpert’s Theorem:

**Theorem 1.4** Let $\ell_{\alpha}, \ell_{\beta}$ be geodesic length functions with $\ell_{\alpha}(X), \ell_{\beta}(X) \leq \ell$. Then

\[
\frac{2}{\pi} \ell_{\alpha}(X) \delta_{\beta}^{\alpha} \leq \langle \nabla \ell_{\alpha}, \nabla \ell_{\beta} \rangle \leq \frac{2}{\pi} \ell_{\alpha}(X) \delta_{\beta}^{\alpha} + k(\ell) \ell_{\alpha}(X) \sinh(\ell_{\alpha}(X)/2) \sinh^2(\ell_{\beta}(X)/2)
\]

where $\delta_{\beta}^{\alpha}$ is the Kronecker delta function and $k$ is an explicit monotonically increasing function with $k(0) = 8/3\pi^2 \simeq 2.7019$ and $k(2\varepsilon_2) \simeq 2.7343$ where $\varepsilon_2 = \sinh^{-1}(1)$ is the Margulis constant.

Note that the estimates in these two results are independent of the topology of the surface. The lower bound (which follows directly from Riera’s formula) is optimal. While the upper bound is not optimal, for small lengths it is close. In particular these estimates give that for the constant $\delta_{1,1}$ we have

\[6.54933 < \delta_{1,1} < 6.63284.\]

**Notation**

In using decimals approximations the expression $a = a_0.a_1a_2a_3\ldots a_n$ where $a_0 \in \mathbb{N}_0$ and $a_i \in \{1,\ldots,9\}$ means that this is the first $n$ decimal places of $a$.

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3 Bounding the gradient

Riera’s formula

The main ingredient in Wolpert’s bound is the following formula of Riera for the Weil-Petersson inner product of length functions $\ell_\alpha$ and $\ell_\beta$. While it holds more generally we will restrict to the case where $\alpha$ and $\beta$ are simple closed curves that don’t intersect.

**Theorem 3.1 (Riera, [Rie])** Let $\ell_\alpha, \ell_\beta$ be geodesic length functions on $\text{Teich}(S)$, with $i(\alpha, \beta) = 0$. For $X \in \text{Teich}(S)$, let $d_i(X)$ be an enumeration of the lengths of the perpendiculars between the geodesic representatives of $\alpha, \beta$ on $X$. Then

$$\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle_X = \frac{2}{\pi} \left( \ell_\alpha(X) \delta_\beta^\alpha + \sum_i R(\cosh(d_i(X))) \right)$$

where

$$R(x) = x \log \left( \frac{x + 1}{x - 1} \right) - 2.$$ 

We briefly describe the strategy for the proof of Theorem 1.4. The function $R(\cosh(t))$ can be approximated by $ae^{-2t}$. To bound the sum in Riera’s formula we compare it to the integral of the function $e^{-2d(\alpha, z)}$ on the annular cover $A_\alpha$ of $X$ associated to $\alpha$ where $d(\alpha, z)$ is the distance be a point $z \in A_\alpha$ and the core geodesic. The integral over the annulus is a straightforward calculation. To compare it to the sum we decompose the annulus into the $r$-neighborhoods $N(h_i, r)$ of the lifts $h_i$ of $\beta$ to $A_\alpha$ where $r$ is an explicit constant given by the collar lemma and then compare the average value of $e^{-2d(\alpha, z)}$ on $N(h_i, r)$ to $e^{-2d_i(X)}$.

While the overall strategy of the proof is the same as Wolpert’s, our estimates within the proof are different. For example Wolpert only estimates the average of $e^{-2d(\alpha, z)}$ on disks rather than over the neighborhoods $N(h_i, r)$.

Preliminary estimates

Before proving the theorem we need to approximate $R$ and implement our averaging estimate. We begin with the former.

**Lemma 3.2** The function

$$a(t) = e^{2t}R(\cosh t)$$

is monotonically decreasing with

$$\lim_{t \to \infty} a(t) = \frac{8}{3}.$$
Proof: We have by [Rie] that for \( x > 1 \)
\[
R(x) = x \log \left( \frac{x+1}{x-1} \right) - 2 = \frac{2}{3x^2} + \frac{2}{5x^4} + \frac{2}{7x^6} + \ldots.
\]

Note that if we replace \( R \) by its series above, the individual terms of \( e^{2R}R(\cosh(t)) \) are not each monotonically decreasing. To prove the lemma we need a different expansion of \( a(t) \). Let \( u = e^{-t} \) and consider
\[
r(u) = u^{-2}R \left( \frac{u + 1/u}{2} \right).
\]

We have
\[
R \left( \frac{u + 1/u}{2} \right) = \left( \frac{u + 1/u}{2} \right) \log \left( \frac{(u + 1/u)+1}{2} \right) - 2
\]
\[
= \left( \frac{u + 1/u}{2} \right) \log \left( \frac{u^2 + 2u + 1}{u^2 - 2u + 1} \right) - 2
\]
\[
= (u + 1/u) \log \left( \frac{1+u}{1-u} \right) - 2
\]
\[
= (u + 1/u) \left( 2u + \frac{2u^3}{3} + \frac{2u^5}{5} + \ldots \right) - 2
\]
\[
= \sum_{n=1}^{\infty} \left( \frac{2}{2n-1} + \frac{2}{2n+1} \right) u^{2n} = \sum_{n=1}^{\infty} \frac{8n}{(2n-1)(2n+1)} u^{2n}
\]

Therefore
\[
r(u) = \sum_{n=0}^{\infty} \frac{8(n+1)}{(2n+1)(2n+3)} u^{2n}.
\]

From the expansion, it follows that \( r(u) \) is monotonically increasing on \([0, 1]\) and therefore \( a(t) = r(e^{-t}) \) is monotonically decreasing on \((0, \infty)\) and
\[
\lim_{t \to \infty} a(t) = r(0) = \frac{8}{3}.
\]

\( \square \)

Let \( d \) denote distance in the hyperbolic plane \( \mathbb{H}^2 \) and \( dA \) the hyperbolic area form. We will use the following lemma to estimate the integral of \( e^{-d(\alpha, z)} \) over \( N(h_i, r) \).

Lemma 3.3 Let \( g, h \) be disjoint geodesics with \( d(g, h) > r \) and let \( N(h, r) \) be the \( r \) neighborhood of \( h \). Then
\[
e^{2d(g, h)} \int_{N(h, r)} e^{-2d(g, w)} dA \geq 2 \tan^{-1}(\sinh(r)) \cosh^2(r) + 2 \sinh(r).
\]
Furthermore if \( d(g,h_n) \to \infty \) then

\[
\lim_{n \to \infty} \left( e^{2d(g,h_n)} \int_{N(h_n,r)} e^{-2d(g,w)} \, dA \right) = 2\tan^{-1}(\sinh(r)) \cosh^2(r) + 2\sinh(r).
\]

**Proof:** We first make a general observation. We consider the triple \((E, p, g)\) where \( E \) is a Borel set in \( \mathbb{H}^2 \), \( p \in E \) and \( g \) is a geodesic such that \( E \) is entirely on one side of \( g \). Note that if \( h \) is a horocycle tangent to \( g \) that is on the other side of \( E \) then \( d(q,h) \geq d(q,g) \) for all \( q \in E \). Therefore

\[
\int_E e^{-2d(g,w)} \, dA \geq \int_E e^{-2d(h,w)} \, dA.
\]

We can estimate the integral on the right by working in the half space model for \( \mathbb{H}^2 \) and normalizing so that \( p = i, \) \( g \) intersects the imaginary axis at \( y_0 > 1 \) and \( h \) is the horizontal line at height \( y_0 \). Then for \( w = (x,y) \in E \) we have

\[
d(h,w) = \log \left( \frac{y_0}{y} \right)
\]

and

\[
\int_E e^{-2d(h,w)} \, dA = \int_{E_0} \frac{y}{y_0} \cdot \frac{dxdy}{y^2} = \frac{A(E)}{y_0^2} = e^{2d(g,p)} A(E).
\]

Let \( g_n \) be a sequence of geodesics such that in the normalized picture \( g_n \) intersect at height \( y_n \) with \( \lim_{n \to \infty} y_n = \infty \). Let \( h_n \) be the horocycle for \( y = y_n \). Then for \( f_n(w) = d(w,h_n) - d(w,g_n) \) we have \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{H}^2 \). Therefore

\[
\lim_{n \to \infty} \left( e^{2d(g_n,p)} \int_{E} e^{-2d(g_n,w)} \, dA \right) = \lim_{n \to \infty} \left( e^{2d(g_n,p)} \int_{E} e^{-2d(h_n,w)} \, dA \right) = A(E).
\]

We now apply this to the geodesics \( g,h \). We consider the triple \((N(h,r), p, g)\) where \( p \) is the nearest point on \( h \) to \( g \). Then from above

\[
\int_{N(h,r)} e^{-2d(g,w)} \, dA \geq e^{-2d(g,h)} A(r)
\]

where \( A(r) \) is the Euclidean area of \( N(h,r) \) when \( h \) is the semicircle of radius 1 about 0.

To calculate \( A(r) \), we do some basic calculus. The boundary of \( N(h,r) \) meet the \( y \)-axis at an angle \( \phi \). Reflecting the bottom boundary component, we obtain a Euclidean circle of radius \( R \) with \( R \cos(\phi) = 1 \). We then consider a Euclidean circle \( C \) of radius \( R \) about the origin and let \( I(t) \) be the area between the vertical line \( x = t \) and \( C \). Then

\[
I(t) = 2 \int_1^R \sqrt{R^2 - x^2} \, dx.
\]
We observe that \( A(r) = \pi R^2 - 2a \left( \sqrt{R^2 - 1} \right) \). Substituting \( x = R \sin \theta \) we have

\[
I \left( \sqrt{R^2 - 1} \right) = 2R^2 \int_\phi^{\pi/2} \cos^2 \theta d\theta = R^2 \left( \frac{\pi}{2} - \phi - \frac{1}{2} \sin(2\phi) \right).
\]

Thus \( A(r) = R^2 (2\phi + \sin(2\phi)) = \frac{2\phi + 2\sin(\phi) \cos(\phi)}{\cos^2(\phi)} \).

By elementary hyperbolic geometry \( \cosh(r) = \sec(\phi), \sinh(r) = \tan(\phi) \) and \( \tanh(r) = \sin(\phi) \). Therefore

\[
A(r) = 2\tan^{-1}(\sinh(r)) \cosh^2(r) + 2\sinh(r).
\]

By elementary hyperbolic geometry we have:

\[
\cosh(r) = \sec(\phi), \quad \sinh(r) = \tan(\phi) \quad \text{and} \quad \tanh(r) = \sin(\phi).
\]

Therefore

\[
A(r) = 2\tan^{-1}(\sinh(r)) \cosh^2(r) + 2\sinh(r).
\]

\[\square\]

Using the above lemmas we will now prove Theorem 1.4.

**Proof of Theorem 1.4** We let \( A_\alpha \) be the annular cover corresponding to geodesic \( \alpha \) in \( X \). We let \( g \) be the core geodesic and \( h_i \) an enumeration of the lifts of \( \beta \) in \( A_\alpha \). We further let \( t_i \) be the distance from \( g \) and \( h_i \).

Then by [Rie] we have:

\[
\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle = \frac{2}{\pi} \left( \ell_\alpha \delta_\beta + \sum_i R(\cosh(t_i)) \right).
\]

The lower bound on \( \langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle \) then follows as \( R(x) > 0 \) for \( x > 1 \). We let \( T \) be the minimum distance between \( \alpha \) and \( \beta \). Then by the collar lemma

\[
T \geq 2 \sinh^{-1} \left( \frac{1}{\sinh(\ell/2)} \right).
\]

As \( d_i(X) \geq T \) for all \( i \), by the Lemma 3.2

\[
\sum_i R(\cosh(t_i)) \leq a(T) \sum e^{-2t_i}
\]

We now bound the expression on the right.

Fix constants \( r > 0 \) and \( s > 0 \) such that \( \sinh(r) = 1/\sinh(\ell_r/2) \) and \( \sinh(s) = 1/\sinh(\ell_\beta/2) \). Define \( N(h_i,s) \) to be the \( s \)-neighborhood of \( h_i \) and \( N(g,r) \) to be the \( r \)-neighborhood of \( g \). Then by the collar lemma the sets \( \{N(h_i,s)\}_{i=1}^\infty \), \( N(g,r) \) are mutually disjoint.

We give \( A_\alpha \) coordinates \( x,t \) where \( t \) is the distance to the core geodesic \( g \) and \( x \) parametrizes the length about the core geodesic. Then

\[
\sum_i \int_{N(h_i,s)} e^{-2t} dA \leq \int_{A_\alpha \setminus \{N(g,r)\}} e^{-2t} dA = 2 \int_0^{\ell_\alpha} \int r e^{-2t} \cosh(t) dt dx = \ell_\alpha \left( e^{-r} + e^{-3r/3} \right).
\]

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To estimate the terms in the sum on the left we note that the integrals can be lifted to the hyperbolic plane and then by Lemma 3.3

\[ \frac{1}{2 \tan^{-1}(\sinh(s)) \cosh^2(s) + 2 \sinh(s)} \int_{N(h, s)} e^{-2t} dA. \]

Substituting \( \sinh(s) = \text{csch}(\ell \beta/2) \) we get

\[ e^{-2t} \leq \frac{\sinh^2(\ell \beta/2)}{2 \left( \tan^{-1}(\text{csch}(\ell \beta/2)) \cosh^2(\ell \beta/2) + \sinh(\ell \beta/2) \right) + e^{-2t}}. \]

Therefore

\[ \sum_i e^{-2t_i} \leq \frac{\ell \alpha \sinh^2(\ell \beta/2)}{2 \left( \tan^{-1}(\text{csch}(\ell \beta/2)) \cosh^2(\ell \beta/2) + \sinh(\ell \beta/2) \right) + e^{-2t}} \left( e^{-t} + \frac{e^{-3t}}{3} \right). \]

As \( \sinh(r) = 1/\sinh(\ell \alpha/2) \) we have

\[ e^{-t} = \frac{\sinh(\ell \alpha/2)}{1 + \cosh(\ell \alpha/2)}. \]

Thus substituting and simplifying we get

\[ \sum_i e^{-2t_i} \leq \frac{\ell \alpha \sinh^2(\ell \beta/2) \sinh(\ell \alpha/2) (2 \cosh(\ell \alpha/2) + 1)}{3(1 + \cosh(\ell \alpha/2))^2 \left( \tan^{-1}(\text{csch}(\ell \beta/2)) \cosh^2(\ell \beta/2) + \sinh(\ell \beta/2) \right)}. \]

Therefore

\[ \sum_i R(\cosh(t_i)) \leq a(T)u(\ell \alpha)v(\ell \beta) \ell \alpha \sinh(\ell \alpha/2) \sinh^2(\ell \beta/2) \]

where

\[ u(\ell \alpha) = \frac{2 \cosh(\ell \alpha/2) + 1}{3(1 + \cosh(\ell \alpha/2))^2} \]

and

\[ v(\ell \beta) = \frac{1}{\tan^{-1}(\text{csch}(\ell \beta/2)) \cosh^2(\ell \beta/2) + \sinh(\ell \beta/2)}. \]

We now show that both \( u, v \) are monotonically decreasing. For \( u \) we implicitly define \( u_1 \) by \( u_1(\cosh(t/2)) = u(t) \). Then

\[ u_1(x) = \frac{2x + 1}{3(x + 1)^2} \]

giving

\[ u_1'(x) = \frac{2(x + 1)^2 - 2(x + 1)(2x + 1)}{3(x + 1)^4} = \left( \frac{2x}{3(x + 1)^3} \right). \]

Therefore \( u_1 \) is monotonically decreasing and by the chain rule so is \( u \).
For \( v \) we implicitly define \( v_1(\sinh(t/2)) = 1/h(t) \). Then

\[
v_1(x) = (1 + x^2)\tan^{-1}\left(\frac{1}{x}\right) + x
\]
giving

\[
v_1'(x) = 2\tan^{-1}\left(\frac{1}{x}\right) + (1 + x^2)\left(\frac{1}{1 + \frac{1}{x^2}}\right)\left(-\frac{1}{x^2}\right) + 1 = 2\tan^{-1}\left(\frac{1}{x}\right).
\]

Therefore \( v_1 \) is monotonically increasing, and \( v \) is monotonically decreasing. Thus we have

\[
u_1(\ell_\alpha) \leq u(0) = 1/4 \text{ and } v(\ell_\beta) \leq v(0) = 2/\pi.
\]

Thus

\[
\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle \leq 2\pi^2 a(T) \ell_\alpha \sinh(\ell_\alpha/2) \sinh^2(\ell_\beta/2).
\]

As \( T \geq 2 \sin^{-1}(1/\sinh(\ell/2)) \) then we define \( k \) by

\[
k(\ell) = \frac{\pi}{a} a(2 \sin^{-1}\left(\frac{1}{\sinh(\ell/2)}\right)) \geq \frac{a(T)}{\ell^2}.
\]

The result follows. \( \square \)

We define \( F(x) = \pi^2 k(x)u(x)v(x) \). Then from above, we have the following:

**Corollary 3.4** Let \( S \) be a finite type hyperbolic surface and \( \ell_\alpha \) be a geodesic length function for \( \alpha \) simple. Then

\[
\frac{2\ell_\alpha(X)}{\pi} \leq ||\nabla \ell_\alpha(X)||^2 \leq \frac{2\ell_\alpha(X)}{\pi} \left(1 + F(\ell_\alpha(X))\sinh^3(\ell_\alpha(X)/2)\right).
\]

### 4 Bounding strata separation

We now give an explicit bound on Wolpert’s strata separation. In the following, as the surface \( S \) is understood, we will denote strata as \( \mathcal{S}_c \) where \( c \) is a multicurve.

Motivated by Theorem 1.4 we define

\[
H(L) = \int_0^L \frac{dx}{\sqrt{\frac{2\pi}{\pi} (1 + F(x)\sinh^3(x/2))}} \quad K(L) = \int_0^L \frac{dx}{\sqrt{\frac{2\pi}{\pi}}} = \sqrt{2\pi L}.
\]

We denote the level sets of the length function \( \ell_\alpha \) by

\[
\mathcal{S}_\alpha^L = \ell_\alpha^{-1}(L) \subseteq \text{Teich}(S).
\]

Then we have
Lemma 4.1 Let $\alpha$ be a simple closed curve on $S$ and $a, b \in [0, \infty)$. Then
\[
|H(b) - H(a)| \leq d_{WP}(\mathcal{S}_{a}^{a}, \mathcal{S}_{a}^{b}) \leq |K(b) - K(a)|.
\]

Proof: We assume that $a \leq b$ and let $c : [0, k] \to \overline{\text{Teich}(S)}$ be a parametrized curve with $c(0) \in \mathcal{S}_{a}^{a}$ and $c(k) \in \mathcal{S}_{a}^{b}$. Then the Weil-Petersson length of $c(t)$ is
\[
L_{WP}(c) = \int_{0}^{k} \|c'(t)\|dt.
\]
We let $x = \ell_{a}(c(t))$. Then from Corollary 3.4
\[
\sqrt{\frac{2x}{\pi}} \leq \|\nabla \ell_{a}(c(t))\| \leq \sqrt{\frac{2x}{\pi}} (1 + F(x) \sinh^{3}(x/2)).
\]
If $c$ is the path given by $\nabla \ell_{a}$ then changing variables to $x$ we have
\[
L_{WP}(c) = \int_{0}^{k} \|c'(t)\|dt = \int_{0}^{k} \frac{d\ell_{a}(c'(t))dt}{\|\nabla \ell_{a}(c(t))\|} \leq \int_{a}^{b} \frac{dx}{\sqrt{\frac{2x}{\pi}}} = K(b) - K(a).
\]
Therefore
\[
d_{WP}(\mathcal{S}_{a}^{a}, \mathcal{S}_{a}^{b}) \leq L_{WP}(c) \leq K(b) - K(a).
\]

For the lower bound, we consider a general curve $c$. By Cauchy-Schwarz,
\[
d\ell_{a}(c'(t)) = \langle c'(t), \nabla \ell_{a}(c(t)) \rangle \leq \|\nabla \ell_{a}(c(t))\| \cdot \|c'(t)\|.
\]
We let $E \subseteq [0, k]$ be the subset where $x = \ell_{a}(c(t))$ is monotonically increasing. Then
\[
L_{WP}(c) = \int_{0}^{k} \|c'(t)\|dt \geq \int_{E} \|c'(t)\|dt \geq \int_{E} \frac{d\ell_{a}(c'(t))dt}{\|\nabla \ell_{a}(c(t))\|}.
\]
On $E$ we change variables to $x$ giving
\[
L_{WP}(c) \geq \int_{E} \frac{d\ell_{a}(c'(t))dt}{\|\nabla \ell_{a}(c(t))\|} \geq \int_{a}^{b} \frac{dx}{\sqrt{\frac{2x}{\pi}} (1 + F(x) \sinh^{3}(x/2))} = H(b) - H(a).
\]
Therefore
\[
d_{WP}(\mathcal{S}_{a}^{a}, \mathcal{S}_{a}^{b}) \geq H(b) - H(a).
\]

\begin{theorem}
If $\mathcal{S}_{\sigma}, \mathcal{S}_{\tau}$ are two strata with geometric intersection number $i(\sigma, \tau) \neq 0$ then
\[
d_{WP}(\mathcal{S}_{\sigma}, \mathcal{S}_{\tau}) \geq 2H(2\varepsilon) \simeq 6.54933.
\]
\end{theorem}
Therefore Reira’s formula gives the lower bound $\alpha$ Thus there are lifts of $X$ Weil-Petersson metric. Therefore the gradient paths from point set of $\iota$ involution. The involution $\alpha$ and $\beta$ opposite sides of the regular ideal quadrilateral. Then $\alpha$ therefore\( -\nabla \ell_\alpha\) (resp. $-\nabla \ell_\beta$) determines a path from $X$ to $\mathcal{S}_\alpha(T)$ (resp. $\mathcal{S}_\beta(T)$). Therefore $\delta_{1,1} \leq 2K(2\varepsilon) \approx 6.65603$.

We note we can slightly improve this bound by making the following observation. The involution $t$ of $\text{Teich}(T)$ interchanging $\alpha, \beta$ is an isometry of the Weil-Petersson metric. Therefore the gradient paths from $X$ are on the fixed point set of $t$ given by $\sinh(\ell_\alpha/2) \sinh(\ell_\beta/2) = 1$. Thus there are lifts of $\alpha$ (resp. $\beta$) with distance is $n\ell_\beta$ for all $n > 0$ (resp. $n\ell_\alpha$). Therefore Reira’s formula gives the lower bound

$$|\nabla \ell_\alpha|^2 \geq \frac{2}{\pi} \left( \ell_\alpha + \sum_{n=1}^{\infty} R(\cosh(n\ell_\beta)) \right).$$

Recall the function $r$ from Lemma 3.2 satisfying $r(e^{-t}) = e^{2R(\cosh(t))}$. Then by the expansion of $r$ we have

$$R_1(e^{-t}) = \sum_{n=1}^{\infty} R(\cosh(nt)) = \sum_{n=1}^{\infty} \frac{8n}{(2n-1)(2n+1)} \left( \frac{e^{-2nt}}{1-e^{-2nt}} \right)$$

Therefore as $\sinh(\ell_\alpha/2) \sinh(\ell_\beta/2) = 1$ we have

$$e^{-\beta} = \frac{\cosh(\ell_\alpha/2) - 1}{\cosh(\ell_\alpha/2) + 1}.$$ 

Thus integrating along the gradient lines we get

$$\delta_{1,1} \leq 2 \int_0^{2\varepsilon_2} \frac{dx}{\sqrt{\frac{2}{\pi} \left( x + R_1 \left( \frac{\cosh(\ell_\alpha/2) - 1}{\cosh(\ell_\alpha/2) + 1} \right) \right)}} \approx 6.63283.$$ 

We note that there is only a 1.3% difference between the two bounds 6.54933 and 6.63284 (compare with [BB]).

**Proof:** Let $c: (0,k) \to \text{Teich}(S)$ is a parametrized curve joining the two strata. Let $\alpha \in \sigma$ and $\beta \in \tau$ be curves such that $i(\alpha, \beta) \neq 0$. Assume $c(0) \in \mathcal{S}_\sigma$ and $c(k) \in \mathcal{S}_\tau$. By the collar lemma, we have $t_0 \in (0,a)$ such that for $X = c(t_0)$ then $\ell_\alpha(X) = 2\varepsilon_2$ and $\ell_\beta(X) \geq 2\varepsilon$. We let $c_1$ be $c$ restricted to $[0,t_0]$ and $c_2$ be $c$ restricted to $[t_0,k]$. Then by Lemma 4.1 above, $L_{WP}(c_1) \geq H(2\varepsilon)$ and $L_{WP}(c_2) \geq H(2\varepsilon)$. Thus $L_{WP}(c) \geq 2H(2\varepsilon)$.

We evaluate $H$ using numerical integration (see Computation section below) and obtain $2H(2\varepsilon) \approx 6.54933$. \(\Box\)

**Lemma 4.3** The constant $\delta_{1,1} \in (6.54933, 6.63284)$.

**Proof:** The lower bound comes from the previous theorem. For the upper bound, we let $T$ be the punctured torus and $X \in \text{Teich}(T)$ be given by identifying opposite sides of the regular ideal quadrilateral. Then $X$ has two simple geodesics $\alpha$ and $\beta$ intersecting perpendicularly with $\ell_\alpha(X) = \ell_\beta(X) = 2\varepsilon_2$. The flow of the gradient $-\nabla \ell_\alpha$ (resp. $-\nabla \ell_\beta$) determines a path from $X$ to $\mathcal{S}_\alpha(T)$ (resp. $\mathcal{S}_\beta(T)$). Therefore $\delta_{1,1} \leq 2K(2\varepsilon) \approx 6.65603$.
Computation

The calculation of $H$ is by numerical integration using Mathematica. The function integrated is

$$G(x) = \frac{1}{\sqrt{\frac{2\pi}{x} (1 + F(x) \sinh^3(x/2))}}$$

where $F(x) = \pi^2 k(x)u(x)v(x)$. The functions $k, u,$ and $h$ are elementary functions involving trigonometric, exponential and log functions. To calculate the function $k$ for $x$ small with precision we cannot use its description in terms of basic functions and must instead use a series expansion. The reason for this is that although $k$ is monotonic and $k(0) = 8/3\pi^2$, the expression for $k$ for small $x$ is the difference of two large numbers with the computation being of the form $(x^{-4} + 8/3\pi^2) - x^{-4}$. To avoid this problem and have arbitrarily high precision, we use the series for the function $r$ introduced in Lemma 3.2 and the relation

$$k(x) = \frac{1}{\pi^2 r} \left( \frac{\cosh(\ell_a/2) - 1}{\cosh(\ell_a/2) + 1} \right).$$

Similarly for the function $R_1$ above, we use the series expansion by $r$.

![Figure 1: Graph of H versus K](image)

Above is a comparison between the upper and lower bounds given by $H$ and $K$.

5 Orthogonal projection onto strata

The Weil-Petersson completion $\overline{\text{Teich}(S)}$ is a CAT(0) space. Let $\tau$ be a multicurve in $S$, $\mathcal{S}_\tau$ the associated strata and $S_\tau = S \setminus \tau$ the noded surface. Then $\mathcal{S}_\tau$ is isometric to $\text{Teich}(S_\tau)$ and the closure $\overline{\mathcal{S}_\tau}$ is convex in $\overline{\text{Teich}(S)}$ (see [Yun], [Wol]). Note that if $S_\tau$ is disconnected then $\text{Teich}(S_\tau)$ is the product of the Teichmüller spaces of each component.
Now, let $\tau_0$ and $\tau_1$ be multicurves in $S$ and $S_{\tau_0}$ and $S_{\tau_1}$ the associated strata. We will show that the infimum of distance between $S_{\tau_0}$ and $S_{\tau_1}$ is attained on any stratum $S_{\sigma}$ for which $\sigma$ is mutually disjoint from both $\tau_0$ and $\tau_1$. Specifically we prove:

**Theorem 5.1** Let $\tau_0, \tau_1$, and $\sigma$ be multicurves with $i(\tau, \sigma) = 0$ for $i = 0, 1$. If $\hat{\tau}_i = \tau_i \cup \sigma$ then

$$d_{WP}(S_{\tau_0}, S_{\tau_1}) = d_{WP}(S_{\hat{\tau}_0}, S_{\hat{\tau}_1}).$$

In a CAT(0) space the nearest point projection to a convex set is 1-Lipschitz (see [BH, Proposition 2.4]). Here we will project to the closure $S_{\sigma}$ and the theorem will follow once we show that this projection maps $S_{\tau_i}$ into $S_{\hat{\tau}_i} \subset S_{\tau_i}$. This will follow quickly from Wolpert’s characterization of tangent cones in the Weil-Petersson metric (see [Wol3]). We begin by reviewing this work.

Given $p, q, r \in \text{Teich}(S)$ we let $\angle(p; q, r)$ be the angle at $p$ in the comparison Euclidean triangle with side lengths $d_{WP}(p, q), d_{WP}(q, r)$ and $d_{WP}(p, r)$. Let $b(t)$ and $c(t)$ be constant speed geodesic segments starting at $p$. The CAT(0) property implies that if $0 < s_0 \leq s_1$ and $0 < t_0 \leq t_1$ then

$$\angle(p; b(s_0), c(t_0)) \leq \angle(p; b(s_1), c(t_1))$$

and therefore

$$\angle(b, c) = \lim_{t \to 0} \angle(p; b(t), c(t))$$

is defined. Let $|b|$ and $|c|$ be the (constant) speed of the two segments. We define an equivalence relation where $b \sim c$ if $|b| = |c|$ and $\angle(b, c) = 0$. If we take all geodesic segments beginning at $p$ and take the quotient under this equivalence relation we have the Alexandrov tangent cone at $p$. At points in $\text{Teich}(S)$ this is the usual tangent space at $p$.

We also define an inner product by

$$\langle b, c \rangle = |b| \cdot |c| \cos(\angle(b, c)).$$

**Theorem 5.2 (Wolpert, [Wol3])** Let $\tau = \{\gamma_1, \ldots, \gamma_k\}$ be a multicurve and assume that $p \in \mathcal{S}_\tau$. The the Alexandrov tangent cone at $p$ is

$$\mathbb{R}^{[\tau]}_\geq \times T_p\mathcal{S}_\tau$$

where the inner product is the product of the standard inner product on $\mathbb{R}^{[\tau]}$ and the Weil-Petersson inner product on $T_p\mathcal{S}_\tau$. Furthermore if $b(t)$ is a constant speed geodesic segment starting at $p$ and $\ell_{\gamma_i}(b(t)) = 0$ then the $i$th coordinate of $b$ in the tangent cone is zero.
Given a multicurve $\sigma$ let
\[
\pi_\sigma : \Teich(S) \to \mathcal{F}_\sigma
\]
be the nearest point projection.

**Lemma 5.3** Let $\sigma$ be a multicurve in $S$ and $p$ and $q$ points in $\Teich(S)$ with $p = \pi_\sigma(q)$. Then $p \in \mathcal{F}_\sigma$ where $\pi_\sigma$ is a (possibly trivial) extension of $\sigma$. Let $b(t)$ be a geodesic segment from $p$ to $q$. Then the image of $b$ in the tangent cone is orthogonal to $\mathbb{R}^{|\hat{\sigma} \setminus \sigma|} \times T_p \mathcal{F}_\sigma$.

**Proof:** Let $c : (-\varepsilon, \varepsilon) \to \mathcal{F}_\sigma \subset \overline{\mathcal{F}_\sigma}$ be a constant speed geodesic with $c(0) = p$. If we let $\bar{c}(t) = c(-t)$ then $\angle(c, \bar{c}) = \pi$. By (3) of [BH] Proposition 2.4 the angles $\angle(b, c)$ and $\angle(b, \bar{c})$ are at least $\pi/2$. Therefore they must be equal to $\pi/2$ and hence $b$ is orthogonal to $T_p \mathcal{F}_\sigma$. In particular, by Theorem 5.2, $b$ lies in $\mathbb{R}^{|\hat{\sigma}|}$.

Every vector in $\mathbb{R}^{|\hat{\sigma} \setminus \sigma|}$ is represented by a geodesic segment $c : [0, \varepsilon) \to \overline{\mathcal{F}_\sigma}$ with $c(0) = p$. In particular $d_{\text{WP}}(q, c(t)) > d_{\text{WP}}(q, p)$ for all $t \in (0, \varepsilon)$. As above, (3) of [BH] Proposition 2.4 implies that $\angle(b, c) \geq \pi/2$. However, as $b$ lies in $\mathbb{R}^{|\hat{\sigma}|}$, we must have that $\angle(b, c) = \pi/2$. \hfill \Box

**Proposition 5.4** Let $\tau$ and $\sigma$ be multicurves with $i(\tau, \sigma) = 0$ and let $\hat{\tau} = \tau \cup \sigma$. Then
\[
\pi_\sigma(\mathcal{F}_\tau) \subset \overline{\mathcal{F}_\tau}.
\]

**Proof:** Let $q$ be a point in $\mathcal{F}_\tau$ and $p = \pi_\sigma(q)$ and $r = \pi_\sigma(q)$ its nearest point projections to $\mathcal{F}_\tau$ and $\overline{\mathcal{F}_\sigma}$. By the previous lemma the angles of the triangle $qpr$ at $p$ and $r$ are $\pi/2$ so in the Euclidean comparison triangles the corresponding angles must be at least $\pi/2$. However, if $p \neq r$ then the angle at $q$ in the comparison triangle will be $> 0$, a contradiction. \hfill \Box

## 6 Topological properties of nearby strata

We now prove Theorem 1.2 which we first restate.

**Theorem 1.2** Let $\mathcal{F}_\sigma, \mathcal{F}_\tau$ be two strata in $\Teich(S)$. Then one of the following holds:

1. $i(\sigma, \tau) = 0$ and $d_{\text{WP}}(\mathcal{F}_\sigma, \mathcal{F}_\tau) = 0$.
2. $i(\sigma, \tau) = 1$ and $d_{\text{WP}}(\mathcal{F}_\sigma, \mathcal{F}_\tau) = \delta_{1,1}$.
3. $i(\sigma, \tau) > 1$ and $d_{\text{WP}}(\mathcal{F}_\sigma, \mathcal{F}_\tau) \geq 7.61138$.

**Proof:** If $i(\sigma, \tau) = 0$ then the closures of the strata intersect and therefore $d_{\text{WP}}(\mathcal{F}_\sigma, \mathcal{F}_\tau) = 0$.

Now assume that $i(\sigma, \tau) = k > 0$ and that for every $\alpha \in \sigma$ we have $i(\alpha, \tau) = 0$ or 1. Note that this implies that for every $\beta \in \tau$ then $i(\beta, \sigma) = 0$ or 1 and if
$i(\sigma, \tau) = 1$ this condition automatically holds. Then the surface filled by $\sigma$ and $\tau$
will be a collection of punctured tori and annuli. Let $\mu$ be a maximal multicurve
such that $i(\sigma, \mu) = i(\tau, \mu) = 0$. Then $S \setminus \mu$ will be the collection of $k$
punctured tori filled by $\sigma$ and $\tau$ along with a collection of thrice punctured spheres. If we
let $\hat{\sigma} = \sigma \cup \mu$ and $\hat{\tau} = \tau \cup \mu$ then by Theorem 5.1

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) = d_{\text{WP}}(\mathcal{I}_{\hat{\sigma}}, \mathcal{I}_{\hat{\tau}}).$$

The strata $\mathcal{I}_{\hat{\sigma}}$ and $\mathcal{I}_{\hat{\tau}}$ are both maximal and hence each are a single point. As
$\mu$ is a multicurve contained in both $\hat{\sigma}$ and $\hat{\tau}$, these strata are in the closure of
$\mathcal{I}_\mu$. Furthermore $\mathcal{I}_\mu$ is the product of $k$ copies of the Weil-Petersson completion
of the Teichmuller space of the punctured torus and when we project to each
factor the image of the strata $\mathcal{I}_\sigma$ and $\mathcal{I}_\tau$ are curves intersecting once. It follows that

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) = \sqrt{k\delta_{1,1}}.$$ 

Therefore if $i(\sigma, \tau) = 1$ we have

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) = \delta_{1,1}$$

and if $i(\sigma, \tau) = k \geq 2$ we have

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) \geq \sqrt{2}(6.54933) \approx 9.26215 > 7.61138.$$

Now we can assume, without loss of generality, that there is a curve $\alpha \in \sigma$
and curves $\beta_1$ and $\beta_2$ in $\tau$ (possibly with $\beta_1 = \beta_2$) and $i(\alpha, \beta_1 \cup \beta_2) \geq 2$. Let
c be any path from $\mathcal{I}_\sigma$ to $\mathcal{I}_\tau$ and choose $t_0$ such that at $c(t_0) = X$ we have
$
\max\{\ell_{\beta_1}(X), \ell_{\beta_2}(X)\} = 2\varepsilon_2$. Therefore the collars about $\beta_1$ and $\beta_2$ have length
at least $2\varepsilon_2$ and as $i(\alpha, \beta) = 2$ this implies $\ell_\alpha(X) \geq 4\varepsilon_2$. Then by Lemma 4.1

$$d_{\text{WP}}(X, \mathcal{I}_\sigma) \geq H(4\varepsilon_2) \text{ and } d_{\text{WP}}(X, \mathcal{I}_\tau) \geq H(2\varepsilon_2).$$

Thus

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) \geq H(4\varepsilon_2) + H(2\varepsilon_2) \geq 7.61138.$$ 

Thus $i(\sigma, \tau) > 1$ and $d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) \geq 7.61138$. \Box

**Topology of supporting surface**

If the subsurface $S(\sigma, \tau) \subset S$ filled by $\sigma$ and $\tau$ has $n$ non-annular components
then by the above

$$d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) \geq \sqrt{n\delta_{1,1}} > 9.26215.$$ 

Thus if $d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) < 9.26215$ then $S(\mu, \tau)$ has a single non-annular component.
Also by the above, if $d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) < 7.61138$ then the non-annular component
is a punctured torus with $i(\mu, \tau) = 1$ and in fact $d_{\text{WP}}(\mathcal{I}_\sigma, \mathcal{I}_\tau) = \delta_{1,1}$. 

15
Separating curves and punctured spheres

The above shows that for any finite type surface $\delta_{1,1}$ is a lower bound on the distance between strata in $\text{Teich}(S)$ whose closures do not intersect. Also it follows that it is attained for any $S$ with a non-separating curve. The only case left is the $n$-punctured sphere $S_{0,n}$ for $n \geq 4$. For completeness, we now consider the punctured sphere case.

In a punctured sphere every curve is non-separating so any two curves with non-trivial intersection will intersect an even number of times. In particular, on the 4-punctured sphere any two distinct curves intersect and the minimal intersection is two. In parallel with the punctured torus case, if $\alpha$ and $\beta$ are simple closed curves in $S_{0,4}$ with $i(\alpha, \beta) = 2$ we define

$$\delta_{0,4} = d_{\text{WP}}(\mathcal{F}_\alpha(S_{0,4}), \mathcal{F}_\beta(S_{0,4})).$$

We note that there is an canonical isomorphism between $\text{Teich}(S_{1,1})$ and $\text{Teich}(S_{0,4})$ and as the area of 4-punctured hyperbolic spheres is twice that of punctured tori this isomorphism scales the Weil-Petersson metric by $\sqrt{2}$. Two noded surfaces in $\text{Teich}(S_{1,1})$ whose nodes intersect once will be taken to noded surfaces in $\text{Teich}(S_{0,4})$ where the nodes intersect twice and therefore

$$\delta_{0,4} = \sqrt{2}\delta_{1,1}.$$

Therefore by the bounds on $\delta_{1,1}$ we have $\delta_{0,4} \in (9.26215, 9.38025)$.

The usual collar lemma states that if $\alpha$ is a simple closed geodesic in a complete hyperbolic surface $X$ then $\alpha$ has an embedded collar of width $r$ with $\sinh(r/2) = 1/\sinh(\ell_\alpha(X))$. If $\alpha$ is non-separating then this result is optimal: for any $\varepsilon > 0$ there is a hyperbolic structure $X$ (on any hyperbolizable surface $S$) such that $\alpha$ doesn’t have a collar of width $r + \varepsilon$. However, for non-separating curves this can be improved. While the proof is elementary we were unable to find a reference so we include one here. (See [Par] for a similar observation.)

**Lemma 6.1** Let $\alpha$ be a non-separating curve on a complete hyperbolic surface $X$. Then $\alpha$ has an embedded collar of width $r$ with

$$\sinh(\ell_\alpha(X)/4) \sinh(r/2) \geq 1.$$  

**Proof:** Let $\beta$ be the shortest non-trivial geodesic arc from $\alpha$ to itself. Then we can choose $r$ to be the length of $\beta$. As $\alpha$ is non-separating, $\beta$ starts and ends on the same side of $\alpha$. Therefore $\alpha$ and $\beta$ are supported on a pair of pants $P$ in $X$. We decompose $P$ into two isometric right-angled hexagons in the standard way by taking perpendiculars between boundary components of $P$. This hexagon has base of length $\ell_\alpha(X)/2$. We extend the sides of $H$ to geodesics in $\mathbb{H}^2$. The sides perpendicular to the base are distance $\ell_\alpha(X)/2$ apart and therefore are the opposite sides of an ideal quadrilateral $Q$ with the two other sides a distance $2\sinh^{-1}(\ell_\alpha(X)/4)$ apart (see Figure 2). The geodesic opposite the base geodesic
is separated from the base geodesic by a side of \( Q \). Therefore the distance from the base to the opposite geodesic is at least \( \sinh^{-1}(1/\sinh(\ell(X)/4)) \).

As \( \beta \) is the union of two geodesic arcs joining the base of \( H \) to its opposite side and \( r \) is the length of \( \beta \), we have

\[
r \geq 2 \sinh^{-1} \left( \frac{1}{\sinh(\ell(X)/4)} \right).
\]

\( \square \)

Figure 2: \( r > 2 \sinh^{-1}(1/\sinh(\ell(X)/4)) \)

In the usual collar lemma, the standard collars are disjoint. We emphasize that this does not hold for the collars we construct here.

Using the above we can improve our gradient bound for separating curves. We have

**Theorem 6.2** Let \( S \) be a finite type surface and \( \ell_\alpha \) be a geodesic length function for \( \alpha \) a simple separating curve on \( S \). Then for \( X \in \text{Teich}(S) \)

\[
\| \nabla \ell_\alpha(X) \|^2 \leq \frac{2\ell_\alpha(X)}{\pi} \left( 1 + F(\ell_\alpha(X)/2) \sinh^3(\ell_\alpha(X)/4) \right).
\]

Furthermore

\[
d_{WP}(\mathcal{F}_\alpha, \mathcal{F}_\alpha^L) \geq H_s(L)
\]

where

\[
H_s(L) = \int_0^L \frac{dx}{\sqrt{\frac{\pi}{2} (1 + F(x/2) \sinh^3(x/4))}}
\]

**Proof:** The proof is the same as in Theorem 1.4. The only difference is that the embedded neighborhood has width \( 2 \sinh^{-1}(1/\sinh(\ell(X)/4)) \) rather
than $2\sinh^{-1}(1/\sinh(\ell_\alpha(X)/2))$. Thus we can substitute $\ell_\alpha(X)/2$ into the lower bound in Corollary 3.4 to obtain the new lower bound. We note the linear factor arises from integrating in the $\alpha$ direction in the collar and therefore remains unchanged. The Weil-Petersson distance bound follows immediately as in Lemma 4.1.

We repeat the proof of Theorem 1.2 for the punctured sphere case.

**Theorem 6.3** Let $\mathcal{S}_\sigma(S), \mathcal{S}_\tau(S)$ be two strata in $\text{Tech}(S)$ for $S$ an $n$-punctured sphere. Then one of the following holds:

1. $i(\sigma, \tau) = 0$ and $d_{WP}(\mathcal{S}_\sigma(S), \mathcal{S}_\tau(S)) = 0$.
2. $i(\sigma, \tau) = 2$ and $d_{WP}(\mathcal{S}_\sigma(S), \mathcal{S}_\tau(S)) = \delta_{0.4}$.
3. $i(\sigma, \tau) > 2$ and $d_{WP}(\mathcal{S}_\sigma(S), \mathcal{S}_\tau(S)) > 10.09656$.

**Proof:** If $i(\sigma, \tau) = 0$ then the closures of the strata intersect and therefore $d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) = 0$.

Now assume that $i(\sigma, \tau) = 2k > 0$ and that for every $\alpha \in \sigma$ we have $i(\alpha, \tau) = 0$ or 2. Then by the same argument as in Theorem 1.2 we can decompose into 4-punctured spheres and get

$$d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) = \sqrt{k}\delta_{0.4}.$$  

Therefore if $i(\sigma, \tau) = 2$ we have

$$d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) = \delta_{0.4}$$

and if $i(\sigma, \tau) = 2k \geq 4$ we have

$$d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) \geq \sqrt{2}\delta_{0.4} \geq 13.09866 > 10.09656.$$  

Now we can assume one of the following:

- there is curve $\alpha \in \sigma$ and curve $\beta \in \tau$ with $i(\alpha, \beta) \geq 4$.
- there is curve $\alpha \in \sigma$ and curves $\beta_1, \beta_2 \in \tau$ and $i(\alpha, \beta_1) = i(\alpha, \beta_2) = 2$.

In the first case, we let $c$ be any path from $\mathcal{S}_\sigma$ to $\mathcal{T}$ and choose $t_0$ such that at $c(t_0) = X$ we have $\ell_{\beta}(X) = L$. Therefore by Lemma 6.1 above, $\alpha$ has an embedded collar of width $2\sinh^{-1}(1/\sinh(L/4))$. Therefore $\ell_{\alpha}(X) \geq 8\sinh^{-1}(1/\sinh(L/4))$. Then by Theorem 6.2

$$d_{WP}(\mathcal{S}_\sigma, \mathcal{T}) \geq H_s(L) + H_s \left(8\sinh^{-1}\left(\frac{1}{\sinh(L/4)}\right)\right) = W_1(L)$$

We choose $L = 3.678$ and evaluating we get

$$d_{WP}(\mathcal{S}_\sigma, \mathcal{T}) \geq W_1(3.678) \simeq 10.76596 > \delta_{0.4}.$$
In the second case, we choose \( t_0 \) such that at \( c(t_0) = X \) and \( L = \max(\ell_{\beta_1}(X), \ell_{\beta_2}(X)) \). Then \( \beta_1 \cup \beta_2 \) split \( \alpha \) into 4 geodesic arcs with endpoints in \( \beta_1 \cup \beta_2 \). Two of the arcs have endpoints in the same component of \( \beta_1 \cup \beta_2 \) and therefore by Lemma 6.1 are both of length at least \( 2\sinh^{-1}\left(1/\sinh(L/4)\right) \). The other two geodesic arcs have one endpoint in \( \beta_1 \) and another in \( \beta_2 \). Then using the fact that the collars about \( \beta_1, \beta_2 \) of width \( 2\sinh^{-1}(1/\sinh(L/2)) \) are disjoint we have each of these arcs are of length at least \( 2\sinh^{-1}(1/\sinh(L/2)) \). Thus

\[
\ell_\alpha(X) \geq 4\sinh^{-1}\left(\frac{1}{\sinh(L/4)}\right) + 4\sinh^{-1}\left(\frac{1}{\sinh(L/2)}\right).
\]

Thus

\[
d_{\text{WP}}(\mathcal{S}_\sigma, \mathcal{S}_\tau) \geq H_4(L) + H_2\left(4\sinh^{-1}\left(\frac{1}{\sinh(L/4)}\right) + 4\sinh^{-1}\left(\frac{1}{\sinh(L/2)}\right)\right) = W_2(L).
\]

We choose \( L = 2.42 \) and get

\[
d_{\text{WP}}(\mathcal{S}_\sigma, \mathcal{S}_\tau) \geq W_2(2.42) \simeq 10.09656 > \delta_{0.4}.
\]

\[\square\]

**Strata distances and gaps**

From the above, if \( S \) has positive genus then the minimal distance between strata \( \mathcal{S}_\sigma, \mathcal{S}_\tau \) with \( i(\sigma, \tau) \neq 0 \) is \( \delta_{1.1} \) and is achieved if and only if \( i(\sigma, \tau) = 1 \). Furthermore if \( i(\sigma, \tau) > 1 \) then the distance between the strata is at least \( H(4\epsilon_2) + H(2\epsilon_2) \). Therefore there is a gap in the distances from \( \delta_{1.1} \) to \( H(4\epsilon_2) + H(2\epsilon_2) \) of size

\[
H(4\epsilon_2) + H(2\epsilon_2) - 2K(2\epsilon) \geq 7.61138 - 6.63284 = 0.97854.
\]

Similarly if \( S \) is an \( n \)-punctured sphere with \( n \geq 4 \), then the minimal distance between strata \( \mathcal{S}_\sigma, \mathcal{S}_\tau \) with \( i(\sigma, \tau) \neq 0 \) is \( \delta_{0.4} \) and is achieved if and only if \( i(\sigma, \tau) = 2 \). Furthermore if \( i(\sigma, \tau) > 2 \) then the distance between the strata is at least \( W_2(2.42) \). Therefore there is a gap in the distances from \( \delta_{0.4} \) to \( W_2(2.42) \) of size

\[
W_2(2.42) - \sqrt{2}(2K(2\epsilon)) \geq 10.09656 - 9.38025 = 0.71631.
\]

**Appendix: Closed geodesics in the moduli space of the punctured torus**

Our methods can also be used to obtain lower bounds on the minimal Weil-Petersson translation length of a pseudo-Anosov mapping class acting on Teichmüller space. We demonstrate the method on the Teichmüller space of punctured tori. For a surface of higher complexity the basic idea will still work but it be harder to get explicit estimates.
Let $T$ be the punctured torus and

$$\psi: T \to T$$

a pseudo-Anosov mapping class. By [DW] there is a unique $\psi$-invariant geodesic $\gamma$ in the Weil-Petersson metric on $\text{Teich}(S_1,1)$. This will descend to a closed geodesic in the moduli space $\mathcal{M}_{1,1}$. We can use our estimates to give a lower bound on the length of the shortest such geodesic.

We identify $\text{Teich}(T)$ so that $\psi$ can be represented by an element of $\text{SL}_2(\mathbb{Z})$:

$$\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

We can conjugate $\psi$ so that the axis $\gamma$ crosses the imaginary axis at some punctured torus $X$. (This is equivalent to $b/c > 0$.) Then $X$ is rectangular: the $(1,0)$-curve and $(0,1)$-curve are represented by geodesics $\alpha$ and $\beta$ that meet orthogonally at a single point. A standard calculation shows that

$$\sinh(\ell_\gamma(X)/2) \sinh(\ell_\beta(X)/2) = 1.$$ 

One of these two curves will be the systole on $X$ (with the other the second shortest curve). In fact this is exactly the situation where the collar lemma is optimal: the width of the collar about $\alpha$ is $\ell_\beta(X)$. In a particular if $i(\alpha, \gamma) = k$ then

$$\ell_\gamma(X) \geq k\ell_\beta(X).$$

We have a similar statement when we switch the roles of $\alpha$ and $\beta$.

As $X$ lies on the axis $\gamma$ the translation length of $\psi$ is $d_{\text{WP}}(X, \psi(X))$. To bound this distance from below we observe that for any curve $\ell_{\psi(\gamma)}(X) = \ell_\gamma(\psi(X))$. We assume that $\alpha$ is the shortest curve.

If $i(\alpha, \psi(\alpha)) \geq 2$ then

$$\ell_\alpha(X) \leq 2\varepsilon_2 \text{ and } \ell_{\psi(\alpha)}(X) \geq 2 \cdot 2\varepsilon_2$$

so by Lemma 4.1

$$d_{\text{WP}}(X, \psi(X)) \geq d_{\text{WP}}(\mathcal{S}_{2\varepsilon_2}^2, \mathcal{S}_{4\varepsilon_2}^4) \geq H(4\varepsilon_2) - H(2\varepsilon_2) \geq 1.06205$$

It follows that for $\psi$ with $i(\alpha, \psi(\alpha)) \geq 2$ then

$$\|\psi\|_{\text{WP}} \geq 1.06205.$$ 

Otherwise as $\psi(\alpha) = (a, c)$ then $|c| = 1$ and $\psi^2(\alpha) = (a^2 + bc, c(a + d))$. As $|a + d| > 2$ then $i(\alpha, \psi^2(\alpha)) = |c(a + d)| = |a + d| \geq 3$ and

$$\ell_\alpha(X) \leq 2\varepsilon_2 \text{ and } \ell_{\psi(\alpha)}(X) \geq 3 \cdot 2\varepsilon_2.$$
Therefore
\[
d_{\text{WP}}(X, \psi^2(X)) \geq d_{\text{WP}}(\mathcal{S}_2^{2\varepsilon_2}, \mathcal{S}_2^{6\varepsilon_2}) \\
\geq H(6\varepsilon_2) - H(2\varepsilon_2) \\
\geq 1.56949
\]

Therefore in general
\[
\|\psi\|_{\text{WP}} \geq \frac{1.56949}{2} \geq .78474
\]

In \([BB]\), the second author and Brock give a lower bound on the systole for of \(\mathcal{M}_{g,n}\) using renormalized volume and the lower bound for the volume of a hyperbolic 3-manifold. They prove

**Theorem 6.4 (Brock-Bromberg, \([BB]\))** Let \(\gamma\) be a closed geodesic for the Weil-Petersson metric on moduli space \(\mathcal{M}_{g,n}\) of the surface \(S_{g,n}\) with \(n > 0\). Then
\[
\ell_{\text{WP}}(\gamma) \geq \frac{4V_3}{3\sqrt{\text{Area}(S_{g,n})}}
\]
where \(V_3\) is the volume of the regular ideal hyperbolic tetrahedron.

We note that for \(\mathcal{M}_{1,1}\), the above theorem gives a bound of \(.53724\) and our bound is \(.78474\). While a more refined analysis could improve this bound, it seems unlikely that these estimates are close to optimal so we do not include them.

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